Approximate fixed points and B-amenable groups

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Abstract

A topological group $G$ is B-amenable if and only if every continuous affine action of $G$ on a bounded convex subset of a locally convex space has an approximate fixed point. Similar results hold more generally for slightly uniformly continuous semigroup actions.

1 Introduction

For a locally compact group $G$, denote by $L_\infty(G)$ the space of (equivalence classes of) bounded Haar-measurable functions, by $C_b(G)$ the space of bounded continuous functions on $G$, and by $U_b(rG)$ the space of bounded right uniformly continuous functions on $G$. The following properties are equivalent characterizations of amenable locally compact groups [2][6][7][9].

(A) There exists a left-invariant mean on $U_b(rG)$.
(B) There exists a left-invariant mean on $C_b(G)$.
(C) There exists a left-invariant mean on $L_\infty(G)$.
(D) Every continuous affine action of $G$ on a compact convex set has a fixed point.

Properties (A) to (D) have been extended in many directions, including these two:

1. Expand the scope from locally compact groups to general topological groups. Then $L_\infty(G)$ is no longer available, but $C_b(G)$ and $U_b(rG)$ are well-defined. Conditions (A) and (D) are still equivalent but (A) and (B) are not [7]: The infinite symmetric group with the topology of pointwise convergence and the unitary group of a separable infinite dimensional Hilbert space with the strong operator topology have property (A) but not (B). Following [2], a topological group $G$ is defined to be amenable iff it satisfies property (A) and B-amenable iff it satisfies (B).

2. Consider affine actions on more general convex sets. Here fixed points are too much to ask for. The next best thing are approximate fixed points, with property (D) replaced by

(E) Every continuous affine action of $G$ on a bounded convex set has an approximate fixed point.

Barroso, Mbombo and Pestov [1] studied property (E). They proved several results relating (A) to the existence of approximate fixed points, and showed that the infinite symmetric group does not have property (E).

By Theorem 5.1 of the current paper, (B) is equivalent to (E) for general topological groups. The proof is not specific to continuous actions by topological groups. It applies equally well to semigroup actions that satisfy a certain quite weak uniform continuity property; that is presented in section 3 after definitions and preliminary results in section 2. The general result covers also the case of amenable groups, due to Schneider and Thom [12], which is derived in section 4.
2 Preliminaries

Throughout the paper, linear spaces are over the scalar field, either the reals or the complex numbers. Functions are mappings to the scalar field. Most of the following notation and terminology is taken from [8]. Several prerequisites are proved in [8] for real-valued functions; the complex case follows simply by considering the real and imaginary part of the function.

Topological, uniform and locally convex spaces are assumed to be Hausdorff. When $S$ is a uniform space, $U_b(S)$ denotes the space of bounded uniformly continuous functions on $S$ with the sup norm. When $G$ is a topological group, $rG$ is the right uniform space of $G$; that is, the set $G$ with the right uniform structure of $G$. For any completely regular topological space $T$, $rT$ denotes the set $T$ with the fine uniform structure [8, 1.25]. Thus $U_b(rT) = C_b(T)$, the space of bounded continuous functions on $T$. When $S$ is a set with the discrete uniformity, $U_b(S)$ is simply $\ell_\infty(S)$.

When $X$ is a locally convex space, $X^*$ denotes its topological dual. In particular, when $S$ is a uniform space, $U_b(S)^*$ is the dual of the Banach space $U_b(S)$.

Let $S$ be a semigroup, $s \in S$, and let $f$ be a function on $S$. Define the functions $sf$ and $fs$ (the left and right translate of $f$) by $sf(t) := f(st)$ and $fs(t) := f(ts)$ for $t \in S$.

I need a short name for a semigroup $S$ with a uniform structure on $S$ such that $sf \in U_b(S)$ and $fs \in U_b(S)$ for all $s \in S$, $f \in U_b(S)$. Such a structure will be called an FIU semigroup, or an FIU group when $S$ is actually a group. (FIU stands for Functionally Invariant Uniform structure.)

Examples.

(a) Let $G$ be a topological group. Then $G$ with any one of the left, right, upper and lower uniformities [10] is an FIU group.

(b) Let $S$ be a semitopological semigroup; that is, a Hausdorff topological space with a separately continuous semigroup operation. Then $S$ with its fine uniformity is an FIU semigroup. In particular, any topological group with its fine uniformity is an FIU group.

(c) Every semiuniform semigroup [8, 3.14] is an FUI semigroup. That includes, for example, the completion of any topological group with its right uniformity, and the unit ball of any Banach algebra with the multiplication operation and the metrizable uniform structure defined by the norm.

(d) Let $S$ be a semigroup, and let $A$ be a linear subspace of $\ell_\infty(S)$ that separates the points of $S$ and is invariant under left and right translations by elements of $S$. Equip $S$ with the uniformity induced by $A$; that is, the coarsest uniformity for which $A \subseteq U_b(S)$. Then $S$ is a precompact FIU semigroup.

When $S$ is an FIU semigroup and $C$ is a subset of a locally convex space $X$, $S \Join C$ means that $S$ acts on $C$ by a mapping $(s, x) \mapsto s \cdot x$ from $S \times C$ to $C$. The action $S \Join C$ is said to be affine if the mapping $x \mapsto s \cdot x$ is affine for every $s \in S$. The action is jointly or separately continuous iff the mapping $(s, x) \mapsto s \cdot x$ is such. The action is slightly continuous [9] iff there is at least one $x \in C$ such that the mapping $s \mapsto s \cdot x$ is continuous from $S$ to $X$. More generally, the action is slightly uniformly continuous iff there is at least one $x \in C$ such that the mapping $s \mapsto s \cdot x$ is uniformly continuous from $S$ to $X$. Obviously every separately continuous action is slightly continuous. An approximate fixed point for the action $S \Join C$ is a net $\{x_\gamma\}_\gamma$ in $C$ such that $\lim_\gamma x_\gamma - s \cdot x_\gamma = 0$ in the topology of $X$ for every $s \in S$.

For $s \in S$, the point mass $\partial(s) \in U_b(S)^*$ at $s$ is defined by $\partial(s)(f) := f(s)$ for $f \in U_b(S)$. The subspace of $U_b(S)^*$ consisting of finite linear combinations of point masses (called molecular measures) is denoted by $\text{Mol}(S)$. The weak* topology on $\text{Mol}(S)$ is the restriction of the weak* topology (the $U_b(S)$-weak topology) on $U_b(S)^*$.
For $s \in S$ and $m \in U_b(S)^*$, define $s \ast m \in U_b(S)^*$ by $s \ast m(f) := m(sf)$ for $f \in U_b(S)$. Clearly $s \ast (t) = (st)$ for $s, t \in S$. Say that $m \in U_b(S)^*$ is left-invariant iff $s \ast m = m$ for all $s \in S$.

A functional $m \in U_b(S)^*$ is a mean iff $m \geq 0$ and $m(1) = 1$. Denote by $2\mathcal{M}^{+1}(S)$ the set of means in $U_b(S)^*$. In the weak$^*$ topology on $U_b(S)^*$, $2\mathcal{M}^{+1}(S)$ is a compact convex set. Write $\text{Mol}^{+1}(S) := 2\mathcal{M}^{+1}(S) \cap \text{Mol}(S)$. Every $m \in \text{Mol}^{+1}(S)$ is of the form $m = \sum_{i=0}^j r_i \partial(s_i)$ where $0 \leq r_i \leq 1$ and $s_i \in S$ for $i = 0, 1, \ldots, j$, and $\sum_{i=0}^j r_i = 1$.

When $S$ is a uniform space, a subset of $U_b(S)$ is a UEB($S$) set iff it is uniformly equicontinuous and bounded in the sup norm [8, 1.19]. The UEB($S$) topology on $U_b(S)^*$ is the topology of uniform convergence on UEB($S$) subsets of $U_b(S)$. In the particular case $S = rT$, where $T$ is a completely regular space, the UEB($rT$) topology on $C_b(T)^*$ is also called the EB($T$) topology; it is the topology of uniform convergence on equicontinuous uniformly bounded subsets of $C_b(T)$.

**Lemma 2.1.** Let $S$ be an FIU semigroup.

(a) When $\text{Mol}(S)$ is equipped with the UEB($S$) topology, the dual of $\text{Mol}(S)$ is $U_b(S)$.

(b) The set $\text{Mol}^{+1}(S)$ is weak$^*$ dense in $2\mathcal{M}^{+1}(S)$.

(c) Let $s \in S$. Equip $U_b(S)^*$ with the weak$^*$ topology. Then the mapping $m \mapsto s \ast m$ is continuous from $U_b(S)^*$ to itself.

(d) Let $m \in \text{Mol}(S)$. Equip $\text{Mol}(S)$ with the weak$^*$ topology. Then the mapping $s \mapsto s \ast m$ is uniformly continuous from $S$ to $\text{Mol}(S)$.

(e) Let $X$ be a locally convex space. Let $\Phi: S \to X$ be a uniformly continuous mapping whose range $\Phi(S)$ is bounded in $X$. Denote by $\tilde{\Phi}: \text{Mol}(S) \to X$ the unique linear mapping such that $\tilde{\Phi} \circ \partial = \Phi$. Then $\tilde{\Phi}$ is continuous from $\text{Mol}(S)$ with the weak$^*$ topology to $X$ with its weak topology.

**Proof.** (a) (See Lemma 6.5 in [8].) If $F \subseteq U_b(S)$ is a UEB($S$) set such that $\|f\| \leq 1$ for $f \in F$ then

$$\Delta(s, t) := \sup \{ |f(s) - f(t)| : f \in F \}, \quad s, t \in S,$$

defines a uniformly continuous pseudometric $\Delta$ on $S$, and $F$ is a subset of $\text{BLip}_b(\Delta) := \{ f \in U_b(S) : \|f\| \leq 1 \text{ and } |f(s) - f(t)| \leq \Delta(s, t) \text{ for } s, t \in S \}$.

Thus the UEB($S$) topology is the topology of uniform convergence on the sets $\text{BLip}_b(\Delta)$ for uniformly continuous pseudometrics $\Delta$. Each such set is compact in the $\text{Mol}(S)$-weak topology on $U_b(S)$. Apply the Mackey–Arens theorem [11, IV.3.2].

(b) is a simple application of the Hahn–Banach theorem: The set $\text{Mol}^{+1}(S)$ is convex. If a real-valued $f \in U_b(S)$ and a real number $r$ are such that $m(f) \leq r$ for all $m \in \text{Mol}^{+1}(S)$ then in particular $f(s) \leq r$ for all $s \in S$, therefore $m(f) \leq r$ for all $m \in \text{Mol}^{+1}(S)$.

(c) follows from the definition of $s \ast m$.

d) Write $m = \sum_{i=0}^j r_i \partial(s_i)$ and fix $f \in U_b(S)$. For $s, t \in S$ we have

$$|s \ast m(f) - t \ast m(f)| = \sum_{i=0}^j r_i |s \ast \partial(s_i)(f) - t \ast \partial(s_i)(f)| \leq \sum_{i=0}^j |r_i| \cdot |f_{s_i}(s) - f_{s_i}(t)|.$$

Since $f_{s_i} \in U_b(S)$ for $i = 0, 1, \ldots, j$, it follows that the function $s \mapsto s \ast m(f)$ is uniformly continuous.

e) Take any $\xi \in X^*$. Then $\xi \circ \Phi$ is uniformly continuous, and it is bounded because $\Phi(S)$ is bounded in $X$. Thus $\xi \circ \Phi \in U_b(S)$. From the linearity of $\tilde{\Phi}$ we get $\xi(\tilde{\Phi}(m)) = m(\xi \circ \Phi)$ for all $m \in \text{Mol}(S)$. □
3 Approximate fixed point property

Let $C$ be a convex subset of a locally convex space $X$. Following Barroso et al. [1], say that a topological group $G$ has the approximate fixed point (AFP) property on $C$ if for every jointly continuous affine action of $G$ on $C$ has an approximate fixed point.

When $C$ is compact, the AFP property is equivalent to the existence of fixed points. Indeed, if $\{x_\gamma\}_\gamma$ is an approximate fixed point for $G \curvearrowright C$ and if the mapping $x \mapsto g \cdot x$ is continuous on $C$ for every $g \in G$ then every cluster point of the net $\{x_\gamma\}_\gamma$ in $C$ is a fixed point.

The following theorem, an abstract version of the “celebrated method of Day” [3] [8], states that an affine action has an approximate fixed point if it has a weak approximate fixed point.

**Theorem 3.1.** Let $S$ be a semigroup, and let $S \curvearrowright C$ be an affine action of $S$ on a convex subset of a locally convex space $X$. Assume there is a net $\{x_\gamma\}_\gamma$ in $C$ such that $\lim_{\gamma} x_\gamma - s \cdot x_\gamma = 0$ in the weak topology of $X$ for every $s \in S$. Then the action has an approximate fixed point.

**Proof.** Let $Y := X^S$ be the space of all families $\{x_s\}_{s \in S}$ of $x_s \in X$; that is, the product of copies of $X$, one for each $s \in S$. Consider two locally convex topologies on $Y$: $t$ is the product topology, and $t_w$ is the product of the weak topologies on the copies of $X$. The weak topology on the product of locally convex spaces is the product of weak topologies [11, IV.4.3]. Hence $t$ and $t_w$ on $Y$ have the same dual and therefore the same closed convex sets [11 II.9.2]. Define the convex set $D := \{\{x - s \cdot x\}_{s \in S} \mid x \in C\}$. By the assumption, $0 \in Y$ is in the $t_w$ closure of $D$ which is also the $t$ closure of $D$, so that the conclusion holds.

Next we apply Theorem 3.1 to the natural affine action $(s, m) \mapsto s \ast m$ of an FIU semigroup $S$ on $M^+(S)$. The set $M^+(S) \subseteq M^+(S)$ is $S$-invariant under the action; in fact, if $m = \sum_{i=0}^\infty r_i \partial(s_i)$ then $s \ast m = \sum_{i=0}^\infty r_i \partial(ss_i)$.

The next theorem is essentially due to Day [2, 3], for whom approximate fixed point were a means for constructing fixed points. The means in $M(S)$ are the finite means of Day. The equivalence (i) $\iff$ (iv) is a variant of a fixed-point result mentioned by Berglund et al. [2, 2.3.30] in the general setting of invariant means on a space $A \subseteq \ell_\infty(S)$.

**Theorem 3.2.** The following properties of an FIU semigroup $S$ are equivalent:

(i) There exists a left-invariant mean in $M^+(S)$.
(ii) There exists a net $\{m_\gamma\}_\gamma$ in $M^+(S)$ such that for every $s \in S$ we have $\lim_{\gamma} m_\gamma - s \ast m_\gamma = 0$ in the weak* topology.
(iii) There exists a net $\{m_\gamma\}_\gamma$ in $M^+(S)$ such that for every $s \in S$ we have $\lim_{\gamma} m_\gamma - s \ast m_\gamma = 0$ in the UEB(S) topology.
(iv) Every slightly uniformly continuous affine action of $S$ on a bounded convex subset of a locally convex space has an approximate fixed point.

**Proof.** First let $m \in M^+(S)$ be a left-invariant mean. By 2.4(b), there is a net $\{m_\gamma\}_\gamma$ in $M^+(S)$ such that $\lim_{\gamma} m_\gamma = m$ in the weak* topology. Then $\lim_{\gamma} s \ast m_\gamma = s \ast m$ for every $s \in S$ by 2.4(c), hence $\lim_{\gamma} m_\gamma - s \ast m_\gamma = 0$. That proves (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (i) let $\{m_\gamma\}_\gamma$ be a net in $M^+(S)$ such that $\lim_{\gamma} m_\gamma - s \ast m_\gamma = 0$ in the weak* topology for every $s \in S$. The set $M^+(S)$ is weak* compact, hence the net $\{m_\gamma\}_\gamma$ has a cluster point $m \in M^+(S)$. Then $s \ast m = m$ for $s \in S$ by 2.4(c).

Obviously (iii) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) follows from 2.1(a) and Theorem 3.1.

The action $(s, m) \mapsto s \ast m$ of $S$ on the convex set $M^+(S) \subseteq M(S)$ is slightly uniformly continuous when $M(S)$ is given the weak* topology. Indeed, by 2.1(d) for every $m \in M^+(S)$ the mapping $s \mapsto s \ast m$ is uniformly continuous. Hence (iv) $\Rightarrow$ (ii).
To prove (ii)→(iv), let $S \subset C$ be a slightly uniformly continuous affine action on a bounded convex subset $C$ of a locally convex space $X$. There is $x \in C$ for which the mapping $s \mapsto \Phi(s) := s \cdot x$ is uniformly continuous from $S$ to $X$. Let $\Phi : \text{Mol}(S) \to X$ be the mapping from 2.1(e). We have $s \cdot \Phi(t) = \Phi(st)$ for $s, t \in S$, hence $s \cdot \Phi(m) = \Phi(s \cdot m)$ for $s \in S$ and $m \in \text{Mol}^+(S)$. Let $m_\gamma$ be as in (ii). Set $x_\gamma := \Phi(m_\gamma)$. Then $x_\gamma - s \cdot x_\gamma = \Phi(m_\gamma) - \Phi(s \cdot m_\gamma) = \Phi(m_\gamma - s \cdot m_\gamma)$, and by 2.1(e) we get $\lim_{\gamma} x_\gamma - s \cdot x_\gamma = 0$ in the weak topology of $X$ for every $s \in S$. Hence (iv) follows by Theorem 3.1. □

4 Amenable groups

When $G$ is a topological group, $rG$ is an FIU group. The group $G$ is said to be amenable [7] iff there is a left-invariant mean on $U_b(rG)$. The following instance of Theorem 3.2 is due to Schneider and Thom (Theorem 3.2 and Corollary 4.8 in [12]).

**Theorem 4.1.** The following properties of a topological group $G$ are equivalent:

(i) $G$ is amenable.
(ii) There exists a net $\{m_\gamma\}_\gamma$ in $\text{Mol}^+(G)$ such that for every $g \in G$ we have $\lim_{\gamma} m_\gamma - g \cdot m_\gamma = 0$ in the $U_b(rG)$-weak topology.
(iii) There exists a net $\{m_\gamma\}_\gamma$ in $\text{Mol}^+(G)$ such that for every $g \in G$ we have $\lim_{\gamma} m_\gamma - g \cdot m_\gamma = 0$ in the UEB$(rG)$ topology.
(iv) Every slightly uniformly continuous affine action of $rG$ on a bounded convex subset of a locally convex space has an approximate fixed point.

5 B-amenable groups

When $G$ is a topological group, $rG$ is an FIU group. The group $G$ is said to be B-amenable [7] iff there is a left-invariant mean on $C_b(G) = U_b(rG)$.

**Theorem 5.1.** The following properties of a topological group $G$ are equivalent:

(i) $G$ is B-amenable.
(ii) There exists a net $\{m_\gamma\}_\gamma$ in $\text{Mol}^+(G)$ such that for every $g \in G$ we have $\lim_{\gamma} m_\gamma - g \cdot m_\gamma = 0$ in the $C_b(G)$-weak topology.
(iii) There exists a net $\{m_\gamma\}_\gamma$ in $\text{Mol}^+(G)$ such that for every $g \in G$ we have $\lim_{\gamma} m_\gamma - g \cdot m_\gamma = 0$ in the EB$(G)$ topology.
(iv) Every slightly continuous affine action of $G$ on a bounded convex subset of a locally convex space has an approximate fixed point.
(v) Every separately continuous affine action of $G$ on a bounded convex subset of a locally convex space has an approximate fixed point.
(vi) Every jointly continuous affine action of $G$ on a bounded convex subset of a locally convex space has an approximate fixed point.

**Proof.** Properties (i) (ii) (iii) and (iv) are equivalent by Theorem 3.2. Obviously (iv)⇒(v)⇒(vi).

To prove (vi)⇒(ii) it is enough to show that the action $\ast$ of $G$ on $\text{Mol}^+(G)$ is jointly continuous when $\text{Mol}^+(G)$ is equipped with the $C_b(G)$-weak topology. By [S 5.16], the $C_b(G)$-weak topology and the $U_b(rG)$-weak topology coincide on $\text{Mol}^+(G)$. It follows that the action is jointly continuous by [S 9.36-3]. □

In the terminology of [11], (vi) states that $G$ has the AFP property on every bounded convex subset of a locally convex space.
6 Concluding remarks

The equivalence $(iv) \iff (v) \iff (vi)$ in Theorem 5.1 stands in contrast to fixed-point properties for affine actions on compact convex sets [4][5][7]: The infinite symmetric group has the fixed-point property for separately continuous actions but not for slightly continuous ones. I do not know if there is a topological group that has the fixed-point property for affine jointly continuous actions but not for separately continuous ones.

Barroso et al [1, 3.4] prove that every amenable locally compact group has the AFP property on every complete convex bounded subset of a locally convex space, and ask if the same holds without the complete restriction. The answer is yes by Theorem 5.1 because every amenable locally compact group is B-amenable [7, 3.10].

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References

[1] Barroso, C. S., Mambono, B. R., and Pestov, V. G. On topological groups with an approximate fixed point property. An. Acad. Brasil. Ciênc. 89, 1 (2017), 19–30.

[2] Berglund, J. F., Junghenn, H. D., and Milnes, P. Analysis on semigroups. John Wiley & Sons Inc., New York, 1989.

[3] Day, M. M. Amenable semigroups. Illinois J. Math. 1 (1957), 509–544.

[4] Day, M. M. Fixed-point theorems for compact convex sets. Illinois J. Math. 5 (1961), 585–590.

[5] Day, M. M. Correction to my paper “Fixed-point theorems for compact convex sets”. Illinois J. Math. 8 (1964), 713.

[6] Greenleaf, F. P. Invariant means on topological groups and their applications. Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York-Toronto-London, 1969.

[7] Grigorchuk, R., and de la Harpe, P. Amenability and ergodic properties of topological groups: from Bogolyubov onwards. Groups, graphs and random walks, London Math. Soc. Lecture Note Ser., Vol. 436. Cambridge Univ. Press, Cambridge, 2017, pp. 215–249.

[8] Pachl, J. Uniform spaces and measures, Fields Institute Monographs, Vol. 30. Springer, New York, 2013. Corrections and supplements: www.fields.utoronto.ca/publications/supplements

[9] Paterson, A. L. T. Amenability, Mathematical Surveys and Monographs, Vol. 29. American Mathematical Society, Providence, RI, 1988.

[10] Roeckele, W., and Dierolf, S. Uniform structures on topological groups and their quotients. McGraw-Hill, New York, 1981.

[11] Schaefer, H. H. Topological vector spaces. Springer-Verlag, New York, 1971. Third printing corrected.

[12] Schneider, F. M., and Thom, A. On Følner sets in topological groups. Compos. Math. 154, 7 (2018), 1333–1361.