A SYMPLECTIC NON-SQUEEZING THEOREM FOR BBM EQUATION

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Abstract. We study the initial value problem for the BBM equation:
\[
\begin{aligned}
  u_t + u_x + uu_x - u_{txx} &= 0 & x \in \mathbb{T}, t \in \mathbb{R} \\
  u(0, x) &= u_0(x)
\end{aligned}
\]
We prove that the BBM equation is globally well-posed on \(H^s(\mathbb{T})\) for \(s \geq 0\) and a symplectic non-squeezing theorem on \(H^{1/2}(\mathbb{T})\). That is to say the flow-map \(u_0 \mapsto u(t)\) that associates to initial data \(u_0 \in H^{1/2}(\mathbb{T})\) the solution \(u\) cannot send a ball into a symplectic cylinder of smaller width.

1. Introduction

In 1877 Joseph Boussinesq proposed a variety of models for describing the propagation of waves on shallow water surfaces, including what is now referred to as the Korteweg-de Vries (KdV) equation. A scaled KdV equation reads
\[
u_t + u_x + \varepsilon (uu_x + u_{xxx}) = 0.
\]
The Benjamin-Bona-Mahony (BBM) equation was introduced in [1] as an alternative of the KdV equation. The main argument to derive the BBM equation is that, to the first order in \(\varepsilon\), the scaled KdV equation is equivalent to
\[
u_t + u_x + \varepsilon (uu_x - u_{txx}) = 0.
\]
Indeed, formally we have \(u_t + u_x = O(\varepsilon)\), hence \(u_{xxx} = -u_{txx} + O(\varepsilon)\).
In this article we shall consider the rescaled BBM equation:
\[
u_t + u_x + uu_x - u_{txx} = 0.
\]

In 2009, Jerry Bona and Nikolay Tzvetkov proved in [2] that BBM equation is globally well-posed in \(H^s(\mathbb{R})\) if \(s \geq 0\), and not even locally well-posed for negative values of \(s\) (see also [3]). The result extends to the periodic case (see section 3 below). Let us denote \(\Phi_t\) the flow map of BBM equation on the circle \(\mathbb{T}\). In this article we prove a symplectic non-squeezing theorem for \(\Phi_t\). That is, the flow map cannot squeeze a ball of radius \(r\) of \(H^{1/2}(\mathbb{T})\) into a symplectic cylinder of radius \(r' < r\). Precisely, let \(H^{1/2}_0(\mathbb{T}) = \{ u \in H^{1/2}(\mathbb{T}) / \int_\mathbb{T} u = 0 \} \) with the Hilbert basis
\[
\varphi^+_n(x) = \sqrt{\frac{n}{\pi(n^2 + 1)}} \cos(nx), \quad \varphi^-_n(x) = \sqrt{\frac{n}{\pi(n^2 + 1)}} \sin(nx).
\]
Set
\[
B_r = \left\{ u \in H^{1/2}_0(\mathbb{T}) / \| u \|_{H^{1/2}} < r \right\},
\]}
The goal of this paper is to prove Theorem 1.1. If $\Phi_t(B_r) \subset C_{R,n_0}$ then $r \leq R$.

S. Kuksin initiated the investigation of non-squeezing results for infinite dimensional Hamiltonian systems (see [7]). In particular he proved that nonlinear wave equation has the non-squeezing property for some nonlinearities. This result were extended to certain stronger nonlinearities by Bourgain [8], and he also proved with a different method that the cubic NLS equation on the circle $\mathbb{T}$ has the non-squeezing property. Using similar ideas Colliander, Keel, Staffilani, Takaoka and Tao obtained the same result for KdV equation on $\mathbb{T}$ (see [4]).

In this article we will use the original theorem of Kuksin. In section 2, we present the construction of a capacity on Hilbert spaces introduced by Kuksin in [7]. This capacity is invariant with respect to the flow of some hamiltonian PDEs provided it has the form “linear evolution + compact”. As a corollary of this result we get a non-squeezing theorem for these PDEs. Then we apply this theorem to the BBM equation in section 3. We prove the global wellposedness of BBM equation on $H^s(\mathbb{T})$ for $s \geq 0$, and some estimates on the solutions.

2. Symplectic capacities in Hilbert spaces and non-squeezing theorem

2.1. The framework and an abstract non-squeezing theorem. Let $(Z, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with $\{\varphi_j^\pm / j \geq 1\}$ a Hilbert basis. For $n \in \mathbb{N}$ we denote $Z_n = \text{Span}(\{\varphi_j^\pm / 1 \leq j \leq n\})$, and $\Pi_n : Z \to Z_n$ the corresponding projector. We also denote $Z_n$ the space such that $Z = Z_n \oplus Z_n$. Then, every $z \in Z$ admits the unique decomposition $z = z_n^+ + z_n^-$ with $z_n^+ \in Z_n^+$ and $z_n^- \in Z_n^-$. We define $J : Z \to Z$ the skewsymmetric linear operator by

$$J\varphi_j^\pm = \mp\varphi_j^\mp$$

and we supply $Z$ with a symplectic structure with the 2-form $\omega$ defined by

$$\omega(\xi, \eta) = \langle J\xi, \eta \rangle.$$

We take a self-adjoint operator $A$, such that

$$\forall j \in Z, A\varphi_j^\pm = \lambda_j\varphi_j^\pm.$$

Define the Hamiltonian

$$f(z) = \frac{1}{2} \langle Az, z \rangle + h(z)$$

where $h$ is a smooth function defined on $Z \times \mathbb{R}$. The corresponding Hamiltonian equation has the form

$$\left\{ \begin{array}{l} \dot{z} = JA\dot{z} + J\nabla h(z) \\ z(0, \cdot) = z_0 \in Z \end{array} \right.$$

If $Z_-$ is a Hilbert space, we denote

$$Z < Z_-.$$
if $Z$ is compactly embedded in $Z_-$ and $\{\varphi^\pm_j\}$ is an orthogonal basis of $Z_-$ (not an orthonormal one!). Clearly $Z$ is dense in $Z_-$. We identify $Z$ and its dual $Z^\ast$. Then $(Z_-)^\ast$ can be identified with a subspace $Z_+$ of $Z$ and we have

$$Z_+ < Z < Z_-.$$ 

Denote $\| \cdot \|_-$ (resp. $\| \cdot \|_+$) the norm of $Z_-$ (resp. $Z_+$).

We also denote $B_R(Z)$ the ball centered at the origin of radius $R$.

We impose the following assumptions:

(H1): The equation (2) defines a $C^1$-smooth global flow map $\Phi$ on $Z$. That is, for all $z_0 \in Z$ the equation (2) has a unique solution $z(t) = \Phi_t(z_0)$ for $t \geq 0$, and the flow map $\Phi_t : z_0 \mapsto z(t)$ is $C^1$-smooth.

(H2): The flow map $\Phi$ is uniformly bounded. That is for each $R > 0$ and $T > 0$, there exists $R' = R'_{R,T}$ such that

$$\Phi_t(B_R(Z)) \subset B_{R'}(Z), \quad \text{for } |t| \leq T.$$ 

(H3): Writing the flow map $\Phi_t = e^{tJA}(I + \tilde{\Phi}_t)$, we also impose the following compactness assumption: fix $R > 0$ and $T > 0$, there exists $C_{R,T}$ such that

$$\forall u_0, u'_0 \in B_R(Z), \quad \left\| \Phi_T(u_0) - \Phi_T(u'_0) \right\|_{Z_+} \leq C_{R,T} \| u_0 - u'_0 \|_Z.$$ 

Under these assumptions, it is well known that the flow maps $\Phi_t$ preserve the symplectic form.

The aim of this section is to show the following non-squeezing theorem

**Theorem 2.1.** Assume $\Phi_T$ is the flow map of an equation of the form (2) and satisfies the previous assumptions. If $\Phi_T$ sends a ball $B_r = \{ z \in Z/ \| z - \bar{z} \| < r \}$, $\bar{z}$ fixed

into a cylinder

$$C_{R,j_0} = \left\{ z = \sum p_j \varphi^+_j + q_j \varphi^-_j \left/ (p_{j_0} - \bar{p}_{j_0})^2 + (q_{j_0} - \bar{q}_{j_0})^2 < R^2 \right. \right\}$$

$j_0, \bar{p}_{j_0}, \bar{q}_{j_0}$ fixed

then $r \leq R$.

In fact, this theorem is a simple version of the conservation of a symplectic capacity on $Z$ by the flow map $\Phi_T$ (see subsection 2.3.2 below)

**Remark 2.2.** This theorem implies the following fact. Fix $\varepsilon > 0$, a time $T > 0$, a Fourier mode $n_0$ and $r > 0$ (no smallness conditions are imposed on $r$ or $T$), then there exists $u_0 \in H^{1/2}(T)$ such that

$$\| u_0 \|_{H^{1/2}} < r$$

and

$$|\hat{u}(T)(n_0)| > \frac{r - \varepsilon}{(n_0^2 + 1)^{1/4}}$$

where $u$ solves (2).
The non-squeezing theorem remains true if we don’t suppose that the flow map is global in (H1), but the conclusion would be:

either

\[ |\hat{u}(T)(n_0)| > \frac{r - \varepsilon}{(n_0^2 + 1)^{1/4}} \]

or

\[ \sup_{0 \leq t \leq T} \|u(t)\|_{H^{1/2}} = +\infty. \]

So we impose the global wellposedness in time for \(\Phi\) in order to rule out the second case.

2.2. An approximation lemma. In order to define a capacity, we will need to approximate the flow by finite-dimensional maps. We shall use the following lemma

**Lemma 2.3.** Let \(\Phi\) the flow at time \(T\) of an equation \(\Phi\) satisfying the previous assumptions. For each \(\varepsilon > 0\) and \(R > 0\), there exists \(N \in \mathbb{N}\) such that for \(u \in B_R\):

\[ \Phi(u) = e^{tJA}(I + \tilde{\Phi}_e)(I + \tilde{\Phi}_N)(u) \]

where \((I + \tilde{\Phi}_e)\) and \((I + \tilde{\Phi}_N)\) are symplectic diffeomorphisms satisfying

\[ \|\tilde{\Phi}_e(u)\| \leq \varepsilon \quad \text{for} \quad u \in (I + \tilde{\Phi}_N)(B_R) \]

\[ (I + \tilde{\Phi}_N)(u^N + u_N) = (I + \tilde{\Phi}_N)(u^N) + u_N \quad \text{for} \quad u^N \in Z^N, u_N \in Z_N. \]

**Proof.** Recall that \(\Phi = e^{tJA}(I + \tilde{\Phi})\). First, we observe that for \(|t| \leq T\), any \(R > 0\) and \(u, v \in B_R(Z)\) we have

\[ \|\tilde{\Phi}(u) - \Pi^N\tilde{\Phi}(u)\|_Z \leq \varepsilon_1(N) \xrightarrow{N \to +\infty} 0. \]

Indeed, as \(K = \bigcup_{|t| \leq T} \tilde{\Phi}(B_r(Z))\) is precompact in \(Z\) (by (H3)), then \(\Phi\) results from the following statement

\[ \sup_{u \in K} \|u - \Pi^N u\|_N \xrightarrow{N \to +\infty} 0. \]

Suppose that the convergence does not hold, then we can find a sequence \((u_n)\) in \(K\) such that \(\|(I - \Pi^N)u_n\| \geq \varepsilon > 0\). As \(K\) is precompact there exists a subsequence \((u_{n_j})\) such that \(u_{n_j} \to u\). For \(n_j\) sufficiently large we have

\[ \|(I - \Pi^N)(u)\| \leq \varepsilon / 2, \quad \|u_{n_j} - u\| \leq \varepsilon / 2. \]

Hence \(\|(I - \Pi^N)(u_{n_j})\| \leq \varepsilon\) and we get a contradiction.

Now we set \(h_N = h \circ \Pi^N\). Then \(\nabla h_N = \Pi^N \nabla h \Pi^N\). We define \(\Phi^N\) the time \(T\) flow of the equation

\[ \dot{v} = J(Av + \nabla h_N(v)) \]

or, equivalently, \(v = v^N + v_N \in Z^N + Z_N\) and

\[
\begin{cases}
\dot{v}^N = J(Av^N + \Pi^N \nabla h(v^N)) \\
\dot{v}_N = JA v_N
\end{cases}
\]

We write \(\Phi^N = e^{tJA}(I + \tilde{\Phi}_N)\).
Since $\Phi_N = 0$ outside $Z^N$, $\Phi_N$ has the desired form \((\ref{eq:phi_n})\). Define
\[
\tilde{\Phi}_\varepsilon = \left( \Phi - \Phi_N \right) \left( I + \Phi_N \right)^{-1},
\]
so we have
\[
e^{TJA} \left( I + \tilde{\Phi}_\varepsilon \right) \left( I + \Phi_N \right) = e^{TJA} \left( I + \tilde{\Phi} \right) = \Phi.
\]

Next we estimate the difference $\tilde{\Phi} - \tilde{\Phi}_N$. For $u \in B_R(Z)$ we have
\[
\left\| \tilde{\Phi}(u) - \tilde{\Phi}_N(u) \right\|_Z \leq \left\| \tilde{\Phi}(u) - \Pi^N \tilde{\Phi}(u) \right\|_Z + \left\| \Pi^N \tilde{\Phi}(u) - \Pi^N \tilde{\Phi}(\Pi^N u) \right\|_Z + \left\| \Pi^N \tilde{\Phi}(\Pi^N u) - \tilde{\Phi}_N(u) \right\|_Z,
\]
Hence by \((\ref{eq:phi_n})\) and assumption (H3), for $u \in B_R(Z)$ we have
\[
\left\| \tilde{\Phi}(u) - \tilde{\Phi}_N(u) \right\|_Z \leq C\varepsilon(N) \xrightarrow{N \to +\infty} 0,
\]
so for $u \in \left( I + \tilde{\Phi}_N \right)(B_R(Z))$
\[
\left\| \tilde{\Phi}_\varepsilon(u) \right\|_Z \leq \varepsilon(N) \xrightarrow{N \to +\infty} 0.
\]

\section{2.3. Symplectic capacities and non-squeezing theorem.}

\subsection{2.3.1. Capacities in finite-dimensional space.}
Consider $\mathbb{R}^{2n}$ supplied with the standard symplectic structure, that is $\omega(x, y) = \langle Jx, y \rangle$ where
\[
J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.
\]

For $f : \mathbb{R}^{2n} \to \mathbb{R}$ a smooth function we define the hamiltonian vectorfield
\[
X_f = J\nabla f.
\]

\begin{definition}
Let $\mathcal{O}$ an open set of $\mathbb{R}^{2n}$, $f \in C^\infty(\mathcal{O})$ and $m > 0$. The function $f$ is called $m$-admissible if
\begin{itemize}
  \item $0 \leq f(x) \leq m$ for $x \in \mathcal{O}$, and $f$ vanishes on a nonempty open set of $\mathcal{O}$, and $f|_{\partial\mathcal{O}} = m$.
  \item The set $\{ z / f(z) < m \}$ is bounded and the distance from this set to $\partial\mathcal{O}$ is $d(f) > 0$.
\end{itemize}

Following \([13]\) we define the capacity $c_{2n}(\mathcal{O})$ of an open set $\mathcal{O}$ of $\mathbb{R}^{2n}$ as
\[
c_{2n}(\mathcal{O}) = \inf \{ m_* \text{ for each } m > m_* \text{ and each } m\text{-admissible function } f \text{ in } \mathcal{O} \}
\]
the vectorfield $X_f$ has a non constant periodic solution of period $\leq 1$.

\begin{theorem}
$c_{2n}$ is a symplectic capacity, that is
\begin{itemize}
  \item if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $c_{2n}(\mathcal{O}_1) \leq c_{2n}(\mathcal{O}_2)$
  \item if $\varphi : \mathcal{O} \to \mathbb{R}^{2n}$ is a symplectic diffeomorphism $c_{2n}(\mathcal{O}) = c_{2n}(\varphi(\mathcal{O}))$.
  \item $c_{2n}(\lambda \mathcal{O}) = \lambda^2 c_{2n}(\mathcal{O})$.
\end{itemize}
\end{theorem}
\[ c_{2n}(B_1) = c_{2n}(C_{r,1}) = \pi \text{ where} \]
\[ B_r = \left\{ (p, q) / \sum (p_j^2 + q_j^2) < r^2 \right\}, \text{ and } C_{r,1} = \left\{ (p, q) / (p_1^2 + q_1^2) < r^2 \right\}. \]

See [6] for a proof. An immediate consequence of this theorem is the non-squeezing theorem of M. Gromov [5].

**Theorem 2.6.** The ball \( B_r \) can be symplecticaly embedded into the cylinder \( C_{R,1} \) if and only if \( r \leq R \).

### 2.3.2. Construction of a capacity on Hilbert spaces.

In this section we define a symplectic capacity on Hilbert spaces which is invariant with respect to the flow of the equation (2). We will follow the construction of S. Kuksin (see [7]).

For \( O \) an open set of \( \mathbb{Z} \) we denote \( O^n = O \cap \mathbb{Z}^n \) and observe that \( \partial O^n \subset \partial O \cap \mathbb{Z}^n \).

**Definition 2.7.** Let \( f \in C^\infty(O) \) and \( m > 0 \). The function \( f \) is called \( m \)-admissible if
- \( 0 \leq f(x) \leq m \) for \( x \in O \), and \( f \) vanishes on a nonempty open set of \( O \), and \( f|_{\partial O} = m \).
- The set \( \{ z / f(z) < m \} \) is bounded and the distance from this set to \( \partial O \) is \( d(f) > 0 \).

**Remark 2.8.** If \( f \) is \( m \)-admissible, denoting \( \text{supp}(f) = \{ z / 0 < f(z) < m \} \), we have
\[ \text{dist}(f^{-1}(0), \partial O) \geq d(f), \]
\[ \text{dist}(\text{supp}(f), \partial O) \geq d(f). \]

Denote \( f_n = f|_{O^n} \) and consider \( X_{f_n} \) the corresponding hamiltonian vectorfield on \( O^n \).

**Definition 2.9.** A \( T \)-periodic trajectory of \( X_{f_n} \) is called fast if it is not a stationary point and \( T \leq 1 \).

A \( m \)-admissible function \( f \) is called fast if there exists \( n_0 \) (depending on \( f \)) such that for all \( n \geq n_0 \) the vectorfield \( X_{f_n} \) has a fast solution.

**Lemma 2.10.** Each periodic trajectory of \( X_{f_n} \) is contained in \( \text{supp}(f) \cap \mathbb{Z}^n \).

**Proof.** Pick \( z \in O^n \setminus \text{supp}(f) \), \( f_n \) takes either its minimal or maximal value in \( z \), hence \( X_{f_n}(z) = 0 \). Therefore \( z \) is a stationary point and a fast trajectory cannot pass through it. \( \blacksquare \)

We are now in position to define a capacity \( c \).

**Definition 2.11.** For an open set \( O \) of \( \mathbb{Z} \) its capacity equals to
\[ c(O) = \inf \{ m_* / \text{each } m \text{-admissible function with } m > m_* \text{ is fast} \}. \]

**Proposition 2.12.** Assume that \( O_1, O_2 \) and \( O \) are open sets of \( \mathbb{Z} \) and \( \lambda \neq 0 \)
- (1) if \( O_1 \subset O_2 \) then \( c(O_1) \leq c(O_2) \);
- (2) \( c(\lambda O) = \lambda^2 c(O) \).
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Proof. (1) Assume \( m < c(\mathcal{O}_1) \), by definition of \( c \) there exists a \( m \)-admissible function \( f \) of \( \mathcal{O}_1 \) which is not fast. Hence, there exists a sequence \( (n_j) \to +\infty \) such that for every \( j \in \mathbb{N} \), \( X_{f_{n_j}} \) has no fast periodic trajectory. Define \( \tilde{f} \) on \( \mathcal{O}_2 \) by

\[
\tilde{f}(x) = \begin{cases} 
  f(x) & \text{if } x \in \mathcal{O}_1 \\
  m & \text{otherwise}
\end{cases}
\]

The function \( \tilde{f} \) is clearly \( m \)-admissible on \( \mathcal{O}_2 \).

By lemma 2.10 for each \( j \in \mathbb{N} \), each fast solution \( x(t) \) of \( X_{\tilde{f}_{n_j}} \) lies in \( \text{supp}\tilde{f} \cap Z^{n_j} = \text{supp}f \cap Z^{n_j} \). Hence \( x(t) \) is a fast trajectory of \( X_{f_{n_j}} \) (\( X_{\tilde{f}_{n_j}} \) and \( X_{f_{n_j}} \) are the same vectorfields on \( \text{supp}(f) \) by definition of \( \text{supp}(f) \)). Therefore, for each \( j \in \mathbb{N} \) the vectorfield \( X_{\tilde{f}_{n_j}} \) of \( \mathcal{O}_2 \) has no fast trajectory.

Hence \( \tilde{f} \) is \( m \)-admissible but is not fast. Thus \( c(\mathcal{O}_2) \geq m \), and the first assertion follows.

(2) Define \( f^\lambda = \lambda^2 f(\lambda^{-1} \cdot) \) on \( \lambda \mathcal{O} \). Clearly \( f \) is \( m \)-admissible on \( \mathcal{O} \) if and only if \( f^\lambda \) is \( \lambda^2 m \)-admissible on \( \lambda \mathcal{O} \). Moreover \( z(t) \in \mathcal{O}^n \) is a \( T \)-periodic trajectory of \( X_{f_n} \) if and only if \( \lambda z(t) \in \lambda \mathcal{O}^n \) is a \( T \)-periodic trajectory of \( X_{f_n} \). Therefore \( c(\lambda \mathcal{O}) = \lambda^2 c(\mathcal{O}) \).

Lemma 2.13. If \( F : Z \to Z \) has the form

\[
F(z^n + z_n) = F^n(z^n) + z_n \quad z = z^n + z_n \in Z = Z^n \oplus Z_n
\]

with \( F^n \) a symplectic diffeomorphism of \( Z^n \), then \( c(\mathcal{O}) = c(F(\mathcal{O})) \), for each open set \( \mathcal{O} \) of \( Z \).

Proof. We observe that if \( f \) is \( m \)-admissible in \( F(\mathcal{O}) \) and \( f \) is fast then \( f \circ F \) is \( m \)-admissible in \( \mathcal{O} \) and \( f \circ F \) is fast. Indeed \( F^n : f \mapsto f \circ F \) clearly sends \( m \)-admissible functions in \( F(\mathcal{O}) \) to similar ones in \( \mathcal{O} \), and for \( p \geq n \) it transforms \( X_{(f \circ F)^p} \) into \( X_{f^p} \). Hence admissible and fast functions are preserved by \( F \) and its inverse (\( F \) is the identity outside of \( Z^n \) which is a finite-dimentional space), and the result follows.

Proposition 2.14. For each open set \( \mathcal{O} \) of \( Z \) and \( \xi \) in \( Z \), we have

\[
c(\mathcal{O}) = c(\mathcal{O} + \xi).
\]

Proof. Denote \( \mathcal{O}_\xi = \mathcal{O} + \xi \). It is sufficient to prove that \( c(\mathcal{O}) \leq c(\mathcal{O} + \xi) \) (\( \lambda \xi \) into \( -\xi \)).

Denote \( \xi = \xi^{n_0} + \xi_{n_0} \in Z^{n_0} + Z_{n_0} \) \( (n_0 \) will be fixed later) and \( \mathcal{O}_1 = \mathcal{O} + \xi^{n_0} \). By lemma 2.13 \( c(\mathcal{O}_1) = c(\mathcal{O}) \). We also remark that \( \mathcal{O}_\xi = \mathcal{O}_1 + \xi_{n_0} \).

Take any \( m \)-admissible function \( f \) on \( \mathcal{O}_\xi \) with \( m > c(\mathcal{O}) \). We wish to check that \( f \) is fast.

Since \( \partial \mathcal{O}_\xi \subset \partial \mathcal{O}_1 + \xi_{n_0} \) and \( \| \xi_n \| \xrightarrow{n \to +\infty} 0 \), we have

\[
\text{dist}(\partial \mathcal{O}_1, \partial \mathcal{O}_\xi) \leq \text{dist}(\partial \mathcal{O}_1, \partial \mathcal{O}_1 + \xi_{n_0}) \leq \| \xi_{n_0} \| \xrightarrow{n_0 \to +\infty} 0.
\]

Pick \( n_0 \) such that

\[
\text{dist}(\partial \mathcal{O}_1, \partial \mathcal{O}_\xi) \leq \| \xi_{n_0} \| < \frac{1}{2} d(f).
\]

We extend \( f \) outside \( \mathcal{O}_\xi \) with \( f(z) = m \) if \( z \notin \mathcal{O}_\xi \) and we denote \( \tilde{f} \) its restriction to \( \mathcal{O}_1 \).
f equals m on a $d(f)$-neighbourhood of $\partial O_1$. By (5), we deduce that $\tilde{f}$ equals m on a $\frac{1}{2}d(f)$-neighbourhood of $\partial O_1$.

By remark 2.8 we have $\text{dist}(f^{-1}(0), \partial O_1) \geq d(f)$. Hence, by (6), we have $\text{dist}(f^{-1}(0), \partial O_1) \geq \frac{1}{2}d(f)$, and in particular $\tilde{f}$ vanishes on a nonempty open set of $O_1 \cap O_2 \subset O_1$. Therefore $\tilde{f}$ is $m$-admissible.

Since $c(O_1) = c(O) < m$, it follows that $X_{f_0}$ has a fast trajectory in $O_1$ if $n \geq n_0$ is sufficiently large. By lemma 2.10 this trajectory lies in $\text{supp}\tilde{f} = \text{supp}f \subset O_1 \cap O$. Hence this trajectory is a fast solution of $X_{f_0}$, and the function $f$ is fast.

If $r = (r_j)_{j \in \mathbb{N}^*}$ is a sequence of $\mathbb{R}_+ \cup \{+\infty\}$ with $0 < r = \inf_{j \in \mathbb{N}^*} r_j < +\infty$, we define

$$D(r) = \left\{ z = \sum_{j=1}^{+\infty} p_j \varphi_j^+ + q_j \varphi_j^- \mid \forall j \in \mathbb{N}, p_j^2 + q_j^2 < r_j^2 \right\}.$$ $$E(r) = \left\{ z = \sum_{j=1}^{+\infty} p_j \varphi_j^+ + q_j \varphi_j^- \mid \sum_{j=1}^{+\infty} \frac{p_j^2 + q_j^2}{r_j^2} < 1 \right\}.$$  

Remark that if $r = (r, +\infty, \ldots, +\infty)$, $D(r)$ is a symplectic cylinder $C_{r,1}$.

**Theorem 2.15.** We have $c(E(r)) = c(D(r)) = \pi r^2$.

**Proof.** We have to check the following inequalities

1. $c(E(r)) \geq \pi r^2$
2. $c(D(r)) \leq \pi r^2$

then we will conclude by proposition 2.12.

- (1) It is sufficient to prove that $c(B_1) \geq \pi$ (then the result follows by proposition 2.12).

Define $m = \pi - \varepsilon$. Choose $f : [0, 1] \to \mathbb{R}_+$ satisfying:

- $0 \leq f(t) < \pi$ for $t \in [0, 1]$
- $f(t) = 0$ for $t$ near 0
- $f(t) = m$ for $t$ near 1

Then, define $H(x) = f(\|x\|) = \pi r^2$ for $x$ in $B(1)$. $H$ is $m$-admissible. We want to prove that $H$ is not fast. Consider

$$H_n(x) = f \left( \sum_{j=1}^{n} (p_j^2 + q_j^2) \right), \quad \text{where } x = \sum_{j=1}^{n} (p_j \varphi_j^+ + q_j \varphi_j^-).$$

Using the variables $I_j = \frac{1}{2} (p_j^2 + q_j^2)$ and $\theta_j = \arctan \left( \frac{p_j}{q_j} \right)$ we observe that non-constant periodic solutions corresponding to this hamiltonian has a period $T > 1$. Hence $X_{H_n}$ has no fast trajectory and $H$ is not fast.

- (2) Denote $O = D(r)$. Pick $m > \pi r^2$ and $f$ a $m$-admissible function in $O$. Since $f^{-1}(0)$ is not empty, there exists $n$ such that $f^{-1}(0) \cap \mathbb{Z}^n \neq \emptyset$. Denote $f_n = f|\mathbb{O}^n$. Since $\partial \mathbb{O}^n \subset \partial O$, we deduce that $f_n$ equals $m$ on a neighbourhood of $\partial \mathbb{O}^n$. Hence $f_n$ is $m$-admissible.
Since \( c_{2n}(O^n) = \pi \min_{1 \leq j \leq n} r_j^2 \), we have
\[
c_{2n}(O^n) \xrightarrow{n \to +\infty} \pi \inf_{j \geq 1} r_j^2 = \pi r^2 < m.
\]
Hence, for \( n \) sufficiently large \( c_{2n}(O^n) < m \). Therefore \( X_{f_n} \) has a fast periodic trajectory and the function \( f \) is fast. \( \blacksquare \)

**Corollary 2.16.** We have \( c(B_r) = c(C_{r,1}) = \pi r^2 \), and for each bounded open set \( O \) of \( Z \) we have \( 0 < c(O) < +\infty \).

The essential property of the capacity \( c \) is its invariance with respect to the flow maps of PDEs satisfying assumptions (H1), (H2) and (H3). In fact the non-squeezing theorem 2.1 is a consequence of the following result.

**Theorem 2.17.** Let \( \Phi_T \) the flow of an equation (2) satisfying the assumptions (H1), (H2) and (H3). For any open set \( O \) of \( Z \) we have
\[
c(\Phi_T(O)) = c(O).
\]

**Proof.** Let us denote \( \Phi = \Phi_T \) and \( Q = \Phi(O) \). One easily checks that \( \Phi^{-1} \) satisfies (H1), (H2) and (H3), therefore it is sufficient to prove that \( c(Q) \leq c(O) \).

Take any \( m > c(O) \) and any \( f \) \( m \)-admissible in \( Q \). We want to prove that \( f \) is fast.

Since \( f \) is \( m \)-admissible there exists \( R > 0 \) such that \( \text{supp}f \subset B_R \). Define \( R_1 = R + d(f) \), \( Q' = Q \cap B_{R'} \) and \( O' = \Phi^{-1}(Q') \). By assumption \( O' \) is bounded, hence there exists \( R' \) such that \( O' \subset B_{R'} \). Moreover we clearly have \( O' \subset O \), thus by proposition 2.12
\[
c(O') \leq c(O).
\]

We apply lemma 2.3 with \( N \) so large that \( \varepsilon < \frac{1}{2} d(f) \), and we use the notations of the lemma 2.3: \( \Phi = e^{TJA}(I + \tilde{F}_2)(I + \tilde{F}_N) \). We denote \( O_1 \) and \( O_2 \) the intermediate domains which arise from the decomposition
\[
O' \xrightarrow{I + \tilde{F}_N} O_1 \xrightarrow{I + \tilde{F}_2} O_2 \xrightarrow{e^{TJA}} Q'.
\]
We also denote
\[
f_2 = (f \circ e^{TJA})\big|_{O_2}.
\]

Observe that \( f_2 \) is \( m \)-admissible on \( O_2 \). Indeed \( f \) is \( m \)-admissible on \( Q \) and also on \( Q' \) (by definition of \( Q' \)). Since \( e^{TJA} \) is an isometry, \( f_2 \) is \( m \)-admissible.

Then, we extend \( f_2 \) as \( m \) outside \( O_2 \), and we denote \( \tilde{f} \) its restriction to \( O_1 \). By (1) the \( \varepsilon \)-neighbourhood of \( \partial O_1 \) is contained in the \( 2\varepsilon \)-neighbourhood of \( \partial O_2 \). Since \( \varepsilon < \frac{1}{2} d(f) \), we deduce that \( \tilde{f} \) equals \( m \) on a neighbourhood of \( \partial O_1 \). Moreover \( f^{-1}(0) = f_2^{-1}(0) \subset O_1 \cap O_2 \). Indeed by remark 2.8
\[
\text{dist}(f_2^{-1}(0), \partial O_2) \geq d(f)
\]
and
\[
\text{dist}(\partial O_1, \partial O_2) \leq \frac{1}{2} d(f).
\]
Hence \( \tilde{f} \) is \( m \)-admissible on \( O_1 \).
Using lemma 2.13 and (9), we deduce that

\[ c(O_1) = c \left( (I + \tilde{\Phi}) (O') \right) = c(O') \leq c(O) < m. \]

Hence \( \tilde{f} \) is m-admissible on \( O_1 \) and \( c(O_1) < m \), thus \( \tilde{f} \) is fast. So for \( n \) sufficiently large, the vectorfield \( X_{f_n} \) (where \( f_n = \tilde{f} |_{O_1} \)) has a fast solution. By lemma 2.10 this solution lies in \( \text{supp} \tilde{f} \) and by remark 2.8, \( \text{supp} \tilde{f} = \text{supp} f_2 \), so this solution is also a fast solution of \( X_{f_2} \) (where \( f_2 = f_2 |_{O_2} \)). Hence \( f_2 \) is fast too. Finally \( f \) is also fast \( (f_2 = (f \circ e^{TJA}) |_{O_2}) \).

3. Application to the BBM equation

In this section we prove that the BBM equation

\[ \left\{ \begin{array}{ll}
  u_t + u_x + uu_x - u_{xxx} = 0, & x \in \mathbb{T} \\
  u(0, x) = u_0(x)
\end{array} \right. \]

is globally well-posed in \( H^s(\mathbb{T}) \) for \( s \geq 0 \) (we will follow the proof given in [2] for \( x \in \mathbb{R} \) and has the non-squeezing property (theorem 1.1).

3.1. Bilinear estimates. We start by two helpful inequalities.

Let \( \varphi(k) = \frac{k}{1 + k^2} \) and \( \varphi(D) \) the Fourier multiplier operator defined by \( \varphi(D) u(k) = \varphi(k) \hat{u}(k) \).

**Lemma 3.1.** Let \( u \in H^r(\mathbb{T}) \) and \( v \in H^{r'}(\mathbb{T}) \) with \( 0 \leq r \leq s \), \( 0 \leq r' \leq s \) and \( 0 \leq 2s - r - r' < 1/4 \). Then

\[ \| \varphi(D)(uv) \|_{H^s} \leq C_{r,r',s} \| u \|_{H^r} \| v \|_{H^{r'}}. \]

**Proof.** We want to prove

\[ \| \langle k \rangle^s \frac{k}{1 + k^2} \hat{uv}(k) \|_{L^2_k} \leq C \| u \|_{H^r} \| v \|_{H^{r'}}. \]

By duality it is sufficient to prove

\[ \left\langle \langle k \rangle^s \frac{k}{1 + k^2} \hat{uv}, \hat{w} \right\rangle_{L^2_k} \leq C \| u \|_{H^r} \| v \|_{H^{r'}} \| w \|_{L^2_k}, \]

that is

\[ I = \sum_{k \in \mathbb{Z}} k \langle k \rangle^{s-2} \hat{uv} \overline{\hat{w}}(k) \leq C \| u \|_{H^r} \| v \|_{H^{r'}} \| w \|_{L^2_k}. \]

Let \( f(k) = \langle k \rangle^r \hat{u}(k) \), \( g(k) = \langle k \rangle^{r'} \hat{v}(k) \) and \( h(k) = k \langle k \rangle^{-2(1+r+r'-2s)} \overline{\hat{u}}(k) \).

Since

\[ \hat{uv}(k) = \sum_{l \in \mathbb{Z}} \hat{u}(l) \hat{v}(k-l) \]

we have

\[ I = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle k \rangle^{-3s+2r+2r'} \langle k-l \rangle^r f(l)g(k-l)h(k). \]

We have \(-2s + r + r' \leq 0\) and \(-s + r \leq 0\) and \(-s + r' \leq 0\) so

\[ -3s + 2r + 2r' = -2s + r + r' + (-s + r') + r \leq r \text{ and } -3s + 2r + 2r' \leq r'. \]
Hence \( \langle k \rangle^{-3s+2r+2r'} \) is bounded for \( k \) and \( l \) in \( \mathbb{Z} \). Then (by Cauchy-Schwarz inequality and Young’s inequality)

\[
I \lesssim \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} f(l) g(k-l) h(k)
\]

\[
\lesssim \|f\|_{\ell^2} \|g \ast h(\cdot)\|_{\ell^2}
\]

\[
\lesssim \|f\|_{\ell^2} \|g\|_{\ell^2} \|h\|_{\ell^1}
\]

\[
\lesssim \|u\|_{H^r} \|v\|_{H^{r'}} \|w\|_{L^2} \left\| \frac{k}{(1+k^2)^{1+r+r'-2s}} \right\|_{\ell_k^2}.
\]

Since \( 2s - r - r' < 1/4 \) we have \( 1 + r + r' - 2s > 3/4 \). Hence

\[
\left\| \frac{k}{(1+k^2)^{1+r+r'-2s}} \right\|_{\ell_k^2} < +\infty.
\]

In subsection 3.3 we will use this lemma in the particular case \( r = r' = s \geq 0 \), that is

\[
\|\varphi(D)(uv)\|_{H^s} \leq C_s \|u\|_{H^r} \|v\|_{H^s}
\]

whereas in subsection 3.4 we need the general case \( 0 \leq r, r' < s \).

**Lemma 3.2.** Let \( u \in H^r(\mathbb{T}) \) and \( v \in H^s(\mathbb{T}) \) with \( 0 \leq s \leq r \) and \( r > \frac{1}{2} \), then

\[
\|\varphi(D)(uv)\|_{H^{s+1}} \leq C \|u\|_{H^r} \|v\|_{H^s}.
\]

**Proof.** Since \( r > \frac{1}{2} \) and \( r \geq s \geq 0 \), the elements of \( H^r(\mathbb{T}) \) are multipliers in \( H^s(\mathbb{T}) \), which is to say

\[
\|uv\|_{H^s} \lesssim \|u\|_{H^r} \|v\|_{H^s}.
\]

Hence

\[
\|\varphi(D)(uv)\|_{H^{s+1}} = \left\| \frac{\langle k \rangle^{s+1} k}{\langle k \rangle^2} \hat{uv} \right\|_{\ell_k^2}
\]

\[
\leq \|\langle k \rangle^s \hat{uv}\|_{\ell_k^2}
\]

\[
= \|uv\|_{H^s}
\]

\[
\lesssim \|u\|_{H^r} \|v\|_{H^s}.
\]

3.2. **Hamiltonian formalism for BBM equation.** Recall that BBM equation reads

\[
u_t + u_x + uu_x - u_{txx} = 0.
\]

Let us prove that BBM equation is a hamiltonian equation \([2]\).

First BBM can be written

\[
u_t = -\partial_x (1 - \partial_x^2)^{-1}(u + \frac{u^2}{2}).
\]
Denote \( Z = H^{1/2}_0(\mathbb{T}) = \{ u \in H^{1/2} / \int_T u = 0 \} \) with the following norm
\[
\| u \|_Z = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1 + k^2}{k} (a_k^2 + b_k^2)
\]
where \( a_k \) and \( b_k \) are the (real) Fourier coefficients of \( u \).

Consider the Hilbert basis of \( Z \) given by
\[
\varphi^+_n(x) = \sqrt{\frac{n}{\pi(n^2 + 1)}} \cos(nx), \quad \varphi^-_n(x) = \sqrt{\frac{n}{\pi(n^2 + 1)}} \sin(nx).
\]
We have \( Z_+ = H^{1/2+\varepsilon}_0 < H^{1/2}_0 < H^{1/2-\varepsilon}_0 = Z_- \), where \( \varepsilon > 0 \) will be fixed later.

Define
\[
H(u) = \int_T \left( \frac{u(x)^2}{2} + \frac{u(x)^3}{6} \right) dx,
\]
we have
\[
\nabla_{L^2} H(u) = u + \frac{u^2}{2}.
\]

Assume
\[
u(t) = \sum_n p_n(t) \varphi^+_n + q_n(t) \varphi^-_n \]
and
\[
\nabla_{L^2} H(u) = \sum_n \alpha_n \varphi^+_n + \beta_n \varphi^-_n.
\]

Denoting \( \tilde{H}(p, q) = H(\sum_n p_n(t) \varphi^+_n + q_n(t) \varphi^-_n) \) we deduce that
\[
\frac{\partial \tilde{H}}{\partial p_n} = \langle \nabla_{L^2} H(u), \varphi^+_n \rangle_{L^2} = \alpha_n \| \varphi^+_n \|_{L^2}^2 = \frac{n\alpha_n}{1 + n^2}
\]
and
\[
\frac{\partial \tilde{H}}{\partial q_n} = \frac{n\beta_n}{1 + n^2}.
\]

Hence
\[
\dot{u} = \sum_n \dot{p}_n \varphi^+_n + \dot{q}_n \varphi^-_n = (1 - \partial_x^2)^{-1} \partial_x (-\nabla_{L^2} H(u))
\]
\[
= \sum_n \frac{-n\alpha_n}{1 + n^2} \varphi^-_n + \frac{n\beta_n}{1 + n^2} \varphi^+_n
\]
so
\[
\begin{cases}
\dot{p}_n = \frac{n\beta_n}{1 + n^2} = \frac{\partial \tilde{H}}{\partial q_n}
\end{cases} \quad \begin{cases}
\dot{q}_n = -\frac{n\alpha_n}{1 + n^2} = -\frac{\partial \tilde{H}}{\partial p_n}
\end{cases}
\]

That is \( \dot{u} = J \nabla_{L^2} H(u) \).

3.3. Verification of (H1).
3.3.1. Local well-posedness. Recall that $\varphi(k) = \frac{k}{1+k^2}$, the equation (10) can be written in the form:

$$
\begin{cases}
iu_t = \varphi(D)u + \frac{1}{2}\varphi(D)u^2 \\
u(0, x) = u_0(x)
\end{cases}
$$

(11)

Let $e^{-it\varphi(D)}$ be the unitary group defining the associated free evolution. That is, $e^{-it\varphi(D)}u_0$ solves the Cauchy problem

$$
\begin{cases}
iu_t = \varphi(D)u \\
u(0, x) = u_0(x)
\end{cases}
$$

(12)

Then, (11) may be rewritten as the integral equation

$$u(t) = e^{-it\varphi(D)}u_0 - \frac{i}{2}\int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(u(\tau)^2)d\tau = A(u)(t, \cdot).$$

Let $X^s_T = C^0([-T, T], H^s(\mathbb{T}))$. The $H^s$ norm is clearly preserved by the free evolution, thus

$$
\left\|e^{-it\varphi(D)}u_0\right\|_{X^s_T} = \|u\|_{H^s}.
$$

(13)

**Theorem 3.3.** Let $s \geq 0$. For any $u_0 \in H^s(\mathbb{T})$, there exist a time $T$ (depending on $u_0$) and a unique solution $u \in X^s_T$ of (10). The maximal existence time $T_s$ has the property that

$$T_s \geq \frac{1}{4C_s\|u_0\|_{H^s}}$$

with $C_s$ the constant from lemma 3.1 (in the special case $r = r' = s$).

Moreover, for $R > 0$, let $T$ denote a uniform existence time for (10) with $u_0 \in B_R(H^s(\mathbb{T}))$, then the map $\Phi : u_0 \mapsto u$ is real-analytic from $B_R(H^s(\mathbb{T}))$ to $X^s_T$.

**Proof.** Let $R = 2\|u_0\|_{H^s}$. For any $u \in B_R(X^s_T)$, by (13) and lemma 3.1 (with $r = r' = s$) we have

$$
\|A(u)\|_{X^s_T} \leq \left\|e^{-it\varphi(D)}u_0\right\|_{X^s_T} + \frac{1}{2}\left\|\int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(u(\tau)^2)d\tau\right\|_{X^s_T},
$$

$$
\leq \|u_0\|_{H^s} + \frac{C_sT}{2}\|u\|_{X^s_T}^2
$$

$$
\leq \|u_0\|_{H^s} + \frac{C_sT}{2}R^2
$$

$$
\leq R \quad \text{for} \quad T = \frac{2}{C_sR}
$$

and for any $u, v \in B_R(X^s_T)$, by lemma 3.1 (with $r = r' = s$) we have

$$
\|A(u) - A(v)\|_{X^s_T} \leq \frac{C_sT}{2}\|u - v\|_{X^s_T}, \|u + v\|_{X^s_T} \leq C_sTR\|u - v\|_{X^s_T}.
$$

Hence, $A$ is a contraction mapping of $B_R(X^s_T)$ for $T = \frac{1}{2C_sR} = \frac{1}{4C_s\|u_0\|_{H^s}}$. Thus $A$ has a unique fixed point which is a solution of (10) on time interval $[-T, T]$. 


Let us consider now the smoothness of $\Phi$. Let $\Lambda : H^s(\mathbb{T}) \times X_T^s \rightarrow X_T^s$ be defined as

$$
\Lambda(u_0, v)(t) = v(t) - e^{-it\varphi(D)}u_0 - \frac{i}{2} \int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(v(\tau)^2)d\tau.
$$

Due to lemma 3.1 (with $r = r' = s$), $\Lambda$ is a smooth map from $H^s(\mathbb{T}) \times X_T^s$ to $X_T^s$. Let $u \in X_T^s$ be the solution of (10) with initial data $u_0 \in H^s(\mathbb{T})$, which is to say $\Lambda(u_0, u) = 0$. Thus, the Fréchet derivative of $\Lambda$ with respect to the second variable is the linear map:

$$
\Lambda'(u_0, u)(t)[h] = h - \int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(u(\tau)h(\tau))d\tau.
$$

Still by lemma 3.1 we get

$$
\left\| \int_0^t e^{-i(t-\tau)\varphi(D)}\varphi(D)(u(\tau)h(\tau))d\tau \right\|_{X_T^s} \leq CT\|u\|_{H^s}\|h\|_{H^s}.
$$

So, for $T'$ sufficiently small (depending only on $\|u\|_{H^s}$), $\Lambda'(u_0, u)(t)$ is invertible since it is of the form $Id + K$ with

$$
\|K\|_{\mathcal{B}(X_T^{s}, X_T^{s})} < 1
$$

where $\mathcal{B}(X_T^{s}, X_T^{s})$ is the Banach space of bounded linear operators on $X_T^{s}$. Thus $\Phi : B_R(H^s(\mathbb{T})) \rightarrow X_T^s$ is real-analytic by Implicit Function Theorem.

### 3.3.2. Global well-posedness.

**Theorem 3.4.** The solution defined in theorem 3.3 is global in time.

**Proof.** Fix $T > 0$. The aim is to show that corresponding to any initial data $u_0 \in H^s$, there is a unique solution of (10) that lies in $X_T^s$. Because of theorem 3.3, this result is clear for data that is small enough in $H^s$, and it is sufficient to prove the existence of a solution corresponding to initial data of arbitrary size (uniqueness is a local issue). Fix $u_0 \in H^s$ and let $N$ be such that

$$
\sum_{|k| \geq N} \langle k \rangle^{2s} |\widehat{u}_0(k)|^2 \leq T^{-2}.
$$

Such values of $N$ exist since $\langle k \rangle^s |\widehat{u}_0(k)|$ is in $\ell^2$. Define

$$
v_0(x) = \sum_{|k| \geq N} e^{ixk} \widehat{u}_0(k).
$$

By theorem 3.3 there exists a unique $v \in X_T^s$ solution of (10) with initial data $v_0$. Split the initial data $u_0$ into two pieces: $u_0 = v_0 + w_0$; and consider the following Cauchy problem (where $v$ is now fixed)

$$
\begin{align*}
\left\{ 
\begin{array}{l}
w_t - w_{xxt} + w_x + ww_x + (vw)_x \\
w(0, x) = w_0(x)
\end{array}
\right.
\end{align*}
$$

If there exists a solution $w$ of (14) in $X_T^s$ then $v + w$ will be a solution of (10) in $X_T^s$. 

First, \( w_0 \) is in \( H^r(\mathbb{T}) \) for all \( r > 0 \), in particular \( w_0 \in H^1(\mathbb{T}) \). And \((14)\) may be rewritten as the integral equation

\[
w(t, x) = e^{-it\varphi(D)}w_0 - \frac{i}{2} \int_0^t e^{-i(t-\tau)\varphi(D)}(vw + w^2)d\tau = \mathcal{K}(w).
\]

This problem can be solved locally in time on \( H^1(\mathbb{T}) \) by the same arguments used to prove theorem 3.3. Indeed for any \( w \in B_R(X^\frac{r}{2}) \), by lemma 3.2 (with \( r = 1 \) and \( s = 0 \)) and lemma 3.1 (with \( r = r' = s = 1 \))

\[
\|\mathcal{K}(w)\|_{X^\frac{r}{2}} \leq \|w_0\|_{H^1} + CS \left( \|\nu\|_{X^0} \|w\|_{X^\frac{r}{2}} + \|w\|^2_{X^\frac{r}{2}} \right) 
\leq CS \|v\|_{X^0} R
\]

and for any \( w_1 \) and \( w_2 \) in \( B_R(X^\frac{r}{2}) \)

\[
\|\mathcal{K}(w_1) - \mathcal{K}(w_2)\|_{X^\frac{r}{2}} 
\leq CS \left( \|v\|_{X^0} \|w_1 - w_2\|_{X^\frac{r}{2}} + \|w_1 - w_2\|_{X^\frac{r}{2}} \|w_1 + w_2\|_{X^\frac{r}{2}} \right) 
\leq CS \left( \|v\|_{X^0} + 2R \right) \|w_1 - w_2\|_{X^\frac{r}{2}}.
\]

Hence, by \((15)\) and \((16)\), \( \mathcal{K} \) has a unique fixed point in \( X^\frac{r}{2} \). Therefore we have a solution \( w \) in \( X^\frac{r}{2} \) for a small time \( S \).

If we have an \( a \) \text{ priori} bound on the \( H^1 \)-norm of \( w \) showing it was bounded on the interval \([-T, T]\) it would follow that a solution on \([-T, T]\) could be obtained.

The formal steps of this inequality are as folllows (the justification is made by regularizing). Multiply the equation \((14)\) by \( w \), integrate over \( \mathbb{T} \), and after integration by parts we get

\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{T} \left( w(t, x)^2 + w_x(t, x)^2 \right) dx - \int_\mathbb{T} v(t, x)w(t, x)w_x(t, x)dx = 0.
\]

By Hölder and Sobolev inequalities we deduce

\[
\left| \int_\mathbb{T} v(t, x)w(t, x)w_x(t, x)dx \right| \leq \|v(t, \cdot)\|_{L^2} \|w(t, \cdot)\|_{L^\infty} \|w_x(t, \cdot)\|_{L^2} 
\leq C \|v(t, \cdot)\|_{L^2} \|w(t, \cdot)\|_{H^1}^2.
\]

Hence

\[
\frac{d}{dt} \|w(t, \cdot)\|_{H^1}^2 \leq 2C \|v(t, \cdot)\|_{L^2} \|w(t, \cdot)\|_{H^1}^2
\]

and by Gronwall’s inequality

\[
\|w(t, \cdot)\|_{H^1} \leq \|w_0\|_{H^1} \exp \left( C \int_0^t \|v(\tau, \cdot)\|_{L^2} d\tau \right).
\]

We deduce from this \( a \) \text{ priori} bound that the solution \( w \) of \((14)\) exists on the interval \([-T, T]\), and \( v + w \) is a solution of \((10)\) in \( X^r \).
3.4. Verification of (H2).

**Proposition 3.5.** For any $T > 0$, $R > 0$, and $s > 0$ there exists $R'$ such that

$$\forall 0 \leq t \leq T, \Phi_t(B_R(H^s)) \subset B_{R'}(H^s).$$

With $s = \frac{1}{2}$ we deduce that $\Phi$ satisfies (H2).

**Proof.** The result is clear for $s \geq 1$, so we assume that $0 < s < 1$. Fix $T > 0$, $R > 0$ and $u_0$ in $H^s$ such that $\|u_0\|_{H^s} \leq R$. Using the same idea as in theorem 3.3, we split $u_0$ into two pieces $u_0 = v_0 + w_0$, where

$$v_0 = \sum_{|k| \geq N} \hat{u}_0(k)e^{ikx}.$$ 

Using the same notations, let $v$ be the solution of BBM equation with the initial data $v_0$ and $w$ the solution of (14). We want to control $v$ and $w$ in $H^s$-norm.

Fix $\varepsilon > 0$ such that $\varepsilon < 1/8$ and $s - \varepsilon > 0$, we have

$$\|v_0\|_{H^{s-\varepsilon}} \leq N^{-\varepsilon} \|v_0\|_{H^s}.$$ 

We choose $N = \left(\frac{4RC}{\varepsilon}\right)^{1/\varepsilon}$ where $C$ is the constant of lemma 3.1. Hence we have

$$\|v_0\|_{H^{s-\varepsilon}} \leq \frac{1}{4CT} = M.$$ 

By local theory (theorem 3.3) the flow map

$$\Phi : B_M(H^{s-\varepsilon}) \to X_T^{s-\varepsilon}$$

is continuous. Since $H^s \cap B_M(H^{s-\varepsilon})$ is precompact in $B_M(H^{s-\varepsilon})$ we have

$$\sup_{v_0 \in H^s \cap B_M(H^{s-\varepsilon})} \|\Phi(v_0)\|_{X^{s-\varepsilon}} = C_1(R, T).$$

By lemma 3.1 with $r = r' = s - \varepsilon$ we have

$$\|v\|_{X^s} \leq \|v_0\|_{H^s} + CT \|v\|_{X^{s-\varepsilon}} \leq R + CTC_1(R, T)^2 = C_2(R, T).$$

The a priori bound on $w$ gives

$$\|w(t)\|_{H^s} \leq \|w(t)\|_{H^1} \leq \|w_0\|_{H^1} \exp \left( C \int_0^t \|v(\tau, \cdot)\|_{L^2} d\tau \right) \leq N^{1-s} \|w_0\|_{H^s} e^{CTC_2(R, T)} \leq C_3(R, T).$$

Hence, we have

$$\|u\|_{X_T^s} \leq C_2(R, T) + C_3(R, T) \quad \blacksquare$$

**Corollary 3.6.** For each $T > 0$ and $s > 0$, the flow map $\Phi : H^s \to X_T^s$ is real analytic.

**Proof.** Let $u_0 \in H^s$, $R = \|u_0\|_{H^s}$, and $T > 0$. By proposition 3.5 there exists $R'$ such that $\Phi_t(B_{R'}(H^s)) \subset B_{R'}(H^s)$, for all $t \in [0, T]$. And by local theory (theorem 3.3) there exists a small time $\tau$ such that $\Phi : B_{R'}(H^s) \to X^s_T$ is real analytic. Splitting the time intervalle $[0, T]$ into $\bigcup_{k} [k\tau, (k+1)\tau]$, we deduce that $\Phi : H^s \to X_T^s$ is real analytic. \[ \blacktriangleleft \]
3.5. Verification of (H3). Recall that $\tilde{\Phi}$ denote the non-linear part of the flow, that is $\Phi_t = e^{-it\Phi(D)}(I + \tilde{\Phi}_t)$. The assumption (H3) results from

**Proposition 3.7.** For any $u_0, v_0 \in B_R(H^{1/2}(\mathbb{T}))$ we have the following estimate

$$\left\| \tilde{\Phi}(u_0) - \tilde{\Phi}(v_0) \right\|_{X^{1/2+\varepsilon}_T} \leq C_{R,T,\varepsilon} \left\| u_0 - v_0 \right\|_{H^{1/2-\varepsilon}}$$

for $0 < \varepsilon < 1/12$.

**Proof.** Let $0 < \varepsilon < \frac{1}{12}$, $u_0$ and $v_0$ in $B_R(H^{1/2})$. Denoting $u$ and $v$ the solutions of BBM equation with initial data $u_0$ and $v_0$. By lemma 3.1 with $s = \frac{1}{2} + \varepsilon$ and $r = \frac{1}{2}$ and $r' = \frac{1}{2} - \varepsilon$ and (H2) we have

$$\left\| \tilde{\Phi}_t(u_0) - \tilde{\Phi}_t(v_0) \right\|_{X^{1/2+\varepsilon}_T} \leq CT \left\| u + v \right\|_{X^{1/2}_T} \left\| u - v \right\|_{X^{1/2-\varepsilon}_T} \leq 2CTR_{R,T} \left\| u - v \right\|_{X^{1/2-\varepsilon}_T}.$$

Since $u_0$ and $v_0$ are in $B_R(H^{1/2})$ and $\Phi$ is $C^1$ on $B_R(H^{1/2})$ which is a relatively compact subset of $H^{1/2-\varepsilon}$ we have

$$\left\| u - v \right\|_{X^{1/2-\varepsilon}_T} = \left\| \Phi_t(u_0) - \Phi_t(v_0) \right\|_{X^{1/2-\varepsilon}_T} \leq \sup_{u_0, v_0 \in B_R(H^{1/2}) \cap H^{1/2-\varepsilon}} \left( \left\| d\Phi_t(u_0) \right\|_{G(H^{1/2-\varepsilon}, X^{1/2-\varepsilon}_T)} \right) \left\| u_0 - v_0 \right\|_{H^{1/2-\varepsilon}} \leq C_{R,T,\varepsilon} \left\| u_0 - v_0 \right\|_{H^{1/2-\varepsilon}}.$$

Hence, we can apply the non-squeezing theorem (theorem 2.1) and that proves the theorem 1.1.

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