A KAM THEOREM WITH APPLICATIONS TO
PARTIAL EQUATIONS OF HIGHER DIMENSION

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Abstract. The existence of lower dimensional KAM tori is shown for a class of
nearly integrable Hamiltonian systems where the second Melnikov’s conditions are
eliminated. As a consequence, it is proved that there exist many invariant tori and
thus quasi-periodic solutions for nonlinear wave equations, Schrödinger equations
and other equations of higher spatial dimension.

1. Introduction and main results.

Let us begin with the non-linear wave (NLW) equation

\[ u_{tt} - u_{xx} + V(x)u + h(x, u) = 0 \]  

subject to Dirichlet boundary conditions. The existence of solutions, periodic in
time, for NLW equations has been studied by many authors. See [B, B-B, B-G, B-P,
Br, L-S] and the references therein, for example. While finding quasi-periodic
solutions, one will inevitably encounter so-called small divisor difficulty. The KAM
(Kolmogorov-Arnold-Moser) theory is a very powerful tool to overcome the diffi-
culty. This theory deals with the existence of invariant tori (and thus quasi-periodic
solutions) for nearly integrable Hamiltonian systems. In order to obtain the quasi-
periodic solutions of a partial differential equation, one may show the existence of
the lower (finite) dimensional invariant tori for the infinitely dimensional Hamilton-
ian system defined by the partial differential equation. Now consider a Hamiltonian
of the form:

\[ H = (\omega, y) + \sum_{j=1}^{\iota} \Omega_j z_j \bar{z}_j + R(x, y, z, \bar{z}), \quad \iota \leq \infty, \]  

with tangential frequency vector \(\omega = (\omega_1, ..., \omega_n)\) and normal frequency vector \(\Omega = (\Omega_1, ..., \Omega_{\iota})\). When \(R \equiv 0\), there is a trivial invariant torus \(x = \omega t, y = 0, z = \bar{z} = 0\).
The KAM theory guarantees the persistence of the trivial invariant torus with
a small deformation under a sufficiently small perturbation \(R\), provided that the
well-known Melnikov conditions are fulfilled:

\[ (k, \omega) - \Omega_j \neq 0 \quad \text{(the first Melnikov’s)} \]  

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for all \( k \in \mathbb{Z}^n \) and \( 1 \leq j \leq \iota \), and
\[
(k, \omega) + \Omega_{j_1} - \Omega_{j_2} \neq 0 \quad \text{(the second Melnikov’s)} \quad (1.4)
\]
for all \( k \in \mathbb{Z}^n \) and \( 1 \leq j_1, j_2 \leq \iota, j_1 \neq j_2 \). See [E,K1,P1,W] for the details. This KAM theorem can be applied to a wide array of Hamiltonian partial differential equations of 1-dimensional spatial variable, including (1.1). In that line, Kuksin[K1,2] shows that there are many quasi-periodic solutions of (1.1), assuming that the potential \( V \) depends on an \( n \)-dimensional external parameter vector in some non-degenerate way. Wayne[W] obtains also the existence of the quasi-periodic solutions of (1.1), when the potential \( V \) is lying on the outside of the set of some “bad” potentials. In [W], the set of all potentials is given some Gaussian measure and then the set of “bad” potentials is of small measure. Bobenko & Kuksin[Bo-K] and Pöschel[P2] get the existence of invariant tori and quasi-periodic solutions for a given potential \( V(x) \equiv m \in (0, \infty) \). By the remark in [P2], the same result holds also true for the parameter values \(-1 < m < 0\). When \( m \in (-\infty, -1) \setminus \mathbb{Z} \), it is shown in [Y1] that there are many hyperbolic-elliptic invariant tori. More recently, the existence of invariant tori (thus quasi-periodic solutions) of (1.1) are shown for any prescribed non-vanishing and smooth potential\(^1\) \( V(x) \) in [Y2] and for \( V(x) \equiv 0 \) in [Y3]. In [C-W], [Bo1], [C-Y] and [Br-K-S], the equation (1.1) subject to periodic boundary conditions is investigated.

For NLW equation (1.1) of spatial dimension 1, the multiplicity of normal frequency \( \Omega_j \) is 1 in Dirichlet boundary condition and 2 in periodic boundary condition. Considering partial differential equations with spatial dimension > 1, a significant new problem arises due to the presence of clusters of normal frequencies of the Hamiltonian system defined by partial differential equations. In this case, the multiplicity of \( \Omega_j \) goes to \( \infty \) as \(|j| \to \infty\); consequently, the second Melnikov’s conditions is destroyed seriously, thus preventing the application of the KAM theorems mentioned above to Hamiltonian partial differential equations of higher spatial dimension. Bourgain[Bo1-4] develops another profound approach, originally proposed by Craig-Wayne in [C-W], and successfully obtains the existence of quasi-periodic solutions of the nonlinear Schrödinger (NLS) equations and NLW equations of higher dimension in space. Instead of KAM theory, the used approach is based on a generalization of Lyapunov-Schmidt procedure and a technique by Fröhlich and Spencer[F-S]. Bambusi calls this approach “C-W-B method”.

The advantage of the KAM approach is, on one hand, to possibly simplify the proof and, on the other hand, to allow the construction of local normal forms closed to the considered torus, which could be useful for the better understanding of the dynamics. For example, in generally, one can easy check the linear stability and the vanishing Lyapunov exponents. As a counterpart of the KAM theory, an advantage of the C-W-B method is more flexible than the KAM scheme to deal with resonant cases where the second Melnikov’s conditions are violated seriously, although a (local) normal form can not be obtained by using the C-W-B method. Therefore, it is expected to find out a method by which the resonant cases can be easily dealt and by which the normal form can be obtained at the same time. In that direction, the first result is due to Bourgain[Bo5], to my knowledge. In order to introduce Bourgain’s idea in [Bo5], let’s recall the basic idea of lower dimensional KAM tori.

\(^1\)This potential \( V \) contains no parameter.
See [E], [K1] and [P1] for details. Consider a finitely dimensional Hamiltonian (1.2) with $\iota < \infty$. Decompose the perturbation $R$ in (1.2) into as follows

$$
R = R^\iota(x) + (R^\iota(x), y) + \langle R^\iota(x), z \rangle + \langle R^{\iota z}(x), \bar{z} \rangle + O(||y||^2 + ||y||z| + |z|^3),
$$

(1.5)

where $(x, y, z, \bar{z})$ is in the some subset of $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}^l \times \mathbb{C}^l$, and where $R^\iota, R^\iota : \mathbb{T}^n \to \mathbb{C}^l$ and $R^{\iota z}, R^{\iota z} : \mathbb{T}^n \to \mathbb{C}^l$ are complex $\iota \times \iota$ matrices. In the classic KAM theory, one finds a series of symplectic transformations to kill the perturbing terms (1.5), (1.6) and (1.7) such that the transformed Hamiltonian is of the form

$$
\tilde{H} = (\omega, y) + \sum_{j=1}^L \Omega_j \bar{z}_j + O(||y||^2 + ||y||z| + |z|^3).
$$

(1.9)

One sees easily that $\mathbb{T}^n \times \{y\}_{y=0} \times \{z, \bar{z}\}_{z=\bar{z}=0}$ is an invariant torus of (1.9). Going back to (1.2), one can get an invariant torus of (1.2). The second Melnikov’s conditions (1.4) comes out while killing the perturbing term (1.7). Bourgain[Bo5] modifies essentially the idea mentioned above. He eliminates the perturbing terms (1.5) and (1.6) and puts the perturbing term (1.7) into the “integrable” part of the Hamiltonian $H$. Consequently, he obtains a Hamiltonian of the form

$$
\tilde{H} = (\omega, y) + \sum_{j=1}^L \Omega_j \bar{z}_j + \langle R^{\iota z}(x), z \rangle + \langle R^{\iota z}(x), \bar{z} \rangle + \langle R^{\iota z}(x), \bar{z}, \bar{z} \rangle + O(||y||^2 + ||y||z| + |z|^3).
$$

(1.10)

$$
\tilde{H} = (\omega, y) + \sum_{j=1}^L \Omega_j \bar{z}_j + \langle R^{\iota z}(x), z \rangle + \langle R^{\iota z}(x), \bar{z} \rangle + \langle R^{\iota z}(x), \bar{z}, \bar{z} \rangle + O(||y||^2 + ||y||z| + |z|^3).
$$

(1.11)

This Hamiltonian $\tilde{H}$ can be regarded as a counterpart of the local normal form of the classic KAM theory. It is clear that $\mathbb{T}^n \times \{y\}_{y=0} \times \{z, \bar{z}\}_{z=\bar{z}=0}$ is still an invariant torus of $\tilde{H}$. Because of without killing (1.7), the second Melnikov’s conditions (1.4) are not required. Therefore, using the Bourgain’s idea, one can deal resonant cases where the second Melnikov’s conditions (1.4) are violated seriously; at the same time, one can get a normal form (1.10+11). However, the “integrable” term (1.10) is not really integrable, since it contains the angle-variable $x$ in $R^{\iota z}(x)$’s. Because of this fact, while eliminating (1.6), one will have to solve a homological equation with variable coefficients

$$
\sqrt{-1}(\omega, \partial_x)F^z + (\Omega^0 + R^{\iota z}(x) + \cdots)F^z = R^z
$$

(1.12)

where $F^z$ is unknown function. By contrast, in the classical KAM theorem, the homological equation is of constant coefficients. To solve (1.12) one needs to investigate the inverse of a “big” matrix $A$ of the form:

$$
A = \text{diag} \left((k, \omega) + \Omega^0_j : k \in \mathbb{Z}^n, j = 1, \ldots, \iota \right) + \left(\mathbb{R}^{\iota z}(k - l) : k, l \in \mathbb{Z}^n\right).
$$

(1.13)

As in the C-W-B method, Bourgain[Bo5] uses and develops the technique by Fröhlich and Spencer[F-S] to investigate the inverse of $A$. Even if one gets the
inverse $A^{-1}$, one has to show that $A^{-1}$ is off-diagonal decay, in order to control the exponent-weight norm of $A^{-1}$. This technique depends heavily on the algebraic structure of $(k, \omega) + \Omega_j$. For nonlinear wave equation of spatial $d$-dimension,

$$\Omega_j^0 \approx \sqrt{|j|^2} = \sqrt{j_1^2 + \ldots + j_d^2}, \quad j = (j_1, \ldots, j_d).$$

This makes the algebraic structure of $(k, \omega) + \Omega_j$ very intricate, if $d > 1$.

In the present paper, we consider Hamiltonian (1.2) with $\iota = \infty$. Following Bourgain’s idea in [Bo5], we put (1.7) into the “integrable” part (1.10) and find a symplectic transformation to eliminate (1.6) and (1.5). This leads to solve the homological equation (1.12) of variable coefficients. Thus, we also have to investigate the inverse of the “big” matrix $A$ as in [Bo5]. However, we do not use the technique by Fröhlich and Spencer [F-S]. We impose a symmetry condition on $R$ as follows

$$R(-x, y, z, \bar{z}) = R(x, y, \bar{z}, z) \quad (1.14)$$

where $(x, y)$ is in some subset of $\mathbb{T}^n \times \mathbb{R}^n$ and $z, \bar{z}$ are in some Hilbert space. With the symmetry condition (1.14) we can prove that the spectra of $A$ are “twisted” with respect to $\omega$, thus that the spectra are different zero if digging out some $\omega$’s of small Lebesgue measure. This implies the existence of $A^{-1}$. In addition, we do not need to know that $A^{-1}$ is off-diagonal decay, by choosing Sobolev space as our working space instead of an exponential weight space. Generally it is very hard to prove that $A^{-1}$ is off-diagonal decay, which involves the intricate algebraic structure of $(k, \omega) + \Omega_j$. It should be noted that the symmetry condition (1.14) is fulfilled by many partial differential equations, such as nonlinear wave equation (1.1), nonlinear Schrödinger equation, nonlinear beam equation and others.

The rest of the present paper is organized as follows. In §2., a KAM theorem to deal Hamiltonian (1.2) with $\iota = \infty$ is given out. This theorem does not require the second Melnikov’s conditions. In §3., the KAM theorem is used to obtain invariant tori and quasi-periodic solution for nonlinear wave equation and nonlinear Schrödinger equation. The §4-7 are devoted to the proof of the KAM theorem. The fifth section is the essential part of the present paper.

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2. A KAM theorem.

2.1. Some notations. Denote by $(\ell_2, |||\cdot|||)$ the usual space of the square summable sequences, and by $(L^2, |||\cdot|||)$ the space of the square integrable functions. By $|\cdot|$ the Euclidian norm. Let $p \geq d/2$. For a sequence $z = (z_j \in \mathbb{C} : j \in \mathbb{Z}^d)$, we define its norm as follows:

$$||z||_p^2 = \sum_{j \in \mathbb{Z}^d} |j|^{2p} |z_j|^2. \quad (2.1)$$

Let $\ell^p$ be the set of all sequences satisfying (2.1). It is easy to see that $\ell^p$ is a Hilbert space with an inner product corresponding to (2.1). Introduce the phase space:

$$\mathcal{P} := (\mathbb{C}^n / \pi \mathbb{Z}^n) \times \mathbb{C}^n \times \ell^p \times \ell^p, \quad (2.2)$$

where $n$ is a given positive integer. We endow $\mathcal{P}$ with a symplectic structure

$$dx \wedge dy + \sqrt{-1} dz \wedge d\bar{z} = dx \wedge dy + \sqrt{-1} \sum_{j \in \mathbb{Z}^d} dz_j \wedge d\bar{z}_j, \quad (x, y, z, \bar{z}) \in \mathcal{P}. \quad (2.3)$$
Let
\[ T^n_0 = (\mathbb{R}^n / 2\pi \mathbb{Z}^n) \times \{ y = 0 \} \times \{ z = 0 \} \times \{ \bar{z} = 0 \} \subset \mathcal{P}. \] (2.4)
Then \( T^n_0 \) is a torus in \( \mathcal{P} \). Introduce a complex neighborhoods of \( T^n_0 \) in \( \mathcal{P} \):
\[ D(s, r) := \{ (x, y, u) \in \mathcal{P} : |\text{Im} x| < s, |y| < r^2, ||z||_p < r, ||\bar{z}||_p < r \} \] (2.5)
where \( r, s > 0 \) are constants.

For \( \tilde{r} > 0 \) we define the weighted phase norms
\[ j|W|_p = |X| + \frac{1}{\tilde{r}^2}|Y| + \frac{1}{r}||Z||_p + \frac{1}{r}||\bar{Z}||_p \] (2.6)
for \( W = (X, Y, Z, \bar{Z}) \in \mathcal{P} \). Let \( \mathcal{O} \subset \mathbb{R}^n \) be compact and of positive Lebesgue measure. For a map \( W : D(s, r) \times \mathcal{O} \rightarrow \mathcal{P} \), set
\[ j|W|_{p, D(s, r) \times \mathcal{O}} := \sup_{(x, \xi) \in D(s, r) \times \mathcal{O}} j|W(x, \xi)|_p \] (2.7)
and
\[ j|W|_{p, D(s, r) \times \mathcal{O}} := \max_{1 \leq j \leq n} \sup_{D(s, r) \times \mathcal{O}} j|\partial_{\xi_j} W(x, \xi)|_p, \quad \xi = (\xi_1, ..., \xi_n). \] (2.8)

Denote by \( \mathcal{L}(\ell^p, \ell^p) \) the set of all bounded linear operators from \( \ell^p \) to \( \ell^p \) and by \( ||| \cdot |||_p \) the operator norm. For any subset \( S \subset \mathbb{Z}^d \) with cardinality \( \text{card} S < \infty \) and a finitely dimensional vector \( u = (u_j \in \mathbb{C} : j \in S) \) and a matrix \( U = (U_{ij} \in \mathbb{C} : i, j \in S) \) of finite order, let
\[ \tilde{u} = (\tilde{u}_j : j \in \mathbb{Z}^d), \quad \text{here } \tilde{u}_j = \begin{cases} u_j, & j \in S \\ 0, & j \in \mathbb{Z}^d \setminus S \end{cases}, \]
and
\[ \tilde{U} = (\tilde{U}_{ij} : i, j \in \mathbb{Z}^d), \quad \text{here } \tilde{U}_{ij} = \begin{cases} U_{ij}, & i, j \in S \\ 0, & i \text{ or } j \in \mathbb{Z}^d \setminus S \end{cases}. \]
Define \( \|u\|_p := \|\tilde{u}\|_p \) and \( \|||u|||_p := \|||\tilde{U}|||_p \). Similarly define \( \|u\| := \|\tilde{u}\| \) and \( \|\|u\|| := \|\|\tilde{U}\|| \). The following lemma states the relation between \( \|\| \cdot \|\|_p \) and \( \|\|| \cdot \|| \| \).

**Lemma 2.1.** (i). Let \( Y = (Y_{ij} : |i|, |j| \leq K) \) be a \( K \times K \) matrix with some \( K > 0 \). Then \( \|\|Y\||\| \leq K^2 \|\|Y\|| \) and \( \|\|Y\|| \leq \|\|Y\||\| \). (ii). Let \( Y = (Y_j : |j| \leq K) \) be a \( K\)-dimensional vector. Then \( \|\|Y\||\| \leq K^\|Y\| \) and \( \|\|Y\|| \leq \|\|Y\||\| \).

**Proof.** The proofs are trivial. We omit it.

In the whole of this paper, by \( C \) or \( c \) a universal constant, whose size may be different in different place. If \( f \leq Cg \), we write this inequality as \( f \ll g \) when we do not care the size of the constant \( C \). Similarly, if \( f \gg g \) we write \( f \gg g \).

**2.2. The statement of the KAM theorem.** For two vectors \( b, c \in \mathbb{C}^k \) or \( \mathbb{R}^k \), we write \( (b, c) = \sum_{j=1}^k b_j c_j \) if \( k < \infty \). If \( k = \infty \), we write \( (b, c) = \sum_{j=1}^\infty b_j c_j \). Consider an infinitely dimensional Hamiltonian in the parameter dependent norm form
\[ N_0 = (\omega^0(\xi), y) + \sum_{j \in \mathbb{Z}^d} \Omega(\xi) z_j \bar{z}_j, \quad (x, y, z, \bar{z}) \in \mathcal{P}. \] (2.9)
The tangent frequencies $\omega^0 = (\omega_1^0, \cdots, \omega_n^0)$ and the normal frequencies $\Omega^0_j$’s ($j \in \mathbb{Z}^d$) depend on $n$ parameters $\xi \in \mathcal{O}_0 \subset \mathbb{R}^n$ where $\mathcal{O}_0$ a given compact set of positive Lebesgue measure. Let

$$\Omega^0_j(\xi) = \text{diag}(\Omega_j^0(\xi) : j \in \mathbb{Z}^d).$$

Let

$$N_0 = (\omega^0(\xi), y) + (\Omega^0(\xi)z, \bar{z}),$$

where $z = (z_j : j \in \mathbb{Z}^d)$. The Hamiltonian equation of motion of $N_0$ are

$$\dot{x} = \omega^0(\xi), \quad \dot{y} = 0, \quad \dot{z} = \sqrt{-1}\Omega^0(\xi)z, \quad \dot{\bar{z}} = -\sqrt{-1}\Omega^0(\xi)\bar{z}.$$  

Hence, for each $\xi \in \mathcal{O}_0$, there is an invariant $n$-dimensional torus

$$\mathcal{T}_0^n = \mathbb{T}^n \times \{ y = 0 \} \times \{ z = \bar{z} = 0 \}$$

with frequencies $\omega^0(\xi)$. The aim is to prove the persistence of the torus $\mathcal{T}_0^n$, for “most” (in the sense of Lebesgue measure) parameter vector $\xi \in \mathcal{O}_0$, under small perturbation $R$ of the Hamiltonian $N_0$. To this end the following assumptions are required.

**Assumption A:** (Multiplicity.) Assume that there are constants $c_1, c_2 > 0$ such that for all $\xi \in \mathcal{O}_0$,

$$|\Omega_j^0| \leq c_1 |j|^{c_2}$$

where we denote by $(\cdot)^j$ the multiplicity of $(\cdot)$.

**Assumption B:** (Non-degeneracy.) There are two absolute constant $c_3, c_4 > 0$ such that

$$\sup_{\xi \in \mathcal{O}_0} |\text{det} \partial_\xi \omega^0(\xi)| \geq c_3, \quad \sup_{\xi \in \mathcal{O}_0} |\partial_\xi^2 \omega^0(\xi)| \leq c_4, \quad j = 0, 1.$$  

**Assumption C:** (Analyticity of parameters.) Assume that both $\omega^0(\xi)$ and $\Omega^0(\xi)$ are analytic in each entry $\xi_j$ of the variable vector $\xi = (\xi_1, \ldots, \xi_l, \ldots, \xi_n) \in \mathcal{O}_0$ and assume that both $\omega^0(\xi)$ and $\Omega^0(\xi)$ are real for real argument $\xi$.

**Assumption D:** (Bounded conditions of Normal frequencies.) Assume that there exists constants $c_5, c_6, c_7 > 0$ and constant $\kappa > 0$ such that

$$\inf_{\xi \in \mathcal{O}_0} \Omega_j^0 \geq c_5 |j|^\kappa + c_6,$$

$$\sup_{\xi \in \mathcal{O}_0} |\partial_\xi \Omega_j^0| \leq c_7 \ll 1, \quad l = 1, \ldots, n,$$

uniformly for all $j$.

**Assumption E:** (Regularity.) Let $s_0, r_0$ given. Assume the perturbation term $R^0(x, y, z, \bar{z}, \xi)$ which is defined on the domain $D(s_0, r_0) \times \mathcal{O}_0$ is analytic in the space coordinates and also analytic in each entry $\xi_l$ ($l = 1, \ldots, n$) of the parameter vector $\xi \in \mathcal{O}_0$, and is real for real argument, as well as, for each $\xi \in \mathcal{O}_0$ its Hamiltonian vector field $X_{R^0} := (R_{y}^0, -R_z^0, \sqrt{-1}\partial_\xi R^0, -\sqrt{-1}\partial_{\bar{z}} R^0)^T$ defines a analytic map

$$X_{R^0} : D(s_0, r_0) \subset \mathcal{P} \rightarrow \mathcal{P}$$

where $T$ is transpose. Also assume that $X_{R^0}$ is analytic in each entry of $\xi \in \mathcal{O}_0$.

**Assumption F:** (Symmetry.) For any $(x, y, z, \bar{z}, \xi) \in D(s_0, r_0) \times \mathcal{O}_0$,

$$R^0(-x, y, z, \bar{z}, \xi) = R^0(x, y, \bar{z}, z, \xi).$$  

(2.18)
Theorem 2.1. Suppose $H = N_0 + R^0$ satisfies assumptions $A-F$ and smallness assumption:

$$r_0 |X_{R^0}|_{p,D(s_0,r_0) \times O_0} < \epsilon, \quad r_0 |X_{R^0}|_{L_p,D(s_0,r_0) \times O_0} < \epsilon^{1/3}. \quad (2.19)$$

Then, for $0 < \epsilon \ll 1$, there is a subset $O_\epsilon \subset O_0$ with

$$\text{Meas } O_\epsilon \geq (\text{Meas } O_0)(1 - O(\epsilon)),$$

and there are a family of torus embedding $\Phi: \mathbb{T}^n \times O_\epsilon \rightarrow \mathcal{P}$ and a map $\omega_* : O_\epsilon \rightarrow \mathbb{R}^n$ where $\Phi(\cdot, \xi)$ and $\omega_*(\xi)$ is analytic in each entry $\xi_j$ of the parameter vector $\xi = (\xi_1, \ldots, \xi_n)$ for other arguments fixed, such that for each $\xi \in O_\epsilon$ the map $\Phi$ restricted to $\mathbb{T}^n \times \{\xi\}$ is a analytic embedding of a rational torus with frequencies $\omega_*(\xi)$ for the Hamiltonian $H$ at $\xi$.

Each embedding is real analytic on $\mathbb{T}^n \times \{\xi\}$, and

$$r_0 |\Phi - \Phi|_{0,\mathbb{T}^n \times O_\epsilon} \leq \epsilon c, \quad r_0 |\Phi - \Phi|_{L_p,\mathbb{T}^n \times O_\epsilon} \leq \epsilon^{1/3},$$

$$|\omega_* - \omega|_{O_\epsilon} \leq \epsilon c, \quad |\omega_* - \omega|_{O_0} \leq \epsilon^{1/3},$$

where $\Phi_0$ is the trivial embedding $\mathbb{T}^n \times O_0 \rightarrow \mathbb{T}^n \times \{y = 0\} \times \{z = \bar{z} = 0\}$, and $c > 0$ is a constant depending on $n$.

3. Application to nonlinear partial differential equations.

3.1 Application to nonlinear wave equations. Consider nonlinear wave equation

$$u_{tt} - \Delta u + M_* u + \epsilon u^3 = 0, \quad \theta \in \mathbb{T}^d, \quad d \geq 1 \quad (3.1)$$

where $u = u(t, \theta)$ and $\Delta = \sum_{j=1}^d \partial_{\theta_j}^2$ and $M_*$ is a real Fourier multiplier

$$M_* \cos(j, \theta) = \sigma_j \cos(j, \theta), \quad M_* \sin(j, \theta) = \sigma_j \sin(j, \theta), \quad \sigma_j \in \mathbb{R}, \quad j \in \mathbb{Z}^d. \quad (3.2)$$

Pick a set $\mathbb{E} = \{e_1, \ldots, e_n\} \subset \mathbb{Z}^d$. Let $\mathbb{Z}^d = \mathbb{Z}^d \setminus \{e_1, \ldots, e_n\}$. Following Bourgain[Bo3], we assume

$$\left\{ \begin{array}{c} \sigma_{e_j} = \sigma_j, \quad (j = 1, \ldots, n) \\ \sigma_j = 0, \quad j \in \mathbb{Z}^d \setminus \mathbb{E} \end{array} \right. \quad (3.3)$$

Let $\lambda_j^\pm (j \in \mathbb{Z}^d)$ be the eigenvalues of the self-adjoint operator $-\Delta + M_*$ subject to periodic b. c. $\theta \in \mathbb{T}^d$, and let $\phi_j^\pm(\theta)$ be the normalized eigenfunctions corresponding to $\lambda_j^\pm$. Then

$$\lambda_j^\pm = |j|^2 + \sigma_j = \sum_{i=1}^d j_i^2 + \sigma_j, \quad j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \quad (3.4)$$

and

$$\phi_j^+ = \frac{\sqrt{2}}{(\sqrt{2\pi})^d} \cos(j, \theta), \quad \phi_j^- = \frac{\sqrt{2}}{(\sqrt{2\pi})^d} \sin(j, \theta). \quad (3.5)$$

Note that $\{\phi_j^\pm : j \in \mathbb{Z}^d\}$ is complete orthogonal system in $L^2(\mathbb{T}^d)$. Let

$$u(t, \theta) = \sum_{j \in \mathbb{Z}^d} q_j(t)\phi_j^\pm(\theta). \quad (3.6)$$
Inserting (3.6) into (3.1), then
\[ \dot{q}_j^\pm + \lambda_j q_j^\pm + \epsilon (u^3, \phi_j^\pm)_{L^2} = 0, \quad j \in \mathbb{Z}^d. \]  

(3.7)

Let \( q_j^\pm = \sqrt{\lambda_j} \tilde{q}_j^\pm \). Then
\[
\begin{aligned}
\dot{q}_j^\pm &= \sqrt{\lambda_j} \tilde{q}_j^\pm \\
\dot{\tilde{q}}_j^\pm &= \frac{1}{\sqrt{\lambda_j}} \tilde{q}_j^\pm = -\sqrt{\lambda_j} q_j^\pm - \frac{\epsilon}{\sqrt{|\lambda_j|}} (u^3, \phi_j^\pm).
\end{aligned}
\]

(3.8)

For simplifying notation, we do not distinguish + sign and − sign in (3.8). For example, we write \( q_j^\pm \) as \( q_j \). However, we should keep in mind that \( q_j \) runs over the set \( \{ q_j^+, q_j^- \} \). The system (3.8) is a Hamiltonian system with its Hamiltonian function
\[ H = H(q, \tilde{q}) = \frac{1}{2} \sum_{j \in \mathbb{Z}^d} \sqrt{\lambda_j} (q_j^2 + q_j^2) + \epsilon G(q), \]

(3.9)

where
\[ G(q) = \sum_{i,j,k,l \in \mathbb{Z}^d} G_{ijkl} q_i q_j q_k q_l \]

(3.10)

and
\[ G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_{\mathbb{T}^d} \phi_i \phi_j \phi_k \phi_l dx. \]

(3.11)

Let
\[ \omega_j^0 = \sqrt{\lambda_j} = \sqrt{|e_j|^2 + \sigma_j}, \quad (j = 1, ..., n), \]

(3.12)

and
\[ \Omega_j^0 = \sqrt{\lambda_j} = \sqrt{|j|^2}, \quad j \in \mathbb{Z}^d. \]

(3.13)

Introduce a symplectic coordinate change
\[
\begin{aligned}
p_{e_j} &= \sqrt{2(1 + y_j)} \sin x_j, \quad q_{e_j} = \sqrt{2(1 + y_j)} \cos x_j, \quad j = 1, ..., n, \\
q_j &= \frac{z_j + \tilde{z}_j}{\sqrt{2}}, \quad p_j = \frac{z_j - \tilde{z}_j}{\sqrt{-1 \sqrt{2}}}, \quad j \in \mathbb{Z}^d.
\end{aligned}
\]

(3.14)

To shorten notation, let
\[ \tilde{y}_i = \sqrt{2(1 + y_i)}. \]

Then (3.9) is changed into
\[ H = H(x, y, z, \tilde{z}) = (\omega, y) + \sum_{j \in \mathbb{Z}^d} \Omega_j^0 z_j \tilde{z}_j + \epsilon R^0 \]

(3.15)
Proof. The proof is very elementary. See \[p.294-295,P2\].

It follows that in a suitable domain of \((x,y,z,\bar{z})\),

\[
R^0(-x,y,z,\bar{z}) = R^0(x,y,\bar{z},z).
\]

This implies the symmetry assumption \(F\) is fulfilled. By (3.5) and (3.11), it is not difficult to verify that

\[
G_{ijkl} = 0, \quad \text{unless } i \pm j \pm k \pm l = 0,
\]

for some combination of plus and minus signs. For \(w,z \in \ell^p\), the convolution \(z \ast w\) is defined by \((z \ast w)_j = \sum_{k \in \mathbb{Z}^d} w_{j-k} z_k\).

**Lemma 3.1.** If \(p > d/2\), then \(\|w \ast z\|_p \leq c\|w\|_p \|z\|_p\) for \(w, z \in \ell^p\) with a constant \(c\) depending only on \(p\).

**Proof.** The proof is very elementary. See [p.294-295,P2].

Let \(s_0 = r_0 = 1\), \(\omega^0 = (\omega_1, \ldots, \omega_n)\) and \(\Omega^0 = (\Omega_j^0 : j \in \mathbb{Z}^d)\). And let parameter \(\sigma = (\sigma_1, \ldots, \sigma_n)\) runs over \(\mathcal{O}_0 := [1,2]^d\). Note that \(R^0\) is independent of \(\sigma\). By (3.18) and Lemma 3.1, we see that \(R^0\) is well defined on \(D(s_0, r_0)\) and the Assumption E holds true; moreover,

\[
r_0 |X_{\epsilon R^0}|_{p,D(s_0,r_0)\times \mathcal{O}} < \epsilon, \quad r_0 |X_{\epsilon R^0}|_{p,D(s_0,r_0)\times \mathcal{O}} = 0.
\]

This implies that (2.19) is fulfilled. It follows from (3.12) that the Assumption B (that is, (2.15) holds true. By (3.13), we see that the assumptions A, C, D are fulfilled. In particular, (2.16) holds true with \(\kappa = 1\). Using Theorem 2.1, we have the following theorem.

**Theorem 3.1.** There is a subset \(\mathcal{O}_0 \subset \mathcal{O}_0\) with Meas \(\mathcal{O} \geq (1 - C\epsilon)\)Meas \(\mathcal{O}\) such that for any \(\sigma \in \mathcal{O}_0\), Eq.(3.1) with small \(\epsilon\) has a rotational invariant torus of frequency vector \(\omega^0 = \omega^0(\sigma)\). The motions on the torus are quasi-periodic with frequency \(\omega^0\).

### 3.2 Application to nonlinear Schrödinger equations

Consider nonlinear Schrödinger equation

\[
\sqrt{-1}u_t - \triangle u + M_\sigma u + cu |u|^2 = 0, \quad \theta \in \mathbb{T}^d, \quad d \geq 1,
\]

where \(M_\sigma\) is a Fourier multiplier

\[
M_\sigma e^{\sqrt{-1}(j,\theta)} = \sigma_j e^{\sqrt{-1}(j,\theta)}, \quad \sigma_j \in \mathbb{R}, \quad j \in \mathbb{Z}^d.
\]
Theorem 3.2. Assume $\sigma_j$'s satisfy (3.3). There is a subset $\mathcal{O}_0 \subset \mathcal{O}$ with

$$\text{Measure } \mathcal{O} \geq (1 - C\epsilon)\text{Measure } \mathcal{O}$$

such that for any $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{O}_0$, the nonlinear Schrödinger equation with small $\epsilon$ has a rotational invariant torus of frequency vector $\omega^0 = \omega^0(\sigma)$ where

$$\omega_j^0 = |e_j|^2 + \sigma_j, \ j = 1, \ldots, n.$$ 

The motions on the torus are quasi-periodic with frequency $\omega^0$.

Proof. The proof is similar to that of Theorem 3.1. Note that in this case, (2.16) is fulfilled for $\kappa = 2$. We omit the details.

4. The linearized equation.

4.1. Unperturbed linear system. Recall $\Omega^0_j$'s satisfy the Assumptions A, C, D. Assume that there is a new tangent frequency vector\(^2\) $\omega$ satisfying the Assumptions B, C. Let $\mathcal{O} \subset \mathcal{O}_0$ be an open set. In this section, we pick two large constants $K_-$ and $\hat{K}$. And let $s = 1/K_-$ and $s' = 4/\hat{K}$. (In the $m$-th KAM step, we will choose $K_0 = K_{m-1} \approx (2^{m-1})^{s'/2}$ and $K = K_m \approx 2^{m^2/2}$.) Let

$$D(s) = \{x \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |3x| < s\}, \ s = 1/K_-.$$ 

Let $X_N$ be a linear Hamiltonian system with Hamiltonian function:

$$\tilde{N} = (\omega(\xi), y) + \sum_{j \in \mathbb{Z}^d} \Omega_j^0(\xi)z_j\tilde{z}_j + \frac{1}{2} \sum_{i,j \in \mathbb{Z}^d} B_{ij}(x;\xi)z_i\tilde{z}_j + \frac{1}{2} \sum_{i,j \in \mathbb{Z}^d} B_{ij}^{xz}(x;\xi)z_i\tilde{z}_j + \frac{1}{2} \sum_{i,j \in \mathbb{Z}^d} B_{ij}^{zx}(x;\xi)\tilde{z}_i z_j \quad (4.1)$$

$$:= (\omega, y) + (\Omega^0 z, \tilde{z}) + \frac{1}{2} (B_{zx} z, \tilde{z}) + \frac{1}{2} (B_{zx}^z z, \tilde{z}) + \frac{1}{2} (B_{zx}^{\tilde{z}} \tilde{z}, z)$$

where $B_{ij}(x;\xi)$'s are analytic in $x \in D(s)$ for any fixed $\xi \in \mathcal{O}$, and all of $\omega(\xi), \Omega_j^0(\xi)$ and $B_{ij}(x;\xi)$ (for fixed $x \in D(s)$) are analytic in each entry $\xi_l (l = 1, \ldots, n)$ of $\xi \in \mathcal{O}$. Assume $B_{ij}$'s satisfy the following conditions:

1. Symmetry.

$$B_{ij}^{zx}(x;\xi, \xi) = B_{ji}^{zx}(x,\xi), \ B_{ij}^{zx}(-x,\xi) = B_{ij}^{zx}(x,\xi) \quad (4.2)$$

$$B_{ij}^{zx}(x,\xi) = B_{ji}^{zx}(x,\xi), \ B_{ij}^{zx}(-x,\xi) = B_{ij}^{zx}(x,\xi) \quad (4.3)$$

for any $(x, \xi) \in D(s) \times \mathcal{O}$.

2. Finiteness of Fourier modes.

$$B_{ij}^{zx}(x;\xi) = \sum_{|k| \leq K} \widehat{B_{ij}^{zx}}(k) e^{i\langle k, x \rangle}, \ldots \quad (4.4)$$

\(^2\)Here $\omega = \omega(\xi)$ is not necessary to be $\omega^0$, due to the well-known phenomenon the shift of frequency in the KAM iteration.
where $K \gg 1$ is a constant independent of $x, \xi, i$ and $j$. (In the $m$-th KAM iteration, we will take $K \approx 2^{m^2/2}$.)

(3.) Boundedness.

$$\sup_{D(s) \times O} |||B(x; \xi)|||_p^* \leq \epsilon \ll 1,$$

(4.) Non-degenerateness. Moreover, we assume

$$c_1 \geq \left| \det \partial \omega(\xi) / \partial \xi \right| \geq c_2 > 0, \ \forall \xi \in O.$$  

(This assumption enables us to regard $\omega$ instead of $\xi$ as a parameter vector.)

2. split and estimate for small perturbation. We now consider a perturbation

$$H = \mathbb{N} + \hat{R}$$

where $\hat{R} = \hat{R}(x, y, u; \xi)$ is a Hamiltonian defined on $D(s, r)$ and depends on the parameter $\xi \in O$. Assume that the vector field $X_{\hat{R}} : D(s, r) \times O \to \mathcal{P}$ is real for real argument, analytic in $(x, y, z, \bar{z}) \in D(s, r)$ and in each entry of $\xi \in O$. We assume that there are quantities $^3 \varepsilon = \varepsilon(r, s, O)$ and $\varepsilon^\mathcal{L} = \varepsilon^\mathcal{L}(r, s, O)$ which depend on $r, s, O$ such that

$$r |X_{\hat{R}}|_{p, D(s, r) \times O} \ll \varepsilon, \quad r |X_{\hat{R}}|^\mathcal{L} \ll \varepsilon^\mathcal{L}, \quad \varepsilon \ll \varepsilon^\mathcal{L} \ll 1.$$  

Let

$$R = \sum_{2|m|+|q_1+q_2| \leq 2} \sum_{k \in \mathbb{Z}^d, m \in \mathbb{Z}^d, q_1, q_2 \in \mathbb{Z}^\infty} \hat{R}_{mq_1q_2}(k) e^{\sqrt{-1}(k,x)y^m z^{q_1} \bar{z}^{q_2}},$$

with the Taylor-Fourier coefficients $\hat{R}_{mq_1q_2}(k)$ of $\hat{R}$ depending on $\xi \in O$, and being analytic in each entry $\xi_j$ of $\xi$. We see that $R$ is a partial Taylor-Fourier expansion of $\hat{R}$. We will approximate $\hat{R}$ by $R$. Now we give some estimates of $R$.

Lemma 4.1.

$$r |X_{\hat{R}}|_{p, D(s, r) \times O} \ll r |X_{\hat{R}}|^* \ll \varepsilon^*,$$

$$\eta r |X_{\hat{R}} - X_{\hat{R}}|^*_{p, D(s, d\eta r) \times O} \ll \eta \varepsilon^*,$$

for any $0 < \eta \ll 1$, where $^* = \text{the blank or } \mathcal{L}$, for example, $\varepsilon^* = \varepsilon$ or $\varepsilon^\mathcal{L}$.

Proof. The proof is similar to that of formula (7) of [P1,129].

With this lemma, we decompose $R = R^0 + R^1 + R^2$, where

$$R^0 = R^x + (R^y, y),$$

$$R^1 = \langle R^z, z \rangle + \langle R^\bar{z}, \bar{z} \rangle,$$

$$R^2 = \frac{1}{2} \langle \langle R^{zz} z, z \rangle + \langle R^{zz} \bar{z}, \bar{z} \rangle + \langle R^{\bar{z}z} z, \bar{z} \rangle \rangle.$$

$^3$We will take $\varepsilon = \varepsilon^{(4/3)m}$ and $\varepsilon^\mathcal{L} = \varepsilon^{1/3}$ in the $m$-th KAM iteration step.
with $R^x, R^y : D(s) \times \mathcal{O} \to \mathbb{C}^n; R^z, R^{\bar{z}} : D(s) \times \mathcal{O} \to \ell^p; R^{zz}, R^{\bar{z}z},$ etc. $D(s) \times \mathcal{O} \to \mathcal{L}(\ell^p, \ell^p).$ For any vector or matrix

$$Y = \sum_{k,m \in \mathbb{Z}^d, q, \bar{q} \in \mathbb{Z}^n} \hat{Y}_{m,q,\bar{q}}(k)e^{\sqrt{-1}(k,x)}y^m z^q \bar{z}^\bar{q}$$

we introduce the cut-off operator $\Gamma_K$ as follows:

$$\Gamma_K Y = \sum_{k,m \in \mathbb{Z}^d, q, \bar{q} \in \mathbb{Z}^n |k| \leq K} \hat{Y}_{m,q,\bar{q}}(k)e^{\sqrt{-1}(k,x)}y^m z^q \bar{z}^\bar{q} \tag{4.11}$$

**Lemma 4.2.** Assume $(s - s')K \geq |\ln \eta|$. Let $R_K = R - \Gamma_K R$. We have

$$r|X_{(\pi, R)}| \leq r|X_R| \leq \varepsilon^*,$$  \hspace{1cm} (4.12)

$$r|X_{R_K}| \leq \eta \varepsilon^* \tag{4.13}$$

where $* = \text{the blank or } \mathcal{L}$.

**Proof.** The proof of (4.12) is obvious. Let us give the proof of (4.13). Write

$$R_K = R^x + (R^y, y) + (R^z, z) + (R^{\bar{z}}, \bar{z}) + (R^{zz}, z, \bar{z}) + (R^{\bar{z}z}, z, \bar{z}).$$

Note that the terms $R^x, R^y$, and so on, are analytic in $x \in D(s)$ for fixed $\xi \in \mathcal{O}$. And observe that

$$R^x = \sum_{|k| > K} R^x(k)e^{\sqrt{-1}(k,x)}, \ldots .$$

Then by Cauchy’s formula, we have $|R^x(k)| \leq e^{-s|k|} \sup_{D(s)} |R^x|$, and so on. Thus,

$$r|X_{R_K}| \leq r|X_R| \sum_{|k| > K} e^{-s|k|} \leq \eta \varepsilon^*.$$

Now we can write

$$\hat{R} = \Gamma_K R + R_K + (\hat{R} - R). \tag{4.14}$$

Let $\mathcal{R} = \Gamma_K R$. Hence we can write

$$\mathcal{R} = R^x + (R^y, y) + (R^z, z) + (R^{\bar{z}}, \bar{z}) + \frac{1}{2}((R^{zz}, z, \bar{z}) + (R^{\bar{z}z}, z, \bar{z})). \tag{4.15}$$

Noting (4.8) and (4.11) we have

$$\mathcal{R}^* = \sum_{|k| \leq K} \hat{R}^*(k)e^{\sqrt{-1}(k,x)}, \tag{4.16}$$

where $* = x, y, z, \bar{z}, zz, z\bar{z}, \bar{z}\bar{z}.$
Lemma 4.3. Under the smallness assumption (4.7) on $\dot{R}$, the following estimates hold true:

\[
\|\partial_z R^z\|_{D(s)\times \mathcal{O}} \leq Kr^2\varepsilon, \quad \|\partial_z R^z\|_{D(s)\times \mathcal{O}} \leq Kr^2\varepsilon \tag{4.17}
\]

\[
\|R^y\|_{D(s)\times \mathcal{O}} \leq K\varepsilon, \quad \|R^y\|_{D(s)\times \mathcal{O}} \leq K\varepsilon \tag{4.18}
\]

\[
\|R^u\|_{p,D(s)\times \mathcal{O}} \leq Kr\varepsilon, \quad \|R^u\|_{p,D(s)\times \mathcal{O}} \leq Kr\varepsilon, \quad u \in \{z, \bar{z}\} \tag{4.19}
\]

\[
\|\partial_z R^u\|_{p,D(s)\times \mathcal{O}} \leq K\varepsilon, \quad \|\partial_z R^v\|_{p,D(s)\times \mathcal{O}} \leq K\varepsilon, \quad u, v \in \{z, \bar{z}\} \tag{4.20}
\]

Proof. Consider $R^{zz}$. Observe that $R^{zz} = \partial_z \partial_z R|_{z=\bar{z}=0}$. By the generalized Cauchy inequality (See Lemma A.3 in [P1]),

\[
\| R^{zz} \|_{p,D(s)\times \mathcal{O}} \leq K\varepsilon \| R^{zz} \|_{p,D(s)\times \mathcal{O}} \leq K\varepsilon \| R^{zz} \|_{p,D(s)\times \mathcal{O}} \leq K\varepsilon \tag{4.21}
\]

The remaining proof is simple. We omit the details. \qed

Lemma 4.4. Assume $\dot{R}$ satisfies the following symmetric condition:

\[
\dot{R}(-x, y, z, \bar{z}; \xi) = \dot{R}(x, y, z, \bar{z}; \xi), \quad \forall (x, y, z, \bar{z}; \xi) \in D(s, r) \times \mathcal{O}. \tag{4.22}
\]

Let $R^{zz}_{ij}, R^{zz}_{ij}, R^{zz}_{ij}$ be the elements of the matrices $R^{zz}, R^{zz}, R^{zz}$, respectively. Then we have

\[
R^{z}(x) = R^{z}(x), \quad R^{z}(x) = R^{z}(x) \tag{4.23}
\]

\[
R^{z}(x) = R^{z}(x), \quad R^{z}(x) = R^{z}(x) \tag{4.24}
\]

\[
R^{z}_{ij}(x) = R^{z}_{ij}(x), \quad R^{z}_{ij}(x) = R^{z}_{ij}(x) \tag{4.25-1}
\]

\[
R^{z}_{ij}(x) = R^{z}_{ij}(x), \quad R^{z}_{ij}(x) = R^{z}_{ij}(x) \tag{4.25-2}
\]

Proof. Noting that

\[
R^{z}_{ij}(x) = \partial_z \partial_z \dot{R}(x, y, z, \bar{z}; \xi)|_{y=0, z=\bar{z}=0},
\]

and $\partial_z \partial_z = \partial_{ij} \partial_{ij}$, we get the first equation of (4.25-1). Applying $\partial_z \partial_z$ to both sides of (4.23), we get

\[
\partial_z \partial_z \dot{R}(-x, y, z, \bar{z}; \xi)|_{y=0, z=\bar{z}=0} = \partial_z \partial_z \dot{R}(x, y, z, \bar{z}; \xi)|_{y=0, z=\bar{z}=0}
\]

\[
= \partial_z \partial_z \dot{R}(x, y, z, \bar{z}; \xi)|_{y=0, z=\bar{z}=0},
\]

that is, the second equation of (4.25-1) holds. The remaining proofs are similar to the previous one. \qed

Recall $\mathcal{R} = \Gamma K R$. Let

\[
B^{zz} = \mathcal{B}^{zz} + R^{zz}, \quad B^{zz} = \mathcal{B}^{zz} + R^{zz}, \quad B^{zz} = \mathcal{B}^{zz} + R^{zz} \tag{4.26}
\]

\[
\omega = \omega + \mathcal{R}^0(0) \tag{4.27}
\]
and

\[ N = (\omega, y) + (\Omega^0 z, \bar{z}) + \frac{1}{2} \langle B^{zz} z, z \rangle + \langle B^{zz} \bar{z}, \bar{z} \rangle + \frac{1}{2} \langle B^{\bar{z}z} \bar{z}, \bar{z} \rangle, \]  

(4.28)

where \( B^{zz} = B^{\bar{z}z} \). Then by (4.25), (4.2) and (4.3),

\[ B_{ij}^z(x, \xi) = B_{ij}^z(x, \xi), \quad B_{ij}^\bar{z}(x, \xi) = B_{ij}^\bar{z}(x, \xi) \]

(4.29)

for any \((x, \xi) \in D(s) \times \mathcal{O}\). Assume \( K\varepsilon < \epsilon \) and \( K\varepsilon \mathcal{L} < \epsilon \). By (4.5) and (4.20), we have

\[ |||B^{zz}|||^a_p, |||B^{\bar{z}z}|||^a_p, |||B^{\bar{z}\bar{z}}|||^a_p \ll \epsilon \]  

(4.30)

where \( * = \) the blank or \( \mathcal{L} \). By (4.18) and \( K\varepsilon < 1 \) as well as (4.6), we get that (4.6) still hold true for \( \omega \):

\[ \hat{c}_1 \geq \left| \det \frac{\partial \omega(\xi)}{\partial \xi} \right| \geq \hat{c}_2 > 0, \quad \forall \xi \in \mathcal{O}. \]  

(4.31)

In view of (4.27, 28, 14, 15),

\[ H = N + \mathcal{R}^x + (\mathcal{R}^y - \hat{\mathcal{R}}^y(0), y) + \langle \mathcal{R}^z, z \rangle + \langle \mathcal{R}^{\bar{z}}, \bar{z} \rangle \]

\[ + (R - \Gamma_R R) + (\hat{R} - R). \]  

(4.32)

4.3. Derivation of homological equations. The KAM theorem is proven by the usual Newton-type iteration procedure which involves an infinite sequence of symplectic coordinate changes. Each coordinate change is obtained as the time-1 map \( X_F^1 \mid_{t=1} \) of a Hamiltonian vector field \( X_F \). Its generating Hamiltonian \( F \) solves the linearized equation

\[ \{F, N\} = \mathcal{R}^x + (\mathcal{R}^y - \hat{\mathcal{R}}^y(0), y) + \langle \mathcal{R}^z, z \rangle + \langle \mathcal{R}^{\bar{z}}, \bar{z} \rangle \]  

(4.33)

where \( \{\cdot, \cdot\} \) is Poisson bracket with respect to the symplectic structure \( dx \wedge dy + \sqrt{-1} dz \wedge d\bar{z} \). Without loss of generality, we assume \( \hat{\mathcal{R}}^z(0) = 0 \) since it does not affect the dynamics. We are now in position to find a solution of the equation (4.33) and to give some estimates of the solution. To this end, we suppose that \( \hat{F} \) is of the same form as the right hand of (4.33), that is, \( F = F^0 + F^1 \), where

\[ \left\{ \begin{array}{l}
F^0 = F^x + (F^y, y), \\
F^1 = \langle F^z, z \rangle + \langle F^{\bar{z}}, \bar{z} \rangle,
\end{array} \right. \]  

(4.34)

with \( F^x, F^y, F^z, F^{\bar{z}} \) depending on \( x, \xi \), and \( \hat{F}^z(0) = 0, \hat{F}^y(0) = 0 \). Let \( \partial_\omega = (\omega, \partial_x) \) where \( (\cdot, \cdot) \) is the usual inner product in \( \mathbb{R}^n \). Using (4.34) and (4.28) we can compute the Poisson Bracket \( \{F, N\} \):

\[ \{F, N\} = -\frac{1}{2} \langle (F^y, \partial_x) B^{zz} z, z \rangle - \langle (F^y, \partial_x) B^{\bar{z}z} \bar{z}, \bar{z} \rangle - \frac{1}{2} \langle (F^y, \partial_x) B^{\bar{z}\bar{z}} \bar{z}, \bar{z} \rangle \]

\[ + \partial_x F^x + \langle \partial_x F^y, y \rangle + \langle \partial_x F^z, z \rangle + \langle \partial_x F^{\bar{z}}, \bar{z} \rangle \]

\[ + \sqrt{-1} (\langle B^{zz} F^z, z \rangle + \langle B^{\bar{z}z} F^{\bar{z}}, \bar{z} \rangle + \langle \Omega^0 F^z, z \rangle) \]

\[ - \sqrt{-1} (\langle B^{\bar{z}z} F^{\bar{z}}, \bar{z} \rangle + \langle B^{zz} F^z, z \rangle + \langle \Omega^0 F^z, \bar{z} \rangle). \]  

(4.35)
From (4.33,35) we derive the homological equations:

$$\partial_\omega F^x = \Gamma_K R^x(x, \xi),$$  \hspace{1cm} (4.36)

$$\partial_\omega F^y = \Gamma_K R^y(x, \xi) - \hat{R}^y(0),$$  \hspace{1cm} (4.37)

$$-\sqrt{-1} \partial_\omega F^z + \Gamma_K (\Omega^0 F^x - B^\omega F^x + B^\omega \bar{F}^x) = -\sqrt{-1} \Gamma_K R^z(x, \xi),$$  \hspace{1cm} (4.38-1)

$$\sqrt{-1} \partial_\omega F^\xi + \Gamma_K (\Omega^0 F^\xi - B^\omega F^\xi + B^\omega \bar{F}^\xi) = \sqrt{-1} \Gamma_K R^\xi(x, \xi),$$  \hspace{1cm} (4.38-2)

We should note that if $F$ solves the equations (4.36-38), then $F$ solves the following equation

$$\{F, N\} = \frac{1}{2}((F^y, \partial_x B^\omega z)z, \bar{z}) + ((F^y, \partial_x B^\omega \bar{z})z, \bar{z}) + \frac{1}{2}((F^y, \partial_x B^\omega \bar{z})z, \bar{z})$$

$$+ \sqrt{-1} (1 - \Gamma_K)(\bar{B}^\omega F^z, \bar{z}) + \sqrt{-1} (1 - \Gamma_K)(\bar{B}^\omega F^\xi, \bar{z}) - \sqrt{-1} (1 - \Gamma_K)(\bar{B}^\omega F^\xi, \bar{z})$$  \hspace{1cm} (4.39)

instead of (4.35).

**Lemma 4.5.** If $F$ in some sub-domain $D(s', r') \times O'$ of $D(s, r) \times O$ is the unique of the homological equations (4.31-34), then $F$ satisfies the skew-symmetric conditions:

$$F(-x, y, z, \bar{z}; \xi) = -F(x, y, z, \bar{z}; \xi), \ \forall (x, y, z, \bar{z}; \xi) \in D(s', r') \times O'. \hspace{1cm} (4.40)$$

**Proof.** By (4.29,30) and (4.38), we see that $-F^\xi(-x)$ solves (4.38-2). Thus,

$$-F^\xi(-x) = F^\xi(x)$$

similarly, we can show that $F^x(-x) = -F^x(x), \ F^y(-x) = -F^y$. Consequently $F(-x, y, z, \bar{z}; \xi) = -F(x, y, z, \bar{z}; \xi). \ \square$

### 4.4. Solutions of the homological equations.

**Proposition 1.** (Solution of (4.36).) There is a subset $O^1_+ \subset O$ with $\text{Meas}O^1_+ \geq (\text{Meas}O)(1 - K^{-1})$ such that for $\xi \in O^1_+$,

$$|(k, \omega(\xi))| \leq K^{n+1}, \ \text{for all} \ 0 \neq k \in \mathbb{Z}^n, |k| \leq K. \hspace{1cm} (4.41)$$

Then, on $D(s') \times O^1_+$, the equation (4.36) has a solution $F^x(x, \xi)$ which is analytic in $x \in D(s')$ for $\xi$ fixed and analytic in each $\xi_j, (j = 1, \ldots, n)$ for other variables fixed, and which is real for real argument, such that

$$|\partial_\xi F^x|_{D(s') \times O^1_+} \leq K^{C_1 r^2 \varepsilon}, \ |\partial_x F^x|_{D(s') \times O^1_+} \leq K^{C_1 r^2 \varepsilon}. \hspace{1cm} (4.42)$$

**Proof.** The existence of the set $O^1_+$ is well-known in KAM theory. We omit the proof of the existence. Recall $X_R : D(s, r) \subset \mathcal{P} \rightarrow \mathcal{P}$ is real analytic in $(x, y, z, \bar{z}) \in D(s, r)$ and each entry of $\xi \in O$. Expanding $\partial_x R^x$ into Fourier series

$$\partial_x R^x = \sum_{0 \neq k \in \mathbb{Z}^n, |k| \leq K} \partial_x R^x(k)e^{\sqrt{-1}k(x)}. \hspace{1cm} (4.43)$$
Since $\partial_x R^x$ is analytic in $x \in D(s)$, we get that the Fourier coefficients $\widehat{\partial_x R^x}(k)$'s decay exponentially in $k$, that is,

$$|\widehat{\partial_x R^x}(k)| < |\partial_x R^x|_{D(s) \times \mathcal{O}_1^d} e^{-s|k|} \ll e^{-s|k|} Kr^2 \varepsilon,$$  \hfill (4.44)

where we have used (4.17). Expanding $\partial_x F^x$ into Fourier series:

$$F^x = \sum_{0 \neq k \in \mathbb{Z}^n, |k| \leq K} \widehat{\partial_x F^x}(k)e^{\sqrt{-1}(k,x)}.$$  \hfill (4.45)

Inserting (4.43,45) into (4.36), we get

$$\partial_x F^x(x, \xi) = \sum_{0 \neq k \in \mathbb{Z}^n, |k| \leq K} \frac{\widehat{\partial_x R^x}(k)}{\sqrt{-1}(k,\omega)} e^{\sqrt{-1}(k,x)}.$$  

By (4.36,41) as well as Lemma A.1, we get that for $x \in D(s')$,

$$|\partial_x F^x(x, \xi)| \ll Kn^2 \varepsilon \sum_{k \in \mathbb{Z}^n} e^{-|k|(s-s')} \ll Kn^2 \varepsilon.$$

Applying $\partial_{\xi_j}$ to both sides of (4.36) and using a method similar to the above, we can get the second inequality of (4.42).

**Proposition 2.** (Solution of (4.37).) On $D(s') \times \mathcal{O}_1^d$, the equation (4.37) has a solution $F^y(x, \xi)$ which is analytic in $x \in D(s')$ for $\xi$ fixed and analytic in each $\xi_j, (j = 1, ..., n)$ for other variables fixed, and which is real for real argument, such that

$$|\partial_x F^y|_{D(s') \times \mathcal{O}_1^d} \ll Kn^2 \varepsilon, \quad |\partial_x F^y|_{D(s') \times \mathcal{O}_1^d} \ll Kn^2 \varepsilon.$$  

**Proof.** The proof follows almost exactly the proof of Prop. 1. We omit it.

**5. Solution of (4.38).**

This section is essential part of the present paper. Let $B_{ij}^{zz}(i, j \in \mathbb{Z}^d)$ be the elements of matrix $B^{zz}$. Note that $B_{ij}^{zz}$ is a function of $(x, \xi)$. We temporally omit the parameter $\xi$ for simplifying notation. Regarding $B_{ij}^{zz}$ as a function of $x$, we let $\widehat{B_{ij}^{zz}}(k)$ be the $k$-Fourier coefficient of $B_{ij}^{zz}$. Let

$$\widehat{B_{ij}^{zz}} = (\widehat{B_{ij}^{zz}}(k-l) : |k|, |l| \leq K).$$

Then $\widehat{B_{ij}^{zz}}$ is a matrix of order $K^d$. Let $\widehat{B}$ be a block matrix whose elements are $\widehat{B_{ij}^{zz}}$ ($i, j \in \mathbb{Z}^d$), that is,

$$\widehat{B} = (\widehat{B_{ij}^{zz}} : i, j \in \mathbb{Z}^d) = (\widehat{B_{ij}^{zz}}(k-l) : |k|, |l| \leq K; i, j \in \mathbb{Z}^d).$$

Similarly we have $\widehat{\Bar{B}}^{zz}$ and $\widehat{\Bar{B}}^{zz}$. Write $F^z$ as a column vector of infinite dimension: $F^z = (F^z_j \in \mathbb{C} : j \in \mathbb{Z}^d)$. Let $\widehat{F}_j^z = (\widehat{F}_j^z(k) : |k| \leq K)$ and $\widehat{\Bar{F}}^z = (\widehat{\Bar{F}}_j^z : j \in \mathbb{Z}^d)$. Similarly we have $\widehat{\Bar{F}}^z$, $\widehat{R}$ and $\widehat{\Bar{R}}$. Set

$$\widehat{\Bar{F}} = \left( \widehat{\Bar{F}}^z \right), \widehat{\Bar{R}} = \left( \frac{-\sqrt{-1}\widehat{\Bar{R}}^z}{\sqrt{-1}\widehat{\Bar{R}}^z} \right), \widehat{\Bar{B}} = \left( \begin{array}{c} -\widehat{\Bar{B}}^{zz} \\ -\widehat{\Bar{B}}^{zz} \end{array} \right).$$
Introduce a tensor product space as follows
\[ \ell^p := \ell^p \otimes C^K = \{ z = (z_j \in C^K : j \in \mathbb{Z}^d) : (|z|_p^*)^2 = \sum_j |z_j|^2 < \infty \}. \]

By abuse use of notation, we use \( || \cdot ||_{\ast p} \) as \( || \cdot ||_p \). For any \( u, v \in \ell^p \), we can regard \((u, v)^T\) as a vector in \( \ell^p \oplus \ell^p \). Define
\[ ||(u, v)^T||_p = \sqrt{||u||_p + ||v||_p}. \]

Therefore,
\[ ||\hat{F}^u||_p^2 = \sum_j \sum_{|k| \leq K} |\hat{F}^u_j(k)|^2 |j|^{2p} = \sum_{|k| \leq K} ||\hat{F}^z(k)||_p^2, u \in \{z, \bar{z}\}, \]
where \( \hat{F}^u(k) \) is the \( k \)-Fourier coefficient of \( F^u(x) \). Similarly,
\[ ||\hat{R}^u||_p^2 = \sum_{|k| \leq K} ||\hat{R}^u(k)||_p^2, u \in \{z, \bar{z}\}. \]

And
\[ ||\hat{F}||_p^2 = ||\hat{F}^z||_p^2 + ||\hat{F}^{\bar{z}}||_p^2. \]

In view of (4.29,30), we see that the linear operator \( \hat{B} \) is self-adjoint in \( \ell^0 \oplus \ell^0 \), i.e., \( \hat{B} = \hat{B}^* \). With these notations, we have the following lemma.

**Lemma 5.1.** (i). The vector \( \hat{R} \) is in \( \ell^p \oplus \ell^p \) with
\[ ||\hat{R}||_p \leq K \varepsilon, \quad ||\hat{R}||_p^C \leq K \varepsilon^C. \] (5.1)

(ii). The linear operator \( \hat{B} \) is self-adjoint in \( \ell^p \oplus \ell^p \) with \( p = 0 \), i.e., \( \hat{B} = \hat{B}^* \); and
\[ ||\hat{B}||_p, ||\hat{B}||_p^C \leq \varepsilon. \] (5.2)

**Proof.** The conclusion (i) comes from (4.19). It follows form (4.30) that the conclusion (ii) holds true. □

Let
\[ \Lambda = \text{diag} \left( \pm (k, \omega) + \Omega_0^0 : |k| \leq K, j \in \mathbb{Z}^d \right), \quad \Lambda = \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix}. \]

Expand \( \hat{B}^z, \hat{B}^{\bar{z}}, \hat{B}^{z\bar{z}}, F^z(x), F^{\bar{z}}(x) \), \( R^z(x) \) and \( R^{\bar{z}}(x) \) in (4.38) into Fourier series, we get a lattice equation which reads
\[ (\Lambda + \hat{B})\hat{F} = \hat{R}. \] (5.3)

Let
\[ M = (1 + 10K \max_{\xi \in \mathcal{O} |\omega|^{1/\kappa}}. \] (5.4)
Write \( \hat{B} = (B_{ij}(k, l) : |k|, |l| \leq K, i, j \in \mathbb{Z}^d) \). Then we see that \( B_{ij}(k, l) \) is one of \( \hat{B}^{zz}_{ij}(k - l), \hat{B}^{z\bar{z}}_{ij}(k - l) \) and \( \hat{B}^{\bar{z}z}_{ij}(k - l) \). Let

\[
\hat{B}_{11} = (B_{ij}(k, l) : |k|, |l| \leq K, |i|, |j| \leq M) \\
\hat{B}_{12} = (B_{ij}(k, l) : |k|, |l| \leq K, |i| \leq M, |j| > M) \\
\hat{B}_{21} = (B_{ij}(k, l) : |k|, |l| \leq K, |i| > M, |j| \leq M) \\
\hat{B}_{22} = (B_{ij}(k, l) : |k|, |l| \leq K, |i| > M, |j| > M).
\]

Then

\[
\hat{B} = \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{pmatrix}.
\]

And by (5.2),

\[
|||\hat{B}_{ij}|||_p, |||\hat{B}_{ij}|||_{L^p} \leq \epsilon, i, j \in \{1, 2\}.
\]

Let

\[
\Lambda_1 = \text{diag} \left( \pm(k, \omega) + \Omega^0 \right) : |k| \leq K, |j| \leq M \\
\Lambda_2 = \text{diag} \left( \pm(k, \omega) + \Omega^0 \right) : |k| \leq K, |j| > M.
\]

Then

\[
\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}.
\]

In view of (5.4),

\[
|\pm(k, \omega) + \Omega^0| \geq K, \forall |j| > M.
\]

Thus by (5.5) and (5.6), we see that there exists the inverse of \( \Lambda_2 + \hat{B}_{22} \) and

\[
||(\Lambda_2 + \hat{B}_{22})^{-1}||_p \leq \sum_{j=0}^{\infty} |||(\Lambda_2^{-1} \hat{B}_{22})^j|||_p \|\Lambda_2^{-1}\|_p < 1.
\]

Set

\[
\hat{B}_{\text{lef}} := \begin{pmatrix} E_1 \\ -(\Lambda_2 + \hat{B}_{22})^{-1} \hat{B}_{21} \\ E_2 \end{pmatrix} \quad (5.8)
\]

\[
\hat{B}_{\text{rig}} := \begin{pmatrix} E_1 \\ -\hat{B}_{12}(\Lambda_2 + \hat{B}_{22})^{-1} \\ E_2 \end{pmatrix}, \quad (5.9)
\]

where \( E_1 (E_2, \text{respectively}) \) is a unit matrix of the same order as that of \( \Lambda_1 (\Lambda_2, \text{respectively}) \). It is easy to verify that

\[
|||\hat{B}_{\text{lef}}^{-1}|||_p, |||\hat{B}_{\text{rig}}^{-1}|||_p < 1. \quad (5.10)
\]

Let

\[
\hat{B}_{11} = \hat{B}_{11} - \hat{B}_{12}(\Lambda_2 + \hat{B}_{22})^{-1} \hat{B}_{21} \quad (5.11)
\]

Then there exists the inverse of \( \Lambda + \hat{B} \):

\[
(\Lambda + \hat{B})^{-1} = \hat{B}_{\text{rig}}^{-1} \left( \begin{pmatrix} \Lambda_1 + \hat{B}_{11} \end{pmatrix}^{-1} 0 \begin{pmatrix} 0 \end{pmatrix} (\Lambda_2 + \hat{B}_{22})^{-1} \right) \hat{B}_{\text{lef}}^{-1} \quad (5.12)
\]
provided that there exists the inverse of $A_1 + \hat{B}_{11}$. And

$$\|(A + \hat{B})^{-1}\|_p^* < \|((A_1 + \hat{B}_{11})^{-1})\|_p^*, \quad (5.13)$$

where $* \equiv$ the blank or $L$. Now we are in position to investigate the inverse of $A_1 + \hat{B}_{11}$. First of all, we would like to point out that the matrix $\hat{B}_{11}$ is of finite order, and the order is bounded by

$$K^* := K_{2}\nu M^d \lesssim K_{2}\nu n + d\kappa^{-1}. \quad (5.14)$$

Secondly, it follows from the self-adjointness of $\hat{B}$ that the matrix $\hat{B}_{11}$ is also self-adjoint. Thirdly, by (5.5,7,11) we have

$$\|\hat{B}_{11}\|_p^* \lesssim \epsilon. \quad (5.15)$$

Fourthly, since each element of $\hat{B}$ is analytic in each entry of $\xi \in \mathcal{O}$, the matrix $\hat{B}_{11}$ is also analytic in each entry of $\xi \in \mathcal{O}$. Without loss of generality, we assume the first entry $\omega_1$ of $\omega$ is in the interval $[1, 2]$. Then the matrix $A_1 + \hat{B}_{11}$ is non-singular if and only if $\omega_1^{-1} A_1 + \omega_1^{-1} \hat{B}_{11}$ is non-singular, and

$$\|((A_1 + \hat{B}_{11})^{-1})\|_p \lesssim \|((\omega_1^{-1} A_1 + \omega_1^{-1} \hat{B}_{11})^{-1})\|_p. \quad (5.16)$$

Let

$$\Xi : \sigma_1 = 1/\omega_1, \sigma_2 = \omega_2/\omega_1, ..., \sigma_n = \omega_n/\omega_1. \quad (5.17)$$

Then it is easy to get

$$\left| \det \left( \frac{\partial(\sigma_1, ..., \sigma_n)}{\partial(\omega_1, ..., \omega_n)} \right) \right| = \omega_1^{-(n+1)} \gtrsim 1. \quad (5.18)$$

Combining (5.18) and (4.31), we can regard $\sigma = (\sigma_1, ..., \sigma_n)$ as a parameter vector instead of $\xi$, and we can show that

$$\text{Meas } \mathcal{O} \ll \text{Meas } \Xi(\mathcal{O}) \ll \text{Meas } \mathcal{O} \quad (5.19)$$

as well as

$$\|\partial_\sigma \hat{B}_{11}(\xi(\sigma))\| \lesssim \epsilon, \quad (5.20)$$

$$\partial_\sigma \Omega^0_j = o(1), \quad (5.21)$$

where (2.17) is used in (5.21). Let

$$\Lambda_\sigma = \text{diag } (\pm (k_1 + \sum_{l=2}^n k_l \sigma_l) + \sigma_1 \Omega^0_j : k = (k_1, ..., k_n) \in \mathbb{Z}^n, |k| \leq K, |j| \leq M). \quad (5.22)$$

Take $\sigma$ as a parameter vector. Then

$$\omega_1^{-1} A_1 + \omega_1^{-1} \hat{B}_{11}(\xi) = \Lambda_\sigma + \sigma_1 \hat{B}_{11}(\xi(\sigma)). \quad (5.23)$$
Using Lemma A.2, there are scalar functions \( \mu_1(\sigma), \cdots, \mu_{K^*}(\sigma) \) and a matrix-valued function \( U(\sigma) \) of order \( K^* \) (See (4.14) for \( K^* \)), which are analytic for each entry of real \( \sigma \) and possess the following properties for every \( \sigma \in \Xi(\mathcal{O}) \):

\[
A_\sigma + \sigma_1 \hat{B}_{11}(\xi(\sigma)) = U(\sigma) \text{diag}(\mu_1(\sigma), \cdots, \mu_{K^*}(\sigma)) U^*(\sigma),
\]

and

\[
U(\sigma)(U(\sigma))^* = E.
\]

It follows from (5.17) that

\[
|||U(\xi)||| = |||U^*||| = 1.
\]

Arbitrarily take \( \mu = \mu(\sigma) \in \{ \mu_1(\sigma), \ldots, \mu_{K^*}(\sigma) \} \). Let \( \phi \) be the normalized eigenvector corresponding \( \mu \). Using Lemma A.3 and noting (5.20) and (5.21), we get

\[
\partial_{\sigma_1} \mu = ((\partial_{\sigma_1}(A_\sigma + \sigma_1 \hat{B}_{11}(\xi(\sigma))))\phi, \phi)
= (\text{diag}(\Omega_0^0 + \sigma_1 \partial_{\sigma_1} \Omega_0^0 : |j| \leq M, |k| \leq K) \phi, \phi) + \partial_{\sigma_1}(\sigma_1 \hat{B}(\xi(\sigma)))\phi, \phi)
= \min_j \Omega_0^0 + o(1).
\]

Thus,

\[
\partial_{\sigma_1} \mu \geq c > 0.
\]

Let

\[
\Xi(\mathcal{O})_l = \{ \sigma \in \Xi(\mathcal{O}) : |\mu_l| < 1/(KK^*) \}, \quad l = 1, \ldots, K^*.
\]

By (5.28), \( \text{Meas}\Xi(\mathcal{O})_l < 1/(KK^*) \). Thus,

\[
\text{Meas} \bigcup_{l=1}^{K^*} \Xi(\mathcal{O})_l < 1/K.
\]

Let

\[
\Xi(\mathcal{O}) = \Xi(\mathcal{O}) \setminus \bigcup_{l=1}^{K^*} \Xi(\mathcal{O})_l.
\]

Therefore,

\[
\text{Meas} \Xi(\mathcal{O}) \geq \text{Meas} \Xi(\mathcal{O})(1 - O(1/K))
\]

and for any \( \sigma \in \Xi(\mathcal{O}) \),

\[
|\mu_l(\sigma)| \geq 1/KK^*.
\]

Moreover,

\[
|||(A_\sigma + \sigma_1 \hat{B}_{11})^{-1}||| \leq |||U||| |||U^*||| \max_l |\mu_l^{-1}| \leq KK^* \leq KC
\]

where \( C = n + dK^{-1} + 1 \). Returning to the parameter \( \xi \), by (5.19), there is a subset \( \mathcal{O}_+^2 \subset \mathcal{O} \) with

\[
\text{Meas} \mathcal{O}_+^2 \geq (\text{Meas} \mathcal{O})(1 - C_1K^{-1}),
\]

\(^4\text{Here } ||| \cdot ||| \text{ is the } \ell_2 \text{ norm of matrix.}\)
and for any $\xi \in \mathcal{O}_+^2$
\[
||(\Lambda_1 + \hat{B}_{11})^{-1}||_p \leq C_2 K^C.
\] (5.37)
Observe that $\mu_l = \mu_l(\xi)$ is continuous in $\xi$. By means of adjusting the constants $C_1, C_2$ in (5.36, 37), we can extend the set $\mathcal{O}_+^2$ to an open set such that (5.36, 37) still hold true for any $\xi$ in the open set. Still denote by $\mathcal{O}_+^2$ the open set. Replacing the $\ell_2$ norm $||| \cdot |||$ by $||| \cdot |||_p$ in (5.37), in view of the order of the matrix $\Lambda_1 + \hat{B}_{11}$ being bounded by $K^*$, we get
\[
|||(\Lambda_1 + \hat{B}_{11})^{-1}||_p \leq K^C(K^*)^p \leq \tilde{K}^C
\] (5.38)
where $\tilde{K}$ is a constant depending on $n, d, \kappa, p$. By (5.13),
\[
|||(\Lambda + \hat{B})^{-1}||_p \leq \tilde{K} C^r \varepsilon, \quad \xi \in \mathcal{O}_+^2.
\] (5.39)
Note
\[
|||\partial_\xi ((\Lambda_1 + \hat{B})^{-1})||_p \leq \tilde{K} C^r \varepsilon, \quad \xi \in \mathcal{O}_+^2.
\] (5.40)
By (5.1) and (5.39, 40),
\[
|||\partial_\xi F^u||_p \leq \tilde{K} C^r \varepsilon, \quad |||\partial_\xi \hat{F}^u||_p \leq \tilde{K} C^r \varepsilon
\] (5.41)
where $\tilde{K}$ is a constant depending on $p, d, n, \kappa$. Recall
\[
|||\hat{F}^u||_p^2 = \sum_{|k| \leq K} |||\hat{F}^u(k)||_p^2, \quad |||\hat{F}^u(k)||_p^2 = \sum_{|k| \leq K} |||\hat{F}^u(k)||_p^2.
\] (5.42)
Then for $u \in \{z, \bar{z}\}$,
\[
\sup_{D(s') \times \mathcal{O}_+^2} |||F^u(x, \xi)||_p^2 = \sup_{D(s') \times \mathcal{O}_+^2} |||\sum_{|k| \leq K} \hat{F}^u(k)e^{\sqrt{-1}v(k, x)}||_p^2 \leq K \sum_{|k| \leq K} |||\hat{F}^u(k)||_p^2 e^{2|k|s'} \leq K e^8 \sum_{|k| \leq K} |||\hat{F}^u(k)||_p^2 \leq e^8 K |||\hat{F}^u||_p^2 \leq (K^C r \varepsilon)^2
\] (5.44)
That is,
\[
\sup_{D(s') \times \mathcal{O}_+^2} |||F^u(x, \xi)||_p \leq K^C r \varepsilon, \quad u \in \{z, \bar{z}\}.
\] (5.45)
Similarly,
\[
\sup_{D(s') \times \mathcal{O}_+^2} |||\partial_\xi F^u(x, \xi)||_p \leq K^C r \varepsilon, \quad u \in \{z, \bar{z}\}.
\] (5.46)
Consequently, we have the following proposition.
Proposition 3. There is a subset $\mathcal{O}^3_+ \subset \mathcal{O}$ with $\text{Meas } \mathcal{O}^3_+ = (\text{Meas } \mathcal{O})(1-O(1/K))$ and there is functions $F^x, F^z: D(s') \times \mathcal{O}^3_+ \to \ell^p$ which are analytic in $x \in D(s')$ and also analytic in each entry of $\xi \in \mathcal{O}^3_+$; moreover the functions $F^x, F^z$ solve (4.38) and satisfy the estimates (5.45,46).

6. Symplectic change of variables.

In this section, our procedure is standard and almost the same as that of Section 3 in [P1,p.128-132]. Here we give out the outline of the procedure. See [P1] for the details. In this section, we denote by $C$ a universal constant depending only on $n, d, p, \kappa$. The constant $C$ might be different in different places.

6.1. Coordinate transformation. By Propositions 1-3, we get a Hamiltonian $F$ on $D(s', r)$ where

$$F = F^x + (F^y, y) + \langle F^z, z \rangle + \langle F^\bar{z}, \bar{z} \rangle$$

and give estimates of $F^x, F^y, F^z$ and $F^\bar{z}$. Let $X_F$ be the vector field corresponding to the Hamiltonian $F$, that is,

$$X_F = (-\partial_y F, \partial_x F, \sqrt{-1} \partial_{\bar{z}} F, -\sqrt{-1} \partial_z F),$$

here $\partial_z$ and $\partial_{\bar{z}}$ are the usual $\ell^2$-gradients. Recall $s = 4/K_-$ and $s' = 4/K$. And let $s'' = 3/K, s''' = 2/K, s'''' = 1/K$. Let

$$\mathcal{O}_+ = \bigcup_{j=1}^{2} \mathcal{O}^j_+.$$ 

It follows from Prop.1,2 and 3 that for $(x, y, z, \bar{z}; \xi) \in D(s', r) \times \subset \mathcal{O}_+$,

$$r |X_F|_{p, D(s', r) \times} \leq K^C \varepsilon,$$

where $C$ is a large constant depending only on $n, \kappa, d$ and $p$. That is,

$$r |X_F|_{p, D(s', r) \times} \leq K^C \varepsilon.$$ (6.1)

Similarly, we have

$$r |X_F|^\mathcal{C}_{p, D(s', r) \times} \leq K^C \varepsilon^\mathcal{C}.$$ (6.2)

where we have used the assumption $\varepsilon < \varepsilon^\mathcal{C}$. As in [P1,p.129], we introduce the operator norm

$$r ||L||_p = \sup_{W \neq 0} \frac{r |LW|_p}{r |W|_p}.$$ (6.3)

Using (6.1), (6.2) and the generalized Cauchy’s inequality (See Lemma A.3 of [P1,p.147]) and the observation that every point in $D(s'', r/2)$ has at least $r \cdot |r|_p$ distance $1/9K$ to the boundary of $D(s', r)$, we get

$$\sup_{D(s', r/2) \times} r |DX_F|_p \leq K r |X_F|_{p, D(s', r) \times} \leq K^C \varepsilon.$$ (6.4)
Recall that (4.39) holds true when information $\Phi = 0$ and $\eta r < \sqrt{K \epsilon}$. Arbitrarily fix $\xi \in \mathcal{O}_+$. By (6.1), the flow $X_F^t$ of the vector field $X_F$ exists on $D(s''', r/4)$ for $t \in [-1, 1]$ and takes the domain into $D(s''', r/2)$, and by Lemma A.4 of [P1, p.147], we have

$$r |X_F^t - id|_{p, D(s''', r/4) \times \mathcal{O}_+} \leq r |X_F|_{p, D(s''', r/2) \times \mathcal{O}_+} \leq K^C \epsilon^L,$$

where $DX_F$ is the differential of $X_F$. Assume that $K^C \epsilon$ and $K^C \epsilon^L$ are small enough. (These assumptions will be fulfilled in the following KAM iterations. In fact, in $m$-th KAM step, $K \approx C2^{m^2/2}$ and $\epsilon = \epsilon^{(4/3)^m}$. It follows that $K^C \epsilon \ll 1$.)

Arbitrarily fix $\xi \in \mathcal{O}_+$. By (6.1), the flow $X_F^t$ of the vector field $X_F$ exists on $D(s''', r/4)$ for $t \in [-1, 1]$ and takes the domain into $D(s''', r/2)$, and by Lemma A.4 of [P1, p.147], we have

$$r |X_F^t - id|_{p, D(s''', r/4) \times \mathcal{O}_+} \leq r |X_F|_{p, D(s''', r/2) \times \mathcal{O}_+} \leq K^C \epsilon^L$$

and

$$r |X_F^t - id|_{p, D(s''', r/4) \times \mathcal{O}_+} \leq K^C \epsilon^L$$

for $t \in [-1, 1]$. Furthermore, by the generalized Cauchy’s inequality,

$$r |DX_F^t - I|_{p, D(s''', r/8) \times \mathcal{O}_+} \leq K^C \epsilon, \quad t \in [0, 1]$$

and

$$r |DX_F^t - I|_{p, D(s''', r/8) \times \mathcal{O}_+} \leq K^C \epsilon^L, \quad t \in [0, 1]$$

6.2. The new error term. Subjecting $H = N + \hat{R}$ to the symplectic transformation $\Phi = X_F^t|_{t=1}$ we get the new Hamiltonian scale $H_+ := H \circ \Phi = H \circ X_F^1$ on $D(s''', \eta r)$ where $0 < \eta \ll 1$. By Taylor’s formula

$$H_+ = (N + R) \circ X_F^1 = (N + R + (\hat{R} - R)) \circ X_F^1$$

$$= (N + R + R_K + (\hat{R} - R)) \circ X_F^1$$

$$= N - \{F, N\} + \int_0^1 \{t\{F, N\}, F\} \circ X_F^t dt$$

$$+ R + \int_0^1 \{R, F\} \circ X_F^t dt + (R_K + (\hat{R} - R)) \circ X_F^1.$$

Recall that (4.39) holds true when $F$ solves (4.36-38). Thus,

$$H_+ = N_+ + \hat{R}_+$$

where

$$N_+ = N + \frac{1}{2} \langle (F^y, \partial_x B^{zz})z, z \rangle + \langle (F^y, \partial_x B^{zz})z, \bar{z} \rangle + \frac{1}{2} \langle (F^y, \partial_x B^{zz})\bar{z}, \bar{z} \rangle$$

$$\hat{R}_+ = \hat{R}_1 + \hat{R}_2 + \hat{R}_3$$

where

$$\hat{R}_1 = \sqrt{-1} \langle (1 - \Gamma_K)(B^{zz}F^{-}), z \rangle + \sqrt{-1} \langle (1 - \Gamma_K)(B^{zz}F^{-})z \rangle$$

$$- \sqrt{-1} \langle (1 - \Gamma_K)(B^{zz}F^{-}), \bar{z} \rangle - \sqrt{-1} \langle (1 - \Gamma_K)(B^{zz}F^{-}), \bar{z} \rangle$$

(6.12)
Hence, the new perturbing vector field is $X_{\tilde{R}_+} = R_K \circ X_{\tilde{F}} + (\tilde{R} - R) \circ X_{\tilde{F}}$ (5.15) 

$$\tilde{R}_+^2 = R_K \circ X_{\tilde{F}} + (\tilde{R} - R) \circ X_{\tilde{F}}$$

$$\tilde{R}_+^3 = \int_0^t \{ (1 + t) R, F \} \circ X_{\tilde{F}} \ dt.$$ (5.16) 

By (4.26,27,28),

$$N_+ = (\omega_+ + y) + (\Omega^0 z, \bar{z}) + \frac{1}{2} (B_{zz}^z z, z) + (B_+^{zz} z, \bar{z}) + \frac{1}{2} (B_{zz}^{zz} z, \bar{z})$$

where

(6.17) 

with

$$B_{zz}^z = B_{zz}^z + R_{zz} + (F^y, \partial_x (B^{zz} + R^{zz})),$$

$$B_+^{zz} = B_+^{zz} + R_{zz} + (F^y, \partial_x (B^{zz} + R^{zz})),$$

and

$$B_{zz}^{zz} = B_{zz}^{zz} + (F^y, \partial_x (B^{zz} + R^{zz})),$$

(6.18) 

Hence, the new perturbing vector field is

$$X_{\tilde{R}_+} = X_{\tilde{R}_+^2} + (X_{\tilde{F}})^* (X_R - X_R + R_K) + \int_0^t (X_{\tilde{F}})^* [X_{(1+t) R}, X_F] \ dt,$$

where $(X_{\tilde{F}})^*$ is the pull-back of $X_{\tilde{F}}$, and $[\cdot, \cdot]$ is the commutator of vector fields. We are now in position to estimate the new perturbing vector field $X_{\tilde{R}_+}$. Let $Y : D(s', r) \subset P \to P$ be a vector field on $D(s', r)$, depending on the parameter $\xi \in \mathcal{O}_+$. Let $U = D(s'', \eta r) \times \mathcal{O}_+$ and $V = D(s'', 2\eta r) \times \mathcal{O}_+$ and $W = D(s'', 4\eta r) \times \mathcal{O}_+$. By (6.6) and the “proof of estimate (12)” of [P1, p.131-132]\(^5\), we have that

$$\eta r [X_{\tilde{F}}]^* Y |_{p, U} \ll \eta r |Y|_{p, V}$$

(6.20) 

and

$$\eta r [X_{\tilde{F}}]^* Y |_{p, U} \ll \eta r |Y|_{p, V} + \frac{K}{\eta^2} \eta r [Y]_{p, W} \cdot \eta r [X_F]^* Y |_{p, V}.$$ (6.21) 

We assume that

$$\varepsilon K C / \eta^2 \ll 1.$$ (6.22) 

These assumptions will be fulfilled in the KAM iterative lemma later. (In fact, in $m$-th KAM iterative step, we will let $K C \approx 2^{cm^2}, \varepsilon = \varepsilon^2, \eta = \varepsilon^{1/3}$. This implies (6.22) is fulfilled.) By (4.10) and (6.20,21),

$$\eta r [X_{\tilde{F}}]^* (X_R - X_R) |_{p, U} \ll \eta r [X_R - X_R]_{p, V} \ll \eta \varepsilon$$

(6.23) 

and

$$\eta r [X_{\tilde{F}}]^* (X_R - X_R) |_{p, U} \ll \eta \varepsilon C + \frac{\varepsilon K C \eta}{\eta^2} \varepsilon \ll \eta \varepsilon.$$ (6.24) 

Recall that (4.9) holds still true after replacing $R$ by $\tilde{R}$. By (4.2) and (5.4,5) and using the generalized Cauchy estimate, following [P1,p.130-131] we get

$$\rho_{(t)} |X_{R(t)}, X_F|_{p, U} \ll K \rho_{(t)} |X_R|_{p, V} \cdot \rho_{(t)} |X_F|_{p, V}$$

$$\ll K \rho_{(t)} |X_R|_{p, W} \cdot \rho_{(t)} |X_F|_{p, W} \ll K^{1+C} \varepsilon^2 < \eta \varepsilon$$

(6.25) 

\(^5\)Let $a = 0$ in (12) of [P1].
and
\[ r\| [X_{R(i)}, X_F] \|_{p,U}^\epsilon \]
\[ < K r\| X_F \|_{p,W} + K r\| X_{R(i)} \|_{p,W} \]
\[ < K^C \epsilon \epsilon \epsilon + K^C \epsilon \epsilon \epsilon < \eta \epsilon \]  
(6.26)

Finally, we have
\[ \eta r \| Y \|_{p,U} < \eta^{-2} r \| Y \|_{p,U}, \quad \eta r \| Y \|_{p,U}^\epsilon < \eta^{-2} r \| Y \|_{p,U}^\epsilon, \]  
(6.27)

for any vector field \( Y \). Note that for any function \( f : D(s) \times \mathcal{O} \to \ell^p \) or \( \mathbb{R}^n \) which is analytic in \( x \in D(s) \) and in each entry of \( \xi \in \mathcal{O} \),
\[
\| (1 - \Gamma_K) f \|_{D(s') \times \mathcal{O}} \leq \sum_{|k| > K} \| \tilde{f}(k) \|_{p \epsilon^{s'[k]}}
\leq \| f \|_{D(s) \times \mathcal{O}} \sum_{|k| > K} \epsilon^{-(s-s')|k|}
\leq \| f \|_{D(s) \times \mathcal{O} \cdot \epsilon^2},
\]
where we have used
\[ (s - s')K \approx 2|\ln \epsilon|. \]  
(6.29)

Applying (6.27,28) to (6.14), we can easily get
\[ \eta r \| X_{R_+} \|_{p,U} < \eta^{-2} \epsilon^2 < \eta \epsilon, \quad \eta r \| X_{R_+} \|_{p,U}^\epsilon < \eta \epsilon \epsilon. \]  
(6.30)

Collecting all terms above, we then arrive at the estimates
\[ r \| X_{R_+} \|_{p,D(s_+,r_+) \times \mathcal{O}^+} < \eta \epsilon, \quad r \| X_{R_+} \|_{p,D(s_+,r_+) \times \mathcal{O}^+}^\epsilon < \eta \epsilon \epsilon. \]  
(6.31)

where
\[ s_+ = s''' \quad r_+ = \eta r. \]  
(6.32)

It follows from (4.22) and (4.40) that the perturbing Hamiltonian \( \hat{R}_+ \) satisfies the same symmetric condition as \( \hat{R} \), that is:
\[ \hat{R}_+(-x, y, z, \bar{z}; \xi) = \hat{R}_+(x, y, \bar{z}, z; \xi), \quad \forall (x, y, z, \bar{z}; \xi) \in D(s_+, r_+) \times \mathcal{O}^+. \]  
(6.33)

Finally we can easily check that \( N_+ \) satisfies the same conditions as \( \mathcal{B} \), that is, the conditions (4.2-6) hold true by replacing \( \mathcal{B} \) by \( B_+ \) and replacing \( \omega \) by \( \omega_+ \). Here we omit the details.

7. Iterative lemma and proof of the theorem.

7.1. Iterative constants. As usual, the KAM theorem is proven by the Newton-type iteration procedure which involves an infinite sequence of coordinate changes. In order to make our iteration procedure run, we need the following iterative constants:
1. \( \epsilon_0 = \epsilon, \epsilon_l = \epsilon_0^l (4/3)^l, l = 1, 2, \ldots; \)
2. \( \eta_l = \epsilon_l^{1/3}, l = 0, 1, 2, \ldots; \)
3. \( K_l = 2^{\sum_{i=1}^l \ln |\epsilon_l|}, l = 1, 2, \ldots, (\text{thus, } K_l = 2^{|l(l+1)/2 \ln \epsilon_l|}); \)
4. \( s_0 > 0 \) is given, \( s_l = 1/K_{l-1}, l = 1, 2, \ldots, (\text{thus, } (s_l - s_{l+1})K_l \geq 2|\ln \epsilon_l|); \)
5. $r_0 > 0$ is given, $r_l = \eta r_0$, $l = 1, 2, \ldots$;
6. $D(s_l) = \{x \in \mathbb{C}^n/(2\pi)^n : |3x| < s_l\}$
7. $D(s_l, r_l) = \{(x, y, z, \bar{z}) \in \mathcal{P} : |3x| < s_l, |y| < r_l^2, ||z||_l < r_l, ||\bar{z}||_l < r_l\}$.

7.2. Iterative Lemma. Consider a family of Hamiltonian functions $H_l (0 \leq l \leq m)$:

$$H_l = (\omega_l(\xi), y) + (\Omega^0(\xi) z, \bar{z}) + \frac{1}{2} (B_l^{zz}(x, \xi) z, z) + (B_l^{\bar{z}\bar{z}}(x, \xi) \bar{z}, \bar{z}) + \frac{1}{2} (B_l^{\bar{z}z}(x, \xi) z, \bar{z}) + \hat{R}_l(x, y, z, \bar{z}, \xi)$$

(7.1)

where $B_l^{zz}(x, \xi)$’s are analytic in $x \in D(s_l)$ for any fixed $\xi \in \mathcal{O}_1$, and all of

$\omega_l(\xi), \Omega^0(\xi)$ and $B_l^{zz}(x, \xi)$’s (for fixed $x \in D(s_l)$) are analytic in each entry $\xi_l (l = 1, \ldots, n)$ of $\xi \in \mathcal{O}_1$, and $\hat{R}_l(x, y, z, \bar{z}, \xi)$ is analytic in $(x, y, z, \bar{z}) \in D(s_l, r_l)$ and analytic in each entry of $\xi \in \mathcal{O}_1$. Assume $B_l^{zz} = (B_l^{zz}_{ij} : i, j \in \mathbb{Z}^d)$’s satisfy the following conditions:

(1.1). Symmetry. For any $(x, \xi) \in D(s_l) \times \mathcal{O}_l$, the operators $B_l^{zz}$, $B_l^{\bar{z}z}$ and $B_l^{\bar{z}\bar{z}}$ are self-adjoint from $L_2$ to $L_2$.

(1.2). Finiteness of Fourier modes.

$$B_l^{zz}(x, \xi) = \sum_{|k| \leq K_l} \hat{B}_l^{zz}(k) e^{\sqrt{-1}(k, x)} \cdots$$

(1.3). Boundedness.

$$\sup_{D(s_l) \times \mathcal{O}_l} |||B_l^{zz}(x, \xi)|||_p \leq \epsilon, \ldots$$

where $\ast$ is the blank and L. Moreover, we assume the parameter sets $\mathcal{O}_l$’s satisfy

(1.4).

$$\mathcal{O}_0 \supset \cdots \supset \mathcal{O}_l \supset \cdots \mathcal{O}_m$$

with

$$\text{Meas } \mathcal{O}_l \geq (\text{Meas } \mathcal{O}_0)(1 - K_l^{-1});$$

(1.5). The map $\xi \mapsto \omega_l(\xi)$ is analytic in each entry of $\xi \in \mathcal{O}_l$, and

$$\inf_{\mathcal{O}_l} \left| \text{det} \frac{\partial \omega_l}{\partial \xi} \right| \geq c_1 > 0, \sup_{\mathcal{O}_l} |\partial^j \omega_l| \leq c_2, j = 0, 1.$$

(1.6). The perturbation $\hat{R}_l(x, y, z, \bar{z}; \xi)$ is analytic in the space coordinate domain $D(s_l, r_l)$ and also analytic in each entry $\xi_k (k = 1, \ldots, n)$ of the parameter vector $\xi \in \mathcal{O}_l$, and is real for real argument; moreover, its Hamiltonian vector field

$$X_{\hat{R}_l} := \left(\partial_y \hat{R}_l, -\partial_x \hat{R}_l, -\partial_z \hat{R}_l, -\sqrt{-1} \partial_{\bar{z}} \hat{R}_l\right)^T$$

defines on $D(s_l, r_l)$ a analytic map

$$X_{\hat{R}_l} : D(s_l, r_l) \subset \mathcal{P} \rightarrow \mathcal{P}.$$

(1.7). In addition, the vector field $X_{\hat{R}_l}$ is analytic in the domain $D(s_l, r_l)$ with small norms

$$r_l |X_{\hat{R}_l}|_{p, D(s_l, r_l) \times \mathcal{O}_l} \leq \epsilon_l, \quad r_l |X_{\hat{R}_l}|_{p, D(s_l, r_l) \times \mathcal{O}_l} \leq \epsilon_l^{1/3}.$$
Then there is an absolute positive constant $\varepsilon^*$ enough small such that, if $0 < \varepsilon_0 < \varepsilon^*$, there is a set $O_{m+1} \subset O_m$, and a change of variables $\Phi_{m+1} : D_{m+1} := D(s_{m+1}, r_{m+1}) \times O_{m+1} \to D(s_m, r_m)$ being real$^6$, analytic in

$$(x, y, z, \bar{z}) \in D(s_{m+1}, r_{m+1})$$

and each entry $\xi \in O_{m+1}$, as well as following estimates holds true:

$$r_m|\Phi_{m+1} - id|_{p, D_{m+1}} < \varepsilon_m^{1/2}$$

and

$$r_m|\Phi_{m+1} - id|_{C^l, D_{m+1}} < \varepsilon_m^{1/4}.$$  

Furthermore, the new Hamiltonian $H_{m+1} := H_m \circ \Phi_{m+1}$ of the form

$$H_{m+1} = (\omega_{m+1}(\xi), y) + (\Omega^0(\xi)z, \bar{z}) + \frac{1}{2}(B^z_{m+1}(x, \xi)z, \bar{z}) + (B^\bar{z}_{m+1}(x, \xi)\bar{z}, z)$$

$$+ \frac{1}{2}(B^{zz}_{m+1}(x, \xi)z, \bar{z}) + \tilde{R}_{m+1}(x, y, z, \bar{z}, \xi)$$

(7.2)

satisfies all the above conditions (l.1–7) with $l$ being replaced by $m + 1$.

7.3. Proof of the Iterative Lemma.

As stated as in the iterative lemma, we have got a family of Hamiltonian functions $H_l$'s ($l = 0, 1, ... , m$) which satisfy the conditions (l.1–7). We now consider the Hamiltonian $H_m$. Let $N = N_m$, $R = R_m$, $N_+ = N_{m+1}$ and $R_+ = R_{m+1}$. Let $s = s_m$, $s^{\mu \nu} = s_{m+1}$, $\eta = \eta_m$, $r = r_m = \eta_m r_0$, $\varepsilon = \varepsilon_m$, $\varepsilon^l = \varepsilon_m^{1/3}$ and $O_{m+1} = O_+^1 \cup O_+^2$. Clearly, $\varepsilon < \varepsilon^l$. By means of the conclusion in Section 4, we got that there is a Hamiltonian $F = F_m$ defined on$^7$ $D(s_{m+1}, r_{m+1}) \times O_{m+1}$ and a symplectic change of variables $\Phi_{m+1} = X^l_m |_{l=1}$. Note $\eta_m \varepsilon_m = \varepsilon_{m+1}$. It is easy to verify that the conditions $(m + 1.1 – 7)$ are fulfilled. We omit the details.

7. Proof of the Theorem 2.1

The proof is similar to that of [P1]. Here we give an outline. By Assumptions A, B, C, D, E and the smallness assumption in Theorem 2.1, the conditions (l.1–7) in the iterative lemma in Section 6.2 are fulfilled with $l = 0$. Hence the iterative lemma applies to $H$. Inductively, we get what as follows:

(i) Domains: for $m = 0, 1, 2, ... ,$

$$D_m := D(s_m, r_m) \times O_m, \quad D_{m+1} \subset D_m;$$

(ii) Coordinate changes:

$$\Psi^m = \Phi_1 \circ \cdots \circ \Phi_{m+1} : D_{m+1} \to D(s_0, r_0);$$

(iii) Hamiltonian functions $\hat{H}_m$ ($m = 0, 1, ...$) satisfy the conditions (l.1, 2, 3) with $l$ replaced by $m$;

$^6$The word “real” means $\Phi_{m+1}(z, \xi) = \Phi_{m+1}(\bar{z}, \xi)$ for any $(z, \xi) \in D_{m+1}$.

$^7$Note $D_{m+1} := D(s_{m+1}, r_{m+1}) \times O_{m+1} \subset D(s_{m+1}, r_{m+1}) \times O_{m+1}$. 

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Let $O_\infty = \cap_{m=0}^\infty O_m$, $D_\infty = \cap D_m$. By the same argument as in [P1, pp.134], we conclude that $\Psi^m, D\Psi^m, H_m, X_{H_m}$ converges uniformly on the domain $D_\infty$, and $X_{\tilde{H}_\infty} \circ \Psi^\infty = D\Psi^\infty \cdot X_{\omega}$, where

\[
\tilde{H}_\infty := \lim_{m \to \infty} H_m = (\omega_\ast(\xi), y) + \langle \Omega(\xi)z, \bar{z} \rangle + \frac{1}{2}\langle B_{zz}(x, \xi)z, z \rangle + \frac{1}{2}\langle B_{\bar{z}\bar{z}}(x, \xi)z, \bar{z} \rangle + \frac{1}{2}\langle B_{\bar{z}}(x, \xi)\bar{z}, \bar{z} \rangle
\]

here $B_{zz}(\lim_{m \to \infty} B^{zz}_m, \cdots, \cdots)$, and $X_{\omega}$ is the constant vector field $\omega_\ast$ on the torus $T^n$. Thus, $T^n \times \{0\} \times \{0\}$ is an embedding torus with rotational frequencies $\omega_\ast(\xi) \in \omega_\ast(O_\infty)$ of the Hamiltonian $\tilde{H}_\infty$. Returning the original Hamiltonian $\tilde{H}$, it has an embedding torus $\Phi^\infty(T^n \times \{0\} \times \{0\})$ with frequencies $\omega_\ast(\xi)$. This proves the Theorem. □

9. Appendix A. Some Technical lemmas.

Lemma A.1. For $\mu > 0, \nu > 0$, the following inequality holds true:

\[
\sum_{k \in \mathbb{Z}^d} e^{-2|k|\mu |k|^\nu} \leq \left( \frac{\nu}{e} \right)^\nu \frac{1}{\mu^{\nu+d}} (1+e)^d.
\]

Proof. This Lemma can be found in [B-M-S].

Lemma A.2. Consider an $n \times n$ complex matrix function $Y(\xi)$ which depends on the real parameter $\xi \in \mathbb{R}$. Let $Y(\xi)$ be a matrix function satisfying conditions:

(i) $Y(\xi)$ is self-adjoint for every $\xi \in \mathbb{R}$; i.e., $Y(\xi) = (Y(\xi))^\ast$, where star denotes the conjugate transpose matrix;

(ii) $Y(\xi)$ is an analytic function of the real variable $\xi$.

Then there exist scalar functions $\mu_1(\xi), \cdots, \mu_n(\xi)$ and a matrix-valued function $U(\xi)$, which are analytic for real $\xi$ and possess the following properties for every $\xi \in \mathbb{R}$:

\[
Y(\xi) = U(\xi)\text{diag}(\mu_1(\xi), \cdots, \mu_n(\xi))U^\ast(\xi), \quad U(\xi)(U(\xi))^\ast = E.
\]

Proof. See [pp.394-396, G-L-R]. □

It is worth to point out that this lemma does not hold true for $\xi \in \mathbb{R}^k$ with $k > 1$. See [Ka].

Lemma A.3. Assume $Y = Y(\xi)$ satisfies the conditions in Lemma A.3. Let $\mu = \mu(\xi)$ be any eigenvalue of $Y$ and $\phi$ be the normalized eigenfunction corresponding to $\mu$. Then

\[
\partial_\xi \mu = ((\partial_\xi Y)(\phi, \phi)).
\]

Proof. The proof can be found in [Ka,p.125].

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