REEB FOLIATIONS ON $S^5$ AND CONTACT 5-MANIFOLDS VIOLATING THE THURSTON-BENNEQUIN INEQUALITY

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Abstract. We obtain the following two results through foliation theoretic approaches including a review of Lawson’s construction of a codimension-one foliation on the 5-sphere:
1) The standard contact structure on the 5-sphere deforms to ‘Reeb foliations’.
2) We define a 5-dimensional Lutz tube which contains a plastikstufe. Inserting it into any contact 5-manifold, we obtain a contact structure which violates the Thurston-Bennequin inequality for a convex hypersurface with contact-type boundary.

1. Introduction and preliminaries

The first aim of this paper is to show that the standard contact structure $D_0$ on $S^5$ deforms via contact structures into spinnable foliations, which we call Reeb foliations (§2). Here a spinnable foliation is a codimension-one foliation associated to an open-book decomposition whose binding is fibred over $S^1$. In 1971, Lawson [16] constructed a spinnable foliation on $S^5$ associated to a Milnor fibration. We construct such a spinnable foliation on $S^5$ as the limit $D_1$ of a family $\{D_t\}_{0\leq t<1}$ of contact structures. Since $S^5$ is compact, the family $\{D_t\}_{0\leq t<1}$ can be traced by a family of diffeomorphisms $\varphi_t : S^5 \to S^5$ with $\varphi_0 = \text{id}$ and $(\varphi_t)_*D_0 = D_t$ (Gray’s stability).

The second aim is to show that any contact 5-manifold admits a contact structure which violates the Thurston-Bennequin inequality for a convex hypersurface (§3). We define a 5-dimensional Lutz tube and explain how to insert it into a given contact 5-manifold to violate the inequality. Moreover a 5-dimensional Lutz tube contains a plastikstufe, which is an obstruction to symplectic fillability found by Niederkrüger [22] and Chekanov. A different Lutz twist on a contact manifold $(M^{2n+1}, \alpha)$ was recently introduced in Etnyre-Pancholi [6] as a modification of the contact structure $D = \ker \alpha$ near an $n$-dimensional submanifold. Contrastingly, the core of our Lutz tube is a codimension-two contact submanifold. We change the standard contact structure on $S^5$ by inserting a Lutz tube along the binding of the open-book decomposition of a certain Reeb foliation.

The author [21] also showed that any contact manifold of dimension $> 3$ violates the Thurston-Bennequin inequality for a non-convex hypersurface. However he conjectures that the inequality holds for any convex hypersurface in the standard $S^{2n+1}$. See §4 for related problems.

The rest of this section is the preliminaries.

1.1. Thurston-Bennequin inequality. A positive (resp. negative) contact manifold consists of an oriented $(2n+1)$-manifold $M^{2n+1}$ and a 1-form $\alpha$ on $M^{2n+1}$ with $\alpha \wedge (d\alpha)^n > 0$ (resp. $\alpha \wedge (d\alpha)^n < 0$). The (co-)oriented hyperplane distribution $D = \ker \alpha$ is called the contact structure on the contact manifold $(M^{2n+1}, \alpha)$. In the case where $(M^{2n+1}, \alpha)$ is positive, the symplectic structure $d\alpha|_{\ker \alpha}$ on the oriented vector bundle $\ker \alpha$ is also positive, i.e., $(d\alpha)^n|_{\ker \alpha} > 0$. Hereafter we assume that all contact structures and symplectic structures are positive.

In this subsection, we assume that any compact connected oriented hypersurface $\Sigma$ embedded in a contact manifold $(M^{2n+1}, \alpha)$ tangents to the contact structure $\ker \alpha$ at finite number of interior points. Note that the hyperplane field $\ker \alpha$ is maximally non-integrable. Let $S_+(\Sigma)$ (resp. $S_-(\Sigma)$) denote the set of the positive (resp. negative) tangent points, and $S(\Sigma)$ the union $S_+(\Sigma) \cup S_-(\Sigma)$. The sign of the tangency at $p \in S(\Sigma)$ coincides with the sign of $\{(d\alpha|_\Sigma)^n\}_p$. Considering on $(\ker \alpha, d\alpha|_{\ker \alpha})$, we see that the symplectic orthogonal of the intersection $T\Sigma \cap \ker \alpha$ forms an oriented line field $L$ on $\Sigma$, where the singularity of $L$ coincides with $S(\Sigma)$.

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**Definition 1.1.** The singular oriented foliation $\mathcal{F}_{\Sigma}$ defined by $T\mathcal{F}_{\Sigma} = L$ is called the characteristic foliation on $\Sigma$ with respect to the contact structure $\ker \alpha$.

Put $\beta = \alpha|\Sigma$ and take any volume form $\nu$ on $\Sigma$. Then we see that the vector field $X$ on $\Sigma$ defined by $\iota_X \nu = \beta \wedge (d\beta)^{n-1}$ is a positive section of $L$. Moreover,

$$\iota_X \{ \beta \wedge (d\beta)^{n-1} \} = -\beta \wedge \iota_X (d\beta)^{n-1} = 0, \quad \beta \wedge (d\beta)^{n-1} \neq 0 \implies \beta \wedge \iota_X d\beta = 0.$$  

Thus the flow generated by $X$ preserves the conformal class of $\beta$. Since $\nu$ is arbitrary, we may take $X$ as any positive section of $L$. Therefore the 1-form $\beta$ defines a holonomy invariant transverse contact structure of the characteristic foliation $\mathcal{F}_{\Sigma}$.

On the other hand, for any volume form $\mu(\neq \nu)$ on $\Sigma$, we see that the sign of $\text{div} X = (L_X \mu)/\mu$ at each singular point $p \in S(\Sigma)$ coincides with the sign of $\mu$. Thus $\mathcal{F}_{\Sigma}$ contains the information about the sign of the tangency to the contact structure $\ker \alpha$. We also define the index $\text{Ind} p = \text{Ind}_p \chi$ of a singular point $p \in S(\Sigma)$ by using the above vector field $X$.

**Definition 1.2.** Suppose that the boundary $\partial \Sigma$ of the above hypersurface $\Sigma$ is non-empty, and the characteristic foliation $\mathcal{F}_{\Sigma}$ is positively (i.e., outward) transverse to $\partial \Sigma$. Then we say that $\Sigma$ is a hypersurface with contact-type boundary. Note that $\beta \partial \Sigma = \alpha \partial \Sigma$ is a contact form.

**Remark.** The Liouville vector field $X$ on a given exact symplectic manifold $(\Sigma, d\lambda)$ with respect to a primitive 1-form $\lambda$ of $d\lambda$ is defined by $\iota_X d\lambda = \lambda$. If $X$ is positively transverse to the boundary $\partial \Sigma$, then $(\partial \Sigma, \lambda|\partial \Sigma)$ is called the contact-type boundary. The above definition is a natural shift of this notion into the case of hypersurfaces in contact manifolds.

Let $D^2$ be an embedded disk with contact-type boundary in a contact 3-manifold. We say that $D^2$ is overtwisted if the singularity $S(D^2)$ consists of a single sink point. Note that a sink point is a negative singular point since it has negative divergence. A contact 3-manifold is said to be overtwisted, or tight depending on whether there exists an overtwisted disk with contact-type boundary in it, or not. We can show that the existence of an overtwisted disk with contact-type boundary is equivalent to the existence of an overtwisted disk with Legendrian boundary, which is an embedded disk $D'$ similar to the above $D^2$ except that the characteristic foliation $\mathcal{F}_{D'}$ tangents to the boundary $\partial D'$, where $\partial D'$ or $-\partial D'$ is a closed leaf of $\mathcal{F}_{D'}$.

Let $\Sigma$ be any surface with contact-type (i.e., transverse) boundary embedded in the standard $S^3$. Then Bennequin[4] proved the following inequality which implies the tightness of $S^3$:

**Thurston-Bennequin inequality.** $\sum_{p \in S_-(\Sigma)} \text{Ind} p \leq 0$.

Eliashberg proved the same inequality for symplectically fillable contact 3-manifolds ([8]), and finally for all tight contact 3-manifolds ([1]). Recently Niederkrüger[22] and Chekanov found a $(n + 1)$-dimensional analogue of an overtwisted disk with Legendrian boundary — a plastikeifte which is roughly the trace $K^{n-1} \times D^2$ of an overtwisted disk $D^2$ with Legendrian boundary travelling along a closed integral submanifold $K^{n-1} \subset M^{2n+1}$. However, in order to create some meaning of the above inequality in higher dimensions, we need a $2n$-dimensional analogue of an overtwisted disk with contact-type boundary.

**Remark.** The Thurston-Bennequin inequality can also be written in terms of relative Euler number: The vector field $X \in T\Sigma \cap \ker \alpha$ is a section of $\ker \alpha|\Sigma$ which is canonical near the boundary $\partial \Sigma$. Thus under a suitable boundary condition we have

$$\langle e(\ker \alpha), [\Sigma, \partial \Sigma] \rangle = \sum_{p \in S_-(\Sigma)} \text{Ind} p - \sum_{p \in S_+(\Sigma)} \text{Ind} p.$$  

Then the Thurston-Bennequin inequality can be expressed as

$$-\langle e(\ker \alpha), [\Sigma, \partial \Sigma] \rangle \leq -\chi(\Sigma).$$  

There is also an absolute version of the Thurston-Bennequin inequality for a closed hypersurface $\Sigma$ with $\chi(\Sigma) \leq 0$, which is expressed as $|\langle e(\ker \alpha), [\Sigma] \rangle| \leq -\chi(\Sigma)$, or equivalently

$$\sum_{p \in S_-(\Sigma)} \text{Ind} p \leq 0 \quad \text{and} \quad \sum_{p \in S_+(\Sigma)} \text{Ind} p \leq 0.$$
The above symplectic manifold \((M,\alpha)\) is said to be convex if there exists a contact vector field transverse to \(\Sigma\).

Let \(Y\) be a contact vector field positively transverse to a closed convex hypersurface \(\Sigma\), and \(\Sigma \times (-\varepsilon,\varepsilon)\) a neighbourhood of \(\Sigma = \Sigma \times \{0\}\) with \(Y = \partial / \partial z\) (\(z \in (-\varepsilon,\varepsilon)\)). We may assume that the contact form \(\alpha\) is \(Y\)-invariant after rescaling it by multiplying a suitable positive function. Note that this rescaling does not change the level set \(\{\alpha(Y) = 0\}\). By perturbing \(Y\) in \(V_\alpha\), if necessary, we can modify the Hamiltonian function \(H = \alpha(Y)\) so that the level set \(\{H = 0\}\) is a regular hypersurface of the form \(\Gamma \times (-\varepsilon,\varepsilon)\) in the above neighbourhood \(\Sigma \times (-\varepsilon,\varepsilon)\), where \(\Gamma \subset M\) is a codimension-2 submanifold. Put \(h = H|\Sigma\).

**Definition 1.4.** The submanifold \(\Gamma = \{h = 0\} \subset \Sigma\) is called the **dividing set** on \(\Sigma\) with respect to \(Y\). \(\Gamma\) divides \(\Sigma\) into the **positive region** \(\Sigma_+ = \{h \geq 0\}\) and the **negative region** \(\Sigma_- = \{-h \leq 0\}\) so that \(\Sigma = \Sigma_+ \cup (-\Sigma_-)\). We orient \(\Gamma\) as \(\partial \Sigma_+ = \partial \Sigma_-\).

Note that \(\pm |\Sigma|\{\pm H > 0\}\) is the Reeb field of \(\alpha / |H| = \beta / |H| \pm dz\), where \(\beta\) is the pull-back of \(\alpha|\Sigma\) under the projection along \(Y\). Since the 2n-form
\[
\Omega = (d\beta)^n - 1 \wedge (H d\beta + n \beta dH)
\]
satisfies \(\Omega \wedge dz = \alpha \wedge (d\alpha)^n > 0\), the characteristic foliation \(F_\Sigma\) is positively transverse to the dividing set \(\Gamma\). Thus \(\Gamma\) is a positive contact submanifold of \((M,\alpha)\). The open set \(U = \{H < \varepsilon'\}\) is of the form \((-\varepsilon',\varepsilon') \times \Gamma \times (-\varepsilon,\varepsilon)\) for sufficiently small \(\varepsilon' > 0\). Let \(\rho(H) > 0\) be an even function of \(H\) which is increasing on \(H > 0\), and coincides with \(1/|H|\) except on \((-\varepsilon',\varepsilon')\). Then we see that \(d(\rho\alpha)|\text{int } \Sigma_\pm\) are symplectic forms.

On the other hand, let \((\Sigma_\pm, d\lambda_\pm)\) be compact exact symplectic manifolds with the same contact-type boundary \((\partial \Sigma_\pm, \mu)\), where we fix the primitive 1-forms \(\lambda_\pm\) and assume that \(\mu = \lambda_\pm |\partial \Sigma_\pm\). Then \(\lambda_i + dz\) is a z-invariant contact form on \(\Sigma_i \times \mathbb{R}\) (\(i = +\) or \(-\), \(z \in \mathbb{R}\)).

**Definition 1.5.** The contact manifold \((\Sigma_i \times \mathbb{R}, \lambda_i + dz)\) is called the **contactization** of \((\Sigma_i, d\lambda_i)\). Take a collar neighbourhood \((-\varepsilon',0] \times \partial \Sigma_i \subset \Sigma_i\) such that
\[
\lambda_i + dz|((-\varepsilon',0] \times \partial \Sigma_i \times \mathbb{R}) = e^s \mu + dz \quad (s \in (-\varepsilon',0]).
\]
We modify \(\lambda_i + dz\) near \((-\varepsilon',0] \times \partial \Sigma_i \times \mathbb{R}\) in a canonical way into a contact form \(\alpha_i\) with
\[
\alpha_i|((-\varepsilon',0] \times \partial \Sigma_i \times \mathbb{R}) = e^{-s^2/\varepsilon'} \mu - \frac{s}{\varepsilon'} dz.
\]
We call the contact manifold \((\Sigma_i \times \mathbb{R}, \alpha_i)\) the **modified contactization** of \((\Sigma_i, d\lambda_i)\).

**Remark.** The above symplectic manifold \((\Sigma_i, d\lambda_i)\) can be fully extended by attaching the half-symplectization \((\mathbb{R} > 0 \times \partial \Sigma_i, d(\varepsilon' \mu))\) to the boundary. The interior of the modified contactization is then contactomorphic to the contactization of the fully extended symplectic manifold.
The modified contactizations $\Sigma_+ \times \mathbb{R}$ and $\Sigma_- \times \mathbb{R}'$ match up to each other to form a connected contact manifold $((\Sigma_+ \cup (-\Sigma_-)) \times \mathbb{R}, \alpha)$ where $\mathbb{R}' = -\mathbb{R}$. Indeed, $\alpha$ can be written near $\Gamma \times \mathbb{R} = \partial \Sigma_+ \times \mathbb{R} = \partial \Sigma_- \times (-\mathbb{R}')$ as

$$\alpha((-\varepsilon', \varepsilon')) \times \mathbb{R} = e^{-s/\varepsilon'} \mu - \frac{s}{\varepsilon} dz \quad (s \in (-\varepsilon', \varepsilon'), z \in \mathbb{R}).$$

**Definition 1.6.** The contact manifold $((\Sigma_+ \cup (-\Sigma_-)) \times \mathbb{R}, \alpha)$ is called the unified contactization of $\Sigma = \Sigma_+ \cup (-\Sigma_-)$. Since $(-\Sigma_+ = \Sigma_-$ and $(-\Sigma)_- = \Sigma_+$, the unified contactization of $-\Sigma = \Sigma_+ \cup (-\Sigma_-)$ can be obtained by turning the unified contactization of $\Sigma_+ \cup (-\Sigma_-)$ upside-down. Note that $-\Sigma \in V_\alpha$.

Clearly, a small neighbourhood of any convex hypersurface $\Sigma_+ \cup (-\Sigma_-)$ is contactomorphic to a neighbourhood of $(\Sigma_+ \cup (-\Sigma_-)) \times \{0\}$ in the unified contactization.

Conceptually, a convex hypersurface in contact topology play the same role as a contact-type hypersurface in symplectic topology — both are powerful tools for cut-and-paste because they have canonical neighbourhoods modeled on the unified contactization and the symplectization. Further Giroux[9] showed that any closed surface in a contact 3-manifold is smoothly approximated by a convex one. This fact closely relates contact topology with differential topology in this dimension.

On the other hand, there exists a hypersurface which cannot be smoothly approximated by a convex one if the dimension of the contact manifold is greater than three (see [21]).

**Definition 1.7.** A compact connected oriented embedded hypersurface $\Sigma$ with non-empty contact-type boundary in a contact manifold $(M, \alpha)$ is said to be convex if there exists a contact vector field $Y$ such that $\alpha(Y)/\Sigma > 0$ and $Y$ is transverse to $\Sigma$.

Put $h = \alpha(Y)/\Sigma$ after perturbing $Y$. Then the dividing set $\Gamma = \{h = 0\}$ divides $\Sigma$ into the positive region $\Sigma_+ = \{h \geq 0\}$ and the (possibly empty) negative region $\Sigma_- = \{h \leq 0\}$ so that $\Sigma = \Sigma_+ \cup (-\Sigma_-)$ and $\partial \Sigma = \partial \Sigma_+ \setminus \partial \Sigma_- \neq \emptyset$.

Note that the above definition avoids touching of $\Gamma$ to the contact-type boundary $\partial \Sigma$. Now the Thurston-Bennequin inequality can be written as

**Thurston-Bennequin inequality for convex hypersurfaces.** $\chi(\Sigma_-) \leq 0$ (or $\Sigma_- = \emptyset$).

Suppose that there exists a convex disk $\Sigma = D^2$ with contact-type boundary in a contact 3-manifold which is the union $\Sigma_+ \cup (-\Sigma_-)$ of a negative disk region $\Sigma_- \leq \Sigma$ and a positive annular region $\Sigma_+$. Then the convex disk $\Sigma$ violates the Thurston-Bennequin inequality and is called a convex overtwisted disk ($\chi(\Sigma_-) = 1 > 0$). Conversely, it is clear that any overtwisted disk with contact-type boundary is also approximated by a convex overtwisted disk.

**Definition 1.8.** A convex overtwisted hypersurface is a connected convex hypersurface $\Sigma_+ \cup (-\Sigma_-)$ with contact-type boundary which satisfies $\chi(\Sigma_-) > 0$.

Note that any convex overtwisted hypersurface $\Sigma$ contains a connected component of $\Sigma_+$ whose boundary is disconnected. This relates to Calabi’s question on the existence of a compact connected exact symplectic 2n-manifold $(n > 1)$ with disconnected contact-type boundary. McDuff[18] found the first example of such a manifold. Here is another example:

**Example 1.9.** (Mitsumatsu[19], Ghys[8] and Geiges[7]) To obtain a symplectic 4-manifold with disconnected contact-type boundary, we consider the mapping torus $T_A = T^2 \times [0, 1]/A \cong ((x, y), z)$ of a linear map $A \in SL(2, \mathbb{Z}) (A : T^2 \times \{1\} \to T^2 \times \{0\})$ with $\det A > 2$. Let $dvol_T^{\mathbb{R}}$ be the standard volume form on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $v_\pm$ eigenvectors of $A$ which satisfy $Av_\pm = \pm a v_\pm$, where $a > 1$ and $dvol_T^{\mathbb{R}}(v_+, v_-) > 0$.

In general, a cylinder $[-1, 1] \times M^3$ admits a symplectic structure with contact-type boundary if $M^3$ admits a co-orientable Anosov foliation (Mitsumatsu[19]). In the case where $M^3 = T_A$, the 1-forms $\beta_\pm = \pm a^{-1} dvol_T^{\mathbb{R}}(v_\pm, \cdot)$ define Anosov foliations. Then the cylinder

$$\left(W_A = [-1, 1] \times T_A, d(\beta_+ + s \beta_-) \right) \quad (s \in [-1, 1])$$

is a symplectic manifold with contact-type boundary $(-T_A) \sqcup T_A$.

Using the above cylinder $W_A$, we construct a convex overtwisted hypersurface in $\Sigma$.
2. CONVERGENCE OF CONTACT STRUCTURES TO FOLIATIONS

First we define a supporting open-book decomposition on a closed contact manifold.

**Definition 2.1.** Let \((M^{2n+1}, \alpha)\) be a closed contact manifold and \(O\) an open-book decomposition on \(M^{2n+1}\) by pages \(P_\theta (\theta \in \mathbb{R}/2\pi \mathbb{Z})\). Suppose that the binding \((N^{2n-1} = \partial P_\theta, \alpha|N^{2n-1})\) of \(O\) is a contact submanifold. Then if there exists a positive function \(\rho\) on \(M^{2n+1}\) such that
\[
d\theta \wedge (d(\rho \alpha))^n > 0 \quad \text{on} \quad M^{2n+1} \setminus N^{2n-1},
\]
the open-book decomposition \(O\) is called a **supporting open-book decomposition** on \((M^{2n+1}, \alpha)\).

The function \(\rho\) can be taken so that \(\rho \alpha\) is axisymmetric near the binding. Precisely, we can modify the function \(\rho\) near a tubular neighbourhood \(N^{2n-1} \times D^2\) except on the binding \(N^{2n-1} \times \{0\}\), if necessary, so that with respect to the polar coordinates \((r, \theta)\) on the unit disk \(D^2\)

- i) the restriction \(\rho \alpha|(N^{2n-1} \times D^2)\) is of the form \(f(r) \mu + g(r)d\theta\),
- ii) \(\mu\) is the pull-back \(\pi^*(\rho \alpha|N^{2n-1})\) under the projection \(\pi : N^{2n-1} \times D^2 \to N^{2n-1}\),
- iii) \(f(r)\) is a positive function of \(r\) on \(N^{2n-1} \times D^2\) with \(f'(r) < 0\) on \((0,1]\), and
- iv) \(g(r)\) is a weakly increasing function with \(g(r) \equiv r^2\) near \(r = 0\) and \(g(r) \equiv 1\) near \(r = 1\).

Next we prove the following theorem.

**Theorem 2.2.** Let \(O\) be a supporting open-book decomposition on a closed contact manifold \((M^{2n+1}, \alpha)\) of dimension greater than three \((n > 1)\). Suppose that the binding \(N^{2n-1}\) of \(O\) admits a non-zero closed 1-form \(\nu\) with \(\nu \wedge (d(\rho \alpha|N^{2n-1}))^{n-1} = 0\) where \(\rho\) is a function on \(M^{2n+1}\) satisfying all of the above conditions. Then there exists a family of contact forms \(\{\alpha_t\}_{0 \leq t < 1}\) on \(M^{2n+1}\) which starts with \(\alpha_0 = \rho \alpha\) and converges to a non-zero 1-form \(\alpha_1\) with \(\alpha_1 \wedge d\alpha_1 \equiv 0\). That is, the contact structure \(\ker \alpha_1\) then deforms to a spinnable foliation.

**Proof.** Take smooth functions \(f_1(r), g_1(r), h(r)\) and \(e(r)\) of \(r \in [0,1]\) such that

- i) \(f_1 \equiv 1\) near \(r = 0\), \(f_1 \equiv 0\) on \([1/2,1]\), \(f'_1 \leq 0\) on \([0,1]\),
- ii) \(g_1 \equiv 1\) near \(r = 1\), \(g_1 \equiv 0\) on \([0,1/2]\), \(g'_1 \geq 0\) on \([0,1]\),
- iii) \(h \equiv 1\) on \([0,1/2]\), \(h \equiv 0\) near \(r = 1\),
- iv) \(e\) is supported near \(r = 1/2\), and \(e(1/2) \neq 0\).

Put \(f_t(r) = (1-t)f(r) + tf_1(r)\), \(g_t(r) = (1-t)g(r) + tg_1(r)\) and
\[
\alpha_t(N \times D^2) = f_t(r)((1-t)\mu + th(r)\nu) + g_t(r)d\theta + te(r)dr,
\]
where \(\nu\) also denotes the pull-back \(\pi^*\nu\). We extend \(\alpha_t\) by
\[
\alpha_t(M \setminus (N \times D^2)) = \tau \rho \alpha + (1-\tau)d\theta \quad \text{where} \quad \tau = (1-t)^2.
\]
Then we see from \(d\nu \equiv 0\) and \(\nu \wedge (d\mu)^{n-1} \equiv 0\) that \(\alpha_t \wedge (d\alpha_t)^n\) can be written as
\[
n_f^{-1}(1-t)^n(g'_1f_1 - f'_1g_1)\mu \wedge (d\mu)^{n-1} \wedge dr \wedge d\theta \quad \text{on} \quad N \times D^2 \quad \text{and}
\]
\[
\tau^{n+1} \rho^{n+1} \alpha \wedge (d\alpha)^n + \tau^n(1-\tau)d\theta \wedge (d(\rho \alpha))^n \quad \text{on} \quad M \setminus (N \times D^2).
\]
Therefore we have
\[
\alpha_t \wedge (d\alpha)^n > 0 \quad (0 \leq t < 1), \quad \alpha_1 \wedge d\alpha_1 \equiv 0 \quad \text{and} \quad \alpha_1 \neq 0.
\]
This completes the proof of Theorem 2.2.

**Remark.**

1) A similar result in the case where \(n = 1\) is contained in the author’s paper: Any contact structure \(\ker \alpha\) on a closed 3-manifold deforms to a spinnable foliation.

2) The orientation of the compact leaf \(\{r = 1/2\}\) depends on the choice of the sign of the value \(e(1/2)\). Here the choice is arbitrary.

We give some examples of the above limit foliations which relate to the following proposition on certain \(T^2\)-bundles over the circle.
Proposition 2.3 (Van Horn[20]). Let $T_{A_{m,0}}$ denote the mapping torus $T^2 \times [0,1]/A_{m,0} \cong \{(x, y), z\}$ of the linear map $A_{m,0} = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} : T^2 \times \{1\} \to T^2 \times \{0\}$ ($m \in \mathbb{N}$). Then $\ker(dy + mxdz)$ is the unique Stein fillable contact structure on $T_{A_{m,0}}$ (up to contactomorphism). Moreover it admits a supporting open-book decomposition $O_{m,0}$ such that

i) the page is a $m$-times punctured torus, and

ii) the monodromy is the right-handed Dehn twist along (the disjoint union of $m$ loops parallel to) the boundary of the page.

Let $\mathbb{C}^3$ be the $\xi\eta\zeta$-space, and $\pi_\xi, \pi_\eta$ and $\pi_\zeta$ denote the projections to the axes.

Example 2.4 (Lawson’s foliation). The link $L$ of the singular point $(0,0,0)$ of the complex surface $\{\xi^4 + \eta^3 + \zeta^3 = 0\} \subset \mathbb{C}^3$ is diffeomorphic to the $T^2$-bundle $T_{A_{m,0}}$. (To see this, consider the projective curve $\{\xi^3 + \eta^3 + \zeta^3 = 0\} \subset \mathbb{CP}^2$ diffeomorphic to $T^2$. Since $L$ is the union of the Hopf fibres over this torus, it is also a $T^2$-bundle over the circle.) Moreover, since $L$ is Stein fillable, it is contactomorphic to $(T_{A_{m,0}}, \mu = dy + 3zdx)$. Indeed the open-book decomposition $O_{m,0}$ in Proposition 2.3 is equivalent to the supporting open-book decomposition $\{\arg(\pi_\xi (L) = \theta)_{\theta \in \mathbb{R}/2\pi \mathbb{Z}}\}$ on $L$. (To see this, regard $\xi$ as a parameter and consider the curve $C_\xi = \{\eta^3 + \zeta^3 = -\xi^4\}$ on the $\eta\zeta$-plane, which is diffeomorphic to $T^2 \setminus \{\text{three points}\}$. Then we can see that the fibration $\{(\xi) \times C_\xi \cap B^6\}_{|\xi| = \varepsilon}$ is equivalent to the page fibration of $O_{3,0}$, where $B^6$ denotes the unit hyperball of $\mathbb{C}^3 \approx \mathbb{R}^6 \ni (x_1, y_1, \ldots, x_3, y_3, \varepsilon)$. We put $\Lambda = 3 \sum_{i=1} (x_i dy_i - y_i dx_i)$ and $V_{\varepsilon, \theta} = \{\varepsilon^3 + \xi^3 = \varepsilon e^{\varepsilon/\pi}B^6 \cap B^6 \ (\theta \in \mathbb{R}/2\pi \mathbb{Z})\}$.

Then Gray’s stability implies that $\partial V_{\varepsilon, \theta} \subset (S^5, \Lambda|S^5)$ is contactomorphic to $L$. Since $\xi^4 + \eta^3 + \zeta^3$ is a homogeneous polynomial, the 1-form $\Lambda|V_{\varepsilon, \theta}$ is conformal to the pull-back of the restriction of $\Lambda$ to $\Sigma_{\varepsilon, \theta} = \{\rho \mid \rho > 0, \rho \in V_{\varepsilon, \theta} \cap S^5 \subset \{\arg(\xi^3 + \eta^3 + \zeta^3 = \theta)\}$ under the central projection. Indeed $\rho x_id(\rho y_i) - \rho y_id(\rho x_i) = \rho^2 (x_i dy_i - y_i dx_i)$ holds for any function $\rho$. Thus the fibration $\{\Sigma_{\varepsilon, \theta}\}_{\theta \in \mathbb{R}/2\pi \mathbb{Z}}$ extends to a supporting open-book decomposition on the standard $S^5$. We put $\nu = dz$ and apply Theorem 2.2 to obtain a limit foliation $F_{m,0}$ of the standard contact structure. This is the memorable first foliation on $S^5$ discovered by Lawson[16].

For other examples, we need the following lemma essentially due to Giroux and Mohns.

Lemma 2.5 (see [11]). Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function with $f(0, \ldots, 0) = 0$ such that the origin $(0, \ldots, 0)$ is an isolated critical point. Take a sufficiently small hyperball $B_\varepsilon = \{|z_1|^2 + \cdots + |z_n|^2 = \varepsilon^2\}$. Then there exists a supporting open-book decomposition on the standard $S^{2n-1}$ such that the binding is contactomorphic to the link $\{f = 0\} \cap \partial B_\varepsilon$ and the page fibration is equivalent to the fibration $\{\{f = \delta\} \cap B_\varepsilon\}_{|\delta| = \varepsilon'}$ ($0 < \varepsilon' < \varepsilon$).

Proof. Take the hyperball $B_\varepsilon' = \{|z_1|^2 + \cdots + |z_{n+1}|^2 = \varepsilon^2\}$ on $\mathbb{C}^{n+1}$ and consider the complex hypersurface $\Sigma_k = \{z_{n+1} = k f(z_1, \ldots, z_n)\} \cap B_\varepsilon'$ with contact-type boundary $\partial \Sigma_k (k \geq 0)$. Then Gray’s stability implies that $\partial \Sigma_k$ is contactomorphic to $\partial \Sigma_0 = \partial B_\varepsilon$. From $dz_{n+1}\Sigma_{\infty} = 0$ and $(x_{n+1}dy_{n+1} - y_{n+1}dx_{n+1})(-y_{n+1}\partial/\partial x_{n+1} + x_{n+1}\partial/\partial y_{n+1}) \geq 0$, we see that $\{\arg(f|\partial \Sigma_k) = \theta\}_{\theta \in S^1}$ is a supporting open-book decomposition of $\partial \Sigma_k$ equivalent to $\{\{f = \delta\} \cap B_\varepsilon\}_{|\delta| = \varepsilon'}$ if $k$ is sufficiently large and $\varepsilon' > 0$ is sufficiently small.

Example 2.6. Consider the polynomials $f_1 = \xi^6 + \eta^3 + \zeta^2$ and $f_2 = \xi^4 + \eta^4 + \zeta^2$. Then the link $L_m$ of the singular point $(0,0,0) \in \{f_m = 0\}$ is contactomorphic to the above $T^2$-bundle $T_{A_{m,0}}$ with the contact form $\mu = dy + mxdz$ ($m = 1, 2$). Indeed $O_{m,0}$ is equivalent to the supporting open-book decomposition $\{\arg(\tau_\xi (L_m) = \theta)_{\theta \in \mathbb{R}/2\pi \mathbb{Z}}\}$ on $L_m$. To see this, regard $\xi$ as a parameter and consider $C_\xi = \{f_m = 0\}$ on the $\eta\zeta$-plane, which is diffeomorphic to $T^2 \setminus \{m \text{ points}\}$. Then we can see that the fibration $\{\{\xi\} \times C_\xi \cap B^6\}_{|\xi| = \varepsilon}$ is equivalent to the page fibration of $O_{m,0}$.

On the other hand, Lemma 2.3 says that there exists a supporting open-book decomposition on the standard $S^5$ which is equivalent to the Milnor fibration with binding $L_m$. We put $\nu = dz$ and apply Theorem 2.2 to obtain a limit foliation $F_{m,0}$ ($m = 1, 2$).
**Definition 2.7.**

1) Let \( f : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \) be a holomorphic function with \( f(0, \ldots, 0) = 0 \) such that the origin is an isolated critical point or a regular point of \( f \). If the origin is singular, the Milnor fibre has the homotopy type of a bouquet of \( n \)-spheres. Suppose that the Euler characteristic of the Milnor fibre is positive, that is, the origin is regular if \( n \) is odd. Then we say that the Milnor fibration is \( PE \) (=positive Euler characteristic).

2) Let \( \mathcal{O} \) be a supporting open-book decomposition of the standard \( S^{2n+1} \). Suppose that the binding is the total space of a fibre bundle \( \pi \) over \( \mathbb{R}/\mathbb{Z} \geq t \), and the Euler characteristic of the page is positive. Then if \( \nu = \pi^* dt \) satisfies the assumption of Theorem 2.2 the resultant limit foliation is called a Reeb foliation.

The above \( \mathcal{F}_{m,0} \) (\( m = 1, 2, 3 \)) are Reeb foliations associated to PE Milnor fibrations. To obtain other examples of foliations associated to more general Milnor fibrations, Grauert’s topological characterization of Milnor fillable 3-manifolds is instructive ([12], see also [2]).

### 3. Five-dimensional Lutz tubes

In this section, we define a 5-dimensional Lutz tube by means of an open-book decomposition whose page is a convex hypersurface. We insert the Lutz tube along the binding of a certain characterization of Milnor fillable 3-manifolds is instructive ([12], see also [2]).

#### 3.1. Convex open-book decompositions

We explain how to construct a contact manifold with an open-book decomposition by convex pages.

**Proposition 3.1.** Let \( (\Sigma_{\pm}, d\lambda_{\pm}) \) be two compact exact symplectic manifolds with contact-type boundary. Suppose that there exists an inclusion \( \iota : \partial\Sigma_- \rightarrow \partial\Sigma_+ \) such that \( \iota^*(\lambda_+|\partial\Sigma_+) = \lambda_-|\partial\Sigma_- \). Let \( \varphi \) be a self-diffeomorphism of the union \( \Sigma = \Sigma_+ \cup_\iota (-\Sigma_-) \) supported in \( \text{int} \Sigma_+ \cup \text{int}(-\Sigma_-) \) which satisfies

\[
(\varphi|\Sigma_{\pm})^*\lambda_{\pm} - \lambda_{\pm} = dh_{\pm}
\]

for suitable positive functions \( h_{\pm} \) on \( \Sigma_{\pm} \). We choose some connected components of \( \partial\Sigma_+ \setminus \iota(\partial\Sigma_-) \) and take their disjoint union \( B \). Then there exists a smooth map \( \Phi \) from the unitedification \( \Sigma \times \mathbb{R} \) to a compact contact manifold \((M, \alpha)\) such that

- i) \( \Phi|(\Sigma \setminus B) \times \mathbb{R} \) is a cyclic covering which is locally a contactomorphism,
- ii) \( P_0 = \Phi(\Sigma \setminus \{0\}) \approx \Sigma \) is a convex page of an open-book decomposition \( \mathcal{O} \) on \((M, \alpha)\),
- iii) \( \Phi(B \times \mathbb{R}) \approx B \) is the binding contact submanifold of \( \mathcal{O} \), and
- iv) \( \varphi \) is the monodromy map of \( \mathcal{O} \).

**Proof.** In the case where \( \Sigma_- = \emptyset \) and \( B = \partial\Sigma_+ \), this proposition was proved in Giroux [10] (essentially in Thurston-Winkelnkemper[25]). Then \( \mathcal{O} \) is a supporting open-book decomposition. In general, let \( \Sigma \times \mathbb{R} \) be the unified contactization, i.e., the union of the modified contactizations of \( \Sigma_{\pm} \) by the attaching map \( (\iota, -id_{\mathbb{R}^c}) : \partial\Sigma_- \times \mathbb{R}^c \rightarrow \partial\Sigma_+ \times \mathbb{R}(=-\mathbb{R}^c) \). Consider the quotient \( \Sigma \times ((\mathbb{R}/2\pi c\mathbb{Z}) \times \mathbb{R}^c) \) of \( \Sigma \times (\mathbb{R}/2\pi c\mathbb{Z}) \times \mathbb{R}^c \), and cap-off the boundary components \( B \times (\mathbb{R}/2\pi c\mathbb{Z}) \) by replacing the collar neighbourhood \( (-\varepsilon, 0] \times B \times (\mathbb{R}/2\pi c\mathbb{Z}) \) with \( (B \times D^2, (\lambda_+|B) + r^2d\theta) \) where \( \theta = z/c \). Adding constants to \( h_{\pm} \) if necessary, we may assume that \( h_{\pm} \) are the restrictions of the same function \( h \). We change the identification \( (x, z + 2\pi c) \sim (x, z) \) to \( (x, z + h) \sim (\varphi(x), z) \) before capping-off the boundary \( B \times S^1 \). This defines the map \( \Phi \) and completes the proof. \( \Box \)

**Remark.** Giroux and Mohsen proved that any symplectomorphism supported in \( \text{int} \Sigma_+ \) is isotopic via such symplectomorphisms to \( \varphi \) with \( \varphi^*\lambda_+ - \lambda_+ = dh_+ \) (\( \exists h_+ > 0 \)). They also proved that there exists a supporting open-book decomposition on any closed contact manifold by interpreting the result of Ibort-Martinez-Presas[14] on the applicability of Donaldson-Auroux’s asymptotically holomorphic methods to complex functions on contact manifolds (see [10]).
3.2. Definition of Lutz tubes. Let $W_A$ be the symplectic manifold with disconnected contact-type boundary $(-T_A)\cup T_A$ in Example 1.9. Then, by a result of Van Horn [26], each of the boundary component is a Stein fillable contact manifold. Note that $\text{tr}(A^{-1}) > 2$ and $(-T_A) \approx T_A^{-1}$. 

Precisely, there exists a supporting open-book decomposition on $T_A$ described as follows.

**Proposition 3.2.**

1) (Honda[13], see also Van Horn[26]) Any element $A \in SL(2;\mathbb{Z})$ with $\text{tr} A > 2$ is conjugate to at least one of the elements

$$A_{m,k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_m \\ 0 & 1 \end{pmatrix} \in SL(2;\mathbb{Z}),$$

where $m \in \mathbb{Z}_{>0}$, $k = (k_1, \ldots, k_m) \in (\mathbb{Z}_{\geq 0})^m$ and $k_1 + \cdots + k_m > 0$.

2) (Van Horn[26]) The contact manifold $(T_{A_{m,k}}, (\beta_- + \beta_+)|T_{A_{m,k}})$ ($k \neq 0$) admits a supporting open-book decomposition which is determined up to equivalence by the following data:

- **Page:** The page $P$ is the $m$-times punctured torus $\bigcup_{i \in \mathbb{Z}_m} P_i$, where $P$ is divided into three-times punctured spheres $P_i$ by mutually disjoint loops $\gamma_i$ with $P_i \cup P_{i+1} = \gamma_i$.

- **Monodromy:** The monodromy is the composition $\tau(\partial P) \circ \prod_{i=1} \{\tau(\gamma_i)\}^{k_i}$, where $\tau(\gamma)$ denotes the right-handed Dehn twist along $\gamma$.

These data determines a PALF (=positive allowable Lefschetz fibration) structure of the canonical Stein filling $V$ of the contact manifold $T_{A_{m,k}}$ if $m \geq 2$ (see Loi-Piergallini [17] and Giroux [10]). Here we see through the PALF structure that attaching 1-handles to the page corresponds to attaching 1-handles to the canonical Stein filling, and adding right-handed Dehn twists along non-null-homologous loops to the monodromy corresponds to attaching 2-handles to the canonical Stein filling. Thus we have

$$\chi(V) = 1 - (\#(0\text{-handles})) - (m+1)(\#(1\text{-handles})) + (m+k_1 + \cdots + k_m)(\#(2\text{-handles})) > 0.$$ 

In the case where $m = 1$, let $\ell_1$ and $\ell_2$ denote simple loops on $P$ which intersect transversely at a single point. It is well-known that $\tau(\partial P)$ is isotopic to $(\tau(\ell_1) \circ \tau(\ell_2))^k$. This new expression determines a PALF structure with $12 + k_1$ singular fibres on the canonical Stein filling $V$ of $T_{A_{1,k_1}}$.

Then we have $\chi(V) = 1 - 2 + 12 + k_1 > 11$. Thus the following corollary holds.

**Corollary.** The contact manifold $(T_A, (\beta_- + \beta_+)|T_A)$ admits a Stein filling $V$ with $\chi(V) > 0$.

Then the unified contactization of $W_A \cup (-V)$ under the natural identification $\{1\} \times T_A \sim \partial V$ contains a convex overtwisted hypersurface $(W_A \cup (-V)) \times \{0\}$.

Now we define a 5-dimensional Lutz tube.

**Definition 3.3.** Putting $\Sigma_+ = W_A$, $\Sigma_- = \emptyset$, $B = -\{\{-1\} \times T_A\} \approx T_A^{-1}$, and $\varphi = \text{id}_\Sigma$, we apply Proposition 3.1 to obtain a contact manifold $T_{A^{-1}} \times D' \approx T_A \times D^2$ ($D' = -D^2$), which we call the 5-dimensional Lutz tube associated to $A$ (tr $A > 2$).

The next proposition explains how to insert a Lutz tube.

**Proposition 3.4.** Let $(V, d\lambda)$ be an exact strong symplectic filling of $T_A$ with $\text{tr} A > 2$, $\psi : V \to V$ a diffeomorphism supported in $\text{int} \ V$. Suppose that $\chi(V) > 0$ and $\psi^* \lambda - \lambda = dh$ ($\exists h > 0$). Putting $\Sigma_+ = V$, $\Sigma_- = \emptyset$, $B = \partial V$ and $\varphi = \psi$, we apply Proposition 3.1 to obtain a closed contact manifold $(M^5, \alpha)$ with an open-book decomposition whose binding is $T_A$. Next we consider the union $W_A \cup (-V)$ with respect to the natural identification $i : \partial V \to \{1\} \times T_A \subset W_A$. Again putting $\Sigma_+ = W_A$, $\Sigma_- = V$, $B = \partial W_A \setminus i(\partial V)$ and $\varphi = (\text{the trivial extension of } \psi^{-1})$, we apply Proposition 3.1 to obtain another contact manifold $(M', \alpha')$ with an open-book decomposition, where the page is a convex overtwisted hypersurface and the binding is $T_{A^{-1}}$. Then there exists a diffeomorphism from $M^5$ to $M'$ which preserves the orientation and sends $T_A$ to $-T_{A^{-1}}$. We may consider that $(M', \alpha')$ is obtained by inserting the 5-dimensional Lutz tube associated to $A$ along the binding $T_A$ of a supporting open-book decomposition on $(M^5, \alpha)$.

**Remark.** 1) Similarly, we can insert a 5-dimensional Lutz tube along any codimension-2 contact submanifold with trivial normal bundle which is contactomorphic to $T_A$ (tr $A > 2$).
Particularly, we can insert the Lutz tube associated to $A^{-1}$ along the core of the Lutz tube associated to $A$. Then we obtain a 5-dimensional analogue of the full Lutz tube.

2) We may consider the original 3-dimensional (half) Lutz tube as a trivial open-book decomposition by positive annuli whose binding is a connected component of the boundary. That is, starting with the exact symplectic annulus $([-1, 1] \times S^1, s \theta)$ ($s \in [-1, 1], \theta \in S^1$) we can construct the 3-dimensional Lutz tube by Proposition 3.1 ($\varphi = \text{id}$).

3) Geiges constructed an exact symplectic 6-manifold $[-1, 1] \times M^5$ with contact-type boundary, where $M^5$ is a certain $T^4$-bundle over the circle. From his example, we can also construct a 7-dimensional Lutz tube. The author suspects that this Lutz tube enables us to change not only the contact structure but also the homotopy class of the almost contact structure of a given contact 7-manifold. See Question 5.3 in Etnyre-Pancholi.

3.3. Exotic contact structures on $S^5$. We can insert a Lutz tube into the standard $S^5$. Namely,

**Theorem 3.5.** In the case where $m \leq 2$ and $k \neq 0$, $T_{A_{m,k}}$ is contactomorphic to the link of the isolated singular point $(0,0)$ of the hypersurface $\{f_{m,k} = 0\} \subset C^3$, where

\[
\begin{align*}
f_{1,k_1} &= (\eta - 2\xi^2)(\eta^2 + 2\eta \xi + \xi^4 - \xi^{4+k_1}) + \xi^2 \quad \text{and} \\
f_{2,k_1,k_2} &= \{(\xi + \eta)^2 - \xi^{2+k_1}\} \{(\xi - \eta)^2 + \xi^{2+k_2}\} + \xi^2.
\end{align*}
\]

Let $O_{m,k}$ denote the Milnor fibration of the singular point $(0,0,0) \in \{f_{m,k} = 0\}$.

**Remark.** From Theorem 2.2 and Lemma 2.5 we obtain a Reeb foliation $F_{m,k}$ associated to $O_{m,k}$.

In order to prove Theorem 3.5 we prepare an easy lemma.

**Lemma 3.6.**

1) The complex curve

\[
C = \{\xi^2 = -(\eta - p_1) \cdots (\eta - p_{m+2})\} \quad (m = 1, 2)
\]

on the $\eta\xi$-plane $C^2$ is topologically an $m$-times punctured torus in $\mathbb{R}^4$ if the points $p_i$ are mutually distinct. These points are the critical values of the branched double covering $\pi_\eta|C$, where $\pi_\eta : C^2 \to C$ denotes the projection to the $\eta$-axis.

2) Let $B : p_1 = p_1(\theta), \ldots, p_{m+2} = p_{m+2}(\theta)$ be a closed braid on $C \times S^1$ ($\theta \in S^1$). Then the above curve $C = C_\theta$ traces a surface bundle over $S^1$. Fix a proper embedding $l : \mathbb{R} \to C$ into the $\eta$-axis such that $l(1) = p_1(0), \ldots, l(m+2) = p_{m+2}(0)$. Suppose that the closed braid $B$ is isotopic to the geometric realization of a composition

\[
\prod_{j=1}^{j} \{\sigma_i(j)^q(j) \text{ where } q(j) \in \mathbb{Z}, i(j) \in \{1, \ldots, m+1\}\},
\]

where $\sigma_i : C \to C$ denotes the right-handed exchange of $p_i$ and $p_{i+1}$ along the arc $l([i,i+1])$ ($i = 1, \ldots, m+1$). Then the monodromy of the surface bundle $C_\theta$ is the composition

\[
\prod_{j=1}^{j} \{\tau(\ell_i(j)) \text{ where } \ell_i = (\pi_\eta|C)^{-1}(l([i,i+1]))\}.
\]

**Proof of Theorem 3.5** Regard $\xi \neq 0$ as a small parameter and take the branched double covering $\pi_\eta|C_\xi$ of the curve $C_\xi = \{f_{m,k} = 0, \xi = \text{const}\} \cap B^6$. Then the critical values of $\pi_\eta|C_\xi$ are

\[
p_1, p_2 = -\xi^2(1 - (\xi^{k_1})^{1/2}) \quad \text{and} \quad p_3 = 2\xi_2 \quad \text{in the case where } m = 1
\]

\[
(p_1, p_2 = -\xi(1 - (\xi^{k_1})^{1/2}), p_3, p_4 = \xi(1 - (\xi^{k_2})^{1/2}) \text{ in the case where } m = 2).
\]

As $\xi$ rotates along a small circle $|\xi| = \varepsilon$ once counterclockwise, the set $\{p_1, \ldots, p_{m+2}\}$ traces a closed braid, which is clearly a geometric realization of the composition

\[
(\sigma_1 \circ \sigma_2)^{q_1}(\sigma_1)^{q_2} \quad (\text{resp. } (\sigma_1 \circ \sigma_2 \circ \sigma_3)^{q_1}(\sigma_1)^{q_2}(\sigma_3)^{q_2}).
\]

From Lemma 3.6 and the well-known relation

\[
\tau(\partial C_\xi) \simeq (\tau(\ell_1) \circ \tau(\ell_2))^{q_1} \quad (\text{resp. } \tau(\partial C_\xi) \simeq (\tau(\ell_1) \circ \tau(\ell_2) \circ \tau(\ell_3))^{q_1}),
\]

we obtain a Reeb foliation $F_{m,k}$ associated to $O_{m,k}$.
we see that the link of the singular point \((0,0,0) \in \{ f_{m,k} = 0 \}\) admits the supporting open-book decomposition in Proposition 3.2. This completes the proof of Theorem 3.5. □

If we insert the Lutz tube associated to \(A_{m,k}\) \(m = 1,2, k \neq 0\) along the binding of the supporting open-book decomposition equivalent to \(\mathcal{O}_{m,k}\), we obtain a contact structure \(\ker(\alpha_{m,k})\) on \(S^5\). Then the page becomes a convex overtwisted hypersurface. The following theorem can be proved in a similar way to the proof of Lemma 2.2.

**Theorem 3.7.** The contact structure \(\ker(\alpha_{m,k})\) \(m = 1, 2, k \neq 0\) deforms via contact structures into a foliation which is obtained by cutting and turbulizing the page leaves of the Reeb foliation \(\mathcal{F}_{m,k}\) along the hypersurface corresponding to the boundary of the Lutz tube.

Let \((M', \alpha')\) be the contact connected sum of any contact 5-manifold \((M^5, \alpha)\) with the above exotic 5-sphere \((S^5, \alpha_{m,k})\). Then we see that the contact manifold \((M' \approx M^5, \alpha')\) contains a convex overtwisted hypersurface. Namely,

**Theorem 3.8.** Any contact 5-manifold admits a contact structure which violates the Thurston-Bennequin inequality for a convex hypersurface with contact-type boundary.

### 3.4. Plastikstufes in Lutz tubes

We show that there exists a plastikstufe in any 5-dimensional Lutz tube. First we define a plastikstufe in a contact 5-manifold.

**Definition 3.9.** Let \((M^5, \alpha)\) be a contact 5-manifold and \(\iota : T^2 \rightarrow M^5\) a Legendrian embedding of the torus which extends to an embedding \(\iota : D^2 \times S^1 \rightarrow M^5\) of the solid torus. Then the image \(\iota(D^2 \times S^1)\) is called a plastikstufe if there exists a function \(f(r)\) such that

\[
(r^2 d\theta + f(r) dr) \wedge (\iota^* \alpha) = 0, \quad \lim_{r \to 0} \frac{f(r)}{r^2} = 0 \quad \text{and} \quad \lim_{r \to 1} |f(r)| = \infty,
\]

where \(r\) and \(\theta\) denote polar coordinates on the unit disk \(D^2\).

**Example 3.10.** Consider the contactization

\[
(\mathbb{R} \times T_A, \mathbb{R}(\mathbb{R}T_A \times \mathbb{R}(\mathbb{R}t)), \alpha = \beta_+ + s\beta_- + dt)
\]

of the exact symplectic manifold \((\mathbb{R} \times T_A, d(\beta_+ + s\beta_-))\), where \(\beta_\pm\) are the 1-forms described in Example 1.10. Take coordinates \(p\) and \(q\) near the origin on \(T^2 = \mathbb{R}^2/\mathbb{Z}^2\) such that \(p = q = 0\) at the origin, \(\partial/\partial p = v_+\) and \(\partial/\partial q = v_-\). Then for small \(\varepsilon > 0\), the codimension-2 submanifold

\[
\mathcal{P} = \{ p = \varepsilon a^{-2}g(s), q = \varepsilon a^2sg(s) \} \subset \mathbb{R} \times T_A \times \mathbb{R}
\]

is compactified to a plastikstufe on the Lutz tube \(T_A \times D^2\), where \(g(s)\) is a function with

\[
g(s) \equiv 0 \quad \text{on} \quad (\infty, 1], \quad \text{and} \quad g(s) \equiv \frac{1}{s \log s} \quad \text{on} \quad [2, \infty).
\]

Note that the boundary \(\{(\infty, (0, 0, z), t) \mid z \in S^1, t \in S^1\} \approx T^2\) of the plastikstufe is a Legendrian torus, and on the submanifold \(\mathcal{P}\) the contact form \(\alpha\) can be written as

\[
\alpha|\mathcal{P} = a^{-2} dq + sa^2 dp + dt = \varepsilon (g(s) + 2sg'(s)) \, ds + dt.
\]

Indeed, as \(s \to \infty,

\[
g(s) \to 0, \quad sg(s) \to 0 \quad \text{and} \quad \int_2^s (g(s) + 2sg'(s))ds \to -\infty.
\]

**Remark.** As \(\varepsilon \to 0\), the above plastikstufe converges to a solid torus \(S^1 \times D^2\) foliated by \(S^1\) times the straight rays on \(D^2\), i.e., the leaves are \(\{ t = \text{const} \}\).

The following theorem is proved in the above example.

**Theorem 3.11.** There exists a plastikstufe in any 5-dimensional Lutz tube.

**Corollary.** (Niederkrüger-van Koert) Any contact 5-manifold \((M^5, \alpha)\) admits another contact structure \(\ker \alpha'\) such that \((M^5, \alpha')\) contains a plastikstufe.
3.5. Topology of the pages. We decide the Euler characteristic of the page of the open-book decomposition given in Theorem 3.5 which is diffeomorphic to

$$ F = \{ f_{m,k}(\xi, \eta, \zeta) = \delta \} \cap \{ |\xi|^2 + |\eta|^2 + |\zeta|^2 \leq \varepsilon \}, $$

where $\delta \in \mathbb{C}, \varepsilon \in \mathbb{R}_{>0}, 0 < |\delta| \ll \varepsilon \ll 1$. Let $\pi_\zeta, \pi_\eta$ and $\pi_\xi$ denote the projections to the axes.

In the case where $m = 1$, the critical values of $\pi_\xi|F$ are the solutions of the system

$$ f_1,(k_1) - \delta = 0, \quad \frac{\partial}{\partial \eta} f_1,(k_1) = 0, \quad \frac{\partial}{\partial \zeta} f_1,(k_1) = 2\zeta = 0 \quad \text{and} \quad |\xi| \ll \varepsilon. $$

Therefore, for each critical value $\xi$ of $\pi_\xi|F$, we have the factorization

$$ (\eta - 2\xi^2)(\eta^2 + 2\xi^2\eta + \xi^4 - \xi^{4+k_1}) - \delta = (\eta - a)^2(\eta + 2a) $$

of the polynomial of $\eta$, where the parameter $a \in \mathbb{C}$ depends on $\xi$. By comparison of the coefficients of the $\eta^1$-terms and the $\eta^0$-terms we have

$$ -4\xi^4 + \xi^4 - \xi^{4+k_1} = a^2 - 2a^2 - 2a^2 \quad \text{and} \quad -2\xi^6 + 2\xi^{6+k_1} - \delta = 2a^3. $$

Eliminating the parameter $a$, we obtain the equation

$$ 4\xi^{12+k_1}(9 - \xi^{k_1})^2 = 108\xi^6(1 - \xi^{k_1})\delta + 27\delta^2. $$

Then we see that $\pi_\xi|F$ has $12 + k_1$ critical points, which indeed satisfy $a \neq -2a$, i.e., the map $\pi_\xi|F$ defines a PALF structure on $F$ with $12 + k_1$ singular fibres. Thus we have

$$ \chi(F) = 1 - 2 + 12 + k_1 = 11 + k_1. $$

In the case where $m = 2$, we have the factorization

$$ \{ (\xi + \eta)^2 - \xi^{2+k_1} \} \{ (\xi - \eta)^2 + \xi^{2+k_2} \} - \delta = (\eta - a)^2(\eta + a + b)(\eta + a + b) \quad (a, b \in \mathbb{C}). $$

By comparison of the coefficients we have

$$ \begin{cases} 
\xi^2(2 + \xi^{k_1} - \xi^{k_2}) = 2a^2 + b^2 \\
\xi^3(\xi^{k_1} + \xi^{k_2}) = ab^2 \\
\xi^4(1 - \xi^{k_1})(1 + \xi^{k_2}) - \delta = a^2(a^2 - b^2). 
\end{cases} $$

In order to eliminate $a, b$, we put $a = u + v$ and $\xi^2(2 + \xi^{k_1} - \xi^{k_2}) = 6uv$. Then we have

$$ \begin{cases} 
6uv - 2(u + v)^2 = b^2 \\
\xi^3(\xi^{k_1} + \xi^{k_2}) = -2(u^3 + v^3) \\
\xi^4(1 - \xi^{k_1})(1 + \xi^{k_2}) - \delta = (u + v)^4 + 2(u^3 + v^3)(u + v). 
\end{cases} $$

Further we put

$$ p = uv, \quad q = u^3 + v^3 \quad \text{and} \quad r = (u + v)^4 + 2(u^3 + v^3)(u + v). $$

Then $p, q$ and $r$ are polynomials of $\xi$. Eliminating $a$ from

$$ q = q(p, a) = 3pa - a^3 \quad \text{and} \quad r = r(p, a) = -6pa^2 + 3a^4, $$

we obtain

$$ (27q^4 - r^3) + 54(pq^2 - p^3q^2) + 18p^2r^2 - 81p^4r = 0, $$

which is a polynomial equation of $\xi$. As $\delta \to 0$, the left hand side converges to

$$ \xi^{12+k_1+k_2} \left\{ 1 - \frac{\xi^{k_1} - \xi^{k_2}}{2} + \frac{(\xi^{k_1} + \xi^{k_2})^2}{16} \right\}. $$

Therefore $\pi_\xi|F$ has $12 + k_1 + k_2$ critical points, which indeed satisfy $4a^2 \neq b^2$ and $b \neq 0$, i.e., the map $\pi_\xi|F$ defines a PALF structure on $F$ with $12 + k_1 + k_2$ singular fibres. Thus we have

$$ \chi(F) = 1 - 3 + 12 + k_1 + k_2 = 10 + k_1 + k_2. $$
3.6. **Symplectic proof.** In this subsection we sketch another proof of the following theorem, which is slightly weaker than Theorem 3.5 and the result of the previous subsection.

**Theorem 3.12.** The contact manifold $T_{A_{m,k}}$ $(m = 1, 2)$ is contactomorphic to the binding of a supporting open-book decomposition on the standard $S^5$ whose page $P_{m,k}$ satisfies $\chi(P_{m,k}) = 11 + k_1$ in the case where $m = 1$ and $\chi(P_{m,k}) = 10 + k_1 + k_2$ in the case where $m = 2$.

We start with the following observation.

**Observation.** 1) We consider the fibre

$$V_\delta = \{\xi^2 + \eta^2 + \zeta^2 = \delta\} \subset \mathbb{C}^3$$

of the singular fibration $f(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 : \mathbb{C}^3 \to \mathbb{C}(\exists \delta)$, which we call the first fibration. If $\delta \neq 0$, the restriction $\pi_\xi|V_\delta$ is a singular fibration over the $\xi$-axis, which we call the second fibration. The fibre of the second fibration is

$$F_\xi = (\pi_\xi|V_\delta)^{-1}(\xi) = \{\eta^2 + \zeta^2 = \delta - \xi^2\}.$$  

If $\xi^2 \neq \delta$, the restriction $\pi_\eta|F_\xi$ has critical values $\pm \gamma = (\delta - \xi^2)^{1/2}$. That is, the second fibre $F_\xi$ is a double cover of the $\eta$-axis branched over $\pm \gamma$. We call $\pi_\eta|F_\xi$ the third fibration. Then the line segment between $\pm \gamma$ lifts to the vanishing cycle

$$\{(0, \gamma x, \gamma y) \mid (x, y) \in S^1 \subset \mathbb{R}^2\} \approx S^1 \subset F_\xi$$

which shrinks to the singular points $(\delta^{1/2}, 0, 0)$ on the fibres $F_{\delta^{1/2}}$. On the other hand, the line segment between $\delta^{1/2}$ on the $\xi$-axis lifts to the vanishing Lagrangian sphere

$$L = \{(\delta^{1/2} x, \delta^{1/2} y, \delta^{1/2} z) \mid (x, y, z) \in S^2 \subset \mathbb{R}^3\} \approx S^2 \subset V_\delta$$

which shrinks to the singular point $(0, 0, 0)$ on $V_0$. The monodromy of the first fibration around $\delta = 0$ is the Dehn-Seidel twist along the Lagrangian sphere $L$ (see [10]).

2) Next we consider the tube

$$B' = \{[\xi + \eta^2 + \zeta^2] \leq \varepsilon\} \cap B^6$$

of the regular fibration $g(\xi, \eta, \zeta) = \xi + \eta^2 + \zeta^2 : \mathbb{C}^3 \to \mathbb{C}$. Let $V'_\delta$ denote the fibre $g^{-1}(\delta)$ of the first fibration $g$ and $F'_\xi$ the fibre of the second fibration $\pi_\xi|V'_\delta$. The third fibration $\pi_\eta|F'_\xi$ has two critical points $(\delta - \xi)^{1/2}$ on the $\eta$-axis. Then the line segment between $(\delta - \xi)^{1/2}$ lifts to the vanishing circle

$$\{((\delta - \xi)^{1/2} x, (\delta - \xi)^{1/2} y) \mid (x, y) \in S^1 \subset \mathbb{R}^2\} \approx S^1 \subset F'_\xi$$

which shrinks to the singular point $N = (\delta, 0, 0)$ on $F'_\delta$. By attaching a symplectic 2-handle to $B'$, we can simultaneously add a singular fibre to each second fibration $\pi_\xi|V'_\delta$ so that the above vanishing cycle shrinks to another singular point $S$ than $N$. Here the vanishing cycle traces a Lagrangian sphere from the north pole $N$ to the south pole $S$. (The attaching sphere can be considered as the equator.) The symplectic handle body $B' \cup (\text{the 2-handle})$ can also be realized as a regular part

$$f^{-1}(U) \cap B^6 \quad (\exists U \approx D^2, U \neq 0)$$

of the singular fibration $f$ in the above 1). Thus we can add a singular fibre $V_0 \cap B^6$ to it by attaching a symplectic 3-handle. That is, the tube $\{|f| \leq \varepsilon\} \cap B^6$ of the singular fibration $f$ can be considered as the result of the cancellation of the 2-handle and the 3-handle attached to the above tube $B'$ of the regular fibration $g$. Note that such a cancellation preserves the contactomorphism-type of the contact-type boundary.

**Proof of Theorem 3.12.** Take the tubes $\{|h_m| \leq \varepsilon\} \cap B^6$ $(m = 1, 2)$ of the regular fibrations

$$h_1(\xi, \eta, \zeta) = \xi + \eta^3 + \zeta^2 \quad \text{and} \quad h_2(\xi, \eta, \zeta) = \xi + \eta^4 + \zeta^2.$$ 

Let $F_{m,\xi}$ denote the fibre of the second fibration $\pi_\xi|h_m^{-1}(\delta)$. Then the third fibration $\pi_\eta|F_{m,\xi}$ has $m + 2$ singular fibres $(m = 1, 2)$. We connect the corresponding critical values on the $\eta$-axis by a simple arc consisting of $m + 1$ line segments $\sigma_1, \ldots, \sigma_{m+1}$, which lift to vanishing cycles.
Remark. 1) Giroux and Mohsen further conjectured that, for any supporting open-book decomposition on a contact manifold \((M^{2n+1}, \alpha)\), we can attach a \(n\)-handle to the page to produce a Lagrangian \(n\)-sphere \(S^n\), and then add a Dehn-Seidel twist along \(S^n\) to the monodromy to obtain another supporting open-book decomposition on \((M^{2n+1}, \alpha)\) \((\text{[10]}))\). We did a similar replacement of the supporting open-book decomposition in the above proof of Theorem 3.12 by means of symplectic handles.

2) Take a triple covering from the three-times punctured torus to the once punctured torus \(F_{1, \xi}\) such that \(\ell_2\) lifts to a long simple closed loop. Then from the relations

\[
\tau(\partial F_{1, \xi}) \simeq (\tau(\ell_1) \circ \tau(\ell_2))^6 \quad \text{(resp. } \tau(\partial F_{2, \xi}) \simeq (\tau(\ell_1) \circ \tau(\ell_2) \circ \tau(\ell_3))^4) \]

we see that the Dehn twist along the boundary of the three-times punctured torus is also isotopic to a composition of Dehn twists along non-separating loops. On the other hand, take a double covering from the four-times punctured torus to the twice punctured torus \(F_{2, \xi}\) such that \(\ell_2\) lifts to a long simple closed loop. Then from the relations

\[
\tau(\partial F_{2, \xi}) \simeq \{\tau(\ell_1)^{-1} \circ \tau(\ell_1) \circ \tau(\ell_3) \circ \tau(\ell_2))^4 \circ \tau(\ell_1) \}
\]

we see that the Dehn twist along the boundary of the four-times punctured torus is also isotopic to a composition of Dehn twists along non-separating loops.

**Problem.** Can we generalize Theorem 3.12 to the case where \(m = 3\) or \(4\)?

4. Further discussions

A (half) Lutz twist along a Hopf fibre in the standard \(S^3\) produces a basic overtwisted contact manifold \(S^3\) diffeomorphic to \(S^3\). This overtwisted contact structure is supported by the negative Hopf band. Indeed any overtwisted contact manifold \(M^3\) is a connected sum with \(S^3\) (i.e., \(M^3 = \# \exists M^3 \# S^3\)). Moreover a typical supporting open-book decomposition on \(M^3\) is the Murasugi-sum (=plumbing) with a negative Hopf band (see [10]). — The author’s original motivation was to find various Lutz tubes in a given overtwisted contact 3-manifold or simply in \(S^3\). (See [20] for the first model of a Lutz tube by means of a supporting open-book decomposition.) Since the binding of the trivial supporting open-book decomposition on \(S^3\) is a Hopf fibre, the above Lutz twist inserts a Lutz tube along the binding. Then the Lutz twist produces a non-supporting trivial open-book decomposition \(O\) by convex overtwisted disks. In 5-dimensional case, the Lutz tube (i.e., the neighbourhood of the binding of \(O\)) is replaced by a 5-dimensional Lutz tube which contains a plastikstufe, and the convex overtwisted disk (i.e., the page) by a convex overtwisted hypersurface violating the Thurston-Bennequin inequality. The idea of placing a Lutz tube around the binding of a non-supporting open-book decomposition is also found in the recent work of Ishikawa [13]. However, in general, the insertion of a 5-dimensional Lutz tube requires only the normal triviality of the contact submanifold \(T_A\) in the original contact manifold.

**Problem 4.1.** Suppose that a contact 5-manifold \((M^5, \alpha)\) contains a Lutz tube. Then does it always contain a convex overtwisted hypersurface?
We also have the basic exotic contact 5-manifold $\mathbb{S}^5$ which is diffeomorphic to $S^5$ and supported by the 5-dimensional negative Hopf band. Here the negative Hopf band is the mirror image of the positive Hopf band which is (the page of) the Milnor fibration of $(0,0,0) \in \{\xi^2 + \eta^2 + \zeta^2 = 0\}$. Thus the monodromy of the negative Hopf band is the inverse of the Dehn-Seidel twist (see Observation 1) in §3.6. The fundamental problem is

**Problem 4.2.** Does $\mathbb{S}^5$ contains a Lutz tube or a plastikstufe? Could it be that $\mathbb{S}^5$ is contactomorphic to $(S^5, \ker(\alpha_{m,k}))$? Note that almost contact structures on $S^5$ are mutually homotopic.

The next problem can be considered as a variation of Calabi’s question (see §1).

**Problem 4.3.** Does the standard $S^{2n+1} (n > 1)$ contains a convex hypersurface with disconnected contact-type boundary?

If there is no such hypersurfaces, the following conjecture trivially holds.

**Conjecture 4.4.** The Thurston-Bennequin inequality holds for any convex hypersurface with contact-type boundary in the standard $S^{2n+1}$.

**References**

[1] D. Bennequin: *Entrelacements et équations de Pfaff*, Astérisque, **107-108** (1983), 83–161.
[2] C. Caubel, A. Nemethi and P. Popescu-Pampu: *Milnor open books and Milnor fillable contact 3-manifolds*, Topology **45**(3) (2006) 673–689.
[3] Y. Eliashberg: *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds 2, London Math. Soc. Lect. Note Ser., **151**(1990), 45–72.
[4] Y. Eliashberg: *Contact 3-manifolds twenty years since J. Martinet’s work*, Ann. Inst. Fourier, **42** (1991), 165–192.
[5] Y. Eliashberg and W. Thurston: *Confoliations*, A.M.S. University Lecture Series, **13** (1998).
[6] J. Etnyre and D. Pancholi: *On generalizing Lutz twists*, preprint (2009), [arXiv:0903.0295 [math.SG]]
[7] H. Geiges: *Symplectic manifolds with disconnected boundary of contact type*, Int. Math. Res. Notices, **1** (1994), 23–30.
[8] E. Ghys: *Déformation de flots d’Anosov et de groupes fuchsiens*, Ann. Inst. Fourier, **42** (1992), 209–247.
[9] E. Giroux: *Convexité en topologie de contact*, Comm. Math. Helv., **66** (1991), 637–677.
[10] E. Giroux: *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proc. ICM-Beijing, **2** (2002), 405–414.
[11] E. Giroux: *Contact structures and symplectic fibrations over the circle*, Notes of the summer school “Holomorphic curves and contact topology”, Berder, 2003.
[12] H. Grauert: *Über Modifikation und exceptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.
[13] K. Honda: *On the classification of tight contact structures II*, J. Diff. Geom., **55** (2000), 83–143.
[14] A. Ibort, D. Martinez and F. Presas: *On the construction of contact submanifolds with prescribed topology*, J. Diff. Geom., **56** (2000), 235–283.
[15] M. Ishikawa: *Compatible contact structures of fibred Seifert links in homology 3-spheres*, preprint (2009).
[16] H. Lawson: *Codimension-one foliations of spheres*, Ann. of Math., **94** (1971), 494–503.
[17] A. Loi and R. Piergallini: *Compact Stein surfaces with boundary as branched covers of $B^4$*, Invent. Math. **143**(2001), 325–348.
[18] D. McDuff: *Symplectic manifolds with contact type boundaries*, Invent. Math. **103**(1991), 651–671.
[19] Y. Mitsumatsu: *Anosov flows and non-Stein symplectic manifolds*, Ann. Inst. Fourier **45**(1995), 1407–1421.
[20] A. Mori: *A note on Thurston-Winkelnkemper’s construction of contact forms on 3-manifolds*, Osaka J. Math. **39**(2002), 1–11.
[21] A. Mori: *On the violation of Thurston-Bennequin inequality for a certain non-convex hypersurface*, preprint (2009).
[22] K. Niederkrüger: *The plastikstufe — a generalization of the overtwisted disk to higher dimensions*, Algerbr. Geom. Topol. **6** (2006), 2473–2508.
[23] K. Niederkrüger and O. van Koert: *Every contact manifold can be given a non-fillable contact structure*, Int. Math. Res. Notices, **23**(2007), rnm115, 22 pages.
[24] W. Thurston: *Norm on the homology of 3-manifolds*, Memoirs of the A. M. S., **339** (1986), 99–130.
[25] W. Thurston and E. Winkelnkemper: *On the existence of contact forms*, Proc. A. M. S., **52** (1975), 345–347.
[26] J. Van Horn-Morris: *Constructions of open book decompositions*, Thesis (2007), Univ. of Texas at Austin.

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