Galerkin Finite Element Method for Nonlinear Riemann-Liouville and Caputo Fractional Differential Equations

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Abstract

In this paper, we study the existence, regularity, and approximation of the solution for a class of nonlinear fractional differential equations. For this aim, suitable variational formulations are defined for a nonlinear boundary value problems with Riemann-Liouville and Caputo fractional derivatives together with the homogeneous Dirichlet condition. We concern the well-posedness and also the regularity of the corresponding weak solutions. Then, we develop a Galerkin finite element approach to proceed the numerical approximation of the weak formulations and prove a priori error estimations. Finally, some numerical experiments are provided to explain the accuracy of the proposed method.

Keywords: fractional differential operators, Caputo derivative, Riemann-Liouville derivative, variational formulation, nonlinear operator, Galerkin method.

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2 Introduction

In the current study, we consider the fractional order nonlinear boundary value problem as follows:

Find $u$ such that

$$
-\mathcal{D}_s^x u(x) + g(x, u(x)) = f(x), \quad x \in \Omega := [0, 1],
$$

(1)

where $s \in (1, 2)$, and $\mathcal{D}_s^x$ refers to either Riemann-Liouville or Caputo fractional derivatives which are detailed below. Furthermore, $f$ and $g$ are known functions chosen from the suitable function spaces defined in Section 3.

Developing the order of differentiation to any real number is an interesting question. To find an affirmative answer, some efforts have been done and different types of so-called fractional derivatives have been introduced [20]. It is notified that the fractional derivative is a concept with attractive applications in science and engineering. It is appeared in the anisotropic diffusion modeling anomalously for cardiac tissue in microscopic and macroscopic levels. Furthermore, fractional derivative models are certain instances of nonlocal models which are introduced in comparison with the classical ones [12].

Similar to the ordinary and partial differential equations, we have two approaches in seeking solutions for Fractional Differential Equations (FDEs); analytic and numeric solutions. The analytical methods such as the Fourier, Laplace and Mellin transform methods and even Green function approach
(fundamental solution) are available for some special types of FDEs \[20, 29\]. In practice, we have to follow numerical methods due to the lack of applicability of analytical methods for a wide range of FDEs. Hence, the study of the numerical approaches for them is of great importance.

It is worthy to mention that in spite of different definitions for fractional differential operators, most of them are defined by Abel’s integral operator. Among the popular numerical approaches for Abel’s integral equation, one could mention collocation method \[7\] and Galerkin method based on piecewise polynomials \[13, 30\] where the different varieties of those approaches, in general perspective, spectral and projection methods could be utilized to find approximation for FDEs \[25, 29\]. Converting to a suitable integral equation and then solving numerically with the above mentioned approaches \[31\] or directly solving by finite difference method \[24\] are based on the adequate regularity assumptions of the strong solution which is not available in general \[14, 18\]. Here, we contribute the study on the numerical solution of one dimensional nonlinear FDEs which involve Riemann-Liouville or Caputo derivative by introducing a convenient weak formulation and describing the Galerkin solution with convergence analysis in some appropriate function spaces.

The fractional operator in \( g \) is non-local and \( g \) is a nonlinear function with respect to \( u \), so the study of the existence, uniqueness, regularity of solution, and furthermore the numerical investigation are challenging. The existence of classical solution for the nonlinear FDEs is considered in \[36\] and for the one dimensional linear case, the regularity of the solution is investigated in \[14\]. In this work, we explore the issue of the existence and uniqueness with the aid of Browder-Minty method of the monotone operators.

In the recent literature, due to their applications in science and engineering \[25\], several types of numerical methods have been proposed for the approximation of FDEs. The theory and numerical solution of a linear Riemann-Liouville and Caputo FDEs with two-point boundary condition have been extensively studied in \[21, 27\]. In those works, the fractional boundary value problem is reformulated appropriately in terms of Volterra integral equation and then the numerical approach is proceed by some suitable schemes such as piecewise polynomial collocation and spectral Galerkin methods. Spectral and pseudo-spectral methods are some of the interesting numerical approaches which are taken into consideration for the FDEs. Among the whole research on this area, we could mention \[26, 35, 33, 34\] wherein the Jacobi polynomials play a crucial role in the construction of the approximation. The attention to this class of orthogonal polynomials is a motivation for introducing a generalization of them with application in the numerical solution of FDEs \[9\].

In this work, we study the Galerkin finite element method for Riemann-Liouville and Caputo nonlinear fractional boundary value problems of Dirichlet type. The finite element method is a popular numerical approach in order to find an approximation for nonlinear differential equations \[17, 32\]. The finite element solution of quasi-linear elliptic problems with non-monotone operator have been considered in \[11, 15\]. Furthermore in \[16\], under the assumptions of the strong monotonicity and Lipschitz continuity of the corresponding second order nonlinear elliptic operator, a linear order approximation by finite element method has been obtained. In this paper, the investigated fractional nonlinear operators have the monotonicity and Lipschitz continuity properties, which are crucial for our analysis.

The reasonable energy space associated with the non-local operators is fractional Sobolev spaces. Also, due to the presence of the nonlinear term in \( g \), we utilize Musielak-Orlicz space in order to introduce a suitable functional space by intersection of two mentioned spaces in a convenient way. Then with the aid of monotone operator theory, the coercivity of the nonlinear variational formulation along with the Riemann-Liouville and Caputo fractional derivatives is investigated. This approach leads to get a unique weak solution which could be approximated by finite element method. Finding a priori error estimation by the generalized version of Céa’s lemma is also proceed.

We organize the reminder of the paper as follows: in Section \[5\] some introduction regarding to the fractional calculus, semi-linear monotone operator and also suitable function spaces are briefly presented. Section \[4\] is devoted to state the variational formulation of nonlinear boundary value problem along with Riemann-Liouville and Caputo derivatives. Furthermore, the regularity of the solution is studied in this section. The numerical approximation of the weak solution is examined by
finite element approximation in Section 5 along with the full study on the existence and uniqueness issues of the discrete equations and the convergence of the method. In numerical experiments section, some FDEs are solved by finite element method. In final, we provide some conclusion and further remarks for the future works.

3 Preliminaries

This section is devoted to express some preliminaries to fractional calculus involving the introduction on the considered problem in the paper. The energy space regarding to fractional operators, fractional Sobolev spaces, are introduced in this section. Then in order to deal with the existence of the weak solution by monotonicity arguments, some preface to nonlinear functions on Orlicz spaces are provided.

3.1 Fractional calculus

Aiming to make the paper self-content, we recall the Riemann-Liouville and Caputo fractional integral and derivatives from [20]. For any \( s > 0 \) with \( n - 1 < s < n, \ n \in \mathbb{N} \), the right and left sided fractional integrals on the bounded interval \([a, b]\) are as follows:

The left-sided Riemann-Liouville fractional integral operator is defined as

\[
(aI^s_x u)(x) = \frac{1}{\Gamma(s)} \int_a^x (x - y)^{s-1} u(y) \, dy, \tag{2}
\]

while the right fractional integral operator is given by

\[
(xI^s_b u)(x) = \frac{1}{\Gamma(s)} \int_x^b (y - x)^{s-1} u(y) \, dy. \tag{3}
\]

Left-sided Riemann-Liouville fractional derivative of order \( s \) for a function \( u \in H^n(\Omega) \) can be defined as

\[
{^R}D^s_x u = D^n {^R}_0I^{n-s}_x u, \tag{4}
\]

where the operator \( D^n \) denotes the classical derivative of order \( n \). The corresponding right-sided Riemann-Liouville fractional derivative is stated as

\[
{^R}I^n_x u = (-1)^n I^n_x {^R}D^n u. \tag{5}
\]

In addition, the left-sided Caputo derivative of order \( s \) is given by

\[
{^C}_0D^s_x u = {^C}_0I^s_x D^n u, \tag{6}
\]

where the following relation defines the right-sided Caputo derivative

\[
{^C}D^s_x u = (-1)^n {^R}_0I^{n-s}_x D^n u. \tag{7}
\]

From the above definitions, it is apparent that the Abel’s integral operator has a significant role in the definitions of fractional derivatives.

3.2 Some properties on semi-linear operators

Let \( V \) be a real Banach space and \( V^* \) be its dual space. We denote by \( \langle y, x \rangle \) the value of a continuous linear functional \( y \in V^* \) on an element \( x \in V \) and \( \| \cdot \| \) and \( \| \cdot \|_* \) are the norms associated with \( V \) and \( V^* \), respectively. We notice that for our semi-linear problem, monotonicity property is central. So in the following, we present a formal definitions for this concept.
Definition 1 \([4, 23]\) Let \(V\) be a separable Banach space. An operator \(F\) is called monotone on \(V\) if
\[
\langle Fx - Fy, x - y \rangle \geq 0, \quad \forall x, y \in V,
\]
it is called strictly monotone if
\[
\langle Fx - Fy, x - y \rangle > 0, \quad \forall x, y \in V, \quad x \neq y,
\]
it is called coercive if
\[
\langle Fx, x \rangle > \gamma \|x\| \|x\|, \quad \text{where } \gamma(s) \to \infty \text{ as } s \to \infty,
\]
it is called hemi-continuous if the real-valued functions
\[
s \mapsto \langle F(u + s \cdot v), w \rangle,
\]
is continuous on \([0, 1]\) for any fixed \(u, v, w \in V\). Also, in the terminology of the article \([19]\), the operator \(F : V \to V^*\) with the domain \(D = D(F)\) is hemi-continuous if for \(u \in D, w \in V\), we get \(F(u + t_n w) \to F(u)\), when the sequence \(t_n\) tends to zero.

In the following, we state Browder-Minty theorem which is utilized to prove the existence and uniqueness of the weak solution.

Theorem 1 (Browder-Minty) Let \(V\) be a real reflexive Banach space and let a hemi-continuous monotone operator \(F : V \to V^*\) be coercive. Then for any \(g \in V^*\), there exists a solution \(u^* \in V\) of the equation
\[
F(u) = g.
\]
This solution is unique if \(F\) is a strictly monotone operator.

Proof. See the details of the proof in \([4]\) or \([22]\). \(\blacksquare\)

3.3 Functional space

To formulate an appropriate function space so that (1) be well-posed, we first recall the definition of fractional-order Sobolev spaces. As usual the standard Lebesgue spaces are denoted by \(L^p(\Omega)\) and their norm by \(\|\cdot\|_{L^p(\Omega)}\). For \(p = 2\), the scalar product is denoted by \(\langle u, v \rangle = \int_{\Omega} u(x)v(x)dx\) and the norm by \(\|\cdot\| = (\cdot, \cdot)^{1/2}\). Let \(\{\lambda_n\}_{n \in \mathbb{N}}\) as the set of all eigenvalue of the boundary-value problem,
\[
D^2u(x) = -\lambda u(x), \quad x \in \Omega,
\]
\[
\frac{du}{dt}(0) = u(1) = 0,
\]
where \(\phi_n\) is an eigen-function related to \(\lambda_n\) for \(n \in \mathbb{N}\). Now for \(s \in \mathbb{R}\), a Hilbert scale \(H^s(\Omega)\) is defined based on \(\{\phi_n\}_{n \in \mathbb{N}}\) with the following scalar products and norms by
\[
(u, v)_{H^s(\Omega)} = \sum_{n=1}^{\infty} \lambda^s(u, \phi_n)(v, \phi_n), \quad u, v \in \text{span}\{\phi_n\}_{n \in \mathbb{N}},
\]
and
\[
\|u\|_{H^s(\Omega)} = (u, u)_{H^s(\Omega)}^{1/2}.
\]
Let \(\lambda_n := \mu_n^2\), then \(\{\mu_n^s \phi_n\}_{n \in \mathbb{N}}\) form an orthonormal basis for \(H^s(\Omega)\). It is well-known that \(\{\phi_n\}_{n \in \mathbb{N}}\) is an orthonormal basis for \(H^0(\Omega) = L^2(\Omega)\), so for \(u_n = (u, \phi_n)\), we have \(u = \sum_{n=1}^{\infty} u_n \phi_n\). For any
$s \geq 0$, the fractional order Sobolev space is defined by the spectral properties of the operator and inner product as follows

$$H^s(\Omega) := \{ u \in L^2(\Omega) \mid \sum_{k=1}^{\infty} \lambda_k^s u_k^2 < \infty \}. \quad (10)$$

Another approach to define the fractional Sobolev space is using the definition of $L^p(\Omega)$ spaces along with the Slobodeckij semi-norm. For our aim, it suffices to set $p = 2$ and let $|s|$ denote the largest integer for which $[s] \leq s$ and define $\lambda \in [0,1]$ by $s = [s] + \lambda$. For $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we introduce the scalar product

$$\langle \varphi, \psi \rangle_{H^s(\Omega)} := \sum_{\alpha \leq |s|} (D^\alpha \varphi, D^\alpha \psi) + \int_{\Omega} \int_{\Omega} \frac{(D^{|s|} \varphi(x) - D^{|s|} \varphi(y))(D^{|s|} \psi(x) - D^{|s|} \psi(y))}{|x - y|^{1+2s}} \, dx \, dy,$$

and the norm $\| \varphi \|_{H^s(\Omega)} := (\varphi, \varphi)^{1/2}_{H^s(\Omega)}$. For $s \in \mathbb{N}$, obviously the second term in (11) is ignored. Then the Sobolev space $H^s(\Omega)$ is given by

$$H^s(\Omega) := \{ u \in L^2(\Omega) \mid \forall 0 \leq k \leq |s|, u^{(k)} \in L^2(\Omega) \text{ and } \| u \|_{H^s(\Omega)} < \infty \}.$$

The dual space of $H^s(\Omega)$ is denoted by $H^{-s}(\Omega)$ and is equipped with the norm

$$\| u \|_{H^{-s}(\Omega)} := \sup_{v \in H^s(\Omega)} \frac{(u,v)}{\| v \|_{H^s(\Omega)}}, \quad (12)$$

where $(\cdot, \cdot)$ denotes the continuous extension of the $L^2$-scalar product to the duality pairing $(\cdot, \cdot)$ in $H^{-s}(\Omega) \times H^s(\Omega)$. Let $\tilde{H}^s(\Omega)$ be the set of functions of $H^s(\Omega)$ extended to $\mathbb{R}$ by zero. This space also could be defined by the intermediate space of order $s \in (0,1)$ given as

$$\tilde{H}^s_0(\Omega) = [H^m_0(\Omega), H^0(\Omega)]_{\theta}, \quad m \in \mathbb{Z}, \quad (13)$$

where $m(1 - \theta) = s$. Indeed for $\phi \in H^m_0(\Omega)$, $\phi$ and its derivatives of order $k \leq m$ have the compact support property. For $m = 1$, $\tilde{H}^s(\Omega) := \tilde{H}^1_0(\Omega)$. In a similar way, the extension by zero on the half intervals $(-\infty, b)$ and $(a, \infty)$ could be defined as $\tilde{H}^s_b(\Omega)$ and $\tilde{H}^s_a(\Omega)$, respectively.

**Theorem 2 (18)** Assume that $n - 1 < s < n$ for $n \in \mathbb{N}$. The operators $R^s_0 D^s_x u$ and $R^s_x D^s_x u$ for $u \in D(\Omega)$ could be extended continuously to operators with the same notations from $\tilde{H}^s(\Omega)$ to $L^2(\Omega)$, i.e.,

$$\| R^s_0 D^s_x u \|_{L^2(\mathbb{R})} \leq c \| u \|_{\tilde{H}^s(\Omega)}, \quad (14)$$

and

$$\| R^s_x D^s_x u \|_{L^2(\mathbb{R})} \leq c \| u \|_{\tilde{H}^s(\Omega)}, \quad (15)$$

where $D(\Omega)$ denotes the set of all infinitely differentiable functions with compact support equipped with the locally convex topology.

The following theorem represents some profitable characteristics of fractional differential and integral operators.

**Theorem 3** The following statements hold:

a) The integral operators $0 I^s_x$ and $x I^s_x$ satisfy the semi-group property.
b) For $\phi, \psi \in L_2^2(\Omega)$, $(a I_x^s \phi, \psi) = (\phi, x I_1^s \psi)$.

c) For any $s > 0$, the function $x^s \in H^\alpha(\Omega)$, where $0 \leq \alpha < \frac{1}{2}$.

d) For any non-negative $\alpha, \gamma$, the Riemann-Liouville integral operator $I^\alpha$ is a bounded map from $H^{\gamma}(\Omega)$ into $H^{\gamma+\alpha}(\Omega)$.

e) The operators $I_x^s \colon H^L(\Omega) \to L_2^2(\Omega)$ and $I_x^s \colon H^L(\Omega) \to L_2^2(\Omega)$ are continuous.

Proof. First item was investigated in [20, Theorem 2.4]. Fubini’s Theorem yields the useful change of integration order formula which proves item b [20, Lemma 2.7]. The proof of other items can be found in [18].

3.3.1 Nonlinear function on Orlicz spaces

Throughout this paper, we require some important properties for the nonlinear part of Eq. (1). A suitable function space to deal with the monotone operators with nonlinear terms are Orlicz spaces or generalized Orlicz space which is called in some relevant texts, Musielak-Orlicz space [2, 5]. We recall some necessary definitions and properties related to the mentioned spaces from [2].

Assumption 1 Assume that a nonlinear function $g(x, t) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies in the following properties

\[
\begin{align*}
&g(x, \cdot) \text{ continuous, odd, strictly monotone, a.e. on } \Omega, \\
&g(x, 0) = 0, \quad \lim_{t \to \infty} g(x, t) = \infty, \quad a.e. \text{ on } \Omega, \\
&g(\cdot, t) \text{ is measurable, } \forall t \in \mathbb{R}.
\end{align*}
\]

Note that the inverse function of $g(x, \cdot)$ exists which follows from the strictly monotone property. Let us denote it by $\tilde{g}(t, \cdot)$. We define $G(x, t)$ and $\tilde{G}(x, t)$ by

\[
G(x, t) := \int_0^{|t|} g(x, s)ds, \quad \tilde{G}(x, t) := \int_0^{|t|} \tilde{g}(x, s)ds.
\]

These functions are complementary Musielak-Orlicz functions that are $N$-functions respect to the second variable [2, 3].

Definition 2 Let $G(x, \cdot)$ be an $N$-function. This function satisfies the global $(\Delta_2)$-condition if there exists a constant $c \in (0, 1]$ such that for $x \in \Omega$ a.e. and for all $t \geq 0$

\[
c t g(x, t) \leq G(x, t) \leq t g(x, t),
\]

where the function $g(x, t)$ is determined by the Assumption [1].

The Musielak-Orlicz space is defined as follows

\[
L_G(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } | G(\cdot, u(\cdot)) \in L^1(\Omega) \right\},
\]

which means that the modular $\rho_G(u) = \int_\Omega G(t, u(t))dt$ is measurable [5]. If this space satisfies the global $(\Delta_2)$-condition, then it is a reflexive Banach space and equipped with the Luxemburg norm given by

\[
\|u\|_{G, \Omega} := \inf \left\{ m > 0 \mid \rho_G \left( \frac{u}{m} \right) \leq 1 \right\}.
\]
Moreover, in our analysis we need the generalized Hölder inequality is given by
\[ \left| \int_{\Omega} u(t)v(t)dt \right| \leq 2\|u\|_{G;\Omega}\|v\|_{\tilde{G};\Omega}, \quad \forall u \in L_G(\Omega), \quad \forall v \in L_{\tilde{G}}(\Omega). \] (16)
Furthermore, the following important result
\[ \lim_{\|u\|_{G;\Omega} \to \infty} \frac{\rho_G(u)}{\|u\|_{G;\Omega}} = \infty, \] (17)
which is obtained in [6] has a significant role in the applicability of the monotone operator theorems for our target.

**Lemma 4** ([6]) If \( g(.,u(.)) \) satisfies in (\( \Delta_2 \))-condition, then for all \( u \in L_G(\Omega) \) one can get \( g(.,u(.)) \in L_{\tilde{G}}(\Omega) \).

Now, we are ready to define the suitable function space which is appropriate for our problem.

**Definition 3** Let \( 1 < s < 2 \). Under the Assumption [7] where fulfilled in Definition [2] consider the following reflective Banach space \( U := \tilde{H}^{\tilde{s}}(\Omega) \) as
\[ U := U(\Omega,G) := \{ \phi \in \tilde{H}^{\tilde{s}}(\Omega) | G(\cdot,\phi) \in L^1(\Omega) \}, \]
where equipped with the norm
\[ \|u\|_U := \|u\|_{\tilde{H}^{\tilde{s}}(\Omega)} + \|u\|_{L_G(\Omega)}. \] (18)
We state the following lemma from [3] which has a crucial role in the investigation of regularity of the solution.

**Lemma 5** Let \( 0 \leq s < 1 \) and assume that for \( M > 0 \), there exists a constant \( l_M \) such that \( f \) satisfies
\[ |g(x,u_1) - g(x,u_2)| \leq l_M|u_1 - u_2|, \quad x,y \in \Omega, \quad u_i \in \mathbb{R} \text{ with } |u_i| \leq M. \] (19)
Then for \( u \in H^s(\Omega) \cap L^\infty(\Omega) \), we have \( g(.,u(.)) \in H^s(\Omega) \).

## 4 Variational formulation and regularity

In this section, we aim to work with the appropriate variational formulation to overcome the difficulty of dealing with the nonlinear and fractional terms of the main problem for both case of Riemann-Liouville and Caputo derivatives separately. The non-local variational problems possess reduced order smoothing properties which is investigated in this section.

### 4.1 The Riemann-Liouville fractional operator

The appropriate variational formulation of the problem [1] with \( g(x,u) = 0 \), introduced in [18] is:
Find \( u \in U := \tilde{H}^{\tilde{s}}(\Omega) \) such that
\[ \mathcal{A}(u,v) = (f,v), \quad v \in V = U, \] (20)
where
\[ \mathcal{A}(u,v) := -\left( R_0D^{\tilde{s}}_x u, \frac{R}{x} D^{\tilde{s}}_1 v \right), \] (21)
and \( f \in L^2(\Omega) \).

Considering the above form for the fractional part has some pros of utilizing the nice properties indicated in Theorem[3] for the approximation procedure. Hence for problem [1], the weak formulation is stated as follows:
Find \( u \in U \) satisfying
\[ \mathcal{L}(u,v) := \mathcal{A}(u,v) + \mathcal{B}(u,v) = (f,v) := F(v), \quad v \in U, \] (22)
where \( f \in U^* \), \( \mathcal{B}(u,v) := (g(x,u),v) \) and \( U^* \) is the dual space of \( U \) introduced in Definition [3].
**Theorem 6** Let $1 < s < 2$ and $u \in \tilde{H}^s(\Omega)$, the operator $A(u, v)$ is coercive and monotone, i.e.,

$$\exists \ c > 0 \ s.t \ A(u, u) \geq c\|u\|_{\tilde{H}^s(\Omega)}^2.$$ 

**Proof.** It is easily verified that

$$A(u, u) = -\left(\frac{D_s}{\Gamma(2-s)}u, u\right).$$

Let us define $S u(x) := -\frac{D_s}{\Gamma(2-s)}u$ and borrow the notation $S^s u$ for $\varepsilon > 0$ from [13] as follows

$$S^s u(x) = \frac{-1}{\Gamma(2-s)} \frac{d^2}{dx^2} \int_0^x (x-t)^{1-s} e^{-\varepsilon(x-t)} u(t) dt, \ x > 0.$$ 

Using the Plancherel theorem, we get

$$(S^s u, u) = \int_{-\infty}^{\infty} (S^s \hat{u})(w)\overline{u(w)}dw,$$  \hspace{1cm} (23)

where the notation $\hat{\cdot}$ refers to the Fourier transform. Let us introduce

$$\hat{a}_\varepsilon(w) = w^2 (\Gamma(2-s))^{-1} \int_0^\infty x^{1-s} e^{-\varepsilon w^2 x} dx,$$

therefore, $(S^s \hat{u})(w) = \hat{a}_\varepsilon(w)$. Now regarding the principal value of the power function, we write

$$\hat{a}_\varepsilon(w) = w^2 (\varepsilon + i w)^{s-2}.$$ 

Hence,

$$\text{Re}(\hat{a}_\varepsilon(w)) > \text{Re} \hat{a}_0(w) = \cos\left(\frac{\pi(2-s)}{2}\right)|w|^s.$$ 

From Eq. (23) and the above equation, we have

$$\text{Re}(u, S^s u) = \text{Re}\left(\int_{-\infty}^{\infty} |\hat{u}(w)|^2 \hat{a}_\varepsilon(w) dw\right) \geq \cos\left(\frac{\pi(2-s)}{2}\right) \int_{-\infty}^{\infty} |\hat{u}(w)|^2 |w|^s dw.$$  \hspace{1cm} (24)

From the contradiction argument investigated in [13] Lemma 4.2, we conclude the coerciveness of the operator $A$. According to this result, one can deduce the monotonicity of the operator $A$ by the Definition [1] i.e.,

$$A(u, u - v) - A(v, u - v) > 0,$$

where $A(u, v) := \langle Au, v \rangle$. $\blacksquare$

In next theorems, we assert the results that guarantee the existence of the unique weak solution for Eq. (1) along with the Riemann-Liouville derivative.

**Theorem 7** Suppose that Assumption [1] holds for the function $g(x, u)$ which satisfies the global ($\Delta_2$)-condition. Consider the variational form (22), then for every $g \in U^*$ and $1 < s < 2$, Eq. (1) has a unique weak solution.

**Proof.** Let $u \in U$ be fixed. Regarding to Lemma [1], $g(., u(\cdot)) \in L_{\tilde{G}}(\Omega)$. Now using Hölder inequality, we obtain

$$|L(u, v)| \leq \|D_{\tilde{G}}^s u\|_{L^2(\Omega)} \|D_1^s v\|_{L^2(\Omega)} + 2\|g(., u)\|_{\tilde{G}} \|v\|_B.$$  \hspace{1cm} (25)

Applying Theorem 2 and Lemma 3, the above inequality could be simplified as

$$|L(u, v)| \leq \left(\|D_{\tilde{G}}^s u\|_{L^2(\Omega)} + 2\|g(., u)\|_{\tilde{G}}\right) \|v\|_U,$$  \hspace{1cm} (26)

which means that the operator $L$ is bounded. Since $L(u, \cdot)$ is linear with respect to the second variable, so $L(u, \cdot) \in U_0^*$ for all $u \in U$. Due to the strictly monotone property of $g(x, \cdot)$ and Theorem 5 the
desired result about the monotonicity of the operator $L$ could be attained. In regard to the previous theorem, we have the coercivity of the operator $A$. Under the assumption about the function $g(x,\cdot)$ and Eq. (17), we have

$$\lim_{\|u\|_{G,\Omega}\to\infty} \frac{\rho_G(u)}{\|u\|_{G,\Omega}} = \infty,$$

therefore $L$ is coercive. Here, we want to show the hemi-continuity of the nonlinear monotone operator $L$. To this end, let $u, w \in \mathbb{R}$ and the sequence $t_n$ tend to zero. The object is to show that $L(u + t_n w, v)$ tends to $L(u, v)$. It is an evident fact that $(R_0D^\alpha_x(u + t_n w)R_0D^\beta_y v) \to (R_0D^\alpha_x u, R_0D^\beta_y v)$ when $t_n$ tends toward zero. Regarding the continuity of the function $g(x,\cdot)$, the claim on the operator $L$ being hemi-continuous is verified. Consequently, the existence of unique weak solution is proved by utilizing Browder-Minty Theorem 1.

We proceed the discussion on the regularity of the solution. To this aim, the following theorem is stated.

**Theorem 8** Consider Eq. (11) along with the Riemann-Liouville fractional derivative where the function $g(x,u)$ satisfies the Lipschitz condition (19). Then, this equation has a solution which fulfills in the nonlinear Volterra-Fredholm integral equation of the form

$$u(x) = x^{s-1} \left( \int_0^x (x-y)^{s-1} \left( g(y,u(y)) - f(y) \right) dy \right).$$

(27)

In addition, let $u \in U$ be a weak solution of Eq. (11). Then $u \in \bar{U}$ where $\bar{U} := \{ u \in H^\alpha(\Omega) \cap \tilde{H}^s(\Omega) \mid G(\cdot, u(\cdot)) \in L^1(\Omega) \}$ for $0 \leq \alpha \leq s - \frac{1}{2}$.

**Proof.** According to the argument about converting the FDEs into integral equation in Chapter 5 of [11] and by adjusting the homogeneous Dirichlet boundary conditions, $u(x)$ takes the following form

$$u(x) = w x^{s-1} - \frac{1}{\Gamma(s)} \int_0^x (x-y)^{s-1} \left( g(y,u(y)) - f(y) \right) dy,$$

(28)

where $w = \left( \int_0^x (g(\cdot,u(\cdot)) - f(\cdot)) \right)$ (1). Moreover, by means of Theorem 3 part (c), we have $x^{s-1} \in H^\alpha(\Omega)$, for $0 \leq \alpha < s - \frac{1}{2}$. On the other hand, Lemma 4 and $u \in U$ insure that $g(\cdot,u(\cdot)) \in H^\frac{s}{2}(\Omega)$ which is a subset of $L^2(\Omega)$. Hence, it is achieved that $g(\cdot,u(\cdot)) - f(\cdot) \in L^2(\Omega)$. In addition, we conclude from part (d) of Theorem 3 that $\int_0^x (g(\cdot,u(\cdot)) - f(\cdot))$ \in $\tilde{H}^s(\Omega)$. Consequently, since $u \in \tilde{H}^s(\Omega)$, one could deduce that $u \in H^\alpha(\Omega) \cap \tilde{H}^s(\Omega)$, for $0 \leq \alpha < s - \frac{1}{2}$.

**Remark 9** Due to the presence of the singular term $x^{s-1}$, it is apparent that the best possible of regularity of the solution (11) could occur in $H^\alpha(\Omega)$. The similar argument about the regularity of the linear form of Eq. (11) reported in Theorem 4.4 and Remark 4.5 of the interesting work [18] verifies the above claim. In that work, $H^\alpha(\Omega)$, $0 \leq \alpha < s - \frac{1}{2}$ is displayed by $H^{s-1+\alpha}(\Omega)$ where $1 - \frac{1}{2} \leq \alpha < \frac{1}{2}$ to show the presence of the singular term better.

### 4.2 The Caputo fractional operator

As discussed in [18], the difference between the variational formulation of Caputo and Riemann-Liouville equations is on their admissible test spaces. It means that the variational formulation of Eq. (11) along with the Caputo derivative is:

Find $u \in U$ such that

$$L(u,v) := A(u,v) + B(u,v) = \langle f,v \rangle, \quad v \in V,$$

(29)

where

$$U := \{ \phi \in \tilde{H}^s(\Omega) \mid G(\cdot,\phi(\cdot)) \in L^1(\Omega) \}.$$
\[ V := \{ \phi \in \dot{H}^s(\Omega) \mid (x^{1-s}, \phi) = 0 \} \quad (30) \]

In order to define an appropriate test space \( V \) for the Caputo case, we assume that \( \phi^*(x) = (1 - x)^{1-s} \), which belongs to \( \dot{H}^s(\Omega) \) and apparently for any \( \phi \in U \), we have \( A(\phi, \phi^*) = 0 \). In the Caputo fractional derivative case, we set \( V = \text{span} \{ \dot{\phi}_i(x) = \phi_i(x) - \gamma_i(1 - x)^{s-1} \mid i = 0, \ldots, N \} \) where

\[ \gamma_i = \frac{(x^{1-s}, \phi_i(x))}{(x^{1-s}, (1 - x)^{s-1})}, \quad (31) \]

and \( \phi_i \in U \). We shall elucidate the above argument in the next section. Note that both Theorems 5 and 7 are valid for the operators including Caputo derivative which have the same variational formulations for the Riemann-Liouville counterpart. Now, we discuss about the regularity of the solution by the following theorem.

**Theorem 10** Let us consider Eq. (11) with Caputo fractional derivative for \( 1 < s < 2 \) in which the function \( g(., u(., .)) \) satisfies the Lipschitz condition (19) and \( f \in H^\alpha(\Omega) \) so that \( \alpha + s \in (\frac{2}{2}, 2) \) and \( \alpha \in [0, \frac{2}{2}) \). Then, this equation has a solution which fulfills in the following nonlinear Volterra-Fredholm integral equation

\[ u(x) = x_0 I_x^s (g(., u(., .)) - f(., .))(1) - a I_x^s (g(., u(., .)) - f(., .))(x). \quad (32) \]

In addition, let \( u(x) \in U \) be a weak solution of Eq. (11). Then \( u \in \tilde{U} \) where \( \tilde{U} := \{ \phi(x) \in H^{\alpha+s}(\Omega) \cap \dot{H}^s(\Omega) \mid G(x, u(x)) \in L^1(\Omega) \} \).

**Proof.** According to [11] Theorem 6. 43, \( u(x) \) has the following form

\[ u(x) = w x - \frac{1}{\Gamma(s)} \int_0^x (x - y)^{s-1} (g(y, u(y)) - f(y)) dy, \quad (33) \]

where \( w = (a I_x^s (g(., u(., .)) - f(., .))(1) \) is determined by adjusting the boundary condition. On the other hand, Lemma 3 and \( u \in U \) imply that \( g(., u(., .)) \in \dot{H}^s(\Omega) \) which is a subset of \( H^\alpha(\Omega) \). Hence, by part (d) of Theorem 3, we have \( a I_x^s (g(., u(., .)) - f(., .)) \in H^{\alpha+s}(\Omega) \). Moreover, by means of Theorem 3 part (c), we have \( x \in H^\beta(\Omega) \), for \( 0 \leq \beta < \frac{s}{2} \). Consequently, from the inclusion argument and \( u \in \dot{H}^s(\Omega) \), we can conclude that \( u \in H^{\alpha+s}(\Omega) \cap \dot{H}^s(\Omega) \) where \( 0 \leq \alpha < \frac{1}{2} \).

**Remark 11** As observed in Theorems 5 and 11, owing to the existence of the intrinsic singular term \( x^{s-1} \) in the solution representation, the solution of differential equation with Riemann-Liouville derivative has less regularity in comparison with the Caputo fractional counterpart. In fact, the best possible regularity in the Riemann-Liouville fractional derivative case belongs to \( \dot{H}^{s-1+\alpha}(\Omega) \). It is worthy to note that the superiority for the Caputo fractional derivative case comes from the fact that the function under the Caputo derivative is supposed to be twice differentiable.

## 5 Finite element approximation

In order to solve weak formulation numerically, we discretize the continuous problem (22) by a Galerkin finite element method. For this aim, piecewise polynomial finite element method is introduced over the interval \( \Omega = [0, 1] \). Let us define \( P_r(\Omega) \) as the space of univariate polynomials of the degree less than or equal to \( r \), for positive integer \( r \). Let \( \chi_h \) be a uniform mesh partition on \( \Omega \), given by

\[ 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \quad N \in \mathbb{N}, \quad (34) \]
with fixed mesh size $h_i = x_i - x_{i-1}$. The set $\chi_h$ induces a mesh $T_h = \{\tau_i | 1 \leq i \leq N\}$ on $\Omega$, where $\tau_i = [x_{i-1}, x_i]$. The length of a subinterval $\tau \in T_h$ is denoted by $h_\tau$ and the maximal mesh width by $h := \max \{h_\tau : \tau \in T_h\}$. We choose standard continuous and piecewise polynomial function space of degree $r \in \mathbb{N}$ on $[0, 1]$ given by

$$S^r_\tau(\Omega) := \{v \in C(\Omega) : v|_\Gamma \in \mathbb{P}_r(\tau), \forall \tau \in T\}.$$  

The nodal points are given by

$$N_\tau := \left\{ \xi_{i,j} := x_{i-1} + j\frac{x_i - x_{i-1}}{r} \mid 1 \leq i \leq N, 0 \leq j \leq r - 1 \right\} \cup \{1\}.$$

We choose the usual standard Lagrange basis functions $\phi_{i,j}^{(r)}$ of $S^r_\tau(\Omega)$. Now with these piecewise functions, one can define the discrete admissible space which is a subspace of $S^r_\tau(\Omega) \cap H^1_0(\Omega)$ denoted by $A_h$. Particularly, we focus on the linear elements in the numerical experiments. Let $\mathbb{L}_h$ be the Lagrange interpolation operator mapping into $A_h$. We denote the finite element test and trial spaces $U_h$ for the Riemann-Liouville fractional derivative with $A_h$ which is described above. In order to investigate the Caputo fractional derivative case, we consider finite dimensional set $U_h = A_h$ as the trial space. In addition, to construct a suitable test space $V_h$, let $V_h = \{\phi_i(x) | i = 0, 1, \ldots, N\}$ where

$$\tilde{\phi}_i(x) = \phi_i(x) - \gamma_i(1-x)^{s-1},$$

where $\gamma_i$ is defined by \[31\]. Finally, the discrete variational formulation released from \[22\] and \[24\] is:

Find $u_h \in A_h$ such that

$$\mathcal{L}(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h. \quad \text{(36)}$$

We notice that in the approximation procedure, one can use the property $(f)$ of Theorem \[3\] which means that for the computation of $R^s_0 D^\frac{s}{\tau} u_h$, one could utilize the relation

$$R^s_0 D^{\frac{s}{\tau}} \phi_i = \int_0^x (x-t)^{-\frac{s}{\tau}} \phi_i'(t)dt$$

$$= \frac{1}{\Gamma(1-\frac{s}{\tau})} \int_0^x (x-t)^{-\frac{s}{\tau}} \left( \frac{x-x_i}{h_i} - \frac{x-x_{i+1}}{h_{i+1}} \right) dt$$

$$= \frac{1}{\Gamma(1-\frac{s}{\tau})} \left[ h_i^{-1} \left( (x-x_i)_{+}^{-\frac{s}{\tau}} - (x-x_i)_{+}^{1-\frac{s}{\tau}} - (x-x_{i+1})_{+}^{1-\frac{s}{\tau}} \right) - h_{i+1}^{-1} \left( (x-x_{i+1})_{+}^{-\frac{s}{\tau}} - (x-x_i)_{+}^{1-\frac{s}{\tau}} - (x-x_{i+1})_{+}^{1-\frac{s}{\tau}} \right) \right],$$

where $a_+ = \max\{a, 0\}$, and analogously for $R^s_x D^{\frac{s}{\tau}} u$, we apply

$$R^s_x D^{\frac{s}{\tau}} \phi_i = -R^s_x I^{1-\frac{s}{\tau}} \phi_i = - \frac{1}{\Gamma(1-\frac{s}{\tau})} \int_x^1 (x-t)^{-\frac{s}{\tau}} \phi_i'(t)dt$$

$$= \frac{1}{\Gamma(1-\frac{s}{\tau})} \int_x^1 (x-t)^{-\frac{s}{\tau}} \left( \frac{x-x_i}{h_i} - \frac{x-x_{i+1}}{h_{i+1}} \right) dt$$

$$= \frac{1}{\Gamma(1-\frac{s}{\tau})} \left[ h_i^{-1} \left( (x-x_i)_{+}^{1-\frac{s}{\tau}} - (x_{i+1}-x)_{+}^{1-\frac{s}{\tau}} - (x-x_{i+1})_{+}^{1-\frac{s}{\tau}} \right) - h_{i+1}^{-1} \left( (x_{i+1}-x)_{+}^{1-\frac{s}{\tau}} - (x_{i+1}-x)_{+}^{1-\frac{s}{\tau}} - (x-x_{i+1})_{+}^{1-\frac{s}{\tau}} \right) \right].$$

Therefore, the term $\mathcal{A}(\phi_i, \phi_j) = (R^s_0 D^{\frac{s}{\tau}} \phi_i, R^s_0 D^{\frac{s}{\tau}} \phi_j)$ could be derived by the above arguments for the Riemann-Liouville derivative case. For the Caputo fractional derivative, we have $\mathcal{A}(\phi_i, \phi_j) = (R^s_0 D^{\frac{s}{\tau}} \phi_i, R^s_x D^{\frac{s}{\tau}} \phi_j)$ which can be simplified as

$$\langle R^s_0 D^{\frac{s}{\tau}} \phi_i, R^s_x D^{\frac{s}{\tau}} \phi_j \rangle - \gamma \langle R^s_0 D^{\frac{s}{\tau}} \phi_i, R^s_x D^{\frac{s}{\tau}} (1-x)^{s-1} \rangle.$$
The first term could be computed using the relations (37) and (38) and the second term is disappeared, because
\[
\left( R_0 D_x^\frac{r}{s} \phi_i, R_x D_t^\frac{s}{r} (1 - x)^{s-1} \right) = - \left( R_0^1 D_x^\frac{r}{s} \phi_i', R_x D_t^\frac{s}{r} (1 - x)^{s-1} \right) \\
= c_\alpha (\phi_i', R_0^1 D_x^\frac{r}{s} (1 - x)^{\frac{s}{r} - 1}) \\
= c_\alpha (\phi_i', 1) \\
= 0,
\]
where \( c_\alpha \) is a constant depending on \( \alpha \).

### 5.1 Convergence analysis

This section is devoted to the study of the achieved approximate solution in previous section. For this end, we consider the existence and uniqueness issue for the discrete equation. In addition, we find an appropriate priori error bound.

**Theorem 12** The discrete problem (30) has a unique solution.

**Proof.** The existence of the discrete solution could be obtained by using Browder-Minty theorem with the same argument pursued in Section 4. For the uniqueness issue, let \( u_1 \) and \( u_2 \) be finite element solutions of (22). Hence,
\[
0 = \mathcal{L}(u_1, v_h) - \mathcal{L}(u_2, v_h) = A(u_1, v_h) - A(u_2, v_h) + B(u_1, v_h) - B(u_2, v_h) \\
= - \int_\Omega R_0 D_x^\frac{r}{s} (u_1 - u_2)(x) R_x D_t^\frac{s}{r} v_h(x) dx + \int_\Omega (g(x, u_1(x)) - g(x, u_2(x))) v_h(x) dx.
\]
(40)

Since the operator \( A(u, v) \) is coercive, so for \( v = u_1 - u_2 \), we get that
\[
- \int_\Omega R_0 D_x^\frac{r}{s} (u_1 - u_2)(x) R_x D_t^\frac{s}{r} (u_1 - u_2)(x) dx \geq c\|u_1 - u_2\|^2_{H^\frac{r}{s} (\Omega)}.
\]
Using above equation and Eq. (40), one can conclude that
\[
c\|u_1 - u_2\|^2_{H^\frac{r}{s} (\Omega)} + \int_\Omega (g(x, u_1(x)) - g(x, u_2(x)))(u_1(x) - u_2(x)) dx \leq - \int_\Omega R_0 D_x^\frac{r}{s} (u_1 - u_2)(x) R_x D_t^\frac{s}{r} (u_1 - u_2)(x) dx \\
+ \int_\Omega (g(x, u_1(x)) - g(x, u_2(x)))(u_1(x) - u_2(x)) dx \\
= 0.
\]

Since \( g(x, u) \) is a monotone function with respect to the second variable, thereby the above inequality, we conclude the uniqueness of the approximate solution. ■

**Lemma 13** Let \( T_h \) be a uniform mesh on \( \Omega \). For real numbers \( s, m \) with \( m \geq \frac{s}{2} \), and also \( S^s_\gamma (\Omega) \) with an integer \( r \geq 0 \), we define \( \hat{r} = \min \{ r + 1, m \} - \frac{s}{2} \). Then there is a constant \( c > 0 \) depending on \( s, m, r \) and \( T_h \) such that
\[
\min_{v_h \in S^s_\gamma (\Omega)} \| u - v_h \|_{H^\frac{r}{2} (\Omega)} \leq c \hat{r}\| u \|_{H^m (\Omega)},
\]
(41)
for all \( u \in H^\frac{r}{2} (\Omega) \cap H^m (\Omega) \). Particularly, for \( r = 1 \) and \( \phi \in H^\frac{1}{2} (\Omega) \cap H^\gamma (\Omega) \) where \( \gamma = \min \{ 2, m \} \) and \( \hat{r} = \gamma - \frac{s}{2} \), one can deduce that
\[
\min_{v_h \in U_h} \| \phi - v_h \|_{H^\frac{r}{2} (\Omega)} \leq c \gamma^{-\frac{s}{2}}\| \phi \|_{H^\gamma (\Omega)}.
\]
(42)
Moreover, if \( \phi \in H^\gamma (\Omega) \cap V \) where \( V \) is defined in (30), the following relation holds
\[
\min_{v_h \in V_h} \| \phi - v_h \|_{H^\frac{r}{2} (\Omega)} \leq c \gamma^{-\frac{s}{2}}\| \phi \|_{H^\gamma (\Omega)}.
\]
(43)
Proof. The relation
\[ \inf_{v \in V_h} \| u - v \|_{H^\alpha(\Omega)} \leq \| u - I_h u \|_{H^\alpha(\Omega)}, \quad 0 \leq \alpha \leq 1, \]
and the similar argument for finding an error estimation of the standard Lagrange finite element for the integer order Sobolev space leads to an error bound for the interpolation error in the intermediate spaces \( \mathcal{H} \) and the special cases \( \mathcal{H}^* \) and \( \mathcal{H}^\# \), for more details see [8, 15]. \( \blacksquare \)

**Theorem 14** Assume that \( u \) be the exact solution of Eq. (1) and \( u_h \) be the approximate solution of the variational formulation (22) or (29). Then
\[ \| u - u_h \|_{H^\gamma(\Omega)} \leq C h^{-\frac{s}{2}} \| u \|_{H^\gamma(\Omega)}, \quad (44) \]
where \( \gamma \) differs for nonlinear boundary value problem with Caputo or Riemann-Liouville fractional derivative. For the case of Caputo differential operator, \( \gamma \) is equal to \( s \). In addition, \( \gamma \) belongs to the interval \( \left[ \frac{s}{2}, s - \frac{1}{2} \right] \) for the Riemann-Liouville fractional counterpart.

**Proof.** Consider \( u_h \in U_h \) be the solution of finite element space of Eq. (1) which satisfies in the following formulation
\[ \mathcal{A}(u_h, v_h) + \mathcal{B}(u_h, v_h) = \langle f, v_h \rangle, \quad v_h \in V_h. \]
Next by subtracting the above equation and Eq. (22), we get that
\[ \mathcal{A}(u, v) - \mathcal{A}(u_h, v_h) + \mathcal{B}(u, v) - \mathcal{B}(u_h, v_h) = \langle f, v \rangle - \langle f, v_h \rangle. \quad (45) \]
Consider the projection operator \( \mathcal{P}_h : H^\gamma(\Omega) \to U_h \) defined by
\[ \mathcal{A}(u, v_h) = \mathcal{A}(\mathcal{P}_h u, v_h). \quad (46) \]
Now by adding and subtracting \( \mathcal{P}_h u \), we have
\[ u - u_h = (u - \mathcal{P}_h u) + (\mathcal{P}_h u - u_h) := \xi + \eta. \quad (47) \]
Then Eq. (45) could be rewritten as follows
\[ \mathcal{A}(u, v) - \mathcal{A}(\mathcal{P}_h u, v_h) + \mathcal{A}(\mathcal{P}_h u, v_h) - \mathcal{A}(u_h, v_h) + \mathcal{B}(u, v) - \mathcal{B}(u_h, v_h) = \langle f, v \rangle - \langle f, v_h \rangle, \quad (48) \]
therefore, from Eq. (46) and setting \( v = v_h \), we get that
\[ \mathcal{A}(\mathcal{P}_h u, v_h) - \mathcal{A}(u_h, v_h) + \mathcal{B}(u, v_h) - \mathcal{B}(u_h, v_h) = 0, \quad (49) \]
or, regarding the bilinearity of the operator \( \mathcal{A} \),
\[ \mathcal{A}(\mathcal{P}_h u - u_h, v_h) + \mathcal{B}(u, v_h) - \mathcal{B}(u_h, v_h) = 0. \quad (50) \]
Let \( v_h = \eta \), we have
\[ \mathcal{A}(\eta, \eta) = \mathcal{B}(u_h, \eta) - \mathcal{B}(u, \eta). \quad (51) \]
Next, by the coercivity of \( \mathcal{A} \), there is a constant \( c_0 \) such that
\[ \mathcal{A}(\eta, \eta) \geq c_0 \| \eta \|_{H^\gamma(\Omega)}. \quad (52) \]
On the other hand,
\[
|\mathcal{B}(u_h, \eta) - \mathcal{B}(u, \eta)| = |(g(\cdot, u) - g(\cdot, u_h), \eta)|
= |(g(\cdot, u) - g(\cdot, \mathcal{P}_h u) + g(\cdot, \mathcal{P}_h u) - g(\cdot, u_h), \eta)|
\leq \| g(\cdot, u) - g(\cdot, \mathcal{P}_h u) \|_{H^\gamma(\Omega)} \| \eta \| + \| g(\cdot, \mathcal{P}_h u) - g(\cdot, u_h) \| \| \eta \|
\leq l_M \| \eta \|_{H^\gamma(\Omega)} (\| \xi \|_{H^\gamma(\Omega)} + \| \eta \|_{H^\gamma(\Omega)}), \quad (53)
\]
where \( l_M \) is verified in [19]. If \( \eta \neq 0 \) and \( c_0 > l_M \), then one could deduce from (52) and (53) that

\[
\|\eta\|_{H^\gamma(\Omega)} \leq \frac{l_M}{c_0 - l_M} \|\xi\|_{H^\gamma(\Omega)}.
\]

Consequently by Eqs. (47), (54) and Lemma 13 we get that

\[
\|u - u_h\|_{H^\gamma(\Omega)} \leq C h^{\gamma - \frac{1}{2}} \|u\|_{H^\gamma(\Omega)}.
\]

It is conspicuous that for each derivative case with different regularity, \( \gamma \) should be different. Using the regularity Theorems 8 and 11 for \( f \in L^2(\Omega) \), one can deduce that \( \gamma \in \left[ \frac{\alpha}{2}, s - \frac{1}{2} \right] \) for Riemann-Liouville fractional derivative operator and \( \gamma = s \) in the Caputo fractional one.

**Remark 15** Let \( f \in H^\alpha(\Omega) \) where \( \alpha \) varies in the interval \( \left[ 0, \frac{1}{2} \right) \) and \( \alpha > s > \frac{1}{2} \). If \( \alpha + s < 2 \), then Theorem 17 yields that \( \gamma = \alpha + s \) arising in the error estimation given for the Caputo fractional operator which means that \( \|u - u_h\|_{H^\gamma(\Omega)} = O(h^{\alpha + s}) \). Otherwise, \( \gamma = 2 \) and \( \|u - u_h\|_{H^\gamma(\Omega)} = O(h^{2 - \frac{1}{2}}) \).

### 6 Numerical Illustrations

The numerical experiments are employed to exhibit the applicability of the Galerkin finite element method for fractional nonlinear boundary value problem with Caputo and Riemann-Liouville derivatives. The experiments are implemented in Mathematica® software platform. We report the absolute error along with the numerical and theoretical rate of convergence for some examples which satisfy in the assumptions considered in the previous sections. Furthermore, the numerical algorithm is examined for some examples with the absence of the mentioned assumptions.

In general, for the numerical experiment with Galerkin method, one of the main issues is the approximation of the integrals. In our examination, the Galerkin finite element solution is obtained from the fully discrete weak form:

Find \( u_h \in A_h \) such that

\[
\mathcal{L}_h(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h.
\]

In the above nonlinear system of equations, we have utilized Gauss-Kronrod quadrature formula to compute the integrals that need to be evaluated numerically. This is happen mainly for the integrals involving the nonlinear term. Furthermore, to solve the nonlinear system, we employ Newton’s iteration method. For this purpose, consider the bilinear form \( N_h(u_h, \cdot, \cdot) \) defined on \( S_F^\gamma(\Omega) \times S_F^\gamma(\Omega) \) by

\[
N_h(u_h; w_h, v_h) = (R_0 D_F^\frac{\alpha}{2} w_h, R_0 D_F^\frac{\alpha}{2} v_h) + (g, (\cdot, u_h) w_h, v_h).
\]

The Newton’s method for approximating \( u_h \) by a sequence \( \{u_h^k\}_{k \in \mathbb{N}} \) in \( S_F^\gamma(\Omega) \) could be written as

\[
N_h(u_h^k; u_h^{k+1} - u_h^k, v_h) = F(v_h) - \mathcal{L}_h(u_h^k, v_h), \quad \forall v_h \in S_F^\gamma(\Omega),
\]

where \( u_0 \in S_F^\gamma(\Omega) \) is an initial guess chosen by the steepest gradient algorithm.

In tables and figures, we notify the experimental and possible theoretical convergence rates (the reported numbers in the parentheses) for the finite element approximation of the nonlinear boundary value problem with the Riemann-Liouville and Caputo fractional derivatives in \( L^2 \) and \( H^\frac{1}{2} \)-norms.

**Example 1** Consider the fractional derivative equation (11) with \( g(x, u(x)) = 3xu^3(x) \). The right hand side function \( f(x) \) is chosen such that

(a) the exact solution with Riemann-Liouville fractional derivative is \( u(x) = \frac{1}{(x+1)}(x^3 - 1) \);

(b) the exact solution is \( u(x) = \frac{1}{(x+1)}(x - x^3) \), where the derivative operator is of Caputo type.
This problem satisfies in the assumptions introduced in Section 5.1. Therefore, the convergence rate of Caputo and Riemann derivative cases in term of $H^2$ error norm are $O(h^2)$ and $O(h^{\frac{1}{2}})$, respectively. This argument directly follows from Theorem 14 for the function $f(x)$ belongs to $L^2(\Omega)$. Tables 1 and 2 report the $L^2$ and $H^2$ error norms for different $s \in (1, 2)$. Moreover, we have provided some figures to exhibit both theoretical and practical rates of convergence. Figures 1 and 2 display the absolute errors for Caputo and Riemann-Liouville fractional derivatives with the above nonlinear term. In figures, the dashed lines show the theoretical convergence rate.

Example 2 In this example, we discuss the approximation of (1) with $g(x, u(x)) = \sin(x)u^5(x)$. The right hand side function $f(x)$ chosen such that

(a) the exact solution is $u(x) = \frac{\Gamma(1.5)}{\Gamma(s+1.5)}(x^{s-1} - x^{s+0.5})$ for the Riemann-Liouville case.

(b) the exact solution is $u(x) = \frac{\Gamma(1.5)}{\Gamma(s+1.5)}(x - x^{s+0.5})$ when Eq. (1) entailed the Caputo fractional derivative.

By a similar reasoning as Example 1 it is seen that the assumptions discussed in the theoretical parts hold. Therefore, we expect $O(h^2)$ and $O(h^{\frac{1}{2}})$ convergence rates for Caputo and Riemann-Liouville fractional differential operators, respectively. This claim is verified by the numerical results reported by Tables 3 and 4 which exhibit the absolute errors in $L^2$ and $H^2$-norms for different $s \in (1, 2)$.
Table 1: The absolute error in $L^2$ and $H^{\frac{s}{2}}$-norms for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$ and mesh size $h = \frac{1}{2^{s+10}}$ for Example 1 with the Caputo fractional derivative operator.

| $s$  | $k$ | $-1$ | 0   | 1   | 2   | 3   | 4   | 5   | Rate |
|------|-----|------|-----|-----|-----|-----|-----|-----|------|
| $\frac{7}{4}$ | $L^2$-norm | 3.08e-03 | 7.44e-04 | 1.80e-04 | 4.33e-05 | 1.04e-05 | 2.06e-06 | 4.93e-07 | 2.047 |
|     | $H^{\frac{s}{2}}$-norm | 2.94e-01 | 1.45e-01 | 7.22e-02 | 3.71e-02 | 1.91e-02 | 1.00e-02 | 5.41e-03 | 0.887 (0.875) |
| $\frac{3}{2}$ | $L^2$-norm | 2.93e-03 | 7.53e-04 | 1.94e-04 | 4.98e-05 | 1.27e-05 | 3.21e-06 | 8.08e-07 | 1.956 |
|     | $H^{\frac{s}{2}}$-norm | 2.05e-01 | 1.11e-01 | 6.08e-02 | 3.42e-02 | 1.93e-02 | 1.10e-02 | 6.49e-03 | 0.769 (0.75) |
| $\frac{4}{3}$ | $L^2$-norm | 3.18e-03 | 8.81e-04 | 2.49e-04 | 8.42e-05 | 2.68e-05 | 8.60e-06 | 2.68e-06 | 1.567 |
|     | $H^{\frac{s}{2}}$-norm | 1.67e-01 | 9.82e-02 | 5.85e-02 | 3.54e-02 | 2.16e-02 | 1.33e-02 | 8.27e-03 | 0.686 (0.67) |

Table 2: The absolute error in $L^2$ and $H^{\frac{s}{2}}$-norms for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$ and mesh size $h = \frac{1}{2^{s+10}}$ for Example 1 with the Riemann-Liouville fractional operator.

| $s$  | $k$ | $-1$ | 0   | 1   | 2   | 3   | 4   | 5   | Rate |
|------|-----|------|-----|-----|-----|-----|-----|-----|------|
| $\frac{7}{4}$ | $L^2$-norm | 7.79e-03 | 2.78e-03 | 1.07e-03 | 4.31e-04 | 1.76e-04 | 7.25e-05 | 3.02e-05 | 1.263 |
|     | $H^{\frac{s}{2}}$-norm | 6.09e-01 | 4.36e-01 | 3.18e-01 | 2.39e-01 | 1.81e-01 | 1.38e-01 | 1.04e-01 | 0.393 (0.375) |
| $\frac{3}{2}$ | $L^2$-norm | 2.73e-02 | 1.31e-02 | 6.43e-03 | 3.17e-03 | 1.56e-03 | 7.68e-04 | 3.78e-04 | 1.020 |
|     | $H^{\frac{s}{2}}$-norm | 7.58e-01 | 6.20e-01 | 5.10e-01 | 4.21e-01 | 3.49e-01 | 2.91e-01 | 2.42e-01 | 0.264 (0.250) |
| $\frac{4}{3}$ | $L^2$-norm | 6.61e-02 | 3.69e-02 | 2.05e-02 | 1.14e-02 | 6.35e-03 | 3.54e-03 | 1.97e-03 | 0.840 |
|     | $H^{\frac{s}{2}}$-norm | 8.87e-01 | 7.76e-01 | 6.81e-01 | 5.99e-01 | 5.28e-01 | 4.66e-01 | 4.13e-01 | 0.173 (0.167) |

Table 3: The absolute error in $L^2$ and $H^{\frac{s}{2}}$-norms for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$ and mesh size $h = \frac{1}{2^{s+10}}$ for Example 2 with the Caputo fractional operator.

| $s$  | $k$ | $-1$ | 0   | 1   | 2   | 3   | 4   | 5   | Rate |
|------|-----|------|-----|-----|-----|-----|-----|-----|------|
| $\frac{7}{4}$ | $L^2$-norm | 2.67e-03 | 6.54e-04 | 1.60e-04 | 3.96e-05 | 9.83e-06 | 2.44e-06 | 6.09e-07 | 2.002 |
|     | $H^{\frac{s}{2}}$-norm | 3.01e-01 | 1.61e-01 | 8.66e-02 | 4.67e-02 | 2.52e-02 | 1.36e-02 | 7.37e-03 | 0.884 (0.875) |
| $\frac{3}{2}$ | $L^2$-norm | 2.64e-03 | 6.44e-04 | 1.59e-04 | 3.94e-05 | 9.79e-06 | 2.54e-06 | 6.16e-07 | 1.992 |
|     | $H^{\frac{s}{2}}$-norm | 2.07e-01 | 1.09e-01 | 5.91e-02 | 3.18e-02 | 1.72e-02 | 9.34e-03 | 5.10e-03 | 0.872 (0.75) |
| $\frac{4}{3}$ | $L^2$-norm | 2.74e-03 | 6.51e-04 | 1.57e-04 | 3.84e-05 | 9.53e-06 | 2.37e-06 | 5.91e-07 | 2.003 |
|     | $H^{\frac{s}{2}}$-norm | 1.70e-01 | 9.20e-02 | 5.00e-02 | 2.74e-02 | 1.52e-02 | 8.67e-02 | 5.00e-03 | 0.793 (0.67) |

Example 3 Consider the nonlinear Riemann-Liouville fractional differential equation \( x' + g(x, u(x)) = \frac{x^s - u(x)}{(s+1)} \) with \( g(x, u(x)) = x \exp(u(x)) \). The right-hand side function \( f(x) \) is chosen such that the exact solution \( u(x) \) is

\[
 u(x) = \frac{1}{\Gamma(s+1)} (x^{s-1} - x^{s+1}) - \frac{2}{\Gamma(s+3)} (x^{s-1} - x^{s+2}).
\]

Tables 3 reports the absolute error in $L^2$ and $H^{\frac{s}{2}}$-norms for different $s \in (1, 2)$ with the above nonlinear term.
Table 4: The absolute errors in $L^2$ and $H^\frac{k}{3}$-norms for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$ and mesh size $h = \frac{1}{2^2 \times 10}$ for Example 2 with the Riemann-Liouville fractional operator.

| $s$  | $k$   | $-1$   | $0$   | $1$   | $2$   | $3$   | $4$   | $5$   | Rate  |
|------|-------|--------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{4}$ | $L^2$-norm | 4.54e-03 | 1.53e-03 | 5.83e-04 | 2.35e-04 | 9.73e-05 | 4.08e-05 | 1.74e-05 | 1.229 |
|      | $H^\frac{k}{3}$-norm | 6.71e-01 | 4.74e-01 | 3.49e-01 | 2.62e-01 | 1.97e-01 | 1.48e-01 | 1.13e-01 | 0.391 (0.375) |
| $\frac{3}{2}$ | $L^2$-norm | 1.56e-02 | 7.55e-03 | 3.73e-03 | 1.85e-03 | 9.21e-03 | 4.60e-04 | 2.28e-04 | 1.002 |
|      | $H^\frac{k}{3}$-norm | 7.64e-01 | 6.27e-01 | 5.18e-01 | 4.30e-01 | 3.58e-01 | 2.98e-01 | 2.49e-01 | 0.261 (0.250) |
| $\frac{1}{3}$ | $L^2$-norm | 4.03e-02 | 2.23e-02 | 1.24e-02 | 6.96e-03 | 3.97e-03 | 2.25e-03 | 1.29e-03 | 0.801 |
|      | $H^\frac{k}{3}$-norm | 8.57e-01 | 7.52e-01 | 6.63e-01 | 5.83e-01 | 5.13e-01 | 4.52e-01 | 3.98e-01 | 0.182 (0.167) |

Table 5: The absolute error in $L^2$ and $H^\frac{k}{3}$-norms for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$ and mesh size $h = \frac{1}{2^3 \times 10}$ for Example 3 with the Riemann-Liouville fractional derivative.

| $s$  | $k$   | $-1$   | $0$   | $1$   | $2$   | $3$   | $4$   | $5$   | Rate  |
|------|-------|--------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{4}$ | $L^2$-norm | 1.17e-03 | 4.24e-04 | 1.58e-04 | 6.03e-05 | 2.37e-05 | 9.42e-06 | 3.82e-06 | 1.301 |
|      | $H^\frac{k}{3}$-norm | 3.09e-01 | 2.23e-01 | 1.64e-01 | 1.21e-01 | 9.10e-02 | 6.92e-02 | 5.28e-02 | 0.389 |
| $\frac{3}{2}$ | $L^2$-norm | 4.59e-03 | 2.19e-03 | 1.10e-03 | 5.46e-04 | 2.73e-04 | 1.38e-04 | 6.95e-05 | 0.987 |
|      | $H^\frac{k}{3}$-norm | 4.00e-01 | 3.27e-01 | 2.68e-01 | 2.20e-01 | 1.82e-01 | 1.51e-01 | 1.26e-01 | 0.266 |
| $\frac{1}{3}$ | $L^2$-norm | 1.13e-02 | 6.27e-03 | 3.24e-03 | 1.71e-03 | 9.10e-04 | 5.06e-04 | 2.82e-05 | 0.845 |
|      | $H^\frac{k}{3}$-norm | 4.72e-01 | 4.12e-01 | 3.62e-01 | 3.19e-01 | 2.81e-01 | 2.48e-01 | 2.19e-01 | 0.178 |

Example 4 We present the nonlinear Caputo fractional differential equations (11) with $g(x, u(x)) = (u(x) - x)^2$, where the exact solution is $u(x) = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{2} + \frac{3}{4}\right)}(x - x^{\frac{3}{4}})$.

Table 4 reports the absolute errors in $L^2$ and $H^\frac{k}{3}$-norms for different $s \in (1, 2)$ with the above nonlinear term.

Table 6: The absolute error in $L^2$ and $H^\frac{k}{3}$-norms for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$ and mesh size $h = \frac{1}{2^3 \times 10}$ for the Caputo fractional operator for Example 4.

| $s$  | $k$   | $-1$   | $0$   | $1$   | $2$   | $3$   | $4$   | $5$   | Rate  |
|------|-------|--------|-------|-------|-------|-------|-------|-------|-------|
| $\frac{7}{4}$ | $L^2$-norm | 3.94e-03 | 1.01e-03 | 2.57e-04 | 6.56e-05 | 1.68e-05 | 4.30e-06 | 1.10e-06 | 1.963 |
|      | $H^\frac{k}{3}$-norm | 3.44e-01 | 1.79e-01 | 9.32e-02 | 4.85e-02 | 2.53e-02 | 1.32e-02 | 6.89e-03 | 0.938 |
| $\frac{3}{2}$ | $L^2$-norm | 3.27e-03 | 9.70e-04 | 2.89e-04 | 8.59e-05 | 2.56e-05 | 7.61e-06 | 2.26e-06 | 1.749 |
|      | $H^\frac{k}{3}$-norm | 2.24e-01 | 1.32e-01 | 7.84e-02 | 4.48e-02 | 2.67e-02 | 1.59e-02 | 9.49e-03 | 0.745 |
| $\frac{1}{3}$ | $L^2$-norm | 1.93e-03 | 6.28e-04 | 2.10e-04 | 7.07e-05 | 2.39e-05 | 8.10e-06 | 2.76e-06 | 1.551 |
|      | $H^\frac{k}{3}$-norm | 1.32e-01 | 8.48e-02 | 5.48e-02 | 3.55e-02 | 2.30e-02 | 1.49e-02 | 9.70e-03 | 0.620 |

Example 5 As the final example, we deal with the linear Caputo fractional differential equation $(g(x, u(x)) = 0)$ by considering $f(x) = x^\theta$ belongs to $H^\alpha(\Omega)$ for $\alpha \in \left[0, \theta + \frac{1}{2}\right)$ and $\theta \in \left\{-\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}\right\}$. The exact solutions for different $\theta$ is $u(x) = c_0(x^{s-1} - x^{s+\theta})$ with $c_0 = \frac{\Gamma(\theta + 1)}{\Gamma(\theta + s + 1)}$.  

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Table 7 displays the theoretical and numerical rates of convergence in $H^{s/2}$-norm for different $s = \frac{7}{4}, \frac{3}{2}, \frac{4}{3}$. This problem satisfies in the Remark 15. For different $\alpha$ and $s$, the value of $\gamma$ is given by $\min\{\alpha + s, 2\}$. For instance, for $s = \frac{7}{4}$ and $\theta = -\frac{1}{5}$, then $\gamma = 2$ and the rate of convergence is $O(h^{2-s}) = O(h^{1.75})$.

| $s$  | $\theta$ | $7/4$       | $3/2$     | $4/3$     |
|------|----------|-------------|-----------|-----------|
| -1/3 | 1.059 (1.042) | 0.916 (0.917) | -- -- --   |           |
| -1/4 | 1.103 (1.125) | 0.974 (1.000) | 0.925 (0.917) |           |
| -1/5 | 1.124 (1.125) | 1.000 (1.050) | 0.970 (0.967) |           |

**Conclusion and future studies**

In this paper, we have studied the Lagrange finite element method for a class of semi-linear FDEs of Riemann-Liouville and Caputo types. To this aim, a weak formulation of the problems have been introduced in the suitable function spaces constructed by considering the fractional Sobolev space and also Musielak-Orlicz space due to the presence of the nonlinear term. In addition, the existence and uniqueness issue of the weak solution together with the regularity is discussed. The weak formulation is discretized by Galerkin method with piecewise linear polynomials basis functions. Finding an error bound in $H^{s/2}$-norm is considered for the Riemann-Liouville and Caputo fractional differential equations. Different examples with the varieties of the nonlinear terms have been examined and the absolute errors are reported in $L^2$ and $H^{s/2}$-norms.

The nature of the nonlinearity and also fractional essence of the problem cause low order convergence of the method. In order to improve the approach for this class of FDEs, one can proceed the idea of splitting method, where the solution is separated into regular and singular parts, which is applicable by utilizing the Taylor expansion of the nonlinear operator and the finite element method accompanying with a quasi-uniform mesh. Also, as discussed in the numerical experiments section, the integrals in the obtained nonlinear system are discretized by a suitable quadrature method. The study on the effect of quadrature method in finite element approximation and a priori error estimation is an idea for future studies. As reported in the numerical section, we have observed the absolute errors in $L^2$-norm which are sharper than the errors in $H^{s/2}$-norm. An interesting question for further study is how to obtain an appropriate theoretical error bound in $L^2$-norm.

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