On SPDE and backward filtering equations for SDE systems (direct approach)

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Abstract

A direct approach to linear backward filtering equations for SDE systems is proposed. This preprint is a corrected version of the paper 1995 in the LMS Lecture Notes combined with another paper by the author on the direct approach to linear SPDEs for SDEs.

1 Introduction

Filtering theory is one of the main sources of stochastic partial differential equations (SPDE’s). In this paper the filtering problem for the simplest two-dimensional stochastic differential equation system is considered,

\[ dX_t = f(X_t)dt + dw_1^t, \quad X_0 = x, \]

\[ dY_t = h(X_t)dt + dw_2^t, \quad Y_0 = y, \]

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where functions $f$ and $h$ are smooth and bounded, $w^1$ and $w^2$ are independent standard Wiener processes on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$; initial data $X_0 = x$ and $Y_0 = y$ are assumed non-random. (In fact, both may be distributed being mutually independent with $(w^1, w^2)$ and for the component $Y$ this may be helpful in filtering, but we do not pursue this goal here.) The problem is to describe the estimate of unobservable process $X_t$ via observable component $Y_t$, $0 \leq s \leq t$, which is optimal in the mean–square sense, i.e.,

$$m_t \equiv m_{0,t} = \mathbb{E}[g(X_t) | \mathcal{F}_t^Y], \quad t \geq 0.$$  

In fact, the answer is known even for more general situations, see [8]. In particular, $m_t$ may be represented via backward stochastic differential equations, which makes sense if we are interested in an optimal estimation for some fixed time $t$; in this case we should find the solution of our backward SPDE and substitute there the trajectory of our observation process.

In this paper we present a direct approach to such a representation, using a similar idea for an equation without filtering, that is, for a completely observed SDE trajectory. This preprint is an improved version of the paper [10] presented along with the main lines of the calculus from [9]. The matter is that the standard way – as in [4, 7, 8] – is to write down the SPDE, then establish existence and uniqueness of solution in appropriate (Sobolev) classes, then apply Ito’s (or Ito–Wentzell’s) formula and, hence, justify that this solution, indeed, coincides with the desired conditional expectation. Apparently, this way assumes that somehow the equation should be known in advance. What the direct approach provides is exactly how to derive the equation “by hand” without reference to any big theory. Note that there is a paper [5] with a very similar title; yet, this is a different direct approach, which also stems from Krylov’s idea of representing solutions of SDEs as solutions of linear SPDEs, see [3], [8], [9].

The paper consists of four sections. Number one is the Introduction; the second one contains the main result about filtering SPDEs as well as two auxiliary Lemmata; the third one is devoted to the proof of the Lemma 1 (the second Lemma is a well known result with a reference provided), and the fourth one contains the proof of the main result – the Theorem 1.

### 2 Main result and auxiliary lemmata

Due to Girsanov’s theorem, process $Y_t$, $0 \leq t \leq T$ is a Wiener process on some the probability space with some new measure $(\Omega, \mathcal{F}, (\mathcal{F}_t, 0 \leq t \leq T), \tilde{\mathbb{P}})$ (see below).
Theorem 1 (backward SPDE) Let \( f, h \in C^3_b \). Then the process \( m_t \) may be represented as follows:

\[
m_T = \frac{v^g(0,x)}{v^1(0,x)},
\]

(2)

where the processes \( v^g \) and \( v^1 \) satisfy the following linear backward stochastic differential equation (the same for both functions):

\[
- dv^g(t,x) = \left[ \frac{1}{2} v^g_{xx}(t,x) + f(x)v^g_x(t,x) \right] dt + h(x)v^g(t,x) \star dY, \quad 0 \leq t \leq T,
\]

(3)

with initial data

\[
v^g(T,x) = g(x), \quad x \in \mathbb{R}^1.
\]

(4)

Note that the denominator in (2) is strictly positive a.s. as a conditional expectation of a strictly positive random variable with respect to some new probability measure. This will be commented in the proof.

Here in (3) \( \int \star dY_t \) means “backward” stochastic Ito integral, i.e., a normal “regular” stochastic Ito integral with inverse time, see [3, 8]. It may be formally defined, for example, by the formula

\[
\int_0^T h(x)v^g(t,x) \star dY_t := \int_0^T h(x)\tilde{v}^g(t,x) d\tilde{Y}_t,
\]

(5)

\[\tilde{Y}_t = Y_T - Y_{T-t}, \quad \tilde{v}^g(t,x) = v^g(T-t,x),\]

where \( \int_0^T h(x)\tilde{v}^g(t,x) d\tilde{Y}_t \) is a standard Itô’s integral. (The only small nuance is that this integral might be naturally defined up to the ± sign – which relates simply to how a Wiener process in the inverse time is defined – and, clearly, this sign would also affect the sign in the last term of the equation (3); this will be commented later.) The function \( v^1 \) has its terminal condition \( v^1(T,x) \equiv 1 \) and satisfies the same SPDE (3). Notice that the random function \( v^g(t,x) \) is, in fact, \( F^{w_1} - \)adapted (not \( F^{w_0} - \)adapted); therefore, the integral above makes sense exactly as a classical standard Itô’s one (cf. [8, Theorem 6.3.1]).

Before the proof we recall another Krylov and Rozovsky’s result – the Lemma 1 below – concerning multidimensional SDEs (see [3], [8], [9]).

Let \((Z^{s,z}_{t}, t \geq s, s \geq 0, z \in \mathbb{R}^d)\) be the family of \( d \)-dimensional processes depending on the parameters \((s,z)\) and satisfying the following multidimensional SDEs:
\[ dZ^s_{t} = b(Z^s_{t})dt + \sigma(Z^s_{t})dw_t, \quad t \geq s, \quad Z^s_{s} = z, \]  

(6)

where \( b \) is a bounded smooth \( d \)-dimensional vector, \( \sigma \) is a matrix \( d \times d_1 \), \( w_t \) is a \( d_1 \)-dimensional Wiener process, \( d, d_1 \geq 1 \); there are neither any other restrictions on the values \( d \) and \( d_1 \), nor any non-degenerability condition is assumed. We will use the following different notations for the same value:

\[ Z^s_{t} \equiv Z(s, t, z), \]

and for \( t = T \) also

\[ Z^{s,z}_{T} = u(s, z). \]

Recall that here \( T \) is fixed throughout the text, and that the multidimensional setting is essential: we will need it in the proof of the Theorem 1 with \( d = 2, d_1 = 1 \).

**Lemma 1** Let \( b, \sigma \in C^3_b \). Then the random field \( Z^{s,z}_{T} \) is continuous in all variables \((s, T, z)\). Moreover, continuous partial derivatives exist, the gradient vector \( \partial_z Z^{s,z}_{T} =: Z_z(s, t, z) \) and the Hessian matrix \( \partial^2_{zz} Z^{s,z}_{T} =: Z_{zz}(s, t, z) \), and the process \( u(s, z) \) satisfies an SPDE

\[
- du(t, z) = \left[ \frac{1}{2}(\sigma \sigma^*)_{ij}(z)u_{z_i z_j}(t, z) + b^i(z)u_{z_i}(t, z) \right] dt \\
+ \sigma_{ij}(z)u_{z_i}(t, z) \star dw^j_t,
\]

(7)

with a terminal condition

\[ u(T, z) \equiv z. \]

Here \( \sigma^* \) means the matrix \( \sigma \) transposed, and the equation (7) holds true for each component of the vector \( u(t, z) = (u^1(t, z), \ldots, u^d(t, z))^* \).

The direct approach to this result may be found in [9], and the main lines of its proof will be recalled below for the convenience of the reader.

Further, we will use the Bayes representation for conditional expectations, also known as Kallianpur–Striebel’s formula, see [8].

**Lemma 2** Let the Borel functions \( h, g \) be bounded. Then the following representation is valid a.s.:

\[
m_T = \frac{\tilde{\mathbb{E}}[g(X_T)|Y]}{\mathbb{E}[\rho^{-1}|F^Y_T]},
\]

4
where $\tilde{E}$ is the expectation with respect to the measure $\tilde{P}$: $d\tilde{P} = \rho P$, with

$$\rho \equiv \rho_{0T} = \exp \left(- \int_0^T h(X_t)dw_t^2 - \frac{1}{2} \int_0^T |h(X_t)|^2 dt \right).$$

Recall that due to Girsanov’s theorem the process $(Y_t, 0 \leq t \leq T)$ is a Wiener process on probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, 0 \leq t \leq T), \tilde{P})$, independent of $w_1$ and, in general, with a non-zero starting value. Denote $\tilde{w}_t = Y_t - y$. Then on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t, 0 \leq t \leq T), \tilde{P})$ our system (1) has the form

$$dX_t = f(X_t)dt + dw_t^1, \quad X_0 = x,$$

$$dY_t = d\tilde{w}_t, \quad Y_0 = y,$$

with two independent Wiener processes $(w^1, \tilde{w})$.

### 3 Direct proof of Lemma [1]

The terminal condition (8) is straightforward: $u(T, z) = Z_{T,t}^{T,z} = z$.

Further, due to [1] (or [6] under relaxed assumptions) the random field $Z_{t}^{s,z}$ is continuous in $(s, t, z)$ for $s \leq t$ and $z \in \mathbb{R}^d$ and, moreover, it admits classical continuous partial derivatives $\partial_z Z_{t}^{s,z} =: Z_z(s, t, z)$ and $\partial_{zz}^2 Z_{t}^{s,z} =: Z_{zz}(s, t, z)$.

Note that continuity in all arguments is required, in particular, for the justification of a substitution of $Z_{0}^{0,z}$ into the initial value of $Z_{t}^{s,r}$, that is,

$$Z_{t}^{0,z} = Z_{t}^{s,z}_0.$$

However, the equation contains two derivatives in the state variable, so we assumed the conditions, which guarantee the existence of these two (also classical) derivatives. The equation (10) itself follows easily from the uniqueness of solution of the related SDEs and, indeed, from the Markov property, which is a standard feature of any SDE with a unique solution in strong or weak sense (see, e.g., [2]).

Further, what is stated in the Theorem, by definition may be written in the integral form with $a = \sigma\sigma^*/2$ as follows (recall about the minus sign in the left hand side of
the equation (7),

\[ Z(t, T, z) - Z(T, T, z) = \int_t^T Z_z(s, T, z) \sigma(x) \ast dW_s \]

\[ + \int_t^T [Z_z(s, T, z)b(z) + \text{Tr} Z_{zz}(s, T, z)a(z)] \, ds. \]

(11)

In the sequel the general case \( d, d_1 \geq 1 \) is presented; we will use it in the case \( d = 2, d_1 = 1 \) in the proof of the Theorem. To show (11), let us split the interval \([0, t]\) by small partitions \( t = t_0 < t_1 < \ldots < t_{n+1} = T \), and let us write down the identity,

\[ Z(t, T, z) - Z(T, T, z) = \sum_{i=0}^n (Z(t_i, T, z) - Z(t_{i+1}, T, z)), \]

and consider each term after substituting

\[ Z(t_i, T, z) = Z(t_{i+1}, T, Z(z, t_i, t_{i+1})), \]

and using Hadamard’s form of Newton–Leibnitz’ formula (also known as the First Theorem of the Calculus),

\[ F(\Delta) = F(0) + \int_0^1 \nabla F(\alpha \Delta) \Delta d\alpha \]

\[ = F(0) + \int_0^1 \left( \nabla F(0) \Delta + \alpha \int_0^1 \Delta^* (\text{div} \nabla F(\alpha \beta)) \Delta \right) d\beta \]

where \( \nabla F(\alpha \Delta) \Delta \) is the inner product of the two vectors, and \( \Delta^* (\text{div} \nabla F(\alpha \beta)) \Delta \) is the Hessian matrix of \( F \) multiplied by the vector \( \Delta \) on the right and by the transposed \( \Delta^* \) on the left. Hence, we write,

\[ Z(t_{i+1}, T, Z(z, t_i, t_{i+1})) - Z(t_{i+1}, T, z) = Z_z(t_{i+1}, T, z) z_i \]

\[ + \int_0^1 \int_0^1 \alpha z_i^* Z_{zz}(t_{i+1}, T, z + \alpha \beta z_i) z_i d\alpha d\beta, \]

or, in the coordinate notations,

\[ Z(t_{i+1}, T, Z(z, t_i, t_{i+1})) - Z(t_{i+1}, T, z) = Z_z^k(t_{i+1}, T, z) z_i^k \]

\[ + \int_0^1 \int_0^1 \alpha z_i^k Z_{zz^k}(t_{i+1}, T, z + \alpha \beta z_i) z_i^k d\alpha d\beta, \]
where summation over repeated indices is assumed (Einstein’s convention),

\[ z_i := Z(t_i, t_{i+1}, z) - z, \]

and the equation is understood component-wise, i.e., for each component of the vector \( Z \). Denote also

\[ \tilde{z}_i := \sigma(z)(W_{t_{i+1}} - W_{t_i}) + b(z)(t_{i+1} - t_i), \]

and let \( \Delta W_t = W_{t_{i+1}} - W_{t_i} \). By virtue of standard estimates in stochastic analysis it follows,

\[ \sup_i E|\tilde{z}_i - z_i|^2 \leq C\frac{n^2}{n^2}. \]

Hence, we get

\[ Z(t_{i+1}, T, Z(z, t_i, t_{i+1})) - Z(t_{i+1}, T, z) = Z_{z^k}(t_{i+1}, T, z)\tilde{z}_i^k \]

\[ + \int_0^1 \int_0^1 \alpha Z_{z^k z^\ell}(t_{i+1}, T, x + \alpha \beta z_i)\tilde{z}_i^k \tilde{z}_i^\ell d\alpha d\beta + o(1/n), \]

where \( o(1/n) \) is understood in the square mean sense. We have, \( Z_z(t_{i+1}, T, z)\tilde{z}_i \approx Z_z(t_{i+1}, T, z)\sigma(z)(\Delta W_{t_i} + b(z)\Delta t_i); \) and \( Z_{zz}(t_{i+1}, T, z)\tilde{z}_i^2 \approx Z_{zz}(t_{i+1}, T, z)\sigma^2(z)(\Delta W_{t_i})^2 \). In all cases the sign “ \( \approx \ldots \)” means “ \( = \ldots + o(\max_i \Delta t_i)^n \)” with \( o(\max_i \Delta t_i) \) in the square mean sense as \( \max_i \Delta t_i \to 0 \).

Recall the definition of the backward integral for \( \xi_t \in \mathcal{F}_t^W \):

\[ \int_0^T \xi(t) \ast dW(t) := \int_0^T \xi_T(s) dW_T(s), \]

where \( \xi_T(s) = \xi(T - s), \) \( W_T(s) = W(T) - W(T - s) \). So, the integral approximations for the right hand side integral here with \( 0 = t_0 < t_1 < \cdots < t_n = T \) read,

\[ \sum_i \xi_T(t_i)(W'_T(t_{i+1}) - W_T(t_i)) \]

\[ = \sum_i \xi(T - t_i)(W(T) - W(T - t_{i+1}) - W(T) + W(T - t_i)) \]

\[ = \sum_i \xi(T - t_i)(W(T - t_i) - W(T - t_{i+1})) \]

\[ = \sum_i \xi(t'_i)(W(t'_i) - W(t'_{i+1})), \]
where $t'_i = T - t_i$. Note that this may be used as a simplified definition of stochastic integral, at least, for continuous $\xi(t)$. Since $0 = t'_n < t'_{n-1} < \cdots < t'_0 = T$, the right way to understand integral approximations in terms of original processes in direct time is

$$
\sum_i \xi(t_{i+1})(W(t_{i+1}) - W(t_i)) = \sum_i \xi(t_{i+1})\Delta W_t
$$

(recall that $\Delta W_t = W_{t_{i+1}} - W_{t_i}$). So, after summation over $i$ in (12), we obtain $Z(t, T, z) - Z(T, T, z)$ in the left hand side, and the following three terms (all component-wise) in the right hand side,

$$
\sum_{i} Z_{zk}(t_{i+1}, T, z)\sigma^{kl}(z)\Delta W_{t_i} \xrightarrow{sq.mean} \int_{t}^{T} Z_{z}(s, T, z)\sigma(s) \ast dW_s,
$$

$$
\sum_{i} Z_{zk}(t_{i+1}, T, z)b^{k}(z)\Delta t_i \xrightarrow{sq.mean} \int_{t}^{T} Z_{z}(s, T, z)b(z)ds,
$$

and

$$
\frac{1}{2} \sum_{i} Z_{z^{kl}}(t_{i+1}, T, z)\sigma^{kl}(z)\sigma^{kj}(z)(\Delta W_{t_i})^2 \xrightarrow{sq.mean} \int_{t}^{T} \text{Tr}(Z_{zz}(s, T, z)a(z))ds,
$$

as $\max_i \Delta t_i \to 0$. Here $\frac{1}{2}$ is due to $\int_{0}^{1} \alpha d\alpha = \frac{1}{2}$. So, we obtain (11), as required. The Lemma 1 is proved.

4 **Direct proof of Theorem 1**

1. Denote

$$
v^g(s, x) = \mathbb{E}[g(X^r_{s,T})|\mathcal{F}_{s,T}^r].
$$

Then,

$$
v^g(T, x) = \mathbb{E}[g(X^r_T)|\mathcal{F}_{T,T}^r] = g(x).
$$

In fact, what we want to establish is exactly the following equality (for each $T > 0$ and any $0 \leq t_0 \leq T$):

$$
v^g(t_0, x) - v^g(T, x) = \int_{t_0}^{T} \left[ \frac{1}{2} v^g_{xx}(t, x) + f(x)v^g_x(t, x) \right] dt + \int_{t_0}^{T} h(x)v^g(t, x) \ast d\tilde{w}_t.
$$
Let us use the identity
\[ v^g(t_0, x) - v^g(T, x) = \sum_{i=1}^{N} (v^g(t_{i-1}, x) - v^g(t_i, x)), \]
for any partition \( t_0 < t_1 < \ldots < t_N = T \). Consider one term from this sum: we have,
\[
v^g(t_{i-1}, x) - v^g(t_i, x) = \mathbb{E}[\rho^{-1}_{t_{i-1},T} g(X(t_{i-1}, T, x)) | \mathcal{F}^Y_{t_{i-1},T}] - \mathbb{E}[\rho^{-1}_{t_i, T} g(X(t_i, T, x)) | \mathcal{F}^Y_{t_i, T}]
\]
\[
= \mathbb{E}[\rho^{-1}_{t_{i-1},T} g(X(t_{i-1}, T, x)) | \mathcal{F}^\tilde{w}_{t_{i-1},T}] - \mathbb{E}[\rho^{-1}_{t_i, T} g(X(t_i, T, x)) | \mathcal{F}^\tilde{w}_{t_i, T}].
\]

2. Using continuity of the family \( X(s, T, x) \) with respect to all variables and existence of two continuous partial derivatives with respect to \( x \) (see [1]) we get a.s. by virtue of Taylor’s expansion,
\[
X(t_{i-1}, T, x) = X(t_i, T, X_{t_i}^{t_{i-1}, x})
\]
\[
= X_T^{t_i, x} + X_x(t_i, T, x)(X_{t_i}^{t_{i-1}, x} - x)
\]
\[
+ \frac{1}{2} X_{xx}(t_i, T, x)(X_{t_i}^{t_{i-1}, x} - x)^2 + \alpha^1_i
\]
\[
= X_T^{t_i, x} + X_x(t_i, T, x)(f(x) \Delta t_i + \Delta w_{t_i}^1)
\]
\[
+ \frac{1}{2} X_{xx}(t_i, T, x) \Delta t_i + \alpha^2_i,
\]
where \( \Delta t_i = t_i - t_{i-1} \), \( \Delta w_{t_i}^j = w_{t_i}^j - w_{t_{i-1}}^j \), and \( |\alpha^1_i| + |\alpha^2_i| = o(\Delta t_i) \) in the mean-square sense. Hence,
\[
g(X(t_{i-1}, T, x)) = g(X(t_i, T, X_{t_i}^{t_{i-1}, x}))
\]
\[
= g \left( X_T^{t_i-1, x} + X_x(t_i, T, x)(X_{t_i}^{t_{i-1}, x} - x) \right)
\]
\[
+\frac{1}{2} X_{xx}(t_i, T, x)(X_{t_i-1,x} - x)^2 + \alpha_i^1
\]

\[
= g \left( X_{t_i,x} + X_{x}(t_i, T, x)(f(x)\Delta t_i + \Delta w_{t_i}^1) + \frac{1}{2} X_{xx}(t_i, T, x)\Delta t_i + \alpha_i^2 \right)
\]

\[
= g(X_{t_i,x}^T)
\]

\[
+ g_x(X_{t_i,x}^T) \left( X_{x}(t_i, T, x)(f(x)\Delta t_i + \Delta w_{t_i}^1) + \frac{1}{2} X_{xx}(t_i, T, x)\Delta t_i \right)
\]

\[
+ \frac{1}{2} g_{xx}(X_{t_i,x}^T)(\Delta w_{t_i}^1)^2 + o(\Delta t_i)
\]

\[
= g(X_{t_i,x}^T)
\]

\[
+ g_x(X_{t_i,x}^T) \left( X_{x}(t_i, T, x)(f(x)\Delta t_i + \Delta w_{t_i}^1) + \frac{1}{2} X_{xx}(t_i, T, x)\Delta t_i \right)
\]

\[
+ \frac{1}{2} g_{xx}(X_{t_i,x}^T)\Delta t_i + o(\Delta t_i).
\]

Denote \( V(s, t, x) = g(X(s, t, x)) \). Then, assuming that \( g \in C^2 \), we have,

\[
V_x = g_x X_x; \quad V_{xx} = g_x X_{xx} + g_{xx} X_x^2,
\]

where we dropped the arguments in \( g_x, g_{xx}, X_x, \) and \( X_{xx} \) for brevity. So,

\[
g(X(t_{i-1}, T, x)) = V(t_{i-1}, T, x)
\]

\[
= g(X_{t_i,x}^T)
\]

\[
+ g_x(X_{t_i,x}^T)X_{x}(t_i, T, x)(f(x)\Delta t_i + \Delta w_{t_i}^1) + \frac{1}{2} X_{xx}(t_i, T, x)\Delta t_i
\]

\[
+ \frac{1}{2}(X_x(t_i, T, x))^2 g_{xx}(X_{t_i,x}^T)\Delta t_i + o(\Delta t_i)
\]
\[ V(t_i, T, x) + V_x(t_i, T, x)(f(x)\Delta t_i + \Delta w_{1,i}^1) + \frac{1}{2}V_{xx}(t_i, T, x)\Delta t_i + o(\Delta t_i). \]

Here and earlier \( o(\Delta t_i) \) is understood in the mean square sense. The obtained relation means that the conditional expectation for \( V = g(X) \) should satisfy the same SPDE as for \( X \) itself, just with another terminal condition.

3. Thus,

\[
\tilde{E}[\rho_{t_{i-1}, T}^{-1}g(X(t_{i-1}, T, x)) | F_{t_{i-1}, T}] = \tilde{E}[\rho_{t_{i-1}, T}^{-1}V(t_{i-1}, T, x) | F_{t_{i-1}, T}] = \tilde{E}[\rho_{t_{i-1}, T}^{-1}V(t_{i-1}, T, x) | F_{t_{i-1}, T}] = \tilde{E}[\rho_{t_{i-1}, T}^{-1}V(t_{i-1}, T, x) | F_{t_{i-1}, T}] = \alpha_i^3.
\]

Here again, \( \alpha_i^3 = o(\Delta t_i) \) in the mean square sense, i.e., \( (\mathbb{E}[\alpha_i^3]^2)^{1/2} = o(\Delta t_i) \).

4. Now, we would like to replace \( \rho_{t_{i-1}, T}^{-1} \) by \( \rho_{t_{i}, T}^{-1} \). For this aim we apply the Lemma \( \Box \) to the process \( (X^{s,x}_{s,t}, \rho_{s,t}^{-1}, t \geq s) \). More precisely, let us note that this two-dimensional process satisfies the following SDE system:

\[
dX^{s,x}_t = f(X^{s,x}_t)dt + dw_{1,t}, \quad X^{s,x}_s = x, \tag{14}
\]

\[
d\rho_{s,t}^{-1} = h(X^{s,x}_t)\rho_{s,t}^{-1}d\tilde{w}_t, \quad \rho_{s,s}^{-1} = 1,
\]

with \( s \leq t \leq T \). Indeed, \( \rho_{s,t}^{-1} \) has the following representation:

\[
\rho_{s,t}^{-1} = \exp \left( \int_s^t h(X^{s,x}_r)dw_r - \frac{1}{2} \int_s^t |h(X^{s,x}_r)|^2dr \right).
\]

Let us consider a bit more general set of processes \( \{(X^{s,x}_t, \rho_{s,t}^{-1})\} \) which satisfy SDE’s
\[ dX_t^{s,x} = f(X_t^{s,x})dt + dw_t, \quad X_s^{s,x} = x, \] 
\[ d\rho_{s,t}^{-1,\xi} = h(X_t^{s,x})\rho_{s,t}^{-1,\xi} d\tilde{w}_t, \quad \rho_{s,s}^{-1,\xi} = \xi, \] 
for \( s \leq t \leq T \), with \( \xi > 0 \). In fact, \( X_t^{s,x} \) here is the same as earlier, and \( \rho_{t}^{-1,\xi} \) has the following representation:
\[ \rho_{s,t}^{-1,\xi} = \xi \exp \left( \int_s^t h(X_r^{s,x}) d\tilde{w}_r - \frac{1}{2} \int_s^t |h(X_r^{s,x})|^2 dr \right) = \xi \rho_{s,t}^{-1}. \]

Then due to the Lemma, we get
\[ -d_s \rho_{s,t}^{-1,\xi} = \left[ \frac{1}{2} h^2(x)(\rho_{s,t}^{-1,\xi})^2(\rho_{s,t}^{-1,\xi}) \xi + \frac{1}{2}(\rho_{s,t}^{-1,\xi})_{xx} 
+ f(x)(\rho_{s,t}^{-1,\xi})_x \right] dt 
+ (\rho_{s,t}^{-1,\xi})_x * dw_t + h(x) \xi (\rho_{s,t}^{-1,\xi})_\xi * d\tilde{w}_t. \]

Note that, in fact, \( (\rho_{s,t}^{-1,\xi})_\xi = \rho_{s,t}^{-1} \) and \( (\rho_{s,t}^{-1,\xi})_{\xi\xi} = 0 \). Hence,
\[ -d_s \rho_{s,t}^{-1,\xi} = \left[ \frac{1}{2}(\rho_{s,t}^{-1,\xi})_{xx} + f(x)(\rho_{s,t}^{-1,\xi})_x \right] dt 
+ (\rho_{s,t}^{-1,\xi})_x * dw_t + h(x) \xi \rho_{s,t}^{-1} * d\tilde{w}_t. \]

So, we get,
\[ \rho_{t_{i-1},T}^{-1,\xi} - \rho_{t_i,T}^{-1,\xi} = -\Delta t_i \rho_{t_i}^{-1,\xi} \]
\[ = \left[ \frac{1}{2}(\rho_{t_i,T}^{-1,\xi})_{xx} + f(x)(\rho_{t_i,T}^{-1,\xi})_x \right] \Delta t_i \]
\[ + \rho_{t_i,T}^{-1,\xi} \Delta w_t + h(x) \xi \rho_{t_i,T}^{-1,\xi} \Delta \tilde{w}_t + \alpha_i^4, \]
with a similar \( o(\Delta t_i) \) property for \( \alpha_i^4 \) as for previous \( \alpha_i^1, \alpha_i^2, \alpha_i^3 \). Below we will use this assertion with \( \xi = 1 \), that is,
\[ \rho_{t_{i-1},T}^{-1} - \rho_{t_i,T}^{-1} = -\Delta t_i \rho_{t_i,T}^{-1} \]
\[
\begin{align*}
&= \left[ \frac{1}{2} \rho_{t_i,T}^{-1} xx + f(x) \rho_{t_i,T}^{-1} x \right] \Delta t_i \\
&+ \rho_{t_i,T}^{-1} \Delta w^1_i + h(x) \rho_{t_i,T}^{-1} \Delta \tilde{w}_i + \alpha_4^i,
\end{align*}
\]

5. Now, we obtain

\[
\tilde{E}[\rho_{t_i,T}^{-1} V(t_{i-1}, T, x)|\mathcal{F}_{t_{i-1},T}]
\]

\[
= \tilde{E} \left[ \left\{ V(t_i, T, x) + (f(x) V_x(t_i, T, x) + \frac{1}{2} V_{xx}(t_i, T, x)) \Delta t_i + V_x(t_i, T, x) \Delta w^1_i \right\} \times \\
\times \left\{ \rho_{t_i,T}^{-1} x + \left( \frac{1}{2} \rho_{t_i,T}^{-1} xx + f(x) \rho_{t_i,T}^{-1} x \right) \Delta t_i \\
+ \rho_{t_i,T}^{-1} x \Delta w^1_i + h(x) \rho_{t_i,T}^{-1} \Delta \tilde{w}_i + \alpha_5^i \right\} |\mathcal{F}_{t_{i},T} \right],
\]

where again, \( \alpha_5^i = o(\Delta t_i) \) in the same sense.

6. Now, note that \( \mathcal{F}_{t_{i-1},T} = \mathcal{F}_{t_{i-1},t_i} \cup \mathcal{F}_{t_{i},T} \) and, moreover this \( \sigma \)-field is independent from \( w^1 \). Using the regular calculus for conditional expectations (cf. [8]), we get

\[
\tilde{E}[V(t_i, T, x)\rho_{t_i,T}^{-1} V(t_{i-1}, T, x)|\mathcal{F}_{t_{i-1},T}]
\]

and in the same manner we can replace \( \sigma \)-fields \( \mathcal{F}_{t_{i-1},T} \) by \( \mathcal{F}_{t_{i-1},t_i} \) in all expressions in the previous step. Also, \( \tilde{E} \left[ \Delta w^1_i |\mathcal{F}_{t_{i-1},T} \right] = 0 \) due to the independence of \( w^1 \) and \( \tilde{w} \) with respect to the measure \( \tilde{P} \), and \((\Delta w^1_i)^2 \approx \Delta t_i \). Hence, we obtain

\[
\tilde{E}[V(t_i, T, x)\rho_{t_i,T}^{-1} V(t_{i-1}, T, x)|\mathcal{F}_{t_{i-1},T}]
\]

\[
= \tilde{E} [V(t_i, T, x)\rho_{t_i,T}^{-1} V(t_{i-1}, T, x)|\mathcal{F}_{t_{i-1},T}]
\]

\[
+ \tilde{E} \left[ \frac{1}{2} V(t_i, T, x)(\rho_{t_i,T}^{-1}) xx + V_x(t_i, T, x)(\rho_{t_i,T}^{-1}) x + \frac{1}{2} V_{xx}(t_i, T, x) \rho_{t_i,T}^{-1} \Delta \tilde{w}_i \right] \Delta t_i
\]

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\[
+\mathbb{E}[f(x)(V_x(t_i, T, x)\rho_{t_i,T}^{-1} + V(t_i, T, x)(\rho_{t_i,T}^{-1})_x)|\mathcal{F}_{t_i,T}]\Delta t_i
\]
\[
+\mathbb{E}[V(t_i, T, x)h(x)\rho_{t_i,T}^{-1}\Delta \tilde{w}_{t_i}|\mathcal{F}_{t_i-1,T}] + \alpha_i^6
\]
\[
= \mathbb{E}[V(t_i, T, x)\rho_{t_i,T}^{-1}|\mathcal{F}_t]\Delta t_i + \mathbb{E}\left[\frac{1}{2}(V(t_i, T, x)\rho_{t_i,T}^{-1})_{xx}|\mathcal{F}_t\right]\Delta t_i
\]
\[
+\mathbb{E}[f(x)(V(t_i, T, x)\rho_{t_i,T}^{-1})_x)|\mathcal{F}_{t_i,T}]\Delta t_i + \Delta \tilde{w}_{t_i}\mathbb{E}[h(x)V(t_i, T, x)\rho_{t_i,T}^{-1}|\mathcal{F}_{t_i,T}] + \alpha_i^6
\]
\[
= v^g(t_i, x) + \frac{1}{2}v^g_{xx}(t_i, x)\Delta t_i + f(x)v^g_x(t_i, x)\Delta t_i + h(x)v^g(t_i, x)\Delta \tilde{w}_{t_i} + \alpha_i^6,
\]
with a similar property for \(\alpha_i^6\): \(\alpha_i^6 = o(\Delta t_i)\) in the mean square sense. The last equality in this calculus holds true because of the possibility to change the order of integration and derivation with respect to the \(x\) variable.

7. Therefore, we obtain the equality

\[
v^g(t_0, x) - v^g(T, x)
\]
\[
= \sum_i \left\{\frac{1}{2}v^g_{xx}(t_i, x) + f(x)v^g_x(t_i, x)\right\}\Delta t_i + \sum_i h(x)v^g(t_i)\Delta \tilde{w}_{t_i} + \alpha_7,
\]
with \(\alpha_7 = o(1)\) in the mean square sense as \(\sup_i \Delta t_i \to 0\). Letting \(\sup_i \Delta t_i \to 0\), we get from here the desired integral equality (13). The Theorem is proved.

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