Support theorem for an SPDE with multiplicative noise driven by a cylindrical Wiener process on the real line

Timur Yastrzhembskiy

Received: 22 February 2019 / Revised: 5 July 2019 / Published online: 18 September 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
We prove a Stroock–Varadhan’s type support theorem for a stochastic partial differential equation on the real line with a noise term driven by a cylindrical Wiener process on $L_2(\mathbb{R})$. The main ingredients of the proof are V. Mackevičius’s approach to support theorem for diffusion processes and N.V. Krylov’s $L_p$-theory of SPDEs.

Keywords SPDE · Stroock–Varadhan’s support theorem · Wong–Zakai approximation · Krylov’s $L_p$-theory of SPDEs

Mathematics Subject Classification 35R60 · 60H15

1 Introduction
Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $(\mathcal{F}_t, t \geq 0)$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all $P$-null sets of $\Omega$. By $\mathcal{P}$ we denote the predictable $\sigma$-field generated by $(\mathcal{F}_t, t \geq 0)$.

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{R}$ be the real line, and $\mathbb{R}_+ = [0, \infty)$. Denote

$$D_x = \frac{\partial}{\partial x}, \quad \partial_t = \frac{\partial}{\partial t}.$$ 

For a function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, the temporal argument is denoted by $t$ (or $\cdot$), and the spatial argument – by $x$ (or $\star$). For a function $f : \mathbb{R} \to \mathbb{R}$, we denote

$$Df(x) = \frac{df}{dx}(x).$$
Let \( W(t), t \in \mathbb{R}_+ \) be an \( \mathcal{F}_t \)-adapted cylindrical Wiener process on \( L_2(\mathbb{R}) \) on the probability space \( (\Omega, \mathcal{F}, P) \) (see Sect. 2 for the definition). We consider the following SPDE:

\[
du(t, x) = [a(t, x)D_x^2\mu(t, x) + b(t, x)D_x\mu(t, x) + f(\mu(t, x))]dt
+ u(t, x)dW(t), \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}. \tag{1.1}
\]

Here, \( a \) and \( b \) are some Hölder space-valued functions, \( a \) is bounded from below by a positive constant, and \( f(\mu, \cdot, \star) \) is a ‘zero-order’ term. We point out that we do not assume continuity in the temporal variable for \( a, b \) and \( f \).

In this paper we adopt N.V. Krylov’s approach to parabolic SPDEs (see [7]), which allows us to treat parabolic SPDEs with relatively mild smoothness assumptions on the coefficients and the initial data. Under certain conditions the Eq. (1.1) has a unique solution \( \mu \) that belongs to some stochastic Banach space \( \mathcal{H}^{1/2-\kappa}_p(T), \ p > 2, \kappa \in (0, 1/2] \) (see Sect. 2), which is a generalization of the parabolic counterpart of the space of Bessel potentials \( \mathfrak{J}^{\gamma}_p(T) \). The other approaches to the regularity theory of SPDEs can be found in [3, 14, 18].

Our goal is to characterize the topological support of the distribution of \( \mu \) in the space \( C^{\gamma}([0, T], H^s_p(\mathbb{R})) \), for some \( \gamma, s \in (0, 1/2) \), \( p > 2 \), where \( H^s_p(\mathbb{R}) \) is the space of Bessel potentials. Let \( \mathcal{H}(T) \) be the set of Borel functions \( h : [0, T] \times \mathbb{R} \to \mathbb{R} \) such that \( \partial_t h \in B([0, T] \times \mathbb{R}) \cap L_2([0, T] \times \mathbb{R}) \), where \( B([0, T] \times \mathbb{R}) \) is the space of bounded Borel functions. In Theorem 2.1 we prove that the support of \( \mu \) coincides with the closure in the aforementioned Hölder–Bessel space of the set \( \mathfrak{R} = \{ \partial h : h \in \mathcal{H}(T) \} \), where \( \partial \mathfrak{R} \) is the unique solution of class \( \mathfrak{J}^{1/2-\kappa}_p(T) \) of the following PDE (see Definition 2.1 and Remark 2.8 (ii)):

\[
\partial_t \nu(t, x) = a(t, x)D_x^2\nu(t, x) + b(t, x)D_x\nu(t, x)
+ f(\nu(t, x)) + \nu(t, x)\partial_t h(t, x), \quad u(0, x) = u_0(x). \tag{1.2}
\]

The support theorem for diffusion processes was first proved by Stroock and Varadhan [15]. A different proof of this result was later given by Mackevičius [11], where the main ingredient was an approximation theorem of a Wong–Zakai type and Girsanov’s theorem (see also [12]). The papers [11, 12], and V. Mackevičius’s proof of the Wong–Zakai theorem for diffusion processes (see [10]) served as an inspiration for this article.

In case of infinitely dimensional stochastic equations the support theorems were established in a number of papers. We will only cite the results related to parabolic SPDEs. Gyöngy [5] adopting methods from [10, 11], proved a support theorem for a linear SPDE on \( \mathbb{R}^d \) with a finite dimensional noise term. In [13, 17] support theorems were proved for SPDEs in a Hilbert space \( H \) with an \( H \)-valued Wiener process. The most relevant result to ours is contained in [1]. In this paper a support theorem was obtained for a one-dimensional nonlinear heat equation on \([0, 1]\) with either Dirichlet or Neumann boundary conditions and with a noise term \( g(\mu(t, x))dW(t) \). Here, \( g \) is a sufficiently smooth function, and \( W \) is a cylindrical Wiener process on \( L_2[0, 1] \).
similar result for a one-dimensional generalized Burgers equation can be found in [2]. In both [1,2] the leading coefficient is equal to 1.

It is well-known that often it is more challenging to work with an SPDE driven by a cylindrical Wiener process on an unbounded domain than on a bounded interval. To the best of this author’s knowledge, there exists only one result in the literature so far that is relevant to characterization of the support of the Eq. (1.1). In [6] a Wong–Zakai type theorem was proved for (1.1) with $a \equiv 1, b \equiv 0 \equiv f$ by means of M. Hairer’s theory of Regularity Structures. The authors showed that the sequence of Wong–Zakai type approximations converges to the unique solution of (1.1) uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}$. However, this result yields only one inclusion in the support theorem.

Let us briefly describe the key steps of the proof of the main theorem of this paper. Our argument is similar to the one used in [5,11]. First, we prove an approximation theorem of a Wong–Zakai type (see Theorem 2.2). We replace $W$ by a ‘finite-dimensional’ approximation $\sum_{k=1}^n \phi_k(x)w^k(t)$ and subtract a Stratonovich type correction term. Here, $\{\phi_k, k \in \mathbb{N}\}$ is the orthonormal basis of $L_2(\mathbb{R})$ consisting of Hermite functions, and $\{w^k, k \in \mathbb{N}\}$ is a sequence of independent standard Wiener processes defined by $w^k(\cdot) = (W(\cdot), \phi_k)_{L_2}$, and $w^k_0$ is a polygonal type approximation of $w^k$ (see Sect. 2) with some ‘small’ mesh size. To prove the approximation result we use V. Mackevičius’s method from [10], which we describe below. We split the noise term into two parts: the first one is an integral with respect to a ‘regular’ part $d(w^k_0 - w^k)$, and the second one is a stochastic integral with respect to $dw^k$. Since $w^k_0 - w^k$ converges to 0 (see Lemma 6.1), it makes sense to integrate by parts in the first integral. Then, following I. Gyöngy in [4], we replace the solution of our approximation scheme by its mollification and integrate by parts one more time. As a result, we find a certain SPDE that is satisfied by the ‘error’ of the approximation. We finish the argument by applying N.V. Krylov’s $L_p$-theory of SPDEs. Next, one of the inclusions of the support theorem follows directly from Theorem 2.2 and Portmanteau theorem. The other inclusion is proved by combining Theorem 2.2 with Girsanov’s theorem for cylindrical Wiener process.

This author used the same method to prove a Wong–Zakai theorem and a support theorem for a parabolic SPDE with a finite dimensional semilinear noise term e.g. $g((u(t, x))dw(t)$, where $w(t)$ is a standard Wiener process (see [19]). There are three main differences between [19] and the present article. First, some terms that we obtain as a byproduct of integration by parts are distributions that do not belong to any $L_p$ space, and additional work should be done to handle them. Second, it can be seen from the proof of Theorem 2.2 that we are forced to choose a very small mesh size for $w^k_n$ because the noise is infinite dimensional. Third, our method fails to work if we replace $u(t, x)$ in the noise term by $g(u(t, x))$, where $g$ is a sufficiently smooth function such that $g(0) = 0$. In particular, to do the integration by parts we need the term $\sum_{k=1}^\infty \int_0^t (g(u(s, \bullet))\phi_k(\bullet), \psi(\bullet))_{L_2(\mathbb{R})} dw^k(s)$, $t \geq 0$ to be a semimartingale, for any $\psi \in C_0^\infty(\mathbb{R})$. However, this might not be true, if $u$ is a solution of the Eq. (1.1) with $u(t, x) dW(t)$ replaced by $g(u(t, x)) dW(t)$. Nevertheless, with some additional work one can use the method described above to prove a Wong–Zakai type theorem for a parabolic equation on $\mathbb{R}$ with the noise term $g(u(t, x)) dW(t)$. This will be done somewhere else.
2 Statement of the main results

Let $X$ be some Banach space, and $\xi$ be an $X$-valued random element on $(\Omega, \mathcal{F}, P)$. Then, by $P \circ \xi^{-1}|_X$ we denote the distribution of $\xi$, and by $\text{supp } P \circ \xi^{-1}|_X$ the support of this probability measure.

Let $C^k = C^k(\mathbb{R})$, $k \in \mathbb{N}$ be the space of real-valued bounded $k$ times differentiable functions with bounded derivatives up to order $k$, $C^\infty_0 = C^\infty_0(\mathbb{R})$ be the space of infinitely differentiable functions with compact support. We denote by $C^{k+\alpha} = C^{k+\alpha}(\mathbb{R})$, $k \in \mathbb{N}, \alpha \in (0, 1)$ the Hölder space of bounded functions such that derivatives up to order $k$ belong to $C^\alpha(\mathbb{R})$. For $T > 0$ finite, by $C^\alpha([0, T], X)$ we mean the Hölder space of $X$-valued functions. For $p \in [1, \infty]$, we denote by $L_p = L_p(\mathbb{R}) (L_p([0, T] \times \mathbb{R}))$ the space of real-valued $L_p$-integrable functions. Next, for $p \in (1, \infty)$, we introduce spaces of Bessel potentials as follows:

$$H^\gamma_p := (1 - D_x^2)^{-\gamma/2}L_p, \quad H^\gamma_p(l_2) := (1 - D_x^2)^{-\gamma/2}L_p(l_2).$$

Here, $\gamma \in \mathbb{R}$, and $l_2$ is the set of all sequences of real numbers $h = \{h^k, k \in \mathbb{N}\}$ such that $|h^k|^2 = \sum_{k=1}^\infty |h^k|^2 < \infty$, and $L_p(l_2)$ is the space of sequences $h$ of $L_p$ functions such that $|h|_{l_2} \in L_p$.

For a distribution $f$, and a sequence of distributions $h = \{h^k, k \in \mathbb{N}\}$, we denote

$$||f||_{\gamma, p} := ||(1 - D_x^2)^{\gamma/2}f ||_p, \quad ||h||_{\gamma, p} := |||(1 - D_x^2)^{\gamma/2}h||_{l_2}||_p,$$

where $|| \cdot ||_p$ stands for the $L_p$ norm. For a distribution $f$, and a test function $g \in C^\infty_0$, we denote the action of $f$ on $g$ by $(f, g)$. For any $f, g \in L_2$, their scalar product is denoted by $(f, g)_{L_2}$.

The following facts about spaces $H^\gamma_p, p \in (1, \infty)$, will be used in the sequel sometimes without mentioning them. First, for any $k \in \mathbb{N}$, the spaces $W^k_p$ and $H^k_p$ coincide as sets and have equivalent norms. Here, $W^k_p = W^k_p(\mathbb{R})$ is the Sobolev space of $L_p$ functions such that the generalized derivatives up to order $k$ belong to $L_p$. Second,

$$||f||_{\gamma_1, p} \leq ||f||_{\gamma_2, p}$$

if $\gamma_1 \leq \gamma_2$. Third, if $\gamma \in \mathbb{R}, f \in H^\gamma_p$, and $\psi \in C^\infty_0$, then

$$(f, \psi) = \int \{(1 - D_x^2)^{\gamma/2}f(x)\}[(1 - D_x^2)^{-\gamma/2}\psi(x)] \, dx.$$

The proof of these facts and a detailed discussion of $H^\gamma_p$ spaces can be found in Chapter 13 of [9].

For any stopping time $\tau$, and $\gamma \in \mathbb{R}, p > 1$, we denote $(0, \tau] := \{(\omega, t) : 0 < t \leq \tau(\omega)\}$,

$$L_p(\tau) := L_p((0, \tau], \mathcal{P}, L_p),$$

\(\mathbb{S}\) Springer
By Remark 2.1 of [7], for any number

It was showed in Theorem 3.7 of [7] that, for any probability.

\[ \sum_{\text{tic integrals}} \]

It follows that, for any bounded stopping time \( \tau \), then, for any \( \gamma \in \mathbb{R} \), \( p \geq 2 \), and any stopping time \( \tau \), we write that \( u \in \mathcal{H}_p^\gamma(\tau) \) if the following holds:

1. \( u \) is a distribution-valued process, and \( u \in \cap_{\gamma > 0} \mathbb{H}_p^\gamma(\tau \wedge t) \);
2. \( D_t^2 u \in \mathbb{H}_p^{\gamma - 2}(\tau), u(0, \star) \in L_p(\Omega, \mathcal{F}_0, \mathbb{H}_p^{\gamma - 2/p}) \);
3. There exist \( f \in \mathbb{H}_p^{\gamma - 2}(\tau) \) and \( g = \{g^k, k \in \mathbb{N}\} \in \mathbb{H}_p^{\gamma - 1}(\tau, l_2) \) such that, for any \( \phi \in C_0^\infty, t \geq 0, \omega \in \Omega \),

\[
(u(t \wedge \tau, \star), \phi(\star)) = (u(0, \star), \phi(\star)) + \int_0^{t \wedge \tau} (f(s, \star), \phi(\star)) \, ds \\
+ \sum_{k=1}^\infty \int_0^{t \wedge \tau} (g^k(s, \star), \phi(\star)) \, dw^k(s).
\] (2.1)

The norm is defined in the following way:

\[
||u||_{\mathcal{H}_p^\gamma(\tau)} = ||D_t^2 u||_{\mathbb{H}_p^{\gamma - 2}(\tau)} \\
+ ||f||_{\mathbb{H}_p^{\gamma - 2}(\tau)} + ||g||_{\mathbb{H}_p^{\gamma - 1}(\tau, l_2)} + (E||u(0, \star)||_{\mathbb{H}_p^{\gamma - 2/p, p}})^{1/p}.
\]

For \( u \in \mathcal{H}_p^\gamma(\tau) \), we denote \( \mathbb{D} u := f, \mathbb{S} u := g \).

By \( \mathcal{S}_p^\gamma(T) \) we denote a subset of \( \mathcal{H}_p^\gamma(T) \) of all functions \( u \) such that \( \mathbb{S} u \equiv 0 \), and \( \mathbb{D} u \) and \( u(0, \cdot) \) are functions independent of \( \omega \).

**Remark 2.1** By Remark 3.2 of [7], for any number \( T > 0 \), the series of stochastic integrals \( \sum_{k=1}^\infty \int_0^T (g^k(s, \star), \phi(\star)) \, dw^k(s) \) converges uniformly in \( t \) on \( [0, T] \) in probability.

**Remark 2.2** It was showed in Theorem 3.7 of [7] that, for any \( \gamma \in \mathbb{R}, p \geq 2, \mathcal{H}_p^\gamma(\tau) \) is a Banach space. In addition, by the same theorem if \( T > 0 \) is finite, and \( \tau \leq T \) is a stopping time, then, for any \( v \in \mathcal{H}_p^\gamma(\tau) \),

\[
||v||_{\mathbb{H}_p^\gamma(\tau)} \leq N(d, T)||v||_{\mathcal{H}_p^\gamma(\tau)}.
\]

It follows that, for any bounded stopping time \( \tau \), we may replace \( ||D_t^2 u||_{\mathbb{H}_p^{\gamma - 2}(\tau)} \) by \( ||u||_{\mathbb{H}_p^\gamma(\tau)} \) in the definition of the norm of \( \mathcal{H}_p^\gamma(\tau) \) and obtain an equivalent norm.
2.1 Assumptions

Fix some numbers $T, h > 0, \kappa \in (0, 1/2], p \geq 2$.

(A1) $(\kappa) a(t, x), b(t, x)$ are real-valued $B([0, T] \times \mathbb{R})$-measurable functions. For any $t \in [0, T], a(t, \star) \in C^{1+1/2+\kappa+\eta}, b(t, \star) \in C^{1/2+\kappa+\eta}$, and

$$||a(t, \star)||_{C^{1+1/2+\kappa+\eta}} + ||b(t, \star)||_{C^{1/2+\kappa+\eta}} \leq L,$$

where $L > 0,$ and $\eta \in (0, 1/2 - \kappa)$ are finite. In addition, there exists a constant $\lambda > 0$ such that, for all $t, x,$

$$\lambda \leq a(t, x) \leq \lambda^{-1}.$$

(A2) $(p, \kappa) f(u, t, x)$ is a real-valued function defined on $\mathbb{R} \times [0, T] \times \mathbb{R}$.

(i) For any $u \in \mathbb{R}$, $f(u, t, x)$ is a Borel measurable function.

(ii) There exists a constant $K > 0$ such that, for any $t, x, u, v$, we have

$$|f(u, t, x) - f(v, t, x)| \leq K|u - v|.$$

(iii) $f(0, \cdot, \star) \in L_p([0, T], H_p^{-3/2-\kappa}).$

(A3) $(p, \kappa) u_0 \in H_p^{1/2-\kappa-2/p}.$

(A4) $(h)$ Denote $\varphi(x) = -1 \vee x \wedge 1, x \in \mathbb{R}.$ For each $i \in \mathbb{N}, w^i(\cdot, h)$ is the polygonal type approximation of $w^i$ with mesh size $h$ defined as follows:

$$w^i(t, h) := w^i((l-1)h) + 1/h (t - lh) \varphi(w^i(lh) - w^i((l-1)h))$$

if $t \in [lh, (l+1)h)$, for some $l \in \mathbb{N} \cup \{0\}$. We assume here that $w^i(t) = 0$, for $t \leq 0$. By $Dw^i(\cdot, h)$ we mean a left-continuous modification of the derivative of $w^i(\cdot, h).$ Note that $Dw^i(\cdot, h)$ is a predictable process. If $\{\gamma_n, n \in \mathbb{N}\}$ is a sequence of positive numbers, then, we denote $w^i_n(t) := w^i(t, \gamma_n)$.

Remark 2.3 There are two reasons why $w^i(\cdot, h)$ is constructed in such a way. First, the need to truncate the slopes of the polygonal approximation of $w^i$ comes from our wish to have a unique solution of (2.8) that does not blow up in finite time (see Remark 2.8 (i)). Second, the delay in the definition of $w^i(\cdot, h)$ makes the processes $w^i(\cdot, h), D^i w^i(\cdot, h), \delta w^i(\cdot, h),$ and $s^{ij}(\cdot, h)$ predictable ($\delta w^i(\cdot, h)$ and $s^{ij}(\cdot, h)$ are defined in Sect. 3). Without these processes being predictable, we would not be able to use Krylov’s $L_p$-theory of SPDEs.

2.2 Statement of the main result

We say that $W(t), t \geq 0$ is an $\mathcal{F}_t$-adapted cylindrical Wiener process on $L_2$ on $(\Omega, \mathcal{F}, P)$ if the following holds:

Springer
(i) For every $\psi \in L^2$ such that $||\psi||_2 = 1$, $(W(t), \psi)_{L^2}, t \geq 0$ is an $\mathcal{F}_t$-adapted standard Wiener process;
(ii) For any $t, s \geq 0$, and $\psi, \phi \in L^2$, we have

$$E(W(t), \psi)L^2(W(s), \phi)L^2 = t \wedge s(\psi, \phi)L^2.$$ 

The Eq. (1.1) can be rewritten as follows:

$$du(t, x) = [a(t, x)D^2_x u(t, x) + b(t, x)D_x u(t, x) + f(u, t, x)]dt$$

$$+ \sum_{k=1}^{\infty} u(t, x)\phi_k(x)dw^k(t), \quad u(0, x) = u_0(x),$$

where $\{\phi_k, k \in \mathbb{N}\}$ is the Hermite orthonormal basis of $L^2$, and

$$\{w^k(\cdot) = (W(\cdot), \phi_k)_{L^2}, k \in \mathbb{N}\}$$

is a sequence of independent $\mathcal{F}_t$-adapted standard Wiener processes. Let us recall the construction of the Hermite basis. First, we define the Hermite polynomials as follows:

$$H_k(x) = (-1)^k e^{x^2} D^k(e^{-x^2}).$$

Then, the $k$-th member of the Hermite basis is given by

$$\phi_k(x) = \frac{H_k(x)e^{-x^2/2}}{(\sqrt{\pi}2^k k!)^{1/2}}.$$  

**Remark 2.4** There are two reasons why we chose the Hermite basis of $L^2$. First, we need $\phi_k$ to be sufficiently smooth for each $k \in \mathbb{N}$ (see Lemma 3.3). The second reason is stated in Remark 2.7.

**Definition 2.2** We say that the Eq. (2.3) has a solution $u$ of class $\mathcal{H}_p^\gamma(T)$ if $u \in \mathcal{H}_p^\gamma(T)$ with

$$D u(t, x) = a(t, x)D^2_x u(t, x) + b(t, x)D_x u(t, x) + f(u, t, x),$$

$$S u(t, x) = \{u(t, x)\phi_k(x), k \in \mathbb{N}\}, \quad u(0, x) = u_0(x).$$

Recall that this implies that $D u \in \mathbb{H}_p^{\gamma - 2}(T)$, $S u \in \mathbb{H}_p^{\gamma - 1}(T, l^2)$, and $u_0 \in L_p(\Omega, \mathcal{F}_0, H_{p}^{\gamma - 2/p}).$

Assume that (A1) $(\kappa)$, (A2) $(p, \kappa)$, (A3) $(p, \kappa)$ hold. Then, by Theorem 8.5 of [7] (see Remark 2.5) the Eq. (2.3) has a unique solution $u$ of class $\mathcal{H}_p^{1/2 - \kappa}(T)$. In addition,
there exists a constant \( N(p, \kappa, \eta, L, K, \lambda, T) > 0 \) such that the following estimate holds:

\[
||u||^p_{\mathcal{H}^{1/2-\kappa}}(T) \leq N \int_0^T ||f(0, t, \star)||^p_{-3/2-\kappa, p} dt + N||u_0||^p_{1/2-\kappa; 2/p, p}.
\]

(2.6)

Remark 2.5 The assumption (A2) \((p, \kappa)\) corresponds to Assumption 8.6 of [7], and the assumption (A3) \((p, \kappa)\) is mentioned in the statement of Theorem 8.5 of [7]. However, the assumption (A1) \((\kappa)\) is weaker than Assumption 8.5. Actually, (A1) \((\kappa)\) corresponds to Assumptions 5.3 and 5.5 of [7] with \( n = -3/2 - \kappa \). The conclusion of Theorem 8.5 of [7] still holds in our case, since it follows from Theorem 5.1 (with \( n = -3/2 - \kappa \)) combined with Lemma 8.4 (both are from [7]).

Remark 2.6 By Theorem 7.2 of [7], for any \( v \in \mathcal{H}^{1/2-\kappa}_p(T) \), there exists a modification of \( v \), such that, for any \( \theta \) and \( \mu \) satisfying \( 1 > \mu > \theta > 2/p \), we have \( v \in C^{\theta/2-1/p}([0, T], H^{1/2-\kappa-\mu}_p) \), for all \( \omega \in \Omega \). Moreover, for any stopping time \( \tau \leq T \),

\[
E||v||^p_{C^{\theta/2-1/p}([0, \tau], H^{1/2-\kappa-\mu}_p)} \leq N(p, \theta, \mu, T)||v||^p_{\mathcal{H}^{1/2-\kappa}(\tau)}.
\]

Furthermore, if \( \delta = 1/2 - \kappa - \mu - 1/p > 0 \), then, by the embedding theorem for \( H^s_p \) spaces (see, for example, Theorem 13.8.1 of [9]) we have \( v \in C^{\theta/2-1/p}([0, T], C^{\delta}) \), for all \( \omega \).

Here is the statement of the main result.

Theorem 2.1 Let \( T > 0 \), and let \( p > 2, \kappa \in (0, 1/2) \) be numbers such that \( 1/2 - \kappa > 3/p \). We assume that (A1) \((\kappa)\), (A2) \((p, \kappa)\), (A3) \((p, \kappa)\) hold. Let \( \mathcal{R}_{cl} \) be the closure of \( \mathcal{R} \) (see Sect. 1 for the definition) in the space \( \mathcal{V}(T) := C^{\theta/2-1/p}([0, T], H^{1/2-\kappa}_p) \), where \( \kappa = 1/2 - \kappa - \mu \), and \( \mu \) and \( \theta \) are any numbers such that \( 1/2 - \kappa - 1/p > \mu > \theta > 2/p \). Let \( u \) be the unique solution of (2.3) of class \( \mathcal{H}^{1/2-\kappa}_p(T) \). Then, \( \text{supp} \ P \circ u^{-1}|_{\mathcal{V}(T)} = \mathcal{R}_{cl} \).

To prove the support theorem we need an approximation result that we present below.

For \( \alpha, \beta \in \mathbb{R} \), we consider the following SPDE:

\[
dv(t, x) = [a(t, x)D^2_x v(t, x) + b(t, x)D_x v(t, x) + f(v, t, x)] dt + (\alpha + \beta) \sum_{k=1}^{\infty} v(t, x)\phi_k(x)dw^k(t), \ v(0, x) = u_0(x).
\]

(2.7)

Also, for any sequence \( \{ \gamma_n, n \in \mathbb{N} \} \) such that \( \gamma_n > 0, n \in \mathbb{N} \), we consider the following equation:

\[
dv_n(t, x) = [a(t, x)D^2_x v_n(t, x) + b(t, x)D_x v_n(t, x)
\]

\( \bigstar \) Springer
\[+ f(v_n, t, x) + \alpha \sum_{k=1}^{n} v_n(t, x) \phi_k(x) Dw_n^k(t)\]

\[-(\alpha^2/2 + \alpha \beta) \sum_{k=1}^{n} v_n(t, x) \phi_k^2(x) \] \[dt\]

\[+ \beta \sum_{k=1}^{\infty} v_n(t, x) \phi_k(x) \, dw^k(t), \quad v_n(0, x) = u_0(x).\] (2.8)

The term

\[(\alpha^2/2 + \alpha \beta) \sum_{k=1}^{n} v_n(t, x) \phi_k^2(x) \, dt\]

is akin to the so-called Stratonovich correction term. In fact, if \(\alpha = 1, \beta = 0\), then, it is exactly the Stratonovich correction term of a Wong–Zakai type approximation scheme of the Eq. (2.3) (see Definition 2.3).

Here is the statement of the approximation theorem.

**Theorem 2.2** Accept the conditions of Theorem 2.1. In addition, assume that either \(\alpha = 1, \beta = 0\) or \(\alpha = -1, \beta = 1\), and let \(v\) be the unique solution of class \(\mathcal{H}_p^{1/2-\kappa}(T)\) of the Eq. (2.7). Then, there exists a sequence \(\gamma_n, n \in \mathbb{N}\) such that, if we additionally assume that \((A4)(\gamma_n)\) holds, and let \(v_n\) be the unique solution of class \(\mathcal{H}_p^{1/2-\kappa}(T)\) of (2.8) (see Remark 2.8 (i)), then, we have

\[||v_n - v||_{\mathcal{V}(T)} \to 0\] (2.9)

in probability as \(n \to \infty\).

**Remark 2.7** It turns out that there exists \(\rho > 0\) such that (2.9) holds and that for each \(n, \gamma_n = n^{-\rho}\) (see (4.15)). It can be seen from the argument of Sect. 4 that this follows from the fact that, for any \(p \in [2, \infty), k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}\),

\[||D^m \phi_k||_p \leq N(m, p) k^{m/2}.\]

The last inequality is proved in Lemma 6.2 (see (6.3)). By the way, in [1] in a Wong–Zakai type theorem the authors used an approximation of the Brownian sheet with the temporal mesh size \(a^n\), for some \(a \in (0, 1)\), and the spatial mesh size \(1/n\).

**Definition 2.3** Under the assumptions of Theorem 2.1, we say that (2.8) is a Wong–Zakai type approximation scheme of (2.3) if \(\alpha = 1, \beta = 0\), and \(\gamma_n, n \in \mathbb{N}\) is a sequence such that

\[||v_n - u||_{\mathcal{V}(T)} \to 0\]

in probability as \(n \to \infty\). Here, \(v_n\) is the unique solution of (2.8) of class \(\mathcal{H}_p^{1/2-\kappa}(T)\), and \(u \in \mathcal{H}_p^{1/2-\kappa}(T)\) is the unique solution of (2.3).
Remark 2.8 Assume the conditions of Theorem 2.1 and let $\alpha, \beta \in \mathbb{R}$.

(i) Let $\{\gamma_n, n \in \mathbb{N}\}$ be any sequence such that $\gamma_n > 0, n \in \mathbb{N}$. We claim that the Eq. (2.8) has a unique solution $v_n$ of class $\mathcal{H}^{1/2-\kappa}_{p} (T)$. For $u, x \in \mathbb{R}$, and $t \geq 0$ we set

\[ \begin{align*}
\tilde{f}(u, t, x) &= f(u, t, x) + \alpha u \sum_{k=1}^{n} \phi_k(x) D w_n^k(t) - (\alpha^2/2 + \alpha \beta) u \sum_{k=1}^{n} \phi_k^2(x), \\
h(u, t, x) &= \beta u,
\end{align*} \]

\[ \tilde{K} = \beta + K + \alpha \gamma_n^{-1} \sum_{k=1}^{n} ||\phi_k||_\infty + (\alpha^2/2 + \alpha \beta) \sum_{k=1}^{n} ||\phi_k||_\infty^2. \]

Here and in (ii), $K$ is the constant from (A2) $(p, \kappa)$. Observe that $\tilde{f}(u, t, x)$ and $h(u, t, x)$ satisfy the Assumption 8.6 of [7] with $K = \tilde{K}, \xi \equiv \beta, \kappa = \infty$. This is where we use predictability of $D w_n^j$ and the fact that $|D w_n^j(t)| \leq \gamma_n^{-1}$, for all $i, \omega$ and $t$. Hence, the claim follows from Theorem 8.5 of [7] (see also Remark 2.5).

(ii) For any $h \in \mathcal{H}(T)$, there exists a unique solution $\mathcal{R} h \in S_{\beta}^{1/2-\kappa} (T)$ of (1.2). This time one needs to set

\[ \tilde{f}(u, t, x) = f(u, t, x) + u \partial_t h(t, x), \quad h(u, t, x) = 0, \]

\[ \tilde{K} = K + ||\partial_t h||_{B([0,T] \times \mathbb{R})} \]

and use the argument of (i).

3 Auxiliary results

For $k_1, k_2 \in \mathbb{N}$, and $l_1, l_2 \in \mathbb{N} \cup \{0\}$, we denote

\[ \Phi_{k_1, k_2}^{l_1, l_2} (x) := \phi_{k_1}^{l_1} (x) \phi_{k_2}^{l_2} (x). \]

We set

\[ \begin{align*}
\delta w^j (t, h) &:= w^j (t) - w^j (t, h), \\
\delta^j (t, h) &:= \int_{0}^{t} \delta w^j (r, h) d_r w^j (r, h) - \delta_i t / 2,
\end{align*} \]

and, for any sequence of positive numbers $\{\gamma_n, n \in \mathbb{N}\}$, we denote

\[ \begin{align*}
\delta w_n^j(t) &:= \delta w^j (t, \gamma_n), \\
\delta^j (t, \gamma_n) &:= \delta^j (t, \gamma_n).
\end{align*} \]

Definition 3.1 Let $\{\gamma_n, n \in \mathbb{N}\}$ be a sequence of positive numbers, and $(\alpha, \beta) \in \{(1, 0), (-1, 1)\}$. We say that a function defined on $\Omega \times [0, T] \times \mathbb{R}$ is of type $\Delta_n$ if
it can be represented as

\[ \sum_{i,j=1}^{n} c_{ij} \phi_{l_{ij},m_{ij}}^i(x)q_{ij}(t) + \sum_{i=1}^{n} d_i D^{k_i} \phi_i(x)\delta w^i_n(t), \]

where
- \( c_{ij}, d_i \) are some constants, depending only on \( \alpha \) and \( \beta \), such that \( |c_{ij}|, |d_i| \leq 2 \);
- \( k_i, l_{ij}, m_{ij} \in \{0, 1, 2\} \);
- \( q_{ij} \) is either \( \delta w^i_n \) or \( s_{ij}^n \).

In the sequel we denote any function of type \( \Delta_n \) by \( \Delta_n \) without specifying the exact expression of \( \Delta_n \).

Remark 3.1 It turns out that if \( \gamma_n \) is small, then, a function of type \( \Delta_n \) is small in certain Banach spaces, for example in \( C([0,T] \times \mathbb{R}) \) (see Lemma 6.2).

Lemma 3.1 Let \( \gamma \in \mathbb{R}, p \geq 2, T > 0 \) be numbers, and \( u \in \mathcal{H}^p_\gamma(T), \psi \in C_0^\infty \). Denote \( f = Du, g = \{g^k, k \in \mathbb{N}\} = \mathcal{S}u \). Then, the following assertions hold.

(i) The process \( (u(t, \star), \psi(\star)), t \geq 0 \) is a semimartingale.
(ii) There exists a set \( \Omega' \) of probability 1 such that, for any \( \omega \in \Omega' \), \( t \in [0,T] \), and \( k \in \mathbb{N} \), we have

\[ \langle (u(\cdot, \star), \psi(\star)), w^k(\cdot) \rangle(t) = \int_0^t (g^k(s, \star), \psi(\star)) \, ds, \quad (3.1) \]

where \( \langle \cdot, \cdot \rangle(\cdot) \) stands for the mutual quadratic variation of two real-valued semimartingales.

Proof (i) For any \( \omega \in \Omega \), \( t \in [0,T] \), we have (see Remark 2.1)

\[ (u(t, \star), \psi(\star)) = (u(0, \star), \psi(\star)) + F(t) + G(t), \]

where

\[ F(t) = \int_0^t (f(s, \star), \psi(\star)) \, ds, \]

\[ G(t) = \sum_{k=1}^{\infty} \int_0^t (g^k(s, \star), \psi(\star)) \, dw^k(s). \]

First, we show that \( F \) has a finite variation on \( [0,T] \) a.s. It suffices to prove that

\[ \hat{F} := \int_0^T |(f(s, \star), \psi(\star))| \, ds < \infty \text{ a.s.} \]
Recall that by the definition of stochastic Banach spaces \( f \in \mathcal{H}_p^{\gamma-2}(T) \). By the properties of \( H^s_p \) spaces (see Sect. 2) and Hölder’s inequality we have

\[
\hat{F} \leq T^{1/p'} \|\psi\|_{2-\gamma,p'} \left( \int_0^T \|f(s, \star)\|_{\gamma-2,p}^p ds \right)^{1/p} < \infty \text{ a.s.,}
\]

where \( p' = p/(p-1) \). By this we only need to show that \( G(t), t \geq 0 \) is a martingale.

Next, denote

\[
G_n(t) := \sum_{k=1}^n \int_0^t (g^k(s, \star), \psi(\star)) \, dw^k(s).
\]

We will show that

\[
\lim_{n \to \infty} E \sup_{t \leq T} |G_n(t) - G(t)|^2 = 0, \tag{3.2}
\]

and, by this \( G \) is a square integrable continuous martingale. First, using Burkholder–Davis–Gundy inequality, for any \( n \in \mathbb{N} \), and any \( m \in \mathbb{N} \cup \{\infty\} \) such that \( m \geq n \), we get

\[
E \sup_{t \leq T} \left| \sum_{k=n}^m \int_0^t (g^k(s, \star), \psi(\star)) \, dw^k(s) \right|^2 \leq V_{n,m} := 3E \sum_{k=n}^m \int_0^T |(g^k(s, \star), \psi(\star))|^2 ds.
\]

Second, by repeating the argument of Remark 3.2 of [7] we obtain

\[
V_{n,m} \leq NE \int_0^T \left\| \left( \sum_{k=n}^m |(1 - D_x^2)^{(\gamma-1)/2} g^k(s, \star)|^2 \right)^{1/2} \right\|_p^2 ds \tag{3.3}
\]

where \( N = 3|||1-D_x^2|||_{L^p_{\gamma-1}(T)}^p ||g||_{\mathcal{H}^\gamma_p(T)}^p \). Then, (3.2) holds, and this implies the assertion (i).

(ii) Using linearity of mutual quadratic variation and Itô’s formula, for all \( \omega \), and \( t \in [0, T] \), and \( n \geq k \), we have

\[
\langle G_n, w^k \rangle(t) = \int_0^t (g^k(s, \star), \psi(\star)) \, ds.
\]
Thus, there exists a set $\Omega'$ of probability 1 such that, for any $\omega \in \Omega'$, and every $t \in [0, T]$, $k \in \mathbb{N}$,

$$
\langle G, w^k \rangle (t) = \lim_{n' \to \infty} \langle G_{n'}, w^k \rangle (t) = \int_0^t (g^k(s, \star), \psi(\star)) \, ds,
$$

where the $n'$ is some subsequence. Here, the passage to the limit is justified by Kunita–Watanabe inequality and (3.3).

\[\square\]

The purpose of the next lemma is to desingularize the integral containing $Dw_i^n$ in (2.8). The idea goes back to V. Mackevičius’s proof of Wong–Zakai theorem for stochastic differential equations (see [10]). Let us split the aforementioned integral into an integral with respect to $d(w_i^n - w_i)$ and a stochastic integral with respect to $dw_i$. Note that by Lemma 6.1 the function $w_i^n - w_i$ is small once the mesh size $\gamma_n$ is small. This is why we integrate by parts in the integral with respect to $w_i^n - w_i$.

This yields a decomposition of this integral into a sum of 3 terms. The first term is a sum of the integrals that contain $\delta w_i^n$, and because of that it can be viewed as a small term. The second one is the term containing $Ds_{ij}^n$ which will be further decomposed in Lemma 3.3. The third one is a Stratonovich type correction term which is a multiple of $\sum_{i=1}^n \int_0^t \sum_{j=1}^n u(s, x) \phi_{ij}^2(x) \, ds$.

The last expression can be considered as a divergent term as $n \to \infty$, and this is why we subtract this integral in (2.8).

**Lemma 3.2** Let $p \geq 2$, $T > 0$, $n \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ be numbers. Let $\{\phi_i, i \in \mathbb{N}\}$ be a basis of $L_2$ consisting of infinitely smooth functions and let $f \in H_{\nu-p}^0$, $u_n \in H_p^\nu(T)$.

We take any sequence of positive numbers $\{\gamma_n, n \in \mathbb{N}\}$ and assume that

$$
\mathbb{D}u_n(t, x) = f(t, x) + \alpha \sum_{j=1}^n u_n(t, x) \phi_j(x) Dw_i^n(t),
$$

$$
\mathbb{S}u_n(t, x) = \{\beta u_n(t, x) \phi_j(x), j \in \mathbb{N}\}.
$$

Then, (a.s.) for any $t \geq 0$, and $\psi \in C_0^\infty$,

$$
J := \int_0^t (u_n(s, \star) \phi_i(\star), \psi(\star)) d(w_i^n(s) - w_i(s))
$$

$$
= - (u_n(t, \star) \phi_i(\star), \psi(\star)) \delta w_i^n(t)
$$

$$
+ \int_0^t (f(s, \star) \phi_i(\star), \psi(\star)) \delta w_i^n(s) \, ds
$$

$$
+ (\alpha/2 + \beta) \int_0^t (u_n(s, \star) \phi_i^2(\star), \psi(\star)) \, ds
$$

$$
+ \alpha \sum_{j=1}^n \int_0^t (u_n(s, \star) \phi_{i,j}(\star), \psi(\star)) Ds_{ij}^n(s) \, ds
$$

\[\square\]
\[ +\beta \sum_{j=1}^{\infty} \int_{0}^{t} (u_{n}(s, \bullet) \phi_{i,j}^{1,1} (\bullet), \psi(\bullet)) \delta w_{n}^{i} (s) \, d w^{j} (s). \quad (3.5) \]

**Proof** For the sake of convenience, we omit the dependence of functions on the spatial variable. In the proof several times we multiply some distributions by \( \phi_{i} \). Since \( \phi_{i} \) is infinitely smooth, those products are well-defined distributions.

By Lemma 3.1 (i) the process \((u(t)\phi_{i}, \psi), t \geq 0\) is a semimartingale. Using integration by parts formula for semimartingales, we get

\[ J = -(u(t)\phi_{i}, \psi) \delta w_{n}^{i} (t) + J_{1} + J_{2}, \quad (3.6) \]

where

\[ J_{1} = \int_{0}^{t} \delta w_{n}^{i} (s) \, d (u(s)\phi_{i}, \psi), \]
\[ J_{2} = \langle (u(\cdot)\phi_{i}, \psi), w^{j}(\cdot) \rangle (t). \]

By Lemma 3.1 (ii) the above mutual quadratic variation yields a portion of a Stratonovich type correction term:

\[ J_{2} = \beta \int_{0}^{t} (u(s)\phi_{i}^{2}, \psi) \, ds. \quad (3.7) \]

Next, by the associativity of stochastic integral, we further decompose \( J_{1} \) in the following way:

\[ J_{1} = \sum_{r=1}^{3} J_{1,r}, \quad (3.8) \]

where

\[ J_{1,1} = \int_{0}^{t} (f(s)\phi_{j}, \psi) \delta w_{n}^{j} (s) \, ds, \]
\[ J_{1,2} = \alpha \sum_{j=1}^{n} \int_{0}^{t} (u(s)\phi_{i,j}^{1,1}, \psi) \delta w_{n}^{i} (s) D w_{n}^{j} (s) \, ds, \]
\[ J_{1,3} = \beta \sum_{j=1}^{\infty} \int_{0}^{t} (u(s)\phi_{i,j}^{1,1}, \psi) \delta w_{n}^{i} (s) \, d w^{j} (s). \]

Observe that due to the fact that \( \delta w_{n}^{i} (s) D w_{n}^{j} (s) = D s_{n}^{ij} (t) + \delta_{ij} / 2 \), we may split the term \( J_{1,2} \) into a sum of an integral containing \( D s_{n}^{ij} \) and a portion of a Stratonovich type correction term. We get
\[ J_{1,2} = \alpha \sum_{j=1}^{n} \int_{0}^{t} (u(s)\phi_{i,j}^{1,1}, \psi) D_{s}^{ij}(s) \, ds + \alpha/2 \int_{0}^{t} (u(s)\phi_{i}^{2}, \psi) \, ds. \]

By this this and (3.7)

\[ J_{1,2} + J_{2} = (\alpha/2 + \beta) \int_{0}^{t} (u_{n}(s)\phi_{i}^{2}, \psi) \, ds. \] (3.9)

Now the assertion follows from (3.6), (3.8), (3.9).

Let us give a reasoning behind the lemma presented below. Our initial idea for the proof of Theorem 2.2 was to subtract the Eq. (2.7) from (2.8), use the decomposition from Lemma 3.2 and then apply the a priori estimate from Theorem 5.1 of [7] to the resulting equation satisfied by \( v_{n} - v \). However, the decomposition from Lemma 3.2 has a couple of problems. First, a Stratonovich type correction term and the integral involving a sum of \( D_{s}^{ij} \) are divergent as \( n \to \infty \). Second, there is a non-integral term on the right hand side of (3.5). We address both of these issues. We will show that a Stratonovich type correction term disappears because of our choice of the Eq. (2.8). We also decompose the integral containing \( D_{s}^{ij} \) into a sum of several terms that we can control (see the proof of Theorem 2.2). Finally, the non-integral terms that come from integration by parts force us to construct a new SPDE satisfied by a function \( \bar{v}_{n} \) defined below which can be seen as \( v_{n} - v \) perturbed by a small error.

**Lemma 3.3** Assume the conditions of Theorem 2.2. Take any sequence \( \{\gamma_{n}, n \in \mathbb{N}\} \) with positive terms, and let \( v_{n} \) be the unique solution of class \( \mathcal{H}^{1/2-\chi}(T) \) of (2.8). Let \( h(t, x) = h(\omega, t, x) \) be a function such that, for any \( \omega \in \Omega \), \( h \in C_{loc}^{2}([0, T] \times \mathbb{R}) \). Denote

\[
\xi^{(1)}_{n}(t, x) = \alpha \sum_{j=1}^{n} \delta_{w_{n}^{i,j}(t)}(x), \quad \xi^{(2)}_{n}(t, x) = \alpha^{2} \sum_{i,j=1}^{n} s_{n}^{ij}(t)\phi_{i,j}^{1,1}(x),
\]

\[
\bar{v}_{n}(t, x) = v_{n}(t, x) - v(t, x) + \xi^{(1)}_{n}(t, x) v_{n}(t, x) - \xi^{(2)}_{n}(t, x) h(t, x).
\]

Then, (a.s.) for all \( t \in [0, T] \), \( \psi \in C_{0}^{\infty} \), the function \( \bar{v}_{n} \) satisfies the following equation:

\[
z(t, x) = \int_{0}^{t} ([a(s, \bullet) D_{x}^{2} z(s, \bullet) + b(s, \bullet) D_{x} z(s, \bullet)], \psi(\bullet)) \, ds \\
+ \sum_{k=1}^{8} \int_{0}^{t} (F_{n}^{(k)}(s, \bullet), \psi(\bullet)) \, ds + \sum_{k=1}^{3} \sum_{i=1}^{\infty} \int_{0}^{t} (G_{n,i}^{(k)}(s, \bullet), \psi(\bullet)) \, dw^{i}(s), \quad (3.10)
\]

where

\[
F_{n}^{(1)}(s, x) = a(s, x) D_{x}(v_{n}(s, x) \Delta_{n}(s, x)), \\
F_{n}^{(2)}(s, x) = (a(s, x) \Delta_{n}(s, x) + b(s, x) \Delta_{n}(s, x)) v_{n}(s, x),
\]
we prove the second assertion of the lemma. The functions $\Delta_{n}(s)$ correspond to the ones introduced in the statement of this lemma. Finally, in Step 6 we subtract (2.7) from (2.8) and formally write the ‘stochastic’ part of $\bar{v}_n - v$ as follows:

$$F_n^{(3)}(s, x) = a(s, x)D_x^2(\xi_n^{(2)}(s, x)h(s, x)) + b(s, x)D_x(\xi_n^{(2)}(s, x)h(s, x)), \quad F_n^{(4)}(s, x) = f(v_n, s, x) - f(v, s, x),$$

$$F_n^{(5)}(s, x) = \Delta_{n}(s, x)f(v_n, s, x), \quad F_n^{(6)}(s, x) = \Delta_{n}(s, x)v_n(s, x), \quad F_n^{(7)}(s, x) = \Delta_{n}(s, x)\partial_nh(s, x),$$

$$F_n^{(8)}(s, x) = \sum_{i,j=1}^{n} \alpha^2(v_n(s, x) - h(s, x))\phi_{i,j}^{1,1}(x)D\varsigma_{n}^{ij}(s),$$

$$G_{n,i}^{(1)}(s, x) = (\alpha + \beta)(v_n(s, x) - v(s, x))\phi_i(x), \quad i = 1, \ldots, n,$$

$$G_{n,i}^{(2)}(s, x) = \beta(v_n(s, x) - v(s, x))\phi_i(x), \quad i > n,$$

$$G_{n,i}^{(3)}(s, x) = 0, \quad i = 1, \ldots, n, \quad G_{n,i}^{(4)}(s, x) = -\alpha v(s, x)\phi_i(x), \quad i > n,$$

$$G_{n,i}^{(5)}(s, x) = \Delta_{n}(s, x)v_n(s, x)\phi_i(x), \quad i \in \mathbb{N}.$$  

Here, all the functions $\Delta_{n}$ are possibly different functions of type $\Delta_{n}$. In addition, if $h, \partial_nh \in H_{p}^{-3/2-s}(T)$, then,

$$F_n^{(k)} \in H_{p}^{-3/2-k}(T), \quad k = 1, \ldots, 8, \quad \{G_{n,i}^{(3)}, i \in \mathbb{N}\} \in H_{p}^{-1/2-k}(T, l_2).$$

**Proof** For the sake of convenience, we omit the dependence of functions on the spatial variable. Also, every time a new function of type $\Delta_{n}$ appears, we explicitly write the constants $d_i, c_{ij}, i, j = 1, \ldots, n$ to demonstrate that the first condition of Definition 3.1 holds.

The proof of the first assertion is split into 5 steps. First, we subtract (2.7) from (2.8) and group the terms. Second, we apply Lemma 3.1 and show that a Stratonovich type correction term disappears. Third, we further decompose the term containing $Ds_{n}^{ij}$ which comes from Lemma 3.2. Fourth, we handle the non-integral terms that appeared in steps 2 and 3 as by-products of integration by parts. We add them to $v_n - v$ and obtain the term $\tilde{v}_n$. Then, we rewrite the original SPDE (2.8) as an SPDE satisfied by $\tilde{v}_n$. In Step 5 we group all of the terms from the previous steps and show how they correspond to the ones introduced in the statement of this lemma. Finally, in Step 6 we prove the second assertion of the lemma.

In the proof sometimes we multiply a distribution by the infinitely smooth function $\phi_i$. It is well-known that this operation yields a well-defined distribution.

**Step 1.** We subtract (2.7) from (2.8) and formally write the ‘stochastic’ part of $v_n - v$ as follows:

$$\alpha \sum_{i=1}^{n} v_n(t)\phi_i d w^i_n(t) + \beta \sum_{i=1}^{\infty} v_n(t)\phi_i d w^i(t)$$

$$- (\alpha + \beta) \sum_{i=1}^{\infty} v(t)\phi_i d w^i(t)$$
\[
\begin{align*}
&= \alpha \sum_{i=1}^{n} v_n(t)\phi_i \, d(w_n^i(t) - w^i(t)) \\
&\quad + (\alpha + \beta) \sum_{i=1}^{n} (v_n(t) - v(t))\phi_i \, dw^i(t) \\
&\quad + \beta \sum_{i=n+1}^{\infty} (v_n(t) - v(t))\phi_i \, dw^i(t) \\
&\quad - \alpha \sum_{i=n+1}^{\infty} v(t)\phi_i \, dw^i(t).
\end{align*}
\]

By the above, for any \(\psi \in C_0^\infty, t \in [0, T], \omega\), the function \(v_n - v\) satisfies the following equation:

\[
(v_n(t) - v(t)), \psi) = \sum_{k=1}^{8} I_n^{(k)}(t), \quad (3.11)
\]

where

\[
I_n^{(1)}(t) = \int_0^t (a(s) D_x^2 [v_n(s) - v(s)], \psi) \, ds,
\]

\[
I_n^{(2)}(t) = \int_0^t (b(s) D_x [v_n(s) - v(s)], \psi) \, ds,
\]

\[
I_n^{(3)}(t) = \int_0^t (f(v_n, s) - f(v, s), \psi) \, ds,
\]

\[
I_n^{(4)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (v_n(s)\phi_i, \psi) \, d(w^i(s) - w_n^i(s)),
\]

\[
I_n^{(5)}(t) = (\alpha + \beta) \sum_{i=1}^{n} \int_0^t ([v_n(s) - v(s)]\phi_i, \psi) \, dw^i(s),
\]

\[
I_n^{(6)}(t) = \beta \sum_{i=n+1}^{\infty} \int_0^t ([v_n(s) - v(s)]\phi_i, \psi) \, dw^i(s),
\]

\[
I_n^{(7)}(t) = -\alpha \sum_{i=n+1}^{\infty} \int_0^t (v(s)\phi_i, \psi) \, dw^i(s),
\]

\[
I_n^{(8)}(t) = -(\alpha^2/2 + \alpha \beta) \sum_{i=1}^{n} \int_0^t (v_n(s)\phi_i^2, \psi) \, ds.
\]

In what follows, all the identities hold a.s., for all \(t \in [0, T]\).
Step 2. We set
\[
f(s) = a(s)D_x^2 v_n(s) + b(s)D_x v_n(s) + f(v_n, s) - \left(\alpha^2/2 + \alpha \beta\right) \sum_{i=1}^{n} v_n(s)\phi_i^2
\]
and apply Lemma 3.2 (note that the equality (3.5) must be multiplied by $\alpha$). Then,
\[
I_n^{(4)} = -(\mathbb{E}^{(1)}(s_n(t), \psi) + \sum_{k=1}^{4} I_n^{(4,k)}(t), \quad (3.12)
\]
where
\[
I_n^{(4,1)}(t) = \alpha \sum_{i=1}^{n} \int_{0}^{t} (f(s)\phi_i, \psi) \delta w_n^i(s) \, ds,
\]
\[
I_n^{(4,2)}(t) = \left(\alpha^2/2 + \alpha \beta\right) \sum_{i=1}^{n} \int_{0}^{t} (v_n(s)\phi_i^2, \psi) \, ds,
\]
\[
I_n^{(4,3)}(t) = \alpha^2 \sum_{i=1}^{n} (v_n(s)\phi_i^{1,1}, \psi)Ds_n^{ij} \, ds,
\]
\[
I_n^{(4,4)}(t) = \alpha \beta \sum_{j=1}^{\infty} \int_{0}^{t} \left( \sum_{i=1}^{n} \phi_i^{1,1} \delta w_n^i(s)v_n(s), \psi \right) d w^i(s)
\]
\[= \sum_{j=1}^{\infty} \int_{0}^{t} (\Delta_n(s)v_n(s)\phi_j, \psi) d w^j(s), \quad c_{ij} = 0, d_i = \alpha \beta.
\]
Note that
\[
I_n^{(4,2)}(t) + I_n^{(8)}(t) = 0 \quad (3.13)
\]
which shows that a Stratonovitch type correction term $I_n^{(4,2)}(t)$ disappears because we subtract it on the level of Eq. (2.8).

Step 3. Here, in the integral $I_n^{(4,3)}(t)$ we split $v_n$ into $v_n - h$ and $h$ and integrate by parts in the integral containing $h$. We get
\[
I_n^{(4,3)}(t) = (\mathbb{E}^{(2)}(s_n(t), \psi) + I_n^{(4,3,1)}(t) + I_n^{(4,3,2)}(t), \quad (3.14)
\]
where
\[
I_n^{(4,3,1)}(t) = -\alpha^2 \sum_{i,j=1}^{n} \int_{0}^{t} (\partial_s h(s)\phi_i^{1,1}, \psi)s_n^{ij}(s) \, ds,
\]
\[= \int_{0}^{t} (\Delta_n(s)\partial_s h(s), \psi) \, ds, \quad d_i = 0, c_{ij} = -\alpha^2,
\]
\[ I_n^{(4,3,2)}(t) = \alpha^2 \sum_{i,j=1}^{n} \int_0^t \left( [v_n(s) - h(s)] \phi_{i,j}^{1,1}, \psi \right) Ds_{n}^{ij}(s) \, ds. \]

We note that this time the mutual quadratic variation term vanishes because \( s_{n}^{ij} \) has a locally bounded variation.

**Step 4.** Here we address the issue of the non-integral terms \( -\xi_{n}^{(1)}(t) v_n(t) \) and \( \xi_{n}^{(2)}(t) h(t) \). First, we write

\[ I_n^{(4,1)}(t) = \sum_{k=1}^{4} I_n^{(4,1,k)}(t), \quad (3.15) \]

where

\[ I_n^{(4,1,1)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (a(s) D_x^2 v_n(s) \phi_i, \psi) \delta w_n^i(s) \, ds, \]

\[ I_n^{(4,1,2)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (b(s) D_x v_n(s) \phi_i, \psi) \delta w_n^i(s) \, ds, \]

\[ I_n^{(4,1,3)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (f(v_n,s) \phi_i, \psi) \delta w_n^i(s) \, ds \]

\[ = \int_0^t (\Delta_n(s)f(v_n,s), \psi) \, ds, \quad d_i = \alpha, c_{ij} = 0, \]

\[ I_n^{(4,1,4)}(t) = -\alpha(\alpha^2/2 + \alpha \beta) \sum_{i,j=1}^{n} \int_0^t (v_n(s) \phi_{i,j}^{1,2}, \psi) \delta w_n^i(s) \, ds \]

\[ = \int_0^t (\Delta_n(s)v_n(s), \psi) \, ds, \quad d_i = 0, c_{ij} = -\alpha(\alpha^2/2 + \alpha \beta). \]

Next, it is easy to check that the following identity holds in the sense of distributions:

\[ \phi_i D_x^2 v_n(s) = D_x^2 (v_n(s) \phi_i) - 2D_x (v_n(s) D\phi_i) + v_n(s) D^2 \phi_i. \]

Then, we get

\[ I_n^{(4,1,1)}(t) = \sum_{k=1}^{3} I_n^{(4,1,1,k)}(t), \quad (3.16) \]
where

\[
I_n^{(4,1,1,1)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (a(s) D_x^2 [v_n(s)] v_i, \psi) \delta w_n^i(s) \, ds
\]

\[
= \int_0^t (a(s) D_x^2 [\xi_n^{(1)}(s) v_n(s)], \psi) \, ds,
\]

\[
I_n^{(4,1,1,2)}(t) = -2\alpha \sum_{i=1}^{n} \int_0^t (a(s) D_x [v_n(s) D\phi_i], \psi) \delta w_n^i(s) \, ds
\]

\[
= \int_0^t (a(s) D_x [\Delta_n(s) v_n(s)], \psi) \, ds, \quad d_i = -2\alpha, c_{ij} = 0,
\]

\[
I_n^{(4,1,1,3)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (a(s) v_n(s) D^2 \phi_i, \psi) \delta w_n^i(s) \, ds
\]

\[
= \int_0^t (a(s) \Delta_n(s) v_n(s), \psi) \, ds, \quad d_i = \alpha, c_{ij} = 0.
\]

By the same argument we have

\[
I_n^{(4,1,2)}(t) = I_n^{(4,1,2,1)}(t) + I_n^{(4,1,2,2)}(t)
\]

with

\[
I_n^{(4,1,2,1)}(t) = \alpha \sum_{i=1}^{n} \int_0^t (b(s) D_x [\phi_i v_n(s)], \psi) \delta w_n^i(s) \, ds
\]

\[
= \int_0^t (b(s) D_x [\xi_n^{(1)}(s) v_n(s)], \psi) \, ds,
\]

\[
I_n^{(4,1,2,2)}(t) = -\alpha \sum_{i=1}^{n} \int_0^t (b(s) v_n(s) D\phi_i, \psi) \delta w_n^i(s) \, ds
\]

\[
= \int_0^t (b(s) \Delta_n(s) v_n(s), \psi), \quad d_i = -\alpha, c_{ij} = 0.
\]

Next, by (3.11), (3.16) and (3.20)

\[
I_n^{(1)}(t) + I_n^{(4,1,1,1)}(t) + I_n^{(2)}(t) + I_n^{(4,1,2,1)}(t)
\]

\[
= \int_0^t (a(s) D_x^2 [v_n(s) + \xi_n^{(1)}(s) v_n(s)] + b(s) D_x [v_n(s) + \xi_n^{(1)}(s) v_n(s)], \psi) \, ds.
\]

By adding and subtracting the integral over \(F_n^{(3)}(s, \psi)\) we obtain

\[
I_n^{(1)}(t) + I_n^{(4,1,1,1)}(t) + I_n^{(2)}(t) + I_n^{(4,1,2,1)}(t)
\]

\[
= \int_0^t (a(s) D_x^2 \tilde{v}_n(s) + b(s) D_x \tilde{v}_n(s) + F_n^{(3)}(s, \psi) \, ds.
\]
Step 5. We combine all the terms that we got from the integration by parts as follows:

\[ I_n^{(4,1,1,2)}(t) = \int_0^t (F_n^{(1)}(s), \psi) \, ds \quad (\text{see } (3.18)), \]
\[ I_n^{(4,1,1,3)}(t) + I_n^{(4,1,2,2)}(t) = \int_0^t (F_n^{(2)}(s), \psi) \, ds \quad (\text{see } (3.19) \text{ and } (3.20)), \]
\[ I_n^{(3)}(t) = \int_0^t (F_n^{(4)}(s), \psi) \, ds \quad (\text{see } (3.11)), \]
\[ I_n^{(4,1,3)}(t) = \int_0^t (F_n^{(5)}(s), \psi) \, ds, \quad (\text{see } (3.15)), \]
\[ I_n^{(4,1,4)}(t) = \int_0^t (F_n^{(6)}(s), \psi) \, ds, \quad (\text{see } (3.15)), \]
\[ I_n^{(4,3,1)}(t) = \int_0^t (F_n^{(7)}(s), \psi) \, ds, \quad (\text{see } (3.14)), \]
\[ I_n^{(4,3,2)}(t) = \int_0^t (F_n^{(8)}(s), \psi) \, ds, \quad (\text{see } (3.14)), \]
\[ I_n^{(5)}(t) + I_n^{(6)}(t) = \sum_{i=1}^{\infty} \int_0^t (G_n^{(1)}(s), \psi) \, dw^i(s), \quad (\text{see } (3.11)), \]
\[ I_n^{(7)}(t) = \sum_{i=1}^{\infty} \int_0^t (G_n^{(2)}(s), \psi) \, dw^i(s), \quad (\text{see } (3.11)), \]
\[ I_n^{(4,4)}(t) = \sum_{i=1}^{\infty} \int_0^t (G_n^{(3)}(s), \psi) \, dw^i(s), \quad (\text{see } (3.12)). \]

Note that the terms \( I_n^{(8)} \) and \( I_n^{(4,2)} \) are missing from the above list because they cancelled each other (see (3.13)).

Step 6. Here we prove the second assertion of the lemma. First, note that by Lemma 5.2 (i) of [7], if we have \( \zeta \in H_p^\gamma \), then, for each \( i, \zeta \varphi_i \in H_p^\gamma \) because the derivatives of \( \varphi_i \) of any order are bounded. Second, recall that, since \( \nu_n \in H_p^{1/2-\kappa}(T) \), one has \( Dv_n \in H_p^{-3/2-\kappa}(T) \), and \( S\nu_n \in H_p^{-1/2-\kappa}(T, l_2) \) by the definition of the stochastic Banach space. Combining these two facts together, we prove the claim. \( \square \)

Lemma 3.4 Let \( \alpha \) and \( \bar{\alpha} \) be numbers such that \( 0 < \alpha < \bar{\alpha} < 1 \), and let \( X \) be a Banach space. For \( \theta \in (0, 1) \), and \( t > 0 \), we denote \( V_T^\theta = C^\theta([0, t], X) \). Then, for any \( f \in V_T^\bar{\alpha} \), the function \( t \to ||f||_{V_T^\alpha} \) is continuous on \([0, T]\).

The proof can be found in [19].
4 Proof of Theorem 2.2

The idea of the proof is to apply the a priori estimate from Theorem 5.1 of [7] to the Eq. (3.10) with \( h = \tilde{v}_n \), where \( \tilde{v}_n \) is a suitable mollification of \( v_n \). Then, we examine all the terms and show that the ones containing functions of type \( \Delta_n \) are small. Special care is needed to handle the terms involving \( \tilde{v}_n \). These terms are \( I_{7,n}, I_{8,n}, I_{12,n}, I_{13,n} \). The integral \( I_{13,n} \) is small provided that the mollification parameter is small, and the terms \( I_{7,n}, I_{8,n}, I_{12,n} \) are small once the mesh size \( \gamma_n \) is small enough.

Take any sequence \( \{\gamma_n, n \in \mathbb{N}\} \) of positive numbers, and let \( v_n \) be the unique solution of class \( \mathcal{H}^{1/2-\kappa}_p(T) \) of (2.8), where \( \kappa \) is introduced in the statement of Theorem 2.1. Later, we will choose \( \{\gamma_n, n \in \mathbb{N}\} \) such that the desired convergence holds.

Fix any \( R > 0 \) and denote

\[
\begin{align*}
\mathcal{V}(t) &:= C^{\theta/2-1/p}([0, t], H^{1/2-\kappa-\mu}_p), \\
\mathcal{W}(t) &:= C^{\theta/2-1/p}([0, t], C^{1/2-\kappa-\mu-1/p}), \\
\sigma_n &:= \inf\{t \geq 0 : ||v_n - v||\mathcal{V}(t) \geq 1\}, \\
\pi(R) &:= \inf\{t \geq 0 : ||v||\mathcal{V}(t) \geq R\}, \\
\tau_n &:= \sigma_n \wedge \pi(R) \wedge T,
\end{align*}
\]

where \( \mu \) and \( \theta \) are the numbers from the statement of Theorem 2.1.

Take any \( \tilde{\theta} \) such that \( \mu > \tilde{\theta} > \theta \). By Remark 2.6 we have

\[
v_n, v \in C^{\tilde{\theta}/2-1/p}([0, T], H^{1/2-\kappa-\mu}_p),
\]

for any \( \omega \). Then, by Lemma 3.4 the functions \( t \to ||z||\mathcal{V}(t) \), for \( z = v, v_n - v \), are \( \mathcal{F}_t \)-adapted processes with continuous sample paths. This implies that \( \pi(R), \sigma_n \) and \( \tau_n \) are stopping times.

By the fact that \( 1/2 - \kappa - \mu > 1/p \) and Remark 2.6, for any \( \omega \), we have

\[
\sup_{t \leq \tau_n} ||z(t, \bullet)||_p + ||z||\mathcal{W}(\tau_n) \leq N, \quad z = v, v_n.
\]

In this proof \( N \) is a constant independent of \( n \) that might change from inequality to inequality.

Let \( \rho \) be a \( C^\infty_0 \) function supported on \((0, 1)\) such that \( \int \rho(x) \, dx = 1 \). Let \( v = (\theta/2 - 1/p) \wedge (1/2 - \kappa - \mu - 1/p) \), and \( \varepsilon = 3/v \). We set

\[
\tilde{v}_n(t, x) = n^{2\varepsilon} \int_{[0,1]} v_n(t - s, x - y) I_{t-s > 0} \rho(n^\varepsilon s) \rho(n^\varepsilon y) \, ds \, dy.
\]

By Lemma 6.3 \( \tilde{v}_n(t, \bullet), t \in [0, T] \) is a predictable \( L_p \)-valued function.

\( \square \) Springer
Step 1. We will use Lemma 3.3 with \( h = \tilde{v}_n \). By the a priori estimate from Theorem 5.1 of [7] with \( m := -3/2 - \kappa \) and Lemma 3.3 we have

\[
||\tilde{v}_n||_{H_{p}^{1/2-\kappa}(\tau_n)}^p \leq N \sum_{k=1}^{13} I_{k,n},
\]  

(4.3)

where

\[
I_{j,n} = E \int_{0}^{\tau_n} ||G_n^{(j)}(t, \star)||_{m+1,p}^p dt, \quad j = 1, 2, 3,
\]

\[
I_{4,n} = E \int_{0}^{\tau_n} ||a(t, \star)D_x(\Delta_n(t, \star)v_n(t, \star))||_{m,p}^p dt,
\]

\[
I_{5,n} = E \int_{0}^{\tau_n} ||a(t, \star)\Delta_n(t, \star)v_n(t, \star)||_{m,p}^p dt,
\]

\[
I_{6,n} = E \int_{0}^{\tau_n} ||b(t, \star)\Delta_n(t, \star)v_n(t, \star)||_{m,p}^p dt,
\]

\[
I_{7,n} = E \int_{0}^{\tau_n} ||a(t, \star)D_x^2(\Delta_n(t, \star)\tilde{v}_n(t, \star))||_{m,p}^p dt,
\]

\[
I_{8,n} = E \int_{0}^{\tau_n} ||b(t, \star)D_x(\Delta_n(t, \star)\tilde{v}_n(t, \star))||_{m,p}^p dt,
\]

\[
I_{9,n} = E \int_{0}^{\tau_n} ||f(v_n, t, \star) - f(v, t, \star)||_{m,p}^p dt,
\]

\[
I_{10,n} = E \int_{0}^{\tau_n} ||\Delta_n(t, \star)f(v_n, t, \star)||_{m,p}^p dt,
\]

\[
I_{11,n} = E \int_{0}^{\tau_n} ||\Delta_n(t, \star)v_n(t, \star)||_{m,p}^p dt,
\]

\[
I_{12,n} = E \int_{0}^{\tau_n} ||\Delta_n(t, \star)\partial_t\tilde{v}_n(t, \star)||_{m,p}^p dt.
\]

\[
I_{13,n} = \sum_{i,j=1}^{n} n^{2\kappa - 2} E \int_{0}^{\tau_n} \||(v_n(t, \star) - \tilde{v}_n(t, \star))\phi_{i,j}^{1,1}(\star)||_{m,p}^p |D\delta_i^{(j)}(t)|^p dt.
\]

Here, \( G_n^{(j)} = \{G_n^{(j,i)}(i \in \mathbb{N}), j = 1, 2, 3 \) are the functions defined in the statement of Lemma 3.3, and all \( \Delta_n \) are possibly different functions of type \( \Delta_n \) (see Definition 3.1).

Step 2. First, by the estimate from Lemma 8.4 of [7]

\[
I_{1,n} \leq NE \int_{0}^{\tau_n} ||v_n(t, \star) - v(t, \star)||_{p}^p dt.
\]

(4.4)

Next, let \( \mathcal{K} \) be the function such that

\[
(1 - D_x^2)^{(m+1)/2}z = \mathcal{K} \ast z, \quad \forall z \in L_p,
\]
where $\ast$ stands for convolution. It turns out (see Sect. 12.9 of [9]) that
\[
\mathcal{K}(x) = \chi |x|^{-(1-2\kappa)/2} \int_0^\infty t^{-(5-2\kappa)/4} e^{-tx^2-1/(4t)} dt,
\]
where $\chi > 0$ is a constant. Then, we have
\[
| (1 - D_x^2)^{(m+1)/2} G_n^{(2)}(t, x) |_{L_2}^2 = \alpha^2 \sum_{i=n+1}^{\infty} \left( \int \mathcal{K}(x-y)v(t, y)\phi_i(y) dy \right)^2.
\]
Again, by Lemma 8.4 of [7] and (4.2), for any $\omega$, and $t \in [0, \tau_n]$, we have
\[
||G_n^{(2)}(t, \ast)||_{m+1, p} \leq N ||v(t, \ast)||_p \leq N.
\]
Hence, by the dominated convergence theorem we obtain
\[
\lim_{n \to \infty} I_{2,n} = 0. \quad (4.5)
\]
Using Lemma 8.4 of [7] and (4.2) once more, we get
\[
I_{3,n} \leq E \int_0^{\tau_n} ||\Delta_n(t, \ast)v_n(t, \ast)||_p^p dt \leq E \int_0^{\tau_n} ||\Delta_n(t, \ast)||_p^p dt \leq Nn^N \gamma_{n}^{p/2}, \quad (4.6)
\]
where the last inequality is due to Lemma 6.2 (i).

**Step 3.** We move to the terms $I_{4,n} - I_{6,n}, I_{11,n}$. First, by the second inequality in (4.6)
\[
I_{11,n} \leq Nn^N \gamma_{n}^{p/2}. \quad (4.7)
\]
Next, using Lemma 5.2 (i) of [7], we get
\[
I_{4,n} + I_{5,n} + I_{6,n} \leq NE \int_0^{\tau_n} ||\Delta_n(t, \ast)v_n(t, \ast)||_{m+1, p}^p dt.
\]
We point out that before applying Lemma 5.2 to $I_{6,n}$ one needs to replace $m$ by $m + 1$. Finally, we replace $m + 1$ by 0 in the last inequality and use the second inequality in (4.6). We obtain
\[
I_{4,n} + I_{5,n} + I_{6,n} \leq Nn^N \gamma_{n}^{p/2}. \quad (4.8)
\]
Step 4. We handle the terms $I_{9,n}$ and $I_{10,n}$. Recall that by Remark 2.6 $v_n, v \in C([0, T], L_p)$, for any $\omega$. By this and (A2) $(p, \kappa)(ii)$ we get

$$I_{9,n} \leq NE \int_0^{\tau_n} ||v_n(t, \star) - v(t, \star)||^p_p \, dt. \quad (4.9)$$

Next, by Lemma 5.2 (i) of [7] we have

$$I_{10,n} \leq NEA_n B_n,$$

where

$$A_n = \sup_{t \leq \tau_n} ||\Delta_n(t, \star)||^p_{C^{1+1/2+\kappa+\eta}},$$

$$B_n = \int_0^{\tau_n} ||f(v_n, t, \star)||^p_{m,p} \, dt,$$

and $\eta \in (0, 1/2 - \kappa)$. Fix any $\delta_1 \in (0, p/2)$. Then, by Lemma 6.2 (ii)

$$EA_n \leq Nn^{N\gamma_n^{p/2 - \delta_1}}.$$

Splitting $f(u, t, x)$ into $f(0, t, x)$ and $f(u, t, x) - f(0, t, x)$ and using (A2) $(p, \kappa)$, we get that, for any $\omega$,

$$B_n \leq N \int_0^{\tau_n} (||v_n(t, \star)||^p_p + ||f(0, t, \star)||^p_{m,p}) \, dt \leq N,$$

where the last inequality is due to (4.2). Hence,

$$I_{10,n} \leq Nn^{N\gamma_n^{p/2 - \delta_1}}. \quad (4.10)$$

Step 5. We deal with $I_{7,n}, I_{8,n}, I_{12,n}, I_{13,n}$. First, by Lemma 5.2 (i) of [7]

$$I_{7,n} \leq NE \int_0^{\tau_n} ||\Delta_n(t, \star)\tilde{v}_n(t, \star)||^p_{m+2,p} \, ds,$$

$$I_{8,n} \leq E \int_0^{\tau_n} ||b(t, \star)D_x(\Delta_n(t, \star)\tilde{v}_n(t, \star))||^p_{m+1,p} \, dt$$

$$\leq NE \int_0^{\tau_n} ||\Delta_n(t, \star)\tilde{v}_n(t, \star)||^p_{m+1,p} \, ds. \quad (4.11)$$

Now by Lemma 6.3 combined with (4.2) we get

$$I_{7,n} + I_{8,n} \leq Nn^{N\gamma_n^{2-\delta_1}},$$

and, similarly,

$$I_{12,n} \leq Nn^{N\gamma_n^{2-\delta_1}}. \quad (4.12)$$
Next, note that, for any \( t \in [0, \tau_n] \), and \( s, y \in (0, 1) \), due to (4.2) we have

\[
|v_n(t - s/n^\varepsilon, x - y/n^\varepsilon) - v_n(t, x)| \leq n^{-\varepsilon} ||v_n||_{W(\tau_n)} \leq Nn^{-\varepsilon}.
\]

Then, by this

\[
I_{13,n} \leq n^{2p-2-\varepsilon p} \sum_{i,j=1}^{n} ||\phi_{i,j}^{1,1}||^p \int_0^{\tau_n} |D\sigma^{ij}_n(t)|^p dt.
\]

By Lemma 1.5.2 of [16], for all \( i, j \in \mathbb{N} \),

\[
||\phi_{i,j}^{1,1}||^p \leq N,
\]

and this combined with Lemma 6.1 (iv) yields

\[
I_{13,n} \leq Nn^{(2-\varepsilon p)} = Nn^{-p},
\]

because \( \varepsilon = 3/\nu \).

Step 6. Combining (4.3)–(4.13), we obtain

\[
||\bar{v}_n||^p_{L_p^{1/2-\kappa}(\tau_n)} \leq NI_{2,n} + Nn^{-p} + Nn^N\gamma_n^{p/2-\delta_1}
\]

\[
+ NE \int_0^{\tau_n} ||v(t, \star) - v_n(t, \star)||^p_{V(\tau_n)} dt.
\]

Clearly, the above inequality holds with \( \tau_n \) replaced by \( t \land \tau_n \), for any \( t \in [0, T] \), and with \( N \) independent of \( n \) and \( t \). In addition, by Remark 2.6, for any \( t \in [0, T] \),

\[
E||\bar{v}_n||^p_{V(t \land \tau_n)} \leq N||\bar{v}_n||^p_{L_p^{1/2-\kappa}(t \land \tau_n)}.
\]

By this and the triangle inequality

\[
E||v_n - v||^p_{V(t \land \tau_n)} \leq E||\Delta_n v_n||^p_{V(\tau_n)} + E||\Delta_n \bar{v}_n||^p_{V(\tau_n)}
\]

\[
+ NI_{2,n} + Nn^{-p} + Nn^N\gamma_n^{p/2-\delta_1}
\]

\[
+ NE \int_0^{t \land \tau_n} ||v(s, \star) - v_n(s, \star)||^p_{V(s \land \tau_n)} ds.
\]

Recall that, since \( 1/2 - \kappa - \mu > 0 \), the space \( H_p^{1/2-\kappa-\mu} \) is embedded in \( L_p \). This yields

\[
E \int_0^{t \land \tau_n} ||v(s, \star) - v_n(s, \star)||^p_{V(s \land \tau_n)} ds \leq E \int_0^t ||v(s, \star) - v_n(s, \star)||^p_{V(s \land \tau_n)} ds.
\]
The last integral is well-defined due to Lemma 3.4 combined with (4.1). Combining
the last two inequalities, we get
\[
E\|v_n - v\|^p_{\mathcal{V}(t \wedge \tau_n)} \leq NI_{2,n} + Nn^{-p} + Nn^N \gamma_n^{p/2-\delta_1} + NE\|\Delta_n v_n\|^p_{\mathcal{V}(\tau_n)}
\]
\[
+ NE\|\Delta_n \tilde{v}_n\|^p_{\mathcal{V}(\tau_n)} + NE \int_0^t \|v_n - v\|^p_{\mathcal{V}(s \wedge \tau_n)} ds.
\] (4.14)

Next, by the product rule inequality in Hölder spaces and Cauchy–Schwartz inequal-
ty we have
\[
E\|\Delta_n z\|^p_{\mathcal{V}(\tau_n)} \leq |E\|\Delta_n\|^2_{\mathcal{V}(\tau_n)} E\|z\|^2_{\mathcal{V}(\tau_n)}|^{1/2}, \quad z = v_n, \tilde{v}_n.
\]
Let us fix some \(\delta_2 < (0, 1/2 - \theta/2 + 1/\gamma_1)\). Then, by Lemma 6.2 (iii) we get
\[
E\|\Delta_n\|^2_{\mathcal{V}(\tau_n)} \leq Nn^N \gamma_n^{2\delta_2 p}.
\]

Recall that
\[
\|v_n\|_{\mathcal{V}(\tau_n)} \leq N, \quad \forall \omega.
\]
Using this and Minkowski inequality for Bochner integral, we conclude that
\[
E\|\tilde{v}_n\|^2_{\mathcal{V}(\tau_n)} \leq N.
\]

Next, we combine the estimates from the previous paragraph with (4.14), and we obtain
\[
E\|v_n - v\|^p_{\mathcal{V}(t \wedge \tau_n)} \leq NI_{2,n} + Nn^{-p} + Nn^N (\gamma_n^{p/2-\delta_1} + \gamma_n^{\delta_2 p})
\]
\[
+ N \int_0^t E\|v_n - v\|^p_{\mathcal{V}(s \wedge \tau_n)} ds, \quad t \in [0, T].
\]
By Gronwall’s inequality
\[
E\|v_n - v\|^p_{\mathcal{V}(\tau_n)} \leq NI_{2,n} + Nn^{-p} + Nn^N (\gamma_n^{p/2-\delta_1} + \gamma_n^{\delta_2 p}).
\]
Since \(N\) is independent of \(n\), we can choose \(\{\gamma_n, n \in \mathbb{N}\}\) such that
\[
\lim_{n \to \infty} n^N (\gamma_n^{p/2-\delta_1} + \gamma_n^{\delta_2 p}) = 0.
\] (4.15)

By this and (4.5) we obtain
\[
\lim_{n \to \infty} E\|v_n - v\|^p_{\mathcal{V}(\tau_n)} = 0.
\] (4.16)
Next, denote $D_n = \{\sigma_n < \pi(R) \land T\}$. Observe that by Lemma 3.4, for any $\omega$,

$$||v_n - v||_{\mathcal{V}(\sigma_n)} = 1.$$  

This combined with (4.16) yields

$$P(D_n) = E||v_n - v||_{\mathcal{V}(\tau_n)}^p I_{D_n} \leq E||v_n - v||_{\mathcal{V}(\tau_n)}^p \to 0 \quad (4.17)$$  

as $n \to \infty$.

**Step 7.** Fix any $\varepsilon > 0$. Since $v \in \mathcal{H}_p^{1/2-\kappa}(T)$, by Remark 2.6 there exists $R > 0$ such that

$$P(\pi(R) < T) \leq \varepsilon. \quad (4.18)$$

Next, for any $c > 0$,

$$P(||v_n - v||_{\mathcal{V}(T)} \geq c) \leq P(||v_n - v||_{\mathcal{V}(\tau_n)} \geq c) + P(\pi(R) < T) + P(\sigma_n < \pi(R) \land T).$$

By Chebyshov’s inequality and (4.16)

$$P(||v_n - v||_{\mathcal{V}(\tau_n)} \geq c) \leq c^{-p} E||v_n - v||_{\mathcal{V}(\tau_n)}^p \to 0$$

as $n \to \infty$. By this and (4.17), and (4.18), for any $c, \varepsilon > 0$ we get

$$\lim_{n \to \infty} P(||v_n - v||_{\mathcal{V}(T)} \geq c) \leq \varepsilon,$$

and this finishes the proof of the theorem.

**5 Proof of Theorem 2.1**

We follow the proof of Theorem 2.9 of [19] very closely, making only a few necessary changes.

**Proof of the inclusion** $\text{supp } P \circ u^{-1}|_{\mathcal{V}(T)} \subset \mathcal{R}_{cl}$. By Theorem 2.2 one can choose a sequence $\{\gamma_n, n \in \mathbb{N}\}$ such that if we denote

$$h_n(t, x) = \sum_{k=1}^{n} (w_n^k(t)\phi_k(x) - t/2 \phi_k^2(x)),$$

then

$$||\mathcal{R}(h_n) - u||_{\mathcal{V}(T)} \to 0.$$

© Springer
as \( n \to \infty \) in probability. Indeed, set \( \alpha = 1, \beta = 0 \). For any sequence of positive numbers \( \{\gamma_n, n \in \mathbb{N}\} \), we denote by \( v_n \) the unique solution of class \( \mathcal{H}_p^{1/2-\kappa}(T) \) of (2.8), and by \( v \) – the unique solution of class \( \mathcal{H}_p^{1/2-\kappa}(T) \) of (2.7). Then, we have \( \mathcal{R}(h_n) \equiv v_n, v \equiv u \) as elements of \( \mathcal{H}_p^{1/2-\kappa}(T) \).

Next, by Portmanteau theorem

\[
1 = \lim_{n \to \infty} P(\mathcal{R}(h_n) \in \mathcal{R}_{cl}) \leq P(\mathcal{R}(h_n) \in \mathcal{R}_{cl}) \leq P(\mathcal{R}(h_n) \in \mathcal{R}_{cl}),
\]

and this yields the desired inclusion.

**Proof of the inclusion** \( \mathcal{R}_{cl} \subset \text{supp} P \circ u^{-1} \mid_{\mathcal{V}(T)} \). Fix any \( h \in \mathcal{H}(T) \). For any sequence of positive numbers \( \{\gamma_n, n \in \mathbb{N}\} \), consider the following SPDE:

\[
dz(t, x) = [a(t, x)D_x^2 z(t, x) + b(t, x)D_x z(t, x) + f(z(t, x), t) + z(t, x) \partial_t h(t, x) - \sum_{k=1}^{n} z(t, x) \phi_k(x) D_w^k(t) + 1/2 \sum_{k=1}^{n} z(t, x) \phi_k^2(x)] \ dt + \sum_{k=1}^{\infty} z(t, x) \phi_k(x) \ dW^k(t), \quad z(0, x) = u_0(x). \tag{5.1}
\]

We set \( \alpha = -1, \beta = 1 \) and consider the Eqs. (2.7) and (2.8) with \( f(z, t, x) \) replaced by \( \hat{f}(z, t, x) = f(z, t, x) + z \partial_t h(t, x) \), \( z, x \in \mathbb{R}, t \in \mathbb{R}_+ \).

Note that \( \hat{f}(z, t, x) \) satisfies the assumption (A2) \( (p, \kappa) \) because \( \partial_t h \in B([0, T] \times \mathbb{R}) \). Then, the Eq. (2.7) has a unique solution \( \hat{v} \in \mathcal{L}_p^{1/2-\kappa}(T) \) (see Remark 2.8 (ii)), and (2.8) has a unique solution \( \hat{v}_n \) of class \( \mathcal{H}_p^{1/2-\kappa}(T) \) (see Remark 2.8 (i)). Observe that \( \hat{v}_n \) satisfies the Eq. (5.1). Also note that \( \hat{v} \) solves (1.2), and, hence, \( \hat{v} \equiv \mathcal{R}(h) \). In what follows, \( \{\gamma_n, n \in \mathbb{N}\} \) is such that, for any \( \varepsilon > 0 \), there exists \( N(\varepsilon) > 0 \) such that, for any \( n > N(\varepsilon) \),

\[
P(||\hat{v}_n - \mathcal{R}(h)||_{\mathcal{V}(T)} \leq \varepsilon) > 0. \tag{5.2}
\]

The existence of such sequence follows from Theorem 2.2.

Next, denote

\[
h_n(t, x) := \sum_{k=1}^{n} (w_n^k(t) \phi_k(x) - t/2 \phi_k^2(x)) - h(t, x),
\]

\[
W_n(t) := W(t) - h_n(t, \bullet), \quad \tilde{w}^{k,n}(t) = (W_n(t), \phi_k(\bullet))_{L_2},
\]

\( \odot \) Springer
and let $P_n$ be a measure on $(\Omega, \mathcal{F})$ defined by

$$
d P_n = \exp \left( \int_0^T (\partial_t h_n(t, \star), dW(t))_{L_2} - 1/2 \int_0^T ||\partial_t h_n(t, \star)||_2^2 dt \right) dP.
$$

We claim that

$$
E \exp(1/2 \int_0^T ||\partial_t h_n(t, \star)||_2^2 dt) < \infty.
$$

This easily follows from the fact that $\partial_t h \in B([0, T] \times \mathbb{R})$, and $|Dw_n^k(t)| < \gamma_n^{-1}$, for any $k, n, \omega, t$. Then, by Proposition 10.17 of [3] Girsanov’s theorem is applicable, and, then, $\{\tilde{w}^{k,n}(t), t \in [0, T], k \in \mathbb{N}\}$ is a sequence of independent $\mathcal{F}_t$-adapted standard Wiener processes on $(\Omega, \mathcal{F}, P_n)$.

For $\gamma \in \mathbb{R}$, we set $\mathcal{H}_p^\gamma(T, n)$ to be a stochastic Banach space defined on $(\Omega, \mathcal{F}, P_n)$ with $\{w^k(\cdot), k \in \mathbb{N}\}$ replaced by $\{\tilde{w}^{k,n}(\cdot), k \in \mathbb{N}\}$. Then, $\hat{v}_n$ is a unique solution of class $\mathcal{H}_p^{1/2-k}(T, n)$ of the following SPDE:

$$
dz(t, x) = [a(t, x)D_x^2z(t, x) + b(t, x)D_xz(t, x)
+ f(z, t, x)] dt + \sum_{k=1}^\infty z(t, x)\phi_k(x)dw^{k,n}(t), \quad z(0, x) = u_0(x).
$$

Next, we will prove that

$$
P_n \circ \hat{v}_n^{-1}|_{\mathcal{V}(T)} = P \circ u^{-1}|_{\mathcal{V}(T)}. \tag{5.4}
$$

To show this we use Theorem 2.2. First, there exists a unique solution $g \in \mathcal{H}_p^{1/2-k}(T)$ of $\partial_t g(t, x) = D_x^2g(t, x)$ with the initial condition $u_0(x)$. By subtracting $g$ from $u$, we may assume that $u_0 \equiv 0$. Second, by Theorem 2.2 one may replace the Eqs. (2.3) and (5.3) by their Wong–Zakai type approximation schemes (see Definition 2.3). Each Wong–Zakai approximation is a fixed point of some contraction operator on $\mathcal{H}_p^{1/2-k}(T)$ (see, for instance Theorem 5.1 and Theorem 6.3 of [7]). Now, (5.4) follows from the embedding theorem for $\mathcal{H}_p^{1/2-k}(T)$ (see Remark 2.6) and Picard iteration in the space $\mathcal{V}(T)$.

Finally, we use (5.2), the fact that $P_n$ is absolutely continuous with respect to $P$, and (5.4). We obtain that, for any $\varepsilon > 0$,

$$
P \circ u^{-1}|_{\mathcal{V}(T)}(\{z \in \mathcal{V}(T) : ||z - \mathcal{R}(h)||_{\mathcal{V}(T)} \leq \varepsilon\}) > 0.
$$

This proves the second inclusion.

Acknowledgements This author would like to express his sincere gratitude to his advisor, N.V. Krylov, for reading a draft of this paper and offering valuable suggestions. This author would also like to thank the anonymous referee for comments that led to the improvement of this paper.
6 Appendix

The following lemma is taken from Appendix A of [19].

**Lemma 6.1** Let \( p > 1, \ T > 0, \ h \in (0, 1 \wedge T), \ \varepsilon > 0, \ \theta \in (0, 1/2), \ \theta' \in (0, \theta) \) be numbers. Assume that \((A4)(h)\) holds. Then, for any \( i, \ j \in \mathbb{N}\), the following assertions hold.

\[
\begin{align*}
(i) \quad & \mathbb{E} \left| \delta w^i (\cdot, h) \right|^p_{C[0,T]} \leq N(p, T, \varepsilon) h^{p/2-\varepsilon}.
(ii) \quad & \mathbb{E} \left| \delta w^i (\cdot, h) \right|^p_{C_{1/2-\theta}[0,T]} \leq N(p, T, \theta, \theta') h^\theta p.
(iii) \quad & \mathbb{E} \left| s^{ij} (\cdot, h) \right|^p_{C[0,T]} \leq N(p, T, \varepsilon) h^{p/2-\varepsilon}.
(iv) \quad & \mathbb{E} \int_0^T (|\delta w^i (t, h)|^p + |s^{ij} (t, h)|^p) \, dt \leq N(p, T) h^{p/2},
(v) \quad & \mathbb{E} \int_0^T |Ds^{ij} (t, h)|^p \, dt \leq N(p, T).
\end{align*}
\]

**Proof** The proof of (i)–(iii), (v) and the proof of the second assertion of (iv) can be found in [19] (see Lemma 5.1). Here we show that the first estimate of (iv) holds. To do that, we only need to prove the following:

\[
\sup_{t \leq T} \mathbb{E} (|\delta w^i (t, h)|^p + |s^{ij} (t, h)|^p) \leq N(p, T) h^{p/2}.
\]

The above inequality for \( \delta w^i (\cdot, h) \) follows from the formulas (5.2) and (5.7) of [19].

Next, we prove the claim for \( s^{ij} (\cdot, h) \). First, consider the case \( i = j \). By Itô’s formula

\[
s^{ij} (t, h) = \int_0^t \delta w^i (s, h) \, dw^i (s) - 1/2 |\delta w^i (t, h)|^2,
\]

and, then, by Burkholder–Davis–Gundy inequality

\[
\sup_{t \leq T} \mathbb{E} |s^{ii} (t, h)|^p \leq N(p, T) \mathbb{E} \sup_{t \leq T} (|\delta w^i (t, h)|^p + |\delta w^i (t, h)|^{2p}) \leq N(p, T) h^{p/2}.
\]

The proof in case \( i \neq j \) was actually given in [19] (see formulas (5.14)–(5.16)). \( \Box \)

**Lemma 6.2** Assume that \((A4)(\gamma^i_n)\) holds for some sequence \( \{\gamma^i_n, n \in \mathbb{N}\} \) of positive numbers. Let \( \theta \in (0, 1), \ T > 0, \ p \geq 2 \) be numbers, and let \( \Delta_n \) be any function of class \( \Delta_\infty \). Then, the following assertions hold.

(i) \[
J_n := \mathbb{E} \int_0^T \left| \Delta_n (t, \cdot) \right|^p \, dt \leq N n^N \gamma^i_n h^{p/2},
\]
where \( N = N(p, T) \).

(ii) For any \( \varepsilon > 0 \), and any \( \delta \in (0, 1) \),

\[
E \sup_{t \leq T} \| \Delta_n(t, \ast) \|_{C^{2-\delta}}^p \leq Nn^N \gamma_n^{p/2-\varepsilon},
\]

where \( N = N(p, T, \delta, \varepsilon) \).

(iii) For any \( \varepsilon > 0 \), and \( \varepsilon' \in (0, \varepsilon) \),

\[
E \| \Delta_n \|_{C^{1/2-\varepsilon}([0, T], H^p_\theta)}^p \leq N_n N_{\varepsilon'}^p,
\]

where \( N = N(p, T, \varepsilon, \varepsilon') \).

\textbf{Proof} (i) Due to Definition 3.1 we have

\[
J_n \leq Nn^{2p-2} \sum_{i,j=1}^n \sum_{k,l,m=0}^2 (\| D^k \phi_i \|_p^p + \| \phi_{i,j}^{l,m} \|_p^p) \times E \int_0^T (|s_{ij}^n(t)|^p + |\delta w_n(t)|^p) \, dt.
\]

Next, it is well-known (see Lemma 1.5.2 of [16]) that, for any \( \rho \in [2, \infty] \), and \( k \in \mathbb{N} \),

\[
\| \phi_k \|_\rho \leq N(\rho).
\]

In addition (see Sect. 1.1 of [16]),

\[
DH_k(x) = 2kH_{k-1}(x).
\]

Then, by formula (2.5) and what was just said we have

\[
\| D^k \phi_j \|_\rho \leq N(k, \rho) j^{k/2}, \quad k \in \mathbb{N} \cup \{0\}, \quad j \in \mathbb{N}.
\]

Combining (6.1) and (6.3) with Lemma 6.1 (i v), we prove the assertion.

(ii) The proof is similar the one above. First, note that by the interpolation inequality for Hölder spaces (see Theorem 3.2.1 in [8]), we may replace \( 2 - \delta \) by 3. Second, by the product rule

\[
\| \phi_{i,j}^{l,m} \|_{C^3} \leq N(\| \phi_i \|_{C^l}^l \| \phi_j \|_{C^m}^m).
\]

Third, by Lemma 6.1 (i), (iii), for any \( i, j \in \mathbb{N} \),

\[
E \| q_{ij} \|_{C[0, T]}^p \leq N(p, T, \varepsilon) \gamma_n^{p/2-\varepsilon},
\]

where \( q_{ij} \in \{ \delta w_n^i, s_{ij}^j \} \). Now the assertion follows from (6.4), (6.3), and (6.5).
(iii) First, by the properties of $H^\theta_p$ spaces, for $k, l, m \geq 0$,

$$||D^k\phi_i||_{\theta,p} + ||\phi^{l,m}_{i,j}||_{\theta,p} \leq ||\phi_i||_{k+1,p} + ||\phi^{l,m}_{i,j}||_{1,p} \leq N(p, k)||\phi_i||_{W^{k+1}} + N(p)||\phi^{l,m}_{i,j}||_{W^{l}}. \tag{6.6}$$

Clearly, we may assume that $m \neq 0$. Note that by (6.3) the right hand side of (6.6) is less than

$$N(p, k)i^{(k+1)/2} + N(p)||\phi_i||_{C_1}||\phi_j||_{C_1}^{-1}(||\phi_j||_{p} + ||D\phi_j||_p) \leq N(p, k, l, m)(i^{(k+1)/2} + i^{l/2}j^{m/2}).$$

This combined with Lemma 6.1 (ii) and (v) proves the claim.

$$\square$$

**Lemma 6.3** Let $\varepsilon \in (0, 1)$ be a number. Let $\rho \in C^\infty_0$ be a function supported on $(0, 1)$, and $h \in L^p(T)$. Denote

$$\tilde{h}(t, x) = \varepsilon^{-2} \int_\mathbb{R}^2 h(t - s, x - y)I_{t-s>0}\rho(s/\varepsilon)\rho(y/\varepsilon) \, dsdy.$$

Then, the following assertions hold.

(i) For any $\gamma > 0$, $\tilde{h}(t, \star)$, $t \geq 0$ is a continuous $\mathcal{F}_t$-adapted $H^\gamma_p$-valued process, and, hence, it is a predictable $H^\gamma_p$-valued process.

(ii) Let $T > 0$, $\gamma \in (0, 2)$, $\delta > 0$, $k \in \mathbb{N} \cup \{0\}$ be numbers, and $\tau \leq T$ be a stopping time. Let $\{\gamma_n, n \in \mathbb{N}\}$ be sequence of positive numbers, and $\Delta_n$ be any function of class $\Delta_n$. Then, there exists a constant $N(p, T, \gamma, k, \delta) > 0$ such that

$$I := E \int_0^\tau ||\Delta_n(t, \star)\tilde{h}(t, \star)||^p_{\gamma \gamma_p} dt \leq Nn^\gamma_p/2-\delta \varepsilon^{-(|\gamma|+k)p} E \int_0^\tau ||h(t, \star)||^p_p dt. \tag{6.7}$$

**Proof** (i) Note that, for any $m \in \mathbb{N} \cup \{0\}$,

$$D_x^m \partial_t^k \tilde{h}(t, x) = \varepsilon^{-2-m-k} \int_0^t \int_\mathbb{R} h(t - s, x - y)(\partial_x^k \rho)(s/\varepsilon)(D_x^m \rho)(y/\varepsilon)I_{t-s>0} \, dsdy. \tag{6.8}$$
Then, by the properties of $H^γ_p$ spaces (see Sect. 2), (6.8) and Minkowski inequality, and Hölder’s inequality, for any $t ≥ 0$,

\[
||\partial_t^k \tilde{h}(t, \star)||_{γ,p} ≤ ||\partial_t^k \tilde{h}(t, \star)||_{[γ], p} ≤ N(γ, p)||\partial_t^k \tilde{h}(t, \star)||_{W_p^{[γ]}}
\]

\[
≤ N(γ, p, ρ)\varepsilon^{-1 - [γ] - k} \int_0^t ||h(s, \star)||_{p}|(\partial_t^k \tilde{h})(s/\varepsilon)| I_{t>s>0} ds (6.9)
\]

\[
≤ N(γ, p, k, ρ)\varepsilon^{1 - [γ] - k} \left( \int_0^t ||h(s, \star)||_p^p ds \right)^{1/p}.
\]

Hence, $\partial_t^k \tilde{h}(t, \star), t ≥ 0$ is an $H^γ_p$-valued function. By a similar argument combined with the dominated convergence theorem, it is also a continuous process taking values in the same space. Finally, by a standard approximation argument combined with (6.9) we conclude that $\partial_t^k \tilde{h}(t, \star)$ is an $F_t$-adapted $H^γ_p$-valued process.

(ii) First, note that by (i) the integral on the left-hand side of (6.7) is a well-defined random variable. Next, by Lemma 5.2 (i) of [7]

\[
I ≤ E \int_0^\tau ||\Delta_n(t, \star)||_{C^γ+n} ||\partial_t^k \tilde{h}(t, \star)||_{γ,p}^p dt,
\]

where $η ∈ (0, 2 - γ)$ is a number. Using Lemma 6.2 (ii),

\[
I ≤ Nn^N γ_n^{p/2 - δ} E \int_0^\tau ||\partial_t^k \tilde{h}(t, \star)||_{γ,p}^p dt.
\]

To estimate the last integral we use the third inequality in (6.9). By a standard argument that uses Minkowski inequality we get

\[
I ≤ Nn^N γ_n^{p/2 - δ} E \int_0^\tau ||h(t, \star)||_p^p dt.
\]

\[
\square
\]

References

1. Bally, V., Millet, A., Sanz-Solé, M.: Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations. Ann. Probab. 23(1), 178–222 (1995)
2. Cardon-Weber, C., Millet, A.: A support theorem for a generalized Burgers SPDE. Potential Anal. 15(4), 361–408 (2001)
3. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications, vol. 152, 2nd edn. Cambridge University Press, Cambridge (2014)
4. Gyöngy, I.: On the approximation of stochastic differential equations. Stochastics 23(3), 331–352 (1988)
5. Gyöngy, I.: The stability of stochastic partial differential equations. II. Stoch. Rep. 27(3), 189–233 (1989)
6. Hairer, M., Labbé, C.: Multiplicative stochastic heat equations on the whole space. J. Eur. Math. Soc. 20(4), 1005–1054 (2018)
7. Krylov, N.V.: An analytic approach to SPDEs. In: Carmona, R., Rozovskii, B. (eds.) Stochastic Partial Differential Equations: Six Perspectives. Mathematical Surveys and Monographs, vol. 64, pp. 185–242. American Mathematical Society, Providence, RI (1999)
8. Krylov, N.V.: Lectures on Elliptic and Parabolic Equations in Hölder Spaces. Graduate Studies in Mathematics, vol. 12. American Mathematical Society, Providence, RI (1996)
9. Krylov, N.V.: Lectures on Elliptic and Parabolic Equations in Sobolev Spaces. Graduate Studies in Mathematics, vol. 96. American Mathematical Society, Providence, RI (2008)
10. Mackevičius, V.: $S^P$-Stability of solutions of symmetric stochastic differential equations. Liet. Matem. Rink. 25(4), 72–84 (1985) (in Russian; English translation in Lithuanian Math. J. 25, 4, 343–352 (1985))
11. Mackevičius, V.: The support of the solution of a stochastic differential equation. Litovsk. Mat. Sb. 26(1), 91–98 (1986) (in Russian; English translation in Lith. Math. J., 26(1), 57–62 (1986))
12. Millet, A., Sanz-Solé, M.: A Simple Proof of the Support Theorem for Diffusion Processes. Séminaire de Probabilités, XXVIII. Lecture Notes in Mathematics, vol. 1583, pp. 36–48. Springer, Berlin (1994)
13. Nakayama, T.: Support theorem for mild solutions of SDE’s in Hilbert spaces. J. Math. Sci. Univ. Tokyo 11(3), 245–311 (2004)
14. Rozovskii, B.L.: Stochastic Evolution Systems: Linear Theory and Applications to Nonlinear Filtering. Mathematics and its Applications, vol. 35. Kluwer Academic Publishers Group, Dordrecht (1990)
15. Stroock, D.W., Varadhan, S.R.S.: On the support of diffusion processes with applications to the strong maximum principle. In: Proceedings of Sixth Berkeley Symposium on Mathematical Statistics and Probability, vol. 3, pp. 333–359. University of California Press (1972)
16. Thangavelu, S.: Lectures on Hermite and Laguerre Expansions. Mathematical Notes, vol. 42. Princeton University Press, Princeton, NJ (1993)
17. Twardowska, K.: On Support Theorems for Stochastic Nonlinear Partial Differential Equations, Stochastic Differential and Difference Equations. Progress in Systems and Control Theory, vol. 23. Birkhäuser, Boston, MA (1997)
18. Walsh, J.: An Introduction to Stochastic Partial Differential Equations. École d’été de probabilités de Saint-Flour, XIV–1984. Lecture Notes in Mathematics, vol. 1180, pp. 265–439. Springer, Berlin (1986)
19. Yastrzhembskiy, T.: Wong–Zakai approximation and support theorem for semilinear SPDEs with finite dimensional noise in the whole space. arXiv:1808.07584

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.