CONTINUOUS DISINTEGRATIONS OF GAUSSIAN PROCESSES

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Abstract. The goal of this paper is to understand the conditional law of a stochastic process once it has been observed over an interval. To make this precise, we introduce the notion of a continuous disintegration: a regular conditional probability measure which varies continuously in the conditioned parameter. The conditioning is infinite-dimensional in character, which leads us to consider the general case of probability measures in Banach spaces. Our main result is that for a certain quantity $M$ based on the covariance structure, $M < \infty$ is a necessary and sufficient condition for a Gaussian measure to have a continuous disintegration. The condition $M < \infty$ is quite reasonable: for the familiar case of stationary processes, $M = 1$.

1. Introduction

Consider a continuous Gaussian process $\xi_t$ on an interval $[0, T]$. Let $S \leq T$, and let $y(s)$ be a continuous function on the sub-interval $[0, S]$. Suppose that we observe $\xi_s = y(s)$ for all $s \leq S$. This paper is a result of asking the following questions:

- Is the conditional law $P^y := P(\cdot \mid \xi_{[0,S]} = y)$ still Gaussian?
- Is there a sufficient condition so that the measures $P^y$ vary continuously in the parameter $y$?

The answer to both these questions is “yes”, as illustrated by the following theorem.

Theorem 1. Let $\xi_t$ be a Gaussian process on $[0, T]$ with mean zero and covariance function $c$:

$$\mathbb{E}\xi_t = 0 \quad \text{and} \quad \mathbb{E}\xi_t\xi_s = c(t, s).$$

Suppose that $\xi_t$ is almost-surely continuous. Let $S \leq T$, and suppose that

$$M = \sup_{s \leq S} \sup_{t \leq T} \sup_{s' \leq S} \sup_{t' \leq T} \frac{|c(s, t)|}{|c(s, s')|} < \infty. \quad (1.1)$$

There exists a closed family $Y_0$ of functions such that with probability one, $\xi_{[0,S]} \in Y_0$, and the regular conditional probability $P^y := P(\cdot \mid \xi_{[0,S]} = y)$ is a well-defined Gaussian measure which varies (weakly) continuously in $y \in Y_0$.

If $\xi_t$ is a stationary process, then $M = 1$.

The case that $\xi_t$ has mean function $\mu(t)$ is handled by applying this theorem to the mean-zero stochastic process $\xi_t - \mu(t)$. This theorem is a special case of Theorem 4 which applies to continuous random fields (stochastic processes) defined over compact parameter spaces.

Since the conditioning is of a function over an entire interval, it is infinite-dimensional in character. This leads us to consider the Banach spaces $X = C([0, T])$ and $Y = C([0, S])$ of continuous functions equipped with the sup norms, as well as the restriction map $\eta : X \to Y$. The main result of the paper, Theorem 2, is simply the general form of the above theorem in the context of arbitrary Banach spaces.

To explain what it means for a regular conditional probability measure to vary (weakly) continuously, we introduce the notion of a continuous disintegration of a probability measure $\mathbb{P}$.

Definition 1.1. Let $X$ and $Y$ be complete metric spaces, with Borel $\sigma$-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, and let $\mathbb{P}$ be a Radon probability measure on $X$. Let $\eta : X \to Y$ be a measurable function, and denote the push-forward measure of $\mathbb{P}$ on $Y$ by $\mathbb{P}_Y = \mathbb{P} \circ \eta^{-1}$. A disintegration (or regular conditional probability) of $\mathbb{P}$ with respect to $\eta$ is a map $Y \times \mathcal{B}(X) \to \mathbb{R}$ (denoted by $(y, B) \mapsto \mathbb{P}^y(B)$) such that:

- For all $y \in Y$, $\mathbb{P}^y$ is a probability measure on $\mathcal{B}(X)$.
- For all $B \in \mathcal{B}(X)$, $y \mapsto \mathbb{P}^y(B)$ is a measurable function of $y \in Y$.
- The measure $\mathbb{P}^y$ is supported on the fiber of $y$. i.e., for $\mathbb{P}_Y$-almost every $y \in Y$, $\mathbb{P}^y(\eta^{-1}(y)) = 1$, and
Theorem 3.11 of [13] gives the existence of a Borel-measurable linear operator $\eta$ continuous, and the measure $\mathbb{P}^y$ is a continuous disintegration given $Y_0$ provided

$$\text{if } y_n \in Y_0 \text{ and } y_n \to y, \text{ then } \mathbb{P}^y \text{ converges weakly to } \mathbb{P}^y.$$  

We remark that disintegration is typically a more general concept than regular conditional probability. In the present work, we ignore the distinction and treat the terms as synonyms.

The notion of a continuous disintegration is a new contribution to the literature, but disintegrations and regular conditional probabilities have been studied in wide generality. For a nice overview of the topic, see the survey [6] by Chang and Pollard, or Sections 10.4 and 10.6 of Bogachev [5]. The typical existence theorem can be found in [7, Section 4.1.c] or [4, Theorem 1.4]. The recent paper [11] contains some very general existence results.

Our main result, Theorem 2, gives a sufficient condition for continuous disintegrations to exist for Gaussian probability measures on Banach spaces. Theorem 3 demonstrates that this condition is also necessary. Theorem 4 is the application of the existence result to the important context of random fields.

Suppose that $X$ and $Y$ are finite-dimensional vector spaces, the map $\eta : X \to Y$ is linear, and the measure $\mathbb{P}$ is Gaussian. It is a simple exercise in linear algebra that the regular conditional probability measure $\mathbb{P}^y = \mathbb{P}(\cdot | \eta^{-1}(y))$ is Gaussian, and that the conditioned covariance matrix does not depend on the actual value $y \in Y$. The conditional mean vector is easily seen to vary continuously in $y$. Since $\mathbb{P}^y$ is Gaussian and depends entirely on its mean and covariance matrix, it follows easily that $\mathbb{P}^y$ is a continuous disintegration.

Now suppose that the spaces $X$ and $Y$ are separable Banach spaces, the map $\eta : X \to Y$ is linear and continuous, and the measure $\mathbb{P}$ is Gaussian. Tarieladze and Vakhania [13] show that $\mathbb{P}$ admits a disintegration $\mathbb{P}^y$ which is a Gaussian measure for all $y$. Furthermore, when the push-forward measure $\mathbb{P}_Y$ has finite-dimensional support in $Y$, it quickly follows from their Theorem 3.11a that $\mathbb{P}^y$ is a continuous disintegration given $\text{supp } \mathbb{P}_Y$.

This fact is quite useful in applications, such as kriging in geosciences and hydrology [2]. In this example, one models a quantity of interest, such as elevation, by a Gaussian random field defined on a domain in $\mathbb{R}^2$. By conditioning the field at finitely many points based on empirical data, the field serves as a reasonable interpolation between the sampled points, with the randomness representing uncertainty. The result of Tarieladze and Vakhania demonstrates that a conditioned Gaussian field is still Gaussian, and that its law varies continuously in the sampled values.

In this paper, we focus on the situation where $X$ and $Y$ are arbitrary Banach spaces, the map $\eta : X \to Y$ is linear and continuous, and $\mathbb{P}$ is a Gaussian measure with mean zero and covariance operator $K$. We need not worry about separability of $X$, as the structure theorem (Theorem 2.1) asserts that the Radon measure $\mathbb{P}$ is supported on the separable subspace $\overline{KX}^*$. The push-forward measure $\mathbb{P}_Y$ has covariance operator $\eta K \eta^*$, and is supported on $Y_0 := \eta K \eta^* Y^*$.

In Lemma 2.2 we show that the map $\eta$ is injective when restricted to $K \eta^* Y^* \subseteq X$. Consequently, the inverse map $\eta^{-1} : \eta K \eta^* Y^* \to X$ is well-defined. Let $M$ denote the operator norm of $\eta^{-1}$, and suppose that $M < \infty$. Then $\eta^{-1}$ extends to a continuous linear map $m : Y_0 \to X$. In Lemma 2.3 we use a Hilbert-space formalism to show that the operator $K := K - \eta m^* \eta$ is well-defined. For each $y \in Y_0$, let $\mathbb{P}^y$ denote the Gaussian measure on $X$ with mean $m(y)$ and covariance operator $K$. In Theorem 2 we show that $\mathbb{P}^y$ is a continuous disintegration of $\mathbb{P}$ given $Y_0$.

Suppose now that $M = \infty$, so that the operator $\eta^{-1}$ does not admit a continuous extension to all of $Y_0$. Theorem 3.11 of [13] gives the existence of a Borel-measurable linear operator $m$ and an operator $K$ so that the Gaussian measure $\mathbb{P}^y$ with mean $m(y)$ and $K$ is a disintegration of $\mathbb{P}$. In Theorem 3 we show that if there exists a continuous disintegration, then it must agree with $\mathbb{P}^y$ on a set of full measure, and that the assumption $M = \infty$ implies that the conditional mean operator $m$ is discontinuous. This results in a contradiction, thus $M < \infty$ is both a necessary and sufficient condition for there to exist a continuous disintegration of a Gaussian measure.
2. Probability Measures on Banach Spaces

We now explore some of the general theory of Radon probability measures on Banach spaces. Let $X$ be a Banach space, and let $B(X)$ denote the Borel $\sigma$-algebra of $X$. Continuous linear functionals of $X$ are measurable functions, hence random variables. Let $\mathbb{P}$ be a Radon probability measure on $X$ with the property that for all $f \in X^*$,

$$E[f]^2 = \int_X |f(x)|^2 \, d\mathbb{P}(x) < \infty. \quad (2.1)$$

This implies that every continuous linear functional has a finite variance and mean. We recall that the support of the measure $\mathbb{P}$ is the largest closed set in $X$ of full measure, and denote it by $\text{supp} \mathbb{P}$.

Theorem 2.1 (Structure Theorem for Radon Probability Measures). If $\mathbb{P}$ is a Radon probability measure on $X$ which satisfies (2.1), then there exist an element $\mu \in X$ and a continuous linear operator $K : X^* \to X$ such that

$$f(\mu) = E(f) \quad \text{and} \quad K f = \int_X f(x) x \, d\mathbb{P}(x) - f(\mu) \mu$$

for all $f \in X^*$. We call $\mu$ the mean of $\mathbb{P}$, and $K$ the covariance operator of $\mathbb{P}$. It follows that

$$f(K g) = E(fg) - f(\mu)g(\mu)$$

for all $f, g \in X^*$.

The space $\mu + KX^*$ is separable, and is dense in the support of $\mathbb{P}$:

$$\text{supp} \mathbb{P} \subseteq \mu + KX^*. \quad (2.4)$$

Consequently, $\mathbb{P}(\mu + KX^*) = 1$. If $\mathbb{P}$ is a Gaussian measure, then $\text{supp} \mathbb{P} = \mu + KX^*$.

Proof. A measure $\mathbb{P}$ which satisfies (2.1) is called weak-order two. The existence of the mean vector $\mu$ is given by the Corollary in Section II.3.1 of [16], and the existence of the covariance operator $K$ is given by Theorem 2.1 of Section III.2.1 of [16]. The separability of the space $KX^*$ is Corollary 1 to that theorem.

The statement about the support of a Gaussian measure is Theorem 1 of [15]. The proof of (2.1) is part (a) of Theorem 1 of [15]. The proof is simple and elegant so we reproduce it.

Without loss of generality, suppose that $\mathbb{P}$ has mean zero. Let $(KX^*)^\perp \subseteq X^*$ denote the annihilator of $KX^*$, defined below in (2.6). If $f \in (KX^*)^\perp$, then $\int f(x)^2 \, d\mathbb{P}(x) = f(Kf) = 0$, so $f(x) = 0$ for $\mathbb{P}$-almost every $x \in X$. Since the set $f^{-1}(0)$ is closed and has full measure, the support $\text{supp} \mathbb{P}$ is a subset of $f^{-1}(0)$. Thus $f \in (\text{supp} \mathbb{P})^\perp$. (2.4) immediately follows. \hfill \Box

In addition to being a powerful technical result, the Structure Theorem presents a useful philosophy when working with Radon probability measures on Banach spaces: many statements about probability can be reformulated in terms of the geometry of the linear space $\mu + KX^*$. This allows us to use linear algebra, functional analysis and, as we will see shortly, the theory of Hilbert spaces.

When $\mathbb{P}$ is a measure on the space of continuous functions with covariance function $c$ (e.g., Wiener measure, whence $c(t, s) = \min\{s, t\}$), the operator $K$ is the integral operator with kernel $c$. This important special case is developed in Theorem 4.

For the remainder of this section, we assume that $\mathbb{P}$ is a Radon probability measure on $X$ with mean zero and covariance operator $K$.

Let $Y$ be a Banach space, and let $B(Y)$ denote the Borel $\sigma$-algebra of $Y$. Let $\eta : X \to Y$ be a continuous linear map from $X$ to $Y$. Let $\mathbb{P}_Y$ be the push-forward measure on $Y$ of $\mathbb{P}$, defined by the equation

$$\mathbb{P}_Y(B) := \mathbb{P}(\eta^{-1}(B))$$

for every Borel set $B \in B(Y)$. This equation implies that the measure $\mathbb{P}_Y$ satisfies the change of variable formula

$$\int_{\eta^{-1}(B)} g(\eta x) \, d\mathbb{P}(x) = \int_B g(y) \, d\mathbb{P}_Y(y), \quad (2.5)$$

for any integrable function $g : Y \to \mathbb{R}$. Consequently, $\mathbb{P}_Y$ has mean zero and covariance operator $\eta K \eta^*$. For a set $B$ of $X^*$, let

$$B^\perp = \{ f \in X^* : f(Kg) = 0 \text{ for all } g \in B \} \quad (2.6)$$

be the annihilator of $B$: the linear space of functionals uncorrelated with $B$. 

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Probability Measures on Banach Spaces
Lemma 2.2. When restricted to the subspace $KX^*$ of $X$, the map $\eta$ has kernel $K(\eta^*Y^*)^\perp$. Consequently, on $K\eta^*Y^*$, $\eta$ is injective. Define

$$ M := \sup_{e \in Y^*} \left\{ \frac{\|K\eta^*e\|_X}{\|\eta K\eta^*e\|_Y} : e(\eta K\eta^*e) \neq 0 \right\}. \quad (2.7) $$

The inverse map $\eta^{-1} : \eta K\eta^*Y^* \to X$ has operator norm $M$.

Proof. Let $f \in X^*$. For all $e \in Y^*$,

$$ e(\eta K f) = f(K\eta^*e) $$

by the symmetry of the operator $K$, thus $f \in (\eta^*Y^*)^\perp$ exactly if $\eta(Kf) = 0$ in $Y$. This proves that

$$ \ker \eta \cap KX^* = K(\eta^*Y^*)^\perp. \quad (2.8) $$

The operator norm of the inverse map $\eta^{-1}$ on $\eta K\eta^*Y^*$ is given by

$$ \|\eta^{-1}\|_{op} = \sup_{e \in Y^*} \left\{ \frac{\|K_0\eta^*e\|_X}{\|\eta K\eta^*e\|_Y} : e(\eta K\eta^*e) \neq 0 \right\}. $$

Let $M$ be as in (2.7). To see that $\|\eta^{-1}\|_{op}$ and $M$ are equal, we apply the Schwarz inequality [8] to the inner product on $Y^*$ generated by $\eta K\eta^*$:

$$ |e'\eta K\eta^*e|^2 \leq |e'\eta K\eta^*e'| |e\eta K\eta^*e|. $$

Thus, $\eta K\eta^*e \neq 0$ exactly if $e(\eta K\eta^*e) \neq 0$. $\square$

Let

$$ Y_0 = \overline{\eta K\eta^*Y^*}. \quad (2.9) $$

Since the measure $\mathbb{P}_Y$ has mean zero and covariance operator $\eta K\eta^*$, Theorem 2.1 implies that it is supported on $Y_0$, so $\mathbb{P}_Y(Y_0) = 1$. Let $M$ be as in (2.7), and suppose henceforth that

$$ M < \infty. \quad (2.10) $$

Define the linear map

$$ m : Y_0 \to X \quad (2.11) $$

first by $m = \eta^{-1}$ on the dense subspace $\eta K\eta^*Y^*$ of $Y_0$, then extend continuously. By Lemma 2.2 the map $m$ is continuous with operator norm $M$. Clearly, $\eta \circ m$ is the identity map on $Y_0$. However, the map $m \circ \eta$ on $KX^* \subset X$ is non-trivial.

The covariance operator $K$ defines a symmetric inner product $\langle f, g \rangle := f(Kg)$ on $X^*$. This inner product is nonnegative-definite, and will be degenerate if the real-valued distribution of some $f \in X^*$ is atomic. Nonetheless, it follows easily from the Schwarz inequality

$$ |g(Kf)|^2 = |\langle g, f \rangle|^2 \leq \langle g, g \rangle \langle f, f \rangle $$

that $\langle f, f \rangle = 0$ if and only if $f \in \ker K$. Thus the inner product is positive-definite on the quotient space $X^*/\ker K$.

Let $H$ be the Hilbert-space completion of the inner product space $X^*/\ker K$, and let $\iota^* : X^* \to H$ be the inclusion map. Define the unitary map $\iota : H \to X$ first on the dense subspace $\iota^*X^*$ by $\iota(\iota^*f) = Kf$, then extend it continuously to all of $H$. The operator $K$ factors as $\iota^*$. We summarize this with the following commutative diagram:

$$ Y^* \xrightarrow{\eta} X^* \xrightarrow{K} X \xrightarrow{\eta} Y \quad (2.13) $$

where the domain of the map $m$ is $Y_0 \subset Y$.

The subspace $\iota H$ of $X$ is called the Cameron-Martin space of $\mathbb{P}$, and is a reproducing kernel Hilbert space [3,10]. The triplet $(\iota, H, X)$ is an abstract Wiener space [11,9]. Since $\iota H$ is dense in the separable Banach space $KX^*$, the Hilbert space $H$ is separable.
Lemma 2.3. The operator $\hat{K} : X^* \rightarrow X$ given by the formula

$$\hat{K} = K - K\eta^*m^*$$

is well-defined. Furthermore,

$$\hat{K} \leq K \quad \text{and} \quad m\eta K\eta^*m^* = K\eta^*m^*. \quad (2.14)$$

The first statement of (2.14) means that $f(\hat{K}f) \leq f(Kf)$ for all $f \in X^*$.

Proof. Let $H_Y$ be the completion of $\iota^*\eta^*Y^*$ in $H$, and let $H_Y^\perp$ be its orthogonal complement. Let $\pi : H \rightarrow H$ be the orthogonal projection map onto the subspace $H_Y$. We claim that the two continuous maps $m\eta\pi$ and $\iota\pi$ from $H$ to $X$ are equal. It suffices to check that that they are equal on the dense subspaces $\iota^*\eta^*Y^* \subseteq H_Y$ and $\iota^*(\eta^*Y^*)^\perp \subseteq H_Y^\perp$. We calculate

$$(m\eta - \iota\pi)\iota^*\eta^*Y^* = m\eta\eta^*Y^* - K\eta^*Y^* = 0$$

since $\pi$ is the identity on $\iota^*\eta^*Y^*$ and $m \circ \eta$ is the identity on $K\eta^*Y^*$; and

$$(m\eta - \iota\pi)(\eta^*Y^*)^\perp = m\eta K(\eta^*Y^*)^\perp - 0 = 0$$

since $\pi$ kills $\iota^*(\eta^*Y^*)^\perp$ and $K(\eta^*Y^*)^\perp = \ker\eta \cap KX^*$ by Lemma 2.2. Thus

$$m\eta\pi = \iota\pi$$

(2.15)

on $H$. By duality, the adjoint maps $\iota^*\eta^*m^*$ and $\pi^*\iota$ from $X^*$ to $H$ are also equal, so

$$\hat{K} = K - \iota^*\eta^*m^* = K - \iota^*\eta^*m^* = K - \iota\pi^*$$

is well-defined. If we write $\pi^\perp : H \rightarrow H$ for the orthogonal projection map onto $H_Y^\perp$, then this shows that

$$\hat{K} = \iota\pi^\perp \iota^*$$

(2.16)

since $K = \iota^*$. This representation implies that $\hat{K} \leq K$.

Finally, since $\pi^2 = \pi$,

$$m\eta K\eta^*m^* = m\eta \circ \iota^*\eta^*m^* = \iota^2\iota^* = \iota\pi^* = K\eta^*m^*.$$

Equation (2.16) and Lemma 2.2 imply that

$$\overline{KX^*} = \ker\eta \cap \overline{KX^*}. \quad (2.17)$$

Consequently,

$$\eta \left(m(y) + \overline{KX^*}\right) = y \quad (2.18)$$

for all $y \in Y_0$, since $\eta \circ m$ is the identity on $Y_0$.

A Radon measure $\mathbb{P}$ on $X$ is Gaussian if every continuous linear functional $f \in X^*$ is a real-valued Gaussian random variable with respect to $\mathbb{P}$. Gaussian measures are completely described by their mean and covariance operators.

Not every continuous operator $\hat{K} : X^* \rightarrow X$ serves as the covariance operator for a Gaussian measure. Nonetheless, the condition $\hat{K} \leq K$ is sufficient, by Proposition 3.9 of [13]. We can now state and prove the main theorem of the paper.

Theorem 2 (Existence of Continuous Disintegrations). Let $X$ and $Y$ be Banach spaces, and let $\eta : X \rightarrow Y$ be a continuous linear map. Let $\mathbb{P}$ be a Gaussian probability measure on $X$ with mean zero and covariance operator $K$. Suppose that

$$M = \sup_{c \in Y^*} \left\{ \frac{\|K\eta^*c\|_X}{\|\eta K\eta^*c\|_Y} : c(\eta K\eta^*c) \neq 0 \right\} < \infty.$$

The operator $\eta^{-1} : \eta K\eta^* \rightarrow X$ is continuous with norm $M$. Let $Y_0 := \overline{\eta K\eta^*Y^*}$, and let $m : Y_0 \rightarrow X$ denote the continuous extension of $\eta^{-1}$ to all of $Y_0$. The map $\eta \circ m$ is the identity operator on $Y_0$.

For any $y \in Y_0$, let $\mathbb{P}^y$ be the Gaussian measure on $X$ with mean $m(y)$ and covariance operator $\hat{K} = K - K\eta^*m^*$. The family of measures $\mathbb{P}^y$ is a continuous disintegration given $Y_0$. 

Proof. Let $H$ be the Hilbert space described by the diagram (2.13) and let $C_H$ denote the cylinder algebra on $H$. Let $\gamma : C_H \to [0, 1]$ be the canonical Gaussian cylindrical measure on the Hilbert space $H$, i.e., with mean zero and covariance operator the identity $I$.

Since the Cameron-Martin space $iH$ is dense in $\overline{KX^*}$, the push-forward cylindrical measure $\gamma \circ \iota^{-1}$ completely determines the measure $\mathbb{P}$:

$$\int_X f(x) \, d\mathbb{P}(x) = \int_H f(h) \, d\gamma(h)$$

(2.20)

for any $f : X \to \mathbb{R}$ measurable with respect to the cylinder algebra $C$ of $X$. If $f$ is a continuous linear functional, then the right side of (2.20) is further equal to $f(\iota^* f, h) \, d\gamma(h)$.

Let $H_Y = \overline{\iota^* \eta^{-1} Y^*}$ denote the subspace of $H$ generated by $Y$, and let $H_Y^\perp$ denote its orthogonal complement in $H$. Let $\pi$ and $\pi^\perp$ be the orthogonal projection maps onto $H_Y$ and $H_Y^\perp$, respectively. Define the push-forward cylindrical measures

$$\gamma_Y = \gamma \circ \pi^{-1} \quad \text{and} \quad \gamma^\perp = \gamma \circ (\pi^\perp)^{-1}$$

(2.21)

on $H$. The cylindrical measure $\gamma_Y$ has mean zero, covariance operator $\pi$ and is supported on $H_Y$. Similarly, $\gamma^\perp$ has mean zero, covariance operator $\pi^\perp$ and is supported on $H_Y^\perp$.

We now exploit a fundamental fact of Gaussians: orthogonality implies independence. Let $k \in H_Y$ and $k' \in H_Y^\perp$. The jointly Gaussian random variables $(k, \cdot)$ and $(k', \cdot)$ each have mean zero, and their covariance is $(\langle k, I k' \rangle = 0$. Consequently, the random variables $(k, \cdot)$ and $(k', \cdot)$ are independent. Extending this analysis shows that for any $C_H$-measurable function $g : H \to \mathbb{R}$,

$$\int_H g(h) \, d\gamma(h) = \int_H \int_H g(k + h) \, d\gamma^\perp(h) \, d\gamma_Y(k).$$

(2.22)

Clearly, $\mathbb{P}_Y$ is the radonification of the push-forward cylindrical measure $\gamma_Y \circ (\eta \iota)^{-1}$ on $Y$. Let $\mathbb{P}^0$ be the radonification of $\gamma^\perp \circ \iota^{-1}$ on $X$. This is the mean-zero Gaussian measure on $X$ with covariance operator $\bar{K} = \iota^* \pi^\perp \iota^*$, using the representation (2.16). Define

$$\mathbb{P}^0(B) = \mathbb{P}^0(m(y) + B)$$

for any $B \in \mathcal{B}(X)$. The measure $\mathbb{P}^y$ is the Gaussian measure on $X$ with mean $m(y)$ and covariance operator $\bar{K}$.

We now verify that $\mathbb{P}^y$ is a disintegration. By the structure theorem (Theorem 2.1), $\supp \mathbb{P}^y = m(y) + \overline{KX^*}$. Thus by (2.18),

$$\mathbb{P}^y(\eta^{-1}(y)) \geq \mathbb{P}^y \left( m(y) + \overline{KX^*} \right) = 1,$$

so $\mathbb{P}^y$ is supported on the fiber $\eta^{-1}(y)$.

The heart of the disintegration equation (1.2) is the fact that $\gamma = \gamma^\perp \ast \gamma_Y$. Let $f : X \to \mathbb{R}$ be measurable with respect to the cylinder algebra $C$ of $X$. Then by equations (2.20) and (2.22),

$$\int_X f(x) \, d\mathbb{P}(x) = \int_H f(\iota h) \, d\gamma(h) = \int_H \int_H f(\iota(k + h)) \, d\gamma^\perp(h) \, d\gamma_Y(k).$$

(2.23)

For $\gamma_Y$-almost every $k$, $k = \pi k$. We apply this to (2.23), as well as the identity (2.15) that $\iota \pi = m \eta \iota$, to get

$$\int_H \int_H f(\iota(k + h)) \, d\gamma^\perp(h) \, d\gamma_Y(k) = \int_H \int_H f(m \eta \iota k + \iota h) \, d\gamma^\perp(h) \, d\gamma_Y(k).$$

(2.24)

We now push forward to the Radon measures $\mathbb{P}^0$ and $\mathbb{P}_Y$, and use the definition $\mathbb{P}^y = \mathbb{P}^0(m(y) \cdot)$, so that (2.24) equals

$$\int_Y \int_X f(m(y) + x) \, d\mathbb{P}^0(x) \, d\mathbb{P}_Y(y) = \int_Y \int_X f(x) \, d\mathbb{P}^y(x) \, d\mathbb{P}_Y(y).$$

(2.25)

Since the cylinder algebra $C$ generates the Borel $\sigma$-algebra $\mathcal{B}(X)$, this proves the disintegration equation (1.2) for arbitrary integrable $f$.

Finally, we show that $\mathbb{P}^y$ satisfies the continuous disintegration property (1.3). Suppose $y_n \to y$ in $Y_0$. The operator $m$ is continuous, so $m(y_n) \to m(y)$. Let $f : X \to \mathbb{R}$ be a bounded, continuous function, so

$$\lim_{n \to \infty} \int_X f(x) \, d\mathbb{P}^y_n(x) = \lim_{n \to \infty} \int_X f(m(y_n) + x) \, d\mathbb{P}^0(x) = \int_X f(m(y) + x) \, d\mathbb{P}^0(x) = \int_X f(x) \, d\mathbb{P}^y(x)$$

(2.26)

by the bounded convergence theorem. This proves that the measures $\mathbb{P}^y_n$ converge weakly to $\mathbb{P}^y$, which completes the proof. \qed
This theorem raises the natural question: is $M < \infty$ a necessary condition for the existence of a continuous disintegration? The next theorem demonstrates that for Gaussian measures, the answer is yes.

In the proof of Theorem 2.3, we used the fact that $M < \infty$ in order to define the conditional mean $m(y)$ and conditional covariance operator $\hat{K}$, as in (2.11) and (2.14), respectively. If we assume that $M = \infty$, then we must define the conditional mean and covariance using a different method. The recent work [13] of Tarieladze and Vakhania does exactly this, by working on the Hilbert space $H$ then using the identities $m\mu = \iota\pi$ and $\hat{K} = \iota\pi \iota^* \pi$ as the definitions of $m$ and $\hat{K}$.

It is likely that the methods of Tarieladze and Vakhania can be adapted to a more general setting. In that case, the Gaussian assumption can be weakened in the following theorem, though our proof does use the fact that the support of a mean-zero Gaussian measure $P$ is the entire linear space $KX^*$, and not a proper subset.

**Theorem 3.** Let $X$ and $Y$ be separable Banach spaces, and let $\eta : X \to Y$ a continuous linear map. Let $P$ be a Gaussian measure on $X$ with mean zero and covariance operator $K$. Suppose that

$$M = \sup_{c \in \mathcal{Y}} \left\{ \frac{\|K\eta^* e\|_X}{\|\eta K\eta^* e\|_Y} : e(\eta K\eta^* e) \neq 0 \right\} = \infty. \tag{2.27}$$

For any closed set $Y_0$ of full $P_Y$-measure, there does not exist a continuous disintegration $P^\eta$ on $Y_0$.

**Proof.** Let $Y_0$ be a closed subset of $Y$ of full $P_Y$-measure, and suppose $P^\eta$ is a continuous disintegration of $P$ on $Y_0$.

The main result of [13] is Theorem 3.11, which states that there exists a map $m : Y \to X$ and a covariance operator $\hat{K}$ such that the Gaussian measure $P^\eta$ with mean $m(y)$ and covariance $\hat{K}$ is a disintegration of $P$. Furthermore, there exists a linear subspace $Y_1$ of $Y$ of full $P_Y$-measure such that the restriction of $m$ to $Y_1$ is Borel-measurable and $\eta(m(y)) = y$ for all $y \in Y_1$.

Disintegrations are unique up to sets of measure zero [13, Theorem 2.4], so there exists a set $Y_2$ of $Y$ of full $P_Y$-measure such that $P^\eta = P^\eta$ for all $y \in Y_2$. Define the closed set

$$Y' = Y_0 \cap Y_1 \cap Y_2,$$

so that $P^\eta$ is a continuous disintegration of $P$ on $Y'$. Since $Y'$ is a closed set of full $P_Y$-measure, it contains the linear space $\text{supp} \ P_Y = \overline{\eta K\eta^* Y^*}$ as a subset.

**Lemma 2.4.** There exists a sequence $y_n \in \eta K\eta^* Y^*$ such that $y_n \to 0$ but $\|m(y_n)\|_X \geq 1$ for all $n$. Consequently, $m$ is discontinuous on $Y'$.

Furthermore, the distance in $X$ from $m(y_n)$ to $KX^*$ is at least 1 for all $n$.

**Proof.** As in the proof of Lemma 2.3, let $H$ be the Hilbert space completion of the space $X^*/\ker K$ under the inner product generated by $K$, and let $\iota^* : X^* \to H$ be the inclusion map. Define the unitary map $\iota : H \to X$ first on the dense subspace $\iota^* X^*$ by $(\iota^* f) = Kf$, then extend it continuously to all of $H$.

Let $H_Y$ be the completion of $\iota^*\eta^* Y^*$ in $H$. Choose $e_i \in Y^*$ so that $h_i = \iota^* \eta^* e_i$ is an orthonormal basis in $H_Y$. For all $y \in Y'$,

$$m(y) = \sum_{i=1}^{\infty} e_i(y) K\eta^* e_i = \iota \left( \sum_{i=1}^{\infty} e_i(y) h_i \right);$$

this follows from the proof of [13, Theorem 3.11, Case 3]. If $\pi : H \to H$ is the orthogonal projection onto $H_Y$ in $H$, this formula implies that $m\pi = \iota\pi$ on $H$. Thus for $g \in Y^*$,

$$\|m(\eta K\eta^* g)\|_X = \|(m\eta g)\iota^*\eta^* g\|_X = \|\iota\pi\iota^*\eta^* g\|_X = \|K\eta^* g\|_X \tag{2.28}$$

since $\pi$ is the identity on $\iota^*\eta^* Y^*$.

Since $M = \infty$, there exist $g_n \in Y^*$ such that

$$\|K\eta^* g_n\|_X \geq n \|\eta K\eta^* g_n\|_Y.$$ 

Setting

$$y_n = \frac{1}{n} \frac{\eta K\eta^* g_n}{\|\eta K\eta^* g_n\|}$$

and applying (2.28) completes the proof that $\|m(y_n)\| \geq 1$.

It also follows from the proof of [13, Theorem 3.11, Case 3] that

$$\hat{K} = K - \iota\pi\iota^*.$$
Let $I$ denote the identity operator in $H$. If $g \in Y^*$ and $f \in X^*$, then for $y = \eta K \eta^* g$,
\[
\|m(y) - \hat{K} f\|_{X^*}^2 = \|\mu^* \eta^* g - (\mu^* f - t \sigma^* f)\|_{X^*}^2
\]
\[
= ||\mu^* \eta^* g - (I - \pi) \sigma^* f||_{H}^2
\]
\[
= ||\mu^* \eta^* g||_{H}^2 + ||(I - \pi) \sigma^* f||_{H}^2
\]
\[
\geq ||m(y)||_{X^*}^2 + 0,
\]
by the Pythagorean Theorem [8], since $I - \pi$ is the orthogonal projection onto $H^\perp$. Plugging in $y_n$ as above completes the proof of the second claim. □

Since $y_n \to 0$ in $\eta K \eta^* Y^* \subseteq Y'$ and $P^y$ is a continuous disintegration given $Y'$, $P^{y_n} \to P^0$ weakly. By Theorem 2.1, $P^0$ is supported on $KX^*$. Thus the open $\frac{1}{2}$-neighborhood of $KX^*$,
\[
U = \left\{x \in X : \|x - x'\| < \frac{1}{2} \text{ for some } x' \in KX^* \right\},
\]
has full $P^0$-measure. Since $P^{y_n} \to P^0$ weakly and $U$ is open, $\lim \inf P^{y_n}(U) \geq P^0(U) = 1$. However, $P^{y_n}$ is supported on $m(y_n) + KX^*$, which is distance at least 1 from $KX^*$ by the preceding lemma, a contradiction. □

We now apply Theorem 2 in the important context of random fields, which are stochastic processes defined on arbitrary parameter sets. Let $T$ be a compact set. Then a (continuous) random field $\xi$ on the parameter set $T$ is simply a random element of the Banach space $X = C(T, \mathbb{R})$ of real-valued continuous functions on $T$.

**Theorem 4.** Let $\xi$ be a Gaussian random field on a compact parameter set $T$ with mean zero and covariance function $c$:
\[
\mathbb{E} \xi = 0 \text{ and } \mathbb{E} \xi \xi^* = c(t, s).
\]
Suppose that $\xi$ is almost-surely continuous. Let $S$ be a closed subset of $T$, and suppose that
\[
M = \sup_{s \in S} \sup_{s' \in S} |c(s, t)| < \infty.
\]  
(2.29)
There exists a closed family $Y_0$ of functions such that with probability one, $\xi |_S \in Y_0$, and the regular conditional probability $P^y := P(\cdot | \xi |_S = y)$ is a well-defined Gaussian measure which varies (weakly) continuously in $y \in Y_0$.

Furthermore, there exists a function $m(y, t)$, linear in $y$ and jointly continuous in $y$ and $t$, such that
\[
m(y, t) = \mathbb{E}(\xi |_{S} \xi |_{S} = y)
\]  
(2.30)
for all $t \in T$, and there exists a covariance function $\hat{c}$ (independent of $y$) such that
\[
\hat{c}(t, s) = \mathbb{E}(\xi |_{S} \xi |_{S} = y) - m(y, t) m(y, s)
\]  
(2.31)
and $\hat{c}(t, s) \leq c(t, s)$ for all $t, s \in T$. The function $m(y, \cdot)$ is a bounded extension of $y$, in the sense that
\[
m(y, s) = y(s) \text{ for all } s \in S, \quad \text{and } \sup_{t \in T} |m(y, t)| \leq M \sup_{s \in S} |y(s)|.
\]  
(2.32)

**Proof.** Consider the Banach space $X = C(T, \mathbb{R})$. By the Riesz representation theorem [8], the dual space $X^*$ has a representation as the space of Radon measures on $T$, so for all $f \in X^*$ there exists a Radon measure $\lambda_f$ so that $f(x) = \int_A x(t) d\lambda_f(t)$. Define the operator $K : X^* \to X$ by
\[
(Kf)(t) = \int_A c(t, s) d\lambda_f(s).
\]
That is, $K$ is the integral operator with kernel $c$. Consequently, the measure $P$ on $X$ is a Radon probability measure with mean zero and covariance operator $K$. Let $\delta_t$ represent the evaluation functional, defined by $\delta_t x = x(t)$; equivalently, $\delta_t$ represents the Dirac point-mass measure with an atom at $t \in T$. Thus $c(t, s) = \delta_t(K \delta_s)$.

Let $Y = C(S, \mathbb{R})$, and let $\eta : X \to Y$ be the restriction map, defined by $(\eta x)(s) = x(s)$ for all $x \in X$ and $s \in S$. Let $\mathbb{P}_Y = \mathbb{P} \circ \eta^{-1}$ denote the push-forward of $\mathbb{P}$ onto $Y$. Using the same notation as above, denote the evaluation functionals on $Y$ by $\delta_y$. Let $M$ be defined by (2.19). Since the linear spans of $\{\delta_t\}_{t \in T}$ and $\{\delta_s\}_{s \in S}$ are dense in $X^*$ and $Y^*$ [12], respectively, $M$ takes the form (2.29).

In this context, the space $Y_0 = \eta K \eta^* Y^*$ takes the form
\[
Y_0 = \text{span}\{\delta(s, \cdot)\} \subseteq Y,
\]  
(2.33)
where the span is over $s \in S$. The space $Y_0$ has full $\mathbb{P}_Y$-measure.
By assumption, $M < \infty$, so Theorem 2 applies. Thus there exists a continuous disintegration $\mathbb{P}^y$ on $Y_0$, and there exist continuous linear operators $m : Y_0 \to X$ and $\hat{K} : X^* \to X$ so that for all $y \in Y_0$, the measure $\mathbb{P}^y$ has mean $m(y)$ and covariance operator $\hat{K}$. Define the functions $m(y, t) := \delta_i(m(y))$ and $\hat{c}(t, s) := \delta_i(\hat{K}\delta_s)$. Since $\eta \circ m$ is the identity operator on $Y_0$, and $m$ has operator norm $M$, the statements (2.32) immediately follow.

Suppose that $T$ is a subset of an abelian group. We say that a random field $\xi_t$ is stationary if its covariance function satisfies

$$c(t, s) = c(t + z, s + z),$$

whenever $t, s, t + z$ and $s + z$ all belong to $T$.

**Corollary 5.** If $\xi_t$ is a stationary Gaussian random field, then $M = 1$, so the above theorem applies to $\xi_t$.

**Proof.** Since the covariance operator $\hat{K}$ defines an inner product, the Schwarz inequality (2.12) implies

$$|c(s, t)|^2 \leq c(s, s) c(t, t).$$

(2.34)

If the field is stationary, then $c(t, t) = c(s, s)$. Consequently, for each $s \in S$,

$$\sup_{t \in T} |c(s, t)| = |c(s, s)| \quad \text{and} \quad \sup_{s' \in S} |c(s, s')| = |c(s, s)|.$$

The ratio of these two quantities is always equal to 1, so $M = 1$.

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