Maximum entropy states of collisionless systems with long-range interaction and different degrees of mixing

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Abstract

Dynamics of many-particle systems with long-range interaction is collisionless and governed by the Vlasov equation. This dynamics is a flow of a six-dimensional incompressible liquid with uncountable integrals of motion. If the flow possesses the statistical property of mixing, each liquid element spreads over the entire accessible space. I derive the equilibrium microcanonical maximum entropy states of this liquid for different degrees of mixing $M$. This $M$ is the number of liquid elements which are statistically independent. To count microstates of a liquid, I develop analog of the discrete combinatorics for continuous systems by introducing the ensemble of phase subspaces and making contact with the Shannon-McMillan-Breiman theorem from the ergodic theory. If $M$ is much larger than the total number of particles $N$, then the equilibrium distribution function (DF) is found to be exactly of the Fermi-Dirac form. If the system is ergodic but without mixing, $M = 0$, the DF is a formal expression which coincides with the famous DF obtained by Lynden-Bell. If the mixing is incomplete and $M \sim N$, the exponentials, which are present in Lynden-Bell’s DF, appear with certain weight given by the entropy of mixing. For certainty, the long-range interaction is taken in the form of the Newton and Coulomb potential in three dimensional space, but the applicability of the method developed in the paper is not restricted to this case. The analogy of the obtained statistics to the Fermi-Dirac statistics allows for expressing the entropy of the system via its total energy and chemical potentials of liquid’s elements. The effect of the long-range interaction to the basic thermodynamic relations is demonstrated.

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I. INTRODUCTION

A. Lynden-Bell’s maximum entropy state of collisionless systems

Interaction potentials which decay not more rapidly than $1/R^D$, where $D$ is the space dimensionality and $R$ is the particles’ separation, are considered as long range ones. The reason is that for a long-range potential, the interaction of a particle with the more remote particles is stronger than that with less remote particles, so that the interaction with closest neighbors in sufficiently large systems becomes negligible. As to a good approximation remote particles constitute a continuous medium, the interaction is determined by their spatially averaged density which is the source of a mean field. Hence systems with long-range interaction can be well-represented by their smooth densities and mean fields disregarding the effects of matter discreetness and, in particular, pair encounters. This is exactly the case of applicability of the collisionless Vlasov equation which describes evolution of a smooth distribution function (DF) in the six-dimensional (6D) space of particles’ coordinates and velocities under the action of the self-consistent mean field. This 6D space is called $\mu$ space. Thus, dynamics of many-particle systems with long-range interaction in the $\mu$ space is collisionless and governed by the Vlasov equation [1–4].

The final state to which any large classical system relaxes is the standard Maxwell-Boltzmann equilibrium as pair collisions will eventually manifest themselves. But in systems with long-range interaction the time when this happens increases approximately as the number of particle to the power which is larger than unity [5–10] so that for a very long time the system is governed by the collisionless Vlasov equation. As a result, the physics of many long-range systems is essentially collisionless. For instance, the relaxation time due to pair collisions in galaxies is estimated to be much longer than the age of our Universe (which is known as Zwicky paradox). At the same time, the remarkable regularity in the characteristics of the gravitational systems, such as galaxies, suggests that these systems have reached some equilibrium states. This was the principal motivation for Lynden-Bell to consider the statistical equilibrium state to which a collisionless gravitational system can relax in its own violently changing mean field [11]. The main challenge of this collisionless statistical approach was that the Vlasov equation describes not particles, but a 6D incompressible liquid, which is a continuous system with the infinite number of conserved
quantities. Lynden-Bell modeled the liquid by dividing it into small stable macroparticles and then incorporated the incompressibility by not allowing two macroparticles to occupy the same cell of the $\mu$ space. Using the standard combinatorics for the discrete macroparticles, Lynden-Bell arrived at his famous Fermi-like DF. This Lynden-Bell’s DF (LBDF) successfully explained the two fundamental experimental facts which cannot be explained by the standard Maxwell-Boltzmann statistics. First, thanks to the smooth macroscopic character of the Vlasov equation which is insensitive to discrete properties of individual stars, the LBDF does not discriminate between stars with different masses as it depends on the average mass density. Second, thanks to the incompressibility of Vlasov’s liquid, the LBDF possesses the Fermi-like degeneracy which naturally explains the dense core-rarefied galo structure of galaxies and which, as eventually turned out, is universal for all systems with long-range interaction [4]. The impressive elegancy of Lynden-Bell’s result, the revealed evidence that the Vlasov equation might have so a unique role in the statistical physics and its relation to the generic systems with long-range interaction had resulted in a wide interest to the equilibrium and dynamical properties of collisionless systems. These studies have developed into the novel branch of statistical physics identified as the statistical mechanics of systems with long-range interaction [1–4].

By now it has been well established that the assumption that the violent relaxation results in certain equilibrium maximum entropy state is an idealization. The collisionless relaxation is incomplete because the efficiency of the mean-field-driven mixing is dying out as the system is approaching states with a smoother and smoother field, so that the state with complete mixing is not attained. In reality, the relaxation can cease at some of dynamically stable states of the Vlasov equation which are numerous and substantially differ from the maximum entropy state [4, 8, 12]. Nevertheless, since its prediction by Lynden-Bell in 1967 for stellar systems, the equilibrium statistical mechanics of this idealized state has been attracting an unrelenting attention of theoretical physicists. The result obtained by Lynden-Bell has been rederived by various technics [13–19], praised and criticized [20], its quantitative predictability has been supported in some cases [4, 21–23] and denied in other cases [24, 25], its validity has been reconsidered and a different form of the equilibrium state has been proposed [26]. One reason is that systems with long-range interaction, hence collisionless and related with the LB statistics, appear in many different fields of physics: gravitational systems in astrophysics, charged beams with Coulomb interaction in the plasma physics, two-dimensional
hydrodynamics where the interaction of vortices is logarithmic, systems of many spherical colloids at liquid surfaces where the attraction scales as $1/R^2$ \[27\], dipolar systems and ferroelectric liquids. Another reason is that the thermodynamics of systems with long-range interaction cannot be based on the fundamental axioms of the standard thermodynamics of systems with short-range interactions such as spatial uniformity, additivity of entropy and energy, equivalence of the canonical and microcanonical description, presence of a thermal bath fixing the system’s temperature \[1–4\]. This brings a fresh breath to the thermodynamics. There is also a very peculiar source of attraction, a challenge unformulated explicitly but making theorists think of collisionless maximum entropy states which under some, even highly idealized assumptions, can result from the collisionless Vlasov dynamics.

The problem is that, as mentioned above, the Vlasov equation describes motion of a 6D incompressible liquid. But, on the one hand, neither a liquid can be approximated by a set of separated shape-preserving fragments, nor any liquid fragment preserves its shape. This is true for any motion, but is particularly fundamental for a chaotic mixing motion which can possibly result in a maximum entropy state in any sense: such a motion is supposed to eliminate correlations between initially neighboring liquid elements which implies producing thinner and thinner liquid filaments. On the other hand, while a verity of technics has been employed to address collisionless statistical equilibrium in different systems with long-range interaction, the common device for counting the number of states of the Vlasov liquid has been either similar or essentially equivalent to the original Lynden-Bell’s device, i.e., dividing the liquid into independent shape-preserving fragments called macroparticles, dividing the $\mu$ space into cells each of which can accommodate exactly one macroparticle or nothing, and using the discrete combinatorics for counting possible distributions of the macroparticles over these cells \[13–19\]. The introduction of macroparticles allows one to substitute the problem of the innumerable liquid degrees of freedom for the well-known problem of a finite number of particles, but the very idea of a nonspreading macroparticle contradicts the idea of the mixing dynamics which can lead to any kind of statistical equilibrium. Indeed, in a mixing flow any small liquid element spreads over the whole accessible space so that its filaments can enter any cell of the $\mu$ space. We see that the main disadvantage of the present theory is the fundamental contradiction between the idea of solid, undiffuse and independent macroparticles and the continuity of a mixing flow of the 6D liquid.

The problem of finding a maximum entropy state or, in other words, a statistical equilib-
rium state, which can result from the Vlasov equation under highly idealized assumptions on the chaotic dynamics, may be described as follows. First of all, one has to deal with an actual 6D liquid rather than with any kind of particles (i.e., not with real particles, macroparticles, vortices). In the statistical equilibrium all the accessible states are equally probable. But any change of liquid’s shape results in its new state. The question is how to describe the macroscopic and microscopic states of this continuous system and how to count its states in order to obtain the entropy. Moreover, it is known that the Vlasov system possesses the uncountable number of conserved quantities, values of the DF at each point of the liquid usually called levels, which do not change along their trajectories. These integrals of motion must be imposed as certain restrictions on the accessible states. It is these integrals that account for the incompressibility and the related Fermi-like exclusion property. However, in contrast to the simple "one cell-one macroparticle" restriction of Lynden-Bell’s model, the actual incompressibility can only mean that the total amount of liquid with all different levels in a given cell of the $\mu$ space cannot exceed its 6D volume $\omega$. Hence the amount of liquid in a given cell, which is either zero or one in Lynden-Bell’s model, is a continuous number ranging from 0 to $\omega$, so that the summation over the occupation numbers must be replaced by some integration. Finally, in principle, the degree of mixing of the 6D liquid can be different, and one has to specify this degree and find its connection to the final equilibrium state.

In this paper I present the theory of the maximum entropy state of a 6D liquid to which systems with a long-range interaction can relax under the assumption of the chaotic dynamics governed by the Vlasov equation. It turns out that the solution to this problems is not unique as it depends on the presupposed statistical properties of the Vlasov dynamics. According to the ergodic theory \cite{28,31}, the chaoticity of a dynamical system ranges from ergodicity without mixing, which is the lowest statistical property, through mixing with zero Kolmogorov-Sinai entropy to the so-called K-mixing with nonzero Kolmogorov-Sinai entropy, which is the highest statistical property. In the next Section I present some information from the ergodic theory which is instructive to our problem, give further details of the approach, and briefly describe the structure and results of this paper.
B. Possible statistical properties of a flow of liquid: ergodicity, mixing, and incomplete mixing

Motion of a liquid in any space is a flow. Flows on different phase spaces and their statistical properties are the subject of the ergodic theory [28–31]. It is therefore natural to resort to the ideas of the ergodic theory for developing a statistical approach to the 6D Vlasov liquid. One of the most important quantities in the language of the ergodic theory is the measure (volume) $\mu(A \cap B)$ of intersection $A \cap B$ of a set $A$ with a set $B$. This quantity determines probability that a point from $A$ can be found in the set $B$ and thus defines the coarse-grained DF of $A$. The flow implies that the set $A$ occupied by liquid changes its shape and is a function of time $t$, i.e., $A = A(t)$. The quantity $\mu(A(t) \cap B)$ is adequate to specify the shape $A(t)$ of a flowing set $A$ at an instant $t$ with respect to the fixed sets $\{B\}$ which can be used as a reference frame for a coarse-grained description of the flow $A(t)$. At last, the fundamental statistical properties of dynamical systems in the ergodic theory can be expressed in terms of the above measure of intersection.

Ergodic and mixing flows preserve the volume of a flowing set: $\mu(A(t)) = \mu(A(0)) = \mu(A)$. Ergodicity is the weakest statistical property of a dynamical system which does not imply its relaxation to some stationary state. In an ergodic (but non-mixing) flow $A(t)$ of a set $A$, its measure of intersection with a fixed set $B$ remains time dependent at all times. The only time independent quantity is the temporal average of $\mu(A(t) \cap B)$ [31]:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mu(A(\tau) \cap B)d\tau = \mu(A)\mu(B).$$

(1)

Hence, measuring the amount of liquid $A$ in the fixed set $B$, $\mu(A(t) \cap B)$, at different times $t$ will give different results as $\mu(A(t) \cap B)$ as a function of $t$ is not converging to any stationary value. We see that the ergodicity alone does not result in a relaxation. The formula (1) tells one that in an ergodic flow, a liquid fragment $A$ does not spread over the accessible space, its points do not decorrelate and, as time elapses, $A$ moves roughly as a single piece. Thus, the chaoticity of the ergodic motion of $A$ consists in the fact that $A$ can be found everywhere with equal probability.

A mixing flow is ergodic, but converse is not necessarily true. Mixing is a much stronger statistical property than ergodicity and can be naturally associated with a relaxation towards certain time independent state. In a mixing flow $A(t)$ of a set $A$, its measure of intersection
with a fixed set \( B \) tends to the time independent value:

\[
\lim_{t \to \infty} \mu(A(t) \cap B) = \mu(A)\mu(B).
\]  

Hence, measuring the amount of liquid \( A \) in the fixed set \( B \), \( \mu(A(t) \cap B) \), at different times \( t \) one will find that after some time the value of \( \mu(A(t) \cap B) \) gets closer and closer to a constant \( \mu(A)\mu(B) \), which can naturally be considered as the equilibrium value. Thus, one can say that in the mixing flow any nonequilibrium distribution relaxes to the equilibrium state. The formula (2) tells one that in a mixing flow, a liquid fragment \( A \) uniformly spreads over the accessible space; As time elapses, \( A \) develops filaments which get thinner and thinner, and its points decorrelate and move more and more independently. Thus, the mixing motion of \( A \) is chaotic indeed. For instance, if \( A \) consists of fragments of different colors which do not interpenetrate, then after the relaxation every one of these colors will be present in any fixed set \( B \) however small it be. The only difference to our case is that we have different levels of the initial DF rather than colors.

The ergodic theory considers flows on a phase space where the state of a system is given by a single point while a set of finite measure represents an ensemble of similar systems. The \( \mu \) space is not the phase space of a 6D liquid (the phase space is introduced in the next Sec.2). In \( \mu \) space the state of a liquid is determined by a set occupied by the liquid or, in other words, by liquid’s shape. A liquid element does spread over the entire \( \mu \) space, but not necessarily homogeneously as in phase space, and it is finding the inhomogeneous distribution in the \( \mu \) space that is the main problem of the theory \[32\]. Therefore, in what follows the homogeneity of smearing expressed by Eq. (2) is not assumed in the \( \mu \) space even in the equilibrium state.

Now we can more instructively describe the fundamental problem one has to face when developing a statistical theory of an actual continuous liquid. Suppose we know the above distribution of the liquid’s levels over the fixed cells \( \{ B \} \) in the \( \mu \) space. Given these coarse-grained macroscopic quantities are fixed, however, the arrangement of different levels inside a single cell can be different at a finer scale. The question is how many states with different inner arrangements exist for the above fixed macroscopic levels’ distribution. The answer to this question can be obtained by using the famous Shannon-McMillan-Breiman theorem. The contact of our problem to this theorem will be established in Sec. III.C.

The statistical property of mixing implies that any small fragment of a liquid spreads over
the total accessible space. This is a complete mixing, but we would like to make a contact with the more real situation. As already stressed above, the Vlasov dynamics cannot result in a complete mixing as the higher the mixing achieved in the system, the smoother the mean field is, and, as a result, the weaker the mixing efficiency becomes. An incomplete mixing implies that points of a liquid that were initially sufficiently remote from one another move independently and can be found at any relative distance. However, the filaments developing in the flow will not be going beyond some minimum size, the mixing stops and finer scale patterns will not be produced. This is because the points of a liquid that were initially separated insufficiently remain correlated and cannot run away from one another. Then the degree of incomplete mixing can be accounted for by the number \( M \) of sets in the partition of the initial liquid’s shape \( A(0) \) which can be considered statistically independent.

Let \( A(0) \) be the initial set occupied by the liquid and \( \{ A_m(0) \} \), \( m = 1, 2, ..., M \), be its partition into disjoint subsets \( A_m \), i.e.,

\[
A(0) = \bigcup_{m=1}^{M} A_m(0). \tag{3}
\]

In the flow, each \( A_m \) changes its shape, and the argument 0 in Eq.(3) changes to \( t \). Let \( \{ B_i \} \) be a partition of the accessible space into fixed disjoint subsets \( B_i \). Obviously, the probability that a point from \( A_m \) is found in \( B_i \) is proportional to the quantity \( \mu(A_m \cap B_i) \).

If the liquid elements \( A_m \) flow independently, then the joint probability \( P \) that a point from \( A_1 \) enters a set \( B_{i_1} \), a point from \( A_2 \) enters a set \( B_{i_2} \), ..., a point from \( A_M \) enters a set \( B_{i_M} \) is proportional to the product of the probabilities:

\[
P\{ A_1 \to B_{i_1} | A_2 \to B_{i_2} | ... | A_M \to B_{i_M} \} \propto \prod_{m=1}^{M} \mu(A_m \cap B_{i_m}). \tag{4}
\]

It is important to stress that there are only \( M \) factors in the joint probability \( P \) as the maximum number of independent events \( A_m(t) \to B_{i_m} \) at a time is \( M \). We will see that it is the length \( M \) of a sequence consisting of the above independent events that determines the number of microscopic states for a given coarse-grained levels’ distribution. As to points within the same set \( A_m \), by the assumption of incomplete mixing, they do not move independently and contribute very little to the number of microscopic states.

I choose the number \( M \) of the statistically independent elements of a liquid as a measure of incomplete mixing. For \( M \) very large, i.e., much larger than the number of particles \( N \) in
the system, the flow is strongly mixing. For $M = 0$, the system is ergodic without mixing. In the intermediate case of $M$ on the order of $N$ or lesser, the mixing is incomplete. We will see that form of the equilibrium DF depends on $M$.

In the next Section, I introduce the phase space of the 6D incompressible Vlasov liquid, its microscopic and macroscopic states, formulate the problem in terms of the intersection measures discussed above, and present the exact constraints on the accessible phase space due to the incompressibility of the flow. In Sec.III, the microcanonical statistical integral for the accessible number of states is introduced and formulated in terms of the macrostates. In particular, in Sections III.B and III.C, analog of the discrete combinatorics for a continuous systems, which consists of two steps, is developed: first, I introduce the ensemble of phase subspaces which accounts for the states of a liquid on all subsets of the phase space; second, the number of microstates in a given macrostate is found by connecting the problem with the Shannon-McMillan-Breiman theorem. In Sec.III.D the energy of the system is expressed in terms of the macroscopic states. Here the interaction is taken in the form of the Newton or Coulomb potentials in the 3D space. This restriction is chosen for certainty, but the method developed in this paper can be applied for any long-range potential. In Sec.IV, the statistical integral is calculated and in Sec.V the results are present for strong mixing, for ergodicity without mixing, and for an incomplete mixing. The end of Sec.III is devoted to some thermodynamics, in particular, the entropy is expressed in terms of the total energy and the initial DF. In the conclusion, Sec.VI, the results are briefly summarized.

The equilibrium DF for strong mixing is found to be exactly of the Fermi-Dirac form: different levels do not appear individually and are present as a perfect mixture which is distributed over energy values. A pure ergodic case without mixing is considered formally since, as explained in Sec.I.B, in this case the system does not relax to a time independent state. The formal summation over possible occupation numbers in a small reference cell results exactly in the LBDF. In the intermediate case of incomplete mixing, the equilibrium DF is more complex than the LBDF: statistically independent elements with different initial densities (levels) have distinguishable contributions in the DF, but, in addition, the contribution of a given value of the (continuous) occupation number in the $\mu$ space appears with the weight depending on the mixing degree.
II. THE MACROSCOPIC AND MICROSCOPIC STATES OF THE VLASOV LIQUID

Denote $\Omega$ the 6D $\mu$ space. Let $\tau$ be a point of $\Omega : \tau = (r, v) \in \Omega$ where $r$ is the vector of position and $v$ is the velocity of a particle. Let at $t = 0$ the initial DF $f_0(\tau_0)$ be nonzero in the bounded area $\Omega_0$ whose points be $\tau_0: \tau_0 \in \Omega_0 \subset \Omega$. Clearly, the set $\Omega_0$ [i.e., support of $f_0(\tau_0)$] describes the initial shape of the 6D liquid while $f_0(\tau_0)$ gives the liquid’s density at each point $\tau_0$. The temporal dynamics of the liquid in $\Omega$, the flow, is governed by the Vlasov equation. After the time $t$ each point $\tau_0$ moves to the point $\tau(\tau_0, t)$ and the liquid’s shape changes, $\Omega_0 \rightarrow \Omega_0(t)$. All liquid elements of $\Omega_0$ also change their shapes but the Vlasov equation preserves their individual volumes which is associated with the incompressibility. As a result, the value of DF does not change along the trajectories of points $\tau_0$:

$$f_0[\tau(\tau_0, t)] = f_0(\tau_0). \ (5)$$

The number of these integrals of motion is uncountable, but they can be enumerated by the points of the initial state $\Omega_0$, one integral per each $\tau_0 \in \Omega_0$.

The state of a liquid at any instant $t$ is fully determined by the trajectories of all the initial points $\tau_0$, thus the function $\tau(\tau_0, t)$ is the liquid’s microstate. To introduce a macrostate that can be specified by a countable amount of information, we resort to the partitions of both moving $\Omega_0(t)$ and immobile $\Omega$ and introduce the measures of intersections between their members in line with the ideas outlined in the previous Section.

We devide the $\mu$ space into $i_m$ small cells $\sigma_i$ of the volume $\mu(\sigma_i) = \mu_i, \ i = 1, 2, \ldots, i_m$:

$$\Omega = \bigcup_{i=1}^{i_m} \sigma_i. \ (6)$$

These cells are fixed and immobile which is indicated by the subscript $i$. We also devide the initial set $\Omega_0$ into $k_m$ small elements $\sigma_{0k}$ of the volume $\mu(\sigma_{0k}) = \mu_{0k}, \ k = 1, 2, \ldots, k_m$. The flow $\Omega_0 \rightarrow \Omega_0(t)$ induces the time evolution of $\sigma_{0k}, \sigma_{0k} \rightarrow \sigma_k(t)$, which preserves the volume, $\mu(\sigma_k(t)) = \mu_{0k}$. The mobile partition has the form

$$\Omega_0(t) = \bigcup_{k=1}^{k_m} \sigma_k(t). \ (7)$$

This partition is moving (which is indicated by the subscript $k$, “kinesis”, movement) so that each initial $\sigma_{0k}$ spreads over the $\mu$ space $\Omega$. Spreading over $\Omega$, the elements $\sigma_k(t)$ intersect
the immobile reference cells $\sigma_i$. Let $\mu_{ik} = \mu(\sigma_k \cap \sigma_i)$ be the volume of intersection of the spreading liquid element $\sigma_k$ with the fixed reference cell $\sigma_i$. The quantity

$$X = \{\mu_{11}, \mu_{12}, ..., \mu_{21}, \mu_{22}, ..., \mu_{ik}, ..., \mu_{im km}\}$$

comprising the measures of all possible intersections is a point of some space $\Phi$ which has dimension $\dim \Phi = im km$.

We assume that $\sigma_{0k}$ is so small that the DF therein is constant: $f_{0}(\tau_{0}) = f_{k}$ for any $\tau_{0} \in \sigma_{0k}$. Hence, by virtue of the incompressibility expressed by the conservation laws (5), one has $f_{0}(\tau) = f_{k}$ for any $\tau \in \sigma_{k}(t)$. This shows that the quantity $X$ gives the coarse-grained distribution of the levels, which can be labelled by the subscript $k$, over the reference cells: $\mu_{ik}$ is the fraction of the liquid with density $f_{k}$ in the cell $\sigma_i$. The quantity $X$ can therefore be considered as the macrostate and the space $\Phi$ as the phase space of the system. The integrals of motion given in Eq.(5) can now be expressed as the following $km$ constraints in $\Phi$:

$$\mu_{0k} = \sum_{l=1}^{im} \mu_{ik}, \quad k = 1, 2, ..., km.$$  

These constraints express the conservation of the volume of the $k$-th element of the liquid: this element spreads over all cells $\sigma_i$, but its total amount in all $\sigma_i$ is equal to its initial volume $\mu_{0k}$. It is more convenient to reformulate the constraints in the form of matter conservation. To this end we notice that the number of particles in $\sigma_k$ is $N_k = f_k \mu_{0k}$. Then multiplying Eq.(9) with $f_k$ one obtains

$$N_k = \sum_{l=1}^{im} \mu_{ik} f_k, \quad k = 1, 2, ..., km.$$  

These constrain do not exhaust all restrictions on the macroscopic states $X$. Indeed, the amount of liquid in a cell $\sigma_i$ cannot exceed its volume $\mu_i$. Then we have the following $im$ inequalities:

$$0 \leq \sum_{k=1}^{km} \mu_{ik} \leq \mu_i, \quad i = 1, 2, ..., im.$$  

The information about the system depends on the partitions \{\sigma_{0k}\} and \{\sigma_i\}. It is known that the information which can be delivered by a partition is given by the so-called partition entropy [31]. The entropies of our partitions are, respectively, $- \sum_k \mu_{0k} \ln \mu_{0k}$ and $- \sum_i \mu_i \ln \mu_i$. Under the constraints $\sum_k \mu_{0k} = \mu(\Omega_0)$ and $\sum_i \mu_i = \mu(\Omega)$ the entropies attain
their maxima for $\mu_{0k} = \mu(\Omega_0)/k_m$ and $\mu_i = \mu(\Omega)/i_m$. Thus, the most informative partitions consist of elements of equal measure. In the following we assume that $\mu(\Omega_0) = 1$ and choose all the partitions’ elements of the same size $\omega$, i.e.,

$$\mu_{0k} = \mu_i = 1/k_m = \omega,$$

$$i = 1, 2, \ldots, i_m; \quad k = 1, 2, \ldots, k_m. \quad (12)$$

We emphasize however that the shape of the elements $\sigma_{0k}$ and $\sigma_i$ is not supposed to be necessarily the same. The only requirement on the shapes is that the partitions are integral ones (so that for small $\omega$ the summation can be changed to integration). Note that in spite of their equal values, the symbols $\mu_{0k}$ and $\mu_i$ will often appear in our consideration as their subscripts are helpful for making formulas clearer and more symmetric. In what follows, making use of the results of this Section, I formulate the statistical integral and then solve it by the steepest descent method.

### III. STATISTICAL INTEGRAL FOR THE 6D INCOMPRESSIBLE VLASOV LIQUID

#### A. Microcanonical ensemble for the Vlasov liquid

The equilibrium states of many systems with long-range interaction, e.g., galaxies and charged beams, are spatially inhomogeneous, confined by their own mean field without a contact with any external body. Such systems are isolated and their statistical equilibrium is microcanonical. Moreover, the derivation of the canonical ensemble is based on system’s additivity and fixed temperature, that is on two assumption invalid for a system with long-range interaction. For this reason we consider the microcanonical statistical integral for an isolated Vlasov system with the energy $E$. In the context of the previous Section, the number of states $\Gamma$ of our system is given by the volume of the fraction of the phase space $\Phi$ determined by the $k_m$ constraints (10), $i_m$ inequalities (11), and the requirement of the constant energy. This can be written as

$$\Gamma = \int d\Sigma \times \delta[E - H(X)] \times W(\Omega)e^{h(X)}, \quad (13)$$

where the integration is over the surface $\Sigma$ in the phase space $\Phi$ which is determined by the constraints (11):
\[
\int d\Sigma(X) = \prod_{k=1}^{k_m} \prod_{i=1}^{i_m} \int_0^\omega d\mu_{ik} \delta(N_k - \sum_{i'=1}^{i_m} \mu_{i'k} f_k).
\] (14)

The function \(H(X)\), which will be considered in Sec. III.D, is the energy of the system as a function of its macrostate \(X\). The product \(W(\Omega) \exp[h(X)]\) is the central problem of the statistical approach to continuous systems. It is the statistical weight of the state \(X\) which is the continuous analog of the statistical weight well-known for discrete systems with a countable number of states and finite number of particles. The factor \(W(\Omega)\) determines what I call the ensemble of phase subspaces, Sec. III.B. It accounts for states with "empty cells" which is the task more elaborated than that for discrete systems. The quantity \(\exp[h(X)]\) gives the number of microstates in a macrostate \(X\) for the given degree \(M\) of incomplete mixing (Sec. III.C). The inequalities (11) are incorporated in the expression for \(W(\Omega)\).

### B. Analog of combinatorics for a continuous Vlasov system: The ensemble of phase subspaces

Using combinatorics, the number of microstates corresponding to a given macrostate can be readily computed for a system with a countable number of states and finite number of particles. The macrostate consists of macroscopic but small boxes which in turn consist of many microscopic states. Some of macroscopic boxes can be empty, and the discrete combinatorics naturally accounts for macroscopic states with different occupation numbers in some boxes while keeping a number of other boxes empty. We need a generalization of this counting to a continuous system with uncountable number of states.

As outlined above, amount of liquid in a fixed macroscopic cell \(\sigma_i\), which plays the role of the occupation number, is a continuous variable ranging between 0 and \(\omega\). Consequently, the summation over the occupation numbers is replaced by the integration over \(d\Sigma(X)\), Eq. (14), which is the measure in the space \(\Phi\) of dimension \(i_m k_m\). Now the states with, say, \(l\) "empty boxes" are those with \(\mu_{ik}\) kept zero for some \(l\) subscripts \(i\) from the set \(\{i\}\), while for the other \((i_m - l)\) subscripts \(i'\) the variables \(\mu_{i'k}\) are varying from 0 to \(\omega\). But such states reside on the subspaces \(\Phi_{(i_m - l)}\) of the dimension \((i_m - l) k_m < \dim \Phi\), hence their measure in the space \(\Phi\) is zero (e.g., 2D surfaces and 1D lines in a 3D volume have measures 0). To account for the possible realizations of a continuous system on the subspaces of \(\Phi\), one
has to insert the singular density \( \prod_k 2\delta(\mu_{ik}) \) for any empty cell \( \sigma_i \) \((\int_0^\infty d\mu_{ik}\delta(\mu_{ik}) = 1/2 \) and the coefficient 2 makes it 1). As \( \{\sigma_i\} \) is the partition of the \( \mu \) space \( \Omega \), summation over all possible distributions of empty cells \( \sigma_i \) is equivalent to a summation over all subspaces of \( \Omega \), which is indicated by the argument of the singular density \( W \) in the integral \( \Gamma \).

Let \( \{i_1, i_2, ..., i_l\} \) be a subset of \( \{i\} \) consisting of some \( l \) possible values of \( i \). Then summing over all possible distributions of empty cells \( \sigma_i \) implies the following form of \( W \):

\[
W(\Omega) = \sum_{l=0}^{i_m} \sum_{i \in \{i_1, i_2, ..., i_l\}} \delta\{i_1, i_2, ..., i_l\},
\]

where

\[
\delta\{i_1, i_2, ..., i_l\} = \prod_{i \in \{i_1, i_2, ..., i_l\}} \prod_{k=1}^{k_m} 2\delta(\mu_{ik}) \times \prod_{i' \notin \{i_1, i_2, ..., i_l\}} \Theta\left(\mu_{i'} - \sum_{k'=1}^{k_m} \mu_{i'k'}\right).
\]

The density \( \delta\{i_1, i_2, ..., i_l\} \) is singular on the single subspace of \( \Omega \) consisting of \( l \) empty cells \( \sigma_i \) with \( i \in \{i_1, i_2, ..., i_l\} \). The inner summation in (15) is over all possible subspaces of \( \Omega \) consisting of \( l \) cells, and the second sum is over all possible \( l \). The structure of the singular density \( \delta\{i_1, i_2, ..., i_l\} \) is as follows: each empty cell \( \sigma_i, i \in \{i_1, i_2, ..., i_l\} \), has the delta functions \( \delta(\mu_{ik}) \) for all \( k \), and each nonempty cell \( \sigma_{i'}, i' \notin \{i_1, i_2, ..., i_l\} \), has the Heaviside step function \( \Theta \) which accounts for the inequality (11). We see that \( W(\Omega) \) is the singular density picking up all possible distributions of empty cells over the \( \mu \) space \( \Omega \). Equivalently, \( W(\Omega) \) provides the contributions from the arrangements of the system on all subspaces of the phase space \( \Phi \) and represents the ensemble of phase subspaces. Of course, the liquid of volume \( \mu(\Omega_0) \) cannot be inserted into a subspace of the volume lesser than \( \mu(\Omega_0) \) as the appropriate states require minimum \( \mu(\Omega_0)/\omega \) cells for its arrangement. But the inappropriate states on insufficient subspaces will be eliminated due to the constrains (9). This is similar to the use of the grand canonical ensemble for a system of \( N \) particles. This ensemble counts for the contributions from any number of particles and then the spare contributions are eliminated by the appropriate choice of the chemical potential. The analogy is even deeper. The grand partition function factorizes because as the restriction on \( N \) is removed, the states of the systems do not correlate. In the ensemble of phase subspaces, there is no restriction on the liquid’s size and, as a result, the density \( W(\Omega) \) factorizes over the cells \( \sigma_i \). It is not difficult to realize that

\[
W(\Omega) = \prod_{i=1}^{i_m} \prod_{k=1}^{k_m} 2\delta(\mu_{ik}) + \Theta\left(\mu_{i'} - \sum_{k'=1}^{k_m} \mu_{i'k'}\right).
\]

(17)
We will also need the expression of the form $W(\Omega) \exp(\sum_i a_i)$ where $a_i = 0$ for $\mu_{ik} = 0$. This expression also factorizes to give

$$W(\Omega) \exp \left( \sum_i a_i \right) = \prod_{i=1}^{i_m} \left[ \prod_{k=1}^{k_m} 2\delta(\mu_{ik}) + \Theta \left( \mu_{i'} - \sum_{k'=1}^{k_m} \mu_{i'k'} \right) e^{a_i} \right].$$  \hspace{1cm} (18)

It is important to stress that it is the accounting for empty cells $\sigma_i$ that results in the Fermi-like properties of the equilibrium states found below: without the delta functions in (17) the resulting DFs can only be Maxwell-Boltzmann-like.

The developed ensemble of phase subspaces solves the problem of picking up the states with empty macroscopic "boxes" for the continuous systems, but this is not yet the full analog of the discrete combinatorics. Now we need to find the number of microscopic states $\exp[h(X)]$ in a given macroscopic state $X$. For continuous systems this problem can be addressed using the ideas and results of the ergodic theory.

C. The Shannon-McMillan-Breiman theorem and the number of microstates in a given macrostate $X$

Assume that the incomplete mixing of the Vlasov flow is described by the parameter $M$ introduced in Sec.I.B. This means that there is a partition of the liquid body into $M$ elements which move independently. Then each mobile liquid element $\sigma_0k$ contains $M/k_m = M_k$ independent subelements $\sigma_{k,m}$, $m = 1, 2, ..., M_k$ (we assume that $M_k$ does not depend on $k$, but keep the subscript $k$ for clarity). In order to apply a statistical approach, we have to assume that a reference cell can have contributions from many different $\sigma_{k,m}$ with the same $k$. Thus we assume that

$$M \gg M/k_m = M_k \gg i_m \geq k_m.$$  \hspace{1cm} (19)

Consider first subelements $\sigma_{k,m}$ of the mobile element with a fixed subscript $k$. In the flow, $\sigma_{k,m}$ spread over the reference cells $\sigma_i$, resulting in nonzero intersections $\mu_{ik,m} = \mu(\sigma_{k,m} \cap \sigma_i)$. The quantity $\mu_{ik,m}$ is proportional to the probability $p_{i,km}$ that a point from the subelement $\sigma_{k,m}$ can be found in the reference cell $\sigma_i$. As all the subelements in the mobile $\sigma_k$ have the same density $f_k$, the probability $p_{i,km}$ is also the same for all the subelements and does not depend on $m$, i.e., $p_{i,km} = p_{ik}$. On the other hand, $\mu_{ik} = \sum_m \mu_{i,km}$, hence $p_{ik} \propto \mu_{ik}/M_k$. To
find the proportionality coefficient, we require that
\[ \sum_{i=1}^{i_m} p_{ik} = 1 \]  
and recall the relations (9) and (12), which gives
\[ p_{ik} = \mu_{ik}/\omega. \]  
Let us choose one point from each subelement: \( \tau_{k,m} \in \sigma_{k,m}, \) \( m = 1, 2, ..., M_k. \) Then, in line with the idea presented in Sec.I.B, the probability of a joint event that \( \tau_{k,1} \) enters \( \sigma_{i_1}, \tau_{k,2} \) enters \( \sigma_{i_2}, ..., \tau_{k,M_k} \) enters \( \sigma_{i_{M_k}} \), is the following product:
\[ P\{\tau_{k,1} \rightarrow \sigma_{i_1}|\tau_{k,2} \rightarrow \sigma_{i_2}|...|\tau_{k,M_k} \rightarrow \sigma_{i_{M_k}}\} = M_k \prod_{m=1}^{M_k} p_{ik}. \]  
Now we formalize the above properties of the \( M_k \) subelements of a mobile element \( \sigma_k \) spreading over the \( \mu \) space \( \Omega \). We have a finite set of states \( \{\sigma_i\} \) consisting of \( i_m \) elements \( \sigma_i \), and the probabilistic measure \( p_{ik} \) on \( \{\sigma_i\} \): \( p(\sigma_i) = p_{ik}. \) The argument of the above joint probability \( P \) is a sequence \( \omega_k \) which can be briefly presented as \( \omega_k = \{\sigma_{i_1}(\omega_k), \sigma_{i_2}(\omega_k), ..., \sigma_{i_{M_k}}(\omega_k)\}. \) One can say that this is a possible realization of a large experiment \( \omega_k \) consisting of \( M_k \) simple experiments: output of the first simple experiment is \( \sigma_{i_1}(\omega_k) \), of the second is \( \sigma_{i_2}(\omega_k) \), ..., and, finally, output of the last simple experiments is \( \sigma_{i_{M_k}}(\omega_k) \). The outputs of different simple experiments are independent so that the probability of the large experiment \( \omega_k \) is given by the product
\[ P\{\omega_k\} = \prod_{m=1}^{M_k} p_{ik}. \]  
The total number of different observable sequences \( \omega_k \) (i.e., with nonzero \( P\{\omega_k\} \)) gives the desired number of the microscopic states for a macroscopic state \( X \) by virtue of the relation (21).

The obtained stochastic process is exactly the well-known Bernulli map and the sequence \( \omega_k \) of an independent random variables with the values from \( \{\sigma_i\} \) is an element of its phase space [30, 31]. Strictly speaking, the phase space of Bernulli map consists of infinite sequences, but this is not an obstacle for our task of finding the total number of different observable sequences of a large length \( M_k \). This number is given by the Shennon-McMillan-Breiman theorem [30, 31] which, for our purpose, can be formulated as follows: for large
M_k, the probability $P\{\omega_k\}$ of any observable sequence $\omega_k$ is close to $\exp(-M_k h_k)$ where $h_k$ is the Kolmogorov-Sinai entropy of a given Bernulli map. The Kolmogorov-Sinai entropy is determined by the probability measures $p_{ik}$ on the space of state $\{\sigma_i\}$ and is given by the following formula:

$$h_k = -\sum_{i=1}^{i_m} p_{ik} \ln p_{ik}.$$  \hfill (24)

The total number of different observable sequences is equal to $1/P$. Thus, the number of microstates $g_k$ corresponding to the same macrostate $X$ with a fixed $k$ is

$$g_k = \exp\left(-M_k \sum_{i=1}^{i_m} p_{ik} \ln p_{ik}\right)$$  \hfill (25)

As subelements of the mobile elements $\sigma_k$ with different $k$ are independent, the total number of microstates corresponding to a given macrostate $X$ with any $k$ is the product $\prod_k g_k$. Then the quantity $h(X)$ in the statistical integral $\Gamma$ (13) is finally obtained in the form

$$h(X) = -M \sum_{i=1}^{i_m} \sum_{k=1}^{k_m} \mu_{ik} \ln(\mu_{ik}/\omega).$$  \hfill (26)

Now we know the $X$ dependence of all of the quantities in the integral $\Gamma$ (13) but the energy $H$. The function $H(X)$ is considered below.

**D. Energy as a function of the macroscopic state $X$**

For further consideration it is convenient to use a simplified notation for the coordinates $\tau$ in the $\mu$ space. A point $\tau$ from the reference element $\sigma_i$ will be denoted as $\tau_i$ or just $i$: $\tau_i \equiv i = (r_i, v_i)$. The 6D volume $\omega$ of $\sigma_i$ is a product $\omega = \omega_r \omega_v$ where $\omega_r$ and $\omega_v$ are its 3D volumes in the $r$ and $v$ spaces. The number of particles $n_i$ in a fixed element $\sigma_i$ of the $\mu$ space is

$$n_i = \sum_{k=1}^{k_m} \mu_{ik} f_k.$$  \hfill (27)

We also need the number of particles $n_{r_i}$ in the cell $\omega_{r_i}$ of the space of coordinates:

$$n_{r_i} = \sum_{v_i} n_i,$$  \hfill (28)

In the mean field approximation, the particle energy depends only on the macroscopic state $X$. Then the total energy $H$ of the system can be written in the form

$$H(X) = \sum_i \varepsilon_{ad,i} n_i + \frac{q^2}{2} \sum_{i,i'} w_{ii'} n_{r_i} n_{r_{i'}}.$$  \hfill (29)
where \( \varepsilon_{ad,i} = m v_i^2/2 + q \varphi_{ext}(r_i) \) is the additive particle energy and \( w_{ii'} = w(r_i - r_{i'}) \) is the pair interaction potential. The function \( q \varphi_{ext}(r_i) \) is the particle energy in the external potential \( \varphi_{ext} \) and \( q \) is the appropriate constant. For instance, \( q \) is the charge for charged particles and \( q \) is the particle’s mass \( m \) for gravitating particles. For certainty, we consider only a Coulomb-like potential. In the 3D geometry, the potential \( w_{ii'} \) has the form

\[
w_{ii'} = -\frac{\gamma}{|r_i - r_{i'}|},
\]

where \( \gamma > 0 \) is Newton’s constant for gravitational systems, and \( \gamma = -1 \) for Coulomb interaction. Now the integrand of \( \Gamma \) is defined as a function of \( \mu_{ik} \) and it remains to perform the integration.

IV. SOLVING THE STATISTICAL INTEGRAL \( \Gamma \)

A. Explicite expression for \( \Gamma \)

In the formulas which follow we omit insignificant factors. We use the following analytical representation of the delta functions entering \( \Gamma \) (13):

\[
\delta(E - H(X)) = \int_{-\infty}^{\infty} d\beta e^{i\beta(E-H(X))},
\]

\[
\delta\left(N_k - \sum_i \mu_{ik} f_k \right) = \int_{-\infty}^{\infty} d\zeta_k \exp i\zeta_k \left(N_k - \sum_i \mu_{ik} f_k \right).
\]

For the Heaviside theta function it is convenient to use the following identity:

\[
\Theta \left( \omega - \sum_k \mu_{i'k'} \right) = \int_{0}^{\omega} d\nu \delta \left( \nu - \sum_k \mu_{i'k'} \right).
\]

To get rid of the terms quadratic in the spatial occupation numbers \( n_{r_i} \) in the function \( \exp(-i\beta H(X)) \), we employ the Habbard-Stratonovitch transformation in the following form (33):

\[
\exp \left(-i\beta \frac{q^2}{2} \sum_{i,i'} w_{ii'} n_{r_i} n_{r_{i'}} \right)
\]

\[
\propto (i\beta)^{-m/2} \prod_{r_i} \int_{-\infty}^{\infty} d\psi_{r_i} \exp \left(q \sum_i n_{r_i} \psi_{r_i} + \frac{1}{2i\beta} \sum_{i,i'} w_{ii'}^{-1} \psi_{r_i} \psi_{r_{i'}} \right),
\]
where $\psi_{ri}$ is a field defined at each spatial point $r_i$ and $w_{ii'}^{-1}$ is the inverse to the matrix (30). We will need the operator $w_{ii'}^{-1}$ in the continuous limit $\omega \rightarrow 0$. In this limit the inverse operator $w^{-1}$ is defined as

$$\int dr'' w^{-1}(r - r'')w(r'' - r') = \delta(r - r')$$

and is known to be

$$w^{-1}(r - r') = \frac{1}{4\pi\gamma} \delta(r - r') \Delta r',$$

(35)

where $\Delta r'$ is Laplace’s operator with respect to $r'$. In the continuous limit the matrix $w_{ii'}^{-1}$ in (34) goes over into $\omega_r w_{ii'}^{-1}(r_i - r_{i'})$.

For brevity, we use the notations $D\zeta_k = \prod_k d\zeta_k$ and $D\psi_{ri} = \prod_i d\psi_{ri}$. Then, making use of the above results of this Section and formulas (18) and (26), the statistical integral (13) can be reduced to the following form:

$$\Gamma = \int_{-\infty}^{\infty} d\beta \beta^{-i m/2} \int D\zeta_k \int D\psi_{ri} \exp \left( i\beta E + i\zeta_k N_k + \frac{1}{8\pi\gamma i\beta} \int dr\psi \Delta \psi \right)$$

(36)

\[ \times \exp \sum_i \ln(1 + Z_i). \]

Here $Z_i$ is a functional of $\mu_{ik}$ and $\nu_i$:

$$Z_i = (1/\omega) \int_0^\omega d\nu_i \left( \prod_k \int_0^\omega d\mu_{ik} \right) \delta \left( \nu_i - \sum_k \mu_{ik} \right) e^{a_i(\alpha_{ik})},$$

(37)

where

$$a_i = \sum_{k=1}^{km} \left[ \alpha_{ik} \mu_{ik} f_k - M \mu_{ik} \ln(\mu_{ik}/\omega) \right],$$

(38)

$$\alpha_{ik} = q\psi_{ri} - i\beta\varepsilon_{ad,i} - i\zeta_k.$$  

(39)

Now we calculate $Z_i$.

**B. The function $Z_i$ for different degrees of mixing**

As $M$ and $\mu_{ik} f_k$ are large numbers, we can perform the $\mu$ integration using the steepest descent method. Eqs.(37) and (38) show that the result is determined by the maximum
of \( a_i \) with respect to \( \mu_{ik} \) under the condition that \( \sum_k \mu_{ik} = \nu_i \) for each \( i \). This extremum problem is solved by

\[
\mu_{ik} = \frac{\nu_i \exp(\alpha_{ik}N_k/M_k)}{z_i},
\]

where \( N_k = \omega f_k \), \( M_k = M/k_m \), and \( z_i \) is the local partition function

\[
z_i = \sum_{k=1}^{k_m} \exp(\alpha_i N_k/M_k).
\]

(41)

Changing to \( x = \nu_i k_m \), the result of the \( \mu \) integration is

\[
Z_i = \int_0^1 dx \exp [a_i(\mu_{ik}) + b_i],
\]

(42)

where

\[
b_i = \ln \sqrt{\frac{(2\pi)^{k_m}}{\det \left| \frac{\partial^2 a_i}{\partial \mu_{ik} \partial \mu_{ik'}} \right|}}.
\]

(43a)

The simple calculation gives

\[
a_i(\mu_{ik}) = M_k x \ln(z_i/x),
\]

(44)

\[
\frac{\partial^2 a_i}{\partial \mu_{ik} \partial \mu_{ik'}} = -\delta_{kk'}/(\mu_{ik} \omega),
\]

and

\[
b_i = -\frac{k_m}{2} \left[ \ln(Mk_m/2\pi) + \ln(z_i/x) \right] + \sum_{k=1}^{k_m} \frac{\alpha_{ik} N_k}{M_k}.
\]

(45)

In order to proceed, we have to make an assumption concerning the relative values of the number \( M_k \) of independent subelements and the number of particles \( N_k \) in a mobile element \( \sigma_k \). Clearly, if \( M_k \gg N_k \), the mixing is strong whereas if \( M_k \ll N_k \) the mixing is practically absent and the system is at best ergodic. The intermediate case \( M_k \sim N_k \) corresponds to an incomplete mixing. We will consider these cases separately. Note that the above steepest descent method can be used only if \( M_k \) is large, hence the ergodic case without mixing, \( M \sim 0 \), has to be considered individually, Sec. V.B.

We first proceed with the case \( M_k \gg N_k \) and \( M_k \sim N_k \). The inequality (19) shows that the last two terms in \( b_i \) can be neglected compared to \( a_i \) so that \( b_i \approx -\frac{k_m}{2} \ln(M/2\pi \omega) \). This constant can be absorbed into \( \zeta_k \) [see Eqs. (38) and (39)], and (42) reduces to the form

\[
Z_i = \int_0^1 dx e^{a_i(x)}.
\]

(46)
Strong mixing. Consider first the case of strong mixing when $M_k \gg N_k$. In this case $a_i$ can be computed using the small parameter $N_k/M_k$. First we consider the function $z_i^{M_k}$ which is present in $a_i$:

$$z_i^{M_k} = \left[ \sum_{k=1}^{k_m} \left( 1 + \frac{\alpha_{ik}N_k}{M_k} \right) \right]^{M_k} \rightarrow k_m^{M_k} \exp \langle \alpha N \rangle_i,$$

where

$$\langle \alpha N \rangle_i = \sum_{k=1}^{k_m} \alpha_{ik} \frac{N_k}{k_m}$$

is the average of $\alpha_{ik}N_k$ over all $k$. Substituting this to $a_i$, one has

$$Z_i = \int_0^1 dx \exp[\langle \alpha N \rangle_i x - M_k x \ln(x/k_m)].$$

The exponent in the above integral does not have a maximum inside the integration interval, but at its right end $x = 1$ the exponent has a large positive derivative and the maximum value. Then, neglecting terms $\sim N_k/M_k$, one obtains:

$$Z_i = \exp(\langle \alpha N \rangle_i + c),$$

where $c = \ln(k_m^{M_k}/M_k \ln k_m)$ is a constant.

Incomplete mixing. For $M_k \sim N_k$, the simplification resulting in the expression (47) does not apply, and one has to deal with the local partition function in the form (41). Then one has:

$$Z_i = \int_0^1 dx z_i^{M_k} x^{-M_k x}.$$

The expressions (50) and (51) for $Z_i$ obtained for different degrees of mixing are essential for the final results. Now we first present the result for strong mixing, then for the ergodic case without mixing, and finally for the intermediate mixing.
V. THE EQUILIBRIUM STATES OF A VLASOV LIQUID

A. Maximum entropy state for strong mixing

In the statistical integral (36) we integrate by part the term \( \psi \Delta \psi \), replace the summation over \( i \) by integration over \( d\tau = drdv \), and substitute expression (50) for \( Z_i \). The result is

\[
\Gamma = \int_{-\infty}^{\infty} d\beta \beta^{-i/2} \int_{-\infty}^{\infty} D\zeta_k \int_{-\infty}^{\infty} D\psi \exp \tilde{S},
\]

(52)

where

\[
\tilde{S} = i\beta E + \sum_k i\zeta_k N_k + \int dr \left[ -\frac{(\nabla \psi)^2}{8\pi\gamma i\beta} + \int d\psi \ln(1 + e^{(\alpha N/c)}) \right].
\]

(53)

This integral can be solved by the steepest descent method because the terms of \( \tilde{S} \) are proportional to the large total number of particles \( N \) (we shall see that the first term in the bracket is proportional to the total interaction energy). In view of the value of \( i\beta \) at the saddle point which is expected to be real, the integration order in (52) is essential. The \( \psi \) integral must be performed first for imaginary \( i\beta \) as then it converges both for positive and negative \( \gamma \). Then the integration contours in \( \beta \) and \( \zeta \) can be deformed down as to cross the imaginary axes at negative values.

The extremum configuration which gives the dominating contribution to \( \Gamma \) is determined by the three equations:

\[
\frac{\delta \tilde{S}}{\delta \psi} = \frac{\partial \tilde{S}}{\partial i\beta} = \frac{\partial \tilde{S}}{\partial (i\zeta_k N_k)} = 0.
\]

(54)

It is straightforward to realize that the third equation for the quantity \( \zeta_k N_k \) is exactly the same for all \( k \), hence \( \zeta_k N_k \) does not depend on \( k \). Then the best way to present and consider the above three equations is to change to the following variables:

\[
i\beta = \frac{1}{T}, \quad \psi = -i\beta \varphi, \quad -i\zeta_k + c/\pi = \frac{\eta}{T},
\]

(55)

where \( \pi = N/k_m \) is the average number of particle per a mobile element \( \sigma_k \). We also introduce the average density of the liquid \( \bar{f}_0 = N/\mu(\Omega_0) \) where \( \mu(\Omega_0) \) is its 6D volume in the natural units, and the total particle energy \( \varepsilon = \varepsilon_{ad} + q\varphi \):

\[
\varepsilon = \frac{mv^2}{2} + q(\varphi_{ext} + \varphi).
\]
Making use of Eqs. (55) and (48) and $\varepsilon$, one has

$$\langle \alpha N \rangle_i + c = \frac{n - \varepsilon_i}{T},$$

(56)

Now the equations (54) can be reduced to the following system:

$$\nabla \phi = 4\pi \gamma q \int d\mathbf{v} f,$$

(57)

$$E = \int d\mathbf{r} \left[ -\frac{(\nabla \phi)^2}{8\pi \gamma} + \int d\mathbf{v} \varepsilon \right],$$

(58)

$$\int d\mathbf{v} d\mathbf{r} f = N.$$  

(59)

where

$$\overline{f}(\mathbf{r}, \mathbf{v}) = f_0 \frac{\exp \frac{n - \varepsilon(\mathbf{r}, \mathbf{v})}{T}}{1 + \exp \frac{n - \varepsilon(\mathbf{r}, \mathbf{v})}{T}}.$$  

(60)

The above equations have a clear interpretation. Eq. (57) is the Poisson equation for the mean field potential $\phi$. Eq. (58) expresses the fact that the energy of the system is equal to $E$. Eq. (59) shows that the function $\overline{f}$ is normalized on the total number of particles $N$. Finally, this function $\overline{f}$ is the equilibrium DF in the $\mu$ space. For $\gamma$ being Newton’s constant and $q$ being mass of a particle, the system is gravitational; While for $\gamma = -1$ and $q$ being the electrostatic charge, this is a Coulomb system.

We see that the DF is exactly of the Fermi-Dirac form, different levels $f_k$ of the initial DF appear only in the mixture $\overline{n} = \sum_k f_k \omega/k_m = N/k_m$. This is in contrast to the LBDF (see Eq. (64) below) which is the sum of contributions from different levels $f_k$ each with its own chemical potential and distribution over the energy values. The obtained result implies that in the case of strong mixing, there are two relaxation processes, fast and slow. The fast consists of creation and maintaining of a perfect mixture with the density $N/\mu(\Omega_0)$, and the second, slow process consists of distributing this mixture over the possible energies without changing the local composition of different levels. The fast relaxation is driven by the fine scale gradients of the mean field which at this scale must be violent indeed. The slow relaxation is driven by large scale gradients. This is a highly idealized relaxation with strong mixing, but if it is assumed, then the DF is given by Eq. (60).

B. Ergodicity without mixing and its relation to the LBDF

Now we consider the result under the assumption that the flow is ergodic without mixing, i.e., $M = 0$. As was explained in Sec.I.B, an ergodic nonmixing system does not tend to a
time independent state. Therefore, all we can do is to find the number of all states through which the system will pass over a very large time. In absence of mixing the shape of area $\Omega_0(0)$ occupied by liquid changes only on a large scale, the small moving elements $\sigma_k$ do not spread, fine scale filaments are not produced, and the boundary of $\Omega_0(t)$ does not develop into a long and intricate line. This means that, with a high accuracy, the occupation number in a reference cell $\sigma_i$, when it is traversed by a $k$-th mobile element $\sigma_k$, is $f_k \omega$. Thus, the occupation number in any reference cell runs over the values $f_k \omega, k = 1, 2, \ldots, k_m$. Hence, summation over these numbers must replace the $\mu$ and $\nu$ integration in the formula for $Z_i^{(37)}$. This gives:

$$Z_i = \sum_{k=1}^{k_m} \exp(\alpha_{ik} f_k \omega).$$

(61)

Now $\zeta_k$ depends on $k$ so that instead of third line in Eq.(54) we put $-i \zeta_k = \frac{\eta_k}{f_k \omega}$. Now the sum over $k$ is transformed to the integral over the initial area $\Omega_0$ of the $\mu$ space occupied by the liquid and the subscript $k$ is changed to the argument $\tau_0 \in \sigma_k$. Then $\eta_k + \text{const} \to \eta(\tau_0)$ and $Z_i^{(61)}$ is replaced by the following integral

$$Z(r, v) = \int_{\Omega_0} d\tau_0 \exp \left[ \frac{\eta(\tau_0) - \epsilon(r, v)}{T} f_0(\tau_0) \omega \right].$$

(62)

Making use of this $Z_i$ in $\tilde{S}$ (53) instead of $e^{(\alpha N) + c}$ and solving the statistical integral (56) as before, one arrives at the following system of equation. The equations $\delta \tilde{S} / \delta \psi = 0$ and $\partial \tilde{S} / \partial i \beta = 0$ are obtained in the form (57) and (58) as in the case of strong mixing, but the last equation and the DF are different. The third equation in (54) is now

$$\int dV dr \frac{\exp \left[ \frac{\eta(\tau_0) - \epsilon(r, v)}{T} f_0(\tau_0) \omega \right]}{1 + \int_{\Omega_0} d\tau_0 \exp \left[ \frac{\eta(\tau_0) - \epsilon(r, v)}{T} f_0(\tau_0) \omega \right]} = 1,$$

(63)

and the equilibrium DF is

$$f_{LB}(r, v) = \frac{\int_{\Omega_0} d\tau_0 f_0(\tau_0) \exp \left[ \frac{\eta(\tau_0) - \epsilon(r, v)}{T} f_0(\tau_0) \omega \right]}{1 + \int_{\Omega_0} d\tau_0 \exp \left[ \frac{\eta(\tau_0) - \epsilon(r, v)}{T} f_0(\tau_0) \omega \right]}.$$

(64)

The DF is presented in the form valid both for the natural units when $\omega = \mu(\Omega_0)/k_m$ and for the dimensionless $\omega = 1/k_m$ when $\mu(\Omega_0) = 1$. The label LB indicates that DF (64) has the form identical to the DF derived by Lynden-Bell. The relation of this result with the assumed ergodicity without mixing is in line with Lynden-Bell’s idea of shape-preserving
macroparticles. In Eqs. (63) and (64), the summation over $k$ is replaced by the integration over the initial domain $\Omega_0$ occupied by the 6D liquid in the $\mu$ space. The quantity $\eta(\tau_0)$ is now the function of $\tau_0 \in \Omega_0$ and plays the role of a chemical potential for the particles from the element $\sigma_k$ which contains the point $\tau_0$.

If $f_0(\tau_0) = \text{const}$, then the LBDF goes over into the Fermi-Dirac distribution (60). However, in both cases of uniform and nonuniform $f_0$ the physical meaning of the DF for ergodicity alone is very different from DF for a mixing flow. In the last case, measuring density of liquid in the $\mu$ space, one will obtain the time independent value $\overline{f}$. In the ergodic case, the same measurements will give results which are different at different times, and the value $\overline{f}_{LB}(r,v)$ can be obtained only by computing the average over a very long time. Thus, our calculation of the equilibrium DF in the pure ergodic case is formal and must be understood in the informal context above.

There is an interesting question, however, whether the spatial density, which is obtained by averaging of $\overline{f}_{LB}(r,v)$ over $v$, is time dependent or not. If on the integration over velocities the time dependence is eliminated, then the results of measurement of the spatial density can be predicted from the unpredictable $\mu$ space density (64). This would make this DF of a high importance even in the ergodic case. In a spatially uniform ergodic system, the average over velocities is equivalent to the time average and therefore is time independent. But this average is a constant independent of the spatial coordinates. I am not aware of the analog of this theorem for a spatially inhomogeneous system. Thus, a possible time independence of the spatial density in the ergodic system without mixing seems to be interesting, but a hypothetical issue.

C. Maximum entropy state for incomplete mixing

In the case of incomplete mixing with $M_k \sim N_k$ we have to use $Z_i$ in the form (51). From the above consideration we know that, in terms of the variables $\varphi$, $\eta_k$, $T$, the local partition function $z_i$ (41) has the form

$$ z_i = \sum_k \exp \left( \frac{\eta_k - \varepsilon_i}{T} \frac{N_k}{M_k} \right) \quad (65) $$

$$ = \int_{\Omega_0} d\tau_0 \exp \left[ \frac{\eta(\tau_0) - \varepsilon(r,v)}{T} \frac{f_0(\tau_0)\omega}{M_k} \right], $$
where \( i \equiv \tau = (r, v) \). Now we use \( Z_i \) with this \( z_i \) in \( \tilde{S} \) instead of \( e^{(\alpha N)_{+c}} \) and solve the statistical integral (52). The equations \( \delta \tilde{S}/\delta \psi = 0 \) and \( \partial \tilde{S}/\partial i \beta = 0 \) have the form (57) and (58) as in the cases of strong mixing and ergodicity. The equation \( \partial \tilde{S}/\partial (i \zeta_k N_k) = 0 \) now is

\[
\int dv dr \int_0^1 dx \exp \left[ \frac{\eta(\tau_0) - \varepsilon(r,v)}{T} f_0(\tau_0) \omega x \right] x^{-M_k x + 1} \frac{M_k x}{x^{-M_k x}} = 1
\]

and has to be satisfied for all \( \tau_0 \in \Omega_0 \). The DF obtains in the form

\[
\tilde{f}_M(r,v) = \frac{\int_0^1 dx \int_{\Omega_0} d\tau_0 f_0(\tau_0) \exp \left[ \frac{\eta(\tau_0) - \varepsilon(r,v) f_0(\tau_0) \omega}{M_k} \right] x^{-M_k x + 1}}{1 + \int_0^1 dx \int_{\Omega_0} d\tau_0 \exp \left[ \frac{\eta(\tau_0) - \varepsilon(r,v) f_0(\tau_0) \omega}{M_k} \right] x^{-M_k x + 1}}.
\]

The DF \( \tilde{f}_M \) desribes the statistics of independent subelements, which are enumerated by \( \tau_0 \in \sigma_k \subset \Omega_0 \) and which have the number of particles \( N_k/M_k = f_0(\tau_0)/M \). We stress, however, that associating these subelements with macroparticles would be incorrect as each subelement gets smeared over the accessible \( \mu \) space.

Let us compare the equilibrium DF (67) for the \( M \) degree mixing with the DF (60) for strong mixing and the DF (64) for a pure ergodic flow. As discussed in Sec. V.A, there are two relaxation processes in the strong mixing flow. The fast relaxation prepares the perfect, energy independent mixture of different density levels and the slow relaxation distributes this mixture over the energy values. In the purely ergodic flow, the mixture is not produced and the frequency of finding a given liquid element at a point \( \tau \) depends solely on its energy at this point. In a not perfectly mixing flow, the energy and mixing entropy compete. This is well seen from the integrand in (67) which, for \( f_0 \sim const \), has the form

\[
\exp \left( \frac{-\varepsilon}{T} N_k x - M_k x \ln x \right),
\]

clearly pointing at this competition.

To illustrate the peculiarity of the DF (67), we resort to the particular case of a constant liquid’s density independent of \( k \), \( N_k = f_0 \omega = n_0 = const \) (known as a ”water bag” initial condition). In this case \( \tilde{f}_M \) reduces to the following form:

\[
\tilde{f}_M = f_0 \frac{\int_0^1 dx \exp \left[ \frac{\eta - \varepsilon(r,v) n_0 x}{T} \right] x^{-M_k x + 1}}{1 + \int_0^1 dx \exp \left[ \frac{\eta - \varepsilon(r,v) n_0 x}{T} \right] x^{-M_k x}}.
\]
As the DF $\bar{f}_M$ describes the statistics of subelements with $n_0/M_k$ particles, it is instructive to compare this DF with the Fermi-Dirac DF $f_{FD}$ for subelements which contain the same number of particles. The two functions to be compared are

$$\bar{f}_M = \frac{\int_0^1 dxe^{\phi x x - M_k x + 1}}{1 + \int_0^1 dxe^{\phi x x - M_k x}},$$  \hspace{1cm} (70)$$

$$f_{FD} = \frac{e^{\phi/M_k}}{1 + e^{\phi/M_k}},$$  \hspace{1cm} (71)$$

where $\phi = \frac{\eta - \varepsilon(r,v)}{T} n_0$. In Fig. 1, the two DFs have been plotted vs $\phi$ for $M_k = 500$. We see that the dependence on the single argument $\phi$ in the DF $\bar{f}_M$ is much sharper than in the Fermi-Dirac DF for the same $M_k$. The smaller $M_k$ the closer to one another the two functions become, but for small $M$ the above theory loses its accuracy and is inapplicable for very small $M_k \sim 1$. For very large $M_k$, the DF $\bar{f}_M$ goes over into the DF (60).

Further analysis of possible physical implications of the maximum entropy states with incomplete mixing deserves a separate study.

D. Some thermodynamics: Entropy, energy, temperature.

The three statistics obtained are Fermi-Dirac-like which allows one to make use of the well-known results from the theory of an ideal Fermi gas [35]. Let us first realize that $\ln \Gamma$, the entropy $S$ of the system, in the limit $N \to \infty$ is equal to the expression for the exponent in formula (36), calculated at the solution of the equation $s (54)$. Indeed, on the $\psi$ integration the prefactor $\beta^{-i m/2}$ cancels out; for large $N$, the additional exponent resulting from the Gauss integral over $d\beta D\zeta_k$, scales as $-\ln N$ and is negligible as compared with the principal term which scales as $N$ (or even as $N^2$ as the potential energy of systems with a long-range interaction is proportional to $N^2$). As a result, the entropy $S$ takes the following form:

$$S(E, N_k) = \frac{E}{T} - \sum_{k=1}^{k_m} \eta_k N_k + \int dr \left[ -\frac{\left(\nabla \varphi\right)^2}{8\pi \gamma T} + \int dv \ln(1 + Z) \right],$$  \hspace{1cm} (72)$$

where $Z$ is defined in (51).

The solution of the equations (54) depends on the parameter $E$ and is a functional of the initial DF $f_0(\tau_0)$ and the external field $\varphi_{ext}(r)$. In the discrete version, the functional
FIG. 1: Dependence of the incomplete mixing DF (70) (solid line) and the Fermi-Dirac DF (71) (dashed line) on $\phi$ for mixing degree $M_k = 500$. (a) - linear plot, (b) - semi logarithmic plot.

dependence on $f_0$ is replaced by the dependence on the $k_m$ parameters $N_k$:

$$
T = T(E, N_k, \varphi_{ext}),
\eta_n = \eta_n(E, N_k, \varphi_{ext}), 
(73)
\varphi(r) = \varphi(r|E, N_k, \varphi_{ext}),
\quad k, n = 1, ..., k_m.
$$
It is now easy to see that, by virtue of the equations (54), one has

$$\frac{\partial S}{\partial E} = \frac{1}{T}, \quad (74)$$

$$\frac{\partial S}{\partial N_k} = -\frac{\eta_k}{T}, \quad (75)$$

$$\frac{\delta S}{\delta \varphi_{ext}(\mathbf{r})} = \frac{q \Delta \varphi(\mathbf{r})}{4\pi \gamma}, \quad (76)$$

where the expression in the rhs of (76) is just the spatial density of particles \(n(\mathbf{r})\) at point \(\mathbf{r}\) times \(q/\gamma\). Thus,

$$TdS = dE - \sum_k \eta_k dN_k + \frac{q}{4\pi \gamma} \int d\mathbf{r} \delta \varphi_{ext} \Delta \varphi$$

$$= dE - \int_{\Omega_0} d\tau_0 \eta(\tau_0) \delta f_0(\tau_0) + \left(\frac{q}{\gamma}\right) \int d\mathbf{r} n \delta \varphi_{ext},$$

(77)

We see that \(T\) can be associated with the global temperature and \(\eta_k\) with the global chemical potential of the particles of the \(k\)-th kind (which is determined by belonging to the liquid element with the density \(N_k/\omega\)). In an additive system with short-range interaction, these two fundamental thermodynamic quantities are the characteristics of any subsystem, but in our nonadditive system they can be attributed only to the total system and in this sense are global. At the same time, the response to a small variation of the external field, \((q/\gamma) n(\mathbf{r})\), is local as in systems with a short-range interaction. The difference is however that the density in systems with short-range interactions is spatially dependent only in a nonzero external field whereas in systems with long-range interaction \(n(\mathbf{r})\) can be inhomogeneous even for zero external field.

It is possible to simplify the expression for entropy. For convenience, let us put \([\eta(\tau_0) - q(\varphi + \varphi_{ext})]/T = u\). Then the general expression for \(Z\) (51) is

$$Z = \int_0^1 dx \left\{ \int_{\Omega_0} d\tau_0 \exp \left[ \left( u - \frac{mv^2}{2T} \right) \frac{f_0(\tau_0)\omega}{M_k} \right] \right\}^{M_kx - M_kx}.$$  

(78)

Consider the integral

$$I = \int d\mathbf{v} \ln(1 + Z).$$  

(79)

Integrating by parts, one obtains

$$I = \frac{2}{3} \int d\mathbf{v} \frac{mv^2}{2T} f_M,$$  

(80)
where \( \bar{f}_M \) is the DF (67). Substituting this expression in \( S \), one gets

\[
TS = E - \sum_{k=1}^{k_m} \eta_k N_k + \int dr \left[ -\frac{(\nabla \varphi)^2}{8 \pi \gamma} + \frac{2}{3} \int dV \frac{mv^2}{2} \bar{f}_M \right].
\]  

(81)

It is now obvious that the spatial integral is equal to the total intrinsic potential energy \( U \) plus \((2/3)\) of the total kinetic energy \( K \). Now we make use of the 3D virial condition for our system \[4\]. The total energy \( E = U + U_{ext} + K \) where \( U_{ext} \) is the system’s energy in the external field:

\[
U_{ext} = \left( \frac{q}{\gamma} \right) \int drr \varphi_{ext}.
\]  

(82)

The most natural assumption is that the interaction of the particles with the external field is also Coulomb-like and thus a homogeneous function of order \(-1\) \[36\]. Then, in a stationary state, the 3D virial relation tells one that \( 2K + U + U_{ext} = 0 \) so that \( K = -E \). Then \( U + (2/3)K = (4/3)E - U_{ext} \) and the entropy reduces to the following form:

\[
TS = \frac{7}{3} E - \int_{\Omega_0} d\tau_0 \eta(\tau_0) f_0(\tau_0) - U_{ext}.
\]  

(83)

This formula is also valid for the ergodic case without mixing when the DF is formally equal to the LBDF (64). In the case of strong mixing, this relation reduces to

\[
TS = \frac{7}{3} E - \eta N - U_{ext}.
\]  

(84)

To demonstrate the effect of nonadditivity of our Coulomb-like long-range interaction in 3 dimensions, we rewrite this formula in the following form:

\[
\eta N - \left( \frac{4}{3} E + 2U_{ext} \right) = E - TS + U_{ext},
\]  

(85)

and compare it with the expression for the Gibbs free energy \( \eta N = E - TS - PV \) (Gibbs-Duham equation) where the work of the external force \(-PV\) corresponds to the energy in the external field \( U_{ext} \) (work done by the external source). We see that the correspondence between the two formulas is violated by the term \(-\left( \frac{4}{3} E + 2U_{ext} \right)\) in the lhs of our formula (85). Very recently Latella et al \[37\] showed that the standard Gibbs-Duham equation is modified by the long-range interaction also in systems which have already attained the final equilibrium state due to the collisions. However, the direct comparison between the collisional systems considered in \[37\] and the quasi-equilibrium collisionless systems considered in this paper cannot be made. The reason is that, in collisional systems, the integrals over
velocities and coordinates are independent whereas, in collisionless systems, they are not since the local conservation laws of Vlasov’s equation impose the connection between volumes of the liquid elements in the velocity and coordinate spaces and, in the end, give rise to Eq. (81).

In the case of a Fermi gas, chemical potential at $T = 0$ is the Fermi energy $\varepsilon_F$. As our statistics are Fermi-like, one obtains the relation between the total energy and the chemical potential $\eta_F(\tau_0)$ of the fully degenerate 6D liquid:

$$E = 3 \frac{7}{7} \sum_{k=1}^{k_m} \eta_{F,k} N_k + U_{ext}$$

(86)

$$= 3 \frac{7}{7} \int_{\Omega_0} d\tau_0 \eta_F(\tau_0) f_0(\tau_0) + U_{ext}.$$

In the case of strong mixing and $U_{ext} = 0$, this reduces to

$$E = \frac{3}{7} \eta_F N,$$

(87)

while, for Fermi gas, this relation is $E = (3/5)\varepsilon_F N$. The difference between our case of the long-range interaction and Fermi-gas is however not just in the coefficients: $\varepsilon_F$ is a function of the particle density whereas $\eta_F$ depends on the total number of particles $N$ (expectedly $\eta_F \sim N$). We note that in the fully degenerate state the 6D liquid is stratified. The liquid is poured in a 6D ”glass” produced by the mean field potential and fills the 6D volume which is exactly equal to the volume of the initial liquid’s volume $\mu(\Omega_0)$. At the very bottom is the liquid element with the maximum $N_k$ which occupies the volume $\omega_k$; the layer of liquid with the next large $N_k$ lies over the liquid with the largest $N_k$, and so on. The quantity $\eta_{F,k}$ is the energy of a particle at the top of the layer with $N_k$. It is worth noting that this state is the absolute minimum energy state and can be realized without any mixing just via precipitation of heavier fractions closer to the bottom of the potential well.

VI. CONCLUSION.

Almost fifty years ago Linden-Bell assumed that the 6D liquid described by the Vlasov equation can relax to some maximum entropy state [11]. This required a very strong and highly idealized assumption of a ”good” mixing in the liquid’s phase space. Moreover, any mixing implies that liquid elements spread over the accessible $\mu$ space and thus do not
preserve their initial shapes. In spite of that Linden-Bell modeled the liquid by a large set of independent shape-preserving macroparticles in an obvious contradiction to the mixing character of the flow. Nevertheless, this result has had a profound influence both on the astrophysics and statistical mechanics. Over the years, there has been a number of papers where the equilibrium maximum entropy state was addressed by means of different technics and in relations to different systems with long-range interaction [13–19]. But the main peculiarity of a liquid motion, a flow, which must be mixing if we expect relaxation to some statistical equilibrium and thus making very small liquid elements spread over the entire space, has not been incorporated. In other words, the actual 6D liquid has not been adequately described. In this paper, based on the ideas of the ergodic theory, I developed an equilibrium statistical approach to an actual 6D liquid in the $\mu$ space and incorporated all the integrals of motion of the Vlasov equation, associated with the point-wise incompressibility of the liquid. I specified the assumption of mixing. It can be very strong, practically a K mixing when all points of a liquid are statistically independent. Or mixing can be incomplete, e.g., approximately one independent liquid element per particle. And mixing can be very weak, equivalent to pure ergodicity when the number of independent liquid elements is very small. In the case of mixing, the flow relaxes to certain equilibrium state which depends on the mixing strength. For a very strong mixing, this state is given exactly by the Fermi-Dirac DF (60), in which different levels of the initial DF are perfectly mixed and do not appear individually. For incomplete mixing, different levels manifest themselves and, in addition, the DF (67) depends on the density of the statistically independent liquid elements. It is found that the LBDF (64) formally corresponds to a pure ergodic flow which does not result in relaxation to a time independent state. The phase space of Vlasov’s liquid where its macrostate is given by a single point (in contrast to the $\mu$ space where the liquid is described by its multipoint shape) is introduced. The integrals of motions are enumerated by the points of the support of the initial DF. The crucial result of the paper is expressed by the formulae (18) and (24) - (26) which give the number of microscopic states in a given macroscopic state of a liquid. This result is based on the two ideas: a) introduction of the ensemble of phase subspaces which is a continuous analog of accounting for distributions of particles realized on subspaces of dimensions smaller than that of the total space of states; b) connecting microscopic states in a given macrostate with the sequences produced by appropriate Bernulli mapping for which the number of observable sequences (i.e., those with
nonvanishing probability) is given by the Shannon-MaMillan-Breiman theorem. The relation between the entropy, total energy, and the initial DF is derived. The result of this paper clarifies status of maximum entropy states which, under any possible assumptions of the chaoticity of Vlasov’s flow, can, in principle, be achieved. The introduced idea of incomplete relaxation connected to the number of statistically independent liquid elements is probably just one approach and does not exhaust all possibilities. Further studies in this direction can clarify the status of the incomplete relaxation and the DF (67) in a more general context. The presented approach can be viewed as a step in development of the statistical mechanical technics adequate for dealing with continuous, distributed, and liquid-like systems. Specific effects that may be derived from the DF (67) will be studied in the future.

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[1] T. Padmanabhan, Phys. Rep. 188, 285 (1990).
[2] Dynamics and Thermodynamics of Systems with Long-Range Interaction, Edited: T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens, Lecture Notes in Physics, Vol. 602 (Springer, Berlin, 2002).
[3] A. Campa, T. Dauxois, and S. Ruffo, Phys. Rep. 480, 57 (2009)
[4] Y. Levin, R. Pakter, F. B. Rizzato, T. N. Teles, F. P. C. Benetti, Phys. Rep. 535, 1 (2014).
[5] M. Antoni and S. Ruffo, Phys. Rev. E 52, 2361 (1995).
[6] V. Latora, A. Rapisarda, S. Ruffo, Phys. Rev. Lett. 80, 792 (1998).
[7] V. Latora, A. Rapisarda, C. Tsallis, Phys. Rev. E 64, 056134 (2001).
[8] Y.Y. Yamaguchi, J. Barr, F. Boucheta, T. Dauxoisc, S. Ruffo, Physica (Amsterdam) 337A, 36 (2004).
[9] K. Jain, F. Bouchet, D. Mukamel, J. Stat. Mech.: Theory and Experiment, P11008 (2007).
[10] Y. Levin, R. Pakter, F. Rizzato, Phys. Rev. E 78 021130 (2008).
[11] D. Lynden-Bell, Mon. Not. Roy. Astr. Soc. 136,101 (1967).
[12] T. M. R. Filho, A. Figueiredo, and M. A. Amato, Phys. Rev. Lett. 95, 190601 (2005).
[13] V.M. Pergamenshchik, Phys. Lett. 113A, 225 (1985).
[14] J. Miller, Phys. Rev. Lett. 65, 2137 (1990).
For instance, the distribution of an ideal gas over its accessible $6N$ dimensional phase space is homogeneous, but the Maxwell-Boltzmann DF, which is the DF in its $6D$ phase space, is inhomogeneous.

Usually, the Hubbard-Stratonovich transformation is based on a real Gauss integral. However, our formula (34) is based on a complex Gauss integral, see, e.g., (34), to incorporate both positive and negative sign of $\gamma$ in the interaction potential (30).

N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).

L.D. Landau and E.M. Lifshitz, *Statistical Physics, Part I* (Pergamon Press, Oxford, 1980).

Of course, it is difficult to imagine how $E$ can be fixed in an external field which can change the system’s energy. The external field for such ”isolated system” plays the role analogous to external pressure. In this sense, the inclusion of an external field is somewhat formal.
[37] I. Latella, A. Pérez-Madrid, A. Campa, L. Casetti, and S. Ruffo, Phys. Rev. Lett. 114, 230601 (2015).