ON APPROXIMATIONS FOR THE DISTRIBUTION OF THE TIME OF FIRST LEVEL CROSSING

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1. Introduction

This paper is an overview of the classical level crossing problem which is studied extensively in the literature and is fundamental in many branches of applied probability. We discuss a number of approximations with an emphasis on their performance, methods of justification and technical conditions which are required in these methods, including a new approximation called “inverse Gaussian”. It is derived by a new method, is fruitful for solving related problems, and is valid under mild regularity conditions. We emphasize its novelty and boons.

1.1. Level crossing by a diffusion process. Let $V_s = \vartheta s + \sigma W_s$, $s > 0$, be a diffusion process with drift coefficient $\vartheta > 0$ and diffusion coefficient $\sigma > 0$, and let $c$ be a positive constant. The random variable $\Upsilon_{u,c} = \inf\{s > 0 : (\vartheta - c)s + \sigma W_s > u\}$, or $+\infty$, if $(\vartheta - c)s + \sigma W_s \leq u$ for all $s > 0$, is the first passage time to level $u > 0$ of the shifted diffusion process $V_s - cs$, $s > 0$. It is well known that with $\Phi(0,1)(x)$ denoting compound distribution function (c.d.f.) of a standard Gaussian distribution, we have

$$P\{\Upsilon_{u,c} \leq t\} = 1 - \Phi_{(0,1)}\left(\frac{u - (\vartheta - c)t}{\sigma \sqrt{t}}\right) + \exp\left\{2\frac{(\vartheta - c)^2}{\sigma^2} \frac{u}{\sigma \sqrt{t}}\right\} \Phi_{(0,1)}\left(-\frac{u - (\vartheta - c)t}{\sigma \sqrt{t}}\right),$$

or

$$P\{\Upsilon_{u,c} \leq t\} = \begin{cases} 
F(t; \mu, \lambda, \frac{1}{2}) \big|_{\mu = \frac{u}{\vartheta - c}, \lambda = \frac{\sigma^2}{\vartheta - c}} & , 0 < c \leq c^* = \vartheta, \\
\exp\left\{-\frac{2\lambda}{\mu}\right\} F(t; \hat{\mu}, \lambda, \frac{1}{2}) \big|_{\hat{\mu} = -\frac{u}{\vartheta - c}, \lambda = \frac{\sigma^2}{\vartheta - c}} & , c > c^* = \vartheta, 
\end{cases}$$

\hspace{1cm} (1.1)

where

$$F\left(x; \mu, \lambda, \frac{1}{2}\right) = \Phi_{(0,1)}\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right) + \exp\left\{\frac{2\lambda}{\mu}\right\} \Phi_{(0,1)}\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right)$$

\hspace{1cm} (1.2)

denotes c.d.f. of (proper) inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$. It is “inverse” in that sense that while the Gaussian distribution refers to a Brownian
motion’s position at a fixed time, the inverse Gaussian distribution refers to the time a diffusion process takes to reach a fixed level.

1.2. Level crossing by a compound renewal process. Let us denote by $f_T(t)$ and $f_T(t)$ the probability density functions (p.d.f.) of a positive random variable $T_1$ and of a set of positive random variables $T_i \overset{d}{=} T$, $i = 2, 3, \ldots$, all distributed identically. Introducing compound renewal process, the random variable $T_1$ is the time interval between starting time zero and time of the first renewal, and the random variables $T_i$ are the inter-renewal times. The distribution of $T_1$ may be different from the distribution of $T$.

By $f_y(t)$ we denote p.d.f. of positive random variables $Y_i \overset{d}{=} Y$, $i = 1, 2, \ldots$, all distributed identically. The random variables $Y_i$ are sizes of jumps which occur only in the moments of renewals. Throughout the entire presentation, p.d.f. $f_T(y)$ and $f_Y(y)$ are assumed bounded from above by a finite constant.

Having assumed that $T_1$, i.i.d. $T_i \overset{d}{=} T$, $i = 2, 3, \ldots$, i.i.d. $Y_i \overset{d}{=} Y$, $i = 1, 2, \ldots$, are all mutually independent, we are within the renewal model, where compound renewal process with time $s \geq 0$ is $V_s = \sum_{i=1}^{N_s} Y_i$, or 0, if $N_s = 0$ (or $T_1 > s$), where $N_s = \max \{ n > 0 : \sum_{i=1}^{n} T_i \leq s \}$, or 0, if $T_1 > s$. The random variable

$$\Upsilon_{t, c}^{\text{ren}} = \inf \{ s > 0 : V_s - cs > u \},$$

or $\pm \infty$, as $V_s - cs \leq u$ for all $s > 0$, is the first passage time to level $u > 0$ of the process $V_s - cs$, $s > 0$.

Let us denote by $P\{\Upsilon_{t, c}^{\text{ren}} \leq t \mid T_1 = v\}$ the distribution of $\Upsilon_{t, c}^{\text{ren}}$ conditioned by $T_1 = v$. It is easily seen that for $0 < v < t$

$$P\{\Upsilon_{t, c}^{\text{ren}} \leq t \} = \int_0^t \int_0^t P\{u + cv - Y < 0\} f_{T_1}(v)dv$$

$$+ \int_0^t \int_0^t P\{v < \Upsilon_{t, c}^{\text{ren}} \leq t \mid T_1 = v\} f_{T_1}(v)dv. \quad (1.3)$$

Furthermore (see Theorem 2.1 in [33] and references therein),

$$P\{v < \Upsilon_{t, c}^{\text{ren}} \leq t \mid T_1 = v\} = \int_v^t \frac{u + cv}{u + cz} \sum_{n=1}^\infty P\{M(u + cz) = n\} f_{T_1}(z-v)dz, \quad (1.4)$$

where $M(x) = \inf \{ k \geq 1 : \sum_{i=1}^{k} Y_i > x \} - 1$.

2. Explicit solutions in level crossing

Similarly to diffusion set-up, in the compound renewal framework with $T$, $Y$ exponential with parameters $\delta$, $\varrho$, the distribution of $\Upsilon_{t, c}^{\text{ren}}$ may be written explicitly. Assuming that $T_1 \overset{d}{=} T$ and writing $I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} (\frac{x}{2})^{n+2k}$ for the modified Bessel function of the first kind of order $n$ (see, e.g., [1]), we have the following equivalent formulas

$$\begin{align*}
\frac{\Upsilon_{t, c}^{\text{ren}}}{\delta, \varrho} \leq t & = e^{-u_0 \delta} \int_0^t e^{-(c+\delta)x} \left( I_0(2 \sqrt{\delta} \varrho x(e^x + u)) \right. \\
& \left. - \frac{c x}{e^x + u} I_2(2 \sqrt{\delta} \varrho x(e^x + u)) \right) dx, \quad (2.1)
\end{align*}$$

\footnote{See, in particular, Theorem 2.2 in [34]. The proof of equivalence of Type I and Type II formulas applies Lommel’s formula.}
Type II formula

\[
P\{T_{u,c|\delta,\varrho}^{[\text{ren}]} \leq t\} = e^{-u\varrho\sqrt{\delta/c\varrho}} \int_0^{ct} e^{-(1+\delta/(c\varrho))x} \times \sum_{n=0}^{\infty} \frac{u^n}{n!} \left( \frac{\delta}{c\varrho} \right)^{n/2} \frac{n+1}{x} I_{n+1} \left(2x\sqrt{\delta/(c\varrho)}\right) \, dx, \tag{2.2}
\]

and Type III formula

\[
P\{T_{u,c|\delta,\varrho}^{[\text{ren}]} \leq t\} = P\{T_{u,c|\delta,\varrho}^{[\text{ren}]} \leq \infty\} \times \frac{1}{\pi} \int_0^{\pi} f(x, u, t) \, dx, \tag{2.3}
\]

where

\[
P\{T_{u,c|\delta,\varrho}^{[\text{ren}]} \leq \infty\} = \begin{cases} 
\frac{\delta}{c\varrho} \exp\left(-u(c\varrho - \delta)/c\right), & \delta/(c\varrho) < 1, \\
1, & \delta/(c\varrho) \geq 1,
\end{cases}
\]

and

\[
f(x, u, t) = \left(\delta/(c\varrho)\right) \left(1 + \delta/(c\varrho) - 2\sqrt{\delta/(c\varrho)} \cos x\right)^{-1} \times \exp\left\{u\varrho\left(\sqrt{\delta/(c\varrho)} \cos x - 1\right) - \delta(c\varrho/\delta)(1 + \delta/(c\varrho) - 2\sqrt{\delta/(c\varrho)} \cos x)\right\} \times \left(\cos \left(u\varrho\sqrt{\delta/(c\varrho)} \sin x\right) - \cos \left(u\varrho\sqrt{\delta/(c\varrho)} \sin x + 2x\right)\right).
\]

The equivalent Type I–Type III explicit formulas (2.1)–(2.3), as well as equation (1.1) in diffusion set-up, may seem very cumbersome and non-informative for intuitive understanding of the level crossing phenomenon. However, the formulas (2.1)–(2.3) do comply with the following observation done in [14], Ch. II, § 7:

surprisingly many explicit solutions in diffusion theory, queuing theory, and other applications involve Bessel functions. It is usually far from obvious that the solutions represent probability distributions, and the analytic theory required to derive their Laplace transforms and other relations is rather complex. Fortunately, the distributions in question (and many more) may be obtained by simple randomization procedures. In this way many relations lose their accidental character, and much hard analysis can be avoided.

To clarify this idea, let us show (see as well [35]) that Type II formula connects the problem of level crossing with random walk with random displacements.

Recall that the random walk with random displacements (see, e.g., [42], Chapter 4, § 22) is defined as follows. Suppose that a particle performs a random walk on the X-axis. Starting at the origin, in each step the particle moves either a unit distance to the right with probability \(p\) or a unit distance to the left with probability \(q\) \((p + q = 1, 0 < p < 1)\). Suppose that the displacements of the particle occur at random times in the time interval \((0, \infty)\). Denote by \(\nu(z)\) the number of steps taken in the interval \((0, z]\). We suppose that \(\nu(z), 0 \leq z < \infty\) is a Poisson process of density \(1/p\) and that the successive displacements are independent of each other and independent of the process \(\nu(z), 0 \leq z < \infty\). Denote by \(\xi_p(z)\) the position of the particle at time \(z\). In this case \(\xi_p(z), 0 \leq z < \infty\) is a stochastic process having stationary independent increments, \(P\{\xi_p(0) = 0\} = 1\) and almost all sample functions of \(\xi_p(z), 0 \leq z < \infty\) are step functions having jumps of magnitude 1 and -1.
In this random walk model, for \( y > 0 \) and integer \( k = 0, \pm 1, \pm 2, \ldots \), we have (see equality (3) in [42])
\[
\mathbb{P}\{\xi_p(y) = k\} = e^{-y/p}(p/q)^{k/2}I_k(2y\sqrt{q/p}) \\
= \frac{y}{k}v_k(y | p),
\]
where \( v_k(y | p) = \left(\frac{p}{q}\right)^{k/2} - \frac{y}{k}\ e^{-y/p}I_k(2y\sqrt{q/p}) \). Direct calculation yields
\[
\mathbb{E}\xi_p(y) = (1 - (q/p))y, \quad \mathbb{D}\xi_p(y) = y/p.
\]
For \( y > 0 \) and integer \( k > 0 \), we have (see equalities (8) and (9) in [42])
\[
\mathbb{P}\left\{\sup_{0 \leq z \leq y} \xi_p(z) < k\right\} = \mathbb{P}\{\xi_p(y) < k\} - (p/q)^k \mathbb{P}\{\xi_p(y) < -k\} \\
= 1 - k(p/q)^{k/2} \int_0^y e^{-z/p}I_k(2z\sqrt{q/p}) \frac{dz}{z} \\
= 1 - k \int_0^y \mathbb{P}\{\xi_p(z) = k\} \frac{dz}{z} = 1 - \int_0^y v_k(z | p) \, dz.
\]
Manipulating with (2.4) and (2.5), we have the equality
\[
k(p/q)^{k/2} \int_0^y e^{-z/p}I_k(2z\sqrt{q/p}) \frac{dz}{z} = e^{-y/p} \sum_{i=k}^{\infty} (p/q)^{i/2}I_i(2y\sqrt{q/p}) \\
+ e^{-y/p} \sum_{i=k+1}^{\infty} (p/q)^{i/2-k}I_i(2y\sqrt{q/p}),
\]
which proof by the methods of Bessel functions is not at all simple.

Denoting by \( \varsigma_k(p) \) the first hitting time of the point \( k \) by the random walk with \( p \in (0,1) \) and bearing in mind (2.6), we have
\[
\mathbb{P}\{\varsigma_k(p) \leq y\} = \mathbb{P}\left\{\sup_{0 \leq z \leq y} \xi_p(z) \geq k\right\} = \int_0^y v_k(z | p) \, dz.
\]
Thus, the function \( v_k(y | p) \) introduced in (2.4) has a clear probabilistic meaning. It is a probability density function of \( \varsigma_k(p) \), i.e., of the first hitting time of the point \( k \) in the model of random walk with random displacements. It is not a surprise that the density \( v_k(y | p) \) is defective for \( p < 1/2 \), i.e., when the random walk drifts to the left, and proper for \( p \geq 1/2 \), i.e., when the drift is absent or to the right.

Making the change of variables \( y = x\delta/(c\delta) \) in Type II formula (2.2), we have
\[
\mathbb{P}\{\Upsilon_{\alpha,c,\delta,0}^{\text{ren}} \leq t\} = e^{-u\delta} \sum_{n=0}^{\infty} \frac{(u\delta)^n}{n!} \int_0^{t\delta} \left(\frac{\delta}{c\delta}\right)^{n+1/2} \frac{n+1}{y} e^{-y(c\delta+\delta)/\delta} I_{n+1}(2y\sqrt{c\delta}/\delta) \, dy \\
= e^{-u\delta} \sum_{n=0}^{\infty} \frac{(u\delta)^n}{n!} \int_0^{t\delta} v_{n+1}(y | p) \bigg|_{p = \delta/(c\delta)} \, dy \\
= e^{-u\delta} \sum_{n=0}^{\infty} \frac{(u\delta)^n}{n!} \mathbb{P}\{\varsigma_{n+1}(p) \leq t\delta\} \bigg|_{p = \delta/(c\delta)},
\]
which makes a link between the random walk with random displacements and the problem of level crossing evident.

\(^2\)See Chapter II, Section 7 and Chapter XIV, Section 6 in [13].
It is noteworthy that under standard assumptions in the renewal model with $T$ and $Y$ exponential we have

$$
P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t \} = \begin{cases} 1, & 0 < c \leq \delta/\varrho, \\ \left(\delta/(\varrho c)\right) \exp \left\{ -u \varrho \left(1 - \delta/(\varrho c)\right) \right\}, & c > \delta/\varrho, \end{cases}
$$

which follows from

$$
\int_0^\infty v_{n+1}(y \mid p) \, dy = \begin{cases} 1, & p \geq 1/2, \\ \left(p/\varrho\right)^{n+1}, & p < 1/2 \end{cases}
$$

and $e^{-x} \sum_{n=0}^\infty \frac{x^n}{n!} (p/\varrho)^{n+1} = (p/\varrho) \exp \left\{ -x \left(1 - (p/\varrho)\right) \right\}$.

3. Approximations in level crossing

Among many types of approximations for $P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t \}$, the most significant are “normal” and “diffusion” which refinement is “corrected diffusion” (see [2], pp. 37 and 42, and [5]). The former name emphasizes the method of the approximation, while the latter name emphasizes the method of its construction.

3.1. “Normal”. Assuming that $T_1 \overset{d}{=} T$ and writing $X = Y - c T$, let us introduce

$$
m_z = E T / E X, \quad D^2_z = E(X ET - T E X)^2 / (E X)^3.
$$

For $0 < c < c^*$, we assume that $0 < D^2_z < \infty$. Then

$$
\sup_{t > 0} \left| P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t \} - \Phi(m_z, u, D^2_z)(t) \right| = O(1), \quad u \to \infty.
$$

If, in addition, $E(Y^3) < \infty$ and $E(T^3) < \infty$, then this supremum is $O(u^{-1/2})$, as $u \to \infty$.

Let us assume that a positive solution $\varkappa$ of the equation $E e^{\varkappa X} = 1$ exists. This assumption entails that the tail of the random variable $Y$ must decrease exponentially fast$^3$. Let us introduce the associated random variables $\bar{X}$, $\bar{T}$ which joint distribution is $F_{\bar{X},\bar{T}}(dz, dw) = e^{\varepsilon z} F_{XT}(dz, dw)$, and

$$
m_{\varepsilon} = E \bar{T} / E \bar{X}, \quad D^2_{\varepsilon} = E(\bar{X} E T - \bar{T} E X)^2 / (E X)^3, \quad C = \frac{1}{\varepsilon \bar{X}} \exp \left\{ -\sum_{n=1}^\infty \frac{1}{n} P\{S_n > 0\} - \sum_{n=1}^\infty \frac{1}{n} P\{\bar{S}_n \leq 0\} \right\},
$$

where $\bar{S}_n = \sum_{i=1}^n \bar{X}_i$, $S_n = \sum_{i=1}^n X_i$. For $c > c^*$, we have $E \bar{X} > 0$ and $m_{\varepsilon} > 0$.

Theorem 3.2. In the renewal model with $c > c^*$, we assume that $0 < D^2_{\varepsilon} < \infty$. Then for $C > 0$ defined in (3.3)

$$
\sup_{t > 0} \left| e^{\varepsilon u} P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t \} - C \Phi(m_z, u, D^2_z)(t) \right| = O(1), \quad u \to \infty.
$$

If, in addition, $E(Y^3) < \infty$ and $E(T^3) < \infty$, then this supremum is $O(u^{-1/2})$, as $u \to \infty$.

$^3$Many other approximations, including Arfwedson’s saddlepoint approximation (see [5], p. 133) are set aside, but may be also discussed in the similar way.

$^4$This clue assumption does not allow extension of this approach to $Y$ with power tail.
Developing the classical results by Lundberg and Cramér, the approximation \((3.4)\) was first obtained by Segerdahl \([38]\). There exist different approaches to the proof (see, e.g., Siegmund \([39]\), von Bahr \([9]\)). Refinements in terms of Edgeworth expansions were first discussed \(5\) in Asmussen \([2]\) and then proved in Malinovskii \([22]\).

For \(Y\) and \(T\) exponential with parameters \(\varrho > 0\) and \(\delta > 0\) respectively, easy calculation yields \(c^* = \frac{\delta}{\varrho}\),

\[
\begin{align*}
C &= \frac{\delta}{(\varrho c)}, \\
m_\varphi &= -\frac{1}{c(1 - \delta/(\varrho c))}, \\
D_\varphi^2 &= \frac{2(\delta/(\varrho c))^2 + (1 - \delta/(\varrho c))^2}{\delta c(1 - \delta/(\varrho c))^3}, \\
m_\delta &= \frac{\delta/(\varrho c)}{c(1 - \delta/(\varrho c))}, \\
D_\delta^2 &= \frac{2(\delta/(\varrho c))}{c^2 \varrho(1 - \delta/(\varrho c))^3}.
\end{align*}
\]

(3.5)

The following result follows straightforwardly from Theorems \([3.1]\) and \([3.2]\).
Corollary 3.1. In the renewal model with \( Y \) and \( T \) exponential with parameters \( \theta > 0 \) and \( \delta > 0 \), for \( 0 < c < c^* = \delta/\theta \) we have
\[
\sup_{t > 0} \left| \mathbb{P}\{ Y_{u,c}^{[\text{ren}],\delta,\theta} \leq t \} - \Phi_{(m_\delta, u, D_\delta^2)}(t) \right| = o(1), \quad u \to \infty,
\]
and for \( c > c^* \) we have
\[
\sup_{t > 0} \left| e^{\kappa u} \mathbb{P}\{ Y_{u,c}^{[\text{ren}],\delta,\theta} \leq t \} - C \Phi_{(m_\delta, u, D_\delta^2)}(t) \right| = o(1), \quad u \to \infty,
\]
with \( m_\delta > 0, D_\delta^2 > 0, 0 < C < 1, \kappa > 0, m_\delta > 0, D_\delta^2 > 0 \) defined in \((3.5)\).

The case \( c = c^* \) is excluded from consideration in both Theorems 3.1 and 3.2. Poor performance of the “normal” approximation around \( c^* \) is illustrated in Figs. 1 and 2.

In applications, the case \( c = c^* \) is known under different names. Related is the term “heavy traffic”. It comes from queueing theory, but has an obvious interpretation also in risk theory: on the average, the premiums \( cs \) within time \( s > 0 \) exceed only slightly the expected claims \( (EY/ET)s \) within the same time \( s > 0 \). That is, heavy traffic conditions mean that the safety loading \( \tau = (ET/EY)c - 1 \) is positive but small.

The approximation of Theorem 3.2 when positive \( \tau \) depends on \( u \) and tends to zero, as \( u \to \infty \), was investigated in \([24]\). In \([35]\), further insight into performance of Theorems 3.1 and 3.2 with rationale of its poorness in the vicinity of \( c^* \), may be found.

3.2. “Diffusion”. The idea behind the “diffusion” approximation is to first approximate the claim surplus process by a Brownian motion with drift by matching the two first moments, and next to note that such an approximation in particular implies that the first passage probabilities are close to those in \((3.5)\).

The idea behind the simple “diffusion” approximation is to replace the risk process by a Brownian motion (by fitting the two first moments) and use the Brownian first passage probabilities as approximation for the ruin probabilities. Since Brownian motion is skip-free, this idea ignores (among other things) the presence of the overshoot, which we have seen to play an important role for example for the Cramér–Lundberg approximation. The objective of the corrected “diffusion” approximation is to take this and other deficits into consideration.

Diffusion approximations of random walks via Donsker’s theorem \([1]\) is a classical topic of probability theory. See for example Billingsley \([10]\). The first application in risk theory is Iglehart \([21]\) and two further standard references in the area are Grandell \([18, 19]\). All material of this section can be found in these references. For claims with infinite variance, Furrer, Michna, Weron \([17]\) suggested an approximation by a stable Lévy process rather than a Brownian motion. Further relevant references in this direction are Furrer \([15]\), Boxma, Cohen \([12]\) and Whitt \([44]\) \((5, p. 139)\)

A “corrected diffusion” approximations were introduced by Siegmund \([40]\) in a discrete random walk setting, with the translation to risk processes being carried out by Asmussen \([2]\); this case is in part simpler than the general random walk case because the ladder height distribution can be found explicitly which avoids the numerical integration involving characteristic functions which was used in \([40]\) to determine the constants. In Siegmund’s book \([41]\) the approach to the finite horizon case is in part different and uses local central limit theorems. The adaptation to risk theory has not been carried out. The “corrected diffusion” approximation was extended to the renewal model in Asmussen, Højgaard \([7]\), and to the Markov-modulated model of Chapter VII in Asmussen \([3]\); Fuh \([16]\) considers the closely related case of discrete time Markov additive processes. Hogan \([20]\) considered

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6This and three following paragraphs are quotations from \([5]\).
7Called also Donsker–Prohorov’s Invariance Principle, with credits to \([37, V.M.\].
a variant of the “corrected diffusion” approximation which does not require exponential moments. His ideas were adapted by Asmussen, Binswanger \[6\] to derive approximations for the infinite horizon ruin probability when claims are heavy-tailed; the analogous analysis of finite horizon ruin probabilities has not been carried out and seems non-trivial. For “corrected diffusion” approximations with higher-order terms, see Blanchet, Glynn \[11\] their results also cover some heavy-tailed cases (\[5\], p. 145).

Assuming that \(T_{\text{infty}} = T\), we focus on the case of \(T\) and \(Y\) exponential\[8\] with parameters \(\delta\) and \(\varrho\), where the distribution of \(T_{\text{ren}}\) may be written explicitly. We follow the idea to use the approximation \(1.1\) and to match the two first moments. We note that for \(R_s = (\vartheta - c)s + \sigma \mathcal{W}_s\) we have \(\mathbb{E}R_s = (\vartheta - c)s\), \(\mathbb{D}R_s = \sigma^2 s\) and for \(R_s = \sum_{i=1}^{N_s} Y_i - cs\) we have \(\mathbb{E}R_s = (\delta/\varrho - c)s\), \(\mathbb{D}R_s = (2\delta/\varrho^2)s\). Matching these two first moments, means taking \(\vartheta = \delta/\varrho\) and \(\sigma^2 = 2\delta/\varrho^2\) in \(1.1\). It yield\[9\]

\[
P\{T_{\text{ren}} \mid b, \varrho \leq t\} \approx \begin{cases} F(t; \mu, \lambda, \frac{1}{2}) |_{\mu = \frac{u^2}{\delta/\varrho - c}, \lambda = \frac{\varrho}{\delta}, \frac{1}{2}} & 0 < c \leq c^*, \\ \exp \left\{ - \frac{2\lambda}{\mu} \right\} F(t; \hat{\mu}, \lambda, \frac{1}{2}) |_{\hat{\mu} = \frac{u}{\delta/\varrho - c}, \lambda = \frac{\varrho}{\delta}, \frac{1}{2}} & c > c^*, \end{cases} \tag{3.6}
\]

where \(c^* = \delta/\varrho\), and

\[
\mu = \frac{u}{\varrho - c} = \frac{u}{(\delta/\varrho) - c}, \quad \lambda = \frac{u^2}{\sigma^2} = \frac{\varrho^2 u^2}{2\delta}, \quad \hat{\mu} = -\frac{u}{\varrho - c} = -\frac{u}{(\delta/\varrho) - c}. \tag{3.7}
\]

The approximation \(3.6\) illustrated in Fig. \[4\] below is further discussed in Section \[5\].

4. New “inverse Gaussian” approximation in level crossing

Being in the renewal framework of Section \[1.2\] let us write \(M = ET/\mathbb{E}Y\), \(D^2 = (ET^2 \mathbb{D}Y + (\mathbb{E}Y)^2 DT)/\mathbb{E}Y^3\), and

\[
\mathcal{M}_{u,c}(t \mid v) = \int_0^{\frac{t(v-c)}{\varrho}} \frac{1}{1 + y} \varphi \left( y; \frac{cM(1+y), \frac{\varrho^2 (1+y)}{s+y+c} \right)(y)dy, \tag{4.1}
\]

\[8\] It is noteworthy that “diffusion” and “corrected diffusion” approximations are identical in this case.

\[9\] Notation \(f(x; z) \mid z=a\) means \(f(x; a)\). Notation \(f(x) \mid x=a\) means the difference \(f(b) - f(a)\).
which is expressed in terms of c.d.f. of inverse Gaussian distribution \([12]\) as

\[
\mathcal{M}_{u,c}(t \mid v) = \begin{cases} 
F(x; \mu, \lambda, -\frac{1}{2}) \left| x = 1 \right|^{\frac{x(t-u)}{x(u)+1}}, & 0 < c \leq c^*, \\
\exp \left\{ -2 \frac{\lambda}{\mu} \right\} F(x; \mu, \lambda, -\frac{1}{2}) \left| x = 1 \right|^{\frac{x(t-u)}{x(u)+1}}, & c \geq c^*,
\end{cases}
\]

(4.2)

where \(c^* = \frac{1}{M}, \lambda = \frac{u}{c^2D^2} > 0, \mu = \frac{1}{1-cM}, \) and \(\hat{\mu} = \frac{1}{c^2M-1}.\) Plainly, \(\mu > 0\) for \(0 < c < c^*\) and \(\hat{\mu} > 0\) for \(c > c^*.\)

4.1. Approximation for conditional distribution. The following result is formulated and proved as Theorem 1.1 in \([33]\).

**Theorem 4.1.** In the renewal model, let p.d.f. \(f_T(y)\) and \(f_Y(y)\) be bounded from above by a finite constant, \(D^2 > 0, E(T^3) < \infty, E(Y^3) < \infty.\) Then for fixed \(c > 0\) and \(0 < v < t\) we have

\[
\sup_{t > v} | P \{ v < T_{u,c}^{\text{ren}} \leq t \mid T_1 = v \} - \mathcal{M}_{u,c}(t \mid v) | = O \left( \frac{\ln(u + cv)}{u + cv} \right),
\]

(4.3)

as \(u + cv \to \infty.\)

Since \(\mathcal{M}_{u,c}(t \mid v)\) in \([12]\) is defined by means of c.d.f. of inverse Gaussian distribution, let us call \([12]\) the approximation of Theorem 4.1, as well as its corollaries put forth below, “inverse Gaussian”.

Bearing in mind easy equality

\[
P \{ M(u + cv + cy) = n \} = P \left\{ \sum_{i=1}^{n} Y_i \leq u + cv + cy < \sum_{i=1}^{n+1} Y_i \right\}
\]

\[
= \int_0^{u+cv+cy} f_Y^n(u + cv + cy - z)P \{ Y_{n+1} > z \} dz,
\]

the idea behind the proof of Theorem 4.1 consists in application of the identity \([14]\).

After the change of variables \(y = z - v\), it writes as

\[
P \{ v < T_{u,c}^{\text{ren}} \leq t \mid T_1 = v \} = \int_0^{t-v} \frac{u + cv}{u + cv + cy} P \left\{ \sum_{i=2}^{M(u+cv+cy)+1} Y_i \right\}(y)dy
\]

\[
= \sum_{n=1}^{\infty} \int_0^{t-v} \frac{u + cv}{u + cv + cy} \int_0^{u+cv+cy} P \{ Y_{n+1} > z \}
\]

\[
\times f_Y^n(u + cv + cy - z) f_T^n(y)dydz,
\]

and non-uniform central limit theorem applied to approximate \(n\)-fold convolutions \(f_Y^n, f_T^n\) yields the approximating and residual terms, both investigated in \([33]\) by direct analytical method. This method is a considerable development of the method used previously in \([22, 23]\) and \([24]\). It applies identical transformations, approximations of integral sums by the corresponding integrals, evaluation of elliptic integrals of the third kind and of sums related to zeta-functions.

---

\(^{10}\)The “diffusion” approximation also involves the inverse Gaussian distribution. Someone will not concur with the proposed name, but to us it seems sensible.
4.2. Approximation for non-conditional distribution. Though approximation for conditional distribution is more convenient to prove because in this case the identity (4.4) is a convenient structure, the traditional object of interest is the unconditional distribution \( \mathcal{P}\{\Upsilon_{u,c}^{\text{ren}} \leq t\} \). Accordingly, let us formulate some corollaries of Theorem 4.1.

**Corollary 4.1.** Under conditions of Theorem 4.1, with p.d.f. \( f_{T_1}(y) \) bounded from above by a finite constant and with \( ET_1 < \infty \), we have

\[
\sup_{t \geq 0} \left| \mathcal{P}\{\Upsilon_{u,c}^{\text{ren}} \leq t\} - \int_0^t \mathcal{P}\{Y > u + cv\} f_{T_1}(v)dv - \int_0^t \mathcal{M}_{u,c}(t \mid v) f_{T_1}(v)dv \right| = O \left( \frac{\ln u}{u} \right),
\]

as \( u \to \infty \).

Having information about “regularity” of \( T_1 \), the insight into the term \( \int_0^t \mathcal{P}\{Y > u + cv\} f_{T_1}(v)dv \) is easy. To be particular, application of the Markov’s inequality yields \( \mathcal{P}\{Y > u + cv\} \leq \mathbb{E}Y^3/(u + cv)^3 \), and \( \int_0^t (u + cv)^{-3} f_{T_1}(v)dv = O(u^{-3}) \), as \( u \to \infty \). This order of magnitude is less than \( O(\ln u/u) \), and this term may be omitted.

Concerning the second term, we bear in mind that for each \( v > 0 \)

\[
\mathcal{M}_{u,c}(t \mid v) = \mathcal{M}_{u,c}(t - v) + O \left( \frac{v}{u} \right), \quad u \to \infty,
\]

where \( \mathcal{M}_{u,c}(t - v) = \mathcal{M}_{u,c}(t - v \mid 0) \). It yields the following result.

**Corollary 4.2.** Under conditions of Corollary 4.1, we have

\[
\sup_{t \geq 0} \left| \mathcal{P}\{\Upsilon_{u,c}^{\text{ren}} \leq t\} - \int_0^t \mathcal{P}\{Y > u + cv\} f_{T_1}(v)dv - \int_0^t \mathcal{M}_{u,c}(t - v) f_{T_1}(v)dv \right| = O \left( \frac{\ln u}{u} \right), \quad (4.4)
\]

as \( u \to \infty \).

The term \( \int_0^t \mathcal{M}_{u,c}(t - v) f_{T_1}(v)dv \) which corresponds to the case \( T_1 < \Upsilon_{u,c}^{\text{ren}} \), is a convolution. It agrees with the probabilistic intuition about the rôle of the first interval \( T_1 \) in the event of first level \( u \) crossing: when \( T_1 \) is fixed and is equal to \( v \), the time is modified from the whole time \( t \) to reduced time \( t - v \).

Until now we did not assume that time \( t \) is large, but deemed that it may be small, moderate, and large. Since the approximation (4.4) is formulated uniformly with respect to \( t > 0 \), the influence of \( T_1 \) can not be eliminated for \( t \) small and moderate. Though \( \mathcal{M}_{u,c}(t - v) \) is rendered in a relatively compact form, the integral \( \int_0^t \mathcal{M}_{u,c}(t - v) f_{T_1}(v)dv \) hardly can be evaluated as a compact explicit expression, even for \( T_1 \) exponential. On the contrary, for large \( t \) and “regular” \( T_1 \), the first interval does not affect the main-term approximation, i.e., for \( t \to \infty \) we have

\[
\int_0^t \mathcal{M}_{u,c}(t - v) f_{T_1}(v)dv = \mathcal{M}_{u,c}(t) (1 + \sigma(1)). \quad (4.5)
\]

**Remark 4.1.** The point \( c^* \) is excluded from consideration in Theorems 3.1 and 3.2. In this respect, it is special. In this point, the approximation (4.4) for \( \mathcal{P}\{\Upsilon_{u,c}^{\text{ren}} \leq t\} \), being neither outstanding, nor especially good, but merely noteworthy, involves the expression

\[
\mathcal{M}_{u,c^*}(t) = 2 \left( \Phi_{(1,1)} \left( \frac{\sqrt{u}}{c^*D} \right) - \Phi_{(1,1)} \left( \frac{u}{c^*D\sqrt{c^*t + u}} \right) \right).
\]

The approximation for \( \mathcal{P}\{\Upsilon_{u,c}^{\text{ren}} \leq \infty\} \) involves the expression

\[
\mathcal{M}_{u,c^*}(\infty) = 2 \Phi_{(1,1)} \left( \frac{M\sqrt{u}}{D} \right) - 1.
\]
4.3. Focus of the approximation and related problems. The approximation (4.4) is informative when \( P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t\} \) tends to a positive value, rather than to zero, as \( u \to \infty \). It is paramount, e.g., in the problem of investigating a solution \( u_{\alpha,t}(c) \) of the equation

\[
P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t\} = \alpha, \tag{4.6}
\]

where \( 0 < \alpha < 1 \). For diffusion process, \( u_{\alpha,t}(c) \) is investigated in [25], [26], and [28]. For compound renewal process with exponential \( T \) and \( Y \), it was done in [27], [29]. Analysis of \( u_{\alpha,t}(c) \) for general compound renewal process was done in [35] and [36]. It was found that the main term in approximation of \( u_{\alpha,t}(c) \), as \( t \to \infty \), is proportional to \( t^{1/2} \).

The focus in (4.4) with the remainder term \( Q(\ln u/u) \) differs from the focus in (3.4) with the remainder term \( \sigma(e^{-\kappa u}) \), as \( u \to \infty \); the latter is highly informative when \( P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t\} \) tends to zero, as \( u \to \infty \), while the former may be not.

**Remark 4.2 (Similarity with CLT).** For approximations (4.4) and (3.4), the difference in focuses is similar to situation known for the normal approximation and large deviations in the common central limit theorem for sums of i.i.d. summands. The former concerns itself with the asymptotic behavior around the mean value of the sum considered, while the latter deals with the exponential decline of remote tails of the distribution of the sum.

4.4. Subexponential distributions. Subexponential distributions are a special class of heavy-tailed distributions prominent in applied probability. First order approximations to ruin probabilities and waiting time distributions are by now called “folklore found in the latter deals with the exponential decline of remote tails of the distribution of the sum.

For approximations (4.4) and (3.4), the difference in focuses is similar to situation known for the normal approximation and large deviations in the common central limit theorem for sums of i.i.d. summands. The former concerns itself with the asymptotic behavior around the mean value of the sum considered, while the latter deals with the exponential decline of remote tails of the distribution of the sum.

4.5. Extensions of (4.4). In (3.4), the asymptotic expansions are constructed with the first correction term given explicitly. It is done by extending the same technique as was used in (3.4). A concurrent objective of (3.4) was to demonstrate the amplitude of the method. The terms in which the correction is found, are the generalized inverse Gaussian distribution. The results of (3.4), (3.4) are illustrated numerically in (3.4).

**Remark 4.3.** There is a difference between asymptotic expansions as a mathematical result, and as a tool to improve numerical performance. Many examples (in a simpler problems, e.g., in the central limit theorem for sums) show that involving a correction term, one not always and not necessary acquire a visible improvement in terms of proximity to the approximated function.

4.6. The asymptotic behavior, as \( t \to \infty \). Assuming that \( N_s, s > 0 \), is a Poisson process and that \( c > c^* = EY/ET \), Teugels [43] obtained\(^{11}\) an approximation of the form

\[
P\{\Upsilon_{u,c}^{[\text{ren}]} \leq t\} \approx P\{\Upsilon_{u,c}^{[\text{ren}]} < \infty\} - Q\left(ut^{-3/2}e^{-\gamma_1u-\gamma_2t}\right), \quad \text{as } t \to \infty, \tag{4.7}
\]

\(^{11}\)Quoting [2], “a relation of this type is highly expected... A rigorous proof and an explicit evaluation of \( Q \) was, however, first provided very recently by Teugels [43]” (2, p. 51).
where $Q$, $\gamma_1$ and $\gamma_2$ are certain positive constants. In [43], the renewal model was considered under the following conditions. The i.i.d. r.v. $T_i \overset{d}{=} T > 0$, $i = 1, 2, \ldots$, are exponential with parameter $\delta$, while i.i.d. r.v. $Y_i \overset{d}{=} Y > 0$, $i = 1, 2, \ldots$, are arbitrarily distributed subject to the following condition: there exists a positive value $v$ inside the region of convergence of $\Lambda(s) = cs - \delta (1 - Ee^{-sY})$, such that $\Lambda(-v) = 0$. This rather restrictive constraint implies that $Ee^{-sY}$ converges as an analytical function in a neighborhood of the origin, and the adjustment coefficient $\varkappa > 0$ exists.

The proof in [43] applies the Laplace transforms method. On p. 174 of [43] it is mentioned that for $Y$ exponential, one can derive (4.7) straightforwardly, “through an asymptotic expansion of the modified Bessel function” in Type II formula (2.2).

Assuming that $t \to \infty$, Corollary 4.2 yields “Teugels-type” approximation, as follows. If $u/t \to 0$ ($u \ll t$), we have

$$M_{u,c}(t) = M_{u,c}(\infty) - Q u \exp \{\gamma_1 u\}
\times t^{-3/2} \exp \{-\gamma_2 t\} \exp \left\{-\frac{\gamma_3 u^2}{t}\right\} (1 + o(1)), \quad t \to \infty, \quad (4.8)$$

where $Q = \sqrt{\frac{2}{\pi}} \frac{D}{c^{3/2}(1 - cM)^2}$, $\gamma_1 = \frac{1 - cM}{cD^2}$, $\gamma_2 = \frac{(1 - cM)^2}{2cD^2}$, $\gamma_3 = \frac{1}{2cD}$ and

$$M_{u,c}(\infty) = \left\{\begin{array}{ll}
\Phi(0, 1) \left(\frac{M}{D} \sqrt{\pi}\right) & - \exp \left\{\frac{2(1 - cM)}{cD} u\right\} \Phi(0, 1) \left(-\frac{2 - cM}{cD} \sqrt{u}\right), \quad 0 \leq c \leq c^*, \\
\exp \left\{-\frac{2(cM - 1)}{cD^2} u\right\} \Phi(0, 1) \left(-\frac{cM - 2}{cD} \sqrt{u}\right) - \Phi(0, 1) \left(-\frac{M}{D} \sqrt{u}\right), \quad c > c^*,
\end{array}\right.$$  

The approximation (4.8) is called “Teugels-type” since it differs from the original Teugels approximation (4.7). First, conditions on the model in these two results are obviously different: “Teugels-type” was obtained under much more general conditions. Second, it is assumed in [43] that $u$ is fixed and $t$ is tending to infinity, while in Corollary 4.2 it is assumed that both $u$ and $t$ are tending to infinity. However, the structure of both approximations (4.7) and (4.8) is the same, as is expected.

Indeed, quoting Teugels’ Remark 8.2 on p. 174 of [43], since “as a function of time the assumption $A(iv)$ essentially introduces an exponential decay”, for $W(x) = P\{\inf_{s \geq 0} (x + cs - W_s) \geq 0\}$ and $W(t, x) = P\{\inf_{0 \leq s \leq t} (x + cs - W_s) \geq 0\}$ with $c = 1$,

$$\text{Figure 4. Graphs (X-axis is $c$) of $P\{\Upsilon_{u,c}^{[\text{ren}]} | s, \delta, \phi \leq t\}$ calculated numerically using Type I exact formula (4.3) (dashed line), of the approximation $M_{u,c}(t) = M_{u,c}(t \mid 0)$ given by (4.2) (red line), and of approximation (4.3) (blue line). Here $t = 200$, $u = 20$, $\delta = 2$, Horizontal line: $P\{\Upsilon_{u,c}^{[\text{ren}]} | s, \delta, \phi \leq t\} = 0.463$.}$$
the extra $t^{-\gamma}$ with $\gamma = 1/2$ for the compound Poisson process and with $\gamma = 3/2$ for its supremum is not surprising in view of an approximating Wiener process. For let us consider the quantity $W(t, x)$ for a Wiener process with drift 1. Then (4.8), (p. 82) . . .

\[ W(t, x) - W(x) = e^{-2x} \left\{ 1 - \Phi(0,1) \left( \frac{t}{\sqrt{t}} \right) \right\} - \left\{ 1 - \Phi(0,1) \left( \frac{t + x}{\sqrt{t}} \right) \right\} \]

... which can be simplified to

\[ W(t, x) - W(x) \sim \sqrt{\frac{2}{\pi}} e^{x-1/2} x t^{-3/2}, \quad t \to \infty. \]

To have an illustration of (4.8) stated in a general renewal framework, let us consider $T$ and $Y$ exponential with positive parameters $\delta$ and $\varrho$. We have

\[ M = \varrho \delta^{-1}, \quad D^2 = 2 \varrho \delta^{-2}, \]

and

\[ Q = \frac{2}{\sqrt{\pi}} \sqrt{\varrho} (1 - c \varrho / \delta)^2, \quad \gamma_1 = \frac{\delta (\delta - c \varrho)}{2c^2 \varrho}, \quad \gamma_2 = \frac{\delta - c \varrho)^2}{4c \varrho}, \quad \gamma_3 = \frac{\delta^2}{4c^3 \varrho}. \]

Performance of Teugels-type approximation (4.8) in this case is illustrated in Fig. 4. It is poor in the vicinity of $c^*$; the same is known for the original Teugels’ approximation in [43]. It is no surprise since $Q = \frac{2}{\sqrt{\pi}} \sqrt{\varrho} (1 - c \varrho / \delta)^2$ turns into infinity for $c = c^*$.

5. “Diffusion” and “inverse Gaussian” approximations vs. exact formula when $T$, $Y$ are exponential

Bearing in mind (4.6) and (4.2), equation (4.4) for large $t > 0$ yields

\[ P\{Y_{u,c}^{\text{ren}} | \delta, \varrho \leq t\} \approx \begin{cases} F\left( \frac{a t}{u} + 1; \mu, \lambda, -\frac{1}{2} \right) - F\left( 1; \mu, \lambda, -\frac{1}{2} \right), & 0 < c \leq c^*, \\ \exp \left\{ -2 \lambda \frac{\lambda}{\mu} \right\} \left( F\left( \frac{a t}{u} + 1; \hat{\mu}, \lambda, -\frac{1}{2} \right) - F\left( 1; \hat{\mu}, \lambda, -\frac{1}{2} \right) \right), & c > c^*, \end{cases} \]

(5.1)

where $c^* = \frac{1}{M}$, $\mu = \frac{1}{1-cM}$, $\lambda = \frac{u}{c^2D^2}$, $\hat{\mu} = \frac{1}{cM-1}$.
For $T, Y$ exponential with parameters $\delta, \varrho$, we have $M = \rho \delta^{-1}$, $D^2 = 2 \rho \delta^{-2}$, and in (5.1) we have $c^* = \delta/\varrho$, and
\[
\mu = \frac{1}{1 - c(\varrho/\delta)} = \frac{\delta/\varrho}{(\delta/\varrho) - c}, \quad \lambda = \frac{\delta^2 u}{2c^2 \varrho} > 0, \quad \hat{\mu} = -\frac{1}{1 - c(\varrho/\delta)}.
\] (5.2)

Having derived explicit expressions (3.6), (3.7) for “diffusion” approximation and (5.1), (5.2) for “inverse Gaussian” approximation, we are able to compare these results with each other. Having Type I–Type III explicit formulas for $P\{\tau_{u,c}^{\text{ren}} \leq t \}$, we can further compare them with the exact formula.

Comparing of (3.6), (3.7) and (5.1), (5.2) with each other clearly shows that the structure of “inverse Gaussian” approximation and the structure of “diffusion” approximation are different, though both (5.1) and (5.2) are written in terms of c.d.f. of inverse Gaussian distribution (1.2).

When the values of $P\{\tau_{u,c}^{\text{ren}} \leq t \}$ are significantly greater than zero (which happens in a region including $c^*$), performance of both “inverse Gaussian” and “diffusion” approximations is compared to exact values of $P\{\tau_{u,c}^{\text{ren}} \leq t \}$ in Fig. 5.

6. Conclusions

The “inverse Gaussian” approximation for $P\{\tau_{u,c}^{\text{ren}} \leq t \}$ considered in Section 4 is structurally different from “normal”, “diffusion”, and “corrected diffusion” approximations (see in particular illustration in Section 5).

The “inverse Gaussian” approximation and its refinements in terms of Edgeworth expansions were rigorously proved by a technique based on an idea different from the ideas lying behind the “normal”, “diffusion”, and “corrected diffusion” approximations (see Sections 3.2 and 4.1). The methodology is said to be creative, and cannot be directly translated to “standard” techniques; it is not very demanding in terms of background.

The conditions which are required in “inverse Gaussian” approximation are weaker, and even look minimal, in comparison with conditions in “normal”, “diffusion”, and “corrected diffusion”; these substantially more restrictive conditions seem immanent to the methods of proof for “normal”, “diffusion”, and “corrected diffusion” approximations.

The “Teugels-type” approximation, as expected, is a corollary of the “inverse Gaussian” approximation; the form of the “Teugels-type” approximation is straightforward under very broad assumptions on the renewal model.

The “inverse Gaussian” approximation is important in many applications, in particular in the problem of finding a solution of non-linear equation (4.6); in the risk context this solution is called “non-ruin capital”; it is used to construct different controls in the models of long-term insurance solvency regulation (see, e.g., [30-32]).

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\[\text{References:} \]
\[13\] Which coincides in this case with “corrected diffusion” approximation.
\[14\] Though “inverse Gaussian”, “diffusion”, and “corrected diffusion” approximations may be written in terms of c.d.f. of inverse Gaussian distribution.
\[15\] Though develops the technique in [22] and [22].
\[16\] Quoting again [5] (see quotation in full in Section 3.2), note that “the analogous analysis of finite horizon ruin probabilities $\psi(u, T)$ has not been carried out and seems non-trivial.”
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