FLOYD MAPS FOR RELATIVELY HYPERBOLIC GROUPS

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JULY 8, 2011

ABSTRACT. Let $\delta_{S,\lambda}$ denote the Floyd metric on a discrete group $G$ generated by a finite set $S$ with respect to the scaling function $f_n = \lambda^n$ for a positive $\lambda < 1$. We prove that if $G$ is relatively hyperbolic with respect to a collection $P$ of subgroups then there exists $\lambda$ such that the identity map $G \to G$ extends to a continuous equivariant map from the completion with respect to $\delta_{S,\lambda}$ to the Bowditch completion of $G$ with respect to $P$.

In order to optimize the proof and the usage of the map theorem we propose two new definitions of relative hyperbolicity equivalent to the other known definitions.

In our approach some “visibility” conditions in graphs are essential. We introduce a class of “visibility actions” that contains the class of relatively hyperbolic actions. The convergence property still holds for the visibility actions.

Let a locally compact group $G$ act on a compactum $\Lambda$ with convergence property and on a locally compact Hausdorff space $\Omega$ properly and cocomactly. Then the topologies on $\Lambda$ and $\Omega$ extend uniquely to a topology on the direct union $T = \Lambda \sqcup \Omega$ making $T$ a compact Hausdorff space such that the action $G \rtimes T$ has convergence property. We call $T$ the attractor sum of $\Lambda$ and $\Omega$.

1. INTRODUCTION

1.1. Floyd. In [F80] W. J. Floyd introduced a class of metrics on finitely generated groups obtained by the “conformal scaling” of the word metric. Namely he regards a group $G$ as the vertex set of a locally finite metric Cayley graph, where the length of an edge depends on the word distance from the origin. The function $f : n \mapsto f_n = (the new length of an edge of word distance $n$ from 1)$ is called the scaling function. Under certain conditions on $f$ the Cauchy completion $\text{Fl}_f G$ with respect to the new path metric is compact and $G$ acts on $\text{Fl}_f G$ by bi-lipschitz homeomorphisms.

Floyd proved that, for any finitely generated geometrically finite Kleinian group $G$ and for the scaling function $f : n \mapsto \frac{1}{n^2+1}$, every orbit map $G \triangleright g \mapsto gp \in \mathbb{H}^3$ extends by continuity to the Floyd map $\phi_f : \text{Fl}_f G \to \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \mathbb{S}^2$ that takes the Floyd boundary $\partial_f G = \text{Fl}_f G \setminus G$ onto the “limit set” $\Lambda G$.

Floyd also calculated the “kernel” of the boundary map $\phi_f|_{\partial_f G}$: the map is one-to-one except for the preimages of the parabolic points of rank one where it is two-to-one.
Our purpose is to generalize Floyd’s result replacing ‘geometrically finite Kleinian’ by ‘relatively hyperbolic’ (r.h. for short). It this paper we prove the existence of the Floyd map. In the next papers [GP09], [GP10] we describe its kernel.

1.2. Relative hyperbolicity. Our strategy depends on the choice of the initial definition of relative hyperbolicity. In [Hr10] some relations between various definitions are discussed. The following two main definitions, the “geometric” and the “dynamical”, reflect two aspects of the subject.

Definition RH fh (‘fh’ stands for ‘fine hyperbolic’). An action of a group $G$ on a connected graph $\Gamma$ is said to be relatively hyperbolic if $\Gamma$ is $\delta$-hyperbolic and “fine” (see 2.6), the action $G \curvearrowright \Gamma^1 = \{\text{the edges of } \Gamma\}$ is proper (i.e., the stabilizers of the edges are finite), cofinite (i.e., $\Gamma^1 / G$ is finite), and non-parabolic (no vertex is fixed by the whole $G$).

A group $G$ is said to be relatively hyperbolic with respect to a finite collection $\mathcal{P}$ of infinite subgroups if it possesses an r.h. action $G \curvearrowright \Gamma$ such that $\mathcal{P}$ is a set of representatives of the orbits of the stabilizers of the vertices of infinite degree.

Definition RH 32 (‘32’ means ‘3-proper and 2-cocompact’). An action of a discrete group $G$ by auto-homeomorphisms of a compactum $T$ is said to be relatively hyperbolic if it is proper on triples, cocompact on pairs and has at least two “limit points” (see 7.3).

A group $G$ is said to be relatively hyperbolic with respect to a finite collection $\mathcal{P}$ of infinite subgroups if it possesses an r.h. action $G \curvearrowright T$ such that $\mathcal{P}$ is a set of representatives of the stabilizers of “non-conical” (see 7.5) limit points.

In both cases (as well as in other equivalent definitions) an r.h. group with respect to $\mathcal{P}$ is defined by means of an r.h. action with a specified set of “parabolic” subgroups which is nontrivial in some sense (“nonparabolic”). The notion of “action” seems more fundamental.

At the present moment no simple proof of $\text{RH}_{fh} \iff \text{RH}_{32}$ is known. To obtain either of the implications one should interpret an r.h. action as an action of the other type. So, for ‘$\Rightarrow$’, one has to construct a compactum $T$ acted upon by $G$ with the desired properties, and, for ‘$\Leftarrow$’ one has to find at least one fine hyperbolic $G$-graph out of the topological information.

This paper is intended to facilitate the translation between the “geometric” and the “dynamical” languages in both directions. In [GP10] and [GP11] we will advance in this program.

Our proof of the Floyd map theorem 3.4.6 requires some information derivable from both of the characteristic properties RH$_*$.

1.3. Alternative hyperbolicity. We call a connected graph $\Gamma$ alternatively hyperbolic if for every edge $e$ of $\Gamma$ there exists a finite set $F \subset \Gamma^1$ such that every geodesic triangle containing $e$ on a side also contains an edge in $F$ on another side.

Definition RH$_{ah}$. An action of a group $G$ on a connected graph $\Gamma$ is said to be relatively hyperbolic if $\Gamma$ is alternatively hyperbolic and the action $G \curvearrowright \Gamma^1$ is proper cofinite and no vertex is fixed by $G$. 

The first implication is some finiteness problem, the second is a deduction ‘dynamics$\Rightarrow$geometry’ and the third is an easy geometry.
1.4. **Perspective divider.** We express relative hyperbolicity as a uniform structure on a group $G$ or on a “connected $G$-set”. Recall that a *uniformity* on a set $M$ is a filter $\mathcal{U}$ on the set $M^2 = M \times M$ whose elements are called *entourages*. Each entourage $u$ should contain the diagonal $\Delta^2 M$ and a symmetric entourage $v$ such that $v^2 \subseteq u$. We often regard an entourage as a set of non-ordered pairs.

Let $G$ be a group. An $G$-set $M$ is said to be connected [Bo97] if there exists a connected $G$-graph $\Gamma$ with the vertex set $M$ such that $\Gamma^1 / G$ is finite. We call such $\Gamma$ a *connecting structure* for the $G$-set $M$.

Let $M$ be a connected $G$-set. A symmetric set $u \subseteq M^2$ containing $\Delta^2 M$ is called a *divider* (see 3.2) if there exists a finite set $F \subseteq G$ such that $(\cap \{ f \circ u : f \in F \})^2 \subseteq u$. The $G$-filter generated by a divider is a uniformity $U_u$ on $M$.

A divider $u$ is said to be *perspective* (see 2.7) if for every pair $\beta$ in $M$ the set $\{ g \in G : g \beta \notin u \}$ is finite.

**Definition** $\text{RH}_{pd}$. A relatively hyperbolic structure on a connected $G$-set $M$ (we also say ‘a relative hyperbolicity’ on $M$) is a “non-parabolic” $G$-uniformity generated by a perspective divider (parabolicity means that the $U$-boundary is a single point).

It is less easy to restore the “parabolic” subgroups for $\text{RH}_{pd}$. We prove in 4.2.2 that the completion $(M, U_u)$ with respect to a r.h. uniformity is a compactum where $G$ acts relatively hyperbolically in the sense of $\text{RH}_{32}$ So we obtain an interpretation $\text{RH}_{pd} \Rightarrow \text{RH}_{32}$.

1.5. **Floyd map theorem for r.h. uniformities.** We use the exponential scaling function $f_n = \lambda^n$ where $0 < \lambda < 1$.

Let $\Gamma$ be a connected graph. For a vertex $v \in \Gamma^0$ define the Floyd metric $\delta_{v, \lambda}$ by postulating that the length of an edge of distance $n$ from $v$ is $\lambda^n$. Change the base vertex $v$ gives a bi-lipschitz-equivalent metric and hence the same uniformity $U_{\Gamma, \lambda}$. We call it the *Floyd uniformity* on $\Gamma$.

**Theorem (map theorem 3.4.6)** Let $G$ be a group, $M$ a connected $G$-set, $\Gamma$ a connecting graph structure for $M$, $\mathcal{U}$ a relatively hyperbolic uniformity on $M$. Then there exists $\lambda \in (0, 1)$ such that $\mathcal{U}$ is contained in the Floyd uniformity $U_{\Gamma, \lambda}$. The inclusion induces a uniformly continuous $G$-equivariant surjective map $(M, U_{\Gamma, \lambda}) \to (M, \mathcal{U})$ between the completions.

This theorem can be applied in particular to either a Cayley graph with respect to a finite generating set or to a Farb’s “conned-off” graph relative with respect to a finite collection of subgroups without any restriction on the cardinality of the “parabolic” subgroups.

**Corollary.** Let $G$ be a group relatively hyperbolic with respect to a collection $\mathcal{P}$ of subgroups. Then, for some $\lambda \in (0, 1)$, there exists a continuous equivariant map from the Floyd boundary $\partial_\lambda G$ to the Bowditch boundary of $G$ with respect to $\mathcal{P}$.

1.6. **Visibility.** For an edge $e$ of a graph $\Gamma$ let $u_e$ denote the set of pairs $(x, y)$ of vertices such that no geodesic segment joining $x$ and $y$ pass through $e$. The filter $\text{Vis}_\Gamma$ on the set of pairs of vertices generated by the sets $u_e$ is called the *visibility filter*. A graph $\Gamma$ is alternatively hyperbolic if and only if $\text{Vis}_\Gamma$ is a uniformity on $\Gamma^0$.

A uniformity $\mathcal{U}$ on $\Gamma^0$ is called a *visibility* on $\Gamma$ if it is contained in $\text{Vis}_\Gamma$. On every graph $\Gamma$ there exists the maximal visibility that contains all other visibilities. It may be smaller than $\text{Vis}_\Gamma$.

The main corollary of the map theorem is the following highly useful fact.

**Generalized Karlsson lemma 3.5.1.** Let $\mathcal{U}$ be a relative hyperbolicity on a connected $G$-set $M$ and let $\Gamma$ be a connected graph with $\Gamma^0 = M$ where $G$ acts on edges properly and cofinitely. Then, for
every entourage \( u \in U \) there exists a finite set \( E \subset \Gamma \) such that \( u \) contains the boundary pair \( \partial I \) of every geodesic segment \( I \) that misses \( E \).

This implies that each relative hyperbolicity is a visibility.

The completion with respect to any visibility \( U \) on a graph \( \Gamma \) is compact \((\text{4.1.1})\). If a group \( G \) acts on \( \Gamma \) properly on edges and keeps \( U \) invariant then the induces action on the completion space \( T \) has the convergence property. This gives a wide class of convergence group actions including the action on the space of ends, Kleinian actions of finitely generated groups, the actions on Floyd completions and many other. The problem is whether there exist convergence actions of other nature.

If \( U \) is a relative hyperbolicity then \( T \) coincides with the Bowditch completion. In this case the induces action on the space of pairs is cocompact.

1.7. Attractor sum. To prove ‘\( \text{RH}_{32} \Rightarrow \text{RH}_{pd} \)’ we need to attach to a compactum \( T \) where a group \( G \) acts properly on triples at least one orbit of isolated points. We do so by a rather general construction that we call attractor sum.

For the sake of future applications we construct this space in an excessive generality of the actions of locally compact groups. However the additional difficulties implied by possible non-discreteness of the acting group \( G \) are not very essential. Moreover they clarify and motivate some aspects of the theory of discrete group actions.

Our main result in this direction is the following.

**Attractor sum theorem (8.3.1).** Let a locally compact group \( G \) act on a compactum \( \Lambda \) properly on triples and on a locally compact Hausdorff space \( \Omega \) properly and cocompactly. Then on the disjoint union \( \Lambda \sqcup \Omega \) there is a unique compact Hausdorff topology \( \tau \) extending the original topologies of \( \Lambda \) and \( \Omega \) such that the \( G \)-action on the space \( X \leftarrow (\Lambda \sqcup \Omega, \tau) \) is proper on triples.

In particular every convergence group action of a discrete group \( G \) on a compactum \( T \) extends to a convergence group action on the attractor sum \( \bar{T} \) of \( G \) and \( T \). Thus the uniformity of \( \bar{T} \) induces a uniformity on \( G \). One could ask whether this uniformity is a visibility or a relative hyperbolicity. The closure \( \bar{G} \) of \( G \) in \( \bar{T} \) can be thought of as an invariant compactification of \( G \) where \( G \) acts with convergence property. The “boundary” \( \bar{G} \setminus G \) is just the limit set of the original action \( G \acts T \).

1.8. The structure of the paper. The following diagram presents the main results illustrating the reasons and dependencies.

The interpretation \( \text{RH}_{32} \Rightarrow \text{RH}_{fh} \) was established in \([\text{Ya04}]\) for finitely generated groups. The argument of \([\text{Ya04}]\) is rather complicated and it is not clear for us whether or not the finite generability
actually used. Anyway it uses the metrisability of the compactum which is equivalent to the assumption that the group is countable.

In [GP10] we will remove any restriction on the cardinality and in [GP11] we will give a conceptually more simple proof of ‘RH_{32} ⇒ RH_{h}’ using “quasigeodesics” with respect to quadratic distortion functions. In [GP09] we will study quasi-isometric maps to r.g. groups.

All the three papers require the map theorem 3.4.6 and the attractor sum theorem 8.3.1 (for discrete groups).

2. Preliminaries

In this section we fix the terminology and notation and recall widely known definitions and facts. For the reader’s convenience we repeat in this section the definitions given in the introduction. The reader should search here for all general definition and notation used elsewhere in this paper. We are trying to collect those and only those definitions and statements that we use. Some conventions are introduced implicitly.

We recommend to browse this section and continue reading farther returning to the preliminaries following the references.

2.1. General notation and conventions. The symbol ‘□’ at the end of line means that the current proof is either completed or leaved to the reader. The reader is supposed to be capable to complete the proof or to find it the common sources.

The single quotes ‘. . . ’ mean that the content is just mentioned, not used. The double quotes “. . . ” mean that the exact interpretation of the content is leaved to the reader. Example: ‘f_n tends to infinity’ means “f_n gets arbitrarily big while n grows”.

The symbol ‘⇒’ means ‘is equal by definition’. We use the italic font for the notions being defined. Example: \( \mathbb{N} = \mathbb{Z}_{\geq 0} \) is the set of positive integers; the elements of \( \mathbb{N} \) are the natural numbers.

For a set \( M \) we denote by \( M^n \) the product of \( n \) copies of \( M \). The quotient of \( M^n \) by the action of the symmetric group transposing the coordinates is denoted by \( S^n M \). The elements of \( S^n M \) are the “subsets of cardinality \( n \) with multiplicity”. The elements of \( M^n \) and of \( S^n M \) are the \( n \)-tuples, either ordered or non-ordered. The 2-tuples are the pairs, the 3-tuples are the triples and so on. An \( n \)-tuple is regular if all its “elements” are distinct. Denote \( \Theta^n M = \{ \text{regular} \ n \text{-tuples in} \ S^n M \} \), \( \Delta^n M = S^n M \setminus \Theta^n M = \{ \text{singular} \ n \text{-tuples} \} \). We identify \( \Theta^n M \) with \( \{ \text{subsets of} \ M \ \text{of cardinality} \ n \} \).

By \( |M| \) we denote the cardinality of a set \( M \). However ‘\( |M|=\infty \)’ means ‘\( M \) is infinite’.

For a set \( M \) we denote by \( \text{Sub}M \) the set of all subsets of \( M \) and \( \text{Sub}^n M = \{ N \subset M : |N|=n \} \), \( \text{Sub}^{\leq n} M = \{ N \subset M : |N|<n \} \) etc. We identify \( \text{Sub}^n M \) with \( \Theta^n M \).

‘\( A \setminus B \)’ means set-theoretical difference. For a subset \( B \) of a set \( A \) we write \( B' \) instead of \( A \setminus B \) when we hope that the reader knows what is \( A \). Example: \( B \) is open if and only if \( B' \) is closed.

We sometimes identify a single-point set \( \{ p \} \) with its unique point. For example, for \( p, q \in T \) we write \( p \times T \cup T \times q \) instead of \( \{ p \} \times T \cup T \times \{ q \} \). Speaking of the one-point compactification \( \hat{L} \) of a space \( L \) we write \( \hat{L} = L \cup \{ \infty \} \) instead of \( \hat{L} = L \cup \{ \{ \} \} \) etc.

If \( M \) is a Cartesian product of sets \( M_\xi \) then a subproduct of \( M \) is a subset which is a product of a family \( \{ M_\xi \} \) of subsets. A subproduct of the form \( \pi_\xi^{-1} N \) where \( \pi_\xi : M \to M_\xi \) is the projection map, is the cylinder over \( N \). If \( N = \{ p \} \) then it is the fiber over the point \( p \).

We use \( \mathbb{R}, \mathbb{Z} \) etc in the common way. For \( a, b \in \mathbb{R} \) by \( [a, b], (a, b), (a, b) \) we denote what the reader expects. For \( m, n \in \mathbb{Z} \) we put \( \overline{m, n} = \{ k \in \mathbb{Z} : m \leq k \leq n \} = [m, n] \cap \mathbb{Z} \).
By $f|_M$ we denote the restriction of a function $f$ onto a set $M$ not necessarily contained in the domain $\text{Dom}(f)$ of $f$. So, $\text{Dom}(f|_M)=M\cap\text{Dom}(f)$.

We consider an equivalence relation on a set $M$ as a subset of either $M^2$ or $S^2M$. The same convention is adopted for other “symmetric” relations and “symmetric” functions. The reader should not be confused.

The kernel of a map $f$ defined on a set $M$ is the equivalence relation ‘$fp=fq$’.

For a function $f$ on a subset of $\mathbb{Z}$ we sometimes write $f_n$ instead of $f(n)$.

2.2. General topology. The information of this subsection is used mainly in the attractor sum theory of section 8.

For a topological space $T$ we denote: $\text{Open}(T)\equiv\{\text{open subsets of } T\}$; $\text{Closed}(T)\equiv\{\text{closed subsets of } T\}$; $\text{Loc}_T S\equiv\{\text{the neighborhoods of a subset } S \text{ of } T\}$. A filter on a set $M$ is a proper subset of $\text{Sub}M$ which is “$\cap$-closed and $\uparrow$-convex”. An ultrafilter on $M$ is a maximal element in the set $\{\text{filters on } M\}$. Every filter is the intersection of a family if ultrafilters. A collection of sets is consistent if any finite subcollection has nonempty intersection. A collection is consistent if and only if it is contained in some filter.

Two collections are inconsistent if their union is not consistent.

A filter on a topological space $T$ converges to a point $p\in T$ if it contains $\text{Loc}_T p$. A space $T$ is compact if every ultrafilter on $T$ converges (to a point). A hausdorff compact space is a compactum.

A subset of a topological space is (topologically) bounded if its closure is compact.

**Proposition 2.2.1.** If a filter $\mathcal{F}$ on a compact space $T$ is generated by $\mathcal{F}\cap\text{Closed}T$ then $\mathcal{F}\supseteq\text{Loc}_T(\cap\mathcal{F})$.

**Proof.** Every ultrafilter containing $\mathcal{F}$ converges to a point that must belong to $\cap\mathcal{F}$.

**Proposition 2.2.2.** Let a space $T$ be compact, $F\in\text{Closed}T$ and let $\mathcal{F}$ be a subfilter of $\text{Loc}_T F$ inconsistent with each $\text{Loc}_T p$, $p\notin F$. Then $\mathcal{F}=\text{Loc}_T F$.

**Proof.** Every ultrafilter that contains $\mathcal{F}$ must converge to a point in $F$.

**Proposition 2.2.3.** If an ultrafilter $\mathcal{F}$ on a space $T$ contains $\text{Loc}_T \Lambda$ for a compact subspace $\Lambda$ then $\mathcal{F}$ converges to a point of $\Lambda$.

**Proof.** The set $\mathcal{G}=\mathcal{F}\cap\text{Closed}T$ is consistent with $\{\Lambda\}$ thus there exists $p\in\Lambda\cap(\cap\mathcal{G})$. Every $\alpha\in\text{Open}T\cap\text{Loc}_T p$ should belong to $\mathcal{F}$ since its complement does not belong to $\mathcal{G}$.

The product topology on a product $X$ of topological spaces is generated by the set $\{\text{cylinders over open sets}\}$.

**Proposition 2.2.4** (“Walles Theorem”). If $T$ is a compact subproduct of a product $X$ of topological spaces then the filter $\text{Loc}_X T$ is generated by $\mathcal{F}\cap\{\text{cylinders}\}$.

**Proposition 2.2.5** (“Aleksandrov Theorem”). The quotient $X/\theta$ of a compact space $X$ by an equivalence $\theta$ is Hausdorff if and only if $\theta$ is closed in $X^2$.

**Proposition 2.2.6** (“Kuratovski Theorem”). A topological space $K$ is compact if and only if for every space $T$ the projection map $T\times K \rightarrow T$ is closed i.e, maps closed sets to closed sets.
A closed correspondence from a space $X$ to a space $Y$ is any closed subset $S$ of $X \times Y$. It is surjective if the restrictions over $S$ of the projections onto $X$ and $Y$ are surjective.

Let $A, B$ be topological spaces and let $K$ be a compactum. For $a \in \text{Closed}(A \times K), b \in \text{Closed}(K \times B)$ the set $a \ast b \equiv a \times b \cap A \times \Delta^2 Y \times B$ is closed in $A \times K^2 \times B$. The composition $a \circ b = \text{pr}_{A \times B} a \ast b$ is closed by Proposition 2.2.6.

Let $\text{Surj}(X \times Y) \equiv \{\text{surjective closed correspondences from } X \text{ to } Y\}$.

**Proposition 2.2.7.** The composition of surjective correspondences is surjective. The operation $\text{Surj}(X \times Y) \times \text{Surj}(Y \times B) \to \text{Surj}(A \times B)$ is associative. $\square$

### 2.3. Metrics and uniformities.

We extend the common notion of a metric by allowing infinite distance and zero distance between points. So, a metric on a set $M$ is a function $\rho : \Delta^2 M \to [0, \infty]$ with $\rho(\Delta^2_M) = 0$ satisfying the $\Delta$-inequality $\rho(a, b) + \rho(b, c) \geq \rho(a, c)$. A metric is finite if it does not take the value $\infty$. A metric $\rho$ on $M$ is exact if $\rho^{-1}(0) = \Delta^2 M$.

Let $\rho$ be a metric on $M$. We extend the function $\rho$ to the pairs of subsets of $M$: $\rho(A, B) = \inf_{a \in A, b \in B} \rho(a, b)$.

The open and closed $r$-neighborhoods of a set $A \subset M$ are the sets $N_\rho(A, r) := \{p \in M : \rho(A, p) < r\}$ and $\overline{N}_\rho(A, r) := \{p \in M : \rho(A, p) \leq r\}$. Sometimes we omit the index ‘$\rho$’.

Any set $u \subset S^2 M$ can be thought as a symmetric binary relation on $M$ and as a set of the edges of a graph whose vertex set is $M$. We call $u$ reflexive if $u \supseteq \Delta^2 M$. Every $u \subset S^2 M$ determines a metric $\delta_u$ on $M$ as the maximal among the metrics $\rho$ on $M$ such that $\rho|_u \leq 1$. The canonical graph metric $d$ discussed in [2.6] is a particular case of this construction. Denote $u^n = \delta_u^{-1}(0, n)$. Clearly

$$\tag{2.3.1} (u \cap v)^n \subset u^n \cap v^n \text{ for every } u, v \subset S^2 M.$$  

A set $m \subset M$ is $u$-small if $S^2 m \subset u \cup \Delta^2 M$ (equivalently: if its $\delta_u$-diameter is $\leq 1$). We denote by $\text{Small}(u)$ the set of $u$-small subsets of $M$. We try to use the convention “small letters denote small sets”. The $u$-neighborhood of a set $m \subset M$ is the set $m_u = \overline{N}_{\delta_u}(m, 1) = m \cup \{p \in M : \exists q \in M \{p, q\} \in u\}$.

Subsets $u, v \subset S^2 M$ are said to be unlinked if $M$ is a union of a $u$-small set and a $v$-small set. We denote this relation by $u \prec v$. If $u$ and $v$ are not unlinked we say that they are linked and denote this relation by $u \# v$. So, $u$ is self-linked ($u \# u$) if $M$ is not a union of two $u$-small sets. A filter $\mathcal{U}$ consisting of reflexive subsets of $S^2 M$ is a uniformity or a uniform structure on $M$ if $\forall u \in \mathcal{U} \forall v \in \mathcal{U} \: v^2 \subset u$.

The elements of a uniformity are called entourages. This notion plays a significant role in our theory. We use the bold font for the entourages and for some sets of pairs that “should be” entourages of some uniformities (see for example, subsections 3.1 and 3.2).

If $u$ is an entourage of a uniformity $\mathcal{U}$ we write $v = \sqrt[\mathcal{U}]{u}$ if $v \in \mathcal{U}$ and $v^n \subset u$. So $\sqrt[\mathcal{U}]{u}$ exists but it is not unique.

An entourage $u$ separates points $x$ and $y$ if $\{x, y\} \not\in u$. A uniformity $\mathcal{U}$ on $M$ is exact if every two distinct points can be separated by an entourage, i.e, $\mathcal{U} = \Delta^2 M$.

Given a uniformity $\mathcal{U}$ a set $m \subset M$ is called an $\mathcal{U}$-neighborhood of a point $p \in M$ if it contains a $u$-neighborhood $pu$ for some entourage $u \in \mathcal{U}$. So $\mathcal{U}$ yields the $\mathcal{U}$-topology on $M$ in which the uniformities of points are the $\mathcal{U}$-neighborhoods. We speak of $\mathcal{U}$-open sets $\mathcal{U}$-closure etc. meaning the $\mathcal{U}$-topology.

A topological space whose topology is determined by a uniformity $\mathcal{U}$ is uniformizable. Every such $\mathcal{U}$ is a uniformity consistent with the topology.

For every compactum (moreover, even for every paracompact Hausdorff space) $T$ the filter $\text{Ent}(T) \equiv \text{Loc} \_ {\mathcal{U}} \Delta^2 T$ of the neighborhoods of the diagonal is an exact uniformity on $T$ consistent with the topology. If $T$ is compact then the uniformity consistent with the topology is unique. Therefore
we have a correct notion of an entourage of a compactum \( T \). It is just a neighborhood of the diagonal \( \Delta^2 T \) in the space \( S^2 T \).

A topological space is uniformisable by an exact uniformity if and only if it is embeddable in a compactum \([\text{Bou58}] \) 1 Prop. 3.

A set endowed with a uniformity is a uniform space \([\text{Bou71}], [\text{We38}]\). Every metric \( \varrho \) on \( M \) determines a uniformity \( \mathcal{U}_\varrho \) generated by the collection \( \{ \varrho^{-1}[0, \varepsilon] : \varepsilon > 0 \} \). A uniformity is determined by a metric (=metrisable) if and only if it is countably generated as a filter \([\text{We38}]\), see also \([3,1]\).

A metric is exact if and only if the corresponding uniformity is exact.

The morphisms of uniformities are the uniformly continuous maps i.e., the maps such that the preimage of an entourage is an entourage. Every subset \( (\{\text{We38}\}, \text{see also } 3.1) \) uniformity is determined by a metric (compactum \([\text{Bou58}], [\text{We38}]\)).

The space is uniformisable by an exact uniformity if and only if it is embeddable in a compactum \([\text{Bou58}], [\text{We38}]\).

2.4. Cauchy-Samuel completion. Let \((M, \mathcal{U})\) be a uniform space. A Cauchy filter \( \mathcal{F} \) on \( M \) is a filter with arbitrarily small elements: \( \forall u \in \mathcal{F} \cap \text{Small}(u) \neq \emptyset \). For \( x \in M \) the filter \( \text{Loc}_{\mathcal{U}} x \) is a Cauchy filter. Moreover, it is minimal element in the set of Cauchy filters ordered by inclusion.

The space is complete if every Cauchy filter converges i.e., contains a filter of the form \( \text{Loc}_{\mathcal{U}} x \) for \( x \in M \). Every closed subset of a complete space is a complete subspace. Every compactum is complete.

Every uniform space \((M, \mathcal{U})\) possesses an initial morphism \( i_\mathcal{U} : (M, \mathcal{U}) \to (\overline{M}, \overline{\mathcal{U}}) \) to a complete space. The points of \( \overline{M} \) are the minimal Cauchy filters. The completion map \( i_\mathcal{U} \) takes \( x \) to \( \text{Loc}_{\overline{\mathcal{U}}} x \). For an entourage \( u \in \mathcal{U} \) the set

\[(2.4.1) \quad \overline{u} = \{p, q \in S^2 \overline{M} : p \cap q \cap \text{Small}(u) \neq \emptyset\}\]

is, by definition, an entourage of \( \overline{M} \). The uniformity \( \overline{\mathcal{U}} \) is the filter generated by \( \{\overline{u} : u \in \mathcal{U}\} \). It is exact.

If \( \mathcal{U} \) is exact then \( i_\mathcal{U} \) is injective and we can identify \( M \) with a subspace of \( \overline{M} \). In this case the remainder \( \partial_\mathcal{U} M = \overline{M} \setminus M \) is sometimes called a \( \mathcal{U} \)-boundary of \( M \).

For every subset of a complete exact uniform space the canonical map from the completion to the closure is an isomorphism of uniform spaces.

An entourage \( u \) is precompact if \( M \) is a union of finitely many \( u \)-small sets. A uniformity is precompact if every its entourage is. Precompactness of a uniformity is equivalent to the compactness of the completion space.

2.5. Dynkin property. Two entourages \( u, v \) of a uniform space \((M, \mathcal{U})\) are said to be unlinked (notation: \( u \bowtie v \)) if \( M \) is a union of an \( u \)-small set and a \( v \)-small set. Otherwise the entourages are linked (notation: \( u \# v \)).

Let a locally compact group \( G \) act on \( M \) keeping \( \mathcal{U} \) invariant. We say that the action \( G \curvearrowright M \) has Dynkin property if for every \( u, v \in \mathcal{U} \) the set \( \{ g \in G : u \bowtie gv \} \) is bounded in \( G \). For compact spaces the Dynkin property is equivalent to the “convergence property”, see \([7,2]\).

**Proposition 2.5.1.** Completion keeps Dynkin property.

**Proof.** It suffices to check that, for unlinked entourages \( u, v \) of a uniform space \((M, \mathcal{U})\), the entourages \( u^3 \) and \( v^3 \) (see \(2.4.1\)), are unlinked.

For a set \( a \subseteq M \) let \( \overline{a} = \{ p \in \overline{M} : a \text{ is consistent with } p \} \) If \( a \) is \( u \)-small then \( \overline{a} \) is \( u^3 \)-small. Indeed if \( p, q \in \overline{a} \) and \( p, q \) are \( u \)-small sets in \( p \cap q \) respectively then the set \( p \cup a \cup q \) is an \( u^3 \)-small set in \( p \cap q \).
If \( u \succ v \) and \( M = a \cup b \) where \( a \in \text{Small}(u) \), \( b \in \text{Small}(v) \) then every filter on \( M \) is consistent with either \( a \) or \( b \). So \( \overline{M} = \overline{a} \cup \overline{b} \).

2.6. **Graphs.** For a graph \( \Gamma \) we denote by \( \Gamma^0 \) and \( \Gamma^1 \) the sets of vertices and edges respectively. We do not interest in graphs with loops and multiple edges. For our purpose a graph is something that is either connected or disconnected. We often identify an edge \( e \in \Gamma^1 \) with its boundary pair \( \partial e \subset \Gamma^0 \) and write \( \cup E \) for the set \( \cup \{ \partial e : e \in E \} \) (\( E \subset \Gamma^1 \)). We consider any set of pairs as a graph.

By \( d \) or by \( d_\Gamma \) we denote the natural metric on \( \Gamma^0 \) it is the maximal among the metrics for which the distance between joined vertices is one.

A **circuit** is a connected graph with exactly two edges at each vertex. An **arc** is a graph obtained from a circuit by removing one edge. By \( \text{Arc}(\Gamma, e) \) we denote the set of all arcs in \( \Gamma \) that contain the edge \( e \in \Gamma^1 \).

A graph \( \Gamma \) is fine \[\text{Bo97}\] if each set of arcs of bounded length with fixed endpoints is finite.

2.7. **Perspectivity.** When something goes farther it looks smaller. This phenomenon is the perspectivity. A uniformity \( \mathcal{U} \) on the set of vertices of a connected graph \( \Gamma \) is said to be **perspective** if every entourage contains all but finitely many edges. We also say that \( \mathcal{U} \) is a perspectivity on \( \Gamma \).

**Proposition 2.7.1.** If a graph \( \Gamma \) possesses an exact perspectivity \( \mathcal{U} \) then it is fine.

**Proof.** Let \( n \) be the smallest positive integer for which there exists an infinite set \( P \) of arcs of length at most \( n \) joining some fixed vertices \( x, y \in \Gamma^0 \). Let an entourage \( u \) separate \( \{ x, y \} \) and let \( v = \sqrt[n]{u} \). We have \( P \subset \cup \{ \text{Arc}(\Gamma, e) : e \in \Gamma^1 \setminus v \} \). So, for some \( e \in \Gamma^1 \setminus v \), the sets \( P \cap \text{Arc}(\Gamma, e) \) is infinite that gives us a counter-example with the smaller \( n \).

Let a locally compact group \( G \) act on a uniform space \((M, \mathcal{U})\). An (ordered or non-ordered) pair \( \beta \) of points of \( M \) is said to be **perspective** if the orbit map \( G \ni g \mapsto g\beta \in \Theta^2 M \) is “proper”, i.e., for every entourage \( u \in \mathcal{U} \) the set \( \{ g \in G : g \not\in u \} \) is bounded in \( G \).

Perspectivity is a property of an orbit in \( \Theta^2 M \). On the other hand it is an equivalence relation. Hence if \( M = \Gamma^0 \) is the vertex set of a connected graph \( \Gamma \) and \( G \) acts by graph automorphisms then the perspectivity on edges implies the perspectivity of the action.

On the other hand perspectivity is a relation between a pair and an orbit of entourages. If a \( G \)-uniformity \( \mathcal{U} \) is generated by a set \( \mathcal{S} \) of entourages and \( \beta \) is perspective with respect to each \( u \in \mathcal{S} \) then \( \beta \) is perspective with respect to \( \mathcal{U} \).

**Proposition 2.7.2.** Let \( \mathcal{U} \) be an invariant exact perspectivity on a connected \( G \)-graph \( \Gamma \) where \( G \) acts properly on edges. Then \( G \) acts properly on pairs.

**Proof.** By 2.7.1 \( \Gamma \) is fine so if \( x, y \) be distinct vertices of \( \Gamma \) then the stabilizer \( \text{St}_G\{x, y\} \) acts properly on the finite set of the geodesic arcs between \( x \) and \( y \).

**Proposition 2.7.3.** Let \( \mathcal{U} \) be an invariant perspectivity on a connected \( G \)-graph \( \Gamma \) where \( G \) acts properly on edges. Let \( \iota_U : \Gamma^0 \to \overline{\Gamma^0} \) be the completion map and let \( \Delta^0 = \iota_U \Gamma^0 \) and \( \Delta^1 = (\iota_U \Gamma^1) \setminus \{ \text{loops} \} \). The action \( G \cap \Delta \) is proper on pairs.

**Proof.** By 2.7.2 it suffices to prove that the action is proper on edges. Let \( e = \{ p, q \} \in \Gamma^1 \). Suppose that the filters \( u(p) = p \), \( u(q) = q \) are distinct. Thus \( p \cap q \) does not contain \( u \)-small sets for some \( u \in \mathcal{U} \). In particular, \( e \) is not \( u \)-small. The pairs \( \{ p, hp \} \) and \( \{ q,hq \} \) belong to \( \mathcal{V} \) and we have a contradiction.
3. FLOYD MAP

3.1. Frink Lemma. The following lemma is a well-known metrisation tool of the general topology ([Bou58, §1 Prop. 2], [Ke75, Lemma 6.2], [En89, Theorem 8.1.10]). J. L. Kelly attributes it to A. H. Frink ([Fr37] noting certain contribution of other authors. R. Engelking cites a paper [Tu40] of another author.

A Frink sequence on a set $M$ is a sequence $v_n$ such that $v_0 = S^2 M$ and $v_n \supset v_{n+1} \supset \Delta^2 M$ for all $n \in \mathbb{N}$. Any Frink sequence $v_n$ determines on $M$ the Frink metric as the maximal among the metrics $\rho$ on $M$ such that $\forall n \rho |v_n| \leq 2^{-n}$ (recall that we do not require for a metric to be exact).

**Proposition 3.1.1** (A. H. Frink). Let $\delta$ denote the Frink metric on a set $M$ determined by a Frink sequence $v_n$. Then $\forall n > 0 \delta^{-1}(0, 2^{-n}) \subset v_{n-1}$. In particular the filter generated by $v_n$ is the uniformity determined by $\delta$.

For the reader’s convenience we adopt the common proof to our notation. The following definitions are valid only within the proof.

An edge of a set $F \subset \mathbb{R}$ is a pair $(x, y) \in F^2$ such that $x < y$ and $F \cap (x, y) = \emptyset$. A path is a map

\[(3.1.2) \gamma : F \to M\]

from a finite $F \subset \mathbb{R}$ such that $y - x = 2^{-\max\{n : \gamma(x, y) \in v_n\}}$ for every edge $(x, y)$ of $F$. Write $\partial \gamma = \{\min F, \max F\}$, $\operatorname{length}(\gamma) = \max F - \min F$.

**LEMMA.** If $\operatorname{length}(\gamma) < 2^{-n}$ for a path $\gamma$ then $\partial \gamma \in v_{n-1}$.

**Proof.** Induction by $|F|$. If $|F| = 1$ then trivially $\partial \gamma \in \Delta^2 M \subset v_{n-1}$. Consider a path $\gamma$ with $0 < \operatorname{length}(\gamma) = l \leq 2^{-n}$. Suppose that the assertion is true for the proper subpaths of $\gamma$.

Let $(f_-, f_+)$ be the edge of $F$, for which $\frac{l}{2}(\max F + \min F) \in [f_-, f_+]$. The length of the restrictions of $\gamma$ over the sets $F \cap [r, l]$ and $F \cap [l, r]$ is at most $\frac{l}{2} < 2^{-n-1}$. By the inductive hypotheses $\gamma \{\min F, f_+\}, \gamma \{f_-, \max F\} \subset v_n$. Furthermore, $f_+ - f_- \leq l < 2^{-n}$, hence, by definition of path, $\gamma \{f_-, f_+\} \subset v_n$. So, by the definition of Frink sequence, $\partial \gamma \subset v_{n-1}$.

The Frink distance between $p, q \in M$ is, clearly, the infimum of $\operatorname{length}(\gamma)$ over all paths $\gamma$ such that $\partial \gamma = \{p, q\}$. The assertion is now follows from the Lemma.

3.2. Frink sequence determined by a divider. For a $G$-set $M$ a set $u \subset S^2 M$ and a finite set $F \subset G$ denote $F\{u\} = \{fu : f \in F\}$. So $\cap(F\{u\})$ is a subset of $S^2 M$. It follows from (3.1.1) that

\[(3.2.1) (\cap(F\{u\}))^n \subset \cap(F\{u^n\}) \text{ for } n \geq 1.\]

We call a set $u \subset S^2 M$ divider, if it contains the diagonal $\Delta^2 M$ and there exists a finite $F \subset G$ such that $(\cap(F\{u\}))^2 \subset u$.

Example: every equivalence relation is a divider for $F = \{1\}$. We will see that every r.h. group contains a divider that determines the relatively hyperbolic structure.

**Proposition 3.2.2.** Let $u$ be a divider for a connected $G$-set $M$. The $G$-filter $U$ on $S^2 M$ generated by a set $\{u\}$ is a uniformity.

**Proof.** The sets of the form $\cap(S\{u\})$, $S \in \text{Sub}^G$ is a base for the $G$-filter $U$. It follows from (3.2.1) that $\forall S \exists S_1 : (\cap(S\{u\}))^3 \subset \cap(S\{u\})$. We say that $U$ is the uniformity generated by $u$. We denote it by $U_u$. 

Every divider \( u \) satisfies a stronger condition
\[
(3.2.3) \quad \forall m > 0 \exists F \subset G \mid |F| < \infty \land (\cap (F\{u\}))^m \subset u.
\]
Indeed, by iterating the inclusion \( (\cap (F\{u\}))^2 \subset u \) we obtain \( 3.2.3 \) for \( m = 2^k \). On the other hand \( (\cap (F\{u\}))^m \subset (\cap (F\{u\}))^n \) for \( m < n \) since \( u \) contains the diagonal. \( \square \)

Note that if \( 3.2.3 \) holds for a fixed \( m \) and \( F \) then it holds for the same \( m \) and every finite \( F \cap F \).

Let \( u \) be a divider and let \( F \) be a finite symmetric (i.e. closed under \( g \mapsto g^{-1} \)) subset of \( G \) containing \( 1 \) and such that
\[
(3.2.4) \quad (\cap (F\{u\}))^3 \subset u.
\]
For \( n \in \mathbb{N} \) let \( F^n = \{ \text{the elements of the group generated by } F \text{ of } F\text{-length } \leq n \} \). The sets \( F^n \) are symmetric. It follows from the associativity that
\[
(3.2.5) \quad F^n \cdot F^m = F^{n+m} \text{ for all } m, n \geq 0.
\]
The sequence
\[
(3.2.6) \quad u_0 = S^2 M, \ u_n = \cap (F^n\{-u\}) \text{ for } n > 0
\]
is a Frink sequence (see \( 3.1 \)) containing the divider \( u = u_1 \). Indeed, ‘\( u_1^3 \subset u_0 \)’ is trivial and ‘\( u_2^3 \subset u_1 \)’ is \( 3.2.4 \). Further, \( 3.2.3 \) implies
\[
(3.2.7) \quad u_{n+m} = \cap (F^n\{u_m\}) \text{ for all } n \geq 0, m \geq 1.
\]
For \( n \geq 3 \), using \( 3.2.7 \) and \( 3.2.1 \) we obtain
\[
u_3^n = (\cap (F^{n-2}\{u_2\}))^3 \subset (\cap (F^{n-2}\{u_3^2\}) \subset \cap (F^{n-2}\{u_1\}) = u_{n-1}.
\]
The Frink metric determined by this sequence is called the dividing metric. It depends on \( u \) and \( F \). The corresponding uniformity is called the dividing uniformity.

### 3.3. Comparing the Floyd metric with the dividing metric.

A divider \( u \) on a connected \( G\)-set \( M \) is said to be perspective if for each pair \( \beta \subset G \) it contains all but finitely many elements of the orbit \( G\{\beta\} \). Suppose that \( M \) is the vertex set \( \Gamma^0 \) of a connected graph \( \Gamma \) where \( G \) acts properly on edges and cofinitely. A divider \( u \) is perspective if and only if it contains all but finitely many \( \Gamma \)-edges.

We fix \( \Gamma \) and a perspective divider \( u \). Let a finite set \( F \) satisfy \( 3.2.4 \) and denote by \( \delta_{u,F} \) the dividing metric.

Denote by \( d \) the canonical graph metric on \( M \) (the “\( d \)-length” of each edge is 1). For a vertex \( v \in M \) and \( \lambda \in (0, 1) \) denote by \( \delta_{v,\lambda} \) the path metric on \( M \) for which the “length” of an edge \( e \) is \( \lambda^{|d(v,e)|} \).

**Proposition 3.3.1.** There exist \( \lambda \in (0, 1) \) and \( C > 0 \) such that \( \delta_{u,F} \leq C \delta_{v,\lambda} \) on \( M \).

**Proof.** Let \( B_n \equiv \{ w \in M : d(v, w) \leq n \} \). The finite set \( F\{v\} \) is contained in some \( B_\rho \). This implies \( F B_n \subset B_{n+\rho} \) for all \( n \geq 0 \). Hence
\[
(3.3.2) \quad F^k B_n \subset B_{n+k \rho} \text{ for } k, n \geq 0.
\]

Let \( \sigma \) be such that \( B_\sigma \) contains all the edges \( \not\in u \). We put
\[
(3.3.3) \quad \lambda = 2^{-1/\rho}, \quad C = 2^{\sigma/\rho}.
\]
Let \( g \in F^k \). Since \( g^{-1} \in F^k \), \( 3.3.2 \) implies \( \not\in = B_\sigma \cap g(\Gamma^0 \setminus B_{\sigma+\rho}) \) and hence \( g(\Gamma^1 \setminus S^2 B_{\sigma+\rho}) \subset u \). We thus have
\[
(3.3.4) \quad \Gamma^1 \setminus S^2 B_{\sigma+\rho} \subset (F^k\{u\}) = u_{k+1}.
\]
We now compare the \( \delta_{u,F} \)-length of an edge \( e \in \Gamma^1 \) with its \( \delta_{\Gamma,\lambda} \)-length for \( \lambda = 2^{-1/\rho} \). Let \( e \) be an edge of \( \Gamma \) not contained in \( B_\sigma \). Then there exists a unique number \( k \geq 0 \) such that \( e \subset B_{\sigma+(k+1)\rho} \setminus B_{\sigma+k\rho} \).

By \( 3.3.4 \) and the definition \( 3.1 \) of the Frink length, we have \( \delta_{u,F}(e) \leq 2^{-k-1} \). The Floyd length of \( e \) is \( \delta_{v,\lambda}(e) \geq \lambda^{\sigma+(k+1)\rho} = 2^{\sigma/\rho} \cdot 2^{-k-1} \). So we have

\[
\delta_{u,F}(e) \leq 2^{\sigma/\rho} \cdot \delta_{v,\lambda}(e)
\]

for every edge outside \( B_\sigma \). The edges inside \( B_\sigma \) also satisfy \( 3.3.5 \) since \( \delta_{u,F} \leq 1 \) everywhere.

Since the Floyd metric is maximal among those having the given length of edges, and \( \Gamma \) is connected, \( 3.3.5 \) holds for every pair \( e \in S^2 M \).

\[ \square \]

3.4. Relatively hyperbolic uniformities and the map theorem. We propose a definition of relatively hyperbolic uniformity equivalent to the other known definitions intended to the proof of the Floyd map theorem. This is a structure including a “geometric part” (graph) and “dynamical part” (divider and uniformity).

**Definition** \( \text{RH}_{pd} \). A uniformity \( \mathcal{U} \) on a connected \( G \)-set \( M \) is called relatively hyperbolic (we also say that \( \mathcal{U} \) is a relative hyperbolicity on \( M \)) if it is generated by a perspective divider (see \( 3.2 \) \( 3.3 \)) and not parabolic i.e, the \( \mathcal{U} \)-boundary \( \partial_M = \overline{M} \setminus M \) has at least two points.

The metrics \( \delta_{v,\lambda}, v \in M \) defined in \( 3.3 \) determine on \( M \) the same uniformity \( \mathcal{U}_{\Gamma,\lambda} \). We call it the exponential Floyd uniformity. It is \( G \)-invariant.

If \( \lambda_1 \leq \lambda_2 \) then \( \mathcal{U}_{\Gamma,\lambda_1} \subset \mathcal{U}_{\Gamma,\lambda_2} \). If \( \Gamma_1, \Gamma_2 \) are different connecting structures for a connected \( G \)-set \( M \) then the identity map is Lipschitz. Hence for every \( \lambda \) one has \( \mathcal{U}_{\Gamma_1,\lambda} \subset \mathcal{U}_{\Gamma_2,\lambda^{1/C}} \) where \( C \) is the maximum of the \( \Gamma_2 \)-length of the \( \Gamma_1 \)-edges.

**Proposition 3.4.6** (Floyd map theorem). Let \( G \) be a group, \( M \) a connected \( G \)-set, \( \Gamma \) a connecting graph structure for \( M \), \( \mathcal{U} \) a relatively hyperbolic uniformity on \( M \). Then there exists \( \lambda \in (0,1) \) such that \( \mathcal{U} \) is contained in the Floyd uniformity \( \mathcal{U}_{\Gamma,\lambda} \). The inclusion induces a uniformly continuous \( G \)-equivariant surjective map \( (M, \mathcal{U}_{\Gamma,\lambda}) \to (M, \mathcal{U}) \) between the completions.

**Proof.** Since the uniformities \( \mathcal{U}(\Gamma, \lambda) \) are \( G \)-invariant it suffices to prove that every perspective divider \( u \) belongs to some \( \mathcal{U}_{\Gamma,\lambda} \).

We fix \( v \in M \) and define \( B_n \) as in \( 3.3.1 \).

Let \( F \) be a finite set from \( 3.2.4 \) let \( \rho \) be such that \( F\{v\} \subset \rho \) and let \( \lambda = 2^{-1/\rho} \) as in \( 3.3.3 \). By Frink lemma \( 3.1.1 \) \( \delta^{-1}_{u,F}[0, \frac{1}{4}) \subset u \). By \( 3.3.1 \) \( \delta^{-1}_{\Gamma,\lambda}[0, \frac{1}{4}) \subset \delta^{-1}_{u,F}[0, \frac{1}{4}) \). Hence \( u \in \mathcal{U}_{\Gamma,\lambda} \).

\[ \square \]

3.5. Generalized Karlsson lemma. The following plays a basic role in our theory of r.h. groups. It is the main application of the map theorem \( 3.4.6 \).

**Proposition 3.5.1.** For every entourage \( \nu \) of the uniformity \( \mathcal{U}_u \) there exists a finite set \( E \subset \Gamma^1 \) such that \( \nu \) contains the boundary pair \( \partial I \) of every geodesic segment \( I \) that misses \( E \).

**Proof.** By \( 3.2.2 \) we can assume that \( \nu = \cap(S\{u\}) \) for a finite \( S \subset G \). Let \( F \) be a finite subset of \( G \) containing \( S \) and satisfying \( 3.2.4 \). Consider the dividing metric \( \delta_{u,F} \) from \( 3.3 \). According to the notation of \( 3.2 \) we have \( \nu = u_2 \). Let \( \lambda \) be a number from \( 3.3.1 \).

By \( 3.3.1 \) and \( 3.1.1 \) there exists \( \varepsilon_2 > 0 \) such that \( \delta_{\Gamma,\lambda} \beta \geq \varepsilon_2 \) for any \( \beta \notin u_2 \). We also need \( \varepsilon_3 > 0 \) such that \( \delta_{\Gamma,\lambda} \beta \geq \varepsilon_3 \) for any \( \beta \notin u_3 \).

Recall that the original Karlsson lemma \( [Ka03] \) claims that the Floyd length of a \( d \)-geodesic \( I \) tends to zero while \( d(v, I) \to \infty \). Applying this result to \( \delta_{\Gamma,\lambda} \) we find a number \( d \) such that if \( d(v, I) \geq d \) then
They form a base for the filter (actually all known) induce a visibility on the Cayley graph of the acting group. We will see that many convergence group actions connected graph with \( v \) to \( x \). The completion with respect to it is just the Freudenthal’s “ends completion” and the boundary is its completion to the completion of each other visibility. Hence if it exact then, by 2.7.1, the graph \( \partial I \in u_2=\mathbf{v} \).

4. Visibility

4.1. Definition and examples. We now turn a weaker version of 3.5.1 into a definition. Let \( \Gamma \) be a connected graph with \( \Gamma^0=M \). For an edge \( e \in \Gamma^1 \) define

\[
(4.1.6) \quad u_e = \{ \{x, y\} \subset \Gamma^0 : d(x, e) + d(y, e) \geq d(x, y) \}
\]

This set consists in all pairs such that no geodesic segment joining them passes through \( e \).

The filter \( \text{Vis}\Gamma \) generated by \( u_e : e \in \Gamma^1 \) is the visibility filter on \( M \).

A uniformity \( U \) on \( M \) is called a visibility on \( \Gamma \) if it is contained in \( \text{Vis}\Gamma \). Any visibility is perspective in the sense of 2.7. Hence if it exact then, by 2.7.1 the graph \( \Gamma \) is fine.

The generalized Karlsson lemma 3.5.1 implies that, for any perspective divider \( u \) for a connected \( G \)-set \( M \) and for any connecting structure \( \Gamma \) for \( M \) the uniformity \( U_u \) is a visibility on \( \Gamma \). Note that 3.5.1 actually claims something stronger: each pair of vertices that can “partially” see each other outside of a big finite set of edges is “small”.

If \( \Gamma \) is a locally finite (i.e., every vertex is adjacent to finitely many edges) \( \delta \)-hyperbolic graph then \( \text{Vis}\Gamma \) is a uniformity and hence the maximal possible visibility.

Every connected graph has the maximal visibility; we call it initial since there is a morphism from its completion to the completion of each other visibility.

If \( \Gamma \) is locally finite then the filter, generated by the collection of the sets of the form \( \mathbf{v}_E = \{ \{v, x\} : \text{the pairs that can be joined by a path not containing the edges in } E \} \), \( E \in \text{Sub} \leq \infty \Gamma \), is a visibility on \( \Gamma \). The completion with respect to it is just the Freudenthal’s “ends completion” and the boundary is the compact totally disconnected “space of ends”. We will see that many convergence group actions (actually all known) induce a visibility on the Cayley graph of the acting group.

For a finite set \( E \subset \Gamma \) we denote \( u_E = \cap \{ u_e : e \in E \} \). The sets of the form \( u_E \) are called principal.

They form a base for the filter \( \text{Vis}\Gamma \).

Proposition 4.1.1. Every visibility \( \mathcal{U} \) on a connected graph \( \Gamma \) is pre-compact.

Proof. Let \( \mathbf{u} = \sqrt{\mathcal{U}} \) for \( \mathbf{v} \in \mathcal{U} \). Since \( \mathbf{u} \in \text{Vis}\Gamma \) it contains a principal set \( u_E \). Let \( S = \cup E = \{ \text{the vertices of the edges in } E \} \). For \( x \in S \) the set \( \{ v \in \Gamma^0 : d(v, x) = d(v, S) \} \) is \( u^2 \)-small since the shortest path from \( v \) to \( x \) does not contain the edges from \( E \). So \( \Gamma^0 \) is a union of finitely many \( \mathbf{v} \)-small sets. \( \square \)
4.2. **Dynkin property.** The following proposition gives a large class of convergence group actions. We do not know examples of convergence actions outside this class. However it seems to be difficult to prove that every convergence action is a visibility action (see section 9). If follows from recent result of Mj Mahan [Mj10] that every Kleinian action of a finitely generated group is.

**Proposition 4.2.1.** Let a group $G$ act on a connected graph properly on edges and let $\mathcal{U}$ be a $G$-invariant visibility on $\Gamma$. Then the action has Dynkin property. Hence the action on the completion has the convergence property.

**Proof.** Consider such an action $G\curvearrowright \Gamma$. By [2.7] it is perspective, so, for each entourage $u \in \mathcal{U}$ and each finite set $S \subseteq \Gamma^0$ the set $\{g \in G : gS \text{ is not } u\text{-small}\}$ is finite.

Let $u, v \in \mathcal{U}$ and let $u_F, u_E$ be principal subsets of $u$ and $v$ respectively. By the above remark it suffices to show that if $\cup E \in \text{Small}(u)$ and $\cup F \in \text{Small}(v)$ then $u^3 \bowtie v^3$.

Let $A=\{x \in \Gamma^0 : d(x, E) \geq d(x, F)\}$, $B=\{x \in \Gamma^0 : d(x, F) \geq d(x, E)\}$. By construction $A$ is $u^3$-small and $B$ is $v^3$-small.

By 4.1.1 the completion $\Gamma^0$ is compact, by [2.5.1] the action $G \curvearrowright \Gamma^0$ has Dynkin property, by [Ge09] 5.3. Proposition P it is discontinuous on triples i.e., has convergence property (see 7.2).

As a corollary we have the implication $\text{RH}_{pd} \Rightarrow \text{RH}_{32}$:

**Proposition 4.2.2.** Let $\mathcal{U}$ be a relative hyperbolicity on a connected $G$-set $M$. Then the action on the completion $\overline{M}$ with respect to $\mathcal{U}$ is relatively hyperbolic in the sense $\text{RH}_{32}$.

**Proof.** By [3.5] $\mathcal{U}$ is a visibility, by 4.2.1 the action $G \curvearrowright \overline{M}$ is discontinuous on triples. It suffices to find a bounded “fundamental domain” for the action $G \curvearrowright \Theta^2 \overline{M}$.

Let $u$ be a perspective divider generating $\mathcal{U}$ and let $v=\sqrt{u}$. Let $\{p, q\} \in \Theta^2 \overline{M}$, so $p \cap q$ has no $w^3$-small sets for $w \in \mathcal{U}$. For every pair $P \subseteq p, Q \subseteq q$ of $w$-small sets we have $P \cap Q = \emptyset$.

Since $u$ generates $\mathcal{U}$ as a $G$-filter, $w$ contain a set $\cap (S\{u\})$ for $S \subseteq \text{Sub}_{<\infty}G$. Let $w_1 = \cap (S\{v\})$ and let $P, Q$ be $w_1$-small sets in $p, q$ respectively. Let $p \subseteq P, q \subseteq Q$. Since $\{p, q\} \notin w$ there exists $g \in S$ such that $\{p, q\} \notin gu$. We claim that the filter $g^{-1}P \cap g^{-1}Q$ does not contain $v$-small sets. Indeed if $R$ is such a set then it intersects $v$-small sets $g^{-1}P$ and $g^{-1}Q$ and hence $\{g^{-1}p, g^{-1}q\} \notin u$.

So the complement of the entourage $\nabla$ (see 2.4) is a bounded fundamental domain for $G \curvearrowright \Theta^2 \overline{M}$. □

A relative hyperbolicity $\mathcal{U}$ is not necessarily exact thus the completion map $t_\mathcal{U} : M \to \overline{M}$ can be not injective. But $\mathcal{U}$ is always a perspective for any connecting graph structure $\Gamma$ and hence $\Gamma$ can be replaced by another graph $\Delta$ as explained in [2.7.3] on which $\mathcal{U}$ induces an exact relative hyperbolicity. We have the following.

**Proposition 4.2.3.** Let $\mathcal{U}$ be a relative hyperbolicity on a $G$-set $M$ and let $\Gamma$ be a connected graph with $\Gamma^0=G$ where $G$ acts properly on edges. Then there exists a $G$-set $N$, an exact relative hyperbolicity $\nabla$ on $N$, a connected graph $\Delta$ with $\Delta^0=N$ and a uniformly continuous $G$-equivariant map $\varphi : M \to N$ such that the induced map $\overline{\varphi} : \overline{M} \to \overline{N}$ is a homeomorphism. □

5. **Alternative Hyperbolicity**

We are going to prove the equivalence between $\text{RH}_{pd}$ and the definition $\text{RH}_{fh}$ given in the introduction.
5.1. **Definition.** A connected graph Γ is said to be *alternatively hyperbolic* (we also say ‘alt-hyperbolic’) if the filter VisΓ is a uniformity and hence a visibility (see [4.1.1]). This means that

\[(5.1.1) \quad \forall e \in \Gamma^1 \exists F \in \text{Sub}^{\leq \Gamma_1} : u_F^2 \subset u_e.\]

In turn this means that for every \(e \in \Gamma^1\) there is finite set \(F(e) \subset \Gamma^1\) such that every geodesic triangle with \(e\) on a side contains an edge from \(F(e)\) on another side.

Every locally finite \(\delta\)-hyperbolic graph is alt-hyperbolic with \(F(e) = \{f \in \Gamma^1 : d(e, f) \leq \delta\}\). On the other hand if Γ is alt-hyperbolic and \(\{d(e, f) : f \in F(e)\}\) is uniformly bounded then it is hyperbolic.

The classes \{hyperbolic graphs\} and \{alt-hyperbolic graphs\} are not included one to the other. We will prove in 5.3.1 that the hyperbolic fine graphs are alt-hyperbolic.

5.2. **Alternative relative hyperbolicity.** We make one more step towards the equivalence of RH\(_{pd}\) to the other RH’s.

**Definition** RH\(_{ah}\). An action of a group \(G\) on a connected graph \(\Gamma\) is *relatively hyperbolic* if \(\Gamma\) is alt-hyperbolic and the action \(G \acts \Gamma^1\) is proper and cofinite and non-parabolic in the sense that no vertex is fixed by the whole \(G\).

To interpret RH\(_{ah}\)-action as an RH\(_{pd}\) we only need to indicate a divider generating the uniformity Vis\(\Gamma\). Let \(E\) be a finite set of edges intersecting each \(G\)-orbit. Then the entourage \(u_E\) is a divider. The corresponding set \(F\) from [3.2] is \(\{F(e) : e \in E\}\). The verification is straightforward.

5.3. **Hyperbolic fine graphs.** B. Bowditch noted that the Farb’s “conned-off” graph for a group relatively hyperbolic (in the “BCP sense” of Farb) with respect to a collection \(\mathcal{P}\) of subgroups, is fine. This was an important step in understanding the relative hyperbolicity.

**Proposition 5.3.1.** Every connected \(\delta\)-hyperbolic fine graph \(\Gamma\) is alt-hyperbolic.

**Proof.** This is an exercise on the common “thin triangle” techniques. Inside the proof we locally change the style and the notation, following [GhH90] Section 2.3.

We regard graphs as CW-complexes, since we need points on the edges other than the endpoints. Actually one additional point in each edge would suffice.

Let \(M\) be a geodesic metric space. We denote \(|ab| = d_M(a, b)\). The word ‘segment’ will mean ‘geodesic segment’. By \([ab]\) we denote a particular segment joining \(a\) and \(b\). The reader should understand which of possible such segments we mean.

For \(a, b, c \in M\) a triangle \(T\) is a union of segments \(S_a, S_b, S_c\) called sides such that \(\partial S_a = \{b, c\}\), \(\partial S_b = \{c, a\}\), \(\partial S_c = \{a, b\}\). The points \(a, b, c\) are the vertices of the triangle.

A tripod is a metric cone over a triple. The points of the triple are the ends and the “vertex” of the cone is its center. We regard a tripod as a triangle whose vertices are the ends. For every triangle \(T = abc\) there exists a unique (up to isometry) comparison tripod \(T’ = a’b’c’\) with center \(t’\) and a comparison map \(x \mapsto x’\) taking isometrically the \(T\)-sides onto \(T’\)-sides. It is short, i.e., \(|x’| \leq |xy|\) for all \(x, y \in T\).

A triangle \(T\) is \(\delta\)-thin if \(|xy| - |x'y'| \leq \delta\) on \(T\).

Now let \(M\) be a graph \(\Gamma\) and let \(T = abc\) be a \(\delta\)-thin triangle in \(\Gamma\) for a positive integer \(\delta\). Let \(E = a_0b_0 \in M\) be an edge on the side \(S_c\) not containing in \(S_a \cup S_b\) such that \(|ab| = |a_0a| + |a_0b_0| + |b_0b|\). We will find a circuit \(H\) in \(\Gamma\) of length \(\leq 24\delta + 6\) containing \(E\) and having an edge in \(S_a \cup S_b\).

Initially we construct the pieces of \(H\) joining \(a_0\) and \(b_0\) with \(S_a \cup S_b\).

Let \(a_1\) be a vertex of \(S_a \cap S_c\) closest to \(a_0\).

In the exceptional case \(|a_0a_1| < \delta\) denote \(a_2 = a_1\). Otherwise \(a_2 = \text{the vertex on } [aa_0] \text{ with } |a_2a_0| = \delta\).
Let $a_3$ be a vertex in $S_a \cap S_b$ closest to $a_2$. In the exceptional case $a_2 = a_3$. Otherwise $|a_2a_3| \leq \delta$ by $\delta$-thinness. We choose and fix a segment $[a_2a_3]$.

Let $a_4$ be the vertex of $[a_2a_3] \cap [a_2a_0]$ closest to $a_0$ (in the exceptional case $a_4 = a_3 = a_2 = a_1$). The arc $L_a = [a_0a_4] \cup [a_4a_3]$ (the thick line on the picture below) of length $\leq 2\delta$ joins $a_0$ with $a_3$. By the construction $L_a \cap (S_a \cup S_b) = \{a_3\}$.

In the exceptional case $L_a = [a_0, a_3] \subset S_c$.

Replacing ‘a’ by ‘b’ we define the points $b_\iota$ for $\iota \in \mathbb{I}$ and the arc $L_b$.

Claim: $L_a \cap L_b = \emptyset$. Indeed if both $L_a$ and $L_b$ are exceptional then they are disjoint segments of $S_c$. If one of them is exceptional, say $L_b$, and the other is not then both pieces $[a_0a_4]$ and $[a_4a_3]$ of $L_a$ are contained in the $\delta$-neighborhood of $a_2$ which is disjoint from $[b_0b]$. If both $L_a, L_b$ are not exceptional then they are contained in the disjoint $\delta$-neighborhoods of $a_2$ and $b_2$ respectively. So $1 \leq |a_3b_3| \leq 4\delta + 1$.

The set $L = L_a \cup E \cup L_b$ is an arc of length $\leq 4\delta + 1$ with $L \cap (S_a \cup S_b) = \emptyset$. $L$

If $a_3$ and $b_3$ belong to $S \in \{S_a, S_b\}$ then the subsegment $[a_3b_3]$ of $S$ has length $\leq 4\delta + 1$ and completes $L$ up to a circuit $H$ of length $\leq 8\delta + 2$.

Suppose now that $a_3$ and $b_3$ belong to different sides of $T$ and do not belong to $S_a \cap S_b$.

Let $t_a, t_b, t_c$ denote the preimages of the center $t'$ of $T'$ in the corresponding sides. For $x \in S_c$ we have

\[(5.3.2) \quad |xt_c| \leq \delta + \max\{d(x, S_a), d(x, S_b)\}\]

Indeed if $x \in [at_c]$ then, for $p \in S_a$, the thinness inequality yields: $|xt_c| = |x't'| \leq |x'p'| \leq |xp| + \delta$.

By \[(5.3.2)\] the distance from $c_t$ to each of $a_0, b_0$ is at most $3\delta + 1$. By the same reason the distance from each of $a_0, b_0$ to the corresponding $t_7$ is at most $5\delta + 1$.

Let $[a_3c]$ be the subsegment of the side in $\{S_a, S_b\}$ that contains $a_3$. Similarly define $[b_3c]$.

Let $a_5 \in [a_3c] \cap [b_3c]$ be the closest to $a_3$. Since $a_5 \neq a_3$ the segment $[a_3a_5]$ contains an edge of $S_a \cup S_b$. If $|a_3a_5| \leq 7\delta + 2$ then $H = L \cup [a_3a_5] \cup [a_5b_3]$ is a circuit of length $\leq 2(7\delta + 2 + 4\delta + 1) = 22\delta + 6$ with the desired properties.

So we can assume that $|a_3a_5| > 7\delta + 2$.

We actually repeat the construction of $L_a$: let $a_6 \in [a_3, a_5]$ be such that $|a_3a_6| = 7\delta + 2$. The comparison image $a_6'$ belongs to $[t'c']$ and $|a_6't'| > 7\delta + 2 - (5\delta + 1) = 2\delta + 1$. Hence $d(a_6, S_c) \geq 2\delta + 1$.

Let $a_7$ be a point in the side containing $b_3$ closest to $a_6$. By thinness, $|a_6a_7| \leq \delta$.

We choose a segment $I = [a_6, a_7]$. The distance from each $x \in I$ to $S_c$ is $\geq 1$ so it does not contain $E$.

Let $a_8 \in [a_3a_6] \cap [a_6a_7]$ be the closest to $a_3$. Since $|a_3a_8| \geq 6\delta + 2$ the segment $[a_3a_8]$ contains an edge in $S_a \cup S_b$. The length of the circuit $H = L \cup [a_3a_8] \cup [a_8a_7] \cup [a_7b_3]$ is $\leq 2(4\delta + 1 + 7\delta + 2 + \delta) = 24\delta + 6$. 

---

\[L \cup L_b \]

\[S_a \cup S_b \]

---

\[S_c \]

\[a_4 \]

\[a_1 \]

---

\[a_3 \]

---

\[a_0 \]

---

\[S_a \]

\[a_2 \]

\[S_b \]

---

\[b_0 \]

---

\[t_a \]

\[t_b \]

\[t_c \]

---

\[S_c \]

\[a_5 \]

---

\[a_7 \]

---

\[a_8 \]

---

\[a_6 \]

---

\[a_6' \]
So $\Gamma$ is alt-hyperbolic for $F(e) = \{e\} \cup \{\text{the edges of the circuits of length } \leq 24\delta + 6 \text{ containing } e\}$. □

The proposition just proved shows that $\text{RH}_{\text{fh}} \Rightarrow \text{RH}_{\text{ah}}$. Since the implication $\text{RH}_{32} \Rightarrow \text{RH}_{\text{fh}}$ is known (see [1.8]) this completes the proof of the equivalence between $\text{RH}_{\text{pd}}$ and the other definitions of relative hyperbolicity.

6. More preliminaries

The main purpose of the rest of the paper is the implication $\text{RH}_{32} \Rightarrow \text{RH}_{\text{pd}}$. We give a proof under the following restriction: the $G$-set $M$ of non-conical points is connected. This is trivially true for finitely generated groups. In [GP10] we will prove that $M$ is always connected.

There is also a “non-main” purpose: to derive a general theory of convergence group actions preparing some tools for the other theorems. So our exposition is not “absolutely minimal”. We are trying to facilitate reading for those readers who is interested only in our main purpose.

This section is a continuation of the preliminary Section 2. Most of the information therein is widely known and can be found in the common sources.

6.1. Actions and representations. Let $A, X, Y$ be sets. Denote by $\text{Mp}(X, Y)$ the set of maps $X \to Y$. We consider an arbitrary map $\rho : A \to \text{Mp}(X, Y)$ as a representation of a set $A$ by maps $X \to Y$ and as families of maps $X \to Y$ indexed by $A$. Our families of maps will be families of homeomorphisms. However this assumption is not always necessary. In most cases $A$ is a group, $X=Y$ (we refer to such cases as symmetric), and $\rho$ is a homomorphism, but sometimes we need to consider non-symmetric cases.

Denote by $\text{Bj}(X, Y)$ the set of bijective maps $X \to Y$. For a family $\rho : A \to \text{Bj}(X, Y)$ a subproduct $U \times V$ of $X \times Y$ is invariant if $\rho(a)$ maps $U$ onto $V$ for each $a \in A$.

We can regard the maps $\alpha : A \times X \to Y$ as actions of a set $A$ of “operators” on a set $X$ of “points” with values in another set $Y$.

Every set $G \subseteq X$ such that $\alpha(A \times G) = Y$ is called generating. A generating set is sometimes called ‘fundamental domain’ for the action.

The “exponential law” $\text{Mp}(A, \text{Mp}(X, Y)) \simeq \text{Mp}(A \times X, Y)$ is a natural bijection
\[
\{\text{representations}\} \leftrightarrow \{\text{actions}\}.
\]
If $\rho$ is a representation and $\alpha$ is an action then by $\rho_\ast, \alpha_\ast$ we denote respectively the corresponding action and the corresponding representation. In the symmetric case $\alpha$ is a group action if and only if $\alpha_\ast$ is a homomorphism of groups.

In case when our sets $A, X, Y$ are topological spaces we suppose by default that the actions are continuous. However sometimes we have to prove the continuity of an action given by some construction. When an action $\alpha$ is continuous then $\alpha_\ast A$ is contained in the set $\text{Top}(X, Y)$ of continuous maps $X \to Y$. Moreover the representation $\alpha_\ast$ is continuous with respect to certain topology on $\text{Top}(X, Y)$. In case when $X$ is locally compact (which is always the case in this paper) this topology is compact-open.

Our default topology on $\text{Top}(X, Y)$ is compact-open. It is well-known (see, e.g. [Du66] Theorem xii.3.1, [McL98] vii.8]) that if $X$ is locally compact then the exponential law gives one-to-one correspondence between continuous actions and continuous representations:
\[
(6.1.3) \quad \text{Top}(A, \text{Top}(X, Y)) \simeq \text{Top}(A \times X, Y)
\]
(Moreover, this is a homeomorphism; we do not need this fact).
For every set \( A \subset \text{Top}(X, Y) \) the inclusion map can be regarded as a representation. The corresponding action of the space \( A \) is an \textit{evaluation} action.

An action \( \alpha : A \times X \to Y \) is \textit{cocompact} if it possesses a compact generating set.

6.2. \textbf{Morphisms.} Let \( \rho_i : L_i \to \text{Homeo}(X_i, Y_i) \) for \( i \in \{0, 1\} \) be continuous locally compact families of homeomorphisms between compactums. A \textit{morphism} \( \rho_0 \to \rho_1 \) is a triple

\[
(6.2.1) \quad (L_0 \xrightarrow{\alpha} L_1, X_0 \xrightarrow{\beta} X_1, Y_0 \xrightarrow{\gamma} Y_1)
\]

of continuous maps with \( \alpha \) \textit{proper} respecting the actions in a natural way. If \( \mu : A \times X \to Y \) is an action then \( (\mu_*, \text{id}_X, \text{id}_Y) \) is the \textit{tautological} morphism from \( \mu \) to the evaluation action \( (\mu_*, A) \times X \to Y \).

6.3. \textbf{Group actions.} For a topological group \( G \) a \textit{G-space} is a topological space \( T \) where \( G \) acts continuously by homeomorphisms. A subspace of a \( G \)-space \( T \) is \textit{G-bounded} if its image in the quotient space \( T/G \) is bounded. In the discrete case \( G \)-bounded sets are \( G \)-finite.

6.4. \textbf{Compactification of a locally compact family.} For a locally compact space \( B \) denote by \( \hat{B} \) the one-point compactification \( \hat{B} = B \cup \{\infty_B\} \).

For a closed subset \( R \) of a compactum \( S \) denote by \( S/R \) the result of collapsing \( R \) to a point. On the categorical language \( S/R \) is the pushout space of the diagram \( 1 \leftarrow R \xrightarrow{i} S \).

The immediate consequence of the above consideration is the following

\textbf{Proposition 6.4.1.} Let \( S \) be a compactum, \( R \subset \text{Closed} S \) and \( f : B \to S\setminus R \) be a continuous map from a locally compact Hausdorff space \( B \). The following properties are equivalent:

\begin{itemize}
\item[a] \( f \) is proper;
\item[b] the map \( \hat{f} : \hat{B} \to S/R \) that maps \( \infty_B \) to the point \( R \) is continuous.
\end{itemize}

A continuous map \( f : B \to S \) is called \( R \)-\textit{compactifiable} if \( \emptyset = R \cap f B \) and, as a map \( f : B \to S\setminus R \), it satisfies conditions a–b of 6.4.1. For such a map the top horizontal arrow of the pullback square

\[
(6.4.2) \quad \begin{array}{ccc}
C & \xrightarrow{f+R} & S \\
\downarrow & & \downarrow \text{pr} \\
B & \xrightarrow{\hat{f}} & S/R
\end{array}
\]

is called the \( R \)-\textit{compactification}. The corresponding space \( C \) is the union of homeomorphic copies of \( B \) and \( R \). Thus we denote it by \( B+fR \).

The following is an easy consequence of the definitions.

\textbf{Proposition 6.4.2.} Let \( S, R, B \) and \( f \) be as in 6.4.1. If \( F \subset \text{Closed} S \) and \( \overline{fB} \subset F \subset R \) then \( f \) is \( R \)-compactifiable if and only if it is \( F \)-compactifiable.

6.5. \textbf{Vietoris topology.} As in [Ge09] we need certain facts about the Vietoris topology. Most of them are simple exercises in general topology.

Let \( Z \) be a compactum. On the set \( Z \equiv \text{Closed} Z \) consider the topology \( OG \) defined by declaring \textit{open} the sets of the form \( o \downarrow = Z \cap \text{Sub}(o) \) (\( o \in \text{Open} Z \)); and the topology \( CG \) defined by declaring \textit{closed} the sets of the form \( c \downarrow \) (\( c \in Z \)). The sum \( Vi \equiv OG+CG \) is the \textit{Vietoris topology}. It will be our default topology on \( Z \).
The space $Z$ is a compactum. If $u \in \text{Ent}(Z)$ then $V_i(u) = \{(c,d) \in S^2Z : c \subseteq du, d \subseteq cu\} \in \text{Ent}Z$. The operator $u \mapsto V_i(u)$ preserves inclusions and maps cofinal subsets of $\text{Ent}Z$ to cofinal subsets of $\text{Ent}Z$.

The union map $Z^2 \to Z$, $(c_0, c_1) \mapsto c_0 \cup c_1$ is continuous with respect to the topologies $\text{OG}, \text{CG}$ and hence with respect to $V_i$.

Every continuous map $f : Z_0 \to Z_1$ between compactums induces the map $f_* : \text{Closed}Z_0 \to \text{Closed}Z_1$ continuous with respect to each of the three topologies. If $f$ is injective then the map $f^* : \text{Closed}Z_0 \leftarrow \text{Closed}Z_1$ of taking preimage is $\text{OG}$-continuous, but not necessarily $V_i$-continuous.

Let $Z$ and $\mathcal{Z}$ be as above.

**Lemma.** The set $\Delta = \{(p,q) \in Z^2 : p \supseteq q\}$ is closed in the topology $\text{OG} \times \text{CG}$ on $Z^2$.

**Proof.** If $p \supseteq q$ then $\exists q \in q' \subseteq p$. Let $o, v$ be disjoint open neighborhoods of $p$ and $q$ respectively. Then $o^i \times (v')^j$ is an $\text{OG} \times \text{CG}$-neighborhood of the point $(p,q) \in Z^2$ disjoint from $\Delta$. \hfill $\Box$

For a set $C \subseteq Z$ let $C^{(i)} = \bigcup\{c^i : c \in C\}$.

**Corollary.** For any $C \subseteq \text{Closed}Z$ the set $\text{Loc}_{(\mathcal{Z}, \text{OG})} C \cap \text{Closed}_{\text{CG}} Z$ generates the filter $\text{Loc}_{V_i}(C^{(i)})$.

**Proof.** If $x \notin C^{(i)}$ then $\emptyset = \Delta \cap C \times \{x\}$. Since $C$ is compact (with respect to $V_i$ and hence with respect to weaker topologies $\text{OG}$ and $\text{CG}$) by Lemma and the Waller Theorem [2.2.1] there exist a $\text{OG}$-open $A \supseteq C$ and a $\text{CG}$-open $B \supseteq \{x\}$ such that $\emptyset = \Delta \cap A \times B$. Thus $B'$ is a $\text{CG}$-closed $\text{OG}$-neighborhood of $C$ and hence of $C^{(i)}$ that does not contain $x$. The result follows from [2.2.1]. \hfill $\Box$

For a collection $S$ of sets denote by $\text{min}S$ the set of all minimal elements of $S$.

**Proposition 6.5.1.** If $C, S \subseteq \text{Closed}_{V_i} Z, C \subseteq \text{min}S$ then every $V_i$-neighborhood of $C$ in $S$ contains an $\text{OG}$-neighborhood of $C$. That is $\text{Loc}_{(S, V_i)} C = \text{Loc}_{(S, \text{OG})} C$.

**Proof.** For a $V_i$-open neighborhood $V$ of $C$ in $S$ the $V_i$-open subset $V \cup S'$ of $Z$ contains $C^{(i)}$ since $S \cap C^{(i)} = C$. By Corollary, $V \cup S'$ contains some $W \in \text{Loc}_{\text{OG}}(C^{(i)})$, so we have $S \cap W \subseteq S \cap V$ as required. \hfill $\Box$

The group $\text{Homeo}(Z, Z)$ acts naturally on $Z = \text{Closed}Z$. This action is continuous since if a homeomorphism is uniformly $u$-close to the identity map then $\{c, \varphi c\} \subseteq V_i(u)$ for every $c \in Z$. Taking into account the correspondence between actions and representations we obtain

**Proposition 6.5.2.** If a topological group acts continuously on a compactum $Z$ then the induced action on the space $\text{Closed}Z$ is continuous.

6.6. **Graphical embeddings and quasi-homeomorphisms.** Let $X, Y$ be compactums and let $Z = \text{Ent}X \times \text{Ent}Y$ the set

$u \cdot v = \{(p_0, q_0), (p_1, q_1)\} : (p_0, p_1) \in u, (q_0, q_1) \in v\}$

is an entourage of $Z$. The operator $(u, v) \mapsto u \cdot v$ preserves inclusions. If $\mathcal{P}$ is cofinal in $\text{Ent}X$ and $\mathcal{Q}$ is cofinal in $\text{Ent}Y$ then $u \cdot v : (u, v) \in \mathcal{P} \times \mathcal{Q}$ is cofinal in $\text{Ent}Z$.

The set $S = \text{Surj}Z$ of all surjective closed subsets is $\text{OG}$-closed (hence $V_i$-closed) in $Z$.

By assigning to a continuous map $f : X \to Y$ its graph $\gamma f = \{(p, fp) : p \in X\}$ we obtain the graphical embedding $\text{Top}(X, Y) \to Z$ which is a homeomorphism onto its image. We identify continuous maps with their graphs. So the set $\mathcal{H} = \text{Homeo}Z \supseteq \text{Homeo}(X, Y)$ of homeomorphisms is a subset of $\text{min}S$ (see [6,5]).
The closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ in $Z$ is contained in $S$ but not in $\min S$ in general. The points of $\overline{\mathcal{H}}$ are the quasi-homeomorphisms from $X$ to $Y$. For a set $H \subset \mathcal{H}$ the points of the remainder $\overline{\mathcal{H}} \setminus H$ are the limit quasi-homeomorphisms of $H$.

A subset $s$ of $Z$ is single-valued at a point $p \in X \sqcup Y$ if the intersection of $s$ with the fiber over $p$ is a single point. A closed set $s$ belongs to $\mathcal{H}$ if and only if it is single-valued at every point of $X \sqcup Y$.

**Proposition 6.6.1.** A set $\Phi \subset \mathcal{H}$ is equicontinuous at a point $p \in X$ if and only if every $s \in \Phi$ is single-valued at $p$.

**Proof.** Assume that $\Phi$ is equicontinuous at $p$. Let $(p, q_0), (p, q_1) \in s \in \Phi$.

For an entourage $v \in \text{Ent} Y$ there exists a neighborhood $p$ of $p$ such that $\varphi p$ is $v$-small for all $\varphi \in \Phi$. Let $u$ be an entourage of $X$ such that $pu \subset p$ and let $\varphi \in \Phi$ be a homeomorphism contained in the $u \times v$-neighborhood of $s$ (see 6.5). Let $(p_0, \varphi_0), (p_1, \varphi_1)$ be points $u \times v$-close to $(p, q_0)$ and $(p, q_1)$ respectively. We have $\{q_0, q_1\} \in v^3$. Since $v$ is an arbitrary entourage of $Y$, we have $q_0 = q_1$.

Assume that $\Phi$ is not equicontinuous at $p$. Thus there exists an entourage $v$ of $Y$ such that for each neighborhood $p$ of $p$ the set $\Phi_p = \{\varphi \in \Phi : \varphi p \notin \text{Small}(v^3)\}$ is nonempty. We have a filtered family $p \mapsto \Phi_p$ in a compactum $Z$. So it possesses an accumulation point $s$. The set $q = s(p)$ is not $v$-small since $qv$ contains a non-$v^3$-small pair.

Remark. The proof of 6.6.1 remains valid in a more general situation when $\Phi$ is a set of continuous maps $X \to Y$ (not necessarily homeomorphisms). We do not need this generalization.

**Proposition 6.6.2.** Let $S$ be a closed set of quasihomeomorphisms single-valued at a point $p \in X$. Then the natural evaluation map $\text{ev}_p : S \ni s \mapsto sp \in Y$ is continuous.

**Proof.** If $f \in \text{Closed} Y$ then $\text{ev}_p^{-1} f = \{s : s \cap p \times f \neq \emptyset\}$ is OG-closed. \]

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### 7. Convergence property

**7.1. Crosses and their neighborhoods**. In this section $X, Y, Z, S, \mathcal{H}$ mean the same as in 6.6.

For $p = (r, a) \in Z$ the set $p^\times \coloneqq r \times Y \cup X \times a = (t' \times a')'$ is a cross with center $p$. The point $r$ is the repeller and $a$ is the attractor for $p^\times$. A cross $(r, a)^\times$ is said to be diagonal if $r = a$.

The cross map $\zeta_p : p \mapsto p^\times$ is a continuous map $Z \to Z$ (see 6.5). So its image $\text{Im} \zeta_Z$ is a closed subset of $S$. A cross is not a minimal element of $S$ if and only if $|Z| > 1$ and its center is an isolated point of $Z$. So the set $C = \{\text{the crosses with nonisolated center}\}$ is closed in $S$. It is contained in $\min S$. Note that it is empty if $Z$ is finite.

**Proposition 7.1.1.** If $|Z| > 1$ and a cross $c$ is a quasihomeomorphism then its repeller and attractor are nonisolated in $X$ and $Y$ respectively. In particular $c$ belongs to $C$. If $|X| \geq 3$ and a cross $c$ contain a quasihomeomorphism $q$ then $c = q$.

**Proof.** If $Z$ is finite then $Z$ is also finite and the topology $V_i$ is discrete. Since $|Z| > 1$ any cross is not a homeomorphism and hence not a quasihomeomorphism. If $|X| \geq 3$ then no homeomorphism is contained in a cross.

Suppose $|Z| = \infty$.

Let $c = (r, a)^\times$. Assume that $r$ is isolated. Let $a$ be a neighborhood of $a$ such that $a' \supseteq Y \setminus a$ is not a single point. Then the neighborhood $\{r\} \times Y \cup X \times a = (t' \times a')'$ of $c$ and hence of $q$ does not contain a homeomorphism.

Thus $r, a$ are isolated hence $c$ is a minimal surjective correspondence. Hence $c = q$. \[
A set $\Phi \subset H$ is $\times$-compactifiable if $\Phi \cup C$ is closed. In terms of [6.4] ‘$\times$-compactifiable’ means that the inclusion $\Phi \hookrightarrow S$ is a $C$-compactifiable family.

Since $H \subset \min S$, it follows from [6.5.1] that each neighborhood in $S$ of any closed subset $D$ of $H \cup C$ contains an $OG\subset H$-neighborhood. In particular the topologies on $D$ induced by $V_1$ and by $OG$ coincide.

Now [2.2.4] implies

**Proposition 7.1.2.** Every neighborhood in $S$ of a cross $(r, a)^\times \subset \min S$ contains a neighborhood of the form $(\{(r' \times a')\}^\perp)$ where $r \in \Loc X r$ and $a \in \Loc Y a$.

Consider now the composition [2.2.7] of crosses as binary correspondences. Note that $(a, b)^\times \circ (c, d)^\times = (a, d)^\times$ if $b \neq c$.

**Proposition 7.1.3.** The composition [2.2.7] of binary correspondences is continuous at each point $p = ((a, b)^\times, (c, d)^\times)$ such that $b \neq c$ and $(a, d)$ is not isolated in $X \times X$.

**Proof.** In view of [7.1.2] consider a neighborhood $U = ((a' \times d')^\perp)$ where $a \in \Loc X a$, $d \in \Loc Y d$. If $b, c$ are disjoint neighborhoods of $b$ and $c$ respectively then

$$(a \times Y \cup X \times b) \circ (c \times U \cup Y \times d) = (a \times U \cup X \times d)$$

hence the composition map takes the neighborhood $((a' \times b')^\perp \times ((c' \times d')^\perp$ of $p$ to $U$. \hfill \Box

### 7.2. Characteristic properties of convergence actions.

Let $X$, $Y$ be compactums of cardinality $\geq 3$. The following properties of a locally compact family $\rho : A \to \Homeo(X, Y)$ of homeomorphisms are equivalent [Ge09] Proposition P, subsection 17]:

- **a**: $\rho$ is 3-proper, i.e. the induced family $\Theta^3 \rho : A \to \Homeo(\Theta^3 X, \Theta^3 Y)$ is proper;
- **b**: $\rho$ is $\times$-compactifiable i.e., $C$-compactifiable in the sense of [6.4.1] as a map $A \to Z$;
- **c**: $\rho$ is Dynkin i.e., for every entourages $u \in \Ent X, v \in \Ent Y$ the set $\{a \in A : v \not\# a' u\}$ is bounded in $A$.

A family satisfying the properties $a$–$c$ is called a $\times$-family or a $\times$-representation. The corresponding action is a $\times$-action or an action with the convergence property.

Remark 1. If $A$ is not compact then the set $\rho A$ possesses limit crosses. Thus $X$ contains nonisolated points, i.e., $|X| = \infty$.

Remark 2. If follows from [6.4.2] that ‘$C$-compactifiable’ family of homeomorphisms is the same as ‘$Z \{\times\}$-compactifiable’. We will consider the compactifications described in [6.4].

### 7.3. Limit set operators.

For a $\times$-action $\mu : L \times X \to Y$ and a set $S \subset L$ denote by $\partial_\mu S$ the set of the centers of the limit crosses of a subset $S \subset L$ and by $\partial_{\mu,0} S$, $\partial_{\mu,1} S$ the projections of $\partial_\mu S$ onto $X$ and $Y$ respectively. We call $\partial_\mu S$, $\partial_{\mu,0} S$, $\partial_{\mu,1} S$ respectively the $\times$-remainder of $S$, the repelling set, and the attracting set. We sometimes omit the index ‘$\mu$’ writing $\partial_\mu, \partial_{0}, \partial_{1}$ for these operators.

**Proposition 7.3.1.** Let $\mu : L \times X \to Y$ be a $\times$-action and let $W = U \times V \in Z$ be an invariant closed subproduct. If $W \supset \partial_\mu L$ then the restriction $\lambda$ onto $U' \times V'$ is proper. If $|U| \geq 2$ then $\partial_\mu L \subset W$. If $|U| \geq 3$ then $\nu = \mu_{|L \times U}$ is an $\times$-action and $\partial_\mu L = \partial_\nu L$.

**Proof.** We can assume that $L$ is not compact thus the $\times$-remainder $\partial_\mu L$ is nonempty. In particular there is at least one homeomorphism $U \to V$ and hence $|V| = |U|$. Let $K \subset U'$, $L \subset V'$ be compact subsets. If $S := \{g \in L : gK \cap L \neq \emptyset\}$ is not bounded in $L$ then it possesses a limit cross $(r, a)^\times$. On the other hand, $X' \setminus K \in \Loc X r$ and $Y \setminus L \in \Loc Y a$ thus there exists $g \in S$ such that $gK \subset Y \setminus L$. A contradiction.

Suppose that $|U| \geq 2$. Let $(r, a) \in \partial_\mu L$ and let $p \in U \setminus r$. 

For an arbitrary neighborhood $a$ of $a$ there exists a homeomorphism $\varphi \in \mu_* L$ contained in the neighborhood $(p \times a')'$ of the cross $(r, a)^\times$. So $\varphi p \in V \cap a$. So $a \in V = V$.

By the same reason we have $r \in U$.

If $|U| \geq 3$ then the restriction $\nu$ of a 3-proper action $\mu$ is obviously 3-proper. The restriction map $Z \to W \Rightarrow \text{Closed } W$ maps limit crosses for $\mu$ to crosses. It is $\text{OG}$-continuous, see [6.5] Thus the restriction over the compactum $\mu_* L \cup C$ is continuous. Hence it maps the closure surjectively onto the closure of the image.

**Corollary.** Let $\mu : G \times T \to T$ be a $\times$-action of a locally compact group on a compactum $T$ and let $U$ be a closed invariant subset of $T$ containing at least two points. Then the limit set $\Lambda G = \partial_0 G = \partial_1 G$ is contained in $U$.

The points of the limit set are the limit points of the action.

For a $\times$-action $\mu : L \times X \to Y$, an entourage $v \in \text{Ent } Y$ and $F \in \text{Closed } X$ put $\text{Big}_\mu(F, v) = \{g \in L : gF \notin \text{Small } (v)\}$.

**Proposition 7.3.2.** $\partial_0 \text{Big}_\mu(F, v) \subset F$

*Proof.* Let $(r, a)^\times$ be a limit cross for $\text{Big}_\mu(F, v)$. Assume that $r \notin F$. Let $a$ be a $v$-small neighborhood of $a$. For some $g \in \text{Big}_\mu(F, v)$ the homeomorphism $\mu_* g$ is contained in the neighborhood $F' \times Y \cup X \times a$ of $(r, a)^\times$. It maps $F$ into a $v$-small set $a$. A contradiction. □

7.4. **Elementary $\times$-actions.** Let $\mu : G \times T \to T$ be a $\times$-action of a locally compact group $G$. It follows from [7.1.3] that the $\times$-remainder $\partial G$ is closed under the **partial** operation

\[
(a, b) \cdot (c, d) = \begin{cases} 
(a, d) & \text{if } b \neq c \\
\text{undefined} & \text{if } b = c
\end{cases}
\]

and symmetric i.e. invariant under the transposition map $(p, q) \mapsto (q, p)$.

The following “algebraic” lemma describes the sets of this types.

**Proposition 7.4.2.** Let $T$ be a set and let $D$ be a symmetric subset of $T^2$ closed under the partial operation [7.4.7]. Then one of the following is true:

a: $D$ has the form $\{(p, q), (q, p)\}$;

b: there exists $p \in T$ such that $(p, p) \in D \subset (p, p)^\times$;

c: $D = L^2$ for $L \subset T$.

*Proof.* Suppose that $D$ is not contained in a diagonal cross. To prove that $D$ is of type (c) it suffices to prove that $(a, b) \notin D \Rightarrow (a, a) \notin D$. Suppose $(a, b) \in D$. Since $D$ is symmetric we have $(b, a) \in D$. Since $D$ is not contained in $(b, b)^\times$ there exists $(r, s) \in D$ such that $b \notin \{r, s\}$. We have $(a, a) = (a, b) \cdot (r, s) \cdot (b, a) \in D$. □

Remark. A similar statement is true without the assumption of symmetry. Only the case (c) should be modified: $D = A \times B$ for $A, B \subset T$. This can be used in the description of the $\times$-remainder of a subsemigroup of a $\times$-acting group. We do not need this in this article.

We use the following simple observation of [Ge09, subsection 10]. A set $D$ of ordered pairs is called **2-narrow**, if the following equivalent conditions hold:

- $D$ contains no 3-matching, i.e., a triple whose both projections are triples;
- $D$ is contained in the union of two fibers.

A $\times$-action $\mu : G \times T \to T$ of a locally compact group $G$ is called **elementary** if $\partial_1 G$ is 2-narrow.
Proposition 7.4.3. If a $\times$-action $\mu : G \times T \to T$ of a locally compact group is not elementary then each point of $\Lambda G$ is non-isolated in $\Lambda G$ and $\partial G = (\Lambda G)^2$.

Proof follows immediately from [7.4.2 7.3.1] and 7.1.1.

Corollary. A $\times$-action $G \cap T$ possesses fixed points if and only if $\partial G$ has type (a) or (b) of [7.4.2] 7.4.2.

Proof. Let $\partial G = \{(p, q), (q, p)\}$ with $p \neq q$. We will prove that the action fixes $p$ and $q$. Indeed, the set $\{p, q\}$ is invariant. Suppose that some $g \in G$ transposes $p$ and $q$. If $h \in G$ is sufficiently close to the cross $(p, q)^x$ then $hgh^{-1}$ is close to the cross $(p, p)^x$ which is impossible.

If $\partial G$ has type (b) then $p$ is the unique fixed point.

On the other hand if $\partial G$ contains $L^2$ with $|L| \geq 2$ and $p \in T$ then there is a limit cross $(r, a)^x$ with $r \neq p$ and $a \neq p$. A homeomorphism close to such cross can not fix the point $p$. □

A $\times$-action with a unique limit cross is called parabolic. The limit cross of a parabolic action has the form $(p, q)^x$ (so it is of type (b)) and the point $p$ is fixed.

Every noncompact locally compact group $G$ possesses a parabolic action: it is the action on the one-point compactification.

7.5. Cones and perspectivity. Let $\mu : L \times X \to Y$ be a $\times$-action. An unbounded set $S \subseteq L$ is a cone if $\partial_1 S \cap \mu(S \times \partial_0 S) = \emptyset$. An unbounded subset of a cone is a cone. Since continuous maps take closure to closure, the following immediately follows:

Proposition 7.5.1. Let $\mu_i : L_i \times X_i \to Y_i$ be $\times$-actions for $i \in \{0, 1\}$ and let $(\alpha, \beta, \gamma)$ be a morphism $\mu_0 \to \mu_1$. If $S \subseteq L_0$ and $\alpha S$ is a cone then $S$ is a cone.

Lemma. If $S$ is a cone for a $\times$-action $\mu : L \times X \to Y$ then $|\partial_0 S| = 1$.

Proof. Otherwise there are $(r, a), (s, b) \in \partial S$ with $r \neq s$. For arbitrary $b \in \text{Loc}_Y b$ there exists $g \in S$ such that $\mu(g, r) \in b$. This implies that $b \in \partial_1 S \cap \mu(S \times \partial_0 S)$. □

The unique point of $\partial_0 S$ of a cone $S$ is its vertex. A point $p$ is conical if it is a vertex of a cone. From [7.5.1] there follows

Proposition 7.5.2. Let $\mu_i : L_i \times X_i \to Y_i$ be $\times$-actions for $i \in \{0, 1\}$ and let $(\alpha, \beta, \gamma)$ be a morphism $\mu_0 \to \mu_1$. If $p \in X_1$ is a conical point for $\mu_1$ then the set $\beta^{-1} p$ is a single point which is conical for $\mu_0$.

In [Ge09, subsection 7] we used another definition of a conical point. We now prove the equivalence of the two definitions.

Proposition 7.5.3. Let $\mu : L \times X \to Y$ be an $\times$-action and let $\beta = \{p, q\} \in \Theta^2 X$. The following properties of a set $S \subseteq L$ are equivalent:

a: the set $\{S\}^\beta = \{g \beta : g \in S\}$ is bounded in $\Theta^2 Y$;

b: $S$ is a union of a bounded set and finitely many cones with vertices in $\beta$.

Proof. $a \Rightarrow b$: If $(r, a) \in \partial S$ then $r \in \beta$ since otherwise $\{S\}^\beta$ contains arbitrarily small pairs. For disjoint closed neighborhoods $P, Q$ of the sets $(p \times Y)^{\times}$ and $(q \times Y)^{\times}$ respectively, $S$ is the union of its intersections with $\mu^{-1} P$ and $\mu^{-1} Q$ and a bounded set.

We can assume that $\partial_0 S = \{p\}$.

Since $\{S\}^\beta$ is bounded in $\Theta^2 Y$ it is contained is a union of finitely many closed subproducts $p \times q \subseteq \Theta^2 Y$. So we can assume that $\{S\} \cap p \times q$ for disjoint $p, q \in \text{Closed}_Y$. Let $a \in \partial_1 S$. Since $S q$ intersects arbitrary neighborhood of $a$ we have $a \in q$. Thus $S$ is a cone.
b\Rightarrow a: It suffices to consider the case when $S$ is a cone with vertex $p$. By definition the closed sets $\mu(S\times p)$ and $\partial_1 S$ are disjoint. Let $p,q$ be disjoint closed neighborhoods of these sets. Then the set $\{s\in S: s(p,q) \notin p\times q\}$ has no limit crosses and therefore is bounded.

**Corollary.** A point $p\in X$ is conical for $\mu$ if and only if there exists an unbounded set $S\subseteq L$ such that $\{S\{p,q\}$ is bounded in $\Theta^2 Y$ for every $q\in X\setminus p$.

So this new definition of a conical point is equivalent to that of [Ge09].

Denote by $NC_{\mu}$ the set of all non-conical points for a $\times$-action $\mu : L\times X \rightarrow Y$.

**Proposition 7.5.4.** For an $\times$-action $\mu : L\times X \rightarrow Y$, each pair $\beta\in NC_{\mu}=\{\text{non-conical points for } \mu\}$ is perspective i.e, the orbit map $L\ni g \mapsto g\beta\in \Theta^2 Y$ is proper.

**Proof.** Follows immediately from 7.5.3.

**Proposition 7.5.5.** For an $\times$-action $\mu : L\times X \rightarrow Y$, each closed set $B \subseteq L\setminus \partial_1 L$ is perspective i.e, for every $u\in \text{Ent} Y$ the set $\text{Big}(B,u)$ (see 7.3.2) is bounded.

**Proof.** by 7.3.3 $\partial_1 \text{Big}(B,u)=\emptyset$. So it is bounded.

7.6. **Image and preimage of a $\times$-action.** A map $f:S\rightarrow T$ is ramified over a point $p\in T$ if $|f^{-1}p|\geq 2$.

**Proposition 7.6.1.** Let a locally compact group $G$ act on compactums $X,Y$ and let $f:X\rightarrow Y$ be a continuous $G$-equivariant map. Then

(a) if $G\backslash X$ is 3-proper, $f$ is surjective, and $|Y|\geq 3$ then $G\backslash Y$ is 3-proper;
(b) if $G\backslash Y$ is 3-proper, $f$ is non-ramified over the limit points, and $|X|\geq 3$ then $G\backslash X$ is 3-proper.

**Proof.** (a) follows immediately from the description of $\times$-actions as Dynkin actions (see 7.2).

To prove (b) consider a limit quasihomeomorphism $s$ for the action $G\backslash X$ that is not a homeomorphism. Since the map $\text{Closed}(X^2)\rightarrow \text{Closed}(Y^2)$ induces by $f$ is continuous (see 6.5) it maps $s$ to a limit cross $(s,a)^X$ for $G\backslash Y$. We will prove that the cross $s=\tau \times X \cup X \cup a$ is a limit quasihomeomorphism for $G\backslash X$. Since $f$ is not ramified over the points $\tau,a$, we have $s\subseteq (f^{-1}\tau,f^{-1}a)^X$. If follows from 7.1.1 that $s=(f^{-1}\tau,f^{-1}a)^X$.

**Proposition 7.6.2.** Let $\mu : L\times X \rightarrow Y$ be a continuous action of a locally compact space $L$ by homeomorphisms. Let $X_i\times Y_i$ for $i\in \{0,1\}$ be invariant closed subproducts such that $X=X_1\cup X_2$ and the restrictions $\mu_i=\mu|_{L\times X_i}$ are $\times$-actions with coinciding $\times$-reminders. Then $\mu$ is a $\times$-action with the same $\times$-remider.

**Proof.** Let $s$ be a limit quasihomeomorphism for $\mu$. Suppose that $s$ contains a regular triple $\gamma$. Let $V$ be a subproduct neighborhood of $\gamma$.

8. **ATTRACTION SUM**

8.1. **Gluing topology.** In this subsection we fix a $\times$-action $\mu : L\times X \rightarrow Y$. For $(g,p)\in L\times X$, $G\times P\subseteq L\times X$ we put $gp=\mu(g,p)$, $GP=\mu(G\times P)$. For $K\times S\subseteq X\times Y$ we denote

$$SK^{-1} = \{g\in L : S\cap gK\neq \emptyset\} = (\mu_*^{-1}((K\times S)''))'.$$

We are going to express the topology of $Y$ in terms of some its restrictions.
Proposition 8.1.2. Let $K \times \Lambda \in \text{Closed}(X \times Y)$. Then

a: If $K \cap \partial L = \emptyset$ then $\partial_1(FK^{-1}) \subset F$ for every $F \in \text{Closed} Y$;

b: If $\Lambda' = LK = LU$ for $U \in \text{Loc}_X K$ then

$\text{Closed} Y \supset S$ $\Rightarrow$ $\{S : S \cap \Lambda \in \text{Closed} \Lambda, S \cap \Lambda' \in \text{Closed} \Lambda', \partial_1(SK^{-1}) \subset S\}$.

Proof. (a). It follows from [8.1.1] that $FK^{-1}$ is a preimage of an OG-closed set. Hence any limit cross $(\tau, a)^x$ for $FK^{-1}$ meets $K \times F$. Since $\tau \notin K$ we have $a \in F$.

(b) For $p \notin S \in S$ we will find $p \in \text{Loc}_Y p$ disjoint from $S$. If $p \in \Lambda'$ then $p = \Lambda' \setminus S \cap \Lambda'$. Assume $p \in \Lambda$. Since $p \notin \partial_1(SK^{-1})$, the set $SK^{-1}$ is equicontinuous at $p$ by [6.6.1]. Hence, for some $p \in \text{Loc}_Y p$, each $g^{-1}p$, for $g \in SK^{-1}$ is small with respect to $u$ $\Rightarrow$ $S^2X \setminus (K \times U') \in \text{Ent} X$. Since $S' \cap \Lambda \in \text{Open} \Lambda$ we may assume that $p \cap \Lambda \cap S = \emptyset$. Claim: $p \subset S'$. Indeed if $q \in p \cap S$ then $q \notin \Lambda$ hence $q = gt$ for $(g, t) \in L \times K$. So $g \in SK^{-1}$, $g^{-1}p \in L, p \in \Lambda'$ contradicting with the assumption `$p \notin \Lambda$'. □

Proposition 8.1.3. Let $\Xi \times \Lambda$ be a closed invariant subproduct of $X \times Y$ with $|\Lambda| \geq 3$ such that $\mu|_{\Xi \times \Lambda}$ is cocompact. Then the topology of $Y$ is determined by the topologies of $\Lambda$ and $\Lambda'$ and by $\nu = \mu|_{\Xi \times \Lambda}$ and $\omega = \mu|_{\Xi \times \Lambda'}$.

Proof. By [7.3.1] $\partial_\nu L = \partial_\omega L \subset \Xi \times \Lambda$. Let $K$ be a generating compact for $\omega$, i.e., $LK = \Lambda'$. It satisfies the conditions a, b of [8.1.2]. Hence $\text{Closed} Y = S$. The definition of $S$ is a desired expression of the topology of $Y$. □

8.2. Auxiliary sum of $L$ and $\Lambda$. We transform the definition of $S$ from [8.1.2] into a definition of a topology.

Let $\nu : L \times \Xi \to \Lambda$ be a $x$-action. On the direct union $T = L \sqcup \Lambda$ define a topology as follows

$\text{Closed} T = \{F \subset T : F \cap L \in \text{Closed} L, F \cap \Lambda \in \text{Closed} \Lambda, \partial_1(F \cap L) \subset F\}$.

Since the operator $\partial_1$ preserves inclusions, the axioms of topology are satisfied. We denote this construction by $L + \nu \Lambda = T$.

Since $L$ is locally compact and $\partial$(bounded set)$= \emptyset$ we have $\text{Open} L \subset \text{Open} T$. Thus $L$ is an open subspace of $T$.

Until the end of this subsection we choose $\gamma \Subset \Lambda$ with $|\gamma| = 3$.

For a set $s \subset \Lambda$ let $\kappa s = \{g \in L : |s \cap g\gamma| > 1\}, \bar{s} = s \cup \kappa s$. The operators $s \mapsto \kappa s \subset L, s \mapsto \bar{s} \subset T$ preserve inclusion and commute with the complement $s \mapsto s'$.

If $s \in \text{Closed} \Lambda$ then $\bar{s} \in \text{Closed} T$. Indeed if, for $(\tau, a) \in \partial(\kappa s)$, the attractor $a$ and were not in $s$ then some $g \in \kappa s$ sufficiently close to $(\tau, a)^x$ would map the set $\gamma \setminus \tau$ of cardinality 2 or 3 to the $\Lambda$-neighborhood $\Lambda \setminus s$ of a contradicting with the definition of $\kappa s$.

This proves that $a \in \text{Open} \Lambda \Rightarrow \bar{s} \in \text{Open} T$. Since $a = \Lambda \setminus \bar{s}$ we have proved the following.

Proposition 8.2.1. $L$ is an open subspace of $T$ and $\Lambda$ is a closed subspace of $T$.

Loc$_T \Lambda = \{S \subset T : S' \text{ is bounded in } L\}$. □

Proposition 8.2.2. $T$ is compact.

Proof. If an ultrafilter $\mathcal{F}$ on $T$ contains a set bounded in $L$ then it converges to a point in $L$ since $L$ is locally compact. Otherwise, by [8.2.1] it contains Loc$_Y \Lambda$ by [8.2.1] and converges by [2.2.3]. □

For $u \in \text{Ent} \Lambda$ let $\kappa u = \emptyset \cup \{S^2(\kappa s) : s \in \text{Small}(u)\}, \bar{u} = \emptyset \cup \{S^2\bar{s} : s \in \text{Small}(u)\}$.

Proposition 8.2.3. Let $w \in T \Rightarrow \{v \cup \bar{u} : v \in \text{Loc}_{S^2} L, u \in \text{Ent} \Lambda\}$. Every $p \in T$ possesses a $w$-small $T$-neighborhood.
Proof. Let \( w = v \cup \tilde{u} \). If \( p \in \Lambda \) and \( p \) is a \( u \)-small \( \Lambda \)-neighborhood of \( p \) then \( \tilde{p} \) is a \( \tilde{u} \)-small \( T \)-neighborhood of \( p \). If \( p \in L \) and \( K \) is a compact \( L \)-neighborhood of \( p \) then any \( v|_K \)-small \( K \)-neighborhood of \( p \) is also a \( v \)-small \( T \)-neighborhood of \( p \).

We have proved that \( T \subset \text{Loc}_{S_2T} \Delta^2 T \). We are going to prove that \( T \) generates this filter.

**Proposition 8.2.4.** Distinct points \( p, q \in T \) possess \( T \)-neighborhoods \( p, q \) such that \( p \times q \) does not meet some \( w \in T \).

Proof. Case 1: \( p, q \in L \). For compact disjoint \( L \)-neighborhoods \( p, q \) of \( p, q \) respectively let \( v = S^2L \setminus p \times q \in \text{Loc}_{S_2L} \Delta^2 L \). There exists \( u \in \text{Ent} \) such that \( g \gamma \) does not contain \( u \)-small proper pairs for every \( g \in p \cup q \). Thus \( p \cup q \) is disjoint from \( \kappa s \) for any \( u \)-small \( s \subset \Lambda \). So \( p \times q \cap (v \cup \tilde{u}) = \emptyset \).

Case 2: \( p \in L \), \( q \in \Lambda \). Let \( p \) be a compact \( L \)-neighborhood of \( p \) and let \( u \) be a \( \Lambda \)-entourage such that \( g \gamma \) contains \( u \)-small proper pairs for every \( g \in p \). Let \( q \) be a \( u \)-small \( \Lambda \)-neighborhood of \( q \). The \( L \)-closed set \( \kappa q \) does not meet \( p \). Hence \( v = S^2L \setminus p \times \kappa q \in \text{Loc}_{S_2L} \Delta^2 L \). We claim that \( p \times \tilde{q} \) is disjoint from \( w = v \cup \tilde{u} \). If \( (g, r) \in p \times q \) then \( (g, r) \) does not belong to \( u \) since \( g \) does not belong to \( \kappa r \) for any \( u \)-small \( r \subset \Lambda \). By the same reason if \( (g, h) \in p \times \kappa q \) then the pair \( (g, h) \) is not in \( \kappa u \). It is not in \( v \) by the choice of \( v \).

Case 3: \( p, q \in \Lambda \). Let \( u \) be a \( \Lambda \)-entourage with \( \{p, q\} \notin u^3 \) and let \( p, q \) be closed \( u \)-small \( \Lambda \)-neighborhoods of \( p, q \) respectively. Since the action \( \nu \) is continuous the set \( e = \{(g, h) \in L^2 : \exists \gamma \in \gamma : (g, h \gamma) \in p \times q \} \) is \( L^2 \)-closed. Let \( v = S^2L \setminus v \in \text{Loc}_{S_2L} \Delta^2 L \). Claim: \( \tilde{p} \times \tilde{q} \) is disjoint from \( w = v \cup \tilde{u} \).

Subcase 1. The set \( p \times q \) does not contain \( u \)-small pairs by the choice of \( u \).

Subcase 2. Let \( g \in \kappa p \) and \( r \in q \). If \( (g, r) \) belongs to \( \tilde{r} \) for a \( u \)-small \( r \subset \Lambda \) then \( g \in \kappa p \cap \kappa r \neq \emptyset \) hence \( p \cap r \neq \emptyset \). On the other hand, \( r \in r \cap q \neq \emptyset \) so \( \{p, q\} \in u^3 \).

Subcase 3. Let \( g \in \kappa p \), \( h \in \kappa q \). Since \( \gamma \in \gamma^{-1} p \cap h^{-1} q \neq \emptyset \) the pair \( (g, h) \) belongs to \( c \) so \( (g, h) \notin v \). If \( (g, h) \in \kappa r \) for \( r \in \text{Small}(u) \) then, as above, \( r \) meets both \( p \) and \( q \) so \( \{p, q\} \in u^3 \).

**Proposition 8.2.5.** \( T \) is Hausdorff. The filter \( \text{Loc}_{S_2T} \Delta^2 T \supset \text{Ent} \) is generated by \( T \).

Proof. The first follows from \ref{8.2.4} The second follows from \ref{8.2.5} and \ref{2.2.2} applied to \( F \subset \Delta^2 T \).

It also follows from \ref{8.2.5} that the uniformity \( \mathcal{U} \) induced in \( L \) by \( \text{Ent} T \) is generated by the set \( S = \{u \cup \kappa u : u \in \text{Loc}_{S_2G} \Delta^2 G, u \in \text{Ent} \} \).

**Proposition 8.2.6.** If \( u, v \in \text{Ent} \Lambda \) and \( u \leadsto v \) then \( \tilde{u} \leadsto \tilde{v} \) on \( T \).

Proof. If \( \Lambda = a \cup b \) where \( a \in \text{Small}(u) \), \( b \in \text{Small}(v) \) then, clearly \( L = \kappa a \cup \kappa b \) and \( T = \tilde{a} \cup \tilde{b} \).

Suppose now that \( X = Y \), \( L = G \) is a group, \( \nu \) is a group action. We consider the action \( G \times G \) by multiplication from the left. Let \( \mu \) denote the resulting action \( G \times T \). The operator \( s \mapsto \kappa s \) commutes with the operators of the action \( \mu \). So \( T \) and hence \( \text{Ent} \) is \( G \)-invariant and \( \mu \) acts by homeomorphisms. Using \ref{8.2.6} we have the following.

**Proposition 8.2.7.** The action \( \mu : G \times (G + y, \Lambda) \) has Dynkin property and hence is a \( \times \)-action.

We have proved \ref{8.3.1} in the particular case ‘\( \Omega = G \)’. It is enough for establishing the Floyd map from any Cayley graph with respect to a finite generating set.

8.3. **Existence of attractor sums.** The following is proved in \ref{8.2.7} in the particular case ‘\( \Omega = G \)’.

**Proposition 8.3.1.** Let a locally compact group \( G \) act on a compactum \( \Lambda \) properly on triples and on a locally compact Hausdorff space \( \Omega \) properly and cocompactly. Then on the disjoint
union $\Lambda \sqcup \Omega$ there is a unique compact Hausdorff topology $\tau$ extending the original topologies of $\Lambda$ and $\Omega$ such that the $G$-action on the space $X = (\Lambda \sqcup \Omega, \tau)$ is proper on triples.

The uniqueness of $\tau$ follows from [8.1.3] so it suffices to indicate $\tau$ with the desired properties.

We call the space $X$ the attractor sum of $\Lambda$ and $\Omega$. We denote it by $\Lambda +_G \Omega'$ or by $\Omega +_G \Lambda$.

Proof of [8.3.1] We initially prove the statement for the spaces $\Omega$ of a special type. We fix a $\times$-action $\nu : G \times \Omega \to \Omega$ and denote by $T$ the $G$-space $G +_G \Omega = G +_\nu \Omega$ of [8.2.7].

Let $K$ be a compactum, and let $\Omega = G \times K$ with the $G$-action induced by the trivial action on $K$: $g(h, p) = (gh, p)$. This action $G \times \Omega$ is proper and cocompact. We construct the attractor sum $\Omega +_G \Lambda$ as follows.

Let $\theta$ be the equivalence relation on $T \times K$ whose classes are the single points of $\Omega$ and the fibers $p \times K$, $p \in \Lambda$. Since $\theta$ is closed the space $X = (T \times K) / \theta$ is a compactum by [2.2.5]. The natural copies of $\Omega$ and $\Lambda$ in $X$ are homeomorphic and $G$-isomorphic to the corresponding spaces. The natural projection $\pi : X \to T$ is continuous $G$-equivariant. By [7.6.11(b)] the action $G \times X$ is $3$-proper. So we can identify $X$ with $(G \times K) +_G \Lambda$.

Now consider the general case of a proper cocompact action $\omega : G \times \Omega \to \Omega$ on a locally compact space. Let $K$ be a compactum in $\Omega$ such that $\Omega = GK$ and let $\theta$ be the kernel of the surjective proper map $\omega|_{G \times K} : G \times K \to \Omega$.

We claim that the equivalence $\Theta = \theta \cup \Delta^2 \Lambda$ is closed in $X^2$.

Let $\theta_1$ be the image of $\theta$ under the map $\pi^2 : X^2 \to T^2$. So $\theta_1 = \cup \{ gb^2 : g \in G \}$ where $B = \{ g \in G : K \cap gK \neq \emptyset \}$. Since $\pi^2|_{(G \times K)^2}$ is a proper map, $\theta_1$ is closed in $G^2$. Since $B$ is bounded and hence perspective (see [7.5.5]) the set $\theta_1 \setminus \theta_1$, where bar means the closure in $T^2$, does not contain non-diagonal pairs. Hence $\overline{\theta_1} = \theta_1 \cup \Delta^2 \Lambda$. The $\pi^2$-preimage of $\overline{\theta_1}$ is contained in $(G \times K)^2 \cup \Delta^2 \Lambda$. Since it is closed and contains $\theta$ we have $\overline{\theta} \subset (G \times K)^2 \cup \Delta^2 \Lambda$. Since $\theta$ is closed in $(G \times K)^2$ we have $\overline{\theta} \subset \theta \cup \Delta^2 \Lambda$.

So $\Theta$ is closed, the space $\Omega + _G \Lambda = (G \times K) / \Theta$ is compact by [2.2.5] and the induced action $G \times (\Omega +_G \Lambda)$ is proper on triples. Since $\omega|_{G \times K}$ is proper the topology of $\Omega$ coincides with the quotient topology, so $\Omega$ is a subspace of $\Omega + _G \Lambda$.

\[ \square \]

8.4. Implication RH$_{32} \Rightarrow$ RH$_{pd}$. Let a discrete group $G$ act on a compactum $T$ proper on triples and cocompact on pairs. Let $\tilde{T} = G +_G T$. By [8.3.1] the action $G \times \tilde{T}$ is proper on triples. If $K$ is a compact fundamental domain for $G \times T$ then $\tilde{K} = K \cup (1 \times (T \setminus 1))$ is a compact fundamental domain for $G \times \tilde{T}$.

Let $\tilde{u}$ be the complement of a compact neighborhood $\tilde{K}$ of $\tilde{K}$ in $\Theta^2 \tilde{T}$. By [2.2.2] the filter generated by $\{ \tilde{u} \}$ coincides with $\text{Ent} \tilde{T}$. If $\tilde{v} = \sqrt{\tilde{u}}$ then the intersection of finitely many $G$-shifts of $\tilde{u}$ is contained in $\tilde{v}$ so $\tilde{u}$ is a divider (see [3.2]).

Let $M$ be the set of all non-conical points of $G \times \tilde{T}$. It contains $G$ and is $G$-finite by [Ge09], Section 7.2. The restriction $u = \tilde{u} \cap \Theta^2 M$ is perspective by [7.5.4]. So we have the following.

Proposition 8.4.1. If the $G$-set $M$ of nonconical points of $\tilde{T}$ is connected then the uniformity of $T$ induces a relative hyperbolicity on $M$.

If $G$ is finitely generated then $G \times G$ is connected since there exists a locally finite Cayley graph. In [GP10] we will prove a theorem which implies that $M$ is always connected.

9. Geometric convergence actions

Let a discrete finitely generated group $G$ act on a compactum $T$ properly on triples. Let $U$ be the uniformity on $G$ induced by the uniformity of $G +_G T$. 
We say that the action \( G \acts T \) is geometric with respect to a finite generating set \( S \) if \( \mathcal{U} \) is a visibility (see 4.1) on the Cayley graph \( \Gamma \) with respect to \( S \).

Every action of 4.2.1 is geometric. By Karlsson lemma [Ka03] every action of a f.g. group on its Floyd completion with respect to any Floyd function on a locally finite Cayley graph is geometric.

The inverse limit of geometric actions is geometric. Hence the action of each f.g. group on its Freudenthal ends completion and on the space of ends is geometric (this action is relatively hyperbolic if and only if the group is accessible in the sense of Dunwoody).

The quotient of a geometric action is geometric.

By 3.4.6 the action of a relatively hyperbolic group on its Bowditch boundary is geometric. In particular the action of a hyperbolic group on its Gromov boundary is geometric. If \( H \) is a hyperbolic subgroup of hyperbolic group \( G \) such that the inclusion \( H \subset G \) induces the “Cannon-Thurston map” \( \partial_{\infty}H \to \partial_{\infty}G \) in particular when \( H \) is normal in \( G \) then [Mi98] the (non-geometricallyfinite) action \( H \acts \Lambda H \) is geometric (as a quotient of the geometric action \( H \acts \partial_{\infty}H \)).

**Question 1.** Does geometricity depend on the choice of finite generating set?

**Question 2.** Is there exist a non-geometric convergence action of an f.g. group?

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