Dynamical programming of continuously observed quantum systems

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We develop dynamical programming methods for the purpose of optimal control of quantum states with convex constraints and concave cost and bequest functions of the quantum state. We consider both open loop and feedback control schemes, which correspond respectively to deterministic and stochastic Master Equation dynamics. For the quantum feedback control scheme with continuous non-demolition observations we exploit the separation theorem of filtering and control aspects for quantum stochastic dynamics to derive a generalized Hamilton-Jacobi-Bellman equation. If the control is restricted to only Hamiltonian terms this is equivalent to a Hamilton-Jacobi equation with an extra linear dissipative term. In this work, we consider, in particular, the case when control is restricted to only observation. A controlled qubit is considered as an example throughout the development of the formalism. Finally, we discuss optimum observation strategies to obtain a pure state from a mixed state of a quantum two-level system.

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I. INTRODUCTION

The dynamical theory of quantum nondemolition observation, developed by Belavkin in the 80’s \cite{1,2,3} resulted in a new class of quantum stochastic equations \cite{4,5,6,7} for quantum dissipative systems under observation. Different quantum jump and quantum diffusive stochastic equations are obtained when the observed quantity has discrete and continuous spectra as, e.g., in photon counting and homodyne detection of optical fields. In the last case, for linear models and initial quantum Gaussian states this allows explicit solutions in terms of a quantum analog of the Kalman linear filter first introduced in Ref.\cite{1} (see also \cite{8,9,10,11,12}). These equations are similar to the classical filtering equation derived by Stratonovich in the 60’s \cite{13} for classical partially observed conditional Markov systems.

In the classical theory of feedback control the so called \textit{Separation Theorem} \cite{14,15,16} applies. This theorem states that the full control problem can be reduced in two separated and independent parts: (i) the estimation of the state of the system; (ii) the optimal control of the system. The same approach can be applied to quantum systems, as first pointed out by Belavkin in Ref.\cite{17} and more recently implemented in the quantum dynamical programming method \cite{18,19,20}. The only difference between classical and quantum systems is related to the concept of “state” given in the two theories. In both mechanics the state carries the necessary information to describe fully the system and this characterizes our knowledge of it. In classical mechanics a system is usually described by its position and momentum phase space variables. In quantum mechanics a state is described by a state vector which belongs to a linear Hilbert space or by a von Neumann density matrix. Contrary to the classical formulation, the linear property of the Hilbert space allows superpositions of quantum states. This, however, does not play a role in the separability of the state estimation and optimal control problems. The optimal control of the state of a quantum system implies the control of a density matrix valued stochastic process and is mathematically equivalent to any classical control problem.

Experimentally, very important achievements have been obtained in the last decade which have led to exciting prospectives to manipulate quantum systems. For instance, high precision quantum measurements at the Heisenberg limit have been implemented \cite{21,22} and quantum feedback has been used to record an externally provided quantum state of light onto an atomic ensemble \cite{23}. At the same time several theoretical proposals for quantum state engineering like spin-squeezing \cite{24,25,26,27}, photon number states \cite{28}, entangled states \cite{29,30,31}, quantum superposition states of optical fields \cite{32} and of two macroscopically distinguishable atomic states \cite{33}, and cooling of either a mechanical resonator \cite{34} or the atomic motion in an optical cavity \cite{35} via continuous measurement have been put forward. Among these activities more related mathematical issues as stability and observability become relevant and have been subject of investigation \cite{18,19,36,37,38,39,40}.

We shall follow the approach of Ref.\cite{18} (which is also taken in most papers on quantum feedback control afterWiseman \cite{41}), where the filtering controlled equation is postulated but not derived as a result of the conditioning of quantum dynamics upon the nondemolition...
observations. To simplify the mathematics, we shall consider only finite-dimensional models (like qubits). However, unlike in [19], we shall consider optimization of not only Hamiltonian control but also of control exercised by the choice among different observations carried out on the system. In the original setup [3, 14, 17], of optimal quantum feedback control theory the cost and target functions were restricted to affine functionals of the state where these functions are therefore expectations of certain observables. The result of minimization of an affine function as expected cost of a controlled observable is not always affine but always concave. This point was argued in Ref. [12], and the use of nonlinear concave cost functions was justified by Jacobs [43], Wiseman and Ralph [44], Wiseman and Bonten [45] for optimal feedback control of qubit purification. We will thus consider concave cost and target functions of the properties of the system controlled by the feedback.

The paper is organized as follows. In Sec. II, we introduce the density matrix notation, time evolution generators, and the notion of derivatives with respect to a quantum state. In Sec. III, we introduce Bellman and Hamilton-Pontryagin optimality, and we introduce cost and bequest functions. In Sec. IV, we turn to the problem of quantum dynamics under observation and present the diffusive quantum filtering equation corresponding to homodyne or heterodyne measurements. In Sec. V, we analyze the optimal control problem with observations and constraints, and we derive a Bellman equation for filtered dynamics. In Sec. VI we discuss the special case of purification of a mixed quantum state by measurements. We conclude the paper with a discussion in Sec. VII.

II. STATES, GENERATORS AND DERIVATIVES

We will assume a complex, finite dimensional, Hilbert space $\mathfrak{h} = \mathbb{C}^d$ for our open (observed and controlled) quantum system. It is convenient to define the quantum state space $\mathcal{S}$ as the compact, convex set of positive and Hermitian matrices. In general every state $\rho$ can be parametrized as $\rho(q) = \rho_0 - q$ with respect to a given state $\rho_0 \in \mathcal{S}$ by a corresponding $q \in \mathcal{L}_0$, where $\mathcal{L}_0 \subseteq \mathcal{L}$ is the space of trace zero matrices.

Example: Throughout this text, we shall illustrate and apply our results and formalism to the example of a single two-level quantum system, also described as a qubit. Any qubit density matrix with respect to the normalized trace $\text{tr}\{\cdot\} := \frac{1}{2} \text{tr}\{\cdot\}$ can be expanded on the Identity and the three Pauli matrices, $\rho = I + \sigma r$, with $\sigma r = r_x \sigma_x + r_y \sigma_y + r_z \sigma_z \equiv \vec{r} \cdot \vec{\sigma}$.

For the qubit we have $q = -\sigma r$ with respect to the central $\rho_0 = I$. Thus $q(\rho)$ can be identified with the Euclidean vector $\vec{q} = -\vec{r}$ globally parametrizing $\rho \in \mathcal{S}$ as $\rho = I - \vec{r} \vec{q}$.

The quantum state Master Equation (ME) describing an open quantum system is defined as

$$\frac{d}{dt} \rho^t + v(u(t), \rho^t) = 0,$$

(1)

where $\rho^t \in \mathcal{S}$ and the generator or drift term $v$ is given by

$$v(u, \rho) = \frac{i}{\hbar} [H(u), \rho] + \sum_j v_{L_j(u)}(\rho),$$

(2)

$$v_{L_j}(\rho) = \frac{L_j^\dagger L_j \rho + \rho L_j^\dagger L_j}{2} - L_j^\dagger \rho L_j^\dagger,$$

(3)

with $H(u) = H(u)^\dagger$ self adjoint, and $L_j(u)$ belongs to the complex space $\mathcal{B}(\mathfrak{h})$ of bounded operators on $\mathfrak{h}$ for each value of the set of control parameters $u$. The parameter $u = u(t)$ is the admissible control trajectory which may be restricted to a domain $\mathcal{U}(t) \subseteq \mathbb{R}^n$ of dimensionality $n$, possibly depending on time $t$.

Example: Assume that the qubit dissipative dynamics is given by a Hamiltonian $H(u) = \frac{\hbar}{2} \sigma_z$ controlled by the magnetic field $u(t) \in \mathbb{R}^d$ for $d \leq 3$, and by a single dissipation operator $L = \frac{1}{\hbar} \sigma_z$. With the Pauli matrix representation $\rho = I - \vec{r} \vec{q}$ for the density matrix, the ME evolution is governed by

$$v(\rho, u) = \vec{r} \cdot (\vec{u} \times \vec{q}) - \frac{|q|^2}{2} (q_x \sigma_x + q_y \sigma_y).$$

(4)

A (nonlinear) functional $\rho \rightarrow F[\rho]$ admits a derivative if there exists a $\mathcal{B}(\mathfrak{h})$-valued function $\nabla_{\rho} F[\cdot]$ such that

$$\lim_{h \rightarrow 0} \frac{1}{h} \{ F[\cdot + h\tau] - F[\cdot] \} = \langle \tau, \nabla_{\rho} F[\cdot] \rangle \quad \forall \tau \in \mathcal{L}_0,$$

(5)

where we have introduced the pairing $\langle \rho, X \rangle := \text{tr}\{\rho X\}$ with $\rho \in \mathcal{S}$ and $X \in \mathcal{L}^*$, where $\mathcal{L}^* = \mathcal{B}(\mathfrak{h})$ is the adjoint space of $\mathcal{L}$.

If $\rho^t$ is a quantum state trajectory controlled by equation (1), then we may apply the chain rule

$$\frac{d}{dt} F[\rho^t] = -\langle v(u(t), \rho^t), \nabla_{\rho} F[\rho^t] \rangle.$$

1 For vectors in $\mathbb{R}^3$ we use the arrow notation, e.g., $\vec{r}$. To avoid misunderstandings and to simplify the notation we use bold Latin characters to indicate the projection $r$ of a three dimensional vector $\vec{r}$ on a subspace $\mathbb{R}^d \subseteq \mathbb{R}^3$ with $d \leq 3$. 

for such a functional $F$.

A Hessian $\nabla_e^{\otimes 2} \equiv \nabla_e \otimes \nabla_e$ is defined as

$$
\lim_{h \to 0} \frac{1}{h} (\tau', \nabla_e F [\rho + h \tau] - \nabla_e F [\rho]) = \langle \tau \otimes \tau', \nabla_e^{\otimes 2} F [\rho] \rangle,
$$

with $\tau, \tau' \in \mathcal{L}_0$ and we say that the functional is twice continuously differentiable whenever $\nabla_e^{\otimes 2} F [\rho]$ exists.

**Example:** Let $F [\rho] = f [\rho]$ be a smooth function of the state, i.e., of $\rho$. Then $\nabla_e F [\rho]$ can be directly identified with $-\nabla f (\rho)$ in the sense that $(\tau, \nabla_e F [\rho]) = -\nabla f (\rho)$ for any $\tau = \sigma T \in \mathcal{L}_0$. Here the minus sign is related to the fact that the state $\rho$ is identified with $\rho^T$, but the gradient $\nabla f (\rho)$ is considered with respect to $\rho = \rho^T$.

Similarly we can write

$$
\nabla_e^{\otimes 2} F [1 - \sigma q] = \left( \sigma \cdot \nabla \right)^{\otimes 2} f (\rho).
$$

**III. BELLMAN AND HAMILTON-PONTRYAGIN OPTIMALITY**

**A. Cost functions**

Let us consider the integral cost for a control function $\{u (t)\}$ of the quantum state $\rho'$ over a time-interval $[t_0, T]$

$$
J \{u (t)\} = \int_{t_0}^{T} C (u (t), \rho') \, dt + G (u (T), \rho^T),
$$

(6)

where $\{\rho' : t \in [t_0, T]\}$ is the solution to a quantum controlled ME with initial condition $\rho'' = \rho_0$, and $C$ is a cost density while $G$ is the terminal cost, or bequest function. Causality implies that for any $t \in [0, T]$ the state $\rho'$ depends only on $u(t')$ with $t' \in [0, t]$ and is independent of the current and future values of $u(t)$. In particular, the choice of $u(T)$ at the terminal time instant $T$ does not affect the state $\rho^T$. We emphasize that the admissible control strategies $u(t)$ are not necessarily continuous but they can be assumed right continuous for all $t$ with left limits $u(t_-)$ not necessarily equal to $u(t)$. One can thus, for any $t$, regard $u(t)$ as entirely separate from earlier values $u(t' < t)$, while any later value $u(t'' \geq t)$, serves as a "postprocessing" control for $\rho'$. Our task is to adapt $u(t)$ to minimize the contribution from the cost density and to use $u(T)$ to "postprocess" the final quantum state or to modify the terminal cost or bequest function in Eq. (6) to most successfully achieve the desired goal. This will be exemplified in the following.

**B. Quantum dynamical programming**

Let us first consider the quantum optimal control theory without observation, assuming that the state $\rho' \in \mathcal{S}$ obeys the ME \(1\). To identify the optimal control strategy $\{u (t)\}$ with the specific cost $J \{u\} : t_0, \rho_0$, we note that for times $t < t + h < T$, one has

$$
S (t, \rho) := \inf_u \left\{ \int_t^{t+h} C (u (r), \rho^r) \, dr + \int_{t+h}^T C (u (r), \rho^r) \, dr + G (u (T), \rho^T) \right\}.
$$

Now, we assume that $\{u^o (r) : r \in (t, T]\}$ is the optimal control when starting in state $\rho$ at time $t$, and denote by $\{\rho^r : r \in (t, T]\}$ the corresponding state trajectory $\rho^r = \rho^o (t, \rho)$. According to Bellman’s optimality principle, the control $\{u^o (r) : r \in (t + h, T]\}$ is then optimal for the evolution starting from $\rho^{t+h}$ at the later time $t + h$, and hence

$$
S (t, \rho) = \inf_u \left\{ \int_t^{t+h} C (u (r), \rho^r) \, dr + S (t + h, \rho^{t+h}) \right\}.
$$

For $h$ small we expand $\rho^{t+h} = \rho - \nu (u (t), \rho) h + o (h)$ and we may apply a Taylor expansion of $S (t, \rho)$. Then, by taking the limit $h \to 0$ we obtain \(19\)

$$
- \frac{\partial}{\partial t} S (t, \rho) = \inf_u \left\{ C (u, \rho) - \nu (u, \rho), \nabla S (t, \rho) \right\}. \quad (7)
$$

This equation should be solved subject to the terminal condition

$$
S (T, \rho) = \inf_u \left\{ G (u, \rho) : u \in \mathcal{U} (T) \right\} = S_T [\rho]. \quad (8)
$$

We recall that when the infimum is reached, then the objective of the optimization problem we aim to solve is obtained. This formalism has been applied in the case of optimal control of the cooling of a quantum dissipative three-level $\Lambda$ system \(1\), and in the classical thermodynamic optimization of the evolutionary Carnot problem \(49\).

**C. Quantum Pontryagin Hamiltonian**

We introduce the Pontryagin Hamiltonian function defined, for $\rho \in \mathcal{L}_0, p \in \mathcal{L}^*$, by

$$
H_v (q, p) := \sup_u \{ \langle v (u, \rho (q)) , p \rangle - C (u, \rho (q)) \}. \quad (9)
$$

We use the parametrization $\rho (q) = \rho_0 - q$ by a zero trace operator $\rho \in \mathcal{L}_0$ and $v$ is the velocity $q$ of $q = \rho_0 - q$.

The equations of motion for the state operator $\rho$ and for the operator $p$ can be expressed with the Pontryagin principle\(2\).

\(2\) The optimality principle states that if the path of a process through the stages $t_o \to t_b \to t_c$ is the optimal path from $t_o$ to $t_c$, then the path from $t_b$ to $t_c$ is optimal as well \(18, 43, 47\).
Hamiltonian in formally the same way as the equations of motion for the canonical coordinates \((q,p)\) in the Hamiltonian formulation of classical mechanics \(^{19}\).

Since \(\langle v \psi, \phi \rangle = 0\), the Pontryagin Hamiltonian does not change if we replace any \(p \in \mathcal{L}^2\) by \(p + \lambda I\) with \(\lambda \in \mathbb{C}\). The mathematical consequences of this equivalence class property and the observation that the operator \(p\) in Eq. (5) is the Legendre-Fenchel transform of the cost function \(C\) are further developed in Ref. \(^{19}\).

We may use the Pontryagin Hamiltonian to rewrite (7) as the (backward) Hamilton-Jacobi-Bellman (HJB) equation

\[-\frac{\partial}{\partial t} S(t,\psi(q)) + H_v(q, p(\nabla_q S(t, \psi))) = 0, \quad (10)\]

which can be simply written as \(\partial_t S(t, \psi) = H_v(q_0 - q, p(\nabla_q S(t, \psi)))\) for any a priori chosen reference state \(q_0 \in \mathcal{S}\).

**Example:** In the case of the Hamiltonian controlled dissipative dynamics \(^1\) we have with \(p = \sigma \bar{p} + CI\)

\[\langle v, p \rangle = (\bar{q} \times \bar{p}) \cdot u - \frac{|\lambda|^2}{2} (q_x p_x + q_y p_y).\]

For the qubit with the density cost

\[C(u, \psi) = O_{B_1}^+(u), \quad O_{B_1}^+(\bar{u}) = \begin{cases} 0, & \bar{u} = u \in B_1 \\ +\infty, & \bar{u} \notin B_1 \end{cases}\]

under the constraint \(B_1 = \{u \in U : |u| \leq 1\}\), the supremum in Eq. (9),

\[H_v(q, p) := \sup \{\langle v(u, q(q)), p \rangle : |u| \leq 1\} = \sup \{u \cdot (\bar{q} \times \bar{p}) \} - \frac{|\lambda|^2}{2} (q_x p_x + q_y p_y),\]

is achieved at the stationary point \(u^*(q)\).

D. Linear, affine, and concave cost and bequest functions

Now, we consider cost density and bequest functions which are linear functions of the quantum state \(\psi\)

\[C(u, \psi) = \langle \psi, C(u) \rangle, \quad G(u, \psi) = \langle \psi, G(u) \rangle, \quad (11)\]

i.e., they can be interpreted as the expectation values of a cost observable \(C(u)\) and a bequest observable \(G(u)\), which may depend on the control parameter \(u \in \mathbb{R}^n\). One can consider, for example, an average energy associated with the control parameter as a cost, say \(C(u) = u^2/2\), and the error probability \(G = \langle \psi_T \mid I - P_T \rangle\) given by the orthonormal projector \(P_T = |\psi_T\rangle \langle \psi_T|\) on a target state-vector \(|\psi_T\rangle\) as the bequest observable. Although the dependence of \(G\) on the final value \(u(T)\) of the control parameter in Eq. (11) is sometimes redundant, for the sake of generality and for reasons which will be clear below we keep this dependence.

It is natural to extend the cost and bequest functions to affine functions

\[C(u, \psi) = \langle \psi, C(u) \rangle + c(u), \quad G(u, \psi) = \langle \psi, G(u) \rangle + g(u), \quad (12)\]

of the state \(\psi\) which can be obtained from the linear functions by replacing \(C \mapsto C + cI\) and \(G \mapsto G + gI\) in Eq. (11). This generalization has no consequences, unless the real-valued functions \(c(u)\) and \(g(u)\) are allowed to take also the infinite cost value \(+\infty\), thus reflecting rigid constraints on \(u\). Indeed, any constraint on the admissible domain \(U(t)\), can be described by the infinite costs

\[c(t, u) = \infty, \quad g(t, u) = \infty, \quad (\forall u \notin U(T)),\]

such that the expected cost is finite only if \(u(t)\) is in the allowed domain \(U(t)\). For instance, in the deterministic preparation of atomic Dicke states of Ref. \(^{54}\) the control \(u\) is the strength of a magnetic field, which in an experiment cannot assume arbitrarily large values.

Of course there is a range of useful cost and bequest functions which cannot be cast into an affine form. For example, the variance of a certain observable, the von Neumann entropy of the state of a quantum system, or sub-system, the purity of a quantum system characterized by the trace of the square of the density matrix, and various entanglement measures are important quantities used to characterize desirable properties of quantum systems, e.g., in precision metrology and quantum information theory.

As described above, if we aim to produce a definite pure target state \(|\psi_T\rangle\), we will maximize the expectation value of the particular pure state projector \(P_T = |\psi_T\rangle \langle \psi_T|\), and we thus have a linear bequest function. If, however, we only wish to maximize the purity, but we do not care precisely which state is produced, for a given \(\psi\), we could look for the nearest pure state, and maximize the expectation value of the corresponding projection operator, and since that projector now depends on \(\psi\), we effectively obtain a non-linear bequest function.

The search for "the nearest pure state" can be parametrized by the "post-processing" \(u(T)\) dependence of a linear bequest observable, and we can more generally write the minimization of quantum state controlled functionals of the type \(^{9}\) as

\[S[\psi] = \inf_u \{\langle \psi, G(u) \rangle : u \in U(T)\} \quad .\]
Example: Consider the affine bequest function with \( \bar{u} \in \mathcal{U} = \mathbb{R}^3 \),
\[
G(u, \varrho) = O_{B_1}^+(\bar{u}) - \vec{q} \cdot \bar{u},
\]
of the qubit state \( \varrho = 1 - \sigma_3 \vec{q} \) corresponding to the generalized qubit cost observable \( G(u) = O_{B_1}^+(\bar{u}) I + \sigma_3 \vec{q} \), including the constraint function \( O_{B_1}^+(\bar{u}) \) for the unit ball \( B_1 = \{ u \in \mathbb{R}^d : |u| \leq 1 \} \) in \( d \leq 3 \). Then \( S[\varrho] = \inf_{\bar{u} \in \mathbb{R}^3} \{ \langle \varrho, \sigma_3 \vec{q} \rangle + O_{B_1}^+(\bar{u}) \} = - \sup_{\bar{u} \in B_1} \vec{q} \cdot \bar{u} = - |\vec{q}| \),
which \( \vec{q} \) is the projection of \( \vec{q} \in \mathbb{R}^3 \) onto \( \mathbb{R}^d \). In this way we recover the concave bequest function \( S(T, \varrho) = - |\vec{q}| \) used as a measure of purity by Wiseman and Boutsen for \( d = 2 \) in Ref. [4].

IV. QUANTUM DYNAMICS UNDER OBSERVATION

A. Quantum measurements and posterior states

The state of an individual continuously measured quantum system does not coincide with the solution of the deterministic ME [11], but instead depends on the random measurement output \( y_\omega^\bullet \) in a causal manner. The posterior \( \varrho_\bullet \) density matrix should be viewed as an \( S \)-valued stochastic process \( \varrho_\bullet : \omega \mapsto \varrho_\bullet(\omega) \), causally depending on the particular observations \( y_\omega^\bullet = \{ y_\omega(r) : r < t \} \), which are, in turn, obtained with a probability distribution determined by the previous posterior states \( \{ \varrho_\bullet : r < t \} \). Here the symbol \( \bullet \) denotes a random variable, when its actual value \( \omega \) is not displayed.

The causal dependence of the posterior state \( \varrho_\bullet \) on the measurement data \( y_\omega^\bullet \) is given by a corresponding quantum filtering equation derived in the general form by Belavkin in [3], [4], [7]. The quantum trajectories, introduced by Carmichael [51], and the Monte Carlo Wave Functions (MCWF), introduced by Dalibard, Castin and Mølmer [52, 53], are stochastic pure state descriptions of dissipative quantum systems. In these approaches the dissipative coupling to a reservoir and resulting mixed state dynamics of a small quantum system is “unravelled” by simulated Gedankenmeasurements on the reservoir. These descriptions are included in Belavkin’s formulation, which, however, does not assume a complete detection of all reservoir degrees of freedom, and hence it retains the density matrix description. More importantly, however, it does not only deal with the simulation of the unavoidable dissipation of a quantum system, but also with the dynamics induced by the probing of the system by coupling to a measurement apparatus. One may, for example, probe atomic internal state populations and coherences in a single atom, or a collection of atoms, by the phase shift or rotation of field polarization experienced by a laser beam interacting with the atoms. This measurement may be turned on and off, and several measurements may go on simultaneously as controlled by the field strengths of different probing laser beams. Here for simplicity we display only the diffuse case corresponding to homodyne or heterodyne detection in optics. These detection schemes were identified as continuous limits of the Monte Carlo Wave Function quantum jump dynamics, associated with photon counting experiments with strong local oscillator fields [54, 55].

The quantum diffusive filtering equation as derived in [4, 6] for probing by coupling to a single set of system observables \( L, L^\dagger \) has the form
\[
d\varrho_\bullet + v(\varrho_\bullet, L + L^\dagger)dt = \theta(\varrho_\bullet) dw(t),
\]
where the time coefficient \( v \) contains the commutator with the Hamiltonian and the damping terms in the deterministic ME [11]. The right hand side of the equation, where \( dw(t) \) denotes an infinitesimal standard Wiener Gaussian process with \( dw^2(t) = dt \), provides the fluctuation innovation term,
\[
dw(t) = dy_\bullet(t) - \langle \varrho_\bullet, L + L^\dagger \rangle dt,
\]
governed by the difference between the random outcome of the measurements and its expectation value. This term acts on the density operator as specified by
\[
\theta(\varrho) = L\varrho + \varrho L^\dagger - \langle \varrho, L + L^\dagger \rangle \varrho.
\]

In optical homodyne detection the term \( dy_\omega(t) \) in Eq. (15) describes the continuous photocurrent, which is the output signal obtained from the detector.

Hereafter we shall use negative integers \( j_- \) to indicate the damping dissipative operators \( L^- \) and positive integers \( j_+ \) to describe the dissipative operators \( L^+ \) due to the coupling of the system with the measurement apparatus. The same notation will be applied to the drift term \( v(\varrho) = \sum_{j_-} v_{j_-}(\varrho) + v_0(\varrho) + \sum_{j_+} v_{j_+}(\varrho) \), where
\[
v_0(\varrho) = \frac{i}{\hbar}[H, \varrho],
\]
\[
v_{j_\pm}(\varrho) = \frac{(L^{j\pm})^\dagger L^{j\pm} \varrho + \varrho (L^{j\pm})^\dagger L^{j\pm}}{2} - L^{j\pm} \varrho (L^{j\pm})^\dagger.
\]

Example: Assume an undamped qubit system with vanishing Hamiltonian, and consider the probing described by the observable \( L^\dagger = \frac{1}{2} \sigma_2 \equiv L \) with \( \lambda \in \mathbb{R} \). Using the Bloch vector notation for the system density matrix we can write \( L\varrho + \varrho L^\dagger = \lambda (\sigma_2 + \vec{q}) \) and \( \langle \varrho, L + L^\dagger \rangle = \lambda \vec{q} \), where we used the fact that \( \vec{q} = - (x, y, z) = -\vec{r} \). Therefore, the drift term \( v_1 \) and the fluctuation coefficient \( \theta \) in the filtering equation (14) are given by:
\[
v_1(\varrho) = \frac{\lambda^2}{2} \sigma_2 \varrho_z,
\]
\[ \theta (\varrho) = \lambda \left[(1-z^2) \sigma_z - z \sigma_r \right] = \lambda \left[(1-z^2) \sigma_z - z (x \sigma_x + y \sigma_y) \right], \]

where \( \vec{r}_n^I = \vec{r} - (\vec{r} \cdot \vec{n}) \vec{n} \) with \( \vec{n} \) being a vector in \( \mathbb{R}^3 \) of unit norm. In our specific case \( \vec{n} = \vec{e}_z = (0,0,1) \), and \( \vec{r}_e^I = (x,y,0) \). The fluctuation coefficient can also be rewritten as \( \theta (\varrho) = \sigma_r \), where \( \vec{l} = \lambda (xz, -yz, 1 - z^2)^t \). The innovation process driving the qubit filtering equation is defined by \( d\gamma_{\omega}(t) + \lambda (\varrho_{\omega}^i, \sigma_r) dt \).

**B. Average change of stochastic functionals**

Let \( \{ \varrho_{\omega}(t, g) : \omega \in \Omega \} \) be the solution of (13) for \( r > t \) starting in state \( \varrho_{\omega}(t, \varrho) \) at \( t = r \) for all \( \omega \in \Omega \). Then, for a smooth functional \( F \) on \( L \), we have the average rate of change

\[ \lim_{h \to 0} \frac{1}{h} \{ \mathbb{E} \left[ F \left[ \varrho_{\omega}^{t+h}(t, g) \right] \right] - F \left[ \varrho(t, g) \right] \} = D(t, g)F[\varrho], \]

where \( \mathbb{E} \). denotes the average of a functional of the stochastic state \( \varrho(t, g) \) at time \( t \). Since the change in \( \varrho \) contains both deterministic terms, linear in \( dt \), and fluctuating terms, scaling with \( \sqrt{dt} \), we apply the It\( \text{ô} \) rule [4] and expand the function to second order in small variations to get the correct average rate of change. Hence, the elliptic operator \( D(t, g) \), in the diffusive case is

\[ D(t, g)F[\varrho] = -\langle \nu(t, \varrho), \nabla_{\varrho}F[\varrho] \rangle + \frac{1}{2} \Delta_{\varrho}F(t, \varrho), \]

where the It\( \text{ô} \) correction is given by

\[ \Delta_{\varrho}F(t, \varrho) = \left< \theta (t, \varrho) \otimes^2, \nabla_{\varrho}^2F[\varrho] \right>. \]

For an \( N \) level system, whose state can be described by a generalized Bloch vector \( \vec{r} \) in \( \mathbb{R}^{N^2 - 1} \), the notation \( \nabla_{\varrho}^2 \) reads as \( \nabla_{\varrho}^2 \equiv (\nabla_{\varrho} \nabla_{\varrho})^T \), where \( \nabla_{\varrho} \) is the \( N^2 - 1 \) column gradient vector operator and \( (\nabla_{\varrho})^T \) is its transpose. The same notation applies for the operator \( \theta(t, \varrho) \otimes^2 \).

**Example:** Let us illustrate the above expression for a functional \( F[\varrho] = f(\varrho) \), where \( r = |\vec{r}| \) is the length of the Bloch vector.

We consider again the situation of the previous example, where \( L^2 = \frac{1}{2} \sigma_z \equiv L \). Then, we obtain

\[ \left< \nu_{1}(\varrho), \nabla_{\varrho}F[\varrho] \right> = \frac{\lambda^2}{2} \vec{r}_{\varrho} \cdot \nabla_{\varrho} f(\varrho), \]

where we used the result of the third example in Sec. [11] for \( \nabla_{\varrho}F[\varrho] \). Then, the operator \( \theta(\varrho) \otimes^2 \) can be written as the matrix

\[ \theta(\varrho) \otimes^2 \equiv \vec{l} \otimes \vec{l} = \lambda^2 \begin{pmatrix} -xz & -zy \\ -zy & 1 - z^2 \end{pmatrix}^T, \]

with \( \vec{l} \) given in the previous example, and \( \nabla_{\varrho}^2F[\varrho] \) can be identified with the Hessian matrix \( \nabla^2F[\varrho] \) as discussed in the third example of Sec. [11]. The It\( \text{ô} \) correction is hence given by

\[ \Delta_{\varrho} F(t, \varrho) = \frac{\lambda^2}{r^2} \begin{pmatrix} -zx & -zy \\ -zy & 1 - z^2 \end{pmatrix} = \begin{pmatrix} -zx & -zy \\ -zy & 1 - z^2 \end{pmatrix}, \]

where \( f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \).

Since \( f \) depends only on \( r \) and \( \hat{\nabla} f(r) = -r^{-1} \vec{l} \partial_r f \), the Hessian matrix \( \nabla^2 F(r) \) can be rewritten as

\[ \nabla^2 F(r) = \frac{1}{r} \frac{\partial f}{\partial r} I + \frac{z^2}{2r^2} \left( \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} \right) \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}. \]

Hence, we have

\[ \Delta_{\varrho} F(t, \varrho) = \frac{\lambda^2}{2r^2} \frac{\partial f}{\partial r} \left( \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} \right) \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}. \]

For instance, if \( f(r) = 1 - r^2 \), then the second line in the above equation disappears and therefore \( D(t, \varrho)F[\varrho] = \lambda^2 (r^2 - 1)(1 - z^2) \).

Simultaneous probing of different observables, represented by operators \( L^j \), coupled to different probing light beams, leads to a vector of random measurement outputs, and is governed by the filtering equation

\[ d\varrho_{\omega} + \nu(\varrho_{\omega}) dt = \sum_{j+1}^{n} \theta_{j+1}(\varrho_{\omega}) dw_{j+1}(t) \]

and with

\[ \theta_{j+1}(\varrho) = L^j \varrho + \varrho(L^j)^\dagger - \langle \varrho, L^j + (L^j)^\dagger \rangle \varrho. \]

Note that both the measurement induced terms and the diffuse operation term contain the relevant probing strengths through the magnitude of the operators \( L^j \).

**Example:** In the above example, we showed the explicit case of probing \( \sigma_z \) with a coupling strength \( \lambda \). By
cyclic permutation of the coordinates \((x, y, z)\) we obtain the equivalent expressions for probing along the other coordinate axes, and by continuous rotation of \((x, y, z)\) the effect of probing along an arbitrary direction can be derived.

Assuming \(L^2 = \frac{\lambda}{2} \sigma_{\hat{n}}^2\) with \(\hat{n}\) of unit norm, we can easily generalise the previous results:

\[
\langle v_{\hat{n}}(q), \nabla_v F (\varrho) \rangle = \frac{\lambda^2}{2r} \frac{\partial f}{\partial r} \left[ r^2 - (\hat{n} \cdot \hat{r})^2 \right] \\
\theta(\varrho) = \sigma^2_{\hat{n}} \text{ with } \hat{r} = \lambda [\hat{n} - (\hat{n} \cdot \hat{r}) \hat{r}], \text{ and}
\]

\[
\Delta_v F (t, \varrho) = \frac{|\hat{f}|^2}{r} \frac{\partial^2 f}{\partial r^2} \left[ 1 - (\hat{n} \cdot \hat{r})^2 \right] \\
\Delta_v F (t, \varrho) = \frac{|\hat{f}|^2}{r} \frac{\partial^2 f}{\partial r^2} \left[ 1 - (\hat{n} \cdot \hat{r})^2 \right]
\]

Hence, the elliptic operator \(\Delta_v\) reads

\[
\begin{aligned}
D (t, \varrho) F [\varrho] &= \frac{\lambda^2}{2r^2} \left[ 1 - r^2 \right] \left\{ \frac{\partial f}{\partial r} \left[ 1 - (\hat{n} \cdot \hat{r})^2 \right] \right. \\
&\quad + \left\{ \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} \right\} (\hat{n} \cdot \hat{r})^2 (1 - r^2) \}
\end{aligned}
\]

where \(|\hat{f}|^2\) has been written explicitly. Since \(D (t, \varrho) F [\varrho]\) provides the average change of \(F [\varrho]\) for the system while monitoring \(L^2\), we are now able, for any \(\varrho\), to make the optimal choice of observable \(L^2\) which, locally in time, gives the largest change. We want \(F [\varrho]\) to decrease as fast as possible, and hence

\[
(\hat{n} \cdot \hat{r})^2 \left[ 1 - r^2 \right] \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r} \frac{\partial f}{\partial r} \leq 0
\]

must be satisfied. If we assume that \(\partial^2 f < 0\), then the coefficient in the square brackets in (23) is positive when

\[
\left| \frac{\partial^2 f}{\partial r^2} \right| < \frac{1}{r(1 - r^2)} \frac{\partial f}{\partial r}
\]

which implies that \(\partial f / \partial r\) must be negative, i.e., \(f(r)\) is monotonic. Then, the minimum is reached for \((\hat{n} \cdot \hat{r}) = 0\). When we consider the function \(f(r) = 1 - r^2\), the above conditions are satisfied and it tells us that it is optimal to perform a measurement in the orthogonal \(\hat{n}\) with respect to the state \(\hat{r}\), when our objective is the purification of the state. This conclusion was also obtained by Jacobs in Ref. [43].

Note that in the example we showed that a proper choice among measurements leads to the highest local increase of the purity, but we remind that this may not necessarily be the optimal way to obtain high purity of the final quantum state after a total probing time \(T\). In the following section we shall treat the measurement strengths as our control parameters, and apply the Bellman principle to identify how a system is optimally controlled by measurements alone.

V. OPTIMAL FEEDBACK CONTROL WITH CONSTRAINTS

We assume a quantum system under continuous observation described by the filtering equation (14). A choice of the control function \(\{u (r) : r \in [t_0, t]\}\) is required before we can solve the filtering equation (14) at the time \(t\) for a given initial state \(\varrho_0\) at time \(t_0\). To this end, we define the optimal average cost on the interval \([t_0, T]\) to be

\[
S (t, \varrho) := \inf_{\{u_t\}} \mathbb{E} \left[ J \left[ \{u_t (t) : t_0, \varrho \} \right] \right]
\]

where the minimum is considered over all admissible measurable control strategies \(\{u_t (t) : t \geq t_0\}\) adapted with respect to the innovation process in Eq. (14). By admissible we mean any stochastic process \(u_t (t)\) for which the controlled filtering equation is well defined and has a unique solution \(\varrho^*_t\) (for more precise mathematical definitions see, for example, [54]). The aim of feedback control theory is then to find an optimal control strategy \(\{u^*_t (t)\}\) and evaluate \(S (t, \varrho)\) on a fixed time interval \([t_0, T]\).

A. Optimality equation for observed systems

We consider the problem of computing the minimum average cost in (24). Even though the cost is random, the Bellman principle can be applied also in this case. As before, we let \(\{u^*_t (t)\}\) be a stochastic control leading to optimality and let \(\varrho^*_t (r)\) be the corresponding state trajectory (now a stochastic process) starting from \(\varrho\) at time \(t\). Again choosing \(t < t + h < T\), we have by the Bellman principle

\[
\begin{aligned}
\mathbb{E} \left[ S (t + h, \varrho|_{t + h}) \right] + o (h) &= S (t, \varrho) \\
+ \inf_{u_t} \left\{ \mathbb{E} \left[ \frac{\partial S}{\partial t} (t, \varrho) + C (u, \varrho) + D (u, \varrho) S (t, \varrho) \right] \right\} h.
\end{aligned}
\]

Taking the limit \(h \to 0\) yields the quantum backward Bellman equation

\[
\begin{aligned}
- \frac{\partial S}{\partial t} (\varrho) &= \inf_{u_t} \left\{ C (u, \varrho) + D (u, \varrho) S (\varrho) \right\} \\
&\text{as derived in [1, 55].}
\end{aligned}
\]

This can be rewritten in the generalized HJB form as

\[
\frac{\partial S}{\partial t} (t, \varrho (q)) = H^0_{\varrho} (q, \nabla_{\varrho} S (t, \varrho (q)))
\]

in terms of the generalized (Bellman) ”Hamiltonian” which takes in the diffusive case the second order derivative form

\[
H^0_{\varrho} (q, \nabla_{\varrho} S) := \sup_{u \in \mathcal{U}} \left\{ \langle v (u, \varrho (q)) , \nabla_{\varrho} S \rangle - C (u, \varrho (q)) \right\} - \frac{1}{2} \Delta_{\varrho} S (u, \varrho (q))
\]
where $\Delta_j S(u, \varrho)$ is defined in (20). This equation is to be solved backwards with the terminal condition $S(T, \varrho) = S_T(\varrho)$.

If $\theta$ in (14) and (16) does not depend on $u$ (the control is only in $H$ and not in $L^j_+, L^j_-$), this gives the diffusive HJB equation with a possible nonlinear dependence only on the first derivative $\nabla_{\varrho} S[\varrho]$

$$-\frac{\partial S}{\partial t} + H_u(q, \nabla_{\varrho} S) = \frac{1}{2} \langle \theta(\varrho)^{\otimes 2}, \nabla_{\varrho}^{\otimes 2} S \rangle.$$  

Exactly as in the deterministic case, the solution $S$ of this diffusive equation defines the optimal strategy through $
abla_{\varrho} S(t, \varrho)$.

In the case we control the strength of different kinds of measurements carried out on the system, $\theta^j_+$ and the associated drift term components depend on the control parameters $u$. Then, the optimality equation is nonlinear only in the Hessian

$$\frac{\partial S}{\partial t} = \langle v_0(\varrho), \nabla_{\varrho} S \rangle + H_0^\varrho(q, \nabla_{\varrho} S, \nabla_{\varrho}^{\otimes 2} S),$$

where $v_0$ is defined in (17), and $H_0^\varrho$ is defined later in Eq. (28). Here we assume the possibility that we can control also the dissipative channels, and therefore their drift terms $u_j-(\varrho)$ are included in the Hamiltonian $H_0^\varrho$.

VI. OPTIMAL CONTROL OF PURIFICATION

A. Generalized Bellman Hamiltonian

A pure Hamiltonian control does not change the eigenvalues of quantum states and therefore does not change the entropy of any state as a natural bequest function of the purification. On the other hand, the filtering dynamics (14) changes the entropy, as it provides the state conditioned on measurements. Thus, both a Hamiltonian feedback together with the continued probing of the system, and a feedback strategy where new measurements are selectively carried out on the system, can be applied to optimize the convergence towards a pure state. The former possibility has been studied by Jacobs [42], Wiseman and Bouten [43]. Instead, we shall assume control of the coupling $\lambda$ to the continuous measurement in the filtering equation as a real function of time $t$ and the information previously obtained. Thus, we are interested in the optimal purification strategy via the feedback measurement control in one or several channels by solving the optimality equation (27) with the Hamiltonian $H_0^\varrho$ containing explicitly all the diffusive measurement terms (22) and the corresponding dissipation drifts $v_{j^{+}}$. We allow, however, also the control of the dissipative operators which correspond to some unobserved modes with velocity term $v_{j^-}$.

We may assume that the operators $L^j_+ = \lambda^j_+ R^j_+$, with $j$ being either $j_+$ or $j_-$, are controlled only by the strengths $u_j = |\lambda|^2$ or by the phases $u_j = \arg \lambda^j$ of the coupling parameters $\lambda^j$ with fixed measurement operators $R^j$. Taking the first (controlled strength) choice with $\arg \lambda^j = 0$, we define the corresponding $v_j = u_j R_j^+$, $\theta^j = \sqrt{u_j} \theta^j_+$ and $\Delta_j^\varrho S = \langle \theta^{\otimes 2}_j, \nabla_{\varrho}^{\otimes 2} S \rangle$ in terms of the rescaled $u_j R^j_+$ and $\theta^j_+$. We then obtain

$$H_0^\varrho = \sup_{\{u_j\leq 0\}} \left\{ \sum_{j \neq 0} u_j \left( \langle v_{R^j_+}, \nabla_{\varrho} S \rangle - \frac{1}{2} \Delta_j^\varrho S \right) - C(u) \right\},$$

explicitly in terms of the measurement strengths $u_j$.

Thus, we have a convex optimization problem under the constraint $u_j^2 \geq 0$. We will consider the optimization problem under the further natural constraint of a given maximum total probing strength, $||u||_1 := \sum_j u_j^2 \leq 1$, which is incorporated by choosing the cost function $C(u, \varrho) = +\infty$ if $||u||_1 > 1$, and $C(u, \varrho) = 0$ otherwise. Equation (28) reduces to

$$H_0^\varrho = \max_{j} \left\{ \left( \langle v_j(\varrho), \nabla_{\varrho} S(t, \varrho) \rangle - \frac{1}{2} \Delta_j^\varrho S(t, \varrho) \right) \right\},$$

if at least one value under the maximum is positive, and $u_j^o(t) = 1$ for any optimal $j = j_0(\varrho, S)$ and $u_j^o(t) = 0 \forall j \neq j_0$. Otherwise, $H_0^\varrho = 0$, no measurement purifies $\varrho$, and the maximum is achieved on the optimal feedback strategy $u^o(t) = 0$. In the case of a single channel measurement this defines a simple two-valued strategy for when the probing should be switched on and off, $u \in \{0, 1\}$, corresponding to the Hamiltonian

$$H_0^\varrho = \left| \langle v(\varrho), \nabla_{\varrho} S(t, \varrho) \rangle - \frac{1}{2} \Delta_1^\varrho S(t, \varrho) \right|_+,$$

where $|x|_+ = \max \{0, x\}$.

B. Purifying a qubit only with measurements

Let us take the cost $c(u) = O^+_H(u)$ of the constraint $U = \{ u^j \geq 0: \sum_j u^j \leq 1 \}$ and the hermitian operators $L^{\lambda}(\Omega) = \frac{\lambda(\Omega)}{2} \sigma_{\Omega \Omega}$, where instead of the integer $j_+$ we use the outward normal unit vector, $\vec{n}_\Omega$, parametrized by the continuous solid angle argument, $\Omega$, along which the diffusive measurement is performed. In the following we set $\lambda(\Omega) = \sqrt{u(\Omega)} = 1$, as discussed in the previous section. Besides, we assume that the system is subject to no dissipation and no Hamiltonian control. In this case, the diffusive filtering equation (14) reduces to

$$d\varrho_t^\varrho + v_{\varrho}(\varrho_t^\varrho) dt = \theta^\Omega(\varrho_t^\varrho) dw(t).$$

This equation can be rewritten in terms of the state vector $\vec{r}$ as
\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt} \\
\frac{dz}{dt}
\end{bmatrix} + \begin{bmatrix}
x - (\vec{n} \cdot \vec{r})x \\
y - (\vec{n} \cdot \vec{r})y \\
z - (\vec{n} \cdot \vec{r})z
\end{bmatrix} \frac{dt}{2} = \begin{bmatrix}
n_x - (\vec{n} \cdot \vec{r})x \\
n_y - (\vec{n} \cdot \vec{r})y \\
n_z - (\vec{n} \cdot \vec{r})z
\end{bmatrix} dw,
\]

where we used the results of the examples in Sec. IV.

As noticed in the last example of Sec. IV about the purification of a qubit state, the elliptic operator (19) with

\[
\frac{\partial S}{\partial t} + \frac{1}{2} \int d\Omega \delta(\vec{n}_\Omega - \vec{n}_{\Omega_{0}}) \left( \vec{r}_{\Omega_{0}} \cdot \nabla \vec{S} \right)
\]

\[
= \frac{1}{2} \int_{4\pi} d\Omega \delta(\vec{n}_\Omega - \vec{n}_{\Omega_{0}}) \left\{ \alpha_{\Omega}^{2} \right\} \left( \vec{r}_{\Omega_{0}} \cdot \nabla \vec{S} \right), \quad \vec{r}_{\Omega_{0}} \cdot \vec{r}_{\Omega_{1}} - \vec{r}_{\Omega_{1}} \cdot \vec{r}_{\Omega_{1}} \right\},
\]

where \( \delta(\vec{n}_\Omega - \vec{n}_{\Omega_{0}}) \) is the Dirac delta function on the surface of the unit sphere, \( \alpha_{\Omega} = \vec{n}_\Omega \cdot \vec{r} \), and \( S_{\Omega_{0}} = \frac{\partial S}{\partial \alpha_{\Omega}} \).

If \( \alpha_{\Omega} = 0 \) is chosen, then \( \vec{r}_{\Omega_{0}} = \vec{r} - \alpha_{\Omega} \vec{n}_{\Omega_{0}} = \vec{r} \) and the above equation simplifies to

\[
- \frac{\partial S}{\partial t} + \frac{1}{2} \frac{d\Omega}{d\Omega_{0}} \cdot \nabla \vec{S} = 0.
\]

If we write \( \vec{r} = r(\sin \varphi \sin \theta, \cos \varphi \sin \theta, \cos \theta) \) this equation can be reduced to

\[
- \frac{\partial S}{\partial t} + \frac{r}{2} \frac{\partial S}{\partial r} = 0. \quad (32)
\]

It is straightforward to check that \( S(t, r) = 1 - r^2 e^{-(T-t)} \) solves (32) with \( S(T, r) = 1 - r^2 \). Note that this solution is obtained under the assumption that we always measure a Bloch-sphere component orthogonal to the current density matrix Bloch vector. It is, however, easy to verify that the supremum according to (29) is in accord with that choice. This confirms the demonstration by Wiseman and Boute in Ref. 42 of the optimality of Jacobs 43 purification protocol.

C. Controlling only the fluctuations \( \theta^{\pm} \)

As a special constraint on our control consider \( u = (u_j) \) indexed by \( j_{\pm} = \pm j \) and add the constraints \( u_{j_{-}} = 1 - u_{j_{+}} \) with \( R_{j_{-}} = R_{j_{+}} \) for all \( j = 1, \ldots, n \) under the constraint \( \sum_{j_{\pm}} u_{j_{\pm}} \leq 1 \) and \( \sum_{j_{-}} u_{j_{-}} = n_{-} \). The new constraint corresponds to keeping the dissipation drifts for each pair \((j_{+}, j_{-})\) independent of the controls, \( v_{j_{-}} + v_{j_{+}} = u_{j_{-}} \). The optimality equation reduces to

\[
- \frac{\partial S}{\partial t} = \langle v(\theta), \nabla \theta S \rangle - \frac{1}{2} \min_{j_{-} > 0} \Delta_{\theta}^{j_{-}} S(t, \theta). \quad (33)
\]

Hence, \( H_{\theta_{\theta}} = \langle v(\theta), \nabla \theta S \rangle + H \), where the new Hamiltonian, \( H \), is defined by the minimal Hessian \( \Delta_{\theta}^{j_{-}} S(t, \theta) \).

Equation (33) becomes linear if one of the Hessians, say \( \Delta_{\theta}^{j_{-}} S(t, \theta) \), is the most negative, \( \Delta_{\theta}^{j_{-}} S \leq \Delta_{\theta}^{j_{-}} S \) for all \( j, \theta \) and \( t \).

The constraint \( u_{j_{-}} = 1 - u_{j_{+}} \) with \( R_{j_{-}} = R_{j_{+}} \) corresponds, for example, to the partial monitoring of a dissipative channel which leaks information to the environment. Such a leakage can be, for example, the one of resonance fluorescence from an atomic quantum system monitored with finite detection efficiency or within a finite solid angle, proportional to \( v_{j_{+}} \).

The qubit purification protocol we have discussed in Sec. VII does not contain dissipation, as in Refs. 46, 47. Those results would be changed in the presence of pure damping terms \( j_{-} < 0 \) and it would be more difficult to obtain a pure state. Here the choice of constraint \( u_{j_{-}} = 1 - u_{j_{+}} \) allows us to reduce the diffusive filtering equation of the observed system to a stochastic master equation where no dissipative terms appear. More precisely, the equation would have only the drift terms \( v_{R_{j_{-}}} \). For instance, the dynamics of a damped qubit, with only a dissipative term \( v_{j_{-}} \) and a measurement observables \( L_{1} \), as before, would be governed by a filtering equation formally identical to (31), but now with the possibility to control only the fluctuation operator \( \theta^{j_{-}} \). Again, however, we can decide to perform a measurement or not by looking at the minimal Hessian \( \Delta_{\theta}^{j_{-}} S(t, \theta) \).

VII. DISCUSSION

We have presented a general formalism for the optimal control of a quantum system subject to measurements, where the control can both be of a suitable feedback Hamiltonian and of the choice of future measurements carried out on the system. The use of measurements to prepare and protect pure and entangled quantum states can thus be made subject of systematic investigation, and optimal schemes can be devised for given physical setups.
It is the philosophy of our work that the quantum state of a controlled system, i.e., its von Neumann density matrix, is treated in the same way as classical control engineers treat the state of their classical systems. The Bellman principle can then be applied in the same way as for classical states. In the present work we derived the corresponding Hamilton-Jacobi-Bellman theory for a wider class of controls and cost functionals than traditionally considered in the literature.

Another interesting problem, which is explicitly solvable but is formulated in the infinite dimensional Hilbert space, is the setup problem for the quantum feedback control with “soft” constraints given by quadratic cost functions $c$ and $g$ in $u$. It reduces to the linear optimal control problem in the finite-dimensional space for the sufficient coordinates of the Gaussian Bosonic states exactly as in the classical linear-quadratic Gaussian case. For a more detailed discussion with proofs we refer to [S], and for a particular case in a more recent work by Yanagisawa [30].

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