Misconceptions in mathematical induction for eleventh-grade students – HOTS categorized items

Y C Adinata*, B Budiyono and D Indriati

Postgraduate School of Mathematics Education, Universitas Sebelas Maret Surakarta, Jl. Ir. Sutarni No. 36A, Surakarta, Jawa Tengah 57126, Indonesia

*candriadinata@yahoo.co.id

Abstract. This study aims to analyse misconceptions made by students regarding mathematical induction. The technic to collection data uses principles in ethnography such as observation, interview, documentation, and field note. Data were collected through student answer documentation based on 40 multiple choice questions with reasons. There are four misconceptions in solving problems related to mathematics induction subject matter. First is a misconception in interpreting the (n) on proof of \( P(n) \). Second is a misconception in generalizations (deduction). The third is a misconception in evidentiary procedures. Fourth is a misconception in manipulating or operating. The results of the findings of the misconceptions are used as material for consideration to improve items. Results construction of instruments to measure higher order thinking skills obtained 40 items that have been corrected based on criteria substance, construction, and language for further validation by experts before being tested in three schools.

1. Introduction

Mathematical induction material was started to be taught to high school students. The conclusions for induction that is still general is very important because knowledge will not develop without conclusions [1]. Proof of mathematics consists of direct and indirect evidence methods and mathematical induction. The results of the study on evidence are known that teacher does not invite students to participate in the proof, so students cannot interpret the evidence that the process is more complex and is often not taught [2]. The process is specifically for mathematical induction which consists of methods of proof of a mathematical statement in the form of sequence, inequality, and division. Based on this, it can be concluded that mathematical induction material is very important to be studied by students because it can develop inductive and deductive conclusions.

Proof in mathematics has a central role to train thinking for students based on the events and facts that exist in their environment. The central role of proof in mathematics is unavoidable so that students must be able to understand what evidence is, build evidence which by that proof it has a function as proof verification, explanation, systematization, discovery, and communication [3]. Specific learning objectives regarding mathematical reasoning include explaining, assessing solutions; comparing, generalizing and developing ideas or guesses; testing ideas or evaluating and proving; trials to form allegations or inductively; developing logical arguments based on an understanding or deductive reasoning [4]. In other words, the ability to analyse, assess, compare, generalize, form allegations, test and prove is part of the goals in learning mathematics at this time. In general, knowing the nature of
misconceptions and their sources helps teachers to understand how to plan the right instructions for students [5]. Therefore, it is necessary to study what misconceptions are carried out by students, so that after knowing the sources of misconceptions that occur in students, action can be taken to improve them.

Misconceptions are misunderstandings and misinterpretations based on incorrect meanings [5]. The misconception is also referred to as alternative conceptions or alternative frames [6]. Based on this, the misconception is the inconsistency between the views of students about a problem and misunderstanding in interpreting carried out by students based on the views of experts. Misconception causes problems at the level of student understanding in solving mathematical induction items. Therefore, the required items will be tested to see what misconceptions are common or found in students. The problem of misconception and lack of understanding of concepts is a major problem that occurs a lot in math subjects. Based on the results of the national exam scores of high school students in the 2017/2018 school year, the mathematics test in the Central Java Province has an average value of 45.97. Particularly in Karanganyar regency, nationally, the total average absorption in the algebraic material which is known from the average percentage of students correct answers is still less than 40%. Misconception can be done by students if the concept is not in accordance with the actual concept [7]. Therefore, the causes of misconception and low understanding must be immediately known, whether misconception occurs in concepts, generalization misconceptions, or operation misconceptions. Improving the quality of education in schools can be achieved in various ways, including through the development of assessment instruments that focus on students’ thinking skills.

Thinking occurs as a result of mental processes experienced by someone in processing a problem or a situation that requires resolution. Thinking as the basis of everything that is done by each individual [8]. The results of the thinking process affect every action, skill, the solution offered and decisions made. Higher order thinking (HOT) is a type of thinking that requires a greater cognitive process than other types of thinking [9]. Higher order thinking skills of students can be trained on questions that require students to be able to use thinking skills effectively in analyzing and evaluating everything both products, ideas and applying ideas based on what is believed to achieve the highest quality in the level of thinking. Thinking skills, better known as higher order thinking skills basically have to know the facts, understand concepts, and apply what is known, so students are able to choose topics separately through analysis, assess and provide solutions to problems. Therefore, Higher Order Thinking Skills (HOTS) require a more complex thinking process in dealing with new challenges and situations, so as to provide solutions to these problems. Critical and creative thinking is an embodiment of higher-order thinking or higher order thinking [10]. Higher order thinking skills include the ability to think critically and creatively. Therefore, higher order thinking skills are thinking skills in solving various problems involving higher order thinking cognitive processes including critical thinking and creative thinking.

The process of thinking by connecting the facts that have been known so that the discovery is made by making guesses towards a general conclusion and then proved the truth using deductive proof or with denial. The items tested on students must contain higher order thinking skills elements. Assessment that includes higher order thinking skills is a standard assessment applied by international testing institutions. Mathematics as a science is very important to be studied by students because it can develop high-level thinking skills [11]. Based on this, in addition to obtaining data on the misconception in mathematical induction questions, the mathematical induction questions must contain higher order thinking skills so that students’ ability to understand a problem becomes more interesting, challenging, and motivating to be solved.

Mathematical questions will be a problem or not a problem depending on the items given to students. This means that the problems proposed are still relative so that items which can measure thinking skills can be developed. From the results of the study, it is known that there are several ways to measure student conceptions, some of which are using interviews, open tests, and multiple choice tests that are used to identify misconceptions [12]. The misconception is very dependent on many factors such as experience, creativity, perception, and textbooks [13]. Based on this, it is necessary to develop the items and study the misconceptions that occur in students in solving mathematical induction questions and provide solutions to these problems.
2. Method
This article is part of the research on the development of cultured-based assessment instruments to measure higher order thinking skills. At the development stage, it consists of four phases, better known as the 4-D development model, namely define, design, develops, and disseminate. Specifically for the development phase, before the actual trial was carried out at the three High Schools, namely SMA 1 Karanganyar, SMA Negeri Colomadu, and SMA 1 Muhammadiyah Karanganyar, then a person test was conducted which was a test applied to nine students to obtain information about constraints, misconception may be experienced by the user in using the product developed. Question items use multiple choices on the grounds that they are used as instruments [13]. The number of instruments used for the trial is 40 items. Therefore the questions answered by students are accompanied by reasons in the form of work descriptions to obtain answers.

The type of this article is descriptive qualitative by using an ethnographic approach to get a description and thorough analysis of information and results on the student's answer sheet. In the technique of collecting data, the research boundaries were chosen based on the principles in ethnography, such as observation, interviews, documentation, and field recording [14]. Interviews with individual trial participants were also conducted to obtain more in-depth information. After the information about the use of individual trials was obtained, then the improvements were conducted to get the products expected by the test developer.

3. Results and discussion
The results of the initial product planning are in the form of cultured-based assessment instruments to measure students' HOTS mathematics which begins with theoretical planning or theoretical drafts. Budiyono argued that in broad outline the test construction steps are as follows [15]: (1) inventorying the material that has been taught, (2) compiling test specifications, (3) compiling the items along with the answer key, (4) reviewing the test items, (5) conduct a trial, (6) conduct a test analysis and item analysis based on the results of the trial, (7) make revisions to the items that are not good, (8) determine the instrument (which consists of good items), (9) carrying out test measurements to the desired subject and (10) interpreting the results obtained. In this article, the results are obtained in steps 1 to 5, namely a person's test on students. The following are the results of the explanation of misconceptions that occur in students working on mathematical induction material.

3.1. The misconception in interpreting the \((n)\) on proof of \(P(n)\)
Misconceptions that occur in students are grouped into 4 common mistakes that occur in the students' work which can be seen on the student's answer sheet. Analysis of misconceptions that occur in students can be done through analysis of conceptual components that have not been mastered by students [16]. The proof uses the principle of mathematical induction of a sequence known \(P(n)\): \(\frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \ldots + \frac{1}{2n(n+1)} = \frac{n^2}{2(n+1)}\). The above question is a Multiple Choices Question with reason. Therefore, the results of students' answers in solving this question can be seen in Figure 1.

\[
\frac{1}{2n(n+1)} = \frac{n^2}{2(n+1)}
\]

Figure 1. Student’s equate the left and right segments.
In figure 1, it can be noted that there is a misconception in interpreting the (n) on proof. Based on the proving procedure using the principle of mathematical induction, the $P(n)$ should be true for $n = 1$, then $P(1)$ is obtained for the left side $P(1)$: $P_1 = \frac{1}{2(1)(1+1)} = \frac{1}{4}$, while the right side is $\frac{12}{2(1)(1+1)} = \frac{1}{4}$, so it is proven correct for $n = 1$. For the left side $P(2)$: $P_2 = \frac{1}{2(2)(2+1)} = \frac{1}{12}$, while for the right side $\frac{2^2}{2(2)(2+1)} = \frac{4}{12}$ so that false evidence is obtained for $n = 2$. This happens because the test participant interprets that $n$ on the left side is the $n^{th}$ term, while for the right side, for example in $P(2)$ is the sum of $\frac{1}{4} + \frac{1}{12} = \frac{4}{12}$ is $\frac{2^2}{2(2)(2+1)}$. The most important thing is if the proof is valid for $n = 1$, then the proof is also valid if it is applied to the next number of the sum of the sequence terms.

3.2. The misconception in making generalizations
Test participants who have proven that it is correct for $P(1)$, $P(2)$, and up to $P(3)$, then they also should conclude that is true for $P(k)$ then it is proven true as well for $P(k + 1)$ without doing proving process in general form. This is shown in the student worksheet in figure 2 as follows.

Figure 2. Student's misconception in making a generalization.

Figure 2 shows that there is a misconception in making generalizing because in the principle of mathematical induction there is a first step and an induction step so that they cannot be separated. Difficulties of test participants in applying proof techniques to move from $P(k)$ to $P(k + 1)$, then the problem is whether students realize or not if mathematical induction works on natural numbers [17]. This happens as it is assumed that they draw conclusions directly from cases or based on examples of proving mathematical induction without continuing on the induction step.

3.3. The misconception in proof procedures
Conceptual changes and misconceptions, changes to this concept include a conception of numbers, providing a principle-based system that students use to make predictive predictions of patterns and explain newly received information [18]. If two misconceptions have been elaborated, namely a misconception of the meaning of proof of $P(n)$ and a misconception in making generalizations, then there is a misconception in the proof procedure as shown in Figure 3.

Figure 3. The misconception of students performing proof procedures.
In figure 3 is the process carried out by the test participant for the induction step (ii) that is: suppose that is true for P(k): \( \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \cdots + \frac{1}{2k(k+1)} = \frac{k^2}{2(k)(k+1)} \), for \( k \in \mathbb{N} \), then it will be proven true also for P(k+1) obtained \( \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \cdots + \frac{1}{2k(k+1)} + \frac{1}{2(k+1)(k+2)} = \frac{(k+1)^2}{2(k+1)(k+2)} \), a procedure like this makes procedures like this the test participant experience a misunderstanding in the procedure of proof. Generally, most students memorizing an equation without understanding, memorizing a rule of operation without enough understanding will only damage the concept of the future [19]. Therefore it is important to understand that if it is true for P(k): \( \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \cdots + \frac{1}{2k(k+1)} = \frac{k^2}{2(k)(k+1)} \), for \( k \in \mathbb{N} \), proven true also for P(k+1) obtained \( \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \cdots + \frac{1}{2k(k+1)} + \frac{1}{2(k+1)(k+2)} = \frac{(k+1)^2}{2(k+1)(k+2)} \), elaborated so that the results are obtained \( \frac{(k+1)^2}{2(k+1)(k+2)} \).

3.4. The misconception in manipulating or operating

Misconceptions that are common to students are related to estimation and calculation [20]. For example, in proof \( \frac{1}{12} + \frac{1}{23} + \frac{1}{34} + \cdots + \frac{1}{n+1} = \frac{n}{(n+1)} \), during the operation step the error is shown as shown in Figure 4 as follows.

\[ \begin{align*}
\text{Andi benar } n &= k \\
\text{Akan dibuktikan } n &= k+1 \\
\frac{1}{12} + \frac{1}{23} + \frac{1}{34} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k+1}{k+2}
\end{align*} \]

Figure 4. The misconception of students manipulating operations.

Figure 4 shows that there are still operation errors. It happens in understanding the operation of the numerator and the denominator has not been equalized. The ability to build evidence is grouped into three, the ability to prove basic steps, the ability to prove the steps of induction, and the ability to present evidence in the correct form [21]. Students with creative thinking also experience misconceptions in solving mathematical problems [22]. The misconception of the deductive proof, incomplete comprehension of the meaning of the basis of induction, a misconception of the increment [23]. In other words, there are only three known misconceptions. Therefore in this article, there are four misconceptions, that is the misconception in interpreting the \( n \) on proof of proof P(n), the misconception in making generalizations (deduction), the misconception in proof procedures, and the misconception in manipulating or operating.

The proof is an ability needed in mathematics to ensure that a proposition or theorem is true. The purpose of proof using mathematical induction is to verify, explain, communicate, and systematize the statement so that it applies in general. Proof by mathematical induction is proof that involves natural numbers. The process of proving it using the first procedure is the first step: P(1) is correct. The second step of induction: if P(k) is correct, then P(k + 1) is true, for every k of the natural numbers. Students consider mathematical induction as a case of circular reasoning that is proof P(n) is true for all positive integers, mathematical induction is a technique in which the drawing of general arguments comes from a number of specific cases and rules determined by the principle of mathematical induction without understanding what they are doing [24]. Therefore misconceptions problems must also be resolved in terms of learning strategies, and the strengthening of concepts conducted especially by math teachers.
and prospective teachers who already know the misconceptions in their students. Some ways to solve this are when the presentation of evidence is made directly for the correct theorem, from the teacher can illustrate the presentation of the evidence for example with images or illustrations such as domino effects that are used as stabilization of concepts to students. The teacher can choose a learning strategy that directs students to find proof forms using the principle of mathematical induction. The students can explore the evidence presented so that they can help to remember and experience themselves based on examples of mathematical induction [25]. The teacher can strengthen the required materials related to mathematical induction that are related to sequence, division, and inequality. Misconceptions can occur in students if the concepts learned by students in a material are weak. Understanding the concept does not mean memorizing, but students are able to analyze consistent and inconsistent information, identify a problem, make an assessment, plan a solution, and integrate ideas, and make presumption which then proves that the statement is applied to all.

3.5. Constructions of instruments

Higher order thinking skills include critical thinking and creative thinking that is elaborated through the indicators. Salmon argues that critical thinking refers to various activities and abilities that specifically express these abilities into four terms, that are [26]: (1) analyze the meaning or information, (2) identify the arguments and evaluate the meaningfulness of the argument, (3) identify and avoid mistakes, and (4) make decisions according to the evidence obtained. Creativity is the ability to create new things [27]. Creative thinking involves the ability to think independently and produce ideas [28]. Based on theories related to higher order thinking skills, the indicators are arranged as in Table 1.

Table 1. Aspects and indicators of higher order thinking skills.

| HOTS Aspect         | HOTS Indicator                                                                 |
|---------------------|-------------------------------------------------------------------------------|
| 1. Critical thinking| 1.1 Analyzing also determining the relevant and irrelevant information from a problem |
|                     | 1.2 Making the right conclusion according to the information that has been obtained |
|                     | 1.3 Checking the consistency or inconsistency of the arguments,               |
|                     | 1.4 Assessing the right operation or product based on criteria or standards  |
| 2. Creative thinking| 2.1 Planning strategies or methods to solve a problem in accordance with the criteria of the problem |
|                     | 2.2 Integrating ideas, thoughts, and strategies to solve problems             |
|                     | 2.3 Create or develop and prove alternative solutions to a problem           |

According to table 1, question items were arranged which contained mathematical induction material. The construction of instrument items was arranged based on aspects consisting of critical thinking including four indicators and creative thinking covering three indicators. Higher order thinking skills include the ability to think critically and creatively [29]. It differs from the taxonomy of blooms which are revised thinking skills through students' cognitive processes such as analyzing, evaluating, and creating [30]. These differences adjust core competencies to be achieved, while critical and creative thinking is found in learning activities. Question items are built based on cultured exploration, which is used to measure HOTS. The stimulus on local culture could be in the form of text, picture, game, motif or pattern, graphic, table, art craft or other pieces of information from students' cultures [31]. Next, after arranging the test specifications, arrange the question items along with the answer key. Test question items were studied by several experts and then revised and conducted individual trials.
3.6. HOTS-Categorized items
After knowing what are the misconceptions of mathematical induction material, it will be tried from the side of the test instrument items arranged based on these findings. The results obtained 40 items that have been improved so that later can be validated by experts in the field of mathematics regarding the substance or material, construction, language and components from culture. The following are displayed several cultured-based questions to measure higher order thinking skills.

Examples of problems that conform to the HOTS Indicators are as follows:

a. In the celebration of Wahyu Kliyu, a participant arranges the Apem cake in the form of pyramid arrangement with consistent sides. The arrangement forms four levels with the first level or the top there is 1 cake and the bottom level is 16 cakes. If there will be formed seven levels of the pyramid, the entire cake that must be prepared is…
   A. 30
   B. 55
   C. 91
   D. 140
   E. 240

b. A batik trader arranges his merchandise based on the type so that it forms a line, which is initially one batik, then four batiks, then seven batiks, and so on. Determine the summation formula for the batik…
   A. \( \frac{(n+1)}{2} \)
   B. \( \frac{(5n-1)}{2} \)
   C. \( \frac{4}{(3n^2-n)} \)
   D. \( \frac{(n^2+3)}{4} \)
   E. \( \frac{n^2+3n+2}{6} \)

c. The profit of a food stall decreases every week, as shown in infinite line in the following \( \frac{1}{4} \), \( \frac{1}{12} \), \( \frac{1}{24} \),…, \( \frac{1}{2n(n+1)} \), if the seller wants to know the total profits obtained, then the exact summation formula is…
   A. \( \frac{n}{n(n+3)} \)
   B. \( \frac{n}{n(2n+2)} \)
   C. \( \frac{n}{2n(n+1)} \)
   D. \( \frac{n^2}{n(2n+2)} \)
   E. \( \frac{n^2}{2n(n+1)} \)

Furthermore, based on the results of interviews with students obtained information relating to mathematical induction material as follows. First, in general, the number pattern in the mathematics induction items, the solution can be calculated using one by one method. Second, students experienced misconception in the operation of using row rules as well as arithmetic sequences. The third most is a misconception in making generalizations and proof procedures. This can happen because in manipulating the operation of the number pattern based on the procedure of proof is still not right, so it immediately concludes that to apply to the first term it also applies to the whole. Fourth, there are a number of questions that confuse students to answer so that they need to be improved based on substance, construction, and language for further validation by experts before being tested in class. The instrument would be ready as an instrument to measure HOTS after revision by experts and all instrument declared valid and worthy to be used.
4. Conclusion
There are four misconceptions in solving the problem of mathematical induction that is, first, the misconception in interpreting the \( P(n) \) on proof of \( P(n) \). Second, the misconception in making generalizations (deduction). Third, the misconception in proof procedures. Fourth, the misconception in manipulating or operating. The results of the construction of a cultured-based assessment instrument to measure higher order thinking skills obtained 40 question items that have been fixed for further validation by experts before testing in the classroom. Therefore, based on the findings of this article, a solution is needed to solve this misconception beside the test instrument, another way that can be done is to establish the principle concept of mathematical induction, select learning strategies that focus findings by students and strengthen understanding of related required materials with mathematical induction like sequence, division, and inequality.

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