EXPLICITLY SOLVABLE SYSTEMS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL RIGHT-HAND SIDES, AND THEIR PERIODIC VARIANTS

Francesco Calogero\textsuperscript{a,b\dagger}, Farrin Payandeh\textsuperscript{c\ddagger\§}

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\textsuperscript{a} Physics Department, University of Rome "La Sapienza", Rome, Italy
\textsuperscript{b} INFN, Sezione di Roma 1
\textsuperscript{c} Department of Physics, Payame Noor University, PO BOX 19395-3697 Tehran, Iran

Abstract

In this Letter we identify special systems of (an arbitrary number) $N$ of first-order Ordinary Differential Equations with homogeneous polynomials of arbitrary degree $M$ on their right-hand sides, which feature very simple explicit solutions; as well as variants of these systems—with right-hand sides no more homogeneous—which feature periodic solutions. A novelty of these findings is to consider special systems characterized by constraints involving both their parameters and their initial data.

The general system of an arbitrary number $N$ of first-order Ordinary Differential Equations (ODEs) with homogeneous polynomials of arbitrary degree $M$ on their right-hand sides reads as follows:

$$
\dot{z}_n(t) = \sum_{m_{\ell}}^{(M)} \left\{ c_{n_1m_2\cdots m_N}[z_1(t)]^{m_1}[z_2(t)]^{m_2} \cdots [z_N(t)]^{m_N} \right\},
$$

where (above and below) the symbol $\sum_{m_{\ell}}^{(M)}$ denotes a sum running over all nonnegative values of the $N$ indices $m_{\ell}$ subject to the restriction

$$
\sum_{\ell=1}^{N} (m_{\ell}) = M,
$$

implying that the polynomials in $N$ variables $z_n(t)$ in the right-hand sides of the $N$ ODEs\textsuperscript{[13]} are all homogeneous of degree $M$.

Notation. Throughout this paper $M$ and $N$ are positive integers larger than unity; the index $n$ takes positive integer values; indices and exponents such as $m_1$, $m_2$, ..., take all the nonnegative integer values consistent with the restriction\textsuperscript{[11]}; the independent variable $t$ can be considered as playing the role of "time", taking all nonnegative real values (but it shall also be eventually convenient to replace it formally with the complex variable $\tau$, see below); a superimposed dot indicates a $t$-differentiation; the coefficients $c_{nm_1m_2\cdots m_N}$ are ($t$-independent) parameters; while of course the dependent variables $z_n(t)$ are functions of the independent variable $t$ and ascertaining their $t$-evolution from the set of $N$ initial data $z_n(0)$ is our main task. The coefficients $c_{nm_1m_2\cdots m_N}$ and the dependent variables $z_n(t)$ might be restricted to be real; but in the last part of this paper we shall assume that they are instead complex, setting

$$
c_{nm_1m_2\cdots m_N} = a_{nm_1m_2\cdots m_N} + ib_{nm_1m_2\cdots m_N};
$$

and we shall as well replace the independent variable $t$ with a complex variable $\tau$, see below eq. \textsuperscript{[5]}; here and below of course $i$ is the imaginary unit, $i^2 = -1$. Finally: below $\omega$ denotes an arbitrary nonvanishing real parameter.\textsuperscript{[\ref{omega}]}
The system (1) has been investigated over time in an enormous number of mainly mathematical, or mainly applicative, papers (more than it is possible to report in an adequate manner; for a seminal paper see, for instance, [1]): although generally for specific, relatively small, values of \( N \) and \( M \). The mathematics behind the findings reported in the present paper is rather elementary; yet these developments may have some interest—perhaps mainly in applicative contexts—because they are based on a somewhat unconventional approach: to identify explicitly solvable cases of the system (1) by introducing constraints involving, in addition to the coefficients \( c_{nm_1m_2...m_M} \), also the initial data \( z_n(0) \) (which, in many applicative contexts, may well play the role of control elements, determining the time evolution of the system).

Our main result is the following

**Proposition.** The system (1) features the special solution

\[
 z_n(t) = z_n(0)(1 + Kt)^{(1-M)/(1-M)} , \quad n = 1, 2, ..., N ,
\]

provided there hold the following \( N \) explicit algebraic constraints on the a priori arbitrary parameter \( K \), the coefficients \( c_{nm_1m_2...m_M} \) and the \( N \) initial data \( z_n(0) \):

\[
 Kz_n(0) = (1 - M) \left( \sum_{m}^{(M)} c_{nm_1m_2...m_M} [z_1(0)]^{m_1} [z_2(0)]^{m_2} ... [z_N(0)]^{m_N} \right) ,
\]

\[
 n = 1, 2, ..., N . \tag{3b}
\]

**Remark 1.** The proof that (3) satisfies the system of ODEs (1) is elementary: just insert (3a) in (1a) and verify that, thanks to (1b) and (3a), the \( N \) ODEs (1a) are satisfied. ■

**Remark 2.** The system of \( N \) algebraic equations (3b) generally determines—for any given assignment of the a priori arbitrary coefficients \( c_{nm_1m_2...m_M} \) —\( N \) out of the \( N + 1 \) quantities \( K \) and \( z_n(0) \); but it is also possible to select ad libitum \( N \) elements out of the complete set of data \( K \), \( c_{nm_1m_2...m_M} \) and \( z_n(0) \), and to then consider these selected elements as those to be determined—by the \( N \) conditions (3b)—in terms of the remaining arbitrarily assigned elements in the complete set of data. If one chooses to satisfy these \( N \) conditions by solving the \( N \) equations (3b) for \( N \) of the coefficients \( c_{nm_1m_2...m_M} \)—or for the parameter \( K \) and \( N - 1 \) of the coefficients \( c_{nm_1m_2...m_M} \)—then this task can be generally performed explicitly, since the relevant algebraic equations to be solved are then linear in the unknown quantities; otherwise, these determinations require the solution of nonlinear equations, a task which can be performed explicitly only rarely in an algebraic setting; but which can generally be performed, with arbitrary approximation, in a numerical context. ■

**Example 1.** Assume for instance \( N = 2 \) and \( M = 4 \), so that the system (1) reads as follows (note below the notational simplification):

\[
 \dot{x}(t) = \sum_{m=0}^{4} c_{nm} [z_1(t)]^{4-m} [z_2(t)]^{m} , \quad n = 1, 2 ,
\]

(4a)

featuring 2 dependent variables \( z_n(t) \) and 10 a priori arbitrary coefficients \( c_{nm} \) \( (n = 1, 2 ; \quad m = 0, 1, 2, 3, 4) \). Then the solution (3a) reads as follows:

\[
 z_n(t) = z_n(0)[1 + Kt]^{-1/3} , \quad n = 1, 2 ,
\]

(4b)

and the 2 conditions (3b) read as follows:

\[
 Kz_n(0) = -3 \sum_{m=0}^{4} c_{nm} [z_1(0)]^{4-m} [z_2(0)]^{m} , \quad n = 1, 2 ,
\]

(4c)

These algebraic constraints can of course be explicitly solved for any 2 of the 10 coefficients \( c_{nm} \) in terms of the other 8 coefficients \( c_{nm} \) and of the 3 arbitrary data \( K \), \( z_1(0) \), \( z_2(0) \); or alternatively for \( K \) and only 1 of the 10 coefficients \( c_{nm} \) in terms of the other 9 coefficients \( c_{nm} \) and of the 2 arbitrary initial data \( z_1(0) \), \( z_2(0) \); with many other possibilities left to the imagination of the interested reader. ■

The periodic variant obtains from the previous results—where we now assume all quantities to be complex and we formally replace the independent variable \( t \) with the complex variable \( \tau \)—via the following well-known trick (amounting to a simple change of dependent and independent variables: see, for instance, [2]):

\[
 x_n(t) + iy_n(t) = \{\exp[i\omega t/(M - 1)]\} z_n(\tau) , \quad \tau = [\exp(i\omega t) - 1]/(i\omega) ,
\]

(5)
implying \( \dot{\tau} (t) = \exp (i \omega t) \) and transforming the system (1a) into the following (still autonomous) system involving now the \( 2N \) real variables \( x_n (t) \) and \( y_n (t) \) (depending of course on the real independent variable \( t \): "time"):

\[
\begin{align*}
\dot{x}_n (t) &= - [\omega / (M - 1)] y_n (t) + Re [Z_n (t)] , \\
\dot{y}_n (t) &= [\omega / (M - 1)] x_n (t) + Im [Z_n (t)] ,
\end{align*}
\]

where (see (5) and (2))

\[
Z_n (t) = \sum_{m}^{(M)} \{(a_{nm} + i b_{nm}) [x_1 (t) + i y_1 (t)]^{m_1} \cdots [x_N (t) + i y_N (t)]^{m_N} \} .
\]

**Remark 3.** The fact that all solutions \( x_n (t) \), \( y_n (t) \) of the system (4) obtained via the definition (5) with \( z_n (\tau) \) defined by (3a) (of course with \( t \) replaced there by \( \tau \), see (4)) are periodic with a period \( T \) which is an (easily identifiable on a case-by-case basis) integer multiple of the basic period \( 2\pi / |\omega| \) is rather obvious; in case of doubt, see [2]. ■

**Example 2.** As an example of solvable system featuring periodic solutions let us display the findings reported in the special case with \( N = 2 \) and \( M = 4 \). Then the system (4) of 4 ODEs reads as follows:

\[
\begin{align*}
\dot{x}_n (t) &= - [\omega / 3] y_n (t) + Re [Z_n (t)] , \quad n = 1, 2 , \\
\dot{y}_n (t) &= [\omega / 3] x_n (t) + Im [Z_n (t)] , \quad n = 1, 2 ,
\end{align*}
\]

\[
Z_n (t) = \sum_{m=0}^{4} \{(a_{nm} + i b_{nm}) [x_1 (t) + i y_1 (t)]^{m_1} \cdots [x_1 (t) + i y_1 (t)]^{m} \} ;
\]

its explicit solutions read as follows:

\[
\begin{align*}
x_n (t) &= Re [\zeta_n (t)] , \quad y_n (t) = Im [\zeta_n (t)] , \quad n = 1, 2 , \\
\zeta_n (t) &= [x_n (0) + i y_n (0)] \exp (i \omega t/3) \cdot \left\{ 1 + (K_R + i K_I) [\exp (i \omega t) - 1] / (i \omega) \right\}^{-1/3} , \quad n = 1, 2 ,
\end{align*}
\]

provided the 2 (a priori arbitrary) real parameters \( K_R \) and \( K_I \), the 4 (a priori arbitrary) real initial data \( x_n (0) \) and \( y_n (0) \) and the 20 (a priori arbitrary) real coefficients \( a_{nm} \) and \( b_{nm} \) \( n = 1, 2 ; m = 0, 1, 2, 3, 4 \) are related to each other by the following 2 complex (i. e., 4 real) constraints:

\[
(K_R + i K_I) [x_n (0) + i y_n (0)]
\]

\[
= -3 \sum_{m=0}^{4} \{(a_{nm} + i b_{nm}) [x_1 (0) + i y_1 (0)]^{m_1} \cdots [x_1 (0) + i y_1 (0)]^{m} \} , \quad n = 1, 2 . ■
\]

**Final Remark.** As already noted above, the mathematics behind the results reported above is rather elementary. Yet these findings do not seem to have been advertised so far, while their applicable potential is clearly vast; so—especially among applied mathematicians and practitioners of the various scientific disciplines where systems of ODEs such as those discussed above play a key role—a wider knowledge of them seems desirable; for instance via their inclusion in standard compilations of solvable ODEs such as [3]. ■

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