The Fourier Transform on the Group $GL_2(\mathbb{R})$ and the Action of the Overalgebra $gl_4$

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Abstract  We define a kind of 'operational calculus' for the Fourier transform on the group $GL_2(\mathbb{R})$. Namely, $GL_2(\mathbb{R})$ can be regarded as an open dense chart in the Grassmannian of 2-dimensional subspaces in $\mathbb{R}^4$. Therefore the group $GL_4(\mathbb{R})$ acts in $L^2$ on $GL_2(\mathbb{R})$. We transfer the corresponding action of the Lie algebra $gl_4$ to the Plancherel decomposition of $GL_2(\mathbb{R})$, the algebra acts by differential-difference operators with shifts in an imaginary direction. We also write similar formulas for the action of $gl_4 \oplus gl_4$ in the Plancherel decomposition of $GL_2(\mathbb{C})$.

Keywords  Unitary representations · Plancherel formula · Differential-difference operators · Grassmannian · Principal series

1 The Statement of the Paper

1.1 The Group $GL_n(\mathbb{R})$

Let $GL_n(\mathbb{R})$ be the group of invertible real matrices of order $n$. We denote elements of $GL_2(\mathbb{R})$ by

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The Haar measure on $GL_2(\mathbb{R})$ is given by

$$(\text{det } X)^{-2} dX = (\text{det } X)^{-2} dx_{11} dx_{12} dx_{21} dx_{22}.$$ 

Recall some basic facts on representations of the group $GL_2(\mathbb{R})$, for systematic exposition of the representation theory of $SL_2(\mathbb{R})$, see, e.g., [4,6]. Denote by $\mathbb{R}^\times$ the multiplicative group of $\mathbb{R}$. Let $\mu \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. We define the function $x^{\mu,\varepsilon}$ on $\mathbb{R}^\times$ by

$$x^{\mu,\varepsilon} := |x|^{\mu} \text{sgn}(x)^\varepsilon,$$

these functions are precisely all homomorphisms from $\mathbb{R}^\times$ to the multiplicative group of $\mathbb{C}$. Denote by $\Lambda$ the set of all collections

$$(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2),$$

i.e.,

$$\Lambda \simeq \mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C} \times \mathbb{Z}_2.$$ 

For each element of $\Lambda$ we define a representation $T_{\mu,\varepsilon}$ of $GL_2(\mathbb{R})$ in the space of functions on $\mathbb{R}$ by

$$T_{\mu_1,\varepsilon_1; \mu_2,\varepsilon_2} \left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right) \varphi(t) = \varphi \left( \begin{array}{c} x_{12} + tx_{22} \\ x_{11} + tx_{21} \end{array} \right)^{-1+\mu_1-\mu_2,\varepsilon_1-\varepsilon_2} \det \left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)^{1/2+\mu_2,\varepsilon_2} \quad (1.1)$$

A space of functions can be specified in various ways, it is convenient to consider the space $C^\infty_{\mu_1,\mu_2,\varepsilon_1,\varepsilon_2}$ of $C^\infty$-functions on $\mathbb{R}$ such that

$$\varphi(-1/t)(-t)^{-1+\mu_1-\mu_2,\varepsilon_1-\varepsilon_2}$$

also is $C^\infty$-smooth.

Thus, for any fixed $X \in GL_2(\mathbb{R})$ we get an operator-valued function $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$ $\mapsto T_{\mu_1,\varepsilon_1; \mu_2,\varepsilon_2}$ on $\Lambda$ holomorphic in the variables $\mu_1, \mu_2$.

Generators of the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ act by formulas

$$L_{11} = -t \frac{d}{dt} + (-1/2 + \mu_1), \quad L_{12} = \frac{d}{dt},$$

$$L_{21} = -t^2 \frac{d}{dt} + t(-1 + \mu_1 - \mu_2), \quad L_{22} = t \frac{d}{dt} + (1/2 + \mu_2).$$

1. This condition means that functions $\varphi$ are smooth as sections of line bundles on the projective line $\mathbb{R} \cup \infty$. 

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The expressions for the generators do not depend on \( \varepsilon_1, \varepsilon_2 \). However the space of the representation depends on \( \varepsilon_1 - \varepsilon_2 \).

Consider integral operators

\[
A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} : C_{\mu_1 - \mu_2, \varepsilon_1 - \varepsilon_2}^\infty \to C_{\mu_2 - \mu_1, \varepsilon_1 - \varepsilon_2}^\infty
\]

defined by

\[
A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} f(t) := \int_{\mathbb{R}} (t-s)^{-1-\mu_1+\mu_2+\varepsilon_1-\varepsilon_2} f(s) \, ds.
\]

The integral is convergent if \( \text{Re}(\mu_1 - \mu_2) < 0 \) and determines a function holomorphic in \( \mu_1, \mu_2 \). As usual (see, e.g., [3], §I.3), the integral admits a meromorphic continuation to the whole plane \( (\mu_1, \mu_2) \in \mathbb{C}^2 \) with poles at \( \mu_1 - \mu_2 = 0, 1, 2, \ldots \).

The operators \( A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \) are intertwining,

\[
A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} = T_{\mu_2, \varepsilon_2; \mu_1, \varepsilon_1} A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}
\] (1.4)

(parameters \( (\mu_1, \varepsilon_1) \) and \( (\mu_2, \varepsilon_2) \) of \( T \) are transposed). If \( \mu_1 - \mu_2 \neq \pm 1, \pm 2, \ldots \), then representations \( T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \) are irreducible and operators \( T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \) are invertible.

Representations \( T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \) form a so-called (nonunitary) principal series of representations. Recall description of some unitary irreducible representations of \( \text{GL}_2(\mathbb{R}) \).

Unitary principal series. If \( \mu_1, \mu_2 \in i \mathbb{R} \), then the representation is unitary in \( L^2(\mathbb{R}) \). Denote by \( \Lambda_{\text{principal}} \subset \Lambda \) the subset consisting of tuples \((i s_1, \varepsilon_1; i s_2, \varepsilon_2)\) with \( s_1 \geq s_2 \).

Discrete series. Notice that the group \( \text{GL}_2(\mathbb{R}) \) acts on the Riemann sphere \( \overline{\mathbb{C}} = \mathbb{C} \cup \infty \) by linear-fractional transformations

\[
X : z \mapsto \frac{x_{12} + z x_{22}}{x_{11} + z x_{21}},
\]

these transformations preserve the real projective line \( \mathbb{R} \cup \infty \) and therefore preserves its complement \( \mathbb{C} \setminus \mathbb{R} \). If \( \det X > 0 \), then (1.5) leave upper and lower half-planes invariant, if \( \det X < 0 \), then (1.5) permutes the half-planes.

Let \( n = 1, 2, 3, \ldots \). Consider the space \( H_n \) of holomorphic functions \( \varphi \) on \( \mathbb{C} \setminus \mathbb{R} \) satisfying

\[
\int_{\mathbb{C} \setminus \mathbb{R}} |\varphi(z)|^2 |\text{Im } z|^{-n-1} \, d\text{Re } z \, d\text{Im } z < \infty.
\]

In fact, \( \varphi \) is a pair of holomorphic functions \( \varphi_+ \) and \( \varphi_- \) determined on half-planes \( \text{Im } z > 0 \) and \( \text{Im } z < 0 \). These functions have boundary values on \( \mathbb{R} \) in distributional sense (we omit a precise discussion since it is not necessary for our purposes). The space \( H_n \) is a Hilbert space with respect to the inner product

\[
\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{C} \setminus \mathbb{R}} \varphi_1(z) \overline{\varphi_2(z)} |\text{Im } z|^{-n-1} \, d\text{Re } z \, d\text{Im } z.
\]
For $s \in \mathbb{R}$, $\delta \in \mathbb{Z}_2$ we define a unitary representation $D_{n,s}$ of $GL_2(\mathbb{R})$ in $H_n$ by

$$D_{n,s} \left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right) \varphi(z) = \varphi \left( \begin{array}{cc} x_{11} + \varepsilon_1 x_{22} \\ x_{11} + \varepsilon_2 \end{array} \right) (x_{11} + z x_{21})^{-1-n} \times \det \left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)^{1/2+n/2+is/\delta}.$$  

In fact, we have operators (1.1) for

$$\mu_1 = -n/2 - is, \quad \varepsilon_1 = -1 + n + \delta; \quad \mu_2 = n/2 + is, \quad \varepsilon_2 = \delta$$

restricted to the subspace generated by boundary values of functions $f_+$ and $f_-$. We denote by $\Lambda_{\text{discrete}}$ the set of all parameters of discrete series.

1.2 The Fourier Transform

Let $F$ be contained in the space $C_0^\infty(GL_2(\mathbb{R}))$ of compactly supported function on $GL_2(\mathbb{R})$. We consider a function that sent each $\tilde{\mu} = (\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$ to an operator in $C_{\mu_1-\mu_2,\varepsilon_1-\varepsilon_2}$ given by

$$T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}(F) = \int_{GL_2(\mathbb{R})} F(X) T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}(X) \frac{dX}{\det(X)^2}.$$  

By definition the Fourier transform is the map $F \mapsto T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}(F)$, for each $F$ we get a function on the space of parameters $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$.

Next, we define a subset $\Lambda_{\text{tempered}} \subset \Lambda$ by

$$\Lambda_{\text{tempered}} := \Lambda_{\text{principal}} \cup \Lambda_{\text{discrete}}.$$  

Let us define Plancherel measure $d\mathcal{P}(\tilde{\mu})$ on $\Lambda_{\text{tempered}}$. On the piece $\Lambda_{\text{principal}}$ it is given by

$$d\mathcal{P}(is_1, 0; is_2, \varepsilon_2) = \frac{1}{16\pi^3} (s_1 - s_2) \tanh \pi (s_1 - s_2) / 2 ds_1 ds_2;$$  

$$d\mathcal{P}(is_1, 1; is_2, \varepsilon_2) = \frac{1}{16\pi^3} (s_1 - s_2) \coth \pi (s_1 - s_2) / 2 ds_1 ds_2.$$  

On $n$-th piece of $\Lambda_{\text{discrete}}$ it is given by

$$d\mathcal{P} = \frac{n}{8\pi^3} ds.$$  

Consider the space $L^2$ of functions $Q$ on $\Lambda_{\text{tempered}}$ taking values in the space of Hilbert–Schmidt operators in the corresponding Hilbert spaces and satisfying the condition
\[ \int_{\Lambda_{\text{tempered}}} \text{tr} \left( Q(\tilde{\mu}) Q^*(\tilde{\mu}) \right) d\mathcal{P}(\tilde{\mu}) < \infty. \]

This is a Hilbert space with respect to the inner product
\[ \langle Q_1, Q_2 \rangle_{L^2} := \int_{\Lambda_{\text{tempered}}} \text{tr} \left( Q_1(\tilde{\mu}) Q_2^*(\tilde{\mu}) \right) d\mathcal{P}(\tilde{\mu}). \]

According the Plancherel theorem, for any \( F_1, F_2 \in C_0^\infty(\text{GL}_2(\mathbb{R})) \) we have
\[ \langle F_1, F_2 \rangle_{L^2(\text{GL}_2(\mathbb{R}))} = \langle T(F_1), T(F_2) \rangle_{L^2(\Lambda_{\text{tempered}})}. \]

Moreover, the map \( F \mapsto T(F) \) extends to a unitary operator \( L^2(\text{GL}_2(\mathbb{R})) \to L^2. \)

### 1.3 Overgroup

Let \( \text{Mat}_2(\mathbb{R}) \) be the space of all real matrices of order 2. By \( \text{Gr}_{4,2}(\mathbb{R}) \) we denote the Grassmannian of 2-dimensional subspaces in \( \mathbb{R}^2 \oplus \mathbb{R}^2 \). For any operator \( \mathbb{R}^2 \to \mathbb{R}^2 \) its graph is an element \( \text{Gr}_{4,2}(\mathbb{R}) \). The set \( \text{Mat}_2(\mathbb{R}) \) of such operators is an open dense chart in \( \text{Gr}_{4,2}(\mathbb{R}) \).

The group \( \text{GL}_{4}(\mathbb{R}) \) acts in a natural way in \( \mathbb{R}^4 \) and therefore on the Grassmannian. In the chart \( \text{Mat}_2(\mathbb{R}) \) the action is given by the formula (see, e.g., [18], Theorem 2.3.2)
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \mapsto (A + XC)^{-1}(B + XD),
\]
where \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is an element of \( \text{GL}_4(\mathbb{R}) \) written as a block matrix of size 2 + 2. The Jacobian of this transformation is (see, e.g., [18], Theorem 2.3.2)
\[
\det(A + XC)^{-4} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2.
\]

For \( \sigma \in i\mathbb{R} \) we define a unitary representation of \( \text{GL}_4(\mathbb{R}) \) in \( L^2(\text{Mat}_2(\mathbb{R})) \) by
\[
R_\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} F(X) = F((A + XC)^{-1}(B + XD)) \times |\det(A + XC)|^{-2+2\sigma} \left|\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|^{1-\sigma}
\]
(these representations are contained in degenerate principal series).

The group \( \text{GL}_2(\mathbb{R}) \) is an open dense subset in \( \text{Mat}_2(\mathbb{R}) \). Therefore, we can identify the spaces \( L^2 \) on \( \text{GL}_2(\mathbb{R}) \) and \( \text{Mat}_2(\mathbb{R}) \). For this we consider a unitary operator
\[
J : L^2(\text{Mat}_2(\mathbb{R})) \to L^2(\text{GL}_2(\mathbb{R})))
\]
given by

\[ J_\sigma F(X) = F(X) \cdot |\det X|^{1-\sigma} \]

This determines a unitary representation \( U_\sigma := J_\sigma R_\sigma J_\sigma^{-1} \) of \( \text{GL}_4(\mathbb{R}) \) in \( L^2(\text{GL}_2(\mathbb{R})) \), the explicit expression is

\[
U_\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} F(X) = F((A + XC)^{-1}(B + XD)) \times \left| \frac{\det(A+XC)\det(B+XD)}{\det X \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}} \right|^{-1+\sigma}.
\]

(1.6)

Consider a subgroup \( \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}) \subset \text{GL}_4(\mathbb{R}) \) consisting of matrices \( \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \). For this subgroup we get the usual left-right action in \( L^2(\text{GL}_2(\mathbb{R})) \),

\[
U_\sigma \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} F(X) = F(A^{-1}XD).
\]

Formula (1.6) extends this formula to the whole group \( \text{GL}_4(\mathbb{R}) \). The Lie algebra \( \mathfrak{gl}_4 \) acts in the space of functions on Mat_2(\mathbb{R}) by first order differential operators, which can be easily written; a list of formulas for all generators \( e_{kl} \), where \( k, l = 1, 2, 3, 4 \), is given below in Sect. 2.5. We restrict this action to the space of smooth compactly supported functions on \( \text{GL}_2(\mathbb{R}) \). Notice that the operators \( i \cdot e_{kl} \) are symmetric on this domain, but some of them are not essentially self-adjoint.

Our purpose is to write explicitly the images \( E_{kl} \) of operators \( e_{kl} \) under the Fourier transform.

1.4 Formulas

Operators \( T_{\mu_1,\epsilon_1;\mu_2;\epsilon_2}(F) \) have the form

\[
T_{\mu_1,\epsilon_1;\mu_2;\epsilon_2}(F)\varphi(t) = \int_{-\infty}^{\infty} K(t, s; \mu_1, \epsilon_1; \mu_2, \epsilon_2) \varphi(t) \, ds.
\]

(1.7)

Recall that functions \( K \) are holomorphic in \( \mu_1, \mu_2 \).

We wish to write operators \( E_{kl} \) on kernels \( K \). The complete list is contained below in Sect. 2.5, here we present two basic expressions.

The algebra \( \mathfrak{gl}_4(\mathbb{R}) \) can be decomposed as a linear space into a direct sum of four subalgebras \( a, b, c, d \) consisting of matrices of the form

\[
\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}
\]
The subalgebras $a$ and $d$ are isomorphic to $\mathfrak{gl}_2(\mathbb{R})$, subalgebras $b$ and $c$ are Abelian. Formulas for the action of $a$ and $d$ immediately follow from the definition of the Fourier transform. To obtain formulas for the whole $\mathfrak{gl}_4$, it is sufficient to write expressions for one generator of $b$, say $E_{14}$, and one generator of $c$, say $E_{32}$. After this other generators can be obtained by evaluation of commutators.

We define shift operators $V_{1}^{\pm}, V_{1}^{-}, V_{2}^{+}, V_{2}^{-}$ by

$$V_{1}^{\pm} K(t, s|\mu_1, \varepsilon_1; \mu_2, \varepsilon_2) = K(t, s|\mu_1 \pm 1, \varepsilon_1 + 1; \mu_2, \varepsilon_2); \quad (1.8)$$

$$V_{2}^{\pm} K(t, s|\mu_1, \varepsilon_1; \mu_2, \varepsilon_2) = K(t, s|\mu_1, \varepsilon_1; \mu_2 \pm 1, \varepsilon_2 + 1). \quad (1.9)$$

**Theorem 1.1** The operators $E_{14}$, $E_{32}$ are given by the formulas

$$E_{14} = -\frac{1}{2} - \sigma + \mu_1 \frac{\partial}{\partial s} V_{1}^{-} + \frac{1}{2} - \sigma + \mu_2 \frac{\partial}{\partial t} V_{2}^{-};$$

$$E_{32} = \frac{1}{2} + \mu_1 + \sigma \frac{\partial}{\partial t} V_{1}^{+} + \frac{1}{2} + \mu_2 + \sigma \frac{\partial}{\partial s} V_{2}^{+}. \quad (1.9)$$

**Remark** We emphasize that these formulas determining unbounded skew-symmetric operators in $\mathcal{L}$ include shifts transversal to the contour of integration in the Plancherel formula (the contour corresponds to pure imaginary $\mu_1, \mu_2$ and (1.8), (1.9) are a shift operators in real directions). □

### 1.5 Remarks on a General Problem

In [17] the author formulated the following question: Assume that we know the explicit Plancherel formula for the restriction of a unitary representation $\rho$ of a group $G$ to a subgroup $H$. Is it possible to write the action of the Lie algebra of $G$ in the direct integral of representations of $H$?

Now it seems that an answer to this question is affirmative.

The initial paper [17] contains a solution for a tensor product of a highest and lowest weight representations of $\text{SL}_2(\mathbb{R})$. In this case the overalgebra acts by differential-difference operators in the space $L^2(S^1 \times \mathbb{R}_+)$ having the form

$$Lf(\varphi, s) = D_1 f(\varphi, s + i) + D_2 f(\varphi, s) + D_3 f(\varphi, s - i),$$

where $D_1, D_2, D_3$ are differential operators in the variable $\varphi$ of orders 0, 1, 2 respectively.

In [9–13] Molchanov solved several rank 1 problems of this type, expressions are similar, but there appear differential operators of order 4. In [20] there was obtained the action of the overalgebra in restrictions from $\text{GL}_{n+1}(\mathbb{C})$ to $\text{GL}_n(\mathbb{C})$, in this case

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2 The problem of a decomposition of a tensor product $\rho_1 \otimes \rho_2$ of representations of a group $G$ is a special case of a decomposition of restrictions. Indeed, $\rho_1 \otimes \rho_2$ is a representation of the group $G \times G$, and we must restrict this representation to the diagonal $G$. 

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differential operators have order \( n \). In all the cases examined by now formulas include shift operators in the imaginary direction.

In the present paper, we write the action of the overalgebra in the restriction of a degenerate principal series of the group \( \text{GL}_4(\mathbb{R}) \) to \( \text{GL}_2(\mathbb{R}) \). Notice that canonical overgroups exist for all 10 series of real classical groups.\(^3\) Moreover overgroups exist for all 52 series of classical semisimple symmetric spaces \( G/H \), see \([8,15]\), see also \([18]\), Addendum D.6. So the problem makes sense for all classical symmetric spaces.

Sturm–Liouville problems for difference operators in \( L^2(\mathbb{R}) \) in the imaginary direction

\[
\lambda f(s) = a(s)f(s + i) + b(s)f(s) + c(s)f(s - i)
\]

arise in a natural way in the theory of hypergeometric orthogonal polynomials, see, e.g., \([1,7]\), apparently a first example (the Meixner–Pollaczek system) was discovered by J. Meixner in 1930s. On such operators with continuous spectra see \([5,16,19]\). See also a multi-rank work of Cherednik \([2]\) on Harish-Chandra spherical transforms.

1.6 The Fourier Transform on \( \text{GL}_2(\mathbb{C}) \)

For a detailed exposition of representations of the Lorentz group, see \([14]\). For \( \nu, \nu' \in \mathbb{C} \) satisfying \( \nu - \nu' \in \mathbb{Z} \) we define the function \( z^{\nu\parallel\nu'} \) on the multiplicative group of \( \mathbb{C} \) by

\[
z^{\nu\parallel\nu'} := z^\nu z^{\nu'} := |z|^{2\nu} z^{\nu' - \nu}.
\]

Denote by \( \Delta \) the set of all \( (\mu_1, \mu_1'; \mu_2, \mu_2') \in \mathbb{C}^4 \) such that \( \mu_1 - \mu_1' \in \mathbb{Z}, \mu_2 - \mu_2' \in \mathbb{Z} \). For \( (\mu_1, \mu_1'; \mu_2, \mu_2') \in \Delta \) we define a representation of \( \text{GL}_2(\mathbb{C}) \) in the space of functions on \( \mathbb{C} \) by

\[
T_{\mu_1,\mu_1';\mu_2,\mu_2'} \left( \begin{array} {cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right) \varphi(t) = \varphi \left( \frac{x_{12} + tx_{22}}{x_{11} + tx_{21}} \right) (x_{11} + tx_{21})^{-1+\mu_1-\mu_2} (x_{11} + tx_{21})^{-1+\mu_1'-\mu_2'} \det \left( \begin{array} {cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)^{1/2+\mu_2} \left( \begin{array} {cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)^{1/2+\mu_2'}
\]

We consider the space of \( C^\infty \)-smooth functions \( \varphi \) on \( \mathbb{C} \) such that

\( \varphi(-t^{-1}) = t^{-1+\mu_1-\mu_2} (1+\mu_1-\mu_2)^{-1+\mu_1'-\mu_2'} \)

is \( C^\infty \)-smooth at 0. These representations form the (nonunitary) principal series.

It is convenient to complexify the Lie algebra of \( \text{GL}_2(\mathbb{C}) \),

\[ \mathfrak{gl}_2(\mathbb{C}) \cong \mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_2(\mathbb{C}). \]

\(^3\) Emphasize that in the present paper we consider groups \( \text{GL} \) and not \( \text{SL} \), also we must consider groups \( \text{U}(p,q) \) and not \( \text{SU}(p,q) \).
Under this isomorphism, the operators of the Lie algebra act in our representation by

\[ L_{11} = -\tau \frac{d}{dt} + (-1/2 + \mu_1), \quad L_{12} = \frac{d}{dt}, \quad (1.10) \]

\[ L_{21} = -\tau^2 \frac{d}{dt} + \tau(-1 + \mu_1 - \mu_2), \quad L_{22} = \tau \frac{d}{dt} + (1/2 + \mu_2), \quad (1.11) \]

\[ \bar{L}_{11} = -\bar{\tau} \frac{d}{dt} + (-1/2 + \mu_1'), \quad \bar{L}_{12} = \frac{d}{dt}, \quad (1.12) \]

\[ \bar{L}_{21} = -\bar{\tau}^2 \frac{d}{dt} + \bar{\tau}(-1 + \mu_1' - \mu_2'), \quad \bar{L}_{22} = \bar{\tau} \frac{d}{dt} + (1/2 + \mu_2'). \quad (1.13) \]

Formally, we have duplicated expressions (1.2)–(1.3).

If

\[ \Re(\mu_1 + \mu_1') = 0, \quad \Re(\mu_2 + \mu_2') = 0, \quad (1.14) \]

then the representation \( T_{\mu_1,\mu_1';\mu_2,\mu_2'} \) is unitary in \( L^2 \). Denote by \( \Delta_{\text{tempered}} \) the set of such tuples \((\mu_1, \mu_1'; \mu_2, \mu_2')\), it is a union of a countable family of parallel 2-dimensional real planes in \( \mathbb{C}^4 \), we equip it by a natural Lebesgue measure \( d\lambda(\mu) \).

For any compactly supported smooth function \( F \) on \( \text{GL}_2(\mathbb{C}) \) we define its Fourier transform as an operator-valued function on \( \Delta \) given by

\[ T_{\mu_1,\mu_1';\mu_2,\mu_2'}(F) = \int_{\text{GL}_2(\mathbb{C})} F(X) |\det X|^{-4} \prod_{k,l=1,2} \text{Re} x_{kl} \text{dIm} x_{kl}. \]

The Plancherel formula is the following identity

\[ \langle F_1, F_2 \rangle_{L^2(\text{GL}_2(\mathbb{C}))} = -C \cdot \int_{\Delta_{\text{tempered}}} \text{tr}(T_{\mu_1,\mu_1';\mu_2,\mu_2'}(F_1)T_{\mu_1,\mu_1';\mu_2,\mu_2'}(F_2))^* \times (\mu_1 - \mu_2)(\mu_1' - \mu_2') d\lambda(\mu), \]

where \( C \) is an explicit constant.

Denote by \( K(t, s | \mu_1, \mu_1'; \mu_2, \mu_2') \) the kernel of the operator \( T_{\mu_1,\mu_1';\mu_2,\mu_2'}(F) \),

\[ T_{\mu_1,\mu_1';\mu_2,\mu_2'}(F)\varphi(t) = \int_{\mathbb{C}} K(t, s | \mu_1, \mu_1'; \mu_2, \mu_2')\varphi(s) \text{d}s. \]

1.7 Overgroup for \( \text{GL}_2(\mathbb{C}) \)

Consider the complex Grassmannian \( \text{Gr}_{4,2}(\mathbb{C}) \) of 2-dimensional planes in \( \mathbb{C}^4 \), again the set \( \text{Mat}_2(\mathbb{C}) \) is an open dense set on \( \text{Gr}_{4,2}(\mathbb{C}) \). For \( \sigma, \sigma' \in i\mathbb{R} \) consider a unitary representation \( R_{\sigma,\sigma'} \) of \( \text{GL}_4(\mathbb{C}) \) in \( L^2(\text{Mat}_2(\mathbb{C})) \) given by
We wish to write the action of the Lie algebra

\[ R_{\sigma,\sigma'}F(X) = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) F(X) = F((A + XC)^{-1}(B + XD)) \]

\[ \times |\det(A + XC)|^{-2+2\sigma -2+2\sigma'} |\det(\begin{array}{cc} A & B \\ C & D \end{array})|^{1-\sigma ||1-\sigma'}. \]

We define a unitary operator \( J : L^2(\text{Mat}_2(\mathbb{C})) \rightarrow L^2(\text{Mat}_2(\mathbb{C})) \) by

\[ J_{\sigma,\sigma'}F(X) = F(X)(\det X)^{-1+\sigma ||1-\sigma'}. \]

In this way we get a unitary representation \( U_{\sigma,\sigma'} = J_{\sigma,\sigma'}R_{\sigma,\sigma'}J_{\sigma,\sigma'}^{-1} \) of GL\(_4(\mathbb{C})\) in \( L^2(\text{Mat}_2(\mathbb{C})) \):

\[ U_{\sigma'} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) F (X) = F((A + XC)^{-1}(B + XD)) \]

\[ \times |\det(A + XC)\det(B + XD) /\det X|^{-1+\sigma ||1-\sigma'}. \]

1.8 Formulas for GL\(_2(\mathbb{C})\)

We wish to write the action of the Lie algebra

\[ \mathfrak{gl}_4(\mathbb{C})_C \simeq \mathfrak{gl}_4(\mathbb{C}) \oplus \mathfrak{gl}_4(\mathbb{C}) \]

in the Plancherel decomposition of GL\(_2(\mathbb{C})\). Denote the standard generators of \( \mathfrak{gl}_4(\mathbb{C}) \oplus 0 \) and \( 0 \oplus \mathfrak{gl}_4(\mathbb{C}) \) by \( E_{kl} \) and \( \overline{E}_{kl} \) respectively. Define the following shift operators

\[ V_1 K(t, s; \mu_1, \mu_1', \mu_2, \mu_2') = K(t, s; \mu_1 + 1, \mu_1', \mu_2, \mu_2'); \]

\[ V_1' K(t, s; \mu_1, \mu_1', \mu_2, \mu_2') = K(t, s; \mu_1, \mu_1' + 1, \mu_2, \mu_2'), \]

and similar operators \( V_2 \) and \( V_2' \) shifting \( \mu_2 \) and \( \mu_2' \).

**Theorem 1.2** The operators \( E_{14}, \overline{E}_{14}, E_{23}, \overline{E}_{23} \) are given by the formulas

\[ E_{14} = -1/2 - \sigma + \mu_1 \frac{\partial}{\partial s} V_1^{-1} - -1/2 - \sigma + \mu_2 \frac{\partial}{\partial t} V_2^{-1}; \]

\[ \overline{E}_{14} = -1/2 - \sigma' + \mu_1' \frac{\partial}{\partial s}(V_1')^{-1} - -1/2 - \sigma' + \mu_2' \frac{\partial}{\partial t}(V_2')^{-1}; \]

\[ E_{32} = 1/2 + \mu_1 + \sigma \frac{\partial}{\partial t} V_1 + 1/2 + \mu_2 + \sigma \frac{\partial}{\partial s} V_2; \]

\[ \overline{E}_{32} = 1/2 + \mu_1' + \sigma' \frac{\partial}{\partial t} V_1' + 1/2 + \mu_2' + \sigma' \frac{\partial}{\partial s} V_2'. \]
2 Calculations

2.1 The Expression for Kernel

Lemma 2.1 The kernel $K(\cdot)$ of an integral operator $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)$ is given by the formula

$$K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) = \int \int \int_{\mathbb{R}^3} F(u - tv, su - stv - tw, v, sv + w) u^{-3/2 + \mu_1 \beta_1} w^{-3/2 + \mu_2 \beta_2} du dv dw. \quad (2.1)$$

For $F \in C_0^\infty(GL_2(\mathbb{R}))$ the integration is actually taken over a bounded domain.

The integral converges if $\text{Re} \mu_1 > 1/2, \text{Re} \mu_2 > 1/2$. For fixed $\varepsilon_1, \varepsilon_2$ it has a meromorphic continuation to the whole complex plane $(\mu_1, \mu_2)$ with poles on the hyperplanes $\mu_1 = -1/2 - k, \mu_2 = -1/2 - k$, where $k = 0, 1, 2, \ldots$ (see, e.g., [3], §I.3).

Proof By the definition

$$\int_{\mathbb{R}} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F) \varphi(t) \psi(t) dt$$

$$= \int \int \int_{\mathbb{R} \text{ Mat}_2(\mathbb{R})} F(x_{11}, x_{12}, x_{21}, x_{22}) \varphi\left(\frac{x_{12} + tx_{22}}{x_{11} + tx_{21}}\right) (x_{11} + tx_{21})^{-1 + \mu_1 - \mu_2 \beta_1 - \beta_2}$$

$$\times (x_{11}x_{22} - x_{12}x_{21})^{1/2 + \mu_2 \beta_2} \frac{dx_{11} dx_{12} dx_{21} dx_{22}}{(x_{11}x_{22} - x_{12}x_{21})^2} dt.$$

In the interior integral, we pass from the variables $x_{11}, x_{12}, x_{21}, x_{22}$ to new variables $u, v, w, s$ defined by

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 - t & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad (2.2)$$

or

$$x_{11} = u - tv, \quad x_{12} = su - stv - tw, \quad x_{21} = v, \quad x_{22} = sv + w.$$

The Jacobi matrix of this transformation is triangular, and the Jacobian is $|u|$. The inverse transformation is

$$u = x_{11} + tx_{21}, \quad v = x_{21}, \quad w = \frac{x_{11}x_{22} - x_{12}x_{21}}{x_{11} + tx_{21}}, \quad s = \frac{x_{12} + tx_{22}}{x_{11} + tx_{21}}.$$

We also have $x_{11}x_{22} - x_{12}x_{21} = uw$. 
After the change of variables we come to

$$\int \int_{\mathbb{R}^2} K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) \varphi(s) \psi(t) \, ds \, dt,$$

where $K(\cdot)$ is given by (2.1).

A function $F$ has a compact support in $\mathbb{R}^4 \setminus \{x_{11}x_{22} - x_{12}x_{21} = 0\}$. So, actually, $x_{21} = v, x_{11} = u - tv, x_{22} = w + sv$ are contained in a bounded domain. This implies the second claim of the lemma. □

### 2.2 Preliminary Remarks

Below $F$ denotes

$$F := F(u - tv, su - stv - tw, v, sv + w).$$

Also, $\partial_{11} F, \partial_{12} F$, etc. denote

$$\partial_{11} F := \frac{\partial}{\partial x_{11}} F(x_{11}, x_{12}, x_{21}, x_{22})\bigg|_{x_{11} = u - tv, x_{12} = su - stv - tw},$$

etc. Partial derivatives of $F$ are

$$\frac{\partial}{\partial u} F = \partial_{11} F + s \partial_{12} F; \quad (2.3)$$

$$\frac{\partial}{\partial v} F = -t \partial_{11} F - st \partial_{12} F + \partial_{21} F + s \partial_{22} F; \quad (2.4)$$

$$\frac{\partial}{\partial w} F = -t \partial_{12} F + \partial_{22} F, \quad (2.5)$$

and

$$\frac{\partial}{\partial s} F = (u - vt) \partial_{12} F + v \partial_{22} F; \quad (2.6)$$

$$\frac{\partial}{\partial t} F = -v \partial_{11} F - (w + sv) \partial_{12} F. \quad (2.7)$$

Also, notice that

$$(y^{v\delta})' = v y^{v-1\delta+1}.$$
2.3 A Verification of the Formula for $E_{14}$

It is easy to verify that the operator $e_{14}$ in $C_0^\infty(\text{GL}_2(\mathbb{R}))$ is given by

$$e_{14} = \frac{\partial}{\partial x_{12}} - (-1 + \sigma)\frac{x_{21}}{x_{11}x_{22} - x_{12}x_{21}}.$$ 

Therefore,

$$T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}(e_{14}F) = \int\int\int \partial_{12} F u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} \, du \, dv \, dw$$

$$- (-1 + \sigma) \int\int\int F \frac{v}{uw} u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} \, du \, dv \, dw, \quad (2.8)$$

(the integration is taken over $\mathbb{R}^3$ on default).

We must verify that (2.8) coincides with

$$E_{14}K = \left( \frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_1^- + \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_2^- \right) K \quad (2.9)$$

Below we establish two formulas

$$\int\int\int \partial_{12} F u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} \, du \, dv \, dw - \frac{\partial}{\partial s} V_1^- K$$

$$= (-3/2 + \mu_2) \int\int\int F \frac{v}{uw} u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} \, du \, dv \, dw, \quad (2.10)$$

$$\int\int\int \partial_{12} F u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} \, du \, dv \, dw + \frac{\partial}{\partial t} V_2^- K$$

$$= (-3/2 + \mu_1) \int\int\int F \frac{v}{uw} u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} \, du \, dv \, dw. \quad (2.11)$$

Considering the sum of (2.10) and (2.11) with coefficients $\frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2}$ and $\frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2}$ we get coincidence of (2.8) and (2.9); for this, we use he identities

$$\frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} - \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} = 1;$$

$$(-3/2 + \mu_2) \cdot \frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} - (-3/2 + \mu_1) \cdot \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} = 1 - \sigma.$$
Now let us check (2.10). The following identity can be verified by a straightforward calculation (with (2.6) and (2.5)):

\[ \partial_1 F - \frac{1}{u} \frac{\partial}{\partial s} F = -\frac{v}{u} \frac{\partial}{\partial w} F. \]

Therefore the left-hand side of (2.10) equals to

\[ \int \int \int \left[ -\frac{v}{u} \frac{\partial}{\partial w} F \right] \cdot u^{-3/2 + \mu_1 \beta_1} w^{-3/2 + \mu_2 \beta_2} du \, dv \, dw. \]

We integrate this expression by parts in the variable \( w \) and come to (2.10).

To check (2.11), we verify the identity

\[ \partial_1 F + \frac{1}{v} \frac{\partial}{\partial t} F = \frac{v}{w} \frac{\partial}{\partial u} F \]

and after this integrate by parts as above.

### 2.4 A Verification of the Formula for \( E_{32} \)

We have

\[ e_{32} = - \left( x_{11} x_{21} \frac{\partial}{\partial x_{11}} + x_{11} x_{22} \frac{\partial}{\partial x_{12}} + x_{21}^2 \frac{\partial}{\partial x_{21}} + x_{21} x_{22} \frac{\partial}{\partial x_{22}} \right) + (-1 + \sigma) x_{21}. \]

Therefore,

\[ T(e_{32} K) = \int \int \int \left( G + (-1 + \sigma) v F \right) \cdot u^{-3/2 + \mu_1 \beta_1} w^{-3/2 + \mu_2 \beta_2} du \, dv \, dw, \]

(2.12)

where

\[ G = (u - vt) v \frac{\partial}{\partial t} F + (u - vt)(w + vs) \frac{\partial}{\partial r} F + v^2 \frac{\partial}{\partial s} F + v(w + vs) \frac{\partial}{\partial u} F + v(w + vs) \frac{\partial}{\partial u} F. \]

On the other hand,

\[ E_{32} K = \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} \int \int \int u \frac{\partial}{\partial t} F \cdot u^{-3/2 + \mu_1 \beta_1} w^{-3/2 + \mu_2 \beta_2} du \, dv \, dw + \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} \int \int \int w \frac{\partial}{\partial s} F \cdot u^{-3/2 + \mu_1 \beta_1} w^{-3/2 + \mu_2 \beta_2} du \, dv \, dw. \]

(2.13)
We must verify that (2.12) and (2.13) are equal. As in the previous subsection, this statement is reduced to a pair of identities

\[
\int\int\int (G + (-1 + \sigma)vF - u\frac{\partial}{\partial t} F) \cdot u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw \\
= (1/2 + \mu_2 + \sigma) \int\int\int vF \cdot u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw;
\]
(2.14)

\[
\int\int\int (G + (-1 + \sigma)vF + w\frac{\partial}{\partial s} F) \cdot u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw \\
= (1/2 + \mu_1 + \sigma) \int\int\int vF \cdot u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw.
\]
(2.15)

Let us verify (2.14). It can be easily checked (with (2.7), (2.4), (2.5)) that

\[
G - u\frac{\partial}{\partial t} F = -v^2 \frac{\partial}{\partial v} F - vw \frac{\partial}{\partial w} F.
\]

We substitute this to the left-hand side of (2.14) and come to

\[
\int\int\int (-v^2 \frac{\partial}{\partial v} F - vw \frac{\partial}{\partial w} F + (-1 + \sigma)vF) \\
\times u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw.
\]
(2.16)

Integrating by parts, we get

\[
\int\int\int F \cdot \frac{\partial}{\partial v}(v^2) \cdot u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw \\
+ \int\int\int vF \cdot u^{-3/2+\mu_1/\varepsilon_1} \frac{\partial}{\partial v}(w^{-1/2+\mu_2/\varepsilon_2+1}) du \, dv \, dw \\
+ (-1 + \sigma) \int\int\int vF \cdot u^{-3/2+\mu_1/\varepsilon_1}w^{-3/2+\mu_2/\varepsilon_2} du \, dv \, dw.
\]

After a summation we come to the right-hand side of (2.14).

A proof of (2.15) is similar, we use the identity

\[
G + v\frac{\partial}{\partial s} F = -v^2 \frac{\partial}{\partial v} F - uv \frac{\partial}{\partial u} F
\]

and repeat the same steps.
First, we present formulas for the action of the Lie algebra \( \mathfrak{gl}_4 \) corresponding to \( U_\sigma \), see (1.6). Denote generators of \( \mathfrak{gl}_4 \) by \( e_{kl} \), where \( 1 \leq k, l \leq 4 \). Denote by \( \partial_{pq} \) the partial derivatives \( \frac{\partial}{\partial x_{pq}} \), where \( p, q = 1, 2 \). The generators \( e_{kl} \) naturally split into 4 groups corresponding to blocks \( A, B, C, D \) in (1.6).

(a) Generators corresponding to the block \( A \) form a Lie algebra \( \mathfrak{gl}_2 \):

\[
\begin{align*}
e_{11} &= -x_{11} \partial_{11} - x_{12} \partial_{12}, & e_{12} &= -x_{21} \partial_{11} - x_{22} \partial_{12}, \\
e_{21} &= -x_{11} \partial_{21} - x_{12} \partial_{22}, & e_{22} &= -x_{21} \partial_{21} - x_{22} \partial_{22}.
\end{align*}
\]

(b) Generators corresponding to the block \( D \) also form a Lie algebra \( \mathfrak{gl}_2 \):

\[
\begin{align*}
e_{33} &= x_{11} \partial_{11} + x_{21} \partial_{21}, & e_{34} &= x_{11} \partial_{12} + x_{21} \partial_{22}, \\
e_{43} &= x_{12} \partial_{11} + x_{22} \partial_{21}, & e_{44} &= x_{12} \partial_{12} + x_{22} \partial_{22}.
\end{align*}
\]

(c) Elements corresponding to the block \( B \) form a 4-dimensional Abelian Lie algebra:

\[
\begin{align*}
e_{13} &= \partial_{11} + (-1 + \sigma) \frac{x_{22}}{\det X}, & e_{14} &= \partial_{12} - (-1 + \sigma) \frac{x_{21}}{\det X}, \\
e_{23} &= \partial_{21} - (-1 + \sigma) \frac{x_{12}}{\det X}, & e_{24} &= \partial_{22} + (-1 + \sigma) \frac{x_{11}}{\det X}.
\end{align*}
\]

(d) Elements corresponding to the block \( C \) also form a 4-dimensional Abelian Lie algebra:

\[
\begin{align*}
e_{31} &= -(x_{11}^2 \partial_{11} + x_{11} x_{12} \partial_{12} + x_{11} x_{21} \partial_{21} + x_{12} x_{21} \partial_{22}) + (-1 + \sigma)x_{11}, \\
e_{32} &= -(x_{11} x_{21} \partial_{11} + x_{11} x_{22} \partial_{12} + x_{21}^2 \partial_{21} + x_{21} x_{22} \partial_{22}) + (-1 + \sigma)x_{21}, \\
e_{41} &= -(x_{11} x_{12} \partial_{11} + x_{12}^2 \partial_{12} + x_{11} x_{22} \partial_{21} + x_{12} x_{22} \partial_{22}) + (-1 + \sigma)x_{12}, \\
e_{42} &= -(x_{12} x_{21} \partial_{11} + x_{12} x_{22} \partial_{12} + x_{21} x_{22} \partial_{21} + x_{22}^2 \partial_{22}) + (-1 + \sigma)x_{22}.
\end{align*}
\]

Denote by \( E_{kl} \) the corresponding operators \( E_{kl} \) on kernels \( K \). Formulas for operators of groups (a), (b) immediately follow from the definition of the Fourier transform,

\[
\begin{align*}
E_{11} &= -t \frac{\partial}{\partial t} - (1/2 - \mu_1), & E_{12} &= \frac{\partial}{\partial t}, \\
E_{21} &= -t^2 \frac{\partial}{\partial t} + (-1 + \mu_1 - \mu_2)t, & E_{22} &= t \frac{\partial}{\partial t} + (1/2 + \mu_2),
\end{align*}
\]

and

\[
\begin{align*}
E_{33} &= -s \frac{\partial}{\partial s} - (1/2 + \mu_1), & E_{34} &= \frac{\partial}{\partial s}, \\
E_{43} &= -s^2 \frac{\partial}{\partial s} + (-1 - \mu_1 + \mu_2)s, & E_{44} &= s \frac{\partial}{\partial s} + (1/2 - \mu_2).
\end{align*}
\]
Next,

\[ E_{13} := \frac{1}{2 - \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} s \frac{\partial}{\partial t} V_2^* + \frac{1}{2 - \mu_1 + \sigma} \left( \frac{1}{\mu_1 - \mu_2} \left( \mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) \right) V_1^* , \]

\[ E_{14} = -\frac{1}{2 - \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} s \frac{\partial}{\partial s} V_1^* - \frac{1}{2 - \mu_1 + \sigma} \frac{1}{\mu_1 - \mu_2} s \frac{\partial}{\partial t} V_2^* , \]

\[ E_{23} := \frac{1}{2 - \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} s \left( -(\mu_1 - \mu_2) + t \frac{\partial}{\partial t} \right) V_2^* + \frac{1}{2 - \mu_1 + \sigma} \frac{1}{\mu_1 - \mu_2} t \left( \mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_1^* , \]

\[ E_{24} = -\frac{1}{2 - \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} s \frac{\partial}{\partial s} V_1^* + \frac{1}{2 - \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} \left( \mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_2^* . \]

and

\[ E_{31} := \frac{1}{2 + \mu_1 + \sigma} \frac{1}{\mu_1 - \mu_2} \left( \mu_1 - \mu_2 - t \frac{\partial}{\partial t} \right) V_1^* - \frac{1}{2 + \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} t \frac{\partial}{\partial s} V_2^* , \]

\[ E_{32} = \frac{1}{2 + \mu_1 + \sigma} \frac{1}{\mu_1 - \mu_2} t \frac{\partial}{\partial t} V_1^* + \frac{1}{2 + \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} s \frac{\partial}{\partial s} V_2^* , \]

\[ E_{41} = \frac{1}{2 + \mu_1 + \sigma} \frac{1}{\mu_1 - \mu_2} s \left( \mu_1 - \mu_2 - t \frac{\partial}{\partial t} \right) V_1^* - \frac{1}{2 + \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} t \left( \mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_2^* , \]

\[ E_{42} = \frac{1}{2 + \mu_1 + \sigma} \frac{1}{\mu_1 - \mu_2} s \frac{\partial}{\partial t} V_1^* + \frac{1}{2 + \mu_2 + \sigma} \frac{1}{\mu_1 - \mu_2} \left( \mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_2^* . \]

2.6 The Case of GL_2(\mathbb{C})

Notice that formulas in Sects. 1.1–1.4 for SL_2(\mathbb{R}) and in Sects. 1.6–1.8 are very similar, except the Plancherel formulas.

The analog of formula (2.1) is

\[
K(t, s | \mu_1, \mu_1'; \mu_2, \mu_2')
= \iiint_{\mathbb{C}^3} F(u - tv, su - stv - tw, v, sv + w) u^{-3/2 + \mu_1} v^{-3/2 + \mu_1'} w^{-3/2 + \mu_2} \bar{w}^{-3/2 + \mu_2'}
\times d\, \text{Re } u \, d\, \text{Im } u \, d\, \text{Re } v \, d\, \text{Im } v \, d\, \text{Re } w \, d\, \text{Im } w. \tag{2.17}
\]

Its derivation is based on the same change of variables (2.2), its real Jacobian is \( u\bar{u} \). A further calculation one-to-one follows the calculation for GL_2(\mathbb{R}).

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