Generalized Hasse invariants for Shimura varieties of Hodge type

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Abstract. We construct canonical Hasse invariants for arbitrary Shimura varieties of Hodge type.

Introduction

The Hasse invariant is a modular form which has been an important tool for constructing congruences between automorphic forms. It is defined for certain Shimura varieties that are endowed with a universal abelian scheme (like Siegel Shimura varieties, Hilbert-Blumenthal varieties, or more generally Shimura varieties of Hodge type). Its non-vanishing locus on the special fiber in positive characteristic $p$ is the locus where the universal abelian scheme is ordinary. It is a section of the $(p−1)$-th power of the Hodge line bundle of the universal abelian scheme.

In this paper we consider the following situation. Let $(G, X)$ be a Shimura datum of Hodge type, $K = K_p K^p \subseteq G(\mathbb{A}_f)$ an open compact subgroup such that $K_p \subseteq G(\mathbb{Q}_p)$ is hyperspecial and such that $K^p \subseteq G(\mathbb{A}_p^f)$ is sufficiently small. Let $S = S_K(G, X)$ be the special fiber of the canonical integral model of a Shimura variety of Hodge type at a place $v$ of the reflex field $E$ lying over $p$. Then usually the ordinary locus is empty, hence the classical Hasse invariant vanishes. Instead, the generic Newton stratum for good reductions of Shimura varieties of Hodge type is the so-called $\mu$-ordinary locus $S^\mu$, introduced by the second author in [Wd1], which has been shown to be open and dense by Wortmann ([Wor]). In this paper we define an integer $N \geq 1$, the Hasse number (Definition 4.11). Then our main result is the following.

Theorem 1. There exists a canonical section $H$ of the $N$-th power of the Hodge line bundle $\omega$ on $S$ such that the non-vanishing locus of $H$ is $S^\mu$.

We refer to the main text (Theorem 4.12) for the precise meaning of “canonical”. If $G$ is split over $\mathbb{Q}_p$ and its derived group is simply connected, then the ordinary locus is non-empty, we obtain $N = p − 1$ (Corollary 2.23) and $H$ is the classical Hasse invariant.

All other generalizations of Hasse invariants in positive characteristic that we are aware of are special cases of our construction. We give two examples: For Hilbert-Blumenthal varieties partial Hasse invariants have been constructed by Goren in [Gor].

\footnote{In fact, in the PEL case the ordinary locus is non-empty if and only if $E_v = \mathbb{Q}_p$ by [Wd1] Theorem 1.6.3.}
This is rephrased in our language in Example 3.22. For Shimura varieties of PEL type such that \( \mathbf{G}(\mathbb{R}) \) is isomorphic to a unitary similitude group has been considered by Goldring and Nicole in [GoNi]. We consider this case in Example 2.25 if \( \mathbf{G}_{\mathbb{Q}_p} \) is non-split; here the Hasse number is \( p^2 - 1 \).

A formal argument using the definition of the minimal compactification and its normality shows that the Hasse invariant \( H \) can be extended to the minimal compactification and that some power of \( H \) can be lifted to characteristic 0. We define the \( \mu \)-ordinary locus \( \mathcal{S}_{\mathsf{min}, \mu} \) of the minimal compactification \( \mathcal{S}_{\mathsf{min}} \) as the non-vanishing locus of the extension of \( H \). As the Hodge line bundle on the minimal compactification is ample, we deduce the following corollary.

**Corollary 2.** The \( \mu \)-ordinary locus \( \mathcal{S}_{\mathsf{min}, \mu} \) of the minimal compactification is affine.

The idea behind our construction is to consider the Ekedahl-Oort stratification of \( S \) which is given by a smooth morphism \( \zeta: S \to G\text{-Zip}^\chi \) constructed by Zhang ([Zha1], see also [Wor]). Here \( G \) is a reductive reduction of \( \mathbf{G} \) over \( \mathbb{F}_p \), \( \chi \) is a certain cocharacter of \( \mathbf{G} \), defined over a finite extension of \( \mathbb{F}_p \), given by the Shimura datum and \( G\text{-Zip}^\chi \) is the algebraic stack of \( G \)-zips of type \( \chi \) introduced by Pink, Ziegler, and the second author in [PWZ2] (see Subsection 4.4 for details). By a result of Wortmann ([Wor]), the inverse image of the generic point of \( G\text{-Zip}^\chi \) is the \( \mu \)-ordinary locus \( \mathcal{S}^\mu \).

We then construct a line bundle \( \omega^b \) on \( G\text{-Zip}^\chi \) whose pull back via \( \zeta \) is the Hodge line bundle on \( S \). Moreover we show that the space of global sections of any power \( (\omega^b)^{\otimes k} \) over \( G\text{-Zip}^\chi \) has dimension at most 1 and that there is a non-vanishing section \( H^k \) for \( k = N \). Then our main technical result (Theorem 3.8) shows that its vanishing locus is precisely the complement of the generic point of \( G\text{-Zip}^\chi \). Hence \( H := \zeta^*(H^k) \) is the desired Hasse invariant.

We now give an overview of the paper. As line bundles on quotient stacks (like \( G\text{-Zip}^\chi \)) are just line bundles with equivariant structures, we recall in Section 1 several results on equivariant Picard groups. None of them are probably new but for many of them we did not find a reference or only references with too restrictive hypotheses (such as that all ground fields are of characteristic 0). Thus we included also some proofs.

In Section 2 we examine the Picard group of the stack of \( G \)-zips and of related stacks like the Bruhat stack. We then state in Section 3 a positivity conjecture and prove the conjecture in a special case which is sufficient for the applications to Shimura varieties stated above. This is the technical heart of our paper.

In the last section we deduce the above applications to Shimura varieties of Hodge type.

**Notation and terminology:** A linear algebraic group \( G \) over a field \( k \) is an affine smooth group scheme over \( k \). We denote by \( X^*(G) \) its group of \( k \)-rational characters.

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1 Equivariant Picard groups

In this section, $G$ denotes an arbitrary connected linear algebraic group over an algebraically closed field $k$. Products of schemes are fiber products over $k$. A variety is an integral $k$-scheme of finite type. A $G$-scheme is a $k$-scheme endowed with a $G$-action $G \times X \to X$.

We always consider left actions. If $G$ is an algebraic group and $H$ is an algebraic subgroup we denote by $G/H$ the quotient of $G$ by the left action $(h,g) \mapsto gh^{-1}$. 

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1.1 $G$-linearization of a line bundle

Let $X$ be a $G$-scheme, where the action is given by a map $a : G \times X \to X$. Let $\mathcal{L}$ be a line bundle on $X$. Define the projections $p_{23} : G \times G \times X \to G \times X$, $(g, h, x) \mapsto (h, x)$ and $p_2 : G \times X \to X$, $(g, x) \mapsto x$. Finally, write $\mu_G$ for the multiplication map $G \times G \to G$.

**Definition 1.1.** A $G$-linearization of $\mathcal{L}$ is an isomorphism $\phi : a^*(\mathcal{L}) \to p_2^*(\mathcal{L})$ satisfying the cocycle condition

$$p_{23}^*(\phi) \circ (\text{id}_G \times a)^*(\phi) = (\mu_G \times \text{id}_X)^*(\phi)$$

If $\pi : L \to X$ is the corresponding geometric line bundle, a linearization of $L$ is an action $a_L : G \times L \to L$ such that:

- The map $\pi : L \to X$ is $G$-invariant.
- The induced isomorphism $G \times L \to a^*(L)$ is a morphism of geometric line bundles.

The first condition ensures that we have a cartesian diagram:

$$\begin{array}{ccc}
G \times L & \xrightarrow{a_L} & L \\
id_G \times \pi \downarrow & & \downarrow \pi \\
G \times X & \xrightarrow{a} & X
\end{array}$$

The second condition says that $G \times L \simeq a^*(L)$ as geometric line bundles. We denote by $\text{Pic}^G(X)$ the group of isomorphism classes of $G$-linearized line bundles on $X$. There is a natural map

$$\text{Pic}^G(X) \to \text{Pic}(X)$$

whose image is the subgroup of $G$-linearizable line bundles, denoted by $\text{Pic}_G(X)$.

The group $\text{Pic}^G(X)$ can be identified with the Picard group of the quotient stack $[G\backslash X]$. Then $\text{Pic}^G(X) \to \text{Pic}(X)$ is the homomorphism given by pull back by the projection $X \to [G\backslash X]$. Thus, if $H \subset G$ are linear algebraic groups, there is an isomorphism $\text{Pic}(G/H) \simeq \text{Pic}^H(G)$. Notice that in this case, $[G/H] = G/H$ is a scheme.

1.2 General results on equivariant Picard groups

Let us first recall the following result by Rosenlicht ([FoM] Corollary 2.2).

**Theorem 1.2 (Rosenlicht).** Let $G$ be a connected linear algebraic group and let $f : G \to \mathbb{G}_m$ be a morphism of $k$-schemes such that $f(1) = 1$. Then $f$ is a character.

For every $k$-scheme $X$, we define:

$$E(X) := \mathcal{O}(X)^\times / k^\times.$$ 

If $X$ is an integral $k$-scheme of finite type, then $E(X)$ is a finitely generated free abelian group ([KKV] 1.3, the general assumption that $k$ is of characteristic 0 is not used in
the proof). This implies that if $X$ carries an action by a connected group $G$, then the induced action of $G$ on $E(X)$ is trivial. Note that if $H$ is an algebraic group, then the inclusion $X^*(H) \subset \mathbb{G}_m(H)$ induces an isomorphism

$$X^*(H) \cong E(H)$$

by Rosenlicht’s theorem.

**Proposition 1.3** ([KKV] Lemma 2.2, Proposition 2.3). Let $G$ be a smooth algebraic group, and let $X$ be an irreducible $G$–variety. Then there are exact sequences:

$$0 \to \frac{(\mathcal{O}(X))^G}{k^\times} \to E(X)^G \to X^*(G) \to H^1_{\text{alg}}(G, \mathcal{O}(X)^*) \to H^1_{\text{alg}}(G/G^0, E(X))$$

$$0 \to H^1_{\text{alg}}(G, \mathcal{O}(X)^*) \to \text{Pic}^G(X) \to \text{Pic}(X)$$

If $X$ is normal and $G$ connected, the second exact sequence has an extension by a map $	ext{Pic}(X) \to \text{Pic}(G)$.

In particular, the two exact sequences of Proposition 1.3 combine into a longer one when $G$ is connected. If we assume that $X$ is normal, we can extend the sequence with the Picard group of $G$:

**Theorem 1.4.** Let $G$ be a smooth connected algebraic group and let $X$ be a normal irreducible $G$-variety over $k$. Then we have an exact sequence of abelian groups

$$1 \to k^\times \to (\mathcal{O}(X)^*)^G \to E(X) \to X^*(G) \to \text{Pic}^G(X) \to \text{Pic}(X) \to \text{Pic}(G)$$

Let us sketch the proof in the connected case, since we will need the explicit construction of these maps.

- If $f \in \mathcal{O}(X)^*$, then for any $g \in G$, there is $\lambda_f(g) \in k^\times$ such that

$$g \cdot f = \lambda_f(g)f.$$  

Then $\lambda_f$ is a character of $G$. This defines the map $E(X) \to X^*(G)$, $f \mapsto \lambda_f$.

- Then, if $\lambda \in X^*(G)$, we define a $G$-linearization of the trivial line bundle on $X$

$$G \times X \times \mathbb{A}^1_k \to X \times \mathbb{A}^1_k$$

by $(g, x, s) \mapsto (g \cdot x, \lambda(g)s)$. This yields a map $X^*(G) \to \text{Pic}^G(X)$.

- Finally, the map $\text{Pic}(X) \to \text{Pic}(G)$ which terminates the sequence depends on the choice of an element $x_0 \in X$. An element $L \in \text{Pic}(X)$ is sent to $a^*(L)|_{X \times \{x_0\}}$.

**Corollary 1.5.** Let $G$ be a smooth connected algebraic group, let $H$ be a smooth connected algebraic subgroup of $G$, and let $\pi: G \to G/H$ be the quotient by left action $(h, g) \mapsto gh^{-1}$. Then there is an exact sequence

$$1 \to X^*(G)^H \to X^*(G) \to X^*(H) \xrightarrow{\lambda} \text{Pic}(G/H) \to \text{Pic}(G) \to \text{Pic}(H)$$
The homomorphism $\chi$ attaches to $\lambda \in X^*(H)$ the invertible $\mathcal{O}_{G/H}$-module $\mathcal{L}_\lambda$ whose sections over $U \subseteq G/H$ are given by

$$(1.1) \quad \Gamma(U, \mathcal{L}_\lambda) = \{ f: \pi^{-1}(U) \to k ; f(gh^{-1}) = \lambda(h)f(g) \text{ for } g \in \pi^{-1}(U), h \in H \}.$$

The corresponding geometric bundle is $(G \times k)/H$, where $H$ acts on $k$ via $\lambda: H \to k^\times$.

**Remark 1.6.** If $\pi: G \to G/H$ has locally for the Zariski topology a section, then $i^*: \text{Pic}(G) \to \text{Pic}(H)$ is surjective ([San Prop. 6.10]).

When the group acting is not connected, we do not have in general a long exact sequence as in Theorem [14]. However, this is true in the case of $G/H$ where $H \subseteq G$ is a subgroup:

**Proposition 1.7.** Let $G$ be a smooth connected algebraic group and $H$ a smooth subgroup of $G$. Then there is an exact sequence:

$$0 \to E(G/H) \to X^*(G) \to X^*(H) \to \text{Pic}^H(G) \to \text{Pic}(G)$$

**Proof.** The group $H$ acts trivially on $E(G) = X^*(G)$, since $\chi(gh^{-1}) = \chi(g)\chi(h)^{-1}$ for all $g \in G$, $h \in H$, $\chi \in X^*(G)$, so $h \cdot \chi = \chi$ in $E(G)$. It follows that $H^1_{\text{alg}}(H, E(G)) = \text{Hom}_{\text{Grp}}(H/H_0, E(G)) = 0$ since $H/H_0$ is finite and $E(G)$ is torsion-free. 

We will also need the following proposition:

**Proposition 1.8.** Let $H \subset G$ be algebraic groups. Then there is a natural isomorphism:

$$\text{Pic}^G(G/H) \simeq X^*(H).$$

**Proof.** One has $\text{Pic}^G(G/H) \simeq \text{Pic}(\![G\backslash(G/H)]\!] \simeq \text{Pic}(\![1/H]\!] \simeq \text{Pic}^H(1) \simeq X^*(H)$. 

### 1.3 Functoriality of the equivariant Picard group

Let $G$, $G'$ be algebraic groups and $f: G \to G'$ a morphism of algebraic groups. Let $X$ be a $G$-scheme, let $X'$ be a $G'$-scheme, and $\pi: X \to X'$ be an $f$-equivariant morphism, i.e. a morphism of $k$-schemes such that

$$(1.2) \quad \pi(g \cdot x) = f(g) \cdot \pi(x) \quad \forall g \in G, x \in X$$

Then $\pi$ induces naturally a homomorphism

$$(1.3) \quad \pi^*: \text{Pic}^{G'}(X') \to \text{Pic}^G(X)$$

by interpreting $\pi^*$ as the pull-back by the induced map of stacks

$$[G\backslash X] \to [G'\backslash X'].$$

A more concrete description of $\pi^*$ is the following. Let $L' \in \text{Pic}^{G'}(X')$ be a (geometric) $G'$-linearized line bundle. Then define a $G$-action on $L = X \times_{X'} L'$ by

$$(g, (x, l')) \mapsto (gx, (gx, f(g)l')).$$

If $X$ and $X'$ are smooth varieties and if $G$, $G'$ are smooth and connected, we obtain a natural morphism between the exact sequences provided by Theorem [13].
1.4 Finite groups of Lie type

We begin with the well-known theorem of Lang \(^\text{[Spr]}\):

**Theorem 1.9** (Lang). Let \( G \) be a connected algebraic group defined over some finite field \( \mathbb{F}_q \). Let \( \varphi: G \to G \) be the \( q \)-th power Frobenius. Then the map \( g: G \to G, x \mapsto \varphi(x)x^{-1} \) is finite étale and surjective.

**Corollary 1.10.** Let \( G \) be a connected algebraic group defined over some finite field \( \mathbb{F}_q \). Then the map \( g \) of theorem 1.9 induces an isomorphism \( G/G(\mathbb{F}_q) \sim \to G \).

**Proof.** Clearly \( g \) induces an injective map \( G/G(\mathbb{F}_q) \to G \) which is surjective and finite étale by Theorem 1.9 and hence an isomorphism. \( \Box \)

**Proposition 1.11.** Let \( G \) be a connected linear algebraic group defined over \( \mathbb{F}_q \). Assume that \( \text{Pic}(G) = 0 \). Then there is an exact sequence:

\[
0 \to X^*(G) \to X^*(G) \to \text{Hom}(G(\mathbb{F}_q), k^\times) \to 0
\]

where the first map is \( \chi \mapsto \sigma \cdot \chi - \chi \).

**Proof.** Write \( H = G(\mathbb{F}_q) \). The map \( g : G \to G, x \mapsto \varphi(x)x^{-1} \) induces an isomorphism \( G/H \to G \), so we deduce that \( \text{Pic}(G/H) = 0 \), so \( X^*(G) \to X^*(H) \) is surjective by proposition 1.7. \( \Box \)

**Remark 1.12.** This result is false without the condition \( \text{Pic}(G) = 0 \). Consider \( G = PGL_n \) over \( \mathbb{F}_q \). Then it follows from the triviality of \( H^1(\text{Gal}(k/\mathbb{F}_q), k^\times) \) that \( G(\mathbb{F}_q) = PGL_n(\mathbb{F}_q) := GL_n(\mathbb{F}_q)/\mathbb{F}_q^\times \). Then the derived group of \( G(\mathbb{F}_q) \) is \( SL_n(\mathbb{F}_q)/\mu_n(\mathbb{F}_q) \) and the abelianization of \( G(\mathbb{F}_q) \) is \( \mathbb{F}_q^\times /\mathbb{F}_q^{\times n} \cong \mathbb{Z}/d\mathbb{Z} \) where \( d = \gcd(q - 1, n) \). It follows that \( \text{Hom}(G(\mathbb{F}_q), k^\times) \cong \mathbb{Z}/d\mathbb{Z} \), whereas \( X^*(G) = 0 \).

**Example 1.13.** For example, if \( G = SL_n \), then the above proposition implies that there are no nontrivial group homomorphisms \( SL_n(\mathbb{F}_q) \to k^\times \).

1.5 \( G \)-varieties and divisors

Let \( X \) be a \( k \)-variety. As usual we denote the free group generated by the irreducible subvarieties of codimension (resp. dimension) \( i \geq 0 \) by \( Z^i(X) \) (resp. \( Z_i(X) \)). Elements of \( Z^1(X) \) are the Weil divisors on \( X \).

If \( X \) is a \( G \)-variety, we define an action of \( G \) on \( Z^i(X) \) in the obvious way: For \( D = \sum n_C [C] \in Z^i(X) \) we set

\[
g \cdot D = \sum n_C [g \cdot C].
\]

Let \( Z^i(X)^G \) be the subgroup of \( G \)-invariant elements of \( Z^i(X) \).

Assume that \( X \) is regular. Recall that the locally free \( \mathcal{O}_X \)-module \( \mathcal{L}(D) \) of rank 1 associated to \( D = \sum n_C [C] \in Z^1(X) \) is defined by

\[
\mathcal{L}(D)(U) = \{ f \in K(X) : v_C(f) + n_C \geq 0 \text{ for all } C \text{ intersecting } U \} \cup \{0\}.
\]
Thus we see that when $D \in Z^1(X)^G$ and $g \in G$, there is a natural isomorphism $a_g: g^*\mathcal{L}(D) \to \mathcal{L}(D)$ defined by $f \mapsto g \cdot f$ (notice that $\text{div}(g \cdot f) = g \cdot \text{div}(f)$). These isomorphisms $(a_g)_{g \in G}$ satisfy the cocycle condition $a_{gh} = a_h \circ (h^*a_g)$ so in turn, they define a $G$-linearization of $\mathcal{L}(D)$. We have constructed a commutative diagram

$$
\begin{array}{ccc}
Z^1(X)^G & \longrightarrow & Z^1(X) \\
\uparrow & & \uparrow \\
\mathcal{L} & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
\text{Pic}^G(X) & \longrightarrow & \text{Pic}(X) \\
\end{array}
$$

Of course this is a very special instance of the theory of equivariant Chow groups. Assume from now on that $X$ is a smooth $G$-variety.

**Proposition 1.14.** Let $G$ be a smooth algebraic group and let $X$ be an irreducible smooth $G$-variety. There is a commutative diagram with exact lines and columns:

$$
\begin{array}{cccc}
0 & \longrightarrow & H^1_{\text{alg}}(G, \mathcal{O}(X)^\times) & \longrightarrow & H^1(G, \mathcal{O}(X)^\times) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z^1(X)^G & \longrightarrow & \text{Pic}^G(X) & \longrightarrow & H^1(G, K(X)^\times) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (K(X)^\times)^G & \longrightarrow & Z^1(X)^G & \longrightarrow & \text{Pic}_G(X) & \longrightarrow & H^1(G, (K(X)^\times)_G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_{\text{alg}}(G, \mathcal{O}(X)^\times) & \longrightarrow & H^1(G, K(X)^\times) & \longrightarrow & 0.
\end{array}
$$

**Proof.** For us only the central square of this diagram will be important. First, let us define the maps. Let $\mathcal{L} \in \text{Pic}_G(X)$ be a $G$-linearizable line bundle on $X$. We may assume that $\mathcal{L} = \mathcal{L}(D)$ for some Weil divisor $D$ on $X$. Then $g \cdot D = D + \text{div}(f_g)$ for some $f_g \in K(X)^\times$. This defines a cocycle, independant of the choice of $D$. We get a map $\text{Pic}_G(X) \to H^1(G, K(X)^\times)$. Now, let $L \in \text{Pic}^G(X)$. Again, assume that the underlying line bundle on $X$ has the form $\mathcal{L}(D)$ for some Weyl divisor $D$. The $G$-linearization of $\mathcal{L}$ gives an isomorphism $\mathcal{L}(gU) \to \mathcal{L}(U)$ for all open $U$ and all $g \in G$. Choose $U \subset X - [D]$. Then $\mathcal{L}(U) = \mathcal{O}_X(U)$, so the element $1 \in \mathcal{L}(U)$ is mapped to some $\lambda_g \in \mathcal{L}(gU) \subset K(X)$. This defines a 1-cocycle $g \mapsto \lambda_g$ in $H^1(G, K(X)^\times)$. The two maps to $Z^1(X)^G$ are given by the divisor of a function. The commutativity of the diagram is an easy exercise.
First, we prove the exactness of the upper horizontal sequence at the term $Z^1(X)^G$. So let $D$ be a $G$–invariant Weil divisor, mapping to zero in $\text{Pic}^G(X)$. In particular, we can write $D = \text{div}(f)$ with some $f \in K(X)^{\times}$. Then the map

$$\gamma : g \mapsto \frac{g \cdot f}{f}$$

is an algebraic cocycle mapping to zero in $\text{Pic}^G(X)$. It follows that this cocycle is a coboundary, so there is a function $h \in \mathcal{O}(X)^{\times}$ such that $\gamma(g) = \frac{g \cdot h}{h}$ for all $g \in G$. It follows that $\frac{1}{f} \in K(X)^{\times G}$ and this shows the first part.

Secondly, we prove exactness of the lower horizontal sequence at the term $\text{Pic}^G(X)$. Let $L = L(D)$ be a $G$–linearizable line bundle on $X$, such that the associated cocycle of $\frac{K(X)^{\times}}{\mathcal{O}(X)^{\times}}$ is a coboundary. By definition, there is $f \in K(X)^{\times}$ such that $g \cdot D = D + \text{div} \left( \frac{g \cdot f}{f} \right)$ and it follows that $D - \text{div}(f)$ is a $G$–invariant Weil divisor mapping to $L$. We will not need the rest of the diagram, so we leave the proofs to the reader.

\[\square\]

1.6 $G$-varieties with open orbit

In this section we assume that $X$ is a smooth irreducible $G$-variety which has an open $G$–orbit $U \subset X$. In this case it is clear that $K(X)^G = k$, so the map $L : Z^1(X)^G \to \text{Pic}^G(X)$ is injective. We assume further that $G$ is connected. Note that in this case the group $Z^1(X)^G$ identifies with the free group generated by the irreducible components of codimension one in $X - U$. Then we have the following commutative diagram with exact lines:

\[
\begin{array}{cccccccc}
0 & \rightarrow & E(U) & \rightarrow & X^*(G) & \rightarrow & \text{Pic}^G(U) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & E(X) & \rightarrow & X^*(G) & \rightarrow & \text{Pic}^G(X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{L}(U) & \rightarrow & \mathcal{L}(X) & \rightarrow & \text{Pic}^G(X) & \rightarrow & 0 \\
\end{array}
\]

The commutativity is clear. The exactness of the two rows is a simple application of Theorem 1.4 using the fact that $\mathcal{O}(X)^G = \mathcal{O}(U)^G = k$. Let us prove the surjectivity
of the last two vertical maps. Note that we only need to prove the surjectivity of \( \text{Pic}^G(X) \to \text{Pic}^G(U) \). Consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & E(X) & \to & X^*(G) & \to & \text{Pic}^G(X) & \to & \text{Pic}(X) & \to & \text{Pic}(G) \\
 & \downarrow & \downarrow & & & \downarrow & & & \downarrow & & \\
0 & \to & E(U) & \to & X^*(G) & \to & \text{Pic}^G(U) & \to & \text{Pic}(U) & \to & \text{Pic}(G) \\
\end{array}
\]

where the rightmost horizontal maps have to be defined using the same choice of an element in \( U \). As \( X \) is smooth, \( \text{Pic}(X) \to \text{Pic}(U) \) is surjective and a simple diagram chase shows the surjectivity of \( \text{Pic}^G(X) \to \text{Pic}^G(U) \).

An element in the kernel of \( \text{Pic}_G(X) \to \text{Pic}_G(U) \) comes from a Weil divisor \( D \) with support in \( X \setminus U \), which is \( G \)-stable since \( Z^1(X)^G = Z_{n-1}(X - U) \). This proves the exactness of the rightmost column. It remains to show that the kernel of \( \text{Pic}^G(X) \to \text{Pic}^G(U) \) is \( Z^1(X)^G \). Let \( L \in \text{Pic}^G(X) \) be mapped to zero in \( \text{Pic}^G(U) \). Since we proved the exactness of the rightmost column, we may assume further that \( L \) is the trivial bundle on \( X \), so it comes from a character of \( G \). Since it is trivial in \( \text{Pic}^G(U) \), this character is attached to a function \( f \in E(U) \). Then the opposite of the divisor of \( f \) is a preimage of \( L \) in \( Z^1(X)^G \).

**Remark 1.15.** From the above diagram, we get a natural map \( E(U) \to X^*(G) \to \text{Pic}^G(X) \), whose image is contained in \( Z^1(X)^G \). As we mentioned in the proof, this map is \( f \mapsto -\text{div}(f) \).

The choice of a point \( x_0 \in U \) yields an isomorphism \( G/G_{x_0} \cong U \), where \( G_{x_0} \) is the (scheme-theoretic) stabilizer of \( x_0 \). If \( G_{x_0} \) is finite (equivalently, \( \dim(G) = \dim(X) \)), the orbit map

\[
u: G \mapsto U, \quad g \mapsto g \cdot x_0
\]

is a finite flat \( G \)-equivariant morphism whose degree is \( N := \dim_k \mathcal{O}(G_{x_0}) \). It is finite étale if and only if \( G_{x_0} \) is reduced.

The morphism \( \nu \) induces a group homomorphism \( E(U) \to X^*(G) \) whose image is the subgroup of \( X^*(G) \) of characters which vanish on \( G_{x_0} \). The natural isomorphism \( X^*(G_{x_0}) \cong \text{Pic}^G(U) \) from Proposition 1.8 induces an isomorphism of exact sequences:

\[
\begin{array}{cccccc}
0 & \to & E(U) & \to & X^*(G) & \to & X^*(G_{x_0}) & \to & \text{im}(X^*(G) \to X^*(G_{x_0})) & \to & 0 \\
 & \downarrow = & \downarrow = & & & \downarrow \cong & \downarrow \cong & & \downarrow & & \\
0 & \to & E(U) & \to & X^*(G) & \to & \text{Pic}^G(U) & \to & \text{Pic}_G(U) & \to & 0 \\
\end{array}
\]

For example, assume that \( G \) is defined over \( \mathbb{F}_q \), acting on itself via Frobenius-conjugation. Then \( U = G \) and one has \( \text{Pic}^G(U) \cong \text{Hom}(G(\mathbb{F}_q), k^\times) \).

### 1.7 The space of global sections

**Proposition 1.16.** Let \( G \) be an algebraic group and let \( X \) be an irreducible \( G \)-variety with an open orbit \( U \subset X \). Denote by \( \pi : X \to [X/G] \) the projection. Let \( \mathcal{L} \) be a line bundle on the stack \([X/G]\) and write \( L = \pi^* \mathcal{L} \). Then
1. $H^0([X/G], \mathcal{L})$ identifies with $H^0(X, L)^G$.

2. The $k$–vector space $H^0([X/G], \mathcal{L})$ has dimension less than 1.

3. If $H^0([X/G], \mathcal{L}) \neq 0$ then $\mathcal{L}$ restricts to the trivial line bundle on $[U/G]$.

4. If $\mathcal{L}$ is trivial, $H^0([X/G], \mathcal{L}) = k$.

Proof. A global section $s \in H^0([X/G], \mathcal{L})$ is the same as a morphism of $O_{[X/G]}-$modules $O_{[X/G]} \to \mathcal{L}$, and this is the same as a morphism of $O_X-$modules $O_X \to L$ which commutes with the $G$–action (with respect to the trivial action on $O_X$). This proves the first assertion.

The map $H^0(X, L) \to H^0(U, L)$ is injective and $G$–equivariant, so in order to prove the rest of the proposition, we may assume $X = U$ and show that only the trivial line bundle has global sections. Now let $H \subset G$ be the stabilizer of some element in $X$. Then $X$ identifies with $G/H$. Now, one has $\text{Pic}^G(G/H) \simeq X^*(H)$. If $\chi \in X^*(H)$ is a character, we denote by $\mathcal{L}(\chi)$ and $L(\chi)$ the corresponding line bundles on $[G\backslash G/H]$ and $G/H$ respectively. The global sections of $L(\chi)$ on $G/H$ is the set:

$$H^0(G/H, L(\chi)) \simeq \{ f : G \to k, \ f(gh) = \chi(h) f(g), \ \forall g \in G, \forall h \in H \}$$

and the $G$–invariant global sections are zero unless $\chi = 1$.  

Let $A \subset \text{Pic}([X/G])$ be the set of line bundles admitting global sections. We have seen that the set $A$ is contained in

$$Z^1(X)^G = \text{Ker} \ (\text{Pic}([X/G]) \to \text{Pic}([U/G]))$$

Then $A$ is equal to the cone of effective divisors in $Z^1(X)^G$. Indeed, if $C \in Z^1(X)^G$ and $\mathcal{L}(C) \in \text{Pic}^G(X)$ is the associated line bundle, then the global sections of $\mathcal{L}(C)$ are zero or the constant functions $k \subset K(X)$ if $C \geq 0$.

On the other hand, if $\chi \in X^*(G)$ and $\mathcal{L}(\chi) \in \text{Pic}^G(X)$ is the associated line bundle, then $\mathcal{L}(\chi) \in A$ if and only if there is a nonzero function $f : X \to k$ such that

$$f(g \cdot x) = \chi(g) f(x), \ \forall g \in G, \forall x \in X$$

In this representation of $\mathcal{L}(\chi)$, the global sections of $\mathcal{L}(\chi)$ identify with the space (of dimension 1) of all such functions $f$. This difference in the interpretation of the space of global functions lies in the fact that we have two separate ways to define an element of $\text{Pic}([X/G])$.

2 The Picard group of the stack of $G$-zips

In this section we will compute the Picard group of the stack $[E\backslash G]$ which was studied in detail in [PWZ1] and [PWZ2]. Let us first fix some notation.
2.1 The Picard group of a flag variety

First recall some general facts about the Picard group of a linear group:

**Proposition 2.1.** Let $G$ be a connected linear algebraic group. Let $G^{\text{red}} = G/R_u(G)$ be its maximal reductive quotient, let $G^{\text{der}}$ the derived group of $G^{\text{red}}$ and let $\pi: \tilde{G} \to G^{\text{der}}$ be the simply connected cover of $G^{\text{der}}$. Then :

$$\text{Pic}(G) = \text{Pic}(G^{\text{red}}) = \text{Pic}(G^{\text{der}}) = X^*(\text{Ker}(\pi)).$$

In particular $\text{Pic}(G)$ is a finite group.

**Proof.** This follows from the description of the Picard group in terms of root datum [Folx Prop. 5.1]

**Corollary 2.2.** Let $G$ be a reductive group. Then the following assertions are equivalent.

(i) $\text{Pic}(G) = 0$.
(ii) The derived group $G^{\text{der}}$ is simply connected.
(iii) For any parabolic subgroup $P$ and any Levi subgroup $L \subseteq P$ one has $\text{Pic}(L) = 0$.

**Proof.** The equivalence of (i) and (ii) is clear by Proposition 2.1 (because $\text{Ker}(\pi)$ is diagonalizable). Moreover, if the derived group of $G$ is simply connected, then this holds for every Levi subgroup $L$ (e.g., [SGA3] Exp. XXI, 6.5.11).

For future use, let us also mention the following consequence of the finiteness of the Picard group of a linear algebraic group :

**Proposition 2.3.** Let $G$ be a linear algebraic group and let $H \subset G$ be an irreducible subvariety of codimension 1. Let $Z \subset G$ be a smooth irreducible closed subvariety not contained in $H$. Then the irreducible components of $Z \cap H$ have codimension 1 in $Z$.

**Proof.** Since the Picard group of $G$ is finite, $H = V(f)$ (set-theoretically) for some function $f$ on $G$. So $G - H = D(f)$ is affine, thus $(G - H) \cap Z$ is affine, and since $Z$ is smooth, $Z - ((G - H) \cap Z) = Z \cap H$ has pure codimension one in $Z$.

For a linear algebraic group $G$ and a parabolic subgroup $P \subset G$, we define the following integer:

$$(2.1) m_P := \text{rk}_Z(X^*(P)) - \text{rk}_Z(X^*(G)).$$

**Proposition 2.4.** Let $G$ be a linear algebraic group and let $P \subset G$ be a parabolic subgroup. Then $\text{Pic}(G/P)$ is a free group of rank $m_P$.

**Proof.** After quotienting out the unipotent radical, we may assume $G$ reductive. Let $\tilde{G}$ be the simply connected cover of the derived group of $G$. Let $\phi: \tilde{G} \to G$ be the canonical homomorphism and let $\tilde{P} := \phi^{-1}(P)$. Then $\tilde{P}$ is a parabolic subgroup of
\( \tilde{G} \) and \( \phi \) induces an isomorphism \( \tilde{G}/\tilde{P} \cong G/P \). As \( \text{Pic}(\tilde{G}) = 0 \) and \( X^*(\tilde{G}) = 0 \), Corollary 1.5 yields an isomorphism

\[
X^*(\tilde{P}) \xrightarrow{\sim} \text{Pic}(\tilde{G}/\tilde{P}) = \text{Pic}(G/P).
\]

so \( \text{Pic}(G/P) \) is free. Finally, we have an exact sequence:

\[
0 \to X^*(G) \to X^*(P) \to \text{Pic}(G/P) \to \text{Pic}(G),
\]

which shows that \( \text{Pic}(G/P) \) has rank \( m_P \).

Fix a Borel pair \( (B, T) \) of \( G \) such that \( T \subset B \subset P \). Let \( (X, \Phi, X^\vee, \Phi^\vee, \Delta) \) be the corresponding based root datum. Denote by \( W = W(G, T) := N_G(T)/T \) the Weyl group and by \( I \subseteq W \) the set of simple reflections defined by \( B \). For \( \alpha \in \Phi \) we denote by \( s_\alpha \in W \) the corresponding reflection. We obtain a bijection

\[
(2.2) \quad \Delta \xrightarrow{\sim} I, \quad \alpha \mapsto s_\alpha.
\]

There are natural bijections between the powerset of \( I \) and the set of parabolic subgroups of \( G \) containing \( B \) (these are called standard). When \( J \subseteq I \) is a subset, the corresponding standard parabolic will be denoted \( P_J \). Let \( M_J \) be the unique Levi subgroup of \( P_J \) containing \( T \). Then we get an inclusion \( W_J := W(M_J, T) \hookrightarrow W(G, T) \) such that \( (W_J, J) \) is a Coxeter system and

\[
J = W_J \cap I.
\]

Every parabolic subgroup \( P \) is conjugate to a unique standard parabolic subgroup \( P_J \) and \( J \subseteq I \) is called the type of \( P \).

**Proposition 2.5.** The integer \( m_P \) is equal to the cardinality of \( I \setminus J \), where \( J \) is the type of \( P \).

**Proof.** Again, we may assume that \( G \) is simply connected. The set of fundamental weights corresponding to \( I \setminus J \) is a basis of \( X^*(P)_Q \) and the result follows.

\( \blacksquare \)

### 2.2 The Picard group of the Bruhat stack

In this section, we fix two parabolic subgroups \( P \) and \( Q \) of a linear algebraic group \( G \). We consider the quotient stack \([P \setminus G/Q]\), which we call the Bruhat stack. It is the quotient stack associated to the action of \( P \times Q \) on \( G \) defined by

\[
(p, q) \cdot x = pxq^{-1}.
\]

**Lemma 2.6.** Let \( X \) be a \( G \times H \)-variety, where \( G, H \) are two linear algebraic groups. Assume that \( (\mathcal{O}(X)^\times)^G = (\mathcal{O}(X)^\times)^H = k^\times \). Then there is an exact sequence

\[
0 \to E(X) \to E(X) \oplus E(X) \to \text{Pic}^{G\times H}(X) \to \text{Pic}^G(X) \oplus \text{Pic}^H(X).
\]

If \( \text{Pic}(X) = 0 \), the last map is surjective. If \( \text{Pic}^G(X) \) and \( \text{Pic}^H(X) \) are free, then \( \text{Pic}^{G\times H}(X) \) is also free.
Proof. The first map is the diagonal map. The second one is the composition of the natural maps

\[ E(X) \oplus E(X) \to X^*(G) \oplus X^*(H) = X^*(G \times H) \to \text{Pic}^{G \times H}(X) \]

The third one is given by forgetting the action on one side. The result follows from diagram chasing in the following commutative diagram with exact lines:

\[
\begin{array}{ccc}
E(X) \oplus E(X) & \to & X^*(G \times H) \\
\downarrow & & \downarrow \cong \\
E(X) & \to & \text{Pic}^P \times Q(G) \\
\downarrow & & \downarrow \\
X^*(G) & \oplus & \text{Pic}^P \times Q(H) \\
\downarrow & & \downarrow \\
\text{Pic}^Q(G \times H) & \to & \text{Pic}^P \times Q(G) \\
\end{array}
\]

\[ \square \]

Remark 2.7. The image of \( \text{Pic}^{G \times H}(X) \to \text{Pic}^G(X) \oplus \text{Pic}^H(X) \) consists of elements of the form \((L,L)\), where \( L \) is a line bundle on \( X \) endowed with a \( G \)-action and an \( H \)-action that commute with each other.

Proposition 2.8. Let \( P, Q \subset G \) be parabolic subgroups. Then \( \text{Pic}^P \times Q(G) \) is free of rank

\[ \text{rk} \mathbb{Z}(X^*(P)) + \text{rk} \mathbb{Z}(X^*(Q)) - \text{rk} \mathbb{Z}(X^*(G)) \]

Proof. Theorem 1.4 gives an exact sequence

\[ 0 \to X^*(G)_Q \to X^*(P \times Q)_Q \to \text{Pic}^P \times Q(G)_Q \to 0 \]

and Lemma 2.6 shows that \( \text{Pic}^P \times Q(G) \) is free, so we are done. \[ \square \]

Example 2.9. Consider the example \( G = \text{GSp}(V, \langle \; , \; \rangle) \) for a symplectic space \((V, \langle \; , \; \rangle)\) of dimension \( 2g \), \( g \geq 1 \). Then \( G \) is of Dynkin type \( C_g \), \( G^{\text{der}} = \text{Sp}(V, \langle \; , \; \rangle) \) is simply connected and hence \( \text{Pic}(G) = 0 \) (Corollary 2.2). Moreover, \( X^*(G) \) is the free abelian group of rank 1 generated by the multiplier character.

Let \( Y, Z \subseteq V \) be Lagrangian subspaces (i.e., maximal totally isotropic subspaces) and let \( P \) and \( Q \) be the stabilizer of \( Y \) and \( Z \) in \( G \), respectively. The types of \( P \) and \( Q \) are equal and are given by the set of those simple reflections that correspond to the vertices \( \alpha_1, \ldots, \alpha_{g-1} \) in Bourbaki’s notation ([?] Chap. VI, Planche III). In particular \( m_P = m_Q = 1 \).

Let \( \mathcal{F} \) be the moduli space of Lagrangian subspaces of \((V, \langle \; , \; \rangle)\). Then

\[ X := G/Q \sim \mathcal{F}, \quad gQ \mapsto g(Z) \]

is an isomorphism of smooth proper varieties.

Let \( \mathcal{X} \) be the category fibered in groupoids over the category of \( k \)-schemes such that \( \mathcal{X}(S) \) is the category of triples \( \mathcal{M} \) consisting of a symplectic \( \mathcal{O}_S \)-module \( \mathcal{M} \) of rank \( 2g \) and Lagrangian submodules \( \mathcal{E} \) and \( \mathcal{F} \) of \( \mathcal{M} \). Morphisms are isomorphisms of such triples. This is a stack with respect to the fpqc-topology as fpqc-descent data are effective for finite locally free modules.

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Let $S$ be a $k$-scheme. Zariski locally on $S$, every triple in $\mathcal{X}(S)$ is isomorphic to $\mathcal{M}_q := ((V, (\ ,\ )) \otimes_k \mathcal{O}_S, Y \otimes_k \mathcal{O}_S, g(Z \otimes_k \mathcal{O}_S))$ for some $g \in G(S)$. Hence $g \mapsto \mathcal{M}_q$ yields a morphism $G \rightarrow \mathcal{X}$ and it is straightforward to check that this morphism induces an isomorphism 

$[P\!\backslash\!G/Q] \xrightarrow{\sim} \mathcal{X}$.

The canonical projection $\pi: \mathcal{F} \cong X \rightarrow [P\!\backslash\!G/Q] \cong \mathcal{X}$ is given by

$$\mathcal{F} \mapsto ((V, (\ ,\ )) \otimes_k \mathcal{O}_S, Y \otimes_k \mathcal{O}_S, \mathcal{F}).$$

Now let $(\mathcal{M}, \mathcal{E}, \mathcal{F})$ be the universal triple over $\mathcal{X}$ and let $\mathcal{L} := \text{det}(\mathcal{E})$. Then $\pi^* \mathcal{E} = Y \otimes_k \mathcal{O}_X$ is a trivial, hence the image of $\mathcal{L} \in \text{Pic}(\mathcal{X})$ in $\text{Pic}(X)$ is trivial. Hence there is a unique character $\lambda \in X^*(P)$ such that $\mathcal{L} \cong \mathcal{L}_\lambda$.

Let $\omega \in \Lambda$ be the unique minuscule fundamental weight. Then $X^*(P) = \mathbb{Z} \omega \oplus X^*(G)$. The image of $\lambda$ in $\mathbb{Z} \omega$ is $-\omega$ (CHECK! – might be $\omega$) and the image in $X^*(G)$ is the multiplier character (CHECK!).

### 2.3 The Picard group of the stack of a Frobenius zip datum

Let $G$ be a reductive group defined over a finite field $\mathbb{F}_q$ with $q = p^m$ elements contained in $k$. Denote by $\sigma$ the $q$-th power Frobenius of $k$. We fix a Borel pair $(B,T)$ of $G$ defined over $\mathbb{F}_q$. Let $P$ be a parabolic of $G$ containing $B^-$ and define $Q = \sigma(P^-)$. Let $\varphi: G \rightarrow G$ be the $q$-th power Frobenius isogeny. This defines a zip datum $(G,P,Q,\varphi)$.

For $x \in P$, we denote by $\pi$ the image of $x$ in $P/R_u(P)$, similarly for $y \in Q$. The associated zip group is defined by

$$E := \{ (x,y) \in P \times Q ; \ \varphi(\pi) = \pi \}$$

and $E$ acts on $G$ by restricting the action of $P \times Q$ to $E$. Note that $\dim(E) = \dim(G)$. The quotient stack $[E\!\backslash\!G]$ is called the stack of $G$-zips. Let $L \subset P$ and $M = \sigma(L) \subset Q$ be the Levi subgroups containing $T$. One has:

$$E = \{ (u\ell,v\varphi(\ell)) ; \ u \in R_u(P), v \in R_u(Q), \ell \in L \}.$$

The subgroup $\tilde{E} = \{ (x,y) \in L \times M ; \ \varphi(x) = y \}$ is a Levi subgroup of $E$, isomorphic to $L$.

**Lemma 2.10.** One has $X^*(G)^E = 0$.

**Proof.** Let $\chi$ be an $E$-invariant character of $G$. In particular, for all $x \in L$, one has $\chi(x) = \chi(\varphi(x))$, so $\chi = \chi \circ \varphi$. But $\chi$ is defined over some finite field, so for some $r \geq 1$, $\chi = \chi \circ \varphi^r = \varphi^r \circ \chi$ (where we also denote by $\varphi$ the Frobenius of $k = \overline{\mathbb{F}_q}$). We conclude that the image of $\chi$ is finite, so $\chi = 1$. \qed

It follows that we have an exact sequence of abelian groups:

$$0 \rightarrow X^*(G) \rightarrow X^*(E) \rightarrow \text{Pic}^E(G) \rightarrow \text{Pic}_E(G) \rightarrow 0.$$
Corollary 2.11. One has \( \dim_Q \langle \text{Pic}^E(G)_Q \rangle = m_P = \#(I \setminus J) \), where \( m_P \) was defined in (2.1) and where \( J \) is the type of \( P \).

Proof. As \( E/R_\alpha(E) \cong P/R_\alpha(P) \cong L \), we have \( X^*(E) \cong X^*(P) \).

Note that the inclusion \( E \subseteq P \times Q \) yields a morphism of quotient stacks \([E\setminus G] \to [P\setminus G/Q]\) and thus a homomorphism \( \beta : \text{Pic}^{P\times Q}(G) \to \text{Pic}^E(G) \). We have a morphism of exact sequences:

\[
\begin{array}{cccccc}
X^*(E) & \rightarrow & X^*(P) \oplus X^*(Q) & \rightarrow & \text{Pic}^{P\times Q}(G) & \rightarrow \text{Pic}^E(G) & 0 \\
\end{array}
\]

(2.3)

It can be checked immediately that this diagram induces isomorphisms \( \text{Ker}(\alpha) \cong \text{Ker}(\beta) \) and \( \text{Coker}(\beta) \cong \text{Coker}(\gamma) \). In particular, the cokernel of \( \beta \) is finite. When \( \text{Pic}(G) = 0 \), the map \( \beta \) is surjective. In this case, any line bundle on the stack of \( G \)-zips arises as the pull-back of a line bundle on the Bruhat stack.

By [PWZ1] Proposition 7.3, \( E \) acts with finitely many orbits on the variety \( G \). These orbits are parametrized as follows ([PWZ1] Theorem 7.5, see also [?] II.6.5). Let \((W,I)\) be the Weyl group of \((G,T)\) with its set of simple reflections given by \( B \). As \( B \) and \( T \) are defined over \( \mathbb{F}_q \), the relative Frobenius \( \varphi : G \to G^{(q)} = G \) induces an isomorphism \( \varphi \) of the Coxeter system \((W,I)\). Let \( J \subseteq I \) and \( K \subseteq I \) be the type of \( P \) and \( Q \), respectively. For every \( w \in W \) we choose a representative \( \hat{w} \in \text{Norm}_G(T) \) such that \( (w_1w_2)\varphi = \hat{w}_1\hat{w}_2 \) if \( \ell(w_1w_2) = \ell(w_1) + \ell(w_2) \). Let \( w_0,J \in W_J \) and \( w_0 \in W \) the longest elements. Then we obtain a bijection

\[
(2.4) \quad \mathcal{O}^w : \rightarrow \{ E\text{-orbits on } G \}, \quad w \mapsto \mathcal{O}^w := E \cdot (\hat{w}_{0,J}\hat{w}_{w_0})
\]

such that \( \dim \mathcal{O}^w = \ell(w) + \dim(P) \).

Let \( \eta = w_{0,J}w_0 \) be the element of maximal length in \( J^\omega \). Then \( \mathcal{O}^\eta = E \cdot 1 \) is the unique open orbit. The stabilizer of \( 1 \in G \) in \( E \) is a finite group scheme \( S \subseteq E \) and \( \mathcal{O}^\eta \cong E/S \) is affine (in fact, every \( E \)-orbit in \( G \) is affine by [WdY3] Theorem 2.2). Hence the complement \( G \setminus \mathcal{O}^\eta \) is a Cartier divisor whose irreducible components are fixed by the \( E \)-action because \( E \) is connected. Hence (2.4) induces bijections

\[
(2.5) \quad I \setminus J \leftrightarrow \{ w \in J^\omega : \ell(w) = \ell(\eta) - 1 \} \leftrightarrow \{ \text{irreducible components of } G \setminus \mathcal{O}^\eta \},
\]

where the first bijection is given by \( s \mapsto s\eta \) and where the second bijection is \( w \mapsto \mathcal{O}^w \).

2.4 Zip datum attached to a cocharacter

Let \( G \) be a reductive group defined over \( \mathbb{F}_q \) and let \( \chi : \mathbb{G}_m \to G \) be a cocharacter of \( G \) defined over \( \mathbb{F}_q^r, r \geq 1 \). Then \( \chi \) gives rise to an algebraic zip datum as follows. Let \( P_\pm = P_{\pm}(\chi) \) be the pair of opposite parabolic subgroups of \( G \) defined by \( \chi \). The Lie algebra of \( P_+ \) (resp. \( P_- \)) is the sum of the weight spaces of weights \( \leq 0 \) (resp. \( \geq 0 \))
in $\text{Lie}(G)$ under the action of $\chi$. The parabolics $P_{\pm}$ have a common Levi subgroup $L$ which is the stabilizer of $\mu$.

We obtain an algebraic zip datum $Z_{G,\chi} := (G, P_+, \sigma(P_-), \varphi)$, where $\sigma$ denotes the $q$-th power absolute Frobenius and where $\varphi : L \to \sigma(L)$ is the relative Frobenius. In particular we obtain an attached zip group $E := E_{G,\chi}$. This setting is particularly convenient with respect to functoriality. Indeed, if $\alpha : G \to G'$ is a morphism of reductive groups, then a cocharacter $\mu : G_m \to G$ induces a cocharacter $\alpha \circ \mu$. Denote by $E$ and $E'$ the associated zip groups. Then we have a natural map of stacks

$$[E \backslash G] \to [E' \backslash G']$$

Zip data arising from small an minuscule cocharacters will be of great importance to us. Let us recall briefly the definitions. Let $G$ be a reductive group with a Borel pair $(T, B)$ and let $(X^*, \Phi, X_+, \Phi^\vee, \Delta)$ be the corresponding based root datum. Let $G^\text{ad}$ be the adjoint group and let $G^\text{ad} = \prod_{i=1}^r G^{(i)}$ be its decomposition into simple adjoint groups. The image of $(T, B)$ in $G^{(i)}$ is a Borel pair $(T^{(i)}, B^{(i)})$. We obtain a corresponding decomposition of the set of roots $\Phi = \bigsqcup_{i=1}^r \Phi^{(i)}$, similarly for the set of coroots and the root basis $\Delta$. Let us call a simple root $\alpha \in \Delta^{(i)}$ special if it lies in the orbit of the affine root under the action of the automorphism group of the extended Dynkin diagram of $(G^{(i)}, B^{(i)}, T^{(i)})$.

**Definition 2.12.** A cocharacter $\chi : G_m \to G$ is called minuscule if it satisfies the following equivalent properties ([?]) 1.2.

(i) The representation $\text{ad} \circ \chi$ of $G_m$ on $\text{Lie}(G)$ has only weights $-1$, 0, and 1.

(ii) It is conjugate to a cocharacter $\chi'$ of $T$ such that $\langle \chi', \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$.

(iii) Let $\chi^{(i)}$ be the unique $B^{(i)}$-dominant cocharacter of $T^{(i)}$ which is conjugate to the image of $\chi$ in $G^{(i)}$. Then there exists at most one simple root $\alpha \in \Delta^{(i)}$ such that $\langle \chi^{(i)}, \alpha \rangle = 1$ and in this case $\alpha$ is special.

Let $\chi \in X_+$ be a dominant minuscule cocharacter and let $P = P_+ (\chi)$ be the associated parabolic subgroup. Then the image $P^{(i)}$ of $P$ in $G^{(i)}$ is either a maximal parabolic subgroup (if there exists $\alpha$ as in (iii)) or equal to $G^{(i)}$ (if $\chi^{(i)}$ is central in $G^{(i)}$). Let $A \subseteq \Delta$ be the set of simple roots $\alpha$ occurring in (iii) and let $A \mapsto \{1, \ldots, r\}$, $\alpha \mapsto i(\alpha)$ be the injective map such that $\alpha \in \Delta_{i(\alpha)}$.

**Definition 2.13.** A cocharacter $\chi : G_m \to G$ is called small if it satisfies Condition (iii) of Definition $[2,12]$ without the assumption that $\alpha$ is special.

### 2.5 Stabilizer of $1 \in G$

First, we need a basic result on the intersection of two parabolic subgroups:

**Proposition 2.14.** Let $P$ and $Q$ be two parabolics of a linear algebraic group $G$ containing a maximal torus $T$, with unipotent radicals $U$ and $V$, respectively. Let $L \subseteq P$ and $M \subseteq Q$ be Levi subgroups with respect to $T$. Then:

(i) The subgroups $P \cap Q$, $L \cap M$, $L \cap V$, $M \cap U$, $U \cap V$ are smooth and connected.
Corollary 2.16. We keep the same notations as in Proposition 2.14. Then:

satisfying $x = abcd$, with $a \in L \cap M$, $b \in L \cap V$, $c \in M \cap U$, $d \in U \cap V$.

Proof. The smoothness of $P \cap Q$ follows from [SGA3] Lemma 4.1.1. This implies the smoothness of the other subgroups. For the rest, see [DM] Proposition 2.1.

The last statement means that $P \cap Q$ is the product of the varieties $L \cap M$, $L \cap V$, $M \cap U$, $U \cap V$. In particular, we have:

Corollary 2.15. We keep the same notations as in Proposition 2.14. If $U \cap V = \{1\}$ and $M \cap U = \{1\}$, then $P \cap Q \subset L$.

If $x \in P$, define $\theta_L^P(x) \in L$ to be the unique element of $L$ such that there exists $u \in U$ satisfying $x = \theta_L^P(x)u$. We will use repeatedly the following immediate corollary:

Corollary 2.16. We keep the same notations as in Proposition 2.14. Then:

(i) For all $x \in P \cap Q$, one has $\theta_L^P(x) \in P \cap Q$.

(ii) For all $x \in P \cap Q$, one has $\theta_L^P(\theta_M^Q(x)) = \theta_M^Q(\theta_L^P(x))$ (note that this makes because of the first assertion).

(iii) Assume $G$ is defined over $\mathbb{F}_q$ and let $\varphi : G \to G$ the $q$-th power Frobenius. Then $\varphi(\theta_L^P(x)) = \theta_{\sigma L}^P(\varphi(x))$.

(iv) Assume $T \subset B \subset P \cap Q$ for some Borel $B$. Then $P \cap Q$ is a parabolic with Levi $L \cap M$ and for all $x \in P \cap Q$, one has $\theta_L^P(\theta_M^Q(x)) = \theta_M^Q(\theta_L^P(x)) = \theta_L^{\sigma Q}(x)$.

Proof. The first assertion follows readily from Proposition 2.14 (iii). To prove the second one, write $\theta_M^Q(x) = \theta_L^P(\theta_{\sigma M}^Q(x))u$ with some $u \in R_u(P)$. Now apply $\theta_L^P$ to this equality. The third assertion is obvious. The first part of the last assertion is Proposition 2.14 (ii). Finally, write $x = \theta_L^P(x)u$ with $u \in R_u(P) \subset R_u(P \cap Q)$. Now $\theta_L^P(x) = \theta_M^Q(\theta_L^P(x)v$ with $v \in R_u(Q) \subset R_u(P \cap Q)$. The result follows.

Let $G$ be a reductive group over $\mathbb{F}_q$ with a Borel pair $(B, T)$ defined over $\mathbb{F}_q$. Let $P$ be a parabolic subgroup containing $B^-$ and $Q := \sigma(P^-)$. So $Q$ is a standard parabolic. Let $L \subset P$ and $M = \sigma(L) \subset Q$ denote the Levi subgroups with respect to the maximal torus $T$. Let $E \subset P \times Q$ denote the zip group. Let $S \subset E$ be the scheme-theoretic stabilizer of $1 \in G$, and let $S_0 := S_{red}$. So $S_0$ is the smooth finite group scheme over $k$ such that $S_0(k) = S(k) \subset E(k)$ is the set-theoretic stabilizer of $1 \in G(k)$. We will identify $S_0$ and $S_0(k)$. We view $S$ as a subgroup of $P \cap Q$. Note that $S$ is usually not smooth, and this is why we need the next few lemmas to circumvent this problem.

We define:

$$P_0 := \bigcap_{i \in \mathbb{Z}} \sigma^i(P) \quad Q_0 := \bigcap_{i \in \mathbb{Z}} \sigma^i(Q) \quad L_0 := \bigcap_{i \in \mathbb{Z}} \sigma^i(L)$$

It is clear that $P_0$ and $Q_0$ are opposite parabolic subgroups of $G$, defined over $\mathbb{F}_q$, with common Levi subgroup $L_0$. We have the following lemma:
Lemma 2.17. One has $Q_0 \cap P \subset L$

Proof. This follows from Corollary 2.16 because $L_0 \cap R_u(P) = \{1\}$ and $R_u(Q_0) \cap R_u(P) = \{1\}$ (since $R_u(Q_0) \subset R_u(B)$ and $R_u(P) \subset R_u(B^{'})$).

Lemma 2.18. The group $S$ is contained in $Q_0 \cap L$. The group $S_0$ is contained in $L_0$. Further, one has $S_0 = L_0(F_q)$ and $S \cap L_0 = S_0$.

Proof. Let $x \in S$. By definition, one has $\varphi(\theta^P_L(x)) = \theta^Q_M(x)$. Since $x \in Q$, one has $\theta^P_L(x) \in Q$ by Corollary 2.16. It follows that $\theta^Q_M(x) \in \sigma(Q)$ and we deduce $x \in \sigma(Q)$ because $R_u(Q) \subset R_u(B) \subset \sigma(Q)$. Now we can apply the same argument to $\sigma(Q)$ to show $x \in \sigma^2(Q)$. Continuing this process, we get $x \in Q_0$. Since $S \subset P$, we also have $S \subset L$ by the previous lemma.

Now, to prove $S_0 \subset L_0$, we need only show that $S_0(k) \subset L_0(k)$, because $S_0$ is smooth. Let $x \in S_0(k)$. By definition, one has $\varphi(\theta^P_L(x)) = \theta^Q_M(x)$. Since $x \in P$, we have $\theta^Q_M(x) \in P$ by Corollary 2.16. We deduce that $\theta^P_L(x) \in \sigma^{-1}(P)$ and then $x \in \sigma^{-1}(P)$. Repeating the argument yields $x \in P_0$. We have showed that $S_0 \subset L_0$. Now for $x \in L_0$, $x$ lies in $S_0$ if and only if $\varphi(x) = x$, so $S_0 = L_0(F_q)$.

Since the Lang-Steinberg map $L_0 \rightarrow L_0$, $x \mapsto x^{-1} \varphi(x)$ is étale, it follows that the algebraic group $S \cap L_0 = \{x \in L_0, \varphi(x) = x\}$ is smooth, equal to the constant group $L_0(F_q)$. So we deduce $S \cap L_0 = S_0$.

In the next lemma, we consider the map $\theta^Q_{L_0} : Q_0 \rightarrow L_0$.

Lemma 2.19. If $x \in S$, then $\theta^Q_{L_0}(x) \in S_0$.

Proof. Let $x \in S$, so by definition one has $\varphi(\theta^P_L(x)) = \theta^Q_M(x)$. Now $x \in L \cap Q_0$ by Lemma 2.18 so we have $\theta^P_L(x) = x$ and $\varphi(x) = \theta^Q_M(x)$. We deduce from this

$$\varphi(\theta^Q_{L_0}(x)) = \theta^Q_{L_0}(\varphi(x)) = \theta^Q_{L_0}(\theta^Q_M(x)) = \theta^Q_M(\theta^Q_{L_0}(x)) = \theta^Q_{L_0}(x)$$

since $L_0 \subset M$. We deduce $\theta^Q_{L_0}(x) \in L_0(F_q) = S_0$.

Proposition 2.20. We have $X^*(E)^S = X^*(E)^{S_0}$ and $X^*(S) = X^*(S_0)$.

Proof. The first assertion follows from the previous lemma, because for all $x \in S$ and $\chi \in X^*(E)$, one has $\chi(x) = \chi(\theta^Q_{L_0}(x))$ so $\chi$ vanishes on $S$ if and only if it does on $S_0$.

For the second assertion, note that we already have proved $X^*(S) \subset X^*(S_0)$. We have an exact sequence

$$1 \rightarrow S^0 \rightarrow S \rightarrow \pi_0(S) \rightarrow 1$$

And since we are working over a perfect field, this exact sequence is split by the group $S_{red} = S_0$. So $S_0 \simeq \pi_0(S)$ and finally $X^*(S) = X^*(S_0)$.

Now, we can rewrite diagram (1.4) taking into account this new information:
Corollary 2.21. The group $\text{Pic}^E(G)$ has no $p$–torsion.

Proof. This follows immediately from the previous diagram, knowing that $X^*(S_0)$ has no $p$–torsion.

Let $\zeta : X^*(L_0) \to X^*(L_0)$ be the map of $\mathbb{Z}$–modules defined by $\zeta(\chi) = \chi - \chi \circ \varphi$. This map is injective, because if $\chi = \chi \circ \varphi$, then there is $d \geq 1$ such that $\chi = \chi \circ \varphi^d = \varphi^d \circ \chi$, and this implies $\chi = 1$.

Corollary 2.22. Assume that $\text{Pic}(L_0) = 0$ (this is the case if $\text{Pic}(G) = 0$). Then we have an exact sequence

$$0 \to X^*(L_0) \xrightarrow{\zeta} X^*(L_0) \to X^*(S_0) \to 0$$

In particular, the order of the group $X^*(S_0)$ is equal to the absolute value of the determinant of $\zeta$.

Proof. This follows from Proposition 1.11 knowing that $S_0 = L_0(F_q)$.

Corollary 2.23. Assume $G$ is split over $F_q$ and $\text{Pic}(L_0) = 0$. Then

$$X^*(S_0) \simeq \left( \frac{\mathbb{Z}}{(q-1)\mathbb{Z}} \right)^d$$

with $d := \text{rk}(X^*(L_0))$.

Example 2.24. Let $G = GL_n, F_q$, with the standard (upper) Borel pair $(B,T)$. Let $r \geq s \geq 0$ be two integers such that $r + s = n$. Let $P$ be the parabolic of matrices of the form

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

with $A \in GL_r(k), B \in GL_s(k), C \in M_{s,r}(k)$.
and define \( Q := P^- \). Then \( L_0 = L \simeq GL_r \times GL_s \) and \( S_0 \simeq GL_r(\mathbb{F}_q) \times GL_s(\mathbb{F}_q) \). Since \( \text{Pic}(GL_n) = 0 \), we deduce that

\[
X^*(S_0) = \left( \frac{\mathbb{Z}}{(q-1)\mathbb{Z}} \right)^2
\]

In particular, \( X^*(S_0) \) is killed by \( q - 1 \).

**Example 2.25.** Let \( G = U(n) \) be the unitary group associated to the \( n \times n \)-matrix

\[
J = \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}
\]

In other words, for every \( \mathbb{F}_q \)-algebra, one has :

\[
G(R) = \{ A \in GL_n(\mathbb{F}_q^2 \otimes_{\mathbb{F}_q} R), \ i\sigma(A)JA = J \}
\]

We make the identification \( G_k = GL_{n,k} \). The Galois action is then given by \( \sigma \cdot A = J^t\sigma(A)^{-1}J \) for all \( A \in GL_n(k) \). Let \( (B,T) \) the standard (upper) Borel pair of \( GL_{n,k} \). One sees immediately that it is defined over \( \mathbb{F}_q \). Let \( r \geq s \geq 0 \) be two integers such that \( r + s = n \) and let \( P \) and \( Q \) be the same parabolics as in the previous example. For an \( \mathbb{F}_q \)-algebra \( R \), the group \( L_0(R) \) is easily seen to be the set of elements in \( G(R) \) of the form

\[
\begin{pmatrix}
A & B \\
C &
\end{pmatrix}
\]

with \( A \in GL_s(\mathbb{F}_q^2 \otimes_{\mathbb{F}_q} R), B \in GL_{r-s}(\mathbb{F}_q^2 \otimes_{\mathbb{F}_q} R), C \in GL_s(\mathbb{F}_q^2 \otimes_{\mathbb{F}_q} R) \)

with the conditions \( \sigma^2(A) = A, C = J^t\sigma(A)^{-1}, \) and \( B \in U(r-s) \). In other words, one has

\[
L_0 \simeq U(r-s) \times \text{Res}_{\mathbb{F}_q^2/\mathbb{F}_q}(GL_s)
\]

And it follows that

\[
S_0 = L_0(\mathbb{F}_q) = U(r-s)(\mathbb{F}_q) \times GL_s(\mathbb{F}_q^2)
\]

For example if \( n = 3 \), it can be shown that one has \( S \simeq S_0 \times \alpha_p \), so \( S \) is in general not reduced. Since \( \text{Pic}(GL_n) = 0 \), we can apply Corollary 2.22 so any group homomorphism \( S_0 \to k^* \) comes from a character of \( L_0 \). We deduce that

\[
X^*(S_0) = \frac{\mathbb{Z}}{(q+1)\mathbb{Z}} \times \frac{\mathbb{Z}}{(q^2-1)\mathbb{Z}}
\]

In particular, \( X^*(S_0) \) is killed by \( q^2 - 1 \).
3 Positivity and ampleness of characters

3.1 Statement of the positivity conjecture

Definition 3.1. Let $X$ be a smooth variety and $U \subset X$ an open subset such that $X - U$ is purely of codimension 1. A function $f \in E(U)$ will be called $U$–ample if it has an effective divisor, and if the support of the divisor of $f$ is exactly the complement of $U$ in $G$. If $f \in E(U)_\mathbb{Q}$, we say that $f$ is $U$–ample if some positive multiple of $f$ in $E(U)$ is $U$–ample.

When no confusion can occur about the open subset $U$, we will call a $U$–ample function simply an ample function.

Definition 3.2. Let $G$ be a group and $H \subset G$ a closed subgroup. Let $\chi \in X^*(H)$ be a character. We say that $\chi$ is ample if the line bundle on $G/H$ defined by $\chi$ is ample. Similarly, we define ampleness for rational characters in $X^*(H)_\mathbb{Q}$.

We define also an antiample character (resp. function) as the inverse of an ample character (resp. function).

Proposition 3.3. Let $G$ be a reductive group over $k$. Let $(B,T)$ be a Borel pair with simple positive roots $I$. Let $P$ be a parabolic subgroup containing $B$, of type $J \subset I$. Let $\chi$ be a character of $P$. The following are equivalent:

1. The character $\chi$ is ample.
2. One has $(\chi, \beta^\vee) > 0$ for all $\beta \in I \setminus J$.
3. The character $\chi$ is a linear combination with $> 0$ coefficients of the fundamental weights of the maximal parabolics containing $P$.

Proof. [BrKu] Exercise 3.1.E (1).

Remark 3.4. Assume $G$ and $P$ decompose as products $G = G_1 \times G_2$ and $P = P_1 \times P_2$. Let $\chi \in X^*(P)$ be a character and let $\chi = (\chi_1, \chi_2)$ be its decomposition. Then $\chi$ is ample if and only if $\chi_1$ and $\chi_2$ are ample.

Remark 3.5. Assume $G \subset G_1$ and $P = P_1 \cap G$ for some parabolic $P_1$ in $G_1$. Then we get a closed immersion $G/P \to G_1/P_1$. It follows that an ample character $\chi \in X^*(P_1)$ restricts to an ample character of $P$.

To formulate the next conjecture, recall that if $U \subset G$ denotes the open $E$–orbit of $G$ in the context of section 2.3, the injection $\iota : E(U) \to X^*(E)$ has finite cokernel, so it induces an isomorphism $\iota : E(U)_\mathbb{Q} \cong X^*(E)_\mathbb{Q}$. Furthermore, the first projection $p_1 : E \to P$ induces an isomorphism $p_1^* : X^*(E) \cong X^*(P)$.

Conjecture 3.6. Let $G$ be a reductive group defined over $\mathbb{F}_q$ which has a Borel pair $(B,T)$ defined over $\mathbb{F}_q$. Let $P$ be a parabolic containing $B^{-}$ and $Q := \sigma(P^{-})$. Through the isomorphism

$$X^*(P)_\mathbb{Q} \xrightarrow{p_1^*} X^*(E)_\mathbb{Q} \xrightarrow{\iota^{-1}} E(U)_\mathbb{Q}$$

an ample character $\alpha \in X^*(P)_\mathbb{Q}$ is mapped to an antiample function in $E(U)_\mathbb{Q}$.
Conjecture 3.6 holds.

It is possible to reduce the proof of Theorem 3.8 to the case when the group $G$ is semisimple adjoint. More precisely:

**Proposition 3.9.** Keep the same notations as in Conjecture 3.6. Let $G' = G^{ad}$, and define $T', B', P', Q'$ as the images of $T, B, P, Q$ by the natural map $G \to G'$. Then $(G', P', Q', \varphi)$ is a zip datum. Further, if Conjecture 3.6 holds for $G'$, then it does for $G$ as well.

**Proof.** The first assertion is obvious. Denote by $E'$ the zip group associated to $(G', P', Q', \varphi)$. We get a map of stacks $[E \backslash G] \to [E' \backslash G']$. The $E$–orbits in $G$ are in one-to-one correspondence with the $E'$–orbits in $G'$, so in particular the natural map $Z^1_{Q}(G')^{E'} \to Z^1_{Q}(G)^{E}$ is an isomorphism, and maps the canonical basis of $Z^1_{Q}(G')^{E'}$ to the canonical basis of $Z^1_{Q}(G)^{E}$. We have a commutative diagram:

\[
\begin{array}{cccc}
Z^1_{Q}(G')^{E'} & \longrightarrow & \Pic_{Q}^{E'}(G') & \longrightarrow & X^*(E')_{Q} & \longrightarrow & X^*(P')_{Q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z^1_{Q}(G)^{E} & \longrightarrow & \Pic_{Q}^{E}(G) & \longrightarrow & X^*(E)_{Q} & \longrightarrow & X^*(P)_{Q}
\end{array}
\]

where all maps are isomorphisms. An ample character of $P$ is the image of an ample character of $P'$ (note that $G/P \simeq G'/P'$), and the result follows.

Now, assume that we have a direct product of two zip data. More precisely, this means that $G = G_1 \times G_2$ where $G_1, G_2$ are reductive groups over $\mathbb{F}_q$, and the groups $T, B, P, Q$ decompose as $T = T_1 \times T_2, B = B_1 \times B_2, P = P_1 \times P_2, Q = Q_1 \times Q_2$, such that $(G_1, P_1, Q_1, \varphi)$ and $(G_2, P_2, Q_2, \varphi)$ are zip data.

**Proposition 3.10.** Assume that Conjecture 3.6 holds for $G_1$ and $G_2$. Then it holds for $G = G_1 \times G_2$.

**Proof.** This follows immediately from Remark 3.4.

As every adjoint group over $\mathbb{F}_q$ is isomorphic to a product of algebraic groups of the form $\text{Res}_{E'/\mathbb{F}_q}(G_0)$, where $G_0$ is a simple adjoint group over some finite extension $\mathbb{F}_q'$ of $\mathbb{F}_q$ ([SGA3] Exp. XXIV, Prop. 5.5 and Prop. 5.9), this reduces the proof of Theorem 3.8 to the case that $G$ is of this form.
3.3 Perfect embeddings of zip data

Let $G \subset G_1$ be two reductive groups defined over $\mathbb{F}_q$. Let $(B,T)$ (resp. $(B_1,T_1)$) be a Borel pair defined over $\mathbb{F}_q$ in $G$ (resp. $G_1$). Let $P$ (resp. $P_1$) be a parabolic in $G$ (resp. $G_1$) containing $B^-$ (resp. $B_1^-$). Define $Q := \sigma(P^-)$ and $Q_1 := \sigma(P_1^-)$. Let $L,M,L_1,M_1$ the associated Levi subgroups of $P,Q,P_1,Q_1$ respectively. Finally, let $U$ (resp. $U_1$) denote the open $E-$orbit (resp. $E_1-$orbit) in $G$ (resp. $G_1$). We make the following assumptions:

1. $B = B_1 \cap G$, $T = T_1 \cap T$, $P = P_1 \cap G$, $Q = Q_1 \cap G$, $L = L_1 \cap G$, $M = M_1 \cap G$, $R_u(P) = R_u(P_1) \cap G$, $R_u(Q) = R_u(Q_1) \cap G$.

2. $U_1 \cap G = U$.

We call this situation a perfect embedding of zip data. In this case, the inclusion $P \times Q \to P_1 \times Q_1$ induces an inclusion $E \to E_1$, where $E$ and $E_1$ are the corresponding zip groups. Furthermore, the inclusion $G \to G_1$ and the group actions of $E$ and $E_1$ are compatible through this map. We get a map of quotient stacks $[E\backslash G] \to [E_1\backslash G_1]$.

**Proposition 3.11.** Assume that Conjecture 3.6 holds for $G_1$. Assume further that the map $X^*(P_1) \to X^*(P)$ defines a surjection on the ample characters of $P$ and $P_1$. Then Conjecture 3.6 holds for $G$.

**Proof.** We have a commutative diagram:

$$
\begin{array}{ccc}
X^*(P_1)Q & \xrightarrow{\sim} & X^*(E_1)Q \\
\downarrow & & \downarrow \\
X^*(P)Q & \xrightarrow{\sim} & X^*(E)Q \\
\end{array}
\xrightarrow{\sim} E(U_1)Q
$$

It suffices to prove that an ample function $f_1 \in E(U_1)$ is mapped to an ample function $f \in E(U)$. It can be seen readily that $f$ is the restriction of $f_1$ to $U$. Since $f_1$ has an effective divisor, it extends to a regular function on $G_1$, so $f$ extends to $G$. Thus $\text{div}(f)$ is effective. Now, $f$ vanishes exactly on $G - (U_1 \cap G) = G - U$, so $f$ is ample. \qed

3.4 The case where $P$ and $Q$ are defined over $\mathbb{F}_q$

In this section we consider the easier case when the parabolics $P$ and $Q$ are defined over $\mathbb{F}_q$. We denote again by $\varphi$ the $q-$th power Frobenius. More precisely, let $G$ be a reductive group over $\mathbb{F}_q$, $(B,T)$ a Borel pair defined over $\mathbb{F}_q$ and $P$ a parabolic containing $B^-$, and which is defined over $\mathbb{F}_q$. This is always the case when $G$ is split. We define $Q := P^-$. Let $L$ be the common Levi subgroup of $P$ and $Q$. We then have the following fact ([Wd2] Proposition 3.1):

**Proposition 3.12.** The open $P \times Q-$orbit in $G$ coincides with the open $E-$orbit in $G$.

In particular, one has $Z^1(G)^{P \times Q} = Z^1(G)^E$. Denoting by $U$ the open $E-$orbit, one has a commutative diagram

\[24\]
which induces a natural section \( j = j_1 \circ j_2^{-1} : X^*(E)_Q \to X^*(P \times Q)_Q \) of the restriction map. To describe it concretely, consider the map:

\[
\zeta : X^*(L) \to X^*(L), \quad \chi \mapsto \chi - \chi \circ \varphi
\]

It induces an automorphism of \( X^*(L)_Q \). We will always make the following identifications: \( X^*(L)_Q = X^*(P)_Q = X^*(Q)_Q \) via the inclusions \( L \subset P \) and \( L \subset Q \), and \( X^*(E)_Q = X^*(P)_Q = X^*(L)_Q \) via the first projection \( E \to P \).

**Lemma 3.13.** There is a commutative diagram:

\[
\begin{array}{ccc}
X^*(E)_Q & \xrightarrow{j} & X^*(P \times Q)_Q \\
\downarrow \cong & & \downarrow \cong \\
X^*(L)_Q \times X^*(L)_Q & \xrightarrow{\cong} & X^*(L)_Q \\
\end{array}
\]

where the vertical maps are the identifications mentioned above, and the lower horizontal map is given by \( \chi \mapsto (\zeta^{-1}(\chi), -\zeta^{-1}(\chi)) \).

**Proof.** Let \( \chi \in X^*(L)_Q \). The corresponding function \( f \in E(U)_Q \) satisfies \( f(a^{-1} \varphi(a)) = \chi(a) f(1) \) for all \( a \in L \). We want to determine the character \( \chi' \in X^*(L \times L)_Q \) such that \( f(a^{-1} b) = \chi'(a, b) f(1) \) for all \( (a, b) \in L \times L \). Writing \( \chi' = (\chi'_1, \chi'_2) \in X^*(L)_Q \times X^*(L)_Q \), we get \( f(a^{-1} b) = \chi'_1(a) \chi'_2(b) f(1) \). We see immediately that \( \chi'_2(a) = \chi'_1(a)^{-1} \) and \( \chi(a) = \chi'_1(a) \chi'_2(\varphi(a)) = \chi'_1(a) \chi'_2(\varphi(a))^{-1} = \zeta(\chi'_1)(a) \). In other words, \( (\chi'_1, \chi'_2) = (\zeta^{-1}(\chi), -\zeta^{-1}(\chi)) \).

**Remark 3.14.** It follows that the image of the map \( E(U)_Q \to X^*(P \times Q)_Q = X^*(L)_Q \times X^*(L)_Q \) consists of elements of the form \( (\alpha, -\alpha), \alpha \in X^*(L)_Q \).

**Remark 3.15.** When \( G \) is split, every character of \( P \) is defined over \( \mathbb{F}_q \), so we deduce that \( \zeta(\chi) = -(q-1)\chi \).

**Lemma 3.16.** Let \( \chi \in X^*(P)_Q \) be an ample character. Then \( \chi \circ \varphi \) is ample.

**Proof.** \( \chi \) is a linear combination with \( > 0 \) coefficients of the fundamental weights of the maximal parabolics containing \( P \). These are permuted by the Galois action, so the result follows.

**Lemma 3.17.** Let \( \chi \in X^*(P)_Q \) be an ample character. Then \( \zeta^{-1}(\chi) \) is antiample.

**Proof.** Write \( \alpha = \zeta^{-1}(\chi) \), so \( \chi = \alpha - \alpha \circ \varphi \). Choose \( d \geq 1 \) such that \( \alpha \) is defined over \( \mathbb{F}_{q^d} \). We deduce that \( \chi + \chi \circ \varphi + \ldots + \chi \circ \varphi^d = \alpha - \alpha \circ \varphi^d = -(q^d - 1)\alpha \) is ample, so \( \alpha = \zeta^{-1}(\chi) \) is antiample.
Remark 3.18. However, if $\chi$ is an ample character, then it is not in general true that $\zeta(\chi)$ is antample. This is the main reason why in Conjecture 3.6, we cannot expect to have an equivalence, but only one implication.

Denote by $p_2 : X^*(P \times Q) \to X^*(Q)$ the projection onto the second factor. We have the following commutative diagram:

\[
\begin{array}{ccc}
X^*(P)_Q & \xrightarrow{\cong} & E(U)_Q \\
\downarrow j_2 & & \downarrow j \\
X^*(E)_Q & \xrightarrow{j_1} & X^*(P \times Q)_Q \\
\downarrow -\text{div} & & \downarrow p_2 \\
Z^1_Q(G)^{P \times Q} & \xrightarrow{\delta} & \text{Pic}^{P \times Q}(G)_Q \\
\end{array}
\]

In this diagram, the map $\text{Pic}^{P \times Q}(G)_Q \to \text{Pic}(G/Q)_Q$ is simply defined by identifying $\text{Pic}(G/Q)$ with $\text{Pic}^Q(G)$, and forgetting the $P$–action. This defines a map $\delta : Z^1_Q(G)^{P \times Q} \to \text{Pic}(G/Q)_Q$.

Lemma 3.19. The map $\delta : Z^1_Q(G)^{P \times Q} \to \text{Pic}(G/Q)_Q$ is an isomorphism. Further, if $C \in Z^1(G)^{P \times Q}$, then $\delta(C)$ is the Weil divisor $C/Q \subset G/Q$.

Proof. These are $\mathbb{Q}$–vector spaces of the same dimension (equal to $\text{rk}(X^*(Q)) - \text{rk}(X^*(G))$), so we need only show that the map is injective. Let $C \in Z^1_Q(G)^{P \times Q}$ mapped to zero in $\text{Pic}(G/Q)_Q$. The image of $C$ in $\text{Pic}^{P \times Q}(G)_Q$ has a preimage in $X^*(P \times Q)_Q = X^*(L)_Q \times X^*(L)_Q$ of the form $(\alpha, 0)$. But $C$ comes from a function $f \in E(U)_Q$, whose character in $X^*(P \times Q)_Q = X^*(L)_Q \times X^*(L)_Q$ has the form $(\beta, -\beta)$, by Remark 3.14. So $\alpha = \beta = 0$, and $C = 0$.

Now consider the following commutative diagram:

\[
\begin{array}{ccc}
Z^1(G)^{P \times Q} & \xrightarrow{\text{Pic}^{P \times Q}(G)} & \text{Pic}^{P \times Q}(G) \\
\downarrow & & \downarrow \\
Z^1(G)^Q & \xrightarrow{\cong} & \text{Pic}^Q(G) \\
\end{array}
\]

The map $Z^1(G)^{P \times Q} \to Z^1(G)^Q$ is simply the inclusion (notice that $Z^1(G)^Q$ has infinite rank). Now the map $Z^1(G)^Q \to \text{Pic}(G/Q)$ is given by $C \mapsto C/Q$, so we are done. 

\[
\[
\]
The Borel case Assume for a moment that $P$ and $Q$ are Borel subgroups, so $P = B^-$ and $Q = B$. For $\alpha$ a simple root, denote by $Z_\alpha$ the closure of the $B^- \times B$–orbit of $\dot{\alpha}$. The $Z_\alpha$ form a basis of $Z^1(G)B^- \times B^-$. The right–$B$–invariant subvariety $Z_\alpha \subset G$ defines the Weil divisor $\delta(Z_\alpha) = \overline{Z_\alpha} = Z_\alpha/B$. Then one has the following ([?] page 99):

**Proposition 3.20.** The natural map $X^*(B) \rightarrow \text{Pic}(G/B)$ is given by

$$\chi \mapsto \sum \langle \chi, \alpha^\vee \rangle \overline{Z_\alpha}.$$ 

We claim that this proves Conjecture $3.6$ in the Borel case. Indeed, take an ample character $\chi \in X^*(B^-)$. By Lemmas $3.13$ and $3.17$ we deduce that in diagram $3.2$ $\chi$ is mapped to the ample line bundle $L = L(\gamma) \in \text{Pic}(G/B)_Q$ with $\gamma = -\zeta^{-1}(\chi)$. But then $\delta^{-1}(L) = \sum \langle \gamma, \alpha^\vee \rangle Z_\alpha$ and since $\gamma$ is ample, all the coefficients are $> 0$.

Let us now return to the general case. Again, we simply need to show that $\delta^{-1}$ maps an ample line bundle in $\text{Pic}(G/Q)$ to a linear combination with $> 0$ coefficients in $Z^1_Q(G)^P \times Q$. We will deduce the result from the Borel case, using an elementary argument. Consider the following commutative diagram :

$$\begin{array}{ccc}
Z^1(G)^P \times Q & \xrightarrow{\sim} & \text{Pic}(G/Q) \\
\downarrow & & \downarrow \\
Z^1(G)B^- \times B & \xrightarrow{\sim} & \text{Pic}(G/B) \leftarrow \quad X^*(B)_Q
\end{array}$$

All the vertical maps are injective. The leftmost one is defined in the following way : If $C \subset G$ is a $P \times Q$–orbit of codimension one, it is mapped to the only $B^- \times B$–orbit of codimension one contained in it. Now, let $\chi \in X^*(Q)$ be an ample character. The restriction to $B$ of $\chi$ is definitely not ample (unless $Q = B$), but it is a linear combination with $> 0$ coefficients of the fundamental weights of the maximal parabolics containing $B$. Using the explicit formula available in the Borel case, we deduce that $\chi$ decomposes with $\geq 0$ coefficients in $Z^1(G)^B \times B$ and thus also in $Z^1(G)^P \times Q$. Now, counting the number of simple roots $\beta$ such that $\langle \chi, \beta^\vee \rangle > 0$ proves that all the $P \times Q$–orbits of codimension one must appear with $> 0$ coefficients. We have just proved :

**Proposition 3.21.** In conjecture $3.6$, assume further that $P$ is defined over $\mathbb{F}_q$. Then the conjecture holds true.

**Example 3.22.** Let $G = \text{Res}_{\mathbb{F}_q'/\mathbb{F}_q}(GL_{2,d})$. Let $(B_0, T_0)$ be the usual upper Borel pair in $GL_{2,d}$, and let $(B, T)$ be its Weyl restriction to $\mathbb{F}_q$. In this example, we take $P = B^-$ and $Q = B$. The group $G_k$ is isomorphic to a product of $d$ copies of $GL_{2,k}$. Let $\alpha : GL_{2,k} \rightarrow k^\times$ be the fundamental weight of $B^{-}_0$, and let $\alpha_i = \alpha \circ \text{pr}_j$ where $\text{pr}_j : G_k \rightarrow GL_{2,k}$ is the $i$-th projection. The $(\alpha_i)_{1 \leq i \leq d}$ form a basis of $X^*(B)/X^*(G)$ and are the fundamental weights of the maximal parabolics containing $B^-$. The map $\zeta$
is given in this basis by

\[
A = \begin{pmatrix}
1 & -q \\
1 & \ddots \\
& \ddots & -q \\
& & 1
\end{pmatrix}
\]

One checks easily that \( A^{-1} \) has negative coefficients, which illustrates Lemma 3.17. The absolute value of the determinant of \( A \) is \( q^d - 1 \), which is the cardinality of \( S_0 = \mathbb{F}_q^\times \), and this illustrates Proposition 2.22. Let \([C_i] \in Z^1_{\mathbb{Q}}(G)^E\) be the closure of the \( E \)-orbit of \((1, \ldots, J, \ldots, 1)\) where \( J = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \). Through the isomorphism

\[
\frac{X^*(P)_{\mathbb{Q}}}{X^*(G)_{\mathbb{Q}}} \simeq \text{Pic}^E(G)_{\mathbb{Q}} \simeq Z^1_{\mathbb{Q}}(G)^E,
\]

the cycle \([C_i]\) corresponds to the character \( \zeta(\alpha_i) = \alpha_i - p\alpha_{i-1} \). This illustrates a result by Goren on Hasse invariants for Hilbert-Blumenthal Shimura varieties.

### 3.5 The almost-simple case

Let \( r \geq 1 \) be an integer. Let \( G_1 \) be a connected reductive group over \( \mathbb{F}_q \) and \( G = \text{Res}_{\mathbb{F}_q'/\mathbb{F}_q}(G_1) \). We denote by \( \sigma \in \text{Gal}(K/\mathbb{F}_q) \) the \( q \)-th power arithmetic Frobenius. Fix a Borel pair \((B_1, T_1)\) defined over \( \mathbb{F}_q \) in \( G_1 \), and define \( B = \text{Res}_{\mathbb{F}_q'/\mathbb{F}_q}(B_1) \) and \( T = \text{Res}_{\mathbb{F}_q'/\mathbb{F}_q}(T_1) \). Over \( k = \overline{\mathbb{F}_q} \), the group \( G \) decomposes as a product

\[
G = G_1 \times \cdots \times G_r
\]

where \( G_i = \sigma^{i-1}(G_1) \). The Frobenius sends \( G_i \) onto \( G_{i+1} \) (indices taken modulo \( r \)). Let \( P \subset G \) be a parabolic subgroup containing \( B^- \), which decomposes as a product

\[
P = P_1 \times \cdots \times P_r
\]

such that for each \( i = 1, \ldots, r \), the parabolic \( P_i \subset G_i \) is either maximal or equal to \( G_i \). Let \( \Delta \subset \{1, \ldots, r\} \) the subset where \( P_i \) is maximal in \( G_i \). The Weil group of \( G \) decomposes naturally into a product : \( W = W_1 \times \cdots \times W_r \) where all factors are isomorphic. If \( J \) denotes the type of \( P \), we have \( W_J = W_{J_1} \times \cdots \times W_{J_r} \) where \( J_i \subset W_i \) is the type of the parabolic \( P_i \). Let \( T \subset L \subset P \) denote the Levi subgroup of \( P \). Define the parabolic \( Q \subset G \) by :

\[
Q := \sigma(P^-) = \sigma(P_r^-) \times \sigma(P_{r-1}^-) \times \cdots \times \sigma(P_1^-)
\]

The Levi subgroup of \( Q \) containing \( T \) is then \( M := \sigma(L) \). Let \( E \) denote the zip group. Recall that the \( E \)-orbits are parametrized by \( J \) through the bijection \( \omega \mapsto \mathcal{O}^{\omega} := E \cdot \omega \). For this convention, we have

\[
\text{codim}(\mathcal{O}^{\omega}) = \ell(\omega)
\]
Thus the $E$–orbits of codimension 1 in $G$ are:

$$C_j := E \cdot \left( 1, \ldots, \beta_j, \ldots, 1 \right)$$

where $j \in \Delta$, $\beta_j \in W_j$ is the element corresponding to the maximal parabolic $P_j \subset G_j$. An element of $E$ can be written in the form

$$((x_1, \ldots, x_r), (y_r, y_1, \ldots, y_{r-1}))$$

with the condition $\varphi(\overline{x}_i) = \overline{y}_i$ for all $i$. For each $i \in \Delta$, let $\alpha_i \in X^\ast(P_i)Q$ denote the inverse of the fundamental weight of $P_i$.

### 3.5.1 First step

Define parabolic subgroups in $G_1$ by:

$$P'_1 := \bigcap_{i=1}^r \sigma^{r-i+1}(P_i) \quad \text{and} \quad Q'_1 := \bigcap_{i=1}^r \sigma^{r-i+1}(Q_i)$$

Since the Borel subgroup $B$ is defined over $\mathbb{F}_q$, we see immediately that $P'$ and $Q'$ are parabolics in $G_1$ containing $B_1$. Their respective Levi subgroups with respect to the torus $T_1$ are:

$$L'_1 := \bigcap_{i=1}^r \sigma^{-(i-1)}(L_i) \quad \text{and} \quad M'_1 := \bigcap_{i=1}^r \sigma^{r-i+1}(L_i) = \sigma^r(L'_1)$$

We may thus consider the quadruple $(G_1, P'_1, Q'_1, \varphi^r)$, since $\varphi^r$ maps $L'_1$ to $M'_1$. As usual, we can associate a zip group $E'_1$ defined by

$$E'_1 := \{(x, y) \in P'_1 \times Q'_1, \varphi^r(\overline{x}) = \overline{y}\}$$

where $\overline{x} = \theta_{P'_1}^{Q'_1}(x)$ and $\overline{y} = \theta_{L'_1}^{Q'_1}(y)$. It operates on $G_1$ in the usual way. For $(x, y) \in E'_1$, consider the elements defined by:

$$u := (x, \varphi(\overline{x}), \varphi^2(\overline{x}), \ldots, \varphi^{r-1}(\overline{x})) \in P$$

$$v := (y, \varphi(\overline{x}), \varphi^2(\overline{x}), \ldots, \varphi^{r-1}(\overline{x})) \in Q$$

(3.6)

It is clear that $(u, v) \in E$. The map $E'_1 \to E$, $(x, y) \mapsto (u, v)$ is an injective algebraic group homomorphism. The image of $E'_1$ is contained (in general strictly) in the stabilizer of $G_1$. One has the following preliminary lemma:

**Lemma 3.23.** Let $a = ((x_1, \ldots, x_r), (y_r, y_1, \ldots, y_{r-1}))$ be an element of $E(k)$ stabilizing $G_1$ (equivalently, $a \cdot 1 \in G_1$). Then for all $1 < j \leq r$, one has

$$x_j \in \bigcap_{i=j}^r \sigma^{-(i-j)}(P_i)$$
Further, for all $1 < j \leq r$, one has

$$y_j \in \bigcap_{i=1}^{j} \sigma^{j-i+1}(P_i^-)$$

In particular, one has $x_1 \in P'_1$ and $y_r \in Q'_1$. Further, one has $(x_1, y_r) \in E'_1(k)$.

**Remark 3.24.** Note that the lemma concerns only the $k$--points of scheme-theoretic stabilizer of $G_1$ in $E$, which may be non reduced. However, this slight restriction will not be an issue.

**Proof.** First, the assumption on $a$ is clearly equivalent to $x_i = y_{i-1}$ for all $1 < i \leq r$. For $j = r$, there is nothing to prove. Now, let us fix an integer $1 \leq j < r$, and let us prove the first assertion using decreasing induction on the integer $j$. By definition, one has $\varphi\left(\theta^{P_j}_{L_j}(x_j)\right) = \theta^{Q_{j+1}}_{M_{j+1}}(y_j)$. By induction, we have at the step $j + 1$:

$$y_j = x_{j+1} \in \bigcap_{i=j+1}^{r} \sigma^{-(i-j-1)}(P_i)$$

Now we apply Corollary 2.16 to the parabolics $\bigcap_{i=j+1}^{r} \sigma^{-(i-j-1)}(P_i)$ and $Q_{j+1}$ in $G_{j+1}$ (both containing the torus $T_{j+1}$). We get:

$$\theta^{Q_{j+1}}_{M_{j+1}}(y_j) \in \bigcap_{i=j+1}^{r} \sigma^{-(i-j-1)}(P_i)$$

We deduce immediately $\theta^{P_j}_{L_j}(x_j) \in \bigcap_{i=j+1}^{r} \sigma^{-(i-j)}(P_i)$ (because we consider only $k$--points). Since the unipotent radical of $P_j$ is contained in the Borel $B_j$ (itself contained in $\bigcap_{i=j+1}^{r} \sigma^{-(i-j)}(P_i)$), one has finally $x_j \in \bigcap_{i=j+1}^{r} \sigma^{-(i-j)}(P_i)$ as claimed. This concludes the first part of the proof. The same applies to the elements $y_j$ and the parabolics $Q_j$.

Take $1 < j \leq r$ and consider the element $y_j \in Q_{j+1}$ (if $j = r$, define $Q_{j+1} = Q_1$). We may write $\varphi\left(\theta^{P_j}_{L_j}(x_j)\right) = \theta^{Q_{j+1}}_{M_{j+1}}(y_j)$. As above, using $x_j = y_{j-1} \in Q_j$, we deduce $y_j \in \sigma(Q_j)$. Continuing in this way, we get:

$$y_j \in \bigcap_{i=2}^{j+1} \sigma^{j-i+1}(Q_i) = \bigcap_{i=1}^{j} \sigma^{j-i+1}(P_i^-)$$

It remains to prove that $\varphi^r(\theta^{P'_1}_{L'_1}(x_1)) = \theta^{Q'_1}_{M'_1}(y_r)$. For this, we use induction to prove:

$$\varphi^j\left(\theta^{\sigma^{-(j-1)}P_j}_{\sigma^{-(j-2)}(\cdot)P_{j-1}} \cdots \theta^{P_1}_{L_1}(x_1)\right) = \theta^{\sigma^{j-1}Q_2}_{\sigma^{j-2}M_2} \cdots \theta^{\sigma_{M_j}Q_{j+1}}_{\sigma_{M_{j+1}}}(y_j)$$

For $j = 1$ this is clear. Assume that the above formula holds and apply the operator $\varphi \circ \theta^{P_{j+1}}_{L_{j+1}}$ to this equality. We get:

30
\[ \varphi^{j+1}(\theta_{\sigma^{-j}L_{j+1}}^{\sigma_{j+1}} \theta_{\sigma_{j+1}}^{\sigma_{j+1}} \theta_{\sigma_{j+1}}^{\sigma_{j+1}} \cdots \theta_{\sigma_{j+1}}^{\sigma_{j+1}}(x_1)) = \varphi(\theta_{\sigma_{j+1}}^{\sigma_{j+1}} \theta_{\sigma_{j+1}}^{\sigma_{j+1}} \theta_{\sigma_{j+1}}^{\sigma_{j+1}} \cdots \theta_{\sigma_{j+1}}^{\sigma_{j+1}}(y_j)) \]

which is the formula for \( j+1 \). The above holds also for \( j = r - 1 \) if we take indices modulo \( r \). Now the formula for \( j = r \) together with Proposition 2.10 (iv) gives exactly \( \varphi^r(\theta_{L_1}^{P_{1'}}(x_1)) = \theta_{M_1}^{Q_{1'}}(y_r) \) as claimed.

\[ \square \]

**Lemma 3.25.** Let \( X_1 \) be the set of \( E \)-orbits in \( G \) intersecting \( G_1 \). Then the map \( \sigma \mapsto \sigma \cap G_1 \) defines a bijection between \( X_1 \) and the set of \( E_1' \)-orbits in \( G_1 \). Furthermore, if \( \sigma \) has codimension 1 in \( G \), so does \( \sigma \cap G_1 \) in \( G_1 \).

**Proof.** First we need to prove that \( \sigma \cap G_1 \) is an \( E_1' \)-orbit. Let \( u, v \in \sigma \cap G_1 \). We can find an element

\[ a = ((x_1, \ldots, x_r), (y_r, y_1, \ldots, y_{r-1})) \in E \]

such that \( a \cdot u = v \). But then \( a \) is in the stabilizer of \( G_1 \), so \( (x_1, y_r) \in E_1' \) by Lemma 3.23. Thus \( \sigma \cap G_1 \) is contained in a \( E_1' \)-orbit.

Now, let \( g \in \sigma \cap G_1 \) and \( (a, b) \in E_1' \). Define \( \overline{a} := \theta_{P_1'}^P(a) \) and \( \overline{b} := \theta_{M_1'}^{Q_1'}(b) \). Consider the pair \( (a, b) \in E \) defined as in (3.8). Then \( agb^{-1} = ugv^{-1} \), so \( \sigma \cap G_1 \) is exactly an \( E_1' \)-orbit. The same argument also shows the injectivity of the map \( \sigma \mapsto \sigma \cap G_1 \). The surjectivity is clear.

To prove the statement on codimensions, first notice that \( G_1 \) is not contained in the open orbit \( U \). Indeed, the \( E_1' \)-orbits are parametrized by \( J_1'W \), where \( J_1' \) is the type of the parabolic \( P_1' \) of \( G_1 \). Since \( P_1' \neq G_1 \), there are at least two \( E_1' \)-orbits. Now, the result follows from Proposition 2.3. \( \square \)

We may define also the groups \( P_j', Q_j', E_j' \) for any \( 1 \leq j \leq r \). Note that we can permute the factors to form a new product

\[ G_j \times G_{j+1} \times \cdots \times G_r \times G_1 \times \cdots \times G_{j-1} \]

and define \( P_j', Q_j', E_j' \) as before with respect to this new numbering. In other words, we define:

\[ P_j' = P_j \cap \sigma^{-1}(P_{j+1}) \cap \cdots \cap \sigma^{-1}(P_r) \cap \sigma^{-1}(P_{j+1})(P_1) \cap \cdots \cap \sigma^{-1}(P_{j-1}) \]

\[ Q_j' = \sigma(P_j') \cap \sigma^{-1}(P_{j+1}) \cap \cdots \cap \sigma^{-1}(P_r') \cap \sigma^{-1}(P_{j+1}) \cap \cdots \cap \sigma(P_{j-1}) \]

The tuple \((G_j, P_j', Q_j', \varphi^r)\) defines a group \( E_j' \) as usual. Then there is a one-to-one correspondence between the \( E \)-orbits in \( G \) intersecting \( G_j \) with the set of \( E_j' \)-orbits in
We may define a map \( G_j \). This bijection is defined by intersecting an \( E \)-orbit with \( G_j \). Let \((x, y) \in E_j'\) and write \( \varphi := \theta_{E_j'}(x) \) and \( \varphi := \theta_{E_j'}(y) \). For all \( 1 \leq j \leq r \), let \( U_j \) denote the open \( E_j'\)-orbit in \( G_j \). Then it is clear that \( U_j = U \cap G_j \) (it is the \( E_j'\)-orbit of \( 1 \in G_j \)). Define:

\[
\begin{align*}
  u &= (\varphi^{r-1}(\varphi), \ldots, \varphi^{r-1}(\varphi), x, \varphi(x), \ldots, \varphi^{r-1}(\varphi)) \\
  v &= (\varphi^{r-1}(\varphi), \ldots, \varphi^{r-1}(\varphi), y, \varphi(y), \ldots, \varphi^{r-1}(\varphi)) 
\end{align*}
\]

It is clear that \((u, v) \in E\). This defines an embedding \( \gamma_j : E_j' \to E \).

**Remark 3.26.** When \( P_1, \ldots, P_r \) are all defined over \( \mathbb{F}_q^* \), then the groups \( E_j' \) are simply Galois translates of each other. In particular, the number of \( E \)-orbits intersecting \( G_j \) is in this case independent of \( j \).

### 3.5.2 Second step

We may define a map \( \gamma_j : P_j' \to P \) compatible with the map \( \gamma_j : E_j' \to E \) defined above. For \( x \in P_j' \), define:

\[
\gamma_j(x) = (\varphi^{r-1}(\varphi), \ldots, \varphi^{r-1}(\varphi), x, \varphi(x), \ldots, \varphi^{r-1}(\varphi))
\]

where \( \varphi := \theta_{E_j'}(x) \). We will now construct a commutative diagram:

\[
\begin{array}{ccc}
X^*(P)_Q & \xrightarrow{\gamma_j^*} & X^*(E)_Q \\
\downarrow{\eta_j^*} & & \downarrow{\eta_j^*} \\
X^*(P'_j)_Q & \xrightarrow{\gamma_j^*} & X^*(E'_j)_Q \\
\end{array}
\]

\[
\begin{array}{ccc}
  & & E(U)_Q \xrightarrow{-\text{div}} Z^1(G)^E_Q \\
\end{array}
\]

The maps \( \gamma_j^* \) is the composition with the embedding \( \gamma_j \). If \( f \in E(U) \), then \( \iota_j(f) \) is the restriction of \( f \) to \( U_j \) via the natural embedding \( \iota_j : G_j \to G, x \mapsto (1, \ldots, x, \ldots, 1) \).

**Lemma 3.27.** The map \( \iota_j^* \) extends uniquely to a map \( u_j : Z^1(G)^E_Q \to Z^1(G_j)^{E_j'}_Q \).

**Proof.** The map \( u_j \) is clearly unique. Now if \( f \in E(U) \) is the restriction of a character \( \chi \in X^*(G) \), then \( f \circ \iota_j \) is the restriction of the character \( \chi \circ \iota_j \) of \( G_j \), so we may define \( u_j(f) = \text{div}(f \circ \iota_j) \) for all \( f \in E(U)_Q \). \hfill \square

If \( C \) is an irreducible component of \( G - U \), then we can write \([C] = \text{div}(f)\) for some \( f \in E(U)_Q \). So \( f \) extends to a non-vanishing function on \( G - C \). The intersection \( C \cap G_j \) is empty or has codimension one in \( G_j \) (Lemma 2.3). We conclude that the divisor of \( f \circ \iota_j \) has support \( C \cap G_j \) if this intersection is nonempty. In any case we may write

\[
(3.8) \quad u_j([C]) = a_j(C) [C \cap G_j], \quad a_j(C) \geq 0
\]

and \( a_j(C) > 0 \) if and only if \( C \cap G_j \neq \emptyset \).
Lemma 3.28. The diagram \( \Delta \) is commutative.

Proof. For \( f \in E(U) \), the corresponding character \( \chi \in X^*(E) \) is defined by the formula \( f(e \cdot x) = \chi(e)^{-1} f(x) \) for all \( e \in E \) and \( x \in U \). Now take \( x = t_j(y) \) and \( e = \gamma_j(u) \) with \( y \in U_j \) and \( u \in E_j' \). Since \( \gamma_j(u) \cdot t_j(y) = t_j(u \cdot y) \), we get \( f(t_j(u \cdot y)) = \chi(\gamma_j(u))^{-1} f(t_j(y)) \) and the result follows.

Lemma 3.29. Assume \( \chi \in X^*(P)_{\mathbb{Q}} \) is ample. Then \( \gamma_j^*(\chi) \in X^*(P'_j)_{\mathbb{Q}} \) is ample.

Proof. To simplify notations, we will assume \( j = 1 \). Recall that \( P'_1 := \bigcap_{i=1}^r \sigma^{-(i-1)}(P) \), so the maximal parabolics of \( G_1 \) containing \( P'_1 \) are the \( \sigma^{-(i-1)}(P) \) for \( i \in \Delta \) (some of them may be equal). For all \( i \in \Delta \), let \( \chi_i \in X^*(P_i)_{\mathbb{Q}} \) be the fundamental weight of \( P_i \). Then \( \chi = \sum_{i \in \Delta} a_i \chi_i \) with \( a_i > 0 \). We deduce that for all \( x \in P'_1 \), one has:

\[
\chi \circ \gamma_1 = \sum_{i \in \Delta} a_i (\chi_i \circ \varphi^{i-1}) = \sum_{i \in \Delta} a_i \varphi^{i-1} \chi_i \sigma^{-(i-1)}
\]

where \( \chi_i^{\sigma^{-(i-1)}} \) is the fundamental weight of \( \sigma^{-(i-1)}(P) \). It follows that \( \chi \circ \gamma_1 \) is a linear combination with \( > 0 \) coefficients of the fundamental weights of the maximal parabolics containing \( P'_1 \), so it is ample.

Lemma 3.30. Let \( D \in Z^1(G)_{\mathbb{F}} \). Assume that \( u_j(D) \geq 0 \) for all \( 1 \leq j \leq r \). Then \( D \geq 0 \).

Proof. Write \( D = \sum_{i \in \Delta} n_i(C_i) \) where \( C_i \) are the codimension one \( E \)-orbits (see equation \( 3.5 \)) and \( C_i \) is its closure. Fix an integer \( j \in \Delta \). Define \( a_{i,j} := a_j(C_i) \) (see equation \( 3.8 \)). Now we have

\[
u_j(D) = \sum_{i \in \Delta} a_{i,j} n_i(C_i \cap G_j)
\]

Clearly \( a_{i,j} > 0 \) because \( C_j \) intersects \( G_j \). To prove the claim, it suffices to show that in this sum, the only \( i \in \Delta \) contributing to \( (C_j \cap G_j) \) is \( i = j \). So assume that \( C_j \cap G_j = C_i \cap G_j, i \in \Delta \). Since the intersection \( C_j \cap G_j \) is nonempty, it follows that \( C_j \cap C_i \neq \emptyset \), and it follows that \( C_j = C_i \), so \( i = j \).

Proposition 3.31. Assume Conjecture 3.10 holds for the zip datum \( (G_j, P'_j, Q'_j, \varphi^r) \) for all \( 1 \leq j \leq r \). Then it holds for \( (G, P, Q, \varphi) \).

Proof. This is a simple application of the previous lemmas.

Corollary 3.32. Assume \( P_1, ..., P_r \) are defined over \( \mathbb{F}_{q^r} \). Then Theorem 3.10 holds.

Proof. Assume that \( P_1, ..., P_r \) are defined over \( \mathbb{F}_{q^r} \). Then \( P'_j \) and \( Q'_j \) are defined over \( \mathbb{F}_{q^r} \), so the result comes from Proposition 3.21.
3.5.3 Third step

In this third step we assume that $P_1, \ldots, P_r$ are defined over some finite field $\mathbb{F}_{q^d}$. We consider the group

$$\tilde{G} = \text{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(G_1 \times \mathbb{F}_{q^d})$$

We embed $G = \text{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(G_1)$ into $\tilde{G}$, such that for any $\mathbb{F}_q$-algebra $R$, the map $G(R) \to \tilde{G}(R)$ is given by the natural map $(R \otimes \mathbb{F}_q^{\mathbb{F}_q}) \to (R \otimes \mathbb{F}_{q^d})$ induced by the inclusion $\mathbb{F}_q \subset \mathbb{F}_{q^d}$. Over the algebraic closure, it decomposes into a product

$$\tilde{G}_k = (G_1 \times \ldots \times G_r) \times \ldots \times (G_1 \times \ldots \times G_r)$$

where the product $G = G_1 \times \ldots \times G_r$ appears $d$ times. Note that the groups $\tilde{G}, G \times \ldots \times G$ and $\text{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(G)$ are all different, although the become isomorphic over $k$. The Galois action of $\tilde{G}(k)$ is given by

$$\sigma \cdot (x_1, \ldots, x_{rd}) = (\sigma \cdot x_{rd}, \sigma \cdot x_1, \ldots, \sigma \cdot x_{rd-1}).$$

Define $\tilde{P} = P \times \ldots \times P$ and $\tilde{Q} = Q \times \ldots \times Q$ inside $\tilde{G}_k = G \times \ldots \times G$.

**Lemma 3.33.** This defines a perfect embedding of zip data.

*Proof.* This is obvious. \(\square\)

Conjecture 3.6 holds for $\tilde{G}$ (see Corollary 3.32). It is clear that the map $X^*(\tilde{P}) \to X^*(P)$ defines a surjective map on the ample characters of $P$ and $P$. It follows from Proposition 3.11 that Conjecture 3.6 also holds for $G$. This terminates the proof. Finally, let us formulate our result:

**Theorem 3.34.** Assume $G$ and $P$ are defined as in 3.3 and 3.4. Then Conjecture 3.6 holds.

4 Hasse invariants for Shimura varieties of Hodge type

4.1 Shimura varieties of Hodge type

Let $(G, X)$ be a Shimura datum. Hence $G$ is a reductive group over $\mathbb{Q}$ and $X$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}} \to G_\mathbb{R}$ satisfying Deligne’s conditions ([De]). We denote by $[\mu]$ the $G(\mathbb{C})$-conjugacy class of the component of $h_\mathbb{C}: \prod_{\text{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m, \mathbb{C}} \to G_\mathbb{C}$ corresponding to $id \in \text{Gal}(\mathbb{C}/\mathbb{R})$. The elements of $[\mu]$ are minuscule by Deligne’s axioms. The field of definition of $[\mu]$ is a finite extension $E$ of $\mathbb{Q}$, the reflex field.

We assume that $(G, X)$ is of Hodge type, i.e., it can be embedded into a Shimura datum of the form $(\text{GSp}(V), S^\pm)$, where $V = (V, \psi)$ is a symplectic space over $\mathbb{Q}$ and where $S^\pm$ is the double Siegel half space. We choose such an embedding $\iota$.

Let $p$ be a prime number such that $G$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}_{(p)}$ (equivalent, $G_{\mathbb{Q}_p}$ has a reductive model over $\mathbb{Z}_p$ [KiII] (2.3.2) which we also denote by $\mathcal{G}$). Hence
$K_p := \mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. We denote by $G$ the special fiber of $\mathcal{G}$. Hence $G$ is a reductive group over $\mathbb{F}_p$.

Choose a place $v$ of the reflex field $E$ of $(G, \mathbf{X})$ over $p$. Let $K = K_pK^p \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. If $K^p$ is sufficiently small (which we assume form now on), Kisin ([Ki1]) and Vasiu ([Va]) have shown the existence of integral canonical models $S_K(G, \mathbf{X})$ for the Shimura variety attached to $(G, \mathbf{X})$ and $K$ (with restrictions for $p = 2$) over $O_{E, v}$. Here we follow Kisin and hence assume that for $p = 2$ Condition (2.3.4) of [Ki1] is satisfied. In particular, $G^\mathrm{ad}$ has no factor of Dynkin type $B$ for $p = 2$. We denote by $S := S_K(G, \mathbf{X})$ the special fiber. It is a smooth quasi-projective scheme over $\kappa := \kappa(v)$ the residue field of the place $v$.

There exists a $\mathbb{Z}_{(p)}$-lattice $\Lambda$ of $V$ such that the embedding $\iota$ in the Siegel Shimura datum is induced by an embedding $G_{\mathbb{Z}_{(p)}} \to \GL(\Lambda)$ ([Ki1] Lemma (2.3.1); see also the remarks in the proof of Lemma 4.6.2 in [Per]). By Zarhin’s trick we may assume after changing $(V, \psi)$ and $\Lambda$ that $\psi$ induces a perfect $\mathbb{Z}_{(p)}$-pairing on $\Lambda$ ([Ki2] (1.3.3)). We obtain an embedding

\begin{equation}
\iota : \mathcal{G} \hookrightarrow \GSp(\Lambda)
\end{equation}

of reductive group schemes over $\mathbb{Z}_{(p)}$ whose generic fiber is an embedding of Shimura data. We call such an embedding a $p$-integral Hodge embedding.

By [Ki1] 1.3.2, $\iota$ identifies $\mathcal{G}$ with the scheme theoretic stabilizer of a finite set $s$ of tensors in $\Lambda^\otimes$. Here for a finite locally free module $M$ over a ring we write $M^\otimes$ for the direct sum of all $R$-modules that one obtains from $M$ by applying the operations of taking duals, tensor products, symmetric powers and exterior powers finitely often. Then we can identify $\Lambda^\otimes$ with $(\Lambda^*)^\otimes$. Moreover we identify $\GL(\Lambda)$ with $\GL(\Lambda^*)$ via $g \mapsto g^\vee := \iota^{-1}g^{-1}$ and hence

$$
\mathcal{G} = \{ g \in \GL(\Lambda^*) : g^\vee(s) = s \}.
$$

We set $\tilde{K}_p := \GSp(\Lambda)(\mathbb{Z}_p)$. By [Ki1] (2.1.2) there exists for $K^p$ sufficiently small an open compact subgroup $\tilde{K}^p \subset \GSp(\Lambda^\vee_p)$ containing $K^p$ such that $\iota$ yields an embedding

$$
\varepsilon^0 : \Sh_K(G, \mathbf{X}) \hookrightarrow \Sh_{\tilde{K}}(\GSp(V), S^\pm),
$$

where $\tilde{K} := \tilde{K}_p\tilde{K}^p$. The left hand side can be identified with a moduli spaces of polarized abelian varieties. More precisely, for $\tilde{K}^p$ sufficiently small, let

$$
\tilde{S} := \mathcal{S}_{\tilde{K}}(\GSp(\Lambda), S^\pm)
$$

be the smooth quasi-projective $\mathbb{Z}_{(p)}$-scheme whose $T$-valued points ($T$ a $\mathbb{Z}_{(p)}$-scheme) the set of isomorphism classes $(\mathcal{A}, \lambda, \eta)$, where

(a) $\mathcal{A}$ is an abelian scheme over $T$ up to prime to $p$ isogeny,
(b) $\lambda$ is an equivalence class of a prime to $p$ quasi-isogeny $\lambda : \mathcal{A} \to \mathcal{A}^\vee$ such that locally on $T$ some multiple is a polarization, and where two such quasi-isogenies are equivalent if they differ by a global section of the constant sheaf with value $\mathbb{Z}^\times_{(p)}$ on $T$,
η is a $\tilde{K}^p$-level structure, i.e., a section over $T$ of $\text{Isom}(V_{\tilde{A}}^p, \tilde{V}^p(A))/\tilde{K}^p$, where

$$\tilde{V}^p(A) := (\lim_{p|n} A[n])_Q.$$  

Then the generic fiber of $S_{\tilde{K}}(\text{GSp}(\Lambda), S^{\pm})$ is naturally identified with $\text{Sh}_{\tilde{K}}(\text{GSp}(V), S^{\pm})$ and the integral canonical model $S_{\tilde{K}}(G, X)$ is defined as the normalization of the closure of $\text{Sh}_{\tilde{K}}(G, X)$ in $S_{\tilde{K}}(\text{GSp}(\Lambda), S^{\pm}) \otimes \mathbb{Z}_{(p)} O_{E,v}$ (K12 (1.3.4)). In particular, one obtains a finite morphism of $O_{E,v}$-schemes

$$(4.2) \quad \varepsilon: S := S_{K}(G, X) \rightarrow S_{\tilde{K}}(\text{GSp}(\Lambda), S^{\pm}) \otimes \mathbb{Z}_{(p)} O_{E,v}$$

extending $\varepsilon_0$.

The conjugacy class of cocharacters $[\mu]$ defines a conjugacy class of cocharacters of $G$ (resp. of $G$) defined over $O_{E,v}$ (resp. over $\kappa$) which we again denote by $[\mu]$. As $G$ is quasi-split, there exists a representative in $[\mu]$ defined over $O_{E,v}$ (resp. over $\kappa$).

### 4.2 The De Rham cohomology and the Hodge line bundle

Let $\tilde{A} \rightarrow S_{\tilde{K}}(\text{GSp}(\Lambda), S^{\pm})$ be the universal abelian scheme. We call its pullback $A$ to $S := S_{K}(G, X)$ via $\varepsilon$ the universal abelian scheme over $S_{K}(G, X)$. We set

$$V_S := H^{1}_{\text{DR}}(A/S)$$

This is a locally free $\mathcal{O}_S$-module canonically endowed with a finite set $s_{4\mathbb{R}}$ of sections of $V_S$ that are horizontal with respect to the Gauss-Manin connection.

Then $e^*\Omega^1_{A/S}$ (where $e$ is the zero section of $A$) is a locally direct summand of $V_S$, the Hodge filtration. We call the highest exterior power of the Hodge filtration

$$\omega_S := \det(e^*\Omega^1_{A/S})$$

the Hodge line bundle on $S$. It depends on the chosen $p$-integral Hodge embedding $\iota$. In case we want to stress this dependency, we write $\omega_S(\iota)$.

**Proposition 4.1.** The line bundle $\omega_S$ is ample.

**Proof.** By [MB], $\omega_S$ is ample. As $\varepsilon$ is finite, $\varepsilon^*\omega_S = \omega_S$ is ample. □

### 4.3 Arithmetic compactifications

Madapusi Pera has constructed in [Per] arithmetic toroidal compactifications $S^\Sigma = S^\Sigma_{K}(G, X)$ (depending on a complete admissible rational partial polyhedral cone decomposition $\Sigma$ for $(G, X, K)$) and the arithmetic minimal compactification $S^{\min} = S^{\min}_{K}(G, X)$. Here we use the following facts about these compactifications.

(1) For every $\Sigma$ as above there exists a flat proper integral model over $O_{E,(v)}$ of the toroidal compactification of $\text{Sh}_{K}(G, X)$ given by $\Sigma$ which contains $S$ as an open dense subscheme ([Per] 4.6.13). It carries a canonical extension $\omega(\iota)^\Sigma$ of the Hodge line bundle ([Per] 4.8.1).
(2) \(S^{\min}\) is a projective \(O_{E,(\nu)}\)-scheme containing \(S\) as an open dense subscheme (Per 4.8.11). It is flat with geometric normal fibers over \(O_{E,(\nu)}\) (in the PEL case this is Lan 7.2.4.3 via the description of the local ring; for Shimura varieties of Hodge type the argument is the same).

(3) For every \(\Sigma\) as above there is a natural proper surjective map \(\int_{\Sigma}: S^\Sigma \to S^{\min}\) with geometrically connected fibers inducing the identity on \(S\) (Per 4.8.11 (3)) such that 
\[(\int_{\Sigma})^*\omega(\iota)^{\min} \cong \omega(\iota)^{\min}.
\]

(4) For every \(p\)-integral Hodge embedding \(\iota\) the Hodge line bundle \(\omega_S(\iota)\) extends to an ample line bundle \(\omega_S(\iota)^{\min}\) over \(S^{\min}\). Given different \(p\)-integral embeddings \(\iota\) and \(\iota'\), there exists \(s,s' \geq 1\) such that \((\omega_S(\iota)^{\min})^{\otimes s} \cong (\omega_S(\iota')^{\min})^{\otimes s}\) (Per 4.8.11 (2)).

(5) If \(\text{PGL}_{2,\Q}\) does not occur as a simple factor of \(G^\text{ad}\), then for any \(k \geq 1\) pullback of sections induces isomorphism

\[(4.4) \quad \Gamma(S^{\min},(\omega(\iota)^{\min})^{\otimes k}) \cong \Gamma(S^\Sigma,(\omega(\iota)^\Sigma)^{\otimes k}) \cong \Gamma(S,\omega(\iota)^{\otimes k}).\]

Indeed, the hypothesis implies that the codimension of \(S^{\min}\) \(\setminus S\) is of codimension at least 2 in \(S^{\min}\). Then the second bijection is the Koecher principle (Per 4.8.12). The composition is given by the pullback of the inclusion \(S \hookrightarrow S^{\min}\). As \(S^{\min}\) is normal, Hartogs' theorem implies that this composition is an isomorphism. The same argument holds for the special fiber and the restriction of \(\omega(\iota)^{\min}\) to the special fiber.

### 4.4 Ekedahl-Oort stratification

The conjugacy class \([\mu^{-1}]\) is defined over \(\kappa\). As \(G\) is quasi-split, there exists a cocharacter \(\chi\) of \(G_\kappa\) whose conjugacy class is \([\mu^{-1}]\). Then \(\chi\) yields an orbitally finite algebraic zip datum as follows. Let \(P_{\pm} = P_{\pm}(\chi)\) be the attached pair of opposite parabolic subgroups of \(G_\kappa\) with common Levi subgroup \(L\) the centralizer of \(\chi\). Let \(U_{\pm}\) be the unipotent radical of \(P_{\pm}\). We obtain an algebraic zip datum \(Z_{G,\chi} := (G,P_+,P^-_\sigma,\varphi)\), where \((\sigma)^\sigma\) denotes the pullback under absolute Frobenius embeddings \(\sigma: x \mapsto x^p\) and where \(\varphi: L \to L^\sigma\) is the relative Frobenius. In particular we obtain an attached zip group \(E := E_{G,\chi}\). We set \(P := P_+\) and \(Q := P^-_\sigma\). Let \(U\) (resp. \(V\)) be the unipotent radical of \(P\) (resp. of \(Q\)). We define the algebraic quotienet stack over \(\kappa\)

\[(4.5) \quad G\text{-Zip}^\chi := [E\setminus G_\kappa].\]

By PWZ2 Proposition 3.11, for a \(\kappa\)-scheme \(T\) the \(T\)-valued points of this quotient stack is the groupoid of tuples \(L = (I,I_+,I_-,\iota)\), where \(I\) is a \(G_\kappa\)-torsor, where \(I_+ \subseteq I\) a \(P\)-torsor, \(I_- \subseteq I\) a \(Q\)-torsor, and where \(\iota: I_+^\sigma/U^\sigma \cong I_-/V\) is an isomorphism of \(L^\sigma\)-torsors.

The following lemma shows that we can assume (after conjugating \(\chi\) over \(\kappa\)) that there exists a Borel pair \((T,B)\) of \(G\) (defined over \(\F_p\)) such that \(\chi\) is a \(B_\kappa\)- antidominant cocharacter of \(T_\kappa\).

**Lemma 4.2.** Let \(T\) be a maximal torus, \(B\) be a Borel subgroup of \(G\) containing \(T\), and let \(B^-\) be the opposite Borel subgroup of \(B\) with respect to \(T\) (all defined over \(\F_p\)). Let
$\chi: \mathbb{G}_{m,k} \to G_\kappa$ be a cocharacter. Then there exists $g \in G(\kappa)$ such that for $\chi' := \text{int}(g) \circ \chi$ the following properties hold.

1. $B_k \subseteq P_-(\chi')$.
2. $\chi'$ factors through $T_\kappa$.
3. $L(\chi')$ is the unique Levi subgroup of $P_-(\chi')$ containing $T_\kappa$. In particular $B^- \subseteq P_+(\chi')$.

The proof of the lemma works for all quasi-split reductive groups over arbitrary base fields.

Proof. By [SGA3] Exp. XXVI, Lemme 3.8 there exists a parabolic subgroup $P'$ of $G_k$ such that $B_k \subseteq P'$ and such that $P'$ has the same type as $P_-(\chi)$. By loc. cit. Corollaire 5.5 (ii) there exists $g \in G(k)$ such that $P_-(\text{int}(g) \circ \chi) = gP_-(\chi) = P'$. Hence we may assume that $B \subseteq P_-(\chi)$. Let $L'$ be the unique Levi subgroup of $P_-(\chi)$ containing $T$. By loc. cit. Corollaire 1.8 there exists (a unique) $g \in U_-(\chi)(k)$ such that $gL(\chi) = L'$. Hence we may assume that $L(\chi)$ is the Levi subgroup $P_-(\chi)$ containing $T_k$. But then $T_k$ is a maximal torus of $L(\chi)$ and hence contains the identity component of the center of $L(\chi)$. By definition of $L(\chi)$, the cocharacter $\chi$ factors through the center of $L(\chi)$ and hence through its identity component.

From now on we choose $T$, $B$, and $\chi$ as in Lemma 4.2.

Zhang has constructed in [Zha1] a $G$-zip of type $\chi$ over $S_K := S_K(G,X)$ and he has shown in loc. cit. that the corresponding classifying morphism $S_K \to G\text{-Zip}^X$ is smooth. Here we use the (slightly different) Construction 5.13 of a $G$-zip $L$ of type $\chi$ given by Wortmann in [Wor] §5 and obtain a smooth morphism

$$\zeta := \zeta_G: S_K \longrightarrow G\text{-Zip}^X.$$ 

The Ekedahl-Oort strata of $S_K$ are the fibers of $\zeta$.

Let us recall Wortmann’s construction. Let $\breve{A} \to S_K$ be the restriction of the universal abelian scheme to the special fiber. Define $\breve{V} := H^1_{DR}(\breve{A}/S_K)$, let $C \subseteq \breve{V}$ be the Hodge filtration, and let $D \subseteq \breve{V}$ be the conjugate filtration. The embedding $\iota$ (4.1) yields an embedding

$$G \hookrightarrow \text{GL}(A_{\breve{F}_p}) \xrightarrow{\sim} \text{GL}(\Lambda_{\breve{F}_p}^\vee),$$

where $(\ )^*$ denotes the dual space and where the isomorphism is given by $g \mapsto g^\vee := t^{-1}g$. Then $(\ )^\vee \circ \chi$ and $(\ )^\vee \circ \chi^\sigma$ define $\mathbb{Z}$-gradings

$$\Lambda_{\kappa}^* = \bigoplus (\Lambda_{\kappa}^*)_n^\chi, \quad \Lambda_{\kappa}^* = \bigoplus (\Lambda_{\kappa}^*)_n^\sigma$$

with $(\Lambda_{\kappa}^*)_n^\chi = (\Lambda_{\kappa}^*)_n^\sigma = 0$ for all $n \neq 0,1$. We obtain a descending resp. ascending filtration

$$\text{Fil}_1^0 := \Lambda_{\kappa}^* \supset \text{Fil}_1^1 := (\Lambda_{\kappa}^*)_1^\chi \supset \text{Fil}_1^2 := 0, \quad \text{Fil}_{\leq 1}^\sigma := 0 \subset \text{Fil}_1^\sigma := (\Lambda_{\kappa}^*)_1^\sigma \subset \text{Fil}_1^\sigma := \Lambda_{\kappa}^*.$$
Then $P_+$ is the stabilizer of $\Fil^\bullet$ in $G_\kappa$, and $P_{-}$ is the stabilizer of $\Fil^\circ$ in $G_\kappa$.

Let $\mathcal{V}_S = H^1_{\text{Dr}}(\mathcal{A}/S_K)$ be the restriction of $\mathcal{V}_S$ to the special fiber. Let $\bar{s}_{\text{dr}} \subset \mathcal{V}^S_\bar{S}$ be the reduction of the tensors $s_{\text{dr}} \subset \mathcal{V}^S_\bar{S}$, and denote by $\bar{s}$ the base change of $s \subset (\Lambda^*)^\circ$ to $(\Lambda^*_\kappa)^\circ$. Define

$$I := \mathcal{I}so_{S_K}(\Lambda^*_\kappa, \bar{s}) \otimes \mathcal{O}_{S_K}(\mathcal{V}_{S}, \bar{s}_{\text{dr}}),$$

$$I_+ := \mathcal{I}so_{S_K}(\Lambda^*_\kappa, \bar{s}, \Fil^\bullet) \otimes \mathcal{O}_{S_K}(\mathcal{V}_{S}, \bar{s}_{\text{dr}}, \mathcal{V}_{S} \not\subset C),$$

$$I_- := \mathcal{I}so_{S_K}(\Lambda^*_\kappa, \bar{s}, \Fil^\circ) \otimes \mathcal{O}_{S_K}(\mathcal{V}_{S}, \bar{s}_{\text{dr}}, \mathcal{D} \subset \mathcal{V}_{S}).$$

Then $G_\kappa$ acts from the right on $I$ by $\beta \cdot g := \beta \circ g^\vee$, inducing right actions of $P_+$ and $P_{-}$ on $I_+$ and $I_-$, respectively. The Cartier isomorphism on $\mathcal{V}_S$ induces an isomorphism $\iota: I^\vee_+/U^\vee_+ \to I_-/U_-\vee$ and $\bar{I} := (I, I_+, I_-, \iota)$ is a $G$-zip of type $\chi$ over $S_K$ [Wor 5.14].

We obtain the morphism $\zeta: S_K \to G\text{-Zip}^\chi(\bar{I})$, which is smooth by [Zha 3.1.2].

**Remark 4.3.** The following properties of the Ekedahl-Oort strata are known to hold.

1. Each Ekedahl-Oort stratum is smooth and quasi-affine ([WdYa]).
2. If it is non-empty, the Ekedahl-Oort $S^w$ has dimension $\ell(w)$ ([Zha Proposition 3.1.6]).
3. The closure of the Ekedahl-Oort stratum $S^w$ is $\bigcup_{w \leq w} S^w$ for a certain refinement $\leq$ of the Bruhat order; see [PWZ1] Definition 6.1 for the precise definition of $\leq$ ([Zha Proposition 3.1.6]).
4. The inclusion $S^w \to \bar{S}^w$ is affine, in particular every irreducible component of $\bar{S}^w \setminus S^w$ is of codimension 1 in $\bar{S}^w$ ([WdYa]).

**Remark 4.4.** For Shimura varieties of PEL type it is shown in [ViWd] Theorem 10.1 that all Ekedahl-Oort strata are non-empty. In general this is expected, but it is not known.

**Example 4.5.** Consider the case $G = \text{GSp}(\Lambda)$. Recall that $\rk_{\mathbb{Z}[\omega]}(\Lambda) = 2g$. We endow $\Lambda^*$ with the symplectic pairing $\psi^*$ corresponding to the symplectic pairing $\psi$ on $\Lambda$ (i.e., if $\psi$ is given by an isomorphism $\Lambda \to \Lambda^\ast$, then $\psi^*$ is given by its inverse $\Lambda^\ast \to \Lambda$). Then $\text{GL}(\Lambda) \to \text{GL}(\Lambda^\ast)$, $g \mapsto g^\vee$ induces an isomorphism $\text{GSp}(\Lambda) \to \text{GSp}(\Lambda^\ast)$

We have $\kappa = \mathbb{F}_p$ and $(\cdot)^\vee \circ \chi = (\cdot)^\vee \circ \chi^\ast$ defines a decomposition $\Lambda^\ast_{\mathbb{F}_p} = (\Lambda^\ast_{\mathbb{F}_p})_0 \oplus (\Lambda^\ast_{\mathbb{F}_p})_1$ in totally isotropic subspaces. Hence $P_+$ and $P_{-}$ are opposite parabolic subgroups whose common Levi subgroup is the stabilizer of the grading of $\Lambda^\ast_{\mathbb{F}_p}$.

By [PWZ2] 8.4, a $\text{GSp}(\Lambda^\ast_{\mathbb{F}_p})$-zip of type $(\cdot)^\vee \circ \chi$ (resp. a $\text{GSp}(\Lambda_{\mathbb{F}_p})$-zip of type $\chi$) over an $\mathbb{F}_p$-scheme $S$ may be interpreted as a triple $(M, \mathcal{L}, E)$ consisting of an $F$-zip $M$ over $S$ of rank $2g$ of type $\tau^\vee$ with $\tau^\vee(i) = g$ for $i = 0, 1$ (resp. of type $\tau$ with $\tau(i) = g$ for $i = -1, 0$), of an $F$-zip $\mathcal{L}$ of $S$ of rank 1 and an admissible epimorphism $E: \Lambda^2 \otimes M \to \mathcal{L}$ such that corresponding morphism of $F$-zips $E: M \to M^\vee \otimes \mathcal{L}$ is an isomorphism. Here $M^\vee$ denotes the dual $F$-zip (in the sense of [PWZ2] Definition 6.5) and the tensor product is in the tensor category of $F$-zips over $S$ ([PWZ2] Definition 6.4).

The correspondence in [PWZ2] 8.4 shows that via functoriality of $G$-zips ([Zha2] Theorem 0.2) the isomorphism $\text{GSp}(\Lambda) \to \text{GSp}(\Lambda^\ast$) yields an equivalence between $\text{GSp}(\Lambda_{\mathbb{F}_p})$-zips of type $\chi$ and $\text{GSp}(\Lambda^\ast_{\mathbb{F}_p})$-zip of type $(\cdot)^\vee \circ \chi$ which is given by attaching to a
GSp(Λ_p)-zip (M, L, E) of type χ the GSp(Λ_p)-zip of type ( )^∨χ given by M^∨, L^∨, E^{-1}).

Here E^{-1} : \bigwedge^2(M^∨) \to L^∨ is the symplectic pairing given by

\[ \tilde{E}^{-1} \circ \text{id}_{L^∨} : M^∨ = M^∨ \otimes L^∨ \otimes L^∨ \cong M \otimes L^∨ = (M^∨)^∨ \otimes L^∨. \]

Let V_S be the first De Rham cohomology of the universal abelian scheme over the special fiber \( \bar{S} := S^p(GSp(\Lambda), S^±) \). The Hodge filtration, the conjugate filtration, and the Cartier isomorphism define a natural structure of an F-zip \( \nabla_S \) on \( V_S \) (MoWd §7).

The universal equivalence class \( \lambda \) of prime to \( p \) quasi-isogenies yields a class of symplectic pairings on \( V_S \) up to multiplication with a locally constant function with values in \( \mathbb{F}_p \).

We choose a pairing \( \gamma \) in this class. This yields by loc. cit. a morphism \( \tilde{\lambda}^\sharp(\nabla_S) \to \mathbb{I}(1) \) of F-zips, where \( \mathbb{I}(1) \) denotes the Tate-F-zip of weight 1 (PWZ2 Example 6.6). One obtains a GSp(Λ_p)-zip of type ( )^∨χ. Its dual (\( \nabla_S^\vee, \mathbb{I}(-1), \gamma^{-1} \)) as explained above is then the GSp(Λ_p)-zips of type χ defining the morphism

\[ \zeta : \bar{S} \longrightarrow \text{GSp}(\Lambda_p) - \text{Zip}^\chi. \]

**Proposition 4.6.** Assume that all Ekedahl-Oort strata are non-empty (e.g., if the Shimura variety is of PEL type, cf. Remark [4]). Then the free rank of Pic(G-Zip^χ) is the number of Ekedahl-Oort strata of codimension one.

**Proof.** The rank of Pic(G-Zip^χ) is the number of irreducible components of \( G - U \), where \( U \) is the open E-orbit in \( G \) (Proposition ??). \( \square \)

**Remark and Definition 4.7.** As \( \zeta \) is open, the preimage of the generic point of G-Zip^χ is open and dense in \( S^\mu(\mathbb{G}, \chi) \). Moreover, Wortmann has shown ([?] Theorem 6.10) that this generic Ekedahl-Oort stratum is the \( \mu \)-ordinary stratum. We denote it by \( S^\mu-\text{ord} = S^\mu-\text{ord}(\mathbb{G}, \chi) \).

### 4.5 Bruhat stratification

We continue to assume \( P \) and \( Q \) contain both a Borel subgroup of \( G_\kappa \) which is already defined over \( \mathbb{F}_p \). We call

\[ \mathcal{B}_G = [P \backslash G_\kappa / Q] \]

the Bruhat stack attached to \((G, \chi)\). Let

\[ (4.8) \quad \beta_G : G-\text{Zip}^\chi \longrightarrow \mathcal{B}_G^\chi \]

be the canonical morphism.

**Example 4.8.** We set \( \bar{\Lambda} := \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p \) endowed with the induced symplectic form and write by \( 2g = \text{dim}_p(\bar{\Lambda}) \). Then \( \bar{\chi} := \iota \circ \chi \) is GSp(\(\bar{\Lambda})\)-conjugate to a cocharacter of GSp(\(\Lambda)\), which is defined over \( \mathbb{F}_p \). As explained in Example [14], \( \bar{\chi} \) defines a cocharacter of GSp(\(\Lambda)\) which has only weights \(-1 \) and \( 0 \) on \( \Lambda_\kappa \) and such that the weight decomposition \( \Lambda_\kappa = \Lambda_{-1} \oplus \Lambda_0 \) is a decomposition into totally isotropic subspaces.

For every \( \kappa \)-scheme, \( \mathcal{B}(\text{GSp}(\Lambda), \bar{\chi})(S) \) can be identified with the groupoid of triples \((\mathcal{M}, L, E, C, D)\), where \( \mathcal{M} \) is a finite locally free \( \mathcal{O}_S \)-module of rank \( 2g \), \( L \) is a finite
locally free $\mathcal{O}_S$-module of rank 1, $E$: $\bigwedge^2(M) \to \mathcal{L}$ is an alternating pairing such that the corresponding homomorphism $\tilde{E}: M \to \mathcal{M}^\vee \otimes \mathcal{L}$ is an isomorphism, and where $\mathcal{C}, \mathcal{D} \subseteq \mathcal{M}$ are Lagrangian submodules (i.e., $\tilde{E}$ induces isomorphisms $\mathcal{C} \to \mathcal{C}^\perp \otimes \mathcal{L}$ and $\mathcal{D} \to \mathcal{D}^\perp \otimes \mathcal{L}$).

Using [Wd2] Example 2.11 one sees that the map

$$\beta_{GSp(\bar{\Lambda})}: GSp(A_F) - \text{Zip}^\chi \to \mathscr{B}(GSp(\bar{\Lambda}), \bar{\chi})$$

is given by attaching to a $GSp(\bar{\Lambda})$-zip $(M, L, E)$ of type $\chi$ (in the sense of [4.5]) the tuple $(M, L, E, C, D)$, where $C = C^0(M)$ and $D = D^{-1}(M)$.

Hence by Example [4.5] the composition

$$\beta_{GSp(\bar{\Lambda})} \circ \zeta_{GSp(\Lambda)}: \bar{S} := S_K(GSp(\Lambda), S^\pm) \to \mathscr{B}^\text{ox}_{GSp(\bar{\Lambda})}$$

is given by the tuple $(V_{\bar{S}}^\vee, \mathbb{I}(-1), \gamma^{-1}, \mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ (resp. $\mathcal{D}$) is the orthogonal complement of the Hodge filtration (resp. the conjugate filtration) in $V_{\bar{S}}$.

We have a diagram of morphisms of algebraic stacks over $\kappa$.

\[
\begin{array}{ccc}
S_K(G, X) & \xrightarrow{\zeta_G} & G\text{-Zip}^\chi \\
\varepsilon \downarrow & & \beta_G \downarrow \\
S_K(GSp(\Lambda), S^\pm) & \xrightarrow{\zeta_{GSp(\Lambda)}} & GSp(\bar{\Lambda}) - \text{Zip}^\text{ox} \\
\end{array}
\]

where the middle and the right vertical arrows are induced by functoriality by the embedding $\iota: G \to GSp(\bar{\Lambda})$.

**Lemma 4.9.** This diagram is commutative.

**Proof.** This is clear for the right square and follows from Theorem 0.2 of [Zha2] for the left square. \qed

### 4.6 Hasse invariants for the $\mu$-ordinary locus

Let $(M, L, E, \mathcal{C}, \mathcal{D})$ be the universal tuple over $\mathscr{B}^\text{ox}_{GSp(\bar{\Lambda})}$. Let

$$\omega_{GSp(\Lambda)}^\mu := \det(\mathcal{C}) \in \text{Pic}(\mathscr{B}^\text{ox}_{GSp(\Lambda)})$$

**Proposition 4.10.** The pullback of $\omega_{GSp(\Lambda)}^\mu$ under $\delta := \beta_{GSp(\Lambda)} \circ \zeta_{GSp(\Lambda)}$ is isomorphic to the Hodge line bundle on $S_K(GSp(\Lambda), S^\pm)$.

**Proof.** Example [4.8] shows that $\delta^* \mathcal{M} \cong V_{\bar{S}}^\vee$. Moreover, $E$ induces an isomorphism $\mathcal{M} \to \mathcal{M}^\vee \otimes \mathcal{L}$ and hence $\det(\mathcal{M}^\vee) \cong \det(\mathcal{M}) \otimes \mathcal{L}^{-1}$. Moreover $\delta^*(\mathcal{C}) = (V_{\bar{S}}/\mathcal{H})^\vee$, where $\mathcal{H} \subset V_{\bar{S}}$ is the Hodge filtration. Hence the Hodge line bundle is given by

$$\det(\mathcal{H}) \cong \det(V_{\bar{S}}) \otimes \det((V/\mathcal{H})^\vee) \cong \delta^*(\det(\mathcal{M}) \otimes \mathcal{L}^{-1} \otimes \det(\mathcal{C})).$$

\[41\]
Now $\det(M)$ is the line bundle attached to the determinant character of $\operatorname{GSp}(\Lambda)$, and $L$ is the line bundle attached to the multiplier character of $\operatorname{GSp}(\Lambda)$. Hence $\det(M) \cong L^g$. Moreover $\delta^* L$ is the underlying line bundle of the Tate $F$-zip $\mathbb{P}(-1)$ and hence is trivial. Therefore $\delta^* (\det M)$ is also trivial. This proves the claim. \hfill $\square$

As the universal abelian scheme $\mathcal{A}$ over $\mathcal{S}_K(G, X)$ is defined as the pull back of the universal abelian scheme over $\mathcal{S}_K^*(\operatorname{GSp}(\Lambda), S^\pm)$, the same holds for the Hodge line bundle of $\mathcal{A}$. Hence the commutativity of (4.9) shows that the pull back of

$$\omega^b_G := \iota^b_G (\omega_{\operatorname{GSp}(\Lambda)})$$

under $\beta_G \circ \zeta_G : S_K(G, X) \to B^X_G$ is the Hodge line bundle $\omega_{\mathcal{S}|S_K(G, X)}$. We define a line bundle on $G\text{-Zip}^X$ by

$$\omega^\lambda_G := \beta_G^*(\omega^b_G).$$

It pullback to $S_K(G, X)$ is the Hodge line bundle on $S_K(G, X)$. By Proposition 1.16 we have

$$(4.10) \quad \dim \Gamma(G\text{-Zip}^X, (\omega^\lambda_G)^{\otimes r}) \leq 1$$

for all $r \in \mathbb{Z}$.

**Definition 4.11.** Let $S_0$ be the reduced stabilizer of $1$ (see Subsection 2.5). The exponent of its character group $X^*(S_0)$ is called the Hasse number.

**Theorem 4.12.** Let $N$ be the Hasse number. Then for every integer $d \geq 1$ we have

$$\dim \Gamma(G\text{-Zip}^X, (\omega^\lambda_G)^{\otimes Nd}) = 1.$$ 

For every non-zero section $H \in \Gamma(G\text{-Zip}^X, (\omega^\lambda_G)^{\otimes Nd})$ the non-vanishing locus of $\zeta_G^*(H) \in \Gamma(S_K(G, X), (\omega^\lambda_G)^{\otimes Nd})$ is the $\mu$-ordinary locus $S^\mu$ of $S_K(G, X)$.

We call any section of the form $\zeta^*_G(H)$ as above a Hasse invariant (of parallel weight $Nd$).

*Proof.* By (4.10) it suffices to consider the case $d = 1$. Let $\bar{P} = P_+ (\iota \circ \chi)$ be the parabolic of $\bar{G} := \operatorname{GSp}(\bar{\Lambda})$ defined by $\iota \circ \chi$. The pullback of $\det(C)$ to $\bar{G}/\bar{P}$ is anti-ample. Hence its restriction to $G/P$ is anti-ample. As $B^- \subseteq P$, $\det(C)$ is defined by a character $\lambda$ such that $(\lambda, \beta^\nu)$ for all simple coroots $\beta^\nu$ corresponding to elements of $I \setminus J$ (Proposition 3.3). Here $J$ is the type of $P$. Equivalently, $\lambda$ is a linear combination with positive coefficients of the fundamental weights of the maximal parabolics containing $P$. Moreover, $P$ is defined by a minuscule cocharacter and in particular by a small cocharacter. Hence we can apply Theorem 3.3, which shows that div$(H)$ is the complement of the generic point in $G\text{-Zip}^X$. As $\zeta_G$ is flat (even smooth) and dominant, the vanishing locus of $\zeta^*_G(H)$ is the complement of the $\mu$-ordinary stratum in $S_K(G, X)$. \hfill $\square$

**Corollary 4.13.** Assume that $S_K(G, X)$ is projective, then the $\mu$-ordinary stratum $S^\mu$ is affine.
Proof. This follows from Theorem 4.12 because $\omega_S$ is ample by Proposition 4.1. \hfill \Box

Remark 4.14. Madapusi Pera has shown that $S_K(G,X)$ is projective if and only if $G^{\text{ad}}$ is an anisotropic group over $\mathbb{Q}$ (Per 4.4.7).

More generally, assume that $G^{\text{ad}}$ has no factor isomorphic to $\text{PGL}_2(\mathbb{Q})$. Then every Hasse invariant $H$ of weight $k$ extends uniquely to a section in $\Gamma(S^{\text{min}} \otimes \kappa, (\omega^{\text{min}}_S)^{\otimes k})$. We call its non-vanishing locus the $\mu$-ordinary locus of $S^{\text{min}}$ and denote it by $S^{\text{min},\mu}$.

Lemma 4.15. The $\mu$-ordinary locus $S^{\text{min},\mu}$ does not depend on the choice of the Hasse invariant.

Proof. Let $H$ and $H'$ are Hasse invariants of weight $k$ and $k'$, respectively. We may assume that $k' = dk$ for some $d \geq 1$. Then $H^\otimes d$ and $H'$ differ only by a non-zero scalar. \hfill \Box

As $\omega^{\text{min}}_S$ is ample and $S^{\text{min}}$ is projective, we deduce again:

Corollary 4.16. The $\mu$-ordinary locus $S^{\text{min},\mu}$ of the minimal compactification is affine.

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