Complete minimal discs in Hadamard manifolds

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Abstract

We solve the asymptotic Plateau problem for disc type minimal surfaces in certain Hadamard manifolds the sectional curvature of which may be unbounded from below.

1 Introduction

The study of complete minimal submanifolds of negatively curved Riemannian manifolds was initiated by Anderson when he showed that each closed submanifold of the sphere at infinity of the hyperbolic $n$–space is the boundary (relative to the compactification of the hyperbolic space) of an area-minimizing variety [1]. In the sequel a considerable number of related and more general results has appeared in the literature. These authors use the geometric measure theory approach like Anderson or they work in the graph setting (see the fairly complete survey [3]). As far as we can see all these results require uniform negative lower and upper bounds for the sectional curvatures of the ambient space.

In this paper we use the methods of the classical Plateau problem to prove the existence of minimal disc type surfaces with prescribed Jordan curve at
infinity as their boundary. The main novelty of our paper, as we believe, lies
in the fact that we allow unbounded sectional curvatures. Instead, we require
that the sectional curvatures approach those of a rotationally symmetric
metric at infinity, in a suitable sense. A precise description of our results
follows immediately.

Let $N^n$, $n \geq 3$, be a Hadamard manifold, that is, $N$ is a connected,
simply connected, complete, $n$–dimensional Riemannian manifold such that
$K_N \leq 0$, where $K_N$ is the supremum of the sectional curvatures of $N$ at
any plane of the tangent space at any point of $N$. For the sake of sim-
plicity, we may assume that $N^n$ is $C^\infty$ smooth. Recall that the asymptotic
boundary $\partial_\infty N$ of $N$ is defined as the set of all equivalence classes of unit
speed geodesic rays in $N$; two such rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are equivalent if
$\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$, where $d$ is the Riemannian distance in $N$. The so
called geometric compactification $\overline{N}$ of $N$ is then given by $\overline{N} := N \cup \partial_\infty N$,
endowed with the cone topology. It is well known that $\overline{N}$ is homeomorphic
to the closed unit ball of $\mathbb{R}^n$ (see [6] or [14], Ch. 2). For any subset $S \subset N$, we define
$\partial_\infty S = \partial_\infty N \cap S$.

Setting $r(x) = d(x, o)$, $x \in N$, where $o$ is a fixed point in $N$, $d$ the
Riemannian distance and

\[
\begin{align*}
k^+(s) &= \sup \{ K(\text{grad } r(x), Y) \mid r(x) = s, Y \in T_xN \} \\
k^-(s) &= \inf \{ K(\text{grad } r(x), Y) \mid r(x) = s, Y \in T_xN \}
\end{align*}
\]  

we prove:

**Theorem 1** Assume that there is a continuous nonincreasing function $k_0$
and an integrable function $\varphi$ on $[0, +\infty)$ such that

(1) \[ k_0(s) - \varphi(s) \leq k^-(s) \leq k^+(x) \leq k_0(s) \leq 0, \ 0 \leq s < +\infty \]

(2) there are numbers $R > 0$, $a > 0$ such that $k_0(s) \leq -a^2$ for $s \geq R$.

Then there is a Riemannian metric $\langle , \rangle_B$ in the unit ball $B \subset \mathbb{R}^n$ such that $(B, \langle , \rangle_B)$ is isometric to $N$ and the asymptotic boundary $\partial_\infty B$ of $B$ is identified with the topological boundary $\partial B$ of $B$. In this model $(B, \langle , \rangle_B)$ of $N$, given a Euclidean rectifiable curve $\Gamma \subset \partial_\infty B$, there is a proper, minimal (possibly branched) immersion $u : D \to B$, where $D$ is the unit disc in $\mathbb{R}^2$, such that $u$ belongs to the Sobolev space $H^1_2(D, \mathbb{R}^n)$ and the trace of $u|\partial D$
parametrizes $\Gamma$ monotonically. In the case $n = 3$ the map $u$ is an embedding
of $D$. 

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It follows from the properties of the Sobolev trace that $\partial_\infty u(D) \supset \Gamma$ but we do not know if $\partial_\infty u(D) = \Gamma$ under the hypothesis of Theorem 1. However we may conclude this equality if one requires additionally that $N$ has some global convexity property. This is the case if $N$ satisfies the strict convexity condition, as defined in [13], namely: Given $x \in \partial_\infty N$ and a relatively open subset $W \subset \partial_\infty N$ containing $x$, there exists a $C^2$-open subset $\Omega \subset \overline{N}$ such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$, where $\text{Int}(\partial_\infty \Omega)$ denotes the interior of $\partial_\infty \Omega$ in $\partial_\infty N$, and $N \setminus \Omega$ is convex.

We remark that under the assumption $K_N \leq -a^2 < 0$, the strict convexity condition is equivalent to the convex conic neighborhood condition as defined by H. Choi in [2]. It is proved in [13] that if $K_N \leq -a^2$ then $N$ satisfies the strict convexity condition either if the metric of $N$ is rotationally symmetric or if the sectional curvature of $N$ decays at most exponentially (Theorems 13 and 14 of [13]).

**Theorem 2** Under the same hypothesis of Theorem 1 if, additionally, $N$ satisfies the strict convexity condition then, besides the conclusions of Theorem 1 it holds $\partial_\infty u(D) = \Gamma$. This holds, in particular, if the metric of $M$ is rotationally symmetric or if the sectional curvature of $N$ decays at most exponentially.

### 2 Differential geometric preliminaries

**Lemma 3** Let $k \in C^0([0, \infty))$, $k \leq 0$, and let $F$ be the solution of the initial value problem

$$F'' + kF = 0, \quad F(0) = 0, \quad F'(0) = 1. \quad (2)$$

Then we have:

(i) For all $s > 0$ it holds

$$s \frac{F'(s)}{F(s)} \geq 1 \quad (3)$$

and, if for some constants $R > 0$, $a > 0$ the inequality $k(s) < -a^2$ holds for $s \geq R$, then it follows that

$$\frac{F'}{F} \geq \min \left\{ \frac{1}{R}, a \right\}. \quad (4)$$

(ii) If $k$ is nonincreasing then $G/(sF)$ is nonincreasing, too, where $G(s) = \int_0^s F(t)dt$. 

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(iii) Let \( F_0 \) be a further solution of (2) with \( k_0 \) replacing \( k \), \( k_0 \leq 0 \). We suppose that, like in (i), there are numbers \( R > 0 \), \( a > 0 \) such that \( k_0(s) \leq -a^2 \) for \( s \geq R \) and, furthermore, that
\[
|k(s) - k_0(s)| \leq \varphi(s), \quad 0 \leq s < +\infty
\]
for some integrable function \( \varphi \). Then the estimate
\[
e^{-C} \leq \frac{F(s)}{F_0(s)} \leq e^C, \quad 0 \leq s < +\infty
\]
holds with \( C = (R + \pi/(2a)) \int_0^{+\infty} \varphi(t)dt \).

**Proof.** (i) Using (2) one computes
\[
\left( \frac{F'}{F} \right)' = \frac{F''F - F'^2}{F^2} = -k - \left( \frac{F'}{F} \right)^2,
\]
so that \( g := F'/F \) solves the initial value problem
\[
g' = -k - g^2, \quad g(0) = +\infty. \tag{4}
\]
Since \( k \leq 0 \) we have \((1/g)' = -g'/g^2 \leq 1\) and hence \( g(s) \geq 1/s \), proving (3): in particular
\[
g(s) \geq \frac{1}{R} \text{ for } 0 \leq s \leq R. \tag{5}
\]
Let us now assume that \( g(R) \leq a \). From (4) and the hypothesis we have the inequality
\[
g'(s) > a^2 - g(s)^2, \quad s \geq R, \tag{6}
\]
from which it would follow that \( g'(R) > 0 \). Let us then consider the maximal interval \( [R, R_1) \) on which \( g' \) is positive. On this interval holds trivially \( g(s) \geq g(R) \geq 1/R \) by (5). If \( R_1 = +\infty \) we are done, if not then \( g'(R_1) = 0 \) and hence \( g(R_1) > a \) by (6). Thus, it remains to consider the case that \( g(R_1) > a \) for some \( R_1 \geq R \). We then consider the maximal interval \( [R_1, R_2) \) on which \( g(s) > a \). On this interval we get from (6)
\[
\frac{1}{2a} \left( \ln \frac{g - a}{g + a} \right)' = \frac{g'}{g^2 - a^2} > -1,
\]
what upon integration yields
\[
\frac{g(s) - a}{g(s) + a} \geq \frac{g(R_1) - a}{g(R_1) + a} e^{-2a(s-R_1)} > 0.
\]

This shows that \( R_2 = +\infty \), i.e. \( g(s) > a \) for \( s \geq R_1 \).

(ii) The statement is equivalent with
\[
\left( \ln \frac{sF}{G} \right)' \geq 0
\]
that is
\[
\left( \ln \frac{sF}{G} \right)' = \frac{1}{s} + \frac{F'}{F} - \frac{G'}{G} = \frac{1}{s} + \frac{F'G - FG'}{FG}.
\]

We compute
\[
(F'G - FG')' = F''G - FG'' = -kFG - FG'' = -F(G'' + kG).
\]

Integrating (2) and using that \( k(s) \) is nonincreasing, we obtain
\[
0 = F'(s) - 1 + \int_0^s k(t)F(t)dt \\
\geq F'(s) - 1 + k(s) \int_0^s F(t)dt = G''(s) + k(s)G(s) - 1.
\]

The last inequality yields
\[
(F'G - FG')' \geq -F
\]
and, upon integration and observing \( G(0) = 0, G'(0) = F(0) = 0, \)
\[
F'G - FG' \geq -G'.
\]

This leads to
\[
\left( \ln \frac{sF}{G} \right)' \geq \frac{1}{s} - \frac{1}{F},
\]
proving (ii) since \( F(s) \geq s \) for all \( s \) because of \( F'(0) = 1 \) and \( F'' \geq 0 \).

(iii) One computes
\[
| (F'F_0 - FF_0')' | = | F''F_0 - FF_0'' | = | - (k - k_0) FF_0 | \leq \varphi FF_0.
\]
By integration we obtain

\[ \left| \frac{F'(s)F_0(s) - F(s)F_0'(s)}{F(s)F_0(s)} \right| \leq \frac{1}{F(s)F_0(s)} \int_0^s \varphi(t) F(t)F_0(t) dt \]

\[ \leq \int_0^s \varphi(t) \frac{F_0(t)}{F_0(s)}dt =: \Phi(s) \quad (7) \]

where we used that \( F \) is non-decreasing. We would like to show that \( \Phi \) is integrable in \([0, +\infty)\) and split the integral in two parts:

\[ \int_0^R \Phi(s)ds \leq \int_0^R \int_0^R \varphi(t)ds \leq R \int_0^{+\infty} \varphi(t)dt, \]

\[ \int_R^{+\infty} \Phi(s)ds = \int_R^{+\infty} \varphi(t) \int_0^{+\infty} \frac{F_0(t)}{F_0(s)}dsdt \leq \int_0^{+\infty} \varphi(t) \int_\tau^{+\infty} \frac{F_0(\tau)}{F_0(s)}dsdt \]

with \( \tau := \max \{t, R\} \). By Taylor expansion and the assumption that \( k_0(s) \leq -a^2 \) for \( s \geq R \) we obtain for \( s \geq \tau \):

\[ F_0(s) = F_0(\tau) + F'_0(\tau)(s-\tau) + \frac{1}{2} F''_0(\theta)(s-\tau)^2 \]

\[ \geq F_0(\tau) - \frac{1}{2} k_0(\theta) F_0(\theta)(s-\tau)^2 \]

\[ \geq F_0(\tau) \left( 1 + \frac{1}{2} a^2 (s-\tau)^2 \right). \]

This implies

\[ \int_\tau^{+\infty} \frac{F_0(\tau)}{F_0(s)}ds \leq \int_0^{+\infty} \frac{ds}{1 + \frac{1}{2} as^2} = \frac{\pi}{\sqrt{2a}} \]

and hence

\[ \int_0^{+\infty} \Phi(s)ds \leq \left( R + \frac{\pi}{\sqrt{2a}} \right) \int_0^{+\infty} \varphi(t)dt =: C. \]

Rewriting \((7)\) in the form

\[ \left| \frac{F'}{F} - \frac{F'_0}{F_0} \right| \leq \Phi \]

we obtain upon integration

\[ \left| \ln \frac{F(s)}{F(\delta)} - \ln \frac{F_0(s)}{F_0(\delta)} \right| \leq C \]
for any $\delta > 0$ or

$$-C \leq \ln \frac{F(s)}{F_0(s)} - \ln \frac{F(\delta)}{F_0(\delta)} \leq C.$$ 

Since

$$\frac{F(\delta)}{F_0(\delta)} \to \frac{F'(0)}{F'_0(0)} = 1$$

as $\delta \to 0$, the statement follows.  

In the following we consider a $n$–dimensional Hadamard manifold $N$, whose Riemannian metric is denoted by $\langle \, , \rangle$. We fix a point $o \in N$ and consider the distance function $r(x) = d(x, o)$ to $o$. We investigate some geometric properties of $N$ by means of the functions $k^+, k^−$ defined by (1).

**Lemma 4** Let $F \leq 0$ be a continuous function such that $k^+(s) \leq k(s)$ for $s \in [0, +\infty)$. Let $F$ be the function defined as in Lemma 3 with the actual $k$. Then the inequality

$$\text{Hess} r(x)(u, u) \geq \frac{F'(r(x))}{F(r(x))} \langle u, u \rangle$$

holds for all $x \in N \setminus \{o\}$ and all $u \in T_xN$ with $u \perp \text{grad} r(x)$.

**Proof.** Let $S$ be the rotationally symmetric manifold with origin $o_S \in S$ such that, in polar coordinates with origin $o_S$, the metric of $S$ is given by $dt^2 = ds^2 + F(s)^2d\Theta^2$. If $z \in S \setminus \{o_S\}$ and $r_S(s) = d_S(z, o_S)$ then

$$K_S(z)(\text{grad} r_S, Z) = -\frac{F''(r_S(z))}{F(r_S(z))} = k(r_S(z)),$$

for any $Z \in T_zS$, $Z \perp \text{grad} r_S$ (see [2]). We notice that since $k(s) \leq 0$, the exponential map of $S$ at $o_S$ is a diffeomorphism and then $r_S$ is smooth on $S \setminus \{o_S\}$.

Let $x \in N \setminus \{o\}$ be given. The same proof as of the usual Hessian Comparison Theorem (see Theorem 1.1 of [14]) gives

$$\text{Hess} r(x)(u, u) \geq \text{Hess} r_S(z)(u_S, u_S) = \frac{F'(r_S(z))}{F(r_S(z))} \langle u_S, u_S \rangle_S$$

$$= \frac{F'(r(x))}{F(r(x))} \langle u, u \rangle,$$

where $z \in S$, $r_S(z) = r(x)$, and $u_S$ is any vector in $T_zS$ orthogonal to $\text{grad} r_S(z)$ such that $\langle u, u \rangle = \langle u_S, u_S \rangle_S$.  


Corollary 5 Setting $G(s) = \int^s F(t)\,dt$ the inequality
\[ \text{Hess } G \circ r (x) (u, u) \geq F'(r(x)) \langle u, u \rangle \]
holds for all $x \in N \setminus \{o\}$ and $u \in T_x N$.

Proof. One has
\[ \text{Hess } G \circ r (u, u) = G'(r) \text{ Hess } r (u, u) + G''(r) \langle \text{grad } r, u \rangle^2, \]
so that for $u \perp \text{grad } r(x)$ we obtain from Lemma 4
\[ \text{Hess } G \circ r (u, u) = F(r) \text{ Hess } r (u, u) \geq F'(r) \langle u, u \rangle. \]
Since $\text{Hess } r (\text{grad } r, \text{grad } r) = 0$ we get
\[ \text{Hess } G \circ r (\text{grad } r, \text{grad } r) = G''(r) = F'(r). \]

We now prove an extension of the classical monotonicity formula for minimal surfaces \[1\]. It will be obvious from the proof that a corresponding result holds for higher dimensional minimal submanifolds. The proof is an adaptation of the proof of Theorem 1 of \[1\] and we refer the reader to this paper for details.

Proposition 6 Let $M$ be a minimal surface in $N$ which has no boundary inside some geodesic ball $B_R$ of $N$ centered at $o$. Assume that $k^+(s) \leq k(s)$ for $s \in [0, +\infty)$ where $k \leq 0$ is a continuous non-increasing function. Let $F$ be the function determined by $k$ as in Lemma 3. Then the function
\[ r \mapsto \frac{\text{Area}(M \cap B_r)}{\int_0^1 F(t)\,dt}, \quad 0 < r \leq R \quad (8) \]
is non-decreasing.

Proof. For any point $z \in M$, let $e_1, e_2$ be an orthonormal basis of $T_z M$ such that $e_2 = \text{grad}^\top r / |\text{grad}^\top r|$, where $\text{grad}^\top r$ is the orthogonal projection of $\text{grad } r$ on $T_z M$. From the first variational formula we have
\[ \sum_{j=1}^2 \int_M \langle \nabla_{e_j} E, e_j \rangle = 0 \]
where $E$ is any smooth vector field with compact support on $M$. Choosing $E$ of the form

$$E = f(r) \chi_s r \operatorname{grad} r,$$

where $f$ is a smooth function satisfying $f' \leq 0$, to be explicitly given later, and $\chi_s$ is a smooth approximation to the characteristic function of $[0, s]$, we obtain

$$\int f \chi_s \sum_{j=1}^{2} \langle \nabla_{e_j} r \operatorname{grad} r, e_j \rangle = -\int (f' \chi_s + f \chi'_s) r |\operatorname{grad} r|^2$$

and then

$$\int \left( f \chi_s \sum_{j=1}^{2} \langle \nabla_{e_j} r \operatorname{grad} r, e_j \rangle + r f' \chi_s \right) \leq -\int r f \chi'_s$$

Using Lemma 4 we obtain

$$\sum_{j=1}^{2} \langle \nabla_{e_j} r \operatorname{grad} r, e_j \rangle \geq \langle e_2, \operatorname{grad} r \rangle^2 + \langle e_2, e_2^\perp \rangle^2 \frac{r F'(r)}{F(r)} + \frac{r F'(r)}{F(r)}$$

where $e_2^\perp$ is the unit vector along the projection of $e_2$ on the tangent plane of the geodesic sphere centered at $o$. From (3) it follows that

$$\langle e_2, \operatorname{grad} r \rangle^2 + \langle e_2, e_2^\perp \rangle^2 \frac{r F'(r)}{F(r)} \geq \langle e_2, \operatorname{grad} r \rangle^2 + \langle e_2, e_2^\perp \rangle^2 = 1$$

and then

$$\int \chi_s \left[ f \left( 1 + \frac{r F'}{F} \right) + r f' \right] \leq -\int r f \chi'_s. \quad (9)$$

Choosing $f$ as a solution of the ODE

$$f(r) \left( 1 + \frac{r F'(r)}{F(r)} \right) + r f'(r) = 1$$

with $f(0) = 1/2$ we obtain

$$f(r) = \frac{\int_0^r F(t) \, dt}{r F(r)}.$$
From Lemma 3 (ii) we have that $f' \leq 0$. Then, setting $v(r) = \text{Vol } (B_r)$ and using (9) we arrive at $v(r) \leq r f(r) v'(r)$ from which we easily obtain (8).

In what follows we want to compare the metric on the given manifold $N$ with the metric of a rotationally symmetric complete background manifold $S_0$ of non-positive sectional curvature $k_0$ given as a function of the distance to the origin $o_0$ in $S_0$. As in Lemma 3 (iii) we assume that $k_0(s) \leq -a^2$, $s \geq R$, for some constants $a >, R > 0$ and, as before, we denote by $F_0$ the solution of $F_0'' + k_0 F_0 = 0$, $F_0(0) = 0$, $F_0'(0) = 1$.

**Lemma 7** Let $\gamma : [0, +\infty) \to N$ be a unit speed geodesic, $\gamma(0) = o$, and $J$ be a normal Jacobi field along $\gamma$, $J(0) = 0$, $\|J'(0)\| = 1$.

(i) If $k^-(s) \geq k(s)$ for some continuous function $k$ and if $F$ is a solution of (2) then it follows that $\|J(s)\| \leq F(s)$, $0 \leq s < +\infty$. Likewise, if $k^+(s) \leq k(s) \leq 0$ then one has $\|J(s)\| \geq F(s)$.

(ii) We suppose that there are non-negative integrable functions $\varphi$ such that

$$k_0(s) - \varphi(s) \leq k^-(s) \leq k^+(s) \leq k_0(s) \leq 0. \quad (10)$$

Then the estimate

$$F_0(s) \leq \|J(s)\| \leq e^{-C} F_0(s)$$

holds with

$$C = \left( R + \frac{\pi}{\sqrt{2a}} \right) \int_0^\infty \varphi(t) dt.$$

**Proof.** (i) It is an immediate consequence of Rauch’s comparison theorem by comparing $\|J(s)\|$ with the norm of a Jacobi field, satisfying the same initial conditions as $J$, in a rotationally symmetric manifold with curvature $k(s)$.

(ii) We may assume that $\varphi$ is continuous and apply part (i) with $k(s) = k_0(s) - \varphi(s)$ and Lemma 3 (iii). ■

**Corollary 8** Let $\text{Exp} : T_o N \to N$ be the exponential map at the base point $o$ and let the condition (10) above be satisfied. Then there is a constant $C$ such that

$$\|w\| \frac{F_0(\|v\|)}{\|v\|} \leq d\text{Exp}(v)w \leq C \|w\| \frac{F_0(\|v\|)}{\|v\|}$$

holds for $v, w \in T_o N$ with $v \perp w$, $v \neq 0$. 
Proof. As is well known, $d \text{Exp}(v)w = J(1)$ where $J$ is the Jacobi field along $t \mapsto \text{Exp}(tv)$ with initial condition $J(0) = 0$ and $J'(0) = w$. Let $J_1$ be the Jacobi field along $t \mapsto \text{Exp}(t\|v\|^{-1}v)$ with $J_1(0) = 0$, $J_1'(0) = \|w\|^{-1}w$. Then

$$\tilde{J} := \frac{\|w\|}{\|v\|} J_1 (\|v\| t)$$

is a Jacobi field along $t \mapsto \text{Exp}(tv)$ with $\tilde{J}(0) = 0$, $\tilde{J}'(0) = \|w\| J_1'(0) = w$. Hence, $\tilde{J} = J$, $J(1) = \frac{\|w\|}{\|v\|} J_1 (\|v\| t)$ and the Corollary follows from Lemma 7 (ii).

In the next lemma we obtain a special metric in the ball model for complete rotationally symmetric metrics with sectional curvature bounded by a negative constant:

**Lemma 9** Let a complete rotationally symmetric metric of radial sectional curvature $k$ be given, where $k$ is a continuous function of arclength $s \in [0, \infty)$. We assume furthermore that $k(s) \leq 0$ and that there are numbers $R > 0$, $a > 0$ such that $k(s) \leq -a^2$ for $s \geq R$. Then there are coordinates defined in the unit ball $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$ in which the metric takes the form

$$f'(\|x\|)^2 dx^2$$

where $dx^2$ stands for the Euclidean metric and $f \in C^2([0,1)) \cap C^3((0,1))$ is the inverse function of

$$g(r) := e^{-\int_r^\infty \frac{dt}{F(t)}}, \quad 0 < r < \infty,$$

and $F$ is the solution of (2) with the given curvature $k$. The function $g$ is of class $C^2([0, +\infty)) \cap C^3((0, +\infty))$ with $g'(r) > 0$ for $r \geq 0$ and hence $f'(t) > 0$ for $t \in [0,1]$.

Proof. It follows from Lemma 3 (i) that $F$ grows at least exponentially so that $\int_r^{+\infty} dt/F(t)$ is finite for each $r > 0$. Since $F \in C^2([0, +\infty))$, $F(0) = 0$ and $F'(0) = 1$ we get

$$\frac{1}{F(t)} = \frac{1}{t} + b(t)$$

for some function $b \in C^0([0, +\infty))$ from which it follows that $g(r) \to 0$ and $g(r)/r \to c$ ($r \to 0$) for some $c > 0$. Hence

$$g'(r) = \frac{g(r)}{F(r)} = \frac{g(r)}{r} \frac{r}{F(r)} \to c, \quad r \to 0$$
and $g \in C^1([0, +\infty))$ and $g'(r) > 0$ for $r \geq 0$. Since
\[ g'' = \frac{g'F - gF''}{F^2} = \frac{g(1 - F')}{F^2} \]
we conclude that $\lim_{r \to 0} g''(r)$ exists and thus $g \in C^2([0, +\infty))$.

Introducing the coordinates $(r, \theta) \in [0, 1) \times S^{n-1}$ by $x = g(r)\theta$ we find by direct computation
\[ f'(|x|)^2 \, dx^2 = dr^2 + F(r)^2 d\theta^2 \]
and, by a well known formula, the radial sectional of this metric is $-F''/F = k$. ■

In the sequel we construct a special ball model of $N$. We denote by $S$ the rotationally symmetric Hadamard manifold with origin $o_S$ with radial sectional curvature $k_0$ given as a function of the distance to $o_S$. We require that $k_0(s) \leq -a^2$ for $s \geq R$, with positive $a, R$. It follows from Lemma 9 that $S$ is isometric to the open unit ball $B \subset \mathbb{R}^n$ endowed with a metric of the form
\[ \langle u, v \rangle_0 = f'(|x|)^2 \langle u, v \rangle, \quad u, v \in T_xB, \] (11)
where $( \cdot, \cdot)$ denotes the Euclidean scalar product and $f$ is given by Lemma 9.

**Proposition 10** $N$ is isometric to an open unit ball $B \subset \mathbb{R}^n$ with a metric of the form
\[ \langle u, v \rangle = f'(|x|)^2 \langle u, v \rangle_b, \quad u, v \in T_xB, \] (12)
where $f$ is the function given in Lemma 9 and $\langle u, v \rangle_b$ is a Riemannian metric on $B$ which is uniformly bounded from above and from below by the Euclidean metric.

The balls with center $0 \in B$ in the metric (12) above are at the same time balls in the metric (11) of the same radius and the geodesics of (12) passing through $0 \in B$ are straight line segments. In particular, it follows that the asymptotic boundary $\partial_\infty B$ of $B$ with respect to the metric (12) is identified with the topological boundary $\partial B$ of $B$ via the map that associates to each point of $x \in \partial B$ the equivalence class of the geodesic ray from 0 to $x$.

**Proof.** Let
\[ \text{Exp} : T_oN \to N, \quad \text{exp} : T_0S \to S \]
be the corresponding exponential maps and let us choose a linear isometry $j : T_o S \to T_o N$. We then define the diffeomorphism

$$\Phi : \text{Exp} \circ j \circ \text{exp}^{-1} : S \to N.$$  

Since the geodesics of $S$ passing through $0 \in B$ are straight line segments and since the exponential maps map straight lines through the origin to geodesics, it is clear that $\Phi$ maps the straight lines segments passing through $0 \in B$ to geodesics of $N$ passing through the base point $o \in N$. The classical Gauss lemma says that the exponential map is an isometry in the radial direction, i.e.,

$$\langle d \text{Exp}(v)(v), d \text{Exp}(v)(w) \rangle = \langle v, w \rangle, \ v, w \in T_o N$$

$$\langle d \text{exp}(v)(v), d \text{exp}(v)(w) \rangle = \langle v, w \rangle_0, \ v, w \in T_0 S. \quad (13)$$

This implies

$$\| d\Phi(x) \text{ grad } r_0(x) \| = 1 \quad (14)$$

and hence $\Phi$ maps balls with center $0 \in S$ onto balls in $N$ centered at $o$ with the same radius.

We now claim that there is a constant $C > 0$ such that

$$\| u \|_0 \leq \| d\Phi(x)u \| \leq C \| u \|_0 \quad (15)$$

holds for $x \in S$ and $u \in T_x S$. If this is shown then Proposition 10 is proved since (15) can be rewritten as

$$(u, u) = \left( \frac{\langle u, u \rangle_0}{f'(|x|)^2} \right) \leq \frac{1}{f'(|x|)^2} \langle d\Phi(x)u, d\Phi(x)u \rangle$$

$$\leq C^2 \left( \frac{\langle u, u \rangle_0}{f'(|x|)^2} \right) = C^2 (u, u).$$

The inequality (15) is already clear for $u := \text{ grad } r_0(x)$ by (14). Let then $u \in T_x S$ with $u \perp \text{ grad } r_0(x)$. It follows from (13) that

$$w := d(\text{exp}^{-1})(x)(u) \perp d(\text{exp}^{-1})(x) \text{ grad } r_0(x) = \lambda \text{ exp}^{-1}(x)$$

for some $\lambda$ and hence $jw \perp j \text{ exp}^{-1}(x)$ so that we obtain from Corollary 8 with $v := \text{ exp}^{-1}(x)$

$$\|w\| \frac{F_0(||v||)}{||v||} = \|jw\| \frac{F_0(||jv||)}{||jv||} \leq \|d\text{Exp}(jv)jw\| = \|d\Phi(x)u\|$$

$$\leq C \|jw\| \frac{F_0(||jv||)}{||jv||} = C \|w\|_0 \frac{F_0(||v||_0)}{||v||_0}.$$
But for the rotationally symmetric metric on \( S \) we have
\[
\|w\|_0 \frac{F_0(\|v\|_0)}{\|v\|_0} = \|d\exp(v)w\| = \|u\|_0.
\]

3 The expanding minimal discs

We remind the reader of the definitions of the functions \( k^+ \) and \( k^- \) given in (1) and the assumptions of Theorem 1, namely:

(1) There are a continuous, nonincreasing, nonpositive function \( k_0 \) and an integrable function \( \varphi \) such that \( k_0(s) - \varphi(s) \leq k^-(s) \leq k^+(x) \leq k_0(s) \), \( 0 \leq s < +\infty \),

(2) there are numbers \( R > 0, a > 0 \) such that \( k_0 \leq -a^2 \) for \( s \geq R \),

will be assumed for the rest of the paper.

Let \( B \) be the ball model of \( N \) given by Proposition 10. Given a rectifiable Jordan curve \( \Gamma \subset \partial\infty B \) let \( \Gamma_1 \) be the radial projections of \( \Gamma \) onto the unit sphere (in the metric of \( N \)) centered at \( 0 \in B \) and let \( \gamma := \text{Exp}^{-1}(\Gamma_1) \). We may assume that \( \|\gamma'\| = 1 \) and define the family of Jordan curves \( \Gamma_R \subset N \), \( 1 < R < +\infty \), by \( \Gamma_R = \text{Exp}(R\gamma) \). Morrey’s existence theorem \([11], [12]\) guarantees, for each \( R \), the existence of a minimizing disc \( M_R \) with boundary \( \Gamma_R \) given by a harmonic, conformal, possibly branched immersion

\[
u_R : D \to N, \quad D = \{ z \in \mathbb{R}^2 \mid |z| < 1 \}
\]

where \( u_R \in C^2(D) \cap C^0(\overline{D}) \) and \( u_R|\partial D \) parametrizes \( \Gamma_R \) one-to-one. We estimate the area of \( M_R \) by comparison with the cone \( c(s,t) = \text{Exp}(t\gamma(s)) \), \( 0 \leq t \leq R, \ 0 \leq s \leq L \). By direct computation and Corollary 8

\[
\text{area}(c) = \int_0^R \int_0^L \|d\text{Exp}(t\gamma(s))(\gamma'(s))\| \|d\text{Exp}(t\gamma(s))(\gamma(s))\| \, ds \, dt \leq CLG_0(R)
\]

with \( G_0(R) = \int_0^R F_0(t) \, dt \), so that

\[
\text{area}(M_R) \leq CLG_0(R).
\]
We now apply the monotonicity formula, Proposition 6 with \( k(s) = k_0(s) \) and obtain

\[
\text{area}(M_R \cap B_s(o)) \leq CLG_0(s).
\] (18)

Recalling our ball model for \( N \), which we use standardly from now on, we translate (18) into a growth condition for Euclidean balls \( B_r^e(0) \subset B \) which have radius \( f(r) \) (see Lemma 9) in \( S \) and as well in \( N \) (Proposition 10):

\[
\text{area}(M_R \cap B_r^e) \leq CLG_0(f(r)).
\] (19)

The next lemma makes the decisive step towards the convergence proof of the family of surfaces \( M_R \).

**Lemma 11** The areas of the family \( M_R \) with respect to the metric \( \langle \cdot, \cdot \rangle_b \) (see Proposition 10) stay bounded independently of \( R \). Moreover, the Euclidean energies of the corresponding mappings \( u_R \) stay bounded as well. There is a radius \( \rho > 0 \) such that each of the surfaces \( M_R \) intersects \( B_\rho(o) \subset N \).

**Proof.** By Proposition 10 the surface area elements \( d\omega_b \) in the metric \( \langle \cdot, \cdot \rangle_b \) and \( d\omega \) in \( N \) stand in the relation

\[
d\omega = f'(|x|)^2 d\omega_b.
\]

If therefore we define

\[
A(r) = \text{area}(M_R \cap B_r^e)
\]

in \( N \) and

\[
A_b(r) = \text{area}(M_R \cap B_r^e)
\]

in the \( \langle \cdot, \cdot \rangle_b \)-metric, it follows from the coarea formula [7, 3.2.12] that

\[
\frac{d}{dr} A_b(r) = f'(r)^{-2} \frac{d}{dr} A(r)
\]

from what we get by integration

\[
A_b(f^{-1}(R)) - A_b(f^{-1}(\rho)) = \int_{f^{-1}(\rho)}^{f^{-1}(R)} f'(r)^{-2} A'(r) dr
\]

\[
= f'(f^{-1}(R))^{-2} A(f^{-1}(R)) - f'(f^{-1}(\rho))^{-2} A(f^{-1}(\rho))
\]

\[
+ 2 \int_{f^{-1}(\rho)}^{f^{-1}(R)} f''(r) f'(r)^{-3} A(r) dr.
\] (20)
From Lemma 9 we recall the following relations

\[ f' = \frac{1}{g' \circ f} = \frac{F_0 \circ f}{g \circ f} \]

\[ f'' = \frac{F_0' g - F_0 g'}{g^2} \circ f' = \frac{(F_0' - 1) F_0}{g^2} \circ f. \]

Since \( F_0'(0) = 1 \) and \( F_0'' \geq 0 \) we see that \( f'' \geq 0 \).

(21)

Lemma 8 (i) implies \( F_0' \geq cF_0 \) for some constant \( c > 0 \) from what we get

\[ G_0 \leq c^{-1} F_0. \]

(22)

By means of (19), (21) and (22) we may continue the estimate (20):

\[ \begin{align*}
A_b(f^{-1}(R)) - A_b(f^{-1}(\rho)) & \leq CL \left( \frac{g(R)^2}{F_0(R)^2} G_0(R) 
+ 2 \int_{f^{-1}(\rho)}^{f^{-1}(R)} \frac{f''(r)}{f'(r)^2} \frac{g(f(r)) G_0(f(r))}{F_0(f(r))} \, dr \right) \\
& \leq CLc^{-1} \left( \frac{g(R)}{F_0(R)} + 2 \int_{f^{-1}(\rho)}^{f^{-1}(R)} \left( -\frac{1}{f'(r)} \right)' \, dr \right) \\
& \leq 2CLc^{-1} \frac{g(\rho)}{F_0(\rho)}
\end{align*} \]

(23)

for arbitrary \( \rho \in (0, R) \). Because of \( 0 \leq g \leq 1 \), \( g(0) = 0 \) and \( g'(0) > 0 \) as we showed in Lemma 9, we see that the areas of the surfaces \( M_R \) in the \( \langle , \rangle_b \)-metric stay bounded independently of \( R \). The maps \( u_R : D \to N \) being conformal in the metric of \( N \) are conformal with respect to the \( \langle , \rangle_b \)-metric as well since this metric differs from the one of \( N \) by a conformal factor. But then it follows that the energies of the mappings \( u_R \) in the \( \langle , \rangle_b \)-metric and, on account of Proposition 10, as well in the Euclidean metric are bounded independently of \( R \). In other words the mapping \( u_R \), considered as mappings into \( \mathbb{R}^n \) are bounded in the norm of the Sobolev space \( H^1_0 \).

Let us now assume that \( M_R \) omits the ball \( B_\rho(o) \) of \( N \). Then one sees from (23) that the area of \( M_R \) in the \( \langle , \rangle_b \)-metric and hence the Euclidean energy of \( u_R \) become arbitrarily small if \( R \) and \( \rho \) are sufficiently large. This
however contradicts the fact that \(u_R|_{\partial D}\) parametrizes a rectifiable Jordan curve \(\Gamma_R \subset B\) and \(\Gamma_R\) converges to a rectifiable curve \(\Gamma \subset \partial_\infty B\) as \(R \to \infty\). Here we used the fact that in the ball model of \(\mathbb{N}\) geodesic cones with center 0 are straight Euclidean cones. This shows that there is a ball \(B_\rho(o) \subset N\) such that \(B_\rho(o) \cap M_R \neq \emptyset\) for all \(R \geq 1\).

In the next lemma we prove local energy and local \(C^0\)-estimates for conformal harmonic maps \(u : D \to N\).

**Lemma 12** Let \(u : D \to N\) be harmonic and conformal.

(i) For any subset \(D_0\) with \(D_0 \subset D\) holds

\[
E(u, D_0) := \frac{1}{2} \int_{D_0} \|du\|^2 \, dx \leq 8 \max \left\{ R^2, a^{-2} \right\} \text{cap}(D_0, D)
\]

with

\[
\text{cap}(D_0, D) = \inf \left\{ \frac{1}{2} \int_{D_0} |d\eta|^2 \, dx \mid \eta \in C_0^\infty(D), \eta = 1 \text{ on } D_0 \right\}
\]

(ii) For any \(z_0 \in D\) and \(s < (1 - |z_0|^2)\) we have the estimate

\[
\text{dist} (u(z), u(z_0)) \leq \left( \sqrt{\frac{8}{\pi} + 4 \sqrt{\frac{\pi}{-\ln s}}} \right) \max \left\{ R^2, a^{-2} \right\} \text{cap}(D_{\sqrt{s}}(z_0), D)^{\frac{1}{2}}
\]

for \(z \in D_s(z_0) := \{ z \in D \mid |z - z_0| < s \}\).

**Proof.** (i) We set \(k = k^+\) in Lemma 3 and Lemma 4 and consider the function

\[w := G \circ r \circ u, \quad r(x) = \text{dist}(x, o).\]

Using the harmonicity of \(u\) we obtain from the Corollary 5

\[\Delta w = \sum_{k=1,2} (\text{Hess } G \circ r)(u) \left( \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_k} \right) \geq F'(r(u)) \|du\|^2.
\]

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We test this inequality with the function \( \varphi = \eta^2/F \circ r \circ u \) where \( \eta \in C_0^\infty (D) \), \( \eta = 1 \) on \( D_0 \) to obtain

\[
0 \geq \int \left( \sum_{k=1,2} \frac{\partial w}{\partial x_k} \frac{\partial \varphi}{\partial x_k} + F' \|du\|^2 \varphi \right) dx
= \int \left( -\eta^2 \frac{G'F'}{F^2} \sum_{k=1,2} \left( \text{grad } r, \frac{\partial u}{\partial x_k} \right)^2 + \eta^2 \frac{F'}{F} \|du\|^2 \right.
+ 2\eta \frac{G'}{F} \sum_{k=1,2} \frac{\partial \eta}{\partial x_k} \left( \text{grad } r, \frac{\partial u}{\partial x_k} \right) dx.
\]

Since \( G' = F \) this simplifies to

\[
\int \left( \eta^2 \frac{F'}{F} \sum_{k=1,2} \left( \| \frac{\partial u}{\partial x_k} \|^2 - \left( \text{grad } r, \frac{\partial u}{\partial x_k} \right)^2 \right) \right) dx
\leq 2 \int \eta |d\eta| \left( \sum_{k=1,2} \left( \text{grad } r, \frac{\partial u}{\partial x_k} \right)^2 \right)^{1/2} dx.
\]

Let now \( x \in D \) be arbitrary and \( (e_1, ..., e_n) \) be an orthonormal basis at \( u(x) \) with \( e_1 = \text{grad } r \). The conformality relations then read

\[
0 = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right) = \sum_{j=1}^n \left( e_j, \frac{\partial u}{\partial x_1} \right) \left( e_j, \frac{\partial u}{\partial x_2} \right),
0 = \left\| \frac{\partial u}{\partial x_1} \right\|^2 - \left\| \frac{\partial u}{\partial x_2} \right\|^2 = \sum_{j=1}^n \left( \left( e_j, \frac{\partial u}{\partial x_1} \right)^2 - \left( e_j, \frac{\partial u}{\partial x_2} \right)^2 \right)
\]

which, in complex notation with \( i = \sqrt{-1} \), may be rewritten in the form

\[
\sum_{j=1}^n \left( e_j, \frac{\partial u}{\partial x_1} \right) + i \left( e_j, \frac{\partial u}{\partial x_2} \right)^2 = 0.
\]

Separating the term \( j = 1 \) in this sum allows to estimate the radial component \( du^{rad} \) of \( du \) by the spherical component \( du^{spher} \), i.e.

\[
\|du^{rad}\|^2 \leq \|du^{spher}\|^2.
\]
We now employ Lemma 3 (i) and (25) to obtain from (24)
\[
\min \left\{ \frac{1}{R}, a \right\} \int \eta^2 \| du^{\text{spher}} \|^2 \, dx
\leq \left( \varepsilon \int \eta^2 \| du^{\text{spher}} \|^2 \, dx + \frac{1}{\varepsilon} \int |d\eta|^2 \, dx \right)
\]
for arbitrary \( \varepsilon > 0 \). Choosing \( \varepsilon = \frac{1}{2} \min \left\{ \frac{1}{R}, a \right\} \) yields
\[
\int \eta^2 \| du^{\text{spher}} \|^2 \, dx \leq 4 \max \left\{ R^2, \frac{1}{a^2} \right\} \int |d\eta|^2 \, dx.
\]
Using (25) once more proves the statement.

(ii) For any \( r < 1 - |z_0| \) the image of \( u|D_r(z_0) \) contains a minimal surface which passes through \( u(z_0) \) and has no boundary inside the geodesic ball centered at \( u(z_0) \) and of radius
\[
\delta(r) := \inf \{ d(u(z), u(z_0)) \mid z \in \partial D_r(z_0) \}.
\]
The classical monotonicity formula for non positively curved metrics gives the estimate
\[
\text{area} \left( u|D_r(z_0) \right) \geq \pi \delta(r)^2.
\]
Part (i) then provides
\[
\delta(r) \leq \sqrt{\frac{8}{\pi}} \max \left\{ R, \frac{1}{a} \right\} \text{cap}(D_r(z_0), D)^{\frac{1}{2}}. \tag{26}
\]

Given now \( s \in (0, (1 - |z_0|)^2) \) the Lemma of Courant-Lebesgue \([4], 4.4\) guarantees the existence of a radius \( r \in (s, \sqrt{s}) \) such that the length of the curve \( u|\partial D_r(z_0) \) is estimated as follows:
\[
L \left( u|\partial D_r(z_0) \right) \leq \sqrt{\frac{8\pi}{-\ln s}} E \left( u|D_{\sqrt{s}}(z_0) \right) \frac{1}{2}
\leq 8 \sqrt{\frac{\pi}{-\ln s}} \max \left\{ R, a^{-1} \right\} \text{cap}(D_{\sqrt{s}}(z_0), D)^{\frac{1}{2}}.
\]
Combining this with (26) we arrive at
\[
\sup \{ \text{dist}(u(z), u(z_0)) \mid z \in \partial D_r(z_0) \}
\leq \left( \sqrt{\frac{8}{\pi}} + 4 \sqrt{\frac{\pi}{-\ln s}} \right) \max \left\{ R, a^{-1} \right\} \text{cap}(D_{\sqrt{s}}(z_0), D)^{\frac{1}{2}}.
\]
The maximum principle for harmonic maps into non-positively curved spaces \cite{9} yields the statement in (ii).

After a suitable conformal reparametrization of $u_R : D \to N$, we may assume that the following important normalization holds

$$u_R(0) \in B_\rho(o) \subset N, \quad 0 < R < +\infty,$$

(27)

where $\rho$ is given by Lemma \[11\].

We are now in position to prove:

**Lemma 13** (i) For some sequence $R_k \to +\infty$ the sequence of conformal harmonic maps $(u_{R_k})$ converges locally in $C^2$ to a proper, conformal harmonic map $u : D \to N$. If $n = 3$ then $u$ is an embedding.

(ii) Considered as a map into $\mathbb{R}^n$, $u$ belongs to the Sobolev space $H^{1,2}(D, \mathbb{R}^n)$ and its trace $u|\partial D$ either is a continuous weakly monotonic parametrization of $\Gamma = \lim_{R \to \infty} \Gamma_R \subset \partial B$ or $u|\partial D$ equals one point of $\Gamma$.

**Proof.** We shall obtain $u$ as a limit of a subsequence of the sequence $u_R : D \to N$ given by (16). Due the normalization condition (27) and by Lemma \[12\], for each subdisc $D_r(0) \subset D$ with $r < 1$ all the maps $u_R$ map $D_r(0)$ into some fixed ball $B_{s(r)}(o) \subset N$ and the energies of $u_R|D_r(0)$ are uniformly bounded as well. This makes Morrey’s H"older estimate for energy minimizing maps applicable \cite{12} so that we get an uniform $C^\alpha-$bound for $u_R$ on each subdisc $D_r(0)$ for some $\alpha(r) \in (0,1)$. By well known regularity estimates for harmonic maps this implies uniform local $C^{2,\alpha}$ bounds for the family $u_R$. Therefore we may find a sequence $R_k \to \infty$ such that $u_{R_k}$ converges locally in $C^2$ to a conformal harmonic map $u : D \to N$. On the other hand, considering the $u_R$ as maps into $\mathbb{R}^n$, we know from Lemma \[11\] that the $u_R$ are uniformly bounded in the $H^{1,2}(D, \mathbb{R}^n)-$norm, so that we may also assume that $u_{R_k}$ converges to $u$ weakly in $H^{1,2}(D, \mathbb{R}^n)$. The trace operator $H^{1,2}_c(D, \mathbb{R}^n) \to L^2(\partial D, \mathbb{R}^n)$ being compact we may then furthermore assume that $u_{R_k}|\partial D \to u|\partial D \quad (k \to +\infty)$ in $L^2(\partial D, \mathbb{R}^n)$.

Let us now choose parametrizations $\gamma_R : [0,2\pi] \to \Gamma_R$, $\gamma : [0,2\pi] \to \Gamma$ which are proportional to Euclidean arclength such that $\gamma_R \to \gamma$ uniformly in the Euclidean metric as $R \to \infty$. We extend $\gamma_R$ and $\gamma$ as periodic functions defined on $\mathbb{R}$. Then we may write

$$u_R(e^{i\theta}) = \gamma_R(\varphi_R(\theta)), \quad 0 \leq \theta \leq 2\pi,$$

(28)
with some monotonic function \( \varphi_R \), \( 0 \leq \varphi_R(0) \leq 2\pi \), \( \varphi_R(2\pi) - \varphi_R(0) = 2\pi \).

A classical theorem of Helly says that any sequence of monotone, uniformly bounded functions has a pointwise convergent subsequence, so that we may also assume that

\[
u_{R_k}(e^{i\theta}) \to \gamma(\varphi(\theta)) \quad (k \to +\infty)
\]

for some monotone function \( \varphi \) with \( \varphi(2\pi) - \varphi(0) = 2\pi \). Together with the \( L_2 \) convergence \( \nu_{R_k}|\partial D \to \nu|\partial D \) this clearly implies that

\[
u(e^{i\theta}) = \gamma(\varphi(\theta)), \quad 0 \leq \theta \leq 2\pi.
\]

This shows that \( \nu|\partial D \) could only have jump discontinuities; however these are not possible for boundary values of an \( H^1 \) function thanks to the lemma of Courant-Lebesgue [4, 4.4]. It follows that \( \varphi \) cannot have jumps of height less than \( 2\pi \) and we arrive at the alternative that either \( \varphi \) is continuous or it makes a jump of \( 2\pi \), in other words, either \( \nu(\partial D) = \Gamma \) or \( \nu|\partial D \) is constant.

As next, let us show that the limit map \( u \) is proper. From what we already showed above we know that \( u(\partial D) \subset \partial_\infty B \) which, in the metric of \( N \), means that the Sobolev trace \( u(\partial D) \) is infinitely far away. Let a ball \( B_R(o) \) be given with arbitrarily large \( R \). Since \( \overline{B_{2R}(o)} \subset N \) is a compact subset of \( B \) and \( u(\partial D) \subset \partial_\infty B \) we can apply Theorem 1 in [5] to find a sequence of radii \( r_k \to 1 \) \( (k \to +\infty) \), \( r_k < r_{k+1} < 1 \), such that \( u(\partial D_{r_k}(0)) \subset B \setminus B_{2R}(o) \). Let us set

\[
A_k := \{ z \in D \mid r_k < |z| < r_{k+1} \}
\]

and let us assume that some \( A_k \) contains points \( z_k \) with \( u(z_k) \in B_R(o) \subset N \). Since \( u(\partial A_k) \) is outside of \( B_{2R}(o) \), \( u(A_k) \cap B_{2R}(o) \) contains a minimal surface which passes through \( u(z_k) \in B_R(o) \) and has no boundary inside \( B_{2R}(o) \) so that the monotonicity formula gives

\[
\text{area} \left( u(A_k) \cap B_{2R}(o) \right) \geq \pi R^2.
\] (29)

Since \( u \in H^1_2(D, \mathbb{R}^n) \) the area of \( u \) inside \( B_{2R}(o) \subset N \) is finite so that (29) can hold only for finitely many \( k \) and hence \( d(u(z), o) \geq R \) for \( |z| \geq r_{k_0} \) for some \( k_0 \), showing that \( u : D \to N \) is proper.

Let us finally consider the case \( n = 3 \). Since the the boundary curves of the surfaces \( u_R \) are contained in the metric spheres of \( N \) and the spheres are convex, it follows from the results in [8], [10] that \( u_R \) is an embedding. Then, as a limit of minimal embeddings, \( u \) is an embedding, too. This concludes the proof of the Lemma.
4 The blowing up procedure and proof of the Theorem

The concentration phenomenon which comes up as a possibility in the limiting process in Lemma 13 and the resulting splitting off of a punctured minimal sphere can be excluded if one can construct suitable foliations of the space by convex hypersurfaces. Such foliations are obvious in the hyperbolic space but do exist also in more general Hadamard manifolds, as explained in the next section. Instead we shall now set up a blow up procedure, magnifying neighborhoods of the point where the concentration happens. The splitting off of punctured minimal spheres may repeat itself, however we can show that after a finitely many split offs a solution to the asymptotic Plateau problem remains.

Proof of Theorem 1. In the proof of Lemma 11 it was already used that there is a positive lower bound for the euclidean area of discs spanned by one of the curves $\Gamma R$, $R \geq 1$. We need a corresponding statement for a family of curves which are obtained from the $\Gamma R$ by the following modifications: One takes out a subarc $\alpha$ from $\Gamma R$ of Euclidean length not exceeding $\varepsilon > 0$ and replaces it by some other rectifiable arc $\beta$ of length at most $\delta$. If $\tilde{M}$ is a disc spanned by such a modified curve $\tilde{\Gamma} R$ one may produce a disc $M$ filling the original $\Gamma R$ by attaching a cone over $\alpha \cup \beta$ along the boundary segment $\beta$ of $\tilde{M}$ and hence

$$\text{area}^e(M) \leq \text{area}^e(\tilde{M}) + (\varepsilon + \delta)^2.$$ 

If therefore $\varepsilon$ and $\delta$ are sufficiently small we see that there is $a_0 > 0$ such that

$$\text{area}^e(M) \geq a_0,$$ 

where $\text{area}^e$ denotes the euclidean area, for all discs spanned by some $\tilde{\Gamma} R$, $R \geq 1$. Let us now return to the representation (28) for the boundary data of the family $u_R$:

$$u_R(e^{i\theta}) = \gamma R(\varphi R(\theta)), \ 0 \leq \theta \leq 2\pi,$$

$\gamma R$ being a proportional-to-arclength parametrization.

If for a sequence $R_k \to +\infty$ the sequence $(\varphi_{R_k})$ converges pointwise to a step function with one jump of height $2\pi$ we may (after a rotation of $D$) assume that the jump occurs at $\theta = \pi$. After passing to a subsequence we
may assume that
\[ \varphi_{R_k} \left( \frac{\pi}{k} + \frac{1}{k} \right) - \varphi_{R_k} \left( \frac{\pi}{k} - \frac{1}{k} \right) > 2\pi - \frac{1}{k} \]
so that \( \gamma_{R_k} \circ \varphi_{R_k} ([\pi - 1/k, \pi + 1/k]) \) represents a subarc of euclidean length at least \( (1 - 1/(2k\pi)) \) length \( (\Gamma_{R_k}) \) and the complementary subarc of \( \Gamma_{R_k} \) has length at most \( (1/(2k\pi)) \) length \( (\Gamma_{R_k}) \). The lemma of Courant-Lebesgue [4, 4.4] provides a radius \( r_k \in \left( \frac{1}{k}, \frac{1}{\sqrt{k}} \right) \) such that
\[ \text{length} \left( u_{R_k} (D \cap \partial D_{r_k} (-1)) \right) \leq \sqrt{\frac{8\pi E_0}{\ln k}} \]
where \( E_0 \) is an upper bound for the euclidean energies of \( u_{R_k} \). For sufficiently large \( k \) the curve \( \tilde{\Gamma}_{R_k} := u_{R_k} (\vartheta (D \cap D_{r_k} (-1))) \) satisfies the conditions required for inequality (30), making it obvious that a concentration of energy takes place near the boundary point \(-1\). Since \( u_{R_k} | D \cap D_{r_k} (-1) \) is part of the surface \( u_{R_k} \), (18) and the estimates of Lemma 11 trivially remain valid for \( u_{R_k} | D \cap D_{r_k} (-1) \), irrespective of the modification of the boundary curve. But then (23) also holds with \( R = R_k \) showing that there is a radius \( \tilde{\rho} > 0 \) only depending on \( \Gamma, a_0 \) and the geometry of \( N \) such that
\[ u_{R_k} (D \cap D_{r_k} (-1)) \cap B_{\tilde{\rho}} (0) \neq \emptyset \]
for all sufficiently large \( k \), unless (30) were violated. Therefore we may now choose conformal maps \( T_k : D \rightarrow D \cap D_{r_k} (-1) \) such that \( \tilde{u}_{R_k} := u_{R_k} \circ T_k \) satisfies
\[ \tilde{u}_{R_k} (0) \in B_{\tilde{\rho}} (0) \subset N. \] (31)

Let us now look at the minimal surface \( u_{R_k} | D \setminus D_{r_k} (-1) \) which tends to a punctured sphere for \( k \rightarrow \infty \). Recalling the condition \( u_{R_k} (0) \in B_{\rho} (0) \subset N \) and observing that
\[ u_{R_k} (\vartheta (D \setminus D_{r_k} (-1))) \cap B_{2\rho} (0) = \emptyset \]
for sufficiently large \( k \) we obtain from the monotonicity formula [1]
\[ \text{area}^N (u_{R_k} ((D \setminus D_{r_k} (-1))) \cap B_{2\rho} (0)) \geq \pi \rho^2, \]
which in view of Proposition [10] leads to an estimate of the euclidean energy of \( u_{R_k} | D \cap D_{r_k} (-1) \) of the form
\[ E (u_{R_k} ((D \setminus D_{r_k} (-1)))) \geq e (\rho), \] (32)

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where \( e (\rho) \) depends only on \( \rho \) and the geometry of \( N \). Recalling that \( E_0 \) was an upper bound for the euclidean energies of the sequence \( u_{R_k} \) we thus see that

\[
E (\tilde{u}_{R_k}) \leq E_0 - e (\rho),
\]

i.e. the splitting off of a punctured minimal sphere reduces the energy by a fixed amount. We may now apply the same analysis as in the proof of Theorem 13 to the sequence \((\tilde{u}_{R_k})\) resulting in the convergence locally in \( C^2 \) and weakly in \( H^1 (D, \mathbb{R}^n) \) of a subsequence of \((\tilde{u}_{R_k})\) towards a conformal, harmonic, proper map from \( D \) to \( N \). Let us now investigate the behavior of the boundary values of \( \tilde{u}_{R_k} |D \) which parametrize the curve \( \tilde{\Gamma}_{R_k} \) monotonically.

We recall that \( \tilde{\Gamma}_{R_k} \) consists of a subarc \( \tilde{\alpha}_k \) of \( \Gamma_{R_k} \) of length at least \((1 - 1/(2k\pi))) \) length \((\Gamma_{R_k}) \) and with endpoints \( u_{R_k} (\partial D \cap \partial D_{R_k} (-1)) \) together with the arc \( \beta_k = u_{R_k} (D \cap \partial D_{R_k} (-1)) \) of length at most \( \sqrt{8\pi E_0 / \ln k} \). We choose proportional-to-arclength parametrizations \( \tilde{\gamma}_k : [0, 2\pi] \to \tilde{\Gamma}_{R_k} \) such that \( \tilde{\gamma}_k (0) \) is an endpoint of \( \alpha_k \). Passing to a subsequence we have \( \tilde{\gamma}_k \to \gamma \) uniformly, where \( \gamma \) is a proportional-to-arclength parametrization of \( \Gamma \). The representation \( \tilde{u}_{R_k} (e^{i\theta}) = \tilde{\gamma}_k (\varphi_k (\theta)) \) holds with monotone functions \( \varphi_k \), \( \varphi_k (2\pi) - \varphi_k (0) = 2\pi \). After a rotation of \( D \) we may assume that \( \varphi_k (0) = 0 \) and hence \( \varphi_k (2\pi) = 2\pi \). Let us choose \( \theta_k \in (0, 2\pi) \) such that \( \tilde{\gamma}_k \circ \varphi_k |[0, \theta_k] \) parametrizes \( \alpha_k \) and \( \tilde{\gamma}_k \circ \varphi_k |[\theta_k, 2\pi] \) parametrizes \( \beta_k \). Then clearly

\[
\varphi_k (\theta_k) \to 2\pi \ (k \to \infty).
\]

After passing to a subsequence we may assume that \( \theta_k \to \tilde{\theta} \in [0, 2\pi] \) \((k \to \infty) \), \( \varphi_k \to \varphi \) pointwise on \([0, 2\pi] \) and \( \tilde{u}_{R_k} \to \tilde{u} \) locally in \( C^2 \) and weakly in \( H^1 (D, \mathbb{R}^n) \) where \( \tilde{u} : D \to N \) is a harmonic, conformal, proper map.

If \( \tilde{\theta} = 0 \) then \( \varphi (\tilde{\theta}) = 2\pi \) on \([0, 2\pi] \) and hence \( \tilde{u} |\partial D = \gamma (2\pi) \), i.e. a punctured minimal sphere has split off. Let us consider the case that \( \tilde{\theta} > 0 \).

It follows from (34) that \( \varphi (\tilde{\theta}) = 2\pi \) for all \( \tilde{\theta} \in (\tilde{\theta}, 2\pi) \) and \( \varphi (\tilde{\theta} - 0) = 2\pi \) so that by monotonicity \( \varphi (\tilde{\theta}) = 2\pi \). Since \( \tilde{u} |\partial D = \gamma \circ \varphi \) exactly as in the proof of Lemma 13 the alternative arises that either \( \varphi \) is continuous and \( \tilde{u} (\partial D) = \Gamma \) or \( \varphi \) is a step function with a jump of height \( 2\pi \). In the first case \( \tilde{u} \) is a solution to the asymptotic Plateau problem in the \( H^2 \) sense and in the second one a punctured minimal sphere has split off again. If the latter happens we can repeat the whole blow-up process, in each step lowering the energy by a fixed amount, see (33). This must stop as soon as the minimal area threshold (30) were violated. This proves the theorem. ■

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5 Proof of Theorem 2

We recall that a Hadamard manifold $N$ satisfies the strict convexity condition if, given $x \in \partial_\infty N$ and a relatively open subset $W \subset \partial_\infty N$ containing $x$, there exists a $C^2$-open subset $\Omega \subset N$ such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$, where $\text{Int}(\partial_\infty \Omega)$ denotes the interior of $\partial_\infty \Omega$ in $\partial_\infty N$, and $N \setminus \Omega$ is convex. Loosely speaking, this means that, as it happens with strictly convex bounded domains in Euclidean spaces, we can take out a neighborhood in $N$ at any point at infinity of $N$, which arbitrarily small asymptotic boundary, and what remains is still convex.

Let $B$ be the model of $N$ as in Theorem 13. We only have to prove that $\partial_\infty u(D) = \Gamma$. Given $x \in \partial_\infty B \setminus \Gamma$ we prove that $x \notin \partial_\infty u(D)$ from what it follows that $\partial_\infty u(D) \subset \Gamma$. Since, by Theorem 13 $u \in H^1_2(D, \mathbb{R}^n)$ it follows that $\partial_\infty u(D) = \Gamma$.

Since $\Gamma$ is compact, there is $W \subset \partial_\infty B$ such that $W \cap \Gamma = \emptyset$. By the strict convex condition there is a $C^2$ convex neighborhood $\Omega$ of $N$ such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$. Let $d : \Omega \to [0, +\infty)$ be the distance to $\partial \Omega$. Then the level hypersurfaces of $d$ determine a foliation of $\Omega$ by equidistant hypersurfaces to $\partial \Omega$. From the Hessian Comparison Theorem if $S$ is a leaf of this foliation at a distance $d$ of $\partial \Omega$ then any principal curvature $\lambda$ of $S$ with respect to the unit normal vector field pointing to the connected component of $B \setminus S$ that does not contain $x$, satisfies $\lambda \geq a \tanh(ad)$. That is, the level hypersurfaces of $d$ provides a foliation $\{S_d\}$ of $\Omega$ which is convex towards the connected component of $B \setminus S_d$ which does not contain $x$. Since $\lim_{R \to \infty} \partial u_R(D) = \Gamma$ for a sufficiently large $R_0$ we have $\partial u_R(D) \cap \Omega = \emptyset$ for all $R \geq R_0$. By the comparison theorem it follows that $u_R(D) \cap \Omega = \emptyset$ for all $R \geq R_0$ and then $x \notin \partial_\infty u(D)$, proving the theorem.

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