Abstract. For polytopes in $\mathbb{R}^d$ with at least as many facets as vertices, we prove that vertices can be mapped injectively to non-incident facets when $1 \leq d \leq 5$ and give counterexamples for $d \geq 7$.

1. Background and notation

Polytopes are present in many fields of mathematics such as poset theory, discrete mathematics, linear optimization, and convex geometry. Unsurprisingly, problems involving polytopes can be addressed by a variety of methods, and different aspects of such problems can be emphasized. A motivation for the contents of the present paper is given by a metrical notion called reducedness introduced for general convex sets in [5]. While the existence of reduced polytopes is clear in the Euclidean plane, the only known examples of reduced polytopes in higher-dimensional Euclidean spaces are those given in [3]. This existence problem for a polytope with a certain metrical property has a combinatorial feature at its core: The authors of [1, Theorem 4] show that, in particular, for any reduced polytope, each vertex of the polytope is (in some sense) opposite to a facet to which it is not incident, and this assignment is injective. Thus, are there combinatorial types of polytopes in $\mathbb{R}^d$ for which their vertices cannot be mapped injectively to non-incident facets and therefore cannot have any reduced realizations? In the following, we proof that the answer is negative for $1 \leq d \leq 5$ and affirmative for $d \geq 7$.

Let us recall some standard notation. For this, we refer to the books [4, 7]. A polytope in $\mathbb{R}^d$ is the convex hull of finitely many points $v_1, \ldots, v_n \in \mathbb{R}^d$, i.e.

$$P = \text{conv}\{v_1, \ldots, v_n\} := \left\{ \sum_{i=1}^n \lambda_i v_i \mid \lambda_1, \ldots, \lambda_n \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$ 

By its dimension, we mean the dimension of its affine hull, or $-1$ if $P$ is empty. A face of $P$ is a subset $f \subseteq P$ with the property, that for all $x, y \in P$ and $\lambda \in (0, 1)$, the condition $\lambda x + (1-\lambda)y \in f$ already implies $x, y \in f$. The faces of $P$ include the empty set and $P$ itself. Each face is also a polytope and therefore has a dimension. The family of $k$-dimensional faces of $P$ is finite and shall be denoted by $\mathcal{F}_k(P)$. Any two faces of $P$ which are not disjoint are said to be incident to each other. Throughout this paper, we use the standing assumption that the polytope $P \subset \mathbb{R}^d$ is of dimension $d$, and that this dimension is $d \geq 1$ (we exclude the empty and singleton polytope). In this case, its $k$-dimensional faces are called vertices, edges and facets when $k = 0$, $k = 1$ and $k = d-1$, respectively. The set $\mathcal{F}_0(P)$ consists of
singletons and we will not distinguish between the point \( v \in \mathbb{R}^d \) and the singleton \( \{v\} \in \mathcal{F}_0(P) \). In particular, we will write \( v \in \mathcal{F}_0(P) \). A polytope is spanned by its vertices: \( P = \text{conv} \mathcal{F}_0(P) \). A face of a face of \( P \) and the intersection of faces of \( P \) is again a face of \( P \). In particular, any face of \( P \) is spanned by some vertices in \( \mathcal{F}_0(P) \). These will be called the vertices of the face. Our central problem now reads as follows:

**Problem 1.1.** Given a polytope \( P \subset \mathbb{R}^d \) with \( |\mathcal{F}_0(P)| \leq |\mathcal{F}_{d-1}(P)| \), is there an injective mapping \( \phi : \mathcal{F}_0(P) \to \mathcal{F}_{d-1}(P) \) for which \( v \) and \( \phi(v) \) are non-incident for all \( v \in \mathcal{F}_0(P) \)?

Obviously, the assumption \( |\mathcal{F}_0(P)| \leq |\mathcal{F}_{d-1}(P)| \) cannot be dropped as it is necessary for the existence of injective mappings \( \phi : \mathcal{F}_0(P) \to \mathcal{F}_{d-1}(P) \) without any extra condition. Polytope duality, which interchanges the roles of \( k \)-dimensional faces and \((d-k-1)\)-dimensional faces, carries over any proof or counterexample to or from the following equivalent problem.

**Problem 1.2.** Given a polytope \( P \subset \mathbb{R}^d \) with \( |\mathcal{F}_0(P)| \geq |\mathcal{F}_{d-1}(P)| \), is there an injective mapping \( \psi : \mathcal{F}_{d-1}(P) \to \mathcal{F}_0(P) \) for which \( f \) and \( \psi(f) \) are non-incident for all \( f \in \mathcal{F}_{d-1}(P) \)?

The injective maps mentioned in Problems 1.1 and 1.2 shall be called *vertex-facet assignments* and *facet-vertex assignments*, respectively.

2. **Graph-theoretic formulation**

Problems involving injective maps between finite sets can be stated conveniently in the language of graph theory, using matchings of bipartite graphs. We refer the reader to [2, Chapter 2]. In order to avoid confusion with the language of polytopes, we refer to the nodes of a graph, instead of its vertices. A bipartite graph is then a graph \( G = (V,E) \) for which there is a decomposition of the set of nodes \( V = V_1 \cup V_2 \) into non-empty disjoint sets called *partition classes*, such that edges (the elements of \( E \)) do not join nodes of the same partition class. For each node \( v \in V_i \), denote by \( N_G(v) \) the *neighborhood* of \( v \), i.e. the set of to \( v \) adjacent nodes in \( G \). Further, set \( N_G(S) := \bigcup_{v \in S} N_G(v) \). A *matching* \( M \) in a bipartite graph \( G = (V_1 \cup V_2, E) \) is a 1-regular subgraph of \( G \). The matching \( M \) of \( G \) is said to be *\( V_1 \)-covering* (resp. *\( V_2 \)-covering*) if every node \( v \in V_1 \) (resp. \( v \in V_2 \)) is a node in \( M \). In this sense, \( V_1 \)-covering matchings of \( G \) correspond to injective single-valued selections of the set-valued mapping \( V_1 \rightrightarrows V_2 \) given by \( v \mapsto N_G(v) \), that is, \( V_1 \)-covering matchings of \( G \) are injective mappings \( \phi : V_1 \to V_2 \) which map nodes to adjacent ones.

For our purposes, a bipartite graph can be assigned to any polytope \( P \subset \mathbb{R}^d \) as follows.

**Definition 2.1.** The *vertex-facet graph* of the polytope \( P \subset \mathbb{R}^d \) is the bipartite graph \( G = (V_1 \cup V_2, E) \) whose partition classes are (disjoint copies of) the vertices \( V_1 = \mathcal{F}_0(P) \) and the facets \( V_2 = \mathcal{F}_{d-1}(P) \) of the polytope. A vertex \( v \in \mathcal{F}_0(P) \) and a facet \( f \in \mathcal{F}_{d-1}(P) \) are adjacent in \( G \) if and only if they are non-incident in \( P \).

With this terminology in place, the vertex-facet assignments (resp. facet-vertex assignments) are just \( \mathcal{F}_0(P) \)-covering (resp. \( \mathcal{F}_{d-1}(P) \)-covering) matchings in the vertex-facet graph.

Hall’s marriage theorem [2, Theorem 2.1.2] gives a necessary and sufficient condition for the existence of \( V_1 \)-covering matchings of bipartite graphs \( G = (V_1 \cup V_2, E) \).
Theorem 2.2. The bipartite graph $G = (V_1 \cup V_2, E)$ possesses a $V_1$-covering matching if and only if $|S| \leq |N_G(S)|$ for all $S \subseteq V_1$.

The condition $|S| \leq |N_G(S)|$ for all $S \subseteq V_1$ appearing in Theorem 2.2 is called Hall condition. While one direction of the theorem is almost trivial, the other one gives the power to construct the desired matchings (provided they exist).

3. Existence in low dimensions

The answers to Problems 1.1 and 1.2 are trivially affirmative for $d \in \{1, 2\}$. Trial and error for polytopes in $\mathbb{R}^3$ with low facet or vertex count does not yield any counterexamples. Using Hall’s marriage theorem (Theorem 2.2) for the vertex-facet graph (see Definition 2.1), we are able to show that there are no counterexamples up to dimension 5. For the proof, we will take the dual point of view, i.e. we address Problem 1.2 which requires finding facet-vertex assignments $\psi : \mathcal{F}_{d-1}(P) \to \mathcal{F}_0(P)$ for polytopes $P \subset \mathbb{R}^d$. This is motivated by the following geometric interpretation of the non-neighborhood $\mathcal{F}_0(P) \setminus N_G(S)$ of a set $S \subseteq \mathcal{F}_{d-1}(P)$ of nodes in the vertex-facet graph $G$.

Proposition 3.1. Let $G$ be the vertex-facet graph of the polytope $P \subset \mathbb{R}^d$. For $S = \{f_1, \ldots, f_k\} \subseteq \mathcal{F}_{d-1}(P)$, the set $\mathcal{F}_0(P) \setminus N_G(S)$ is the set of vertices of $f_1 \cap \cdots \cap f_k$.

Proof. Note that a vertex $v \in \mathcal{F}_0(P)$ is in $N_G(S)$ if and only if there is at least one facet $f \in S$ to which it is non-incident (since then they are adjacent in $G$). Hence, a vertex lies in the complement $\mathcal{F}_0(P) \setminus N_G(S)$ if and only if it incident to all the facets $f_1, \ldots, f_k$, i.e. it lies in the intersection $f_1 \cap \cdots \cap f_k$. \hfill $\square$

In the sequel, we denote by $n = |\mathcal{F}_0(P)|$ and $m = |\mathcal{F}_{d-1}(P)|$ the vertex count and the facet count of the polytope $P \subset \mathbb{R}^d$ under consideration. Also, we denote by $G$ the vertex-facet graph of $P$. As a corollary of Theorem 2.2 and Proposition 3.1, we obtain the following statement.

Corollary 3.2. A polytope $P \subset \mathbb{R}^d$ possesses a facet-vertex assignment if and only if for all $k \in \{1, \ldots, m\}$ and facets $f_1, \ldots, f_k \in \mathcal{F}_{d-1}(P)$, we have

\begin{equation}
|\mathcal{F}_0(f_1 \cap \cdots \cap f_k)| \leq n - k.
\end{equation}

In order to construct a counterexample to Problem 1.2, it is required to find a polytope $P \subset \mathbb{R}^d$ and facets $f_1, \ldots, f_k \in \mathcal{F}_{d-1}(P)$ such that $|\mathcal{F}_0(f_1 \cap \cdots \cap f_k)| > n - k$. The possible existence of counterexamples is constrained by the affine dimension of $f_1 \cap \cdots \cap f_k$.

Theorem 3.3. Let $P \subset \mathbb{R}^d, d \geq 1$, be a polytope with $m \leq n$ and $f_1, \ldots, f_k \in \mathcal{F}_{d-1}(P)$. If $\dim(f_1 \cap \cdots \cap f_k) \in \{0, 1, 2, d-2, d-1\}$, then condition (1) holds.

Proof. We deal with each case for $\delta := \dim(f_1 \cap \cdots \cap f_k) \in \{0, 1, 2, d-2, d-1\}$ separately.

Case $\delta = 0$: This means that $f_1 \cap \cdots \cap f_k$ is a vertex of $P$. Assume that condition (1) is violated, i.e. $1 = |\mathcal{F}_0(f_1 \cap \cdots \cap f_k)| > n - k$. This implies $n = k$, and we conclude $m \leq n = k \leq m$, i.e. $m = k$. Thus the intersection $f_1 \cap \cdots \cap f_k$ uses all facets of $P$. But no vertex of $P$ is incident to all facets of $P$, a contradiction.

Case $\delta = 1$: This means that $f_1 \cap \cdots \cap f_k$ is an edge of $P$. Assume that condition (1) is violated, i.e. $2 = |\mathcal{F}_0(f_1 \cap \cdots \cap f_k)| > n - k$. This implies $n \leq k + 1$, and we conclude $m \leq n \leq k + 1$, i.e. $k \geq m - 1$. The affine hull of the edge $f_1 \cap \cdots \cap f_k$ is
a straight line which is contained in the hyperplanes obtained as the affine hulls of the facets \( f_1, \ldots, f_k \). Two additional hyperplanes (or facets of \( P \)) are required to “cut out” the edge from its affine hull. However, since \( k \geq m - 1 \), there is at most one facet of \( P \) left which is not one of \( f_1 \cap \cdots \cap f_k \). This is a contradiction.

Case \( \delta = 2 \): This means that \( f_1 \cap \cdots \cap f_k \) is a polygon with, say, \( N \) vertices and edges. Assume that condition (1) is violated, i.e. \( N = |F_0(f_1 \cap \cdots \cap f_k)| > n - k \). This implies \( n \leq k + N - 1 \), and we conclude \( m \leq n \leq n + N - 1 \), i.e. \( k \geq m - N + 1 \). Thus the intersection \( f_1 \cap \cdots \cap f_k \) uses all but at most \( N - 1 \) facets of \( P \). However, in order to “cut out” the \( N \) edges of the polygon, we need \( N \) more facets not already used in the intersection, a contradiction.

Case \( \delta = d - 2 \): This means that \( f_1 \cap \cdots \cap f_k \) is a \((d - 2)\)-face of \( P \). As such, it is incident to exactly two facets of \( P \), see [7, Theorem 2.7]. Hence \( k = 2 \). Assume that condition (1) is violated, i.e. \( N := |F_0(f_1 \cap f_2)| > n - 2 \). Thus the two facets \( f_1 \) and \( f_2 \) must intersect in all vertices, or all but one. Since the intersection \( f_1 \cap f_2 \) is \((d - 2)\)-dimensional, the convex hull of the union of \( f_1 \cap f_2 \) and the (possibly) remaining vertex of \( P \) is at most \((d - 1)\)-dimensional. This contradicts the assumption of \( P \) being \( d \)-dimensional.

Case \( \delta = d - 1 \): This means that the intersection consists of a single facet, i.e. \( k = 1 \). Assume that condition (1) is violated, i.e. \( |F_0(f_1)| > n - 1 \). Hence, the facet \( f_1 \) contains all vertices, which contradicts the assumption of \( P \) being \( d \)-dimensional.

An immediate consequence of Theorem 3.3 is the existence of facet-vertex assignments for polytopes up to dimension 5.

**Corollary 3.4.** Every polytope \( P \subset \mathbb{R}^d \) of dimension \( d \in \{1, 2, 3, 4, 5\} \) possesses a facet-vertex assignment.

Therefore, the smallest possible counterexample to Problem 1.2 might be of dimension 6, where some facets \( f_1, \ldots, f_k \in F_{d-1}(P) \) intersect in a face of dimension 3.

### 4. Counterexamples in high dimensions

In this section, we construct for any \( d \geq 7 \) polytopes \( P \subset \mathbb{R}^d \) which do not possess vertex-facet assignments. This leaves open only the case of dimension \( d = 6 \). Our counterexamples are based on a standard construction for polytopes. Given two polytopes \( P_i \subset \mathbb{R}^{d_i}, i = 1, 2 \), their free join \( P_1 \bowtie P_2 \) is defined as the convex hull of these polytopes after embedding them into skew affine subspaces of \( \mathbb{R}^{d_1+d_2+1} \). If the vertices of the two polytopes are given as \( F_0(P_i) = \{p_i^1, \ldots, p_{n_i}^i\} \), then the free join can be defined as

\[
P_1 \bowtie P_2 := \text{conv} \left\{ \begin{bmatrix} p_1^1 \\ 0 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} p_{n_1}^1 \\ 0 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ p_1^2 \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ p_{n_2}^2 \\ -1 \end{bmatrix} \right\}.
\]

Most importantly, if the polytopes \( P_i \) have \( n_i \) vertices and \( m_i \) facets, then their free join has \( n := n_1 + n_2 \) vertices and \( m := m_1 + m_2 \) facets. This is quite obvious for the vertices. For the facets note the following: each facet of \( P_i \bowtie P_2 \) is spanned either by \( P_1 \) and a facet of \( P_2 \), or by \( P_2 \) and a facet of \( P_1 \). For details regarding the free join and its properties see [6, Corollary 2].
Theorem 4.1. For \( i \in \{1, 2\} \), let \( P_i \subset \mathbb{R}^d \) be a polytope with \( n_i \) vertices and \( m_i \) facets. Assume that \( n_1 > m_1 \) and \( n_2 < m_2 \), and that \( n_1 + n_2 \leq m_1 + m_2 \). Then \( P_1 \bowtie P_2 \) does not possess any vertex-facet assignment.

Proof. Note that \( n := n_1 + n_2 \leq m_1 + m_2 \), i.e. \( P_1 \bowtie P_2 \) has more facets than vertices, and this is precisely the minimal requirement for the existence of vertex-facet assignments. Let \( f_1, ..., f_{m_2} \) be the facets of \( P_1 \bowtie P_2 \) which contain \( P_1 \) (there are \( m_2 \) many because one for each facet of \( P_2 \)). Then

\[
|\mathcal{F}_0(f_1 \cap \cdots \cap f_{m_2})| \geq |\mathcal{F}_0(P_1)| = n_1 > m_1 = m_1 + m_2 - m_2 \geq n - m_2.
\]

The claim now follows from Theorem 2.2 and Corollary 3.2. \( \square \)

A more direct reasoning (without using the detour over the matching formulation) might go like this: In a vertex-facet assignment, each vertex \( v \) originally from \( P_1 \) must be assigned to a unique non-incident facet. The only facets not incident to \( v \) are the ones spanned by \( P_2 \) and a (non-incident) facet of \( P_1 \). However, since \( m_1 < n_1 \), there are not enough facets of \( P_1 \) to provide such non-incident facets in \( P \) for all vertices of \( P_1 \).

Note that the condition \( n_2 < m_2 \) is not used in the proof but follows from \( n_1 > m_1 \) and \( n_1 + n_2 \leq m_1 + m_2 \) anyway. Furthermore, the conditions \( n_1 > m_1 \) and \( n_2 < m_2 \) cannot be satisfied for polytopes in dimension \( \leq 2 \). This is the reason why this construction only yields counterexamples for Problem 1.1 in dimensions \( \geq 3 + 3 + 1 = 7 \).

Example 4.2. We list some concrete counterexamples for \( d \geq 7 \).

(a) For odd dimensions \( d = 2j + 1 \geq 7 \), let \( P_1 \subset \mathbb{R}^j \) be any polytope with more vertices than facets, and let \( P_2 \subset \mathbb{R}^j \) be the dual of \( P_1 \). The free join \( P_1 \bowtie P_2 \) then is a self-dual \( d \)-dimensional polytope, see [6, Corollary 2]. By Theorem 4.1, the polytope \( P_1 \bowtie P_2 \) does not possess vertex-facet assignments, and by self-duality, it does not possess facet-vertex assignments either.

(b) We can construct counterexamples for Problem 1.1 from any polytope \( P_1 \subset \mathbb{R}^d \) with \( d \geq 3 \) and \( n_1 := |\mathcal{F}_0(P_1)| > |\mathcal{F}_{d-1}(P_1)| := m_1 \). Take \( P_2 \subset \mathbb{R}^3 \) to be a bipyramid with \( n_2 := |\mathcal{F}_0(P_2)| \geq n_1 - m_1 + 4 \) vertices. Then \( m_2 := |\mathcal{F}_2(P_2)| = 2(n_2 - 2) \) and \( n_1 + n_2 \leq m_1 + m_2 \). The free join \( P_1 \bowtie P_2 \) then satisfies the assumptions of Theorem 4.1, and thus is a polytope without vertex-facet assignments in dimension \( d + 4 \geq 7 \).

5. Open problems

Question 5.1. Does every 6-dimensional polytope possess a vertex-facet assignment, or are there any counterexamples?

As noted in Section 3, any possible counterexample \( P \) in six dimensions must have facets \( f_1, ..., f_k \in \mathcal{F}_{d-1}(P) \) that intersect in a 3-dimensional face with more than \( n - k \) vertices.

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