Geometric Complexity Theory III: on deciding positivity of Littlewood-Richardson coefficients

Dedicated to Sri Ramakrishna

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February 1, 2008

Abstract

We point out that the remarkable Knutson and Tao Saturation Theorem [9] and polynomial time algorithms for linear programming [14] have together an important, immediate consequence in geometric complexity theory [15, 16]: The problem of deciding positivity of Littlewood-Richardson coefficients belongs to P; cf. [10].

Specifically, for $GL_n(\mathbb{C})$, positivity of a Littlewood-Richardson coefficient $c_{\alpha,\beta,\gamma}$ can be decided in time that is polynomial in $n$ and the bit lengths of the specifications of the partitions $\alpha$, $\beta$ and $\gamma$. Furthermore, the algorithm is strongly polynomial in the sense of [14].

The main goal of this article is to explain the significance of this result in the context of geometric complexity theory. Furthermore, it is also conjectured that an analogous result holds for arbitrary symmetrizable Kac-Moody algebras.

The fundamental Littlewood-Richardson rule in the representation theory of $GL_n(\mathbb{C})$ [14] states that the tensor product of two irreducible representations (Weyl modules) $V_\alpha$ and $V_\beta$ of $GL_n(\mathbb{C})$ decomposes as follows:

$$V_\alpha \otimes V_\beta = \bigoplus_{\gamma} c_{\alpha,\beta,\gamma} V_\gamma,$$

(1)
where \( c_{\alpha, \beta, \gamma} \) are Littlewood-Richardson coefficients. Here \( \alpha, \beta \) are partitions (Young diagrams) with at most \( n \) rows. The sum is over all Young diagrams \( \gamma \) of height at most \( n \), and size equal to the sum of the sizes of \( \alpha \) and \( \beta \).

This rule has been studied intensively in representation theory; cf. Fulton \[6, 4\]. But the problem of deciding positivity of \( c_{\alpha, \beta, \gamma} \) efficiently did not receive much attention, perhaps because there was really no motivation for studying it. The problem arises naturally in geometric complexity theory \[15, 16, 17\], which is an approach to the fundamental problems in complexity theory (GCT), such as \( P \) vs. \( NP \), through algebraic geometry and representation theory. The basic philosophy of this approach is the flip from hard nonexistence to easy existence. Specifically, the approach first reduces the hard nonexistence problems in complexity theory, such as \( P \) vs. \( NP \), in characteristic zero, to showing existence of certain obstructions, which are certain gadgets with algebro-geometric and representation theoretic properties. The central geometric invariant theoretic \[18\] results of GCT \[16, 17\] pave the road for proving easiness of this and related existence problems in geometric invariant theory, once certain existence problems in representation theory are shown to be easy. The transition from nonexistence to existence was proposed in \[15\]. The stronger transition from hard nonexistence to easy existence was proposed in \[16\], which is an extended abstract of \[17\].

By divine justice, as was to be expected for the \( P \) vs. \( NP \) problem, showing that these representation theoretic existence problems are easy turned out to be extremely hard. Because they are intimately related to the century old, fundamental unsolved problems of representation theory, such as the plethysm problem \[19, 4\]. As such, when the flip philosophy was first proposed in \[15, 16\], it went against the common belief among mathematicians. Deciding positivity of a Littlewood-Richardson coefficient \( c_{\alpha, \beta, \gamma} \), i.e. deciding if the the Weyl module \( V_{\gamma} \) occurs (exists) within \( V_{\alpha} \otimes V_{\beta} \) is the simplest instance of a general existence problem, called the subgroup restriction problem in \[16\] described below. Its membership in \( P \) (Theorem \[4\]) provides the first concrete evidence in support of the flip philosophy of GCT.

It is a direct consequence of the Saturation Theorem of Knutson and Tao \[9\] and polynomial time algorithms for linear programming \[14\]. After a preliminary version of this note was written, it was communicated to us by Prof. Tao that actually they had thought briefly about the polynomial time algorithm question for positivity in the different context of the Honeycomb model \[10\], and asked Peter Shor about it. He basically gave the same response that is in this note. See page 180-181 of \[10\]; though there is
a slight error in that paper, in asserting that the simplex method takes polynomial time. Nevertheless, as we point out, for the LP that arises here even a strongly polynomial time algorithm exists.

The Saturation Theorem itself was proved in an entirely different context: as a step in the proof of Horn’s conjecture \[20\] \[5\], which arose from the work of H. Weyl in 1912 and I. M. Gelfand in 1940’s. After several attempts, finally Klyachko \[5\] proved some remarkable results in the study of stability criterion for toric vector bundles on the projective plane. Zelevinsky observed \[20\] that Horn’s conjecture would follow from these results if the Saturation Conjecture were proved; as happened soon after in \[9\]. For the sake of a computer scientist not familiar with these developments, we give a self-contained proof of Theorem 1 here, assuming only the statement of the Saturation Theorem.

Theorem 1 was stated in \[16\] as known, implicitly assuming integrality of the polytope \(P\) defined below. We recently realized that \(P\) need not be integral, in view of \[12\], which disproved a conjecture of Berenstein and Kirillov \[1\] that Gelfand-Tsetlin polytopes are integral. Fortunately, the Saturation Theorem, which had come to our attention just then, provided a sufficient relaxation of integrality.

Let \(\lambda = (\lambda_1, \cdots, \lambda_k)\), where \(\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > 0\), be a partition (Young diagram). By its bit length, we mean the bit length of its specification, which is \(\sum_i \log_2(\lambda_i)\). Observe that the dimension of the Weyl module \(V_{\lambda}\) can be exponential in \(n, k\) and the bit lengths of \(\lambda_i\)’s. Because the dimension of \(V_{\lambda}\) is the total number of semistandard tableau of shape \(\lambda\) with entries in \([1, n]\) \[4\].

**Theorem 1** Given partitions \(\alpha, \beta\) and \(\gamma\), deciding if \(V_{\gamma}\) exists within \(V_{\alpha} \otimes V_{\beta}\)—i.e., if \(c_{\alpha, \beta, \gamma}\) is positive—can be done in polynomial time; i.e., in time that is polynomial in \(n\) and the bit lengths of \(\alpha, \beta\), and \(\gamma\). Furthermore, the algorithm is strongly polynomial in the sense of \[14\].

This is remarkable, since the dimensions of \(V_{\alpha}, V_{\beta}, V_{\gamma}\) can be exponential in \(n\) and the bit lengths of \(\alpha_i, \beta_j\) and \(\gamma_k\)’s. What the result says is that whether an exponential dimensional object \(V_{\gamma}\) can be embedded in another exponential dimensional object \(V_{\alpha} \otimes V_{\beta}\) can be decided in time that is polynomial in \(n\) and the bit lengths of just their labels \(\alpha, \beta\) and \(\gamma\).

\[1\]If we assume that a partition \(\lambda\) is specified as \((\lambda_1, \cdots, \lambda_n)\), with \(\lambda_1 \geq \cdots \geq \lambda_n\), where \(\lambda_i = 0\) for \(i\) higher than the height of \(\lambda\), then the term \(n\) can be subsumed in the bit length of the input.
Strong polynomiality stated in the theorem means that: (1) The number of arithmetic steps in the algorithm is polynomial in \(n\). It does not depend on the bit lengths of \(\alpha_i, \beta_j, \text{ and } \gamma_k\)'s. (3) The bit length of every intermediate operand that arises in the algorithm is polynomial in the total bit length of \(\alpha, \beta\) and \(\gamma\).

Subgroup restriction problem

The fundamental problems and conjectures in representation theory that arise in geometric complexity theory are instances of the following subgroup restriction problem. Suppose \(G\) is a reductive group over \(\mathbb{C}\). In complexity theory, we shall only be interested in nice reductive groups such as: \(SL_n(\mathbb{C})\), the classical simple groups, the group \(\mathbb{C}^*\) of nonzero complex numbers, finite simple groups, and the groups obtained from these by standard groups theoretic constructions such as products, wreath products etc. Suppose \(H \subseteq G\) is a nicely embedded, nice subgroup of \(G\). Two important examples of nice embeddings are:

1. \(H \rightarrow G = H \times H\) (diagonal map). In this case, the subgroup restriction problem will reduce to decomposing the tensor product of two representations of \(H\), together with the associated decision problem.

2. \(GL(\mathbb{C}^n) \times GL(\mathbb{C}^n) \rightarrow GL(\mathbb{C}^n \otimes \mathbb{C}^n)\). In this case, the subgroup restriction problem will become equivalent to finding a positive decomposition of the tensor product of two irreducible representations (Specht modules) of the symmetric group \(\mathfrak{S}_n\)–a fundamental, century old unsolved problem in the representation theory of the symmetric groups–together with the associated decision problem.

3. \(U\) is a representation of \(H\), \(G = GL(U)\), and \(H \rightarrow G\) is the representation homomorphism. In this case, the subgroup restriction problem will reduce to the (generalized) plethysm problem \(\mathfrak{S}_n\)–a fundamental, century old unsolved problem in the representation theory of the general linear group–together with the associated decision problem.

Let \(V = V_\alpha\) be a representation of \(G\), where \(\alpha\) is a label that completely specifies \(V\). For example, if \(G = GL_n(\mathbb{C})\), and \(V_\alpha\) is its irreducible representation (Weyl module) then the label \(\alpha\) is the Young diagram. If \(V = V_\beta \otimes V_\gamma\), where \(V_\beta\) and \(V_\gamma\) are irreducible, then the label \(\alpha\) is the composite \(\beta \otimes \gamma\), and so on.
Since $H$ is a subgroup of $G$, $V$ is also a representation of $H$. The classical result of H. Weyl says that $V$ has an essentially unique decomposition as an $H$-module:

$$V_\alpha = \bigoplus_{\beta} m(\beta) W_\beta,$$

(2)

where $\beta$ is the label ranging over irreducible representations of $H$, $W_\beta$ is the corresponding irreducible representation, and $m(\beta)$ is its multiplicity.

The subgroup restriction problem is find an explicit efficient positive decomposition rule for (2) akin to the Littlewood-Richardson rule for (1). The associated existence problem is: given labels $\alpha$ and $\beta$ of $H$ and $G$ respectively, does $W_\beta$ occur within $V_\alpha$? That is, is $m(\beta)$ positive? The goal is to show that this problem belongs to the complexity class $P$. Here by polynomial, we mean polynomial in the numeric parameters associated with $G$ and $H$ and the bit lengths of the labels $\alpha, \beta$. For example, the numeric parameter associated with $GL_n(\mathbb{C})$ is $n$, with the symmetric group $S_n$ is $n$, and if $G$ is built using products etc. then they are the numeric parameters of the building blocks.

When $H = GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$, the decomposition (2) coincides with the tensor product decomposition (1), and the decision problem is simply deciding positivity of a Littlewood-Richardson coefficient. Though the general problem is far harder than the latter, it is qualitatively similar. Hence, Theorem 1 supports the conjecture that the general problem also belongs to $P$.

Once this representation theoretic existence problem is shown to be in $P$, and a sufficiently concrete form of the decomposition (2) is found, the central algebro-geometric results of GCT in [16, 17] give a lead on the (harder) geometric invariant theoretic existence problems in GCT. The road ahead is undoubtedly long and arduous, but, at least, the journey has begun.

Proof

The proof of Theorem 1 follows easily from the following three results:

1. Littlewood-Richardson rule: specifically, a polyhedral interpretation of the Littlewood-Richardson coefficients. The polytope we use here is more elementary than Berenstein-Zelevinsky polytope [2] and the Hive polytope [9]—the latter two have some stronger properties not used here.

2. Saturation Theorem [9].
3. Polynomial time algorithm for linear programming: e.g. the ellipsoid
or the interior point method, and the related strongly polynomial time
algorithm for combinatorial linear programming due to Tardos [14].

Let us begin with a polyhedral interpretation; this should be well known.
Recall that the Littlewood-Richardson coefficient $c_{\alpha,\beta}^\gamma$ has the following combi-

Let us say that a word $w = w_1 \cdots w_r$ is a reverse lattice word if, when
read backwards from the end to any letter $w_s$, $s < r$, the sequence $w_r \cdots w_s$
contains at least as many 1’s as 2’s, at least as many 2’s as 3’s, and so on
for all positive integers. The row word $w(T)$ of a skew tableau $T$ is defined
to be the word obtained by reading its entries from bottom to top, and left
to right. A skew-tableau $T$ of shape $\gamma/\alpha$ is called a Littlewood-Richardson
skew tableau if its row word $w(T)$ is a reverse lattice word.

Then $c_{\alpha,\beta}^\gamma$ is the number of Littlewood-Richardson skew tableaux of
shape $\gamma/\alpha$ of content $\beta$.

Let $r^i_j(T)$, $i \leq n$, $j \leq n$, denote the number of $j$’s in the $i$-th row of $T$.
These are integers satisfying the constraints:

1. Nonnegativity: $r^i_j \geq 0$.

2. Shape constraints: For $i \leq n$,

$$\alpha_i + \sum_j r^i_j = \gamma_i.$$

3. Content constraints: For $j \leq n$:

$$\sum_i r^i_j = \beta_j.$$

4. Tableau constraints: No $k \leq j$ occurs in the row $i + 1$ of $T$ below a $j$
or a higher integer in the row $i$ of $T$:

$$\alpha_{i+1} + \sum_{k \leq j} r^{i+1}_k \leq \alpha_i + \sum_{k' \leq j} r^i_{k'}.$$

5. Reverse lattice word constraints: $r^i_j = 0$ for $i < j$, and for $i \leq n$,

$$\sum_{i' \leq i} r^{i'}_j \leq \sum_{i' < i} r^{i'}_{j-1}.$$
Let \( r \) denote the vector with the entries \( r^j_i(T) \). These constraints can be written in the form of a linear program:

\[
Ar \leq b,
\]

where the entries of \( A \) are 0, 1 or \(-1\), and the entries of \( b \) are homogeneous, integral, linear forms in \( \alpha_i, \beta_j \), and \( \gamma_k \)’s. Thus \( c^{\gamma}_{\alpha, \beta} \) is the number of integer points in the polytope \( P \) determined by these constraints.

**Claim 1** The polytope \( P \) contains an integer point iff it is nonempty.

**Proof:** One direction is trivial.

Suppose \( P \) is nonempty. Since \( b \) is homogeneous in \( \alpha, \beta \) and \( \gamma \), it follows that, for any positive integer \( q \), \( c^{q\gamma}_{q\alpha,q\beta} \) is the number of integer points in the scaled polytope \( qP \). All vertices of \( P \) have rational coefficients. Hence, for some positive integer \( q \), the scaled polytope \( qP \) has an integer point. It follows that, for this \( q \), \( c^{q\gamma}_{q\alpha,q\beta} \) is positive. Saturation Theorem [9] says that, in this case, \( c^{\gamma}_{\alpha, \beta} \) is positive. Hence, \( P \) contains an integer point. Q.E.D.

Whether \( P \) is nonempty can be determined in polynomial time using either the ellipsoid or the interior point algorithm for linear programming. Since the linear program (3) is combinatorial [14], this can also be done in strongly polynomial time using Tardos’ algorithm [14]. This proves Theorem [14].

It is of interest to know if there is a purely combinatorial algorithm for this problem that does not use linear programming; i.e., one similar to the max-flow or weighted matching problems in combinatorial optimization. The polytopes that arise in these combinatorial optimization problems are unimodular and integral—i.e., their vertices are integral [14]. In contrast, \( P \) need not be integral. This is known for the Gelfand-Tsetlin polytope that arises in the study of Kostka numbers [12], and also for the Hive polytope [8]. Unlike the hive polytope [9], \( P \) need not even have an integral vertex. It is reasonable to conjecture that there is a polynomial time algorithm that provides an integral proof of positivity of \( c^{\gamma}_{\alpha, \beta} \), in the form of an integral point in \( P \). The above algorithm, as also the one in [10], only provides a rational proof; i.e., a rational point in \( P \).

There is a generalization of the Littlewood-Richardson rule for arbitrary classical Lie algebras, and also for symmetrizable Kac-Moody algebras [7, 13, 11]. It was erroneously stated in [16] as known that the problem of deciding positivity of generalized Littlewood-Richardson coefficients also belongs to \( P \). But now we conjecture that this is so.
Specifically, let $G$ be a symmetrizable generalized Kac-Moody algebra with rank $r$, which is the dimension of the corresponding symmetrizable generalized Cartan matrix. Let $V_\alpha, V_\beta$ be two irreducible integrable representations of $G$ with highest weights $\alpha$ and $\beta$. Then it is known that

$$V_\alpha \otimes V_\beta = \bigoplus c^\gamma_{\alpha,\beta} V_\gamma,$$

where $c^\gamma_{\alpha,\beta}$ are generalized Littlewood-Richardson coefficients, as defined in [7, 13, 11].

**Conjecture 2** Given fixed $\alpha, \beta, \gamma$, positivity of $c^\gamma_{\alpha,\beta}$ can be decided in polynomial time; i.e. in time that is polynomial in the rank $r$ and the bit lengths of the specifications of $\alpha, \beta, \gamma$. Furthermore, there exists a strongly polynomial time algorithm for the same.

The proof here does not generalize, since the saturation conjecture is known to be false for the type $B, C, D$ [3, 12]. Hari Narayanan pointed out to us that J. De Loera and T. McAllister [12] have recently made some conjectures for the hive polytopes for types $B, C, D$. These may provide a step towards the proof of Conjecture 2 for types $B, C, D$.

Given the fundamental importance of the $P \text{ vs. } NP$ question, we hope this note will bring to a computer scientist’s attention, similar, but far more formidable conjectures of geometric complexity theory [16]. The first few instances of these conjectures say that the decision versions of the well known representation theoretic positivity problems [19], such as the plethysm problem, have (strongly) polynomial time algorithms. It also makes sense to make similar conjectures for other positivity problems in [19] not considered in [16], such as the ones concerning Kazdan-Lusztig polynomials.

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