On sums of powers of cosecs

J.S. Dowker

Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England

The finite sums of powers of cosecs occur in numerous situations, both physical and mathematical, examples being the Casimir effect, Rényi entropy, Verlinde’s formula and Dedekind sums. I here present some further discussion which consists mainly of a reprise of early work by H.M. Jeffery in 1862-64 which has fallen by the wayside and whose results are being reproduced up to the present day. The motivation is partly historical justice and partly that, because of the continuing appearance of the sums, his particular methods deserve re-exposure. For example, simple trigonometric generating functions are found and these have a field theoretic, Green function significance and I make a few comments in the topic of Rényi entropies.
1. Introduction

The finite summations of powers of cosecs occur in a number of different areas which, mathematically, are associated with image constructions, in one way or another, and involve the regular subdivisions or discretisations of the circle, a very old topic.

Relatively recent papers that detail some of the history behind these, and like trigonometric summations, are Berndt and Yeap, [1], and Cvijović and Srivastava, [2,3] and I refer to them for motivation, both physical and mathematical. I will not repeat the references in these useful works, unless they are directly relevant, but I might add a few more. These summations continue to appear in the physics literature.

My intention here to give some historical and calculational details which I hope will be interesting and/or useful. I have elaborated the algebra because the main reference I employ (Jeffery, [4]) is, perhaps, a little obscure and certainly unknown. Also the often quite simple methods have application today. I have also included an example that is not in [4] as a slight novelty as it combines bosonic and fermionic elements.

The next section introduces the sums in their general form and a (known) computable answer. Section 3 starts with Jeffery’s generating function for the simplest, untwisted sum for which an explicit expression is thence found.

2. The summations

The summations in question are the modulated cosec sums,

$$C_\nu(n, w) \equiv \sum_{l=1}^{n-1} \cos \left( \frac{2\pi wl}{n} \right) \cosec^{2\nu} \left( \frac{\pi l}{n} \right), \quad \nu \in \mathbb{Z},$$

(1)

with twisting $w \in \mathbb{Z}$, In fact here I will deal mostly with the untwisted case $w = 0$

The sums were evaluated in terms of generalised Bernoulli polynomials by a contour method, [5] and [6],

$$C_\nu(n, w) = \frac{2^{2\nu}}{(2\nu)!} B_{2\nu+1}^{(2\nu+1)}(w + \nu \mid n, 1),$$

(2)

which can be expanded as a polynomial in $n$.

This could be taken as the final answer, and it is, but it is a little awkward to expand hence I now pass to another expression, by a different approach, which yields a form amenable to hand calculation.
3. Jeffery’s generating function

In fairly recent times generating functions for the (untwisted) summations have been given by Fisher, [7], and Zagier, [8], (and, more generally, for the twisted ones in [6]) but the main point I wish to make in the present paper is that a much earlier version was derived by Jeffery, [4], in his discussion of certain classes of integrals associated with the derivatives of the Gamma function.

Amongst other things, Jeffery derives the following result in §36, \(^2\)

\[
\frac{\pi}{2n} \sum_{l=1}^{n-1} \left( \cot \frac{\pi}{n}(l-x) - \cot \frac{\pi}{n}(l+x) \right) = \int_0^1 dy \frac{1 + y + \ldots + y^{n-2}}{1 - y^n} (y^{-x} - y^x) 
\]

\[
= \int_0^1 dy \left( \frac{1}{1-y} - \frac{y^{n-1}}{1-y^n} \right) (y^{-x} - y^x) = \int_0^1 dy \left( \frac{y^{-x} - y^x}{1-y} - \frac{y^{-x/n} - y^{x/n}}{n(1-y)} \right) 
\]

\[
= 2 \left( 1 - \frac{1}{n^2} \right) \zeta(2) x + 2 \left( 1 - \frac{1}{n^4} \right) \zeta(4) x^3 + \ldots 
\]

\[
= \frac{\pi}{n} \cot \frac{\pi x}{n} - \pi \cot \pi x , \tag{3}
\]

where \(n\) can be even or odd.

The third line results from expanding \(y^x\) and then using,

\[
\int_0^1 dt \log^q \frac{1}{1-t} = \sum_{p=0}^{\infty} \int_0^1 dt \ t^q \log^p t = (-1)^q q! \sum_{p=0}^{\infty} \frac{1}{(p+1)^{q+1}} 
\]

\[
= (-1)^q q! \zeta(q + 1) .
\]

I amplify Jeffery’s derivation of the first line of the identity, (3), which he does not give in detail. I will assume that any formula in (9) is known, although Jeffery derives many of them anew. Substitution of the integral,

\[
\psi(z) = \int_0^1 dt \frac{1-t^{z-1}}{1-t} + \Gamma'(1) ,
\]

into the standard result,

\[
-\pi \cot \pi z = \psi(z) - \psi(1 - z)
\]

gives

\[
-\pi \cot \pi z = \int_0^1 dt \frac{-t^{z-1} + t^{-z}}{1-t} . \tag{4}
\]

\(^2\) I enlarge on his algebra. Be aware that there are a number of bad misprints in this tersely written, disjointed, but very interesting paper.
Therefore, \( z = (l \pm x)/n \),

\[
\pi \cot \frac{\pi}{n}(l - x) - \pi \cot \frac{\pi}{n}(l + x) = \int_0^1 dt \frac{-t^{(l+x)/n-1} + t^{-(l+x)/n} + t^{(l-x)/n-1} - t^{-(l-x)/n}}{1 - t} = n \int_0^1 dy \frac{(y^{n-l-1} + y^{l-1})(y^{-x} - y^x)}{1 - y^n},
\]

where \( t = y^{-n} \).

Hence, writing out the sum at length,

\[
\pi \sum_{l=1}^{n-1} \left( \cot \frac{\pi}{n}(l - x) - \cot \frac{\pi}{n}(l + x) \right) = 2n \int_0^1 dy \frac{1 + y + \ldots + y^{n-2}}{1 - y^n} (y^{-x} - y^x)
\]

which is the first line in (3).

Equation (3) constitutes a generating function and is used as such by Jeffery to evaluate the sums \( C_\nu(m, 0) \) by finding the (odd) derivatives with respect to \( x \) at 0.

One might as well take the first derivative at once to give,

\[
\frac{1}{2} \left( \frac{\pi}{n} \right)^2 \sum_{l=1}^{n-1} \left( \cosec^2 \frac{\pi}{n}(l - x) + \cosec^2 \frac{\pi}{n}(l + x) \right) = 2 \left( 1 - \frac{1}{n^2} \right) \zeta(2) + 2.3 \left( 1 - \frac{1}{n^4} \right) \zeta(4) x^2 + 2.5 \left( 1 - \frac{1}{n^6} \right) \zeta(6) x^4 + \ldots
\]

\[
= \left( \pi^2 \cosec^2 \frac{\pi x}{n} - \frac{\pi^2}{n^2} \cosec^2 \frac{\pi x}{n} \right).
\]

Setting \( x \) to zero yields the easiest sum,

\[
C_1(n, 0) = \frac{1}{2} \sum_{l=1}^{n-1} \cosec^2 \left( \frac{l\pi}{n} \right) = \frac{n^2 - 1}{\pi^2} \zeta(2) = \frac{n^2 - 1}{6}.
\]

Jeffery’s paper is the earliest to which I can trace this result.

Equation (5) is used in the following way. The sums \( C_2(n, 0) \) and \( C_4(n, 0) \) are given explicitly in [4] as further examples and the construction of the numerical coefficients exhibited there indicates that [4] has employed, without comment, the
recursion formula, \(^3\)

\[
cosec^\nu y = \frac{1}{(\nu - 1)(\nu - 2)} (D_z^2 + (\nu - 2)^2) cosec^{\nu-2}(y + z) \bigg|_{z=0}, \quad \nu > 2, \quad (6)
\]

for even \(\nu\), and \(D_z = \partial / \partial z\), which must have been common knowledge, probably going back to Euler. Much later references are Ely, [13], and Saalschütz, [14].

The complete iteration of this yields,

\[
cosec^{2p} y = \frac{1}{(2p - 1)!} (D_z^2 + (2p - 2)^2) \ldots (D_z^2 + 2^2) cosec^2(y + z) \bigg|_{z=0}
= \frac{1}{(2p - 1)!} \sum_{i=0}^{p-1} U_i^p D_z^{2i} cosec^2(y + z) \bigg|_{z=0}, \quad (7)
\]

so that equation (5) can now be employed to give the required even derivatives at zero,

\[
D_z^{2i} \sum_{l=1}^{n-1} cosec^2\left(\frac{\pi l}{n} \pm z\right) \bigg|_{z=0} = 2 (2i + 1)! \left( n^{2i+2} - 1 \right) \frac{\zeta(2i+2)}{\pi^{2i+2}}.
\]

Therefore, finally, setting \(y = \pi l/n\) in (7) and summing,

\[
\frac{1}{2} \sum_{l=1}^{n-1} cosec^{2p}\left(\frac{\pi l}{n}\right) = \frac{2^{2p-2}}{(2p - 1)!} \sum_{i=0}^{p-1} \frac{(2i + 1)!}{2^{2i}} W_i^p \left( n^{2i+2} - 1 \right) \frac{\zeta(2i+2)}{\pi^{2i+2}}, \quad (8)
\]

which I will refer to as Jeffery’s form. The constants \(U_i^p\) equal the sum of the products of the squared first \(p - 1\) integers (excluding zero) taken \(p - 1 - i\) at a time, but, for later numerical convenience, I have extracted a factor of a power of two to give the more usual constants, \(W_i^p\).

A quite recent paper, [15], has again considered these sums and reaches the same expression as (8) by a similar method. It appears also in a recent work, [16].

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\(^3\) It is interesting to note that this recursion is an early manifestation of Hadamard’s technique, of increasing the dimension of the manifold by two by applying an intertwining operator to a propagation quantity. Hadamard worked in flat space. An application to spheres was made by ourselves, [10], and in a relevant conical context in [11]. See also [12].
4. The calculation of the coefficients

The combinatorial expression for the coefficients, $U_i^p$, directly applied, is satisfactory for a numerical hand calculation of the lower summations (Jeffery stops at $p = 4$) but is arduous for high $p$. For this reason consider the product occurring in (7). It is clear that the coefficient of $D^{2i}$ equals $2^{2p-2-2i}$ times the coefficient of $x^{2i}$, up to a sign, in the product

$$(x^2 - (p - 1)^2) \cdots (x^2 - 1) = x^{[2p]-2}$$

where $x^{[2p]}$ is an even central factorial, see e.g. Steffensen, [17]. I have called this coefficient, $W_i^p$.

The expansion of the central factorial in terms of the central factorial coefficients (essentially just a Taylor expansion),

$$x^{[2p]-2} = \sum_{\nu=1}^{p} \frac{D^{2\nu} 0^{[2p]}}{(2\nu)!} x^{2\nu - 2},$$

is here expressed as central derivatives of nothing. These satisfy a recursion which allows machine calculation. This is, [17],

$$\frac{D^{2\nu} 0^{[2p+2]}}{(2\nu)!} = \frac{D^{2\nu-2} 0^{[2p]}}{(2\nu-2)!} - p^2 \frac{D^{2\nu} 0^{[2p]}}{(2\nu)!}$$

with the starting values

$$\frac{D^{2} 0^{[2p]}}{(2)!} = (-1)^{p-1} ((k - 1)!)^2; \quad \frac{D^{2p} 0^{[2p]}}{(2p)!} = 1.$$

Hence the coefficients in (7) are,

$$W_i^p = (-1)^{p-1-i} \frac{D^{2\nu} 0^{[2p]}}{(2\nu)!}. \quad (9)$$

Early tabulations can be found in Steffensen, [17], and Thiele, [19] p.35 sufficient to reach $p = 8$. Later tabulations exist (e.g. in [20]). Table 1 in the more recent discussion of these cosec sums by Grabner and Prodinger, [16], is the same as the one in Thiele and equivalent to that in [15].

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4 Jeffery, [18], gives a symbolic form for the combinatorial description but it is expressed in terms of forward differences and is more complicated.
I table some values taken from Thiele,

\[
\begin{array}{cccccc}
 p & 1 & 2 & 3 & 4 & 5 \\
 i & 1 & 1 & 4 & 36 & 576 \\
 & 1 & 5 & 49 & 820 & 1 \\
 & 1 & 14 & 273 & 1 \\
 & 1 & 30 & 3 \\
 & 1 & 4 \\
\end{array}
\]

Substitution of these numbers into the answer, (8), produces agreement with existing results and, of course, with (2). There is no need to write out any specific cases.

5. Jeffery’s twisted generating function

Jeffery also gives expressions for the (fermionic) summations,

\[
-C_{m+1}(n, \frac{n-1}{2}) = \sum_{l=1}^{n-1} (-1)^l \cos \left( \frac{\pi l}{n} \right) \csc^{2m+2} \left( \frac{\pi l}{n} \right)
\]

\[
= \sum_{l=1}^{n-1} (-1)^l \cot \left( \frac{\pi l}{n} \right) \csc^{2m+1} \left( \frac{\pi l}{n} \right),
\]

which, again, he derives from a generating function,

\[
\frac{\pi}{2n} \sum_{l=1}^{n-1} (-1)^{l+1} \left( \csc \frac{\pi}{n} (l-x) - \csc \frac{\pi}{n} (l+x) \right) = \pi \csc \frac{\pi x}{n} - \frac{\pi}{n} \csc \frac{\pi x}{n},
\]

if \( n \) is odd.

The proof proceeds as for the untwisted case and begins with the representation,

\[
\pi \csc \pi x = \int_0^1 dt \frac{t^{x-1} + t^{-x}}{t + 1} = \frac{1}{x} + \int_0^1 dt \frac{t^x - t^{-x}}{t + 1}, \quad |x| < 1,
\]

which follows by differentiating Kummer’s integral, (re–proved by Jeffery),

\[
\log \tan \frac{\pi x}{2} = \int_0^1 dy \frac{y^{x-1} - y^{-x}}{(1 + y) \log y}.
\]

\footnote{I give with elaborations the, sometimes elementary, algebra because the reference may not be available. The journal can actually be found in the Göttinger Digitalisierungszentrum.}
Hence,
\[
\pi \csc \frac{\pi}{n} (l \pm x) = \int_0^1 dt \frac{t^{(l \pm x)/n - 1} - t^{-(l \pm x)/n}}{t + 1}, \quad |x| < 1,
\]
if \(l\) lies between 0 and \(n\).

Set \(t = y^{-n}\) then the integral leads to,
\[
\frac{\pi}{n} \left( \csc \frac{\pi}{n} (l - x) - \csc \frac{\pi}{n} (l + x) \right) = 
- \int_0^1 dy \frac{y^{l-x-1+n} - y^{-l-x-1} - y^{-l-x-1+n} - y^{l+x-1}}{1 + y^n}
+ \int_0^1 dy \frac{(y^{-l-1+n} - y^{-l-1})(y^{-x} - y^x)}{1 + y^n}.
\]

This allows the summation to be done.

Now put in the values of \(l = 1, 2, \ldots, n - 1\) in (11) with the correct parity. The numerator of (13) contains \((n \text{ is odd})\),
\[
y^{n-2} - y^{n-3} + \ldots - y + 1 + 1 - y + y^2 - \ldots + y^{n-2},
\]
so that (13) becomes,
\[
\frac{\pi}{2n} \sum_{l=1}^{n-1} (-1)^{l+1} \left( \csc \frac{\pi}{n} (l - x) - \csc \frac{\pi}{n} (l + x) \right) = 
\int_0^1 dy \frac{(1 - y + y^2 - \ldots + y^{n-2})(y^{-x} - y^x)}{1 + y^n}
+ \int_0^1 dy \frac{1}{1 + y} \left[ \frac{1}{1 + y} - \frac{1}{1 + y^n} \right] (y^{-x} - y^x)
= \int_0^1 dy \frac{y^{-x} - y^x}{1 + y} \left[ \frac{1}{1 + y/n} - \frac{1}{1 + y} \right]
= \pi \csc \pi x - \frac{\pi}{n} \csc \frac{\pi x}{n}
\]
after using the second representation in (12) for \(\csc\). We have proved (11).

The procedure is exactly as previously. The power series of the right–hand side of equation (11) is obtained by expanding \(y^t\) and using,
\[
\int_0^1 dt \frac{\log^q t}{1 + t} = \sum_{p=0}^\infty \int_0^1 dt (-1)^p t^p \log^q t = (-1)^q q! \sum_{p=0}^\infty (-1)^p \frac{1}{(p + 1)^{q+1}}
= (-1)^q q! \eta(q + 1),
\]
where \( \eta \) is the Dirichlet \( \eta \)–function, related to the Riemann \( \zeta \)–function by \( \eta(q+1) = (1 - 2^{-q}) \zeta(q + 1) \). (Note that the trigonometric closed form of the generating function has not actually been used.)

This results in,

\[
\frac{\pi}{2n} \sum_{l=1}^{n-1} (-1)^{l+1} \left( \csc \frac{\pi}{n}(l - x) - \csc \frac{\pi}{n}(l + x) \right) = 2 \sum_{j=1}^{\infty} \left( 1 - \frac{1}{n^{2j}} \right) \eta(2j) x^{2j-1},
\]

which I differentiate with respect to \( x \) to get,

\[
\frac{1}{2} \left( \frac{\pi}{n} \right)^2 \sum_{l=1}^{n-1} (-1)^{l+1} \left( \cot \frac{\pi}{n}(l - x) \csc \frac{\pi}{n}(l - x) + \cot \frac{\pi}{n}(l + x) \csc \frac{\pi}{n}(l + x) \right) = 2 \sum_{j=1}^{\infty} (2j - 1) \left( 1 - \frac{1}{n^{2j}} \right) \eta(2j) x^{2j-2},
\]

For the left–hand side, note that

\[
D_z \csc^{2m+1}(y + z) = -(2m + 1) \csc^{2m+1}(y + x) \cot(y + z)
\]

and the iteration for \( \csc^{2m+1} \cot \) is determined by that for \( \csc^{2m+1} \), which is,

\[
\csc^{2m+1} y = \pm \frac{1}{(2m)!} \left( D_z^2 + (2m - 1)^2 \right) \ldots \left( D_z^2 + 1 \right) \csc(y \pm z) \bigg|_{z=0}
\]

and can again be expanded in terms of central factorial numbers. I write out the expansion in the form,

\[
\csc^{2m+1} y = \pm \frac{1}{(2m)!} \sum_{i=0}^{m} V_i^m D_z^{2i} \csc(y \pm z) \bigg|_{z=0}
\]

Differentiating (18) with respect to \( y \), using (16), setting \( y = \pi l/n \), \( z = \pm \pi x/n \) and taking the sum,

\[
\sum_{l=1}^{n-1} (-1)^{l+1} \cot \frac{\pi l}{n} \csc^{2m+1} \frac{\pi l}{n}
\]

\[
= \frac{1}{(2m+1)!} \sum_{i=0}^{m} V_i^m \left( \frac{n}{\pi} \right)^{2i} D_x^{2i} \sum_{l=1}^{n-1} (-1)^{l+1} \left( \cot \frac{\pi}{n}(l \pm x) \csc \frac{\pi}{n}(l \pm x) \right) \bigg|_{x=0}
\]
Substituting in the derivatives at $x = 0$, which can be read off from (11), delivers the final result, for $n$ odd,

$$\sum_{l=1}^{n-1} (-1)^{l+1} \cot \frac{\pi l}{n} \csc^{2m+1} \frac{\pi l}{n} = \frac{2}{(2m+1)!} \sum_{i=0}^{m} (2i+1)! \ V_i^m \left( n^{2i+2} - 1 \right) \frac{\eta(2i+2)}{\pi^{2i+2}},$$

which should be compared with the bosonic sum, (8). One notices now the appearance of the Dirichlet function, typical for fermionic quantities.

From the form of the full iteration, (17), the coefficients, $V_i^m$, are given by the sums of products of the squares of the odd natural numbers, and this is how Jeffery, [4] §41, computes them. Another way is to employ a recursion relation but the easiest option is to use Thiele, [19], p.36 who conveniently tabulates them as the positive integers they are. These are sufficient to reach $m = 8$.

For the convenience of the reader I lift a few lower values from [19],

| $m$ | 0   | 1   | 2   | 3   | 4   |
|-----|-----|-----|-----|-----|-----|
| $i$ |     | 1   | 9   | 225 | 11025 | 0  |
|     | 1   | 10  | 259 | 12916 | 1  |
|     | 1   | 35  | 1974 | 2  |
|     | 1   | 84  | 3   |
|     | 1   | 4   |

The tables in Steffensen, [17], and in [20] can also be consulted.

As an example, Jeffery writes out the specific case $m = 2$ and I here repeat it, putting in the numerical values of the coefficients to give an explicit polynomial,

$$\sum_{l=1}^{n-1} (-1)^{l+1} \cot \frac{\pi l}{n} \csc^{5} \frac{\pi l}{n} = \frac{1}{80} (n^2 - 1) + \frac{7}{720} (n^4 - 1) + \frac{31}{15120} (n^6 - 1),$$

if $n$ is odd.

6. A mixed summation of alternating cosecs

To ring the changes I compute, using Jeffery’s method, the alternating sum

$$-C_\nu(n, n/2) \equiv \sum_{i=1}^{n-1} (-1)^{l+1} \csc^{2\nu} \left( \frac{\pi l}{n} \right), \quad \nu \in \mathbb{Z},$$

(20)
for even \( n \), which is neither bosonic nor fermionic. The algebra is a mixture of that in sections 3 and 5 and starts with the representation (4) for the cotan, so that

\[
\frac{\pi}{2n} \sum_{l=1}^{n-1} (-1)^{l+1} \left( \cot \frac{\pi}{n}(l-x) - \cot \frac{\pi}{n}(l+x) \right) 
\]

\[
= \int_0^1 dy \frac{1-y+y^2-\ldots+y^{n-2}}{1-y^n}(y^{-x}-y^x) 
\]

\[
= \int_0^1 dy \left( \frac{1}{1+y} + \frac{y^{-n}}{1-y^n} \right)(y^{-x}-y^x) = \int_0^1 dy \left( \frac{y^{-x}-y^x}{1+y} + \frac{y^{-x/n}-y^{x/n}}{n(1-y)} \right) 
\]

\[
= 2 \left( \eta(2) + \frac{\zeta(2)}{n^2} \right)x + 2 \left( \eta(4) + \frac{\zeta(4)}{n^4} \right)x^3 + \ldots 
\]

\[
= \pi \csc \pi x \frac{\pi}{n} - \cot \frac{\pi x}{n} . 
\]

(21)

Differentiating with respect to \( x \),

\[
\frac{1}{2} \left( \frac{\pi}{n} \right)^2 \sum_{l=1}^{n-1} (-1)^{l+1} \left( \csc \frac{2\pi}{n}(l-x) + \csc \frac{2\pi}{n}(l+x) \right) 
\]

\[
= 2 \left( \eta(2) + \frac{1}{n^2}\zeta(2) \right) + 2.3 \left( \eta(4) + \frac{1}{n^4}\zeta(4) \right)x^2 + 2.5 \left( \eta(6) + \frac{1}{n^6}\zeta(6) \right)x^4 + \ldots 
\]

(22)

Again, the recursion for \( \csc 2\pi x \) is brought in and gives, as before, for \( n \) even,

\[
\frac{1}{2} \sum_{l=1}^{n-1} (-1)^{l+1} \csc \frac{2\pi}{n} \frac{\pi l}{n} 
\]

\[
= \frac{2^{2p-2}}{(2p+1)!} \sum_{i=0}^p \frac{(2i+1)!}{2^{2i}} W_i^p \left( n^{2i+2}\eta(2i+2) + \zeta(2i+2) \right) \frac{1}{\pi^{2i+2}} , 
\]

(23)

which has both bosonic and fermionic aspects. The purely fermionic summation, (10), contains an extra ‘phase factor’ which is just the trace of the spin-1/2 rotation matrix through angle \( 2\pi/n \) which is necessary because of the rotation of the zwei-beine, [21].

Formula (23) is easily programmed. Numerical examples agree with the general expression, (2). Chu and Marini have computed some examples, [22] p.149, using a method somewhat less convenient than the one employed here, and I have found agreement.
7. Comments, connections and conclusion

As mentioned in the Introduction, the polynomials crop up in several seemingly disparate topics, examples being statistical mechanics, vector bundle theory, Dedekind sums, interpolation, quantum field theory on multiply connected manifolds, such as lens spaces, or on orbifolds with conical structures, real or artificial.

In the latter category is the construction of the entanglement and Rényi entropies using the replica technique which introduces a conical singularity into a codimension 2 submanifold. This process is usually accomplished by using integer coverings but can equally well be done with images as is well known. In this connection, Cardy, [23], has also recently recalculated the classic summations, $C_1(n, 0)$ and $C_2(n, 0)$. Furthermore in a related subject, Herzog and Nian, [24], have employed the recursion (6).

In this regard it is interesting to note that the simple rewriting of the generating function (5),

$$\sum_{l=0}^{n-1} \csc^2 \frac{\pi}{n} (l \pm x) = n^2 \csc^2 2\pi x,$$

(24)
gives the Green function in four dimensions as in [24] eqn.(33), to a constant factor.

More general summations (in higher dimensions) of the same form appear in the appendix to [6].

It should be remarked that, for the requirements of the present paper, the construction of generating functions such as (3) is somewhat of an afterthought. Only the, prior derived, power series are used.

The relation between coverings and images can be illustrated by Lubbock’s summation formulae, [17], in interpolation theory as described in [26]. All I point out here is that the Lubbock approximation involves a series of polynomials which, in one variant, are shown, [26], by contours, to be given by,

$$P_{2\nu}(n) = \frac{1}{(2\nu)!} \frac{1}{n} B^{(2\nu+1)}_{2\nu} (\nu + (n - 1)/2 | n, 1),$$

in terms of generalised Bernoulli polynomials, or,

$$P_{2\nu}(n) = \frac{1}{2^{2\nu} n} C_{\nu} (n, (n - 1)/2),$$

(25)

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6 This is a by now standard identity, [25] p.211.
in terms of the fermionic summations. Here, \( n \) would have to be an integer for the sum to make sense. After the summation has been effected, \( n \) could be anything and in the context of Lubbock’s formula \( n \) is set equal to \( 1/h \) with \( h \) integral providing the alternative form (see Steffensen, [17]) in terms of central factorials,

\[
P_{2\nu}(1/h) = \frac{1}{(2\nu)!} \sum_{j=-(h-1)/2}^{(h-1)/2} \binom{1}{j}^{2\nu}\]

where the sum is over \( j \) in steps of one.

Steffensen lists a few examples of the \( P \). These check with (25) which is no surprise in view of the image relation,

\[
\sum_{s=0}^{h-1} B^{(n+1)}(a + \frac{s}{h} | 1) = h B^{(n+1)}(a | \frac{1}{h}, 1). \tag{26}
\]

It might be helpful to spell things out. The left-hand side is the pre-image sum giving the quantity (here \( B \)) on the (multiply connected) circle of circumference \( 1/h \) in terms of that on a (bigger) covering circle of unit circumference consisting of \( h \) copies of the smaller circle in the form of circumference intervals of length \( 1/h \) with interval ends. This can be referred to as a subdivision of the bigger circle.

By contrast, an integer \( n \)-cover of the unit circle means \( n \) copies of the unrolled unit circle joined end to end and the two boundary points identified giving a wrapped up circle of circumference \( h \). The left-hand side of the sum, (26), then does not make sense but the right-hand side does.

When embedded in the plane, all circles have the same radius and give conical singularities with positive and negative deficit angles for subdivisions and integer coverings respectively.

The specific summations considered here all fall into the general twisted class, (1), and if one is looking for just a numerical polynomial then the general expression, (2), in terms of Bernoulli polynomials provides it. Jeffery’s forms, (8) and (19) (and (23) ) arrange these polynomials in a different, more explicit manner. This arrangement has advantages when considering the heat-kernel coefficients in the presence of conical singularities, [27].

Everything that has been given can be transcribed by replacing cot by tan and cosec by sec because sec and cosec obey the same recursion. (Some adjustment of the summation is needed.)

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7 Another Lubbock variant gives polynomials, \( Q \), which are the untwisted sums.
8 The term ‘covering’ is meant in the projection sense.
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