COMPLETE 3-DIMENSIONAL $\lambda$-TRANSLATORS IN THE MINKOWSKI SPACE $\mathbb{R}^4_1$

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ABSTRACT. In this paper, we obtain the classification theorem for three-dimensional complete space-like $\lambda$-translators $x : M^3 \to \mathbb{R}^4_1$ with constant norm of the second fundamental form and constant $f_4$ in the Minkowski space $\mathbb{R}^4_1$.

1. INTRODUCTION

Let $x : M^n \to \mathbb{R}_1^{n+1}$ be an immersed space-like hypersurface in the Minkowski space $\mathbb{R}_1^{n+1}$. Fix a constant vector $T \neq 0$ in $\mathbb{R}_1^{n+1}$ and $\lambda$ a real number. In this paper we study orientable hypersurface $M^n$ in $\mathbb{R}_1^{n+1}$ whose mean curvature vector $\vec{H}$ satisfies

\begin{equation}
\vec{H} + T^\perp = \lambda \mathbf{n},
\end{equation}

where $\vec{H} = H \mathbf{n}$ and $\mathbf{n}$ is the unit normal vector. Then $x$ is called a $\lambda$-translating soliton or simply a $\lambda$-translator of the mean curvature flow (MCF). The constant vector $T$ will be called the corresponding translating vector or density vector. In particular, if we denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}_1^{n+1}$, then the equation (1.1) is equivalent to

\begin{equation}
H - \langle T, \mathbf{n} \rangle = \lambda, \quad \langle \mathbf{n}, \mathbf{n} \rangle = -1,
\end{equation}

where $x$ is space-like.

The interest of this equations is due to its relation with manifolds with density. So it naturally makes sense to study the $\lambda$-translator in $\mathbb{R}_1^{n+1}$. Indeed, as described in [18], considering $\mathbb{R}_1^{n+1}$ with a positive density function $e^\phi, \phi \in C^\infty(\mathbb{R}_1^{n+1})$, which serves as a weight for the volume and the surface area. The first variation of the weighted volume $V_\phi(t)$ with density $e^\phi$ under compactly supported variations of $M^n$ is

\begin{equation}
\frac{d}{dt} \bigg|_{t=0} V_\phi(t) = -\int_{M^n} H_\phi \langle \mathbf{n}, \xi \rangle dV_\phi,
\end{equation}

where $\xi$ is the variation vector and $H_\phi = H - \langle \nabla \phi, \mathbf{n} \rangle$. So that $X$ is a critical point of the functional $V_\phi(t)$ for a given weighted volume if and only if $H_\phi$ is a constant function $H_\phi \equiv \lambda$: see (17, 19). In particular, if we take $\phi : \mathbb{R}_1^{n+1} \to \mathbb{R}$ to be the height function $\phi(x) := \langle x, T \rangle$, then the expression $H_\phi \equiv \lambda$ is exactly the equation (1.2). Secondly, a special case of (1.1) is when $\lambda = 0$. In such a case the

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immersion $x$ is called a translating soliton of the mean curvature flow, or simply a translator \((26)\). Translators play an important role in the study of mean curvature flow. On the one hand, a translating soliton is a solution of the mean curvature flow that evolves purely by translations along the direction $T$. On the other hand, they arise as blow-up solutions of MCF at type II singularities\((11, 13)\). For instance, Huisken and Sinestrari \((12)\) proved that at type II singularity of a mean convex flow, there exists a blow-up solution which is a convex translating solution. Besides, in the nonparametric form, the equation $H_\phi = 0$ appeared in the classical article of Serrin \((24)\) and it was studied in the context of the maximum principle of elliptic equations. As we know, translating soliton have been widely studied and various interesting results have been obtained in recent years. For more information about translating soliton, please refer to the literatures \((1, 6, 9, 10, 14, 20, 21, 22, 23, 25, 27, 28)\).

In \((17)\), López classified all $\lambda$-translators in $\mathbb{R}^3$ that are invariant by a one-parameter group of translations and a one-parameter group of rotations. He also studied in \((18)\) the shape of a compact $\lambda$-translator of $\mathbb{R}^3$ in terms of the geometry of its boundary, obtaining some necessary conditions for the existence of two-dimensional compact $\lambda$-translators with a given closed boundary curve. In particular, he proved that there do not exist any closed $\lambda$-translators of dimension two. In fact, just as that $\lambda$-translating solitons are generalization of the translators of mean curvature flow, $\lambda$-hypersurfaces defined by Cheng and Wei in \((4)\) are generalization of the self-shrinkers of mean curvature flow. Self-shrinking solutions are important in the study of type-I singularities of MCF. For instance, by proving the monotonicity formula, at a given type-I singularity of the MCF, Huisken \((8)\) proved that the flow is asymptotically self-similar, which implies that in this situation the flow can be modeled by self-shrinking solutions. As is known, there have been many rigidity theorems and classification theorems for self-shrinkers in the Euclidean space and the pseudo-Euclidean space. Furthermore, there have been, up to now, several interesting and important results in the study of $\lambda$-hypersurfaces. In particular, Cheng and Wei recently obtained a classification theorem using their own generalized maximum principle \((2)\) specially for $\lambda$-hypersurfaces, which generalizes an interesting classification theorem in \((2)\) for self-shrinkers.

The classification theorem also exists for $\lambda$-translators. For example, canonical examples of $\lambda$-translators in $\mathbb{R}^{n+1}_1$ are the space-like affine hyperplanes, and the right hyperbolic cylinders $\mathbb{H}^k(r) \times \mathbb{R}^{n-k}$ with $1 \leq k \leq n-1$, where $\mathbb{H}^k(r)$ is the hyperbolic $k$-space defined by

$$\mathbb{H}^k(r) = \{ x \in \mathbb{R}^{k+1}_1; \langle x, x \rangle = -r^2 \}.$$  

Recently, Li, Qiao and Liu \((15)\) have classified complete $\lambda$-translators in the Euclidean space $\mathbb{R}^3$ and the Minkowski space $\mathbb{R}^3_1$ with second fundamental form of constant length $S$. For the higher dimension $n$, it is not easy to classify $\lambda$-translator in $\mathbb{R}^n$ and $\mathbb{R}^n_1$ with constant squared norm $S$ of the second fundamental form. In this paper, under the assumption that $f_4$ is constant, we give a complete classification for 3-dimensional complete $\lambda$-translators in $\mathbb{R}^4_1$ with constant squared norm $S$ of the second fundamental form. In fact, we prove the following result.
Theorem 1.1. Let \( x : M^3 \to \mathbb{R}^4 \) be a 3-dimensional complete space-like \( \lambda \)-translator in \( \mathbb{R}^4 \). If the squared norm \( S \) of the second fundamental form and \( f_4 \) are constant, then \( x : M^3 \to \mathbb{R}^4 \) is isometric to one of

1. \( \mathbb{R}^4 \),
2. \( \mathbb{H}^1(\frac{1}{\lambda}) \times \mathbb{R}^2 \),
3. \( \mathbb{H}^2(\frac{2}{\lambda}) \times \mathbb{R}^1 \).

In particular, \( S \) must be 0, \( \lambda^2 \) and \( \frac{1}{2}\lambda^2 \); \( f_4 \) must be 0, \( \lambda^4 \) and \( \frac{1}{8}\lambda^4 \), where \( \lambda \neq 0 \), \( S = \sum_{i,j} h_{ij}^2 \) and \( f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} \).

Remark 1.1. We also obtain a similar classification for 3-dimensional complete \( \lambda \)-translators in \( \mathbb{R}^4 \) (see [16]). That is, for a 3-dimensional complete \( \lambda \)-translator in the Euclidean space \( \mathbb{R}^4 \), if the squared norm \( S \) of the second fundamental form and \( f_4 \) are constant, then hypersurface is isometric to one of \( \mathbb{R}^3 \); \( S^1(\frac{1}{\lambda}) \times \mathbb{R}^2 \); \( S^2(\frac{2}{\lambda}) \times \mathbb{R}^1 \).

2. Preliminaries

Let \( x : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional space-like hypersurface of the \((n + 1)\)-dimensional Minkowski space \( \mathbb{R}^{n+1} \). Around each point of \( M^n \), we choose a local orthonormal frame field \( \{e_A\}_{A=1}^{n+1} \) in \( \mathbb{R}^{n+1} \) with dual coframe field \( \{\omega_A\}_{A=1}^{n+1} \), such that, restricted to \( M^n \), \( e_1, \ldots, e_n \) are tangent on \( M^n \).

From now on, we use the following conventions on the ranges of indices:

\[
1 \leq i, j, k, l \leq n
\]

and \( \sum_i \) means taking summation from 1 to \( n \) for \( i \). Then we have

\[
dx = \sum_i \omega_i e_i,
\]

\[
de_i = \sum_j \omega_{ij} e_j + \omega_{i,n+1} e_{n+1},
\]

\[
d e_{n+1} = \omega_{n+1;i} e_i, \quad \omega_{n+1} = \omega_{i,n+1},
\]

where \( \omega_{ij} = -\omega_{ji} \) is the Levi-Civita connection of the hypersurface.

By restricting these forms to \( M^n \), we get

\[
(2.1) \quad \omega^{n+1} = 0.
\]

Taking exterior derivatives of (2.1), we obtain

\[
0 = d\omega_{n+1} = \sum_i \omega_{n+1;i} \wedge \omega_i.
\]

By Cartan’s lemma, we know that there exist local smooth functions \( h_{ij} \), \( 1 \leq i, j \leq n \), such that

\[
(2.2) \quad \omega_{i,n+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji},
\]

\[
h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad H = \sum_i h_{ii}
\]
are called the second fundamental form and the mean curvature of \( x : M \to \mathbb{R}^{n+1} \), respectively. Let \( S = \sum_{ij} (h_{ij})^2 \) be the squared norm of the second fundamental form of \( x : M \to \mathbb{R}^{n+1} \). The induced structure equations of \( M^n \) are given by
\[
d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji},
\]
\[
d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\]
where \( R_{ijkl} \) denotes components of the curvature tensor of the hypersurface. Hence, the Gauss equations of the space-like hypersurface \( x \) in \( \mathbb{R}^{n+1} \) are as follows:
\[
R_{ijkl} = -(h_{ik} h_{jl} - h_{il} h_{jk}).
\]
Defining the covariant derivative of \( h_{ij} \) by
\[
\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kji} \omega_k + \sum_k h_{ikj} \omega_k,
\]
we obtain the Codazzi equations
\[
h_{ijk} = h_{ikj}.
\]
By taking exterior differentiation of (2.4), and defining
\[
\sum_l h_{ijkl} \omega_l = dh_{ijkl} + \sum_l h_{ijlk} \omega_l + \sum_l h_{ijkl} \omega_l + \sum_l h_{ijl} \omega_l,
\]
we have the following Ricci identities:
\[
h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}.
\]
Defining
\[
\sum_m h_{ijklm} \omega_m = dh_{ijkl} + \sum_m h_{mjkl} \omega_m + \sum_m h_{imkl} \omega_m + \sum_m h_{ijml} \omega_m + \sum_m h_{ijkm} \omega_ml
\]
and taking exterior differentiation of (2.6), we get
\[
h_{ijkl} - h_{ijkl} = \sum_m h_{mj} R_{mikln} + \sum_m h_{im} R_{mjkl} + \sum_m h_{ijm} R_{mkln}.
\]
For a smooth function \( f \), we define
\[
\sum_i f_i \omega_i = df,
\]
\[
\sum_j f_{ij} \omega_j = df, + \sum_j f_{ji} \omega_j,
\]
\[
|\nabla f|^2 = \sum_i (f_i)^2, \quad \Delta f = \sum_i f_{ii}.
\]
Let $V$ be a tangent $C^1$-vector field on $M^n$, and denote by $Ric_V := Ric - \frac{1}{2}L_V g$ the Bakry-Emery Ricci tensor with $L_V$ to be the Lie derivative along the vector field $V$. Define a differential operator

$$\Delta_V f = \Delta f + \langle V, \nabla f \rangle,$$

where $\Delta$ and $\nabla$ denote the Laplacian and the gradient operator, respectively. Then we have the following maximum principle of Omori-Yau type which was proved by Chen-Qiu [3] and Li-Qiao-Liu[15]:

**Lemma 2.1.** Let $(M^n, g)$ be a complete Riemannian manifold, and $V$ is a $C^1$ vector field on $M^n$. If the Bakry-Emery Ricci tensor $Ric_V$ is bounded from below, then for any $f \in C^2(M^n)$ bounded from above, there exists a sequence $p_m \subset M^n$, such that

$$\lim_{m \to \infty} f(p_m) = \sup f, \quad \lim_{m \to \infty} |\nabla f|(p_m) = 0, \quad \lim_{m \to \infty} \Delta_V f(p_m) \leq 0.$$

Suppose that the given hypersurface $x : M \to \mathbb{R}^{n+1}_{1}$ is a $\lambda$-translator with a translating vector $T$, and let $\{e_i\}$ be an orthonormal tangent frame on $M^n$. Then from the definitions (1.2) and (1.3) of $\lambda$-translators in $\mathbb{R}^{n+1}_{1}$, we have the following basic formulas for covariant derivatives:

$$\nabla_i H = \sum_k h_{ik} \langle T, e_k \rangle,$$

$$\nabla_j \nabla_i H = \sum_k h_{ijk} \langle T, e_k \rangle + (H - \lambda) \sum_k h_{ik} h_{kj},$$

$$\nabla_i \nabla_j \nabla_i H = \sum_k h_{ijkl} \langle T, e_k \rangle + (H - \lambda) \sum_k (h_{ikl} h_{kj} + h_{ik} h_{kjl} + h_{ijk} h_{kl})$$

$$+ \nabla_i H \sum_k h_{ik} h_{kj}.$$ (2.13)

Moreover, we define three functions $f_3$, $f_4$ and $f_5$ as follows:

$$f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}, \quad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}, \quad f_5 = \sum_{i,j,k,l,m} h_{ij} h_{jk} h_{kl} h_{im} h_{mi}.$$

If we denote $V = T^T$, the tangent component of the translating vector $T$ when restricted to $M^n$, then direct computations using above formulas and the Ricci identities easily give the following Lemma (cf. [4] and [15]):

**Lemma 2.2.** Let $x : M^n \to \mathbb{R}^{n+1}_{1}$ be an $n$-dimensional complete $\lambda$-translator in $\mathbb{R}^{n+1}_{1}$, we have

$$\Delta_{-V} H = S(H - \lambda).$$ (2.14)

$$\frac{1}{2} \Delta_{-V} H^2 = |\nabla H|^2 + S(H - \lambda) H.$$ (2.15)

$$\frac{1}{2} \Delta_{-V} S = \sum_{i,j,k} h_{ijk}^2 + S^2 - \lambda f_3.$$ (2.16)
\[
\frac{1}{4} \Delta_{-V} f_4 = 2 \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{kl} h_{li} + \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{klm} h_{li} + S f_4 - \lambda f_5. 
\]

Lemma 2.3. Let \( x : M^n \to \mathbb{R}^{n+1}_1 \) be an \( n \)-dimensional complete \( \lambda \)-translator in \( \mathbb{R}^{n+1}_1 \). If \( S \) is constant, we have

\[
\frac{1}{2} \Delta_{-V} \sum_{i,j,k} (h_{ijk})^2 = \sum_{i,j,k,l} (h_{ijkl})^2 + S \sum_{i,j,k} (h_{ijk})^2 - 6 \sum_{i,j,k,l,p} h_{ijk} h_{ijl} h_{kp} h_{klp} + 3 \sum_{i,j,k,l,p} h_{ijk} h_{ijl} h_{kp} h_{klp} - 3 \lambda \sum_{i,j,k,l} h_{ijk} h_{ijl} h_{kl}. 
\]

Furthermore, for \( n = 3 \), we have

\[
f_3 = \frac{H}{2} (3S - H^2) - 3h_{11} h_{23}^2 - 3h_{22} h_{13}^2 - 3h_{33} h_{12}^2 + 6h_{11} h_{22} h_{33} + 6h_{12} h_{13} h_{23}. 
\]

Then,

\[
\frac{1}{2} \Delta_{-V} \sum_{i,j,k} (h_{ijk})^2 \\
= - \frac{3}{2} \lambda H |\nabla H|^2 + \frac{3}{4} \lambda S (S - H^2)(H - \lambda) - 3\lambda \sum_k (h_{11} h_{23k}^2) \\
+ h_{22} h_{13k}^2 + h_{33} h_{12k}^2 + \frac{9}{2} \lambda S h_{11} h_{22} h_{33} - \frac{3}{2} \lambda^2 \sum_k (h_{22} h_{33} h_{1k}^2 + h_{11} h_{33} h_{2k}^2) \\
+ h_{11} h_{22} h_{33k}^2 + 3\lambda \sum_k (h_{11} h_{22k} h_{33k} + h_{22} h_{11k} h_{33k} + h_{33} h_{11k} h_{22k}).
\]

Proof. By making use of the Ricci identities (2.7), (2.9) and a direct calculation, we can obtain (2.17). Besides, from (2.16) in Lemma 2.2 we have

\[
\sum_{i,j,k} h_{ijk}^2 = -(S^2 - \lambda f_3). 
\]
Then, by making use of the Ricci identities [2.7], we obtain
\[
- \frac{1}{2} \Delta_{-V} (S^2 - \lambda f_3) = \frac{1}{2} \lambda \Delta_{-V} f_3
\]
\[
= - \frac{3}{2} \lambda H|\nabla H|^2 + \frac{3}{4} \lambda S (S - H^2)(H - \lambda) - \frac{9}{2} \lambda S (h_{11} h_{23}^2 + h_{22} h_{13}^2 + h_{33} h_{12}^2)
\]
\[
+ \frac{3}{2} \lambda^2 \sum_k (h_{23}^2 h_{1k}^2 + h_{13}^2 h_{2k}^2 + h_{12}^2 h_{3k}^2 + 2 h_{11} h_{2k} h_{23} h_{3k} + 2 h_{23} h_{12} h_{13} h_{3k})
\]
\[
+ 2 h_{33} h_{12} h_{1k} h_{2k}) - 3 \lambda \sum_k (h_{11} h_{23}^2 + h_{22} h_{13}^2 + h_{33} h_{12}^2 + 2 h_{23} h_{12} h_{13} h_{3k})
\]
\[
+ 2 h_{13} h_{12} h_{22} h_{33}) + \frac{9}{2} \lambda S h_{11} h_{22} h_{33} - \frac{3}{2} \lambda^2 \sum_k (h_{22} h_{33} h_{1k}^2)
\]
\[
+ h_{11} h_{33} h_{2k}^2 + h_{11} h_{22} h_{3k}^2) + 3 \lambda \sum_k (h_{11} h_{22} h_{3k} + h_{22} h_{11} h_{3k} + h_{33} h_{11} h_{2k})
\]
\[
+ 9 \lambda S h_{13} h_{12} h_{23} - 3 \lambda^2 \sum_k (h_{12} h_{13} h_{2k} + h_{12} h_{23} h_{1k} + h_{13} h_{23} h_{1k} h_{2k})
\]
\[
+ 6 \lambda \sum_k (h_{12} h_{13} h_{23} + h_{13} h_{12} h_{23} + h_{23} h_{12} h_{13} k).
\]
If diagonalized \((h_{ij})\) at some point, it is easy to get (2.19). \(\square\)

**Lemma 2.4.** Let \(x : M^3 \rightarrow \mathbb{R}^4\) be an 3-dimensional complete \(\lambda\)-translator in \(\mathbb{R}^4\). Then we can choose a local field of orthonormal frames on \(M^3\) such that, at the point, \(h_{ij} = \lambda_i \delta_{ij}\),

\[
f_3 = \frac{H}{2} (3S - H^2) + 3 \lambda_1 \lambda_2 \lambda_3,
\]
\[
f_4 = \frac{4}{3} H f_3 - H^2 S + \frac{1}{6} H^4 + \frac{1}{2} S^2,
\]
\[
f_5 = \frac{5}{6} H^2 f_3 + \frac{5}{6} S f_3 - \frac{5}{6} H^3 S + \frac{1}{6} H^5,
\]
\[
\nabla_l f_3 = 3 \sum_{i,j,k} h_{ijl} h_{jk} h_{ki}, \text{ for } l = 1, 2, 3,
\]
\[
\nabla_p \nabla_l f_3 = 3 \sum_{i,j,k} h_{ijlp} h_{jk} h_{ki} + 6 \sum_{i,j,k} h_{ijl} h_{jk} h_{kp} h_{ki}, \text{ for } l, p = 1, 2, 3,
\]
and
\[
\nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijml} h_{jk} h_{kl} h_{li}, \text{ for } m = 1, 2, 3,
\]
\[
\nabla_p \nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijmp} h_{jk} h_{kl} h_{li}
\]
\[
+ 4 \sum_{i,j,k,l} h_{ijm} (2 h_{jk} h_{kl} h_{li} + h_{jk} h_{kl} h_{pi}), \text{ for } m, p = 1, 2, 3.
\]

(2.20)
\[
\nabla_k f_4 = \frac{4}{3} f_3 H_{,k} + \frac{4}{3} H \nabla_k f_3 - 2 S H H_{,k} + \frac{2}{3} H^3 H_{,k},
\]
\begin{equation}
\n\nabla_l \nabla_k f_4 = \frac{4}{3} f_3 H_{,kl} - 2 SH H_{,kl} + \frac{2}{3} H^3 H_{,kl} + \frac{4}{3} H \nabla_l \nabla_k f_3 + \frac{4}{3} \nabla_l f_3 H_{,k} \\
+ \frac{4}{3} H_j \nabla_k f_3 - 2 SH H_{,j} + 2 H^2 H_{,j},
\end{equation}

for \( k, l = 1, 2, 3 \).

To make use of the maximum principle of Omori-Yau type, we prove the following lemma.

**Lemma 2.5.** For a space-like complete \( \lambda \)-translator \( x : M^n \to \mathbb{R}^{n+1}_1 \) with the translating vector \( T \) and non-zero constant squared norm \( S \) of the second fundamental form, the Bakry-Emery Ricci tensor \( \text{Ricc}_{-V} \) is bounded from below, where \( V = TT^T \).

**Proof.** Let \( e \) be an arbitrary unit eigenvector of the symmetric two-tensor \( \text{Ricc}_{-V} \). Choose an orthonormal tangent frame field \( \{ e_i \}_{1}^{n} \) such that \( e_1 = e \). Then, by the definition of \( \lambda \)-translator, we have

\[
-\frac{1}{2} L_{-V} g(e, e) = \frac{1}{2} V(g(e_1, e_1)) - g([V, e_1], e_1) \\
= \frac{1}{2} \{ g(\nabla_V e_1, e_1) + g(e_1, \nabla_V e_1) \} - g(\nabla_V e_1 - \nabla_{e_1} V, e_1) \\
= g(\nabla_{e_1} (T - T^\perp), e_1) \\
= - g(\nabla_{e_1} T^\perp, e_1) \\
= (H - \lambda) g(\nabla_{e_1} \bar{N}, e_1),
\]

and

\[
g(\nabla_{e_1} \bar{N}, e_1) = g(d \bar{N}(e_1), e_1) \\
= g(\omega_{n+1}^i (e_1) e_i, e_1) \\
= g(\omega_{n+1}^n (e_1) e_i, e_1) \\
= h_{11}.
\]

Therefore,

\[
-\frac{1}{2} L_{-V} g(e, e) = (H - \lambda) h_{11},
\]

\[
\text{Ricc}_{-V}(e, e) = \text{Ricc}(e, e) - \frac{1}{2} L_{-V} g(e, e) \\
= - (H h_{11} - \sum h_{1k}^2) + (H - \lambda) h_{11} \\
= \sum h_{1k}^2 - \lambda h_{11} \\
\geq \sum h_{1k}^2 - \frac{1}{2} h_{11}^2 - \frac{1}{2} \lambda^2 \geq -\frac{1}{2} \lambda^2.
\]

The proof of Lemma 2.5 is finished. \( \square \)
3. Proof of the main result

If $S = 0$, we know that $x : M^3 \to \mathbb{R}^4_1$ is $\mathbb{R}^3_1$, obviously. Next, we assume that $S > 0$. From Lemma 2.4, it is sufficient to prove that $\inf H^2 > 0$. We now prove the following theorems.

**Theorem 3.1.** For a 3-dimensional complete $\lambda$-translator $x : M^3 \to \mathbb{R}^4_1$ with non-zero constant squared norm $S$ of the second fundamental form and constant $f_4$, then $\inf H^2 > 0$, where

$$S = \sum_{i,j} h_{ij}^2$$

and

$$f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}.$$

**Proof.** If $\inf H^2 = 0$, there exists a sequence $\{p_t\}$ in $M^3$ such that

$$\lim_{t \to \infty} H^2(p_t) = \inf H^2 = \bar{H}^2 = 0.$$

From (2.16), (2.18), (2.19) and $S$ being constant, we know that $\{h_{ij}(p_t)\}$, $\{h_{ijk}(p_t)\}$ and $\{h_{ijkl}(p_t)\}$ are bounded sequences, one can assume

$$\lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}_i \delta_{ij}, \quad \lim_{t \to \infty} h_{ijk}(p_t) = \bar{h}_{ijk}, \quad \lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}$$

for $i, j, k, l = 1, 2, 3$. Then,

$$\bar{H} = \sum_i \bar{h}_{ii} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0, \quad S = \sum_{i,j} \bar{h}_{ij}^2 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2 = 2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_1 \bar{\lambda}_2).$$

From

$$H, i = \sum_k h_{ik}(T, e_k), \quad \text{for } i = 1, 2, 3,$$

we have

$$(3.1) \quad \bar{h}_{11k} + \bar{h}_{22k} + \bar{h}_{33k} = \bar{\lambda}_k \lim_{t \to \infty} \langle T, e_k(p_t) \rangle, \quad \text{for } k = 1, 2, 3.$$ 

Since

$$(3.2) \quad \nabla_j \nabla_i H = \sum_k h_{ijk}(T, e_k) + (H - \lambda) \sum_k h_{ik} h_{kj},$$

we conclude

$$\bar{H}_{ij} = \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle T, e_k(p_t) \rangle + (\bar{H} - \lambda) \bar{\lambda}_i \bar{\lambda}_j \delta_{ij},$$
that is,

\[
\begin{align*}
\bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} &= \sum_k \bar{h}_{11k} \lim_{t \to \infty} \langle T, e_k \rangle (p_t) - \lambda \bar{\lambda}_1^2, \\
\bar{h}_{1122} + \bar{h}_{2222} + \bar{h}_{3322} &= \sum_k \bar{h}_{22k} \lim_{t \to \infty} \langle T, e_k \rangle (p_t) - \lambda \bar{\lambda}_2^2, \\
\bar{h}_{1133} + \bar{h}_{2233} + \bar{h}_{3333} &= \sum_k \bar{h}_{33k} \lim_{t \to \infty} \langle T, e_k \rangle (p_t) - \lambda \bar{\lambda}_3^2, \\
\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} &= \sum_k \bar{h}_{12k} \lim_{t \to \infty} \langle T, e_k \rangle (p_t), \\
\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} &= \sum_k \bar{h}_{13k} \lim_{t \to \infty} \langle T, e_k \rangle (p_t), \\
\bar{h}_{1123} + \bar{h}_{2223} + \bar{h}_{3323} &= \sum_k \bar{h}_{23k} \lim_{t \to \infty} \langle T, e_k \rangle (p_t).
\end{align*}
\]  

(3.3)

Since \( S \) is constant, we know

\[ \sum_{i,j} h_{ij} h_{ijk} = 0, \quad \text{for } k = 1, 2, 3. \]

Thus,

\[ \sum_{i,j} \bar{h}_{ij} h_{ijk} = 0, \quad \text{for } k = 1, 2, 3. \]

Specifically,

\[
\bar{\lambda}_i \bar{h}_{11k} + \bar{\lambda}_2 \bar{h}_{22k} + \bar{\lambda}_3 \bar{h}_{33k} = 0, \quad \text{for } k = 1, 2, 3.
\]

(3.4)

Now we consider three scenarios.

1. **The principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are all equal.**

   From \( \bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0, \bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3 = 0 \), we get \( S = 0 \). It is impossible since \( S > 0 \).

2. **Two of the values of the principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are equal.**

   Without loss of generality, we assume that \( \bar{\lambda}_1 = \bar{\lambda}_2 \neq \bar{\lambda}_3 \).

   From \( \bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0 \), we infer that \( \bar{\lambda}_1 = \bar{\lambda}_2 \neq 0 \) and \( \bar{\lambda}_3 \neq 0 \).

   From (2.20) in Lemma 2.4, we obtain

   \[ \lim_{t \to \infty} f_3(p_t) \neq 0, \quad \bar{H}_k = 0 \quad \text{for } k = 1, 2, 3. \]

   By (3.1) and \( \bar{H}_k = 0 \) for \( k = 1, 2, 3 \), we have

   \[ \lim_{t \to \infty} \langle T, e_k \rangle (p_t) = 0, \quad \text{for } k = 1, 2, 3. \]

   From (3.3), we have that

\[
\begin{align*}
\bar{H}_{11} &= \bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} = -\lambda \bar{\lambda}_1^2, \\
\bar{H}_{22} &= \bar{h}_{1122} + \bar{h}_{2222} + \bar{h}_{3322} = -\lambda \bar{\lambda}_2^2, \\
\bar{H}_{33} &= \bar{h}_{1133} + \bar{h}_{2233} + \bar{h}_{3333} = -\lambda \bar{\lambda}_3^2.
\end{align*}
\]

(3.5)
From (2.21) in Lemma 2.4 and $\bar{H}_{k} = 0$ for $k = 1, 2, 3$, we have

$$\lim_{t \to \infty} \nabla_{l} \nabla_{k} f_{4}(p_{t}) = 0, \quad \bar{H}_{kl} = 0, \quad \text{for} \quad k, l = 1, 2, 3. \tag{3.6}$$

Then, it follows from (3.5) that $\bar{H}_{kk} = -\lambda \bar{\lambda}_{k}^{2} = 0$ for $k = 1, 2, 3$. It is a contradiction.

3. The values of the principal curvature $\bar{\lambda}_{1}$, $\bar{\lambda}_{2}$ and $\bar{\lambda}_{3}$ are not equal to each other.

Case 1: $\bar{\lambda}_{1} \bar{\lambda}_{2} \bar{\lambda}_{3} = 0$.

Without loss of generality, we assume that $\bar{\lambda}_{3} = 0$. That is, $\bar{\lambda}_{1} \neq 0$, $\bar{\lambda}_{2} \neq 0$ and $\bar{\lambda}_{1} \neq \bar{\lambda}_{2}$.

From $\bar{H} = 0$, $\bar{\lambda}_{3} = 0$ and $f_{3} = \frac{H}{2}(3S - H^{2}) + 3\lambda_{1}\lambda_{2}\lambda_{3}$, we have

$$\lim_{t \to \infty} f_{3}(p_{t}) = 0.$$

From (2.16) in Lemma 2.2, we have

$$\sum_{i,j,k} h_{ijk}^{2} + S^{2} = 0.$$

Since $\lim_{t \to \infty} f_{3}(p_{t}) = 0$, we have

$$\sum_{i,j,k} \bar{h}_{ijk}^{2} + S^{2} = 0,$$

and then,

$$S = 0.$$

It is a contradiction.

Case 2: $\bar{\lambda}_{1} \bar{\lambda}_{2} \bar{\lambda}_{3} \neq 0$.

From $S > 0$, $\bar{H} = 0$ and $f_{3} = \frac{H}{2}(3S - H^{2}) + 3\lambda_{1}\lambda_{2}\lambda_{3}$, we have

$$\lim_{t \to \infty} f_{3}(p_{t}) \neq 0.$$

Since $f_{4} = \frac{1}{3}Hf_{3} - H^{2}S + \frac{1}{6}H^{4} + \frac{1}{2}S^{2}$ and $\lim_{t \to \infty} f_{3}(p_{t}) \neq 0$, we get

$$0 = \nabla_{k} f_{4} = \frac{4}{3} f_{3} H_{k} + \frac{4}{3} H \nabla_{k} f_{3} - 2SH H_{,k} + \frac{2}{3} H^{3} H_{,k},$$

$$0 = \nabla_{l} \nabla_{k} f_{4} = \frac{4}{3} f_{3} H_{kl} - 2SH H_{,kl} + \frac{2}{3} H^{3} H_{,kl} + \frac{4}{3} H \nabla_{l} \nabla_{k} f_{3} + \frac{4}{3} \nabla_{l} f_{3} H_{k}$$

$$+ \frac{4}{3} H \nabla_{k} f_{3} - 2SH_{,k} H_{,l} + 2H^{2} H_{,k} H_{,l}, \quad \text{for} \quad k, l = 1, 2, 3.$$

Then, $\bar{H}_{k} = 0$ and $\bar{H}_{kl} = 0$ for $k, l = 1, 2, 3$.

Especially,

$$\begin{cases}
\bar{H}_{1} = \bar{\lambda}_{1} \lim_{t \to \infty} \langle T, e_{1} \rangle(p_{t}) = 0, \\
\bar{H}_{2} = \bar{\lambda}_{2} \lim_{t \to \infty} \langle T, e_{2} \rangle(p_{t}) = 0, \\
\bar{H}_{3} = \bar{\lambda}_{3} \lim_{t \to \infty} \langle T, e_{3} \rangle(p_{t}) = 0,
\end{cases} \tag{3.7}$$
and

\[
\begin{align*}
\bar{H}_{11} &= \sum_k h_{11k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) - \lambda_1^2 = 0, \\
\bar{H}_{22} &= \sum_k h_{22k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) - \lambda_2^2 = 0, \\
\bar{H}_{33} &= \sum_k h_{33k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) - \lambda_3^2 = 0.
\end{align*}
\]

\tag{3.8}

From (3.7) and \(\lambda_k \neq 0\) for \(k = 1, 2, 3\), one has

\[\lim_{t \to \infty} \langle T, e_k \rangle(p_t) = 0, \quad \text{for } k = 1, 2, 3.\]

By (3.8), we know that \(\lambda_k = 0\) for \(k = 1, 2, 3\). It is a contradiction. \qed

**Theorem 3.2.** For a 3-dimensional complete \(\lambda\)-translator \(x : M^3 \to \mathbb{R}_1^4\) with non-zero constant squared norm \(S\) of the second fundamental form and constant \(f_4\), where \(S = \sum_{i,j} h_{ij}^2\) and \(f_4 = \sum_{i,j,k,l} h_{ijk} h_{kil} h_{lii}\), we have either

1. \(\lambda^2 = S\) and \(\sup H^2 = S\), or
2. \(\lambda^2 = 2S\) and \(\sup H^2 = 2S\), or
3. \(\lambda^2 = 3S\) and \(\sup H^2 = 3S\).

**Proof.** From Lemma 2.2, we have

\[\frac{1}{2} \Delta_{\nabla} H^2 = |\nabla H|^2 + S(H - \lambda)H.\]

At each point \(p \in M^3\), we choose \(e_1, e_2\) and \(e_3\) such that

\[h_{ij} = \lambda_i \delta_{ij}.\]

From \(2ab \leq \frac{a^2}{\alpha} + \frac{1}{\alpha} b^2\), we obtain

\[S = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad H^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 \leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = 3S.\]

Hence, we have on \(M^3\)

\[H^2 \leq 3S\]

and the equality holds if and only if \(\lambda_1 = \lambda_2 = \lambda_3\).

From Lemma 2.5, we know that the Bakry-Emery Ricci tensor \(\text{Ric}_{\nabla}\) of \(x : M^3 \to \mathbb{R}_1^4\) is bounded from below. We can apply the generalized maximum principle for the operator \(\Delta_{\nabla}\) to the function \(H^2\). Thus, there exists a sequence \(\{p_t\}\) in \(M^3\) such that

\[\lim_{t \to \infty} H^2(p_t) = \sup H^2, \quad \lim_{t \to \infty} |\nabla H^2(p_t)| = 0, \quad \lim_{t \to \infty} \Delta_{\nabla} H^2(p_t) \leq 0.\]

For \(S \neq 0\), from Theorem 3.1, we know that \(\sup H^2 \geq \inf H^2 > 0\). Without loss of the generality, at each point \(p_t\), we can assume \(H(p_t) \neq 0\). From (2.10), (2.17), (2.18) and \(S = \text{constant}\), we know that \(\{h_{ij}(p_t)\}, \{h_{ijk}(p_t)\}\) and \(\{h_{ijkl}(p_t)\}\) are bounded sequences for \(i, j, k, l = 1, 2, 3\). We can assume

\[\lim_{t \to \infty} f_3(p_t) = \bar{f}_3, \quad \lim_{t \to \infty} f_5(p_t) = \bar{f}_5, \quad \lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}_i \delta_{ij}, \quad \lim_{t \to \infty} h_{ijk}(p_t) = \bar{h}_{ijk}, \quad \lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad \text{for } i, j, k = 1, 2, 3.\]
From Lemma 2.2, we get
\[
\left\{ \begin{array}{l}
\lim_{t \to \infty} H^2(p_t) = \sup H^2 = \bar{H}^2, \\
\lim_{t \to \infty} |\nabla H^2(p_t)| = 0,
\end{array} \right.
0 \geq \lim_{t \to \infty} |\nabla H^2(p_t) + S(\bar{H} - \lambda)\bar{H}|.
\]
From \(\lim_{t \to \infty} |\nabla H^2(p_t)| = 0\) and \(|\nabla H^2|^2 = 4 \sum_k (HH_k)^2\), we have
\[
\bar{H}_k = 0, \text{ for } k = 1, 2, 3,
\]
that is,
\[
\bar{h}_{11k} + \bar{h}_{22k} + \bar{h}_{33k} = 0, \text{ for } k = 1, 2, 3.
\]
Since \(x\) is a \(\lambda\)-translator, from (2.13), we have
\[
H_{i,j} = \sum_k h_{ijk}(T, e_k), \text{ for } i, j = 1, 2, 3,
\]
\[
\nabla_j \nabla_i H = \sum_k h_{ijk}(T, e_k) + (H - \lambda) \sum k h_{ijk}, \text{ for } i, j = 1, 2, 3.
\]
Thus,
\[
\bar{H}_{i} = \bar{h}_{11i} + \bar{h}_{22i} + \bar{h}_{33i} = \bar{\lambda}_i \lim_{t \to \infty} \langle T, e_i \rangle(p_t), \text{ for } i = 1, 2, 3,
\]
\[
\bar{H}_{ij} = \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) + (\bar{H} - \lambda)\bar{\lambda}_i \bar{\lambda}_j \delta_{ij}, \text{ for } i, j = 1, 2, 3.
\]
Especially,
\[
\left\{ \begin{array}{l}
\bar{H}_1 = \bar{\lambda}_1 \lim_{t \to \infty} \langle T, e_1 \rangle(p_t) = 0,
\bar{H}_2 = \bar{\lambda}_2 \lim_{t \to \infty} \langle T, e_2 \rangle(p_t) = 0,
\bar{H}_3 = \bar{\lambda}_3 \lim_{t \to \infty} \langle T, e_3 \rangle(p_t) = 0,
\end{array} \right.
\]
and
\[
\left\{ \begin{array}{l}
\bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} = \sum_k \bar{h}_{11k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) + (\bar{H} - \lambda)\bar{\lambda}_1^2,
\bar{h}_{1122} + \bar{h}_{2222} + \bar{h}_{3322} = \sum_k \bar{h}_{22k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) + (\bar{H} - \lambda)\bar{\lambda}_2^2,
\bar{h}_{1133} + \bar{h}_{2233} + \bar{h}_{3333} = \sum_k \bar{h}_{33k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t) + (\bar{H} - \lambda)\bar{\lambda}_3^2,
\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} = \sum_k \bar{h}_{12k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t),
\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} = \sum_k \bar{h}_{13k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t),
\bar{h}_{1123} + \bar{h}_{2223} + \bar{h}_{3323} = \sum_k \bar{h}_{23k} \lim_{t \to \infty} \langle T, e_k \rangle(p_t).
\end{array} \right.
\]
Since \(S\) is constant, we know
\[
\sum_{i,j} h_{ij} h_{ijk} = 0, \text{ for } k = 1, 2, 3,
\]
\[ \sum_{i,j} h_{ij} h_{ijkl} + \sum_{i,j} h_{ijk} h_{ij} = 0, \text{ for } k, l = 1, 2, 3. \]

Thus,

\[ \sum_{i,j} \bar{h}_{ij} \bar{h}_{ijk} = 0, \text{ for } k = 1, 2, 3, \]

\[ \sum_{i,j} \bar{h}_{ij} \bar{h}_{ijkl} + \sum_{i,j} \bar{h}_{ijk} \bar{h}_{ijl} = 0, \text{ for } k, l = 1, 2, 3. \]

Specifically,

\[ \bar{h}_{1111} + \bar{h}_{2222} + \bar{h}_{3333} = \frac{1}{4} h_{1111} - \frac{1}{2} h_{2222} - \frac{1}{2} h_{3333} \]

From Ricci identities \((2.7)\), we obtain

\[ \bar{h}_{ijkl} - \bar{h}_{ijlk} = -\lambda_i \lambda_j \lambda_k \delta_{ij} \delta_{jk} - \lambda_i \lambda_j \lambda_k \delta_{ik} \delta_{jl} - \lambda_i \lambda_j \lambda_k \delta_{il} \delta_{jk}, \]

that is,

\[ \bar{h}_{1212} - \bar{h}_{2121} = -\lambda_1 \lambda_2 (\lambda_1 - \lambda_2), \quad \bar{h}_{1313} - \bar{h}_{1131} = -\lambda_1 \lambda_3 (\lambda_1 - \lambda_3), \quad \bar{h}_{2323} - \bar{h}_{2332} = -\lambda_2 \lambda_3 (\lambda_2 - \lambda_3), \quad \bar{h}_{iilk} - \bar{h}_{iilk} = 0, \text{ for } i, k, l = 1, 2, 3. \]

Since \( f_4 \) is constant, we know from the Lemma \((2.4)\)

\[ 0 = \nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijm} h_{jk} h_{kl} h_{ti}, \]

\[ 0 = \nabla_p \nabla_m f_4 = 4 \sum_{i,j,k,l} h_{ijmp} h_{jk} h_{kl} h_{ti} + 4 \sum_{i,j,k,l} h_{ijm} (2 h_{jkp} h_{kl} h_{ti} + h_{jkm} h_{kl} h_{ti}), \]

for \( m, p = 1, 2, 3 \). Thus,

\[ \lambda_1^3 \bar{h}_{111} + \lambda_2^3 \bar{h}_{222} + \lambda_3^3 \bar{h}_{333} = 0, \text{ for } k = 1, 2, 3, \]
1. The principal curvature $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ are all equal.  

From $\bar{H} = \lambda_1 + \lambda_2 + \lambda_3 \neq 0$, $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$, we get

$$\bar{H}^2 = 3S.$$  

From (3.11) and $\tilde{\lambda}_k \neq 0$ for $k = 1, 2, 3$, we have

$$\bar{H}_k = \tilde{\lambda}_k \lim_{t \to \infty} \langle T, e_k \rangle(p_t) = 0, \quad \lim_{t \to \infty} \langle T, e_k \rangle = 0, \quad \text{for } k = 1, 2, 3.$$  

From (3.14), (3.17) and $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3$, we have

$$\begin{align*}
\lambda_1 \sum_i h_{i11} &= - (\bar{h}_{111}^2 + \bar{h}_{221}^2 + \bar{h}_{331}^2) - 2(\bar{h}_{121}^2 + \bar{h}_{131}^2 + \bar{h}_{231}^2), \\
\lambda_1 \sum_i h_{i22} &= - (\bar{h}_{112}^2 + \bar{h}_{222}^2 + \bar{h}_{332}^2) - 2(\bar{h}_{122}^2 + \bar{h}_{132}^2 + \bar{h}_{232}^2), \\
\lambda_1 \sum_i h_{i33} &= - (\bar{h}_{113}^2 + \bar{h}_{223}^2 + \bar{h}_{333}^2) - 2(\bar{h}_{123}^2 + \bar{h}_{133}^2 + \bar{h}_{233}^2),
\end{align*}$$

Now we consider three scenarios.

1. The principal curvature $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ are all equal.
and

\[
\begin{cases}
\lambda_1^3 \sum_i h_{i11} = -3\lambda_1^2(h_{i12}^2 + h_{i21}^2 + h_{i33}^2) - 6\lambda_1^2(h_{121}^2 + h_{131}^2 + h_{231}^2), \\
\lambda_1^3 \sum_i h_{i22} = -3\lambda_1^2(h_{i12}^2 + h_{i22}^2 + h_{i33}^2) - 6\lambda_1^2(h_{122}^2 + h_{132}^2 + h_{232}^2), \\
\lambda_1^3 \sum_i h_{i33} = -3\lambda_1^2(h_{i13}^2 + h_{i23}^2 + h_{i33}^2) - 6\lambda_1^2(h_{123}^2 + h_{133}^2 + h_{233}^2).
\end{cases}
\]

From (3.19) and (3.20), we have

\[
(3.21) \bar{h}_{ijk} = 0, \quad H_{kk} = 0, \quad \text{for } i, j, k = 1, 2, 3.
\]

From (3.12), (3.18) and (3.21), we have

\[
\lambda^2 = \bar{H}^2 = \sup H^2, \quad \lambda^2 = 3S.
\]

2. Two of the values of the principal curvature \(\lambda_1, \lambda_2, \lambda_3\) are equal.

Without loss of generality, we assume that \(\lambda_1 \neq \lambda_2 = \lambda_3\), and then,

\[
\bar{H} = \lambda_1 + 2\lambda_2 \neq 0.
\]

From (3.11) and (3.13), we get

\[
(3.22) \bar{h}_{11k} = 0, \quad \bar{h}_{22k} + \bar{h}_{33k} = 0, \quad \text{for } k = 1, 2, 3.
\]

Case 1: \(\lambda_1\lambda_2\lambda_3 = 0\).

Subcase 1.1: \(\lambda_1 \neq 0, \quad \lambda_2 = \lambda_3 = 0\).

Since \(\bar{H}^2 \neq 0\), we have that \(\bar{H}^2 = S\) and \(f_3 = \lambda_1 S\).

From the first equation in (3.17) and (3.22), we have

\[
\bar{h}_{1111} = 0,
\]

and then, by (3.14), we know

\[
0 = \lambda_1 \bar{h}_{1111} + \lambda_2 \bar{h}_{2211} + \lambda_3 \bar{h}_{3311}
\]

\[
= -\bar{h}_{111}^2 - \bar{h}_{221}^2 - \bar{h}_{331}^2 - 2\bar{h}_{121}^2 - 2\bar{h}_{131}^2 - 2\bar{h}_{231}^2,
\]

where \(\lambda_2 = \lambda_3 = 0\). Thus,

\[
\bar{h}_{221} = \bar{h}_{331} = \bar{h}_{231} = 0.
\]

From \(\bar{h}_{221} = \bar{h}_{331} = \bar{h}_{231} = 0\), the second equation in (3.17) and (3.22), we have

\[
\bar{h}_{1122} = 0,
\]

and then, by (3.14), we know

\[
0 = \lambda_1 \bar{h}_{1122} + \lambda_2 \bar{h}_{2222} + \lambda_3 \bar{h}_{3322}
\]

\[
= -\bar{h}_{112}^2 - \bar{h}_{222}^2 - \bar{h}_{332}^2 - 2\bar{h}_{122}^2 - 2\bar{h}_{132}^2 - 2\bar{h}_{232}^2,
\]

where \(\lambda_2 = \lambda_3 = 0\). Thus,

\[
\bar{h}_{222} = \bar{h}_{332} = \bar{h}_{232} = \bar{h}_{333} = 0.
\]
That is, 
\[ \bar{h}_{ijk} = 0, \quad \text{for } i, j, k = 1, 2, 3. \]

From (2.16) in Lemma 2.2, we have 
\[ 0 = S^2 - \lambda \bar{f}_3, \]
then, we obtain 
\[ \lambda^2 = \bar{H}^2 = \sup H^2, \quad \lambda^2 = S. \]

**Subcase 1.2:** \( \bar{\lambda}_1 = 0, \bar{\lambda}_2 = \bar{\lambda}_3 \neq 0. \)

Since \( \bar{H}^2 \neq 0, \) we have that \( \bar{H}^2 = 2S \) and \( \bar{f}_3 = \bar{\lambda}_2 S. \)

From (3.11), we have 
\[ \bar{H}_k = \bar{\lambda}_k \lim_{t \to \infty} \langle T, e_k \rangle (p_t) = 0, \quad \text{for } k = 1, 2, 3, \]
and then,
\[ \lim_{t \to \infty} \langle T, e_2 \rangle (p_t) = 0, \quad \lim_{t \to \infty} \langle T, e_3 \rangle (p_t) = 0. \]

From (3.22), we have 
\[ \bar{h}_{111} = \bar{h}_{112} = \bar{h}_{113} = 0, \quad \bar{h}_{221} = -\bar{h}_{331}, \quad \bar{h}_{222} = -\bar{h}_{332}, \quad \bar{h}_{223} = -\bar{h}_{333}. \]

By (3.14) and (3.17), we have that 
\[ \begin{cases} \bar{\lambda}_2(\bar{h}_{221} + \bar{h}_{331}) = -2\bar{h}_{221}^2 - 2\bar{h}_{123}^2, \\
\bar{\lambda}_2^2(\bar{h}_{221} + \bar{h}_{331}) = -6\lambda_2^2\bar{h}_{221}^2 - 6\lambda_2^2\bar{h}_{123}^2, \\
\bar{\lambda}_2(\bar{h}_{222} + \bar{h}_{332}) = -2\bar{h}_{222}^2 - 2\bar{h}_{223}^2 - 2\bar{h}_{123}^2, \\
\bar{\lambda}_2^2(\bar{h}_{222} + \bar{h}_{332}) = -2\lambda_2^2(\bar{h}_{222}^2 + 3\bar{h}_{222}^2 + 3\bar{h}_{223}^2 + \bar{h}_{123}^2), \end{cases} \]
then,
\[ \bar{h}_{221} = 0, \quad \bar{h}_{123} = 0, \quad \bar{h}_{222} = 0, \quad \bar{h}_{223} = 0. \]

Therefore,
\[ \bar{h}_{ijk} = 0, \quad \text{for } i, j, k = 1, 2, 3. \]

From (2.16) in Lemma 2.2, we have 
\[ 0 = S^2 - \lambda \bar{f}_3, \]
then, we obtain 
\[ \lambda^2 = \bar{H}^2 = \sup H^2, \quad \lambda^2 = 2S. \]

**Case 2:** \( \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 \neq 0. \)

According to the hypothesis, we have that 
\[ \bar{H} = \bar{\lambda}_1 + 2\bar{\lambda}_2 \neq 0, \quad \bar{\lambda}_1 \neq \bar{\lambda}_2 = \bar{\lambda}_3, \quad \bar{\lambda}_k \neq 0, \quad \text{for } k = 1, 2, 3. \]

From (3.11) and \( \bar{\lambda}_k \neq 0 \) for \( k = 1, 2, 3, \) we get
\[ (3.23) \quad \bar{H}_k = \bar{\lambda}_k \lim_{t \to \infty} \langle T, e_k \rangle (p_t) = 0, \quad \lim_{t \to \infty} \langle T, e_k \rangle (p_t) = 0, \quad \text{for } k = 1, 2, 3, \]

From (3.12), (3.14), (3.17), (3.22) and (3.23), we know that
\[
\begin{aligned}
\begin{cases}
\bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} = (\bar{H} - \lambda)\bar{\lambda}_1^2, \\
\bar{h}_{1122} + \bar{h}_{2222} + \bar{h}_{3322} = (\bar{H} - \lambda)\bar{\lambda}_2^2, \\
\bar{h}_{1133} + \bar{h}_{2233} + \bar{h}_{3333} = (\bar{H} - \lambda)\bar{\lambda}_3^2, \\
\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} = 0, \\
\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} = 0,
\end{cases}
\end{aligned}
(3.24)
\]

\[
\begin{aligned}
\begin{cases}
\bar{\lambda}_1\bar{h}_{1111} + \bar{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1\bar{h}_{1122} + \bar{\lambda}_2(\bar{h}_{2222} + \bar{h}_{3322}) = -2(\bar{h}_{222}^2 + \bar{h}_{2233}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1\bar{h}_{1133} + \bar{\lambda}_2(\bar{h}_{2233} + \bar{h}_{3333}) = -2(\bar{h}_{222}^2 + \bar{h}_{2233}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1\bar{h}_{1112} + \bar{\lambda}_2(\bar{h}_{2212} + \bar{h}_{3312}) = -2(\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123}), \\
\bar{\lambda}_1\bar{h}_{1113} + \bar{\lambda}_2(\bar{h}_{2213} + \bar{h}_{3313}) = -2(\bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123}),
\end{cases}
\end{aligned}
(3.25)
\]

and

\[
\begin{aligned}
\begin{cases}
\bar{\lambda}_1^3\bar{h}_{1111} + \bar{\lambda}_2^3(\bar{h}_{2211} + \bar{h}_{3311}) = -6\bar{\lambda}_2^2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1^3\bar{h}_{1122} + \bar{\lambda}_2^3(\bar{h}_{2222} + \bar{h}_{3322}) = -6\bar{\lambda}_2^2(\bar{h}_{222}^2 + \bar{h}_{2233}^2) - 2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) + \bar{\lambda}_1\bar{\lambda}_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1^3\bar{h}_{1133} + \bar{\lambda}_2^3(\bar{h}_{2233} + \bar{h}_{3333}) = -6\bar{\lambda}_2^2(\bar{h}_{222}^2 + \bar{h}_{2233}^2) - 2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) + \bar{\lambda}_1\bar{\lambda}_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1^3\bar{h}_{1112} + \bar{\lambda}_2^3(\bar{h}_{2212} + \bar{h}_{3312}) = -6\bar{\lambda}_2^2(\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123}), \\
\bar{\lambda}_1^3\bar{h}_{1113} + \bar{\lambda}_2^3(\bar{h}_{2213} + \bar{h}_{3313}) = -6\bar{\lambda}_2^2(\bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123}).
\end{cases}
\end{aligned}
(3.26)
\]

From \(\bar{\lambda}_1 \neq \bar{\lambda}_2\), \(3.24\), \(3.25\) and \(3.26\), we get

\[
\bar{h}_{2212} + \bar{h}_{3312} = -\bar{h}_{1112}, \quad \bar{h}_{2213} + \bar{h}_{3313} = -\bar{h}_{1113},
\]

and then,

\[
\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123} = 0, \quad \bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123} = 0, \quad \bar{h}_{1112} = 0, \quad \bar{h}_{1113} = 0.
(3.27)
\]

Besides, by \(3.25\) and \(3.26\), we get

\[
\begin{aligned}
\begin{cases}
\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)\bar{h}_{1111} = 4\bar{\lambda}_2^2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)\bar{h}_{1112} = 4\bar{\lambda}_2^2(\bar{h}_{222}^2 + \bar{h}_{223}^2) + 2(\bar{\lambda}_1^2 + \bar{\lambda}_1\bar{\lambda}_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)\bar{h}_{1113} = 4\bar{\lambda}_2^2(\bar{h}_{222}^2 + \bar{h}_{223}^2) + 2(\bar{\lambda}_1^2 + \bar{\lambda}_1\bar{\lambda}_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2).
\end{cases}
\end{aligned}
(3.28)
\]

Now we consider four subcases.

**Subcase 2.1:** \(\bar{h}_{221}^2 + \bar{h}_{123}^2 \neq 0, \quad \bar{h}_{222} + \bar{h}_{223}^2 \neq 0\).

From \(3.27\), it is a contradiction.

**Subcase 2.2:** \(\bar{h}_{221}^2 + \bar{h}_{123}^2 = 0, \quad \bar{h}_{222} + \bar{h}_{223}^2 = 0\).

From \(3.22\), we know

\[
\bar{h}_{ijk} = 0, \quad \text{for } i, j, k = 1, 2, 3,
(3.29)
\]

and then, by \(2.16\) in Lemma 2.2, we have

\[
0 = S^2 - \lambda\bar{f}_3.
(3.30)
\]
If $\bar{\lambda}_1 + \bar{\lambda}_2 = 0$, we have
\[(3.31) \quad \bar{H} = -\bar{\lambda}_1, \quad S = 3\bar{\lambda}_1^2, \quad \bar{f}_3 = -\bar{\lambda}_1^3.\]

From (3.30) and (3.31), we know
\[(3.32) \quad \lambda = -9\bar{\lambda}_1 = 9\bar{H}.\]

From (2.17), (2.18), (3.29), (3.31) and (3.32), we know
\[
\frac{1}{2} \lim_{t \to \infty} \Delta_V \sum_{i,j,k} (h_{ijkl})^2(p_{lt}) = \sum_{i,j,k,l} (\bar{h}_{ijkl})^2, \\
\frac{1}{2} \lim_{t \to \infty} \Delta_V \sum_{i,j,k} (h_{ijkl})^2(p_{lt}) = \frac{3}{4} \lambda S(S - \bar{H}^2)(\bar{H} - \lambda) + \frac{9}{2} \lambda S\bar{h}_{11}\bar{h}_{22}\bar{h}_{33} - \frac{3}{2} \lambda^2 \sum_k (\bar{h}_{22}\bar{h}_{33}\bar{h}_{1k} + \bar{h}_{11}\bar{h}_{33}\bar{h}_{2k} + \bar{h}_{11}\bar{h}_{22}\bar{h}_{3k})
\]
\[
= \frac{3}{4} \lambda S(S - \bar{H}^2)(\bar{H} - \lambda) + \frac{9}{2} \lambda \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 S - \frac{3}{2} \lambda^2 \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 \bar{H}
\]
\[
= -324\bar{\lambda}_1^6,
\]
and then,
\[
\sum_{i,j,k,l} (\bar{h}_{ijkl})^2 = -324\bar{\lambda}_1^6 < 0.
\]

It is a contradiction.

If $\bar{\lambda}_1 + \bar{\lambda}_2 \neq 0$, from (3.25) and (3.26), we know that
\[
\begin{cases}
\bar{h}_{1111} = 0, & \bar{h}_{2211} + \bar{h}_{3311} = 0, \\
\bar{h}_{1122} = 0, & \bar{h}_{2222} + \bar{h}_{3322} = 0, \\
\bar{h}_{1133} = 0, & \bar{h}_{2233} + \bar{h}_{3333} = 0,
\end{cases}
\]
and then, by (3.24), we have
\[(3.33) \quad \lambda = \bar{H}.
\]

From (3.30) and (3.33), we know
\[
\bar{\lambda}_1 = \bar{\lambda}_2,
\]
where $\bar{H} = \bar{\lambda}_1 + 2\bar{\lambda}_2$, $S = \bar{\lambda}_1^2 + 2\bar{\lambda}_2^2$ and $\bar{f}_3 = \bar{\lambda}_1^3 + 2\bar{\lambda}_2^3$. It is a contradiction.

**Subcase 2.3:** $\bar{h}_{2211}^2 + \bar{h}_{123}^2 = 0, \quad \bar{h}_{2222}^2 + \bar{h}_{2233}^2 \neq 0$.

From (3.28), we know
\[
\begin{cases}
\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)\bar{h}_{1111} = 0, \\
\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)\bar{h}_{1122} = 4\bar{\lambda}_2^2(\bar{h}_{2222} + \bar{h}_{2233}), \\
\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)\bar{h}_{1133} = 4\bar{\lambda}_2^2(\bar{h}_{2222} + \bar{h}_{2233}),
\end{cases}
\]
and then,
\[(3.34) \quad \bar{\lambda}_1 + \bar{\lambda}_2 \neq 0, \quad \bar{h}_{1111} = 0, \quad \bar{h}_{1122} = \frac{4\bar{\lambda}_2^2}{\bar{\lambda}_1(\bar{\lambda}_2^2 - \bar{\lambda}_1^2)}(\bar{h}_{2222}^2 + \bar{h}_{2233}^2).\]
From $\bar{h}_{1111} = 0$ and the first equation in (3.25), we know
\[ \bar{h}_{2211} + \bar{h}_{3311} = 0, \]
and then, by (3.24), we have
\[ (3.35) \quad \bar{H},_{11} = 0, \quad \lambda = \bar{H}, \quad \bar{H},_{22} = 0. \]
From (3.25) and (3.35), we know
\[ (3.36) \quad \bar{h}_{1122} = \frac{2}{\lambda_2 - \lambda_1}(\bar{h}_{222}^2 + \bar{h}_{223}^2), \]
From (3.34) and (3.36), we have
\[ \bar{\lambda}_1 = \bar{\lambda}_2. \]
It is a contradiction.

**Subcase 2.4:** $\bar{h}_{221}^2 + \bar{h}_{123}^2 \neq 0, \quad \bar{h}_{222}^2 + \bar{h}_{223}^2 = 0.$
From (3.28), we know
\[
\begin{cases}
\bar{\lambda}_1(\lambda_2^2 - \lambda_1^2)\bar{h}_{1111} = 4\lambda_2^2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1(\lambda_2^2 - \lambda_1^2)\bar{h}_{1122} = 2(\lambda_2^2 + \bar{\lambda}_1\lambda_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1(\lambda_2^2 - \lambda_1^2)\bar{h}_{1133} = 2(\lambda_1^2 + \bar{\lambda}_1\lambda_2)(\bar{h}_{221}^2 + \bar{h}_{123}^2),
\end{cases}
\]
and then,
\[ (3.37) \quad \bar{\lambda}_1 + \bar{\lambda}_2 \neq 0, \quad \bar{h}_{1111} = \frac{4\lambda_2^2}{\lambda_1(\lambda_2^2 - \lambda_1^2)}(\bar{h}_{221}^2 + \bar{h}_{123}^2), \]
\[ \bar{h}_{1122} = \bar{h}_{1133} = \frac{2}{\lambda_2 - \lambda_1}(\bar{h}_{221}^2 + \bar{h}_{123}^2). \]
From (3.24), (3.28) and (3.37), we know
\[
-2(\bar{h}_{221}^2 + \bar{h}_{123}^2) = \bar{\lambda}_1\bar{h}_{1122} + \bar{\lambda}_2((\bar{H} - \lambda)\lambda_2^3 - \bar{h}_{1122})
= (\bar{H} - \lambda)\lambda_2^3 + (\bar{\lambda}_1 - \bar{\lambda}_2)\bar{h}_{1122}
= (\bar{H} - \lambda)\lambda_2^3 + (\bar{\lambda}_1 - \bar{\lambda}_2)\cdot \frac{2}{\lambda_2 - \lambda_1}(\bar{h}_{221}^2 + \bar{h}_{123}^2)
= (\bar{H} - \lambda)\lambda_2^3 - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2),
\]
and then, by (3.24), we have
\[ (3.38) \quad \lambda = \bar{H}, \quad \bar{H},_{11} = 0. \]
From (3.25) and (3.38), we have
\[ (3.39) \quad \bar{h}_{1111} = \frac{2}{\lambda_2 - \lambda_1}(\bar{h}_{221}^2 + \bar{h}_{123}^2). \]
From (3.37) and (3.39), we know
\[ \bar{\lambda}_1 = \bar{\lambda}_2. \]
It is a contradiction.

**3. The values of the principal curvature $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are not equal to each other.**
**Case 1:** $\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 = 0$.

Without loss of generality, we assume that $\bar{\lambda}_3 = 0$, we know that $\bar{\lambda}_1 \neq 0$, $\bar{\lambda}_2 \neq 0$, $\bar{\lambda}_1 - \bar{\lambda}_2 \neq 0$ and $\bar{H} = \bar{\lambda}_1 + \bar{\lambda}_2 \neq 0$.

From (3.13) and (3.16), we have that

\begin{equation}
\bar{h}_{11k} = \bar{h}_{22k} = 0, \text{ for } k = 1, 2, 3.
\end{equation}

From (3.11) and (3.40), we have

\begin{equation}
\bar{h}_{33k} = 0, \text{ for } k = 1, 2, 3.
\end{equation}

By (3.14), (3.17), (3.40) and (3.41), we have

\begin{align*}
\lambda_1 h_{1111} + \lambda_2 h_{2211} &= -2\bar{h}_{123}^2, \\
\lambda_1 h_{1112} + \lambda_2 h_{2212} &= -2\bar{h}_{123}^2,
\end{align*}

and then,

\begin{equation}
\bar{h}_{1111} = 0, \quad \bar{h}_{2211} = -\frac{2\bar{h}_{123}^2}{\lambda_2}, \quad \bar{h}_{1122} = -\frac{2\bar{h}_{123}^2}{\lambda_1}, \quad \bar{h}_{2222} = 0.
\end{equation}

From (3.15) and (3.42), we know

$$
\bar{h}_{1122} - \bar{h}_{2211} = -\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 - \bar{\lambda}_2) = \frac{2(\bar{\lambda}_1 - \bar{\lambda}_2)\bar{h}_{123}^2}{\lambda_1 \lambda_2},
$$

and then,

$$
\bar{h}_{123}^2 = -\frac{\bar{\lambda}_1^2 \bar{\lambda}_2^2}{2}, \quad \bar{h}_{123}^2 = 0, \quad \bar{\lambda}_1 \bar{\lambda}_2 = 0.
$$

It is a contradiction.

**Case 2:** $\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 \neq 0$.

From (3.11), we have that

\begin{equation}
\lim_{m \to \infty} \langle T, e_k(t) \rangle (p_t) = 0, \text{ for } k = 1, 2, 3.
\end{equation}

From (3.11), (3.13) and (3.16), we have that

\begin{equation}
\bar{h}_{11k} = \bar{h}_{22k} = \bar{h}_{33k} = 0, \text{ for } k = 1, 2, 3.
\end{equation}

From (3.12), (3.14), (3.17), (3.43) and (3.44), we have that

\begin{equation}
\begin{cases}
\bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} = (\bar{H} - \bar{\lambda})\bar{\lambda}_1^2, \\
\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} = (\bar{H} - \bar{\lambda})\bar{\lambda}_2^2, \\
\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} = (\bar{H} - \bar{\lambda})\bar{\lambda}_3^2, \\
\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} = 0, \\
\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} = 0, \\
\bar{h}_{1123} + \bar{h}_{2223} + \bar{h}_{3323} = 0,
\end{cases}
\end{equation}
\[
\begin{align*}
\lambda_1 \tilde{h}_{1111} + \lambda_2 \tilde{h}_{2211} + \lambda_3 \tilde{h}_{3311} &= -2\tilde{h}_{231}^2, \\
\lambda_1 \tilde{h}_{1122} + \lambda_2 \tilde{h}_{2222} + \lambda_3 \tilde{h}_{3322} &= -2\tilde{h}_{132}^2, \\
\lambda_1 \tilde{h}_{1133} + \lambda_2 \tilde{h}_{2233} + \lambda_3 \tilde{h}_{3333} &= -2\tilde{h}_{123}^2, \\
\lambda_1 \tilde{h}_{1112} + \lambda_2 \tilde{h}_{2212} + \lambda_3 \tilde{h}_{3312} &= 0, \\
\lambda_1 \tilde{h}_{1113} + \lambda_2 \tilde{h}_{2213} + \lambda_3 \tilde{h}_{3313} &= 0, \\
\lambda_1 \tilde{h}_{1123} + \lambda_2 \tilde{h}_{2223} + \lambda_3 \tilde{h}_{3323} &= 0,
\end{align*}
\]

(3.46)

and

\[
\begin{align*}
\lambda_1^3 \tilde{h}_{1111} + \lambda_2^3 \tilde{h}_{2211} + \lambda_3^3 \tilde{h}_{3311} &= -2(\lambda_2^2 + \lambda_3^2 + \lambda_2 \lambda_3)\tilde{h}_{123}^2, \\
\lambda_1^3 \tilde{h}_{1122} + \lambda_2^3 \tilde{h}_{2222} + \lambda_3^3 \tilde{h}_{3322} &= -2(\lambda_1^2 + \lambda_3^2 + \lambda_1 \lambda_3)\tilde{h}_{123}^2, \\
\lambda_1^3 \tilde{h}_{1133} + \lambda_2^3 \tilde{h}_{2233} + \lambda_3^3 \tilde{h}_{3333} &= -2(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2)\tilde{h}_{123}^2, \\
\lambda_1^3 \tilde{h}_{1112} + \lambda_2^3 \tilde{h}_{2212} + \lambda_3^3 \tilde{h}_{3312} &= 0, \\
\lambda_1^3 \tilde{h}_{1113} + \lambda_2^3 \tilde{h}_{2213} + \lambda_3^3 \tilde{h}_{3313} &= 0, \\
\lambda_1^3 \tilde{h}_{1123} + \lambda_2^3 \tilde{h}_{2223} + \lambda_3^3 \tilde{h}_{3323} &= 0.
\end{align*}
\]

(3.47)

Therefore,

\[
\begin{align*}
\tilde{h}_{1111} &= \frac{\lambda_2 \lambda_3 (\lambda_2 + \lambda_3)}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{2211} &= \frac{-2\tilde{h}_{123}^2}{\lambda_2 - \lambda_3} + \frac{\tilde{\lambda}_1 \tilde{\lambda}_3 (\tilde{\lambda}_1 + \tilde{\lambda}_3)}{H(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{3311} &= \frac{-2\tilde{h}_{123}^2}{\lambda_3 - \lambda_2} + \frac{\tilde{\lambda}_1 \tilde{\lambda}_2 (\tilde{\lambda}_1 + \tilde{\lambda}_2)}{H(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{2222} &= \frac{\tilde{\lambda}_1 \tilde{\lambda}_3 (\tilde{\lambda}_1 + \tilde{\lambda}_3)}{H(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{1122} &= \frac{-2\tilde{h}_{123}^2}{\lambda_1 - \lambda_3} + \frac{\tilde{\lambda}_2 \tilde{\lambda}_3 (\tilde{\lambda}_2 + \tilde{\lambda}_3)}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{3322} &= \frac{-2\tilde{h}_{123}^2}{\lambda_3 - \lambda_1} + \frac{\tilde{\lambda}_1 \tilde{\lambda}_2 (\tilde{\lambda}_1 + \tilde{\lambda}_2)}{H(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{3333} &= \frac{\tilde{\lambda}_1 \tilde{\lambda}_2 (\tilde{\lambda}_1 + \tilde{\lambda}_2)}{H(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{1112} &= \frac{-2\tilde{h}_{123}^2}{\lambda_1 - \lambda_2} + \frac{\tilde{\lambda}_2 \tilde{\lambda}_3 (\tilde{\lambda}_2 + \tilde{\lambda}_3)}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{2233} &= \frac{-2\tilde{h}_{123}^2}{\lambda_2 - \lambda_1} + \frac{\tilde{\lambda}_1 \tilde{\lambda}_3 (\tilde{\lambda}_1 + \tilde{\lambda}_3)}{H(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot (\tilde{H} - \lambda)\tilde{\lambda}_1^2, \\
\tilde{h}_{1113} &= \tilde{h}_{2212} = \tilde{h}_{3312} = 0, \\
\tilde{h}_{1113} &= \tilde{h}_{2213} = \tilde{h}_{3313} = 0, \\
\tilde{h}_{1123} &= \tilde{h}_{2223} = \tilde{h}_{3323} = 0.
\end{align*}
\]

(3.48)
From (3.15) and (3.48), we have that

\[
\begin{align*}
2\tilde{h}_{123}^2(\bar{\lambda}_1 - \bar{\lambda}_2) &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_3}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \left( \bar{\lambda}_1^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_2}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \left( \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) - \bar{\lambda}_1^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
= -\bar{\lambda}_1\bar{\lambda}_2(\lambda_1 - \lambda_2), \\
-2\tilde{h}_{123}^2(\lambda_1 - \lambda_3) &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_3}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \left( \bar{\lambda}_3^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) + \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_3}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \left( \bar{\lambda}_3^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) - \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
= -\bar{\lambda}_2\bar{\lambda}_3(\lambda_2 - \lambda_3), \\
2\tilde{h}_{123}^2(\bar{\lambda}_2 - \bar{\lambda}_3) &- \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_1}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \left( \bar{\lambda}_1^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) + \bar{\lambda}_3^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) \right) \\
(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_3) &- \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_1}{H(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \left( \bar{\lambda}_1^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) - \bar{\lambda}_3^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) \right) \\
= -\bar{\lambda}_3(\lambda_3 - \lambda_3),
\end{align*}
\]

and then,

\[
\begin{align*}
2\tilde{h}_{123}^2 \cdot \left( \bar{\lambda}_3(\bar{\lambda}_1 - \bar{\lambda}_2) + \bar{\lambda}_2(\bar{\lambda}_1 - \bar{\lambda}_3) \right) &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_3}{H} \left( \bar{\lambda}_1^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
+ \frac{\bar{\lambda}_2^2\bar{\lambda}_3^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_1^3\bar{\lambda}_2^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)}{\bar{\lambda}_1 - \lambda_3} &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_2}{H} \left( \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) - \bar{\lambda}_1^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
+ \frac{\bar{\lambda}_1^2\bar{\lambda}_3^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_1^3\bar{\lambda}_2^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)}{\bar{\lambda}_3 - \lambda_1} &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_3}{H} \left( \bar{\lambda}_3^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) + \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
+ \frac{\bar{\lambda}_1^2\bar{\lambda}_3^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_3^3\bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2)}{\bar{\lambda}_2 - \lambda_3} &+ \frac{(\bar{\lambda} - \lambda)\bar{\lambda}_3}{H} \left( \bar{\lambda}_3^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2) - \bar{\lambda}_2^3(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) \right) \\
= 0,
\end{align*}
\]

That is,

\[
AX = 0,
\]

where

\[
A = \begin{pmatrix}
\bar{\lambda}_3(\bar{\lambda}_1 - \bar{\lambda}_2) + \bar{\lambda}_2(\bar{\lambda}_1 - \bar{\lambda}_3) & \frac{\bar{\lambda}_1^3\bar{\lambda}_3^2(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_3^3\bar{\lambda}_2^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)}{\bar{\lambda}_1 - \lambda_3} \\
\bar{\lambda}_1(\bar{\lambda}_3 - \bar{\lambda}_2) + \bar{\lambda}_2(\bar{\lambda}_3 - \lambda_1) & \frac{\bar{\lambda}_2^3\bar{\lambda}_3^2(\bar{\lambda}_3^2 - \bar{\lambda}_2^2) + \bar{\lambda}_3^3\bar{\lambda}_2^3(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)}{\bar{\lambda}_3 - \lambda_2}
\end{pmatrix},
\]

and

\[
X = \begin{pmatrix}
2\tilde{h}_{123}^2 \\
\frac{H - \lambda}{\lambda} 
\end{pmatrix},
\]
By a direct calculation, we have

\[
det(A) = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \cdot (2\lambda_1^3 \lambda_2^4 - 2\lambda_1^6 \lambda_2^5 - 2\lambda_1^5 \lambda_2^6 + 2\lambda_1^4 \lambda_2^7 - 2\lambda_1^7 \lambda_2^3 \\
+ 4\lambda_1^6 \lambda_2^5 \lambda_3 - 2\lambda_1^4 \lambda_2^6 \lambda_3 + 4\lambda_1^5 \lambda_2^4 \lambda_3^2 - 4\lambda_1^6 \lambda_2^3 \lambda_3^2 + 2\lambda_1^5 \lambda_2^4 \lambda_3^3 + 2\lambda_1^4 \lambda_2^5 \lambda_3^3 \\
- 4\lambda_1^3 \lambda_2^6 \lambda_3^3 + 4\lambda_1^4 \lambda_2^5 \lambda_3^3 - 2\lambda_1^5 \lambda_2^4 \lambda_3^4 - 4\lambda_1^6 \lambda_2^3 \lambda_3^4 - 2\lambda_1 \lambda_2^7 \lambda_3^4 \\
+ 2\lambda_1^5 \lambda_2^3 + 2\lambda_1^6 \lambda_2^4 + 2\lambda_1^7 \lambda_2^5 + 2\lambda_1^8 \lambda_2^6 - 2\lambda_1^9 \lambda_2^7 + 4\lambda_1^5 \lambda_2 \lambda_3^5 + 2\lambda_1^4 \lambda_2^2 \lambda_3^5 \\
+ 2\lambda_1^5 \lambda_2^2 \lambda_3^5 + 4\lambda_1 \lambda_2 \lambda_3^6 - 2\lambda_1^5 \lambda_3^6 - 4\lambda_1^4 \lambda_2^2 \lambda_3^6 - 4\lambda_1^3 \lambda_2^3 \lambda_3^6 - 4\lambda_1^2 \lambda_2^4 \lambda_3^6 \\
- 2\lambda_1^5 \lambda_3^6 + 2\lambda_1^4 \lambda_2^5 \lambda_3^6 - 2\lambda_1^3 \lambda_2^2 \lambda_3^6 + 4\lambda_1^2 \lambda_2^3 \lambda_3^6 - 2\lambda_1 \lambda_2^4 \lambda_3^6 + 2\lambda_2^5 \lambda_3^6)
= \frac{2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \cdot (\lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_2 \lambda_3 - \lambda_1 \lambda_2^2 \lambda_3 + \lambda_1^2 \lambda_3^2 - \lambda_1 \lambda_2 \lambda_3^2 \\
+ \lambda_2^2 \lambda_3^2) \cdot (\lambda_1^2 \lambda_2^2 - \lambda_1^4 \lambda_3^2 - \lambda_1^2 \lambda_2^5 + \lambda_1^3 \lambda_2^2 + \lambda_2^5 \lambda_3^2 - \lambda_1 \lambda_2^3 \lambda_3^2 - \lambda_1^3 \lambda_3^4 - \lambda_2^4 \lambda_3^4 \\
- \lambda_1 \lambda_2 \lambda_3^4 + \lambda_1^2 \lambda_3^4 + \lambda_2^4 \lambda_3^4)
= \frac{2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \cdot \left((\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1^2 + \lambda_2 \lambda_3 + \lambda_3^2) - (\lambda_1^2 + \lambda_2^2 \\
+ \lambda_3^2)(\lambda_1^2 + \lambda_2 \lambda_3 + \lambda_3^2)\right) \cdot \left((\lambda_1^2 + \lambda_2 \lambda_3)^2 + \lambda_2^2(\lambda_1 + \lambda_3)^2 + \lambda_3^2(\lambda_2 + \lambda_1)^2\right).
\]

When

\[(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1^2 + \lambda_2 \lambda_3 + \lambda_3^2) - (\lambda_1^3 + \lambda_2^3 + \lambda_3^3)(\lambda_1^4 + \lambda_2^4 + \lambda_3^4) \neq 0,
\]
that is

\[Sf_5 - \bar{f}_3 f_4 \neq 0,
\]
we have that the matrix \(A\) is nondegenerate, and then

\[\bar{h}_{123}^2 = 0, \quad \lambda = \bar{H}.
\]
That is,

\[\bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\]
From (2.16) and (2.17) in lemma 2.2 we obtain

\[
\sum_{i,j,k} \bar{h}_{ijk}^2 + S^2 - \lambda f_3 = 0, \\
2 \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{ki} h_{li} + \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{klm} h_{li} + S f_4 - \lambda f_5 = 0.
\]

Specifically,

\[
\sum_{i,j,k} \bar{h}_{ijk}^2 + S^2 - \lambda \bar{f}_3 = 0, \quad S^2 - \lambda \bar{f}_3 = 0,
\]

\[
2 \sum_{i,j,k,l,m} \bar{h}_{ijm} \bar{h}_{jkm} \bar{h}_{kl} \bar{h}_{li} + \sum_{i,j,k,l,m} \bar{h}_{ijm} \bar{h}_{jkm} \bar{h}_{klm} \bar{h}_{li} + S f_4 - \lambda \bar{f}_5 = 0, \quad S f_4 - \lambda \bar{f}_5 = 0,
\]
and then, \(S \bar{f}_5 - \bar{f}_3 f_4 \neq 0\). This contradicts the hypothesis.

When

\[(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2)(\bar{\lambda}_1^5 + \bar{\lambda}_2^5 + \bar{\lambda}_3^5) - (\bar{\lambda}_1^3 + \bar{\lambda}_2^3 + \bar{\lambda}_3^3)(\bar{\lambda}_1^4 + \bar{\lambda}_2^4 + \bar{\lambda}_3^4) = 0,
\]
that is
\[(3.50) \quad S\bar{f}_5 - \bar{f}_3 f_4 = 0.\]

From (2.16) and (2.17) in Lemma 2.2, we have
\[
\sum_{i,j,k} h_{ijk}^2 + S^2 - \lambda f_3 = 0,
\]
\[
2 \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{kl} h_{li} + \sum_{i,j,k,l,m} h_{ijm} h_{jkm} h_{klm} h_{li} + S f_4 - \lambda f_5 = 0.
\]

Thus,
\[
\sum_{i,j,k} \bar{h}_{ijk}^2 + S^2 - \lambda \bar{f}_3 = 0,
\]
\[
2 \sum_{i,j,k,l,m} \bar{h}_{ijm} \bar{h}_{jkm} \bar{h}_{kl} \bar{h}_{li} + \sum_{i,j,k,l,m} \bar{h}_{ijm} \bar{h}_{jkm} \bar{h}_{klm} \bar{h}_{li} + S \bar{f}_4 - \lambda \bar{f}_5 = 0.
\]

Especially,
\[
\bar{h}_{123}^2 = -\frac{1}{6} (S^2 - \lambda \bar{f}_3),
\]
\[
\bar{h}_{123}^2 = -\frac{(S f_4 - \lambda \bar{f}_5)}{H^2 + 3S}.
\]

From (3.51), we obtain
\[
\lambda \left( 6\bar{f}_5 - \bar{f}_3 (H^2 + 3S) \right) = 6S f_4 - S^2 (H^2 + 3S).
\]

Supposing
\[
6\bar{f}_5 - \bar{f}_3 (H^2 + 3S) = 0,
\]
we obtain
\[(3.52) \quad 6f_4 = S(H^2 + 3S).
\]

From Lemma 2.3, we have
\[
f_4 = \frac{4}{3} \bar{H} \bar{f}_3 - \bar{H}^2 S + \frac{1}{6} \bar{H}^4 + \frac{1}{2} S^2,
\]
\[
\bar{f}_5 = \frac{5}{6} \bar{H}^2 \bar{f}_3 + \frac{5}{6} S \bar{f}_3 - \frac{5}{6} \bar{H}^3 S + \frac{1}{6} \bar{H}^5.
\]

From (3.51) and (3.53), we obtain
\[(3.54) \quad 8\bar{H} \bar{f}_3^2 + (\bar{H}^4 - 11\bar{H}^2 S - 2S^2) \bar{f}_3 + 5\bar{H}^3 S^2 - \bar{H}^5 S = 0.
\]

From (3.52) and (3.53), we obtain
\[
8\bar{H} \bar{f}_3 - 7\bar{H}^2 S + \bar{H}^4 = 0,
\]
that is,
\[
(3.55) \quad \bar{f}_3 = \frac{7}{8} \bar{H} S - \frac{1}{8} \bar{H}^3.
\]
From (3.54) and (3.55), we obtain
\[ \bar{H}S(2\bar{H}^4 - 7\bar{H}^2S + 7S^2) = \bar{H}S \left( 2(H^2 - \frac{7}{4}S)^2 + \frac{7}{8}S^2 \right) = 0, \]
which is impossible. Then we have
\[ (3.56) \quad 6\bar{f}_5 - \bar{f}_3(\bar{H}^2 + 3S) \neq 0, \quad \lambda = \frac{6Sf_4 - S^2(\bar{H}^2 + 3S)}{6f_5 - \bar{f}_3(\bar{H}^2 + 3S)}. \]

From (3.51) and (3.56), we obtain
\[ \bar{h}_{123} = -\frac{1}{6}(S^2 - \lambda \bar{f}_3) \]
\[ = -\frac{1}{6} \left( S^2 - \bar{f}_3 \cdot \frac{6Sf_4 - S^2(\bar{H}^2 + 3S)}{6f_5 - \bar{f}_3(\bar{H}^2 + 3S)} \right) \]
\[ = -S \left( \frac{S\bar{f}_5 - f_4\bar{f}_3}{6f_5 - \bar{f}_3(\bar{H}^2 + 3S)} \right) = 0, \]
where \( S\bar{f}_5 - \bar{f}_3f_4 = 0. \)
That is,
\[ \bar{h}_{123} = 0, \quad \bar{h}_{ijk} = 0, \quad \text{for} \quad i, j, k = 1, 2, 3. \]

Supposing \( \bar{H} - \lambda = 0, \) from \( \bar{h}_{123} = 0 \) and (3.48), we obtain
\[ \bar{h}_{ijkl} = 0, \quad \text{for} \quad i, j, k = 1, 2, 3. \]

From (2.18) and (2.19) in lemma 2.3, we have
\[ 0 = \lim_{t \to \infty} \frac{1}{2} \Delta_{-V} \sum_{i,j,k}(h_{ijk})^2(p_t) \]
\[ = \frac{9}{2} \lambda S\bar{h}_{11}\bar{h}_{22}\bar{h}_{33} - \frac{3}{2} \lambda^2 \sum_k (\bar{h}_{22}\bar{h}_{33}\bar{h}_{1k}^2 + \bar{h}_{11}\bar{h}_{33}\bar{h}_{2k}^2 + \bar{h}_{11}\bar{h}_{22}\bar{h}_{3k}^2) \]
\[ = \frac{3}{2} \lambda \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 (3S - \lambda \bar{H}) \]
\[ = \frac{3}{2} \lambda \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 (3S - H^2), \]
where \( \bar{H} - \lambda = 0 \) and \( \bar{h}_{ijk} = 0, \quad \bar{h}_{ijkl} = 0, \quad i, j, k, l = 1, 2, 3. \)
Therefore,
\[ 3S - H^2 = 0, \quad \bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3. \]
This contradicts the hypothesis. We have
\[ \bar{H} - \lambda \neq 0. \]

From \( \bar{h}_{123} = 0 \) and (3.51), we have
\[ (3.57) \quad \lambda = \frac{S^2}{f_3}, \quad \frac{\bar{H} - \lambda}{\bar{H}} = \frac{\bar{H}f_3 - S^2}{\bar{H}f_3}. \]
From $S = \text{constant}$ and (2.16) in Lemma 2.2, we have

\[
2 \sum_{i,j,k} h_{ijk} h_{ijkl} - \lambda \nabla_l f_3 = 0,
\]
\[
2 \sum_{i,j,k} h_{ijkm} h_{ijkl} - \lambda \nabla_m \nabla_l f_3 = 0, \quad \text{for } l, m = 1, 2, 3.
\]

Thus,

\[
\sum_{i,j,k} \bar{h}_{ijk} \bar{h}_{ijklm} + \sum_{i,j,k} \bar{h}_{ijkm} \bar{h}_{ijkl} - \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_m \nabla_l f_3(p_t) = 0, \quad \text{for } l, m = 1, 2, 3.
\]

Especially,

\[
\begin{align*}
\bar{h}_{1111}^2 + 3\bar{h}_{2211}^2 + 3\bar{h}_{3311}^2 - \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_1 \nabla_1 f_3(p_t) &= 0, \\
\bar{h}_{2222}^2 + 3\bar{h}_{1122}^2 + 3\bar{h}_{3322}^2 - \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_2 \nabla_2 f_3(p_t) &= 0, \\
\bar{h}_{3333}^2 + 3\bar{h}_{1133}^2 + 3\bar{h}_{2233}^2 - \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_3 \nabla_3 f_3(p_t) &= 0.
\end{align*}
\]

(3.58)

From $f_4 = \text{constant}$ and (2.21) in Lemma 2.4, we have

\[
\left( \frac{4}{3} \bar{f}_3 - 2S \bar{H} + \frac{2}{3} \bar{H}^3 \right) \bar{H},kl + \frac{4}{3} \bar{H} \lim_{t \to \infty} \nabla_l \nabla_k f_3(p_t) = 0,
\]

and then,

\[
\lim_{t \to \infty} \nabla_l \nabla_k f_3(p_t) = -\frac{\bar{f}_3 - \frac{3}{2} S \bar{H} + \frac{1}{2} \bar{H}^3}{\bar{H}} \cdot \bar{H},kl = -\frac{3\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3}{H} \bar{H}_{kl}.
\]

Therefore,

\[
\begin{align*}
- \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_1 \nabla_1 f_3(p_t) &= \lambda \frac{3\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3}{2H} \bar{H}_{,11} = \frac{3\lambda(\bar{H} - \lambda)}{2H} \bar{\lambda}_1^3 \bar{\lambda}_2 \bar{\lambda}_3, \\
- \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_2 \nabla_2 f_3(p_t) &= \lambda \frac{3\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3}{2H} \bar{H}_{,22} = \frac{3\lambda(\bar{H} - \lambda)}{2H} \bar{\lambda}_1 \bar{\lambda}_2^3 \bar{\lambda}_3, \\
- \frac{1}{2} \lambda \lim_{t \to \infty} \nabla_3 \nabla_3 f_3(p_t) &= \lambda \frac{3\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3}{2H} \bar{H}_{,33} = \frac{3\lambda(\bar{H} - \lambda)}{2H} \bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3^3.
\end{align*}
\]
From $\tilde{h}_{123} = 0$ and (3.48), we have

$$
\begin{align*}
\tilde{h}_{1111} &= \frac{\bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_2^2}{H}, \\
\tilde{h}_{2211} &= \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_1^2}{H}, \\
\tilde{h}_{3311} &= \frac{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_1^2}{H}, \\
\tilde{h}_{2222} &= \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_2^2}{H}, \\
\tilde{h}_{1122} &= \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_3^2}{H}, \\
\tilde{h}_{3322} &= \frac{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_3^2}{H}, \\
\tilde{h}_{3333} &= \frac{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + \bar{\lambda}_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_2^2}{H}, \\
\tilde{h}_{1133} &= \frac{\bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 + \bar{\lambda}_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_2^2}{H}, \\
\tilde{h}_{2233} &= \frac{\bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 + \bar{\lambda}_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \cdot \frac{(\bar{H} - \lambda)\bar{\lambda}_3^2}{H}.
\end{align*}
$$

From (3.57), (3.58), (3.59) and (3.60), we have

$$
\begin{align*}
\tilde{\lambda}_1 (\bar{H}f_3 - S^2) &= \left( \frac{\bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2} + \frac{3\bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2} \right) \frac{H f_3}{2f_3} \\
+ \left( \frac{3\bar{\lambda}_1^2 \bar{\lambda}_2^2 (\bar{\lambda}_1 + \bar{\lambda}_2)^2}{(\lambda_3 - \lambda_1)^2(\lambda_3 - \lambda_2)^2} + \frac{3\bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2} \right) = 0, \\
\tilde{\lambda}_2 (\bar{H}f_3 - S^2) &= \left( \frac{\bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2} + \frac{3\bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2} \right) \frac{H f_3}{2f_3} \\
+ \left( \frac{3\bar{\lambda}_1^2 \bar{\lambda}_2^2 (\bar{\lambda}_1 + \bar{\lambda}_2)^2}{(\lambda_3 - \lambda_1)^2(\lambda_3 - \lambda_2)^2} + \frac{3\bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2} \right) = 0, \\
\tilde{\lambda}_3 (\bar{H}f_3 - S^2) &= \left( \frac{\bar{\lambda}_1^2 \bar{\lambda}_2^2 (\bar{\lambda}_1 + \bar{\lambda}_2)^2}{(\lambda_3 - \lambda_1)^2(\lambda_3 - \lambda_2)^2} + \frac{3\bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2 + \bar{\lambda}_3)^2}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2} \right) \frac{H f_3}{2f_3} \\
+ \left( \frac{3\bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_1 + \bar{\lambda}_3)^2}{(\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2} + \frac{3\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 S^2}{2f_3} \right) = 0.
\end{align*}
$$
And then,
\[
\begin{align*}
2\lambda_1(\bar{H}\bar{f}_3 - S^2) & \cdot \left(\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2\right) \\
+ 3\bar{\lambda}_2\bar{\lambda}_3\bar{H}S^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2(\bar{\lambda}_1 - \bar{\lambda}_3)^2(\bar{\lambda}_2 - \bar{\lambda}_3)^2 &= 0, \\
(1) \\
2\lambda_2(\bar{H}\bar{f}_3 - S^2) & \cdot \left(\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2\right) \\
+ 3\bar{\lambda}_2\bar{\lambda}_3\bar{H}S^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2(\bar{\lambda}_1 - \bar{\lambda}_3)^2(\bar{\lambda}_2 - \bar{\lambda}_3)^2 &= 0, \\
(2) \\
2\lambda_3(\bar{H}\bar{f}_3 - S^2) & \cdot \left(\bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2\right) \\
+ 3\bar{\lambda}_1\bar{\lambda}_2\bar{H}S^2(\bar{\lambda}_1 - \bar{\lambda}_2)^2(\bar{\lambda}_1 - \bar{\lambda}_3)^2(\bar{\lambda}_2 - \bar{\lambda}_3)^2 &= 0. \\
(3)
\end{align*}
\]

By computing $\lambda_1 \times (1) - \bar{\lambda}_2 \times (2)$, we have
\[
2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)(\bar{H}\bar{f}_3 - S^2) \left(3\bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 + 3\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 + 2\bar{\lambda}_1^2\bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_1^2 + \bar{\lambda}_2^2 - 2\bar{\lambda}_3^2)\right) = 0.
\]

Supposing $\bar{H}\bar{f}_3 - S^2 = 0$, from \(3.54\), we obtain
\[
0 = 8\bar{H}\bar{f}_3^2 + (\bar{H}^4 - 11\bar{H}^2S - 2S^2)\bar{f}_3 + 5\bar{H}^3S^2 - \bar{H}^5S
\]
\[
= \frac{S}{\bar{H}}(6S^3 - 11\bar{H}^2S^2 + 6\bar{H}^4S - \bar{H}^6)
\]
\[
= \frac{S}{\bar{H}}(S - \bar{H}^2)(2S - \bar{H}^2)(3S - \bar{H}^2).
\]

From $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, we obtain
\[
H^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 < 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = 3S.
\]

Hence,
\[
H^2 < 3S.
\]

From \(3.62\), we obtain that $S - \bar{H}^2 = 0$ or $2S - \bar{H}^2 = 0$. Besides, for $n = 3$, we have $\bar{f}_3 = \frac{S}{\bar{H}}(3S - \bar{H}^2) + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3$.

When $S - \bar{H}^2 = 0$, we have that
\[
\bar{f}_3 = \frac{S^2}{\bar{H}} = \bar{H}^3,
\]
\[
\bar{f}_3 = \frac{\bar{H}}{2}(3S - \bar{H}^2) + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = \bar{H}^3 + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3.
\]

And then, $\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = 0$. This contradicts the hypothesis.

When $2S - \bar{H}^2 = 0$, we have that
\[
\bar{f}_3 = \frac{S^2}{\bar{H}} = \frac{\bar{H}^3}{4},
\]
\[
\bar{f}_3 = \frac{\bar{H}}{2}(3S - \bar{H}^2) + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = \frac{\bar{H}^3}{4} + 3\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3.
And then, $\bar{\lambda}_1 \bar{\lambda}_2 \bar{\lambda}_3 = 0$. This contradicts the hypothesis. Hence,

$$\bar{H} f_3 - S^2 \neq 0.$$ 

Supposing $\bar{\lambda}_1^2 - \bar{\lambda}_2^2 = 0$, that is $\bar{\lambda}_1 = -\bar{\lambda}_2$.

From (3.50), we obtain

$$0 = S f_5 - f_3 f_4$$

$$= \left( 2 \bar{\lambda}_1^2 + \bar{\lambda}_3^2 \right) \bar{\lambda}_5^2 - \left( 2 \bar{\lambda}_1^4 + \bar{\lambda}_3^4 \right) \bar{\lambda}_3^2$$

$$= 2 \bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2),$$

which implies $\bar{\lambda}_1^2 = \bar{\lambda}_3^2$. Then $\bar{\lambda}_1 = \bar{\lambda}_3$ or $\bar{\lambda}_1 = -\bar{\lambda}_3 = -\bar{\lambda}_2$, which is a contradiction. Hence,

$$0 = S f_5 - f_3 f_4$$

$$= (2 \bar{\lambda}_1^2 + \bar{\lambda}_3^2) \bar{\lambda}_5^2 - (2 \bar{\lambda}_1^4 + \bar{\lambda}_3^4) \bar{\lambda}_3^2$$

$$= 2 \bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2).$$

Similarity, by computing $\bar{\lambda}_2 \times (2) - \bar{\lambda}_3 \times (3)$, we have

$$2 (\bar{\lambda}_2^2 - \bar{\lambda}_3^2) (\bar{H} f_3 - S^2) \left( 3 \bar{\lambda}_1^2 \bar{\lambda}_2^2 (\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 + 3 \bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3 \bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2 \right.$$ 

$$+ 2 \bar{\lambda}_1^2 \bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 + \bar{\lambda}_3^2 - 2 \bar{\lambda}_1^2) = 0,$$

which implies

$$3 \bar{\lambda}_1^2 \bar{\lambda}_2^2 (\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2 + 3 \bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_1^2 - \bar{\lambda}_3^2)^2 + 3 \bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 - \bar{\lambda}_3^2)^2$$

$$+ 2 \bar{\lambda}_1^2 \bar{\lambda}_2^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 + \bar{\lambda}_3^2 - 2 \bar{\lambda}_1^2) = 0.$$ 

From (3.63) and (3.64), we have

$$\bar{\lambda}_1^2 + \bar{\lambda}_2^2 - 2 \bar{\lambda}_3^2 = \bar{\lambda}_2^2 + \bar{\lambda}_3^2 - 2 \bar{\lambda}_1^2.$$ 

That is, $\bar{\lambda}_1 = -\bar{\lambda}_3$.

From (3.50) and $\bar{\lambda}_1 = -\bar{\lambda}_3$, we obtain

$$0 = S f_5 - f_3 f_4$$

$$= (2 \bar{\lambda}_1^2 + \bar{\lambda}_3^2) \bar{\lambda}_5^2 - (2 \bar{\lambda}_1^4 + \bar{\lambda}_3^4) \bar{\lambda}_3^2$$

$$= 2 \bar{\lambda}_1^2 \bar{\lambda}_3^2 (\bar{\lambda}_2^2 - \bar{\lambda}_1^2).$$

Then $\bar{\lambda}_1 = -\bar{\lambda}_3 = -\bar{\lambda}_2$ and $\bar{\lambda}_2 = \bar{\lambda}_3$, which is a contradiction. \qed

**Theorem 3.3.** For a 3-dimensional complete $\lambda$-translator $x : M^3 \to \mathbb{R}^4$ with non-zero constant squared norm $S$ of the second fundamental form and constant $f_4$, where $S = \sum h_{ij}^2$ and $f_4 = \sum h_{ijk}h_{ilj}h_{kli}$, we have either

1. $\lambda^2 = S$ and $\inf H^2 = S$, or
2. $\lambda^2 = 2S$ and $\inf H^2 = 2S$, or
3. $\lambda^2 = 3S$ and $\inf H^2 = 3S$.

**Proof.** We apply the generalized maximum principle for the operator $\Delta_{-V}$ to the function $-H^2$. Thus, there exists a sequence $\{p_t\}$ in $M^3$ such that

$$\lim_{t \to \infty} H^2(p_t) = \inf H^2 = \bar{H}^2, \quad \lim_{t \to \infty} |\nabla H^2(p_t)| = 0, \quad \lim_{t \to \infty} \Delta_{-V} H^2(p_t) \geq 0,$$
that is,
\[
\begin{align*}
\lim_{t \to \infty} H^2(p_t) &= \sup H^2 = \bar{H}^2, \\
\lim_{t \to \infty} |\nabla H^2(p_t)| &= 0, \\
0 &\leq \lim_{t \to \infty} |\nabla H|^2(p_t) + S(\bar{H} - \lambda)\bar{H}.
\end{align*}
\]

By taking the limit and making use of the same assertion as in Theorem 3.1, we can prove \( \inf H^2 > 0 \). Hence, without loss of the generality, we can assume
\[
\lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad \lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}\delta_{ij}, \quad \lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl},
\]
for \( i, j, k, l = 1, 2, 3 \).

By making use of the same assertion as in the proof of the Theorem 3.2, we will discuss three cases.

Case 1: The values of the principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are not equal to each other.

This case does not exist.

Case 2: Two of the values of the principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are equal.

There are two scenarios:

First, one is not zero and the other two are equal to zero. We have
\[
\lambda^2 = \bar{H}^2 = \inf H^2, \quad \lambda^2 = S;
\]

Second, one is zero and the other two are equal and not zero. We have
\[
\lambda^2 = H^2 = \inf H^2, \quad \lambda^2 = 2S;
\]

Case 3: The values of the principal curvature \( \bar{\lambda}_1, \bar{\lambda}_2 \) and \( \bar{\lambda}_3 \) are all equal.

We have
\[
\inf H^2 = (\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3)^2 = 3S.
\]

Since
\[
0 \leq 3S - H^2 \leq \sup(3S - H^2) = 3S - \inf H^2 = 0.
\]

Namely, we obtain \( H \) is constant. Hence, we conclude from (2.15)
\[
\lambda = H, \quad \lambda^2 = 3S.
\]

The proof of Theorem 3.3 is finished.

**Proof of Theorem 1.1.** If \( S = 0 \), we know that \( x : M^3 \to \mathbb{R}_4^1 \) is a space-like affine plane \( \mathbb{R}_3^1 \), not necessarily passing through the origin. If \( S \neq 0 \), from Theorem 3.2 and Theorem 3.3 we have

1. \( \lambda^2 = S \) and \( \sup H^2 = \inf H^2 = S \), or
2. \( \lambda^2 = 2S \) and \( \sup H^2 = \inf H^2 = 2S \), or
3. \( \lambda^2 = 3S \) and \( \sup H^2 = \inf H^2 = 3S \).

It follows that the mean curvature \( H \) and the principal curvature must be a constant. From (1.2) and (2.14), we have
\[
\lambda = H, \quad \langle T, \vec{N} \rangle = 0.
\]

So the nonzero constant vector \( T = T^T \) is tangent to \( x(M^3) \) at each point of \( M^3 \). It follows that \( x(M^3) \) consists of a family of parallel planes in \( \mathbb{R}_4^1 \) and thus, up to an isometry of \( \mathbb{R}_4^1 \), it is a cylinder \( \mathbb{H}^1(a_1) \times \mathbb{R}^2 \) or \( \mathbb{H}^2(a_2) \times \mathbb{R}^1 \) for \( a_1 > 0 \) and...
\( a_2 > 0 \), where \( \mathbb{H}^1(a_1) \) and \( \mathbb{H}^2(a_2) \) are hyperbolic curve and hyperboloid respectively. Besides, the parameters \( a_1 \) and \( a_2 \) can be determined by \( \lambda \) via the defining equation \( (1.2) \). By an easy computation, we have that \( \lambda > 0, a_1 = \frac{1}{\lambda} \) and \( a_2 = \frac{2}{\lambda} \). Theorem 1.1 is proved.

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