LINE GRAPHS OF SIMPLICIAL COMPLEXES

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ABSTRACT. We consider the line graph of a simplicial complex. We prove that, as in the case of line graphs of simple graphs, one can compute the second graded Betti number of the facet ideal of a pure simplicial complex in terms of the combinatorial structure of its line graph. We also give a characterization of those graphs which are line graphs of some pure simplicial complex. In the end, we consider pure simplicial complexes which are chordal and study their relation to chordal line graphs.

INTRODUCTION

Defined by Whitney [23] line graphs have been intensively studied in graph theory. This concept has been introduced under different names by many authors but the term ‘line graph’ was later introduced by Hoffmann [16]. A characterization of those graphs which are line graphs of some graph can be found for instance in [1] and [11]. Due to their properties, a lot of work has been done for generalizing them for hypergraphs, [2, 17, 21]. For instance, Tyshkevich and Zverovich defined the line graph of a hypergraph as the graph whose vertex set is the edge set of \( H \), and two vertices are adjacent in \( L(H) \) if the corresponding edges are adjacent or intersecting edges in \( H \), [21]. Moreover, in [2] Bermond, Heydemann, and Sotteau considered the \( k \)-line graph of a hypergraph as being the graph with the vertex set given by the set of edges of the hypergraph and two vertices are adjacent if the intersection of the corresponding edges of \( H \) has at least \( k \) elements.

From commutative algebra point of view, line graphs of simple graph appear in the computation of the second graded Betti number of its edge ideal as Eliashou and Villarreal proved [7]. We are mainly interested in simplicial complexes and their facet ideals, which can be viewed as edge ideals of hypergraphs. Therefore we consider pure simplicial complexes of dimension \( d - 1 \), \( d \geq 2 \), and we use the definition of \( k \)-line graphs given in [2], with \( k = d - 1 \). Since this is the only graph that we consider, we will call it the line graph of the simplicial complex. Under this definition, line graphs appears also in the study of polytopes as facet-ridge incidence graphs. For instance, Blind and Mani proved that simplicial polytopes are completely determined by their facet-ridge graphs [3].

For line graphs of simplicial complexes we prove several results which are similar to the case of simple graphs. We give a similar result for the second graded Betti number of the facet ideal of a pure simplicial complex [Theorem 3.2] and we characterize those graphs which are line graphs of some simplicial complexes, [Theorem 4.5]. Moreover, we pay attention to the Alexander dual of the simplicial complex and derive some properties related to the linearity of the resolution.
The paper is structured in five sections. In the first section we recall basic notions of simplicial complexes, clutters, graphs and edge ideals. The second section is devoted to the definition of the line graph of a simplicial complex and combinatorial properties such as being connected or a formula for the number of edges are determined.

In the third section we consider applications to the resolutions of facet ideals of pure simplicial complexes. More precisely, we prove Theorem 3.2 which is similar to the one given by Eliahou and Villarreal [7, Proposition 2.1].

In the forth section we pay attention to combinatorial properties of those graphs which are line graphs of some simplicial complex. In Theorem 4.5 we give a characterization of them similar to the one from [1]. We also determine necessary conditions that a graph should fulfill in order to be the line graph of a pure simplicial complex.

The last section is devoted to the study of properties of the line graph of being chordal. Since also exists the notion of chordal simplicial complexes, we are interested in the relations between them. We prove that a pure simplicial complex of dimension $d$ is chordal if its line graph is chordal and it has diameter at most $d$ (Theorem 5.14).

Through the paper, we pointed out several problems that naturally arose.

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1. Background

In this section we recall the notions and properties that will be used later. For more details, one may see [1, 9, 10, 12, 18, 22].

1.1. Simplicial complexes. A simplicial complex $\Delta$ on the vertex set $\{1, \ldots, n\}$, where $n \geq 1$ is an integer, is a collection of subsets (called faces) such that any vertex is in $\Delta$ and, if $F$ is a face of $\Delta$ and $G \subseteq F$, then $G$ is also a face of $\Delta$. Maximal faces (with respect to the inclusion) are called facets. We usually write $F(\Delta)$ for the set of facets. Moreover, if $F(\Delta) = \{F_1, \ldots, F_r\}$, then $\Delta = \langle F_1, \ldots, F_r \rangle$ is just another way to write the simplicial complex with facets $F(\Delta)$. A simplex is a simplicial complex with only one facet. The dimension of the simplicial complex is denoted by $\text{dim}(\Delta)$ and is defined as $\text{dim}(\Delta) = \max\{|F| - 1 : F \in \Delta\}$. A simplicial complex is pure if all its facets have the same dimension and a $d$-simplex is a simplex of dimension $d$, that is $\langle\{1, \ldots, d + 1\}\rangle$.

Let $\Delta_1$ and $\Delta_2$ be simplicial complexes on the vertex set $V_1$ and $V_2$ respectively, with $V_1 \cap V_2 = \emptyset$. The simplicial complex

$$\Delta_1 \ast \Delta_2 = \{F \cup G : F \in \Delta_1, G \in \Delta_2\}$$

is the join of $\Delta_1$ and $\Delta_2$. 

The Alexander dual of a simplicial complex $\Delta$, denoted by $\Delta^\vee$, is the simplicial complex with the faces given by the complementary of non-faces of $\Delta$, that is

$$\Delta^\vee = \{ F^c : F \not\in \Delta \} .$$

For a simplicial complex $\Delta$, let $\Delta^c$ be the simplicial complex with the facet set

$$\mathcal{F}(\Delta^c) = \{ F^c : F \in \mathcal{F}(\Delta) \} .$$

A simplicial complex is called shellable if there is an ordering of its facets $F_1, \ldots, F_r$ such that for all $i, j$ with $1 \leq j < i \leq r$, there exist a vertex $v \in F_i \setminus F_j$ and an integer $k < i$ such that $F_i \setminus F_k = \{ v \}$.

1.2. Clutters. A clutter $\mathcal{C}$ on the vertex set $V$ is a set of subsets of $V$ (called circuits of $\mathcal{C}$) such that if $e_1$ and $e_2$ are distinct circuits of $\mathcal{C}$ then $e_1 \not\subseteq e_2$. Clutters are simple hypergraphs and have also been referred to in the literature as Sperner families, or as antichains of sets. For more details on clutters and relations to simplicial complexes, one may check [18] for instance. A $d$-circuit is a circuit consisting of exactly $d$ vertices, and a clutter is $d$-uniform if every circuit has exactly $d$ vertices. An independent set of $\mathcal{C}$ is a subset of $V$ containing no circuit. Clutters and simplicial complexes are linked via the independence complex

$$\text{Ind}(\mathcal{C}) = \{ F \subseteq V : F \text{ is an independent set of } \mathcal{C} \} .$$

Note that the Stanley–Reisner complex of the edge ideal of a clutter is the independence complex of the clutter [18].

Through this paper, to any simplicial complex $\Delta$, we will associate a clutter with the same vertex set as $\Delta$ and with the circuits given by the facet set $\mathcal{F}(\Delta)$. We denote by $\mathcal{C}(\Delta)$ this clutter. Since we will not be mainly interested in the structure of the simplicial complex, but more on the combinatorics of the associated clutter, we will simply say $\Delta$, but we will understand $\mathcal{C}(\Delta)$ whenever the confusion is unlikely.

1.3. Squarefree monomial ideals associated to simplicial complexes and clutters. Let $\Delta$ be a simplicial complex on the vertex set $V = \{1, \ldots, n\}$ and $k$ a field. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over the field $k$. To a set $F = \{ i_1, \ldots, i_t \} \subseteq V$, one may associate the squarefree monomial $x_F = x_{i_1} \cdots x_{i_t} \in S$ and we will refer to $F$ as the support of the monomial $x_F$.

For the simplicial complex $\Delta$ two squarefree monomial ideals are of interest:

- the Stanley–Reisner ideal $I_\Delta$ which is generated by the squarefree monomials which correspond to the minimal non-faces of $\Delta$,

$$I_\Delta = (x_F : F \not\in \Delta)$$

- the facet ideal $I(\Delta)$ which is generated by the squarefree monomials which correspond to the facets of $\Delta$,

$$I(\Delta) = (x_F : F \in \mathcal{F}(\Delta))$$

We will write $k[\Delta]$ for the Stanley–Reisner ring of $\Delta$, that is $k[\Delta] = S/I_\Delta$.

If we consider the Stanley–Reisner ideal of the Alexander dual of $\Delta$, then one has $I_{\Delta^\vee} = I(\Delta^c)$ according to [13, Lemma 1.2].
If $C$ is a clutter, then its edge ideal is

$$I(C) = (x_e : e \text{ is a circuit of } C).$$

Note that, if $C$ is given by the facets of a simplicial complex $\Delta$, then $I(C) = I(\Delta)$. Let $I \subseteq S = \mathbb{k}[x_1, \ldots, x_n]$ be an ideal and $F$ is the minimal graded free resolution of $S/I$ as an $S$-module:

$$F : 0 \rightarrow \bigoplus_j S(-j)^{\beta_{ij}} \rightarrow \cdots \rightarrow \bigoplus_j S(-j)^{\beta_{ij}} \rightarrow S \rightarrow S/I \rightarrow 0,$$

then the numbers $\beta_{ij}$ are the graded Betti numbers of $S/I$, the projective dimension of $S/I$ is $\text{proj dim } S/I = \max\{i : \beta_{ij} \neq 0\}$ and the Castelnuovo–Mumford regularity is $\text{reg } S/I = \max\{j - i : \beta_{ij} \neq 0\}$. Let $d > 0$ be an integer. An ideal $I$ of $S$ has a $d$-linear resolution if the minimal graded free resolution of $S/I$ is of the form

$$\cdots \rightarrow S(-d - 2)^{\beta_2} \rightarrow S(-d - 1)^{\beta_2} \rightarrow S(-d)^{\beta_1} \rightarrow S \rightarrow S/I \rightarrow 0.$$

Equivalently, an ideal $I$ has a $d$-linear resolution if and only if it is minimally generated in degree $d$ and $\text{reg } S/I = d - 1$. If $d = 2$, we simply say that the ideal has a linear resolution.

In between the combinatorics of simplicial complexes and the homological properties of the associate squarefree monomial ideals there are strong connections.

**Theorem 1.1.** (Eagon–Reiner) \cite{er} Let $\mathbb{k}$ be a field and $\Delta$ a simplicial complex. Then $\mathbb{k}[\Delta]$ is Cohen–Macaulay if and only if $I_{\Delta^\vee}$ has a linear resolution.

We recall that a simplicial complex is Cohen–Macaulay if its Stanley–Reisner ring has this property.

**Definition 1.2.** \cite{14} A monomial ideal $I$ of $S$ is called an ideal with linear quotients if there is an ordering of its minimal monomial set of generators $u_1, \ldots, u_r$, satisfying the following property: for all $2 \leq i \leq r$ and for all $j < i$, there exist $l$ and $k$, $l \in \{1, \ldots, n\}$ and $k < i$, such that $u_k / \gcd(u_k, u_i) = x_l$ and $x_l$ divides $u_j / \gcd(u_j, u_i)$.

In between the combinatorics of simplicial complexes and the homological properties of the associate squarefree monomial ideals there are strong connections as the following two results show.

**Theorem 1.3.** \cite[Theorem 1.4]{13} Let $\mathbb{k}$ be a field and $\Delta$ a pure simplicial complex. Then $\Delta$ is shellable if and only if $I_{\Delta^\vee}$ has linear quotients.

### 1.4. Graphs.

Through this paper, all the graphs will be assumed to be simple, that is without loops or multiple edges. Let $G$ be a finite simple graph with the vertex set $V(G)$ and the set of edges $E(G)$. Two vertices $u, v \in V(G)$ are called adjacent (or neighbors) if they form an edge in $G$. For a vertex $u$ of $G$, we denote by $\mathcal{N}(u)$ the set of all the neighbors of $u$, also called the neighborhood of $u$. More precisely, $\mathcal{N}(u) = \{v \in V(G) : \{u, v\} \in E(G)\}$. The closed neighborhood of $u$ is denoted by $\mathcal{N}[u]$ and is defined as $\mathcal{N}[u] = \mathcal{N}(u) \cup \{u\}$. The degree of the vertex $u$, denoted by $\deg u$, is defined to be the size of the neighborhood set of $u$, that is $\deg u = |\mathcal{N}(u)|$. By a free vertex we mean a vertex of degree 1. A graph is called complete if any
two vertices are adjacent. We denote by $K_n$ the complete graph with $n$ vertices. Moreover, we denote by $K_{1,d}$ the star graph on $d + 1$ vertices, that is the graph with the vertex set $V = \{u, v_1, \ldots, v_d\}$ and the edges $\{u, v_i\}$, $1 \leq i \leq d$.

By a subgraph $H$ of $G$ we mean a graph with the property that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. One says that a subgraph $H$ of $G$ is induced if whenever $u, v \in V(H)$ so that $\{u, v\} \in E(G)$ then $\{u, v\} \in E(H)$.

A path of length $t \geq 2$ in $G$ is, by definition, a set of distinct vertices $x_0, x_1, \ldots, x_t$ such that $\{x_i, x_{i+1}\}$ are edges in $G$ for all $i \in \{0, \ldots, t - 1\}$. The distance between two vertices $x$ and $y$, denoted by $\text{dist}_G(x, y)$, is defined to be the length of a shortest path joining $x$ and $y$. If there is no path joining $x$ and $y$, then $\text{dist}_G(x, y) = \infty$. We will drop the subscript when the confusion is unlikely. The diameter of the graph $G$, denoted by $\text{diam}(G)$, is the maximum of all the distances between any two vertices in $G$, namely

$$\text{diam}(G) = \max\{\text{dist}(x, y) : x, y \in V(G)\}.$$  

A cycle of length $n \geq 3$, usually denoted by $C_n$, is a graph with the vertex set $[n] = \{1, \ldots, n\}$ and the set of edges $\{i, i + 1\}$, $1 \leq i \leq n + 1$ by convention. A graph is chordal if it does not have any induced cycles of length strictly greater than 3. A graph is called a tree if it is connected and it does not have cycles.

For a graph $G$, let $\overline{G}$ be the complement of the graph $G$, that is the graph with the same vertex set as $G$ and $\{u, v\}$ is an edge of $\overline{G}$ if it is not an edge of $G$.

Remark 1.4. We emphasize that we distinguish between the simplicial complex $\Delta^c$ and the complement of a graph and use different notations for them.

Let $G$ be a finite simple graph. The line graph of the graph $G$, denoted by $L(G)$, is defined to have as its vertices the edges of $G$, and two vertices in $L(G)$ are adjacent if the corresponding edges in $G$ share a vertex in $G$.

Given a finite simple graph $G$ with the vertex set $V(G) = \{1, \ldots, n\} = [n]$ and the set of edges $E(G)$, one may consider its edge ideal which is the squarefree monomial ideal denoted by $I(G) \subseteq S = \mathbb{k}[x_1, \ldots, x_n]$, where $\mathbb{k}$ is a field, defined by

$$I(G) = (x_i x_j : \{i, j\} \in E(G)).$$

One may note that graphs are simplicial complexes of dimension 1 and the edge ideals are simply the facet ideals of these simplicial complexes.

Fröberg’s Theorem gives a combinatorial characterization of the property of an edge ideal to have a linear resolution:

**Theorem 1.5 (Fröberg).** Let $G$ be a finite simple graph. The edge ideal $I(G)$ has a linear resolution if and only if $\overline{G}$ is a chordal graph.

2. **The line graph of a simplicial complex**

In the literature, there are various generalizations of line graphs of graphs to line graphs of hypergraphs. In [2], the authors defined the notion of $k$-line graph of a hypergraph $\mathcal{H}$ as being the graph with the vertex set given by the set of edges of the hypergraph, $\mathcal{E}(\mathcal{H})$, and two vertices are adjacent if the intersection of the
corresponding edges of $\mathcal{H}$ has at least $k$ elements. They denote the $k$-line graph of the hypergraph $\mathcal{H}$ by $L_k(\mathcal{H})$.

We will consider the above definition for the case of pure simplicial complexes, where the hypergraph has the vertex set given by the vertex set of the simplicial complex and the edges are the facets. More precisely, let $\Delta$ be a pure simplicial complex of dimension $d - 1$, $d \geq 2$, on the vertex set $V = \{x_1, \ldots, x_n\}$, with the facet set $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$, $r \geq 1$. We will consider $\mathcal{C}(\Delta)$ as hypergraph $H$. The $(d - 1)$-line graph of $\mathcal{C}(\Delta)$ is the graph with the vertex set given by the facets of $\Delta$ and the set of edges $\{\{F_i, F_j\} : |F_i \cap F_j| = d - 1\}$ (we must have equality due to the fact that simplicial complex is pure of dimension $d - 1$). Since this is the only line graph that we will consider through this paper, we will simply refer to it as the line graph of the simplicial complex $\Delta$ and we will denote it by $L(\Delta)$.

In order to avoid confusions, we will denote by $v_1, \ldots, v_r$ the vertices of $L(\Delta)$, where the vertex $v_i$ corresponds to the facet $F_i$. Moreover, we will denote the edges of the hypergraph $\mathcal{C}(\Delta)$ by $E(\mathcal{C}(\Delta))$, while the edges of the graph $G$ will be simply denoted by $E(G)$.

We will pay attention to those properties of the line graphs of graphs which are preserved by the line graph of a pure simplicial complex.

**Remark 2.1.** It is easily seen that the graph $L(\Delta)$ does not depend (up to a relabeling of the vertices) on the labels of the facets of $\Delta$.

Note that both $\Delta$ and $\Delta^c$ have the same line graph, as the next result shows.

**Proposition 2.2.** If $\Delta$ is a pure simplicial complex of dimension $d - 1$, then $L(\Delta)$ and $L(\Delta^c)$ coincide (up to the labeling of the vertices).

**Proof.** The proof is straightforward since $\Delta$ is pure of dimension $d - 1$, $\Delta^c$ is pure of dimension $n - d - 1$ and $F^c \cap G^c = (F \cup G)^c$, therefore if $|F \cap G| = d - 1$ then $|F^c \cap G^c| = |(F \cup G)^c| = n - (d + 1) = n - d - 1$.

For the converse, one may note that $(F^c)^c = F$. Therefore

$$|F \cap G| = |(F^c)^c \cap (G^c)^c| = |(F^c \cup G^c)^c| = n - (n - d + 1) = d - 1.$$ 

□

We start by considering the number of edges of the line graph. We recall that, for line graphs of graphs, the number of edges is known:

**Proposition 2.3.** [22, Proposition 7.6.2] If $G$ is a graph with vertices $x_1, \ldots, x_n$ and edge set $E(G)$, then the number of edges of the line graph $L(G)$ is given by

$$|E(L(G))| = \sum_{i=1}^{n} \left(\frac{\deg(x_i)}{2}\right) = -|E(G)| + \sum_{i=1}^{n} \frac{\deg^2 x_i}{2}$$

We will determine the number of edges of the line graph of a pure simplicial complex. Let $\Delta$ be a pure simplicial complex of dimension $d - 1$ with the facet set $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$ For each $i$, let

$$s_i = |\{F_j : j > i, |F_j \cap F_i| = d - 1\}|.$$
Remark 2.4. Note that $s_i$ is just the number of neighbors of $v_i$ which were not counted before.

Proposition 2.5. Under the above assumptions, $|E(L(\Delta))| = \sum_{i=1}^{r} s_i$

Proof. Let $F_1, \ldots, F_r$ be a labeling of the facets of $\Delta$. An edge of the graph $L(\Delta)$ is given by a pair of facets $F_i, F_j$ such that $|F_i \cap F_j| = d - 1$. Therefore, the number of edges induced by the facet $F_i$ is given by all its neighbors except the ones which were considered before (in order to skip the overlaps).

Corollary 2.6. $\sum_{i=1}^{r} \deg(v_i) = 2 \sum_{i=1}^{r} s_i$

Proof. The equality follows from the Euler’s inequality $2|E(G)| = \sum_{i=1}^{n} \deg(x_i)$ and the previous result.

3. FACET IDEALS AND LINE GRAPHS

For edge ideals of graphs one can describe the second Betti number in terms of the combinatorial structure of its line graph.

Proposition 3.1. [7, Proposition 2.1] Let $I \subset R$ be the edge ideal of the graph $G$, let $V$ be the vertex set of $G$, and let $L(G)$ be the line graph of $G$. If

$$\cdots \longrightarrow R^e(-4) \oplus R^h(-3) \longrightarrow R^4(-2) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is the minimal graded resolution of $R/I$, then $b = |E(L(G))| - N_t$, where $N_t$ is the number of triangles of $G$ and $e$ is the number of unordered pairs of edges $\{f, g\}$ such that $f \cap g = \emptyset$ and $f$ and $g$ cannot be joined by an edge.

We obtain a similar result for pure simplicial complexes by using the line graph. Let $\Delta$ be a pure simplicial complex of dimension $d - 1$, $d \geq 2$ on the vertex set $[n] = \{1, \ldots, n\}$, with the facet set $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$, $r \geq 1$. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $k$, and $I(\Delta) = (x_{F_1}, \ldots, x_{F_r})$ its facet ideal.

Theorem 3.2. Let $I \subset R$ be the facet ideal of $\Delta$ and $L(\Delta)$ its line graph. Let $N_t$ be the number of all the triangles in $L(\Delta)$ which are disjoint (their vertex sets are disjoint) and don’t arise from facets $F, G, H$ with $|F \cap G \cap H| = d - 1$. Then $\beta_{2,d+1}(R/I) = |E(L(\Delta))| - N_t$.

Proof. The proof is similar to [22, Proposition 7.6.3]. Let $F_1, \ldots, F_r$ be the facets of $\Delta$. We may assume that $\psi(e_i) = F_i$. Let $Z'_1$ be the set of elements in ker($\psi$) of degree $d + 1$. We regard $F_i$'s as the vertices of $G'_1(\Delta)$. Every edge $e = \{F_i, F_j\}$ in $L(\Delta)$ determines a syzygy $\text{syz}(e) = ve_i - ue_j$, where $F_i = \{u\} \cup (F_i \cap F_j)$ and $F_j = \{v\} \cup (F_i \cap F_j)$ for some vertices $u, v \in V$. By [22, Theorem 3.3.19] the set of those syzygies generates $Z'_1$. Given any triangle $C_3 = \{F_i, F_j, F_k\}$ in $L(\Delta)$ one has that $|F_i \cap F_j \cap F_k| = d - 1$ or $|F_i \cap F_j \cap F_k| = d - 2$. 


If $|F_i \cap F_j \cap F_k| = d - 1$, then one must have $F_i = \{ u \} \cup (F_i \cap F_j \cap F_k)$, $F_j = \{ v \} \cup (F_i \cap F_j \cap F_k)$, and $F_k = \{ w \} \cup (F_i \cap F_j \cap F_k)$ for some vertices $u,v,w$. Therefore

$$\phi(C_3) = \{ ve_i - we_j, \ we_j - ve_k, \ we_i - ve_k \}$$

and all the elements from this set are linear independent.

If $|F_i \cap F_j \cap F_k| = d - 2$, one must have $F_i = \{ u,v \} \cup (F_i \cap F_j \cap F_k)$, $F_j = \{ v,w \} \cup (F_i \cap F_j \cap F_k)$, and $F_k = \{ u,w \} \cup (F_i \cap F_j \cap F_k)$ for some vertices $u,v,w$. Therefore

$$\phi(C_3) = \{ ve_i - we_j, \ we_j - ve_k, \ we_i - ve_k \}.$$ One may note that, in this case $we_i - ve_k = we_i - we_j + we_j - ve_k$, hence they are linear dependent.

Since $C_3$ and $C'_3$ are disjoint triangles, then $\phi(C_3) \cap \phi(C'_3) = \emptyset$. Let $T$ be the set of all the triangles in $\mathcal{L}(\Delta)$ which are disjoint and don’t arise from facets $F,G,H$ with $|F \cap G \cap H| = d - 1$. From every triangle from $T$, choose an element $\rho(C_3) \in \phi(C_3)$. Then

$$B = \{ syz(e)|e \in E(\mathcal{L}(\Delta)) \} \setminus \{ \rho(C_3) : C_3 \in T \}$$

is a minimal generating set for $Z'_1$. The statement follows. \hfill \Box

**Remark 3.3.** Note that the formula obtained does not depend on the characteristic of the ground field since the second Betti number of a Stanley–Reisner is independent of the ground field [15].

**Problem 3.4.** We cannot obtain a similar result for the other graded Betti numbers $\beta_{2,i}(R/I)$ in terms of $\mathcal{L}(\Delta)$, but examples shows that their description is encoded in the combinatorics of the other $k$-line graphs, with $k < d - 1$. Therefore, taking into account Proposition 3.1 is there a similar formula for $\beta_{2,d+2}(R/I)$ in terms of the combinatorics of the $k$-line graphs?

### 4. Combinatorial properties of the line graph of a simplicial complex

Through this section we pay attention to properties of the line graph of a simplicial complex which are similar to the ones of line graphs of graphs. It is easy to see that the line graph is not connected, in general, even if the simplicial complex $\Delta$ is connected. Therefore, we give a sufficient condition for the connectivity of the line graph of a simplicial complex.

**Proposition 4.1.** If $\Delta$ is a pure shellable simplicial complex, then $\mathcal{L}(\Delta)$ is connected.

**Proof.** Since $\Delta$ is shellable, there is an order of the facets $F_1, \ldots, F_r$ such that for all $1 \leq i < j \leq r$ there is a vertex $v \in F_j \setminus F_i$ and some $l \in \{1,\ldots,j-1\}$ with $F_j \setminus F_l = \{ v \}$. In particular $|F_j \cap F_l| = d - 1$, therefore $\{ F_j, F_l \}$ is an edge in $\mathcal{L}(\Delta)$. Thus $G_1(\Delta)$ is connected. \hfill \Box

One may note that the converse does not hold. There are simplicial complexes which are not even Cohen–Macaulay, but their line graph is connected, as the following example shows:
Example 4.2. Let $\Delta$ be the simplicial complex on the vertex set $\{1, \ldots, 7\}$ with the set of facets $\mathcal{F}(\Delta) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{5, 6, 7\}\}$. Therefore $\Delta$ and its line graph are

Note that $\mathcal{L}(\Delta)$ is connected, but $\Delta$ is not even Cohen–Macaulay since the Stanley–Reisner ideal of its Alexander dual does not have a linear resolution as one may easy check with Singular, for instance [5].

The next problem naturally arise:

**Problem 4.3.** Is there any characterization of those simplicial complexes whose line graph is connected?

Now we pay attention to those graphs which are line graphs of a pure simplicial complexes. Since not every graph is the line graph of a graph and taking into account that graphs are particular classes of simplicial complexes, one has to determine the properties that a graph should have in order to be the line graph of a simplicial complex. For line graphs of graphs there is the following characterization:

**Theorem 4.4.** [1] The following statements are equivalent for a graph $G$.

(i) $G$ is the line graph of some graph

(ii) The edges of $G$ can be partitioned into complete subgraphs in such a way that no vertex belongs to more than two of the subgraphs.

(iii) The graph $K_{1,3}$ is not an induced subgraph of $G$; and if abc and bcd are distinct odd triangles, then a and d are adjacent (we recall that a triangle is odd if there is a vertex of $G$ which is adjacent to an odd number of vertices of the triangle).

(iv) None of the nine graphs given bellow is an induced subgraph of $G$
We determine several conditions that the line graph of a pure simplicial complex should fulfill. These conditions are similar to the ones from the above proposition. Firstly, we obtain a characterization similar to the equivalence “i)⇒ii)” from Theorem 4.4. In the proof, we will use Proposition 5.2 which will be proved in the next section.

**Theorem 4.5.** The following statements are equivalent:

(i) The graph $G$ is the line graph of a pure simplicial complex of dimension $d-1$.

(ii) The edges of each connected component of $G$ can be partitioned into complete subgraphs in such a way that no vertex belongs to more than $d$ of the subgraphs.

**Proof.** “(i)⇒(ii)” Let $F_i = \{i_1, \ldots, i_{d-1}, i_d\}$ be a facet. One may note that any complete graph which contains the facet $F_i$ is given by the facets of the form \(\{i_1, \ldots, i_{d-1}, j\}\) for some integer $j \notin F_i$. Since there are \(\binom{d}{d-1} = d\) subsets of $F_i$ with $d-1$ elements, the statement follows.

“(ii)⇒(i)” Let $V = \{v_1, \ldots, v_d, u_1, \ldots, u_{d-1}\}$ be a set of vertices. We consider the following simplicial complexes:

\[
\Delta_1 = \{\{v_1\}, \ldots, \{v_d\}\}, \quad \Delta_2 = \{\{v_1, v_2\}, \ldots, \{v_1, v_d\}\}, \\
\Delta_3 = \{\{v_2, v_3\}, \ldots, \{v_2, v_d\}\}, \quad \Gamma = \{\{u_1, \ldots, u_{d-1}\}\}
\]

\[
\Gamma_i = \{\{u_1, \ldots, \hat{u}_i, \ldots, u_{d-1}\}\}, \quad 1 \leq i \leq d-1.
\]

Let

\[
\Delta = (\Delta_1 * \Gamma) \cup \left(\bigcup_{i=1}^{d-1} (\Delta_2 * \Gamma_i)\right) \cup \left(\bigcup_{i=1}^{d-1} (\Delta_3 * \Gamma_i)\right).
\]

One may note that $\Delta$ is pure simplicial complex oh dimension $d-1$. Moreover, its line graph $\mathcal{L}(\Delta)$ can be partitioned into complete subgraphs induced by $\Delta_1 * \Gamma$, $\Delta_2 * \Gamma_i$, and $\Delta_3 * \Gamma_i$. Therefore, the statement follows.
The vertex which corresponds to the facet \( \{v_1, u_1, \ldots, u_{d-1}\} \) is contained in exactly \( d \)-complete subgraphs induced by \( \Delta_1 * \Gamma \) and \( \Delta_2 * \Gamma_i, 1 \leq i \leq d - 1 \). The vertex which corresponds to the facet \( \{v_2, u_1, \ldots, u_{d-1}\} \) is contained in exactly \( d \)-complete subgraphs induced by \( \Delta_1 * \Gamma, \langle\{v_1, v_2\}\rangle * \Gamma_1, \) and \( \Delta_3 * \Gamma_i, 1 \leq i \leq d - 1 \). Any other vertex is contained in at most \( d - 1 \) complete subgraphs. The statement follows. 

**Proposition 4.6.** If \( G \) is the line graph of a pure simplicial complex of dimension \( d - 1 \), then \( G \) does not contain \( K_{1,d+1} \) as an induced subgraph.

**Proof.** Assume by contradiction that \( K_{1,d+1} \) is an induced subgraph of \( G \). Then there is a vertex \( v_i \) with \( d + 1 \) neighbours. Since \( F_i \) has \( d \) elements, there are two facets \( F_l \) and \( F_k \) such that \( F_l \cap F_k = F_j \cap F_i \) and \( v_l, v_k \) are neighbors of \( v_i \). Therefore \( |F_l \cap F_k| = d - 1 \), hence \( \{v_l, v_k\} \) is an edge in \( L(\Delta) \), a contradiction. 

Until now, we don’t have any property related to forbidden subgraphs in a line graph of a pure simplicial complex. As before, the next two problems naturally appears:

**Problem 4.7.** Can be extended the result form Proposition 4.6 to a characterization similar to the one from Theorem 4.4 “(i) \Rightarrow (iii)”?

**Problem 4.8.** Is there a characterization of the line graphs of simplicial complexes similar to the one from Theorem 4.4 “(i) \Rightarrow (iv)”?

## 5. Chordal line graphs and simplicial complexes

Through this section, we aim at determining characterizations of line graphs of pure simplicial complexes which are complete graphs or cycles. Moreover, we establish a connection between the property of line graphs of being a chordal graph and the chordality of the simplicial complex.

**Proposition 5.1.** Let \( \Delta \) be a pure simplicial complex of dimension \( d - 1 \) with three facets. Then \( L(\Delta) \) is \( C_3 \) if and only if one of the following holds

\begin{enumerate} \setlength\itemsep{0em} 
\item \( \Delta = \langle\{1\}, \{2\}, \{3\}\rangle * \Gamma, \) where \( \Gamma \) is a \( (d - 2) \)-simplex such that \( 1, 2, 3 \notin V(\Gamma) \)
\item \( \Delta = \langle\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\}\rangle * \Gamma, \) where \( \Gamma \) is a \( (d - 3) \)-simplex such that \( i_1, i_2, i_3 \notin V(\Gamma) \).
\end{enumerate}

**Proof.** “\( \Rightarrow \)” Let’s assume that \( L(\Delta) \) is \( C_3 \), therefore, the facets of \( \Delta \), say \( F_1, F_2, \) and \( F_3 \) have the property that \( |F_1 \cap F_2| = d - 1, \) \( |F_1 \cap F_3| = d - 1 \) and \( |F_2 \cap F_3| = d - 1 \). By easy set operations, one may note that \( |F_1 \cup F_2 \cup F_3| - |F_1 \cap F_2 \cap F_3| = 3 \). Hence we must have the facets of \( \Delta \) of the form \( \{1\} \cup H, \{2\} \cup H, \{3\} \cup H \) where \( H \) is a set of cardinality \( d - 1 \) or \( \Delta = \langle\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\}\rangle * \Gamma, \) where \( \Gamma \) is a \( (d - 3) \)-simplex. 

“\( \Leftarrow \)” It is easy to see that, in both cases, the line graph is the cycle \( C_3 \). 

We aim at characterizing those simplicial complexes whose line graph is complete.

**Proposition 5.2.** Let \( \Delta \) be a pure simplicial complex of dimension \( d - 1 \) and \( L(\Delta) \) its line graph. Then \( L(\Delta) \) is a complete graph with more than four vertices if and only if one of the following holds

\begin{enumerate} \setlength\itemsep{0em} 
\item \( \Delta \) is a complete graph with more than four vertices.
\item \( \Delta \) is a complete graph with at most four vertices, and \( \Delta = \langle\{1\}, \{2\}, \{3\}\rangle * \Gamma, \) where \( \Gamma \) is a \( (d - 2) \)-simplex such that \( 1, 2, 3 \notin V(\Gamma) \).
\end{enumerate}
i) \( \Delta = (\{u_1\}, \ldots, \{u_r\}) \ast \Gamma, \) where \( \Gamma \) is a \((d-2)\)-simplex such that \( \{u_1, \ldots, u_r\} \cap V(\Gamma) = \emptyset \)

ii) the facets of \( \Delta \) are subsets of cardinality \( d \) of a \( d \)-simplex.

Proof. “\( \Leftarrow \)” If \( \Delta \) is one of the above simplicial complexes, then the intersection of any two facets has cardinality \( d-1 \), hence \( \mathcal{L}(\Delta) \) is a complete graph.

“\( \Rightarrow \)” Assume now that \( \mathcal{L}(\Delta) \) is a complete graph and \( \mathcal{L}(\Delta) \) is the line graph of a pure simplicial complex of dimension \( d-1 \). Let \( v_1 \) and \( v_2 \) be two vertices of \( \mathcal{L}(\Delta) \). Since \( \mathcal{L}(\Delta) \) is a complete graph, they are adjacent. Moreover, the corresponding facets in \( \Delta \) are \( F_1 \) and \( F_2 \). Hence \( F_1 \cup F_2 = \{i_1, \ldots, i_{d+1}\} \). Let \( v_3 \) and \( v_4 \) be two different vertices. If \( |F_1 \cap F_2 \cap F_3| = d-1 \), then \( |F_1 \cap F_2 \cap F_3 \cap F_4| = d-1 \) and \( \Delta \) is of the first type. Indeed, let \( H = F_1 \cap F_2 \cap F_3 \). Then \( F_1 = \{i_1\} \cup H, F_2 = \{i_2\} \cup H \) and \( F_3 = \{i_3\} \cup H \), up to a relabeling. If we assume that \( H \not\subseteq F_4 \), then \( F_4 \) must contain \( i_1, i_2, i_3 \) due to the requirements on the cardinality of the intersection, which is a contradiction.

Let’s assume now that \( F_1 \cap F_2 \cap F_3 \neq H \). Since \( |F_1 \cap F_3| = d-1 \) and \( |F_2 \cap F_3| = d-1 \), one has that \( \{i_1, i_2\} \subset F_3 \). Due to the requirements, \( |F_1 \cup F_2 \cup F_3| = d+1 \), that is \( F_1 \cup F_2 \cup F_3 = H \cup \{i_1, i_2\} \). Since \( \mathcal{L}(\Delta) \) contains at least 4 vertices, the same discussion holds for \( F_4 \). Hence all the facets should be subsets of cardinality \( d \) in \( H \cup \{i_1, i_2\} \), where \( |H| = d-1 \). \( \square \)

Remark 5.3. Since there will be used often in the paper, we will refer to a simplicial complex from the Proposition 5.2(i) as to the \( r \)-cone. We recall that a cone is the join of a simplicial complex with a vertex \( v \) not in the simplicial complex.

Next we will characterize those simplicial complexes whose line graph is a cycle. We recall that a path of length \( t \geq 2 \) in \( G \) (or \( t \)-path, for short) is, by definition, a set of distinct vertices \( u_0, u_1, \ldots, u_t \) such that \( \{u_i, u_{i+1}\} \) are edges in \( G \) for all \( i \in \{0, \ldots, t-1\} \). Note that, in a \( t \)-path there are \( t+1 \) vertices.

Proposition 5.4. Let \( \Delta \) be a pure simplicial complex of dimension \( d-1 \) and \( \mathcal{L}(\Delta) \) its line graph. Then \( \mathcal{L}(\Delta) \) is a cycle of length \( r \geq 4 \) if and only if one of the following holds:

\begin{itemize}
  \item if \( d < r-1 \), the facets of \( \Delta \) are the \((d-1)\)-paths of the cycle of length \( r \).
  \item if \( d \geq r-1 \) the facets of \( \Delta \) are the \((r-2)\)-paths of the cycle of length \( r \) union a set \( H \) of cardinality \( d-r+1 \).
\end{itemize}

Proof. “\( \Leftarrow \)” It is clear by the shape of the facets that every facet has exactly two neighbors whose intersection is of cardinality \( d-1 \).

“\( \Rightarrow \)” Since \( \mathcal{L}(\Delta) \) is the line graph of a pure simplicial complex, its vertices correspond to some facets \( F_1, \ldots, F_r \) of cardinality \( d \). Each facet is obtained by removing one vertex from the previous facet and adding a new one. If \( d < r-1 \), then one obtains the \((d-1)\)-path of a cycle of length \( r \). If \( d \geq r-1 \), there will be some vertices which will be in the intersection of all the facets. Since only \( r \) vertices are changing (that is we consider \((r-2)\)-paths in the cycle \( C_r \)), there will be other \( d-r+1 \) common vertices. The statement follows. \( \square \)
In [24], R. Woodroofe defined the notion of chordal clutter. We recall his definition, but we will consider in the particular case of simplicial complexes.

**Definition 5.5.** [24] Let $\Delta$ be a simplicial complex. A vertex $v$ of $\Delta$ is simplicial if for every two facets $F_1$ and $F_2$ of $\Delta$ that contain $v$, there is a third facet such that $F_3 \subseteq (F_1 \cup F_2) \setminus \{v\}$.

In order to define the notion of chordal simplicial complex, we have to recall a few concepts. Let $v \in V(\Delta)$. The deletion $\Delta \setminus v$ is the subcomplex of $\Delta$ on the vertex set $V(\Delta) \setminus \{v\}$ with facets $\{F : F$ a facet of $\Delta$ with $v \notin F\}$. The contraction $\Delta/v$ is the simplicial complex on the vertex set $V(\Delta) \setminus \{v\}$ with facets the maximal sets of $\{F \setminus \{v\} : F$ a facet of $\Delta\}$. Thus, $\Delta \setminus v$ deletes all facets containing $v$, while $\Delta/v$ removes $v$ from each facet containing it (and then removes any redundant facets). A simplicial complex $\Delta'$ obtained from $\Delta$ by repeated deletion and/or contraction is called a minor of $\Delta$.

**Remark 5.6.** One may easily note that the minors of a simplicial complex are not necessary pure, even if we start with a pure simplicial complex.

**Definition 5.7.** [24, Definition 4.3] A simplicial complex $\Delta$ is chordal if every minor of $\Delta$ has a simplicial vertex.

In [20], the authors considered the notion of clutters with the free-vertex property. As before, we will use the definitions and properties from [20], but we will consider the particular case of simplicial complexes. A simplicial complex $\Delta$ has the free vertex property if every minor of $\Delta$ has a free vertex, that is, a vertex appearing in exactly one facet of $\Delta$.

**Remark 5.8.** [24, Example 4.5] A free vertex is simplicial, so simplicial complexes with the free vertex property are chordal.

The next remark is now straightforward.

**Remark 5.9.** An $r$-cone is chordal since it has the free vertex property.

We pay attention to the property of the line graph of being chordal. For graphs, the following result is known:

**Proposition 5.10.** [19, Lemma 3.1] The graph $G$ contains a cycle $C$ of length at least four if and only if $L(G)$ contains an induced cycle of the same length.

One may note that the above result does not require that the cycle in $G$ is induced, as one can see in the next example

**Example 5.11.** Let $G$ be the graph with the edges from the next figure in the left part and its line graph. Its line graph contains an induced cycle $C_4$, while $G$ contains a cycle $C_4$ which is not an induced subgraph.

According to the above proposition we obtain that the chordality of the line graph is sufficient for the chordality of the graph.
Corollary 5.12. Let $G$ be a finite simple graph. If $L(G)$ is chordal, then $G$ is chordal.

A similar result does not hold for the line graph of a simplicial complex as the following example shows.

Example 5.13. Let $\Delta$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta) = \{ F_1 = \{1, 2, 3\}, F_2 = \{2, 3, 4\}, F_3 = \{3, 4, 5\}, F_4 = \{4, 5, 6\}, F_5 = \{1, 5, 6\} \}$. Then $\Delta$ does not have any simplicial vertex, therefore it is not chordal, but $L(\Delta)$ is chordal since it is the path on 5 vertices.

Still, we can consider additional conditions in order to obtain the chordality of the simplicial complex:

Theorem 5.14. Let $\Delta$ be a pure simplicial complex of dimension $d - 1$. If $L(\Delta)$ is connected, chordal, and of diameter at most $d$, then $\Delta$ is chordal.

Proof. Let $\mathcal{F}(\Delta) = \{ F_1, \ldots, F_r \}$. We prove by induction on the number of facets. If $r = 2$, then $\Delta$ is chordal (it has the free vertex property). Assume now that the statement is true for all the simplicial complexes with at most $r - 1$ facets whose line graphs satisfy the conditions from the statement.

Let $\Delta$ be a pure simplicial complex with $r$ facets such that $L(\Delta)$ is connected, chordal and $\text{diam}(L(\Delta)) \leq d$.

If $\Delta$ has a free vertex $v$, then $v$ is a simplicial vertex, hence $\Delta$ is chordal ($\Delta \setminus v = \Delta / v$ are chordal by induction).

Assume that $\Delta$ does not have free vertices. Since $L(\Delta)$ is chordal it has a simplicial vertex. Let $x_F$ be a simplicial vertex, that is $N(x_F)$ is a complete graph. According to Proposition 5.2, we have to split the proof into two cases.

Case 1: Assume that the minor induced by $\mathcal{N}[x_F]$ is an $t$-cone, $t \geq 2$, that is there is a set $H$ such that $|H| = d - 1$ and $x_{F_i} = x_i \cup H$, for some vertex $x_i$, $1 \leq i \leq t$ and some $x_{F_i}$ a neighbor of $x_F$. Let $F = \{v\} \cup H$. Since $\Delta$ does not have any free vertices, there is a facet $G$ of $\Delta$ such that $v \in F \cap G$ and $|F \cap G| < d - 1$. Therefore there is a path $x_{G_0}, \ldots, x_{G_s}$ such that $F = G_0$, $G = G_s$ and $s = \text{dist}(x_F, x_G)$. Since $s \leq d$, $G_s$ contains all the new vertices and $F$ the old ones, therefore there is a facet $G_i \subseteq (F \cap G) \setminus \{v\}$. Thus $v$ is a simplicial vertex of $\Delta$. Now $\Delta \setminus \{v\}$ is chordal by induction and it is easy to see that $\Delta / v = \Delta \setminus v$, therefore it is chordal.

Case 2: Assume that the minor induced by $\mathcal{N}[x_F]$ has as edges subsets of cardinality $d$ of a $d$-simplex. As before, $F$ does not contain a free vertex. Let $v \in F$ be a vertex and $F_i$ a facet such that $x_{F_i} \in \mathcal{N}(x_F)$ and $v \in F \cap F_i$. Therefore $F = \{u\} \cup (F \cap F_i)$ and $F_i = \{w\} \cup (F \cap F_i)$ and $|(F \cup F_i) \setminus \{v\}| = d$. 

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If we assume that there is not any facet $G$ such that $G \subseteq (F \cup F_i) \setminus \{v\}$, then $x_F$ has only one neighbor in $L(\Delta)$. In this case, since $u \notin F_i$ and cannot be a free vertex, there is a facet $H$ such that $u \in F \cap H$ and $|F \cap H| < d - 1$. As in the previous case, the distance between $x_F$ and $x_H$ is at most $d$ and by using the same arguments, $u$ is a simplicial vertex. One may note that $\Delta \setminus u$ is chordal by induction and $\Delta \setminus u = \Delta \setminus \{v\}$, therefore it is chordal.

If we assume that there is a facet $G$ such that $G \subseteq (F \cup F_i) \setminus \{v\}$, then $x_F$ has at least two neighbors in $L(\Delta)$ and all of them are subsets of cardinality $d$ of a $d$-simplex. It can also happen that there is some other facet $H$ such that $u \in F \cap H$ and $|F \cap H| < d - 1$ but due to the fact that the distance between $x_F$ and $x_H$ is at most $d$, there is a facet contained in $(F \cup H) \setminus \{v\}$. Hence $v$ is a simplicial vertex of $\Delta$. Note that $\Delta \setminus u$ is chordal by induction and $\Delta \setminus u = \Delta \setminus \{v\}$, therefore $\Delta$ is chordal. \hfill $\square$

Remark 5.15. Note that the converse does not hold. Indeed, if $\Delta$ is the graph from Example 5.11 then its line graph is not chordal since it contains the cycle $C_4$ as an induced subgraph.

The above example is the source for constructing simplicial complexes which are chordal, but their line graphs are not a chordal graphs.

Remark 5.16. The above example can be extended to a more general case. Indeed, let $\Delta_1 = \Delta * \Gamma$, where $\Delta$ is the graph from the previous remark and $\Gamma$ is a $(d - 3)$-simplex. It is easy to see that $\Delta_1$ is chordal and its line graph is (up to the labels of the vertices) the same as before.

Since both $\Delta$ and $\Delta^c$ have the same line graph, according to Proposition 2.2 we also get

**Corollary 5.17.** Let $\Delta$ be a pure simplicial complex. If $L(\Delta)$ is connected, chordal, and of diameter at most $d$, then $\Delta^c$ is chordal.

The property of a clutter of being chordal implies the shellability of its independence complex, as the following theorem shows:

**Theorem 5.18.** [24, Theorem 1.1] If $\mathcal{C}$ is a chordal clutter then the independence complex $\text{Ind}(\mathcal{C})$ is shellable and hence sequentially Cohen–Macaulay.

The above theorem has several important consequences related to properties of the simplicial complexes. We mention here several of them:

**Corollary 5.19.** Let $\Delta$ be a pure simplicial complex. If $L(\Delta)$ is connected, chordal, and of diameter at most $d$, then:

1. the independence complex of $\text{Ind}(\mathcal{C}(\Delta))$ is shellable and hence is Cohen–Macaulay,
2. $I_\Delta$ is Cohen–Macaulay,
3. $\Delta^\vee$ is shellable and hence it is Cohen–Macaulay,
4. $I_\Delta$ has linear quotients, thus a linear resolution.
Proof. (i) Is a direct consequence of Theorem \ref{thm:linear-resolution}.  
(ii) For a simplicial complex, the Stanley–Reisner ideal of $\text{Ind}(\mathcal{C}(\Delta))$ is $I_{\Delta}$, therefore, the statement holds.
(iii) This follows easily since $\mathcal{L}(\Delta) = \mathcal{L}(\Delta^c)$ and $I_{\Delta^c} = I(\Delta^c)$
(iv) By using (iii) and Theorem \ref{thm:bounded-diameter}, the statement follows. □

By using the Fröberg’s theorem, we also get

**Corollary 5.20.** Let $\Delta$ be a pure simplicial complex. If $\mathcal{L}(\Delta)$ is connected, of diameter at most $d$ and $I(\mathcal{L}(\Delta))$ has a linear resolution, then:

(i) the independence complex of $\mathcal{C}(\Delta)$ is shellable and hence is Cohen–Macaulay.
(ii) $I_{\Delta}$ has linear quotients, thus a linear resolution.

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