String Theory Symmetries

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Abstract

A brief review of the status of duality symmetries in string theory is presented. The evidence is accumulating rapidly that an enormous group of duality symmetries, including perturbative T dualities and non-perturbative S-dualities, underlies string theory. It is my hope that an understanding of these symmetries will suggest the right way to formulate non-perturbative string theory. Whether or not this hope is realized, it has already been demonstrated that this line of inquiry leads to powerful new tools for understanding gauge theories and new evidence for the uniqueness of string theory, as well as deep mathematical results.

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1 Introduction

Duality symmetries in string theory have gained more and more attention in recent years. As our understanding increases, it is becoming clear that they are trying to tell us something important about the structure of string theory. At first, the focus was on $T$ duality $^1$ – or target-space duality – which holds order-by-order in string perturbation theory, though it is nonperturbative on the string world sheet. Now the focus is shifting towards $S$ duality $^2$ – or coupling-constant duality – which is truly nonperturbative. At the same time, evidence is coming from several different directions that $S$ and $T$ duality are part of a larger unified symmetry structure, for which the name $U$ duality has been coined.$^3$

The history of these ideas goes back to studies of supergravity theories in the mid 1970s on the one hand,$^4,5$ and to conjectures concerning electric-magnetic dualities of supersymmetric gauge theories on the other.$^6$ Both of these themes have been pursued, and carried a great deal further, in the last couple of years.

Supergravity theories generically contain non-compact global symmetry groups. The general rule is that the scalar fields of the theory in question parametrize a symmetric space. Thus, if the non-compact symmetry group is $G$, and its maximal compact subgroup is $H$, the scalar fields map the space-time into the symmetric space $G/H$, and the number of scalar fields is $\dim G - \dim H$. The first supergravity example of this type to be found, $N = 4$ supergravity in four dimensions, is one of the most interesting. In this case there are two scalar fields and the symmetric space is $SL(2, \mathbb{R})/SO(2)$. By now, all the possibilities for any number of supersymmetries and any dimension of space-time have been enumerated.$^5$

It has been clear for some time that instanton effects modify the story in the quantum theory in a significant way. For example, in four dimensions certain of the scalar fields appear as coefficients of $F\tilde{F}$ terms, and, therefore, the Peccei–Quinn translational symmetry of the classical theory is broken to (at most) a discrete subgroup in the quantum theory. For the specific example of $N = 4, D = 4$ supergravity, this suggests that the $SL(2, \mathbb{R})$ symmetry is broken to an $SL(2, \mathbb{Z})$ subgroup. Now, as everyone should know, four-dimensional gravity and supergravity theories are not consistent quantum theories. But they do represent low-energy approximations to string theory (for suitable classes of vacua), which probably is a consistent quantum theory. These considerations led Font, $et\ al.$, to conjecture that the $SL(2, \mathbb{Z})$ group should be an exact symmetry of non-perturbative quantum string theory, which they called $S$ duality.$^2$

The conjecture that $S$ duality is an exact symmetry of string theory was a bold conjecture, because it includes an electric-magnetic duality transformation as a special case. Such a transformation sends the electric charge to the magnetic charge, which is

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inversely proportional to the electric charge, because of the Dirac quantization condition. Thus, it relates weak and strong coupling, and it is inherently nonperturbative. This represents an extension of the Montonen–Olive duality conjecture, which relates one theory with a particular coupling constant to a (possibly different) theory with a different coupling constant. In the string theory context, the coupling constants are given by the expectation values of scalar fields, the duality transformations are realized as field transformations, and the conjectured duality can be a symmetry of a single theory.

Once the conjecture of S duality is formulated, we are faced with a paradoxical situation, which arises from the limitations of our knowledge about string theory. What is known, and reasonably well understood, are recipes for constructing “classical solutions” and for adding quantum corrections to any finite order in perturbation theory. What is not known, is the equation that the classical “solutions” solve, or a non-perturbative formulation of the quantum theory. Given this state of affairs, a natural reaction to the S-duality conjecture would be to declare it “premature” on the grounds that non-perturbative symmetries cannot be tested in a theory that is only known perturbatively. Strictly speaking, this is indeed true. However, if one is willing to restrict attention to vacua with lots of supersymmetry and make some mild assumptions about the non-perturbative physics, then some interesting and non-trivial tests of S duality can be carried out. All such tests to date have been successful. Thus, while S duality has certainly not been proved to be a string theory symmetry, the situation is quite encouraging.

My point of view is that the purpose of these studies should not be viewed as verifications of the symmetry, but rather as explanations of how it operates. They can provide important clues about the structure of non-perturbative string theory, which could eventually provide guidance in attempts to formulate non-perturbative string theory. Perhaps the most important information to be learned is the fundamental underlying symmetry group of the theory. Finding this symmetry is a non-trivial challenge because, for any particular choice of vacuum (or classical solution), there is a great deal of spontaneous symmetry breaking. As we have said, duality symmetries are realized by non-linear transformations of scalar fields that describe symmetric spaces. Models of this type (sometimes called non-linear sigma models) are prototypes for Goldstone bosons and spontaneous symmetry breaking phenomena.

As has already been indicated, larger pieces of the hidden duality symmetries of string theory become visible by restricting attention to classical backgrounds with lots of supersymmetry. It is also advantageous to consider low space-time dimensions. The reason for this is intuitively clear. The analysis of duality symmetries is carried out in terms of effective field theories containing scalar fields associated with symmetric spaces. However, in the underlying string theory it is clear that all string modes should have a democratic status, irrespective of their spins. By considering
ground states in low dimension, one limits the possibilities for the spins and forces a large portion of the spectrum to reside in scalar fields, and therefore more of the symmetry can be characterized by symmetric spaces. In the extreme case where all spatial dimensions have been suitably compactified (or eliminated), one may have a chance of identifying a very large symmetry group such as $E_{10}$ or the monster Lie algebra, or discrete subgroups of these.\textsuperscript{9,10} This kind of information could prove crucial in formulating string theory. The key point is that the symmetries that are being identified are properties of the underlying theory, irrespective of the choice of any classical background or quantum vacuum. The only purpose of the background or vacuum is to make the symmetry visible.

Recent work has made it dramatically clear that these studies are valuable even if they do not serve the purpose that I am advocating. Studies of electric-magnetic duality transformations of supersymmetric gauge theories have clarified the Montonen–Olive conjecture a great deal and lent additional support to the $S$-duality conjecture in string theory.\textsuperscript{11,12} Perhaps more importantly, they have provided remarkable exact non-perturbative results concerning fundamental issues in gauge theory. With a slight twist, they have also had a major impact in the mathematical world.\textsuperscript{13} These achievements should convince any sceptic that this is a useful line of inquiry. It is important to understand that in these gauge theories the scalar fields of string theory have been frozen out, \textit{i.e.}, replaced by their expectation values. As a consequence, the string theory duality symmetry, which was described by a field transformation, is replaced by a transformation of parameters. Thus, what one wishes to demonstrate in these cases is the equivalence of a family of theories rather than the symmetry of a single theory. This viewpoint is clearest in the case of $N = 4$ Yang–Mills theories, where the parameters really are fixed constants (since the theory is finite). For renormalizable theories in which parameters run with scale, and dimensional transmutation occurs, there are additional issues. These have been addressed in the $N = 2$ case in Ref. 12.

## 2 Duality Symmetries in Four Dimensions

When the heterotic string theory is toroidally compactified to four dimensions, in the manner described by Narain,\textsuperscript{14} two distinct duality groups appear. One way of understanding this is to focus attention on the massless scalar modes of the low-energy effective supergravity theory. These fields describe the product of two symmetric spaces. The $T$ duality symmetric space is $O(6,22)/O(6) \times O(22)$, and the $S$ duality one is $SL(2,\mathbb{R})/SO(2)$. The 132 scalars that parametrize the $T$ space have their origin in internal components of the ten-dimensional metric tensor $g_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$, and 16 $U(1)$ gauge fields $A_\mu^I$ (corresponding to the Cartan subalgebras
E_8 \times E_8 \text{ or } SO(32)). The two scalars that parametrize the S space correspond to the dilaton \( \phi \) (the ten-dimensional dilaton redefined by a term involving the other scalars) and the axion \( \chi \) (obtained by replacing the four-dimensional \( B_{\mu \nu} \) by its dual). The associated duality symmetries are restricted to the subgroups \( G_T = O(6, 22; \mathbb{Z}) \) and \( G_S = SL(2, \mathbb{Z}) \).

The continuous symmetries \( O(6, 22) \) and \( SL(2, \mathbb{R}) \) are global symmetries of the classical effective field theory, but the duality subgroups are gauge symmetries of the string theory. Thus, the underlying gauge symmetry of string theory, whether continuous or discrete, should contain these groups as subgroups. The meaning of gauge symmetry in string theory ultimately must be that configurations related by symmetry transformations are identified as physically identical, and should be counted only once in the path integral that defines the theory. Since space-time must ultimately be a derived concept in string theory, not built into the basic (background-independent) formulation, it would be meaningless to speak about space-time dependent transformations, which is the way we are accustomed to describing gauge symmetries in ordinary gauge theories.

The complete massless spectrum of the toroidally compactified classical heterotic string contains the scalars discussed above, as well as various other fields, which will now be enumerated. The 132 \( T \) moduli are conveniently described by a \( 28 \times 28 \) matrix \( M \), belonging to the group \( O(6, 22) \). This means that if \( L \) is the metric matrix of \( O(6, 22) \),

\[
M^T LM = L. \tag{1}
\]

The restriction to the coset space \( O(6, 22)/O(6) \times O(22) \) is achieved by requiring in addition that

\[
M^T = M. \tag{2}
\]

This procedure for describing a non-compact symmetric space is completely general. If (for some other problem) the matrix \( M \) is complex, then the restriction to the coset is given by \( M^\dagger = M \). One can introduce an analogous \( 2 \times 2 \) matrix \( M \) belonging to \( SL(2, \mathbb{R}) \), also taken to be symmetric, to describe the \( S \) moduli. Alternatively, since there are only two \( S \) moduli, and the coset space is Kähler, one can describe it by a single complex scalar field

\[
\lambda = \lambda_1 + i\lambda_2 = \chi + ie^{-\phi}, \tag{3}
\]

where \( \chi \) and \( \phi \) are the axion and dilaton.

The rest of the massless spectrum depends on the values of the \( T \) moduli. Since \( G_T \) is a gauge symmetry of the string theory, the \( T \) moduli space is actually \( O(6, 22)/O(6) \times O(22) \times G_T \). The division by \( G_T \) introduces orbifold points, which correspond to points of enhanced gauge symmetry where the massless spectrum is
enlarged. We choose to avoid these points, restricting $M$ to “generic points in moduli space.” Then the massless spectrum is as follows: The graviton, described by the canonically normalized Einstein metric, is invariant under both $S$ and $T$ transformations. (Note that the “string metric,” which differs by a dilaton-dependent factor transforms under $S$ transformations.) The axion-dilaton field $\lambda$ is invariant under $T$ transformations and undergoes linear fractional transformations

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d},$$

(4)

under the $S$ group. The 132 moduli described by $M$, transform as a symmetric $28 \times 28$ representation of the $T$ group and are invariant under the $S$ group. In addition, there are 28 abelian gauge fields $A_a^\mu$, giving a gauge group $[U(1)]^{28}$, which form a 28-dimensional representation of $O(6,22)$. Finally, there are all the fermions required by supersymmetry, which we will omit from further consideration. (They are not an essential complication.)

We can now write down a Lagrangian that has manifest $T$ symmetry – the $S$ structure will require additional discussion:

$$\frac{1}{\sqrt{-g}} L = R - \frac{1}{2\lambda_2} g^{\mu\nu} \partial_\mu \lambda \partial_\nu \lambda + \frac{1}{8} g^{\mu\nu} \text{tr} (M^{-1} \partial_\mu MM^{-1} \partial_\nu M) + \frac{1}{4} L_{ab} F^a_{\mu\nu} \tilde{G}^{\mu\nu b},$$

(5)

where

$$G^a_{\mu\nu} = \lambda_1 F^a_{\mu\nu} + \lambda_2 (ML)^a_b F^b_{\mu\nu},$$

(6)

and tilde represents the covariant dual. The first three terms in the lagrangian are invariant under the $S$ group $SL(2,\mathbb{R})$, but the last one is not. However, the equations of motion do have $SL(2,\mathbb{R})$ symmetry, so that it is a classical symmetry. To see this one, one must require that $G^a_{\mu\nu}$ and $F^a_{\mu\nu}$ from an $SL(2,\mathbb{R})$ doublet, and that

$$\begin{pmatrix} G \\ F \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}.$$

(7)

Note that this means that $F$ and $\tilde{F}$ get mixed (in a way that depends on all the moduli), which is the hallmark of an electric-magnetic duality.

The structure of the theory described above can be described in greater generality, without choosing specific groups. The discussion that follows is based on the paper of Hull and Townsend,\textsuperscript{3} who built on earlier work by Gaillard and Zumino.\textsuperscript{15} The lagrangian above has a number of scalars $\phi^i$ and the structure

$$\frac{1}{\sqrt{-g}} L = R - \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{4} F^I_{\mu\nu} \tilde{G}^{\mu\nu I},$$

(8)
where
\[ G_{\mu
u I} = a_{IJ}(\phi) F^I_{\mu\nu} + m_{IJ}(\phi) \tilde{F}^J_{\mu\nu}. \] (9)

There are \( n \) equations of motion and \( n \) Bianchi identities for the gauge fields. These
\( 2n \) equations can be succinctly written as \( d \mathcal{F} = 0 \), where \( \mathcal{F} \) is the \( 2n \)-component
vector \((\mathcal{F}^I, G_I)\).

Gauss’s law can be used to define \( 2n \) conserved charges
\[ Z = \oint_{\Sigma} \mathcal{F} = (p^I, q_I). \] (10)

Here \( \Sigma \) is a large two-sphere, and all fields are assumed to approach constant valus at
spatial infinity. \( p^I \) and \( q_I \) encode convenient linear combinations of the \( 2n \) conserved
electric and magnetic charges. In terms of these combinations the most general version
of the Dirac quantization conditions for a pair of dyons with charges \((p^I, q_I)\) and
\((p'^I, q'_I)\) in the presence of vacuum \( \theta \) angles takes the form
\[ p^I q'_I - p'^I q_I \in \mathbb{Z}. \] (11)

The significant fact about this formula is its symplectic structure, \( i.e., \) it is invariant
under \( Sp(2n) \) transformations. This implies that the allowed charges \((q_I, p^I)\) form
a self-dual lattice that is invariant under \( Sp(2n; \mathbb{Z}) \). Without the restriction to integers,
the quantization condition would be violated. Therefore, the most general
global symmetry group that can arise in a theory of this type must be a subgroup of
\( Sp(2n; \mathbb{Z}) \). In the heterotic string example we had \( n = 28 \), and it is true that
\[ SL(2, \mathbb{R}) \times O(6, 22) \subset Sp(56). \] (12)

In fact, the fundamental 56 of \( Sp(56) \) corresponds to \((2, 28)\) for the subgroup. Moreover, as required by the Dirac quantization condition,
\[ SL(2, \mathbb{Z}) \times O(6, 22; \mathbb{Z}) \subset Sp(56, \mathbb{Z}). \] (13)

3 Other Examples

Let us start by describing one more example in four dimensions, and then discuss
what happens in other dimensions. The type II superstring has twice as much
supersymmetry as the heterotic string, and so when toroidally compactified to four
dimensions, the low-energy effective field theory is \( N = 8 \) supergravity. The classical
global symmetry of this theory is \( E_{7,7} \), which is the maximally non-compact form of
\( E_7 \). The fundamental representation of \( E_7 \) is 56-dimensional, and the \( E_{7,7} \) matrices
in this representation are symplectic. Thus, \( E_{7,7} \subseteq Sp(56) \), as required. It may not be obvious at first how to define a discrete subgroup \( E_7(\mathbb{Z}) \), but Hull and Townsend point out that the appropriate definition is clearly

\[
E_7(\mathbb{Z}) \equiv E_{7,7} \cap Sp(56, \mathbb{Z}).
\]

(14)

This is the four-dimensional duality of the type II string in four dimensions, and the expected continuous \( S \) and \( T \) duality groups are \( SL(2, R) \) and \( O(6, 6) \). In fact, we are finding more, since

\[
SL(2, R) \times O(6, 6) \subset E_{7,7}
\]

(15)

is a proper subgroup. So we have \( S \times T \subset U \), where the \( U \) duality group is \( E_7(\mathbb{Z}) \).

It is a curious coincidence that both the heterotic and type II theories in four dimensions have gauge group \([U(1)]^{28}\). However, there are also some interesting differences. In the heterotic case, there are 6 compactified right-movers and 22 compactified left-movers, so that all 28 electric charges arise in the elementary string spectrum as Kaluza–Klein and winding-mode excitations, \( i.e. \) as discrete internal left- and right-moving momenta. In the type II case, on the other hand, there are just 6 right-movers and 6 left-movers, so that only 12 of the 28 types of electric charges arise in the spectrum of elementary string excitations. Group theoretically, the 56 of \( E_{7,7} \) decomposes under \( S \times T \) as \((2, 12) + (1, 32)\), where 32 is a spinor representation of \( O(6, 6) \). The \((2, 12)\) describes the 12 electric charges which are excited in the elementary string spectrum as well as the corresponding magnetic charges, while \((1, 32)\) describes the remaining electric and magnetic charges. Thus, just as for the magnetic charges, these latter electric charges only appear in the soliton spectrum.

Let us now briefly discuss what happens in other dimensions \((d)\), always assuming that we are considering toroidally compactified strings. (Other geometries will generally give smaller groups and be more difficult to analyze.) First of all, the \( T \) duality groups, whose origin is directly tied to the compactified dimensions, is known to be \( O(10 - d, 26 - d; \mathbb{Z}) \) in the heterotic case and \( O(10 - d, 10 - d; \mathbb{Z}) \) in the type II case. \( S \) duality on the other hand depended on the fact that a duality transformation applied to \( B_{\mu \nu} \) gives rise to a scalar axion, so it has no counterpart in dimensions greater than four. However, this is often not the whole story. The complete \( U \) duality group is larger than \( S \times T \) for type II strings in any dimension and for heterotic strings in dimensions less than four.

In the case of the type II theory, the relevant symmetry is determined by the corresponding maximal supergravity theory. The global symmetry of these theories in \( d \) dimensions turns out to be \( E_{11-d,11-d} \). This has been demonstrated in detail for the cases \( d = 3, 4, 5 \) corresponding to usual Cartan \( E \) groups. For other values of \( d \), the meaning of \( E_{11-d} \) is given by adding or removing dots from the Dynkin diagram in the standard way. Thus, for example, \( E_{5,5} = SL(5, \mathbb{R}) \) and \( E_{1,1} = SL(2, \mathbb{R}) \). The
latter is a symmetry only of the type IIB theory in ten dimensions. What is more interesting to contemplate is the larger symmetries that this formula suggests for $d < 3$. $E_9$ is the affine extension of $E_8$ and $E_{10}$ is a poorly understood hyperbolic Lie algebra. Nicolai has presented some evidence that $E_9$ is a global symmetry of the classical two-dimensional supergravity theory, though work that I am currently doing suggests that this may not be precisely correct. In any case, what is of greater importance is to identify the discrete subgroup that should describe a symmetry of the quantum theory. Understanding $E_{10}$ and figuring out whether it has something to do with the symmetry of string theory is an even greater challenge.

In the case of the heterotic string theory, dimensions less than four are also interesting. It was shown a long time that the three-dimensional supergravity theory has an $O(8, 24)$ symmetry. This is an extension of the expected $S \times T$ symmetry:

$$O(8, 24) \supset SL(2, \mathbb{R}) \times O(7, 23).$$  \hspace{1cm} (16)$$

Recently, Sen has explained that the string theory duality symmetry in this case is $O(8, 24; \mathbb{Z})$, which can be viewed as a $U$ duality extension of $SL(2, \mathbb{Z}) \times O(7, 23; \mathbb{Z})$. The next step is to investigate what happens on reduction to two dimensions. The general rule for symmetric space models, at least roughly, is believed to be that if the symmetry group in three dimensions is $G$, reduction to two dimensions will give a classical theory whose symmetry is $\hat{G}$, the affine extension of $G$. I find that actually the classical symmetry is a large subgroup of $\hat{G}$, which I call $\hat{G}_H$, where $H$ is the maximal compact subgroup of $G$. Sen has recently reported impressive progress in identifying the discrete duality group of the toroidally compactified heterotic string in two dimensions. Once a satisfactory understanding of the situation in two dimensions is achieved, we may build up the courage to investigate the hyperbolic algebra and its discrete subgroup that are expected to appear upon reduction to one dimension.

## 4 Solitons and String Excitations

Let us now return to the heterotic string theory toroidally compactified to four dimensions, for which the duality group is $SL(2; \mathbb{Z}) \times O(6, 22; \mathbb{Z})$ and consider the spectrum of charged excitations. The great advantage of having $N = 4$ supersymmetry is that the supersymmetry algebra contains central charges whose origin can be understood in terms of internal components of the ten-dimensional momenta. Thus, since supersymmetry is associated with right-movers, there are six complex central charges corresponding to right-moving electric and magnetic charges. The consequence of this is that one can derive a lower bound on the $(mass)^2$ of any charged state, known as
the Bogomol'nyi bound. In terms of the charges \( p^I \) and \( q_I \) introduced in section two, it takes the form

\[
M^2 \geq \frac{1}{2} (p^I, q_I) \mathcal{M}^{(0)} (M^{(0)})^T + L_{IJ} \left( \frac{p^I}{q_J} \right),
\]

(17)

where an overall constant factor can be absorbed in the definition of the string scale \( \alpha' \). \( \mathcal{M}^{(0)} \) is a \( 2 \times 2 \) matrix representing the asymptotic values of the \( S \) moduli and \( M_{IJ}^{(0)} \) is a \( 28 \times 28 \) matrix representing the asymptotic values of the \( T \) moduli. The combination \( M^{(0)} + L \) projects out those charge contributions that come from right-movers.

This inequality must hold so long as the \( N = 4 \) supersymmetry remains unbroken. What is more interesting is that when it is realized as an equality the algebra has “short representations,” which are not possible otherwise. The basic “massive” representative of \( N = 4 \) supersymmetry is 256-dimensional, whereas the basic “massless” one is 16-dimensional. The latter is very familiar from \( N = 4 \) super Yang–Mills theory and from the ground state structure of superstrings (8 NS states and 8 R states). So, if we identify a set of states that in some approximation saturate the Bogomol’nyi bound, then they must continue to do so in the exact quantum theory. The only assumptions that enter this argument are that \( N = 4 \) supersymmetry remains an exact symmetry and that the full theory does not have different phases (e.g., one that confines), which could invalidate the argument. These mild assumptions allow us to draw exact non-perturbative conclusions about a theory we don’t even know! (G. ’t Hooft found this remark to be very provocative.)

Let us now examine the spectrum of elementary string excitations. These are entirely electric, of course. As usual, because of the phenomenon of level-matching, there are two formulas for the mass of each state, one in terms of left-movers and one in terms of right-movers. These are

\[
M^2 = \frac{1}{\lambda_2^{(0)}} \left( \frac{1}{2} p^2_L + N_L - 1 \right) = \frac{1}{\lambda_2^{(0)}} \left( \frac{1}{2} p^2_R + N_R - \delta \right),
\]

(18)

where \( \lambda_2^{(0)} \) is the asymptotic value of \( \lambda_2 = e^{-\phi} \), and the internal momentum contributions are

\[
p^2_L = \frac{1}{2} p^I (M^{(0)} - L)_{IJ} p^J \]

(19)

\[
p^2_R = \frac{1}{2} p^I (M^{(0)} + L)_{IJ} p^J.
\]

(20)

\( N_L \) and \( N_R \) are the usual oscillator-excitation eigenvalues, and \( \delta \) is the superstring zero-point energy (\( \frac{1}{2} \) in the NS sector and 0 in the R sector). The factor \( \lambda_2^{(0)} \), which is essential (and possibly unfamiliar), appears because masses are being measured with respect to the Einstein metric rather than the string metric, as is often assumed.
Now, the way to saturate the Bogomol’nyi bound is clear. One must take $N_R = 0$, since only by taking the ground state of the right-moving oscillators can one obtain a “short” 16-dimensional representation of the supersymmetry algebra. Of course, this gets tensored with whatever left-moving contributions occur.

Setting $N_R = 0$ and using the level-matching condition, we see that the most general elementary string excitations saturating the Bogomol’nyi bound are characterized by the equation

$$N_L = 1 + \frac{1}{2}(p^2_R - p^2_L) = 1 + \frac{1}{2}p^I L_{IJ} p^J. \quad (21)$$

Since the charge vectors $p^I$ belong to an even self-dual lattice, with metric $L_{IJ}$, this is guaranteed to be an integer. There are string states corresponding to each distinct way of solving this equation (tensored with the 16-dimensional supermultiplet of right-movers). $S$ duality, if it really is a symmetry, requires the existence of a rich spectrum of magnetically charged partners for these particles. Since none of these are in the spectrum of elementary excitations, they must all arise as solitons.

The simplest case is when there are no excitations of left-moving oscillators. In this case $p^2 = p^I L_{IJ} p^J = -2$, and the complete charge vector is $(p^I, 0)$. However, $S$ duality transforms this to the vector $(a p^I, c p^I)$, where $a$ and $c$ are relatively prime integers. Since these will also belong to a short supermultiplet, they must also saturate the Bogomol’nyi bound and, therefore, eq. (17) tells us what their masses must be. The existence of these states has been discussed in detail by Sen. He argues that the states with $N_L = 0$ and $c = 1$ can be identified with BPS monopoles and dyons (appropriately generalized to the present context), and that they occur with precisely the required multiplicities and other properties. The required states with $c > 1$ are much more difficult to analyze. Sen has argued that the $S$-duality conjecture requires that the moduli space of $c$ BPS monopoles should have one, and only one, normalizable harmonic form. Such a form is necessarily self-dual (or anti-self-dual). He then proceeded to prove that this is the case for $c = 2$. I am informed that the case $c > 2$ is being investigated by Segal, and that he is making good progress towards establishing the desired result.

The preceding is mathematically very challenging, even though it only concerns $N_L = 0$ states. The next level, $N_L = 1, p^2 = 0$ states include excitations with spins ranging up to $J = 2$, and certainly probe aspects of the theory that go well beyond ordinary field theory. Some of these states, which are identified with $H$ monopoles, have been investigated by Gauntlett and Harvey. They have made impressive progress, but their work still appears to be somewhat inconclusive.
5 Conclusion

The study of string duality symmetries is proving to be a fascinating subject that is opening up many new avenues of research. These have far-reaching implications for four-dimensional gauge theories and fundamental mathematics, as well as for the study of string theory itself. The latter is certainly the most challenging problem, however. The hope is that when the complete group of duality symmetries is identified and understood it will provide the key to obtaining a complete formulation of string theory. Whether or not this goal will ever be achieved remains to be seen, but it has already been demonstrated that we will learn a great deal in the attempt.

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