On quantum key distribution in decoherence-free subspace

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We propose easy implementable protocols for robust quantum key distribution with the collective dephasing channel or collective rotating channel. In these protocols, Bob only takes passive photon detection to measure the polarization qubits in the random bases. The source for the protocol with collective rotating channel is made by type 2 spontaneous parametric down conversion with random unitary rotation and phase shifter, no quantum disentangler is required. A simple proof for unconditionally security is shown. Consider the following collective dephasing channel to polarization qubits [1]:

\[
|0\rangle \rightarrow e^{i\phi_0}|0\rangle; |1\rangle \rightarrow e^{i\phi_1}|1\rangle
\]
\[
|0\rangle|1\rangle \rightarrow e^{i\Delta}|0\rangle|1\rangle; |1\rangle|0\rangle \rightarrow e^{i\Delta}|1\rangle|0\rangle
\]

and \(\Delta = \phi_0 + \phi_1\). Here \(|0\rangle, |1\rangle\) represent for horizontal and vertical polarization states respectively. Obviously, 2-qubit state \(|01\rangle, |10\rangle\) and their arbitrary linear superposition will be robust via the above defined noise, since they will only obtain an unimportant overall phase shift which is not detectable. Naturally, one may consider to replace the original single qubit BB84 states [2] by the 2-qubit states randomly chosen from the set \(\{|01\rangle, |10\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}\) to do the quantum key distribution (QKD) [2,3] if the dominant channel noise is the above defined collective dephasing noise. There has been a proposal on how to experimentally realize the decoherence-free subspace QKD very recently [4,5]. However, no security proof for any of such type of protocols is given so far. Also, the existing protocol [5] with collective rotating error uses 4-photon state which is technically very difficult in practice. In this paper, we propose easy implementable robust QKD protocols with both collective dephasing channel [4] and collective rotating channel [5] and we also give a simple proof for the unconditional security in both cases. We first give a QKD protocol that tolerates the collective phase error as the following:

**protocol 1**: Alice creates \((4 + \delta)n\) single qubit state randomly chosen from \(\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}\) \(\{\pm\} = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\) and \((4 + \delta)n\) ancilla which are all in state \(|0\rangle\). She then encodes each individual qubit with an ancilla into a 2-qubit code through \(|00\rangle \rightarrow |01\rangle; |10\rangle \rightarrow |10\rangle\) where the second qubit states in the left hand-side of the arrows are for the ancilla. After this operation, she now has made \((4 + \delta)n\) 2-qubit quantum codes with each of them randomly chosen from the set \(\{|01\rangle, |10\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}\). For each 2-bit code, she puts down the “preparation basis” as “Z basis” \(\{|0\rangle, |1\rangle\}\) basis if it is in state \(|01\rangle\) or \(|10\rangle\) and the preparation basis as “X basis” \(\{|\pm\rangle\}\) if it is in one of the states \(\{\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}\). For those code states of \(|01\rangle\) or \(|10\rangle\), she denotes a bit value 0; for those code states of \(|10\rangle\) or \(|01\rangle\), she denotes a bit value 1.

3: Alice sends the 2-qubit codes to Bob. 4: Bob receives the \((4 + \delta)n\) 2-qubit codes. To each code, He measures the first qubit (the one in the right position of the state) in X basis and then measures the other qubit (say, qubit 2) in either X basis or Z basis after taking a unitary transformation \(U\) to it. Unitary \(U\) to qubit 2 is dependent on the measurement outcome of qubit 1: If the outcome is \(|+\rangle\), \(U\) is unity, \(I\); if it is \(|-\rangle\), \(U\) is \(\sigma_z\) which is defined by \(\sigma_z|0\rangle = |0\rangle; \sigma_z|1\rangle = -|1\rangle\). Bob denotes his “measurement basis” just the same as his measurement basis to qubit 2. And, if the outcome to qubit 2 is \(|0\rangle\) or \(|+\rangle\), he puts down a bit value 0; if the outcome to qubit 2 is \(|1\rangle\) or \(|-\rangle\), he puts down a bit value 1. 5: Alice announces her “preparation basis” for each codes. 6: Alice and Bob discards those bits on which Bob has measured in a basis different from Alice’s preparation basis. With high probability, there are at least \(2n\) bits left (if not, abort the protocol). Alice decides randomly on a set of \(n\) bits to use for the protocol, and the rest to be check bits. 7: Alice and Bob announce the values of their check bits. If too few of these values agree, they abort the protocol. 8: Alice and Bob distill the final key by using the classical CSS code [6].

Before we make the security proof, we first take a look at its fault tolerance property. Since we have already assumed that the main error of the physical channel is the collective dephasing error as defined by eq.1, all the 2-qubit codes will be sent robustly over the physical channel, since the collective dephasing error will now only offer a trivial overall phase factor \(e^{i\Delta}\). After Bob receives the 2-qubit codes, he decodes them in step 4. One may check it easily that after decoding, the original BB84 are recovered provided that Bob’s measurement basis to qubit 2 is same with Alice’s “preparation basis”. We now give a very simple security proof for the above protocol in the way that if the protocol above is insecure, then the known BB84 protocol with CSS code proposed by Shor and Preskill [6] is also insecure. We first compare our protocol here with Shor-Preskill protocol [6]. The only difference between them is the encoding and

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decoding in our protocol. Without these encoding and decoding, our protocol is exactly just Shor-Preskill protocol. Suppose our protocol is insecure then eavesdropper, Eve must have a certain intercept-and-resend scheme $S$ to obtain significant information to the final key in our protocol. Most generally, $S$ may contain the intercepting the qubits from Alice, operation $\hat{A}$ to all qubits and ancilla, resending the qubits to Bob and finally a certain operation $\hat{O}$ to the rest qubits with Eve. Operation $\hat{O}$ can include certain type of measurement which optimizes Eve’s information to the final key. We now show that if scheme $S$ can help Eve to obtain significant information to the final key of our protocol, then we can always construct a scheme $S'$ for Eve to obtain the same amount of information to the final key of Shor-Preskill protocol.

Since the encoding procedure in our protocol requires no information to the original single qubit state and the decoding procedure requires no information to the 2-qubit code state, the encoding and decoding can actually be done by anybody. Let’s consider the Shor-Preskill protocol. Eve now uses attack $S'$. In scheme $S'$, Eve replaces operation $\hat{A}$ by

$$\text{Encoding} \rightarrow \hat{A} \rightarrow \text{Decoding}$$

and everything else identical to that in scheme $S$. Note that in scheme $S'$, Eve resends the decoded single qubits to Bob and discards all “qubit 1” in each code after decoding. If Eve uses this $S'$ to Shor-Preskill’s protocol, everything will be identical with that in our protocol with Eve’s attack $S$. In other words, to Eve, the game of attacking Shor-Preskill protocol with scheme $S'$ is exactly identical to the game of attacking our protocol with $S$ since she may just regard her encoding and decoding as operations done by someone else. More specifically, given Shor-Preskill protocol with Eve’s attack $S'$ and our protocol with Eve’s attack $S$, everything of the two protocols are identical therefore Eve’s information to the final key of the two protocols must be exactly the same. This is to say, if our protocol is insecure, Shor-Preskill protocol must be also insecure. However, the security of Shor-Preskill protocol has been proven [6] already, therefore our protocol must be secure.

Actually, we can give a more general theorem about the security of any protocol with quantum code. Consider two protocols, protocol $P_0$ and protocol $P$. In protocol $P_0$, Alice directly sends Bob each individual qubits. In protocol $P$, Alice first encodes each individual qubits by a certain quantum code and then sends each quantum codes to Bob. Bob will decode the transmitted codes. After decoding, Bob obtains one single qubit state from each code and discarded all other qubits in the same code. Alice and Bob continue the protocol. Suppose except for the extra steps of encoding and decoding, everything else in protocol $P_0$ and protocol $P$ are identical and the encoding and decoding do not require any information of the quantum state of the original qubit or the codes, then we have the following theorem: If protocol $P_0$ is secure then protocol $P$ is also secure. The proof can obviously be done by the similar argument: Eve can also do the encoding and decoding, if $P$ is insecure under attacking scheme $S$, we can always construct an attack $S'$ for Eve to attack $P_0$ effectively.

Note that there are two important points here. First, the encoding must be independent of the original quantum state itself therefore Eve can also do so. Also this shows that the encoding does not offer extra information to the qubit state. The simple repetition code does not work because the encoding needs the information to the original quantum state and Eve is not able to do so. Also the repetition code will offer more information to the bit value. Secondly, Alice and Bob test the error rate after the decoding, that is to say, they directly test the error rate to the decoded qubits instead of testing the error rate of each qubits before decoding and then mathematically deduce the error rate after decoding by using the collective dephasing property of the physical channel. Note that Eve may change all properties of the physical channel when the QKD task is being done. But in our protocol the error test is done after the decoding, this will defeat any type of Eve’s attack. Since Eve does not want to be detected, she has to consider the error test done by Alice and Bob latter on. Therefore she must limit herself to those types of attack which will not affect the result of error test to be done by Alice and Bob. (Here Eve does not have to respect the error rate of the physical channel since that rate will not be tested in the protocol, but Eve has to respect the error rate after decoding since this is the issue that will be tested.) That is to say, if Eve is not detected by the error test, the protocol will work as effectively as the case that there is no Eve, since the key rate of the final key is uniquely determined by the error rate in test.

After the above security proof, we can now further simplify our protocol. The encoding procedure in step 1 is just the local operation by Alice. She does not have to really first prepare the single qubit state and then encodes it by controlled-not (CNOT) operation. Instead, she may directly prepare the 2-qubit codes with each of them randomly chosen from $\{\vert 01 \rangle, \vert 10 \rangle, \frac{1}{\sqrt{2}} (\vert 01 \rangle + \vert 10 \rangle), \frac{1}{\sqrt{2}} (\vert 01 \rangle - \vert 10 \rangle)\}$ . These random codes can be produced with type II spontaneous parametric down conversion (SPDC) [8] with a certain filter [4]. In step 4 of our protocol, Bob measures qubit 2 in either X basis or Z basis. This requires Bob to change measurement basis rapidly. However, this task can be done by passive detection with a 50:50 beam splitter, see figure 1. In step 4, Bob takes unitary transformation $U$ to qubit 2 according to the measurement result of qubit 1. However, since we have already limited ourselves to the 4 BB84 states, instead of taking $U$ before the measurement to qubit 2, he may directly measure qubit 2 and then determine the corresponding bit value $b$ according to the measurement result of both qubit 1 and qubit 2 through the following rule:

$$\{\vert + \rangle\langle 0 \vert, \vert - \rangle\langle 0 \vert, \vert + \rangle\langle + \vert, \vert - \rangle\langle - \vert\} \rightarrow b = 0;$$
\{\ket{\pm}|\ket{1}, \ket{-}|\ket{1}, \ket{+}|\ket{-}, \ket{-}|\ket{+}\} \rightarrow b = 1. \tag{2}

Note that the state preparation and the bit value detection are all local operations, nobody outside is able to know how they have actually completed the tasks.

![Diagram](image)

Intuitively, the collective errors could happen more often to qubits transmitted in the same line. This type of source can be produced by figure 1. Note that the nonlinear crystal there can emit either fully entangled state or product state, dependent on the polarization of pump beam [7]. We can use the following figure to realize the QKD protocol in decoherence free subspace:

![Diagram](image)

The security is obvious if we regard the operation by the dashed rectangular as part of the decoding. This is now a probabilistic decoding with a rate 75% to discard the code. Consider the case of Shor-Preskill protocol where Eve uses scheme S' for attacking. Eve detects the right vertical beam in \{\ket{\pm}\} basis and also observes the photon number of left vertical beam in the same time. If there is no photon, he blocks the beam and sends Bob nothing. (Even though there is no photon at that time point, there could be a delayed photon in the same beam. Bob blocks the possible delayed photon.) If there is one photon, he takes unitary transformation U to that photon and sends it to Bob. U is defined in step 4 of our protocol. Here Eve has used a very sophisticated photon detector to detect the photon number but not disturb the polarization on the left vertical beam. If Eves takes such operation, to Alice and Bob, they are carrying out Shor-Preskill protocol with a lossy channel with the lossy rate of 75%. However, we know that Shor-Preskill protocol is secure even the channel is lossy, since the corresponding entanglement purification also works with lossy channel. Therefore the protocol in fig.2 must be secure since otherwise Shor-Preskill protocol would be insecure with lossy channel.

Having considered the QKD protocol with the collective dephasing channel, we now consider the case with collective equatorial rotating channel. Such a channel adds collective rotation \(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\) to each individual qubits with \(\theta\) value being random but same to all qubits simultaneously transmitted. A B92-like protocol was proposed recently [5], the protocol tolerates the collective rotation error [5]. However the security for that protocol is not proven [5]. Also in that protocol, they use 4 qubits to encode one bit, this will be technically very difficult and greatly decrease the efficiency because of the small emission probability of SPDC process. (Even though the probability can be improved in the future, one still cannot arbitrarily use large emission probability because this will cause the unwanted higher order states.) To overcome these drawbacks, we propose a new protocol here. Our new protocol has the following advantages: 1) the security proof is given; 2) we use a 2-qubit state to encode one bit 3) the protocol is BB84-like and the efficiency is further improved compared with the one given in ref. [5]. Consider the fact that both 2-qubit state \(\ket{\phi^+}\) and state \(\ket{\psi^-} = \frac{1}{\sqrt{2}}(\ket{01} - \ket{10})\) are unchanged after any collective equatorial rotation to both qubits. We can use these two states and their 2 linear superpositions to replace the BB84 state in QKD. Specifically, the states transmitted from Alice to Bob are randomly chosen from \(\{\ket{\phi^+}, \ket{\psi^-}\}, \frac{1}{\sqrt{2}}(\ket{\phi^+} + \ket{\psi^-}), \frac{1}{\sqrt{2}}(\ket{\phi^+} - \ket{\psi^-})\). Note that \(\frac{1}{\sqrt{2}}(\ket{\phi^+} \pm \ket{\psi^-}) = \frac{1}{\sqrt{2}}(\ket{0} \pm \ket{1})\). Therefore this type of source can be easily made through combining the type II SPDC emission [8] and an electrically driven unitary crystal to make random unitary transformations to one of the two emitted photons. The state from type II SPDC emission is \(\ket{\phi^+}\). The random unitary transformations produced by the unitary crystal include unitarity, joint phase-flip and bit-flip, Hadamard transform and Hadamard transform with a flipping in \(\{\ket{\pm}\}\) basis. Therefore the required 4 2-bit states will be produced randomly with equal distribution. Unlike protocol in the case of dephasing channel [4], here we don’t have to use a disentangler to generate the source.

According to our theorem, to show the security of this protocol, we need only to show that these 4 2-bit states can be in principle generated from BB84 states without any information of the BB84 state and Bob can recover the original BB84 state on a single qubit by decoding the 2-bit code. If we can construct the encoding and
decoding, the security of the protocol is equivalent to that of BB84 protocol. The encoding indeed exists. To encode, Alice makes the following unitary transformation on the BB84 qubit (the first qubit) and the ancilla (the second qubit) which is in state $|0\rangle$: \[
|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle); |10\rangle \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \quad (3)
\]
In such a way, BB84 state $|\pm\rangle$ with ancilla $|0\rangle$ will become $\frac{1}{\sqrt{2}}(|\phi^+\rangle \pm |\psi^-\rangle)$. So in principle, there is indeed an encoding scheme to produce the requested 4 2-bit states from BB84 states. Now we show Bob can recover the outcome of the first qubit requires overall phase factor on the second qubit. However, such

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\[
\{ |0\rangle|0\rangle, |1\rangle|1\rangle, |0\rangle|+, |1\rangle|\rangle \} \rightarrow b = 0;
\{ |0\rangle|1\rangle, |1\rangle|0\rangle, |0\rangle|-, |1\rangle|\rangle \} \rightarrow b = 1. \quad (4)
\]

Note that Bob's record on "measurement basis" to each encoded bit is same as his measurement basis to the second qubit. He will discard those results obtained from a wrong measurement basis through classical communication with Alice.

Moreover, if we exchange the two qubits before any measurement done by Bob, all results of the protocol is unchanged. That is to say, here Bob can chose any qubit as "qubit 1". Thus in the application Bob just make sure the measurement basis to one qubit is Z, while the measurement to the other qubit can be either X or Z and then use the rule of eq.(4) to determine the bit value. The experimental scheme is given by the following figure 3. Bob will use the results of any 2-fold clicking events in $\{(D_1, D_i), (D_3, D_i); i = 1, 2, 3, 4\}$. He determines the bit value according to eq.(4). Note that the 2-fold event $(D_1, D_1)$ or $(D_3, D_3)$ means that detector D1 or D3 clicks 2 times at different time. Those events of 2-fold clicking which contain neither D1 nor D3 will be discarded.

![FIG. 3. Robust quantum key distribution with collective equatorial rotating channel. $U$ is a crystal driven electrically to produce random unitary transformation selected from $\{I, \Sigma, H, \sigma_x H\}$. $H$ is a Hadamard transform which interchanges $\{\langle 0 |, \langle 1 |\}$ and $\{\langle + |, \langle - |\}$, respectively. The devices inside the dashed rectangular are denoted by C.

Foe completeness, we give the following protocol with collective rotating channel:

**Protocol 2.** 1: Alice creates $(4 + \delta)n$ 2-qubit quantum codes with each of them randomly chosen from the set $\{\langle \phi^+ \rangle, \langle \psi^- \rangle, \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\psi^-\rangle), \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\psi^-\rangle)\}$. 2: For each 2-bit code, if it is in state $|\phi^+\rangle$ or $|\psi^-\rangle$ she puts down the "preparation basis" as "Z basis" otherwise she puts down the "preparation basis" as "X basis". For those code states of $|\phi^+\rangle$ or $|\psi^-\rangle$, she denotes a bit value 0; for those code states of $|\psi^-\rangle$ or $\frac{1}{\sqrt{2}}(|\phi^+\rangle - |\psi^-\rangle)$, she denotes a bit value 1. 3: Alice sends the 2-qubit codes to Bob. 4: Bob receives the $(4 + \delta)n$ 2-qubit codes. To each code, He measures one qubit (say, qubit 1) in Z basis and measures the other qubit (say, qubit 2) in either X basis or Z basis. Bob denotes his "measurement basis" just the same of his measurement basis to qubit 2. And he determines the bit value corresponding to each code by eq.(4). 5: Alice announces her "preparation basis" for each codes. 6: Alice and Bob discards those bits on which Bob has measured in a basis different from Alice’s preparation basis. With high probability, there are at least 2n bits left (if not, abort the protocol). Alice decides randomly on a set of n bits to use for the protocol, and the rest to be check bits. 7: Alice and Bob announce the values of their check bits. If too few of these values agree, they abort the protocol. 8: Alice and Bob distill the final key by using the classical CSS code [6].

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