Some Properties on Sums of Element Orders in Finite $p$-groups

Nil Mansuroğlu*

Abstract

In literature, there are many papers on the sum of element orders of a finite group. In this study, in particular, we deal with the cases in finite $p$-groups. Our main aim is to investigate the sums of element orders in finite $p$-groups and to give some properties of such sums. Let $\psi(G)$ denote the sum of element orders of a finite group $G$. As an immediate consequence, we proved that $\psi(G) < \frac{1}{p - 1}\psi(C)$ and $\psi(G) < \frac{1}{p^r - 1}\psi(C)$, where $G$ is a non-cyclic finite $p$-group of order $p^r$ and $C$ is a cyclic group of order $p^r$ for some prime $p$.

Keywords: Finite $p$-group; group element orders; sum of element orders.

AMS Subject Classification (2020): Primary: 20D60; Secondary: 20E34.

*Corresponding author

1. Introduction

Our main starting point is given by the papers (see H. Amiri [2], H. Amiri and S.M.J. Amiri [1], Herzog et al. [5]) which studied on the sums of element orders in finite groups. Given a finite group $G$, we denote the sum of element orders in $G$ by $\psi(G)$. Historically, the most enlightening in this area is due [5], who introduced the function $\psi(G)$ for a finite group $G$ in [2] and proved that $\psi(G) < \frac{1}{p - 1}\psi(C)$, where $C$ denotes a cyclic group of the same order with the order of $G$. Then, in [5], by improving the results obtained by S.M. Jafarian Amiri and M. Amiri in [4] and by R. Shen, G. Chen and C. Wu in [10], M. Herzog, P. Longobardi and M. Maj found an exact upper bound for sums of element orders in non-cyclic finite groups. In [8], N. Mansuroğlu derived the exact formula for the sums of element orders in symmetric groups. Throughout this paper, we assume that $G$ is a finite $p$-group of order $p^r$ for a prime $p$. In this note we will focus on the study of $\psi(G)$. Our main aim is to investigate the sum of element orders in finite $p$-groups and to give some properties of the sum of element orders in finite $p$-groups. We investigate to find an exact upper bound for sums of element orders in non-cyclic $p$-finite groups.

2. Preliminaries

This section contains necessary preliminary results and notation. We use standard notation. We define the function $\psi(G) = \sum_{x \in G} o(x)$, where as usual, $o(x)$ is the order of the element $x$. Basic concepts and some results on group theory can be found in [6, 9]. Specially, more details for finite $p$-groups can be found in [7]. An important ingredient in our proofs is the following two lemmas which are particular case of Lemma 2.9 in the paper [5].

Lemma 2.1. If $C$ is a cyclic group of order $p^r$ for some prime $p$, then

$$\psi(C) = \frac{p^{2r+1} + 1}{p + 1}.$$
Proof. Let $\varphi(p)$ be Euler’s function.

\[
\psi(C) = 1 + p\varphi(p) + p^2\varphi(p^2) + \ldots + p^r\varphi(p^r) \\
= 1 + pp\left(\frac{p-1}{p}\right) + p^2p^2\left(\frac{p-1}{p}\right) + \ldots + p^r p^r\left(\frac{p-1}{p}\right) = \frac{p^{2r+1} + 1}{p + 1}.
\]

\[\square\]

Lemma 2.2. If $C$ is a cyclic group of order $p^r$ for some prime $p$, then

\[
\psi(C) > p^{2r-1}(p - 1).
\]

Proof. By Lemma 2.1,

\[
\psi(C) = \frac{p^{2r+1} + 1}{p + 1} = p^{2r} - p^{2r-1} + p^{2r-2} - \ldots + 1 > p^{2r-1}(p - 1).
\]

\[\square\]

3. The main results

Now, in this section we give our main results.

Theorem 3.1. Let $G$ be a non-cyclic finite $p$-group of order $p^r$ for some prime $p$. Then

\[
\psi(G) < p^{2r-1}.
\]

Proof. Since $G$ is a non-cyclic finite $p$-group of order $p^r$, for each element $x \in G$, $o(x) \leq p^{r-1}$. But the order of the identity element 1 is 1, as a result of this, we have

\[
\psi(G) \leq (p^r - 1)p^{r-1} + 1 = \frac{(p^r - 1)p^r}{p} + 1 < \frac{p^{2r}}{p} = p^{2r-1},
\]

as required.

\[\square\]

Theorem 3.2. Let $G$ be a non-cyclic finite $p$-group of order $p^r$ and $C$ be a cyclic finite group of order $p^r$. Then

\[
\psi(G) < \frac{1}{p-1}\psi(C).
\]

Proof. Suppose that $\psi(G) \geq \frac{1}{p-1}\psi(C)$. By Lemma 2.2,

\[
\psi(G) \geq \frac{1}{p-1}\psi(C) > \frac{1}{p-1}p^{2r-1}(p - 1) = p^{2r-1}.
\]

This implies that there exists $x \in G$ with $o(x) > p^{r-1}$. Thus $|G : \langle x \rangle| < p$ and $\langle x \rangle$ is a $p$-group. As a consequence, $|G| = o(x)$, namely $G = \langle x \rangle$. But $G$ is non-cyclic group, which is a contradiction. Hence, $\psi(G) < \frac{1}{p-1}\psi(C)$.

\[\square\]

We now provide a quantitative version of Theorem 3.2.

Corollary 3.1. Suppose that $p$ is odd prime. Let $G$ be a non-cyclic finite $p$-group of order $p^r$ and $C$ be a cyclic finite group of order $p^r$. Then

\[
\psi(G) < \frac{1}{2}\psi(C).
\]

Proof. By Theorem 3.2, we have $\psi(G) < \frac{1}{p-1}\psi(C)$. Since $p - 1 \geq 2$, we obtain

\[
\psi(G) < \frac{1}{p-1}\psi(C) \leq \frac{1}{2}\psi(C).
\]

This competes the proof.

\[\square\]
The next result is the analogous of the result obtained by Theorem 1 in [5] for finite $p$-groups.

**Theorem 3.3.** Let $G$ be a non-cyclic finite $p$-group of order $p^r$ and $C$ be a cyclic finite group of order $p^r$. Then

$$\psi(G) \leq \frac{7}{11}\psi(C).$$

**Proof.** Assume that $G$ is a non-cyclic finite $p$-group of order $p^r$ satisfying $\psi(G) > \frac{7}{11}\psi(C)$. By Lemma 2.1,

$$\psi(G) > \frac{7}{11}\psi(C) = \frac{7}{11}\frac{p^{2r+1}+1}{p+1} > \frac{7}{11}\frac{2p^{2r}}{p+1} = \frac{14}{11}\frac{p^{2r}}{p+1}.$$

There exists $x \in G$ with $o(x) > \frac{14}{11}\frac{p+1}{11}$. It follows that

$$|G : \langle x \rangle| < \frac{11(p+1)}{14}.$$

Now first we suppose that $p = 2$, then $G$ is 2-group. Therefore, we have

$$|G : \langle x \rangle| < \frac{33}{14}.$$

Thus $|G : \langle x \rangle| = 2, 2^r \geq 4, r \geq 2$. Let $C_{2^{r-1}}$ be a cyclic group of order $2^{r-1}$. Hence

$$\psi(G) \leq \psi(C_{2^{r-1}}) + 2^{2(r-1)} = \frac{2^{2r-1}+1}{3} + \frac{2^{2r}}{4} = \frac{5}{12}2^{2r} + \frac{1}{3} \leq \frac{7}{11}\left(\frac{2^{2r+1}+1}{3}\right) = \frac{7}{11}\psi(C),$$

which contradicts to the fact that $\psi(G) > \frac{7}{11}\psi(C)$.

Now we investigate the case that $p \geq 3$. Since $p$ is odd prime, it follows from Corollary 3.1 that

$$\psi(G) < \frac{1}{2}\psi(C) < \frac{7}{11}\psi(C),$$

which is a contradiction. This completes the proof. \qed

Our main result is the following theorem.

**Theorem 3.4.** Let $G$ be a non-cyclic finite $p$-group of order $p^r$ and $C$ be a cyclic finite group of order $p^r$. Then

$$\psi(G) < \frac{3}{4}\psi(C).$$

**Proof.** Assume that $G$ is a non-cyclic finite $p$-group of order $p^r$ satisfying $\psi(G) > \frac{2}{3}\psi(C)$. Since $G$ is a non-cyclic, for each element $x \in G$, $o(x) \leq p^{r-1}$. But the order of the identity element 1 is 1, as a result of this, we have

$$\psi(G) \leq (p^r - 1)p^{r-1} + 1.$$

Now first we suppose that $p = 2$, then $G$ is 2-group. Therefore, we have

$$\psi(G) \leq (2^r - 1)2^{r-1} + 1 = 2^{2r-1} - 2^{r-1} + 1 < \frac{3}{4}\left(\frac{2}{3}2^{2r} + \frac{1}{3}\right) = \frac{3}{4}\psi(C).$$

Now we investigate the case that $p \geq 3$. Since $p$ is odd prime, by Corollary 3.1, we have

$$\psi(G) < \frac{1}{2}\psi(C) < \frac{3}{4}\psi(C),$$

a contradiction. This completes the proof. \qed

The following examples hold all our main consequences.

**Example 3.1.** Let $G = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ be a non-cyclic 2-group of order 8 and $C$ be a cyclic group of order 8.

**Example 3.2.** Let $G = \langle a, b \mid a^8 = 1, a^4 = b^2, b^{-1}ab = a^{-1} \rangle$. Then $G$ is a generalized quaternional group of order 16. It is easy to see that $G$ satisfies $\psi(G) < \frac{2}{3}\psi(C)$ where $C$ is a cyclic group of order 16.
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Affiliations

NIL MANSUROĞLU
ADDRESS: Kırşehir Ahi Evran University, Dept. of Mathematics, 40100, Kırşehir-Turkey.
E-MAIL: nil.mansuroglu@ahievran.edu.tr
ORCID ID: 0000-0002-6400-2115