Slow-roll, acceleration, the Big Rip and WKB approximation in NLS-type formulation of scalar field cosmology

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Aspects of non-linear Schrödinger-type (NLS) formulation of scalar (phantom) field cosmology on slow-roll, acceleration, WKB approximation and Big Rip singularity are presented. Slow-roll parameters for the curvature and barotropic density terms are introduced. We reexpress all slow-roll parameters, slow-roll conditions and acceleration condition in NLS form. WKB approximation in the NLS formulation is also discussed when simplifying to linear case. Most of the Schrödinger potentials in NLS formulation are very slowly-varying, hence WKB approximation is valid in the ranges. In the NLS form of Big Rip singularity, two quantities are infinity in stead of three. We also found that approaching the Big Rip, \(w_{\text{eff}} \to -1 + 2/3q\), \((q < 0)\) which is the same as effective phantom equation of state in the flat case.

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I. INTRODUCTION

Cosmology with scalar field is one of today research mainstreams. Although the scalar field has not yet been observed, it is motivated from many ideas in high energy physics and quantum gravities. Near future TeV scale experiments at LHC and Tevatron might discover its existence. It has been widely accepted in theoretical frameworks especially in model building of contemporary cosmology that the field sources acceleration expansion at early time, i.e. inflation, in order to solve horizon and flatness problems \([1]\) and it also plays similar role in explaining present acceleration observed and confirmed from cosmic microwave background \([2]\), large scale structure surveys \([3]\) and supernovae type Ia \([4, 5, 6]\). In the late acceleration, it plays the role of dark energy (see Ref. \([7]\) for reviews). Both inflation and acceleration are convinced by recent combined results \([8]\) with possibility that the scalar field could be phantom, i.e. having equation of state coefficient \(w_\phi < -1\). The phantom equation of state is attained from negative kinetic energy term in its Lagrangian density \([9, 10]\). Using BBN constraint of limit of expansion rate \(11, 12\), with most recent WMAP five-year result \(13\), \(w_{\phi,0} = -1.09 \pm 0.12\) at 68% CL. While WMAP five-year result combined with Baryon Acoustic Oscillation of large scale structure survey (from SDSS and 2dFGRS) \(14\) and type Ia supernovae data (from HST \([3]\), SNLS \([4]\) and ESSENCE \([15]\)) assuming dynamical \(\omega\) with flat universe yields \(-1.38 < w_{\phi,0} < -0.86\) at 95% CL and \(w_{\phi,0} = -1.12 \pm 0.13\) at 68% CL. Although the phantom field has its room from observation, in flat universe the idea suffers from unwanted Big Rip singularity \([16, 17]\). However there have been many attempts to resolve the singularity from both phenomenological and fundamental inspirations \([18]\).

Inflationary models in presence of other field behaving barotropic-like apart from having only single scalar field were considered such as in \([19]\) where the scale invariant spectrum in the cosmic microwave background was claimed to be generated not only from fluctuation of scalar field alone but rather from both scalar field and interaction between gravity to other gauge fields such as Dirac and gauge vector fields. This is similar to the situation in the late universe in which the acceleration happens in presence of both dark matter fluid and scalar fluid (as dark energy). Proposal of mathematical alternatives to the standard Friedmann canonical scalar field cosmology with barotropic perfect fluid, was raised, such as non-linear Ermakov-Pinney equation \([20, 21]\). There are also other applications of Ermakov-Pinney equation, for example in \([22]\), a link from standard cosmology with \(k > 0\) in Ermakov system to Bose-Einstein condensates was shown. Another example is a connection from generalized Ermakov-Pinney equation with perturbative scheme to generalized WKB method of comparison equation \([23]\). Recently a link from standard canonical scalar field cosmology in Friedmann-Lemaître-Robertson-Walker (FLRW) background with barotropic fluid to quantum mechanics is established. It was realized from the fact that solutions of generalized Ermakov-Pinney equation are correspondent to solutions of the non-linear Schrödinger-type equation, hereafter NLS equation \([21, 24]\). Connection from the NLS-type formulation to Friedmann scalar field cosmology formulation is concluded in Ref. \([25]\) where standard cosmological quantities are reinterpreted in the language of quantum mechanics assuming power-law expansion, \(a \sim t^n\) and the phantom field case is included. The quantities in the new form satisfies a non-linear Schrödinger-
type equation. In most circumstance, the scalar field exact solution $\phi(t)$ can be solved analytically only when assuming flat geometry ($k = 0$) and scalar field fluid domination. When $k \neq 0$ with more than one fluid component, the system is not always possible to solve analytically in standard Friedmann formulation. Transforming standard Friedmann cosmological quantities into NLS forms could help solving for the solution \[26, 27\]. In the NLS formulation, the independent variable $t$ in standard formulation is re-scaled to variable $x$. However, pre-knowledge of scale factor as function of time, $a(t)$, must be presumed in order to express NLS quantities. It is interesting to see the other features of field velocity, $\dot{\phi}$, e.g. acceleration condition, slow-roll approximation, written in NLS formulation. Mathematical tools such as WKB approximation in quantum mechanics might also be interesting for application in standard scalar field cosmology. It is worthwhile to investigate this possibility. It is worth noting that Schrödinger-type equation in scalar field cosmology was previously considered in different procedure to study inflation and phantom field problems \[28\].

We introduce the NLS formulation in Sec. II. The slow-roll conditions in both formulations are discussed in Sec. II where we define slow-roll parameters for barotropic fluid and curvature terms. Then in Sec. IV we show acceleration conditions in NLS form. The WKB approximation is performed in Sec. V. The NLS form of Big Rip singularity is in Sec. VI and finally conclusion is made in Sec. VII.

II. SCALAR FIELD COSMOLOGY IN NLS FORMULATION

Two perfect fluids are considered in FLRW universe: barotropic fluid and scalar field. The barotropic equation of state is $p_\gamma = w_\gamma \rho_\gamma$ with $w_\gamma$ expressed as $n$ where $n = 3(1+w_\gamma)$. The scalar field pressure obeys $p_\phi = \omega_\phi \rho_\phi$. Total density and pressure of the mixture are sum of the two components. Evolution of barotropic density is governed by conservation equation, $\dot{\rho}_\gamma = -nH\rho_\gamma$, with solution, $\rho_\gamma = D/a^n$, where $a$ is scale factor, the dot denotes time derivative, $D \geq 0$ is a proportional constant. Using scalar field Lagrangian density, $\mathcal{L} = (1/2)\dot{\phi}^2 - V(\phi)$, i.e. minimally coupling to gravity,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

(1)

The branch $\epsilon = 1$ is for non-phantom case and $\epsilon = -1$ is for phantom case \[17\]. Note that the phantom behavior ($\rho_\phi < -p_\phi$) can also be obtained in non-minimal coupling to gravity case \[29\]. Dynamics of the field is controlled by conservation equation

$$\epsilon \left( \dot{\phi} + 3H\phi \right) = \frac{dV}{d\phi}.$$ 

(2)

The spatial expansion of the universe sources friction to dynamics of the field in Eq. \[2\] via the Hubble parameter $H$. The Hubble parameter is governed by Friedmann equation,

$$H^2 = \frac{k^2}{3}\rho_{tot} - \frac{k}{a^2},$$

(3)

where here $\rho_{tot} = (1/2)\epsilon\dot{\phi}^2 + V + D/a^n$ and by acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{k^2}{6}(\rho_{tot} + 3p_{tot}),$$

(4)

which does not depend on $k$. This gives acceleration condition

$$p_{tot} < -\frac{\rho_{tot}}{3}.$$ 

(5)

Here $p_{tot} = w_{eff}\rho_{tot}, k^2 \equiv 8\pi G = 1/M_p^2$, $G$ is Newton’s gravitational constant, $M_p$ is reduced Planck mass, $k$ is spatial curvature and $w_{eff} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{tot}$. Using these facts, it is straightforward to show that

$$\dot{\phi}(t)^2 = -\frac{2}{\kappa} \left[ H - \frac{k}{\kappa^2 a^2} \right] - \frac{nD}{3a^n}.$$ 

(6)

$$V(\phi) = \frac{3}{\kappa^2} \left[ H^2 + \frac{H}{3} + \frac{2k}{3a^2} \right] + \left( \frac{n-6}{6} \right) \frac{D}{a^{n}}.$$ 

(7)

The Friedmann formulation of scalar field cosmology above can be transformed to the NLS formulation as one defines NLS quantities \[24\],

$$u(x) = a(t)^{-n/2},$$

(8)

$$E \equiv -\frac{\kappa^2 n^2 D}{12},$$

(9)

$$P(x) = \frac{\kappa^2 n}{4} a(t)^{n} \epsilon \dot{\phi}(t)^2.$$ 

(10)

In the NLS formulation, there is no such analogous equations to Friedmann equation or fluid equation since both of them are written together in form of a non-linear Schrödinger-type equation

$$u''(x) + [E - P(x)] u(x) = -\frac{nk}{2} u(x)^{(4-n)/n},$$

(11)

where $'$ denotes d/dx. Independent variable $t$ is scaled to NLS independent variable $x$ as $x = \sigma(t)$, such that

$$\dot{\phi}(t) = u(x),$$

(12)

$$\epsilon \psi'(x)^2 = \frac{4}{\kappa^2 n} P(x).$$ 

(13)

Using Eq. \[10\] and $\epsilon \dot{\phi}(t)^2 = \epsilon \dot{x}'(x)^2$, we get \[25\]

$$\epsilon \psi'(x)^2 = \frac{4}{\kappa^2 n} P(x),$$ 

(14)

1 NLS equation considered here is only $x$-dependent hence it is not partial differential equation with localized soliton-like solution as in \[30\].
hence

\[ \psi(x) = \pm \frac{2}{\kappa \sqrt{n}} \int_x^\infty \frac{P(x)}{\epsilon} \, dx. \]  

(15)

Inverse function \( \psi^{-1}(x) \) exists for \( P(x) \neq 0 \) and \( n \neq 0 \). In this circumstance, \( x(t) = \psi^{-1} \circ \phi(t) \) and the scalar field potential, \( V \circ \sigma^{-1}(x) \) and \( e \dot{\phi}(t)^2 \) can be expressed in NLS formulation as

\[ e \dot{\phi}(x)^2 = \frac{4}{\kappa^2 n^2} uu'' + \frac{2k}{\kappa^2 n^2} u^{4/n} + \frac{4E}{\kappa^2 n^2} u^2 = \frac{4P}{\kappa^2 n^2} u^2, \]  

(16)

\[ V(x) = \frac{12}{\kappa^2 n^2} (u')^2 - \frac{2P}{\kappa^2 n^2} u^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}. \]  

(17)

From Eqs. (16) and (17), we can find

\[ \rho_\phi = \frac{12}{\kappa^2 n^2} (u')^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}, \]  

(18)

\[ p_\phi = -\frac{12}{\kappa^2 n^2} (u')^2 + \frac{4P}{\kappa^2 n^2} u^2 - \frac{12E}{\kappa^2 n^2} u^2 - \frac{3k}{\kappa^2} u^{4/n}. \]  

(19)

We know that \( \rho_\gamma = Du^2 = -12Ev^2/(\kappa^2 n^2) \) from Eq. (10) and the barotropic pressure is \( p_\gamma = [(n-3)/3]\rho_\gamma \), therefore

\[ \rho_{tot} = \frac{12}{\kappa^2 n^2} (u')^2 + \frac{3k}{\kappa^2} u^{4/n}, \]  

(20)

\[ p_{tot} = -\frac{12}{\kappa^2 n^2} (u')^2 + \frac{4P}{\kappa^2 n^2} u^2 - \frac{k}{\kappa^2} u^{4/n}. \]  

(21)

Using the Schrödinger-type equation (11), then

\[ p_{tot} = -\frac{12}{\kappa^2 n^2} (u')^2 + \frac{4}{\kappa^2 n^2} uu'' - \frac{k}{\kappa^2} u^{4/n}. \]  

(22)

This results in an approximation \( |\dot{H}| \ll H^2 \) from which the slow-roll parameter,

\[ \varepsilon \equiv -\frac{\dot{H}}{H^2} \]  

(26)

is defined. Then the condition \( |\dot{\phi}^2/2| \ll V(\phi) \) is equivalent to \( |\varepsilon| \ll 1 \), i.e. \(-1 \ll \varepsilon < 0 \) for phantom field case and \( 0 < \varepsilon \ll 1 \) for non-phantom field case. For the non-phantom field, this condition is necessary for inflation to happen (though not sufficient) \[32\] but for the phantom field case, the slow-roll condition is not needed because the negative kinetic term results in acceleration with \( w_\phi \leq -1 \). The other slow-roll parameter is defined by balancing magnitude of the field friction and acceleration terms in Eq. (14). This is independent of \( k \) or \( \rho_\gamma \).

When friction dominates \( |\dot{\phi}| \ll |3H\phi| \), then

\[ \eta \equiv -\frac{\dot{\phi}}{H\phi}. \]  

(27)

is defined \[32\]. The condition is then \( |\eta| \ll 1 \) and the fluid equation is approximated to \( \dot{\phi} \sim -V_\phi/3eH \) which allows the field to roll up the hill when \( \varepsilon = -1 \). Using both conditions, e.g. \( |\dot{\phi}^2/2| \ll V \) and \( |\dot{\phi}| \ll |3H\phi| \) together, one can derive \( \varepsilon = (1/2\kappa^2\epsilon)(V_\phi/V)^2 \) and \( \eta = (1/\kappa^2)(V_\phi/V) \) as well-known.

### III. SLOW-ROLL CONDITIONS

#### A. Slow-roll conditions: flat geometry and scalar field domination

In flat universe with scalar field domination (\( k = 0, \rho_\gamma = 0 \)), the Friedmann equation \( H^2 = \kappa^2 \rho_\phi/3 \), together with the Eq. (2) yield \( \dot{H} = -\kappa^2 \dot{\phi}^2 \epsilon/2 \). For \( \epsilon = -1 \), we get \( \dot{H} > 0 \) and

\[ 0 < \alpha H^2 < \bar{a}, \]  

(23)

i.e. the acceleration is greater than speed of expansion per Hubble radius, \( a/cH^{-1} \). On the other hand, for \( \epsilon = 1 \), we get \( \dot{H} < 0 \) and

\[ 0 < \bar{a} < a H^2. \]  

(24)

Slow-roll condition in \[31,32\] assumes negligible kinetic term hence \( |\dot{\phi}/2| \ll V(\phi) \), therefore \( \rho_\phi \simeq V(\phi) \) hence \( H^2 \simeq \kappa^2 V/3 \). With this approximation,

\[ H^2 = \frac{\dot{H}}{3} + \frac{\kappa^2}{3} V, \Rightarrow \quad H^2 \simeq -\frac{\dot{H}}{3} + H^2. \]  

(25)

These slow-roll conditions are:

1. **Friedmann formulation**

When considering the case of \( k \neq 0 \) and \( \rho_\gamma \neq 0 \), then

\[ \dot{H} = -\frac{\kappa^2}{2} \dot{\phi}^2 \epsilon + \frac{k}{a^2} - \frac{nk^2 D}{6 a^n}. \]  

(28)

We can then write slow-roll condition as: \( |\kappa^2 \dot{\phi}^2/6| \ll (\kappa^2 V/3) - (k/a^2) + (\kappa^2 D/3a^n) \) and hence \( H^2 \simeq (\kappa^2 V/3) + (\kappa^2 D/3a^n) - (k/a^2) \). Using this approximation and Eq. (28) in (3),

\[ H^2 \simeq -\frac{\dot{H}}{3} + \frac{k}{3a^2} - \frac{nk^2 D}{18 a^n} + H^2, \]  

(29)

which implies \( -(\dot{H}/3) + (k/3a^2) - (nk^2 D/18a^n) \ll H^2 \).

We can reexpress this slow-roll condition as

\[ |\varepsilon + \varepsilon_k + \varepsilon_D| \ll 1, \]  

(30)

where \( \varepsilon_k \equiv k/a^2 H^2 \) and \( \varepsilon_D \equiv -nk^2 D/6a^n H^2 \). Another slow-roll parameter \( \eta \) is defined as \( \eta \equiv -\dot{\phi}/H \dot{\phi} \), i.e. the same as the flat scalar field dominated case since the condition \( |\dot{\phi}| \ll |3H\dot{\phi}| \) is derived from fluid equation of the field which is independent of \( k \) and \( \rho_\gamma \).
2. NLS formulation

In NLS formulation, the Hubble parameter takes the form

\[ H = -\frac{2}{n} u', \quad (31) \]

with

\[ \dot{H} = -\frac{2}{n} uu'' = \frac{2}{n} u^2 [E - P(x)] + ku^{4/n}. \quad (32) \]

The slow-roll condition \( |\epsilon_0^2/2| \ll V \) using Eqs. (10) and (17) in NLS form, is then

\[ |P(x)| \ll \frac{3}{n} \left[ \left( \frac{u'}{u} \right)^2 + E \right] + \frac{3}{4} k n u^{(4-2n)/n}. \quad (33) \]

If the absolute sign is not used, the condition is then \( \epsilon_0^2/2 \ll V \), allowing fast-roll negative kinetic energy. Then Eq. (33), when combined with the NLS equation (11), yields

\[ u'' \ll \frac{3}{n} \frac{u'^2}{u} + \left( \frac{3}{n} - 1 \right) E u + \frac{k n}{4} u^{(4-n)/n}. \quad (34) \]

Friedmann formulation analog of this condition can be obtained simply by using Eqs. (33) and (7) in the condition. Consider another aspect of slow-roll in the fluid equation, the field acceleration can be written in NLS form:

\[ \dot{\phi} = \frac{2Puu' + P'u^2}{\kappa \sqrt{Pcn}}, \quad (35) \]

while the friction term in NLS form is

\[ 3H \dot{\phi} = -\frac{12u'u}{nk} \sqrt{\frac{P}{cn}}. \quad (36) \]

The second slow-roll condition, \( |\dot{\phi}| \ll |3H \dot{\phi}| \) hence corresponds to

\[ \left| \frac{P'}{P} \right| \ll -2 \left( \frac{6 + n}{n} \right) \frac{u'}{u}. \quad (37) \]

This condition yields the approximation \( 3H \epsilon \dot{\phi}^2 \simeq -dV/d\phi \). Using Eqs. (10), (17), (31) and (32), one can express the approximation, \( 3H \epsilon \dot{\phi}^2 \simeq -dV/d\phi \), in NLS form as

\[ \frac{P'}{P} \simeq -\frac{2n}{u} = nH a^{n/2}. \quad (38) \]

and finally the slow-roll parameters \( \epsilon, \epsilon_k, \) and \( \epsilon_D \), introduced previously, become

\[ \epsilon = \frac{n uu''}{2u'^2}, \quad \epsilon_k = \frac{n^2 k a^{4/n}}{4u'^2}, \quad \epsilon_D = \frac{n E}{2} \left( \frac{u'}{u} \right)^2. \quad (39) \]

in NLS form. With help of NLS equation (11), summation of the slow-roll parameters takes simple form,

\[ \epsilon_{tot} = \epsilon + \epsilon_k + \epsilon_D = \frac{n}{2} \left( \frac{u'}{u} \right)^2 P(x). \quad (40) \]

Finally the slow-roll condition, \( |\epsilon_{tot}| < 1 \) (Eq. (30)), in NLS form, is

\[ \left| \left( \frac{u'}{u} \right)^2 P(x) \right| < 1. \quad (41) \]

Another slow-roll parameter \( \eta = -\dot{\phi}/H \dot{\phi} \) can be found as follow. First considering \( \psi(x) = \phi(t) \) (Eq. (6)), using relation \( d/dt = \dot{x} d/dx \) and Eq. (31), we obtain

\[ \eta = n \frac{2}{k} \left( \frac{u}{u'} \right)^2 P(x) + 1. \quad (42) \]

The Eq. (35) yields

\[ \psi' = \pm \frac{2}{\kappa} \sqrt{\frac{P}{n\epsilon}} \quad \text{and} \quad \psi'' = \pm \frac{P'}{\kappa \sqrt{n P \epsilon}}. \quad (43) \]

Hence

\[ \eta = \frac{n}{2} \left( \frac{\frac{P'}{u'} P + 1}{2P + 1} \right). \quad (44) \]

At last, the slow-roll condition \( |\eta| < 1 \) then reads

\[ \left| \frac{u}{u'} \frac{P'}{2P + 1} \right| < 1. \quad (45) \]

IV. ACCELERATION CONDITION

The slow-roll condition is useful for non-phantom field because it is a necessary condition for inflating acceleration. However, in case of phantom field, the kinetic term is always negative and could take any large negative values hence slow-roll condition is not necessary for acceleration condition. More generally, to ensure acceleration, the Eq. (4) must be positive. It is straightforward to show that, obeying acceleration condition, \( \ddot{a} > 0 \), the Eq. (5), takes the form,

\[ \epsilon \dot{\phi}(x)^2 < - \left( \frac{3}{2} \right) \frac{D}{a^n} + V. \quad (46) \]

With Eqs. (31), (9), (10), (17) and (11), the acceleration condition (10) in NLS-type formulation is

\[ E - P > -\frac{2}{n} \left( \frac{u'}{u} \right)^2 - \frac{2nk}{2} \left( \frac{u^{2/n}}{u} \right)^2. \quad (47) \]

With help of non-linear Schrödinger-type equation (11), it is simplified to

\[ u'' < \frac{2 u'^2}{n u}. \quad (48) \]

Using Eqs. (31) and (32), the acceleration condition is just \( \epsilon < 1 \) without using any slow-roll assumptions.
V. WKB APPROXIMATION

WKB approximation can be assumed when the coefficient of highest-order derivative term in the Schrödinger equation is small or when the potential is very slowly-varying. The Eq. (11), when $k = 0$, is linear. It is then

$$- \frac{1}{n} u'' + \left[ \tilde{P}(x) - \tilde{E} \right] u = 0 , \quad (49)$$

where $\tilde{P}(x) \equiv P(x)/n$ and $\tilde{E} \equiv E/n$. For a slowly-varying $P(x)$ with assumption of $n \gg 1$, the solution of Eq. (49) can be written as $u(x) \simeq A \exp(\pm i n W_0(x))$, where $W_0(x) = W(x_0)$ is the lowest-order term in Taylor expansion of the function $W(x)$ in $(1/n)$ about $x = x_0$.

$$W(x) = W(x_0) + W'(x_0) \frac{(x-x_0)}{n} + \ldots , \quad (50)$$

Then an approximation

$$W(x) = \pm \frac{1}{n} \int_{x_1}^{x_2} k(x) \, dx \simeq W_0(x) , \quad (51)$$

is made in analogous to the method in time-independent quantum mechanics. The Schrödinger wave number is hence

$$k(x) = \frac{2\pi}{\lambda(x)} = \sqrt{n \left[ \tilde{E} - \tilde{P}(x) \right]} , \quad (52)$$

and small variation in $\lambda(x)$ is

$$\frac{\delta \lambda}{\lambda(x)} = \left| \frac{\pi \tilde{P}'}{\sqrt{n \left[ \tilde{E} - \tilde{P}(x) \right]^{3/2}}} \right| = \left| \frac{\pi P'}{|E - P(x)|^{3/2}} \right| , \quad (53)$$

For WKB approximation, $\delta \lambda/\lambda(x) \ll 1$. In real universe, we have $n = 3$ (dust) or $n = 4$ (radiation) which is not much greater than one. However, if considering a range of very slowly-varying potential, $P' \simeq 0$ implying $\delta P/\delta x \sim 0$, hence $\delta k/\delta x \sim 0 \sim W'(x)$. Therefore $W'(x) \simeq W_0'(x)$ still holds in this range. Since $u(x) = a^{-n/2}$, using WKB approximation, we get

$$a \sim A \exp \left[ \pm (2/n) i \int_{x_1}^{x_2} \sqrt{E - P(x)} \, dx \right] , \quad (54)$$

where $A$ is a constant. Examples of Schrödinger potentials for exponential, power-law and phantom expansions are derived in [25, 26, 27]. These potentials are steep only in some small particular region but very slowly-varying in most regions, especially at large value of $|x|$ which are WKB-well valid.

VI. BIG RIP SINGULARITY

When the field becomes phantom, i.e. $\epsilon = -1$, in a flat FRLW universe, it leads to future Big Rip singularity. In flat universe, when $w_{\text{eff}} < -1$, i.e. being phantom, the expansion obeys $a(t) \sim (t_a - t)^q$, where $q = 2/3(1 + w_{\text{eff}}) < 0$ is a constant in time and $t_a$ is a finite time. The NLS phantom expansion was studied in Ref. [27] with inclusion of non-zero $k$ case. Therein, the same expansion function is assumed with constant $q < 0$ and $x$ is related to cosmic time scale, $t$ as $x(t) = (1/\beta) (t_a - t)^{-\beta} + x_0$, so that $u(x) = [\beta(x-x_0)]^\alpha$. Here $\alpha \equiv qn/(qn-2)$ and $\beta \equiv (qn-2)/2$ with conditions $0 < \alpha < 1$ and $\beta < -1$ since $n > 0$ always. The first and second $x$-derivative of $u$ are

$$u'(x) = \alpha \beta [\beta(x-x_0)]^{\alpha - 1} , \quad (55)$$

$$u''(x) = \alpha(\alpha - 1)\beta^2 [\beta(x-x_0)]^{\alpha - 2} , \quad (56)$$

where exponents $\alpha - 1$ and $\alpha - 2$ are always negative. Using Eqs. (20) and (22), then

$$\rho_{\text{tot}} = \frac{12\alpha^2 \beta^2}{\kappa^2 n^2} [\beta(x-x_0)]^{2(\alpha - 1)} + \frac{3k}{\kappa^2} \beta^2 [\beta(x-x_0)]^{4\alpha/n} , \quad (57)$$

$$p_{\text{tot}} = \frac{4\alpha^2}{\kappa^2 n^2} [\beta(x-x_0)]^{2(\alpha - 1)} \left[ \left( 1 - \frac{3}{n} \right) \alpha^2 - \alpha \right]$$

$$- \frac{k}{\kappa^2} \beta^2 [\beta(x-x_0)]^{4\alpha/n} . \quad (58)$$

The Big Rip: $(a, \rho_{\text{tot}}, |p_{\text{tot}}|) \to \infty$ happens when $t \to t_a^-$. In NLS formulation, if $a \to \infty$, then $u \to 0^+$ (Eq. 59). From above, we see that conditions of the Big Rip equation are

$$t \to t_a^- \quad \Leftrightarrow \quad x \to x_0^- , \quad (59)$$

$$a \to \infty \quad \Leftrightarrow \quad u(x) \to 0^+ , \quad (60)$$

$$\rho_{\text{tot}} \to \infty \quad \Leftrightarrow \quad u'(x) \to \infty , \quad (61)$$

The effective equation of state $w_{\text{eff}} = p_{\text{tot}}/\rho_{\text{tot}}$ can also be stated in NLS language as a function of $x$. Approaching the Big Rip, $x \to x_0^-$ and the effective equation of state approaches a value

$$\lim_{x \to x_0^-} w_{\text{eff}} = \frac{n}{3} \left( 1 - \frac{1}{\alpha} \right) - 1 = -1 + \frac{2}{3q} , \quad (62)$$

which is similar to the equation of state in flat case.

VII. CONCLUSIONS

We feature cosmological aspects of NLS formulation of scalar field cosmology such as slow-roll conditions, acceleration condition and the Big Rip. We conclude these

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2 The relation $q = 2/3(1 + w_{\text{eff}}) < 0$ holds only when $k = 0$.
3 Note that $(x - x_0)$ and $\beta$ are negative hence $(x - x_0)^\alpha$, $\beta^\alpha$, $(x - x_0)^{\alpha - 1}$ and $\beta^{\alpha - 1}$ are imaginary.
aspects in standard Friedmann formulation before deriv-
ing them in the NLS formulation. We consider a non-
flat FRW universe filled with scalar (phantom) field
and barotropic fluid because, in presence of barotropic
fluid density, the NLS-type formulation is consistent [23].
We obtain all NLS version of slow-roll parameters, slow-
roll conditions and acceleration condition. This provides
such analytical tools in the NLS formulation. For phan-
tom field, due to its negative kinetic term, the slow-roll
condition is not needed. When the NLS system is sim-
plified to a constant exponent of the expansion
$\omega_{\text{eff}} = 0$. In a flat universe with phantom expansion,
the Big Rip singularity is its final fate. When the Big Rip
happens, three quantities ($a(t), p(t)$ and $\rho(t)$) become
infinite. Rewriting the singularity in NLS form (Eq. (50)),
we can remove one infinite (see Eq. (50)). We found that
at near the Big Rip, $\omega_{\text{eff}} \rightarrow -1 + 2/3q$ where $q < 0$ is
a constant exponent of the expansion $a(t) \sim (t_a - t)^{\delta}$. This limit is the same as the effective phantom equation of state in the case $k = 0$.

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