Remarks on non-Gaussian fluctuations of the inflaton and constancy of $\zeta$ outside the horizon

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Abstract. We have pointed out that the non-Gaussianity arising from cubic self interactions of the inflaton field is proportional to $\xi N_e$. For scales of interest $N_e = 60$, and for models such as new inflation, natural inflation, and running mass inflation $\xi$ is large compared to the slow roll parameter. Therefore, the contribution from self interactions should not be outrightly ignored while retaining other terms in the non-Gaussianity parameter $f_{NL}$. But the $N_e$ dependent term seems to imply the growth of non-Gaussianities outside the horizon. Therefore, we have briefly discussed the issue of the constancy of correlations of the curvature perturbation $\zeta$ outside the horizon. We have then presented our results on the 3-point function of $\zeta$, and found that the $N_e$ dependent contribution to $f_{NL}$ from self interactions of the inflaton field is cancelled by contributions from other terms associated with non-linearities in cosmological perturbation theory.

1. Introduction
Non-Gaussianities in the curvature perturbation $\zeta$ in single field inflation can arise from self interactions of the inflaton and from non-linearities in cosmological perturbation theory. The contribution arising from self interactions of the inflaton field is usually treated as negligible compared to that from non-linearities in cosmological perturbation theory [1, 2], because it is proportional to higher order slow roll parameters. For example, consider a cubic self interaction term $\mu \phi^3$, for the inflaton field $\phi$. The contribution from this term to $\langle \zeta^3 \rangle \sim \langle \phi^3 \rangle$ is proportional to $\mu$ and thus, to the slow roll parameter $\xi \sim V''$. This is ignored when compared to terms proportional to slow roll parameters $\epsilon \sim V'^{-2}$ and $\eta \sim V''$. But for many models of inflation such as new inflation, small field natural inflation and running mass inflation $\xi \gg \epsilon$. Moreover, as we have pointed out below, the contribution of the self interaction term is actually proportional to $\xi N_e$, where $N_e(t)$ is the number of e-foldings, since the mode of interest crossed the horizon, and varies from 0 to 60 by the end of inflation for our current horizon scale. The above arguments indicate that one should not outrightly ignore the contribution of the self interactions of the inflaton in generating non-Gaussianities in small field inflation models.

If the expression for the 3-point function of the curvature perturbation is proportional to $N_e(t)$, then it indicates that correlations of the curvature perturbation are growing outside the horizon. This is contrary to one's expectations. One may wish to question: Whether the time dependent result is itself correct? However, it has been obtained independently in [3, 4, 5, 6] in different contexts. So, we have now first considered the literature on the constancy of correlations of $\zeta$ outside the horizon in the context of cosmological perturbation theory, and discussed the
arguments related to constancy of n-point functions of the curvature perturbation outside the horizon.

2. Discussion and results
Salopek and Bond [7] first introduced the generalization of the Bardeen-Steinhardt-Turner variable \( \zeta \) [8] for the case when one wishes to consider non-Gaussianities in the curvature perturbation. The Salopek-Bond \( \zeta(x) \) is constant outside the horizon. In other works [9, 10, 11, 12, 13, 14], the constancy of \( \zeta \) is shown for a \( \zeta_k \) mode. The above works deal with classical \( \zeta \), and the constancy of classical \( \zeta(x) \) or \( \zeta_k \) implies that the quantum 2-point function \( \langle \hat{\zeta}_k \hat{\zeta}_{\bar{k}} \rangle = (2\pi)^3 |\zeta_k|^2 \delta(k_1 - k_2) \) is constant outside the horizon (at lowest order)\(^1\). But, it is not obvious that other higher \( n \)-point functions of \( \zeta \) are constant outside the horizon.

The \( n \)-point functions of \( \zeta \) are given by [15, 1]:

\[
\langle \hat{O}(t) \rangle = \sum_{N=0}^{\infty} i^N \int_{t_0}^t dt_N \int_{t_0}^{t_N} dt_{N-1} \cdots \int_{t_0}^{t_2} dt_1 \times \langle \left[ \hat{H}_I(t_1), \left[ \hat{H}_I(t_2), \cdots \left[ \hat{H}_I(t_N), \hat{O}(t) \right] \cdots \right] \right] \rangle,
\]

where \( \hat{O}(t) \) can be any product of \( \hat{\zeta} \) operators, \( \hat{O}_I(t) \) is \( \hat{O}(t) \) in the interaction picture generated by the quadratic part of the Hamiltonian, and \( t_0 \) is some early time. Note that the expectation values are obtained in the in-in formalism and so the bra-s and kets refer to \( \langle \langle {}_i \rangle \rangle \) and \( \langle \langle {}_o \rangle \rangle \) respectively. \( \hat{H}_I \) is the interaction Hamiltonian and includes terms that are third or higher order in \( \zeta \). For the 3-point function at lowest order, this reduces to:

\[
\langle \hat{\zeta}^3(t) \rangle = i \int_{t_0}^t dt' \langle \left[ \hat{H}_I(t'), \hat{\zeta}^3(t) \right] \rangle.
\]

If \( \zeta_k \) is constant, \( \hat{\zeta} \sim e^{ik \cdot x} \zeta_k c_k + e^{-ik \cdot x} \zeta^*_k c_k^\dagger \) is constant. But for \( \langle \hat{\zeta}^3(t) \rangle \) to be constant outside the horizon, one must ensure that the contribution to the integral above from \( t_{ex} \) to \( t \) is suppressed, where \( t_{ex} \) is the time of horizon exit for the scale of interest.

The convergence of the integral for large \( t \) and certain other conditions for the constancy of \( \langle \hat{\zeta}^n \rangle \) outside the horizon have been discussed in general in [16]. But in [16] (see Eq. (29)), only Gaussian fluctuations of the inflaton are considered\(^2\).

One may argue that after horizon exit, the curvature perturbations are classical and so the constancy of \( \zeta \) should imply constancy of \( n \)-point function for \( n > 2 \) also. But as argued by Weinberg in [15], the perturbations are classical in the sense that commutators involving \( \zeta \) and its time derivative go to 0 for large \( t \), i.e.,

\[
[\zeta, \zeta], \ [\zeta, \dot{\zeta}] \rightarrow 0, \quad \text{as} \ t \rightarrow \infty.
\]

However, this implies that quantum \( n \)-point functions of zeta can nevertheless grow outside the horizon, though only as powers of \( \ln a \) and not as powers of \( a \). It is the time dependence due to \( \ln a \sim N_e \) for the 3-point function, which is the focus of our study.

In the light of the time dependent contribution from self interactions and the above discussion, we explicitly verify whether or not \( \langle \hat{\zeta}^n \rangle \) is, indeed, constant outside the horizon when one includes non-Gaussian fluctuations of the inflaton.

\(^1\) Higher order corrections from loops can give time dependent or \( N_e \) dependent contributions as mentioned in section VI of [15].

\(^2\) Note that the authors in [7] have also considered only Gaussian fluctuations of the inflaton, since \( \delta \phi \) is set equal to \( H/(2\pi) \) in the evaluation of \( \zeta \) in sections IIB and IID.
In [17], we have first calculated $\langle \delta \phi^3 \rangle$ in the $\delta \phi \neq 0$ gauge using the canonical formalism for cubic inflation with $V(\phi) = V_0 - \mu \phi^3$. We have then related $\xi_\epsilon(t)$ to $\delta \phi(k, t)$ and calculated $\langle \xi^3(t) \rangle$, and the non-Gaussianity parameter $f_{\text{NL}}$. For us, $t$ is arbitrary, unlike in the $\delta N$ formalism in which $t$ is the time of horizon exit. This allows us to study possible time dependent growth outside the horizon.

After a detailed calculation [17], we have found that:

$$\frac{6}{5} f_{\text{NL}} = \xi \left[ \frac{1}{3} + \gamma - N_e + \frac{3}{\sum_i k_i^3} \left( k_i \sum_{i<j} k_i k_j - \frac{4}{9} k_i^3 \right) \right]$$

$$+ \frac{3}{2\epsilon} - \eta + \frac{\epsilon}{\sum_i k_i^4} \left( \frac{4}{k_i} \sum_{i<j} k_i^2 k_j^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 \right),$$

where $f_{\text{NL}}$ is defined by:

$$\langle \xi(k_1, t) \xi(k_2, t) \xi(k_3, t) \rangle \equiv (2\pi)^3 \delta(k_1 + k_2 + k_3) \frac{6}{5} f_{\text{NL}} \sum_{i<j} P_\xi(k_i) P_\xi(k_j).$$

Our expression for $f_{\text{NL}}$ is similar to that in Eq. (38) of [6] with their $t_s$ replaced by $t$, and we have replaced $N_e$ by $-N_e = -H(t - t_\text{ex})$. $\gamma \approx 0.577216$ is Euler’s constant, and $k_i = \sum_j k_j$ and all magnitudes $k_i$ are presumed to be approximately equal.

For our potential $V = V_0 - \mu \phi^3 (\phi > 0)$, $\xi = 0.5 \eta^2$. Also, $\eta = 0.5 (n_s - 1) + 3 \epsilon \approx 0.5 (n_s - 1)$ [18]. If we take $n_s = 0.96$ [19], then $\eta = 0.02$ and $\xi N_e$ is 0.012. $\epsilon$ is much smaller. Thus, we have seen that $\xi N_e$ is comparable to $\eta$ and $\epsilon \ll \xi N_e$ and so the self interaction contribution proportional to $\xi$ should not be automatically ignored$^3$.$^4$.

Now, we have considered the constancy of the 3-point function of $\xi$. To determine how $f_{\text{NL}}$ changes during inflation after a mode crosses the horizon, we have taken the derivative of $f_{\text{NL}}$ with respect to $t$. Now, using $d/dt = \phi d/d\phi$, we get:

$$\frac{de}{dt} \simeq [4\epsilon^2 - 2\eta \epsilon] H,$$

$$\frac{d\eta}{dt} \simeq [2\eta \epsilon - \xi] H, \quad \text{and}$$

$$\frac{d\xi}{dt} \simeq [4\epsilon \xi - \eta \xi - \sigma] H,$$

where $\sigma = V'' V''' / V^3$. Then $df_{\text{NL}}/dt \approx (5/6) d/\xi N_e - \eta) / dt = 0$. Here, we have kept terms to first order in slow roll parameters as in $f_{\text{NL}}$. Thus, for the terms considered above $df_{\text{NL}}/dt$ is zero indicating that $f_{\text{NL}}(t) = f_{\text{NL}}(t_\text{ex})$, i.e., the $N_e$ factor in $\xi N_e$, which corresponds to growth outside the horizon, is cancelled by changes in other terms and does not contribute to the bispectrum$^5$.

(The argument above for the constancy of $\xi$ closely follows that in [6].)

$^3$ Interestingly, Appendix B of [20] obtains a similar result, but erroneously concludes that the $N_e$ dependent result of [3] and the $N_e$ independent result of [1] are the same because they are of the same order.

$^4$ In [6], it is argued that evaluating expectation values at the end of inflation may not be valid for large $N_e \approx 60$, because of divergences of the form $e^{\xi N_e^2} N_e^m (m \geq 1)$. However, for the potential, we are considering $\epsilon$ is much smaller than 1/60.

$^5$ Note that these arguments are for adiabatic fluctuations of the inflaton and do not apply to any non-adiabatic fluctuations or fluctuations of other scalar fields considered in [3, 4, 5].
3. Conclusion

We have argued that it is not proper to a priori, ignore the contribution of inflaton self interactions to the correlations of the curvature perturbation $\zeta_k(t)$ in the case of small field inflation. In fact, the contribution of these terms grows linearly outside the horizon. Explicitly calculating the total 3-point correlation function of $\zeta$, we have found that the time dependent contribution from the self interaction term to the non-Gaussianity parameter $f_{NL}$ is cancelled by the growth of other terms in $f_{NL}$.

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