Stability analysis of squashed Kaluza–Klein black holes with charge

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Abstract
We study gravitational and electromagnetic perturbation around the squashed Kaluza–Klein black holes with charge. Since the black hole spacetime focused on in this paper has the $SU(2) \times U(1) \simeq U(2)$ symmetry, we can separate the variables of the equations for perturbations by using the Wigner function $D_{JM}^{J}$ which is the irreducible representation of the symmetry. In this paper, we mainly treat $J = 0$ modes which preserve the $SU(2)$ symmetry. We derive the master equations for the $J = 0$ modes and discuss the stability of these modes. We show that the modes of $J = 0$ and $K = 0, \pm 2$ and the modes of $K = \pm (J + 2)$ are stable against small perturbations from the positivity of the effective potential. As for $J = 0, K = \pm 1$ modes, since there are domains where the effective potential is negative except for the maximally charged case, it is difficult to show the stability of these modes in general. To show the stability for $J = 0, K = \pm 1$ modes in general is an open issue. However, we can show the stability for $J = 0, K = \pm 1$ modes in the maximally charged case where the effective potential is positive outside of the horizon.

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1. Introduction
In recent years, the studies on higher dimensional black holes have attracted much attention in the context of the brane world scenario. One of the main reasons is that mini black hole creation in a particle collider was suggested [1–6] based on a brane world model. Such physical phenomena are expected to give us evidence for the existence of extra dimensions and to draw some information toward quantum gravity.

It is known that higher-dimensional black objects admit various topology of horizons unlike the four-dimensional case. In fact, there are many exotic solutions of the Einstein equation in higher dimensions in addition to black holes, e.g. black rings, strings, Starns and
di-rings [7–16]. To predict what type of black objects are created in a particle collider, one of the important criterion is their stability against small gravitational perturbations. In [17], Kodama and Ishibashi showed that the equation of motion for perturbation of the maximally symmetric higher-dimensional black holes reduces to a single master equation for each mode. They also showed the stability of the higher-dimensional Schwarzschild black holes. Recently, much effort has been devoted to revealing the stability of higher-dimensional black holes with a more complicated structure by many authors [18–26].

From a realistic point of view, the extra dimensions need to be compactified to reconcile the higher-dimensional gravity theory with our apparently four-dimensional world. We call the higher-dimensional black holes on the spacetime with compact extra dimensions Kaluza–Klein black holes. In general, it is difficult to construct exact solutions describing Kaluza–Klein black holes because of less symmetry than the asymptotically flat case. However, if we consider the spacetime with twisted extra-dimensions, we can construct such exact solutions, i.e. squashed Kaluza–Klein (SqKK) black holes [27, 28] in the class of the cohomogeneity-one symmetry. The topology of the horizon of SqKK black holes is $S^3$, while it looks like four-dimensional black holes with a circle as an internal space in the asymptotic region.

Recently, much effort has been devoted to reveal the properties of squashed Kaluza–Klein black holes. In [29–47] generalizations of SqKK black holes are studied. Several aspects of SqKK black holes are also discussed, e.g. thermodynamics [48–50], Hawking radiation [51–53], gravitational collapse [54], behavior of the Klein–Gordon equation [55], stability for the neutral case [56], quasinormal modes [57, 58], geodetic precession [59], Kerr/CFT correspondence [60] and gravitational lensing effects [61].

In this paper, we study gravitational and electromagnetic perturbation around the squashed Kaluza–Klein black holes with charge [31] by extending the analysis in [56]. SqKK black holes with charge have the $SU(2) \times U(1)$ symmetry as in the neutral case, so we can use the same techniques used in [20, 56, 62] to analyze the perturbations of SqKK black holes with charge. We expand the perturbation variables in terms of the Wigner function $D^{JM}_{KM}$, which is the irreducible representation of the symmetry characterized by three indices $J, M, K$. Since the instability empirically appears in the lower modes, we mainly focus on the perturbations in $J = 0$ modes which preserve the $SU(2)$ symmetry. In this paper, we derive the master equations for $(J = 0, M = 0, K = 0, \pm 1, \pm 2)$ modes which have the $SU(2)$ symmetry and the highest modes $(K = \pm (J + 2))$. Using the effective potential functions in the master equations, we discuss the stability for these modes.

There are several motivations to study the perturbation for SqKK black holes with charge. One is to understand the property of the spacetime deeper by using small perturbations as a probe such as quasinormal modes. Next, the stability of SqKK black holes is needed for the arguments in the physics around the black holes [48–53, 57–59, 61] to be meaningful. Finally, it is also useful to consider the perturbation for SqKK black holes with charge to understand the effective theories reduced from unified theories based on string theory which usually contains not only gravity but also gauge fields.

The organization of this paper is as follows. In section 2, we review the geometry of SqKK black holes with charge and the formalism to classify metric perturbations based on the symmetry. In section 3, we derive the master equations for master variables. By analyzing these equations, we discuss the stability of SqKK black holes with charge. The final section is devoted to discussion.

1 Note that the stability of $J = 0, K = \pm 1$ modes were not shown analytically since there are some mistakes in the discussion in $J = 0, K = \pm 1$ modes in [56]. It is difficult to find the $S$-deformation function to show stability for these modes analytically. To find the $S$-deformation function analytically is an open issue. However, the stability for these modes have been confirmed by the numerical method in [57].
2. Squashed Kaluza–Klein black holes with charge

In this section, we review the geometry of SqKK black holes with charge [31] and the formalism to classify metric perturbations based on the symmetry [20, 56, 62].

2.1. Geometry of squashed Kaluza–Klein black holes with charge

We start with the five-dimensional Einstein–Maxwell system whose action is given by

\[ S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} (R - F_{\mu\nu}F^{\mu\nu}), \]  

(1)

where \( G, R \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) are the five-dimensional gravitational constant, the Ricci scalar curvature and the Maxwell field strength with the gauge potential \( A_\mu \). From this action, we obtain the Einstein equation

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2T_{\mu\nu}, \]  

(2)

with

\[ T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^\lambda - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}, \]  

(3)

and the Maxwell equation

\[ \nabla_\mu F^{\mu\nu} = 0. \]  

(4)

SqKK black holes with charge [31] are solutions of equations (2) and (4) whose metric and gauge potential are given by

\[ ds^2 = -F(\rho) dt^2 + K(\rho)^2 \frac{F(\rho)}{\rho} d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \rho^2 \left( d\psi + \cos \theta d\phi \right)^2, \]  

(5)

\[ A_\mu dx^\mu = \sqrt{3} \frac{\sqrt{\rho_+ \rho_-}}{\rho} dt, \]  

(6)

where the functions \( F(\rho) \) and \( K(\rho) \) are

\[ F(\rho) = \frac{(\rho - \rho_+)(\rho - \rho_-)}{\rho^2}, \quad K(\rho)^2 = \frac{\rho + \rho_0}{\rho}, \]  

(7)

and \( \rho_0 \) and \( \rho_{\pm} \) are constants which satisfy

\[ \rho_+ \geq \rho_- \geq 0, \quad \rho_- + \rho_0 \geq 0, \]  

(8)

and the invariant one-forms \( \sigma^a (a = 1, 2, 3) \) of \( SU(2) \) are given by

\[ \sigma^1 = -\sin \psi \, d\theta + \cos \psi \sin \theta \, d\phi, \quad \sigma^2 = \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi, \quad \sigma^3 = d\psi + \cos \theta \, d\phi. \]  

(9)

The domains of angular coordinates are \( 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 4\pi \) and the radial coordinate \( \rho \) runs in the range \( 0 < \rho < \infty \). Horizons locate at \( \rho = \rho_{\pm} \).

In the region far from the horizon, the metric (5) becomes

\[ ds^2 \simeq -dt^2 + d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{r_\infty^2}{4} (d\psi + \cos \theta \, d\phi)^2 + O(1/\rho), \]  

(10)

where \( r_\infty = 2\sqrt{(\rho_+ + \rho_0)(\rho_- + \rho_0)} \) is the size of the extra-dimension. So we can see that the spacetime behaves effectively as four-dimensional Minkowski spacetime if we focus on physical phenomena whose typical size is much larger than \( r_\infty \). On the other hand, in the
vicinity of the horizon, the metric (5) has a five-dimensional nature. To see this we introduce
new coordinates \( \eta, r \) and new parameters \( r_\pm \) as
\[
t = \frac{4 \rho_0^2}{r_\infty^2} \eta, \quad \rho = \frac{\rho_0 r^2}{r_\infty^2 - r^2}, \quad r_\pm = \frac{\rho_0 r^2_\pm}{r_\infty^2 - r^2_\pm},
\]
and then the metric (5) becomes
\[
ds^2 = - f(r) \, d\eta^2 + \frac{k(r)^2}{f(r)} \, dr^2 + r^2 [k(r)(\sigma^1)^2 + (\sigma^2)^2] + (\sigma^3)^2],
\]
with
\[
f(r) = \frac{(r^2 - r_\infty^2)(r^2 - r_\pm^2)}{r^4}, \quad k(r) = \frac{(r^2_\infty - r_\infty^2)(r^2_\infty - r_\pm^2)}{(r^2_\infty - r^2_\pm)^2}.
\]
In this coordinate, the horizons locate at \( r = r_\pm \) and the shape of horizon is a squashed \( S^3 \) which is characterized by the function \( k(r) \). Especially, if the size of the horizon is much smaller than that of the extra-dimension \( r_\infty \gg r_\pm \), then we can see that the function \( k(r) \approx 1 \) and the metric (12) behaves almost as five-dimensional Reissner–Nordström black holes in the vicinity of the horizon.

The Komar mass \( M \) and electric charge \( Q \) for this SqKK black hole are given by [31, 49]
\[
M = \frac{3 \pi r_\infty}{4 G} (\rho_0 - \rho_0 - \rho_\pm), \quad Q = \sqrt{3 \pi r_\infty} \sqrt{\rho_0 + \rho_\pm}.
\]
For later calculations, it is convenient to define the new invariant forms:
\[
\sigma^\pm = \frac{1}{2} (\sigma^1 \mp i \sigma^2).
\]
We treat the metric perturbation \( g_{\mu\nu} + h_{\mu\nu} \) and the electromagnetic perturbation \( A_\mu + \delta A_\mu \), where \( g_{\mu\nu} \) and \( A_\mu \) are the background quantities. Because of the background symmetry, these perturbations can be expanded by using Wigner functions which are known as the irreducible representation of \( SU(2) \times U(1) \) [20, 56, 62]. The scalar Wigner function is defined by
\[
L_2^2 D_{KM} = J(J+1)D^J_{KM}, \quad L_x D^I_{KM} = MD^I_{KM}, \quad W_3 D^J_{KM} = KD^J_{KM},
\]
where \( L_\alpha := i \xi_\alpha (\alpha = x, y, z) \), \( W_3 := i e_3 \), \( L_2^2 := L_x^2 + L_y^2 + L_z^2 \) and \( J, K, M \) are integers or half-integers satisfying \( J \geq 0, |K| \leq J, |M| \leq J \). The vector Wigner functions are
\[
2 \, K, M \text{ take integers iff } J \text{ takes integers.}
\]
constructed by the action of invariant forms $\sigma^a$ on scalar Wigner functions:
\begin{align*}
D^a_{\ell,K} &= \sigma^a_{\ell} D^-_{K-1}, \quad (|K-1| \leq J), \\
D^a_{\ell,K} &= \sigma^a_{-\ell} D^+_{K+1}, \quad (|K+1| \leq J), \\
D^a_{\ell,K} &= \sigma^a_{+\ell} D_{K}. \quad (|K| \leq J).
\end{align*}
Here, we have omitted the subscript $J, M$. The vector Wigner functions satisfy
\begin{align}
L^2 D^a_{\ell,K} &= J(J+1) D^a_{\ell,K}, \quad L_+ D^a_{\ell,K} = M D^a_{\ell,K}, \quad W_+ D^a_{\ell,K} = K D^a_{\ell,K},
\end{align}
where $a = \pm, 3$ and operations are defined by Lie derivatives: $W_\ell D^a_{\ell,K} := \mathcal{L}_{\ell a} D^a_{\ell,K}$ and $L_\ell D^a_{\ell,K} := \mathcal{L}_{L_\ell} D^a_{\ell,K}$. The tensor Wigner functions $D_{ij,K}^{ab}$ are defined by
\begin{align}
D_{ij,K}^{\sigma^a_{\ell} \sigma^b_{\ell}} &= \sigma^a_{\sigma^b_{\ell}} D_{K-2}, \quad (|K-2| \leq J), \\
D_{ij,K}^{\sigma^a_{\ell} \sigma^b_{-\ell}} &= \sigma^a_{\sigma^b_{-\ell}} D_{K}, \quad (|K| \leq J), \\
D_{ij,K}^{\sigma^a_{\ell} \sigma^b_{+\ell}} &= \sigma^a_{\sigma^b_{+\ell}} D_{K+2}, \quad (|K+2| \leq J), \\
D_{ij,K}^{\sigma^a_{-\ell} \sigma^b_{\ell}} &= \sigma^a_{\sigma^b_{\ell}} D_{K+1}, \quad (|K+1| \leq J), \\
D_{ij,K}^{\sigma^a_{-\ell} \sigma^b_{+\ell}} &= \sigma^a_{\sigma^b_{+\ell}} D_{K}, \quad (|K| \leq J).
\end{align}

One can check that $D_{ij,K}^{ab}$ forms the irreducible representation
\begin{align}
L^2 D_{ij,K}^{ab} &= J(J+1) D_{ij,K}^{ab}, \quad L_\ell D_{ij,K}^{ab} = M D_{ij,K}^{ab}, \quad W_\ell D_{ij,K}^{ab} = K D_{ij,K}^{ab}.
\end{align}
The tensor field $h_{ij}$ can be divided into three parts, $h_{AB}, h_{\delta A}, h_{ij}(A, B = t, \rho)$ which behave as the scalar, vector and tensor within the submanifold $\theta, \phi, \psi$. Similarly, the vector field $\delta A_\mu$ can be divided into two parts, $\delta A_A$ and $\delta A_\mu$. These perturbations can be expanded using the Wigner functions as
\begin{align}
&h_{\mu A} \, dx^\mu \, dx^\nu = dx^A \, dx^B \sum_k h_{AB}^k(x^A) D_{K}(x^A) + 2 \, dx^A \sum_k h_{Aa}^k(x^A) D_{i,j,K}^{ab}(x^A) \\
&\quad + dx^i \sum_k h_{ij,K}^{ab}(x^A) D_{ij,K}^{ab}(x^A),
\end{align}
\begin{align}
&\delta A_\mu \, dx^\mu = dx^A \sum_k \delta A_A^k(x^A) D_{K}(x^A) + dx^i \sum_k \delta A_\mu(x^A) D_{ij,K}^{ab}(x^A).
\end{align}
Because perturbed quantities are expanded in terms of the representation of the spacetime symmetry, no coupling appears between coefficients with different sets of indices $(J, M, K)$ in the perturbed equations. In addition, we utilize the Fourier expansion with respect to the time coordinate $t$.

Interestingly, without the explicit calculation, we can reveal the structure of couplings between coefficients with the same $(J, M, K)$. First, since the index $K$ is shifted in the definition of vector and tensor harmonics, the coefficients $h_{AB}^k, h_{Aa}^k$ and $h_{ij}^{ab}$ exist only if $K$ satisfies the inequality listed in the following table:

| $h_{++}$ | $h_{A+,+3}$, $\delta A_+$ | $h_{AB}, h_{A3}, h_{+-}, h_{33}, \delta A_A, \delta A_3$ | $h_{A-,-3}$, $\delta A_-$ | $h_{--}$ |
|----------|--------------------------|---------------------------------|--------------------------|----------|
| $|K-2| \leq J$ | $|K-1| \leq J$ | $|K| \leq J$ | $|K+1| \leq J$ | $|K+2| \leq J$ |

Therefore, for zero mode $J = 0$, we can classify the coefficients by possible $K$ as follows:

$J = 0$:

| $h_{++}$ | $h_{A+,+3}$, $\delta A_+$ | $h_{AB}, h_{A3}, h_{+-}, h_{33}, \delta A_A, \delta A_3$ | $h_{A-,-3}$, $\delta A_-$ | $h_{--}$ |
|----------|--------------------------|---------------------------------|--------------------------|----------|
| $K = 2$ | | | | |
| $K = 1$ | | | | |
| $K = 0$ | | | | |
| $K = -1$ | | | | |
| $K = -2$ | | | | |
Apparently, for \( h_{++} \) and \( h_{--} \), we can obtain equations for a single variable, respectively. For other sets of coefficients \((h_A, h_{A+}, \delta A_+), (h_{AB}, h_{A3}, h_{33}, \delta A_3, \delta A_A), (h_{A-}, h_{-3}, \delta A_-)\) they are coupled to the coefficients in each set. As we will see later, after fixing the gauge symmetry, we have the coupled master equations for one or two master variables in each set. We can also easily see that \( h_{++} \) in \((J, M, K = J + 2)\) modes and \( h_{--} \) in \((J, M, K = -(J + 2))\) modes are always decoupled. The perturbed equations for these modes can be reduced to the master equations for the single variables, respectively. We discuss the stability for \( J = 0 \) modes and \( K = \pm (J + 2) \) modes in the next section.

3. Stability analysis

The linearized Einstein equation is

\[
\delta G_{\mu\nu} - 2\delta T_{\mu\nu} = 0,
\]

where the linearized Einstein tensor \( \delta G_{\mu\nu} \) and the linearized stress tensor \( \delta T_{\mu\nu} \) are given by

\[
\delta G_{\mu\nu} = \frac{1}{2} \left[ \nabla^\rho \nabla^\sigma h_{\rho\sigma} + \nabla^\rho \nabla^\sigma h_{\rho\sigma} - \nabla^2 h_{\mu\nu} - \nabla^\mu \nabla^\nu \right] h_{\rho\sigma} - \nabla^\rho h_{\rho\mu} - \nabla^\rho h_{\rho\nu} + h_{\mu\rho} R_{\rho\sigma} - h_{\mu\nu} R,
\]

\[
\delta T_{\mu\nu} = -h_{\alpha\beta} F_{\mu}^{\alpha} F_{\nu}^{\beta} - \frac{1}{4} g_{\mu\nu} h_{\alpha\beta} F_{\alpha\beta}^{\alpha\beta} + \frac{1}{2} g_{\mu\nu} \delta F_{\alpha\beta} F_{\alpha\beta} + \delta F_{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} g_{\mu\nu} \delta F_{\alpha\beta} F_{\alpha\beta},
\]

where

\[
h = g^{\mu\nu} h_{\mu\nu}, \quad \delta F_{\mu\nu} = \partial_{\mu}(\delta A_{\nu}) - \partial_{\nu}(\delta A_{\mu}),
\]

and \( R_{\mu\nu}, F_{\mu\nu} \) are the background quantities. Note that \( \nabla_{\mu} \) denotes the covariant derivative with respect to the background metric \( g_{\mu\nu} \). The Maxwell equation for the perturbations is

\[
\delta (\nabla_{\mu} F^{\mu\nu}) = g^{\mu\nu} \nabla_{\sigma} \delta F_{\mu\sigma} - \nabla_{\sigma} (h_{\beta\sigma} F^{\alpha\nu}) - g^{\mu\nu} \nabla_{\nu} (h_{\alpha\beta} F^{\mu\beta}) + \frac{1}{2} F^{\alpha\nu} \nabla_{\alpha} h - \frac{1}{2} F^{\alpha\beta} \nabla_{\gamma} h_{\alpha\beta} = 0.
\]

We treat two kinds of gauge transformations. One is related to infinitesimal coordinate transformation:

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu},
\]

\[
\delta A_{\mu} \rightarrow \delta A_{\mu} + \xi_{\nu} \nabla_{\nu} A_{\mu} + A_{\nu} \nabla_{\mu} \xi_{\nu},
\]

where \( \xi_{\mu} \) is an arbitrary vector field. The other is related to the \( U(1) \) gauge:

\[
\delta A_{\mu} \rightarrow \delta A_{\mu} + \nabla_{\mu} \chi,
\]

where \( \chi \) is an arbitrary scalar field.

3.1. Zero-mode perturbations \((J = 0, M = 0)\)

In the case of \( J = 0, M = 0 \), there are five modes, \( K = \pm 2, \pm 1, 0 \). We analyze these modes separately. These modes correspond to perturbations with the \( SU(2) \) symmetry.
3.1.1. $K = \pm 2$ modes. In $K = \pm 2$ modes, there are two coefficients $h_{++}$ and $h_{--}$, and these are gauge invariant. Because the perturbed metric $h_{\mu\nu}(x') \, dx'^\mu \, dx'^\nu$ is real, $h_{--}$ must be the complex conjugate of $h_{++}$, i.e. $h_{++} = \bar{h}_{--}$ where the bar denotes the complex conjugate. So, it is sufficient to treat only $h_{++}$. We set $h_{\mu\nu}$ as

$$h_{\mu\nu}(x') \, dx'^\mu \, dx'^\nu = h_{++}(\rho) \, e^{-\text{int}} \, \sigma^+ \, \sigma^+.$$  \hspace{1cm} (32)

Inserting equation (32) into the ($+$) component of equation (24), we obtain the perturbation equation for $h_{++}$ whose explicit form is written in the appendix. By introducing the master variable

$$\varphi_2(\rho) := \frac{1}{\rho^2(\rho + \rho_0)^2} h_{++}(\rho),$$  \hspace{1cm} (33)

and switching to the tortoise coordinate $\rho_*$ defined by

$$\frac{d\rho_*}{d\rho} = K(\rho)/F(\rho),$$  \hspace{1cm} (34)

we get the perturbation equation in the Schrödinger form:

$$-\frac{d^2}{d\rho_*^2} \varphi_2 + V_2(\rho) \varphi_2 = \omega^2 \varphi_2,$$  \hspace{1cm} (35)

where the potential $V_2(\rho)$ is defined by

$$V_2(\rho) = \frac{(\rho - \rho_+)(\rho - \rho_+ + \bar{\rho}_-) \, \rho^2(\rho + \bar{\rho}_-) \, \rho_0}{16 \rho(\rho + \bar{\rho}_-)^2 \, \rho_0 \, (\rho + \rho - \rho_+)^3}
\times \left(4 \rho_+(3 \rho_+ + 8 \rho_+ + 4 \rho_+ \rho_+(9 \rho_+ + 41 \rho_+) \rho_+^3 + 4 \rho_+^2 \rho_+(9 \rho_+ + 74 \rho_+) \rho_+^5
\hspace{1cm} + 12 \rho_+^3 \rho_+(\rho + 19 \rho_+) \rho_0 + 64 \rho_+^4 \rho_+^2 + (188 \rho_+ - 9 \rho_-) \rho_0^4
\hspace{1cm} + (-27 \rho_+^2 + 418 \rho_+ \rho_+ - 200 \rho_+^2) \rho_0^3 + \rho_+(-27 \rho_+^2 + 700 \rho_+ \rho_+ + 655 \rho_+^2) \rho_0^5
\hspace{1cm} + (-9 \rho_+^2 + 498 \rho_+ \rho_+^3 + 711 \rho_+^2 \rho_+^2) \rho_0 + 128 \rho_+^3 \rho_+ \rho_+^2 + 2 \rho_+)(\rho - \rho_+)
\hspace{1cm} + (35 \rho_+^3 + (149 \rho_+ + 442 \rho_+) \rho_+^3 + (257 \rho_+^2 + 1400 \rho_+ \rho_+ + 355 \rho_+^2) \rho_+^5
\hspace{1cm} + \rho_+(-207 \rho_+^2 + 1470 \rho_+ \rho_+ + 379 \rho_+^2) \rho_0 + 64 \rho_+^2 (\rho_+^2 + 8 \rho_+ \rho_+ - 6 \rho_+^2)
\hspace{1cm} + (200 \rho_+^3 + 8(80 \rho_+ + 91 \rho_+) \rho_+^5 + 8(87 \rho_+^2 + 187 \rho_+ \rho_+ + 32 \rho_+^2) \rho_+^5
\hspace{1cm} + 256 \rho_+^2 (\rho_+^2 + 3 \rho_+ \rho_+ + \rho_+^2)(\rho - \rho_+)^2
\hspace{1cm} + (352 \rho_+^3 + 736 \rho_+ + 512 \rho_+) \rho_0 + 64 (6 \rho_+^2 + 8 \rho_+ \rho_+ + \rho_+^2) \rho_0^3)(\rho - \rho_+)^4
\hspace{1cm} + (256 \rho_0^2 + 128(2 \rho_+ + \rho_+))(\rho - \rho_+)^5 + 64(\rho - \rho_+)^6\right].$$  \hspace{1cm} (36)

Here, we introduced new parameters $\rho_*$ and $\rho_0$ defined by

$$\rho_* := \rho_+ - \rho_-, \hspace{1cm} \rho_0 := \rho_0 + \rho_-. \hspace{1cm} (37)$$

From equation (8), these parameters satisfy the following inequalities:

$$0 \leq \rho_* \leq \rho_+, \hspace{1cm} \rho_0 > 0. \hspace{1cm} (38)$$

Since the terms in the square bracket in equation (36) are given in a power series expansion of $(\rho - \rho_+)$ and the expansion coefficients of these terms are positive, we can see $V_2 > 0$ in the region $\rho_+ < \rho < \infty$. Typical profiles of the potential $V_2$ are plotted in figures 1–3. One should note that we have equality $\sqrt{3} Q/(2M) = 1$ in the maximally charged case $\rho_+ = \rho_-$ from (14), and we normalized the depicted effective potential by using the size of an extra dimension. The desired solution for $\varphi_2$ is square integrable in the domain
Because the boundary term vanishes, the positivity of $V_2$ means $\omega^2 > 0$. Therefore, we conclude that the background spacetime is stable with respect to the $K = \pm 2$ perturbations.

3.1.2. $K = \pm 1$ modes. Because of the relations $\bar{h}_{\lambda+} = h_{\lambda-}$, $\bar{h}_{+3} = h_{-3}$ and $\delta \bar{A}_+ = \delta A_-$, it is sufficient to consider only $h_{\lambda+}$, $h_{+3}$ and $\delta A_+$. Thus, we assume

\begin{align*}
    h_{\mu\nu} \, dx^\mu \, dx^\nu &= 2h_{\lambda+}(\rho) \, e^{-\text{i} \omega t} \, dx^\lambda \sigma^+ + 2h_{+3}(\rho) \, e^{-\text{i} \omega t} \sigma^3, \\
    \delta A_\mu \, dx^\mu &= \delta A_+(\rho) \, e^{-\text{i} \omega t} \sigma^+.
\end{align*}

Here, we set the gauge $\xi_\mu \, dx^\mu$ as

\begin{equation}
    \xi_\mu \, dx^\mu = \xi_+(\rho) \, e^{-\text{i} \omega t} \sigma^+,
\end{equation}

where $\rho_+ > 0$, $\xi_+ < \rho_+ < \infty$ with real $\omega^2$. Multiplying both sides of equation (35) by $\Phi_2$, and integrating it, we obtain

\begin{equation}
    \int d\rho_+ \left[ \left| \frac{d\Phi_2}{d\rho_+} \right|^2 + V_2 |\Phi_2|^2 \right] - \left[ \Phi_2 \frac{d}{d\rho_+} \Phi_2 \right]_{-\infty}^{\infty} = \omega^2 \int d\rho_+ |\Phi_2|^2. \tag{39}
\end{equation}

Because the boundary term vanishes, the positivity of $V_2$ means $\omega^2 > 0$. Therefore, we conclude that the background spacetime is stable with respect to the $K = \pm 2$ perturbations.
and under the gauge transformations (29), the metric perturbations transform as

\[ h_{t^+} \rightarrow h_{t^+} - i \omega \xi_+, \]
\[ h_{\rho^+} \rightarrow h_{\rho^+} - \frac{2\rho + \rho_0}{\rho (\rho + \rho_0)} \xi_+ + \frac{d \xi_+}{d \rho}, \]
\[ h_{3^+} \rightarrow h_{3^+} - \frac{i(\rho^2 + 2\rho_0 \rho - \rho_- \rho_+ - (\rho_+ + \rho_0)\rho_0)}{(\rho + \rho_0)^2} \xi_+. \]

So we can impose the gauge condition

\[ h_{3^+} = 0, \]

which completely fixes the gauge freedom. Substituting equations (40), (41) and (46) into equations (24) and (28), we obtain perturbation equations whose explicit forms are given in the appendix. Eliminating \( h_{t^+} \) from these equations and defining new variables

\[ \phi_{1G} := \frac{(\rho - \rho_+)(\rho - \rho_-)(\rho_+\rho_- + \rho_0(\rho_+ + \rho_- - 2\rho) - \rho^2)}{\rho^{7/4}(\rho + \rho_0)^{7/4}} h_{\rho^+}, \]
\[ \phi_{1E} := \frac{\rho^{1/4}}{(\rho + \rho_0)^{1/4}} \frac{\delta A_+}{i \omega}, \]

we have the coupled master equations for the gravitational and the electromagnetic perturbations

\[ - \frac{d^2}{d\rho_+^2} \phi_{1G} + V_{11}(\rho)\phi_{1G} + V_{12}(\rho)\phi_{1E} = \omega^2 \phi_{1G}, \]
\[ - \frac{d^2}{d\rho_+^2} \phi_{1E} + V_{22}(\rho)\phi_{1E} + V_{21}(\rho)\phi_{1G} = \omega^2 \phi_{1E}. \]
where the effective potentials $V_{11}$, $V_{12}$, $V_{21}$ and $V_{22}$ are defined by

$$V_{11} = \frac{(\rho - \rho_0)(\rho - \rho_5)}{16\rho_5(\rho + \rho_0)(\rho + \rho_5)(\rho + \rho_0)^3} \left[ 16\rho_0^4 + 128\rho_0^3 \rho_9 + 32\rho_0(\rho_0 - \rho_5)(\rho_0 + \rho_5) \right] \rho^6$$

$$+ 8(119\rho_0^7 - 9(\rho_0 + \rho_5)\rho_0^2 - (16\rho_0^2 + 25\rho_0 + 16\rho_5^2)\rho_0$$

$$+ 6\rho_0(\rho_0 + \rho_5)\rho_5^4 + (1103\rho_0^4 - 501(\rho_0 + \rho_5)\rho_5^3$$

$$+ (484\rho_0^2 + 697\rho_0 + \rho_5 + 484\rho_5^2)\rho_5^2 - 196\rho_0(\rho_0 + \rho_5)\rho_0 + 288\rho_0^2\rho_5^2)\rho_5^6$$

$$+ \rho_0(764\rho_0^9 - 959(\rho_0 + \rho_5)\rho_5^3 - (827\rho_5^2 + 634\rho_0 + \rho_5 + 827\rho_5^2)\rho_0^2$$

$$+ 325\rho_0(\rho_0 + \rho_5)\rho_5 + 1152\rho_5^2\rho_5^2)\rho_5^5$$

$$+ (252\rho_0^9 - 990(\rho_0 + \rho_5)\rho_5^3 - (722\rho_5^2 + 193\rho_0 + \rho_5 - 722\rho_5^2)\rho_0^4$$

$$+ (\rho_0 + \rho_5)(72\rho_0^2 + 1159\rho_0 + \rho_5 + 72\rho_5^2)\rho_5^3$$

$$+ 3\rho_0(16\rho_0^2 + 611\rho_0 + \rho_5 + 16\rho_5^2)\rho_5^5 - 120\rho_0^2\rho_5^2(\rho_0 + \rho_0)\rho_5 - 96\rho_0^3\rho_5^3)\rho_5^4$$

$$- 2\rho_0(228(\rho_0 + \rho_0)\rho_5^3 + (53\rho_0^2 + 55\rho_0 + \rho_5 + 53\rho_5^2)\rho_5^3$$

$$+ (\rho_0 + \rho_5)(111\rho_0^2 + 709\rho_0 + \rho_5 + 111\rho_5^2)\rho_5^3$$

$$- 6\rho_0(7\rho_0^2 + 104\rho_0 + \rho_5 + 7\rho_5^2)\rho_5^5 + 249\rho_0^2\rho_5^3(\rho_0 + \rho_0)\rho_5 + 180\rho_0^3\rho_5^3)\rho_5^3$$

$$+ (-4(\rho_0 + \rho_0)^3(5\rho_0 - 4\rho_0)\rho_0^4 + \rho_0(\rho_0 + \rho_0)^3(28\rho_0^2 - 55\rho_0\rho_0 + 151\rho_0^3)\rho_0^3$$

$$+ \rho_0(\rho_0 + \rho_0)(12\rho_0 + 297\rho_0^2 + 45\rho_0^2\rho_0 + 187\rho_0^3)\rho_0^5$$

$$+ 3\rho_0^3(\rho_0 + \rho_0)(-44\rho_5^2 - 213\rho_0 + \rho_0 + 85\rho_0^2)\rho_0$$

$$+ \rho_0^2\rho_0^2(-20\rho_0^2 + 151\rho_0 + \rho_0 + 187\rho_0^3)\rho_0^2 - 3\rho_0(\rho_0 + \rho_0)(\rho_0 + \rho_0)$$

$$\times (\rho_0 + \rho_0)(\rho_0 + \rho_0)(\rho_0 + \rho_0)\rho_0$$

$$+ 3(\rho_0^2 + 34\rho_0 + \rho_0 + 3\rho_0^2)\rho_0^2)\rho_0$$

$$+ 39\rho_0(\rho_0 + \rho_0)(\rho_0 + \rho_0)(\rho_0 + \rho_0)(\rho_0 + \rho_0)\rho_0^2)\rho_0^2]\right].$$

$$V_{12} = \frac{2\sqrt{3}(\rho + \rho_0)(\rho - \rho_0)^2 + 2\rho_0\rho - \rho_0\rho_0 - (\rho_0 + \rho_0)\rho_0}{\rho^4(\rho + \rho_0)^2},$$

$$V_{21} = \frac{\sqrt{3}(\rho + \rho_0)(\rho - \rho_0)^2 + 2\rho_0\rho - \rho_0\rho_0 - (\rho_0 + \rho_0)\rho_0}{2\rho^4(\rho + \rho_0)(\rho + \rho_0)(\rho + \rho_0)^2},$$

$$V_{22} = \frac{(\rho - \rho_0)(\rho - \rho_0)}{16\rho_5^4(\rho + \rho_0)(\rho + \rho_0)(\rho + \rho_0)} \left[ 16\rho_0^6 + 64\rho_0^5 \rho_9 + 96\rho_0^2 \rho_5^4$$

$$+ 8\rho_0(7\rho_0^2 - (\rho_0 + \rho_0)\rho_0 - \rho_0\rho_0)^3 + (15\rho_0^4 + 11(\rho_0 + \rho_0)\rho_5^3$$

$$+ (12\rho_0^2 + 71\rho_0\rho_0 + 12\rho_0^2 + 60(\rho_0 + \rho_0)\rho_0 + 48\rho_0^2\rho_0^2)\rho_0^2$$

$$+ 5\rho_0(\rho_0 + \rho_0)(\rho_0 + \rho_0)(16\rho_0\rho_0 + (\rho_0 + \rho_0)\rho_0)\rho$$

$$+ 39\rho_0(\rho_0 + \rho_0)(\rho_0 + \rho_0)(\rho_0 + \rho_0)\rho_0^3)\right].$$

In order to discuss the stability for $K = \pm 1$ modes, we rewrite equations (49) and (50) in the following form:

$$-\frac{d^2}{d\rho^2} \Phi + V(\rho) \Phi = \alpha^2 \Phi.$$
where $\Phi$ and $V$ represent the master variables and the potential given by

$$
\Phi = \begin{pmatrix} \phi_1G \\ \phi_1E \end{pmatrix}, \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.
$$

Multiplying both sides of equation (55) by $\Phi^\dagger$ and integrating it, we obtain

$$
\int d\rho_* \left[ \left| \frac{d\Phi}{d\rho_*} \right|^2 + \Phi^\dagger V \Phi \right] = \omega^2 \int d\rho_* |\Phi|^2,
$$

where $|\Phi|^2 := \Phi^\dagger \Phi$. Because the boundary term vanishes, if the second term of the integrand on the left-hand side of equation (57) is real and positive everywhere, there is no $\omega^2 < 0$ mode.

We transform the term $\Phi^\dagger V \Phi$ as follows. We diagonalize the matrix $V$ using the unitary transformation

$$
\Phi^\dagger V \Phi = \Phi^\dagger Q^\dagger QVQ^\dagger Q \Phi = \Psi^\dagger \tilde{V} \Psi
$$

where

$$
\tilde{V} = \begin{pmatrix} \tilde{V}_{11} & 0 \\ 0 & \tilde{V}_{22} \end{pmatrix},
$$

and $\Psi$ is defined by

$$
\Psi = Q \Phi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{V} = QVQ^\dagger = \begin{pmatrix} \tilde{V}_{11} & 0 \\ 0 & \tilde{V}_{22} \end{pmatrix}.
$$

From equations (57) and (58), we can see that the sufficient condition for $\omega^2 > 0$ is that both $\tilde{V}_{11}$ and $\tilde{V}_{22}$ are real and positive in the domain $-\infty < \rho_* < \infty$. Here, $\tilde{V}_{11}$ and $\tilde{V}_{22}$ are written by using the components of the matrix $V$ as

$$
\tilde{V}_{11} = \frac{V_{11} + V_{22} + \sqrt{(V_{11} - V_{22})^2 + 4V_{12}V_{21}}}{2},
\tilde{V}_{22} = \frac{V_{11} + V_{22} - \sqrt{(V_{11} - V_{22})^2 + 4V_{12}V_{21}}}{2}.
$$

One can easily check that both $\tilde{V}_{11}$ and $\tilde{V}_{22}$ are real everywhere. Then, we split the condition both $\tilde{V}_{11} > 0$ and $\tilde{V}_{22} > 0$ to two conditions: (i) $\det V = \det \tilde{V} > 0$ in the domain $-\infty < \rho_* < \infty$, and (ii) $\tilde{V}_{11} > 0$ and $\tilde{V}_{22} > 0$ at one point outside of the horizon.

First, we show the condition (ii). Far from the horizon, $\tilde{V}_{11}$ and $\tilde{V}_{22}$ behave as

$$
\tilde{V}_{11} = \frac{16(\rho - \rho_*)(\rho + \rho_0)^3 \rho^{10}}{16\rho^5(\rho_- + \rho)(\rho_+ + \rho_0)(\rho + \rho_0)^3(\rho^2 + 2\rho_0\rho - \rho_-\rho_+ - (\rho_- + \rho_0)^2) + \mathcal{O}\left(\frac{1}{\rho}\right)},
\tilde{V}_{22} = \frac{16(\rho - \rho_*)(\rho - \rho_0)^3 \rho^{6}}{16\rho^5(\rho_- + \rho)(\rho_+ + \rho_0)(\rho + \rho_0)^3 + \mathcal{O}\left(\frac{1}{\rho}\right)}.
$$

Since the dominant terms of (61) and (62) are positive, the condition (ii) is fulfilled at great distance.

Next, we consider the condition (i). Typical profiles of $\det V$ are plotted in figures 4–6. In the case $\sqrt{3}Q/(2M) < 1$, because there are domains in which the $\det V$ is negative (see figures 5 and 6), we have not show the stability for this case. On the other hand, we can show the positivity of $\det V$ in the maximally charged case $\sqrt{3}Q/(2M) = 1$ (see figure 4). In the
maximally charged case, we can write $\det V$ as

$$
\begin{align*}
\det V &= \frac{(\rho - \rho_*)^5}{256\rho_0^5\rho^9(\rho - \rho_*)^6(2\rho_0 - \rho_0 + \rho)^2} [256(\rho - \rho_*)^{13} \\
+ & (3072\rho_0 + 1024\rho_*)(\rho - \rho_*)^{12} + (17408\rho_0^2 + 12288\rho_*\rho_0 + 1536\rho_0^2)(\rho - \rho_*)^{11} \\
+ & (59136\rho_0^3 + 67840\rho_*\rho_0^2 + 18432\rho_*^2\rho_0 + 1024\rho_*^3)(\rho - \rho_*)^{10} \\
+ & (132064\rho_0^4 + 219712\rho_*\rho_0^3 + 99552\rho_*^2\rho_0^2 + 12288\rho_*^3\rho_0 + 256\rho_*^4)(\rho - \rho_*)^9 \\
+ & (203008\rho_*^5 + 459968\rho_*\rho_0^4 + 309120\rho_*^2\rho_0^3 \\
+ 65216\rho_*^3\rho_0^2 + 3072\rho_*^4\rho_0)(\rho - \rho_*)^8 \\
+ & (219328\rho_*^6 + 656000\rho_*\rho_0^5 + 611232\rho_*^2\rho_0^4 \\
+ 195648\rho_*^3\rho_0^3 + 16096\rho_*^4\rho_0^2)(\rho - \rho_*)^7 \\
+ & (165520\rho_*^7 + 659024\rho_*\rho_0^6 + 819120\rho_*^2\rho_0^5 \\
+ 366448\rho_*^3\rho_0^4 + 47104\rho_*^4\rho_0^3)(\rho - \rho_*)^6 \\
+ & (83521\rho_*^8 + 472188\rho_*\rho_0^7 + 788454\rho_*^2\rho_0^6)
\end{align*}
$$

**Figure 4.** The determinant of the potential matrix $r_+^4 \det V$ in the maximally charged case.

**Figure 5.** The determinant of the potential matrix $r_+^4 \det V$ in the $\sqrt{3}Q/(2M) = 2/3$ case.
Figure 6. The determinant of the potential matrix $r_A^4 \det V$ in the $\sqrt{3} Q/2M = 1/3$ case.

\begin{align}
+ 449980 \rho_3^3 \rho_0^5 + 84641 \rho_4^4 \rho_0^4 \right)(\rho - \rho_5)^5 \\
+ (255720 \rho_5^6 + 234624 \rho_4 \rho_0^8 + 574056 \rho_5^2 \rho_0^7 \\
+ 378400 \rho_5^3 \rho_0^6 + 96020 \rho_4^4 \rho_0^5 \right)(\rho - \rho_5)^4 \\
+ (3780 \rho_5^{10} + 73616 \rho_4 \rho_0^9 + 313560 \rho_5^2 \rho_0^8 \\
+ 233104 \rho_5^3 \rho_0^7 + 69188 \rho_4^4 \rho_0^6 \right)(\rho - \rho_5)^3 \\
+ (11520 \rho_5^{10} + 113280 \rho_4^2 \rho_0^9 + 32640 \rho_5^4 \rho_0^7 \right)(\rho - \rho_5)^2 \\
+ (19968 \rho_5^{10} + 36864 \rho_4^2 \rho_0^9 + 10752 \rho_5^4 \rho_0^7 \right)(\rho - \rho_5) \\
+ 2048 \rho_5 \rho_0^9 \rho_4^4 \rho_0^6 + 6144 \rho_5 \rho_0^{10} \rho_4^3 \rho_0^7 \right], \tag{63}
\end{align}

where $\tilde{\rho}_0$ is defined by equation (37). From (38) and (63), we can see the positivity of $\det V$ outside of the horizon. Since we have shown conditions (i) and (ii), we conclude that the background spacetime is stable against perturbation for $K = \pm 1$ modes in the maximally charged case.

3.1.3. $K = 0$ mode. For the $K = 0$ mode, we set $h_{\mu\nu}$ and $\delta A_\mu$ as

$$h_{\mu\nu} \, dx^\mu \, dx^\nu = h_{A\beta}(\rho) \, e^{-i\omega t} \, dx^A \, dx^\beta + 2h_{A3}(\rho) \, e^{-i\omega t} \, dx^A \sigma^3$$

$$+ 2h_{+\rho}(\rho) \, e^{-i\omega t} \sigma^\rho + h_{33}(\rho) \, e^{-i\omega t} \sigma^3 \sigma^3,$$

$$\delta A_\mu \, dx^\mu = \delta A_\lambda (\rho) \, e^{-i\omega t} \, dx^A + \delta A_3 (\rho) \, e^{-i\omega t} \sigma^3. \tag{64}$$

We set the gauge $\xi_\mu \, dx^\mu$ as

$$\xi_\mu \, dx^\mu = \xi_\lambda (\rho) \, e^{-i\omega t} \, dx^A + \xi_3 (\rho) \, e^{-i\omega t} \sigma^3,$$

and under the gauge transformation, the metric perturbations transform as

$$h_{tt} \rightarrow h_{tt} - 2i\omega \xi_t - \frac{(\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) \xi_t}{\rho^6 (\rho + \rho_5)^2} \xi_t, \tag{67}$$

$$h_{tp} \rightarrow h_{tp} - \frac{(\rho_5 + \rho_5) (\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) \xi_t}{\rho (\rho - \rho_5) (\rho - \rho_5)} \xi_t - i\omega \xi_\rho + \frac{\partial \xi_t}{\partial \rho}, \tag{68}$$

$$h_{pp} \rightarrow h_{pp} + \frac{(\rho_5 + \rho_5) (\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) (\rho - \rho_5) \xi_t}{\rho (\rho - \rho_5) (\rho - \rho_5) (\rho + \rho_5)} \xi_t + 2 \frac{\partial \xi_\rho}{\partial \rho}. \tag{69}$$
\[ h_{ij} \to h_{ij} - i\omega \xi, \]  
\[ h_{\rho\rho} \to h_{\rho\rho} - \frac{\rho_0}{\rho(\rho + \rho_0)} \xi + \frac{d\xi}{d\rho}, \]  
\[ h_{++} \to h_{++} + \frac{2(\rho - \rho_+)(\rho - \rho_+)(2\rho + \rho_0)}{\rho(\rho + \rho_0)} \xi, \]  
\[ h_{33} \to h_{33} + \frac{(\rho - \rho_+)(\rho - \rho_+)\rho_0(\rho_+ + \rho_0)}{\rho(\rho + \rho_0)^3} \xi. \]  

So we can choose the condition\(^3\)
\[ h_{++} = 0, \quad h_{tt} = 0, \quad h_{ij} = 0. \]  

In addition, for the gauge transformation (31), the electromagnetic perturbations transform as
\[ \delta A_t \to \delta A_t - i\omega \chi(\rho), \]  
\[ \delta A_\rho \to \delta A_\rho + \frac{d\chi}{d\rho}, \]  
\[ \delta A_3 \to \delta A_3. \]  

We can choose the gauge condition
\[ \delta A_t = 0. \]  

Substituting equations (64), (65), (74) and (78) into equations (24) and (28), we obtain perturbation equations whose explicit forms are given in the appendix. Fortunately, in these modes, the metric and the electromagnetic perturbations are decoupled. For the metric perturbation equations, the master variable
\[ \Phi_{0G}(\rho) + V_{0G}(\rho) \Phi_{0G} = \omega^2 \Phi_{0G}, \]  
where the master variable \(\Phi_{0G}(\rho)\) and the potential \(V_{0G}\) are defined by
\[ \Phi_{0G}(\rho) := \frac{(\rho + \rho_0)^{5/4}(2\rho + \rho_0)}{\rho^{1/4}(4\rho + 3\rho_0)} h_{33}(\rho), \]  
\[ V_{0G} = \frac{(\rho - \rho_+)(\rho - \rho_+)\rho_0}{16\rho^5(\rho_0 + \rho_+ - \rho_+ + \rho)^3(3\rho_0 + \rho_+ - \rho_+ + 4\rho)^2} \times \left(128\rho_0 + 128(-\rho_+ + 3\rho_+)(\rho - \rho_+)^3 + 528\rho_0^3 + 32(-\rho_+ + 55\rho_+ - \rho_+ + 16(-3\rho_+^2 + 98\rho_+ - 17\rho_+^2))(\rho - \rho_+)^4 + (720\rho_0^3 + 48(13\rho_+ + 63\rho_+\rho_0^2 + 16(-57\rho_+^2 + 330\rho_+ - 79\rho_+^2)\rho_0 + 16(-51\rho_+^2 + 141\rho_+\rho_0^2 + 63\rho_+\rho_+ - 7\rho_+^2))(\rho - \rho_+)\right)^3 + (315\rho_0^3 + 60(8\rho_+ + 41\rho_+\rho_0^2 + (-450\rho_+^2 + 6376\rho_+\rho_+ - 1850\rho_+^2)\rho_0^2 + (-1080\rho_+^3 + 5372\rho_+\rho_0^2 + 3056\rho_+\rho_+ + 460\rho_+^2)\rho_0 - 465\rho_+^3 + 35\rho_+^4 + 328\rho_+\rho_+^3 + 1206\rho_+^2\rho_+ + 1456\rho_+^3\rho_+)(\rho - \rho_+)^2 + (792\rho_+ - 81\rho_+\rho_0^2 + 12(-27\rho_+^2 + 239\rho_+\rho_+ + 98\rho_+^2)\rho_0^2 + (-486\rho_+^3 + 3852\rho_+\rho_0^2 + 2834\rho_+\rho_+ + 520\rho_+^4)\rho_0^2$}

\(^3\) Note that we cannot choose this gauge condition for static perturbation.
The master equation for the electromagnetic perturbations is given by

\[ \frac{d^2}{d\rho^2} \Phi_{0E} + V_{0E}(\rho)\Phi_{0E} = \omega^2 \Phi_{0E}, \]  

where \( \Phi_{0E}(\rho) \) and \( V_{0E} \) are defined by

\[ \Phi_{0E}(\rho) := \rho^{1/4}(\rho + \rho_0)^{3/4} A_3(\rho), \]  

\[ V_{0E} = \frac{(\rho - \rho_0)(\rho - \rho_+ + \rho_-)}{16\rho^3(\rho_0 + \rho_- - \rho_+ + \rho)^3} \left[ (8\rho_0 - 8\rho_+ + 24\rho_+) (\rho - \rho_+)^3 \right. \]
\[ + (15\rho_0^2 + 2\rho_- + 50\rho_+\rho_0 - 13\rho_+^2 + 63\rho_-^2 - 2\rho_-\rho_+)(\rho - \rho_+)^2 \]
\[ + (5(-\rho_- + 8\rho_+))\rho_0^2 + 10(-\rho_-^2 - 3\rho_-\rho_+ + 12\rho_-^2)\rho_0 \]
\[ + \rho_-(5\rho_-^2 - 70\rho_-\rho_+ + 123\rho_-^2)(\rho - \rho_+) \]
\[ + 4\rho_-\rho_0(15\rho_- - 11\rho_+ + 4\rho_0\rho_-\rho_+)(31\rho_- - 22\rho_+ + \rho_0^2(64\rho_-^2 - 44\rho_-\rho_+)). \]  

From (38), (81) and (84), we can see \( V_{0G} > 0 \) and \( V_{0E} > 0 \) outside of the horizon. The stability has been shown against the perturbation for the \( K = 0 \) mode. Typical profiles of the potentials are plotted in figures 7–10.

3.2. \( K = \pm (J + 2) \) modes perturbation

As noted in the previous section, the highest modes for \( h_{++,} \) and \( h_{--} \) are always decoupled for arbitrary \( J \). Since these are gauge invariant, it is straightforward to get the perturbation...
equation for $h_{++}$, whose explicit form is given in the appendix. Defining the new variable

$$\Phi_J(\rho) := \frac{1}{\rho^{3/4} (\rho + \rho_0)^{3/4}} h_{++}(\rho),$$

we obtain the master equation

$$-\frac{d^2}{d\rho_*^2} \Phi_J + V_J(\rho) \Phi_J = \omega^2 \Phi_J,$$

where the potential $V_J(\rho)$ is defined by

$$V_J = \frac{\rho_0 (\rho - \rho_+)(\rho - \rho_* + \rho_-)}{16 \rho_0 (\rho_0 + \rho_-) \rho^5 (\rho_0 + \rho_- + \rho - \rho_*)^3} \left[ 16 \rho_*^2 (2 + J)^2 \rho_*^2 
+ 4 \rho_*^3 \rho_0 (3 \rho_- + 2(\rho_0 + \rho_- + \rho - \rho_*)^3 \rho_0 (249 + 240 J + 64 J^2) \rho_*^4 
+ 32 \rho_*^2 (2 + J)^2 \rho_* (\rho_- + 2 \rho_*) + \rho_*^2 (-9 \rho_*^2 + 2 \rho_- (249 + 240 J + 64 J^2) \rho_*^2 \right].$$
the master equations for variables in terms of the Wigner function given in [56] to the SqKK black holes with charge [31]. We have expanded the perturbation

In this paper, we have extended the stability analysis for perturbations of the SqKK black holes shown against perturbation for

\[ V_{\text{eff}} \]

Since we can see \( V_J > 0 \) outside of the horizon from (38) and (87), the stability has been shown against perturbation for \( K = \pm (J + 2) \) modes.

### 4. Summary and discussion

In this paper, we have extended the stability analysis for perturbations of the SqKK black holes given in [56] to the SqKK black holes with charge [31]. We have expanded the perturbation variables in terms of the Wigner function \( D^{J}_{K,M} \) which have three indices \( J, M, K \), and derived the master equations for \( J = 0, M = 0, K = 0, \pm 1, \pm 2 \) modes which have the \( SU(2) \) symmetry and the highest modes \( (K = \pm (J + 2)) \). Using the effective potential functions in the master equations, we have discussed the stability for these modes.
Since the modes of $J = 0, M = 0, K = \pm 2$ and the highest modes contain only the gravitational perturbation, the perturbation equation for each mode can be reduced to a single master equation. In the mode of $J = 0, M = 0, K = 0$, the perturbed equations can be reduced to two decoupled master equations. As for the modes of $J = 0, M = 0, K = \pm 1$, we have obtained the coupled master equations for the gravitational and the electromagnetic perturbations, in contrast to the case for the Reissner–Nordström black holes of which these perturbations are decoupled against all modes [18]. We guess that the coupling for the gravitational and the electromagnetic perturbations comes from the differences of the asymptotic structures.

We have shown from the positivity of the effective potential functions of the master equations that the modes of $J = 0, M = 0, K = 0, \pm 2$ and $K = \pm (J + 2)$ are stable, the same as in the neutral case. In the case of $J = 0, M = 0, K = \pm 1$, the effective potential functions form a matrix. In the maximally charged case, we have shown the stability for $J = 0, K = \pm 1$ by using the matrix of effective potential functions. In general, because there are domains in which the effective potential functions are negative, we must find an appropriate $S$-deformation function similar to the discussions in [17, 18] to show the stability for $J = 0, M = 0, K = \pm 1$ modes. Even in neutral cases where the gravitational and the electromagnetic perturbations are decoupled, the $S$-deformation function have not been found yet. It is difficult to show the stability of these modes in the case of charged black holes. To show stability for $J = 0, K = \pm 1$ modes in general is an open issue.

Since the stability for $J = 0$ modes is suggested in the limit of both the neutral case [56, 57] and the extremal case, and the instability empirically appears in the lower modes, we expect that $SqKK$ black holes with charge are stable.

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Appendix. The perturbation equations

In this section, the precise form of the perturbation equations are given for each mode.

A.1. $K = \pm 2$ modes

In the $K = 2$ mode, there is only the $(++)$ component of the linearized Einstein equation which is given by

$$
\delta G_{++} - 2\delta T_{++} = \frac{h_{++}}{2\rho^3(\rho + \rho_0)^3(\rho_0 + \rho_+)(\rho_0 + \rho_-)} \\
\times \left[ 4\rho_0^6 + 16\rho_0^5\rho_0 - 2\rho_0^2\rho_+\rho_-(\rho_0 + \rho_+)(\rho_0 + \rho_-) \\
+ 4\rho^4(5\rho_0^3 - \rho_+\rho_- - \rho_0(\rho_+ + \rho_-)) \\
+ \rho_0(\rho_0 + \rho_+)(\rho_0 + \rho_-)(-6\rho_+\rho_- + \rho_0(\rho_+ + \rho_-)) \\
+ 4\rho^3(3\rho_0^3 + \rho_+\rho_-(\rho_+ + \rho_-) + \rho_0(\rho_+^2 + \rho_+\rho_- + \rho_-^2)) \\
+ \rho^2(2\rho_0^5 - 6\rho_0^2\rho_+^2 + \rho_0^3(\rho_+ + \rho_-))
\right]
$$
\[-3\rho_0\rho_-\rho_-\rho_- + \rho_0^2(3\rho_0^2 + 3\rho_+^2 - 2\rho_+\rho_-)\] 
\[+2\rho^2 + 2\rho_0\rho_- + \rho_- - 3\rho^2(\rho_+ - \rho_-) - \rho(-4\rho_+\rho_- + \rho_0(\rho_+ + \rho_-))\frac{d\dot{h}_{++}}{d\rho}\] 
\[= \frac{(\rho - \rho_+)(\rho - \rho_-)}{2\rho(\rho + \rho_0)} \frac{d^2h_{++}}{d\rho^2} - \frac{\rho^2}{2(\rho - \rho_+)(\rho - \rho_-)}\omega^2h_{++} = 0. \quad (A.1)\]

The equation for the $K = -2$ mode can be obtained by just replacing $h_{++}$ to $h_{--}$ in equation (A.1).

### A.2. $K = \pm 1$ modes

In the $K = 1$ mode, there are $(t \pm)$, $(\rho \pm)$ and $(+3)$ components of the linearized Einstein equation and there is the $(\pm)$ component of the Maxwell equation for perturbations which are given by

\[
\delta G_{t \pm} - 2\delta T_{t \pm} = \frac{1}{2\rho(\rho + \rho_0)(\rho_+ + \rho_-)(\rho + \rho_0)^2} \times \left[ \rho^2 + 3\rho_0^2 + 3\rho_+^2 + (\rho_0^2 - 2(\rho - \rho_+)^2\rho_0 \right.
\[\left. - 2\rho_+\rho_- + (\rho_+ + \rho_-)\rho^2 - (\rho_+ + \rho_-)\rho(\rho_+ + \rho_-)\rho_0 \right.
\[\left. - 3\rho_+\rho_- + \rho_+\rho_0(\rho_+ + \rho_-)\rho_0\right]h_{tt} = \frac{\rho^2}{2\rho(\rho + \rho_0)^2} \frac{d^2h_{tt}}{d\rho^2} - \frac{\rho}{2(\rho - \rho_+)(\rho - \rho_-)}\omega^2h_{tt} = 0, \quad (A.2)\]

\[
\delta G_{\rho \pm} - 2\delta T_{\rho \pm} = \frac{\rho^2 + 2\rho_0\rho_+-\rho_- - (\rho_+ + \rho_-)\rho_0^2}{2\rho(\rho_+ + \rho_-)(\rho_+ + \rho_-)\rho_0}h_{\rho \pm} - \frac{i\omega(2\rho + \rho_0)}{2(\rho - \rho_+)(\rho - \rho_-)(\rho + \rho_0)}h_{\rho \pm} 
\[\left. - \frac{i\rho^2}{(\rho - \rho_+)(\rho - \rho_-)}\delta A_{\pm} + \frac{2\omega^2\rho^2}{2(\rho - \rho_+)(\rho - \rho_-)}\frac{d\delta A_{\pm}}{d\rho} \right] 
\[\omega^2\rho^2h_{\rho \pm} = 0, \quad (A.3)\]

\[
\delta G_{+3} - 2\delta T_{+3} = \frac{i}{2\rho^2(\rho + \rho_0)^4} \left[ (2\rho^5 + (-\rho_- - \rho_+ + 5\rho_0)^2) + 2\rho_0^2(\rho_--\rho_+)^2 \right.
\[\left. - (5(\rho_+ + \rho_-)^2 + 3(\rho_+ + \rho_-)^2 + 3\rho_0^2(\rho_+ + \rho_-))\rho_0 \right.
\[\left. + 2\rho_+\rho_-\rho_0(\rho_+ + \rho_-)(\rho_+ + \rho_-)\rho_0 \right. 
\[\left. + 2\rho_0\rho_\rho_+\rho_- - (\rho_+ + \rho_-)\rho_0 \right. 
\[\left. + \rho_0^2(\rho_+^2 + 2\rho_0\rho_- + \rho_-^2)(\rho + \rho_0)^2 \right]h_{\rho \pm} 
\[\left. + \frac{i\rho^2(\rho_+^2 + 2\rho_0\rho_- - (\rho_+ + \rho_-)\rho_0 \right. 
\[\left. + \frac{2(\rho - \rho_+)(\rho - \rho_-)(\rho + \rho_0)^2}{2(\rho - \rho_+)(\rho - \rho_-)(\rho + \rho_0)^2} \frac{d\delta A_{\pm}}{d\rho} \right] 
\[\omega^2\rho^2h_{\rho \pm} = 0, \quad (A.4)\]

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\[ \delta \left( \nabla_{\mu} F^{\mu\nu} \right) = -\frac{1}{\sin^2 \theta} \frac{\delta A_{\nu}}{\rho^2 (\rho - \rho_0)(\rho + \rho_0)} - \frac{(\rho_- + \rho_+ \rho_0 - \delta A_{\nu})}{\rho^3 (\rho + \rho_0)^3} \delta \frac{\delta A_{\nu}}{d\rho} \]

\[ = \frac{\omega^2 \rho}{(\rho - \rho_\nu)(\rho - \rho_\nu)(\rho + \rho_\nu)} \delta A_{\nu} - \frac{\sqrt{3} \sqrt{(\rho - \rho_\nu)(\rho - \rho_\nu)(\rho + \rho_\nu)(2 \rho + \rho_0)}}{2 \rho^3 (\rho - \rho_\nu)(\rho + \rho_\nu)^3} \delta h_{t\nu}
\]

\[ + \frac{\sqrt{3} \sqrt{(\rho - \rho_\nu)(\rho - \rho_\nu)(\rho_\nu + \rho_\nu)(\rho + \rho_\nu)^2}}{2 \rho^2 (\rho - \rho_\nu)(\rho + \rho_\nu)^2} \frac{\delta h_{\nu \nu}}{d\rho} + \frac{\frac{2 \rho^2 (\rho - \nu)(\rho + \rho_\nu)^3}{d\rho} + \frac{(\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)(\rho + \rho_\nu)^2}{d\rho} + \frac{(\rho_\nu - \rho_\nu)(\rho_\nu - \rho_\nu)^2}{d\rho} - \frac{(\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)}{d\rho} = 0. \]

(A.5)

The equations for the \( K = -1 \) mode can be obtained by taking equations (A.2), (A.3), (A.4) and (A.5) to its complex conjugate equations, and using the relations \( \dot{h}_{A\nu} = h_{A\nu}, \dot{h}_{t\nu} = h_{t\nu} \) and \( \dot{\delta A}_{\nu} = \delta A_{\nu} \).

### A.3. \( K = 0 \) mode

In the \( K = 0 \) mode, there are \((t t), (\nu \nu), (\rho \rho), (t \nu), (\nu \nu), (\nu \nu)\) components of the linearized Einstein equation and there are \((t t), (\rho \rho), (t \nu)\) and \((\nu \nu), (++)\) and (33) components of the Maxwell equation for perturbations which are given by

\[
\delta G_{tt} - 2\delta T_{tt} = -\frac{(\rho - \rho_\nu)(\rho - \rho_\nu)(\rho_\nu + \rho_\nu) h_{t3} + (\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)^2}{4 \rho^3 (\rho + \rho_\nu)^3} + \frac{(\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)(\rho_\nu + \rho_\nu)^2}{d\rho} = 0, \quad \text{(A.6)}
\]

\[
\delta G_{\rho\rho} - 2\delta T_{\rho\rho} = \frac{\omega(\rho - \rho_\nu)(\rho - \rho_\nu)(\rho_\nu + \rho_\nu)^2}{d\rho} = 0, \quad \text{(A.7)}
\]
\[ \delta G_{33} - 2\delta T_{33} = - \frac{i\omega}{2\rho(\rho + \rho_0)^2} \delta A_{\rho} = 0, \tag{A.11} \]

\[ \delta G_{\rho\rho} - 2\delta T_{\rho\rho} = - \frac{i\omega\sqrt{3}(\rho - \rho_\rho)}{\rho(\rho - \rho_\rho)} \delta A_{\rho} = 0. \tag{A.12} \]
\[ \delta(\nabla_\mu F^{\mu\nu}) = \frac{-1}{4\rho^3(\rho + \rho_0)(\rho+\rho_\pm)} \left[ \sqrt{3} \sqrt{\delta A_\rho} \rho_0(\rho + \rho_0)^2 \delta h_{33} \right. \\
+ \sqrt{3} \delta A_\rho (\rho_+ + \rho_0) \rho_0^2 (\rho_+ + \rho_0) \rho_0 \rho_0^2 \\
- 2 \rho_+ \rho_\pm - \rho_\pm \rho_0 \delta h_{\rho \rho} - \sqrt{3} \rho_+ \rho_0 (\rho + \rho_0)^2 \frac{d\delta h_{33}}{d\rho} \\
+ \sqrt{3} \delta A_\rho (\rho_+ - \rho)(\rho_+ + \rho_0) \delta h_{\rho \rho} \\
+ i \rho^3 (\rho_+ + \rho_0) (\rho_+ + \rho_0) (\rho + \rho_0)^2 \delta A_\rho \\
+ i \rho^4 (\rho_+ + \rho_0) (\rho_+ + \rho_0) (\rho + \rho_0)^2 \frac{d\delta A_\rho}{d\rho} \left] = 0; \quad (A.13) \right. \]

\[ \delta(\nabla_\mu F^{\mu\rho}) = \frac{-1}{4\rho^2(\rho + \rho_0)(\rho+\rho_\pm)} \left[ -i \rho^2 \sqrt{3} \delta A_\rho \rho_0 (\rho + \rho_0)^2 \delta h_{33} \right. \\
+ i \rho^2 \sqrt{3} \delta A_\rho (\rho_+ - \rho)(\rho_+ + \rho_0) \delta h_{\rho \rho} \\
- \alpha^2 \rho^4 (\rho_+ + \rho_0) (\rho_+ + \rho_0) (\rho + \rho_0) \delta A_\rho \right] = 0, \quad (A.14) \]

\[ \delta(\nabla_\mu F^{\mu3}) = - \frac{i \rho^2 \sqrt{3} \delta A_\rho}{2\rho^2(\rho_+ + \rho_0)(\rho+\rho_0)} \delta h_{33} - \frac{1}{\rho^2(\rho + \rho_0)^2} \delta A_3 \\
+ \frac{2 \rho^4}{\rho^3(\rho_+ + \rho_0)(\rho+\rho_0)(\rho + \rho_0) \rho_0} \delta A_3 \\
+ \frac{(\rho_+ - \rho_\pm - \rho_0) \rho_0}{\rho_3(\rho_+ + \rho_0)(\rho_\pm + \rho_0)(\rho + \rho_0) \rho_0} \delta A_3 \\
+ \frac{\rho^2}{\rho_3(\rho_+ + \rho_0)(\rho_\pm + \rho_0)(\rho + \rho_0) \rho_0} \frac{d^2 \delta A_3}{d\rho^2} \\
+ \frac{\alpha^2 \rho^4}{(\rho_+ - \rho_\pm)(\rho_\pm + \rho_0)(\rho + \rho_0)} \delta A_3 = 0. \quad (A.15) \]

A.4. \( K = \pm (J + 2) \) modes

In the \( K = (J + 2) \) mode, there is only the (+) component of the linearized Einstein equation which is given by

\[ \delta G_{++} - 2\delta T_{++} = \frac{h_{++}}{2 \rho^3(\rho_+ + \rho_0)(\rho_+ + \rho_0)(\rho + \rho_0)^2} \left[ (J + 2)^2 \rho^6 + 4(\rho + \rho_0)^3 \right. \\
+ \rho^3((2J + 5)(J + 4)\rho_0 - (J + 4)(\rho_+ + \rho_0)\rho_0 - (J + 4)\rho_+ - \rho_0) \\
+ 2 \rho^3(J + 2)\rho_0(3J + 3)\rho_0^3 - J(\rho_+ + \rho_0)\rho_0^3 + 2(\rho_+ - \rho_0)\rho_0^3 \\
+ 2 \rho_\pm(\rho_+ + \rho_0) + \rho^2((J + 4)(\rho_+ + \rho_0)\rho_0 - (J + 4)\rho_+ - \rho_0) \\
+ 3(\rho_+ - \rho_0)\rho_0(\rho_+ + \rho_0)\rho_0^3 - 3 \rho_\pm(\rho_+ + \rho_0)\rho_0^3 - 6 \rho_\pm^2 \rho_0 \\
+ \rho^2((\rho_+ + \rho_0)(\rho_+ + \rho_0)(\rho_+ + \rho_0) - 2 \rho_\pm(\rho_+ + \rho_0)(\rho_+ + \rho_0)) \\
+ 2 \rho^3 - 3 \rho_\pm^2 \rho_0^2 - ((\rho_+ + \rho_0)\rho_0 - 4 \rho_\pm \rho_0 - 2 \rho_\pm\rho_0\rho_0 \frac{d\delta h_{++}}{d\rho} \\
- \frac{(\rho_+ - \rho_\pm)\rho_0^2}{2 \rho^2(\rho + \rho_0)^2} \frac{d^2 \delta h_{++}}{d\rho^2} - \frac{\rho^2}{2 \rho^2(\rho_+ + \rho_0)} \frac{d^2 \delta h_{++}}{d\rho^2} \right] = 0. \quad (A.16) \]

The equation for the \( K = -(J + 2) \) mode can be obtained by replacing \( h_{++} \) to \( h_{--} \) in equation (A.16).
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