Rigidity results for quotient almost Yamabe solitons

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Abstract
In this paper we investigate the structure of certain solutions of the fully nonlinear Yamabe flow, which we call quotient almost Yamabe solitons because they extend quite naturally those called quotient Yamabe solitons. We then present sufficient conditions for a compact quotient almost Yamabe soliton to be either trivial or isometric with an Euclidean sphere. We also characterize noncompact quotient gradient almost Yamabe solitons satisfying certain conditions on both its Ricci tensor and potential function.

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1. Introduction and main results

The Yamabe flow
\[
\frac{\partial g^t}{\partial t} = -(R_{g^t} - r_{g^t})g^t, \quad g^0 = g_0, \tag{1}
\]
where $R_{g^t}$ is the scalar curvature of $g^t$ and
\[
r_{g^t} = \frac{\int_M R_{g^t} dv_{g^t}}{\int_M dv_{g^t}},
\]

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is the mean value of $R_g$, along $M^n$, was introduced by R. Hamilton [16] and has become one of the standard tools of recent differential geometry. Yamabe solitons arise as self-similar solutions of (1).

**Definition 1.** A solution $g^t$ of (1) is a Yamabe soliton if there exist a smooth function $\alpha : [0, \varepsilon) \to (0, \infty)$, $\varepsilon > 0$, and a 1-parameter family $\{\psi_t\}$ of diffeomorphisms of $M^n$ such that

$$g^t = \alpha(t)\psi_t^*(g_0), \quad \alpha(0) = 1 \quad \text{and} \quad \psi_0 = id_M.$$ 

One gets

$$\frac{1}{2} L_X g = (R_g - \lambda) g,$$

by substituting $g^t = \alpha(t)\psi_t^*(g_0)$ into (1) and evaluating the resulting expression at $t = 0$, where $L_X g$ is the Lie derivative of $g$ with respect to the field $X$ of directions associated with the 1-parameter family $\{\psi_t\}$ and $\lambda = \alpha'(0) + r_g$. Equation (2) is the fundamental equation of Yamabe solitons. Since their beginning, a lot has been proved about the nature of Yamabe solitons. For example, Chow [10] proved that compact Yamabe solitons have constant scalar curvature (see also [13, 18]). Daskalopoulos and Sesum [11] proved that complete locally conformally flat Yamabe solitons with positive sectional curvature are rotationally symmetric and must belong to the conformal class of flat Euclidean space.

A new soliton is born if one replaces the scalar curvature in (1) by functions of the higher order scalar curvatures. As is the case with any generalization, it’s hoped that one recovers the old objects as particular instances of the new ones, while opening room for new and exciting phenomena to happen. In what follows we give formal definitions and even before we state out main results, we examine a few examples. We included a section containing the lemmas that we have used in the text for the convenience of the reader and a separate section with the proofs to our statements can be found right after it.

The Riemann curvature tensor $Rm$ of $(M^n, g)$ admits the following decomposition

$$Rm = W_g + A_g \oplus g,$$

where $W_g$ and $A_g$ are the tensors of Weyl and Schouten, respectively, and $\oplus$ is the Kulkarni-Nomizu product of $(M^n, g)$. Recall that the Schouten tensor is given by

$$A_g = \frac{1}{n-2} \left( Ric_g - \frac{R_g}{2(n-1)} g \right).$$

The $\sigma_k$-curvature of $g$ is defined as the $k$-th elementary symmetric function of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the endomorphism $g^{-1}A_g$, that is,

$$\sigma_k(g) = \sigma_k(g^{-1}A_g) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n.$$
Here we set $\sigma_0(g) = 1$ for convenience. A simple calculation shows that $\sigma_1(g) = \frac{R}{2(n-1)}$, which indicates that the $\sigma_k$-curvature is a reasonable substitute for the scalar curvature of $(M^n, g)$ in (1).

Guan and Guofang introduced [15] the fully nonlinear flow

$$
\frac{\partial g^t}{\partial t} = -\left( \log \frac{\sigma_k(g^t)}{\sigma_l(g^t)} - \log r_{k,l}(g^t) \right) g^t, \quad g^0 = g_0, \quad (3)
$$

where

$$
\log r_{k,l}(g^t) = \frac{\int_M \sigma_l(g^t) \log \frac{\sigma_k(g^t)}{\sigma_l(g^t)} dv_{g^t}}{\int_M \sigma_l(g^t) dv_{g^t}},
$$

was defined as to make the flow preserve the quantities

$$
E_l(g^t) = \int_M \sigma_l(g^t) dv_{g^t}, \quad l \neq \frac{n}{2},
$$

$$
= -\int_0^1 dt \int_M u \sigma_l(g^t) dv_{g^t}, \quad l = \frac{n}{2},
$$

where $u \in C^\infty(M)$, $g = e^{-2u}g_0$ and $g^t = e^{-2tu}g_0$. The convergence of the fully nonlinear flow was then proved under certain conditions to be satisfied by the eigenvalues of the Schouten tensor. The authors also provided geometric inequalities such as the Sobolev-type inequality in case $0 \leq l < k < \frac{n}{2}$, the conformal quasimass-integral-type inequality for $\frac{n}{2} \leq k \leq n$, $1 \leq l < k$ and the Moser–Trudinger-type inequality for $k = \frac{n}{2}$.

Bo et al. [7] presented quotient Yamabe solitons as self-similar solutions of the flow (3) and stated rigidity results for the existence of such objects on top of locally conformally flat manifolds. For example, it was shown that any compact and locally conformally flat manifold with the structure of a quotient Yamabe soliton, where both $\sigma_k > 0$ and $\sigma_l > 0$, must have constant quotient curvature $\sigma_k \sigma_l$. Also, for the so called gradient $k$-Yamabe soliton ($l = 0$) they proved that, for $k > 1$, any compact gradient $k$-Yamabe soliton with negative constant scalar curvature necessarily has constant $\sigma_k$-curvature. Almost Yamabe solitons were introduced by Barbosa and Ribeiro [3] as generalizations of self-similar solutions of the Yamabe flow. Essentially, they allowed the parameter $\lambda$ in (2) to be a function on $M$. The authors then stated rigidity results for almost Yamabe solitons on compact manifolds. We refer the reader to [3, 12, 22, 24] for further information.

In [9] Catino et al. proposed the study of conformal solitons. A conformal soliton is a Riemannian manifold $(M^n, g)$ together with a nonconstant function $f \in C^\infty(M)$ satisfying $\nabla^2 f = \lambda g$ for some $\lambda \in \mathbb{R}$. They provided classification results according to the number of critical points of $f$. It should be noticed that solitons of Yamabe, $k$-Yamabe and quotient Yamabe types are examples of conformal solitons.
We introduce quotient almost Yamabe solitons in extension to the quotient Yamabe solitons.

**Definition 2.** A solution $g^t$ of (3) is a quotient almost Yamabe soliton if there exist a function $\alpha : M \times [0, \varepsilon) \to (0, \infty)$, $\varepsilon > 0$, and a 1-parameter family $\{\psi_t\}$ of diffeomorphisms of $M^n$ such that

$$g^t = \alpha(x, t)\psi_t^*(g_0), \quad \alpha(\cdot, 0) \equiv 1 \text{ on } M^n \quad \text{and} \quad \psi_0 = \text{id}_M.$$ 

Equivalently, $(M^n, g)$ is a quotient almost Yamabe soliton if there exists a pair $X \in \mathfrak{X}(M), \lambda \in C^\infty(M)$ satisfying

$$\frac{1}{2} L_X g = \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) g, \quad \sigma_k \cdot \sigma_l > 0. \quad (4)$$

We will write the soliton in (4) as $(M^n, g, X, \lambda)$ for the sake of simplicity.

Following the terminology already in use with almost Yamabe solitons, a soliton $(M^n, g, X, \lambda)$ will be called:

- **expanding** if $\lambda < 0$,
- **steady** if $\lambda = 0$,
- **shrinking** if $\lambda > 0$ and, finally,
- **indefinite** if $\lambda$ change signs on $M^n$.

**Definition 3.** A quotient gradient almost Yamabe soliton is a quotient almost Yamabe soliton $(M^n, g, X, \lambda)$ such that $X = \nabla f$ is the gradient field of a function $f \in C^\infty(M)$.

Since

$$\frac{1}{2} L_{\nabla f} g = \nabla^2 f,$$

it follows from (4) that a quotient gradient almost Yamabe soliton $(M^n, g, \nabla f, \lambda)$ is characterized by the equation

$$\nabla^2 f = \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) g, \quad \sigma_k \cdot \sigma_l > 0. \quad (5)$$

Quotient almost Yamabe solitons, gradient or not, are regarded as **trivial** if their defining equation vanishes identically. Thus, $(M^n, g, X, \lambda)$ is trivial if $L_X g = 0$ and $(M^n, g, \nabla f, \lambda)$ if $\nabla^2 f = 0$. In either case, $\log \frac{\sigma_k}{\sigma_l} - \lambda = 0$.

Before we state our main results, let’s take a look at some examples.

**Example 1.** The product manifold $(\mathbb{R} \times S^n, g = dt^2 + g_{S^n})$ alongside the function

$$f : \mathbb{R} \times S^n \to \mathbb{R}, \quad (t, x) \mapsto f(t, x) = at + b \quad (a, b \in \mathbb{R}),$$

is, for $k = l = 1$, a trivial quotient gradient almost Yamabe soliton with $\lambda = 0$, since $\sigma_1(g^{-1}A_g) = \frac{n}{2}$ and $\nabla^2 f = 0$. 

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Example 2. Identities

\[ \text{Ric}_{g_{S^n}} = (n - 1)g_{S^n}, \quad \text{R}_{g_{S^n}} = n(n - 1) \quad \text{and} \quad A_{g_{S^n}} = \frac{1}{2}g_{S^n}, \]

rule the Ricci tensor, scalar curvature and Schouten tensor, respectively, of the Euclidean sphere \((S^n, g_{S^n})\). Therefore, we have that

\[ \sigma_k(g_{S^n}^{-1}A_{g_{S^n}}) = \frac{1}{2^k} \binom{n}{k}, \quad 1 \leq k \leq n. \]

Consider the height function

\[ h_v : S^n \to \mathbb{R}, \quad x \mapsto h_v(x) = \langle x, v \rangle, \]

on \(S^n\) with respect to a given \(v \in S^n\). It then follows that

\[ \nabla^2 h_v = -h_v g_{g_{S^n}}, \]

showing that \((S^n, g_{S^n}, \nabla h_v, \lambda)\) is a compact quotient almost Yamabe soliton with

\[ \lambda : S^n \to \mathbb{R}, \quad x \mapsto h_v(x) + \log \frac{\sigma_k}{\sigma_l}. \]

Example 3. On the hyperbolic space \((\mathbb{H}^n, g_{\mathbb{H}^n})\) one has

\[ \text{Ric}_{g_{\mathbb{H}^n}} = -(n - 1)g_{\mathbb{H}^n}, \quad \text{R}_{g_{\mathbb{H}^n}} = -n(n - 1) \quad \text{and} \quad A_{g_{\mathbb{H}^n}} = -\frac{1}{2}g_{\mathbb{H}^n}, \]

for its Ricci tensor, scalar curvature and Schouten tensor, respectively. Therefore, we have that

\[ \sigma_k(g_{\mathbb{H}^n}^{-1}A_{g_{\mathbb{H}^n}}) = \frac{(-1)^k}{2^k} \binom{n}{k}, \quad 1 \leq k \leq n. \]

We consider the model \(\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle_0 = -1, x_1 > 0 \}\) of the hyperbolic space, where \(\mathbb{R}^{n+1}\) is nothing but the Euclidean space \(\mathbb{R}^{n+1}\) endowed with lorentzian inner product \(\langle x, x \rangle_0 = -x_1^2 + x_2^2 + \cdots + x_{n+1}^2\). As in our previous example, we consider the height function

\[ h_v : \mathbb{H}^n \to \mathbb{R}, \quad x \mapsto h_v(x) = \langle x, v \rangle_0, \]

on \(\mathbb{H}^n\) with respect to a given \(v \in \mathbb{H}^n\). Because

\[ \nabla^2 h_v = h_v g_{g_{\mathbb{H}^n}}, \]

we conclude that \((\mathbb{H}^n, g_{\mathbb{H}^n}, \nabla h_v, \lambda)\) is a quotient almost Yamabe soliton with

\[ \lambda : \mathbb{H}^n \to \mathbb{R}, \quad x \mapsto -h_v(x) + \log \frac{\sigma_k}{\sigma_l}, \]

as long as we have \(k \equiv l \pmod{2}\).
Example 4. Consider $\mathbb{R}^n$ endowed with a metric tensor of the form

$$g_{ij} = e^{2u_i} \delta_{ij}, \quad 1 \leq i, j \leq n,$$

in cartesian coordinates $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ where $u_1, \ldots, u_n \in C^\infty(\mathbb{R}^n)$. Then, the Ricci tensor of $(\mathbb{R}^n, g)$ is ruled \cite{20} by the formulas

$$\text{Ric}_g(\partial_j, \partial_k) = \sum_{l \neq k,j} U^l_{jk} + u_j,k u_{l,j}, \quad j \neq k,$$

$$\text{Ric}_g(\partial_k, \partial_k) = \sum_{l \neq k} e^{2(u_k - u_l)} U^l_{kk} + U^l_{kk} - \sum_{m \neq k,l} e^{2(u_k - u_m)} u_{k,m} u_{l,m},$$

where

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}, \quad \text{and} \quad u_{i,j,k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j}, \quad 1 \leq i, j, k \leq n,$$

and $U^l_{jk} = u_{l,k}(u_k - u_l)_j - u_{l,j,k}$. Assume that $4 \leq n \in \mathbb{Z}$. Also, let $\tau$ be the $n$-cycle $(1, 2, 3, \ldots, n)$ in the symmetric group $S_n$. It turns out that by choosing $u_i(x_1, \ldots, x_n) = \log \cosh(x_{\tau(i)})$, $i \equiv 0 \pmod{2}$, $u_i = 0$, $i \equiv 1 \pmod{2}$,

we simplify the situation a little bit and the Ricci tensor of $(\mathbb{R}^n, g)$ ends up being a constant multiple of the metric, $\text{Ric}_g = -g$. Therefore, $(\mathbb{R}^n, g)$ is a complete Einstein manifold and, as such, $A_g = -\frac{1}{2(n-1)}g$. Then, we have that

$$\sigma_k(g^{-1}A_g) = \frac{(-1)^k}{2^k(n-1)^k} \binom{n}{k}, \quad 1 \leq k \leq n.$$

Because $X = (0, 1, \ldots, 0, 1)$ is a Killing field on $(\mathbb{R}^n, g)$ we know that $(\mathbb{R}^n, g, X, \lambda)$ is a trivial quotient almost Yamabe soliton with whenever $k \equiv l \pmod{2}$. It should be mentioned that $X$ is not a gradient field with respect to the metric $g$.

Any smooth vector field $X$ on a compact Riemannian manifold $(M^n, g)$ can be written in the form

$$X = \nabla h + Y,$$

where $Y \in \mathfrak{X}(M)$ is divergence free and $h \in C^\infty(M)$. In fact, by the Hodge-de Rham Theorem \cite{28} we have that

$$X^\flat = d\alpha + \delta\beta + \gamma.$$

Now, take $Y = (\delta\beta + \gamma)^k$, $\nabla h = (d\alpha)^l$ and we are done. The function $h$ is called the Hodge-de Rham potential of $X$. Our first theorem states the triviality of a compact quotient almost Yamabe soliton under certain integral assumptions.

**Theorem 5.** A compact quotient almost Yamabe soliton $(M^n, g, X, \lambda)$ is trivial if one of the following is true:

a) $\int_M e^\lambda \sigma_1(\nabla \lambda, X)dv_g = -\int_M e^\lambda (\nabla \sigma_1, X)dv_g$ plus any of these:
i. $\nabla \text{Ric}_g = 0$

ii. $\text{div} C_g = 0$ where $C_g$ is the Cotton tensor of $(M^n, g)$

iii. $X = \nabla f$ is a gradient vector field;

b) $\int_M \langle \nabla h, X \rangle d\nu_g \leq 0$ where $h$ is the Hodge-de Rham potential of $X$.

The next two corollaries deal with quotient Yamabe solitons ($\lambda$ is a real constant) and constitute direct applications of Theorem 5. In [7] Bo et al. proved that $\sigma_k/\sigma_l$ must be constant on any compact and locally conformally flat quotient Yamabe soliton. We extend Bo’s result.

**Corollary 6.** Let $(M^n, g, X, \lambda)$ be any compact quotient Yamabe soliton with a null cotton tensor. Then, $\sigma_k/\sigma_l$ is constant and, as such, the soliton is trivial.

In [8] Catino et al. proved that any compact gradient $k$-Yamabe soliton with nonnegative Ricci tensor is trivial. Bo et al. [7] also proved that any compact gradient $k$-Yamabe soliton with constant negative scalar curvature is trivial. In [26] it was shown that any compact gradient $k$-Yamabe soliton must be trivial. We extend all these results at once.

**Corollary 7.** Let $(M^n, g, \nabla f, \lambda)$ be any compact quotient gradient Yamabe soliton. Then, $\sigma_k/\sigma_l$ is constant and, as such, the soliton is trivial.

Yet another triviality result hold for quotient almost Yamabe solitons if one drops compacity on $M^n$ in favor of a decaimient condition on the norm of the soliton field $X$.

**Theorem 8.** Let $(M^n, g, X, \lambda)$ be a complete and noncompact quotient almost Yamabe soliton satisfying

$$\int_{M^n \setminus B_r(x_0)} \frac{|X|}{d(x, x_0)} d\nu_g < \infty \quad \text{and} \quad \mathcal{L}_X g \geq 0,$$

where $d$ is the distance function with respect to $g$ and $B_r(x_0)$ is the ball of radius $r > 0$ centered at $x_0$. Then, $(M^n, g, X, \lambda)$ is trivial.

Next, we give a sufficient condition for a compact quotient gradient almost Yamabe soliton to be isometric with an Euclidean sphere.

**Theorem 9.** Let $(M^n, g, \nabla f, \lambda)$ be a nontrivial compact quotient gradient almost Yamabe soliton with constant scalar curvature $R_g = R > 0$. Then $(M^n, g)$ is isometric to the Euclidean sphere $S^n(\sqrt{r})$, $r = R/n(n-1)$. Moreover, up to a rescaling the potential $f$ is given by $f = h_v + c$ where $h_v$ is the height function on the sphere and $c$ is a real constant.

**Remark 10.** A similar result concerning almost Ricci solitons is found in [4].

Another situation in which a quotient gradient almost Yamabe soliton must be isometric with an Euclidean sphere is described below.
Theorem 11. Let \((M^n, g, \nabla f, \lambda)\) be a nontrivial compact quotient gradient almost Yamabe soliton with constant \(\sigma_k\)-curvature, for some \(k = 2, \ldots, n\), and \(A_g > 0\). Then, \((M^n, g)\) is isometric with an Euclidean sphere \(S^n\).

Finally we investigate the structure of noncompact quotient gradient almost Yamabe solitons satisfying reasonable conditions on its potential function and both Ricci and scalar curvatures.

Theorem 12. Let \((M^n, g, \nabla f, \lambda)\) be a nontrivial and noncompact quotient gradient almost Yamabe soliton. Assume that
\[
\mathcal{L}_{\nabla f^2} R \geq 0, \quad \Ric \geq 0 \quad \text{and} \quad |\Ric(\nabla f^2)| \in L^1(M).
\]
Then, \((M^n, g)\) has constant scalar curvature \(R_g = R \leq 0\) and \(f\) has at most one critical point. Moreover, we have that:

a) If \(R = 0\), then \((M^n, g)\) is isometric with a Riemannian product manifold \((\mathbb{R} \times \mathbb{R}^{n-1}, dt^2 + g_{\mathbb{R}})\) such that \(\Ric_{g_{\mathbb{R}}} \geq 0\);

b) If \(R < 0\) and \(f\) has no critical points, then \((M^n, g)\) is isometric with a warped product manifold \((\mathbb{R} \times \mathbb{R}^{n-1}, dt^2 + \xi(t)^2 g_{\mathbb{R}})\) such that
\[
\xi'' + \frac{R}{n(n-1)} \xi = 0;
\]

c) If \(R < 0\) and \(f\) has only one critical point, then \((M^n, g)\) is isometric with a hyperbolic space.

2. Key Lemmas

Lemma 13. (\[2, 14]\)) Let \((M^n, g)\) be a compact Riemannian manifold with a possibly empty boundary \(\partial M\). Then,
\[
\int_M X(\text{tr} T) dv_g = n \int_M \text{div} T(X) dv_g + \frac{n}{2} \int_M \langle T, \mathcal{L}_{Xg} T \rangle dv_g - n \int_{\partial M} \mathcal{L}_X \nu ds_g,
\]
for every symmetric \((0,2)\)-tensor \(T\) and every vector field \(X\) on \(M\), where
\[
\text{tr} T = g^{ij}T_{ij} \quad \text{and} \quad \mathcal{L}_X = \frac{\text{tr} T}{n} - g,
\]
and \(\nu\) is the outward unit normal field on \(\partial M\).

Proof. First notice that integration by parts yields
\[
\int_{\partial M} T(X, \nu) dA_g = \int_M \nabla^i(T_{ij}X^j) dv_g,
\]
and because
\[
\nabla^i (T_{ij} X^j) = \nabla^i T_{ij} X^j + T_{ij} \nabla^i X^j \\
= \nabla^i T_{ij} X^j + \frac{1}{2} T_{ij} (\nabla^i X^j + \nabla^j X^i) \\
= \text{div} T(X) + \frac{1}{2} (T, \mathcal{L}_X g),
\]
we get that
\[
\int_{\partial M} T(X, \nu) dA_g = \int_M \nabla^i (T_{ij} X^j) dv_g \\
= \int_M \text{div} T(X) dv_g + \frac{1}{2} \int_M \langle T, \mathcal{L}_X g \rangle dv_g \\
= \int_M \text{div} T(X) dv_g + \frac{1}{2} \int_M \langle T, \mathcal{L}_X g \rangle dv_g + \frac{1}{2} \int_M \frac{\text{tr} T}{n} (g, \mathcal{L}_X g) dv_g \\
= \int_M \text{div} T(X) dv_g + \frac{1}{2} \int_M \langle T, \mathcal{L}_X g \rangle dv_g + \frac{1}{2} \int_M \text{tr} T \cdot \text{div} X dv_g. \\
\]

(7)

On the other hand, we have that
\[
\int_M \text{tr} T \cdot \text{div} X dv_g = \int_{\partial M} \text{tr} T \cdot \langle X, \nu \rangle dA_g - \int_M X (\text{tr} T) dv_g. \\
\]

(8)

The result now follows from (7) and (8) above.

**Lemma 14.** Let \((M^n, g)\) be a Riemannian manifold and \(T\) be a symmetric \((0, 2)\)-tensor field on \(M^n\). Then,
\[
\text{div} (\mathcal{L}_X g) = \frac{1}{2} \mathcal{L}_X g = \varphi g,
\]
for every \(k = 1, 2, \ldots, n\).

**Proposition 15.** If \(X\) is a conformal vector field on a compact Riemannian manifold \((M^n, g)\) with null Cotton tensor, then
\[
\int_{M^n} \langle X, \nabla \sigma_k \rangle dv_g = 0,
\]
for every \(k = 1, 2, \ldots, n\).
Recall that the $k$-Newton tensor field associated with $g^{-1}A_g$ is defined by

$$T_k(g^{-1}A_g) = \sum_{j=0}^{k} (-1)^j \sigma_{k-j}(g)(g^{-1}A_g)^j, \quad 1 \leq k \leq n.$$  

Among the identities satisfied by $T_k(g^{-1}A_g)$ one finds (see [4])

$$\text{tr} T_k(g^{-1}A_g) = (n-k)\sigma_k(g) \quad \text{and} \quad \text{div} T_k(g^{-1}A_g) = 0,$$

for every $1 \leq k \leq n$.

Proof. Let $\varphi \in C^\infty(M)$ be such that

$$\frac{1}{2} \mathcal{L}_X g = \varphi g,$$

and take $T_k = T_k(g^{-1}A_g)$ where $k \in \{1, 2, \ldots, n-1\}$. Now a direct application of Lemma 13 yields

$$\int_M X(\text{tr} T_k) dv_g = n \int_M \text{div} T_k(X) dv_g + n \int_M \varphi(T_k, g) dv_g. \quad (9)$$

It follows from Corollary 1 of [4] that $\text{div} T_k = 0$ and because

$$\circ T_k = T_k - \frac{\text{tr} T_k}{n} g = T_k - \frac{n-k}{n} \sigma_k g,$$

equation (9) can rewritten in the simpler form

$$(n-k) \int_M \langle X, \nabla \sigma_k \rangle dv_g = 0,$$

which proves the proposition in case $k \neq n$. As for the remaining case, it follows from [17] that

$$n(X, \nabla \sigma_n) = \nabla_a \left[ T^n_b \nabla^b (\text{div} X) + 2n \sigma_n X^a \right],$$

where $T^n_b$ are the components of $T_{n-1}(g^{-1}A_g)$. Therefore, if we go there and write

$$Y^a = T^n_b \nabla^b (\text{div} X) + 2n \sigma_n X^a,$$

we get that

$$n \int_M \langle X, \nabla \sigma_k \rangle dv_g = \int_M \nabla_a Y^a dv_g = 0,$$

which proves the proposition also for $k = n$. \qed

Our next Lemma states structural equations for quotient gradient almost Yamabe solitons.

Lemma 16. Let $(M^n, g, \nabla f, \lambda)$ be a quotient gradient almost Yamabe soliton. Then, we have that:
Proof. a) Simply take traces at equation (5);
b) Next, we differentiate (5) to get
\[
\nabla_j \nabla_r \nabla_i f = \nabla_j \left( \frac{\sigma_k}{\sigma_l} - \lambda \right) g_{ri},
\]
from what we see that
\[
\nabla_i \nabla_j \nabla_r f + \sum_s R_{tij} s \nabla_s f = \nabla_j \left( \frac{\sigma_k}{\sigma_l} - \lambda \right) g_{ri},
\]
with the help of the Ricci identity that can be found in [1], pg. 4. Now we only need to contract this equation on the indices \(j, r\) in order to get
\[
\nabla_i \Delta f + \sum_s \text{Ric}_{is} \nabla_s f = \nabla_i \left( \frac{\sigma_k}{\sigma_l} - \lambda \right),
\]
then yielding
\[
(n - 1) \nabla_i \left( \frac{\sigma_k}{\sigma_l} - \lambda \right) + \sum_{s} \text{Ric}_{is} \nabla_s f = 0,
\]
by a), which proves b);
c) As for the remaining identity, we apply the divergence operator on both sides of (10) and use the twice contracted second Bianchi identity to obtain
\[
(n - 1) \Delta \left( \frac{\sigma_k}{\sigma_l} - \lambda \right) + \frac{1}{2} (\nabla R, \nabla f) + \sum_{s} \text{Ric}_{sl} \nabla_s \nabla_l f = 0,
\]
which is equivalent to
\[
(n - 1) \Delta \left( \frac{\sigma_k}{\sigma_l} - \lambda \right) + \frac{1}{2} (\nabla R, \nabla f) + \left( \frac{\sigma_k}{\sigma_l} - \lambda \right) R = 0,
\]
by the fundamental equation (5), which proves c).
This concludes the proof.
3. Proofs of the main results

**Proof of Theorem 5.**

a) We get

\[ \int_M \operatorname{Ric}_{jk} \nabla_i C_{ijk} \, dv_g = - \int_M \nabla_i \operatorname{Ric}_{jk} C_{ijk} \, dv_g = 0, \]

if either \( \nabla \operatorname{Ric}_g = 0 \) or \( \operatorname{div} C_g = 0 \) and because

\[ \int_M \nabla_i \operatorname{Ric}_{jk} C_{ijk} \, dv_g = \int_M |C_g|^2 \, dv_g + \frac{1}{2(n-1)} \int_M (C_{ijk} g_{jk} \nabla_i R_g - C_{ijk} g_{ij} \nabla_j R_g) \, dv_g \]

\[ = \int_M |C_g|^2 \, dv_g, \]

(11)

we conclude that \( C_g = 0 \). Equation (5) implies that \( X \) is a conformal field and so we can apply Proposition 15 to conclude that

\[ \int_M n \sigma_k (\log \frac{\sigma_k}{\sigma_l} - \lambda) \, dv_g = - \frac{1}{n} \int_M \langle \nabla \sigma_k, X \rangle \, dv_g = 0. \]

Therefore, we have that

\[ \int_M \sigma_l \left( \frac{\sigma_k}{\sigma_l} - e^\lambda \right) \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) \, dv_g = - \int_M \frac{e^\lambda \sigma_l}{n} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) \, dv_g \]

\[ = \int_M e^\lambda \sigma_l (\nabla \lambda, X) \, dv_g + \int_M e^\lambda (\nabla \sigma_l, X) \, dv_g = 0, \]

(12)

by our hypothesis on the nullity of the integral at the right hand of (12). Because \( \sigma_l \neq 0 \) does not change signs on \( M^n \) we must then admit that \( \sigma_k/\sigma_l = e^\lambda \), which proves our assertion in case \( \operatorname{Ric}_g \) is parallel or \( C_g \) is divergence free. Instead, if \( X = \nabla f \), we argue by contradiction to show that \( f \) is a constant function. Should \( f \) not be constant on \( M^n \), the manifold \( (M^n, g) \) could not lie in any conformal class other than that of the Euclidean sphere \((S^n, g_{S^n})\), by Theorem 1.1 of [9]. So, just as it happens with any locally conformally flat manifold, the Cotton tensor of \( (M^n, g) \) would then vanish identically and by what has been said above \((M^n, g, \nabla f, \lambda)\) ought to be trivial. This contradiction shows that \( f \) is indeed a constant function, now concluding a);

b) Because the fields \( \nabla h, Y \) in the Hodge-de Rham decomposition \( X = \nabla h + Y \) of \( X \) are orthogonal to one another in \( L^2(M) \) we get that

\[ \int_{M^n} |\nabla h|^2 \, dv_g = \int_{M^n} \langle \nabla h, \nabla h + Y \rangle \, dv_g = \int_{M^n} \langle \nabla h, X \rangle \, dv_g \leq 0, \]
the inequality being a part of the hypothesis. Then, \( \nabla h = 0 \) and \( X = Y \). Since \( Y \) is divergence free, we conclude that

\[
n \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) = \text{div} \, X = 0,
\]

and, as such, the soliton is trivial.

This proves the Theorem. \( \Box \)

**Proof of Theorem 8.** As we already know, the fundamental equation

\[
\frac{1}{2} \mathcal{L} g = \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) g,
\]

leads to

\[
\text{div} \, X = n \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right), 
\tag{13}
\]

and because we suppose that \( \mathcal{L} g \geq 0 \) we must then admit that \( \log \frac{\sigma_k}{\sigma_l} - \lambda \geq 0 \).

So, if we now take a cut-off function \( \psi : M \to \mathbb{R} \) satisfying

\[
0 \leq \psi \leq 1 \text{ on } M, \quad \psi \equiv 1 \text{ in } B_r(x_0), \quad \text{supp}(\psi) \subset B_{2r}(x_0) \quad \text{and} \quad |\nabla \psi| \leq \frac{K}{r},
\]

where \( K > 0 \) is a real constant, we are in place to conclude that

\[
n \int_{B_r(x_0)} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) dv_g \leq \int_{B_r(x_0)} n \psi \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) dv_g \\
\leq \int_{B_{2r}(x_0)} \psi \text{div} \, X dv_g \\
= - \int_{B_{2r}(x_0)} g(\nabla \psi, X) dv_g \\
\leq \int_{B_{2r}(x_0) \setminus B_r(x_0)} |\nabla \psi| |X| dv_g \\
\leq K \int_{B_{2r}(x_0) \setminus B_r(x_0)} \frac{|X|}{r} dv_g, \\
\leq 2K \int_{M \setminus B_r(x_0)} \frac{|X|}{d(x, x_0)} dv_g,
\]

from what it follows that

\[
0 \leq \int_M \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) dv_g = \lim_{r \to \infty} \int_{B_r(x_0)} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) dv_g \\
\leq \frac{2K}{n} \lim_{r \to \infty} \int_{M \setminus B_r(x_0)} \frac{|X|}{d(x, x_0)} dv_g = 0.
\]

Henceforth, we have that \( \mathcal{L} g = \log \frac{\sigma_k}{\sigma_l} - \lambda = 0 \) which proves the Theorem. \( \Box \)
Proof of Theorem 9. It follows from Lemma 16(c) that if the scalar curvature of \((M^n, g, \nabla f, \lambda)\) is a constant function on \(M^n\), then
\[
\Delta \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) + \frac{R}{n-1} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) = 0, \tag{14}
\]
and, by the min-max principle, we must have \(R > 0\). Because we know that
\[
\Delta f = n \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right), \tag{15}
\]
we then get
\[
\Delta \left( \log \frac{\sigma_k}{\sigma_l} - \lambda + \frac{R}{n(n-1)} f \right) = 0,
\]
and since \((M^n, g)\) is a compact Riemannian manifold, we see that
\[
\log \frac{\sigma_k}{\sigma_l} - \lambda + \frac{R}{n(n-1)} f = c \text{ on } M^n,
\]
for a certain \(c \in \mathbb{R}\), by a Theorem of E. Hopf. But, then
\[
\nabla \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) + \frac{R}{n(n-1)} \nabla f = 0,
\]
and so
\[
\nabla_X \nabla \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) = -\frac{R}{n(n-1)} \nabla_X \nabla f = -\frac{R}{n(n-1)} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) X.
\]
We can now apply Theorem A from Obata [21] to conclude that \((M^n, g)\) is isometric with an Euclidean sphere of radius \(\sqrt{r}\), \(r = R/n(n-1)\). To prove our last claim we notice that we can assume that \(R = n(n-1)\) possibly at the cost of rescaling the metric \(g\). From equations (14) and (15) it’s seen that \(\Delta f\) is an eigenfunction of the Laplacian on \((S^n, g)\) and so there must exist a \(v \in S^n\) such that
\[
\frac{1}{n} \Delta f = h_v = -\frac{1}{n} \Delta h_v. \text{ Hence, } \Delta (f + h_v) = 0 \text{ but then } f = h_v + c \text{ for some real } c. \tag*{\square}
\]

Proof of Theorem 11. By Theorem 1.1 of [9] the only nontrivial compact quotient gradient almost Yamabe solitons reside in the conformal class of the Euclidean sphere and because of that we can assume that
\[
M^n = S^n \text{ and } \varphi^{-2} g = g_{S^n},
\]
where \(\varphi \in C^\infty(S^n)\) is strictly positive. Then, the Ricci tensors of \(g\) and \(g_{S^n}\) are correlated by the equation
\[
\text{Ric}_{S^n} = \text{Ric}_g + \frac{1}{\varphi^n} \left\{ (n-2) \nabla \varphi \nabla \varphi + [\varphi \Delta \varphi - (n-1) \nabla \varphi]^2 \right\} g,
\]

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which we algebraically manipulate in order to get the similar equation
\[ A_{g;\alpha} = A_g + \frac{\nabla^2 \varphi}{\varphi} - \frac{1}{2} \frac{|\nabla \varphi|^2}{\varphi^2} g, \]  
(16)
for the Schouten tensors. But then we have
\[ \frac{1}{2} \left( \varphi^2 + \frac{|\nabla \varphi|^2}{\varphi^2} \right) g = A_g + \nabla^2 \varphi, \]
from what it follows that
\[ \nabla^2 \varphi = \varphi \left[ -A_g + \frac{1}{n} \left( \sigma_1(g) + \frac{\Delta \varphi}{\varphi} \right) g \right]. \]
(17)
Notice that Lemma 13 applied to \( T = T_k(g^{-1}A_g) \) and \( X = \nabla \varphi \) gives
\[ \int_M \langle T_k(g^{-1}A_g), \nabla^2 \varphi \rangle dv_g = 0, \]
(18)
because \( \text{tr} T_k(g^{-1}A_g) = (n - k)\sigma_k(g) \) is constant on \( S^n \) by hypothesis and \( \text{div} T_k(g^{-1}A_g) = 0 \). A combination of (18) and (17) above leads to
\[ 0 = \int_M \langle T_k(g^{-1}A_g), -A_g + \frac{\sigma_1(g)\varphi + \Delta \varphi}{n} g \rangle dv_g = 0 \]
\[ = \int_M \varphi \left[ \frac{n-k}{n} \right] \sigma_1(g)\sigma_k(g) - (k+1)\sigma_{k+1}(g) \] 
\[ \int_M \varphi \left[ \frac{n-k}{n} \right] \sigma_1\sigma_k = (k+1)\sigma_{k+1}, \]
where we have used the identity \( \text{tr} T_k(g^{-1}A_g \circ A_g) = (k+1)\sigma_{k+1}(g) \). By Lemma 23 of [27] we conclude that
\[ \left( \frac{n-k}{n} \right) \sigma_1\sigma_k = (k+1)\sigma_{k+1}, \]
implying that \((S^n, g)\) is an Einstein manifold. In particular, the scalar curvature of \( g \) is constant on \( S^n \) and by Theorem 9 there is even an isometry between \((S^n, g)\) and \((S^n, g_{S^n})\) which proves the Theorem. \( \square \)

**Proof of Theorem 12.** Lemma 13 applied to the data \( T = \text{Ric}_g, \ X = \nabla f \) and \( \varphi = f \) gives
\[ \text{div} \circ \text{Ric}_g(f \nabla f) = f(\text{div} \circ \text{Ric}_g)(\nabla f) + f(\nabla^2 f, \text{Ric}_g) + \circ \text{Ric}_g(\nabla f, \nabla f), \]
(19)
and it then follows from the second contracted Bianchi identity that
\[ (\text{div} \circ \text{Ric}_g)(\nabla f) = \frac{n-2}{2n}(\nabla f, \nabla R). \]
(20)
A straightforward computation shows that
\[ f(\nabla^2 f, \Ric_g) = f \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) \langle g, \Ric_g \rangle = 0, \tag{21} \]
and equations (19), (20) and (21) together give
\[ \frac{1}{2} \text{div} \Ric_g(\nabla f^2) = \frac{n-2}{4n} \langle \nabla R_g, \nabla f^2 \rangle + \Ric_g(\nabla f, \nabla f). \tag{22} \]
Proposition 1 of [8] tells us that \( \text{div} \Ric_g(\nabla f^2) = 0 \) because \( |\Ric_g(\nabla f^2)| \in L^1(M) \). Consequently,
\[ \langle \nabla R_g, \nabla f^2 \rangle = 0 \quad \text{and} \quad \Ric_g(\nabla f, \nabla f) = 0. \]
As \( (M^n, g, \nabla f, \lambda) \) is a nontrivial quotient gradient almost Yamabe soliton, any regular level set \( \Sigma \) of the potential function \( f \) admits a maximal open neighborhood \( U \subset M \) in which \( g \) can be written like
\[ g = dr \otimes dr + (f'(r))^2 g^\Sigma, \tag{23} \]
where \( g^\Sigma \) is the restriction of \( g \) to \( \Sigma \) (see [9]). Since \( M \) is noncompact, \( f \) has at most one critical point. As the Ricci tensor of a warped product metric, \( \Ric_g \) now admits the following decomposition
\[ \Ric_g = \Ric^\Sigma - (n-1)\frac{f''}{f} dr \otimes dr - [(n-2)(f'')^2 + f'f'''] g^\Sigma, \tag{24} \]
thus giving \( \frac{R_g}{n} = -(n-1)\frac{f''}{f} \) because \( \Ric_g(\nabla f, \nabla f) = 0 \). Equation (24) can also be manipulated to show that
\[ \Ric_g(\nabla f) = \frac{R_g}{n} \nabla f, \]
of which
\[ \nabla \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) + \frac{R_g}{n(n-1)} \nabla f = 0, \tag{25} \]
is a consequence by Lemma 16 b). The divergence of equation (25) is
\[ \Delta \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) + \frac{1}{n(n-1)} \langle \nabla R_g, \nabla f \rangle + \frac{R_g}{n-1} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) = 0. \tag{26} \]
which we compare with the expression in Lemma 16 c) to see that \( \langle \nabla R_g, \nabla f \rangle = 0 \). Since \( R_g \) only depends on \( r \) we get that
\[ f'R_g = f' \langle \nabla R_g, \partial r \rangle = \langle \nabla R_g, \nabla f \rangle = 0, \]
implies that the scalar curvature \( R_g = R \) is constant. We claim that \( R \leq 0 \). As a matter of fact, if we had \( R > 0 \), we would then have
\[ \Ric_g \geq \frac{R}{n} g > \frac{R}{2n} g, \]
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because $\overset{\circ}{\operatorname{Ric}} \geq 0$ by hypothesis and the manifold $M^n$ would then be compact by the Bonnet-Myers Theorem. Therefore, $R \leq 0$.

a) It follows from (25) that $\log \frac{\sigma_k}{\sigma_l} - \lambda = c$ for some $c \in \mathbb{R}$ because we now have $R = 0$. By Theorem 2 of [25] $(M^n, g)$ must be isometric with flat Euclidean space $\mathbb{R}^n$ in case $c \neq 0$. Since this would leave us with $\sigma_1(g) = \sigma_2(g) = \cdots = \sigma_n(g) = 0$, the function $\log \frac{\sigma_k}{\sigma_l}$ could not be defined. Then, $c = 0$ and so $\nabla^2 f = 0$ by the fundamental equation (5). Theorem B of Kanai [19] then implies that $(M^n, g)$ is isometric with a Riemannian product manifold $\mathbb{R} \times F^{n-1}$. Notice that $\operatorname{Ric}_g \geq 0$ forces $F^{n-1}$ to have a nonnegative Ricci curvature;

b) If $f$ has no critical points and $R < 0$ then once more by (25) we get that $\log \frac{\sigma_k}{\sigma_l} - \lambda$ is not constant on $M^n$ and satisfies

$$
\nabla_X \nabla \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) = -\frac{R}{n(n-1)} \nabla_X \nabla f = -\frac{R}{n(n-1)} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) X,
$$
on $M^n$ for every $X \in \mathfrak{X}(M)$. In virtue of Theorem D in [19] the manifold $(M^n, g)$ is isometric with a warped product manifold $(\mathbb{R} \times F^{n-1}, dr^2 + \xi(r)^2 g_F)$ in which the warping function $\xi$ solves the second order linear ODE with constant coefficients $\xi'' + \left( n(n-1) \right) \xi = 0$;

c) In our last call to equation (25) we observe that if $f$ has exactly one critical point and $R < 0$ then $\log \frac{\sigma_k}{\sigma_l} - \lambda$ is not constant on $M^n$ and must satisfy

$$
\nabla_X \nabla \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) = -\frac{R}{n(n-1)} \nabla_X \nabla f = -\frac{R}{n(n-1)} \left( \log \frac{\sigma_k}{\sigma_l} - \lambda \right) X,
$$
on $M^n$ for every $X \in \mathfrak{X}(M)$. We then apply Theorem C in [19] to conclude that $(M^n, g)$ is isometric with a hyperbolic space.

□

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