Radial symmetry and partially overdetermined problems in a convex cone

Jihye Lee, Keomkyo Seo

1Department of Mathematics, Sookmyung Women’s University, Seoul, South Korea
2Department of Mathematics and Research Institute of Natural Science, Sookmyung Women’s University, Seoul, South Korea

Correspondence
Keomkyo Seo, Department of Mathematics and Research Institute of Natural Science, Sookmyung Women’s University, Cheongpa-ro 47-gil 100, Yongsan-ku, Seoul 04310, Korea. Email: kseo@sookmyung.ac.kr

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Abstract
We obtain the radial symmetry of the solution to a partially overdetermined boundary value problem in a convex cone in space forms by using the maximum principle for a suitable subharmonic function P and integral identities. In dimension 2, we prove Serrin-type results for partially overdetermined problems outside a convex cone. Furthermore, we obtain a Rellich identity for an eigenvalue problem with mixed boundary conditions in a cone.

KEYWORDS
convex cone, eigenvalue problem, overdetermined problem, P-function

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1 INTRODUCTION

In a celebrated paper [14], James Serrin obtained the following remarkable result. Let $\Omega$ be a smooth, bounded, open, connected domain in $\mathbb{R}^n$ and let $\nu$ be the outward unit normal to $\partial \Omega$. If $u$ is a smooth solution to the overdetermined problem

$$\begin{align*}
\Delta u &= -1 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} &= \text{const} = c & \text{on } \partial \Omega,
\end{align*}$$

then $\Omega$ is a ball and the solution $u$ is radially symmetric. His proof is based on the moving plane method (or Alexandrov [1] reflection method). After Serrin’s work, his symmetry result has been generalized to space forms (see [3, 4, 6, 8, 9, 11, 13], for instance, and references therein) Figure 1 and 2.

On the other hand, Pacella–Tralli [10] studied a partially overdetermined problem for a domain in a convex cone in the Euclidean space. In order to describe precisely, let us introduce some notations. Let $C$ be an open convex cone with vertex at the origin $O$ in $\mathbb{R}^n$, $n \geq 2$, that is,

$$C = \{tx : x \in \omega, t \in (0, \infty)\}$$

for some domain $\omega$ in the unit sphere $\mathbb{S}^{n-1}$. Recall that a $C^2$ domain $\Omega$ is convex in an $n$-dimensional Riemannian manifold $M^n$ if the normal curvature on $\partial \Omega$ with respect to the inward normal direction is nonnegative. In particular, if the domain is convex and $n \geq 3$, the second fundamental form $II$ is nonnegative at every point on the boundary $\partial \Omega$. Given an open convex cone $C$ such that $\partial C \setminus \{O\}$ is smooth and a domain $\Omega \subset C$, we denote by $\Gamma_0$ its relative boundary, that is, $\Gamma_0 = \partial \Omega \cap \partial C \setminus \{O\}$.
C and let \( \Gamma_1 = \partial \Omega \setminus \overline{\Gamma_0} \). Assume that \( H^{n-1}(\Gamma_i) > 0 \) for \( i = 0, 1 \), where \( H^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure. Moreover, we assume that \( \Gamma_0 \) is an \((n-1)\)-dimensional smooth manifold, while \( \partial \Gamma_0 = \partial \Gamma_1 \subset \partial C \setminus \{O\} \) is an \((n-2)\)-dimensional smooth manifold. Following [10], such a domain \( \Omega \) is called a sector-like domain. Given a sector-like domain \( \Omega \) in an open convex cone \( C \subset \mathbb{R}^n \), consider the partially overdetermined mixed boundary value problem

\[
\begin{aligned}
\Delta u &= -1 & & \text{in } \Omega, \\
u &= 0, & & \frac{\partial u}{\partial \nu} = \text{const} = -c < 0 & & \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} &= 0 & & \text{on } \Gamma_1 \setminus \{O\},
\end{aligned}
\tag{1.1}
\]

where \( \nu = \nu(x) \) denotes the outward unit normal to \( \partial \Omega \) wherever it is defined (i.e., for \( x \in \Gamma_0 \cup \Gamma_1 \setminus \{O\} \)). Pacella–Tralli [10] proved the following.

**Theorem** ([10]). Let \( \Omega \) be a sector-like domain in an open convex cone \( C \) in \( \mathbb{R}^n \). Assume that there exists a classical solution \( u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \) to the partially overdetermined problem (1.1) such that \( u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \). Then,

\[
\Omega = C \cap B_R(p) \quad \text{and} \quad u(x) = \frac{R^2 - |x - p|^2}{2n}
\]

for some point \( p \in \partial C \). Here, \( B_R(p) \) denotes the ball centered at a point \( p \in \mathbb{R}^n \) and \( R = nc \).

Note that the point \( p \) may not be the origin \( O \) in the above theorem. Recently, Ciraolo–Roncoroni [2] extended the above theorem into space forms. Indeed, they considered the partially overdetermined problem in space forms. In [2], Ciraolo–Roncoroni obtained that if \( \Omega \) is a sector-like domain in a convex cone in space forms, then \( \Omega = C \cap B_R(p) \) and the solution \( u \) is radially symmetric with respect to the point \( p \), where \( B_R(p) \) denotes a geodesic ball of radius \( R \) centered at \( p \).

This paper is organized as follows. In Section 2, we investigate two partially overdetermined problems for domains on a convex cone with vertex at \( p \) in the unit sphere \( S^n \). First, we consider the equation

\[
\Delta u = -n \cos r,
\]

where \( \Delta \) denotes the Laplace–Beltrami operator on \( S^n \) and \( r \) denotes the geodesic distance from the vertex \( p \). In fact, using the above equation, Molzon [9] extended Serrin’s symmetry result to the upper unit hemisphere \( S^n_+ \). In Theorem 2.2, we obtain an analog of Molzon’s result for domains in a convex cone. Second, we consider the same partially overdetermined problem (1.2) for a domain in a cone in the unit sphere as in [2]. However, we do not assume that the domain \( \Omega \) is contained in the upper hemisphere \( S^n_+ \), but assume that \( \Omega \) is star-shaped with respect to the vertex \( p \). Using the maximum principle for a suitable subharmonic function \( P \) and some integral identities (which are originated by Weinberger [15]), we are able to prove a rigidity result of Serrin type for a star-shaped domain in a convex cone with vertex at \( p \) in Theorem 2.5.

In the two-dimensional case, it turns out that it is not necessary to assume that a domain is contained in a convex cone for the partially overdetermined problem (1.2). Indeed, we prove that if \( C \subset S^2 \) is a convex cone with vertex at \( p \) and \( \Omega \) is a star-shaped domain with respect to \( p \) outside a convex cone \( C \) and if (1.2) admits a solution, then \( \Omega = B_R(p_0) \setminus \overline{C} \) for some \( p_0 \in \partial C \) and the solution \( u \) is radially symmetric in Section 3 (see Theorem 3.3). In Section 4, we study an
eigenvalue problem with mixed boundary conditions in a cone. Given the Dirichlet eigenvalue problem for a bounded domain \( \Omega \subset \mathbb{R}^n \)

\[
\begin{cases}
\Delta u + \lambda u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Rellich [12] obtained the following identity:

\[
\lambda = \frac{1}{4} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 \frac{r^2}{\partial r} d\sigma,
\]

where \( r \) denotes the distance from the origin and \( u \) is normalized so that \( \int_{\Omega} u^2 dV = 1 \). Equation (1.3) is called the Rellich identity. In 1991, Molzon [9] extended (1.3) to space forms. Motivated by his result, we consider the mixed boundary eigenvalue problem for a domain \( \Omega \) in a cone \( C \) with vertex at \( p \)

\[
\begin{cases}
\Delta u + \lambda u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega \cap C, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \setminus (\{p\} \cup (\partial \Omega \cap C)),
\end{cases}
\]

and obtain a similar result for domains in a cone in Theorem 4.1. Note that the cone \( C \) needs not to be convex.

## 2 PARTIALLY OVERDETERMINED PROBLEMS INSIDE A CONVEX CONE IN THE UNIT SPHERE

Let \((M^n, g)\) be an \( n \)-dimensional space form, that is, an \( n \)-dimensional complete simply connected Riemannian manifold with constant sectional curvature \( K \). Up to homotheties, we may assume that \( K = 0, 1, \) and \( -1 \): The corresponding spaces are the Euclidean space \( \mathbb{R}^n \), the unit upper hemisphere \( \mathbb{S}^n_+ \), and the hyperbolic space \( \mathbb{H}^n \), respectively. These three spaces can be represented as the warped product space \( M = I \times \mathbb{S}^{n-1} \), which is equipped with the rotationally symmetric metric

\[
g = dr^2 + h(r)^2 g_{\mathbb{S}^{n-1}},
\]

where \( g_{\mathbb{S}^{n-1}} \) denotes the round metric on the \((n - 1)\)-dimensional unit sphere \( \mathbb{S}^{n-1} \) and

1. \( h(r) = r \) on \( I = [0, \infty) \) in \( \mathbb{R}^n \),
2. \( h(r) = \sin r \) on \( I = [0, \frac{\pi}{2}) \) in \( \mathbb{S}^n_+ \), and
3. \( h(r) = \sinh r \) on \( I = [0, \infty) \) in \( \mathbb{H}^n \).

Here, \( r(\cdot) \) denotes the distance dist(\( \cdot, p \)) from the pole \( p \) of the model space. Define a cone \( C \) with vertex at \( p \) as follows:

\[
C := \{ tx : x \in \omega, t \in I \}
\]

for some domain \( \omega \subset \mathbb{S}^{n-1} \). Note that \( C \subset M \) is convex if \( \omega \) is convex in \( \mathbb{S}^{n-1} \).

**Definition 2.1.** A connected bounded open set \( \Omega \subset C \) is an admissible interior domain (see Figure 1) if the boundary \( \partial \Omega \) satisfies the following:

1. \( \partial \Omega \) contains the vertex \( p \).
2. \( \Gamma_0 := \partial \Omega \setminus \partial C \neq \emptyset \) is an \((n - 1)\)-dimensional smooth manifold.
3. \( \Gamma_1 := \partial \Omega \setminus \Gamma_0 \neq \emptyset \) and \( \partial \Gamma_0 = \partial \Gamma_1 \subset \partial C \setminus \{p\} \) is an \((n - 2)\)-dimensional smooth manifold.
4. \( H^{n-1}(\Gamma_i) > 0 \) for \( i = 0, 1 \), where \( H^{n-1} \) denotes the \((n - 1)\)-dimensional Hausdorff measure.
Following [2, 10], we remark that if the boundary of a sector-like domain contains the vertex $p$, then such a domain is an admissible interior domain. Modifying Molzon's argument in [9], we are able to prove the following partially overdetermined problem for domains in a convex cone in the upper unit hemisphere $\mathbb{S}^n_+$. 

**Theorem 2.2.** Let $C \subset \mathbb{S}^n_+$ be an open convex cone with vertex at $p$ such that $\partial C \setminus \{p\}$ is smooth and $\Omega$ be an admissible interior domain in $C$. Suppose there exists a solution $u \in C^2(\Omega) \cap C^1(\Gamma_0 \cup \Gamma_1 \setminus \{p\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ to the partially overdetermined problem

\[
\begin{cases}
\Delta u = -n h'(r) = -n \cos r & \text{in } \Omega, \\
u = 0, \quad \frac{\partial u}{\partial \nu} = \text{const} = c & \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \setminus \{p\},
\end{cases}
\]

where $\nu$ is the outward unit normal to $\Gamma_0 \cup \Gamma_1 \setminus \{p\}$ and $r(x) = \text{dist}(p, x)$. Then $\Omega$ is part of the ball centered at the vertex $p$ of the cone $C$, that is,

\[
\Omega = C \cap B_R(p),
\]

where $B_R(p)$ denotes the geodesic ball centered at the vertex $p$ with radius $R$ in the upper hemisphere $\mathbb{S}^n_+$. Moreover, the solution $u$ is radial and it is given by

\[
u(x) = \cos r - \cos R,
\]

where $R = \sin^{-1}(-c)$.

**Proof.** We first claim that $u > 0$ in $\Omega$. To see this, let

\[
u^-(x) := \begin{cases} 0 & \text{if } u(x) \geq 0, \\
u(x) & \text{if } u(x) < 0.\end{cases}
\]
Then, \( u^-(x) \leq 0 \) and \( \Delta u = -n \cos r \leq 0 \) on \( \Omega \subset S^n_+ \). Using the divergence theorem, we obtain

\[
0 \leq \int_{\Omega} u^- \Delta u dV = \int_{\partial \Omega} u^- \frac{\partial u}{\partial v} d\sigma - \int_{\Omega} (\nabla u^-, \nabla u) dV
= -\int_{\Omega \cap \{u < 0\}} |\nabla u|^2 dV \leq 0,
\]

which yields \( u \geq 0 \) in \( \Omega \). Since \( u \) is not constant, it follows from the maximum principle that \( u > 0 \) in \( \Omega \). Thus, we may assume that \( u \) is positive in \( \Omega \).

It is well known that

\[
(\Delta u)^2 \leq n \text{tr}(\text{Hess}^2 u),
\]

where \( \text{Hess} u \) denotes the Hessian of \( u \) and \( \text{Hess}^2 u = \text{Hess} u \circ \text{Hess} u \). Thus,

\[
h'^2 \leq \text{tr}(\text{Hess}^2 u). \tag{2.1}
\]

A straightforward calculation shows that

\[
\text{Hess} h' = -h' g \quad \text{and} \quad \Delta h' = -nh',
\]

where \( g \) is the standard metric of \( S^n \). Applying the polarized Bochner formula on a Riemannian manifold

\[
\Delta \langle \nabla \phi, \nabla \psi \rangle = \nabla \phi(\Delta \psi) + \nabla \psi(\Delta \phi) + 2\text{tr}(\text{Hess} \phi \circ \text{Hess} \psi) + 2\text{Ric}(\nabla \phi, \nabla \psi)
\]

for any smooth function \( \phi \) and \( \psi \), we have

\[
\Delta \langle \nabla(u - h'), \nabla u \rangle = \langle \nabla(\Delta(u - h')), \nabla u \rangle + \langle \nabla(u - h'), \nabla(\Delta u) \rangle
+ 2\text{tr}(\text{Hess}(u - h') \circ \text{Hess} u) + 2\text{Ric}(\nabla(u - h'), \nabla u), \tag{2.2}
\]

where \( \text{Ric}(\cdot, \cdot) \) is the Ricci tensor of \( g \). Using (2.1), we get

\[
\text{tr}(\text{Hess}(u - h') \circ \text{Hess} u) = \text{tr}(\text{Hess}^2 u) + h' \Delta u = \text{tr}(\text{Hess}^2 u) - nh'^2 \geq 0.
\]

Thus, (2.2) becomes

\[
\Delta \langle \nabla(u - h'), \nabla u \rangle \geq -n \langle \nabla(u - h'), \nabla h' \rangle + 2(n - 1) \langle \nabla(u - h'), \nabla u \rangle.
\]

Since \( u > 0 \) in \( \Omega \), we obtain

\[
\int_{\Omega} u \Delta \langle \nabla(u - h'), \nabla u \rangle dV \geq -n \int_{\Omega} u \langle \nabla(u - h'), \nabla h' \rangle dV
+ 2(n - 1) \int_{\Omega} u \langle \nabla(u - h'), \nabla u \rangle dV. \tag{2.3}
\]

Note that

\[
\frac{\partial h'}{\partial v} = \langle \nabla h', v \rangle = -\sin r \langle \nabla r, v \rangle = 0
\]
on $\Gamma_1$. Using the divergence theorem, we get
\[
\int_{\Omega} \langle \nabla(u - h'), \nabla(u^2) \rangle \, dV = \int_{\Omega} \text{div}(u^2 \nabla(u - h')) \, dV - \int_{\Omega} u^2 \Delta(u - h') \, dV
\]
\[
= \int_{\partial \Omega} u^2 \frac{\partial}{\partial \nu}(u - h') \, d\sigma
\]
\[
= 0,
\]
which yields
\[
\int_{\Omega} u \langle \nabla(u - h'), \nabla u \rangle \, dV = 0. \tag{2.4}
\]
From (2.4), we see that
\[
\int_{\Omega} u \langle \nabla(u - h'), \nabla h' \rangle \, dV = -\int_{\Omega} u \langle \nabla(u - h'), -\nabla h' \rangle \, dV
\]
\[
= -\int_{\Omega} u |\nabla(u - h')|^2 \, dV. \tag{2.5}
\]
Combining (2.3), (2.4), and (2.5), we get
\[
\int_{\Omega} u \Delta \langle \nabla(u - h'), \nabla u \rangle \, dV \geq n \int_{\Omega} u |\nabla(u - h')|^2 \, dV \geq 0. \tag{2.6}
\]
On the other hand, Green's identity gives
\[
\int_{\Omega} u \Delta \langle \nabla(u - h'), \nabla u \rangle \, dV = \int_{\Omega} \langle \nabla(u - h'), \nabla u \rangle \Delta u \, dV + \int_{\partial \Omega} u \frac{\partial}{\partial \nu} \langle \nabla(u - h'), \nabla u \rangle \, d\sigma
\]
\[
- \int_{\partial \Omega} \langle \nabla(u - h'), \nabla u \rangle \frac{\partial u}{\partial \nu} \, d\sigma. \tag{2.7}
\]
Using the divergence theorem and the boundary conditions, we get
\[
\int_{\Omega} \langle \nabla(u - h'), \nabla(u h') \rangle \, dV = \int_{\Omega} \text{div}(uh' \nabla(u - h')) \, dV - \int_{\Omega} uh' \Delta(u - h') \, dV
\]
\[
= \int_{\partial \Omega} uh' \frac{\partial}{\partial \nu}(u - h') \, d\sigma
\]
\[
= 0. \tag{2.8}
\]
Using (2.5) and (2.8), we have
\[
\int_{\Omega} \langle \nabla(u - h'), \nabla u \rangle \Delta u \, dV = -n \int_{\Omega} h' \langle \nabla(u - h'), \nabla u \rangle \, dV
\]
\[
= n \int_{\Omega} u \langle \nabla(u - h'), \nabla h' \rangle \, dV
\]
\[
= -n \int_{\Omega} u |\nabla(u - h')|^2 \, dV
\]
\[
\leq 0. \tag{2.9}
\]
Since $\Gamma_0 \subset \{u = 0\}$ is a level set of $u$,

$$\nabla u = cv \quad \text{on } \Gamma_0.$$  

Thus,

$$\int_{\partial \Omega} \langle \nabla (u - h'), \nabla u \rangle \frac{\partial u}{\partial \nu} d\sigma = c^2 \int_{\Gamma_0} \langle \nabla (u - h'), \nu \rangle d\sigma$$

$$= c^2 \int_{\partial \Omega} \langle \nabla (u - h'), \nu \rangle d\sigma$$

$$= c^2 \int_{\Omega} \Delta (u - h') dV$$

$$= 0. \quad (2.10)$$

Substituting (2.9) and (2.10) into (2.7), we obtain

$$\int_{\Omega} u \Delta \langle \nabla (u - h'), \nabla u \rangle dV \leq \int_{\Gamma_1} u \frac{\partial}{\partial \nu} \langle \nabla (u - h'), \nabla u \rangle d\sigma. \quad (2.11)$$

One can evaluate the right-hand side of (2.11). We note that

$$\frac{\partial}{\partial \nu} \langle \nabla (u - h'), \nabla u \rangle = \langle \nabla \nabla h' \nabla u, \nu \rangle + \langle \nabla u, \nabla \nabla h' \nu \rangle = 2\text{Hess}_u(\nabla h', \nu) - \text{Hess}_u(\nabla u, \nu). \quad (2.12)$$

Since $\frac{\partial u}{\partial \nu} = 0$ and $\frac{\partial h'}{\partial \nu} = 0$ on $\Gamma_1$, $\nabla u$ and $\nabla h'$ are tangent vectors on $\Gamma_1$. Observe that

$$\nabla \nabla h' \nu = 0 \quad \text{on } \Gamma_1.$$  

On $\Gamma_1$,

$$0 = \nabla \nabla h' \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla \nabla h' \nabla u, \nu \rangle + \langle \nabla u, \nabla \nabla h' \nu \rangle = \text{Hess}_u(\nabla h', \nu). \quad (2.13)$$

Moreover, the convexity of the cone $C$ tells us that

$$\Pi(\nabla u, \nabla u) = \langle \nabla \nabla u \nu, \nabla u \rangle \geq 0 \quad \text{on } \Gamma_1,$$

where $\Pi(\cdot, \cdot)$ is the second fundamental form. Thus,

$$0 = \nabla \nabla u \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla \nabla u \nu, \nabla u \rangle + \langle \nabla u, \nabla \nabla u \nu \rangle$$

$$= \text{Hess}_u(\nabla u, \nu) + \Pi(\nabla u, \nabla u)$$

$$\geq \text{Hess}_u(\nabla u, \nu) \quad \text{on } \Gamma_1. \quad (2.14)$$

Plugging (2.13) and (2.14) into (2.12), we get

$$\frac{\partial}{\partial \nu} \langle \nabla (u - h'), \nabla u \rangle \leq 0 \quad \text{on } \Gamma_1.$$
By the continuity of $u$, we see that $u \geq 0$ on $\Gamma_1$, since $u > 0$ in $\Omega$. Thus,

$$\int_{\Gamma_1} u \frac{\partial}{\partial \nu} \left\langle \nabla (u - h'), \nabla u \right\rangle d\sigma \leq 0. \tag{2.15}$$

Therefore (2.11) and (2.15) shows

$$\int_{\Omega} u\Delta \left\langle \nabla (u - h'), \nabla u \right\rangle dV \leq 0. \tag{2.16}$$

By (2.6) and (2.16),

$$\int_{\Omega} u\Delta \left\langle \nabla (u - h'), \nabla u \right\rangle dV = 0.$$

Equality in (2.6) shows

$$\nabla (u - h') \equiv 0 \text{ in } \Omega,$$

which implies that

$$u(x) = h' + a = \cos r + a$$

for some constant $a$. Moreover, the constant $a$ can be expressed in terms of the constant $c$. To see this, note that the function $u$ vanishes on $\Gamma_0$ by the boundary condition. Thus, $\Gamma_0$ is part of the boundary of the geodesic ball $B_R(p)$ of radius $R = \cos^{-1}(-a)$ centered at $p$. This shows that

$$u(x) = \cos r - \cos R$$

and

$$\Omega = B_R(p) \cap C.$$

The boundary condition on $\Gamma_0$ gives

$$c = \frac{\partial u}{\partial \nu} = \langle -\sin R \nabla r, \nabla r \rangle = -\sin R \quad \text{on } \Gamma_0.$$

Finally we obtain

$$u(x) = \cos r - \cos R,$$

where $R = \sin^{-1}(-c).$ \hfill \Box

Remark 2.3. It should be mentioned that for the mixed boundary value problem as in Theorem 2.2, the regularity of the solution up to the boundary $\partial \Omega$ depends on the way $\Gamma_0$ and $\Gamma_1$ intersect (see [5, 7] for instance). In particular, the regularity on the boundary $\partial \Omega$ is related with the angles formed at the intersection between $\Gamma_0$ and $\Gamma_1$. Due to this phenomenon, we require that the solution $u$ is contained in $W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$.

Remark 2.4. Using the divergence theorem, we get

$$-n \int_{\Omega} h' dV = \int_{\Omega} \Delta u dV = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} d\sigma = \int_{\Gamma_0} \frac{\partial u}{\partial \nu} d\sigma = c |\Gamma_0|. $$
Since \( h' \geq 0 \) on \( \Omega \subset S^+_n \),

\[
c = -\frac{n \int_{\Omega} h' dV}{|\Gamma_0|} < 0.
\]

Recently, Ciraolo–Roncoroni [2] obtained the radial symmetry of the solution to a partially overdetermined problem inside a convex cone in \( S^+_n \), considering the equation \( \Delta u + nu = -n \). In the following, we shall consider the same problem in \( S^n \) without the assumption that \( \Omega \) is contained in \( S^+_n \). Instead, we add an assumption that \( \Omega \) is a star-shaped domain with respect to the pole \( p \). A domain \( \Omega \subset S^n \) is called star-shaped with respect to \( p \) if each component of the boundary \( \partial \Omega \) can be written as a graph over a geodesic sphere centered at \( p \). Now consider the unit sphere \( S^n = I \times S^{n-1} \) with the warped product metric \( g = dr^2 + h(r)^2 g_{S^{n-1}} \) as before. Note that the interval \( I \) is given by \( I = [0, \pi) \), which is different from the hemisphere case. For the pole \( p \) of the model space and a convex domain \( \omega \subset S^{n-1} \), we can define a convex cone \( C \subset S^n \) with vertex at \( p \) in the same manner:

\[
C = C_p(\omega) := \{ tx : x \in \omega, t \in I \}.
\]

Geometrically, \( C = C_p(\omega) \) is the set of all the unique great semicircles from \( p \) to \( -p \) passing through \( x \) for any \( x \in \omega \). Thus, given a convex domain \( \omega \subset S^{n-1} \), it follows that the cone with vertex \( -p \) coincides with the cone with vertex at \( p \), that is,

\[
C_{-p}(\omega) = C_p(\omega).
\]

Adopting the \( P \)-function method used in [4, 11, 15], we are able to prove the following theorem, which can be seen as a generalization of the results by Ciraolo–Roncoroni [2] and Pacella–Tralli [10].

**Theorem 2.5.** Let \( C \subset S^n \) be an open convex cone with vertex at \( p \) and \( \Omega \subset C \) be an admissible interior domain. Assume that \( \Omega \) is a star-shaped domain with respect to \( p \) and \( -p \notin \partial \Omega \). Suppose that there exists a solution \( u \in C^2(\Omega) \cap C^1(\Gamma_0 \cup \Gamma_1 \setminus \{p\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \) to the partially overdetermined problem

\[
\begin{cases}
\Delta u + nu = -n & \text{in } \Omega, \\
u = 0, & \frac{\partial u}{\partial \nu} = \text{const} = c & \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \setminus \{p\},
\end{cases}
\]

where \( \nu \) is the outward unit normal to \( \Gamma_0 \cup \Gamma_1 \setminus \{p\} \). Then, \( \Omega \) is part of the geodesic ball \( B_R(p_0) \) of radius \( R \) centered at \( p_0 \) in the cone \( C \), that is,

\[
\Omega = C \cap B_R(p_0)
\]

and the solution \( u \) is given by

\[
u(x) = \frac{1}{\cos R} (\cos r(x) - \cos R)
\]

with \( r(x) = \text{dist}(p_0, x) \) and \( R = \tan^{-1}(-c) \). Moreover, one of the following two possibilities holds:

(I) \( p_0 = p \);

(II) \( p_0 \in \partial C \) and \( \partial \Omega \cap \partial C \) is totally geodesic.

**Proof.** It is well known that

\[
(\Delta u)^2 \leq n \text{tr}(\text{Hess}^2 u).
\]

(2.17)
Note that equality in (2.17) holds if and only if \( \text{Hess } u \) is proportional to the metric \( g \). Using the Bochner formula,

\[
\Delta |\nabla u|^2 = 2 \langle \nabla (\Delta u), \nabla u \rangle + 2 \text{tr}(\text{Hess}^2 u) + 2 \text{Ric}(\nabla u, \nabla u) \geq -2n|\nabla u|^2 + \frac{2}{n}(\Delta u)^2 + 2(n - 1)|\nabla u|^2
\]

\[
= \frac{2}{n}(-n - nu)\Delta u - 2|\nabla u|^2
\]

\[
= -2\Delta u - 2u\Delta u - 2|\nabla u|^2
\]

\[
= -2\Delta u - \Delta u^2. \quad \text{(2.18)}
\]

Define the function \( P \) by

\[
P(u) := |\nabla u|^2 + 2u + u^2.
\]

Then, (2.18) implies

\[
\Delta P \geq 0.
\]

We also define another function \( \tilde{P} \) by

\[
\tilde{P} := \langle \nabla u, \nabla h' \rangle + uh' + h',
\]

where \( h(r) = \sin r \) and \( r(x) = \text{dist}(p, x) \). Then,

\[
\text{Hess } h' = -h'g \quad \text{and} \quad \Delta h' = -nh',
\]

where \( g \) is the metric of \( S^n \). By the polarized Bochner formula, we get

\[
\Delta \langle \nabla u, \nabla h' \rangle = \langle \nabla (\Delta u), \nabla h' \rangle + \langle \nabla u, \nabla (\Delta h') \rangle + 2\text{tr}(\text{Hess } u \circ \text{Hess } h') + 2\text{Ric}(\nabla u, \nabla h')
\]

\[
= -n\langle \nabla u, \nabla h' \rangle - n\langle \nabla u, \nabla h' \rangle - 2h'\Delta u + 2(n - 1)\langle \nabla u, \nabla h' \rangle
\]

\[
= -2h'(-nu - n) - 2\langle \nabla u, \nabla h' \rangle
\]

\[
= 2nuh' + 2nh' - 2\langle \nabla u, \nabla h' \rangle.
\]

Since

\[
\Delta (uh') = u\Delta h' + h'\Delta u + 2\langle \nabla u, \nabla h' \rangle = -2nuh' - nh' + 2\langle \nabla u, \nabla h' \rangle,
\]

we obtain

\[
\Delta \langle \nabla u, \nabla h' \rangle + uh' = nh' = -\Delta h',
\]

which shows

\[
\Delta \tilde{P} = 0.
\]

Note that

\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad \frac{\partial h'}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1,
\]

which implies that \( \nabla u \) and \( \nabla h' \) are tangent vectors on \( \Gamma_1 \). On \( \Gamma_1 \),

\[
\nabla_{\nu h'} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} \equiv \text{const.}
\]
Thus,

\[ 0 = \nabla_{\nabla h'} \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla_{\nabla h'} \nabla u, \nu \rangle + \langle \nabla u, \nabla_{\nabla h'} \nu \rangle = \text{Hess } u(\nabla h', \nu) \quad (2.19) \]

on \( \Gamma_1 \). Moreover, by the convexity of the cone \( C \), we have

\[ \Pi(\nabla u, \nabla u) \geq 0 \quad \text{on } \Gamma_1, \]

where \( \Pi(\cdot, \cdot) \) is the second fundamental form. Thus,

\[
0 = \nabla_{\nabla u} \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla_{\nabla u} \nabla u, \nu \rangle + \langle \nabla u, \nabla_{\nabla u} \nu \rangle \\
= \text{Hess } u(\nabla u, \nu) + \Pi(\nabla u, \nabla u) \\
\geq \text{Hess } u(\nabla u, \nu) \quad \text{on } \Gamma_1.
\]

Hence, we obtain

\[
\frac{\partial P}{\partial \nu} = 2 \text{Hess } u(\nabla u, \nu) + 2 \frac{\partial u}{\partial \nu} + 2u \frac{\partial u}{\partial \nu} \leq 0 \quad \text{on } \Gamma_1.
\]

Suppose that neither \( P \) nor \( \tilde{P} \) is constant. We claim that \( P \leq c^2 \) in \( \Omega \). To see this, we note that \( P \) satisfies the following:

\[
\begin{cases}
\Delta P \geq 0 & \text{in } \Omega, \\
P \equiv c^2 & \text{on } \Gamma_0, \\
\frac{\partial P}{\partial \nu} \leq 0 & \text{on } \Gamma_1.
\end{cases}
\]

Using the divergence theorem, we get

\[
0 \leq \int_{\Omega} (P - c^2)^+ \Delta P dV \\
= \int_{\Omega} \text{div}((P - c^2)^+ \nabla P) dV - \int_{\Omega} \langle \nabla (P - c^2)^+, \nabla P \rangle dV \\
= \int_{\partial \Omega} (P - c^2)^+ \frac{\partial P}{\partial \nu} d\sigma - \int_{\Omega \cap \{P > c^2\}} |\nabla P|^2 dV \leq 0,
\]

where \( (P - c^2)^+ = \max\{P - c^2, 0\} \). Thus, we see that \( P \leq c^2 \) in \( \Omega \) and \( P \) attains its maximum value on \( \Gamma_0 \).

Let \( \{e_1, \ldots, e_{n-1}, \nu\} \) be a local orthonormal frame of \( \Omega \) at \( x \in \Gamma_0 \), where \( \{e_i\}_{i=1}^{n-1} \) is tangent to \( \Gamma_0 \) and \( \nu \) is orthogonal to \( \Gamma_0 \).

Since \( \Gamma_0 \) is a level set of \( u \), we have

\[
u_i = 0 \quad \text{and} \quad u_{ij} = 0 \quad \text{on } \Gamma_0
\]

for all \( i, j = 1, \ldots, n-1 \). Moreover, since \( \frac{\partial u}{\partial \nu} \) is constant on \( \Gamma_0 \),

\[
u_{ij} = 0 \quad \text{on } \Gamma_0
\]

for all \( i = 1, \ldots, n-1 \). Thus,

\[
\text{Hess } u(\nabla u, \nu) = u_{\nu\nu} \frac{\partial u}{\partial \nu} \quad \text{and} \quad \text{Hess } u(\nabla h', \nu) = u_{\nu\nu} \frac{\partial h'}{\partial \nu} \quad \text{on } \Gamma_0.
\]

(2.20)
Applying the Hopf boundary point lemma on $\Gamma_0$,

$$0 < \frac{\partial P}{\partial \nu} = 2\text{Hess } u(\nabla u, \nu) + 2u \frac{\partial u}{\partial \nu} + 2u \frac{\partial u}{\partial \nu} = 2 \frac{\partial u}{\partial \nu} (u_{\nu \nu} + 1) = 2c(u_{\nu \nu} + 1).$$

Since $c$ is constant, we obtain

$$u_{\nu \nu} + 1 > 0 \quad \text{or} \quad u_{\nu \nu} + 1 < 0 \quad \text{on } \Gamma_0. \quad (2.21)$$

Note that

$$\langle \nabla h', \nu \rangle = -\sin r \langle \nabla r, \nu \rangle = 0 \quad \text{on } \Gamma_1$$

and

$$\frac{\partial \tilde{P}}{\partial \nu} = \text{Hess } u(\nabla h', \nu) + \text{Hess } h'(\nabla u, \nu) + u \frac{\partial h'}{\partial \nu} + h' \frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu}.$$ 

Thus, it follows from (2.19) and the boundary condition that

$$\frac{\partial \tilde{P}}{\partial \nu} = 0 \quad \text{on } \Gamma_1.$$

On the other hand, it follows from (2.20) that

$$\frac{\partial \tilde{P}}{\partial \nu} = \text{Hess } u(\nabla h', \nu) + \text{Hess } h'(\nabla u, \nu) + u \frac{\partial h'}{\partial \nu} + h' \frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu} = (u_{\nu \nu} + 1) \frac{\partial h'}{\partial \nu} \quad \text{on } \Gamma_0.$$ 

Since $\Omega$ is a star-shaped domain with respect to $p$, we have

$$\frac{\partial h'}{\partial \nu} = -\sin r \langle \nabla r, \nu \rangle < 0 \quad \text{on } \Gamma_0.$$

From (2.21), we deduce that $\frac{\partial \tilde{P}}{\partial \nu} < 0$ or $\frac{\partial \tilde{P}}{\partial \nu} > 0$ on $\Gamma_0$. Applying the divergence theorem, we have

$$0 = \int_{\Omega} \Delta \tilde{P} dV = \int_{\partial \Omega} \frac{\partial \tilde{P}}{\partial \nu} d\sigma = \int_{\Gamma_0} \frac{\partial \tilde{P}}{\partial \nu} d\sigma < 0 \quad \text{or} \quad > 0,$$

which is a contradiction. Therefore, either $P$ or $\tilde{P}$ is a constant function in $\Omega$.

Suppose $\tilde{P}$ is a constant function. Then,

$$\frac{\partial \tilde{P}}{\partial \nu} = 0 \quad \text{and} \quad u_{\nu \nu} + 1 = 0 \quad \text{on } \Gamma_0,$$

which implies that

$$\frac{\partial P}{\partial \nu} = 0 \quad \text{on } \Gamma_0.$$
Since $P$ has the maximum value on $\Gamma_0$, $P$ is constant in $\Omega$ by the Hopf boundary point lemma. Thus, we may assume that $P$ is a constant function in $\Omega$. In particular, $\Delta P = 0$ in $\Omega$. We see that equality holds in (2.18), which implies that $\text{Hess } u$ is proportional to the metric $g$. Thus,

$$\text{Hess } u = \frac{\Delta u}{n} g = (-u - 1)g.$$  

(2.22)

Since $u = 0$ on $\overline{\Gamma_0}$, the function $u$ defined on $\overline{\Omega}$ cannot attain simultaneously both its maximum and minimum values on $\overline{\Gamma_0}$, which shows that $u$ attains either its maximum or minimum value at some point $p_0 \in \Omega \cup \Gamma_1$. By the elliptic regularity theory, the solution $u$ of our overdetermined problem has classical derivatives on $\Gamma_0 \cup \Gamma_1 \cup \{p_0\}$. Thus, we have $\nabla u(p_0) = 0$.

Let $\gamma(s)$ be a unit-speed geodesic passing through $p_0$ satisfying

$$\gamma(0) = p_0, \nabla\gamma'(s) \gamma'(s) = 0, \text{ and } |\gamma'(s)|^2 = 1.$$  

Let $f(s) := u(\gamma(s))$. Then,

$$f'(s) = \langle \nabla u, \gamma'(s) \rangle$$  

and by (2.22)

$$f''(s) = \langle \nabla\gamma'(s) \nabla u, \gamma'(s) \rangle + \langle \nabla u, \nabla\gamma'(s) \gamma'(s) \rangle$$

$$= \text{Hess } u(\gamma'(s), \gamma'(s))$$

$$= -1 - f(s).$$  

From the fact that $\nabla u(p_0) = 0$, we obtain an initial value problem:

$$f''(s) + f(s) = -1, \quad f'(0) = 0.$$  

A general solution to this ordinary differential equation (ODE) is given by

$$f(s) = c_1 \cos s + c_2 \sin s - 1,$$

where $c_1$ and $c_2$ are constants. Using the initial condition,

$$f(s) = c_1 \cos s - 1,$$

which shows that the solution $u$ depends only on the geodesic distance because $\gamma$ was arbitrarily chosen to be a geodesic passing through $p_0$. Therefore,

$$u(x) = c_1 \cos r(x) - 1,$$  

(2.23)

where $r(x) = \text{dist}(p_0, x)$. Since $u = 0$ on $\Gamma_0$ and $\cos r$ is injective in $r \in [0, \pi)$, $\Gamma_0$ is part of the geodesic sphere centered at $p_0$ with radius $R = \cos^{-1}\left(\frac{1}{c_1}\right)$. Since $\Omega$ is connected,

$$u(x) = \frac{1}{\cos R} (\cos r - \cos R)$$

and $\Omega = C \cap B_R(p_0)$, where $B_R(p_0)$ denotes the geodesic ball centered at $p_0$ with radius $R$. Since

$$c = \frac{\partial u}{\partial \nu} = -\frac{\sin R}{\cos R} \langle \nabla r, \nu \rangle = -\tan R \quad \text{on } \Gamma_0,$$
we get
\[ R = \tan^{-1}(-c). \]

Observe that \( \nabla u(x) \) is parallel to \( \nabla r(x) \) by (2.23). Moreover, \( \nabla u(x) \) lies on the tangent space of \( \Gamma_1 \) for all \( x \in \Gamma_1 \) by the boundary condition on \( \Gamma_1 \). Therefore, the point \( p_0 \) satisfies one of the following two possibilities:

(I) \( p_0 \) is the vertex \( p \).
(II) \( p_0 \in \partial C \) and \( \partial \Omega \cap \partial C \) is totally geodesic.

For (I), we see that \( \Omega = C \cap B_R(p) \). For (II), \( \Omega \) is clearly a half geodesic ball centered at \( p_0 \) lying over a totally geodesic portion of \( C \).

\[ \square \]

### 3 TWO-DIMENSIONAL PARTIALLY OVERDETERMINED PROBLEMS OUTSIDE A CONVEX CONE

In Section 2, we studied partially overdetermined PDE problems for a domain inside a convex cone. One may ask whether the similar results as Theorem 2.2 and Theorem 2.5 are still valid for a domain outside a convex cone. In this section, we give a partial answer to this question.

**Definition 3.1.** Let \( C \) be a convex cone with vertex at \( p \) in a space form \( M \). A connected bounded open set \( \Omega \subset M \setminus \overline{C} \) is an **admissible exterior domain** (see Figure 2) if the boundary \( \partial \Omega \) satisfies the following:

1. \( \partial \Omega \) contains the vertex \( p \).
2. \( \Gamma_0 := \partial \Omega \setminus \partial C \neq \emptyset \) is an \((n-1)\)-dimensional smooth manifold.
3. \( \Gamma_1 := \partial \Omega \setminus \Gamma_0 \neq \emptyset \) and \( \partial \Gamma_1 = \partial C \setminus \{p\} \) is an \((n-2)\)-dimensional smooth manifold.
4. \( H^{n-1}(\Gamma_i) > 0 \) for \( i = 0, 1 \), where \( H^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure.

Using the same argument as in the proof of Theorem 2.2, we consider the following partially overdetermined problem outside a convex cone in a two-dimensional case.

**Theorem 3.2.** Let \( M \) be a two-dimensional space form \( \mathbb{R}^2 \) or \( \mathbb{S}^2 \). Let \( C \subset M \) be an open convex cone with vertex at \( p \) and \( \Omega \) be an admissible exterior domain in \( M \setminus \overline{C} \). If \( M = \mathbb{S}^2 \), assume that either \( \Omega \) is contained in \( \mathbb{S}^2_+ \) or \( u \) is positive on \( \Omega \) and assume that \(-p \notin \partial \Omega \). Suppose there exists a solution \( u \in C^2(\Omega) \cap C^1(\Gamma_0 \cup \Gamma_1 \setminus \{p\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \) to the partially...
overdetermined problem

\[
\begin{aligned}
\Delta u &= -2h' \quad \text{in } \Omega, \\
u &= 0, \quad \frac{\partial u}{\partial \nu} = \text{const} = c \quad \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_1 \setminus \{p\},
\end{aligned}
\]

where \(\nu\) is the outward unit normal to \(\Gamma_0 \cup \Gamma_1 \setminus \{p\}\) and the function \(h(r)\) is the same as in Section 2 with \(r(x) = \text{dist}(p, x)\). Then,

\[
\Omega = B_R(p) \setminus \overline{C},
\]

where \(B_R(p)\) denotes the geodesic ball centered at the vertex \(p\) with radius \(R\) in \(M\). In particular, the solution \(u\) is given by

\[
u(x) = \begin{cases} 
\frac{R^2 - r^2}{2} & \text{in } \mathbb{R}^2, \\
\cos r - \cos R & \text{in } \mathbb{S}^2.
\end{cases}
\]

**Proof.** First let us assume that \(M = \mathbb{S}^2\). The proof uses the same argument as in the proof of Theorem 2.2. However, we have a simpler situation in dimension 2. On \(\Gamma_1\), the boundary condition \(\frac{\partial u}{\partial \nu} = 0\) and \(\frac{\partial h'}{\partial \nu} = 0\) implies

1. \(\nabla_{vh'}\nu = 0\) and \(\nabla_{vv}\nu = 0\),
2. \(0 = \nabla_{vu} \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla_{vu} \nabla u, \nu \rangle + \langle \nabla u, \nabla_{vu} \nu \rangle = \text{Hess } u(\nabla u, \nu)\),
3. \(0 = \nabla_{v'h'} \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla_{v'h'} \nabla u, \nu \rangle + \langle \nabla u, \nabla_{v'h'} \nu \rangle = \text{Hess } u(\nabla h', \nu)\).

Using this observation and the argument as in the proof of Theorem 2.2, we can obtain the conclusion. Furthermore, if \(M = \mathbb{R}^2\), then we can prove Theorem 3.2 in the same manner. \(\square\)

Using the same functions \(P\) and \(\tilde{P}\) as in Theorem 2.5, we obtain a similar Serrin-type symmetry result for domains outside a convex cone as follows.

**Theorem 3.3.** Let \(C \subset \mathbb{S}^2\) be a convex cone with vertex at the pole \(p\) and \(\Omega\) be an admissible exterior domain in \(\mathbb{S}^2 \setminus \overline{C}\). Assume that \(\Omega\) is a star-shaped domain with respect to \(p\). Suppose that there exists a solution \(u \in C^2(\Omega) \cap C^1(\Gamma_0 \cup \Gamma_1 \setminus \{p\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)\) to the partially overdetermined problem

\[
\begin{aligned}
\Delta u + 2u &= -2 \quad \text{in } \Omega, \\
u &= 0, \quad \frac{\partial u}{\partial \nu} = \text{const} = c \quad \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_1 \setminus \{p\},
\end{aligned}
\]

where \(\nu\) is the outward unit normal to \(\Gamma_0 \cup \Gamma_1 \setminus \{p\}\). Then,

\[
\Omega = B_R(p_0) \setminus \overline{C},
\]

where \(B_R(p_0)\) denotes the geodesic ball centered at \(p_0\) with radius \(R\) and the solution \(u\) is given by

\[
u(x) = \frac{1}{\cos R}(\cos r(x) - \cos R),
\]

where \(r(x) = \text{dist}(p_0, x)\). Moreover, one of the following two possibilities holds:
(I) $p_0 = p$;
(II) $p_0 \in \partial C$ and $\partial \Omega \cap \partial C$ is totally geodesic.

Proof. As in the proof of Theorem 2.5, define two $P$-functions as follows:

$$P(u) = |\nabla u|^2 + 2u + u^2 \quad \text{and} \quad \bar{P}(u) = \langle \nabla u, \nabla h' \rangle + uh' + h'.$$

Then,

$$\Delta P \geq 0 \quad \text{and} \quad \Delta \bar{P} = 0.$$

Since $\frac{\partial u}{\partial v} = 0$ and $\frac{\partial h'}{\partial v} = 0$ on $\Gamma_1$, $\nabla u$ and $\nabla h'$ are tangent vectors of $\Gamma_1$. A direct computation gives

$$\nabla_{\nabla h'} v = 0 \quad \text{on} \quad \Gamma_1.$$

Since $\frac{\partial u}{\partial v}$ is constant on $\Gamma_1$,

$$0 = \nabla_{\nabla h'} \left( \frac{\partial u}{\partial v} \right) = \langle \nabla_{\nabla h'} \nabla u, v \rangle + \langle \nabla u, \nabla_{\nabla h'} v \rangle = \text{Hess} u(\nabla h', v) \quad \text{on} \quad \Gamma_1. \quad (3.1)$$

We note that $\frac{\partial u}{\partial v} = 0$ on $\Gamma_1$ implies that $\nabla u$ is a radial direction along $\Gamma_1$. This leads to

$$\text{Hess} u(\nabla u, v) = 0 \quad \text{on} \quad \Gamma_1.$$

Thus, we obtain

$$\frac{\partial P}{\partial v} = 2\text{Hess} u(\nabla u, v) + 2\frac{\partial u}{\partial v} + 2u\frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \Gamma_1.$$

Suppose neither $P$ nor $\bar{P}$ is constant. We claim that $P \leq c^2$ in $\Omega$. To see this, we note that $P$ satisfies the following:

$$\begin{cases}
\Delta P \geq 0 & \text{in} \quad \Omega, \\
P \equiv c^2 & \text{on} \quad \Gamma_0, \\
\frac{\partial P}{\partial v} = 0 & \text{on} \quad \Gamma_1.
\end{cases}$$

Using the divergence theorem, we get

$$0 \leq \int_{\Omega} (P - c^2)^+ \Delta P dV$$

$$= \int_{\Omega} \text{div}((P - c^2)^+ \nabla P) dV - \int_{\Omega} \langle \nabla (P - c^2)^+, \nabla P \rangle dV$$

$$= \int_{\partial \Omega} (P - c^2)^+ \frac{\partial P}{\partial v} d\sigma - \int_{\Omega \cap \{P > c^2\}} |\nabla P|^2 dV \leq 0,$$

where $(P - c^2)^+ = \max\{P - c^2, 0\}$. Thus, $P \leq c^2$ in $\Omega$ and $P$ attains its maximum value on $\Gamma_0$.

Let $\{e_1, v\}$ be a local orthonormal frame at $x \in \Gamma_0$. Since $\Gamma_0$ is a level set of $u$, we obtain

$$u_1 = 0 \quad \text{and} \quad u_{11} = 0.$$
Since $\frac{\partial u}{\partial \nu}$ is constant on $\Gamma_0$, we obtain

$$u_{\nu 1} = 0 \quad \text{on } \Gamma_0.$$ 

Then, we deduce that on $\Gamma_0$,

$$\text{Hess } u(\nabla u, \nu) = u_{\nu \nu} \frac{\partial u}{\partial \nu} \quad \text{and} \quad \text{Hess } u(\nabla h', \nu) = u_{\nu \nu} \frac{\partial h'}{\partial \nu}. \quad (3.2)$$

By the Hopf boundary lemma, we have

$$0 < \frac{\partial P}{\partial \nu} = 2 \text{Hess } u(\nabla u, \nu) + 2 \frac{\partial u}{\partial \nu} + \frac{\partial u}{\partial \nu} + 2 u_{\nu \nu} \frac{\partial u}{\partial \nu} + 2 \frac{\partial u}{\partial \nu} = 2 c (u_{\nu \nu} + 1)$$

on $\Gamma_0$. Since $c$ is constant, we obtain

$$u_{\nu \nu} + 1 > 0 \quad \text{or} \quad u_{\nu \nu} + 1 < 0 \quad \text{on } \Gamma_0. \quad (3.3)$$

We note that $\langle \nabla h', \nu \rangle = -\sin r \langle \nabla r, \nu \rangle = 0$ on $\Gamma_1$. Moreover,

$$\frac{\partial \tilde{P}}{\partial \nu} = \text{Hess } u(\nabla h', \nu) + \text{Hess } h'(\nabla u, \nu) + u \frac{\partial h'}{\partial \nu} + h' \frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu}.$$ 

Thus, it follows from (3.1) that

$$\frac{\partial \tilde{P}}{\partial \nu} = 0 \quad \text{on } \Gamma_1.$$ 

By the boundary conditions on $\Gamma_0$ and (3.2), we get

$$\frac{\partial \tilde{P}}{\partial \nu} = \text{Hess } u(\nabla h', \nu) + \text{Hess } h'(\nabla u, \nu) + u \frac{\partial h'}{\partial \nu} + h' \frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu}$$

$$= u_{\nu \nu} \frac{\partial h'}{\partial \nu} + \frac{\partial h'}{\partial \nu} - h' \frac{\partial u}{\partial \nu} + u \frac{\partial h'}{\partial \nu} + h' \frac{\partial u}{\partial \nu} + \frac{\partial h'}{\partial \nu}$$

$$= (u_{\nu \nu} + 1) \frac{\partial h'}{\partial \nu} \quad \text{on } \Gamma_0.$$ 

Since $\Omega$ is a star-shaped domain with respect to $p$,

$$\frac{\partial h'}{\partial \nu} = -\sin r \langle \nabla r, \nu \rangle < 0 \quad \text{on } \Gamma_0.$$ 

Then, we deduce that

$$\frac{\partial \tilde{P}}{\partial \nu} < 0 \quad \text{or} \quad \frac{\partial \tilde{P}}{\partial \nu} > 0 \quad \text{on } \Gamma_0$$

from (3.3). Using the divergence theorem, we have

$$0 = \int_{\Omega} \Delta \tilde{P} dV = \int_{\Omega} \frac{\partial \tilde{P}}{\partial \nu} d\sigma = \int_{\Gamma_0} \frac{\partial \tilde{P}}{\partial \nu} d\sigma < 0 \quad (\text{or} > 0),$$

which is a contradiction.
Thus, either $P$ or $\tilde{P}$ is a constant function in $\Omega$. Suppose $\tilde{P}$ is a constant function. Then,

$$\frac{\partial \tilde{P}}{\partial \nu} = 0 \quad \text{and} \quad u_{\nu} + 1 = 0 \quad \text{on } \Gamma_0,$$

which implies that

$$\frac{\partial P}{\partial \nu} = 0 \quad \text{on } \Gamma_0.$$

Since $P$ has the maximum value on $\Gamma_0$, $P$ is a constant function in $\Omega$ by the Hopf boundary point lemma. Therefore, we may assume that $P$ is constant. Using the same argument as in the proof of Theorem 2.5, we can show that the solution $u$ is radially symmetric with respect to some point $p_0$ and it is given by

$$u(x) = \frac{1}{\cos R}(\cos r(x) - \cos R),$$

where $r(x) = \text{dist}(p_0, x)$. Moreover, $\Omega$ is the intersection of $C$ and the geodesic ball $B_R(p_0)$ of radius $R$ centered at $p_0$. □

### 4 AN EIGENVALUE PROBLEM WITH MIXED BOUNDARY CONDITIONS IN A CONE

Let $M^n(K)$ be one of the space forms $\mathbb{R}^n$, $\mathbb{S}^n_+$, and $\mathbb{H}^n$ of constant sectional curvature $K = 0, 1,$ and $-1$, respectively. Given the Dirichlet eigenvalue problem for a bounded domain $\Omega \subset M^n(K)$

$$\begin{cases}
\Delta u + \lambda u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

it is known that the following Rellich identity holds (see [12] for $K = 0$ and [9] for $K = 1$ or $-1$): When $K = 0$,

$$\lambda = -\frac{\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma}{2 \int_{\Omega} u^2 dV},$$

and when $K = 1$ or $-1$,

$$\lambda = \frac{-n(n - 2)K}{4} - \frac{1}{2K} \frac{\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma}{\int_{\Omega} f u^2 dV}.$$ 

Here, the function $f(r)$ is defined by

$$f(r) = \begin{cases}
\frac{r^2}{2} & \text{if } K = 0, \\
\cos r & \text{if } K = 1, \\
\cosh r & \text{if } K = -1,
\end{cases}$$

where $r(x) = \text{dist}(p, x)$. Motivated by this, we prove an analog for an eigenvalue problem with mixed boundary conditions for domains inside a (not necessarily convex) cone in the following.

**Theorem 4.1.** Let $M^n(K)$ be one of space forms $\mathbb{R}^n$, $\mathbb{S}^n_+$, and $\mathbb{H}^n$ of constant sectional curvature $K = 0, 1,$ and $-1$, respectively. Let $C \subset M$ be a cone with vertex at $p$ and let $\Omega \subset C$ be an admissible interior domain. Suppose there exists a function $u \in$
\( C^2(\Omega) \cap C^1(\Gamma_0 \cup \Gamma_1 \setminus \{p\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \) such that

\[
\begin{aligned}
\Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_1 \setminus \{p\}.
\end{aligned}
\]

If \( K = 0 \), then

\[
\lambda = -\frac{\int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\sigma}{2 \int_{\Omega} u^2 \, dV}.
\]

If \( K = 1 \) or \(-1\), then

\[
\lambda = \frac{-n(n-2)K}{4} - \frac{1}{2K} \frac{\int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\sigma}{\int_{\Omega} f^2 \, dV}.
\]

Here, the function \( f(r) \) is defined as above.

**Proof.** We first prove the case where \( K = 1 \) or \(-1\). The function \( f \) satisfies

\[
\Delta f = -nKf \quad \text{and} \quad \text{Hess } f = -Kg,
\]

where \( g \) denotes the metric on \( M \). By the polarized Bochner formula, we get

\[
\Delta \langle \nabla u, \nabla f \rangle = \langle \nabla (\Delta u), \nabla f \rangle + \langle \nabla u, \nabla (\Delta f) \rangle + 2 \text{tr} (\text{Hess } u \circ \text{Hess } f) + 2 \text{Ric} (\nabla u, \nabla f)
\]

\[
= -\lambda \langle \nabla u, \nabla f \rangle - nK \langle \nabla u, \nabla f \rangle - 2Kf \Delta u + 2(n-1)K \langle \nabla u, \nabla f \rangle
\]

\[
= (-\lambda + nK - 2K) \langle \nabla u, \nabla f \rangle + 2Kf u.
\]

Thus,

\[
\int_{\Omega} u \Delta \langle \nabla u, \nabla f \rangle \, dV = (-\lambda + nK - 2K) \int_{\Omega} u \langle \nabla u, \nabla f \rangle \, dV
\]

\[
+ 2K \lambda \int_{\Omega} f u^2 \, dV.
\]

(4.1)

On \( \Gamma_0 \), \( \nabla u \) is parallel to \( \nu \) because \( \Gamma_0 \) is a level set of \( u \). Using Green's identity,

\[
\int_{\Omega} u \Delta (\nabla u, \nabla f) \, dV
\]

\[
= \int_{\Omega} \langle \nabla u, \nabla f \rangle \Delta u \, dV + \int_{\partial \Omega} u \frac{\partial}{\partial \nu} \langle \nabla u, \nabla f \rangle \, d\sigma - \int_{\partial \Omega} \langle \nabla u, \nabla f \rangle \frac{\partial u}{\partial \nu} \, d\sigma
\]

\[
= -\lambda \int_{\Omega} \langle \nabla u, \nabla f \rangle \, dV + \int_{\Gamma_1} u \frac{\partial}{\partial \nu} \langle \nabla u, \nabla f \rangle \, d\sigma - \int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\sigma.
\]
Moreover, we note that $\nabla_{\nabla f} \nu = 0$ on $\Gamma_1$. Since $\frac{\partial u}{\partial \nu}$ is constant on $\Gamma_1$ and $\nabla f$ is a tangent vector at $x \in \Gamma_1$, we see that

$$0 = \nabla_{\nabla f} \left( \frac{\partial u}{\partial \nu} \right) = \langle \nabla_{\nabla f} \nabla u, \nu \rangle + \langle \nabla u, \nabla_{\nabla f} \nu \rangle = \text{Hess } u(\nabla f, \nu) \text{ on } \Gamma_1,$$

which implies that

$$\frac{\partial}{\partial \nu} \langle \nabla u, \nabla f \rangle = \text{Hess } u(\nabla f, \nu) + \text{Hess } f(\nu, \nabla u) = -K_f \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1.$$

Thus,

$$\int_{\Omega} u \Delta \langle \nabla u, \nabla f \rangle dV = -\lambda \int_{\Omega} u \langle \nabla u, \nabla f \rangle dV - \int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma. \quad (4.2)$$

By (4.1) and (4.2), we obtain

$$(n - 2)K \int_{\Omega} u \langle \nabla u, \nabla f \rangle dV + 2K \lambda \int_{\Omega} f u^2 dV = - \int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma. \quad (4.3)$$

Applying the divergence theorem,

$$\int_{\Omega} u \langle \nabla u, \nabla f \rangle dV = \frac{1}{2} \int_{\Omega} \langle \nabla (u^2), \nabla f \rangle dV$$

$$= \frac{1}{2} \int_{\Omega} \operatorname{div}(u^2 \nabla f) dV - \frac{1}{2} \int_{\Omega} u^2 \Delta f dV$$

$$= \frac{1}{2} \int_{\partial \Omega} u^2 \frac{\partial f}{\partial \nu} d\sigma - \frac{1}{2} \int_{\Omega} u^2 (-nKf) dV$$

$$= \frac{nK}{2} \int_{\Omega} f u^2 dV.$$

Plugging the above equality into (4.3), we have

$$\frac{(n - 2)nK^2}{2} \int_{\Omega} f u^2 dV + 2K \lambda \int_{\Omega} f u^2 dV = - \int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma.$$

Therefore,

$$\lambda = \frac{-n(n - 2)K}{4} - \frac{1}{2K} \int_{\Gamma_0} \frac{\partial f}{\partial \nu} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma.$$

Now let us consider the case where $K = 0$. In this case,

$$\Delta f = -n \quad \text{and} \quad \text{Hess } f = -\text{Id}.$$

Applying the same argument as in the above gives the conclusion.

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