The experimental and theoretical status of the inclusive decay $\bar{B} \rightarrow X_s \gamma$ is briefly summarized. Results from a very recent theoretical analysis are reported. An $\sim 11\%$ increase in the SM prediction for $\text{BR}[\bar{B} \rightarrow X_s \gamma]$ is found after replacing $m_c^{\text{pole}}/m_b^{\text{pole}}$ by $m_c(\mu)/m_b^{S}$ in the NLO QCD correction. The well-known enhancement of the branching ratio by QCD logarithms is identified as an effect of $m_b$-evolution in the top-quark contribution to the amplitude. This observation helps controlling the residual scale-dependence. The present prediction for the “total” branching ratio differs by $1.4\sigma$ from the experimental world average.

The inclusive decay $\bar{B} \rightarrow X_s \gamma$ is well known as a good testing ground for extensions of the SM. It arises mainly at one loop in the SM, so it is naturally sensitive to electroweak-scale exotica. All the parameters that are relevant for the SM prediction are well measured in other processes. Moreover, there is no overall non-perturbative factor in the theoretical expression for the decay amplitude, contrary e.g. to the $B\bar{B}$ and $K\bar{K}$ mixing or to $\bar{B}_s \rightarrow \mu^+\mu^-$ that require lattice inputs at present. In $\bar{B} \rightarrow X_s \gamma$ (within certain range of photon energy cuts), non-perturbative effects enter only as corrections, in analogy to the inclusive semileptonic decay $\bar{B} \rightarrow X_c e\bar{\nu}_e$. Last but not least, the suppression of $\text{BR}_\gamma \equiv \text{BR}[\bar{B} \rightarrow X_s \gamma]$ by $m_b/M_W \ll 1$ in the SM can be relaxed in many popular extensions of the SM, e.g. in the MSSM with large tan $\beta$ or in the left-right symmetric models. Then, the sensitivity of $\text{BR}_\gamma$ to exotic particles extends much above the electroweak scale (up to $\Lambda \sim M_W^2/m_b \simeq 1.3$ TeV), even if the CKM matrix remains the only source of flavour violation.

Of course, the power of $\text{BR}_\gamma$ for testing new physics crucially depends on how accurate its measurements are and how accurate the theoretical prediction is. The current experimental results read: $(3.03 \pm 0.47) \times 10^{-4}$ (CLEO), $(3.11 \pm 0.80_{\text{stat}} \pm 0.72_{\text{sys}}) \times 10^{-4}$ (ALEPH), $[3.36 \pm 0.53_{\text{stat}} \pm 0.42_{\text{sys}} (+0.50 -0.54)_{\text{model}}] \times 10^{-4}$ (BELLE). Their weighted average

$$\text{BR}^{\text{exp}}_\gamma = (3.11 \pm 0.39) \times 10^{-4}$$ (1)
has an error of around 13%. New results from CLEO, BELLE and BABAR are expected soon. However, our limited knowledge of the photon energy spectrum may restrict the accuracy of comparing theory with experiment. The $\bar{B} \to X_s \gamma$ photon spectrum in the $\bar{B}$-meson rest frame is shown in Fig. 1. The solid and dashed lines describe the spectrum without the intermediate $\psi$ contribution (i.e. the contribution from $\bar{B} \to X_s \psi$ followed by $\psi \to X' \gamma$). The dotted line shows how the spectrum changes when the intermediate $\psi$ contribution is included. This contribution has been effectively treated as background in all the existing analyses of $\bar{B} \to X_s \gamma$, both on the experimental and theoretical sides. This convention will be followed below.

The thickness of the solid and dashed lines in Fig. 1 reflects the degree of confidence with which the shape of the spectrum is theoretically known. The prediction is quite solid where the line is solid. For higher energies, it is only and “artist view” how the spectrum could look like. We know that there is a peak there, and we can determine the size of this peak, because the total inclusive decay rate is calculable within the Heavy Quark Effective Theory. However, the shape of the peak can be determined only experimentally. In this respect, the recent results of CLEO are very interesting. Unfortunately, their present energy cut-off $E_\gamma > 2 \text{ GeV}$ is still quite high. Consequently, the present comparison of theory and experiment must rely on a model-dependent extrapolation of the photon energy spectrum. This issue might become less problematic once the spectrum above the cut-off is more precisely measured.

In discussing the theoretical predictions below, I will assume that the cut-off is already low enough, e.g. $E_\gamma > 1.6 \text{ GeV}$ in the $\bar{B}$-meson rest frame. In such a case, the dominant contribution to $\text{BR}_{\gamma}$ is given by the partonic decay $b \to X_s \gamma$ of the $b$-quark. The electroweak one-loop diagrams that are relevant for this decay were calculated 20 years ago. Seven years later, existence of very large logarithmic QCD effects was realized. An enhancement of $\text{BR}_{\gamma}$ by a factor of 2.6 (for $m_t = 175 \text{ GeV}$) was found after resummation of $(a_s \ln M^2_{W}/m_b^2)^n$ to all orders in $n$ with the help of renormalization-group techniques. Since the perturbative uncertainties at LO were large, a calculation of NLO QCD corrections was necessary. It was completed in 1996, up to small two-loop matrix elements of the so-called penguin four-quark operators. The NLO QCD corrections enhanced $\text{BR}_{\gamma}$ by another $\sim 20\%$. The electroweak and non-perturbative corrections that were calculated later had smaller effects.

The overall uncertainty in the NLO prediction for $\text{BR}_{\gamma}$ is still dominated by perturbative QCD. It has been estimated in several papers. However, only the latter article properly accounts for errors due to $m_c/m_b$. In consequence, the predicted value of $\text{BR}_{\gamma}$ is significantly higher than in the previous analyses. The uncertainty can be maintained at the level of around 10% thanks to an observation that $m_b(\mu)$ in the top-quark contribution to the decay amplitude is the main source of logarithmic QCD effects. Below, I will discuss those very recent developments.

The $(m_c/m_b)$-dependence of the $b \to s \gamma$ amplitude arises in the diagrams shown in Fig. 2.
where the W-boson propagator has been contracted to a point. We have to ask what renormalization scheme should be used for quark masses. Should we use $m_c^\text{pole}/m_b^\text{pole} = 0.29 \pm 0.02$ or $m_c^\text{MS} (\mu)/m_b^\text{pole} \approx 0.22 \pm 0.04$ (with $\mu \in [m_c, m_b]$)? In principle, such a question is a NNLO issue, i.e. it is as relevant as three-loop corrections to the considered diagrams. However, it is numerically very important, because changing $m_c/m_b$ from 0.29 to 0.22 implies an increase of $\text{BR}_\gamma$ by 11%, i.e. by as much as the present experimental and theoretical uncertainties.

Since calculating three-loop corrections to Fig. 2 would be a very difficult task at present, we have to guess what the optimal choice of $m_c$ and $m_b$ is, on the basis of our experience from other calculations. All the factors of $m_c$ in Fig. 2 originate from explicit mass factors in the charm-quark propagators. In the real part of the considered amplitude, those charm quarks are dominantly off-shell, with momentum scale $\mu$. Therefore, it seems reasonable to vary $\mu$ between $m_c \sim \frac{1}{3} m_b$ and $m_b$, and to use $m_c^\text{MS} (\mu)$ in the ratio $m_c/m_b$.

Factors of $m_b$ in Fig. 2 originate either from the overall momentum release in $b \to s\gamma$ or from the explicit appearance of $m_b$ in the $b$-quark propagators. In the first case, the appropriate choice of $m_b$ is a low-virtuality mass. In the second case, there is no intuitive argument that could tell us whether $m_b^\text{pole}$ or $m_b(m_b)$ is preferred. However, so long as the three-loop diagrams remain unknown, setting all the factors of $m_b$ equal to $m_b^\text{pole}$ seems to be a good choice. Even a better choice is the so-called 1S-mass of the $b$-quark defined as half of the perturbative contribution to the $\Upsilon$ mass. It is leading-renormalon free and differs from $m_b^\text{pole}$ only by 1% at one loop.

Once $m_c^\text{MS} (\mu)/m_b^\text{1S}$ with $\mu \in [m_c, m_b]$ is used in Fig. 2, the uncertainty in $\text{BR}_\gamma$ significantly increases. This is due in part to a strong scale-dependence of $m_c(\mu)$. Moreover, in all the previous analyses, the $m_c$-dependence of $\Gamma[b \to s\gamma]$ cancelled partially against that of the semileptonic decay rate that is conventionally used for normalization. Once the different nature of the charm mass in the two cases is appreciated, the cancellation no longer takes place.

Fortunately, it is possible to make several improvements in the calculation, which allows us to maintain the theoretical uncertainty at the level of around 10%. In particular, good control over the behaviour of QCD perturbation series is achieved by splitting the charm- and top-quark-loop contributions to the $b \to s\gamma$ amplitude. The overall factor of $m_b$ is frozen at the electroweak scale in the top contribution to the effective vertex $m_b(\bar{s}_L\sigma^{\mu\nu}b_R)F_{\mu\nu}$. All the remaining factors of $m_b$ are expressed in terms of the bottom 1S-mass.

Splitting the charm and top contributions to the amplitude allows us to better understand the origin of the well-known factor of $\sim 3$ enhancement of $\text{BR}_\gamma$ by QCD logarithms. The charm contribution is found to be extremely stable under logarithmic QCD effects. The QCD enhancement of the branching ratio appears to be almost entirely due to the $\mu$-dependence of $m_b(\mu)$ in the top-quark sector.

$\text{BR}_\gamma$ with an energy cut-off $E_0$ in the $\bar{B}$-meson rest frame can be expressed as follows:

$$
\text{BR}[\bar{B} \to X_{s\gamma}]_{E_0 > E_0}^{\text{subtracted } \psi, \psi'} = \text{BR}[\bar{B} \to X_{c\ell\bar{\nu}}]_{\exp} \left| \frac{V_{ts}V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{em}}{\pi C} \frac{1}{P(E_0) + N(E_0)},
$$

where $C = \left| V_{ub}/V_{cb} \right|^2 \Gamma[\bar{B} \to X_{c\ell\bar{\nu}}]/\Gamma[\bar{B} \to X_{u\ell\bar{\nu}}] \approx 0.575$ is the phase-space factor for $\bar{B} \to X_{c\ell\bar{\nu}}$. $N(E_0)$ is the non-perturbative correction. The perturbative quantity $P(E_0)$ reads

$$
P(E_0) = \left| \frac{V_{ub}}{V_{ts}V_{tb}} \right|^2 \frac{\pi}{6\alpha_{em}} \left[ \frac{\Gamma[b \to X_{s\gamma}]_{E_0 > E_0}}{\Gamma[b \to X_{u\ell\bar{\nu}}]} \right] = |K_c + r(m_t)K_t + \varepsilon_{ew}|^2 + B(E_0).
$$
Table 1: Numerical results.

|                  | “naive” | LO        | NLO        |
|------------------|---------|-----------|------------|
| Re $K_c$ ($\mu_0 = M_W$) | -0.639  | -0.631 ± 0.003 | -0.611 ± 0.002 |
| Re $K_t$ ($\mu_0 = m_t$)   | 0.450   | 0.434 ± 0.005 | 0.397 ± 0.003  |
| BR$_{E\gamma > 1.6 \text{ GeV}} \times 10^4$ | 3.53    | 3.56 ± 0.14    | 3.60 ± 0.05    |

Here, $K_t$ contains the top contributions to the $b \to s \gamma$ amplitude. $K_c$ contains the remaining contributions, among which the charm loops are by far dominant. The electroweak correction is denoted by $\varepsilon_{\text{ew}}$. The ratio $r(m_t) = m_b^{\text{MS}}(m_t)/m_b^{1S} \approx 0.578$ appears in Eq. (3) because we keep $m_b$ renormalized at $m_t$ in the top-quark contribution to $b \to s \gamma$, while all the kinematical factors of $m_b$ are expressed in terms of the bottom 1S-mass. The bremsstrahlung function $B(E_0)$ contains the effects of $b \to s \gamma g$ and $b \to s \gamma q\bar{q}$ ($q = u, d, s$) transitions. It is the only $E_0$-dependent part in $P(E_0)$. Its influence on BR$_\gamma$ is less than 4% when $1 \text{ GeV} < E_0 < 2 \text{ GeV}$.

In Table 1, the numerical results are presented at various orders of the renormalization-group-improved perturbation theory. In the “naive” approach, the difference of $r(m_t)$ from unity is the only included QCD effect. At LO, all the QCD logarithms $(\alpha_s \ln M_W^2/m_b^2)^n$ are taken into account. At NLO, we add the non-logarithmic $\mathcal{O}(\alpha_s)$ corrections together with the electroweak and non-perturbative ones. The indicated errors correspond to varying the low-energy scale $\mu_b$ between $m_b/2$ and $2m_b$. One can see that the behaviour of the QCD perturbation series for all the considered quantities is good, and that their residual $\mu_b$-dependence is quite weak. Such a weak $\mu_b$-dependence is not caused by any accidental cancellations, contrary to what was observed previously. In the present approach, there is no indication that the unknown NNLO corrections could be much larger than $(\alpha_s(m_b)/\pi)^2 \approx 0.5\%$ times a factor of order unity. Consequently, our estimate of the overall uncertainty in the final prediction for BR$_\gamma$ is not larger than in the previous analyses, despite taking the problems with $m_c/m_b$ into account here.

When all the errors are included and added in quadrature, we find

$$\text{BR}[\bar{B} \to X \gamma]_{E\gamma > 1.6 \text{ GeV}}^{\text{subtracted } \psi, \psi'} = (3.60 \pm 0.30) \times 10^{-4}. \quad (4)$$

The experimental weighted average (4) for the “total” branching ratio should be compared with the theoretical result for $E_0 \approx 1/20 m_b \approx 0.23 \text{ GeV}$ (i.e. $\delta \equiv 1 - 2E_0/m_b \approx 0.9$). Then, Eq. (2) gives $\text{BR}[\bar{B} \to X \gamma]_{E\gamma > m_b/20} = 3.73 \times 10^{-4}$ with an error roughly comparable to the one in Eq. (4). Thus, the difference between theory and experiment is at the level of 1.4$\sigma$. However, one should remember that the theoretical errors have no statistical interpretation, which implies that the value of $1.4\sigma$ has only an illustrative character.

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