Extremal black hole entropy from conformal string sigma model

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Abstract

We present string-theory derivation of the semiclassical entropy of extremal dyonic black holes in the approach based on conformal sigma model (NS-NS embedding of the classical solution). We demonstrate (resolving some puzzles existed in previous related discussions) that the degeneracy responsible for the entropy is due to string oscillations in four transverse dimensions ‘intrinsic’ to black hole: four non-compact directions of the D=5 black hole and three non-compact and one compact (responsible for embedding of magnetic charges) dimensions in the D=4 black hole case. Oscillations in other compact internal directions give subleading contributions to statistical entropy in the limit when all charges are large. The dominant term in the statistical entropy is thus universal (i.e. is the same in type II and heterotic string theory) and agrees with Bekenstein-Hawking expression.

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1. Introduction

One of the first systematic attempts to give a statistical interpretation of the Bekenstein-Hawking (BH) black hole entropy on the basis of string theory was made in [1] where it was proposed that entropy of extremal electric black holes can be understood in terms of counting of the corresponding elementary supersymmetric (BPS) string states. This idea was clarified and put on a firmer ground in [2,3] by identifying degenerate extremal electric black hole states with oscillating modes of underlying macroscopic string. While the statistical entropy of elementary BPS string states is given (in the limit of large charges) by

$$S_{\text{stat}} = 2\pi \sqrt{\frac{c_{\text{eff}}}{6} N_L}, \quad N_L = Q_1 Q_2,$$

(1.1)

(where we assumed the simplest case of the two electric charges corresponding to a circular dimension and $c_{\text{eff}} = 12$ in type II superstring and $c_{\text{eff}} = 24$ for the heterotic string), the semiclassical BH entropy of extremal electric black holes vanishes when computed at the singular $r = 0$ horizon. It was suggested [1] that the account of string $\alpha'$ corrections may modify the black hole solution leading to a new ‘stretched’ [4] position of horizon at $r \sim \sqrt{\alpha'}$ with the entropy $S_{BH} = k \sqrt{Q_1 Q_2}$. The latter has the right form to match $S_{\text{stat}}$ (see also [5]), but the coefficient $k$ which should depend on detailed structure of $\alpha'$ corrections in a particular string theory (and, indeed, should be different in type II and heterotic string theories in order for $S_{\text{stat}}$ and $S_{BH}$ to have chance to agree) is hard to compute explicitly.

The important further suggestion [6] was that a precise test of the idea about the string statistical origin of the BH entropy should be possible in the case of the $D = 4$ supersymmetric dyonic extremal black holes [7,8,9] for which $S_{BH}$ is non-zero already at the semiclassical level

$$S_{BH} = 2\pi \sqrt{Q_1 Q_2 P_1 P_2}.$$

(1.2)

The idea [6] was to interpret the latter as the statistical entropy (1.1) of supersymmetric oscillating states of a free string with tension renormalised by the product $P_1 P_2$ of magnetic charges.

The reason for this magnetic renormalisation was explained in [10] (see also [11]) starting with the conformal $\sigma$-model which describes the embedding [9] of the dyonic black holes into string theory. The marginal supersymmetric deformations of the conformal $\sigma$-model were interpreted (in the spirit of [4,5]) as describing degenerate black holes with the same asymptotic charges but different short distance structure. Since these perturbations

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1 In this Section we shall assume for simplicity that the electric and magnetic charges $Q_i$ and $P_i$ are normalised to take integer values at the quantum level. $Q_i$ and $P_i$ in the following Sections will differ from integers by certain factors discussed below.
are important only at small scales they can be effectively counted near the horizon ($r = 0$). The crucial observation is that for $r \to 0$ the model reduces to a 6-dimensional WZW theory with level $\kappa = P_1 P_2$ (similar models were discussed in [12,13]).

The original suggestions of [6,10,14] suffer, however, from the two (related) problems. First, it appears that if one rescales the free string oscillator level $N_L = Q_1 Q_2$ by $\kappa = P_1 P_2$ and uses the standard expression (1.1) for the statistical entropy with $c_{eff} = 24$ corresponding to the heterotic string one finds the result which is twice the BH expression (1.2). For type II theory choice $c_{eff} = 12$ one is off by factor of $\sqrt{2}$. The second problem is why, in fact, the expected result should at all depend on a choice of a particular string theory in which black hole is assumed to be a solution. In contrast to the ‘quantum’ stretched horizon entropy of [1] the BH entropy (1.2) is a semiclassical one. It should depend only on data contained in the classical solution, i.e. values and types of asymptotic charges and the number of space-time dimensions. The entropy (1.2) certainly cannot ‘feel’ the precise value of all internal compact directions of a particular critical string theory. All it may implicitly ‘know’ about is how the four charges are embedded into string theory in a way preserving supersymmetry and that relevant microstates should also be supersymmetric.

Our aim below will be to resolve these problems and to argue that the approach initiated in [6,10,14] does indeed lead to the quantitative explanation of the BH entropy (1.2) and in that sense is consistent with and complementary to the approaches based on D-brane counting of BPS states (see [13,14,18,20,21,22] and [23] for a review). Its advantage is a direct space-time interpretation (in particular, a clear identification of oscillation modes responsible for the BH entropy).

An important observation is that the statistical entropy corresponding to all possible supersymmetric marginal perturbations of a string theory solution (non-trivial conformal $\sigma$-model) which represents the dyonic black hole is a complicated function of the charges which reproduces (1.2) only in the limit when all charges are large. Assuming that $Q_1, Q_2, P_1, P_2$ are of order $Q \gg 1$ we may expect to find, symbolically (ignoring possible $Q^n \ln Q$ terms, etc.)

$$S_{stat} = a_1 \sqrt{Q^4} + a_2 \sqrt{Q^3} + a_3 \sqrt{Q^2} + \ldots + O(Q^{-1}) . \quad (1.3)$$

Different types of perturbations may contribute to different terms in the exact expression for $S_{stat}$. In the limit when magnetic charges vanish, the exact expression should reduce to the standard free string BPS entropy (1.1). While the $\sqrt{Q^2}$ term in (1.3) should thus depend on embedding into a particular string theory, the leading $\sqrt{Q^4}$ term should be universal. For this to be possible, this leading term should be built out of contributions of only certain universal types of perturbations which are common to heterotic and type II embeddings. Other perturbations, in particular, string oscillations in internal toroidal directions should contribute only to subleading terms in the entropy.
Indeed, the relevant perturbations turn out to correspond to string oscillations in three non-compact spatial directions and one compact direction responsible for the Kaluza-Klein-type embedding of the two magnetic charges. These dimensions are indeed ‘intrinsic’ to the black hole and do not depend on a choice of a superstring theory. It is only these four directions that get multiplied by $P_1 P_2$ in the near-horizon region. In terms of free-oscillator description of perturbations (which indeed applies near $r = 0$ for large $P_1 \sim P_2$, i.e. large level $P_1 P_2$ of the underlying WZW model) this corresponds to the effective number of degrees of freedom

$$c_{eff} = 4(1 + \frac{1}{2}) = 6$$ \hspace{1cm} (1.4)

where we have included the contributions of superpartners of the four bosonic string coordinates. This result is the same in type II and heterotic string theory (provided the black hole solution is embedded into the heterotic theory in the manifestly conformally-invariant ‘symmetric’ way). The statistical entropy (1.1) corresponding to supersymmetric string oscillations in the four transverse directions with the tension or oscillator level rescaled by $P_1 P_2$ exactly matches the BH expression (1.2).

To make the discussion more transparent, in what follows we shall consider in detail the very similar case of $D = 5$ dyonic black holes. An advantage of the present conformal $\sigma$-model approach is that the treatment of the $D = 4$ and $D = 5$ dyonic black hole cases is parallel since both are described by closely related $D = 6$ supersymmetric conformal $\sigma$-models \cite{19,21}. Once the main conceptual issues are clarified on the $D = 5$ example, generalisation to the $D = 4$ case is straightforward.

The conformal model \cite{21} describing $D = 5$ dyonic black hole with two electric and one magnetic charges $Q_1, Q_2, P$ can be interpreted as representing a BPS ‘bound state’ of a fundamental string \cite{25} and solitonic 5-brane \cite{22,26} or as a ‘boosted’ $D = 6$ dyonic string \cite{27} (Section 2). The analogs of the expression (1.2) and (1.3) are

$$S_{BH} = 2\pi \sqrt{Q_1 Q_2 P}$$ \hspace{1cm} (1.5)

and $(Q_1, Q_2, P \sim Q \gg 1)$

$$S_{stat} = a_1 \sqrt{Q^3} + a_2 \sqrt{Q^2} + ... + O(Q^{-1})$$ \hspace{1cm} (1.6)

We shall argue in Section 3 that the supersymmetric marginal perturbations which reproduce (1.3) as the leading term in the statistical entropy (1.6) correspond to the transversal

\footnote{This is an obvious consequence of the discussion in \cite{10,11} but was overlooked there: it was expected, following the conjecture of \cite{8}, that all the transverse string coordinates, including the internal toroidal ones get magnetically renormalised tension. The possibility that only perturbations in four of transverse directions are actually renormalised by $P_1 P_2$ was suggested to me by F. Larsen \cite{24}.}
oscillations of $D = 6$ dyonic string, i.e. oscillations in the four non-compact directions (spatial dimensions of $D = 5$ black hole). Only these four string coordinates get their tension effectively rescaled by the magnetic charge $P$ (level $\kappa$) in the near-horizon (throat) region described by a WZW theory. Oscillations in the compact toroidal directions (4 in type II and 4+16 in the heterotic theory) contribute only to subleading terms in the statistical entropy. As in the $D = 4$ case the effective number of degrees of freedom responsible for (1.3) is thus $c_{eff} = 6$.\(^3\)

To relate the number of oscillations to charges of the classical solution and to fix the quantisation of $Q_1, Q_2$ we shall follow [3,2] and consider matching of deformed supersymmetric dyonic string solutions onto oscillating string source (Section 4). Although trying to match onto a string source may seem surprising in view of non-singular nature of the dyonic solution we shall suggest that source interpretation is still consistent for solutions which carry both magnetic and electric charges (sources are not needed only in the case of pure magnetic solitons).\(^4\)

The resulting analogue of the free-string level matching condition makes possible to express the oscillation number corresponding to the four non-compact transverse directions in terms of the product of charges $Q_1 Q_2 P$ and thus to reproduce the BH entropy (1.5) as the statistical one (1.1). The generalisation of the discussion to the case of $D = 4$ dyonic black hole is straightforward and will be briefly sketched in Section 5.

2. $D = 5$ dyonic black hole and $D = 6$ dyonic string behind it

We shall consider the following $D = 5$ extremal dyonic black hole with the Einstein-frame metric [11]\(^5\)

$$ds_E^2 = -\lambda^2(r) dt^2 + \lambda^{-1}(r)(dr^2 + r^{-2} d\Omega_3^2) ,$$

$$\lambda = \left(\frac{r^2}{(r^2 + Q_1)(r^2 + Q_2)(r^2 + P)}\right)^{1/3} .$$

Supplemented by two electric and one magnetic vector fields and scalar fields it represents an $N = 1$ supersymmetric solution of extended $D = 5$ supergravity. We shall assume that

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\(^3\) Not surprisingly, the value $c_{eff} = 6$ (same for $D = 4$ and $D = 5$ black hole cases) appears also in related D-brane (in particular, [22]) and M-brane [28,29] approaches. It is interesting to note, however, that the four relevant directions in the present NS-NS description are orthogonal to the solitonic 5-brane, while in the R-R (D-brane) and M-brane descriptions they are the tangential R-R 5-brane directions which are transversal to the R-R string lying within the 5-brane [16].

\(^4\) This seems to apply not only to the $D = 6$ dyonic string but also to $D = 10$ dyonic 3-brane [30,31] (even though it is also non-singular [32]).

\(^5\) Non-extremal generalisation of this solution was constructed in [33,34].
all three charges are positive so that \( r = 0 \) represents a regular horizon. For \( Q_1 = Q_2 = P \) this metric is equivalent to \( D = 5 \) Reissner-Nordström one \([35,15]\). The mass of this \( D = 5 \) black hole and the Bekenstein-Hawking entropy proportional to the 3-area of the horizon are given by \((G_N \text{ is the 5-dimensional Newton’s constant})\)

\[
M = \frac{\pi}{4G_N}(Q_1 + Q_2 + P), \tag{2.2}
\]

\[
S_{BH} = \frac{A}{4G_N} = \frac{\pi^2}{2G_N} \sqrt{Q_1 Q_2 P}. \tag{2.3}
\]

This black hole background can be embedded into string theory in several different ways, depending on the interpretation (NS-NS or R-R) of the corresponding charges or vector fields. Here we shall follow \([11]\) and discuss the purely NS-NS embedding (R-R embeddings were considered in \([15,16]\)). The corresponding background is an exact solution in both heterotic and type II superstring theories \([36,9]\) and may be interpreted as a BPS ‘bound state’ of a closed macroscopic string \([25]\) and a solitonic 5-brane \([12,26]\) (5-brane will be assumed to be wrapped around 5-torus with the string wound around one of its cycles).\(^6\) It can be represented by the \(D = 10\) supersymmetric conformal \(\sigma\)-model with the following bosonic part \([36,9,11]\)

\[
L = (G_{\mu\nu} + B_{\mu\nu})(x)\partial X^\mu \bar{\partial} X^\nu + \mathcal{R}\Phi(x) \tag{2.4}
\]

\[
= F(x)\partial u \left[ \bar{\partial} v + K(x)\bar{\partial} u \right] + (g_{mn} + B_{mn})(x)\partial x^m \bar{\partial} x^n + \partial y_a \bar{\partial} y_a + \mathcal{R}(\phi + \frac{1}{2} \ln F),
\]

where \(u = y_5 - t, \ v = y_5 + t, \ x_m \ (m = 1, 2, 3, 4)\) are non-compact spatial coordinates, \(y_a \ (a = 1, 2, 3, 4)\) and \(y_5\) are coordinates of 5-torus, and \([12]\)

\[
g_{mn} = f(x)\delta_{mn}, \quad H^{mnk} = -\frac{2}{\sqrt{g}}\epsilon^{mnkp} \partial_p \Phi, \quad e^{2\phi} = f, \quad \partial^m \partial_m f = 0. \tag{2.5}
\]

where \(H_{mnk} \equiv 3\partial_{[m}B_{nk]}\). Note that \(\sqrt{Ge^{-2\Phi}} = \sqrt{g} e^{-2\phi} = f\) and \(\sqrt{g} e^{-2\phi} g^{mn} = \delta^{mn}\). The conformal invariance conditions reduce to the flat-space ones

\[
\partial^m \partial_m F^{-1} = 0, \quad \partial^m \partial_m K = 0, \tag{2.6}
\]

i.e. the model is parametrised by the three harmonic functions \(f, F^{-1}, K\) depending on four non-compact transverse coordinates \(x^m\). This is a reflection of the BPS-saturated nature of the solution. The special one-center choice \((r^2 \equiv x_m x_m)\) \(^7\)

\[
f = 1 + \frac{P}{r^2}, \quad F^{-1} = 1 + \frac{Q_1}{r^2}, \quad K = \frac{Q_2}{r^2}, \tag{2.7}
\]

\(^6\) Under \(SL(2)\) duality of \(D = 10\) type IIB theory it becomes a solution with RR-charges which can be described as a superposition of a D1-brane and D5-brane \([16]\).

\(^7\) We change notation slightly compared to \([11]\): in \([11]\) \(K\) had asymptotic value 1 and at the same time \(v\) was \(2t\), not \(v = y + t\) as here. We also flip the definitions of \(Q_1\) and \(Q_2\).
leads upon compactification along \(y_1, \ldots, y_5\) to the \(D = 5\) black hole (2.1). The corresponding \(D = 10\) dilaton and ‘radius’ of \(y_5\)

\[
e^{2\Phi} = F f = \frac{r^2 + P}{r^2 + Q_1}, \quad e^{2\sigma} = F(1 + K) = \frac{r^2 + Q_2}{r^2 + Q_1},
\]

are regular both at \(r = 0\) and \(r = \infty\).

Setting \(Q_2 = 0\) and omitting the trivial 4-torus part the model (2.4) becomes

\[
L = F(x)\partial u \bar{\partial} v + f(x)\partial x^m \bar{\partial} x_m + B_{mn}(x)\partial x^m \bar{\partial} x^n + \frac{1}{2} R \ln[F(x)f(x)],
\]

and may be also interpreted as describing the dyonic \(D = 6\) string [27]: \(B_{\mu\nu}\) has both electric \(Q_1\) and magnetic \(P\) charges. Switching on \(Q_2\) corresponds effectively to adding momentum along the string direction. One can also generalise this conformal model by introducing a rotational parameter [11] (see also Section 3).

The non-singular solitonic nature of the 5-brane model [12] is responsible for the regularity of the dyonic theory at \(r = 0\). Here the throat limit \(r \to 0\) is described by the direct product of \(SL(2,R) (u,v,\rho)\) and \(SU(2) (\theta,\varphi,\psi)\) WZW models with equal levels proportional to \(P\).

\[
I = \frac{1}{\pi \alpha'} \int d^2 \sigma L_{r \to 0} = \frac{1}{\pi \alpha'} \int d^2 \sigma \left( e^{-2\rho} \partial u \bar{\partial} v + Q_2 Q_1^{-1} \partial u \bar{\partial} u \right)
\]

\[
+ \frac{\kappa}{\pi} \int d^2 \sigma \left[ \partial \rho \bar{\partial} \rho + \partial \theta \bar{\partial} \theta + \sin^2 \theta \partial \varphi \bar{\partial} \varphi + \cos^2 \theta \partial \psi \bar{\partial} \psi + \frac{1}{2} \cos 2\theta (\partial \varphi \bar{\partial} \psi - \bar{\partial} \varphi \partial \psi) \right],
\]

\[
\rho \equiv \ln \frac{\sqrt{Q_1}}{r} \to \infty, \quad P \equiv \alpha' \kappa. \tag{2.11}
\]

We have omitted the constant dilaton term. The \((1,1)\) supersymmetric version of this model has free-theory central charge: \(c = [3(1 + \frac{2}{\kappa}) + \frac{3}{\kappa}] + [3(1 - \frac{2}{\kappa}) + \frac{3}{\kappa}] = 6(1 + \frac{1}{2})\). Since \(P\) is related to the coefficient of a Wess-Zumino term in the string action, i.e. is proportional to the integer level \(\kappa\), it is quantised in units of \(\alpha'\). The quantisation of the electric charge \(Q_1\) can then be deduced from the Dirac condition applied to the antisymmetric tensor in \(D = 6\). The quantisation of \(Q_2\) is the same as in the fundamental string case [25,3] and will be discussed in Section 4 by matching on a string source (\(Q_1\) is then related to the winding number and \(Q_2\) to the momentum of a string source).

8 The ‘transverse’ \((\rho,\theta,\varphi,\psi)\) part of the throat region model is exactly the same as in the 5-brane case [12] (where \(Q_1 = Q_2 = 0\) except for the fact that here the dilaton \(\Phi\) is constant in the \(r \to 0\) region, i.e. the string coupling is not blowing up. The angular coordinates \(\theta,\varphi,\psi\) (appearing in \(dx^m dx_m = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2)\)) are related to the Euler angles of \(SU(2) (0 \leq \theta' \leq \pi, \ 0 \leq \varphi' \leq 2\pi, \ 0 \leq \psi' \leq 4\pi)\) by \(\theta = \frac{1}{2} \theta', \ \varphi = \frac{1}{2} (\varphi' + \psi'), \ \psi = \frac{1}{2} (\psi' - \varphi'). \)
3. Oscillating $D = 6$ dyonic string states as degenerate dyonic $D = 5$ black holes

Our aim will be to follow the previous suggestions [2,4,8] and to try to reproduce the BH entropy (2.3) as a statistical entropy related to existence of an infinite family of more general $D = 5$ black hole solutions which asymptotically look the same as (2.1), i.e. have the same charges, but differ at scales of order of compactification scale (equal to the radius of the string direction $y_5$).

3.1. Deformed sigma model and conformal invariance conditions

Demanding preservation of the same amount of supersymmetry, i.e. the BPS property (or related property of manifest exactness of the solution), these more general backgrounds are described by the following extension of (2.4) [36,10]

$$L = F(x)\partial u [\partial v + K(u, x)\partial u + 2A_n(u, x)\partial x^n + 2A_a(u, x)\partial y^a]$$

$$+ (g_{mn} + B_{mn})(x)\partial x^m\partial x^n + \partial y_a\partial y_a + R(\phi + \frac{1}{2}\ln F) ,$$

where $A_m$ and $A_a$ correspond to ‘deformations’ in four non-compact $x^m$ and compact $y^a$ directions respectively. The requirement of supersymmetry is directly related to the chiral null structure of (3.1) [36,38], i.e. to the dependence of deformation functions on only one null (string-like) direction.

The conditions of exact conformal invariance of this model [10,11] are again

$$\partial^m \partial_m F^{-1} = 0$$

$$\nabla^m (e^{-2\phi} \partial_m K) - 2\partial_u \nabla^m (e^{-2\phi} A_m) = 0 , \text{ i.e. } \partial^m (\partial_m K - 2\partial_u A_m) = 0 ,$$

$$\partial^m \partial_m A_a = 0 ,$$

$$\nabla_m (e^{-2\phi} F^{mn}) - e^{-2\phi} H^{nkl} F_{kl} = 0 ,$$

where $g_{ij}$, $B_{ij}$ and $\phi$ are the couplings of the transverse conformal theory. Since the latter is trivial in $y_a$-directions and $A_a$ does not depend on $y_a$ one concludes that $A_a$ should satisfy the free Laplace equation in 4 non-compact directions.

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9 There are other marginal deformations of the chiral null model which break supersymmetry [37]. These are expected to suffer from string $\alpha'$ (and loop) corrections so that their contribution to the entropy is hard to determine precisely (presumably they give subleading contributions to statistical entropy in the limit when all three charges are large). It is also likely that these deformations are unstable and cannot be interpreted as true microscopic black hole states.

10 If $x^i$ denotes the set of all transverse directions (both compact and non-compact) the condition of marginality of the perturbation $A_i(u, x)\partial x^i$ (i.e. $A_n(u, x)\partial x^n + A_a(u, x)\partial y^a$ in the present case) is, in general, $\nabla_{+i}(e^{-2\phi} F^{ij}) = 0$ ($\Gamma^i_{+jk} = \Gamma^i_{jk} + \frac{1}{2}H^i_{jk}$), i.e. $\partial_i(e^{-2\phi}\sqrt{g}F^{ij}) - \frac{1}{2}e^{-2\phi}\sqrt{g}H^{kij}F_{kl} = 0$, where $g_{ij}$, $B_{ij}$ and $\phi$ are the couplings of the transverse conformal theory. Since the latter is trivial in $y_a$-directions and $A_a$ does not depend on $y_a$ one concludes that $A_a$ should satisfy the free Laplace equation in 4 non-compact directions.
where $F_{mn} \equiv \partial_m A_n - \partial_n A_m$. This system is invariant under $K \to K + \partial_u \sigma(u, x)$, $A_m \to A_m + \frac{1}{2} \partial_m \sigma(u, x)$ induced by $v \to v + \sigma(u, x)$ in (3.1) (the $u$-dependent part of $K$ can be absorbed into $A_n$ by a redefinition of $v$). One may assume, for example, that $\partial_u \partial^m A_m = 0$ as a gauge choice so that the equation for $K$ becomes again the flat Laplace one $\partial^m \partial_m K = 0$.  

The equation for $A_m$ (3.4) can be put in the following simple form \[11\]

$$
\partial_m (e^{-2\phi} \sqrt{g} F^{mn}) = 0 , \quad \text{i.e.} \quad \partial^m (f^{-1} F_{mn}) = 0 ,
$$

(3.5)

$$
F_{+mn} \equiv F_{mn} + \frac{1}{2\sqrt{g}} g_{mp} g_{nq} \epsilon^{pql} F_{kl} = F_{mn} + \frac{1}{2} \epsilon^{mnkl} F_{kl} .
$$

(3.6)

3.2. Perturbations in compact and non-compact directions and entropy

There are several important conclusions which follow from these exact marginality conditions. First, they impose essentially no restrictions on $u$-dependence of $K, A_a, A_m$, i.e. as in the case of the ‘free’ fundamental string [39,40,36,2,3] they represent various ‘left-moving’ waves propagating along the string or oscillating string modes. Assuming Kaluza-Klein compactification along the string direction $y_5$, the $\sigma$-model (3.1) describes a family of BPS saturated $D = 5$ black hole backgrounds which are the same as (2.1) at distances larger that the radius of $y_5$ but are different at smaller scales [2,3,1].

Second, deformations in the compact $y_a$ directions described by $A_a$ (charge waves along the string) satisfy the free Laplace equation and thus are effectively decoupled from the non-trivial non-compact part of the model, i.e. do not depend on the ‘magnetic’ function $f$. At the same time, the string oscillations in non-compact directions described by $A_m$ (in particular, rotational perturbations) are non-trivially coupled to the function $f$, i.e. are, in general, sensitive to the value of the magnetic charge $P$.

The fact that perturbations in compact directions do not ‘feel’ $P$ suggests that counting of the corresponding oscillating string states should give exactly the same result as in the case of the ‘free’ fundamental string, i.e. the associated statistical entropy should have nothing to do with the leading-order semiclassical BH entropy (2.3). Indeed, it turns out that the matching condition relating ‘compact’ oscillation number to charges has the form $N_L \sim Q_1 Q_2$, i.e. also does not involve $P$.

This is a natural conclusion: the entropy related to oscillations in $D_{int}$ compact toroidal directions ($\sim \sqrt{c_{eff}^D} Q_1 Q_2$, $c_{eff} = D_{int}(1 + \frac{1}{2})$) depends on their number, i.e. on a particular string theory ($D_{int} = 4$ in type II theory and $D_{int} = 4 + 16 = 20$ in heterotic theory) while the black hole solution (2.1) and its semiclassical BH entropy (2.3) are certainly universal. These oscillations do contribute to the entropy (as in the purely

\[11\] Alternatively, one may define $A'_m = A_m - \frac{1}{2} \partial_m \int du K(u, x)$ and integrate (3.2) to get $\partial^m A'_m = h(x)$.  

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electric extremal black hole case \([\text{1,5]}\), but they produce a subleading contribution when all three charges \(Q_1, Q_2\) and \(P\) are large and of the same order (cf. \([\text{1,0}])\). The \(P = 0\) limit of the exact expression for the statistical BPS entropy is expected to reproduce the entropy associated with the stretched horizon of the extremal electric black hole \([\text{1}]\) with the coefficient which is different in heterotic and type II theories (since the structure of \(\alpha'\) corrections and thus the position of the stretched horizon is different in the two theories).

At the same time, the counting of perturbations in non-compact dimensions should depend only on their number (four) and the values of all three charges – the data intrinsic to \(D = 5\) black hole, i.e. independent of embedding into a particular string theory (assuming only that supersymmetry is maintained). The corresponding statistical entropy should thus be universal as is the BH entropy \([\text{2,3}].\) Below we shall confirm that the two entropies indeed match. The conclusion is thus that only the supersymmetric transverse perturbations of the dyonic \(D = 6\) string are responsible for the semiclassical BH entropy \([\text{2,3}].\)

### 3.3. Examples of conformal deformations

The equation \((3.3)\) is solved by

\[
\mathcal{A}_a(u, x) = \frac{q_a(u)}{r^2} \quad .
\]

(3.7)

Since \(u = y_5 - t\) and \(y_5 \equiv y_5 + 2\pi R\), the ‘charge wave’ functions \(q_a(u)\) are periodic, i.e. are given by Fourier series expansions, \(q_a(u) = \bar{q}_a + \check{q}_a(u), \bar{q}_a(u) = b_a \sin(R^{-1}u) + \ldots.\) If the constant parts \(\bar{q}_a\) are non-vanishing, they produce additional asymptotic (‘left’) electric charges of the \(D = 6\) string and thus of the corresponding more general \(D = 5\) extremal black hole. The BH entropy of the latter can be obtained by making the shift

\[
Q_1 Q_2 \rightarrow Q_1 Q_2 - \bar{q}_a^2
\]

(3.8)

of the product of the two electric charges in \([\text{2,3}].\) (see also Section 4).

A particular solution of \((3.3)\) is found by imposing the flat-space anti-selfduality condition

\[
\mathcal{F}_{+mn} = 0 \quad .
\]

(3.9)

In terms of the angular coordinates \((\theta, \varphi, \psi)\) in \((2.10)\) which are related to \(x^m\) by

\[
x^1 + ix^2 = r \sin \theta \ e^{i\varphi}, \quad x^3 + ix^4 = r \cos \theta \ e^{i\psi},
\]

\[dx^m dx_m = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2) \quad ,
\]

9
we find that (3.3), (3.9) are satisfied by
\[ A_m(u, x)dx^m = \frac{\gamma(u)}{r^2} (\sin^2 \theta d\varphi + \cos^2 \theta d\psi) .\] (3.10)

Here \( \gamma(u) \) is an arbitrary periodic function, \( \gamma(u) = \bar{\gamma} + \tilde{\gamma}(u), \tilde{\gamma}(u) = b \sin(R^{-1}u) + .... \)

The constant part \( \bar{\gamma} \) has the meaning of a rotational parameter. In fact, dimensionally reducing along \( y_5 \) (i.e. averaging over \( u \)) one finds the following modification of the D = 5 metric (2.1)
\[ ds_E^2 = -\lambda^2(r) (dt + \bar{A}_m dx^m)^2 + \lambda^{-1}(r) dx^m dx_m , \] (3.11)

which is the supersymmetric rotating generalisation of the 3-charge \( (Q_1, Q_2, P) \) extreme dyonic D = 5 black hole [11] (the special case \( Q_1 = Q_2 = P \) of this solution was found in [18]). In addition to the mass (still given by (2.2)) and three charges there are also two equal angular momenta in the two orthogonal planes
\[ J_\varphi = J_\psi \equiv J = \frac{\pi}{4G_N} \bar{\gamma} . \] (3.12)

The corresponding BH entropy is found to be given by (2.3) with the product of all three charges shifted by \( \tilde{\gamma}^2 \)
\[ Q_1Q_2P \rightarrow Q_1Q_2P - \bar{\gamma}^2 . \] (3.13)

Comparison of (3.8) and (3.13) illustrates the difference between parameters associated with supersymmetric marginal deformations in internal and non-compact directions.

On may assume for simplicity that the mean values of the above deformations vanish \( (\bar{q}_a = 0, \bar{\gamma} = 0, \text{etc.}) \), i.e. that they do not introduce new asymptotic parameters so that the BH entropy is given by the original expression (2.3).

3.4. Perturbations in the throat region

Since we would like to count such non-compact deformations \( A_m(u, x) \) which decay at large \( r \) this can be done near \( r = 0 \), i.e. at the ‘throat’ (or \( D = 5 \) horizon). Indeed, it is in this region that the degeneracy between different members of the family of \( D = 5 \) black holes with the same asymptotic charges is lifted.

This is an important simplification since the \( r \rightarrow 0 \) limit of the basic \( \sigma \)-model (2.4) is given by the WZW theory (2.10). The problem can then be reformulated in terms of counting of supersymmetric (chiral null) marginal deformations of the model (2.10)
\[ I' = I + \frac{1}{\pi \alpha'} \int d^2 \sigma \left( F(x)[K'(u, x) \partial u \partial u + 2A_m(u, x) \partial u \partial x^m] \right)_{r \rightarrow 0} \] (3.14)
\[ = I + \frac{1}{\pi \alpha'} \int d^2 \sigma Q_1^{-1}[k(u) \partial u \partial \rho + 2a(u) \partial u \partial \beta + 2a_s(u, \beta) \partial u \partial \beta^s] , \]
where $\beta_s (s = 1, 2, 3)$ denote the three angular coordinates $\theta, \varphi, \psi$. The contribution of $u$-dependent part $K'(u, x)$ of $K(u, x)$ can be omitted without loss of generality. Indeed, if $K'(u, x) = h_m(u)x^m + \tilde{Q}_2(u), \tilde{Q}_2(u) = Q_2(u) - Q_2$, then (in contrast to the fundamental string case [3]) the oscillation part $h_m(u)x^m$ of $K$ drops out while $k(u) = \tilde{Q}_2(u)$ has zero mean value (see also below).

Another crucial point is that we are actually interested in the limit of large charges. For large $P$, i.e. for large level $\kappa$ of WZW model, the count of perturbations should be the same as in the theory of four free fields $\rho, \beta_s$ (interactions between the fields in WZW action in (2.10) are suppressed by $1/\kappa$). The only difference as compared to the free string case [3] will be due to the fact that the kinetic terms of the four fields $\rho, \beta_s$ have the factor $P$ while such factor is absent in front of the perturbations in (3.14). As a result, integrating out the four transverse fields in the limit of large $P$ we find from (2.10), (3.14)

$$I' = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ e^{-2\rho} \partial u \bar{\partial} v + E(u) \partial u \bar{\partial} u \right],$$

$$E(u) = Q_1^{-1} Q_2 + Q_1^{-1} k(u) - P^{-1} Q_1^{-2} \gamma_m(u) + O(P^{-2}) \quad (3.15)$$

Here $\gamma_m = (\gamma_s, \gamma_4), \gamma_s(u)$ is the linearized ('free-theory') part of $a_s(u, \beta)$ and $\gamma_4(u) \equiv a(u)$. As we shall find in Section 4, the analogue of the level matching condition for the free fundamental string [2,3] which relates the oscillation level numbers to the charges of the background solution is

$$\overline{E(u)} = 0, \quad \overline{E(u)} \equiv \frac{1}{2\pi R} \int_0^{2\pi R} du E(u),$$

i.e., to the leading order in $1/P$,

$$Q_1 Q_2 P = \overline{\gamma_m(u)} \quad (3.16)$$

This is to be compared with the condition one finds after adding the perturbations (3.7) in the compact internal directions and integrating over $y_a$

$$I'' = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ \partial y_a \bar{\partial} y_a + 2 |F(x)A_a(u, x)|_{r \to 0} \partial u \bar{\partial} y^a + ... \right] \quad (3.17)$$

$$= \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ \partial y_a \bar{\partial} y_a + 2 Q_1^{-1} q_a(u) \partial u \bar{\partial} y^a + ... \right] \to \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ E(u) \partial u \bar{\partial} u + ... \right],$$

$\text{Note that the time-like } SL(2, R) \text{ part of the model (2.10) does not enter the discussion in any essential way.}$
\[ E(u) = Q_1^{-1}Q_2 - Q_1^{-2}q_a^2(u) + \ldots . \]  \hspace{1cm} (3.20)

Then (3.17) leads to the relation
\[ Q_1Q_2 = \overline{q_a^2(u)}, \]  \hspace{1cm} (3.21)

implying that the contribution of \( q_a \)-oscillations to statistical entropy compared to that of \( \gamma_m \)-oscillations is subleading in the limit of large \( P \).

The above discussion can be illustrated on the example of the special rotational perturbation (3.10) which corresponds to the chiral \( SU(2) \) Cartan current deformation
\[ I' = I + \frac{1}{\pi \alpha'} \int d^2 \sigma \, Q_1^{-1} \gamma(u) \, \partial u \, \bar{J}_3 , \]  \hspace{1cm} (3.22)

\[ \bar{J}_3 = 2(\sin^2 \theta \partial \bar{\phi} + \cos^2 \theta \partial \bar{\psi}) . \]

It is possible to decouple \( u \) from \( SU(2) \) angles (or, equivalently, to integrate the latter out) for generic \( P \) by making a field redefinition. Introducing
\[ \bar{\phi} = \phi + P^{-1}Q_1^{-1} \hat{\gamma}(u) , \quad \bar{\psi} = \psi + P^{-1}Q_1^{-1} \hat{\gamma}(u) , \quad \hat{\gamma}(u) \equiv \int_0^u du' \gamma(u') , \]  \hspace{1cm} (3.23)

we can represent the action in the form similar to (2.10)
\[ I' = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ e^{-2\rho} \partial u \bar{\psi} + E(u) \partial u \bar{\phi} \right] \]  \hspace{1cm} (3.24)

\[ + \frac{P}{\pi \alpha'} \int d^2 \sigma [ \partial \bar{\rho} \bar{\phi} + \partial \bar{\theta} \bar{\phi} + \sin^2 \theta \partial \bar{\phi} \bar{\psi} + \cos^2 \theta \partial \bar{\psi} \bar{\psi} + \frac{1}{2} \cos 2\theta (\partial \bar{\phi} \bar{\bar{\psi}} - \partial \bar{\psi} \bar{\bar{\phi}})] , \]

\[ E(u) = Q_1^{-1}Q_2 - P^{-1}Q_1^{-2} \gamma^2(u) = P^{-1}Q_1^{-2} [Q_1Q_2 P - \gamma^2(u)] . \]  \hspace{1cm} (3.25)

In this special case the level matching condition (3.17) is (cf. (3.13))
\[ Q_1Q_2 P = \overline{\gamma^2(u)} = \bar{\gamma}^2 + \bar{\gamma}^2(u) . \]  \hspace{1cm} (3.26)

When \( \gamma(u) \) is assumed to contain a constant part (i.e. \( \hat{\gamma}(2\pi Rn) = 2\pi R \bar{\gamma} \neq 0 \)) one finds that the corresponding rotational parameter should be quantised as demanded that \( \bar{\phi}, \bar{\psi} \) should have the same \( 2\pi \) periodicity as \( \phi, \psi \) we get \( \bar{\gamma} R P^{-1} Q_1^{-1} = l = \text{integer} \). Then the quantisation of \( Q_1 = \frac{4G_N}{\pi} \frac{wR}{\alpha'} \) (see Section 4) and \( P = \kappa \alpha' \) leads to the quantization of \( \bar{\gamma} = R^{-1} PQ_1 l \), i.e. to the conclusion that the angular momentum \( J = \frac{\pi}{4G_N} \bar{\gamma} \) should take integer values, \( J = \kappa w l = n \).
4. String sources and level matching condition

To establish a relation between the charges (parameters of the classical solution) and the oscillator level numbers (and also to determine the quantisation condition for $Q_1$ and $Q_2$) we shall consider the analogue of the procedure of matching on a string source used in [25,3].

The backgrounds corresponding to the $D = 5$ dyonic black hole and $D = 6$ dyonic string model (2.4) are non-singular at $r = 0$. Contrary to what one might expect, we believe that it is consistent to assume that having both electric and magnetic aspects, the solitonic $D = 6$ string, like purely ‘electric’ fundamental string, still needs to be supported by a source at the origin. This interpretation is important in order to be able to relate the macroscopic charges to microscopic string oscillations.

A systematic approach is to start with equations always containing source terms and to see whether the sources contribute or not for specific choices of backgrounds. This makes possible to discuss both ‘solitonic’ (source-free) and ‘elementary’ (supported by sources) solutions as well as intermediate ‘dyonic’ ones from a unified point of view.

4.1. Fundamental string

Let us first review the source interpretation of the fundamental string solution [25].

While the equations of conformal invariance of the $\sigma$-model corresponding to the fundamental string (i.e. (2.4) with $F \neq 1, K = 0, f = 1$) are formally satisfied without need to introduce a source at the origin ($R_{\mu\nu} + \ldots = 0$ implies $F^2 \partial^m \partial_m F^{-1} = 0$ which is satisfied at all points, see [11]) there is also an alternative ‘string source’ interpretation [25,3] consistent with singularity of the background at the origin. One starts with the ‘combined’ action

$$-\frac{1}{16\pi G_{N}^{(D)}} \int d^D x \sqrt{G} e^{-2\Phi} (R + \ldots) + \frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{g} g^{pq} G_{\mu\nu}(x) \partial_p x^\mu \partial_q x^\nu + \ldots \ , \ (4.1)$$

and thus obtains the set of equations containing (see [3] for details)

$$\sqrt{G} e^{-2\Phi} (R_{\mu\nu} + \ldots) = \frac{4G_{N}^{(D)}}{\alpha'} \int d^2 \sigma \partial_p x_0^\mu \partial_p x_0^\nu \delta^{(D)}(x - x_0(\sigma)) \ . \ (4.2)$$

In the simplest case of flat transverse space and $K = 0$ one finds that the classical string source equations and the condition of vanishing of the 2-d stress tensor (following from variation over 2-d metric $g_{pq}$ which is set equal to $\delta_{pq}$ in (4.2)) are satisfied by the static

\[13\] The dilaton (or conformal anomaly) equation $R - \frac{1}{12} (H_{\mu\nu\lambda})^2 - 4(\partial_\mu \Phi)^2 + 4 \nabla^2 \Phi = 0$ is not modified since the (classical) string source is not coupled to the dilaton. This implies that the $O(G^{\mu\nu})$ term in the above equation coming from the variation of $\sqrt{G}$ vanishes separately.
winding string configuration $x_0^i$: $u_0 = 2Rw\sigma_+, v_0 = 2Rw\sigma_-$, $x_0^i = 0$, where $w = \pm 1, \pm 2, \ldots$ and $\sigma_\pm \equiv \sigma \pm \tau$, $0 < \sigma \leq \pi$. Then (4.2) implies that (note that here $\sqrt{Ge^{-2\Phi}} = 1$)

$$\partial^m \partial_m F^{-1} = -\mu \delta^{(D-2)}(x) , \quad (4.3)$$

$$\mu = -\frac{8G_N^{(D)}}{\alpha'} \int d^2 \sigma \partial_0 u_0 \partial_0 v_0 \delta(u_0) \delta(v_0) = \frac{8G_N^{(D)}}{\alpha'} w ,$$

so that in the 1-center case ($G_N^{(D-1)} = G_N^{(D)}/2\pi R$)

$$F^{-1} = 1 + \frac{Q_1}{r^{D-4}} , \quad Q_1 = \frac{16\pi G_N^{(D-1)}}{(D-4)\omega_D \cdot \frac{wR}{\alpha'} , \quad (4.4)$$

i.e. for the $D = 6$ string ($G_N \equiv G_N^{(5)}$)

$$Q_1 = \frac{4G_N}{\pi} \cdot \frac{wR}{\alpha'} . (4.5)$$

The $vv$-component of (4.2) gives the equation for $K$ in the form

$$K \partial^m \partial_m F^{-1} + F^{-1} \partial^m \partial_m K = -\frac{8G_N^{(D)}}{\alpha'} \int d^2 \sigma \partial_+ v_0 \partial_- v_0 \delta(u_0) \delta(v_0) \delta^{(D-2)}(x) . \quad (4.6)$$

The solution $K = Q_2/r^{D-4}$ of $\partial^2 K = 0$ cannot be matched onto string source [3]. If one requires that all solutions should be supported by sources one thus arrives at the condition

$$E(u) \equiv [F(x)K(u, x)]_{x \to 0} = 0 . \quad (4.7)$$

A point of view which we shall adopt here is to admit the possibility that the ‘basic’ macroscopic solution may still be described by $K = Q_2/r^{D-4}$. Even though this term cannot be directly matched onto source it may be considered as an effective description of momentum flow since there is a related oscillating solution that gives the same space-time background after averaging over compact direction [3]. Q_2 can then be identified with momentum $p = m/R$ along the string direction (this interpretation is also consistent with

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14 Analogous condition was suggested in [2] from the requirement that the singularity at $r = 0$ should be null.

15 The oscillating fundamental string can be described either by $A_m = g_m(u)$, or equivalently, by $K = h_m(u)x^m$ (both choices are obvious solutions of conformal invariance equations and are related by a redefinition of $v$), where $g_m$ and $h_m$ are related to the profile functions $f_m(u)$ of oscillating string, $g_m = f_m'(u)$, $h_m = 2f_m''(u)$ [2].
T-duality in $y_5$ direction which interchanges $F(x)$ and $K(x)$ and thus $Q_1$ and $Q_2$ as well as the winding and momentum numbers). In $D = 6$ one finds

$$Q_2 = \frac{4G_N}{\pi} \cdot \frac{m}{R} .$$

(4.8)

The ‘matching on source’ condition may be imposed only on oscillating generalisations of the basic background. The minimal requirement is that (4.7) should be satisfied in the average sense

$$\int_0^{2\pi R} du \, E(u) = 0 .$$

(4.9)

This condition can be also interpreted as the standard classical Virasoro level matching condition $L_0 - \bar{L}_0 = 0$ or $\int_0^\pi d\sigma (T_{++} - T_{--}) = 0$ applied to the solitonic string described by (collective-coordinate) $\sigma$-model $L = F(x)\partial_+ u[\partial_- v + K(u,x)\partial_- u] + \partial_+ x_m \partial_- x_m$ for the basic static winding source configuration $u_0 = 2Rw\sigma_-, \ v_0 = 2Rw\sigma_+ , \ x_0^m = 0$.

4.2. Dyonic string

In the case of the 5-brane or solitonic $D = 6$ ‘magnetic’ string described by (2.4) with $F = 1, \ K = 0, \ f \neq 1$, the equations with source terms which follow from (4.1) are satisfied with the left and right parts of (4.2) being separately zero. This is in agreement with expectation that a solitonic solution need not be supported by a source. For example, the l.h.s. part of $(mn)$ component of (4.2) reduces to (see (2.5))

$$f^{-1} \Delta_m f , \ \ \ \Delta_m \equiv \frac{1}{\sqrt{g_{e^{-2\phi}}} g^{mn} \partial_n} = f^{-1} \partial^m \partial_m ,$$

(4.10)

and thus vanishes automatically for the harmonic function $f$ (the same is true also if we start with (4.2) divided by $\sqrt{G e^{-2\Phi}}$).

When both $F$ and $f$ are assumed to be non-trivial, i.e. for a ‘bound state’ of fundamental string and 5-brane or the $D = 6$ dyonic string (2.9), the equation for $f$ is still satisfied automatically while the equation for $F$ ((uv) component of (4.2), cf.(4.3))

$$f \Delta_m F^{-1} = \partial^m \partial_m F^{-1} = -\mu \delta^{(4)}(x) ,$$

(4.11)

does not change its form compared to the free fundamental string case and thus still needs a source for its support. Note that if we started with (4.2) written in a different form with both parts divided by $\sqrt{G e^{-2\Phi}}$ (here equal to $f$) the conclusion would be different: we would get

$$\Delta_m F^{-1} = f^{-1} \partial^m \partial_m F^{-1} = -\mu f^{-1} \delta^{(4)}(x) ,$$

(4.12)
so that the left and right parts would vanish separately for harmonic \( f \) and \( F \) in (2.7). It seems, however, that it is (4.2) (and thus (4.11)) that is the consistent form of the equation with source since it directly follows from (4.1).

Once the source interpretation of the dyonic string solution is accepted, one is able (as in the free fundamental string case) to relate \( Q_1 \) in \( F \) to the string source winding number \( w \) and \( Q_2 \) in \( K \) to the quantised momentum number \( m \), i.e. to get again the expressions for the quantised charges (4.5),(4.8). One can also obtain again the level matching condition (4.3), i.e. the condition of the vanishing of the mean value of the total coefficient \( E(u) \) of the \( du^2 \) term in the metric (put into the form without off-diagonal \( du \)-terms) at the core. This condition (3.17) was already applied to perturbations in the near-horizon region in Section 3.4.

5. Statistical entropy

As was already mentioned above, for the purpose of counting perturbations relevant for reproducing the leading term in the black-hole entropy it is sufficient to consider the region near the origin \( r = 0 \). Returning to the discussion of the throat region perturbations in Section 4.1 let us express the level matching condition in terms of the quantised charges. Using the relations between the charges and integer quantum numbers (4.5),(4.8),(2.11) we thus find for the ‘compact’ and ‘non-compact’ perturbations (see (3.21),(3.18))

\[
wm = N_L^{(q)} \quad N_L^{(q)} = \frac{\pi^2 \alpha'}{16G_N^2} q_a(u),
\]

\[
wm\kappa = N_L^{(\gamma)} \quad N_L^{(\gamma)} = \frac{\pi^2}{16G_N^2} \gamma_m(u).
\]

The normalisation coefficient \( \frac{\pi^2}{16G_N^2} \) relating \( \gamma_m(u) \) to integer oscillator number can be checked by considering the special case of rotational perturbation (3.22),(3.26) with \( \gamma(u) = \bar{\gamma} \) and using that the angular momentum \( J = \frac{\pi}{4G_N} \bar{\gamma} \) should take integer values as discussed after eq.(3.26).

In general, the r.h.s. of the relation (5.2) will contain contributions of all possible near-horizon perturbations in all transverse directions, but as clear from the above discussion the oscillations in the four ‘external’ spatial dimensions have dominant statistical weight

\[16\] This action may be understood as resulting from a resummation of string loop corrections [12] which implies that there should be no extra factors in front of the string metric in the source term.
for $w, m, \kappa \gg 1$. Taking into account the superpartners of the four bosonic oscillation directions the leading term in the statistical entropy is thus given by (cf. (1.3), (1.4))

$$S_{\text{stat}} = 2\pi \sqrt{N_L^{(\gamma)}} = 2\pi \sqrt{wm\kappa} = \frac{\pi^2}{2G_N} \sqrt{Q_1Q_2P}, \quad (5.3)$$

i.e. reproduces the BH entropy (2.3).

In above discussion we were assuming that the world sheet theory is $(1,1)$ supersymmetric as in type II theory. The same conclusions apply to the case of the heterotic string since we may assume the ‘symmetric’ embedding of the 5-brane solution [12] (i.e. with the spatial magnetic part of the spin connection being identified with the gauge field background). Then the relevant 4-dimensional part of the heterotic $\sigma$-model becomes identical to that of type II theory.

All the steps leading to statistical derivation of the $D = 5$ black hole entropy (5.3) have straightforward generalisation to the case of the $D = 4$ dyonic black hole considered in [9,10,11]. The corresponding conformal $\sigma$-model is very similar to (2.4), i.e. has again the non-trivial transverse part represented by the $((4,4)$ supersymmetric) 4-dimensional model (cf. (2.4))

$$L_\perp = (g_{mn} + B_{mn})(x)\partial x^m \bar{\partial} x^n + R\phi(x)$$

$$= f(x)k(x) [\partial x_4 + a_s(x)\partial x^s] [\bar{\partial} x_4 + a_s(x)\bar{\partial} x^s] + f(x)k^{-1}(x)\partial x^s \bar{\partial} x^s + b_s(x)(\partial x_4 \bar{\partial} x^s - \bar{\partial} x_4 \partial x^s) + R\phi(x),$$

$$\partial_p b_q - \partial_q b_p = -\epsilon_{pqrs} \partial^s f, \quad \partial_p a_q - \partial_q a_p = -\epsilon_{pqrs} \partial^s k^{-1}, \quad p, q, s = 1, 2, 3,$$

$$f = 1 + \frac{P_1}{r}, \quad k^{-1} = 1 + \frac{P_2}{r}, \quad \phi = \frac{1}{2} \ln f.$$

Here $x_4$ is a periodic coordinate and $r^2 = x_s x_s$. $F$ and $K$ in (2.4) are now given by $F^{-1} = 1 + \frac{Q_1}{r}$, $K = \frac{Q_2}{r}$. The short-distance limit of the $D = 6$ conformal model is again described by a WZW model with level $\kappa = 4P_1P_2/\alpha'$. Dimensional reduction along $x_4$ and the string direction $y_5$ leads to extremal dyonic black hole with the corresponding BH entropy

$$S_{\text{BH}} = \frac{\pi}{G_N} \sqrt{Q_1Q_2P_1P_2} = 2\pi \sqrt{wm\kappa}, \quad (5.5)$$

which is reproduced by counting supersymmetric marginal perturbations in the *four* relevant ‘external’ directions $x_s, x_4$. Contributions of oscillations in other compact internal directions are subleading, in complete analogy with the $D = 5$ case.

In conclusion, it is clear that there are many similarities between the present NS-NS conformal $\sigma$-model approach and the approach based on R-R embedding and D-brane

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17 The embedding of the time-like ‘fundamental string’ components of the spin connection into the gauge group is not possible, and, indeed, is not needed for exact conformal invariance [36].
count of BPS states (especially in the formulation proposed in [22] where the effective central charge is fixed). We believe that establishing a direct translation between the two approaches will be illuminating for better understanding solitons and black holes in string theory.

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