Effects of aging and links removal on epidemic dynamics in scale-free networks

K. P. Chan\textsuperscript{1}, Dafang Zheng\textsuperscript{2}, P. M. Hui\textsuperscript{1}

\textsuperscript{1}Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong
\textsuperscript{2}Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, People’s Republic of China

Abstract

We study the combined effects of aging and links removal on epidemic dynamics in the Barabási-Albert scale-free networks. The epidemic is described by a susceptible-infected-refractory (SIR) model. The aging effect of a node introduced at time $t_i$ is described by an aging factor of the form $(t - t_i)^{-\beta}$ in the probability of being connected to newly added nodes in a growing network under the preferential attachment scheme based on popularity of the existing nodes. SIR dynamics is studied in networks with a fraction $1 - p$ of the links removed. Extensive numerical simulations reveal that there exists a threshold $p_c$ such that for $p \geq p_c$, epidemic breaks out in the network. For $p < p_c$, only a local spread results. The dependence of $p_c$ on $\beta$ is studied in detail. The function $p_c(\beta)$ separates the space formed by $\beta$ and $p$ into regions corresponding to local and global spreads, respectively.

PACS numbers: 89.75.-k, 89.75.Fb, 87.23.Ge, 87.19Xx
I. INTRODUCTION

It has been discovered in recent years that many real network systems, while showing different levels of complexity of their own, possess novel common structural or topological properties. These networks include, for example, the Internet [1], World-Wide-Web [2], scientific citations [3], cells [4], the web of collaborations among actors and actresses [5], and the web of human sexual contacts [6]. Typically, networks are characterized by nodes and links. The nodes may represent websites on the world-wide-web, a paper in scientific journal, or an actor, etc., depending on the network concerned. The links represent the interaction between the nodes. The interaction may represent the links from a webpage to another, references cited in a journal to papers in the literature, collaboration among actors and actresses, etc., again depending on the system concerned. These structures are found to show the small-world effect, i.e., a node can reach any randomly chosen node through only a few links; the high-clustering effect, i.e., nodes connected to a chosen node have high probability of having links among themselves; and a well-defined degree distribution [7, 8]. These universal features observed in real systems have led to the intensive studies of complex networks in recent years. Physicists have made major contributions to the understanding of the physics of networks. On the structural properties, physicists have proposed a number of models in which the observed statistical features in real networks are reproduced. The most representative of these models are the small-world model proposed by Watts and Strogatz [9]; and the scale-free growing network proposed by Barabási and Albert (BA) [10]. The latter model gives a better description of the scale-free or broad-scale distributions of degrees found in many real-life networks and hence has been extensively studied recently. While structural properties are important in the geometrical description of networks, many applications of the physics of networks rely on our understanding of dynamical processes on networks. Dynamical processes, such as percolation, searchability problems, and especially the spread of diseases on complex networks have been investigated only recently. It has been found that the BA networks are highly susceptible to large scale epidemics [11].

In the present work, we study how aging phenomena [12] and links removal combined may affect the spreading of a disease or rumor in BA growing networks. Aging refers to situations in which the older nodes in a network become increasingly unattractive to the newly added nodes. We use a susceptible-infected-refractory (SIR) model for the dynamics
of epidemics. Using extensive numerical simulations, we study the SIR dynamics on BA networks with a fraction of $1 - p$ of the links removed or turned ineffective. It is found that a threshold $p_c(\beta)$ exists, where $\beta$ is an aging parameter. For $p \geq p_c$, a global spread results; while for $p < p_c$, only a local spread results. For the full BA network, i.e., $p = 1$, global outbreak can take place only if the value of $\beta$ is smaller than a critical value.

II. MODEL

The underlying network can be set up as follows. Initially there are $m_0$ fully connected nodes. We take $m_0 = 5$ in the present work. At each time step, a new node is introduced. Each newly added node establishes $m$ ($\leq m_0$) outgoing links to existing nodes. Each new link introduced at time step $t$ has a probability

$$\Pi_i = \frac{k_i/(t - t_i)^\beta}{\sum_l k_l/(t - t_l)^\beta}$$

(1)

to be connected to an existing node $i$ with degree $k_i$, i.e., having $k_i$ links. Here $t_i$ is the time at which node $i$ was added to the network and $\beta$ is an aging parameter characterizing how rapidly an existing node becomes unattractive to a newly added node. Eq. (1) states that newly established links are preferentially attached to the younger and highly connected existing nodes in the system. For $\beta = 0$, there is no aging effect and the model reduces to the BA model [10]. The aging factor thus models the situation where older persons in a society tend to isolate themselves or being isolated by the younger ones and lose their influence in the younger persons.

As an undiluted BA network is highly susceptible to large scale global spread of diseases, it is interesting to study the effects of removing a fraction of links in a fully grown BA network. These diluted networks can be constructed as follows. Having set up an aging network of $N$ nodes, each link has a probability $1 - p$ of being removed. Hence, on the average, a fraction $p$ of the links in a fully grown aging network consisting of $N$ nodes are kept. The removed links in a diluted network can be regarded as ineffective links through which a disease or a rumor cannot spread. In the present work, we study aging networks of size up to $N = 50,000$ nodes.

To study epidemiological processes on aging and diluted networks, we use the three-state Susceptible-Infected-Refractory (SIR) model [13]. The model is a standard model
for studying epidemics and the spread of a rumor in a connected population. Initially, all nodes are in the S(susceptible)-state, and one node is randomly chosen to be infected, i.e., in the I-state. The SIR dynamics proceeds as follows. At a time step $t$, a node, say $i$, is randomly chosen among all the infected (I) nodes. A neighbor or friend $j$ is then selected randomly among all the neighbors of node $i$, i.e., those with a link connected to $i$. If node $j$ is susceptible, it becomes infected and the chosen node $i$ remains in the I-state; otherwise (i.e., node $j$ is either I or R) the state of node $j$ remains unchanged and the chosen node $i$ becomes refractory (R) at the end of the time step. Within the context of a spread of a rumor, the I-state nodes refer to persons who want to spread the rumor, the S-state nodes are persons who have not heard the rumor, and the R-state nodes refer to persons who lost interest in spreading the rumor after knowing it. As time evolves, the number of R-nodes (S-nodes) increases (decreases); while the number of I-nodes increases initially and then eventually drops to zero. The number of R-nodes at the end of the dynamical evolution is denoted by $N_R$. We are interested in how $N_R/N$ varies as the fraction of effective links $p$ and the aging parameter $\beta$.

III. RESULTS AND DISCUSSION

We performed extensive numerical simulations to explore the combined effects of aging and links removal on the SIR dynamics. In all the simulations, each data point is obtained by averaging over 100 realizations of the network structure and 100 different initially infected nodes for each realization. In Fig.1 we show the mean fraction of R-nodes, $r \equiv <N_R>/N$, on networks of different sizes up to $N = 50,000$ on a log-log plot for different values of the aging parameter (a) $\beta = 0$, (b) $\beta = 0.5$, (c) $\beta = 1.0$, and (d) $\beta = 1.5$. For each value of $\beta$, we carried out simulations for different levels of dilution $p$. For global spread, we expect that the number of refractory sites $N_R$ to be directly proportional to the network size $N$ and hence the fraction of refractory nodes $r$ becomes independent of $N$. For local spreads, $N_R$ is finite for a sufficiently large network and hence $r$ decreases as $1/N$. We thus search for a threshold $p_c$ that separate these two behavior of $r$ for a given value of $\beta$. The results in Fig. 1 show that $r = N_R/N$ does exhibit different behavior as $p$ decreases. Precise determination of $p_c$ is quite difficult, due to finite size effect and the inhomogeneous nature of the problem resulting from the random removal of links from a originally randomly
established BA network. To proceed, we study the slope of the lines in Fig. 1 as \( p \) decreases. Typical results for \( \beta = 0, 0.5, \) and 1.0 are shown in Fig. 2. Notice that, for given value of \( \beta \), the slope starts to drop from zero, i.e., \( N_R \sim N \) for global spread, for some value of \( p \). Notice that there is a transitional range of \( p \) for which \( N_R \) neither scales linearly with \( N \) nor independent of \( N \). We adopt the value of \( p \) at which the slope drops to the value of \( -0.1 \) as \( p_c(\beta) \). According to this criteria, we found \( p_c(\beta = 0) = 0.2, p_c(\beta = 0.5) = 0.25, p_c(\beta = 1) = 0.3, \) and \( p_c(\beta = 1.5) = 0.35 \) for the values of \( \beta \) in Fig. 1 and Fig. 2.

Carrying out the calculations for different values of the aging parameter \( \beta \), we obtained the dependence of the threshold \( p_c(\beta) \) on the aging parameter \( \beta \). The results of \( p_c(\beta) \) are shown in Fig. 3 as the phase boundary in a phase space formed by the aging parameter \( \beta \) and fraction of effective links \( p \). For small \( \beta \) (0 < \( \beta < 1.25 \)), \( p_c \) increases gradually as \( \beta \) increases. For \( \beta > 1.25 \), \( p_c \) increases more sensitively with \( \beta \). As \( \beta \) approaches 2, \( p_c \) rapidly increases and takes on a value close to 1, indicating that a rapidly aging network has the slightly older nodes effectively isolated and hence only local spread of a disease or rumor is possible, regardless of whether links are further removed. The phase boundary \( p_c(\beta) \) separate the phase space into two regions, one corresponds to local spread of a disease and another corresponds to a global spread of a disease. Note that for \( \beta > \beta_c \), where \( \beta_c \approx 2.0 \), \( p_c = 1 \) implying that aging effect alone is sufficient to keep a disease or rumor from spreading globally.

The effect of aging can be understood qualitatively as follows. A BA network without aging, i.e. \( \beta = 0 \), is highly resilient to random damages in the form of link removal [14]. The results in Fig. 1(a) show that one needs to remove more than 80% of the links to prevent global spread of an epidemic. This feature is similar to previous studies of site percolation [15], where it was shown that for networks with a broad-scale degree distribution, one needs to remove a large fraction of nodes (about 99%) before the network falls apart. The sensitivity of \( p_c \) to the aging parameter \( \beta \) is related to the form of the degree distribution as \( \beta \) increases. For small values of \( \beta \) (\( \beta < 1 \)), the degree distribution still carries scale-free features that lead to a small \( p_c \) and the weak dependence of \( p_c \) on \( \beta \). For larger values of \( \beta \), the older nodes lose their attractiveness rapidly giving rise to a narrow degree distribution. The scale-free nature of the network is lost and the degree distribution becomes increasingly dominated by an exponential form rather than a power law. In this case, local spread is more probable unless the network carries a larger fraction of effective links, giving rise to
the sensitive dependence of $p_c$ on $\beta$. For rapidly aging networks, the small world effect vanishes and the “regular lattice” behavior dominates. The high clustering feature of a rapidly aging network results in only local spread of a disease.

In summary, we have studied the combined effects of aging and links removal on SIR dynamics in the Barabási-Albert scale-free networks. A threshold $p_c$ exists for the fraction of effective links in a network below which only local spreads of disease or rumor take place. For small aging effect, $p_c$ is low. As aging effect increases, $p_c$ increases. For rapidly aging network, $p_c = 1$ implying that only local spread results, regardless of the number of effective links in the network.

Acknowledgments

K.P.C. and P.M.H. would like to thank K. H. Chung for useful discussions in the initial stage of this work. One of us (D.F. Zheng) acknowledges the support from the National Natural Science Foundation of China under grant number 70371069.
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FIGURE CAPTIONS

Fig. 1: The fraction of refractory nodes $r = N_R/N$ as a function of the size of network $N$ for different fractions of effective links in BA networks on a log-log scale. The aging parameter is (a) $\beta = 0$, (b) $\beta = 0.5$, (c) $\beta = 1.0$ and (d) $\beta = 1.5$. Note that for global spreads, $N_R \sim N$ and a horizontal line results.

Fig. 2: The slope of log $r$ against log $N$ as a function of fraction $p$ of effective links in a network. Note the change from vanishing slope to negative slopes as $p$ decreases, signifying a change from global to local spread of an epidemic.

Fig. 3: The thresholds $p_c(\beta)$ constitute a boundary in the phase space formed by the aging parameter $\beta$ and fraction of effective links $p$. In the region below (above) the boundary, local (global) spreads occur.
The graph shows the slope of log $r$ against log $N$ as a function of the fraction of effective links $p$. The graph includes lines for different values of $\beta$: $\beta = 0$, $\beta = 0.5$, and $\beta = 1.0$. The lines represent the relationship between the slope of log $r$ and log $N$ for each value of $\beta$. The x-axis represents the fraction of effective links $p$, ranging from 0.0 to 1.0, while the y-axis represents the slope of log $r$. The legend indicates the symbols used for each $\beta$ value: $\beta = 0$ is represented by square symbols, $\beta = 0.5$ by circle symbols, and $\beta = 1.0$ by triangle symbols.
