HOM-ALGEBRAS AND HOM-COALGEBRAS

ABDENACER MAKHLOUF AND SERGEI SILVESTROV

ABSTRACT. The aim of this paper is to develop the theory of Hom-coalgebras and related structures. After reviewing some key constructions and examples of quasi-deformations of Lie algebras involving twisted derivations and giving rise to the class of quasi-Lie algebras incorporating Hom-Lie algebras, we describe the notion and some properties of Hom-algebras and provide examples. We introduce Hom-coalgebra structures, leading to the notions of Hom-bialgebra and Hom-Hopf algebras, and prove some fundamental properties and give examples. Finally, we define the concept of Hom-Lie admissible Hom-coalgebra and provide their classification based on subgroups of the symmetric group.

1. INTRODUCTION

In [17, 26, 27], the authors have developed a general quasi-deformation scheme for Lie algebras of vector fields based on use of general $\sigma$-derivations, and shown that this natural quasi-deformation scheme leads to a new broad class of non-associative algebras with twisted six terms generalized Jacobi identities instead of the usual Jacobi identities of Lie algebras. The main initial motivation for this investigation was the goal of creating a unified general approach to examples of $q$-deformations of Witt and Virasoro algebras constructed in 1990-1992 in pioneering works [1, 2, 3, 4, 5, 6, 8, 24, 35], where in particular it was observed that in these examples some $q$-deformations of ordinary Lie algebra Jacobi identities hold. Motivated by these and the new examples arising as application of the general quasi-deformation construction of [17, 26, 27] on the one hand, and the desire to be able to treat within the same framework such well-known generalizations of Lie algebras as the color and super Lie algebras on the other hand, quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras were introduced in [17, 26, 27, 28]. In the subclass of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map. The main feature of quasi-Lie algebras and quasi-Hom-Lie algebras is that both the skew-symmetry and the Jacobi identity are twisted by several deforming twisting maps and also the Jacobi identity in quasi-Lie and quasi-Hom-Lie algebras in general contains six twisted triple bracket terms. The three terms twisted Jacobi identity of Hom-Lie algebras is obtained when the...
six terms in Jacobi identity of the quasi-Lie or of the quasi-Hom-Lie algebras can be combined pairwise in a suitable way. That possibility depends deeply on how the twisting maps interact with each other and with the bracket multiplication.

Quasi-Lie algebras is a broad class of non-associative algebras, defined in such a way that it encompasses Lie algebras, Lie superalgebras, color Lie algebras as well as numerous deformed algebras arising in connection with twisted, discrete or deformed generalizations and modifications of derivatives and corresponding generalizations, discrete versions of vector fields and differential calculus. It turns out that, among examples fitting within the framework of quasi-Lie algebras and subclasses of quasi-Lie and Hom-Lie algebras belong also various quantum deformations of Lie algebras, such as deformations and quasi-deformations of the Heisenberg Lie algebra, \( \mathfrak{sl}_2(\mathbb{K}) \), oscillator algebras and of other finite-dimensional Lie algebras and infinite-dimensional Lie algebras of Witt and Virasoro type important in Physics within the string theory, vertex operator models, quantum scattering, lattice models and other contexts, as well as various classes of quadratic and sub-quadratic algebras arising in connection to non-commutative geometry, twisted derivations and deformed difference operators and non-commutative differential calculi. Many such examples of algebras, which can in fact be shown to fit within the framework of quasi-Lie algebras in one or another way, can be found for instance in 

\[
1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 17, 21, 22, 23, 24, 25, 31, 32, 33, 34, 35, 45
\]

and in references cited therein.

In the paper [39], we provided a different way for constructing a subclass of quasi-Lie algebras, the Hom-Lie algebras, by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. To this end we defined the notion of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted by a linear map, and showed that the commutator product defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. We introduced also Hom-Lie-admissible algebras and more general \( G \)-Hom-associative algebras with subclasses of Hom-Vinberg and pre-Hom-Lie algebras, generalizing to the twisted situation Lie-admissible algebras, \( G \)-associative algebras, Vinberg and pre-Lie algebras respectively, and shown that for these classes of algebras the operation of taking commutator leads to Hom-Lie algebras as well. We constructed also all the twistings so that the brackets

\[
[x_1, x_2] = 2x_2, \ [x_1, x_3] = -2x_3, \ [x_2, x_3] = x_1
\]

determine a three dimensional Hom-Lie algebra. Finally, we provided for a subclass of twistings, the list of all three-dimensional Hom-Lie algebras. This list contains all three-dimensional Lie algebras for some values of structure constants. The notion, constructions and properties of the enveloping algebras of Hom-Lie algebras are yet to be properly studied in full generality. An important progress in this direction has been made in the recent work by D. Yau [46]. Also, in this connection it may be appropriate to mention that in [2], for so-called their \( q \)-Lie algebras, which are a special subclass of Hom-Lie algebras, a universal enveloping algebra has been defined, Poincare-Birkhoff-Witt basis was constructed and corresponding Poincare-Birkhoff-Witt type theorem has been proved in detail using the reduction system technique of Bergman’s Diamond Lemma. It is important challenging problem to develop further the proper notion and theory of universal enveloping algebras for general Hom-algebras and for general quasi-Hom-Lie and Quasi-Lie algebras. The recent works by Hellström [18, 19], pertaining to generalizations of Diamond Lemma and of reduction system technics to more general algebraic structures of operadic type, might be useful in this respect. The fundamentals of the formal deformation theory and associated cohomology structures for Hom-Lie algebras have been considered recently by the
authors in [41]. Simultaneously, D. Yau has developed elements of homology for Hom-Lie algebras in [47]. These directions of future investigation promise to be very fruitful.

Further development of the area requires a broader insight in various new Hom-algebraic structures, generalizing the corresponding key structures from the context of associative and Lie algebras. In the present paper we review some constructions and examples of quasi-Lie and Hom-Lie algebras and develop the coalgebra counterpart of the notions and results of [39], extending in particular in the framework of Hom-associative, Hom-Lie algebras and Hom-coalgebras, the notions and results on associative and Lie admissible coalgebras obtained in [15]. In this context, we also define structures of Hom-bialgebras, generalized Hom-bialgebras and Hom-Hopf algebras and describe some of their properties extending properties of bialgebras and Hopf algebras. More specifically, in Section 2 we recall the definitions of quasi-Lie algebras and their subclass of quasi-Hom-Lie algebras defined in [17, 26, 27, 28], and then review the method of constructing the quasi-Hom-Lie algebras via discrete modifications of vector fields using twisted derivations, and describe two classes of examples of multi-parameter families of algebras arising as application of this method, the quasi-Lie quasi-deformations of \( \mathfrak{sl}_2(\mathbb{K}) \) on the algebra of polynomials in nilpotent indeterminate and quasi-Lie algebras generalizing Witt (centerless Virasoro) algebras via discretizations by \( \sigma \)-derivations with general endomorphism \( \sigma \) of the algebra of Laurent polynomials. In Section 3 we summarize the relevant definitions of Hom-associative algebra, Hom-Lie algebra, Hom-Leibniz algebra and Hom-Poisson algebra. In this section we define the notions and describe some of basic properties of Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras which generalize the classical coalgebra, bialgebra and Hopf algebra structures. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra. We also consider a generalization of bialgebra structure in the spirit of Loday [36], where the algebra in no longer unital and the coalgebra is no longer counital, and extend it to our context. In [47], it is shown that starting from an algebra and algebra endomorphism, one can construct a Hom-associative algebra. We extend this result to Hom-associative coalgebras and generalized Hom-bialgebras and use it to provide some examples. The structures of Hom-coalgebra and the two kinds of Hom-bialgebra structures were already introduced in [37]. Recently, some developments on Hom-bialgebras were made in [48]. In Section 4, we introduce the concept of Hom-Lie admissible Hom-coalgebra, describe some useful relations between comultiplication, opposite comultiplication, the cocommutator defined as their difference, and their \( \beta \)-twisted coassociators and \( \beta \)-twisted co-Jacobi sums. We also introduce the notion of \( G \)-Hom-coalgebra for any subgroup \( G \) of permutation group \( S_3 \). We show that \( G \)-Hom-coalgebras are Hom-Lie admissible Hom-coalgebras, and also establish duality based correspondence between classes of \( G \)-Hom-coalgebras and \( G \)-Hom-algebras.

2. QUASI-HOM-LIE ALGEBRAS ASSOCIATED WITH \( \sigma \)-DERIVATIONS

Throughout this paper \( \mathbb{K} \) denotes a field of characteristic zero.

In this section we first recall the definitions of quasi-Lie algebras and their subclass of quasi-Hom-Lie algebras defined first in [17, 26, 27], and then review the construction of quasi-Hom-Lie algebras of discretizations of vector fields by twisted derivations. Let \( \mathcal{L}_\mathbb{K}(V) \) be the set of linear maps of the linear space \( V \) over the field \( \mathbb{K} \).

**Definition 2.1.** (Larsson, Silvestrov [27]) A quasi-Lie algebra is a tuple \((V, [-, -]_V, \alpha, \beta, \omega, \theta)\) where

- \( V \) is a linear space over \( \mathbb{K} \);
- \([-,-]_V : V \times V \to V\) is a bilinear map called a product or bracket in \( V \);
Definition 2.2. Let \( V \) be a \( \mathbb{K} \)-linear space, \( \{ \cdot, \cdot \} : \mathbb{K} \times V \rightarrow V \) is a bilinear map called a \emph{product}, and \( \alpha, \beta : V \rightarrow \mathbb{K} \) are linear maps, such that the following conditions hold:

- \( \alpha, \beta : V \rightarrow \mathbb{K} \) are linear maps;
- \( \omega : D_\theta \rightarrow \mathcal{L}_\mathbb{K}(V) \) and \( \theta : D_\omega \rightarrow \mathcal{L}_\mathbb{K}(V) \) are maps with domains of definition \( D_\omega, D_\theta \subseteq V \times V \),

such that the following conditions hold:

- (\( \omega \)-symmetry) The product satisfies a generalized skew-symmetry condition
  \[
  [x, y]_V = \omega(x, y)[y, x]_V, \quad \text{for all } (x, y) \in D_\omega;
  \]
- (quasi-Jacobi identity) The bracket satisfies a generalized Jacobi identity
  \[
  \bigotimes_{x,y,z} \{ \theta(z, x) \left( [\alpha(x), [y, z]_V] + \beta(x, [y, z]_V) \right) \} = 0,
  \]
  for all \( (z, x), (y, z), (y, z) \in D_\beta \) and where \( \bigotimes_{x,y,z} \) denotes summation over the cyclic permutation on \( x, y, z \).

Note that the twisting maps in the definition of quasi-Lie algebras are not arbitrary. For example the axioms of quasi-Lie algebra above imply some properties like

\[
(\omega(x, y)\omega(y, x) - \text{id})[x, y] = 0, \quad \text{if } (x, y), (y, x) \in D_\omega,
\]

which follows from the computation

\[
[x, y] = \omega(x, y)[y, x] = \omega(x, y)\omega(y, x)[x, y].
\]

The class of algebras introduced in Definition 2.1 incorporates as special cases \emph{Hom-Lie algebras} and \emph{quasi-Hom-Lie algebras} which appear naturally in the algebraic study of \( \sigma \)-derivations and related deformations of infinite-dimensional and finite-dimensional Lie algebras. To get the class of quasi-Hom-Lie algebras one specifies \( \theta = \omega \) and restricts to \( \alpha \) and \( \beta \) satisfying the twisting condition \( [\alpha(x), \alpha(y)] = \beta \circ \alpha[x, y] \).

**Definition 2.2.** A \emph{quasi-Hom-Lie algebra} is a tuple \((V, [\cdot, \cdot]_V, \alpha, \beta, \omega)\) where

- \( V \) is a \( \mathbb{K} \)-linear space,
- \([\cdot, \cdot]_V : V \times V \rightarrow V \) is a bilinear map called a \emph{product} or a \emph{bracket in} \( V \),
- \( \alpha, \beta : V \rightarrow \mathbb{K} \) are linear maps,
- \( \omega : D_\omega \rightarrow \mathcal{L}_\mathbb{K}(V) \) is a map with domain of definition \( D_\omega \subseteq V \times V \),

such that the following conditions hold:

- (\( \beta \)-twisting) The map \( \alpha \) is a \( \beta \)-twisted algebra homomorphism, i.e.
  \[
  [\alpha(x), \alpha(y)]_V = \beta \circ \alpha[x, y]_V, \quad \text{for all } x, y \in V;
  \]
- (\( \omega \)-symmetry) The product satisfies a generalized skew-symmetry condition
  \[
  [x, y]_V = \omega(x, y)[y, x]_V, \quad \text{for all } (x, y) \in D_\omega;
  \]
- (quasi-Hom-Lie Jacobi identity) The bracket satisfies a generalized Jacobi identity
  \[
  \bigotimes_{x,y,z} \{ \omega(z, x) \left( [\alpha(x), [y, z]_V] + \beta(x, [y, z]_V) \right) \} = 0,
  \]
  for all \( (z, x), (y, z), (y, z) \in D_\omega \).

Specifying in the definition of quasi-Lie algebras \( D_\omega = V \times V \), \( \beta = \text{id}_V \) and \( \theta(x, y) = \omega(x, y) = -\text{id}_V \) for all \( (x, y) \in D_\omega = D_\theta \), we get a subclass of Hom-Lie algebras with twisting linear map \( \alpha \), which is moreover an algebra homomorphism due to \( \beta \)-twisting axiom. This class of Hom-Lie algebras includes Lie algebras when \( \alpha = \text{id} \). We will come back to general Hom-Lie algebras in the forthcoming sections.

Important classes and examples of Quasi-Hom-Lie algebras and Hom-Lie algebras are obtained using a quasi-deformation procedure of discretizing vector fields by twisted
derivations. In this quasi-Lie deformation procedure we start with the Lie algebra $\mathfrak{g}$ we wish to deform, and let $\rho : \mathfrak{g} \to \text{Der}(\mathcal{A}) \subseteq \mathfrak{gl}(\mathcal{A})$ be a representation of $\mathfrak{g}$ in terms of derivations on some commutative, associative algebra with unity. The Lie structure on $\text{Der}(\mathcal{A})$ is given by the usual commutator bracket between linear operators. The quasi-deformation procedure changes first the involved derivations to $\sigma$-derivations, i.e., linear maps $\partial_{\sigma} : \mathcal{A} \to \mathcal{A}$ satisfying a generalized Leibniz rule: $\partial_{\sigma}(ab) = \partial_{\sigma}(a)b + \sigma(a)\partial_{\sigma}(b)$, for all $a, b \in \mathcal{A}$, and for an algebra endomorphism $\sigma$ on $\mathcal{A}$. The usual commutator of the obtained $\sigma$-derivations might be in general not a $\sigma$-derivation. Thus in the course of this deformation we also deform the commutator to a $\sigma$-twisted bracket $[\cdot, \cdot]$. The new product $[\cdot, \cdot]$ is defined and closed on the left $\mathcal{A}$-submodules $\mathcal{A} \cdot \partial_{\sigma}$ of $\text{Der}_{\sigma}(\mathcal{A})$, for each choice of $\partial_{\sigma} \in \text{Der}_{\sigma}(\mathcal{A})$. This is the content of Theorem 2.3 which establishes also a canonical Jacobi-like relation on $\mathcal{A} \cdot \partial_{\sigma}$ for $[\cdot, \cdot]$, reducing to the ordinary Jacobi identity when $\sigma = \text{id}$, i.e., in the "limit" case of this deformation scheme corresponding to the Lie algebra $\mathfrak{g}$. We remark that in some cases, for instance when $\mathcal{A}$ is a unique factorization domain, $\mathcal{A} \cdot \partial_{\sigma} = \text{Der}_{\sigma}(\mathcal{A})$ for suitable $\partial_{\sigma} \in \text{Der}_{\sigma}(\mathcal{A})$ (see [17]). In this scheme, we have two "deformation parameters" namely, $\mathcal{A}$ and $\sigma$. Note, however, that they are not independent. Indeed, $\sigma$ certainly depends on $\mathcal{A}$. The algebra structure on $\mathcal{A} \cdot \partial_{\sigma}$ is then pulled back to an abstract algebra which is then to be viewed as the quasi-deformed version of $\mathfrak{g}$. It might be so that one do not actually retrieve the original $\mathfrak{g}$ by performing appropriate (depending on the case considered) limit procedure. This is because for some "values" of the involved parameters the representation or specific operators might collapse, and sometimes taking the limit might even become meaningless in these circumstances. This is why we choose to call our deformations quasi-deformations. Another interesting phenomena that arises is that the pull-back might "forget relations". That is to say that the operators in $\mathcal{A} \cdot \partial_{\sigma}$ may satisfy relations, for instance coming from the twisted Leibniz rules, that the abstract algebra does not satisfy.

Let $\mathcal{A}$ be a commutative, associative $\mathbb{K}$-algebra with unity $1$. Furthermore $\sigma$ will denote an endomorphism on $\mathcal{A}$. Then by a twisted derivation or $\sigma$-derivation on $\mathcal{A}$ we mean an $\mathbb{K}$-linear map $\partial_{\sigma} : \mathcal{A} \to \mathcal{A}$ such that a $\sigma$-twisted Leibniz rule holds:

\begin{equation}
\partial_{\sigma}(ab) = \partial_{\sigma}(a)b + \sigma(a)\partial_{\sigma}(b).
\end{equation}

The best known $\sigma$-derivations are $(\partial a)(t) = a'(t)$, the ordinary differential operator with the ordinary Leibniz rule, i.e., $\sigma = \text{id}$; and $(\partial_{\sigma} a)(t) = (D_{\sigma} a)(t)$, the Jackson $q$-derivation operator with $\sigma$-Leibniz rule $(D_{\sigma} ab)(t) = (D_{\sigma} a)(t)b(t) + a(qt)(D_{\sigma} b)(t)$, where $\sigma = t_{\sigma}$ and $t_{\sigma}f(t) := f(qt)$. (See [20] and references there.)

We let $\mathcal{D}_{\sigma}(\mathcal{A})$ denote the set of $\sigma$-derivations on $\mathcal{A}$. Fixing a homomorphism $\sigma : \mathcal{A} \to \mathcal{A}$, an element $\partial_{\sigma} \in \mathcal{D}_{\sigma}(\mathcal{A})$, and an element $\delta \in \mathcal{A}$, we assume that these objects satisfy the following two conditions:

\begin{align}
\sigma (\text{Ann}(\partial_{\sigma})) &\subseteq \text{Ann}(\partial_{\sigma}), \\
\sigma \partial_{\sigma}(\sigma(a)) &= \delta \sigma(\partial_{\sigma}(a)), & \text{for } a \in \mathcal{A},
\end{align}

where $\text{Ann}(\partial_{\sigma}) := \{ a \in \mathcal{A} \mid a \cdot \partial_{\sigma} = 0 \}$. Let $\mathcal{A} \cdot \partial_{\sigma} = \{ a \cdot \partial_{\sigma} \mid a \in \mathcal{A} \}$ denote the cyclic $\mathcal{A}$-submodule of $\mathcal{D}_{\sigma}(\mathcal{A})$ generated by $\partial_{\sigma}$ and extend $\sigma$ to $\mathcal{A} \cdot \partial_{\sigma}$ by $\sigma(a) \cdot \partial_{\sigma} = \sigma(a) \cdot \partial_{\sigma}$. The following theorem, from [17], introducing an $\mathbb{K}$-algebra structure on $\mathcal{A} \cdot \partial_{\sigma}$ making it a quasi-Hom-Lie algebra, provides a method for construction of various classes and examples of quasi-Lie algebras using twisted derivations.
Theorem 2.3 (Hartwig, Larsson, Silvestrov [17]). If $\sigma(\operatorname{Ann}(\partial_\sigma)) \subseteq \operatorname{Ann}(\partial_\sigma)$, then the map $\lbrack \cdot , \cdot \rbrack_\sigma$ defined by

$$\lbrack a \cdot \partial_\sigma, b \cdot \partial_\sigma \rbrack_\sigma = (\sigma(a) \cdot \partial_\sigma) \circ (b \cdot \partial_\sigma) - (\sigma(b) \cdot \partial_\sigma) \circ (a \cdot \partial_\sigma),$$

for $a, b \in \mathcal{A}$ and where $\circ$ denotes composition of maps, is a well-defined $\mathbb{K}$-algebra product on the $\mathbb{K}$-linear space $\mathcal{A} \cdot \partial_\sigma$. It satisfies the following identities for $a, b, c \in \mathcal{A}$:

$$\lbrack a \cdot \partial_\sigma, b \cdot \partial_\sigma \rbrack_\sigma = (\sigma(a) \cdot \partial_\sigma)(b) - (\sigma(b) \cdot \partial_\sigma)(a) \cdot \partial_\sigma,$$

and if, in addition, (2.3) holds, we have the deformed six-term Jacobi identity

$$\lbrack a, b, c \rbrack_\sigma \equiv (\sigma(a) \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma])_\sigma + \delta \cdot [a \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]]_\sigma = 0.$$

The algebra $\mathcal{A} \cdot \partial_\sigma$ in the theorem is then a quasi-Hom-Lie algebra with $\alpha = \sigma, \beta = \delta$ and $\omega = -\operatorname{id}_\mathcal{A} \cdot \partial_\sigma$.

As example of application of the method in Theorem 2.3 we review the results in [28, 29] concerned with this quasi-deformation scheme when applied to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{K})$.

The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ can be realized as a vector space generated by elements $H, E$ and $F$ with the bilinear bracket product defined by the relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Our basic starting point is the following representation of $\mathfrak{sl}_2(\mathbb{K})$ in terms of first order differential operators acting on some vector space of functions in the variable $t$:

$$E \mapsto \partial, \quad H \mapsto -2t\partial, \quad F \mapsto -t^2\partial.$$

To quasi-deform $\mathfrak{sl}_2(\mathbb{K})$ means that we firstly replace $\partial$ by $\partial_\sigma$ in this representation. At our disposal are now the deformation parameters $\mathcal{A}$ (the "algebra of functions") and the endomorphism $\sigma$. After computing the bracket on $\mathcal{A} \cdot \partial_\sigma$ by Theorem 2.3 the relations in the quasi-Lie deformation are obtained by pull back.

Let $\mathcal{A}$ be a commutative, associative $\mathbb{K}$-algebra with unity 1, $t$ an element of $\mathcal{A}$, and let $\sigma$ denote an $\mathbb{K}$-algebra endomorphism on $\mathcal{A}$. Also, let $D_\sigma(\mathcal{A})$ denote the linear space of $\mathcal{A}$-derivations on $\mathcal{A}$. Choose an element $\partial_\sigma$ of $D_\sigma(\mathcal{A})$ and consider the $\mathbb{K}$-subspace $\mathcal{A} \cdot \partial_\sigma$ of elements on the form $a \cdot \partial_\sigma$ for $a \in \mathcal{A}$. We will usually denote $a \cdot \partial_\sigma$ simply by $a\partial_\sigma$. Notice that $\mathcal{A} \cdot \partial_\sigma$ is a left $\mathcal{A}$-module, and by Theorem 2.3 there is a skew-symmetric algebra structure on $\mathcal{A} \cdot \partial_\sigma$ given by

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma] = \sigma(a) \cdot \partial_\sigma(b \cdot \partial_\sigma) - \sigma(b) \cdot \partial_\sigma(a \cdot \partial_\sigma)$$

(2.9)

$$= (\sigma(a) \cdot \partial_\sigma)(b) - (\sigma(b) \cdot \partial_\sigma)(a) \cdot \partial_\sigma,$$

where $a, b \in \mathcal{A}$. The elements $e := \partial_\sigma, h := -2t\partial_\sigma$ and $f := -t^2\partial_\sigma$ span an $\mathbb{K}$-linear subspace $\mathcal{A} := \operatorname{LinSpan}_\mathbb{K}\{\partial_\sigma, -2t\partial_\sigma, -t^2\partial_\sigma\} = \operatorname{LinSpan}_\mathbb{K}\{e, h, f\}$ of $\mathcal{A} \cdot \partial_\sigma$. We restrict the multiplication (2.9) to $\mathcal{A}$ without, at this point, assuming closure. Now, $\partial_\sigma(t^2) = \partial_\sigma(t \cdot t) = \sigma(t) \partial_\sigma(t) + \partial_\sigma(t)t = (\sigma(t) + t) \partial_\sigma(t)$. Under the natural assumptions $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$ (see [27]), (2.9) leads to

$$[h, f] = 2\sigma(t)\partial_\sigma(t)\partial_\sigma$$

(2.10a)

$$[h, e] = 2\sigma(t)\partial_\sigma$$

(2.10b)

$$[e, f] = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma.$$

Remark 2.4. Note that when $\sigma = \operatorname{id}$ and $\partial_\sigma(t) = 1$ we retain the classical $\mathfrak{sl}_2(\mathbb{K})$ with relations (2.3).
In [28], we studied mostly some of the algebras appearing in the quasi-deformation scheme in the case when \( \mathcal{A} = \mathbb{K}[t] \). This resulted in new multi-parameter families of sub-quadratic algebras in particular containing for special choices of parameters known examples of Lie algebras, color Lie algebras, and \( q \)-deformed Lie algebras. One of such quasi-deformations of \( sl_2(\mathbb{K}) \) is Jackson \( sl_2(\mathbb{K}) \) corresponding to discretization of vector fields by Jackson \( q \)-derivative and thus to a corresponding special choice of deformation parameters defined by the choice of \( \mathcal{A}, \sigma \) and \( \partial_\sigma \). But we also have constructed there quasi-Lie deformations in the case \( \mathcal{A} = \mathbb{K}[t]/(t^2) \) yielding new interesting unexpected parametric families of algebras. In [29], we have constructed quasi-Lie deformations when \( \mathcal{A} = \mathbb{K}[t]/(t^4) \). This case leads typically to six relations instead of three which might have been thought as natural as \( sl_2(\mathbb{K}) \) only has three relations. We will now review an extension of the construction to the general class of quasi-Lie deformations when \( \mathcal{A} = \mathbb{K}[t]/(t^N) \) (see also [30]). We believe that this is a new, very interesting and rich multi-parameter family of sub-quadratic algebras that, when studied from various points of view, may reveal remarkable properties. It would be of interest to determine the ring-theoretic properties of these algebras e.g., for which values of the parameters are they domains, noetherian, PBW-remarkable properties.

Let \( \mathbb{K} \) include all \( N \)-th roots of unity and \( \mathcal{A} \) be the algebra \( \mathbb{K}[t]/(t^N) \) for a positive integer \( N \geq 2 \). This is obviously an \( N \)-dimensional \( \mathbb{K} \)-vector space and a finitely generated \( \mathbb{K}[t] \)-module with basis \( \{ 1, t, \ldots, t^{N-1} \} \). For \( i = 0, \ldots, N-1 \), let \( g_i = c_i t^i \partial_\sigma, \ c_i \neq 0 \). Put

\[
(2.11) \quad \partial_\sigma(t) = p(t) = \sum_{k=0}^{N-1} p_k t^k, \quad \sigma(t) = \sum_{k=0}^{N-1} q_k t^k,
\]

considering these as elements in the ring \( \mathbb{K}[t]/(t^N) \). The equalities (2.11) have to be compatible with \( t^N = 0 \). This means in particular that (if \( s(t) = (\sigma(t) - q_0)/t \))

\[
\sigma(t^N) = \sigma(t)^N = (q_0 + s(t)t)^N = \sum_{v=0}^{N} \binom{N}{v} q_0^v s(t)^{N-v} t^{N-v} = \sum_{v=1}^{N} \binom{N}{v} q_0^v (s(t))^{N-v} t^{N-v} = q_0 \sum_{v=1}^{N} \binom{N}{v} q_0^{v-1} (s(t))^{N-v} t^{N-v} = 0
\]

implying (and actually equivalent to) \( q_0^N = 0 \) and hence \( q_0 = 0 \). Furthermore,

\[
(2.12) \quad \partial_\sigma(t^N) = \sum_{j=0}^{N-1} \sigma(t)^j t^{N-j-1} \partial_\sigma(t) = p(t) \sum_{j=0}^{N-1} s(t)^j t^{N-j-1} = p(t) \sum_{j=0}^{N-1} s(t)^j = p_0 t^{N-1} \sum_{j=0}^{N-1} s(t)^j = p_0 \{N\}_q t^{N-1} = 0
\]

where \( \{N\}_q = \sum_{j=0}^{N-1} q_j^j \). It follows thus that

\[
(2.13) \quad (1 + q_1 + q_1^2 + \ldots + q_1^{N-1}) p_0 = 0.
\]

In other words, when \( p_0 \neq 0 \) we generate deformations at the zeros of the polynomial \( u^{N-1} + \ldots + u^2 + u + 1 \), that is at \( N \)’th roots of unity; whereas if \( p_0 = 0 \) then \( q_1 \) is a true formal deformation parameter.

As before we make the assumptions that \( \sigma(1) = 1, \partial_\sigma(1) = 0 \) and so relations (2.10a), (2.10b) and (2.10c) still hold. Moreover, since for \( k \geq 0 \) we have

\[
\partial_\sigma(t^{k+1}) = \sum_{j=0}^{k} \sigma(t)^j t^{k-j} \partial_\sigma(t) = p(t) t^k \sum_{j=0}^{k} s(t)^j.
\]
Using (2.9), we get
\[
[g_i, g_j] = c_i c_j [t^i \partial_\sigma, t^j \partial_\sigma] = c_i c_j [\sigma(t^i) \partial_\sigma(t^j) - \sigma(t^j) \partial_\sigma(t^i)] \partial_\sigma
\]
(2.14) \hspace{1cm} c_i c_j [\sigma(t^i) \partial_\sigma(t^j) - \sigma(t^j) \partial_\sigma(t^i)] \partial_\sigma = c_i c_j [\partial_\sigma(t^i) - \partial_\sigma(t^j)] \sigma(t^i) \partial_\sigma.

By (2.4) the bracket can be computed abstractly on generators \( g_i, g_j \) as
\[
[g_i, g_j] = [c_i t^i \partial_\sigma, c_j t^j \partial_\sigma] = c_i c_j \left( (\sigma(t^i) \partial_\sigma) \circ (t^j \partial_\sigma) - (\sigma(t^j) \partial_\sigma) \circ (t^i \partial_\sigma) \right)
\]
(2.15) \hspace{1cm} = c_i \sigma(t^j) \partial_\sigma \circ g_j - c_j \sigma(t^i) \partial_\sigma \circ g_i.

Expanding according to the multinomial formula \( s(t)^k = (q_1 + qt + \ldots + qt^{N-2})^k \) and \( \sigma(t^k) = \sigma(t^k) \sigma(t^k) = (q_1 + qt + \ldots + qt^{N-2})^k \) we obtain
\[
[g_i, g_j] = c_i (s(t)^i \partial_\sigma) \circ g_j - c_j (s(t)^j \partial_\sigma) \circ g_i;
\]
\[
= c_i ! \left( \sum_{\substack{i_1, \ldots, i_N \geq 0 \\text{ such that } i_1 + \ldots + i_N = j \times \text{ min}(i, j) \times \text{ gcd}(i, j) \text{ and } i_1 + \ldots + i_N - 1 < N}} \frac{q_1^{i_1} \cdots q_N^{i_N}}{i_1! \cdots i_N!} g_{i_1 + i_2 + \ldots + i_N - 1} \right) \cdot c_j ! \left( \sum_{\substack{j_1, \ldots, j_N \geq 0 \\text{ such that } j_1 + \ldots + j_N = j \times \text{ min}(i, j) \times \text{ gcd}(i, j) \text{ and } j_1 + \ldots + j_N - 1 < N}} \frac{q_1^{j_1} \cdots q_N^{j_N}}{j_1! \cdots j_N!} g_{j_1 + j_2 + \ldots + j_N - 1} \right).
\]

The bracket is closed on linear span of \( g_i \)'s as for \( N - 1 \geq i, j \geq 0 \), by (2.9), we get
\[
[g_i, g_j] = c_i c_j [\partial_\sigma(t^i) - \sigma(t^{i-j}) \partial_\sigma(t^j)] \sigma(t^i) \partial_\sigma
\]
\[
= c_i c_j \sum_{k=0}^{j-i-1} \text{sign}(j - i) \sum_{\substack{k_1, k_2, \ldots, k_{N-1} \geq 0 \\text{ such that } k_1 + k_2 + \ldots + k_{N-1} = (k + \text{ min}(i, j))! \times \text{ gcd}(i, j) \text{ and } 2k_1 + \ldots + (N-2)k_{N-1} < N}} \frac{1}{k_1! \cdots k_N!} \cdot \sum_{\substack{i_1, \ldots, i_N \geq 0 \\text{ such that } i_1 + \ldots + i_N = (k + \text{ min}(i, j))! \times \text{ gcd}(i, j) \text{ and } i_1 + \ldots + i_N - 1 < N}} \frac{q_1^{i_1} \cdots q_N^{i_N}}{i_1! \cdots i_N!} g_{i_1 + i_2 + \ldots + i_N - 1} \partial_\sigma.
\]
(2.16) \hspace{1cm} = c_i c_j \sum_{i=0}^{N-1} \sum_{k=0}^{j-i-1} \text{sign}(j - i) \sum_{\substack{k_1, k_2, \ldots, k_{N-1} \geq 0 \\text{ such that } k_1 + k_2 + \ldots + k_{N-1} = (k + \text{ min}(i, j))! \times \text{ gcd}(i, j) \text{ and } 2k_1 + \ldots + (N-2)k_{N-1} \leq N - i - j - 1}} \frac{1}{k_1! \cdots k_N!} \cdot \sum_{\substack{i_1, \ldots, i_N \geq 0 \\text{ such that } i_1 + \ldots + i_N = (k + \text{ min}(i, j))! \times \text{ gcd}(i, j) \text{ and } i_1 + \ldots + i_N - 1 < N - i - j - 1}} \frac{q_1^{i_1} \cdots q_N^{i_N}}{i_1! \cdots i_N!} g_{i_1 + i_2 + \ldots + i_N - 1} \partial_\sigma.
\]
(2.17) \hspace{1cm} \times \sum_{\substack{k_1, k_2, \ldots, k_{N-1} \geq 0 \\text{ such that } k_1 + k_2 + \ldots + k_{N-1} = (k + \text{ min}(i, j))! \times \text{ gcd}(i, j) \text{ and } 2k_1 + \ldots + (N-2)k_{N-1} \geq (N-1)}} \frac{q_1^{k_1} \cdots q_N^{k_{N-1}}}{c_{i_1 + i_2 + \ldots + i_N - 1}} g_{i_1 + i_2 + \ldots + i_N - 1 + k_1 + k_2 + \ldots + (N-2)k_{N-1}}.

where \( \text{sign}(x) = -1 \text{ if } x < 0, \text{sign}(x) = 0 \text{ if } x = 0 \text{ and } \text{sign}(x) = 1 \text{ if } x > 0 \).

Now we turn to the other example, quasi-Lie deformation of Witt algebra using \( \sigma \)-derivations on \( \mathbb{C}[t, t^{-1}] \), where \( \sigma \) is arbitrary endomorphism of the algebra \( \mathbb{C}[t^{-1}] \). These algebras were constructed by Hartwig, Larsson and Silvestrov in [17, Theorem 31] as an outcome of application of Theorem [2.3]. Let \( \mathcal{A} = \mathbb{K}[t, t^{-1}] \), the algebra of Laurent polynomials. Any endomorphism of \( \mathcal{A} = \mathbb{K}[t, t^{-1}] \) is uniquely determined by its action on the generator \( t \). So, assume that \( \sigma(t) = p(t) \neq 0 \in \mathcal{A} \). Note that \( \sigma(1) = 1 \) and \( \sigma(t^{-1}) = \sigma(t)^{-1} \) since \( \mathcal{A} \) has no zero-divisors. Hence, since \( \sigma(t) \) is invertible, \( \sigma(t) = p(t) = q^t \), for some \( q \in \mathbb{K} \setminus \{0\} \) and \( s \in \mathbb{Z} \). Since \( \mathcal{A} = \mathbb{K}[t, t^{-1}] \) is a unique factorization domain, if \( \sigma : \mathcal{A} \to \mathcal{A} \) is a homomorphism different from the identity map \( \text{id} - \sigma \), and the outer
twisting multiplier in 6 term Jacobi identity (2.7) can be computed then as \( \delta = \sigma(g)/g \) (see [17]). It suffices to compute a greatest common divisor of \((\text{id} - \sigma)(\mathcal{A})\) on the generator \( t \) since \( \sigma(t^{-1}) \) is determined by \( \sigma(t) \) and any gcd is only determined up to a multiple of an invertible element. Thus \( g = t^{-1}k^{-1}(t - qt) \) is a perfectly general gcd and \( D = t^{-k} \frac{\text{id} - \sigma}{1 - qt} \) is a generator for \( \mathcal{D}_\sigma(A) \) as a left \( \mathcal{A} \)-module. The \( \sigma \)-derivations on \( \mathbb{K}[t, t^{-1}] \) are on the form \( f(t) \cdot D \) for \( f \in \mathbb{K}[t, t^{-1}] \) and so, given that \( t^{s} \) is a linear basis of \( \mathbb{K}[t, t^{-1}] \) (over \( \mathbb{K} \)), \( -t^s \cdot D \) is a linear basis (over \( \mathbb{K} \) again) for \( \mathcal{D}_\sigma(\mathbb{K}[t, t^{-1}]) \). By Theorem 2.3 the linear space \( \mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{K} \cdot d_n \), where \( d_n = -t^n D, n \in \mathbb{Z} \) can be equipped with the bilinear bracket product defined on generators (by (2.4)) as \([d_n, d_m]_\sigma = q^m d_n d_m - q^n d_m d_n \) and satisfying defining commutation relations

\[
[d_n, d_m]_\sigma = \alpha \text{sign}(n - m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{(n+m-1)-(l-1)-(l-1)}
\]

for \( n, m \geq 0 \);

\[
[d_n, d_m]_\sigma = \alpha \left( \sum_{l=0}^{m-1} q^{n+l} d_{(m-l)(s-1)+n+s-m-k} + \sum_{l=0}^{n-1} q^{m+l} d_{(s-1)+n+ms-k-l} \right)
\]

for \( n \geq 0, m < 0 \);

\[
[d_n, d_m]_\sigma = -\alpha \left( \sum_{l=0}^{m-1} q^{n+l} d_{(s-1)+n+s-m-k} + \sum_{l=0}^{n-1} q^{m+l} d_{(s-1)+n+ms-k-l} \right)
\]

for \( m \geq 0, n < 0 \);

\[
[d_n, d_m]_\sigma = \alpha \text{sign}(n - m) \sum_{l=\min(n,m)}^{\max(-n,-m)-1} q^{n+m+l} d_{(m+n)s+(s-1)(l-k)}
\]

for \( n, m < 0 \),

skew-symmetry \([d_n, d_m]_\sigma = -[d_m, d_n]_\sigma\) and a twisted Jacobi identity

\[
\circ_{n,m,l}(q^n [d_n, [d_m, d_l]_\sigma]_\sigma + \delta [d_n [d_m, d_l]_\sigma]_\sigma) = 0,
\]

where \( \delta = q^k t^{(s-1)} \sum_{i=0}^{s-1} (qt^{-1})^i \in \mathbb{K}[t, t^{-1}] \). We get a family of quasi-Hom-Lie algebras deforming the Witt algebra. Here the twisting map \( \alpha \) is a linear map on \( \mathcal{D}_\sigma(\mathcal{A}) \) sending \( d_n \) to \( q^n d_n \), the twisting map \( \beta \) is a linear map on \( \mathcal{D}_\sigma(\mathcal{A}) \) acting as the multiplication by \( \delta \) from the left, and \( \omega = \theta = -\text{id} \).

A Hom-Lie algebra is obtained when \( \delta \in \mathbb{K} \), and this can be achieved only when \( s = 1 \), that is, when \( \sigma(t) = qt \) (See Example 3.13). If \( s = 1 \) then \([d_n, d_m]_q = q^n d_n d_m - q^m d_m d_n \) which is the bracket for the usual \( q \)-Witt algebra associated to discretizations by Jackson \( q \)-derivative [17, Theorem 27]), reducing further to the usual commutator for Witt Lie algebra if \( s = 1 \) and \( q = 1 \). For further analysis of these quasi-Lie algebras quasi-deformations of Witt algebra we refer to [43, 44].

3. Hom-Algebra and Hom-Coalgebra Structures

The notions of Hom-associative, Hom-Leibniz, and Hom-Lie-admissible algebraic structures was introduced in [39], generalizing the well known associative, Leibniz and Lie-admissible algebras. The Hom-Poisson algebra was introduced in [41], it is suitable for deformation theory of commutative Hom-associative algebras.

By dualization of Hom-associative algebra we define in the sequel the Hom-coassociative coalgebra structure.

Definition 3.1. A Hom-associative algebra is a triple \((V, \mu, \alpha)\) where \( V \) is a \( \mathbb{K} \)-linear space, \( \mu : V \otimes V \to V \) is a bilinear multiplication and \( \alpha : V \to V \) is a \( \mathbb{K} \)-linear space homomorphism satisfying the Hom-associativity condition

\[
\mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z)).
\]
The Hom-associativity condition \((3.1)\) may be expressed by the following commutative diagram.

\[
\begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{\mu \otimes \alpha} & V \otimes V \\
\downarrow{\alpha \otimes \mu} & & \downarrow{\mu} \\
V \otimes V & \xrightarrow{\mu} & V
\end{array}
\]

The Hom-associative algebra is said to be unital if there exists a homomorphism \(\eta : \mathbb{K} \to V\) such that the following diagrams are commutative

\[
\begin{array}{ccc}
\mathbb{K} \otimes V & \xrightarrow{\eta \otimes \text{id}} & V \otimes V \\
\downarrow{\sim} & & \downarrow{\sim} \\
\mathbb{K} & \xrightarrow{\mu} & V \otimes \mathbb{K}
\end{array}
\]

Let \((V, \mu, \alpha)\) and \((V', \mu', \alpha')\) be two Hom-associative algebras. A linear map \(f : V \to V'\) is said to be a morphism of Hom-associative algebras if

\[\mu' \circ (f \otimes f) = f \circ \mu, \quad f \circ \alpha = \alpha' \circ f\]

and \(f \circ \eta = \eta'\) if the Hom-algebras are unital with units \(\eta\) and \(\eta'\).

The tensor product of two Hom-associative algebras \((V_1, \mu_1, \alpha_1)\) and \((V_2, \mu_2, \alpha_2)\) is defined in an obvious way as the Hom-associative algebra \((V_1 \otimes V_2, \mu_1 \otimes \mu_2, \alpha_1 \otimes \alpha_2)\). If \(\eta_1\) and \(\eta_2\) are the units of these Hom-associative algebras, then the tensor product is also unital with the unit \(\eta_1 \otimes \eta_2\).

**Example 3.2.** Let \(\{x_1, x_2, x_3\}\) be a basis of a 3-dimensional linear space \(V\) over \(\mathbb{K}\). The following multiplication \(\mu\) and linear map \(\alpha\) on \(V\) define a Hom-associative algebra over \(\mathbb{K}^3\):

\[
\begin{align*}
\mu(x_1, x_1) &= ax_1, \\
\mu(x_1, x_2) &= \mu(x_2, x_1) = ax_2, \\
\mu(x_1, x_3) &= \mu(x_3, x_1) = bx_3, \\
\mu(x_2, x_2) &= bx_3, \\
\mu(x_2, x_3) &= \mu(x_3, x_2) = \mu(x_3, x_3) = 0.
\end{align*}
\]

\[\alpha(x_1) = ax_1, \quad \alpha(x_2) = ax_2, \quad \alpha(x_3) = bx_3\]

where \(a, b\) are parameters in \(\mathbb{K}\). This algebra is not associative when \(a \neq b\) and \(b \neq 0\), since

\[\mu(\mu(x_1, x_1), x_3) - \mu(x_1, \mu(x_1, x_3)) = (a-b)bx_3.\]

In \([47]\), D. Yau shows that one can construct a Hom-associative algebra starting from an associative algebra and an algebra endomorphism. Therefore using the following theorem, one can provide examples of Hom-associative algebras.

**Theorem 3.3** \([47]\). Let \((V, \mu)\) be an associative algebra and let \(\alpha : V \to V\) be an algebra endomorphism. Then \((V, \mu_\alpha, \alpha),\) where \(\mu_\alpha = \alpha \circ \mu\), is a Hom-associative algebra.

Moreover, suppose that \((V', \mu')\) is another associative algebra and \(\alpha' : V' \to V'\) is an algebra endomorphism. If \(f : V \to V'\) is an algebra morphism that satisfies \(f \circ \alpha = \alpha' \circ f\) then

\[f : (V, \mu_\alpha, \alpha) \longrightarrow (V', \mu'_\alpha, \alpha')\]

is a morphism of Hom-associative algebras.
Proof. We show that \((V, \mu_\alpha, \alpha)\) satisfies the Hom-associativity. Indeed
\[
\mu_\alpha(\alpha(x) \otimes \mu_\alpha(y \otimes z)) = \alpha(\mu(\alpha(x) \otimes \alpha(\mu(y \otimes z))))
\]
\[
= \alpha(\mu(\alpha(x) \otimes \alpha(y) \otimes \alpha(z)))
\]
\[
= \alpha(\alpha(\mu(\alpha(x) \otimes \alpha(y)) \otimes \alpha(z)))
\]
\[
= \alpha(\mu(\alpha(x \otimes y) \otimes \alpha(z)))
\]
\[
= \mu_\alpha(\mu_\alpha(x \otimes y) \otimes \alpha(z))
\]
The second assertion is proved similarly. \(\square\)

The free Hom-associative algebra was constructed in [46]. In the following, we give some identities satisfied by a Hom-associative algebra \((V, \cdot, \alpha)\) and a pentagonal diagram under certain assumptions.

**Lemma 3.4.** Let \((V, \cdot, \alpha)\) be a Hom-associative algebra. Then, for \(n \geq 2, x_0, x_1, \ldots, x_n \in V\) the following is true
\[
\alpha^{n-1}(x_0) \cdot (\alpha^{n-2}(x_1) \cdot (\alpha^{n-3}(x_2) \cdot \cdots (\alpha(x_{n-2}) \cdot (x_{n-1} \cdot x_n) \cdots )) =
\]
\[
((\cdots ((x_0 \cdot x_1) \cdot \alpha(x_2)) \cdots \alpha^{n-3}(x_{n-2})) \cdot \alpha^{n-2}(x_{n-1})) \cdot \alpha^{n-1}(x_n)
\]

**Proof.** The case \(n = 1\) corresponds to the associativity condition. For \(n = 2\), we have
\[
\alpha^2(x_0) \cdot (\alpha(x_1) \cdot (x_2 \cdot x_3)) = \alpha^2(x_0) \cdot ((x_1 \cdot x_2) \cdot \alpha(x_3))
\]
\[
= (\alpha(x_0) \cdot (x_1 \cdot x_2)) \cdot \alpha^2(x_3)
\]
\[
= ((x_0 \cdot x_1) \cdot \alpha(x_2)) \cdot \alpha^2(x_3)
\]

Similarly, one may obtain the complete proof by induction on \(n\). \(\square\)

**Remark 3.5.** If the homomorphism \(\alpha\) is invertible, the previous identity is equivalent to
\[
x_0 \cdot (x_1 \cdot (x_2 \cdot \cdots (x_{n-2} \cdot (x_{n-1} \cdot x_n) \cdots ))) =
\]
\[
((\cdots ((\alpha^{-1}(x_0) \cdot \alpha^2(x_1)) \cdot \alpha^3(x_2)) \cdots \alpha^{n-3}(x_{n-2})) \cdot \alpha^{n-2}(x_{n-1})) \cdot \alpha^{n-1}(x_n)
\]

In particular,
\[
x_0 \cdot (x_1 \cdot x_2) = (\alpha^{-1}(x_0) \cdot \alpha(x_1) \cdot \alpha(x_2))
\]
\[
x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)) = ((\alpha^{-2}(x_0) \cdot \alpha^{-1}(x_1)) \cdot \alpha(x_2)) \cdot \alpha^2(x_3)
\]
\[
x_0 \cdot (x_1 \cdot (x_2 \cdot (x_3 \cdot x_4))) = ((\alpha^{-3}(x_0) \cdot \alpha^{-2}(x_1)) \cdot \alpha(x_2) \cdot \alpha^2(x_3)) \cdot \alpha^2(x_4)
\]

To construct a pentagonal diagram, similar to Mac Lane pentagon, we start with \(\alpha^2(x_0) \cdot (\alpha(x_1) \cdot (x_2 \cdot x_3))\) and by Hom-associativity we have
\[
\alpha^2(x_0) \cdot (\alpha(x_1) \cdot (x_2 \cdot x_3)) = \alpha^2(x_0) \cdot ((x_1 \cdot x_2) \cdot \alpha(x_3))
\]
\[
= (\alpha(x_0) \cdot (x_1 \cdot x_2)) \cdot \alpha^2(x_3)
\]
\[
= ((x_0 \cdot x_1) \cdot \alpha(x_2)) \cdot \alpha^2(x_3)
\]
\[
= (\alpha(x_0) \cdot (x_1 \cdot x_2)) \cdot \alpha^2(x_3)
\]

In another hand
\[
\alpha^2(x_0) \cdot (\alpha(x_1) \cdot (x_2 \cdot x_3)) = (\alpha(x_0) \cdot \alpha(x_1)) \cdot \alpha(x_2 \cdot x_3)
\]
Therefore
\[ \alpha(x_0 \cdot x_1) \cdot (\alpha(x_2) \cdot \alpha(x_3)) = (\alpha(x_0) \cdot \alpha(x_1)) \cdot \alpha(x_2 \cdot x_3) \]

Then, if \( \alpha \) is an algebra homomorphism, that is \( \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y) \), the previous identity is satisfied and therefore we obtain the following commutative pentagonal diagram
\[
\begin{array}{c}
\alpha^2(x_0) \cdot (\alpha(x_1) \cdot (x_2 \cdot x_3)) \\
\alpha^2(x_0) \cdot ((x_1 \cdot x_2) \cdot \alpha(x_3)) \\
(\alpha(x_0) \cdot (x_1 \cdot x_2)) \cdot \alpha^2(x_3) \\
(\alpha(x_0) \cdot (x_1 \cdot x_2)) \cdot \alpha^2(x_3) \\
(\alpha(x_0) \cdot (x_1 \cdot x_2)) \cdot \alpha^2(x_3)
\end{array}
\]

The Hom-Lie algebras which is a particular case of quasi-Lie and quasi-Hom-Lie algebras is defined as follows.

**Definition 3.6.** A Hom-Lie algebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a linear space \(V\), bilinear map \([\cdot, \cdot] : V \times V \to V\) and a linear space homomorphism \(\alpha : V \to V\) satisfying
\[ [x, y] = -[y, x] \quad \text{(skew-symmetry)} \\
\bigcirc_{x,y,z} [\alpha(x), [y, z]] = 0 \quad \text{(Hom-Jacobi condition)} \]

for all \(x, y, z\) from \(V\).

**Example 3.7.** Let \(\{x_1, x_2, x_3\}\) be a basis of a 3-dimensional linear space \(V\) over \(\mathbb{K}\). The following bracket and linear map \(\alpha\) on \(V\) define a Hom-Lie algebra over \(\mathbb{K}^3\):
\[
\begin{align*}
[x_1, x_2] &= ax_1 + bx_3, & \alpha(x_1) &= x_1 \\
[x_1, x_3] &= cx_2, & \alpha(x_2) &= 2x_2 \\
[x_2, x_3] &= dx_1 + 2ax_3, & \alpha(x_3) &= 2x_3
\end{align*}
\]

with \([x_2, x_1]\), \([x_3, x_1]\) and \([x_3, x_2]\) defined via skew-symmetry. It is not a Lie algebra if and only if \(a \neq 0\) and \(c \neq 0\), since
\[ [x_1, [x_2, x_3]] + [x_3, [x_1, x_2]] + [x_2, [x_3, x_1]] = acx_2. \]

We call a triple \((V, \mu, \alpha)\) Hom-Lie admissible algebra if the commutator defined for \(x, y \in V\) by \([x, y] = \mu(x, y) - \mu(y, x)\) defines a Hom-Lie algebra \((V, [\cdot, \cdot], \alpha)\). The Hom-Lie admissible algebras were studied in [39], where we have shown among other things, that the Hom-associative algebras are Hom-Lie admissible, thus generalizing the well known fact that the commutator on associative algebra defines a Lie algebra structure.

**Proposition 3.8.** To any Hom-associative algebra defined by the multiplication \(\mu\) and a homomorphism \(\alpha\) over a \(\mathbb{K}\)-linear space \(V\), one may associate a Hom-Lie algebra defined for all \(x, y \in V\) by the bracket \([x, y] = \mu(x, y) - \mu(y, x)\).

Thus, we have a functor from the category HomAss of Hom-associative algebras into a category HomLie of Hom-Lie algebras. Its left adjoint functor was constructed by Yau in [40]. It corresponds to the enveloping algebra of a Hom-Lie algebra. The construction makes use of the combinatorial objects of weighted binary trees.

Removing the skew-symmetry and rearranging the Hom-Jacobi condition we get similarly the class of Hom-Leibniz algebra.

**Definition 3.9.** A Hom-Leibniz algebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a linear space \(V\), bilinear map \([\cdot, \cdot] : V \times V \to V\) and a homomorphism \(\alpha : V \to V\) satisfying
\[
[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]]. \]
Note that if a Hom-Leibniz algebra is skew-symmetric then it is a Hom-Lie algebra.

We introduce in the following definition Hom-Poisson structure involved naturally in the deformation theory of Hom-Lie algebras [41].

**Definition 3.10.** A Hom-Poisson algebra is a quadruple \((V, \mu, \{\cdot,\cdot\}, \alpha)\) consisting of a linear space \(V\), bilinear maps \(\mu : V \times V \rightarrow V\) and \(\{\cdot,\cdot\} : V \times V \rightarrow V\), and a linear space homomorphism \(\alpha : V \rightarrow V\) satisfying

1. \((V, \mu, \alpha)\) is a commutative Hom-associative algebra,
2. \((V, \{\cdot, \cdot\}, \alpha)\) is a Hom-Lie algebra,
3. for all \(x, y, z\) in \(V\),

\[
\{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\alpha(z), \{x, y\}).
\]

The condition (3.3) expresses the compatibility between the multiplication and the Poisson bracket. It can be reformulated equivalently as

\[
\{\mu(x, y), \alpha(z)\} = \mu(\{x, z\}, \alpha(y)) + \mu(\alpha(x), \{y, z\})
\]

for all \(x, y, z\) in \(V\). Note that in this form it means that \(ad_z(\cdot) = \{\cdot, z\}\) is a sort of generalization of derivation of associative algebra defined by \(\mu\), and also it resembles the identity (3.2) in the definition for Leibniz algebra.

It is good place now to give some examples of Hom-Lie algebras. It is well known that \(\mathfrak{sl}_2(\mathbb{K})\) is a rigid Lie algebra, that is every formal deformation is equivalent to trivial deformation. In the following, we provide examples of Hom-Lie algebras as deformations of the classical Lie algebra \(\mathfrak{sl}_2(\mathbb{K})\) defined by \([x_1, x_2] = 2x_2\), \([x_1, x_3] = -2x_3\), \([x_2, x_3] = x_1\) and \(q\)-deformed Witt algebra.

**Example 3.11.** We construct Hom-Lie infinitesimal formal deformations of \(\mathfrak{sl}_2(\mathbb{K})\) which are not Lie algebras. We consider the 3-dimensional Hom-Lie algebras with the skew-symmetric bracket \([\cdot, \cdot]\) and linear map \(\alpha_t\) defined as follows

\[
\begin{align*}
[x_1, x_2]_t &= a_1 tx_1 + (2 - a_2 t)x_2 \\
[x_1, x_3]_t &= a_3 tx_1 + a_4 tx_2 + (-2 + a_2 t)x_3, \\
[x_2, x_3]_t &= (1 - \frac{a_3}{a_2} t)x_1,
\end{align*}
\]

where \(a_1, a_2, a_3, a_4, b_1, b_2\) are parameters in \(\mathbb{K}\).

These Hom-Lie algebras become Lie algebras for all \(t\) if and only if \(a_1 = 0\) and \(a_3 = 0\), as follows from

\[
[x_1, [x_2, x_3]] + [x_3, [x_1, x_2]] + [x_2, [x_3, x_1]] = (2a_3 t - (a_2 a_3 + a_1 a_4) t^2)x_2 + (2a_1 t - a_1 a_2 t^2)x_3 = 0.
\]

**Example 3.12 (Jackson \(\mathfrak{sl}_2(\mathbb{K})\)).** We consider the Hom-Lie algebra Jackson \(\mathfrak{sl}_2(\mathbb{K})\) which is a Hom-Lie deformation of the classical Lie algebra \(\mathfrak{sl}_2(\mathbb{K})\). The Jackson \(\mathfrak{sl}_2(\mathbb{K})\) is related to Jackson derivations. As linear space, it is generated by \(x_1, x_2, x_3\) with the skew-symmetric brackets defined by

\[
[x_1, x_2] = 2x_2, \quad [x_1, x_3] = -2x_3 - 2t x_3, \quad [x_2, x_3] = x_1 + \frac{t}{2} x_1.
\]

The linear map \(\alpha_t\) is defined by

\[
\alpha_t(x_1) = x_1, \quad \alpha_t(x_2) = \frac{2 + t}{2(1 + t)} x_2 = x_2 + \sum_{k=0}^{\infty} \frac{(-1)^k}{2} t^k x_2, \quad \alpha_t(x_3) = x_3 + \frac{t}{2} x_3.
\]
The Hom-Jacobi identity is proved as follows. It is enough to consider it on $x_2$, $x_3$ and $x_1$:

$$\{\alpha(x_2), \{x_3, x_1\}_\sigma\} + \{\alpha(x_3), \{x_1, x_2\}_\sigma\} + \{\alpha(x_1), \{x_2, x_3\}_\sigma\} = 0.$$ 

(3.6)

Therefore, $\{\cdot, \cdot\}_\sigma$ defined on generators as

$$\{x_n, x_m\} = q^n x_n x_m - q^{-m} x_m x_n = \{x_n, x_m\} = (\{n\}_q - \{m\}_q)x_{n+m},$$

where $\{n\}_q = (q^n - 1)/(q - 1)$ for $q \neq 1$ and $\{n\}_q = n$. This bracket is skew-symmetric and satisfies the $\sigma$-deformed Jacobi identity

$$\{q^n x_n, \{x_m, x_l\}_\sigma\} + \{q^l x_l, \{x_m, x_n\}_\sigma\} + \{q^m x_m, \{x_n, x_l\}_\sigma\} = 0.$$ 

(3.6)

Example 3.13. Let $\mathcal{A}$ be the unique factorization domain $\mathbb{K}[z, z^{-1}]$, the Laurent polynomials in $z$ over the field $\mathbb{K}$. Then the space $\mathcal{P}_\sigma(\mathcal{A})$ can be generated by a single element $D$ as a left $\mathcal{A}$-module, that is, $\mathcal{P}_\sigma(\mathcal{A}) = \mathcal{A} : D$ (This is a special case of [17, Theorem 4].) When $\sigma(z) = qz$ with $q \neq 0$ and $q \neq 1$, one can take $D$ as $z$ times the Jackson $q$-derivative

$$D = \frac{id - \sigma}{1-q} : f(z) \mapsto \frac{f(z) - f(qz)}{1-q}.$$ 

The $\mathbb{K}$-linear space $\mathcal{P}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{K} \cdot x_n$, with $x_n = -\sigma^n D$ can be equipped with the skew-symmetric bracket $\{\cdot, \cdot\}_\sigma$ defined on generators as

(3.5)

$$\{x_n, x_m\} = q^n x_n x_m - q^{-m} x_m x_n = \{x_n, x_m\} = (\{n\}_q - \{m\}_q)x_{n+m},$$

where $\{n\}_q = (q^n - 1)/(q - 1)$ for $q \neq 1$ and $\{n\}_q = n$. This bracket is skew-symmetric and satisfies the $\sigma$-deformed Jacobi identity

$$\{q^n x_n, \{x_m, x_l\}_\sigma\} + \{q^l x_l, \{x_m, x_n\}_\sigma\} + \{q^m x_m, \{x_n, x_l\}_\sigma\} = 0.$$ 

Therefore, $(V, \{\cdot, \cdot\}_\sigma)$ is a $\mathcal{A}$-graded $\mathbb{K}$-algebra with the bilinear bracket defined on generators as $\{x_n, x_m\} = (\{n\}_q - \{m\}_q)x_{n+m}$ and the linear twisting map $\sigma : V \to V$ acting on generators as $\sigma(x_n) = (q^n + 1)x_n$ is an example of Hom-Lie algebra [22, 39] (see Definition 3.4 in this paper). Obviously this Hom-Lie algebra can be viewed as a $q$-deformation of Witt algebra in the sense that for $q = 1$ indeed one recovers the bracket and the commutation relations for generators of the Witt algebra. The definition of its generators using first order differential operators is recovered if one assumes that $D = \frac{x}{z^2}$ for $q = 1$ as one would expect from passing to a limit in the definition of the operator $D$.

It can be also shown that there is a central extension $\text{Vir}_q$ of this deformation in the category of Hom-Lie algebras with the deformed Jacobi identity (3.6), which is a natural $q$-deformation of the Virasoro algebra. The algebra $\text{Vir}_q$ is spanned by elements $\{x_n | n \in \mathbb{Z}\} \cup \{c\}$ where $c$ is central with respect to Hom-Lie bracket, i.e., $[\text{Vir}_q, c] = [c, \text{Vir}_q] = 0$. The bracket of $x_n$ and $x_m$ is computed according to

$$\{x_n, x_m\} = (\{n\}_q - \{m\}_q)x_{n+m} + \delta_{n+m,0} \frac{q^{-m}(n-1)\{n\}_q\{n+1\}_q}{6(1+q^n)}.$$ 

Note that when $q = 1$ we retain the classical Virasoro algebra

$$\{x_n, x_m\} = (n-m)x_{n+m} + \delta_{n+m,0} \frac{1}{12} (n-1)\{n\}_q\{n+1\}_q c.$$ 

from conformal field and string theories. Note also that when specializing $c$ to zero, or equivalently rescaling $c$ by extra parameter and then letting the parameter degenerate to zero, one recovers the $q$-deformed Witt algebra. Because of this the Witt algebra is called also, primarily in the physics literature, a centerless Virasoro algebra. In a similar way the $q$-deformed Witt algebra could be called a centerless $q$-deformed Virasoro algebra but of course with the word "central" used in terms of the Hom-Lie algebra bracket.

In [41] it was shown how the $q$-deformed Witt algebras can be viewed in the framework of deformation theory of Hom-Lie algebras.
Remark 3.14. In [17], the Example 3.13 has been presented as the simplest example of application of the general method for quasi-Lie quasi-deformation and discrete modification of Lie algebras of vector fields based on general $\sigma$-derivations. To our knowledge, analogous examples of $q$-deformed Witt algebras and $q$-Virasoro algebras associated to ordinary $q$-derivatives have been constructed for the first time in 1990 in [1, 2, 4, 6] where also the $q$-deformed Jacobi identities for these algebras have been discovered. The earliest paper we know, approaching such example of $q$-deformed Witt algebra and of the associated with it $q$-Virasoro algebra in the way closest to Hom-Lie algebras approach by systematically using the $q$-deformed Jacobi identity as the identity for a non-associative algebra for obtaining central extension, was the paper by Aizawa and Sato [1] from 1990-1991. In that paper, the authors achieve an important insight by defining a class of non-associative algebras basically almost as general as Hom-Lie algebras and use it systematically for construction of the $q$-Virasoro algebra as central extension in terms of this class of non-associative algebras. The only minor difference of their definition from Hom-Lie algebras (see Definition 3.6) is that they put extra condition that in addition to skew-symmetry and a Hom-Jacobi identity there must be a ”special limit” so that in the limit a Lie algebra is obtained. At the same time no precise meaning what ”special limit” should mean for the general class of non-associative algebras satisfying the skew symmetry and the Hom-Jacobi identity (in our terminology) is given. When imposed on this specific example, the meaning they use is exactly that by passing to the limit $q \to 1$ in the defining commutation relations one must recover the defining commutation relations of the classical Witt and Virasoro Lie algebras. In 1999 in [21], the Example 3.13 (up to minor change of deformation parameter and generators) of $q$-Witt and $q$-Virasoro algebras associated to Jackson $q$-derivative has been presented in a way close to the way it appeared in [17], namely using Jackson $q$-derivative by stressing the fact that it is a $\sigma$-derivation (skew-derivation), presenting the bracket multiplication and the same $q$-Jacobi identity as described in the Example 3.13 and then systematically using it for obtaining the relations for $q$-Virasoro algebra as a central extension in terms of this bracket and $q$-Jacobi identity. The article [21] is mainly concerned with this specific example and some modifications and with the specific for this example non-associative structure, thus fitting perfectly within the framework and the general method developed in [17]. The author of [21] also used the specific $q$-deformed Jacobi condition (3.6) satisfied in this particular example, the skew-symmetry, and the extra condition for the non-associative algebra being graded (but then not necessarily with one dimensional homogeneous components) to define what he called $q$-Lie algebras. He also provided more examples of such $q$-Lie algebras obtained as a sum of the $q$-Witt algebra and of quantum spaces. The $q$-Jacobi condition (3.6) is of cause a very specific example of the general operator twisted Jacobi condition of Hom-Lie algebras, and the condition that $q$-Lie algebras are graded is an additional restriction. Therefore, the class of $q$-Lie algebras from [21] is obviously just a specific subclass of Hom-Lie algebras. Also clearly, in the limit when $q \to 1$ this particular $q$-deformed Jacobi condition converges to Lie algebra Jacobi condition which means that the extra ”special limit” condition in the 1990-1991 paper of Aizawa and Sato is obviously satisfied for $q$-Lie algebras in this sense, thus making them a subclass with respect to the definition of Aizawa and Sato as well.

3.1. Hom-coalgebra, Hom-bialgebra and Hom-Hopf algebra structures. Now we introduce the notions of Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras and describe some properties of those structures extending the classical structures of coalgebras, bialgebra and Hopf algebra. We also define notions of modules and comodules over Hom-associative algebra and Hom-coassociative coalgebra. Also, by removing the unit and the
counit from the bialgebra structure, we obtain the notion of generalized bialgebra which we extend to our context. We prove, in the spirit of the theorem 3.3, that we can construct a Hom-associative coalgebra and a generalized Hom-bialgebra from a coalgebra (resp. generalized bialgebra) together with coalgebra morphism (resp. generalized bialgebra morphism).

**Definition 3.15.** A *Hom-coalgebra* is a triple $(V, \Delta, \beta)$ where $V$ is a $\mathbb{K}$-vector space and $\Delta : V \rightarrow V \otimes V$, $\beta : V \rightarrow V$ are linear maps.

A *Hom-coassociative coalgebra* is a Hom-coalgebra satisfying

$$ (\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta. $$

A Hom-coassociative coalgebra is said to be counital if there exists a map $\varepsilon : V \rightarrow \mathbb{K}$ satisfying

$$ (id \otimes \varepsilon) \circ \Delta = id \quad \text{and} \quad (\varepsilon \otimes id) \circ \Delta = id $$

The conditions 3.7 and 3.8 are respectively equivalent to the following commutative diagrams:

$$
\begin{array}{ccc}
V & \xrightarrow{\Delta} & V \otimes V \\
\downarrow{\Delta} & & \downarrow{\beta \otimes \Delta} \\
V \otimes V & \xrightarrow{\Delta \otimes \beta} & V \otimes V \otimes V
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{K} \otimes V & \xleftarrow{\varepsilon \otimes id} & V \otimes V \\
\uparrow{\Delta} & & \uparrow{\Delta} \\
V & \xrightarrow{id \otimes \varepsilon} & V \otimes \mathbb{K}
\end{array}
$$

Let $(V, \Delta, \beta)$ and $(V', \Delta', \beta')$ be two Hom-coalgebras (resp. Hom-associative coalgebras). A linear map $f : V \rightarrow V'$ is a *morphism of Hom-coalgebras* (resp. *Hom-coassociative coalgebras*) if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad f \circ \beta = \beta' \circ f.$$ 

If furthermore the Hom-coassociative coalgebras admit counits $\varepsilon$ and $\varepsilon'$, we have moreover $\varepsilon = \varepsilon' \circ f$.

The following theorem shows how to construct a Hom-coassociative Hom-coalgebra starting from a coalgebra and a coalgebra endomorphism. It is a coalgebra version of the theorem 3.3. We need only the coassociative comultiplication of the coalgebra. The existence of counit is not necessary.

**Theorem 3.16.** Let $(V, \Delta)$ be a coalgebra and let $\beta : V \rightarrow V$ be a coalgebra endomorphism. Then $(V, \Delta_{B}, \beta)$, where $\Delta_{B} = \Delta \circ \beta$, is a Hom-coassociative coalgebra.

Moreover, suppose that $(V', \Delta')$ is another coalgebra and $\beta' : V' \rightarrow V'$ is a coalgebra endomorphism. If $f : V \rightarrow V'$ is a coalgebra morphism that satisfies $f \circ \beta = \beta' \circ f$ then

$$ f : (V, \Delta_{B}, \beta) \longrightarrow (V', \Delta'_{B'}, \beta') $$

is a morphism of Hom-coassociative coalgebras.

**Proof.** We show that $(V, \Delta_{B}, \beta)$ satisfies the axiom 3.7.
Indeed, using the fact that \((\beta \otimes \beta) \circ \Delta = \Delta \circ \beta\), we have

\[
(\beta \otimes \Delta \beta) \circ \Delta \beta = (\Delta \otimes \beta) \circ \Delta \beta \\
= ((\Delta \circ \beta) \otimes \beta) \circ \Delta \circ \beta \\
= (((\beta \otimes \beta) \circ \Delta) \otimes \beta) \circ \Delta \circ \beta \\
= (\beta \otimes (\beta \otimes \beta)) \circ (id \otimes \Delta) \circ \Delta \circ \beta \\
= (\beta \otimes \beta \otimes \beta) \circ (id \otimes \Delta) \circ \Delta \circ \beta \\
= (\Delta \otimes \beta) \circ \Delta \beta.
\]

The second assertion is proved similarly:

\[
f \circ \Delta \beta = f \circ \Delta \circ \beta = \Delta' \circ f \circ \beta = \Delta' \circ \beta' \circ f = \Delta' \circ f.
\]

\[\square\]

We introduce in the following, the structures of module and comodule over Hom-associative algebras and Hom-coassociative coalgebras and Hom-bialgebras.

Let \(\mathcal{A} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)\) be a Hom-associative \(K\)-algebra, an \(\mathcal{A}\)-module (left) is a triple \((M, f, \gamma)\) where \(M\) is \(K\)-vector space and \(f, \gamma\) are \(K\)-linear maps, \(f: M \to M\) and \(\gamma: V \otimes M \to M\), such that the following diagram commutes:

\[
\begin{array}{ccc}
V \otimes V \otimes M & \overset{\mu \otimes f}{\longrightarrow} & V \otimes M \\
\downarrow_{\alpha \otimes \gamma} & & \downarrow_{\gamma} \\
V \otimes M & \overset{\gamma}{\longrightarrow} & M
\end{array}
\]

The dualization leads to comodule definition over a Hom-coassociative coalgebra.

Let \(C = (V, \Delta, \beta)\) be a Hom-coassociative coalgebra. A \(C\)-comodule (right) is a triple \((M, g, \rho)\) where \(M\) is \(K\)-vector space and \(g, \rho\) are \(K\)-linear maps, \(g: M \to M\) and \(\rho: M \to M \otimes V\), such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \overset{g \circ \Delta}{\longrightarrow} & M \otimes V \\
\downarrow_{\rho} & & \downarrow_{g \otimes \Delta} \\
M \otimes V & \overset{\rho \otimes \beta}{\longrightarrow} & M \otimes V \otimes V
\end{array}
\]

**Remark 3.17.** A Hom-associative \(K\)-algebra \(\mathcal{A} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)\) is a left \(\mathcal{A}\)-module with \(M = V\), \(f = \alpha\) and \(\gamma = \mu\). Also, a Hom-coassociative coalgebra \(C = (V, \Delta, \beta)\) is a right \(C\)-comodule with \(M = V\), \(g = \beta\) and \(\rho = \Delta\).

**Definition 3.18.** A **Hom-bialgebra** is a 7-uple \((V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)\) where

1. \((V, \mu, \alpha, \eta)\) is a Hom-associative algebra with unit \(\eta\).
2. \((V, \Delta, \beta, \varepsilon)\) is a Hom-coassociative coalgebra with counit \(\varepsilon\).
3. The linear maps \(\Delta\) and \(\varepsilon\) are compatible with the multiplication \(\mu\), that is

\[
\begin{align*}
\Delta(e_1) &= e_1 \otimes e_1 & \text{where } e_1 &= \eta(1) \\
\Delta(\mu(x \otimes y)) &= \Delta(x) \bullet \Delta(y) &= \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\
\varepsilon(e_1) &= 1 \\
\varepsilon(\mu(x \otimes y)) &= \varepsilon(x) \varepsilon(y)
\end{align*}
\]

where the bullet \(\bullet\) denotes the multiplication on tensor product and by using the Sweedler’s notation \(\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}\). If there is no ambiguity we denote the multiplication by a dot.
Remark 3.19. One can consider a more restrictive definition where linear maps $\Delta$ and $\varepsilon$ are morphisms of Hom-associative algebras that is the condition (B3) becomes equivalent to

$$\begin{aligned}
\Delta(e_1) &= e_1 \otimes e_1 \quad \text{where } e_1 = \eta(1) \\
\Delta(\mu(x \otimes y)) &= \Delta(x) \bullet \Delta(y) = \sum_{(x,y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\
\varepsilon(e_1) &= 1 \\
\varepsilon(\mu(x \otimes y)) &= \varepsilon(x) \varepsilon(y) \\
\Delta(\alpha(x)) &= \sum_{(x)} \alpha(x^{(1)}) \otimes \alpha(x^{(2)}) \\
\varepsilon \circ \alpha(x) &= \varepsilon(x)
\end{aligned}$$

Given a Hom-bialgebra $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, we show that the vector space $\text{Hom}(V, V)$ with the multiplication given by the convolution product carries a structure of Hom-associative algebra.

**Proposition 3.20.** Let $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra. Then the algebra $\text{Hom}(V, V)$ with the multiplication given by the convolution product defined by

$$f \ast g = \mu \circ (f \otimes g) \circ \Delta$$

and the unit being $\eta \circ \varepsilon$ is a unital Hom-associative algebra with the homomorphism map defined by $\gamma(f) = \alpha \circ f \circ \beta$.

**Proof.** Let $f, g, h \in \text{Hom}(V, V)$.

$$\begin{aligned}
\gamma(f) \ast (g \ast h) &= \mu \circ (\gamma(f) \otimes (g \ast h)) \Delta \\
&= \mu \circ (\gamma(f) \otimes (\mu \circ (g \otimes h) \circ \Delta)) \Delta \\
&= \mu \circ (\alpha \otimes \mu) \circ (f \otimes g \otimes h) \circ (\beta \otimes \Delta) \Delta.
\end{aligned}$$

Similarly

$$(f \ast g) \ast \gamma(h) = \mu \circ (\mu \otimes \alpha) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \beta) \Delta.$$

Then, the Hom-associativity of $\mu$ and a Hom-coassociativity of $\Delta$ lead to the Hom-associativity of the convolution product.

The map $\eta \circ \varepsilon$ is the unit for the convolution product. Indeed, let $f \in \text{Hom}(V, V)$ and $x \in V$,

$$\begin{aligned}
(f \ast (\eta \circ \varepsilon))(x) &= \mu \circ (f \otimes \eta \circ \varepsilon) \circ \Delta(x) = \sum_{(x)} \mu(x^{(1)} \otimes \eta \circ \varepsilon(x^{(2)})) = \\
\sum_{(x)} \epsilon(x^{(2)}) \mu(f(x^{(1)}) \otimes \eta(1)) &= \sum_{(x)} \epsilon(x^{(2)}) f(x^{(1)}) = \sum_{(x)} f(x^{(1)} \epsilon(x^{(2)})) = f(x).
\end{aligned}$$

Similar calculation shows that $(\eta \circ \varepsilon) \ast f = f$. \hfill $\square$

**Definition 3.21.** An endomorphism $S$ of $V$ is said to be an antipode if it is the inverse of the identity over $V$ for the Hom-associative algebra $\text{Hom}(V, V)$ with the multiplication given by the convolution product defined by

$$f \ast g = \mu \circ (f \otimes g) \Delta$$

and the unit being $\eta \circ \varepsilon$.

The condition being antipode may be expressed by the condition:

$$\mu \circ S \otimes Id \circ \Delta = \mu \circ Id \otimes S \circ \Delta = \eta \circ \varepsilon.$$

**Definition 3.22.** A Hom-Hopf algebra is a Hom-bialgebra with an antipode.
Then, a Hom-Hopf algebra over a $K$-vector space $V$ is given by

$$\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$$

where the following homomorphisms

$$\begin{align*}
\mu : V \otimes V &\to V, \quad \eta : K \to V, \quad \alpha : V \to V \\
\Delta : V &\to V \otimes V, \quad \varepsilon : V \to K, \quad \beta : V \to V \\
S : V &\to K
\end{align*}$$

satisfy the following conditions

1. $(V, \mu, \alpha, \eta)$ is a unital Hom-associative algebra.
2. $(V, \Delta, \beta, \varepsilon)$ is a counital Hom-coalgebra.
3. $\Delta$ and $\varepsilon$ are compatible with the multiplication $\mu$, that is

$$\begin{align*}
\Delta(e_1) &= e_1 \otimes e_1 \quad \text{where } e_1 = \eta(1) \\
\Delta(x \cdot y) &= \Delta(x) \bullet \Delta(y) = \sum_{(x,y)} x^{(1)} \cdot y^{(1)} \otimes x^{(2)} \cdot y^{(2)} \\
\varepsilon(e_1) &= 1 \\
\varepsilon(x \cdot y) &= \varepsilon(x) \varepsilon(y)
\end{align*}$$

4. $S$ is the antipode:

$$\mu \circ S \otimes Id \circ \Delta = \mu \circ Id \otimes S \circ \Delta = \eta \circ \varepsilon.$$

Now we consider some antipode’s properties of Hom-Hopf algebras.

Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ be a Hom-Hopf algebra. For any element $x \in V$, using the counity and Sweedler notation, one may write

$$(3.9) \quad x = \sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes x^{(2)}.$$

Then, for any $f \in \text{End}_K(V)$, we have

$$(3.10) \quad f(x) = \sum_{(x)} f(x^{(1)}) \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes f(x^{(2)}).$$

Let $f \ast g = \mu \circ (f \otimes g) \Delta$ be the convolution product of $f, g \in \text{End}_K(V)$. One may write

$$(3.11) \quad (f \ast g)(x) = \sum_{(x)} \mu(f(x^{(1)}) \otimes g(x^{(2)})).$$

Since the antipode $S$ is the inverse of the identity for the convolution product then $S$ satisfies

$$(3.12) \quad \varepsilon(x) \eta(1) = \sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)}) = \sum_{(x)} \mu(x^{(1)} \otimes S(x^{(2)})).$$

**Proposition 3.23.** We have the following properties of the antipode when it exists :

1. The antipode $S$ is unique,
2. $S(\eta(1)) = \eta(1),$
3. $\varepsilon \circ S = \varepsilon.$

**Proof.** 1) We have $S \ast \text{id} = \text{id} \ast S = \eta \circ \varepsilon$. Thus, $(S \ast \text{id}) \ast S = S \ast (\text{id} \ast S) = S$. If $S'$ is another antipode of $\mathcal{H}$ then

$$S' = S' \ast \text{id} \ast S' = S' \ast \text{id} \ast S = S \ast \text{id} \ast S = S.$$

Therefore, the antipode when it exists is unique.

2) Setting $e_1 = \eta(1)$ and since $\Delta(e_1) = e_1 \otimes e_1$ one has

$$(S \ast \text{id})(e_1) = \mu(S(e_1) \otimes e_1) = S(e_1) = \eta(\varepsilon(e_1)) = e_1.$$
3) Applying (3.10) to $S$, we obtain $S(x) = \sum_{(x)} S(x^{(1)})\varepsilon(x^{(2)})$.

Applying $\varepsilon$ to (3.12), we obtain

$$\varepsilon(x) = \varepsilon(\sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)})).$$

Since $\varepsilon$ is compatible with the multiplication $\mu$, one has

$$\varepsilon(x) = \sum_{(x)} \varepsilon(S(x^{(1)}))\varepsilon(x^{(2)}) = \varepsilon(\sum_{(x)} S(x^{(1)})\varepsilon(x^{(2)})) = \varepsilon(S(x)).$$

Thus $\varepsilon \circ S = \varepsilon$. 

Group-like and primitive elements. Next, we describe some properties of group-like and primitive elements in a Hom-bialgebra. We also define generalized primitive elements. It turns out that the sets of primitive and generalized primitive elements carry a natural structure of Hom-Lie algebra.

Let $H = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit.

**Definition 3.24.** An element $x \in H$ is called **group-like** if $\Delta(x) = x \otimes x$ and primitive if $\Delta(x) = e_1 \otimes x + x \otimes e_1$.

The coassociativity of $\Delta$ on a group-like element $x \in H$ leads to the condition

$$\beta(x) \otimes x \otimes x = x \otimes x \otimes \beta(x).$$

The condition is satisfied, in particular, if $\beta(x) = \lambda x$ with $\lambda \in K$. Let $V$ be an $n$-dimensional vector space, assume that $\{x_i\}_{i=1, \ldots, n}$ be a basis of $V$, the group-like element $x = \sum_{i=1}^n a_i x_i$ and $\beta(x) = \sum_{i=1}^n \beta_i x_i$, then the coassociativity condition is equivalent to the system of equations

$$a_i \beta_j - a_j \beta_i = 0, \quad \text{for } i, j = 1, \ldots, n.$$

For the unit $e_1$, which is also a group-like element, the coassociativity implies

$$\beta(e_1) = \lambda e_1 \quad \lambda \in K.$$

Now, let $x \in H$ be a primitive element, the coassociativity of $\Delta$ on $x$ implies

$$e_1 \otimes e_1 \otimes (\beta(x) - \lambda x) + (\beta(x) - \lambda x) \otimes e_1 \otimes e_1 = 0.$$

Therefore, for a primitive element $x \in H$, we have

$$\beta(x) = \lambda x.$$

Also, for a primitive element $x \in H$, one has

$$(\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)$$

where $\tau_{13}$ is a permutation in the symmetric group $S_3$. Thus, the coassociativity condition becomes

$$(\Delta \otimes \beta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)$$

**Lemma 3.25.** Let $x$ be a primitive element in $H$, then $\varepsilon(x) = 0$.

**Proof.** By counity property, we have $x = (id \otimes \varepsilon) \circ \Delta(x)$. If $\Delta(x) = e_1 \otimes x + x \otimes e_1$, then $x = \varepsilon(x)e_1 + \varepsilon(e_1)x$, and since $\varepsilon(e_1) = 1$ it implies $\varepsilon(x) = 0$. 

**Proposition 3.26.** Let $H = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If $x$ and $y$ are two primitive elements in $H$. Then we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a primitive element.

The set of all primitive elements of $H$, denoted by $\text{Prim}(H)$, has a structure of Hom-Lie algebra.
Proof. By a direct calculation one has
\[
\Delta([x,y]) = \Delta(\mu(x \otimes y) - \mu(y \otimes x)) \\
= \Delta(x) \bullet \Delta(y) - \Delta(y) \bullet \Delta(x) \\
= (e_1 \otimes x + x \otimes e_1) \bullet (e_1 \otimes y + y \otimes e_1) - (e_1 \otimes y + y \otimes e_1) \bullet (e_1 \otimes x + x \otimes e_1) \\
= e_1 \otimes (\mu(x \otimes y) + y \otimes x + x \otimes y + \mu(x \otimes y) \otimes e_1) \\
- e_1 \otimes (\mu(y \otimes x) - x \otimes y - y \otimes x - \mu(y \otimes x) \otimes e_1) \\
= e_1 \otimes [x,y] + [x,y] \otimes e_1
\]
which means that \(\text{Prim}(\mathcal{H})\) is closed under the bracket multiplication \([\cdot, \cdot]\).

We have seen in [39] that there is a natural map from the Hom-associative algebras to Hom-Lie algebras. The bracket \([x,y] = \mu(x \otimes y) - \mu(y \otimes x)\) is obviously skewsymmetric and one checks that the Hom-Jacobi condition is satisfied:

\[
\begin{align*}
[\alpha(x), [y,z]] &- [[x,y], \alpha(z)] - [\alpha(y), [x,z]] = \\
\mu(\alpha(x) \otimes \mu(y \otimes z)) &- \mu(\alpha(x) \otimes \mu(z \otimes y)) - \mu(\mu(y \otimes z) \otimes \alpha(x)) + \mu(\mu(z \otimes y) \otimes \alpha(x)) \\
- \mu(\mu(x \otimes y) \otimes \alpha(z)) &+ \mu(\mu(y \otimes x) \otimes \alpha(z)) + \mu(\alpha(z) \otimes \mu(x \otimes y)) - \mu(\alpha(z) \otimes \mu(y \otimes x)) \\
- \mu(\alpha(y) \otimes \mu(x \otimes z)) &+ \mu(\alpha(y) \otimes \mu(z \otimes x)) + \mu(\mu(x \otimes z) \otimes \alpha(y)) - \mu(\mu(z \otimes x) \otimes \alpha(y)) = 0
\end{align*}
\]

We introduce now a notion of generalized primitive element.

**Definition 3.27.** An element \(x \in \mathcal{H}\) is called generalized primitive element if it satisfies the conditions

\[
(\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)
\]

\[
\Delta^{op}(x) = \Delta(x)
\]

where \(\tau_{13}\) is a permutation in the symmetric group \(\mathcal{S}_3\).

In particular, a primitive element in \(\mathcal{H}\) is a generalized primitive element.

**Remark 3.28.** The condition (3.13) may be written

\[
(\Delta \otimes \beta) \circ \Delta(x) = \tau_{13} \circ (\beta \otimes \Delta) \circ \Delta(x).
\]

**Proposition 3.29.** Let \(\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)\) be a Hom-bialgebra and \(e_1 = \eta(1)\) be the unit. If \(x\) and \(y\) are two generalized primitive elements in \(\mathcal{H}\). Then, we have \(\varepsilon(x) = 0\) and the commutator \([x,y] = \mu(x \otimes y) - \mu(y \otimes x)\) is also a generalized primitive element.

The set of all generalized primitive elements of \(\mathcal{H}\), denoted by \(\text{GPrim}(\mathcal{H})\), has a structure of Hom-Lie algebra.

**Proof.** Let \(x\) and \(y\) be two generalized primitive elements in \(\mathcal{H}\). In the following the multiplication \(\mu\) is denoted by a dot. The following equalities hold:

\[
(\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) = (\Delta \otimes \beta) \circ \Delta(x \cdot y) - (\Delta \otimes \beta) \circ \Delta(y \cdot x)
\]

\[
= (\Delta \otimes \beta)(\Delta(x) \bullet \Delta(y)) - (\Delta \otimes \beta)(\Delta(y) \bullet \Delta(x))
\]

\[
= \Delta(x^{(1)}, y^{(1)}) \otimes \beta(x^{(2)} \cdot y^{(2)}) - \Delta(y^{(1)} \cdot x^{(1)}) \otimes \beta(y^{(2)} \cdot x^{(2)})
\]

\[
= (x^{(1)} \cdot y^{(1)}) \otimes (x^{(2)} \cdot y^{(2)}) \otimes \beta(x^{(2)} \cdot y^{(2)})
\]

\[= \Delta(x^{(1)}) \otimes \beta(y^{(2)} \cdot x^{(2)}).
\]
Then, using the fact that $\Delta^{op} = \Delta$ for generalized primitive elements one has:
\[
\tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) = \beta((x^{(2)} \cdot y^{(2)}) \otimes (x^{(1)(2)} \cdot y^{(1)(2)}) \otimes (x^{(1)(1)} \cdot y^{(1)(1)}) - \beta((y^{(2)} \cdot x^{(2)}) \otimes (y^{(1)(2)} \cdot x^{(1)(2)}) \otimes (y^{(1)(1)} \cdot x^{(1)(1)})) = (\beta \otimes \Delta) \circ \Delta(x \cdot y - y \cdot x).
\]

The structure of Hom-Lie algebra follows from the same argument as in the primitive elements case. 

Generalized Hom-bialgebras. We introduce in the following a more general definition of Hom-bialgebra, where we do not consider that the Hom-associative algebra is unital and the Hom-coassociative coalgebra is counital. This notion extend the notion of generalized bialgebra in the associative case introduced by Loday [36] in a more general framework. A generalized bialgebra is a triple $(V, \mu, \Delta)$ of an associative multiplication $\mu$, a coassociative comultiplication $\Delta$ together with a compatibility condition.

**Definition 3.30.** A generalized Hom-bialgebra is a 5-uple $(V, \mu, \alpha, \Delta, \beta)$ where

1. $(V, \mu, \alpha)$ is a Hom-associative algebra.
2. $(V, \Delta, \beta)$ is a Hom-coassociative coalgebra.
3. The linear maps $\Delta$ is compatible with the multiplication $\mu$, that is
   \[
   \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}).
   \]

We call generalized bialgebra a generalized Hom-bialgebra where $\alpha$ and $\beta$ are the identity map. Any bialgebra is in particular a generalized bialgebra. A morphism of generalized Hom-bialgebras is just a morphism of Hom-associative algebras and a morphism of Hom-coassociative coalgebras.

**Proposition 3.31.** Let $(V, \mu, \Delta)$ be a generalized Hom-bialgebra and let $\alpha : V \to V$ be a generalized bialgebra endomorphism. Then $(V, \mu, \alpha, \Delta_{\alpha}, \alpha)$, where $\mu_{\alpha} = \alpha \circ \mu$ and $\Delta_{\alpha} = \Delta \circ \alpha$, is a generalized Hom-bialgebra.

Moreover, suppose that $(V', \mu', \Delta')$ is another generalized bialgebra and $\alpha' : V' \to V'$ is a generalized bialgebra endomorphism. If $f : V \to V'$ is a generalized bialgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then
\[
f : (V, \mu, \alpha, \Delta_{\alpha}, \alpha) \longrightarrow (V', \mu', \alpha', \Delta_{\alpha'}, \alpha')
\]
is a morphism of generalized Hom-bialgebras.

**Proof.** The condition GB1 follows from the theorem 3.3 and GB2 from the theorem 3.16.

It remains to prove the compatibility condition (GB3). The condition may be written
\[
\Delta_{\alpha} \circ \mu_{\alpha} = (\mu_{\alpha} \otimes \mu_{\alpha}) \circ \Upsilon \circ (\Delta_{\alpha} \otimes \Delta_{\alpha})
\]
where $\Upsilon$ is the usual flip, $\Upsilon(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_1 \otimes x_3 \otimes x_2 \otimes x_4$. We have
\[
\Delta_{\alpha} \circ \mu_{\alpha} = \Delta \circ \alpha \circ \alpha \circ \mu = (\alpha \circ \alpha) \circ \Delta \circ \mu \circ (\alpha \otimes \alpha)
= (\alpha \otimes \alpha) \circ ((\mu \circ \mu) \circ \Upsilon \circ (\Delta \circ \Delta)) \circ (\alpha \otimes \alpha)
= ((\alpha \circ \mu) \otimes ((\alpha \circ \mu) \circ \Upsilon \circ ((\Delta \circ \alpha) \otimes \Delta \circ \alpha))
= (\mu_{\alpha} \otimes \mu_{\alpha}) \circ \Upsilon \circ (\Delta_{\alpha} \otimes \Delta_{\alpha})
\]
The second assertion follows also from theorems 3.3 and 3.16. 

Now, we provide some examples of generalized Hom-bialgebra.
Example 3.32. Let $K[G]$ be the group-algebra over the group $G$. As a vector space, $K[G]$ is generated by $\{e_g : g \in G\}$. If $\alpha : G \to G$ is a group homomorphism, then it can be extended to an algebra endomorphism of $K[G]$ by setting

$$\alpha(\sum_{g \in G} a_g e_g) = \sum_{g \in G} a_g \alpha(e_g).$$

A structure of Hom-associative algebra over $K[G]$ was defined in [47] using theorem 3.3. Consider the usual bialgebra structure on $K[G]$ and $\alpha$ a generalized bialgebra morphism. Then, following the proposition [3.31], we define over $K[G]$ a generalized Hom-bialgebra $(K[G], \mu, \alpha, \Delta, \alpha)$ by setting:

$$\mu(e_{g} \otimes e_{g'}) = \alpha(e_{g} \cdot e_{g'}),$$

$$\Delta(e_{g}) = \alpha(e_{g}) \otimes \alpha(e_{g}).$$

Example 3.33. Consider the polynomial algebra $A = \mathbb{K}[\{X_{ij}\}]$ in variables $(X_{ij})_{i,j=1}^{n}$. It carries a structure of generalized bialgebra with the comultiplication defined by $\delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$ and $\delta(1) = 1 \otimes 1$. Let $\alpha$ be a generalized bialgebra morphism, it is defined by $n$ polynomials $\alpha(X_{ij})$. We define a generalized Hom-bialgebra $(A, \mu, \alpha, \Delta, \alpha)$ by

$$\mu(f \otimes g) = f(\alpha(X_{11}), \ldots, \alpha(X_{nn}))g(\alpha(X_{11}), \ldots, \alpha(X_{nn})), $$

$$\Delta(X_{ij}) = \sum_{k=1}^{n} \alpha(X_{ik}) \otimes \alpha(X_{kj}),$$

$$\Delta(1) = \alpha(1) \otimes 1.$$ 

Example 3.34. Let $X$ be a set and consider the set of non-commutative polynomials $A = \mathbb{K}[X]$. It carries a generalized bialgebra structure with a comultiplication defined for $x \in X$ by $\delta(x) = 1 \otimes x + x \otimes 1$ and $\delta(1) = 1 \otimes 1$. Let $\alpha$ be a generalized bialgebra morphism. We define a generalized Hom-bialgebra $(A, \mu, \alpha, \Delta, \alpha)$ by

$$\mu(f \otimes g) = f(\alpha(X))g(\alpha(X)),$$

$$\Delta(x) = \alpha(1) \otimes \alpha(x) + \alpha(x) \otimes \alpha(1),$$

$$\Delta(1) = \alpha(1) \otimes \alpha(1).$$

Remark 3.35. The previous constructions show that Hom-associative algebra (resp. the Hom-coassociative coalgebra and the generalized Hom-bialgebra) may be viewed as a deformation of the associative algebra (resp. coalgebra and generalized bialgebra). We recover the first structure when the endomorphism $\alpha$ becomes the identity map.

4. HOM-LIE ADMISSIBLE HOM-COALGEBRAS

Let $(V, \Delta, \beta)$ be a Hom-coalgebra where $V$ is a vector space over $\mathbb{K}$, $\Delta : V \to V \otimes V$ and $\beta : V \to V$ are linear maps and $\Delta$ is not necessarily coassociative or Hom-coassociative. By a $\beta$-coassociator of $\Delta$ we call a linear map $c_{\beta}(\Delta)$ defined by

$$c_{\beta}(\Delta) := (\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta.$$ 

Let $\mathcal{S}_3$ be the symmetric group of order 3. Given $\sigma \in \mathcal{S}_3$, we define a linear map

$$\Phi_{\sigma} : V^{\otimes 3} \to V^{\otimes 3}$$

by

$$\Phi_{\sigma}(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$ 

Recall that $\Delta^{op} = \tau \circ \Delta$ where $\tau$ is the usual flip that is $\tau(x \otimes y) = y \otimes x$. 

**Definition 4.1.** A triple \((V, \Delta, \beta)\) is a *Hom-Lie admissible Hom-coalgebra* if the linear map

\[ \Delta_L : V \rightarrow V \otimes V \]

defined by \(\Delta_L = \Delta - \Delta^{\text{op}}\), is a Hom-Lie coalgebra multiplication, that is the following condition is satisfied

\[ c_\beta(\Delta_L) + \Phi_{(213)} \circ c_\beta(\Delta_L) + \Phi_{(231)} \circ c_\beta(\Delta_L) = 0 \]  

(4.1)

where \((213)\) and \((231)\) are the two cyclic permutations of order 3 in \(S_3\).

**Remark 4.2.** Since \(\Delta_L = \Delta - \Delta^{\text{op}}\), the equality \(\Delta^{\text{op}}L = -\Delta_L\) holds.

**Lemma 4.3.** Let \((V, \Delta, \beta)\) be a Hom-coalgebra where \(\Delta : V \rightarrow V \otimes V\) and \(\beta : V \rightarrow V\) are linear maps and \(\Delta\) is not necessarily coassociative or Hom-coassociative, then the following relations are true

\[ c_\beta(\Delta_{\text{op}}) = -\Phi_{(13)} \circ c_\beta(\Delta) \]  

(4.2)

\[ (\beta \otimes \Delta_{\text{op}}) \circ \Delta = \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta_{\text{op}} \]  

(4.3)

\[ (\beta \otimes \Delta) \circ \Delta_{\text{op}} = \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta_{\text{op}} \]  

(4.4)

\[ (\Delta \otimes \beta) \circ \Delta_{\text{op}} = \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta \]  

(4.5)

\[ (\Delta_{\text{op}} \otimes \beta) \circ \Delta = \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \]  

(4.6)

**Lemma 4.4.** The \(\beta\)-coassociator of \(\Delta_L\) is expressed using \(\Delta\) and \(\Delta_{\text{op}}\) as follows:

\[ c_\beta(\Delta_L) = c_\beta(\Delta) + c_\beta(\Delta_{\text{op}}) - (\Delta \otimes \beta) \circ \Delta_{\text{op}} - (\Delta_{\text{op}} \otimes \beta) \circ \Delta - \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta_{\text{op}} + \Phi_{(13)} \circ (\Delta_{\text{op}} \otimes \beta) \circ \Delta \]  

(4.7)

\[ = c_\beta(\Delta) - \Phi_{(13)} \circ c_\beta(\Delta) - \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta + \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta. \]  

(4.8)

**Proposition 4.5.** Let \((V, \Delta, \beta)\) be a Hom-coalgebra. Then one has

\[ c_\beta(\Delta_L) + \Phi_{(213)} \circ c_\beta(\Delta_L) + \Phi_{(231)} \circ c_\beta(\Delta_L) = 2 \sum_{\sigma \in S_3} (-1)^{\varepsilon(\sigma)} \Phi_{\sigma} \circ c_\beta(\Delta) \]  

(4.9)

where \((-1)^{\varepsilon(\sigma)}\) is the signature of the permutation \(\sigma\).
Proof. By (4.8) and multiplication rules in the group $\mathcal{S}_3$, it follows that

\[
\Phi_{(213)} \circ c_\beta (\Delta_L) = \Phi_{(213)} \circ c_\beta (\Delta) - \Phi_{(13)} \circ c_\beta (\Delta) \\
- \Phi_{(213)} \circ (\beta \circ \Delta) \circ \Delta - \Phi_{(12)} \circ (\Delta \circ \beta) \circ \Delta \\
+ \Phi_{(213)} \circ (\beta \circ \Delta) \circ \Delta + \Phi_{(13)} \circ (\Delta \circ \beta) \circ \Delta
\]

(4.10)

\[
\Phi_{(231)} \circ c_\beta (\Delta_L) = \Phi_{(213)} \circ (\beta \circ \Delta) \circ \Delta + \Phi_{(13)} \circ (\Delta \circ \beta) \circ \Delta \\
+ \Phi_{(12)} \circ (\beta \circ \Delta) \circ \Delta + \Phi_{(213)} \circ (\Delta \circ \beta) \circ \Delta
\]

(4.11)

After summing up the equalities (4.8), (4.10) and (4.11) the terms on the right hand sides may be pairwise combined into the terms of the form $(-1)^{\varepsilon(\sigma)} \Phi_{\sigma} \circ c_\beta (\Delta)$ with each one being present in the sum twice for all $\sigma \in \mathcal{S}_3$. \qed

Definition 4.7 together with (4.9) yields the following corollary.

Corollary 4.6. A triple $(V, \Delta, \beta)$ is a Hom-Lie admissible Hom-coalgebra if and only if

\[
\sum_{\sigma \in \mathcal{S}_3} (-1)^{\varepsilon(\sigma)} \Phi_{\sigma} \circ c_\beta (\Delta) = 0
\]

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation $\sigma$.

Next we introduce the notion of $G$-Hom-coalgebra where $G$ is any subgroup of the symmetric group $\mathcal{S}_3$.

Definition 4.7. Let $G$ be a subgroup of the symmetric group $\mathcal{S}_3$, A Hom-coalgebra $(V, \Delta, \beta)$ is called $G$-Hom-coalgebra if

\[
\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi_{\sigma} \circ c_\beta (\Delta) = 0
\]

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation $\sigma$.

Proposition 4.8. Let $G$ be a subgroup of the permutations group $\mathcal{S}_3$. Then any $G$-Hom-Coalgebra $(V, \Delta, \beta)$ is a Hom-Lie admissible Hom-coalgebra.

Proof. The skew-symmetry follows straightforward from the definition. Take the set of conjugacy classes $\{gG\}_{g \in I}$ where $I \subseteq G$, and for any $\sigma_1, \sigma_2 \in I, \sigma_1 \neq \sigma_2 \Rightarrow \sigma_1 G \cap \sigma_2 G = \emptyset$.

Then

\[
\sum_{\sigma \in \mathcal{S}_3} (-1)^{\varepsilon(\sigma)} \Phi_{\sigma} \circ c_\beta (\Delta) = \sum_{\sigma_1 \in I} \sum_{\sigma_2 \in \sigma_1 G} (-1)^{\varepsilon(\sigma)} \Phi_{\sigma} \circ c_\beta (\Delta) = 0.
\]

The subgroups of $\mathcal{S}_3$ are

\[
G_1 = \{Id\}, \ G_2 = \{Id, \tau_{12}\}, \ G_3 = \{Id, \tau_{23}\},
\]
where $A_3$ is the alternating group and where $\tau_{ij}$ is the transposition between $i$ and $j$.

We obtain the following type of Hom-Lie-admissible Hom-coalgebras.

- The $G_1$-Hom-coalgebras are the Hom-associative coalgebras defined above.
- The $G_2$-Hom-coalgebras satisfy the condition
  \[ c_\beta(\Delta) + \Phi_{(12)}c_\beta(\Delta) = 0. \]
- The $G_3$-Hom-coalgebras satisfy the condition
  \[ c_\beta(\Delta) + \Phi_{(23)}c_\beta(\Delta) = 0. \]
- The $G_4$-Hom-coalgebras satisfy the condition
  \[ c_\beta(\Delta) + \Phi_{(13)}c_\beta(\Delta) = 0. \]
- The $G_5$-Hom-coalgebras satisfy the condition
  \[ c_\beta(\Delta) + \Phi_{(213)}c_\beta(\Delta) + \Phi_{(231)}c_\beta(\Delta) = 0. \]
- The $G_6$-Hom-coalgebras are the Hom-Lie-admissible coalgebras.

The $G_2$-Hom-coalgebras may be called Vinberg-Hom-coalgebraand $G_3$-Hom-coalgebras may be called preLie-Hom-coalgebras.

**Definition 4.9.** A triple $(V, \Delta, \beta)$ consisting of a linear space $V$, a linear map $\mu : V \to V \times V$ and a homomorphism $\beta$ is called a Vinberg-Hom-coalgebra if it satisfies
\[ c_\beta(\Delta) + \Phi_{(12)}c_\beta(\Delta) = 0. \]
and is called a preLie-Hom-coalgebra if it satisfies
\[ c_\beta(\Delta) + \Phi_{(23)}c_\beta(\Delta) = 0. \]

More generally, by dualization we have a correspondence between $G$-Hom-associative algebras introduced in [39] and $G$-Hom-coalgebras for a subgroup $G$ of $S_3$.

Let $G$ be a subgroup of $S_3$, and let $(V, \mu, \alpha)$ be a $G$-Hom-associative algebra that is $\mu : V \otimes V \to V$ and $\alpha : V \to V$ are linear maps and the following condition is satisfied
\[ \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_{\alpha, \mu} \circ \Phi_{\sigma} = 0. \]
where $a_{\alpha, \mu}$ is the $\alpha$-associator that is $a_{\alpha, \mu} = \mu \circ (\mu \otimes \alpha) - \mu \circ (\alpha \otimes \mu)$

Setting
\[ (\mu \otimes \alpha)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\mu \otimes \alpha) \circ \Phi_{\sigma} \quad \text{and} \quad (\alpha \otimes \mu)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\alpha \otimes \mu) \circ \Phi_{\sigma} \]
the condition (4.13) is equivalent to the following commutative diagram
\[ \begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{(\mu \otimes \alpha)_G} & V \otimes V \\
\downarrow{(\alpha \otimes \mu)_G} & & \downarrow{\mu} \\
V \otimes V & \xrightarrow{\mu} & V
\end{array} \]
By the dualization of the square one may obtain the following commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\Delta} & V \otimes V \\
\downarrow \Delta & & \downarrow (\beta \otimes \Delta_G) \\
V \otimes V & \xrightarrow{(\Delta \otimes \beta)_G} & V \otimes V \otimes V
\end{array}
\]

where

\[
(\beta \otimes \Delta)_G = \sum_{\sigma \in G} (-1)^{e(\sigma)} \Phi_G \circ (\beta \otimes \Delta) \quad \text{and} \quad (\Delta \otimes \beta)_G = \sum_{\sigma \in G} (-1)^{e(\sigma)} \Phi_G \circ (\Delta \otimes \beta).
\]

The previous commutative diagram expresses that \((V, \Delta, \beta)\) is a \(G\)-Hom-coalgebra. More precisely we have the following connection between \(G\)-Hom-coalgebras and \(G\)-Hom-associative algebras.

**Proposition 4.10.** Let \((V, \Delta, \beta)\) be a \(G\)-Hom-coalgebra where \(G\) is a subgroup of \(S_3\). Its dual vector space \(V^*\) is provided with a \(G\)-Hom-associative algebra \((V^*, \Delta^*, \beta^*)\) where \(\Delta^*, \beta^*\) are the transpose map.

**Proof.** Let \((V, \Delta, \beta)\) be a \(G\)-Hom-coalgebra, and let \(V^*\) be the dual space of \(V\), that is \(V^* = Hom(V, K)\).

Consider the map

\[
\lambda_n : (V^*)^n \longrightarrow (V^*)^n
\]

\[
f_1 \otimes \cdots \otimes f_n \longrightarrow \lambda_n(f_1 \otimes \cdots \otimes f_n)
\]

such that for \(v_1 \otimes \cdots \otimes v_n \in V^\otimes n\)

\[
\lambda_n(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)
\]

and set

\[
\mu := \Delta^* \circ \lambda_2 \quad \alpha := \beta^*
\]

where the star \(\ast\) denotes the transpose linear map. Then, the quadruple \((V^*, \mu, \eta, \alpha)\) is a \(G\)-Hom-associative algebra. Indeed, \(\mu(f_1, f_2) = \mu_G \circ \lambda_2(f_1 \otimes f_2) \circ \Delta\) where \(\mu_G\) is the multiplication of \(K\) and \(f_1, f_2 \in V^*\). One has

\[
\mu \circ (\mu \otimes \alpha)(f_1 \otimes f_2 \otimes f_3) = \mu(f_1 \otimes f_2) \otimes (f_3) = \mu_G \circ \lambda_2(\mu(f_1 \otimes f_2) \otimes (f_3)) \circ \Delta
\]

\[
= \mu_G \circ \lambda_2(\lambda_2((f_1 \otimes f_2) \circ \Delta) \otimes (f_3)) \circ \Delta = \mu_G \circ (\mu_G \otimes id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes \beta) \circ \Delta.
\]

Similarly

\[
\mu \circ (\alpha \otimes \mu)(f_1 \otimes f_2 \otimes f_3) = \mu_G \circ (id \otimes \mu_G) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\beta \otimes \Delta) \circ \Delta.
\]

Using the associativity and the commutativity of \(\mu_G\), the \(\alpha\)-associator may be written as

\[
a_{\alpha, \mu} = \mu_G \circ (id \otimes \mu_G) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ ((\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta).
\]

Then we have the following connection between the \(\alpha\)-associator and \(\beta\)-coassociator

\[
a_{\alpha, \mu} = \mu_G \circ (id \otimes \mu_G) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ c_\beta(\Delta).
\]

Therefore if \((V, \Delta, \beta)\) is a \(G\)-Hom-coalgebra, then the \((V^*, \Delta^*, \beta^*)\) is a \(G\)-Hom-associative algebra.

\[\square\]
**Proposition 4.11.** Let \((V, \mu, \alpha)\) be a finite dimensional \(G\)-Hom-associative algebra where \(G\) is a subgroup of \(S_n\). Its dual vector space \(V^*\) is provided with a \(G\)-Hom-coalgebra \((V^*, \mu^*, \alpha^*)\), where \(\mu^*, \alpha^*\) are the transpose map.

**Proof.** Let \(A = (V, \mu, \alpha)\) be a \(n\)-dimensional Hom-associative algebra \((n\ finite)\). Let \(\{e_1, \cdots, e_n\}\) be a basis of \(V\) and \(\{e_1^*, \cdots, e_n^*\}\) be the dual basis. Then \(\{e_i^* \otimes e_j^*\}_{i,j}\) is a basis of \(A^* \otimes A^*\). The comultiplication \(\Delta = \mu^*\) on \(A^*\) is defined for \(f \in A^*\) by

\[
\Delta(f) = \sum_{i,j=1}^n f(e_i \otimes e_j) e_i^* \otimes e_j^*
\]

Setting \(\mu(e_i \otimes e_j) = \sum_{k=1}^n C_{ij}^k e_k\) and \(\alpha(e_i) = \sum_{k=1}^n \alpha_i^k e_k\), then \(\Delta(e_i^*) = \sum_{i,j=1}^n C_{ij}^k e_i^* \otimes e_j^*\) and \(\beta(e_i) = \alpha^*(e_i) = \sum_{i,j=1}^n \alpha_i^k e_k\).

The condition (4.12) of \(G\)-Hom-coassociativity of \(\Delta\), applied to any element \(e_k^*\) of the basis, is equivalent to

\[
\sum_{p,q,s=1}^n (-1)^{\varepsilon(\sigma)} (\sum_{i,j=1}^n \alpha_i^k C_{ij}^p C_{pq} - \alpha_i^k C_{ij}^e C_{q}s) e_{\sigma^{-1}(p)} \otimes e_{\sigma^{-1}(q)} \otimes e_{\sigma^{-1}(s)} = 0
\]

Therefore \(\Delta\) is \(G\)-Hom-coassociative if for any \(p,q,s,k \in \{1, \cdots, n\}\) one has

\[
\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\sum_{i,j=1}^n \alpha_i^k C_{ij}^p C_{pq} - \alpha_i^k C_{ij}^e C_{q}s) = 0
\]

The previous system is exactly the condition (4.12) of \(G\)-Hom-associativity of \(\mu\), written on \(e_i^* \otimes e_j^* \otimes e_k^*\) and setting \(p = \sigma(p')\), \(q = \sigma(q')\), \(s = \sigma(s')\). \(\square\)

**Corollary 4.12.** The dual vector space of a Hom-coassociative coalgebra \((V, \Delta, \beta, \varepsilon)\) is a Hom-associative algebra \((V^*, \Delta^*, \beta^*, \varepsilon^*)\), where \(V^*\) is the dual vector space and the star for the linear maps denotes the transpose map. The dual vector space of finite-dimensional Hom-associative algebra is a Hom-coassociative coalgebra.

**Proof.** It is a particular case of the previous Propositions (\(G = G_1\)). \(\square\)

**Remark 4.13.** Let \(V\) be a finite-dimensional \(K\)-vector space. If \(H = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)\) is a Hom-Hopf algebra, then

\[
H^* = (V^*, \Delta^*, \beta^*, \varepsilon^*, \mu^*, \alpha^*, \eta^*, S^*)
\]

is also a Hom-Hopf algebra.

**Remark 4.14.** An earlier version of this article has appeared as a Preprint in Mathematical Sciences in Lund University, Center for Mathematical Sciences in 2008. The results in this paper were also presented in the talks given by the authors at the European Science Foundation Mathematics Conference “Algebraic Aspects in Geometry”. Mathematical Research and Conference Center, Bedlewo, Poland, October 2007; AGMF Baltic-Nordic network conferences, Göteborg, Sweden, October 2007 and Tartu, Estonia, October 2008; International Conference on Noncommutative Rings and Geometry, Almería, Spain, September 2007 and Seminar Sophus Lie XXXV conference, Budapest, Hungary, March, 2008.
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ABDENCER MAKHLOUF, UNIVERSITÉ DE HAUTE ALSACE, LABORATOIRE DE MATHEMATIQUES, INFORMATIQUE ET APPLICATIONS,, 4, RUE DES FRÈRES LUMIÈRE F-68093 MULHOUSE, FRANCE

E-mail address: Abdenacer.Makhlouf@uha.fr

SERGEI SILVESTROV, CENTRE FOR MATHEMATICAL SCIENCES, LUND UNIVERSITY, BOX 118, SE-221 00 LUND, SWEDEN

E-mail address: sergei.silvestrov@math.lth.se