On Efficient Domination for Some Classes of $H$-Free Chordal Graphs

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Abstract

A vertex set $D$ in a finite undirected graph $G$ is an efficient dominating set (e.d.s. for short) of $G$ if every vertex of $G$ is dominated by exactly one vertex of $D$. The Efficient Domination (ED) problem, which asks for the existence of an e.d.s. in $G$, is known to be $\mathbb{NP}$-complete even for very restricted graph classes such as for 2$P_3$-free chordal graphs while it is solvable in polynomial time for $P_6$-free chordal graphs (and even for $P_5$-free graphs). A standard reduction from the $\mathbb{NP}$-complete Exact Cover problem shows that ED is $\mathbb{NP}$-complete for a very special subclass of chordal graphs generalizing split graphs. The reduction implies that ED is $\mathbb{NP}$-complete e.g. for double-gem-free chordal graphs while it is solvable in linear time for gem-free chordal graphs (by various reasons such as bounded clique-width, distance-hereditary graphs, chordal square etc.), and ED is $\mathbb{NP}$-complete for butterfly-free chordal graphs while it is solvable in linear time for 2$P_2$-free graphs.

We show that (weighted) ED can be solved in polynomial time for $H$-free chordal graphs when $H$ is net, extended gem, or $S_{1,2,3}$.

Keywords: Weighted efficient domination; $H$-free chordal graphs; $\mathbb{NP}$-completeness; net-free chordal graphs; extended-gem-free chordal graphs; $S_{1,2,3}$-free chordal graphs; polynomial time algorithm; clique-width.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph. A vertex $v$ dominates itself and its neighbors. A vertex subset $D \subseteq V$ is an efficient dominating set (e.d.s. for short) of $G$ if every vertex of $G$ is dominated by exactly one vertex in $D$; for any e.d.s. $D$ of $G$, $|D \cap N[v]| = 1$ for every $v \in V$ (where $N[v]$ denotes the closed neighborhood of $x$). Note that not every graph has an e.d.s.; the Efficient Dominating Set (ED) problem asks for the existence of an e.d.s. in a given graph $G$.

The Exact Cover Problem (X3C [SP2] in [17]) asks for a subset $F'$ of a set family $F$ over a ground set, say $V$, containing every vertex in $V$ exactly once. As shown by Karp [19], the problem is $\mathbb{NP}$-complete even for set families containing only 3-element subsets of $V$ (see [17]). Clearly, ED is the Exact Cover problem for the closed neighborhood hypergraph of $G$. The notion of efficient domination was introduced by Biggs [3] under the name perfect code. The ED problem is motivated by various applications, including coding theory and resource allocation in parallel computer networks; see e.g. [1][3][12][20][22][26][20][28][29].
In [1,2], it was shown that the ED problem is \(\mathsf{NP}\)-complete. Moreover, ED is \(\mathsf{NP}\)-complete for \(2P_3\)-free chordal unipolar graphs [14,27,29].

In this paper, we will also consider the following weighted version of the ED problem:

### Weighted Efficient Domination (WED)

**Instance:** A graph \(G = (V, E)\), vertex weights \(\omega : V \to \mathbb{N} \cup \{\infty\}\).

**Task:** Find an e.d.s. of minimum finite total weight, or determine that \(G\) contains no such e.d.s.

The relationship between WED and ED is analyzed in [6].

For a set \(\mathcal{F}\) of graphs, a graph \(G\) is called \(\mathcal{F}\)-free if \(G\) contains no induced subgraph isomorphic to a member of \(\mathcal{F}\). In particular, we say that \(G\) is \(H\)-free if \(G\) is \(\{H\}\)-free. Let \(H_1 + H_2\) denote the disjoint union of graphs \(H_1\) and \(H_2\), and for \(k \geq 2\), let \(kH\) denote the disjoint union of \(k\) copies of \(H\). For \(i \geq 1\), let \(P_i\) denote the chordless path with \(i\) vertices, and let \(K_i\) denote the complete graph with \(i\) vertices (clearly, \(P_2 = K_2\)). For \(i \geq 4\), let \(C_i\) denote the chordless cycle with \(i\) vertices.

For indices \(i, j, k \geq 0\), let \(S_{i,j,k}\) denote the graph with vertices \(u, x_1, \ldots, x_i, y_1, \ldots, y_j, z_1, \ldots, z_k\) such that the subgraph induced by \(u, x_1, \ldots, x_i\) forms a \(P_{i+1}\) \((u, x_1, \ldots, x_i)\), the subgraph induced by \(u, y_1, \ldots, y_j\) forms a \(P_{j+1}\) \((u, y_1, \ldots, y_j)\), and the subgraph induced by \(u, z_1, \ldots, z_k\) forms a \(P_{k+1}\) \((u, z_1, \ldots, z_k)\), and there are no other edges in \(S_{i,j,k}\). Thus, claw is \(S_{1,1,1}\), chair is \(S_{1,1,2}\), and \(P_k\) is isomorphic to e.g. \(S_{0,0,k-1}\). \(H\) is a linear forest if every component of \(H\) is a chordless path, i.e., \(H\) is claw-free and cycle-free.

\(H\) is a co-chair if it is the complement graph of a chair. \(H\) is a \(P\) if \(H\) has five vertices such that four of them induce a \(C_4\) and the fifth is adjacent to exactly one of the \(C_4\)-vertices. \(H\) is a co-\(P\) if \(H\) is the complement graph of a \(P\). \(H\) is a bull if \(H\) has five vertices such that four of them induce a \(P_4\) and the fifth is adjacent to exactly the two mid-points of the \(P_4\). \(H\) is a net if \(H\) has six vertices such that five of them induce a bull and the sixth is adjacent to exactly the vertex of the bull with degree 2. \(H\) is a gem if \(H\) has five vertices such that four of them induce a \(P_3\) and the fifth is adjacent to all of the \(P_3\) vertices. \(H\) is a co-gem if \(H\) is the complement graph of a gem.

For a vertex \(v \in V\), \(N(v) = \{u \in V : uv \in E\}\) denotes its (open) neighborhood, and \(N[v] = \{v\} \cup N(v)\) denotes its closed neighborhood. A vertex \(v\) sees the vertices in \(N(v)\) and misses all the others. The non-neighborhood of a vertex \(v\) is \(\overline{N}(v) := V \setminus N[v]\). For \(U \subseteq V\), \(N(U) := \bigcup_{u \in U} N(u) \setminus U\) and \(\overline{N}(U) := V \setminus (U \cup N(U))\).

We say that for a vertex set \(X \subseteq V\), a vertex \(v \notin X\) has a join (resp., co-join) to \(X\) if \(X \subseteq N(v)\) (resp., \(X \subseteq \overline{N}(v)\)). Join (resp., co-join) of \(v\) to \(X\) is denoted by \(v \upharpoonright X\) (resp., \(v \upharpoonright \overline{N}(X)\)). Correspondingly, for vertex sets \(X, Y \subseteq V\) with \(X \cap Y = \emptyset\), \(X \upharpoonright Y\) denotes \(x \upharpoonright Y\) for all \(x \in X\) and \(X \upharpoonright \overline{Y}\) denotes \(x \upharpoonright \overline{Y}\) for all \(x \in X\). A vertex \(x \notin U\) contacts \(U\) if \(x\) has a neighbor in \(U\). For vertex sets \(U, U'\) with \(U \cap U' = \emptyset\), \(U\) contacts \(U'\) if there is a vertex in \(U\) contacting \(U'\).

If \(v \notin X\) but \(v\) has neither a join nor a co-join to \(X\), then we say that \(v\) distinguishes \(X\). A set \(H\) of at least two vertices of a graph \(G\) is called homogeneous if \(H \neq V(G)\) and every vertex outside \(H\) is either adjacent to all vertices in \(H\), or to no vertex in \(H\). Obviously, \(H\) is homogeneous in \(G\) if and only if \(H\) is homogeneous in the complement graph \(\overline{G}\). A graph is prime if it contains no homogeneous set.

A graph \(G\) is chordal if it is \(C_4\)-free for any \(i \geq 4\). \(G = (V, E)\) is unipolar if \(V\) can be partitioned into a clique and the disjoint union of cliques, i.e., there is a partition \(V = A \cup B\) such that \(G[A]\) is a complete subgraph and \(G[B]\) is \(P_3\)-free. \(G\) is a split graph if \(G\) and its complement...
graph are chordal. Equivalently, \( G \) can be partitioned into a clique and an independent set. It is well known that \( G \) is a split graph if and only if it is \((2P_2, C_4, C_5)\)-free [14].

It is well known that ED is \( \text{NP} \)-complete for claw-free graphs (even for \((K_{1,3}, K_4 - e)\)-free perfect graphs [24]) as well as for bipartite graphs (and thus for triangle-free graphs) [25] and for chordal graphs [14, 27, 29]. Thus, for the complexity of ED on \( H \)-free graphs, the most interesting cases are when \( H \) is a linear forest. Since \((W)ED\) is \( \text{NP} \)-complete for \( 2P_3 \)-free graphs and polynomial for \((P_5 + kP_2)\)-free graphs [7], \( P_6 \)-free graphs were the only open case finally solved in [10, 11] by a direct polynomial time approach (and in [23] by an indirect one).

It is well known that for a graph class with bounded clique-width, ED can be solved in polynomial time [13]. Thus we only consider ED on \( H \)-free chordal graphs for which the clique-width is unbounded. For example, the clique-width of \( H \)-free chordal graphs is unbounded for claw-free chordal graphs while it is bounded if \( H \in \{ \text{bull, gem, co-gem, co-chair} \} \). In [4], for all but two stubborn cases, the clique-width of \( H \)-free chordal graphs is classified.

For graph \( G = (V, E) \), the square \( G^2 \) has the same vertex set \( V \), and two vertices \( x, y \in V \), \( x \neq y \), are adjacent in \( G^2 \) if and only if \( d_G(x, y) \leq 2 \). The WED problem on \( G \) can be reduced to Maximum Weight Independent Set (MWIS) on \( G^2 \) (see [6, 8, 9, 26]).

While the complexity of ED for \( 2P_3 \)-free chordal graphs is \( \text{NP} \)-complete (as mentioned above), it was shown in [5] that WED is solvable in polynomial time for \( P_6 \)-free chordal graphs, since for \( P_6 \)-free chordal graphs \( G \) with e.d.s., \( G^2 \) is chordal. It is well known [16] that MWIS is solvable in linear time for chordal graphs.

For \( H \)-free chordal graphs, however, there are still many open cases. Motivated by the \( G^2 \) approach in [5], and the result of Milanič [23] showing that for \((S_{1,2,2}, \text{net})\)-free graphs \( G \), its square \( G^2 \) is claw-free, we show in the next section that \( G^2 \) is chordal for \( H \)-free chordal graphs with e.d.s. when \( H \) is a net or an extended gem (see Figure 1 - extended gem generalizes \( S_{1,2,2} \)), and thus, WED is solvable in polynomial time for these two graph classes.

## 2 For Net-Free Chordal Graphs and Extended-Gem-Free Chordal Graphs with e.d.s., \( G^2 \) is Chordal

![net and extended gem](image)

**Figure 1:** net and extended gem

Let \( G \) be a chordal graph and \( G^2 \) its square.

**Claim 2.1.** Let \( v_1, \ldots, v_k, \ k \geq 4 \), form a \( C_k \) in \( G^2 \) with \( d_G(v_i, v_{i+1}) \leq 2 \) and \( d_G(v_i, v_j) \geq 3 \), \( i, j \in \{1, \ldots, k\}, \ |i - j| > 1 \) (index arithmetic modulo \( k \)). Then

(i) for each \( i \in \{1, \ldots, k\}, \ d_G(v_i, v_{i+1}) = 2 \); let \( x_i \) be a common neighbor of \( v_i \) and \( v_{i+1} \) in \( G \) (an auxiliary vertex).

(ii) for each \( i, j \in \{1, \ldots, k\}, \ i \neq j \), we have \( x_i \neq x_j \), and \( x_i x_{i+1} \in E(G) \).
Proof. (i): If without loss of generality, \(d_G(v_1, v_2) = 1\) then, since \(d_G(v_1, v_3) \geq 3\) and \(d_G(v_1, v_2) \geq 3\), we have \(d_G(v_2, v_3) = 2\) and \(d_G(v_2, v_1) = 2\); let \(x_2\) be a common neighbor of \(v_2, v_3\) and \(x_k\) be a common neighbor of \(v_2, v_1\). Clearly, \(x_2 \neq x_k\) since \(d_G(v_k, v_2) \geq 3\). Moreover, \(x_2x_1 \notin E\) since \(d_G(v_1, v_3) \geq 3\) and \(x_2v_1 \notin E\) since \(d_G(v_1, v_2) \geq 3\). Now, \(x_2x_1 \notin E\) since otherwise \(x_k, v_1, v_2, x_2\) would induce a \(C_4\) in \(G\) but now in any case, the \(P_4\) induced by \(x_k, v_1, v_2, x_2\) leads to a chordless cycle in \(G\) which is a contradiction.

(ii): Clearly, as above, we have \(x_i \neq x_j\) for any \(i \neq j\), and a non-edge \(x_1x_2 \notin E\) would lead to a chordless cycle in \(G\).

\[\square\]

Lemma 1. If \(G\) is a net-free chordal graph with e.d.s., then \(G^2\) is chordal.

Proof. Let \(G = (V, E)\) be a net-free chordal graph and assume that \(G\) contains an e.d.s. \(D\).

Case 1. First suppose to the contrary that \(G^2\) contains \(C_4\), say with vertices \(v_1, v_2, v_3, v_4\) such that \(d_G(v_i, v_{i+1}) \leq 2\) and \(d_G(v_i, v_{i+2}) \geq 3\), \(i \in \{1, 2, 3, 4\}\) (index arithmetic modulo 4). By Claim 2 we have \(d_G(v_i, v_{i+1}) = 2\) for each \(i \in \{1, 2, 3, 4\}\); let \(x_i\) denote a common neighbor of \(v_i, v_{i+1}\) (an auxiliary vertex). By Claim 2, \(x_i \neq x_j\) for \(i \neq j\). Since \(G\) is chordal, \(x_1, x_2, x_3, x_4\) either induce a diamond or \(K_4\) in \(G\).

Assume first that \(x_1, x_2, x_3, x_4\) induce a diamond in \(G\), say with \(x_2x_4 \notin E\).

Case 1.1 \(D \cap \{x_1, x_2, x_3, x_4\} \neq \emptyset\).

If \(x_1 \in D\) then by the e.d.s. property, \(v_3, v_4 \notin D\). Let \(d_4 \in D\) with \(d_4v_4 \in E\). Clearly, \(d_4 \neq x_3\) and \(d_4 \neq x_4\), and by the e.d.s. property, \(v_1d_4 \notin E, x_3d_4 \notin E, x_4d_4 \notin E\), and since \(G\) is chordal, \(v_3d_4 \notin E\) but then \(v_1, x_4, v_4, d_4, x_3, v_3\) induce a net in \(G\). Thus, \(x_1 \notin D\) and correspondingly, \(x_3, x_4 \notin D\).

If \(x_4 \in D\) then \(v_2, v_3, x_2 \notin D\). Let \(d_2 \in D\) with \(d_2v_2 \in E\) and \(d_3 \in D\) with \(d_3v_3 \in E\); clearly, \(d_2 \neq x_2\) and \(d_3 \neq x_2\). Since \(G\) is chordal, \(d_2 \neq d_3\) (and in particular, \(v_3d_2 \notin E\) and \(v_2d_3 \notin E\)).

Clearly, by the e.d.s. property, \(x_1d_2 \notin E, x_3d_3 \notin E\).

Now, if \(d_2x_2 \notin E\) then \(x_1, x_2, d_2, x_2, v_3\) induce a net in \(G\), and if \(d_2x_2 \in E\) then \(d_3x_2 \notin E\) and now, \(v_2, x_2, v_3, d_3, x_3, x_4\) induce a net in \(G\). Thus, \(x_4 \notin D\) and correspondingly, \(x_2 \notin D\). This implies \(D \cap \{x_1, x_2, x_3, x_4\} = \emptyset\).

Case 1.2. \(D \cap \{v_1, v_2, v_3, v_4\} \neq \emptyset\) and \(D \cap \{x_1, x_2, x_3, x_4\} = \emptyset\).

If without loss of generality, \(v_1 \in D\) then \(v_3, v_4, x_2, x_3 \notin D\). Let \(d_2 \in D\) with \(d_2v_2 \in E\) and \(d_4 \in D\) with \(d_4v_4 \in E\). Clearly, since \(G\) is chordal, we have \(d_2v_3 \notin E\) and \(d_4v_3 \notin E\).

If \(v_3 \in D\) then \(d_2, v_2, x_1, v_1, x_2, v_3\) induce a net. Thus, \(v_3 \notin D\). Let \(d_3 \in D\) with \(d_3v_3 \in E\). If \(d_3x_2 \notin E\) and \(d_3x_3 \notin E\) then \(v_2, x_2, v_3, d_3, x_3\) induce a net. If \(d_3x_2 \in E\) then \(v_1, x_1, v_2, d_2, x_2, d_3\) induce a net. Finally, if \(d_3x_3 \in E\) then \(v_1, x_4, v_4, d_4, x_3, d_3\) induce a net. Thus, also Case 1.2 is excluded, and \(D \cap \{v_1, v_2, v_3, v_4\} = \emptyset\).

Case 1.3. \(D \cap \{v_1, v_2, v_3, x_1, x_2, x_3, x_4\} = \emptyset\).

Let \(d_i \in D\) be the \(D\)-neighbor of \(v_i\); \(d_i \neq v, x_j\). Clearly, since \(G\) is chordal and since \(d_G(v_i, v_{i+2}) > 2\), \(d_1, d_2, d_3, d_4\) are pairwise distinct.

If \(d_1x_1 \notin E\) and \(d_1x_4 \notin E\) then \(d_1, v_1, x_1, v_4, v_2, v_3\) induce a net in \(G\), and correspondingly by symmetry, for \(d_i, x_{i-1}, x_i, i \neq 1\). Thus, we can assume that for each \(i \in \{1, \ldots, 4\}\), \(d_i\) sees at least one of \(x_1, x_i\).

If \(d_1x_1 \in E\) and \(d_1x_4 \in E\) then clearly, \(d_1x_1 \notin E\) and \(d_4x_4 \notin E\) and thus, by the above, we can assume that \(d_2x_2 \in E\) and \(d_4x_4 \in E\) but now, \(d_2, x_2, v_3, x_3, d_3, d_4\) induce a net in \(G\).

Thus, assume that \(d_1\) is adjacent to exactly one of \(x_1, x_4\), say \(d_1x_1 \in E\) (which implies \(d_2x_2 \notin E\)) and \(d_1x_4 \notin E\). By symmetry, this holds for \(d_2, d_3, d_4\) as well, i.e., \(d_2x_2 \in E, d_3x_3 \in E, d_1x_4 \in E\). Then \(d_1, x_1, d_2, x_2, d_3, x_3\) induce a net in \(G\).
In a very similar way, we can show that we can exclude a $C_4$ in $G^2$ when $x_1, x_2, x_3, x_4$ induce a $K_4$ in $G$.

**Case 2.** Now suppose to the contrary that $G^2$ contains $C_k$, $k \geq 5$, say with vertices $v_1, \ldots, v_k$ such that $d_G(v_i, v_{i+1}) \leq 2$ and $d_G(v_i, v_{i+2}) \geq 3$, $i \in \{1, 2, 3, 4\}$ (index arithmetic modulo $k$). By Claim 2.1, we have $d_G(v_j, v_{i+1}) = 2$ for each $i \in \{1, \ldots, k\}$; let $x_i$ denote a common neighbor of $v_i, v_{i+1}$. Again, by Claim 2.1, the auxiliary vertices are pairwise distinct and $x_i x_{i+1} \in E$ for each $i \in \{1, \ldots, k\}$. Since $k \geq 5$ and $G$ is chordal, there is an edge $x_i x_{i+1} \in E$ having a common neighbor $x_j, j \neq i, i-1$, say without loss of generality, $x_1, x_2, x_4$ induce a $K_3$ in $G$. Then $v_1, x_1, v_3, x_2, v_4, x_4$ induce a net in $G$ (note that for $k \geq 5$, we do not need the existence of an e.d.s. in $G$).

Similarly, we get a net for any $C_k$ in $G^2$, $k > 5$. Thus, Lemma 1 is shown. □

By [6], and since MWIS is solvable in linear time for chordal graphs, we obtain:

**Corollary 1.** $WED$ is solvable in time $O(n^3)$ for net-free chordal graphs.

Lemma 1 generalizes the corresponding result for AT-free chordal graphs (i.e., interval graphs).

**Lemma 2.** If $G$ is an extended-gem-free chordal graph with e.d.s., then $G^2$ is chordal.

**Proof.** Let $G = (V, E)$ be an extended-gem-free chordal graph and assume that $G$ contains an e.d.s. $D$.

**Case 1.** First suppose to the contrary that $G^2$ contains $C_4$, say with vertices $v_1, v_2, v_3, v_4$ such that $d_G(v_i, v_{i+1}) \leq 2$ and $d_G(v_i, v_{i+2}) \geq 3$, $i \in \{1, 2, 3, 4\}$ (index arithmetic modulo 4). By Claim 2.1, we have $d_G(v_i, v_{i+1}) = 2$ for each $i \in \{1, 2, 3, 4\}$; let $x_i$ denote a common neighbor of $v_i, v_{i+1}$ (an auxiliary vertex). By Claim 2.1, $x_i \neq x_j$ for $i \neq j$. Since $G$ is chordal, $x_1, x_2, x_3, x_4$ either induce a diamond or $K_4$ in $G$.

Assume first that $x_1, x_2, x_3, x_4$ induce a diamond in $G$, say with $x_2 x_4 \notin E$ and $x_1 x_3 \in E$.

**Case 1.1** $D \cap \{x_1, x_2, x_3, x_4\} \neq \emptyset$.

If $x_1 \in D$ then by the e.d.s. property, $v_3, v_4 \notin D$. Let $d_3 \in D$ with $d_3 v_3 \in E$ and $d_4 \in D$ with $d_4 v_4 \in E$. Clearly, since $G$ is chordal, $d_3 \neq d_4$, and by the e.d.s. property, $d_3 x_3 \notin E$, and $d_4 x_3 \notin E$, but then $v_1, x_1, x_3, v_4, x_2, v_3, d_3$ induce an extended gem. Thus, $x_1 \notin D$ and correspondingly, $x_3 \notin D$.

If $x_2 \in D$ then by the e.d.s. property, $v_1, v_4, x_4 \notin D$. Let $d_1, d_4 \in D$ with $d_1 v_1 \in E$ and $d_4 v_4 \in E$. Clearly, since $G$ is chordal, $d_1 \neq d_4$, and in particular, $d_1 v_4 \notin E$ and $d_1$ is nonadjacent to all neighbors of $x_2$ but then $d_1, v_1, x_1, v_2, x_2, v_3, x_3, v_4$ induce an extended gem. Thus, $x_2 \notin D$ and correspondingly, $x_4 \notin D$.

**Case 1.2** $D \cap \{x_1, x_2, x_3, x_4\} = \emptyset$ and $D \cap \{v_1, v_2, v_3, v_4\} \neq \emptyset$.

Without loss of generality, assume that $v_1 \in D$; then by the e.d.s. property, $v_2, v_4, x_2, x_3 \notin D$.

Let $d_2 \in D$ with $d_2 v_2 \in E$ and $d_4 \in D$ with $d_4 v_4 \in E$.

First assume that $v_3 \in D$. Then clearly, $d_2, v_2, x_1, v_1, x_4, v_4, x_3, v_3$ induce an extended gem. Thus, $v_3 \notin E$: let $d_3 \in D$ with $v_3 d_3 \in E$. Note that $d_2 x_1 \notin E$ since $v_1 \in D$ and $d_2 x_3 \notin E$ since otherwise, $d_2, v_2, x_1, x_3$ would induce a $C_4$ in $G$ but now again, $d_2, v_2, x_1, v_1, x_4, v_4, x_3, v_3$ induce an extended gem. Thus, also $D \cap \{v_1, v_2, v_3, v_4\} = \emptyset$.

**Case 1.3** $D \cap \{x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\} = \emptyset$.

For $i \in \{1, 2, 3, 4\}$, let $d_i \in D$ be the $D$-neighbor of $v_i$; $d_i \neq v_j, j \in \{1, 2, 3, 4\}$. Clearly, since $d_G(v_i, v_{i+2}) > 2$ and since $G$ is chordal, $d_1, d_2, d_3, d_4$ are pairwise distinct.
By the e.d.s. property, \( d_3x_3 \notin E \) or \( d_4x_3 \notin E \); without loss of generality, assume that \( d_3x_3 \notin E \). Since \( d_3v_3, x_3, x_1 \) do not induce a \( C_4 \), we have \( d_3x_1 \notin E \), and correspondingly, \( d_3x_4 \notin E \) but now, \( v_1, x_1, x_3, v_4, x_4, v_2, v_3, d_3 \) induce an extended gem.

Thus, \( G^2 \) is \( C_4 \)-free.

In a very similar way, we can show that we can exclude a \( C_4 \) in \( G^2 \) when \( x_1, x_2, x_3, x_4 \) induce a \( K_4 \) in \( G \).

**Case 2.** Now suppose to the contrary that \( G^2 \) contains \( C_k, k \geq 5 \), say with vertices \( v_1, \ldots, v_k \) such that \( d_G(v_i, v_{i+1}) \leq 2 \) and \( d_G(v_i, v_j) \geq 3 \), \( i, j \in \{1, \ldots, k\} \), \( |i - j| > 1 \) (index arithmetic modulo \( k \)). By Claim 2.1, we have \( d_G(v_i, v_{i+1}) = 2 \) for each \( i \in \{1, \ldots, k\} \); let \( x_i \) denote a common neighbor of \( v_i, v_{i+1} \). Again, by Claim 2.1, the auxiliary vertices are pairwise distinct and \( x_ix_{i+1} \in E \) for each \( i \in \{1, \ldots, k\} \).

Clearly, since \( G \) is chordal, there is an edge \( x_ix_{i+2} \in E \).

**Claim 2.2.** If \( x_ix_{i+2} \in E \) then \( x_i, x_{i+1}, x_{i+2} \notin D \) and \( v_{i+1}, v_{i+2} \notin D \).

**Proof.** Without loss of generality, let \( x_ix_3 \in E \). If \( x_2 \in D \) then clearly, \( v_1 \notin D \) and \( x_k \notin D \); let \( d_1 \in D \) with \( d_1v_1 \in E \). Clearly, \( d_1x_1 \notin E \) since \( x_2 \in D \) but now, \( x_1, v_1, x_2, v_3, x_3, v_4, v_1, d_1 \) induce an extended gem. Thus, \( x_2 \notin D \).

If \( x_1 \in D \) then clearly, \( v_4 \notin D \); let \( d_4 \in D \) with \( d_4v_4 \in E \) but now, \( v_1, x_1, v_2, x_2, v_3, x_3, v_4, d_4 \) induce an extended gem. Thus, \( x_1 \notin D \) and correspondingly, \( x_3 \notin D \).

If \( v_2 \in D \) then clearly, \( v_1 \notin D \); let \( d_1 \in D \) with \( d_1v_1 \in E \) but now, \( d_1, v_1, x_1, v_2, x_2, v_3, x_3, v_4 \) induce an extended gem. Thus, \( v_3 \notin D \) and correspondingly, \( v_3 \notin D \) which shows Claim 2.2.

**Claim 2.3.** If \( x_ix_{i+2} \in E \) then \( x_{i+2}x_{i+4} \notin E \) and \( x_{i-1}x_i \notin E \).

**Proof.** Without loss of generality, let \( x_ix_3 \in E \) and suppose to the contrary that \( x_3x_5 \in E \). Then by Claim 2.2 there are new vertices \( d_3, d_4 \in D \), \( d_3, d_4 \notin \{v_3, v_1, x_2, x_3, x_4\} \), with \( d_3v_3 \in E \) and \( d_4v_4 \in E \). By the e.d.s. property, \( d_3x_3 \notin E \) or \( d_4x_3 \notin E \); say without loss of generality, \( d_3x_3 \notin E \). Then by the chordality of \( G \), \( d_3x_3 \notin E \) and \( d_3x_1 \notin E \) but now, \( v_1, x_1, v_2, x_2, v_3, x_3, v_4, d_4 \) induce an extended gem. Thus, Claim 2.3 is shown.

For a \( C_5 \) in \( G^2 \), Claim 2.3 leads to a \( C_4 \) in \( G \) induced by \( x_1, x_3, x_4, x_5 \) if \( x_1x_3 \in E \). Thus, from now on, let \( k \geq 6 \).

**Claim 2.4.** If \( x_ix_{i+2} \in E \) then \( x_{i+1}x_{i+3} \notin E \) and \( x_{i-1}x_{i+1} \notin E \).

**Proof.** Without loss of generality, let \( x_ix_3 \in E \) and suppose to the contrary that \( x_2x_4 \in E \). Then by Claim 2.3 \( x_3x_5 \notin E \) and \( x_4x_6 \notin E \) as well as \( x_1x_{k-1} \notin E \) and \( x_2x_k \notin E \), and since \( G \) is chordal, \( x_3x_6 \notin E \) and \( x_2x_{k-1} \notin E \).

Since \( v_2, x_2, v_3, x_3, v_4, x_4, v_5, v_6 \) does not induce an extended gem, we have \( x_2x_5 \in E \). For \( k = 6 \) this contradicts the fact that \( x_2x_{k-1} \notin E \). Thus, from now on, let \( k \geq 7 \).

Since \( v_2, x_2, x_3, v_4, x_4, v_5, x_5, v_7 \) does not induce an extended gem, we have \( x_2x_6 \in E \). For \( k = 7 \), again, this contradicts the fact that \( x_2x_{k-1} \notin E \). Thus, from now on, let \( k \geq 8 \).

Now, \( x_2, v_3, x_3, v_4, x_4, v_5, x_5, v_6 \) induce an extended gem. Thus, Claim 2.4 is shown.

Now we can assume that \( k \geq 6 \); without loss of generality, let \( x_1x_3 \in E \). Then by Claims 2.3 and 2.4 we have \( x_2x_4 \notin E \), \( x_4x_2 \notin E \), and \( x_3x_5 \notin E \), \( x_{k-1}x_1 \notin E \). Since \( G \) is chordal, we have \( x_2x_5 \notin E \).

Since \( v_2, x_2, x_3, v_3, x_4, v_4, v_5, v_1 \) does not induce an extended gem, we have \( x_1x_4 \in E \).

Since \( x_2, x_1, v_1, x_3, v_3, x_5, v_6, v_1 \) does not induce an extended gem, we have \( x_1x_5 \in E \) (which, for \( k = 6 \) contradicts the fact that \( x_{k-1}x_1 \notin E \)) but now, \( v_2, x_1, x_3, v_3, x_5, v_5, v_4 \) induce an extended gem.
Thus, Lemma 2 is shown.

By [6], and since MWIS is solvable in linear time for chordal graphs [16], we obtain:

**Corollary 2.** \(\text{WED is solvable in time } O(n^3)\) for extended-gem-free chordal graphs.

## 3 WED is \(\text{NP-complete}\) for a Special Class of Chordal Graphs

As mentioned in the Introduction, the reduction from X3C to Efficient Domination shows that ED is \(\text{NP-complete}\). For making this manuscript self-contained, we describe the reduction here:

Let \(H = (V, \mathcal{E})\) with \(V = \{v_1, \ldots, v_n\}\) and \(\mathcal{E} = \{e_1, \ldots, e_m\}\) be a hypergraph with \(|e_i| = 3\) for all \(i \in \{1, \ldots, m\}\). Let \(G_H\) be the following reduction graph:

\[
V(G_H) = V \cup X \cup Y
\]

such that \(X = \{x_1, \ldots, x_m\}\), \(Y = \{y_1, \ldots, y_m\}\) and \(V, X, Y\) are pairwise disjoint. The edge set of \(G_H\) consists of all edges \(v_ix_j\) whenever \(v_i \in e_j\). Moreover \(V\) is a clique in \(G_H\), and every \(y_i\) is only adjacent to \(x_i\).

Clearly, \(H = (V, \mathcal{E})\) has an exact cover if and only if \(G_H\) has an e.d.s. \(D\): For an exact cover \(\mathcal{E}'\) of \(H\), every \(e_i \in \mathcal{E}'\) corresponds to vertex \(x_i \in D\), and every \(e_i \notin \mathcal{E}'\) corresponds to vertex \(y_i \in D\). Conversely, if \(D\) is an e.d.s. in \(G_H\) we can assume without loss of generality that \(D \cap V = \emptyset\), and now, \(D \cap X\) corresponds to an exact cover of \(H\).

Figure 2: \(2P_3, K_3 + P_3, 2K_3, \text{butterfly, extended butterfly, extended co-P, extended chair, and double-gem}\)

Clearly, \(G_H\) is chordal and unipolar. The reduction shows that \(G_H\) is not only \(2P_3\)-free but also \(H\)-free for various other graphs \(H\) such as \(K_3 + P_3, 2K_3, \text{butterfly, extended butterfly, extended co-P, extended chair, and double-gem}\) as shown in Figure 2; actually, it corresponds to a slight generalization of split graphs which was described by Zverovich in [30] as satgraphs.

**Proposition 1.** \(\text{ED is } \text{NP-complete}\) for \((2P_3, K_3 + P_3, 2K_3, \text{butterfly, extended butterfly, extended co-P, extended chair, double-gem})\)-free chordal graphs

The reduction implies that ED is \(\text{NP-complete}\) e.g. for double-gem-free chordal graphs while it is solvable in linear time for gem-free chordal graphs (by various reasons such as bounded clique-width, distance-hereditary graphs, chordal square etc.), and ED is \(\text{NP-complete}\) for butterfly-free chordal graphs while it is solvable in linear time for \(2P_2\)-free graphs [9].

The clique-width of co-P-free chordal graphs, of \((K_3 + P_2)\)-free chordal graphs, and of claw-free chordal graphs is unbounded (see [2]). Since co-P and \(K_3 + P_2\) are subgraphs of extended gem, Lemma 2 implies the following result:

**Lemma 3.** For \((K_3 + P_2)\)-free chordal graphs and for co-P-free chordal graphs, WED is solvable in polynomial time.
Corollary 3. For every proper induced subgraph $H'$ of any graph $H \in \{2P_3, K_3 + P_3, 2K_3, \text{butterfly}, \text{extended butterfly}, \text{extended co-P}, \text{extended chair}, \text{double-gem}\}$, WED is solvable in polynomial time for $H'$-free chordal graphs.

Proof. By [4], the clique-width of co-chair-free chordal graphs is bounded, and by [18], the clique-width of gem-free chordal graphs is bounded. By Lemma 2 WED is solvable in polynomial time for chair-free chordal graphs since chair is a subgraph of extended gem, and similarly, by Lemma 3 WED is solvable in polynomial time for $(K_3 + P_3)$-free chordal graphs and for co-P-free chordal graphs. In all other cases, we can use the fact that if WED is solvable in polynomial time for $H$-free graphs then it is solvable in polynomial time for $(H + P_2)$-free graphs (see [7]) and the fact that WED is solvable in polynomial time (even in linear time) for $P_3$-free graphs (and thus also for $2P_2$-free graphs).
4 WED for $S_{1,2,3}$-Free Chordal Graphs - a Direct Approach

The forbidden induced subgraph $H$=extended gem in Lemma 2 contains $S_{1,2,2}$ as subgraph. In this section, we generalize the polynomial-time solution of WED for $S_{1,2,2}$-free chordal graphs as well as for $P_t$-free chordal graphs by a direct approach for $S_{1,2,3}$-free chordal graphs. Let $G = (V, E)$ be $S_{1,2,3}$-free chordal. For any $v \in V$, let

$$Z^+(v) := \{u \in V : N[u] \subset N[v]\},$$

and

$$Z^-(v) := \{u \in V : N[u] \subset N[v]\}.$$

Clearly such sets $Z^+(v)$, $Z^-(v)$ may be empty and, for any $x, y \in V$, $x \in Z^-[y]$ if and only if $y \in Z^+[x]$. Then for any $x, y \in V$, let us say that $x \leq y$ if $x = y$ or $x \in Z^-(y)$, i.e., $N[x] \subset N[y]$. Clearly, $(V, \leq)$ is a partial order (in particular, anti-symmetric and transitive). A vertex $v \in V$ is maximal if there is no $y \in V$, $y \neq v$, with $v \leq y$, i.e., $Z^+(v) = \emptyset$. Clearly, $(V, \leq)$ has such a maximal element.

**Lemma 4.** Let $v \in V$ be a maximal element of $(V, \leq)$. Then a minimum (finite) weight e.d.s. containing $v$ in the connected component of $G[V]$ with $v$ (if it exists) can be computed in polynomial time.

**Proof.** Let us assume without loss of generality that $G$ is connected. As usual, let $N_1, \ldots, N_t$ (for some natural $t$) denote the distance levels of $v$ in $G$. Then $\{\{v\}, N_1, \ldots, N_t\}$ is a partition of $V$. Clearly, $(N_1 \cup N_2) \cap D = \emptyset$. Since $G$ is chordal, we have:

**Claim 4.1.** For every $i \in \{2, \ldots, t\}$ and every vertex $x \in N_i$, $N(x) \cap N_{i-1}$ is a clique, and in particular, $x$ contacts exactly one component of $G[N_{i-1}]$.

**Claim 4.2.** For any vertex $u_1 \in N_1$, there is a non-neighbor $z_1 \in N_1$ of $u_1$. For any vertex $u_2 \in N_2$, with neighbor $u_1 \in N_1$, there is a vertex $z_1 \in N_1$ nonadjacent to both $u_1, u_2$. For any vertex $u_i \in N_i$, for $i \in \{3, \ldots, t\}$, there is an induced path of at least three vertices from $u_i$ to $v$.

**Proof.** The first statement holds since $v$ is a maximal element of $(V, \leq)$. The second statement holds by the first one and since $G$ is chordal. The third statement trivially holds by construction. 

Assume that $D$ is a (possible) e.d.s. of finite weight of $G$ containing $v$. For any fixed $i$, $i \in \{2, \ldots, t\}$, let

$$X := \{x \in N_i : x \text{ has a neighbor in } D \cap N_{i+1}\},$$

and let

$$C_X := \{Y_1, \ldots, Y_q\} \text{ (for some natural $q$) be the family of components of } G[N_{i+1}] \text{ contacting } X.$$

**Claim 4.3.** For any $x \in X$, $x$ contacts exactly one component of $G[N_{i+1}]$.

**Proof.** Suppose to the contrary that $x$ contacts two components of $G[N_{i+1}]$, say $H, H'$, and let us assume without loss of generality that the neighbor of $x$ in $D \cap N_{i+1}$, say $d$, belongs to $H$. Then let $x'$ be any neighbor of $x$ in $H'$; then by the e.d.s. property $x'$ has a neighbor in $D$, say $d'$, with $d' \neq d$. By Claim 4.1 $d' \notin N_i$, i.e., $d' \in N_{i+1} \cup N_{i+2}$. Then by Claim 4.2 $d', x', d, x$, and three further vertices of $G$ (with respect to $x$) induce an $S_{1,2,3}$, which is a contradiction. 

**Claim 4.4.**

(i) $X$ admits a partition $\{X_1, \ldots, X_q\}$ such that $Y_h$ contacts $X_h$ and does not contact $X_k$ (for $h, k = 1, \ldots, q$, $k \neq h$).
(ii) \(|D \cap Y_h| = 1, for h = 1, \ldots, q\), say \(D \cap Y_h = \{d_h\}\), and \(d_h\) dominates \(X_h \cup Y_h\).

\textbf{Proof.} (i) follows directly by Claim 4.3 and by definition of \(X\) and \(C_X\).

(ii): First we claim that \(|D \cap Y_h| = 1\) (clearly, \(D \cap Y_h \neq \emptyset\): Suppose to the contrary that there are \(d, d' \in D \cap Y_h, d \neq d'\). Since \(G\) is connected and by definition of \(X\), there are \(x \in X\) with \(xd \in E\) and \(x' \in X\) with \(x'd' \in E\). By the e.d.s. property, the shortest path, say \(P\) in \(Y_h\) from \(d\) to \(d'\) has at least two internal vertices, i.e., there exist \(a, b \in P\) with \(da \in E\) and \(bd' \in E\). Since \(G\) is \(S_{1,2,3}\)-free, by Claim 4.2 and by the e.d.s. property, \(x\) is nonadjacent to all vertices of \(P \setminus \{a\}\), while \(x'\) is nonadjacent to all vertices of \(P \setminus \{b\}\) which contradicts the fact that \(G\) is chordal.

Let \(D \cap Y_h = \{d_h\}\). Next we claim that \(d_h\) dominates \(X_h\): This follows by definition of \(X\), by statement (i), and by the e.d.s. property.

Finally we claim that \(d_h\) dominates \(Y_h\): Suppose to the contrary that \(d_h\) does not dominate \(Y_h\), i.e., there is a vertex \(y \in Y_h\) with \(yd_h \notin E\). Then there is \(d \in D\), \(d \neq d_h\), with \(yd \in E\). Let \(P'\) be a shortest path in \(Y_h\) between \(d_h\) and \(y\), and let \(x \in X\) be adjacent to \(d_h\) (by the above, \(d_h\) dominates \(X_h\)). Clearly, by the e.d.s. property, \(xd \notin E\).

\textbf{Case 1:} \(xy \in E\).

Then by Claim 4.1 \(d \notin N_i\), i.e., \(d \in N_{i+1} \cup N_{i+2}\); then by Claim 4.2 \(d, y, d_h, x\), and three further vertices of \(G\) (with respect to \(x\)) induce a \(S_{1,2,3}\), which is a contradiction.

\textbf{Case 2:} \(xy \notin E\). Then the only possible subcases are:

\textbf{Case 2.1:} \(d \in N_i\).

Then, by considering the (not necessarily induced) path formed by \(x, d_h, P', y, d\), one gets a contradiction to the fact that \(G\) is chordal.

\textbf{Case 2.2:} \(d \in N_{i+1} \cup N_{i+2}\). Then let \(y'\) be a neighbor of \(y\) in \(N_i\); clearly, by the e.d.s. property, \(y' \notin D\).

Note that \(y'\) is nonadjacent to \(d_h\) (else one would be again in Case 1 in which \(x\) is adjacent to \(y\)), and \(y'\) is adjacent to \(x\) (else by an argument similar to that of the previous paragraph one would get a contradiction since \(G\) is chordal). Then there is \(d' \in D\) adjacent to \(y'\). Clearly, \(d' \neq d_h\) by the above. Furthermore \(d' \neq d\); Otherwise \(d \in N_{i+1}\), and then by considering the path in \(Y_h\) between \(d\) and \(d_h\), we get a contradiction by an argument similar to the one above for showing that \(|D \cap Y_h| = 1\).

Then let us consider the following subcases:

\textbf{Case 2.2.1:} \(d' \in N_{i-1}\).

Then by Claim 4.2 \(d_h, x, y, y', d'\), and two further vertices of \(G\) (with respect to \(d'\)) induce an \(S_{1,2,3}\), which is a contradiction.

\textbf{Case 2.2.2:} \(d' \in N_i\) (which implies \(i \geq 3\)).

Then, since \(G\) is chordal, \(y'\) and \(d'\) have a common neighbor in \(N_{i-1}\), say \(z\), and then \(z\) is adjacent to \(x\) (since otherwise, by Claim 4.2 \(d_h, x, y, y', z\), and two further vertices of \(G\) (with respect to \(z\)) induce an \(S_{1,2,3}\)). Now, since \(xz \in E\), the vertices \(d', d_h, x, z\) and three further vertices of \(G\) (according to Claim 4.2 with respect to \(z\)) induce an \(S_{1,2,3}\), which is a contradiction.

\textbf{Case 2.2.3:} \(d' \in N_{i+1}\).

Then by Claim 4.2 \(d, y, d', y'\), and three further vertices of \(G\) (with respect to \(y'\)) induce an \(S_{1,2,3}\), which is a contradiction.

Thus, Claim 4.4 is shown. \(\diamond\)
Claim 4.5. For every component \( K \) of \( G[N_i] \), \( i \in \{3, \ldots, t\} \), we have

(i) \( |D \cap K| \leq 1 \), and

(ii) if \( |D \cap K| = 1 \), say \( D \cap K = \{d\} \) then \( d \) dominates \( K \).

Proof. (i): The first statement can be proved similarly to the first paragraph of the proof of Claim 4.4 (ii).

(ii): The second statement follows by Claim 4.4 (ii) since \( y \) (and thus \( K \)) contacts a set of vertices of \( N_{i-1} \) which consequently have a neighbor in \( D \cap N_i \).

Now let us consider the problem of checking whether such an e.d.s. \( D \) does exist. According to Claim 4.4, graph \( G \) can be viewed as a tree \( T \) rooted at \( \{v\} \), whose nodes are the components of \( G[N_i] \) for \( i \in \{0,1,\ldots,t\} \) (for \( N_0 := \{v\} \)), such that two nodes are adjacent if and only if the corresponding components contact each other. Then for any component \( K \) of \( G[N_i] \), \( i \in \{0,1,\ldots,t\} \), let us denote as \( G[T(K)] \) the induced subgraph of \( G \) corresponding to the subtree of \( T \) rooted at \( K \). In particular, \( G[T(N_0)] = G \).

According to Claim 4.5 let us say that a vertex \( d \) of \( G \) of finite weight, belonging to a component say \( K \) of \( G[N_i] \), \( i \in \{0,1,\ldots,t\} \), is a D-candidate (or equivalently let us say that \( K \) admits a D-candidate \( d \) if

(i) \( d \) dominates \( K \), and

(ii) there is an e.d.s. \( D \) in \( G[T(K)] \) containing \( d \).

Claim 4.6. \( D \) does exist if and only if \( v \) is a D-candidate.

Proof. It directly follows by the above.

Claim 4.7. Let \( K \) be a component of \( G[N_i] \), \( i \in \{1,\ldots,n\} \), and let \( d \) be a vertex of finite weight of \( K \). Then let us write: \( H_j := T(K) \cap N_j \) for \( j = i+1,\ldots,t \). Then let \( X := \{ x \in H_{i+1} : x \) is nonadjacent to \( d \} \), \( A := \{ x \in H_{i+1} : x \) is adjacent to \( d \} \). Then let \( C_X \) be the family of components of \( G[H_{i+2}] \) contacting \( X \), and let \( C_0 \) be the family of components of \( G[H_{i+2}] \) not contacting \( X \). Finally let \( K_0 \) be the family of components of \( G[N_{i+3}] \) contacting components in \( C_0 \). Then the following statements hold:

(i) If \( X = \emptyset \) and \( C_0 = \emptyset \) then \( d \) is a D-candidate if and only if \( d \) dominates \( K \).

(ii) If \( X \neq \emptyset \) and \( C_0 = \emptyset \) then \( d \) is a D-candidate if and only if \( d \) dominates \( K \) and each component of \( C_X \) admits a D-candidate.

(iii) If \( X = \emptyset \) and \( C_0 \neq \emptyset \) then \( d \) is a D-candidate if and only if \( d \) dominates \( K \) and each component of \( K_0 \) admits a D-candidate.

(iv) If \( X \neq \emptyset \) and \( C_0 \neq \emptyset \) then \( d \) is a D-candidate if and only if \( d \) dominates \( K \) and each component of \( C_X \) admits a D-candidate and each component of \( K_0 \) admits a D-candidate.

Proof. It directly follows by definition of a D-candidate and by the e.d.s. property.

Then by Claims 4.6 and 4.7 one can check if e.d.s. \( D \) with \( v \in D \) does exist by the following procedure which can be executed in polynomial time:
Procedure $v$-Maximal-WED

**Input:** A maximal element $v$ of $(V, \leq)$.

**Output:** An e.d.s. $D$ of $G$ containing $v$ (if it exists).

**begin**

Let $N_0, N_1, \ldots, N_t$ (for some natural $t$), with $N_0 = \{v\}$, be as usual the distance levels of $v$ in $G$.

**for** $i = t, t-1, \ldots, 1, 0$ **do**

**begin**

for each component $K$ of $G[N_i]$, check whether $K$ admits a $D$-candidate

**end**

**if** $v$ is a $D$-candidate **then** return “$D$ does exist”;

**else** return “$D$ does not exist”

**end**

Let us observe that, if $D$ does exist, then:

(i) the vertices of an e.d.s. $D$ containing $v$ can be easily obtained according to Claim 4.7 (by iteratively choosing, for each component $K$ containing a $D$-candidate, any $D$-candidate in $K$), and

(ii) the vertices of a minimum weight e.d.s. $D$ containing $v$ can be easily obtained according to Claim 4.7 (by iteratively choosing, for each component $K$ containing a $D$-candidate, a $D$-candidate in $K$ of minimum weight).

This completes the proof of Lemma 4.

**Theorem 1.** For $S_{1,2,3}$-free chordal graphs, WED is solvable in polynomial time.

**Proof.** Let us observe that, if all vertices $G$ are maximal elements of $(V, \leq)$, then by Lemma 4.1, the WED problem can be solved for $G$ by computing a minimum finite weight e.d.s. $D$ containing $v$ in the connected component of $G[V]$ with $v$ (if $D$ exists), for all $v \in V$.

Then let us focus on those vertices $x$ which are not maximal elements, i.e., there is a vertex $y$ with $N[x] \subset N[y]$ (which means $x \in Z^-(y)$). Thus, there is a maximal vertex $v$ such that $x \in Z^-(v)$. In particular removing such maximal vertices $v$ leads to new maximal vertices in the reduced graph. Recall that for any graph $G = (V, E)$ and any e.d.s. $D$ of $G$, $|D \cap N[x]| = 1$ for every $x \in V$.

**Claim 4.8.** Let $v \in V$ be a maximal element of $(V, \leq)$, with $Z^-(v) \neq \emptyset$, and let $x \in Z^-(v)$. Then, if $G$ has an e.d.s., say $D$, then $D \cap (N(v) \setminus N(x)) = \emptyset$. In particular, one can define a reduced weighted graph $G^*$ from $G$ as follows:

(i) For each vertex $x \in Z^-(v)$, assign weight $\infty$ to all vertices in $N(v) \setminus N(x)$, and

(ii) remove $v$, i.e., $V(G^*) = V \setminus \{v\}$ (and reduce $G^*$ to its prime connected components).

In particular, the problem of checking if $G$ has an (minimum weight) e.d.s. not containing $v$ can be reduced to that of checking if $G^*$ has a finite (minimum weight) e.d.s.
Proof. The reduction is correct by the e.d.s. property and by definition of $Z^-(v)$. Moreover, by the e.d.s. property, by definition of $Z^-(v)$ and by construction of $G^*$, every (possible) e.d.s. of finite weight of $G^*$ contains exactly one vertex which is a neighbor of $v$ in $G$ since $|D \cap N[x]| = 1$ for a vertex $x \in Z^-(v)$.

According to the fact that the above holds in a hereditary way for any subgraph of $G$, and to the fact that WED for any graph $H$ can be reduced to the same problem for the connected components of $H$, let us introduce a possible algorithm to solve WED for $G$ in polynomial time.

Algorithm WED-$S_{1,2,3}$-Free-Chordal-Graphs

**Input:** Graph $G = (V, E)$.

**Output:** A minimum (finite) weight e.d.s. of $G$ (if it exists).

**begin**

Set $W := \emptyset$;

while $V \neq W$ do

begin

take any maximal element of $(V, \leq)$, say $v \in V$, and set $W := W \cup \{v\}$;

compute a minimum (finite) weight e.d.s. containing $v$ in the connected component of $G[V]$ with $v$ (if it exists) \{according to Lemma 4 and Procedure $v$-Maximal-WED\};

if $Z^-(v) \neq \emptyset$ then \{according to Claim 4.8\}

begin

for each vertex $x \in Z^-(v)$, assign weight $\infty$ to all vertices in $N(v) \setminus N(x)$;

remove $v$ from $V$, i.e., set $V := V \setminus \{v\}$

end

end

if there exist some e.d.s. of finite weight of $G$ (in particular, for each resulting set of e.d.s. candidates, check whether this is an e.d.s. of $G$) then choose one of minimum weight, and return it else return “$G$ has no e.d.s.”

**end**

The correctness and the polynomial time bound of the algorithm is a consequence of the arguments above and in particular of Lemma 2 and Claim 4.8. This completes the proof of Theorem 1.

5 Conclusion

The results described in Lemmas 1, 2, and 3 are still far away from a dichotomy for the complexity of ED on $H$-free chordal graphs. For chordal graphs $H$ with four vertices, all cases are solvable in polynomial time as described in Lemma 6 below.

For chordal graphs $H$ with five vertices, the complexity of ED on $H$-free chordal graphs is still open for the following examples as described in Lemma 5.

**Lemma 5 ([7]).** If WED is solvable in polynomial time for $F$-free graphs then WED is solvable in polynomial time for $(P_2 + F)$-free graphs.
This clearly implies the corresponding fact for \((P_1 + F)\)-free graphs.

**Lemma 6.**

(i) For every chordal graph \(H\) with exactly four vertices, WED is solvable in polynomial time for \(H\)-free chordal graphs.

(ii) For every chordal graph \(H\) with exactly five vertices, the four cases described in Figure 4 are the only cases for which the complexity of WED is open for \(H\)-free chordal graphs.

**Proof.** (i): It is well known that for \(H \in \{K_4, K_4-e, paw, P_4\}\), the clique-width is bounded for \(H\)-free chordal graphs and thus, WED is solvable in polynomial time. By Lemma ??, or more generally, by Theorem 1, WED is solvable in polynomial time for claw-free chordal graphs.

By Lemma 5, WED is solvable in polynomial time for all other graphs \(H\) with four vertices (see Figure 3 for all such graphs; clearly, \(C_4\) is excluded).

(ii): For graphs \(H\) with five vertices, let \(v\) be one of its vertices. We consider the following cases for \(N(v)\) (and clearly exclude the cases when \(H\) is not chordal):

**Case 1.** \(|N(v)| = 4\) (i.e., \(v\) is universal in \(H\)):

Clearly, if \(H[N(v)]\) is a 2\(P_2\) then \(H\) is a butterfly and thus, WED is \(\text{NP}\)-complete. If \(H[N(v)]\) is a \(K_4\), or paw, or \(P_4\), or \(K_3+P_1\), then the clique-width is bounded \[4\]; in particular, if \(H[N(v)]\) is a paw or \(K_3+P_1\) then \(H\) is a subgraph of \(K_{1,3}+2P_1\), and according to Theorem 1 of \[4\], the clique-width is bounded. If \(H[N(v)]\) is \(P_3+P_1\) then it is a special case of Lemma 2 where it is shown that this case can be solved in polynomial time. The other cases correspond to graphs \(H_1, \ldots, H_4\) of Figure 4 (by Theorem 1 of \[4\], their clique-width is unbounded).

**Case 2.** \(|N(v)| = 0\) (i.e., \(v\) is isolated in \(H\)): By Lemma 5 and by Lemma 6 (i), WED is solvable in polynomial time.

In particular, for the same reason, WED is solvable in polynomial time whenever \(H\) is not connected (since in that case, at least one connected component of \(H\) has at most two vertices). Thus, from now on, we can assume that \(H\) is connected.

**Case 3.** \(|N(v)| = 3\) (and thus, \(|\overline{N}(v)| = 1\)):
If \( v \) has exactly one non-neighbor in \( K_4 \) then \( H = H_4 \). If \( v \) has exactly one non-neighbor in \( K_{1,3} \) with midpoint \( w \), namely one of degree 1, then \( H[N(w)] = P_3 + P_1 \) according to Case 1 (the special case of Lemma 2).

If \( v \) has has exactly one non-neighbor in diamond, namely one of degree 2, or exactly one non-neighbor in paw, namely one of degree 1, then \( H \) is a subgraph of \( K_{1,3} + 2P_1 \). Moreover, if \( v \) has has exactly one non-neighbor in paw, namely one of degree 2, then \( H \) is a gem, and if \( v \) has exactly one non-neighbor in \( P_4 \), namely one of degree 1, then \( H \) is a co-chair. If \( v \) has exactly one non-neighbor in \( P_1 + P_3 \), namely one of degree 1, then \( H \) is a bull. In all these cases, the clique-width is bounded according to Theorem 1 of [4].

In the remaining cases, \( H \) is a chair or co-P, and thus, WED is solvable in polynomial time.

**Case 4.** \( |N(v)| = 2 \) (and thus, \( |\overline{N(v)}| = 2 \)):

In one of the cases, namely if \( v \) is adjacent to the two vertices with degree 1 and with degree 3 in paw, \( H \) is a butterfly and thus, WED is NP-complete.

If \( v \) has exactly two neighbors in \( K_4 \) or if \( v \) is adjacent to degree 2 and degree 3 vertices in diamond or if \( v \) is adjacent to the two degree 2 vertices in paw or if \( v \) is adjacent to the two degree 2 vertices (midpoints) in \( P_4 \), then by Theorem 1 of [4], the clique-width is bounded.

If \( v \) is adjacent to the two vertices of degree 3 of diamond then \( H = H_3 \). If \( v \) is adjacent to degree 2 vertex \( u \) and degree 3 vertex \( w \) in paw then for the degree 3 vertex \( w \), \( H[N(w)] = P_3 + P_1 \) as above. If \( v \) is adjacent to degree 1 and degree 3 vertices in claw then \( H = H_2 \).

In all other cases, \( H \) is \( P_5 \), chair or co-P, and thus, WED is solvable in polynomial time (by Lemma 3 for co-P-free chordal graphs, and by Theorem 1 for \( P_5 \)-free chordal graphs, and for chair-free chordal graphs).

**Case 5.** \( |N(v)| = 1 \) (and thus, \( |\overline{N(v)}| = 3 \)):

Now \( v \) is adjacent to exactly one vertex of \( V \setminus \{v\} \).

If \( v \) is adjacent to a degree 3 vertex \( w \) of diamond then \( H[N(w)] = P_3 + P_1 \) as above. If \( v \) is adjacent to a degree 3 vertex of paw then \( H = H_2 \). If \( v \) is adjacent to a degree 3 vertex of claw then \( H = H_1 \).

If \( v \) is adjacent to one vertex of \( K_4 \) or one vertex of diamond of degree 2 (co-chair) or one vertex of paw of degree 2 (bull) then by Theorem 1 of [4], the clique-width is bounded.

In all other cases, \( H \) is \( P_5 \), chair or co-P, and thus, WED is solvable in polynomial time as above.  

Of course there are many larger examples of graphs \( H \) for which ED is open for \( H \)-free chordal graphs such as the two variants in Figure 5.

![Figure 5: Two variants of an extended P_6](image-url)
In general, one can restrict $H$ by various conditions such as diameter (if the diameter of $H$ is at least 6 then $H$ contains $2P_3$) and size of connected components (if $H$ has at least two connected components of size at least 3 then $H$ contains $2P_3$, $K_3 + P_3$, or $2K_3$). It would be nice to classify the open cases in a more detailed way.

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