Optimal Multi-Modes Switching Problem in Infinite Horizon

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Abstract

This paper studies the problem of the deterministic version of the Verification Theorem for the optimal $m$-states switching in infinite horizon under Markovian framework with arbitrary switching cost functions. The problem is formulated as an extended impulse control problem and solved by means of probabilistic tools such as the Snell envelop of processes and reflected backward stochastic differential equations. A viscosity solutions approach is employed to carry out a fine analysis on the associated system of $m$ variational inequalities with inter-connected obstacles. We show that the vector of value functions of the optimal problem is the unique viscosity solution to the system. This problem is in relation with the valuation of firms in a financial market.

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1 Introduction

First let us deal with an example in order to introduce the problem we consider in this paper:

Assume we have a power station/plant which produces electricity and which has several modes of production, e.g., the lower, the middle and the intensive modes. The price of electricity in the market, given by an adapted stochastic process $(X_t)_{t \geq 0}$, fluctuates in reaction to many factors such as demand level, weather conditions, unexpected outages and so on. On the other hand, electricity is non-storable, once produced it should be almost immediately consumed. Therefore, as a consequence, the station produces electricity in its instantaneous most profitable mode known that when the plant is in mode $i \in \mathcal{I}$, the yield per unit time $dt$ is given by means of $\psi_i(X_t)dt$ and, on the other hand, switching the plant from the mode $i$ to the mode $j$ is not free and generates expenditures given by $g_{ij}(X_t)$ and

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possibly by other factors in the energy market. The switching from one regime to another one is realized sequentially at random times which are part of the decisions. So the manager of the power plant faces two main issues:

(i) when should she decide to switch the production from its current mode to another one?

(ii) to which mode the production has to be switched when the decision of switching is made?

In other words she faces the issue of finding the optimal strategy of management of the plant. This issue is in relation with the price of the power plant in the energy market.

Optimal switching problems for stochastic systems were studied by several authors (see e.g. [1, 2, 3, 4, 9, 10, 11, 12, 13, 17, 20, 23, 24] and the references therein). The motivations are mainly related to decision making in the economic sphere. Several variants of the problem we deal with here, including finite and infinite horizons, have been considered during the recent years. In order to tackle those problems, authors use mainly two approaches. Either a probabilistic one [10, 11, 17] or an approach which uses partial differential inequalities (PDIs for short) [1, 2, 4, 12, 20, 24, 23].

In the finite horizon framework Djehiche et al. have studied the multi-modes switching problem in using probabilistic tools. For general stochastic processes, they have shown that a value of the problem and an optimal strategy exits. The partial differential equation version of this work has been carried out by El-Asri and Hamadène [13]. They showed that when the price process $X_t$ is solution of a Markovian standard differential equation, then with this problem is associated a system of variational inequalities with interconnected obstacles for which they provide a solution in viscosity sense. This solution is bind to the value function of the problem. The solution of the system is unique.

In the case when the horizon is infinite, there still much to do and this is the novelty of this paper. Actually, authors treat mainly the case when the price process $X_t$ is of Markovian Itô type, the switching costs are deterministic functions of time $t$ and the profit functions are deterministic functions of $(t, X_t)$ and have linear growth at most (see e.g. [1, 2, 12, 20, 24]). Therefore the main objective of this paper is to fill in the gap between finite and infinite horizon by providing a complete treatment of the optimal multiple switching problem in infinite horizon when the price is only a continuous process. This is what we did in the first part of this paper. Actually inspired by the work of Djehiche et al. [11], using probabilistic tools such the Snell envelope of processes and BSDEs we provide a verification theorem which shapes the problem and then we have constructed a solution for this latter. This solution provides an optimal strategy for the switching problem. Later on, in the Markovian framework of randomness, i.e. in the case when $X$ is a solution of a SDE, we show that with the value function of the problem is associated an uplet of deterministic functions $(\nu^1, \ldots, \nu^m)$ which is the unique solution of the following
system of partial differential inequalities (PDIs for short):

$$\begin{align*}
\min\{v_i(x) - \max_{j \in I^{-i}} (-g_{ij}(x) + v_j(x)), rv_i(x) - \mathcal{A}v_i(x) - \psi_i(x)\} = 0
\end{align*}$$

\forall x \in \mathbb{R}^k, i \in I = \{1, \ldots, m\}, \quad (1.1)$$

where $\mathcal{A}$ an infinitesimal generator associated with a diffusion process and $I^{-i} := \{1, \ldots, i - 1, i + 1, \ldots, m\}$. This system is the deterministic version of the Verification Theorem of the optimal multi-modes switching problem in infinite horizon.

This paper is organized as follows: In Section 2, we formulate the problem and we give the related definitions. In Section 3, we introduce the optimal switching problem under consideration and give its probabilistic Verification Theorem. It is expressed by means of a Snell envelope of processes. Then we introduce the approximating scheme which enables to construct a solution for the Verification Theorem. Moreover we give some properties of that solution, especially the dynamic programming principle. Section 4 is devoted to the connection between the optimal switching problem, the Verification Theorem and the associated system of PDIs. This connection is made through backward stochastic differential equations with one reflecting obstacle in the case when randomness comes from a solution of a standard stochastic differential equation. Further we provide some estimate for the optimal strategy of the switching problem which, in combination with the dynamic programming principle, plays a crucial role in the proof of existence of a solution for (1.1) which we address. In Section 5, we show that the solution of PDIs is unique in the class of continuous functions which satisfy a polynomial growth condition. In Section 6, we give some numerical examples.\end{proof}

2 Assumptions and formulation of the problem

Throughout this paper $k$ is a fixed integer positive constant. Let us now consider the followings assumption:

H1: $b : \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : \mathbb{R}^k \to \mathbb{R}^{k \times d}$ are two continuous functions for which there exists a constant $C \geq 0$ such that for any $x, x' \in \mathbb{R}^k$

$$|b(x)| + |\sigma(x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma(x) - \sigma(x')| + |b(x) - b(x')| \leq C|x - x'| \quad (2.1)$$

H2: for $i, j \in I = \{1, \ldots, m\}$, $g_{ij} : \mathbb{R}^k \to \mathbb{R}$ is a continuous function. Moreover we assume that there exists a constant $\alpha > 0$ such that for any $x \in \mathbb{R}^k$,

$$\frac{1}{\alpha} \leq g_{ij}(x) \leq \alpha, \quad \forall i, j \in I, \quad i \neq j. \quad (2.2)$$

H3: for $i \in I\; \psi_i : \mathbb{R}^k \to \mathbb{R}$ is a continuous function of polynomial growth, i.e., there exist a constant $C$ and $\gamma$ such that for each $i \in I$:

$$|\psi_i(x)| \leq C(1 + |x|^\gamma), \forall x \in \mathbb{R}^k. \quad (2.3)$$
We now consider the following system of $m$ variational inequalities with inter-connected obstacles:

$$\forall \; i \in \mathcal{I}$$

$$\min \{ v_i(x) - \max_{j \in \mathcal{I}^{-i}} (-g_{ij}(x) + v_j(x)), rv_i(x) - A v_i(x) - \psi_i(x) \} = 0.$$  \hspace{1cm} (2.4)

where $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$, $r$ is a positive discount factor and $A$ is the following infinitesimal generator:

$$A = \frac{1}{2} \sum_{i,j=1,k} (\sigma \sigma^*)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1,k} b_i(x) \frac{\partial}{\partial x_i};$$  \hspace{1cm} (2.5)

hereafter the superscript (*) stands for the transpose, $Tr$ is the trace operator and finally $<x, y>$ is the inner product of $x, y \in \mathbb{R}^k$.

The main objective of this paper is to focus on the existence and uniqueness of the solution in viscosity sense of (2.4) whose definition is:

**Definition 1** Let $(v_1, ..., v_m)$ be a m-uplet of continuous functions defined on $\mathbb{R}^k$, $\mathbb{R}$-valued. The m-uplet $(v_1, ..., v_m)$ is called:

(i) a viscosity supersolution (resp. subsolution) of the system (2.4) if for each fixed $i \in \mathcal{I}$, for any $x_0 \in \mathbb{R}^k$ and any function $\varphi_i \in C^{1,2}(\mathbb{R}^k)$ such that $\varphi_i(x_0) = v_i(x_0)$ and $x_0$ is a local maximum of $\varphi_i - v_i$ (resp. minimum), we have:

$$\min \bigg\{ v_i(x_0) - \max_{j \in \mathcal{I}^{-i}} (-g_{ij}(x_0) + v_j(x_0)),$$

$$rv_i(x_0) - A v_i(x_0) - \psi_i(x_0) \bigg\} \geq 0 \quad \text{(resp.} \leq 0).$$  \hspace{1cm} (2.6)

(ii) a viscosity solution if it is both a viscosity supersolution and subsolution. \hspace{1cm} \Box

There is an equivalent formulation of this definition (see e.g. [6]) which we give since it will be useful later. So firstly we define the notions of superjet and subjet of a continuous function $v$.

**Definition 2** Let $v \in C(\mathbb{R}^k)$, $x$ an element of $\mathbb{R}^k$ and finally $S_k$ the set of $k \times k$ symmetric matrices. We denote by $J^{2,+}v(x)$ (resp. $J^{2,-}v(x)$), the superjets (resp. the subjets) of $v$ at $x$, the set of pairs $(q, X) \in \mathbb{R}^k \times S_k$ such that:

$$v(y) \leq v(x) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)$$

(resp. $v(y) \geq v(x) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)$). \hspace{1cm} \Box

Note that if $\varphi - v$ has a local maximum (resp. minimum) at $x$, then we obviously have:

$$\big( D_x \varphi(x), D_{xx}^2 \varphi(x) \big) \in J^{2,-}v(x) \quad \text{(resp.} J^{2,+}v(x)). \hspace{1cm} \Box$$

We now give an equivalent definition of a viscosity solution of the elliptic system with inter-connected obstacles (2.4).
Definition 3 Let \( (v_1, ..., v_m) \) be a \( m \)-uplet of continuous functions defined on \( \mathbb{R}^k \) and \( \mathbb{R} \)-valued. The \( m \)-uplet \( (v_1, ..., v_m) \) is called a viscosity supersolution (resp. subsolution) of (2.3) if for any \( i \in I \), \( x \in \mathbb{R}^k \) and \( (q, X) \in J^2 \), \( v_i(x) \geq 0 \) (resp. \( v_i(x) \leq 0 \)).

It is called a viscosity solution if it is both a viscosity sub solution and supersolution .

As pointed out previously we will show that system (2.4) has a unique solution in viscosity sense.

3 The optimal \( m \)-states switching problem

3.1 Setting of the problem

Let \( (\Omega, \mathcal{F}, P) \) be a fixed probability space on which is defined a standard \( d \)-dimensional Brownian motion \( B = (B_t)_{t \geq 0} \) whose natural filtration is \( \mathcal{F}_t := \sigma(B_s, s \leq t) \). Let \( \mathbf{F} = (\mathcal{F}_t)_{t \geq 0} \) be the completed filtration of \( (\mathcal{F}_t)_{t \geq 0} \) with the \( P \)-null sets of \( \mathcal{F} \), hence \( (\mathcal{F}_t)_{t \geq 0} \) satisfies the usual conditions, i.e., it is right continuous and complete. Furthermore, let:

- \( \mathcal{P} \) be the \( \sigma \)-algebra on \([0, +\infty) \times \Omega \) of \( \mathbf{F} \)-progressively measurable sets;
- \( \mathcal{M}^{2,k} \) be the set of \( \mathcal{P} \)-measurable and \( \mathbb{R}^k \)-valued processes \( w = (w_t)_{t \geq 0} \) such that \( E[\int_0^{+\infty} |w_s|^2 ds] < \infty \) and \( \mathcal{S}^2 \) be the set of \( \mathcal{P} \)-measurable, continuous processes \( w = (w_t)_{t \geq 0} \) such that \( E[\sup_{t \geq 0} |w_t|^2] < \infty \);
- for any stopping time \( \tau \in \mathbb{R}^+ \), \( \mathcal{T}_\tau \) denotes the set of all stopping times \( \theta \) such that \( \tau \leq \theta \);
- for any stopping time \( \tau \), \( \mathcal{F}_\tau \) is the \( \sigma \)-algebra on \( \Omega \) which contains the sets \( A \) of \( \mathcal{F} \) such that \( A \cap \{ \tau \leq t \} \in \mathcal{F}_t \) for every \( t \geq 0 \).

A decision (strategy) of the problem of multiple switching, on the one hand, consists of the choice of a sequence of nondecreasing stopping times \( (\tau_n)_{n \geq 1} \) (i.e. \( \tau_n \leq \tau_{n+1} \)) where the manager decides to switch the activity from its current mode to another one. On the other hand, it consists of the choice of the mode \( \xi_n \), a r.v. \( \mathcal{F}_{\tau_n} \)-measurable with values in \( I \), to which the production is switched at \( \tau_n \). Therefore the admissible management strategies are the pairs \( (\delta, \xi) := ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1}) \) and we denote by \( \mathcal{D} \) the set of these strategies.

Let now \( X := (X_t)_{t \geq 0} \) be an \( \mathcal{P} \)-measurable, \( \mathbb{R}^k \)-valued continuous stochastic process which stands for the market price of \( k \) factors which determine the market price of the commodity. On the other hand, assuming that the production activity is in mode 1 at the initial time \( t = 0 \), let \( (u_t)_{t \geq 0} \) denote
the indicator of the production activity’s mode at time $t \in \mathbb{R}^+$:

$$u_t = \mathbb{I}_{[0,\tau]}(t) + \sum_{n \geq 1} \xi_n \mathbb{I}_{(\tau_n,\tau_{n+1}]}(t).$$

(3.1)

Then for any $t \geq 0$, the state of the whole economic system related to the project at time $t$ is given by the vector:

$$(t,X_t,u_t) \in \mathbb{R}^+ \times \mathbb{R}^k \times \mathcal{I}.$$  

(3.2)

Finally, let $\psi_i(X_t)$ be the instantaneous profit when the system is in state $(t,X_t,i)$, and for $i,j \in \mathcal{I}$ $i \neq j$, let $g_{ij}(X_t)$ denote the switching cost of the production at time $t$ from the current mode $i$ to another mode $j$. When the plant is run under the strategy $(\delta,\xi) = ((\tau_n)_{n \geq 1},(\xi_n)_{n \geq 1})$ the expected total profit is given by:

$$J(\delta,\xi) = E[\int_0^{+\infty} e^{-rs} \psi_s(X_s)ds - \sum_{n \geq 1} e^{-r\tau_n} g_{u_{\tau_n-1}u_{\tau_n}}(X_{\tau_n})].$$

Then the problem we are interested in is to find an optimal strategy, i.e., a strategy $(\delta^*,\xi^*)$ such that $J(\delta^*,\xi^*) \geq J(\delta,\xi)$ for any $(\delta,\xi) \in \mathcal{D}$.

Note that in order that the quantity $J(\delta,\xi)$ makes sense we assume throughout this paper that for any $i \in \mathcal{I}$ the processes $(e^{-rt}\psi_i(X_t))_{t \geq 0}$ belong to $\mathcal{M}^{2,1}$. On the other hand there is a bijective correspondence between the pairs $(\delta,\xi)$ and the pairs $(\delta,u)$. Then throughout this paper one refers indifferently to $(\delta,\xi)$ or $(\delta,u)$.

### 3.2 The Verification Theorem

To tackle the problem described above in the finite horizon case, Djehiche et al. \[11\] have introduced a Verification Theorem which is expressed by means of Snell envelope of processes which we describe briefly now. The Snell envelope of a stochastic process $(\eta_t)_{t \geq 0}$ of $\mathcal{S}^2$ (with a possible positive jump at $+\infty$ and $\lim_{t \to \infty} \eta_t = M \in L^2(\Omega,\mathcal{F},P)$) is the lowest supermartingale $R(\eta) := (R(\eta)_t)_{t \geq 0}$ of $\mathcal{S}^2$ such that for any $t \geq 0$, $R(\eta)_t \geq \eta_t$. It has the following expression:

$$\forall t \geq 0, R(\eta)_t = \text{esssup}_{\tau \geq t} E[\eta_{\tau}|\mathcal{F}_t] \quad (\text{then it satisfies } \lim_{t \to +\infty} R(\eta)_t = M.)$$

For more details on the Snell envelope notion one can see e.g. \[7, 14, 16\].

The Verification Theorem for the $m$-states optimal switching problem in infinite horizon is the following:

**Theorem 1.** Assume that there exist $m$ processes $(Y^i := (Y^i_t)_{t \geq 0}, i = 1, ..., m)$ of $\mathcal{S}^2$ such that:

$$\forall t \geq 0, e^{-rt}Y^i_t = \text{ess sup}_{\tau \geq t} E[\int_\tau^t e^{-rs}\psi_s(X_s)ds + e^{-r\tau} \max_{j \in \mathcal{I}} (-g_{ij}(X_{\tau}) + Y^j_\tau)|\mathcal{F}_t],$$

$$\lim_{t \to +\infty} (e^{-rt}Y^i_t) = 0.$$  

(3.3)

Then:
(i) \( Y_0^1 = \sup_{(\delta, u) \in \mathcal{D}} J(\delta, u) \).

(ii) Define the sequence of \( F \)-stopping times \( \tau^* = (\tau^*_n)_{n \geq 1} \) as follows:

\[
\begin{align*}
\tau^*_1 &= \inf \{ s \geq 0, \ Y_1 s = \max_{j \in \mathcal{I}^{-1}} \{-g_{1j}(X_s) + Y^j_s\} \}, \\
\tau^*_n &= \inf \{ s \geq \tau^*_{n-1}, \ Y_s u^*_n \geq \max_{k \in \mathcal{I} \setminus \{ u^*_n \}} \{-g_{u^*_n,k}(X_s) + Y^k_s\} \}, \text{ for } n \geq 2,
\end{align*}
\]

where:

- \( u^*_n = \sum_{j \in \mathcal{I}} j \mathbb{I}_{\{ \max_{k \in \mathcal{I}^{-1}} \{-g_{jk}(X_{\tau^*_n}) + Y^k_{\tau^*_n}\} = -g_{\mathbb{I}_{\{ u^*_n \}}}(X_{\tau^*_n}) + Y^\mathbb{I}_{\{ u^*_n \}}_{\tau^*_n} \}} \);
- for any \( n \geq 1 \) and \( t \geq \tau^*_n \), \( Y_t u^*_n = \sum_{j \in \mathcal{I}} \mathbb{I}_{\{ u^*_n = j \}} Y^j_t \);
- for any \( n \geq 2 \), \( u^*_n = l \) on the set

\[
\left\{ \max_{k \in \mathcal{I} \setminus \{ u^*_n \}} \{-g_{u^*_n,k}(X_{\tau^*_n}) + Y^k_{\tau^*_n}\} \right\}
\]

with \( g_{u^*_n,k}(X_{\tau^*_n}) = \sum_{j \in \mathcal{I}} \mathbb{I}_{\{ u^*_n = j \}} g_{jk}(X_{\tau^*_n}) \) and \( \mathcal{I} \setminus \{ u^*_n \} = \sum \mathbb{I}_{\{ u^*_n = j \}} \mathcal{I}^{-j} \).

Then the strategy \((\delta^*, u^*)\) satisfies \( E[\sum_{n \geq 0} e^{-rt_{\tau^*_n}}] < +\infty \) and it is optimal i.e. \( J(\delta^*, u^*) \geq J(\delta, u) \) for any \((\delta, u) \in \mathcal{D} \). □

**Proof.** The arguments of this proof are standard, based on the properties the Snell envelope. We defer the proof in the Appendix. □

The issue of existence of the processes \( Y^1, \ldots, Y^m \) which satisfy \((3.3)\) is also addressed in \([11]\). For \( n \geq 0 \) let us define the processes \((Y^{n,1}, \ldots, Y^{n,m})\) recursively as follows: for \( i \in \mathcal{I} \) we set,

\[
e^{-rt}Y_{t}^{0,i} = E[\int_{t}^{+\infty} e^{-rs} \psi_i(X_s) ds | \mathcal{F}_t], \ t \geq 0,
\]

and for \( n \geq 1 \),

\[
e^{-rt}Y_{t}^{n,i} = \esssup_{t \geq t} E[\int_{t}^{T} e^{-rs} \psi_i(X_s) ds + e^{-rt} \max_{k \in \mathcal{I}^{-1}} \{-g_{ik}(X_r) + Y_{r}^{n-1,k}\} | \mathcal{F}_t], \ t \geq 0.
\]

Then the sequence of processes \(((Y^{n,1}, \ldots, Y^{n,m}))_{n \geq 0}\) have the following properties:

**Proposition 1** \(([11], \text{Pro.3 and Th.2})\)

(i) for any \( i \in \mathcal{I} \) and \( n \geq 0 \), the processes \( Y^{n,1}, \ldots, Y^{n,m} \) are well-posed, continuous and belong to \( S^2 \), and verify

\[
\forall t \geq 0, \ e^{-rt}Y_{t}^{n,i} \leq e^{-rt}Y_{t}^{n+1,i} \leq E[\int_{t}^{+\infty} e^{-rs} \max_{i=1,m} |\psi_i(X_s)| ds | \mathcal{F}_t];
\]

\(7\)
(ii) there exist \( m \) processes \( Y^1, \ldots, Y^m \) of \( S^2 \) such that for any \( i \in I \):

(a) \( \forall t \geq 0, Y^i_t = \lim_{n \to \infty} Y^{n,i}_t \)

(b) \( \forall t \geq 0, \)

\[
e^{-rt}Y^i_t = \text{ess sup}_{\tau \geq t} E \left[ \int_t^\tau e^{-rs} \psi_0(X_s) ds + e^{-rr} \max_{k \in I^{-i}} (-g_{ik}(X_\tau) + Y^k_\tau) |F_\tau \right]
\]  

(i.e. \( Y^1, \ldots, Y^m \) satisfy the Verification Theorem 1)

(c) \( \forall t \geq 0, \)

\[
e^{-rt}Y^i_t = \text{ess sup}_{(\delta,\xi) \in D_t} E \left[ \int_t^{+\infty} e^{-rs} \psi_0(X_s) ds - \sum_{n \geq 1} e^{-rr_n} g_{u_{n-1}u_n}(X_{\tau_n}) |F_\tau \right]
\]  

where \( D_t = \{ (\delta,\xi) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1}) \text{ such that } u_0 = i \text{ and } \tau_1 \geq t \} \). This characterization means that if at time \( t \) the production activity is in its regime \( i \) then the optimal expected profit is \( Y^i_t \).

(d) the processes \( Y^1, \ldots, Y^m \) verify the dynamical programming principle of the \( m \)-states optimal switching problem, i.e., \( \forall t \leq T, \)

\[
e^{-rt}Y^i_t = \text{ess sup}_{(\delta,u) \in D_t} E \left[ \int_t^{T_n} e^{-rs} \psi_0(X_s) ds - \sum_{1 \leq k \leq n} e^{-rr_k} g_{u_{k-1}u_k}(X_{\tau_k}) + e^{-rr_n} Y_{u_n}^{u_n} |F_\tau \right].
\]  

(3.9)

Note that except (ii – d), the proofs of the other points are the same as in [11] in the framework of finite horizon. The proof of (ii – d) can be easily deduced from using relation (3.7). Actually from (3.7) for any \( i \in I, t \geq 0 \) and \( (\delta,\xi) \in D_t \) we have:

\[
e^{-rt}Y^i_t \geq E \left[ \int_t^{\tau_n} e^{-rs} \psi_0(X_s) ds - \sum_{1 \leq k \leq n} e^{-rr_k} g_{u_{k-1}u_k}(X_{\tau_k}) + e^{-rr_n} Y_{u_n}^{u_n} |F_\tau \right].
\]  

(3.10)

Next using the optimal strategy we obtain the equality instead of inequality in (3.10). Therefore the relation (3.9) holds true. □

**Remark 1** The characterization (3.8) implies that the processes \( Y^1, \ldots, Y^m \) of \( S^2 \) which satisfy the Verification Theorem are unique.

4 Existence of a solution for the system of variational inequalities

4.1 Connection with BSDEs with one reflecting barrier

Let \( x \in \mathbb{R}^k \) and let \( X^x \) be the solution of the following standard SDE:

\[
dX^x_t = b(X^x_t)dt + \sigma(X^x_t)dB_t, \quad X^x_0 = x
\]  

(4.1)
where the functions $b$ and $\sigma$ are the ones of $\textbf{H1}$. These properties of $\sigma$ and $b$ imply in particular that $X^x$ solution of the standard SDE (4.1) exists and is unique in $\mathbb{R}^k$. The operator $A$ defined in (2.5) is the infinitesimal generator associated with $X^x$.

In the following result we collect some properties of $X^x$.

**Proposition 2** (see e.g. [22]) The process $X^x$ satisfies the following estimates:

(i) For any $q \geq 2$ there exists $C_q$ such that,

$$E[|X^x_t|^q] \leq C_q e^{C_q t} (1 + |x|^q) \quad \forall t \geq 0. \quad (4.2)$$

(ii) There exists a constant $C$ such that for any $x, x' \in \mathbb{R}^k$ and $T \geq 0$,

$$E[\sup_{0 \leq s \leq T} |X^x_s - X^{x'}_s|^2] \leq C e^{CT} |x - x'|^2. \quad (4.3)$$

In the sequel we consider the following condition:

**H4**: Assume $\gamma \geq 2$ and

$$-r + C_\gamma < 0, \quad (4.4)$$

where $\gamma$ is the growth exponent of the functions $\psi_i$ and $C_\gamma$ is the constant in (4.2). □

**Remark 2**: If $\gamma < 2$, there exists a constant $\gamma_1 \geq 2$ such that $\gamma_1$ verifies the growth exponent of the functions $\psi_i$.

We are going now to introduce the notion of a BSDE with one reflecting barrier considered in [19]. This notion will allow us to make the connection between the variational inequalities system (2.4) and the $m$-states optimal switching problem described in the previous section.

Let us introduce the pair of process $(Y^x, Z^x) \in \mathcal{S}^2 \times \mathcal{M}^{2,d}$ solution of the following BSDE:

$$Y^x_s = Y^x_T + \int_s^T F(X^x_l, Y^x_l, Z^x_l)dl - \int_s^T Z^x_l dB_l, \quad \text{for all } T \geq 0 \text{ and } t \leq T, \quad (4.5)$$

where $F : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and satisfies: there exist a continuous increasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and constant $K, K', \mu < 0, p > 0$ such that,

$$|F(x, y, z)| \leq K'(1 + |x|^p + \phi(|y|) + |z|),$$

$$\langle y - y', F(x, y, z) - F(x, y', z) \rangle \leq \mu |y - y'|^2,$$

$$|F(x, y/z, z') - F(x, y, z')| \leq K||z - z'||. \quad (4.6)$$

We assume moreover that for some $\lambda > 2\mu + K2$,

$$E[\int_0^{+\infty} e^{\lambda s} |F(X^x_s, 0, 0)|^2 ds] < +\infty, \quad (4.7)$$

We are going now to introduce the notion of a BSDE with one reflecting barrier considered in [19].
which essentially implies that $\lambda + C_{2\gamma} < 0$.

Let us consider the following semilinear elliptic PDE in $\mathbb{R}^k$:

$$Au(x) + F(x, u(x), \sigma(x)\nabla u(x)) = 0, \quad x \in \mathbb{R}^k. \quad (4.8)$$

Then we have the following result:

**Theorem 2** ([21], Th. 5.2) Under the above assumptions, $u(x) = Y^x_0$ is a continuous function and it is a viscosity solution of (4.8) which satisfies,

$$|Y^x_0| \leq C \sqrt{E\left[\int_0^{+\infty} e^{\lambda s}|F(X^x_s, 0, 0)|^2 ds\right]}, \quad (4.9)$$

for any $\lambda > 2\mu + K_2$.

Let us now introduce the following functions:

(i) $f : \mathbb{R}^k \to \mathbb{R}$ is continuous and of polynomial growth, i.e., there exist some positive constants $C$ and $\gamma$ such that:

$$|f(x)| \leq C(1 + |x|^\gamma), \quad \forall x \in \mathbb{R}^k. \quad (4.10)$$

(ii) $h : \mathbb{R}^k \to \mathbb{R}$ is continuous and bounded.

Then we have the following result related to BSDEs with one reflecting barrier:

**Theorem 3** For any $x \in \mathbb{R}^k$, there exists a unique triple of processes $(Y^x, Z^x, K^x)$ such that:

$$\begin{cases} 
Y^x, K^x \in \mathcal{S}^2 \text{ and } Z^x \in \mathcal{M}^{2d}; 
K^x \text{ is non-decreasing and } K^x_0 = 0, \\
\int_s^{+\infty} e^{-rl}f(X^x_l)dl - \int_s^{+\infty} Z^x_l dB_l + K^x_{l+} - K^x_s, \\
\int_s^{+\infty} (e^{-rl}Y^x_l - e^{-rl}h(X^x_l))dK^x_l = 0.
\end{cases} \quad (4.11)$$

Moreover the following characterization of $Y^x$ as a Snell envelope holds true:

$$\forall s \geq 0, \quad e^{-rs}Y^x_s = \operatorname{esssup}_{\tau \in \mathcal{F}_s} E\left[\int_s^\tau e^{-rl}f(X^x_l)dl + e^{-r\tau}h(X^x_\tau)|\mathcal{F}_s\right]. \quad (4.12)$$

On the other hand there exists a deterministic continuous with polynomial growth function $u : \mathbb{R}^k \to \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}^k \quad Y^x_0 = u(x).$$

Moreover the function $u$ is the viscosity solution in the class of continuous function with polynomial growth of the following PDE with obstacle:

$$\min\{u(x) - h(x), ru(x) - Au(x) - f(x)\} = 0. \quad (4.13)$$
Proof: Existence and uniqueness of the triple \((Y_t^x, Z_t^x, K_t^x)_{t \geq 0}\) follow from Theorem 3.2 in [19]. Now we consider the infinite horizon BSDE:

\[
\begin{align*}
nY_s^x e^{-rs} &= \int_{s}^{+\infty} e^{-rt} f(X_t^x) dW_t - \int_{s}^{+\infty} Z_t^x dB_t + \int_{s}^{+\infty} n e^{-rl(nY_t^x - h(X_t^x))} dl. \tag{4.14}
\end{align*}
\]

From Theorem 1 in [5] there exists a unique solution \((nY_t^x, Z_t^x) \in \mathcal{S}^2 \times \mathcal{M}^{2,d}\) satisfying the BSDE (4.14).

Next let us define

\[
K_s^{n,x} = \int_{0}^{s} n e^{-rl(nY_t^x - h(X_t^x))} dl,
\]

then

\[
\int_{0}^{+\infty} e^{-rl(nY_t^x - h(X_t^x) \wedge nY_t^x)} dK_t^{n,x} = n \int_{0}^{+\infty} e^{-rl(nY_t^x - h(X_t^x) \wedge nY_t^x)} e^{-rl(nY_t^x - h(X_t^x))} dl = 0.
\]

Since \(K^{n,x}\) is non-decreasing and \(K_0^{n,x} = 0\), we rewrite Eq. (4.14) in RBSDE form

\[
\begin{align*}
nY_s^x e^{-rs} &= \int_{s}^{+\infty} e^{-rt} f(X_t^x) dW_t - \int_{s}^{+\infty} Z_t^x dB_t + K_s^{n,x} - K_0^{n,x}; \tag{4.15}
nY_s^x e^{-rs} &\geq e^{-rs}(h(X_s^x) \wedge nY_s^x), \forall s \geq 0 \text{ and } \int_{0}^{+\infty} e^{-rl(nY_t^x - h(X_t^x) \wedge nY_t^x)} dK_t^{n,x} = 0.
\end{align*}
\]

Then from property (4.12) we have:

\[
nY_s^x e^{-rs} = \text{esssup}_{t \in \mathcal{T}} \mathbb{E} \left[ \int_{s}^{\tau} e^{-rt} f(X_t^x) dW_t + e^{-r\tau} (nY_{\tau}^x \wedge h(X_{\tau}^x)) | \mathcal{F}_s \right]. \tag{4.16}
\]

Note that if we define

\[
f_n(t, x, y, z) = e^{-rt} f(x, y, z) + n e^{-rl}(y - h(x))^{-},
\]

\[
f_n(t, x, y, z) \leq f_{n+1}(t, x, y, z).
\]

Then it follows from the comparison Theorem 2.2 in [19] \(nY_s^x e^{-rs} \leq nY_s^x e^{-rs}, s \geq 0\), a.s. and from (4.12) and (4.16) \(nY_s^x e^{-rs} \leq Y_s^x e^{-rs}\). This implies that there exists a càdlàg process \((\tilde{Y}_s^x)_{s \geq 0}\) such that \(P - a.s.\) for any \(s \geq 0\),

\[
nY_s^x e^{rs} \uparrow e^{rs} \tilde{Y}_s^x, \quad a.s.
\]

Let us actually show that \(\tilde{Y}_s^x\) is càdlàg. By (4.16), for any \(n \geq 1\), the process \((nY_t^x + \int_{t}^{\tau} e^{-rs} f(X_s^x) ds)_{t \geq 0}\) is an \(\mathcal{F}\)-supermartingale which converges increasingly and pointwisely to \((\tilde{Y}_t^x + \int_{t}^{\tau} e^{-rs} f(X_s^x) ds)_{t \geq 0}\). Therefore, the limit is also a càdlàg \(\mathcal{F}\)-supermartingale (see e.g. Dellacherie and Meyer (1980), pp. 86). Hence, the process \(Y_s^x\) is càdlàg.

Then it follows from Proposition 2 in [11], as \(n \to +\infty\),

\[
\tilde{Y}_s^x e^{-rs} = \text{esssup}_{t \in \mathcal{T}} \mathbb{E} \left[ \int_{s}^{\tau} e^{-rt} f(X_t^x) dW_t + e^{-r\tau} (\tilde{Y}_{\tau}^x \wedge h(X_{\tau}^x)) | \mathcal{F}_s \right]. \tag{4.17}
\]
From \(4.14\) we have:
\[
E[\int_s^{+\infty} e^{-r t} (n Y_t^x - h(X_t^x))^{-} \, dl] = \frac{1}{n} E[\int_s^{+\infty} e^{-r s} f(X_t^x) \, dl] \leq \frac{1}{n} E[|Y_s^x e^{-r s} - f(X_t^x)|] + C \int_s^{+\infty} e^{-r t} C_{\gamma t} |x|^\gamma \, dl,
\]
for a constant \(C\) independent of \(n\) and \(H4.\) Then
\[
E[\int_s^{+\infty} e^{-r t} (n Y_t^x - h(X_t^x))^{-} \, dl] \leq \frac{C_x}{n},
\]
Hence as \(n \to +\infty\) we obtain, \(E[\int_s^{+\infty} e^{-r t} (\tilde{Y}_t^x - h(X_t^x))^{-} \, dl] = 0\), and since \((\tilde{Y}_s^x)_{s \geq 0}\) (resp. \(h(x)\)) is a càdlàg process (resp. continuous), we have
\[
\tilde{Y}_t^x \geq h(X_t^x).
\]  
(4.18)

From \(4.12, 4.17\) and \(4.18\) we get:
\[
\tilde{Y}_t^x = Y_t^x \quad \forall t \geq 0.
\]

Now rewrite Eq. \(4.14\) in differential form
\[
d(n Y_s^x e^{-r s}) = -[e^{-r s} f(X_s^x) + n e^{-r s} (n Y_s^x - h(X_s^x))^{-}] ds + Z_s^{n,x} dB_s.
\]
So for arbitrary \(T > 0\) and \(0 \leq s \leq T\), Eq. \(4.14\) is equivalent to
\[
n Y_s^x = Y_s^x + \int_s^T [(f(X_t^x) + n (n Y_t^x - h(X_t^x))^{-}) - n Y_s^x] \, dl - \int_s^T Z_t^{n,x} dB_t,
\]
with \(Z_s^{n,x} = Z_s^{n,x} e^{r s}.\) Let us set \(F_n(x, y, z) = f(x) + n(y - h(x))^{-} - ry.\)

In order that it satisfies the assumptions of Theorem \(2\) we just need to verify that \(F_n\) satisfy condition (4.6) and (4.7). It is obvious that \(F_n\) satisfy (4.6) where \(\mu > -r\), and we show that \(F_n\) satisfy (4.7).

From the polynomial growth of \(f\) and since \(h\) bounded and estimate (4.2), we deduce
\[
E[\int_0^{+\infty} e^{\lambda s} |F_n(X_s^x, 0, 0)|^2 \, ds] = E[\int_0^{+\infty} e^{\lambda s} |f(X_s^x)|^2 \, ds] \leq 2 E[\int_0^{+\infty} e^{\lambda s} ((1 + |X_s^x|^\gamma) 2 + n^2 C2) \, ds] \leq C \int_0^{+\infty} e^{\lambda s} e^{C_{2\gamma}} (|x|^\gamma + n^2) \, ds,
\]
for \(\lambda + C_{2\gamma} < 0\). This proves assumption (4.7). Then
\[
u_n(x) = n Y_0^x,
\]
and is a viscosity solution of the elliptic PDE
\[
\mathcal{A} u_n(x) + F_n(x, u_n(x), \sigma(x)^* \nabla u_n(x)) = 0.
\]

We now define
\[
u(x) = Y_0^x, \quad \forall x \in \mathbb{R}^k,
\]
which is a deterministic quantity. Let us admit for a moment the following Lemma:
Lemma 1 The function $u$ is continuous in $\mathbb{R}^k$. □

From the previous results we have, for each $x \in \mathbb{R}^k$,

$$u_n(x) \uparrow u(x) \quad \text{as} \quad n \to +\infty.$$  

Since $u_n$ and $u$ are continuous, it follows from Dini’s theorem that the above convergence is uniform on compacts.

We now show that $u$ is a subsolution of (4.13). Let $x$ be a point at which $u(x) > h(x)$, and let $(q, X) \in J^{2,+}u(x)$. From Lemma 6.1 in [6], there exists sequences:

$$n_j \to +\infty, \quad x_j \to x, \quad (q_j, X_j) \in J^{2,+}u_{n_j}(x_j),$$

such that

$$(q_j, X_j) \to (q, X).$$

But for any $j$,

$$-\frac{1}{2} Tr[\sigma^* X_j \sigma] - \langle b, q_j \rangle - F_n(x_j, u_{n_j}(x_j), \sigma(x_j)^* \nabla u_{n_j}(x_j)) \leq 0,$$

$$-\frac{1}{2} Tr[\sigma^* X_j \sigma] - \langle b, q_j \rangle - f(x_j) - n_j(u_{n_j}(x_j) - h(x_j))^- + ru_{n_j}(x_j) \leq 0.$$

From the assumption that $u(x) > h(x)$ and the uniform convergence of $u_n$, it follows that for $j$ large enough $u_{n_j}(x_j) > h(x_j)$. Hence, taking the limit as $j \to +\infty$ in the above inequality yields:

$$-\frac{1}{2} Tr[\sigma^* X \sigma] - \langle b, q \rangle - f(x) + ru(x) \leq 0,$$

and we have proved that $u$ is a subsolution of (4.13).

We now show that $u$ is a supersolution of (4.13). Let $x$ be arbitrary in $\mathbb{R}^k$, and $(q, X) \in J^{2,-}u(x)$. We already know that $u(x) \geq h(x)$. By the same argument as above, there exist sequences:

$$n_j \to +\infty, \quad x_j \to x, \quad (q_j, X_j) \in J^{2,-}u_{n_j}(x_j),$$

such that

$$(q_j, X_j) \to (q, X).$$

But for any $j$,

$$-\frac{1}{2} Tr[\sigma^* X_j \sigma] - \langle b, q_j \rangle - F_n(x_j, u_{n_j}(x_j), \sigma(x_j)^* \nabla u_{n_j}(x_j)) \geq 0,$$

$$-\frac{1}{2} Tr[\sigma^* X_j \sigma] - \langle b, q_j \rangle - f(x_j) - n_j(u_{n_j}(x_j) - h(x_j))^- + ru_{n_j}(x_j) \geq 0.$$

Hence,

$$-\frac{1}{2} Tr[\sigma^* X \sigma] - \langle b, q \rangle - f(x) + ru_{n_j}(x_j) \geq 0,$$
and taking the limit as \( j \to +\infty \), we conclude that:

\[
-\frac{1}{2} \text{Tr}[\sigma^* X \sigma] - \langle b, q \rangle - f(x) + ru(x) \geq 0.
\]

We conclude by showing that \( u \) is of polynomial growth. From (4.12) we have,

\[
|Y_0^x| \leq \sup_{t \geq 0} E[\int_0^t e^{-\rho s}|f(X_s^x)| \, ds + |h(X_s^x)| \, I_{[\tau < +\infty]}] \\
\leq \sup_{t \geq 0} E[\int_0^t e^{-\rho s}|f(X_s^x)| \, ds + e^{-\rho \tau}|h(X_t^x)|] \\
\leq E[\int_0^{+\infty} e^{-\rho s}|f(X_s^x)| \, ds + C_1].
\]

From polynomial growth of \( f \) and \( u(x) = Y_0^x \), we deduce that \( u \) is of polynomial growth. Now we proceed to the proof of Lemma.

**Proof** of Lemma 2. It suffices to show that whenever \( x_n \to x \), \( |Y_0^{x_n} - Y_0^x| \to 0 \).

From (4.12) we have,

\[
Y_0^x = \sup_{\tau \in T_0} E[\int_0^\tau e^{-rt} f(X_t^x) \, dt + e^{-r\tau} h(X_t^x)], \\
Y_0^{x_n} = \sup_{\tau \in T_0} E[\int_0^\tau e^{-rt} f(X_t^{x_n}) \, dt + e^{-r\tau} h(X_t^{x_n})]
\]

then,

\[
|Y_0^{x_n} - Y_0^x| \leq \sup_{\tau \in T_0} E[\int_0^\tau e^{-rt}|f(X_t^{x_n}) - f(X_t^x)| \, dt + e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] \\
\leq E[\int_0^{+\infty} e^{-rt}|f(X_t^{x_n}) - f(X_t^x)| \, dt] + E[\sup_{t \geq 0} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|].
\]

In the right-hand side of (4.21) the first term converges to 0 as \( x_n \to x \). Next let us show that,

\[
E[\sup_{t \geq 0} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] \to 0 \quad \text{as} \quad x_n \to x.
\]

For any \( T \geq 0 \) we have

\[
E[\sup_{t \geq 0} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] \leq E[\sup_{0 \leq t \leq T} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] + E[\sup_{t \geq T} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|].
\]

Since \( h \) is bounded there exists \( C \) such that,

\[
E[\sup_{t \geq 0} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] \leq E[\sup_{0 \leq t \leq T} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] + Ce^{-rT}.
\]

For any \( \rho > 0 \) we have:

\[
E[\sup_{0 \leq t \leq T} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)|] = E[\sup_{0 \leq t \leq T} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)| I_{[\sup_{t \leq T} |X_t^{x_n}| + \sup_{t \leq T} |X_t^x| \leq \rho]}] \\
+ E[\sup_{0 \leq t \leq T} e^{-r\tau}|h(X_t^{x_n}) - h(X_t^x)| I_{[\sup_{t \leq T} |X_t^{x_n}| + \sup_{t \leq T} |X_t^x| > \rho]}].
\]

But since \( h \) is continuous then it is uniformly continuous on compact subsets, then there exists \( \pi : R^k \to R \) increasing with \( \pi(0) = 0 \), such that:

\[
|h(X_t^{x_n}) - h(X_t^x)| \leq \pi(|X_t^{x_n} - X_t^x|),
\]

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Moreover we have
\[
E[\sup_{0 \leq t \leq T} e^{-rt}|h(X_t^{x_n}) - h(X_t^x)|\mathbb{I}_{[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x] \leq \rho}] \leq E[\sup_{0 \leq t \leq T} \pi(|X_t^{x_n} - X_t^x|)\mathbb{I}_{[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x] \leq \rho}] \\
\leq E[\pi(\sup_{0 \leq t \leq T} |X_t^{x_n} - X_t^x|)\mathbb{I}_{[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x] \leq \rho}].
\]

Using the continuity property \[4.23\], \(\pi(0) = 0\) and the Lebesgue dominated convergence theorem to obtain that
\[
E[\sup_{0 \leq t \leq T} e^{-rt}|h(X_t^{x_n}) - h(X_t^x)|\mathbb{I}_{[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x] \leq \rho}] \rightarrow 0 \quad \text{as} \quad x_n \rightarrow x. \tag{4.22}
\]
The second term satisfies:
\[
E[\sup_{0 \leq t \leq T} e^{-rt}|h(X_t^{x_n}) - h(X_t^x)|\mathbb{I}_{[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x] > \rho}] \\
\leq E[\sup_{0 \leq t \leq T} e^{-2rt}|h(X_t^{x_n}) - h(X_t^x)|^2] \frac{1}{2} \{E[\mathbb{I}_{[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x] > \rho}]\}^{\frac{1}{2}} \\
\leq E[\{\sup_{0 \leq t \leq T} e^{-2rt}|h(X_t^{x_n}) - h(X_t^x)|^2\} \frac{1}{2} \{\rho^{-1}E[\sup_{t \leq T} X_t^{x_n}] + \sup_{t \leq T} X_t^x]\}^{\frac{1}{2}}.
\]
Since \(h\) is bounded, it follows that, when \(x_n \rightarrow x\), the right-hand side of the last inequality is smaller than \(\rho^{-\frac{1}{2}}C_x\). However, from previous results we have,
\[
\lim_{x_n \rightarrow x} \sup_{t \geq 0} E[|h(X_t^{x_n}) - h(X_t^x)|] \leq \rho^{-\frac{1}{2}}C_x + Ce^{-rT}.
\]
As \(\rho\) and \(T\) are arbitrary then making \(\rho \rightarrow +\infty\) and \(T \rightarrow +\infty\) to obtain that,
\[
\lim_{x_n \rightarrow x} E[\sup_{t \geq 0} e^{-rt}|h(X_t^{x_n}) - h(X_t^x)|] = 0. \tag{4.23}
\]
From \[4.21\] and \[4.23\], we deduce
\[
|Y_t^{x_n} - Y_t^x| \rightarrow 0 \quad \text{as} \quad x_n \rightarrow x. \tag{4.24}
\]

### 4.2 Existence of a solution for the system of variational inequalities

Let \((Y_s^{1,x}, \ldots, Y_s^{m,x})\)\(s \geq 0\) be the processes which satisfy the Verification Theorem \[1\] in the case when the process \(X \equiv X^x\). Therefore using the characterization \[4.12\], there exist processes \(K_i^{i,x}\) and \(Z_i^{i,x}\), such that the triples \((Y_i^{i,x}, Z_i^{i,x}, K_i^{i,x})\) are unique solutions (thanks to Remark \[2\]) of the following reflected BSDEs: for any \(i = 1, \ldots, m\) we have,
\[
\begin{cases}
Y_t^{i,x}, K_i^{i,x} \in S^2 \text{ and } Z_i^{i,x} \in \mathcal{M}^{d,2}; K_i^{i,x} \text{ is non-decreasing and } K_0^{i,x} = 0, \\
e^{-rs}Y_s^{i,x} = \int_s^{+\infty} e^{-rt}\psi_i(X_t^x)ds - \int_s^{+\infty} Z_t^{i,x}dB_t + K_t^{i,x} - K_s^{i,x}, \quad s \in \mathbb{R}^+, \lim_{s \rightarrow +\infty} (e^{-rs}Y_s^{i,x}) = 0, \\
e^{-rs}Y_s^{i,x} \geq -e^{-rs} \max_{j \in I^{-i}} (-g_{ij}(X_s^x) + Y_s^{j,x}), \quad s \in \mathbb{R}^+, \\
\int_0^{+\infty} e^{-rt}(Y_t^{i,x} - \max_{j \in I^{-i}} (-g_{ij}(X_t^x) + Y_t^{j,x}))dK_t^{i,x} = 0.
\end{cases}
\]

Moreover we have the following result.
Proposition 3 There are deterministic functions \( v^1, \ldots, v^m : \mathbb{R}^k \rightarrow \mathbb{R} \) such that:

\[
\forall x \in \mathbb{R}^k, Y_0^{i,x} = v^i(x), \ i = 1, \ldots, m.
\]

Moreover the functions \( v^i, i = 1, \ldots, m, \) are of polynomial growth.

Proof: For \( n \geq 0 \) let \( (Y_s^{n,1,x}, \ldots, Y_s^{n,m,x})_{s \geq 0} \) be the processes constructed in \([3.4]-[3.5]\). Therefore using an induction argument and Theorem 2 there exist deterministic continuous with polynomial growth functions \( v^{n,i} (i = 1, \ldots, m) \) such that for any \( x \in \mathbb{R}^k, Y_0^{n,i,x} = v^{n,i}(x) \). Using now inequality \([3.6]\) we get:

\[
Y_t^{n,i,x} \leq Y_t^{n+1,i,x} \leq C E \left[ \int_0^{+\infty} \left\{ \max_{i=1,m} |e^{-rs} \psi_i(X_s^x)| \right\} ds \right]
\]

since \( Y_t^{n,i,x} \) is deterministic. Therefore combining the polynomial growth of \( \psi_i \) and estimate \([4.2]\) for \( X^x \) we obtain:

\[
v^{n,i}(x) \leq v^{n+1,i}(x) \leq C(1 + |x|^\gamma)
\]

for a constant \( C \) independent of \( n \). In order to complete the proof it is enough now to set \( v^i(x) := \lim_{n \rightarrow \infty} v^{n,i}(x), x \in \mathbb{R}^k \) since \( Y^{n,i,x} \not\rightarrow Y^{i,x} \) as \( n \rightarrow \infty \). \( \square \)

We are now going to focus on the continuity of the functions \( v^1, \ldots, v^m \). But first let us deal with some properties of the optimal strategy which exist thanks to Theorem 1.

Proposition 4 Let \((\delta, u) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})\) be an optimal strategy, then there exists a constant \( C \) which does not depend on \( t \) and \( x \) such that:

\[
\forall n \geq 1, \ E[e^{-r\tau_n}] \leq \frac{C(1 + |x|^\gamma)}{n}.
\]

Proof: Recall the characterization of \([3.8]\) that reads as:

\[
Y_0^{i,x} = \sup_{(\delta, u) \in D} E\left[ \int_0^{+\infty} e^{-rs} \psi_{u_s}(X^x_s) ds - \sum_{k \geq 1} e^{-r\tau_k} g_{u_{\tau_k-1}u_{\tau_k}}(X^x_{\tau_k}) \right].
\]

Now if \((\delta, u) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})\) is the optimal strategy then we have:

\[
Y_0^{i,x} = E\left[ \int_0^{+\infty} e^{-rs} \psi_{u_s}(X^x_s) ds - \sum_{k \geq 1} e^{-r\tau_k} g_{u_{\tau_k-1}u_{\tau_k}}(X^x_{\tau_k}) \right].
\]

Taking into account that \( g_{ij} \geq \frac{1}{\alpha} > 0 \) for any \( i \neq j \) we obtain:

\[
\frac{1}{\alpha} E[\sum_{k=1,n} e^{-r\tau_k}] + Y_0^{i,x} \leq E\left[ \int_0^{+\infty} e^{-rs} \psi_{u_s}(X^x_s) ds - \sum_{k \geq n+1} e^{-r\tau_k} g_{u_{\tau_k-1}u_{\tau_k}}(X^x_{\tau_k}) \right].
\]

But for any \( k \leq n, e^{-r\tau_k} \leq e^{-r\tau_n} \) then:

\[
\frac{n}{\alpha} E[e^{-r\tau_n}] + Y_0^{i,x} \leq E\left[ \int_0^{+\infty} e^{-rs} \psi_{u_s}(X^x_s) ds - \sum_{k \geq n+1} e^{-r\tau_k} g_{u_{\tau_k-1}u_{\tau_k}}(X^x_{\tau_k}) \right]
\]

\[
\leq E\left[ \int_0^{+\infty} e^{-rs} \psi_{u_s}(X^x_s) ds \right].
\]

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Finally taking into account the facts that $\psi$ and $Z^{0,i,x}$ are of polynomial growth, estimate (4.2) for $X^x$ and $H4$ to obtain the desired result. Note that the polynomial growth of $Z^{0,i,x}$ stems from Proposition 3.

**Remark 3** The estimate (4.25) is also valid for the optimal strategy if at the initial time the state of the plant is an arbitrary $i \in I$. □

We are now ready to give the main result of this article.

**Theorem 4** The functions $(v^1, ..., v^m) : \mathbb{R}^k \to \mathbb{R}$ are continuous and solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.4).

**Proof:** First let us focus on continuity and let us show that $v^1$ is continuous. The same proof will be valid for the continuity of the other functions $v^i$ $(i = 2, ..., m)$. First the characterization (3.8) implies that:

$$Y_0^{1,x} = \sup_{(\delta,\xi) \in \mathcal{D}} E \left[ \int_0^{\infty} e^{-rs} \psi_{u,s}(X^x_s) ds - \sum_{n \geq 1} e^{-r\tau_n} g_{u_{\tau_n-1}u_{\tau_n}}(X^x_{\tau_n}) \right]$$

On the other hand an optimal strategy $(\delta^*, \xi^*)$ exists and satisfies the estimates (4.25) with the same constant $C$. Next let $\epsilon > 0$ and $x' \in B(x, \epsilon)$ and let us consider the following set of strategies:

$$\tilde{D} := \{(\delta^*, \xi^*) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 0}) \in \mathcal{D} \text{ such that } \forall n \geq 1, E[e^{-r\tau_n}] \leq \frac{C(1 + (\epsilon + |x|^\gamma))}{n}\}.$$ 

Therefore the strategy $(\delta^*, \xi^*)$ belongs to $\tilde{D}$ and then we have:

$$Y_0^{1,x} = \sup_{(\delta,\xi) \in \tilde{D}} E \left[ \int_0^{\infty} e^{-rs} \psi_{u,s}(X^x_s) ds - \sum_{n \geq 1} e^{-r\tau_n} g_{u_{\tau_n-1}u_{\tau_n}}(X^x_{\tau_n}) \right]$$

and

$$Y_0^{1,x'} = \sup_{(\delta,\xi) \in \tilde{D}} E \left[ \int_0^{\infty} e^{-rs} \psi_{u,s}(X^{x'}_s) ds - \sum_{n \geq 1} e^{-r\tau_n} g_{u_{\tau_n-1}u_{\tau_n}}(X^{x'}_{\tau_n}) \right].$$

The second equalities it due to the dynamical programming principle. It follows that:

$$|Y_0^{1,x} - Y_0^{1,x'}| \leq \sup_{(\delta,\xi) \in \tilde{D}} E \left[ \int_0^{\tau_n} e^{-rs} |\psi_{u,s}(X^x_s) - \psi_{u,s}(X^{x'}_s)| ds \right.$$ 

$$+ \sum_{1 \leq k \leq n} e^{-r\tau_k} \left| g_{u_{\tau_k-1}u_{\tau_k}}(X^x_{\tau_k}) - g_{u_{\tau_k-1}u_{\tau_k}}(X^{x'}_{\tau_k}) \right|$$ 

$$+ e^{-r\tau_n} \left| Y_{\tau_n}^{u_{\tau_n},x} - Y_{\tau_n}^{u_{\tau_n},x'} \right| \right]$$

$$\leq E \left[ \int_0^{\infty} \max_{j=1,m} e^{-rs} |\psi_j(X^x_s) - \psi_j(X^{x'}_s)| ds \right.$$ 

$$+ n \max_{i \neq j \in I} \left\{ \sup_{s \geq 0} e^{-rs} |g_{ij}(X^x_s) - g_{ij}(X^{x'}_s)| \right\}$$ 

$$+ \sup_{(\delta,\xi) \in \tilde{D}} E \left[ e^{-2r\tau_n} \right] \frac{2}{2} \left( 2E[(Y_{\tau_n}^{u_{\tau_n},x'})^2 + (Y_{\tau_n}^{u_{\tau_n},x})^2] \right)^{\frac{1}{2}}.$$

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In the right-hand side of (4.26) the first and the second term converges to 0 as \( x' \to x \).

Now let us focus on the last one. Since \((\delta, u) \in \tilde{D}\) then:

\[
\sup_{(\delta, u) \in \tilde{D}} (E[e^{-2\tau n}] \frac{1}{2} (2E[(Y_{\tau n}^{u,x'})^2 + (Y_{\tau n}^{u,x})^2])^{\frac{1}{2}} \leq \sup_{(\delta, u) \in \tilde{D}} (E[e^{-\tau n}] \frac{1}{2} (2E[(Y_{\tau n}^{u,x'})^2 + (Y_{\tau n}^{u,x})^2])^{\frac{1}{2}} \\
\leq n^{-\frac{1}{2}} \sup_{(\delta, u) \in \tilde{D}} (2E[(Y_{\tau n}^{u,x'})^2 + (Y_{\tau n}^{u,x})^2])^{\frac{1}{2}} \\
\leq Cn^{-\frac{1}{2}} (1 + |x|^\gamma + |x'|^\gamma)
\]

where \(C\) an appropriate constant which comes from the polynomial growth of \(\psi_i, i \in \mathcal{I}\), estimate (4.2) for the process \(X^x\) and inequality (3.6). Going back now to (4.26), taking the limit as \(x' \to x\) to obtain:

\[
\lim_{x' \to x} |Y_0^{1,x'} - Y_0^{1,x}| \leq Cn^{-\frac{1}{2}} (1 + 2|x|^\gamma).
\]

As \(n\) is arbitrary then putting \(n \to +\infty\) to obtain:

\[
Y_0^{1,x'} \to Y_0^{1,x}.
\]

Therefore \(v^1\) is continuous. In the same way we can show that \(v^2,...,v^m\) are continuous. As they are of polynomial growth then taking into account Theorem 2 to obtain that \((v^1,\ldots,v^m)\) is a viscosity solution for the system of variational inequalities with inter-connected obstacles (2.4). □

5 Uniqueness of the solution of the system

We are going now to address the question of uniqueness of the viscosity solution of the system (2.4).

We have the following:

**Theorem 5** The solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.4) is unique in the space of continuous functions on \(\mathbb{R}^k\) which satisfy a polynomial growth condition, i.e., in the space

\[
\mathcal{C} := \{\varphi : \mathbb{R}^k \to \mathbb{R}, \text{ continuous and for any } x, \ |\varphi(x)| \leq C(1 + |x|^\gamma) \text{ for some constants } C \text{ and } \gamma\}.
\]

**Proof.** We will show by contradiction that if \(u_1,...,u_m\) and \(w_1,...,w_m\) are a subsolution and a supersolution respectively for (2.4) then for any \(i = 1,\ldots,m, u_i \leq w_i\). Therefore if we have two solutions of (2.4) then they are obviously equal. Actually for some \(R > 0\) suppose there exists \((x_0,i_0) \in BR \times \mathcal{I}\) \((BR := \{x \in \mathbb{R}^k; |x| \leq R\})\) such that:

\[
\max_{(x,i)} (u_i(x) - w_i(x)) = u_{i_0}(x_0) - w_{i_0}(x_0) = \eta > 0.
\]

Then, for a small \(\epsilon > 0\), and \(\theta, \lambda \in (0, 1)\) small enough, let us define:

\[
\Phi^i_{\epsilon}(x, y) = u_i(x) - (1 - \lambda)w_i(y) - \frac{1}{2\epsilon} |x - y|^{2\gamma} - \theta(|x - x_0|^{2\gamma + 2} + |y - x_0|^{2\gamma + 2}).
\]

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By the polynomial growth assumption on $u_i$ and $w_i$, there exists a $(x, y, i) \in B_R \times B_R \times \mathcal{I}$, such that:

$$
\Phi^i(x, y) = \max_{(x, y, i)} \Phi^i(x, y).
$$

On the other hand, from $2\Phi^i(x, y) \geq \Phi^i(x, x, x) + \Phi^i(y, y, y)$, we have

$$
\frac{1}{2\varepsilon}|x - y|^{2\gamma} \leq (u_{i_0}(x) - u_{i_0}(y)) + (1 - \lambda)(w_i(x, x) - w_i(y, y)) \\
\leq \sum_{i \in \mathcal{I}} |u_i(x, x) - u_i(y, y)| + (1 - \lambda) \sum_{i \in \mathcal{I}} |w_i(x, x) - w_i(y, y)| \tag{5.3}
$$

and consequently $\frac{1}{2\varepsilon}|x - y|^{2\gamma}$ is bounded, and as $\varepsilon \to 0$, $|x - y| \to 0$. Since $u_i$ and $w_i$ are uniformly continuous on $B_R$, then $\frac{1}{2\varepsilon}|x - y|^{2\gamma} \to 0$ as $\varepsilon \to 0$.

Since

$$
u_{i_0}(0) - (1 - \lambda)w_{i_0}(0) \leq \Phi^i(x, y) \leq u_{i_0}(x) - (1 - \lambda)w_{i_0}(y),
$$

it follow as $\lambda \to 0$ and the continuity of $u_i$ and $w_i$ that, up to a subsequence,

$$(x, y, i) \to (x_0, x_0, i_0). \tag{5.4}$$

We now claim that:

$$u_i(x) - \max_{j \in \mathcal{I} - i_0} \{-g_{i, j}(x) + u_j(x)\} > 0. \tag{5.5}$$

Indeed if

$$u_i(x) - \max_{j \in \mathcal{I} - i_0} \{-g_{i, j}(x) + u_j(x)\} \leq 0
$$

then there exists $k \in \mathcal{I} - i_0$ such that:

$$u_i(x) \leq -g_{i, k}(x) + u_k(x).$$

From the supersolution property of $w_{i_0}(y)$, we have

$$w_{i_0}(y) \geq \max_{j \in \mathcal{I} - i_0} (-g_{i, j}(y) + w_j(y))
$$

then

$$w_i(y) \geq -g_{i, k}(y) + w_k(y).$$

It follows that:

$$u_i(x) - (1 - \lambda)w_i(y) - (u_k(x) - (1 - \lambda)w_k(y)) \leq (1 - \lambda)g_{i, k}(y) - g_{i, k}(x).
$$

Now since $g_{ij} \geq \alpha > 0$, for every $i \neq j$, and taking into account of (5.2) to obtain:

$$\Phi^i(x, y) - \Phi^k(x, y) < -\alpha \lambda + g_{i, k}(y) - g_{i, k}(x)
$$

But this contradicts the definition of $i$, since $g_{i, k}$ is uniformly continuous on $B_R$ and the claim (5.5) holds.
Next let us denote
\[
\varphi_\epsilon(x,y) = \frac{1}{2\epsilon} |x - y|^{2\gamma} + \theta(|x - x_0|^{2\gamma+2} + |y - x_0|^{2\gamma+2}).
\] (5.6)

Then we have:
\[
\begin{align*}
D_x\varphi_\epsilon(t,x,y) &= \frac{\gamma}{\epsilon}(x - y)|x - y|^{2\gamma-2} + \theta(2\gamma + 2)(x - x_0)|x - x_0|^{2\gamma}, \\
D_y\varphi_\epsilon(t,x,y) &= -\frac{\gamma}{\epsilon}(x - y)|x - y|^{2\gamma-2} + \theta(2\gamma + 2)(y - y_0)|y - y_0|^{2\gamma}, \\
B(t,x,y) &= D^2_{x,y}\varphi_\epsilon(t,x,y) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x,y) & -a_1(x,y) \\ -a_1(x,y) & a_1(x,y) \end{pmatrix} + \begin{pmatrix} a_2(x) & 0 \\ 0 & a_2(y) \end{pmatrix}
\end{align*}
\] (5.7)

with
\[
a_1(x,y) = \gamma |x - y|^{2\gamma - 2}I + \gamma(2\gamma - 2)(x - y)(x - y)^*|x - y|^{2\gamma - 4}
\]
and
\[
a_2(x) = \theta(2\gamma + 2)|x - x_0|^{2\gamma}I + 2\theta\gamma(2\gamma + 2)(x - x_0)(x - x_0)^*|x - x_0|^{2\gamma - 2}.
\]

Taking into account (5.5) then applying the result by Crandall et al. (Theorem 3.2, [3]) to the function
\[
u_i(x) - (1 - \lambda)w_i(y) - \varphi_\epsilon(x,y)
\]
at the point \((x_, y_\epsilon)\), for any \(\epsilon_1 > 0\), we can find \(X, Y \in S_k\), such that:
\[
\begin{align*}
\left(\frac{\gamma}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma - 2} + \theta(2\gamma + 2)(x_\epsilon - x_0)|x_\epsilon - x_0|^{2\gamma}, X\right) &\in J^{2,+}(u_i(x_\epsilon)), \\
\left(\frac{\gamma}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma - 2} - \theta(2\gamma + 2)(y_\epsilon - y_0)|y_\epsilon - y_0|^{2\gamma}, Y\right) &\in J^{2,-}(1 - \lambda)w_i(y_\epsilon), \\
-\left(\frac{1}{\epsilon_1} + ||B(x_\epsilon, y_\epsilon)||\right) &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq B(x_\epsilon, y_\epsilon) + \epsilon_1B(x_\epsilon, y_\epsilon)I.
\end{align*}
\] (5.8)

Taking now into account (5.5), and the definition of viscosity solution, we get:
\[
r u_i(x_\epsilon) - \frac{1}{2}Tr[\sigma^*(x_\epsilon)X\sigma(x_\epsilon)] - \frac{\gamma}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma - 2} \\
+ \theta(2\gamma + 2)(x_\epsilon - x_0)|x_\epsilon - x_0|^{2\gamma}, b(x_\epsilon)) - \psi_i(x_\epsilon) \leq 0 \text{ and }
\]
\[
r(1 - \lambda)w_i(y_\epsilon) - \frac{1}{2}Tr[\sigma^*(y_\epsilon)Y\sigma(y_\epsilon)] - \frac{\gamma}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma - 2} \\
- \theta(2\gamma + 2)(y_\epsilon - x_0)|y_\epsilon - x_0|^{2\gamma}, b(y_\epsilon)) - (1 - \lambda)\psi_i(y_\epsilon) \geq 0
\]
which implies that:
\[
r u_i(x_\epsilon) - r(1 - \lambda)w_i(y_\epsilon) \leq \frac{1}{2}Tr[\sigma^*(x_\epsilon)X\sigma(x_\epsilon) - \sigma^*(y_\epsilon)Y\sigma(y_\epsilon)] \\
+ (\frac{\gamma}{\epsilon}(x_\epsilon - y_\epsilon)|x_\epsilon - y_\epsilon|^{2\gamma - 2}, b(x_\epsilon) - b(y_\epsilon)) \\
+ (\theta(2\gamma + 2)(x_\epsilon - x_0)|x_\epsilon - x_0|^{2\gamma}, b(x_\epsilon) + (\theta(2\gamma + 2)(y_\epsilon - x_0)|y_\epsilon - x_0|^{2\gamma}, b(y_\epsilon) \\
+ \psi_i(x_\epsilon) - (1 - \lambda)\psi_i(y_\epsilon).
\] (5.9)

But from (5.7) there exist two constants \(C\) and \(C_1\) such that:
\[
||a_1(x_\epsilon, y_\epsilon)|| \leq C|x_\epsilon - y_\epsilon|^{2\gamma - 2} \text{ and } (||a_2(x_\epsilon)|| \vee ||a_2(y_\epsilon)||) \leq C_1\theta.
\]
As
\[
B = B(x_\epsilon, y_\epsilon) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x_\epsilon, y_\epsilon) & -a_1(x_\epsilon, y_\epsilon) \\ -a_1(x_\epsilon, y_\epsilon) & a_1(x_\epsilon, y_\epsilon) \end{pmatrix} + \begin{pmatrix} a_2(x_\epsilon) & 0 \\ 0 & a_2(y_\epsilon) \end{pmatrix}
\]
then
\[
B \leq \frac{1}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta I.
\]
It follows that:
\[
B + \epsilon_1 B^2 \leq C(\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^{2\gamma - 2} + \frac{\epsilon_1}{\epsilon^2}|x_\epsilon - y_\epsilon|^{4\gamma - 4}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta I
\]
where $C$ and $C_1$ which hereafter may change from line to line. Choosing now $\epsilon_1 = \epsilon$, yields the relation
\[
B + \epsilon_1 B^2 \leq C(\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^{2\gamma - 2} + |x_\epsilon - y_\epsilon|^{4\gamma - 4}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \theta I.
\]
Next
\[
\frac{1}{2}Tr[\sigma^*(x_\epsilon)X_\sigma(x_\epsilon) - \sigma^*(y_\epsilon)Y_\sigma(y_\epsilon)] \leq C(\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^{2\gamma} + |x_\epsilon - y_\epsilon|^{4\gamma - 2}) + C_1 \theta(1 + |x_\epsilon|^2 + |y_\epsilon|^2).
\]
and finally,
\[
\langle (2\gamma + 2)(x\epsilon - x_0)|x\epsilon - x_0|^{2\gamma}, b(x\epsilon) - b(y_\epsilon) \rangle \leq \frac{C2 \theta}{\epsilon}|x\epsilon - y_\epsilon|^{2\gamma}
\]
and finally,
\[
\langle (2\gamma + 2)(y\epsilon - x_0)|y\epsilon - x_0|^{2\gamma}, b(y_\epsilon) \rangle \leq \theta C(1 + |y_\epsilon|^2)|y_\epsilon - x_0|^{2\gamma+1}
\]
So that by plugging into (5.9) we obtain:
\[
u_{i_{\epsilon}}(x_\epsilon) - r(1 - \lambda)w_{i_{\epsilon}}(y_\epsilon) \leq C(\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^{2\gamma} + |x_\epsilon - y_\epsilon|^{4\gamma - 2}) + C_1 \theta(1 + |x_\epsilon|^2 + |y_\epsilon|^2) + \frac{C2 \theta}{\epsilon}|x_\epsilon - y_\epsilon|^{2\gamma} + 
\theta C(1 + |x_\epsilon|)|x_\epsilon - x_0|^{2\gamma+1} + \theta C(1 + |y_\epsilon|)|y_\epsilon - x_0|^{2\gamma+1} + \psi_{i_{\epsilon}}(x_{\epsilon}) - (1 - \lambda)\psi_{i_{\epsilon}}(y_\epsilon).
\]
By sending $\epsilon \rightarrow 0$, $\lambda \rightarrow 0$, $\theta \rightarrow 0$ and taking into account of the continuity of $\psi_{i_{\epsilon}}$, we obtain $u_{i_{\epsilon}}(x_0) - w_{i_{\epsilon}}(x_0) < 0$ which is a contradiction. The proof of Theorem 5 is now complete. □

As a by-product we have the following Corollary:

**Corollary 1** Let $(v^1, ..., v^n)$ be a viscosity solution of (2.4) which satisfies a polynomial growth condition then for $i = 1, ..., m$ and $(t, x) \in \mathbb{R}^k$,
\[
v^i(x) = \sup_{(\delta, \xi) \in \mathcal{D}_0} E\left[ e^{-r_s}u_{a\delta}(X_s^\xi)ds - \sum_{n \geq 1} e^{-r_{n-1}}u_{a\delta}(X_{r_n}^\xi) \right].
\]
6 Numerical results

We consider now some numerical examples of the optimal switching problem (2.4).

Example 1: In this example we consider an optimal switching problem with two modes, where 
\[ r = 100, \ b = x, \ \sigma = \sqrt{2}x, \ g_{12}(x) = \frac{1}{2}|x| + 0.1, \ g_{21}(t,x) = |x| + 0.48, \ \psi_1(x) = \frac{1}{2}x^2 - 0.3x + 1, \]
\[ \psi_2(t,x) = x^2 + 1. \]

Example 2: We now consider the case of 3 modes where 
\[ r = 100, \ b = x, \ \sigma = \sqrt{2}x, \ g_{12}(t,x) = 0.5|x| + 1, \ g_{13}(t,x) = x^2 + 0.5, \ g_{21}(t,x) = |x| + 4, \ g_{23}(t,x) = |x| + 5, \ g_{31}(t,x) = 0.001|x| + 0.1, \]
\[ g_{32}(t,x) = x^2 + |x| + 0.5, \ \psi_1(t,x) = x + 1, \ \psi_2(t,x) = -x - 2 \text{ and finally } \psi_3(t,x) = -x - 2. \]

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Appendix: proof of Theorem 1

The proof consists in showing that for any \( t \leq T \), \( Y^i_t \), as defined by (3.3), is nothing but the expected total profit or the value function of the optimal problem, given that the system is in mode \( i \) at time \( t \). More precisely,

\[
e^{-rt} Y^i_t = \operatorname{ess} \sup_{(\delta,u) \in D_t} E \left[ \int_t^{+\infty} e^{-rs} \psi^i(X_s) ds - \sum_{k \geq 1} e^{-rt} g_{u_{\tau_k-1} u_{\tau_k}}(X_{\tau_k}) |\mathcal{F}_t] \right],
\]

where \( D_t \) is the set of strategies such that \( \tau_1 \geq t \), P-a.s. if at time \( t \) the system is in the mode \( i \).

Let us admit for a moment the following Lemma.

**Lemma 2** For every \( t \geq \tau^*_1 \).

\[
e^{-rt} Y^i_t = \operatorname{ess} \sup_{r \geq t} E \left[ \int_t^T e^{-rs} \psi^i_{u_{\tau_1}^r} (X_s) ds + e^{-rt} \max_{j \in I} (-g_{ij}(X_r) + Y^j_r) |\mathcal{F}_t] \right] \tag{6.1}
\]

From properties of the Snell envelope and at time \( t = 0 \) the system is in mode 1, we have:
Now from Lemma 2 and the definition of $\tau_2^*$ we have:
\[
e^{-rt_2} Y_{\tau_1^*} = E\left[\int_{\tau_1^*}^{\tau_2^*} e^{-rs} \psi_{u_1}(X_s) ds + e^{-rt_2} \max_{j \in I_{\tau_1^*}} (-g_{u_1 j}(X_{\tau_1^*}) + Y_{\tau_1^*}^j)\right] = E\left[\int_{\tau_1^*}^{\tau_2^*} e^{-rs} \psi_{u_1}(X_s) ds + e^{-rt_2} \max_{j \in I_{\tau_2^*}} (-g_{u_1 j}(X_{\tau_2^*}) + Y_{\tau_2^*}^j)\right].\]

It implies that
\[
Y_0^1 = E\left[\int_0^{\tau_2^*} e^{-rs} \psi(X_s, u_1) ds - e^{-rt_2} g_{u_1}(X_{\tau_1^*}) - \sum_{1 \leq k \leq n} e^{-rt_2} g_{u_{k-1} u_k}(X_{\tau_2^*}) + e^{-rt_2} Y_{\tau_2^*}^{u_{\tau_2^*}}\right],
\]
since between 0 and $\tau_1^*$ (resp. $\tau_1^*$ and $\tau_2^*$) the production is in regime 1 (resp. regime $u_{\tau_1^*}$) and then $u_t = 1$ (resp. $u_t = u_{\tau_1^*}$) which implies that

\[
\int_0^{\tau_1^*} e^{-rs} \psi(X_s, u_1) ds = \int_0^{\tau_1^*} e^{-rs} \psi(X_s) ds + \int_{\tau_1^*}^{\tau_2^*} e^{-rs} \psi_{u_1}(X_s) ds.
\]

Now repeating this reasoning as many times as necessary we obtain that for any $n \geq 0$,
\[
Y_0^1 = E\left[\int_0^{\tau_2^*} e^{-rs} \psi(X_s, u_1) ds - \sum_{1 \leq k \leq n} e^{-rt_2} g_{u_{k-1} u_k}(X_{\tau_2^*}) + e^{-rt_2} Y_{\tau_2^*}^{u_{\tau_2^*}}\right].
\]

Then, the strategy $(\delta^*, u^*)$ verify $E[\sum_{n>0} e^{-rt_2}] < +\infty$, otherwise $Y_0^1$ would be equal to $-\infty$ contradicting the assumption that the processes $Y^1$ belong to $S^2$. Therefore, taking the limit as $n \to +\infty$ we obtain $Y_0^1 = J(\delta^*, u^*)$.

To complete the proof it remains to show that the strategy $(\delta^*, u^*)$ it is optimal i.e. $J(\delta^*, u^*) \geq J(\delta, u)$ for any $(\delta, u) \in D$.

The definition of the Snell envelope yields
\[
Y_0^1 \geq E\left[\int_0^{\tau_1^*} e^{-rs} \psi(X_s) ds + e^{-rt_1} \max_{j \in I_{\tau_1^*}} (-g_{1 j}(X_{\tau_1^*}) + Y_{\tau_1^*}^j)\right] \geq E\left[\int_0^{\tau_1^*} e^{-rs} \psi(X_s) ds + e^{-rt_1} (-g_{1 u_1}(X_{\tau_1^*}) + Y_{\tau_1^*}^{u_1})\right].
\]

But, once more using a similar characterization as [6.1], we get
\[
e^{-rt_1} Y_{\tau_1^*}^{u_{\tau_1^*}} \geq E\left[\int_{\tau_1^*}^{\tau_2^*} e^{-rs} \psi_{u_{\tau_1^*}}(X_s) ds + e^{-rt_2} \max_{j \in I_{\tau_2^*}} (-g_{u_{\tau_1^*} j}(X_{\tau_2^*}) + Y_{\tau_2^*}^j)\right] \geq E\left[\int_{\tau_1^*}^{\tau_2^*} e^{-rs} \psi_{u_{\tau_1^*}}(X_s) ds + e^{-rt_2} (-g_{u_{\tau_1^*} u_2}(X_{\tau_2^*}) + Y_{\tau_2^*}^{u_2})\right].
\]
Therefore,

\[
Y_0^1 \geq E\left[\int_0^{\tau_1} e^{-r_s} \psi_1(X_s) ds - e^{-r_{\tau_1}} g_{1u_{\tau_1}}(X_{\tau_1})\right]
+ E\left[\int_{\tau_1}^{\tau_2} e^{-r_s} \psi_{u_{\tau_1}}(X_s) ds + e^{-r_{\tau_2}} (-g_{u_{\tau_1}u_{\tau_2}}(X_{\tau_2}) + Y_{\tau_2}^u)\right]
= E\left[\int_0^{\tau_2} e^{-r_s} \psi(X_s, u_s) ds - e^{-r_{\tau_2}} g_{1u_{\tau_1}}(X_{\tau_1}) - e^{-r_{\tau_2}} g_{u_{\tau_1}u_{\tau_2}}(X_{\tau_2}) + e^{-r_{\tau_2}} Y_{\tau_2}^u\right].
\]

Repeat this argument \( n \) times to obtain

\[
Y_0^1 \geq E\left[\int_0^{\tau_n} e^{-r_s} \psi(X_s, u_s) ds - \sum_{1 \leq k \leq n} e^{-r_{\tau_k}} g_{u_{\tau_{k-1}}u_{\tau_k}}(X_{\tau_k}) + e^{-r_{\tau_n}} Y_{\tau_n}^u\right].
\]

Finally, taking the limit as \( n \to +\infty \) yields

\[
Y_{10} \geq E\left[\int_0^{+\infty} e^{-r_s} \psi(X_s, u_s) ds - \sum_{k \geq 1} e^{-r_{\tau_k}} g_{u_{\tau_{k-1}}u_{\tau_k}}(X_{\tau_k})\right].
\]

Hence, the strategy \((\delta^*, u^*)\) is optimal. We proceed to the proof of Lemma 2.

**Proof of Lemma 2.** From (6.3) we have for any \( i \in I \) and \( t \geq 0 \)

\[
e^{-rt} Y_t^i = \text{ess sup}_{t \geq t_0} E\left[\int_t^{t_1} e^{-r_s} \psi_i(X_s) ds + e^{-r_t} \max_{j \in I^{-1}_t} (-g_{ij}(X_t) + Y_t^j)\right].
\]

(6.2)

This also means that the process \((e^{-rt} Y_t^i + \int_0^t e^{-r_s} \psi_i(X_s) ds)_{t \geq 0}\) is a supermartingale which dominates

\[
\left(\int_0^t e^{-r_s} \psi_i(X_s) ds + e^{-rt} \max_{j \in I^{-1}_t} (-g_{ij}(X_t) + Y_t^j)\right)_{t \geq 0}.
\]

This implies that the process \((\mathbb{I}_{[u_{\tau_1} = i]}(e^{-rt} Y_t^i + \int_{\tau_1}^t e^{-r_s} \psi_i(X_s) ds))_{t \geq \tau_1^*}\) is a supermartingale which dominates

\[
(\mathbb{I}_{[u_{\tau_1} = i]}(\int_{\tau_1}^t e^{-r_s} \psi_i(X_s) ds + e^{-rt} \max_{j \in I^{-1}_t} (-g_{ij}(X_t) + Y_t^j))_{t \geq \tau_1^*}.
\]

Since \( I \) is finite, the process \((\sum_{i \in I} \mathbb{I}_{[u_{\tau_1} = i]}(e^{-rt} Y_t^i + \int_{\tau_1}^t e^{-r_s} \psi_i(X_s) ds))_{t \geq \tau_1^*}\) is also a supermartingale which dominates \((\sum_{i \in I} \mathbb{I}_{[u_{\tau_1} = i]}(\int_{\tau_1}^t e^{-r_s} \psi_i(X_s) ds + e^{-rt} \max_{j \in I^{-1}_t} (-g_{ij}(X_t) + Y_t^j))_{t \geq \tau_1^*}\).

Thus, the process \((e^{-rt} Y_t^{u_{\tau_1}^*} + \int_{\tau_1^*}^t e^{-r_s} \psi_{u_{\tau_1}^*}(X_s) ds)_{t \geq \tau_1^*}\) is a supermartingale which is greater than

\[
\left(\int_{\tau_1^*}^t e^{-r_s} \psi_{u_{\tau_1}^*}(X_s) ds + e^{-rt} \max_{j \in I^{-1}_{u_{\tau_1}^*}} (-g_{u_{\tau_1}^*j}(X_t) + Y_t^j)\right)_{t \geq \tau_1^*}.
\]

To complete the proof it remains to show that it is the smallest one which has this property and use the characterization of the Snell envelope see e.g. [7, 14, 16].

Indeed, let \((Z_t)_{t \geq \tau_1^*}\) be a supermartingale of class \([D]\) such that, for any \( t \geq \tau_1^*\),

\[
Z_t \geq \int_{\tau_1^*}^t e^{-r_s} \psi_{u_{\tau_1}^*}(X_s) ds + e^{-rt} \max_{j \in I^{-1}_{u_{\tau_1}^*}} (-g_{u_{\tau_1}^*j}(X_t) + Y_t^j).
\]

It follows that for every \( t \geq \tau_1^*\),

\[
\mathbb{I}_{[u_{\tau_1} = i]} Z_t \geq \mathbb{I}_{[u_{\tau_1} = i]}(\int_{\tau_1^*}^t e^{-r_s} \psi_i(X_s) ds + e^{-rt} \max_{j \in I^{-1}_t} (-g_{ij}(X_t) + Y_t^j)).
\]
But, the process \( \mathbb{I}_{[u^*_1]} Z_t \) \( t \geq \tau^*_1 \) is a supermartingale and for every \( t \geq \tau^*_1 \),

\[
\mathbb{I}_{[u^*_1]} e^{-r Y_t^i} = \text{ess sup}_{t \geq t} E[\mathbb{I}_{[u^*_1]}(\int_t^\tau e^{-r s} \psi_i(X_s) ds + e^{-r \tau} \max_{j \in \mathcal{I}^{-u^*_1}} (g_{ij}(X_\tau) + Y^j_t)) | \mathcal{F}_t].
\]

It follows that, for every \( t \geq \tau^*_1 \),

\[
\mathbb{I}_{[u^*_1]} Z_t \geq \mathbb{I}_{[u^*_1]} e^{-r Y_t^i} + \int_{\tau^*_1}^t e^{-r s} \psi_i(X_s) ds.
\]

Summing over \( i \), we get, for every \( t \geq \tau^*_1 \),

\[
Z_t \geq e^{-r Y_t^{u^*_1}} + \int_{\tau^*_1}^t e^{-r s} \psi_{u^*_1}(X_s) ds.
\]

Hence, the process \( e^{-r Y_t^{u^*_1}} + \int_{\tau^*_1}^t e^{-r s} \psi_{u^*_1}(X_s) ds \) \( t \geq \tau^*_1 \) is the Snell envelope of

\[
(\int_{\tau^*_1}^t e^{-r s} \psi_{u^*_1}(X_s) ds + e^{-r \tau} \max_{j \in \mathcal{I}^{-u^*_1}} (g_{ij}(X_\tau) + Y^j_t))_{t \geq \tau^*_1},
\]

whence Lemma 2. □

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