ADJOINT QUOTIENTS OF REDUCTIVE GROUPS

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Abstract. — Let G be a reductive group over a commutative ring k. In this article, we prove that the adjoint quotient G//G is stable under base change. Moreover, if G has a maximal torus T, then the adjoint quotient of the torus T by its Weyl group will be isomorphic to G//G. Then we focus on the semisimple simply connected group G of the constant type. In this case, G//G is isomorphic to the Weil restriction \( \prod_{D/\text{Spec }k} A_D^1 \), where D is the Dynkin scheme of G. Then we prove that for such G, the Steinberg’s cross-section can be defined over k if G is quasi-split and without \( A_{2m} \)-type components.

Résumé. — Soit k un anneau commutatif et G un groupe réductif sur k. Dans cet article, on va définir le quotient adjoint G//G de G sur k et démontrer que la construction est stable par changement de base. En plus, si G possède un tore maximal T, le quotient adjoint de T par son groupe de Weyl est isomorphe à G//G. Dans la dernière section, on se concentre sur le cas G semi-simple simplement connexe de type constant. Dans ce cas, G//G est isomorphe à la restriction de Weil \( \prod_{D/\text{Spec }k} A_D^1 \), où D est le schéma de Dynkin. Si G est de plus quasi-déployable et sans composantes de type \( A_{2m} \), on peut construire la cross-section de Steinberg sur k.

1. Introduction

Let k be a commutative ring and G be a reductive group over k. In this article, we want to discuss the adjoint quotient of G which is denoted by G//G. Roughly speaking, the adjoint quotient of G is determined by those regular functions of G which are constants on the conjugacy classes of G. Suppose

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that G contains a maximal torus T and let W be the corresponding Weyl group. Then the G-conjugation action on the regular functions of G induces a W-conjugation action on the regular functions of T. Let T//W be the adjoint quotient of T by W. The natural restriction on regular functions induces a natural morphism

\[ \iota : T//W \to G//G. \]

When k is an algebraically closed field, \( \iota^{\#} \) is an isomorphism. One can find the classical treatment about adjoint quotients over algebraically closed fields in Steinberg’s paper [St65] §6, or in Humphreys’s book [Hum95] Chap. 3.

In this article, we will show that the same result holds for any commutative ring k. Namely, we will prove the following theorem:

**Theorem.** — Let k be a commutative ring and G be a reductive group defined over k. Suppose that G contains a maximal k-torus T. Let W be the corresponding Weyl group of T. Then T//W \( \iso \to \) G//G.

The strategy we take here is actually the same one used for k an algebraically closed field. In § 3, we will prove the Theorem over \( \mathbb{Z} \) and generalize the result to arbitrary commutative rings in § 4.

In § 5, we focus on the adjoint quotient of the semisimple simply connected group scheme of constant type. For such group G, G//G is isomorphic to the Weil restriction \( \prod_{D/\text{Spec}k} A_D^1 \), where D is the Dynkin scheme of G. Moreover, we prove that the Steinberg’s cross-section ([St65], Thm. 1.4) can be defined over arbitrary commutative ring k if G is quasi-split and without A\(_2\)m-type components. At the end of the article, we prove that for a semisimple simply connected group scheme G of constant type, it always contains a semisimple regular element over a semi-local ring k, and the centralizer of a semisimple regular element is a maximal torus of G.

**2. Notations and Definitions**

Let k be a commutative ring. For an affine k-scheme X, we let \( k[X] \) be the ring of regular functions of X. For a k-scheme X, a k-algebra A, we let \( X_A := X \times_k \text{Spec} A \). Let \( \mathbb{G}_m \) (resp. \( \mathbb{G}_a \)) be the multiplicative (resp. additive) group defined over \( \mathbb{Z} \).

For a k-module V, we regard it as a functor by defining

\[ V(A) = V \otimes_k A, \text{ for all } k\text{-algebras } A. \]

In order to define the adjoint quotient of G over an arbitrary commutative ring k, we first define a G-conjugation action on the k-module \( \mathbb{V} := k[G] \). Let A be a k-algebra and \( g \in G(A), f \in \mathbb{V}(A), \) we define

\[ (g.f)(x) = f(g x g^{-1}), \text{ for all } A\text{-algebras } A', \text{ and for all } x \in G(A'). \]
Let $c : V \to k[G] \otimes V$ be the comodule map corresponding to the conjugation action defined above. We define

$$k[G]^G := \{ f \in V(k) | \sigma.f_A = f_A, \forall \sigma \in G(A), \forall -algebras A\},$$

Let $G//G = \text{Spec}(k[G]^G)$ be the adjoint quotient of $G$. Suppose that $G$ contains a maximal torus $T$ and let $W$ be the corresponding Weyl group. Then the $G$-conjugation action on $k[G]$ induces a $W$-conjugation action on $k[T]$ and let $T//W = \text{Spec}(k[T]^W)$. The natural restriction from $k[G]$ to $k[T]$ induces a natural homomorphism

$$i : k[G]^G \to k[T]^W.$$

When $k$ is an algebraically closed field, $i$ is an isomorphism. Namely,

**Theorem 2.1.** — Let $k$ be an algebraically closed field. Let $G$ be a semisimple $k$-group and $T$ be a maximal torus of $G$. Let $W$ be the Weyl group with respect to $T$. Then the restriction map $i : k[G]^G \to k[T]^W$ is an isomorphism. Furthermore, if $G$ is semisimple simply connected, then $k[G]^G$ is freely generated as a commutative $k$-algebra by the characters of the irreducible representations with respect to the fundamental highest weights.

**Proof.** — The injectivity relies on the fact that the semisimple regular elements in $G$ form an open dense subset.

The idea to prove the surjectivity is to find a set of representations $\rho : G \to \text{GL}_{n,k}$ such that the corresponding set of characters restricted to $T$ generates $k[T]^W$. For more details, one can refer to [Hum95] 3.2, [St65] §6 and [Jan03] Part II, 2.6.

In the following section, we want to repeat some arguments in the standard proof to show that those techniques fit quite well for reductive groups. Moreover, we will generalize these arguments from fields to $\mathbb{Z}$ when $G$ is a split reductive group scheme over $\mathbb{Z}$, and $T$ is a maximal $\mathbb{Z}$-torus of $G$.

### 3. The adjoint quotient over $\mathbb{Z}$

In this section, we will show that a result similar to Theorem 2.1 also holds over $\mathbb{Z}$. Namely,

**Theorem 3.1.** — Let $G$ be a split reductive $\mathbb{Z}$-group and $T$ be a maximal $\mathbb{Z}$-torus of $G$. Let $W$ be the Weyl group with respect to $T$. Then the restriction map $i : \mathbb{Z}[G]^G \to \mathbb{Z}[T]^W$ is an isomorphism.

As the first step, we want to generalize the techniques used to prove Theorem 2.1.
3.1. The W-conjugation action on tori. — Let $k$ be a commutative ring and $T$ be a split torus over $k$. Let $M$ be the character group of $T$ which can be regarded as an additive group, and $M^\vee$ be the dual of $M$ considered as $\mathbb{Z}$-module. Let $\mathcal{R} = (M, M^\vee, R, R^\vee)$ be a reduced root datum with respect to $T$ and $W$ be the corresponding Weyl group. Let $\Pi$ be a system of simple roots of $R$. Let $M^+$ be the set of characters $\lambda$ which satisfy $(\alpha^\vee, \lambda) \geq 0$ for all $\alpha^\vee \in \Pi^\vee$. A character $\lambda$ is called dominant if $\lambda \in M^+$.

Here we want to look at $k[T]^W$ more closely. Since $k[T] = k[M]$ and $W$ permutes $M$, we observe that $k[T]$ is a $W$-permutation module under the conjugation action. Let $e^\lambda$ be the element in $k[T]$ corresponding to $\lambda$ in $M$.

Then $k[T]^W$ is generated by elements of the form $\text{Sym}(e^\lambda) := \sum_{w \in W/W_\lambda} e^{w(\lambda)}$, where $\lambda \in M$ and $W_\lambda$ is the stabilizer of $\lambda$. Since for each $\lambda \in M$ we can find a $w \in W$ such that $w\lambda$ is in $M^+$, $k[T]^W$ is a free $k$-module generated by the set $\{\text{Sym}(e^\lambda) \mid \lambda \in M^+\}$ (ref. [Bou1], Chap. VI, §3, Lemma 3), which in turn means that $k[T]^W$ is determined by the $W$-action on $M$ and therefore is stable under arbitrary base change. We rephrase this fact as a lemma:

**Lemma 3.2.** — Let $k$, $T$ and $W$ be defined as above. Then the ring $k[T]^W$ is a free $k$-module generated by the set $\{\text{Sym}(e^\lambda) \mid \lambda \in M^+\}$ and hence is determined by the $W$-action on $M$. In particular, we have $k[T]^W \otimes k' = k'[T]^W$, for any $k$-algebra $k'$.

However, besides the basis $\{\text{Sym}(e^\lambda) \mid \lambda \in M^+\}$, we sometimes need to choose an alternative basis to simplify our proof. The next two lemmas are useful for this purpose:

**Lemma 3.3.** — Let $I$ be an ordered set satisfying the following condition:

(Min) Each nonempty subset of $I$ contains a minimal element.

Let $A$ be a commutative ring, $E$ be an $A$-module, and $\{e_i\}_{i \in I}$ be a basis of $E$. Let $\{x_i\}_{i \in I}$ be a family of elements such that $x_i = e_i + \sum_{j < i} a_{i,j} e_j$, where $a_{i,j} \in A$ and only finitely many $a_{i,j}$ are nonzero. Then also $\{x_i\}_{i \in I}$ is a basis of $E$.

**Proof.** — [Bou1], Chap. VI, §3, Lemma 4.

Let us define a partial order on $M$ with respect to $\Pi$ by $\lambda \geq 0$ if $\lambda = \sum a_s s$ where $a_s$’s are nonnegative integers.

**Lemma 3.4.** — Given an element $\lambda \in M^+$, the set $I(\lambda) := \{\mu \in M^+ \mid \mu \leq \lambda\}$ is finite.

**Proof.** — If the root datum $\mathcal{R}$ is semisimple, then one can find a proof of the above lemma in [Bou1], Chap. VI, §3, Prop. 3. For the reduced root datum $\mathcal{R}$, let $\text{corad}(\mathcal{R})$ be the coradical of $\mathcal{R}$, and $\text{ss}(\mathcal{R})$ be the semisimple part of $\mathcal{R}$.
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The partial order on \( R \) induces a partial order on \( \text{ss}(R) \), and we define a partial order on \( \text{corad}(R) \) as \( x \leq y \) if \( x = y \). Let \( p \) be the canonical isogeny \( p : \text{corad}(R) \times \text{ss}(R) \to R \), and \( d \) be the degree of the isogeny. If \( \mu \leq \lambda \) in \( R \), then \( d\mu \leq d\lambda \) in \( \text{corad}(R) \times \text{ss}(R) \). Hence we reduce the case to the semisimple case, and the lemma follows.

3.2. Representations associated to dominant weights. — Let \( K \) be a field, \( G \) be a split \( K \)-reductive group and \( T \) be a maximal \( K \)-torus of \( G \) which splits. Let \((M, M^\vee, R, R^\vee)\) be the root datum with respect to \( T \), and fix a system of simple roots \( \Pi \) of \( R \). Let \( B \) be the Borel subgroup containing \( T \) determined by \( \Pi \) and \( B^- \) be the opposite Borel subgroup of \( B \). Let \( U \) be the unipotent radical of \( B \).

In this subsection, we want to associate to each character \( \lambda \) which is dominant with respect to \( B \) a representation \( \rho_\lambda \). As we have mentioned in § 1, this is a crucial point to prove the surjectivity of \( i : K[G]^G \to K[T]^W \).

In the beginning, we would like to recall some well-known facts over a field. Let \( \lambda \in M^+ \). Then canonical homomorphism \( B^- \to T \) allows us to view \( \lambda \) as a homomorphism from \( B^- \) to \( \mathbb{G}_{m,K} \). Thus we can define a \( B^- \)-action on \( G \times \mathbb{G}_{a,K} \) as \( a(g, x) = (ga^{-1}, \lambda(a)x) \), for all \( a \in B^-_K \), \( g \in G(A) \), \( x \in \mathbb{G}_{a,K}(A) \), where \( A \) is a \( K \)-algebra. In this way, \( (G \times \mathbb{G}_{a,K})/B^- \) becomes a line-bundle over \( G/B^- \), and we denote it by \( L(\lambda) \). We then identify the global sections \( H^0(G/B^-, L(\lambda)) \) in a canonical way with the morphisms of schemes \( f : G \to \mathbb{G}_{a,K} \) satisfying \( f(\beta) = \lambda(\beta)^{-1}f(g) \), for all \( g \in G(A) \), \( \beta \in B^-(A) \), for any \( K \)-algebra \( A \) (ref. [Bo75] 4.4). Since \( G/B^- \) is projective, \( H^0(G/B^-, L(\lambda)) \) is finite dimensional. Let \( V(\lambda) = H^0(G/B^-, L(\lambda)) \).

Now we consider the following left regular \( G \)-action on the functor \( \text{Hom}_{K-sch}(G, \mathbb{G}_{a,K}) \) as below. For a \( K \)-algebra \( A \), an element \( \sigma \in G(A) \) and \( f \in \text{Hom}_{K-sch}(G, \mathbb{G}_{a,K})(A) \), we define \( l_\sigma f : G_A \to \mathbb{G}_{a,A} \) as \( (l_\sigma f)(g) = f(\sigma^{-1}g) \) for any \( A \)-algebra \( R \) and any \( g \in G(R) \). It is known that \( V(\lambda) \) is a \( G \)-module under the left regular \( G \)-action (ref. [Jan03] I, 5.12). Let \( V(\lambda)^U \) be the subspace where \( U \) acts trivially. For \( \lambda \) is dominant, we have the following proposition:

**Proposition 3.5.** — Let \( \lambda \) be a character of \( T \) which is dominant with respect to the Borel subgroup \( B \). Then we have the following:

1. \( V(\lambda) \neq 0 \).
2. \( \dim_K V^U = 1 \).
3. \( T \) acts on \( V(\lambda)^U \) by character \( \lambda \).

**Proof.** — [Jan03] Part II, 2.2 and 2.6.
From Proposition 3.5, we get the following corollary immediately:

**Corollary 3.6.** — For each $\lambda \in M^+$, there exists a representation 

$$\rho_\lambda : G \to \text{GL}_{n_\lambda}$$

such that the character $\chi_\lambda$ associated to $\rho_\lambda$ restricted to $T$ takes the form 

$$\text{Sym}(e^\lambda) + \sum_{\substack{\mu \in M^+ \lessdot \lambda \geq \mu}} a_\mu \text{Sym}(e^\mu).$$

**Proof.** — For each $\lambda \in M^+$, $V(\lambda)$ defined above is nonempty by Proposition 3.5 (1). We claim that the $V(\lambda)$’s do the job. Let $V(\lambda) = \bigoplus \mu V(\lambda)_\mu$, where $T$ acts on $V(\lambda)_\mu$ by the character $\mu$. Note that $W$ permutes the $\mu$’s, so $\dim_k V(\lambda)_\mu = \dim_k V(\lambda)_{\mu_2}$ if $\mu_1, \mu_2$ are in the same $W$-orbit. Let $\mu$ be a maximal weight of $V(\lambda)$ and by Proposition 3.5 (2), $\dim_k V(\lambda)_\mu = 1$. Hence the character $\chi_\lambda$ restricted to $T$ takes the form $\text{Sym}(e^\lambda) + \sum_{\substack{\mu \in M^+ \lessdot \lambda \geq \mu}} a_\mu \text{Sym}(e^\mu)$. □

Now we consider a split $\mathbb{Z}$-group $G$ equipped with a maximal $\mathbb{Z}$-torus $T$. We keep all the notations defined above. What we want to show next is that Corollary 3.6 is also true over $\mathbb{Z}$. We start with a lemma.

**Lemma 3.7.** — Let $k$ be a commutative ring and $G$ be an affine $k$-group scheme. Let $V$ be a $G$-module and $\Delta : V \to k[G] \otimes V$ be the corresponding comodule map. Let $V^G = \{v \in V | \Delta(v) = 1 \otimes v\}$. Then for any flat $k$-algebra $A$, $V^G \otimes_k A = (V \otimes_k A)^{G \times_k A}$.

**Proof.** — We refer to [Se77] Lemma 2. □

**Remark 3.8.** — When $A$ is not a flat $k$-algebra, the above lemma may be false. For example, let $k = \mathbb{Z}$, $A = \mathbb{Z}/p\mathbb{Z}$, where $p$ is an integer and $G = \text{Spec}(k[t])$ defined as the additive group. Let $V = k \oplus k$, and $e_1, e_2$ be its standard basis. Define $\Delta : V \to k[G] \otimes V$ as $\Delta(e_1) = 1 \otimes e_1$ and $\Delta(e_2) = pt \otimes e_1 + 1 \otimes e_2$. Then $\Delta$ is a comodule map and $V^G \otimes_k A = Ae_1$ while $(V \otimes_k A)^{G \times_k A} = V \otimes_k A$.

Let $V(\lambda)$ be the $G$-module $H^0(G_Q/B_{Q^*}, \mathcal{L}(\lambda)_Q)$ defined above. Then there is a lattice $N(\lambda) \subseteq V(\lambda)$ which is a $G$-module (ref. [Ser68], §2.4, Lemme 2). By Lemma 3.7, we have 

$$N(\lambda)^U \otimes_k \mathbb{Q} \simeq (N(\lambda) \otimes_k \mathbb{Q})^{U_Q}.$$ 

By Proposition 3.5, $\dim_k Q(N(\lambda) \otimes_k \mathbb{Q})^{U_Q} = \dim_k Q(V(\lambda))^{U_Q} = 1$, so the $\mathbb{Z}$-module $N(\lambda)^U$ is free of rank one and $T$ acts on it by $\lambda$. Hence $\lambda$ is the unique
maximal weight of $N(\lambda)$ and $N(\lambda) = N(\lambda)^U$. Now we get all the ingredients at hand to prove Theorem 3.1 over $\mathbb{Z}$.

**Proof of Theorem 3.1.** — The injectivity part is the easy one. Note that for a split reductive $\mathbb{Z}$-group, $\mathbb{Z}[G]$ is a free $\mathbb{Z}$-module, so $\mathbb{Z}[G]^G$ is torsion free and is flat over $\mathbb{Z}$. Therefore the map $\mathbb{Z}[G]^G \hookrightarrow (\mathbb{Z}[G] \otimes _\mathbb{Z} \mathcal{O})^G$ is injective. By Lemma 3.7, $\mathbb{Z}[G]^G \otimes _\mathbb{Z} \mathcal{O} = (\mathbb{Z}[G] \otimes _\mathbb{Z} \mathcal{O})^G$. Now the injectivity follows from Theorem 2.1.

For the surjectivity, let $\lambda$ be a dominant character of $T$ with respect to the Borel subgroup $B$ and $\rho_\lambda$ be the homomorphism $\rho_\lambda : G \to \text{GL}(N(\lambda))$ defined as above. Let $\chi_\lambda$ be the character of $\rho_\lambda$. Since $N(\lambda)_\lambda$ is free of rank one, $\chi_\lambda$ restricted to $T$ will take the form $\text{Sym}(e^\lambda) + \sum_{\mu < \lambda, \mu \in M^+} c_{\mu} \text{Sym}(e^\mu)$, where $n_\mu$ is the rank of $N(\lambda)_\mu$. By Lemma 3.3 and Lemma 3.4, the $\chi_\lambda$’s restricted to $T$ form a basis of $\mathbb{Z}[T]^W$ and the surjectivity follows.

Let $\lambda$ in $M^+$ and define the character associated to the weight $\lambda$ to be $\chi_\lambda$ as in the above proof. Then we have the following Corollary:

**Corollary 3.9.** — Let $G$, $T$ and $W$ be defined as in Theorem 3.1. In case $G$ is semisimple simply connected, $\mathbb{Z}[G]^G$ is freely generated as a commutative $\mathbb{Z}$-algebra by the characters associated to the fundamental weights. Especially, we have $G//G \simeq \mathbb{A}^n[\mathbb{Z}]$, where $n$ is the rank of $G$.

**Proof.** — Let $\{\lambda_1, ..., \lambda_n\}$ be the fundamental weights of $T$, and $\chi_i$’s be the characters associated to $\lambda_i$’s respectively. Since $G$ is semisimple simply connected, the fundamental highest weights generate $M^+$. From the proof of Theorem 3.1, we know that $\lambda := \sum_i m_i \lambda_i$ is the unique maximal weight occurring in $\prod_{i=1,...,n} \chi_i^{m_i}$ and the monomial $e^\lambda$ is with coefficient 1. Since $\chi_i$’s are in $\mathbb{Z}[T]^W$, $\prod_{i=1,...,n} \chi_i^{m_i}$ takes the form $\text{Sym}(e^\lambda) + \sum_{\mu < \lambda} c_{\mu} \text{Sym}(e^\mu)$. By Lemma 3.3 and Lemma 3.4, $\{\prod_{i=1,...,n} \chi_i^{m_i}\}_{m_i \in \mathbb{Z}_{\geq 0}}$ restricted to $T$ form a basis of $\mathbb{Z}[T]^W$ and hence $\mathbb{Z}[G]^G \simeq \mathbb{Z}[T]^T \simeq \mathbb{Z}[\chi_1, ..., \chi_n]$. The rest of the Corollary follows.

4. **Stability under base change**

Let $k$ be a commutative ring and $G$ be a reductive $k$-group with a maximal torus $T$. Let $W$ be the corresponding Weyl group of $T$. In the previous section, we have proved that $G//G \simeq T//W$ when $k = \mathbb{Z}$. Here, we want to show that this result holds over an arbitrary commutative ring $k$. To be more precise, we have the following theorem:
Theorem 4.1. — Let \( k \) be a commutative ring and \( G \) be a reductive \( k \)-group. Then:

1. For any \( k \)-algebra \( A \), \( G_A \sim (G/G)_A \).
2. Suppose that \( G \) has a maximal torus \( T \). Let \( W \) be the corresponding Weyl group of \( T \). Then \( T/W \sim G/G \).

Instead of proving Theorem 4.1 directly, we prove the following special case first:

Lemma 4.2. — Let \( A \) be a commutative ring. Let \( G \) be a split reductive group over \( \mathbb{Z} \) and \( T \) be a maximal torus of \( G \). Let \( W \) be the Weyl group with respect to \( T \). Then \( G_A \sim (G/G)_A \) and \( T_A \sim G_A \).

Proof. — As we have shown in Theorem 3.1, \( T/W \sim G/G \) and hence \( (T/W)_A \sim (G/G)_A \). On the other hand, from Lemma 3.2, we have \( (T_A/W_A) \sim (T/W)_A \). Therefore, the isomorphism between \( T_A/W_A \) and \( G_A/G \) follows from the isomorphism between \( G_A/G \) and \( (G/G)_A \). So it is enough to prove \( G_A/G \sim (G/G)_A \).

Let \( c \) be the comodule map corresponding to the \( G \)-conjugation action on \( \mathbb{Z}[G] \). Define the map \( \gamma \) as \( \gamma(f) = c(f) - 1 \otimes f \). Then we have the following exact sequences:

\[
0 \rightarrow \mathbb{Z}[G]^{G} \rightarrow \mathbb{Z}[G] \rightarrow \text{Im}(\gamma) \rightarrow 0,
\]
\[
0 \rightarrow \text{Im}(\gamma) \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \rightarrow Q \rightarrow 0.
\]

Since \( \mathbb{Z}[G] \) is a free \( \mathbb{Z} \)-module and \( \text{Im}(\gamma) \) is a submodule of \( \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \), \( \text{Im}(\gamma) \) is torsion-free and hence a flat \( \mathbb{Z} \)-module. Therefore, for an arbitrary commutative ring \( A \), we have the following exact sequence

\[
0 = \text{Tor}_{\mathbb{Z}}(A, \text{Im}(\gamma)) \rightarrow A \otimes \mathbb{Z}[G]^{G} \rightarrow A[G] \rightarrow A \otimes \text{Im}(\gamma) \rightarrow 0.
\]

If we can prove that \( Q \) is also flat, then the multiplication map

\[
\text{Im}(\gamma) \otimes_{\mathbb{Z}} A \rightarrow A[G] \otimes_{A} A[G]
\]

will be injective, which in turn means that \( A \otimes \mathbb{Z}[G]^{G} = A[G] \).

Now take a prime \( p \) and consider the following diagram:

\[
\begin{array}{ccc}
\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}[G]^{G} & \xrightarrow{id_{p} \otimes i} & \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}[T]^{W} \\
\downarrow m & & \downarrow m \\
\mathbb{Z}/p\mathbb{Z}[G]^{G} & \xrightarrow{ip} & \mathbb{Z}/p\mathbb{Z}[T]^{W}.
\end{array}
\]
By Lemma 3.2, we have
\[ Z/pZ[T]^W \simeq Z/pZ \otimes Z[T]^W; \]
by Theorem 3.1, we have
\[ Z[G]^G \simeq Z[T]^W; \]
and by Theorem 2.1, \( i_p \) is an isomorphism. Therefore,
\[ Z/pZ \otimes Z[G]^G \simeq Z/pZ[G]^G, \]
which implies \( \text{Im}(\gamma) \otimes Z/pZ \rightarrow Z/pZ[G] \otimes Z/pZ[Z/pZ[G]] \) is an injection and we get that \( \text{Tor}_Z(Z/pZ, Q) = 0 \) for any prime \( p \). Hence, \( Q \) is a flat \( Z \)-module and \( A \otimes Z[G]^G = A[G]^G \).

Since every reductive \( k \)-group is étale locally split (ref. [SGA3], Exp. XXII, Cor. 2.3), we can deduce Theorem 4.1 from the above lemma.

**Proof of Theorem 4.1.** — We prove (1) first. We have a natural morphism
\[ m^a : G_A // G_A \rightarrow (G // G)_A \]
corresponding to the multiplication \( m : A \otimes_k k[G]^G \rightarrow A[G]^G \).

If we can find an étale covering \( \{ \text{Spec}(A_n) \}_n \rightarrow \text{Spec}(A) \) such that \( m^a \times_A A_n \) is an isomorphism, then \( m^a \) is an isomorphism ( [DG70] Chap. III, §1, 2.6).

Since the reductive group \( G \) splits étale locally (ref. [SGA3], Exp. XXII, Cor. 2.3), we can find an étale covering \( \{ \text{Spec}(k_n) \}_n \rightarrow \text{Spec}(k) \) such that \( G_{k_n} \) is split with respect to a split maximal torus \( T_n \) for each \( n \in I \). Then over \( k_n \), there exists a split reductive \( Z \)-group \( G_0 \) such that \( G_{k_n} \simeq G_{0,k_n} \) ( [SGA3] Exp. XXV, Cor. 1.2 and Exp. XXIII, Cor. 5.10). Let \( A_n = A \otimes_k k_n \). We want to prove that \( m^a \times_A A_n : (G_A // G_A)_{A_n} \rightarrow (G // G)_{A_n} \) is an isomorphism. Since \( k_n \) is flat over \( k \), by Lemma 3.7, we have
\[ (G//G) \times_k k_n = G_{k_n} // G_{k_n}, \]
and
\[ (G_A // G_A) \times_A A_n = G_{A_n} // G_{A_n}. \]
Since \( G_{k_n} \simeq G_{0,k_n} \), we have \( G_{A_n} \simeq G_{0,A_n} \). By Lemma 4.2, we have
\[ (G_A // G_A) \times_A A_n \simeq G_{A_n} // G_{A_n} \]
\[ \simeq G_{0,A_n} // G_{0,A_n} \]
\[ \simeq (G_0 // G_0)_{A_n} \]
\[ \simeq (G_0 // G_0)_{k_n} \times_{k_n} A_n \]
\[ \simeq (G_{0,k_n} // G_{0,k_n}) \times_{k_n} A_n \]
\[ \simeq (G_{k_n} // G_{k_n}) \times_{k_n} A_n. \]
On the other hand, we have
\[(G//G)_{A_n} \simeq (G//G)_{k_n} \times_{k_n} A_n = (G_{k_n}//G_{k_n}) \times_{k_n} A_n.\]
Therefore, \(m^a \times A_n\) is an isomorphism and (1) follows.

We prove (2) now. As we have mentioned in the introduction, there is a natural morphism \(i : T//W \to G//G\) corresponding to the restriction map \(i : k[G]^G \to k[T]^W\). As in the proof of (1), to verify that \(i\) is an isomorphism, it is enough to prove that there exists an étale covering \(\{U_n\}_{n \in I} \to \text{Spec}(k)\) such that \(i \times U_n\) is an isomorphism for all \(n \in I\). Since the reductive group \(G\) splits étale locally with respect to \(T\) (ref \([\text{SGA3}]\), Exp. XXII, Cor. 2.3), we can find an étale covering \(\{\text{Spec}(k_n)\}_{n \in I} \to \text{Spec}(k)\) such that \(G_{k_n}\) splits with respect to \(T_{k_n}\) for each \(n \in I\). Then there exists a split reductive \(\mathbb{Z}\)-group \(G_0\) such that \(G_{k_n} \simeq G_{0,k_n}\) (\([\text{SGA3}]\) Exp. XXV, Cor. 1.2 and Exp. XXIII, Cor. 5.10). Then over \(k_n\), by Lemma 4.2, we have
\[(*) \ T_{k_n}//W_{k_n} \simeq G_{k_n}//G_{k_n}.\]
Since \(k_n\) is flat over \(k\), from (*), we get \((T//W)_{k_n} \simeq (G//G)_{k_n}\). Therefore, \(i\) is an isomorphism over \(k_n\) for each \(n\) and hence an isomorphism.

5. Generalized Steinberg’s cross-section

In this section, we let \(G\) be a semisimple simply-connected group over a commutative ring \(k\). Moreover, we assume that \(G\) is of constant type, i.e., there exists a split semisimple group \(G_0\) over \(\mathbb{Z}\) such that \(G\) is étale locally isomorphic to \(G_{0,k}\). In this case, \(\text{Isom}(G_{0,k}, G)\) is a right \(\text{Aut}(G_{0,k})\)-torsor, and \(G\) is a form of \(G_{0,k}\) twisted by a right \(\text{Aut}(G_{0,k})\)-torsor \(\text{Isom}(G_{0,k}, G)\).

Here we want to discuss the adjoint quotient \(G//G\) in this special case and show that the Steinberg’s cross-section can be defined over arbitrary \(k\) in this case.

5.1. Adjoint quotients of semisimple simply connected groups. —
As we have mentioned in Corollary 3.9, if \(G\) splits, then the adjoint quotient of \(G\) is isomorphic to the affine space \(A^n_k\), where \(n\) is the rank of \(G\). In general, since \(G\) is locally split in étale topology, \(G//G\) is a \(k\)-form of \(A^n_k\). In the following, we want to show:

**Proposition 5.1.** — Let \(k\) be an arbitrary commutative ring. Let \(G\) be as in the beginning of this section. Then \(G//G\) is isomorphic to \(\prod_{D/\text{Spec} k} (A^1_D)\), where \(D\) is the Dynkin scheme of \(G\) (\([\text{SGA3}]\), Exp. XXIV, 3.2, 3.3), and \(\prod\) stands for the Weil restriction (\([\text{DG70}]\), Chap. I, §1, 6.6).
Before we prove Proposition 5.1, we recall some facts about split semisimple groups.

Let $T_0$ be a maximal torus in $G_0$, $(M_0, M_0^\vee, R_0, R_0^\vee)$ be the root datum with respect to $T_0$ and $\Pi_0$ be a fixed base of $R_0$. Let us fix a pinning $E_0$ of $G_0$ with respect to the chosen base $\Pi_0$ ([SGA3], Exp. XXIV, 1.0). Let $\text{Centr}(G_0,k)$ be the center of $G_0,k$ and $\text{ad}(G_0,k)$ be the adjoint group associated to $G_0,k$ which is defined by $G_0,k/\text{Centr}(G_0,k)$. Then we have the following exact sequence of group schemes which splits ([SGA3], Exp. XXIV, Thm. 1.3):

\[(*) \quad 1 \longrightarrow \text{ad}(G_0,k) \longrightarrow \text{Aut}_k^{\text{grp}}(G_0,k) \longrightarrow \text{Out}(G_0,k) \longrightarrow 1.\]

Moreover, we can choose a splitting $s : \text{Out}(G_0,k) \rightarrow \text{Aut}_k^{\text{grp}}(G_0,k)$ of (*) with respect to $E_0$, i.e., $\text{Out}_0,k$ acts on $E_0$ through $s$. We denote the image of $s$ as $\text{Out}_0,k(E_0)$. The splitting $s$ also allows us to regard $\text{Out}(G_0,k)$ as the automorphism group of the Dynkin scheme of $G_0,k$, because $G_0,k$ is simply connected ([SGA3], Exp. XXIV, 3.6).

Let $\lambda_i$ be the fundamental weight corresponding to the coroot $\alpha_i^\vee \in \Pi_0^\vee$ and $\chi_i$ be the character of the fundamental representation associated to the fundamental weight $\lambda_i$. Let $\Lambda_0$ be the set of fundamental weights. Note that $\text{Out}(G_0,k)$ also acts on $\Lambda_0$ through $s$.

Proof of Proposition 5.1. — Note that the $\text{Aut}_k^{\text{grp}}(G_0,k)$-action on $G_0$ induces an $\text{Aut}_A^{\text{grp}}(G_0,A)$-action on $k[G_0]$. Namely, for a $k$-algebra $A$, $\sigma \in \text{Aut}_A^{\text{grp}}(G_0,A)$, $f \in A[G_0]$, we have $\sigma f \in A[G_0]$ defined as $(\sigma f)(g) = f(\sigma^{-1}(g))$, for all $g \in G_0,A(A')$, for all $A$-algebra $A'$. Then by the definition of $k[G_0]^{G_0}$, $\text{ad}(G_0,k)$ acts on $k[G_0]^{G_0}$ trivially. Therefore, $\text{Aut}_k^{\text{grp}}(G_0,k)$ acts on $k[G_0]^{G_0}$ through $\text{Out}(G_0,k)$, and $G//G$ is just a form of $G_0,k//G_0,k$ twisted by a right $\text{Out}(G_0,k)$-torsor $\text{Isom}(\text{Dyn}(G_0,k),\text{Dyn}(G))$ in our case ([SGA3], Exp. XXIV, 3.6). Therefore, we can assume $G$ is quasi-split.

Let $j : \Lambda_0 \rightarrow \Pi_0$ be the map between sets defined by $j(\lambda_i) = \alpha_i$. Then $j$ is compatible with the $\text{Out}(G_0,k)$ action on $\Lambda_0$ and $\Pi_0$, i.e., $\sigma \lambda_i$ is the fundamental weight corresponding to the coroot $(\sigma \alpha_i)^\vee$. If we regard $\sigma$ as an element of the symmetric group which permutes $\{1, \ldots, n\}$ and maps $\sigma \alpha_i$ to $\sigma(\alpha_i)$, then $\sigma \chi_i = \chi_{\sigma(i)}$. Therefore $\text{Out}(G_0,k)$ acts on $k[G_0]^{G_0} \simeq k[\chi_1, \ldots, \chi_n]$ by permuting the parameters, where $\chi_i$'s are characters of fundamental representations.

Let $S$ be the $\text{Out}_k^{\text{grp}}(G_0,k)$-torsor corresponding to $[\xi] \in H_1^{\text{et}}(k, \text{Out}_k^{\text{grp}}(G_0,k))$. Then

$$(G//G)_S \simeq (G_0,k//G_0,k)_S = (A_0^\Lambda)^{\Lambda_0}.$$  

Since $\text{Out}_k^{\text{grp}}(G_0,k)$ acts on $(A_0^\Lambda)^{\Lambda_0}$ by permuting $\Lambda_0$, and $\Lambda_0$ and $\Pi_0$ are isomorphic as $\text{Out}_k^{\text{grp}}(G_0,k)$-set, we have that $(G//G) \simeq \prod D_{\text{Spec} k} A_0^\Lambda$.  \qed
For a scheme $S$, let $\mathcal{O}_S$ be the structure sheaf of $S$, and $\mathcal{E}$ be an $\mathcal{O}_S$-module. We define two functors $\mathcal{V}(\mathcal{E})$ and $\mathcal{W}(\mathcal{E})$ over $S$-schemes as the following:

\[
\mathcal{V}(\mathcal{E})(T) := \text{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_T);
\]

\[
\mathcal{W}(\mathcal{E})(T) := \Gamma(T, \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T),
\]

where $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ means the inverse image of $\mathcal{E}$ in $T$ and

\[
\Gamma(T, \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T)
\]

means the global sections of $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ over $T$. Note that if $\mathcal{E}$ is a locally free coherent $\mathcal{O}_S$-module, then $\mathcal{V}(\mathcal{E}) \simeq \mathcal{W}(\mathcal{E}^\vee)$.

With the above notation, we have the following Corollary:

**Corollary 5.2.** — Let $k$ be a semi-local ring and $G$ be defined as in Proposition 5.1. Then $G//G \simeq \mathbb{A}^n_k$, where $n$ is the rank of $G$.

**Proof.** — Let $\mathcal{E}$ be the free $\mathcal{O}_D$-module of rank one. Then $\mathbb{A}^1_D \simeq \mathcal{V}(\mathcal{E}) \simeq \mathcal{W}(\mathcal{E}^\vee)$ as $D$-functors. Let $\pi : D \to \text{Spec}(k)$ be the structure morphism. Then $\pi_* \mathcal{E}^\vee$ is just $\mathcal{E}^\vee$ viewed as a $k$-module, and we have

\[
\mathcal{W}(\pi_* \mathcal{E}^\vee)(A) = \mathcal{E}^\vee \otimes_k A \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_D} (\mathcal{O}_D \otimes_k A) = (\prod_{D/\text{Spec}k} \mathcal{W}(\mathcal{E}^\vee))(A),
\]

for all $k$ algebra $A$. Therefore, $\mathcal{W}(\pi_* \mathcal{E}^\vee) \overset{\sim}{\to} \prod_{D/\text{Spec}k} \mathcal{W}(\mathcal{E}^\vee)$. Since $\pi$ is a finite étale morphism, $\mathcal{O}_D$ is a projective $k$-module of finite type, so is $\pi_* \mathcal{E}^\vee$. Therefore, $\pi_* \mathcal{E}^\vee$ is a free $k$-module for $k$ semi-local. By Proposition 5.1, $G//G \simeq \mathbb{A}^n_k$. \qed

### 5.2. Generalized Steinberg’s cross-section.

Let $p : G \to G//G$ be the natural map. Recall that a cross-section of $p$ is a closed subscheme $N$ of $G$ such that $p$ is a bijection between functors $N$ and $G//G$. Suppose $k$ is a perfect field. Then $G//G$ is isomorphic to $\mathbb{A}^n_k$, and Steinberg proved that if $G$ has a Borel subgroup, then there exists a cross-section of $p$. In particular, for $G$ without type $A_{2m}$, the Steinberg’s cross-section is contained in $G^{\text{reg}}$, where $G^{\text{reg}}$ denotes the open subset of $G$ which consists of regular elements ( [St65], Thm. 1.4, 1.5, and 1.6). In the following, we will show that the similar result holds for $k$ arbitrary commutative ring and $G$ quasi-split without $A_{2m}$ components.

We start with the definition of a regular element over an arbitrary base scheme $S$:

**Definition 5.3.** — Let $G$ be a reductive group with constant type over a scheme $S$. Let $n$ be the rank of $G$ and $S'$ be an $S$-scheme. An element $g \in G(S')$ is called regular if its centralizer $C_G(g)$ is of minimal dimension in all fibers, i.e. $\dim(C_G(g))_{s'} < n + 1$, for all $s' \in S'$. 

**Remark 5.4.** — Keep all the notation above. Since \( \dim(C_G(g))_{s'} \geq n \) for all \( s' \in S' \), the condition \( \dim(C_G(g))_{s'} < n + 1 \) is equivalent to the condition \( \dim(C_G(g))_{s'} = n \).

Let \( G \) be as in Definition 5.3 and \( \eta \in G(G) \) be the identity map on \( G \). Define
\[
G^{reg} = \{ g \in G \mid \dim C_G(\eta)_g < n + 1 \}.
\]
Then by Chevalley’s semi-continuity Theorem ([EGA4], 13.1.3), \( G^{reg} \) is an open subscheme of \( G \). Moreover, we have \( g \in G(S') \) is regular if and only if \( g \in G^{reg}(S') \). To see this, note that we can regard \( S' \) as a \( G \)-scheme with structure morphism \( g \), and under this morphism, the image of \( \eta \) in \( G(S') \) is \( g \). Then we have \( C_G(g) = C_G(\eta) \times_G S' \) and \( C_G(g)_{s'} = C_G(\eta)_{g(s')} \times_{g(s')} s' \). Therefore, \( \dim(C_G(g))_{s'} < n + 1 \) if and only if \( \dim C_G(\eta)_{g(s')} < n + 1 \), which means that \( g \in G(S') \) is regular if and only if \( g \in G^{reg}(S') \).

**Remark 5.5.** — An element \( g \in G(k) \) is said to be semisimple regular if it is semisimple regular on each geometrical fibre. Note that for \( G \) a reductive group, the definition of a regular element in [SGA3] is the definition of a semisimple regular element here ([SCC], Exp. 7, Def. 2). The functor of semi-simple regular elements is also representable by an open subscheme of \( G \) ([SGA3], Exp. XIII, 3.1, 3.2).

**Theorem 5.7.** — Let \( k \) be a commutative ring, \( G \) be a semisimple simply connected group of constant type with rank \( n \) over \( k \). Let \( G_0 \) be the Chevalley group scheme \( G_0 \) associated to \( G \). If \( G \) is quasi-split and without components of type \( A_{2m} \), then there exists a cross section \( C : \prod_{D/\text{Spec}(k)} \mathbb{A}^1_D \to G^{reg} \) of \( p \), i.e., \( p \circ C \) is an isomorphism of \( \prod_{D/\text{Spec}(k)} \mathbb{A}^1_D \).

**Proof.** — We can fix a quasi-pinning \( E \) of \( G \), and a pinning \( E_0 \) of \( G_0 \). Since \( (G,E) \) is a form of \((G_0,k,E_0)\) twisted by a right \( \text{Aut}(G_0,k,E_0) \)-torsor \( \text{Isomext}(G_0,k,G) \), we only need to prove the theorem for \( G_0 \), and show that the section \( C_0 \) which we construct is \( \text{Aut}(G_0,k,E_0) \)-equivariant.

Let \( T_0 \) be a maximal torus in \( G_0 \), \((M_0, M_0', R_0, R_0')\) be the root datum with respect to \( T_0 \) and \( \Delta_0 \) be a fixed base of \( R_0 \) with respect to the pinning \( E_0 \). Let \( \mathfrak{g}_0 \) be the Lie algebra of \( G_0 \), and \( \mathfrak{t}_0 \) be the Lie algebra of \( T_0 \). Let \( \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{g}_0^\alpha \) be the decomposition with respect to the adjoint action of \( T_0 \) on \( \mathfrak{g}_0 \). For each \( \alpha \in \Delta_0 \), let \( X_\alpha \in \Gamma(\text{Spec}(k), \mathfrak{g}_0^\alpha) \) which is defined in \( E_0 \). We know that for each \( \alpha \in R_0 \), there is an unique morphism \( \exp_\alpha : \mathcal{W}(\mathfrak{g}_0^\alpha) \to G_0 \) which induces the canonical inclusion over the Lie algebra \( \mathfrak{g}_0^\alpha \to \mathfrak{g}_0 \). Moreover, \( \exp_\alpha \) is a closed
immersion ([SGA3] Exp. XXII, Thm. 1.1), and we let $U_\alpha$ be the image of $\exp_\alpha$. Let $p_\alpha : G_{a,k} \to G_0$ be defined as $p_\alpha(a) = \exp_\alpha(aX_\alpha)$. For each $\alpha \in \Delta$, let $w_\alpha$ be the element defined by $\exp_\alpha(X_\alpha)\exp_{-\alpha}(-X_\alpha^{-1})\exp_\alpha(X_\alpha)$. Note that the image of $w_\alpha$ in the Weyl group is exactly the reflection with respect to $\alpha$ ([SGA3] Exp. XXII, 1.5). Let us number the roots in $\Delta$ in the order such that roots in the same orbit under $\Aut(G_0,k,E_0)$ are given consecutive numbers.

Let $\chi_i$ be the fundamental weight associated to $\alpha_i$ and the $i$-th coordinate of $A_n^k$ correspond to $\chi_i$. Define $C_0 : A_n^k \to G_0$ as $C_0(a_1, ..., a_n) = \prod_{i=1}^{n} p_{\alpha_i}(a_i X_{\alpha_i}) w_{\alpha_i}$.

Let $f \in \Aut(G_0,k,E_0)$. By the definition of $\Aut(G_0,k,E_0)$, $f$ permutes the roots in $\Delta$, and $f(\exp_\alpha(X_\alpha)) = \exp_{f(\alpha)}(X_{f(\alpha)})$. Thus, $f$ also permutes the $\sigma'_\alpha$s. We can regard $f$ as an element of the symmetric group which permutes $\{1, ..., n\}$ and maps $\alpha_i$ to $f(\alpha_i)$. Since $G_0$ has no $A_{2m}$ components, there are no edges between $\alpha_i$ and $\alpha_{f(i)}$ in the Dynkin diagram of $G_0$ and $U_\alpha$ commutes with $U_{\alpha_{f(i)}}$. From the way we number the roots, we have that $U_{\alpha_i}$ is next to $U_{\alpha_{f(i)}}$. Therefore,

$$C_0(f(a_1, ..., a_n)) = C_0(a_{f^{-1}(1)}, ..., a_{f^{-1}(n)})$$

$$= \prod_{i=1}^{n} p_{\alpha_i}(a_{f^{-1}(i)} X_{\alpha_i}) w_{\alpha_i}$$

$$= \prod_{i=1}^{n} p_{\alpha_{f(i)}}(a_i X_{\alpha_{f(i)}}) w_{\alpha_{f(i)}}$$

$$= f(\prod_{i=1}^{n} p_{\alpha_i}(a_i X_{\alpha_i}) w_{\alpha_i}),$$

This shows that $C_0$ is stable under $\Aut(G_0,k,E_0)$. Since $p \circ C_0$ is an isomorphism at each point $x \in \Spec(k)$ ([St65], Thm. 1.4), $p \circ C_0$ is an isomorphism ([EGA4], 17.9.5). Moreover, for a $k$-algebra $R$ and $y \in (\prod_{D/\Spec(k)} A^1_D)(R)$, $C_0(y)$ is regular at each fiber ([St65], 7.9, 7.14), so $C_0$ factors through $G_0^{reg}$. The theorem then follows. \hfill \Box

Recall that for an infinite field $k$, any reductive group has a semisimple regular element over $k$ ([SGA3], Exp. XIV, 6.8). If $k$ is a finite field, Lehrer proves that any semisimple simply connected group contains a semisimple regular conjugacy class ([Leh92], Cor. 3.5), and by Lang’s Theorem, it implies the existence of a semisimple regular element over $k$. Furthermore, for a semisimple simply connected group over a field, the centralizer of a semisimple regular element is a maximal torus. Here, we want to show that the same properties also holds for $k$ a semilocal ring.
**Proposition 5.8.** — Let $G$ be as in the beginning of this section. Suppose that $k$ is a semilocal ring and $G$ is quasi-split. Then $G$ has a semisimple regular elements $g \in G(k)$. Moreover, $\text{Centr}_G(g)$ is a maximal torus.

**Proof.** — Let $\{s_i\}$ be the set of closed points of $k$ and $\{\kappa_i\}$ be the set of corresponding residual fields. Then from the above discussion, we know that $G//G$ has a semisimple regular element over $\kappa_i$. Let us fix a Borel subgroup $B$ of $G$ and let $U^+$, $U^-$ be the unipotent radical of $B$ and the opposite of $B$. Then $U^+(k) \to \prod_i U^+(\kappa_i)$ (resp. $U^-$) is surjective ([SGA3], Exp. XXII, 5.9.10). On the other hand, for each $\kappa_i$, we can find a semisimple regular element generated by $(U^+(\kappa_i))$ and $(U^-(\kappa_i))$ (for finite fields, see [St67], Lemma 64). Therefore, we can find a semisimple regular element $g \in G(k)$. Since every semisimple regular element is contained in a maximal torus ([SGA3], Exp. XIII, Thm. 3.1), $g$ is contained in a maximal torus $T$ of $G$ which is in turn contained in $\text{Centr}_G(g)$, and on each geometrical fibre, they are isomorphic, so $T = \text{Centr}_G(g)$ ([EGA4], 17.9.5).

**Remark 5.9.** — For $G$ as in Proposition 5.8. If $G$ is without $A_{2m}$ components, then we can show the above Proposition using Steinberg’s cross section. Note that $G//G$ is isomorphic to $\prod_{D/\text{Spec}k} A^1$ which is isomorphic to $A^n$, so $(G//G)(k) \to \prod_i (G//G)(\kappa_i)$ is surjective. Pick $x \in (G//G)(k)$ which is mapped to the semisimple regular class of $(G//G)(\kappa_i)$ for each $i$. Then by Theorem 5.7, there is $g \in G(k)$ which is mapped to the semisimple regular class $x$. This implies $g$ is semisimple regular.

**Remark 5.10.** — Note that if $G$ is a semisimple $k$-group which is not simply connected, then the connected component of the centralizer of a semisimple regular element is not necessarily a torus. If $\text{Centr}_G(g)$ is smooth, then $\text{Centr}_G(g)^\circ$ will be representable ([SGA3], Exp. VI_B, Thm 3.10) and hence will be a torus. However, $\text{Centr}_G(g)$ is not always smooth. For example, let $k = \mathbb{C}[[x]]$ and $G = \text{PGL}_{2,k}$. Let $g$ be the matrix $\begin{pmatrix} 1 + x & 0 \\ 0 & -1 \end{pmatrix}$. Then $g$ is semisimple regular and contained in a unique maximal torus $T$. Therefore $\text{Centr}_G(g) \subseteq \text{Norm}_G(T)$ and we can define $W_g$ as the closed subgroup scheme of the Weyl group which fixes $g$. In our case, $T$ is a split torus, and the corresponding Weyl group is a constant group scheme. Suppose $\text{Centr}_G(g)$ is smooth. Then $\text{Centr}_G(g)^\circ$ is a torus and we have the following exact sequence:

$$
\begin{array}{c}
0 \\ \text{Centr}_G(g)^\circ \\ \text{Centr}_G(g) \\ W_g \\ 0
\end{array}
$$

Since $\text{Centr}_G(g)$ and $\text{Centr}_G(g)^\circ$ are flat, $W_g$ is also flat ([SGA3], Exp. VI_B, Prop 9.2 (xi)). So $W_g$ is a constant group scheme. However, $W_g$ is trivial at
the generic point and has an order 2 element \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) at the closed point, which means \( \text{Centr}_r(g) \) cannot be smooth!

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