Parallelisms and translations of (affine) SL(2, q)-unitals

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Abstract. Unitals can be obtained as closures of affine unitals via parallelisms. Affine SL(2, q)-unitals are affine unitals of order q admitting a regular action of SL(2, q). The construction of those affine unitals is due to Grundhöfer, Stroppel and Van Maldeghem and motivated by the action of SL(2, q) on the classical (Hermitian) unital. For affine SL(2, q)-unitals, we introduce a class of parallelisms for odd order and one for square order and compute their stabilizers. For each of the known parallelisms of affine SL(2, q)-unitals, we compute all translations with centers on the block at infinity.

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1. Introduction

The strategy of building the affine part of a geometry first and then completing it by using a parallelism on the blocks can successfully be applied to other incidence structures than affine and projective planes. We apply this approach to unitals. As opposed to the construction of projective planes from affine planes, a unital is not determined uniquely by its affine part.

SL(2, q)-unitals were first constructed by Grundhöfer, Stroppel and Van Maldeghem [3]. Inspired by the action of the special unitary group of degree 2 on the classical (Hermitian) unital, they give a general construction for unitals of prime power order q, where the points outside one block—i.e. the points of an affine part—are given by the elements of SL(2, q). Any such affine SL(2, q)-unital

Most of the results in the present paper have been obtained in the author’s Ph.D. thesis [7], where detailed arguments can be found for some statements that we leave to the reader here.
can be completed to an $SL(2,q)$-unital via a parallelism on the set of short blocks. In [3], Grundhöfer, Stroppel and Van Maldeghem introduce a non-classical affine $SL(2,4)$-unital and complete it via two different parallelisms to non-isomorphic $SL(2,4)$-unitals. In [8], we introduce three non-classical affine $SL(2,8)$-unitals which we complete to six pairwise non-isomorphic $SL(2,8)$-unitals.

In the present paper, we consider parallelisms of affine $SL(2,q)$-unitals. For each prime power $q$, there are two parallelisms known, namely $♭$ and $♮$ (see Page 4 for the definitions). Starting with the classical affine $SL(2,q)$-unital, those two parallelisms lead to the Grünig unital and to the classical unital, respectively. We introduce a class of parallelisms for each odd order (Theorem 3.3) and one for each square order (Theorem 3.7). We determine their stabilizers in the automorphism group of affine $SL(2,q)$-unitals (Theorems 3.4 and 3.11). By an exhaustive computer search using GAP [1], we find all possible parallelisms for the orders 3, 4 and 5, respectively (Sect. 3.3).

Of special interest are unitals of order $q$ where two points are centers of translation groups of order $q$ (see Definition 2.2). In the classical unital of order $q$, any two such translation groups generate a group isomorphic to $SL(2,q)$; see [2, Main Theorem] for further possibilities. We compute all possible translations with centers on the block at infinity for each of the known parallelisms of affine $SL(2,q)$-unitals. For the parallelism $♭$, each point on the block at infinity is a translation center (Lemma 4.2). For the parallelism $♮$, there are at most $q + 1$ non-trivial translations (Theorem 4.6). For our new classes of parallelisms for odd order and square order, respectively, there are no non-trivial translations with centers on the block at infinity (Theorem 4.6).

2. Basic definitions

A unital of order $n$ is a 2-$(n^3 + 1, n + 1, 1)$ design, i.e. an incidence structure with $n^3 + 1$ points, $n + 1$ points on each block and through any two points there is exactly one block. An affine unital is obtained from a unital by removing a block and the points incident with it. Any affine unital has two types of blocks, called short and long, respectively. A short block contains $n$ points and a long block contains $n + 1$ points. Any affine unital can be completed to a unital using a parallelism on the set of short blocks. We give an axiomatic description:

Definition 2.1. Let $n \in \mathbb{N}$, $n \geq 2$. An incidence structure $U = (\mathcal{P}, \mathcal{B}, I)$ is called an affine unital of order $n$ if:

(AU1) There are $n^3 - n$ points.

(AU2) Each block is incident with either $n$ or $n + 1$ points. The blocks incident with $n$ points will be called short blocks and the blocks incident with $n + 1$ points will be called long blocks.

(AU3) Each point is incident with $n^2$ blocks.
(AU4) For any two points there is exactly one block incident with both of them.

(AU5) There exists a parallelism on the short blocks, meaning a partition of the set of all short blocks into \( n + 1 \) parallel classes of size \( n^2 - 1 \) such that the blocks of each parallel class are pairwise non-intersecting.

Axioms (AU1)–(AU4) do not imply the existence of a parallelism as in (AU5) (see [7, Example 3.10]). We must hence explicitly require the existence of such a parallelism if we want an affine unital to be extendable to a unital. An affine unital \( U \) of order \( n \) with parallelism \( \pi \) can be completed to a unital \( U^{\pi} \) of order \( n \) as follows: For each parallel class, add a new point that is incident with each short block of that class. Then add a single new block \([\infty]^\pi\) (the block at infinity), incident with the \( n + 1 \) new points (see [7, Proposition 3.9]). We call \( U^{\pi} \) the \( \pi \)-closure of \( U \).

Note that though we must require the existence of a parallelism in the definition of an affine unital, this parallelism need not be unique. It is therefore reasonable not to require that isomorphisms of affine unitals respect certain parallelisms and we will only ask them to be isomorphisms of incidence structures. We call two parallelisms \( \pi \) and \( \pi' \) of an affine unital \( U \) equivalent if there is an automorphism of \( U \) which maps \( \pi \) to \( \pi' \).

Given an affine unital \( U \) with parallelisms \( \pi \) and \( \pi' \), the closures \( U^\pi \) and \( U^{\pi'} \) are isomorphic with \([\infty]^\pi \mapsto [\infty]^{\pi'}\) exactly if \( \pi \) and \( \pi' \) are equivalent (see [7, Proposition 3.12]).

**Definition 2.2.** A translation with center \( c \) of a unital \( U \) is an automorphism of \( U \) that fixes the point \( c \) and each block through \( c \). The group of all translations with center \( c \) will be denoted by \( G_{[c]} \).

We call \( c \) a translation center if \( G_{[c]} \) acts transitively on the set of points different from \( c \) on any block through \( c \).

### 3. Parallelisms of affine \( \text{SL}(2,q) \)-unitals

From now on let \( p \) be a prime and \( q := p^e \) a \( p \)-power. We define affine \( \text{SL}(2,q) \)-unitals:

Let \( S \subseteq \text{SL}(2,q) \) be a subgroup of order \( q + 1 \) and let \( T \subseteq \text{SL}(2,q) \) be a Sylow \( p \)-subgroup. Recall that \( T \) has order \( q \) (and thus trivial intersection with \( S \)), that any two conjugates \( T^h := h^{-1}Th, \ h \in \text{SL}(2,q) \), have trivial intersection unless they coincide and that there are \( q + 1 \) conjugates of \( T \).

Consider a collection \( D \) of subsets of \( \text{SL}(2,q) \) such that each set \( D \in D \) contains \( 1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), that \( \#D = q + 1 \) for each \( D \in D \), and the following properties hold:

(Q) For each \( D \in D \), the map

\[
(D \times D) \setminus \{(x, x) \mid x \in D\} \to \text{SL}(2,q), \quad (x, y) \mapsto xy^{-1},
\]
is injective, i.e. the set $D^* := \{ xy^{-1} \mid x, y \in D, x \neq y \}$ contains $q(q+1)$ elements.

(P) The system consisting of $S \setminus \{ \mathbb{1} \}$, all conjugates of $T \setminus \{ \mathbb{1} \}$ and all sets $D^*$ with $D \in D$ forms a partition of $SL(2, q) \setminus \{ \mathbb{1} \}$.

Set

$$\mathcal{P} := SL(2, q),$$

$$\mathcal{B} := \{ Sg \mid g \in SL(2, q) \} \cup \{ T^h g \mid h, g \in SL(2, q) \}$$

$$\cup \{ Dg \mid D \in D, g \in SL(2, q) \}$$

and let the incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$ be containment.

Then we call the incidence structure $U_{\mathcal{S}, \mathcal{D}} := (\mathcal{P}, \mathcal{B}, I)$ an affine $SL(2, q)$-unital. Each affine $SL(2, q)$-unital is indeed an affine unital of order $q$, see [7, Prop. 3.15]. If $\pi$ is a parallelism on the short blocks of an affine $SL(2, q)$-unital $U_{\mathcal{S}, \mathcal{D}}$, we call the $\pi$-closure $\mathcal{U}_{\mathcal{S}, \mathcal{D}}$ an $SL(2, q)$-($\pi$-)unital.

For the investigation of parallelisms of affine unitals, we only need to know what the short blocks look like. In any affine $SL(2, q)$-unital, the set of short blocks is the set of all right cosets of the Sylow $p$-subgroups. A parallelism as in (AU5) is a partition of the set of short blocks into $q + 1$ sets of $q^2 - 1$ pairwise non-intersecting blocks.

**Definition 3.1.** Let $\mathfrak{P}$ denote the set of all Sylow $p$-subgroups of $SL(2, q)$.

(a) We denote by $\mathfrak{G} := \{ Tg \mid T \in \mathfrak{P}, g \in SL(2, q) \}$ the set of short blocks of any affine $SL(2, q)$-unital.

(b) A parallelism on $\mathfrak{G}$ is a partition of $\mathfrak{G}$ into $q + 1$ sets of $q^2 - 1$ pairwise non-intersecting cosets.

Note that each right coset $Tg$ is a left coset $gT^g$ of a conjugate of $T$. Hence, for each prime power $q$, there are two obvious parallelisms on $\mathfrak{G}$: One parallelism is obtained by partitioning the set of short blocks into the sets of right cosets of the Sylow $p$-subgroups, and another one by partitioning the set of short blocks into the sets of left cosets of the Sylow $p$-subgroups. We name those two parallelisms “flat” and “natural”, respectively, and denote them by the corresponding musical signs

$$\flat := \{ \{ Tg \mid g \in SL(2, q) \} \mid T \in \mathfrak{P} \}$$

and

$$\natural := \{ \{ gT \mid g \in SL(2, q) \} \mid T \in \mathfrak{P} \}.$$

**Remark 3.2.** Inspired by the construction of $SL(2, q)$-unitals from affine $SL(2, q)$-unitals via different parallelisms, Nagy and Mezőfi created a method of constructing new incidence structures from old ones by removing a block and attaching it again in a different way, see [5]. They call this method paramodification and used it to construct many new unitals of orders 3 and 4, respectively. Their new unitals also comprise the unitals introduced in Sect. 3.3, since these are obtained from the classical unital of order 4 by paramodification.
On any affine unital, the full automorphism group is a subgroup of the group of permutations of the point set. For affine SL(2, q)-unitals, having SL(2, q) as point set, we consider two subgroups $R$ and $A$ of Sym(SL(2, q)), as follows. For $h \in$ SL(2, q), let

$$\rho_h : \text{SL}(2, q) \to \text{SL}(2, q), \ x \mapsto \rho h,$$

be the permutation induced by right multiplication with $h$ and let

$$R := \{ \rho_h \mid h \in \text{SL}(2, q) \} \leq \text{Sym}(\text{SL}(2, q)).$$

Let further $A \leq \text{Sym}(\text{SL}(2, q))$ be the permutation group induced by all automorphisms of SL(2, q). For each $a \in \text{GL}(2, q)$, conjugation by $a$ induces a permutation $\gamma_a \in A$:

$$\gamma_a : \text{SL}(2, q) \to \text{SL}(2, q), \ x \mapsto a^{-1}xa.$$

Note that each permutation in $A$ fixes 1, while no non-trivial permutation in $R$ fixes any point in SL(2, q). Note further that on any affine SL(2, q)-unital $U_{S, D}$, the group $R$ acts as group of automorphisms (regularly on the point set) while not every $\alpha \in A$ acts as automorphism on $U_{S, D}$. Consider the holomorph (see e. g. [10, p. 37])

$$A \rtimes R \leq \text{Sym}(\text{SL}(2, q))$$

of SL(2, q). We know that for $q \geq 3$, the full automorphism group of $U_{S, D}$ is a subgroup of $A \rtimes R$, more precisely a subgroup of $A_S \rtimes R$ (see [6, Theorem 3.3]). We are therefore interested in the stabilizers of our parallelisms under the action of $A \rtimes R$.

Regarding the parallelisms $♭$ and $♮$, we see that both are stabilized by $A \rtimes R$. Note that right multiplication with $h \in$ SL(2, q) fixes each parallel class of $♭$ while it acts on the parallel classes of $♮$ via conjugation on the Sylow $p$-subgroups $T \in \mathcal{P}$ (since $gTh = ghT^h$).

Example. (a) For each prime power $q$ we may choose $S = C$ to be cyclic of order $q + 1$ and $\mathcal{H}$ a set of blocks through 1 such that $U_{C, \mathcal{H}}$ is isomorphic to the affine part of the classical unital and the closure $U_{C, \mathcal{H}}^S$ is isomorphic to the classical unital. We call $U_{C, \mathcal{H}}$ the classical affine SL(2, q)-unital and $U_{C, \mathcal{H}}^S$ the classical SL(2, q)-unital. See [3, Example 3.1] or [7, Sect. 3.2.2] for details.

(b) Several non-classical affine SL(2, q)-unitals are known, namely one of order 4, described in [3], and three of order 8, described in [8, Sect. 3].

3.1. A class of parallelisms for odd order

For each odd prime power $q$, there is at least one class of parallelisms on $\mathcal{S}$ apart from $♭$ and $♮$. Let $q$ be odd throughout this section.

Let $T$ be a fixed Sylow $p$-subgroup of SL(2, q), namely

$$T := \{(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) \mid x \in \mathbb{F}_q \}.$$
The normalizer of $T$ in $\text{GL}(2, q)$ is the set of upper triangular matrices. Let $\mathbb{F}_q^\square$ denote the set of all squares in $\mathbb{F}_q$, let $\mathbb{F}_q^{\times, \square}$ denote the set of all squares in $\mathbb{F}_q^{\times} = \mathbb{F}_q \setminus \{0\}$ and $\mathbb{F}_q^{\times, \square}$ the set of all non-squares in $\mathbb{F}_q^{\times}$. Note that $\#\mathbb{F}_q^{\times, \square} = \#\mathbb{F}_q^\square = \frac{1}{2}(q - 1)$ since $q$ is odd. Let

$$\Omega := \{ g \in \text{SL}(2, q) \mid (0, 1) \cdot g \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \in \mathbb{F}_q^\square \} \quad \text{and} \quad \Omega^c := \text{SL}(2, q) \setminus \Omega.$$ 

Note that $\Omega$ is a union of $T$-cosets (left as well as right cosets) since

$$(0, 1) \cdot Tg \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = \{(0, 1) \cdot g \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \} = (0, 1) \cdot gT \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right).$$

Let further

$$A := \{ Tg \mid g \in \Omega \} \cup \{ gT \mid g \in \Omega^c \},$$

$$A' := \{ Tg \mid g \in \Omega^c \} \cup \{ gT \mid g \in \Omega \}$$

and $^1$

$$\pi_{\text{odd}} := \{ A^h \mid h \in \text{SL}(2, q) \}, \quad \pi'_{\text{odd}} := \{ A'^h \mid h \in \text{SL}(2, q) \}.$$

**Theorem 3.3.** For odd $q$, the sets $\pi_{\text{odd}}$ and $\pi'_{\text{odd}}$ are parallelisms on $\mathcal{S}$. With $v \in \mathbb{F}_q^{\times, \square}$, conjugation by $\left( \begin{smallmatrix} 1 \\ 0 \\ v \end{smallmatrix} \right)$ maps $\pi_{\text{odd}}$ to $\pi'_{\text{odd}}$.

**Proof.** Since $\Omega$ is a union of $T$-cosets, the cosets in $A$ are pairwise non-intersecting and we get

$$\#A = \frac{\#\Omega}{\#T} + \frac{\#\Omega^c}{\#T} = \frac{\#\text{SL}(2, q)}{\#T} = q^2 - 1.$$ 

Hence, $A$ is a set of $q^2 - 1$ pairwise non-intersecting short blocks.

Let $h \in N := N_{\text{SL}(2, q)}(T)$. Then $h$ is upper triangular, say $h := \left( \begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix} \right)$ with $a \in \mathbb{F}_q^{\times}$. For any $g \in \text{SL}(2, q)$, we have

$$(0, 1) \cdot g^h \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = (0, a) \cdot g \cdot \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) = a^2(0, 1) \cdot g \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right).$$

Since $a^2 \in \mathbb{F}_q^{\times, \square}$, we get $g^h \in \Omega$ exactly if $g \in \Omega$. Hence, conjugation by $h$ stabilizes $A$.

Now let $h \in \text{SL}(2, q) \setminus N$ and $Tg \in A$. Then $(Tg)^h = T^h g^h$ is not a right coset of $T$. Compute $T^h g^h = g^h (T^h) g^h = g^h T g^h$ and consider $gh \in N$. Then $h = g^{-1}n$ for an $n \in N$ and we have $g^h = g^n$. But since $g^n \in \Omega$ exactly if $g \in \Omega$ (as shown above), the coset $(Tg)^h$ is not contained in $A$. A similar consideration shows that $(gT)^h$ is not contained in $A$ if $gT \in A$ and $h \notin N$.

Let $g, h \in \text{SL}(2, q)$ and assume $A^h \cap A^g \neq \emptyset$. Then $A^{hg^{-1}} \cap A \neq \emptyset$ and we get $hg^{-1} \in N$, $A^{hg^{-1}} = A$ and hence $A^h = A^g$. Thus, $\pi_{\text{odd}} = \{ A^h \mid h \in \text{SL}(2, q) \}$ is indeed a partition of $\mathcal{S}$ into $\#\text{SL}(2, q)/\#N = q + 1$ sets of $q^2 - 1$ pairwise non-intersecting cosets, hence a parallelism on $\mathcal{S}$.

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$^1$ In [7], the parallelism $\pi_{\text{odd}}$ is called $\pi^\square$ and $\pi'_{\text{odd}}$ is called $\square \pi$.  

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Now let \( v \in \mathbb{F}_q^\times \sqcup \mathbb{I} \) and \( f := (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \). Then \( f \in N_{GL(2,q)}(T) \) and \( g^f \in N \) exactly if \( g \in N \). Further, for any \( g \in SL(2,q) \setminus N \), we have
\[
(0,1) \cdot g^f \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = (0, v^{-1}) \cdot g \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = v^{-1}(0,1) \cdot g \cdot \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right).
\]
Hence, \( g^f \in \Omega \setminus N \) exactly if \( g \in \Omega^c \) and we get \( A^f = A' \) (note that \( N \subseteq \Omega \)).

For each \( A^h \in \pi_{\text{odd}} \), we get
\[
(A^h)^f = A^h = (A')^{(h\cdot f)} = A'^{(h\cdot f)} \in \pi'_{\text{odd}}.
\]
Hence, we obtain \( \pi'_{\text{odd}} \) from \( \pi_{\text{odd}} \) via conjugation by \( f \), and \( \pi'_{\text{odd}} \) is a parallelism on \( \mathcal{S} \) since \( \pi_{\text{odd}} \) is a parallelism on \( \mathcal{S} \).

As mentioned above and according to \([6, \text{Theorem 3.3}]\), the full automorphism group of any affine \( SL(2,q) \)-unital of order \( q \geq 3 \) is a subgroup of \( \mathfrak{A} \times R \). We compute the stabilizer of the parallelism \( \pi_{\text{odd}} \) in \( \mathfrak{A} \times R \cong Aut(SL(2,q)) \ltimes SL(2,q) \) (note that \( q \) is odd and hence \( q \geq 3 \)). Recall that \( Aut(SL(2,q)) = PGL(2,q) := PGL(2,q) \rtimes Aut(\mathbb{F}_q) \), where automorphisms of \( \mathbb{F}_q \) act entrywise on matrices. We denote by \( \varphi \) the Frobenius automorphism \( x \mapsto x^q \) on \( \mathbb{F}_q \).

**Theorem 3.4.** Let \( c := (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \). The stabilizer of \( \pi_{\text{odd}} \) in the group \( \mathfrak{A} \times R \) equals
\[
\begin{align*}
(a) & \quad \PSL(2,q) \rtimes \langle \rho_{-1} \rangle \quad \text{if } q \equiv 1 \mod 4 \text{ and } \\
(b) & \quad \PSL(2,q) \rtimes \langle \gamma_c \cdot \rho_{-1} \rangle \quad \text{if } q \equiv 3 \mod 4,
\end{align*}
\]
where \( \PSL(2,q) := PSL(2,q) \rtimes Aut(\mathbb{F}_q) \).

**Proof.** Note first that the Frobenius automorphism \( \varphi \) stabilizes \( \Omega \) and \( T \) and hence \( A \) and \( A' \). Thus, for each \( A^h \in \pi_{\text{odd}} \), we have
\[
A^h \cdot \varphi = (A \cdot \varphi)^{h \cdot \varphi} = A^{h \cdot \varphi} \in \pi_{\text{odd}}
\]
and \( \langle \varphi \rangle = Aut(\mathbb{F}_q) \) stabilizes \( \pi_{\text{odd}} \) and equally \( \pi'_{\text{odd}} \).

The action of \( PSL(2,q) \) obviously stabilizes \( \pi_{\text{odd}} \) and \( \pi'_{\text{odd}} \) by construction. Since the index of \( PSL(2,q) \) in \( PGL(2,q) \) equals 2, the orbit of \( \pi_{\text{odd}} \) under the action of \( PGL(2,q) \) has length 1 or 2. From Theorem 3.3, we know that conjugation by \( (\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}) \) maps \( \pi_{\text{odd}} \) to \( \pi'_{\text{odd}} \), and hence the stabilizer of \( \pi_{\text{odd}} \) in the action of \( PGL(2,q) \) equals \( PSL(2,q) \) and conjugation by any element in \( PGL(2,q) \setminus PSL(2,q) \) interchanges \( \pi_{\text{odd}} \) and \( \pi'_{\text{odd}} \).

Since we know now that \( PTL(2,q) \) stabilizes \( \{ \pi_{\text{odd}}, \pi'_{\text{odd}} \} \), we need to find those elements in \( R \) which also stabilize the set of these two parallelisms. Let \( \rho_g \in R \) and assume \( \pi_{\text{odd}} \cdot \rho_g \in \{ \pi_{\text{odd}}, \pi'_{\text{odd}} \} \). Since for each \( A^h \in \pi_{\text{odd}} \) the automorphism \( \rho_g \) maps the set of right cosets of \( T^h \) in \( A^h \) on a set of right cosets of \( T^h \), it must then also map the set of left cosets of \( T^h \) in \( A^h \) on a set of left cosets of \( T^h \). Hence, \( g \) is contained in the normalizer of every Sylow \( p \)-subgroup of \( SL(2,q) \) and thus \( g \in \{ \pm 1 \} \). We see immediately that
\[
\pi_{\text{odd}} \cdot \rho_{-\mathbb{I}} = \begin{cases} 
\pi_{\text{odd}} & \text{if } -1 \in \mathbb{F}_q^\times, \\
\pi'_{\text{odd}} & \text{if } -1 \in \mathbb{F}_q^\times.
\end{cases}
\]
Let $\pi_{\text{odd}}$ normalizes $P\Sigma L(2,q)$. Since $\rho_{-1}$ does not fix $1$—while every automorphism in $\mathfrak{A}$ does—and since $\rho_{-1}$ commutes with every automorphism in $\mathfrak{A}$, statement (a) follows.

If $q \equiv 3 \mod 4$, then both $\rho_{-1}$ and conjugation by $c$ interchange $\pi_{\text{odd}}$ and $\pi'_{\text{odd}}$ and hence the product stabilizes $\pi_{\text{odd}}$. Again, $\gamma_c \cdot \rho_{-1}$ does not fix $1$—while every automorphism in $\mathfrak{A}$ does—and hence the product $P\Sigma L(2,q) \cdot (\gamma_c \cdot \rho_{-1})$ is semidirect, since $\rho_{-1}$ commutes with every automorphism in $\mathfrak{A}$ and $\gamma_c$ normalizes $P\Sigma L(2,q) \leq \mathfrak{A}$. □

**Remark 3.5.** Knowing the full automorphism group of $\cup_{C,H}$, we use Theorem 3.4 to compute that the stabilizer of $[\infty]_{\pi_{\text{odd}}}$ in the full automorphism group of the closure $\cup_{C,H}^{\pi_{\text{odd}}}$ has order $2e(q + 1)$ (see [7, Corollary 5.3]).

### 3.2. A class of parallelisms for square order

For each square order, there also is at least one class of parallelisms on $\mathfrak{S}$ apart from $\mathfrak{b}$ and $\mathfrak{g}$. Consider the quadratic field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ and the unique involutory field automorphism

$$\tau: \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}, x \mapsto x^q,$$

with fixed field $\mathbb{F}_q$. We let this automorphism act entrywise on matrices over $\mathbb{F}_{q^2}$. Let again $T:=\{(0, x) \mid x \in \mathbb{F}_{q^2}\}$ be a fixed Sylow $p$-subgroup of $\text{SL}(2,q^2)$. Let further

$$\Omega:=\{g \in \text{SL}(2,q^2) \mid (0,1) \cdot g \cdot (0,1) \in \mathbb{F}_q\} \quad \text{and} \quad \Omega^\mathfrak{c}:=\text{SL}(2,q^2) \setminus \Omega.$$

As in Sect. 3.1, $\Omega$ is a union of $T$-cosets. Consider the action

$$\eta: \mathbb{F}_{q^2}^{2\times2} \times \text{GL}(2,q^2) \to \mathbb{F}_{q^2}^{2\times2}, (x,h) \mapsto x \cdot \eta_h:=h^{-1}xh,$$

and note that $\eta_h$ equals conjugation with $h$ if $h \in \text{GL}(2,q)$.

**Lemma 3.6.** Let $h \in N_{\text{SL}(2,q^2)}(T)$. Then $\Omega \cdot \eta_h = \Omega$ and $\Omega^\mathfrak{c} \cdot \eta_h = \Omega^\mathfrak{c}$.

**Proof.** Since $h$ normalizes $T$, we have $h$ upper triangular, say $h:=\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ with $a \in \mathbb{F}_q^\times$. For any $g \in \text{SL}(2,q^2)$, we have

$$(0,1) \cdot (g \cdot \eta_h) \cdot (0,1) = (0,a) \cdot g \cdot (0,1) = a\overline{a} \cdot (0,1) \cdot g \cdot (0,1).$$

Since $a\overline{a} \in \mathbb{F}_q^\times$, we get $g \cdot \eta_h \in \Omega$ exactly if $g \in \Omega$. □

**Theorem 3.7.** Let

$$A:=\{Tg \mid g \in \Omega\} \cup \{gT \mid g \in \Omega^\mathfrak{c}\} \quad \text{and} \quad \pi_{\text{sq}}:=\{A \cdot \eta_h \mid h \in \text{SL}(2,q^2)\}.$$

Then $\pi_{\text{sq}}$ is a parallelism on $\mathfrak{S}$.

**Proof.** Note first that the cosets in $A$ are pairwise non-intersecting and that

$$\#A = \frac{\#\Omega}{\#T} + \frac{\#\Omega^\mathfrak{c}}{\#T} = \frac{\#\text{SL}(2,q^2)}{\#T} = q^4 - 1.$$
Hence, $A$ is indeed a set of $(q^2)^2 - 1$ non-intersecting short blocks.

Let $h \in N := N_{\text{SL}(2,q^2)}(T)$ and $T g \in A$. Then

$$(T g) \cdot \vartheta_h = h^{-1} T g h = h^{-1} T h h^{-1} g h = T h (g \cdot \vartheta_h) = T (g \cdot \vartheta_h) \in A,$$

since $\vartheta_h$ leaves $\Omega$ invariant (Lemma 3.6). Analogously, $(gT) \cdot \vartheta_h \in A$ if $gT \in A$ and hence $A \cdot \vartheta_h = A$ for $h \in N$.

Now let $h \in \text{SL}(2,q^2) \setminus N$ and $T g \in A$. Then $(T g) \cdot \vartheta_h = T h (g \cdot \vartheta_h)$ is not a right coset of $T$. Assume $(T g) \cdot \vartheta_h = (g \cdot \vartheta_h) T g h$ is a left coset of $T$, i.e. $n := g h \in N$. Hence,

$$g \cdot \vartheta_h = h^{-1} g h = \pi^{-1} g n = g \cdot \vartheta_\pi,$$

which is in $\Omega$ exactly if $\overline{g} \in \Omega$. But since obviously $\overline{\Omega} = \Omega$ and $g \in \Omega$ since $T g \in A$, we have $g \cdot \vartheta_h \in \Omega$ and hence $(T g) \cdot \vartheta_h \notin A$. A similar consideration shows that $(gT) \cdot \vartheta_h \notin A$ if $gT \in A$. Thus, $(A \cdot \vartheta_h) \cap A = \emptyset$ if $h \in \text{SL}(2,q^2) \setminus N$.

Finally, let $g, h \in \text{SL}(2,q^2)$ and assume $(A \cdot \vartheta_g) \cap (A \cdot \vartheta_h) \neq \emptyset$. Then $(A \cdot \vartheta_{g^{-1}h^{-1}}) \cap A \neq \emptyset$ and hence $g h^{-1} \in N$, $A \cdot \vartheta_{g^{-1}h^{-1}} = A$ and $A \cdot \vartheta_g = A \cdot \vartheta_h$. Thus, $\pi_{sq} = \{A \cdot \vartheta_h \mid h \in \text{SL}(2,q^2)\}$ is indeed a partition of $\mathcal{S}$ into $\# \text{SL}(2,q^2)/\# N = q^2 + 1$ sets of $(q^2)^2 - 1$ pairwise non-intersecting cosets, hence a parallelism on $\mathcal{S}$.

**Remark 3.8.** (a) For each parallel class $X$ of $\pi_{sq}$, there exists a unique Sylow $p$-subgroup $P \in \mathfrak{P}$ such that $X$ consists of $q^3 - 1$ right cosets of $P$ (of which $q^2 - 1$ are left cosets of $P$) and of $q^4 - q^3$ left cosets of $P$, none of which is a right coset of $P$.

(b) Applying inversion on $\text{SL}(2,q^2)$ to the parallelism $\pi_{sq}$ yields another parallelism $\pi_{sq}^{-1}$. Since no element of $\mathfrak{A} \rtimes R$ maps $\pi_{sq}$ to $\pi_{sq}^{-1}$, the two parallelisms are not equivalent in any affine $\text{SL}(2,q^2)$-unitil.

(c) If $q^2$ is odd, neither $\pi_{sq}$ nor $\pi_{sq}^{-1}$ are equivalent to $\pi_{\text{odd}}$.

As for $\pi_{\text{odd}}$, we compute the stabilizer of $\pi_{sq}$ in $\mathfrak{A} \rtimes R$. We abbreviate $\Gamma := \text{Stab}_{\mathfrak{A} \rtimes R}(\pi_{sq})$.

**Lemma 3.9.** Let $x \in \mathbb{F}_q^\times$ and $c \in \mathbb{F}_q^\times$ such that $x \overline{x} = c^2$. Then $x \in \mathbb{F}_{q^2}^\times$.

**Proof.** If $q$ is even, then every element in $\mathbb{F}_q$ is a square. Let $q$ be odd and let $z$ be a generator of $\mathbb{F}_q^\times$. Note that $\mathbb{F}_q^\times = \{z^{l(q+1)} \mid l \in \mathbb{Z}\}$ and let $k, l \in \mathbb{Z}$ such that $x = z^k$ and $c = z^l$. Then

$$z^{k(q+1)} = x \overline{x} = c^2 = z^{2l(q+1)},$$

and we get $k - 2l \equiv 0 \mod q^2 - 1$. Since $q^2 - 1$ is even, we get that $k$ is even and hence $x$ is a square.

**Lemma 3.10.** (a) $\text{Aut}(\mathbb{F}_{q^2}) \leq \Gamma$. 

Proof. If $q$ is even, then every element in $\mathbb{F}_q$ is a square. Let $q$ be odd and let $z$ be a generator of $\mathbb{F}_q^\times$. Note that $\mathbb{F}_q^\times = \{z^{l(q+1)} \mid l \in \mathbb{Z}\}$ and let $k, l \in \mathbb{Z}$ such that $x = z^k$ and $c = z^l$. Then

$$z^{k(q+1)} = x \overline{x} = c^2 = z^{2l(q+1)},$$

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$$z^{k(q+1)} = x \overline{x} = c^2 = z^{2l(q+1)},$$

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$$z^{k(q+1)} = x \overline{x} = c^2 = z^{2l(q+1)},$$

and we get $k - 2l \equiv 0 \mod q^2 - 1$. Since $q^2 - 1$ is even, we get that $k$ is even and hence $x$ is a square. 

□
Theorem 3.11. Let $q := p^e$ and $\Gamma := \text{Stab}_{\mathfrak{S} \times R}(\pi_{\text{sq}})$. Then

(a) $\Gamma = (\langle \vartheta_s \mid s \in \text{SL}(2, q^2) \rangle \times \text{Aut}(\mathbb{F}_{q^2}) \times \langle \rho_{-1} \rangle) \cong P\Sigma L(2, q^2) \times \langle \rho_{-1} \rangle$.
(b) \( \#\Gamma = 2e(q^2 - 1)q^2(q^2 + 1) \).

**Proof.** Directly from Lemma 3.10. \qed

**Remark 3.12.** (a) Let \( \pi \) be a parallelism on \( \mathcal{S} \) and \( \pi^{-1} \) the parallelism on \( \mathcal{S} \) obtained by inversion. The stabilizers \( \text{Stab}_{\mathfrak{A} \rtimes R}(\pi) \) and \( \text{Stab}_{\mathfrak{A} \rtimes R}(\pi^{-1}) \) are isomorphic via \( \psi: \alpha \rho_r \mapsto \alpha \gamma_r \rho_{r^{-1}} \).

(b) We have \( \Gamma \cdot \psi = \Gamma \) and thus \( \text{Stab}_{\mathfrak{A} \rtimes R}(\pi_{sq}^{-1}) = \text{Stab}_{\mathfrak{A} \rtimes R}(\pi_{sq}) \).

**Remark 3.13.** For the classical affine \( SL(2,q^2) \)-unital \( U_{C,H} \), the stabilizer \( \text{Aut}(U_{C,H})_{[\infty]} = \text{Aut}(U_{C,H}^\pi_{sq})_{[\infty]} \) is isomorphic to \( (P\Sigma L(2,q^2)_C) \times \langle \rho^{-1} \rangle \). The order of \( \text{Aut}(U_{C,H})_{[\infty]} \) equals \( 4e(q^2 + 1) \). For each affine \( SL(2,q^2) \)-unital \( U_{S,D} \), the order of \( \text{Aut}(U_{S,D})_{[\infty]} \) equals the order of \( \text{Aut}(U_{S,D}) \).

### 3.3. A class of unitals of order four

For small orders, parallelisms on \( \mathcal{S} \) can be found using a computer. For \( q \in \{3, 5\} \), the classical affine \( SL(2,q) \)-unital \( U_{C,H} \) represents the only isomorphism type of affine \( SL(2,q) \)-unitals (see [3, Theorem 3.3] and [7, Theorem 6.1]). An exhaustive search using GAP [1] shows that up to equivalence, the three known parallelisms \( b, z \) and \( \pi_{\text{odd}} \) are the only ones existing for \( q \in \{3, 5\} \) (see [7, Sect. 6.2.1] for details).

For \( q = 4 \), there exist two isomorphism types of affine \( SL(2,q) \)-unitals, represented by the classical affine \( SL(2,4) \)-unital \( U_{C,H} \) and the non-classical affine \( SL(2,4) \)-unital described in [3], which we denote by \( U_{C,E} \). The full automorphism group of \( U_{C,H} \) is \( \mathfrak{A}_C \ltimes R \cong (C_5 \times C_4) \ltimes SL(2,4) \) and the full automorphism group of \( U_{C,E} \) is of type \( C_4 \ltimes SL(2,4) \) and a subgroup of \( \text{Aut}(U_{C,H}) \).

For order 4, an exhaustive computer search yields 182 parallelisms on \( \mathcal{S} \). We consider the actions of the automorphism groups \( \text{Aut}(U_{C,H}) \) and \( \text{Aut}(U_{C,E}) \) on the set of 182 parallelisms found by GAP. The orbit lengths of these actions are listed in Table 1.

Apart from \( b \) and \( z \), which are invariant under the action of \( \mathfrak{A} \ltimes R \), there are hence seven pairwise non-equivalent parallelisms in \( U_{C,E} \). We name representatives of those parallelisms \( \pi_1, \ldots, \pi_7 \) such that \( \pi_1 \) and \( \pi_2 \) as well as \( \pi_3 \) and \( \pi_4 \) are equivalent in \( U_{C,H} \). Following the numbering in [7, Table 6.2], the parallelism \( \pi_7 \) is equivalent to \( \pi_{sq} \), and \( \pi_6 \) is equivalent to \( \pi_{sq}^{-1} \). In fact, the construction of the parallelism \( \pi_{sq} \) was inspired by the discovery of those parallelisms and partly answers one of the open problems stated in the author’s Ph.D. thesis ([7, Question 6 in Chapter 7]). Completing the two affine
Table 1 Orbit lengths on the set of 182 parallelisms for order 4

| Aut($U_{C,H}$) | Aut($U_{C,E}$) |
|----------------|----------------|
| 1              | 1              |
| 1              | 1              |
| 30             | 24             |
| 25             | 20             |
| 5              | 5              |
| 55             | 60             |
| 60             | 60             |

unitals with the seven parallelisms $\pi_1, \ldots, \pi_7$, we obtain twelve pairwise non-isomorphic $\text{SL}(2,4)$-unitals, namely$^2$

$$U_1 := U^\pi_{C,H}, U_2 := U^\pi_{C,H}, U_3 := U^\pi_{C,H}, U_4 := U^\pi_{C,H}, U_5 := U^\pi_{C,H}, U_6 := U^\pi_{C,E},$$

$$U_7 := U^\pi_{C,E}, U_8 := U^\pi_{C,E}, U_9 := U^\pi_{C,E}, U_{10} := U^\pi_{C,E}, U_{11} := U^\pi_{C,E}, U_{12} := U^\pi_{C,E}.$$ Knowing all affine $\text{SL}(2,4)$-unitals and all parallelisms on $\mathcal{S}$ for order 4, we obtain the following

**Theorem 3.14.** (by exhaustive computer search) There are exactly 16 isomorphism types of $\text{SL}(2,4)$-unitals, represented by $U_{C,H}^\flat$, $U_{C,H}^\sharp$, $U_{C,E}^\flat$, $U_{C,E}^\sharp$ and the twelve unitals $U_1, \ldots, U_{12}$. □

Recall that the stabilizer of the block at infinity in any $\text{SL}(2,q)$-unital $U^\pi_{S,D}$ equals the group of those automorphisms of the affine unital $U_{S,D}$ which stabilize the parallelism $\pi$. We use the GAP package UnitalSZ [9] by Nagy and Mezőfi to compute that in each of the unitals $U_1, \ldots, U_{12}$ there are indeed no automorphisms moving the block at infinity. Thus, we are able to compute their full automorphism groups as subgroups of Aut($U_{C,H}$) or Aut($U_{C,E}$), respectively. See Table 2 for the isomorphism types of the full automorphism groups of the unitals $U_1, \ldots, U_{12}$.

**Remark 3.15.** Regarding the orders of the full automorphism groups of the twelve unitals in Table 2, we see that the order of Aut($U^\pi_{C,H}$) is notably greater than the other orders. Indeed, although $R$ is not contained in its full automorphism group, the unital $U^\pi_{C,H}$ admits a group of automorphisms (of isomorphism type $A_4 \times C_5$) which acts regularly on the affine points.

---

$^2$ These parallelisms for order 4 were found during the Leonid meteor shower in November 2018. Since they resulted in twelve new unitals at once, the author was reminded of a unital shower and hence called the twelve unitals $U_1, \ldots, U_{12}$ the Leonids unitals in her PhD thesis [7].
Table 2. Isomorphism types of the full automorphism groups of the unitals $U_1, \ldots, U_{12}$

| $U$          | $\text{Aut}(U)$            | $\# \text{Aut}(U)$ |
|--------------|----------------------------|---------------------|
| $U_{C,H}^2$  | $C_4 \ltimes D_5$          | 40                  |
| $U_{C,H}^4$  | $C_4 \ltimes A_4$          | 48                  |
| $U_{C,H}^5$  | $C_4 \ltimes (A_4 \times C_5)$ | 240                |
| $U_{C,H}^6$  | $C_5 \rtimes C_4$          | 20                  |
| $U_{C,H}^7$  | $C_5 \rtimes C_4$          | 20                  |
| $U_{C,E}^1$  | $D_5$                      | 10                  |
| $U_{C,E}^2$  | $C_4 \ltimes D_5$          | 40                  |
| $U_{C,E}^3$  | $A_4$                      | 12                  |
| $U_{C,E}^4$  | $C_4 \ltimes A_4$          | 48                  |
| $U_{C,E}^5$  | $C_4 \ltimes A_4$          | 48                  |
| $U_{C,E}^6$  | $C_4$                      | 4                   |
| $U_{C,E}^7$  | $C_4$                      | 4                   |

4. Translations

We consider translations of $\text{SL}(2,q)$-unitals (recall Definition 2.2). Since in any closure of an affine $\text{SL}(2,q)$-unital, the respective parallelism determines the possible translations with centers on the block at infinity, the study of translations is closely related to the study of parallelisms.

Remark 4.1. Let $U$ be a unital of order $q$ and $c$ a point of $U$. Then the group $G_{[c]}$ of all translations of $U$ with center $c$ acts semiregularly on the points of $U$ different from $c$ and hence semiregularly on the blocks of $U$ that are not incident with $c$ (see [4, Theorem 1.3]). Thus, $\#G_{[c]} \leq q$ and $c$ is a translation center exactly if $\#G_{[c]} = q$. Further, if a block $B$ of $U$ is fixed by $\text{Aut}(U)$, then the center of any translation of $U$ lies on $B$.

In the classical unital, each point is a translation center. In any non-classical $\text{SL}(2,q)$-$\pi$-unital and in any $\text{SL}(2,q)$-$\delta$-unital of order $q \geq 3$, the block $[\infty]$ is fixed by the full automorphism group (see [6, Proposition 3.11 and Theorem 3.16]) and hence the center of every translation lies on $[\infty]$.

We label the points at infinity of any $\text{SL}(2,q)$-$\pi$-unital with the Sylow $p$-subgroups, such that each (affine short) block $P \in \mathfrak{P}$ through $1$ is incident with the point $P \in [\infty]$. We first compute translations of $\text{SL}(2,q)$-$\pi$-unitals.

Lemma 4.2. Let $U_{S,D}^S$ be an $\text{SL}(2,q)$-$\pi$-unital. For each $P \in \mathfrak{P}$, the point $P \in [\infty]$ is a translation center with $G_{[P]} = R_P := \{ \rho_x \mid x \in P \}$.

Proof. For each $P \in \mathfrak{P}$, the set of blocks through the point $P$ in $U_{S,D}^S$ equals $\{[\infty]\} \cup \{gP \mid g \in \text{SL}(2,q)\}$. Hence, $R_P$ obviously is a group of translations
of \( U_{S,D}^\natural \) with center \( P \). Since \( \#R_P = q \), the statement follows (recall Remark 4.1).

If \( U_{S,D}^\natural \) is not classical, then the translations given in Lemma 4.2 are all translations of \( U_{S,D}^\natural \). We need some preparation to compute all possible translations of \( \text{SL}(2, q) - \pi \)-unital rings with centers on \([\infty] \) and \( \pi \in \{ \varphi, \pi_{\text{odd}}, \pi_{\text{sq}}, \pi_{\text{sq}^{-1}} \} \) (Theorem 4.6).

**Lemma 4.3.** Let \( P \in \mathcal{P} \) and let \( \tau \in \mathfrak{A} \ltimes R \) such that \( P \cdot \tau = P \). Then \( \tau = \alpha \rho_h \) with \( \alpha \in \mathfrak{A}_P \) and \( x \in P \).

**Proof.** Let \( \tau = \alpha \cdot \rho_h \in \mathfrak{A} \ltimes R \). Then \( P = P \cdot \tau = (P \cdot \alpha)h \) and thus \( \alpha \) stabilizes \( P \) and we have \( h \in P \).

Let in the following \( T := \{ (\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) \mid x \in \mathbb{F}_q \} \) be a fixed Sylow \( p \)-subgroup of \( \text{SL}(2, q) \).

**Lemma 4.4.** Let \( M := \{ Tg \mid g \in N(T) \} \) and \( f := (\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}) \) with \( c \in \mathbb{F}_q^\times \).

(a) Let \( \tau \in \mathfrak{A} \ltimes R \) such that \( X \cdot \tau = X \) for all \( X \in M \). Then \( \tau = \gamma_a \rho_t \) for some \( a \in N_{\text{GL}(2, q)}(T) \) and \( t \in T \).

(b) Let \( \tau \in \mathfrak{A} \ltimes R \) such that \( X \cdot \tau = X \) for all \( X \in M \cup \{ Tf \} \). Then \( \tau = \gamma_{t^{-1}} \rho_t \) for some \( t \in T \).

**Proof.** (a) Since \( T \in M \), we have \( \tau = \alpha \rho_t \) with \( \alpha \in \mathfrak{A}_T \cong \text{P}(N_{\text{GL}(2, q)}(T)) \ltimes \text{Aut}(\mathbb{F}_q) \) and \( t \in T \), according to Lemma 4.3. Note first that right multiplication with any element \( t \in T \) fixes each coset \( Tg = gT \in M \). If \( g = (\begin{smallmatrix} b & x \\ 0 & b^{-1} \end{smallmatrix}) \in N(T) \), the coset \( Tg \) equals \( \{ (\begin{smallmatrix} b & x \\ 0 & b^{-1} \end{smallmatrix}) \mid x \in \mathbb{F}_q \} \). Let \( [a] \in \text{P}(N_{\text{GL}(2, q)}(T)) \). Then we may choose \( a = (\begin{smallmatrix} 1 & y \\ 0 & d \end{smallmatrix}) \) and for each \( Tg \) with \( g \in N(T) \), we have \( (Tg) \cdot \gamma_a = (Tg)^a = Tg \). Applying a power \( \varphi^l \) of the Frobenius automorphism \( \varphi \) stabilizes the block \( Tg = \{ (\begin{smallmatrix} b & x \\ 0 & b^{-1} \end{smallmatrix}) \mid x \in \mathbb{F}_q \} \) exactly if \( b^{(p^l)} = b \). Thus, if \( \tau = \alpha \rho_t \) stabilizes each block in \( M \), then \( \alpha = \gamma_a \) with \( a \in N_{\text{GL}(2, q)}(T) \).

(b) We already know \( \tau = \gamma_a \rho_t \) for some \( a \in N_{\text{GL}(2, q)}(T) \) and \( t \in T \). We need to show \( \gamma_a = \gamma_{t^{-1}} \). Since \( \alpha \) normalizes \( T \), we have \( (Tf) \cdot \gamma_a \rho_t = (Tf)^a t = Tf^a t \). Hence, \( (Tf) \cdot \tau = Tf \) exactly if \( f^a t f^{-1} \in T \). Let \( t := (\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) \) and choose without restriction \( a = (\begin{smallmatrix} 1 & y \\ 0 & d \end{smallmatrix}) \). Then

\[
(0, 1) \cdot f^a t f^{-1} = (c(\frac{1}{d}(c + y) - 1), 1 + \frac{c(x + y)}{d}).
\]

Since \( f^a t f^{-1} \) is in \( T \) and \( c \neq 0 \), we get \( y = -x \) and \( d = 1 \) and hence \( a = t^{-1} \).

□

If \( \pi \) is a parallelism on \( \mathfrak{S} \) and \( A \in \pi \) a parallel class, we denote

\[ T_A^\pi := \{ \tau \in \mathfrak{A} \ltimes R \mid \forall X \in A : X \cdot \tau = X \} \]

If \( B \in \mathfrak{S} \) is a short block, we denote by \([B]\) the parallel class of \( \pi \) containing \( B \).
Lemma 4.5. For any $P \in \mathcal{P}$, we have:

(a) $T_{[P]}^\gamma = \{ \gamma x^{-1} \rho x \mid x \in P \}$.
(b) $T_{[P]}^\pi$ is trivial for $\pi \in \{ \pi_{\text{odd}}, \pi_{\text{sq}}, \pi_{\text{sq}}^{-1} \}$.

Proof. Let $\pi \in \{ b, \pi_{\text{odd}}, \pi_{\text{sq}}, \pi_{\text{sq}}^{-1} \}$. Following the notation in Lemma 4.4, the parallel class $[T] \in \pi$ contains $M$ and at least one coset $T f$ with $f = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ and $c \in \mathbb{F}_\gamma^\times$. According to Lemma 4.4, we thus get $T_{[T]}^\pi \subseteq \{ \gamma t^{-1} \rho t \mid t \in T \}$. Note that $\gamma t^{-1} \rho t$ equals left multiplication with $t$ and that $tgT = gT$ exactly if $g \in N(T)$ or $t = I$. The parallel class $[T] \in b$ equals the set of right cosets of $T$, while in $\pi_{\text{odd}}$, $\pi_{\text{sq}}$, and $\pi_{\text{sq}}^{-1}$, the parallel class $[T]$ contains at least one left coset $gT$, respectively, with $g \notin N(T)$. Hence, we get $T_{[T]}^\gamma = \{ \gamma t^{-1} \rho t \mid t \in T \}$ and $T_{[T]}^\pi$ is trivial if $\pi \in \{ \pi_{\text{odd}}, \pi_{\text{sq}}, \pi_{\text{sq}}^{-1} \}$.

For any $\pi \in \{ b, \pi_{\text{odd}}, \pi_{\text{sq}}, \pi_{\text{sq}}^{-1} \}$, the stabilizer of $\pi$ in $\mathcal{A} \rtimes R$ acts transitively on the parallel classes of $\pi$. Hence, statement (4.5) follows. If $\pi = b$, then each parallel class $[T^h]$ equals $[T]^h$ and we have $(Tg)^h \cdot \tau = (Tg)^h$ exactly if $\tau = \gamma t^{-h} \rho t^h$ for some $t \in T$.

Theorem 4.6. Let $\mathbb{U} := \mathbb{U}^\pi_{S, D}$ be an $\text{SL}(2, q)$-$\pi$-unital.

(a) If $\pi = b$ and $q \geq 3$, then:

(i) If $p = 2$, then every non-trivial translation of $\mathbb{U}$ is given by left multiplication with an involution contained in $N(S)$. For each Sylow $2$-subgroup $P \in \mathcal{P}$, the normalizer $N(S)$ contains exactly one non-trivial element of $P$.

(ii) If $q$ is odd, then $\mathbb{U}$ does not admit any non-trivial translation.

(b) If $\pi \in \{ \pi_{\text{odd}}, \pi_{\text{sq}}, \pi_{\text{sq}}^{-1} \}$, then $\mathbb{U}$ admits no non-trivial translation with center on $[\infty]$.

Proof. The translations of $\mathbb{U}$ with center on the block $[\infty]$ are automorphisms of $\mathbb{U}$ stabilizing $[\infty]$ and are hence contained in $\mathcal{A}_S \rtimes R$. Thus, statement (b) follows directly from Lemma 4.5.

Now let $\pi = b$ and $q \geq 3$. Then the block $[\infty]$ is fixed by the full automorphism group of $\mathbb{U}$ (see [6, Theorem 3.16]) and hence the center of every translation of $\mathbb{U}$ lies on $[\infty]$ (recall Remark 4.1). Fix $P \in \mathcal{P}$. According to Lemma 4.5, we have

$$
\Gamma_{[P]} \leq T_{[P]}^\gamma \cap (\mathcal{A}_S \rtimes R) = \{ \gamma x^{-1} \rho x \mid x \in P \text{ such that } \gamma x \in \mathcal{A}_S \}.
$$

Thus, every possible translation of $\mathbb{U}$ with center $P$ is given by left multiplication with $x \in P \cap N(S)$. The possible isomorphism types of $S$ and its stabilizer in $\text{Aut}(\text{SL}(2, q))$ are given in [7, Proposition 2.5 and Theorem 2.11]. According to those, we get that $x = 1$ or $\text{ord}(x) = p$ divides $2(q + 1)$ and hence $p = 2$.

If $p = 2$, then $S$ is cyclic (see e.g. [8, Proposition 2.2] or [7, Proposition 2.5]) and the normalizer $N(S)$ is a dihedral group of order $2(q + 1)$ containing $S$ as normal subgroup of order $q + 1$. There are thus $q + 1$ involutions in $N(S)$,
contained in one coset of $S$. For any $P \in \mathfrak{P}$, the intersection of $S$ and $P$ is trivial and hence no two non-trivial elements of $P$ are contained in $N(S)$. □

**Corollary 4.7.** For $q$ even, the unital $U_{C,H}^{1}$ admits exactly $q + 1$ non-trivial translations, each of order 2. These translations generate a dihedral group of order $2(q + 1)$.

For the unitals of order 4 introduced in Sect. 3.3, all translations can explicitly be computed (e. g. with GAP).

**Proposition 4.8** ([7], Proposition 6.17). All translations of the unitals $U_1, \ldots, U_{12}$ are known:

(i) $U_{C,H}^{\pi_2}$ and $U_{C,E}^{\pi_2}$ admit exactly one non-trivial translation each, of order 2.

(ii) $U_{C,H}^{\pi_4}, U_{C,H}^{\pi_5}, U_{C,E}^{\pi_4}, U_{C,E}^{\pi_5}$ admit exactly three non-trivial translations each. These translations, respectively, have a common center (which is thus a translation center) and generate an elementary abelian group of order 4. □

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