Anticaustics in a Fabry-Perot interferometer

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We address the response of a Fabry-Perot interferometer to a monochromatic point source. We calculate the anticaustics (that is, the virtual wavefronts of null path difference) resulting from the successive internal reflections occurring in the system. They turn to be a family of ellipsoids (or hyperboloids) of revolution, which allows us to reinterpret the operation of the Fabry-Perot from a geometrical point of view that facilitates comparison with other apparently disparate arrangements, such as Young’s double slit. © 2022 Optica Publishing Group

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1. INTRODUCTION

Young’s double slit experiment [1] is the epitome of the phenomenon of interference and it is still found in any modern optics textbook [2]. Initially, the experiment was a crucial step in establishing the wave theory of light. Ironically, it played a quintessential role in elaborating the concept of wave-particle duality, the “central mystery” of quantum theory [3].

The standard interpretation of Young’s experiment is based on the superposition of the wavefronts emanating from two coherent point sources. Similar arrangements have also been developed during the XIXth century, including Fresnel’s biprism [4], Lloyd’s mirrors [5] and Billet’s split lens [6], among others. In all of these cases, two point sources (real or virtual) are required to explain the observed intensity pattern.

One might rightly wonder how this picture is modified when many mutually coherent waves are superposed. The Fabry-Perot (FP) interferometer is the paradigm of this situation. Apart from its luminosity, the distinctive feature of this setup is its narrow resonances, which is the basis for its extensive use in high-resolution spectroscopy, interferometry, and laser resonators [7, 8]. However, the ordinary approach assumes plane waves; that is, the source is at infinity. Therefore, it seems clear that to appreciate the analogies and differences between two- and multiple-source interference, one should look at the response of the FP to a point source.

In this paper, we take this unorthodox viewpoint. To this end, we analyze the wavefronts generated at each of the multiple reflections occurring in the FP. Although methods for obtaining the local properties of the refracted wavefronts are well established, the global properties are more elusive. The reason is that, in general, wavefronts may be very intricate surfaces, particularly in the neighborhood of a focus, and their analytic expressions might become quite involved [9].

However, it is often possible to extract from a family of wavefronts one that is simpler in form, whose global properties are more comprehensible, but which, nonetheless, is representative of the entire family. It corresponds to the privileged wavefront of zero algebraic optical path difference. This concept has been reinvented under a variety of names; an indicative list of dates where it appears with different designations has been compiled by Ouellette [10]: anticaustic (Bernouilli, 1692 [11]), secondary caustic (Quetelet, 1829 [12]), orthogonal trajectory (Cayley, 1857 [13]), simplest orthotomic curve (Her- man, 1900 [14]), zero phase-front (Eaton, 1952 [15]), emerging wavefront of null optical path (Damien, 1952 [16]), archetypal wavefront (Stavroudis, 1969 [17]), zero-distance phase front (Cornbleet, 1984 [18, 19]), and phase front (Avendaño-Alejo et al, 2015 [20]).

In a comprehensive historical review, Chastang and Farouki [21] endorse the use of anticaustic, returning to the original Bernouilli’s suggestion. This was strongly supported by the late Emil Wolf [22]. We believe that reintroducing the term is appropriate, and constitutes a tribute to a great geometer and to a great physicist.

The anticaustic has been explicitly calculated in a few simple cases, including refraction by a plane (ellipse/hyperbola) and refraction (Cartesian ovals) and reflection (limaçon of Pascal) by a sphere [10, 23]. Albeit the concept has been extensively used in microwaves [24], it has not received the attention it deserves in optics. We hope that this neglect will be repaired with our discussion of the FP and the notion of anticaustic will experience a merited revival, given its beauty and potential usefulness.
We will examine a simplified model of a FP interferometer, which consists of a plane parallel transparent plate of refractive index $n$ and thickness $d$ surrounded by a medium of refractive index $n'$. As heralded in the Introduction, we consider that the system is illuminated by the spherical wavefronts arising from an ideal point source located at the origin of the coordinate system. A typical ray defining the anticaustic is marked in bold blue, while the faint lines indicate other rays generated in the multiple reflections.

2. ANTICAUSTICS IN A FABRY-PEROT

We will examine a simplified model of a FP interferometer, which consists of a plane parallel transparent plate of refractive index $n$ and thickness $d$ surrounded by a medium of refractive index $n'$. As heralded in the Introduction, we consider that the system is illuminated by the spherical wavefronts arising from an ideal point source located at the origin of the coordinate system. A typical ray defining the anticaustic is marked in bold blue, while the faint lines indicate other rays generated in the multiple reflections.

Consider a typical ray from $P_0$ striking the first interface at an angle of incidence $\theta'$ after traveling a distance $\ell$. Inside the plate, it experiences multiple reflections: after $2k+1$ (with $k = 0, 1, \ldots$) of these reflections, the ray is transmitted, as indicated in the figure. Finally, let the transmitted ray travel a distance $\ell'$ in the second medium (of index $n'$).

By direct inspection, one can check that the coordinates of the endpoint $(x, y)$ of this ray can be expressed as

$$
\begin{align*}
x &= h + d + \ell' \cos \theta', \\
y &= h \tan \theta' + (2k + 1)d \tan \theta + \ell' \sin \theta',
\end{align*}
$$

(1)

where $\theta$ is the angle of incidence in the medium $n$, which is related to the angle of incidence according to Snell’s law $n' \sin \theta' = n \sin \theta$.

The optical path length of the considered ray, denoted by $\Delta$ and computed from the source $P_0$, is

$$
\Delta(x, y) = n' \ell + (2k + 1) \frac{nd}{\cos \theta} + n' \ell'.
$$

(2)

To ensure that $(x, y)$ define a wavefront, we have to impose that $\Delta(x, y)$ takes a constant value, much in the spirit of Malus-Dupin. This gives a one-parameter family of wavefronts which constitutes the image of the family of object wavefronts centered at $P_0$.

Out of this family, we extract a single wavefront, the anticaustic, for which the optical path length $\Delta$ in Eq. (2) is zero: this fixes at once

$$
\ell' = -\frac{h}{\cos \theta} - (2k + 1) \frac{nd}{\cos \theta'},
$$

(3)

where, for simplicity, we have introduced the relative index $n = n'/n$ and we have taken into account that $\ell = h/\cos \theta'$. The negative sign in the optical path length indicates that we are dealing with a virtual ray, marked with broken lines in Fig. 2.

Replacing this value of $\ell'$ in Eq. (1) we get

$$
\begin{align*}
X &= d - (2k + 1)nd \cos \theta' \cos \theta, \\
Y &= (2k + 1)d \tan \theta - (2k + 1)nd \sin \theta' \cos \theta',
\end{align*}
$$

(4)

where we denote by $(X, Y)$ the coordinates of the anticaustic point to distinguish them from the coordinates of the endpoint of the ray.

Finally, we use Snell’s law to eliminate $\theta'$; the result reads

$$
\begin{align*}
X &= d - (2k + 1)nd \sqrt{1 + (1 - n^2) \tan^2 \theta}, \\
Y &= (2k + 1)(1 - n^2)d \tan \theta.
\end{align*}
$$

(5)

This is the parametric form of the anticaustic we were looking for [25]. To make things crystal clear, the explicit equation can be easily obtained by eliminating $\theta$ between these two equations:

$$
\frac{(X - d)^2}{(2k + 1)^2n^2d^2} + \frac{Y^2}{(2k + 1)^2(n^2 - 1)d^2} = 1.
$$

(6)
we have plotted the first two anticaustics obtained
and thus, at a distance \( d \) of the source \( P_0 \). The respective
semiaxes are given by
\[
\begin{align*}
 a_k &= (2k + 1) d, \\
 b_k &= (2k + 1) \sqrt{n^2 - 1} d.
\end{align*}
\]
(7)

The distance from each focus to the center is
\( f_k = \sqrt{a_k^2 - b_k^2} = (2k + 1) d \). The eccentricity of all these
ellipses is the same; viz, \( \varepsilon = f_k/a_k = 1/n < 1 \). Note that these
elliptical wavefronts are independent of the distance \( d \), so they do not change in shape
nor position when the plate undergoes a translation; showing a
remarkable invariance.

In Fig. 2 we have plotted the first two anticaustics obtained
by a simple transmission \( (k = 0) \) and a transmission and two
internal reflections \( (k = 1) \). The physical meaning of these surfaces
is apparent from the figure. In practice, only a certain cap
is effective in each ellipse. In Fig. 2 we have marked these caps
for the plotted rays. The larger the plate, the greater the extension
of the cap. The limit of the effective region can be found
by looking at the ray that emerges forming a greater angle with
the normal to the interfaces; when virtually prolonged, it gives,
on the corresponding ellipse, the point that serves as the limit.

Consider, for example, the case \( k = 0 \). Suppose a spherical
wavefront originates from \( P_0 \) at \( t = 0 \) and, after refractions,
takes the form \( W \) at time \( t \). By propagating \( W \) backward in time
in the medium \( n' \) we get an initial wavefront \( W_0 \) at \( t = 0 \). Mutationis mutandis, the propagation of \( W_0 \) (without any refraction)
yields the true wavefront \( W \) at time \( t \).

As stressed in the Introduction, the wavefront \( W_0 \) was investi-
gated by Bernoulli. The term anticaustic, he concocted, has a
mathematical origin: actually the evolute of \( W_0 \) (i.e., the locus of
all its centers of curvature) is precisely the caustic. Conversely,
the wavefront \( W_0 \) is an involute of the caustic [26].

In consequence, the evolutes of Eq. (6) are the caustics
generated in the successive internal reflections. The calculation is
straightforward, with the result [27]
\[
\frac{n^2}{2/n} (X - d)^{2/3} + (n^2 - 1)^{1/3} y^{2/3} = (2k + 1)^{2/3} d^{2/3}.
\]
(8)
This constitutes a series of astroids, all centered at \((d, 0)\), in
turn the common center of the ellipses. The amazing properties
of these curves have been discussed in detail in the literature
[28, 29]. In Fig. 3 we plot the first anticaustic \((k = 0)\) and
its corresponding caustic. Although the astroid is a 4-cusped
curve, we concentrate on the horizontal left cusp; they appear
at positions
\[
P_k = d - (2k + 1) \frac{d}{n},
\]
(9)
as one can check directly from Eq. (8). They are thus separated
by a distance \( 2d/n \), as it shown in Fig. 4.

Once the anticaustic \( W_0 \) is known, the propagated wave-
front \( W \) is equidistant from \((or offset from)\) \( W_0 \). To obtain the
equidistant curve, we consider the \((oriented)\) normals of \( W_0 \):

3. DISCUSSION

The action of the FP can be seen as producing the superposition
of this set of ellipses (or hyperbolas) labeled by the integer \( k \).
In what follows, we concentrate on the case of elliptical
wavefronts, although the results can be directly translated to
the hyperbolic ones. All these wavefronts are concentric and
have common axes of directions, the major ones being coinci-
dent with the \( X \) axis. The center of all of them is at the point
\((d, 0)\) and thus, at a distance \( d \) of the source \( P_0 \). The respective
semiaxes are given by
\[
\begin{align*}
 a_k &= (2k + 1) d, \\
 b_k &= (2k + 1) \sqrt{n^2 - 1} d.
\end{align*}
\]
(7)

This represents a series of ellipses for the usual case \( n > 1 \)
\((n' < n)\), whereas for \( n < 1 \) \((n' > n)\) they are hyperbolas. These
wavefronts are unique and provide exact information about every
ray emanating from \( P_0 \) and refracted by the FP.

Fig. 3. (Left) First anticaustic of the FP as in Fig. 2, showing the corresponding caustic, which is the envelope of the refracted rays. We have used \( n = 1.3 \) and \( d = 2 \) (in arbitrary units). (Right) Propagation of the previous anticaustic at four different times. We have included spherical wavefronts (dotted circles), corresponding to a single point source located at the position \( P_0 \).
we have also included the spherical wavefronts as-
we can appreciate that
This is represented in Fig.
the equidistant curve is formed by the points at a distant \( vt \) (with \( v \) the velocity of light in the medium) in the forward di-
In Fig. 3 we have also included the spherical wavefronts as-
significant differences with the propagated anticaustics. For small aperture angles \( \theta' \), we are in the paraxial regime and the effective caps in the ellipses can be considered locally as spheres. In this case, the FP appears as the interference of the virtual point sources \( P_1, P_2, \ldots \), sepa-
In Fig. 4 we can appreciate that these virtual sources are precisely at the cusps of the caustics previously discussed. This makes clear contact with Young's ex-
In summary, we have made extensive use of the notion of anti-
As a final curiosity, we quote that this picture in terms of an-
the equidistant curve is smooth. But from some value of \( t \) on (namely, this critical value being the minimal curvature radius of the curve), the equidi-
successive images by the interfaces of the point \( P_0 \) are no single points, but the system presents aberrations. This is also appar-
the propagating anticaustics. For small \( \ell \) the equidistant curve is smooth. But from some value of \( t \) on (namely, this critical value being the minimal curvature radius of the curve), the equidistant curve ac-
these virtual sources get fainter farther away from
This number will depend on the reflectivity of the interfaces.

Fig. 4. First three anticaustics of the same FP as in Fig. 3, with

4. CONCLUDING REMARKS
In summary, we have made extensive use of the notion of anti-
We stress that the benefit of this approach lies not in any in-
Disclosures. The authors declare no conflicts of interest.

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