SHARP VALUE FOR THE NORM OF COMPOSITION
AND MULTIPLICATIVE OPERATORS BETWEEN
TWO DIFFERENT GRAND LEBESGUE SPACES

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Abstract.

We calculate in this paper the norm of composition, multiplicative and product
operator, generated by multiplicative and measurable argument transformation
between two different ordinary Lebesgue-Riesz and Grand Lebesgue spaces.
We set ourselves the aim to obtain the exact expression for the norm of the
considered operators by means of building of appropriate examples.

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product linear operators; Hölder’s inequality, absolutely continuity of measures,
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1 Introduction

Let \((X = \{x\}, M, \mu)\) and \((Y = \{y\}, N, \nu)\) be two measurable spaces equipped with
a non-zero sigma-finite measures \(\mu\) and \(\nu\) correspondingly. Denote by \(N_0 = N_0(Y)\)
(and correspondingly \(M_0 = M_0(X)\)) the linear set of all numerical measurable
functions \(f : Y \to \mathbb{R}\). The case \(X = Y\) can not be excluded.

Let also \(\xi = \xi(x)\) be measurable function from the set \(X\) to \(Y : \xi : X \to Y\).

Definition 1.1.

The linear operator \(U_\xi[f] = U_\xi[f](x)\) defined may be on some Banach invariant
subspace \(B_1\) of the space \(M_0(X)\) to another one, in general case, subspace \(B_2\) of
the space \(N_0(Y)\) by the formula
\[ U_\xi[f](x) = f(\xi(x)) \]  

is said to be composition operator generated by \( \xi(\cdot) \).

Many important properties on these operators acting on different spaces \( B_1 \) with values in \( B_2 \): Lebesgue, Lorentz, mainly Orlicz spaces etc., namely: boundedness, compactness, the exact values of norm are investigated, e.g. in [3], [6], [8], [10], [11], [12], [22], [26], [43], [16], [42], [45] - [48], [50].

Applications other than those mentioned above appears in ergodic theory, see, e.g., in [2], [5], [39], [49] etc.

We study first of all the action of these operators between the classical Lebesgue-Riesz spaces \( L^p(X, \mu) = L^p,X \) consisting on all the measurable functions with finite norm

\[ |\xi|L^p(X, \mu) = |\xi|L^p,X = |\xi|_p := \left[ \int_X |\xi(x)|^p \mu(dx) \right]^{1/p}, \ p = \text{const} \geq 1, \quad (1.2a) \]

and correspondingly

\[ |g|L^q(Y, \nu) = |g|L^q,Y = |g|_q := \left[ \int_Y |g(y)|^q \nu(dy) \right]^{1/q}, \ q = \text{const} \geq 1. \quad (1.2b) \]

\[ |f|L^r(Y, \nu) = |f|L^r,Y = |f|_r := \left[ \int_Y |f(y)|^r \nu(dy) \right]^{1/r}, \ r = \text{const} \geq 1, \quad (1.2c) \]

As ordinary,

\[ |\xi|L^\infty(X, \mu) = |\xi|L^\infty,X = |\xi|_\infty := \text{vraisup}_{x \in X} |\xi(x)|, \quad (1.2d) \]

and analogously may be defined the norm \( |g|L^\infty(Y, \nu) = |g|L^\infty,Y = |g|_\infty \).

Our purpose in this short article is calculation of the exact value of the norm for these operators in ordinary Lebesgue-Riesz as well as in Grand Lebesgue spaces (GLS):

\[ |U_\xi|_{q \rightarrow p} = R(q, p) = R(q, p; Y, X) \overset{def}{=} \sup_{0 \neq f \in L^q(Y, \nu)} \frac{|U_\xi[f]|_{p,X}}{|f|_{q,Y}}. \quad (1.3) \]

We consider further at the same problems for some another operators: multiplicative and product of these operators.

The case of the so-called Grand Lebesgue Spaces (GLS) will be also considered further.

Some estimations of the norm of these operators acting between Orlicz spaces are obtained in the articles [10], [11], [12]. A particular case \( \xi(x) = 1/x, \ X = (0, \infty) \) is considered in a famous monograph [28], pp. 220 - 221. See also [9], chapter n 7, pp. 660-666, where is considered the case \( X = Y \) and \( q = p \).
The classical (kernel) integral operators, including singular, acting in Lebesgue-Riesz, Orlicz, Grand Lebesgue spaces are investigated in [28], [40], p. 198-220, [38] etc.

Another operators acting in these spaces: Hardy, Riesz, Fourier, maximal, potential etc. are investigated, e.g. in [24], [31], [32], [34], [35].

Other notations. Denote by \( F(\xi) = F_{\xi}(\cdot) \) the distribution of the (measurable) function \( \xi : A \in \mathcal{N} \),

\[
F(A) = F_{\xi}(A) = \mu(x : \xi(x) \in A),
\]

then \( F_{\xi}(\cdot) \) is sigma-additive and sigma-finite measure in \( \mathcal{N} \).

Introduce also the Radon-Nikodym derivative \( z = z(y) = dF/d\nu \) of the measure \( F_{\xi} \) relative the source measure \( \nu \) on the \( Y \), i.e. such that for arbitrary measurable function \( h : Y \rightarrow \mathbb{R} \)

\[
\int_{X} h(\xi(x)) \mu(dx) = \int_{Y} h(y) \, z(y) \nu(dy).
\]

If the measure \( F_{\xi} \) is not absolutely continue relative the measure \( \nu \), we define formally \( z(y) = +\infty \).

In particular,

\[
\mid \mid U_{\xi}[f] \mid \mid_{p}^{p} = \int_{X} |f(\xi(x))|^{p} \mu(dx) = \int_{Y} |f(y)|^{p} \, z(y) \nu(dy).
\]

2 Main result. The case of ordinary Lebesgue-Riesz spaces.

Define also the following important functional

\[
K_{z}(p, q) \overset{\text{def}}{=} \mid z \mid^{1/p}_{q/(q-p),Y} = \left[ \int_{Y} \, z(y)^{q/(q-p)} \nu(dy) \right]^{1/p-1/q},
\]

in the case when \( q = \text{const} > p = \text{const} \geq 1 \) and

\[
K_{z}(p, p) \overset{\text{def}}{=} \mid z \mid_{\infty,Y}^{1/p} = \left[ \text{vraisup}_{y \in Y} z(y) \right]^{1/p},
\]

if \( p = \text{const} \geq 1 \).

Remark 2.1. We does not exclude the case when the integral in (2.1) divergent; then evidently \( K_{z}(p, q) = +\infty \).

Theorem 2.1. \( \mid U_{\xi} \mid_{q \rightarrow p} =: R(q, p; Y, X) = K_{z}(p, q) \).

Proof. Upper bound. It is sufficient to consider the case \( 1 \leq p < q < \infty \). Suppose \( K_{r,z}(p, q) < \infty \); the opposite case will be considered further.
We apply the Hölder’s inequality to the right-hand side of inequality (1.6):

\[
|U_\xi[f]|_p^p \leq \left[ \int_Y |f(y)|^{\alpha p} \nu(dy) \right]^{1/\alpha} \cdot \left[ \int_Y z^\gamma(y) \nu(dy) \right]^{1/\gamma} = |f|_{\alpha p, Y}^p \cdot |z|_{\gamma, Y},
\]

where \(\alpha = \text{const} > 1\) and \(\gamma\) is its conjugate number \(\gamma = \alpha/(\alpha - 1)\).

If we choose in (2.3) \(\alpha = q/p > 1\), then

\[
\gamma = \frac{q}{q - p}.
\]

Substituting into (2.3), we get after simple calculations to the estimate

\[
|U_\xi[f]|_{p, X} \leq K_z(p, q) \cdot |f|_{q, Y}.
\]

(2.4)

It remains to prove that the condition \(F_\xi(\cdot) \ll \nu(\cdot)\) is necessary for the inequality

\[
|U_\xi|_{q \to p} = R(q, p) = R(q, p; Y, X) < \infty
\]
at last for some pairs of positive numbers \(p\) and \(q\). Suppose there exists a finite constant \(W = W(p, q)\) such that

\[
\left[ \int_Y |f(y)|^p F_\xi(dy) \right]^{1/p} \leq W(p, q) \left[ \int_Y |f(y)|^q \nu(dy) \right]^{1/q}
\]

(2.5)

for arbitrary function \(f(\cdot) \in L_q(Y, \nu)\).

We can and will assume as the capacity of the value \(W(p, q)\) for all the positive values \(p, q\) its minimal value, indeed

\[
W(p, q) := \sup_{0 < |f|_{q, Y} < \infty} \left\{ \left[ \int_Y |f(y)|^p F_\xi(dy) \right]^{1/p} \right\} / |f|_{q, Y}.
\]

We deduce choosing \(f(y) = I_B(y)\), where \(I(\cdot)\) is ordinary indicator function of the measurable set \(B \in N\) and \(0 \leq \nu(B) < \infty\):

\[
[ F_\xi(B) ]^{1/p} \leq W(p, q) \left[ \nu(B) \right]^{1/q}.
\]

Therefore, every time when \(\nu(B) = 0\), then right here \(F_\xi(B) = 0\). Thus, the distribution measure \(F_\xi(\cdot)\) is absolutely continuous relative the measure \(\nu(\cdot)\):

\(F_\xi(\cdot) \ll \nu(\cdot)\).

Note in addition that in the case \(q = p\) the correspondent Radon-Nikodym derivative is bounded:

\[
\frac{dF_\xi}{d\nu}(y) \leq W(p, p)
\]

and following in this case

\[
\| U_\xi \|(L_{p, Y} \to L_{p, X}) = W(p, p).
\]
This fact was proved first in the famous article of Cui, Hudzik, Kumar, Maligranda [6].

We will see further that in the general case $1 \leq p < q < \infty$ the Radon-Nikodym derivative $dF_{\xi}/d\nu$ may be unbounded.

**Proof. Lower bound.** Let us choose as a capacity of the trial function

$$f(y) := z^\beta(y), \quad \beta = \text{const}.$$  \hspace{1cm} (2.6)

We have

$$|f|_{q,Y} = \left[ \int_Y z^{\beta q}(y) \, \nu(dy) \right]^{1/q} = |z|_{\beta q,Y}^\beta,$$

$$|u_\xi[f]|_{p,X} = \int_X |f(y)|^p z(y) \, \nu(dy) = \int_Y z^{\beta p+1} \, \nu(dy) = |z|_{\beta p+1,Y}^\beta;$$

$$Q[f] := \|u_\xi[f]|_{p,X} \|_{f,q,Y} = \|z|_{\beta p+1,Y}^\beta \|.$$  \hspace{1cm} (2.7)

If we choose the value $\beta$ such that $\beta p + 1 = \beta q$, i.e. $\beta := 1/(q - p)$, then

$$Q[f] = |z|_{1/(q-(q-p))}^{1/p} = K_z(p,q).$$  \hspace{1cm} (2.8)

The case when $K_z(p,q) = \infty$ may be investigated analogously. Namely, let us choose the *sequence* of truncated trial functions

$$f_n(y) := z^\beta(y) \, I(z(y) \leq n), \quad n = 1, 2, \ldots; \quad \beta = \text{const} = 1/(q - p),$$

where $I(\cdot)$ is again indicator function. We have as before as $n \to \infty$

$$Q[f_n] = |u_\xi[f_n]|_{p,X} \|_{f_n,q,Y} = |z \, I(z(y) \leq n)|_{\beta p+1,Y} \| = |z \, I(z(y) \leq n)|_{\beta q,Y} \|.$$  \hspace{1cm} (2.9)

Thus, and in this case

$$|U_\xi|_{q-p} = \infty = K_z(p,q).$$

This completes the proof of theorem 2.1.

**Example 2.1.** Let $X = Y = [0,1]$ with ordinary Lebesgue measure and let $r = \text{const} > 0$. We exclude also the trivial case $r = 1$.

Define the linear composition operator of the form

$$U_{(r)}[f](x) = f(x^r).$$  \hspace{1cm} (2.10)

The correspondent Radon-Nikodym derivative $z(y) = z^{(r)}(y)$ is equal here

$$z(y) = z^{(r)}(y) = r^{-1} \, y^{1/r-1}, \quad y > 0.$$
The expression for the "constant" \( K_z(p,q) = K_z^{(r)}(p,q) \) has here a form
\[
K_z^{(r)}(p,q) = r^{-1/p} \left( \frac{q-p}{q-pr} \right)^{1/p - 1/q} \overset{\text{def}}{=} S_r(p,q). \tag{2.11}
\]

We apply the proposition of theorem 2.1
\[
\| U(r) \| (L_q[0,1] \to L_p[0,1]) = S_r(p,q). \tag{2.12}
\]

Let us investigate in detail the variable \( S_r(p,q) \). We can distinguish the following two cases: a case A) \( 0 < r < 1 \), (a "good case") and a case B) \( r > 1 \) ("a hard case").

In the first case A) the "transfer" function \( K_z^{(r)}(p,q) \) is bounded relative the variable \( q \) in the closed domain \( q \in [p, \infty] \):
\[
\sup_{q \in [p,\infty]} K_z^{(r)}(p,q) \leq r^{-1/p} \tag{2.13}
\]
and wherein
\[
\lim_{q \to p+0} K_z^{(r)}(p,q) = r^{-1/p}, \tag{2.14}
\]
\[
K_z^{(r)}(p,p) = r^{-1/p} = \left[ \text{vraisup}_{y \in (0,1)} z^{(r)}(y) \right]^{1/p}. \tag{2.14a}
\]

Let us now consider the opposite case \( r > 1 \). Then the "transfer" function \( K_z^{(r)}(p,q) \) is finite only in the open interval \( q > pr \) and we have as \( q \to pr + 0 \)
\[
K_z^{(r)}(p,q) \sim r^{-1/p} \left[ \frac{p(r-1)}{q-pr} \right]^{(r-1)/pr}. \tag{2.15}
\]
Thus, in this case the "transfer" function \( K_z^{(r)}(p,q) \) is really unbounded.

3 **Multiplicative operators.**

Define the so-called *multiplicative* operator \( V = V_g \) acting from the space \( N_0(Y) \) into itself
\[
V_g[f](y) \overset{\text{def}}{=} g(y) \cdot f(y), \tag{3.1}
\]
where a factor \( g = g(y) \) is measurable function ("weight"): \( g : Y \to R \).

The acting of these operators between Orlicz’s spaces, satisfying as a rule the \( \Delta_2 \) condition, was investigated in many works, see e.g. [1], [16], [27]; see also a recent article [15] and reference therein.

Denote also
\[
Q_g(p,q) := \| g \|_{pq/(q-p)} = \| g \|_{pq/(q-p),Y}. \]
Proposition 3.1.

\[ \| V_g \|_q \to p = Q_g(p, q) = |g|_{pq/(q-p)}, \quad q > p, \quad (3.2) \]

and \( \| V_g \|_q \to p = \infty \) if \( q \leq p \) or when \( g \notin L_{pq/(q-p)} \).

**Proof** is quite analogous to ones in theorem 2.1; we omit some non-essential details. Note first of all that again by virtue of Hölder’s inequality

\[ |g \cdot f|_p \leq |f|_q \cdot |g|_r, \]

where

\[ \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \quad p, q, r > 1. \]

Since \( r = pq/(q-p), \quad q > p \), we conclude

\[ \| V_g \|_{q \to p} \leq |g|_{pq/(q-p)}, \quad q > p. \quad (3.3) \]

It remains to obtain the lower estimate. Suppose without loss of generality \( g(x) \geq 0 \). One can choose the correspondent trial function as follows

\[ f(y) := g^\beta(y), \quad \beta = \text{const} > 0. \]

Then

\[ |V_g[f]|_p = \int_Y g^{(\beta+1)p}(y) \nu(dy) = |g(\cdot)|_{(\beta+1)p,Y}^{(\beta+1)p}; \]

\[ |f|_{q,Y} = \left[ \int_Y g^{\beta q}(y) \nu(dy) \right]^{1/q} = |g|^\beta_{\beta q,Y}; \]

\[ \frac{|V_g[f]|_p}{|f|_{q,Y}} = \frac{|g|^\beta_{\beta q,Y}}{|g|^\beta_{\beta q,Y}}; \]

and when we choose \( \beta = p/(q-p) \), we obtain what is desired:

\[ \| V_g \|_{q \to p} \geq |g|_{pq/(q-p)}, \quad q > p. \quad (3.4) \]

**Example 3.1.** Let again \( Y = (0, 1) \) with ordinary Lebesgue measure and define the following multiplicative operator

\[ V^{(t)}[f](y) \stackrel{df}{=} y^{-t} \cdot f(y), \quad t = \text{const} \in (0, 1); \quad (3.5) \]

the case \( t \leq 0 \) is trivial for us. I.e. in this case

\[ g(y) = g^{(t)}(y) := y^{-t}. \]

We obtain after simple calculations
\[ \| V_{g(t)} \|_{q \to p} = Q_{g(t)}(p, q) := \left(1 - \frac{tpq}{q - p}\right)^{1/q - 1/p} \]  

(3.6)

in the case when

\[ 1 \leq p < q, \quad \frac{tpq}{q - p} < 1 \]

and

\[ \| V_{g(t)} \|_{q \to p} = +\infty \]  

(3.7)

otherwise.

**Remark 3.1.** Note that the case when the function \( g(\cdot) \) is bounded is trivial; in this case one can take \( p = q \).

**Remark 3.2.** Many another examples may be found further in the seventh section.

4 Product operators.

Let us consider in this section the so-called "product" operator \( W_{g,\xi}[f] \) of the form

\[ W_{g,\xi}[f](x) = g(x) \cdot f(\xi(x)), \quad x \in X, \]  

(4.1)

where as above \( \xi : X \to Y, \ f : Y \to R, \ g : X \to R \) be measurable functions.

On the other words, the operator \( W_{g,\xi}(\cdot) \) is a product of two non commuting, in general case, operators \( W_{g,\xi}(\cdot) = V_{g}(\cdot) \circ U_{\xi}(\cdot) \).

These operators play a very important role in the study of linear isometries acting in the Lebesgue-Riesz spaces \( L_{p} \), see \[28\], p. 176-177.

We retain all the assumptions and notations of all the foregoing sections.

A. General case.

Let \( f(\cdot) \in L_{q(1),Y}, \ q(1) > 1 \); and let \( q(2) < q(1) \). We conclude by virtue of theorem 2.1 \( U_{\xi}[f] \in L_{q(2),Y} \) and herewith

\[ | U_{\xi}[f] |_{L_{q(2),Y}} \leq K_{z}(q(1), q(2)) \cdot | f |_{L_{q(1),Y}}. \]

We have further applying theorem 3.1 for the certain value \( q(3) < q(2) \)

\[ | W_{g,\xi}[f] |_{L_{q(3),Y}} \leq Q_{g}(q(2), q(3)) \cdot K_{z}(q(1), q(2)) \cdot | f |_{L_{q(1),Y}}. \]  

(4.2)

The estimate (4.2) may be issued as follows. Let us introduce the following function

\[ T(p, q) = T_{g,\xi}(p, q) \equiv \inf_{l \in (p,q)} [K_{z}(p, l) Q_{g}(l, q)], \ 1 \leq p < q < \infty. \]
Theorem 4.1. Let $1 \leq p < q < \infty$. Then

$$|| W_{g,\xi} \|(L_p(Y) \rightarrow L_q(X)) \leq T_{g,\xi}(p,q) = \inf_{t \in (p,q)} [K_z(p,t) Q_g(t,q)].$$

(4.3)

Note that at the same result may be obtained by means of Hölder’s inequality applying to the right-hand side of the relation (4.1).

B. Particular case.

Suppose here that the weight function (factor) $g(x)$ has a form $g(x) = h(\xi(x))$, where as before $\xi : X \rightarrow Y$, $h : Y \rightarrow R$ are measurable functions. On the other words, here

$$W_{g,\xi}[f](x) = h(\xi(x)) \cdot f(\xi(x)).$$

(4.4)

We deduce using umpteenth time Hölder’s inequality for three multipliers

$$| W_{g,\xi}[f] |_{p,X} \leq | h |_{p\theta,Y} | f |_{q,Y} \left[ | z |_{r,Y} \right]^{1/p},$$

(4.5)

where

$$q > p, \quad \theta, \tau > 1$$

(4.6a)

and

$$\frac{1}{\theta} + \frac{1}{\tau} = \frac{q - p}{q}.$$

(4.6b)

Eventually:

Theorem 4.2. We propose under formulated in this pilcrow definitions and conditions

$$|| W_{g,\xi} \|(L_q,Y \rightarrow L_p,X) = \inf_{\theta,\tau} \left\{ | h |_{p\theta,Y} \cdot \left[ | z |_{r,Y} \right]^{1/p} \right\},$$

(4.7)

where "inf" in the inequality (4.7) is calculated over all the variables satisfying the relations (4.6a) and (4.6b).

C. Independent case.

Suppose in this subsection that both the functions $f(\xi(\cdot))$ and $g(\cdot)$ are independent in the theoretical probability sense, i.e.

$$\mu\{x : g(x) \in A_1, f(\xi(x)) \in A_2\} = \mu\{x : g(x) \in A_1\} \times \mu\{x : f(\xi(x)) \in A_2\}, \quad A_1, A_2 \in M.$$
\[ g f(\xi(\cdot)) \big|_{p,X} = g \big|_{p,X} \cdot f(\xi(\cdot)) \big|_{p,X}. \] (4.8)

This statement may be issued in the considered here independent case as follows.

**Theorem 4.3.** We propose under conditions of independent case by virtue of theorem 2.1

\[ \| W_{g,\xi} \| (L_{q,Y} \to L_{p,X}) = |g|_p \cdot K_z(p, q). \] (4.9)

We note in conclusion that the lower bounds in the last two theorems are trivial. They follow immediately from ones in theorems 2.1 and 3.1.

## 5 Grand Lebesgue Spaces (GLS).

We recall here first of all for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [13], [14], [17], [18], [19], [20], [21], [24], [29], [30] etc. appear the so-called Grand Lebesgue Spaces (GLS)

\[ G(\psi) = G(\psi, X) = G(\psi, X; A; B); A; B = \text{const}; A \geq 1, B \leq \infty \]

spaces consisting on all the measurable functions \( f : X \to R \) with finite norms

\[ \| f \|_{G(\psi)} \overset{\text{def}}{=} \sup_{p \in (A; B)} \left[ \frac{|f|_p}{\psi(p)} \right], \] (5.1)

where as above

\[ |f|_{p,X} = |f|_p = \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}. \]

Here \( \psi = \psi(p), p \in (A, B) \) is some continuous positive on the open interval \( (A; B) \) function such that

\[ \inf_{p \in (A; B)} \psi(p) > 0. \] (5.2)

We define formally \( \psi(p) = +\infty \) for the values \( p \notin [A, B] \).

We will denote also

\[ \text{supp}(\psi) \overset{\text{def}}{=} (A; B). \]

The set of all such a functions with support \( \text{supp}(\psi) = (A, B) \) will be denoted by \( \Psi = \Psi(A, B) \).

The GLS space \( G(\zeta, Y) = G(\zeta, Y; A_1, B_1) \) based on the measurable space \( (Y, N, \nu) \) may be introduced quite analogously.
These spaces are complete and rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, theory of Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue-Riesz spaces $L^r$ are extremal cases of the Grand Lebesgue Spaces. Indeed, define the degenerate $\Psi -$ function of the form $\psi(r)(p), \; r = \text{const} \geq 1, \; p \geq 1$ as follows:

$$
\psi(r)(p) := 1, \; p = r; \quad \psi(r)(p) = +\infty, \; p \neq r;
$$

and define formally $C/\infty = 0, \; C = \text{const}$.

Then

$$
||f||_{G(\psi(r), X)} = ||f||_{r,X}.
$$

The so-called exponential Orlicz spaces are also the particular cases of the GLS, see for instance [30], [29], chapter 1.

More detail, let for simplicity $\mu(X) = 1$, i.e. let the measure $\mu$ be probabilistic, and let the measurable function (random variable) $\psi(\cdot) \in G\Psi = G\Psi_{\infty}$ be such that the new generated by $\psi$ function

$$
\nu(p) = \nu(\psi)(p) := p \ln \psi(p), \; p \in [1, \infty)
$$

is convex. The measurable function $\eta = \eta(x)$ belongs to the space $G\psi$ if and only if it belongs to the Orlicz’s space $L(N_\psi)$ with the correspondent exponential continuous Young-Orlicz function

$$
N_\psi(u) := \exp \left( -\nu^*_\psi(\ln |u|) \right), \; |u| \geq e,
$$

and herewith of course both the Banach spaces norms: $|| \cdot ||_{L(N_\psi)}$ and $|| \cdot ||_{G\psi}$ are equivalent.

Here the $\nu^*(\cdot)$ denotes as usually the Young-Fenchel, or Legendre transform of the function $\nu(\cdot)$:

$$
\nu^*(v) \overset{\text{def}}{=} \sup_{u>0} (u \cdot v - \nu(u)), \; v > 0.
$$

One can also complete characterize (under formulated here conditions) the belonging of the non-zero function $f : X \rightarrow R$ to the space $G\psi$ by means of its tail behavior:

$$
f \in G\psi \iff \exists K = \text{const} \in (0, \infty), \; \max(\mu\{x : f(x) > u\}, \mu\{x : f(x) < -u\}) \leq \exp \left( -\nu^*_\psi(\ln |u|/K) \right), \; u \geq Ke,
$$

see [21], [29], p. 33 - 35.
For instance, the random variable $\eta$ defined on some probability space $(\Omega, B, P)$, has a finite GLS norm of the form 

$$|| \eta ||_m \overset{def}{=} \sup_{p \geq 1} \left[ \frac{|\eta|^p}{p^{1/m}} \right] < \infty,$$

where $m$ is positive constant not necessary to be integer, if and only if

$$\max(P(\eta > u), P(\eta < -u)) \leq \exp \left( -C(m) \ u^m \right), \ u \geq 1.$$

The case when the supremum in (5.1) is calculated over finite interval is investigated in [24], [38]:

$$G_b \psi = \{ \xi, ||\xi||_{G_b \psi} < \infty \}, \ ||\xi||_{G_b \psi} \overset{def}{=} \sup_{1 \leq p < b} \left[ \frac{|\xi|^p}{\psi(p)} \right], \ b = \text{const} > 1,$$

but here $\psi = \psi(p)$ is continuous function in the semi-open interval $1 \leq p < b$ such that $\lim_{p \uparrow b} \psi(p) = \infty$; the case when $\psi(b-0) < \infty$ is trivial. Indeed, if $\psi(b-0) < \infty$, then the space $G_b \psi$ coincides up to norm equivalence with Lebesgue - Riesz space $L_b$.

We define formally in the case when $b < \infty \ \psi(p) := +\infty \ \text{for all the values } p > b$.

Let a function $f : X \rightarrow R$ be such that

$$\exists (A, B) : 1 \leq A < B \leq \infty, \ \forall p \in (A, B) \Rightarrow |f|_p < \infty.$$

Then the function $\psi = \psi(p) = \psi_f(p)$ may be naturally defined by the following way:

$$\psi_f(p) := |f|_p, \ p \in (A, B). \quad (5.7)$$

Evidently, $||f||_{G\psi_f} = 1$.

6 Acting of the composition operator on GLS.

Statement of problem. Assume the function $f(\cdot)$ belongs to some Grand Lebesgue pace $G(\psi, Y) = G(\psi, Y; A_1, B_1)$, where $1 \leq A_1 < B_1 \leq \infty$. Let also $G(\zeta, X) = G(\zeta, X; A_2, B_2)$ be another GLS builded on the measurable space $(X, M, \mu)$. We set ourselves the problem of the norm estimate of product operator $W_{g, \xi}[f]$ acting between two Grand Lebesgue spaces

$$D(\zeta, \psi) = D_{g, \xi}(\zeta, \psi) \overset{def}{=} || W_{g, \xi} ||( G(\psi, Y) \rightarrow G(\zeta, X) ). \quad (6.1)$$

So, let $f \in G(\psi, Y; A_1, B_1)$; we can and will suppose without loss of generality $||f||_1 = 1$. This imply in particular
∀q ∈ (A₁, B₁) ⇒ |f|_{q,Y} ≤ ψ(q).

For instance, the function ψ = ψ(q) may be selected as a natural function for the function f(x).

We apply theorem 4.1:

\[ | W_{g,ξ}[f] |_{p,X} ≤ T_{g,ξ}(p,q) |f|_{q,Y} ≤ T_{g,ξ}(p,q) ψ(q). \] (6.2)

Introduce the following Ψ function

\[ Θ(p) := \inf_{q>p,q∈(A₁,B₁)} [T_{g,ξ}(p,q) ψ(q)] \] (6.3)

with correspondent support

\[ (A₂, B₂) := \text{supp} Θ(p), \] (6.4)

then the inequality (6.2) may be rewritten as follows

\[ | W_{g,ξ}[f] |_{p} ≤ Θ(p), \quad p ∈ (A₂, B₂). \] (6.5)

To summarize:

**Theorem 6.1.**

\[ || W_{g,ξ} ||(G(ψ,Y) → G(Θ,X)) ≤ 1, \] (6.6)

where the constant ”1” in the right-hand side (6.6) is the best possible.

The last assertion follows immediately from the main result of the article [31], see also [33].

**Remark 6.1.** The multiplicative operators between two Orlicz’s spaces are investigated in many works, see e.g. [7], [23], [25], [41].

7 Examples.

**Example 7.1.** Consider the following product operator

\[ W^{(r,t)}[f](x) = x^{-1/t} f(x^r). \] (7.1)

Here X = Y = (0, 1), t = const ∈ (0, 1), r = const > 0, r ≠ 1.

We find using the proposition of theorem 4.1: || W^{(r,t)} ||(L_q → L_p) = φ(p, q; r, t),

where

\[ φ(p, q; r, t) \overset{def}{=} \inf_{l∈(\max(p,pr),q/(tq+1))} \left\{ r^{-1/p} \times \left( \left( \frac{l - p}{l - pr} \right)^{1/p-1/l} \cdot \left( 1 - \frac{tlq}{q - l} \right)^{1/q-1/l} \right) \right\}. \] (7.2)
Recall that here $q \geq \max(p, pr)$. It will be presumed of course that $\max(p, pr) < q/(tq + 1)$.

Let at first $r < 1$; we find then by some calculations

$$\phi(p, q; r, t) \leq r^{-1/p} \cdot e^{-1/q}.$$ 

The opposite case $r > 1$ is more complicated. Denote

$$l_0 = l_0(p, q; r, t) := \frac{p^2 r (1 + qt) + q(r - 1)}{(1 + qt)(r - 1 + tpr)}.$$  \hspace{1cm} (7.3)

It is easily to verify that $pr < l_0 < q/(qt + 1)$. Our statement:

$$\phi(p, q; r, t) \leq r^{-1/p} \times \left\{ \left( \frac{l_0 - p}{l_0 - pr} \right)^{1/p - 1/l_0} \cdot \left( 1 - \frac{tl_0 q}{q - l_0} \right)^{1/q - 1/l_0} \right\}.$$ 

Thus, the value $l_0 = l_0(p, q; r, t)$ from (7.3) is asymptotically optimal in both the cases $l \to pr + 0$ and $l \to q/(qt + 1) - 0$.

The expression for the function $\phi(p, q; r, t)$ allows a simplification. One can use the following identity

$$\min_{x \in (a, b)} [(x - a)^{-\gamma} (b - x)^{-\beta}] = \frac{(\beta + \gamma)^{\beta + \gamma}}{\beta^\beta \gamma^\gamma} \cdot (b - a)^{-(\beta + \gamma)}.$$ 

Here $0 < a < b < \infty$, $\beta, \gamma = \text{const} > 0$.

We propose after some calculations in the case when $q/(1 - qt) - pr \to 0$, say $0 < q/(1 - qt) - pr < 1$, and when $0 < t < 1$, $r > 1$,

$$r_0 \leq \min(p, q) \leq \max(p, q) \leq R_0, \quad 0 < r_0 = \text{const} < R_0 = \text{const} < \infty :$$

$$\phi(p, q; r, t) \asymp \left\{ \frac{q}{1 - qt - pr} \right\}^{-(1/q + t)}.$$ 

**Example 7.2.A.** Multiplicative operator between Grand Lebesgue Spaces.

Let now both the spaces $(X, M, \mu) = (Y, N, \nu) = ([0, 1], B, dx)$ be probability spaces. Assume for simplicity $r = 1$ and $t \in (0, 1)$ in the examples (3.1), (and, after, (2.1)). Let also $\psi = \psi(q), \ 1 \leq q < \infty$ be certain $\Psi$ – function with unbounded support. Introduce one still the following (linear) multiplicative operator acting on arbitrary function $f(\cdot)$ from the GLS space $G\psi : f \in G\psi$

$$V^{(t)}[f](y) \overset{df}{=} y^{-t} \cdot f(y).$$

Introduce for the values $p \in [1, 1/t)$ the following $\Psi$ – function
\[\tau(p) := \inf_{q > p/(1 - pt)} \left[ \left( 1 - \frac{tpq}{q - p} \right)^{(p-q)/pq} \psi(q) \right]. \quad (7.4)\]

We derive by virtue of the example 3.1 the following non-improvable in general case estimate of the form

\[\| V^{(t)}[f] \|_{G\tau} \leq \| f \|_{G\psi}. \quad (7.5)\]

Let us choose in (7.4)

\[q = \frac{2p}{1 - pt},\]

we find then

\[\tau(p) \leq \left[ \frac{1 + pt}{1 - pt} \right]^{2p/(1 + pt)} \cdot \psi\left( \frac{2p}{1 - pt} \right). \quad (7.6)\]

In particular, if \(\psi(q) \asymp C_1 q^{1/m}, \ m = \text{const} > 0,\) then

\[\tau(p) \asymp C_2 (1/t - p)^{-(1/t + 1/m)}, \ 1 \leq p < 1/t. \quad (7.7)\]

It is interest by our opinion to note that the function \(\tau(p)\) has a bounded support, despite the source \(\Psi - \text{function } \psi(\cdot)\) has unbounded one.

**Example 7.2.B.** Composite operator between Grand Lebesgue Spaces.

Let again both the spaces \((X, M, \mu) = (Y, N, \nu) = ([0, 1], B, dx)\) be probability spaces. Assume for simplicity \(r > 1,\) in the example (2.1), the so-called "hard case".

Let also \(\psi = \psi(q), \ 1 \leq q < \infty\) be certain \(\Psi - \text{function}.\) Consider as before the following (linear) composite operator acting on arbitrary function \(f(\cdot)\) from the GLS space \(G\psi:\ f \in G\psi\)

\[U_{(r)}[f](x) := f(x^r).\]

Introduce for all the values \(p \in [1, \infty)\) the following \(\Psi - \text{function}\)

\[\sigma(p) = \sigma_r[\psi](p) := r^{-1/p} \inf_{q > pr} \left( \frac{q - p}{q - pr} \right)^{1/p - 1/q} \psi(q) \bigg) \bigg] \bigg). \quad (7.8)\]

We derive by virtue of theorem 2.1 the following non-improvable in general case estimate of the form

\[\| U_{(r)}[f] \|_{G\sigma} \leq \| f \|_{G\psi}. \quad (7.9)\]

In particular, let us choose in (7.8) \(q := q_0 = \lambda p r, \ \lambda = \text{const} > r;\) then

\[\sigma_r[\psi](p) \leq C_r(\lambda) \psi(\lambda p)\]

or more precisely
\[
\sigma_r[\psi](p) \leq r^{-1/p} \cdot \inf_{\lambda > r} \left\{ \frac{(\lambda - 1)}{\lambda - r} \right\}^{1/(1/\lambda - 1/p)} \cdot \psi(\lambda p). \tag{7.10}
\]

If for instance \(\psi(p) = \psi^{(m)}(p) = p^{1/m}\) or more generally if the function \(\psi = \psi(p)\) satisfies the so-called weak \(\Delta_2\) condition at the infinity:

\[\exists \lambda > r \Rightarrow \sigma_r[\psi](p) \leq C_{2,r}(\lambda) \psi(p),\]

then

\[
\| U_r[\cdot] \|(G\psi \rightarrow G\psi) = C_{3,r} < \infty. \tag{7.11}
\]

On the other words, the function \(U_r[f](x)\) may belong at the same space as the source function \(f(\cdot)\), in contradiction to the foregoing example.

Notice that despite the function \(\psi(p) = \psi^{(m)}(p) = p^{1/m}\) satisfies the weak \(\Delta_2\) condition at the infinity, the correspondent Young - Orlicz function

\[
N_{\psi^{(m)}}(u) \asymp \exp(C(m, r) |u|^m), \quad |u| \geq 1
\]
does not. Since the Orlicz’s norm \(L(N_{\psi^{(m)}})\), as we know, is equivalent to the \(G\psi^{(m)}\) norm, we conclude that

\[
\| U_r[\cdot] \| \{L(N_{\psi^{(m)}}) \rightarrow L(N_{\psi^{(m)}})\} = C\{4, r, m\} < \infty.
\]

Note that this estimate does not follows from the main result of the article [6].

**Example 7.3.** (Counterexample).

Let again \(X = Y = (0, 1)\) and define

\[
f(x) = x^{-1/2}, \quad \xi(x) = x^3. \tag{7.12}
\]

Then

\[
|f|_p = \left[\frac{2}{2 - p}\right]^{1/p} =: \psi(p), \quad 1 \leq p < 2;
\]

\[
z(x) = 3^{-1}x^{-2/3}, \quad 0 < x \leq 1; \quad |z|_q = 3^{1/p - 1} \cdot (3 - 2p)^{-1/p} =: \theta(p), \quad 1 \leq q < 3/2;
\]
or equally \(f(\cdot) \in G\psi, \ z(\cdot) \in G\theta\), but the superposition function \(g(x) = f(\xi(x)) = x^{-3/2}\) does not belongs to any \(L_p(X)\) space with \(p \geq 1\).

The cause of seeming contradiction with theorem 2.1 is following: the function \(f(\cdot)\) does not belongs to arbitrary Lebesgue - Riesz space \(L_q\) with \(q > pr, p \geq 1\), as long as here \(r = 3\).

**Example 7.4.** (Linear substituting).

Let here \(X = R^d\) with ordinary Lebesgue measure and \(f : R^d \rightarrow R\) be some function belonging to the space \(G\psi\). Let also \(A\) be non degenerate linear map (matrix) from \(R^d\) to itself.
Define an operator of a view
\[ V_A[f] = f(Ax). \] (7.13)

Obviously,
\[ |V_A[f]|_p^p = \int_{\mathbb{R}^d} |f(Ax)|^p \, dx = \int_{\mathbb{R}^d} |\det(A)|^{-1} |f(y)|^p \, dy = |\det(A)|^{-1} |f|_p^p, \]
or equally
\[ |V_A[f]|_p = |\det(A)|^{-1/p} |f|_p, \]
and following
\[ |V_A[f]|_p \leq |\det(A)|^{-1/p} \cdot ||f||G\psi \cdot \psi(p). \] (7.14)

Let the function \( \psi(\cdot) \) be factorable:
\[ \psi(p) = \frac{\zeta(p)}{\tau(p)}, \quad p \in (A, B), \]
where both the functions \( \zeta(\cdot), \tau(\cdot) \) are from the set \( G\Psi \), i.e. satisfy all the conditions imposed on the function \( \psi(\cdot) \). We deduce after dividing the inequality (7.14) on the function \( \zeta(p) \):
\[ \frac{|V_A[f]|_p}{\zeta(p)} \leq ||f||G\psi \cdot \frac{|\det(A)|^{-1/p}}{\tau(p)}. \] (7.15)

Recall now that the fundamental function \( \phi(G\tau, \delta) \), \( 0 \leq \delta \leq \mu(X) \) for the Grand Lebesgue Space \( G\tau \) may be calculated by the formula
\[ \phi(G\tau, \delta) = \sup_{p \in (A, B)} \left[ \frac{\delta^{1/p}}{\tau(p)} \right]. \]

This notion play a very important role in the theory of operators, Fourier analysis etc., see [4]. The detail investigation of the fundamental function for GLS is done in [24], [30].

Taking the maximum over \( p; \ p \in (A, B) \) from both the sides of inequality (7.15), we get to the purpose of this subsection: under our condition
\[ ||V_A[f]||G\zeta \leq ||f||G\psi \cdot \phi(G\tau, |\det(A)|^{-1}). \] (7.16)

We note in conclusion that the multivariate case, for instance, the operator of the form
\[ W^{(r_1, r_2; t_1, t_2)}[f](x_1, x_2) := x_1^{-t_1} x_2^{-t_2} f(x_1^{r_1}, x_2^{r_2}), \]
where \( r_1, r_2 = \text{const} > 0, \ t_1, t_2 = \text{const} \in (0, 1) \), may be investigated analogously.
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