Spherical T-duality

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Recent work:

P. Bouwknegt, J. Evslin and V. Mathai, *Spherical T-duality*,
[arXiv:1405.5844 [hep-th]].

P. Bouwknegt, J. Evslin and V. Mathai, *Spherical T-duality II: An infinity of spherical T-duals for non-principal SU(2)-bundles*,
[arXiv:1409.1296 [hep-th]].

Review based on:

P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: Topology Change from H-flux*, Comm. Math. Phys. 249 (2004) 383-415,
[arXiv:hep-th/0306062].
Fourier Transform

Fourier series for $f : S^1 \to \mathbb{R}$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-inx} \, dx$$

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$$

Fourier transform for $f : \mathbb{R} \to \mathbb{R}$

$$\hat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ipx} \, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx} \, dp$$
More generally, for $G$ a locally compact, abelian group, we have a Fourier transform $\mathcal{F} : \text{Fun}(G) \to \text{Fun}(\hat{G})$

$$\hat{f}(p) = \int_G f(x) e^{-ipx} \, dx = \mathcal{F}(f)(p)$$

$$f(x) = \int_{\hat{G}} \hat{f}(p) e^{ipx} \, dp$$

where

$$\hat{G} = \text{Hom}(G, U(1)) = \text{char}(G)$$

is the Pontryagin dual of $G$. I.e. a character is a $U(1)$ valued function on $G$, satisfying $\chi(x + y) = \chi(x)\chi(y)$. The characters form a locally compact, abelian group $\hat{G}$ under pointwise multiplication.
Fourier transform expresses the fact that the characters of $G$ span $\text{Fun}(G \times \hat{G})$.

We can think of $\chi(x, p) = e^{ipx} \in \text{Fun}(G \times \hat{G})$ as the universal character.
I.e. we have the following “correspondence”

\[ \mathcal{F}f = \hat{\pi}_*(\pi^*(f) \times \chi(x, p)) \]
T-duality is a geometric version of harmonic analysis, i.e. by replacing functions by geometric objects (such as bundles, sheaves, D-modules, ...) or, as an intermediate step, by topological characteristics associated to these objects (cohomology, K-theory, derived categories, ...).
Consider a manifold $P = M \times S^1$. By the Künneth theorem we have

$$H^\bullet(P) \cong H^\bullet(M) \otimes H^\bullet(S^1)$$

i.e.

$$H^n(P) \cong H^n(M) \oplus H^{n-1}(M)$$

We have a similar decomposition at the level of forms

$$\Omega^n(P)^{\text{inv}} \cong \Omega^n(M) \oplus \Omega^{n-1}(M).$$

i.e. invariant degree $n$ forms on $P$ are of the form $\omega$ or $\omega \wedge d\theta$, where $\omega$ is an $n$, respectively $n - 1$, form on $M$.

Consider $\hat{P} = M \times \hat{S}^1$. We have an isomorphism

$$\mathcal{F} : H^i(P) \cong H^{i+1}(\hat{P})$$
where

\[ H^0(P) = \bigoplus_{i \geq 0} H^{2i}(P), \quad H^1(P) = \bigoplus_{i \geq 0} H^{2i+1}(P), \]

Explicitly

\[ \omega \mapsto d\hat{\theta} \wedge \omega, \quad d\theta \wedge \omega \mapsto \omega \]

or

\[ \mathcal{F}\Omega = \int_{S^1} (1 + d\theta \wedge d\hat{\theta}) \Omega = \int_{S^1} e^{d\theta \wedge d\hat{\theta}} \Omega = \int_{S^1} e^{F} \Omega \]
I.e. \( \mathcal{F} \) is given by a correspondence

\[
\mathcal{F}\Omega = p_* (\hat{p}^* \Omega \wedge e^F)
\]
Once we recognize that $F = d\theta \wedge d\hat{\theta}$ is the curvature of a canonical linebundle $\mathcal{P}$ (the Poincaré linebundle) over $S^1 \times \hat{S}^1$, in fact $e^F = ch(\mathcal{P})$, this immediately suggests a ‘geometrization’ in terms of vector bundles over $P$ and $\hat{P}$.

$$\mathcal{F}E = p_*(\hat{p}^* E \otimes \mathcal{P})$$

This gives rise to the so-called Fourier-Mukai transform

$$\mathcal{F} : K^i(P) \xrightarrow{\cong} K^{i+1}(\hat{P})$$

which has many of the properties of the Fourier transform discussed earlier.

The discussion can be generalized to complexes of vector bundles (complexes of sheaves) and thus gives rise to a Fourier-Mukai correspondence between derived categories $D(P)$ and $D(\hat{P})$. 

Peter Bouwknegt

Spherical T-duality
Closed strings on $M \times S^1$ are described by

$$X : \Sigma \rightarrow M \times S^1$$

where $\Sigma = \{ (\sigma, \tau) \}$ is the closed string worldsheet. Upon quantization, we find

- **Momentum modes**: $p = \frac{n}{R}$
- **Winding modes**: $X(0, \tau) \sim X(1, \tau) + mR$

$$E = \left( \frac{n}{R} \right)^2 + (mR)^2 + \text{osc. modes}$$

We have a duality $R \rightarrow 1/R$, such that ST on $M \times S^1$ is equivalent to ST on $M \times \hat{S}^1$ (or a duality between IIA and IIB ST, for susy ST)
Suppose we have a pair \((P, H)\), consisting of a principal circle bundle

\[
\begin{array}{ccc}
S^1 & \longrightarrow & P \\
\downarrow & & \downarrow \\
\pi & & \pi \\
M & \downarrow & M \\
\end{array}
\]

and a so-called H-flux \(H\) on \(P\), a Čech 3-cocycle.

Topologically, \(P\) is classified by an element in \(F \in H^2(M, \mathbb{Z})\) while \(H\) gives a class in \(H^3(P, \mathbb{Z})\).
The (topological) T-dual of $(P, H)$ is given by the pair $(\hat{P}, \hat{H})$, where the principal $S^1$-bundle

\[
\begin{array}{ccc}
\hat{S}^1 & \rightarrow & \hat{P} \\
\downarrow & & \downarrow \hat{\pi} \\
M & &
\end{array}
\]

and the dual H-flux $\hat{H} \in H^3(\hat{P}, \mathbb{Z})$, satisfy

\[
\hat{F} = \pi_* H, \quad F = \hat{\pi}_* \hat{H}
\]

where $\pi_* : H^3(P, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})$, is the pushforward map ('integration over the $S^1$-fibre').
The ambiguity in the choice of $\hat{H}$ is (almost) removed by requiring that

$$\hat{p}^*H - p^*\hat{H} \equiv 0 \quad \in H^3(P \times_M \hat{P}, \mathbb{Z})$$

where $P \times_M \hat{P}$ is the correspondence space

$$P \times_M \hat{P} = \{(x, \hat{x}) \in P \times \hat{P} \mid \pi(x) = \hat{\pi}(\hat{x})\}$$
Gysin sequences

\[ \cdots \to H^3(M) \overset{\pi^*}{\to} H^3(P) \overset{\pi^*}{\to} H^2(M) \overset{\cup F}{\to} H^4(M) \to \cdots \]

\[ \cdots \to H^3(M) \overset{\hat{\pi}^*}{\to} H^3(\hat{P}) \overset{\hat{\pi}^*}{\to} H^2(M) \overset{\cup \hat{F}}{\to} H^4(M) \to \cdots \]
T-duality - Principal $S^1$-bundles

\[
\begin{array}{cccccccc}
0 & \rightarrow & H^1(M) & \rightarrow & H^1(\hat{P}) & \rightarrow & H^0(M) & \rightarrow & \cdots \\
\Uparrow \cup \hat{F} & & \Uparrow \hat{\pi}^* & & \Uparrow \hat{\pi}^* & & \Uparrow \hat{F} & \\
H^1(M) & \rightarrow & H^3(M) & \rightarrow & H^3(\hat{P}) & \rightarrow & H^2(M) & \rightarrow & \cdots \\
\Uparrow \hat{\pi}^* & & \Uparrow \hat{\pi}^* & & \Uparrow \hat{\pi}^* & & \Uparrow \hat{\pi}^* & \\
H^1(P) & \rightarrow & H^3(P) & \rightarrow & H^3(P \times_M \hat{P}) & \rightarrow & H^2(P) & \rightarrow & \cdots \\
\Uparrow \pi^* & & \Uparrow \pi^* & & \Uparrow \pi^* & & \Uparrow \pi^* & \\
H^0(M) & \rightarrow & H^2(M) & \rightarrow & H^2(\hat{P}) & \rightarrow & H^1(M) & \rightarrow & \cdots \\
\Uparrow \hat{F} & & \Uparrow \hat{\pi}^* & & \Uparrow \hat{\pi}^* & & \Uparrow \hat{F} & \\
\cdots & & \cdots & & \cdots & & \cdots & \\
\end{array}
\]
Consider principal $S^1$-bundles $P$ over $M = S^2$, then

$$H^2(M, \mathbb{Z}) \cong \mathbb{Z}, \quad H^3(P, \mathbb{Z}) \cong \mathbb{Z}$$

and we have, for example,

$$(S^2 \times S^1, 0) \longrightarrow (S^2 \times S^1, 0)$$

$$(S^2 \times S^1, 1) \longrightarrow (S^3, 0)$$

or more generally

$$(L_p, k) \longrightarrow (L_k, p)$$

where $L_p = S^3/\mathbb{Z}_p$ is the lens space.
Using $\Omega^k(P)^{inv} \cong \Omega^k(M) \oplus \Omega^{k-1}(M)$

$$F = dA, \quad H = H(3) + A \wedge H(2)$$

we find

$$\hat{F} = H(2) = d\hat{A}, \quad \hat{H} = H(3) + \hat{A} \wedge F$$

such that

$$\hat{H} - H = \hat{A} \wedge F - A \wedge \hat{F} = d(A \wedge \hat{A}).$$

**Theorem**

We have an isomorphism of ($\mathbb{Z}_2$-graded) differential complexes

$$T_* : (\Omega(P)^{inv}, d_H) \longrightarrow (\Omega(\hat{P})^{inv}, d_{\hat{H}})$$

where $d_H = d + H \wedge$. 
Proof.

Define

\[ T_\ast \omega = \int_{S^1} e^{A \wedge \hat{A}} \omega \]

then

\[ d_H T_\ast = T_\ast d_{\hat{H}}. \]

and consequently, we have isomorphisms

\[ T_\ast : H^i(P, H) \xrightarrow{\cong} H^{i+1}(\hat{P}, \hat{H}) \]
as well as

\[ T_* : K^i(P, H) \xrightarrow{\cong} K^{i+1}(\hat{P}, \hat{H}) \]

For example,

\[ K^i(L_p, k) \cong \begin{cases} \mathbb{Z}_k & i = 0 \\ \mathbb{Z}_p & i = 1 \end{cases} \]
Much of the above can be generalized to principal SU(2)-bundles:

Gysin sequence for principal SU(2)-bundles $\pi : P \rightarrow M$

\[
\cdots \rightarrow H^7(M) \xrightarrow{\pi^*} H^7(P) \xrightarrow{\pi^*} H^4(M) \xrightarrow{\cup c_2(P)} H^8(M) \rightarrow \cdots
\]

where

\[
c_2(P) = \frac{1}{8\pi^2} \text{Tr}(F \wedge F) \in H^4(M)
\]

is (a de Rham representative of) the 2nd Chern class of $P$. However, in this case,

\[
[M, BSU(2)] \rightarrow H^4(M, \mathbb{Z})
\]

is, in general, neither surjective nor injective.
SU(2) and quaternions

Recall that

\[
SU(2) = \left\{ U(a,b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \colon a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}
\]

can be identified with the unit sphere \( S(\mathbb{H}) = Sp(1) = S^3 \) in the quaternions

\[
\mathbb{H} = \{ \alpha + \beta i + \gamma j + \delta k : ij = k = -ji, \text{ cyclic} \}
\]

The isomorphism is given explicitly as

\[
SU(2) \ni U(a,b) \mapsto a + jb \in Sp(1) = S^3
\]

The relationship of principal SU(2)-bundles to quaternionic line bundles is analogous to the relationship of principal U(1)-bundles to complex line bundles.
Recall that a \textbf{quaternionic line bundle} over a manifold \( M \) is a complex rank 2 vector bundle \( V \rightarrow M \) together with a reduction of structure group to \( \mathbb{H} \setminus \{0\} \). Note that the unit sphere bundle \( S(V) \rightarrow M \) is an \( S^3 \)-bundle together with the inherited group structure, i.e. a principal SU(2)-bundle.

Conversely, given a principal SU(2)-bundle \( P \rightarrow M \), then the associated vector bundle

\[
V = P \times_{\text{SU}(2)} \mathbb{H} \rightarrow M
\]

is a quaternionic line bundle.
Principal SU(2)-bundles on $S^4$ are described by smooth maps $g : SU(2) \to SU(2)$. Let $g(z) = z$, $z \in SU(2)$, which is a degree 1 map. Then $g(z) = z^r$, $r \in \mathbb{Z}$ is a degree $r$ map. Let $P(r) \to S^4$ be the corresponding principal SU(2)-bundle on $S^4$. Then $c_2(P(r)) = r \in \mathbb{Z} \cong H^4(S^4, \mathbb{Z})$.

The principal SU(2)-bundle $S^7 = P(1) \to S^4$ is known as the Hopf bundle.
Let $M$ be a compact, connected, oriented 4-dimensional manifold. Then one can show fairly easily that isomorphism classes of principal $SU(2)$-bundles $P$ on $M$ is canonically identified with homotopy classes $[M, S^4] \cong H^4(M; \mathbb{Z})$ given by $c_2(P)$.

More precisely, given a degree 1 map $h : M \to S^4$, then $h^*(P(r)) \to M$ is a principal $SU(2)$-bundle on $M$ with $c_2(h^*(P(r))) = r \in \mathbb{Z} \cong H^4(M, \mathbb{Z})$. 
Recall the Gysin sequence for principal $\text{SU}(2)$-bundles $\pi : P \to M$

\[ \cdots \to H^7(M) \xrightarrow{\pi^*} H^7(P) \xrightarrow{\pi^*} H^4(M) \xrightarrow{\cup c_2(P)} H^8(M) \xrightarrow{} \cdots \]

We consider pairs of the form $(P, H)$ consisting of a principal $\text{SU}(2)$-bundle $P \to M$ and a 7-cocycle $H$ on $P$.

The Gysin sequence implies that $\pi_*$ is a canonical isomorphism $H^7(P, \mathbb{Z}) \cong H^4(M, \mathbb{Z}) \cong \mathbb{Z}$, and intuitively spherical T-duality exchanges $H$ with the second Chern class $c_2$. 

Spherical T-duality
More precisely, the **spherical T-dual** bundle $\hat{\pi} : \hat{P} \to M$ is defined by $c_2(\hat{P}) = \pi_* H$ while the dual 7-cocycle $\hat{H} \in H^7(\hat{P})$ is related to $c_2(P)$ by the isomorphism $\hat{\pi}_*$, via a similar Gysin sequence for $\hat{P} \to M$. 
Let $M$ be a connected compact, oriented, 4 dimensional manifold, and consider the principal $\text{SU}(2)$-bundle $P(r)$ over $M$ with $c_2(P(r)) = r \in \mathbb{Z} \cong H^4(M, \mathbb{Z})$, together with the 7-cocycle $H = s \text{ vol}$ on $P(r)$.

Since $H \cup H = 0$ for dimension reasons, we can define integer-valued $H$-twisted cohomology as

$$H^\bullet(P(r), H; \mathbb{Z}) \equiv H^\bullet(H^\bullet(P(r); \mathbb{Z}), H \cup).$$
Use the Gysin sequence to calculate the cohomology groups $H^{even/odd}(F(p); \mathbb{Z})$, and obtain for $p \neq 0$

\[
H^j(P(r); \mathbb{Z}) = H^{4-j}(M; \mathbb{Z}), \quad j = 0, 1, 2, 3
\]

\[
H^4(P(r); \mathbb{Z}) = \mathbb{Z}_r \oplus H^1(M; \mathbb{Z})
\]

\[
H^{7-j}(P(r); \mathbb{Z}) = H^{4-j}(M; \mathbb{Z}), \quad j = 0, 1, 2, 3
\]

Therefore there is an isomorphism of 7-twisted cohomology groups over the integers with a parity change,

**Theorem**

\[
H^{even}(P(r), s; \mathbb{Z}) \cong H^{odd}(P(s), r; \mathbb{Z}),
\]

\[
H^{odd}(P(r), s; \mathbb{Z}) \cong H^{even}(P(s), r; \mathbb{Z}).
\]

There is a similar isomorphism of 7-twisted K-theories.
Beyond dimension 4 the situation becomes more complicated as not all integral 4-cocycles of $M$ are realized as $c_2$ of a principal SU(2)-bundle $\pi : P \to M$ and moreover multiple bundles can have the same $c_2(P)$.

More precisely, principal SU(2)-bundles are classified up to isomorphism by homotopy classes of maps into the classifying space $M \to BSU(2)$. However, the complete homotopy type of $S^3 = SU(2)$ is still unknown, and therefore also for $BSU(2)$.

However Serre’s theorem tells us that $\pi_j(BSU(2)) \otimes \mathbb{Q} \cong \pi_j(K(\mathbb{Z}, 4)) \otimes \mathbb{Q}$, i.e. the homotopy groups of degree higher than 4 are all torsion.
For example, recall that principal SU(2)-bundles over $S^5$ are classified by $\pi_4(\text{SU}(2)) \cong \mathbb{Z}_2$, while $H^4(S^5, \mathbb{Z}) = 0$.

By a theorem of Granja, there is a natural number $N(d)$ where $d = \dim(M)$, such that if $\alpha \in N(d) \times H^4(M, \mathbb{Z})$, then it is the 2nd Chern class of a principal SU(2)-bundle over $M$. Therefore a pair $(P, H)$ is spherical T-dualizable if $\pi_*(H) \in N(d) \times H^4(M; \mathbb{Z})$. Then $\pi_*(H) = c_2(\hat{P})$ where $\hat{P}$ is a principal SU(2)-bundle over $M$. However, this does not necessarily uniquely specify $\hat{P}$. But at most, there are finitely many choices.

We will simply assert that a spherical T-dual $\hat{\pi} : \hat{P} \to M$ be any SU(2)-bundle with $c_2(\hat{P}) = \pi_* H$, with $\hat{H}$ defined such that $\hat{\pi}_* \hat{H} = c_2(P)$ with $\hat{p}^* H = p^* \hat{H}$ on the correspondence space $P \times_M \hat{P}$. 
T-duality induces an isomorphism on twisted cohomologies with real or rational coefficients.

Theorem

\[ H^{\text{even}}(P, H; \mathbb{Q}) \cong H^{\text{odd}}(\hat{P}, \hat{H}; \mathbb{Q}), \]
\[ H^{\text{odd}}(P, H; \mathbb{Q}) \cong H^{\text{even}}(\hat{P}, \hat{H}; \mathbb{Q}). \]

There is a similar isomorphism of 7-twisted K-theories with parity shift, upto \( \mathbb{Z}_2 \)-extensions.
T-duality for non-principal SU(2)-bundles (non-uniqueness, even for $S^4$)

A generalised geometry counterpart of spherical T-duality?

What is the physics behind spherical T-duality?

What are useful geometric realisations of integral 7-cocycles?

Is there a useful geometric description of 7-twisted K-theory?

When $\dim M \geq 4$, then it is known that not every spherical pair $(P, H)$ has a spherical T-dual. Can the missing spherical T-duals be obtained some other way?

Is there a C*-algebra version of spherical T-duality?
THANK YOU