Supplying bells and whistles in symmetric monoidal categories
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AMS Fall Western Section
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Sometimes we want to add additional icons to string diagrams:
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Every object is equipped with an algebraic structure, compatible with $\otimes$. 
Example: **Set** supplies comonoids

In \((\textbf{Set}, \times)\), we have commutative comonoids:

- **terminal**: \(\epsilon\)
- **diagonal**: \(\delta\)

Moreover, morphisms are comonoid homomorphisms.

(In fact, we can recover products from this perspective.)
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There are compatible with \(\times\):

\[
\begin{align*}
\epsilon_{X \otimes Y} &= \epsilon_X \otimes \epsilon_Y \\
\delta_{X \otimes Y} &= \delta_X \otimes \delta_Y
\end{align*}
\]

Moreover, morphisms are comonoid homomorphisms.

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f \epsilon_X &= \epsilon_Y \\
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  ![Terminal](image1)

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  ![Diagonal](image2)

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Outline

I. Props and theories
II. Definition
III. Examples
IV. An equivalent definition
V. Some fun facts
A **prop** is a symmetric strict monoidal category where the monoid of objects is \((\mathbb{N}, +)\).

**Examples:**

- **Bij**: the prop of bijections
- **Cob**: the prop of unoriented 1-cobordisms
- **FinSet**: the prop of functions
- **Cospan**: the prop of cospans of functions
**Key idea:** Props describe algebraic (monoidal) theories

For example, \( \text{FinSet} \) is the theory of commutative monoids.

\[
\text{SymMonCat}_{\text{strong}}(\text{FinSet}, \mathcal{C}) \cong \text{CommMon}(\mathcal{C})
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\[\text{SymMonCat}_{\text{strong}}(\text{FinSet}, \mathcal{C}) \cong \text{CommMon}(\mathcal{C})\]

\textbf{Bij}: the prop for objects
\textbf{Cob}: the prop for self-duality
\textbf{FinSet}: the prop for commutative monoids
\textbf{Cospan}: the prop for special commutative frobenius monoids
Definition of Supply

Let $\mathbb{P}$ be a prop, and $\mathcal{C}$ be a SMC.

A **supply** $s$ of $\mathbb{P}$ in $\mathcal{C}$ is
- for each $c \in \mathcal{C}$, a strong SMF $s_c: \mathbb{P} \to \mathcal{C}$
Definition of Supply

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- for each $c \in \mathcal{C}$, a strong SMF $s_c : \mathbb{P} \to \mathcal{C}$

such that for all $m, n \in \mathbb{N}$, $c, d \in \mathcal{C}$, $\mu : m \to n \in \mathbb{P}$

(i) $s_c(m) = c^\otimes m$

(ii) the strongators are the unique coherence maps

$$c^\otimes m \otimes c^\otimes n \to c^\otimes(m+n)$$

(iii)

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
c^\otimes m \otimes d^\otimes m \\
\downarrow\sigma
\end{array}
\xrightarrow{s_c(\mu) \otimes s_d(\mu)}
\begin{array}{c}
c^\otimes n \otimes d^\otimes n \\
\downarrow\sigma
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(c \otimes d)^\otimes m \\
\downarrow\sigma
\end{array}
\xrightarrow{s_{c \otimes d}(\mu)}
\begin{array}{c}
(c \otimes d)^\otimes n \\
\downarrow\sigma
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
I^\otimes m \\
\downarrow\sigma
\end{array}
\xrightarrow{s_I(\mu)}
\begin{array}{c}
I^\otimes n \\
\downarrow\sigma
\end{array}
\end{array}
\end{array}
\end{array}$$
A morphism \( f: c \to d \) is an \( s \)-homomorphism if for all \( \mu: m \to n \) in \( \mathbb{P} \):

\[
\begin{align*}
&c \otimes m \xrightarrow{s_c(\mu)} c \otimes n \\
&f \otimes m \downarrow \quad \downarrow f \otimes n \\
&d \otimes m \xrightarrow{s_d(\mu)} d \otimes n
\end{align*}
\]
Examples

- Every symmetric monoidal category uniquely supplies $\text{Bij}$. Moreover, this unique supply is homomorphomic.
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- A category $\mathcal{C}$ supplies $\text{Cospan}$ iff it is hypergraph. For example, $(\text{Rel}, \times)$ supplies $\text{Cospan}$. Supply homomorphisms are bijections. (The homomorphisms of $\text{FinSet}^{\text{op}}$ are functions.)
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- A category homomorphically supplies $\text{FinSet}^{\text{op}}$ iff the monoidal product is a categorical product.
A key theorem

**Theorem**

Let $s$ be a supply of $\mathbb{P}$ in $\mathcal{C}$. Then all coherence maps of $\mathcal{C}$ are $s$-homomorphisms.

For example,

\[
\begin{align*}
((a \otimes b) \otimes c)^{\otimes m} & \xrightarrow{s(a \otimes b) \otimes c(\mu)} ((a \otimes b) \otimes c)^{\otimes n} \\
(a \otimes (b \otimes c))^{\otimes m} & \xrightarrow{s_{a \otimes (b \otimes c)}(\mu)} (a \otimes (b \otimes c))^{\otimes n}
\end{align*}
\]
Define $\text{inc}: C_0 \hookrightarrow C$ to be the smallest subcategory of $C$ containing all the coherence maps.

**Corollary**
The following are equivalent:

(a) A supply $s$ of $P$ in $C$.

(b) A strong SMF $s: P \to \text{SymMonCat}_{\text{strong}}(C_0, C)$ such that

(i) $m \mapsto \text{inc} \otimes m$

(ii) strongators are coherence maps
Preservation of supply

Let $s$, $t$ respectively supply $\mathbb{P}$ in $\mathcal{C}$, $\mathcal{D}$.

Morphisms of categories supplying $\mathbb{P}$ are defined as follows.

A strong monoidal functor $(F, \varphi): \mathcal{C} \to \mathcal{D}$ preserves supply
iff for all $\mu$ in $\mathbb{P}$, $c \in \mathcal{C}$:

$$
\begin{array}{c}
F(c) \otimes m \\ \varphi \downarrow \cong
\end{array}
\xrightarrow{t_{F(c)}(\mu)}
\begin{array}{c}
F(c) \otimes n \\ \varphi \downarrow \cong
\end{array}
\cong
\begin{array}{c}
F(c \otimes m) \\ \cong
\end{array}
\xrightarrow{F(s_c(\mu))}
\begin{array}{c}
F(c \otimes n) \\ \varphi \downarrow \cong
\end{array}
$$
Some fun facts

- If $A: \mathbb{P} \to \mathbb{Q}$ is a morphism of props, and $s$ supplies $\mathbb{Q}$ in $\mathcal{C}$, then $A; s$ supplies $\mathbb{P}$ in $\mathcal{C}$.

- If $F: \mathcal{C} \to \mathcal{D}$ is a symmetric, essentially surjective strict monoidal functor, then if $\mathcal{C}$ supplies $\mathbb{P}$, so does $\mathcal{D}$.

- If $\mathcal{C}$ supplies $\mathbb{P}$, so does its strictification $\tilde{\mathcal{C}}$, and the equivalence $\mathcal{C} \simeq \tilde{\mathcal{C}}$ preserves supply.

- If $F$ preserves supply, it sends supply homomorphisms to supply homomorphisms.

- More in the paper...