BRAKKE’S FORMULATION OF VELOCITY AND
THE SECOND ORDER REGULARITY PROPERTY

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ABSTRACT. Suppose that a family of $k$-dimensional surfaces in $\mathbb{R}^n$ evolves by the motion law of $v = h + u^\perp$ in the sense of Brakke’s formulation of velocity, where $v$ is the normal velocity vector, $h$ is the generalized mean curvature vector and $u^\perp$ is the normal projection of a given vector field $u$ in a dimensionally sharp integrability class. When the flow is locally close to a time-independent $k$-dimensional plane in a weak sense of measure in space-time, it is represented as a graph of a $C^{1,\alpha}$ function over the plane. On the other hand, it is not known if the graph satisfies the PDE of $v = h + u^\perp$ pointwise in general. For this problem, when $k = n - 1$ and under the additional assumption that the distributional time derivative of the graph is a signed Radon measure, it is proved that the graph satisfies the PDE pointwise. An application to a short-time existence theorem for a surface evolution problem is given.

1. Introduction

A family of $k$-dimensional surfaces $\{M_t\}_{t \geq 0}$ in $\mathbb{R}^n$ is called the mean curvature flow (abbreviated hereafter as MCF) if the velocity $v$ is equal to the mean curvature vector $h$ of $M_t$ at each point on $M_t$. Given a smooth compact surface $M_0$, the MCF as the initial value problem is well-posed until some singularities such as vanishing and pinching appear. To extend the flow past singularities, a number of generalized formulations of MCF have been proposed: we mention among others, the level-set flow [3, 6], Brakke flow [2] and BV solution [10]. The properties of these generalized MCFs and their relations have been studied by numerous researchers for the last 40 years or so. The present paper is concerned with a subtle aspect on the formulation of velocity in the definition of the Brakke flow. Within this framework, the velocity of the flow is characterized by the so-called Brakke’s inequality which dictates the rate of change of surface measure. We may, for example, characterize a normal vector field $v$ to be the velocity of $M_t$ in the sense of Brakke if

$$\frac{d}{dt} \left( \int_{M_t} \phi \, d\mathcal{H}^k \right) \leq \int_{M_t} \left\{ (\nabla \phi - \phi \, h) \cdot v + \partial_t \phi \right\} \, d\mathcal{H}^k \text{ for all non-negative } \phi = \phi(x,t). \quad (1.1)$$

Here, the symbol $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure and (1.1) is understood in the sense of distributions. If $\{M_t\}_{t \geq 0}$ is any smooth family of surfaces (which need not be MCF), one can prove that the inequality (1.1) is satisfied if and only if the normal vector field $v$ is the usual velocity of $M_t$ (see [15] Chapter 2). Omitting all the details on what $\{M_t\}_{t \geq 0}$ should additionally satisfy, if we can take $v = h$ in (1.1), then the family $\{M_t\}_{t \geq 0}$ may be roughly called a MCF in the sense of Brakke (or Brakke flow).

A special feature of the Brakke flow compared to other formulations is Brakke’s local regularity theorem [2, 8, 14], namely, if a Brakke flow is locally close to a time-independent $k$-dimensional plane in a weak sense of measure, then it is a $C^\infty$ MCF in the space-time...
interior. The a priori regularity of the Brakke flow is just rectifiability for almost all time with square integrable generalized mean curvature, so it is highly nontrivial to prove such a high degree of regularity. The regularity theorem has two parts, the $C^{1,\alpha}$ part [8] which shows the flow is represented as a $k$-dimensional graph of $C^{1,\alpha}$ function among other things, and the $C^{2,\alpha}$ part [14] which shows the flow satisfies $v = h$ pointwise. At this point, the standard regularity theory for parabolic PDE can be applied to obtain $C^\infty$ regularity. Just as usual for parabolic problems, the regularity with respect to the time variable is halved, that is, when we talk about $C^{2,\alpha}$, it means that the time derivative of the graph is $\alpha/2$-Hölder continuous in the $t$ direction, so at least, the classical pointwise notions of velocity and mean curvature are well-defined once the $C^{2,\alpha}$ regularity is available. The regularity theorem of [8] [14] is actually more general than Brakke’s original version in that the motion law can be $v = h + u^\perp$ with a given vector field $u$ in a suitable regularity class. Here, $u^\perp$ is the projection of $u$ to the orthogonal complement of the tangent space of $M_t$. Just to familiarize the reader, note that the motion law is a geometric analogue of the heat equation with inhomogeneous term: “$\partial_t f = \Delta f + u^\perp$”. To relate to the time-independent case, the result [8] is the precise parabolic extension of the Allard regularity theorem [11] and gives a $C^{1,\alpha}$ regularity under the dimensionally sharp integrability assumption on $u$. If $u$ is assumed to be Hölder continuous, then [14] proves the $C^{2,\alpha}$ regularity with the conclusion that $v = h + u^\perp$ holds classically in the space-time interior.

One question not addressed in [8] and the main subject of the present paper may be explained as follows. In the situation considered in [8] where $u$ satisfies

$$\|u\|_{L^{p,q}} := \left( \int_0^T \left( \int_{M_t} |u(x,t)|^p \, dH^k(x) \right)^{q/p} \, dt \right)^{1/q} < \infty$$

(1.2)

with $p, q \in [2, \infty)$ satisfying

$$\alpha := 1 - \frac{k}{p} - \frac{2}{q} > 0,$$

(1.3)

if $M_t$ is locally close to a $k$-plane in measure and $v = h + u^\perp$ is satisfied in the sense of Brakke as in [11], then [8] proves that $M_t$ is represented as a $C^{1,\alpha}$ graph with a suitable estimate (see [8] Theorem 8.7). It is then reasonable to ask if the graph belongs additionally to $L^q((0,T); W^{2,p}) \cap W^{1,q}((0,T); L^p)$, a natural class in view of the parabolic PDE regularity theory. Furthermore, one may ask if the motion law of $v = h + u^\perp$ is satisfied almost everywhere pointwise. In other words, the question is: is the $C^{1,\alpha}$ graph of $M_t$ a strong PDE solution for $v = h + u^\perp$? Though the affirmative conclusion may sound reasonable, the answer is not known in general at present. A quick remark on the reason is that the formulation (1.1) is based on the inequality and fails to give a PDE with equality to work with even when $M_t$ is a $C^{1,\alpha}$ graph. The comparison with the time-independent situation of the Allard regularity theorem is interesting in that, once $C^{1,\alpha}$ regularity is established for the weak formulation of $0 = h + u^\perp$ (see [8] 10.2), where (1.2) and (1.3) in the time-independent situation require $p > k$ (with $q$ arbitrary) and $h = -u^\perp \in L^p$, then the standard elliptic PDE regularity theory shows that the graph is in $W^{2,p}$ and a strong solution for $h = -u^\perp$. Thus, showing $C^{1,\alpha}$ leads automatically to the strong solution in the time-independent case of Allard regularity theory. As stated already, if $u$ is in $C^\alpha$, then [14] shows that the graph is a classical $C^{2,\alpha}$ solution satisfying the PDE, thus there is a certain gap between the Hölder case and $L^{p,q}$ case presently.
Towards this question in this paper, for the hypersurface case \( k = n - 1 \), we prove that \( M_t \) is a strong PDE solution of \( v = h + u^\perp \) if we additionally assume a certain regularity with respect to \( t \):

**Theorem 1.1.** Suppose that \( M_t \) is represented as a \( C^{1,\alpha} \) graph \( x_n = f(x_1, \ldots, x_{n-1}, t) \) locally in space-time, and (1.1) is satisfied in the sense of distributions with \( v = h + u^\perp \) for \( u \) satisfying (1.2) and (1.3). If the time derivative \( \partial_t f \) exists as a signed Radon measure, then \( \partial_t f \) and \( \nabla^2 f \) belong to \( L^{p,q} \) and the function \( f \) satisfies \( v = h + u^\perp \) pointwise for almost all point, that is, \( f \) is a strong solution of the PDE, \( v = h + u^\perp \).

The precise statement with detailed descriptions on \( M_t \) will be given later, but the reader may think that \( \{ M_t \}_{t \geq 0} \) is a “Brakke-like flow” satisfying (1.1) with \( v = h + u^\perp \). The result shows that Brakke’s formulation of the velocity \( v = h + u^\perp \) gives a strong solution if it is supplemented by the additional assumption on the time derivative. Though this additional assumption itself does not appear to follow from Brakke’s formulation of (1.1), it is interesting to note that the flow obtained as a limit of the Allen-Cahn equation with transport term in [13] satisfies \( \partial_t f \in L^2 \) in addition to (1.1), so in particular a signed Radon measure. Roughly speaking, \( \partial_t f \in L^2 \) follows from the property that the distributional derivative of the limit phase function with respect to \( t \) is \( L^2 \) with respect to the surface measure of interface. The latter property is strongly related to the MCF in the sense of BV solutions [10], where the existence of \( L^2 \) velocity is a part of the definition. Combined with [13], we prove a local-in-time existence of the strong solution for the surface evolution problem \( v = h + u^\perp \) where \( u \) is a given vector field in a Sobolev space whose trace on \( \{ M_t \}_{t \geq 0} \) is \( L^{p,q} \), see Theorem 2.3.

For Theorem 1.1 we first give a proof under a stronger assumption of \( \partial_t f \in L^2 \). There are three reasons for doing this: (1) The proof is simpler compared to the case of \( \partial_t f \) being a signed Radon measure. (2) The proof should work in principle for general \( k \)-dimensional case. (3) The application to the Allen-Cahn equation with transport term falls in this situation and it is good to have a simpler proof in this case separately. If \( \partial_t f \in L^2 \), the graph can be approximated by a smooth function and some appropriate scaling argument shows the desired PDE. For the case of signed Radon measure, we use the fact that Brakke’s inequality gives rise to a Radon measure with which Brakke’s inequality is turned into the equality. Then using the signed distance function to the suitably mollified smooth graph and estimating the errors coming from the approximation, we show that the graph has a weak \( L^2 \) \( t \)-derivative. We remark that, because of the use of signed distance function, the proof seems to be limited to the hypersurface case. One natural question is that whether the additional assumption on \( \partial_t f \) is necessary or not and presently we do not know the answer.

The organization of the paper is as follows. In Section 2, detailed assumptions and main results are presented. In Section 3, the proof of the main regularity theorem under the stronger assumption of \( \partial_t f \in L^2 \) is given, and that of general case is given in Section 4. In the final Section 5, the proof of Theorem 2.3 is given.

2. MAIN RESULTS

2.1. Basic notation. For \( 0 < r < \infty \) and \( a \in \mathbb{R}^k \) (\( 1 \leq k \leq n \)), we define \( B^k_r(a) := \{ x \in \mathbb{R}^k : |x - a| < r \} \) and \( B^k_r := B^k_r(0) \). We often use \( B^{n-1}_r \) throughout the paper, so we write \( B_r \) for \( B^{n-1}_r \). The symbols \( \mathcal{L}^k \) defined in \( \mathbb{R}^k \) and \( \mathcal{H}^k \) defined in \( \mathbb{R}^n \) are the \( k \)-dimensional Lebesgue measure and the Hausdorff measure, respectively. Notation for the functional spaces such as \( L^p(B_r) \) and \( W^{k,p}(B_r) \) are the same as in [7].
2.2. Setting of the problem. Suppose that a function $f : (x, t) \in B_1 \times (0, 1) \to \mathbb{R}$ is a $C^{1,\alpha}$ function in the parabolic sense, where $\alpha \in (0, 1)$ is defined as in \([1,3]\) with given $p, q \in [2, \infty)$. Here, parabolic $C^{1,\alpha}$ means that

$$\left[ f \right]_{C^{1,\alpha}} := \sup_{(y_j,s_j) \in B_1 \times (0,1), j=1,2} \frac{|\nabla f(y_1,s_1) - \nabla f(y_2,s_2)|}{\max\{|y_1 - y_2|^\alpha, |s_1 - s_2|^\alpha/2\}}$$

$$+ \sup_{(y_j,s_j) \in B_1 \times (0,1), j=1,2} \frac{|f(y_1,s_1) - f(y_2,s_2)|}{|s_1 - s_2|^{(1+\alpha)/2}} < \infty. \quad (2.1)$$

The symbol $\nabla f$ is the gradient of $f$ with respect to the space variables. With this $f$, define the $(n-1)$-dimensional hypersurface $M_t$ by

$$M_t := \{(x, f(x,t)) : x \in B_1 \} \subset \mathbb{R}^n \quad (2.2)$$

for $t \in (0,1)$. We assume that $M_t$ has the generalized mean curvature vector (see \([11,12]\) for the definition) $h = h(\cdot, t)$ for $L^1$-a.e. $t \in (0,1)$. In the case that $M_t$ is represented as a graph, we may consider either that $h$ is defined on $M_t$ or $B_1$, and we may use the same notation $h$ with no fear of confusion. From the definition of the generalized mean curvature vector and since $h = (h \cdot \nu)\nu$ by the perpendicularity theorem of Brakke \([2, \text{Chapter 5}]\), $h$ satisfies

$$\int_{B_1} \nabla \psi \cdot \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) \, dx = - \int_{B_1} \psi \, h \cdot \nu \, dx \quad (2.3)$$

for all $\psi \in C^1_c(B_1)$. Here, $\nu := (-\nabla f, 1)/\sqrt{1 + |\nabla f|^2}$ is the unit normal vector of $M_t$. We assume that $h$ is in $L^2$, namely,

$$\int_0^1 \int_{M_t} |h|^2 \, d\mathcal{H}^{n-1} \, dt = \int_0^1 \int_{B_1} |h|^2 \sqrt{1 + |\nabla f|^2} \, dx \, dt < \infty. \quad (2.4)$$

Next, suppose that a vector field $u(\cdot, t) : M_t \to \mathbb{R}^n$ defined for $L^1$-a.e. $t \in (0,1)$ satisfies (just like $h$, we may use the same notation $u(\cdot, t)$ as a function defined on $M_t$ or $B_1$)

$$\int_0^1 \left( \int_{M_t} |u|^p \, d\mathcal{H}^{n-1} \right)^{\frac{2}{p}} \, dt = \int_0^1 \left( \int_{B_1} |u|^p \sqrt{1 + |\nabla f|^2} \, dx \right)^{\frac{2}{p}} \, dt < \infty. \quad (2.5)$$

For all non-negative test function with compact support $\phi \in C^1_c((B_1 \times \mathbb{R}) \times (0,1); \mathbb{R}^+)$, assume that we have the inequality

$$0 \leq \int_0^1 \int_{M_t} (\nabla \phi - \phi h) \cdot \{h + (u \cdot \nu)\nu\} + \partial_t \phi \, d\mathcal{H}^{n-1} \, dt. \quad (2.6)$$

Here note that $\nabla \phi$ is the gradient of $\phi$ in $\mathbb{R}^n$ and the formula \((2.6)\) corresponds to Brakke’s formulation for “$v = h + u_1^+”$ as in \([1,1]\).

Remark 2.1. The above assumptions are locally satisfied after a suitable change of variables once the regularity result in \([8]\) is applied under the assumptions \([8, (A1)-(A4)]\), see the statement of \([8, \text{Theorem 8.7}]\).
2.3. Statement of main result. The following is the main theorem of the present paper.

**Theorem 2.2.** Suppose that $f$, $h$ and $u$ are as discussed in Section 2.2 and assume additionally that $\partial_t f$ is a signed Radon measure on $B_1 \times (0, 1)$. Then, we have

$$f \in L^p((s^2, 1); W^{1,p}(B_1-s)) \cap W^{1,q}((s^2, 1); L^q(B_1-s))$$

for all $s \in (0, 1)$ and $f$ satisfies the motion law of $v = h + u^\perp$, that is,

$$\frac{\partial_t f}{\sqrt{1 + |\nabla f|^2}} = \text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) + u \cdot \frac{-(\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$$

(2.8)

$L^n$-a.e. on $B_1 \times (0, 1)$.

We give one application of Theorem 2.2 to an existence theorem of surface evolution problem studied in [13]. The assumptions on $u$ are the same as [13] Theorem 2.2] but to avoid confusion, $p, q$ in [13] are denoted by $\beta, \gamma$. The claim is that, whenever the $C^{1,\alpha}$ regularity theorem of [8] is applied to the solution established in [13] in some space-time neighborhood, then it is a strong solution in the same neighborhood. Since the paper [13] shows the short-time existence of solution for which the $C^{1,\alpha}$ regularity theorem is applicable for all points, we obtain the following.

**Theorem 2.3.** Suppose $n \geq 2$,

$$2 < \gamma < \infty, \quad \frac{n\gamma}{2(\gamma - 1)} < \beta < \infty \quad \left(\frac{4}{3} \leq \beta \text{ in addition if } n = 2\right)$$

and $\Omega = \mathbb{R}^n$ or $\mathbb{T}^n$. Given any time-dependent Sobolev vector field

$$u \in L^\gamma_{xoc}([0, \infty); (W^{1,\beta}(\Omega))^{n})$$

and a non-empty bounded domain $\Omega_0 \subset \Omega$ with $C^1$ boundary $M_0 = \partial \Omega_0$, there exist $T > 0$ and a family of $C^{1,\alpha}$ hypersurfaces $\{M_t\}_{t \in (0,T)}$ whose motion law is $v = h + u^\perp$ as a strong solution and $\lim_{t \to 0^+} M_t = M_0$ in $C^1$ topology. Here, $\alpha = 2 - n/\beta - 2/\gamma$ if $\beta < n$, and if $\beta \geq n$ one may take any $\alpha$ with $0 < \alpha < 1 - 2/\gamma$. In addition, the vector field $u$ is defined as a trace on $M_t$ for a.e. $t \in (0, T)$ and $\|u\|_{L^{p,q}}$ (as in (1.2)) is finite with $p = \beta(n-1)/(n-\beta)$ and $q = \gamma$ if $\beta < n$.

To be clear about being a strong solution here, for each $t \in (0, T) \times x \in M_t$, there exists a space-time neighborhood in which $\cup_{t \in (0,T)}(M_t \times \{t\})$ is represented as a graph of a function $f$ with the regularity of (2.7) and satisfying the equation (2.8) for $L^n$-a.e. in the neighborhood. Note that the conditions on $\beta$ and $\gamma$ imply (1.2) for $p$ and $q$. If $\beta > n$, then the Sobolev embedding shows $u \in L^\gamma_{loc}([0, \infty); (C^{1,\frac{\beta}{n}}(\Omega))^{n})$ and $\|u\|_{L^{p,q}} < \infty$ for any $p > 2$ and $q = \gamma$ and the corresponding regularity result follows for $f$ (the last part is also true for the case of $\beta = n$).

3. Proof of Theorem 2.2: $\partial_t f \in L^2$ case

First we note that (2.3) combined with the standard argument (for example, see [4] Section 6.3.1) that

$$\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = h \cdot \nu \text{ a.e. in } B_1$$

(3.1)

$$\|\nabla^2 f(\cdot, t)\|_{L^2(B_1-s)} \leq C(\|f(\cdot, t)\|_{L^2(B_1)} + \|h(\cdot, t)\|_{L^2(B_1)})$$
for $L^1$-a.e. $t \in (0, 1)$ and $s \in (0, 1)$ for $C = C(n, \|\nabla f\|_{C^0}, s)$ and thus, combined with (2.4), we have

$$f \in L^2((0, 1); W^{2, 2}(B_{1-s})) \quad \text{for all } s \in (0, 1).$$

(3.2)

For the rest of this section, assume the setting explained in Section 2.2.

**Lemma 3.1.** Suppose that $\partial_t f \in L^2(B_1 \times (0, 1))$. Then for all $\psi \in C^1_c(B_1 \times \mathbb{R} \times (0, 1))$, we have

$$\int_0^1 \int_{M_t} \{(\nabla \psi - \psi h) \cdot v + \partial_t \psi\} dH^{n-1} dt = 0,$$

(3.3)

where $\nabla \psi$ is the gradient of $\psi$ in $\mathbb{R}^n$ and $v = \frac{\partial_t f}{\sqrt{1 + |\nabla f|^2}}$.

**Proof.** In the following calculations, we write the gradient of $\psi$ with respect to the first $n - 1$ variables as $\nabla' \psi$ and the derivative with respect to $x_n$ as $\partial_{x_n} \psi$. First we assume that $f$ is in $C^\infty(B_1 \times (0, 1))$ and $M_t$ is defined as in (2.2). Then, the direct computation shows

$$\frac{d}{dt} \int_{M_t} \psi dH^{n-1} = \frac{d}{dt} \int_{B_1} \psi(x, f(x, t), t) \sqrt{1 + |\nabla f|^2} dx
= \int_{B_1} (\partial_t \psi + \partial_{x_n} \psi \partial_t f) \sqrt{1 + |\nabla f|^2} dx + \int_{B_1} \psi \frac{\nabla f \cdot \nabla \partial_t f}{\sqrt{1 + |\nabla f|^2}} dx.$$

Integrating by part, the second term is rewritten as follows.

$$\int_{B_1} \psi \nabla f \cdot \nabla \partial_t f \sqrt{1 + |\nabla f|^2} dx = -\int_{B_1} \partial_t f \psi (\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}) dx = -\int_{B_1} \psi \partial_t f (\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}) dx$$

$$-\int_{B_1} \partial_t f \frac{\nabla' \psi \cdot \nabla f}{\sqrt{1 + |\nabla f|^2}} dx - \int_{B_1} \partial_t f \frac{\partial_{x_n} \psi |\nabla f|^2}{\sqrt{1 + |\nabla f|^2}} dx.$$

Thus

$$\frac{d}{dt} \int_{M_t} \psi dH^{n-1} = \int_{B_1} \partial_t \psi \sqrt{1 + |\nabla f|^2} dx + \int_{B_1} \partial_t f \frac{\partial_{x_n} \psi}{\sqrt{1 + |\nabla f|^2}} dx$$

$$- \int_{B_1} \psi \partial_t f (\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}) dx - \int_{B_1} \partial_t f \frac{\nabla' \psi \cdot \nabla f}{\sqrt{1 + |\nabla f|^2}} dx$$

$$= \int_{B_1} \partial_t \psi \sqrt{1 + |\nabla f|^2} dx + \int_{B_1} (\nabla \psi - \psi h) \cdot v \sqrt{1 + |\nabla f|^2} dx,$$

where $h = \text{div}(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}})\nu$, $v = \frac{\partial_t f}{\sqrt{1 + |\nabla f|^2}}\nu$ with $\nu = \frac{1}{\sqrt{1 + |\nabla f|^2}}(-\nabla f, 1)$. Integrating over $t \in (0, 1)$, we obtain (3.3).

For general case, we have $\nabla^2 f, \partial_t f \in L^2_{\text{loc}}(B_1 \times (0, 1))$, the first one by (3.2) and the second one from the assumption. Then we can take a sequence $\{f_k\}$ of smooth functions such that, as $k \to \infty$,

$$f_k \to f, \ \nabla f_k \to \nabla f \text{ in } C^{\alpha'/2}_{\text{loc}}(B_1 \times (0, 1)) \quad \text{for any } \alpha' \in (0, \alpha),$$

$$\nabla^2 f_k \to \nabla^2 f, \ \partial_t f_k \to \partial_t f \text{ in } L^2_{\text{loc}}(B_1 \times (0, 1)).$$

Since each $f_k$ satisfies (3.3), $f = \lim_{k \to \infty} f_k$ also satisfies (3.3) and the proof is completed. □
**Proof of Theorem 2.2** with $\partial_tf \in L^2$. The idea of the proof is similar to [15 Section 2.1] for the smooth case. First we prove that $f$ is a strong solution of $v = h + u$. By (2.6) and (3.3), it holds for any $\psi \in C_c^1(B_1 \times \mathbb{R} \times (0, 1); \mathbb{R}^+)$ that

$$0 \leq \int_0^1 \int_{M_t} (\nabla \psi - \psi h) \cdot \{h + (u \cdot \nu)v - v\} \, d\mathcal{H}^{n-1} \, dt.$$  \hspace{1cm} (3.4)

We use the Lebesgue differentiation theorem with respect to the parabolic ball. The following can be proved by adapting the proof in [11 Chapter 7] for parabolic balls and $L^2$ norm: For $g \in L^2(\mathbb{R}^{n-1} \times \mathbb{R})$ and $P_r(y, s) \equiv \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \max \{|x - y|, |t - s|^{1/2}\} < r\}$, for $\mathcal{L}^n$-a.e. $(y, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^n(P_r(y, s))} \int_{P_r(y, s)} |g(y, s) - g(x, t)|^2 \, dx \, dt = 0.$$  

We note that the well-known version of the Lebesgue differentiation theorem does not cover the use of the parabolic balls since they are not "nicely shrinking sets" in [11 7.9], but the same proof using $P_r(y, s)$ works. Using this, let $(y, s)$ be a Lebesgue point of the functions $\partial_t f$, $\nabla^2 f$ and $u$, which is true $\mathcal{L}^n$-a.e. on $B_1 \times (0, 1)$. For any $\tilde{\psi} \in C_c^1(\mathbb{R}^n; \mathbb{R}^+)$ and $\eta \in C^1_c((-1, 1); [0, 1])$ with $\int_{\mathbb{R}} \eta(t) \, dt = 1$, define $\psi(x_1, \ldots, x_n, t) := \tilde{\psi}(x_1, \ldots, x_n) \eta(t)$. Then for all small $\lambda > 0$, define

$$\psi_\lambda(x_1, \ldots, x_n, t) = \lambda^{-n}\psi\left(\frac{x_1 - y_1}{\lambda}, \ldots, \frac{x_{n-1} - y_{n-1}}{\lambda}, \frac{x_n - f(y, s)}{\lambda}, \frac{t - s}{\lambda^2}\right).$$

Here, the reason that we use $\lambda^{-n}$ and not $\lambda^{-n-1}$ is that, in the following, we want $\int_{P_1} \psi_\lambda \to 0$ while $\int_{\mathbb{R}} \nabla \psi_\lambda = O(1)$ as $\lambda \to 0$. If $\text{spt} \tilde{\psi} \subset B^R_1$ for $R \geq 1$, we have $\text{spt} \psi_\lambda \subset B^{\lambda R}_1(y, f(y, s)) \times B^{1 \lambda^2}_1(s)$, thus for all sufficiently small $\lambda$, $\psi_\lambda \in C^1_c(B_1 \times \mathbb{R} \times (0, 1); \mathbb{R}^+)$. We estimate as

$$\left| \int_0^1 \int_{M_t} \psi_\lambda h \cdot \{h + (u \cdot \nu)v - v\} \, d\mathcal{H}^{n-1} \, dt \right|$$

$$\leq \lambda^{-n} \sup \|\psi\| \int_{B_{\lambda R}(y) \times B^{1 \lambda^2}_1(s)} (|h|^2 + |h||u| + |h||v|) \sqrt{1 + |\nabla f|^2} \, dx \, dt$$

$$\leq 2\lambda \omega_{n-1} \sup \|\psi\| R^{n+1} (1 + \sup \|\nabla f\|) \int_{P_{\lambda R}(y, s)} (|h|^2 + |h||u| + |h||v|) \, dx \, dt,$$

where we note $\mathcal{L}^n(P_{\lambda R}(y, s)) = 2\omega_{n-1}(\lambda R)^{n+1}$. Since $(y, s)$ is a Lebesgue point, (3.5) shows

$$\lim_{\lambda \to 0} \int_0^1 \int_{M_t} \psi_\lambda h \cdot \{h + (u \cdot \nu)v - v\} \, d\mathcal{H}^{n-1} \, dt = 0.$$  \hspace{1cm} (3.6)

Next, we estimate the term involving $\nabla \psi_\lambda$ in (3.4). To do so, we estimate the difference of $\nabla \psi$ evaluated on $M_t$ and on the tangent plane of $M_t$ at $(y, s)$. For $(x, t) \in B_{\lambda R}(y) \times B^{1 \lambda^2}_1(s)$ and by (2.1), we have

$$|f(x, t) - f(y, s) - \nabla f(y, s) \cdot (x - y)| \leq [f]_{C^{1, \alpha}}(1 + R^{1+\alpha})^{1+\alpha}.\hspace{1cm} (3.7)$$

Writing $\tilde{f}(x) := f(y, s) + \nabla f(y, s) \cdot (x - y)$, we can use (3.7) to estimate

$$|\nabla \psi_\lambda(x, f(x, t)) - \nabla \psi_\lambda(x, \tilde{f}(x))| \leq \lambda^{-n-1+\alpha}[\nabla^2 \psi]\|f\|_{C^{1, \alpha}}(1 + R^{1+\alpha}).\hspace{1cm} (3.8)$$
Write $w := \{ h + (u \cdot \nu) \nu - v \}$ (evaluated on $M_t$) and using (3.8),
\[
\left| \int_0^1 \int_{M_t} \nabla \psi \cdot w \, dH^{n-1} \, dt - \int_{B_{2R}(y) \times B_{2\lambda}^1(s)} \nabla \psi(x, \tilde{f}(x), t) \cdot w \sqrt{1 + |\nabla f|^2} \, dx \, dt \right| \\
\leq \lambda^{-n-1+\alpha c(\alpha, R, \|\psi\|_{C^2}, \|f\|_{C^{1,\alpha}})} \int_{B_{2R}(y) \times B_{2\lambda}^1(s)} |w| \, dx \, dt \\
\leq \lambda^\alpha c(\alpha, R, \|\psi\|_{C^2}, \|f\|_{C^{1,\alpha}}) \int_{P_{2R}(y, s)} |w| \, dx \, dt \rightarrow 0 \quad (\lambda \rightarrow 0),
\]
where we used the fact that $(y, s)$ is a Lebesgue point of $w$. Let $\tilde{x} := (x - y)/\lambda$ and $\tilde{t} := (t - s)/\lambda^2$ and we see that (writing $\tilde{w}(\tilde{x}, \tilde{t}) := w(x, t)$)
\[
\int_{B_{2\lambda}(y, s)} \nabla \psi(x, \tilde{f}(x), \tilde{t}) \cdot w \sqrt{1 + |\nabla f|^2} \, dx \, dt = 0 \quad (\lambda \rightarrow 0),
\]
where we again used (2.1). Because of the property of the Lebesgue point, the last quantity converges to (note $\int_{-1}^1 \eta dt = 1$)
\[
\int_{B_{2\lambda}} \nabla \tilde{w}(\tilde{x}, \nabla f(y, s) \cdot \tilde{x}) \cdot w(y, s) \sqrt{1 + |\nabla f(y, s)|^2} \, d\tilde{x} = \left( \int_{T_{\lambda}M_s} \nabla \tilde{w} \, dH^{n-1} \right) \cdot w(y, s),
\]
where $T_{\lambda}M_s$ denotes the tangent space of $M_s$ at $Y = (y, f(y, s))$. Combining (3.4), (3.6) and (3.9)-(3.11), we obtain
\[
0 \leq \left( \int_{T_{\lambda}M_s} \nabla \tilde{w} \, dH^{n-1} \right) \cdot \{ h + (u \cdot \nu) \nu - v \}(y, s)
\]
for any $\tilde{w} \in C^1_c(\mathbb{R}^n; \mathbb{R}^+).$ By integration by parts, the integral is perpendicular to $T_{\lambda}M_s$. We may still choose a non-negative $\tilde{w}$ so that the integral is equal to $\pm \nu(y, s)$, and since $\{ h + (u \cdot \nu) \nu - v \}(y, s)$ is parallel to $\nu(y, s)$, it has to be 0. This ends the proof that $v = h + (u \cdot \nu) \nu$ $\mathcal{L}^n$-a.e. on $B_1 \times (0, 1)$.

Once we establish the PDE, the regularity of the solution is standard. For the completeness, we present the proof. Fix $s \in (0, 1)$ and let $\tilde{\zeta} \in C^\infty(B_1 \times (0, 1))$ be a non-negative cutoff function such that it vanishes on the parabolic boundary and $\tilde{\zeta} = 1$ on $B_{1-s} \times [s^2, 1)$. Then $\tilde{f} = \zeta \tilde{f}$ satisfies
\[
\partial_h \tilde{f} - \sum_{i,j=1}^{n-1} a_{ij}(x, t) \partial^2 f_{x_ix_j} = F(x, t) \quad \text{for } \mathcal{L}^n\text{-a.e. } (x, t) \in B_1 \times (0, 1).
\]

Here the coefficients $a_{ij} = \delta_{ij} - \frac{\partial_x f \partial_{x_j} f}{1 + |\nabla f|^2}$ are uniformly elliptic and Hölder continuous and we put
\[
F = \zeta u \cdot \nu \sqrt{1 + |\nabla f|^2} + f(\partial_h \zeta - \sum_{i,j=1}^{n-1} a_{ij} \partial^2 f_{x_ix_j} \zeta) - 2 \sum_{i,j=1}^{n-1} a_{ij} \partial_x f \partial_{x_j} \zeta.
\]
By (2.1) and (2.5), we have $F \in L^2((0, 1); L^p(B_1))$. Thus it holds from [9, Theorem 7.3.9] that there exists a unique solution $f_1$ of (3.13) with $f_1(x, 0) = 0$ such that
\[
f_1 \in W^{1,q}((0, 1); L^p(B_1)) \cap L^q((0, 1); W^{2,p}(B_1) \cap W^{1,p}_0(B_1)).
Since both $f_1$ and $\tilde{f}$ are also the unique solution of $\text{(3.13)}$ starting from 0 in $W^{1,2}((0,1); L^2(B_1)) \cap L^2((0,1); W^{2,2}(B_1) \cap W_0^{1,2}(B_1))$, it holds that $f_1 = \tilde{f}$, and since $\tilde{f} = f$ on $B_{1-s} \times [s^2, 1)$, the proof is completed.

4. Proof of Theorem 2.2: The General Case

In this section, we assume that $\partial_t f$ is a signed Radon measure. We first observe that we may regard the right-hand side of $\text{(2.6)}$ as a positive operator defined on $C_0^1((B_1 \times \mathbb{R}) \times (0,1))$. Then, it is well-known (see, for example, [5, Corollary 1.8.1]) that it extends uniquely to a nonnegative bounded linear operator on $C_c((B_1 \times \mathbb{R}) \times (0,1))$. By the Riesz representation theorem, there exists a Radon measure $\xi$ whose support is contained in $\cup_{t \in (0,1)} (M_t \times \{t\})$ and

$$\int_0^1 \int_{M_t} (\nabla \phi - \phi h) \cdot \{h + (u \cdot \nu) \nu\} + \partial_t \phi d\mathcal{H}^{n-1} dt = \int_{\cup_{t \in (0,1)} (M_t \times \{t\})} \phi d\xi \quad \text{(4.1)}$$

for all $\phi \in C_0^1((B_1 \times \mathbb{R}) \times (0,1))$. Here, note that $\phi$ need not be non-negative.

Before we present the rigorous proof, we give a formal proof. Suppose for a moment that $f$ is smooth and consider the signed distance function $d(\cdot, t) := \pm \text{dist}(\cdot, M_t)$ depending above/below of $M_t$ in $B_1 \times \mathbb{R}$. Given $\phi \in C^1_c(B_1 \times \{0,1\})$, we use $\phi d$ in $\text{(4.1)}$. Since $d = 0$ on $M_t$, the right-hand side is 0. Moreover, $\partial_t (\phi d) = \phi \partial_t d = -\phi - \frac{\partial_t f}{\sqrt{1 + |\nabla f|^2}}$ and $\nabla (\phi d) = \phi \nu$ on $M_t$. Plugging these in, we obtain

$$\int_0^1 \int_{B_1} \phi \nu \cdot \{h + (u \cdot \nu) \nu\} \sqrt{1 + |\nabla f|^2} - \phi \partial_t f \, dx \, dt = 0$$

and by the arbitrariness of $\phi$, we see that the motion law $\text{(2.8)}$ is satisfied. Since the distance function is not smooth for $M_t$ in general, this is a formal argument, but we show that this approach works if $f$ is approximated properly with careful estimates on the errors.

Let $\rho(x,t)$ defined on $\mathbb{R}^n$ be the standard radially symmetric mollifier with spt $\rho \subset B_1^n$ and define $\rho^\varepsilon(x,t) := \varepsilon^{-n-1} \rho(x/\varepsilon, t/\varepsilon^2)$ for $\varepsilon > 0$ so that $\int_{\mathbb{R}^{n-1} \times \mathbb{R}} \rho^\varepsilon \, dx \, dt = 1$. Define

$$f^\varepsilon(x,t) := (\rho^\varepsilon \ast f)(x,t), \quad M_t^\varepsilon := \{(x, f^\varepsilon(x,t)) \in \mathbb{R}^n : x \in B_{1-\varepsilon}\}, \quad \text{(4.2)}$$

$$\tilde{d}^\varepsilon(X,t) := \begin{cases} \text{dist}(X, M_t^\varepsilon) & \text{if } x_n \geq f^\varepsilon(x_1, \ldots, x_{n-1}, t), \\ -\text{dist}(X, M_t^\varepsilon) & \text{if } x_n < f^\varepsilon(x_1, \ldots, x_{n-1}, t), \end{cases} \quad \text{(4.3)}$$

where $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let us fix a function $\eta^\varepsilon \in C^\infty_c(\mathbb{R}; [-2\varepsilon, 2\varepsilon])$ such that

$$\eta^\varepsilon(s) = s \text{ if } |s| \leq \varepsilon, \quad \eta^\varepsilon(s) = 0 \text{ if } |s| \geq 2\varepsilon$$

and define

$$d^\varepsilon(X,t) := \eta^\varepsilon(\tilde{d}^\varepsilon(X,t)). \quad \text{(4.4)}$$

In the following, we fix $\phi \in C^1_c(B_1 \times \{0,1\})$ and use $\phi \, d^\varepsilon$ in $\text{(4.1)}$, so we are interested in the values of $d^\varepsilon$, $\nabla d^\varepsilon$, $\nabla^2 d^\varepsilon$ and $\partial_t d^\varepsilon$ on $M_t$. We regard $\phi$ to be a function defined on $B_1 \times \mathbb{R} \times (0,1)$ which is independent of $x_n$ direction as well as a function on $B_1 \times (0,1)$. First, we show that $d^\varepsilon$ is $C^\infty$ on a small neighborhood of $M_t$. The following two lemmas are easy but we include the proofs for the reader’s convenience:

Lemma 4.1. For $\varepsilon \in (0,1)$, we have

$$\sup_{(x,t) \in B_{1-\varepsilon} \times [\varepsilon^2, 1-\varepsilon^2]} |f^\varepsilon(x,t) - f(x,t)| \leq 2[f]_{C^{1,\alpha}} \varepsilon^{1+\alpha}. \quad \text{(4.5)}$$
Proof. For \((y, s) \in B_{\varepsilon} \times (-\varepsilon^2, \varepsilon^2)\) and \((x, t) \in B_{1-\varepsilon} \times (\varepsilon^2, 1-\varepsilon^2)\), by \((2.1)\), we have
\[
|f(x + y, t + s) - f(x, t) - \nabla f(x, t) \cdot y| \leq 2|f|_{C^{1, \alpha}} \varepsilon^{1+\alpha}.
\] (4.6)
Since \(\int_{B_{\varepsilon}} y\rho^\varepsilon(y, s) \, dy = 0\), we have
\[
f^\varepsilon(x, t) - f(x, t) = \int_{B_{\varepsilon} \times (-\varepsilon^2, \varepsilon^2)} \rho^\varepsilon(y, s)(f(x + y, t + s) - f(x, t) - \nabla f(x, t) \cdot y) \, dy \, ds. \tag{4.7}
\]
Then, \((4.6)\) and \((4.7)\) give \((4.5)\). □

**Lemma 4.2.** For \(\varepsilon \in (0, 1)\), we have
\[
\sup_{(x, t) \in B_{1-\varepsilon} \times (\varepsilon^2, 1-\varepsilon^2)} |\nabla^2 f^\varepsilon(x, t)| \leq c(\rho)\varepsilon^{\alpha-1}. \tag{4.8}
\]
Proof. For \(i, j = 1, \ldots, n-1\), by the symmetry of \(\rho^\varepsilon\), we have
\[
\partial^2_{x_i x_j} f^\varepsilon(x, t) = \int \partial_{x_i} \rho^\varepsilon(x - y, t - s) \partial_{y_j} f(y, s) \, dy \, ds \tag{4.9}
\]
\[
= \int \partial_{x_i} \rho^\varepsilon(x - y, t - s) (\partial_{y_j} f(y, s) - \partial_{y_j} f(x, t)) \, dy \, ds.
\]
Since \(\int |\nabla \rho^\varepsilon| \leq c(\rho)\varepsilon^{-1}\), \((2.1)\) and \((4.9)\) show \((4.8)\). □

By \((4.5)\), for all sufficiently small \(\varepsilon\) and for any \(X \in M_t \cap \text{spt} \phi(\cdot, t)\), \(\text{dist}(X, M^\varepsilon_t) \leq 2|f|_{C^{1, \alpha}} \varepsilon^{1+\alpha} < \varepsilon\). Thus we have
\[
d^\varepsilon(X, t) = d^\varepsilon(X, t) \text{ for } X \in M_t \cap \text{spt} \phi(\cdot, t). \tag{4.10}
\]
Also by computing the first and second fundamental forms of the graph of \(f^\varepsilon\), one finds that the principal curvatures correspond to the solutions \(\lambda\) of
\[
\det \left( \frac{\nabla^2 f^\varepsilon}{\sqrt{1 + |\nabla f^\varepsilon|^2}} - \lambda (I + \nabla f^\varepsilon \otimes \nabla f^\varepsilon) \right) = 0.
\]
Thus the principal curvatures of \(M^\varepsilon_t\) are bounded by \(\|\nabla^2 f^\varepsilon\| \leq c(\rho)\varepsilon^{\alpha-1}\) in particular by \((4.8)\). The signed distance function \(d^\varepsilon\) is then known to be smooth in the \(\varepsilon^{-\alpha} / c(\rho)\)-neighborhood of \(M^\varepsilon_t\). Since \(M_t \cap \text{spt} \phi(\cdot, t)\) is in \(2|f|_{C^{1, \alpha}} \varepsilon^{1+\alpha}\)-neighborhood of \(M^\varepsilon_t\), it is also contained there and thus \(d^\varepsilon = d^\varepsilon\) is smooth on \(M_t \cap \text{spt} \phi(\cdot, t)\). By construction, then, \(\phi d^\varepsilon \in C^1_t (B_1 \times \mathbb{R} \times (0, 1))\) and we may justify using it in \((4.1)\). Since \(d^\varepsilon\) and \(\nabla d^\varepsilon\) respectively converge to \(0\) and \(\nu\) uniformly on \(\text{spt} \phi \cap \cup_{t \in (0, 1)} (M_t \times \{t\})\), we can deduce that
\[
0 = \lim_{\varepsilon \to 0} \int_{\cup_{t \in (0, 1)} (M_t \times \{t\})} \phi d^\varepsilon \, d\xi = \lim_{\varepsilon \to 0} \int_0^1 \int_{M_t} \{\nabla (\phi d^\varepsilon) - \phi d^\varepsilon \, h\} \cdot \{h + (u \cdot \nu) \, \nu\} + \partial_t (\phi d^\varepsilon) \, dH^{n-1} \, dt \tag{4.11}
\]
\[
= \int_0^1 \int_{M_t} \phi \, \nu \cdot \{h + (u \cdot \nu) \, \nu\} \, dH^{n-1} \, dt + \lim_{\varepsilon \to 0} \int_0^1 \int_{M_t} \phi \, \partial_t d^\varepsilon \, dH^{n-1} \, dt.
\]
Here we prove

**Lemma 4.3.**
\[
\lim_{\varepsilon \to 0} \int_0^1 \int_{M_t} \phi \, \partial_t d^\varepsilon \, dH^{n-1} \, dt = \int_0^1 \int_{B_1} \partial_t \phi \, f \, dx \, dt. \tag{4.12}
\]
Once this is proved, with (4.11), it implies that $f$ has the weak $L^2$ derivative $\partial_t f = \{(h + u) \cdot \nu \} \sqrt{1 + |\nabla f|^2}$ on $B_1 \times (0, 1)$, which is (2.8), and the argument for (2.7) is the same as the previous section.

**Proof.** We aim to change the domain of integration from $M_t$ to $M^\varepsilon_t$ by using the nearest point projection as follows. For each $X \in M_t \cap \text{spt } \phi(\cdot, t)$, there exists a unique $X^* \in M^\varepsilon_t$ such that $\text{dist}(X, M^\varepsilon_t) = |X - X^*| \leq 2[f]_{C^{1, \alpha}} \varepsilon^{1+\alpha}$. By writing these two points as $X = (x, f(x, t))$ and $X^* = (x^*, f^\varepsilon(x^*, t))$, note that these points are related by the following equation:

$$X^* + d^\varepsilon(X, t) \nabla d^\varepsilon(X^*, t) = X.$$  \tag{4.13}

Since $X^*$ is the nearest point to $X$ in $M^\varepsilon_t$, $X - X^*$ is perpendicular to $M^\varepsilon_t$ at $X^*$ and the unit normal vector at $X^*$ is given by $\nabla d^\varepsilon(X^*, t)$. Then it is clear that (4.13) holds. Since $\nabla d^\varepsilon(X, t) = \nabla d^\varepsilon(X^*, t)$, we have

$$X^* = X - d^\varepsilon(X, t) \nabla d^\varepsilon(X, t).$$  \tag{4.14}

We next claim that

$$\partial_t d^\varepsilon(X, t) = \partial_t d^\varepsilon(X^*, t) = -\partial_t f^\varepsilon(x^*, t)/\sqrt{1 + |\nabla f^\varepsilon(x^*, t)|^2}. \tag{4.15}$$

By differentiating $d^\varepsilon(X - d^\varepsilon(X, t) \nabla d^\varepsilon(X, t), t) = 0$ (which follows from $d^\varepsilon(X^*, t) = 0$) with respect to $t$, we obtain

$$\nabla d^\varepsilon(X^*, t) \cdot (-\partial_t d^\varepsilon(X, t) \nabla d^\varepsilon(X, t) - d^\varepsilon(X, t) \partial_t \nabla d^\varepsilon(X, t)) + \partial_t d^\varepsilon(X^*, t) = 0. \tag{4.16}$$

Since $\nabla d^\varepsilon(X^*, t) \cdot \nabla d^\varepsilon(X, t) = 1$ and

$$\nabla d^\varepsilon(X^*, t) \cdot \partial_t \nabla d^\varepsilon(X, t) = \nabla d^\varepsilon(X, t) \cdot \partial_t \nabla d^\varepsilon(X, t) = \frac{1}{2} \partial_t |\nabla d^\varepsilon(X, t)|^2 = 0,$$

we have the first equality of (4.15) from (4.16). The second equality of (4.15) is obtained by differentiating $d^\varepsilon(x, f^\varepsilon(x, t), t) = 0$ (which follows from $(x, f^\varepsilon(x, t)) \in M^\varepsilon_t$) with respect to $t$ and by using $\partial_x d^\varepsilon(x, f^\varepsilon(x, t), t) = 1/\sqrt{1 + |\nabla f^\varepsilon|^2}$.

Next, consider the map $F^\varepsilon(\cdot, t)$ from $B_1 \cap \text{spt } \phi(\cdot, t)$ to $B_1$ defined by

$$x \mapsto X = (x, f(x, t)) \in M_t \mapsto X^* = (x^*, f^\varepsilon(x^*, t)) \in M^\varepsilon_t \mapsto x^* =: F^\varepsilon(x, t).$$  \tag{4.17}

As indicated in Figure 1, $x$ is lifted up to $M_t$ first, and mapped to the nearest point on $M^\varepsilon_t$ (indicated by $(\triangle)$), and then projected down to $\mathbb{R}^{n-1}$ (indicated by $(\square)$). More explicitly, by projecting the equation (4.14) to $\mathbb{R}^{n-1}$, we have

$$F^\varepsilon(x, t) = x^* = x - d^\varepsilon(x, f(x, t), t) \nabla d^\varepsilon(x, f(x, t), t), \tag{4.18}$$

![Figure 1](image.png)
where we recall that $\nabla' = (\partial_{x_1}, \ldots, \partial_{x_{n-1}})$. Since $d^\varepsilon$ is $C^\infty$ as a function of $(X, t)$ near $M_t \cap \text{spt} \phi(\cdot, t)$ and $f$ is $C^{1,\alpha}$, $F^\varepsilon$ is also $C^{1,\alpha}((B_1 \times (0,1)) \cap \text{spt} \phi)$. We next compute $\nabla F^\varepsilon(x, t)$ as

$$\nabla F^\varepsilon(x, t) = I - \nabla' d^\varepsilon \otimes \nabla' d^\varepsilon - \partial_{x_n} d^\varepsilon \nabla f \otimes \nabla' d^\varepsilon - (\partial_{x_n} \nabla' d^\varepsilon \otimes \nabla f + \nabla^2 d^\varepsilon) d^\varepsilon. \quad (4.19)$$

Since the principal curvatures of $M^\varepsilon$ are bounded by $c\varepsilon^{a-1}$ and $M_t$ is within $c\varepsilon^{1+\alpha}$-neighborhood of $M^\varepsilon$, $\nabla^2 d^\varepsilon$ on $M_t$ is bounded by $c\varepsilon^{\alpha-1}$ as well (see [7] 14.6 for the expression of $\nabla^2 d^\varepsilon$). Since $|d^\varepsilon| \leq c\varepsilon^{1+\alpha}$ on $M_t$, the last two terms of (4.19) involving $\nabla^2 d^\varepsilon$ are bounded by $c\varepsilon^{2\alpha}$ and vanish as $\varepsilon \to 0$. To estimate the second and third terms of (4.19), we compute

$$\left|\nabla' d^\varepsilon + \partial_{x_n} d^\varepsilon \nabla f\right| = \left| -\frac{\nabla f^\varepsilon}{\sqrt{1 + |\nabla f^\varepsilon|^2}} (x^*, t) + \frac{1}{\sqrt{1 + |\nabla f^\varepsilon|^2}} \nabla f(x, t) \right|
\leq |\nabla f^\varepsilon(x^*, t) - \nabla f(x, t)| \leq |\nabla f^\varepsilon(x^*, t) - \nabla f(x^*, t)| + |\nabla f(x^*, t) - \nabla f(x, t)|$$

and since $|x - x^*| \leq c\varepsilon^{1+\alpha}$, this also vanishes as $\varepsilon \to 0$. Thus we have

$$\lim_{\varepsilon \to 0} \nabla F^\varepsilon(x, t) = I$$

uniformly on $(B_1 \times (0,1)) \cap \text{spt} \phi$. By the Inverse Function Theorem, for all sufficiently small $\varepsilon$, $F^\varepsilon(\cdot, t)$ has the $C^1$ inverse function $G^\varepsilon(\cdot, t) : F^\varepsilon(\cdot, t)(\text{spt} \phi(\cdot, t)) \to \text{spt} \phi(\cdot, t)$. One can also check that $G^\varepsilon$ is Hölder continuous in the direction of $t$ and

$$\lim_{\varepsilon \to 0} \nabla G^\varepsilon(x^*, t) = I. \quad (4.20)$$

Now we compute (evaluating $\phi$ and $d^\varepsilon$ at $(x, f(x, t), t)$) using (4.15)

$$\int_0^1 \int_{M_t} \phi \partial_t d^\varepsilon \, d\mathcal{H}^{n-1} \, dt = \iint_{B_1 \times (0,1)} \phi \partial_t d^\varepsilon \sqrt{1 + |\nabla f|^2} \, dx \, dt
= -\iint_{B_1 \times (0,1)} \phi \partial_t f^\varepsilon(F^\varepsilon(x, t), t) \frac{\sqrt{1 + |\nabla f(x, t)|^2}}{\sqrt{1 + |\nabla f^\varepsilon(F^\varepsilon(x, t), t)|^2}} \, dx \, dt. \quad (4.21)$$

By our assumption, $\partial_t f$ is a signed Radon measure, so that the functional

$$\mathcal{R}[\varphi] := -\iint_{B_1 \times (0,1)} \partial_t \varphi f \, dx \, dt$$

defined for $\varphi \in C^1_c(B_1 \times (0,1))$ is continuous with respect to the topology of uniform convergence. We also have

$$\partial_t f^\varepsilon(z, t) = \iint_{B_1 \times (0,1)} \partial_t \rho^\varepsilon(z - y, t - s) f(y, s) \, dy \, ds = \mathcal{R}[\rho^\varepsilon(\cdot - z, \cdot - t)]$$

and the Fubini theorem shows (the domain of integration $B_1 \times (0,1)$ omitted)

$$\iint \varphi(x, t) \mathcal{R}[\rho^\varepsilon(\cdot - F^\varepsilon(x, t), \cdot - t)] \, dx \, dt = \mathcal{R} \left[ \iint \varphi(x, t) \rho^\varepsilon(\cdot - F^\varepsilon(x, t), \cdot - t) \, dx \, dt \right].$$
Using these, we see that
\[- \int \phi \partial_t f^\varepsilon (F^\varepsilon(x,t),t) \sqrt{1 + |\nabla f(x,t)|^2 \over 1 + |\nabla f^\varepsilon(F^\varepsilon(x,t),t)|^2} \, dx \, dt \]
\[- = - \int \phi \Re[\rho^\varepsilon(\cdot - F^\varepsilon(x,t),\cdot - t)] \sqrt{1 + |\nabla f(x,t)|^2 \over 1 + |\nabla f^\varepsilon(F^\varepsilon(x,t),t)|^2} \, dx \, dt \]
\[- = - \Re \left[ \int \rho^\varepsilon(\cdot - F^\varepsilon(x,t),\cdot - t) \phi(x,t) \sqrt{1 + |\nabla f(x,t)|^2 \over 1 + |\nabla f^\varepsilon(F^\varepsilon(x,t),t)|^2} \, dx \, dt \right] \quad (4.22) \]
\[- = - \Re \left[ \int \rho^\varepsilon(\cdot - x^*,\cdot - t) \phi(G^\varepsilon(x^*,t),t) \sqrt{1 + |\nabla f(G^\varepsilon(x^*,t),t)|^2 \over 1 + |\nabla f^\varepsilon(x^*,t)|^2} |\det \nabla G^\varepsilon| \, dx^* \, dt \right]. \]
In the last line, we changed variables \(x = G^\varepsilon(x^*,t)\). It is clear from (4.20) that the function
\[ (y,s) \mapsto \int \rho^\varepsilon(y - x^*, s - t) \phi(G^\varepsilon(x^*,t),t) \sqrt{1 + |\nabla f(G^\varepsilon(x^*,t),t)|^2 \over 1 + |\nabla f^\varepsilon(x^*,t)|^2} |\det \nabla G^\varepsilon| \, dx^* \, dt \]
converges to \(\phi(y,s)\) as \(\varepsilon \to 0\) uniformly for \((y,s) \in B_1 \times (0,1)\). Therefore (4.21) and (4.22) show
\[ \lim_{\varepsilon \to 0} \int_0^1 \int_{M_t} \phi \partial_\varepsilon d^\varepsilon \, dH^{n-1} \, dt = - \Re[\phi] \]
and the proof is completed. \( \square \)

5. PROOF OF THEOREM 2.3

Since the short-time existence of the \(C^1,\alpha\) solution is already established in [13, Theorem 2.5(1)(3)], we only need to consider the situation described in Section 2.2 after a suitable change of variables. To apply Theorem 2.2, the only missing piece is that \(\partial_t f\) is a signed Radon measure. As we alluded in the introduction, we show that \(\partial_t f \in L^2\) in this case of the limit of Allen-Cahn equation with a transport term. In [13], the method of the proof is to approximate \(v = h + u^\perp\) by the Allen–Cahn equation with a transport term coming from \(u\) as follows. With an appropriate initial datum \(\varphi^\varepsilon|^\varepsilon=0\) derived from \(M_0\), one solves
\[ \partial_t \varphi^\varepsilon + u^\varepsilon \cdot \nabla \varphi^\varepsilon = \Delta \varphi^\varepsilon - {W''(\varphi^\varepsilon) \over \varepsilon^2} \quad \text{in} \; \Omega \times (0,\infty), \quad (5.1) \]
where \(\varepsilon > 0\) is a small parameter tending to 0 and \(u^\varepsilon\) is a smooth approximation of \(u\). Moreover \(W \in C^3(\mathbb{R};\mathbb{R}^+)\) satisfies
\[ W(\pm 1) = 0, \quad W'' < 0 \quad \text{on} \; (\gamma, 1) \quad \text{and} \quad W'' > 0 \quad \text{on} \; (-1, \gamma) \quad \text{for some} \; \gamma \in (-1, 1) \]
and
\[ W''(x) \geq \kappa \quad \text{for all} \; 1 \geq |x| \geq \alpha \quad \text{for some} \; \alpha \in (0,1), \; \kappa > 0. \]
Define a constant \(\sigma := \int_{-1}^1 \sqrt{2W(s)} \, ds\). Then it is proved that there exists a sequence \(\varepsilon_i \downarrow 0\) such that
\[ \mu_{\varepsilon_i} := {1 \over \sigma} \left( \varepsilon_i \left| \nabla \varphi^\varepsilon_i(x,t) \right|^2 + {W(\varphi^\varepsilon_i(x,t)) \over \varepsilon_i} \right) \, dx \rightharpoonup \mu_t \quad \text{as} \; i \to \infty \]
for all \(t \geq 0\) as Radon measures and that \(\mu_t\) is rectifiable and integral measure for \(L^{1,\text{a.e.}}\). It is proved that \(\{V_t\}_{t \geq 0}\) is a weak solution of \(v = h + u^\perp\) (see [13, Theorem 2.2] for the precise
Then there exists $w \in BV_{\text{loc}}(\Omega \times [0, \infty)) \cap C^{3,1}_{\text{loc}}([0, \infty); L^1(\Omega))$ such that $w^\varepsilon_i := \Phi \circ \varphi^\varepsilon_i(\cdot, t) \rightarrow w(\cdot, t)$ in $L^1_{\text{loc}}(\Omega)$ as $i \rightarrow \infty$ for all $t \geq 0$.

Here $\Phi(s) := \frac{1}{\varepsilon} \int_{-1}^{s} \sqrt{2W(y)} \, dy$. Using the estimates in the proof of [13, Theorem 2.2], one can prove the following.

**Lemma 5.1.** Let $\varphi^\varepsilon$ be the solution of (5.1) constructed in [13]. Then there exists a constant $C$ independent of $\varepsilon > 0$ such that $w^\varepsilon = \Phi \circ \varphi^\varepsilon$ satisfies

$$
\int_0^T \int_\Omega |\psi \partial_t w^\varepsilon| \, dx \, dt \leq C \left( \int_0^T \int_\Omega \psi^2 \, d\mu^\varepsilon_t \, dt \right)^{\frac{1}{2}}
$$

for all $\psi \in C_c(\Omega \times (0, T))$.

**Proof.** By the Cauchy-Schwarz inequality, we have

$$
\int_0^T \int_\Omega |\psi \partial_t w^\varepsilon| \, dx \, dt = \int_0^T \int_\Omega |\psi| \sigma^{-1} \sqrt{2W(\varphi^\varepsilon)} |\partial_t \varphi^\varepsilon| \, dx \, dt
$$

$$
\leq 2\sigma^{-1} \left( \int_0^T \int_\Omega \psi^2 \left( \frac{W(\varphi^\varepsilon)}{\varepsilon} \right) \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega \left( \frac{\varepsilon (\partial_t \varphi^\varepsilon)^2}{2} \right) \, dx \, dt \right)^{\frac{1}{2}}.
$$

As in the proof of [13, Theorem 2.2], the second term of the right-hand side is bounded by a constant independent of $\varepsilon$ and the first term is bounded by $(\int_0^T \int_\Omega \psi^2 \, d\mu^\varepsilon_t \, dt)^{1/2}$, which shows the desired inequality.

**Lemma 5.2.** Let $\{\mu_t\}_{t \geq 0}$ and $w$ be a family of measures and a limit phase function as above. Then there exists $v \in L^2_{\text{loc}}(d\mu_t \, dt)$ such that for any $T > 0$ and $\psi \in C^1_c(\Omega \times (0, T))$,

$$
0 = \int_0^T \int_\Omega \psi \, v \, d\mu_t \, dt + \int_0^T \int_\Omega w \, \partial_t \psi \, dx \, dt.
$$

(5.2)

**Proof.** Let $w^i = w^\varepsilon_i$ be as above. From Lemma 5.1, by extracting a subsequence, there exists a signed measure $\eta$ on $\Omega \times (0, T)$ such that

$$
\lim_{i \rightarrow \infty} \int_{\Omega \times (0, T)} \psi \, \partial_t w^i \, dx \, dt = \int_{\Omega \times (0, T)} \psi \, d\eta.
$$

Again by Lemma 5.1 and by $d\mu^\varepsilon_t \, dt \rightarrow d\mu_t \, dt$ as $i \rightarrow \infty$,

$$
\left| \int_{\Omega \times (0, T)} \psi \, d\eta \right| \leq C \left( \int_0^T \int_\Omega \psi^2 \, d\mu_t(x) \, dt \right)^{\frac{1}{2}}.
$$

Thus there exists $v \in L^2_{\text{loc}}(d\mu_t \, dt)$ such that for any $\psi \in C^1_c(\Omega \times (0, T))$,

$$
\lim_{i \rightarrow \infty} \int_{\Omega \times (0, T)} \psi \, \partial_t w^i \, dx \, dt = \int_{\Omega \times (0, T)} \psi \, d\eta = \int_{\Omega \times (0, T)} \psi \, v \, d\mu_t \, dt.
$$

Integrating

$$
\frac{d}{dt} \int_\Omega \psi \, w^i \, dx = \int_\Omega w^i \, \partial_t \psi \, dx + \int_\Omega \psi \, \partial_t w^i \, dx
$$
Thus we obtain

\[ 0 = \int_0^T \int_\Omega w^i \partial_t \psi \, dx \, dt + \int_0^T \int_\Omega \psi \partial_t w^i \, dx \, dt. \]

Thus, letting \( i \to \infty \), we have \((5.2)\) and complete the proof. \( \square \)

Now we are ready to finish the proof of Theorem 2.3. Suppose that \( \text{spt} \mu_t \) is locally represented as a \( C^{1,\alpha} \) graph \( f \) as in Section 2.2 and assume without loss of generality that \(-1 < f < 1\) on \( B_1 \times (0,1) \). Thus, \( \text{spt} \mu_t = \{(x, f(x,t)): x \in B_1\} \) and \( \{(x,y) \in B_1 \times (-1,1): y < f(x,t)\} = \{(x,y) \in B_1 \times (-1,1): w(x,y,t) = 1\} \) for \( t \in (0,1) \). Here, we implicitly use the fact established in [13] that the phase function \( w \) has the boundary of \{\( w = 1 \)\} on the support of \( \mu_t \) (see [13, Theorem 2.3(2)]) and we assume without loss of generality that \{\( w = 1 \)\} lies below the graph here. Let \( \psi \in C_c^1(B_1 \times (0,1)) \) be arbitrary, and let \( \omega(x_n) \) be a function such that \( \omega = 1 \) for \( |x_n| < 1 \) and \( \omega = 0 \) if \( |x_n| > 2 \) and smooth otherwise with \( |\omega| \leq 1 \). Define \( \tilde{\psi}(x_1, \ldots, x_n, t) = \omega(x_n)\psi(x_1, \ldots, x_{n-1}, t) \) so that \( \tilde{\psi} \in C_c^1(B_1 \times \mathbb{R} \times (0,1)) \). By Lemma 5.2 we have

\[ 0 = \int_0^1 \int_{B_1 \times (-2,2)} \tilde{\psi} v \, d\mu_t \, dt + \int_0^1 \int_{\{(x,y) \in B_1 \times (-2,2): y \leq f(x,t)\}} \partial_t \tilde{\psi} \, dx \, dt. \]

(5.3)

Since \( \omega = 1 \) on the support of \( \mu_t \), the first term of the right-hand side of \((5.3)\) is equal to

\[ \int_0^1 \int_{B_1} \tilde{\psi}(x, f(x,t), t) v \sqrt{1 + |\nabla f|^2} \, dx \, dt = \int_0^1 \int_{B_1} \psi(x,t) v \sqrt{1 + |\nabla f|^2} \, dx \, dt. \]

The second term of the right-hand side of \((5.3)\) is equal to

\[ \int_0^1 \int_{B_1} \partial_t \psi(x,t) \int_{-2}^{f(x,t)} \omega(y) \, dy \, dx \, dt = \int_0^1 \int_{B_1} f(x,t) \partial_t \psi(x,t) \, dx \, dt. \]

Thus we obtain

\[ 0 = \int_0^1 \int_{B_1} \psi v \sqrt{1 + |\nabla f|^2} \, dx \, dt + \int_0^1 \int_{B_1} f \partial_t \psi \, dx \, dt. \]

Hence, there exists the weak derivative \( \partial_t f \) of \( f \) and it holds that

\[ \partial_t f = v \sqrt{1 + |\nabla f|^2} \in L^2_{\text{loc}}(B_1 \times (0,1)) \]

and the proof is completed.

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