Classical Scale of Quantum Gravity

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Abstract

Characteristic length scale of the post-Newtonian corrections to the gravitational field of a body is given by its gravitational radius $r_g$. The role of this scale in quantum domain is discussed in the context of the low-energy effective theory. The question of whether quantum gravity effects appear already at $r_g$ leads to the question of correspondence between classical and quantum theories, which in turn can be unambiguously resolved considering the issue of general covariance. The $O(h^0)$ loop contributions turn out to violate the principle of general covariance, thus revealing their essentially quantum nature. The violation is $O(1/N)$, where $N$ is the number of particles in the body. This leads naturally to a macroscopic formulation of the correspondence principle.
Since the early days of quantum field theory it was realized that pursuing general quantization program in the case of gravity would result in a theory which is very difficult to test directly, because of the extremely small value of the characteristic quantum length scale

\[ l_P = \sqrt{\frac{G\hbar}{c^3}} \]

that can be built from the three relevant fundamental constants of Nature – the Newton gravitational constant \( G \), the speed of light \( c \), and the Planck constant \( \hbar \). Playing the role of a coupling constant, \( l_P \) is also the root of ultraviolet pathology in quantum gravity none of which models has yet succeeded in reconciliation of renormalizability with unitarity and causality. It is important, on the other hand, that smallness of \( l_P \) fully justifies application of methods of the effective field theory to the case of gravitational interaction. The model independence of this approach \[1\] implies that the low-energy properties of quantum gravity are completely determined by the lowest-order Einstein theory whatever the ultimate theory be. It is thus an excellent theoretic laboratory for investigation of synthesis of quantum theory and gravitation.

In combining the characteristic parameter with dimension of length above, one does not take into account other dimensional parameters which can enter the theory, such as masses \( m \) of matter field quanta. This is certainly legitimate as far as spacetime is quantized on its own, since the quantity \( l_P^2 \) is the only parameter appearing in the quantum theory of pure gravitational field. Inclusion of matter, however, brings in another parameter with dimension of length, namely the gravitational radius

\[ r_g = \frac{2Gm}{c^2} \]

This parameter appears, of course, already in classical theory. In this connection, an important question arises as to whether \( r_g \) has an independent meaning in quantum domain, representing a scale of specifically quantum effects. This question may seem strange at first sight, as \( r_g \) does not contain the Planck constant \( \hbar \), an inalienable attribute of quantum theory. However, well known is the fact that gravitational radiative corrections do contain pieces independent of \( \hbar \). In the framework of the effective theory, they appear as a power series in \( r_g/r \), just like post-Newtonian corrections in classical general relativity. This fact was first clearly stated by Iwasaki \[2\]. The reason for the appearance of \( \hbar^0 \) terms through the loop contributions is that in the case of gravitational interaction, the mass and “kinetic”
terms in a matter Lagrangian determine not only the properties of matter quanta propagation, but also their couplings. Thus the mass term of, e.g., scalar field Lagrangian generates the vertices proportional to

\[
\left( \frac{mc}{\hbar} \right)^2
\]

containing inverse powers of \( \hbar \). Naively, one expects these be cancelled by \( \hbar \)'s coming from the propagators when combining an amplitude. One should remember, however, that such counting of powers of \( \hbar \) in Feynman diagrams is a bad helpmate in the presence of massless particles. Virtual propagation of self-interacting gravitons results in a root non-analyticity of the massive particle form factors at zero momentum transfer (\( p \)). For instance, the low-energy expansion of the diagram in Fig. 1 begins with terms proportional to \( \sqrt{-p^2} \), rather than integer powers of \( p^2 \). It is this singularity which is responsible for the appearance of \( r_\delta/r \)-terms. The question we ask is of what nature, classical or quantum, these pieces are.

Evidently, one can answer this question only after one established a limiting procedure of transition from quantum to classical theory. Thus, this is actually the question of correspondence in quantum gravity.

In classical Einstein theory, gravitational field is completely described by the metric tensor \( g_{\mu\nu} \). To establish a correspondence between classical and quantum theories, one has to find a quantum field quantity to be traced back to \( g_{\mu\nu} \). For this purpose, one could try to use the scattering matrix to define a potential of particle interaction, and then compare it with the corresponding classical quantity (the Einstein-Infeld-Hoffman potential, in the lower orders). In fact, this is the way followed by Iwasaki in [2]. Unfortunately, this method
turned out to be highly ambiguous \[3, 4, 5\]. There are infinitely many potentials which lead to the same S-matrix, and one has to invent certain *ad-hoc* prescriptions at each order of the post-Newtonian expansion to achieve an agreement with classical theory. Clearly, no valuable correspondence can be established in this way, which could help us to elucidate nature of the post-Newtonian radiative corrections.

There is, however, a much more direct approach to this problem, based on calculation of the expectation value \( \langle g_{\mu\nu} \rangle \). Diagrammatically, \( \langle g_{\mu\nu} \rangle \) is represented by the sum of all diagrams having one external graviton line and an arbitrary number of external matter lines. For instance, contribution of the order \( G^2 \) is given by the sum of the tree diagrams shown in Fig. 2, and the one-loop diagrams like that in Fig. 1. Suppose that the matter producing gravitational field satisfies the usual quantum mechanical quasi-classical conditions, *e.g.*, consider sufficiently heavy particles. Then the quasi-classical conditions for the gravitational field can be inferred from the requirement that \( \langle g_{\mu\nu} \rangle \) coincides with the corresponding classical solution of the Einstein equations. Practically, the most decisive way of looking for these conditions is to compare *transformation properties* of the quantities involved under deformations of the reference frame, thus avoiding explicit calculation of the expectation values. The latter point of view takes advantage of the fact that the transformation law of classical solutions is known in advance. Hence, we have to check whether \( \langle g_{\mu\nu} \rangle \) transforms covariantly with respect to transitions between different reference frames. In other words, we have to consider the question of *general covariance in quantum gravity*. This question might also seem strange, since the principle of general covariance is what the whole theory is based upon. One should remember, however, that in quantum domain, this principle is a quite formal operator relation expressing degeneracy of the quantum action. Transformation properties of *observable* quantities is what we are interested in instead.

A choice of the reference frame is equivalent to imposition of an appropriate set of gauge conditions on the metric field. As far as the tree contribution to \( \langle g_{\mu\nu} \rangle \) is considered, dependence on the choice of gauge can be easily determined in a quite general form using the anti-canonical formalism \[6\]. The result is that gauge variations induce spacetime diffeomorphisms, just like in classical theory. This is as it should be, since the effective action of quantum theory coincides with the initial classical action at the tree level. Investigation of the post-Newtonian contributions coming from radiative corrections is a much more difficult task because formal manipulations using standard techniques do not give a definite answer.
An explicit calculation reveals the following remarkable fact: gravitational radiative corrections to $\langle g_{\mu\nu} \rangle$ transform non-covariantly under transitions between different reference frames. Specifically, let the gravitational field be produced by a scalar particle with mass $m$, and the reference frame be fixed by the following conditions depending on a parameter $\rho$

$$\eta^{\mu\nu} \partial_\mu g_{\nu\gamma} - \left( \frac{\rho - 1}{\rho - 2} \right) \eta^{\mu\nu} \partial_\gamma g_{\mu\nu} = 0.$$ 

Violation of general covariance is most conveniently expressed in terms of classically invariant quantities, e.g., the scalar curvature $R$. Consider deformations of the reference frame, induced by variations of $\rho$. Under such a deformation, the scalar curvature measured at a given point of the reference frame acquires a non-zero variation resulting from the one-loop diagram in Fig. 1:

$$\delta R = \frac{G^2 m^2}{c^4 r^4} (1 - 2\rho) \delta \rho.$$ 

Thus, despite their independence of the Planck constant, the post-Newtonian loop contributions turn out to be of a purely quantum nature.

We are now in a position to ask for conditions to be imposed on a system in order to allow classical consideration of its gravitational field. Such a condition providing vanishing of the $\hbar^0$ loop contributions can easily be found out by examining their dependence on the number of particles in the system. Let us consider a body with mass $M$, consisting of a large number $N = M/m$ of elementary particles with mass $m$. Then it is readily seen that the $n$-loop contribution to the effective gravitational field of the body turns out to be suppressed by a factor $1/N^n$ in comparison with the tree contribution. For instance, at the first post-Newtonian order, the tree diagrams in Fig. 2 are *bilinear* in the energy-momentum tensor $T^{\mu\nu}$ of the particles, and therefore proportional to $(m \cdot N) \cdot (m \cdot N) = M^2$. On the other hand,
the post-Newtonian contribution of the diagram in Fig. 1 is proportional to $m^2 \cdot N = M^2 / N$, since it has only two external matter lines.

Thus, we are led to the following formulation of the correspondence principle in quantum gravity: *the effective gravitational field produced by a macroscopic body of mass $M$ consisting of $N$ particles turns into corresponding classical solution of the Einstein equations in the limit $N \to \infty$*. In particular, the principle of general covariance is to be considered as approximate, valid only for the description of macroscopic phenomena.

The quantum gravity effects characterized by the scale $r_g$ are normally highly suppressed. For the solar gravitational field, their relative value is $m_{\text{proton}} / M_\odot \approx 10^{-57}$. However, they are the larger the more gravitating body resembles an elementary particle, and can become noticeable for a sufficiently massive compact body. Black hole physics is likely the most promising place to search for manifestations of the classical scale in quantum gravity

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