Point-like mechanical singularity in the method of boundary states

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Abstract. This paper presents an effective technique for solving some spatial problems in the theory of elasticity, particularly for a bounded set of point-like physical singularities. This technique development involves the advanced energy method of boundary states (MBS) for a class of solutions with singularities. A test problem for a single singularity such as the center of expansion complying with the proposed technique calculations reliability has been solved and analyzed. The use of the method of boundary states demonstrates the efficiency of the technique in solving problems with many centers of expansion and concentrated forces in an unbounded medium. The results of problem solving are displayed in a convenient graphical form.

1. Introduction

The theory of elasticity gives special attention to the problems, which solutions satisfy all the defining relations but normalize displacements or stresses to infinitely large values in the singular points. The singularities of the concentrated force and the center of expansion (and contraction) need special consideration.

In the classical theory of elasticity, there is a concept that characterizes the combination of three double forces with zero moments acting along the coordinate axes and characterized by \( P \) value. Usually, this feature is called the center of contraction (figure 1a), and \( P \) with the opposite sign is called the center of expansion (figure 1b). The corresponding point location can be in a cavity inside the body [1, 2].

The concept of the concentrated force is an idealization of a force applied to a point in space, by virtue of the fact that a point is defined as a dimensionless and infinitely small unit of space. This concept is often used to solve various problems in continuum mechanics. The singular point on the boundary, to which \( P \) force is applied, can be represented as a small and limited surface on which the surface force is distributed (figure 1c). Its resultant corresponds to the concentrated force.
Figure 1. Determination of a) the center of contraction; b) the center of expansion; c) the concentrated force.

In the theory of elasticity, some works were devoted to the plane problems in rigid deformable body mechanics complicated by the presence of concentrated forces. [3, 4]. Several works dedicated to the method of boundary states (MBS) development considered the stress-strain state problems in concentrated force presence [5, 6].

The other works solved special problems in the spatial theory of elasticity in an unbounded elastic medium for force point-like singularities [3, 7, 8]. The work [7] solves the problem of concentrated force acting on a spherical body in the three-dimensional case.

Within the framework of the MBS, the concentrated force [9, 10] and the center of expansion were considered by the works of [11]. The singularities were taken into account in the classical (regular) way, with the help of special solutions directly included in the initial basis of internal and boundary states, but small neighborhoods of singular force action were excluded from the elastic body. The approach proposed below avoids artificial complications in describing the shape of the body and seeks solutions not in the class of regular functions but of singular ones. This technique is based on the method of boundary states.

2. Method of boundary states
The fundamental concept of MBS is the condition of the medium, which is a particular solution to the defining equations of the medium, regardless of the conditions determined at the boundary of a body [12]. Defining relations in the mathematical model of a homogeneous elastostatic body are presented in tensor-index notation (a point in the index means differentiation, repetition of indexes means summation) and are enclosed in Cauchy relations

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \],

(1)
generalized Hooke’s law

\[ \sigma_{ij} = \lambda \varepsilon_{ij} + 2\mu \varepsilon_{ij} \],

(2)
equilibrium equation

\[ \sigma_{ij,j} + X^0_i = 0 \],

(3)

where \( u_i \) – displacement vector components, \( \varepsilon_{ij} \) – strain tensor components \( \sigma_{ij} \) – stress tensor components, \( \Theta \) – volumetric deformation \( \delta_{ij} \) – Kronecker symbol, \( \lambda, \mu \) – Lame parameters, \( X^0_i \) – volumetric forces. With fixed values \( \lambda, \mu \) the set of relations (1)-(3) is reduced to a system of Lame’s equations

\[ \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + X^0_i = 0 \].
Their general solution was developed by Papkovic and Neuber and presented in the form of Arzhanykh-Slobodyansky (body forces absence case) for a limited simple-connected body

$$u_i = 4(1-\nu)B_i + x_j B_{i,j} - x_i B_{j,i},$$  \hfill (4)

where $\nu$ – Poisson’s ratio, $B_i$ – component of an arbitrary harmonic vector. General solutions (4) are an effective tool for building the basis of a state space representation for a body with no singular factors [12].

The concept of the medium condition is transformed into the notions of internal $\xi$ and boundary $\gamma$ conditions, if it is the case of a specific body $V$ with a boundary $\partial V$

$$\xi = \{u_i, \varepsilon_{ij}, \sigma_{ij}\}, \quad \gamma = \{u_i \mid_{\partial V}, p_i\},$$  \hfill (5)

where $p_i = \sigma_{ij} \mid_{\partial V} \cdot n$.

A set of all possible conditions $\xi \leftrightarrow \gamma$ forms isomorphic Hilbert spaces of internal $\Xi$ and boundary $\Gamma$ conditions with scalar products,

$$(\xi^{(k)}, \xi^{(m)})_\Xi = \int_V \sigma_{ij}^{(k)} \varepsilon_{ij}^{(m)} \partial V, \quad (\gamma^{(k)}, \gamma^{(m)})_\Gamma = \int_{\partial V} p_i^{(k)} u_i^{(m)} \partial S,$$

which are equal due to the principle of virtual displacements

$$(\xi^{(k)}, \xi^{(m)})_\Xi = (\gamma^{(k)}, \gamma^{(m)})_\Gamma.$$  

After orthogonalization, the attributes of resulting internal and boundary conditions are presented in Fourier series by elements of orthonormal bases

$$u_i = \sum_k c_k u_i^{(k)}, \quad \sigma_{ij} = \sum_k c_k \sigma_{ij}^{(k)}, \quad \varepsilon_{ij} = \sum_k c_k \varepsilon_{ij}^{(k)}, \quad u_i \mid_{\partial V} = \sum_k c_k u_i^{(k)} \mid_{\partial V}, \quad p_i = \sum_k c_k p_i^{(k)}.$$  

Thus, the solution to the primal problems for any linear media and bodies of arbitrary outlines is reduced to elementary calculation of quadratures. The convergence of a series of factors $c_k$ was proven [13].

3. Stating problems for a body with a mechanical singularity

Let the only $L \subset V$ mechanical singularity be localized in $V$ region (figure 2).

The figure indicates:

$\xi$ – a singular internal state from a mechanical singularity;

$\xi_0$ – a regular state that compensates for the disturbance of the boundary conditions (BC) by a mechanical singularity;

$\xi$ – a regular condition caused only by BC on $\partial V$ without considering the mechanical singularity impacts.

Figure 2. Localizing a single mechanical singularity.

Due to the linear properties of states $\xi, \xi_0, \xi$ it is fair to assert that the real state (resulting state) is a superposition of all the above mentioned factors.
\( \xi = \tilde{\xi} + \xi_0 + \tilde{\xi} \).

and due to the homogeneity of these states, we obtain

\( \xi = p_0 \tilde{\xi}_0 + p_0 \xi_0 + \tilde{\xi} \),

where \( p_0 \) is the specified value of the mechanical singularity parameter, \( \tilde{\xi} \) is the singular field from the reference impact, \( \xi_0 \) – is the regular field compensating the reference (single) impact of the mechanical singularity in terms of the perturbation of the boundary conditions by the singular impact.

For the case with a finite set of mechanical singularities \( L, \ldots, L' \) in \( L' \subseteq V \) localization and characterized in \( L' \) values of \( p_j \), in which \( j,k \in \{1,\ldots,J\} \).

From superposition of states we obtain

\[
\xi = \sum_{k=1}^{J} p_0 \left( \xi^{k} + \xi^{k}_0 \right) + \tilde{\xi}.
\]  \( \text{(6)} \)

Superposition of fields causes distortion of the required value of the singularity parameter \( p \) in \( L \) localization. To provide the required value as the singularity parameter, we should use such \( p_0 \), that it is \( p \) that is observed in the resulting state.

Let the linear functional be responsible for estimating the value of \( p \) parameter in \( L \) localization

\( p = F(\xi, L) \).

Then from its application to the resulting field \( \xi \) we get

\[
F(\xi, L) = F \left( p_0 \tilde{\xi}_0 + p_0 \xi_0 + \tilde{\xi} \right) = p_0 \left[ 1 + F\left( \xi_0, L \right) \right] + F \left( \tilde{\xi}, L \right).
\]

For the given boundary conditions and the reference impact in the localization of the singularity, the values of the functionals on the right side of the equation are calculated, and on the left side they must be equal to \( p \). Then it follows from the equation that

\[
p_0 = \frac{p - F(\tilde{\xi}, L)}{1 + F(\xi_0, L)}. \]  \( \text{(7)} \)

By setting \( p_0 \) parameter value in the \( \tilde{\xi} \) special solution, we provide the required value in \( p \) resulting state.

**Note 1.** In case of the center of expansion in \( L \) localization, we can assume \( F(\xi, L) = \frac{R_n}{K} p_n |_{L_c} \),

where \( p_n = \sigma_0 n_j n_j \), \( K = -\frac{4}{3\pi} \).

**Note 2.** In case of \( P \) concentrated force, we denote the neighborhood of \( L \) localization by \( L' \) to estimate it. Then the functional \( F_i(\xi, L) = \int_{L} \sigma_0 n_i ds \) returns \( p_i \) component of \( P \) vector. Each force concentrated in \( L \), localization can be characterized with a set of three singularity parameters \( p_i \).

Let \( L \subseteq V \) be the sets of points of \( V \) region that determine the spatial position of mechanical
singularities (localization of a mechanical singularity). For convenience, we introduce $F(\xi, L^j)$, functional that returns the value of the value corresponding to the selected context parameter for $\xi$ state in $L^j$ localization.

Let $\{L^j\}$ finite set of mechanical singularities be localized in $V \in \mathbb{R}^3$ region with $\partial V$ boundary (figure 3).

The vector of displacement of $M$ observation point in an unbounded elastic medium under the action of $P$ concentrated force at $Q$ source point is formed in accordance with the Kelvin – Somigliana tensor (8), and is determined by the formula from the center of expansion with intensity $q$ (9)

$$u(M, Q) = \frac{1}{16\pi\mu(1-\nu)} R \left[ (3-4\nu) \hat{E} + \frac{\mathbf{R} \cdot \mathbf{R}}{R^2} \right] \times P,$$

$$u(M, Q) = \frac{q}{4\pi\mu(1-\nu)} \frac{R}{R^3},$$

where $R = r_M - r_Q$, $R = |r_M - r_Q|$.

The formation of the sought components of the stress and strain tensors is performed in accordance with relations (1)-(3).

The $L^j$ mechanical singularities set definition domain will be called the localization

$$L = \bigcup_{j=1}^{J} L^j, \quad J = \left|\{L^j\}\right|,$$

where $\left|\cdots\right|$ defines the $\{L^j\}$ finite set cardinality.

In the general case, each $j$ mechanical singularity is characterized with its own $p_j$, scalar parameter, which is significant for it. In $V$ region, in general case, it is conditioned by $\xi_j = p_j\xi_0^j$, singular internal state, where $\xi_0^j$ is a “reference” state corresponding to a single parameter value: $p_j = 1$.

The isomorphic boundary state corresponds to $\xi_j$ state

$$\bar{\xi}_j = p_j\xi_0^j,$$

where $\bar{\xi}_j, \bar{\xi}_0^j$ meet to corresponding $p_j$. The $\bar{\xi}_0^j$ boundary state allows restoring the $\xi_0^j$ regular state.
corresponding to \((-\bar{\gamma}_0^j\)). The amount \(\bar{x}_0^j + \bar{x}_0^j\) compensates for impacts at the border \(\partial V\) to a trivial level.

Let us denote the value of the functional that returns the value of some \(p\) scalar parameter corresponding to \(\bar{x}\) state over \(L^j\) localization by \(F(\bar{x}, L^j)\). In particular, we take the notation

\[\bar{I}_{ij} = F(\bar{x}_0^j, L^j).\]

Hence, due to the “standard” impact for \(\bar{x}_0^j\) state, we obtain

\[\bar{I}_{kk} = F(\bar{x}_0^k, L^k) = 1.\]

The next statements hold for

\[F(\bar{x}, L^j) = p_i I_{ij}, \quad F(\bar{x}, L^k) = p_k.\]

The introduced designations make it possible to solve the question of assigning to special solutions responsible for mechanical singularities, the values of \(p_j\), parameters that provide the required level of \(p_j\) for all possible combinations of fields of mechanical singularities and compensating reactions to them.

To correct \(p_j\) singularity parameter required in \(L^j\) localization, we use \(F(\bar{x}, L^j)\) functional

\[F(\bar{x}, L^j) = \sum_{k=1}^J \left[ F(\bar{x}_0^k, L^j) + F(\bar{x}_0^k, L^j) \right], p_{ij} + F(\bar{x}, L^j),\]

which leads us to the system of linear algebraic equations

\[\sum_{k=1}^J \left( I_{ij} + I_{ij} \right) p_{0}^k = p_j - \bar{p}_j, \quad j \in \{1, \ldots, J\},\]

the solution of which allows us to assign the value of the corresponding parameters in each localization.

4. Test problem for a single mechanical singularity

The MBS approach testing for solving problems complicated by the presence of a physical singularity such as the center of expansion was performed within the first main problem (according to the classification of N.I. Muskhelishvili [3]) for a homogeneous elastostatic medium enclosed inside a hemisphere of \(T\) radius and \(T\) height located along \(z\) axis within \(z \in [-T/2; T/2]\). The center of the extension was placed at the origin. In the calculations, a non-dimensionalization was carried out by means of \(\mu, T\) scales; after the non-dimensionalization \(T = 1\) value was taken.

The nature of convergence of the solution can be concluded from the results of calculations presented in figure 4. Figure 4a graphically displays the Fourier coefficients, where the number of the coefficient is indicated on the horizontal axis and the value of this coefficient on the vertical axis. The convergence of the Fourier series to the solution can be indirectly reflected by the saturation of \(Bessel sum\) shown in figure 4b, where the number of the summed coefficients is indicated along the horizontal axis, and the sum of the square roots of the values of these coefficients along the vertical axis. The nature of saturation is a key indicator when deciding on the size of value \(n\).
The obtained solution reliability’s main criterion is the quadratic residual between the boundary conditions and the on-surface solution results. Figure 5 presents the root-mean-square residual (horizontal axis) dependence on the number of used basis elements (vertical axis) detailed analysis in the form of a bar chart with grouping; the parentheses indicate the maximum degree for independent harmonic polynomials used linearly.

To conclude the approach efficiency, 426 orthonormal elements were used in isomorphic spaces of Ξ internal and Γ boundary states (eleventh order of polynomials).

5. Problems of interaction for concentrated forces in unbounded medium
The first two main problems of the theory of elasticity for a homogeneous elastostatic medium enclosed inside a parallelepiped in equilibrium were solved using the MBS in a non-dimensionalized formulation, within: $x \in [-2;3]$, $y \in [-2;2]$, $z \in [-2;3]$. According to the conditions of the first problem, two oppositely directed concentrated forces with the coordinates of the “source points”: $O_1 \left( \frac{3}{2}, 0, 2 \right)$ and $O_2 \left( -\frac{1}{2}, 0, -2 \right)$ were placed inside this medium. At the point $O_1$ and $O_2$ a concentrated force is applied with the $P_1 = \{-\mu,0,0\}$ and $P_2 = \{\mu,0,0\}$ intensity, respectively (figure 6a).
According to the conditions of the second problem, three concentrated forces are located inside the medium with the coordinates of the “source points”: \( O_1 \left( 0, 1, 1/2 \right) \), \( O_4 \left( -1/2, -\sqrt{3}/4, 1/2 \right) \) and \( O_5 \left( 1/2, -\sqrt{3}/4, 1/2 \right) \). At these points the intensity is set accordingly: \( P_1 = \{ 0, -\mu, 0 \} \), \( P_4 = \{ \mu \sqrt{3}/2, \mu/2, 0 \} \) and \( P_5 = \{-\mu \sqrt{3}/2, \mu/2, 0 \} \) (figure 6b).

![Figure 6. Statement of problem: a) with two forces; b) with three forces.](image)

In the calculation, all faces of the parallelepiped are free of loads. The non-dimensionalized version considers the geometric and physical aspects of the problem. Poisson’s ratio, \( \nu \) is assumed to be 0.25. The concentrated force parameter was taken as \( \mu = 1 \) (the resulting stress and force fields change proportionally at any other value). The parallelepiped size is rather big so that we can model an unbounded medium.

In the case of the first main problem, \( \{ \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(i)}, \ldots \} \in \Xi \) orthonormal basis presence makes it possible to expand the desired condition of the space of \( \Xi \) internal states in series and calculate the Fourier coefficients from the definition of the scalar product along the boundary of the body [14]

\[
c_k = \langle \gamma, \gamma^{(i)} \rangle = \int_{\partial \Omega} p, u^{(i)} dS,
\]

where \( p \) – the components of the surface forces \( u^{(i)} \) – axis \( X_i \) translation from the orthonormal basis of the boundary conditions space \( \Gamma \).

The system of Fourier coefficients is subordinated to Bessel’s inequality (the left part of Bessel’s inequality which correlates Fourier coefficients with the Euclidean norm of a split element)

\[
\sum_{j=1}^{n} c_j^2 \leq \xi_{\Xi}^2 = \gamma_{\Gamma}^2,
\]

where \( n \) is the dimension of the truncated basis.

The quadratic residual of the boundary conditions with the on-surface solution results, which, at \( n = 102 \), was 0.636, can characterize the obtained results’ reliability.

The obtained solution responsible for the stress-strain state has a numerical-analytical form. We do not consider it here due to the visual immensity of the internal state components. For brevity, figure 7 shows the display of the saturation of stress fields in the form of isolines for the first problem in the section of \( z = 2 \), and figure 8-9 does the same for the second problem in the section of \( z = 1/2 \) section.
Figure 7. Isolines of stresses in the section of $z = 2$ for the problem with two forces a) $\sigma_{xx}$; b) $\sigma_{yy}$; c) $\sigma_{zz}$; d) $\sigma_{xy}$; e) $\sigma_{xz}$; f) $\sigma_{yz}$.

In the figures, a darker layer corresponds to more compression. The figures show the values of two adjacent layers, and we can easily calculate the remaining values in this step. The components of the stress tensors within the region have finite values. When approaching the coordinates of the “source points”, they increase and highlight the stress concentration zone, and when moving away from the coordinates, they decrease.

Figure 8. Isolines of stresses in the section of $z = 1/2$ for the problem with three forces a) $\sigma_{xx}$; b) $\sigma_{yy}$. 
6. Problem of interaction of three centers of expansion

The first main problem of the theory of elasticity for a homogeneous elastostatic medium enclosed inside a cube in equilibrium was solved using the MBS in a non-dimensionalized formulation, within:

\[ x \in [-3;3], \; y \in [-3;3], \; z \in [-3;3]. \]

Three centers of expansion with coordinates and intensity were placed inside this medium.

\[ P_1 = P_2 = P_3 = \mu, \]

in the calculation, all faces of the cube are free of loads. The non-dimensionalized version considers the geometric and physical aspects of the problem. Poisson’s ratio \( \nu \), is assumed to be 0.25. The concentrated force parameter was taken as \( \mu = 1 \).

Figure 11 shows the display of the saturation of stress fields in the form of isolines in the section of \( z = 2 \).
Figure 11. Isolines of stresses in the section of $z = 2$ a) $\sigma_{xx}$; b) $\sigma_{yy}$; c) $\sigma_{zz}$; d) $\sigma_{xy}$; e) $\sigma_{xz}$; f) $\sigma_{yz}$.

The conclusions on isolines for this problem solution coincide with those of the solved above problem of interaction for concentrated forces in an unbounded medium. The quadratic residual of the boundary conditions with the on-surface results at $n = 10^2$ was 0.705.

6. Conclusions
The main conclusions on the work done are as follows:

1. A technique of numerical and analytical construction of the stress-strain state for spatial problems of solid mechanics, including a finite set of point-like physical singularities, was developed based on the method of boundary states.

2. The proposed technique has shown its effectiveness for solving specific problems.

3. By means of the MBS, the following problems were solved: a test problem for a single mechanical singularity, a problem of interaction for concentrated forces in an unbounded medium in two versions, and a problem of interaction of three centers of expansion.

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