Remote Empirical Coordination

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Abstract—We apply the framework of imperfect empirical coordination to a two-node setup where the action $X$ of the first node is not observed directly but via $L$ agents who observe independently impaired measurements $\hat{X}$ of the action. These $L$ agents, using a rate-limited coordination that is available to all of them, help the second node to generate the action $Y$ in order to establish the desired coordinated behaviour. When $L < \infty$, we prove that it suffices $R_i \geq I(\hat{X}; \hat{Y})$ for at least one agent whereas for $L \to \infty$, we show that it suffices $R_i \geq I(\hat{X}; \hat{Y}|X)$ for all agents where $\hat{Y}$ is a random variable such that $X - \hat{X} - \hat{Y}$ and $\|p_{X,Y}(x,y) - p_{X,Y}(x,y)\|_{TV} \leq \Delta$ ( $\Delta$ is the pre-specified fidelity).

I. INTRODUCTION

The development of machine to machine communication and the Internet of Things has enabled a renewed interest in further investigating heterogeneous network topologies where various objects are allowed to be interconnected. Such objects may be for instance computers with different operating systems and protocols, embedded sensors, medical devices, smart meters, and autonomous vehicles. A key factor to elucidate further insights of such network topologies is to study the cooperation and coordination of the different devices in the network on the level of information theory.

In many practical scenarios, there is no direct access to the source data of some phenomenon due to possible technical limitations. In this case, multiple agents can be deployed to collect noisy measurements of the source. Examples include the remote source coding problem introduced in [1] (see also [2],[3]) and the CEO problem introduced in [4]. Here, we adopt the concept of the “remote source” to the framework of “imperfect” empirical coordination [5] using also ideas from the framework of “perfect” empirical coordination [6].

The notion of empirical coordination in information theory was formalized in [6]. According to [6], when we are given the actions of some nodes by nature, empirical coordination is achieved if the joint type, measured by total variation distance, of the actions of all nodes in a network is close to the desired distribution, in probability. The literature on empirical coordination is vast. For instance, the authors in [7], [8] studied empirical coordination for various network topologies, whereas in [9] empirical coordination was established using polar coding and distributed approximation. This type of coordination is also used with ideas from other fields, such as game theory [10], optimal control [11] and networked control systems [12]. The framework of empirical coordination of [6] was recently extended to the more general framework of imperfect empirical coordination in [5] who was inspired by [13]. According to [5], imperfect empirical coordination is established if the total variation between the joint type of the actions in a network comes close, on average, to a desired distribution within distance pre-specified by a threshold $\Delta$. The choice of $\Delta$ regulates the coordination rates between the agents and therefore the system’s designer can choose to coordinate in a range of rates depending on the available rate budget. Clearly, if we choose $\Delta = 0$, then, we obtain as a special case the perfect empirical coordination of [6]. The result in [5] was applied to a multiple description problem with two channels in [14].

In this work, we consider the setup illustrated in Fig. 1. In this setup, the action of the first node, which is distributed according to $p_X$, is partially observed via multiple agents who then communicate via multiple rate-limited links to the second node. In particular, the $L$ agents collect independently noisy versions of the action, distributed according to $p_X$, and, by applying the coordination code, communicate to the second node. Based on the messages that it receives, the second node produces the action $Y$. Through our framework, we claim that imperfect empirical coordination is an appropriate approach to study coordination of nodes which do not directly communicate. This is because, by definition, the metric to achieve perfect empirical coordination can only be satisfied if the desired distributions satisfy the Markov chain $X - \hat{X} - \hat{Y}$.
which is not a necessary requirement in imperfect empirical cooperation due to the flexibility of our achievable performance criterion. Our achievability results rely on [5] Theorem 1] and we break our derivations in two parts. First, in sections III and IV, we give a lower bound of the coordination capacity region for the problem of perfect empirical coordination. Second, in section V, we apply [5] Theorem 1 to get a lower bound of the rate-distortion-coordination region. It is noteworthy to point out that our results are obtained for 

\[ L < \infty \] and when \( L \to \infty \).

II. GENERAL DEFINITIONS

We begin with some basic mathematical concepts and the definition of the coordination code i.e., the protocol which is used to coordinate the nodes of the network. We denote as \( X \) the (common) alphabet of random variables \( X \) and \( X \) and as \( Y \) the (common) alphabet of \( Y \) and \( Y \).

**Definition 1 (Joint type):** The joint type \( P_{x^n,y^n} \) of a tuple of sequences \( (x^n, y^n) \) is the empirical probability mass function, given by

\[ P_{x^n,y^n} (x,y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}((x_i, y_i) = (x, y)), \]

for all \( (x, y) \in X \times Y \), where \( 1 \) is the indicator function.

**Definition 2 (Total variation):** The total variation between two probability mass functions (PMF) is given by

\[ \|p(x) - q(x)\|_{TV} = \frac{1}{2} \sum_{x} |p(x) - q(x)|. \]

**Definition 3 (\( \Delta \)-neighborhood):** The \( \Delta \)-neighborhood of a PMF \( p(x) \) is defined as

\[ N_{\Delta}(p(x)) = \{q(x) : \|p(x) - q(x)\|_{TV} \leq \Delta\}. \]

**Definition 4 (Coordination code):** The \( (2^nR_1, 2^nR_2, \ldots, 2^nR_L, n) \) coordination code for our set-up consists of \( L + 1 \) functions-L encoding functions

\[ i_l : X^n \to \{1, \ldots, 2^nR_l\}, l = 1, \ldots, L, \]

and a decoding function

\[ y^n : \{1, \ldots, 2^nR_1\} \times \cdots \times \{1, \ldots, 2^nR_L\} \to Y^n. \]

In our set-up, the actions \( X^n \) and \( \hat{X}_l^n \) for \( l = 1, \ldots, L \) are chosen by nature to be i.i.d according to \( p_{X,\hat{X}_1,\ldots,\hat{X}_L}(x, \hat{x}_1, \ldots, \hat{x}_L) = p_0(x) \prod_{l=1}^{L} p_{X|X}(\hat{x}_l|x). \) Thus, \( X^n \) and \( \hat{X}_l^n \) for \( l = 1, \ldots, L \) are distributed according to a product distribution \( (X^n, \hat{X}_1^n, \ldots, \hat{X}_L^n) \sim \prod_{l=1}^{n} p_0(x_i) \prod_{l=1}^{L} p_{X|X}(\hat{x}_l|x_i). \) The action \( Y^n \) is function of \( \hat{X}_1^n, \ldots, \hat{X}_L^n \) given by \( Y^n = y^n(i_1(\hat{X}_1^n), \ldots, i_L(\hat{X}_L^n)). \)

III. FINITE NUMBER OF AGENTS

In this section, we give and discuss an inner bound of the coordination capacity region for the case of finite \( L \). For a proof, see Appendix A. We begin with the required definitions.

**Definition 5 (Achievability for perfect coordination and \( L \) finite):** A desired PMF \( p_{X,Y}(x,y) \) is achievable for empirical coordination with the rates \( (R_1, \ldots, R_L) \) if there exists a sequence \( (2^nR_1, \ldots, 2^nR_L, n) \) \( L \)-encoding codes such that as \( n \to \infty \)

\[ \|P_{x^n,y^n}(x,y) - p_0(x)p_{Y|X}(y|x)\|_{TV} \to 0, \]

in probability.

**Definition 6 (Coordination capacity region for \( L \) finite):** The coordination capacity region \( C_{p_{X,\hat{X}_1,\ldots,\hat{X}_L}} \) for the source-agent joint PMF \( p_{X,\hat{X}_1,\ldots,\hat{X}_L}(x,\hat{x}_1,\ldots,\hat{x}_L) = p_0(x) \prod_{l=1}^{L} p_{X|X}(\hat{x}_l|x) \) is the closure of the set of rate-coordination tuples \( (R_1, R_2, \ldots, R_L, p_{Y|X}(y|x)) \) that are achievable:

\[ C_{p_{X,\hat{X}_1,\ldots,\hat{X}_L}} = \{ (R_1, \ldots, R_L, p_{Y|X}(y|x)) : \text{is achievable at rates} (R_1, \ldots, R_L) \}. \]

**Theorem 1:**

\[ C_{p_{X,\hat{X}_1,\ldots,\hat{X}_L}} = \{ (R_1, \ldots, R_L, p_{Y|X}(y|x)) : \exists l \text{ such that} R_l \geq I(\hat{X};Y) \}. \]

**Proof:** See Appendix A.

**Remark 1:** According to Theorem 1 in the case of \( L < \infty \), the PMFs \( p_{X,Y}(x,y) = p_0(x)p_{Y|X}(y|x) \) which form a markov chain \( X - \hat{X} - Y \), are achievable if the rate of at least one agent exceeds the mutual information between \( \hat{X} \) and \( Y \). Although we do not prove an outer bound, it seems to us that, if the number of agents is finite and the rate of all of them is under the threshold of \( I(\hat{X};Y) \), the establishment of perfect empirical coordination is impossible i.e., that is optimal to deactivate all but one agent with rate at least equal to \( I(\hat{X};Y) \). On the other hand, as we will see in the next section, if the number of agents allowed to become arbitrarily large, then, the joint decoding is becoming gainful and we can distribute the rate among the different agents in order to satisfy the coordination criterion.

IV. INFINITE NUMBER OF AGENTS

In this section, we give and discuss an inner bound of the coordination capacity region for the case of \( L \to \infty \). For a proof, see Appendix A. We begin with the required definitions.

**Definition 7 (Achievability for perfect coordination and \( L \to \infty \):** A desired PMF \( p_{X,Y}(x,y) = p_0(x)p_{Y|X}(y|x) \) is achievable for empirical coordination with the rate per
agent $R_{ag}$ if there exists a sequence of $(2^n R_{ag}, \ldots, 2^n R_{ag}, n)$ coordination codes such that as $L \to \infty$ and $n \to \infty$,
\[
\| P_{x^n,y^n} (x, y, z) - p_0 (x) p_{Y|X} (y|x) \|_{TV} \to 0, \tag{2}
\]
in probability.

Remark 2: The double convergence in Definition 7 should be interpreted as $L \to \infty$ first, followed by $n \to \infty$. See proof of Theorem 2 (in Appendix A).

Definition 8 (Coordination capacity region for $L \to \infty$): The coordination capacity region $C^P_{p_X,X}$ for the source-agent PMF $p_{X,Y} (x, y) \triangleq p_0 (x) p_{Y|X} (y|x)$ is the closure of the set of rate-coordination tuples $(R_{ag}, p_{Y|X} (y|x))$ that are achievable:
\[
C^P_{p_X,X} \triangleq \text{Cl} \left\{ (R_{ag}, p_{Y|X} (y|x)) : R_{ag} \geq I \left( \hat{X}; Y|X \right) \right\}.
\]

Theorem 2:
\[
C^P_{p_X,X} \supseteq \text{Cl} \left\{ (R_{ag}, p_{Y|X} (y|x)) : X - \hat{X} - Y,\right\}
\]

Proof: See Appendix A.

Remark 3: According to Theorem 2 in the case of $L \to \infty$, the PMFs $p_{X,Y} (x, y) \triangleq p_0 (x) p_{Y|X} (y|x)$ which form a Markov chain $X - \hat{X} - Y$, are achievable if every agent has rate at least equal to $I \left( \hat{X}; Y|X \right)$, which of course is smaller or equal to $I \left( \hat{X}; Y \right)$ due to the Markovian property. In other words, the arbitrarily large number of agents allows us to get rid of the constraint $R_i \geq I \left( \hat{X}; Y \right)$ for at least one agent.

V. IMPERFECT EMPIRICAL COORDINATION

In this section, we combine the inner bounds from the previous two sections with [5, Theorem 1] in order to get inner bounds for the rate-distortion-coordination region, both in the cases of $L$ finite and $L \to \infty$.

A. Finite number of agents

Definition 9 (Achievability for $\Delta$-empirical coordination and $L$ finite): A desired PMF $p_{X,Y} (x, y) \triangleq p_0 (x) p_{Y|X} (y|x)$ is achievable for $\Delta$-empirical coordination with the rate-pair $(R_1, \ldots, R_L)$ if there is an $N$ such that for all $n > N$, there exists a coordination code $(2^n R_1, \ldots, 2^n R_L, n)$ such that
\[
\mathbb{E} \{ \| P_{x^n,y^n} (x, y) - p_0 (x) p_{Y|X} (y|x) \|_{TV} \} \leq \Delta.
\]

Definition 10 (Rate-distortion-coordination region for $L$ finite): The rate-distortion-coordination region $R^I_{p_X,X_1,\ldots,X_L}$ for the source-agent PMF $p_{X,X_1,\ldots,X_L} (x, \hat{x}_1, \ldots, \hat{x}_L) \triangleq p_0 (x) \prod_{i=1}^{L} p_{X|X} (\hat{x}_i|x)$ and for a fixed conditional distribution $p_{Y|X} (y|x)$ is defined as:
\[
R^I_{p_X,X_1,\ldots,X_L} \left( p_{Y|X} (y|x) \right) \triangleq \text{Cl} \left\{ (R_1, \ldots, R_L, \Delta) : p_0 (x) p_{Y|X} (y|x) \right\}.
\]

Lemma 1: For every source-agent PMF $p_{X,X_1,\ldots,X_L} (x, \hat{x}_1, \ldots, \hat{x}_L) = p_0 (x) \prod_{i=1}^{L} p_{X|X} (\hat{x}_i|x)$ and for every fixed conditional PMF $p_{Y|X} (y|x)$:
\[
R^I_{p_X,X_1,\ldots,X_L} \left( p_{Y|X} (y|x) \right) \supseteq \left\{ (R_1, \ldots, R_L, \Delta) : \right\}
\]
\[
\left\{ \begin{array}{l}
R \left( p_0 q_{Y|X} \right) \cup \bigcup_{p_0 (x) q_{Y|X} (y|x) \in N\Delta \left( p_0 (x) p_{Y|X} (y|x) \right)}
\end{array} \right\}.
\]

Proof: This Lemma is a direct consequence of a more general result which is explained and proved in [5]. See also, Fig. 2 and Fig. 3.

Theorem 3: For every source-agent PMF $p_{X,X_1,\ldots,X_L} (x, \hat{x}_1, \ldots, \hat{x}_L) = p_0 (x) \prod_{i=1}^{L} p_{X|X} (\hat{x}_i|x)$ and for every fixed conditional PMF $p_{Y|X} (y|x)$:
\[
R^I_{p_X,X_1,\ldots,X_L} \left( p_{Y|X} (y|x) \right) \supseteq \begin{array}{l}
\begin{array}{l}
R \left( p_0 q_{Y|X} \right) \cup \bigcup_{p_0 (x) q_{Y|X} (y|x) \in N\Delta \left( p_0 (x) p_{Y|X} (y|x) \right)}
\end{array} \right\}
\end{array}.
\]

Proof: From Theorem 1 and Lemma 1 we obtain the characterization of the theorem.

Remark 4: For $L = 1$, the previous theorem together with a simple converse give [13, Theorem 1].

B. Infinite number of agents

Definition 11 (Achievability for $\Delta$-empirical coordination and $L \to \infty$): A desired PMF $p_{X,Y} (x, y) \triangleq p_0 (x) p_{Y|X} (y|x)$ is achievable for $\Delta$-empirical coordination with the rate per agent $R_{ag}$ if there is an $L$ and an $N$ such that for all $L > \bar{L}$ and $n > N$, there exists a coordination code $(2^n R_1, \ldots, 2^n R_L, n)$ such that
\[
\mathbb{E} \{ \| P_{x^n,y^n} (x, y) - p_0 (x) p_{Y|X} (y|x) \|_{TV} \} \leq \Delta.
\]
For every source-agent PMF $p_{X,\hat{X}}(x,\hat{x}) = p_0(x)p_{\hat{X}|X}(\hat{x}|x)$ and for every fixed conditional PMF $p_{Y|X}(y|x)$,

$$R^I_{p_{X,\hat{X}}}(p_{Y|X}(y|x)) \triangleq \min_{\Delta} \left\{ (R_{ag}, \Delta) : p_0(x)p_{Y|X}(y|x) \text{ is achievable for } \Delta \text{-empirical coordination at rate per agent } R_{ag} \right\}.$$
can bound the probability of Case (a) as \( \Pr(\text{Case a}) \leq \sum_{l=1}^{L} \delta_l (n, \epsilon', \mathbb{X} \times \mathbb{X}) \). In case (b), we get

\[
\Pr(\text{Case b}) \leq \prod_{l=1}^{L} \Pr \left( \left\{ (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right\} \right)
\]

\[
\cap \left\{ \exists l : (\hat{x}_l, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right\} \right) 
\]

\[
= \prod_{l=1}^{L} \Pr \left( (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right)
\]

\[
\cdot \Pr \left( \exists l : (\hat{x}_l, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right)
\]

\[
\leq \prod_{l=1}^{L} \prod_{w(l)=1}^{\left\lceil e^{(R_{l}+\epsilon_{l})} \right\rceil} \Pr \left( \hat{x}_l, Y(l)^{(w(l))} \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right)
\]

\[
\left\{ \hat{x}_l \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X}) \right\} \cap \left\{ x \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}} | \hat{x}_l) \right\} \right) 
\]

\[
= \prod_{l=1}^{L} \prod_{w(l)=1}^{\left\lceil e^{(R_{l}+\epsilon_{l})} \right\rceil} \Pr \left( Y(l)^{(w(l))} \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y} | x_l) \right)
\]

\[
\left\{ \hat{x}_l \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X}) \right\}
\]

\[
= \prod_{l=1}^{L} \prod_{w(l)=1}^{\left\lceil e^{(R_{l}+\epsilon_{l})} \right\rceil} \left( 1 - e^{-n(I(\hat{X};Y)+2\epsilon_{m})} \right) \tag{6}
\]

\[
\leq \prod_{l=1}^{L} \left( 1 - e^{-n(I(\hat{X};Y)+2\epsilon_{m})} \right) \tag{7}
\]

\[
= \prod_{l=1}^{L} \exp \left( - e^{n(R_{l}+\epsilon_{l})} \right)
\cdot \left( 1 - e^{-n(I(\hat{X};Y)+2\epsilon_{m})} \right) \tag{8}
\]

\[
= \prod_{l=1}^{L} \exp \left( - e^{n(R_{l}+I(\hat{X};Y)+\epsilon_{l}-\delta_{l})} \right),
\]

where \( \delta_{l} \) accounts for the rounding mistake and includes the \( 2\epsilon_{m} \)-term and the \( (1 - \delta_{l}) \)-factor. So, we see that as long as \( n \) is large enough, \( R_{l} \geq I(\hat{X};Y) \) for at least one \( l \) and \( \epsilon \) small enough such that \( \delta_{l} < \epsilon_{l} \) for this \( l \), the probability \( \Pr(\text{Case b}) \) tends to zero double-exponentially fast in \( n \). Here, (4) results from Lemma 4 (in Appendix B), (5) follows again from Lemma 4 (in Appendix B) and because we discard irrelevant information, (6) follows from Lemma 8 (in Appendix B), (7) holds because the factor in the product does not depend on \( w \) anymore and (8) follows from Lemma 5 (in Appendix B). In case (c),

\[
\Pr(\text{Case c}) \leq \Pr \left( \left\{ (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \mid \forall l \right\} \cap \{ L \text{ is not empty} \}
\]

\[
\cap \left\{ \exists l : (x, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right\} \right) \right) 
\]

\[
\leq \Pr \left( \exists : (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right)
\]

\[
\cap \left\{ (x, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right\} \right) 
\]

\[
\leq \Pr \left( \exists : (x, \hat{x}_l, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right)
\]

\[
\cap \left\{ (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right\} \right) 
\]

\[
\leq \sum_{l=1}^{L} \Pr \left( \left\{ (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right\} \right)
\]

\[
\leq 1 - \Pr \left( (x, \hat{x}_l, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right)
\]

\[
\left\{ (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right\} \right) 
\]

\[
= \sum_{l=1}^{L} \left( 1 - \Pr \left( (x, \hat{x}_l, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right) \right)
\]

\[
\leq \sum_{l=1}^{L} \Pr \left( (x, \hat{x}_l, Y(l)^{(w(l))}) \notin A_{e_{\epsilon_{l}}}^{(n)}(p_{X,Y}) \right)
\]

\[
\left\{ (x, \hat{x}_l) \in A_{e_{\epsilon_{l}}}^{(n)}(p_{X, \hat{X}}) \right\} \right) 
\]

\[
\leq \sum_{l=1}^{L} \delta_l (n, \epsilon/2, \mathbb{X} \times \mathbb{X} \times \mathbb{Y}),
\]
where (9) follows because, due to the fact that (strong) joint typicality implies pairwise typicality, we conflate the set and the last step follows from Lemma 7 (in Appendix B).

**Proof of Theorem 2:**

- **Setup:** We assume that $\epsilon_{ag} > 0$ is given. We fix some rates per agent $R_{ag}$ and $R'_{ag}$, some blocklength $n$, some $\epsilon > 0$, $\epsilon_0 > 0$ and for every PMF $p_{X,Y}^n = p_X p_Y | X p_Y | X$ compute the marginal $p_Y$.

- **Codebook design:** Generate $[e^{n(R_{ag} + \epsilon_0)}] [e^{n(R'_{ag} - \epsilon_0)}]$ length-$n$ codewords $Y(i) = 1, \ldots, e^{n(R_{ag} + \epsilon_0)}, v(i) = 1, \ldots, e^{n(R'_{ag} - \epsilon_0)}$, by choosing each of the $[e^{n(R_{ag} + \epsilon_0)}] [e^{n(R'_{ag} - \epsilon_0)}]$ symbols $Y(i) = v(i)$ independently at random according to $p_Y$ for $l = 1, \ldots, L$.

- **Encoder Design:** For given sequences $x, x_1, \ldots, x_L$, the $l$-th encoder tries to find a pair $(w(i), v(i))$ such that

$$\left( \hat{x}_l, Y(i) \left( w(i), v(i) \right) \right) \in A_{x}^{(nL)}(p_{X,Y}) . \tag{10}$$

If it finds several possible choices, it picks the first. If it finds none, it declares an error. The $l$-th encoder puts out $w(i)$. (10)

- **Decoder Design:** The decoder $y^n$ based on the bin numbers $(w(1), \ldots, w(L))$ that receives, it tries to find a tuple $(v(1), \ldots, v(L))$ and an $x$ such that $Y^n = \left( Y(1) \left( w(1), v(1) \right), \ldots, Y(L) \left( w(L), v(L) \right) \right)$ and $x^n = \left( x, \ldots, x \right)$ to be jointly typical i.e.,

$$\left( x^n, Y^n \right) \in A_{x}^{(nL)}(p_{X,Y}) . \tag{11}$$

If it finds more than one $(v(1), \ldots, v(L))$ or none, it declares an error. Otherwise, it chooses some $j$ and puts out $Y(j) \left( w(j), v(j) \right)$.

- **Performance Analysis:** We define $\epsilon' = \frac{\epsilon}{2n^2}$ and partition the error space into four disjoint cases: (a) $(x, \hat{x}_l) \notin A_{x}^{(nL)}(p_{X,Y})$ for some $l$ (b) $(x, \hat{x}_l) \in A_{x}^{(nL)}(p_{X,Y})$ for every $l$ but at least one encoder declares an error (c) $(x, \hat{x}_l) \in A_{x}^{(nL)}(p_{X,Y})$ for every $l$, all encoders do not declare an error but the decoder finds none $(v(1), \ldots, v(L))$ (event $C_b$) or more than one (event $C_b$) (d) $(x, \hat{x}_l) \in A_{x}^{(nL)}(p_{X,Y})$ for every $l$, all encoders do not declare an error and the decoder finds exactly one $(v(1), \ldots, v(L))$ but (2) is not satisfied. By the Union Bound and Lemma 5 (in Appendix B), we get $\Pr \left( \text{Case a} \right) \leq \sum_{l=1}^{L} \delta_l (n, \epsilon, \epsilon, X \times X)$. Easily, it follows that $\Pr \left( \text{Case b} \right) \leq \sum_{l=1}^{L} \exp \left( -e^{n(R_{ag} + R'_{ag} - l) (X,Y) + \epsilon_0 - \delta_l} \right)$, where $\delta_l$ accounts for the rounding mistake and includes the $2\epsilon_0$-term and the $(1 - \delta_l)$-factor. Hence, we see that as long as $n$ is large enough, $R_{ag} + R'_{ag} \geq I(\hat{X}; Y)$ and $\epsilon$ small enough such that $\delta < \epsilon_{ag}$, the probability $\Pr \left( \text{Case b} \right)$ tends to zero double-exponentially fast in $n$. In case (c), we have $\Pr \left( \text{Case c} \right) = \Pr \left( C_b \cup C_b \right) = \Pr \left( C_b \right) + \Pr \left( C_b - C_b \right)$. By (10), the fact that $(x, \hat{x}_l) \in A_{x}^{(nL)}(p_{X,Y})$ for every $l$ and the simple properties $P_{\hat{X}}^{nL \times Y}(Y^n_l) = \frac{1}{L} \sum_{l=1}^{L} P_{\hat{X}}^{nL \times Y}(Y^n_l)$, $P_{\hat{X}}^{nL \times Y}(Y^n_l) = \frac{1}{L} \sum_{l=1}^{L} P_{\hat{X}}^{nL \times Y}(Y^n_l)$, it follows that $(\hat{x}^n_l, Y^n_l) \in A_{x}^{(nL)}(p_{X,Y})$ and $(\hat{x}^n_l, \hat{x}^n_l) \in A_{x}^{(nL)}(p_{X,Y})$ where $\hat{x}^n_l = (\hat{x}_1, \ldots, \hat{x}_L)$. Lemma 7 (in Appendix B) gives us that $\Pr \left( C_b \right) \leq \delta_l (n, \epsilon, \epsilon, X \times X \times Y)$. We proceed with the event $C_b - C_b$. The cardinality of the set $Y^n_l \subseteq nL \subseteq Y^n_l$ of the codewords which satisfy (11) is bounded as $|Y^n_l| \leq |A_{x}^{(nL)}(p_{X,Y})| \max_{x^n \in \hat{A}_{x}^{(nL)}}(p_{X,Y})$ $\leq e^{n(H(X)+\epsilon_0)} e^{nL(H(Y)+\epsilon_0)} \leq e^{nL(H(Y)+4\epsilon_0)}$, where the second inequality follows from Lemma 5 and Lemma 6 (in Appendix B) and the last inequality is true for $L \geq H(X)/\epsilon_0$. The probability for each element of this set to be chosen to a specific bin-tuple $(w(1), \ldots, w(L))$ is due to Lemma 7 (in Appendix B) at most $e^{-nL(H(Y)+\epsilon_0)} e^{nL(R_{ag} - \epsilon_0)} \leq e^{-nL(H(Y)+\epsilon_0)} e^{nL(R_{ag} - \epsilon_0)}$. Hence, combining these two gives us that $\Pr \left( C_b - C_b \right) \leq e^{-nL(1/H(X) - R_{ag} - 5\epsilon_0)}$. Therefore, we see that as long as $n$ is large enough, $L \geq H(X)/\epsilon_0, R_{ag} \leq I(\hat{X}; Y)$ and $\epsilon$ small enough such that $\epsilon_0 < \epsilon_0/5$, the probability $\Pr \left( \text{Case c} \right)$ tends to zero exponentially. Lemma 7 (in Appendix B) guarantees again that $\Pr \left( \text{Case d} \right)$ decays. So, collecting all the cases together gives us the desired result.

**APPENDIX B**

**TYPICAL SETS**

**Lemma 3:** $\forall \theta > 0, \forall \xi \leq 1 : (1 - \xi)^{\theta} \leq e^{\theta \xi}$.

**Definition 13 (Strongly $\epsilon$-typical sets [24]):**

$$A_{x}^{(n)}(p_{X,Y}) \triangleq \left\{ (x,y) \in \mathbb{X}^n \times \mathbb{Y}^n : \left| p_{x,y} (a,b) - p_{x,y} (a,b) \right| < \frac{\epsilon}{|\mathbb{X}|^2}, \forall(a,b) \in \mathbb{X} \times \mathbb{Y} \right\} .$$

$$A_{x}^{(n)}(p_{x,y}|x) \triangleq \left\{ y \in \mathbb{Y}^n : (x,y) \in A_{x}^{(n)}(p_{X,Y}) \right\} .$$

**Lemma 4:** $X \in A_{x}^{(n)}(p_{X,Y}) \Rightarrow \{ (X,Y) \in A_{x}^{(n)}(p_{X,Y}) \}$

$$\{ X \in A_{x}^{(n)}(p_{X,Y}) \} \cap \{ Y \in A_{x}^{(n)}(p_{X,Y}|X) \} .$$

**Definition 14:** $\epsilon_m, \epsilon_0, \epsilon, \mathbb{X} \times \mathbb{Y}$

$$- \log \left( p_{x,y}^{(m)} \right), \delta_l (n, \epsilon, \mathbb{X} \times \mathbb{Y}) \triangleq \frac{(n + 1) |\mathbb{X}|^2 e^{-n \frac{\epsilon}{|\mathbb{X}|^2} \log \epsilon}}{|\mathbb{X}|^2 |\mathbb{Y}|^2} ,$$

where $p_{x,y}^{(m)}$ is the smallest value of $p_{x,y}(x,y)$.
Lemma 5 (\cite{14}): Let \((x, y) \in A^*(x, y)\). Then, 
\[
e^{-n(H(X; Y) + \epsilon_m(p_{X,Y}(x,y)))} < p^\epsilon_{X,Y}(x, y) < e^{-n(H(X; Y) - \epsilon_m(p_{X,Y}(x,y)))}.
\]
Moreover, 
\[
1 - \delta_1(n, \epsilon, X \times Y) \leq \Pr[(x, y) \in A^*(x, y)] \leq 1 \text{ and } |A^*(x, y)| < e^{n(H(Y|X) + \epsilon_m(p_{X,Y}(x,y)))}.
\]

Lemma 6 (\cite{15}): Let \(A^*(n)\) be a joint PMF with marginals \(p_X(x), p_Y(y)\). Let \((x, y)\) be generated, then, we obtain 
\[
\Pr[Y \in A^*(n)(p_{X,Y}|x)] > (1 - \delta_1(n, \epsilon, X \times Y)) e^{-n(I(X; Y) + \epsilon_3)} \text{ where } \epsilon_3 \triangleq \epsilon_m(p_{X,Y}(x, y)) \leq 2\epsilon_m(p_{X,Y}(x, y)).
\]

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