Group Theory/Lie Algebras

Exterior powers of the reflection representation in the cohomology of Springer fibres

Les puissances extérieures de la représentation géométrique dans la cohomologie des fibres de Springer

Anthony Henderson

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

ABSTRACT

Let $H^*(\mathcal{B}_e)$ be the cohomology of the Springer fibre for the nilpotent element $e$ in a simple Lie algebra $\mathfrak{g}$. Let $\Lambda^i V$ denote the $i$th exterior power of the reflection representation of $W$. We determine the degrees in which $\Lambda^i V$ occurs in the graded representation $H^*(\mathcal{B}_e)$, under the assumption that $e$ is regular in a Levi subalgebra and satisfies a certain extra condition which holds automatically if $\mathfrak{g}$ is of type A, B, or C. This partially verifies a conjecture of Lehrer and Shoji.

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1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank $\ell$. Let $W$ denote the Weyl group of $\mathfrak{g}$, and let $V$ be the reflection representation of $W$. It is well known that the exterior powers $\Lambda^i V$, for $i = 0, 1, \ldots, \ell$, are inequivalent irreducible representations of $W$, each of which is self-dual.

Let $e$ be a nilpotent element of $\mathfrak{g}$. The Springer fibre $\mathcal{B}_e$ is the variety of Borel subalgebras of $\mathfrak{g}$ containing $e$. Let $H^*(\mathcal{B}_e)$ denote the graded cohomology ring of $\mathcal{B}_e$ with complex coefficients; the cohomology lives solely in even degrees, so $H^*(\mathcal{B}_e)$ is commutative. We have the Springer representation of $W$ on each $H^{2j}(\mathcal{B}_e)$ (see [5], [2, Chapter 9]). Let $s$ (depending on $e$) denote the multiplicity of the irreducible representation $V$ in the total representation $H^*(\mathcal{B}_e)$. Let $m_1, m_2, \ldots, m_s$ be the multiset of nonnegative integers, listed in increasing order, which are the halved degrees of the occurrences of $V$ in the graded representation $H^*(\mathcal{B}_e)$. That is, we have by definition $\sum_j \dim(H^{2j}(\mathcal{B}_e) \otimes V)^W q^j = q^{m_1} + q^{m_2} + \cdots + q^{m_s}$.

E-mail address: anthony.henderson@sydney.edu.au.

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In the special case $e = 0$, it is well known (see [5, §5]) that $H^*(B)$ is isomorphic to the coinvariant algebra $C^*(W)$ of $W$, that $s = \ell$, and that $m_1, \ldots, m_\ell$ are the exponents of $W$. More generally, if $e$ is a regular nilpotent in a Levi subalgebra of semisimple rank $r$, then it was proved by Lehrer and Shoji in [3, Theorem 2.4] (see also [9]) that $s = \ell - r$ and that $m_1, \ldots, m_\ell$ are the coexponents of the corresponding parabolic hyperplane arrangement, in the sense of Orlik and Solomon. See [8] for some other related interpretations of these coexponents. For $g$ of classical type and general $e$, the numbers $m_1, m_2, \ldots, m_\ell$ were calculated by Spaltenstein in [9, Propositions 1.6–19].

Lehrer and Shoji conjectured that, at least in the parabolic case which they considered, the occurrences of each exterior power $\Lambda^i V$ in $H^*(B_e)$ were also controlled in a natural way by $m_1, m_2, \ldots, m_\ell$.

**Conjecture 1.1.** (See [3, Conjecture 8.3]) Suppose that $e$ is a regular nilpotent in a Levi subalgebra. Then for any $i = 0, 1, \ldots, \ell$, we have $\sum_j \dim(H^j(B_e) \otimes \Lambda^i V)^W q^j = e_i(q^{m_1}, q^{m_2}, \ldots, q^{m_\ell})$ (the $i$th elementary symmetric polynomial in $q^{m_1}, q^{m_2}, \ldots, q^{m_\ell}$, which is defined to be zero if $i > s$).

The $e = 0$ case of this conjecture had already been proved by Solomon in [6]; indeed, he proved the stronger statement that the algebra $(C^*(W) \otimes \Lambda^i V)^W$ is a free exterior algebra on $(C^*(W) \otimes V)^W$.

The main result of this Note is the following generalization of Solomon’s result, which implies various cases of Conjecture 1.1:

**Theorem 1.2.** Suppose that $e$ is regular in a Levi subalgebra of $g$, and define $s$ and $m_1, \ldots, m_\ell$ as above. Also suppose that there is a parabolic subgroup $W_K$ of $W$ such that the following two conditions hold:

1. There exist invariant polynomials $f_1, f_2, \ldots, f_\ell \in (S^* V)^W$, homogeneous of degrees $m_1 + 1, m_2 + 1, \ldots, m_\ell + 1$, whose restrictions to the reflection representation $V_K$ of $W_K$ form a set of fundamental invariants for $W_K$.
2. The nilpotent orbit of $e$ intersects the nilradical of the parabolic subalgebra $p_K$ associated to $W_K$.

(See Section 2 for the definitions of $V_K$ and $p_K$.) Then the algebra $(H^*(B_e) \otimes \Lambda^i V)^W$ is a free exterior algebra on $(H^*(B_e) \otimes V)^W$.

More precisely, the natural homomorphism $\psi : \Lambda^i((H^*(B_e) \otimes V))^W \to (H^*(B_e) \otimes \Lambda^i V)^W$ is an isomorphism.

Here the domain and codomain of $\psi$ are $(\mathbb{N} \times \mathbb{N})$-graded algebras over $\mathbb{C}$, where the $(i, j)$-components are $\Lambda^i((H^j(B_e) \otimes V))^W$ and $(H^j(B_e) \otimes \Lambda^i V)^W$ respectively, and in both cases the algebra multiplication is graded-commutative with respect to the $\mathbb{N}$-grading labelled by $i$; the homomorphism $\psi$ is induced by the inclusion of the subspace $(H^*(B_e) \otimes V)^W$ in $(H^*(B_e) \otimes \Lambda^i V)^W$. Since the graded degrees of this subspace are $(1, m_1), (1, m_2), \ldots, (1, m_\ell)$, the statement that $\psi$ is an isomorphism implies Conjecture 1.1.

Simple calculations\footnote{The details may be found in the preprint version of this Note, arXiv:1001.3164.} verify the following results:

**Proposition 1.3.** If $g$ is of type $A$, then the assumptions of Theorem 1.2 hold for any $e$.

**Proposition 1.4.** If $g$ is of type $B$ or $C$, then the assumptions of Theorem 1.2 hold for any $e$ which is regular in a Levi subalgebra.

Hence Conjecture 1.1 is proved in types $A$–$C$. By contrast, suppose that $g$ is of type $D_4$ and $e$ has Jordan type $(3^2 1^2)$ in the natural representation on $\mathbb{C}^8$. Then $e$ is regular in a Levi subalgebra of type $A_3$, but we have $m_2 = 2$ and there are no $W$-invariant polynomials of degree 3, so condition (1) of Theorem 1.2 cannot be satisfied.

**2. Proof of Theorem 1.2**

Continue the notation of the introduction. Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra and Borel subalgebra of $g$, and let $P \subset \Phi^+ \subset \Phi$ be the corresponding set of simple roots, positive roots, and roots. We identify $W$ with the subgroup of $GL(\mathfrak{h})$ generated by the simple reflections $s_\alpha$ for $\alpha \in P$; the reflection representation $V$ of $W$ is merely $\mathfrak{h}$ itself.

Let $J \subset P$ be a subset of size $r$, and set $s = \ell - r$. We have a Levi subalgebra $J$ and parabolic subalgebra $P_J$ containing $\mathfrak{h}$ and $\mathfrak{b}$ respectively, a parabolic subsystem $\Phi_J$ of $\Phi$, and a parabolic subgroup $W_J$ of $W$. Define $V_J = \bigcap_{\alpha \in P_J} \ker(\alpha) = V^{W_J}$. We write $V_J$ for the unique $W_J$-invariant complement to $V_J$ in $V$, which is the reflection representation of $W_J$. Note that $\dim V_J = s$ and $\dim V_J = r$. Let $\mathcal{A}_J$ and $\mathcal{A}_J$ be the hyperplane arrangements in $V_J$ and $V_J$ respectively induced by the root hyperplanes in $V$.

We assume for the remainder of the section that $e$ is parabolic of type $J$, meaning that the orbit of $e$ contains the regular nilpotent elements of $\mathfrak{g}_J$. As mentioned in the introduction, Lehrer and Shoji proved in this case that $\sum_j \dim(H^j(B_e) \otimes V)^W q^j = q^{m_1} + q^{m_2} + \cdots + q^{m_\ell}$, where $m_1, \ldots, m_\ell$ are the coexponents of the arrangement $\mathcal{A}_J$ (see [3, Theorem 2.4]); the missing case in type $D$ is covered by the results of Spaltenstein [9].

An important special feature of the parabolic case is Lusztig’s Induction Theorem for Springer representations.
Theorem 2.1. (See [4].) The representation of \( W \) on \( H^*(B_e) \), neglecting the grading, is isomorphic to the induction \( \text{Ind}^W_{W_J}(\mathbb{C}) \) of the trivial representation of \( W_J \).

Corollary 2.2. We have \( \dim(H^*(B_e) \otimes \Lambda^*V)^W = 2^\ell \), so the domain and codomain of \( \psi \) have the same dimension.

**Proof.** By Frobenius Reciprocity, we know that \( \dim(\text{Ind}^W_{W_J}(\mathbb{C}) \otimes \Lambda^*V)^W = \dim(\Lambda^*V)^W \). From the fact that \( (\Lambda^*V)^{\alpha} = \Lambda^*(\ker(\alpha)) \) for all \( \alpha \in \ell \) one deduces \( (\Lambda^*V)^W = \Lambda^*(V^I) \), and the corollary follows. \( \square \)

So to prove Theorem 1.2, it suffices to show that the homomorphism \( \psi \) is injective. Since the domain is an exterior algebra, this will follow if we can show that \( \psi(\Lambda^I((H^*(B_e) \otimes V)^W)) \neq 0 \).

Using the \( W \)-equivariant isomorphism of \( V \) with its dual, we will identify the symmetric algebra \( S^*V \) with the ring of polynomial functions on \( V \). It is well known that the invariant subring \( (S^*V)^W \) is freely generated by \( \ell \) homogeneous polynomials, called fundamental invariants for \( W \). The coinvariant algebra \( C^*(W) \) of \( W \) is the quotient \( S^*V/I \), where \( I \) is the ideal generated by these fundamental invariants.

Now there is a canonical (degree-doubling) \( W \)-equivariant algebra homomorphism \( S^*V \rightarrow H^*(B) \) which identifies \( H^*(B) \) with \( C(W) \) (see [5, §5]). Composing this with the natural homomorphism \( H^*(B) \rightarrow H^*(B_e) \), which is \( W \)-equivariant by [9, Lemma 1.4] (this fact in characteristic \( p \) was [1, Theorem 1.1]), we obtain a \( W \)-equivariant homomorphism \( \varphi : S^*V \rightarrow H^*(B_e) \). Note that the image of \( \varphi \) is contained in the subspace \( H^*(B_e)^{\Lambda^*} \) of invariants for the component group of the centralizer of \( e \) in the adjoint group of \( B_e \); in particular, \( \varphi \) is not surjective in general. However, it may happen that the induced map \( (S^*V \otimes V)^W \rightarrow (H^*(B_e) \otimes V)^W \) is surjective even if \( \varphi \) is not. We will see that this occurs under the assumptions of Theorem 1.2, which means that a calculation with polynomials on \( V \) suffices to prove what we want.

Henceforth we let \( K \subseteq \mathcal{P} \) be a subset satisfying conditions (1) and (2) of Theorem 1.2. Note that condition (1) entails \( |K| = s \). Choose a basis \( v_1, v_2, \ldots, v_\ell \) of \( V \) such that \( v_1, \ldots, v_s \) is a basis of \( V_K \). Since the exterior derivative \( S^*V \rightarrow S^*V \otimes V \) is \( W \)-equivariant, we have the following \( s \) elements of \( (H^*(B_e) \otimes V)^W \):

\[
\sum_{j=1}^n \varphi \left( \frac{\partial f_1}{\partial v_j} \right) \otimes v_j, \sum_{j=1}^n \varphi \left( \frac{\partial f_2}{\partial v_j} \right) \otimes v_j, \ldots, \sum_{j=1}^n \varphi \left( \frac{\partial f_n}{\partial v_j} \right) \otimes v_j.
\]

We can prove both that these form a basis of \( (H^*(B_e) \otimes V)^W \), and that \( \psi(\Lambda^I((H^*(B_e) \otimes V)^W)) \neq 0 \) as required, by proving the single fact that \( \varphi(\Delta) \neq 0 \), where \( \Delta \) is the determinant of the \( s \times s \) matrix \( (\varphi(\frac{\partial f_i}{\partial v_j}))_{i,j=1}^s \), with \( a \) and \( b \) ranging from 1 to \( s \). Since \( v_1, \ldots, v_s \) span the reflection representation of \( W_K \), we have \( w \Delta = \varepsilon(w) \Delta \) for all \( w \in W_K \). This forces \( \Delta \) to be divisible by the polynomial \( \pi_K = \prod_{\beta \in \Phi_+^s} \beta \). Condition (1) tells us that on restricting to \( V_K \), \( \Delta \) becomes the Jacobian matrix of the fundamental invariants of \( W_K \), which is well known to be a nonzero scalar multiple of the restriction to \( V_K \) of \( \pi_K \) (see [10]). So \( \Delta \) is a nonzero scalar multiple of \( \pi_K \), and it suffices to prove that \( \varphi(\pi_K) \neq 0 \).

By condition (2), we may suppose that \( e \) lies in the nilradical of \( p_K \). Then any Borel subalgebra contained in \( p_K \) must contain \( e \), so we have an inclusion \( B^K \rightarrow B_e \), where \( B^K \) denotes the variety of Borel subalgebras contained in \( p_K \), which can be identified with the flag variety of \( \mathfrak{g}_K \). Hence it suffices to prove that \( \pi_K \) is not in the kernel of the composition \( S^*V \rightarrow H^*(B) \rightarrow H^*(B^K) \). But this composition is the canonical homomorphism identifying \( C^*(W_K) \) with \( H^*(B^K) \), which maps \( \pi_K \) to a generator of the top-degree cohomology of \( B^K \) (compare [1, Proposition 1.4], which uses exactly this argument in the case when \( e \) lies in the Richardson orbit of \( p_K \)). This completes the proof of Theorem 1.2.

**Acknowledgements**

The main result of this Note, Theorem 1.2, dates from 1997, when I was a student at the University of Sydney, supervised by Gus Lehrer. As the reader will observe, it is indebted to Lehrer’s ideas, and I thank him for his help and encouragement. I did not publish this result at the time, since it did not prove the motivating Conjecture 1.1 in general. Recently Eric Sommers [7] has completed the proof of Conjecture 1.1 by a different method, and also removed the assumption that \( e \) is regular in a Levi subalgebra. I thank him for his interest in my old result, and for the suggestion that it be published to supply part of the general proof.

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