Simple mass relations for bulk fermions

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Abstract

Relations between bulk mass parameters for fermions propagating in higher dimensions are studied in analogy with the empirical mass relation for charged leptons. Masses of three generation of four-dimensional charged leptons are achieved from the bulk mass parameters of the same-order values. We find that the observed pattern of charged lepton mass spectrum is accommodated approximately in a set of simple relations for bulk and physical masses.
1 Introduction

The hierarchical structure such as macroscopic and microscopic pictures has provided an intuitive and predictable way to understand physics as effective field theory. The type of dynamical variables to describe phenomena in the theory depends on the energy scale of interest. The standard model of elementary particles is effective field theory for weak, strong and electromagnetic interactions. It is placed in a specific region of expansive energy scales. One can effectively focus on the energy scale of interest without knowing details at other scales. In this aspect, the hierarchy is favorable. On the other hand, the dynamical variables of the standard model possess yet another hierarchy between parameters of the theory itself. It may be expected that parameters of a theory would be of the same order if the theory gives a physical description in a certain energy region. For the variables of the standard model, fermion masses are not of the same order.

Effective field theory should have a typical energy scale which is stabilized. The weak scale of the standard model is not protected from radiative corrections because masses of fields can receive ultraviolet sensitive quantum corrections. Candidates to stabilize energy scales of a theory have been presented based on a possibility of extra dimensions [1, 2]. As for the other hierarchy, an explanation for generating fermion masses from order $O(1)$ quantities was proposed in an extra-dimensional model where massive fermions propagate in the bulk [3]. Here bulk masses of fermions are of the same order and the resulting low energy fermion masses are hierarchical via hyperbolic trigonometric functions for overlapping wave functions, as we will write down explicitly in the subsequent section. This picture is applicable for an extension of the standard model. It would be straightforward to construct a model, if the values of bulk mass parameters are chosen by hand like the standard model. In treating extra-dimensional models, it would be important to identify what becomes to constrain the original parameters in the bulk.

From a viewpoint of effective theory, focusing on relations between values of masses is as important as individual absolute values of the masses. For example, in hadron physics, the masses of pion, kaon and $\eta$ approximately satisfy a version of the Gell-Mann–Okubo relation [4]:

$$3m_{\eta}^2 + 2m_{\pi}^2 - m_{\pi_0}^2 = 2m_{K_+}^2 + 2m_{K_0}^2.$$  \hspace{1cm}(1.1)

The $SU(3)$ properties gave insight to leading to quantum chromodynamics (QCD). The pion, kaon and $\eta$ are not fundamental dynamical variables of QCD. This implies that to find mass relations in a theory can be a clue to discover a property in a more fundamental theory. It is well known that charged lepton masses satisfy the relation [5]:

$$m_e + m_\mu + m_\tau = \frac{2}{3} (\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2.$$  \hspace{1cm}(1.2)

For the central values of the pole masses in Particle Data Group [6]:

$$m_e = 0.510998910, \quad m_\mu = 105.658367, \quad m_\tau = 1776.84,$$  \hspace{1cm}(1.3)

in unit of MeV, the factor $2/3$ in Eq. (1.2) is given by $1.999978/3$. Mass relations for quarks can be taken into account and they would receive large QCD corrections. It needs to be examined if there are simple relations between the same-order parameters for bulk masses similar to Eq. (1.2).
In this paper, we consider a five-dimensional toy model for charged leptons on the orbifold \(S^1/Z_2\) where the five-dimensional spacetime is flat. The gauge group is \(SU(2)_L \times U(1)_Y\) to describe weak and electromagnetic interactions. The leptons propagating in five dimensions are Dirac fermions and have Dirac masses that are not hierarchical originally. By orbifolding, components with zero modes which are left-handed or right-handed survive in low energies and low energy theory becomes chiral. For chiral anomaly cancellation, quarks could be introduced in a parallel way. The low energy chiral leptons combine with each other to form mass terms through Yukawa couplings given on a single boundary. The four-dimensional Yukawa couplings are multiplied by wave functions of the leptons and depend on their bulk masses. The hierarchical masses are due to large suppression factors of the wave functions depending on bulk masses. Instead of choosing reasonable bulk masses by hand, we study a possibility that their bulk masses fulfill simple relations like Eq. (1.2). In other words, we search for mass relations to specify how the fundamental variables of our effective theory are constrained by substructural properties. We find that a set of simple relations accommodates the observed pattern of charged lepton mass spectrum.

2 Model

The field content is \(SU(2)_L\) gauge boson \(A^i_M(x, y)\), \(U(1)_Y\) gauge boson \(B_M(x, y)\), \(SU(2)_L\)-doublet Higgs boson \(H(x)\), \(SU(2)_L\)-doublet Dirac fermion \(E^i(x, y)\), and \(SU(2)_L\)-singlet Dirac fermion \(e^i(x, y)\). The fermions are written in terms of chiral components as

\[
E^i(x, y) = \begin{pmatrix} E^i_L \cr E^i_R \end{pmatrix} (x, y), \quad e^i(x, y) = \begin{pmatrix} e^i_L \cr e^i_R \end{pmatrix} (x, y), \quad (2.1)
\]

where \(i = 1, 2, 3\). Capitalized indices \(M, N\) run over \(0, 1, 2, 3, 5\), Greek indices \(\mu\) run over \(0, 1, 2, 3\) and fifth index is also denoted as \(y\). We use a metric \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). The extra-dimensional space is compactified on \(S^1/Z_2\), where the fundamental region is \(0 \leq y \leq L\). The five-dimensional spacetime is flat. The charged leptons are included as

\[
E^i_L = \begin{pmatrix} \nu^i_L \\ e^i_L \end{pmatrix} = \begin{pmatrix} (\nu_e)_{L} \\ (e^-)_{L} \end{pmatrix}, \quad (2.2)
\]

In a parallel way, it is possible to include quarks for anomaly cancellation. We will not discuss quarks further. The covariant derivative is given by

\[
D_M = \partial_M - igA^i_M T^a - ig' \frac{Y}{2} B_M. \quad (2.3)
\]

The hypercharges are assigned as \(Y/2 = 1/2\) for the Higgs \(H\), \(Y/2 = -1/2\) for the doublet \(E^i\) and \(Y/2 = -1\) for the singlet \(e^i\), which are combined with \(T^3 = \pm 1/2\) to give the correct electric charges.

The fermion kinetic energy terms and mass terms are

\[
\mathcal{L} = \bar{E}^i \left( i \gamma^M \partial_M - M^i_E \epsilon(y) \right) E^i + \bar{e}^i \left( i \gamma^M \partial_M - M^i_e \epsilon(y) \right) e^i, \quad (2.4)
\]

where Dirac conjugate is taken as \(\bar{E}^i = E^{i\dagger} \gamma^0\) and \(\bar{e}^i = e^{i\dagger} \gamma^0\). The bulk masses are denoted as \(M^i_E\) and \(M^i_e\). For \(E^i\) and \(e^i\), we assume the economical case \(M^i_E = M^i_e \equiv M^i\).
for each $i$. To examine relations between masses for generations $i$ is our main analysis. The basis for the Dirac matrices is

$$\gamma^M = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{array} \right),$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and the sign function is denoted as $\epsilon(y) = 1$ for $0 < y < L$ mod $2L$ and $\epsilon(y) = -1$ for $-L < y < 0$ mod $2L$.

The parity assignments at $y = 0, L$ are taken as

$$A_\mu(x, -y) = A_\mu(x, y), \quad A_\mu(x, L - y) = A_\mu(x, L + y),$$

$$A_5(x, -y) = -A_5(x, y), \quad A_5(x, L - y) = -A_5(x, L + y),$$

$$B_\mu(x, -y) = B_\mu(x, y), \quad B_\mu(x, L - y) = B_\mu(x, L + y),$$

$$B_5(x, -y) = -B_5(x, y), \quad B_5(x, L - y) = -B_5(x, L + y),$$

$$E^i(x, -y) = i\gamma^5 E^i(x, y), \quad E^i(x, L - y) = i\gamma^5 E^i(x, L + y),$$

$$e^i(x, -y) = -i\gamma^5 e^i(x, y), \quad e^i(x, L - y) = -i\gamma^5 e^i(x, L + y).$$

The $SU(2)_L$ gauge boson $A_\mu$ has the parity $+$ at $y = 0$ and $+$ at $y = L$, which are denoted as $(++)$. The parity $(++)$ are for the fields $A_\mu$, $B_\mu$, $E^i_L$ and $e^i_R$. The other fields $A_5$, $B_5$, $E^i_R$ and $e^i_L$ have the parity $(-)$. The scalars $A_5$ and $B_5$ do not develop vacuum expectation values. The gauge group $SU(2)_L \times U(1)_Y$ is broken to the electromagnetic $U(1)_{em}$ by a vacuum expectation value of the Higgs boson $H$ on a boundary.

From Eq. (2.4), the equations of motion are

$$\left( \begin{array}{cc} \partial_5 - M^i \epsilon(y) & i\sigma \cdot \partial \\ i\bar{\sigma} \cdot \partial & -\partial_5 - M^i \epsilon(y) \end{array} \right) \left( \begin{array}{c} \psi^i_L \\ \psi^i_R \end{array} \right) = 0. \quad (2.12)$$

Here $\psi = E, e$. This equation is written as second differential equations as

$$[\partial^2 - (\partial_5 + M^i \epsilon)(\partial_5 - M^i \epsilon)] \psi^i_L = 0, \quad (2.13)$$

$$[\partial^2 - (\partial_5 - M^i \epsilon)(\partial_5 + M^i \epsilon)] \psi^i_R = 0. \quad (2.14)$$

With the mode expansion

$$\psi^i_L(x, y) = \sum_{n=0}^{\infty} \psi^i_{L(n)}(x) f_{\psi, n}(y), \quad \psi^i_R(x, y) = \sum_{n=0}^{\infty} \psi^i_{R(n)}(x) g_{\psi, n}(y), \quad (2.15)$$

the wave functions obey

$$(\partial_5 + M \epsilon)(\partial_5 - M \epsilon)f_{\psi, n}(y) = -M_n^2 f_{\psi, n}(y), \quad (2.16)$$

$$(\partial_5 - M \epsilon)(\partial_5 + M \epsilon)g_{\psi, n}(y) = -M_n^2 g_{\psi, n}(y). \quad (2.17)$$

where $M_n$ indicates the Kaluza-Klein mass. For simplicity for reading, $i$ has been abbreviated. These equations are written as

$$(\partial_5 - M \epsilon)f_{\psi, n}(y) = M_n g_{\psi, n}(y), \quad (\partial_5 + M \epsilon)g_{\psi, n}(y) = -M_n f_{\psi, n}(y), \quad (2.18)$$

The solutions of mode functions are summarized in [S], for various parity assignment. The relevant part of our analysis is only zero mode. For zero mode, the equations of motion for $f_{\psi, 0}$ and $g_{\psi, 0}$ are the two independent equations,

$$(\partial_5 - M \epsilon)f_{\psi, 0}(y) = 0, \quad (\partial_5 + M \epsilon)g_{\psi, 0}(y) = 0. \quad (2.19)$$
For the parity \((++\) for \(f_{\psi,0}\), the mode function is

\[
f_{\psi,0}^{(++)} = \sqrt{\frac{M}{\sinh(LM)}} \times \begin{cases} 
e^{-M|y-L|/2}, & M > 0, \\
e^{M|y-L|/2}, & M < 0, \end{cases}
\]

(2.20)

for \(0 \leq y \leq L\). The normalization constant is fixed by \(\int_0^L dy (f_{\psi,0}^{(++)})^2 = 1\). For \(M > 0\), \(\psi_L^{++}\) is localized at \(y = L\) and for \(M < 0\), it is localized at \(y = 0\). For the parity \((-\) for \(f_{\psi,0}\), \(f_{\psi,0}^{(-)} = 0\). For the parity \((++)\) for \(g_{\psi,0}\), the mode function is

\[
g_{\psi,0}^{(++)} = \sqrt{\frac{M}{\sinh(LM)}} \times \begin{cases} e^{-M|y|/2}, & M > 0, \\
e^{M|y|/2}, & M < 0, \end{cases}
\]

(2.21)

for \(0 \leq y \leq L\). The normalization constant is fixed by \(\int_0^L dy (g_{\psi,0}^{(++)})^2 = 1\). For \(M > 0\), \(\psi_R^{++}\) is localized at \(y = 0\) and for \(M < 0\), it is localized at \(y = L\). For the parity \((-\) for \(g_{\psi,0}\), \(g_{\psi,0}^{(-)} = 0\). These mean that the fields \(\psi_L^{++}\) and \(\psi_R^{++}\) are localized at the opposite fixed points. The product \(f_{\psi,0}^{(++)} g_{\psi,0}^{(++)}\) is written as

\[
f_{\psi,0}^{(++)} g_{\psi,0}^{(++)} = \frac{M}{\sinh(LM)},
\]

(2.22)

for arbitrary \(M\) including \(M = 0\).

The Yukawa interaction is given by

\[
\mathcal{L}_Y = -y_{ij} \bar{E}_i \cdot H e^j \delta(y) + h.c.
\]

(2.23)

If neutrinos were regarded as massless, leptons conserve \(CP\) exactly and the lepton number of each generation is conserved. The charged leptons may be diagonalized. In this case, the Yukawa couplings can be taken as \(y_{ij} = y_{0ij}\). The hierarchy with respect to the generation for the observed masses occurs from the overlapping of wave functions. Throughout the present section and the next section, this economical case will be treated. Correspondingly to nonzero neutrino masses, the large mixing for leptons will be taken into account in Section 4. From the Yukawa interaction (2.23) with \(\langle H \rangle = v\), the mass terms are obtained as

\[
\mathcal{L}_Y = -y^u \left( \frac{M^1}{\sinh(LM^1)} \bar{e}_L e_R + \frac{M^2}{\sinh(LM^2)} \bar{\mu}_L \mu_R + \frac{M^3}{\sinh(LM^3)} \bar{\tau}_L \tau_R \right) + h.c.
\]

\[
= -\frac{y}{L} \left( \frac{c_e}{\sinh(c_e)} \bar{e}_L e_R + \frac{c_{\mu}}{\sinh(c_{\mu})} \bar{\mu}_L \mu_R + \frac{c_{\tau}}{\sinh(c_{\tau})} \bar{\tau}_L \tau_R \right) + h.c.
\]

(2.24)

Here the dimensionless bulk mass parameters are defined as

\[
(c_e, c_\mu, c_\tau) = (LM^1, LM^2, LM^3).
\]

(2.25)

The dimensionless Yukawa coupling is given by \(Y = y_M M\), where \(M\) denotes the fundamental scale. The running of gauge coupling constants is significantly enhanced beyond the scale of size of extra dimensions so that \(M L\) is not very large \(\mathcal{O}(10)\). According to \(\mathcal{O}\), the dimensionless quantities can be taken as \(Y \sim \mathcal{O}(1) - \mathcal{O}(10)\) and \(M L \sim \mathcal{O}(10)\).
We choose $Y = 1$ and $M_*L = 10$. From Eq. (2.24), the charged lepton masses are written in terms of bulk mass parameters as

$$m_e = \left( \frac{Y}{M_*L} \right) \frac{c_e}{\sinh(c_e)} \cdot v, \quad (2.26)$$

$$m_\mu = \left( \frac{Y}{M_*L} \right) \frac{c_\mu}{\sinh(c_\mu)} \cdot v, \quad (2.27)$$

$$m_\tau = \left( \frac{Y}{M_*L} \right) \frac{c_\tau}{\sinh(c_\tau)} \cdot v. \quad (2.28)$$

For $v$, we will adopt $v = (2\sqrt{2}G_F)^{-1/2} = 174.104$ GeV with $G_F = 1.16637 \times 10^{-5}$ GeV$^{-2}$.

## 3 Mass relations

We consider our five-dimensional model an effective theory where hierarchical numbers can be generated from order $O(1)$ numbers. It still remains to be unknown what the origin of the masses of charged leptons is. A clue to fundamental theory beyond our model would be to find the existence of simple mass relations for the dynamical variables of our model, that is, bulk fermions.

In this section, we analyze relations between $m_e$, $m_\mu$, $m_\tau$ and between $c_e$, $c_\mu$, $c_\tau$. The charged lepton mass relation (1.2) has a geometrical expression given by [7],

$$\cos \theta = \frac{(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})}{|(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})|} \cdot \bar{n}, \quad (3.1)$$

where $\theta = 45^\circ$ and the unit vector $\bar{n} = (1, 1, 1)/\sqrt{3}$. We would like to look for a similar simple relation between the bulk mass parameters $c_e$, $c_\mu$ and $c_\tau$. The mass relations required here are equations to give a clue to more fundamental theory. As a principle, they should be simple. As the model is an effective theory, the relations are not necessarily exact but can be approximate as in Eq. (1.1). A candidate of the value of $\theta$ may be such a rough and a simple value as $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$. If three masses $m_e$, $m_\mu$, and $m_\tau$ were the same value, the right-hand side of Eq. (3.1) would be 1. This corresponds to $\theta = 0^\circ$. Because $c_e$, $c_\mu$, and $c_\tau$ are less hierarchical than $m_e$, $m_\mu$, and $m_\tau$, the degree $\theta$ for a mass relation like Eq. (3.1) would not necessarily be $\theta = 45^\circ$. On the other hand, the relation (3.1) includes the unit vector $(1, 1, 1)/\sqrt{3}$. For $c_e$, $c_\mu$ and $c_\tau$, we consider four independent vectors with components being the same size, $\sqrt{3}\bar{n} = (1, 1, 1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)$. Thus we search for a possibility of two mass relations for each $\theta$ and $\bar{n}$,

$$\cos 45^\circ = \frac{(\sqrt{c_e}, \sqrt{c_\mu}, \sqrt{c_\tau})}{|(\sqrt{c_e}, \sqrt{c_\mu}, \sqrt{c_\tau})|} \cdot (1, 1, 1)/\sqrt{3}, \quad (3.2)$$

which is the relation (1.2) with Eqs. (2.26)-(2.28) and

$$\cos \theta = \frac{(\sqrt{c_e}, \sqrt{c_\mu}, \sqrt{c_\tau})}{|(\sqrt{c_e}, \sqrt{c_\mu}, \sqrt{c_\tau})|} \cdot \bar{n}. \quad (3.3)$$
For the equations (3.2) and (3.3), the vector (1, −1, 1) is equivalent to (1, 1, −1) up to the naming of $c_\mu$ and $c_\tau$. Therefore we can take independent unit vectors as $\vec{n} = (n_1, 1, n_3)/\sqrt{3}$ with $n_1 = \pm 1$ and $n_3 = \pm 1$.

A practical way to make extra-dimensional models phenomenologically realistic is that three charged lepton masses are an input in obtaining three bulk mass parameters. As seen from the equations (2.26)-(2.28), the bulk mass parameters can be regarded as functions of the charged lepton masses as $c_\mu = c_\mu(m_e, m_\mu, m_\tau; \frac{Y}{M_{3L}} v)$, $c_\tau = c_\tau(m_e, m_\mu, m_\tau; \frac{Y}{M_{3L}} v)$. Instead of choosing such parameters by hand, we connect two charged lepton masses to the two mass relations (3.2) and (3.3) by employing one of the charged lepton masses as a dimensionful quantity. If the observed value $m_e = 0.510998910$ MeV in Eq. (1.3) is employed as an input, from Eq. (2.26) the bulk mass parameter $c_\mu$ is fixed as $c_\mu = 13.7504$. The other $c_\mu$ and $c_\tau$ are completely solved by Eqs. (3.2) and (3.3). Thus $m_\mu$ and $m_\tau$ are derived.

For the numerical analysis, it is convenient to rewritten the equations (3.2) and (3.3) as an equation for $c_\tau$ as

$$\left(\sqrt{\frac{c_\tau}{\sinh(c_\tau)}} \right) - 2 \left(\sqrt{\frac{c_\mu}{\sinh(c_\mu)}} + \sqrt{\frac{c_\tau}{\sinh(c_\tau)}} \right)^2 - 3 \left(\frac{c_\tau}{\sinh(c_\tau)} \sinh(c_\mu) + 4 \sqrt{\frac{c_\mu}{\sinh(c_\mu)}} \sinh(c_\mu) \right) = 0, \quad (3.4)$$

with

$$\sqrt{c_\mu} = \frac{-n_1\sqrt{c_\mu} + n_3\sqrt{c_\tau}}{1 - 3\cos^2 \theta} + k \frac{\sqrt{3} \cos \theta}{1 - 3\cos^2 \theta} \sqrt{(c_\tau + c_\mu)(2 - 3\cos^2 \theta) + 2n_1n_3\sqrt{c_\tau}}, \quad (3.5)$$

where $k = \pm 1$. From these equations, the solutions are obtained as in Table 1. Here the mass parameters are shown for $c_i < 50$, otherwise the corresponding masses $m_i$ are clearly small. It is found that for $\theta = 60^\circ$, $k = +1$ and $n_1 = -n_3 = 1$, the values of $m_\mu$ and $m_\tau$ are close to the observed values $m_\mu = 105.658367$ MeV and $m_\tau = 1776.84$ MeV in Eq (1.3). Therefore there exists a set of simple relations for bulk and physical masses. The relations (3.2) and (3.3) with $\theta = 60^\circ$ and $\vec{n} = (1, 1, -1)/\sqrt{3}$ approximately accommodate the observed values of charged lepton masses.

### Table 1: Masses in unit of MeV and bulk masses for the relations (3.2) and (3.3) with a dimensionful quantity $m_e$.

| $\theta$ | $k$ | $n_1$ | $n_3$ | $m_\mu$ | $m_\tau$ | $c_\mu$ | $c_\tau$ |
|----------|-----|------|------|--------|--------|--------|--------|
| 30°      | +1  | +1   | +1   | 17002.5 | 1149.4 | 0.378  | 5.026  |
| 45°      | +1  | +1   | −1   | 1153.1  | 17056.4 | 5.022  | 0.352  |
| 60°      | +1  | +1   | −1   | 116.38  | 1941.1 | 7.749  | 4.359  |
| 90°      | ±1  | −1   | +1   | 711.83  | 10697.9 | 5.616  | 1.791  |
| 90°      | ±1  | +1   | −1   | 7.118   | 8.124 × 10^{-16} | 10.883 | 49.098 |

$m_\tau$ are close to the observed values $m_\mu = 105.658367$ MeV and $m_\tau = 1776.84$ MeV in Eq (1.3). Therefore there exists a set of simple relations for bulk and physical masses. The relations (3.2) and (3.3) with $\theta = 60^\circ$ and $\vec{n} = (1, 1, -1)/\sqrt{3}$ approximately accommodate the observed values of charged lepton masses.

### 4 Lepton flavor mixing

As a large mixing for the lepton sector has been observed, it would be indispensable to take into account this effect. In this section, we discuss how to include neutrino masses...
and flavor mixing.

Similarly to Eq. (2.1), neutrinos are introduced as

\[ N^i(x, y) = \left( \begin{array}{c} N^l_i \\ N^r_i \end{array} \right)(x, y), \quad \nu^i(x, y) = \left( \begin{array}{c} \nu^l_i \\ \nu^r_i \end{array} \right)(x, y), \] (4.1)

where \( i = 1, 2, 3 \). Here the neutrinos are included in 

\[ N^l_i = \left( \begin{array}{c} \nu^l_i \\ e^- N^l_i \end{array} \right), \quad \nu^l_i = \left( \begin{array}{c} \nu^e_i \\ \nu^\mu_i \\ \nu^\tau_i \end{array} \right) \] (4.2)

and in Eq. (2.2). In order to distinguish the components of \( N^l_i \) from the components of the lepton doublet \( E^i_L \), the subscript \( N \) has been attached in Eq. (4.2). In other words, the number of lepton doublets is two for each generation \( i \). The setup presented here is that one of the linear combination is a light doublet and the other becomes a heavy doublet with brane couplings. Due to the linear combinations with complex numbers, \( CP \) phases are included. The kinetic energy terms and mass terms for the fields \( N^i(x, y) \) and \( \nu^i(x, y) \) are given by

\[ \mathcal{L} = \bar{N}^i \left( i\gamma^\mu \partial_\mu - M^2_N \epsilon(y) \right) N^i + \bar{\nu}^i \left( i\gamma^\mu \partial_\mu - M^2_\nu \epsilon(y) \right) \nu^i, \] (4.3)

as in the charged-lepton Lagrangian (2.4). The small neutrino masses are achieved via an exponential suppression for a appropriate \( M^2_N = M^2_\nu \) for each \( i \).

The brane couplings to make a part of linear combinations heavy are given by

\[ (\bar{E}, \bar{N}) \left( \begin{array}{c} M_1 \\ M_2 \end{array} \right) D \delta(y) + \text{h.c.}, \] (4.4)

with a doublet brane field \( D(x) \). Here two mass matrices are denoted as \( M_1 \) and \( M_2 \) and the flavor index \( i \) has been suppressed. We define heavy doublets \( H^i \) and light doublets \( L^i \) as

\[ (\bar{H}, \bar{L}) = (\bar{E}, \bar{N})U^{-1}, \quad U = \left( \begin{array}{ccc} U_1 & U_2 \\ U_3 & U_4 \end{array} \right). \] (4.5)

where \( U_1, \ldots, U_4 \) are unitary matrices. For these fields, the equation (4.4) is written as

\[ (\bar{H}, \bar{L}) \left( \begin{array}{cc} U_1M_1 + U_2M_2 \\ U_3M_1 + U_4M_2 \end{array} \right) D \delta(y) + \text{h.c.} \] (4.6)

The condition that \( L^i \) are light yields \( M_2 = -U_4^{-1}U_3M_1 \). Then the heavy fields have the coupling

\[ \bar{H}(U_1 - U_2U_4^{-1}U_3)M_1 D \delta(y) + \text{h.c.} \] (4.7)

With the degrees of freedom for \( U, M_1 \) and \( D \), this coupling can be diagonalized.

After the heavy fields are decoupled, the lepton doublets are given by \( (E, N) = (U_3^{-1}L, U_4^{-1}L) \). Then the charged lepton mass terms are given by

\[ \mathcal{L}_Y = -yv \sum_{i,j=1}^3 \left( \frac{M^j}{\sinh(LM^j)} \bar{e}^l_i U_{3ij} e^l_j \right) + \text{h.c.}, \] (4.8)
instead of Eq. (2.24). Here $U_{3ij} = (U_{3}^{-1})_{ji}^\dagger$ and $e_{L}^{i}$ denote the charged leptons included in the doublets $L^{i}$. The neutrino mass terms are derived similarly. Consequently the masses for charged leptons and neutrinos are obtained as

\[
\begin{align*}
\text{diag}(m_{e}, m_{\mu}, m_{\tau})_{j_{1}j_{2}} &= y_{e}(V_{eL}^{\dagger})_{j_{1}i}U_{3ij} \frac{M_{j}^{\dagger}}{\sinh(LM_{j})}(V_{eR})_{j_{2}i}, \\
\text{diag}(m_{\nu_{1}}, m_{\nu_{2}}, m_{\nu_{3}})_{j_{1}j_{2}} &= y_{\nu}(V_{\nu L}^{\dagger})_{j_{1}i}U_{4ij} \frac{M_{\nu}^{\dagger}}{\sinh(LM_{\nu}^{j})}(V_{\nu R})_{j_{2}i},
\end{align*}
\]  

(4.9) (4.10)

where $V_{eL}$, $V_{eR}$, $V_{\nu L}$ and $V_{\nu R}$ are mixing matrices for leptons. The physical lepton mixing is given by $V_{MNS} = V_{\nu L}^{\dagger}V_{eL}$. We can choose $U_{3} = V_{eL}V_{eR}^{\dagger}$ with keeping the physical value for $V_{MNS}$. While this choice may affect the value of $U_{4}$, the corresponding charged lepton masses are given in Eqs. (2.26)-(2.28). Therefore combining this discussion with the result in the previous section, a set of relations for bulk and physical masses accommodates the observed pattern of charged lepton masses compatibly with the large mixing for flavor in lepton sector. To fix all the matrices appearing here, information on the empirical neutrino mass relation seems necessary.

5 Conclusion

We have studied a possibility to connect masses of charged leptons to two simple mass relations. One is the empirical charged lepton mass relation given in Eq. (1.2). The other is a similar relation imposed between bulk mass parameters being of the same order. When the bulk mass parameters obey

\[
c_{e} + c_{\mu} + c_{\tau} = \frac{4}{3}(\sqrt{c_{e}} + \sqrt{c_{\mu}} - \sqrt{c_{\tau}})^{2},
\]  

(5.1)

which is the case of $\theta = 60^\circ$ as in Eq. (3.2), we have found that the observed pattern of charged lepton mass spectrum is approximately reproduced. If $Y/(M_{e}L) = 1/11$ and the observed $m_{e}$ are chosen as an input, the other charged lepton masses are derived as $m_{\mu} = 106.439$ MeV and $m_{\tau} = 1788.94$ MeV, which are very close to the observed values given in (1.3). It has also been shown that these mass relations are viable including the large mixing for lepton flavors. In addition to this conclusion, there are several open questions as follows.

As discussed in Introduction, the desired thing is to induce a more fundamental theory beyond the standard model from a relation like Eq. (5.1). Our result means that there is a set of simple relations for bulk masses as well as the observed lepton masses. For the observed charged leptons, the origin of the mass relation (1.2) has been studied based on family gauge symmetry in [11]. As an extension of our model, it would be useful to apply the idea of family symmetry to our bulk mass parameter relation.

The present model is a toy model for charged leptons. Our procedure is applicable for models with quarks and neutrinos. If they are incorporated in grand unified models, the parameter space should be constrained further. In such a case, our economical choices $M_{E}^{j} = M_{E}^{i}$ and $y_{ij} = y_{ji}$ might be fixed rather than assumptions. It should be clarified whether matter including quarks and neutrinos is accommodated in simple mass relations.

While utilizing extra dimensions for generating hierarchical fermion masses, we have not carefully taken into account extra-dimensional effects to stabilize the typical scale of the model. In our model, the role of extra dimensions is regarded as an ingredient for a
possible extension of the standard model analogously to universal extra dimensions [12]. It needs to be examined whether stabilization of the scale can be realized simultaneously. In the Randall-Sundrum model [2], cutoff scales of theories are separated in a spatially transparent way. When the typical scale of the theory is stabilized in a warped spacetime, mass relations might be found in a clearly compatible way with the stabilization.

Finally, it would be theoretically desirable that the radius of an extra dimension is stabilized. A radius stabilization has been found in a model where the mass parameter of a bulk scalar field has a critical value [13]. Because the model is supersymmetrically constructed, a bulk fermion as the superpartner of this scalar field has the identical mass parameter. When such a possibility is taken into account, the existence of the critical value might add a constraint on bulk mass parameters of fermions.

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