A MODIFIED VARIATIONAL PRINCIPLE IN
RELATIVISTIC HYDRODYNAMICS
II. Variations of the vector field and the projection tensor in the
general case and under definite assumptions
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Abstract

The purpose of the paper is to develop further a projection variational
approach in relativistic hydrodynamics. The approach, previously proposed
in [gr-qc/9908032], is based on the variation of the vector field and the projec-
tion tensor (instead of the given metric tensor) and their first partial deriva-
tives. The previously proved property of non-commutativity of the variation
and the partial derivative in respect to the projection tensor has been used
to find all the variations. Subsequently, motivated by some analogy with the
well-known (3+1) ADM projection formalism, an assumption has been made
about a zero-covariant derivative of the projection tensor in respect to the
projection connection. The combination of the equations for the variations of
the projective tensor with covariant and contravariant indices has lead to the
derivation of an important and concisely written relation: the derivative of
the vector field length is equal to the "twice" projected along the vector field
initial Christoffell connection. The result is of interest due to the following
reasons: 1. It is a more general one and contains in itself a well-known for-
formulae in affine differential geometry for the so called equiaffine connections
(admitting covariantly conserved tensor fields), for which the trace of the con-
nection is equal to the gradient of the logarithm of the vector field length.
2. The additional term is the projected (with the projection tensor) initially
given connection and accounts for the influence of the reference system on
the change of the vector field’s length, measured in this system. 3. The for-
formulae has been obtained within the proposed formalism of non-commuting
variation and partial derivative.

I. INTRODUCTION.

A key object in relativistic hydrodynamics is the energy -momentum ten-
sor, which by the nature of its physical foundations should embody in itself
not only the properties of the gravitational field, but also the mechanical properties of matter (in the monograph [1] called also "a continuum" and having a more concrete meaning - matter should not be described by means of a discrete model). For example, if according to the "continuous" model to each particle of the matter a four-velocity vector \( u \) can be assigned, then the energy-momentum tensor is defined as

\[
T_{ij} \equiv \mu u_i u_j - S_{ij},
\]

where \( \mu \) is the proper density of energy (or mass) and \( S_{ij} \) is the stress tensor, accounting for the action of tension (or surface forces). The idea that such tension forces can contribute to the energy of the system "gravitational field and matter" is an old one. In [2], even in the case of Post-Newtonian approximation, the tension forces have been accounted by taking into consideration the elastic energy of potential deformation, and relating this energy to the large-scale translational and rotational motion of matter in the potential field, created by the gravitational field. In [3] it has been suggested that the stress tensor is nothing else, but the "twice" projected energy-momentum tensor

\[
S_{ik} \equiv p_{il} T_{lm} p_{mk}
\]

where \( p_{il} \) denotes the so-called projection tensor

\[
p_{il} \equiv \delta_{il} + \frac{1}{c^2} u_i u_k
\]

Since the four-vector \( u \) is defined as \( u_i \equiv \frac{dx_i}{dt} \) and \( dt \equiv \sqrt{1 - \frac{v^2}{c^2}} \) is the proper time in the comoving frame of the particle, \( u \) can naturally be assumed to be a time-like, unit-normalized vector. In [4], the more interesting consequence from this assumption has been shown - the projection tensor \( p_{il} \) can be regarded as a metric tensor in the subspace, orthogonal to the vector \( u \).

It is important to realize that although this is conceived as something very natural, the metricity of the projection tensor turns out to play a crucial role, and not surprisingly, it is present in many papers, including most recent ones. For example, in [5] a set of three Lagrangian space-dimensional coordinates \( \xi^a = \xi^a(x) \) \( (a = 1, 2, 3) \) had been assigned, and a metric in the three-dimensional "material" space is introduced

\[
G^{ab} := g^{\mu \nu} \xi_\mu \xi_\nu, \mu, \nu = 1, 2, 3, 4
\]
where $\xi^a_\mu \equiv \partial_\mu \xi^a$. The metric $G^{ab}$ enables one to measure the distances between adjacent particles in the medium. In fact, the idea about the "material" metric is an old one [6] and the analogy originates from continuous mechanics, where Christoffel symbols have been introduced to describe the medium’s mechanical properties in curvilinear coordinates. In the case of gravitational theory, this turns out to be a second metric, besides the initially given space-time metric. Similar to [4], in [5] it has also been assumed that the velocity vector field $u$ is a \textit{future oriented, time-like and unit-normalized vector}, orthogonal to $\xi^a_\mu$:

$$u^\mu \xi^a_\mu \equiv 0 \quad \text{and} \quad u^\mu u_\mu = -1$$

Note, however that although $\xi^a_\mu$ is defined on a three-dimensional (coordinate) subspace, orthogonal to the vector field, it is not yet a projection tensor. But this might happen if an idempotent endomorphism of the tangent space onto the same space is defined, according to which

$$p^\alpha_\beta \equiv p^\alpha_\gamma$$

which is of course a typical property of the projection operator and is also a consequence of the defining equation (3). In the case when the vector field is not unit-normalized and has a length $e = u^\mu u_\mu$, the projection tensor should be defined as

$$p_{\mu\nu} \equiv g_{\mu\nu} - \frac{1}{e} u_{\mu} u_{\nu}$$

and the above definition will be used further in the paper.

It is important to mention that the projection tensor definition in [7] and in the subsequent papers [8-11] is no longer related to any time-like velocity vector field, unlike the approach for example in [12-13]. In these papers, in view of tracing the time dependence of the relativistic internal energy and projecting the energy-momentum conservation condition $\nabla_\beta T^\alpha_\beta \equiv 0$ onto a three-dimensional space-like hypersurface, such an assumption has naturally been preserved. But in a more general context and from a mathematical point of view [8, 9], the definition is given in terms of a $(4 + n)$—dimensional manifold $M$ with a tensor field $\gamma$, assigning a mapping of the tangent space $T_x$ onto the same space, a set of linearly independent deformation form fields $\theta^{(A)}$, and also an additional tensor field $H$, called a gauge. The entity $(M, \gamma, \theta^{(A)})$
is known as a deformation. For all of them the following conditions are fulfilled for $\gamma$ and $\theta^{(A)}$:

$$\gamma \theta^{(A)} \equiv 0$$  
(8)

$$d\theta^{(A)} \equiv 0$$  
(9)

and the following conditions - for the gauge $H$ (can be identified with the projection tensor):

$$H^a_b H^b_c \equiv H^c_a$$  
and  
$$H_{ab} H^{bc} \equiv H^c_a$$  
(10)

$$H^a_b \gamma^{bc} \equiv \gamma^{ac}.$$  
(11)

More important, however, is that only in a special coordinate chart - the aligned chart, it is possible to have

$$H^\alpha_\beta \equiv \delta^\alpha_\beta.$$  
(12)

In other words, if $H_{\alpha\gamma} \equiv p_{\alpha\gamma}$, because of (10) and (12) the equality

$$p_{\alpha\gamma} \gamma^{\gamma\beta} \equiv p^\beta_\alpha \equiv H^\beta_\alpha \equiv \delta^\beta_\alpha$$  
(13)

will be fulfilled, meaning that only in the particular aligned coordinate chart the projection tensor has a well-defined inverse one. In the general case (as can be seen from (7)), an inverse projection tensor will not exist, but in some special case it can be achieved. A typical example is the well-known Arnowitt-Deser-Misner (ADM) (3+1) decomposition of space-time [14, 15]. The ADM approach is nothing else but a special kind of a projection approach, in which identification of components of the vector field (in ADM notations - $N_i$) with certain components of the initial metric or the projection tensor is made according to the following substitutions [14, 15] ($i, j = 1, 2, 3$)

$$g_{\alpha\alpha} \equiv - (N^2 - N_i N^i)$$  
$$g^{\alpha\alpha} \equiv - \frac{1}{N^2}$$  
(14)

$$g_{ij} \equiv p_{ij}$$  
$$g^{ij} \equiv p^{ij} - \frac{N^i N^j}{N^2}$$  
(15)
Since the formulae are well-known and widely applicable, it is more important to understand the advantages of such a substitution, from the standpoint of a more general projection theory, which is believed that should exist (for some classical aspects, see also [16, 17]). First, note that from (15) $p_{ij}p^{jk} \equiv \delta^k_i$, which follows also from $N_iN^i \equiv N^2$ and $N_iN^k \equiv \delta^k_i$. As a result, the projection tensor has a well defined inverse one. The last fact, although not commented at all, has been correctly noted also in [18]. Second, from (15) it follows that if the initially given metric has zero covariant derivative, then the projection tensor has also this property, and moreover, this property is valid for the projection tensor with covariant and contravariant indices. But it should be stressed that this is a consequence of the substitutions (14-16), due to which the ADM formalism should be regarded as a partial case of a more general formalism.

Note also that in the most general case, the definition of a metric (also - the projection metric) is not related to the existence of an inverse tensor and in this sense it has a more restricted meaning. For example, within the class of the so called theories with covariant and contravariant metrics and affine connections [19], a covariant and a contravariant projection metric can be defined, in spite of the fact, that an inverse projection tensor may not exist. Perhaps it should be mentioned that in [20-22] in an analogous to (4) way a projection metric on a three dimensional subspace has been introduced, but with assigned on it space-like Lagrangian coordinates. Of course, this assumption is needed in this particular case of application of the formalism of Lagrangian coordinates, but it is not a consequence of the projection approach in gravitational theory in its most general aspects. As it can be seen [19], for the introduction of a projection metric, defined by means of the action of the contraction operator on two vector field from one and the same vector space, the only assumption that is needed is about the existence of a non-null vector field, defined over some differentiable manifold. The applied projectional approach in this paper will use this definition, allowing a vector field of a more general type. Moreover, the manifold may be assumed to be of arbitrary dimension, and the projection formalism can be therefore generalized for the case of a $p-$ dimensional submanifold, orthogonal to the complimentary $(n - p)-$ submanifold [23]. Such a model can turn to be
useful in multidimensional cosmological models, including "brane" physics. However, this remains out of the scope of the present paper.

II. OUTLINE OF THE METHOD IN THE PRESENT PAPER

The above analyses aims to show the necessity of a more general projection approach, based on the following assumptions:

1. The vector field $u$ is assumed to be an arbitrary, non-null vector field.

2. The projection tensor does not have an inverse one, which means that as a result of (7) and the existence of an inverse initial metric tensor $g^{\alpha\beta}$, the following relation between the covariant and the contravariant components of the projection tensor and the vector field is fulfilled:

$$p_{mk}p^{ik} \equiv \delta_m^i - \frac{1}{e} u^i u_m. \quad (17)$$

Two more relations shall constitute the basic equations, extensively used further in the text - the relation, expressing the orthogonality of the vector field $u$ in respect to the projective tensor, written in the form:

$$\frac{1}{e} u^k u_i p_{km} \equiv 0, \quad (18)$$

and also the relation for the Riemannian initial metric $g_{ik}$ - zero covariant derivative $\nabla_{\alpha} g_{\mu\nu} \equiv 0$ ($\nabla_{\alpha}$ - a covariant derivative in respect to the initial Christoffel connection $\Gamma_{ij}^s$). It can be expressed as an nonlinear equation between the vector field $u$, the projective tensor field $p_{ik}$ and their first partial derivatives:

$$\partial_j g_{ik} \equiv g_{s(k} \Gamma_{ij)}^s \quad (19)$$

and a separate equation for the metric tensor with contravariant components:

$$\partial_j g^{ki} \equiv -g^{s(k} \Gamma_{sj)}^i. \quad (20)$$

It follows also that

$$\nabla_{\alpha} p_{\mu\nu} = -\nabla_{\alpha}(\frac{1}{e} u_\mu u_\nu) \neq 0. \quad (21)$$
Evidently, this covariant derivative will be zero if $\nabla_{\alpha} u_{\mu} = 0$, but the last would mean that a special kind of transport of the vector field $u$ has been assumed. Of course, the covariant derivative of $p_{\mu\nu}$ (denoted by $\nabla$) in respect to the so called projection connection $\tilde{\Gamma}_{\alpha\mu}^{\gamma}$ is also different from zero, where $\tilde{\Gamma}_{\alpha\mu}^{\gamma}$ is defined in the standard way as

$$\tilde{\Gamma}_{\alpha\mu}^{\gamma} \equiv \frac{1}{2} p^{rs} (\partial_{\alpha} p_{\mu s} + \partial_{\mu} p_{\alpha s} - \partial_{r} p_{\alpha \mu}).$$

Note that the projection connection does not have the properties of the Christoffell connection because it is defined by means of a non-metrical projection tensor, not having an inverse one.

In [24-preceeding paper] a projection approach has been proposed, and also some reasoning from a physical point of view for constructing a variational formalism for a Lagrangian of the kind

$$L = L(p_{mk}, p^{mk}, \partial_j p_{mk}, \partial_j p^{mk}, u_k, u^k, \partial_j u_k, \partial_j u^k)$$

has been given. The Lagrangian (23) is in fact derived from the standard gravitational Lagrangian, decomposed according to (7). A basic feature of this projection variational approach is that an account has been taken of the form variations of all vector and tensor quantities with covariant and contravariant indices and their first partial derivatives. The approach is similar to that in [25-27], where variations have been taken also of (generally - non-metric) tensor fields of a mixed type with covariant and contravariant indices. In the present paper, the role of the non-metric tensor field is played by the projection tensor and its first partial derivatives, but another choice is made of taking the variations of the projection tensor with covariant and separately - with contravariant components. The choice obviously is dictated by the gravitational Lagrangian decomposition. Unlike the investigation in [25-27], where the non-metric tensor fields were not specified or just assumed to be the components of the affine connection, here the non-metric fields are the projected components $p_{\mu\nu}$ of the metric tensor $g_{\mu\nu}$ in respect to the vector field $u$, which is related to matter and therefore to the reference system. In [25-27] also variations of the metric tensor (naturally, only with covariant components) and its partial derivatives have been accounted. In the present case, however, variations of $g_{\mu\nu}$ are not to be taken account, because the variation itself is applied after the Lagrangian decomposition has been performed. Instead of $g_{\mu\nu}$, variations of the vector field with covariant and
contravariant components will be taken. From a physical point of view, the inclusion of the vector field $u$ from the gravitational part of the Lagrangian in the variational approach may have serious physical implications, at least because there will be an additional equation of motion for the vector field.

Performing the form (or functional) variation means that the variational operator acts on the Lagrangian action functional just by acting only on the under-integral expression, and not on the volume element

$$\delta L \equiv \delta \left[ \int L(p_{mk}, p^{mk}, \partial_{j}p_{mk}, u_k, u^k, \partial_{j}u_k, \partial^i u^k) d^4x \right] =$$

$$= \int \delta L(p_{mk}, \ldots, \partial^i u^k)d^4x = 0. \quad (24)$$

This is a peculiar feature of the form-variational operator, which is understood as the difference between the functional values, taken at one and the same point.

$$\delta p_{ij} \equiv p_{ij}'(x) - p_{ij}(x). \quad (25)$$

This is unlike the total variational operator, defined as

$$\delta u_i(x) \equiv u_i'(x') - u_i(x), \quad (26)$$

where the prime sign "'" means that both the argument $x$ and the functional values are being varied. The total variational operator acts in a more special way on the whole action functional and on the volume element particularly. In spite of the fact that in gravitational theory mainly the form-variation is applied (for some mathematical aspects - see [28]), the total variational operator may also play a significant role in a theory, where a unit volume (matter) element is subjected to an expansion, motion and deformation. Since these physical processes are related to the introduced in the theory rotation tensor, deformation and expansion tensors [4], it is clear that in such a "form non-invariant" theory after performing a total variation of the gravitational Lagrangian there will be an additional contribution (compared with the form-variated Lagrangian), expressable in terms of the above mentioned variables. An interesting definition of the form-variation from a purely mathematical point of view as a sequence of the "non-commuting" functional, Lie and covariant variations is given also in [29]. But since this is a more subtle
and complicated subject, this will be treated in another paper. In this paper, for convenience the symbol $\delta$ will denote everywhere a form (functional) variation.

The result of the form-variation in (24) will be the equation

$$\bar{\delta}L(p_{mk}, \ldots, \partial^i u^k) \equiv 0 \text{ (i.e. equal to zero under-integral expression ),}$$

written as:

$$\frac{\delta L}{\delta p_{mk}} \delta p_{mk} + \frac{\delta L}{\delta \partial_j p_{mk}} \delta \partial_j p_{mk} + \frac{\delta L}{\delta p_{mr}} \delta p_{mr} + \frac{\delta L}{\delta \partial_j p_{mr}} \delta \partial_j p_{mr} +$$

$$+ \frac{\delta L}{\delta u_k} \delta u_k + \frac{\delta L}{\delta \partial_j u_k} \delta \partial_j u_k + \frac{\delta L}{\delta u_k} \delta u^k + \frac{\delta L}{\delta \partial_j u_k} \delta \partial_j u^k \equiv 0. \quad (27)$$

In order to write down the corresponding equations of motion for the independent variables $p_{ik}$ and $u_i$, all variations should be explicitly written. In the usual variational approach of gravitational theory, this is trivial, since it is normal to assume that the variation and the partial derivative commute, provided however that connection form-variation is zero [24]. But also in [24] it has been proved that in the case of projection gravitational theory, this is much more complicated since the variation and the partial derivative do not commute. The exact expression for the different from zero commutator $[\delta, \partial_j] p_{ik}$ has also been found, and it will be used in Section III.

It is the purpose of the present paper to express all the other variations in (27), and this requires the computation of all the variations not only of the projection tensor and the vector field with covariant indices, but also with contravariant ones. This has been performed in Section III, using the initial set of defining equations (17-18), and also the formulae for the vector field’s length $e = u_k u^k$. Moreover, combining all the variations and noticing that some of them can be expressed through others, it turned out to be possible to express all of them only through variations of $\delta u_t$, $\delta p_{ik}$ and $\partial_j \delta u_t$ (or $\partial_j \delta u_t$, which is the same, since $\partial_j$ and $\delta$ commute in respect to the vector field $u$). The derived expressions will be the starting point for constructing the adequate variational formalism and finding the conserved quantities in the subsequent paper.

Motivated from the given in the Introduction reasoning about the importance to consider the case of zero covariant derivative of the projection tensor (in respect to the projection connection) and the analogy made with the three-dimensional projection tensor in the ADM formalism, being defined on a three dimensional Riemannian subspace of the initially given four
dimensional space, some relations between the variations have been found in section IV, V and VI under the above particular assumption. Note that there is no full analogy with the ADM case, because no substitutions like those in (16-18) have been made. Due to this the result obtained in the end of section VI is not known from the ADM formalism. In a sense, it may be admitted that the ADM approach admits a more general freedom in the theory, because the vector field is not assumed to be necessarily orthogonal to the three-dimensional hypersurface (subspace), as it is required in the projection approach, used here. The formulae are valid also for an arbitrary-dimensional space-time, and yet no 3+1 decomposition has been performed. But in the case of a (3+1) decomposition, the derived formulae can be used to check whether the ADM formalism can be obtained as a limiting case of the more general approach, presented so far. This is an interesting investigation, also not presented in this paper.

Section IV will deal with the variations of the projection tensor with covariant components, making an extensive use of all the formulae in Section III. At the end, a relation between the projection and initial connections will be obtained.

In Section V the projection connection variation will be found, and the approach will be based on the fact that in the proposed projection variation approach different results may be obtained if at first two systems of equations are combined, and after that the variation is taken, in comparison with an approach, when first the variations of the two equations are taken, and only after that the obtained equations are combined. The obtained result will be possible to be written in two ways. In the second way, no divergent term will be present but just the variations of the initial connection, the vector field and the projection tensor.

Section VI, similarly to Section IV, will be devoted to finding the variations also of the projection tensor, but with contravariant components. Although the approach will be the same as that in IV, the result about the expressed projection connection will make it possible to obtain an equation, expressing the relation between only the vector field and the projection tensor variations. Since the two variations are independent, the expressions before them have to be zero, and from one of the expressions an important relation for the vector field derivative will be obtained in a very concise form. The significance of the obtained relation will be discussed in the conclusion part of the paper.
III. VARIATIONS OF FIRST-ORDER DERIVATIVES OF THE VECTOR FIELD AND THE PROJECTION TENSOR

In order to implement the projectional variational formalism in the investigated gravitational theory, using as basic variables the projection tensor field with covariant and contravariant components, it is necessary to find all the variations $\delta \partial_j u^k$ and $\delta \partial_j p^{ik}$ of the first derivatives of the vector field $u^k$ and the projection tensor $p^{ik}$ with contravariant components. In order to find the variation $\delta \partial_j u^k$ for example, the commutator $[\delta, \partial_j]$ will be applied to the equation $u^k p_{mk} \equiv 0$ and as a result it can be derived

\[ p_{mk} \delta \partial_j u^k \equiv -(\partial_j u^k) \delta p_{mk} - (\partial_j p_{mk}) \delta u^k - u^k \delta \partial_j p_{mk}. \quad (28) \]

Our aim will be to find also an expression for $\frac{1}{e} u_m u_k \delta \partial_j u^k$, which, if summed up with (28) and subsequently contracted, will give the required expression for $\delta \partial_j u^k$. This can be done by applying the commutator $[\delta, \partial_j]$ to the equation

\[ p_{mr} p^{rk} \equiv \delta^k_m - \frac{1}{e} u^k u_m. \quad (29) \]

The resulting equation, multiplied by $u_k$, is

\[ \frac{1}{e} u_m u_k \delta \partial_j u^k \equiv -u_k \delta \partial_j (p_{mr} p^{rk}) - u_k \partial_j (\frac{1}{e} u_m) \delta u^k - e \delta \partial_j (\frac{1}{e} u_m) - u_k (\partial_j u^k) \delta (\frac{1}{e} u_m). \quad (30) \]

If (28) and (30) are summed up and contracted with $g^{ ms}$, an expression can be obtained, which contains the other undetermined yet variation $\delta \partial_j p^{ik}$. In order to avoid this, a more reasonable choice can be performed by applying the commutator $[\delta, \partial_j]$ to the equation

\[ e = u_k u^k \quad (31) \]

Taking into account that $\delta$ and $\partial_j$ commute when applied to the scalar quantity $e$, i.e.

\[ [\delta, \partial_j] e \equiv 0 \quad (32) \]

the following expression can be obtained (multiplied with $\frac{1}{e} u_m$)
\[
\frac{1}{e} u_m u_k \delta \partial_j u^k \equiv -\frac{1}{e} u_m u^k \delta \partial_j u_k + \frac{1}{e} u_m u^k \partial_j \delta u_k + \frac{1}{e} u_m u_k \partial_j \delta u^k. \tag{33}
\]

Now, summing up (28) and (32) and contracting with \(g^{ms}\), we obtain the final expression for \(\delta \partial_j u^s\):

\[
\delta \partial_j u^s \equiv -g^{ms} u^k \delta \partial_j p_{mk} - g^{ms} \delta \partial_j u^k \delta p_{mk} - (\partial_j p_{mk}) g^{ms} \delta u^k - \frac{1}{e} u^s u^k \delta \partial_j u_k + \frac{1}{e} u^s \partial_j \delta e. \tag{34}
\]

As can be noted, this expression does not contain variations of first-order derivatives of the projective tensor with contravariant indices.

Following the same described above procedure and applying the commutator \([\delta, \partial_j]\) to the equations

\[
p^{mk} u_k \equiv 0 \quad \text{and} \quad p^{mk} p_{ks} \equiv \delta^m_s - \frac{1}{e} u^m u_s, \tag{35}
\]

an expression for the other variation \(\delta \partial_j p^{mr}\) can be derived:

\[
\delta \partial_j p^{mr} \equiv -\partial_j \left[ (\frac{1}{e}) u^m u^r \right] + \left[ (\partial_j u^r) u^m - g^{sr} u^m \partial_j u_s \right] \delta (\frac{1}{e}) -
\]

\[
-\left( \frac{1}{e} u^r \partial_j u_k + g^{sr} \partial_j p_{ks} \right) \delta p^{mk} - g^{sr} p^{mk} \delta \partial_j p_{ks} -
\]

\[
- g^{sr} \partial_j p^{mk} \delta p_{ks} - g^{sr} \partial_j (\frac{1}{e} u_s) \delta u^m - \frac{1}{e} u^r \delta \partial_j u^m -
\]

\[
- \left[ g^{kr} \partial_j (\frac{1}{e} u_m) + \frac{1}{e} u^r \partial_j p^{mk} \right] \delta u_k - \frac{1}{e} \left[ u^r g^{mk} + \frac{1}{e} u^m u^k \right] \delta \partial_j u_k. \tag{36}
\]

Now it remains to find the variations \(\delta p^{mr}\) and \(\delta u^k\). Again, taking the variations of equations (35) and then summing up, it can be found

\[
\delta p^{mr} \equiv -\frac{2}{e} u^r g^{mk} \delta u_k - \frac{1}{e} u^r (g^m_k - \frac{1}{e} u^m u_k) \delta u^k - g^{sr} p^{mk} \delta p_{ks}. \tag{37}
\]

In the same way, after taking the variations of the equations
\[ p_{mk} u^k \equiv 0 \quad \text{and} \quad e \equiv u_k u^k, \]  
\( \delta u^k \equiv -u^r (g^{mk} + \frac{1}{e} u^l u^m) \delta p_{mr} \equiv -u^r g^{mk} \delta p_{mr}. \)  
(38)

the following formulae can be obtained

\[ u^r u^m \delta p_{mr} \equiv 0. \]  
(40)

Formulae (39) is convenient since it expresses a variation of a contravariant quantity only through the variation of a covariant projection tensor. If (39) is substituted into (37) for \( \delta p_{mr} \), again in the right-hand side only variations of covariant quantities will be present

\[ \delta p_{mr} \equiv -\frac{2}{e} u^r g^{mk} \delta u^k - p^r s p^{mk} \delta p_{ks}. \]  
(41)

In the preceding paper [24] the following expression has been found for the commutator \([\delta, \partial_j] p_{mr}\)

\[ [\delta, \partial_j] p_{mr} \equiv g_{ir} g_{mk} \partial_j \left( \delta p^{ik} + \delta \left( \frac{1}{e} u^l u^k \right) \right) + \partial_j \delta p_{mr} + \]
\[ + g_{kr} \left[ p^s (g_m)_{ri} + \frac{1}{e} u_m u^s \right] \delta \Gamma^k_{sj} + \frac{1}{e} g_l (r u_m) \Gamma^l_{kj} \delta u^k + \]
\[ + g_{sj} g_l (u^k u_m) \delta u^k + \frac{1}{e} u^s u_n g_m) \right] \delta p^{nk} - \]
\[ - g_l (r u^m p_{pm}) \Gamma^l_{sj} u^k \delta p_{nk} - \frac{1}{e^2} \Gamma^l_{nj} g_l (r u_m) u^n u^k \delta u_k. \]  
(42)

This formulae contains also variations of the contravariant quantities \( \delta u^k \) and \( \delta p^{nk} \), but one can substitute formulae (39) and (41) to find the expression for \( \delta \partial_j p_{mr} \), expressed also through only variations of covariant quantities

\[ \delta \partial_j p_{mr} \equiv -\partial_j \left[ \frac{1}{e} u_m (2 g^i_r + \frac{1}{e} u^l u_r) \delta u^k \right] + \Gamma^s_{jk} p^{nk} \delta u^k + g_{s(r} g_{m)} \delta \Gamma^s_{kj} + \Gamma^s_{jk} G^{pk}_{srm} \delta p_{pq}, \]  
(43)
where $P^r_{rms}$ is expression (A2) in Appendix A and $G^p_{srm}$ is the following expression:

$$G^p_{srm} \equiv g^k_p g^q_{(m} g^r_{m)} s + g^p_s g^q_r g^r_m s - g^p_s g^q_r g^q_m - \frac{1}{2} g^p_s (u_m u^q) g^q_r.$$  \hfill(44)

Since the expression for $\delta \partial_j p_{mr}$ enters equation (36) for $\delta \partial_j p^{mr}$, the last one can easily be found to be

$$\delta \partial_j p^{mr} \equiv W^t_{jm} \delta u_t + V^t_{jm} \partial_j \delta u_t + Y^{pqmr}_j \delta p_{pq} + Z^r_{pq} \delta \Gamma^p_{qr},$$  \hfill(45)

where $W^t_{jm}$, $V^t_{jm}$, $Y^{pqmr}_j$ and $Z^r_{pq}$ are given in Appendix A - formulaes (A1) and (A3-A5) respectively.

And the last expression to be given is the one for $\delta \partial_j u^m$ (34), which is found to be (after using all the above derived expressions)

$$\delta \partial_j u^m \equiv \partial_j \left[ \frac{3}{e} u^r u^m \delta u_t \right] - (g^{mr} u_s + g^m_s u^r \delta \Gamma^s_{jr} + A^{pqm}_j \delta p_{pq} + B^t_{jm} \delta u_t,$$  \hfill(46)

where $A^{pq}$ and $B^t$ in (46) denote the following expressions:

$$A^{pqm}_j \equiv \frac{1}{e} u^m \partial_j u^r u^q g^{pr} + \Gamma^s_{jr} \left[ -g^p_s u^r (g^q m) + \frac{e}{2} g^{pq} (g^m_s + \frac{1}{e} u^m u_s) \right]$$  \hfill(47)

$$B^t_{jm} \equiv \frac{3}{e} (u_p g^{mq} + u^q g^m_p) u^r \Gamma^p_{qj} - \Gamma^p_{qj} g^s_p g^{mk} u^r P^{pq}_{rks} - \frac{1}{e} \partial_j u^r u^m (2 g^r_{jr} + \frac{1}{e} u^r u_r).$$  \hfill(48)

IV. VARIATIONS IN THE CASE OF ZERO-COVARIANT DERIVATIVE OF THE PROJECTION TENSOR (WITH COVARIANT INDICES)

The assumption about a projection tensor with a zero-covariant derivative in respect to the projection connection means that besides all the variation formulaes in the preceeding section, two more conditions are being imposed, from which again some relations between the variations can be found. The first condition is for zero covariant derivative of the projection tensor with covariant indices

$$\tilde{\nabla}_j p_{mr} \equiv 0 \equiv \partial_j p_{mr} - p_s (\tilde{\Gamma}^s_{jr}),$$  \hfill(49)
and the second condition is the analogous one, but for the projection tensor field with contravariant indices

\[ \tilde{\nabla}_j p^m r = 0 \equiv \partial_j p^m r - p^s \Gamma^m_{js} \tilde{r}^s. \]  

(50)

In this section only the first condition shall be investigated. Note that while in standard Riemannian geometry (49) will be satisfied if the usual formulae for the Christoffell connections is used (the two formulae are a consequence of one another), here this will not happen, because the projection tensor does not have an inverse one. Consequently, after formulae (22) for the projection connection \( \tilde{\Gamma}^s_{rj} \) is substituted into (49), one obtains:

\[ \frac{1}{2} u^l u^m \left[ \partial_r p_{jl} + \partial_j p_{rl} - \partial_l p_{rj} \right] \equiv 0, \]  

(51)

which is fulfilled when

\[ u^l \left[ \partial_r p_{jl} + \partial_j p_{rl} - \partial_l p_{rj} \right] \equiv 0. \]  

(52)

After taking the variation of this equation, with account also of the variations (39) and (43), the following equation can be obtained:

\[
\left[-2 u^9 \tilde{\Gamma}^p_{rj} + u^l F^{mna}_{jlr} G^{pkq}_{smn} \right] \partial_p q + u^l F^{mna}_{jlr} g_{s(m} g_{n)} \delta \Gamma^s_{ka} + H^t_{rj} \delta u_t -
\left[ \frac{1}{e} u^l u^m \left( \partial_r p_{jl} + \partial_j p_{rl} - \partial_l p_{rj} \right) \right] \equiv 0,
\]

(53)

where \( F^{mna}_{jlr} \) denotes the following expression:

\[ F^{mna}_{jlr} \equiv g^m_{jr} g^l_{gr} + g^m_{jl} g^l_{gr} - g^m_{jr} g^l_{jl}, \]  

(54)

\( G^{pkl}_{smn} \) is given by (44) and \( H^t_{rj} \) is the expression

\[ H^t_{rj} \equiv u^l \left( \Gamma^s_{rk} P^l_{tjs} + \Gamma^s_{jkr} P^l_{ts} - \Gamma^s_{lrs} P^t_{jrs} \right) + \tilde{H}^t_{rj}, \]  

(55)

\[ \tilde{H}^t_{rj} \equiv \frac{1}{e} \left[ (\partial_r u^t) u_j + (\partial_j u^t) u^t u^m u_n - 2 (\partial_\alpha u^\alpha) u_r g^t_j - \frac{1}{e} (\partial_\alpha u^\alpha) u^t u_r u_j \right]. \]  

(56)

Simple calculations give also that

\[ u^l F^{mna}_{jlr} g_{s(m} g_{n)} \delta \Gamma^s_{ka} \equiv 2 u_s \delta \Gamma^s_{jr}. \]  

(57)
and the divergent term (the last term in (53) is found to be 

\[ S_{rj} \equiv -\partial_\alpha \left[ \frac{1}{\epsilon}(-2p_j^\alpha u^\alpha u_r + 3p_r^\alpha u^\alpha u_j + 3g_j^\alpha u_r u^\alpha)\delta u_t \right]. \] (58)

The expression, standing in front of the variation \( \delta p_{pq} \) in (53) (the first term) can be calculated to be 

\[ u^\alpha \left[ 2\Gamma_j^{\rho\alpha} - 2\tilde{\Gamma}_j^{\rho\alpha} + \frac{1}{2}g^{\rho kl}(\Gamma_s^{kl}g_s(ju_r) - \Gamma_s^{kr}g_s(lu_j) - \Gamma_s^{kj}g_s(lu_r)) \right] \] (59)

If we assume that the variation and the partial derivative commute in respect to the metric tensor and therefore \( \delta \tilde{\Gamma}_j^{\rho\alpha} \equiv 0 \), then in view of the independent variations \( \delta u_t \) and \( \delta p_{pq} \) the expressions (55), (58) and (59), standing in front of them should be equal to zero. From the last expression, particularly, the projection connection may be expressed through the full connection and its projections along the vector field as

\[ \Gamma_j^{\rho\alpha} \equiv \tilde{\Gamma}_j^{\rho\alpha} + \frac{1}{4}g^{\rho kl}(\Gamma_s^{kl}g_s(ju_r) - \Gamma_s^{kr}g_s(lu_j) - \Gamma_s^{kj}g_s(lu_r)) \equiv \Gamma_j^{\rho\alpha} + \frac{1}{2}g^{\rho kl}u_r u^\alpha \Gamma_s^{kl}(\delta g_{jk}) \] (60)

Note that the second term is a projected antisymmetric (in the indices \( l \) and \( j \)) expression. The last equation, however, should be viewed only as a possible, but not obligatory choice of the projection connection in the case of zero-covariant derivative of the projection tensor field and commutation of the derivative and the variation. The reason for this is simple: Since the variation \( \delta p_{pq} \) does not enter the divergent term (58) and only the variation \( \delta u_t \) enters it, equation (53) can be considered as a system of differential equations in respect to the variation \( \delta u_t \). This equation will have a solution for arbitrary variations \( \delta p_{pq} \), and therefore there will be no need (60) to be fulfilled. This will be discussed also in the following section.

Nevertheless, a way can be found to express the projective connection variation through the initially given one. For that purpose, from the equation, resulting from the initial Riemannian metric

\[ \partial_j g_{mk} - g_{s(m} \Gamma_s^{k)j} \equiv 0, \] (61)

equation (49) is substracted. As a result, the following equation is obtained
\[ \partial_j \left( \frac{1}{e} u_m u_k \right) - 2 g_{sm} H_{kj} \equiv 0. \] (62)

It is understood that the connection decomposition is
\[ \Gamma^s_{kj} \equiv \tilde{\Gamma}^s_{kj} + H_{kj}^s \] (63)
and the variations of the covariant and contravariant metric tensors are
\[ \delta g_{pq} \equiv \delta p_{pq} + \frac{2}{e} u_p (p_q^t - \frac{1}{e} u^t u_q) \delta u_t \] (64)
\[ \delta g^{ms} \equiv -g^{mp} g^{sq} \delta g_{pq}. \] (65)

Performing the variation of (62), contracting afterwards with \( g^{ml} \) and making use of (63-65), it can be obtained after some calculations that
\[ \delta \tilde{\Gamma}^s_{kj} \equiv \delta \Gamma^s_{kj} - \frac{1}{2} \partial_j \left[ g^{ms} \delta \left( \frac{1}{e} u_m u_k \right) \right] + \frac{1}{2} \partial_j \left( \frac{1}{e} u_m u_k \right) g^{mp} g^{sq} \delta p_{sq} + \bar{P}_t^l \delta u_t, \] (66)
where \( \bar{P}_k^l \) is the expression
\[ \bar{P}_k^l \equiv \frac{1}{e} \left[ g^{sq} u^m \partial_j \left( \frac{1}{e} u_m u_k \right) - g^k_l u_m g^{l(m)} g^{s)}_{ij} \right] (p_q^t - \frac{1}{e} u^t u_q). \] (67)

Unlike equation (60) and the connection variation, which can be derived from it, equation (66) is always fulfilled, provided the two starting assumptions about a Riemannian initial metric and zero-covariant derivative of the projection tensor are fulfilled.

V. FINDING THE CONNECTION VARIATION AND EXPRESSING THE DIVERGENT TERM

It would have been very convenient and useful if there is a way to find the variations \( \delta \Gamma^s_{kj} \) and \( \delta \tilde{\Gamma}^s_{kj} \). For the purpose, the following observation can be made. In (49-52) use is made first of the equation for the zero-covariant derivative (49), as a second step - the formulae for the projection connection is substituted and finally - the variation operator is applied. As a result equation (53) is obtained. Now, let us proceed in a different way. Let us first find the variations of equation (49) and of the defining equation (22) for the projection connection, and as a second step, let us combine the two obtained
(after the variation) equations. The claim which will be made is that the result which is to be obtained from the two equations will be different from the already derived equation (53). In such a way, the two independently derived equations may be combined to obtain the final result.

Indeed, the variation of the projection connection, using the defining formulae (22), is

$$\delta \tilde{\Gamma}^k_{ln} \equiv \frac{1}{2} \delta p^{kt}(\partial_n p_{lt} + \partial_t p_{nt} - \partial_t p_{ln}) + \frac{1}{2} p^{kt} F_{ltn}^{pq\alpha} \delta \partial_{\alpha} p_{pq}$$  

(68)

where $F_{ltn}^{pq\alpha}$ is the familiar expression (54) and $\delta p^{kt}$ is written again for convenience

$$\delta p^{kt} \equiv -\frac{2}{e} u^t g^{ks} \delta u_s - p^{ts} p^{kr} \delta p_{sr}.$$  

(69)

Next, the variation $\delta \partial_{\alpha} p_{pq}$ is to be expressed after performing the variation of the equation $\partial_{\alpha} p_{pq} - p_r(\tilde{\Gamma}^r_{pq})_{\alpha} \equiv 0$, as a result of which

$$\delta \partial_{\alpha} p_{pq} \equiv \delta p_r(\tilde{\Gamma}^r_{pq})_{\alpha} + p_r(\partial_{\alpha} p_{pq} \equiv 0).$$  

(70)

Substituting the last equation into (68), using (69) and transferring all the terms with $\delta \tilde{\Gamma}^k_{ln}$ in the left-hand side, it is obtained

$$K^{kgh}_{fln} \delta \tilde{\Gamma}^f_{gh} \equiv N^{kab}_{ln} \delta p_{ab} - \frac{1}{e} u^t g^{ks} F_{ltn}^{pq\alpha} (\partial_{\alpha} p_{pq}) \delta u_s,$$

(71)

where the tensor expressions $N^{kab}_{ln}$ and $K^{kgh}_{fln}$ are expressed as follows

$$N^{kab}_{ln} \equiv p^{kt} F_{ltn}^{pq\alpha} g_{r}^{a}(\tilde{\Gamma}^r_{pq})_{\alpha} - p^{ta} p^{kb} F_{ltn}^{pq\alpha} \partial_{\alpha} p_{pq}$$  

(72)

$$K^{kgh}_{fln} \equiv g^{g}_{f} g^{h}_{l} g^{a}_{n} - \frac{1}{e^2} p^{q}_{f} F_{ltn}^{pq\alpha} g_{r}^{a}(\tilde{\Gamma}^r_{pq})_{\alpha} \equiv \frac{1}{2} \left[ \frac{1}{e} g_{(l}^{g} g_{n)}^{h} u^{k} u^{f} + g_{(l}^{u} p_{n)} f^{h} p^{k} \right].$$  

(73)

where the square brackets in the contravariant indices mean antisymmetrization, i.e. $[gh] = gh - hg$. Because of the assumption about symmetric initial and projection connection and the antisymmetrization of the indices $g$ and $h$ in the second term of (73), this term will not give contribution to the left-hand side of (71). Therefore, upon contraction with $u^k$ from (71) it can be found
\[ u_s \delta \Gamma^s_{kj} \equiv -\frac{1}{e} u^m u^t F^{pq\alpha}_{kmj} \partial_\alpha p_{pq} \delta u_t. \]  
(74)

Substituting the last expression into (66), a relation is obtained only between the (initial) connection variation and the projection and vector field variations

\[ 2u_s \delta \Gamma^s_{kj} \equiv \partial_j \delta u_k - u^q g^{mp} \partial_j \left( \frac{1}{e} u_m u_k \right) \delta p_{pq} - \left[ \frac{1}{e} (\partial_j u_s) (u_k p_{ts} + u^s g^t_k) + 2u_s P^s_{kj} + \frac{2}{e} u^m u^t F^{pq\alpha}_{kmj} \partial_\alpha p_{pq} \right] \delta u_t. \]  
(75)

The expression for \( u_s P^s_{kj} \) is easily calculated to be

\[ u_s P^s_{kj} \equiv \frac{1}{e^2} (\partial_j u_s) (u_k u^t u_k) - \frac{1}{e} u^t \partial_j u_k - \frac{2}{e} u_s u^t \Gamma^s_{lj} (p^l_k - \frac{1}{e} u^t u_k). \]  
(76)

Notice that in deriving (75) use has not been made of the previously derived equation (54) for the variations. That is why (75) can be substituted into (53) to obtain

\[ K^{pq}_{rj} \delta p_{pq} + O^t_{rj} \delta u_t + \text{divergent terms} \equiv 0, \]  
(77)

where

\[ K^{pq}_{rj} \equiv u^q p^{pk} u_r \left[ u^t \Gamma^s_{kl} - u_s \Gamma^s_{kj} \right] \equiv p^{pk} u^q u^l u_r (\partial_t g_{kj} - \partial_j g_{kl}). \]  
(78)

\[ O^t_{rj} \equiv H^t_{rj} - \frac{1}{e} (\partial_j u_s) (u_t p_{ts} + u^s g^t_s) - 2u_s P^s_{kj} - \frac{2}{e} u^m u^t F^{pq\alpha}_{kmj} \partial_\alpha p_{pq} \]  
(79)

and the divergent terms are found to be

\[ S_{rj} + \partial_j \delta u_t. \]  
(80)

As usual, \( S_{rj} \) is given by the expression (58). The obtained equation (77) contains variations only of the vector and the projective fields, but also a rather complicated divergent term. However, from (53) and (77) a more simplified expression can be found, in which the part \( S_{rj} \) of the divergent term does not participate.
\[ 2u_s \delta \Gamma^s_{jr} + 2u^q (\Gamma^p_{jr} - \bar{\Gamma}^p_{jr}) \delta p_{pq} + (H^t_{rj} - O^t_{rj}) \delta u_t - \partial_j \delta u_r \equiv 0. \]  

Unlike (77), here the variation \( \delta \Gamma^s_{jr} \) is present, but in the following section this will turn out to be a useful property.

Now let us assume that the variation and the partial derivative commute in respect to the initial Riemannian metric (for which \( \nabla_\alpha g_{\mu\nu} \equiv 0 \)). Then, according to a proved in [24-preceeding paper] proposition, this is possible if and only if \( \delta \Gamma^s_{jr} \equiv 0 \), provided also that \( \nabla_\alpha (\delta g_{\mu\nu}) \equiv 0 \). The reason is that only upon fulfillment of these conditions the expression for the commutator \([\delta, \partial_j] g_{\mu\nu} \) will be equal to zero

\[
[\delta, \partial_\alpha] g_{\mu\nu} = 0.
\]  

So, if \( \delta \Gamma^s_{jr} \equiv 0 \) and also the independence of the variations \( \delta u_t \) and \( \delta p_{pq} \) are assumed, then from the expression before \( \delta p_{pq} \) in (81) it follows that \( \Gamma^p_{jr} = \bar{\Gamma}^p_{jr} \). The last is possible only when \( u = 0 \) (then the remaining in (81) terms will also be zero), but this has to be rejected as contradictory to the performed projective decomposition.

However, another opportunity exists. Since the partial derivative of the variation \( \delta u_r \) also enters (81), it can be considered as a (matrix) system of four differential equations in respect to the four vector \( \delta u \). In matrix notations equation (81) can be written as

\[
\partial_j \delta u \equiv \tilde{K}_j \delta u + \tilde{T}_j
\]  

where \( \tilde{K}_j \equiv K^t_{rj} \) is a 4×4 matrix \( (r, t = 1, 2, 3, 4) \)

\[
K^1_{1j} \quad K^2_{1j} \quad K^3_{1j} \quad K^4_{1j} \\
K^1_{2j} \quad K^2_{2j} \quad K^3_{2j} \quad K^4_{2j} \\
K^1_{3j} \quad K^2_{3j} \quad K^3_{3j} \quad K^4_{3j} \\
K^1_{4j} \quad K^2_{4j} \quad K^3_{4j} \quad K^4_{4j}
\]  

and \( K^t_{rj} \) denotes the tensor expression, standing before the variation \( \delta u_t \) in (81)

\[
K^t_{rj} \equiv H^t_{rj} - O^t_{rj}
\]  

This formulae can be expressed from (79), and \( \tilde{T}_j \) denotes the four-vector (column) \( T_{jr} \) (when \( j \) is fixed)

20
\[ \widetilde{T}_j \equiv T_{jr} \equiv 2u_s \delta \Gamma^s_{jr} + 2u^q (\Gamma^p_{jr} - \widetilde{\Gamma}^p_{jr}) \delta p_{pq}. \]  

(86)

Note that the matrix system (83) is for each \( j \) and \( j = 1, 2, 3, 4 \), so in fact there are four such systems. More important, this system is a linear one since the operation of variation of the four vector \( u \) gives another four-vector \( \delta u \), not dependent on \( u \). Although \( \widetilde{K}_j \) and \( \widetilde{T}_j \) depend on \( u \) and its derivatives, they are independent of \( \delta u \).

The solution of the system (83) for each \( j \) can also be written in matrix notations

\[ \delta u \equiv \int e^{\widetilde{K}_j} dx_j \left[ \text{const.} + \int \widetilde{T}_j e^{-\int \widetilde{K}_j dx_j} dx_j \right]. \]  

(87)

The solution is also for each \( j \), so the couple of indices in (87) has nothing to do with summation.

It is important to realize that (87) in fact expresses a complicated relation between the variations \( \delta u_t \), \( \delta p_{pq} \) and \( \delta \Gamma^s_{jr} \), and therefore, only in the case of the fulfillment of the last relation, one may define a covariant derivative of the covariant projection tensor. But in the general case, when the assumption about independent variations \( \delta u_t \) and \( \delta p_{pq} \) is explicitly present, this is not possible, and one is obliged to investigate the other case with the contravariant projection tensor. The above formulae shall not be further used because use will be made of the expression (81) for the divergent term \( \partial_j \delta u_t \).

However, since there will be no such (divergent) terms in the other case in the next section, such formulae will no longer be written.

### VI. VARIATIONS IN THE CASE OF ZERO-COVARIANT DERIVATIVE OF THE PROJECTION TENSOR (WITH CONTRAVARIANT INDICES)

In general, the approach in this section will be the same as in the preceding section. Some of the formulae in it will turn out to be related with the formulae in the present section.

The defining equation for the zero-covariant derivative of the projective tensor with contravariant indices is

\[ \nabla^p_{jr} \equiv \partial_j p^{mr} + p^s (\Gamma^r_{jm} - \widetilde{\Gamma}^r_{jm}) = 0. \]  

(88)
Performing the variation of this equation and taking into account expressions (45) and (41) for the variations $\delta \partial_j p^{mr}$ and $\delta p^{sr}$ respectively, the following equation can be obtained

$$-p^s(r\delta^{m}_{sj}) \equiv \partial_j \left[ V^{r\text{rm}} \delta u_t \right] + \left[ W_{rj}^{\text{rtm}} - \frac{2}{e} u^{(r)g^{st}\tilde{\Gamma}^{m}_{sj}} - \partial_j V^{t\text{rm}} \right] \delta u_t +$$

$$+ Z_{r}^{mq} \delta \Gamma^p_{jq} + \left[ V^{p\text{qm}} - p^{(rq)p\tilde{\Gamma}^{m}_{sj}} \right] \delta p_{pq}. \quad (89)$$

On the other hand, from the equation for the Riemannian initial metric

$$\nabla_j g^{mr} \equiv \partial_j g^{mr} + g^{s(r\Gamma^m_{sj})} \quad (90)$$

after substracting (88) and performing the variation, it can be derived that

$$\delta \partial_j \left( \frac{1}{e} u^m u^r \right) + \delta \left( \frac{1}{e} u^s u^{(r)\Gamma^m_{sj}} \right) + \frac{1}{e} u^s u^{(r)\delta \Gamma^m_{sj}} +$$

$$+ \delta p^s(r(\Gamma^m_{sj} - \tilde{\Gamma}^m_{sj})) + p^s(r\delta(\Gamma^m_{sj} - \tilde{\Gamma}^m_{sj})) \equiv 0. \quad (91)$$

Making use of the formulae for the variations in the previous sections, the variations of the expressions in the first two terms of (91) can be found. Omitting some cumbersome transformations, the final form for the transformed equation (91) can be explicitly given

$$\partial_j \left[ \frac{5}{e^2} u^r u^t u^m \delta u_t \right] + \left[ \frac{1}{e} u^{(r)\Gamma^m_{sj}} \delta u_s + g^{(m)p^s_{jq}} \delta \Gamma^s_{ij} \right]$$

$$-p^s(r\delta^{m}_{sj}) + M^{pqmr} \delta p_{pq} + N^{t\text{rm}} \delta u_t \equiv 0, \quad (92)$$

where $M^{pqmr}$ and $N^{mrt}$ are expressions (A10) and (A11), given in Appendix A. So far the two independent equations (89) and (92) have been obtained and they follow naturally from the assumptions about the zero covariant derivatives of the metric and the projective tensor.

The term $-p^s(r\delta^{m}_{sj})$ from (89) can be substituted into (92) and in such a way the two equations can be combined. The resulting equation is

$$\partial_j \left[ \frac{5}{e^2} u^r u^t u^m + V^{r\text{rtm}} \delta u_t \right] + \left[ N^{rtm} + W_{rj}^{r\text{rtm}} - \frac{2}{e} u^{(r)g^{st}\tilde{\Gamma}^{m}_{sj}} - \partial_j V^{t\text{rtm}} \right] \delta u_t +$$

$$+ \left[ M^{pqmr} + Y^{pqmr} - p^{(rq)p\tilde{\Gamma}^{m}_{sj}} \right] \delta p_{pq} \equiv 0 \quad (93)$$
Surprisingly, in (93) the variation $\delta \Gamma^s_{ij}$ is absent, because it has turned out that the expression before $\delta \Gamma^s_{ij}$ is

$$Z^r_{sml} - \frac{1}{e} u^r g^{ml} u_s + g_s^{(m} p^{r)l} \equiv 0. \quad (94)$$

The first (divergent) term in (93) can be rewritten, using the expression (81) for the partial derivative of the variation $\partial_j \delta u$. Further, after this term is substituted in (93), the derived equation is contracted with $u_r u_m$ and use is made of equation (A6) from Appendix A, according to which

$$5u^t + u_r u_m V^{tmr} \equiv 0. \quad (95)$$

Due to (95) some terms in the transformed equation (93) will become zero, and the expression before the variation $\delta \Gamma^s_{jl}$ will also turn out to be zero

$$2u_s (5u^l + u_r u_m V^{lmr}) \delta \Gamma^s_{jl} \equiv 0. \quad (96)$$

Therefore, the transformed equation (93) turns out to be without any divergent term, and moreover, it contains only the variations $\delta u_t$ and $\delta p_{pq}$ and not the variation $\delta \Gamma^s_{jl}$

$$\delta u_t \left[ W_j^{tm} u_r u_m - u_r u_m \partial_j V^{tmr} - 6\partial_j \left( \frac{1}{e} u^r \right) u_r u^t - 4\partial_j u^t - \frac{2}{e} (\partial_j u^r) u_r u^t \right] +$$

$$+ \delta u_t \Gamma_{j}^{p} \left[ \frac{6}{e} u_p u^q u^t - 2u^a u^\beta P_{\beta\alpha p} - 4u_p p^{gt} \right] +$$

$$+ \delta p_{pq} \left[ u_r u_m V^{pmr} + 2(\partial_j u_s) u^q p^{ps} - 4\Gamma_{pq}^{l} u^s u^q \right] \equiv 0. \quad (97)$$

Since the variations $\delta u_t$ and $\delta p_{pq}$ are independent, the expressions, standing in front of them must be zero. For example, using the formulae (A8) for the expression $u_r u_m V^{pmr}$, the equation in front of $\delta p_{pq}$ is found to be

$$(\partial_j u^q) u^p - u^q \Gamma_{sj}^{l} (3u^s g_{il} + 2g^{ps} u_t) + g^{kp} u^q (\partial_j u_k) \equiv 0. \quad (98)$$

Contracting with $u_p u_q$, the following important relation is obtained

$$\partial_j e = 5u^s u_l \Gamma_{js}^{l}. \quad (99)$$

The obtained result can be formulated as follows: in a projection gravitational theory with zero covariant derivatives of the projection tensor in respect to
the projection connection the partial derivative of the vector field length \(e\) can be expressed (up to a number factor, which as insignificant may be omitted) as the "twice" projected along the vector field initial connection. In other words, provided the connection and the vector field are known, the zero covariant derivative condition sets up a "certain law", according to which the vector field length may change in space and time. This of course means that under the above assumption for zero covariant derivative the variation of the vector field length turns out to be an important characteristic.

VII. CONCLUSION

In this paper a formalism has been developed for finding the variations of the vector field and the projection tensor with covariant and contravariant indices, and including only the first and not the second partial derivatives. An essential feature of the developed approach of non-commuting variation and partial derivative is that the variations are assumed to satisfy a system of equations, obtained after the variation of the defining system of equations - two systems of algebraic equations (35) and two systems (19-20) of differential equations for the derivative of the metric tensor with covariant and contravariant indices. Of course, in the last system the projective decomposition (7) is assumed to be performed. After finding all the variations and expressing all of them through variations of the vector field and the projection tensor with covariant indices only, the case with zero covariant derivative of the projection tensor has been worked out. This has been motivated by the presented argumentation in the Introduction for the necessity to compare the applied here more general approach with the already known and widely used ADM (3+1) projection approach. As noted, the basic feature of this approach is the existence of a three-dimensional Riemannian projection metric tensor with a well defined inverse one. This is of course not characteristic for the more general approach presented here, but mainly due to this reason the partial and limiting cases such as the ADM approach deserve particular attention, if the validity of the more general approach has to be tested.

It should be noted that in the present case, because of the defining equation (17) \(p_{mk}p^{ik} \equiv \delta^i_m - \frac{1}{e} u^i u_m\), the zero-covariant derivative assumption requires also that the equality

\[
\nabla (\frac{1}{e} u^i u_m) \equiv 0
\]

should be fulfilled. Indeed, equation (100), after contraction with \(u_i u_m\), gives
\[- \partial_j e + (\tilde{\nabla}_j u^i) u_i + u^m (\tilde{\nabla}_j u_m) \equiv 0, \tag{101}\]
which of course is always fulfilled, so there is no contradiction.

The applied approach in Sections IV, V and VI in the paper for considering the variations of the projection tensor with covariant and contravariant indices (under the zero-covariant derivative assumption) clearly demonstrates how important is to combine and relate the results in the two cases. In particular, it is interesting to see how the divergent term in equation (93) in Section VI (for the variations of $p^{ij}$) is expressed by means of the equation for the variations (81), obtained for the previous case of variations of the projection tensor with covariant indices. Also, for some particular cases it is of importance which equations are first combined and then - variated, and which - first variated, and afterwards - combined together. The results in the both cases turn out to be different, but this gives an opportunity to make use of all the derived equations.

The main result in this paper is formulae (99), expressing the derivative of the vector field’s length as a twice projected along the vector field connection. Now let us write down (99) in another way, using the projective decomposition (7), i.e. $u^s u_l = g^s_l e - p^s_l e$. The new obtained formulae is

\[\partial_j e = - p^s_l \Gamma^l_{js} e + \Gamma^s_{sj} e = P^s_{sj} e \tag{102}\]

and expressing the fact that the vector field’s length is proportional to the partial derivative of the logarithm of the new connection’s trace $P^s_{sj}$, i.e.

\[e = \partial_j \ln P^s_{sj} \tag{103}\]

where $P^s_{sj} = - p^s_l \Gamma^l_{js} + \Gamma^s_{sj}$. The first term in (102), more exactly - the expression before $e$, is the projected with the projection tensor $p^s_l$ initial connection $\Gamma^l_{js}$, thus accounting the change of the vector field’s length in a reference system, connected with the vector field $u$. The second term in (102) is a familiar one from affine differential geometry, where the notion of an equiaffine connection is introduced [30] - this is the connection, which presumes the existence of a covariantly constant vector or tensor field (in [30] called a “n-vector”). Since the last means that the covariant derivative of the tensor field is zero in respect to this connection, an unit volume element does not change under a parallel displacement, performed in a space with an equiaffine connection. More important, the necessary and sufficient condition for a
connection to be equiaffine is just formulae (103), but in the sense of the
definition in [30] with $P^s_{sj} \equiv \Gamma^s_{sj}$. Evidently, in the present case $P^s_{sj}$ plays the
role of an ”equiaffine” connection. Note also that when $p^s_{sj} \equiv \delta^s_{sj} \equiv g^s_{sj}$ and
there is no vector field (see (7)), from the obtained relation (102) it follows
that $\partial_s e = 0$. The equality is fulfilled also when $e = 0$, so it is seen that the
formulae is consistent and not contradictory.

Although in our case we have used the assumption about zero covariant
derivative, similar to the case of the equiaffine connection, the present formulae
is derived within the formalism of non-commuting variation and partial
derivative in respect to the projection tensor. Therefore, the derivation of a
familiar result, but with some physically justified modification, within a dif-
ferent variational approach sets up some interesting questions and problems.

The first problem is that the derived formulae (99) can also be used, after
performing a variation, to find new relations between the variations. Again
an equation for the variations will be obtained, and again from the (equal to
zero) expressions before the variations a new formulae may be obtained for
example for the relation between the projection and the initial connection.
Afterwards, the variation again may be performed and so on, following the
same procedure. In such a way the number of the derived equations may
become infinite. Since the formalism of non-commuting variation and partial
derivative depends on the equations used in the system for the variations,
each time different results will be obtained. That is why here only the defining
equations for the vector field and the projection tensor are used.

The second problem is whether the usual variational formalism with com-
muting $\delta$ and $\partial_j$ is a limiting case of a more general case of non-commuting
ones. Fortunately, there is a simple way to check this, making use of the found in
Section III variations. Also, a 3+1 decomposition should be performed,
and the already mentioned defining system (35) and (19-20) should be solved
also in the 3+1 decomposition approach. Since the variational formalism in
the ADM approach is based on commutativity, it may be expected that after
the substitution of all the relations in the formulaes for the variations, the
commutator $[\delta, \partial_j] p_{kl}$ will turn out to be zero.

It is worth mentioning also that in classical mechanics long time ago in-
vestigations on the commutativity of the variation and the partial derivative
have also been carried out. For example, in [31] it was proved that the varia-
tion (understood as a space displacement) and the time derivative commute,
provided however that the positions of the material point after and before
the variation should be referred to one and the same moment of time. Since
in a gravitational theory space and time transformations are related, it is perhaps natural to speak about non-commutativity of the variation and the partial derivative in some cases.

**APPENDIX A**

In this Appendix the exact expressions for $W_{j_{tmr}}$, $V_{j_{tmr}}$, $Y_{j_{pqmr}}$ and $Z_{p_{rms}}$ will be given. These formulae are the expressions, standing before the variations $\delta u_t, \partial_j \delta u_t, \delta p_{pq}$ and $\delta \Gamma_{jq}^p$ respectively in formulae (45).

The exact expression for $W_{j_{tmr}}$ is

$$W_{j_{tmr}} \equiv \frac{2}{e} u^s g^{mt} g^{kr} \partial_j p_{ks} + \frac{1}{e} u_k \partial_j p^{mk}(2g^{tr} + \frac{1}{e} u^t u^r) - \partial_j (\frac{1}{e} u^m p^r) - \frac{1}{e} u^t u^r \partial_j (\frac{1}{e} u^m) - g^{ms}(p^{rk} - \frac{1}{e} u^r u^k)\Gamma_{jq}^p P_{ksp} + \frac{3}{e^3} u^r u^m u^t u^s (\partial_j u_s)$$

$$+ \frac{2}{e^3} u^r u^m u^t u_s \partial_j g^{sp} + \frac{1}{e^2} u^r (\partial_j u^m) u^t + \frac{2}{e^2} u^r g^{tr} u^m (\partial_j u_s).$$

(104)

The expression $P_{rkms}$ in (A1) is

$$P_{rkms} = \frac{2}{e} (p^s_r p_{ms} u^k + p^s_r p^k_{ms}) + \frac{3}{e^2} (p_{ms} u^k u^t u_r + p^k_{ms} u^t u_r u_s) - \frac{4}{e^3} u_m u^t u^k u_r u_s + \frac{u_m}{e^2} \left[4 p^k_{ts} u^k u_s - 4 u_s u_r p^{kt} - 10 p_{sr} u^k u^t + 2 p^k_{tq} u^t u_r + 3 p^k_{tq} u^t u_s \right] +$$

$$+ \frac{u_m}{e^2} \left[-2 p_{sr} p^{kt} + p^t_{ps} p^k_r + \frac{3}{e^2} u^k u^t u_r u_s \right].$$

(105)

The square brackets $\llbracket \rrbracket$ in (A1) in respect to the tensor indices mean antisymmetrization, i.e. $\llbracket mt \rrbracket = mt - tm$.

The expressions for $V_{j_{pqmr}}$, $Y_{j_{pqmr}}$ and $Z_{p_{rms}}$ are given in the formulae (A3 - A5) below

$$V_{j_{tmr}} \equiv - \frac{1}{e} u^r \left[ g^{mt} + \frac{3}{2e} u^m u^t \right]$$

(106)

$$Y_{j_{pqmr}} \equiv -\Gamma_{jn}^t (\frac{1}{e} u^r u^a g^{mt} u^m u_t + g^t_{pq} g^{r(nq)m}) + g^{ms}(p^{r(p} p^{k)p} - \frac{1}{e} g^{kp} u^r u^a) (\partial_j p_{ks}) +$$

$$+ \frac{2}{e^3} u^s u^m u^t u_r \partial_j g^{sp} + \frac{1}{e^2} u^r (\partial_j u^m) u^t + \frac{2}{e^2} u^r g^{tr} u^m (\partial_j u_s).$$

(107)

The expression $\Gamma_{jq}^p$ in (A1) is

$$\Gamma_{jq}^p = \frac{2}{e} (p^s_r p_{ms} u^k + p^s_r p^k_{ms}) + \frac{3}{e^2} (p_{ms} u^k u^t u_r + p^k_{ms} u^t u_r u_s) - \frac{4}{e^3} u_m u^t u^k u_r u_s + \frac{u_m}{e^2} \left[4 p^k_{ts} u^k u_s - 4 u_s u_r p^{kt} - 10 p_{sr} u^k u^t + 2 p^k_{tq} u^t u_r + 3 p^k_{tq} u^t u_s \right] +$$

$$+ \frac{u_m}{e^2} \left[-2 p_{sr} p^{kt} + p^t_{ps} p^k_r + \frac{3}{e^2} u^k u^t u_r u_s \right].$$

(108)

The square brackets $\llbracket \rrbracket$ in (A1) in respect to the tensor indices mean antisymmetrization, i.e. $\llbracket mt \rrbracket = mt - tm$.
\[ \partial_j \left( \frac{1}{e} u^r u^q \right) g^{mp} + \frac{1}{e} p^{mp} p^{kq} u^r \partial_j u_k \]  

(107)

\[ Z_{p}^{rmq} \equiv -g_{pr}^{m} p^{mq} - g^{rq} p_{p}^{m} + \frac{1}{e} u^r g^{mq} u_p + \frac{1}{e} u^r g_{p}^{m} u^q . \]  

(108)

Since the twice projected along the vector field \( u \) expressions \( u_r u_m W_{j}^{tmr} \), \( u_r u_m V_{j}^{tmr} \), \( u_r u_m Y_{j}^{pqmr} \) and \( u_r u_m \partial_j V_{j}^{rmn} \) are also used in the calculations in the preceding sections, the exact expressions are also given below.

\[ u_r u_m W_{j}^{tmr} \equiv \frac{3}{e} u^t (\partial_j e) - u_s (\partial_j g^{qs}) + \frac{2}{e} u^t u^s (\partial_j u_s) - u^s u^k \Gamma_{j}^{k} P_{ksp} \]  

(109)

\[ u_r u_m V_{j}^{tmr} \equiv -5 u^t \]  

(110)

\[ u_r u_m (\partial_j V_{j}^{tmr}) \equiv \frac{5}{e} u^t u^r \partial_j u_r - 5 \partial_j u^t \]  

(111)

\[ u_r u_m Y_{j}^{pqmr} \equiv -u^q \Gamma_{jn}^{l} [e g^{pm} + 2 u^n g_{l}^{p}] - u^s g^{kp} u^q \partial_j p_{ks} + (\partial_j u^q) u^p - \frac{1}{e} u^r (\partial_j u_r) u^q u^p \]  

(112)

Note that the last term will not give contribution in the expression for \( u_r u_m Y_{j}^{pqmr} \delta_{pq} \) because of the property \( u^p u^q \delta_{pq} = 0 \).

Further, the expressions \( M_{pqmr} \) and \( N_{mrt} \) in (92) can be written as follows

\[ M_{pqmr} \equiv \frac{2}{e^2} (\partial_j u_s) u^q p^{ps} u^m u^r - \partial_j \left( \frac{1}{e} u^r (u^q u^p) \right) + \Gamma_{sj}^{(m)} p_{j}^{r} q + \]  

\[ + \Gamma_{sj}^{l} \left( g_{l}^{(m) p} q + \frac{1}{e} g_{l}^{p} u^s g^{q(m)u_r} - \frac{1}{e} u^q u^r \left( g_{l}^{p} g^{sm} + \frac{1}{2} g_{ps} p_{j}^{m} \right) \right) \]  

(113)

\[ N_{mrt} \equiv \frac{2}{e} u^r (g^{m}_{p} g^{pt} q) u^{qt} - \frac{3}{e} u^t u^m \partial_j \left( \frac{1}{e} u^r \right) - \frac{4}{e^2} (\partial_j u^t) u^r u^m - \frac{2}{e^2} (\partial_j u^t) u_s u^t u^r u^m + \]  

\[ + \Gamma_{jq}^{p} \left[ \frac{3}{e^2} u^r u_n g^{m}_{p} g^{nt} q - \frac{1}{e} u^r g^{mk} u^n p_{j}^{lt} q - \frac{2}{e} u^r g^{m}_{p} p_{j}^{lt} q \right] \]  

(114)
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