A NEW OPTIMAL BOUND ON LOGARITHMIC SLOPE OF ELASTIC HADRON-HADRON SCATTERING

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In this paper we prove a new optimal bound on the logarithmic slope of the elastic slope $b$ when: $\sigma_{el}$ and $\frac{d\sigma}{d\Omega}(1)$ and $\frac{d\sigma}{d\Omega}(-1)$, are known from experimental data. The results on the experimental tests of this new optimal bound are presented in Sect. 3 for the principal meson-nucleon elastic scatterings: $(\pi^\pm P \rightarrow \pi^\pm P$ and $K^\pm P \rightarrow K^\pm P)$ at all available energies. Then we show that the saturation of this optimal bound is observed with high accuracy practically at all available energies in meson-nucleon scattering.

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1. Introduction

Recently, in Ref. [1], by using reproducing kernel Hilbert space (RKHS) methods [2-4], we described the quantum scattering of the spinless particles by a principle of minimum distance in the space of the scattering quantum states (PMD-SQS). Some preliminary experimental tests of the PMD-SQS, even in the crude form [1], when the complications due to the particle spins are neglected, showed that the actual experimental data for the differential cross sections of all $PP$, $\overline{P}P$, $K^\pm P$, $\pi^\pm P$, scatterings at all energies higher than 2 GeV, can be well systematized by PMD-SQS predictions. Moreover, connections between the optimal states [1], the PMD-SQS in the space of quantum states and the maximum entropy principle for the statistics of the scattering channels was also recently established by introducing quantum scattering entropies [5-8].

The aim of this paper is to prove a new optimal bound on the logarithmic slope of the elastic hadron-hadron scattering by solving the following optimization problem: to find an lower bound on the logarithmic slope $b$ when: $\sigma_{el}$, $\frac{d\sigma}{d\Omega}(+1)$ and $\frac{d\sigma}{d\Omega}(-1)$, including spin effects, are given. The results on the experimental tests of this new optimal bound are presented for the principal meson-nucleon elastic scatterings: $(\pi^\pm P \rightarrow \pi^\pm P$ and $K^\pm P \rightarrow K^\pm P)$ at all available energies. Then it was shown that the saturation of this optimal bound is observed with high accuracy practically at all available energies in meson-nucleon scattering.

2. Optimal helicity amplitudes for spin $(0^{-1}/2^+ \rightarrow 0^{-1}/2^+)$ scatterings

First we present some basic definitions and results for the optimal states in the meson-nucleon scattering when the integrated elastic cross section $\sigma_{el}$ and differential cross sections $\frac{d\sigma}{d\Omega}(\pm 1)$ are known from experiments. Therefore, let $f_{++}(x)$ and $f_{+-}(x)$, $x \in [-1, 1]$, be the scattering helicity amplitudes of the meson-nucleon scattering process:

$$M(0^-) + N(1/2^+) \rightarrow M(0^-) + N(1/2^+) \quad (1)$$

$x = \cos \theta, \theta$ being the c.m. scattering angle. The formalizations of the helicity amplitudes $f_{++}(x)$ and $f_{+-}(x)$ are chosen such that the differential cross section $\frac{d\sigma}{d\Omega}(x)$ is given by

$$\frac{d\sigma}{d\Omega}(x) = |f_{++}(x)|^2 + |f_{+-}(x)|^2 \quad (2)$$

Then, the elastic integrated cross section $\sigma_{el}$ is given by

$$\frac{\sigma_{el}}{2\pi} = \int_{-1}^{+1} \frac{d\sigma}{d\Omega}(x)dx = \int_{-1}^{+1} |f_{++}(x)|^2 + |f_{+-}(x)|^2 dx \quad (3)$$

Since we will work at fixed energy, the dependence of $\sigma_{el}$ and $\frac{d\sigma}{d\Omega}(x)$ and of $f(x)$, on this variable was suppressed. Hence, the helicities of incoming and outgoing nucleons are denoted by $\mu, \mu'$, and was written as $(+, -)$, corresponding to $(\frac{1}{2})$ and $(-\frac{1}{2})$, respectively. In terms of the partial waves amplitudes $f_{J^+}$ and $f_{J^-}$ we have

$$\left\{ \begin{array}{l}
 f_{++}(x) = \sum_{J=\frac{1}{2}}^{max} (J + \frac{1}{2}) (f_{J^+} + f_{J^-}) d_{J^+}^1(x) \\
 f_{+-}(x) = \sum_{J=\frac{1}{2}}^{max} (J + \frac{1}{2}) (f_{J^-} - f_{J^+}) d_{J^-}^1(x)
 \end{array} \right. \quad (4)$$

where the $d_{J^\pm}^0(x)$-rotation functions are given by

$$\left\{ \begin{array}{l}
 d_{J^+}^1(x) = \frac{1}{J+1} \cdot \left[ \frac{1}{\sqrt{2}} P_{J+1}(x) - \frac{1}{\sqrt{2}} P_{J}(x) \right] \\
 d_{J^-}^1(x) = \frac{1}{J+1} \cdot \left[ \frac{1}{\sqrt{2}} P_{J+1}(x) + \frac{1}{\sqrt{2}} P_{J}(x) \right]
 \end{array} \right. \quad (5)$$

and prime indicates differentiation of Legendre polynomials $P_{J}(x)$ with respect to $x \equiv \cos \theta$.

$$\frac{\sigma_{el}}{2\pi} = \sum (2J + 1) |f_{J^+}|^2 + |f_{J^-}|^2 \quad (6)$$

Now, let us consider the optimization problem

$$\left\{ \begin{array}{l}
 \text{min} \left[ \sum (2J + 1) |f_{J^+}|^2 + |f_{J^-}|^2 \right], \text{subject to:} \\
 \frac{d\sigma}{d\Omega}(+1) = \text{fixed}, \text{and} \frac{d\sigma}{d\Omega}(-1) = \text{fixed}
 \end{array} \right. \quad (7)$$
which will be solved by using Lagrange multiplier method [9] where
\begin{align}
L &= \left\{2(2J+1)(|f_{++}|^2 + |f_{--}|^2) + a \frac{df_{++}}{d\Omega}(+1) - \left[\sum_{J+1/2}(f_{++} + f_{--})\right]^2\right\} \\
&+ \beta \left[\frac{df_{++}}{d\Omega}(-1) - \left[\sum_{J+1/2}(f_{++} - f_{--})\right]^2\right] \\
&+ \frac{\lambda}{2} \frac{d^2f_{++}}{d\Omega}(+1) - \frac{\lambda}{2} \frac{d^2f_{++}}{d\Omega}(-1)
\end{align}
(8)

So, we prove that the solution of the problem (7) - (8) as follows
\begin{align}
f^o_{++}(x) &= f_{++}(+1) \cdot \frac{K_{++}(x,+1)}{K_{++}(+1,+1)} \\
f^o_{--}(x) &= f_{--}(-1) \cdot \frac{K_{--}(x,-1)}{K_{--}(-1,-1)}
\end{align}
(9)

where the reproducing kernel functions are defined as
\begin{align}
K_{++}(x,y) &= \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) \frac{dJ}{dx} \left(x^J \frac{dJ}{dx}\right) \\
K_{--}(x,y) &= \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) \frac{dJ}{dx} \left(x^J \frac{dJ}{dx}\right) \\
2K_{++}(+1,+1) &= (J_o + 1)^2 - 1/4 \\
2K_{--}(-1,-1) &= (J_o + 1)^2 - 1/4 \\
(J_o + 1)^2 - 1/4 &= \frac{4\pi}{\sigma cl} \left[\frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1)\right] \\
J_o &= \frac{4\pi}{\sigma cl} \left[\frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1)\right] - 1
\end{align}
(10)

Proof: Let us consider the complex partial amplitudes
\[ f_{++} \equiv r_{++} + ia_{++} \]
where \( r_{++} \) and \( a_{++} \) are real and imaginary parts, respectively. Then, Eq.(8) can be expressed completely in terms of the variational variables \( r_{++} \) and \( a_{++} \). Therefore, by calculating the first derivative we obtain
\begin{align}
\left(\frac{1}{(2J+1)} \frac{\partial L}{\partial r_{++}} = r_{++} - \alpha R^{++}(+1) \pm \beta R^{--}(-1) = 0\right) \\
\left(\frac{1}{(2J+1)} \frac{\partial L}{\partial a_{++}} = a_{++} - \alpha A^{++}(+1) \pm \beta A^{--}(-1) = 0\right)
\end{align}
(11)

where we have defined \( f^{++}(x) = R^{++}(x) + iA^{++}(x) \), and \( f^{--}(x) = R^{--}(x) + iA^{--}(x) \), respectively, where
\begin{align}
R^{++}(+1) &= \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) (r_{++} + r_{--}) \\
A^{++}(+1) &= \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) (a_{++} + a_{--}) \\
R^{--}(-1) &= \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) (r_{--} - r_{++}) \\
A^{--}(-1) &= \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) (a_{--} - a_{++})
\end{align}
(12)

Therefore, from Eqs (11) we get
\begin{align}
\begin{cases}
r_{++} = \alpha R^{++}(+1) - \beta R^{--}(-1) \\
r_{--} = \alpha R^{++}(+1) + \beta R^{--}(-1) \\
a_{++} = \alpha A^{++}(+1) - \beta A^{--}(-1) \\
a_{--} = \alpha A^{++}(+1) + \beta A^{--}(-1)
\end{cases}
\end{align}
(13)

Then, using the definitions (2) and (3), we get
\begin{align}
\alpha^{-1} = \beta^{-1} = (J_o + 1)^2 - 1/4 = \frac{4\pi}{\sigma cl} \left[\frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1)\right]
\end{align}
(14)

and, consequently we obtain that the optimal solution of the problem (7) can be written in the form
\begin{align}
f^o_{++}(x) &= \frac{2(J_o+1)}{(J_o+1)^2 - 1/4} \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) \frac{dJ}{dx} \left(x^J \frac{dJ}{dx}\right) (x) d^J_{++}(+1) \\
f^o_{--}(x) &= \frac{2(J_o+1)}{(J_o+1)^2 - 1/4} \sum_{J=1}^{J_{max}} \left(J + \frac{1}{2}\right) \frac{dJ}{dx} \left(x^J \frac{dJ}{dx}\right) (x) d^J_{--}(-1)
\end{align}
(15)

Now from Eqs. (14) and (15) we obtain the optimal solution (9) in which the reproducing functions \( K_{++} \) and \( K_{--} \) are defined by (10).

3. Optimal bound on logarithmic slope

We recall the definition of the elastic slope \( b \), and the relation
\[ b = \frac{d}{dt} \left[ \ln \frac{d\sigma}{d\Omega}(s,t) \right] \bigg|_{s=0} = \frac{\lambda^2}{2} \frac{d}{dx} \left[ \ln \frac{d\sigma}{d\Omega}(x) \right] \bigg|_{x=1} \]
(16)

where transfer momentum is defined by \( t = -2q^2(1-x), \) \( \lambda = 1/q, \) and \( q \) is the c.m. momentum.

Now, let us assume that \( \sigma_{cl}, \frac{d\sigma}{d\Omega}(+1), \) and \( \frac{d\sigma}{d\Omega}(-1) \) are known from the experimental data. Then, taking into account the solution (9)-(10) of the optimization problem (7), it is easy to prove that the elastic slope \( b \) defined by (16) must obey the optimal inequality:
\[ b \geq b_o = \frac{\lambda^2}{4} \left\{ \frac{4\pi}{\sigma cl} \left[\frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1)\right] - 1 \right\} \]
(17)

Proof: Indeed a proof of the optimal inequality (17) can be obtained as singular solution of the following optimization problem
\[ \min \{ b \}, \text{ subject to: } \sigma cl = \text{fixed}, \quad \frac{d\sigma}{d\Omega}(+1) = \text{fixed}, \quad \frac{d\sigma}{d\Omega}(-1) = \text{fixed} \]
(18)

So, the lower limit of the elastic slope \( b \) is just the elastic of the differential cross section given by the result (9)-(10). Consequently, we obtain that the optimal slope \( b_o \) is given by
\[ b_o = \frac{\lambda^2}{4} \frac{d}{dx} \left[ \frac{K_{++}(x,+1)}{K_{++}(+1,+1)} \right] \bigg|_{x=1} = \frac{\lambda^2}{4} \left\{ \left[ J_o(J_o+2) - \frac{3}{4} \right] \right\} \]
(19)

Then, using the second part of (14) we obtain the inequality (17).

An important model independent result obtained Ref. [1], via the description of quantum scattering by the principle of minimum distance in space of states (PMD-SS), is the following optimal lower bound on logarithmic slope of the forward diffraction peak in hadron-hadron elastic scattering:
\[ b \geq b_o \geq \frac{\lambda^2}{4} \left\{ \frac{4\pi}{\sigma cl} \left[\frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1)\right] - 1 \right\} \]
(20)

In is important to remark, the optimal bound (17) improves in a more general and exact form not only the unitarity bounds derived by MacDowell and Martin [10] for the logarithmic slope \( b_A \) of absorptive contribution \( \frac{d\sigma}{d\Omega}(s,t) \) to the elastic differential cross sections but also the unitarity lower bound derived in Ref. [1] (see also Ref. [11], [12]) for the slope \( b \) of the entire \( \frac{d\sigma}{d\Omega}(s,t) \) differential cross section. Therefore, it would be important to make an experimental detailed investigation of the saturation
of this bond in the hadron-hadron scattering, especially in the low energy region.

4. Experimental tests of the bound (17)

A comparison of the experimental elastic slopes $b$ with the optimal slope $b_o(17)$ is presented in Figs. 1 for ($\pi^\pm P$ and $K^\pm P$)-scatterings: The values of the $\chi^2 = \sum_j (b_j - b_{o,j})^2/\sigma_{o,j}^2$, (where $\epsilon_{b,j}$ and $\epsilon_{b,o,j}$ are the experimental errors corresponding to $b$ and $b_o$, respectively) are used for the estimation of departure from the optimal PMD-SS-slope $b_o$, and then, we obtain the statistical parameters presented in Table 1. For $\pi^\pm P$-scattering the experimental data on $b$, $\frac{d\sigma}{d\Omega}(+1)$, $\frac{d\sigma}{d\Omega}(-1)$, and $\sigma_{el}$, for the laboratory momenta in the interval $0.2\text{ GeV} \leq \text{PLAB} \leq 10\text{ GeV}$ are calculated directly from the phase shifts analysis (PSA) of Hohler et al. \[12\]. To these data we added some values of $b$ from the linear fit of Lasinski et al. \[14\] and also from the original fit of authors quoted in some references in \[13\]. Unfortunately, the values of $b_o$ corresponding to the Lasinski’s data \[14\] was impossible to be calculated since the values of $\frac{d\sigma}{d\Omega}(1)$ from their original fit are not given. For $K^\pm P-$scatterings the experimental data on $b$, $\frac{d\sigma}{d\Omega}(+1)$, $\frac{d\sigma}{d\Omega}(-1)$ and $\sigma_{el}$, in the case of $K^-P$, are calculated from the experimental (PSA) solutions of Arndt et al. \[13\]. To these data we added those collected from the original fit of data from references of \[15\] which the approximation $\frac{d\sigma}{d\Omega}(-1) = 0$. For $K^+P$-scattering, we added some values of $b$ from the linear fit of Lasinski et al. \[14\] and also those pairs $(b, b_o)$ calculated directly from the experimental (PSA) solutions of Arndt et al. \[13\]. All these results can be compared with those presented in Ref. \[15\].

5. Summary and Conclusion

The main results and conclusions obtained in this paper can be summarized as follows:

(i) In this paper we proved the optimal bound (17) as the singular solution ($\lambda_0 = 0$) of the optimization problem to find a lower bound on the logarithmic slope $b$ with the constraints imposed when $\sigma_{el}$ and $\frac{d\sigma}{d\Omega}(+1)$ and $\frac{d\sigma}{d\Omega}(-1)$ are fixed from experimental data. This result is similar with that obtained recently in Refs. \[1, 12\] for the problem to find an upper bound for the scattering entropies when $\sigma_{el}$ and $\frac{d\sigma}{d\Omega}(+1)$ are fixed.

(ii) We find that the optimal bound (17) is verified experimentally with high accuracy at all available energies for all the principal meson-nucleon scatterings. \[3-4\].

(iii). From mathematical point of view, the PMD-SQS-optimal states (9)-(10), are functions of minimum constrained norm and consequently can be completely described by reproducing kernel functions (see also Ref. \[3-4\]). So, with this respect the PMD-SQS-optimal states from the reproducing kernel Hilbert space (RKHS) of the scattering amplitudes are analogous to the coherent states from the RKHS of the wave functions.

(iv) The PMD-SQS-optimal state (9)-(10) have not only the property that is the most forward-peaked quantum state but also possesses many other peculiar properties such as maximum Tsallis-like entropies, as well as the scaling and the s-channel helicity conservation properties, etc., that make it a good candidate for the description of the quantum scattering via an optimum principle. In fact the validity of the principle of least distance in space of states in hadron-hadron scattering is already well illustrated in Fig. 1 and Table 1.

All these important properties of the optimal helicity amplitudes (9)-(10) will be discussed in more detail in a forthcoming paper.

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TABLE I: $\chi^2$ -statistical parameters of the principal hadron-hadron scattering. In these estimations for $P_{LAB} \leq 2$ GeV/c the errors $\epsilon^{PSA}(\pi \pm P) = 0.1 \, b^{PSA}$ and $\epsilon^{PSA}(K \pm P) = 0.1b^{PSA}$ are taken into account while for the errors to the optimal slopes $b_o$ calculated from phase shifts analysis and $[13]$. For $P_{LAB} \geq 2$ GeV/c and $P_{LAB} \geq 0.2$ GeV/c.

| Statistical parameters | For $P_{LAB} \geq 2$ GeV/c | For all $P_{LAB} \geq 0.2$ GeV/c |
|------------------------|-----------------------------|----------------------------------|
| $\pi^+P \rightarrow \pi^+P$ | 28 | 1.02 | 90 | 3.37 |
| $\pi^-P \rightarrow \pi^-P$ | 31 | 0.92 | 93 | 8.00 |
| $K^+P \rightarrow K^+P$ | 37 | 1.15 | 73 | 1.91 |
| $K^-P \rightarrow K^-P$ | 37 | 1.52 | 73 | 7.84 |
| $PP \rightarrow PP$ | 29 | 5.01 | 32 | 5.06 |
| $P\bar{P} \rightarrow P\bar{P}$ | 27 | 0.56 | 45 | 1.86 |

FIG. 1: The experimental values (black circles) of the logarithmic slope $b$ for the principal meson-nucleon scatterings are compared with the optimal PMD-SQS-predictions $b_o$ (white circles). The experimental data for $b_o$, $\frac{d\sigma}{d\Omega}(+1)$ and $\sigma_{el}$, are taken from Refs. $[12]-[14]$. (see the text).