COMPOUND BASIS ARISING FROM THE BASIC $A_1^{(1)}$-MODULE

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Abstract. A new basis for the polynomial ring of infinitely many variables is constructed which consists of products of Schur functions and $Q$-functions. The transition matrix from the natural Schur function basis is investigated.

1. Introduction

This note concerns with realizations of the basic representation of the affine Lie algebra of type $A_1^{(1)}$ (cf. [6]). The most well-known realization is $PU$, principal, untwisted, whose representation space is

$$\mathcal{F}^{PU} = \mathbb{C}[t_j; j \geq 1, \text{odd}].$$

In the context of nonlinear integrable systems, this space appears as that of the KdV hierarchy. The second one is $HU$, homogeneous, untwisted, which is on

$$\mathcal{F}^{HU} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}(m); \quad \mathcal{F}(m) = \mathbb{C}[t_j; j \geq 1] \otimes q^m.$$  

This space is for the NLS (nonlinear Schrödinger) hierarchy and also for the Fock representation of the Virasoro algebra (cf. [5]). The third one is $PT$, principal, twisted, on $\mathcal{F}^{PT}$ which coincides with $\mathcal{F}^{PU}$. And the fourth one is $HT$, homogeneous, untwisted, on $\mathcal{F}^{HT}$ which is the same as $\mathcal{F}^{HU}$. The Lie algebra of type $A_1^{(1)}$ is isomorphic to that of type $D_2^{(2)}$. One can discuss twisted realization of $A_1^{(1)}$-modules via this isomorphism.

The purpose of this note is to give a weight basis for $\mathcal{F}^{HT}$ and compare it with a standard Schur function basis for $\mathcal{F}^{HU}$. We will show that the transition matrix has several interesting combinatorial features. This is a detailed version of our announcement [4].

2. A Quick Review of Realizations

Let us first consider the principal untwisted realization on $\mathcal{F}^{PU} = \mathbb{C}[t_j; j \geq 1, \text{odd}]$. To describe a weight basis for this space we need Schur functions and Schur’s $Q$-functions in our setting. Let $P_n$ be the set of all partitions of $n$ and put $P = \bigcup_{n \geq 0} P_n$. For $\lambda \in P_n$, the Schur function $S_\lambda(t)$ is defined by

$$S_\lambda(t) = \sum_{\rho = (1^{m_1}2^{m_2} \ldots) \in P_n} \frac{\chi_\rho^{t_1^{m_1}t_2^{m_2} \ldots}}{m_1!m_2! \ldots},$$

where the summation runs over all partitions $\rho = (1^{m_1}2^{m_2} \ldots)$ of $n$, and $\chi_\rho^\lambda$ is the irreducible character of the symmetric group $\mathfrak{S}_n$, indexed by $\lambda$ and evaluated at the conjugacy class $\rho$. The Schur functions are the ordinary irreducible characters of the general linear groups. If the group element $g$ has eigenvalues $x_1, x_2, \ldots$, then
the original irreducible character is recovered by putting $p_j := j t_j$ ($j \geq 1$), where $p_j = \sum_{i \geq 1} x_i^j$ is the $j$-th power sum of the eigenvalues.

The 2-reduction of a polynomial $f(t)$ is to “kill” the even numbered variables $t_2, t_4, \ldots$, i.e.,

$$f^{(2)}(t) = f(t)|_{t_2 = t_4 = \ldots = 0} \in F^{PU}.$$ 

The 2-reduced Schur functions are linearly dependent in general. However all linear relations among them are known, and one can choose certain set $P' \subset P$ so that $\{ S^{(2)}_\lambda; \lambda \in P' \}$ forms a basis for $F^{PU}$ (cf. \cite{2}).

The space $F^{PU}$ also affords the principal twisted realization. A weight basis is best described by Schur’s $Q$-functions. Let $SP_n$ (resp. $OP_n$) be the set of all strict (resp. odd) partitions of $n$ and put $SP = \bigcup_{n \geq 0} SP_n$, $OP = \bigcup_{n \geq 0} OP_n$. For $\lambda \in SP_n$, the $Q$-function $Q_\lambda(t)$ is defined by

$$Q_\lambda(t) = \sum_{\rho = (1^{m_1} 3^{m_3} \ldots) \in OP_n} 2^{\ell(\lambda) - \ell(\rho)} \zeta^\lambda_\rho \frac{\ell^{m_1} t_3^{m_3} \cdots}{m_1! m_3! \cdots},$$

where the summation runs over all odd partitions $\rho = (1^{m_1} 3^{m_3} \ldots)$ of $n$, $\epsilon = 0$ or 1 according to that $\ell(\lambda) - \ell(\rho)$ is even or odd and $\zeta^\lambda_\rho$ is the irreducible spin character of $\mathfrak{S}_n$, indexed by $\lambda$ and evaluated at the conjugacy class $\rho$. For the $Q$-functions, we set $p_j := \frac{j}{2} t_j$ ($j \geq 1$, odd) as the relation with the “eigenvalues”. A more detailed account is found in \cite{9}. Here we remark the relation of $Q$-functions and the $P$-functions. We define inner product $\langle \cdot , \cdot \rangle_q$ on $F(0)$ by $\langle p_\lambda, p_\mu \rangle_q = z_\lambda q^\lambda \delta_{\lambda \mu}$, where $z_\lambda(q) = z_\lambda \prod_{i \geq 1} (1 - q^i)^{-1}$. Note that $z_\lambda(-1)$ cannot be defined for $\lambda$ which has even parts. Therefore we have to re-define $\langle \cdot , \cdot \rangle$ by setting $\langle p_\lambda, p_\mu \rangle_{-1} = 2^{-\ell(\lambda)} z_\lambda \delta_{\lambda \mu}$. The $P$-functions are dual to the $Q$-functions with respect to the inner product $\langle \cdot , \cdot \rangle_{-1}$ on $F^{PU}$. For a strict partition $\lambda$, we see that $P_\lambda(t) = 2^{-\ell(\lambda)} Q_\lambda(t)$ (cf. \cite{3}).

In order to give the homogeneous, twisted realization we employ a combinatorics of strict partitions. We introduce the following h-abacus. For example, the h-abacus of $\lambda = (11, 10, 5, 3, 2)$ is shown below.

```
  1  3
  2
  4  5  7
  6
  8  9  13
 10
 12 13 15
```

From this h-abacus of $\lambda$ we read off a triplet $(\lambda^{hc}; \lambda^{h[0]}, \lambda^{h[1]})$ of partitions. Firstly $\lambda^{h[0]} = (5, 1)$, a strict partition obtained just by taking halves of the circled positions of the leftmost column.

For obtaining $\lambda^{h[1]}$, we need the following process:

1. For the third column, the circled positions correspond to the vacancies “•”.
2. For the second column, the circled positions correspond to being occupied “o”.

...
(3) Read the third column from infinity to the position 3 and consequently the second column from the position 1 to infinity, and draw the Maya diagram
\[
\cdots \ 15 \ 11 \ 7 \ 3 \ 1 \ 5 \ 9 \ \cdots
\]

(4) For each \( \bullet \), count the number of vacancies which are on the left of that \( \bullet \), and get a partition 
\( \lambda h[1] = (3, 1) \).

Next the h-core \( \lambda hc \) is obtained by the following moving and removing:
(1) Remove all circles on the leftmost column.
(2) Move a circle one position up along the second or the third column.
(3) Remove the two circles at the positions 1 and 3 simultaneously.
(4) The “stalemate” determines the partition 
\( \lambda hc = (3) \).

Note that \( \lambda hc \) is always of the form 
\( \Delta h(m) = (4m - 3, 4m - 7, \ldots, 5, 1) \) or \( \Delta h(-m) = (4m - 1, 4m - 5, \ldots, 7, 3) \) for some \( m \in \mathbb{N} \) (\( \Delta h(0) = \emptyset \)). Let \( HC \) be the set of all such \( \lambda hc \)-s. In this way we have a one-to-one correspondence between \( \lambda \in SP \) and \( (\lambda hc; \lambda h[0], \lambda h[1]) \in HC \times SP \times P \) with the condition 
\[ |\lambda| = |\lambda hc| + 2(|\lambda h[0]| + 2|\lambda h[1]|).\]

By making use of this one-to-one correspondence, we define the linear map \( \eta : F^{PT} \rightarrow F^{HT} \) by 
\[ \eta(Q_{\lambda}(t)) = Q_{\lambda h[0]}(t)S_{\lambda h[1]}(t') \otimes q^{m(\lambda)}. \]

Here 
\[ m(\lambda) = (\text{number of circles on the second column}) - (\text{number of circles on the third column}) \]
and \( S_{\nu}(t') = S_{\nu}(t)|_{t\rightarrow t^2} \) for any \( j \geq 1 \). For any integer \( m \), the set 
\[ \{ \eta(Q_{\lambda}); \lambda \in SP, m(\lambda) = m \} \]
forms a basis for \( F(m) = \mathbb{C}[t_j; j \geq 1] \otimes q^m \) (cf. [4]). Under the condition \( m = 0 \), there is a one-to-one correspondence between the following two sets for any \( n \geq 0 \):
(i) \( \{ \lambda \in SP_{2n}; \lambda hc = \emptyset \} \),
(ii) \( \{ (\mu, \nu) \in SP_{n_0} \times P_{n_1}; n_0 + 2n_1 = n \} \).

3. Compound Basis

We begin with some bijections between sets of partitions. The first one is
\[ \phi : P_n \rightarrow \bigcup_{n_0 + 2n_1 = n} SP_{n_0} \times P_{n_1} \]
deﬁned by \( \lambda \mapsto (\lambda^v, \lambda^d) \). Here the multiplicities \( m_i(\lambda^v) \) and \( m_i(\lambda^d) \) of \( i \geq 1 \) are given respectively by
\[ m_i(\lambda^v) = \begin{cases} 
1 & m_i(\lambda) \equiv 1 \pmod{2} \\
0 & m_i(\lambda) \equiv 0 \pmod{2},
\end{cases} \]
For example, if $\lambda = (5^4 2^7 1)$, then $\lambda^r = (521)$ and $\lambda^d = (542^2 3)$. We set 
\[ P_{n_0, n_1} = \phi^{-1}(SP_{n_0} \times P_{n_1}). \]

The second bijection is defined by $\psi : \lambda \mapsto (\lambda^o, \lambda^v)$. Here $\lambda^o$ is obtained by picking up the odd parts of $\lambda$, while $\lambda^v$ is obtained by taking halves of the even parts. For example, if $\lambda = (5^4 2^7 1)$, then $\lambda^o = (5^3 1)$ and $\lambda^v = (2^4 1^7)$.

The third bijection is called the Glaisher map. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a strict partition of $n$. Suppose that $\lambda_i = 2^p q_i$ ($i = 1, 2, \ldots$), where $q_i$ is odd. Then an odd partition $\tilde{\lambda}$ of $n$ is defined by 
\[ m_{2j-1}(\tilde{\lambda}) = \sum_{q_i = 2j-1, j \geq 1} 2^{p_i}. \]

For example, if $\lambda = (8, 6, 4, 3, 1)$, then $\tilde{\lambda} = (3^3, 1^3)$. This gives a bijection between $SP_n$ and $OP_n$.

**Proposition 3.1.** Let $(n_0, n_1)$ be fixed. Then we have

\[
\sum_{\lambda \in P_n} \ell(\lambda) = \sum_{\lambda \in P_n} (\ell(\lambda^r) + 2\ell(\lambda^d)) = \sum_{\lambda \in P_n} (\ell(\lambda^x) + \ell(\lambda^y)) = \sum_{\lambda \in P_n} (\ell(\lambda^x) + \ell(\lambda^y)),
\]

\[
\sum_{\lambda \in P_{n_0, n_1}} \ell(\lambda) = \sum_{\lambda \in P_{n_0, n_1}} (\ell(\lambda^r) + 2\ell(\lambda^d)) = \sum_{\lambda \in P_{n_0, n_1}} (\ell(\lambda^o) + \ell(\lambda^v)),
\]

\[
\sum_{\lambda \in P_n} 2\ell(\lambda^d) = \sum_{\lambda \in P_n} 2\ell(\lambda^v) = \sum_{\lambda \in P_n} (\ell(\lambda^o) + \ell(\lambda^v) - \ell(\lambda^r)) = \sum_{\lambda \in P_n} (\ell(\lambda^v) + \ell(\lambda^v) - \ell(\lambda^r)),
\]

and

\[
\sum_{\lambda \in P_{n_0, n_1}} 2\ell(\lambda^d) = \sum_{\lambda \in P_{n_0, n_1}} (\ell(\lambda^o) + \ell(\lambda^v) - \ell(\lambda^r)).
\]

Looking at the representation spaces $\mathcal{F}^{HU}$ and $\mathcal{F}^{HT}$, we have the following two natural bases for the space 
\[ \mathcal{F}(0)_n = \mathbb{C}[t_j : j \geq 1]_n \]
consisting of the homogenous polynomials of degree $n$ subject to $\deg t_j = j$. Namely we have

(i) $\{S_\lambda(t) : \lambda \in P_n\},$

(ii) $\{Q_{\lambda^v}(t)S_{\lambda^o}(t^2) : \lambda \in P_n\}.$

For simplicity we write 
\[ W_\lambda(t) = Q_{\lambda^v}(t)S_{\lambda^o}(t^2) \]
for $\lambda \in P_n$ and call the set (ii) the compound basis for $\mathcal{F}(0)_n$. 

Our problem is to determine the transition matrix between these two bases. Let $A_n = (a_{\lambda\mu})$ be defined by

\[(1) \quad S_\lambda(t) = \sum_{\mu \in P_n} a_{\lambda\mu} W_\mu(t)\]

for $\lambda \in P_n$.

Here we remark the relation between our basis and the $Q'$-functions. Lascoux, Leclerc and Thibon (cf. [7]) introduced the $Q'$-functions as the basis for $\mathcal{F}(0)_n$ dual to $P$-functions with respect to the inner product

$$\langle F(t), G(t) \rangle_0 := \left. F(\tilde{\partial}) G(t) \right|_{t=0},$$

where $\tilde{\partial} = (\frac{\partial}{\partial x_1}, \frac{1}{2} \frac{\partial}{\partial x_2}, \frac{1}{3} \frac{\partial}{\partial x_3}, \ldots)$. For a strict partition $\mu$ we see that $Q'_\mu(t) = Q_\mu(2t)$. For a partition $\lambda$ which is not necessarily strict, we see that $Q'_\lambda(t) = Q_\lambda(r(2t)) h_\lambda d(t')$ where $h_\lambda d$ is the complete symmetric function indexed by $\lambda^d$. Therefore the transition from $W_\lambda$ to $Q'_\mu$ is essentially given by the Kostka numbers.

### 4. Transition Matrices

In the previous section, functions are expressed in terms of the “time variables” $t = (t_1, t_2, \ldots)$ of the soliton equations. However, for the description and the proof of our formula, it is more convenient to use the “original” variables of the symmetric functions, i.e., the eigenvalues $x = (x_1, x_2, \ldots)$.

The definition (1) of $a_{\lambda\mu}$ is rewritten as

$$S_\lambda(x, x) = \sum_{\mu \in P_n} a_{\lambda\mu} Q_{\mu^r}(x) S_{\mu^d}(x^2),$$

where $(x, x) = (x_1, x_1, x_2, x_2, \ldots)$ and $x^2 = (x_1^2, x_2^2, \ldots)$. Hereafter we will denote $W_\lambda(x) = Q_{\lambda^r}(x) S_{\lambda^d}(x^2), \quad V_\lambda(x) = P_{\lambda^r}(x) S_{\lambda^d}(x^2)$.

Also we set the following spaces of symmetric functions

$$\Lambda = \mathbb{C}[p_r(x); r \geq 1], \quad \Gamma = \mathbb{C}[p_r(x); r \geq 1, \text{odd}],$$

and

$$\Gamma' = \mathbb{C}[p_r(x); r \geq 2, \text{even}]$$

so that

$$\Lambda \cong \Gamma \otimes \Gamma'.$$

We have two bases for $\Lambda$:

$$W = (W_\lambda(x))_\lambda \quad \text{and} \quad V = (V_\lambda(x))_\lambda.$$

First we notice the following Cauchy identity.

**Proposition 4.1.**

$$\prod_{i,j \geq 1} \frac{1}{(1 - x_i y_j)^2} = \sum_{\lambda \in P} W_\lambda(x) V_\lambda(y).$$
Proof. We compute
\[ \sum_{\lambda \in P} W_{\lambda}(x)V_{\lambda}(y) = \sum_{\lambda \in P} Q_{\lambda'}(x)S_{\lambda}(x^2)P_{\lambda'}(y)S_{\lambda'}(y^2) = \sum_{\mu \in SP} Q_{\mu}(x)P_{\mu}(y) \sum_{\nu \in P} S_{\nu}(x^2)S_{\nu}(y^2). \]

Taking the inner products \( \langle \cdot, \cdot \rangle_{-1} \) and \( \langle \cdot, \cdot \rangle_0 \) on \( \Lambda \), we obtain
\[ \sum_{\mu \in SP} Q_{\mu}(x)P_{\mu}(y) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j}, \]
and
\[ \sum_{\nu \in P} S_{\nu}(x^2)S_{\nu}(y^2) = \prod_{i,j} \frac{1}{1 - x_i^2 y_j^2}. \]
We have
\[ \sum_{\lambda \in P} W_{\lambda}(x)V_{\lambda}(y) = \prod_{i,j} \frac{1}{(1 - x_i y_j)^2}. \]

By a standard argument, we have

**Corollary 4.2.**
\[ \langle W_{\lambda}(x), V_{\mu}(x) \rangle_{-1} = \delta_{\lambda\mu}. \]

**Theorem 4.3.** The matrix \( A_n \) is integral.

*Proof.* We have
\[ \sum_{\lambda \in P} W_{\lambda}(x)V_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} \sum_{\lambda \in P} S_{\lambda}(x,x)S_{\lambda}(y). \]
Taking the inner product \( \langle \cdot, \cdot \rangle_0 \) with \( S_{\mu}(y) \), we obtain
\[ S_{\lambda}(x,x) = \sum_{\mu \in P} \langle W_{\mu}(x)V_{\mu}(y), S_{\lambda}(y) \rangle_0 = \sum_{\mu \in P} \langle V_{\mu}(y), S_{\lambda}(y) \rangle_0 W_{\mu}(x). \]
Thus we know
\[ a_{\lambda\mu} = \langle V_{\mu}(y), S_{\lambda}(y) \rangle_0. \]
The numbers \( g_{\mu'\nu} \) defined by
\[ P_{\mu'}(y) = \sum_{\nu \in P} g_{\mu'\nu} S_{\nu}(y) \]
are called the Stembridge coefficients and are known to be non-negative integers.
Also one finds the following formula in [3].
\[ S_{\mu'}(y^2) = \sum_{\xi \in P} \delta(\xi)\xi'_{\mu}^d S_{\xi}(y), \]
where \( \delta(\xi) \) is the 2-sign of \( \xi, (\xi[0], \xi[1]) \) is the 2-quotient of \( \xi \) (cf. [10]) and \( c_{\xi[0],\xi[1]}^{\mu} \) is the Littlewood-Richardson coefficient. Hence

\[
V_{\mu}(y) = P_{\mu^r}(y)S_{\mu^r}(y^2) = \sum_{\nu, \xi} \delta(\xi)g_{\mu^r, \nu^r}c_{\xi[0],\xi[1]}^{\mu^r}S_{\nu}(y)S_{\xi}(y)
= \sum_{\lambda} \left( \sum_{\nu, \xi} \delta(\xi)g_{\mu^r, \nu^r}c_{\xi[0],\xi[1]}^{\mu^r}c_{\xi[0],\xi[1]}^{\nu^r} \right) S_{\lambda}(y).
\]

Therefore

\[
a_{\lambda\mu} = \sum_{\nu, \xi} \delta(\xi)g_{\mu^r, \nu^r}c_{\xi[0],\xi[1]}^{\lambda}c_{\xi[0],\xi[1]}^{\mu^r}
\]
is an integer. \( \square \)

**Example 4.4.**

\[
A_3 = \begin{pmatrix}
(3) & (21, 0) & (1, 1) \\
(21) & 1 & 0 & 1 \\
(1^2) & 1 & 0 & -1
\end{pmatrix}
\]

\[
A_4 = \begin{pmatrix}
(4) & (31, 0) & (0, 2) & (0, 1^2) & (2, 1) \\
(31) & 1 & 1 & -1 & 0 & 1 \\
(2^2) & 0 & 1 & 1 & 1 & 0 \\
(1^4) & 1 & 0 & 0 & 1 & -1 \\
(21^2) & 1 & 1 & 0 & -1 & -1
\end{pmatrix}
\]

As for the columns corresponding to \((\mu, \emptyset)\) with \(\mu \in SP_n\), entries are non-negative integers. The submatrix consisting of these columns will be denoted by \(\Gamma_n\). The entries of \(\Gamma_n\) are the Stembridge coefficients, whose combinatorial nature has been known ([11], [8]).

Here we recall the definition of decomposition matrices for the \(p\)-modular representations of the symmetric group \(S_n\). Let \(p\) be a fixed prime number. A partition \(\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)\) is said to be \(p\)-regular of there are no parts satisfying \(\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1} \geq 1\). Note that a 2-regular partition is nothing but a strict partition. The set of \(p\)-regular partitions of \(n\) is denoted by \(P^{r(p)}_n\). A partition \(\rho = (1^{m_1}, 2^{m_2}, \cdots)\) is said to be \(p\)-class regular if \(m_p = m_{2p} = \cdots = 0\). Note that a 2-class regular partition is nothing but an odd partition. The set of \(p\)-class regular partitions of \(n\) is denoted by \(P^{e(p)}_n\). The \(p\)-Glaisher map \(\lambda \mapsto \tilde{\lambda}\) is defined in a natural way. This gives a bijection between \(P^{r(p)}_n\) and \(P^{e(p)}_n\). For \(\lambda \in P^{r(p)}_n\), we define the Brauer-Schur function \(B_{\lambda}^{(p)}(t)\) indexed by \(\lambda\) as follows.

\[
B_{\lambda}^{(p)}(t) = \sum_{\rho \in P^{r(p)}_n} \varphi_\lambda^p \frac{t_{m_1}^{m_1}t_{m_2}^{m_2} \cdots}{m_1!m_2! \cdots} \in \mathcal{F}(0)_n,
\]

where \(\varphi_\lambda^p\) is the irreducible Brauer character corresponding to \(\lambda\), evaluated at the \(p\)-regular conjugacy class \(\rho\). These functions form a basis for the space \(\mathcal{F}^{(p)}_n\) =
Given a Schur function $S_\lambda(t)$, define the $p$-reduced Schur function $S_\lambda^{(p)}(t)$ by “killing” all variables $t_1, t_2, \ldots$:

$$S_\lambda^{(p)}(t) = S_\lambda(t)|_{t_p=0}.$$ These $p$-reduced Schur functions are no longer linearly independent. All linear relations among these polynomials are known (cf. [2]). The $p$-decomposition matrix $D_n^{(p)} = (d_{\mu\lambda})$ is defined by

$$S_\lambda^{(p)}(t) = \sum_{\mu \in P_n^{(p)}} d_{\mu\lambda} B_\mu^{(p)}(t)$$

for $\lambda \in P_n$, and are known to satisfy the properties: $d_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$, $d_{\lambda\mu} = 0$ unless $\mu \geq \lambda$ and $d_{\lambda\lambda} = 1$. Here “$\geq$” denotes the dominance order.

Now let us go back to the case of $p = 2$. By definition, the Stembridge coefficients $\gamma_{\lambda\mu}$ ($\lambda \in P_n, \mu \in SP_n$) appear as

$$S_\lambda^{(2)}(t) = \sum_{\mu \in SP_n} \gamma_{\lambda\mu} Q_\mu(t).$$

Looking at the matrices $D_n^{(2)} = (d_{\mu\lambda})$ and $\Gamma_n = (\gamma_{\lambda\mu})$, one observes that they are “very similar”. We consider the Cartan matrix $C_n^{(2)} = tD_n^{(2)}D_n^{(2)}$ and the correspondent $G_n = \Gamma_n \Gamma_n$. There is a compact formula for the elementary divisors of $C_n^{(2)}$ (12): $2^{\ell(\lambda)} - t(\lambda)$ for $\lambda \in SP_n$.

**Theorem 4.5.** The elementary divisors of $C_n^{(2)}$ and $G_n$ coincide.

**Proof.** We put $\tilde{Z}_n = (2^{\ell(\lambda)} - t(\lambda))_{\lambda \in SP_n, \rho \in OP_n}$, $\Phi_n^{(2)} = (\varphi^\lambda_{\rho})_{\lambda \in SP_n, \rho \in OP_n}$ and $X_n^{(2)} = (\chi^\lambda_{\rho})_{\rho \in OP_n, \rho \in OP_n}$. The transition matrix $T_n = (t_{\lambda\mu})_{\mu, \lambda \in SP_n}$ is defined by

$$B_\lambda^{(2)}(t) = \sum_{\mu \in SP(n)} t_{\lambda\mu} Q_\mu(t).$$

By definition of $\Gamma_n$ and $D_n^{(2)}$, we have $X_n^{(2)} = \Gamma_n \tilde{Z}_n = D_n^{(2)} \Phi_n^{(2)}$ and $\Phi_n^{(2)} = T_n \tilde{Z}_n$. Hence, we have $\Gamma_n = X_n^{(2)} \tilde{Z}_n^{-1} = D_n^{(2)} \Phi_n^{(2)} \tilde{Z}_n^{-1} = D_n^{(2)} T_n$.

The matrix $\Gamma_n$ has the following properties: $\gamma_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$, $\gamma_{\lambda\mu} = 0$ unless $\mu \geq \lambda$ and $\gamma_{\lambda\lambda} = 1$ (11). Fix a total order in the set of partitions which is compatible with the dominance order, and we shall write $(d_{ij}), (\gamma_{ij})$ and $(t_{ij})$ in place of $(d_{\lambda\mu}), (\gamma_{\lambda\mu})$ and $(t_{\lambda\mu})$, respectively. Looking at the first row of $D_n^{(2)} T_n$, we have

$$\delta_{1j} = \gamma_{1j} = \sum_{k=1} d_{1kt_k j}.$$ This shows that $t_{1j} = \delta_{1j}$. As for the second row of $D_n^{(2)} T_n$, we have

$$\delta_{2j} = \gamma_{2j} = \sum_{k=1} d_{2kt_k j} = \sum_{k=2} d_{2kt_k j} \quad (j \geq 2).$$ This shows that $t_{2j} = \delta_{2j}$. Inductively, we can see that $T_n$ is a lower unitriangular integral matrix. Therfore the matrix $D_n^{(2)}$ and $\Gamma_n$ are transformed to each other by
column operations. By a standard argument we see that the elementary divisors of $C_n^{(2)}$ and $G_n$ coincide.

Our transition matrix $A_n = (a_{\lambda\mu})_{\lambda,\mu \in P_n}$ can be regarded as a common extension of the matrix $\Gamma_n$ of Stembridge coefficients and the decomposition matrix $D_n^{(2)}$.

**Theorem 4.6.**

\[
|\det A_n| = 2^{k_n},
\]

where $k_n = \sum_{\lambda \in P_n} \ell(\lambda^e) = \sum_{\lambda \in P_n} (\ell(\lambda^r) - \ell(\lambda^e))$.

**Proof.** We have four bases of $F(0)_n$: $S = (S_{\lambda}(x))_{\lambda \in P_n}$, $\tilde{S} = (S_{\lambda}(x))_{\lambda \in P_n}$, $V = (P_{\lambda^r}(x)S_{\lambda^e}(x^2))_{\lambda \in P_n}$ and $W = (Q_{\lambda^r}(x)S_{\lambda^e}(x^2))_{\lambda \in P_n}$. From Corollary 4.2, $W$ and $V$ are dual to each other with respect to the inner product $\langle , \rangle_{-1}$. Likewise, $\tilde{S}$ and $S$ are dual to each other. Hence we obtain

\[ t\, M(S,V)_n M(\tilde{S},W)_n = I, \]

where $M(S,V)_n$ denotes the transition matrix from the basis $S$ to the basis $V$ for $F(0)_n$. Since

\[ M(S,V)_n = M(S,\tilde{S})_n A_n M(W,V)_n, \]

we see that

\[ (\det A_n)^2 = \frac{1}{\det M(S,\tilde{S})_n \det M(W,V)_n}. \]

Let $X_n = (\chi_{\lambda})_{\lambda \rho}$ be the character table of $G_n$. We put $R_n = \text{diag}(z_{\rho}; \rho \in P_n)$ and $L_n = \text{diag}(2^{\ell(\rho)}; \rho \in P_n)$. Then we see that

\[ \det M(S,\tilde{S})_n = \det M(S,p)_n \det M(p,\tilde{S})_n \]

\[ = \det X_n R_n^{-1} \det L_n^{-1} X_n \]

\[ = \det L_n^{-1}, \]

and

\[ \det M(W,V)_n = \prod_{\lambda \in P_n} 2^{\ell(\lambda^e)}. \]

Hence we have

\[ \det A_n^2 = \prod_{\lambda \in P_n} 2^{\ell(\lambda) - \ell(\lambda^e)} = \prod_{\lambda \in P_n} 2^{2\ell(\lambda^e)} = \prod_{\lambda \in P_n} 2^{2\ell(\lambda^r)}. \]

Here is a small list of $k_n$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \cdots |
|-----|---|---|---|---|---|---|---|---|\cdots |
| $k_n$ | 0 | 1 | 1 | 4 | 5 | 11 | 15 | 28 | \cdots |

Next we consider the “Cartan-like” matrix $t\, A_n A_n$. The Frobenius formula for $W_\lambda$ reads

\[ p_\sigma p_\rho = \sum_{\lambda \in P_{\mu_0,\mu_1}} 2^{-\ell(\lambda^r)} X^r_{\sigma} x^e_{\rho} W_\lambda(x) \]
for \( \sigma \in OP_{n_0} \) and \( \rho \in P_{n_1} \), where the Green function \( X_{\sigma}^\lambda \) is defined by

\[
Q_{\lambda}(x) = \sum_{\sigma} \alpha^{\ell(\sigma)} z^{-1}_{\sigma} X_{\sigma}^\lambda p_{\sigma}
\]

for \( \lambda \in SP_n \). This formula shows that the transition matrix \( M(p, W)_n \) is, after a suitable sorting of rows and columns, decomposed into diagonal blocks, each block indexed by the pair \((n_0, n_1)\) with \( n_0 + 2n_1 = n \). We have

\[
\begin{aligned}
^{t}A_n A_n &= ^{t}M(p, W)_n^{t}M(\tilde{S}, p)_n M(\tilde{S}, p)_n M(p, W)_n \\
&= {^{t}M(p, W)_n}({^{t}L_nX_n^{-1}})(X_n R_n^{-1}L_n) M(p, W)_n \\
&= {^{t}M(p, W)_n}L_n R_n^{-1}L_n M(p, W)_n.
\end{aligned}
\]

Since \( L_n R_n^{-1} \) is diagonal matrix, \( ^{t}A_n A_n \) is block diagonal matrix, each block indexed by the pair \((n_0, n_1)\). Let denote \( B_{n_0, n_1} \) the corresponding block in \( ^{t}A_n A_n \). Note that the “principal” block \( B_{n,0} \) is nothing but the matrix \( G_n \).

**Example 4.7.**

\[
^{t}A_3 A_3 = \begin{pmatrix}
(3, \emptyset) & (21, \emptyset) & (1, 1) \\
3 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

\[
^{t}A_4 A_4 = \begin{pmatrix}
(4, \emptyset) & (31, \emptyset) & (\emptyset, 2) & (\emptyset, 1^2) & (2, 1) \\
4 & 2 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

**Theorem 4.8.**

\[
| \det B_{n_0, n_1} | = 2^{\sum_{\lambda \in \mathbb{P}_{n_0, n_1}} (\ell(\tilde{\lambda}) + \ell(\lambda^d) - \ell(\lambda^r))}.
\]

**Proof.** We have

\[
^{t}A_n A_n = ^{t}M(p, W)_n^{t}L_n R_n^{-1}L_n M(p, W)_n \\
= ^{t}M(p, W)_n^{t}L_n R_n^{-1}L_n M(\tilde{p}, V)_n M(V, W)_n \\
= ^{t}M(p, W)_n^{t}L_n R_n^{-1}M(\tilde{p}, V)_n M(V, W)_n,
\]

where \( \tilde{p}_\rho(x) = p_\rho(x, x) \). Note that \( ^{t}L_n, R_n^{-1}, M(V, W)_n \) are diagonal matrices. By requiring that \( \langle p_\rho, p_\sigma \rangle_{-1} = 2^{-\ell(\rho)} z_\rho \delta_{\rho \sigma} \), we obtain \( \langle p_\rho, \tilde{p}_\rho \rangle_{-1} = 2^{\ell(\rho) - \ell(\rho)} z_\rho \delta_{\rho \sigma} \). Hence,

\[
\det(M(\tilde{p}, V)_n) ^{t}M(p, W)_n Z_n^{-1}) = \det I.
\]

We recall that \( ^{t}A_n A_n \) is block diagonal. We have

\[
| \det B_{n_0, n_1} | = 2^{\sum_{\lambda \in \mathbb{P}_{n_0, n_1}} (\ell(\tilde{\lambda}) + \ell(\lambda^d) - \ell(\lambda^r))}.
\]

\[\square\]
For the principal block $B_{n,0}$, we have

$$|\det B_{n,0}| = 2\sum_{\lambda \in S_n} (\ell(\lambda) - \ell(\bar{\lambda})).$$

We conclude this note with an inner product expression of $t A_n A_n$.

**Proposition 4.9.**

$$t A_n A_n = \left( \langle P_{\lambda^r}(x), P_{\mu^r}(x) \rangle_0 \langle S_{\lambda^d}(x^2), S_{\mu^d}(x^2) \rangle_0 \right)_{\lambda,\mu}.$$

**Proof.** We have already given

$$t A_n A_n = t M(p,W)_n L_n R_n^{-1} M(p,W)_n.$$

Hence

$$\sum_{\sigma,\rho} 2^{-\ell(\lambda^r) - \ell(\mu^r)} X^\lambda_{\sigma} X^\mu_{\rho} \chi^\rho \chi^\mu \frac{2\ell(\sigma) + 2\ell(\rho)}{2} z_{\sigma}^{-1} z_{\rho}^{-1}$$

$$= 2^{-\ell(\lambda^r) - \ell(\mu^r)} \sum_{\sigma,\rho} \left( 2^{\ell(\sigma)} X^\lambda_{\sigma} X^\mu_{\rho} \chi^\rho \chi^\mu \frac{2\ell(\sigma) + 2\ell(\rho)}{2} z_{\sigma}^{-1} z_{\rho}^{-1} \right)$$

$$= 2^{-\ell(\lambda^r) - \ell(\mu^r)} \langle Q_{\lambda^r}, Q_{\mu^r} \rangle_0 \langle S_{\lambda^d}(x^2), S_{\mu^d}(x^2) \rangle_0.$$

□

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