THE PASCAL RHOMBUS AND THE GENERALIZED GRAND MOTZKIN PATHS

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ABSTRACT. In the present article, we find a closed expression for the entries of the Pascal rhombus. Moreover, we show a relation between the entries of the Pascal rhombus and a family of generalized grand Motzkin paths.

1. Introduction

The Pascal rhombus was introduced by Klostermeyer et al. [6] as a variation of the well-known Pascal triangle. It is an infinite array \( R = [r_{i,j}]_{i=0,j=-\infty}^{\infty,\infty} \) defined by

\[ r_{i,j} = r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} + r_{i-2,j}, \quad i \geq 2, \ j \in \mathbb{Z}, \]  

(1.1)

with the initial conditions

\[ r_{0,0} = r_{1,-1} = r_{1,0} = r_{1,1} = 1, \quad r_{0,j} = 0 \ (j \neq 0), \quad r_{1,j} = 0, \ (j \neq -1,0,1). \]

The first few rows of \( R \) are

|   | 1 | 1 | 1 |
|---|---|---|---|
| 1 | 1 | 2 | 4 | 2 | 1 |
| 1 | 3 | 8 | 9 | 8 | 3 | 1 |
| 1 | 4 | 13 | 22 | 29 | 22 | 13 | 4 | 1 |
| 1 | 5 | 19 | 42 | 72 | 82 | 72 | 42 | 19 | 5 | 1 |
|   |   |   |   |   |   |   |   |   |   |   |

Table 1. Pascal Rhombus.

Klostermeyer et al. [6] studied several identities of the Pascal rhombus. Goldwasser et al. [4] proved that the limiting ratio of the number of ones to the number of zeros in \( R \), taken modulo 2, approaches zero. This result was generalized by Mosche [7]. Recently, Stockmeyer [9] proved four conjectures about the Pascal rhombus modulo 2 given in [6].

The Pascal rhombus corresponds with the entry A059317 in the On-Line Encyclopedia of Integer Sequences (OEIS) [8], where it is possible to read: There does not seem to be a simple expression for \( r_{i,j} \).

In the present article, we find an explicit expression for \( r_{i,j} \). In particular, we prove that

\[ r_{i,j} = \sum_{m=0}^{i} \sum_{l=0}^{i-j-2m} \binom{2m+j}{m} \binom{l+j+2m}{l} \binom{l}{i-j-2m-l}. \]

For this we show that \( r_{i,j} \) is equal to the number of 2-generalized grand Motzkin paths.
2. The Main Result

A Motzkin path of length \( n \) is a lattice path of \( \mathbb{Z} \times \mathbb{Z} \) running from \((0,0)\) to \((n,0)\) that never passes below the \( x \)-axis and whose permitted steps are the up diagonal step \( U = (1,1) \), the down diagonal step \( D = (1,-1) \) and the horizontal step \( H = (1,0) \), called rise, fall and level step, respectively. The number of Motzkin paths of length \( n \) is the \( n \)-th Motzkin number \( m_n \), (sequence A001006). Many other examples of bijections between Motzkin numbers and others combinatorial objects can be found in [1]. A grand Motzkin path of length \( n \) is a Motzkin path without the condition that never passes below the \( x \)-axis. The number of grand Motzkin paths of length \( n \) is the \( n \)-th grand Motzkin number \( g_n \), sequence A002426. A 2-generalized Motzkin path is a Motzkin path with an additional step \( H_2 = (2,0) \). The number of 2-generalized Motzkin paths of length \( n \) is denoted by \( m^{(2)}_n \). Analogously, we have 2-grand generalized Motzkin paths, and the number of these paths of length \( n \) is denoted by \( g^{(2)}_n \).

Lemma 2.1. The generating function of the 2-generalized Motzkin numbers is given by

\[
B(x) := \sum_{i=0}^{\infty} m^{(2)}_i x^i = \frac{1 - x - x^2 - \sqrt{1 - 2x - 5x^2 + 2x^3 + x^4}}{2x^2}
\]

(2.1)

\[
= \frac{F(x)}{x} C(F(x)^2),
\]

(2.2)

where \( F(x) \) and \( C(x) \) are the generating functions of the Fibonacci numbers and Catalan numbers, i.e.,

\[
F(x) = \frac{x}{1-x-x^2}, \quad C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.
\]

Proof. From the first return decomposition any nonempty 2-generalized Motzkin path \( T \) may be decomposed as either \( UT' DT'' \), \( HT' \), or \( H_2 T' \), where \( T', T'' \) are 2-generalized Motzkin paths (possible empty). Making use of the Flajolet’s symbolic method (cf. [3]) we obtain

\[
B(x) = 1 + (x + x^2)B(x) + x^2B(x)^2.
\]

Therefore Equation (2.1) follows. Moreover,

\[
B(x) = \frac{1 - x - x^2 - \sqrt{(1 - x - x^2)^2 - 4x^2}}{2x^2} = \frac{1 - \sqrt{1 - 4 \left( \frac{x}{1-x-x^2} \right)^2}}{2x^2}
\]

\[
= \frac{1}{1-x-x^2} \frac{1 - \sqrt{1 - 4F(x)^2}}{2F(x)^2} = \frac{F(x)}{x} C(F(x)^2).
\]

□

The height of a 2-generalized grand Motzkin path is defined as the final height of the path, i.e., the stopping \( y \)-coordinate. The number of 2-generalized grand Motzkin paths of length \( n \) and height \( j \) is denoted by \( g^{(2)}_{n,j} \).

Theorem 2.2. The generating function of the 2-generalized grand Motzkin paths of height \( j \) is

\[
M^{(j)}(x) := \sum_{i=0}^{\infty} g^{(2)}_{n,j} x^i = \frac{F(x)^{j+1} C(F(x)^2)^j}{x(1 - 2F(x)^2C(F(x)^2))},
\]
where $F(x)$ and $C(x)$ are the generating function of the Fibonacci numbers and Catalan numbers. Moreover,

$$g_{ij}^{(2)} = \sum_{m=0}^{i} \sum_{l=0}^{i-j-2m} \binom{2m+j}{m} \binom{l+j+2m}{l} \binom{1}{i-j-2m-l}, \quad (0 \leq j \leq i).$$

**Proof.** Consider any 2-generalized grand Motzkin path $P$. Then any nonempty path $P$ may be decomposed as either

$UMDP'$, $DMUP'$, $HP'$, $H_2P'$, or $UM_1UM_2\cdots UM_j$,

where $M, M_1, \ldots, M_j$ are 2-generalized Motzkin paths (possible empty), $P'$ is a 2-generalized grand Motzkin path (possible empty).

Schematically,

![Diagram](image)

**Figure 1.** Factorizations of any 2-generalized grand Motzkin path.

From the Flajolet’s symbolic method we obtain

$$M^{(j)}(x) = 2x^2B(x)M^{(j)}(x) + (x + x^2)M^{(j)}(x) + x^j(B(x))^j, \quad j \geq 0.$$ 

Therefore

$$M^{(j)}(x) = \frac{x^jB(x)^j}{1 - x - x^2 - 2x^2B(x)}.$$ 

From Lemma 2.1 we get

$$M^{(j)}(x) = \frac{x^j(F(x)/x)C(F(x)^2)}{1 - x - x^2 - 2x^2F(x)/xC(F(x)^2)} = \frac{F(x)^j+1C(F(x)^2)^j}{x(1 - 2F(x)^2C(F(x)^2))}.$$ 

On the other hand, from the following identity (see Ec. 2.5.15 of [10])

$$\frac{1}{\sqrt{1-4x}} \left(1 - \sqrt{1-4x} \frac{x}{4} \right)^k = \sum_{m=0}^{\infty} \binom{2m+k}{m} x^m$$

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we obtain
\[ \frac{C(x^2)^j}{1 - 2x^2C(x^2)} = \sum_{m=0}^{\infty} \binom{2m + j}{m} x^{2m} \]

Therefore
\[ M^{(j)}(x) = \frac{F(x)^{j+1}(x)}{x} \sum_{m=0}^{\infty} \binom{2m + j}{m} F(x)^{2m} = \frac{1}{1 - x - x^2} \sum_{m=0}^{\infty} \binom{2m + j}{m} F(x)^{2m+j} \]
\[ = \sum_{m=0}^{\infty} \binom{2m + j}{m} \frac{x^{2m+j}}{(1 - x - x^2)^{2m+1+j}} \]
\[ = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \binom{2m + j}{m} \binom{l + j + 2m}{l} \binom{l}{s} x^{2m+j+l+s}, \]

Put \( t = 2m + j + l + s \)
\[ M^{(j)}(x) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \binom{2m + j}{m} \binom{l + j + 2m}{l} \binom{l}{s} x^{2m+j+l+s}, \]

The result follows by comparing the coefficients. \( \square \)

**Theorem 2.3.** The number of 2-generalized grand Motzkin paths of length \( n \) and height \( j \) is equal to the entry \((n, j)\) in the Pascal rhombus, i.e.,
\[ r_{n,j} = g_{n,j}^{(2)}. \]

**Proof.** The sequence \( g_{n,j}^{(2)} \) satisfies the recurrence \([11]\) and the same initial values. It is clear, by considering the positions preceding to the last step of any 2-generalized grand Motzkin path. \( \square \)

**Corollary 2.4.** The generating function of the \( j \)th column of the Pascal rhombus is
\[ L_j(x) = \frac{F(x)^{j+1}C(F(x)^2)^j}{x(1 - 2F(x)^2C(F(x)^2))}, \]
where \( F(x) \) and \( C(x) \) are the generating function of the Fibonacci numbers and Catalan numbers. Moreover,
\[ r_{i,j} = \sum_{m=0}^{i} \sum_{l=0}^{i-j-2m} \binom{2m + j}{m} \binom{l + j + 2m}{l} \binom{l}{i-j-2m-l} \quad (0 \leq j \leq i). \]

The convolved Fibonacci numbers \( F_j^{(r)} \) are defined by
\[ (1 - x - x^2)^{-r} = \sum_{j=0}^{\infty} F_j^{(r)} x^j, \quad r \in \mathbb{Z}^+. \]

If \( r = 1 \) we have the classical Fibonacci sequence.

Note that
\[ F_{m+1}^{(r)} = \sum_{j_1 + j_2 + \cdots + j_r = m} F_{j_1+1} F_{j_2+1} \cdots F_{j_r+1}. \]
Moreover, using a result of Gould [5, p. 699] on Humbert polynomials (with \(n = j, m = 2, x = 1/2, y = -1, p = -r\) and \(C = 1\)), we have
\[
F_{j+1}^{(r)} = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j + r - l - 1}{j - l} \binom{j - l}{l}.
\]

**Corollary 2.5.** The following equality holds
\[
F_{i,j} = \sum_{m=0}^{\lfloor i-j/2 \rfloor} \binom{2m + j}{m} F_{i-j-2m+1}^{(j+2m+1)},
\]
where \(F_{i}^{(r)}\) are the convolved Fibonacci numbers.

**Proof.**
\[
L_n(x) = \sum_{m=0}^{\infty} \binom{2m + n}{m} \frac{x^{2m+n}}{(1 - x - x^2)^{n+2m+1}} = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \binom{2m + n}{m} F_{j+1}^{(n+2m+1)} x^{2m+n+j},
\]
Put \(t = 2m + n + j\)
\[
L_n(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{2m + n}{m} F_{i-j-2m+1}^{(n+2m+1)} x^t.
\]
The result follows by comparing the coefficients. □

**Example 2.6.** The generating function of the central column of the Pascal rhombus (sequence A059345) is
\[
L_0(x) = \frac{1}{\sqrt{1 - 2x - 5x^2 + 2x^3 + x^4}} = 1 + x + 4x^2 + 9x^3 + 29x^4 + 82x^5 + 255x^6 + \cdots.
\]
The generating function of the first few columns \((j = 1, 2, 3)\) of the Pascal rhombus are:
\[
L_1(x) = x + 2x^2 + 8x^3 + 22x^4 + 72x^5 + 218x^6 + 691x^7 + 2158x^8 + \cdots, \quad \text{(A106053)}
\]
\[
L_2(x) = x^2 + 3x^3 + 13x^4 + 42x^5 + 146x^6 + 476x^7 + 1574x^8 + \cdots, \quad \text{(A106050)}
\]
\[
L_3(x) = x^3 + 4x^4 + 19x^5 + 70x^6 + 261x^7 + 914x^8 + 3177x^9 + \cdots
\]

**Remark:** The results of this article were discovered by using the Counting Automata Methodology, [2].

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