ON THE VIRTUALLY-CYCLIC DIMENSION OF SURFACE Braid Groups AND RIGHT-ANGLED ARTIN GROUPS

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Abstract. We give a bound for the virtually cyclic dimension of groups with a normal subgroup of finite index which satisfies that every infinite virtually-cyclic subgroup is contained in a unique maximal such subgroup. As an application we provide a bound for the virtually-cyclic dimension for the braid group of a closed surface with genus greater than 2 and for right-angled Artin groups.

1. Introduction

Let $G$ be a group. A family $\mathcal{F}$ of subgroups of $G$ is a set of subgroups of $G$ which is closed under conjugation and taking subgroups. A model for the classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$ is a $G$-CW-complex $X$ such that the fixed point set $X^H$ is contractible for $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$.

Let $\mathcal{FIN}_G$ and $\mathcal{VC}_G$ be the families of finite and virtually cyclic subgroups of $G$, respectively. We abbreviate $\mathcal{E}_G := E_{\mathcal{FIN}_G}G$ and $\mathcal{E}_G := E_{\mathcal{VC}_G}G$. The study of models for $\mathcal{E}_G$ and $\mathcal{E}_G$ has been motivated by the Baum-Connes and Farrel-Jones Conjectures.

A model for $E_{\mathcal{F}}G$ always exists and two models for $E_{\mathcal{F}}G$ are $G$-homotopy equivalent (see [9]), however the model may not have finite dimension. The smallest possible dimension of a model for $E_{\mathcal{F}}G$ is called the geometric dimension of $G$ for the family $\mathcal{F}$ and is usually denoted as $gd_{\mathcal{F}}G$. We abbreviate $gdG := gd_{\mathcal{FIN}_G}G$ and $gdG := gd_{\mathcal{VC}_G}G$, they are called proper and virtually-cyclic dimension respectively, and for the trivial family $\{1\}$, denote $gdG := gd_{\{1\}}G$.

We remark that in general $gdG \leq gdG + 1$ (see [10]), and for many families of subgroups the following inequality hold:

$$gdG \leq gdG + 1.$$  

In particular, Lück and Weiermann showed in [10] that if a group $G$ has property Max$_{\mathcal{VC}_G}$ (which states that every infinite virtually-cyclic subgroup is contained in a unique maximal such subgroup), then $G$ satisfies inequality (1). In [10] it was showed that the mapping class group of an orientable compact surface has normal subgroups of finite index $\Gamma$ with the property Max$_{\mathcal{VC}_G}$.

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The principal result of this paper is the following:

**Theorem 1.** Let \( 1 \to H \to G \to F \to 1 \) be a short exact sequence of groups where \( F \) is a non-trivial finite group. If \( G \) is torsion free, \( H \) has property \( \text{Max}_{\mathcal{VC}_G^\infty} \) and \( \text{gd} H < \infty \), then \( \text{gd} G \leq \max\{3, \text{gd} H + 1\} \).

Also we will give a bound for \( \text{gd} G \) when \( G \) has torsion. We remark that Theorem 1 extends some of the results that we obtained in \([1]\), in which we showed that in many cases, inequality \([1]\) holds for the whole mapping class group, we used that the mapping class groups have the property of uniqueness of roots and an extra condition.

Below are some applications of Theorem 1. Let \( S \) be a surface, denote by \( B_n(S) \) the \( n \)-braid group of \( S \).

**Proposition 2.** If \( n \geq 1 \) and \( S \) is an orientable closed surface of genus \( g \geq 2 \), then \( \text{gd} B_n(S) \leq n + 2 \).

**Proposition 3.** If \( A \) is a right-angled Artin group, then
\[
\text{gd} A \leq \text{gd} A + 1.
\]
In addition, for any subgroup \( H \subseteq A \), \( \text{gd} H \leq \text{gd} H + 1 \).

## 2. Classifying spaces for families

A method for constructing a model for \( \overline{E}G \) is to start from a model for \( E\overline{G} \) and then promote the action to a larger space to get a model for \( E\overline{G} \). In this section we will describe the model of Lück and Weiermann given in \([10]\).

Define an equivalence relation on the set \( \mathcal{VC}_G^\infty = \mathcal{VC}_G - \mathcal{FIN}_G \) of infinite virtually cyclic subgroups of \( G \). Given \( V, U \in \mathcal{VC}_G^\infty \)
\[
V \sim U \text{ if and only if } |V \cap U| = \infty.
\]
Let \([\mathcal{VC}_G^\infty] \) denote the set of equivalence classes and by \([V]\) the equivalence class of \( V \in \mathcal{VC}_G^\infty \). For \([V] \in \mathcal{VC}_G^\infty \) define
\[
\text{Comm}_G[V] := \{ g \in G \mid [gV^g^{-1}] = [V] \},
\]
the commensurator of \( V \) in \( G \). Define a family of subgroups of \( \text{Comm}_G[V] \) as
\[
\mathcal{G}_G[V] := \{ U \in \mathcal{VC}_{\text{Comm}_G[V]}^\infty \mid [V] = [U] \} \cup \mathcal{FIN}_{\text{Comm}_G[V]}.
\]

**Theorem 2.1.** \([10]\) Thm.2.3 Let \( \mathcal{VC}_G^\infty \) and \( \sim \) be as above. Let \( I \) be a complete system of representatives \([V]\) of the \( G \)-orbits in \([\mathcal{VC}_G^\infty]\) under the \( G \)-action coming from conjugation. Choose arbitrary \( \text{Comm}_G[V] \)-CW-models for \( E(\text{Comm}_G[V]), E_G[V](\text{Comm}_G[V]) \) and an arbitrary \( G \)-CW-model for
$\mathcal{E}(G)$. Define $X$ a $G$-CW-complex by the cellular $G$-pushout

\[
\begin{array}{c}
\bigsqcup_{[V] \in I} G \times_{\text{Comm}_G[V]} E(\text{Comm}_G[V]) \xrightarrow{i} \mathcal{E}G \\
\bigsqcup_{[V] \in I} \text{id}_G \times_{\text{Comm}_G[V]} f_{[V]} \downarrow \downarrow
\end{array}
\]

such that $f_{[V]}$ is a cellular $\text{Comm}_G[V]$-map for every $[V] \in I$ and $i$ is an inclusion of $G$-CW-complexes, or such that every map $f_{[V]}$ is an inclusion of $\text{Comm}_G[V]$-CW-complexes for every $[V] \in I$ and $i$ is a cellular $G$-map. Then $X$ is a model for $\mathcal{E}G$.

Remark 2.2. [10, Rem. 2.5] Suppose that there exists $n$ such that $\text{gd}_G \leq n$, and for each $[V] \in \mathcal{I}$,

\[
\text{gd} \text{Comm}_G[V] \leq n - 1 \quad \text{and} \quad \text{gd}_{[V]} \text{Comm}_G[V] \leq n,
\]

then $\text{gd} G \leq n$.

2.1. Proper dimension. The following Theorems will be used for the proof of the main Theorem, which is given in Section 3.

Let $G$ be a group, and $F \in \mathcal{FIN}_G$ a finite subgroup. The length $l(F)$ of $F$ is defined as the largest natural number $k$ for which there is a chain $1 = F_0 < F_1 < \cdots < F_k = F$. The length of $G$ is

\[
l(G) = \sup\{l(F) \mid F \in \mathcal{FIN}_G\}.
\]

Theorem 2.3. [12, Thm. 3.10, Lem. 3.9] Suppose that $\text{gd} G < \infty$. If $l(G)$ is finite, then

\[
\text{gd} G \leq \max\{3, \text{vcd} G + l(G)\},
\]

where $\text{vcd}(G)$ denotes virtual cohomological dimension of $G$.

Theorem 2.4. [11, Thm. 2.5] Let $G$ be a group such that any finite subgroup is nilpotent and $\text{vcd} G < \infty$. Let $n = \max_{F \in \mathcal{FIN}_G} \{\text{vcd} G + \text{rk}(W_G F)\}$, where $\text{rk}(\cdot)$ denotes the biggest rank of a finite elementary abelian subgroup, then $\text{gd} G \leq \max\{3, n\}$.

3. Virtually-cyclic dimension

We say that $G$ has property $\text{Max}_{\mathcal{VC}_G}^\infty$ if every $V \in \mathcal{VC}_G^\infty$ is contained in a unique maximal $V_{\text{max}G} \in \mathcal{VC}_G^\infty$.

Suppose $G$ satisfies $\text{Max}_{\mathcal{VC}_G}^\infty$, let $V, U \in \mathcal{VC}_G^\infty$, thus $V \sim U$ if only if $V_{\text{max}G} = U_{\text{max}G}$, therefore

\[
\text{Comm}_G[V] = N_G(V_{\text{max}G})
\]

is the normalizer of $V_{\text{max}G}$ in $G$ (see [10]).

Lemma 3.1. If $G$ has property $\text{Max}_{\mathcal{VC}_G}^\infty$ and $U \in \mathcal{VC}_G^\infty$, then

\[
(gUg^{-1})_{\text{max}G} = gU_{\text{max}G}g^{-1}.
\]
Proof. We first claim that $gU_{\text{max}_G}g^{-1}$ is maximal in $\mathcal{VC}_G^\infty$: let $V \in \mathcal{VC}_G^\infty$ such that $gU_{\text{max}_G}g^{-1} \subseteq V$, hence $U_{\text{max}_G} \subseteq g^{-1}Vg$ and by maximality we have that $U_{\text{max}_G} = g^{-1}Vg$. In this way, $gU_{\text{max}_G}g^{-1} = V$. So we conclude that $(gUg^{-1})_{\text{max}_G} = gU_{\text{max}_G}g^{-1}$ by the uniqueness requirement of the property Max$\mathcal{VC}_G^\infty$.

Lemma 3.2. Let $1 \to H \to G \xrightarrow{\psi} F \to 1$ be a short exact sequence of groups, such that $H$ satisfies property Max$\mathcal{VC}_H^\infty$ and $F$ is finite. Let $V \in \mathcal{VC}_G^\infty$ and $(V \cap H)_{\text{max}_H} \in \mathcal{VC}_H^\infty$ be the maximal containing $(V \cap H)$, then

$$\text{Comm}_G[V] = \mathcal{N}_G((V \cap H)_{\text{max}_H}).$$

Proof. Let $g \in G$, we have that

$$[gVg^{-1}] = [V] \text{ if and only if } [g(V \cap H)g^{-1}] = [V \cap H]$$

if and only if $[g(V \cap H)g^{-1}]_{\text{max}_H} = (V \cap H)_{\text{max}_H}$, by Lemma 3.1 $(g(V \cap H)g^{-1})_{\text{max}_H} = g(V \cap H)_{\text{max}_H}g^{-1}$, then equality (6) holds if and only if $g(V \cap H)_{\text{max}_H}g^{-1} = (V \cap H)_{\text{max}_H}$. □

Theorem 3.3. Let $1 \to H \to G \xrightarrow{\psi} F \to 1$ be a short exact sequence of groups where $F$ is a non-trivial finite group. Suppose that $H$ has property Max$\mathcal{VC}_H^\infty$ and $\text{gd}H < \infty$, then $\text{gd}G \leq \max\{3, \text{gd}H + l(F)\}$. If in addition $G$ is torsion free, then $\text{gd}G \leq \max\{3, \text{gd}H + 1\}$.

Proof. Let $V \in \mathcal{VC}_G^\infty$ and $U = (V \cap H)_{\text{max}_H}$, by Lemma 3.2 we have that $\text{Comm}_G[V] = \mathcal{N}_G(U)$. Thus a model for $EN_G(U)/U$ is a model for $E_{\mathcal{N}_G(U)}/\mathcal{N}_G(U)$ with the $\mathcal{N}_G(U)$-action induced by the projection $p: \mathcal{N}_G(U) \to \mathcal{N}_G(U)/U$. Hence $\text{gd}_{\mathcal{N}_G(U)}\mathcal{N}_G(U) \leq \text{gd}\mathcal{N}_G(U)/U$.

We will use Theorem 2.3 to give a bound for $\text{gd}\mathcal{N}_G(U)/U$. From the exact sequence

$$1 \to N_H(U) \xrightarrow{\psi} N_G(U) \xrightarrow{\psi'} F' \to 1,$$

where $\psi|$ is the restriction of $\psi$ and $F' \subseteq F$, we have

$$1 \to N_H(U)/U \xrightarrow{\psi} N_G(U)/U \to F'/U \to 1,$$

so $\text{vcd}(N_G(U)/U) = \text{vcd}(N_H(U)/U) \leq \text{gd}(N_H(U)/U)$. Since $H$ satisfies property Max$\mathcal{VC}_H^\infty$ and $\text{gd}H < \infty$, by the proof of [10] Thm. 5.8 we have that $\text{gd}(N_H(U)/U) \leq \text{gd}N_H(U)$, hence

$$\text{vcd}(N_G(U)/U) \leq \text{gd}N_H(U) \leq \text{gd}H,$$

we remark that $\text{gd}H \leq \text{gd}H + 1 < \infty$.

Further, $N_H(U)/U$ is torsion free because $U$ is virtually cyclic maximal in
\( N_H(U) \), therefore the finite subgroups of \( N_G(U)/U \) embeds in \( F' \), so that
\[
l(N_G(U)/U) \leq l(F') \leq l(F) < \infty.
\]
Applying Theorem 2.3 we have that
\[
\text{gd} N_G(U)/U \leq \max\{3, \text{gd} H + l(F)\}.
\]
By Theorem 2.1 and Remark 2.2 we may conclude that
\[
\text{gd} G \leq \max\{3, \text{gd} H + l(F)\}.
\]
If \( G \) is torsion free, then the finite subgroups of \( N_G(U)/U \) are cyclic. Applying Theorem 2.4 we have
\[
\text{gd} N_G(U)/U \leq \max\{3, \text{vcd} N_G(U)/U + 1\}
\]
\[
\leq \max\{3, \text{gd} H + 1\},
\]
again by Theorem 2.1 and Remark 2.2 we obtain the following inequality
\[
\text{gd} G \leq \max\{3, \text{gd} H + 1\}.
\]
\[\square\]

4. Applications

We will use the following Lemma and the fact that surface braid groups and right-Artin groups can be embeded in some mapping class group of a surface, to show that they have normal subgroups of finite index with property of maximality in the family of virtually cyclic subgroups, and finally apply Theorem 3.3.

**Lemma 4.1.** Let \( G \) be a group that satisfies property \( \text{Max}_\infty \), if \( H \subseteq G \), then \( H \) has property \( \text{Max}_\infty \).

The proof of the Lemma is left to the reader.

4.1. **Mapping class groups.** Let \( S \) be a connected, compact, orientable surface with a finite set of points removed from the interior (punctures). Denote by \( \text{Mod}(S) \) the mapping class group of \( S \).

Let \( m > 1 \) be an integer number, there is a natural homomorphism
\[
\tau: \text{Mod}(S) \to \text{Aut}(H_1(S; \mathbb{Z}_m))
\]
defined by the action of diffeomorphisms on the homology group. The subgroup
\[
\text{Mod}(S)[m] = \ker \tau,
\]
is called the \( m \)-congruence subgroup of \( \text{Mod}(S) \). Note that this subgroup has finite index in \( \text{Mod}(S) \).
**Theorem 4.2.** [6] Prop. 5.11, Thm. 5.3] Let $S$ be an orientable compact surface with finitely many punctures and $\chi(S) \leq 0$, then $\mathrm{gd}(\mathrm{Mod}(S))$ is finite. Furthermore, for $m \geq 3$ the group $\mathrm{Mod}(S)[m]$ satisfies property $\mathrm{Max}_c \in \mathrm{Mod}(S)[m]$.

**Corollary 4.3.** Let $S$ be an orientable compact surface with finitely many punctures and $\chi(S) \leq 0$. If $H$ is a torsion free subgroup of $\mathrm{Mod}(S)$, then $\mathrm{gd}(H) \leq \mathrm{gd}(H) + 1$.

**Proof.** Let $H$ be a torsion free subgroup of $\mathrm{Mod}(S)$ and $H_m = H \cap \mathrm{Mod}(S)[m]$, with $m \geq 3$. By Theorem 4.2 and Lemma 4.1, $H_m$ has property $\mathrm{Max}_c \in H_m$. Since $H_m$ is a normal finite index subgroup of $H$, by applying Theorem 3.3, we conclude that $\mathrm{gd}(H) \leq \mathrm{gd}(H) + 1$. \hfill \Box

**4.2. Surface braid groups.** Let $S$ be a compact orientable surface. The $n$-configuration space of $S$ is defined as follows,

$$F_n(S) = \{(y_1, \ldots, y_n) \in S^n \mid y_i \neq y_j \text{ for all } i, j \in \{1, \ldots, n\}, i \neq j\}.$$ 

We endow $F_n(S)$ with the topology induced by the product topology from the space $S^n$. The configuration space $F_n(S)$ is a connected $2n$-dimensional open manifold. There is a natural free action of the symmetric group $\Sigma_n$ on $F_n(S)$ by permuting the coordinates. We will denote the quotient by the action as $D_n(S) = F_n(S)/\Sigma_n$ and it may be thought of as the configuration space of $n$ unordered points.

**Definition 4.4.** Let $n \in \mathbb{N}$. The $n$-braid group of $S$ is defined as

$$B_n(S) = \pi_1(D_n(S)).$$

If $S$ is a closed orientable surface of genus $> 1$, $D_n(S)$ is a model for $E \in \mathbb{R}$, [5]. We remark that braid groups $B_n(S)$ are torsion free if and only if $S$ is different from $S^2$ (see [13]), in that case $\mathrm{gd}(B_n(S))$ coincides with the cohomological dimension $\mathrm{cd}(B_n(S))$, except possibly for the case where $\mathrm{cd}(B_n(S)) = 2$ and $\mathrm{gd}(B_n(S)) = 3$.

**Theorem 4.5.** [8] Thm. 1.2] If $n \geq 1$ and $S$ is a closed orientable surface of genus $g \geq 1$, then $\mathrm{cd}(B_n(S)) = n + 1$.

The mapping class groups are closely related to braid groups, see [2]. Let $S$ be a closed orientable surface of genus $\geq 2$, and $Q_n = \{x_1, \ldots, x_n\}$ be a set with $n \geq 1$ different points in the interior of $S$, then

$$(7) \quad 1 \longrightarrow B_n(S) \longrightarrow \mathrm{Mod}(S - Q_n) \xrightarrow{\rho} \mathrm{Mod}(S) \longrightarrow 1.$$ 

**Proposition 4.6.** If $n \geq 1$ and $S$ is an orientable closed surface of genus $g \geq 2$, then $\mathrm{gd}(B_n(S)) \leq n + 2$.

**Proof.** If $n = 1$, then $B_1(S) = \pi_1(S)$ is hyperbolic and $\mathrm{gd}(B_1(S)) = 2$. If $n \geq 2$, then $\mathrm{cd}(B_n(S)) \geq 3$ and $\mathrm{cd}(B_n(S)) = \mathrm{gd}(B_n(S))$. We consider $B_n(S)$.
as a subgroup of $\text{Mod}(S - Q_n)$ via the inclusion of the exact sequence \cite{7}. By Corollary 4.3 and Theorem 4.5 we have
\[
\text{gd}B_n(S) \leq \text{gd}B_n(S) + 1 = n + 2.
\]

4.3. **Right angled Artin groups.** Artin groups arose as a natural generalization of braid groups $B_n = B_n(\mathbb{D}^2)$. A *right-angled Artin group* $A$ is a group with presentation of the form
\[
A = \langle s_1, ..., s_n \mid s_is_js_i... = s_js_is_j... \text{ for all } i \neq j \rangle,
\]
where $m_{ij} = m_{ji} \in \{2, \infty\}$ for all $i, j$, when $m_{ij} = \infty$ we omit the relation between $s_i$ and $s_j$.

We remark that right-angled Artin groups contain many interesting groups: some 3-manifold groups, surface groups \cite{3} and graph braid groups.

It is well known that for any right-angled Artin group $A$, the geometric dimension $\text{gd}A$ is finite, and $A$ is torsion free.

**Proposition 4.7.** \cite{7} Every right-angled Artin group embeds in the mapping class group of any surface of sufficiently high genus.

**Proposition 4.8.** If $A$ is a right-angled Artin group, then
\[
\text{gd}A \leq \text{gd}A + 1.
\]
In addition, for any subgroup $H \subseteq A$, $\text{gd}H \leq \text{gd}H + 1$.

**Proof.** The proof follows directly by Corollary 4.3 and Proposition 4.7. □

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