A family of slice-torus invariants from the divisibility of reduced Lee classes

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Abstract
We give a family of slice-torus invariants, one defined for each prime element \( c \) in a principal ideal domain \( R \), from the \( c \)-divisibility of the reduced Lee class in a variant of reduced Khovanov homology. It is proved that this family contains the Rasmussen invariant \( s^F \) over any field \( F \). Moreover, computational results show that the invariants corresponding to \( (R, c) = (\mathbb{Z}, 2), (\mathbb{Z}, 3) \) and \( (\mathbb{Z}[i], 1 + i) \) are distinct from \( s^F \).

Contents

1 Introduction ........................................... 2

2 Khovanov homology ................................. 5
  2.1 Khovanov homology ............................... 5
  2.2 Lee classes ........................................ 6
  2.3 Reduction of parameters .......................... 10

3 Module structures and reduced homology ........... 12
  3.1 Module structures ................................. 12
  3.2 Cobordism maps .................................. 14
  3.3 Reduced homology ............................... 16
  3.4 Reduced Lee classes ............................. 19
  3.5 Connected sums and mirrors ..................... 20

4 The invariant \( \tilde{s}_c \) ............................... 23
  4.1 Setup ............................................. 24
  4.2 Divisibility of reduced Lee class ................. 25
  4.3 Definition and properties of \( \tilde{s}_c \) ............ 27
  4.4 Refined Lee classes .............................. 29
  4.5 Coincidence with \( s^F \) ........................... 31
  4.6 Classification .................................... 31

5 Computations ...................................... 33
1 Introduction

Rasmussen’s \( s \)-invariant is an integer valued knot invariant obtained from a variant of Khovanov homology \([\text{Kho00; Lee05; Ras10}]\). Its major applications are Rasmussen’s alternative proof of the Milnor conjecture \([\text{Mil68}]\) and Piccirillo’s proof of the non-sliceness of the Conway knot \([\text{Pic20}]\). Both are problems that arose in the intersection of knot theory and 4-dimensional topology and remained unsolved for decades. The \( s \)-invariant belongs to a class of knot concordance invariants called the slice-torus invariants \([\text{Liv04; Lew14}]\).

**Definition 1.1.** A slice-torus invariant \( \nu \) is an abelian group homomorphism

\[
\nu : \text{Conc}(S^3) \to \mathbb{R},
\]

satisfying the following two conditions:

(Slice) \(|\nu(K)| \leq 2g_4(K)\) for any knot \( K \),

(Torus) \( \nu(T_{p,q}) = (p - 1)(q - 1) \) for the positive \((p, q)\)-torus knot \( T_{p,q} \).

Here \( \text{Conc}(S^3) \) denotes the smooth concordance group of knots in \( S^3 \), and \( g_4(K) \) the slice genus of a knot \( K \).

The existence of a slice-torus invariant is itself nontrivial, for it immediately implies the Milnor conjecture. Other examples of slice-torus invariants are (i) the \( \tau \)-invariant obtained from knot Floer homology \([\text{OS03}]\), (ii) the Rasmussen invariant \( s^F \) generalized over any field \( F \) using Bar-Natan homology \([\text{MTV07; LS14}]\), (iii) the \( s_n \)-invariants (\( n \geq 2 \)) from \( \mathfrak{s}_n \) Khovanov–Rozansky homologies \([\text{Wu09; Lob09; Lob12}]\), (iv) the \( \tau^\# \)-invariant from framed instanton Floer homology \([\text{BS21}]\), and (v) the \( \tilde{s} \)-invariant from equivariant singular instanton Floer homology \([\text{Dae+22}]\). More studies on general slice-torus invariants are given in \([\text{CC20; FLL22}]\).

In this paper we introduce a family of slice-torus invariants, each obtained from the divisibility of the reduced Lee class (or the ‘canonical class’) in a variant of reduced Khovanov homology.

**Theorem 1.** For each prime element \( c \) in a principal integral domain \( R \), the value defined by

\[
\tilde{s}_c(K) = 2\tilde{d}_c(D) + w(D) - r(D) + 1
\]

is a slice-torus invariant. Here \( K \) is a knot with diagram \( D \), \( \tilde{d}_c(D) \) the \( c \)-divisibility of the reduced Lee class of \( D \) in \( \tilde{H}_c(D) \) (modulo torsion), \( w(D) \) the writhe of \( D \) and \( r(D) \) the number of Seifert circles of \( D \).
Note that the definition of \( \tilde{s}_c \) is formally identical to that of the invariant \( \bar{s}_c \) given in [San20], except that here we consider the reduced homology. Arguments of [San20] run in parallel, and in addition from the simplicity that \( \tilde{H}_c(D) \) has rank 1, the additivity of \( \tilde{s}_c \) follows from a straightforward argument (whereas the unreduced homology \( H_c(D) \) has rank 2 and the additivity of \( \bar{s}_c \) was only proved for some specific \( (R,c)'s) \).

The following theorem states that our family contains all \( s^F \), including \( \text{char } F = 2 \) which was excluded in [San20].

**Theorem 2.** For any field \( F \), the invariant \( \tilde{s}_c \) for the pair \( (R,c) = (F[H],H) \) coincides with the Rasmussen invariant \( s^F \) over \( F \), i.e.

\[
s^F(K) = \tilde{s}_H(K;F[H])
\]

for any knot \( K \).

A question arises naturally whether or not our family contains an invariant that is independent from \( s^F \). First, we have the following classification result.

**Theorem 3.** Suppose \( R \) is a PID and \( c \) is a prime in \( R \).

1. If \( \text{char } R = p \neq 0 \), then \( \tilde{s}_c \) coincides with \( s^{Fr} \).
2. If \( \text{char } R = 0 \) and \( c \nmid n \cdot 1_R \) for every \( n \in \mathbb{Z} \setminus 0 \), then \( \tilde{s}_c \) coincides with \( s^Q \).
3. If \( \text{char } R = 0 \) and \( c \sim p \cdot 1_R \) for some prime \( p \in \mathbb{Z} \), then \( \tilde{s}_c \) coincides with \( \tilde{s}_p(-;\mathbb{Z}) \).

Therefore an invariant \( \tilde{s}_c \) that is potentially distinct from \( s^F \) can be given only by the third case \( (R,c) = (\mathbb{Z},p) \) for each prime \( p \), or by one in the complementary case:

4. \( \text{char } R = 0 \), \( c \mid p \cdot 1_R \) and \( c \not\sim p \cdot 1_R \) for some prime \( p \in \mathbb{Z} \).

Examples of the fourth case are \( (R,c) = (\mathbb{Z}[i], 1+i) \) where \( 1+i \mid 2 \), or \( (\mathbb{Z}[\omega], 1+\omega) \) where \( \omega = e^{2\pi i/3} \) and \( 1+\omega \mid 3 \). Currently we do not know whether there exists \( \tilde{s}_c \) that is independent from \( s^F \). In Table 1 we list some of the notable results obtained by direct computations performed by a computer program [San22]. The results are summarized as

**Proposition 1.2.** Denote \( \tilde{s}_2 = \tilde{s}_2(-;\mathbb{Z}) \), \( \tilde{s}_3 = \tilde{s}_3(-;\mathbb{Z}) \) and \( \tilde{s}_{1+i} = \tilde{s}_{1+i}(-;\mathbb{Z}[i]) \).
1. There are two prime knots with crossing number 14 such that
\[ s^Q = s^F_3 = \tilde{s}_3 \neq s^F_2 = \tilde{s}_2 = \tilde{s}_{1+i}. \]

2. There are two prime knots with crossing number 18 such that
\[ s^Q = s^F_2 = \tilde{s}_2 \neq s^F_3 = \tilde{s}_3. \]

The former two knots in Table 1 are the first discovered examples satisfying
\[ s^Q \neq s^F_2 \] by Seed [See]. The latter two are those satisfying
\[ s^Q = s^F_2 \neq s^F_3 \] discovered by Schütz in [Sch22]. The above results give rise to the following questions:

**Question 1.3.** Is \( s^F_p = \tilde{s}_p \) for every prime number \( p \)?

**Question 1.4.** Is \( \tilde{s}_p = \tilde{s}_c \) for every prime number \( p \) and every \( c \) such that \( c \mid p \) and \( c \not\sim p \)?

The properties of \( \tilde{s}_c \) are obtained from the fact that the behavior of the Lee classes under Reidemeister moves and cobordisms can be described explicitly (Propositions 2.16 and 2.19). As a byproduct, we get a reformulation of \( \tilde{s}_c \) analogous to (yet more simple than) the one for \( s^Q \) given by Kronheimer–Mrowka in [KM13].

**Proposition 1.5.** Let \( S \) be a connected cobordism from the unknot \( U \) to a knot \( K \). Then
\[ \tilde{s}_c(K) = 2d_c(z) + \chi(S) \]
where \( z \) is the image of 1 under the cobordism map
\[ \phi_S : R = \tilde{H}_c(U) \to \tilde{H}_c(K), \]
\( d_c(z) \) the \( c \)-divisibility of \( z \) (modulo torsion), and \( \chi(S) \) the Euler number of \( S \).
(Restated more precisely in Proposition 4.33.)

Recall that in [KM13] an instanton Floer homology analogue \( s^# \) of \( s \) is introduced. It was originally thought that \( s \) and \( s^# \) are equal, but turned out to be distinct (in fact \( s^# \) is not even additive, see [Gon21]). We expect that our reformulation leads us to a better understanding of the relation between \( s \) and \( s^# \), possibly via the aforementioned invariant \( \tilde{s} \) [Dae+22] which is defined by the divisibility of the ‘special cycles’ in equivariant singular instanton Floer theory.

This paper is organized as follows. In Section 2 we briefly review the basics of Khovanov homology theory, including the construction of the (unreduced) Lee classes and related results obtained in [San20]. In Section 3 we setup the foundation for reduced Khovanov homologies, generalizing the ones given by Khovanov in [Kho03; KWZ19; AZ22]. Here we also introduce the reduced Lee classes and study their behavior under Reidemeister moves and cobordisms. Section 4 is the main part of this paper, where we introduce the \( c \)-divisibility of the reduced Lee class, and derive the link invariant \( \tilde{s}_c \). Then the three main theorems follow easily from the results obtained in the previous sections. In Section 5 we briefly explain how the direct computations were performed, and finally in Section 6 we compare \( \tilde{s}_c \) with the unreduced counterpart \( \bar{s}_c \) of [San20].
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2 Khovanov homology

Throughout this paper we work in the smooth category, and assume that all links and link diagrams are oriented.

2.1 Khovanov homology

Definition 2.1. Let \( R \) be a commutative ring with unity. A Frobenius algebra over \( R \) is a quintuple \((A, m, \iota, \Delta, \varepsilon)\) such that:

1. \((A, m, \iota)\) is an associative \( R \)-algebra with multiplication \( m \) and unit \( \iota \),
2. \((A, \Delta, \varepsilon)\) is a coassociative \( R \)-coalgebra with comultiplication \( \Delta \) and counit \( \varepsilon \),
3. the Frobenius relation holds:
   \[\Delta \circ m = (id \otimes m) \circ (\Delta \otimes id) = (m \otimes id) \circ (id \otimes \Delta).\]

Let \( R \) be a commutative ring with unity, and \( h, t \) be elements in \( R \). Define \( A_{h,t} = R[X]/(X^2 - hX - t) \). \( A_{h,t} \) is endowed a Frobenius algebra structure as follows: the \( R \)-algebra structure is inherited from \( R[X] \). Regarding \( A_{h,t} \) as a free \( R \)-module with basis \( \{1, X\} \), the counit \( \varepsilon : A_{h,t} \to R \) is defined by
\[\varepsilon(1) = 0, \quad \varepsilon(X) = 1.\]

Then the comultiplication \( \Delta \) is uniquely determined so that \((A_{h,t}, m, \iota, \Delta, \varepsilon)\) becomes a Frobenius algebra. Explicitly, the operations \( m \) and \( \Delta \) are given by
\[m(1 \otimes 1) = 1, \quad m(X \otimes 1) = m(1 \otimes X) = X, \quad m(X \otimes X) = hX + t, \quad \Delta(1) = X \otimes 1 + 1 \otimes X - h(1 \otimes 1), \quad \Delta(X) = X \otimes X + t(1 \otimes 1).\]  

Given a link diagram \( D \), a complex \( C_{h,t}(D; R) \) over \( R \) is defined by the construction given in [Kho00], except that the defining Frobenius algebra \( A = R[X]/(X^2) \) is replaced by \( A_{h,t} = R[X]/(X^2 - hX - t) \).

Definition 2.2. The complex \( C_{h,t}(D; R) \) is called the Khovanov complex, and its homology denoted \( H_{h,t}(D; R) \) is called the Khovanov homology of \( D \) (with respect to the triple \((R, h, t)\)).
Recall that Khovanov’s original theory [Kho00] is given by \((h, t) = (0, 0)\), Lee’s theory [Lee05] by \((h, t) = (0, 1)\), and (the filtered version of) Bar-Natan’s theory [Bar05] by \((h, t) = (1, 0)\). The universal one among all triples \((R, h, t)\) is given by \(X^2 - hX - t\) over \(R = \mathbb{Z}[h, t]\), which is called the \(U(2)\)-equivariant theory (see [Kho06; KR22]). The following proposition justifies referring to the isomorphism class of \(H_{h,t}(D)\) as the Khovanov homology of the corresponding link \(L\).

**Proposition 2.3** ([Bar05, Theorem 1], [Kho06, Proposition 6]). Suppose \(D, D’\) are link diagrams related by a Reidemeister move. Then there is a chain homotopy equivalence
\[
\rho: C_{h,t}(D) \rightarrow C_{h,t}(D').
\]

The explicit descriptions of \(\rho\) for the three Reidemeister moves are given in [Bar05, Section 4.3]. We call each of these maps an \(R\)-move map. We introduce a few more terms and notations that will be used throughout this paper.

**Definition 2.4.** A state \(u\) of a diagram \(D\) is an assignment of 0 or 1 to each crossing of \(D\).

When a total ordering of the crossings of \(D\) is given, a state \(u\) is identified with an element \(u \in \{0, 1\}^n\). We denote by \(D(u)\) the crossingless diagram obtained from \(D\) by resolving all crossings accordingly.

**Definition 2.5.** For an arbitrary set \(S\), an \(S\)-enhanced state of a diagram \(D\) is a pair \(x = (u, a)\) such that \(u\) is a state of \(D\) and \(a\) is an assignment of an element in \(S\) to each circle of \(D(u)\).

When \(S\) is a subset of \(A_{h,t}\), an \(S\)-enhanced state is identified with an element of \(C_{h,t}(D)\) by the corresponding tensor product of the elements of \(S\). In particular for \(S = \{1, X\} \subset A_{h,t}\), the set of all \(1X\)-enhanced states of \(D\) forms a basis of \(C_{h,t}(D)\), and is called the standard generators of \(C_{h,t}(D)\).

### 2.2 Lee classes

Lee [Lee05] proved that for any link diagram \(D\), its \(Q\)-Lee homology has dimension \(2^{\#D}\) and is generated by the explicitly constructed Lee classes (or the canonical classes, as called in [Ras10]). Their argument is generalized in [MTV07] and in [Tur20], provided that the Frobenius algebra \(A_{h,t}\) is diagonalizable. Here we give an equivalent condition in terms of the defining quadratic polynomial of \(A_{h,t}\).

**Definition 2.6.** We say \((R, h, t)\) is a factorable triple (or simply is factorable) if the quadratic polynomial \(X^2 - hX - t \in R[X]\) factors as linear polynomials.

Khovanov’s original theory \((h, t) = (0, 0)\), Lee’s theory \((h, t) = (0, 1)\) and Bar-Natan’s theory \((h, t) = (1, 0)\) are examples of such triples. The universal theory among all factorable triples is given by \(X^2 - hX - t = (X - a)(X - b)\) over \(R = \mathbb{Z}[a, b]\), which is called the \(U(1) \times U(1)\)-equivariant theory (see [KR22]).
Figure 1: Coloring the Seifert circles by $a$, $b$.

For the remainder of this section we assume $(R, h, t)$ is factorable. Fix two roots $a, b \in R$ of $X^2 - hX - t$ and put $c = b - a$. Here $(h, t)$ and $(a, b, c)$ are related as

$$h = a + b, \quad t = -ab, \quad c^2 = h^2 + 4t.$$ 

Define two elements in $A_{h, t}$ by

$$X_a = X - a, \quad X_b = X - b.$$ 

Then we have

$$m(X_a \otimes X_a) = cX_a, \quad m(X_b \otimes X_b) = -cX_b, \quad m(X_a \otimes X_b) = m(X_b \otimes X_a) = 0,$$

$$\Delta(X_a) = X_a \otimes X_a, \quad \Delta(X_b) = X_b \otimes X_b.$$ 

Thus the operations $m$ and $\Delta$ are diagonalized in the submodule spanned by $\{X_a, X_b\}$. Moreover, from

$$\begin{pmatrix} X_a & X_b \end{pmatrix} = \begin{pmatrix} 1 & X \end{pmatrix} \begin{pmatrix} -a & -b \\ 1 & 1 \end{pmatrix},$$

we see that $\{X_a, X_b\}$ form a basis of $A_{h, t}$ if and only if $c = b - a$ is invertible in $R$. Thus the Frobenius algebra $A_{h, t}$ is diagonalizable\(^1\) if and only if $(R, h, t)$ is factorable and $c$ is invertible in $R$.

**Definition 2.7.** For a link $L$, let $O(L)$ be the set of all orientations on the underlying unoriented link of $L$. The set $O(D)$ for a link diagram $D$ is defined likewise.

For each $o \in O(L)$, let $L_o$ denote the link $L$ with its orientation replaced by $o$. A diagram $D_o$ is defined likewise for each $o \in O(D)$.

**Algorithm 2.8.** Given a link diagram $D$, the ab-coloring on its Seifert circles are obtained as follows: separate $\mathbb{R}^2$ into regions by the Seifert circles of $D$, and color the regions in the checkerboard fashion, with the unbounded region colored white. For each Seifert circle, let it inherit the orientation from $D$. Assign to it $a$ if it sees a black region to the left, otherwise $b$.

---

\(^1\)As defined in [Tur20], a rank 2 Frobenius algebra is $A$ over $R$ is diagonalizable if there exists a basis $\{e_1, e_2\}$ of $A$ such that $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = 0$. The equivalence (for one direction) can be seen by putting $e_1 = c^{-1}X_a$, $e_2 = -c^{-1}X_b$. 

For any oriented link diagram $D$, there is a unique state $u$ that gives the orientation preserving resolution of $D$, so that $D(u)$ consists of the Seifert circles of $D$. The $ab$-coloring on $D$ determines a unique $X_a X_b$-enhanced state $\alpha(D) \in C_{h,t}(D)$ given by the corresponding tensor products of $X_a$ and $X_b$. Similarly for each $o \in O(D)$, we obtain an $X_a X_b$-enhanced state $\alpha(D,o) \in C_{h,t}(D)$ from the $ab$-coloring on $D_o$.

**Definition 2.9.** The above constructed elements $\alpha(D,o)$ are called the Lee cycles of $D$.

For any diagram $D$ there are $2|D|$ distinct Lee cycles. For convenience we also write $\beta(D,o)$ for $\alpha(D,-o)$. Then $\beta(D,o)$ is obtained by flipping all $X_a$’s and $X_b$’s for the tensor factors of $\alpha(D,o)$.

**Example 2.10.** For the diagram $D$ given in Figure 1, the Lee cycles are given by $\alpha(D,o) = X_b \otimes X_a$ and $\beta(D,o) = X_a \otimes X_b$, where the first factor corresponds to the outer circle.

**Proposition 2.11.** $\alpha(D,o)$ is indeed a cycle, i.e. $d\alpha(D,o) = 0$.

*Proof.* From the procedure of the $ab$-coloring, we see that each crossing connects differently colored Seifert circles of $D_o$, so all outgoing edge maps annihilates $\alpha(D,o)$. \hfill \Box

By abuse of notation, we write $\alpha(D,o)$ and $\beta(D,o)$ for the corresponding homology classes. The following Propositions 2.12 and 2.13 generalize [Lee05, Theorem 4.2], which are proved in [Tur20, Theorem 4.2] and in [San20, Proposition 2.9].

**Proposition 2.12.** If $c$ is invertible, then $H_{h,t}(D; R)$ is freely generated over $R$ by the Lee classes. In particular $H_{h,t}(D; R)$ has rank $2|D|$.

**Proposition 2.13.** Let $D$ be an $l$-component link diagram and $D_1, \cdots, D_l$ be the component diagrams. For any orientation $o$ on the underlying unoriented diagram of $D$, let $I \subset \{1, \ldots, l\}$ be the set of indices $i$ such that $o$ is opposite to the given orientation on $D_i$. The homological grading of $\alpha(D,o)$ is given by

$$
gr_h(\alpha(D,o)) = 2 \sum_{i \in I, j \notin I} \text{lk}(D_i, D_j),$$

where $\text{lk}$ denotes the linking number. In particular, $\gr_h(\alpha(D)) = \gr_h(\beta(D)) = 0$.

Thus when $c$ is invertible the graded module structure of $H_{h,t}(D; R)$ is completely known. This is the case for $\mathbb{Q}$-Lee homology ($c = 2$) and $\mathbb{Z}$-Bar-Natan homology ($c = 1$). We will see in Section 2.3 that, even if $c$ is not invertible, the graded module structure of $H_{h,t}(D; R)$ is determined by $c$.

Finally we state the variances of the Lee classes under the Reidemeister moves and cobordisms. The following Proposition 2.16 was the key to defining the link invariant $\tilde{s}_c$ in [San20], which in particular implies that the Lee classes are not invariant under the Reidemeister moves.

---

In [Ras10] the element $\alpha(D,o)$ is denoted $\tilde{s}_o$. 

8
Definition 2.14. For any unary function \( f \), its difference function \( \delta f \) is defined by
\[
\delta f(x, y) = f(y) - f(x).
\]

Definition 2.15. For an oriented link diagram \( D \), let \( w(D) \) denote the writhe of \( D \) and \( r(D) \) denote the number of Seifert circles of \( D \).

Proposition 2.16 ([San20, Proposition 2.13]). Suppose \( D, D' \) are related by a single Reidemeister move. The corresponding isomorphism \( \rho \) maps the Lee classes as
\[
\begin{align*}
\alpha(D) & \xrightarrow{\rho} \varepsilon c^j \alpha(D'), \\
\beta(D) & \xrightarrow{\rho} \varepsilon' c^j \beta(D').
\end{align*}
\]
Here \( j \in \{-1, 0, 1\} \) given by
\[
 j = \frac{\delta w(D, D') - \delta r(D, D')}{2}
\]
and \( \varepsilon, \varepsilon' \in \{\pm 1\} \) are signs satisfying
\[
\varepsilon \varepsilon' = (-1)^j.
\]

Remark 2.17. Here it is not assumed that \( c \) is invertible. When \( j < 0 \), equations (2.3) should be understood as
\[
\alpha(D') \xrightarrow{\rho^{-1}} \varepsilon c^{-j} \alpha(D),
\]
\[
\beta(D') \xrightarrow{\rho^{-1}} \varepsilon' c^{-j} \beta(D).
\]

Remark 2.18. For each orientation \( o \in O(D) \) and the corresponding orientation \( o' \in O(D') \), the relations given in Proposition 2.16 also hold between \( \alpha(D, o) \in H(D) \) and \( \alpha(D', o') \in H(D') \) (with \( j, \varepsilon, \varepsilon' \) depending on \( (o, o') \)). This is because \( C(D) \) and \( C(D_o) \) only differ by some bigrading shift, and the cycles and the map \( \rho \) correspond relevantly.

The proof of [San20, Proposition 2.13] is based on the element-wise description of \( \rho \). We give a more simple proof in Appendix A, based on the diagrammatic description of \( \rho \).

Proposition 2.19 ([San20, Proposition 3.17]). Suppose \( (R, h, t) \) is factorable and \( c \) is invertible. Let \( S \) be an oriented cobordism between links \( L, L' \) that has no closed components. Let \( D, D' \) be the diagrams of \( L, L' \) respectively, and \( \phi \) be the cobordism map corresponding to \( S \)
\[
\phi : H_{h,t}(D; R) \to H_{h,t}(D'; R).
\]
Then \( \phi \) maps the Lee classes as
\[
\begin{align*}
\alpha(D) & \xrightarrow{\phi} \varepsilon c^j \alpha(D') + \cdots, \\
\beta(D) & \xrightarrow{\phi} \varepsilon' c^j \beta(D') + \cdots.
\end{align*}
\]
where $j \in \mathbb{Z}$ is given by

$$j = \frac{\delta w(D, D') - \delta r(D, D') - \chi(S)}{2}$$

and $\varepsilon, \varepsilon' \in \{\pm 1\}$ are signs satisfying

$$\varepsilon \varepsilon' = (-1)^j.$$

Moreover if every component of $S$ has a boundary in $L$, then the $(\cdots)$ terms vanish.

### 2.3 Reduction of parameters

Here we continue to assume that $(R, h, t)$ is factorable. It will be convenient to consider another basis $\{1, X_a\}$ for $A_{h,t}$, so that the operations on $A_{h,t}$ are described as

$$m(1 \otimes 1) = 1, \quad m(X_a \otimes 1) = m(1 \otimes X_a) = X_a, \quad m(X_a \otimes X_a) = cX_a,$$

$$\Delta(1) = X_a \otimes 1 + 1 \otimes X_a - c(1 \otimes 1), \quad \Delta(X_a) = X_a \otimes X_a,$$

$$\iota(1) = 1, \quad \varepsilon(1) = 0, \quad \varepsilon(X_a) = 1. \tag{2.4}$$

**Definition 2.20.** Let $A$ be a Frobenius algebra and $\theta$ an invertible element in $A$. The twisting $A_\theta$ of $A$ by $\theta$ is another Frobenius algebra $(A, m, \iota, \Delta_\theta, \varepsilon_\theta)$ with the same algebra structure as $A$ but with a different coalgebra structure given by

$$\Delta_\theta(x) = \Delta(\theta^{-1} x), \quad \varepsilon_\theta(x) = \varepsilon(\theta x).$$

**Lemma 2.21.** Suppose $(R, h, t)$, $(R, h', t')$ are both factorable, and that $c = \sqrt{h^2 + 4t}$ and $c' = \sqrt{h'^2 + 4t'}$ are related as $c' = \theta c$ for some invertible $\theta \in R$. Then there is a Frobenius algebra isomorphism

$$\psi : A_{h,t} \longrightarrow A_{h',t',\theta}$$

mapping

$$X_a \mapsto \theta^{-1} X_{a'}, \quad X_b \mapsto \theta^{-1} X_{b'}.$$

Here $A_{h',t',\theta}$ denotes the $\theta$-twisting of $A_{h',t'}$. Moreover these maps satisfy the cocycle condition, i.e. the following diagram consisting of the above described maps commute.

$$\begin{array}{ccc}
A_{h,t} & \xrightarrow{\psi''} & A_{h'',t'',\theta''} \\
\downarrow \psi & & \downarrow \psi' \\
A_{h',t',\theta} & & \end{array} \tag{2.5}
$$

**Proof.** Define a ring isomorphism

$$\psi : R[X] \rightarrow R[X], \quad X \mapsto \theta^{-1}(X - a') + a.$$

Using (2.4) it is easy to check that $\psi$ induces the desired Frobenius algebra isomorphism. \qed
**Proposition 2.22.** Suppose \((R, h, t), (R', h', t')\) satisfy the condition of Lemma 2.21. Then for any link diagram \(D\), there is a chain isomorphism 

\[ \psi : C_{h,t}(D; R) \to C_{h',t'}(D; R). \]

Moreover when \(\theta = 1\), the above map is natural with respect to \(D\), i.e. if two diagrams \(D, D'\) are related by a single Reidemeister move, the following diagram commutes.

\[
\begin{array}{ccc}
C_{h,t}(D) & \xrightarrow{\psi} & C_{h',t'}(D) \\
\downarrow \rho & & \downarrow \rho \\
C_{h,t}(D') & \xrightarrow{\psi} & C_{h',t'}(D')
\end{array}
\]

Here \(\rho\) is the corresponding R-move map of Proposition 2.3.

**Proof.** Let \(C_{h',t';\theta}(\cdot; R)\) denote the Khovanov complex corresponding to the Frobenius algebra \(A_{h',t';\theta}\). The Frobenius algebra isomorphism \(\psi\) of Lemma 2.21 induces a chain isomorphism \(C_{h,t}(D; R) \to C_{h',t'}(D; R)\). Postcomposing the chain isomorphism \(C_{h',t';\theta}(D; R) \to C_{h',t'}(D; R)\) corresponding to the \(\theta\)-twisting (see [Kho06, Proposition 3]) gives the desired chain isomorphism \(\psi\). Naturality follows from the explicit definition of \(\psi\) and \(\rho\), together with the identity

\[ X \otimes 1 - 1 \otimes X = X_a \otimes 1 - 1 \otimes X_a \]

for the case of R1-move.

**Corollary 2.23.** If \((R, h, t)\) is factorable, then \(C_{h,t}(D; R)\) is isomorphic to \(C_{c,0}(D; R)\). If in addition \(c/2 \in R\), it is isomorphic to \(C_{0,(c/2)^2}(D; R)\).

**Corollary 2.24.** \(C_{Kh}(\cdot; \mathbb{F}_2) \cong C_{Lee}(\cdot; \mathbb{F}_2)\).

Proposition 2.22 implies that \(c\) determines the isomorphism class of \(C_{h,t}(D; R)\). The isomorphism class can be visualized by the hyperbola \(h^2 + 4t = c^2\) on the \(ht\)-coordinate space as in Figure 2.
Proposition 2.25. Under the assumption of Proposition 2.22, the chain isomorphism \( \psi \) maps the Lee cycles of \( C_{h,t}(D; R) \) to that of \( C'_{h',t'}(D; R) \) multiplied by a power of \( \theta \). In particular when \( \theta = 1 \), the Lee cycles of \( D \) and \( D' \) correspond exactly.

Proof. \( \psi \) maps \( X_a \) to \( \theta^{-1}X'_a \) and \( X_b \) to \( \theta^{-1}X'_b \), and the \( \theta \)-twisting are given by vertex-wise multiplications of powers of \( \theta \).

Thus when considering Lee classes and its behavior under Reidemeister moves and cobordisms, it suffices to consider the case \( h = c \) and \( t = 0 \). We occasionally denote \( H_c(D) \) for \( H_{c,0}(D) \). The universal theory among all such triples is given by \( X^2 -HX = X(X-H) \) over \( R = \mathbb{Z}[H] \), which is the bigraded Bar-Natan theory, or what is called the \( U(1) \)-equivariant theory (see [Kho06; KR22]).

3 Module structures and reduced homology

A module structure on Khovanov homology was first defined in [Kho03], and on other variants in [HN13; Ali19; AD19]. The reduced version of Khovanov homology was also defined by Khovanov in [Kho03], and for other variants, the reduced Bar-Natan homology is given by Kotelskiy–Watson–Zibrowius in [KWZ19] and the reduced \( U(1) \times U(1) \)-equivariant Khovanov homology by Akhmechet–Zhang in [AZ22]. Here we generalize these structures for a general triple \((R, h, t)\).

3.1 Module structures

Definition 3.1. A pointed link \((L, p)\) is a link \( L \) with a marked point \( p \in L \). A pointed link diagram \((D, p)\) is defined likewise, where the point \( p \) lies on an arc of \( D \).

Let \((D, p)\) be a pointed link diagram. First we define an endomorphism \( x_p \) on \( C_{h,t}(D) \) as follows: Take a small circle \( \bigcirc \) near \( p \). Merging \( \bigcirc \) into a neighborhood of \( p \) corresponds to the multiplication

\[
m_p : A_{h,t} \otimes C_{h,t}(D) \to C_{h,t}(D).
\]

Define

\[
x_p = m_p(X \otimes -) : C_{h,t}(D) \to C_{h,t}(D).
\]

The following proposition is a generalization of [HN13, Lemma 2.3], [Ali19, Lemma 2.1] and [AD19, Lemma 2.1] and [AD19, Lemma 2.1].

Proposition 3.2. Suppose \( p, q \) are two marked points on \( D \) separated by a crossing \( c \). Then \( x_p \) and \( h - x_q \) are chain homotopic.

Proof. Let \( D_0, D_1 \) be the diagrams obtained from \( D \) by 0-, 1-resolving the crossing \( c \) respectively. There are chain maps between \( C(D_0) \) and \( C(D_1) \) corresponding to the saddle moves in both ways

\[
C(D_0) \xrightarrow{f} C(D_1) \xleftarrow{g} C(D_0).
\]
We may view $C(D)$ as the mapping cone of $f$ with differential
\[ d = \begin{pmatrix} -d_0 & 0 \\ f & d_1 \end{pmatrix}. \]
We claim that $x_p + x_q - h$ are null homotopic by the chain homotopy
\[ H = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}. \]
First we have
\[ dH + Hd = \begin{pmatrix} gf & -d_0g + gd_1 \\ 0 & fg \end{pmatrix} = \begin{pmatrix} gf & 0 \\ 0 & fg \end{pmatrix}. \]
Take any state $u$ and focus on the $u$-summand of $C(D)$. Here we assume $u(c) = 0$ since the proof for the other case is identical. In this case it suffices to prove that
\[ gf = x_p + x_q - h. \]
If the points $p, q$ belong to the same circle of $D(u)$, then $gf$ is a merge-after-split, while $x_p, x_q$ are both multiplication by $X$ on the corresponding tensor multipicand, so
\[ gf = m \Delta = 2X - h. \]
Otherwise if $p, q$ belong to different circles of $D(u)$, then $gf$ is a split-after-merge and
\[ gf = \Delta m = x_p + x_q - h \]
can be checked directly (or by the neck-cutting relation of [KR22, Equation 13]).

Remark 3.3. In particular if $h = 2 = 0$ in $R$, which is the case for $\mathbb{F}_2$-Khovanov homology, it follows that $x_p$ and $x_q$ are chain homotopic, which is the case proved in [HN13].

In view of Proposition 3.2 we define an endomorphism $U_p$ on $C_{h,t}(D)$ as follows: color the arcs of $D$ according to Algorithm 2.8, and define
\[ U_p = \begin{cases} x_p & \text{if } p \text{ is colored } a, \\ h - x_p & \text{if } p \text{ is colored } b. \end{cases} \]
Obviously $U_p$ commutes with the differential $d$, and $U_p^2 = hU_p + t$ holds. Regarding $A_{h,t} = R[U]/(U^2 - hU - t)$, we obtain an $A_{h,t}$-module structure on $C_{h,t}(D)$ and on $H_{h,t}(D)$. Proposition 3.2 implies that the module structure on $H_{h,t}(D)$ only depends on the component on which $p$ lies. Moreover,

Proposition 3.4. Suppose $D, D'$ are pointed link diagrams related by a Reidemeister move that does not contain the marked points in the changing disk. Then the corresponding R-move map $\rho$ commutes with $U_p$.  

13
Proof. Consider the diagram $\tilde{D}$ obtained from $D$ by adding a positive twist near $p$. We may regard $C(\tilde{D})$ as the mapping cone of

$$m_p : A \otimes C(D) \to C(D).$$

Consider the similar diagram for $D'$, then the corresponding R-move map is given by

$$\left( \begin{array}{cc}
1 \otimes \rho & 0 \\
0 & \rho
\end{array} \right).$$

From the fact that it is a chain map, we get

$$m_p \circ (1 \otimes \rho) = \rho \circ m_p$$

and hence the desired result. (Alternatively, in view of [Bar05; KR22], we may regard $x_p$ as a cobordism that merges a dotted cup near $p$. The R-move map $\rho$ is also represented by a cobordism, and the composition of the two cobordisms is obviously commutative.)

Proposition 3.5. Let $D$ be a pointed link diagram. Suppose $(R, h, t), (R, h', t')$ are both factorable with $c = c'$. Then with the isomorphism $\psi$ of Proposition 2.22 the following diagram commutes

$$
\begin{array}{ccc}
A_{h,t} \otimes C_{h,t}(D) & \xrightarrow{m_p} & C_{h,t}(D) \\
\downarrow \psi \otimes \psi & & \downarrow \psi \\
A_{h',t'} \otimes C_{h',t'}(D) & \xrightarrow{m_p} & C_{h',t'}(D)
\end{array}
$$

Proof. $\psi$ is a Frobenius algebra isomorphism.

Thus we conclude that, for a pointed link $L$, there is an $A_{h,t}$-module structure on $H_{h,t}(L)$ whose isomorphism class only depends on $c$ and on the component on which the marked point $p$ lies.

3.2 Cobordism maps

Here we consider cobordisms between pointed links, and define cobordism maps between the corresponding chain complexes as $A_{h,t}$-module homomorphisms. First we define an explicit chain homotopy equivalence that represents the change of marked points. We follow the standard argument given in [Kho03], which is also used for other variants in [KWZ19; AZ22].

Proposition 3.6. Suppose $D, D'$ are diagrams (in $\mathbb{R}^2$) related by an isotopy in $S^2$ that passes an arc ‘through infinity’. Then there is a chain isomorphism

$$I : C_{h,t}(D) \to C_{h,t}(D')$$

such that the Lee cycles of the two diagrams correspond exactly (up to sign). Moreover if $p, p'$ are marked points of $D, D'$ related by the isotopy, then

$$I \circ U_p = U_{p'} \circ I.$$
Proof. There is an obvious isomorphism
\[ I_1 : C(D) \to C(D') \]
that maps the enhanced states identically to the corresponding ones. Consider another chain isomorphism \( I_2 \) induced from the Frobenius algebra isomorphism, induced from the ring isomorphism
\[ R[X] \to R[X], \quad X \mapsto h - X. \]
Note that \( I_2 \) exchanges \( X_a \) and \( X_b \) up to sign. Define \( I = I_2 \circ I_1 \). One sees from Algorithm 2.8 that the \( ab \)-colorings of the Seifert circles of \( D \) gets flipped as an arc passes through infinity. This effect is taken care by \( I_2 \) and hence \( I \) commutes with \( U \). The latter statement is also obvious since \( p \) and \( p' \) are colored differently.

**Proposition 3.7.** Suppose \( p,q \) are two marked points on \( D \) separated by a crossing. Then there is a self-chain homotopy equivalence \( \tau_{p,q} \) on \( \mathcal{C}_{h,t}(D) \) such that
\[ \tau_{p,q} \circ U_p = U_q \circ \tau_{p,q}. \]

**Proof.** Instead of moving the marked point over or under the strand, the desired move can be realized by a sequence of planer isotopies and Reidemeister moves as follows: pull the strand that separates the two points to the outermost region in \( \mathbb{R}^3 \). Pass this arc through infinity, and then slide it back close to the original position so that the resulting diagram is identical to the original one, except that the marked point \( p \) is moved to \( q \). The desired map \( \tau_{p,q} \) is given by the composition of the corresponding R-move maps, with the isomorphism \( I \) of Proposition 3.6 placed in between. Commutativity with \( U \) follows from Propositions 3.4 and 3.6.

**Definition 3.8.** A **cobordism** between pointed links \( L,L' \) in \( \mathbb{R}^3 \) is a compact oriented surface in \( \mathbb{R}^3 \times I \) with boundary \( -L \times \{0\} \cup L' \times \{1\} \), together with an embedded curve \( \gamma \subset S \) that connects the marked points of \( L,L' \).

Given a cobordism \( S \) as above, we may isotope \( S \) (rel boundary, together with \( \gamma \)) so that \( \gamma \) intersects each slice \( \mathbb{R}^3 \times \{t\} \) at a single point and that \( S \) can be represented by a finite sequence of local moves between pointed link diagrams, each of which is either (i) a Reidemeister move, (ii) a Morse move or (iii) a marked point crossing move. This induces a homomorphism
\[ \phi : \mathcal{C}_{h,t}(D) \to \mathcal{C}_{h,t}(D') \]
by the composition of corresponding homomorphisms.

**Proposition 3.9.** The above constructed \( \phi \) is an \( \mathcal{A}_{h,t} \)-module homomorphism.

**Proof.** That \( U \) commutes with the R-move maps and \( \tau_{p,q} \) is already proved. Analogous result for the Morse moves also holds.

**Remark 3.10.** Although not necessary in this paper, it is interesting to ask whether \( \phi \) is independent (up to sign) of the choice of the isotopy of \( S \).
3.3 Reduced homology

For a pointed link diagram \((D, p)\), let \((X_a)_p C_{h,t}(D)\) denote the subcomplex of \(C_{h,t}(D)\) which is generated by the \(1X_a\)-enhanced states whose tensor factor corresponding to the marked circle is restricted to \(X_a\). This is indeed a subcomplex since the operations involving the marked circles are given by

\[
X_a \cdot 1 = X_a, \quad X_a \cdot X = bX_a, \quad \Delta X_a = X_a \otimes X_a.
\]

Here the underline indicates the factor corresponding to the marked circle. We also declare \(\text{deg}(X_a) = 0\) so that the subcomplex inherits the quantum grading (which does not agree with the one on \(C_{h,t}(D)\)). The subcomplex \((X_b)_p C_{h,t}(D)\) is defined similarly.

**Definition 3.11.** The reduced Khovanov complex \(\tilde{C}^\pm_{h,t}(D)\) of a pointed link \(D\) is defined as follows: color the arcs of \(D\) according to Algorithm 2.8, and define

\[
\tilde{C}^+_h(D) = \begin{cases} (X_a)_p C_{h,t}(D) & \text{if } p \text{ is colored } a, \\ (X_b)_p C_{h,t}(D) & \text{if } p \text{ is colored } b \end{cases}
\]

and

\[
\tilde{C}^-_{h,t}(D) = \begin{cases} (X_b)_p C_{h,t}(D) & \text{if } p \text{ is colored } a, \\ (X_a)_p C_{h,t}(D) & \text{if } p \text{ is colored } b. \end{cases}
\]

The corresponding homology groups are denoted \(\tilde{H}^\pm_{h,t}(D)\).

With the endomorphism \(U_p\) given in Section 3.1, we may alternatively define

\[
\tilde{C}^+_h(D) = \text{Im}(U_p - a), \quad \tilde{C}^-_{h,t}(D) = \text{Im}(U_p - b).
\]

**Proposition 3.12.** There is a bigrading preserving involution \(I\) on \(C_{h,t}(D)\) that maps \(C^+_h(D)\) isomorphically onto \(C^-_{h,t}(D)\).

**Proof.** Let \(I\) be the chain isomorphism \(\psi\) of Proposition 2.22 for the case \(\theta = -1\). The statement can be seen from the fact that \(I\) is induced from the ring isomorphism \(X \mapsto h - X\). \qed

**Proposition 3.13.** There is a short exact sequence of complexes

\[
0 \to \tilde{C}^+_h(D) \xrightarrow{i} C_{h,t}(D) \xrightarrow{\pi} \tilde{C}^-_{h,t}(D) \to 0.
\]

Moreover this sequence splits when \(c\) is invertible in \(R\).

To prove this, we first give alternative descriptions of \(C^\pm_{h,t}(D)\). Define a quotient complex

\[
C_{h,t}(D)/(X_a)_p = C_{h,t}(D)/(X_a)_p C_{h,t}(D).
\]
We may regard $C_{h,t}(D)/(X_a)_p$ as a chain complex generated by the enhanced states whose tensor factor corresponding to the marked circle is 1. The operations involving the marked circle are given by

$$1 \cdot 1 = 1, \quad 1 \cdot X = a1, \quad \Delta 1 = 1 \otimes X_b.$$ 

We declare $\deg(1) = 0$ so that the quotient complex inherits the quantum grading. Similarly define $C_{h,t}(D)/(X_b)_p$.

**Proposition 3.14.** There are bigraded preserving chain isomorphisms

$$(X_a)_p C_{h,t}(D) \cong C_{h,t}(D)/(X_b)_p,$$

$$(X_b)_p C_{h,t}(D) \cong C_{h,t}(D)/(X_a)_p.$$ 

**Proof.** The first isomorphism is given by

$$X_a \otimes x \mapsto 1 \otimes x$$

and similarly for the second. □

**Proof of Proposition 3.13.** We assume that $p$ is colored $a$. There is a short exact sequence

$$0 \to (X_a)_p C(D) \xrightarrow{i_a} C(D) \xrightarrow{\pi_a} C(D)/(X_a)_p \cong (X_b)_p C(D) \to 0.$$ 

The other inclusion

$$i_b : (X_b)_p C(D) \hookrightarrow C(D)$$

postcomposed with $\pi_a$ is the multiplication by $a - b = -c$. Thus when $c$ is invertible, $-c^{-1}i_b$ gives a splitting. □

**Corollary 3.15.** If $c$ is invertible, $\tilde{H}^\pm_{h,t}(D)$ is a free $R$-module of rank $2|D| - 1$. □

Next we prove that the isomorphism class of the reduced homology is an invariant of pointed links.

**Proposition 3.16.** Suppose $D, D'$ are pointed link diagrams related by a Reidemeister move that does not contain the marked point in the changing disk. Then the corresponding R-move map $\rho$ induces chain homotopy equivalences

$$\rho : \tilde{C}^\pm_{h,t}(D) \to \tilde{C}^\pm_{h,t}(D').$$

**Proof.** From the explicit descriptions of the R-move maps the corresponding chain homotopies given in [Bar05, Section 4.3] we see that the $ab$-coloring on the marked circles remains unchanged by these maps. □

**Proposition 3.17.** Suppose $D, D'$ are pointed link diagrams related by a Morse move that does not contain the marked point in the changing disk. Then the corresponding map $\rho$ induces homomorphisms

$$\rho : \tilde{C}^\pm_{h,t}(D) \to \tilde{C}^\pm_{h,t}(D').$$
Proof. Similar to the proof of Proposition 3.16.

**Proposition 3.18.** Suppose $p,q$ are two marked points on $D$ separated by a crossing. Then the chain homotopy equivalence $\tau_{p,q}$ of Proposition 3.7 induces chain homotopy equivalences

$$\tau_{p,q} : \tilde{C}_{h,t}^\pm(D,p) \to \tilde{C}_{h,t}^\pm(D,q).$$

Proof. That $\tau_{p,q}$ induces homomorphisms between the reduced complexes can be seen by the commutative diagram

$$\begin{array}{ccc}
C(D) & \xrightarrow{U_{\theta} - \eta} & \tilde{C}^+(D,p) \\
\downarrow{\tau_{p,q}} & & \downarrow{\tau_{p,q}} \\
C(D) & \xrightarrow{U_{\theta} - \eta} & \tilde{C}^+(D,q)
\end{array}$$

and similarly for $\tilde{C}^-$. That $\tau_{p,q}$ is a chain homotopy equivalence can be seen from the construction together with Proposition 3.16.

**Proposition 3.19.** If $(R,h,t), (R,h',t')$ are both factorable with $c = c'$, then the isomorphism $\psi$ of Proposition 2.22 induces chain isomorphisms

$$\psi : \tilde{C}_{h,t}^\pm(D) \to \tilde{C}_{h',t'}^\pm(D').$$

Proof. Immediate from the observation given in the proof of Proposition 2.25 with $\theta = 1$.

Assembling the above obtained results, we conclude that the reduced Khovanov homology $\tilde{H}_{h,t}^\pm(L)$ of a pointed link $L$ is well-defined, whose isomorphism class depends only on $c$ and on the component on which the marked point lies. In particular the reduced Khovanov homology of an (unpointed) knot $K$ is well-defined. We also have cobordism maps between the reduced complexes.

**Proposition 3.20.** Suppose $S$ is a cobordism between pointed links $L,L'$. The cobordism map

$$\phi : C_{h,t}(D) \to C_{h,t}(D')$$

given in Section 3.2 restricts to the reduced complexes

$$\phi : \tilde{C}_{h,t}^\pm(D) \to \tilde{C}_{h,t}^\pm(D').$$

Proof. Immediate from Proposition 3.9 and Propositions 3.16 to 3.18.

In the following we only consider $\tilde{C}^+$ and omit the $+$ symbol when there is no need to consider both complexes at the same time. Finally we remark that the unreduced theory can be recovered from the reduced theory.

**Definition 3.21.** Given an unpointed link $L$, a pointed link $L_+$ is defined by adding a disjoint pointed unknot to $L$. For an unpointed link diagram $D$, a pointed link diagram $D_+$ is defined likewise, with the added circle oriented counterclockwise.
Proposition 3.22. There is an isomorphism
\[ C_{h,t}(D) \cong \tilde{C}_{h,t}(D_+) \]
given by the correspondence \( x \mapsto X_o \otimes x \).

3.4 Reduced Lee classes

Definition 3.23. Define \( \tilde{O}(D) \) as the subset of \( O(D) \) consisting of orientations \( o \) whose color at \( p \) by the \( ab \)-coloring of Algorithm 2.8 coincides with the one obtained from the given orientation of \( D \).

Definition 3.24. For each \( o \in \tilde{O}(D) \) define the reduced Lee cycle
\[ \tilde{\alpha}(D,o) \in \tilde{C}^+_{h,t}(D) \]
by the preimage of the Lee cycle \( \alpha(D,o) \in C_{h,t}(D) \) under the inclusion \( \tilde{C}^+_{h,t}(D) \hookrightarrow C_{h,t}(D) \). Similarly define
\[ \tilde{\beta}(D,o) \in \tilde{C}^-_{h,t}(D) \]
by the preimage of \( \beta(D,o) = \alpha(D,-o) \in C_{h,t}(D) \) under the inclusion \( \tilde{C}^-_{h,t}(D) \hookrightarrow C_{h,t}(D) \).

Proposition 3.25. If \( c \) is invertible, then \( \tilde{H}^+_{h,t}(D) \) is freely generated by the classes \( \tilde{\alpha}(D,o) \) and \( \tilde{H}^-_{h,t}(D) \) is freely generated by the classes \( \tilde{\beta}(D,o) \) over \( R \).

Proof. Immediate from Propositions 2.12 and 3.13. \( \square \)

Corollary 3.26. If \( c \) is invertible and \( D \) is a knot diagram, \( \tilde{H}_{h,t}(D) \) is freely generated by the single class \( \tilde{\alpha}(D) \).

The following two propositions are reduced versions of Propositions 2.16 and 2.19.

Proposition 3.27. Suppose \( D, D' \) are pointed link diagrams related by a Reidemeister move that does not contain the marked point in the changing disk. Then the corresponding R-move map \( \rho \) of Proposition 3.16 sends the reduced Lee classes as
\[ \tilde{\alpha}(D) \xrightarrow{\rho} \varepsilon c^j \tilde{\alpha}(D'), \]
\[ \tilde{\beta}(D) \xrightarrow{\rho} \varepsilon' c^j \tilde{\beta}(D'). \]

Here \( j \in \{-1, 0, 1\} \) given by
\[ j = \frac{\delta w(D, D') - \delta r(D, D')}{2} \]
and \( \varepsilon, \varepsilon' \in \{\pm 1\} \) are signs satisfying
\[ \varepsilon \varepsilon' = (-1)^j. \]
Proof. The proof of Proposition 2.16 works without change (see Appendix A). □

Proposition 3.28. Suppose $(R, h, t)$ is factorable and $c$ is invertible. Let $S$ be an oriented cobordism between pointed links $L, L'$ that has no closed components. The corresponding cobordism map $\phi$ of Proposition 3.20 sends the reduced Lee classes as

$$
\hat{\alpha}(D) \xrightarrow{\phi} \varepsilon c^j \hat{\alpha}(D') + \cdots,
$$
$$
\hat{\beta}(D) \xrightarrow{\phi} \varepsilon' c^j \hat{\beta}(D') + \cdots
$$

where $j \in \mathbb{Z}$ is given by

$$
j = \frac{\delta w(D, D') - \delta r(D, D') - \chi(S)}{2}
$$

and $\varepsilon, \varepsilon' \in \{\pm 1\}$ are signs satisfying

$$
\varepsilon \varepsilon' = (-1)^j.
$$

Moreover if every component of $S$ has a boundary in $L$, then the $(\cdots)$ terms vanish.

Proof. Again the proof of Proposition 2.19 works without change, except that we need to consider the effect of $\tau_{p,q}$, which in fact needs no care since it is defined by a composition of R-move maps and the map $I$ of Proposition 3.6. □

Proposition 3.29. Under the identification given in Proposition 3.22, the unreduced and the reduced Lee classes correspond as

$$
\alpha(D, o) = \hat{\alpha}(D_+, o_+)
$$

for each $o \in \hat{\mathcal{O}}(D)$. □

3.5 Connected sums and mirrors

3.5.1 Connected sums

Proposition 3.30. Suppose $D, D'$ are pointed link diagrams. Then there is a chain isomorphism

$$
\check{C}_{h,t}(D \# D') \cong \check{C}_{h,t}(D) \otimes_R \check{C}_{h,t}(D').
$$

Under this identification, the Lee cycles correspond as

$$
\hat{\alpha}(D \# D', o \# o') = \hat{\alpha}(D, o) \otimes \hat{\alpha}(D', o')
$$

for any $o \in \hat{\mathcal{O}}(D)$ and $o' \in \hat{\mathcal{O}}(D')$. Here it is assumed that, both $D, D'$ have the marked points on outermost arcs, and that $D \# D'$ can be realized by a surgery along a coherently oriented untwisted band that connects the two marked points.
Proof. From the assumption, the two marked points are colored the same by the \(ab\)-colorings of \(D\) and \(D'\). The isomorphism is given by mapping \(1 \otimes x \otimes y \mapsto (1 \otimes x) \otimes (1 \otimes y)\).

Here we used the quotient description of the reduced complexes. \(\square\)

Remark 3.31. For unpointed link diagrams \(D, D'\), we have \((D \sqcup D')_+ = D_+ \# D'_+\) and hence obtain the well known formula

\[C_{h,t}(D \sqcup D') \cong C_{h,t}(D) \otimes_R C_{h,t}(D')\]

and

\[\alpha(D \sqcup D') = \alpha(D) \otimes \alpha(D').\]

3.5.2 Mirrors

For any Frobenius algebra \(A = (A, m, \iota, \Delta, \varepsilon)\) over \(R\), its dual Frobenius algebra is defined by \(A^* = (A^*, \Delta^*, \varepsilon^*, m^*, \iota^*)\) where \(A^* = \text{Hom}_R(A, R)\) and other maps are the dual maps. For a link diagram \(D\), let \(D^*\) denote its mirror.

Lemma 3.32. There is a Frobenius algebra isomorphism

\(\varphi : A_{h,t} \rightarrow A_{h,t}^*\)

given by

\[\varphi(1) = X^*, \quad \varphi(X) = 1^* + hX^*\]

where \(\{1^*, X^*\}\) is the basis of \(A_{h,t}^*\) dual to the basis \(\{1, X\}\) for \(A_{h,t}\).

Proof. The desired \(\varphi\) is given by the composition of two isomorphisms: first we have \(A_{h,t} \cong A_{-h,t}^*\) by the correspondence

\[1 \mapsto X^*, \quad X \mapsto 1^*.\]

Next the ring isomorphism

\(R[X] \rightarrow R[X]; \quad X \mapsto X + h\)

induces \(A_{h,t} \cong A_{-h,t}\) and hence \(A_{h,t}^* \cong A_{h,t}^*\) by

\[1^* \mapsto 1^* + hX^*, \quad X^* \mapsto X^*\].

\(\square\)

Proposition 3.33. The isomorphism \(\varphi\) of Lemma 3.32 induces a chain isomorphism

\[C_{h,t}(D^*) \cong C_{h,t}(D)^*.\]

This gives a perfect pairing

\[\langle \cdot, \cdot \rangle : C_{h,t}(D) \otimes C_{h,t}(D^*) \rightarrow R\]

defined by

\[\langle z, w \rangle = \langle z, \varphi(w) \rangle_{\text{std}}\]

where the right hand side \(\langle \cdot, \cdot \rangle_{\text{std}}\) is the standard pairing between \(C\) and \(C^*\).
Proof. The composition of chain isomorphisms

\[ C_{h,t}(D^*) \cong C_{-h,t}(D)^* \cong C_{h,t}(D)^* \]

is realized by applying \( \varphi \) to the tensor factors. \( \square \)

Now assume \((R, h, t)\) is factorable.

**Lemma 3.34.** The isomorphism \( \varphi \) of Lemma 3.32 maps

\[ \varphi(X_a) = 1^* + bX^*, \quad \varphi(X_b) = 1^* + aX^* \]

and

\[ \langle X_a, \varphi(X_a) \rangle = c, \quad \langle X_b, \varphi(X_b) \rangle = -c, \]

\[ \langle X_a, \varphi(X_b) \rangle = \langle X_b, \varphi(X_a) \rangle = 0. \]

\( \square \)

**Proposition 3.35.** The Lee cycles of \( D \) and \( D^* \) pair as

\[ \langle \alpha(D), \alpha(D^*) \rangle = \varepsilon \varepsilon' r, \quad \langle \beta(D), \beta(D^*) \rangle = \varepsilon' \varepsilon r, \]

\[ \langle \alpha(D), \beta(D^*) \rangle = \langle \beta(D), \alpha(D^*) \rangle = 0 \]

where \( r = r(D) \) and \( \varepsilon, \varepsilon' \in \{ \pm 1 \} \) are signs such that \( \varepsilon \varepsilon' = (-1)^r \).

Proof. Immediate from Lemma 3.34 together with the observation that the Seifert circles of \( D \) and \( D^* \) are identical. \( \square \)

Next we relate the reduced complex of \( D^* \) with the dual of the reduced complex of \( D \).

**Proposition 3.36.** The isomorphism \( \varphi \) of Proposition 3.33 induces isomorphisms \( \tilde{\varphi} \) such that the following diagram commutes

\[ (X_a)_p C(D^*) \xrightarrow{\sim} C(D^*)/(X_b)_p \]

\[ \downarrow \tilde{\varphi} \quad \downarrow \tilde{\varphi} \]

\[ (C(D)/(X_b)_p)^* \xrightarrow{\sim} ((X_a)_p C(D))^*. \]

Here the horizontal arrows are the isomorphisms of Proposition 3.14. Similar isomorphisms with \( a, b \) exchanged also exist. Thus we get isomorphisms

\[ \hat{C}_{h,t}^{\pm}(D^*) \cong (\hat{C}_{h,t}^{\pm}(D))^*. \]

Proof. The desired maps are obtained from the following diagram

\[ (X_a)_p C(D^*) \xrightarrow{\iota_a} C(D^*) \xrightarrow{\pi_b} C(D^*)/(X_b)_p \]

\[ \downarrow \tilde{\varphi} \quad \downarrow \varphi \quad \downarrow \tilde{\varphi} \quad \downarrow \varphi \]

\[ (C(D)/(X_b)_p)^* \xrightarrow{\pi^*_b} C(D^*) \xrightarrow{\iota^*_a} ((X_a)_p C(D))^*. \]
The unique existence of the dashed arrows follows from Lemma 3.34, for \( \varphi(X_a) \) annihilates \( X_b \). One can check that the correspondences are given by
\[
\begin{array}{c}
X_a \otimes x & \longrightarrow & 1 \otimes x \\
\downarrow & & \downarrow \\
1^* \otimes \varphi(x) & \longrightarrow & X^*_a \otimes \varphi(x)
\end{array}
\]
using \( \langle 1, \varphi(X_a) \rangle = \langle X_a, \varphi(1) \rangle = 1. \)

\[ \square \]

**Proposition 3.37.** There are perfect pairings
\[
\langle \cdot, \cdot \rangle \sim : \tilde{C}^\pm_{h,t}(D) \otimes \tilde{C}^\pm_{h,t}(D^*) \rightarrow R
\]
such that the following diagrams commute up to sign:
\[
\begin{array}{c}
\tilde{C}^\pm(D) \otimes \tilde{C}^\pm(D^*) & \longrightarrow & C(D) \otimes C(D^*) \\
\downarrow \langle \cdot, \cdot \rangle \sim & & \downarrow \langle \cdot, \cdot \rangle \\
R & \longrightarrow & R
\end{array}
\]

**Proof.** Unraveling the definition of \( \tilde{\phi} \) we get the following commutative diagram
\[
\begin{array}{c}
(X_a)pC(D) \otimes (X_a)pC(D^*) & \longrightarrow & C(D) \otimes C(D^*) \\
\downarrow 1 \otimes \tilde{\phi} & & \downarrow 1 \otimes \varphi \\
(X_a)pC(D) \otimes (C(D)/(X_b)p)^* & \longrightarrow & C(D) \otimes C(D^*) \\
\downarrow \sim & & \downarrow \sim \langle \cdot, \cdot \rangle_{\text{std}} \\
(X_a)pC(D) \otimes ((X_a)pC(D))^* & \longrightarrow & R \\
\downarrow \langle \cdot, \cdot \rangle_{\text{std}} & & \downarrow c \\
R & \longrightarrow & R
\end{array}
\]
There is a similar commutative diagram with \( a, b \) exchanged and the bottom horizontal arrow replaced with \(-c\). The composition of the right vertical maps gives \( \langle \cdot, \cdot \rangle \). We define \( \langle \cdot, \cdot \rangle_{\sim} \) to be the composition of the left vertical maps. \( \square \)

**Proposition 3.38.** The reduced Lee cycles of \( D \) and \( D^* \) pair as
\[
\langle \tilde{\alpha}(D), \tilde{\alpha}(D^*) \rangle_{\sim} = \varepsilon \varepsilon'^{-1}, \quad \langle \tilde{\beta}(D), \tilde{\beta}(D^*) \rangle_{\sim} = \varepsilon' \varepsilon'^{-1}
\]
where \( r = r(D) \) and \( \varepsilon, \varepsilon' \in \{ \pm 1 \} \) are signs such that \( \varepsilon \varepsilon' = (-1)^{r-1} \). \( \square \)

### 4 The invariant \( \tilde{s}_c \)

Now we are ready to define the link invariant \( \tilde{s}_c \) from the \( c \)-divisibility of the reduced Lee class. This is the reduced counterpart of the invariant \( s_c \) given in [San20].
4.1 Setup

Definition 4.1. Let $R$ be an integral domain, $M$ an $R$-module, and $c$ a non-zero, non-invertible element in $R$. Define the $c$-divisibility of an element $z$ in $M$ by

$$d_c(z) = \max\{ k \geq 0 \mid z \in c^k M \}.$$ 

We say $z$ is $c^k$-divisible if $k \leq d_c(z)$.

The following lemmas will be used in the coming sections. It is assumed that the assumptions of Definition 4.1 remain valid.

Lemma 4.2. For any $z \in M$ and $n \geq 0$,

$$d_c(c^n z) \geq n + d_c(z).$$ 

Moreover if $M$ has no $c$-torsions, the equality holds.

Proof. The inequality is obvious. Suppose $M$ has no $c$-torsion. Put $c^n z = c^k z'$ with $k = d_c(c^n z)$ and some $z' \in M$. From $k \geq n$ we have $c^n (z - c^{k-n} z') = 0$ and hence $z = c^{k-n} z'$ from the assumption. Thus $d_c(z) \geq k - n$. \qed

Lemma 4.3. For any $a \in R$ and $z \in M$,

$$d_c(az) \geq d_c(a) + d_c(z).$$ 

Moreover if $M$ is free and $c$ is prime, the equality holds.

Proof. The inequality is obvious. For the latter statement, we may assume $M = R^n$ and put $z = (z_i)$. From the previous lemma we may also assume $d_c(a) = d_c(z) = 0$. If $az$ is $c$-divisible, then $az_i$ is so for all $i$. Since $c$ is prime, it follows that either $a$ is $c$-divisible or otherwise all $z_i$ are $c$-divisible. Both contradict the assumption and hence $d_c(az) = 0$. \qed

Lemma 4.4. Suppose $M'$ is another $R$-module. For any $z \in M$ and $w \in M'$,

$$d_c(z \otimes w) \geq d_c(z) + d_c(w).$$ 

Moreover if $M, M'$ are free and $c$ is prime in $R$, the equality holds.

Proof. The inequality is obvious. For the latter statement, we may assume $M = R^n$, $M' = R^m$ and identify $M \otimes M'$ with $R^{mn}$. We may also assume $d_c(z) = d_c(w) = 0$. If $z \otimes w$ is $c$-divisible, then $z_i w \in R^n$ is $c$-divisible for each $i$. Then from Lemma 4.3, either $w$ is $c$-divisible or otherwise all $z_i$ are $c$-divisible. Both contradict the assumption and hence $d_c(z \otimes w) = 0$. \qed

Lemma 4.5. Let $R'$ be another ring and $M'$ be an $R'$-module. Suppose there is a ring homomorphism $f : R \to R'$ and an $R$-module homomorphism $\phi : M \to f^* M'$. Then for any $z \in M$,

$$d_c(z) \leq d_{f(c)}(\phi(z)).$$ 

Moreover if $f, \phi$ are isomorphisms, the equality holds.
Proof. If \( z = c^k z' \), then \( \phi(z) = f(c)^k \phi(z') \) and hence
\[
d_{f(c)}(\phi(z)) \geq k + d_{f(c)}(\phi(z')) \geq k.
\]
\[\square\]

Lemma 4.6. Let \( c^{-1}R \) denote the localization of \( R \) away from \( c \) (i.e. the minimal extension of \( R \) such that \( c \) is invertible), and \( c^{-1}M \) the module \( M \otimes_R c^{-1}R \). If \( M \) has no \( c \)-torsions and two elements \( z, w \in M \) are related as \( z \otimes 1 = c^n (w \otimes 1) \) in \( c^{-1}M \) for some \( n \in \mathbb{Z} \), then \( d_c(z) = n + d_c(w) \).

Proof. The natural map \( M \rightarrow c^{-1}M \) is injective. If \( n \geq 0 \) then \( z = c^n w \) in \( M \), otherwise if \( n < 0 \) then \( c^{-n}z = w \).
\[\square\]

Lemma 4.7. Let \( R_{(c)} \) denote the localization of \( R \) at \( c \) (i.e. the minimal extension of \( R \) such that all elements in \( R \setminus \{c\} \) are invertible), and \( M_{(c)} \) the module \( M \otimes_R R_{(c)} \). For any \( z \in M \), we have
\[
d_c(z) \leq d_c(z/1).
\]
Moreover if \( M \) is free and \( c \) is prime, the equality holds.

Proof. The inequality is obvious from the existence of the natural map
\[
M \rightarrow M_{(c)}, \quad z \mapsto z/1.
\]
For the latter statement, put \( z/1 = c^k (w/s) \) for some \( w \in M \) and \( s \in R \setminus \{c\} \) with maximal \( k \). Then \( sz = c^k w \) in \( M \). We have \( d_c(s) = 0 \) and also \( d_c(w) = 0 \) from the maximality of \( k \). Thus from Lemma 4.3 we have \( d_c(z) = k \).
\[\square\]

4.2 Divisibility of reduced Lee class

For the remainder of this section, we assume \( R \) is an integral domain, \((R, h, t)\) is factorable, and \( c \) is non-zero, non-invertible in \( R \).

Definition 4.8. Let \( D \) be a pointed link diagram. Define \( \tilde{d}_c(D) \) by the \( c \)-divisibility of the reduced Lee class \( \tilde{\alpha}(D) \) in \( \tilde{H}^+_R(D)/\text{Tor} \),
\[
\tilde{d}_c(D) = d_c(\tilde{\alpha}(D)).
\]

Example 4.9. \( D = \bigcirc \) has \( \tilde{C}(D) = R \) and \( \tilde{\alpha}(D) = 1 \) hence \( \tilde{d}_c(D) = 0 \).

Example 4.10. Consider the unknot diagram \( D \) with one negative crossing. Suppose \( p \) lies on an arc colored \( a \) with respect to the given orientation. Then
\[
\tilde{C}(D) = \{ R \xrightarrow{\Delta} R \otimes A \}
\]
and \( \tilde{\alpha}(D) = 1 \otimes X_b \). From \( \Delta 1 = 1 \otimes X_a \),
\[
\tilde{\alpha}(D) \sim 1 \otimes (X_b - X_a) = -c(1 \otimes 1).
\]
Since \( [1 \otimes 1] \) generates \( \tilde{H}(D) \cong R \), we have \( \tilde{d}_c(D) = 1 \).
The above examples show that $\tilde{d}_c(D)$ is not a pointed link invariant.

**Proposition 4.11.** $\tilde{d}_c(D) = \tilde{d}_c(-D)$.

*Proof.* The involution $I$ of Proposition 3.12 sends $\tilde{\alpha}(D)$ in $\tilde{C}^+(D)$ to $\tilde{\beta}(D) = \tilde{\alpha}(-D)$ in $\tilde{C}^-(D) = \tilde{C}^+(D)$. Thus the result follows from Lemma 4.5. \qed

**Proposition 4.12.** $\tilde{d}_c(D \sqcup \odot) = \tilde{d}_c(D)$.

*Proof.* Suppose $\odot$ is oriented counterclockwise. Then $\tilde{\alpha}(D \sqcup \odot) = \tilde{\alpha}(D) \otimes X_a$. We have maps in both directions

$$\tilde{C}(D) \xrightarrow{\text{id} \otimes X_a} \tilde{C}(D \sqcup \odot)$$

such that

$$\tilde{\alpha}(D) \leftrightarrow \tilde{\alpha}(D \sqcup \odot).$$

Thus the result follows from Lemma 4.5. \qed

**Proposition 4.13.** If $D$ is a positive diagram, then $\tilde{d}_c(D) = 0$.

*Proof.* The orientation preserving state of $D$ is $s = (0, \ldots, 0)$. By 0-resolving the crossings one by one, we get a sequence of quotient maps

$$\tilde{C}(D) \to \tilde{C}(D_0) \to \cdots \to \tilde{C}(D_{0^0}).$$

Since the rightmost diagram is a disjoint union of circles, we have

$$0 \leq \tilde{d}_c(D) \leq \tilde{d}_c(D_{0^0}) = 0.$$  \qed

**Proposition 4.14.** Suppose $D, D'$ are pointed diagrams that are related by a saddle move that splits a Seifert circle of $D$ into two Seifert circles of $D'$ and that does not contain the marked points of $D$, $D'$. Then

$$\tilde{d}_c(D) \leq \tilde{d}_c(D') \leq \tilde{d}_c(D) + 1.$$  

*Proof.* The sequence of two saddle moves

$$D \to D' \to D$$

induces a sequence of chain maps that send the Lee cycles as

$$\tilde{\alpha}(D) \xrightarrow{\Delta} \tilde{\alpha}(D') \xrightarrow{m} \pm c \cdot \tilde{\alpha}(D).$$

Thus the result follows from Lemmas 4.2 and 4.5. \qed

**Proposition 4.15.** For pointed link diagrams $D, D'$,

$$\tilde{d}_c(D \# D') \geq \tilde{d}_c(D) + \tilde{d}_c(D').$$

Moreover when $R$ is a PID and $c$ is prime, this becomes an equality.

26
Proof. From Proposition 3.30 there is a natural map
\[ \tilde{H}(D)/\text{Tor} \otimes \tilde{H}(D')/\text{Tor} \to \tilde{H}(D\#D')/\text{Tor} \]
that maps
\[ [\tilde{\alpha}(D)] \otimes [\tilde{\alpha}(D')] \mapsto [\tilde{\alpha}(D\#D')] \].
Thus the inequality holds from Lemmas 4.4 and 4.5. When \( R \) is a PID, this map is an isomorphism and hence the reverse inequality follows from the latter statement of Lemma 4.4.

**Proposition 4.16.** For pointed link diagrams \( D, D' \),
\[ \hat{d}_c(D\#D') \leq \hat{d}_c(D \sqcup D') \leq \hat{d}_c(D\#D') + 1. \]
Here \( D \sqcup D' \) is regarded as a pointed link by taking any point on its arc.

Proof. This is a special case of Proposition 4.14.

**Proposition 4.17.** If \( D, D' \) are related by a Reidemeister move, then
\[ \hat{d}_c(D) = \hat{d}_c(D') + j \]
where
\[ j = \frac{\delta w(D, D') - \delta r(D, D')}{2}. \]

Proof. Immediate from Proposition 3.27.

**Proposition 4.18.** Suppose \( S \) is a cobordism between pointed links \( L, L' \), such that each component of \( S \) has a boundary in \( L \). Then
\[ \hat{d}_c(D) \leq \hat{d}_c(D') + j \]
where
\[ j = \frac{\delta w(D, D') - \delta r(D, D') - \chi(S)}{2}. \]

Proof. Immediate from Proposition 3.28 with Lemma 4.6.

### 4.3 Definition and properties of \( \tilde{s}_c \)

The following definition is justified from Proposition 4.17.

**Definition 4.19.** For any pointed link \( L \), define
\[ \tilde{s}_c(L) = 2\hat{d}_c(D) + w(D) - r(D) + 1 \]
where \( D \) is any pointed diagram representing \( L \).

**Proposition 4.20.** \( \tilde{s}_c(L) \equiv |L| - 1 \mod 2. \)
Proof. Take any diagram $D$ of $L$, and let $S$ be the Seifert surface of $L$ obtained by applying Seifert’s algorithm to $D$. Then from

$$\chi(S) = 2 - 2g(S) - |L| = r(D) - n(D),$$

we have

$$\tilde{s}_c(L) \equiv n(D) + r(D) + 1 \equiv |L| + 1 \mod 2.$$ 

The following properties of $\tilde{s}_c$ immediately follow from those of $\tilde{d}_c$.

**Proposition 4.21.**

1. $\tilde{s}_c(\bigodot) = 0$.
2. $\tilde{s}_c(L \sqcup \bigodot) = \tilde{s}_c(L) - 1$.
3. $\tilde{s}_c(L) = \tilde{s}_c(-L)$.
4. $\tilde{s}_c(L\#L') \geq \tilde{s}_c(L) + \tilde{s}_c(L')$.
5. $\tilde{s}_c(L\#L') - 1 \leq \tilde{s}_c(L \sqcup L') \leq \tilde{s}_c(L\#L') + 1$.

When $R$ is a PID and $c$ is prime in $R$, 4. becomes an equality.

The following proposition states the behavior of $\tilde{s}_c$ under cobordisms, which is the key to proving the main theorem.

**Proposition 4.22.** Suppose $S$ is a cobordism between pointed links $L, L'$, such that each component of $S$ has a boundary in $L$. Then

$$\tilde{s}_c(L) \leq \tilde{s}_c(L') - \chi(S).$$

Moreover, if every component $S$ has boundary in both $L$ and $L'$, then

$$|\tilde{s}_c(L) - \tilde{s}_c(L')| \leq -\chi(S).$$

**Proof.** Immediate from Proposition 4.18. \qed

**Corollary 4.23.** $\tilde{s}_c$ is invariant under link concordance. \qed

**Corollary 4.24.** If $K$ is a slice knot, then $\tilde{s}_c(K) = 0$. \qed

**Corollary 4.25.** For a knot $K$, we have

$$|\tilde{s}_c(K)| \leq 2g_4(K).$$

Here $g_4(K)$ denotes the slice genus of $K$. \qed

**Corollary 4.26.** For a positive knot $K$,

$$\tilde{s}_c(K) = 2g_4(K) = 2g(K).$$
Proof. Take any positive diagram $D$ of $K$, and let $S$ be a Seifert surface of $K$ obtained by applying Seifert’s algorithm to $D$. Then

$$\tilde{s}_c(K) = n(D) - r(D) + 1 = 2g(S)$$

and

$$2g_4(K) \leq 2g(K) \leq 2g(S) = \tilde{s}_c(K) \leq 2g_4(K).$$

Thus we conclude,

**Theorem 1.** For any PID $R$ and a prime $c$ in $R$, the invariant $\tilde{s}_c$ is a slice-torus invariant.

The following property is what all slice-torus invariants have in common.

**Corollary 4.27** ([Lew14, Corollary 5.9]). Suppose $R$ is a PID and $c$ is prime in $R$. For any alternating knot $K$, the invariant $\tilde{s}_c(K)$ coincides with the knot signature $\sigma(K)$ of $K$.

### 4.4 Refined Lee classes

Here we prove that the Lee classes (modulo torsion) can be refined so that it becomes invariant (up to sign) under the Reidemeister moves.

**Definition 4.28.** Let $D$ be a pointed link diagram. Define the **refined Lee class** $\tilde{\zeta}(D)$ in $\tilde{H}_{h,t}(D)/\text{Tor}$ by

$$\tilde{\zeta}(D) = c^{-k} \tilde{\alpha}(D)$$

where $k = \tilde{d}_c(D)$.

**Proposition 4.29.** The class $\tilde{\zeta}(D)$ is invariant (up to sign) under the Reidemeister moves.

**Proof.** Immediate from Proposition 3.27.

**Proposition 4.30.** Suppose $R$ is a PID and $c$ is prime. For a knot diagram $D$, the refined Lee class $\tilde{\zeta}(D)$ is a generator of $\tilde{H}_{h,t}(D)/\text{Tor}$.

**Proof.** From Proposition 3.38 we have

$$\langle \tilde{\alpha}(D), \tilde{\alpha}(D^*) \rangle^\sim = \pm e^{c(D)-1}.$$  
From Theorem 1 it follows that

$$\tilde{d}_c(D) + \tilde{d}_c(D^*) = r(D) - 1$$
and hence

$$\langle \tilde{\zeta}(D), \tilde{\zeta}(D^*) \rangle^\sim = \pm 1.$$  
Since $\langle \cdot , \cdot \rangle^\sim$ is a perfect pairing, the classes $\tilde{\zeta}(D), \tilde{\zeta}(D^*)$ must be generators of $\tilde{H}_{h,t}(D)/\text{Tor}$ and $\tilde{H}_{h,t}(D^*)/\text{Tor}$ respectively.
Thus for a link $L$, we may define the refined Lee class $\tilde{\zeta}(L)$ in $\tilde{H}_{h,t}(L)/\text{Tor}$ (up to sign). In particular when $K$ is a knot (and when the assumption of Proposition 4.30 holds) it is justified to call $\zeta(K)$ the canonical generator of $\tilde{H}_{h,t}(K)/\text{Tor}$.

**Proposition 4.31.** Let $S$ be a cobordism between pointed links $L, L'$, such that each component of $S$ has a boundary in $L$. Then the corresponding cobordism map\[\phi : \tilde{H}_{h,t}(D)/\text{Tor} \to \tilde{H}_{h,t}(D')/\text{Tor}\]

sends\[\tilde{\zeta}(D) \xrightarrow{\phi} \pm c \tilde{\zeta}(D')\]

where\[l = \frac{\delta \tilde{s}_{\epsilon}(L, L') - \chi(S)}{2}.\]

**Proof.** Immediate from Proposition 3.28. \qed

**Corollary 4.32.** The class $\zeta(L)$ is invariant (up to sign) under link concordance. \qed

The following characterization of $\tilde{s}_{\epsilon}(L)$ via a cobordism is analogous to the reformulation of the $s$-invariant given in [KM13, Section 2.2].

**Proposition 4.33.** Suppose $R$ is a PID and $c$ is prime. Let $S$ a connected cobordism from the pointed unknot $U$ to a pointed link $L$. Let $\phi$ be the corresponding cobordism map\[\phi : R = \tilde{H}_{h,t}(U) \to \tilde{H}_{h,t}(L)/\text{Tor}.\]

Put\[z = \phi(1) \in \tilde{H}_{h,t}(L)/\text{Tor}.\]

Then\[\tilde{s}_{\epsilon}(L) = 2d_{\epsilon}(z) + \chi(S).\]

**Proof.** Obviously $\tilde{\alpha}(U) = \tilde{\zeta}(U) = 1$. From Proposition 4.31,

\[z = \pm c \tilde{\zeta}(L)\]

where\[l = \frac{\tilde{s}_{\epsilon}(L) - \chi(S)}{2}.\]

Now the result is follows from $d_{\epsilon}(\tilde{\zeta}(L)) = 0$. \qed

**Question 4.34.** For a link diagram $D$ we can define classes $\tilde{\zeta}(D, o)$ for any orientation $o \in \hat{O}(D)$ and similarly prove that they are invariant (up to sign) under the Reidemeister moves. Do these classes form a basis for $\tilde{H}_{h,t}(D)/\text{Tor}$?
4.5 Coincidence with $s^F$

**Theorem 2.** For any field $F$ and any knot $K$,

$$s^F(K) = \hat{s}_H(K; F[H]).$$

Here $s^F$ is Rasmussen’s $s$-invariant over $F$.

**Proof.** Let $\hat{H}_{BN}$ denote the reduced bigraded Bar-Natan homology over $F$, given the triple $(R, h, t) = (F[H], H, 0)$ with $\deg H = -2$. From [KWZ19, Proposition 3.8], $s^F(K)$ is given by the $q$-grading of the generator of $\hat{H}_{BN}(K) / \text{Tor}$. Thus with Proposition 4.30 we get

$$s^F(K) = \text{gr}_q(\hat{\zeta}(K)).$$

On the other hand, for any diagram $D$ of $K$, we have

$$\text{gr}_q(\hat{\alpha}(D)) = w(D) - r(D) + 1$$

and hence

$$\text{gr}_q(\hat{\zeta}(D)) = 2\hat{d}_H(D) + w(D) - r(D) + 1$$

which is precisely the definition of $\hat{s}_H(K)$. \qed

**Remark 4.35.** This result implies that in general $\hat{s}_c$ is not a homomorphism, in particular when $(R, c) = (Z[H], H)$. This can be seen as follows. Take any $(R, c)$ and consider the ring homomorphism

$$Z[H] \to R, \quad H \mapsto c.$$ 

This gives an equality

$$\hat{d}_H(D; Z[H]) \leq \hat{d}_c(D; R)$$

and hence

$$\hat{s}_H(K; Z[H]) \leq \hat{s}_c(K; R).$$

Now if both $\hat{s}_H$ and $\hat{s}_c$ are homomorphisms, we get the reverse inequality and hence $\hat{s}_H = \hat{s}_c$. With Theorem 2 we get that all $s^F$ are equal, but we know from [LS14; See] that $s^Q \neq s^{F_2}$, hence a contradiction.

4.6 Classification

Here we assume $R$ is a PID and $c \in R$ is prime. Such pairs $(R, c)$ are classified into four types by the following mutually exclusive conditions:

(A) $\text{char } R \neq 0$.

(B) $\text{char } R = 0$ and $c \not| n$ for every $n \in Z \setminus 0$.

(C) $\text{char } R = 0$ and $c \sim p \cdot 1_R$ for some prime $p \in Z$.

(D) $\text{char } R = 0$, $c | p \cdot 1_R$ and $c \not\sim p \cdot 1_R$ for some prime $p \in Z$.
Example 4.36. Typical examples of \((R, c)\) belonging to the four types are:

(A) \((R, c) = (\mathbb{F}_p[h], h),\) \(p: \) prime.

(B) \((R, c) = (\mathbb{Q}[h], h).\)

(C) \((R, c) = (\mathbb{Z}, p),\) \(p: \) prime.

(D) \((R, c) = (\mathbb{Z}[i], 1 + i)\) where \(c\) divides 2, or
\((R, c) = (\mathbb{Z}[\omega], 1 + \omega)\) where \(\omega = e^{\frac{2\pi i}{3}}\) and \(c\) divides 3.

The following proposition states that for \((R, c)\)'s of type (A) - (C), the knot invariants \(\tilde{s}_c\) are reduced to considering the ones for the above typical cases.

Theorem 3.

1. If \((R, c)\) is of type (A), then \(\tilde{s}_c = s^{F_p},\) where \(p = \text{char } R.\)

2. If \((R, c)\) is of type (B), then \(\tilde{s}_c = s^{Q}.\)

3. If \((R, c)\) is of type (C), then \(\tilde{s}_c = \tilde{s}_p(-; \mathbb{Z}).\)

We first prove the following two lemmas.

Lemma 4.37. Suppose \((R_0, c_0)\) is another pair such that \(R_0\) is a PID and \(c_0\) is prime. If there is a ring homomorphism
\[\psi: R_0 \to R\]
such that \(\psi(c_0) \sim c,\) then the two corresponding knot invariants coincide,
\[\tilde{s}_{c_0}(-; R_0) = \tilde{s}_c(-; R).\]

Proof. From Lemma 4.5, we have
\[\tilde{d}_{c_0}(D; R_0) \leq \tilde{d}_c(D; R)\]
and hence
\[\tilde{s}_{c_0}(K; R_0) \leq \tilde{s}_c(K; R).\]
From Theorem 1, both invariants satisfy the mirror formula, so the reverse inequality also holds.

Lemma 4.38. Let \(R_{(c)}\) denote the localization of \(R\) at \(c.\) The two knot invariants corresponding to \((R, c)\) and \((R_{(c)}, c)\) coincide,
\[\tilde{s}_c(-; R) = \tilde{s}_c(-; R_{(c)}).\]

Proof. Immediate from Lemma 4.7.
Proof of Theorem 3. If \((R, c)\) is of type (A), there is a ring homomorphism
\[
\mathbb{F}_p[H] \to R, \quad H \mapsto c
\]
and hence \(\tilde{s}_c = s^p\). If \((R, c)\) is of type (B), every \(n \in \mathbb{Z} \setminus \{0\}\) is invertible in \(R_{(c)}\).
Thus from the universal property of localizations, the ring homomorphism
\[
\mathbb{Z}[H] \to R, \quad H \mapsto c
\]
duces
\[
\mathbb{Q}[H] \to R_{(c)}, \quad H \mapsto c
\]
and hence \(\tilde{s}_c = s^\mathbb{Q}\). Finally if \((R, c)\) is of type (C), the unique ring homomorphism
\[
\mathbb{Z} \to R
\]
sends \(p \mapsto p \sim c\) and hence \(\tilde{s}_c = \tilde{s}_p(-; \mathbb{Z})\).

5 Computation

Direct computations of \(\tilde{s}_c\) were performed for some specific \((R, c)\)'s of type (C) and (D) by a computer program developed by the first author [San22]. Some of the notable results are listed in Table 1, Section 1.

The algorithm of the program is straightforward: it takes as input a value \(c\) in a (computer-representable) Euclidean domain \(R\) and a knot diagram \(D\), and computes the 0-th homology group \(H_c^0(D)\) by performing Gaussian elimination on the differential matrices, while simultaneously producing the transition matrices so that the Lee class \(\tilde{\alpha}(D)\) can be expressed as an \(R\)-valued vector. Then we can easily compute \(\tilde{d}_c(D)\) by dropping the factors corresponding to the torsions. This computation is obviously time consuming. An improvement can be certainly made by adopting the ‘divide-and-conquer’ method proposed by Bar-Natan in [Bar07] and by Schütz in [Sch21]. This is left as a future work.

Several conjectures regarding the independence of \(s^E\) are proposed by Schütz in [Sch22] and by Lewark–Zibrowius in [LZ22], based on their theoretical and computational results. It would be worthwhile to try to compute \(\tilde{s}_c\) for the specific classes of knots mentioned therein, for example Whitehead doubles of torus knots, so that we might find counterexamples to Questions 1.3 and 1.4.

6 Unreduced counterparts

Finally we study the relations between \(\tilde{s}_c\) and the unreduced counterpart \(\bar{s}_c\) given in [San20].

Definition 6.1. For a link diagram \(D\), define
\[
d_c(D) = \tilde{d}_c(D_+).
\]
Definition 6.2. For a link $L$, define
\[ s_c(L) = \tilde{s}_c(L_+) . \]

It follows from Proposition 3.22 that the definition of $d_c(D)$ coincides with the one given in [San20, Definition 3.5]. For $s_c(L)$, we have
\[ s_c(L) = \tilde{s}_c(L_+) \]
\[ = 2\tilde{d}_c(D_+) + w(D_+) - r(D_+) + 1 \]
\[ = 2d_c(D) + w(D) - r(D) \]
\[ = \bar{s}_c(L) - 1 . \]

Here $\bar{s}_c$ is the invariant defined in [San20, Theorem 1]. The above defined $s_c$ suites better for as link invariants, for instance $s_c(\emptyset) = 0$, whereas $\bar{s}_c$ suits better as knot invariants, for instance $\bar{s}_c(\bigodot) = 0$. All of the basic properties of $d_c$ and $\bar{s}_c$ given in [San20] immediately follows from the results obtained in this paper. It is natural to ask whether there is an essential difference between the reduced version and the unreduced invariants.

Definition 6.3. For a pointed link diagram $D$, define
\[ \delta_c(D) = d_c(D) - \tilde{d}_c(D) . \]

Equivalently, we may define
\[ \delta_c(D) = \frac{1}{2}(\bar{s}_c(D) - \tilde{s}_c(D)) \]
which in particular shows that $\delta_c$ is a link invariant.

Proposition 6.4. $\delta_c$ takes values in $\{0, 1\}$.

Proof. The injection $\tilde{C}(D) \hookrightarrow C(D)$ maps $\tilde{\alpha}(D)$ to $\alpha(D)$, and hence
\[ \tilde{d}_c(D) \leq d_c(D) . \]

On the other hand, from Proposition 4.16,
\[ d_c(D) = \tilde{d}_c(D \cup D) \leq \tilde{d}_c(D) + 1 . \]

Proposition 6.5. $\delta_c(L) = 0$ for all positive links.

Proposition 6.6. $\delta_c(K) = 0$ if both $\tilde{s}_c$ and $\bar{s}_c$ preserve additive inverses of $K$.

In particular if both $\tilde{s}_c$ and $\bar{s}_c$ are abelian group homomorphisms, then $\tilde{s}_c = \bar{s}_c$ as knot invariants. The following three cases give such examples, with the assumption that $R$ is a PID and $c$ is prime:
1. \((R, c) = (F[H], H)\),
2. \(c \sim 2\),
3. \(\text{char } R = 2\).

The first case (restricted to \(\text{char } F \neq 2\)) is proved in [San20, Section 3.3]. The second case is also stated in [San20, Remark 3.37]. The third can be proved similarly using the splitting

\[ C_{h,t}(D) \cong \tilde{C}_{h,t}^+(D) \oplus \tilde{C}_{h,t}^-(D) \]

given by Wigderson in [Wig16]. (Hence the condition \(\text{char } F \neq 2\) can be dropped from the first case.)

**Question 6.7.** Is there a pair \((R, c)\) and a link \(L\) such that \(\delta_w(L) = 1\)?

## Appendix: Proof of Proposition 2.16

Here we prove Proposition 2.16 for each Reidemeister move using the explicit descriptions of the chain homotopy equivalences

\[ F : C(D) \xrightarrow{\sim} C(D') : G \]

given in [Bar05, Section 4.3]. Note that the proof also works for Proposition 3.27 by simply replacing every object with its reduced counterpart.

**Reidemeister move 1**

See [Bar05, Figure 5]. The map \(G : C(D') \to C(D)\) is given by a cap, so it is obvious that \(G(\alpha(D')) = \alpha(D)\) and \(G(\beta(D')) = \beta(D)\). On the other hand we have \(\delta_r(D', D) = -1\), so the result holds.

**Reidemeister move 2**

See [Bar05, Figure 6]. If the two strands are oriented in the same direction, then the orientation preserving resolutions of \(D\) and \(D'\) are identical. In this case \(G(\alpha(D')) = \alpha(D)\), \(G(\beta(D')) = \beta(D)\) by definition of \(G\), while \(\delta_r(D', D) = 0\).

If the two strands are oriented in the opposite direction, then the orientation preserving resolution \(D'\) yields a circle inside the changing disk. Again by definition of \(G\), this circle is capped and a saddle move is performed to the other two strands. The saddle is either a merge or a split, depending on how the strands are connected outside the disk. If it is a merge, then \(G(\alpha(D')) = \pm c\alpha(D)\), \(G(\beta(D')) = \mp c\beta(D)\) while \(\delta_r(D', D) = -2\). If it is a split, then \(G(\alpha(D')) = \alpha(D)\), \(G(\beta(D')) = \beta(D)\) while \(\delta_r(D', D) = 0\).
Reidemeister move 3

See [Bar05, Figure 7, 8]. The equivalences are given by the compositions of equivalences

\[ C(D) \xrightarrow{\tilde{G}} C \xleftarrow{\tilde{F}'} C(D') \]

where \( C \) is the mapping cone of two identical maps \( \Psi_L = \Psi \circ F_0 \) and \( \Psi_R = \Psi' \circ F_0' \). It suffices to prove that the desired relations hold in \( C \) under the maps \( \tilde{G} \) and \( \tilde{G}' \). If the center crossing is negative, then \( \alpha(D) \) and \( \beta(D) \) lies in the codomain of \( \Psi \) and are mapped identically into \( C \). The same is true for \( \alpha(D') \) and \( \beta(D') \) while \( \delta(D, D') = 0 \) so the result holds. If the center crossing is positive, then \( \alpha(D) \) and \( \beta(D) \) lies in the domain of \( \Psi \), and the cycles are mapped as

\[
\left( \begin{array}{c} z \\ 0 \end{array} \right) \xrightarrow{\tilde{G}} \left( \begin{array}{c} G_0 z \\ \Psi h_0 z \end{array} \right).
\]

Here the maps \( G_0, h_0 \) are the maps corresponding to the RM2 performed on the other two crossings. When \( h_0(z) = 0 \) the result reduces to the case of RM2. The only case where \( h_0(z) \neq 0 \) is when the cycle \( z \) belongs to the state that contains a circle in the changing disk, which happens only when the three strands are oriented symmetrically with respect to the rotation by \( \pi/3 \). However, by a sequence of moves described in the beginning of [Pol10, Section 3], we may avoid this move and assume that \( h_0 = 0 \). This deformation is valid since we know from [Bar05, Theorem 4] that the cobordism maps are invariant (up to sign) under isotopies. Thus the proof is complete.

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