Sex-oriented stable matchings of the Marriage Problem with correlated and incomplete information

Guido Caldarelli\textsuperscript{a,1}, Andrea Capocci\textsuperscript{b,2} and Paolo Laureti\textsuperscript{b,3}

\textsuperscript{a}INFM Unità ROMA1 and Dip. di Fisica - Università di Roma ”La Sapienza”, P.le A.Moro 2, 00185 Roma, Italy
\textsuperscript{b}Institut de Physique Théorique, Université de Fribourg, Perolles CH-1700, Switzerland

Abstract

In the Stable Marriage Problem two sets of agents must be paired according to mutual preferences, which may happen to conflict. We present two generalizations of its sex-oriented version, aiming to take into account correlations between the preferences of agents and costly information. Their effects are investigated both numerically and analytically.

Key words: Stable Marriage, Bounded Rationality, Game Theory

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Game theory [1] has become a fruitful source of inspiration for both physicists and economists in recent times. The idea of Nash equilibria [2], as opposed to the minimisation of an Hamiltonian, is a powerful tool to investigate the behaviour of selfish agents aiming to maximise their individual utility. A Nash Equilibrium is a state in which any variation of an agent’s strategy results in a worse performance for herself, the other agents’ strategies being unchanged. A growing effort is being devoted to analyzing situations in which only a subset of possible strategies can be explored, due to the limited capabilities of real agents [3], that is a source of market inefficiency [4]. The Stable Marriage Problem provides a simple and natural environment to represent agents with bounded rationality.

\textsuperscript{1} gcalda@pil.phys.uniroma1.it
\textsuperscript{2} andrea.capocci@unifr.ch
\textsuperscript{3} paolo.laureti@unifr.ch
The Stable Marriage Problem [5] describes a system where two classes of \( N \) players (e.g., men and women) have to be matched pairwise. Each player is been assigned his/her list of preferred partners. In the original model, the lists are drawn at random and are independent from one another. If man \( m \) marries woman \( w \) we attribute him an energy equal to the ranking of \( w \) in \( m \)’s list. This way each player tends to minimise his/her energy, or unhappiness. Let us define the matrices \( f \) (for women) and \( h \) (for men), such that the element \( f(w, m) \) denotes the rank of man \( m \) in \( w \)’s list and \( h(m, w) \) the rank of woman \( w \) in \( m \)’s list. We call stable state a collection of couples \( M = \{(m, w)_i\}_{i=1,...,N} \) where there is no man \( m_i \) and no woman \( w_j \), members of two different couples \((i \text{ and } j)\), who would both prefer to marry each other rather than staying with their respective partners. The energies per person for women (\( f \)) and men (\( h \)) in a given matching are defined by

\[
\epsilon_f = \frac{1}{N} \sum_{w=1}^{N} f(w, m) 
\]

(1)

and

\[
\epsilon_h = \frac{1}{N} \sum_{m=1}^{N} h(m, w),
\]

(2)

and we will consider their averages over many instances of the preference lists.

Many numerical studies are based on the Gale-Shapley Algorithm (GSA) [6]. According to the GSA, the agents of one class propose to marry to the agents of the other class, who judge whether to accept or not. This algorithm finds the stable state with the minimal proponents’ energy. In the man-oriented GSA, man \( m \) makes a proposal to the first woman on his list. If she accepts, they get engaged; if she refuses \( m \) goes on proposing to the next woman on his list. Conversely, woman \( w \) accepts a proposal if she is not engaged or if she is engaged with a man \( m' \) worse ranked (in her list) than the proponent. In this case, \( m' \) restarts proposing to the following women in his list.

When all men have run through their lists and all women are engaged, the algorithm stops and the resulting matching \( M_h \) is stable. Of course, the algorithm could also be run reversing the roles and proponents are always better rewarded. Indeed, it is possible to derive a general relation between the energies of women and men in a stable state. Through a mean-field approach, it has been shown in [7] and [8] that the energy of men is

\[
\epsilon_h = \log N + 0.5772
\]

(3)

and that the energy of women \( \epsilon_f \) can be determined by the relation \( \epsilon_f = N/\epsilon_h \).
which holds for all stable matchings.

A more general case is represented in [9] where the preference lists are correlated in such a way that certain agents rank, on average, higher than others. This amounts to provide them with an intrinsic characteristic, which we shall call beauty. Lists are built as follows: for each man \( m \), woman \( w \) is assigned a score \( \mu_m + U \mu_w \), where \( \mu_m \) and \( \mu_w \) are random variables uniformly distributed in the range \([0, 1]\) and \( U \) is a parameter tuning the effect of beauty. The wish list of player \( m \) results from ordering all the scores thus obtained. Setting \( U = 0 \), one recovers the original uncorrelated model. One observes [9] that, as \( U \) is increased, the average total energy grows. Moreover, the energy gap between proponents and judges decreases, while an inequality is introduced in the energies of ”beautiful” and ”ugly” agents, regardless their role.

The Stable Marriage Problem has been widely studied as a model of economical systems, since it mimics agents maximising their individual utility in a competing environment. As pointed out by behavioural economists [3], the cost of information has to be taken into account. Since agents can exert a finite effort, the exploration of all possible alternatives is bounded. The effects of limited information in matching problems are investigated in a following paper [10] with a more general approach.

In our sex-oriented marriage problem, one can bound the rationality of agents by arranging them on a regular lattice of linear dimension \( L \) and assuming that they only know partners within a given euclidean distance \( d \). Unreachable individuals (who may happen to be the best ranked ones) figure as holes in the wish lists. Under this assumption, each agent only knows a fraction \( \alpha \sim \left( \frac{d}{L} \right)^K \) of the total number of agents, where \( K \) is the lattice dimension.

In a GSA with ”full” lists, the energy of a man equals the number of proposals needed to get married. If lists are ”sparse”, each subsequent proposal results in adding \( \frac{1}{\alpha} \), on average, to his energy. The energy of men, in this case, equals the number of proposals divided by \( \alpha \). If we follow the same procedure of [7], neglecting correlations between the position of the holes, we find

\[
\epsilon_h(\alpha) = \frac{\alpha \epsilon_f(\alpha)}{N[1 - (1 - \frac{\epsilon_f(\alpha)}{N})^\alpha]^2} \tag{4}
\]

where

\[
\epsilon_H(\alpha) = \frac{1}{\alpha}(\log N + 0.5772) \tag{5}
\]

These relations are verified by numerical simulations, as shown in figure 1 and 2.
We investigated both numerically and analytically a generalised Stable Marriage Problem. First we assigned an individual beauty to the agents, which results in correlated preference lists. This correlation smoothes the favorable edge of proponents but introduces an energy gap related to the intrinsic feature of agents. Then, we studied a model where agents have an incomplete information about potential partners. We show that a richer information implies a greater competition, but also a lower average energy.

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Fig. 2. Energy of men and women in stable matchings in a one dimensional model for different values of $\alpha$ plotted on log-log scale. Symbols correspond to simulation data, lines are obtained from eq. (4). The solid line and “plus” symbols refer to $\alpha = 0.6$; the dashed line and triangles refer to $\alpha = 0.7$; the long-dashed line and the squares refer to $\alpha = 0.8$.

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