Elastic Membrane Equation with Dynamic Boundary Conditions and Infinite Memory

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Abstract: In this paper, we study the elastic membrane equation with dynamic boundary conditions, source term and a nonlinear weak damping localized on a part of the boundary and past history. Under some appropriate assumptions on the relaxation function the general decay for the energy have been established using the perturbed Lyapunov functionals and some properties of convex functions.

Key Words: Elastic membrane equation, Energy decay, Balakrishnan-Taylor damping, Dynamic boundary conditions, Infinite memory.

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1. Introduction

The objective of this work is to study the following problem

\begin{align}
\begin{cases}
    u_{tt} - M(t) \Delta u + \int_0^\infty g(s) u(t-s) ds = 0, & \text{in } \Omega \times (0, +\infty), \\
    u = 0, & \text{on } \Gamma_0 \times (0, +\infty), \\
    u_{tt}(t) + \frac{\partial u(t)}{\partial \nu} - \int_0^\infty g(t-s) \frac{\partial u(t)}{\partial \nu} (t-s) ds - h(u_t) - f(u), & \text{on } \Gamma_1 \times (0, +\infty), \\
    u(x, -t) = u_0(x), u_t(x, 0) = u_1(x), & \text{in } \Omega.
\end{cases}
\end{align}

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ such that $\partial \Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\Gamma_0$, $\Gamma_1$ have positive measure $\lambda_{n-1}(\Gamma_i)$, $i = 0, 1$, $\nu$ denotes the unit outer normal vector pointing towards the exterior of $\Omega$ and $M(t) = \xi_0 + \xi_1 \| \nabla u(t) \|^2 + \sigma (\nabla u(t), \nabla u_t(t))$, where $u$ is the plate transverse displacement, $x$ is the spatial coordinate in the direction of the fluid flow, and $t$ is the time. The viscoelastic structural damping terms are denote by $\sigma$, $\xi_1$ is the nonlinear stiffness of the membrane, $\xi_0$ is an in-plane tensile load. All quantities are physically non-dimensionalized $\xi_0, \xi_1, \sigma$ and $\alpha$ are fixed positive. Equation (1.1) is related to the flutter panel equation with memory term this equation arises in a wind tunnel experiment for a panel at supersonic speeds. For a derivation of this model see, for instance, Dowell [14], Holmes [24, 25], Bass [5].

For more results concerning Balakrishnan-Taylor equation, one can refer to Zaraï and Tatar [2, 3]. For viscoelastic wave equation with Dirichlet boundary condition, the problems are truly overworked. Many existence and stability results have been established, Cavalcanti and Oquendo [10], Fabrizio and Polidoro [15], Messaoudi [30, 34]. For linear Cauchy viscoelastic problem, one can refer to Kafini and Mustafa [26]. With respect to viscoelastic wave equation with boundary stabilization, Cavalcanti [8 – 11] considered the following system

\begin{align}
\begin{cases}
    u_{tt} - \Delta u(t) + \int_0^\infty g(s) u(t-s) ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\
    u = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\
    \frac{\partial u(t)}{\partial \nu} - \int_0^\infty g(t-s) \frac{\partial u(t)}{\partial \nu} (t-s) ds + h(u_t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+, \\
    u(0) = u_0(x), u_t(0) = u_1(x), & x \in \Omega, \\
    x \in \Omega.
\end{cases}
\end{align}
Under the following assumptions on functions $h$,
\[
\begin{align*}
C_1 |s|^p &\leq |h(s)| \leq C_2 |s|^p, \quad \text{if } |s| \leq 1 \\
C_3 |s| &\leq |h(s)| \leq C_4 |s|, \quad \text{if } |s| > 1
\end{align*}
\]
the authors first proved the global existence of solutions, and obtained the energy decays exponentially if $p = 1$ and decays polynomially if $p > 1$. The results were generalized by Cavalcanti et al. [9]. They obtained the same results without imposing a growth condition on $h$ and under a weaker assumption on $g$. Messaoudi and Mustafa [26] extended these results and established an explicit and general decay rate result by exploiting some properties of convex functions. Recently, using the same method as in [26], Messaoudi et al. [33] considered the above wave system with infinite memory \( \int_0^\infty g(s)\Delta u(t-s)ds \) and obtained a general decay result using multiplier method. Gerbi and Said-Houari [21] studied a viscoelastic wave equation with dynamic boundary conditions of the form
\[
\begin{align*}
u_{tt}(t) - \Delta u(t) - \alpha \Delta \xi_t + \int_0^\infty g(t-s)\Delta u(s)ds = |u|^{p-2}u, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
u = 0, \quad &\text{on } \Gamma_0 \times \mathbb{R}^+, \\
u_{tt}(t) = -\left[ \frac{\partial u}{\partial \nu} - \alpha \frac{\partial u}{\partial \nu} \int_0^\infty g(t-s)\frac{\partial u(s)}{\partial \nu}ds + \frac{\partial \xi}{\partial \nu} \right], \quad &\text{on } \Gamma_1 \times \mathbb{R}^+, \\
u(0) = u_0(x), u_t(0) = u_1(x), \quad &x \in \Omega,
\end{align*}
\]
Using the Faedo-Galerkin method and fixed point theorem, they proved the existence and uniqueness of a local in time solution, and proved the solution exists globally in time under some restrictions on the initial data. They also proved if $\alpha > 0$, the solution is unbounded and grows as an exponential function, if $\alpha = 0$, then the solution ceases to exist and blows up in finite time. Ferhat and Hakem [11] considered the same viscoelastic wave equation as in [14] but with the following dynamic boundary conditions
\[
u_{tt}(t) = -\left[ \frac{\partial u}{\partial \nu} \int_0^\infty g(t-s)\frac{\partial u(s)}{\partial \nu}ds + \frac{\partial \xi}{\partial \nu} \right], \quad \text{on } \Gamma_1 \times \mathbb{R}^+
\]
They established a general decay result by introducing suitable energy and Lyapunov functionals and some properties of convex functions. Ferhat and Hakem [12] investigated the following system
\[
\begin{align*}
u_{tt}(t) - \Delta u(t) - \alpha \Delta \xi_t + \delta(t) \int_0^\infty g(t-s)\Delta u(s)ds = |u|^{p-2}u, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
u = 0, \quad &\text{on } \Gamma_0 \times \mathbb{R}^+, \\
u_{tt}(t) = -\left[ \frac{\partial u}{\partial \nu} \int_0^\infty g(t-s)\frac{\partial u(s)}{\partial \nu}ds + \alpha \frac{\partial \xi}{\partial \nu} \right], \quad &\text{on } \Gamma_1 \times \mathbb{R}^+, \\
u(0) = u_0(x), u_t(0) = u_1(x), \quad &x \in \Omega,
\end{align*}
\]
They proved the global existence and energy decay of solutions for this system. Ferhat and Hakem [18] considered a weak viscoelastic wave equation with dynamic boundary conditions and Kelvin Voigt damping and delay term acting on the boundary in a bounded domain, and proved the asymptotic behavior by making an appropriate Lyapunov functional. Recently, Benahiss and Ferhat [6] considered a viscoelastic wave equation with dynamic boundary conditions and infinite memory
\[
\begin{align*}
u_{tt}(t) - \Delta u(t) + \int_0^\infty g(s)\Delta u(t-s)ds = 0, \quad &\text{in } \Omega \times \mathbb{R}^+, \\
u = 0, \quad &\text{on } \Gamma_0 \times \mathbb{R}^+, \\
u_{tt}(t) = -\left[ \frac{\partial u(t)}{\partial \nu} \int_0^\infty g(s)\frac{\partial u(t-s)}{\partial \nu}ds \right], \quad &\text{on } \Gamma_1 \times \mathbb{R}^+, \\
u(0) = u_0(x), u_t(0) = u_1(x), \quad &x \in \Omega,
\end{align*}
\]
and established an exponential decay result of energy by exploiting the frequency domain method which consists in combining a contradiction argument and a special analysis for the resolvent of the operator under the assumption $-\zeta_0 g(t) \leq g'(t) \leq \zeta_0 g(t)$. For more results concerned with wave equation with boundary stabilization, one can refer to Doronin and Larkin [13], Muñoz Rivera and Andrade [35], Gerbi and Said-Houari [20–22], Liu and Yu [29]. Since there are few works on wave equation with dynamic boundary conditions, source term and a nonlinear weak damping localized on a part of the boundary.
and past history, motivated by above scenario, we study in the present work the stability of solutions to problem (1.1) – (1.4). The main objective of the present work is to establish an explicit and general decay result using multiplier method and some properties of convex functions. Our result is obtained without imposing any restrictive growth assumption on the damping term. We end this section by establishing the usual history setting of problem (1.1) – (1.4). Following the same arguments of Dafermos [12], we introduce a new variable

\[ \eta(x, t, s) = u(t) - u(t - s), \]

which gives us

\[ \eta_t + \eta_s = u_t. \]

Assuming \( \xi_0 - \int_0^1 g(s)ds = l \) then we can get a new system, which is equivalent to problem (1.1)

\[
\begin{cases}
\quad u_{tt}(t) - \left( l + \xi_1 \|u\|^2 + \sigma (\nabla u, \nabla u_t) \right) \Delta u(t) - \int_0^1 g(s)\Delta \eta(s)ds = 0 & \text{in } \Omega \times \mathbb{R}^+, \\
u = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\
u_{tt}(t) = \left( l + \xi_1 \|u\|^2 + \sigma (\nabla u, \nabla u_t) \right) \frac{\partial u(t)}{\partial n} & \text{on } \Gamma_1 \times \mathbb{R}^+, \\
u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]

The rest of this paper is as follows. In Sect.2, we give some assumptions and our main results. In Sect.3, we establish the general decay result of the energy. In this paper we will use a lot of concepts and techniques contained in Feng [16].

2. Assumptions and Main Results

In this section, we present some materials and assumptions used in this paper. \( L^q(\Omega), 1 \leq q \leq \infty \), and \( H^1(\Omega) \) denote Lebesgue integral and Sobolev spaces \( \| \cdot \|_q \) and \( \| \cdot \|_{q, \Gamma_1} \) are the norm in the space \( L^q(\Omega) \) and \( L^q(\Gamma_1) \), respectively. For simplicity, we write \( \| \cdot \| \) and \( \| \cdot \|_{\Gamma_1} \) instead of \( \| \cdot \|_2 \) and \( \| \cdot \|_{2, \Gamma_1} \), respectively \( C \) is used to denote a generic positive constant. Denote

\[
H^1_{\Gamma_0}(\Omega) = \{ u \in H^1(\Omega) : u|\Gamma_0 = 0 \}
\]

then we have the embedding \( H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Gamma_1) \). We will usually use the following Green’s formula

\[
\int_\Omega \nabla u(x) \nabla w(x) dx = - \int_\Omega \Delta u(x) w(x) dx + \int_{\Gamma_1} \frac{\partial u}{\partial \nu}(x) w(x) d\Gamma, \forall w \in H^1_{\Gamma_0}(\Omega).
\]

To deal with the new variable \( \eta \), we introduce a weighted \( L^2 \) space

\[
M = L^2_g(\mathbb{R}^+; H^1_{\Gamma_0}(\Omega)) = \left\{ \zeta : \mathbb{R}^+ \rightarrow H^1_{\Gamma_0}(\Omega) : \int_0^\infty g(s) \| \nabla \zeta(s) \|^2 ds < \infty \right\},
\]

which is Hilbert space endowed with inner product and norm

\[
\langle \zeta, \vartheta \rangle_M = \int_0^\infty g(s) \left( \int_\Omega \nabla \zeta(s) \nabla \vartheta(s) dx \right) ds,
\]

and

\[
\| \zeta \|_M^2 = \int_0^\infty g(s) \| \nabla \zeta(s) \|^2 ds.
\]

The phase space \( \hat{H} \) is defined by

\[
\hat{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times M.
\]

In the sequel, we shall give some assumptions. For the relaxation function \( g \), we assume:

(A1) \( g(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a nonincreasing \( C^1 \) function satisfying

\[
g(0) > 0 \text{ and } \xi_0 - \int_0^\infty g(s)ds = l > 0.
\]
In addition, there exists an increasing strictly convex function \( G: \mathbb{R}^+ \mapsto \mathbb{R}^+ \) of class \( C^1 (\mathbb{R}^+) \cap C^2 (\mathbb{R}^+) \) satisfying
\[
G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty
\] (2.2)
such that
\[
\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty
\] (2.3)

(A2) \( h: \mathbb{R} \mapsto \mathbb{R} \) is a nondecreasing \( C^0 \) function such that there exists a strictly increasing function \( h_0 \in C^1 (\mathbb{R}^+) \) with \( h_0(0) = 0 \) and positive constants \( c_1, c_2 \) and \( \varepsilon \) such that
\[
\begin{aligned}
& h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|); \quad \text{if } |s| \leq \varepsilon \\
& c_1 |s| \leq |h(s)| \leq c_2 |s| \quad \text{if } |s| > \varepsilon
\end{aligned}
\] (2.4)
Moreover we suppose that the function \( H(s) = \sqrt{s}h_0 (\sqrt{s}) \) is a strictly convex \( C^2 \) function on \((0, r^2] \) for some \( r > 0 \) when \( h_0 \) is nonlinear.

(A3) We assume \( f: \mathbb{R} \mapsto \mathbb{R} \) that for some \( c_0 > 0 \),
\[
|f(u) - f(v)| \leq c_0 (1 + |u|^p + |v|^p) |u - v|, \quad \forall u, v \in \mathbb{R}
\] (2.5)
where
\[
0 < p < \frac{2}{n - 2} \quad \text{if } n \geq 3 \quad \text{and} \quad p > 0 \quad \text{if } n = 1, 2.
\]
In addition, we assume that
\[
f(u)u \geq F(u) \geq 0; \quad \forall u \in \mathbb{R}
\] (2.6)
where \( F(z) = \int_0^z f(s) ds \). Assumptions (2.5) – (2.6) include nonlinear terms of the form
\[
f(u) \approx |u|^p u + |u|^\alpha u, \quad 0 < \alpha < p.
\]

(A4) There exists a positive constant \( m_0 \) such that
\[
\| \nabla u_0 (., s) \| \leq m_0.
\] (2.7)
The same arguments as in [5], [15] and [32], we can prove the global existence of solutions to problem (1.5) – (1.8) given in the following theorem.

**Theorem 2.1.** Suppose assumptions (A1) – (A4) hold. If the initial data \((u_0 (., 0), u_1, \eta_0) \in \dot{H} \) then problem (1.5) – (1.8) has a unique weak solution such that for any \( T > 0 \),
\[
u(t) \in L^\infty ([0, T]; H^1_{\Sigma_0} (\Omega)), \quad u_t (t) \in L^\infty ([0, T]; L^2(\Omega)) \quad \text{and} \quad \eta \in L^\infty ([0, T]; M).
\]
The energy functional of problem (1.5) – (1.8) is defined by
\[
E(t) = \frac{1}{2} \left\{ \| u_t (t) \|^2 + \| u_t (t) \|^2_{\Gamma_1} + \int_{\Gamma_1} F(u) d\Gamma + \| \eta \|^2_M + l \| \nabla u \|^2 + \frac{b}{2} \| \nabla u \|^4 \right\}
\] (2.8)
Then we can get the stability result of energy to problem (1.5) – (1.8) given in following theorem.

**Theorem 2.2.** Suppose (A1) – (A4) hold, let \((u_0 (., 0), u_1, \eta_0) \in \dot{H}, \) then there exist positive constants \( k_2, k_3, k_4, \varepsilon_1, \varepsilon_0 \) such that the energy \( E(t) \) defined by (2.8) satisfies
\[
E(t) \leq k_1 W_1^{-1} (k_2 t + k_3), \forall t \in \mathbb{R}^+,
\] (2.9)
where
\[
W_1(t) = \int_0^1 \frac{1}{W_2(s)} ds \quad \text{and} \quad W_2(t) = tG' (\varepsilon_1 t) H' (\varepsilon_0 t).
\]
3. General Decay

In this section, we shall study the general decay of energy to problem (1.5) – (1.8) to prove Theorem 2. For this purpose, we need the following technical lemmas.

Lemma 3.1. Under the assumptions of Theorem 2, the energy functional $E(t)$ is non-increasing and satisfies that for any $t \geq 0$,

$$E'(t) \leq -\sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u \|^2 \right)^2 - \int_{\Gamma_1} h(u_t)u_t d\Gamma + \frac{1}{2} \int_{\Gamma} g'(s) \| \nabla \eta(s) \|^2 ds \quad (3.1)$$

Proof. Multiplying (1.5) by $u_t$, and using integration by parts, boundary conditions (1.6) – (1.7) and Green’s formula, we can obtain that

$$\frac{1}{2} \frac{d}{dt} \left( \| u_t(t) \|^2 + \| u_t(t) \|^2_{\Gamma_1} + t \| \nabla u \|^2 + \frac{b}{2} \| \nabla u \|^4 + \int_{\Gamma_1} F(u) d\Gamma \right)$$

$$+ \sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u \|^2 \right)^2 + \int_{\Gamma_1} h(u_t)u_t d\Gamma$$

$$+ \int_{\Omega} \nabla u_t \int_0^\infty g(s) \nabla \eta(s) ds dx = 0 \quad (3.2)$$

Note that

$$\int_{\Omega} \nabla u_t \int_0^\infty g(s) \nabla \eta(s) ds dx = \frac{1}{2} \frac{d}{dt} \| \eta(t) \|^2_M - \frac{1}{2} \int_{\Gamma} g'(s) \| \nabla \eta(s) \|^2 ds \quad (3.3)$$

Inserting (3.3) into (3.2), we can get the desired estimate (3.1). Using (A2), we know that $h(u_t)u_t \geq 0$. Then $E'(t) \leq 0$. The proof is complete. $\square$

Lemma 3.2. Under the assumptions of Theorem 2, the functional $\phi(t)$ defined by

$$\phi(t) = \int_{\Omega} u_t(t)u(t) dx + \int_{\Gamma_1} u_t(t)u(t) d\Gamma + \frac{\sigma}{4} \| \nabla u \|^4 \quad (3.4)$$

satisfies that for any $t \geq 0$,

$$\phi'(t) \leq \| u_t(t) \|^2 + \| u_t(t) \|^2_{\Gamma_1} - (I - \delta_1 (1 + c)) \| \nabla u \|^2 - \xi_1 \| \nabla u \|^4$$

$$+ C_1 \int_{\Gamma} \int_0^\infty g(s) \| \nabla \eta(s) \|^2 ds + C_2 \int_{\Gamma_1} h^2(u_t) d\Gamma \quad (3.5)$$

Proof. Differentiating $\phi(t)$ with respect to $t$, we obtain

$$\phi'(t) = \| u_t(t) \|^2 + \| u_t(t) \|^2_{\Gamma_1} + \int_{\Omega} u_{tt}(t)u(t) dx + \int_{\Gamma_1} u_{tt}(t)u(t) d\Gamma + \sigma \| \nabla u \|^2 (\nabla u(t), \nabla u_t(t)) \quad (3.6)$$

We infer from (1.7) and Green’s formula that

$$\int_{\Omega} u_{tt}(t)u(t) dx + \int_{\Gamma_1} u_{tt}(t)u(t) d\Gamma$$

$$= \int_{\Omega} \left[ \left( I + \xi_1 \| \nabla u \|^2 + \sigma (\nabla u, \nabla u_t) \right) \Delta u(t) + \int_0^\infty g(s) \Delta \eta(s) ds \right] u dx$$

$$+ \int_{\Gamma_1} \left[ \left( I + \xi_1 \| \nabla u \|^2 + \sigma (\nabla u, \nabla u_t) \right) \frac{\partial u}{\partial \nu} \right] - \int_0^\infty g(s) \frac{\partial \eta}{\partial \nu}(s) ds - h(u_t) - f(u) \right] u d\Gamma$$

$$= - \left( I + \xi_1 \| \nabla u \|^2 \right) \| \nabla u \|^2 - (\sigma (\nabla u, \nabla u_t)) \| \nabla u \|^2 - \int_{\Omega} \nabla u \int_0^\infty g(s) \nabla \eta(s) ds dx$$

$$- \int_{\Gamma_1} uh(u_t) d\Gamma - \int_{\Gamma_1} uf(u) d\Gamma$$
which, combined with (3.6), gives us
\[
\varphi'(t) = \|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1} - \left(1 + \xi_1 \|\nabla u\|^2\right) \|\nabla u\|^2
\]
\[
- \int_\Omega \nabla u \int_0^\infty g(s) \nabla \eta(s) \, ds \, dx - \int_{\Gamma_1} uh(u_t) \, d\Gamma - \int_{\Gamma_1} uf(u) \, d\Gamma
\]  
(3.7)

Using Young’s inequality, Holder’s inequality and Poincaré’s inequality, we know that for any \(\delta_1 > 0\),
\[
- \int_\Omega \nabla u \int_0^\infty g(s) \nabla \eta(s) \, ds \, dx
\]
\[
\leq \delta_1 \|\nabla u\|^2 + \frac{1}{4\delta_1} \int_\Omega \left[\int_0^\infty g(s) \nabla \eta(s) \, ds\right]^2 dx
\]
\[
\leq \delta_1 \|\nabla u\|^2 + \frac{1}{4\delta_1} \int_\Omega \left[\int_0^\infty g(s)ds\right] \left[\int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds\right] dx
\]
\[
\leq \delta_1 \|\nabla u\|^2 + \frac{(\xi_0 - l)}{4\delta_1} \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds
\]  
(3.8)

and
\[
- \int_{\Gamma_1} uh(u_t) \, d\Gamma \leq C_1 \delta_1 \|\nabla u\|^2 + \frac{1}{4\delta_1} \int_{\Gamma_1} h^2(u_t) \, d\Gamma
\]
(3.9)

Inserting (3.8) – (3.9) into (3.7), we get for any \(\delta_1 > 0\)
\[
\varphi'(t) \leq \|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1} - (l - \delta_1 (1 + C_1)) \|\nabla u\|^2 - \xi_1 \|\nabla u\|^4
\]
\[
+ \frac{(\xi_0 - l)}{4\delta_1} \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds
\]
\[
+ \frac{1}{4\delta_1} \int_{\Gamma_1} h^2(u_t) \, d\Gamma - \int_{\Gamma_1} uf(u) \, d\Gamma
\]  
(3.10)

Now, we take \(\delta_1 > 0\) so small that
\[
l - \delta_1 (1 + C_1) > \frac{l}{2}.
\]

Thus, (3.5) follows from (3.10) and (2.6) with
\[
C = \max \left\{ \frac{1}{4\delta_1}, \frac{\xi_0 - l}{4\delta_1} \right\}.
\]

The proof is complete. \(\square\)

**Lemma 3.3.** Define the functional \(\psi(t)\) as
\[
\psi(t) = - \int u_t(t) \int_0^\infty g(s) \eta(s) \, ds \, dx - \int_{\Gamma_1} u_t(t) \int_0^\infty g(s) \eta(s) \, ds \, d\Gamma.
\]

Under the assumptions of Theorem 2, then the functional \(\psi(t)\) satisfies for any \(\delta_2 > 0\),
\[
\psi'(t) \leq -3 \left(\frac{\xi_0 - l}{4}\right) \|u_t\|^2 - 3 \left(\frac{\xi_0 - l}{4}\right) \|u_t\|^2_{\Gamma_1} + \delta_2 \|\nabla u\|^2 + \sigma^2 E(0) \left(\frac{1}{2\frac{dt}{dt}} \|\nabla u\|^2\right)^2
\]
\[
+ K_1 \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds + K_2 \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds + \frac{1}{2} \int_{\Gamma_1} h^2(u_t) \, d\Gamma.
\]  
(3.11)
Proof. Differentiating \( \psi(t) \) with respect to \( t \), we can obtain that

\[
\psi'(t) = -\int_\Omega u_{tt}(t) \int_0^\infty g(s) \eta(s) \, ds \, dx - \int_{\Gamma_1} u_{tt}(t) \int_0^\infty g(s) \eta(s) \, ds \, d\Gamma \\
\quad - \int_\Omega u_t(t) \int_0^\infty g(s) \eta_t(s) \, ds \, dx - \int_{\Gamma_1} u_t(t) \int_0^\infty g(s) \eta_t(s) \, ds \, d\Gamma \\
\quad =: I_1,
\]

\[
 \quad - \int_\Omega u_t(t) \int_0^\infty g(s) \eta_t(s) \, ds \, dx
\]

which, using (1.7), yields

\[
I_1 = \int_\Omega \left( l + \xi_1 \|\nabla u\|^2 + \sigma (\nabla u, \nabla u_t) \right) \nabla u(t) \int_0^\infty g(s) \nabla \eta(s) \, ds \, dx \\
\quad + \int_\Omega \left( \int_0^\infty g(s) \nabla \eta(s) \, ds \right)^2 \, dx + \int_{\Gamma_1} f(u) \int_0^\infty g(s) \eta(s) \, ds \, d\Gamma \\
\quad + \int_{\Gamma_1} h(u_t) \int_0^\infty g(s) \eta(s) \, ds \, d\Gamma.
\]

(3.13)

Since \( E(t) \) is nonincreasing, then we can infer from (2.9) that

\[
\frac{l}{2} \|\nabla u\|^2 \leq E(t) \leq E(0)
\]

which gives us

\[
\|\nabla u\|^2 \leq \frac{2}{l} E(0).
\]

(3.14)

Performing Hölder’s and Young’s inequalities, (2.5) and (2.8), we infer that for any \( \delta_3 > 0 \)

\[
\int_\Omega \left( l + \xi_1 \|\nabla u\|^2 + \sigma (\nabla u, \nabla u_t) \right) \nabla u(t) \int_0^\infty g(s) |\nabla \eta(s)| \, ds \, dx \\
\leq \left( l + \frac{2 \xi_1}{l} E(0) \right) \int_\Omega \nabla u(t) \int_0^\infty g(s) |\nabla \eta(s)| \, ds \, dx \\
\leq \delta_3 \|\nabla u\|^2 + \frac{1}{4 \delta_3} \left( l + \frac{2 \xi_1}{l} E(0) \right)^2 \int_\Omega \int_0^\infty g(s) |\nabla \eta(s)| \, ds \, dx \\
\leq \delta_3 \|\nabla u\|^2 + \frac{\xi_0 - l}{4 \delta_3} \left( l + \frac{2 \xi_1}{l} E(0) \right)^2 \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds
\]

(3.15)
\[
\sigma (\nabla u, \nabla u) \int_\Omega \nabla u(t) \int_0^\infty g(s) \nabla \eta(s) \, ds \, dx \\
\leq \sigma^2 \left( \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 \right)^2 E(0) + \frac{\xi_0 - l}{2l} \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds 
\]

Combining (3.15) - (3.19) with (3.13), we have for any, \( \delta_3 > 0 \)

\[
I_1 \leq \left( \frac{\xi_0 - l}{4\delta_3} \left( l + \frac{2b}{l} E(0) \right) \right)^2 + \frac{\xi_0 - l}{2l} + (\xi_0 - l) + \frac{\xi_0 - l}{2} C_1 + \frac{\xi_0 - l}{4\delta_3} C_1 \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds \\
+ \sigma^2 \left( \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 \right)^2 E(0) + \delta_3 \|\nabla u\|^2 + \frac{1}{2} \int_{\Gamma_1} h^2(u_t) \, d\Gamma. 
\]

Noting that

\[
\int_0^\infty g(s) \eta_t(s) \, ds = - \int_0^\infty g(s) \eta_t(s) \, ds + \int_0^\infty u_t(t) g(s) \, ds \\
= \int_0^\infty g'(s) \eta(s) \, ds + (1 - l) u_t
\]

we can derive that

\[
I_2 = - (\xi_0 - l) \|u_t\|^2 - \int_\Omega u_t(t) \int_0^\infty g'(s) \eta(s) \, ds \, dx \\
\leq - \frac{3}{4} (\xi_0 - l) \|u_t\|^2 - \frac{1}{\xi_0 - l} \int_\Omega \left( \int_0^\infty -g'(s) \, ds \right)^2 \left( \int_0^\infty -g'(s) \eta(s) \, ds \right) \, dx \\
\leq - \frac{3}{4} (\xi_0 - l) \|u_t\|^2 - \frac{g(0)C_1^2}{\xi_0 - l} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 \, ds. 
\]

The same arguments give us

\[
I_3 \leq - \frac{3}{4} (\xi_0 - l) \|u_t\|^2 - \frac{g(0)C_1^2}{\xi_0 - l} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 \, ds. 
\]

Inserting (3.20) - (3.22) into (3.12), we can get (3.11) with

\[
K_1 = \frac{\xi_0 - l}{4\delta_3} \left( l + \frac{2\xi_1}{l} E(0) \right) \right)^2 + \frac{\xi_0 - l}{2l} + (1 - l) + \frac{\xi_0 - l}{2} C_1 + \frac{\xi_0 - l}{4\delta_3} C_1 \\
K_2 = \frac{2g(0)C_1^2}{\xi_0 - l}. 
\]
The proof is done. □

Define the functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := E(t) + \varepsilon_1 \phi(t) + \varepsilon_2 \psi(t),$$

where $\varepsilon_1$ and $\varepsilon_2$ are positive constants will be chosen later. It is easy to verify that for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough,

$$\frac{1}{2} E(t) \leq \mathcal{L}(t) \leq \frac{3}{2} E(t).$$

Lemma 3.4. There exists a positive constant $m$ such that for any $t \geq 0$,

$$\mathcal{L}'(t) \leq -m E(t) + C \int_0^\infty g(s) ||\nabla \eta(s)||^2 ds + C \int_{\Gamma_1} h^2(u_t) d\Gamma.$$  \hspace{1cm} (3.24)

Proof. It follows from (3.1), (3.5) and (3.11) that for any $t \geq 0$,

$$\mathcal{L}'(t) \leq - \left[ \frac{3}{4} (\xi_0 - l) \varepsilon_2 - \varepsilon_1 \right] ||u_t||^2 - \left[ \frac{3}{4} (\xi_0 - l) \varepsilon_2 - \varepsilon_1 \right] ||u_t||_{\Gamma_1}^2$$

$$- ||\varepsilon_1 + \delta_1 (1 + C) \varepsilon_1 - \delta_2 \varepsilon_2|| \nabla u||^2$$

$$- \xi_1 \varepsilon_1 ||\nabla u||^4 - \sigma (1 - \sigma E(0) \varepsilon_2) \left( \frac{1}{2} \frac{d}{dt} ||\nabla u||^2 \right)^2$$

$$+ \left( \frac{1}{2} - K_2 \varepsilon_2 \right) \int_0^\infty g'(s) ||\nabla \eta(s)||^2 ds$$

$$+ \left( \left( \frac{\xi_0 - l}{4 \delta_1} \right) \varepsilon_1 + K_2 \varepsilon_2 \right) \int_0^\infty g(s) ||\nabla \eta(s)||^2 ds$$

$$+ \left( \frac{1}{4 \delta_1} \varepsilon_1 + \frac{\varepsilon_2}{2} \right) \int_{\Gamma_1} h^2(u_t) d\Gamma.$$ \hspace{1cm} (3.25)

At this point we choose $\delta_2 > 0$ satisfying $\epsilon$

$$\delta_2 < \frac{3l}{8} (\xi_0 - l)$$

which gives us

$$\frac{2}{l} \delta_2 \varepsilon_2 < \frac{3}{4} (\xi_0 - l) \varepsilon_2.$$  

For any fixed $\delta_2 > 0$ we take $\varepsilon_2 > 0$ small enough so that (3.23) remains valid and further. For fixed $\delta_2$ and $\varepsilon_2$, we pick $\varepsilon_1 > 0$ so small that (3.23) remains valid and further

$$\frac{2}{l} \delta_2 \varepsilon_2 < \varepsilon_1 < \min \left\{ \frac{1}{2C}, \frac{3}{4} (\xi_0 - l) \varepsilon_2 \right\},$$

which gives us

$$\frac{3}{4} (\xi_0 - l) \varepsilon_2 > 0, (\varepsilon_1 + \delta_1 (1 + C) \varepsilon_1 - \delta_2 \varepsilon_2) > 0, \left( \frac{1}{2} - K_2 \varepsilon_2 \right) > 0.$$  

Therefore, there exists a positive constant $m$ such that for any $t \geq 0$,

$$\mathcal{L}'(t) \leq -m E(t) + C \int_0^\infty g(s) ||\nabla \eta(s)||^2 ds + C \int_{\Gamma_1} h^2(u_t) d\Gamma.$$  

which completes the proof. □

The same arguments as in [17], we can get the following lemma.
Lemma 3.5. Under the assumptions of Theorem 2, there exists a positive $\gamma > 0$ such that for any $\varepsilon_0 > 0$

$$G'(\varepsilon_0 E(t)) \int_0^\infty g(s) \|\nabla \eta(s)\|^2 ds \leq -\gamma_1 E'(t) + \gamma_1 E(t)G'(\varepsilon_0 E(t)).$$

(3.26)

Proof. of Theorem 2. We distinguish the following two cases to prove Theorem 2.

Case 1. The function $h_0$ is linear. It follows from (A2) that

$$c_1 |s| \leq |h(s)| \leq c_2 |s|, \forall s \in \mathbb{R},$$

which implies

$$h^2(s) \leq c_2 sh(s), \forall s \in \mathbb{R}^+. \quad (3.27)$$

Combining (3.1) and (3.27) with (3.24), we arrive at for any $t > 0$,

$$L'(t) \leq -mE(t) + C \int_0^\infty g(s) \|\nabla \eta(s)\|^2 ds - CE'(t). \quad (3.28)$$

Let $\epsilon(t) = L(t) + CE(t)$. Using (3.23), we know that $\epsilon(t) \sim E(t)$. Then (3.28) gives us

$$\epsilon'(t) \leq -mE(t) + C \int_0^\infty g(s) \|\nabla \eta(s)\|^2 ds. \quad (3.29)$$

Multiplying (3.29) by $G'(\varepsilon_0 E(t))$ and using (3.26), we obtain

$$G'(\varepsilon_0 E(t)) \epsilon'(t) \leq -mG'(\varepsilon_0 E(t)) E(t) - C\gamma_1 E'(t)$$

$$+ C\gamma_1 \varepsilon_0 E(t)G'(\varepsilon_0 E(t))$$

$$= - (m - C\gamma_1 \varepsilon_0) E(t)G'(\varepsilon_0 E(t)) - C\gamma_1 E'(t).$$

Now we take $\varepsilon_0 > 0$ so small that $m - C\gamma_1 \varepsilon_0 > 0$, and denote $\epsilon'_1(t) = G'(\varepsilon_0 E(t)) \epsilon'(t) + C\gamma_1 E'(t)$ we can get there exists some $K_1 > 0$ such that

$$\epsilon_1(t) \sim E(t) \text{ and } \epsilon'_1(t) \leq -K_1 G'(\varepsilon_1 \epsilon_1(t)) \epsilon_1(t) \quad (3.30)$$

which yields $(W_1(\epsilon_1))' \geq K_1$, where

$$W_1(\tau) = \int_0^1 \frac{1}{C s G'(\epsilon_1 s)} ds$$

for $0 < \tau \leq 1$. Integrating the last inequality in (3.30) over $[0, t]$, we have for any $t > 0$,

$$\epsilon_1(t) \leq W_1^{-1}(K_1 t + K_2). \quad (3.31)$$

Then (2.9) follows from (3.31), (2.8) and $\epsilon_1 \sim E$.

Case 2. The function $h_0$ is nonlinear on $[0, \varepsilon]$.

Following the arguments as in [20], we first suppose that $\max\{r, h_0(r)\} < \varepsilon$ otherwise we choose $r$ smaller.

Let $\varepsilon_1 = \min\{r, h_0(r)\}$ It follows from (A2) that for $\varepsilon_1 \leq |s| \leq \varepsilon$

$$|h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|}|s| \leq \frac{h_0^{-1}(|\varepsilon|)}{|\varepsilon_1|}|s|,$$

and

$$|h(s)| \geq \frac{h_0(|s|)}{|s|}|s| \leq \frac{h_0(|\varepsilon_1|)}{|\varepsilon_1|}|s|.$$
Then we have
\[
\begin{cases}
    h_0(|s|) \leq |h(s)| \leq h^{-1}_0(|s|), \text{ for } |s| < \varepsilon_1, \\
    c_1 |s| \leq |h(s)| \leq c_2 |s|, \text{ for } |s| \geq \varepsilon_1,
\end{cases}
\] (3.32)
which gives us for all \(|s| \leq \varepsilon_1,
\]
\[H(h^2(s)) = |h(s)| h_0(|h(s)|) \leq sh(s).
\]
Then
\[h^2(s) \leq H^{-1}(sh(s)), \forall |s| \leq \varepsilon_1.
\] (3.33)
As in [19], we denote
\[\Gamma_{12} = \{x \in \Gamma_1 : |u_t(t)| > \varepsilon_1\}, \Gamma_{12} = \{x \in \Gamma_1 : |u_t(t)| \leq \varepsilon_1\}.
\]
It follows from (3.32) that on \(\Gamma_{12}\),
\[u_t h(u_t) \leq \varepsilon_1 h_0^{-1}(\varepsilon_1) \leq h_0(r) r = H^2(r).
\] (3.34)
We define \(J(t)\) by
\[J(t) = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t h(u_t) \, d\Gamma.
\]
Since \(H^{-1}\) is concave, we infer from Jensen’s inequality that
\[H^{-1}(J(t)) \geq C \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) \, d\Gamma.
\] (3.35)
Using (3.33) and (3.35), we conclude that
\[
\int_{\Gamma_{12}} h^2(u_t) \, d\Gamma \leq \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) \, d\Gamma + \int_{\Gamma_{11}} h^2(u_t) \, d\Gamma \leq CH^{-1}(J(t)) - CE'(t),
\]
which, together with (3.24), yields for any \(t > 0,\)
\[k'(t) \leq -mE(t) + C \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds + CH^{-1}(J(t))
\]
where \(k(t) = L(t) + CE(t)\) and \(k(t) \sim E(t)\) For \(\varepsilon_0 \leq r^2\) and \(C_0 > 0\) and the fact \(E' < 0, H' > 0\) and \(H'' > 0\), we obtain that the functional \(k_1(t)\) defined by
\[k_1(t) = H' \left(\frac{\varepsilon_0 E(t)}{E(0)}\right) k(t) + C_0 E(t),
\]
is equivalent \(E(t)\) to and
\[
k'_1(t) = \frac{\varepsilon_0 E'(t)}{E(0)} H'' \left(\frac{\varepsilon_0 E(t)}{E(0)}\right) k(t) + H' \left(\frac{\varepsilon_0 E(t)}{E(0)}\right) k'(t) + C_0 E'(t)
\]
\[
\leq -mE(t) H' \left(\frac{\varepsilon_0 E(t)}{E(0)}\right) + CH^{-1}(J(t)) H' \left(\frac{\varepsilon_0 E(t)}{E(0)}\right) + C_0 E'(t)
\]
\[+ CH' \left(\frac{\varepsilon_0 E(t)}{E(0)}\right) \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \, ds.
\] (3.36)
Now, we denote the conjugate function of the convex function \(H\) by \(H^*\) see, for example, Arnold [1], and Lasiecka and Tataru [28], i.e.,
\[H^*(s) = \sup_{t \in \mathbb{R}^+} (st - H(t)).
\]
Then
\[ H^* (s) = s (H')^{-1} (s) - H \left( (H')^{-1} (s) \right), \]
is the Legendre transform of \( H \), which satisfies
\[ AB \leq H^* (A) + H(B). \]

For \( A = H' \left( \frac{\varepsilon E(t)}{E(0)} \right) \) and \( B = H^{-1} (J(t)) \), and noting the fact \( H^* (s) \leq s (H')^{-1} (s) \) and using (3.36), we shall see that
\[
\varepsilon_0 \leq - (mE(0) - C\varepsilon_0) E(t) \frac{E(t)'}{E(0)} + C \frac{E(t)'}{E(0)} \int_0^\infty g(s) \| \nabla \eta (s) \|^2 \, ds
\]
\[
- (mE(0) - C\varepsilon_0) E(t) \frac{E(t)'}{E(0)} - (C - C_0) E'(t)
\]
\[
+ CH' \left( \frac{\varepsilon E(t)}{E(0)} \right) \int_0^\infty g(s) \| \nabla \eta (s) \|^2 \, ds. \tag{3.37}
\]

In (3.37), we choose \( \varepsilon_0 > 0 \) so small that \( mE(0) - C\varepsilon_0 > 0 \) and \( C_0 \) so large that \( C - C_0 < 0 \) to get for any \( t > 0 \),
\[
\varepsilon_0 \leq - K E(t) \frac{E(t)'}{E(0)} \varepsilon_0 E(t) + C H' \left( \frac{\varepsilon E(t)}{E(0)} \right) \int_0^\infty g(s) \| \nabla \eta (s) \|^2 \, ds. \tag{3.38}
\]

Multiplying (3.38) by \( G' (\varepsilon_0 E(t)) \) and using (3.26), we obtain
\[
G' (\varepsilon_0 E(t)) \varepsilon_0 E(t) \leq - K E(t) \frac{E(t)'}{E(0)} G' \left( \varepsilon_0 E(t) \right) H' \left( \varepsilon_0 E(t) \right)
\]
\[
- \gamma \left( \varepsilon_0 E(t) \right) H' \left( \varepsilon_0 E(t) \right)
\]
\[
+ \gamma \varepsilon_0 E(t) G' \left( \varepsilon_0 E(t) \right) H' \left( \frac{\varepsilon E(t)}{E(0)} \right)
\]
\[
- K E(t) \frac{E(t)'}{E(0)} G' \left( \varepsilon_0 E(t) \right) H' \left( \frac{\varepsilon E(t)}{E(0)} \right) - C E'(t)
\]
\[
+ \gamma \varepsilon_0 E(t) G' \left( \varepsilon_0 E(t) \right) H' \left( \frac{\varepsilon E(t)}{E(0)} \right). \tag{3.39}
\]

Define the functional \( \varepsilon_1 E(t) \) by
\[
\varepsilon_1 E(t) = G' \left( \varepsilon_0 E(t) \right) \varepsilon_0 E(t) + C E(t).
\]

It is easy to verify that \( \varepsilon_1 E(t) \sim E(t) \). i.e., there exist two positive constants \( \beta_1 \) and \( \beta_2 \) such that
\[
\beta_1 \varepsilon_1 E(t) \leq E(t) \leq \beta_2 \varepsilon_1 E(t). \tag{3.40}
\]

Noting the fact \( E'(t) \leq 0 \) and \( G'' > 0 \) we infer from (3.39) that
\[
\varepsilon_0 \leq - \left( K - \gamma \varepsilon_1 E(t) \right) E(t) \frac{E(t)'}{E(0)} \frac{E(t)}{E(0)} \right)
\]
\[
\varepsilon_1 \leq \varepsilon_0 E(t). \tag{3.41}
\]

with \( \varepsilon_1 = \varepsilon_0 E(0) \). For a suitable choice of \( \varepsilon_0 \), we get from (3.41) that for some constant \( K_1 > 0 \),
\[
\varepsilon_0 \leq - K_1 E(t) \frac{E(t)'}{E(0)} \frac{E(t)}{E(0)} \right)
\]
\[
- K_1 W_2 \left( \frac{E(t)}{E(0)} \right), \tag{3.42}
\]
where \( W_2(t) = tH'(\varepsilon_0 t)G'(\varepsilon_1 t) \)

Denote \( R(t) = \frac{\beta_k(t)}{E(0)} \)

It follows from (3.40) that

\[
R(t) \sim E(t).
\]  

(3.43)

By (3.42), we get for some \( k_2(t) > 0, \)

\[
R'(t) \leq -K_2 W_2(R(t)),
\]  

(3.44)

which implies \((W_1(R(t)))' \geq K_2, \)

where

\[
W_1(t) = \int_t^1 W_2(s)ds, \text{ for } t \in (0,1].
\]

Integrating (3.44) over \([0,t]\) we have for any \( t > 0, \)

\[
R(t) \leq W_1^{-1}(K_2 t + K_3).
\]  

(3.45)

Then (2.9) follows from (3.43) and (3.45). The proof is done.

\[\square\]

References

1. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Springer, New York (1989)

2. A. Zarai and N-e. Tatar, Global existence and polynomial decay for a problem with Balakrishnan-Taylor damping. Arch. Math. (Brno) 46 (2010), 47-56.

3. A. Zarai and N-e. Tatar, Non-solvability of Balakrishnan-Taylor equation with memory term in \( \mathbb{R}^N \). In: Anastassiou G., Duman O. (eds) Advances in Applied Mathematics and Approximation Theory, Springer Proceedings in Mathematics & Statistics, vol 41. Springer, New York, NY 2013.

4. A. Zarai and N-e. Tatar and S. Abdelmalek, Elastic membrane equation with memory term and nonlinear boundary damping: global existence, decay and blowup of the solution. Acta Math. Sci. 33B (2013), 84-106.

5. R. W. Bass and D. Zes, Spillover, nonlinearity and flexible structures, in The Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems, NASA Conference Publication 10065 (ed. L.W.Taylor), 1991, 1-14.

6. Benaissa, A., Ferhat, M.: Stability results for viscoelastic wave equation with dynamic boundary conditions. arXiv: 1801.02988v1

7. Berrimi, S., Messaoudi, S.: Existence and decay of solutions of a viscoelastic equation with a nonlinear source. Nonlinear Anal. TMA 64, 2314-2331 (2006)

8. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Prates Filho, J.S., Soriano, J.A.: Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. Differ. Integr. Equ. 14, 85–116 (2001)

9. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: Exponentnal decay for the solution of semilinear viscoelastic wave equations with localized damping. Electron. J. Differ. Equ. 44, 1–14 (2002)

10. Cavalcanti, M.M., Oquendo, H.: Frictional versus viscoelastic damping in a semilinear wave equation. SIAM J. Control Optim. 42(4), 1310–1324 (2003)

11. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Martinez, P.: General decay rate estimates for viscoelastic dissipative systems. Nonlinear Anal. TMA 68, 177–193 (2008)

12. Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. 37, 297–308 (1970)

13. Doronin, G.G., Larkin, N.A.: Global solvability for the quasilinear damped wave equation with nonlinear second-order boundary conditions. Nonlinear Anal. TMA 8, 1119–1134 (2002)

14. E. H. Dowell, Aeroelasticity of plates and shells, Groninger, NL, Noordhoff Int. Publishing Co. (1975).

15. Fabrizio, M., Polidoro, S.: Asymptotic decay for some differential systems with fading memory. Appl. Anal. 81(6), 1245–1264 (2002)

16. Feng, Baowei. General decay rates for a viscoelastic wave equation with dynamic boundary conditions and past history. Mediterr. J. Math. 15 (2018), no. 3, Art. 103, 17 pp.

17. Ferhat, M., Hakem, A.: On convexity for energy decay rates of a viscoelastic wave equation with a dynamic boundary and nonlinear delay term. Facta Univ.Ser. Math. Inform. 30, 67–87 (2015)
18. H. Medekhel, S. Boulaaras, Existence of positive solutions for a class of Kirchhoff parabolic systems with multiple parameters Appl. Math. E-Notes, 18(2018), 295-306

19. Ferhat, M., Hakem, A.: Asymptotic behavior for a weak viscoelastic wave equations with a dynamic boundary and time varying delay term. J. Appl. Math.Comput. 51, 509-526 (2016)

20. Gerbi, S., Said-Houari, B.: Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions. Adv. Differ.Equ. 13, 1051–1074 (2008)

21. S. Boulaaras, A well-posedness and exponential decay of solutions for a coupled Lamé system with viscoelastic term and logarithmic source terms, Applicable Analysis (2019) DOI: 10.1080/00036811.2019.1648793

22. Gerbi, S., Said-Houari, B.: Global existence and exponential growth for a viscoelastic wave equation with dynamic boundary conditions. Adv. Nonlinear Anal. 2, 163–193 (2013)

23. S. Otmani, S. Boulaaras, A.Allahem. The maximum norm analysis of a nonmatching grids method for a class of parabolic $p(x)$-Laplacien equation, Boletim Sociedade Paranaense de Matemática, (2019) doi:10.5269/bspm.45218

24. Guesmia, A.: Asymptotic stability of abstract dissipative systems with infinite memory. J. Math. Anal. Appl. 382, 748–760 (2011)

25. P. Holmes, Bifurcations to divergence and flutter in flow-induced oscillations-a finite dimensional analysis, Journal of Sound and Vibration, 53(1977), pp. 471-503.

26. P. Holmes, J. E. Marsden, Bifurcation to divergence and flutter flow induced oscillations; an infinite dimensional analysis, Automatica, Vol. 14 (1978).

27. Kafini, M., Mustafa, M.I.: On the stabilization of a non-dissipative Cauchy viscoelastic problem. Mediterr. J. Math. 13, 5163–5176 (2016)

28. Komornik, V.: Exact Controllability and Stabilization, the Multipiles Method. RMA, vol. 36. Masson, Paris (1994)

29. Lasiecka, I., Tataru, D.: Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping. Differ. Integr. Equ. 6, 507–533(1993)

30. Liu, W.J., Yu, J.: On decay and blow-up of th esolution for a viscoelastic wave equation with boundary damping and source terms. Nonlinear Anal. TMA 74,2175–2190 (2011)

31. Messaoudi, S.A.: Blow up and global existence in nonlinear viscoelastic wave equations. Math. Nachr. 260, 58–66 (2003)

32. S. Boulaaras , A. Zarai and A. Draifia, Galerkin method for nonlocal mixed boundary value problem for the Moore-Gibson-Thompson equation with integral condition, Mathematical Methods in the Applied Sciences, https://doi.org/10.1002/mma.5540

33. Messaoudi, S.A., Tatar, N.E.: Global existence and uniform stability of solutions for a quasilinear viscoelastic problem. Math. Methods Appl. Sci. 30, 665–680 (2007)

34. S. Boulaaras, A. Draifia and A. Alnegga, Polynomial Decay Rate for Kirchhoff Type in Viscoelasticity with Logarithmic Nonlinearity and Not Necessarily Decreasing Kernel, Symmetry 2019, 11(2), 226; https://doi.org/10.3390/sym11020226

35. Messaoudi, S.A., Said-Houari, B.: Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. J. Math. Anal. Appl. 365, 277–287 (2010)

36. Munoz Rivera, J.E., Andrade, D.: Exponential decay of non-linear wave equation with a viscoelastic boundary condition. Math. Methods Appl. Sci. 23,41–61 (2000)

37. M. Maizi, S. Boulaaras, M. Haiour, A. Mansour, Existence of positive solutions of Kirchhoff hyperbolic systems with multiple parameters, Boletim Sociedade Paranaense de Matemática, (2019), doi:10.5269/bspm.45418

38. Mustafa, M.I.: Optimal decay rates for the viscoelastic wave equation. Math.Methods Appl. Sci. 41, 192–204 (2018)

39. N. Mezouar, S. Boulaaras: Global existence of solutions to a viscoelastic non-degenerate Kirchhoff equation, Applicable Analysis, (2018) DOI: 10.1080/00036811.2018.1544621

40. N. Mezouar, S. Boulaaras Global existence and decay of solutions for a class of viscoelastic Kirchhoff equation, Bulletin of the Malaysian Mathematical Sciences Society DOI: 10.1007/s40840-018-0708-2 (2018)

41. N-e. Tatar and A. A. Zarai, Exponential stability and blow up for a problem with Balakrishnan-Taylor damping. Demonstratio Math. 44(2011), 67-90.

42. N-e. Tatar and A. Zarai, On a Kirchhoff equation with Balakrishnan-Taylor damping and source term. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 18(2011), 615-627.

43. S. Boulaaras, R.Guefaifia, Existence of positive weak solutions for a class of Kirrchoff elliptic systems with multiple parameters, Mathematical Methods in the Applied Sciences, Volume 41, Issue 13, 5205-5210

44. S. Boulaaras and R.Guefaifia, S. Kabli: An asymptotic behavior of positive solutions for a new class of elliptic systems involving of $(p(x),q(x))$-Laplacian systems. Bol. Soc. Mat. Mex. (2017). https://doi.org/10.1007/s40590-017-0184-4

45. R. Guefaifia and S. Boulaaras Existence of positive radial solutions for $(p(x),q(x))$-Laplacian systems Appl. Math. E-Notes, 18(2018), 209-218
46. R. Guefaifia and S. Boulaaras, Existence of positive solution for a class of \((p(x),q(x))-\)Laplacian systems, Rend. Circ.
Mat. Palermo, II. Ser 67 (2018), 93–103

47. S. Boulaaras and R. Guefaifia, Existence of positive weak solutions for a class of Kirchhoff elliptic systems with multiple
parameters, Mathematical Methods in the Applied Sciences, Volume 41, Issue 13, 5203-5210

48. S. Boulaaras and Ghfaifia, S. Kabli, An asymptotic behavior of positive solutions for a new class of elliptic systems
involving of \((p(x),q(x))-\)Laplacian systems. Bol. Soc. Mat. Mex. (2017). https://doi.org/10.1007/s40590-017-0184-4

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