Visualizing WSPDs and Their Applications

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Abstract

Introduced by Callahan and Kosaraju back in 1995, the concept of well-separated pair decomposition (WSPD) has occupied a special significance in computational geometry when it comes to solving distance problems in $d$-space. We present an in-browser tool that can be used to visualize WSPDs and several of their applications in 2-space. Apart from research, it can also be used by instructors for introducing WSPDs in a classroom setting. The tool will be permanently maintained by the third author at https://wisno33.github.io/VisualizingWSPDsAndTheirApplications/.

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1 Introduction

Let $P$ and $Q$ be two finite pointsets in $d$-space and $s$ be a positive real number. We say that $P$ and $Q$ are well-separated with respect to $s$, if there exist two congruent disjoint balls $B_P$ and $B_Q$, such that $B_P$ contains the bounding-box of $P$, $B_Q$ contains the bounding-box of $Q$, and the distance between $B_P$ and $B_Q$ is at least $s$ times the common radius of $B_P$ and $B_Q$. The quantity $s$ is referred to as the separation ratio of the decomposition. Using this idea of well-separability, one can define a well-separated decomposition of a pointset (WSPD) [4] in the following way. Let $P$ be a set of $n$ points in $d$-space and $s$ be a positive real number. A well-separated pair decomposition for $P$ with respect to $s$ is a collection of pairs of non-empty subsets of $P$, $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}$ for some integer $m$ (referred to as the size of the WSPD) such that

- for each $i$ with $1 \leq i \leq m$, $A_i$ and $B_i$ are well-separated with respect to $s$, and
- for any two distinct points $p, q \in P$, there is exactly one index $i$ with $1 \leq i \leq m$, such that $p \in A_i, q \in B_i$, or $p \in B_i, q \in A_i$.

Note that in some cases, $m = C(n, 2) = \Theta(n^2)$. Refer to [5, 6, 7] for a detailed discussion on WSPDs and their uses. In this work, we consider WSPDs in 2-space only. Our implementations are based on the algorithms presented in the book by Narasimhan and Smid [6, Chapters 9 and 10]. These algorithms were originally presented in [2, 3, 4] by Callahan and Kosaraju.
Algorithms implemented

We have implemented the algorithms using the JSXGraph library. Some code segments have been borrowed from the tool presented in [1].

2.1 Constructing WSPDs

Given a pointset $P$ and a positive real number $s$, a WSPD of $P$ can be constructed using a split-tree. Our implementation is based on the naive quadratic time approach presented in [6]. It accepts $P$ and $s$, and returns the WSPD pairs in the WSPD decomposition. Refer to Algorithm 1. An advanced linearithmic construction is also presented in [6].

Notations. Let $x$ be a split-tree node. Let $S_x$ denote the points stored in the subtree rooted at $x$ and $R(x)$ denotes the bounding-box of $S_x$. Further, $L_{\text{max}}(R(x))$ denotes the length of the longer side of $R(x)$.

Algorithm 1

ConstructWspd($P, s > 0$).

1. Construct a split-tree $T$ on $P$ in the following way:
   - If $|P| = 1$, then the split-tree consists of one single node that stores that point. Otherwise, split the bounding-box of $P$ into two rectangles by cutting the longer side of the bounding-box into two equal parts. Let $P_1$ and $P_2$ be the two subsets of $P$ that are contained in these two new rectangles. The split-tree for $P$ consists of a root having two subtrees, which are recursively defined for $P_1$ and $P_2$.
   - For each internal node $u$ of $T$, find WSPD pairs using $v$ and $w$, the left and right child of $u$, respectively, in the following way:
     a. Compute $S_v, S_w, L_{\text{max}}(R(v))$ and $L_{\text{max}}(R(w))$.
     b. If $S_v, S_w$ are well-separated with respect to $s$, then node pair $\{v, w\}$ is a WSPD pair.
        Otherwise, if $L_{\text{max}}(R(v)) \leq L_{\text{max}}(R(w))$, recursively find WSPD pairs using $v$, LEFTCHILD($w$) and then recursively find WSPD pairs using $v$, RIGHTCHILD($w$).
        Else, recursively find WSPD pairs using LEFTCHILD($v$), $w$, and then recursively find WSPD pairs using RIGHTCHILD($v$), $w$.

2.2 Applications of WSPDs

Construction of $t$-Spanners. Given a pointset $P$ and $t \geq 1$, a $t$-spanner on $P$ is a Euclidean geometric graph $G$ on $P$ such that for every pair of points $p, q \in P$, the length of the shortest-path between $p, q$ in $G$ is at most $t$ times the Euclidean distance between them. Refer to Algorithm 2. It returns the set of spanner edges and can be implemented to run in $O(n \log n)$ time [6].

Algorithm 2

Construct-$t$-Spanner($P, t > 1$).

Let $s = 4(t + 1)/(t - 1)$. Construct a WSPD of $P$ with separation ratio $s$. For every pair $(A_i, B_i)$ of the decomposition do the following: include the edge $\{a_i, b_i\}$ in the spanner where $a_i$ is an arbitrary point in $A_i$ and $b_i$ is an arbitrary point in $B_i$. 
Finding Closest Pairs. The problem asks to find two distinct points of $P$ whose distance is minimum among the $C(n, 2)$ point pairs. The idea of well-separatedness can be used to design an algorithm for this problem. See Algorithm 3. It can be implemented to run in $O(n \log n)$ time \([6]\).

Algorithm 3 ClosestPair($P$).

Construct a 2-spanner using Algorithm 2. Since the closest pair is connected by an edge of the spanner, find the pair by iterating over all the edges.

Finding k-Closest Pairs. It is a generalization of the closest-pair problem. Given a positive integer $k$ such that $k \leq C(n, 2)$, the goal is to find the $k$ closest pairs among the $C(n, 2)$ pairs. See Algorithm 4. It can be implemented to run in $O(n \log n + k)$ time \([6]\).

Algorithm 4 $k$-ClosestPairs($P$).

1. Create a WSPD with some $s > 0$. For every pair $(A_i, B_i)$ in the decomposition, let $R(A_i)$ and $R(B_i)$ be the bounding boxes of $A_i$ and $B_i$, respectively. Further, by $|R(A_i) R(B_i)|$, we denote the minimum distance between the two bounding-boxes $R(A_i), R(B_i)$. Rename the $m$ pairs in the decomposition such that $|R(A_1) R(B_1)| \leq |R(A_2) R(B_2)| \leq \ldots \leq |R(A_m) R(B_m)|$.
2. Compute the smallest integer $\ell \geq 1$, such that $\sum_{i=1}^{\ell} |A_i| \cdot |B_i| \geq k$.
3. Let $r := |R(A) R(B)|$.
4. Compute the integer $\ell'$, which is defined as the number of indices with $1 \leq i \leq m$, such that $|R(A_i) R(B_i)| \leq (1 + 4/s)r$.
5. Compute the set $L$ consisting of all pairs $\{p, q\}$ for which there is an index $i$ with $1 \leq i \leq \ell'$, such that $p \in A_i, q \in B_i$ or $q \in A_i, p \in B_i$.
6. Compute and return the $k$ smallest distances determined by the pairs in the set $L$.

Finding All-Nearest Neighbors. In this problem, for every point $p$ in $P$, we need to find its nearest neighbor $q$ in $P \setminus \{p\}$. Refer to Algorithm 5 for a description of the algorithm. It can be implemented to run in $O(n \log n)$ time \([6]\).

Algorithm 5 AllNearestNeighbors($P$).

Choose $s > 2$ and obtain the pairs of WSPD. For every $p \in P$, compute its nearest neighbor in the following way: Find all such pairs of the WSPD, for which at least one of their sets is a singleton containing $p$. For every such pair $(A_i, B_i)$, if $A_i = \{p\}$, then $S_p = S_p \cup B_i$, else if $B_i = \{p\}$, then $S_p = S_p \cup A_i$. The nearest neighbor of $p$ is the point in $S_p$ closest to $p$ (found by exhaustive search).

t-Approximate Minimum Spanning Trees. Let $t > 1$, be a real number. A tree connecting the points of $P$ is called a $t$-approximate minimum spanning tree of $P$, if its weight is at most $t$ times the weight of the Euclidean minimum spanning tree of $P$. Refer to Algorithm 6. In $d$-space, it runs in $O(n \log n + n/(t - 1)^d)$ time \([6]\).

Algorithm 6 $t$-ApproximateMinimumSpanningTree($P, t > 1$).

Compute the $t$-spanner $G$ using Algorithm 2. Using Prim’s algorithm compute a minimum spanning tree $T$ of $G$. Return $T$. 
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