Lattice solutions in a Ginzburg-Landau model for a chiral magnet

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Abstract

We examine micromagnetic pattern formation in chiral magnets, driven by the competition of Heisenberg exchange, Dzyaloshinskii-Moriya interaction, easy-plane anisotropy and thermodynamic Landau potentials. Based on equivariant bifurcation theory we prove existence of lattice solutions branching off the zero magnetization state and investigate their stability. We observe in particular the stabilization of quadratic vortex-antivortex lattice configurations and instability of hexagonal skyrmion lattice configurations, and we illustrate our findings by numerical studies.

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1 Introduction and main results

Dzyaloshinskii-Moriya interaction (DMI) is the antisymmetric counterpart of Heisenberg exchange. It arises from the lack of inversion symmetry in certain magnetic system, induced by the underlying crystal structures or by the system geometry in the presence of interfaces. In micromagnetic models DMI arises in form of linear combinations of the so-called Lifshitz invariants, i.e. the components of the chirality tensor $\nabla m \times m$, and is therefore sensitive with respect to reflections and independent rotation in the domain and the target space, respectively. It is well-known that DMI gives rise to
modulated phases. The basic phenomenon is that the energy of the homogeneous magnetization state \( m = \text{const.} \) can be reduced below zero by means of spiralization in form of periodic domain wall arrays, the helical phase. A prominent form of doubly periodic lattice state arises from the stabilization of topological structures, so-called chiral skyrmions, in two-dimensional chiral ferromagnets. Chiral skyrmions are localized structures which are topologically characterized by a unit \( S^2 \) degree and a well-defined helicity depending on the specific form of DMI. In the presence of sufficiently strong perpendicular anisotropy and/or Zeeman field interaction, chiral skyrmions occur as local energy minimizers in form of isolated topological solitons. Zeeman field interaction enables the possibility of an intermediate regime where chiral skyrmion embedded into a hexagonal lattice are expected to be globally energy minimizing. Micromagnetic theories including DMI have been proposed in \cite{11} with the idea that skyrmion lattice configurations represent a magnetic analogue of the mixed state in type-II superconductors. Corresponding phase diagrams and stability questions have been examined analytically and numerically in the seminal work \cite{9,10}, see \cite{22} for a recent review. A fully rigorous functional analytic theory on the existence, stability, asymptotics, internal structure and exact solvability of isolated chiral skyrmions has recently started to emerge \cite{25,16,23,20,21,7,8}.

Lattice states in two-dimensional Ginzburg-Landau models for chiral magnets including thermodynamic effects have been proposed theoretically in \cite{29}, see also \cite{26,34}. Mathematically one may expect a close analogy with Abrikosov’s vortex lattice solutions in Ginzburg-Landau models for superconductors \cite{1} with a well-established theory in mathematical analysis. The occurrence of Abrikosov lattices in the framework of gauge-periodic solutions of Ginzburg-Landau equations and the optimality of hexagonal lattices have been thoroughly investigated by means of variational methods and bifurcation theory \cite{6,28,32,5,2,30}.

Ginzburg-Landau models in micromagnetics are, in contrast to superconductivity, directly formulated in terms of the physically observable magnetization field. A class of such models has been proposed and examined computationally in \cite{29,26}. Compared to the purely ferromagnetic case \( |m| = \text{const.} \), Ginzburg-Landau models offer a larger variety of patterns, including vortex and half-skyrmion arrays of opposite in-plane winding on square lattices, and skyrmions on hexagonal lattices, see Figure \cite{1}.

We shall examine the occurrence of periodic solutions near the paramagnetic state of zero magnetization, i.e., in a high temperature regime. Our starting point are magnetization fields \( m : \mathbb{R}^2 \to \mathbb{R}^3 \) governed by energy densities of
the form
\[ A|\nabla m|^2 + D m \cdot (\nabla \times m) + f(m) + K(m \cdot \hat{e}_3)^2. \]
The Dirichlet term with \( A > 0 \) is referred to as Heisenberg exchange interaction. The helicity term with \( D \neq 0 \) is a prototypical form of DMI arising in the context of noncentrosymmetric cubic crystals. Note that for \( m = (m, m_3) \) defined on a two-dimensional domain
\[ \nabla \times m = \begin{pmatrix} -\nabla \perp m_3 \\ \nabla \perp \cdot m \end{pmatrix}. \]
A key analytical feature induced by DMI is the loss of independent rotational symmetry in magnetization space \( \mathbb{R}^3 \) and the domain \( \mathbb{R}^2 \). Finally, the Landau term \( f \) is an even polynomial in the modulus \( |m| \)
\[ a(T - T_C)|m|^2 + b|m|^4 + c|m|^6 + d|m|^8 + \cdots \]
where \( T - T_C \) is the deviation from the Curie temperature. We shall focus on a minimal model with
\[ a(T - T_C)|m|^2 + b|m|^4 \quad \text{where} \quad a, b > 0. \]
For stability reasons we also include the easy-plane anisotropy \( K(m \cdot \hat{e}_3)^2 \) with \( K > 0 \), which typically emerges as a reduced form of magnetostatic stray-field interaction in thin-film geometries, see e.g. [8, 15, 24].

We are interested in magnetization fields \( m \) which are periodic with respect to a two-dimensional lattice \( r\Lambda \) (\( r > 0 \)) with
\[ \Lambda = 2\pi(\mathbb{Z} \oplus r\mathbb{Z}). \]
where $\tau$ is a complex number in the fundamental domain of the modular group, referred to as the lattice shape parameter, see Section 2.1. Rescaling space we may assume $r = 1$. The rescaled energy density reads

$$e(m) = \frac{1}{2} |\nabla m|^2 + \kappa m \cdot (\nabla \times m) + \frac{\lambda}{2} |m|^2 + \frac{\alpha}{4} |m|^4 + \frac{\beta}{2} (m \cdot \hat{e}_3)^2$$  \hspace{1cm} (1)

with dimensionless constants

$$\kappa = \frac{Dr}{2A}, \quad \lambda = \frac{a(T - T_C)r^2}{A}, \quad \alpha = \frac{2br^2}{A} \quad \text{and} \quad \beta = \frac{Kr^2}{A}. \quad (2)$$

The energy density is invariant with respect to translations and joint rotations by $R \in SO(3)$

$$m(x) \mapsto Rm(R^{-1}x) \quad \text{for} \quad x \in \mathbb{R}^2$$

taking into account $O(2) \hookrightarrow SO(3)$. Since a sign reversal of the DM density can be achieved by reflections such as $m_3 \mapsto -m_3$, we may assume w.l.o.g that $\kappa > 0$.

We shall investigate the occurrence and stability of non-trivial periodic solutions $m$ to the Euler-Lagrange equation associated to (1)

$$-\Delta m + 2\kappa \nabla \times m + \lambda m + \alpha |m|^2 m + \beta m_3 \hat{e}_3 = 0.$$  \hspace{1cm} (3)

We first discuss energy minimizing solutions.

**Theorem 1.** Suppose $\alpha > 0$, $\beta \geq 0$, and $\kappa > 0$.

(i) If $\lambda > \kappa^2$, then $m \equiv 0$ is the unique energy minimizer on every lattice.

(ii) If $\lambda < \kappa^2$ and $\beta = 0$, then the helix

$$m(x) = M (0, \cos(\kappa x_1), \sin(\kappa x_1)) \quad \text{where} \quad M = \sqrt{\frac{\kappa^2 - \lambda}{\alpha}} \quad (4)$$

(see Figure 2) is up to a joint rotation the unique energy minimizer on suitable lattices.

In the isotropic case $\beta = 0$, Theorem 1 indicates the existence of only two phases, paramagnetic or helical, while the picture in the anisotropic case $\beta > 0$ is incomplete. The occurrence of helical phases is common to other mathematically related theories for condensed matter such as the Oseen-Frank model for chiral liquid crystals (see e.g. [33]) or the Gross-Pitaevskii
model for spin-orbit coupled Bose-Einstein condensates (see e.g. [3]). Helical
structures in chiral ferromagnets are also discussed in [14, 27].

Here we are interested in doubly periodic solutions. Given a lattice \( \Lambda \), we
aim to find \( \lambda \) and a non-trivial \( \Lambda \)-periodic solution \( m \) of (3) at \( \lambda \). We call
such pairs \((m, \lambda)\) \( \Lambda \)-lattice solutions and will prove the following:

**Theorem 2.** Suppose \( \alpha > 0, \beta \geq 0, \kappa > 0 \), and \( \Lambda = 2\pi(Z \oplus \tau Z) \). Then (3)
has a branch of \( \Lambda \)-lattice solution \((m_s, \lambda_s)\) in a neighbourhood of \( m_0 \equiv 0 \) and

\[
\lambda_0 = -\left( \frac{1}{\text{Im} \tau} \right)^2 - \frac{\beta}{2} \pm \sqrt{\left( \frac{2\kappa}{\text{Im} \tau} \right)^2 + \frac{\beta^2}{4}}
\]

(5)
satisfying the non-resonances condition

\[
\lambda_0 \neq -|\omega|^2 - \frac{\beta}{2} \pm \sqrt{4\kappa^2|\omega|^2 + \frac{\beta^2}{4}}
\]

(6)
for any \( \omega \) in the dual lattice \( \Lambda^* \), see Sec. 2.3, with \( |\omega| \neq \frac{1}{\text{Im} \tau} \).
The branch \((m_s, \lambda_s)\) is analytically parameterized by a real parameter \( s \) near
0 and has the form

\[
m_s = s\varphi_1 + O(s^3), \quad \lambda_s = \lambda_0 + s^2\lambda_2 + O(s^4)
\]
as \( s \to 0 \) with \( \lambda_2 < 0 \) and \( \varphi_1 \) explicitly determined.

The morphology of bifurcation solutions is related to symmetry properties of
the underlying lattice. Depending on this, the first-order bifurcation solution
\( \varphi_1 \), arising from the first critical wave number, indicates a threefold pattern
formation.

1. **helical pattern**: on non-equilateral lattices, the first-order bifurcation
   solution (26) is given by a single helical mode (see Figure 2);

2. **vortex-antivortex pattern**: on square or rhombic lattices, the first-order
   bifurcation solution (27) is a superposition of two helices propagating
   in different directions, see Figure 1(a);

3. **skyrmionic pattern**: on hexagonal lattices, the first-order bifurcation
   solution (28) is a superposition of three helices propagating in distinct
   directions, see Figure 1(b).
Figure 2: The one-dimensional modulations on a non-equilateral lattice.

We say that a bifurcation solution \((m_s, \lambda_s)\) is *linearly stable* under \(\Lambda\)-periodic perturbations if the linearized operator \(L_s = D_mF(m_s, \lambda_s)\) is non-negative with a kernel that is only induced by translations, i.e.,

\[
\ker L_s = \text{span} \{\partial_1 m_s, \partial_2 m_s\}.
\]

Reducing the domain of the bifurcation parameter \(s\) if necessary we shall prove the following stability properties of the bifurcation solutions obtained in Theorem 2:

**Theorem 3.** Let \(\alpha > 0\), \(\beta \geq 0\) and \(\kappa > 0\) satisfying

\[
\lambda_0 = -\left(\frac{1}{\text{Im}\tau}\right)^2 - \frac{\beta}{2} + \sqrt{\left(\frac{2\kappa}{\text{Im}\tau}\right)^2 + \frac{\beta^2}{4}} > 0 \tag{7}
\]

and

\[
4\kappa^2 \leq \sqrt{\left(\frac{2\kappa}{\text{Im}\tau}\right)^2 + \frac{\beta^2}{4}} + \sqrt{\left(\frac{2\kappa}{\text{Im}\tau}\right)^2 \gamma^2 + \frac{\beta^2}{4}} \tag{8}
\]

where \(\gamma\) is the quotient of the second and first critical wave number, i.e.,

\[
\gamma^2 = \begin{cases} 
\min\{4, \tau^2 - 2\tau \cos \theta + 1\} & \text{for } |\tau| > 1, \\
2 - 2 \cos \theta, & \text{for } |\tau| = 1, \frac{\pi}{3} < \theta \leq \frac{\pi}{2}, \\
3 & \text{for } |\tau| = 1, \theta = \frac{\pi}{3},
\end{cases}
\]

depending on the lattice shape \(\tau = |\tau|e^{i\theta}\).

(i) The bifurcation solution on non-equilateral lattices is linearly stable under \(\Lambda\)-periodic perturbations for any \(\beta \geq 0\);

(ii) The bifurcation solution on square lattices, *i.e.* the vortex-antivortex lattice solution, is linearly stable under \(\Lambda\)-periodic perturbations provided \(\beta > \frac{4}{\sqrt{3}} \kappa\).
(iii) The bifurcation solution on hexagonal lattices is unstable for any $\beta \geq 0$.

If $\lambda_0 \leq 0$ bifurcation solutions are unstable, independently of $\beta \geq 0$ and (8).

**Corollary 1.** The quadratic vortex-antivortex lattice configuration exists and is stable if

$$\beta > \frac{4}{\sqrt{3}} \kappa, \quad \beta \geq \sqrt{16\kappa^4 - 24\kappa^2 + 1} \quad \text{and} \quad \beta < 4\kappa^2 - 1.$$

The admissible set of $(\kappa, \beta)$ is not empty, see Figure 3.

Figure 3: The admissible set of $(\kappa, \beta)$ for a stable quadratic vortex-antivortex lattice (indicated by the grey shaded area).

The existence and stability results are only an initial step towards understanding the stabilization of two-dimensional lattice solutions in chiral magnets. In particular, stability of lattice solutions is only examined under the simplest perturbations which preserve lattice periodicity. A more general stability result in the style of [31] is beyond the scope of this work and requires a different approach.

The mathematical framework for our construction of lattice solutions is the equivariant branching lemma [12, 17], a concept of symmetry-breaking bifurcation based on a particular type of (axial) symmetry group. For each lattice we identify an isotropy subgroup $\Sigma \subset SO(3)$ so that the fixed subspace of $\Sigma$ in the kernel of linearized operator $D_mF(m_0, \lambda_0)$ is one-dimensional. By means of an equivariant Lyapunov-Schmidt procedure, (3) reduces to a one-dimensional bifurcation equation. The implicit function theorem provides a solutions to the bifurcation equation in the one-dimensional fixed subspace.
of Σ, from which a solution to (3) can be reconstructed. This solution is the bifurcation solution and inherits the symmetries featured by Σ.

In Section 2 we shall briefly recall the representation of lattices in the plane with an emphasis on symmetry and Fourier series which are key to our bifurcation argument. In Section 3 we shall derive energy bounds proving Theorem 1. Solving the linearized equation of (3) explicitly by Fourier methods is the key ingredient to the proof of Theorem 2 in Section 4 and provides insight about the morphology and topology of bifurcation solutions. In Section 5 we investigate the stability of bifurcation solutions under Λ-periodic perturbations proving Theorem 3. Finally in Section 6 we validate our analytical results by a series of numerical simulations of gradient flows using a modified Crank-Nicolson scheme.

2 Preliminaries

2.1 Representation of lattices

Recall that a planar lattice Λ is the integer span of two linearly independent vectors \( t_1, t_2 \in \mathbb{R}^2 \), i.e.,

\[
\Lambda = \{ x_1 t_1 + x_2 t_2 : x \in \mathbb{Z}^2 \}.
\]

Given \( x \in \mathbb{R}^2 \), a primitive cell of \( \Lambda \) is a set of the form

\[
\Omega_\Lambda = \{ x + a_1 t_1 + a_2 t_2, a_1, a_2 \in [0,1] \}.
\]

The lattice basis \( \{ t_1, t_2 \} \) is clearly non-unique. Identifying \( \mathbb{R}^2 \cong \mathbb{C} \), however, the complex ratio \( \tau \in \mathbb{C} \) of two basis vectors of \( \Lambda \) contained in the fundamental domain

\[
T = \left\{ |\tau| \geq 1 : \text{Im} \tau > 0, \ -\frac{1}{2} < \text{Re} \tau \leq \frac{1}{2} \text{ and } \text{Re} \tau \geq 0 \text{ if } |\tau| = 1 \right\}
\] (9)

parametrizes the lattice shape uniquely, see e.g. [1]. Writing \( \tau = |\tau|e^{i\theta} \), the range of \( |\tau| \) and \( \theta \) corresponding to fundamental domain (9) is

\[
\frac{\pi}{3} \leq \theta < \frac{2\pi}{3} \text{ if } |\tau| \geq 1 \quad \text{and} \quad \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \text{ if } |\tau| = 1.
\] (10)

A lattice is called equilateral if \( |\tau| = 1 \), where the borderline cases \( \theta = \pi/2 \) and \( \theta = \pi/3 \) are referred to as square and hexagonal lattice, respectively; other equilateral lattices are called rhombic.
There are two distinct types of symmetries preserving the lattice: the translation $\mathbb{R}^2/\Lambda$ and the holohedry, which is a finite subgroup of $O(2)$. For planar lattices there are four basic holohedries: non-equilateral lattices have holohedry $D_1$ (oblique) or $D_2$ (rectangular); rhombic lattices have holohedry $D_2$; square lattices have holohedry $D_4$; hexagonal lattices have holohedry $D_6$, see e.g. [12].

![Figure 4: The lattice cell $\Omega_\tau$ determined by lattice basis $\{2\pi\hat{e}_1, 2\pi\tau\}$](image)

### 2.2 Function spaces on lattices

As the governing energy densities and the Euler-Lagrange equations are invariant under translation and joint rotation we can fix one basis vector of the lattice $\Lambda$ as $2\pi r\hat{e}_1$ so that $\Lambda = 2\pi r(\mathbb{Z} \oplus \tau \mathbb{Z})$, is uniquely characterized by $r$ and $\tau$. Upon rescaling we can arrange $\Lambda = 2\pi (\mathbb{Z} \oplus \tau \mathbb{Z})$ spanned by $\{2\pi\hat{e}_1, 2\pi\tau\}$ and consider the rescaled density [1] containing only dimensionless parameters.

For $\Lambda$-periodic functions or fields $f, g$ on $\mathbb{R}^2$ we denote the average by

$$\langle f \rangle := \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} f(x) \, dx,$$

the $L^2$ scalar product by

$$\langle f, g \rangle := \int_{\Omega_\Lambda} f(x) \cdot g(x) \, dx,$$

and the $L^2$ norm by $\|f\| := \sqrt{\langle f, f \rangle}$ once existent. Accordingly we define

$$L^2_\Lambda := \{ f : \mathbb{R}^2 \to \mathbb{R}^3 \text{ $\Lambda$-periodic with } \|f\| < \infty \}$$
and for \( k \in \mathbb{N} \) the Sobolev spaces
\[
H^k_\Lambda := \{ f \in L^2_\Lambda : \partial^\nu f \in L^2_\Lambda \text{ for all } |\nu| = k \}
\]
which are subspaces of \( L^2_{\text{loc}} := L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^3) \) and \( H^k_{\text{loc}} := H^k_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^3) \), respectively. Thanks to Sobolev embedding the average energy
\[
E_\Lambda(m) := \frac{1}{\Omega_\Lambda} \int_{\Omega_\Lambda} \frac{1}{2} |\nabla m|^2 + \kappa m \cdot (\nabla \times m) + \frac{\lambda}{2} |m|^2 + \frac{\alpha}{4} |m|^4 + \frac{\beta}{2} (m \cdot \hat{e}_3)^2 \ dx
\]
defines an analytic functional on \( H^1_\Lambda \). Critical points \( m \in H^2_\Lambda \) of \( E_\Lambda \) satisfy
\[
F(m, \lambda) = 0
\]
where \( F : H^2_{\text{loc}} \times \mathbb{R} \to L^2_{\text{loc}} \) is the nonlinear operator given by
\[
F(m, \lambda) = -\triangle m + 2\kappa \nabla \times m + \lambda m + \alpha |m|^2 m + \beta m_3 \hat{e}_3. \tag{12}
\]

### 2.3 Dual lattice and Fourier series

Fourier expansion on \( \Lambda \) requires the notion of dual lattice given by
\[
\Lambda^* = \{ \nu \in \mathbb{R}^2 : u \cdot \nu \in 2\pi \mathbb{Z} \text{ for all } u \in \Lambda \}.
\]
In particular \( \Lambda^* = A^{-T} \mathbb{Z}^2 \) for \( \Lambda = 2\pi A \mathbb{Z}^2 \) where in our setting
\[
A = \begin{pmatrix} 1 & |\tau| \cos \theta \\ 0 & |\tau| \sin \theta \end{pmatrix} \quad \text{and} \quad A^{-T} = \begin{pmatrix} |\tau| \sin \theta \\ -|\tau| \cos \theta \\ 1 \end{pmatrix}.
\]
In the equilateral case \( |\tau| = 1 \), dual lattices remain square if \( \theta = \pi/2 \) and hexagonal if \( \theta = \pi/3 \).

For \( f \in L^2_\Lambda \) and \( \nu \in \Lambda^* \) Fourier coefficients are defined as
\[
\tilde{f}(\nu) = \int_{\Omega_\Lambda} f(x) e^{-i\nu \cdot x} \ dx,
\]
and the following Fourier expansion
\[
f(x) = \sum_{\nu \in \Lambda^*} \tilde{f}(\nu) e^{i\nu \cdot x}
\]
holds true in the \( L^2 \) sense along with Parseval’s identity
\[
\int_{\Omega_\Lambda} |f(x)|^2 \ dx = \sum_{\nu \in \Lambda^*} |\tilde{f}(\nu)|^2.
\]
2.4 Equivariance and lattice symmetry

We consider the following action of \( \sigma \in SO(3) \) on a field \( m : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)
\[
(\sigma \bullet m)(x) = \sigma m(\sigma^{-1}x) \quad \text{for} \quad x \in \mathbb{R}^2
\]
under which \( \Sigma \) is equivariant in the sense that
\[
F(\sigma \bullet m, \lambda) = \sigma \bullet F(m, \lambda).
\]

In a periodic setting we need to restrict to certain discrete subgroups \( \Sigma \) of the holohedry that preserve the lattice \( \Lambda = 2\pi(\mathbb{Z} \oplus \tau \mathbb{Z}) \) and suitable subspaces of the kernel of the linearized operator \( D_mF(0,\lambda_0) \). Three cases are to be distinguished: For \( |\tau| > 1 \) we consider the symmetry group consisting of the identity and rotation through \( \pi \)
\[
\Sigma_1 = \{ I, R \} = D_1
\]
where
\[
I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Rhombic and square lattices (\( |\tau| = 1 \) and \( \pi/3 < \theta \leq \pi/2 \)) have an additional symmetry given by reflections across the diagonals of the lattice cell. Therefore, in this case, we consider the symmetry group
\[
\Sigma_2 = \{ I, R, R^+_\theta, R^-_\theta \} = D_2 \subset D_4
\]
where
\[
R^+_\theta = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & -\cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad R^-_\theta = \begin{pmatrix} -\cos \theta & -\sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
For hexagonal lattices \( |\tau| = 1 \) and \( \theta = \pi/3 \), the symmetry group considered is the subgroup of \( D_6 \) containing the rotations by an angle of \( \frac{2\pi}{3} \)
\[
\Sigma_3 = \{ R_k, k = 0, 1, \cdots, 5 \} \subset D_6
\]
where
\[
R_k = \begin{pmatrix} \cos \frac{k\pi}{3} & -\sin \frac{k\pi}{3} & 0 \\ \sin \frac{k\pi}{3} & \cos \frac{k\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
In each of the three cases $\Sigma = \Sigma_i$, the group action is an isometry on $H^k_\Lambda$ for every $k \in \mathbb{N}_0$, and the operator (12) is $\Sigma$-equivariant in the sense that

$$F(\sigma \cdot m, \lambda) = \sigma \cdot F(m, \lambda)$$

for all $\sigma \in \Sigma$, $m \in H^2_\Lambda$ and $\lambda \in \mathbb{R}$. Given a subspace $X \subset H^2_\Lambda$, the fixed subspace of $\Sigma$ in $X$ is

$$\text{Fix}_X(\Sigma) = \{ \phi \in X : \sigma \cdot \phi = \phi \}.$$  

$L^2_\Lambda$ orthogonal projections on such invariant subspaces of $H^2_\Lambda$ are equivariant. Equivariance therefore propagates to the Lyapunov-Schmidt decomposition enabling a reduction of the bifurcation equation to $\text{Fix}_{X_0}(\Sigma)$ where $X_0$ is the kernel of the linearization of $F$ at a bifurcation point $(0, \lambda_0)$, see e.g. [12]. The symmetry of a field $\phi \in H^2_\Lambda$ is given in terms of the isotropy subgroup

$$\Sigma_\phi = \{ \sigma \in SO(3) : \sigma \cdot \phi = \phi \},$$

i.e., the largest subgroup of $SO(3)$ which fixes $\phi$. Bifurcation solutions arising from the equivariant branching lemma turn out to have full $\Sigma$ symmetry and are unique in this class.

### 3 Energy bounds on lattices

For a lattice $\Lambda$ and $e(m)$ given by (1), we examine ansatz-free lower bounds

$$E_\Lambda(m) = \int_{\Omega_\Lambda} e(m) \, dx. \quad (17)$$

We start by expressing the total exchange density

$$e_0(m) = \frac{1}{2} |\nabla m|^2 + \kappa m \cdot (\nabla \times m)$$

as a sum of sign definite terms and a null Lagrangian also know as Frank’s formula in the theory of liquid crystals, see e.g. [33] Chapter 3.

**Lemma 1.** For $m \in H^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ the following holds

$$e_0(m) = \frac{1}{2} \left( (\nabla \cdot m)^2 + |\nabla \times m + \kappa m|^2 - \kappa^2 |m|^2 + \nabla \cdot |(m \cdot \nabla)m - m(\nabla \cdot m)| \right)$$

in the sense of distributions. In particular for $m \in H^1_\Lambda$

$$E_\Lambda(m) \geq \int_{\Omega_\Lambda} \frac{|m|^2}{2} \left( \lambda - \kappa^2 + \frac{\alpha}{2} |m|^2 \right) \, dx$$

with equality if and only if $\nabla \times m + \kappa m = 0$ and $\beta = 0$.  

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From the lemma we obtain immediately:

**Proposition 1.** If \( \lambda > \kappa^2 \), then \( m \equiv 0 \) is the unique energy minimizer on every lattice. If \( \lambda < \kappa^2 \) the energy admits a lower bound

\[
E_\Lambda(m) \geq -\frac{(\lambda - \kappa^2)^2}{4\alpha}
\]

which for \( \beta = 0 \) is precisely attained for \( m \) of constant modulus

\[
|m| = \sqrt{\frac{\kappa^2 - \lambda}{\alpha}}
\]

and such that \( \nabla \times m + \kappa m = 0 \).

The unimodular Beltrami fields being parallel to their curl have been classified by Ericksen within the variational theory of liquid crystals, see e.g. [33] and references therein. For the present case of constant \( \kappa \) those are helices of pitch \( 2\pi/|\kappa| \), i.e., (4). For the convenience of the reader we present this fundamental result in the Appendix Lemma 6. Thus in order to realize the lower energy bound, the underlying lattice \( \Lambda \) is required to accommodate such a helix. In this case the zero state loses its linear stability at \( \lambda = \kappa^2 \). In fact, the Hessian

\[
H_\Lambda(m)(\phi, \phi) = \frac{d^2}{ds^2} \bigg|_{s=0} E_\Lambda(m + s\phi)
\]

for \( m, \phi \in H^1_\Lambda \) at \( m \equiv 0 \) reads

\[
H_\Lambda(0)(\phi, \phi) = \int_{\Omega_\Lambda} |\nabla \phi|^2 + 2\kappa \phi \cdot (\nabla \times \phi) + \lambda|\phi|^2 \, dx.
\]

By the preceding arguments it satisfies

\[
H_\Lambda(0)(\phi, \phi) \geq (\lambda - \kappa^2) \langle |\phi|^2 \rangle
\]

and has a helical instability at \( \lambda = \kappa^2 \).

### 4 Bifurcation on lattices

In this section we will prove the existence of the bifurcation solutions, i.e., the following extended version of Theorem 2.
Theorem 4. For every \( \Lambda \) with \( \tau \) satisfying \( (10) \), \( \alpha > 0 \), \( \beta \geq 0 \), \( \kappa > 0 \), and \( \lambda_0 = - \left( \frac{1}{\Im \tau} \right)^2 - \frac{\beta}{2} \pm \sqrt{\left( \frac{2\kappa}{\Im \tau} \right)^2 + \frac{\beta^2}{4}} \) satisfying \( (6) \), \((0, \lambda_0)\) is a bifurcation point of the equation

\[- \Delta \mathbf{m} + 2\kappa \nabla \times \mathbf{m} + \lambda \mathbf{m} + \alpha |\mathbf{m}|^2 \mathbf{m} + \beta \mathbf{m}_3 \hat{e}_3 = 0.\]  

More precisely, there is a branch of non-trivial \( \Sigma \)-invariant solutions \((\mathbf{m}_s, \lambda_s)\) in a neighbourhood of \((0, \lambda_0)\) in \( H^2_\Lambda \times \mathbb{R} \), which has the form

\[\lambda_s = \lambda_0 + s^2 \lambda_2 + s^4 \psi\lambda(s), \quad \mathbf{m}_s = s\varphi_1 + s^3 \varphi_3 + s^5 \psi\mathbf{m}(s),\]  

where \( s \in (-\delta, \delta) \) for some \( \delta > 0 \), \( \varphi_1, \varphi_3 \in H^2_\Lambda \), \( \psi\lambda : (-\delta, \delta) \to \mathbb{R} \) and \( \psi\mathbf{m} : (-\delta, \delta) \to H^2_\Lambda \) are analytic in \( s \). Furthermore, we have

\[\lambda_2 = -\alpha \frac{\langle |\varphi_1|^4 \rangle}{\langle |\varphi_1|^2 \rangle^2} < 0\]  

and

\[E_\Lambda(\mathbf{m}_s, \lambda_s) = \frac{s^4}{4} \left( -\alpha \frac{\langle |\varphi_1|^4 \rangle}{\langle |\varphi_1|^2 \rangle^2} \right) + O(s^6).\]  

For the proof we examine the linearization

\[L^{(\lambda)} = D_mF(0, \lambda) : H^2_\Lambda \to L^2_\Lambda\]  

of \( F \) at \( \mathbf{m} = 0 \) for arbitrary \( \lambda \) given by

\[L^{(\lambda)} \phi = -\Delta \phi + 2\kappa \nabla \times \phi + \lambda \phi + \beta \phi_3 \hat{e}_3.\]  

We need to find non-trivial solutions \( \Lambda \)-periodic solutions \( \phi \) of the equation

\[- \Delta \phi + 2\kappa \nabla \times \phi + \lambda \phi + \beta \phi_3 \hat{e}_3 = 0\]  

for \( \lambda = \lambda_0 \) depending on \( \kappa, \beta \) and \( \tau \).

Lemma 2. Eq. \((22)\) admits non-constant solutions \( \phi \in H^2_\Lambda \) if and only if

\[\lambda = -|\nu|^2 - \frac{\beta}{2} \pm \sqrt{4\kappa^2 |\nu|^2 + \frac{\beta^2}{4}}\]  

for some \( \nu \in \Lambda^* \setminus \{0\} \).
Proof. We expand $\phi \in H^2_\Lambda$ in Fourier series

$$\phi(x) = \sum_{\nu \in \Lambda^*} \phi_\nu e^{i\nu \cdot x} = \sum_{\nu \in \Lambda^*} \begin{pmatrix} a_\nu \\ b_\nu \\ c_\nu \end{pmatrix} e^{i\nu \cdot x}$$

for $x \in \mathbb{R}^2$, where, recalling (13),

$$\nu = (\nu_1, \nu_2) = A^{-T}k = \frac{1}{\text{Im } \tau} \begin{pmatrix} \text{Im } \tau k_1 \\ -\text{Re } \tau k_1 + k_2 \end{pmatrix}.$$ (23)

Since

$$-\Delta \phi(x) = \sum_{\nu \in \Lambda^*} |\nu|^2 \begin{pmatrix} a_\nu \\ b_\nu \\ c_\nu \end{pmatrix} e^{i\nu \cdot x}$$ and $\nabla \times \phi = \sum_{\nu \in \Lambda^*} \begin{pmatrix} iv_2 c_\nu \\ -iv_1 c_\nu \\ i(\nu_1 b_\nu - \nu_2 a_\nu) \end{pmatrix} e^{i\nu \cdot x},$

the linearized equation (22) is equivalent to the system

$$|\nu|^2 a_\nu + 2\kappa|\nu|c_\nu + \lambda a_\nu = 0$$
$$|\nu|^2 b_\nu - 2\kappa|\nu|c_\nu + \lambda b_\nu = 0$$
$$|\nu|^2 c_\nu + 2\kappa(\nu_1 b_\nu - \nu_2 a_\nu) + \lambda c_\nu + \beta c_\nu = 0,$$

for all $\nu \in \Lambda^*$. Constant solutions with $\nu = 0$ exist only if $\lambda = 0$ or $\lambda + \beta = 0$. For $\nu \neq 0$, we have

$$a_\nu = -\frac{2\kappa \nu_2}{|\nu|^2 + \lambda} c_\nu, \quad b_\nu = \frac{2\kappa \nu_1}{|\nu|^2 + \lambda} c_\nu$$ (24)

and

$$\left( |\nu|^2 + \lambda + \beta - \frac{4\kappa^2 |\nu|^2}{|\nu|^2 + \lambda} \right) c_\nu = 0$$

which is possible for $c_\nu \neq 0$ only if $\lambda = -|\nu|^2 - \frac{\beta}{2} \pm \sqrt{4\kappa^2 |\nu|^2 + \frac{\beta^2}{4}}$. \qed

We focus on the wave vectors $\nu \in \Lambda^* \setminus \{0\}$ of shortest length. Bifurcations at $\lambda$ corresponding to wave vectors with longer length, if they exist, turn out to be unstable, see proof of Lemma 4 below.

A straightforward analysis of the minimizing problem

$$|\nu|^2 = \left( \frac{1}{\text{Im } \tau} \right)^2 \min_{k \in \mathbb{Z}^2 \setminus \{0\}} \left( \text{Im } \tau k_1 \right)^2 + (\text{Re } \tau k_1 - k_2)^2$$

yields (see Figure 5):
Lemma 3. Let \( \nu \in \Lambda^* \setminus \{0\} \). Then \( |\nu| \) is minimized by

(i) \( k = (0, \pm 1) \) if \( |\tau| > 1 \),
(ii) \( k = (\pm 1, 0), (0, \pm 1) \) if \( |\tau| = 1, \frac{\pi}{3} < \theta \leq \frac{\pi}{2} \), and
(iii) \( k = (\pm 1, 0), (0, \pm 1), \pm (1, 1) \) if \( |\tau| = 1, \theta = \frac{\pi}{3} \)

with minimal value

\[
|\nu| = \frac{1}{\text{Im} \tau}.
\]

Figure 5: The critical circle with radius \( \frac{1}{\text{Im} \tau} \) around 0 and the critical wave vectors (red points) on the dual lattice of a (a) non-equilateral lattice, (b) rhombic lattice, (c) square lattice and (d) hexagonal lattice.

Hence, in this case, we shall consider the bifurcation occurring at

\[
\lambda_0 = -\left( \frac{1}{\text{Im} \tau} \right)^2 - \frac{\beta}{2} \pm \sqrt{\left( \frac{2\kappa}{\text{Im} \tau} \right)^2 + \frac{\beta^2}{4}}
\]

satisfying (6), which guarantees that only the first non-trivial wave number contributes to the kernel of the linearization.

Proposition 2. Under the assumptions of Theorem 2, the linearization

\[
L_0 := D_m F(0, \lambda_0) : H^2 \subset L^2_{\Lambda} \to L^2_{\Lambda}
\]

is a self-adjoint Fredholm operator with \( \dim \ker L_0 = N \), where \( N = 2 \) on non-equilateral lattices, \( N = 4 \) on rhombic and square lattices and \( N = 6 \) on hexagonal lattices. The fixed subspace of the corresponding symmetry group \( \Sigma \) (see section 2.4) in \( X_0 = \ker L_0 \) is one-dimensional,

\[
\text{Fix}_{X_0}(\Sigma) = \text{span} \{ \varphi_1 \}.
\]
Proof. The normalized Fourier coefficients \([24]\) obtained in the proof of Lemma \([2]\) are

\[
\phi_\nu = \frac{1}{\sqrt{1 + A^2}} \left( A \left( \frac{-i\nu_2}{|\nu|} \right) \right)
\]

where

\[
A = \frac{2\kappa \Im \tau}{-\frac{\beta}{2} \pm \sqrt{\left( \frac{2\kappa}{\Im \tau} \right)^2 + \frac{\beta^2}{4}}}. \quad (25)
\]

The corresponding real-space solutions of \([22]\) to the wave vector \(\nu\) are

\[
\phi_{1,\nu}(x) = \phi_\nu e^{i\nu \cdot x} + \phi_{-\nu} e^{-i\nu \cdot x} = \frac{1}{\sqrt{1 + A^2}} \left( A \left( \frac{i\nu_2}{|\nu|} \sin(\nu \cdot x) \right) \right)
\]

and

\[
\phi_{2,\nu}(x) = i\phi_\nu e^{i\nu \cdot x} + i\phi_{-\nu} e^{-i\nu \cdot x} = \frac{1}{\sqrt{1 + A^2}} \left( A \left( \frac{-i\nu_2}{|\nu|} \cos(\nu \cdot x) \right) \right).
\]

We need to find all solutions of \(L_0 \phi = 0\) for a given wave number \(|\nu|\). According to Lemma \([3]\) we consider following cases separately.

Case 1: Non-equilateral lattice, \(|\tau| > 1\)

The wave vector \([23]\) corresponding to \(k = (0, \pm 1)\) are

\[
\nu^{(1)} = \left( \begin{array}{c} 0 \\ \frac{1}{|\tau| \sin \theta} \end{array} \right) \in \Lambda^*.
\]

Therefore

\[
\ker L_0 = \text{span}\{\phi_{1,\nu^{(1)}}, \phi_{2,\nu^{(1)}}\}.
\]

Since only \(\phi_{1,\nu^{(1)}}\) is fixed under the group \(\Sigma = \Sigma_1\) given in \([14]\), we have

\[
\text{Fix}_{X_0}(\Sigma) = \text{span}\{\varphi_1\}
\]

with the \(L^2_\Lambda\) normalized

\[
\varphi_1 = \sqrt{2}\phi_{1,\nu^{(1)}} = \sqrt{\frac{2}{1 + A^2}} \left( A \sin \left( \frac{\nu_2}{|\tau| \sin \theta} \right) \right)
\]

\[
\cos \left( \frac{\nu_2}{|\tau| \sin \theta} \right).
\]

(26)

Case 2: Rhombic or square lattice, \(|\tau| = 1\), \(\frac{\pi}{3} < \theta \leq \frac{\pi}{2}\)
The wave vectors (23) corresponding to \( k = (\pm 1, 0), (0, \pm 1) \)

\[
\nu^{(2)} = \left( \frac{1}{\cos \theta}, \frac{1}{\sin \theta} \right) \quad \text{and} \quad \nu^{(3)} = \left( 0, \frac{1}{\sin \theta} \right).
\]

Therefore

\[
\ker L_0 = \text{span}\{\phi_{i,\nu(j)}, \ i = 1, 2, \ j = 2, 3\}.
\]

The fixed subspace of \( \Sigma = \Sigma_2 \) (see (15)) in the kernel \( X_0 \) is

\[
\text{Fix}_{X_0}(\Sigma) = \text{span}\{\varphi_1\}
\]

with the \( L_2^\Lambda \)-normalized field

\[
\varphi_1 = \phi_{1,\nu(2)} - \phi_{1,\nu(3)} = \frac{1}{\sqrt{1 + A^2}} \left( A \left( -\cos \theta \sin(x_1 - \frac{\cos \theta}{\sin \theta} x_2) + \sin(x_2) \right) - \sin \theta \sin(x_1 - \frac{\cos \theta}{\sin \theta} x_2) \right) - \cos(x_1 - \frac{\cos \theta}{\sin \theta} x_2) - \cos \left( \frac{x_2}{\sin \theta} \right)
\]

(27)

Case 3: Hexagonal lattice, \( |\tau| = 1, \ \theta = \frac{\pi}{3} \)

The wave vectors (23) corresponding to \( k = (\pm 1, 0), (0, \pm 1), \pm(1, 1) \) are

\[
\nu^{(4)} = \left( 0, 1, \frac{1}{\sqrt{3}} \right), \quad \nu^{(5)} = \left( 0, \frac{2}{\sqrt{3}} \right), \quad \nu^{(6)} = \left( 1, \frac{1}{\sqrt{3}} \right).
\]

Therefore

\[
\ker L_0 = \text{span}\{\phi_{i,\nu(j)}, \ i = 1, 2, \ j = 4, 5, 6\}.
\]

The fixed subspace of \( \Sigma = \Sigma_3 \) (see (16)) in the kernel \( X_0 \) is

\[
\text{Fix}_{X_0}(\Sigma) = \text{span}\{\varphi_1\}
\]

with the \( L_2^\Lambda \)-normalized field

\[
\varphi_1 = \frac{\sqrt{2}}{3} (\phi_{1,\nu(4)} + \phi_{1,\nu(5)} + \phi_{1,\nu(6)})
\]

\[
= \sqrt{\frac{2}{3(A^2 + 1)}} \left( A \left( \frac{1}{\sqrt{3}} \sin(x_1 - \frac{1}{\sqrt{3}} x_2) + \sin(x_1 + \frac{1}{\sqrt{3}} x_2) \right) - \frac{\sqrt{3}}{2} \sin(x_1 - \frac{1}{\sqrt{3}} x_2) - \frac{\sqrt{3}}{2} \sin(x_1 + \frac{1}{\sqrt{3}} x_2) \cos(x_1 - \frac{1}{\sqrt{3}} x_2) + \cos \left( \frac{x_2}{\sqrt{3}} \right) + \cos \left( x_1 + \frac{1}{\sqrt{3}} x_2 \right) \right).
\]

(28)
Remark 1. The first order bifurcation solution on non-equilateral lattices is an exact solution of the nonlinear equation at $\beta = 0$, i.e.

$$F(s\phi_1, \lambda_0 - as^2) = 0 \quad \text{for any} \quad s \in \mathbb{R}.$$  

Proof of Theorem 4. Clearly $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. According to Proposition 2, the fixed subspace of the symmetry group $\Sigma = \Sigma_i, i = 1, 2, 3$, is 1-dimensional in all three cases

$$\text{Fix}_{X_0}(\Sigma) = \text{span}\{\varphi_1\},$$

and

$$D_\lambda D_m F(0, \lambda_0) \langle \phi \rangle = \phi \notin \text{ran} L_0 \quad \text{for any nonzero} \ \phi \in \text{Fix}_{X_0}(\Sigma).$$

Invoking the equivariant branching lemma (see [12, 17]) we conclude the existence of bifurcation solutions in the form of

$$\lambda_s = \lambda_0 + \varphi_\lambda(s), \quad m_s = s\varphi_1 + \varphi_m(s)$$

where $s \in (-\delta, \delta)$ for some $\delta > 0$, $\varphi_\lambda : (-\delta, \delta) \to \mathbb{R}$ and $\varphi_m : (-\delta, \delta) \to H^2_\Lambda$ are analytic in $s$ satisfying $\varphi_\lambda(0) = 0, \varphi_m(0) = 0$ and $\langle \varphi_m(s), \varphi_1 \rangle = 0$.

Inserting the analytic expansion of bifurcation solutions into (18) and (17) we obtain (19)-(21). Explicit calculations are carried out in Appendix B.

Topology of bifurcation solutions On hexagonal lattices, the bifurcation solution is nowhere vanishing and features in every primitive cell a skyrmion, i.e. a vortex-like structure with the magnetizations pointing upwards at the core and downwards at the perimeter, see Figure 6(a).

On rhombic and square lattices, the horizontal component of $\varphi_1$ has a finite number of isolated zeros in a primitive cell and forms a vortex or an antivortex around each zero. The antivortices are half-skyrmions (sometimes referred to as merons, see e.g. [34]) and have magnetizations pointing upwards or downwards at the core; while the center of vortices are singularity points ($m = 0$) due to the continuity and the $\Sigma$-invariance of bifurcation solutions, see Figure 6(b).

5 Linear stability of the bifurcation solutions

In this section we discuss the stability of bifurcation solutions following the standard perturbation argument, see e.g. [19]. Suppose $(m_s, \lambda_s)$ is a
Figure 6: Top view of $\phi_1$ on (a) hexagonal lattice and (b) square lattice. The cones indicate the direction and length of the in-plane magnetizations; the color indicates the out-of-plane magnetizations (red: upwards, green: in-plane, blue: downwards). The dashed lines in (a) and (b) define a primitive cell of the hexagonal and square lattice, respectively.

We focus on the larger root as positivity turns out to be necessary for the stability of bifurcation solutions. The linearization of (3) at $(m_s, \lambda_s)$

$L_s = D_m F(m_s, \lambda_s) : H^2_\Lambda \subset L^2_\Lambda \rightarrow L^2_\Lambda$

is given by

$L_s \phi = -\Delta \phi + 2\kappa \nabla \times \phi + \lambda_s \phi + \alpha \left( |m_s|^2 \phi + 2(m_s \cdot \phi)m_s \right) + \beta \phi_3 \hat{e}_3.$

We first investigate the spectrum of $L_0$.

**Lemma 4.** The spectrum of $L_0$ is discrete and non-negative if (7) and (8) hold. Otherwise $L_0$ has negative eigenvalues.

**Proof.** The equation $(L_0 - \mu)\phi = 0$ admits constant nonzero solutions at $\mu = \lambda_0$. So $L_0$ has negative eigenvalues with constant eigenfunctions precisely if $\lambda_0 < 0$. As in Lemma 2 the eigenvalues of $L_0$ with non-constant
eigenfunctions at $\lambda_0$ are

$$
\mu_{0,\omega} = |\omega|^2 - \left(\frac{1}{\text{Im} \tau}\right)^2 + \sqrt{\left(\frac{2\kappa}{\text{Im} \tau}\right)^2 + \frac{\beta^2}{4}} \pm \sqrt{4\kappa^2|\omega|^2 + \frac{\beta^2}{4}}, \quad \omega \in \Lambda^* \setminus \{0\}.
$$

We have $|\omega| \geq 1/\text{Im} \tau$ for all $\omega \in \Lambda^* \setminus \{0\}$, which, together with (8), ensures that $\mu_{0,\omega+} \geq 0$ and

$$
\mu_{0,\omega-} = \left(|\omega|^2 - |\nu|^2\right) \left(1 - \frac{4\kappa^2}{\sqrt{\left(\frac{2\kappa}{\text{Im} \tau}\right)^2 + \frac{\beta^2}{4}} + \sqrt{4\kappa^2|\omega|^2 + \frac{\beta^2}{4}}}ight) \geq 0
$$

for all $\omega \in \Lambda^* \setminus \{0\}$ and any $\beta \geq 0$.

From now on we focus on the case $\lambda_0 > 0$. It follows from Lemma 4 and the standard perturbation theory of eigenvalue [18] that the spectrum of $L_s$ consists of eigenvalues of the same multiplicities in a neighbourhood of the eigenvalues of $L_0$. Thus the stability of $(m_s, \lambda_s)$ depends on the perturbation of the critical eigenvalue 0. It follows from Proposition 2 that there exist the following topological decompositions

$$
L^2_\Lambda = \ker L_0 \oplus \text{ran} L_0 \quad \text{and} \quad H^2_\Lambda = \ker L_0 \oplus X^1,
$$

where $X^1 = \{\phi \in H^2_\Lambda : \phi \perp \ker L_0 \text{ in } L^2\}$.

We first consider the perturbation of the zero eigenvalue corresponding to the eigenvector $\varphi_1$ spanning $\text{Fix}_{X^0}(\Sigma)$.

**Lemma 5.** After reducing $\delta > 0$ if necessary, there exists a smooth map

$$(\mu, \psi) : (-\delta, \delta) \to \mathbb{R} \times X^1 \quad \text{with} \quad (\mu(0), \psi(0)) = (0, 0)$$

such that

$$L_s(\varphi_1 + \psi(s)) = \mu(s)(\varphi_1 + \psi(s)). \quad (29)$$

Furthermore, there exists a $C > 0$ so that $\mu(s) = C\alpha s^2$ for $s \in (-\delta, \delta)$.

**Proof.** We introduce the smooth operator

$$G : (\text{Fix}_{X^0}(\Sigma) \oplus X^1) \times \mathbb{R} \times X_1 \times \mathbb{R} \to \text{Fix}_{X^0}(\Sigma) \oplus \text{ran} L_0$$

given by

$$G(m_s, \lambda_s, \psi, \mu) = L_s(\varphi_1 + \psi) - \mu(\varphi_1 + \psi).$$

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Since $L_\phi \in \text{Fix}_{X_0}(\Sigma) \oplus \text{ran} L_0$ for $\phi \in \text{Fix}_{X_0}(\Sigma) \oplus X_1$, this operator is well-defined. As $G(0, \lambda_0, 0, 0) = 0$ and the differential
\[ \partial_{(\phi, \mu)} G(0, \lambda_0, 0, 0) : X_1 \times \mathbb{R} \to \text{Fix}_{X_0}(\Sigma) \oplus \text{ran} L_0 \]
given by
\[ \partial_{(\phi, \mu)} G(0, \lambda_0, 0, 0) = \begin{pmatrix} L_0 \\ -\phi_1 \end{pmatrix} \]
is invertible, the implicit function theorem provides a smooth map
\[ (\mu, \psi) : (-\delta, \delta) \to \mathbb{R} \times X_1 \] with $(\mu(0), \psi(0)) = (0, 0)$ so that $G(m_s, \lambda_s, \psi(s), \mu(s)) = 0$ for $s \in (-\delta, \delta)$.

It remains to examine the properties of $\mu(s)$. Differentiating (29) with respect to $s$ in $s = 0$ yields
\[ \dot{\mu}(0) \phi_1 = D_\mu^2 F(m_0, \lambda_0) \langle \phi_1, \phi_1 \rangle + D_\lambda D_m F(m_0, \lambda_0) \langle \phi_1 \rangle \dot{\lambda} |_{s=0} = L_0 \dot{\psi}(0) \]
since $\dot{\lambda} = 2s \lambda_2 + O(s^2)$ and
\[ D_\mu^2 F(m_0, \lambda_0) \langle \phi, \psi \rangle = 2\alpha((\phi \cdot \psi) m_0 + (m_0 \cdot \phi) \psi + (m_0 \cdot \psi) \phi) = 0 \]
with $m_0 = 0$. Testing (30) with $\phi_1$ yields
\[ \dot{\mu}(0) = \frac{\langle L_0 \dot{\psi}(0), \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = 0. \]

Calculating the second derivative of $\mu(s)$ at $s = 0$, we obtain
\[ \ddot{\mu}(0) \phi_1 = D_\mu^2 F(m_0, \lambda_0) \langle \phi_1, \phi_1 \rangle + D_m F(m_0, \lambda_0) \langle \dot{\psi}(0) \rangle + D_\lambda D_m F(m_0, \lambda_0) \langle \phi_1 \rangle \ddot{\lambda}(0). \]

Taking into account
\[ D_\lambda D_m F(m_0, \lambda_0) \langle \phi \rangle = \phi, \quad \ddot{\lambda}(0) = 2\lambda_2 \]
\[ D_m^3 F(m_0, \lambda_0) \langle \phi, \phi, \psi \rangle = 2\alpha((\phi \cdot \phi) \psi + (\phi \cdot \psi) \phi + (\phi \cdot \psi) \phi) \]
and using Theorem [4] yields
\[ \ddot{\mu}(0) = \frac{6\alpha \langle |\phi_1|^2 \phi_1, \phi_1 \rangle + 2\lambda_2 \langle \phi_1, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = 4\alpha \langle |\phi_1|^4 \rangle. \]

Hence $\mu(s) = Cas^2$ with a positive constant $C$. Provided $\alpha > 0$, we have $\mu(s) > 0$ for nonzero $s \in (-\delta, \delta)$. \hfill $\Box$
The helical solution on non-equilateral lattices is stable.

**Proposition 3.** For $\beta \geq 0$ and every nonzero $s \in (-\delta, \delta)$, the bifurcation solution on non-equilateral lattices is linearly stable in the sense that

$$L_s \geq 0 \quad \text{with} \quad \ker L_s = \text{span}\{\partial_2 m_s\}.$$  

**Proof.** Recall that on non-equilateral lattices

$$\ker L_0 = \text{span}\{\phi_{1,\nu(1)} \} \oplus \text{span}\{\phi_{2,\nu(1)} \},$$

where both $\varphi_1 = \phi_{1,\nu(1)}$ and $\phi_{2,\nu(1)}$ depend only on the spatial variable $x_2$ and $\partial_2 \varphi_1 = \frac{1}{\text{Im } \tau} \phi_{2,\nu(1)}$. By translational invariance $\partial_2 m_s = s\partial_2 \varphi_1 + O(s^3)$ is, for small $s$, a non-trivial element of $\ker L_s$, and the claim follows with Lemma 5. \qed

The quadratic vortex-antivortex lattice is stable under large enough anisotropy.  

**Proposition 4.** For every nonzero $s \in (-\delta, \delta)$, the bifurcation solution on square lattices is linearly stable in the sense that

$$L_s \geq 0 \quad \text{with} \quad \ker L_s = \text{span}\{\partial_1 m_s, \partial_2 m_s\}.$$  

provided $\alpha > 0$ and

$$\beta > \frac{4}{\sqrt{3}} \kappa \approx 2.3 \kappa.$$  

For $\beta < \frac{4}{\sqrt{3}} \kappa$ there exists $\phi \in L^2_{\Lambda}$ such that $\langle L_s \phi, \phi \rangle < 0$.

**Proof.** Recall that on square lattices

$$\ker L_0 = \text{span}\{\phi_{1,\nu(2)}, \phi_{1,\nu(3)}, \phi_{2,\nu(2)}, \phi_{2,\nu(3)}\}$$

$$= \text{span}\{\phi_{1,\nu(2)} - \phi_{1,\nu(3)}\} \oplus \text{span}\{\phi_{1,\nu(2)} + \phi_{1,\nu(3)}\} \oplus \text{span}\{\phi_{2,\nu(2)}, \phi_{2,\nu(3)}\}.$$  

Note that $\varphi_1 = \phi_{1,\nu(2)} - \phi_{1,\nu(3)}$ is $\Sigma$-invariant, while $\tilde{\varphi}_1 = \phi_{1,\nu(2)} + \phi_{1,\nu(3)}$ is invariant under another symmetry group $\tilde{\Sigma}$

$$\tilde{\Sigma} = \{R_k, k = 0, 1, 2, 3\} \quad \text{where} \quad R_k = \begin{pmatrix}
\cos \frac{k\pi}{4} & -\sin \frac{k\pi}{4} & 0 \\
\sin \frac{k\pi}{4} & \cos \frac{k\pi}{4} & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  

Repeating the argument from Lemma 5, there exists a smooth map

$$(\tilde{\mu}, \tilde{\psi}) : (-\delta, \delta) \rightarrow \mathbb{R} \times X_1 \quad \text{with} \quad (\tilde{\mu}(0), \tilde{\psi}(0)) = (0, 0).$$  

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such that
\[ L_s(\tilde{\varphi}_1 + \tilde{\psi}) = \tilde{\mu}(s)(\tilde{\varphi}_1 + \tilde{\psi}(s)). \]
Moreover, we obtain \( \tilde{\mu}(s) = \tilde{C} s^2 \) with
\[ \tilde{C} = 4\langle (\tilde{\varphi}_1 \cdot \varphi_1)^2 \rangle + 2\langle (|\tilde{\varphi}_1|^2)\varphi_1|^2 \rangle + 2\frac{\lambda_2}{\alpha} \langle |\tilde{\varphi}_1|^2 \rangle = \frac{A^2 - 3}{A^2 + 1} \]
where \( A \) is the amplitude given in (25) depending on \( \kappa \) and \( \beta \). \( \tilde{C} \) is positive provided \( A^2 > 3 \), which is fulfilled if \( \beta > \frac{4}{\sqrt{3}} \kappa \).

Finally \( \partial_i \mathbf{m}_s = s \partial_i \varphi_1 + O(s^3), \ i = 1, 2, \) are linearly independent for small \( s \) due to the linear independence of \( \partial_1 \varphi_1 = \phi_{2,\nu(2)} \) and \( \partial_2 \varphi_1 = -\phi_{2,\nu(3)} \), and annihilate \( L_s \) by translational invariance of \( F \). We conclude that \( L_s \geq 0 \) and
\[ \ker L_s = \text{span}\{\partial_1 \mathbf{m}_s, \partial_2 \mathbf{m}_s\}. \]
If \( \beta < \frac{4}{\sqrt{3}} \kappa \), then \( \langle L_s \phi, \phi \rangle < 0 \) for any \( \phi \in \text{Fix}_{X_0}(\tilde{\Sigma}) \).

The hexagonal skyrmion lattice is unstable independently of any additional easy-plane anisotropy: For example, for any nonzero \( s \in (-\delta, \delta) \)
\[ \langle L_s \phi, \phi \rangle = -\frac{\alpha(2A^2 + 3)}{3(A^2 + 1)} s^2 + O(s^4) < 0 \]
for \( \phi = \phi_{1,\nu(4)} - \phi_{1,\nu(5)} \).

6 Numerical simulations

6.1 Numerical scheme
To examine critical points we consider the \( L^2 \)-gradient flow equation for the energy functional \( E_\Lambda \)
\[ \partial_t \mathbf{m} + \text{grad}_{L^2} E_\Lambda(\mathbf{m}) = 0. \tag{31} \]
Decomposing the energy gradient into a linear (second-order elliptic)
\[ \mathcal{L} \mathbf{m} := -\Delta \mathbf{m} + 2\kappa \nabla \times \mathbf{m} + \lambda \mathbf{m} + \beta m_3 \hat{e}_3 \]
and a nonlinear part
\[ \mathcal{N}(\mathbf{m}) := -\alpha |\mathbf{m}|^2 \mathbf{m}, \]
the gradient flow equation \( (31) \) reads

\[
\partial_t \mathbf{m} + \mathcal{L} \mathbf{m} = \mathcal{N}(\mathbf{m}).
\]  

(32)

We aim to find equilibria of the energy functional \( E_\Lambda \) by solving \( (32) \) numerically on the primitive cell \( \Omega_\tau \) induced by lattice spanned by \( \{ 2\pi \hat{e}_1, 2\pi \tau \} \).

Eq. \( (32) \) is discretized by a modified Crank-Nicolson approximation for the time variable and a Fourier collocation method for the space variable. We denote \( m_N \) the trigonometric interpolation function of \( m \) on the discretized grid by \( N^2 \) collocation points

\[
x_{ij} := 2\pi \left( \frac{i}{N}, -\tau \cos \theta i + j \frac{\tau}{N} \sin \theta \right), \quad (i, j) \in \mathbb{N}_N^2,
\]

for \( \mathbb{N}_N = \{0, \ldots, N-1\} \), \( N \in \mathbb{N} \) and odd. For continuous fields \( u, v \) on \( \Lambda_\tau \), we define the discrete \( L^2 \) scalar product

\[
\langle u, w \rangle_N := \left( \frac{2\pi}{N} \right)^2 \frac{1}{|\Omega_\tau|} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} u(x_{ij}) \cdot w(x_{ij}),
\]

the associated norm \( \| \cdot \|_N \), defined by \( \| u \|_N^2 := \langle u, u \rangle_N \) and the discrete energy

\[
E_N(\mathbf{m}) := \frac{1}{2} \langle \mathbf{m}, \mathcal{L} \mathbf{m} \rangle_N + \langle 1, W(\mathbf{m}) \rangle_N \quad \text{with} \quad W(\mathbf{m}) = \frac{\alpha}{4} |\mathbf{m}|^4.
\]

Our numerical scheme at time iteration \( n + 1 \) reads: find \( m^{n+1}_N \) such that

\[
\frac{m^{n+1}_N - m^n_N}{\Delta t} + \mathcal{L} \left( \frac{m^{n+1}_N + m^n_N}{2} \right) = I_N \mathcal{N}(m^n_N, m^{n+1}_N),
\]

where \( I_N \) denotes the trigonometric interpolation operator and

\[
\mathcal{N}(u, w) = -\alpha \frac{u + w}{4} \left( |u|^2 + |w|^2 \right).
\]

At each time iteration, in order to find the solution \( m^{n+1}_N \), we use a fixed point iteration

\[
m^{n+1}_{N,k+1} = \left( \text{Id} + \frac{\Delta t}{2} \mathcal{L} \right)^{-1} \left[ \left( \text{Id} - \frac{\Delta t}{2} \mathcal{L} \right) m^n_N + \Delta t \mathcal{N}(m^{n+1}_{N,k}, m^n_N) \right]
\]

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for some time step $\Delta t$. Well-posedness and a-priori error bounds of this numerical scheme follow analogously as in [13]. Given initial data $m^0_N$, the corresponding sequence $(m^N_n)_{n \in \mathbb{N}_0}$ satisfies the following energy law

$$\frac{1}{\Delta t} \|m^{n+1}_N - m^n_N\|^2_N + E_N(m^{n+1}_N) = E_N(m^n_N), \quad n \in \mathbb{N}_0.$$

We have implemented this numerical scheme in MATLAB. At each time-step the iteration process stops if a certain norm of the difference of two successive iterations becomes smaller than a chosen stopping tolerance. In our case we choose the $L^\infty$-norm and set the stopping tolerance to $10^{-8}$. The discrete energy is evaluated in each time step, and the terminal time is controlled through a smallness condition for the discrete energy gradient, i.e.

$$\frac{E(m^n_N) - E(m^{n+1}_N)}{\Delta t} < 10^{-7}.$$

After the termination of the scheme, an equilibrium configuration is reached approximately.

### 6.2 Numerical experiments

We have implemented the method on a lattice of $275 \times 275$ grid points and with a time increment $\Delta t = 0.1$ for different parameters and a randomly distributed initial field with modulus between 0 and 0.1 as initial condition.

**Bifurcation without anisotropy** We first performed the simulations at $\beta = 0$ on equilateral lattices, i.e. $|\tau| = 1$ with different angle $\theta$ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$ by setting $\lambda = \lambda_0 + \delta \lambda_2$, where $\delta = 0.001$, $\lambda_0$ and $\lambda_2$ are calculated according to (7) and (20), respectively. Despite our instability results at $\beta = 0$ (see Section 5) doubly periodic solutions emerge numerically. As displayed in Figure 7 vortex-antivortex lattice configurations were observed for different $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$, while the skyrmion lattice was observed for $\theta = \frac{\pi}{3}$.

Then we implemented the method on lattices $|\tau|e^{i\frac{\pi}{2}}$ with different $|\tau| \geq 1$. For $|\tau| \approx 1$ we obtained the vortex-antivortex lattice configuration, indicating its stability under perturbations of the lattice shape, while for $|\tau|$ large enough only the helix state as shown in Figure 8 was observable.

**Bifurcation with easy-plane anisotropy** Next we included easy-plane anisotropy and implemented the simulations on a square lattice for different parameters $\kappa$ and $\beta$ near the bifurcation point $\lambda_0$. 

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Figure 7: Simulation results on equilateral lattices $e^{i\theta}$ for $\kappa = 1$, $\alpha = 1$, $\beta = 0$ and different values of the lattice angle, respectively (a) $\theta = 60^\circ$, $\lambda = 0.975$, (b) $\theta = 75^\circ$, $\lambda = 0.998$, (c) $\theta = 90^\circ$, $\lambda = 0.999$. The colour indicated the out-of-plane magnetization $m_3$.

Figure 8: Simulation results on lattices $|\tau|e^{i\pi/2}$ for $\kappa = 1$, $\alpha = 1$, $\beta = 0$ and different values of $\tau$, respectively (a) $|\tau| = 1$, $\lambda = 0.9988$, (b) $|\tau| = 1.02$, $\lambda = 0.9986$, (c) $|\tau| = 1.1$, $\lambda = 0.9907$. 
For $\kappa \in (0, 0.5)$, the bifurcation point $\lambda_0$ is negative for any $\beta \geq 0$. In this case, we obtained an almost homogeneous field.

When the value of $\kappa$ was increased over 0.5, vortex-antivortex lattice configurations were observed for $\beta \geq 0$ small enough so that $\lambda_0 > 0$. For $\beta$ large enough so that $\lambda_0 < 0$, the vortex-antivortex lattice configuration decayed to an almost homogeneous field, as shown in Figure 9.

![Figure 9](image)

Figure 9: Simulation results on square lattice for $\alpha = 1$ and different values of $\kappa$ and $\beta$, respectively (a) $\kappa = -0.6$, $\beta = 0$, $\lambda = 0.1875$, (b) $\kappa = -0.6$, $\beta = 1$, $\lambda = -0.2125$, (c) $\kappa = 0.8$, $\beta = 1$, $\lambda = 0.1638$, (d) $\kappa = 0.8$, $\beta = 2$, $\lambda = -0.1257$.

Other patterns emerged for $\kappa > 1.2$ and small $\beta \geq 0$. For example, at $\kappa = 1.4$ and small $\beta \geq 0$ the solution converged to a stripe pattern, i.e. helices with a pitch smaller than $2\pi$. Vortex-antivortex lattice configurations were observed for $\beta$ larger than the stability threshold (about $2.3\kappa$) and $\lambda_0 > 0$. Increasing $\beta$ further so that $\lambda_0 < 0$, we obtained the almost homogeneous field again, as shown in Figure 10.

![Figure 10](image)

Figure 10: Simulation results on square lattice for $\kappa = 1.4$, $\alpha = 1$ and different values of $\beta$, respectively (a) $\beta = 3$, $\lambda = 0.6640$, (b) $\beta = 4$, $\lambda = 0.1116$, (c) $\beta = 5$, $\lambda = 0.4284$, (d) $\beta = 7$, $\lambda = -0.0303$. 

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A Helical state

Lemma 6. Suppose \( \kappa \neq 0 \) and \( \nabla \times \mathbf{m} + \kappa \mathbf{m} = 0 \) for some measurable \( \mathbf{m} \) field with \( |\mathbf{m}| = M \) on a connected domain in \( \mathbb{R}^3 \). Then \( \mathbf{m} \) is a helix of pitch \( 2\pi/\kappa \), i.e., there exists \( R \in SO(3) \) such that

\[
\mathbf{m}(x) = M(R \cdot \mathbf{h})(\kappa x) \quad \text{where} \quad \mathbf{h}(x) = (0, \cos x_1, \sin x_1).
\]

Proof. Upon rescaling one may assume \( M = 1 \) and \( \kappa = 1 \). Taking the divergence it follows that \( \nabla \cdot \mathbf{m} = 0 \) and hence \( \Delta \mathbf{m} + \mathbf{m} = 0 \) in the sense of distributions by taking the curl. So \( \mathbf{m} \) is smooth by virtue of standard elliptic regularity theory. In particular, it is enough to prove the claim locally. Denoting the componentwise gradient by \( (\nabla \mathbf{m})_{jk} := (\partial_j m_k) \) we claim that

\[
\text{rank}(\nabla \mathbf{m}) = 1 \quad \text{and} \quad (\nabla \mathbf{m})^2 = 0. \tag{33}
\]

In fact, we observe that for arbitrary smooth fields

\[
\nabla \cdot ((\mathbf{m} \cdot \nabla)\mathbf{m}) = \text{tr}(\nabla \mathbf{m})^2 + (\mathbf{m} \cdot \nabla)(\nabla \cdot \mathbf{m}) \tag{34}
\]

and by using the assumptions

\[
(\mathbf{m} \cdot \nabla)\mathbf{m} = (\nabla \mathbf{m})^T \mathbf{m} = ((\nabla \mathbf{m})^T - (\nabla \mathbf{m})) \mathbf{m} = (\nabla \times \mathbf{m}) \times \mathbf{m} = 0.
\]

Collecting all these facts we obtain

\[
\mathbf{m} \in \ker(\nabla \mathbf{m}) \cap \ker((\nabla \mathbf{m})^T) \tag{35}
\]

and

\[
\text{tr}(\nabla \mathbf{m}) = \text{tr}(\nabla \mathbf{m})^2 = 0. \tag{36}
\]

Fixing a point \( x \) we may assume after rotation \( \mathbf{m}(x) = \hat{e}_3 \), so that by (35) the matrix \( \nabla \mathbf{m}(x) \) is given by a \( 2 \times 2 \) matrix \( A \) such that \( \text{tr} \ A = 0 \) and \( \text{tr} \ A^2 = 0 \) by (36). It is easy to see that \( A^2 = 0 \) which implies \( \det A = 0 \).

According to (33) there exist local smooth unit vector fields \( X \) and \( Y \) and a function \( \lambda \) such that

\[
\nabla \mathbf{m} = \lambda X \otimes Y \quad \text{and} \quad \mathbf{m} = X \times Y.
\]
Now $\nabla \times \mathbf{m} = \lambda \mathbf{X} \times \mathbf{Y} = \lambda \mathbf{m}$, hence $\lambda = -1$ and $\nabla \mathbf{m} = -\mathbf{X} \otimes \mathbf{Y}$.

Assuming $\mathbf{X}$ is constant so that after rotation $\mathbf{X} = \hat{e}_1$, it follows that $\mathbf{m} = \mathbf{m}(x_1)$ and $m_1 = 0$. Now the equation implies for the remaining components $m'_2 = -m_3$ and $m'_3 = m_2$, so that after a rotation around the $\hat{e}_1$ axis, $\mathbf{m} = \mathbf{h}$.

To show that $\mathbf{X} = \text{const}$, one may invoke the spectral theorem. In fact, symmetry of $\nabla \mathbf{X}$ follows from the symmetry of $\nabla^2 \mathbf{m} = (\nabla \otimes \nabla) \mathbf{m}$ since

$$-\nabla^2 \mathbf{m} = \nabla \mathbf{X} \otimes \mathbf{Y} + \mathbf{X} \otimes \nabla \mathbf{Y} \quad \text{so that} \quad -\nabla^2 \mathbf{m} \cdot \mathbf{Y} = \nabla \mathbf{X}.$$ 

Next we use the identity

$$\mathbf{m} = -\Delta \mathbf{m} = (\nabla \cdot \mathbf{X}) \mathbf{Y} + (\mathbf{X} \cdot \nabla) \mathbf{Y},$$

which, after multiplication by $\mathbf{Y}$, implies that $\text{tr}(\nabla \mathbf{X}) = \nabla \cdot \mathbf{X} = 0$. Now since $\mathbf{X} \in \ker \nabla \mathbf{m}$

$$(\nabla \mathbf{X}) \mathbf{m} = \nabla (\mathbf{X} \cdot \mathbf{m}) = 0$$

so that $\mathbf{X}, \mathbf{m} \in \ker(\nabla \mathbf{X})$, and in turn $\nabla \mathbf{X} = 0$. \hfill \square

### B Higher order terms in the bifurcation solution

We have the following analytic expansion for the bifurcation solution

$$\mathbf{m}_s = \sum_{k=1}^{\infty} s^k \varphi_k, \quad \lambda_s = \lambda_0 + \sum_{k=1}^{\infty} s^k \lambda_k$$

where the $\varphi_k \in H^2_\Lambda$ satisfy

$$\langle \varphi_k, \varphi_1 \rangle = \int_{\Omega_\Lambda} \varphi_k(x) \cdot \varphi_1(x) \, dx = 0 \quad \text{for } k \geq 2. \quad (37)$$

Inserted into (18), we obtain a hierarchy of equations in orders of $s$ that can be solved successively. Solutions, which are fixed under the symmetry group, can be found by the Fourier method used for solving (22).

The first order equation is the linearized equation

$$-\Delta \varphi_1 + 2\kappa \nabla \times \varphi_1 + \lambda_0 \varphi_1 + \beta(\varphi_1)_3 = 0$$
which is certainly fulfilled. By normalizing in terms of $\langle |\varphi_1|^2 \rangle^{1/2}$, the second order equation becomes

$$-\Delta \varphi_2 + 2\kappa \nabla \times \varphi_2 + \lambda_0 \varphi_2 + \beta(\varphi_2)_3 + \lambda_1 \frac{\varphi_1}{\langle |\varphi_1|^2 \rangle^{1/2}} = 0. \quad (38)$$

Multiplying this equation with $\varphi_1$ and integrating over $\Omega_\Lambda$ yields

$$\langle L_0 \varphi_2, \varphi_1 \rangle + \lambda_1 \langle |\varphi_1|^2 \rangle^{1/2} = 0$$

hence $\lambda_1 = 0$ and in turn $L_0 \varphi_2 = 0$ by (38) which implies $\varphi_2 \equiv 0$ by (37).

The third order equation is

$$-\Delta \varphi_3 + 2\kappa \nabla \times \varphi_3 + \lambda_0 \varphi_3 + \beta(\varphi_3)_3 + \alpha \frac{\varphi_1^2}{\langle |\varphi_1|^2 \rangle^{3/2}} + \lambda_2 \frac{\varphi_1}{\langle |\varphi_1|^2 \rangle^{1/2}} = 0. \quad (39)$$

Multiplying this equation with $\varphi_1$ and integrating over $\Omega_\Lambda$ yields

$$\lambda_2 = -\alpha \frac{\langle |\varphi_1|^4 \rangle}{\langle |\varphi_1|^2 \rangle^2}$$

where $\varphi_3$ is a complicated function in $H^2_\Lambda$ satisfying $\langle \varphi_1, \varphi_3 \rangle = 0$.

The forth order equation

$$-\Delta \varphi_4 + 2\kappa \nabla \times \varphi_4 + \lambda_0 \varphi_4 + \beta(\varphi_4)_3 + \lambda_3 \frac{\varphi_1}{\langle |\varphi_1|^2 \rangle^{1/2}} = 0$$

is identical to (38), so that $\lambda_3 = 0$ and $\varphi_4 \equiv 0$ by the same argument.

Substituting the bifurcation solution into the average energy (11) yields

$$E_\Lambda(m_s, \lambda_s) = s^2 \langle L_0 \varphi_1, \varphi_1 \rangle + s^4 \langle L_0 \varphi_1, \varphi_3 \rangle + \frac{s^4}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \lambda_2 \frac{\varphi_1^2}{\langle |\varphi_1|^2 \rangle} + \frac{\alpha |\varphi_1|^4}{4 \langle |\varphi_1|^2 \rangle^2} + O(s^6) \, dx$$

$$= s^4 \left( -\alpha \frac{\langle |\varphi_1|^4 \rangle}{\langle |\varphi_1|^2 \rangle^2} \right) + O(s^6).$$

References

[1] A. A. Abrikosov. On the Magnetic properties of superconductors of the second group. *Sov. Phys. JETP*, 5:1174–1182, 1957. [Zh. Eksp. Teor. Fiz.32,1442(1957)].
A. Aftalion, X. Blanc, and F. Nier. Lowest Landau level functional and Bargmann spaces for Bose-Einstein condensates. *J. Funct. Anal.*, 241(2):661–702, 2006.

A. Aftalion and R. Rodiac. One dimensional phase transition problem modelling striped spin orbit coupled bose-einstein condensates. *Journal of Differential Equations*, 2019.

L. V. Ahlfors. Complex analysis: an introduction to the theory of analytic functions of one complex variable. *New York, London*, page 177, 1953.

H. Aydi and E. Sandier. Vortex analysis of the periodic Ginzburg-Landau model. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(4):1223–1236, 2009.

E. Barany, M. Golubitsky, and J. Turski. Bifurcations with local gauge symmetries in the Ginzburg-Landau equations. *Physica D: Nonlinear Phenomena*, 56(1):36–56, 1992.

B. Barton-Singer, C. Ross, and B. J. Schroers. Magnetic skyrmions at critical coupling. *Communications in Mathematical Physics*, 2020.

A. Bernand-Mantel, C. B. Muratov, and T. M. Simon. A quantitative description of skyrmions in ultrathin ferromagnetic films and rigidity of degree ±1 harmonic maps from $\mathbb{R}^2$ to $S^2$. *arXiv preprint arXiv:1912.09854*, 2019.

A. Bogdanov and A. Hubert. Thermodynamically stable magnetic vortex states in magnetic crystals. *Journal of magnetism and magnetic materials*, 138(3):255–269, 1994.

A. Bogdanov and A. Hubert. The stability of vortex-like structures in uniaxial ferromagnets. *Journal of magnetism and magnetic materials*, 195(1):182–192, 1999.

A. N. Bogdanov and D. Yablonskii. Thermodynamically stable “vortices” in magnetically ordered crystals. the mixed state of magnets. *Zh. Eksp. Teor. Fiz*, 95(1):178, 1989.

P. Chossat and R. Lauterbach. *Methods in equivariant bifurcations and dynamical systems*, volume 15 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
[13] N. Condette, C. Melcher, and E. Süli. Spectral approximation of pattern-forming nonlinear evolution equations with double-well potentials of quadratic growth. *Mathematics of Computation*, 80(273):205–223, 2011.

[14] E. Davoli and G. Di Fratta. Homogenization of chiral magnetic materials: A mathematical evidence of dzyaloshinskii’s predictions on helical structures. *Journal of Nonlinear Science*, 2020.

[15] A. DeSimone, R. Kohn, S. Müller, and F. Otto. Recent analytical developments in micromagnetics. the science of hysteresis ii: Physical modeling, micromagnetics, and magnetization dynamics, g. bertotti and i. mayergoyz eds, 2006.

[16] L. Döring and C. Melcher. Compactness results for static and dynamic chiral skyrmions near the conformal limit. *Calculus of Variations and Partial Differential Equations*, 56(3):60, 2017.

[17] M. Golubitsky, I. Stewart, and D. G. Schaeffer. *Singularities and groups in bifurcation theory*, volume 2. Springer Science & Business Media, 2012.

[18] T. Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 2013.

[19] H. Kielhöfer. *Bifurcation Theory: An Introduction with applications to partial differential equations*, volume 156. Springer Science & Business Media, 2011.

[20] S. Komineas, C. Melcher, and S. Venakides. The profile of chiral skyrmions. *arXiv preprint arXiv:1904.01408*, 2019.

[21] S. Komineas, C. Melcher, and S. Venakides. The profile of chiral skyrmions of large radius. *arXiv preprint arXiv:1910.04818*, 2019.

[22] A. Leonov, T. Monchesky, N. Romming, A. Kubetzka, A. Bogdanov, and R. Wiesendanger. The properties of isolated chiral skyrmions in thin magnetic films. *New Journal of Physics*, 18(6):065003, 2016.

[23] X. Li and C. Melcher. Stability of axisymmetric chiral skyrmions. *J. Funct. Anal.*, 275(10):2817–2844, 2018.

[24] C. Melcher. Thin-film limits for Landau-Lifshitz-Gilbert equations. *SIAM J. Math. Anal.*, 42(1):519–537, 2010.
[25] C. Melcher. Chiral skyrmions in the plane. *Proc. R. Soc. A*, 470(2172), 2014.

[26] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni. Skyrmion lattice in a chiral magnet. *Science*, 323(5916):915–919, 2009.

[27] C. B. Muratov and V. V. Slastikov. Domain structure of ultrathin ferromagnetic elements in the presence of Dzyaloshinskii-Moriya interaction. *Proc. A.*, 473(2197):20160666, 19, 2017.

[28] F. Odeh. Existence and bifurcation theorems for the Ginzburg-Landau equations. *Journal of mathematical physics*, 8(12):2351–2356, 1967.

[29] U. Rößler, A. Bogdanov, and C. Pfleiderer. Spontaneous skyrmion ground states in magnetic metals. *Nature*, 442(7104):797, 2006.

[30] E. Sandier and S. Serfaty. From the ginzburg-landau model to vortex lattice problems. *Communications in Mathematical Physics*, 313(3):635–743, 2012.

[31] I. M. Sigal and T. Tzaneteas. On stability of Abrikosov vortex lattices. *Adv. Math.*, 326:108–199, 2018.

[32] T. Tzaneteas and I. Sigal. Abrikosov lattice solutions of the Ginzburg-Landau equations. *Spectral Theory and Geometric Analysis, in: Contemp. Math.*, 535:195–213, 2009.

[33] E. G. Virga. *Variational theories for liquid crystals*, volume 8. CRC Press, 1995.

[34] X. Yu, W. Koshibae, Y. Tokunaga, K. Shibata, Y. Taguchi, N. Nagaosa, and Y. Tokura. Transformation between meron and skyrmion topological spin textures in a chiral magnet. *Nature*, 564(7734):95, 2018.

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