REGULARITY AND H-POLYNOMIALS OF BINOMIAL EDGE IDEALS

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ABSTRACT. Let $G$ be a finite simple graph on the vertex set $[n] = \{1, \ldots, n\}$ and $K[x, y] = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring in $2n$ variables over a field $K$ with each degree $x_i = \deg y_j = 1$. The binomial edge ideal of $G$ is the binomial ideal $J_G \subset K[x, y]$ which is generated by those binomials $x_iy_j - x_jy_i$ for which $\{i, j\}$ is an edge of $G$. The Hilbert series $H_{K[x, y]/J_G}(\lambda)$ of $K[x, y]/J_G$ is of the form $H_{K[x, y]/J_G}(\lambda) = h_{K[x, y]/J_G}(\lambda)/(1 - \lambda)^d$, where $d = \dim K[x, y]/J_G$ and $h_{K[x, y]/J_G}(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s$ with each $h_i \in \mathbb{Z}$ and with $h_s \neq 0$ is the $h$-polynomial of $K[x, y]/J_G$. It is known that, when $K[x, y]/J_G$ is Cohen–Macaulay, one has $\deg h_{K[x, y]/J_G}(\lambda)$, where $\deg h_{K[x, y]/J_G}(\lambda)$ is the (Castelnuovo–Mumford) regularity of $K[x, y]/J_G$. A lot of computational experience supports the conjecture that $\deg h_{K[x, y]/J_G}(\lambda) = s$ will be constructed.

INTRODUCTION

The binomial edge ideal of a finite simple graph was introduced in [2] and in [9] independently. (Recall that a finite graph $G$ is simple if $G$ possesses no loop and no multiple edge.) Let $G$ be a finite simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ and $K[x, y] = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring in $2n$ variables over a field $K$ with each degree $x_i = \deg y_j = 1$. The binomial edge ideal $J_G$ of $G$ is the binomial ideal of $K[x, y]$ which is generated by those binomials $x_iy_j - x_jy_i$ for which $\{i, j\}$ is an edge of $G$.

Let, in general, $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over a field $K$ with each degree $x_i = 1$ and $I \subset S$ a homogeneous ideal of $S$ with $\dim S/I = d$. The Hilbert series $H_{S/I}(\lambda)$ of $S/I$ is of the form $H_{S/I}(\lambda) = (h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s)/(1 - \lambda)^d$, where each $h_i \in \mathbb{Z}$ ([1] Proposition 4.4.1). We say that $h_{S/I}(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s$ with $h_s \neq 0$ is the $h$-polynomial of $S/I$. Let $\deg h_{S/I}(\lambda)$ denote the (Castelnuovo–Mumford) regularity ([1] p. 168) of $S/I$. It is known (e.g., [13] Corollary B.4.1]) that, when $S/I$ is Cohen–Macaulay, one has $\deg h_{S/I}(\lambda) = \deg h_{S/I}(\lambda)$. Furthermore, in [3] and [4], for given integers $r$ and $s$ with $r, s \geq 1$, a monomial ideal $I$ of $S = K[x_1, \ldots, x_n]$ with $n \gg 0$ for which $\deg h_{S/I}(\lambda) = s$ was constructed.

Let, as before, $G$ be a finite simple graph on the vertex set $[n]$ with $d = \dim K[x, y]/J_G$ and $h_{K[x, y]/J_G}(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s$ the $h$-polynomial of $K[x, y]/J_G$. A lot of computational experience supports the conjecture that

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Conjecture 0.1. One has \( \text{reg}(K[x,y]/J_G) \leq \deg h_{K[x,y]/J_G}(\lambda) \).

Now, in the present paper, given arbitrary integers \( r \) and \( s \) with \( 2 \leq r \leq s \), a finite simple graph \( G \) on \( [n] \) with \( n \gg 0 \) for which \( \text{reg}(K[x,y]/J_G) = r \) and \( \deg h_{K[x,y]/J_G}(\lambda) = s \) will be constructed.

Theorem 0.2. Given arbitrary integers \( r \) and \( s \) with \( 2 \leq r \leq s \), there exists a finite simple graph \( G \) on \( [n] \) with \( n \gg 0 \) for which \( \text{reg}(K[x,y]/J_G) = r \) and \( \deg h_{K[x,y]/J_G}(\lambda) = s \).

1. Proof of Theorem 0.2

Our discussion starts in the computation of the regularity and the h-polynomial of the binomial edge ideal of a path graph.

Example 1.1. Let \( P_n \) be the path on the vertex set \([n]\) with \( \{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\} \) its edges. Since \( K[x,y]/J_{P_n} \) is a complete intersection, it follows that the Hilbert series of \( K[x,y]/J_{P_n} \) is \( H(K[x,y]/J_{P_n}, \lambda) = (1 + \lambda)^{n-1}/(1 - \lambda)^{n+1} \) and that \( \text{reg}(K[x,y]/J_{P_n}) = \deg h_{K[x,y]/J_{P_n}}(\lambda) = n - 1 \).

Let \( G \) be a finite simple graph on the vertex set \([n]\) and \( E(G) \) its edge set. The suspension of \( G \) is the finite simple graph \( \hat{G} \) on the vertex set \([n+1]\) whose edge set is \( E(G) = E(G) \cup \{\{i, n+1\} : i \in [n]\} \). Given a positive integer \( m \geq 2 \), the \( m \)-th suspension of \( G \) is the finite simple graph \( \hat{G}^m \) on \([n+m]\) which is defined inductively by \( \hat{G}^m = \hat{G}^{m-1} \), where \( \hat{G}^1 = \hat{G} \).

Lemma 1.2. Let \( G \) be a finite connected simple graph on \([n]\) which is not complete. Suppose that \( \dim K[x,y]/J_G = n + 1 \) and \( \deg h_{K[x,y]/J_G}(\lambda) \geq 2 \). Then

\[
\text{reg} \left( K[x,y,x_{n+1},y_{n+1}]/J_{\hat{G}} \right) = \text{reg}(K[x,y]/J_G),
\]

\[
\deg h_{K[x,y,x_{n+1},y_{n+1}]/J_{\hat{G}}}(\lambda) = \deg h_{K[x,y]/J_G}(\lambda) + 1.
\]

In particular, if \( \text{reg}(K[x,y]/J_G) \leq \deg h_{K[x,y]/J_G}(\lambda) \), then

\[
\text{reg} \left( K[x,y,x_{n+1},y_{n+1}]/J_{\hat{G}} \right) < \deg h_{K[x,y]/J_G}(\lambda).
\]

Proof. The suspension \( \hat{G} \) is the join product ((11, p.3)) of \( G \) and \( \{n+1\} \), and \( \hat{G} \) is not complete. Hence, by virtue of [10, Theorem 2.1] and [11, Theorem 2.1 (a)], one has

\[
\text{reg} \left( K[x,y,x_{n+1},y_{n+1}]/J_{\hat{G}} \right) = \max \{\text{reg}(K[x,y]/J_G), 2\} = \text{reg}(K[x,y]/J_G).
\]

Furthermore, [7, Theorem 4.6] says that

\[
H(K[x,y,x_{n+1},y_{n+1}]/J_{\hat{G}}, \lambda) = H(K[x,y]/J_G, \lambda) + \frac{2\lambda + (n-1)\lambda^2}{(1-\lambda)^{n+2}} - \frac{h_{K[x,y]/J_G}(\lambda)}{(1-\lambda)^{n+1}} + \frac{2\lambda + (n-1)\lambda^2}{(1-\lambda)^{n+2}} - \frac{h_{K[x,y]/J_G}(\lambda)}{(1-\lambda)^{n+2}}.
\]
Thus \( \deg h_{K[x,y,s_1,y_1]}(\hat{\lambda}) = \deg h_{K[x,y]}(\hat{\lambda}) + 1 \), as desired. \( \square \)

We are now in the position to give a proof of Theorem 0.2.

\textbf{Proof.} (Proof of Theorem 0.2) Each of the following three cases is discussed.

(Case 1) Let \( 2 \leq r = s \). Let \( G = P_{r+1} \). As was shown in Exampleodel 1.1 one has

\[ \text{reg}(K[x,y]/J_G) = \deg h_{K[x,y]}(\hat{\lambda}) = r. \]

(Case 2) Let \( r = 2 \) and \( 3 \leq s \). Let \( G = K_{s-1,s-1} \) denote the complete bipartite graph on the vertex set \( [2s-2] \). By using [12, Theorem 1.1 (c) together with Theorem 5.4 (a)], one has \( \text{reg}(K[x,y]/J_G) = 2 \) and

\[ F(K[x,y]/J_G,\hat{\lambda}) = \frac{1 + (2s-3)\hat{\lambda} + 2}{(1-\hat{\lambda})^{2s-1}} - \frac{2(1+(s-2)\hat{\lambda})}{(1-\hat{\lambda})^{s}} \]

\[ = \frac{1 + (2s-3)\hat{\lambda} + 2(1-\hat{\lambda}) - 2(1+(s-2)\hat{\lambda})(1-\hat{\lambda})^{s-1}}{(1-\hat{\lambda})^{2s-1}}. \]

Hence \( \deg h_{K[x,y]}(\hat{\lambda}) = s \), as required.

(Case 3) Let \( 3 \leq r < s \). Let \( G = \bar{P}_{r+1}^{s-r} \) be the \( (s-r) \)-th suspension of the path \( P_{r+1} \). Applying Lemma [1.2] repeatedly shows \( \text{reg}(S/J_G) = r \) and

\[ H(K[x,y]/J_G,\hat{\lambda}) \]

\[ = (1+\hat{\lambda})^r(1-\hat{\lambda})^{s-r} + 2\lambda \left\{ \sum_{i=0}^{s-r-1}(1-\hat{\lambda})^i \right\} + \lambda^2 \sum_{i=0}^{s-r-1}(s-1-i)(1-\hat{\lambda})^i \]

\[ = (1+\hat{\lambda})^r(1-\hat{\lambda})^{s-r} + 2\lambda \cdot \frac{1-(1-\hat{\lambda})^{s-r}}{\hat{\lambda}} + \lambda^2 \cdot \frac{1+(1-\hat{\lambda})^{s-r} + \lambda(s-r(1-\hat{\lambda})^{s-r})}{\hat{\lambda}^2} \]

\[ = (1+\hat{\lambda})^r(1-\hat{\lambda})^{s-r} + 2 \left\{ 1 - (1-\hat{\lambda})^{s-r} \right\} - 1 + (1-\hat{\lambda})^{s-r} + \lambda \left\{ s - r(1-\hat{\lambda})^{s-r} \right\} \]

\[ = (1+\hat{\lambda})^r(1-\hat{\lambda})^{s-r} + 1 - (1-\hat{\lambda})^{s-r} + \lambda \left\{ s - r(1-\hat{\lambda})^{s-r} \right\} \]

\[ = \frac{1 + s\lambda + (1-\hat{\lambda})^{s-r}\{(1+\hat{\lambda})^r - 1 - r\hat{\lambda}\}}{(1-\hat{\lambda})^{s+2}}. \]

Hence \( \deg h_{K[x,y]/J_G}(\hat{\lambda}) = s \), as desired. \( \square \)

\section{2. Examples}

A few classes of finite simple graphs which support Conjecture 0.1 will be demonstrated.
Proposition 2.1. The cycle $C_n$ of length $n \geq 3$ satisfies

$$\text{reg}(K[x,y]/J_{C_n}) \leq \deg h_{K[x,y]}/J_{C_n}(\lambda).$$

**Proof.** Since the length of the longest induced path of $C_n$ is $n-2$, it follows from [8 Theorem 1.1] and [6 Theorem 3.2] that reg$(K[x,y]/J_{C_n}) = n - 2$. Furthermore, [14 Theorem 10 (b)] says that

$$\deg h_{K[x,y]/J_{C_n}} = \begin{cases} 1 & (n = 3), \\ n - 1 & (n > 3). \end{cases}$$

Hence the desired inequality follows. \qed

Let $k \geq 1$ be an integer and $p_1, p_2, \ldots, p_k$ a sequence of positive integers with $p_1 \geq p_2 \geq \cdots \geq p_k \geq 1$ and $p_1 + p_2 + \cdots + p_k = n$. Let $V_1, V_2, \ldots, V_k$ denote a partition of $[n]$ with each $|V_i| = p_i$. In other words, $[n] = V_1 \cup V_2 \cup \cdots \cup V_k$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Suppose that

$$V_i = \left\{ \sum_{j=1}^{i-1} p_j + 1, \sum_{j=1}^{i-1} p_j + 2, \ldots, \sum_{j=1}^{i-1} p_j + p_i - 1, \sum_{j=1}^{i} p_j \right\}$$

for each $1 \leq i \leq k$. The complete multipartite graph $K_{p_1, \ldots, p_k}$ is the finite simple graph on the vertex set $[n]$ with the edge set

$$E(K_{p_1, \ldots, p_k}) = \{ \{k, \ell\} : k \in V_i, \ell \in V_j, 1 \leq i < j \leq k \}.$$

Proposition 2.2. The complete multipartite graph $G = K_{p_1, \ldots, p_k}$ satisfies

$$\text{reg}(K[x,y]/J_G) \leq \deg h_{K[x,y]}/J_{G}(\lambda).$$

**Proof.** We claim reg$(K[x,y]/J_G) \leq \deg h_{K[x,y]}/J_{G}(\lambda)$ by induction on $k$. If $k = 1$, then $G = K_{p_1}$ is the complete graph and reg$(K[x,y]/J_G) = \deg h_{K[x,y]}/J_{G}(\lambda) = 1$.

Let $k > 1$. If $p_k = 1$, then $G = \tilde{G}'$, where $G' = K_{p_1, \ldots, p_{k-1}}$. Lemma [2] as well as the induction hypothesis then guarantees that reg$(K[x,y]/J_{G}) \leq \deg h_{K[x,y]}/J_{G}(\lambda)$. Hence one can assume that $p_k > 1$. In particular $G$ is not complete. It then follows from [11, Theorem 2.1(a)] that reg$(K[x,y]/J_G) = 2$. Furthermore, [7 Corollary 4.14] says that

$$\deg h_{K[x,y]}/J_{G}(\lambda) = \begin{cases} n - p_k + 1 & (2p_1 < n + 1), \\ 2p_1 - p_k & (2p_1 \geq n + 1). \end{cases}$$

Since $k > 1$ and $p_k > 1$, one has $\deg h_{K[x,y]}/J_{G}(\lambda) \geq n - p_k + 1 \geq p_1 + 1 \geq 3$. Thus the desired inequality follows. \qed

Let $t \geq 3$ be an integer and $K_{1,t}$ the complete bipartite graph on $\{1, v_1, \ldots, v_t\}$ with the edge set $E(K_{1,t}) = \{ \{1, v_i\} : 1 \leq i \leq t \}$. Let $p_1, p_2, \ldots, p_t$ be a sequence of positive integers and $P^{(i)}$ the path of length $p_i$ on the vertex set $\{w_{i,1}, w_{i,2}, \ldots, w_{i,p_i+1}\}$ for each
1 \leq i \leq t. Then the t-starlike graph $T_{p_1,p_2,...,p_t}$ is defined as the finite simple graph obtained by identifying $v_i$ with $w_{i,1}$ for each $1 \leq i \leq t$. Thus the vertex set of $T_{p_1,p_2,...,p_t}$ is

$$\{1\} \cup \bigcup_{i=1}^{t}\{w_{i,1},w_{i,2},...,w_{i,p_i+1}\}$$

and its edge set is

$$E(T_{p_1,p_2,...,p_t}) = \bigcup_{i=1}^{t}\{\{w_{i,j},w_{i,j+1}\} \mid 0 \leq j \leq p_i\},$$

where $w_{i,0} = 1$ for each $1 \leq i \leq t$.

**Proposition 2.3.** The t-starlike graph $G = T_{p_1,p_2,...,p_t}$ satisfies

$$\text{reg}(K[x,y]/J_G) < \deg h_{K[x,y]/J_G}(\lambda).$$

**Proof.** It follows from [5, Corollary 3.4 (2)] that $\text{reg}(K[x,y]/J_G) = 2 + \sum_{i=1}^{t} p_i$. Furthermore, [12, Theorem 5.4 (a)] guarantees that

$$F(K[x,y]/J_{K_{1,t}}) = \frac{1}{(1-\lambda)^2} - \frac{1 + (t-1)\lambda}{(1-\lambda)^{t+1}} + \frac{1 + t\lambda}{(1-\lambda)^{t+2}}$$

$$= \frac{1 - \{1 + (t-1)\lambda\}(1-\lambda)^{t-1} + (1 + t\lambda)(1-\lambda)^{t-2}}{(1-\lambda)^2}$$

$$= \frac{1 + (1-\lambda)^{t-2}\{2\lambda + (t-1)\lambda^2\}}{(1-\lambda)^2}$$

Hence, by virtue of [7, Corollary 3.3], one has

$$h_{K[x,y]/J_G}(\lambda) = \left[1 + (1-\lambda)^{t-2}\{2\lambda + (t-1)\lambda^2\}\right] \cdot (1-\lambda)^{\sum_{i=1}^{t} p_i}.$$

Thus

$$\deg h_{K[x,y]/J_G}(\lambda) = t + \sum_{i=1}^{t} p_i > 2 + \sum_{i=1}^{t} p_i = \text{reg}(K[x,y]/J_G),$$

as required. \hfill \square

**Example 2.4.** Let $m \geq 0$ be an integer and $G_m$ the finite simple graph on the vertex set $[m+9]$ drawn below:

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Then $\mathbb{K}[x, y]/J_{G_m}$ is not unmixed. In fact, for each subset $S \subset [m+9]$, we define

$$P_S = \left( \bigcup_{i \in S} \{x_i, y_i\}, J_{G_1}, \ldots, J_{\tilde{G}_{c(S)}} \right),$$

where $G_1, \ldots, G_{c(S)}$ are connected components of $G_{[m+9]\setminus S}$ and where $\tilde{G}_1, \ldots, \tilde{G}_{c(S)}$ is the complete graph on the vertex set $V(G_1), \ldots, V(G_{c(S)})$, respectively. It then follows from [2, Lemma 3.1 and Corollary 3.9] that $P_0$ and $P_{\{3,8\}}$ are minimal primes of $J_{G_m}$ and that $\text{height} P_0 = m + 8 < \text{height} P_{\{3,8\}} = m + 9$. Thus $\mathbb{K}[x, y]/J_{G_m}$ is not unmixed. In particular, $\mathbb{K}[x, y]/J_{G_m}$ is not Cohen-Macaulay. However, one has $\text{reg}(\mathbb{K}[x, y]/J_{G_m}) = \deg h_{\mathbb{K}[x, y]/J_{G_m}}(\lambda) = m + 6$.

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