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New results on the \( q \)-generalized Bernoulli polynomials of level \( m \)

Abstract: This paper aims to show new algebraic properties from the \( q \)-generalized Bernoulli polynomials \( B_{n}^{[m-1]}(x; q) \) of level \( m \), as well as some others identities which connect this polynomial class with the \( q \)-generalized Bernoulli polynomials of level \( m \), as well as the \( q \)-gamma function, and the \( q \)-Stirling numbers of the second kind and the \( q \)-Bernstein polynomials.

Keywords: \( q \)-generalized Bernoulli polynomials, \( q \)-gamma function, \( q \)-Stirling numbers, \( q \)-Bernstein polynomials

MSC 2010: 33E12

1 Introduction

Fix a fixed \( m \in \mathbb{N} \), the generalized Bernoulli polynomials of level \( m \) are defined by means of the following generating function \([1]\)

\[
\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} B_{n}^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < 2\pi,
\]

where the generalized Bernoulli numbers of level \( m \) are defined by \( B_{n}^{[m-1]}(0) \), for all \( n \geq 0 \). We can say that if \( m = 1 \) in (1.1), then we obtain the definition via a generating function, of the classical Bernoulli polynomials \( B_{n}(x) \) and classical Bernoulli numbers, respectively, i.e. \( B_{n}(x) = B_{n}^{[0]}(x) \) and \( B_{n} = B_{n}^{[0]} \), respectively.

The \( q \)-analogue of the classical Bernoulli numbers and polynomials were initially investigated by Carlitz [2]. More recently, J. Choi, T. Ernst, D. kim, S. Nalci, C.S. Ryoo [3–8] defined the \( q \)-Bernoulli polynomials using different methods and studied their properties. There are numerous recent investigations on \( q \)-generalizations of this subject by many others author; see [9–17]. More recently, Mahmudov et al. [18] used the \( q \)-Mittag-Leffler function

\[
E_{1,m+1}(z; q) := \frac{z^m}{e^z_q - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q}!}, \quad m \in \mathbb{N},
\]

to define the generalized \( q \)-Apostol Bernoulli numbers and \( q \)-Apostol Bernoulli polynomials in \( x, y \) of order \( \alpha \) and level \( m \) using the following generating functions, respectively

\[
\frac{z^m}{(e^z_q + T_{m-1, q}(z))_\alpha} = \sum_{n=0}^{\infty} B_{n, \alpha}^{[m-1]}(\lambda) \frac{z^n}{[n]_q!},
\]

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The \( q \)-analogue of the \( \alpha \)-binomial coefficient is defined by

\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!},
\]

where \( [n]_q! = \prod_{k=1}^{n} [k]_q \) and \( [0]_q! = 1 \). The \( q \)-shifted factorial is defined as

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N},
\]

\[
(a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a), \quad a, q \in \mathbb{C}; \quad |q| < 1.
\]

The \( q \)-binomial coefficient is defined by

\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}, \quad (n, k \in \mathbb{N}; 0 \leq k \leq n).
\]

The \( q \)-analogue of the function \( (x + y)^n \) is defined by

\[
(x + y)_q^n := \sum_{k=0}^{n} \binom{n}{k}_q q^k x^{n-k} y^k, \quad n \in \mathbb{N}_0,
\]

\[
(1 - a)_q^n = (a; q)_n = \sum_{k=0}^{n} \binom{n}{k}_q q^k (-1)^k a^k = \prod_{j=0}^{n-1} (1 - q^j a).
\]
The \(q\)-derivative of a function \(f(z)\) is defined by
\[
D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(qz) - f(z)}{(q - 1)z}, \quad 0 < |q| < 1, \quad \mathbb{C}.
\]
The \(q\)-analogue of the exponential function is defined in two ways
\[
e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1 - q|} \tag{2.1}
\]
\[
E_q^z = \sum_{n=0}^{\infty} q^{\frac{1}{n} n(n-1) \frac{z^n}{[n]_q!}} = \prod_{k=0}^{\infty} (1 + (1 - q^k z)), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.
\]
In this sense, we can see that
\[
e_q^z \cdot E_q^z = 1, \quad e_q^z \cdot E_q^z = e_q^{z+w}.
\]
Therefore,
\[
D_q e_q^z = e_q^z, \quad D_q E_q^z = E_q^{z+1}.
\]

**Definition 2.1.** For any \(t > 0\)
\[
\Gamma_q(t) = \int_{0}^{\infty} x^{t-1} E_q^{-qx} d_q x
\]
is called the \(q\)-gamma function.

The Jackson’s \(q\)-gamma function is defined in \([20, 24]\) as follows
\[
\Gamma_q(x) = \frac{(q; q)^{x}}{(q^x; q)^{\infty}} (1 - q)^{1-x}, \quad 0 < |q| < 1,
\]
replacing \(x\) by \(n + 1\) we have
\[
\Gamma_q(n + 1) = \frac{(q; q)^{n+1}}{(q^{n+1}; q)^{\infty}} (1 - q)^{-n} = (q; q)_n (1 - q)^{-n} = [n]_q!, \quad n \in \mathbb{N}.
\]
Furthermore, it satisfies the following relations
\[
\Gamma_q(1) = 1, \quad \Gamma_q(n) = [n - 1]_q!, \quad \Gamma_q(n + 1) = [x]_q \Gamma_q(x).
\]

**Definition 2.2.** \([25]\) For \(\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0\) and \(|q| < 1\) the function \(E_{\alpha, \beta}^\gamma(z; q)\) is defined as
\[
E_{\alpha, \beta}^\gamma(z; q) = \sum_{n=0}^{\infty} (q; q)_n \frac{z^n}{(q^n; q)^{\infty}} \Gamma_q(an + \beta).
\]
Note that when \(\gamma = 1\) the equation above is expressed as
\[
E_{\alpha, \beta}(z; q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(an + \beta)} \tag{2.2}
\]
From (2.2), setting \(\alpha = 1\) and \(\beta = m + 1\), we can deduce that
\[
\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n + m + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{[n + m]_q!} = \frac{1}{z^m} \sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} = \left( \frac{e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_q!}}{z^m} \right). \tag{2.3}
\]
The $q$-Stirling number of the first kind $s(n, k)_q$ and the $q$-Stirling number of the second kind $S(n, k)_q$ are the coefficients in the expansions, (see [26, p.173])

\[
(x)_{n,q} = \sum_{k=0}^{n} s(n, k)_q x^k, \\
x^n = \sum_{k=0}^{n} S(n, k)_q (x)_{k,q}, \tag{2.4}
\]

where

\[
(x)_{k,q} = \prod_{n=0}^{k-1} (x - [n]_q).
\]

Let $C[0, 1]$ denote the set of continuous functions on $[0, 1]$. For any $f \in C[0, 1]$, the $q$-Bernstein operator of order $n$ for $f$ and is defined as (see [15, p.3 Eq. (28)])

\[
\mathcal{B}_n(f; x) = \sum_{r=0}^{n} f_r \binom{n}{r} \left( \frac{x^r}{q^r} \right) \prod_{s=0}^{n-r-1} \left( 1 - q^s x \right) = \sum_{r=0}^{n} f_r b_{n,r}(x),
\]

where $f = f([r]_q/[n]_q)$. The $q$-Bernstein polynomials of degree $n$ or a $q$-Bernstein basis are defined by

\[
b_{n,r}(x) = \binom{n}{r} x^r \prod_{s=0}^{n-r-1} \left( 1 - q^s x \right).
\]

We know that \(\sum_{k=0}^{n-j} b_{n-j,k}(x) = 1\), and so

\[
x^j = \sum_{k=0}^{n-j} \binom{n-j}{k} \left( \frac{x^k}{q^k} \right) \prod_{t=0}^{n-k-1} \left( 1 - q^t x \right).
\]

By using the identity

\[
\binom{n-j}{k} \left( \frac{x}{q} \right)^j = \binom{n}{j} \binom{k}{j} \left( \frac{x}{q} \right)^j,
\]

we have

\[
x^j = \sum_{k=j}^{n-j} \binom{n}{j} \left( \frac{x}{q} \right)^j b_{n,k}(x). \tag{2.5}
\]

Otherwise, setting $\alpha = \lambda = 1$ in the equation (1.2), we have the following definition:

**Definition 2.3.** Let $m \in \mathbb{N}$, $q, z \in \mathbb{C}$, $0 < |q| < 1$. The $q$-generalized Bernoulli polynomials $B_n^{[m-1]}(x; q)$ of level $m$ are defined in a suitable neighborhood of $z = 0$ by means of the generating function

\[
\left( \frac{z^m}{e^{qz} - \sum_{l=0}^{m-1} \frac{z^l}{[l]_q}} \right) e^{xz} e^{yz} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x + y; q) \frac{z^n}{[n]_q!}, \quad |z| < 2\pi, \tag{2.6}
\]

where the $q$-generalized Bernoulli numbers of level $m$ are defined by

\[
B_n^{[m-1]}(q) := B_n^{[m-1]}(0; q).
\]

Furthermore,

\[
B_n^{[m-1]}(x, y; q) := B_n^{[m-1]}(x + y; q),
\]

\[
B_n^{[m-1]}(x, 0; q) := B_n^{[m-1]}(x; q),
\]

\[
B_n^{[m-1]}(0, y; q) := B_n^{[m-1]}(y; q).
\]
The first three $q$-generalized Bernoulli polynomials of level $m$ (cf. [18, p.7]) are

\[
B_0^{[m-1]}(x; q) = [m]_q!, \\
B_1^{[m-1]}(x; q) = [m]_q! \left( x - \frac{1}{[m+1]_q} \right), \\
B_2^{[m-1]}(x; q) = [m]_q! \left( x^2 - \frac{2}{[m+1]_q} + \frac{2q^m}{[m+2]_q[m+1]_q^2} \right).
\]

Also, the first three $q$-generalized Bernoulli numbers of level $m$ are

\[
B_0^{[m-1]}(q) = [m]_q!, \\
B_1^{[m-1]}(q) = \frac{[m]_q!}{[m+1]_q}, \\
B_2^{[m-1]}(q) = \frac{2}[m]_q! [q]_q^{m+1} \frac{1}{[m+2]_q[m+1]_q^2}.
\]

**Definition 2.4.** [14] Let $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$. The $q$-Bernoulli polynomials in $x, y$ of order $\alpha$ are defined by means of the generating function

\[
\left( \frac{ze^{xz}}{e^z-1} \right)^\alpha e^{yz} E_q^x = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; q) \frac{z^n}{[n]_q!}, \quad |z| < 2\pi,
\]

where the $q$-Bernoulli numbers of order $\alpha$ are defined by

\[ B_n^{(\alpha)}(q) := B_n^{(\alpha)}(0; q). \]

Furthermore

\[
B_n^{(\alpha)}(x, q) := B_n^{(\alpha)}(x, 0; q), \\
B_n^{(\alpha)}(y, q) := B_n^{(\alpha)}(0, y; q).
\]

## 3 Properties of the $q$-generalized Bernoulli polynomials of level $m$

In this section, we show some properties of the $q$-generalized Bernoulli polynomials $B_n^{[m-1]}(x; q)$ of level $m$. We demonstrated the facts for one of them. Obviously, by applying a similar technique, other ones can be determined. The following proposition summarizes some properties of the polynomials $B_n^{[m-1]}(x; q)$. We will only show in details the proofs to (2), (5) and (7).

**Proposition 3.1.** Let a fixed $m \in \mathbb{N}$, $n, k \in \mathbb{N}_0$ and $q \in \mathbb{C}$, $0 < |q| < 1$. Let \( \left\{ B_n^{[m-1]}(x; q) \right\}_{n=0}^{\infty} \) be the sequence of $q$-generalized Bernoulli polynomials of level $m$. Then the following statements hold.

1. **Summation formula.** For every $n \geq 0$

\[
B_n^{[m-1]}(x; q) = \sum_{k=0}^{n} \binom{n}{k}_q B_k^{[m-1]}(q)x^{n-k}.
\]

2. For $n \geq 1$

\[
\sum_{k=0}^{n} \binom{n+m}{k}_q B_k^{[m-1]}(q) = 0.
\]
3. Addition formulas

\[ B_n^{[m-1]}(x + y; q) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]}(x; q) y^{n-k}, \quad (3.2) \]

\[ B_n^{[m-1]}(x + y; q) = \sum_{k=0}^{n} \binom{n}{k} \frac{q^{\frac{k}{2}(n-k)(n-k-1)}}{q} B_k^{[m-1]}(y; q)x^{n-k}, \quad (3.3) \]

\[ B_n^{[m-1]}(x + y; q) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]}(q)(x + y)^{n-k}, \]

\[ B_n^{[m-1]}(x + y; q) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]}(y; q)x^{n-k}. \]

4. Inversion formulas

\[ x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{[k]_q!}{[k + m]_q!} B_{n-k}^{[m-1]}(x; q), \quad (3.4) \]

\[ y^n = \frac{[n]_q!}{q^{2(n-1)} [n + m]_q!} \sum_{k=0}^{n} \binom{n + m}{k} B_k^{[m-1]}(y; q), \quad (3.5) \]

\[ x^n = \sum_{k=0}^{n} \frac{[n]_q! B_k^{[m-1]}(x; q)}{[k]_q!T_q(n-k + m + 1)}. \quad (3.6) \]

5. Difference equations

\[ B_n^{[m-1]}(1 + y; q) - B_n^{[m-1]}(y; q) = [n]_q \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{p_{k}^{[m-1]}(y, q)B_{n-1-k}^{(-1)}(q)}{q}. \]

6. Differential relations. For \( m \in \mathbb{N} \) and \( n, j \in \mathbb{N}_0 \), where \( 0 \leq j \leq n \), we have

\[ D_q B_n^{[m-1]}(x; q) = [n + 1]_q B_n^{[m-1]}(x; q), \quad (3.7) \]

\[ D_q^j B_n^{[m-1]}(x; q) = \frac{[n]_q!}{[n-j]_q!} B_n^{[m-1]}(x; q). \]

7. Integral formulas

\[ \int_{x_0}^{x} B_n^{[m-1]}(x; q)d_q x = \frac{B_n^{[m-1]}(x; q) - B_n^{[m-1]}(x_0; q)}{[n+1]_q}, \quad n \in \mathbb{N}_0, \quad (3.8) \]

\[ B_{n-1}^{[m-1]}(x; q) = [n]_q \int_{0}^{x} B_{n-1}^{[m-1]}(x; q)d_q x + B_{n}^{[m-1]}(q), \quad n \in \mathbb{N}, \]

\[ \int_{x_0}^{x} B_n^{[m-1]}(x; q)d_q x = \sum_{k=0}^{n} \frac{n}{k} \binom{n}{k} B_k^{[m-1]}(q) \left( \frac{x^{n+1-k} - x_0^{n+1-k}}{[n+1]_q} \right). \]

Proof. To prove (2), we start with (2.1) and (2.6), from which it follows that

\[ z^m = \left( \sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left( \sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} \right) = \left( \sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left( \sum_{j=0}^{\infty} \frac{z^j}{[j+m]_q!} \right), \]

and therefore

\[ 1 = \left( \sum_{n=0}^{\infty} B_n^{[m-1]}(q) \frac{z^n}{[n]_q!} \right) \left( \sum_{j=0}^{\infty} \frac{z^j}{[j+m]_q!} \right). \]
By multiplying (2.6) and (2.7), we have

\[ B_0^{[m-1]}(q) = \frac{1}{[m]_q!} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{B_k^{[m-1]}(q)}{[k]_q!} \frac{z^n}{[n-k+m]_q!}, \]

Therefore

\[ I = \sum_{n=0}^{\infty} \frac{B_n^{[m-1]}(1+y; q) z^n}{[n]_q!} B_n^{[m-1]}(y; q) \frac{z^n}{[n]_q!} = \frac{z^m}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} E_q^m(e_q^z - 1) \]

By comparing coefficients of \( \frac{z^n}{[n]_q!} \) on both sides we obtain the result.

**Proof.** Proof of (5). Considering the expression \( B_n^{[m-1]}(1+y; q) - B_n^{[m-1]}(y; q) \) and using the generating functions (2.6) and (2.7), we have

\[ I := \sum_{n=0}^{\infty} \frac{B_n^{[m-1]}(1+y; q) z^n}{[n]_q!} - \sum_{n=0}^{\infty} \frac{B_n^{[m-1]}(y; q) z^n}{[n]_q!} = \left( \frac{z^m}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right) E_q^m(e_q^z - 1) \]

By comparing coefficients of \( \frac{z^n}{[n]_q!} \) on both sides we obtain the result.

**Proof.** Proof of (7). From (3.7) we have

\[ B_n^{[m-1]}(x; q) = \frac{1}{[n+1]_q} D_q B_n^{[m-1]}(x; q). \]

Now, by integrating on both sides of the equation above, we get

\[ \int_{x_0}^{x_1} B_n^{[m-1]}(x; q) dx = \frac{1}{[n+1]_q} \int_{x_0}^{x_1} D_q B_n^{[m-1]}(x; q) dx \]
From identities (2.4), (2.5) and Proposition 3.1 we can deduce some interesting algebraic relations between polynomials of level \( q \)-generalized Bernoulli polynomials of level \( m \). Setting \( x_0 = 0 \) and \( x_1 = x \) in (3.8), we have

\[
\int_0^x B_n^{[m-1]}(x; q) \, dq = \frac{B_n^{[m-1]}(x; q) - B_n^{[m-1]}(0; q)}{[n + 1]_q},
\]

and so

\[
\int_0^x B_n^{[m-1]}(x; q) \, dq = \frac{B_n^{[m-1]}(x; q) - B_n^{[m-1]}(0; q)}{[n]_q}.
\]

Finally, we get

\[
B_n^{[m-1]}(x; q) = [n]_q \int_0^x B_n^{[m-1]}(x; q) \, dq + B_n^{[m-1]}(q).
\]

\[\square\]

\section{Some connection formulas for the polynomials \( B_n^{[m-1]}(x + y; q) \)}

From identities (2.4), (2.5) and Proposition 3.1 we can deduce some interesting algebraic relations between the \( q \)-generalized Bernoulli polynomials of level \( m \) with the \( q \)-gamma function, the \( q \)-Stirling numbers of the second kind and the \( q \)-Bernstein polynomials.

\textbf{Proposition 4.1.} For \( n, j, k \in \mathbb{N}_0 \), \( q \in \mathbb{C} \) where \( 0 < |q| < 1 \) and where \( m \in \mathbb{N} \), the \( q \)-generalized Bernoulli polynomials of level \( m \) are related with the \( q \)-gamma function by the means of the following identity

\[
B_n^{[m-1]}(x + y; q) = [n]_q! \sum_{j=0}^n \sum_{k=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! [n]_q! [n-j-k+m+1]_q} B_j^{[m-1]}(y; q).
\]

\textbf{Proof.} By substituting (3.6) in (3.3), we have

\[
B_n^{[m-1]}(x + y; q) = \sum_{j=0}^n \sum_{k=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{[n]_q! [j]_q! [n-j-k+m+1]_q}{[n-j-k+m+1]_q} B_j^{[m-1]}(y; q)
\]

\[
= \sum_{j=0}^n \sum_{k=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{[n]_q! [j]_q! [n-j-k+m+1]_q}{[n-j-k+m+1]_q} B_j^{[m-1]}(y; q)
\]

\[
= [n]_q! \sum_{j=0}^n \sum_{k=0}^{n-k} \frac{B_k^{[m-1]}(x; q)}{[k]_q!} \frac{[n]_q! [j]_q! [n-j-k+m+1]_q}{[n-j-k+m+1]_q} B_j^{[m-1]}(y; q).
\]

\textbf{Corollary 4.1.} For \( n, j, k \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \), we have

\[
B_n^{[m-1]}(x; q) = [n]_q! \sum_{j=0}^n \sum_{k=0}^{n-k} \frac{B_j^{[m-1]}(q) B_k^{[m-1]}(x; q)}{[j]_q! [k]_q! [n-j-k+m+1]_q}.
\]
Proof. By replacing equation (3.6) in (3.1), we obtain

\[
B_n^{[m-1]}(x; q) = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] q^\frac{(n-j)(n-j-1)}{2} B_j^{[m-1]}(x; q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! [n-j-k+m+1]_q! [n-j-k+m+1]_q!}.
\]

By substituting equation (3.6) in (3.1), we obtain

\[
= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{[n]_q! B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! [n-j-k+m+1]_q!}.
\]

Corollary 4.2. For \(n, j, k \in \mathbb{N}_0\) and \(m \in \mathbb{N}\), we have

\[
B_n^{[m-1]}(x; q) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{[n]_q! B_j^{[m-1]}(x; q)}{[k+m]_q! [n-j-k]_q! [j]_q!}.
\]  

Proof. By substituting (3.4) in equation (3.1), we obtain

\[
B_n^{[m-1]}(x; q) = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] q^\frac{(n-j)(n-j-1)}{2} B_j^{[m-1]}(x; q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! [n-j-k+m+1]_q!}.
\]

By substituting the equation (3.5) in (3.2), we obtain

\[
= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{[n]_q! B_j^{[m-1]}(x; q)}{[k+m]_q! [n-j-k]_q! [j]_q!}.
\]

Corollary 4.3. For \(n, j, k \in \mathbb{N}_0\) and \(m \in \mathbb{N}\)

\[
B_n^{[m-1]}(x+y; q) = [n]_q! \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{B_j^{[m-1]}(x; q) B_k^{[m-1]}(y; q)}{[k+m]_q! [n-j-k+m]_q! [j]_q!}.
\]

Proof. By substituting the equation (3.5) in (3.2), we obtain

\[
B_n^{[m-1]}(x+y; q) = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] q^\frac{(n-j)(n-j-1)}{2} B_j^{[m-1]}(x; q) \sum_{k=0}^{n-j} \frac{[n-j]_q! B_k^{[m-1]}(x; q)}{[k]_q! [j]_q! [n-j-k+m]_q!}.
\]

Proposition 4.2. For \(n, j, k \in \mathbb{N}_0\), \(q \in \mathbb{C}\) where \(0 < |q| < 1\) and where \(m \in \mathbb{N}\), the \(q\)-generalized Bernoulli polynomials of level \(m\) are related with the \(q\)-Stirling numbers of the second kind \(S(n, k; q)\) by means of the following identities

\[
B_n^{[m-1]}(x+y; q) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \left[ \begin{array}{c} n \\ j \end{array} \right] q^\frac{(n-j)(n-j-1)}{2} B_j^{[m-1]}(y; q) S(n-j, k; q) (x)_q; k.
\]

\[
B_n^{[m-1]}(x; q) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \left[ \begin{array}{c} n \\ j \end{array} \right] B_j^{[m-1]}(q) S(j, k; q) (x)_q; k.
\]
Proof. Proof of (4.5). By replacing (2.4) in (3.1), we have
\[
B_n^{[m-1]}(x + y; q) = \sum_{n=0}^{\infty} \left[ \sum_{j=0}^{n} \binom{n}{j}_q \right] q^k x^{n-j} y^j B_j^{[m-1]}(y; q) S(n-j, k; q)(x)_q k.
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{n-k} \binom{n}{j}_q q^k x^{n-j} y^j B_j^{[m-1]}(y; q) S(n-j, k; q)(x)_q k.
\]

\[
\text{Corollary 4.4. For } n, k \in \mathbb{N}_0 \text{ and } m \in \mathbb{N}, \text{ we obtain}
\]
\[
\sum_{k=0}^{\infty} S(n, k) q(x)_q = \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{n-k} \binom{n}{j}_q \right] \frac{[k]_q!}{[k+m]_q!} B_n^{[m-1]}(x; q).
\]

Prooposition 4.3. For \( n, j, k \in \mathbb{N}_0, q \in \mathbb{C} \) where \( 0 < |q| < 1 \) and where \( m \in \mathbb{N} \) the \( q \)-generalized Bernoulli polynomials of level \( m \) are related with the \( q \)-Bernstein polynomials \( b_{n,k}(x; q) \) by means of the following identities
\[
B_n^{[m-1]}(x, q) = \sum_{j=0}^{n} \binom{n}{j}_q \sum_{k=j}^{n-j} \binom{k}{j}_q b_{n,k}(x; q), \tag{4.7}
\]
\[
B_n^{[m-1]}(x + y; q) = \sum_{j=0}^{n} \binom{n}{j}_q \sum_{k=j}^{n-j} \binom{k}{j}_q \frac{q^k}{[k]_q} B_{n-j}^{[m-1]}(y; q) b_{n,k+j}(x; q). \tag{4.8}
\]

Proof. Proof of (4.7). By replacing (2.5) in (3.1), we have
\[
B_n^{[m-1]}(x, q) = \sum_{j=0}^{n} \binom{n}{j}_q B_{n-j}^{[m-1]}(q) \sum_{k=j}^{n-j} \binom{k}{j}_q b_{n,k}(x; q)
\]
\[
= \sum_{j=0}^{n} \sum_{k=j}^{n-j} B_{n-j}^{[m-1]}(q) \binom{k}{j}_q b_{n,k}(x; q)
\]
\[
= \sum_{j=0}^{n} \sum_{k=j}^{n-j} \binom{k}{j}_q \frac{q^k}{[k]_q} B_{n-j}^{[m-1]}(y; q) b_{n,k+j}(x; q).
\]

Proof. Proof of (4.8). By replacing (2.5) in Equation (3.3), we obtain
\[
B_n^{[m-1]}(x + y; q) = \sum_{j=0}^{n} \binom{n}{j}_q q^k \frac{q^k}{[k]_q} B_{n-j}^{[m-1]}(y; q) b_{n,k}(x; q)
\]
\[
= \sum_{j=0}^{n} \sum_{k=j}^{n-j} \binom{k}{j}_q q^k \frac{q^k}{[k]_q} B_{n-j}^{[m-1]}(y; q) b_{n,k+j}(x; q)
\]
\[
= \sum_{j=0}^{n} \sum_{k=j}^{n-j} \binom{k}{j}_q q^k \frac{q^k}{[k]_q} B_{n-j}^{[m-1]}(y; q) b_{n,k+j}(x; q).
\]
Corollary 4.5. For $n, j \in \mathbb{N}_0$ and $x \in [0, 1]$, we have
\[
\sum_{k=0}^{n} \frac{[k]_q}{[j]_q} b_{n,k}(x; q) = \sum_{k=0}^{n} \frac{[j]_q}{[k]_q} \frac{[k]_q!}{[k+m]_q!} B_{j-k}(x; q).
\]

Proposition 4.4. For $n, j, k \in \mathbb{N}_0$ and $n \geq j \geq k \geq 0$, we have
\[
b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \binom{n-j}{k} \frac{[j]_q!B_{n-k-j}^{[m-1]}(1-x; q)}{[j]_q [j+m]_q!}.
\]

Proof. To prove (4.9), we used the following equality [14, Theorem 19, p. 10]
\[
\frac{x^ke_q^z E_q^{xz}}{|k|_q!} = \sum_{n=0}^{\infty} b_{n,k}(x; q) \frac{z^n}{|n|_q!}.
\]
We see that
\[
\frac{x^ke_q^z E_q^{xz}}{|k|_q!} = \frac{x^ke_q^z}{|k|_q!} \left( e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{|h|_q!} \right)
\]
\[
\frac{z^m}{|k|_q!} \left( e_q^z - \sum_{h=0}^{m-1} \frac{z^h}{|h|_q!} \right).
\]
Next, by using the equations (2.3) and (2.6), we get
\[
\frac{x^ke_q^z E_q^{xz}}{|k|_q!} = \frac{x^ke_q^z}{|k|_q!} \sum_{n=0}^{\infty} \frac{z^n}{|n|_q!} \sum_{j=0}^{\infty} B_{j-k}^{[m-1]}(1-x; q) \frac{z^j}{[j]_q!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^k}{|k|_q!} \frac{z^j}{|j|_q!} \frac{B_{j-k}^{[m-1]}(1-x; q) z^n}{[n-j]_q!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{n-k} \frac{x^k}{|k|_q!} \frac{B_{j-k}^{[m-1]}(1-x; q) z^n}{[n-j]_q!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{n-k} \binom{n-j}{k} \frac{[j]_q!B_{n-k-j}^{[m-1]}(1-x; q) z^n}{[j+m]_q!}.
\]
By comparing coefficients of $\frac{z^n}{|n|_q!}$ on both sides, we obtain
\[
b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \binom{n-j}{k} \frac{[j]_q!B_{n-k-j}^{[m-1]}(1-x; q)}{[j+m]_q!}.
\]
To demonstrate (4.10) we used the identity $\Gamma_q(j+m+1) = [j+m]_q!$ and Equation (4.9). Continuing this process, we get
\[
b_{n,k}(x; q) = x^k \sum_{j=0}^{n-k} \binom{n-j}{k} \frac{[j]_q!B_{n-k-j}^{[m-1]}(1-x; q)}{[j+m]_q!}.
\]
References

[1] Natalini P., Bernardini A., A generalization of the Bernoulli polynomials, J. Appl. Math., 2003, 3, 155–163
[2] Carlitz L., q-Bernoulli numbers and polynomials, Duke Math., 1948, 15, 987–1000
[3] Choi J., Anderson P., Srivastava H. M., Carlitz’s q-Bernoulli and q-Euler numbers and polynomials and a class of q-Hurwitz zeta functions, Appl. Math. Comput., 2009, 215, 1185–1208
[4] Ernst T., q-Bernoulli and q-Euler polynomials, an umbral approach, Int. J. Difference Equ., 2006, 1, 31–80
[5] Hegazi A. S., Mansour M., A note on q-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys., 2006, 13(1), 9–18
[6] Kim D., Kim M.-S., A note on Carlitz q-Bernoulli numbers and polynomials, Adv. Difference Equ., 2012, 2012:44
[7] Quintana Y., Ramírez W., Uriel E., Generalized Apostol-type polynomials matrix and its algebraic properties, Math. Rep., 2019, 21, 249–264
[8] Ryoo C. S., A note on q-Bernoulli numbers and polynomials, Appl. Math. Lett, 2017, 20(5), 524–531
[9] Garg M., Alha S., A new class of q-Apostol-Bernoulli polynomials of order a, Revi. Tecn. URU, 2014, 6, 67–76
[10] Hernandes P., Quintana Y., Uriel E., About extensions of generalized Apostol-type polynomials, Res. Math., 2015, 68, 203–225
[11] Kurt B., A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials B_{2D}(x, y), Appl. Math. Sci., 2010, 4(47), 2315–2322
[12] Kurt B., Some relationships between the generalized Apostol-Bernoulli and Apostol-Euler polynomials, Turk. J. Anal. Num. The., 2013, 1(1), 54–58
[13] Luo Q.-M., Guo B.-N., Qi F., Debnath L., Generalizations of Bernoulli numbers and polynomials, Int. J. Math. Math. Sci., 2003, 59, 3769–3776
[14] Mahmudov N. I., On a class of q-Bernoulli and q-Euler polynomials, Adv. Difference Equ., 2013, 1, 108–125
[15] Ramírez W., Castilla L., Uriel E., An extended generalized q-extensions for the Apostol type polynomials, Abstr. Appl. Anal., 2018, Article ID 2937950, DOI: 10.1155/2018/2937950
[16] Tremblay R., Gaboury S., Fugere J., A further generalization of Apostol-Bernoulli polynomials and related polynomials, Hon. Math. Jou., 2012, 34, 311–326
[17] Quintana Y., Ramírez W., Uriel E., On an operational matrix method based on generalized Bernoulli polynomials of level m, Calcolo, 2018, 55, 30
[18] Mahmudov N. I., Eini Keleshteri M., q-extensions for the Apostol type polynomials, J. Appl. Math., 2014, Article ID 868167, http://dx.doi.org/10.1155/2014/868167
[19] Ernst T., The history of q-calculus and a new method, Licentiate Thesis, Dep. Math. Upps. Unive., 2000
[20] Gasper G., Rahman M., Basic Hypergeometric Series, Cambr. Univ. Press, 2004
[21] Kac V., Cheung P., Quantum Calculus, Springer-Verlag New York, 2002
[22] Araci S., Duran U., Acikgoz M., (p, q)-Volkenborn integration, J. Number Theory, 2017, 171, 18–30
[23] Araci S., Duran U., Acikgoz M., Srivastava H. M., A certain (p, q)-derivative operator and associated divided differences, J. Ineq. Appl., 2016, 2016:301, DOI: 10.1186/s13660-016-1240-8
[24] Srivastava H. M., Choi J., Zeta and q-zeta functions and associated series and integrals, Editorial Elsevier, Boston, 2012, DOI: 10.1016/C2010-0-67023-4
[25] Sharma S., Jain R., On some properties of generalized q-Mittag Leffler, Math. Aeterna, 2014, 4(6), 613–619
[26] Ernst T., A comprehensive treatment of q-calculus, Birkhäuser, 2012
[27] Ostrovskaya S., q-Bernstein polynomials and their iterates, J. Approx. Theory, 2003, 123(2), 232–255