SCHUBERT UNIONS IN GRASSMANN VARIETIES

JOHAN P. HANSEN, TRYGVE JOHNSEN AND KRISTIAN RANESTAD

Abstract. We study subsets of Grassmann varieties $G(l, m)$ over a field $F$, such that these subsets are unions of Schubert cycles, with respect to a fixed flag. We study the linear spans of, and in case of positive characteristic, the number of $F_q$-rational points on such unions. Moreover we study a geometric duality of such unions, and give a combinatorial interpretation of this duality. We discuss the maximum number of $F_q$-rational points for Schubert unions of a given spanning dimension, and we give some applications to coding theory. We define Schubert union codes, and study the parameters and support weights of these codes and of the well-known Grassmann codes.

1. Introduction

Let $G(l, m) = G_F(l, m)$ be the Grassmann variety of $l$-dimensional subspaces of a fixed $m$-dimensional vector space $V$ over a field $F$. By the standard Plücker coordinates $G(l, m)$ is embedded into $\mathbf{P}^{k-1} = \mathbf{P}^{k-1}_F$ as a non-degenerate smooth subvariety, where $k = \binom{m}{l}$.

This paper was motivated by the problem: "What is the maximal intersection of a linear subspace of a given dimension in $\mathbf{P}^{k-1}$ with $G(l, m)$?" Here "maximal" may refer to the number of $F_q$-rational points, or to Krull dimension. We answer the last question, concerning Krull dimension, in Theorem 7.2 while for the former we only give partial answers. The unions of Schubert cycles with respect to a fixed flag, called Schubert unions, turn out to play a key role in this problem, so the main body of this paper concerns them with the hope that this treatment may have some independent interest.

First we point out that the spanning dimension and number of $F_q$-rational points of the Schubert unions are natural generalizations of the corresponding formulas for Schubert cycles, and that each union is precisely the intersection of its linear span with $G(l, m)$.

Furthermore we identify the Schubert unions as fixed points for the action of a certain Borel group acting on natural (derived) Grassmann varieties.

This is instrumental in applying Borel’s Fixed Point Theorem to solve the maximality problem for the Krull dimension.

The set of Schubert unions enjoy an involution given by a natural duality. The dual of a Schubert union $U$ arises from taking the set of $(m-l)$-dimensional subspaces of $V$ intersecting all $l$-dimensional subspaces parametrized by points of $U$ non-trivially. By the standard duality between $G(l, m)$ and $G(m-l, m)$ we interpret the dual set as a subset of $G(l, m)$. It then turns out that the dual of a Schubert union is a Schubert union with complementary spanning dimension in the Plücker space (and that we have biduality justifying the terminology).

We proceed to study a natural point grid in $\mathbf{Z}^l$ corresponding to the Plücker coordinates. It turns out that the duality just described corresponds to a duality of subsets of this grid, where the cardinality of a subset $G_U$ corresponding to $U$ is equal to the spanning dimension of the affine cone of $U$ in the affine space (of dimension $\binom{m}{k}$) over the Plücker space. In

1991 Mathematics Subject Classification. 14M15 (05E15, 94B27).

Key words and phrases. Schubert cycles, Grassmann codes.
the special case \( l = 2 \) we describe an additional structure on the set of Schubert unions for fixed \( m \), as a power set \( \mathcal{P}(M) \) for a set \( M \) with \( m - 1 \) elements.

In the case where \( F \) is the finite field \( F_q \), we study the (equivalence class of) Grassmann code(s) \( C(l, m) \) obtained by using the Plücker coordinate tuples of each point of \( G(l, m) \) as columns of the generator matrix. The weight hierarchy of these codes is in a natural manner computed by linear sections of \( G(l, m) \), and finding the weights amounts to finding the linear sections, for each spanning dimension, with the maximum number of \( F_q \)-rational points. It then easily turns out that the highest weights, corresponding to sections of small spanning dimension, are computed by projective spaces that are Schubert cycles and therefore Schubert unions. Furthermore the lowest weights, found in \([N]\), corresponding to linear sections with large spanning dimension, are computed by the respective dual Schubert unions. This observation makes it natural to ask whether the linear sections, for each spanning dimension, with the maximum number of \( F_q \)-rational points, can be taken to be Schubert unions. Moreover we ask whether the dual \( U^* \) of a Schubert union \( U \) with the maximum number of \( F_q \)-rational points (for a given spanning dimension) has the maximal number of \( F_q \)-rational points for the complementary spanning dimension. It turns out that these two statements are true for certain low values of \( l \) and \( m \). On the other hand we show that for \( l = 2 \), both statements cannot hold simultaneously, for large enough \( m \). In fact we show that if \( m \) is large enough, then at least one of them fails for spanning dimensions between ca. 36\% and ca. 64\% of \( \binom{m}{2} \). In contrast, the percentage of number of weights described in \([N]\), will approach zero. For these code-theoretical questions associated to \( G(l, m) \), see also \([N]\), \([GL]\), and \([GT]\).

In general \( F \) will be just a field, unless otherwise specified. The paper is organized as follows.

In Section 2 we will fix an ordered basis for an \( m \)-dimensional vector space \( V \), and recall the well-known cell-decomposition of \( G(l, m) \) and the definition of Schubert cycles with respect to this flag, as described by many other authors. Then we will proceed to study unions of such cycles, and determine the dimensions of their linear spans in the Plücker space \( \mathbb{P}^{k-1} \), and find concrete equations for their linear spans. For the case \( F = F_q \) we will use the cell decomposition to determine the number of \( F \)-rational points on the given unions.

In Section \( 2.1 \) we will describe the geometric duality of Schubert unions, and show how this duality corresponds to a quite natural duality or symmetry of a certain “diagram of \( l \)-cubes (or just points)”\( \)”. Each \( l \)-cube in the diagram corresponds to a Plücker coordinate \( X_{i_1, \ldots, i_l} \), and the center of the \( l \)-cube is located in the point \( (i_1, \ldots, i_l) \). In the case \( F = F_q \) the number of \( F \)-rational points on a given union will be easily obtainable from that of its dual union. Our duality, which interchanges dimensions of linear spans of, and not Krull dimension of, Schubert unions, is different from, but closely related to, Poincaré duality.

In Section \( 3 \) we will study the particular case \( l = 2 \). We will show how the set of Schubert unions for a fixed \( m \) in a natural way corresponds to a Boolean algebra \( \mathcal{P}(M) \), where \( M \) is a set of \( m - 1 \) elements, say \( \{1, 2, \ldots, m - 1\} \). Moreover duality of Schubert unions will correspond to complementarity of subsets of \( M \), and there will be an easy way to describe the “diagram of 2-cubes”(squares) from a subset of \( m \) and vice versa.

In Section \( 4 \) and \( 6 \) we will discuss applications to coding theory. The Schubert unions will be used to tell us as much as possible about the support weights of the Grassmann codes \( G(l, m) \) described for example in \([N]\), \([GL]\), and \([GT]\). We introduce the upper Schubert union bound for the support weights of these codes, and show how this bound gives the true value in two ranges, that described by Nogin in \([N]\), and that described by
Schubert unions that are dual to those used to reproduce Nogi’s result. We will discuss some natural questions for the properties of the support weights and the properties of the Schubert union bounds, and investigate these properties for the $C(2,m)$ for some low values of $m$. In particular we will determine all ten support weights for $C(2,5)$ and show how they coincide with the Schubert union bound(s).

In Section 5 we assume $F = F_q$ and $l = 2$. We describe a systematic algorithm for finding the maximum number of $F_q$-rational points on a Schubert union spanning an $K$-space in the Plücker space, at least for large $q$. This amounts to finding the Schubert union bound for $d_r$ for $r = k - K$ for each $K$ in the range $0 \leq K \leq k$, where $k = \binom{m}{2}$ as usual. It turns out that for $m \geq 10$ the optimal codes for the different spanning dimensions do not always exhibit the same symmetries as for $m \leq 9$. We reveal the asymptotic behaviour of the optimal Schubert unions of $G(G(2,m))$ as $m$ goes to infinity.

In Section 7 we treat the question: “What is the maximal Krull dimension of a component of an intersection of $G(G(l,m))$ with a linear space of dimension $K$ in the Plücker space, for each possible $K$?” We use Borel’s Fixed Point Theorem, and show that the maximal Krull dimension is always attained by linear intersections spanned by Schubert unions. So far we have not been able to prove the corresponding fact for the maximal number of $F_q$-rational points, (except for the smallest and the largest linear spaces).

In Section 8 we treat codes made from the points of a Schubert union. We determine the minimum distance and some higher weights of such codes, in the case $l = 2$. Our results rely heavily on a result from [HC], where the minimum distances for codes from Schubert cycles in $G(2,m)$ are found.

In the appendix of Section 9 we list some known facts about $d_r$ for the $G(2,m)$ for some low $m$, and demonstrate how the Schubert unions of $G(2,m)$ for $m \leq 9$, and also $G(3,6)$, exhibit some nice symmetries.

We thank Rita Vincenti for an enlightening correspondence at the start of this work, introducing us to [MV], and thereby enabling us to prove Proposition 4.6. We also thank Torsten Ekedahl for suggesting to use Borel’s fixed point theorem to prove the results in Section 7. The second author was supported in part by the Norwegian Research Council and thanks for this support and for the kind hospitality extended by Aarhus University, Denmark, where the support was spent.

2. BASIC DESCRIPTION OF SCHUBERT UNIONS

In this section we will recall the well known definition of a Schubert cycle $\alpha = (a_1, ..., a_l)$ in the Grassmann variety $G(l,m)$ over a field $F$, and describe unions of such cycles. Let $B = \{e_1, ..., e_m\}$ be a basis of a $m$-dimensional vector space $V$ over $F$, and let $A_i = \text{Span}\{e_1, ..., e_{i}\}$ in $V$, for $i = 1, ..., m$. Then $A_1 \subset A_2 \subset \ldots \subset A_m = V$ form a complete flag of subspaces of $V$. With respect to the basis $B$ there is the following canonical cell decomposition of $G(l,m)$.

For a given $l$-subspace $W$ of $V$ form an $(l \times m)$-matrix $M_W$ where the rows form a set of basis vectors for $W$, each row expressed in terms of the basis $B$.

We choose a basis for $W$ such that the matrix $M_W$ have reduced lower left triangular form, i.e. the last nonzero entry in each row is 1, each of these 1’s are the only nonzero entries in their column, and each of these 1’s lie in a column to the right of the trailing 1 in the previous row. The trailing 1 in row $i$ is then in column $a_i(W)$ where

$$a_i(W) = \min\{j | \dim(W) \cap A_j = i\}.$$
Obviously
\[ 1 \leq a_1(W) < a_2(W) < \ldots < a_l(W) \leq m. \]
For \( \alpha = (a_1, \ldots, a_l) \) with \( 1 \leq a_1 < a_2 < \ldots a_l \leq m \) let
\[ C_\alpha = \{ [W] \in G(l, m) | a_i(W) = a_i, i = 1, \ldots, l \} \]
Since the reduced lower left triangular form of \( M_W \) is unique, \( C_\alpha \) is an affine space of dimension
\[ \dim C_\alpha = \sum_{i=1}^{l} (a_i - i) = \sum_{i=1}^{l} a_i - \frac{l(l + 1)}{2}. \]
Therefore \( C_\alpha \) is called a cell, and when \( \alpha \) varies these cells are pairwise disjoint and form a decomposition of \( G(l, m) \).

Note that the ordered \( l \)-uples \( \alpha \) belong to the grid
\[ G_{G(l,m)} = \{ \beta = (b_1, \ldots, b_l) \in \mathbb{Z}^l | 1 \leq b_1 < b_2 < \ldots b_l \leq m \} \]
and that this grid is partially ordered by \( \alpha \leq \beta \) if \( a_i \leq b_i \) for \( i = 1, \ldots, l \).

For each \( \alpha \in G_{G(l,m)} \) the Schubert cycle \( S_\alpha \) is defined as:
\[ S_\alpha = \{ W | \dim(W \cap A_\alpha) \geq i, \quad i = 1, \ldots, l \} = \cup_{\beta \leq \alpha} C_\beta. \]
Note that \( S_\alpha \) inherits a cell-decomposition from \( G(l, m) \).

Next, we choose coordinates for the Plücker space \( \mathbb{P}^{k-1} = \mathbb{P}(\wedge^l V) \), with respect to the chosen basis \( B \). Our choice of Plücker coordinates are the maximal minors of the matrix \( M_W \) (with alternating signs). The Plücker coordinates from another basis for \( W \) would differ from these only by a nonzero scalar factor, so they are welldefined as projective coordinates. Since the maximal minors of \( M_W \) are indexed by the grid \( G_{G(l,m)} \) these Plücker coordinates are denoted by \( \{ X_\alpha(W) | \alpha \in G_{G(l,m)} \} \). For the unique basis of \( W \) above we get \( X_\alpha = 1 \) when \([W] \in C_\alpha \).

These Plücker coordinates for subspaces belonging to the Schubert cycle \( S_\alpha \) become particularly simple:
\[ S_\alpha = \{ p \in G(l, m) | X_\beta = 0 \text{ for all } \beta \text{ with } b_i > a_i \text{ for some } i \}. \]

We therefore collect all the indices of nonzero Plücker coordinates on the Schubert cycle \( S = S_\alpha \) in a subgrid of \( G_{G(l,m)} \), called the \( S \)-grid or \( \alpha \)-grid:

**Definition 2.1.** \( G_S = G_\alpha = \{ \beta \in G_{G(l,m)} | \beta \leq \alpha \} \).

In the special case \( \alpha = (m - l + 1, \ldots, m - 1, m) \), we get \( S_\alpha = G(l, m) \) and \( G_\alpha = G_{G(l,m)} \).

**Definition 2.2.** A subset \( M \subset G_{G(l,m)} \) is called Borel fixed if it enjoys the property that \( \beta \in M \) whenever \( \alpha \in M \) and \( \beta \leq \alpha \).

Clearly \( G_\alpha \) is Borel fixed for each Schubert cycle \( S_\alpha \). Notice that \( M \) is Borel fixed if and only if the union of the cells \( \cup_{\alpha \in M} C_\alpha \) is closed.

For each \( \alpha \) we also define:
\[ H_\alpha = G_{G(l,m)} - G_\alpha, \]
in other words the complement of the \( \alpha \)-grid in the entire \( G(l, m) \)-grid. We then see that
\[ S_\alpha = \{ [W] \in G(l, m) | X_\beta(W) = 0, \forall \beta \in H_\alpha \}, \]
- For a subset \( M \) of \( G(l, m) \subset \mathbb{P}(\wedge^l V) \), let \( L(M) \) be its linear span in the projective Plücker space \( \mathbb{P}(\wedge^l V) \).
Proof. The first statement is already proven above. The span of \( l \) in the linear span of the union of each \( M \) be the entries of the matrix \( \beta \).

Let \( l \) then \( \beta \) matrix \( I \) linear generators of the ideal \( S \) flag. Let \( S \) and set \( \gamma \).

Let Proposition 2.4.

\[ G \]

and all finite intersections of these Schubert cycles are again Schubert cycles, the union of \( S \) equals with respect to our fixed flag. Set \( S \) decomposition of \( S \).

\[ \gamma \]

The intersection \( S_\gamma \) is itself a Schubert cycle with \( \gamma \).

Thus the intersection of a finite set of Schubert cycles \( S_\alpha \) is again a Schubert cycle. In particular \( \dim L(\cap S_\alpha) \) is equal to the cardinality of \( G_\gamma \).

For a union \( S_U = \bigcup_{i=1}^s S_{\alpha_i} \) of Schubert cycles, denote by \( V_U \) the union \( G_U = \bigcup_{i=1}^s G_{\alpha_i} \), and set \( H_U = G_{G(l,m)} - G_U \). Since all Schubert cycles \( S_\alpha \) has a decomposition of cells \( C_\beta \), and all finite intersections of these Schubert cycles are again Schubert cycles, the union \( S_U \) also has a cell-decomposition inherited from \( G(l,d) \):

\[ S_U = \bigcup_{\gamma \in G_U} C_\gamma \]

Proposition 2.4. Let \( S_{\alpha_1}, \ldots, S_{\alpha_s} \) be finitely many Schubert cycles with respect to our fixed flag. Let \( S_\gamma = \bigcap_{i=1}^s S_{\alpha_i} \) be their intersection, and let \( S_U = \bigcup_{i=1}^s S_{\alpha_i} \) be their union.

1. The intersection \( S_\gamma \) is itself a Schubert cycle with \( S \)-grid \( G_\gamma = \bigcap_{i=1}^s G_{\alpha_i} \).
2. \( L(S_U) \cap G(l,m) = S_U \).
3. \( \dim L(S_U) \) equals the cardinality of the grid \( G_U \).
4. The number of \( F_q \)-rational points on \( S_U \) is \( \Sigma_{(x_1,\ldots,x_l) \in G_U} q^{x_1+\ldots+x_l-l(l+1)/2} \).

Proof. The first statement is already proven above. The ideal of the linear span of the union \( S_U \) is defined by the intersection of the ideals of the \( S_{\alpha_i} \). Therefore \( \{X_\beta|\beta \in H_U\} \) form the linear generators of the ideal \( I_U \) of \( S_U \). Looking more closely at \( H_U \), notice that if \( \alpha \in H_U \), then \( \beta \in H_U \) whenever \( \alpha \leq \beta \). We interpret this condition on the Plücker coordinates of a \( l \)-dimensional subspace \( W \subset V \), coming from a reduced lower left triangular form for the matrix \( M_W \) as above. We need the following characterisation of the coordinates \( X_\alpha \) for \( \alpha \in H_U \):

Lemma 2.5. Let \( W \subset V \) and let

\[ \{(w_{ij})|1 \leq i \leq l, 1 \leq j \leq m\} \]

be the entries of the matrix \( M_W \) in reduced lower triangular form. Then \( X_\alpha(W) = 0 \) for each \( \alpha = (a_1, \ldots, a_l) \in H_U \) if and only if one of the following is satisfied

1. \( w_{1j} = 0 \) when \( j \geq a_1 \)
2. \( w_{1j} = w_{2j} = 0 \) when \( j \geq a_2 \)

\ldots
(3) \( w_{ij} = 0 \) for \( i = 1, \ldots, l \) when \( j \geq a_i \)

**Proof.** None of the criteria are satisfied, if and only if the trailing 1’s in each row appear in columns \((b_1, \ldots, b_l)\) where \( b_i \geq a_i \) for each \( i \). But equivalently the Plücker coordinate \( X_\beta(W) = 1 \) for \( \beta = (b_1, \ldots, b_l) \in H_U \), contrary to our assumption. \( \square \)

Now, each itemized condition in the lemma is the condition that \([W]\) belong to a Schubert cycle:

\[
w_{1j} = \ldots = w_{rj} = 0 \text{ implies that } \dim W \cap A_{j-1} \geq r.
\]

Therefore the collection of conditions given by the element \( \alpha = (a_1, \ldots, a_l) \in H_U \) implies that \([W]\) belongs to a union of Schubert cycles with respect to the given flag. Since the intersection of two Schubert cycles with respect to the given flag is a Schubert cycle, the intersection of a union of Schubert cycles is again a Schubert union. In particular, it is now straightforward to check that if \( X_\beta(W) = 0 \) for each \( \beta \in H_U \), then \([W]\) belongs to the Schubert union \( S_U \), i.e. the linear span of \( S_U \) intersects \( G(l,m) \) precisely in \( S_U \).

The cardinality of \( H_U \) is the codimension of \( L(S_U) \), so the dimension of \( L(S_U) \) equals the cardinality of the complement \( G_U \). Finally the number of \( F_q \)-rational points is counted using the cell-decomposition. Note that one may also use the exclusion-inclusion principle using Proposition 2.3 since all intersections are Schubert cycles. \( \square \)

Notice that when \( S_U \) is a Schubert union, then the grid \( G_U \) is Borel fixed. In fact

**Proposition 2.6.** The Borel fixed subsets of the grid \( G_{G(l,m)} \) are precisely the grids \( G_U \) of Schubert unions. Similarly, the closed unions of cells \( C_\alpha \) are precisely the Schubert unions.

**Proof.** Let \( M \) be a Borel fixed subset of \( G_{G(l,m)} \). Since \( M \) is finite, it has finitely many maximal elements, \( \alpha_1, \alpha_2, \ldots, \alpha_r \), say. Then \( M = \cup_{\alpha_i} G_{\alpha_i} \). In particular \( M \) is the grid of a Schubert union. Similarly, the second statement follows considering the fact that the cell \( C_\beta \) lies in the closure of the cell \( C_\alpha \), precisely when \( \beta \leq \alpha \). \( \square \)

The name “Borel fixed” originates from the a natural action of a Borel subgroup \( B \) on the Plücker space. Extend the scalars of \( V \) to the algebraic closure \( \overline{F}_q \) of the field \( F_q \). Then \( GL(m, \overline{F}_q) \) acts (from the left) on \( V \), on the exterior product \( \wedge^l V \) and on \( G(l,m) \). Let \( B \subset GL(m, \overline{F}_q) \) be the subgroup of upper triangular matrices with respect to the basis \( e_1, \ldots, e_m \) of \( V \). Then \( B \) is precisely the subgroup that fixes the given flag \( A_1 \subset A_2 \subset \cdots \subset A_m \). Let \( X \) be a set of Plücker coordinates, and let \( M(X) = \{ \beta X \in X \} \) be the corresponding grid in \( G_{G(l,m)} \). Then the linear span of \( X \) is fixed by \( B \) if and only \( M(X) \) is Borel fixed: Let \( V_l = \wedge^l V \). Then \( V_l \) is a \( \binom{m}{l} \)-dimensional vector space. Let \( r \leq k \) and consider \( V_{l,r} = \wedge^n V_l \), and the Grassmannian \( G(r, V_l) \). The linear action of \( B \) on \( V_{l,r} \), clearly induces a linear action on \( V_{l,r} \), and on \( G(r, V_l) \).

Let \[
\{ e_\alpha = e_{a_1} \wedge e_{a_2} \wedge \ldots \wedge e_{a_l} | 1 \leq a_1 < a_2 \ldots < a_l \} \]
be the basis of \( V_l \) with Plücker coordinates \( X_\alpha \). Order this basis lexicographically. Then \[
\{ e_{a_1} \wedge e_{a_2} \wedge \ldots \wedge e_{a_r} | (1, 2, \ldots, l) \leq a_1 < a_2 < \ldots < a_r \leq (m-l+1, \ldots, m) \} \]
form a basis for \( V_{l,r} \). The only subspaces of \( V \) that are stable under \( B \) are the subspaces \( A_i \), i.e. those spanned by \( e_1, e_2, \ldots, e_i \) for some \( i \). Therefore the only subspaces of \( V_l \) that are stable under \( B \) are those spanned by \( \{ e_\alpha | \alpha \in I \} \), for some finite index set \( I \subset G(l,m) \), with the property that \( \beta = (b_1, \ldots, b_l) \in I \) whenever \( \alpha = (a_1, \ldots, a_l) \in I \) and \( b_i \leq a_i \) for all \( i \). But these index sets are precisely the Borel fixed subsets of \( G_{G(l,m)} \). On the other hand, the stable subspaces of \( V_l \) of dimension \( r \) are the only fixed points on \( G(r, V_l) \) under the action of \( B \).
**Proposition 2.7.** The Schubert unions of spanning dimension r define the fixed points under the action of the Borel subgroup $B \subset GL(m, F)$ on $G(r, V_l)$.

### 2.1. Duality of Schubert Unions

There are various natural forms of duality for Schubert unions. One is combinatorial and one is geometric. We will define both and show that they coincide.

First we will describe a geometrical duality, valid for Schubert unions, but not for general linear sections of the Grassmann variety $G(l, m)$.

This duality is a restriction of ordinary duality between the Plücker space $\mathbf{P}(\wedge^l V)$ and its natural dual $\mathbf{P}(\wedge^V V^*)$. Just as $G(l, m)$ has a natural embedding by Plücker coordinates in $\mathbf{P}(\wedge^l V)$, the Grassmannian $G(l, m)^*$ parametrizing rank $l$-subspaces of $V^*$, i.e. $l$-dimensional subspaces of linear forms on $V$, has an embedding in $\mathbf{P}(\wedge^V V^*)$.

For a subspace $L \subset V$, we set

$$L^\perp = \{ H \in V^* | L \subset \ker H \}.$$ 

Let $M$ be a subset of $G(l, m)$. Then we define:

**Definition 2.8.** The geometric dual set $D(M)$ is the subset of $G(l, m)^* \subset \mathbf{P}(\wedge^V V^*) = \mathbf{P}(\wedge^l V)\mathbf{P}(\wedge^V V^*)$ parametrizing $l$-subspaces of linear forms on $V$ whose common kernel simultaneously intersect all $l$-spaces represented by points of $M$ in more than 0.

In terms of natural duality between $\mathbf{P}(\wedge^l V)$ and $\mathbf{P}(\wedge^V V^*)$ one finds

**Lemma 2.9.** $D(M) = \mathcal{L}(M)^\perp \cap G(l, m)^*$, where $\mathcal{L}(M)$ is the linear span of $M$.

**Proof.** A point $P \in G(l, m)^*$ lies in $D(M)$ if and only if the hyperplane $H_P \subset \mathbf{P}(\wedge^l V)$ defined by $P$ contains $M$, i.e. the span $\mathcal{L}(M)$.

Therefore, if $M \subset G(l, m)$ is a general subset that spans a linear space of dimension larger than that of $G(l, m)$, then the geometric dual $D(M) = \emptyset$.

But for any Schubert union $S_U \not= G(l, m)$, the geometric dual $D(S_U)$ is nonempty. In fact we will show that $D(S_U)$ is again a Schubert union with respect to the dual flag $0 = A^l_m \subset A^l_{m-1} \subset A^l_{m-2} \supset \cdots \supset A^l_1 \subset V^*$.

We denote the Plücker coordinates on $\mathbf{P}(\wedge^l V)$ with respect to this flag by

$$\{ X^*_\alpha | \alpha = (a_1, a_2, \ldots, a_l), \quad 0 < a_1 < \ldots < a_l \leq m \}.$$ 

Notice that these coordinates are natural dual to the coordinates $X_\alpha$ defined on $\mathbf{P}(\wedge^l V)$.

In fact

$$X^*_\alpha (X_\beta) = 0 \quad \text{if and only if} \quad \alpha \not= \beta$$

As above we may define a grid $G_{G(l, m)^*} \subset \mathbb{Z}^l$ for these Plücker coordinates:

$$G_{G(l, m)^*} = \{ \alpha = (a_1, a_2, \ldots, a_l) | 1 \leq a_1 < \ldots < a_l \leq m \}.$$ 

A Schubert union $S_U^* \subset G(l, m)^*$ with respect to this flag is associated to a $G$-grid $G_U^* \subset G_{G(l, m)^*}$.

**Definition 2.10.** Let the natural map

$$\text{rev} : G_{G(l, m)} \rightarrow G_{G(l, m)^*}$$

be defined as

$$(a_1, a_2, \ldots, a_l) \mapsto (m + 1 - a_1, \ldots, m + 1 - a_2, m + 1 - a_l).$$
Remark 2.11. Clearly the map $\text{rev}$ has a natural inverse

$$\text{rev}^* : G_{G(l,m)}^* \to G_{G(l,m)}.$$ 

It therefore sets up a duality between the two grids that we naturally call Grid duality. For the relation to Poincare duality, see Remark 2.20.

Definition 2.12. Let $M$ be an arbitrary subset of the $G_{G(l,m)}$-grid \{$(a_1, ..., a_l) \in Z^l | 1 \leq a_1 < a_2 < ... < a_l \leq m$\}. Then the grid-dual of $M$ is

$$M^{\text{rev}} = \{\text{rev}(\alpha) | \alpha \in M\} \subset G_{G(l,m)}^*.$$ 

We are now ready to define the grid dual of a Schubert union $U \subset G(l, m)$.

Definition 2.13. Let $S_U$ be a Schubert union in $G(l, m)$ with $G$-grid $G_U$. Then the grid dual of $S_U$ is the Schubert union $S_U^* \subset G(l, m)^*$ whose $G$-grid $G_U^*$ is the grid-dual $H_U^{\text{rev}}$ of $H_U$.

Roughly speaking this means that one finds the dual of a Schubert union by “turning its $H$-grid around with the map $\text{rev}$” and use it as $G$-grid. (“Turning around” just means taking its mirror image relative to the level linear space $d = i_1 + i_2 - 3 = \delta/2 = m - 2$ if $l = 2$). The key lemma that links grid-duality to geometric duality is:

Lemma 2.14. Let $M \subset G_{G(m,l)}$, and let $\mathcal{L}(M) \subset P(\wedge^i V)$ be the linear space defined by the vanishing of the Plücker coordinates $\{X_\alpha | \alpha \in G_{G(m,l)} \setminus M\}$. Then $\mathcal{L}(M)^\perp \subset P(\wedge^i V^*)$ is defined by the vanishing of the Plücker coordinates $\{X^*_\alpha | \alpha \in M^{\text{rev}}\}$.

Proof. This is simply the orthogonality induced by the duality of the two basis $\{X_\alpha | \alpha \in G_{G(l,m)}\}$ and $\{X^*_\alpha | \alpha \in G_{G(l,m)^*}\}$. □

We may now state the promised result:

Theorem 2.15. For a Schubert union $S_U$ in $G(l, m)$ its geometric dual $D(S_U)$, and its grid-dual $S_U^*$, are equal.

Proof. The homogeneous ideal $I_{S_U}$ of $S_U$ modulo the ideal of $G(l, m)$ is generated by

$$\{X_\beta | \beta \in H_U\}.$$ 

By lemmas 2.13 and 2.14 a linear form in this ideal correspond to a point $P$ on the geometric dual $D(S_U)$ if and only if $P \in G(l, m)^*$ and the nonzero Plücker coordinates of $P$ lie in $H_U^{\text{rev}}$, i.e. if and only if they lie in the grid-dual $S_U^*$. □
Example 2.16. (1) Let $l = 2$, $m = 7$, and consider the Schubert cycle $S_{(3,5)}$. Its $G$-grid is

$$G_{(3,5)} = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(3,4),(3,5)\},$$

and

$$H_{(3,5)} = \{(1,6),(1,7),(2,6),(2,7),(3,6),(3,7),(4,5),(4,6),(4,7),\ldots,(6,7)\}.$$

Its grid-dual $S_U^*$ has $G$-grid

$$G_{(3,5)^*} = \{(1,2),(1,3),\ldots,(1,7),(2,3),\ldots,(2,7),(3,4)\}.$$

Its geometric dual is therefore the Schubert union

$$D(S_{(3,5)}) = S_{(2,7)}^* \cup S_{(3,4)^*}.$$  

In Figure 2, we have marked the points of the $G$-grid of $S_{(3,5)}$ by 0 and the points of its $H$-grid by $x$. In Figure 3, we have marked the points of the $G$-grid of the dual union of $S_{(3,5)}$ by 0 and the points of its $H$-grid by $x$.

(2) The geometric dual of $S_{(3,7)}$ is $S_{(3,4)^*}$.

(3) The geometric dual of $S_{(5,6)}$ is $S_{(1,7)^*}$.

It would of course be nice also to have an explicit expression for the dual of a Schubert union.

Let $U$ be the Schubert union

$$S_{(a_{1,1},a_{1,2},\ldots,a_{1,l})} \cup S_{(a_{2,1},a_{2,2},\ldots,a_{2,l})} \cup \ldots \cup S_{(a_{s,1},a_{s,2},\ldots,a_{s,l})},$$

or phrased differently:

$$\bigcup_i S_{(a_{i,1},a_{i,2},\ldots,a_{i,l})},$$

with $i \in I = \{1,\ldots,s\}$. The dual Schubert union is then given as the union over all disjoint partitions $\{A_1,\ldots,A_l\}$ of $I$ of the Schubert cycles that are described as follows:

$$X_1 \leq m - \max\{a_{i,l} | i \in A_1\},$$

$$X_2 \leq m - \max\{a_{i,l-1} | i \in A_2\},$$

$$\ldots$$

$$X_{l-1} \leq m - \max\{a_{i,2} | i \in A_{l-1}\},$$

$$X_l \leq m - \max\{a_{i,1} | i \in A_l\}.$$  

Each such collection of $l$ simple conditions give the Schubert union $S_{(f_1,f_2,\ldots,f_l)}$, where

$$f_1 = \min\{f_2 - 1, m - \max\{a_{i,l} | i \in A_l\}\}.$$
\[ f_2 = \min\{f_3 - 1, m - \max\{a_{i,l-1} \mid i \in A_{i-1}\}\}, \]

\[ f_{l-1} = \min\{f_l - 1, m - \max\{a_{i,2} \mid i \in A_2\}\}, \]

\[ f_l = m - \max\{0, a_{i,1} \mid i \in A_1\}. \]

This must be interpreted such that \( S_{(f_1, f_2, \ldots, f_l)} = \emptyset \), if \( f_i < i \) for some \( i \).

### 2.2. Further properties of dual Schubert unions.

Theorem 2.15 obviously gives the following:

**Corollary 2.17.** The geometrically dual Schubert union of the geometrically dual Schubert union of the Schubert union \( U \) is \( U \).

**Proof.** The analogous result for the grid dual, that is: grid-biduality, obviously holds, and using Theorem 2.15 we also have geometric biduality. \( \square \)

It would be nice to be able to count the Schubert unions of \( G(l, m) \) (relatively to a fixed flag, as always). For general \( l \) and \( m \) it is a combinatorial challenge to do this, a challenge we have not yet met. For \( l = 2 \), however, we will see in the next section how one can find the number of Schubert unions for fixed \( m \).

**Definition 2.18.** For a Schubert union \( U \) (or \( S_U \)) denote by \( U^* \) (or \( S_U^* \)) its dual union.

**Remark 2.19.** For the sake of completeness we will now will give a slight variation of the representation of a Schubert union by its \( G \)-grid or \( H \)-grid. This will not, however, bring any essentially new. Look at a rectangular \( l \times (m - l) \)-box \( B \) with \( l \) columns and \( m - l \) rows, formed by \( l(m - l) \) squares of sidelength 1. We are now interested in partitions, whose Young diagrams (or Ferrers diagrams, see [Fu], p. 2-3 for a discussion of notation) can be placed inside \( B \). Obviously, these are some of the partitions of the integers \( N \) that range from 0 to \( l(m - l) \). The Young diagram of \( s_1 + s_2 + \ldots + s_r \), for non-increasing \( s_j \), consists of \( s_i \) consecutive squares in row \( i \), for \( i = 1, \ldots, r \), where all left ends of the sequences of squares start just below each other, and row \( i \) is above row \( j \) if \( i < j \).

The set of points in the \( G_{G(l,m)} \)-grid can in a natural way be identified with the set \( \mathcal{P}_{(l,m)} \) of partitions whose Young diagram can be placed inside \( B \). For a given partition \( P \), let \( c_j \) be the number of summands \( s_i \) in the partition such that \( s_i = j \). Hence, if the partition for example is \( 2 + 2 + 3 = 7 \), then \( c_2 = 2, c_3 = 1 \) and \( c_j = 0 \) for all other \( j \). Obviously \( P \) is characterized by the \( c_i \).

(i) Put

\[ X_1 = c_1 + 1, X_2 = c_1 + 1 + c_2 + 2, \ldots, X_{l-1} = c_2 + c_3 + \ldots + c_l + l - 1, X_l = c_1 + c_2 + \ldots + c_l + l. \]

Then \( h(P) = (X_1, X_2, \ldots, X_l) \) gives the natural bijection from \( \mathcal{P}_{(l,m)} \) to \( G_{G(l,m)} \).

(ii) For the Schubert cycle \( S_\alpha \) with \( \alpha = (a_1, a_2, \ldots, a_l) \) we have \( h^{-1}(G_\alpha) = \mathcal{P}_\alpha \), where \( \mathcal{P}_\alpha \) is the set of partitions \( P \) with

\[ c_1 \leq a_1 - 1, c_1 + c_2 \leq a_2 - 2, \ldots, c_2 + c_3 + \ldots + c_l + l - 1 \leq a_{l-1} - (l-1), c_1 + c_2 + \ldots + c_l \leq a_l - l. \]

(iii) For a Schubert union \( U \) we see that \( h^{-1}(G_\alpha) \) is a corresponding union of sets of type \( \mathcal{P}_\alpha \).

(iv) If we represents a partition \( P \) by its \( l \)-tuple \( (c_1, \ldots, c_l) \), then

\[ \mathcal{P}_{(l,m)} = h^{-1}(G_{G(l,m)}) = \{(c_1, \ldots, c_l) \in \mathbb{Z}^l \mid c_1 + \ldots + c_l \leq m - l, \text{ and } c_i \geq 0, \text{ all } i \}, \]

in other words an alternative, “twisted” grid with \( \binom{l}{m} \) elements.

(v) Of course \( h^{-1}(H_U) \) is the complement of \( h^{-1}(G_U) \) in \( h^{-1}(G_{G(l,m)}) \). The relationship between \( h^{-1}(H_U) \) and \( h^{-1}(G_U^*) \) is, however, not as striking as that between \( H_U \) and \( G_U^* \).
a slightly more complicated operation than just “turning $h^{-1}(H_U)$ around” is necessary to obtain $h^{-1}(G_U^*)$.

(vi) In Corollary 2.4 we determined the number of ($F_q$-rational) points on $S_U$ as the sum of terms $q^{x_1 + \ldots + x_l - l(l+1)/2}$, where the sum is taken over all tuples $(x_1, \ldots, x_l)$ in the (usual) grid $G_U$. We convert $x_i$ to $c_j$ and observe:

$$x_1 + \ldots + x_l - l(l+1)/2 = c_1 + 2c_2 + \ldots + lc_l.$$  

But this is the number, say $N(P)$, of which the partition $P$ in question is a partition. Hence we see that the number of ($F_q$-rational) points on $S_U$ is the sum of terms $q^{N(P)}$, where the sum is taken over all partitions represented by tuples in the (usual) grid $G_U$.

**Remark 2.20.** For Schubert cycles there is of course another, well-known duality, Poincare duality. We know that the Poincare dual of $S_{(a_1, a_2, \ldots, a_l)}$ is

$$S_{(m+1-a_1, m+1-a_2-1, \ldots, m+1-a_2, m+1-a_1)}.$$  

Using the map rev defined earlier in this section this means that the Poincare dual of $S_{\alpha}$ is $S_{\text{rev}(\alpha)}$. Hence one can view Poincare duality as a map (rev) that sends individual points of the $G_{(l,m)}$-grid to their image points, while our duality of Schubert unions sends configurations of points ($H$-grids) to their corresponding configurations of image points, and in addition interchanges the roles of $G$-grids and $H$-grids.

The sum of the Krull dimensions of a Schubert cycle and its Poincare dual cycle is $\delta = l(m - l)$, the Krull dimension of $G(l, m)$. The sum of the (affine) spanning dimension of a Schubert cycle/union and its geometric/grid dual union is $k = \binom{m}{l}$, the (affine) spanning dimension of $G(l, m)$. While geometrical or grid duality will play an important role for us, we will not be concerned with Poincare duality in this paper (apart from the fact we have now pointed out, that the map rev that we use, is essentially Poincare duality).

Let us finish this section by some remarks concerning duality of Schubert unions over finite fields. Let $U$ be a Schubert union, and let $g_U(q)$ be its number of $F_q$-rational points, as given by Corollary 2.4. Let $\delta = l(m - l)$ be the Krull dimension of $G(l, m)$. Denote by $n(q)$ the number of $F_q$-rational points of $G(l, m)$, and set $h_U(q) = n(q) - g(q)$.

**Proposition 2.21.** Let $U^*$ denote the dual of a Schubert union $U$. Then the number of $F_q$-rational points of $U^*$ is $q^{\delta}h_U(q^{-1})$.

**Proof.** This is clear since $n(q)$ is the sum of terms $q^{i_1 + \ldots + i_l - l(l+1)/2}$, where the sum is taken over all points $(i_1, \ldots, i_l)$ in the grid $G_{G(l,m)}$, and $g_U(q)$ is the corresponding sum over all points of $G_U$, and $h_U(q)$ is the corresponding sum over all points of $H_U$. Passing from $H_U$ to $\text{rev}(H_U)$ gives rise to the passage from $h_U(q)$ to $q^{\delta}h_U(q^{-1})$.  

\[\square\]

### 3. Schubert Unions in $G(2, m)$

In this section we will make a special study of Schubert unions in $G(2, m)$ for $m \geq 3$. We will show that for each $m$ there are $2^{m-1}$ such unions (for fixed flag) and that they in a natural way correspond to the set of subsets $P(M)$ of a given set $M$ with $m - 1$ elements., we may assume $M = \{1, 2, \ldots, m - 1\}$. Moreover, taking complements in $M$ corresponds to the duality of Schubert unions described in general in Section 2.1. We will also interpret Schubert unions in other ways, including a more “physical” one.
3.1. **Subsets of $M$ and increasing sequences of $2s$ numbers.** We call a union of two Schubert cycles proper if it not equal to a Schubert cycle. Let $m = 2$. We start out with the following, but important technical result:

**Lemma 3.1.** The union of two Schubert cycles $S_{(a,b)}$ and $S_{(c,d)}$ is proper if and only if $a < c < d < b$ or $c < a < b < d$.

**Proof.** The result follows from $S_{(a,b)} \cap S_{(c,d)} = S_{(e,f)}$, where $e = \min\{a, c\}$ and $f = \min\{b, d\}$. \hfill \Box

This means that specifying a proper union of two Schubert cycles amounts to specifying four integers $1 \leq a < c < d < b \leq m$, and then we obtain the union $S_{(a,b)} \cup S_{(c,d)}$. In general, specifying a Schubert union of $s$ Schubert cycles, which is not a union of $s - 1$ Schubert cycles (a proper union of $s$ Schubert cycles), amounts to specifying $2s$ integers $1 \leq a_1 < a_2 < ... < a_s < b_s < ... < b_1 \leq m$, and then we obtain the union $S_{(a_1,b_1)} \cup ... \cup S_{(a_s,b_s)}$. This means that there is $\binom{m}{0} = 1$ empty set, there are $\binom{m}{2}$ unions that are just Schubert cycles, and in general there are $\binom{2m}{s}$ unions that are proper unions of $s$ cycles, for each $s$ up to $\frac{2m}{2}$. Since

$$\binom{m}{0} + \binom{m}{2} + \binom{m}{4} + ... = \binom{m-1}{0} + \binom{m-1}{2} + \binom{m-1}{4} + \binom{m-1}{6} + ... + \binom{m-1}{s-1} = 2^{m-1},$$

we see that there is a potential for expressing the set of Schubert unions as a power set $P(M)$ as described above. Concretely, we choose to do it as follows: Think of the $G_{(2,m)}$-grid as consisting of squares with sidelengths 1 centered at the points of $\mathbb{Z}^2$ in the $(x,y)$-plane considered earlier. Then, specifying the $G$-grid of a Schubert union corresponds to picking $m_1$ squares for $x = 1$, $m_2$ squares for $x = 2$, ..., $m_r$ squares for $x = r$, for some $r \leq m - 1$. Moreover $m_1 > m_2 > ... > m_r$. This gives rise to:

**Definition 3.2.** For a Schubert union $S_U$ the (unordered) subset $M_U$ of $M = \{1, 2, ..., m-1\}$ is $\{m_1, m_2, ..., m_r\}$ if there are exactly $m_i$ points $(x, y)$ in $G_U$ with $x = i$, for $i = 1, ..., r$, and no points with $x = i$, for $i > r$.

If one prefers to list the numbers in increasing order, which in fact will be strictly increasing, we have $M_U = \{m_r, m_{r-1}, ..., m_1\}$. A simple look at the plane diagram of $G_{(2,m)}$, represented as $\binom{m}{2}$ squares forming a triangle, reveals that the complement of $M_U$ is equal to $M_U^\ast$.

3.2. **Different descriptions of Schubert unions for $l = 2$.** Look at the set of sequences $1 \leq a_1 < a_2 < ... < a_d < b_d < ... < b_1 \leq m$. Given such a sequence that represents the Schubert union $S_U = S_{(a_1,b_1)} \cup ... \cup S_{(a_s,b_s)}$, we give the following:

**Definition 3.3.** For a Schubert union $S_U$ we set $\sigma_U = a_1 < a_2 < ... < a_d < b_d < ... < b_1$

We now will give the function $f$ that, with $\sigma_U$ as input, gives the subset $M_U = f(\sigma_U)$ as output.

**Proposition 3.4.** If $\sigma_U$ is the sequence representing $S_U$, then

$$M_U = f((a_1 < a_2 < ... < a_s < b_s < ... < b_1)) =$$

$$\{b_s - a_s, b_s - a_s + 1, ..., b_s - a_{s-1} - 1, b_{s-1} - a_{s-1}, ..., b_1 - a_1, b_1 - 1\}.$$
where all integers \( \{d_1, \ldots, d_{s-1}\} \) are at least two and represent the “jumps” in \( M_U \). (The cardinality of \( M_U \) is then \( c_1 + \ldots + c_s + s \).) The formula \( f^{-1} \) which gives the sequence \( \sigma_U \) is now:

\[
    f^{-1}(\{c_0, \ldots, c_0 + \ldots + c_s - 1 + d_1 + \ldots + d_{s-1} + c_s\}) =
    \begin{align*}
        c_s + 1 < c_{s-1} + c_s + 2 &< c_{s-2} + c_{s-1} + c_s + 3 < \ldots < c_s + c_{s-1} + c_s + s - 1 < c_1 + \ldots + c_s + s < c_0 + c_1 + \ldots + c_s + s < c_0 + c_1 + c_2 + \ldots + c_s + d_1 + s - 1 < c_0 + c_1 + c_2 + \ldots + c_s + d_1 + d_2 + s - 2 < \ldots <
        c_0 + \ldots + c_{s-1} + c_s + d_1 + \ldots + d_{s-2} + 2 < c_0 + \ldots + c_{s-1} + c_s + d_1 + \ldots + d_{s-1} + 1.
    \end{align*}
\]

We leave it to the reader to check that \( f(f^{-1}(M_U)) = M_U \) and that \( f^{-1}(f(\sigma_U)) = \sigma_U \).

On the level of sequences \( \sigma_U \) we have:

**Lemma 3.5.** If \( \sigma_U = \{a_1 < a_2 < \ldots < a_s < b_s < \ldots < b_1\} \) then the dual sequence \( \sigma_{U^*} \) is obtained by listing the \( 2s + 2 \) integers \( m - b_1, m - b_2, \ldots, m - b_s, m - a_s - 1, m - a_s, \ldots, m - a_1, m \), with the convention that if and only if \( b_1 = m \), then we remove the outer pair \( m - b_1 \) and \( m \), and if and only if \( m - a_s - 1 = m - a_s \), then is: \( a_s = a_{s-1} + 1 \), then we remove the midpair \( m - a_s - 1 \) and \( m - a_s \).

**Proof.** Calculate \( f(\sigma_U) \), take its complement \( C \) in \( M \), and find \( g(C) \). We leave the calculations to the reader.

We then immediately get the following result, which also follows from a direct inspection of the grid diagrams involved:

**Corollary 3.6.** The dual of a proper Schubert union of \( s \) Schubert cycles is a proper union of \( s - 1, s \) or \( s + 1 \) Schubert cycles.

We now give a simple example, demonstrating the various ways of representing a Schubert union and its dual.

**Example 3.7.** Look at \( S_U = S_{(1,7)} \cup S_{(3,5)} \) in \( G(2,7) \). We see that \( G_{G(2,7)} \) roughly speaking consists of a triangular grid of integral points with corners \((1, 2), (1, 7), (6, 7)\). One might embed these points in squares with side length 1 if one prefers. We see that \( S_U \) corresponds to 6 squares in the first column, 3 squares in the second one, and 2 squares in the third column. Hence we have \( M_U = \{2, 3, 6\} \). The complement of \( M_U \) in \( M \) is \( \{1, 4, 5\} \). We observe that both these sets have one “jump” each (from 3 to 6 and 1 to 4, respectively), and hence each of them correspond to a proper union of exactly two Schubert cycles. Furthermore we see that \( H_U \) consists of 5 squares in the upper row, 4 squares in the row below, and 1 square in the row that is third from the top. “Turning this around” (using the operation \((a, b) \rightarrow (8, 8) - (b, a)\) from Definition 2.12) we get 5 squares in column 1 and 4 squares in column 2, and 1 square in column 3, in other words we get \( G_{U^*} \). We observe that \( G_{U^*} \) is the union of \( S_{(2,6)} \) and \( S_{(3,4)} \). Hence \( S_{U^*} = S_{(2,6)} \cup S_{(3,4)} \).

Moreover we see that \( \sigma_U = \{1 < 3 < 5 < 7\} \), while \( \sigma_{U^*} = \{2 < 3 < 4 < 6\} \). We encourage the reader to check the various duality formulas and transformations \( f \) and \( g \) described above in this example.

Using the terminology of Remark 2.19 and the “alternative grid” introduced there, we see that \( S_{(1,7)} \) corresponds to \( \{(c_1, 0)|0 \leq c_1 \leq 5\} \), while \( S_{(3,5)} \) corresponds to \( \{(c_1, 0)|0 \leq c_1 \leq 3\} \cup \{(c_1, 1)|0 \leq c_1 \leq 2\} \cup \{(c_1, 2)|0 \leq c_1 \leq 2\} \), and the union \( S_U \) thus corresponds to the union of these 2 sets, that is: a set of 6 grid points for \( c_2 = 0 \), 3 grid points for \( c_2 = 1 \), and 2 grid points for \( c_2 = 2 \). Likewise the dual union \( S_{U^*} \) corresponds to 5 gridpoints for \( c_2 = 0 \), 4 grid points for \( c_2 = 1 \), and one grid point for \( c_2 = 2 \). Likewise the dual union
\( S_U \) corresponds to 5 gridpoints for \( c_2 = 0,4 \) grid points for \( c_2 = 1 \), and one grid point for \( c_2 = 2 \).

We end this section with a more “physical” remark.

**Remark 3.8.** We represent the \( G_{G(2,m)} \) by \( \binom{m}{2} \) squares in a triangle, as described. Rotate this configuration of squares an angle \( \frac{\pi}{3} \) counterclockwise, and upscale by a factor \( \sqrt{2} \) in each direction, so that the point \((1,2)\) stays fixed, the corner point \((m-1,m)\) is moved to \((1,2m-4)\), and the third corner point \((1,m)\) is moved to \((3-m,m)\). Assume now that you start with any collection of \( K \leq \binom{m}{2} \) squares that you put inside the “frame” \( G_{G(2,m)} \). After we have rotated the frame, we let vertical gravity work, and we assume that the \( K \) squares can move in a frictionless way inside the frame and relatively to each other. Then the configuration of \( K \) squares stays in equilibrium if and only of they form a \( G_U \)-grid for a Schubert union \( S_U \). The function \( x + y - 3 \) before, and simply \( (y-2) \) after, rotating, can be thought of as a potential. The highest peak(s) of \( G_U \) has(have) height equal to the Krull dimension of \( S_U \), which by definition is the highest Krull dimension among the Schubert cycles, of which \( S_U \) is a union. If we let inverse gravity work, then a configuration stays in equilibrium if and only if it forms a \( H_U \)-grid.

An obvious analogous formulation can be given for \( l = 3 \), and using slightly less concrete formulations, also for \( l \geq 4 \).

### 4. Applications to Codes

Let \( C \) be the Grassmann code \( C(l, m) \) over a finite field \( F_q \) as described for example in \[N\], \[GL\], and \[GT\]. Here \( l \) and \( m \) are natural numbers with \( l < m \). The code \( C(l, m) \) is defined as follows. One starts with \( G(l, m) \), which is embedded in the Plücker space \( \mathbb{P}^{k-1} \) with \( k = \binom{m}{l} \). It is well known that \( G(l, m) \) contains \( n \) points, where

\[
(2) \quad n = \frac{(q^m - 1)(q^{m-1} - 1)\ldots(q^{m-l+1} - 1)}{(q^l - 1)(q^{l-1} - 1)\ldots(q - 1)}.
\]

For the special case \( l = 2 \), which we will often study, this is clearly:

\[
(3) \quad n = \frac{(q^m - 1)(q^{m-1} - 1)}{(q^2 - 1)(q - 1)}.
\]

Of course both these formulas are special cases of the last point of Corollary \[2\]. Pick a representative of each of the \( n \) points as a column vector in \((F_q)^k\), and form a \( k \times n \)-matrix \( M \) with these \( n \) vectors as columns (in any preferred order). The code \( C(l, m) \) is then the code with \( M \) as generator matrix. Hence \( C \) is a linear \([n, k]\)-code (only defined up to code equivalence, since we have not specified which representative in \((F_q)^k\) we choose for each point, but this ambiguity will not affect the questions we will study, concerning code parameters and higher weights).

We will now recall and establish some facts about the weight hierarchy \( d_1 < d_2 < \ldots < d_k \) of the codes \( C(l, m) \), using what we know from the previous sections about Schubert unions.

It has been shown in \[N\] that the higher weights \( d_r \) satisfy

**Proposition 4.1.** \( d_r = q^\delta + q^{\delta-1} + \ldots + q^{\delta-r+1} \), for \( r = 1, \ldots s \), where \( s = \max(l, m-l)+1 \), and \( \delta = \dim G(l, m) = l(m-l) \).

Now \( s \) is in almost all cases much smaller than \( k \), so there still remains a lot to be shown. The proof in \[N\] involves 3 elements, the first one is a special proof, using multilinearity.
algebra, for \( d_1 = q^\delta \). The second ingredient is the so-called Griesmer bound, valid for all linear codes, in our case it gives (for all \( r \) in the range \( 2, \ldots, k \), and in particular \( 2, \ldots, s \)):

\[
d_r \geq \sum_{0}^{r-1} \frac{d_1}{q^i}.
\]

This gives:

\[
d_r \geq q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1},
\]

for \( r = 1, \ldots, s \). The third ingredient is the usage of the following well known fact: Let \( S \) be the set of column vectors in a generator matrix for a linear code \( C \). Then its higher weights satisfy:

(4)

\[
d_r = n - H_r,
\]

for all \( r = 1, \ldots, \dim C \), where \( H_r \) is the maximum number of points from \( S \) contained in a codimension \( r \) subspace of \((F_q)^k\). In our case the columns are the points of \( G(l, m) \), so once and for all we define:

**Definition 4.2.** \( H_r \) (or if necessary to specify \( l \) and \( m \): \( H_r^{l,m} \)) is the maximum number of points from \( G(l, m) \) contained in a codimension \( r \) subspace of \( \mathbb{P}^{k-1} \).

In \([N]\) one exhibits concrete codimension \( r \) subspaces of \( \mathbb{P}^{k-1} \) containing \( n - (q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}) \) points, for \( r = 1, \ldots, s \). Equation \([3]\) then gives:

\[
d_r \leq q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1},
\]

which in conjunction with the Griesmer bound gives the result. The third ingredient, or step, has been given in an alternative way in \([GL]\), using so-called close families (of what we would call grid points). In \([GL]\) one looks at the zero set within \( G(l, m) \) of \( X_{\beta_1}, \ldots, X_{\beta_r} \), where the \( \beta_i \) are grid points with \( l-1 \) common coordinates (this is what it means that the family is close). One then shows that this zero set contains \( n - (q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}) \) points, in analogy with \([N]\), and that you can find close families up for \( r = 1, \ldots, s \).

We will now use Schubert unions to give a third (admittedly similar) variant of this third step. We will, however, first show a dual result:

**Theorem 4.3.** For the \( q \)-ary code \( C(l, m) \) defined in the introduction we have \( d_k = n \), and

\[
d_{k-a} = n - (1 + q + \cdots q^{a-1}),
\]

for \( a = 1, \ldots, s \).

**Proof.** Without loss of generality we assume \( s = m - l \). We use the notation and flag \( A_0 < A_1 < \ldots < A_m \) defined in the start of Section \([2]\). Look at the set of \( l \)-dimensional subspaces of \( V \) that contain \( A_{l-1} \). This behaves under the Plücker embedding as the set of lines through the origin in \( V' = (F_q)^{m-l+1} \), in other words it is a projective space of dimension \( s - 1 = m - l \). This space then of course contains projective subspaces of dimension \( i \), for \( i = 0, \ldots, m - l \). In particular we look at the set \( S \) of those \( l \)-spaces \( V \) that contain \( A_{l-1} \) and are contained in \( A_{l+i} \), for \( i = 0, \ldots, m - l \). But this is by definition the Schubert cycle \( S_{\alpha_i} \), where \( \alpha_i = (1, 2, \ldots, l - 1, l + i) \). In particular it is a Schubert union \( U_i \) with \( G \)-grid

\[
G_i = \{(1, 2, \ldots, l - 1, l), (1, 2, \ldots, l - 1, l + 1, (1, 2, \ldots, l - 1, l + 2), \ldots, (1, 2, \ldots, l - 1, l + i)\}.
\]

It contains \( 1 + q + q^2 + \cdots + q^i \) points in virtue of being a projective space or alternatively, by using Proposition \([2.3]\) (or Corollary \([2.4]\)). This is a priori the maximum number of points (from \( G(l, m) \) or even from all of the Plücker space) a \( (k - 1 - i) \)-codimensional subspace of
the Plücker space may contain. This means that these Schubert unions compute \( H_r \), and therefore \( d_r \), for \( r = k - 1 - m + l, \ldots, k - 1 \), in other words for \( m - l = s - 1 \) consecutive values of \( r \). This gives the result. In addition we have, trivially, \( d_k = n \), computed by the particular Schubert union \( \emptyset \).

\[ \square \]

**Remark 4.4.** Theorem 4.3 uses only the well-known fact of the so-called index of the Grassmann variety, the maximum dimension of linear subspace of \( G(l, m) \). We chose to present a detailed proof here in order to demonstrate how in some cases \( H_r \) is computed by Schubert unions, and to give an easy variant of Step 3 of the proofs in [N] and [GL] of Proposition 4.4 as follows:

For each \( i = 1, \ldots, m-l \), we study the Schubert unions \( U_i^* \) that are dual to the ones with \( G \)-grid \( G_i \) as just described. That is the ones with \( H \)-grid equal to \( G_i^{rev} \). Set \( \delta = l(m-l) \).

Proposition 2.21 now gives that the number of points on \( U_i \) is

\[ q^{l(m-l)}(n(q^{-1}) - (1 + q^{-1} + \ldots + q^{i-1})) \]

for \( i = 1, \ldots, m-l \), and this is equal to:

\[ n(q) - (q^\delta + q^{\delta-1} + \ldots + q^{\delta-i}) \]

Since the affine spanning dimension of \( U_i \) is \( i+1 \) for each \( i \), the spanning dimension of \( U_i^* \) is \( k - (i + 1) \), and hence, by Corollary 2.4, \( U_i^* \) is in fact equal to a linear section of \( G(l, m) \) of codimension \( r = i+1 \). As a consequence we obtain: \( H_r \geq n(q) - (q^\delta + q^{\delta-1} + \ldots + q^{\delta-(r-1)}) \),

for \( r = 2, \ldots, m-l+1 = s \), and therefore:

\[ d_r \leq q^\delta + q^{\delta-1} + \ldots + q^{\delta-(r-1)} \]

This gives an alternative proof of Step 3. Even if all the \( U_i \) are Schubert cycles, many of the \( U_i^* \) are proper unions of more than one cycle, in codimension 2, for example, one needs more than one cycle of Krull dimension \( \delta - 2 \). One sees that the points of the \( H \)-grids of the \( U_i^* \) satisfy the requirements of being a close family, in the sense of [GL], since the points of \( G_i \) do, and the \( H \)-grids in question are obtained by turning around the \( G_i \) as described (the only coordinate that varies for the points in these \( H \)-grids will be the first one, since the only one varying in each \( G_i \) is the last one). Hence one could use the results in [GL] instead of Proposition 2.21 to find the number of points on the \( U_i^* \) also.

**Example 4.5.** For \( C(2,4) \) we have \( k = 6 \) and \( s = 3 \). Nogin’s result gives \( d_1 = q^4, d_2 = q^4 + q^3, d_3 = q^4 + q^3 + q^2 \). Theorem 4.3 gives \( d_6 = n, d_5 = n - 1, d_4 = n - 1 - q, d_3 = n - 1 - q - q^2 \). Comparing the two formulas for \( d_3 \), we obtain \( n = q^4 + q^3 + 2q^2 + q + 1 \), which is equal to the well-known formula \( \frac{(q^4-1)(q^3-1)}{(q^2-1)(q-1)} \). We remark that the Griesmer bound gives: \( d_4 \geq q^4 + q^3 + q^2 + q = d_3 + q \), while the true value is \( q^4 + q^3 + 2q^2 = d_3 + q^2 \). This code has been studied in much greater detail in [MV] where one has not only calculated the higher weights, but also the higher spectra (how many subspaces of dimension \( r \) of the given code \( C(2,4) \) have support weight \( s \) for each conceivable \( r \) and \( s \)).

For \( C(2,5) \) we have \( k = 10 \) and \( s = 4 \). Nogin’s result gives \( d_1 = q^6, d_2 = q^6 + q^5, d_3 = q^6 + q^5 + q^4, d_4 = q^6 + q^5 + q^4 + q^3 \). Theorem 4.3 gives \( d_{10} = n, d_9 = n - 1, d_8 = n - 1 - q, d_7 = n - 1 - q - q^2, d_6 = n - 1 - q - q^2 - q^3 \). Here \( n = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1 \). The only remaining case is \( d_5 \). The Griesmer bound gives: \( d_5 \geq d_4 + q^2 \), while we know that \( d_6 = d_4 + q^4 + q^2 \). We have:

**Proposition 4.6.** For the code \( C(2,5) \) we have \( d_5 = n - (q^3 + 2q^2 + q + 1) = d_4 + q^4 = d_6 - q^2 \).
Proof. The Grassmannian $G = G(2, 5)$ is embedded by the Plücker embedding in $\mathbb{P}^9$. Let $H_5$ be the maximal number of points in the intersection of $G$ with a 4-space (codimension 5) in the Plücker 9-space $P$. We use the formula $d_5 = n - H_5$ to prove

$$d_5 \leq d_4 + q^4 = d_6 - q^2$$

since $S_{(1,5)} \cup S_{2,3}$ contains $q^3 + 2q^2 + q + 1$ points, and spans a codimension 5 space by Proposition 2.4. To complete the proof we now prove the supplementary inequality $d_5 \geq d_4 + q^4 = d_6 - q^2$, or equivalently: $H_5 \leq q^3 + 2q^2 + q + 1$. We know that $G$ is cut out by the Plücker quadrics $Q_1, \ldots, Q_5$ in $P$ since there are in general $\binom{m}{4}$ such relations for $G(2, m)$, for $m \geq 3$. A 4-space $W$ in $P$ cannot be contained in $G$, since the maximum number $s$ such that $G$ contains an $s$-space, is 3. In fact, any line in $G$ is formed by all projective lines through a point in a plane in $\mathbb{P}^4$, so a linear subspace of $G$ is formed by either all lines in a plane, or by all lines through a point in a linear subspace of $\mathbb{P}^4$. Therefore the maximal dimension is obtained by the family of all lines through a point in $\mathbb{P}^4$, which of course is 3-dimensional.

Hence at least one of the Plücker quadrics does not contain $W$, and so $W_G = W \cap G$ is contained in a quadric in $W = \mathbb{P}^4$. The maximal rank of the restriction of the quadrics $Q_i$ to $W$ is between 1 and 5. In case the maximal rank is 1 or 2, each quadric decomposes in linear factors when restricted to $W$, so the intersection $W_G$ is a union of linear subspaces. As above the maximal dimension of a linear component is 3, and in that case it corresponds to the lines through a fixed point $P$. If this is one of the components then the residual is also linear, and from the description above we conclude in the following way that it has dimension at most 2, and the two components intersect in a line: If the residual component had dimension 3, then that would also correspond to the lines through (another) fixed point $Q$. On one hand the intersection between the 3-dimensional components is a plane, since we are in $\mathbb{P}^4$. On the other hand it only consists of one point, the one corresponding to the line between the two fixed points $P$ and $Q$. This is a contradiction, and hence the residual component is at most a plane, and it intersects the three-dimensional component in codimension one in the residual component. In this case the cardinality of $W_G$ is at most $q^3 + 2q^2 + q + 1$. If the dimension of the linear components of $W_G$ are smaller than 3 and 2 then $W_G$ is always contained in the union of a 3-space and a 2-space so the cardinality is always smaller than $q^3 + 2q^2 + q + 1$.

In case the maximal rank of the restriction of the quadrics $Q_i$ to $W$ is at least 3, then we get the desired upper bound from the cardinality of points on an irreducible quadric. By projecting to $\mathbb{P}^4$ from a smooth point one gets maximal cardinality $q^3 + q^2 + q + 1$ when the quadric has rank 3 or 5, while the maximal cardinality is $q^3 + 2q^2 + q + 1$ when the quadric has rank 4 (cf. [HT], p 4-5 for details).

\[\square\]

Remark 4.7. We have observed that $d_5$ is computed by fourspaces spanned by Schubert unions $S_{(1,5)} \cup S_{2,3}$ corresponding to pairs $(a, H)$, with $a$ in $H$ in $\mathbb{P}^4$. In addition, using Proposition 2.4 we see that $d_5$ is computed by Schubert unions (in fact cycles) of type $S_{(2,4)}$, which happen to be duals of the $S_{(1,5)} \cup S_{(2,3)}$. To be concrete: For each 3-space $F$ in $\mathbb{P}^4$, and each line $L$ in $F$, we look at the set of lines contained in $F$, intersecting $L$. All in all, for each pair $(a, H)$ with $a \in H$, and each pair $(L, F)$, with $L \in F$, we get a 4-space $W$ that computes $d_5$. There are $(q^4 + q^3 + 2q^2 + q + 1)(q^3 + 2q^2 + q + 1)$ of both kinds, and we conjecture that no other 4-spaces compute $d_5$, and that the number of 5-subspaces of $C(2, 5)$ with minimal support weight $d_5$ therefore is $2(q^4 + q^3 + 2q^2 + q + 1)(q^3 + 2q^2 + q + 1)$.

Definition 4.8. For given $l, m$, set $\Delta_r = d_r - d_{r-1}$ for $r = 1, \ldots, k$. ($\Delta_0 = 0,$)
Example 4.5 and Proposition 4.6 enable us to make a complete table for all the true values of $\Delta_r$ for $(l, m) = (2, 5)$. Although only $\Delta_5, \Delta_6$ for $m = 5$ gives something which could not be concluded from Proposition 4.1 and Theorem 4.3, we also, to illustrate, give corresponding tables for $(l, m) = (2, 3)$ and $(2, 4)$:

$$C(2, 3) : \begin{array}{c|ccc} r : & 1 & 2 & 3 \\ \hline \Delta_r : & q^2 & q & 1 \end{array}$$

$$C(2, 4) : \begin{array}{c|cccccc} r : & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \Delta_r : & q^4 & q^3 & q^2 & q & 1 \end{array}$$

$$C(2, 5) : \begin{array}{c|cccccccc} r : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \hline \Delta_r : & q^6 & q^5 & q^4 & q^3 & q^4 & q^3 & q^2 & q & 1 \end{array}$$

All values of $d_r$ in these cases are computed by Schubert unions.

This motivates the following definitions:

**Definition 4.9.** For given $l, m$, let $J_r$ be the maximum number of points in a Schubert union spanning a linear space of codimension at least $r$ in the Plücker space, and set $D_r = n - J_r$, and $E_r = D_r - D_{r-1}$, for $r = 1, ..., k$. ($D_0 = 0$.)

We end this section with the following result:

**Proposition 4.10.** For all $l, m$, and $r$ we have

$$d_r \leq D_r.$$  

The fact that the result is obvious does not prevent that it is useful, since we have Schubert unions for any spanning dimension, and since we can calculate the numbers of points contained in them, using Corollary 2.4. In Section 5 we will give general methods to calculate the upper bound $D_r$. Nevertheless, it is an open question whether the upper bound $D_r$ is equal to the true value $d_r$ in the cases not determined by Proposition 4.1, Theorem 4.3 and Proposition 4.6.

# 5. SCHUBERT UNIONS WITH A MAXIMAL NUMBER OF POINTS

In this section we will make significant steps toward finding the $J_r$ and $D_r$ for all $m$ when $l = 2$.

**Definition 5.1.** Fix a dimension $0 \leq K \leq \binom{m}{l}$, and consider the set of Schubert unions $\{U\}_K$ in $G(l, m)$ with spanning dimension $K$.

Then we order the elements $U$ in $\{U\}_K$ according to the lexicographic order on the polynomials $g_U$. In other words $U > V$ if $\deg g_U > \deg g_V$ or $\deg g_U = \deg g_V$, and the coefficient of $g_U$ is larger than that of $g_V$ in the largest degree where the coefficients differ. We call this the order with respect to $g_U$.

For $l = 2$ the elements in $\{U\}_K$ can also be ordered in another way. Recall Definition 3.2 of $M_U = \{m_r, \ldots, m_2, m_1\}$ with $m_r < \ldots < m_2 < m_1$ for each $U$. Then $U > V$ if the largest element $m_1$ of $U$ is larger than that of $V$, or if these elements are equal, if the second largest, $m_2$, is larger than that of $V$, and so on. We call this the order with respect to $M_U$.

**Remark 5.2.** It is clear that for given $m$ and $K$, we obtain the $G$-grid of the maximal element of $\{U\}_K$ with respect to $M_U$ by “filling up as many columns of the $G_{Gl(2,m)}$-grid as we can from the left”. Likewise we obtain the $G$-grid of the minimal element of $\{U\}_K$ with respect to $M_U$ by “filling up as many rows of the $G_{Gl(2,m)}$-grid as we can from the bottom”.
Each non-empty Schubert union, maximal with respect to maximal $M_U$, is a unions of two Schubert cycles as follows:

- $S_{(x,m)} \cup S_{(x+1,y)}$, with $1 \leq x \leq m - 1$ and $1 \leq y \leq m$.
- The non-empty $U$ minimal with respect to $M_U$ are unions of type $S_{(x,x+1)} \cup S_{(a,x+2)}$, with $1 \leq x \leq m - 1$ and $1 \leq a \leq x + 1$.

We have:

**Proposition 5.3.** Assume $l = 2$. Fix a dimension $0 \leq K \leq \binom{m}{2}$, and consider the set of Schubert unions $\{U\}_K$ with spanning dimension $K$. Let $U_1$ and $U_2$ be the maximal and minimal elements in $\{U\}_K$ with respect to $M_U$. Then $U_1$ or $U_2$ is maximal in $\{U\}_K$ with respect to $g_U$. Furthermore, the one(s) that is(are) maximal with respect to $g_U$, also has(have) the maximum number of points over $F_q$ for all large enough $q$.

**Proof.** Given the spanning dimension $K$, let $d = d(K)$ be the maximal Krull dimension for the Schubert unions $\{U\}_K$. This Krull dimension is the crucial ingredient in our argument, since the Krull dimension is the degree of the polynomial $g_U$. We will find the maximal polynomial $g_U$ in the lexicographic order. The fact that the union(s) that is(are) maximal with respect to $g_U$, also has(have) the maximum number of points over $F_q$ for all large enough $q$, is obvious. Our argument is visualized by the $G_{G(2,m)}$-grid, arranged as a set of squares in a triangle defined by

$$G_{G(2,m)} = \{(x,y) | 1 \leq x < y \leq m\}.$$ 

Each point $(a, b) \in G_{G(2,m)}$ defines a Schubert cycle $S_{(a,b)}$ with Krull dimension $d(a,b) = a + b - 3$. Therefore the Schubert cycles with a fixed Krull dimension lie on the diagonal $D_d = \{(x,y) | 1 \leq x < y \leq m, \ x + y - 3 = d\}$.

Let as above

$$G_{a,b} = \{(x,y) \in G_{G(2,m)} | x \leq a, y \leq b\},$$

and

$$G_U = \cup_{S_{(a,b)} \subset U} G_{a,b}.$$ 

By definition of $d = d(K)$, there is a Schubert union $U$ of spanning dimension $K$ with a $G_U$ that contains a point $(a,b)$ on the diagonal $D_d$, i.e. $a + b - 3 = d$, but there is no such union with $G$-grid that contains a point on the diagonal $D_{d+1}$.

The cardinality $c(x,y)$ of a $G$-grid $G_{x,y}$ defines the function

$$c : G_{G(2,m)} \rightarrow \mathbb{Z}, \ (x,y) \mapsto xy - \frac{x(x+1)}{2}.$$ 

The restriction of this function to the diagonal $D_d$ is defined by

$$c(x,d-x+3) = x(d-x+3) - \frac{x(x+1)}{2}, \text{ for } \max\{d+2-m,0\} < x < \frac{d+3}{2}$$

which is clearly quadratic and concave. Therefore it attains its minimum $C(d)$, when $x$ is minimal or maximal, i.e. at one of the end points of the diagonal $D_d$.

Clearly

$$C(d(K)) \leq K \leq C(d(K) + 1) - 1.$$ 

We say that a point $(a,b) \in D_{d(K)}$ is admissible, if $G_{a,b} \subset G_U$ for some Schubert union $U$ of spanning dimension $K$. Equivalently, $(a,b) \in D_d$ is admissible if

$$c(a,b) < C(d + 1),$$

i.e. has less cardinality than any point in the next diagonal.

Next, we characterize the admissible points by which diagonal $D_d$ they belong to.
Lemma 5.4. Consider the diagonal

\[ D_d = \{(x,y)|x+y-3=d\} = \{(x,d-x+3) \mid \max\{d+2-m,0\} < x < \frac{d+3}{2}\} \]

(i) Let \(d \leq m-3\), then the only admissible point on \(D_d\) is \((1,d+2)\), except when \(d = 2\), since the point \((2,3)\) is also admissible.

(ii) Let \(d > m-3\), then \((x,d-x+3)\) is an admissible point on the diagonal \(D_d\) only if \(d + 3 - m \leq x \leq d + 4 - m\) or \(\frac{4}{5} \leq x \leq \frac{d+4}{2}\), i.e. only if it is among the two points with the smallest value of \(x\), or the two points with largest value \(x\), with one exception, namely when \(m = 11\) and \(d = 10\), then the point \((4,9)\) is also admissible.

(iii) If \(m > 10\), then the point \((x,m)\) is admissible, only if \(x \geq m-3\) or \(x \leq \frac{m}{5} + 2\) if \(x + m\) is odd, and only if \(x \geq m-3\) or \(x \leq \frac{m}{5} + 1\) if \(x + m\) is even.

(iv) If \(m > 10\), then the point \((x,m-1)\) is admissible if \(x \geq m-4\) or \(x \leq \frac{m}{5} + 1\) if \(x + m\) is odd, and only if \(x \geq m-4\) or \(x \leq \frac{m}{5} + 2\) if \(x + m\) is even.

Remark 5.5. (a) The lemma implies, in very rough terms, that if you cross out the leftmost column, the two uppermost rows, and the two right-lowest points on each diagonal (it’s really only necessary to cross out the right half of them, in addition to \((2,3)\)) then the “interior” grid that remains contains no admissible points (except \((4,9)\) for \(m = 11\)).

(b) If \(d\) is odd, then part (ii) of the lemma implies that the point of the diagonal \(D_d\) with the next to largest \(x\), is non-admissible, so that the admissible ones must be among the two to the left, and the one to the right.

Proof. (i) This follows from an easy application of the concavity of the cardinality function \(c(x,y)\) along the diagonals. For the smaller values of \(x\) we get

\[ c(2,d+1) = 2d - 1 > c(1,d+3) = d + 1 \quad d \geq 2 \]

except when \(d = 2\), while for the larger values of \(x\) we have \(c(\frac{d+2}{2}, \frac{d+4}{2}) = \frac{d^2+6d+8}{8} > d + 1\) when \(d\) is even, and \(c(\frac{d+1}{2}, \frac{d+5}{2}) = \frac{d^2+8d+7}{8} > d + 1\) when \(d\) is odd.

(ii) Again, by the concavity of the function \(c(x,y)\) restricted to the diagonal \(D_d\), it suffices to compare a few points on each diagonal with the end points on the next. Furthermore the diagonal \(D_d\) has length at least 5 only if \(m/2 > 5\) or \(d/2 - (d - m + 3) \geq 5\), i.e. when \(m > 9\) or \(d < 2m - 14\). So from here on we restrict to these values of \(d\) and \(m\). Technically there is a difference between \(d\) odd and even. More precisely, if \(d\) is even, we compare the value of \(c\) at the endpoints \((d - m + 4, m)\) and \((d + 2, d + 4)\) on \(D_{d+1}\), with its value on \((d-m+5, m-2)\) and \((d-2, d+8)\), namely the points number three from the endpoints on \(D_d\). Thus \((d-m+5, m-2)\) is admissible only if

\[ c(d-m+5, m-2) < c(d-m+4, m) \quad \text{and} \quad c(d-m+5, m-2) < c(\frac{d+2}{2}, \frac{d+6}{2}) \]

The first inequality reduces to \(4m - 15 < 3d\), while the second one yields

\[ 5d^2 + (70 - 16m)d + 12m^2 - 100m + 216 > 0, \]

hence

\[ (2m - 10 - d)(5d - 6m + 20) < 16. \]

In this latter inequality, either the factors have different sign or both factors have small positive value. On the one hand \(d < 2m - 14\), means \(2m - 10 - d > 0\), so only the second factor can be negative. In this case, \(4m - 15 < 3d\) and \(5d - 6m + 20 < 0\), which means \(m \leq 7\), contrary to the above. Thus both factors have small value, i.e. \(2m - 26 \leq d \leq 2m - 14\) and \(\frac{6}{5}m - 4 \leq d \leq \frac{6}{5}m - \frac{4}{5}\). These inequalities are both satisfied only if \(2m - 26 < \frac{6}{5}m - \frac{4}{5}\).
i.e. $m < 32$. It is easily checked that the assertions hold for these small values of $m$ except $m = 11$ and $d = 10$, when the point $(4, 9)$ is also admissible.

The point $(\frac{d-2}{2}, \frac{d+8}{2})$ is admissible only if $$c(\frac{d-2}{2}, \frac{d+8}{2}) < c(d-m+4, m) \quad \text{and} \quad c(\frac{d-2}{2}, \frac{d+8}{2}) < c(\frac{d+2}{2}, \frac{d+6}{2}).$$

The latter inequality yields $d < 12$, and hence $m < d + 3 = 15$, in which case the first inequality is satisfied only if $m = 11$ and $d = 10$, when the point $(4, 9)$ is also admissible.

If $d$ is odd, we compare the value of $c$ at the endpoints $(d-m+4, m)$ and $(\frac{d+3}{2}, \frac{d+5}{2})$ on $D_{d+1}$, with its value at $(d-m+5, m-2)$ and $(\frac{d-1}{2}, \frac{d+7}{2})$, namely the points number three and two from the endpoints on $D_d$. Thus $(d-m+5, m-2)$ is admissible only if $$c(d-m+5, m-2) < c(d-m+4, m) \quad \text{and} \quad c(d-m+5, m-2) < c(\frac{d+3}{2}, \frac{d+5}{2}).$$

The first inequality reduces to $4m - 15 < 3d$, while the second one yields $$5d^2 + (68 - 16m)d + 12m^2 - 100m + 215 > 0,$$ hence $$(5d - 10m + 23)(5d - 6m + 45) > -40.$$ In this last inequality, when $d < 2m - 14$ the first factor is negative, so either the second factor is also negative, or they both have small value. The second factor is negative if $d < \frac{6}{5}m - 9$, which is never satisfied together with $4m - 15 < 3d$ for positive $m$. The second factor is positive with value $5d - 6m + 45 < 41$, while $4m - 15 < 3d$ only if $m < 32$. Again for values $m > 32$, it is straightforward to check the assertion directly.

(iii) The point $(x, m)$ with $x < m$ is the endpoint with minimal $x$ of the diagonal $D_d$ with $d = x + m - 3$. It has cardinality $x_m - \frac{x(x+1)}{2}$, and is admissible only if the cardinality of the other endpoint of the diagonal $D_{x+m-2}$ is strictly bigger. If $x + m$ is even, then this endpoint is $(\frac{x+m}{2}, \frac{x+m+2}{2})$ and the inequality becomes

$$x_m - \frac{x(x+1)}{2} < \frac{x+m}{2} \cdot \frac{x+m+2}{2} - \frac{1}{2} \frac{x+m}{2} \cdot \frac{x+m+2}{2} = \frac{1}{8}(x+m)(x+m+2)$$

which means that $$(x+3)(5x-m-5)+15 > 0.$$ Now the first factor is negative unless $m-3 \leq x \leq m-1$, while the second factor is negative when $x < \frac{1}{5}m + 1$. With $x = \frac{m}{5} + 2$ the inequality is satisfied only if $m \leq 10$. Likewise, with $x = m - 4$, the inequality is satisfied only if $m < 10$.

If $x + m$ is odd, then the other endpoint of $D_{x+m-2}$ is $(\frac{x+m-1}{2}, \frac{x+m+3}{2})$ and the inequality becomes

$$x_m - \frac{x(x+1)}{2} < \frac{x+m-1}{2} \cdot \frac{x+m+3}{2} - \frac{1}{2} \frac{x+m-1}{2} \cdot \frac{x+m+1}{2} = \frac{1}{8}(x+m-1)(x+m+5)$$

which means $$(x+3)(5x-m-7)+16 > 0.$$ The first factor is negative unless $m-3 \leq x \leq m-1$, while the second factor is negative when $x < \frac{1}{5}(m+7)$. Since $x + m$ is odd, $x < m-3$ means $x \leq m-5$. If we set $x = m - 5$, the inequality is satisfied only if $m < 10$. Likewise, if $x = \frac{m}{5} + 2$ the inequality is satisfied only if $m < \frac{15}{12}$, so the result follows for $m \geq 13$. A special check reveals that the inequality does not hold for $m = 11$ and 12, either. By concavity of the function $(x-m+3)(5x-m-7)+16$ then gives that it is negative for all $m$ in the range $\frac{7}{3} < x < m-3$.  


(iv) In this case the computation is similar. The point \((x, m - 1)\) in the next to upper-row has cardinality \(c((x, m - 1)) = x(m - 1) - \frac{x(x + 1)}{2}\) and is admissible only if it has lower cardinality than the lower endpoint of the diagonal above. If \(x + m\) is odd, then the lower end of the diagonal above is \((\frac{x+m-1}{2}, \frac{x+m+1}{2})\), and its cardinality is \(\frac{(x+m+1)(x+m-1)}{8}\). Now the condition
\[
x(m - 1) - \frac{x(x + 1)}{2} < \frac{(x + m + 1)(x + m - 1)}{8}
\]
translates to:
\[
(x - m + 3)(5x - m - 3) + 8 > 0.
\]
The first factor is negative unless \(m - 3 \leq x \leq m - 1\), while the second factor is negative when \(x < \frac{1}{5}(m + 3)\). Since \(x + m\) is odd, \(x < m - 3\) means \(x \leq m - 5\). If we set \(x = m - 5\), the inequality is satisfied only if \(m < 8\). Likewise, if \(x = \frac{m}{5} + 1\) the inequality is satisfied only if \(m < 10\). By the concavity argument the result follows.

If \(x + m\) is even, then the lower end of the diagonal above is \((\frac{x+m-2}{2}, \frac{x+m+2}{2})\), and its cardinality is \(\frac{(x+m-2)(x+m+4)}{8}\). The necessary condition for \((x, m - 1)\) being admissible is
\[
x(m - 1) - \frac{x(x + 1)}{2} < \frac{(x + m - 2)(x + m + 4)}{8}.
\]
This becomes
\[
(x - m + 4)(5x - m - 6) + 16 > 0.
\]
We insert \(m - 6\) which is the largest \(x\)-value smaller than \(m - 4\) making \(x + m\) even and obtain \(m \leq 10\). Likewise, we insert \(x = \frac{m}{5} + 2\) and obtain \(m \leq 12\). Hence the statement holds for \(m \geq 13\). A special check reveals that it holds for \(m = 11, 12\) also. \(\square\)

We return to the proof of Proposition 5.3 and assume that \(U\) is a Schubert union with spanning dimension \(K\), and that \(U\) has the maximal number of points among such unions, i.e. \(g_U\) is maximal in the lexicographical order. Therefore the grid \(G_U\) contains an admissible point \(\alpha = (x, y)\) in the \((d(K))\)-diagonal, i.e. \(x + y - 3 = d = d(K)\). By Lemma 5.4, it suffices to study the following eight cases.

(a) \(d \leq m - 3\)

(b) \(m > 10\) and \(\alpha = (d - m + 3, m)\) with and \(2 \leq d - m + 3 \leq \frac{m}{5} + 2\), i.e. \(m - 1 \leq d \leq \frac{6m - 1}{5}\).

(c) \(m > 10\) and \(\alpha = (d - m + 4, m - 1)\), with \(2 \leq d - m + 4 \leq \frac{m}{5} + 2\), i.e. \(m - 2 \leq d \leq \frac{6m - 2}{5}\).

(d) \(d\) is even and \(\alpha = (\frac{d+2}{2}, \frac{d+1}{2})\).

(e) \(d\) is odd and \(\alpha = (\frac{d+1}{2}, \frac{d+3}{2})\).

(f) \(d\) is even and \(\alpha = (\frac{d}{2}, \frac{d+4}{2})\).

(g) \(m = 11\), and \(G_U\) intersects the \((d(K))\)-diagonal in \((4, 9)\).

(h) \(m \leq 10\).

In each case we consider the residual grid \(\Delta = G_U \setminus G_\alpha\), and find the diagonal \(D_{d'}\) with largest \(d'\) that \(\Delta\) intersects. By the lexicographical ordering of \(g_U\), the value of \(d'\) is determined in a similar fashion as \(d = d(K)\) by the cardinality of \(\Delta\) and the shape of the grid \(G_{(2,m)} \setminus G_\alpha\). Notice that \(U\) is a finite union of irreducible components that are all Schubert cycles. Furthermore, the point \(\alpha\) correspond to a Schubert cycle component \(S_\alpha\) of maximal Krull dimension in \(U\), and that a point \(\beta \in \Delta \cap D_{d'}\) correspond to a Schubert cycle \(S_\beta\) of maximal Krull-dimension among the rest of the irreducible components of \(U\). First of all the cardinality of \(\Delta\) is \(e = K - c(\alpha)\), and by definition of \(d = d(K)\), the Krull dimension \(d'\) is at most \(d(K)\).
Starting with (a), when \( d \leq m - 3 \), then \( \alpha = (1, d+2) \). If \( K \geq d+2 \), then \( d(K) > d+1 = c(\alpha) \), contrary to the assumption, so \( K = c(\alpha) = d+1 \). In particular \( e = K - c(\alpha) = 0 \) and \( U = S_\alpha \).

In case (b), the grid \( G_{G(2,m)} \setminus G_\alpha \) is similar to the original grid \( G_{G(2,m)} \), but with \( \{(x,y)|(d-m+4 \leq x < y \leq m)\} \). Since \( d(K) = d \), the cardinality \( e = K - c(\alpha) \) of \( \Delta \) is the cardinality of the first column of \( G_{G(2,m)} \setminus G_\alpha \), i.e. at most \( m - (d-m+5) = 2m-d-5 \). Therefore we may use the argument of (a) to conclude that \( \Delta = \{(d-m+4,y)|d-m+4 < y \leq e\} \). Notice furthermore that \( U \) clearly is maximal with respect to the lexicographical order on \( M_U \).

In case (c), with \( \alpha = (d-m+4, m-1) \). Since \( d(K) = d \), we first see that the cardinality of \( \Delta \) is less than the cardinality of the upper row of \( G_\alpha \), i.e. \( e = K - c(\alpha) < d - m + 4 \). Compare now the row of points \( R = \{(x,m)|1 \leq x < d - m + 4\} \) with the column \( C = \{(d-m+5,y)|d-m+5 < y < m\} \), both in \( G_{G(2,m)} \setminus G_\alpha \). Notice that both have cardinality at least \( e \), so that for \( \Delta \) to reach the maximal diagonal \( D_{d'} \), it must be contained in one of these. The row \( R \) starts on the diagonal \( D_{m-2} \), while the columns \( C \) starts on the diagonal \( D_{2d-2m+8} \). When \( m > 10 \) and \( d \leq 6m - 2 \), the highest of these diagonals is the first one, since then \( 2d-2m+8 \leq \frac{2m}{3} + 4 < m - 2 \), so in that case \( \Delta \) must be completely contained in the row \( R \).

To see whether \( U \) is maximal with respect to the lexicographical order of \( M_U \), there are essentially two different situations: \( e = d-m+3 \) (maximum possible), and \( e \leq d - m + 2 \). If \( e = d-m+3 \), we are already in case a), since \( (d-m+3, m) \) and \( (d-m+4, m-1) \) are on the same diagonal.

Assume \( e \leq d - m + 2 \). Then we collectively remove the \( d-m+3-e \) top squares of the right column of \( G_\alpha \), and reinstall them horizontally as points \( (e+1, m), (e+2, m), \ldots, (d-m+3, m) \). This amounts to moving squares along \( d-m+3-e \) diagonals, and does not alter the number of \( F_q \)-rational points of the Schubert unions represented by the two grids. But after moving, we have a union which is maximal with respect to the lexicographic order on the \( M_U \), and we are done. (We have \( d-m+3 \) columns to the left filled up completely, and \( K - c(d-m+3, m) \) squares in column \( d-m+4 \)).

Starting with (d), with \( \alpha = \left(\frac{d+2}{2}, \frac{d+4}{2}\right) \) the first row in \( G_{G(2,m)} \setminus G_\alpha \) has length larger than \( d' \), since \( d(K) > d' \). Therefore \( \Delta \) is completely contained in this first row. Furthermore \( U \) is clearly minimal with respect to the lexicographic order of \( M_U \).

Starting with (e) we see that \( K - c(\alpha) = 0 \), since if \( \Delta \) is non-empty, then it could lie to the right of \( \alpha \) in the top row, and \( d(K) \) would have been larger. On the other hand \( U = S_\alpha \) is clearly minimal with respect to the lexicographic order of \( M_U \).

Starting with (f), we see that \( K - C(d(K)) = 0 \) or 1, since if it was at least 2, \( \Delta \) could lie to the right of \( \alpha \), and then \( d(K) \) would have been larger. If \( \Delta \) is empty, there is nothing to do, and if it is one point the optimal solution is that it lies to the right of \( \alpha \). In both cases \( U \) is minimal with respect to the lexicographic order of \( M_U \).

Starting with (g), we have \( m = 11 \), and \( \alpha = (4,9) \). Since \( \alpha \) lies on the diagonal \( D_{10} \) and \( c(4,9)+1 = 27 = C(11) \), we see that this is only an issue when \( K = c(4,9) = 26 \), i.e. \( Delta \) is empty. But a quick calculation reveals that \( S(4,9) \) is not optimal at all for \( K = 26 \). An optimal choice in this case is \( U = S_{(2,11)} \cup S_{(3,10)} \), with \( M_U = \{10, 9, 7\} \), which is maximal with respect to the lexicographical order. Hence the assertion holds in all cases.

The cases \( m \leq 10 \) are entirely similar. The check that no case occurs that is not of one of the kinds above is left to the reader. \( \square \)

Remark 5.6. From Proposition 5.3 it is clear that for each spanning (co)dimension we only need to check two Schubert unions to find one which is maximal with respect to \( g_U \).
In the tables below we utilize this fact, and indicate with an \( L \) (go left) if we may use the union maximal with respect to \( M_U \), with an \( R \) (go right) if we may use the union minimal with respect to \( M_U \), and with \( LR \) if and only if we may use both. The spanning codimension is \( r = \binom{m}{2} - K \).

C(2,7):

| Codim  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|----|
| Direction | LR | LR | LR | LR | R | R | R | R | LR | L |

| Codim. | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|--------|----|----|----|----|----|----|----|----|----|----|----|
| Direction | R | LR | L | L | L | L | L | LR | LR | LR | LR |

C(2,9):

| Codim. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|--------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| Direction | LR | LR | LR | LR | R | R | R | R | R | R | R | R | R |

| Codim. | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|
| Direction | R | R | R | R | R | LR | L | L | L | L | L | L |

| Codim. | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|
| Direction | L | L | L | L | L | L | L | LR | LR | LR | LR | LR |

C(2,10):

| Codim. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|--------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| Direction | LR | LR | LR | LR | R | R | R | R | R | R | R | R | R | R | R | R |

| Codim. | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Direction | R | R | R | R | LR | L | R | R | R | LR | L | L | L | L |

| Codim. | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Direction | L | L | L | L | L | L | L | L | L | L | L | L | LR | LR | LR | LR |

Each table starts and ends with 4 occurrences of \( LR \). This is because in the three largest and the three smallest spanning dimensions \( K \) there is only one Schubert union, and because we have only two Schubert unions with spanning dimension 3, namely \( S_{(2,3)} \), in projective terms a \( \beta \)-plane, or \( S_{(1,3)} \), an \( \alpha \)-plane. Both have \( q^2 + q + 1 \) points. In codimension 3 we have the duals of these two, of course also with the same number of points.

From the tables for \( G(2,8) \) (which is not listed) and \( G(2,9) \) one can conclude without further computations that the \( E_r \) are always monomials of type \( q^i \) in these cases (See Question (Q7) of Section 5). That is because we never jump directly from an \( R \) to an \( L \) or vice versa in these cases, we always go via an \( LR \). For \( m = 7 \) there is a jump between \( L \) and \( R \) between codimensions 10 and 11, but a calculation reveals that \( J_{10} - J_{11} = q^5 \). For \( m = 10 \) we observe the fatal jump from \( R \) to \( L \), passing from codimension 22 to 21. Here \( E_{22} \) is not a monomial in \( q \). As opposed to the tables above it, the one for \( (l, m) = (2, 10) \) is not symmetric in \( L \) and \( R \).

**Corollary 5.7.** For \( G(2, m) \) and any \( m \): There are no Schubert unions that are proper unions of more than two Schubert cycles, and that contain the maximal number of \( F_q \)-rational points, given its spanning dimension.
Proof. This is not a corollary of Proposition 5.3, which only says that for each spanning dimension \( K \), there exists a Schubert union of a special kind containing the maximum number of points. On the other hand the proof of Proposition 5.3 gives the corollary as follows: For an optimal Schubert union we have to be in one of the situations (a), (b), (c), (d), (e), (f) or (g) described in that proof. In each case the refined analysis gave that an optimal union would have to look in a specific way. Tracing the arguments, we see that it is obvious that the optimal unions are proper unions of one or two cycles in all cases (a), (b), (c), (d), (e) and (f).

Case (g) \((m = 11, K = 26)\) is checked explicitly. \(\square\)

Remark 5.8. One might ask: For a given \( l, m \) and spanning dimension \( K \): Do there exist Schubert unions with a maximal number of points among those of that spanning dimension that are proper Schubert unions of at least \( l + 1 \) Schubert cycles. Corollary 5.7 says: “No, if \( l = 2 \).” On the other hand it is clear that if \( l \geq 3 \), then there are optimal Schubert unions that are proper unions of more than two Schubert cycles. We will see, in the tables for \( G(3, 6) \) in Section 9 that \( S_{(1,5,6)} \cup S_{(2,3,6)} \cup S_{(3,4,5)} \), is among the ones with a maximal number of points for \( G(3, 6) \) and \( K = 15 \).

We now will investigate, for each \( m \), the range of those spanning dimensions \( K \), where we will use the Schubert union \( U \) with maximal \( M_U \) to compute the maximum number of \( F_q \)-rational points, and the range where we will use the \( U \) with minimal \( M_U \) to do the same. We will give some general results. In fact, we are well on our way, through Lemma 5.4 above. We never used part (iii) of that result to prove Proposition 5.3, but we will use part (iii) now.

Proposition 5.9. (i) If \( d(K) > 1.2m - 1 \), then the Schubert union \( U \) with minimal \( M_U \) will have a maximal number of \( F_q \)-rational points.

(ii) If \( d(K) \leq 1.2m - 5 \), then the Schubert union \( U \) with maximal \( M_U \) will have a maximal number of \( F_q \)-rational points.

Proof. In case (i) we will prove that the union with maximal \( M_U \) is impossible, unless \( d(K) \geq 2m - 7 \), and for these \( d(K) \) the matter is obvious. In case (ii) we will prove that the union with minimal \( M_U \) is impossible, for diagonals that intersect the top row of \( G_U \). For the remaining ones the result is obvious.

In case (i), to be able to use the union with maximal \( M_U \), a necessary condition is that the point \( \alpha = (x, m) \) where \( G_U \) intersects the diagonal \( x + m - 3 = d(K) \) is admissible. But from Lemma 5.4 (iii) we know that \( \alpha \) is admissible only if \( x \leq 0.2m + 2 \), or \( x \geq m - 3 \). Hence, among the two unions in question, only the \( U \) with minimal \( M_U \) is possible for \( d(K) > (0.2m + 2) + (m - 3) = 1.2m - 1 \), and \( d(K) \leq (m - 4) + m - 3 = 2m - 7 \). But for \( d(K) \geq 2m - 6 \) a quick glance at a diagram reveals that the \( U \) with minimal \( M_U \) is optimal.

In case (ii) we perform the same sort of calculation. To handle diagonals that intersect the top row we compare the cardinality \( (x + 1)m - \frac{(x + 1)(x + 2)}{2} \) of a point \( (x + 1, m) \) in the top row with that of the bottom-right point in the diagonal below it. That cardinality is \( \frac{(x + m + 1)(x + m - 1)}{8} \) if we have a point which is rightmost in its row, and \( \frac{(x + m + 2)(x + m)}{8} - 1 \) if it isn’t. In the first case we obtain that the cardinality \( c(x + 1, m) \) is the smaller one if \( x \leq 0.2m - 1.4 \) which gives \( d(K) = (x + m - 3 \leq 1.2m - 4.4 \). In the second case we obtain that \( x \leq 0.2m - 1.4 \), corresponding to \( d(K) = (x + m - 3 \leq 1.2m - 5 \) is enough For diagonals that intersect the left side of \( G_{G(2,m)} \) the result is obvious. \(\square\)
Remark 5.10. (i) The results in Proposition 5.9 are not best possible, direct calculations for each \( m \) will often give results that are a little sharper. The rough, but essential, picture is that we change from Schubert unions that are maximal to unions that are minimal, with respect to \( M_U \), where we reach a diagonal which intersects the upper edge of the \( G_{G(2,m)} \) about 20\% of the way from the left corner to the right one.

(ii) If one prefers bounds on \( K \) instead of on \( d(K) \), one can do as follows. The argument above showed: For \( x \leq 0.2m - 2 \) (corresponding to \( d \leq 1.2m - 5 \)) the points \( (x, m) \) in the top row of \( G_{G(2,m)} \) are contained in \( G_U \) for unions \( U \) which are optimal for spanning dimensions \( K \) with \( d(K) = x + m - 3 \). Let \( a \) be the integral value \( \lfloor 0.2m - 2 \rfloor \). Then we can fill up the \( a \) first columns with squares, and be certain that the grids thus formed, are associated with unions, maximal with respect to \( g_U \). This gives: For \( K \leq \sum_{i=0}^{a-1} m - (a + 1) \) we can always find maximal unions with respect to \( g_U \) that are maximal with respect to \( M_U \). Since \( 0.2m - 3 \leq a \leq 0.2m - 2 \), this number is at least \( 0.18m^2 - 2.7m - 1 \). But \( k = \binom{m}{2} \) is \( 0.5m^2 - 0.5m \), so we see that for about 36\% of the \( K \) we can be sure to “stick to the left” when \( m \) is big. But a similar argument can be used, using Proposition 5.9 (ii), and then we see that for about 64\% of the \( K \) we can be sure to “stick to the right”.

This gives rise to the following result for the \( G(2,m) \):

Proposition 5.11. For every \( \epsilon > 0 \), there exists a natural number \( M \), such that if \( m > M \), then

(i) If \( K \leq 0.36k - \epsilon \), then he \( U \) with maximal \( M_U \) is maximal with respect to \( g_U \), and the \( U \) minimal with respect to \( M_U \) is not maximal with respect to \( g_U \) unless \( K \geq k - 3 \).

(ii) If \( K \geq 0.36k + \epsilon \), then he \( U \) minimal with respect to \( M_U \) is maximal with respect to \( g_U \), and the \( U \) maximal with respect to \( M_U \) is not maximal with respect to \( g_U \) unless \( K \leq 3 \).

Remark 5.12. (i) Let \( l = 2 \). For every \( K \) between 0 and \( k = \binom{m}{2} \) we know that if \( U \) has spanning dimension \( K \) and is minimal with respect to \( M_U \), then the dual of \( U \) has spanning dimension \( k - K \) and \( U^* \) is maximal with respect to \( M_U \). From Proposition 5.11 we see that in the range between \( 0.36k + \epsilon \) and \( 0.64m - \epsilon \), say in the interval \([0.37k, 0.63k]\), the \( U \) which is minimal with respect to \( M_U \), but not its dual, will be maximal with respect to \( g_U \) if \( m \) is large enough. For \( l = 2 \), the smallest \( m \) such that there exists an \( K \) with a Schubert union \( U \), minimal or maximal with respect to \( M_U \) and maximal with respect to \( g_U \), such that its dual \( U^* \) is not maximal with respect to \( g_U \) (for spanning dimension \( k - K \)), is \( m = 10 \) (and \( K = 22 \)).

(ii) Set \( l = 2 \). Proposition 5.11 gives that the percentage of the \( r \) such that \( E_r \) is not of the form \( q^r \), goes to zero, as \( m \) grows to infinity, (\( r \) must be very close to \( 0.64k \), for smaller \( r \) we “stick to the left side”, for bigger ones we “stick to the right side”).

Remark 5.13. Let us interpret Proposition 5.11 in a continuous setting. Study the triangle with corners \((0,0),(0,1),(1,1)\). This is the “limit” of \( G_{G(2,m)} \) scaled down a factor \( m \) in all directions as \( m \) goes to infinity. Look at the tetragon \( T_x \) with corners \((0,0),(x,x),(x,1),(0,1)\). This tetragon has area \( A = x - x^2 \). We also study the triangle \( P_y \) with corners \((0,0),(y,y),(0,y)\) and area \( A = \frac{y^2}{2} \). Reversing the area formulas we get \( x = 1 - \sqrt{1 - 2A} \) for the tetragon, and \( y = \sqrt{2A} \) for the triangle. Now we look at diagonals of the form \( x + y = d \). The largest \( d \) for which the trapes \( T_x \) intersects such a diagonal is \( d_1(A) = 1 + x = 2 - \sqrt{1 - 2A} \), where \( A \) is the area of the trapes. The largest \( d \) for which the triangle \( P_y \) intersects such a diagonal is \( d_2(A) = 2y = 2\sqrt{2A} \), where \( A \) is the area of the triangle. The decisive criterion for whether it is optimal (with respect to the lexicographical ordering on the \( g_U \)) is “how much \( d \)” you can obtain with a given area.
A. It is now easy to see that $d_1(A) > d_2(A)$ for $0 \leq A < 0.18$, and the values are equal for $A = 0.18$ and $A = 0.5$, while $d_1(A) < d_2(A)$ for $0.18 < A < 0.5$. Since the area of the whole triangle is $\frac{1}{2}$, the (exact) value 0.18 of course corresponds to 36% of the area.

6. Some questions and answers about the Grassmann codes $C(l, m)$

For fixed $l, m, q$ let $C(l, m)$ be the Grassmann code over $F_q$ described in Section 3. Recall the invariants $H_r, \Delta_r, J_r, D_r, E_r$ introduced in Definitions 4.2, 4.8, and 4.9 inspired by Proposition 4.1 Theorem 4.3 Example 4.5 Proposition 4.6 and the results of Section 5 we now will formulate some natural questions, which we will also comment on briefly:

For each $l, m, q$ we obviously have:

$$\sum_{r=1}^{k} \Delta_r = d_k = n.$$  

Here $n$ and $k$ are the word length and dimension of $C(l, m)$ as before. Moreover it is clear that $n$ is the sum of $k = \binom{m}{l}$ monomials of type $q^i$. For each $l, m$ one may raise the following questions:

- (Q1) Are the $d_r$ always sums of $r$ monomials of type $q^i$, for $r = 1, \ldots, k$?
- (Q2) Is $\Delta_r$ always a monomial of the form $q^i$?
- (Q3) Is it true that:

$$\Delta_r(q) = q^{l(m-l)} \Delta_{k+1-r}(q^{-1}),$$

for all $C(l, m)$, and all $r$? This in turn implies that if the answer to question (Q2) is (partly) positive, and $\Delta_r = q^i$ for some $i$, then $\Delta_{k+1-r} = q^{l(m-l)-i}$.

- Answers to (Q1), (Q2), (Q3): Affirmative for $(l, m) = (2, 3), (2, 4), (2, 5)$ by Proposition 4.1 Theorem 4.3 and Proposition 4.6. In other cases we do not know the answers for all $r$ (affirmative for the smallest and biggest $r$).

Question (Q3) is inspired by Proposition 2.21 and the following question:

- (Q4) If $H_r$ is computed by a linear section $H$ of $G(l, m)$, is then $H_{k-r}$ always computed by $D(H)$ in the sense of Definition 2.3?

- Answer: We do not know.

- (Q5) Is it true that $J_r = H_r$, and therefore $D_r = d_r$, and $E_r = \Delta_r$, for all $l, m, r$?

- Answer: Affirmative for $(l, m) = (2, 3), (2, 4), (2, 5)$. In other cases we do not know the answers for all $r$ (affirmative for the smallest and biggest $r$).

Taking into account the possibility that the answer to question (Q5) is no, we may phrase similar questions as (Q1,4) with the $J_r, D_r, E_r$ replacing $H_r, d_r, \Delta_r$, respectively:

- (Q6) Are the $D_r$ and $J_r$ always sums of $r$ monomials of type $q^i$, for $r = 1, \ldots, k$?

  - Answer: Affirmative for all $(l, m)$ by Proposition 2.4.

- (Q7) Is $E_r$ always a monomial of the form $q^i$?

  - Answer: Affirmative for $(l, m) = (2, m)$, for $m \leq 9$, and $(l, m) = (3, 6)$. Negative for some $r$ for $(2, m)$, and $m = 10$ or $m$ big enough. We have not performed further investigations.

- (Q8) If $J_r$ is computed by a Schubert union $S_U$, is $J_{k-r}$ then computed by $S_{U}$?

  - Answer: Affirmative for $(l, m) = (2, m)$, for $m \leq 8$, and $(l, m) = (3, 6)$. Negative for $(2, m)$, and $m = 10$ or big enough. We have not performed further investigations.

- (Q9) Is it true that:
For all $C(l, m)$, and all $r$? This in turn would imply that if the answer to question (Q7) is (partly) positive, and $\Delta_c = q^i$ for some $i$, then $\Delta_{k+1-r} = q^{l(m-l)-i}$.

- Answer: Affirmative for $(l, m) = (2, m)$, for $m \leq 9$, and $(l, m) = (3, 6)$. Negative for $(2, m)$, and $m = 10$. We have not performed further investigations.

**Remark 6.1.** It follows from the results of Section 5 that all questions, except possibly (Q3) have affirmative answers for $l = 2$ and $m \leq 5$. The affirmative answers to (Q7), (Q8), (Q9) for $(l, m) = (2, 6), (2, 7), (2, 8), (3, 6)$ are due to explicit investigations. See the tables in the appendix. For $(l, m) = (2, 9)$ it is at least clear that (Q7) and (Q9) have affirmative answers. See Remark 5.6. The negative parts of the answers to these questions follow essentially from Remark 5.12. See Remark 5.12(ii) though, concerning (Q7). For $(l, m) = (2, 10)$ we see from Remark 5.6 and explicit calculations that $E_22 = J_{21} - J_{22} = q^9 + q^8 - q^6$, so (Q7) has a negative answer. Moreover $E_24 = J_{23} - J_{24} = q^6$, and hence (Q9) also has a negative answer.

Detailed descriptions of Schubert unions for $(l, m) = (2, 4), (2, 5), (2, 6), (3, 6)$ are given in the Appendix as illustrations.

From the observations above we may conclude:

**Proposition 6.2.** Neither of the questions question (Q7), (Q8), and (Q9) do always have affirmative answers, and questions (Q1), (Q2), (Q3), (Q4), and (Q5) do therefore not simultaneously have affirmative answers for all $l, m, r, q$.

### 7. Linear section with maximal Krull dimension

Inspired by the questions in the preceding paragraphs we may ask 3 addition questions for fixed $(l, m)$, and $K$ in the range between 1 and $k$:

- (q1) What is the maximal Krull dimension of a component of a linear section of $G(l, m)$ with a subspace of Plücker space of (projective) dimension $K - 1$?
- (q2) What is the maximal Krull dimension $d(K)$ of a component of a linear section of $G(l, m)$ with a subspace of Plücker space of (projective) dimension $K - 1$, spanned by a Schubert union?
- (q3) Are the answers to (q1) and (q2) identical?

The last question is obviously an analogue of, and weaker version of, (Q5) from Section 6. It is clear that the analysis in Section 5 in particular the proof of Proposition 5.3 gives the answer to (q2) for $l = 2$.

For each $d \geq m - 2$ let $c_1(d)$ be the cardinality of the upper point of the diagonal corresponding to Krull dimension $d$, that is of $(d - m + 3, m)$. This is:

$$c_1(d) = (d - m + 3)m - \frac{(d - m + 3)(d - m + 4)}{2} = \frac{4dm - 3m^2 - d^2 + 13m - 7d - 12}{2}.$$

Moreover we let $c_2(d)$ be the cardinality of the lower point of the diagonal corresponding to Krull dimension $d$. This point is $\left(\frac{d+2}{2}, \frac{d+1}{2}\right)$ if $d$ is even and $\left(\frac{d+1}{2}, \frac{d+5}{2}\right)$ if $d$ is odd. Hence $c_2(d) = \frac{d^2 + 6d + 8}{8}$, if $d$ is even, and $c_2(d) = \frac{d^2 + 8d + 12}{8}$, if $d$ is odd. Furthermore we set

$$C(d) = \min\{c_1(d), c_2(d)\}.$$

For $d \leq m - 2$ we set $C(d) = d + 1$.

We obtain from the arguments in Section 5
Proposition 7.1. The maximal Krull dimension $d(K)$ of a component of a linear section of $G(l,m)$ with a subspace of Plücker space of (projective) dimension dimension $K - 1$, spanned by a Schubert union, is the largest $d$ such that $C(d) \leq K$.

Now we will argue that the answer to question (q3) is affirmative, thereby also answering question (q1): Recall the notation from Propositions 2.6 and 2.7 and the text between these two results. For $1 \leq r \leq k$ let $G(r, V_l)$ be the Grassmann variety parametrizing projective $(r - 1)$-spaces in the Plücker space $P$ containing $G(l, m)$. Let $I$ be the incidence variety in $G(l, m) \times G(r, V_l)$ parametrizing inclusion relations of points in $G(l, m)$ and linear sections of $P$ corresponding to points of $G(r, V_l)$. Let $f$ be the projection to the second factor. Let $E$ be the subset of $I$ corresponding to those $x$ of $I$ whose inverse image $f^{-1}(f(x))$ contains a component of maximal dimension among the fibres of $f$. By Exercise II, 3.22 of [H], we see that $E$ is closed in $I$. Moreover the set $F = f(E)$ is closed in $G(r, V_l)$ by Exercise II, 4.4, since $E$ is closed and proper. By the argument preceding Proposition 2.7 the solvable Borel group $B$ acts on $V_{l,r}$, and the set $F$ is closed in $P(V_{l,r})$ in virtue of being closed in $G(r, V_l)$. The set $F$ is stable under the action of $B$ since the property of having a component with maximal fibre dimension under $f$ is invariant under the action of $B$. Since $B$ is irreducible each irreducible component of $F$ is therefore stable under $B$. Hence $B$ acts on each component of $F$, which is closed in $P(V_{l,r})$ where the action is induced by a linear action on $V_{l,r}$. Borel’s Fixed Point Theorem, as quoted for example in [FH], p. 384 (or Theorem 7.2.5 of [SI], or [FH], p. 155, see also Remark 7.3 below), then gives that $B$ must have a fixed point in each component of $F$, so at least one fixed point. But by Proposition 2.7 the fixed points correspond precisely to the projective $(K - 1)$-planes spanned by Schubert unions.

Hence at least one of these special $(K - 1)$-planes have a component of maximal fibre (Krull) dimension under $f$. This gives an affirmative answer to question (q3), and also gives the following result for $l = 2$:

Theorem 7.2. The maximal Krull dimension $d(K)$ of a component of a linear section of $G(l,m)$ with a subspace of Plücker space of (projective) dimension dimension $K - 1$, is the largest $d$ such that $C(d) \leq K$.

Remark 7.3. Borel’s Fixed Point Theorem is valid for algebraically closed fields $F$, and we have now shown that for such fields the linear sections of $G(l,m)$ with a component of the maximal possible Krull dimension can be found among Schubert unions. These are given by linear equations of type $X_{i,j} = 0$, in other words defined over $\mathbb{Z}$ and hence over any subfield of $F$ (here we take only those linear sections given by equations with coefficients in the subfield). Hence Theorem 7.2 is valid also for non-algebraically closed fields.

The affirmative answer to question (q3) is an indication that the following conjecture holds.

Conjecture 7.4. The answer to Question (Q5) of Section 6 is always affirmative. Hence the higher weights of the Grassmann codes are always computed by Schubert unions.

8. Codes from Schubert unions

In earlier sections we have studied the impact of Schubert unions to Grassmann codes in order to make the bound $d_r \leq D_r$ explicit. Now we will study codes made from a Schubert union $S_U$ in the same way as the codes $C(l,m)$ are made from the $G(l,m)$. In other words; For a given Schubert union $S_U$ and prime power $q$ denote the (affine) spanning dimension of $S_U$ by $K_U = K$. Then the Plücker coordinates of all points of $S_U$ have only zeroes in
all the coordinates corresponding to the $k - K$ points of $H_U$, so we delete them. Choose coordinates for each points, and make the corresponding $K$-tuples columns of a $k \times g_U(q)$-matrix $G$. This matrix will be the generator matrix of a code. If we change coordinates for a point by multiplying by a factor, the code changes, but its equivalence class and code parameters do not, so by abuse of notation we denote all equivalent codes appearing this way by $C_U$.

In [HC] it was shown that if $l = 2$, and we simply have a Schubert cycle $S_\alpha$, then the minimum distance $d_1 = d$ of the code is $q^\delta$, where $\delta$ is the Krull dimension of the Schubert cycle. We will use this result to give the following generalization:

**Proposition 8.1.** For a Schubert union $S_U$ in $G(2, m)$, which is the proper union of $s$ Schubert cycles $S_i$ with Krull dimensions $\delta_i$, for $i = 1, \ldots, s$, the minimum distance of $C_U$ is the smallest number among the $\delta_i$.

**Proof.** Let $S_\alpha$ be one of the cycles in the given union with minimal Krull dimension $\delta$. We now intersect $S_\alpha$ with the coordinate hyperplane $X_\alpha$ (restricted to the $K$-space in which $S_U$ sits, if one prefers). Since $\alpha$ is not contained in the $G_\beta$ of any Schubert union $S_\beta$ different from $S_\alpha$ appearing in the union, this coordinate hyperplane contains all these $S_\beta$. By standard arguments there are exactly $q^\delta$ points from $S_\alpha$ that are not contained in this hyperplane (all these points are then of course outside all the other $S_\beta$). Standard argument: If $\alpha = (a, b)$, then this hyperplane cuts out $S_{(a-1, b)} \cup S_{(a, b-1)}$, with exactly one point $(a, b)$ less in its $G$-grid, and by Corollary 2.4 we must then subtract $q^{a+b-3}$ to obtain the number of points. On the other hand it is clear that if we intersect $S_U$ with an arbitrary hyperplane $H$ in $K$-space (or an arbitrary hyperplane in the Plücker space, not containing $S_U$), then there is at least one $S_i$, which is not contained in $H$. Now the maximal number of points of any hyperplane section of $S_i$ is equal to the cardinality of $S_i$ minus $\delta_i$, so there are at least $q^{\delta_i}$ points of $S_i - H$. Hence there are at least $\delta_i$ points of $S_U - H$ also. Hence the maximal number of points of $S_U \cap H$ is $g_U(q) - \delta$, where $\delta$ is the smallest $\delta_i$, and $d = d_1$ is computed by $X_{\alpha_i}$ for such a corresponding $i$. \hfill \Box

We may also mimic the contents of Proposition 4.1 and Theorem 4.3. Let $\alpha$ be such that $S_\alpha$ is one of the Schubert cycles $S_i$ with minimal Krull dimension in $S_U$, and set $\delta = \delta_i$ (not necessarily the degree of $g_U$).

**Proposition 8.2.** $d_r = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}$, for $r = 1, \ldots, s$, where $s$ is largest natural number such that $(a - s + 1), (a - s + 2, b), \ldots, (a - 1, b), (a, b)$ all are contained in $G_U$.

**Proof.** $d_r \geq q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}$ is the Griesmer bound. The opposite inequalities follow if we can exhibit linear spaces with increasing codimension, which intersect $S_U$ in an appropriate number of points. We intersect with:

$$X_{(a,b)} = X_{(a-1,b)} = X_{(a-2,b)} = \ldots = X_{(a-r+1,b)} = 0$$

Then, as intersections we obtain smaller successive Schubert unions. Their cardinalities are determined by Corollary 2.4 and the fact that we peel off points one by one to obtain the successive $G$-grids. \hfill \Box

We also have:

**Proposition 8.3.** Let $b = b_s$ be the largest number appearing in the sequence $\sigma_U = a_1 < a_2 < \ldots < a_s < b_s < \ldots < b_2 < b_1$. Then $d_K = g_U(q)$, and $d_{k-a} = g_U(q) - (1 + q + \ldots + q^{a-1})$, for $a = 1, \ldots, b_s - 1$. 

Proof. $S_U$ contains a projective space of dimension $b_s - 2$. See the proof of Theorem 4.3. □

Of course we also have a relative bound of type $d_r \leq D_r$.

**Proposition 8.4.** Let $S_U$ be a Schubert union in $G(l, m)$, and let $M_r$ be the maximum cardinality of a Schubert union that is contained in $U$, and whose spanning dimension is $r$ less than that of $S_U$. Then $d_r \leq g_U(q) - M_r$.

The proof is obvious.

**Example 8.5.** In the appendix (Section 9) we will list Schubert unions that compute the $d_r$ for the Grassmann code $C(2, 5)$ from $G(2, 5)$. We leave it to the reader to find the full weight hierarchy for all $C_U$, for all 15 non-empty Schubert unions $U$ of $G(2, 5)$, using the results above and the table for $G(2, 5)$ in the appendix.

For $l \geq 3$ the expected result $d = d_1 = q^\delta$ for Schubert cycles has not yet been shown. If it is shown, we see that we can extend it to Schubert unions as in the case $l = 2$, and also a variant of Proposition 8.2 will then follow. A variant of Proposition 4.3 holds already, and we leave it to the reader to formulate it.
9. APPENDIX: TABLES OF SCHUBERT UNIONS

We first give tables of the $E_r$ for $C(2,m)$, for $m = 6, 7, 8$, and for $C(3,6)$. The values are determined using Corollary 2.4.

$C(2,6)$:

$$C(2,6) : \begin{array}{cccccccccccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
 E_r & q^8 & q^7 & q^6 & q^5 & q^4 & q^6 & q^5 & q^4 & q^3 & q^2 & q & 1 \\
\end{array}$$

$C(2,7)$:

$$C(2,7) : \begin{array}{cccccccccccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 E_r & q^{10} & q^9 & q^8 & q^7 & q^6 & q^5 & q^4 & q^3 & q^2 & q & 1 \\
 q^5 & q^4 & q^3 & q^2 & q^5 & q^4 & q^3 & q^2 & q & 1 \\
\end{array}$$

$C(2,8)$:

$$C(2,8) : \begin{array}{cccccccccccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 E_r & q^{12} & q^{11} & q^{10} & q^9 & q^8 & q^7 & q^6 & q^5 & q^4 & q^3 & q^2 & q & 1 \\
 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
 q^5 & q^4 & q^3 & q^2 & q^5 & q^4 & q^3 & q^2 & q & 1 \\
\end{array}$$

$C(3,6)$:

$$C(3,6) : \begin{array}{cccccccccccccccc}
 r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 E_r & q^9 & q^8 & q^7 & q^6 & q^5 & q^4 & q^3 \\
 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
 q^6 & q^5 & q^4 & q^3 & q^2 & q^3 & q^2 & q & 1 \\
\end{array}$$

The expressions in boldface indicate values where $E_r = \Delta_r$ because of Theorem 4.3 (to the right) or Proposition 4.6 (to the left). The expressions not in boldface contribute to upper bounds for “the true values” $d_r$, when adding monomials from left.

As further illustration we give some more detailed tables of Schubert unions:

$G(2,5)$:

| U     | Span | Krull       | $M_U$ | number of points | Maximal |
|-------|------|-------------|-------|------------------|---------|
| $\emptyset$ | 0    | $-1$        | $\emptyset$ | 0                | Yes     |
| $(1,2)$ | 1    | 0           | $\{1\}$ | 1                | Yes     |
| $(1,3)$ | 2    | 1           | $\{2\}$ | $q + 1$          | Yes     |
| $(1,4)$ | 3    | 2           | $\{3\}$ | $q^2 + q + 1$    | Yes     |
| $(1,5)$ | 4    | 3           | $\{4\}$ | $q^3 + q^2 + q + 1$ | Yes |
| $(2,3)$ | 3    | 2           | $\{1,2\}$ | $q^4 + q + 1$ | Yes |
| $(1,4) \cup (2,3)$ | 4    | 2           | $\{1,3\}$ | $2q^2 + q + 1$ | No |
| $(1,5) \cup (2,3)$ | 5    | 3           | $\{1,4\}$ | $q^4 + 2q^2 + q + 1$ | Yes |
| $(2,4)$ | 5    | 3           | $\{2,3\}$ | $q^4 + 2q^2 + q + 1$ | Yes |
| $(1,5) \cup (2,4)$ | 6    | 3           | $\{2,4\}$ | $2q^4 + 2q^2 + q + 1$ | No |
| $(2,5)$ | 7    | 4           | $\{3,4\}$ | $q^4 + 2q^3 + q + 1$ | Yes |
| $(3,4)$ | 6    | 4           | $\{1,2,3\}$ | $q^4 + q^3 + 2q^2 + q + 1$ | Yes |
| $(1,5) \cup (3,4)$ | 7    | 4           | $\{1,2,4\}$ | $q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(2,5) \cup (3,4)$ | 8    | 4           | $\{1,3,4\}$ | $2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(3,5)$ | 9    | 5           | $\{2,3,4\}$ | $q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(4,5)$ | 10   | 6           | $\{1,2,3,4\}$ | $q^6 + q^5 + 2q^3 + 2q^2 + q + 1$ | Yes |
In this table we have listed all non-trivial Schubert unions (for a fixed flag) for $G(2, 5)$. Below we give similar tables for $G(2, 6)$ and $G(3, 6)$. In the column to the right we indicate whether the Schubert union in question has the maximum possible of points among the Schubert unions of that spanning dimension. The (affine) spanning dimension is given in the column marked “Span”.

The dual of a Schubert union with a given $M_U$ in $G(2, m)$ is the Schubert union $V$ with $M_V = \{1, ..., m - 1\} - M_U$. We also remark that even in the cases where $\binom{m}{2}$ is even, there can be no self-dual Schubert unions in $G(2, m)$ (of spanning dimension $\binom{m}{2}/2$), since a set $M_U$ is never equal to its own complement. We shall see below that the situation may be different for $G(l, m)$ with $l = 3$.

$G(2, 6)$:

| U       | Span | Krull | $M_U$       | number of points | Max |
|---------|------|-------|-------------|-----------------|-----|
| $\emptyset$ | 0    | $-1$  | $\emptyset$ | 0               | Yes |
| $(1, 2)$ | 1    | 0     | $\{1\}$    | 1               | Yes |
| $(1, 3)$ | 2    | 1     | $\{2\}$    | $q + 1$         | Yes |
| $(1, 4)$ | 3    | 2     | $\{3\}$    | $q^2 + q + 1$   | Yes |
| $(1, 5)$ | 4    | 3     | $\{4\}$    | $q^3 + q^2 + q + 1$ | Yes |
| $(1, 6)$ | 5    | 4     | $\{5\}$    | $q^4 + q^3 + q^2 + q + 1$ | Yes |
| $(2, 3)$ | 3    | 2     | $\{1, 2\}$ | $q^2 + q + 1$   | Yes |
| $(1, 4) \cup (2, 3)$ | 4    | 2     | $\{1, 3\}$ | $2q^2 + q + 1$  | No  |
| $(1, 5) \cup (2, 3)$ | 5    | 3     | $\{1, 4\}$ | $q^3 + 2q^2 + q + 1$ | No  |
| $(1, 6) \cup (2, 3)$ | 6    | 4     | $\{1, 5\}$ | $q^4 + q^3 + 2q^2 + q + 1$ | Yes |
| $(2, 4)$ | 5    | 3     | $\{2, 3\}$ | $q^3 + 2q^2 + q + 1$ | No  |
| $(1, 5) \cup (2, 4)$ | 6    | 3     | $\{2, 4\}$ | $2q^3 + 2q^2 + q + 1$ | No  |
| $(1, 6) \cup (2, 4)$ | 7    | 4     | $\{2, 5\}$ | $q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(2, 5)$ | 7    | 4     | $\{3, 4\}$ | $q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(1, 6) \cup (2, 5)$ | 8    | 4     | $\{3, 5\}$ | $2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(2, 6)$ | 9    | 5     | $\{4, 5\}$ | $q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(3, 4)$ | 6    | 4     | $\{1, 2, 3\}$ | $q^4 + q^3 + 2q^2 + q + 1$ | Yes |
| $(1, 5) \cup (3, 4)$ | 7    | 4     | $\{1, 2, 4\}$ | $q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(1, 6) \cup (3, 4)$ | 8    | 4     | $\{1, 2, 5\}$ | $2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(2, 5) \cup (3, 4)$ | 9    | 4     | $\{1, 3, 4\}$ | $2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(1, 6) \cup (2, 5) \cup (3, 4)$ | 9    | 4     | $\{1, 3, 5\}$ | $3q^4 + 2q^3 + 2q^2 + q + 1$ | No  |
| $(2, 6) \cup (3, 4)$ | 10   | 5     | $\{1, 4, 5\}$ | $q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ | No  |
| $(3, 5)$ | 9    | 5     | $\{2, 3, 4\}$ | $q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(1, 6) \cup (3, 5)$ | 10   | 5     | $\{2, 3, 5\}$ | $q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ | No  |
| $(2, 6) \cup (3, 5)$ | 11   | 5     | $\{2, 4, 5\}$ | $2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ | No  |
| $(3, 6)$ | 12   | 6     | $\{3, 4, 5\}$ | $2q^6 + 3q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(4, 5)$ | 10   | 6     | $\{1, 2, 3, 4\}$ | $q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(1, 6) \cup (4, 5)$ | 11   | 6     | $\{1, 2, 3, 5\}$ | $q^6 + q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(2, 6) \cup (4, 5)$ | 12   | 6     | $\{1, 2, 4, 5\}$ | $q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(3, 6) \cup (4, 5)$ | 13   | 6     | $\{1, 3, 4, 5\}$ | $2q^6 + 3q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(3, 6) \cup (4, 5)$ | 14   | 7     | $\{2, 3, 4, 5\}$ | $q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$ | Yes |
| $(5, 6)$ | 15   | 8     | $\{1, 2, 3, 4, 5\}$ | $n$ | Yes |
G(3, 6):

| U          | Span | Krull | Max. | Number of points |
|------------|------|-------|------|------------------|
| ∅          | 0    | -1    | Yes  | 0                |
| (1, 2, 3)  | 1    | 0     | Yes  | 1                |
| (1, 2, 4)  | 2    | 1     | Yes  | q + 1            |
| (1, 2, 5)  | 3    | 2     | Yes  | q\(^2\) + q + 1  |
| (1, 2, 6)  | 4    | 3     | Yes  | q\(^a\) + q\(^b\) + q + 1 |
| (1, 3, 4)  | 3    | 2     | Yes  | q\(^2\) + q + 1  |
| (1, 3, 5)  | 5    | 3     | Yes  | q\(^3\) + 2q\(^2\) + q + 1 |
| (1, 3, 6)  | 7    | 4     | Yes  | q\(^a\) + 2q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 4, 5)  | 6    | 4     | Yes  | q\(^a\) + 2q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 4, 6)  | 9    | 5     | Yes  | q\(^a\) + 2q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 4, 6)  | 10   | 6     | Yes  | q\(^b\) + q\(^a\) + 2q\(^d\) + 2q\(^2\) + q + 1 |
| (2, 3, 4)  | 4    | 3     | Yes  | q\(^a\) + q\(^2\) + q + 1 |
| (2, 3, 5)  | 7    | 4     | Yes  | q\(^a\) + 2q\(^b\) + 2q\(^2\) + q + 1 |
| (2, 3, 6)  | 10   | 5     | No   | q\(^a\) + 2q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (2, 4, 5)  | 9    | 5     | Yes  | q\(^a\) + 2q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (2, 4, 6)  | 14   | 6     | No   | q\(^a\) + 3q\(^b\) + 3q\(^d\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (2, 5, 6)  | 16   | 7     | Yes  | q\(^a\) + 2q\(^b\) + 3q\(^d\) + 3q\(^2\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (3, 4, 5)  | 10   | 6     | Yes  | q\(^a\) + q\(^b\) + q\(^d\) + 2q\(^2\) + q + 1 |
| (3, 4, 5)  | 16   | 7     | Yes  | q\(^a\) + 2q\(^b\) + 3q\(^d\) + 3q\(^2\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (3, 5, 6)  | 19   | 8     | Yes  | q\(^d\) + 2q\(^a\) + 3q\(^b\) + 3q\(^2\) + 3q\(^2\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (4, 5, 6)  | 20   | 9     | Yes  | q\(^a\) + q\(^b\) + 2q\(^d\) + 3q\(^e\) + 2q\(^2\) + q\(^2\) + 3q\(^2\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (1, 2, 5) ∪ (1, 3, 4) | 4    | 2     | No   | 2q\(^2\) + q + 1 |
| (1, 2, 5) ∪ (2, 3, 4) | 5    | 3     | Yes  | q\(^a\) + 2q\(^b\) + q + 1 |
| (1, 2, 6) ∪ (1, 3, 4) | 5    | 3     | Yes  | q\(^b\) + q\(^a\) + q + 1 |
| (1, 2, 6) ∪ (1, 3, 5) | 6    | 3     | No   | 2q\(^a\) + q\(^2\) + q + 1 |
| (1, 2, 6) ∪ (1, 4, 5) | 7    | 4     | Yes  | q\(^a\) + 2q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 2, 6) ∪ (2, 3, 4) | 6    | 3     | No   | 2q\(^a\) + 2q\(^2\) + q + 1 |
| (1, 2, 6) ∪ (2, 3, 5) | 8    | 4     | No   | q\(^a\) + 3q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 2, 6) ∪ (2, 4, 5) | 10   | 5     | No   | q\(^a\) + 2q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (1, 3, 5) ∪ (2, 3, 4) | 6    | 3     | No   | 2q\(^a\) + 2q\(^b\) + q + 1 |
| (1, 3, 6) ∪ (1, 4, 5) | 8    | 4     | Yes  | 2q\(^a\) + 2q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 3, 6) ∪ (3, 4, 5) | 12   | 6     | Yes  | q\(^a\) + q\(^b\) + 3q\(^d\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (1, 3, 6) ∪ (2, 3, 4) | 8    | 4     | No   | q\(^a\) + 3q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 3, 6) ∪ (2, 3, 5) | 9    | 4     | No   | 2q\(^a\) + 3q\(^b\) + 2q\(^2\) + q + 1 |
| (1, 3, 6) ∪ (2, 4, 5) | 11   | 5     | No   | q\(^a\) + 3q\(^b\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (1, 4, 6) ∪ (2, 3, 4) | 10   | 5     | No   | q\(^a\) + 2q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (1, 4, 6) ∪ (2, 3, 5) | 11   | 5     | No   | q\(^a\) + q\(^b\) + 3q\(^d\) + 3q\(^2\) + 2q\(^2\) + q + 1 |
| (1, 4, 6) ∪ (2, 3, 6) | 12   | 5     | No   | 2q\(^a\) + 3q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (1, 4, 6) ∪ (2, 4, 5) | 12   | 5     | No   | 2q\(^a\) + 3q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (1, 4, 6) ∪ (3, 4, 5) | 13   | 6     | Yes  | q\(^a\) + 3q\(^b\) + 3q\(^d\) + 2q\(^2\) + q + 1 |
| (1, 5, 6) ∪ (2, 3, 4) | 11   | 6     | Yes  | q\(^a\) + q\(^b\) + 2q\(^d\) + 3q\(^e\) + 2q\(^2\) + q + 1 |
| (1, 5, 6) ∪ (2, 3, 5) | 12   | 6     | Yes  | q\(^a\) + q\(^b\) + 3q\(^d\) + 3q\(^e\) + 2q\(^2\) + q + 1 |
All the tables made so far have been produced, mainly by using Corollary 2.4. The Krull dimension (of the component cycle with biggest such dimension) is of course equal to the degree of \( g_U(q) \), interpreted as a polynomial in \( q \), and this is the polynomial appearing in the column marked “number of points”. Moreover it is well known that the Krull dimension of a Schubert cycle \( S_{(a_1, \ldots, a_t)} \) is \( a_1 + a_2 + \ldots + a_t - \frac{t(t+1)}{2} \), so the Krull dimension can be “read off” both from the leftmost and the rightmost column.

In the table for \( G(3,6) \) above study the 16 rows corresponding to unions of cycles \( S_{(a,b,c)} \) with \( c \leq 5 \). This gives rise to the corresponding table for \( G(3,5) \). But this is isomorphic to \( G(2,5) \). It is an amusing exercise to translate all unions in \( G(3,5) \) to corresponding ones in \( G(2,5) \) and check that the relevant columns of the tables coincide.

**Remark 9.1.** Given two Schubert unions \( U_1, U_2 \) with corresponding polynomials \( g_{U_1}(q) \) and \( g_{U_2}(q) \). The issue of which of the two that gives the highest value for given \( q \) is in principle a different one, for each \( q \). On the other hand, if we order the Schubert unions lexicographically with respect to \( g_U \), then as remarked in Section 5 it is clear that the lexicographic order is the same as the “number of point”-order for all large enough \( q \). In all the examples we have seen up to now, it is clear by inspection that these orders are the same for all prime powers \( q \). Hence the “Yes” and “No” in the “Max.” column can be interpreted in two ways simultaneously (counting points, and ordering with respect to \( g_U \)).
Table of dual pairs of Schubert unions for $G(3, 6)$:

| $U$ | Span | Dual Schubert union | Max. |
|-----|------|---------------------|------|
| $\emptyset$ | 0    | $(4, 5, 6)$         | Yes  |
| $(1, 2, 3)$ | 1    | $(3, 5, 6)$         | Yes  |
| $(1, 2, 4)$ | 2    | $(2, 5, 6) \cap (3, 4, 6)$ | Yes  |
| $(1, 2, 5)$ | 3    | $(1, 5, 6) \cap (3, 4, 6)$ | Yes  |
| $(1, 3, 4)$ | 3    | $(2, 5, 6) \cap (3, 4, 5)$ | Yes  |
| $(1, 2, 6)$ | 4    | $(3, 4, 6)$         | Yes  |
| $(2, 3, 4)$ | 4    | $(2, 5, 6)$         | Yes  |
| $(1, 2, 5) \cup (1, 3, 4)$ | 4    | $(1, 5, 6) \cup (2, 4, 6) \cup (3, 4, 5)$ | No  |
| $(1, 3, 5)$ | 5    | $(1, 5, 6) \cup (2, 3, 6) \cup (3, 4, 5)$ | Yes  |
| $(1, 2, 5) \cup (2, 3, 4)$ | 5    | $(1, 5, 6) \cup (2, 4, 6)$ | Yes  |
| $(1, 2, 6) \cup (1, 3, 4)$ | 5    | $(2, 4, 6) \cup (3, 4, 5)$ | Yes  |
| $(1, 4, 5)$ | 6    | $(1, 5, 6) \cup (3, 4, 5)$ | Yes  |
| $(1, 2, 6) \cup (1, 3, 5)$ | 6    | $(1, 4, 6) \cup (2, 3, 6) \cup (3, 4, 5)$ | No  |
| $(1, 2, 6) \cup (2, 3, 4)$ | 6    | $(2, 4, 6)$         | No  |
| $(1, 3, 5) \cup (2, 3, 4)$ | 6    | $(1, 5, 6) \cup (2, 3, 6) \cup (2, 4, 5)$ | No  |
| $(1, 3, 6)$ | 7    | $(2, 3, 6) \cup (3, 4, 5)$ | Yes  |
| $(2, 3, 5)$ | 7    | $(1, 5, 6) \cup (2, 3, 6)$ | Yes  |
| $(1, 2, 6) \cup (1, 4, 5)$ | 7    | $(1, 4, 6) \cup (3, 4, 5)$ | Yes  |
| $(1, 4, 5) \cup (2, 3, 4)$ | 7    | $(1, 5, 6) \cup (2, 4, 5)$ | Yes  |
| $(1, 2, 6) \cup (1, 3, 5) \cup (2, 3, 4)$ | 7    | $(1, 4, 6) \cup (2, 3, 6) \cup (2, 4, 5)$ | No  |
| $(1, 3, 6) \cup (1, 4, 5)$ | 8    | $(1, 3, 6) \cup (3, 4, 5)$ | Yes  |
| $(1, 4, 5) \cup (2, 3, 5)$ | 8    | $(1, 5, 6) \cup (2, 3, 5)$ | Yes  |
| $(1, 2, 6) \cup (2, 3, 5)$ | 8    | $(1, 4, 6) \cup (2, 3, 6)$ | No  |
| $(1, 3, 6) \cup (2, 3, 4)$ | 8    | $(2, 3, 6) \cup (2, 4, 5)$ | No  |
| $(1, 2, 6) \cup (1, 4, 5) \cup (2, 3, 4)$ | 8    | $(1, 4, 6) \cup (2, 4, 5)$ | No  |
| $(1, 4, 5) \cup (2, 3, 4)$ | 8    | $(1, 2, 6) \cup (3, 4, 5)$ | Yes  |
| $(2, 4, 5)$ | 9    | $(1, 5, 6) \cup (2, 3, 4)$ | Yes  |
| $(1, 3, 6) \cup (2, 3, 5)$ | 9    | $(1, 4, 5) \cup (2, 3, 6)$ | No  |
| $(1, 3, 6) \cup (1, 4, 5) \cup (2, 3, 4)$ | 9    | $(1, 3, 6) \cup (2, 4, 5)$ | No  |
| $(1, 2, 6) \cup (1, 4, 5) \cup (2, 3, 5)$ | 9    | $(1, 4, 6) \cup (2, 3, 5)$ | No  |
| $(1, 5, 6)$ | 10   | $(3, 4, 5)$         | Yes  |
| $(2, 3, 6)$ | 10   | $(2, 3, 6)$         | No  |
| $(1, 2, 6) \cup (2, 4, 5)$ | 10   | $(1, 4, 6) \cup (2, 3, 4)$ | No  |
| $(1, 3, 6) \cup (1, 4, 5) \cup (2, 3, 5)$ | 10   | $(1, 3, 6) \cup (1, 4, 5) \cup (2, 3, 5)$ | No  |

Above we also present a table of Schubert unions and their associated dual Schubert unions for $G(3, 6)$. We have used the methods described in Subsection 2.2 to make the table. All Schubert unions with spanning dimension at most 9 can be found in the left half of the table, and unions with spanning dimension at least 11 can be found on the right side (as duals). For spanning dimension 10 all 6 unions are listed on at least one side.

Remark 9.2. (i) The table reveals a situation different from the case $l = 2$ and \( \binom{n}{l} \) even, where no Schubert union is self-dual. Here we see that both $(2, 3, 6)$ and $(1, 3, 6) \cup (1, 4, 5) \cup (2, 3, 5)$ are self-dual Schubert unions.
(ii) Another fact that can be found from the table is that the conclusion of Corollary 3.6 fails for \( l = 3 \). We see that the dual of \( S_{(1,3,5)} \) is the proper triple union \( S_{(1,5,6)} \cup S_{(2,3,6)} \cup S_{(3,4,5)} \) (and vice versa).

(iii) For this Grassmann varieties we have described in the tables above, a Schubert union has a maximal number of points, given its spanning dimension, if and only if its dual union enjoys the same property, so the “Yes” and “No” in the “Max.”-column of the last table apply to the left and right half of the table simultaneously. The same nice property also holds for \((l, m) = (2, 7)\) and \((2, 8)\), but for reasons of space we do not give the full tables here, from which the shorter lists of the \( E_r \) at the start of this section were deduced.

References

[GL] Ghorpade, S., Lachaud, G. Higher weights of Grassmann Codes, in Coding Theory, Cryptography, and Related Areas (Guanajuato, 1998), Springer Verlag, Berlin/Heidelberg, 122-31 (2000).

[Fu] W. Fulton, Young Tableaux, Student Texts, 35, London Math. Soc. (1991).

[FH] W. Fulton, J. Harris, Representation Theory, Graduate Texts in Mathematics, 129, Springer Verlag (1991).

[GT] Ghorpade, S., Tsfasman, M. Schubert Varieties, Linear Codes, and Enumerative Combinatorics, Preprint (2003).

[HC] H. Chen, On the minimum distance of Schubert codes, IEEE Trans. of Inform. Theory, 46, 1535-38 (2000).

[H] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, Springer Verlag, (1977).

[HT] J.P. Hirschfeld, J. A.Thas, General Galois Geometries, Oxford: Clarendon Press(1991).

[MV] Elisa Montanucci, Rita Vinscenti Characterization of linear codes and classical varieties, Preprint (2004).

[N] Nogin, D.Yu., Codes associated to Grassmannians, in Arithmetic Geometry and Coding theory (Luminy 1993), R. Pellikaan, M. Perret, S.G. Vladut, Eds. Walter de Gruyter, Berlin/New York, 145-54 (1996).

[Sp] T. A. Springer, Linear Algebraic Groups, Progress in Mathematics, 9, Birkhäuser (1981).

Johan P. Hansen, Dept. of Mathematics, University of Aarhus, Bygn. 530, DK-8000 C Aarhus, Denmark

E-mail address: matjp@imf.au.dk

Trygve Johnsen, Dept. of Mathematics, University of Bergen, Johs. Bruns gt 12., N-5008 Bergen, Norway

E-mail address: johnsen@mi.uib.no

Kristian Ranestad, Dept. of Mathematics, University of Oslo, P.O. 1053., N-316 Oslo, Norway

E-mail address: ranestad@math.uio.no