FROM QUANTUM QUASI-SHUFFLE ALGEBRAS TO BRAIDED
ROTA-BAXTER ALGEBRAS

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Abstract. In this letter, we use quantum quasi-shuffle algebras to construct
Rota-Baxter algebras, as well as tridendriform algebras. We also propose the
notion of braided Rota-Baxter algebras, which is the relevant object of Rota-
Baxter algebras in a braided tensor category. Examples of such new algebras
are provided by using quantum multi-brace algebras in a category of Yetter-
Drinfeld modules.

1. Introduction

Rota-Baxter operators originate from a work on probability theory more than
four decades ago. In the paper [4], G. Baxter deduced many well-known identi-
ties in the theory of fluctuations for random variables from a simple relation of
operators on a commutative algebra. Later, based on this work and others, G.-C.
Rota introduced the notion of Baxter algebra, which is called Rota-Baxter algebra
nowadays, in his fundamental papers [24]. Since then, this new algebraic object is
investigated by many mathematicians with various motivations. Besides their own
interest in mathematics, Rota-Baxter algebras have many significant applications
in mathematical physics. For instance, they play essential role in the Hopf alge-
braic approach to the Connes-Kreimer theory of renormalization in perturbative
quantum field theory. There the theory of non-commutative Rota-Baxter algebras
with idempotent Rota-Baxter operator is used to provide an algebraic setting for
the formulation of renormalization (cf. [7], [9], [10]). Rota-Baxter algebras are
also related to the works of pre-Poisson algebras [1] and Loday-type algebras [8].

There is a seminal but implicit connection between Rota-Baxter algebras and
another important kind of algebras. This is the quasi-shuffle algebra. Quasi-shuffle
algebras first arose in the study of the cofree irreducible Hopf algebra built on
an arbitrary associative algebra by K. Newman and D. E. Radford ([21]). In 2000,
they were rediscovered independently by M. E. Hoffman in an inductive form ([16]).
In the same year, L. Guo and W. Keigher ([12] and [13]) defined a new algebra
structure called mixable shuffle algebra and used it to construct free commutative
Rota-Baxter algebras. Nevertheless, the mixable shuffle algebra is equivalent to
the quasi-shuffle algebra in some sense. Recently, quasi-shuffle algebras have many
important applications in other areas of mathematics, such as multiple zeta values

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algebra, braided Rota-Baxter algebra, quantum multi-brace algebra.
(13 and 17), Rota-Baxter algebras (7), and commutative tridendriform algebras (22). They also appear in mathematical physics. For instance, D. Kreimer used quasi-shuffle algebras to study shuffle identities between Feynman graphs (21).

On the other hand, after the birth of quantum groups, mathematicians and mathematical physicists begin to be interested in braided tensor categories and the quantization of classical algebraic structures. These braided or quantized objects not only provide new subjects but also bring deeper interpretation of the classical ones. They also provide new examples and tools in the research of noncommutative geometry and conformal field theory (cf. 3). For the importance of quasi-shuffle algebras and as a subsequent work of his quantum shuffles, M. Rosso constructed the quantum quasi-shuffle algebra which is the quantization of quasi-shuffle algebras. Some interesting properties and applications of quantum quasi-shuffle algebras have been discovered during the last few years (cf. 19 and 11). To our surprise, by applying a similar construction of Guo and Keigher (14) to quantum quasi-shuffle algebras, we can still provide Rota-Baxter algebras and tridendriform algebras. This phenomenon does not happen usually since the complicated action of the braiding makes quantum objects behave quite differently from the usual ones. Based on this construction, we can also construct idempotent Rota-Baxter operators. This discovery extremely enlarges the families of Rota-Baxter algebras and tridendriform algebras. Meanwhile, we consider the relevant object of Rota-Baxter algebras in a braided tensor category. In order to define such an object, we have to impose compatibility between the braiding and the multiplication, as well as the Rota-Baxter operator. Examples of these new algebras are constructed from quantum multi-brace algebras in a category of Yetter-Drinfeld modules.

This paper is organized as follows. In Section 2, we construct Rota-Baxter algebras and tridendriform algebras by using quantum quasi-shuffle algebras. In Section 3, we define the notion of braided Rota-Baxter algebra and provide examples from quantum multi-brace algebras in a category of Yetter-Drinfeld modules.

**Notation.** In this paper, we denote by $K$ a ground field of characteristic 0. All the objects we discuss are defined over $K$. For a vector space $V$, we denote by $T(V)$ the tensor algebra of $V$, by $\otimes$ the tensor product within $T(V)$, and by $\otimes$ the one between $T(V)$ and $T(V)$.

We denote by $S_n$ the symmetric group acting on the set $\{1, 2, \ldots, n\}$ and by $s_i$, $1 \leq i \leq n-1$, the standard generators of $S_n$ permuting $i$ and $i+1$.

A braiding $\sigma$ on a vector space $V$ is an invertible linear map in $\text{End}(V \otimes V)$ satisfying the quantum Yang-Baxter equation on $V^{\otimes 3}$:

$$(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

A braided vector space $(V, \sigma)$ is a vector space $V$ equipped with a braiding $\sigma$. For any $n \in \mathbb{N}$ and $1 \leq i \leq n-1$, we denote by $\sigma_i$ the operator $\text{id}_{V^{i-1}} \otimes \sigma \otimes \text{id}_{V^{n-i-1}} \in \text{End}(V^{\otimes n})$. For any $w \in S_n$, we denote by $T^\sigma_w$ the corresponding lift of $w$ in the braid group $B_n$, defined as follows: if $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression of $w$, then $T^\sigma_w = \sigma_{i_1} \cdots \sigma_{i_l}$. This definition is well-defined (see, e.g., Theorem 4.12 in 20).
We define $\beta : T(V) \otimes T(V) \to T(V) \otimes T(V)$ by requiring that the restriction of $\beta$ on $V^\otimes i \otimes V^\otimes j$, denoted by $\beta_{ij}$, is $T_{\chi_{ij}}$, where

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & 2 & \cdots & j \end{pmatrix} \in \mathfrak{S}_{i+j},$$

for any $i, j \geq 1$. For convenience, we interpret $\beta_{0i}$ and $\beta_{i0}$ as the identity map of $V^\otimes i$.

Let $(C, \Delta, \varepsilon)$ be a coalgebra. We denote $\Delta^{(0)} = \text{id}_C$, $\Delta^{(1)} = \Delta$, and $\Delta^{(n+1)} = (\Delta^{(n)} \otimes \text{id}_C) \circ \Delta$ recursively for $n \geq 1$. We adopt Sweedler’s notation for coalgebras and comodules: for any $c \in C$,

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)},$$

for a left $C$-comodule $(M, \rho)$ and any $m \in M$,

$$\rho(m) = \sum m_{(-1)} \otimes m_{(0)}.$$

2. Rota-Baxter algebras coming from quantum quasi-shuffle algebras

We start by recalling some basic notions. In this letter, by an algebra we always mean an associative $K$-algebra which is not necessarily unital.

**Definition 2.1.** Let $\lambda$ be an element in $K$. A pair $(R, P)$ is called a *Rota-Baxter algebra of weight $\lambda$* if $R$ is an algebra and $P$ is a linear endomorphism of $R$ satisfying that for any $x, y \in R$,

$$P(x)P(y) = P(xy) + P(xP(y)) + P(P(x)y) + \lambda P(xy).$$

The map $P$ is called a *Rota-Baxter operator*.

For a systematic and detailed introduction of these algebras, we refer the readers to the book [13].

By the works [12] and [14], Rota-Baxter algebras are closely related to mixable shuffle algebras. More precisely, mixable shuffle algebras provide the free object in the category of commutative Rota-Baxter algebras. Now we generalize this construction to the quantized version of mixable shuffle algebras. Our framework is the braided category. We need to work with the relevant object of associative algebras in braided categories.

**Definition 2.2.** Let $A = (A, m)$ be an algebra with product $m$, and $\sigma$ be a braiding on $A$. We call the triple $(A, m, \sigma)$ a *braided algebra* if it satisfies the following conditions:

$$(\text{id}_A \otimes m)\sigma_1 \sigma_2 = \sigma(m \otimes \text{id}_A),$$

$$(m \otimes \text{id}_A)\sigma_2 \sigma_1 = \sigma(\text{id}_A \otimes m).$$

Moreover, if $A$ is unital and its unit $1_A$ satisfies that for any $a \in A$,

$$\sigma(a \otimes 1_A) = 1_A \otimes a,$$

$$\sigma(1_A \otimes a) = a \otimes 1_A,$$

then $A$ is called a *unital braided algebra*. 
Remark 2.3. 1. Let $\tau$ be the usual flip map which switches two arguments $a \otimes b \mapsto b \otimes a$. Obviously, it is a braiding. For any algebra $(A, m)$, it is evident that the first two identities in the definition of braided algebras hold automatically for $m$ and $\tau$. Hence any associative algebra is a braided algebra with respect to $\tau$. On the other hand, for any braided vector space $(V, \sigma)$, there is a trivial braided algebra structure on it whose multiplication is the trivial one $m = 0$.

2. The braided algebra structure is very crucial. Given a braided vector space $(V, \sigma)$, it is not reasonable that there should be a non-trivial braided algebra structure on $V$. For instance, let $V$ be a vector space with basis $\{e_1, e_2\}$. Consider the following two braidings on $V$:

$$\sigma(e_1 \otimes e_1) = e_1 \otimes e_1,$$
$$\sigma(e_1 \otimes e_2) = q e_2 \otimes e_1,$$
$$\sigma(e_2 \otimes e_1) = q e_1 \otimes e_2 + (1 - q^2)e_2 \otimes e_1,$$
$$\sigma(e_2 \otimes e_2) = e_2 \otimes e_2,$$

and

$$\sigma'(e_i \otimes e_j) = q e_j \otimes e_i, \quad \forall i, j,$$

where $q \in \mathbb{K}$ is nonzero and not equal to $\pm 1$.

We mention that $\sigma$ is of Hecke type. It comes originally from the R-matrix of the quantum enveloping algebra $U_q\mathfrak{sl}_2$. Both of these braidings play important roles in the theory of quantum groups. But by an easy argument on the structure coefficients of the multiplication, one can show that the only product on $V$ which is compatible with $\sigma$ is just the trivial one. The case is the same for $\sigma'$.

Since $\sigma$ is of Hecke type, we can ask the following interesting question: Does there exist a non-trivial braided algebra whose braiding is of Hecke type?

3. For any braided algebra $(A, m, \sigma)$, one can embed it into a unital braided algebra $(\tilde{A}, \tilde{m}, \tilde{\sigma})$ in the following way. First of all, we set $\tilde{A} = \mathbb{K} \oplus A$. Then we define the multiplication $\tilde{m}$ and the braiding $\tilde{\sigma}$ by: for any $\lambda, \mu \in \mathbb{K}$ and $a, b \in A$,

$$\tilde{m}((\lambda + a) \otimes (\mu + b)) = \lambda \mu + \lambda \cdot b + \mu \cdot a + m(a \otimes b),$$

and

$$\tilde{\sigma}((\lambda + a) \otimes (\mu + b)) = \mu \otimes \lambda + b \otimes \lambda + \mu \otimes a + \sigma(a \otimes b).$$

It is easy to verify that $(\tilde{A}, \tilde{m}, \tilde{\sigma})$ is a braided algebra with unit $1 \in \mathbb{K}$.

Given a braided algebra $(A, m, \sigma)$, we define a map $\otimes_\sigma; T(A) \otimes T(A) \to T(A)$ as follows. For any $\lambda \in \mathbb{K}$ and $x \in T(A)$,

$$\lambda \otimes_\sigma x = x \otimes_\sigma \lambda = \lambda \cdot x.$$
(a_1 \otimes \cdots \otimes a_i) \ltimes_{\sigma} b_1
\begin{align*}
  &= \ a_1 \otimes \left((a_2 \otimes \cdots \otimes a_i) \ltimes_{\sigma} b_1\right) + \beta_{i,1}(a_1 \otimes \cdots \otimes a_i \otimes b_1) \\
  &\quad + (m \otimes \text{id}_{A}^{\otimes i-1})(\text{id}_{A} \otimes \beta_{i-1,1})(a_1 \otimes \cdots \otimes a_i \otimes b_1),
\end{align*}
and
\begin{align*}
  (a_1 \otimes \cdots \otimes a_i) \ltimes_{\sigma} (b_1 \otimes \cdots \otimes b_j)
  &= a_1 \otimes \left((a_2 \otimes \cdots \otimes a_i) \ltimes_{\sigma} (b_1 \otimes \cdots \otimes b_j)\right) \\
  &\quad + (\text{id}_{A} \otimes \ltimes_{\sigma(i,j-1)})(\beta_{i,1} \otimes \text{id}_{A}^{\otimes j-1})(a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j) \\
  &\quad + (m \otimes \ltimes_{\sigma(i-1,j-1)})(\text{id}_{A} \otimes \beta_{i-1,1} \otimes \text{id}_{A}^{\otimes j-1})(a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j),
\end{align*}
where \ltimes_{\sigma(i,j)} denotes the restriction of \ltimes_{\sigma} on \ A^{\otimes i} \otimes A^{\otimes j}.

Then \ T^{sh}_{\sigma}(A) = (T(A), \ltimes_{\sigma}) is an associative algebra with unit 1 \in \mathbb{K}, and called the \textit{quantum quasi-shuffle algebra} built on \ ((A, m, \sigma)) is the quantization of the usual quasi-shuffle algebra. We observe that \ T^{sh}_{\sigma}(A) is a filtered algebra with the filtration \ T^{sh}_{\sigma}(A)^{n} = \bigoplus_{i=0}^{n} A^{\otimes i} (\text{see Propssion 4.22 in [18]}). For more information about quantum quasi-shuffle algebras, one can see [19].

Let \ (A, m, 1_{A}, \sigma) be a unital braided algebra. Then manifestly \ (A, \lambda \cdot m, \sigma) is a braided algebra for any \ \lambda \in \mathbb{K}. We denote by \ltimes_{\lambda} the quantum quasi-shuffle product with respect to \ ((A, \lambda \cdot m, \sigma)) by Lemma 3 in [2], the space \ A^{\otimes} \ T^{sh}_{\sigma}(A) is a unital associative algebra with the product
\[ \diamond_{\sigma,\lambda} = (m \otimes \ltimes_{\sigma,\lambda})(\text{id}_{A} \otimes \beta \otimes \text{id}_{T(A)}). \]
We denote by \mathcal{R}_{\sigma,\lambda}(A) the pair \ (A^{\otimes} \ T^{sh}_{\sigma}(A), \diamond_{\sigma,\lambda}). We can view \mathcal{R}_{\sigma,\lambda}(A) as \ T^{+}(A) = \bigoplus_{i \geq 1} A^{\otimes i} at the level of vector spaces. We have two products \ltimes_{\sigma,\lambda} and \diamond_{\sigma,\lambda} on \ T^{+}(A). This is an example of 2-braided algebras which produce quantum multi-brace algebras (for the definitions, see [18]). We define an endomorphism \ P : \mathcal{R}_{\sigma,\lambda}(A) \to \mathcal{R}_{\sigma,\lambda}(A) by
\begin{align*}
P(a_{0} \otimes u) &= 1_{A} \otimes a_{0} \otimes u, \text{ if } u \in T^{+}(A), \\
P(a_{0} \otimes \nu) &= 1_{A} \otimes \nu \cdot a_{0}, \text{ if } \nu \in \mathbb{K}.
\end{align*}

**Theorem 2.4.** Under the assumptions above, the pair \ ((\mathcal{R}_{\sigma,\lambda}(A), P)) is a Rota-Baxter algebra of weight \ \lambda.

**Proof.** Observe that \ \beta(u \otimes 1_{A}) = 1_{A} \otimes u for any \ u \in T(A). Therefore for any \ a, b \in A and \ x \in A^{\otimes i}, y \in A^{\otimes j}, we have
\begin{align*}
P((a \otimes x) \diamond_{\sigma,\lambda} P(b \otimes y))
  &= P((a \otimes x) \ltimes_{\sigma,\lambda}(1_{A} \otimes b \otimes y)) = P\left(a \otimes (x \ltimes_{\sigma,\lambda} (b \otimes y))\right) \\
  &= 1_{A} \otimes (x \ltimes_{\sigma,\lambda} (b \otimes y)),
\end{align*}
\begin{align*}
P(P(a \otimes x) \diamond_{\sigma,\lambda} (b \otimes y))
  &= P((1_{A} \otimes a \otimes x) \diamond_{\sigma,\lambda} (b \otimes y)) \\
  &= 1_{A} \otimes ((\text{id}_{A} \otimes \ltimes_{\sigma,\lambda(i+1,j)})(\beta_{i+1,1} \otimes \text{id}_{A}^{\otimes j})(a \otimes x \otimes b \otimes y)),
\end{align*}
and
\[
\lambda P((a \otimes x) \check{\alpha}_{\sigma,\lambda} (b \otimes y))
\]
\[
= 1_A \otimes (\lambda \cdot m \otimes \check{\alpha}_{\sigma,\lambda}) (\text{id}_A \otimes \beta_{i+1} \otimes \text{id}_A^\otimes)(a \otimes b \otimes y).
\]
By taking a summation, we get
\[
P((a \otimes x) \check{\alpha}_{\sigma,\lambda} P(b \otimes y)) + P(P(a \otimes x) \check{\alpha}_{\sigma,\lambda} (b \otimes y)) + \lambda P((a \otimes x) \check{\alpha}_{\sigma,\lambda} (b \otimes y))
\]
\[
= 1_A \otimes \left( a \otimes (x \check{\alpha}_{\sigma,\lambda} (b \otimes y)) + (\text{id}_A \otimes \check{\alpha}_{\sigma,\lambda}(i+1,j)) (\beta_{i+1} \otimes \text{id}_A^\otimes)(a \otimes b \otimes y) + (\lambda \cdot m \otimes \check{\alpha}_{\sigma,\lambda}) (\text{id}_A \otimes \beta_{i+1} \otimes \text{id}_A^\otimes)(a \otimes b \otimes y) \right)
\]
\[
= 1_A \otimes ((a \otimes x) \check{\alpha}_{\sigma,\lambda} (b \otimes y))
\]
\[
= P(a \otimes x) \check{\alpha}_{\sigma,\lambda} P(b \otimes y).
\]
□

Here we mention that if \( \sigma \) is the usual flip map, the resulting Rota-Baxter algebra \((\mathcal{R}_{\sigma,\lambda}(A), P)\) in the above theorem is the free commutative Rota-Baxter algebra constructed in [14] and [12]. Hence this result provides concrete examples of Rota-Baxter algebras in a much bigger framework.

Besides the above construction, we can also provide idempotent Rota-Baxter operators on \( \mathcal{R}_{\sigma,\lambda}(A) \) even \( A \) is not unital. In order to construct such operators, we need the following proposition. For a proof, one can see Theorem 1.1.13 in [14] or verify it directly.

**Proposition 2.5.** Suppose \( R \) is an algebra and \( R_1 \) and \( R_2 \) are two subalgebras of \( R \) such that \( R = R_1 \oplus R_2 \) as vector space. Then the projection \( P \) from \( R \) onto \( R_1 \) is an idempotent Rota-Baxter operator of weight -1.

Note that \( \mathcal{R}_{\sigma,\lambda}(A) = A \otimes T(A) = (A \otimes \mathbb{K}) \bigoplus (\bigoplus_{i \geq 1} A \otimes A^\otimes)^i \) as a vector space. Since \( \beta_{01} \) is the identity map and \( T^\text{sh}(A) \) is a filtered algebra, both of \( A = A \otimes \mathbb{K} \) and \( \bigoplus_{i \geq 1} A \otimes A^\otimes \) are subalgebras of \( \mathcal{R}_{\sigma,\lambda}(A) \). Therefore we have immediately the following.

**Theorem 2.6.** Suppose \((A, m, \sigma)\) is a braided algebra. Then the projection \( P_1 \) from \( \mathcal{R}_{\sigma,\lambda}(A) \) onto its subalgebra \( A \) is an idempotent Rota-Baxter operator of weight -1. So is the projection \( P_2 \) from \( \mathcal{R}_{\sigma,\lambda}(A) \) onto its subalgebra \( \bigoplus_{i \geq 1} A \otimes A^\otimes \).

Now we turn to tridendriform algebras which were introduced by Loday and Ronco ([23]).

**Definition 2.7.** Let \( V \) be a vector space, and \( \prec, \succ \) and \( \cdot \) be three binary operations on \( V \). The quadruple \((V, \prec, \succ, \cdot)\) is called a tridendriform algebra if the following
relations are satisfied: for any \( x, y, z \in V \),
\[
(x \prec y) \prec z = x \prec (y \ast z),
(x \succ y) \prec z = x \succ (y \prec z),
(x \ast y) \succ z = x \succ (y \succ z),
(x \succ y) \cdot z = x \succ (y \cdot z),
(x \prec y) \cdot z = x \cdot (y \succ z),
(x \cdot y) \prec z = x \cdot (y \prec z),
(x \cdot y) \cdot z = x \cdot (y \cdot z),
\]
where \( x \ast y = x \prec y + x \succ y + x \cdot y \).

**Theorem 2.8.** Let \((A, m, \sigma)\) be a braided algebra. We define three operations \(\cdot, \prec, \succ\) on \(T(A)\) recursively by: for any \(a, b \in A\) and any \(x \in A^\otimes_i, y \in A^\otimes_i\),
\[
(a \otimes x) \cdot (b \otimes y) = (m \otimes \kappa_{(i,j)})( \id_A \otimes \beta_{i,1} \otimes \id_A^{\otimes 2} )(a \otimes x \otimes b \otimes y),
(a \otimes x) \prec (b \otimes y) = a \otimes (x \kappa_\sigma (b \otimes y)),
(a \otimes x) \succ (b \otimes y) = (\id_A \otimes \kappa_{(i+1,j)})(\beta_{i+1,1} \otimes \id_A^{\otimes 2})(a \otimes x \otimes b \otimes y).
\]

Then \((T(A), \prec, \succ, \cdot)\) is a tridendriform algebra.

**Proof.** All the verifications are direct. We just need that \(\kappa_\sigma\) is associative and compatible with the braiding \(\beta\) in the sense of braided algebras. For instance, we show the third condition. For any \(a \in A\) and \(x, y, z \in T(A)\),
\[
(x \kappa_\sigma \cdot z) \succ (a \otimes z)
\]
\[
= (\id_A \otimes \kappa_\sigma)(\beta_{i,1} \otimes \id_{T(A)})(\kappa_\sigma \otimes \id_A \otimes \id_{T(A)})(x \otimes y \otimes a \otimes z)
= (\id_A \otimes \kappa_\sigma)(\beta_{i,1} \otimes \id_{T(A)})(x \otimes y \otimes a \otimes z)
= (\id_A \otimes \kappa_\sigma)(\beta_{i,1} \otimes \id_{T(A)})(x \otimes y \otimes a \otimes z)
= (\id_A \otimes \kappa_\sigma)(\beta_{i,1} \otimes \id_{T(A)})(x \otimes y \otimes a \otimes z)
= x \succ (y \succ (a \otimes z)).
\]

Here we denote by \(\beta_{i,1}\) the action of \(\beta\) on \(A^\otimes_i \otimes A\) where \(?\) is a number determined by the corresponding computation. \(\square\)

**Remark 2.9.** Let \((R, \cdot, P)\) be a Rota-Baxter algebra of weight 1. Define \(a \prec b = a \cdot P(b)\) and \(a \succ b = P(a) \cdot b\). Then \((R, \prec, \succ, \cdot)\) is a tridendriform algebra (see \(\Re\)). Using this fact, we can prove the above theorem by an earlier argument: embed \(A\) into the unital braided algebra \((\tilde{A}, \tilde{m}, 1, \tilde{\sigma})\), then the tridendriform algebra
structure in Theorem 2.8 comes from the Rota-Baxter algebra \((R_{\varsigma,1}(\tilde{A}), P)\) defined in Theorem 2.4.

3. Braided Rota-Baxter algebras and their examples

As we mentioned in the introduction, braided categories play an important role in mathematical physics. So we would like to extend the notion of Rota-Baxter algebras in braided categories.

**Definition 3.1.** A triple \((R, P, \sigma)\) is called a **braided Rota-Baxter algebra of weight** \(\lambda\) if \((R, \sigma)\) is a braided algebra and \(P\) is an endomorphism of \(R\) such that \((R, P)\) is a Rota-Baxter algebra of weight \(\lambda\) and \(\sigma(P \otimes P) = (P \otimes P)\sigma\).

**Example 3.2.** Let \((A, m, 1_A, \sigma)\) be a unital braided algebra and \((\mathcal{R}_{\sigma, \lambda}(A), P)\) be the Rota-Baxter algebra defined before. Then \((\mathcal{R}_{\sigma, \lambda}(A), P, \beta)\) is a braided Rota-baxter of weight \(\lambda\).

Indeed, the only thing we need to verify is that \(\beta(P \otimes P) = (P \otimes P)\beta\). For any \(a, b \in A\) and \(x \in A^{\otimes i}, y \in A^{\otimes j}\), we have

\[
\beta(P_A \otimes P)((a \otimes x) \otimes (b \otimes y)) = \beta((1_A \otimes (a \otimes x)) \otimes (1_A \otimes (b \otimes y))) = 1_A \otimes (\beta_{1,j+1} \otimes \text{id}_{A^{\otimes i+1}})(1_A \otimes \beta_{i+1,j+1}((a \otimes x) \otimes (b \otimes y))) = (P \otimes P)\beta((a \otimes x) \otimes (b \otimes y)).
\]

**Example 3.3.** Let \((A, m, \sigma)\) be a braided algebra and \(P_1\) be the projection defined in Theorem 2.6. Then \((\mathcal{R}_{\sigma, \lambda}(A), P_1, \beta)\) is a braided Rota-baxter of weight \(-1\).

Again, we only need to verify \(\beta(P_1 \otimes P_1) = (P_1 \otimes P_1)\beta\). For any \(x_1, y_1 \in A\) and \(x_2, y_2 \in \bigoplus_{i \geq 1} A^{\otimes i}\),

\[
(P_1 \otimes P_1)\beta((x_1 + x_2) \otimes (y_1 + y_2)) = (P_1 \otimes P_1)(\beta(x_1 \otimes y_1) + \beta(x_1 \otimes y_2) + \beta(x_2 \otimes y_1) + \beta(x_2 \otimes y_2)) = \beta(\sigma(x_1 \otimes y_1)) = \beta(P_1 \otimes P_1)((x_1 + x_2) \otimes (y_1 + y_2)),
\]

where the second equality follows from the fact that \(\beta(A^{\otimes i} \otimes A^{\otimes j}) \subset A^{\otimes j} \otimes A^{\otimes i}\).

Similarly, \((\mathcal{R}_{\sigma, \lambda}(A), P_2, \beta)\) is also such an algebra.

**Proposition 3.4.** Let \((R, P)\) be a Rota-Baxter algebra of weight \(\lambda\) and \(\sigma\) be a braiding on \(R\) such that \((R, \sigma)\) is a braided algebra and

\[
\sigma(P \otimes \text{id}) = (\text{id} \otimes P)\sigma,
\]

\[
\sigma(\text{id} \otimes P) = (P \otimes \text{id})\sigma.
\]

Then \((R, P, \sigma)\) is a braided Rota-Baxter algebra of weight \(\lambda\). Moreover, if we define

\[
x \ast_P y = xP(y) + P(x)y + \lambda xy,
\]

for any \(x, y \in R\), then \((R, \ast_P, P, \sigma)\) is again a braided Rota-Baxter algebra of weight \(\lambda\).
**Proof.** Note that
\[
\sigma(P \otimes P) = \sigma(P \otimes \text{id})(\text{id} \otimes P)
\]
\[
= (\text{id} \otimes P)\sigma(\text{id} \otimes P)
\]
\[
= (\text{id} \otimes P)(P \otimes \text{id})\sigma
\]
\[
= (P \otimes P)\sigma.
\]
So \((R, P, \sigma)\) is a braided Rota-Baxter algebra of weight \(\lambda\).

It is well-known that \((R, \star_p, P)\) is a Rota-Baxter algebra of weight \(\lambda\). We denote by \(m\) the multiplication of \(R\). Then we have
\[
\sigma(\star_p \otimes \text{id})
\]
\[
= \sigma((m \otimes \text{id})(P \otimes \text{id}) + (m \otimes \text{id})(P \otimes \text{id} \otimes \text{id} + \lambda m \otimes \text{id})
\]
\[
= (\text{id} \otimes m)\sigma_1\sigma_2(\text{id} \otimes P \otimes \text{id}) + (\text{id} \otimes m)\sigma_1\sigma_2(P \otimes \text{id} \otimes \text{id})
\]
\[
+ (\text{id} \otimes \lambda m)\sigma_1\sigma_2
\]
\[
= ((\text{id} \otimes m)(\text{id} \otimes P \otimes \text{id}) + (\text{id} \otimes m)(\text{id} \otimes P \otimes \text{id}) + (\text{id} \otimes \lambda m))\sigma_1\sigma_2
\]
\[
= (\text{id} \otimes \star_p)\sigma_1\sigma_2.
\]
The another condition \(\sigma(\text{id} \otimes \star_p) = (\star_p \otimes \text{id})\sigma_2\sigma_1\) can be verified similarly. □

Given any triple \((R, P, \sigma)\) described as above, this proposition provides another example of 2-braided algebras.

We provide more examples of braided Rota-Baxter algebras by using quantum multi-brace algebras introduced in [18].

**Definition 3.5.** A *quantum multi-brace algebra* \((V, M, \sigma)\) is a braided vector space \((V, \sigma)\) equipped with a family of operations \(M = \{M_{pq}\}_{p, q \geq 0}\), where
\[
M_{pq} : V^\otimes_p \otimes V^\otimes_q \to V, \quad p \geq 0, \quad q \geq 0,
\]
satisfying
(i)
\[
M_{00} = 0,
\]
\[
M_{10} = \text{id}_V = M_{01},
\]
\[
M_{n0} = 0 = M_{0n}, \text{ for } n \geq 2,
\]
(ii) for any \(i, j, k \geq 1\),
\[
\beta_{1k}(M_{ij} \otimes \text{id}_V^\otimes k) = (\text{id}_V^\otimes k \otimes M_{ij})\beta_{i+j,k},
\]
\[
\beta_{11}(\text{id}_V^\otimes i \otimes M_{jk}) = (M_{jk} \otimes \text{id}_V^\otimes j)\beta_{i+j+k},
\]
(iii) for any triple \((i, j, k)\) of positive integers,
\[
\sum_{r=1}^{i+j} M_{rk} \circ ((M^\otimes \circ \Delta_{\beta}^{(r-1)}) \otimes \text{id}_V^\otimes k)
\]
\[
= \sum_{l=1}^{j+k} M_{hl} \circ (\text{id}_V^\otimes l \otimes (M^\otimes \circ \Delta_{\beta}^{(l-1)})).
Let $\delta$ be the deconcatenation coproduct on $T(V)$, i.e.,

$$\delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n),$$

and $\varepsilon$ be the projection from $T(V)$ onto $\mathbb{K}$. We define $\Delta_\beta = (\text{id}_{T^\varepsilon(V)} \otimes \beta \otimes \text{id}_{T^\varepsilon(V)}) \circ (\delta \otimes \delta)$, and

$$\ast = (\varepsilon \otimes \varepsilon) + \sum_{n \geq 1} M^\otimes n \circ \Delta_\beta^{(n-1)} : T(V) \otimes T(V) \to T(V).$$

**Proposition 3.6 (IQ).** Let $(V, M, \sigma)$ be a quantum multi-brace algebra. Then $(T(V), \ast, \beta)$ is a braided algebra.

From now on, we focus on the category of Yetter-Drinfeld modules. Let $H$ be a Hopf algebra. The category $^H\mathcal{YD}$ of Yetter-Drinfeld modules over $H$ consists of the following data: objects in $^H\mathcal{YD}$ are vector spaces $V$ having simultaneously a left $H$-module structure and a left $H$-comodule structure such that whenever $h \in H$ and $v \in V$,

$$\sum h(1)v(-1)h(2) \cdot v(0) = \sum (h(1) \cdot v)(-1)h(2) \otimes (h(1) \cdot v)(0),$$

and morphisms in $^H\mathcal{YD}$ are linear maps which are module and comodule homomorphisms. It is well-known that $^H\mathcal{YD}$ is a braided tensor category. Let $V$ be an object in $^H\mathcal{YD}$. Then $V$ is a braided vector space with the natural braiding defined by $\sigma(v \otimes w) = \sum v(-1) \cdot w \otimes v(0)$. An algebra in the category $^H\mathcal{YD}$ is an object $A$ in $^H\mathcal{YD}$ together with an associative multiplication $m$ which is a morphism in this category. Clearly, in this case, $(A, m)$ is a braided algebra with respect to the natural braiding.

**Proposition 3.7.** Let $V$ be an object in $^H\mathcal{YD}$ and $\sigma$ be its natural braiding. If $M$ is a quantum multi-brace algebra structure on $(V, \sigma)$ such that all $M_{pq}$ are morphism in $^H\mathcal{YD}$, then $(T(V), \ast)$ is an algebra in $^H\mathcal{YD}$.

**Proof.** Since both of the $H$-module and $H$-comodule structures are diagonal, and all maps involving in the formula of $\ast$ are module and comodule homomorphisms, the result follows from an easy verification. \qed

Consider the bosonization $T(V)\# H$ of $(T(V), \ast)$ by $H$. It is an associative algebra with underlying vector space $T(V) \otimes H$ and multiplication defined by

$$(x \# h)(y \# h') = \sum x \ast (h(1) \cdot y) \# h(2)h', x, y \in T(V), h, h' \in H.$$ 

Here, we use the symbol $x \# h$, instead of $x \otimes h$, to indicate the new structure.

Furthermore, $T(V)\# H$ is a braided algebra with respect to the natural braiding $\Sigma$ defined by

$$\Sigma((x \# h) \otimes (y \# h')) = \sum ((x_{(-3)}h(1)) \cdot y \# x_{(-2)}h(2)h'S(x_{(-1)}h(3))) \otimes (x(0) \# h(4)),$$

where $S$ is the antipole of $H$.

Observe that both of $H = \mathbb{K}\# H$ and $T^+(V)\# H$ are subalgebras of $T(V)\# H$ and $T(V)\# H = H \oplus T^+(V)\# H$. By Proposition 2.5, the projection $P_0$ from $T(V)\# H$ onto $H$ is an idempotent Rota-Baxter operator of weight -1. In addition, since
\( \Sigma((V \otimes p \# H) \otimes (V \otimes q \# H)) \subset (V \otimes q \# H) \otimes (V \otimes p \# H) \), we obtain that \( \Sigma(P_0 \otimes \text{id}) = (\text{id} \otimes P_0)\Sigma \) and \( \Sigma(\text{id} \otimes P_0) = (P_0 \otimes \text{id})\Sigma \). By summarizing the above discussion and Proposition 3.4, we have

**Theorem 3.8.** The triple \((T(V) \# H, P_0, \Sigma)\) is a braided Rota-Baxter algebra of weight -1.

**Remark 3.9.** The algebra \( T(V) \# H \) is isomorphic, as a Hopf algebra, to the quantum multi-brace cotensor Hopf algebra introduced in [11].

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