DELOCALIZATION FOR RANDOM LANDAU HAMILTONIANS WITH UNBOUNDED RANDOM VARIABLES
François Germinet, Abel Klein, Benoît Mandy

To cite this version:
François Germinet, Abel Klein, Benoît Mandy. DELOCALIZATION FOR RANDOM LANDAU HAMILTONIANS WITH UNBOUNDED RANDOM VARIABLES. 2009. hal-00363145

HAL Id: hal-00363145
https://hal.science/hal-00363145
Preprint submitted on 20 Feb 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
DELOCALIZATION FOR RANDOM LANDAU HAMILTONIANS
WITH UNBOUNDED RANDOM VARIABLES

FRANÇOIS GERMINET, ABEL KLEIN, AND BENOIT MANDY

Abstract. In this note we prove the existence of a localization/delocalization
transition for Landau Hamiltonians randomly perturbed by an electric potential
with unbounded amplitude. In particular, with probability one, no Landau
gaps survive as the random potential is turned on; the gaps close, filling up
partly with localized states. A minimal rate of transport is exhibited in the
region of delocalization. To do so, we exploit the a priori quantization of the
Hall conductance and extend recent Wegner estimates to the case of unbounded
random variables.

1. Introduction

In this note we prove the existence of a dynamical localization/delocalization
transition for Landau Hamiltonian randomly perturbed by an electric potential
with unbounded amplitude, extending results from [GKS1, GKS2]. In [GKS1] the
perturbation had to be sufficiently small compared to the strength of the magnetic
field: the amplitude of the random potential was such that the Landau gaps sur-
vived after adding the perturbation. In [GKS2] the Landau gaps where allowed to
close, but the random potentials were bounded. In this article we consider random
potentials such that, with probability one, all the Landau gaps close as the ran-
dom potential is turned on, and are shown to be (partially) filled up with localized
states. As in [GKS1, GKS2], a minimal rate of transport is exhibited in the region
of delocalization.

These results exploit the a priori quantization of the Hall conductance proved in
[GKS2]. Many of the results we will need rely on [GK1, GK4], where the random
potential was assumed to be bounded. Such a strong assumption is not necessary,
and can be replaced by weaker hypotheses, satisfied by the random Landau Hamil-
tonian with unbounded random couplings studied in this paper. We will require
Wegner estimates for these random operators, which are obtained by extending the
analysis of [CHK1, CHK2] to the case of unbounded random variables, a result of
independent interest.

We now describe the model and the results. We consider a \( \mathbb{Z}^2 \)-ergodic Landau
Hamiltonian

\[
H_{B, \lambda, \omega} = H_B + \lambda V_\omega \quad \text{on} \quad L^2(\mathbb{R}^2, dx),
\]

(1.1)

where \( H_B \) is the (free) Landau Hamiltonian,

\[
H_B = (-i \nabla - A)^2 \quad \text{with} \quad A = \frac{B}{2}(x_2, -x_1).
\]

(1.2)
(A is the vector potential and $B > 0$ is the strength of the magnetic field, we use the symmetric gauge and incorporated the charge of the electron in the vector potential), $\lambda \geq 0$ is the disorder parameter, and $V_\omega$ is an unbounded ergodic potential: there is a probability space $(\Omega, \mathbb{P})$ equipped with an ergodic group $\{\tau(a); a \in \mathbb{Z}^2\}$ of measure preserving transformations, a potential-valued map $V_\omega$ on $\Omega$, measurable in the sense that $\langle \phi, V_\omega \phi \rangle$ is a measurable function of $\omega$ for all $\phi \in C_0^\infty(\mathbb{R}^d)$. We assume that
\begin{equation}
V_\omega(x) = \sum_{j \in \mathbb{Z}^2} \omega_j u(x - j),
\end{equation}
where the single site potential $u$ is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin, and the $\omega_j$’s are independent, identically distributed random variables, whose common probability distribution $\mu$ has a bounded density $\rho$ with $\text{supp} \rho = \mathbb{R}$ and fast decay:
\begin{equation}
\rho(\omega) \leq \rho_0 \exp(-|\omega|^\alpha),
\end{equation}
for some $\rho_0 \in [0, +\infty]$ and $\alpha > 0$. We fix constants for $u$ by
\begin{equation}
C_{-1} \chi_{\Lambda_-} (0) \leq u \leq C_{+1} \chi_{\Lambda_+} (0),
\end{equation}
with $C_{\pm1}, \delta_{\pm} \in ]0, \infty[$, and normalize $u$ so that we have $\| \sum_{j \in \mathbb{Z}^2} \omega_j \|_1 \leq 1$. (We write $\Lambda_L (x) := x + \left[ -\frac{L}{2}, \frac{L}{2} \right]^d$, for the box of side $L > 0$ centered at $x \in \mathbb{R}^d$, with $\chi_{\Lambda_L(x)}$ being its characteristic function. We also write $\chi_x = \chi_{\Lambda_1(x)}$.)

Under these hypotheses, $H_{B,\lambda,\omega}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ with probability one, with the bound
\begin{equation}
H_{B,\lambda,\omega} \geq -c_\omega \langle \log(x) \rangle^\beta, \quad \text{for all } x \in \mathbb{R}^d,
\end{equation}
for any given $\beta > \alpha^{-1}$, with $c_\omega$ depending also on $\alpha, \beta, d$. (See Lemma [3.1].)

Moreover, the unbounded random potential $V_\omega$ satisfies the probability estimate of Lemma [A.1] namely [A.1], the condition that replaces the boundedness of the potential in [GK1] [GK4]. Note that [A.1] is similar to the condition given in [U Eq. (3.2)]. Using the Wegner estimate given in Theorem [B.1] we can conclude, similarly to the results in [U] for a continuous Gaussian random potential, that the results of [GK1] [GK4], and hence also [GK2] [GK5], hold for $H_{B,\lambda,\omega}$. (See also Appendix [A]) This condition also suffices for the validity of [GKS2] Theorems 1.1 and 1.2. Thus we just refer to [GK1] [GK2] [GK3] [GK5] [GKS2] where appropriate.

The spectrum $\sigma(H_B)$ of the Landau Hamiltonian $H_B$ consists of a sequence of infinitely degenerate eigenvalues, the Landau levels:
\begin{equation}
B_n = (2n - 1)B, \quad n = 1, 2, \ldots.
\end{equation}
For further reference, we also set
\begin{equation}
B_1 = [\infty, 2B], \quad \text{and} \quad B_n = [B_n - B, B_n + B], \quad n = 2, 3, \ldots.
\end{equation}
On the other hand, as soon as $\lambda > 0$, the spectrum fills the Landau gaps and we have [BCH]
\begin{equation}
\sigma(H_{B,\lambda,\omega}) = \mathbb{R}, \quad \mathbb{P} \text{-a.s.}
\end{equation}
The fact that the Landau gaps are immediately filled up as soon as the disorder is turned on implies that the approach used in [GKS1] is non applicable. More properties of the Hall conductance are needed in order to perform the simple reasoning
that provides the existence of a dynamical transition. More precisely, it becomes crucial to know a priori that the Hall conductance is an integer in the region of complete localization (which includes the spectral gaps), a fact that was circumvented in [GKS1] by resorting to an open gap condition. That the Hall conductance for ergodic models is integer valued in the region of complete localization (which includes the spectral gaps), a fact that was circumvented in [GKS1] by resorting to an open gap condition. That the Hall conductance for ergodic models is integer valued in the region of complete localization was known for discrete Anderson type models since [BeES, AG]. For ergodic Schrödinger operators in the continuum, it was first established in [AvSS] for energies in gaps and extended to the region of complete localization in [GKS2], where the analysis of [AG] has been carried over to the continuum. This property has to be combined with the continuity of the Hall conductance for arbitrary small $\lambda$ (in order to let $\lambda$ go to zero). In [GKS2] it is shown that it is actually enough to prove the same continuity property but for the integrated density of states; see [GKS2, Lemma 3.1]. This is done in this note by revisiting the article [HiKS]; see Theorem B.2. But first, we extend the Wegner estimate given in [CHK2] to unbounded random variables; the estimate is given in terms of the concentration function of a measure which is a modification of the single-site probability measure $\mu$. (See Theorem B.1, which has independent interest.)

We state the main result of this note and its corollary. Following [GK4, GK5, GKS2], we set $\Xi_{DL}^{B,\lambda}$ to be the region of complete localization (gaps included), that is, the set of energies where the multiscale analysis applies (or, if applicable, the fractional moment method of [AENSS]). Its complement is the set of dynamical delocalization $\Xi_{DD}^{B,\lambda}$. An energy $E \in \Xi_{DD}^{B,\lambda}$ such that for any $\varepsilon > 0$, $[E - \varepsilon, E + \varepsilon] \cap \Xi_{DL}^{B,\lambda} \neq \emptyset$, is called a dynamical mobility edge.

**Theorem 1.1.** Let $H_{B,\lambda,\omega}$ be a random Landau Hamiltonian as above. For each $n = 1, 2, \ldots$, if $\lambda$ is small enough (depending on $n$) there exist dynamical mobility edges $E_{j,n}(B, \lambda) \in B_n$, $j = 1, 2$, such that

$$\max_{j=1,2} \left| E_{j,n}(B, \lambda) - B_n \right| \leq K_n(B) \lambda |\log \lambda|^{\frac{1}{2}} \to 0 \quad \text{as} \ \lambda \to 0, \quad (1.10)$$

with a finite constant $K_n(B)$. (It is possible that $E_{1,n}(B, \lambda) = E_{2,n}(B, \lambda)$, i.e., dynamical delocalization occurs at a single energy.)

By the characterization of the region of complete localization established in [GK4], Theorem 1.1 has a consequence in terms of transport properties of the Hall system. Indeed, to measure “dynamical delocalization” as stated in the theorem, we introduce

$$M_{B,\lambda,\omega}(p, X', t) = \left\| \langle x \rangle \frac{\partial}{\partial t} e^{-itH_{B,\lambda,\omega}X'(H_{B,\lambda,\omega})\chi_0} \right\|_2^2, \quad (1.11)$$

the random moment of order $p \geq 0$ at time $t$ for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the square of side one around the origin (with characteristic function $\chi_0$), and “localized” in energy by the function $X' \in C_{c,1}^{\infty}(\mathbb{R})$. Its time averaged expectation is given by

$$M_{B,\lambda}(p, X, T) = \frac{1}{T} \int_0^T \mathbb{E} \left\{ M_{B,\lambda,\omega}(p, X, t) \right\} e^{-\frac{t}{T}} dt. \quad (1.12)$$

**Corollary 1.2.** The random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits dynamical delocalization in each Landau band $B_n(B, \lambda)$: For each $n = 1, 2, \ldots$ there exists at
we prove that for any \( (B, \lambda, E) \), such that for every \( \mathcal{X} \in C_{C^1}(\mathbb{R}) \) with \( \mathcal{X} \equiv 1 \) on some open interval \( J \ni E_n(B, \lambda) \) and \( p > 0 \), we have
\[
\mathcal{M}_{B, \lambda}(p, \mathcal{X}, T) \geq C_{p, \mathcal{X}} T^{\frac{p}{p} - 6},
\]
for all \( T \geq 0 \) with \( C_{p, \mathcal{X}} > 0 \).

As mentioned above, to prove Theorem 1.1, we extend the Wegner estimate of [CHK2] to measures \( \mu \) with unbounded support. More precisely, the finite volume operator \( H_\lambda^{(\Lambda)} \) satisfies extensions of the Wegner estimates of [CH, CHK1, CHK2]. As in [CHK2], we do not require the probability measure \( \mu \) to have a density. Precise statements and proofs are given in Appendix B.

2. Hall conductance and dynamical delocalization

We start by introducing some notation. Given \( p \in [1, \infty) \), \( T_p \) will denote the Banach space of bounded operators \( S \) on \( L^2(\mathbb{R}^2, dx) \) with \( \|S\|_{T_p} = \|S\|_p \equiv \langle \text{tr}|S|^p \rangle^{\frac{1}{p}} < \infty \). A random operator \( S_\omega \) is a strongly measurable map from the probability space \((\Omega, \mathcal{F})\) to bounded operators on \( L^2(\mathbb{R}^2, dx) \). Given \( p \in [1, \infty) \), we set
\[
\|S_\omega\|_p \equiv \left\{ \mathbb{E}\left\{ \|S_\omega\|_p^p \right\} \right\}^{\frac{1}{p}} = \|S_\omega\|_{L^p(\Omega, \mathcal{F})},
\]
and
\[
\|S_\omega\|_\infty \equiv \|S_\omega\|_{L^\infty(\Omega, \mathcal{F})}.
\]

We define the \((B, \lambda, E)\) parameter set by
\[
\Xi = \{(0, \infty) \times [0, \infty) \times \mathbb{R}\} \setminus \bigcup_{B \in (0, \infty)} \{(B, 0) \times \sigma(H_B)\};
\]
that is, we exclude the Landau levels at no disorder. We set
\[
P_{B, \lambda, E, \omega} = \chi_{\{\mathcal{X}\} = \infty, E} (H_{B, \lambda, \omega}).
\]

The Hall conductance \( \sigma_H(B, \lambda, E) \) is given by (e.g. [BeES, AvSS, AG, BoGKS])
\[
\sigma_H(B, \lambda, E) = -2\pi i \mathbb{E} \{ \text{tr} \{ \chi_0 P_{B, \lambda, E, \omega} \left( [P_{B, \lambda, E, \omega}, X_1] + [P_{B, \lambda, E, \omega}, X_2] \right) \chi_0 \} \}, \tag{2.3}
\]
defined for \((B, \lambda, E) \in \Xi\) such that
\[
\|\langle \chi_0 P_{B, \lambda, E, \omega} \left( [P_{B, \lambda, E, \omega}, X_1] + [P_{B, \lambda, E, \omega}, X_2] \right) \chi_0 \rangle\|_1 < \infty. \tag{2.4}
\]

\(X_i\) denotes the operator given by multiplication by the coordinate \( x_i, i = 1, 2\), and \(|X|\) the operator given by multiplication by \(|x|\). In particular, \( \sigma_H(B, \lambda, E) \) is well-defined for all \((B, \lambda, E)\) such that \( E \in \Xi_{B, \lambda}^{DL}\). Moreover it is proved in [GKS2] that \( \sigma_H(B, \lambda, E) \) is integer valued for all \((B, \lambda, E)\) such that \( E \in \Xi_{B, \lambda}^{DL}\). We need to investigate the continuity properties of \( \sigma_H(B, \lambda, E) \), as \( \lambda \) tends to zero. In [GKS2] we prove that for any \((B, \lambda, E)\) such that \( E \in \Xi_B^{DL}\), for any \( p > 1 \), there exists a constant \( C(p, B, \lambda, E) \) such that \( \sigma_H(B', \lambda', E', \omega) \leq C(p, B, \lambda, E) \sigma_H(B, \lambda, E) \).

We shall combine this fact with the following proposition, a consequence from Theorem 1.2 which includes an extension of [HK3] to unbounded random variables.
Proposition 2.1. Let $I$ be an open interval in a spectral gap of $H_B$. Then for all $\lambda \geq 0$ the Hall conductance is Hölder continuous in $E \in I$, and for any $E \in I$ the Hall conductance at Fermi energy $E$ is Hölder continuous in the disorder parameter $\lambda \geq 0$.

Proof. The proposition is a direct consequence of Theorem (2.2) and (2.6).

Proof of Theorem (1.7). We set

$$L_B = K_B \sqrt{\frac{4\pi}{B}}, \quad N_B = L_B \mathbb{N}, \quad \text{and} \quad Z_B^2 = L_B Z^2.$$  \hfill (2.6)

Note that $L_B \geq 1$ may not be an integer. We consider squares $A_L(0)$ with $L \in \mathbb{N}_B$ and identify them with the torii $T_L := \mathbb{R}^2 / (L \mathbb{Z})$ in the usual way. We further let $\tilde{A}_L(x) = \mathbb{Z}^2 \cap A_L(x)$. Given $L \in \mathbb{N}_B$ we define finite volume Landau Hamiltonians $H_{B,0,L}$ on $L^2(\Lambda_L(0))$ as in [GKS1] Section 5, and set

$$H_{B,\lambda,0,L,\omega} = H_{B,0,L} + \lambda V_{0,L,\omega} \quad \text{on} \quad L^2(\Lambda_L(0)),$$

$$V_{0,L,\omega}(x) = \sum_{i \in \tilde{A}_L - \delta_\alpha(0)} \omega_i u(x - i),$$ \hfill (2.7)

It follows from (1.7) that

$$\mu(\{|u| \geq \varepsilon\}) \leq C_\alpha \exp\left(-\frac{1}{2} |\varepsilon|^\alpha\right) \quad \text{for all} \quad \varepsilon > 0.$$ \hfill (2.8)

Let $\tilde{L} \in \mathbb{N}_B$ (see (2.6)), and let $H_{B,\lambda,0,L,\omega}$ and $V_{0,L,\omega}$ be as in (2.7). A straightforward computation shows that uniformly in $\lambda \in [0,1]$,\textcolor{red}{See the revised version for the correct formula and explanation.}

$$\mathbb{P}\left\{\sigma(H_{B,\lambda,0,L,\omega}) \subset \bigcup_{n=1}^\infty |B_n - \lambda \varepsilon, B_n + \lambda \varepsilon|\right\} \geq \mathbb{P}\left\{|\omega_i| \leq \varepsilon \quad \text{if} \quad i \in \tilde{A}_L - \delta_\alpha(0)\right\}$$

\[\geq (1 - C_\alpha \exp\left(-\frac{1}{2} |\varepsilon|^\alpha\right))(L - \delta_\alpha)^2 \geq 1 - C_2 C_\alpha \exp\left(-\frac{1}{2} |\varepsilon|^\alpha\right) L^2.\] \hfill (2.9)

We now apply the finite volume criterion for localization given in [GK2 Theor. 2.4], in the same way as in [GK2 Proof of Theorem 3.1], with parameters (we fix $q \in [0,1]$) $\eta_{B,1,q} = \frac{1}{2} \eta_{B,1,L,q}$ and $Q_{B,1,L} \leq \tilde{Q}$, for some $\tilde{Q} < \infty$ independent of $\lambda \in [0,1]$. As it follows from Theorem [GK2] (Note that the fact that we work with length scales $L \in \mathbb{N}_B$ instead of $L \in 6 \mathbb{N}$ only affects the values of the constants in [GK2 Eqs. (2.16) -(2.18)].)

To conduct the multiscale analysis of [GK1, GK2], we note that in finite volume we have, for any given $\eta < 1$, and uniformly in $\lambda \in [0,1]$,\textcolor{red}{See the revised version for the correct formula and explanation.}

$$\mathbb{P}\left\{|\lambda V_{\omega}(x)| \leq \varepsilon L^\eta, \text{for all} \quad x \in \Lambda_L(y)\right\} \leq \mathbb{P}\left\{|V_{\omega}(x)| \leq \varepsilon L^\eta, \text{for all} \quad x \in \Lambda_L(y)\right\} \geq 1 - C_\alpha \exp\left(\frac{1}{2} L^{\eta_0}\right) L^2,$$ \hfill (2.10)

which is as close to 1 as wanted, provided $L$ is large enough (independently of $\lambda$). Probabilistic bounds on the constant in SLI and EDI follow, with constants bounded by $L^{\eta/2}$. Since we are working in spectral gaps, we use the Combes-Thomas estimate of [BCH] Proposition 3.2) (see also [KK1] Theorem 3.5)—its proof, based on [BCH] Lemma 3.1], also works for Schrödinger operators with magnetic fields), adapted to finite volume as in [GK2 Section 3].

Now fix $n \in \mathbb{N}$, take $I = I_n(B)$, and set $B = \tilde{L}(n,B)$ to be the smallest $L \in \mathbb{N}_B$ satisfying (GK2 Eq. (2.16)). Let $E \in I_n(B)$, $|E - B_n| \geq 2\lambda \varepsilon$, where $\varepsilon = \ldots$. \hfill \boxed
ε(n, B, λ) > 0 will be chosen later. Then, using (2.9) and the Combes-Thomas estimate, we conclude that condition [GK2, Eq. (2.17)] will be satisfied at energy E if

\[ \varepsilon \geq C_3 (\log L)^{\frac{1}{4}}, \]  

(2.13)

\[ C_4 (\lambda \varepsilon)^{-1} L^n e^{-C_5 \sqrt{\lambda L}} < 1, \]

(2.14)

for appropriate constants \( C_j = C_j(n, B), j = 3, 4, 5 \), with \( C_5 > 0 \). This can be done by choosing (in view of (2.9))

\[ \varepsilon = C_3 (\log L)^{\frac{1}{4}}, \]

(2.15)

and taking \( L \) large enough to satisfy (2.14) depending on \( \lambda \leq 1 \). We conclude from [GK2, Theorem 2.4] that

\[ \{ E \in \mathcal{I}_n(B); |E - B_n| \geq C_5 \lambda |\log \lambda|^{\frac{1}{2}} \} \subset \Xi_{B,\lambda}^{DL}. \]

(2.16)

for all \( \lambda \leq 1 \). In particular, for all \( n \in \mathbb{N} \) there is \( \lambda_n > 0 \) such that \( B_n - B \in \Xi_{B,\lambda}^{DL} \) for all \( \lambda \in [0, \lambda_n] \).

The existence at small disorder of dynamical mobility edges \( \bar{E}_{j,n}(B, \lambda), j = 1, 2, \) satisfying (1.10) now follows from [GKS2] and (2.16). Indeed, since \( B_n - B \in \Xi_{B,\lambda}^{DL} \) for all \( \lambda \in [0, \lambda_n] \), the Hall conductance is constant at energy \( B_n - B \) for all \( \lambda \in [0, \lambda_n] \). Since for \( \lambda = 0 \), its value is \( n - 1 \), we can conclude that there is an energy of delocalization between \( B_n - B \) and \( B_n + B = B_{n+1} - B \) for all \( \lambda \in [0, \min \{ \lambda_n, \lambda_{n+1} \}] \). Then (2.13) and the constancy of the Hall conductance on sub-intervals of \( \Xi_{B,\lambda}^{DL} \) imply the estimate (1.10).

\[ \Box \]

**APPENDICES**

In these appendices we extend results known for Anderson-type random Schrödinger operator to unbounded random variables. These appendices are of separate interest and independent of the rest of the paper.

We consider a random Schrödinger operator of the form \( H_{\lambda,\omega} = H_0 + \lambda V_\omega \) on \( L^2(\mathbb{R}^d, dx) \), where the random potential \( V_\omega \) is as in (1.3) and \( \lambda \geq 0 \). The unperturbed Hamiltonian \( H_0 \) will be either the Landau Hamiltonian \( H_B \) on \( L^2(\mathbb{R}^2, dx) \), as in (1.2), or it will have the general form \( H_0 = (-i \nabla - A_0)^2 + V_0 \) on \( L^2(\mathbb{R}^d, dx) \), \( d \in \mathbb{N} \), where both \( A_0 \) and \( V_0 \) are regular enough so that \( H_0 \) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^d) \) and bounded from below by some constant \( \Theta \in \mathbb{R} \). As a sufficient condition, it is enough to require that the magnetic potential \( A_0 \) and the electric potential \( V_0 \) satisfy the Leinfelder-Simader conditions (cf. [BoGKS]):

- \( A_0(x) \in L^4_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) \) with \( \nabla \cdot A_0(x) \in L^2_{\text{loc}}(\mathbb{R}^d) \).
- \( V_0(x) = V_{0+}(x) - V_{0-}(x) \) with \( V_{0+}(x) \in L^2_{\text{loc}}(\mathbb{R}^d), V_{0+}(x) \geq 0, \) and \( V_{0-}(x) \) relatively bounded with respect to \( \Delta \) with relative bound \( < 1 \), i.e.,

\[ \| V_{0-} \psi \| \leq \alpha \| \Delta \psi \| + \beta \| \psi \| \quad \text{for all } \psi \in \mathcal{D}(\Delta). \]

We will say that \( H_0 \) is periodic if \( A_0 \) and \( V_0 \) are \( \mathbb{Z}^d \)-periodic. It has the property (UCP) if it satisfies the unique continuation principle. \( H_0 \) has the (UCP) if \( A_0 \) and \( V_0 \) are sufficiently regular; see the discussion in [CHKT].
Appendix A. Applicability of the multiscale analysis

We provide here estimates that are needed for extending the multiscale analysis, more precisely results of [GK1, GK2, GK4, GK5, GKS1, GKS2], from bounded to unbounded random variables, as mentioned in the introduction. Finite volume operators are as defined in those papers. We fix the disorder \( \lambda \geq 0 \) and omit it from the notation. Note that the constants are all uniform in \( \lambda \leq \lambda_0 \).

Lemma A.1. Given a box \( \Lambda \), there exists \( L^* \), such that for any \( L \geq L^* \) we have, for any \( \beta > \alpha^{-1} \),
\[
\mathbb{P}\{ \| \chi_{\Lambda}\omega \|_\infty \leq C_+ (\log L)^{\beta} \} \geq 1 - C(\alpha, \delta_+, d) \rho_0 \exp(-C(\alpha, \beta, \delta_+, d)|\log L|^{\alpha \beta}).
\]
(A.1)

Then for \( \mathbb{P} \)-a.e. \( \omega \) we have
\[
V_\omega(x) \geq -c_\omega(\log(x))^\beta \quad \text{for all} \quad x \in \mathbb{R}^d,
\]
(A.2)

where \( c_\omega > 0 \) (depending also on \( d, \alpha, \beta \)). As a consequence \( H_\omega \) satisfies the lower bound
\[
H_\omega \geq -c_\omega(\log(x))^\beta, \quad \text{for all} \quad x \in \mathbb{R}^d,
\]
(A.3)

for any given \( \beta > \alpha^{-1} \) and is essentially self-adjoint on \( C_\infty^c(\mathbb{R}^d) \) with probability one.

Proof. To get (A.1), we note that
\[
\mathbb{P}\{ \| \chi_{\Lambda}\omega \|_\infty \leq C_+ (\log L)^{\beta} \} \geq 1 - C(2L)^d \mathbb{P}\{ |\omega| \geq (\log L)^{\beta} \}.
\]
(A.4)

The bound (A.2), then follows from the Borel-Cantelli Lemma. Now in view of (A.2), \( H_{B,\omega} \) satisfies the lower bound (A.3) and thus \( H_\omega \) is essentially self-adjoint on \( C_\infty^c(\mathbb{R}^d) \) with probability one by the Paris-Levine Theorem [RS, Theorem X.38]. □

Bounds on the constant in SLI and EDI follow from (A.1). GEE follows from heat kernel estimates, as given in [BrLM]. As for SGEE, the bound has been derived by Ueki [U] for Gaussian random variables. For the reader’s convenience we provide a short proof in the next theorem. Recall that \( H_0 \geq \Theta \). We write \( E_{H_\omega}(I) = \chi_I(H_\omega) \).

Theorem A.2. There exist \( m(d) > 0 \) such that if \( \mathbb{E}(|\omega_0|^{m(d)+\alpha}) < \infty \), with \( \alpha \geq 0 \), then for any bounded interval \( I \) we have
\[
\mathbb{E}\{ |\omega_0|^{\alpha} \quad \text{tr} \quad \chi_0 E_{H_\omega}(I)} \chi_0 \} \leq C(H_0, d, I, \alpha)
\]
(A.5)

for some constant \( C(H_0, d, I, \alpha) < \infty \). Moreover, \( m(1) = 1 \) and \( m(2) = 2 \) for \( d = 2, 3 \).

Proof. For simplicity, we assume that the support of \( u_0 \) is included in the unit cube centered at the origin. If not, straightforward modifications of the argument (as in [CHK2]) yield the result as well. We write \( H = H_\omega = H_0 + V_\omega \), with \( H_0 \) bounded from below, say \( H_0 \geq 0 \). We denote by \( E \) the center of the interval \( I \). We set \( \tilde{I} \) to be the interval \( I \) but enlarged by a distance \( d := 2|I| \) from above and below: \( I \subset \tilde{I} \) and dist\( (I, \tilde{I}) = d \). We have
\[
\text{tr} \chi_0 E_H(I) = \text{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I}) + \text{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I})
\]
(A.6)
\[
\leq C(|E| + 3|I|^d) + \text{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I}).
\]
(A.7)
Now, with $R_0(z) = (H_0 - z)^{-1}$,
\[
\text{tr} \chi_0 E_H(I) E_{H_0}(\tilde{F}) = \text{tr} \chi_0 E_H(I)(H_0 - E - V_\omega)R_0(E)E_{H_0}(\tilde{F}) \quad (A.8)
\]
\[
\leq \frac{|I|}{d} \text{tr} \chi_0 E_H(I)\chi_0 + |\text{tr} \chi_0 E_H(I)V_\omega R_0(E)E_{H_0}(\tilde{F})\chi_0| \quad (A.9)
\]
\[
\leq \frac{1}{2} \text{tr} \chi_0 E_H(I)\chi_0 + \sum_{j \neq 0} ||\omega_j u_j R_0(E)E_{H_0}(\tilde{F})\chi_0||_1 \quad (A.10)
\]
\[
+ ||\omega|| \text{tr} \chi_0 E_H(I)u_0 R_0(E)E_{H_0}(\tilde{F})\chi_0|, \quad (A.11)
\]
so that, for $p > d$ given, taking advantage of $u_j \chi_0 = 0$ if $j \neq 0$ (use Helffer-Sjöstrand formula plus resolvent identities to get trace class operators),
\[
\sum_{j \neq 0} ||\omega_j u_j R_0(E)E_{H_0}(\tilde{F})\chi_0||_1 \leq \mathbb{E}|\omega_0| \sum_{j \neq 0} C_p (1 + |j|)^{-p}. \quad (A.12)
\]
Next, if $d = 1$ then $u_0 R_0(E)E_{H_0}(\tilde{F})$ is trace class, and $\mathbb{E}|\omega_0| < \infty$ is a sufficient condition. If $d = 2, 3$ (in the present application $d = 2$), then Cauchy-Schwartz inequality leads to
\[
|\text{tr} \chi_0 E_H(I)\omega_0 u_0 R_0(E)E_{H_0}(\tilde{F})| \leq ||\chi_0 E_H(I)||_2 ||\omega_0 R_0(\Theta - 1)\chi_0||_2 ||(H_0 + \Theta + 1)R_0(E)E_{H_0}(\tilde{F})||_\infty \quad (A.13)
\]
\[
\leq \left(1 + \frac{|E| + |\Theta| + 1}{d} \right) ||\chi_0 E_H(I)||_2 ||\omega_0 R_0(\Theta - 1)\chi_0||_2 \quad (A.14)
\]
\[
\leq \frac{1}{4} \text{tr} \chi_0 E_H(I) + \left(1 + \frac{|E| + |\Theta| + 1}{d} \right)^2 \omega_0^2 \text{tr} \chi_0 R_0(\Theta - 1)^2. \quad (A.15)
\]
The latter trace is finite in dimension $d = 2, 3$, finishing the proof provided $\mathbb{E}\omega_0^2 < \infty$. In higher dimensions, one repeats the very last step as many times as necessary, as in [CHK2].

**Appendix B. Optimal Wegner estimate with unbounded random variables**

In this appendix we extend the analyses of [CHK2] and [HiKS] to unbounded random variables.

Given a finite box $\Lambda \subset \mathbb{R}^d$, we denote by $H_{\lambda,\omega}^{(\Lambda)}$ an appropriate self-adjoint restriction of $H_{\lambda,\omega}$ to $\Lambda$, in which case $H_{\lambda,\omega}^{(\Lambda)}$ has a compact resolvent (see [CHK1] [CHK2] [GKS1]). There is no other restriction on the boundary condition in Theorem [B.1](b),(c) below. When we use the (UCP) for $H_0$ periodic, as in Theorem [B.1](a), we assume periodic boundary condition as in [CHK2]. If $H_0 = H_B$, the Landau Hamiltonian, in Theorem [B.1](a) we assume finite volume operators as defined in [GKS1] Section 4) and used in [CHK2] Section 4).

If $\Delta$ is a Borelian, $E_{H_{\lambda,\omega}^{(\Lambda)}}(\Delta)$ denotes the associated spectral projection for $H_{\lambda,\omega}^{(\Lambda)}$.

In this appendix we assume $0 \leq \lambda \leq 1$ since we are mostly interested in small values of the coupling constant, but arguments easily extend to $\lambda \leq \lambda_0$ for any given $\lambda_0$. 

□
Given an arbitrary Borel measure $\nu$ on the real line, we set $Q_\nu(s)$ to be a multiple of its concentration function:

$$Q_\nu(s) := 8 \sup_{a \in \mathbb{R}} \nu([a, a + s]) \quad (B.1)$$

Note that $Q_\nu(s) < \infty$ if $\nu$ is a finite measure. The Wegner estimate in [CHK2] is stated in terms of $Q_\nu$; in our extension to unbounded measures $Q_\mu$ is replaced by $Q_{\mu^{(a)}}$, for an appropriate $a \geq 1$, where $d\mu^{(a)}(s) := |s|^a d\mu(s)$ for $q > 0$.

**Theorem B.1.** Consider $H_{\lambda, \omega}$ with $0 < \lambda \leq 1$. There exists $1 \leq m(d) < \infty$, such that if $E \{ |\omega_0|^{m(d)} \} < \infty$, given $E_0 \in \mathbb{R}$:

(a) Assume either $H_0 = H_B$ or $H_0$ is periodic with the (UCP). Then there exists a constant $K_W(\lambda)$, depending also on $d$, $E_0$, $\delta_\pm$ and $C_\pm$, such that for any compact interval $\Delta \subset - \infty, E_0[$ we have

$$E \left\{ \operatorname{tr} E_{H_{\lambda, \omega}}^{(\lambda)}(\Delta) \right\} \leq K_W(\lambda)Q_{\mu^{(m(d)}}(|\Delta||\Delta|). \quad (B.2)$$

(b) Assume the IDS of $H_0$ is Hölder continuous with exponent $\delta > 0$ in some open interval $\Delta_0 \subset - \infty, E_0[$, then there exists a constant $K_W$ depending on $d$, $E_0$, $\delta_\pm,C_\pm$, such that for any $\lambda \leq 1$, $\Delta \subset \Delta_0$ compact, $|\Delta|$ small enough, and any $0 < \gamma < 1$,

$$E \left\{ \operatorname{tr} E_{H_{\lambda, \omega}}^{(\lambda)}(\Delta) \right\} \leq K_W \max \left( |\Delta|^{\delta \gamma}, |\Delta|^{-\gamma m(d)}Q_{\mu^{(m(d)}}(|\Delta|) \right) |\Delta|. \quad (B.3)$$

In particular, if $Q_{\mu^{(m(d)}}(\varepsilon) \leq C\varepsilon^\lambda$, for some $\varepsilon \in [0,1]$, then

$$E \left\{ \operatorname{tr} E_{H_{\lambda, \omega}}^{(\lambda)}(\Delta) \right\} \leq K_W |\Delta|^{\frac{\lambda}{\gamma m(d)}} |\Delta|. \quad (B.4)$$

(c) Assume $E \in \Delta_0 \subset (\mathbb{R} \setminus \sigma(H_0)) \cap - \infty, E_0[$, $\Delta_0$ compact, then there exists a constant $K_W$, depending on $d$, $E_0$, $\delta_\pm,C_\pm$ and $\Delta_0$, such that for any $\lambda \leq 1$ and any $\Delta \subset \Delta_0$ centered at $E$, $|\Delta|$ small enough,

$$E \left\{ \operatorname{tr} E_{H_{\lambda, \omega}}^{(\lambda)}(\Delta) \right\} \leq K_W \lambda Q_{\mu^{(m(d)}}(|\Delta|) |\Delta|. \quad (B.5)$$

We adapt the proof of [CHK2], using the basic spectral averaging estimate proved in [CHK2]: Let $H_0$ and $W$ be self-adjoint operators on a Hilbert space $\mathcal{H}$, with $W \geq 0$ bounded. Let $H_s := H_0 + sW$ for $s \in \mathbb{R}$. Then, given $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, for all Borel measures $\nu$ on $\mathbb{R}$ and all bounded intervals $I \subset \mathbb{R}$ we have ([CH] Corollary 4.2), [CHK2] Eq. (3.16)\footnote{The estimate \eqref{B.6} is stated with $W$ instead of $\sqrt{W}$, with the additional hypothesis that $W \leq 1$. But a careful reading of their proof shows that they actually prove \eqref{B.6} as stated here.}

$$\int d\nu(s) \langle \varphi, \sqrt{W} \chi_I(H_s) \sqrt{W} \varphi \rangle \leq Q_\nu(|I|). \quad (B.6)$$

The result is stated in [CHK2] for a probability measure $\nu$ with compact support, but their proof works for an arbitrary Borel measure $\nu$. In particular, for $H_\omega$ as in Theorem B.1 we get, for any $\phi \in L^2(\mathbb{R}^d)$, $j \in \mathbb{Z}^d$, $\alpha > 0$, and any interval $I_\varepsilon$ of length $\varepsilon > 0$,

$$\mathbb{E} \{ |\omega_j|^{\alpha} \phi, j \delta_j \} \leq Q_{\mu^{(m(d)}}(\varepsilon) \|\phi\|^2. \quad (B.7)$$
As a consequence, for any trace class operator $S \geq 0$,
\[
E \left\{ |\omega_j|^{\alpha} \text{ tr} \left\{ \sqrt{\tau_j} E_{H_{\lambda}(\Delta)}(I_{\epsilon}) \sqrt{\tau_j} S \right\} \right\} \leq \frac{1}{\epsilon} (\text{tr} S) Q_{\mu, \alpha}(\epsilon). \tag{B.8}
\]

Proof of Theorem B.7. Recall that $H_{\lambda, \omega} = H_0 + \lambda \mathcal{V}_{\omega}$, $\lambda \in [0, 1]$, and to alleviate notations we write $E_{\lambda}(\Delta) := E_{H_{\lambda, \omega}(\Delta)}$ and $E_{\lambda}^{0}(\Delta) := E_{H_{\lambda, \omega}^{0}(\Delta)}$. To simplify the exposition we assume that the support of $u$ is smaller than the unit cube; if not the case, the proof can be modified in a straightforward way, as in [CHK2]. In particular, $u_i u_j = 0$ if $i \neq j$. We also introduce $\chi$ to be the characteristic function of a cube containing the support of $u$, contained in the unit cube, such that $\chi_i \chi_j = 0$ if $i \neq j$, where $\chi_j(x) = \chi(x-j)$. With $\Delta \subset \Delta$, and denoting $d_{\Delta} = \text{dist}(\Delta, \Delta^c)$, we get
\[
\text{tr}(E_{\lambda}(\Delta)) = \text{tr}(E_{\lambda}(\Delta) E_{\lambda}^{0}(\Delta)) + \text{tr}(E_{\lambda}(\Delta) E_{\lambda}^{0}(\Delta^c)). \tag{B.9}
\]

We first consider the term $\text{tr}(E_{\lambda}(\Delta) E_{\lambda}^{0}(\Delta^c))$ and take care of the unboundedness of the random variables. We have,
\[
\text{tr}(E_{\lambda}(\Delta) E_{\lambda}^{0}(\Delta^c)) \leq C_d(\Delta) \lambda^2 \sum_{i,j \in \Lambda} |\omega_j| |\text{tr}(u_j E_{\lambda}(\Delta) u_i K_{ij})| \tag{B.10}
\]
\[
\leq C_d(\Delta) \lambda^2 \sum_{i,j \in \Lambda, i \neq j} |\omega_i| |\omega_j| |\text{tr}(u_j E_{\lambda}(\Delta) u_i K_{ij})| \tag{B.11}
\]
\[
+ C_d(\Delta) \lambda^2 \sum_{i \in \Lambda} |\omega_i|^2 |\text{tr}(u_i E_{\lambda}(\Delta) u_i K_{ii})| \tag{B.12}
\]
where
\[
K_{ij} = \chi_j (H_{0}^{\lambda} + M)^{-2} \chi_j, \tag{B.13}
\]
and
\[
\left\| \left( \frac{H_{0}^{\lambda} + M}{H_{0}^{\lambda} - \mathcal{E}_m} \right)^{2} E_{\lambda}^{0}(\Delta^c) \right\| \leq \left( 1 + \frac{2(M + \Delta_{\lambda})}{d_{\Delta}} + \frac{(M + \Delta_{\lambda})^2}{d_{\Delta}^2} \right) = C_d(\Delta) \tag{B.14}
\]
for some $M < \infty$ such that $H_0 + M \geq 1$, for example $M = 1$ is enough, and where the $\chi_i, \forall i \in \mathbb{Z}^d$ are compactly supported functions, with support slightly larger than the $u_i$’s one such that $\chi_i u_i = u_i$. Note that $K_{ij}$ is trace class as soon as $i \neq j$ (since we assume $\text{supp } u_j \subset \Lambda_{ij}(j)$), as can be seen by a successive use of the resolvent identity, and by Combes-Thomas its trace class norm satisfies $||K_{ij}||_1 \leq C_d e^{-|x-j|}$, for $i \neq j$. It follows, as in [CGK] Eqs (4.1)-(4.4), that
\[
\sum_{i \neq j} |\omega_i| |\omega_j| |\text{tr}(u_j E_{\lambda}(\Delta) u_i K_{ij})| \tag{B.15}
\]
\[
\leq \sum_{i \neq j} \frac{1}{2} \left( |\omega_i|^2 |\text{tr}(u_i E_{\lambda}(\Delta) u_j K_{ij})| + |\omega_j|^2 |\text{tr}(u_j E_{\lambda}(\Delta) u_j K_{ij}^*)| \right) \tag{B.16}
\]
\[
= \sum_{i} |\omega_i|^2 |\text{tr}(u_i E_{\lambda}(\Delta) u_i S_j)|, \tag{B.17}
\]
where
\[
S_j = \frac{1}{2} \sum_{i \neq j} (|K_{ij}| + |K_{ij}^*|) \geq 0, \tag{B.18}
\]
with
\[
\max_{j \in \Lambda} \text{tr} S_j \leq Q_2 < \infty. \tag{B.19}
\]
It remains to consider the diagonal term $i = j$, that is $|\omega|^2 \text{tr}(u_i E_{\lambda}(\Delta) u_i \tilde{K}_n)$. Note that $K_n$ is trace class in dimension $d = 1, 2, 3$ but not higher. To deal with the general case of arbitrary dimension we proceed as in [CHK2] and perform successive Cauchy-Schwartz inequalities, getting, for any integer $m \geq 1$, for some constant $K_{d,m} < \infty$,

$$C_d(|\omega|^2 \text{tr}(u_i E_{\lambda}(\Delta) u_i \tilde{K}_n) \leq \frac{1}{4} \text{tr}(u_i E_{\lambda}(\Delta) u_i) + K_{d,m}(C_d(|\omega|^2)^m \text{tr}(u_i E_{\lambda}(\Delta) u_i \tilde{K}_n^{2^m-1}). \quad (B.20)$$

We chose $m$ so that $K_n^{2^m-1}$ is trace class, that is, we take $m(d) := 2^m+1 > d$, i.e., $m = \lfloor \log d / \log 2 \rfloor$, where $\lfloor x \rfloor$ stands for the integer part of $x$. It follows that, using $\sum_j u_j \leq 1$, uniformly in $\lambda \leq 1$,

$$\text{tr}(E_{\lambda}(\Delta) E_{\lambda}^\Delta(\tilde{\Delta}^\ell)) \leq \frac{1}{4} \sum_i \text{tr}(u_i E_{\lambda}(\Delta) u_i) + K_{d,m(d)} \lambda^2 \sum_i (C_d(|\omega|^2)^m \text{tr}(u_i E_{\lambda}(\Delta) u_i \tilde{S}_i) \quad (B.22)$$

$$\leq \frac{1}{4} \text{tr} E_{\lambda}(\Delta) + K_{d,m(d)} \lambda^2 \sum_i \left( \frac{|\omega|}{d\Delta} \right)^m \text{tr}(u_i E_{\lambda}(\Delta) u_i \tilde{S}_i), \quad (B.23)$$

where

$$\tilde{S}_i = S_i + K_n^{2^m-1} \geq 0, \quad (B.24)$$

is a trace class operator. We apply (B.23) to finish the bound:

$$\mathbb{E} \text{tr}(E_{\lambda}(\Delta) E_{\lambda}^\Delta(\tilde{\Delta}^\ell)) \leq \frac{1}{4} \mathbb{E} \text{tr} E_{\lambda}(\Delta) + C_A \lambda^2 \frac{Q_{\mu(m(d))}}{d\lambda^{-m(d)}(|\Delta|/\lambda| \lambda. \quad (B.25)$$

We now turn to the first term of the right hand side in (B.24), that is $\text{tr}(E_{\lambda}(\Delta) E_{\lambda}^\Delta(\tilde{\Delta})).$ To get the general Wegner estimate [CHK2], the latter is treated as in [CHK2], using either the unique continuation principle for the free Hamiltonian, or, in the Landau case, explicit properties of the Landau Hamiltonian. Note that we then incorporate $d\Delta$ into the constant. To get (B.26), we control $\text{tr}(E_{\lambda}(\Delta) E_{\lambda}^\Delta(\tilde{\Delta}))$ using the hypothesis on the IDS of $H_0$, that is $\text{tr} E_{\lambda}(\Delta) \leq C |\Delta|^{\beta} |\lambda|$. In this case, we need $d\Delta$ to be small enough and it then remains to control the growth of the constant in the second term of the r.h.s. of (B.24). Taking $d\Delta = \varepsilon^\gamma$, with $0 < \gamma < 1$, and using $Q_{\mu(m(d))}(|\Delta|) \leq C \varepsilon^\zeta$ if $\mu$ is $C_\zeta$-Hölder continuous, we get, with a new constant $K_{\varepsilon}$, and $\varepsilon$ small enough so that $\Delta \subset \Delta_0$,

$$\mathbb{E} \text{tr} E_{\lambda}(\Delta) \leq K_{\varepsilon} \max \left( \eps^{\gamma \delta}, \frac{1}{\varepsilon^{m(d)}} Q_{\mu(m(d))}(\varepsilon) \right) |\lambda| \quad (B.27)$$

$$\leq K_{\varepsilon} \max \left( \varepsilon^{m(d) \varepsilon^{\gamma \delta} \varepsilon^{-m(d) \gamma}} \right) |\lambda| \quad (B.28)$$

$$\leq K_{\varepsilon} \varepsilon^{\gamma \delta + m(d) \gamma} |\lambda| \quad (B.29)$$

where we have chosen $\gamma$ such that $\gamma \delta = \varepsilon - m(d) \gamma$.

Finally, in the particular case of (B.23), $\text{tr} E_{\lambda}(\Delta) = 0$ as long as $\Delta \subset \Delta_0$. \quad \Box

The following theorem contains an extension of [HiKS] to unbounded random variables. We set, for $E \in \mathbb{R}$, $P_{\lambda, E; \omega} = \chi_{|E| < \infty, E \in \mathcal{H}_{\lambda, \omega}}$, the Fermi projection.
Theorem B.2. Consider $H_{\lambda, \omega}$ with $0 < \lambda \leq 1$. Assume that the IDS of $H_0$ is H"older continuous in $E \in \Delta_0$ an open interval. Then for some $\nu > 0$ and $C_{\Delta_0} < \infty$, for any $E, E' \in \Delta_0$, $|E - E'|$ small enough, we have uniformly in $0 \leq \lambda \leq 1$,

$$\max_{u \in \mathbb{Z}^d} E \left\{ \| \chi_0 \left( P_{\lambda, E, \omega} - P_{\lambda, E', \omega} \right) \chi_u \|_1 \right\} \leq C_{\Delta_0} |E - E'|^{\nu},$$

(B.30)

and for some $\nu' > 0$, for all $E \in \Delta_0$, for all $\lambda', \lambda'' \in [0, 1]$, $|\lambda'' - \lambda'|$ small enough,

$$\max_{u \in \mathbb{Z}^d} E \left\{ \| \chi_0 \left( P_{\lambda', E, \omega} - P_{\lambda'', E, \omega} \right) \chi_u \|_1 \right\} \leq C_{\Delta_0} |\lambda'' - \lambda'|^{\nu'}.$$  

(B.31)

Proof. Eq. (B.30) follows from Cauchy-Schwarz and the continuity of the Integrated Density of States of $H_{\lambda, \omega}$ given by Theorem B.1 Eq. (B.4). We turn to (B.31). Let $E \in \Delta_0$ and $\lambda', \lambda'' \in [\lambda_1, \lambda_2]$ possibly containing 0. We let $\gamma = |\lambda' - \lambda''|^{\alpha}$, where $\alpha \in (0, 1)$ will be chosen later. Let $f(t)$ be a smooth decaying switch function, equal to 1 for $t \leq 0$ and 0 for $t \geq 1$. We set $g(t) = f \left( \frac{t - (\con) - 1}{\gamma} \right)$; note $g \in C^\infty(\mathbb{R})$, with $0 \leq g(t) \leq 1$, $g(t) = 1$ if $t \leq -\gamma$, $g(t) = 0$ if $t \geq E$. We write

$$P_{\lambda', E, \omega} - P_{\lambda'', E, \omega} = \left\{ P_{\lambda', E, \omega} - g^2(H_{\lambda', \omega}) \right\}$$

(B.32)

$$+ \left\{ g^2(H_{\lambda', \omega}) - g^2(H_{\lambda'', \omega}) \right\} + \left\{ g^2(H_{\lambda'', \omega}) - P_{\lambda'', E, \omega} \right\}.$$

By construction, for any $\lambda' \geq 0$ we have

$$0 \leq P_{\lambda, E, \omega} - g^2(H_{\lambda, \omega}) \leq P_{\lambda, E, \omega} - P_{\lambda, E - \gamma, \omega},$$

(B.33)

and thus, for $\lambda'' = \lambda'$, $\lambda''$ and any $u \in \mathbb{Z}^d$, we have

$$\left\| \chi_0 \left( P_{\lambda, E, \omega} - g^2(H_{\lambda, \omega}) \right) \chi_u \right\|_1$$

(B.34)

$$\leq \left\| \chi_0 \left( P_{\lambda, E, \omega} - g^2(H_{\lambda, \omega}) \right) \chi_u \right\|_1 \leq C_{\Delta_0} |\lambda'' - \lambda'|^{\nu'}.$$

To control the middle term in the r.h.s. of (B.32), we proceed as in [HKKS Eq. (3.8)] and sequel. In the Helffer-S"{o}jstrand formula, one needs to go to the $(4+2d)$th order. The term corresponding to [HKKS Eq. (3.15)] is controlled as follows (we denote by $R_{\lambda, \omega}(z)$ the resolvent of $H_{\lambda, \omega}$):

$$\| R_{\lambda, \omega}(z) V_{\omega} R_{\lambda', \omega}(z) V_{\omega} R_{\lambda, \omega}(z) \chi_0 \|$$

(B.35)

$$\leq \sum_{j, k \in \mathbb{Z}^d} |\omega_j \omega_k| \| R_{\lambda, \omega}(z) u_j R_{\lambda', \omega}(z) u_k R_{\lambda, \omega}(z) \chi_0 \|$$

(B.36)

$$\leq \sum_{j, k \in \mathbb{Z}^d} |\omega_j \omega_k| \| z \|^{-|\omega|} e^{-c|\omega| |j-k|} e^{-c|\omega| |k|}.$$  

(B.37)

It follows, using the Combes-Thomas inequality that

$$E \| \chi_0 g(H_{\lambda, \omega}) \|_1 \| R_{\lambda, \omega}(z) V_{\omega} R_{\lambda', \omega}(z) V_{\omega} R_{\lambda, \omega}(z) \chi_0 \|$$

(B.38)

$$\leq \sum_{j, k \in \mathbb{Z}^d} \left( E \| \omega_j \omega_k \| \| \chi_0 g(H_{\lambda, \omega}) \|_1 |z \|^{-|\omega|} e^{-c|\omega| |j-k|} e^{-c|\omega| |k|} \right)$$

(B.39)

$$\leq C(I, d)|\omega| \| z \|^{-|\omega| - 2d} \sum_{j, k \in \mathbb{Z}^d} e^{-c|\omega| |j-k|} e^{-c|\omega| |k|}$$

(B.40)

$$\leq C(I, d)|\omega| \| z \|^{-|\omega| - 2d}.$$  

(B.41)
by Theorem A.2. The term corresponding to [HiKS, Eqs. (3.16)-(3.18)] is controlled in a similar way using Theorem A.2.

\[ \square \]

REFERENCES

[AENSS] Aizenman, M., Elgart, A., Naboko, S., Schenker, J., Stolz, G.: Moment analysis for localization in random Schrödinger operators. Inv. Math. 163, 343-413 (2006)

[AG] Aizenman, M., Graf, G.M.: Localization bounds for an electron gas. J. Phys. A: Math. Gen. 31, 6783-6806, (1998)

[AvSS] Avron, J., Seiler, R., Simon, B.: Charge deficiency, charge transport and comparison of dimensions. Comm. Math. Phys. 159, 399-422 (1994).

[BCH] Barbaroux, J.M., Combes, J.M., Hislop, P.D.: Localization near band edges for random Schrödinger operators. Helv. Phys. Acta 70, 16-43 (1997)

[BeES] Bellissard, J., van Elst, A., Schulz-Baldes, H.: The non commutative geometry of the quantum Hall effect. J. Math. Phys. 35, 5373-5451 (1994).

[BoGKS] Bouclet, J.M., Germinet, F., Klein, A., Schenker, J.: Linear response theory for magnetic Schrödinger operators in disordered media. J. Funct. Anal. 226, 301-372 (2005)

[BrLM] Broderix, K., Leschke, H., Müller, P.: Continuous integral kernels for unbounded Schrödinger semigroups and their spectral projections. J. Funct. Anal. 212, 287-323 2004

[CGK] Combes, J.-M., Germinet, F., Klein, A.: Poisson Statistics for Eigenvalues of Continuous Random Schrödinger Operators. Preprint 2008.

[CH] Combes, J.M., Hislop, P.D.: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. 177, 603-629 (1996)

[CHK1] Combes, J.M., Hislop, P.D., Klopp, F.: Hölder continuity of the integrated density of states for some random operators at all energies. IMRN 4, 179-209 (2003)

[CHK2] Combes, J.M., Hislop, P.D., Klopp, F.: An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. Duke Math. J. 140, 469-498 (2007)

[GK1] Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in random media. Commun. Math. Phys. 222, 415-448 (2001)

[GK2] Germinet, F., Klein, A.: Explicit finite volume criteria for localization in continuous random media and applications. Geom. Funct. Anal. 13, 1201-1238 (2003)

[GK3] Germinet, F, Klein, A.: The Anderson metal-insulator transport transition. Contemporary Mathematics 339, 43-57 (2003)

[GK4] Germinet, F, Klein, A.: A characterization of the Anderson metal-insulator transport transition. Duke Math. J. 124 (2004)

[GK5] Germinet, F, Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. J. Stat. Phys. 122, 73-94 (2006)

[GKS1] Germinet, F, Klein, A., Schenker, J.: Dynamical delocalization in random Landau Hamiltonians. Annals of Math. 166, 215-244 (2007).

[GKS2] Germinet, F, Klein, A., Schenker, J.: Quantization of the Hall conductance and delocalization in ergodic Landau Hamiltonians. preprint 2008

[H] Halperin, B.: Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. Phys. Rev B 25, 2185-2190 (1982)

[HiKS] Hislop, P., Klopp, F., Schenker, J.: Continuity with respect to disorder of the integrated density of states, Illinois J. Math. 49, 893-904 (2005)

[HuLMW1] Hupfer, T., Leschke, H., Müller, P., Warzel, S.: Existence and uniqueness of the integrated density of states for Schrödinger operators with magnetic fields and unbounded random potentials. Rev. Math. Phys. 13, 1547-1581 (2001)

[HuLMW2] Hupfer, T., Leschke, H., Müller, P., Warzel, S.: The absolute continuity of the integrated density of states for magnetic Schrödinger operators with certain unbounded potentials. Commun. Math. Phys. 221, 229-254 (2001)

[KM] Kirsch, W., Martinelli, F.: On the ergodic properties of the spectrum of general random operators. J. Reine Angew. Math. 334, 141-156 (1982)
[KIK1] Klein, A., Koines, A.: A general framework for localization of classical waves: I. Inhomogeneous media and defect eigenmodes. Math. Phys. Anal. Geom. 4, 97-130 (2001)

[RS] Reed, M., Simon, B.: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness. New York: Academic Press, 1975

[U] Ueki, N.: Wegner estimates and localization for Gaussian random potentials. Publ. Res. Inst. Math. Sci. 40, 29-90 (2004)

[W] Wang, W.-M.: Microlocalization, percolation, and Anderson localization for the magnetic Schrödinger operator with a random potential. J. Funct. Anal. 146, 1-26 (1997)

(Germinet) Université de Cergy-Pontoise, CNRS UMR 8088, IUF, Département de Mathématiques, F-95000 Cergy-Pontoise, France
E-mail address: germinet@math.u-cregy.fr

(Klein) University of California, Irvine, Department of Mathematics, Irvine, CA 92697-3875, USA
E-mail address: aklein@uci.edu

(Mandy) Université de Cergy-Pontoise, CNRS UMR 8088, Département de Mathématiques, F-95000 Cergy-Pontoise, France
E-mail address: mandy@math.u-cregy.fr