COMPLEXES FROM COMPLEXES: FINITE ELEMENT COMPLEXES IN THREE DIMENSIONS

LONG CHEN AND XUEHAI HUANG

ABSTRACT. In the realm of solving partial differential equations (PDEs), Hilbert complexes have gained paramount importance, and recent progress revolves around devising new complexes using the Bernstein-Gelfand-Gelfand (BGG) framework, as demonstrated by Arnold and Hu [Complexes from complexes. Found. Comput. Math., 2021]. This paper significantly extends this methodology to three-dimensional finite element complexes, surmounting challenges posed by disparate degrees of smoothness and continuity mismatches. By incorporating techniques such as smooth finite element de Rham complexes, the $t-n$ decomposition, and trace complexes with corresponding two-dimensional finite element analogs, we systematically derive finite element Hessian, elasticity, and divdiv complexes. Notably, the construction entails the incorporation of reduction operators to handle continuity disparities in the BGG diagram at the continuous level, ultimately culminating in a comprehensive and robust framework for constructing finite element complexes with diverse applications in PDE solving.

1. INTRODUCTION

Hilbert complexes play a crucial role in the development of robust numerical methods for solving partial differential equations (PDEs) [4, 5, 1, 10]. Leveraging the Bernstein-Gelfand-Gelfand (BGG) framework, Arnold and Hu [6] have recently introduced a systematic methodology for generating new complexes from well-established differential complexes, including but not limited to the Hessian complex, the elasticity complex, and the divdiv complex.

Let $\Omega$ be a domain in $\mathbb{R}^3$. The de Rham complex reads as

$$
\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0,
$$

where Sobolev spaces

$$
H^1(\Omega) := \{ \phi \in L^2(\Omega) : \text{grad} \phi \in L^2(\Omega; \mathbb{R}^3) \},
$$

$$
H(\text{curl}, \Omega) := \{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^3) : \text{curl} \mathbf{u} \in L^2(\Omega; \mathbb{R}^3) \},
$$

$$
H(\text{div}, \Omega) := \{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^3) : \text{div} \mathbf{u} \in L^2(\Omega) \}.
$$

By stacking copies of de Rham complexes to form a BGG diagram, several complexes can be derived from the BGG framework [6] including but not limited to the Hessian complex, the elasticity complex, and the divdiv complex.

---

2020 Mathematics Subject Classification. 65N30; 58J10; 65N12.

The first author was supported by NSF DMS-2012465 and DMS-2309785.

The second author is the corresponding author. The second author was supported by the National Natural Science Foundation of China (Grant No. 12171300) and the Natural Science Foundation of Shanghai (Grant No. 21ZR146050).
Recently, there have been constructive developments in the realm of finite element Hessian complexes, elasticity complexes, and divdiv complexes. These constructions have been carried out on a case-by-case basis in prior works \([11, 17, 15, 18, 21, 22, 23, 25]\). Our objective is to extend the application of the BGG construction to finite element complexes, thereby unifying these dispersed outcomes and systematically generating more results. This endeavor has been accomplished in our recent work \([13]\), focused on two dimensions.

The transition to three dimensions presents non-trivial challenges. One notable complication arises from the development of finite element de Rham complexes with varying degrees of smoothness in three dimensions. We have successfully addressed this issue in our recent work \([14]\), which we will briefly overview below.

Given an integer vector \( \mathbf{r} = (r^v, r^e, r^f)^T \), it is called a smoothness vector if it satisfies \( r^f \geq -1, r^e \geq \max\{2r^f, -1\}, \) and \( r^v \geq \max\{2r^e, -1\} \). For a smoothness vector \( \mathbf{r} \), we define \( r \oplus 1 := \max\{r - 1, -1\} \) and \( r_+ = \max\{r, 0\} \). Through the utilization of a simplicial lattice decomposition, we are enabled to construct scalar finite elements \( \mathcal{V}_{k-1}^{\text{grad}}(\mathbf{r}) \) and \( \mathcal{V}_{k}^{\text{div}}(\mathbf{r}) \) that adhere to \( C^r \) conforming criteria. Such smooth finite elements are firstly constructed in \([24]\) by Hu, Lin and Wu.

By skillfully combining \( t - n \) decompositions across different sub-simplexes, we establish finite element descriptions for the spaces:

\[
\mathcal{V}_{k+1}^{\text{curl}}(r_1, r_2) := \{ v \in \mathcal{V}_{k+1}^3(r_1) \cap H(\text{curl}, \Omega) : \text{curl} v \in \mathcal{V}_k^{\text{div}}(r_2) \}, \\
\mathcal{V}_{k}^{\text{div}}(r_2, r_3) := \{ v \in \mathcal{V}_{k}^3(r_2) \cap H(\text{div}, \Omega) : \text{div} v \in \mathcal{V}_{k-1}^{L^2}(r_3) \}.
\]

Let \( r_0 \geq 0, r_1 = r_0 - 1, r_2 \geq r_1 \oplus 1, r_3 \geq r_2 \oplus 1 \). Assume \( r_2, r_3, k \) is div stable, and \( k \geq \max\{2r_1^v + 1, 2r_2^v + 1, 2r_3^v + 2, 1\} \). We have managed to construct the finite element de Rham complex as follows:

\[
\mathbb{R} \leftarrow_{\text{grad}} \mathcal{V}_{k+2}^{\text{grad}}(r_0) \xrightarrow{\text{grad}} \mathcal{V}_{k+1}^{\text{curl}}(r_1, r_2) \xrightarrow{\text{curl}} \mathcal{V}_{k}^{\text{div}}(r_2, r_3) \xrightarrow{\text{div}} \mathcal{V}_{k-1}^{L^2}(r_3) \rightarrow 0.
\]

Another intricate challenge emerges from the mis-match in the continuity of Sobolev spaces, specifically \( H^1(\Omega), H(\text{curl}, \Omega), \) and \( H(\text{div}, \Omega) \). This discrepancy can be visualized using the following diagram:

\[
H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{grad}} H(\text{curl}, \Omega; \mathbb{M}) \xrightarrow{\text{curl}} H(\text{div}, \Omega; \mathbb{M}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \rightarrow 0 \\
\xrightarrow{\iota} H^1(\Omega) \xrightarrow{\text{mskw}} H(\text{div}, \Omega) \xrightarrow{\text{id}} \xrightarrow{\text{mskw}} \xrightarrow{\text{id}} L^2(\Omega) \rightarrow 0,
\]

where

- The mapping \( \iota : H^1(\Omega) \rightarrow H(\text{curl}, \Omega; \mathbb{M}), \) where \( \iota v = v I, \) is well-defined. However, its right inverse \( \text{tr} : H(\text{curl}, \Omega; \mathbb{M}) \rightarrow H^1 \) is not, as functions in \( H(\text{curl}, \Omega) \) possess only tangential continuity.
- The inclusion \( \text{mskw} H(\text{curl}, \Omega) \subset H(\text{div}, \Omega; \mathbb{M}) \) is justifiable (refer to Section 2.1), yet its right inverse \( \text{vskw} : H(\text{div}, \Omega; \mathbb{M}) \rightarrow H(\text{curl}, \Omega) \) is not attainable.
- The obvious inclusion \( H(\text{div}, \Omega) \subset L^2(\Omega, \mathbb{R}^3) \) is not surjective.

In the work of Arnold and Hu \([6]\), the domain Sobolev spaces \( H^s(\Omega), \) where \( d = \text{curl} \) or \( \text{div}, \) are substituted with \( H^s \) Sobolev spaces having matching indices \( s, \) as depicted in diagram \((5)\). For finite element spaces that solely conform to \( H(\text{curl}, \Omega) \) or \( H(\text{div}, \Omega), \) the challenge posed by the mis-match in tangential or normal continuity serves as the primary obstacle to extending the BGG framework to the discrete context.
Our strategy involves identifying sub-complexes within the finite element de Rham complex (2) through the imposition of suitable subspace restrictions. The $\sim$ and $\wedge$ operations applied to a short exact sequence are discussed in detail in Section 2.2.

Returning to the diagram (3), we proceed to construct the corresponding finite element counterpart:

$$
\begin{align*}
\mathcal{V}_{k+2}^{\text{div}}(r_0, (r_1)_+) & \xrightarrow{\text{grad}} \mathcal{V}_{k+1}^{\text{curl}}(r_1; \mathbb{M}) \xrightarrow{\text{curl}} \mathcal{V}_{k}^{\text{div}}(r_2; \mathbb{M}) \xrightarrow{\text{div}} \mathcal{V}_{k-1}^{\text{div}}(r_2 \ominus 1) \rightarrow 0 \\
\mathcal{V}_{k+1}^{\text{grad}} ((r_1)_+) & \xrightarrow{\text{grad}} \mathcal{V}_{k}^{\text{curl}}(r_1 \ominus 1) \xrightarrow{\text{curl}} \mathcal{V}_{k-1}^{\text{div}}(r_2 \ominus 1) \xrightarrow{\text{div}} \mathcal{V}_{k-2}^{\text{L}^2}(r_3) \rightarrow 0,
\end{align*}
$$

where the spaces on the top are reduced so that operators in the $\sim$ direction are well-defined. For example, the two spaces connected by id operator are the same and using $\text{tr} \grad v = \text{div} v$, one can show $\text{tr} \mathcal{V}_{k+1}^{\text{div}}(r_1; \mathbb{M}) = \mathcal{V}_{k+1}^{\text{grad}} ((r_1)_+)$. See Section 5.3 for more details.

Then we can apply BGG process to derive the finite element div div complex:

$$
\begin{align*}
RT \xrightarrow{\subseteq} \mathcal{V}_{k+2}^{\text{div}}(r_0, (r_1)_+) & \xrightarrow{\text{dev grad}} \mathcal{V}_{k+1}^{\text{sym curl}^+}(r_1; \mathbb{T}) \\
& \xrightarrow{\text{sym curl}} \mathcal{V}_{k}^{\text{div}^+}(r_2; \mathbb{S}) \xrightarrow{\text{div div}} \mathcal{V}_{k-2}^{\text{L}^2}(r_3) \rightarrow 0,
\end{align*}
$$

where sub-script $^+$ or sub-script $^\dagger$ in $\text{sym curl}^+$ and $\text{div}^+$ denotes extra smoothness. By relaxing the extra smoothness, we have the finite element div div complex

$$
\begin{align*}
RT \xrightarrow{\subseteq} \mathcal{V}_{k+2}^{\text{div}}(r_0; \mathbb{R}^3) & \xrightarrow{\text{dev grad}} \mathcal{V}_{k+1}^{\text{sym curl}}(r_1; \mathbb{H}) \\
& \xrightarrow{\text{sym curl}} \mathcal{V}_{k}^{\text{div}^+}(r_2; \mathbb{S}) \xrightarrow{\text{div div}} \mathcal{V}_{k-2}^{\text{L}^2}(r_3) \rightarrow 0.
\end{align*}
$$

Examples include the finite element div div complex in [17]: $r_0 = (1, 0, 0)^\top$, $r_1 = r_0 - 1$, $r_2 = (0, -1, -1)^\top$, and $r_3 = r_2 \ominus 2$, and the finite element div div complex in [22]: $r_0 = (2, 0, 0)^\top$, $r_1 = r_0 - 1$, $r_2 = r_1 \ominus 1$, and $r_3 = r_2 \ominus 2$.

Similarly, we can use BGG approach to construct the finite element elasticity complex

$$
\begin{align*}
RM \xrightarrow{\subseteq} \mathcal{V}_{k+2}^{\text{curl}}(r_0, (r_1)_+) & \xrightarrow{\text{def}} \mathcal{V}_{k+1}^{\text{inc}^+}(r_1; \mathbb{S}) \\
& \xrightarrow{\text{inc}} \mathcal{V}_{k}^{\text{div}^+}(r_2; \mathbb{S}) \xrightarrow{\text{div}} \mathcal{V}_{k-2}^{\text{L}^2}(r_3; \mathbb{R}^3) \rightarrow 0.
\end{align*}
$$

By relaxing the extra smoothness, we obtain the finite element elasticity complex

$$
\begin{align*}
RM \xrightarrow{\subseteq} \mathcal{V}_{k+2}^{\text{grad}}(r_0; \mathbb{R}^3) & \xrightarrow{\text{def}} \mathcal{V}_{k+1}^{\text{inc}}(r_1; \mathbb{S}) \xrightarrow{\text{inc}} \mathcal{V}_{k}^{\text{div}^+}(r_2; \mathbb{S}) \xrightarrow{\text{div}} \mathcal{V}_{k-2}^{\text{L}^2}(r_3; \mathbb{R}^3) \rightarrow 0.
\end{align*}
$$

The finite element elasticity complex constructed in [15] is a variant of the case: $r_0 = (2, 1, 0)^\top$, $r_1 = r_0 - 1$, $r_2 = (0, -1, -1)^\top$ and $r_3 = r_2 \ominus 2$.

We can also construct the finite element Hessian complex

$$
\mathcal{P}_1(\Omega) \xrightarrow{\subseteq} \mathcal{V}_{k+2}^{\text{curv}}(r_0) \xrightarrow{\text{hess}} \mathcal{V}_{k}^{\text{curl}}(r_1; \mathbb{S}) \xrightarrow{\text{curl}} \mathcal{V}_{k-1}^{\text{div}}(r_2; \mathbb{T}) \xrightarrow{\text{div}} \mathcal{V}_{k-2}^{\text{L}^2}(r_3; \mathbb{R}^3) \rightarrow 0
$$

using the BGG approach. The finite element Hessian complex in [21] is recovered by setting: $r_0 = (4, 2, 1)^\top$, $r_1 = (2, 0, -1)^\top$, $r_2 = (1, -1, -1)^\top$, $r_3 = (0, -1, -1)^\top$ and $k \geq 7$.

As demonstrated earlier, after modification, the BGG framework is instrumental in deriving finite element complexes from finite element de Rham complexes. However, the derivation of finite element descriptions for these subspaces, which encompass element-wise DoFs, presents a more intricate challenge and necessitates substantial effort. We shall
provide DoFs for these tensor finite element spaces through the utilization of three key methodologies:

1. Smooth finite element de Rham complexes. As previously mentioned and discussed in detail in [14], these provide a foundational basis.
2. The \( t - n \) decomposition approach. This approach, introduced in [12], is crucial in constructing \( H(\text{div}) \)-conforming elements.
3. Trace complexes and 2D finite element complexes. We utilize two trace complexes on each face and leverage insights from 2D finite element complexes [13] to motivate DoFs on edges.

| \( R \) | \( H^1 \) | \( \text{grad} \) | \( H(\text{curl}) \) | \( \text{curl} \) | \( H(\text{div}) \) | \( \text{div} \) | \( L^2 \) | 0 |
|----------|---------|------------|----------------|--------|-------------|--------|-------|-----|

**FIGURE 1.** Organization of Sections 3 - 6.

The remainder of this paper is structured as follows. We commence with a comprehensive presentation of the BGG framework and the two reduction operators in Section 2, which serves as foundational background. In Section 3, we revisit the smooth finite element de Rham complexes, providing a comprehensive review. Subsequently, Section 4 delves into the construction of face finite elements. The BGG construction of finite element Hessian complexes, elasticity complexes, and divdiv complexes is expounded upon in Section 5. Moving forward, Section 6 focuses on the construction of edge elements.

A visual depiction of the organization can be found in Figure 1, which provides a clear overview of the various sections.

## 2. Preliminary

In this section we shall briefly review the framework developed in [6] to derive more complexes from the Bernstein-Gelfand-Gelfand (BGG) construction [8]. Earlier advancements in this field can be explored in references such as [20, 7, 3]. Additionally, we draw attention to a significant challenge encountered during the process of generalizing these concepts to finite element complexes. To address this challenge, we propose two reduction operations applied to a short exact sequence, which effectively mitigates the issue and enhances the applicability of the framework.

Throughout the whole paper, we assume \( \Omega \) is topologically trivial and thus the de Rham complex (1) is exact. Consequently all derived complexes based on de Rham complexes are also exact. We will skip \( \Omega \) in the space notation. For example, \( H^* = H^*(\Omega) \) is the standard Sobolev space with real index \( s \).
2.1. Notation. We define the dot product and the cross product from the left, denoted as $b \cdot A$ and $b \times A$ respectively. These operations are applied column-wise to the matrix $A$. Conversely, when the vector is on the right of the matrix, i.e., $A \cdot b$ and $A \times b$, the operations are defined row-wise. The order of performing the row and column products is interchangeable, resulting in the associative rule of triple products:

$$b \times A \cdot c := (b \times A) \cdot c = b \times (A \cdot c).$$

Similar rules apply for $b \cdot A \cdot c$ and $b \cdot A \times c$, allowing parentheses to be omitted.

For a given plane $F$ with a normal vector $n$, we define the projection matrix as

$$\Pi_F := I - nn^\top,$$

and its rotation as

$$\Pi^F_F := n \times \Pi_F.$$

We treat the Hamilton operator $\nabla = (\partial_1, \partial_2, \partial_3)^\top$ as a column vector and introduce the following definitions:

$$\nabla_F := \Pi_F \nabla, \quad \nabla^1_F := \Pi^F_F \nabla.$$

For a scalar function $v$, we have:

$$\text{grad} \; Fv = \nabla Fv = \Pi_F (\nabla v) = -n \times (n \times \nabla v),$$

$$\text{curl} \; Fv = \nabla^1_F v = n \times \nabla v = n \times \nabla Fv,$$

where $\text{grad} \; Fv$ is the surface gradient of $v$, and $\text{curl} \; Fv$ is the surface curl. For a vector function $v$, the surface divergence is defined as:

$$\text{div}_F \; v := \nabla_F \cdot v = \nabla_F \cdot (\Pi_F v).$$

Furthermore, the surface rot operator is given by:

$$\text{rot} \; Fv := \nabla^1_F \cdot v = (n \times \nabla) \cdot v = n \cdot (\nabla \times v),$$

representing the normal component of $\nabla \times v$.

We denote by $\mathbb{M}$ the space of $3 \times 3$ matrices, $\mathbb{S}$ the subspace of symmetric matrices, $\mathbb{T}$ the subspace of traceless matrices, and $\mathbb{K}$ the subspace of skew-symmetric matrices. A matrix $\tau$ can be decomposed into $\tau = \text{sym} \; \tau + \text{skw} \; \tau$ with the symmetric part $\text{sym} \; \tau := (\tau + \tau^\top)/2$ and the skew-symmetric part $\text{skw} \; \tau := (\tau - \tau^\top)/2$. For a matrix function, differential operators $\text{curl}$, $\text{div}$ in letter are applied row-wise.

2.2. BGG Construction. A bounded Hilbert complex is a sequence of Hilbert spaces connected by a sequence of linear bounded operators satisfying the property: the composition of two consecutive operators vanishes. Assume we have two bounded Hilbert complexes $(\mathcal{V}_\bullet \otimes \mathbb{E}_\bullet, d_\bullet)$, $(\mathcal{Y}_\bullet \otimes \mathbb{E}_\bullet, \tilde{d}_\bullet)$ and bounded linking maps $S_i = \text{id} \otimes s_i : \mathcal{V}_{i+1} \otimes \mathbb{E}_i \rightarrow \mathcal{V}_{i+1} \otimes \mathbb{E}_{i+1}$ for $i = 0, \ldots, n - 1$

(4)

$$
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{V}_0 \otimes \mathbb{E}_0 & d_0 & \rightarrow & \mathcal{V}_1 \otimes \mathbb{E}_1 & d_1 & \rightarrow & \cdots & \cdots & d_{n-1} & \rightarrow & \mathcal{V}_n \otimes \mathbb{E}_n & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{V}_1 \otimes \mathbb{E}_0 & d_0 & \rightarrow & \mathcal{V}_2 \otimes \mathbb{E}_1 & d_1 & \rightarrow & \cdots & \cdots & d_{n-1} & \rightarrow & \mathcal{V}_{n+1} \otimes \mathbb{E}_n & \rightarrow & 0,
\end{array}
$$

in which $s_i : \mathbb{E}_i \rightarrow \mathbb{E}_{i+1}$ is a linear operator between finite-dimensional spaces. The operators in (4) satisfy anti-commutativity: $S_{i+1} \tilde{d}_i = -d_{i+1} S_i$, and $J$-injectivity/surjectivity
condition: for some particular \( J \) with \( 0 \leq J < n \), \( s_i \) is injective for \( i = 0, \ldots, J \) and is surjective for \( i = J + 1, \ldots, n - 1 \). The output complex is
\[
0 \rightarrow \mathcal{Y}_0 \xrightarrow{\partial_0} \mathcal{Y}_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{J-1}} \mathcal{Y}_J \xrightarrow{\partial_J} \mathcal{Y}_{J+1} \xrightarrow{\partial_{J+1}} \cdots \xrightarrow{\partial_{n-1}} \mathcal{Y}_n \rightarrow 0,
\]
where \( \mathcal{Y}_i \) is \( \mathcal{V}_i \otimes (\mathcal{E}_i / \ker(s_i)) \) for \( i = 0, \ldots, J \) and \( \mathcal{V}_{i+1} \otimes \ker(s_i) \) for \( i = J+1, \ldots, n \), \( \partial_i \) is the projection of \( \partial_i \) onto \( \mathcal{Y}_{i+1} \) for \( i = 0, \ldots, J - 1 \), \( \partial_i \) for \( i = J + 1, \ldots, n - 1 \) and \( d_j(S_j)^{-1}d_j \) for \( i = J \). By the BGG framework, many new complexes can be generated from old ones.

The following example is presented in [6]. In three dimensions, we stack copies of de Rham complexes to form the diagram
\[
\begin{array}{ccccccccc}
\mathbb{R} & \rightarrow & H^s & \xrightarrow{\text{grad}} & H^{s-1}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & H^{s-2}(\mathbb{R}^3) & \xrightarrow{\text{div}} & H^{s-3} & \rightarrow 0 \\
\text{id} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{R}^3 & \rightarrow & H^{s-1}(\mathbb{R}^3) & \xrightarrow{\text{grad}} & H^{s-2}(M) & \xrightarrow{\text{curl}} & H^{s-3}(M) & \xrightarrow{\text{div}} & H^{s-4}(\mathbb{R}^3) & \rightarrow 0 \\
\text{mskw} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{R}^3 & \rightarrow & H^{s-2}(\mathbb{R}^3) & \xrightarrow{\text{grad}} & H^{s-3}(M) & \xrightarrow{\text{curl}} & H^{s-4}(M) & \xrightarrow{\text{div}} & H^{s-5}(\mathbb{R}^3) & \rightarrow 0 \\
\text{id} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{R} & \rightarrow & H^{s-3} & \xrightarrow{\text{grad}} & H^{s-4}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & H^{s-5}(\mathbb{R}^3) & \xrightarrow{\text{div}} & H^{s-6} & \rightarrow 0,
\end{array}
\]

where \( H^s(\mathbb{X}) = H^s \otimes \mathbb{X} \) for \( \mathbb{X} = \mathbb{R}^3 \) or \( M \), and operators
\[
S\tau = \tau^T - \text{tr}(\tau)I, \quad \iota : \mathbb{R} \rightarrow M, \quad \iota w = wI,
\]
\[
\text{mskw} \omega := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ -\omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \text{vskw} := \text{mskw}^{-1} \circ \text{skw}.
\]

Considering operators in the \( \nearrow \) direction, the three operators in the diagonal are one-to-one, the lower triangular part is injective, and the upper triangular part is surjective. By direct calculation, the parallelogram formed by the north-east diagonal \( \nearrow \) and the horizontal operators is anticommutative.

Then applying the BGG construction, several complexes can be derived from the BGG framework including but not limited to the Hessian complex, the elasticity complex, and the \( \text{div} \text{div} \) complex; see the three zigzag paths in Figure 1.

Recent endeavors have led to the individual construction of finite element Hessian complexes, elasticity complexes, and \( \text{div} \text{div} \) complexes, as evidenced by works such as [11, 17, 15, 18, 21, 22, 23, 25]. However, these accomplishments have materialized in a case-specific manner, prompting an intriguing inquiry: can the overarching BGG framework be employed to establish a unified foundation for these diverse constructions?

Expanding the BGG methodology to construct finite element complexes presents a challenging task. A notable hurdle arises in the construction of finite element de Rham complexes with varying degrees of smoothness. We have successfully addressed this challenge, as detailed in [14], and we revisit the specific techniques in Section 2.

Another intricate difficulty that arises from the diagram (5) becomes evident when we shift from the Sobolev space \( H^s \) to the domain spaces \( H(\text{grad}) \), \( H(\text{curl}) \), or \( H(\text{div}) \)
within the diagram:

\[ \mathbb{R} \rightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \rightarrow 0 \]
\[ \mathbb{R}^3 \rightarrow H^1(\mathbb{R}^3) \xrightarrow{\text{grad}} H(\text{curl};\mathbb{M}) \xrightarrow{\text{curl}} H(\text{div};\mathbb{M}) \xrightarrow{\text{div}} L^2(\mathbb{R}^3) \rightarrow 0 \]
\[ \mathbb{R}^3 \rightarrow H^1(\mathbb{R}^3) \xrightarrow{\text{grad}} H(\text{curl};\mathbb{M}) \xrightarrow{\text{curl}} H(\text{div};\mathbb{M}) \xrightarrow{\text{div}} L^2(\mathbb{R}^3) \rightarrow 0 \]

The Hamilton operator \( \nabla \) can be employed to represent the differential operators: \( \text{grad} = \nabla \), \( \text{curl} = \nabla \times \), and \( \text{div} = \nabla \cdot \). Symbolically, by substituting \( \nabla \) with the outward unit normal vector \( n \), these operations maintain their respective anticommutative properties. For example,

\[ 2n \cdot (\text{vskw} \sigma) = -\text{tr}(\sigma \times n), \]
\[ (S\sigma)n = -2 \text{vskw}(\sigma \times n), \]
\[ n \times u = 2 \text{vskw}(un^T). \]

Therefore, the operators in the \( \searrow \) direction retain their well-defined nature. To illustrate, consider the operator \( S : H(\text{curl};\mathbb{M}) \rightarrow H(\text{div};\mathbb{M}) \) positioned at the center of (6). Through the utilization of integration by parts and identity (7), it becomes evident that in the distributional sense,

\[ \langle \text{div} S\tau, \phi \rangle = (2 \text{vskw} \text{curl} \tau, \phi), \quad \phi \in C_0^\infty(\Omega; \mathbb{R}^3). \]

Given that \( \tau \in H(\text{curl};\mathbb{M}) \), leading to \( \text{curl} \tau \in L^2(\mathbb{M}) \), we can conclude that \( S\tau \in H(\text{div};\mathbb{M}) \).

Nonetheless, the converse direction, i.e., the \( \nearrow \) direction, is not well-defined due to inherent continuity mismatches. For example, all three operators along the diagonal of diagram (6) are obviously not one-to-one, impeding a straightforward application of the BGG procedure.

In instances where finite element spaces exhibit sufficient smoothness, the framework corresponds to the diagram (5), yet for certain finite element spaces, the scenario aligns with (6) rather than (5). This discrepancy represents the primary obstacle impeding the extension of the BGG construction into the discrete realm. To surmount this challenge, we introduce a pair of operations for an exact sequence.

2.3. Two reduction operators. Let \( V_i, i = 0, 1, 2 \) be Hilbert spaces forming an exact sequence:

\[ \ker(d_0) \xrightarrow{\subseteq} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \rightarrow 0. \]

The exactness implies \( \ker(d_1) = \text{img}(d_0) \) and \( d_1V_1 = V_2 \). When they are finite-dimensional, the following dimension identity holds:

\[ \dim \ker(d_0) - \dim V_0 + \dim V_1 - \dim V_2 = 0. \]

Introduce a subspace \( \bar{V}_2 \subseteq V_2 \), and define \( \bar{V}_1 \) as the preimage of \( \bar{V}_2 \), namely,

\[ \bar{V}_1 = \{ v \in V_1 : d_1v \in \bar{V}_2 \} \subseteq V_1. \]
Since \( d_1 d_0 = 0 \), the exact sequence remains intact through this construction:

\[
\ker(d_0) \hookrightarrow V_0 \xrightarrow{d_0} \tilde{V}_1 \xrightarrow{d_1} V_2 \rightarrow 0.
\]

We shall refer to the process of transitioning from (8) to (10) as a \( \sim \) (tilde) operation.

Suppose we have two additional operators, \( d_2 \) acting on \( V_0 \) and \( s^* \) acting on \( V_1 \) respectively. Consider a subspace \( W \) contained within the intersection of \( d_2(V_0) \) and \( s^*(V_1) \). Define the preimage spaces of \( W \) in \( V_0 \) and \( V_1 \) as:

\[
\begin{align*}
\hat{V}_0 &= \{ u \in V_0 : d_2 u \in W \}, \\
\hat{V}_1 &= \{ v \in V_1 : s^* v \in W \}.
\end{align*}
\]

Assume we have the following triangular commutative diagram

\[
\begin{array}{ccc}
\hat{V}_0 & \xrightarrow{d_0} & \hat{V}_1 \\
\downarrow{d_2} & & \downarrow{s^*} \\
W & & \\
\end{array}
\]

That is

\[
s^* d_0 = d_2.
\]

**Lemma 2.1.** Assume we have the exact sequence (8) with finite-dimensional spaces and triangular commutative diagram (12). Define \( \hat{V}_0 \) and \( \hat{V}_1 \) by (11). Additionally assume that \( \ker(d_0) \subset \hat{V}_0 \) and the equation \( \dim V_0 - \dim \hat{V}_0 = \dim V_1 - \dim \hat{V}_1 \) holds. Then

\[
\ker(d_0) \hookrightarrow \hat{V}_0 \xrightarrow{d_0} \hat{V}_1 \xrightarrow{d_1} V_2 \rightarrow 0
\]

is also an exact sequence.

**Proof.** By construction, \( d_0 \hat{V}_0 \subset \hat{V}_1 \) and thus (13) is indeed a complex. To show its exactness, we start by demonstrating that \( \ker(d_1) \cap \hat{V}_1 = d_0 \hat{V}_0 \). Taking a \( \hat{v} \in \hat{V}_1 \) and \( d_1 \hat{v} = 0 \), by the exactness of (8), there exists a \( u \in V_0 \) s.t. \( d_0 u = \hat{v} \). Recall that \( d_2 = s^* d_0 \). So \( d_2 u = s^* d_0 u = s^* \hat{v} \in W \), i.e., \( u \in \hat{V}_0 \).

For the second part, it is evident that \( d_1(\hat{V}_1) \subset V_2 \). To prove \( d_1(\hat{V}_1) = V_2 \), we count the dimensions:

\[
\begin{align*}
\dim d_1(\hat{V}_1) &= \dim \hat{V}_1 - \dim d_0(\hat{V}_0) \\
&= \dim V_1 - \dim \hat{V}_0 + \dim \ker(d_0) \\
&= \dim V_1 - \dim V_0 + \dim \ker(d_0) = \dim V_2,
\end{align*}
\]

where in the last step, we have used the identity (9). \( \square \)

The procedure described above, which transforms the sequence from (8) to the sequence presented in (13), is referred to as a \( \hat{\sim} \) (hat) operation. This operation specifically modifies the domain and co-domain of the operator \( d_0 \). By combining the hat and tilde operations, it becomes possible to create a further reduced exact sequence:

\[
\ker(d_0) \hookrightarrow \hat{V}_0 \xrightarrow{d_0} \hat{\tilde{V}}_1 \xrightarrow{d_1} V_2 \rightarrow 0.
\]

To illustrate these operations and their interplay, please refer to Figure 2. Throughout the process, the spaces \( \hat{V}, \tilde{V}, \) or \( \hat{\tilde{V}} \) can be renamed as needed, depending on the context and the spaces being constructed.
2.4. Derived complexes. By diagram (5), the surjectivity \( \text{tr} H(\text{div}; M) = L^2 \) follows from \( \text{div} H^1(\mathbb{R}^3) = L^2 \), and \(-2 \text{ vskw } H(\text{curl}; M) = H(\text{div}) \) follows from the regularity decomposition \( H(\text{div}) = H^1(\mathbb{R}^3) + \text{curl} H^1(\mathbb{R}^3) \). Then applying the hat operation \( \hat{\cdot} \) to \( V_0 = H^1 \) and \( V_1 = H(\text{curl}; M) \) with \( \hat{V}_0 = H^2 \), \( \hat{V}_1 = H^1(\mathbb{R}^3) \) and \( W = H^1(\mathbb{R}^3) \), we obtain

\[
\begin{align*}
\mathbb{R} & \longrightarrow H^2 \\
\mathbb{R}^3 & \rightarrow H^1(\mathbb{R}^3) \\
H^1(\text{curl}) & \longrightarrow \hat{H}(\text{curl}; M) \\
H^1(\mathbb{R}^3) & \longrightarrow \hat{H}(\text{div}; M) \\
H^1(\text{curl}) & \rightarrow \hat{H}(\text{div}; M) \\
\text{div} & \rightarrow L^2(\mathbb{R}^3) \longrightarrow 0
\end{align*}
\]

which leads to the Hessian complex \([6, 29]\)

\[
\begin{align*}
\mathbb{R}^3 & \rightarrow H^1(\mathbb{R}^3) \\
\mathbb{R}^3 & \rightarrow H(\text{curl}; M) \\
\text{curl} & \rightarrow H(\text{div}; M) \\
\text{div} & \rightarrow L^2(\mathbb{R}^3) \rightarrow 0.
\end{align*}
\]

Now we look at the second and third rows of diagram (6). As we mentioned before \( S: H(\text{curl}; M) \rightarrow H(\text{div}; M) \) is well-defined but \( S^{-1}: H(\text{div}; M) \rightarrow H(\text{curl}; M) \) is not. To fix it, we apply the \( \sim \) operation to reduce space \( H(\text{div}; M) \) to \( \hat{H}(\text{div}; M) \). Then \( S: H(\text{curl}; M) \rightarrow \hat{H}(\text{div}; M) \) is one-to-one.

To apply the \( \sim \) operation, we use \( 2 \text{ vskw } \text{grad } u = \text{curl } u \) and the triangular diagram

\[
\begin{align*}
H^1(\text{curl}) & \longrightarrow \tilde{H}(\text{curl}; M) \\
\text{curl} & \longrightarrow \tilde{H}(\text{div}; M) \\
H^1(\mathbb{R}^3) & \longrightarrow \tilde{H}(\text{div}; M)
\end{align*}
\]

where

\[
\begin{align*}
H^1(\text{curl}) & := \{ v \in H^1(\mathbb{R}^3) : \text{curl } v \in H^1(\mathbb{R}^3) \}, \\
\tilde{H}(\text{curl}; M) & := \{ \tau \in H(\text{curl}; M) : \text{vskw } \tau \in H^1(\mathbb{R}^3), \text{curl } \tau \in \hat{H}(\text{div}; M) \}.
\end{align*}
\]

By applying the two reductions, we obtain

\[
\begin{align*}
\mathbb{R}^3 & \rightarrow H^1(\text{curl}) \\
\mathbb{R}^3 & \rightarrow H(\text{curl}; M) \\
\text{curl} & \rightarrow \tilde{H}(\text{div}; M) \\
\text{div} & \rightarrow L^2(\mathbb{R}^3) \rightarrow 0
\end{align*}
\]

This leads to the elasticity complex

\[
(14) \quad \mathbf{R} \mathbf{M} \sqsubset H^1(\text{curl}) \longrightarrow H(\text{inc}^+; \mathbb{S}) \longrightarrow H(\text{div}; \mathbb{S}) \longrightarrow L^2(\mathbb{R}^3) \rightarrow 0,
\]
where \( \text{def} \, \mathbf{u} = \text{sym grad} \, \mathbf{u} \), \( \text{inc} \, \mathbf{\tau} = -\text{curl} \, (\text{curl} \, \mathbf{\tau})^\top \), \( \mathbf{RM} = \{ \mathbf{a} \times \mathbf{x} + \mathbf{b} : \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \} \) is the space of the linearized rigid body motion, and the space

\[
H(\text{inc}^+; S) := H(\text{inc}; S) \cap H(\text{curl}; S) = \{ \mathbf{\tau} \in H(\text{curl}; S) : \text{inc} \, \mathbf{\tau} \in L^2(S) \}.
\]

The elasticity complex (14) is slightly smoother than the elasticity complex [20, 4]

\[
\mathbf{RM} \overset{\text{def}}{\rightarrow} H^1(\mathbb{R}^3) \overset{\text{sym}}{\rightarrow} H(\text{inc}, S) \overset{\text{inc}}{\rightarrow} H(\text{div}, S) \overset{\text{div}}{\rightarrow} L^2(\mathbb{R}^3) \rightarrow 0.
\]

It is worth noting that (14) can be obtained from (15) by one \( \sim \) operation.

Similarly by applying the two reductions to the third and fourth rows of diagram (6), we obtain

\[
\begin{align*}
\mathbb{R}^3 & \rightarrow L^2(\mathbb{R}^3) \overset{\text{grad}}{\rightarrow} \tilde{H}(\text{curl}; \mathbb{M}) \overset{\text{curl}}{\rightarrow} \tilde{H}(\text{div}; \mathbb{M}) \overset{\text{div}}{\rightarrow} H(\text{div}) \rightarrow 0 \\
\mathbb{R} & \rightarrow H^1 \overset{\text{grad}}{\rightarrow} H(\text{curl}) \overset{\text{curl}}{\rightarrow} H(\text{div}) \overset{\text{div}}{\rightarrow} L^2 \rightarrow 0,
\end{align*}
\]

where

\[
\begin{align*}
H^1(\text{div}) & := \{ \mathbf{v} \in H^1(\mathbb{R}^3) : \text{div} \, \mathbf{v} \in H^1 \}, \\
\tilde{H}(\text{curl}; \mathbb{M}) & := \{ \mathbf{\tau} \in H(\text{curl}; \mathbb{M}) : \text{tr} \, \mathbf{\tau} \in H^1, \text{curl} \, \mathbf{\tau} \in \tilde{H}(\text{div}; \mathbb{M}) \}, \\
\tilde{H}(\text{div}; \mathbb{M}) & := H(\text{div div}^+; S) \oplus \text{mskw} \, H(\text{curl}), \\
H(\text{div div}^+; S) & := \{ \mathbf{\tau} \in H(\text{div}; S) : \text{div} \, \mathbf{\tau} \in H(\text{div}) \}.
\end{align*}
\]

This leads to the div div complex

\[
\mathbf{RT} \overset{\text{def}}{\rightarrow} H^1(\text{div}) \overset{\text{sym}}{\rightarrow} H(\text{sym curl}^+_\mathbb{T}; \mathbb{T}) \overset{\text{sym}}{\rightarrow} H(\text{div div}^+; S) \overset{\text{div}}{\rightarrow} L^2 \rightarrow 0,
\]

where \( \text{dev} \, \mathbf{\sigma} = \mathbf{\sigma} - \text{tr}(\mathbf{\sigma}) \mathbb{I} / 3 \), \( \mathbf{RT} = \{ a \mathbf{x} + \mathbf{b} : a \in \mathbb{R}, \mathbf{b} \in \mathbb{R}^3 \} \), and the space

\[
H(\text{sym curl}^+_\mathbb{T}; \mathbb{T}) := \{ \mathbf{\tau} \in H(\text{curl}; \mathbb{T}) : \text{curl} \, \mathbf{\tau} \in \tilde{H}(\text{div}; \mathbb{M}) \}.
\]

The derived div div complex is slightly smoother than the div div complex [6, 29]

\[
\mathbf{RT} \overset{\text{def}}{\rightarrow} H^1(\mathbb{R}^3) \overset{\text{sym}}{\rightarrow} H(\text{sym curl} \mathbb{T}; \mathbb{T}) \overset{\text{sym}}{\rightarrow} H(\text{div div}, \mathbb{T}) \overset{\text{div}}{\rightarrow} L^2 \rightarrow 0.
\]

It is worth noting that Lemma 2.1 cannot be directly employed to deduce the exactness of the reduced de Rham complexes due to the fact that these spaces are not finite-dimensional. A rigorous proof would involve employing the regular decomposition of \( H(d) \) spaces. However, since our primary focus is on the finite element spaces, we omit this proof here. Additionally, we would like to highlight that the complexes derived from the BGG framework typically encompass spaces that possess a slightly higher degree of smoothness.

3. Smooth finite element de Rham Complexes

In this section we shall review the smooth finite element de Rham complexes developed in [14]. We construct finite element spaces with different smoothness at vertices, edges, and faces which is summarized as a smoothness vector \( \mathbf{r} = (r^v, r^e, r^f) \). An integer vector \( \mathbf{r} = (r^v, r^e, r^f) \) is called a smoothness vector if \( r^f \geq -1 \), \( r^e \geq \max\{2r^f, -1\} \) and \( r^v \geq \max\{2r^e, -1\} \). Its restriction \( (r^v, r^e)^\top \) is a two-dimensional smoothness vector. For a smoothness vector \( \mathbf{r} \), define \( \mathbf{r} \ominus 1 := \max\{\mathbf{r} - 1, -1\} \) and \( r_+ = \max\{\mathbf{r}, 0\} \).
3.1. **Smooth bubble functions.** For edge $e$, let $r^e \geq 0$ and $k \geq 2(r^e + 1)$, define edge bubble polynomial space

$$\mathbb{B}_k(e; r^e) := \{ u \in \mathbb{P}_k(e) : \partial_t^j u \text{ vanishes at all vertices of } e \text{ for } j = 0, \ldots, r^e \},$$

where $\partial_t$ is the tangential derivative along $e$. This bubble space can be easily characterized as $\mathbb{B}_k(e; r^e) = b_{e, r}^{r^e + 1} \mathbb{P}_{k-2(r^e+1)}(e)$, where $b_{e, r}$ is a polynomial of degree $r$ vanishing at two vertices of $e$.

For triangle $f$ and a smoothness vector $r = (r^e, r^c)^T$, define face bubble polynomial space

$$\mathbb{B}_k(f; \begin{pmatrix} r^e \\ r^c \end{pmatrix}) := \{ u \in \mathbb{P}_k(f) : \nabla_j^e u \text{ vanishes at all vertices of } f \text{ for } j = 0, \ldots, r^e, \nabla_j^c u \text{ vanishes on all edges of } f \text{ for } j = 0, \ldots, r^c \},$$

where $\nabla_j$ is the surface gradient on $f$. All polynomials defined on $e$ and $f$ can be naturally extended to the whole tetrahedron using the Bernstein representation in the barycentric coordinate.

For tetrahedron $T$ and a smoothness vector $r = (r^e, r^c, r^f)^T$, define bubble polynomial space

$$\mathbb{B}_k(T; r) := \{ u \in \mathbb{P}_k(T) : \nabla_j^e u \text{ vanishes at all vertices of } T \text{ for } j = 0, \ldots, r^e, \nabla_j^c u \text{ vanishes on all edges of } T \text{ for } j = 0, \ldots, r^c, \nabla_j^f u \text{ vanishes on all faces of } T \text{ for } j = 0, \ldots, r^f \}.$$ 

Notice that when $r^f = -1$, the bubble function may not vanish on the boundary of $T$.

Precise characterization of bubble polynomial spaces $\mathbb{B}_k(f; r)$ and $\mathbb{B}_k(T; r)$ can be given by decompositions of simplicial lattice points; see [13, 14] for details.

To simplify notation, for a three-dimensional smoothness vector $r = (r^e, r^c, r^f)^T$, $\mathbb{B}_k(f; r) := \mathbb{B}_k(f; (r^e, r^c)^T)$ is the face bubble using the restriction of $r$ on $f$. Similarly $\mathbb{B}_k(e; r) = \mathbb{B}_k(e; r^e)$.

For a vector space $V$, we abbreviate $V \otimes \mathbb{R}^3$ as $V^3$. Define bubble spaces

$$\mathbb{B}_{k}^{\text{curl}}(T; r) := \{ v \in \mathbb{B}_k^3(T; r) : v \times n|_{\partial T} = 0 \},$$

$$\mathbb{B}_{k}^{\text{div}}(T; r) := \{ v \in \mathbb{B}_k^3(T; r) : v \cdot n|_{\partial T} = 0 \},$$

$$\mathbb{B}_{k}^{L^2}(T; r) := \mathbb{B}_k(T; r) \cap L^2(T).$$

Usually $T$ will be skipped in the notation, i.e., $\mathbb{B}_k(r) = \mathbb{B}_k(T; r)$. When $r^f \geq 0$, functions in $\mathbb{B}_k(r)$ vanish on $\partial T$ and thus $\mathbb{B}_k^{\text{curl}}(r) = \mathbb{B}_k^{\text{div}}(r) = \mathbb{B}_k^1(r)$. When $r^f = -1$, $\mathbb{B}_k^3(r^f) \subset \mathbb{B}_k^3(r) \subset \mathbb{B}_k^3(r)$ for $\mathbb{B}_k^3$ as only tangential or normal direction vanishes respectively. Precise characterization of bubble spaces $\mathbb{B}_k^{\text{curl}}(r)$ and $\mathbb{B}_k^{\text{div}}(r)$ can be also found in [14].

3.2. **Smooth scalar finite elements.** For each edge $e$, we choose two normal vectors $n_1^e$ and $n_2^e$, which we abbreviate as $n_1$ and $n_2$. For each face $f$, we choose a normal vector $n_f$, abbreviated as $n$ when $f$ is clear from the context. In a conforming mesh $\mathcal{T}_h$, $n_1^e$, $n_2^e$, or $n_f$ depends on $e$ or $f$, not the element containing it. In partial derivatives like $\partial_n u$, we use regular font $n$ and not boldface $n$. 

Let \( r = (r^v, r^e, r^f) \) be a smoothness vector, and nonnegative integer \( k \geq 2r^v + 1 \). The shape function space \( \mathbb{P}_k(T) \) is determined by the DoFs

\[
\nabla^j u(\nu), \quad j = 0, 1, \ldots, r^v, \nu \in \Delta_0(T),
\]

\[
\int_e \frac{\partial^j u}{\partial n_i^e \partial n_j^e} \, ds, \quad q \in \mathbb{B}_{k-j}(e; r^{i-j}), 0 \leq i \leq j \leq r^e, e \in \Delta_1(T),
\]

\[
\int_f \frac{\partial^j u}{\partial n_i^f} \, q \, dS, \quad q \in \mathbb{B}_{k-j}(f; r - j), 0 \leq j \leq r^f, f \in \Delta_2(T),
\]

\[
\int_T u q \, dx, \quad q \in \mathbb{B}_k(T; r).
\]

As \( b_e \geq 0 \), the test function space in (16b) can be changed to \( q \in \mathbb{P}_{k-2(r^v+1)+j}(e) \). For the sake of simplifying notation, we use DoF \( \mathcal{V}_k \) to represent the set of DoFs as defined in (16), and DoF \( s(r) \) for the subset corresponding to the sub-simplex \( e \). The unisolvence property can be succinctly expressed as:

\[
\mathbb{P}_k(T) \iff \text{DoF}_k(r).
\]

Namely a function \( u \in \mathbb{P}_k(T) \) is uniquely determined by the associated set of DoFs \( \text{DoF}_k(r) \).

When considering a mesh \( T_h \), the DoFs defined by (16) define the global \( C^{r^f} \)-continuous finite element space as follows:

\[
\mathcal{V}_k(T_h; r) = \{ u \in C^{r^f}(\Omega) : u|_T \in \mathbb{P}_k(T) \text{ for all } T \in T_h, \text{ and all the DoFs (16) are single-valued} \}.
\]

In cases where \( \mathcal{V}_k(T_h; r) \) is used as a subspace of \( H^1(\Omega) \) and \( L^2(\Omega) \), notation \( \mathcal{V}_k^{\text{grad}}(T_h; r) \) and \( \mathcal{V}_k^{\text{L}}(T_h; r) \) are employed respectively. The reference to the mesh \( T_h \) will subsequently be omitted in the notation.

3.3. \( H(\text{div}) \)-conforming finite elements. Let \( r_3 \geq r_2 \oplus 1 \) and positive integer \( k \geq \max\{2r_2^v + 1, 2r_2^e + 2\} \). We introduce

\[
\mathcal{V}_k^{\text{div}}(r_2, r_3) := \{ v \in \mathcal{V}_k^{\text{grad}}(r_2) \cap H(\text{div}, \Omega) : \text{div } v \in \mathcal{V}_k^{\text{L}}(r_3) \},
\]

and give finite element description of \( \mathcal{V}_k^{\text{div}}(r_2, r_3) \) in [14, Section 4.4]. We will abbreviate \( \mathcal{V}_k^{\text{div}}(r_2, r_2 \oplus 1) \) as \( \mathcal{V}_k^{\text{div}}(r_2) \).

The \( \text{div} \) stability, specifically \( \text{div} : \mathcal{V}_k^{\text{div}}(r_2, r_3) \rightarrow \mathcal{V}_k^{\text{L}}(r_3) \) being surjective, is established under certain restrictions on \( (r_2, r_3) \) and for sufficiently large values of \( k \).

**Lemma 3.1** (Theorem 4.10 in [14]). Assume

\[
\begin{cases}
    r^f_2 \geq 0, & r^e_2 \geq 2r^f_2 + 1, & r^v_2 \geq 2r^e_2, \\
    r^f_2 = -1, & r^e_2 \geq 1, & r^v_2 \geq 2r^e_2, \\
    r^f_2 \in \{0, -1\}, & r^v_2 \geq 2r^e_2 + 1,
\end{cases}
\]

and \( r_4 \geq r_2 \oplus 1 \). Assume \( k \geq \max\{2r_2^v + 1, r_2^v + 2, 3(r_2^v + 1), 2r_3^v + 2, 4r_3^v + 5, (r_3^v + r_3^f + 5)(r_3^v = 0)\} \). It holds that

\[
\text{div } \mathcal{V}_k^{\text{div}}(r_2, r_3) = \mathcal{V}_k^{\text{L}}(r_3).
\]
When (17) holds, we shall call \((r_2, r_3, k)\) div stable. We emphasize that for continuous div element, i.e., \(r^f \geq 0\), the minimal stable pair is \(r_2 = (2, 1, 0)^T, r_3 = (1, 0, -1)^T\) and \(k \geq 6\) which is the Stokes element constructed by Neilan [28].

### 3.4. \(H(\text{curl})\)-conforming finite elements.
Let \(r_2 \geq r_1 \oplus 1\) be two smoothness vectors. Next we introduce
\[ V^\text{curl}_{k+1}(r_1, r_2) := \{ v \in V^3_{k+1}(r_1) \cap H(\text{curl}, \Omega) : \text{curl} v \in V^\text{div}_{k}(r_2) \} , \]
and will abbreviate \(V^\text{curl}_{k+1}(r_1, r_1 \oplus 1)\) as \(V^\text{curl}_{k+1}(r_1)\). Its finite element description, i.e., local DoFs for the shape function space \(\mathbb{P}_{k+1}(T; \mathbb{R}^3)\), is given in [14, Section 5.3].

### 3.5. Finite element de Rham complexes in three dimensions.

**Theorem 3.2** (Theorem 5.9 in [14]). \(\text{Let } r_0 \geq 0, r_1 = r_0 - 1, r_2 \geq r_1 \oplus 1, r_3 \geq r_2 \oplus 1\) be smoothness vectors. Assume \((r_2, r_3, k)\) is div stable. Assume \(k \geq \max \{2r_1^2 + 1, 2r_2^2 + 1, 2r_3^2 + 2, 4r_3^2 + 5, (r_3^2 + r_3^2 + 5)[r_3^2 = 0]\}\). Then the finite element de Rham complex
\[ (18) \quad \mathbb{R} \subset V^\text{grad}_{k+2}(r_0) \xrightarrow{\text{grad}} V^\text{curl}_{k+1}(r_1, r_2) \xrightarrow{\text{curl}} V^\text{div}_k(r_2, r_3) \xrightarrow{\text{div}} V^L_{k-1}(r_3) \rightarrow 0 \]
is exact.

We will refer to a parameter sequence \((r_0, r_1, r_2, r_3, k)\) as a valid de Rham parameter sequence if (18) holds with exactness.

When \(r^f_2 \geq 0\), (18) transforms into a finite element Stokes complex, given that the space \(V^\text{div}_{k}(r_2, r_3) \subset H^1(\Omega; \mathbb{R}^3)\). This enables the discretization of Stokes equation. Notably, existing works on finite element Stokes complexes [28] and finite element de Rham complexes [19] exemplify instances of (18) achieved by selecting different vectors of smoothness.

In contrast to the two-dimensional findings in [13], the areas requiring focused attention are as follows:

- The establishment of div stability, specifically \(V^\text{div}_{k}(r_2, r_3) = V^L_{k-1}(r_3)\);
- The formulation of DoFs for edge elements \(V^\text{curl}_{k+1}(r_1, r_2)\).

In our preceding 2D work [13], we demonstrated div stability using dimension counts and observed that an edge element is essentially a rotation of a face element.

### 4. Face Elements

In this section, our objective is to construct finite elements conforming to \(H(\text{div}; \mathbb{X})\) for either \(\mathbb{X} = \mathbb{S}\) or \(\mathbb{T}\). When \(r \geq 0\), the space \(V_k(r)\) is continuous and the construction is straightforward using the tensor product \(V_k(r) \otimes \mathbb{X}\). However, for cases where \(r^f = -1\) or \(r^e = -1\), we only enforce normal continuity for the \(H(\text{div})\)-conforming element. We will employ the \(t - n\) decomposition technique introduced in [12] to construct the finite elements, subsequently leveraging the BGG framework to establish the div stability property.

#### 4.1. \(H(\text{div}; \mathbb{T})\) and \(H(\text{div}; \mathbb{S})\) finite elements.
Given a smoothness vector \(r\) with \(r^c \geq 0\), we examine the space associated with \(r^c \geq 0\), yielding the unisolvence relationship
\[ (19) \quad \mathbb{P}_k(T) \otimes \mathbb{X} \leftrightarrow \text{DoF}_k(r^+_c) \otimes \mathbb{X}, \]
and its global extension, \(V_k(r^+_c) \otimes \mathbb{X}\). However, in situations where \(r^f = -1\) or \(r^e = -1\), we need to modify the continuity of the element corresponding to \(r^+_c\) and transfer the
tangential component into the bubble space. Recall that for $\sigma \in \mathbb{P}_k(T) \otimes X$, $\text{tr}^{\text{div}} \sigma = \sigma n$ and to be $H(\text{div})$-conforming, $\sigma n$ should be continuous across faces of the triangulation.

The crux lies in an appropriate $t - n$ decomposition of the tensor $X$ at a sub-simplex $s$

$$X = \mathcal{T}^s(X) \oplus \mathcal{N}^s(X),$$

where $\mathcal{T}^s$ and $\mathcal{N}^s$ denote the tangential and normal planes of $s$.

For a sub-simplex $s$, we use the following decomposition of DoFs associated to $s$

$$\text{DoF}(s; r_+ \otimes X) = \begin{cases} 
\text{DoF}(s; r) \otimes X & \text{when } r^s \geq 0, \\
\text{DoF}(s; r_+ \otimes \mathcal{N}^s(X)) \oplus & \text{normal trace, when } r^s = -1 \\
\text{DoF}(s; r_+ \otimes \mathcal{T}^s(X)) \oplus & \text{div bubbles.}
\end{cases}$$

We relocate the tangential component into the bubble space and introduce

$$\mathcal{B}^\text{div}_k(r; X) := \mathcal{B}_k(r_+ \otimes X)$$

where $[\cdot]$ is Iverson bracket. This concept extends the Kronecker delta function to a statement $P$

$$[P] = \begin{cases} 
1 & \text{if } P \text{ is true,} \\
0 & \text{otherwise.}
\end{cases}$$

Different frames will be employed for distinct sub-simplices $s$. On an edge $e$, one possible frame is $\{n_1, n_2, t\}$, where $f_1$ and $f_2$ denote the two faces containing $e$. Another option is $\{n^1_1, n^2_2, t\}$, where $n^1_1$ and $n^2_2$ represent two orthogonal normal vectors of $e$. Notably, $n^1_1$ depends only on $e$, while $n^2_2$ is contingent on face $f$. On a face $f$, an orthonormal frame $\{t^f_1, t^f_2, n^f\}$ is utilized, comprising two tangential vectors $t^f_1$ and a face normal $n^f$, both of which depend solely on face $f$.

Here are the explicit decompositions for $X = T$ or $S$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{The $t - n$ decompositions of $T$ on edges and faces. Red blocks are associated to bubbles and green blocks for the normal traces which are redistributed to faces.}
\end{figure}
Decompositions for $\mathcal{T}$ on edge $e$ and face $f$ are

\[ \mathcal{S}^e(\mathcal{T}) := \text{span}\{n^e_t \otimes t_e, i = 1, 2\}, \]
\[ \mathcal{A}^e(\mathcal{T}) := \text{span}\{t \otimes n_f, n_f \otimes n_f - (n_f \cdot n_f) t_e \otimes t_e, i, j = 1, 2\}, \]
\[ \mathcal{F}^f(\mathcal{T}) := \text{span}\{n_f \otimes t'_1, n_f \otimes t'_2, t'_2 \otimes t'_2 - t'_1 \otimes t'_1\}, \]
\[ \mathcal{A}^f(\mathcal{T}) := \text{span}\{n^f_t \otimes n^f_t - t'_1 \otimes t'_1, t'_1 \otimes n^f_t, i = 1, 2\}. \]

The tangential component is integrated into the bubble space. As an example, consider a function $b_e n^e_t \otimes t_e \in 0(\mathcal{E}_k; r^v) \otimes \mathcal{S}^e(\mathcal{T})$. For two faces $f$ that include the edge $e$, $t_e \cdot n_f = 0$ for the other two faces $f$ that do not contain $e$, the quadratic edge bubble function $b_e$ vanishes on $f$, i.e., $b_e|_f = 0$. Consequently, $(b_e n^e_t \otimes t_e)n_{\partial T} = 0$, which falls within $0(\mathcal{E}_k; r; \mathcal{T})$. A less apparent fact is that $0(\mathcal{E}_k; r; \mathcal{T})$ defined in (21) encompasses all $\text{div}$ bubble polynomials $\mathbb{B}^\text{div}_k(\mathcal{R}; r; \mathcal{T}) \cap \ker(\text{tr}_\text{div})$.

The normal component can be reallocated to each face to enforce the desired normal continuity. Further details will be elucidated in the proof of Lemma 4.1.

Take $\mathbb{P}_k(T; \mathcal{T}) := \{ \mathcal{T} \}_{\mathcal{T}}$ as the space of shape functions. When $r^f \geq 0$, DoFs are simply tensor product of DoF$_{\mathcal{E}}(r)$ in (16) and $\mathcal{T}$. We thus focus on the case $r^f = -1$. The DoFs are

\[ \nabla^\text{v}_1(\mathcal{T}) := \text{span}\{n^f_t \otimes n^f_t, i = 1, 2\}, \]
\[ \int_{\mathcal{T}} \frac{\partial^j \tau}{\partial n_i\partial n_k} : q \, ds, \quad q \in 0_{\mathcal{E}_k}(\mathcal{E}_k; r^v - j) \otimes \mathcal{T}, 0 \leq i \leq j \leq r^e, \quad e \in \Delta_1(T), \]
\[ \int_{\mathcal{T}} (\nabla^\text{v}_1 \tau) \cdot n \, ds, \quad q \in (\mathbb{B}^\text{div}_k(\mathcal{R}; r)) \oplus \mathbb{RT}(f), f \in \Delta_2(T), \]
\[ \int_{\mathcal{T}} \tau : q \, dx, \quad q \in 0_{\mathcal{E}_k}(\mathcal{E}_k; r) \otimes \mathcal{T}. \]

**Lemma 4.1.** Let $r$ be a smoothness vector with $r^f = -1, r^v \geq 0$, and let $k \geq 2r^v + 1$. DoFs (22) are unsolvent for $\mathbb{P}_k(T; \mathcal{T})$. Given a triangulation $\mathcal{T}_h$ of $\Omega$, define $\mathbb{V}^\text{div}_k(\mathcal{R}; \mathcal{T}) := \{ \tau \in L^2(\mathcal{T}; \mathcal{T}) : \tau|_T \in 0_k(T; \mathcal{T}) \text{ for all } T \in \mathcal{T}_h, \}

and all the DoFs (22) are single-valued).

Then $\mathbb{V}^\text{div}_k(\mathcal{R}; \mathcal{T}) \subset H(\text{div}, \Omega; \mathcal{T})$.

**Proof.** First consider the case $r^e \geq 0$. The continuous element $\mathbb{V}_k(r^+; \mathcal{T})$ is determined by DoFs (22a)-(22b) plus

\[ \nabla^\text{v}_1(\mathcal{T}) := \text{span}\{n^f_t \otimes n^f_t, i = 1, 2\}, \]
\[ \int_{\mathcal{T}} \tau : q \, dx, \quad q \in 0_{\mathcal{E}_k}(\mathcal{E}_k; r^v) \otimes \mathcal{T}. \]

For (23), we decompose $\mathcal{T} = \mathcal{A}^f(\mathcal{T}) \oplus \mathcal{F}^f(\mathcal{T})$ and move $\mathbb{B}_k(f; r) \otimes \mathcal{F}^f(\mathcal{T})$ into the volume DoFs (22c) by utilizing $q \in 0_{\mathcal{E}_k}(\mathcal{E}_k; r; \mathcal{T})$. For the normal component, we employ the idea of Petrov-Galerkin method. The function $\tau$ is in the trial space containing basis $n^f_t \otimes n^f_t - t'_1 \otimes t'_1$ for which the test function could be just $n^f_t \otimes n^f_t$ as $(n^f_t \otimes n^f_t - t'_1 \otimes t'_1)n^f_t = n^f_t$, corresponding to DoF (22d). We then combine this with the other two components $t'_1 \otimes n$, i.e. DoF (22c), to determine the vector $\tau n$. 
The test function space is further decomposed, e.g. \( \mathbb{B}_k(f; r) = (\mathbb{B}_k(f; r)/\mathbb{P}_0(f)) \oplus \mathbb{P}_0(f) \) so that the average \( \int_f n^T \tau n \) is included in DoF which is crucial for the \( \text{div} \) stability. Similar modification is applied in (22c) to include \( RT(f) \) in the test function space.

Consequently, (22c)-(22e) are essentially a rearrangement of (23)-(24). The unisolvence then follows from that for tensor product spaces; see (19).

Now let us turn our attention to the case where \( r^v \geq 0, r^e = -1, r^f = -1 \), and \( r_+ = (r^v, 0, 0)^T \). The set of DoFs \( \text{DoF}_k(r_+) \otimes T \) includes vertex DoF (22a), volume DoF (24), as well as the following edge and face DoFs:

\[
\int_e \tau : q \, ds, \quad q \in \mathbb{B}_k(e; r^v) \otimes T, \quad e \in \Delta_1(T),
\]

\[
\int_f \tau : q \, dS, \quad q \in \mathbb{B}_k(f; \begin{pmatrix} r^v \\ 0 \end{pmatrix}) \otimes T, \quad f \in \Delta_2(T).
\]

As previously mentioned, on each edge \( e \), we employ the frame \( \{ n_{f_1}, n_{f_2}, t_e \} \) where \( f_1, f_2 \) are two faces containing \( e \). The tangential component \( \mathbb{B}_k(e; r^v) \otimes \mathcal{F}^v(T) \) is moved into the bubble space \( \mathbb{B}_k^{\text{div}}(r; T) \). The normal components will be redistributed to the two faces \( f_i, i = 1, 2 \), containing \( e \). More precisely, we can first modify the DoF (25) with \( q \in \mathbb{B}_k(e; r^v) \otimes \mathcal{N}^v(T) \) to

\[
\int_e (\tau n_{f_i})|_e \cdot q \quad \text{for} \quad q \in \mathbb{B}_k^3(e; r^v), \quad i = 1, 2.
\]

Then redistribute this edge DoF to the face \( f_i, i = 1, 2 \) containing \( e \):

\[
\int_e (\tau n_{f_i})|_e \cdot q \rightarrow \int_e (\tau n_{f_i})|_{f_i} \cdot q \quad \text{for} \quad q \in \mathbb{B}_k^3(e; r^v), \quad i = 1, 2,
\]

so that in (26)

\[
\mathbb{B}_k(f; \begin{pmatrix} r^v \\ 0 \end{pmatrix}) \oplus e \in \Delta_1(f) \mathbb{B}_k(e; r^v) = \mathbb{B}_k(f; \begin{pmatrix} r^v \\ -1 \end{pmatrix})
\]

for \( r^v \geq 0, r^e = -1 \), which leads to the establishment of (22c)-(22d). Thus, the unisolvence is proven.

The conclusion \( \mathcal{V}_k^{\text{div}}(r; T) \subset H(\text{div}; \Omega; T) \) is obvious as \( \tau n \) is continuous on each face \( f \) due to the single-valued DoFs (22a)-(22d). \( \square \)

When \( r^v = 0 \) and \( r^e = r^f = -1 \), the element \( \mathcal{V}_k^{\text{div}}(r; T) \) exhibits continuity at vertices, which is reminiscent of the Stenberg element [30] designed for \( H(\text{div}) \)-conforming vector functions. If \( \tau \in \mathcal{M} \) at vertices, the vertex DoFs can be further reorganized across each face, yielding the second family of \( \text{Nédélec} \) face elements [27] or Brezzi-Douglas-Marini (BDM) face element [9].

The construction of an \( H(\text{div}; S) \)-conforming element follows a similar approach, albeit with additional complexities introduced by \( \mathcal{N}^v(S) \). Decompositions on edge \( e \) and \( f \) are

\[
\mathcal{I}^e(S) := \text{span} \{ t_e \otimes t_e \},
\]

\[
\mathcal{N}^v(S) := \text{span} \{ \text{sym}(t_e \otimes n_{f_i}), i = 1, 2 \} \oplus \text{span} \{ \text{sym}(n_{f_i}^e \otimes n_{f_j}^e), 1 \leq i \leq j \leq 2 \},
\]

\[
\mathcal{F}^f(S) := \text{span} \{ \text{sym}(t_i^e \otimes t_j^e), 1 \leq i \leq j \leq 2 \},
\]

\[
\mathcal{N}^f(S) := \text{span} \{ n_i^e \otimes n_j^e, \text{sym}(t_i^e \otimes n_j^e), i = 1, 2 \}.
\]

Once again, the tangential component will be incorporated into the bubble space. However, the redistribution of certain normal components to faces might be constrained by symmetry conditions. On an edge \( e \), for instance, the symmetry constraint demands that the normal plane of \( e \) must obey \( \text{span} \{ \text{sym}(n_{f_i}^e \otimes n_{f_j}^e), 1 \leq i \leq j \leq 2 \} \), which is a global
requirement, indicating that the two normal vectors \( \{ n_1^e, n_2^e \} \) are independent of the elements containing \( e \). We refer to the blue blocks in Figure 4 for clarification. Conversely, in \( \mathcal{N}(T) \), all the components can be effectively redistributed to faces, as demonstrated by the green blocks in Figure 4.

Take \( \mathbb{P}_k(T; S) \) as the space of shape functions. Again we focus on the case \( r^f = -1 \).

The DoFs are

\[
(27a) \quad \nabla^i \tau(v), \quad i = 0, \ldots, r^\nu, v \in \Delta_0(T),
\]

\[
(27b) \quad \int_e \frac{\partial^j \tau}{\partial n_1^{j-1} n_2^{j-1}} : q \, ds, \quad q \in \mathbb{B}_k(e; r^\nu - j) \otimes S, 0 \leq i \leq j \leq r^\nu, e \in \Delta_1(T),
\]

\[
(27c) \quad \int_e (n_1^e \tau n_j) \, dq, \quad q \in \mathbb{B}_k(e; r^\nu), 1 \leq i \leq j \leq 2, e \in \Delta_1(T), \text{ if } r^\nu = -1,
\]

\[
(27d) \quad \int_f (\Pi_f \tau n) \cdot q \, dS, \quad q \in (\mathbb{B}_k^{\text{div}}(f; r) / RM(f)) \oplus RM(f), f \in \Delta_2(T),
\]

\[
(27e) \quad \int_f (n_1^f \tau n) \, dq, \quad q \in (\mathbb{B}_k(f; r_+) / \mathbb{P}_1(f)) \oplus \mathbb{P}_1(f), f \in \Delta_2(T),
\]

\[
(27f) \quad \int_T \tau : q \, dx, \quad q \in \mathbb{B}_k^{\text{div}}(r; S).
\]

**Lemma 4.2.** Let \( r \) be a valid smoothness vector with \( r^f = -1, r^\nu \geq 0 \), and let \( k \geq 2r^\nu + 1 \). DoFs (27) are unisolvent for \( \mathbb{P}_k(T; S) \). Given a triangulation \( T_h \) of \( \Omega \), define

\[
\forall_k^{\text{div}}(r; S) := \{ \tau \in L^2(\Omega; S) : \tau|_T \in \mathbb{P}_k(T; S) \text{ for all } T \in T_h, \quad \text{and all the DoFs (27) are single-valued} \}.
\]

Then \( \forall_k^{\text{div}}(r; S) \subset H(\text{div}, \Omega; S) \).

**Proof.** Certainly, the core approach of the proof aligns with that of Lemma 4.1. We will highlight the differences here. The case where \( r^\nu \geq 0 \) remains unchanged. When \( r^\nu = -1 \), the components \( t_e \otimes n_{f_i} \) can be redistributed to faces, resulting in the expression:

\[
\mathbb{B}_k^{\text{div}}(f; r) = \mathbb{B}_k^2(f; r_+) \bigoplus_{e \in \Delta_1(f)} \mathbb{B}_k(e, r^\nu) t_e,
\]
which leads to the form in (27d). The components $\mathbf{n}^e \otimes \mathbf{n}^e$ cannot be redistributed to faces and are preserved in (27c). Therefore, in (27e), the notation $\mathbf{r}^+$ is still retained in $\mathcal{B}_k(f; \mathbf{r}^+)$ while in (22d), $\mathcal{B}_k(f; \mathbf{r})$ is used.

Remark 4.3. When $r^e = -1$, if we do not redistribute the tangential-normal component of edge DoFs (i.e. change green blocks in Figure 4 (a) to blue), we obtain the Hu-Zhang element [26]

\begin{align}
\nabla^i \tau(v), & \quad i = 0, \ldots, r^e, v \in \Delta_0(T), \\
\int_e (n^e_i \mathbf{n}^e_j) q \, ds, & \quad q \in \mathcal{B}_k(e; \mathbf{r}^e), 1 \leq i \leq j \leq 2, e \in \Delta_1(T), \\
\int_e (\mathbf{t}^e_i \mathbf{n}^e_j) q \, ds, & \quad q \in \mathcal{B}_k(e; \mathbf{r}^e), j = 1, 2, e \in \Delta_1(T), \\
\int_f (\Pi_f \mathbf{n}) \cdot \mathbf{q} \, dS, & \quad \mathbf{q} \in \mathcal{B}_k^2(f; \mathbf{r}^+), f \in \Delta_2(T), \\
\int_f (\mathbf{n}^e_i \mathbf{r}^e) q \, dS, & \quad q \in \mathcal{B}_k(f; \mathbf{r}^+), f \in \Delta_2(T), \\
\int_T \mathbf{r} : \mathbf{q} \, dx, & \quad \mathbf{q} \in \mathcal{B}_k^{\text{div}}(r; \mathcal{S}).
\end{align}

The unisolvence can be proved similarly as DoFs (28c)-(28d) are just rearrangement of the tangential-normal component (27d). We prefer (27d) as it is more close to the vector face elements.

Remark 4.4. The continuity at vertices is enforced due to the constraints—traceless conditions in $T$ or symmetry conditions in $S$. This constraint-driven continuity at vertices cannot be relaxed. To elucidate, let $v_0, v_1, v_2, v_3$ be the four vertices of a tetrahedron $T$, with corresponding barycentric coordinates $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. Selecting $v_0$ as the origin, we define $t_{0i} := v_i - v_0$ for $i = 1, 2, 3$, which serve as three basis vectors. For a smooth traceless tensor $\tau$, due to the duality between $\{t_{01}, t_{02}, t_{03}\}$ and $\{\nabla \lambda_1, \nabla \lambda_2, \nabla \lambda_3\}$, we can represent

$$\tau(v_0) = \sum_{i=0}^{3} (\tau \nabla \lambda_i)|_{f_i}(v_0) t_{0i}^T.$$  

The traceless property of $\tau$ implies

$$\sum_{i=0}^{3} t_{0i}^T (\tau \nabla \lambda_i)|_{f_i}(v_0) = 0,$n
indicating that $(\tau n_1)|_{f_1}, (\tau n_2)|_{f_2}$, and $(\tau n_3)|_{f_3}$ are linearly dependent at vertex $v_0$. Consequently, the vertex DoFs in equation (22a) cannot be reallocated to the faces, which underscores the inalterable nature of the constraint $r^e \geq 0$.

4.2. Div stability. Due to the similarity, we use $H(\text{div}; \mathcal{S})$-conforming finite element to illustrate the BGG procedure. Consider the diagram

$$
\begin{array}{c}
\Psi_{k+1}^{\text{div}}(r + 1; \mathcal{M}) \xrightarrow{\text{div}} \Psi_k^{L^2}(r; \mathbb{R}^3) \xrightarrow{0} \\
\Psi_{k+1}^{\text{curl}}(r + 1; \mathcal{M}) \xrightarrow{\text{curl} - 2 \text{vschw}} \Psi_k^{\text{div}}(r; \mathcal{M}) \xrightarrow{\text{div}} \Psi_{k-1}^{L^2}(r \otimes 1; \mathbb{R}^3) \xrightarrow{0}.
\end{array}
$$
We require that both \((r + 1, r, k + 1)\) and \((r, r ⊕ 1, k)\) are div stable. Then
\[
\begin{align*}
  r + 1 & \geq \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \\
  r & \geq \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \\
  r ⊕ 1 & \geq \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.
\end{align*}
\]
In particular \(r^c \geq 0\).

**Lemma 4.5.** Let \(r\) be a smoothness vector and \(k\) large enough satisfying: both \((r + 1, r, k + 1)\) and \((r, r ⊕ 1, k)\) are div stable. Then we have the \((\text{div}; S)\) stability
\[
\text{div } V^\text{div}_k(r; S) = V^L_{k-1}(r ⊕ 1; \mathbb{R}^3).
\]

**Proof.** As \(r + 1 \geq 0\), both \(V^\text{div}_{k+1}(r + 1; \mathbb{M}) = V^\text{curl}_{k+1}(r + 1; \mathbb{M}) = V^\text{grad}_{k+1}(r + 1) \otimes \mathbb{M}\). Therefore, \(S\) is one-to-one.

Given \(u \in V^L_k(r; \mathbb{R}^3)\), since \((r + 1, r, k + 1)\) is div stable, we can find \(σ \in V^\text{div}_{k+1}(r + 1; \mathbb{M})\) such that \(\text{div } σ = u\). Then, by defining \(τ := \text{curl} S^{-1}σ\), we have \(τ \in V^\text{div}_k(r; \mathbb{M})\) and
\[
2 \text{vskw } τ = 2 \text{vskw } \text{curl } (S^{-1}σ) = \text{div } S(S^{-1}σ) = u.
\]
Thus, \(\text{vskw} : V^\text{div}_k(r; \mathbb{M}) \to V^L_k(r; \mathbb{R}^3)\) is surjective.

We can now apply the BGG construction to conclude that \(\text{div} : V^\text{div}_k(r; \mathbb{M}) \cap \ker(\text{vskw}) \to V^L_{k-1}(r ⊕ 1; \mathbb{R}^3)\) is surjective. Our next step is to establish the relationship
\[
(29) \quad V^\text{div}_k(r; S) = V^\text{div}_k(r; \mathbb{M}) \cap \ker(\text{vskw}).
\]
Namely, we need to show that the subspace \(V^\text{div}_k(r; \mathbb{M}) \cap \ker(\text{vskw})\) derived via BGG corresponds to the constructive finite element space \(V^\text{div}_k(r; S)\).

The inclusion \(V^\text{div}_k(r; S) \subseteq V^\text{div}_k(r; \mathbb{M}) \cap \ker(\text{vskw})\) is evident. To establish their equality, it suffices to demonstrate that
\[
\dim V^\text{div}_k(r; S) = \dim (V^\text{div}_k(r; \mathbb{M}) \cap \ker(\text{vskw})),
\]
which is equivalent to showing
\[
(30) \quad \dim V^\text{div}_k(r; \mathbb{M}) - \dim V^\text{div}_k(r; S) = \dim V^L_k(r; \mathbb{R}^3),
\]
since we have proved that \(\text{vskw} : V^\text{div}_k(r; \mathbb{M}) \to V^L_k(r; \mathbb{R}^3)\) is surjective.

In the case where \(r^f ≥ 0\), we have \(V^\text{div}_k(r; \mathbb{X}) = V^\text{div}_k(r; \mathbb{X}) \otimes \mathbb{X}\) for \(\mathbb{X} = \mathbb{M}, S,\) or \(\mathbb{R}^3\). Consequently, \((30)\) trivially holds. Let us now consider the case where \(r^f = -1\) and \(r^c ≥ 0\). For the vertex and edge DoFs \((27a)-(27b)\), we find that \(\dim \mathbb{M} - \dim S = \dim \mathbb{R}^3\). The face DoFs \((27d)-(27e)\) remain the same. The only remaining dimension change is within the bubble spaces, and this can be computed as follows:
\[
\begin{align*}
  \dim B^\text{div}_k(r; \mathbb{M}) - \dim B^\text{div}_k(r; S) \\
  = \dim B_k(r_+; \mathbb{M}) - \dim B_k(r_+; S) \\
  & + 4 \dim B_k(f; r) \otimes \mathcal{F}(\mathbb{M}) - 4 \dim B_k(f; r) \otimes \mathcal{F}(S) \\
  = 3(\dim B_k(r_+) + 4 \dim B_k(f; r)) \\
  = \dim B_k(r; \mathbb{R}^3).
\end{align*}
\]
Hence, \((30)\) holds, and consequently, \((29)\) is confirmed.

**Remark 4.6.** Discussion on \((\text{div}; T)\) stability is similar. The dimension identity for traceless matrices
\[
\dim V^\text{div}_{k-1}(r; \mathbb{M}) - \dim V^\text{div}_{k-1}(r; T) = \dim V^L_k(r)
\]
Corollary 4.7. Assume the smoothness vector $r$ and polynomial degree $k$ satisfy either:

1. $r^\tau \geq 1$, $r^e \geq 0$, both $(r+1, r, k+1)$ and $(r, r \ominus 1, k)$ are div stable, or
2. $r^\tau \geq 0$, $r^e = r^f = -1$,

\[ k \geq \max\{2r^\tau + 1, 4\} \text{ for } S, \begin{cases} k \geq \max\{2r^\tau + 1, 4\}, & \text{if } r^\tau \geq 1, \text{ for } T, \\ k \geq 2, & \text{if } r^\tau = 0 \end{cases} \]

Then we have the (div; $X$) stability, for $X = S$ or $T$,

\[ \text{(31)} \quad \text{div } \mathcal{V}^\text{div}_k(r; S) = \mathcal{V}^2_{k-1}(r \ominus 1; \mathbb{R}^3). \]

When (31) is satisfied, we will refer to the triple $(r, r \ominus 1, k)$ as being (div; $X$) stable. The conditions presented in Corollary 4.7 are sufficient to establish this stability, although they might not be necessary in all cases. It is important to note that due to the redistribution of edge DoFs to faces, when $r^\tau = 0$, for $X = T$, the face DoFs (22c)-(22d) include $P_1(f)$ for $k \geq 2$, which is necessary to prove the div stability. On the other hand, for $X = S$, $k \geq 4$ is required, since the face DoF (27e) contains a smaller face bubble $B_k(f; r_\perp)$ that demands higher-order continuity.

4.3. Inequality constraints. We can apply one $\ominus$ operation to get the div stability with an inequality constraint.

Corollary 4.8. Let $(r_2, r_2 \ominus 1, k)$ be (div; $X$) stable and $r_3 \geq r_2 \ominus 1$. Define

\[ \mathcal{V}^\text{div}_k(r_2, r_3; X) = \{ \tau \in \mathcal{V}^\text{div}_k(r_2; X) : \text{div } \tau \in \mathcal{V}^{L^2}_{k-1}(r_3; \mathbb{R}^3) \}. \]

Then we also have the (div; $X$) stability

\[ \text{div } \mathcal{V}^\text{div}_k(r_2, r_3; X) = \mathcal{V}^{L^2}_{k-1}(r_3; \mathbb{R}^3). \]

The subspace $\mathcal{V}^\text{div}_k(r_2, r_3; X)$ always exists by definition. However, the central challenge lies in formulating local DoFs for this subspace. In this pursuit, we draw insights from our recent work, as outlined in [14, Section 4.4]. We add DoFs to determine div $\tau$ first but remove non-free index (white blocks in Figure 3 and 4) in the $t-n$ decomposition. For example, for face DoFs, we remove component $\partial_n(t_1 \tau t_1)$ from vector $\partial_n(t_1 \tau t_1)$ as $t_1 \otimes t_1 \not\in \mathcal{T}(T)$. Similarly remove $t^T \tau t$ from the edge DoF.
Lemma 4.9. Let \( (r_2, r_2 \oplus 1, k) \) be \((\text{div;} S)\) stable and \( r_3 \geq r_2 \oplus 1 \). The DoFs (32) are uni-solvent for \( \mathbb{P}_k(T; S) \).

Proof. The introduced DoFs given by (32b), (32h), and (32i)-(32m) play a crucial role in characterizing the divergence of \( \tau \). The total number of these DoFs, along with (32a), is independent of \( r_3 \), specifically given by the expression:

\[
6 \left( \frac{r_2^\gamma + 3}{3} \right) + \dim \mathbb{P}^3_{k-1}(T) - \dim \text{RM} - 3 \left( \frac{r_2^\gamma + 2}{3} \right).
\]

This count remains unaffected by variations in \( r_3 \). For convenience, we can proceed with \( r_3 = r_2 \oplus 1 \). Then by making comparisons with (27), we deduce that the number of DoFs (32) is equal to \( \dim \mathbb{P}_k(T; S) \).

Suppose we have \( \tau \in \mathbb{P}_k(T; S) \) satisfying the vanishing conditions for all DoFs (32). It then follows that \( \tau n|_{\partial T} = 0 \). By applying integration by parts and utilizing the vanishing

\[
\gamma \left( \frac{r_2^\gamma + 3}{3} \right) + \dim \mathbb{P}^3_{k-1}(T) - \dim \text{RM} - 3 \left( \frac{r_2^\gamma + 2}{3} \right).
\]
DoFs (32i)-(32j), we deduce the critical relation:

\[
\int_T (\text{div} \, \mathbf{\tau}) \cdot \mathbf{q} \, dx = 0, \quad \mathbf{q} \in \mathbb{R}M.
\]

This result, combined with DoFs (32b), (32h), and (32l)-(32m), which pertain to the divergence of \( \mathbf{\tau} \), leads to the conclusion that \( \text{div} \, \mathbf{\tau} = 0 \).

The situation is somewhat analogous when dealing with edges. By expressing \( \text{div} \) in the frame \( \{ \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2 \} \), we uncover a representation of \( \text{div} \, \mathbf{\tau} \) that encompasses the partial derivatives along tangential (\( \mathbf{t} \)) and normal (\( \mathbf{n}_1, \mathbf{n}_2 \)) directions:

\[
\text{div} \, \mathbf{\tau} = \partial_t (\mathbf{\tau t}) + \partial_{n_1} (\mathbf{\tau n}_1) + \partial_{n_2} (\mathbf{\tau n}_2).
\]

Taking into account (32a)-(32h), it becomes evident that \( \nabla^j \mathbf{\tau} \) vanishes along edges, where \( 0 \leq j \leq r_2 \). A similar reasoning applies to faces, where the decomposition into \( \{ \mathbf{n}, \mathbf{t}_1, \mathbf{t}_2 \} \) aids in expressing

\[
\text{div} \, \mathbf{\tau} = \partial_n (\mathbf{\tau n}) + \partial_{t_1} (\mathbf{\tau t}_1) + \partial_{t_2} (\mathbf{\tau t}_2),
\]

and highlighting that \( \nabla^j \mathbf{\tau} \) vanishes along faces for \( 0 \leq j \leq r_f^j \). This collective analysis demonstrates that \( \mathbf{\tau} \in B^{\text{div}}_{k} (r_2; S) \cap \ker(\text{div}) \). This observation, coupled with DoF (32n), solidifies the conclusion that \( \mathbf{\tau} = 0 \).

Example 4.10. The space \( \mathbb{V}^{\text{div}}_k (r_2, r_3; S) \) for \( r_2 = r_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \) and \( k \geq 4 \) has been constructed recently in [23].

4.4. \( H(\text{div} \, \text{div}^+; S) \) and \( H(\text{div} \, \text{div}; S) \) elements. In this subsection, we proceed to construct various \( H(\text{div} \, \text{div}) \)-conforming finite elements characterized by a smoothness vector \( r \). Define the spaces:

\[
H(\text{div} \, \text{div}; S) := \{ \mathbf{\tau} \in L^2(\Omega; S) : \text{div} \, \mathbf{\tau} \in L^2(\Omega) \},
\]

\[
H(\text{div} \, \text{div}^+; S) := \{ \mathbf{\tau} \in L^2(\Omega; S) : \text{div} \, \mathbf{\tau} \in H(\text{div}, \Omega) \}.
\]

It is evident that the inclusion \( H(\text{div} \, \text{div}^+; S) \subset H(\text{div} \, \text{div}; S) \) holds.

We will now proceed to construct finite elements that are \( H(\text{div} \, \text{div}^+; S) \)-conforming. Specifically, when \( r^j \geq 1 \), we define \( \mathbb{V}^{\text{div}}_k (r; S) \) as \( \mathbb{V}_k(r) \otimes S \). For the cases where \( r^j = -1 \) or \( r^j = 0 \), we will make use of a recent approach presented in [25] and [16]. The space of shape functions is still \( \mathbb{P}_k(T; S) \). By modifying the DoFs (27a)-(27f), which are originally designed for \( H(\text{div}; S) \)-conforming finite elements, we ensure that \( \text{div} \, \mathbf{\tau} \) belongs to \( \mathbb{V}^{\text{div}}_{k-1} (r \otimes 1) \), satisfying the \( H(\text{div}; S) \)-conforming conditions.
Lemma 4.11. The DoFs are

\[ (33a) \quad \nabla^j \tau (v), \quad j = 0, 1, \ldots, r^v, \]
\[ (33b) \quad \int_C \frac{\partial^j \tau}{\partial n^i} : q \, ds, \quad q \in B_{k-j}(e; r^v - j) \otimes S, 0 \leq i \leq j \leq r^e, \]
\[ (33c) \quad \int_C (n^i \tau n_j) q \, ds, \quad q \in B_k(e; r^v), 1 \leq i \leq j \leq 2, \text{ if } r^e = -1, \]
\[ (33d) \quad \int_f \tau : q \, dS, \quad q \in B_k(f; r) \otimes S, \text{ if } r^f = 0, \]
\[ (33e) \quad \int_f (\Pi_f \tau n) \cdot q \, dS, \quad q \in B_k^{\text{div}}(f; r) / RM(f) \oplus RM(f), \text{ if } r^f = -1, \]
\[ (33f) \quad \int_f (n^i \tau n_j) q \, dS, \quad q \in B_k(f; r_+) / P_1(f) \oplus P_1(f), \text{ if } r^f = -1, \]
\[ (33g) \quad \int_f \text{div} \tau q \, dS, \quad q \in B_{k-1}(f; r \ominus 1), \]
\[ (33h) \quad \int_T (\text{div} \tau) \cdot q \, dx, \quad q \in B_k^{\text{div}}(r \ominus 1) / RM, \]
\[ (33i) \quad \int_T \tau : q \, dx, \quad q \in B_k^{\text{div}}(r; S) \cap \ker(\text{div}), \]

for each \( v \in \Delta_0(T), e \in \Delta_1(T) \) and \( f \in \Delta_2(T) \).

**Lemma 4.11.** The DoFs (33) are uni-solvent for \( F_k(T; S) \).

**Proof.** We first consider the case when \( r^f = -1 \). If we compare the DoFs (27a)-(27f) for constructing \( V_k^{\text{div}}(r; S) \) with the DoFs required for \( H(\text{div} \text{div}^+; S) \)-conforming finite elements, the primary distinction lies in the volume DoF (27f) for \( B_k^{\text{div}}(r; S) \). In the new context, this particular DoF is replaced by three alternative DoFs: (33g), (33h), and (33i).

Let \( E_0 = B_k^{\text{div}}(r; S) \cap \ker(\text{div}) \) and let \( E_{0}^{\perp} \) denote its \( L^2 \)-orthogonal complement of \( B_k^{\text{div}}(r; S) \). By performing a decomposition of the dual space, we arrive at:

\[ (B_k^{\text{div}}(r; S))' = (E_0)' \oplus (E_0^{\perp})'. \]

DoF (33i) is exactly a basis of \( (E_0)' \). The subspace \( \text{div} E_{0}^{\perp} \) can be uniquely determined through the DoFs (33g) and (33h), both of which are consistent with the requirements for constructing \( H(\text{div}) \)-conforming elements.

As we count the dimensions, it is essential to note that \( \text{div} B_k^{\text{div}}(r; S) = B_{k-1}^{\text{div}}(r \ominus 1) / RM \). The difference in the number of DoFs between (27f) and the newly introduced DoFs (33g)-(33i) is given by:

\[ \dim B_{k-1}^{\text{div}}(T; r \ominus 1) - \dim B_k^{\text{div}}(T; r \ominus 1) - 4 \dim B_{k-1}^{\perp}(f; r \ominus 1) = 0. \]

So the sum of number of DoFs (33) is equal to \( P_k(T; S) \).

Now, let \( \tau \in F_k(T; S) \) and assume that all the DoFs (33a)-(33i) vanish. Due to the vanishing DoFs (33a)-(33b) and (33g), we can infer that \( \text{div} \tau \in B_{k-1}^{\text{div}}(r \ominus 1) \). By considering the vanishing DoFs (33d)-(33f) and (33h), we deduce that \( \text{div} \tau = 0 \). Finally, utilizing the uniqueness of the DoFs (27a)-(27f) for \( H(\text{div}; S) \)-conforming finite elements, we conclude that \( \tau = 0 \).

This completes the explanation of the construction for \( H(\text{div} \text{div}^+; S) \)-conforming finite elements for the case \( r^f = -1 \). The case for \( r^f = 0 \) follows a similar logic, with the primary difference being in the structure of the bubble space \( B_k^{\text{div}}(r; S) = B_k(r) \otimes S \). \( \square \)
When \( r^f = 1 \), \( \mathcal{V}_k^{\div \div^+}(r; S) = \mathcal{V}(r; S) \otimes S \). When \( r^f = -1, 0 \), define
\[
\mathcal{V}_k^{\div \div^+}(r; S) = \{ \tau \in L^2(\Omega; S) : \tau|_T \in P_k(T; S) \text{ for all } T \in \mathcal{T}_h, \\
\text{and all the DoFs (33) are single-valued} \}.
\]

Due to DoFs (33d)-(33g), \( \mathcal{V}_k^{\div \div^+}(r; S) \subset H(\text{div} \ \text{div}^+; S) \).

Next we use the following BGG diagram
\[
\begin{array}{cccc}
\mathcal{V}_k^{\div \div^+}(r; S) & \overset{\text{id}}{\longrightarrow} & \mathcal{V}_{k-1}(r \ominus 1) & \overset{0}{\longrightarrow} \\
\mathcal{V}_{k-1}(r \ominus 1) & \overset{\text{id}}{\longrightarrow} & \mathcal{V}^L_{k-2}(r \ominus 2) & \overset{0}{\longrightarrow}
\end{array}
\]
to prove the divdiv stability.

**Lemma 4.12.** Assume \((r, r \ominus 1, k)\) is \((\text{div}; S)\) stable, and \(r^f = -1, 0\). It holds that
\[
\text{div} \ \mathcal{V}_k^{\div \div^+}(r; S) = \mathcal{V}_{k-1}(r \ominus 1).
\]

**Proof.** Clearly \( \text{div} \ \mathcal{V}_k^{\div \div^+}(r; S) \subseteq \mathcal{V}_{k-1}^{\div}(r \ominus 1) \), then it suffices to count the dimensions. Both \( \dim \mathcal{V}_k^{\div}(r; S) - \dim \mathcal{V}_{k-1}^{\div}(r \ominus 1) \) and \( \dim \mathcal{V}^L_{k-2}(r \ominus 1; \mathbb{R}^3) - \dim \mathcal{V}_{k-1}^{\div}(r \ominus 1) \) equal
\[
(4|\Delta_3(\mathcal{T}_h)| - |\Delta_2(\mathcal{T}_h)|) \dim \mathbb{B}_{k-1}(f; r \ominus 1),
\]
that is
\[
\dim \mathcal{V}_k^{\div}(r; S) - \dim \mathcal{V}_{k-1}^{\div}(r \ominus 1) \overset{\text{id}}{\longrightarrow} \dim \mathcal{V}^L_{k-2}(r \ominus 1; \mathbb{R}^3) - \dim \mathcal{V}_{k-1}^{\div}(r \ominus 1).
\]

As \( \text{div}(\mathcal{V}_k^{\div}(r; S)) = \mathcal{V}^L_{k-1}(r \ominus 1; \mathbb{R}^3) \) and the modification will not change \( \ker(\text{div}) \), we get \( \dim \text{div} \ \mathcal{V}_k^{\div \div^+}(r; S) = \dim \mathcal{V}_{k-1}^{\div}(r \ominus 1) \) and (34) follows. \( \square \)

Combing the div stability for \((r \ominus 1, r \ominus 2, k - 1)\), we conclude the divdiv stability.

**Corollary 4.13.** Assume \((r, r \ominus 1, k)\) is \((\text{div}; S)\) stable and \((r \ominus 1, r \ominus 2, k - 1)\) is div stable. Then it holds that
\[
\text{div} \ \text{div} \ \mathcal{V}_k^{\div \div^+}(r; S) = \mathcal{V}^L_{k-2}(r \ominus 2).
\]

Next we modify the DoFs (33) slightly to get an \( H(\text{div} \ \text{div}; S)\)-conforming element. Take \( \mathcal{P}_k(T; S) \) as the space of shape functions. When \( r^f \geq 1 \), define \( \mathcal{V}_k^{\div \div}(r; S) := \)
The modification is introduced in (35f), where we now enforce the continuity condition:

\[ \Pi \]

associated with \( \Pi \) different elements containing the shared face contributes to the divdiv bubble space. This implies that (35i) can take different values in \( V \) instead of enforcing continuity for both \( \Pi \), since the face DoF (35i) associated with \( \Pi \) is zero in the bubble space. This implies that (35i) can take different values in different elements containing the shared face \( f \).

For convenience, we introduce the following notation:

The modification is introduced in (35f), where we now enforce the continuity condition:

\[ \text{tr}_2^{\text{div div}}(\tau) = n^\top \text{div} \tau + \text{div} f(\Pi f \tau n) \]

instead of enforcing continuity for both \( n^\top \text{div} \tau \) and \( \Pi f \tau n \). Notably, the face DoF (35i) associated with \( \Pi f \tau n \) has been moved to the end to signify its role as a local DoF that contributes to the divdiv bubble space. This implies that (35i) can take different values in different elements containing the shared face \( f \).

For convenience, we introduce the following notation:

By definition, we have the inclusion relationship:

\[ B_k^{\text{div div}}(r; S) \subset B_k^{\text{div div}}(r; S), \]

since the face DoF (35i) associated with \( \Pi f \tau n \) is zero in the bubble space \( B_k^{\text{div div}}(r; S) \) and non-zero in \( B_k^{\text{div div}}(r; S) \).

Defining the spaces:

we find that for the case \( r^f = 0 \), \( r_k^{\text{div div}}(r; S) \) and \( r_k^{\text{div div}+}(r; S) \) are indistinguishable. However, in the scenario where \( r^f = -1 \), again due to (35i), we can deduce that:

Consequently, this inclusion relationship ensures the divdiv stability.
Corollary 4.14. Assume $(r, r \oplus 1, k)$ is $(\text{div}; S)$ stable and $(r \oplus 1, r \oplus 2, k - 1)$ is div stable. Then it holds that

$$\text{div div } V_k^{\text{div}}(r; S) = V_{k-2}^{L^2}(r \oplus 2).$$

In a similar vein to the inequality constraint that ensures $(\text{div}; S)$ stability, we also have a comparable flexibility when it comes to div div elements. Consider the scenario where $(r_2, r_3, k)$ are $(\text{div}; S)$ stable, where $r_3 \geq r_2 \oplus 1$, and further assume that $(r_3, r_3 \oplus 1, k - 1)$ exhibit div stability. In such cases, following the diagram

$$V_k^{\text{div}}(r_2, r_3; S) \xrightarrow{\text{div}} V_{k-1}^{\text{div}}(r_3) \xrightarrow{\text{div}} V_{k-2}^{L^2}(r_3 \oplus 1) \rightarrow 0,$$

and the approach in Section 4.3, we can similarly define the spaces $V_k^{\text{div div}^+}(r_2, r_3 \oplus 1; S)$ and $V_k^{\text{div div}^-}(r_2, r_3 \oplus 1; S)$, with the constraint $r_3 \geq r_2 \oplus 1$.

Regarding the changes in DoFs, when $r_3^f \geq 0$, $V_{k-1}^{\text{div}}(r_3) = V_{k-3}^{\text{grad}}(r_3; \mathbb{R}^3)$. This implies that $V_k^{\text{div div}^+}(r_2, r_3 \oplus 1; S) = V_k^{\text{div div}^-}(r_2, r_3; S)$, and therefore, there is no need to modify the DoFs (32). On the other hand, when $r_3^f = -1$, we partition the DoF (32m) associated with the bubble component of div $\tau$ into:

$$(36a) \quad \int_f n^T \text{div } \tau \, q \, dS, \quad q \in \mathbb{B}_{k-1}(r_3), f \in \Delta_2(T),$$

$$(36b) \quad \int_T (\text{div } \tau) \cdot q \, dx, \quad q \in \mathbb{B}_{k-1}^{\text{div}}(r_3) / \mathbb{R}M,$$

while retaining all other DoFs from (32). This gives DoFs for $V_k^{\text{div div}^+}(r_2, r_3 \oplus 1; S)$.

To construct $V_k^{\text{div div}}(r_2, r_3 \oplus 1; S)$, we further replace DoFs (36a) by

$$\int_f \text{tr} \, V_k^{\text{div div}}(r_2, r_3 \oplus 1; S) \rightarrow 0,$$

and treat (32i) local. The procedure is the same as before and thus the details are skipped.

Corollary 4.15. Let $(r_2, r_3, k)$ be $(\text{div}; S)$ stable with $r_3 \geq r_2 \oplus 1$ and $(r_3, r_3 \oplus 1, k - 1)$ be div stable. Then it holds that

$$\text{div div } V_k^{\text{div div}^+}(r_2, r_3 \oplus 1; S) = \text{div div } V_k^{\text{div div}}(r_2, r_3 \oplus 1; S) = V_{k-2}^{L^2}(r_3 \oplus 1).$$

Example 4.16. The space $V_k^{\text{div div}^+}(r_2, r_3 \oplus 1; S)$ for $r_2 = r_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ and $k \geq 4$ has been constructed recently in [23].

5. Finite Element Complexes

In the preceding section, our focus was primarily on the last two columns within the diagram (6), where we established the $(\text{div}; X)$ stability. In this section, our attention turns to the first three columns of the diagram (6) to derive finite element complexes.

5.1. Finite element Hessian complexes. Let

$$r_0 \geq 1, \quad r_1 = r_0 - 2, \quad r_2 = r_1 \oplus 1, \quad r_3 = r_2 \oplus 1.$$
Assume both $\mathbf{r}_0, \mathbf{r}_0 - 1, \mathbf{r}_1, \mathbf{r}_2$ and $(\mathbf{r}_0 - 1, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ are valid de Rham smoothness sequences. We will justify the following BGG diagram

\[
\begin{align*}
\nabla_{k+2}^{\text{grad}} & \xrightarrow{\text{grad}} \nabla_{k+1}^{\text{curl}}(\mathbf{r}_0 - 1) & \nabla_{k}^{\text{curl}}(\mathbf{r}_1) & \nabla_{k-1}^{\text{div}}(\mathbf{r}_2; \mathbb{T}) & \xrightarrow{\text{div}} 0 \\
\nabla_{k+1}^{\text{grad}}(\mathbf{r}_0 - 1; \mathbb{R}^3) & \xrightarrow{\text{grad}} \nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M}) & \nabla_{k-1}^{\text{div}}(\mathbf{r}_2; \mathbb{T}) & \xrightarrow{\text{div}} 0,
\end{align*}
\]

where $\nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M}) = \{ \tau \in \nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M}) : \nabla^{\text{curl}} \tau \in \nabla_{k-1}^{\text{div}}(\mathbf{r}_2; \mathbb{T}) \cap \ker(\text{div}) \}$.

**Lemma 5.1.** The mapping $\text{vskw} : \nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M}) \to \nabla_{k-1}^{\text{div}}(\mathbf{r}_1) \cap \ker(\text{div})$ is well-defined and surjective.

**Proof.** Recall that the parallelogram formed by the north-east diagonal $\n$ and the horizontal operators is anticommutative. By substituting the differential operators with the face normal vector, we derive the following relationship:

\[
2(\text{vskw} \, \tau) \cdot \mathbf{n} = - \text{tr}(\tau \times \mathbf{n}).
\]

Since $\tau \times \mathbf{n}$ remains continuous for an $H(\text{curl})$ function $\tau$, it follows that $(\text{vskw} \, \tau) \mathbf{n}$ also maintains continuity across each face. In other words, $\text{vskw} \, \tau \in H(\text{div}, \Omega)$. Moreover, due to the traceless property of $\nabla^{\text{curl}} \tau$:

\[
\text{div} \, 2(\text{vskw} \, \tau) = \text{tr} \nabla^{\text{curl}} \tau = 0.
\]

This implies that $\text{vskw}(\nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M})) \subseteq \nabla_{k-1}^{\text{div}}(\mathbf{r}_1) \cap \ker(\text{div})$.

For the surjectivity proof, we select a function $u \in \nabla_{k}^{\text{div}}(\mathbf{r}_1) \cap \ker(\text{div})$. Since $\text{div} \, u = 0$, we can find a $v \in \nabla_{k}^{\text{curl}}(\mathbf{r}_0 - 1)$ such that $\nabla^{\text{curl}} v = u$. Consequently, $\nabla^{\text{grad}} v \in \nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M})$ and $2 \text{vskw} \, \nabla^{\text{grad}} v = \nabla^{\text{curl}} v = u$. This completes the argument for surjectivity.

Now define

\[
\nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{S}) := \nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{M}) \cap \ker(\text{vskw}).
\]

With this space in place, we can proceed to apply the BGG construction to (37) and subsequently deduce the finite element Hessian complex.

**Theorem 5.2.** Let $\mathbf{r}_0 \geq 1, \mathbf{r}_1 = \mathbf{r}_0 - 2, \mathbf{r}_2 = \mathbf{r}_1 \oplus 1, \mathbf{r}_3 = \mathbf{r}_2 \oplus 1$ be valid smoothness vectors. Assume $(\mathbf{r}_2, \mathbf{r}_3, k - 1)$ is $(\text{div}; \mathbb{T})$ stable and $k + 2 \geq 2\mathbf{r}_0^\vee + 1$. Then the finite element Hessian complex

\[
\mathbb{P}_1 \xrightarrow{\nabla_{k+2}^{\text{grad}}(\mathbf{r}_0)} \nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{S}) \xrightarrow{\nabla_{k}^{\text{curl}}(\mathbf{r}_1)} \nabla_{k-1}^{\text{div}}(\mathbf{r}_2; \mathbb{T}) \xrightarrow{\text{div}} \nabla_{k-2}^{L^2}(\mathbf{r}_3; \mathbb{R}^3) \to 0
\]

is exact.

While we have established finite element descriptions for several spaces, including $\nabla_{k+2}^{\text{grad}}(\mathbf{r}_0), \nabla_{k-1}^{\text{div}}(\mathbf{r}_2; \mathbb{T}),$ and $\nabla_{k-2}^{L^2}(\mathbf{r}_3; \mathbb{R}^3)$, we are currently faced with a challenge in providing a finite element description for $\nabla_{k}^{\text{curl}}(\mathbf{r}_1; \mathbb{S})$. The local DoFs on each element for this space are not easily derived using the BGG construction.

**Example 5.3.** Taking $\mathbf{r}_0 = (4, 2, 1)^T, r_1 = \mathbf{r}_0 - 2, \mathbf{r}_2 = \mathbf{r}_1 \oplus 1, \mathbf{r}_3 = \mathbf{r}_2 \oplus 1,$ and $k \geq 7$, we obtain the finite element Hessian complex in [21]

\[
\mathbb{P}_1 \xrightarrow{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \xrightarrow{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} 0.
\]
By modifying the smoothness constraint for $V^\text{curl}_k(r_1; \mathbb{S})$ to $r_2 \geq r_1 \odot 1$ and then introducing the space $V^\text{curl}_k(r_1, r_2; \mathbb{S}) = \{ \tau \in V^\text{curl}_k(r_1; \mathbb{S}), \text{curl} \tau \in V^\text{div}_{k-1}(r_2; \mathbb{T}) \}$, we can derive a more general finite element Hessian complex.

**Corollary 5.4.** Let $r_0 \geq 1$, $r_1 = r_0 - 2$, $r_2 \geq r_1 \odot 1$, $r_3 \geq r_2 \odot 1$. Assume $(r_2, r_3, k-1)$ is $(\text{div}; \mathbb{T})$ stable and $k \geq \max\{2r_0^2 - 1, 2r_3^2 + 2, 2r_3^3 + 3\}$. Then the finite element Hessian complex

$$
\mathbb{P}_1 \subset \nabla_{k+2}^\text{grad}(r_0) \xrightarrow{\text{hess}} V^\text{curl}_k(r_1, r_2; \mathbb{S}) \xrightarrow{\text{curl}} V^\text{div}_{k-1}(r_2, r_3; \mathbb{T}) \xrightarrow{\text{div}} V^2_{k-2}(r_3; \mathbb{R}^3) \rightarrow 0
$$

is exact.

**5.2. Finite element elasticity complexes.** Let

$$
(39) \quad r_0 \geq \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad r_1 = r_0 - 1 \geq \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad r_2 = \max\{r_1 \odot 2, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}\}, \quad r_3 = r_2 \odot 1.
$$

Assume $(r_1 \odot 1, r_2, k)$ is div-stable, and that $(r_2, r_3, k-1)$ exhibits $(\text{div}; \mathbb{S})$ stability. The requirement $r_2^2 \geq 0$ in (39) is essential for the attainment of $(\text{div}; \mathbb{S})$ stability.

In the pursuit of constructing a BGG diagram, specifically (44), our endeavor initiates with the bottom complex. Evidently, $(r_1)_+ - 1 = r_1 \odot 1$ and $(r_2)_+ \geq (r_1 \odot 1) \odot 1$. Moreover, predicated on our assumptions, $(r_2, r_3, k-1)$ is already div-stable. Consequently, the sequence $((r_1)_+, r_1 \odot 1, r_2, r_3)$ forms a valid de Rham smoothness sequence, culminating in the establishment of the exact sequence:

$$
\mathbb{R}^3 \subset \nabla_{k+1}^\text{grad}((r_1)_+; \mathbb{R}^3) \xrightarrow{\text{grad}} V^\text{curl}_k(r_1, r_2; \mathbb{M}) \xrightarrow{\text{curl}} V^\text{div}_{k-1}(r_2; \mathbb{M}) \cap \ker(\text{div}) \xrightarrow{\text{div}} 0.
$$

Employing a $\sim$ operation to effectuate the transition from $\mathbb{M}$ to $\mathbb{S}$, the resultant exact sequence can be presented as:

$$
\mathbb{R}^3 \subset \nabla_{k+1}^\text{grad}((r_1)_+; \mathbb{R}^3) \xrightarrow{\text{grad}} V^\text{curl}_k(r_1 \odot 1; \mathbb{M}) \xrightarrow{\text{curl}} V^\text{div}_{k-1}(r_2; \mathbb{S}) \cap \ker(\text{div}) \xrightarrow{\text{div}} 0,
$$

where

$$
\nabla^\text{curl}_k(r_1 \odot 1; \mathbb{M}) := \{ \tau \in V^\text{curl}_k(r_1 \odot 1; \mathbb{M}) : \text{curl} \tau \in V^\text{div}_{k-1}(r_2; \mathbb{S}) \}.
$$

Moreover, a definition emerges for the space:

$$
\nabla^\text{div}_k(r_1 \odot 1; \mathbb{M}) := S(\nabla^\text{curl}_k(r_1 \odot 1; \mathbb{M})).
$$

**Lemma 5.5.** It holds

$$
\nabla^\text{div}_k(r_1 \odot 1; \mathbb{M}) \subseteq V^\text{div}_k(r_1 \odot 1; \mathbb{M}) \cap \ker(\text{div}).
$$

**Proof.** By $(S\text{curl} \times n) = -2 \text{vskw}(\tau \times n)$, $\nabla^\text{div}_k(r_1 \odot 1; \mathbb{M}) \subseteq V^\text{div}_k(r_1 \odot 1; \mathbb{M})$. And $\text{div} \nabla^\text{div}_k(r_1 \odot 1; \mathbb{M}) = 0$ follows from $\text{div}(S\text{curl}) = 2 \text{vskw}(\text{curl} \tau)$. □

We then proceed to address the top complex of (44). It is straightforward to demonstrate that the sequence $(r_0, r_1, r_1 \odot 1, r_2)$ is also a valid de Rham smoothness sequence. With the application of a $\sim$ operation to this finite element de Rham complex, an exact sequence unfolds as follows:

$$
\mathbb{R}^3 \subset \nabla^\text{curl}_{k+2}(r_0) \xrightarrow{\text{grad}} \nabla^\text{curl}_{k+1}(r_1; \mathbb{M}) \xrightarrow{\text{curl}} \nabla^\text{div}_k(r_1 \odot 1; \mathbb{M}) \xrightarrow{\text{div}} 0,
$$

where

$$
\nabla^\text{curl}_{k+1}(r_1; \mathbb{M}) := \{ \tau \in V^\text{curl}_{k+1}(r_1; \mathbb{M}) : \text{curl} \tau \in V^\text{div}_k(r_1 \odot 1; \mathbb{M}) \},
$$

and in light of the fact that $r_0 \geq 0$, $V^\text{curl}_{k+2}(r_0) = V^\text{grad}_{k+2}(r_0; \mathbb{R}^3)$. 

We further apply a \( \widehat{\nabla} \) operation to the exact sequence (41). Consider the diagram

\[
\begin{align*}
\nabla_{k+2}(r_0, (r_1)_+) & \xrightarrow{\text{grad}} \widehat{\nabla}_{k+1}(r_1; M) & \xrightarrow{\text{curl}} \nabla_{k}^{\text{div}}(r_1 \ominus 1; M) & \xrightarrow{\text{div}} 0 \\
\text{curl} & \downarrow & \text{2 vskw} & \\
\nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3) & \\
\end{align*}
\]

where

\[
\widehat{\nabla}_{k+1}(r_1; M) := \{ \tau \in \nabla_{k+1}^{\text{curl}}(r_1; M) : \text{vskw } \tau \in \nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3), \text{curl } \tau \in \nabla_{k}^{\text{div}}(r_1 \ominus 1; M) \}.
\]

**Lemma 5.6. The de Rham complex**

(42) \( \mathbb{R}^3 \subseteq \nabla_{k+2}^{\text{curl}}(r_0, (r_1)_+) \xrightarrow{\text{grad}} \widehat{\nabla}_{k+1}(r_1; M) \xrightarrow{\text{curl}} \nabla_{k}^{\text{div}}(r_1 \ominus 1; M) \xrightarrow{\text{div}} 0 \)

is exact.

**Proof.** By invoking Lemma 2.1, the task at hand simplifies to confirm the dimension identity. Given that \( r_1^f \geq 0 \), it becomes sufficient to focus solely on the alteration of the face degrees of freedom. For any \( u \in \nabla_{k+2}^{\text{curl}}(r_0) \) and \( \tau \in \widehat{\nabla}_{k+1}^{\text{curl}}(r_1; M) \), it holds true that both \( n \cdot \text{curl } u \) and \( n \cdot \text{vskw } \tau \) retain their continuity across faces. With the aim of conforming to the space \( \nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3) \), it becomes necessary to impose the tangential component. Consequently, the dimensions \( \dim \nabla_{k+2}^{\text{curl}}(r_0) - \dim \nabla_{k+1}^{\text{curl}}(r_0, (r_1)_+) \) and \( \dim \widehat{\nabla}_{k+1}^{\text{curl}}(r_1; M) - \dim \nabla_{k+1}^{\text{curl}}(r_1; M) \) equate to

\[
[r_1^f = -1](4|\Delta_3(T_h)| - |\Delta_2(T_h)|) \dim \mathbb{B}_{k+1}^2(f; r_1).
\]

This completes the verification process. \( \square \)

We further give characterization of the space \( \widehat{\nabla}_{k+1}(r_1; M) \).

**Lemma 5.7.** We have

(43) \( \widehat{\nabla}_{k+1}^{\text{curl}}(r_1; M) = \nabla_{k+1}^{\text{inc}}(r_1; \mathbb{S}) \oplus \text{mskw } \nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3) \),

where

\[
\nabla_{k+1}^{\text{inc}}(r_1; \mathbb{S}) := \{ \tau \in \nabla_{k+1}^{\text{curl}}(r_1; M) : \tau \in \mathbb{S} \}.
\]

**Proof.** For \( \tau = \text{mskw } v \in \text{mskw } \nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3) \), it follows

\[
\text{curl } \tau = \text{curl } \text{mskw } v = -S(\text{grad } v) \in \nabla_{k}^{\text{div}}(r_1 \ominus 1; M),
\]

and \( \text{vskw } \tau = v \in \nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3) \). We thus have proved \( \text{mskw } \nabla_{k+1}^{\text{grad}}((r_1)_+; \mathbb{R}^3) \subseteq \nabla_{k+1}^{\text{curl}}(r_1; M) \). Then decomposition (43) holds from

\[
\tau = \text{sym } \tau + \text{skw } \tau = \text{sym } \tau + \text{mskw}(\text{vskw } \tau).
\]

\( \square \)
Theorem 5.8. Let \((r_0, r_1, r_2, r_3)\) be given by (39) and assume \((r_1 \ominus 1, r_2, k)\) is \(\text{div}\) stable, and \((r_2, r_3, k - 1)\) is \((\text{div}; \mathbb{S})\) stable. Let \(k + 2 \geq 2r_0' + 1\). We have the BGG diagram (44)

\[
\begin{align*}
V_{k+2} \rightarrow \nabla \text{curl} (r_0, (r_1)_+) & \xrightarrow{\text{grad}} \nabla \text{curl} (r_1; \mathbb{M}) \\
\text{curl} & \xrightarrow{\text{curl}} \nabla \text{div} (r_1 \ominus 1; \mathbb{M}) \\
\text{div} & \xrightarrow{\text{div}} 0
\end{align*}
\]

which leads to the finite element elasticity complex

(45) \(RM \subseteq V_{k+2} (r_0, (r_1)_+) \xrightarrow{\text{def}} \nabla \text{inc}^+(r_1; \mathbb{S}) \xrightarrow{\text{inc}} \nabla \text{div} (r_2; \mathbb{S}) \xrightarrow{\text{div}} \nabla L^2 (r_3; \mathbb{R}^3) \rightarrow 0\).

Proof. The bijectiveness of the mapping \(S : \nabla \text{curl} (r_1 \ominus 1; \mathbb{M}) \rightarrow \nabla \text{div} (r_1 \ominus 1; \mathbb{M})\) is a direct outcome of its definition. With complex (42), complex (40), and the decomposition (43) at our disposal, we arrive at our desired conclusion by applying the BGG framework.

Example 5.9. Taking \(r_0 = (2, 1, 0)^T\), \(r_1 = r_0 - 1\), \(r_2 = (0, -1, -1)^T\), \(r_3 = r_2 \ominus 1\), we have

\[
RM \subseteq V_{k+2} (r_0, (r_1)_+) \xrightarrow{\text{def}} \nabla \text{inc}^+(r_1; \mathbb{S}) \xrightarrow{\text{inc}} \nabla \text{div} (r_2; \mathbb{S}) \xrightarrow{\text{div}} \nabla L^2 (r_3; \mathbb{R}^3) \rightarrow 0,
\]

which is the finite element discretization of the elasticity complex (14).

In the notation \(\text{inc}^+\), the super-script \(^+\) means \(\tau \in H(\text{curl})\) which is not necessary for \(\tau \in H(\text{inc})\). Next we relax the smoothness of \(\nabla \text{inc}^+(r_1; \mathbb{S})\) to \(\nabla \text{inc}^+(r_1; \mathbb{S})\). Define

(46) \(\nabla \text{inc}^+(r_1; \mathbb{S}) := \{\tau \in \nabla k(k_1; \mathbb{S}) : \text{inc} \tau \in \nabla \text{div} (r_2; \mathbb{S})\}\).

Obviously \(\nabla \text{inc}^+(r_1; \mathbb{S}) \subseteq \nabla \text{inc}^+(r_1; \mathbb{S})\).

Corollary 5.10. Let \((r_0, r_1, r_2, r_3)\) be given by (39) and assume \((r_1 \ominus 1, r_2, k)\) is \(\text{div}\) stable, and \((r_2, r_3, k - 1)\) is \((\text{div}; \mathbb{S})\) stable. Let \(k + 2 \geq 2r_0' + 1\). We have the finite element elasticity complex

(47) \(RM \subseteq V_{k+2} (r_0; \mathbb{R}^3) \xrightarrow{\text{def}} \nabla \text{inc}^+(r_1; \mathbb{S}) \xrightarrow{\text{inc}} \nabla \text{div} (r_2; \mathbb{S}) \xrightarrow{\text{div}} \nabla L^2 (r_3; \mathbb{R}^3) \rightarrow 0\).

Proof. Clearly (47) is a complex. By the exactness of complex (45), we have

\[
\text{div} \nabla \text{div} (r_2; \mathbb{S}) = \nabla L^2 (r_3; \mathbb{R}^3), \quad \text{inc} \nabla \text{inc}^+(r_1; \mathbb{S}) = \nabla \text{inc}^+(r_1; \mathbb{S}) \cap \ker(\text{div}).
\]

Then we get from \(\nabla \text{inc}^+(r_1; \mathbb{S}) \subseteq \nabla \text{inc}^+(r_1; \mathbb{S})\) that \(\text{inc} \nabla \text{inc}^+(r_1; \mathbb{S}) = \nabla \text{inc}^+(r_1; \mathbb{S}) \cap \ker(\text{div})\).

Next we prove \(\text{def} \nabla \text{grad} (r_0; \mathbb{R}^3) = \nabla \text{inc}^+(r_1; \mathbb{S}) \cap \ker(\text{inc})\). For \(\tau \in \nabla \text{inc}^+(r_1; \mathbb{S}) \cap \ker(\text{inc})\), by the elasticity complex it follows that \(\tau = \text{def}(v)\) with \(v \in H^1(\Omega; \mathbb{R}^3)\), which implies \(v \in \nabla \text{grad} (r_0; \mathbb{R}^3)\) as \(\tau\) is a piecewise polynomial tensor and \(r_1 = r_0 - 1\). □
It is worth noting that (45) can be obtained from (47) by applying one \( \sim \) operation. Equivalently (47) is obtained from (45) by applying an inverse of the \( \sim \) operation.

By the finite element elasticity complex (47), we have the dimension identity

\begin{align}
-6 + \dim \nabla^{\text{grad}}_{k+2}(r_0; \mathbb{R}^3) - \dim \nabla^{\text{inc}}_{k+1}(r_1; \mathcal{S}) \\
+ \dim \nabla^{\text{div}}_{k-1}(r_2; \mathcal{S}) - \dim \nabla_{k-2}^{L^2}(r_3; \mathbb{R}^3) = 0.
\end{align}

So far we have finite element descriptions for \( \nabla^{\text{grad}}_{k+2}(r_0; \mathbb{R}^3) \), \( \nabla^{\text{div}}_{k-1}(r_2; \mathcal{S}) \), and \( \nabla_{k-2}^{L^2}(r_3; \mathbb{R}^3) \) but not for \( \nabla^{\text{inc}}_{k+1}(r_1; \mathcal{S}) \).

**Example 5.11.** Consider the choice \( r_0 = (2,1,0)^T, r_1 = r_0 - 1, r_2 = (0,-1,-1)^T, r_3 = r_2 \oplus 1 \) and \( k + 1 \geq 6 \). From the finite element elasticity complex (47) we get

\[
\begin{align*}
RM \ &
\begin{array}{c}
\begin{array}{c}
(2,1,0) \\
(1,0,-1)
\end{array}
\end{array}
\xrightarrow{\text{def}}
\begin{array}{c}
\begin{array}{c}
(1,0) \\
(-1,-1)
\end{array}
\end{array}
\xrightarrow{\text{inc}}
\begin{array}{c}
\begin{array}{c}
(0) \\
(-1)
\end{array}
\end{array}
\xrightarrow{\text{div}}
\begin{array}{c}
\begin{array}{c}
(-1) \\
(-1)
\end{array}
\end{array}
\rightarrow 0.
\end{align*}
\]

This sequence presents a variation of the finite element elasticity complex in the work by Chen and Huang [15], where the Hu-Zhang \( H(\text{div}; \mathcal{S}) \)-conforming element is used. \( \square \)

**Example 5.12.** For the case of \( k + 1 \geq 8 \) and \( r_0 = (4,2,1)^T \), we arrive at a discrete elasticity complex (49) originating from a subspace of \( H^2(\Omega) \):

\[
\begin{align*}
RM \ &
\begin{array}{c}
\begin{array}{c}
(4,2,1) \\
(1,0,-1)
\end{array}
\end{array}
\xrightarrow{\text{def}}
\begin{array}{c}
\begin{array}{c}
(3,1,0) \\
(-1,-1)
\end{array}
\end{array}
\xrightarrow{\text{inc}}
\begin{array}{c}
\begin{array}{c}
(1) \\
(-1)
\end{array}
\end{array}
\xrightarrow{\text{div}}
\begin{array}{c}
\begin{array}{c}
(0) \\
(-1)
\end{array}
\end{array}
\rightarrow 0.
\end{align*}
\]

In the scenario where \( r_1^{f} = 0 \), considering \( \tau \in \nabla^{\text{inc}}_{k+1}(r_1; \mathcal{S}) \), we observe that \( \tau n \) remains continuous. With the additional continuity of \( n \times \tau \times n \), it follows that \( \tau \in H^1(\Omega; \mathcal{S}) \).

The corresponding continuous version manifests as the elasticity complex that initiates with \( H^2(\Omega; \mathbb{R}^3) \):

\begin{align}
RM \ &
\begin{array}{c}
\begin{array}{c}
(2,1,0) \\
(1,0,-1)
\end{array}
\end{array}
\xrightarrow{\text{def}}
\begin{array}{c}
\begin{array}{c}
(0) \\
(-1)
\end{array}
\end{array}
\xrightarrow{\text{inc}}
\begin{array}{c}
\begin{array}{c}
(0) \\
(-1)
\end{array}
\end{array}
\xrightarrow{\text{div}}
\begin{array}{c}
\begin{array}{c}
(-1) \\
(-1)
\end{array}
\end{array}
\rightarrow 0.
\end{align}
\]

Here, the space

\[
H^1(\text{inc}, \Omega; \mathcal{S}) := \{ \tau \in H^1(\Omega; \mathcal{S}) : \text{inc} \tau \in L^2(\Omega; \mathbb{R}^3) \}.
\]

It is worth noting that a finite element elasticity complex has been recently established for the Alfeld split of a tetrahedron [18], presenting another discrete counterpart of (49). \( \square \)

**Remark 5.13.** The discrete elasticity complex introduced in [18] corresponds, within our notation, to the following sequence:

\[
\begin{align*}
RM \ &
\begin{array}{c}
\begin{array}{c}
(2,1,0) \\
(1,0,-1)
\end{array}
\end{array}
\xrightarrow{\text{def}}
\begin{array}{c}
\begin{array}{c}
(0) \\
(-1)
\end{array}
\end{array}
\xrightarrow{\text{inc}}
\begin{array}{c}
\begin{array}{c}
(0) \\
(-1)
\end{array}
\end{array}
\xrightarrow{\text{div}}
\begin{array}{c}
\begin{array}{c}
(-1) \\
(-1)
\end{array}
\end{array}
\rightarrow 0.
\end{align*}
\]

This particular complex cannot be derived using our framework due to the fact that \( r_0 = (2,1,1)^T \) does not constitute a valid smoothness vector for a \( C^1 \)-element. In [18], the space \( \nabla^{\text{grad}}((2,1,1)^T; \mathbb{R}^3) \) is constructed on Alfeld splits of tetrahedra. \( \square \)
5.3. Finite element divdiv complexes. Let

\[ r_0 \geq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad r_1 = r_0 - 1, \quad r_2 = \max\{r_1 \otimes 1, (0, -1), -1\}, \quad r_3 = r_2 \otimes 2. \]

Assume both \((r_2, r_2 \otimes 1, k)\) and \((r_2 \otimes 1, r_3, k - 1)\) are div stable. Consequently both \((r_0, r_1, r_2, r_2 \otimes 1)\) and \(((r_1)_+, r_1 \otimes 1, r_2 \otimes 1, r_3)\) are valid de Rham smoothness sequences.

For the \((r_0, r_1, r_2, r_3)\) de Rham complex, define

\[
\begin{align*}
\nu^\text{div}_{k+1}(r_1; \mathbb{R}) &:= \nu^{\text{div}\text{div}^+}_{k+1}(r_1; \mathbb{R}) \\
\nu^\text{curl}_{k+1}(r_1; \mathbb{R}) &:= \{ \tau \in \nu^\text{curl}_{k+1}(r_1; \mathbb{R}) : \text{curl} \tau \in \nu^\text{div}_{k+1}(r_1; \mathbb{R}) \}, \\
\nu^\text{curl}_{k+1}(r_1; \mathbb{R}) &:= \{ \tau \in \nu^\text{curl}_{k+1}(r_1; \mathbb{R}) : \text{tr} \tau \in \nu^{\text{grad}}_{k+1}((r_1)_+), \text{curl} \tau \in \nu^\text{div}_{k+1}(r_1; \mathbb{R}) \}.
\end{align*}
\]

**Lemma 5.14.** We have

\[
\text{tr} \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) = \nu_{k+1}^L((r_1)_+), \quad \text{tr} \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) = \nu_{k+1}^L(r_1).
\]

**Proof.** Since \(\nu_{k+2}^\text{div}(r_0, (r_1)_+) = \nu_{k+1}^L((r_1)_+)\), by the diagram

\[
\begin{array}{c}
\nu_{k+2}^\text{div}(r_0, (r_1)_+) \downarrow \text{grad} \quad \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) \quad \text{curl} \quad \nu_{k}^\text{div}(r_2; \mathbb{R}) \cap \ker(\text{div}) \quad \text{div} \quad 0 \\
\nu_{k+1}^L((r_1)_+) \quad \text{tr}
\end{array}
\]

we have \(\text{tr} \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) = \nu_{k+1}^L((r_1)_+)\). Similarly, \(\text{tr} \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) = \nu_{k+1}^L(r_1)\) follows from \(\text{div} \nu_{k+2}^\text{grad}(r_0; \mathbb{R}) = \nu_{k+1}(r_1)\). \hfill \square

**Lemma 5.15.** The complex

\[ \mathbb{R}^3 \xhookrightarrow{\nu_{k+2}^\text{div}(r_0, (r_1)_+)} \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) \xrightarrow{\text{curl}} \nu_k^\text{div}(r_2; \mathbb{R}) \cap \ker(\text{div}) \xrightarrow{\text{div}} \nu_{k-1}^\text{div}(r_2 \otimes 1) \rightarrow 0 \]

is exact.

**Proof.** By Lemma 2.1, it suffices to count the dimension difference. Employing the fact \(\dim \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) \cap \ker(\text{tr}) = \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) \cap \ker(\text{tr})\), it follows from Lemma 5.14 that

\[
\dim \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) - \dim \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) = \dim \text{tr} \nu_{k+1}^\text{curl}(r_1; \mathbb{R}) - \dim \text{tr} \nu_{k+1}^\text{curl}(r_1; \mathbb{R})
\]

which equals \(\dim \text{div} \nu_{k+2}^\text{grad}(r_0; \mathbb{R}) - \dim \text{div} \nu_{k+2}^\text{div}(r_0, (r_1)_+)\), i.e. \(\dim \nu_{k+2}^\text{grad}(r_0; \mathbb{R}) - \dim \nu_{k+2}^\text{div}(r_0, (r_1)_+).\) Then we conclude the sequence (51) is exact. \hfill \square

We construct a finite element divdiv complex by the BGG procedure.
**Theorem 5.16.** Let \((r_0, r_1, r_2, r_3)\) be given by (50). Assume both \((r_2, r_2 \ominus 1, k)\) and 
\((r_2 \ominus 1, r_3, k - 1)\) are div stable. Then we have the BGG diagram

\[
\begin{array}{cccccc}
\uparrow \text{grad} & \uparrow \text{curl} & \uparrow \text{div} & \uparrow \text{id} \\
V_{k+1}^\text{div}(r_0, (r_1)_+) & V_{k+1}^\text{curl}(r_1; M) & V_k^\text{div}(r_2; M) & V_{k-1}^\text{div}(r_2 \ominus 1) & \rightarrow 0 \\
V_k^\text{grad}((r_1)_+) & \downarrow \text{mskw} & \downarrow \text{curl} & \downarrow \text{div} & \uparrow \text{div} & \uparrow \text{div} & \uparrow L^2 & V_{k-2}^L(r_3) & \rightarrow 0,
\end{array}
\]

which leads to the exact finite element divdiv complex

\[
(52) \quad RT \subseteq V_{k+2}^\text{div}(r_0, (r_1)_+) \xrightarrow{\text{dev grad}} V_{k+1}^\text{sym curl}+(r_1; T) \xrightarrow{\text{sym curl}} V_k^\text{div div}+(r_2; S) \xrightarrow{\text{div div}} V_{k-2}^L(r_3) \rightarrow 0,
\]

where

\[
V_{k+1}^\text{sym curl}+(r_1; T) := \{ \tau \in V_{k+1}^\text{curl}(r_1; M) : \text{tr} \tau = 0, \text{curl} \tau \in V_k^\text{div div}(r_2; M) \}.
\]

**Proof.** By definition of spaces, both \(\iota\) and \(\text{mskw}\) are injective. In addition, we have the decomposition

\[
\begin{align*}
\hat{\nabla}_{k+1}^\text{curl}(r_1; M) &= V_{k+1}^\text{sym curl}+(r_1; T) \oplus \iota V_k^\text{grad}((r_1)_+) , \\
\hat{\nabla}_k^\text{div}(r_2; M) &= V_k^\text{div div}+(r_2; S) \oplus \text{mskw} V_{k+1}^\text{curl}(r_1 \ominus 1).
\end{align*}
\]

We conclude the result by employing the BGG framework. \(\square\)

We can relax the smoothness of \(\tau \in V_{k+1}^\text{curl}(r_1; M)\) and define

\[
V_{k+1}^\text{sym curl}+(r_1; T) := \{ \tau \in V_{k+1}^\text{curl}(r_1; T) : \text{sym curl} \tau \in V_k^\text{div div}(r_2; S) \}.
\]

**Corollary 5.17.** Let \((r_0, r_1, r_2, r_3)\) be given by (50). Assume both \((r_2, r_2 \ominus 1, k)\) and 
\((r_2 \ominus 1, r_3, k - 1)\) are div stable. We have the exact finite element divdiv complex

\[
(53) \quad RT \subseteq V_{k+2}^\text{grad}(r_0; \mathbb{R}^3) \xrightarrow{\text{dev grad}} V_{k+1}^\text{sym curl}+(r_1; T) \xrightarrow{\text{sym curl}} V_k^\text{div div}+(r_2; S) \xrightarrow{\text{div div}} V_{k-2}^L(r_3) \rightarrow 0.
\]

Consequently

\[
(54) \quad -4 + \dim V_{k+2}^\text{grad}(r_0; \mathbb{R}^3) - \dim V_{k+1}^\text{sym curl}+(r_1; T) + \dim V_k^\text{div div}+(r_2; S) - \dim V_{k-2}^L(r_3) = 0.
\]

**Proof.** Similar to the proof of Corollary 5.10. \(\square\)

We can further enlarge the space \(V_k^\text{div div}+(r_2; S)\) to \(V_k^\text{sym div}(r_2; S)\) and define

\[
V_{k+1}^\text{sym curl}(r_1) := \{ \tau \in V_{k+1}^\text{curl}(r_1) \otimes T : \text{sym curl} \tau \in V_k^\text{sym div}(r_2; S) \}
\]

to get the finite element divdiv complex

\[
(55) \quad RT \subseteq V_{k+2}^\text{grad}(r_0; \mathbb{R}^3) \xrightarrow{\text{dev grad}} V_{k+1}^\text{sym curl}(r_1; T) \xrightarrow{\text{sym curl}} V_k^\text{sym div}(r_2; S) \xrightarrow{\text{div div}} V_{k-2}^L(r_3) \rightarrow 0.
\]

We can also define

\[
V_{k+1}^\text{sym curl}+(r_1; T) := \{ \tau \in V_{k+1}^\text{curl}(r_1; M) : \text{tr} \tau = 0, \text{sym curl} \tau \in V_k^\text{sym div}(r_2; S) \},
\]
and obtain another finite element divdiv complex
\[
\begin{align*}
RT & \subseteq \mathcal{V}_{k+2}^{\text{div}}(r_0, (r_1)_+) \xrightarrow{\text{dev grad}} \mathcal{V}_{k+1}^{\text{sym curl}^+}(r_1; T) \\
& \xrightarrow{\text{sym curl}} \mathcal{V}_k^{\text{div div}}(r_2; S) \xrightarrow{\text{div div}} \mathcal{V}_{k-2}^{L^2}(r_3) \to 0.
\end{align*}
\]

Those variants are summarized in Figure 5. Again the complexes derived from the BGG

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Diverse configurations of finite element divdiv complexes.}
\end{figure}

framework typically encompass spaces that possess a slightly higher degree of smoothness.

**Example 5.18.** We recover the finite element divdiv complex in [17]
\[
RT \subseteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{dev grad}} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\text{sym curl}} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\text{div div}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \to 0,
\]

and the finite element divdiv complex in [22]
\[
RT \subseteq \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{dev grad}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\text{sym curl}} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\text{div div}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \to 0.
\]

Verification of the exactness for the complexes (55) and (56) becomes intricate when

\[
\text{the space } \mathcal{V}_k^{\text{div div}}(r_2; S) \text{ is introduced. A rigorous proof will be presented subsequently,}
\]

following the constructive characterization of these spaces.

6. Edge Elements

In this section we shall construct finite element spaces for \(H(\text{curl}; S), H(\text{inc}; S), \) and \(H(\text{sym curl}; T)\) spaces. For \(H(\text{curl}; S),\) we use the \(t - n\) decomposition approach and for the other two, we use the trace bubble complexes and the knowledge for div elements
to determine the face DoFs and the edge traces to determine the edge DoFs.

6.1. \(H(\text{curl}; S)\)-conforming elements. When \(r^f \geq 0,\) it is simply the tensor product:
\[
\mathcal{V}_k^{\text{curl}}(r, S) := \mathcal{V}_k(r) \otimes S \quad \text{when } r \geq 0.
\]

Recall that the Hessian complex starts from an \(H^2\)-conforming element \(\mathcal{V}_k^{\text{hess}}(r + 2),\) with \(r + 2 \geq (4, 2, 1)^\top\) and consequently \(r^c \geq 0.\) Hence, in the remainder of this subsection, we shall focus exclusively on the cases where \(r^f = -1\) and \(r^e \geq 0.\)

Since we have \(r^v \geq 2r^e \geq 0,\) the vertex and edge DoFs take on a tensor product structure: \(\text{DoF}_k(s; r) \otimes S,\) where \(s = v \text{ or } e.\) However, the determination of face DoFs
requires a distinct approach. We continue to utilize the diagram (20), but this time, with a modified \( t - n \) decomposition, as the trace of the curl operator takes on a tangential form, \((\cdot) \times \mathbf{n} \). Consequently, the normal component will contribute to the bubble space. To clarify this process, we define

\[
\mathcal{B}_{k}^{\text{curl}}(r; \mathbb{S}) := \mathcal{B}_{k}(r_{+}; \mathbb{S}) \oplus [r^{f} = -1] \bigoplus_{f \in \Delta_{2}(T)} (\mathcal{B}_{k}(f; r) \otimes \text{span}\{ \mathbf{n} \otimes \mathbf{n} \}).
\]

\[\begin{array}{c}
\text{face } f \\
t_{1} \\
t_{2} \\
\end{array}\]

\[\begin{array}{c}
t_{1} \\
t_{2} \\
\mathbf{n}
\end{array}\]

**Figure 6.** A \( t - n \) decomposition of \( \mathbb{S} \) on a face for \( H(\text{curl}; \mathbb{S}) \) element.

Take \( \mathcal{P}_{k}(T; \mathbb{S}) \) as the space of shape functions. Let \( r \) be a smoothness vector with \( r^{f} = -1 \) and \( r^{e} \geq 0 \). The degrees of freedom are

\[
\begin{align*}
(57a) \quad & \nabla^{i} \tau(\nu), \quad i = 0, \ldots, r^{v}, \nu \in \Delta_{0}(T), \\
(57b) \quad & \int_{e} \frac{\partial^{j} \tau}{\partial n_{1}^{i} \partial n_{2}^{j}} : q \, ds, \quad q \in \mathbb{B}_{k-j}(e; r^{v} - j) \otimes \mathbb{S}, 0 \leq i \leq j \leq r^{e}, e \in \Delta_{1}(T), \\
(57c) \quad & \int_{f} (\mathbf{n} \times \tau \times \mathbf{n}) : q \, dS, \quad q \in \mathbb{B}_{k}(f; r) \otimes \mathbb{S}(f), f \in \Delta_{2}(T), \\
(57d) \quad & \int_{f} (\mathbf{n}^{\top} \Pi_{f}) \cdot q \, dS, \quad q \in \mathbb{B}_{k}^{2}(f; r), f \in \Delta_{2}(T), \\
(57e) \quad & \int_{T} \tau : q \, dx, \quad q \in \mathcal{B}_{k}^{\text{curl}}(r; \mathbb{S}),
\end{align*}
\]

where \( \mathbb{S}(f) := \mathcal{F}^{f}(\mathbb{S}) = \text{span}\{ \text{sym}(\mathbf{t}_{i}^{f} \otimes \mathbf{t}_{j}^{f}), 1 \leq i \leq j \leq 2 \} \).

**Lemma 6.1.** Let \( r \) be a smoothness vector with \( r^{f} = -1, r^{e} \geq 0 \), and let \( k \geq 2r^{v} + 1 \). DoFs (57) are unisolvent for \( \mathcal{P}_{k}(T; \mathbb{S}) \). Given a triangulation \( \mathcal{T}_{h} \) of \( \Omega \), define

\[
\mathbb{V}_{k}^{\text{curl}}(r_{1}; \mathbb{S}) := \{ \tau \in L^{2}(\Omega; \mathbb{S}) : \tau|_{T} \in \mathcal{P}_{k}(T; \mathbb{S}) \text{ for all } T \in \mathcal{T}_{h}, \text{ and all the DoFs (57) are single-valued} \}.
\]

Then \( \mathbb{V}_{k}^{\text{curl}}(r_{1}; \mathbb{S}) \subset H(\text{curl}; \mathbb{S}) \).

**Proof.** The unisolence property can be established by leveraging the unisolence of the tensor product space \( \mathbb{V}_{h}(r_{+}) \otimes \mathbb{S} \) and strategically relocating the face DoF \( \int_{n^{\top} \mathbf{n} q} \) to \( \mathcal{B}_{k}^{\text{curl}}(r; \mathbb{S}) \). The \( H(\text{curl}) \)-conformity is a direct consequence of the continuity exhibited by \( \tau \times \mathbf{n} \), which arises from the single-valued DoFs (57a)-(57d). \( \square \)

Recall that in Section 5.1, we have defined a space \( \mathbb{V}_{k}^{\text{curl}}(r_{1}; \mathbb{S}) = \overline{\mathbb{V}_{k}^{\text{curl}}(r_{1}; \mathbb{M})} \cap \ker(\text{vskw}) \) from the BGG construction. Next we show they are equal.
Lemma 6.2. The space $\mathcal{V}_{k}^{\text{curl}}(r_1; S)$ defined by (58) is equal to the space defined by (38).

Proof. Certainly, we have the inclusion $\mathcal{V}_{k}^{\text{curl}}(r_1; S) \subseteq \mathcal{V}_{k}^{\text{curl}}(r_1; M) \cap \ker(\text{vskw})$. Thus, it is enough to demonstrate the dimension equality:

$$\dim \mathcal{V}_{k}^{\text{curl}}(r_1; S) = \dim(\mathcal{V}_{k}^{\text{curl}}(r_1; M) \cap \ker(\text{vskw})).$$

Let us compare this with $\dim(\mathcal{V}_{k}^{\text{curl}}(r_1; M))$. Using the definition of the $\cap$ operation, we can express:

$$\dim \mathcal{V}_{k}^{\text{curl}}(r_1; M) = \dim \mathcal{V}_{k}^{\text{curl}}(r_1; M) - (\dim \mathcal{V}_{k-1}^{\text{div}}(r_2; M) - \dim \mathcal{V}_{k-1}^{\text{div}}(r_2; T))$$

$$= \dim \mathcal{V}_{k}^{\text{curl}}(r_1; M) - \dim \mathcal{V}_{k-1}^{\text{div}}(r_2).$$

Also, observe that $\dim(\mathcal{V}_{k}^{\text{div}}(r_1) \cap \ker(\text{div})) = \dim \mathcal{V}_{k}^{\text{div}}(r_1) - \dim \mathcal{V}_{k-1}^{\text{div}}(r_2)$. With the surjectiveness of vskw, we can write:

$$\dim(\mathcal{V}_{k}^{\text{curl}}(r_1; M) \cap \ker(\text{vskw})) = \dim \mathcal{V}_{k}^{\text{curl}}(r_1; M) - \dim \mathcal{V}_{k}^{\text{div}}(r_1).$$

Now, let us delve into the construction of $\mathcal{V}_{k}^{\text{curl}}(r_1; S)$ and scrutinize the dimension reduction as we transition from $M$ to $S$.

When $r_f^1 \geq 0$, it is evident that $\dim \mathcal{V}_{k}^{\text{curl}}(r_1; S) = \dim \mathcal{V}_{k}^{\text{curl}}(r_1; M) - \dim \mathcal{V}_{k}^{\text{div}}(r_1)$. For the scenario where $r_f^1 = -1$, a more intricate analysis is needed. On vertices and edges, since $r_f^1 \geq 2r_f^2 \geq 0$, the change is a net decrease of 3 DoFs. In the case of faces, referring to Figure 6, the component $t_1 t_2$ goes missing. In terms of the bubble space, transitioning from $B_k(r_1; M)$ to $B_k(r_1; S)$ incurs a reduction of 3 DoFs. Additionally, for the face bubbles, we observe a reduction of 2 DoFs (namely, the components $t_1 n$ and $t_2 n^T$). This reduction perfectly aligns with the dimension decrease required to define $\mathcal{V}_{k}^{\text{div}}(r_1)$. Therefore, we have successfully demonstrated that $\dim \mathcal{V}_{k}^{\text{curl}}(r_1; S) = \dim \mathcal{V}_{k}^{\text{curl}}(r_1; M) - \dim \mathcal{V}_{k}^{\text{div}}(r_1)$, which ultimately implies the validity of (59) by referring back to (60).

A finite element description of $\mathcal{V}_{k}^{\text{curl}}(r_1, r_2; S)$ can also be derived by utilizing the DoFs introduced for $\mathcal{V}_{k+1}^{\text{curl}}(r_1, r_2)$ in Section 3.4. The overarching concept involves the addition of DoFs for curl $\tau \in \mathcal{V}_{k+1}^{\text{div}}(r_2; T)$, followed by a careful elimination of redundancies in the DoFs that involve derivatives. We skip the lengthy details here.

6.2. $H(\text{inc}; S)$-conforming elements. We proceed to provide a detailed and explicit characterization of the $H(\text{inc})$-conforming element space $\mathcal{V}_{k+1}^{\text{inc}}(r_1; S)$, as defined in (46). Our focus will be primarily on scenarios where $r_f^1 = 0$ or $r_f^2 = -1$, as the cases with $r_f^1 \geq 1$ are simply tensor product $\mathcal{V}_{k+1}^{\text{inc}}(r_1; S) = \mathcal{V}_{k+1}(r_1) \otimes S$. As $r_0 \geq (2, 1, 0)^T$ in the finite element elasticity complex, our subsequent analysis is restricted to $r_f^1 \geq 1$ and $r_f^2 \geq 0$.

To motivate the edge DoFs, we first recall the trace complexes. For a smooth and symmetric tensor $\sigma \in H(\text{inc}, T; S)$, define two trace operators as

$$t_1^{\text{inc}}(\tau) = n \times \tau \times n,$$

$$t_2^{\text{inc}}(\tau) = n \times (\text{curl} \tau)^T + \text{grad}_f(\Pi_f \tau \cdot n).$$

Then in [15, Section 4.2] we have obtained the following trace complexes

$$a \times x + b \subseteq v \xrightarrow{\text{def}} \tau \xrightarrow{\text{inc}} \sigma \xrightarrow{\text{div}} p$$

$$a_f x_f + b_f \subseteq v \times n \xrightarrow{\text{sym curl}_f} n \times \tau \xrightarrow{\text{div}_{f \text{div}}f} n \cdot \sigma \cdot n \rightarrow 0.$$
and

\[ \mathbf{a} \times \mathbf{x} \mathbf{a}_f \cdot \mathbf{x}_f + b \mathbf{v} = \mathbf{v} \rightarrow \mathbf{r} \rightarrow \mathbf{\tau} \rightarrow \mathbf{\sigma} \rightarrow \text{div} \rightarrow \mathbf{p} \]

Take \( \mathbb{P}_{k+1}(T; S) \) as the shape function space. Let \((r_0, r_1, r_2, r_3)\) be given by (39). The degrees of freedom are

\[
\begin{align*}
\text{(61a)} & \quad \nabla^i \mathbf{\tau}(v), \quad i = 0, \ldots, r_1^v, \\
\text{(61b)} & \quad \text{inc} \mathbf{\tau}(v), \quad \text{if } r_1^v = 1, \\
\text{(61c)} & \quad \int_e \frac{\partial^j \mathbf{\tau}}{\partial n_1 \partial n_2^j} : \mathbf{q} \, ds, \quad \mathbf{q} \in \mathbb{B}_{k+1-k}(e; r_1^v - j) \otimes S, 0 \leq i \leq j \leq r_1^v, \\
\text{(61d)} & \quad \int_e \left(\text{curl} \mathbf{\tau}\right)^T \mathbf{t} \cdot \mathbf{q} \, ds, \quad \mathbf{q} \in \mathbb{B}^1_{k}(e; r_1^v - 1), \quad \text{if } r_1^v = 0, \\
\text{(61e)} & \quad \int_e \left(\nabla \text{inc} \mathbf{\tau}\right) \mathbf{n}_j \cdot \mathbf{q} \, ds, \quad \mathbf{q} \in \mathbb{B}_{k-1}(e; r_2^v), 1 \leq i \leq j \leq 2, \quad \text{if } r_2^v = -1, \\
\text{(61f)} & \quad \int_f \left(\mathbf{n} \times \mathbf{\tau} \times \mathbf{n}\right) \cdot \mathbf{q} \, dS, \quad \mathbf{q} \in \text{sym curl}_f \mathbb{B}_{k+2}^2(f; r_0), \\
\text{(61g)} & \quad \int_f \text{tr}^\text{inc}_2(\mathbf{\tau}) : \mathbf{q} \, dS, \quad \mathbf{q} \in \nabla^2 \mathbb{B}_{k+2}(f; r_0), \\
\text{(61h)} & \quad \int_f \left(\mathbf{\tau}^T \mathbf{n}\right) \cdot \mathbf{q} \, dS, \quad \mathbf{q} \in \mathbb{B}^1_{k+1}(f; r_1), \quad \text{if } r_1^v = 0, \\
\text{(61i)} & \quad \int_f \Pi_f(\text{inc} \mathbf{\tau}) \mathbf{n} \cdot \mathbf{q} \, dS, \quad \mathbf{q} \in \mathbb{B}^\text{div}_{k-1}(f; r_2) / RT(f), \\
\text{(61j)} & \quad \int_f \mathbf{n}^T(\text{inc} \mathbf{\tau}) \mathbf{n} \mathbf{q} \, dS, \quad \mathbf{q} \in \mathbb{B}_{k-1}(f; (r_2)_+)/\mathbb{P}_1(f), \\
\text{(61k)} & \quad \int_T \mathbf{\tau} \cdot \mathbf{q} \, dx, \quad \mathbf{q} \in \text{def} \mathbb{B}^3_{k+2}(r_0), \\
\text{(61l)} & \quad \int_T \text{inc} \mathbf{\tau} \cdot \mathbf{q} \, dx, \quad \mathbf{q} \in \mathbb{B}^\text{div}_{k-1}(r_2; S) \cap \ker(\text{div})
\end{align*}
\]

for each \( v \in \Delta_0(T), e \in \Delta_1(T) \) and \( f \in \Delta_2(T) \).

The motivation behind incorporating DoFs such as (61b),(61e),(61i),(61j), and (61l) lies in their role in enforcing the condition \( \text{inc} \mathbf{\tau} \in \nabla^\text{div}_{k-1}(r_2; S) \), which mirrors the purpose of DoFs (27a)-(27f). The inclusion of DoFs (61k)-(61l) serves the distinct purpose of determining the bubble component. By the trace complexes, the face bubble complexes would be

\[
\begin{align*}
\text{tr}_1 : & \quad 0 \rightarrow \mathbb{B}^2_{k+2}(f; r_0) \xrightarrow{\text{sym curl}_f} \mathbb{B}^1_{k+1} \xrightarrow{\text{div}_f} \mathbb{B}_{k-1}(f; r_2) / \mathbb{P}_1(f) \rightarrow 0, \\
\text{tr}_2 : & \quad 0 \rightarrow \mathbb{B}_{k+2}(f; r_0) \xrightarrow{\text{hess}_f} \mathbb{B}^1_{k} \xrightarrow{\text{rot}_f} \mathbb{B}_{k-1}(f; r_2) / RT(f) \rightarrow 0.
\end{align*}
\]

However the face DoFs (61i) and (61j) imply the face bubble complexes are

\[
\begin{align*}
\text{tr}_1 : & \quad 0 \rightarrow \mathbb{B}^2_{k+2}(f; r_0) \xrightarrow{\text{sym curl}_f} \mathbb{B}^1_{k+1} \xrightarrow{\text{div}_f} \mathbb{B}_{k-1}(f; (r_2)_+) / \mathbb{P}_1(f) \rightarrow 0, \\
\text{tr}_2 : & \quad 0 \rightarrow \mathbb{B}_{k+2}(f; r_0) \xrightarrow{\text{hess}_f} \mathbb{B}^1_{k} \xrightarrow{\text{rot}_f} \mathbb{B}_{k-1}(f; (r_2)_+) / RT(f) \rightarrow 0.
\end{align*}
\]
\[
\text{tr}_2 : 0 \rightarrow B_{k+2}(f; r_0) \xrightarrow{\text{hess}} B_{k}^{\text{rot}}(f; r_1 \ominus 1, S) \xrightarrow{\text{rot}} B_{k-1}^{\text{rot}}(f; r_2) / \text{RM}(f) \rightarrow 0. 
\]

These modifications have been accounted in the face DoFs for inc \(\tau\), without affecting the components stemming from \(B_{k+2}(f; r_0)\) and \(B_{k+2}(f; r_0)\), as described by DoFs (61f)-(61g). For further insight into the specifics of these two-dimensional bubble polynomial spaces and finite element complexes, we refer to our recent work [13].

In the event that \(r_1^i = 0\), the inclusion of (61d) is rooted in the aim of enforcing \((\text{curl } \tau)^i \in \nabla_k^{\text{curl}}(r_1 \ominus 1; \mathbb{M})\). Conversely, if \(r_1^i \geq 1\), the same condition is inherently encompassed by (61c), given that \(\tau\) exhibits \(C^1\) continuity across edges. A similar rationale underpins the inclusion of (61e), which is exclusively required when \(r_2^e = -1\). Importantly, all traces of \(\tau\) are confined to its tangential component. In instances where \(r_1^i = 0\), the introduction of (61h) becomes crucial to ensure the continuous nature of the normal component \(\tau n\).

We present the following lemma for the ease of the dimension count.

**Lemma 6.3. The polynomial elasticity complex**

\[
RM \overset{\subset}{\hookrightarrow} \mathbb{P}_{k+1}(T; \mathbb{R}^3) \xrightarrow{\text{def}} \mathbb{P}_{k}(T; \mathbb{S}) \xrightarrow{\text{inc}} \mathbb{P}_{k-2}(T; \mathbb{S}) \xrightarrow{\text{div}} \mathbb{P}_{k-3}(T; \mathbb{R}^3) \rightarrow 0
\]

is exact for \(k \geq 3\). Consequently, for integer \(k \geq 0\),

\[
-6 + 3 \left( \begin{array}{c} k + 4 \\ 3 \end{array} \right) - 6 \left( \begin{array}{c} k + 3 \\ 3 \end{array} \right) + 6 \left( \begin{array}{c} k + 1 \\ 3 \end{array} \right) - 3 \left( \begin{array}{c} k \\ 3 \end{array} \right) = 0.
\]

**Proof.** The polynomial elasticity complex is comprehensively presented in [2, (2.6)], or can be systematically derived via polynomial de Rham complexes within the BGG framework; see [15, Section 2.2].

Upon investigating cases where \(k \geq 3\), the exactness of (64) implies that (65) can be deduced by taking the alternating sum of the dimensions of the involved spaces. For the instances of \(k = 0, 1, 2\), the validity of (65) remains intact and can be readily confirmed through direct verification.

**Lemma 6.4. For \(r_1^i = 0, -1\), the sum of the number of DoFs (61) equals \(\dim \mathbb{P}_{k+1}(T; \mathbb{S})\).**

**Proof.** At each vertex, when \(r_1^i \geq 2\), only DoFs (61a) exists with dimension \(6(r_1^i + 3)\). By (65), we have

\[
|\text{DoF}^{\text{inc}}_{k+1}(v; r_1)| = -6 + 3 \left( \begin{array}{c} r_1^0 + 3 \\ 3 \end{array} \right) + 6 \left( \begin{array}{c} r_1^2 + 3 \\ 3 \end{array} \right) - 3 \left( \begin{array}{c} r_1^3 + 3 \\ 3 \end{array} \right).
\]

When \(r_1^i = 1\), \((r_1^0, r_1^1, r_1^2, r_1^3) = (2, 1, 0, -1)\), and 6 DoFs (61b) are added. But the sum of number of DoFs (61a)-(61b) is still equal to (66) by direct calculation.

On tetrahedron \(T\), the number of DoFs (61k)-(61l) is

\[
|\text{DoF}^{\text{inc}}_{k+1}(T; r_1)| = 6 + \dim B_{k+2}^2(r_0) + \dim B_{k-1}^{\text{div}}(r_2; S) - \dim B_{k-2}^3(r_3),
\]

as \(\text{div } B_{k-2}(r_2; S) = B_{k-2}(r_3)/RM\).

On each face \(f\), we first consider the case \(r_1^f = -1\). By (62)-(63), the number of DoFs (61f)-(61j) is

\[
\dim B_{k+2}^2(f; r_0) + \dim B_{k-1}(f; r_2)_+ - 3 \\
+ \dim B_{k+2}(f; r_0) + \dim B_{k-1}^\text{rot}(f; r_2) - 3 \\
= -6 + |\text{DoF}^{\text{grad}}_{k+2}(f; r_0)| + |\text{DoF}^{\text{div}}_{k-1}(f; r_2)|.
\]
No DoFs for $\text{DoF}^\text{grad}_{k+2}(f; r_3)$ as $r_3^f = -1$. When $r_4^f = 0$, $r_0^f = 1$, we have one more layer in $\text{DoF}^\text{grad}_{k+2}(f; r_0)$ for $\partial_n v$: $B^3_{k+2-1}(f; r_0 - 1)$ which matches the number of DoF (61h) added for $r_1^f = 0$. So we conclude the the number of face DoFs (61f)-(61j) satisfies

$$\text{DoF}^\text{inc}_{k+1}(f; r_1) = -6 + |\text{DoF}^\text{grad}_{k+2}(f; r_0)| + |\text{DoF}^\text{div}_{k-1}(f; r_2)|.$$  

On each edge $e$, when $r_1^e \geq 2$, only the (61c) exists and its number satisfies

$$6 + 3 \sum_{j=0}^{r_0^e} (j + 1)(k - 2r_1^e - 1 + j) - 6 \sum_{j=0}^{r_0^e} (j + 1)(k - 2r_1^e + j)$$

$$+ 6 \sum_{j=0}^{r_2^e} (j + 1)(k - 2r_1^e + 2 + j) - 3 \sum_{j=0}^{r_2^e} (j + 1)(k - 2r_1^e + 3 + j).$$

Let $m = k - 2r_1^e - 1 \geq 0$. We split (69) into terms containing $m$ or not:

$$I_1 = m \left[ 3 \left( \frac{r_0^e + 3}{2} \right) - 6 \left( \frac{r_0^e + 2}{2} \right) + 6 \left( \frac{r_0^e}{2} \right) - 3 \left( \frac{r_0^e - 1}{2} \right) \right],$$

$$I_2 = 6 + 3 \sum_{j=0}^{r_0^e} (j + 1)j - 6 \sum_{j=0}^{r_0^e} (j + 1)^2 + 6 \sum_{j=0}^{r_0^e} (j + 1)(j + 3) - 3 \sum_{j=0}^{r_0^e} (j + 1)(j + 4).$$

By symbolical calculation, $I_1 = 0$ and $I_2 = 0$ for all integers $r_0^e \geq 2$ even for the boundary case $(r_0^e, r_1^e, r_2^e) = (3, 2, 0, -1)$.

When $r_1^e = 1$, $r_2^f = -1$, (61e) is added. The added number of DoFs (61e) matches that of (27c) for $\text{DoF}^\text{div}_{k-1}(e; S)$. When $r_1^e = 0$, (61d) is further added. Recall that $m = k - 2r_1^e - 1$. By direct calculation, $|\text{DoF}^\text{grad}_{k-1}(e; r_0)| = 9m + 6$. Sum of number of DoFs (61c)-(61d) is

$$6 \dim B_{k+1}(e; r_1) + 3 \dim B_k(e; r_1 - 1) = 9m + 12 = 6 + |\text{DoF}^\text{grad}_{k+2}(e; r_0)|.$$  

So for all cases $r_0^e \geq 0$ the sum of number of edge DoFs (61c)-(61e) satisfies

$$|\text{DoF}^\text{inc}_{k}(e; r_1)| = 6 + |\text{DoF}^\text{grad}_{k+2}(e; r_0)| + |\text{DoF}^\text{div}_{k-1}(e; r_2)| = |\text{DoF}^\text{L}^2_{k-2}(e; r_3)|.$$  

Then by the DoFs (16a)-(16d) of spaces $V_{k+2}(r_0; \mathbb{R}^3)$ and $V^L_{k-2}(r_3; \mathbb{R}^3)$, Euler’s formula $|\Delta_0| - |\Delta_1| + |\Delta_2| - |\Delta_3| = 1$, and the DoFs (27a) - (27f) of space $V^\text{div}_{k-1}(r_2; S)$, and identities (66), (70), (68), and (67), the number of DoFs (61) is

$$-6 + \dim P_{k+2}(T; \mathbb{R}^3) + \dim P_{k-1}(T; S) - \dim P_{k-2}(T; \mathbb{R}^3),$$

which equals $\dim P_{k+1}(T; S)$ in view of the polynomial elasticity complex (64). □

**Lemma 6.5.** The DoFs (61) are uni-olvent for $P_{k+1}(T; S)$.

**Proof.** By leveraging Lemma 6.4, our objective is to establish $\tau = 0$ when $\tau \in P_{k+1}(T; S)$ adheres to the condition that all the specified DoFs (61) vanish.

To commence, the vanishing (61a) implies that $\tau(v) = 0$ and $(\text{curl} \tau)^T(v) = 0$. This, combined with the vanishing DoFs (61c)-(61d), leads to $\tau|_e = 0$ and $(\text{curl} \tau)|_e = 0$. Employing integration by parts further yields:

$$\int_f \Pi_f(\text{inc} \tau) n \cdot q \, dS = \int_f \text{rot}_f(n \times (\text{curl} \tau)^T \Pi_f) \cdot q \, dS = 0, \quad q \in RT(f),$$  

$$\int_f n^T(\text{inc} \tau) n \cdot q \, dS = \int_f \text{div}_f \text{div}_f(n \times \tau \times n) q \, dS = 0, \quad q \in P_1(f).$$
More detailed descriptions of edge traces can be found in [15, Lemma 4.8]. This, combined
with the nullified DoFs (61a)-(61c), (61e), (61i)-(61j), and (61i), yields inc $\tau = 0$.

Consequently it implies that $\tau = \text{def}(u)$ for $u \in P_{k+2}(T; \mathbb{R}^3)$. Furthermore, the
nullification of DoFs (61a) and (61e) leads to $(\nabla^i u)(v) = 0$ for $v \in \Delta_0(T)$ and $i = 0, \ldots, r^i_1 + 1$, and $(\nabla^i u)|_e = 0$ for $e \in \Delta_1(T)$ and $i = 0, \ldots, r^i_1 + 1$. Similarly, from
the nullification of (61f)-(61h), we deduce that $(\nabla^i u)|_f = 0$ for $f \in \Delta_2(T)$ and $i = 0, \ldots, r^i_1 + 1$.

Combining these outcomes, it is apparent that $v \in \mathbb{B}_{k+2}(r_0)$. Consequently, $v = 0$ is
deduced from the vanishing DoF (61k).

Next we show the constructed $H(\text{inc})$-conforming finite element space is indeed the
space $V_{k+1}^{\text{inc}}(r_1; S)$ defined by (46) and used in the finite element elasticity complex (47).

**Lemma 6.6.** For $r^i_1 = -1, 0$, let

$$V_{k+1}^{\text{inc}}(r_1; S) := \{ \tau \in L^2(\Omega; S) : \tau|_T \in P_{k+1}(T; S) \text{ for all } T \in \mathcal{T}_h, \}
\text{ and all the DoFs (61) are single-valued} \}.$$

Then it is equal to the space $\{ \tau \in V_{k+1}(r_1) \otimes S : \text{inc } \tau \in V_{k-1}^{\text{div}}(r_2; S) \}$.

**Proof.** Clearly $V_{k+1}^{\text{inc}}(r_1; S) \subseteq \{ \tau \in V_{k+1}(r_1) \otimes S : \text{inc } \tau \in V_{k-1}^{\text{div}}(r_2; S) \}$. By (48) and
the proof of Lemma 6.4, their dimensions are equal. $\square$

There exist various variations of the finite element elasticity complexes. To illustrate
one of these variations, we construct the space $V_{k+1}^{\text{inc}}(r_1; S)$. In cases where $r^i_1 = -1$, we
introduce an additional face degree of freedom:

$$\int_f (n^T \tau f) \cdot q \, dS, \quad q \in \mathbb{B}_{k+1}^2(f; r_1). \tag{71}$$

Moreover, we modify (61k) to:

$$\int_T \tau : q \, dx, \quad q \in \text{def}(\mathbb{B}_{k+2}^\text{curl}(r_0, (r_1)_+)). \tag{72}$$

Recall that $r_0 \geq (2, 1, 0)^T$. For $u \in V_{k+1}^{\text{curl}}(r_0)$, the normal continuity of curl $u$ is always
maintained. To achieve continuity for all components, we require $\int_f n \times \text{curl } u \cdot q \, dS$ for
$q \in \mathbb{B}_{k+1}^2(f; r_1)$. This ensures a balance between the added DoFs in (71) and the reduced
DoFs from (61k) to (72), maintaining unisolvence in a similar manner.

An advantage of utilizing the space $V_{k+1}^{\text{inc}}(r_1; S)$ is the dimension reduction:

$$\dim V_{k+1}^{\text{inc}}(r_1; S) - \dim V_{k+1}^{\text{inc}}(r_1; S) = (4 |\Delta_3(\mathcal{T}_h)| - |\Delta_2(\mathcal{T}_h)|) \dim \mathbb{B}_{k+1}^2(f; r_1).$$

Another avenue is to relax the constraint to: $r_2 \geq \max\{r_1 \ominus 2, (0, -1, -1)^T\}$. With
this relaxed constraint, we define the space as follows:

$$V_{k+1}^{\text{inc}}(r_1, r_2; S) := \{ \tau \in V_{k+1}^{\text{inc}}(r_1; S) : \text{inc } \tau \in V_{k-1}^{\text{div}}(r_2; S) \}.$$

A finite element description of this space can be derived by first introducing the necessary
DoFs to determine inc $\tau$, and then eliminating any redundant DoFs. However, due to the
complexity of these variations, a detailed explanation is omitted here.
6.3. $H(\text{sym curl}; \mathbb{T})$-conforming elements. We will now provide a comprehensive description of the $H(\text{sym curl})$-conforming element space $V^\text{sym curl}_{k+1}(r_1; \mathbb{T})$. When $r_1^f \geq 1$, it is simply $V_{k+1}(r_1) \otimes \mathbb{T}$. So our focus is $r_1^f = -1$ or $r_1^f = 0$.

It is important to recall the trace complexes that were previously established in [17]:

\begin{align*}
\text{RT} & \subset v \xrightarrow{\text{grad}} \tau \xrightarrow{\text{curl}} \sigma \xrightarrow{\text{div div}} p \\
\mathbb{R} & \subset v \cdot n \xrightarrow{-\text{curl}_f} n \cdot \tau \xrightarrow{\text{div}_f} n \cdot \sigma \cdot n \to 0,
\end{align*}

and

\begin{align*}
\text{RT}_f & \subset \Pi_f v \xrightarrow{-\text{sym curl}} \tau \xrightarrow{\text{curl}} \sigma \xrightarrow{\text{div div}} p \\
\Pi_f & \xrightarrow{\Pi_f \text{sym}(\tau \times n) \Pi_f} \Pi_f \text{sym}(\tau \times n) \Pi_f \xrightarrow{\text{div div}_f} \Pi_f / \text{dev grad}_f(\sigma) \to 0.
\end{align*}

The trace complexes above play a crucial role in guiding the design of edge and face DoFs to ensure the necessary continuity. As shown in [17, Lemma 6.1], the expression

$$\text{tr}^{\text{div}_f \text{div}_j}(\Pi_f \text{sym}(\tau \times n) \Pi_f) = n^f_{r,e}(\text{curl} \tau)n_f + \partial_i (t^T \tau_t),$$

provides the motivation for introducing DoFs involving terms like $n^f_{r,e}(\text{curl} \tau)n_f + \partial_i (t^T \tau_t)$ on edges. In cases where $r_1^e = 0$, the focus is on enforcing the continuity of $n^e_{r_1}(\text{curl} \tau)n_1$ on edges, which aligns with the requirement skw curl $\bigcup V_{k+1}(r_1; M) \subseteq \text{mskw} V^\text{curl}_{k+1}(r_1 \otimes 1)$. The other edge traces further ensure the continuity of terms like $n^e_{r_1} \tau_t$ on edges.

The shape function space is $P_{k+1}(T; \mathbb{T})$. The degrees of freedom are

\begin{align*}
&\nabla^i \tau(v), \quad i = 0, \ldots, r_1^e, \\
&\text{sym curl} \tau(v), \quad \text{if } r_1^e = 0, \\
&\int_e \frac{\partial^j \tau}{\partial n^i_1 \partial n^{2-i}_e} : q \, ds, \quad q \in \mathbb{B}_{k+1-j}(e; r_1^e - j) \otimes \mathbb{T}, 0 \leq i \leq j \leq r_1^e, \\
&\int_e (n^e_{r_1} \tau_t)q \, ds, \quad q \in \mathbb{B}_{k+1}(e; r_1^e), \quad \text{if } r_1^e = -1, \\
&\int_e (n^e_{r_1}(\text{sym curl} \tau)n_j)q \, ds, \quad q \in \mathbb{B}_k(e; r_2^e), 1 \leq i \leq j \leq 2, \quad \text{if } r_2^e = -1, \\
&\int_f \tau n \cdot q \, dS, \quad q \in \mathbb{B}_{k+1}(f; r_1), \quad \text{if } r_1^f = 0, \\
&\int_f (n \cdot \tau \times n) \cdot q \, dS, \quad q \in \text{curl}_f \mathbb{B}_{k+2}(f; r_0), \\
&\int_f \text{div}_f(n \cdot \tau \times n) \cdot q \, dS, \quad q \in \mathbb{B}_k(f; (r_2)_+)/\mathbb{R}, \\
&\int_f \Pi_f \text{sym}(\tau \times n) \Pi_f : q \, dS, \quad q \in \text{sym curl}_f \mathbb{B}^2_{k+2}(f; r_0),
\end{align*}
The following polynomial

\( \text{(73)} \int_{f} \text{div} \text{div}(\text{sym}(\tau \times n)) \cdot q \, dS, \quad q \in \mathbb{B}_{k-1}(f; r_2 \ominus 1)/\mathbb{P}_1(f), \)

\( \text{(73i)} \int_{f} \Pi_{f}(\text{sym curl} \tau) \cdot n \cdot q \, dS, \quad q \in \mathbb{B}_{k}^{\text{div}}(f; r_2), \)

\( \text{(73m)} \int_{T} \tau : q \, dx, \quad q \in \text{dev grad } (\mathbb{B}_{k+2}^{3}(r_0)), \)

\( \text{(73n)} \int_{T} (\text{sym curl} \tau) : q \, dx, \quad q \in \mathbb{B}_{k}^{\text{div div}^+}(r_2; S) \cap \ker(\text{div div}) \)

for each \( v \in \Delta_0(T), e \in \Delta_1(T) \) and \( f \in \Delta_2(T) \).

\textbf{Lemma 6.7.} The following polynomial \( \text{div div complex} \)

\( \text{(74)} \quad \mathbb{R}T \xrightarrow{\text{dev grad}} \mathbb{P}_{k+1}(T; \mathbb{R}^3) \xrightarrow{\text{sym curl}} \mathbb{P}_k(T; S) \xrightarrow{\text{div div}} \mathbb{P}_{k-2}(T) \rightarrow 0 \)

is exact. Consequently, for integer \( k \geq 2 \),

\( -4 + \dim \mathbb{P}_{k+2}(T; \mathbb{R}^3) - \dim \mathbb{P}_{k+1}(T; T) + \dim \mathbb{P}_k(T; S) - \dim \mathbb{P}_{k-2}(T) = 0. \)

And, for all integers \( m \geq 0 \),

\( 4 + 3 \left( \frac{m + 4}{3} \right) - 8 \left( \frac{m + 3}{3} \right) + 6 \left( \frac{m + 2}{3} \right) - \left( \frac{m}{3} \right) = 0. \)

\textbf{Proof.} A proof of (74) can be found in [17]. A consequence of the exactness of (74) is the dimension identity (75). Identity (76) follows from (75) when \( m \geq 3 \) and can be verified directly for \( m = 0, 1, 2 \).

\textbf{Lemma 6.8.} The sum of the number of DoFs (73) equals \( \dim \mathbb{P}_{k+1}(T; T) \).

\textbf{Proof.} At each vertex, when \( r_1^v \geq 1 \), only DoFs (73a) exists with dimension \( 8\left(\frac{r_1^v + 3}{3}\right) \).

By (76), we have

\( \text{DoF}^{\text{sym curl}^+}(v; r_1) = -4 + 3 \left( \frac{r_1^v + 3}{3} \right) + 6 \left( \frac{r_1^2 + 3}{3} \right) - \left( \frac{r_1^3 + 3}{3} \right). \)

When \( r_1^v = 0 \), additional 6 DoFs (73b) are added but now \( r_1^2 = 0, r_1^3 = -1 \) and so (77) still holds.

On tetrahedron \( T \), the number of DoFs (73m)-(73n) is

\( 4 + \dim \mathbb{B}_{k+2}^3(r_0) + \dim \mathbb{B}_k^{\text{div div}^+}(r_2; S) - \dim \mathbb{B}_{k-2}(r_3), \)

as \( \text{div div } \mathbb{B}_k^{\text{div div}^+}(r_2; T) = \mathbb{B}_{k-2}(r_3)/\mathbb{R}T. \)

On each face, we consider \( r_1^f = -1 \) first. Sum of number of DoFs (73h) and (73j) is \( \dim \mathbb{B}_{k+2}^3(f; r_0) = |\text{DoF}_{k+1}^{\text{grad}}(f; r_0)|. \) Sum of (73i) and (73k) is \( \dim \mathbb{B}_k(f; (r_2)_+) + \dim \mathbb{B}_{k-1}(f; r_2 \ominus 1) - 4. \) Comparing with the face DoFs (33e)-(33g) for \( \mathbb{V}_k^{\text{div div}^+}(r_2; S) \), plus (73l), we conclude the dimension identity

\( |\text{DoF}_{k+1}^{\text{sym curl}^+}(f; r_1)| = -4 + |\text{DoF}_{k+2}^{\text{grad}}(f; r_0)| + |\text{DoF}_k^{\text{div div}^+}(f; r_2)|. \)

When \( r_1^f = 0, r_2^f = -1 \) and thus no change of \( |\text{DoF}_k^{\text{div div}^+}(f; r_2)|. \) But \( r_0^f = 1 \), one more layer in \( \text{DoF}_{k+1}^{\text{grad}}(f; r_0) \) is added for \( \partial_n \mathbf{v}: \dim \mathbb{B}_{k+2}^3(f; r_0 - 1) \), which matches the number of DoF (73g) added for \( r_1^f = 0. \) So (79) holds for both \( r_1^f = -1, 0 \). No face DoFs for \( \mathbb{V}_{k-2}^2(f; r_3) \) as \( r_3^f = -1. \)
On each edge, we separate into three cases.

1. When \( r_1^r \geq 1 \), only (73c) exists. We write out the dimension of edge DoFs for spaces in the complex

\[
4 + 3 \sum_{j=0}^{r_0^r} (j + 1)(k - 2r_1^r - 1 + j) - 8 \sum_{j=0}^{r_1^r} (j + 1)(k - 2r_1^r + j)
\]

(80)

\[
+ 6 \sum_{j=0}^{r_2^r} (j + 1)(k - 2r_1^r + 1 + j) - \sum_{j=0}^{r_4^r} (j + 1)(k - 2r_1^r + 3 + j).
\]

Let \( m = k - 2r_1^r - 1 \geq 0 \). We split (80) into terms containing \( m \) or not:

\[
I_1 = m \left[ 3 \left( r_1^r + 3 \right) - 8 \left( r_1^r + 2 \right) + 6 \left( r_4^r + 1 \right) - \left( r_1^r - 1 \right) \right],
\]

\[
I_2 = 4 + 3 \sum_{j=0}^{r_0^r} (j + 1) - 8 \sum_{j=0}^{r_1^r} (j + 1)^2 + 6 \sum_{j=0}^{r_2^r} (j + 1)(j + 2) - \sum_{j=0}^{r_4^r} (j + 1)(j + 4).
\]

By symbolical calculation, \( I_1 = 0 \) and \( I_2 = 0 \) for all integers \( r_1^r \geq 1 \) even for the case \((r_0^r, r_1^r, r_2^r, r_4^r) = (2, 1, 0, -1)\).

2. When \( r_1^r = r_2^r = 0, r_4^r = -1, r_0^r = -1 \), in (80) the last two terms disappeared. DoFs (73e) and (73f) are added for \( \text{DoF}^{\text{sym}}_{k+1}(e; r_1) \). The number of DoFs (73) is \( |\text{DoF}^{\text{div}}_{k+1}(e; r_1)| \).

Recall that \( m = k - 2r_1^r - 1 \). By direct calculation, \( |\text{DoF}^{\text{grad}}_{k+2}(e; r_0)| = 9m + 6 \). Sum of number of DoFs (73c) and (73e) is \( 8 \dim \mathbb{B}_{k+1}(e; r_1^r) + \dim \mathbb{B}_k(e; r_1^r + 1) = 9m + 10 \). Therefore we conclude

(81)

\[ |\text{DoF}^{\text{sym}}_{k+1}(e; r_1)| = 4 + |\text{DoF}^{\text{grad}}_{k+2}(e; r_0)| + |\text{DoF}^{\text{div}}_{k+1}(e; r_1)|. \]

No \( |\text{DoF}^{L^2}_{k-2}(e; r_3)| \) is in (81) as \( r_3^r = -1 \).

3. When \( r_1^r = -1, r_2^r = -1, r_0^r = 0 \), (73c) is further removed. Now (73d), (73e), and (73f) are present. The number of DoFs (73) is still \( |\text{DoF}^{\text{div}}_{k+1}(e; r_2)| \). The number of DoFs (73d)-(73e) is

\[ 2 \dim \mathbb{B}_{k+1}(e; r_1^r) + \dim \mathbb{B}_k(e; r_1^r + 1) = 3m + 1 = 4 + |\text{DoF}^{\text{grad}}_{k+2}(e; r_0)|. \]

So (81) still holds.

In summary, for all cases the number of DoFs (73c)-(73f) at an edge satisfies

(82)

\[ |\text{DoF}^{\text{sym}}_{k+1}(e; r_1)| = 4 + |\text{DoF}^{\text{grad}}_{k+1}(e; r_0)| + |\text{DoF}^{\text{div}}_{k+1}(e; r_2)| - |\text{DoF}^{L^2}_{k-2}(e; r_3)|. \]

Then combining (77), (82), (79), and (78), by the DoFs of spaces \( \mathbb{V}_{k+2}(r_1 + 1; \mathbb{R}^3) \), \( \mathbb{V}^{\text{div}}_{k+2}(r_2; \mathbb{S}) \) and \( \mathbb{V}^{\Delta}_{k-2}(r_3) \) and the Euler’s formulae \(|\Delta_0(T)| - |\Delta_1(T)| + |\Delta_2(T)| - |\Delta_3(T)| = 1\), the number of DoFs (73a)-(73n) is

\[-4 + \dim \mathbb{P}_{k+2}(T; \mathbb{R}^3) + \dim \mathbb{P}_k(T; \mathbb{S}) - \dim \mathbb{P}_{k-2}(T), \]

which equals \( \dim \mathbb{P}_{k+1}(T; T) \) in view of (75).

\[ \square \]

Lemma 6.9. The DoFs (73) are uni-solvent for \( \mathbb{P}_{k+1}(T; T) \).
Proof. In light of Lemma 6.8, we only need to prove \( \tau = 0 \) for \( \tau \in \mathbb{P}_{k+1}(T; \mathbb{T}) \) that all the specified DoFs (73) are nullified.

The vanishing DoF (73c) implies
\[
\int_e \frac{\partial^j (\text{sym curl } \tau)}{\partial n_1^i \partial n_2^{j-1}} : q \, dt = 0, \quad q \in \mathbb{B}_{k-j}(e; r_2^i - j) \otimes S, 0 \leq i \leq j \leq r_2^c.
\]

The vanishing DoF (73e) implies
\[
\int_{\tau} \text{tr}_2^{\text{div}} \text{tr}_2^{\text{div}} (\Pi_f \text{sym}(\tau \times n)) q \, dt = 0, \quad q \in \mathbb{B}_k(e; r_2^i - 1), \text{ if } r_2^e = -1, 0.
\]

Detailed expressions for these formulations are provided in [17, Lemma 6.1].

Applying the integration by parts, it follows from the vanishing DoFs (73c)-(73e) that
\[
\int_f \text{div}_f (n \cdot \tau \times n) \, dS = 0,
\]
\[
\int_f \text{div}_f \text{div}_f (\text{sym}(\tau \times n)) q \, dt = 0, \quad q \in \mathbb{P}_1(f).
\]

Using the identities
\[
\text{tr}_2^{\text{div}} \text{tr}_2^{\text{div}} (\text{sym curl } \tau) = \text{div}_f \text{div}_f \text{sym}(\tau \times n),
\]
\[
\text{tr}_2^{\text{div}} \text{tr}_2^{\text{div}} (\text{sym curl } \tau) = (\text{div} \text{sym curl } \tau) \cdot n + \text{div}_f (\Pi_f (\text{sym curl } \tau)n),
\]
the linear combination of DoFs (73k) and (73l) implies the continuity of \((\text{div} \text{sym curl } \tau) \cdot n\), i.e., \(\text{sym curl } \tau \in H(\text{div}, \Omega; S)\).

This, combined with the vanishing (73a)-(73c),(73e),(73f),(73i),(73k),(73l) and (73n), lead us to the conclusion that \(\text{sym curl } \tau = 0\). As a result, we find that \(\tau = \text{dev grad } v\) where \(v \in \mathbb{P}_{k+2}(T; \mathbb{R}^3)\).

Then the vanishing DoFs (73a) and (73c)-(73e) imply \((\nabla^i v)(v) = 0\) for \(v \in \Delta_0(T)\) and \(i = 0, \ldots, r_2^0\), and \((\nabla^i v)|_e = 0\) for \(e \in \Delta_1(T)\) and \(i = 0, \ldots, r_2^e\). By the vanishing DoFs (73h) and (73j), we get \((\nabla^i v)|_f = 0\) for \(f \in \Delta_2(T)\) and \(i = 0, \ldots, r_2^f\). Combining these results indicates \(v \in \mathbb{B}_k^+(r_0)\). Therefore \(v = 0\) holds from the vanishing DoF (73m).

Lemma 6.10. For \(i = -1, 0\), define
\[
\mathcal{V}_{k+1}^\text{sym curl} (r_1; T) := \{ \tau \in L^2(\Omega; \mathbb{T}) : \tau|_T \in \mathbb{P}_{k+1}(T; \mathbb{T}) \text{ for all } T \in \mathcal{T}_h, \text{ and all the DoFs (73) are single-valued} \}.
\]

Then it is equal to the space \{ \(\tau \in \mathcal{V}_{k+1}(r_1) \otimes \mathbb{T} : \text{sym curl } \tau \in \mathcal{V}_{k}^\text{div} \mathcal{V}_{k}^\text{div}^+ (r_2; S)\} \}

Proof. Clearly \(\mathcal{V}_{k+1}^\text{sym curl} (r_1; T) \subseteq \{ \tau \in \mathcal{V}_{k+1}(r_1) \otimes \mathbb{T} : \text{sym curl } \tau \in \mathcal{V}_{k}^\text{div} \mathcal{V}_{k}^\text{div}^+ (r_2; S)\} \}

By (54) and the proof of Lemma 6.8, their dimensions are equal.

To construct the space \(\mathcal{V}_{k+1}^\text{sym curl} (r_1; T)\), we will modify the DoFs (73). The changes involve removing the DoF (73l), and extending the DoF (73n) to a more general form:
\[
(83) \quad \int_T (\text{sym curl } \tau) : q \, dx, \quad q \in \mathbb{B}_k^\text{div} \mathcal{V}_{k}^\text{div}^+ (r_2; S) \cap \ker(\text{div div}).
\]

These modifications maintain the sum of DoFs unchanged, as determined by the bubble space definition. The unisolvence of the modified DoFs can be proven in a manner similar to the original ones.
For the case when \( r_1 = -1 \) or 0, we need to redefine the space as follows:

\[
\mathbb{V}_{k+1}^{\text{sym curl}}(r_1; T) := \{ \tau \in L^2(\Omega; T): \tau|_T \in \mathbb{P}_{k+1}(T; T) \text{ for all } T \in \mathcal{T}_h, \}
\]

and all DoFs (73a)-(73m), removing (73l) adding (83), are single-valued. By this construction, we ensure that sym curl \( \mathbb{V}_{k+1}^{\text{sym curl}}(r_1; T) \subset \mathbb{V}_k^{\text{div}}(r_2; \mathbb{S}) \). Thus, we obtain the complex (55). To establish its exactness, we verify the dimension identity:

\[
-4 + \dim \mathbb{V}_{k+2}^{\text{grad}}(r_0) - \dim \mathbb{V}_{k+1}^{\text{sym curl}}(r_1; T)
+ \dim \mathbb{V}_k^{\text{div}}(r_2; \mathbb{S}) - \dim \mathbb{V}_{k-2}^{L^2}(r_3) = 0,
\]

which can be derived from (54) by noting:

\[
\begin{align*}
\dim \mathbb{V}_{k+1}^{\text{sym curl}}(r_1; T) - \dim \mathbb{V}_{k+1}^{\text{sym curl}}(r_1; T) \\
= \dim \mathbb{V}_k^{\text{div}}(r_2; \mathbb{S}) - \dim \mathbb{V}_k^{\text{div}}(r_2; \mathbb{S}) \\
= (4|\Delta_3(\mathcal{T}_h)| - |\Delta_2(\mathcal{T}_h)|) \dim \mathbb{H}_k^{\text{div}}(f; r_2).
\end{align*}
\]

The removed DoF (73l) will contribute to the bubble functions.

**Remark 6.11.** When considering the div \( \text{div}^+ \) element, one can explore various variants such as the Hu-Zhang type element. This may lead to an increase in the number of edge DoFs for \( \mathbb{V}_k^{\text{div}^+}(r_2) \) when \( r_2 = -1 \). One can refer to (28c) in Remark 4.3 for details on these additional DoFs. However, the modifications introduced will not affect the dimension count for edge DoFs, as the added edge DoFs will correspond to \( |\text{DoF}_k^{\text{div}^+}(e; r_2)| \), thus preserving the relationship stated in (81).

There are more variants of sym curl elements. We can add edge continuity of \( t^t \tau \) and face continuity \( \int_f \text{skw}(\Pi_f \tau \times n) \) so that \( \tau \times n \) is continuous. Then we can construct finite element spaces \( \mathbb{V}_{k+1}^{\text{sym curl}^+}(r_1; T) \) and \( \mathbb{V}_{k+1}^{\text{sym curl}^+}(r_1; T) \). We can also relax to \( r_3 \geq r_2 \ominus 1 \) and impose condition

\[
\text{sym curl} \{ \tau \in \mathbb{V}_k^{\text{div}^+}(r_2, r_3; \mathbb{S}),
\]

which require additional DoFs for \( \text{div} \text{sym curl} \tau \in \mathbb{V}_{k-1}^{\text{div}}(r_3; \mathbb{S}) \). The divd complex in [23] belongs to this type of variant. Furthermore \( r_2 \) can be relaxed to \( r_2 \geq r_1 \ominus 1 \) but the modification of DoFs will be more involved and the lengthy detail is skipped here.

**References**

[1] D. N. Arnold. *Finite element exterior calculus*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018. 1
[2] D. N. Arnold, G. Awanou, and R. Winther. Finite elements for symmetric tensors in three dimensions. *Math. Comp.*, 77(263):1229–1251, 2008. 38
[3] D. N. Arnold, R. S. Falk, and R. Winther. Differential complexes and stability of finite element methods. II. The elasticity complex. In *Compatible spatial discretizations*, volume 142 of *IMA Vol. Math. Appl.*, pages 47–67. Springer, New York, 2006. 4
[4] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006. 1, 10
[5] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.),* 47(2):281–354, 2010. 1
[6] D. N. Arnold and K. Hu. Complexes from complexes. *Found. Comput. Math.*, 21(6):1739–1774, 2021. 1, 2, 4, 6, 9, 10
[7] D. N. Arnold and R. Winther. Mixed finite elements for elasticity. *Numer. Math.*, 92(3):401–419, 2002. 4
[8] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Differential operators on the base affine space and a study of \( g \)-modules. In *Lie groups and their representations (Proc. Sammer School, Bohai Janos Math. Soc., Budapest, 1971)*, pages 21–64, 1975. 4
[9] F. Brezzi, J. Douglas, Jr., and L. D. Marini. Recent results on mixed finite element methods for second order elliptic problems. In Vistas in applied mathematics, Transl. Ser. Math. Engrg., pages 25–43. Optimization Software, New York, 1986.

[10] L. Chen and X. Huang. Decoupling of mixed methods based on generalized Helmholtz decompositions. SIAM J. Numer. Anal., 56(5):2796–2825, 2018.

[11] L. Chen and X. Huang. Finite elements for div-div-conforming symmetric tensors. arXiv preprint arXiv:2005.01271, 2020.

[12] L. Chen and X. Huang. Geometric decomposition of div-conforming finite element tensors. arXiv preprint arXiv:2112.14351, 2021.

[13] L. Chen and X. Huang. Finite element complexes in two dimensions. arXiv preprint arXiv:2206.00851, 2022.

[14] L. Chen and X. Huang. Finite element de Rham and Stokes complexes in three dimensions. Math. Comp., 2022.

[15] L. Chen and X. Huang. A finite element elasticity complex in three dimensions. Math. Comp., 2022.

[16] L. Chen and X. Huang. Finite elements for div- and div-div-conforming symmetric tensors in arbitrary dimension. SIAM J. Numer. Anal., 2022.

[17] L. Chen and X. Huang. Finite elements for div div conforming symmetric tensors in three dimensions. Math. Comp., 2022.

[18] S. H. Christiansen, J. Gopalakrishnan, J. Guzmán, and K. Hu. A discrete elasticity complex on three-dimensional Alfeld splits. arXiv preprint arXiv:2009.07744, 2020.

[19] S. H. Christiansen, J. Hu, and K. Hu. Nodal finite element de Rham complexes. Numer. Math., 2018.

[20] M. Eastwood. A complex from linear elasticity. In The Proceedings of the 19th Winter School “Geometry and Physics” (Srní, 1999), pages 23–29, 2000.

[21] J. Hu and Y. Liang. Conforming discrete Gradgrad-complexes in three dimensions. Math. Comp., 2021.

[22] J. Hu, Y. Liang, and R. Ma. Conforming finite element divdiv complexes and the application for the linearized Einstein–Bianchi system. SIAM J. Numer. Anal., 2022.

[23] J. Hu, Y. Liang, R. Ma, and M. Zhang. New conforming finite element divdiv complexes in three dimensions. arXiv preprint arXiv:2204.07895, 2022.

[24] J. Hu, T. Lin, and Q. Wu. A construction of $C^r$ conforming finite element spaces in any dimension. Found. Comput. Math., 2023.

[25] J. Hu, R. Ma, and M. Zhang. A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids. Sci. China Math., 2021.

[26] J. Hu and S. Zhang. A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids. Sci. China Math., 2015.

[27] J.-C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. Numer. Math., 1986.

[28] M. Neilan. Discrete and conforming smooth de Rham complexes in three dimensions. Math. Comp., 2015.

[29] D. Pauly and W. Zulehner. The divDiv-complex and applications to biharmonic equations. Appl. Anal., 2020.

[30] R. Stenberg. A nonstandard mixed finite element family. Numer. Math., 2010.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT IRVINE, IRVINE, CA 92697, USA
Email address: chenlong@math.uci.edu

SCHOOL OF MATHEMATICS, SHANGHAI UNIVERSITY OF FINANCE AND ECONOMICS, SHANGHAI 200433, CHINA
Email address: huang.xuehai@sufe.edu.cn