Exclusion Statistics in Conformal Field Theory
– generalized fermions and spinons for level-1 WZW theories –

Peter Bouwknegt
Department of Physics and Mathematical Physics
University of Adelaide
Adelaide, SA 5005, AUSTRALIA

Kareljan Schoutens
Institute for Theoretical Physics
University of Amsterdam
Valckenierstraat 65
1018 XE Amsterdam, THE NETHERLANDS

Abstract

We systematically study the exclusion statistics for quasi-particles for Conformal Field Theory spectra by employing a method based on recursion relations for truncated spectra. Our examples include generalized fermions in $c_{\text{CFT}} < 1$ unitary minimal models, $\mathbb{Z}_k$ parafermions, and spinons for the $\mathfrak{su}(n)_1$, $\mathfrak{so}(n)_1$ and $\mathfrak{sp}(2n)_1$ Wess-Zumino-Witten models. For some of the latter examples we present explicit expressions for finitized affine characters and for the $N$-spinon decomposition of affine characters.

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1. Introduction

Among the most spectacular features of quantum many body systems in low dimensions are phenomena such as spin-charge separation and quantum number fractionalization. These terms refer to situations where fundamental excitations over a many-body ground state carry quantum numbers (for spin, charge, etc.) which are fractions of the quantum numbers carried by the microscopic degrees of freedom in the system. Examples are fractionally charged (Laughlin) quasi-particles in fractional quantum Hall systems, spinons in antiferromagnetic spin-chains and the spinons and holons for $d = 1$ itinerant electrons with repulsive interactions. Recent experiments on fractional quantum Hall systems and on spin-chain compounds have directly probed various aspects of these ‘fractional quasi-particles’. It has been suggested that the physics of quantum critical points in dimension higher than one may involve the same phenomenon of quantum number fractionalization [1].

An essential aspect of quasi-particles carrying unusual (fractionalized) quantum numbers is that their statistics will be equally unusual. Depending on the context, one may wish to consider braid statistics (in 2 spatial dimensions) or exclusion statistics, the two notions being closely related. Prototypical examples are quasi-hole excitations over the Laughlin ground states for the $\nu = \frac{1}{m}$ fractional quantum Hall effect. Such quasi-holes satisfy fractional (anyonic) braid statistics [2], and obey a specific form of fractional exclusion statistics (see below).

The term ‘fractional exclusion statistics’ was introduced in Haldane’s 1991 paper [3], which proposed a particular generalization of the Pauli principle. This definition leads to the notion of a ‘quantum gas of non-interacting particles satisfying fractional exclusion statistics’. The thermodynamic properties of such gases have been analyzed in, e.g., [4].

There exist a number of different approaches by which the ‘fractional exclusion statistics’ (Haldane’s or more general) of (quasi-)particles in specific quantum many body systems can be investigated.

In models that are solvable by the Bethe Ansatz, the Bethe equations for a set of fundamental rapidities can sometimes be interpreted in terms of exclusion statistics. This approach has been followed for the $SU(2)$ Haldane-Shastry spin-chain [3] and for a specific integrable 3-state Potts chain [3]. The character expressions that resulted from the latter example have been generalized to a large category of ‘fermionic sum
formulas’ for the characters in a variety of models of Conformal Field Theory (see, e.g., [7, 8, 9, 10, 11, 12, 13, 14] and references therein).

If scattering (S-matrix) data for the quasi-particles of choice are known, one may employ the Thermodynamic Bethe Ansatz (TBA) to determine the corresponding exclusion statistics. For a particular class of S-matrices, the TBA statistics agree with ‘fractional exclusion statistics’ in the sense of Haldane [15]. A spectacular application of this TBA approach has been the exact computation of a universal conductance curve for edge-to-edge tunneling of quasi-particles in the fractional quantum Hall effect (FQHE) [16].

A third and independent approach to ‘fractional exclusion statistics’ is possible for systems that are described by an (effective) Conformal Field Theory (CFT). This approach will be central to the work presented in this paper. In general, a (rational) CFT comes with a precise list of primary field operators from which one may extract creation and annihilation operators for various quasi-particles. In this manner, many important examples of quantum number fractionalization are elegantly described. In addition, the exclusion statistics properties of quasi-particles can conveniently be studied in the CFT setting, by following a method proposed in [17]. This method, which employs recursion relations for truncations (‘finitizations’) of the chiral CFT spectrum, leads to explicit expressions for single-level partition sums and hence exposes the underlying exclusion statistics. We stress that this approach does not rely on underlying Bethe equations and/or scattering data for the quasi-particles, but is instead directly based on algebraic properties of CFT (primary) fields. This observation is important as in many cases of interest Bethe equations or scattering data are simply not available.

In [18, 19], the recursion method was applied to CFT’s describing edge theories for a variety of FQHE states and used to determine the exclusion statistics of edge quasi-particles. Of particular interest are the so-called pfaffian QHE states, where the edge quasi-holes satisfy what is called non-abelian exclusion statistics [19]. In [18], it was demonstrated how the exclusion statistics properties of edge quasi-particles manifest themselves in equilibrium properties (notably: the Hall conductance) and in transport properties.

To further illustrate the ‘CFT approach’ to quasi-particle statistics, we recall that the excitation spectrum for critical spin-S spin chains (with $S \geq 1$) can be understood in terms of fundamental spinons carrying spin-$\frac{1}{2}$. This remarkable result, which can
be derived from a Bethe Ansatz exact solution \[20\], is immediately clear when one recognizes that the effective CFT is a \(\mathfrak{su}(2)_{k=2S}\) Wess-Zumino-Witten (WZW) model, which has a \(j = \frac{1}{2}\) primary ‘spinon’ field in its spectrum. A spinon basis for the \(\mathfrak{su}(2)_k\) spectrum was presented in \[21\] and the corresponding spinon statistics were described in \[22\].

In some special cases, the exclusion statistics of CFT quasi-particles (as obtained from the recursion method) turn out to agree with ‘fractional exclusion statistics’ as defined by Haldane. In such cases, the quasi-particle character formulas assume the form of ‘fermionic sum formulas’. Comparing these with the results of the Stony Brook group \[4, 8\], one concludes that in these special cases an analysis using ‘quasi-particles associated to conformal fields’ and an approach based on ‘Bethe Ansatz quasi-particles’ lead to identical results. The correspondence between ‘fermionic sum formulas’ and Haldane’s statistics was first made in a large number of special cases (see \[23, 24, 18, 25\]) and was recently put on a general footing in \[26\] (see also \[27\]). We would like to stress that the most general ‘fermionic sum formulas’ (see Eq. (2.23)), with finite values for some of the parameters \(u_a\), correspond to a form of non-abelian exclusion statistics. (An example is provided by the abovementioned \(\mathfrak{su}(2)_k\) spinons, see \[21, 22\].) For many of the examples that we treat in this paper, notably the spinons for level-1 WZW models, the quasi-particle character formulas are not of the ‘fermionic sum’ type and appear to be more general than what has been considered in the literature.

In the present paper we present a systematic study of some of the more interesting examples of exclusion statistics for CFT quasi-particles. Throughout this paper, we shall be employing the recursion method of \[17\] to determine the exclusion statistics and the associated thermodynamics of specific CFT quasi-particles. In section 2.1 we review this method, paying special attention to the possibility of non-abelian exclusion statistics. In section 2.2 we briefly review Haldane’s approach, and we derive equations that determine the CFT central charge for a given choice of statistics matrix. In section 2.3 we treat two proto-typical examples, which are the Majorana fermion and \(\mathfrak{su}(2)_1\) spinons, and in section 2.4 we discuss in general terms character expressions that are associated to a CFT quasi-particle basis. In section 3 we discuss generalized fermions in \(c_{\text{CFT}} < 1\) minimal models and section 4 is devoted to \(\mathbb{Z}_k\) parafermions. In section 5 we discuss spinons for \(\mathfrak{su}(n)_1\) WZW models. Section 6 is devoted to \(\mathfrak{so}(n)_1\) WZW models, which we analyze in terms of quasi-particles transforming in the spinor
representation(s) of $\mathfrak{so}(n)$. In section 7 we briefly discuss the case $\mathfrak{sp}(2n)_1$ WZW. For some of the examples that we treat, we provide explicit character formulas, both for the finitized characters and the $N$-particle truncated characters. Section 8 contains some further remarks on the WZW theories, and in section 9 we offer a brief outlook.

Possible physical applications of the results and approach discussed in this paper have been discussed in, e.g., [18, 19, 22, 28].

2. General structure

2.1 Introducing the method

We start by introducing in general terms the ‘recursion method’, which will be applied throughout this paper.

The subject of study is what is called the chiral spectrum or chiral Hilbert space of a conformal field theory. In a Rational Conformal Field Theory the chiral spectrum consists of a finite number of irreducible modules of the Chiral Algebra. (To be precise, these will be the modules that participate in the modular invariant partition function.) Depending on the choice of theory, the Chiral Algebra can be, e.g., the Virasoro algebra, a $\mathcal{W}$-type extended algebra or an affine Lie algebra. We shall encounter examples of all three possibilities in this paper.

By a quasi-particle approach towards a RCFT we mean a formulation where we interpret the chiral spectrum as a collection of states, each of which is generated by the repeated action of creation operators for CFT quasi-particles on a suitable, finite, set of reference states. The creation operators are nothing else than the Fourier modes of a selected set of (primary) field operators. Denoting these modes by $\phi^{(a)}_{-s}$, we are thus considering states of the type

$$|a\rangle_{J} = \phi^{(a_1)}_{-s_1} \phi^{(a_2)}_{-s_2} \ldots \phi^{(a_N)}_{-s_N} |0\rangle_{J}$$

(2.1)

In this notation, the label $a$ enumerates the selected primary fields and the index $J$ labels the various reference states.

For a quasi-particle formulation to be complete we require that the collection of states (2.1) spans the complete chiral spectrum. When this condition is met, one likes
to reduce the collection (2.1) in such a manner that the reduced set precisely forms a basis for the chiral Hilbert space. In first approximation, the restriction will amount to ordering the \( s_i, i = 1, 2, \ldots \), in ascending order, the precise details depending on the case at hand. Obviously, the prototype for constructions of this kind is a theory of free fermions, where the canonical anti-commutation relations of the fermion fields determine the systematics. For quasi-particles satisfying abelian braiding statistics, one may derive ‘generalized commutation relations’, which may then be employed in the reduction of (2.1) to a basis set. The \( su(2)_1 \) WZW theory provides an example where this procedure has been carried out in explicit detail [24]. In theories with non-abelian braiding, simple algebraic relations do not seem to be available and a systematic procedure for obtaining a quasi-particle basis is presently not known (except after \( q \)-deformation in the crystal limit, i.e., the \( q \to 0 \) limit [29]). Examples of non-abelian theories where explicit quasi-particle bases have been proposed are the \( su(2)_k \) WZW theories [21], and the CFT for the so-called pfaffian quantum Hall state [19].

Having obtained an explicit quasi-particle basis for a given chiral spectrum, one may go ahead and try to translate the result into a statement about the exclusion statistics of the quasi-particles. In [17], one of us proposed the following procedure: one starts by restricting a basis of quasi-particle states of the form (2.1) by requiring that the participating momenta (modes) satisfy \( s \leq l \). One then defines truncated partition sums as expressions of the type

\[
P_l^{(\alpha)}(x_a, q) = \text{tr}^\alpha \left( q^{L_0} \prod_a x_a^{N_a} \right),
\]

(2.2)

where \( x_a = e^{\beta \mu_a} \) denotes the fugacity of the particle of species \( a \) and the superscript \( \alpha \) denotes the restriction that the full quasi-particle state belongs to a sector specified by \( \alpha \). Typically, the label \( \alpha \) will run over the irreducible representations of the Chiral Algebra. Having defined the truncated partition sums \( P_l^{(\alpha)}(x_a, q) \), one is interested in their rate of growth,

\[
P_{l+1}^{(\alpha)}(x_a, q)/P_l^{(\alpha)}(x_a, q) \sim \lambda(z_a = x_a q^l)
\]

(2.3)

where, typically, \( \lambda(z_a) \) will not depend on the sector \( \alpha \). In practice, the functions \( \lambda(z_a) \) can be obtained from recursion relations satisfied by the \( P_l^{(\alpha)} \). In fact, for this purpose

\[
\text{In some cases, such as } so(n)_1, \text{ we allow for an extra shift in } l \text{ of order one if this makes the recursion look simpler.}
\]
it suffices to consider the recursion relations at the point \( q = 1 \). We can identify 
\( \lambda(z_a = e^{\beta(\mu_a - \epsilon)}) \) with the grand partition function for a single energy level, hence from 
\( \lambda(z_a) \) one derives thermodynamic quantities such as the appropriate generalization of

\[
    n_a(\epsilon) = z_a \frac{\partial}{\partial z_a} \log \lambda(z) \bigg|_{z_b = e^{\beta(\mu_b - \epsilon)}}.
\]

(2.4)

It is an easy exercise to express the specific heat \( C \) of the CFT in terms of \( \lambda(z_a) \). In the 1-component case, the result is

\[
    C = \gamma \rho k_B^2 T, \quad \gamma = 2 \int_0^1 dy \frac{1}{y} \log \lambda(y),
\]

(2.5)

with \( \rho \) the density of states, and comparing with the well-known result in terms of the
central charge \( c_{\text{CFT}} \),

\[
    \gamma = \frac{\pi^2}{3} c_{\text{CFT}},
\]

(2.6)

one obtains identities, which in concrete cases lead to new proofs for certain
dilogarithm identities (see \([30, 31]\) and references therein). In many cases, the central
charge identity serves as a first check on a conjectured quasi-particle basis and/or
recursion relation.

We now focus on some interesting limits of the expressions \( \lambda(z_a) \). A first limit of
interest is \( z_a \ll 1 \), where one may expand

\[
    \lambda(z_a) = 1 + \sum_b \alpha_b z_b + \mathcal{O}(z^2).
\]

(2.7)

The quantities \( \alpha_a \) manifest themselves as prefactors in the Boltzmann tails

\[
    n_a(\epsilon) \xrightarrow{\epsilon \to \infty} \alpha_a e^{-\beta \epsilon},
\]

(2.8)

of the generalized distribution functions. As a general result, a factor \( \alpha_a \neq 1 \) signals
non-abelian braiding statistics of the associated CFT primary fields \( \phi^a(z) \). A precise
statement is that \( \alpha_a \) equals the largest eigenvalue of the incidence matrix associated to
the fusion rules of the primary field $\phi^a(z)$ with the other primary fields in the theory. This can be seen as follows. Suppose the chiral algebra has a set of fusion rules

$$a \times b = \sum_c N_{ab}^c c,$$

(2.9)

then the recursion relations for the truncated characters $P_i^{(i)}(x_a, q)$ (at $q = 1$) will be of the form

$$P_{l+1}^{(i)} = P_l^{(i)} + \sum_{a,j} x_a N_{ai}^j P_{l+1-a_j}^{(j)} + O(x^2),$$

(2.10)

for some set $a_j \in \mathcal{Q}_{\geq 0}$. Asymptotically, $P_l^{(i)} \sim \mu_i(x)\lambda(x)^l$. Substituting this in (2.10) gives

$$\mu_i(x)\lambda(x)^l(\lambda(x) - 1) = \sum_{a,j} x_a N_{ai}^j \mu_j(x)\lambda(x)^{l+1-a_j} + O(x^2).$$

(2.11)

Expanding $\mu_i(x) = \mu_i + O(x)$ and $\lambda(x) = 1 + \sum_a \alpha_a x_a + O(x^2)$ as in (2.7), we find the following equation for the $O(x)$ term

$$\sum_j N_{ai}^j \mu_j = \alpha_a \mu_i,$$

(2.12)

which proves that $\alpha_a$ is the largest (real) eigenvalue (and $\mu_i$ the eigenvector) of the fusion matrix $(N_a)^j_i = N_{ai}^j$ corresponding to the primary field $\phi_a(z)$. Below we encounter concrete examples of this statement.

A second interesting limit is $z_a \to \infty$ in some fixed ratio. For example, setting all $z_a'$ with $a' \neq a$ equal to zero and keeping just $z_a$, one typically finds $\lambda \sim z^\beta_a$. The number $\beta_a$ then represents the maximum $n_a^{\text{max}}$ of the distribution function for particles of species $a$ in the absence of any others.

In the example of level-1 WZW models, the set of quasi-particles transform in an irreducible finite dimensional representation of the underlying Lie algebra (or two irreducibles in the case of $\mathfrak{so}(2n)$). In that case we will often associate the same fugacity $x_a = x$ to all particles in the representation and derive an equation for the total grand partition function $\lambda_{\text{tot}}(x) = \prod_a \lambda_a(x)$. The small $x$ expansion of $\lambda_{\text{tot}}(x)$ is then given by $\lambda_{\text{tot}}(x) = 1 + D \alpha x + O(x^2)$, where $D$ is the dimension of the irreducible representation and $\alpha$ the largest eigenvalue of the fusion matrix, while for the large $x$
behavior one typically finds \( \lambda_{\text{tot}} \sim x^\beta \). We define \( n_{\text{tot}}^{\text{max}} \) as this value of \( \beta \). The large \( x \) behavior of the average partition function \( \lambda_{\text{av}}(x) = \lambda_{\text{tot}}(x)^{1/D} \) defines similarly the number \( n_{\text{av}}^{\text{max}} = n_{\text{tot}}^{\text{max}} / D \).

### 2.2 Haldane’s exclusion principle

We conclude this general introduction with a brief review of Haldane’s notion of (abelian) fractional exclusion statistics [3]. It is based on the idea that the number of accessible states \( d_a \) for a particle of species \( a \) depends on the particle numbers \( N_b \) of all the other particles through a statistical interaction matrix \( G_{ab} \) by

\[
\frac{\partial d_a}{\partial N_b} = -G_{ab}.
\]  

(2.13)

For a ‘generalized ideal gas of fractional statistics particles’ this leads the following equations for the one-particle grand canonical partition functions \( \lambda_a(z) \) ([4], see the last paper of this reference for an explicit discussion of the multi-component case)

\[
\left( \frac{\lambda_a - 1}{\lambda_a} \right) \prod_b \lambda_b^{G_{ab}} = z_a,
\]  

(2.14)

from which the 1-particle distribution functions can be recovered by eqn. (2.4). The system (2.14) always leads to a small \( x \) expansion \( \lambda_a(x) = 1 + x_a + O(x^2) \), i.e., corresponds to abelian exclusion statistics.

Starting from the eqns. (2.5) and (2.6), one may compute the central charge of a generalized ideal gas satisfying Haldane exclusion statistics

\[
\left( \frac{\pi^2}{6} \right) c_{\text{CFT}} = \sum_a \int_0^1 \frac{dz_a}{z_a} \log \lambda_a(z_a).
\]  

(2.15)

From (2.14) we find, for each \( a \),

\[
\frac{d\lambda_a}{\lambda_a(\lambda_a - 1)} + \sum_b G_{ab} \frac{d\lambda_b}{\lambda_b} = \frac{dz_a}{z_a}.
\]  

(2.16)

Thus

\[
\left( \frac{\pi^2}{6} \right) c_{\text{CFT}} = \sum_a \int_{y_a}^1 \frac{d\lambda_a}{\lambda_a(\lambda_a - 1)} \log \lambda_a + \sum_{a,b} G_{ab} \int_{y_b}^1 \frac{d\lambda_b}{\lambda_b} \log \lambda_a,
\]  

(2.17)
where the $y_a = \lambda_a(z_a = 1)$ are determined as a solution of (2.14) with all $z_a = 1$. Substituting (2.14) again, we find
\[
\left(\frac{\pi^2}{6}\right)c_{\text{CFT}} = \sum_a \int_{y_a}^1 \left(\frac{d\lambda_a}{\lambda_a - 1} \log \lambda_a - \frac{d\lambda_a}{\lambda_a} \log(\lambda_a - 1)\right) + \sum_a \int_{y_a}^1 \frac{d\lambda_a}{\lambda_a} \log z_a . \tag{2.18}
\]
The last term on the right hand side of (2.18) equals the left hand side (upto a sign) by partial integration. Changing variables to $\mu_a = (\lambda_a - 1)/\lambda_a$ in the remaining terms finally gives
\[
\left(\frac{\pi^2}{6}\right)c_{\text{CFT}} = \sum_a L(x_a) , \tag{2.19}
\]
where
\[
L(x) = -\frac{1}{2} \int_0^x dy \left(\frac{\log y}{1 - y} + \frac{\log(1 - y)}{y}\right) = \text{Li}_2(x) + \frac{1}{2} \log x \log(1 - x) ,
\]
\[
\text{Li}_2(x) = -\int_0^x dy \frac{dy}{y} \log(1 - y) , \tag{2.20}
\]
are Rogers’ and Euler’s dilogarithm functions, respectively [32], and the $x_a$ are a solution of the TBA system
\[
x_a = \prod_b (1 - x_b)^{G_{ab}} . \tag{2.21}
\]
In the context of Haldane’s statistics, the result (2.19), (2.21) was first given in [33], where a slightly different derivation was presented.

The eqn. (2.19) is the same as the equation that determines the central charge of the quasi-particle character (see, e.g., [34, 30, 35])
\[
\text{ch}(x_a, q) = \sum_{\text{relations}} \left(\prod_a x_a^{n_a}\right) \frac{q^{\frac{1}{2}n_aG_{ab}n_b + C_{ab}n_a}}{\Pi_a(q)n_a} . \tag{2.22}
\]
Indeed, it has been conjectured [26] (see also [27]) that the quasi-particle character of a generalized ideal gas of particles with abelian exclusion statistics is precisely of the type (2.22). This connection has been established in [23] for $g$-ons (i.e., the 1 component case of (2.14)), and it has been observed in several multi-component cases [24, 17, 18, 25].
The character (2.22) is the \( u \to \infty \) limit of a more general ‘Universal Chiral Partition Function’ [26]

\[
\text{ch}(x_a, q) = \sum_{n_a} q^{\frac{1}{2} n_a G_{ab} n_b + C_a n_a} \left( \prod_a x_a^{n_a} \left( \frac{((1 - G) n + \frac{u}{2})}{n_a} \right) \right),
\]  
(2.23)

with

\[
\left[ \begin{array}{c}
  m \\
  n
\end{array} \right] = \left( \frac{(q)_m}{(q)_{m-n}(q)_n} \right), \quad (q)_n = \prod_{k=1}^n (1 - q^k).
\]  
(2.24)

The exclusion statistics of CFT quasi-particles that correspond to a character formula (2.23) with some of the \( u_a \) finite, are more general than Haldane’s. The CFT quasi-particles are associated to those values \( a = A \) for which \( u_A = \infty \), while the other values \( a = i \), with \( u_i < \infty \), correspond to auxiliary or pseudo-particles. After eliminating the auxiliary quantities \( \lambda_i \) from the thermodynamic equations, one finds that \( \lambda_A(x) = 1 + \alpha_A x + O(x^2) \) with \( \alpha_A \neq 1 \), showing that the CFT quasi-particles obey non-abelian exclusion statistics. Fundamental examples are spinons in \( k > 1 \) \( su(2)_k \) WZW theories [21, 22] and the generalized fermions that we discuss in Section 3. We refer to a forthcoming publication [36] for a detailed discussion.

It should be emphasized that not all examples of exclusion statistics are most naturally described by characters of the form (2.23). Examples are the \( su(n)_1 \) WZW models which are based on ‘Gentile parastatistics’ rather than fermionic statistics [17]. We refer to Section 5 for a discussion.

### 2.3 Prototypes: free fermions and \( su(2)_1 \) spinons

To get started we review the (trivial) example of the free Majorana fermion and the case of \( su(2)_1 \) spinons, which we developed elsewhere [17]. The CFT for a chiral Majorana fermion \( \psi(z) \) with Neveu-Schwarz boundary conditions has two Virasoro sectors, with leading conformal dimensions \( h_{(1,1)} = 0, h_{(2,1)} = \frac{1}{2} \). We write the truncated characters as \( P_l^1(x, q) \equiv P_l^{(1,1)}(x, q) \) and \( P_{l+\frac{1}{2}}^2(x, q) \equiv P_l^{(2,1)}(x, q) \) (\( l = 0, 1, \ldots \)). In the limit \( l \to \infty \), these reproduce the corresponding characters of the \( c_{\text{CFT}} = \frac{1}{2} \) Virasoro algebra. If we realize that the two sectors are generated by modes \( \psi_{l-l-\frac{1}{2}} \), with \( l = 0, 1, \ldots \), satisfying canonical anti-commutation relations, we immediately obtain the recursion

...
relation

\[
\begin{pmatrix}
P_{l+1}^1 \\
P_{l+1}^2
\end{pmatrix} = \begin{pmatrix}
1 & xq^{l+\frac{1}{2}} \\
xq^{l+\frac{1}{2}} & 1
\end{pmatrix} \begin{pmatrix}
P_{l}^1 \\
P_{l}^2
\end{pmatrix}, \quad l = 0, 1, 2, \ldots .
\] (2.25)

Clearly, the associated one-particle partition function \(\lambda(z)\) (cf. (2.3)) is in this case simply given by the largest eigenvalue of the recursion matrix in (2.25), i.e., \(\lambda(z) = 1 + z\), where \(z = xq^{l+\frac{1}{2}}\), and the usual Fermi-Dirac distribution is obtained.

The example of spinons for the \(su(2)_1\) theory has been treated elsewhere. In [5, 37, 38, 24] the spinon basis for the affine modules was discussed in great detail, and in [17] the systematics of this basis were translated into a recursion relation for the truncated characters \(P_{l}^{(2j)}(x, t, q)\) with the \(su(2)\) spin taking the values \(j = 0, \frac{1}{2}\) appropriate for the \(k = 1\) WZW model. It should be stressed that in the limit \(l \to \infty\), these characters reproduce characters of the affine Lie algebra \(su(2)_1\). In a general RCFT, one expects characters of the Chiral Algebra of that theory. With the notation \(\chi_{2j+1}\) for the \(su(2)\) characters, e.g., \(\chi_2(t) = t + t^{-1}\) and \(\chi_3(t) = t^2 + 1 + t^{-2}\), the recursion relations take the form

\[
\begin{pmatrix}
P_{l+1}^{(0)} \\
P_{l+1}^{(1)}
\end{pmatrix} = \begin{pmatrix}
1 + x^2q^{2l+1} & \chi_3 & xq^{l+\frac{3}{2}}(1 - x^2q^{2l}) & \chi_2 \\
xq^{l+\frac{3}{2}} & 1 - x^2q^{2l} & \chi_2 & xq^{l+\frac{3}{2}}
\end{pmatrix} \begin{pmatrix}
P_{l}^{(0)} \\
P_{l}^{(1)}
\end{pmatrix} .
\] (2.26)

It is interesting to note that the subtractions that are part of the recursion have their origin in the symmetrization prescription which is part of the generalized Pauli Principle for spinons [24]. It has been demonstrated [17] (see also [25]) that the thermodynamic distribution functions that follow from this recursion relation are identical to those associated to fractional exclusion statistics in the sense of Haldane, with statistical interaction matrix \(G = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{pmatrix}\).

### 2.4 Character identities

The procedure outlined in section 2.1 relies on recursion relations for truncated characters \(P_{l}^{(\alpha)}(x, q)\), and does not need closed form results for these characters. Nevertheless, it is interesting and often illuminating to consider exact character formulas, both for truncated partition sums \(P_{l}^{(\alpha)}(x, q)\) and their \(l \to \infty\) limit, \(ch^{(\alpha)}(x, q)\), i.e.,
the character of the full CFT Hilbert space. We have

\[ \text{ch}^{(\alpha)}(x_a, q) = \sum_{\text{restrictions}} \left( \prod_a x_a^{N_a} \right) \text{ch}^{(N_a)}(q), \quad (2.27) \]

where \( \text{ch}^{(N_a)}(q) \) is referred to as the \( N \)-particle cut of the quasi-particle basis \((2.1)\). In the explicit examples of this paper we will often see that there is a remarkable ‘duality’ (involving \( q \rightarrow q^{-1} \)) between the truncated characters \( P^{(\alpha)}_l(x_a = 1, q) \) and the \( N \)-particle cuts \( \text{ch}^{(N_a)}(q) \).

For the example of the Majorana fermion, the truncated partition sums can be written as

\[ P^1_l(x, q) = \sum_{n \text{ even}} x^n q^{\frac{1}{2}n^2} \left[ \begin{array}{c} l \\ \frac{1}{n} \end{array} \right], \]
\[ P^2_{l+\frac{1}{2}}(x, q) = \sum_{n \text{ odd}} x^n q^{\frac{1}{2}n^2} \left[ \begin{array}{c} l + 1 \\ \frac{1}{n} \end{array} \right], \quad (2.28) \]

whereas the \( N \)-fermion cut of the full chiral Hilbert space simply reads

\[ \text{ch}^{(N)}(q) = \frac{q^{\frac{1}{2}N^2}}{(q)_N}. \quad (2.29) \]

Clearly, both lead to the well-known Virasoro characters

\[ \chi_{(1,1)}(q) = \lim_{l \rightarrow \infty} P^1_l(x = 1, q) = \sum_{N \text{ even}} \text{ch}^{(N)}(q), \]
\[ \chi_{(2,1)}(q) = \lim_{l \rightarrow \infty} P^2_{l+\frac{1}{2}}(x = 1, q) = \sum_{N \text{ odd}} \text{ch}^{(N)}(q). \quad (2.30) \]

Upon contemplating expressions for ‘truncated partition sums’ and ‘\( N \)-particle cuts’ one clearly wants to look for guidance in the extensive literature on CFT character formulas. In addition to the canonical ‘bosonic sum formulas’ for the characters irreducible highest weight modules of affine Lie algebras, of the Virasoro algebra, and and of \( \mathcal{W} \)-algebras, there exist a variety of ‘fermionic sum formulas’ and ‘bosonic product formulas’. The relations among the three types take the form of so-called Rogers-Ramanujan (RR) identities, which go back to the 19th century. In the process of proving some of the RR identities, several groups \([39, 40, 41]\) introduced so-called
$L$-finitizations of fermionic and bosonic character expressions and considered recursion relations satisfied by such finitized characters (in the affine Lie algebra case a natural finitization of the character is provided by Demazure modules, see, e.g., [41]). In special cases, the $L$-finitized characters agree with the ‘truncated partition sums’ of section 2.1, and closed form expressions, both of the ‘fermionic sum’ and of the ‘bosonic product’ type are immediately available. We would like to stress that the existing literature on $L$-finitizations only covers some special cases of the ‘CFT statistics’ program that we are pursuing here.

The truncated partition sums for various level-1 WZW models are special polynomials, whose structure is largely dictated by the underlying Lie algebra structure. For the spinon formulation of the $\mathfrak{su}(n)_1$ WZW CFT, the truncated characters have been identified with the full partition sum of a so-called $\mathfrak{su}(n)$ Haldane-Shastry spin chain on a finite number of sites. In earlier papers [42], we provided explicit formulas for truncated characters and $N$-particle cuts for $\mathfrak{su}(n)_1$ spinons. Further character formulas for the $\mathfrak{su}(n)_1$ and $\mathfrak{so}(n)_1$ WZW models will be presented in sections 5 and 6.

3. Generalized fermions in minimal models

A first generalization of the Majorana fermion is encountered in the unitary minimal model $\mathcal{M}^m$ of central charge $c_{\text{CFT}}(m) = 1 - \frac{6}{m(m+1)}$ with $m = 3, 4, \ldots$. We choose $\Phi_{(2,1)}$, of conformal dimension $h_{(2,1)} = \frac{m+3}{4m}$ as the fundamental quasi-particle and study the Virasoro sectors labeled as $(r, s) = (1, 1), (2, 1), \ldots, (m-1, 1)$, that are generated by the repeated action of the modes of $\Phi_{(2,1)}(z)$ on the vacuum (cf. [43]). In the example $m = 4$ we obtain the recursion relations

$$
\begin{pmatrix}
P_{l+1}^1 \\
P_{l+\frac{1}{2}}^2 \\
P_{l+1}^3
\end{pmatrix}
= \begin{pmatrix}
1 & x q^{l+\frac{3}{16}} & x^2 q^{2l+\frac{1}{2}} \\
x q^{l+\frac{5}{16}} & 1 & x q^{l-\frac{1}{16}} \\
0 & x q^{l+\frac{3}{16}} & 1
\end{pmatrix}
\begin{pmatrix}
P_{l}^1 \\
P_{l-\frac{1}{2}}^2 \\
P_{l+\frac{1}{2}}^3
\end{pmatrix},
$$

where $P_{l}^r(x, q) = P_{l}^{r,(1)}(x, q)$. For $q = 1$, the associated eigenvalue $\lambda(x)$ satisfies

$$(1 - \lambda)^3 + x^4 - 2x^2(1 - \lambda) = 0,$$

(3.2)
and we read off that (i) \( \lambda(x) = 1 + \sqrt{2}x + O(x^2) \) for small \( x \) and (ii) \( \lambda(x) \sim x^{\frac{4}{3}} \) for \( x \) large. As explained in section 2, the result (i) has its origin in the non-abelian statistics of the field \( \Phi_{(2,1)} \): the degeneracy factor \( \alpha = \sqrt{2} \) can be understood, as explained in section 2.1, as the largest eigenvalue of the fusion matrix \( N \) of the field \( \Phi_{(2,1)} \)

\[
N = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\] (3.3)

The exponent \( \frac{4}{3} \) in (ii) gives the maximal occupation of the effective one-particle levels for \( \Phi_{(2,1)} \) and sets the maximum \( n_{\text{max}} \) of the associated generalization of the Fermi-Dirac distribution.

For general \( m \), the recursion matrix (for \( q = 1 \)) has the form

\[
\begin{pmatrix}
1 & x & x^2 \\
x & 1 & x \\
0 & x & 1 & x & x^2 \\
x^2 & x & 1 & x & 0 \\
0 & x & 1 & x & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},
\] (3.4)

and by inspecting the characteristic equation one obtains

\[
\alpha_m = 2 \cos\left(\frac{\pi}{m}\right), \quad n_{\text{max}}^m = 2 \left(\frac{m-2}{m-1}\right). \quad (3.5)
\]

Note that in the limit \( m \to \infty \), \( \alpha = 2 \) and \( n_{\text{max}}^m = 2 \), in agreement with the values for the \( \mathfrak{su}(2)_1 \) spinons. In this sense, the \( \Phi_{(2,1)} \) quasi-particles interpolate between the Majorana fermion and the \( \mathfrak{su}(2)_1 \) spinons.

We remark that the recursion relations (3.4) are mathematically identical to an iteration of the recursion relations used by Andrews, Baxter and Forrester (ABF) in their analysis of local height probabilities in specific RSOS models [44]. Interestingly, the two situations are dual in the sense that, where the ABF recursions are in terms of the system size \( m \), the recursion relations (3.1), (3.4) are in terms of a momentum variable \( l \).

The systematics that led us to the recursion matrix (3.4) are reminiscent of the structure underlying the ‘fermionic sum’ expressions for the characters of the minimal
model $\mathcal{M}^m$. Indeed, the truncated characters, which solve the recursion relations set by (3.4), can be represented as ‘fermionic sums’. We refer to [36] for an alternative point of view on the results presented in this section.

4. $Z_k$ parafermions

We move on to an alternative generalization of the Majorana fermion, namely the $Z_k$ parafermion. This generalization arises if we replace the Virasoro algebra by a $\mathcal{W}_k$ algebra and focus on the simplest unitary minimal model with that symmetry. This model, at central charge $c_k = \frac{2(k-1)}{k+2}$, contains a primary field of conformal dimension $h = \frac{k-1}{k}$ and this provides a natural generalization of the Majorana fermion at $k = 2$.

One possible quasi-particle formulation uses as fundamental quanta the modes of a set of $k-1$ parafermion fields $\psi^{(i)}$, $i = 1, \ldots, k-1$, of conformal dimension $h^{(i)} = \frac{i(k-i)}{k}$ and $Z_k$ charge equal to $i$. On the basis of the Lepowski-Primc character formulas [15] (cf. (4.16)), it has been argued that these quanta satisfy Haldane statistics with matrix $G$ equal to twice the inverse of the Cartan matrix of the Lie algebra $A_{k-1}$ [25]. Indeed, we have argued in section 2.2 that both lead to the same central charge $c_{\text{CFT}}$. In this case the explicit solution of (2.21) is given by

$$x_a = \left( \frac{\sin \left( \frac{\pi \alpha}{k+2} \right)}{\sin \left( \frac{\pi (\alpha+1)}{k+2} \right)} \right)^2, \quad a = 1, \ldots, k-1,$$

(4.1)

and indeed

$$\sum_{a=1}^{k-1} L(x_a) = \left( \frac{\pi^2}{6} \right) \left( \frac{2(k-2)}{k+2} \right) = \left( \frac{\pi^2}{6} \right) c_{\text{CFT}},$$

(4.2)

by a well-known identity for dilogarithms (see, e.g., [32, 34]).

The parafermion fields $\psi^{(i)}$ satisfy abelian braiding statistics and it is therefore possible to derive a set of generalized commutation relations for the modes $\psi^{(i)}$ [16]. By exploiting these algebraic relations one may eliminate all fields with $i > 1$ and generate the chiral spectrum by using the modes $\psi_{-s} \equiv \psi^{(1)}_{-s}$ only. For definiteness we first discuss the case $k = 3$ where the systematics of the construction are as follows.
Denoting the modes of the single parafermion field by $\psi_s$ we consider the combinations

$$\phi^{(1)}_{-s} = \psi_s, \quad \phi^{(2)}_{-s} = \psi_s \psi_{-s-\frac{2}{3}}. \quad (4.3)$$

The states that we allow are of the form

$$\phi^{(i_N)}_{-s_N} \cdots \phi^{(i_2)}_{-s_2} \phi^{(i_1)}_{-s_1} |0\rangle, \quad (4.4)$$

with minimal spacing specified as

$$\begin{align*}
&\text{if } i_{l+1} = 1, \ i_l = 1 \text{ then } s_{l+1} - s_l \in \mathbb{Z}_{\geq 0} + \frac{1}{3} \\
&\text{if } i_{l+1} = 2, \ i_l = 1 \text{ then } s_{l+1} - s_l \in \mathbb{Z}_{\geq 0} + \frac{2}{3} \\
&\text{if } i_{l+1} = 1, \ i_l = 2 \text{ then } s_{l+1} - s_l \in \mathbb{Z}_{\geq 0} + \frac{4}{3} \\
&\text{if } i_{l+1} = 2, \ i_l = 2 \text{ then } s_{l+1} - s_l \in \mathbb{Z}_{\geq 0} + \frac{2}{3}
\end{align*} \quad (4.5)$$

and where $s_1 \in \mathbb{Z}_{\geq 0} + \frac{2}{3}$ if $i_1 = 1$ and $s_1 \in \mathbb{Z}_{\geq 0}$ if $i_1 = 2$.

The truncated characters $X_l^{(i)}$ are defined by the restriction that the highest occupied mode is of type $\phi^{(i)}_{-s}$ with $l-s \in \mathbb{Z}_{\geq 0}$. The above rules lead to the recursion relations

$$\begin{align*}
X_{l+1}^{(1)} - X_l^{(1)} &= xq^{l+1} \left( X_{l+\frac{2}{3}}^{(1)} + X_{l-\frac{1}{3}}^{(2)} \right), \\
X_{l+1}^{(2)} - X_l^{(2)} &= x^2 q^{2l+\frac{2}{3}} \left( X_{l+\frac{1}{3}}^{(1)} + X_{l+\frac{2}{3}}^{(2)} \right), \quad (4.6)
\end{align*}$$

with starting point $X_{l}^{(1)} = X_{l}^{(2)} = 0$, for $l \leq -1$, and $X_{-\frac{2}{3}}^{(1)} = 0, X_{-\frac{2}{3}}^{(2)} = 1$. Defining $Y_l = X_{l}^{(1)} + X_{l}^{(2)}$, we obtain

$$Y_{l+1} = xq^{l+1} Y_{l+\frac{2}{3}} + x^2 q^{2l+\frac{2}{3}} Y_{l+\frac{1}{3}} + (1 - x^3 q^{3l+3}) Y_l, \quad (4.7)$$

which, for $q = 1$ and with $Y_l \sim \lambda^l$, leads to

$$\mu^3 - x\mu^2 - x^2 \mu - (1 - x^3) = 0, \quad (4.8)$$

where $\mu = \lambda\frac{1}{3}$. The associated statistics are abelian ($\alpha_{k=3} = 1$), allow a maximal occupation of $n_{k=3}^{\text{max}} = 3$ and lead to the correct central charge of $c_{k=3} = \frac{4}{5}$. In [23], the
eqn. (4.8) for \( \lambda \) was recovered in an approach which starts from the TBA equations for the two \( \mathbb{Z}_3 \) parafermions \( \psi^{(1)} \) and \( \psi^{(2)} \), leading to one particle partition functions \( \lambda_1 \) and \( \lambda_2 \), and then performing the reduction to a single quasi-particle partition function by \( \lambda = \lambda_1 \lambda_2^2 \) where, moreover, we have to take \( z_2 = z_1^2 = x^2 \).

Note that, for \( x = 1 \) and \( q = 1 \), the recursion relation (4.7) is solved by \( Y_{l/3} = F_l \), where \( F_l \) is the \( l \)-th Fibonacci number given explicitly, e.g., by Lucas’ expression

\[
F_l = \left[ \frac{l}{2} \right] \sum_{k=0}^{l/2} \left( \begin{array}{c} l-k \\ k \end{array} \right).
\] (4.9)

For general \( q \), the solution is a deformation of \( F_l \).

Using the generalized commutation relations for the parafermions \( \psi^{(i)} \) one may write down yet another basis of the irreducible parafermion module in terms of the \( \phi^{(i)} \). One possible choice for such a basis is

\[
\phi^{(1)}_{-s_1} \cdots \phi^{(1)}_{-s_{N_1}} \phi^{(2)}_{-t_{N_2}} \cdots \phi^{(2)}_{-t_1} |0\rangle,
\] (4.10)

where the sequences \( \{s_i\} \) and \( \{t_j\} \) each satisfy the conditions (4.5), \( t_1 \in \mathbb{Z}_{\geq 0} \), and \( s_1 \in \mathbb{Z}_{\geq 0} + \frac{2}{3}(N_2 + 1) \). The basis (4.10) immediately leads to the following parafermion character

\[
\text{ch}(x, q) = \sum_{n_1, n_2 \geq 0} x^{n_1+2n_2} \frac{q^{\frac{1}{2}(n_1^2+4n_1n_2+4n_2^2+3n_1)}}{(q)_{n_1}(q^2)_{n_2}},
\] (4.11)

which can be shown to equal the Lepowski-Primc formula [45]

\[
\text{ch}(x, q) = \sum_{n_1, n_2 \geq 0} x^{n_1+2n_2} \frac{q^{\frac{1}{2}(n_1^2+n_1n_2+n_2^2)}}{(q)_{n_1}(q)_{n_2}}.
\] (4.12)

For \( \mathbb{Z}_k \) parafermions with general \( k \geq 2 \) we similarly combine the \( \psi = \psi^{(i)} \) modes into combinations \( \phi^{(i)} \), \( i = 1, \ldots, k-1 \), according to

\[
\phi^{(i)}_{-s} = \psi_{-s} \psi_{-s-\frac{2}{k}} \cdots \psi_{-s-\frac{s(i-1)}{k}}.
\] (4.13)

The truncated partition sums \( X^{(i)}_l \), \( i = 1, \ldots, k-1 \), \( l \in \mathbb{Z}/k \), correspond to all states in a quasi-particle basis built from the \( \phi^{(i)} \) modes such that the highest occupied mode
is of type $\varphi_{-s}^{(i)}$ with $s - l \in \mathbb{Z}_{\geq 0}$. We find the following recursion relation generalizing (4.6)

$$X_{l+1}^{(i)} - X_l^{(i)} = x^i q^{j(l+1)+\frac{j(l-1)}{2}} \sum_{j=1}^{k-1} \frac{X_{l+1-\frac{j(k-1)}{k}}^{(j)}}{x^{j+i+j\leq k-1}},$$

(4.14)

with starting point $X_l^{(i)} = 0$, for $l \leq -1$, and $X_{-1+\frac{1}{k}}^{(i)} = \delta_{i,k-1}$. As for $k = 3$ the recursion relation for the $X_l^{(i)}$ can be cast into a single recurrence relation for $Y_l = \sum_i X_l^{(i)}$. Note that, in contrast to the case $k = 3$, the different ‘sectors’ $l \mod \frac{1}{k}$ are no longer in 1–1 correspondence to the sectors with fixed $\mathbb{Z}_k$ charge. We have checked numerically that the solution for $Z_l = \sum_{j=0}^{k-1} Y_{l-\frac{j}{k}}$,

(4.15)

in the limit $l \to \infty$ indeed approaches the Lepowski-Prime character

$$\text{ch}(x, q) = \sum_{n_1, \ldots, n_{k-1} \geq 0} \left( \prod x^{in_i} \right) \frac{q^\frac{1}{2} \sum n_i G_{ij} n_j}{\prod_{i=1}^{k-1} (q)_{n_i}},$$

(4.16)

where $G_{ij}$ is twice the inverse Cartan matrix of $A_{k-1}$.

The equation for $Y_l$ leads to an equation for $\mu = \lambda^{\frac{1}{k}}$ defined through $Y_l \sim \lambda^l$ as in (4.8), e.g.,

$$k = 4 \quad \mu^4 - x(1+2x^2)\mu^2 - (1+x^2)(1-x^4) = 0,$$

$$k = 5 \quad \mu^5 - x^2 \mu^4 - 2x^4 \mu^3 - x(1-2x^5)\mu^2 - x^3(1-x^5)\mu - (1-x^5)^2 = 0.$$  

(4.17)

We find, for arbitrary $k \geq 2$,

$$\alpha_k = 1, \quad n_{k}^{\max} = \frac{k(k-1)}{2}.$$  

(4.18)

We have checked, for $k = 4, 5$, that the equations (4.17) agree with the equation for $\lambda = \lambda_1 \lambda_2^2 \ldots \lambda_{k-1}^{k-1}$ starting from the TBA equation (2.14) for $k - 1$ parafermions $\psi^{(i)}$.  

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with a statistical interaction matrix $G_{ij}$ given by twice the inverse Cartan matrix of the Lie algebra $A_{k-1}$ and fugacities $z_i = x^i$, thus confirming the conjecture of [23].

5. $\mathfrak{su}(n)_1$ WZW model

We start the discussion of exclusion statistics for conformal field theories based on WZW models with the case of $\mathfrak{su}(n)_1$ (see also section 2.3 for $\mathfrak{su}(2)_1$).

Let $\Lambda_a$, $a = 1, \ldots, n-1$, denote the fundamental weights of $\mathfrak{su}(n)$. We denote by $\chi_{\Lambda_a}$ the (formal) character of the finite dimensional irreducible representation $L(\Lambda_a)$ of $\mathfrak{su}(n)$ with highest weight $\Lambda_a$. Evaluating the character $\chi_{\Lambda_a}$ at the identity gives the dimension of $L(\Lambda_a)$

$$\dim L(\Lambda_a) = \binom{n}{a}, \quad a = 1, \ldots, n-1. \quad (5.1)$$

The affine Lie algebra $\mathfrak{su}(n)_1$ has $n$ integrable highest weight modules corresponding to highest weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$ with conformal dimensions

$$h(\Lambda_a) = \frac{a(n-a)}{2n}, \quad (5.2)$$

while the central charge of $\mathfrak{su}(n)_1$ is given by

$$c_{\text{CFT}} = n-1. \quad (5.3)$$

In accordance with the conventions in [12] let us take the fundamental quasi-particle (spinon) to transform in the irrep $L(\Lambda_{n-1}) = \bar{n}$ of $\mathfrak{su}(n)$. Let us define the truncated characters $P_l^{(a)}(x, q)$, $a = 0, \ldots, n-1$, as in (2.2), where we have assigned the same fugacity $x_i = x$ to all spinons in the $\bar{n}$. We will keep track of the $\mathfrak{su}(n)$ weights as well so, strictly speaking, the $P_l^{(a)}(x, q)$ are character valued polynomials in both $x$ and $q$. The recursion relations for the $P_l^{(a)}(x, q)$ follow straightforwardly from the spinon basis constructed in [12]. In terms of the character valued polynomials $X_l(x, q)$, $l \in \mathbb{Z}/n$, where

$$X_l = P_l^{(a)}, \quad \text{for} \quad nl \equiv a \mod 1, \quad (5.4)$$
they are given by

\[ X_l = \sum_{m=1}^{n} x^{n-m} q^{(n-m)(l-\frac{m}{n})} \left( \prod_{i=1}^{m-1} (1 - x^n q^{n-l-i}) \right) \chi_{\Lambda_m} X_{l-m}, \]  

(5.5)

with \( \Lambda_n \equiv \Lambda_0 \). The starting point for the recursion is \( X_l = 0 \) for \( l < 0 \) and \( X_0 = 1 \). Specializing the character \( \chi_{\Lambda_m} \) to the dimension of \( L(\Lambda_m) \) (cf. eqn. (5.1)) and putting \( q = 1 \), gives the following equation for \( \lambda_{av}(x) \), where \( X_l(x;1) \sim \lambda_{tot}(x)^l = \lambda_{av}(x)^n \)

\[ \lambda_{av}^{n} - \sum_{m=1}^{n} \left( \frac{n}{m} \right) (1 - x^n)^{m-1} x^{n-m} \lambda_{av}^{n-m} = 0, \]  

(5.6)

or, equivalently,

\[ 1 - (x + (1 - x^n)\lambda_{av}^{-1})^n = 0. \]  

(5.7)

The physical solution of (5.7) is given by

\[ \lambda_{av}(x) = \frac{1-x^n}{1-x} = 1 + x + \ldots + x^{n-1}. \]  

(5.8)

The large and small \( x \) limits can be immediately read off and lead to

\[ \alpha = 1, \quad n_{av}^{\text{max}} = n - 1. \]  

(5.9)

The statistics going with the distribution (5.8) generalize the fermionic statistics of section 2.3 in the sense that a state can contain at most \( n_{av}^{\text{max}} = n - 1 \) excitations with the same quantum numbers. These kind of statistics were proposed by Gentile as early as 1940 [47].

Along the same lines one can show that a single quasi-particle species, in the absence of the others, behaves as a \( g \)-on with \( g = (n - 1)/n \), i.e., we have (cf. section 2.2)

\[ n_{a}^{\text{max}} = \frac{1}{g} = \frac{n}{n - 1}. \]  

(5.10)

As a consistency check on the recursion (5.5) we can compute the central charge \( c_{\text{CFT}} \) through eqns. (2.5) and (2.6). Indeed, we find

\[ \frac{\pi^2}{6} c_{\text{CFT}} = \int_{0}^{1} dx \frac{x}{x} \log \lambda_{tot}(x) = n \int_{0}^{1} dx \frac{x}{x} \log \lambda_{av}(x) \]  

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\[
n \int_0^1 \frac{dx}{x} \log(1 - x^n) - n \int_0^1 \frac{dx}{x} \log(1 - x) = n \left( -\frac{1}{n} + 1 \right) \text{Li}_2(1) = \frac{\pi^2}{6} (n - 1), \tag{5.11}
\]

where Li\(_2\)(x) was defined in (2.20) and we have used the value Li\(_2\)(1) = \(\frac{\pi^2}{6}\).

By putting \(x = 1 = q\) in (5.5) we find

\[
X_l(1, 1) = (\chi_{\Lambda_1})^{nl}, \quad nl \in \mathbb{Z}_{\geq 0}. \tag{5.12}
\]

Note that \(X_l(1, 1)\) is precisely the representation content of a one-dimensional spin chain of length \(L = nl\) where the spins transform in the irrep \(L(\Lambda_1) = \mathfrak{n}\) of \(\mathfrak{su}(n)\). For general \((x, q)\) the solution \(X_l(x, q)\) of (5.5) will be a deformation of (5.12). One might wonder whether \(X_l(1, q)\) can be identified with the partition function of such a spin chain. This indeed turns out to be the case, \(X_l(1, q)\) corresponds precisely to the partition function of the \(\mathfrak{su}(n)\) Haldane-Shastry spin chain of length \(L = nl\) \([18, 19]\) (this partition function was computed in, e.g., \([14]\), eqn. (3.12)).

A natural deformation of (5.12) is the (dual) Milne polynomial \(M_\lambda(x_a, q)\) defined in Appendix A, i.e.,

\[
M_\lambda(x_a, q) = \sum_{\mu} \left( \prod_{a} x_a^{(\mu, \alpha_a^\vee)} \right) \tilde{K}_{\mu\lambda}(q) \chi_{\mu}, \tag{5.13}
\]

where \(\tilde{K}_{\mu\lambda}(q)\) is the dual Kostka polynomial defined in Appendix A. Indeed, we find the following solution

\[
X_l(x, q) = q^{\frac{n(n-1)}{2}} M_{\lambda_1}(x, q^{-1}), \quad nl \in \mathbb{Z}_{\geq 0}, \tag{5.14}
\]

where \(x_a = x^{n-a}, a = 1, \ldots, n - 1\). An explicit formula for the dual Milne polynomial entering in (5.14), at \(x = 1\), is (cf. \([35]\), section 2.4.1)

\[
M_{m\Lambda_1}(1, q) = \sum_{m_1, \ldots, m_n \geq 0 \atop m_1 + \cdots + m_n = m} \left[ \begin{array}{c} m \\ m_1, \ldots, m_n \end{array} \right] e^{m_1 \epsilon_1 + \cdots + m_n \epsilon_n} = (q)_m \sum_{m_1, \ldots, m_n \geq 0 \atop m_1 + \cdots + m_n = m} \frac{1}{(q)_{m_1} \cdots (q)_{m_n}} e^{m_1 \epsilon_1 + \cdots + m_n \epsilon_n}, \tag{5.15}
\]

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where the \( \epsilon_i, i = 1, \ldots, n \), are the weights of the \( n \)-dimensional irreducible representation \( L(\Lambda_1) \) of \( \mathfrak{su}(n) \). Clearly, (5.13) has an interpretation as \( (q)_m \times \) the \( m \)-particle cut of the Fock space character of a set of \( n \) quasi-particles transforming in the irreducible representation \( L(\Lambda_1) \) of \( \mathfrak{su}(n) \). It is possible to give a formula analogous to (5.13) for the Milne polynomial \( M_\lambda(1, q) \) corresponding to a general weight \( \lambda \) by introducing a set of quasi-particles for each fundamental irreducible representation \( L(\Lambda_a) \) of \( \mathfrak{su}(n) \) (see, e.g., section 5.1 for \( \mathfrak{su}(3) \)). In terms of Milne polynomials, the \( N \)-particle decomposition of the affine \( \mathfrak{su}(n)_1 \) characters is given by (cf. [50] for \( x = 1 \))

\[
\text{ch}(x, q) = \sum_{N \geq 0} x^N \text{ch}^{(N)}(q) = \sum_{\mu = \sum m_a \Lambda_a} \left( \prod_a x^{(n-a)m_a} \right) \frac{q^{\frac{1}{2}|\mu|^2}}{\prod_{a=1}^n (q)^{m_a}} M_\mu(q). \quad (5.16)
\]

The \( N \)-particle cut \( \text{ch}^{(N)}(q) \) contributes to the affine character in the sector \( L(\Lambda_{-N \mod n}) \). Equation (5.16) emphasizes the fact that the quasi-particles transforming in representations \( L(\Lambda_a) \) should be viewed as composites of \( n-a \) elementary ‘spinons’ transforming in the representation \( L(\Lambda_{n-1}) \). Indeed, the \( \mathfrak{su}(n)_1 \) modules can be built up using the \( L(\Lambda_{n-1}) \) spinons only (cf. section 4). An explicit formula for \( \text{ch}(x, q) \), using only the \( L(\Lambda_{n-1}) \) spinons, was given in [42]

\[
\text{ch}(x, q) = \sum_{\mu = -\sum m_i \epsilon_i} \left( \prod_i x^{m_i} \right) \sum_{p \geq 0} (-1)^p q^{\frac{1}{2}p(p-1)} \frac{q^{\frac{1}{2}|\mu|^2}}{\prod_{i} (q)^{m_i-p}} e^\mu. \quad (5.17)
\]

Finally, we recall that both the \( \mathfrak{su}(n)_1 \) modules and their truncations (whose characters are given by \( X_l(1, q) \)) admit an action of the Yangian \( Y(\mathfrak{su}(n)) \). This action finds its origin in the Haldane-Shastry spin chain [37, 51]. The decomposition of both the affine characters and their truncations \( X_l(1, q) \) under the action of \( Y(\mathfrak{su}(n)) \) were discussed in [42, 52].

### 5.1 \( \mathfrak{su}(3)_1 \)

In the previous section we remarked that an explicit formula for the Milne polynomial \( M_\lambda(q) \) entering the \( N \)-particle characters (5.16) can be given by introducing quasi-particles for each fundamental irreducible representation \( L(\Lambda_a) \) of \( \mathfrak{su}(n) \). Here we make this more explicit for \( \mathfrak{su}(3) \) and remark on the origins of these formulas.
For \text{su}(3) we introduce two sets of three quasi-particles, transforming in the \(L(\Lambda_1) = 3\) and the \(L(\Lambda_2) = 3^*\), respectively. The character of the subspace of the total Fock space containing \(m^a\) particles of type \(a\) \((a = 1, 2)\) is given by

\[
M(m_1, m_2)(q) = \sum_{m_a} \frac{1}{\prod_{a=1}^2 \prod_{i=1}^3 (q)_m} e^{\sum(m^1_i - m^2_i) \kappa_i}. \tag{5.18}
\]

In terms of the quasi-particle Fock space characters \(M(m_1, m_2)(q)\) the most general \text{su}(3) Milne polynomial at \(x_a = 1\) can be expressed as

\[
M_{m_1 \Lambda_1 + m_2 \Lambda_2}(1, q) = (q)_{m_1}(q)_{m_2} \sum_{p \geq 0} (-1)^p q^{\frac{p(p-1)}{2}} (q)_p M_{m_1-p, m_2-p}(q). \tag{5.19}
\]

The equality of (5.16) and (5.17) for \text{su}(3) (using eqn. (5.19)) was demonstrated in [42].

A similar formula holds for the \text{su}(n) Milne polynomial \(M_{m_1 \Lambda_1 + m_2 \Lambda_2}(1, q)\). The quasi-particle expression for general \text{su}(n) weight \(\lambda\) is considerably more complicated.

In subsequent sections we will see other examples of formulas of the type (5.15) and (5.19). Let us briefly remark on the origins of these formulas. Consider a finite dimensional simple (complex) Lie algebra \(\mathfrak{g}\) of rank \(\ell\) and with fundamental weights \(\Lambda_a, a = 1, \ldots, \ell\). Introduce a set of coordinates \(x_i^{(a)}, i = 1, \ldots, \dim L(\Lambda_a)\), for each fundamental irreducible representation \(L(\Lambda_a)\). The Lie algebra \(\mathfrak{g}\) acts on the polynomial ring \(\mathbb{C}[x_i^{(a)}]\) by linear differential operators, preserving the subspaces \(\mathbb{C}[x_i^{(a)}]_{\{m^a\}}\) consisting of homogeneous polynomials of order \(m^a\) in the \(x_i^{(a)}\). In general, though, the algebra does not act irreducibly on \(\mathbb{C}[x_i^{(a)}]_{\{m^a\}}\), but preserves an ideal \(I\) generated by homogeneous relations. (For example, in the case of \text{su}(3) above the ideal \(I\) is generated by \(\sum_i x_i^{(1)} x_i^{(2)}\).) In fact, \(\mathbb{C}[x_i^{(a)}]/I\) is the coordinate ring of a flag manifold associated to \(\mathfrak{g}\). The action of \(\mathfrak{g}\) on \(\mathbb{C}[x_i^{(a)}]_{\{m^a\}}/(\mathbb{C}[x_i^{(a)}]_{\{m^a\}} \cap I)\) is irreducible and the irreducible representation is isomorphic to \(L(\sum m^a \Lambda_a)\). The character of \(L(\sum m^a \Lambda_a)\) follows easily once we have a free resolution of \(\mathbb{C}[x_i^{(a)}]/I\) by applying the Euler-Poincaré principle. Equations like (5.15) and (5.19) can be interpreted as ‘affinizations’ of the above constructions where, instead of the polynomial ring \(\mathbb{C}[x_i^{(a)}]\), we have a Fock space of quasi-particles \(\phi_i^{(a)}(z)\). We refer to [53] for more details, see also [11] for closely related ideas.
5.2 \( su(2)_k, \ k \geq 1 \)

The spinon basis for \( su(2)_k, \ k \geq 1 \), has been worked out in \([21]\). Here also the \( N \)-spinon cuts of the affine characters were found. These were subsequently proved in \([29, 54]\). The \( su(2)_k \) recursion relations were written down in \([22]\). For completeness we briefly review these results. We refer to \([21, 22]\) for more details.

The affine Lie algebra \( su(2)_k \) has \( k + 1 \) integrable highest weight modules, with highest weights \((k - i)\Lambda_0 + i\Lambda_1, \ i = 0, \ldots, k\), and conformal dimension

\[
h(i) \equiv h((k - i)\Lambda_0 + i\Lambda_1) = \frac{i(i + 2)}{4(k + 2)}. \tag{5.20}
\]

The central charge of \( su(2)_k \) is given by

\[
c_{\text{CFT}} = \frac{3k}{k + 2}. \tag{5.21}
\]

As our fundamental quasi-particle we again take the spinon transforming in the irrep \( L(\Lambda_1) = 2 \) of \( su(2) \) and we denote by \( P^{(i)}_l(x, q), \ i = 0, \ldots, k, \ l \in \mathbb{Z}/2 \), the truncated spinon character in the sector \( L((k - i)\Lambda_0 + i\Lambda_1) \). If we denote the character of the \( su(2) \) irrep \( L(m\Lambda_1) = m + 1 \) by \( \chi_m \) (note that \( \chi_m = \chi_{m+1} \) in the notation of section 2.3) the recursion relations take the following form

\[
P^{(i)}_{l+1} = x^i q^{l+h(i)} \chi_i P^{(0)}_l + (1 - x^2 q^{2l}) \sum_{j=1}^{i} x^{i-j} q^{(i-j)(l+h(i)-h(j))} \chi_{i-j} P^{(j)}_l,
\]

\[
P^{(i)}_{l+1} = (1 - x^2 q^{2l+1}) \sum_{j=i}^{k-1} x^{j-i} q^{(j-i)(l+1)+h(i)-h(j)} \chi_{j-i} P^{(j)}_{l+\frac{1}{2}} + x^{k-i} q^{(k-i)(l+1)+h(i)-h(k)} \chi_{k-i} P^{(k)}_{l+\frac{1}{2}}, \tag{5.22}
\]

where \( l \in \mathbb{Z}_{\geq 0} \). Alternatively, one may write the equations (5.22) in matrix form (cf. (2.26)) \([22]\)

\[
\vec{P}_{l+1}(x, q) = \mathcal{R}_l(x, q) \vec{P}_l(x, q), \tag{5.23}
\]

from which it is clear that the grand partition function \( \lambda_{\text{tot}}(x) \) can be obtained as the largest eigenvalue of the matrix \( \mathcal{R}_l(x, 1) \). Explicitly, after specializing the characters...
to the dimensions, one finds for $\mu = \lambda_{\text{tot}}^{\frac{1}{2}}$ (see also [22] for $k = 2$)

$$
\begin{align*}
    k = 2 & \quad (\mu - 1)^2 - x^2(\mu + 1) = 0, \\
    k = 3 & \quad (\mu - 1)^2 - x(\mu - 1) - x^3(\mu + 1) - x^2 = 0, \\
    k = 4 & \quad (\mu - 1)^3 - x^2(\mu - 1)(\mu + 2) - x^4(\mu + 1)^2 = 0,
    \text{etc.}
\end{align*}
$$

The asymptotics of $\lambda_{\text{tot}}(x)$ yield

$$
\alpha = 2 \cos \left( \frac{\pi}{k+2} \right), \quad n_{\text{tot}}^{\max} = 2k, \quad \text{(5.25)}
$$

signaling the presence of non-abelian exclusion statistics for $k \geq 2$.

Clearly, the recursion (5.22) for $x = 1 = q$ is solved by

$$
\begin{align*}
P^{(i)}_{l+\frac{1}{2}}(1, 1) &= \chi_1 (\chi_k)^{2l}, \\
P^{(i)}_l(1, 1) &= \chi_{k-i} (\chi_k)^{2l-1}, \quad l \in \mathbb{Z}_{\geq 0}.
\end{align*}
$$

The solution for general $q$, $x = 1$, and $l \in \mathbb{Z}_{\geq 0}$, turns out to be given by

$$
\begin{align*}
P^{(i)}_{l+\frac{1}{2}} &= q^{h(i)-kl(l-1)-il} Q'_{(k^2i)}(q), \\
P^{(i)}_l &= q^{h(i)-kl(l-1)+i(l-1)} Q'_{(k^2i-1, k-i)}(q), \quad \text{(5.27)}
\end{align*}
$$

where we use the standard notation $(1^{m_1} 2^{m_2} \ldots)$ for partitions, and where $Q'_\lambda(q)$ is the Milne polynomial defined in Appendix A, eqn. (A.11). Namely,

$$
Q'_\lambda(q) = \sum_{\mu=(\mu_1, \mu_2)} K_{\mu\lambda}(q) \chi_\mu. \quad \text{(5.28)}
$$

Here the sum is over all partitions $\mu$ with at most two parts (in order for the $\mathfrak{su}(2)$ character $\chi_\mu$ to be nonvanishing), and $K_{\lambda\mu}(q)$ is the Kostka polynomial. Explicit expressions for the Milne polynomials $Q'_\lambda(q)$ can be found in [35], Theorem 14. That the $l \to \infty$ limit of the expressions (5.27) produces the $\mathfrak{su}(2)_k$ affine characters has been established in [33, 34] for $i = 0$. 

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6. \( \mathfrak{so}(n)_1 \) WZW model: free fermion CFT

It is well-known that the integrable highest weight modules of affine \( \mathfrak{so}(n) \) at level-1 can be realized in terms of \( n \) free Majorana fermions transforming in the vector representation of \( \mathfrak{so}(n) \). In this section we will show that, alternatively, one can realize these modules in terms of quasi-particles transforming in the spinor representation(s) of \( \mathfrak{so}(n) \) – these quasi-particles are referred to as spinons. In a sense they are more fundamental than the fermions since the latter can be expressed in terms of composites of spinons. It will be necessary to discuss the \( n \) even or odd case separately. First we discuss the \( n \) odd case, which is a generalization of the known \( \mathfrak{so}(3)_1 \cong \mathfrak{su}(2)_2 \) result, then we discuss \( n \) even. Several low rank cases not covered by this analysis, nevertheless interesting in their own right, will be discussed separately.

6.1 \( \mathfrak{so}(2n+1)_1, n \geq 2 \)

Let \( \Lambda_a, a = 1, \ldots, n \), denote the fundamental weights of \( \mathfrak{so}(2n+1) \). The dimension of the finite dimensional irreducible representation \( L(\Lambda_a) \) of \( \mathfrak{so}(2n+1) \) with highest weight \( \Lambda_a \) is given by

\[
\dim L(\Lambda_a) = \begin{cases} 
\binom{2n+1}{a} & \text{for } a = 1, \ldots, n-1, \\
2^n & \text{for } a = n.
\end{cases} \tag{6.1}
\]

The affine Lie algebra \( \mathfrak{so}(2n+1)_1 \) has three integrable highest weight modules corresponding to highest weights \( \Lambda = \Lambda_0, \Lambda_1 \) and \( \Lambda_n \), referred to as the singlet (I), vector (v) and spinor (s), respectively. Here, and in the rest of this section, we take \( n \geq 2 \). The cases \( n = 0, 1 \) will be treated separately. The conformal dimensions are given by

\[
h(\Lambda_0) = 0, \quad h(\Lambda_1) = \frac{1}{2}, \quad h(\Lambda_n) = \frac{2n+1}{16}, \tag{6.2}\]

the central charge of \( \mathfrak{so}(2n+1)_1 \) is

\[
c_{\text{CFT}} = \frac{2n+1}{2}, \tag{6.3}\]
and the fusion rules are given by
\[
    s \times v = s, \quad s \times s = 1 + v, \quad v \times v = 1. \quad (6.4)
\]

Consider the \(\mathfrak{so}(2n + 1)_1\) module spanned by the modes \(\phi^{(i)}_a\) of the \(2^n\) spinon operators and let \(P^{(a)}_l(x, q)\) \((a = 1, v, s)\) be the truncated partition function \((2.2)\), where we have assigned the same fugacity \(x_a = x\) to all spinons. The recursion relations look most elegant in terms of the character valued polynomials \(X_l\) and \(Y_l\) \((l \in \mathbb{Z}/2)\)
\[
    X_l = \begin{cases} 
        P^{(1)}_l & \text{for } l \text{ integer}, \\
        P^{(v)}_l & \text{for } l \text{ half odd integer}, 
    \end{cases} \\
    Y_l = P^{(s)}_l \text{ for all } l. \quad (6.5)
\]

We obtain the following recursion relations
\[
    Y_{l+\frac{1}{2}} = (1 - x^2 q^{2l}) Y_l + x q^{2n+1+l} \chi_{\Lambda_l} X_l, \\
    X_{l+\frac{1}{2}} = x^2 q^{2l+\frac{1}{2}} \chi_{\Lambda_l} X_l + (1 - x^2 q^{2l}) \chi_{\Lambda_0} Y_{l-\frac{1}{2}} \\
    \quad + x^2 \sum_{i=2}^{n-1} q^{2l+1-i} \left( \prod_{k=0}^{i-2} (1 - x^2 q^{2l-k}) \chi_{\Lambda_{i-2j}} \right) X_{l-i-\frac{1}{2}} \\
    \quad + x q^{-\frac{2n+1}{2}+l+\frac{1}{2}} \prod_{k=0}^{n-2} (1 - x^2 q^{2l-k}) \chi_{\Lambda_n} Y_{l-\frac{n-2}{2}} \\
    \quad - x^2 \sum_{i=1}^{n} q^{2l-n-\frac{3+i}{2}} \left( \prod_{k=0}^{n-3-i} (1 - x^2 q^{2l-k}) \chi_{\Lambda_{n-i-2j}} \right) X_{l-n+i-2}.
\]

(6.6)

After substituting \(X_l \sim \lambda_{\text{tot}}^l, Y_l \sim \mu \lambda_{\text{tot}}^{l-1}\) in the recursion relation \((6.6)\), putting \(q = 1\) and specializing the characters to the dimension, we find that the equation for the grand partition function \(\lambda_{\text{tot}}(x)\) possesses an (unphysical) root \(\lambda_{\text{tot}}^2 = -(1 - x^2)\) for each \(n\). After dividing out this root, the equation can most succinctly be written as
\[
    (1 - \zeta)^2 = x^2 (1 + \zeta)^{2n-1}, \quad (6.7)
\]

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\[ \zeta = (1 - x^2)\lambda_{\text{tot}}^{\frac{1}{2}}. \]  
(6.8)

It follows immediately that

\[ \alpha = \sqrt{2}, \quad n_{\text{tot}}^{\max} = 4. \]  
(6.9)

Note that the factor \( \sqrt{2} \) arises indeed as the largest eigenvalue of the fusion matrix corresponding to (6.4).

As a check on the recursion relations (6.6) we can compute the resulting central charge through eqns. (2.5) and (2.6).

\[ \pi^2 c_{\text{CFT}} = \int_0^1 \frac{dx}{x} \log \lambda = 2 \int_0^1 \frac{dx}{x} \log(1 - x^2) - 2 \int_0^1 \frac{dx}{x} \log \zeta = 2 \int_0^1 \frac{dx}{x} \log(1 - x^2) - 2 \int_0^1 \frac{d\zeta}{1 - \zeta} \log \zeta - (2n - 1) \int_0^1 \frac{d\zeta}{1 + \zeta} \log \zeta = (-1 + 2 + \frac{1}{2}(2n - 1)) \frac{\pi^2}{6}, \]  
(6.10)

in accordance with (2.3).

The recursion relations (6.6) can be solved exactly. Observe, first of all, that for \( x = 1 = q \) we have

\[ X_l = (\lambda_{\Lambda_1})^{2l}, \quad Y_l = \lambda_{\Lambda_n} (\lambda_{\Lambda_1})^{2l-1}. \]  
(6.11)

For general \( q \) the solution is a \( q \)-deformation of (6.11). As for \( su(n) \) it turns out that the correct \( q \)-deformation is again the Milne polynomial \( M_\lambda(x_a; q) \) defined in Appendix A, i.e.,

\[ X_l = q^{2l^2} M_{2l\Lambda_1}(x, q^{-1}), \quad Y_l = q^{2\lambda + (2l-1)\Lambda_n} M_{2l-1,\Lambda_1+\Lambda_n}(x, q^{-1}), \]  
(6.12)

where \( x_a = x^2 \) for \( a = 1, \ldots, n-1 \) and \( x_n = x \). An explicit expression for the Milne polynomials entering in (6.12), at \( x = 1 \), can be given, namely (cf. (5.15))

\[ M_{\lambda,\Lambda_1}(1, q) = (q)_{\lambda} \sum_{p \geq 0} (-1)^p \frac{q^p\lambda^{-p-1}}{(q)_{p}} M_{(l-2p,0)}(q). \]
\[ M_{l\Lambda_1+\Lambda_n}(1,q) = (q)_1 \left( \sum_{p \geq 0} (-1)^p q^{\frac{1}{2}p(p-1)} (q)_p \right) M_{l-2p,1}(q) \]

\[ - \sum_{p \geq 0} (-1)^p q^{\frac{1}{2}p(p+1)} (q)_p (q)_p \chi_{\Lambda_n} \]

\[ + \sum_{p \geq 0} (-1)^p q^{\frac{1}{2}p(p+1)} (q)_p (q)_p \chi_{l-2p-2,0}(q) \chi_{\Lambda_n} \] (6.13)

where \( M_{(l,m)}(q) \) is the Fock space character for quasi-particles transforming in the \((2n+1)\)-dimensional vector- and \(2^n\)-dimensional spinor representation of \( so(2n+1) \), i.e.,

\[ M_{(l,m)}(q) = \sum_{t_1+\cdots+t_{2n+1}=l \atop m_1+\cdots+m_{2n}=m} \frac{1}{(q)_{t_1} \cdots (q)_{t_{2n+1}}} \frac{1}{(q)_{m_1} \cdots (q)_{m_{2n}}} e^{\sum_i t_i \lambda_i + \sum_n m_n \tilde{\lambda}_n} . \] (6.14)

Here \( \lambda_i \) and \( \tilde{\lambda}_n \) denote the weights of the vector and spinor representation, respectively. The second expression in (6.13) can be simplified, but we have left it in this form to elucidate its origin, i.e. it arises from the resolution of the coordinate ring of a (partial) flag manifold associated to \( so(2n+1) \) (see the remarks in section 5.1).

As for \( su(n) \) (cf. eqn. (5.16)) it should be possible to write a formula for the \( N \)-particle cut of the affine \( so(2n+1) \) characters in terms of Milne polynomials. However, due to the more complicated fusion rules of \( so(2n+1) \) this formula is somewhat more involved. In fact, it is only known for \( so(5) \) (see [56] and section 6.7).

6.2 \( so(2n)_1, n \geq 3 \)

Let \( \Lambda_a, a = 1, \ldots, n \), denote the fundamental weights of \( so(2n) \). The dimension of the finite dimensional irreducible representation \( L(\Lambda_a) \) of \( so(2n) \) with highest weight \( \Lambda_a \) is given by

\[ \dim L(\Lambda_a) = \begin{cases} \binom{2n}{a} & \text{for } 1 \leq a \leq n-2 , \\ 2^{n-1} & \text{for } a = n-1, n . \end{cases} \] (6.15)

The affine Lie algebra \( so(2n)_1 \) has four integrable highest weight modules corresponding to highest weights \( \Lambda_0, \Lambda_1, \Lambda_{n-1} \) and \( \Lambda_n \). Here, and in the rest of this section, we
take $n \geq 3$. The cases $n = 1, 2$ will be treated separately. The conformal dimensions are given by

$$h(\Lambda_0) = 0, \quad h(\Lambda_1) = \frac{1}{2}, \quad h(\Lambda_{n-1}) = h(\Lambda_n) = \frac{n}{8},$$

(6.16)

and the central charge is

$$c_{\text{CFT}} = n.$$

(6.17)

For convenience, let $s = L(\Lambda_{n-1})$ denote the spinor, $c = L(\Lambda_n)$ the conjugate spinor and $v = L(\Lambda_1)$ the vector representation of $\mathfrak{so}(2n)$. As our fundamental quasi-particles we will now take both the $s$ and $c$ spinors. The fusion rules for $\mathfrak{so}(2n)_1$ depend on whether $n$ is even or odd. For $n = 2p$ the relevant fusion rules are

$$s \times s = 1, \quad c \times c = 1, \quad s \times c = v, \quad s \times v = c, \quad c \times v = s,$$

(6.18)

while for $n = 2p + 1$ we have

$$s \times s = v, \quad c \times c = v, \quad s \times c = 1 \quad s \times v = c, \quad c \times v = s.$$

(6.19)

The recursion relations are most elegantly written in terms of character valued polynomials $X_l, Y_l$ and $Z_l$ ($l \in \mathbb{Z}/2$) defined by

$$X_l = \begin{cases} P_l^{(s)} & \text{for } l \text{ integer,} \\ P_l^{(c)} & \text{for } l \text{ half odd integer,} \end{cases}$$

$$Y_l = \begin{cases} P_l^{(s)} & \text{for } l \text{ integer,} \\ P_l^{(c)} & \text{for } l \text{ half odd integer,} \end{cases}$$

$$Z_l = \begin{cases} P_l^{(c)} & \text{for } l \text{ integer,} \\ P_l^{(s)} & \text{for } l \text{ half odd integer.} \end{cases}$$

(6.20)

Note in particular that we have chosen to define $Y_l$ and $Z_l$ alternatingly as the truncated character of the $s$ and $c$ to incorporate the different fusion rules for $n$ even or odd. In

*Note that, even though $\mathfrak{su}(4)_1 \cong \mathfrak{so}(6)_1$, our description of $\mathfrak{so}(6)_1$ differs from the one for $\mathfrak{su}(4)_1$ discussed in section 5.
terms of a single fugacity $x_s = x_c = x$ the recursion relations are

\[
\begin{align*}
Y_{l+\frac{1}{2}} &= (1 - x^2 q^{2l})Z_l + xq^{\frac{n}{2}+l}\chi_{\Lambda_{n-1}}X_l, \\
Z_{l+\frac{1}{2}} &= (1 - x^2 q^{2l})Y_l + xq^{\frac{n}{2}+l}\chi_{\Lambda_n}X_l, \\
X_{l+\frac{1}{2}} &= x^2 q^{2l+\frac{1}{2}}\chi_{\Lambda_1}X_l + (1 - x^2 q^{2l})\chi_{\Lambda_0}X_{l-\frac{1}{2}} \\
&
\quad + x^2 \sum_{i=2}^{n-2} q^{2l+1-i} \left( \prod_{k=0}^{i-2} (1 - x^2 q^{2l-k}) \right) \sum_{j=0}^{[i/2]} \chi_{\Lambda_{i-2j}} X_{l-\frac{i+1}{2}} \\
&
\quad + xq^{-\frac{n}{2}+l+\frac{1}{2}} \left( \prod_{k=0}^{n-3} (1 - x^2 q^{2l-k}) \right) \chi_{\Lambda_{n-1}}Y_{l-n-\frac{3}{2}} \\
&
\quad + xq^{-\frac{n}{2}+l+\frac{1}{2}} \left( \prod_{k=0}^{n-2} (1 - x^2 q^{2l-k}) \right) \chi_{\Lambda_n}Y_{l-n-2} \\
&
\quad - x^2 \sum_{i=2}^{n} q^{2l-i+\frac{1}{2}} \left( \prod_{k=0}^{n-i-4} (1 - x^2 q^{2l-k}) \right) \sum_{j=0}^{[(n-i)/2]} \chi_{\Lambda_{n-i-2j}} X_{l-n+\frac{i+3}{2}}.
\end{align*}
\]  

(6.21)

It is trivial to generalize these relations to account for different fugacities $x_s$ and $x_c$ for the $s$ and $c$ spinons, respectively. One simply replaces $x^2$ by either $x_s^2$ ($x_c^2$) or $x_s x_c$ depending on the fusion rules (6.18) and (6.19). For examples, see sections 6.4 and 6.6.

After putting $X_l \sim \chi_{\Lambda_{\text{tot}}}^l$, $Y_l \sim \mu \chi_{\Lambda_{\text{tot}}}^{l-\frac{1}{2}}$ and $Z_l \sim \mu \chi_{\Lambda_{\text{tot}}}^{l+\frac{1}{2}}$, putting $q = 1$ and specializing the characters to the dimensions we find, as in section 6.1, the equation

\[
(1 - \zeta)^2 = x^2 (1 + \zeta)^{2n-2},
\]

(6.22)

where

\[
\zeta = (1 - x^2)\chi_{\Lambda_{\text{tot}}}^{-\frac{1}{2}}, \quad \mu = \frac{x^{2n-1}}{1 - \zeta}.
\]

(6.23)

It follows (for $n > 1$)

\[
\alpha = 1, \quad n_{\text{tot}}^\text{max} = 4,
\]

(6.24)

while $c_{\text{CFT}} = n$, by a computation similar to the one in (6.10).
We have the following solution to the recursion relations for $\mathfrak{so}(2n)_1$ in terms of the $\mathfrak{so}(2n)$ Milne polynomials

$$
X_l = q^{2l^2} M_{2l\Lambda_1}(x, q^{-1}),
Y_l = q^{\frac{1}{2}l(l+1)} M_{(2l-1)\Lambda_1+\Lambda_{n-1}}(x, q^{-1}),
Z_l = q^{\frac{1}{2}l(l+1)} M_{(2l-1)\Lambda_1+\Lambda_n}(x, q^{-1}),
$$

(6.25)

where $x_a = x^2$ for $a = 1, \ldots, n-2$ and $x_a = x$ for $a = n-1, n$. Explicit expressions for the Milne polynomials entering in (6.25) at $x = 1$ in terms of quasi-particle Fock space characters can be given. Their forms are similar to those for $\mathfrak{so}(2n+1)$ (cf. (6.13)) and will therefore be omitted.

### 6.3 $\mathfrak{so}(1)_1$ and the Ising model

Formally, the Ising model can be viewed as the $\mathfrak{so}(2n+1)_1$ WZW model for $n = 0$. The three primary fields of the Ising model, the identity $1$, the energy operator $\psi$ and the spin operator $\sigma$, correspond to the three ‘integrable’ representations of $\mathfrak{so}(1)_1$, i.e., the identity, the vector and the spinor representation, respectively. The fusion rules, central charge and conformal dimensions are given by the $\mathfrak{so}(2n+1)_1$ expressions (6.1) – (6.4) for $n = 0$. In particular, the dimension of the $\mathfrak{so}(2n+1)_1$ vector and spinor representation are both 1 for $n = 0$. Therefore, it is interesting to see if we can find a basis of the Ising representations in terms of the quasi-particle $\sigma$. With the same definition for $X_l$ and $Y_l$ as in (5.3), the appropriate recursion relations for $n = 0$ read (cf. (6.4))

$$
Y_{l+\frac{1}{2}} = (1 - x^2 q^{2l}) Y_l + x q^{\frac{1}{2}l+1} X_l,
$$

$$
X_{l+\frac{1}{2}} = (1 - x^2 q^{2l}) X_{l-\frac{1}{2}} + x^2 q^{2l+\frac{1}{2}} X_l + x q^{l+\frac{1}{2}} (1 - x^2 q^{2l}) Y_l.
$$

(6.26)

The solution to these $\mathfrak{so}(1)_1$ recursion relations behaves, however, crucially different than the corresponding ones for $\mathfrak{so}(2n+1)_1$, $n \geq 1$. While the latter are polynomials in $x$ and $q$ with positive integer coefficients, the solution to (6.26) has negative coefficients as well. In fact, by construction, we always have $X_l = Y_l = 1$ for $x = q = 1$. As $l \to \infty$ the negative coefficients get pushed off to infinity, however, and the solutions $X_l$ and $Y_l$ (for $x = 1$) do approach the correct Ising model characters. We find the following
The characteristic equation for $\lambda(x)$ at $q = 1$

$$\lambda^\frac{3}{2} - \lambda - \lambda^\frac{1}{2}(1 - x^2) + (1 - x^2)^2 = 0,$$  \hspace{1cm} (6.27)

which, in terms of $\zeta$ defined as in (6.8), can be written as (5.7) with $n = 0$. The small $x$ expansion is

$$\lambda(x) = 1 + \sqrt{2}x + O(x^2).$$ \hspace{1cm} (6.28)

In fact, it is possible to show that (6.26), for $x = 1$, is solved by

$$X_l = q^{2l^2}M_{(2l,0)}(q^{-1}),$$

$$Y_l = q^{\frac{l}{4} + l(2l-1)}M_{(2l-1,1)}(q^{-1}),$$ \hspace{1cm} (6.29)

where $M_{(l,m)}(q)$, for $m = 0, 1$, are given by (6.13) and (6.14) for $n = 0$. Explicitly, after simplification,

$$M_{(m,0)}(q) = \sum_{p \geq 0} (-1)^p q^{p(p-1)} \frac{(q)_m}{(q)_p(q)_m-2p},$$

$$M_{(m,1)}(q) = \sum_{p \geq 0} (-1)^p q^{\frac{1}{2}p(p-1)+(m-p)} \frac{(q)_m}{(q)_p(q)_m-2p}.$$  \hspace{1cm} (6.30)

Or, after inverting and redefining the summation variable $p$,

$$X_l = \sum_{p \text{ even/odd}} q^{l^2} q^{p(p-1)} \frac{(q)_2l}{(q)_p(q)_{l-\frac{p}{2}}},$$

$$Y_{l+\frac{1}{2}} = q^{\frac{l}{16}} \sum_{p \text{ even/odd}} q^{\frac{1}{2}p(p-1)} \frac{(q)_2l}{(q)_p(q)_{l-\frac{p}{2}}},$$ \hspace{1cm} (6.31)

where the sums are over even $p$ for $l \in \mathbb{Z}$ and odd $p$ for $l \in \mathbb{Z} + \frac{1}{2}$. By taking the $l \to \infty$ limit in (6.31) we reproduce well-known fermionic expressions for the Ising model characters.

The finitized Ising characters (6.31) are different from the finitizations obtained from the Majorana fermion description of the Ising model (cf. eqn. (2.28)). In fact, one can show that (6.31) correspond to an alternative, so-called Bailey or $M$-finitization,
of the Ising model Virasoro characters \[57\]. For the Virasoro characters \(\chi_{(1,s)}(q)\), the latter take the form

\[
\chi_{(1,s)}^{(l)}(q) = q^{h_{(1,s)}} \sum_j \left( q^{j(12j+4-3s)} \left[ \frac{2l}{l-3j} \right] - q^{(3j+1)(4j+s)} \left[ \frac{2l}{l-3j-1} \right] \right),
\]

(6.32)

and we have \(X_l = \chi_{(1,1)}^{(l)}\), \(X_{l+\frac{1}{2}} = \chi_{(1,3)}^{(l)}\) and \(Y_{l+\frac{1}{2}} = \chi_{(1,2)}^{(l)}\) for \(l \in \mathbb{Z}\). The identity between the expressions (6.31) and (6.32) follows from a single iteration of one of Slater’s identities (eqn. (A.5) in \[58\]).

As argued above, we expect that the truncated characters are somehow related to treating the spin operator \(\sigma\) as the fundamental quasi-particle. The precise meaning of this statement has, due to the negative coefficients in the truncated characters \(X_l\) and \(Y_l\), so far eluded us.

6.4 \(\mathfrak{so}(2)_1\)

The \(n \to 1\) limit of the \(\mathfrak{so}(2n)_1\) WZW model can be thought of as a \(c_{\text{CFT}} = 1\) model with four representations \(\mathbb{1}, v, s\) and \(c\) with fusion rules as in (6.18). Here the primary fields corresponding to the \(\mathbb{1}, s\) and \(c\) transform in a 1-dimensional representation of \(\mathfrak{so}(2)\) while the primary field for the \(v\) transforms as a doublet.

The appropriate recursion relations for the characters built from the \(s\) and \(c\) quasi-particles read (cf. (6.21))

\[
\begin{align*}
Y_{l+\frac{1}{2}} &= (1 - x_s x_c q^{2l})Z_l + x_s q^{\frac{3}{8}+\frac{1}{2}} X_{l}, \\
Z_{l+\frac{1}{2}} &= (1 - x_s x_c q^{2l})Y_l + x_c q^{\frac{3}{8}+\frac{1}{2}} X_{l}, \\
X_{l+\frac{1}{2}} &= (x_s^2 + x_c^2)q^{2l+\frac{1}{2}} X_{l} + (1 - x_s x_c q^{2l}) X_{l-\frac{1}{2}} + q^{\frac{3}{8}+\frac{1}{2}}(1 - x_s x_c q^{2l})(x_c Y_l + x_s Z_l).
\end{align*}
\]

(6.33)

Here we have introduced separate fugacities \(x_s\) and \(x_c\) for the \(s\) and \(c\) quasi-particles according to the remarks in section 6.2.

With the asymptotic behavior \(X_l \sim \lambda_{\text{tot}}^l\), \(Y_l \sim \mu_s \lambda_{\text{tot}}^{l-\frac{1}{2}}\), and \(Z_l \sim \mu_c \lambda_{\text{tot}}^{l+\frac{1}{2}}\), the equation for \(\lambda_{\text{tot}}\) at \(q = 1\) becomes

\[
\lambda_{\text{tot}}^2 - (x_s^2 + x_c^2)\lambda_{\text{tot}}^\frac{3}{8} - (1 - x_s x_c)(2 + x_s x_c)\lambda_{\text{tot}} + (1 - x_s x_c)^3 = 0.
\]

(6.34)
Note that for \( x_s = x_c = x \) this equation reduces to eqn. (6.22) for \( n = 1 \), with, however, \( n_{\text{tot}}^{\text{max}} = 2 \). Equation (6.34) is the same as that obtained for a system of two particles with Haldane statistics and statistical interaction matrix \( G = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \). Indeed, eqn. (2.14) leads to

\[
\left( \lambda_s - 1 \right) \left( \frac{\lambda_c}{\lambda_s} \right)^{\frac{3}{4}} = x_s,
\]

\[
\left( \lambda_c - 1 \right) \left( \frac{\lambda_s}{\lambda_c} \right)^{\frac{3}{4}} = x_c,
\]

which reduces to eqn. (6.34) for \( \lambda_{\text{tot}} = \lambda_s \lambda_c \). This result shows that the exclusion statistics of \( \mathfrak{so}(2)_1 \) spinors are of the type proposed by Haldane.

6.5 \( \mathfrak{so}(3)_1 \)

This case coincides with \( \mathfrak{su}(2)_2 \), which has been investigated elsewhere \[21, 22\] (see also section 5.2). The recursion reads

\[
Y_{l+\frac{1}{2}} = (1 - x^2 q^{2l}) Y_l + x q^{\frac{3}{4}} + l \chi_2 Y_l,
\]

\[
X_{l+\frac{1}{2}} = x^2 q^{2l+\frac{3}{4}} \chi_3 X_l + (1 - x^2 q^{2l}) X_{l-\frac{1}{2}}
\]

\[+ x q^{\frac{3}{4}} + l (1 - x^2 q^{2l}) \chi_2 Y_l.\]  

(6.36)

where the \( \mathfrak{su}(2) \) characters \( \chi_{2j+1} \) are explicitly given by \( \chi_2 = t + t^{-1} \) and \( \chi_3 = t^2 + 1 + t^{-2} \). At \( q = 1 \), the equation for \( \lambda_{\text{tot}}(x) \) reads

\[
\lambda_{\text{tot}}^{\frac{3}{4}} - (1 + x^2 (t^2 + t^{-2})) \lambda_{\text{tot}} - (1 - x^2)(1 + x^2) \lambda_{\text{tot}}^{\frac{1}{4}} + (1 - x^2)^3 = 0.
\]

(6.37)

Note that for \( t = 1 \) this equation reduces to eqns. (6.4) and (6.8) for \( n = 1 \).

6.6 \( \mathfrak{so}(4)_1 \)

Since \( \mathfrak{so}(4)_1 \cong \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1 \) and the \( \mathfrak{so}(4) \) spinors \( s \) and \( c \) transform as the \( (2, 0) \) and \( (0, 2) \) under \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) \), respectively, one would expect the solution to the \( \mathfrak{so}(4)_1 \) recursion relations to exhibit a similar factorization in terms of the solution to the
\(\mathfrak{su}(2)_1\) recursion relations (cf. section 2.3). This is indeed the case as we will now show.

Denoting \(\chi_s = \chi_{\Lambda_1}\), \(\chi_c = \chi_{\Lambda_2}\), and introducing separate fugacities \(x_s\) and \(x_c\) for the spinors, the \(\mathfrak{so}(4)_1\) recursion relations are (cf. (6.21))

\[
\begin{align*}
Y_{l+\frac{1}{2}} &= (1 - x_s^2 q^{2l})Z_l + x_s q^{\frac{1}{2} + l}\chi_s X_l, \\
Z_{l+\frac{1}{2}} &= (1 - x_c^2 q^{2l})Y_l + x_c q^{\frac{1}{2} + l}\chi_c X_l, \\
X_{l+\frac{1}{2}} &= x_s x_c q^{2l + \frac{1}{2}} \chi_s \chi_c X_l + (1 - x_s^2 q^{2l})(1 - x_c^2 q^{2l}) X_{l-\frac{1}{2}} \\
&
+ q^{l+\frac{1}{2}} \left[ x_s (1 - x_c^2 q^{2l}) \chi_s Y_l + x_c (1 - x_s^2 q^{2l}) \chi_c Z_l \right].
\end{align*}
\]

(6.38)

In terms of \(\mathfrak{su}(2)_1\) quantities \(R^s_l(x_s)\) and \(R^c_l(x_c)\) satisfying

\[
R^s_{l+\frac{1}{2}} = (1 - x_s^2 q^{2l}) R^s_{l-\frac{1}{2}} + x_s q^{l+\frac{1}{2}} \chi_s R^s_l,
\]

(6.39)

(equivalent to the matrix recursion eqn. (2.26)) and similar for \(R^c_l(x_c)\), the recursion (6.38) is solved by

\[
X_l = R^s_l R^c_l, \quad Y_l = R^s_l R^c_{l-\frac{1}{2}}, \quad Z_l = R^s_{l-\frac{1}{2}} R^c_l.
\]

(6.40)

This factorization expresses the decomposition \(\mathfrak{so}(4)_1 \cong \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1\) at the level of truncated partition sums.

Note that upon specializing the \(\mathfrak{su}(2)\) characters by their dimension, the recursion (6.39) is solved by \(R^s(x_s) = (1 + x_s^2 q^{2l})\). Thus, for \(\mathfrak{so}(4)_1\) and equal chemical potentials for all components, the \(s\) and \(c\) spinons behave as four real fermions. Also, note that for \(x_s = x_c = x\) the \(\mathfrak{so}(4)\) results are consistent with (6.22) and (6.23). We conclude that, despite having to separate \(\mathfrak{so}(n)\) for \(n\) even or odd, and having to single out several low \(n\) cases, the results for \(\lambda(x)\) are universal for \(n \geq 2\).

### 6.7 \(\mathfrak{so}(5)_1\)

The case of \(\mathfrak{so}(5)_1\) is covered by the analysis in section 6.1. In addition to the results mentioned there an explicit formula is known for the \(N\)-spinon decomposition. It is given by (cf. [56] for \(x = 1\))

\[
\text{ch}_{\lambda}(x, q) = q^{h(\lambda)} \sum_{\mu=m_1 \Lambda_1 + m_2 \Lambda_2} x^{2m_1 + m_2} \frac{1}{(q)_m_1(q)_m_2} M_{\lambda\mu}^{(1)}(q) M_{\mu}(1, q). \tag{6.41}
\]

37
where $\lambda$ labels the $\mathfrak{so}(5)_1$ sector, i.e., $\lambda = r\Lambda_1 + s\Lambda_2$ ($(r, s) = (0, 0), (1, 0)$ or $(0, 1)$), $h(\lambda)$ is given in eqn. (6.2), and $M_\mu(q)$ is the Milne polynomial of Appendix A. An explicit quasi-particle expression for some of the Milne polynomials was given in (6.13). The fusion rule factor, $M_\mu^{(1)}(q)$, for which a recipe is given in [56], is explicitly given by

$$M_{(0,0),(m,n)}^{(1)}(q) = \begin{cases} q^{1/4(2m^2+2mn+n^2)} \sum_{p \text{ even}} q^{\frac{1}{2}p^2} \left[ \frac{1}{2}n \right] & m, n \text{ even} \\ q^{1/4(2m^2+2mn+n^2)} \sum_{p \text{ odd}} q^{\frac{1}{2}p^2} \left[ \frac{1}{2}n \right] & m \text{ odd}, n \text{ even} \end{cases}$$

$$M_{(1,0),(m,n)}^{(1)}(q) = \begin{cases} q^{1/4(2m^2+2mn+n^2-2)} \sum_{p \text{ odd}} q^{\frac{1}{2}p^2} \left[ \frac{1}{2}n \right] & m, n \text{ even} \\ q^{1/4(2m^2+2mn+n^2-2)} \sum_{p \text{ even}} q^{\frac{1}{2}p^2} \left[ \frac{1}{2}n \right] & m \text{ odd}, n \text{ even} \end{cases}$$

$$M_{(0,1),(m,n)}^{(1)}(q) = \begin{cases} q^{1/4(2m^2+2mn+n^2-1)} \sum_{p \text{ even}} q^{\frac{1}{2}p(p-1)} \left[ \frac{1}{2}(n+1) \right] & m \text{ even}, n \text{ odd} \\ q^{1/4(2m^2+2mn+n^2-1)} \sum_{p \text{ even}} q^{\frac{1}{2}p(p-1)} \left[ \frac{1}{2}(n+1) \right] & m, n \text{ odd} \end{cases}$$

(6.42)

7. $\mathfrak{sp}(2n)_1$ WZW model

Let $\Lambda_a, a = 1, \ldots, n$, denote the fundamental weights of $\mathfrak{sp}(2n)$. The dimension of the irreducible finite dimensional representation $L(\Lambda_a)$ with highest weight $\Lambda_a$ of $\mathfrak{sp}(2n)$ is given by

$$\dim L(\Lambda_a) = \binom{2n}{a} - \binom{2n}{a-2}$$

(7.1)

The affine Lie algebra $\mathfrak{sp}(2n)_1$ has $n+1$ integrable representations with highest weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ with conformal dimension

$$h(\Lambda_a) = \frac{a(2n + 2 - a)}{4(n+2)}, \quad a = 0, \ldots, n$$

(7.2)
while the central charge is given by

\[
c_{\text{CFT}} = \frac{n(2n + 1)}{n + 2}. \tag{7.3}
\]

We will take the fundamental quasi-particle for \(\mathfrak{sp}(2n)\) to transform in the \(2n\) dimensional irrep \(L(\Lambda_1)\) of \(\mathfrak{sp}(2n)\). The relevant \(\mathfrak{sp}(2n)\) fusion rules are

\[
\begin{align*}
\Lambda_1 \times \Lambda_1 & = \Lambda_0 + \Lambda_2, \\
\Lambda_1 \times \Lambda_2 & = \Lambda_1 + \Lambda_4, \\
& \vdots \\
\Lambda_1 \times \Lambda_{n-1} & = \Lambda_{n-2} + \Lambda_n, \\
\Lambda_1 \times \Lambda_n & = \Lambda_{n-1}.
\end{align*}
\]  

(7.4)

Denote the truncated characters in the sector \(L(\Lambda_a), a = 0, \ldots, n\), by \(P^{(a)}(l)(x, q)\), \(l \in \mathbb{Z}/2\). The recursion relations for \(\mathfrak{sp}(2) \cong \mathfrak{su}(2)\) and \(\mathfrak{sp}(4) \cong \mathfrak{so}(5)\) have been worked out in earlier sections. Their generalization to \(\mathfrak{sp}(2n)\), \(n > 2\), will be of the following form: for \(l\) integer

\[
\begin{align*}
P^{(0)}_{l+\frac{1}{2}} & = P^{(0)}_l, \\
P^{(1)}_{l+\frac{1}{2}} & = q^{h(\Lambda_1) + l} x \chi_{\Lambda_1} P^{(0)}_l + (1 - q^{2l} x^2) P^{(1)}_l, \\
P^{(i)}_{l+\frac{1}{2}} & = q^{h(\Lambda_i) + il} x^i \chi_{\Lambda_i} P^{(0)}_l + \sum_{j=1}^{i} q^{h(\Lambda_i) - h(\Lambda_j) + (i-j)l} x^{i-j} (1 - q^{2l} x^2) \chi_{\Lambda_{i-j}} P^{(j)}_l \\
& \quad + \ldots, \quad i \geq 2,
\end{align*}
\]  

(7.5)

and for \(l\) half odd integer

\[
\begin{align*}
P^{(i)}_{l+\frac{1}{2}} & = \sum_{j=1}^{n-1} q^{h(\Lambda_i) - h(\Lambda_j) + (j-i)(l+\frac{1}{2})} x^{j-i} (1 - q^{2l} x^2) P^{(j)}_l \\
& \quad + q^{h(\Lambda_n) - h(\Lambda_i) + (n-i)(l+\frac{1}{2})} x^{n-i} P^{(n)}_l + \ldots, \quad i \leq n - 2, \\
P^{(n-1)}_{l+\frac{1}{2}} & = (1 - q^{2l} x^2) P^{(n-1)}_l + q^{h(\Lambda_n - 1 - h(\Lambda_n) + l + \frac{1}{2})} x \chi_{\Lambda_1} P^{(n)}_l, \\
P^{(n)}_{l+\frac{1}{2}} & = P^{(n)}_l,
\end{align*}
\]  

(7.6)
where the ... stand for additional correction terms of $O(x^2)$ and such that the representation content of the truncated characters at $x = 1 = q$ is given by

$$P^{(a)}_l = \begin{cases} 
\chi_{\Lambda_{n-a}}(\chi_{\Lambda_n})^{2l-1}, & l \text{ integer}, \\
\chi_{\Lambda_{a}}(\chi_{\Lambda_n})^{2l-1}, & l \text{ half odd integer}.
\end{cases} \quad (7.7)$$

Unfortunately, we have not been able to find the complete recursion relations. They turn out to be considerably more complicated than the the other ones analyzed in this paper, in particular non-fundamental representations enter the recursion relations. In principle one might deduce the recursion relations by demanding that they should be solved by the $sp(2n)$ Milne polynomials, i.e.,

$$P^{(a)}_l = \begin{cases} 
q^{a(l-\frac{1}{2})^2 + a(l-\frac{1}{2})} M_{\Lambda_{n-a} + (2l-1)\Lambda_n} (x, q^{-1}), & l \text{ integer}, \\
q^{a(l-\frac{1}{2})^2 + a(l-\frac{1}{2})} M_{\Lambda_{a} + (2l-1)\Lambda_n} (x, q^{-1}), & l \text{ half odd integer},
\end{cases} \quad (7.8)$$

where $x_a = x^a, a = 1, \ldots, n$. The recursion relations (7.4) lead to a small $x$-expansion for $\lambda_{tot}$ with

$$\alpha = 2 \cos \left( \frac{\pi}{n + 1} \right), \quad (7.9)$$

corresponding to the largest eigenvalue of the fusion matrix related to (7.4) (cf. (3.3)).

8. Further remarks on level-1 WZW models

The first remark concerns the status of the various claims made in this paper, in particular on the recursion relations, the fact that their solutions approach the correct CFT characters in the $l \to \infty$ limit, their solutions in terms of Milne polynomials for the level-1 affine Lie algebra case and the various quasi-particle expressions for these Milne polynomials. While we have been able to prove isolated cases of some of these claims, in most cases a rigorous proof is still lacking. However, all the claims are substantiated by extensive Mathematica calculations. This, together with various consistency checks, such as the computation of the resulting central charge $c_{CFT}$, sheds little doubt on the validity of all these claims.

The second remark concerns our construction of the recursion relations for the classical affine Lie algebras at level 1. This construction bears a close resemblance to
the reproduction scheme of Yangian representations in \[59\]. In fact, the quasi-particles that we use precisely transform in the minimal representations that are used to generate all Yangian representations through the reproduction scheme. Moreover, for \(\mathfrak{su}(n)\) we have seen that the Yangian \(Y(\mathfrak{su}(n))\) acts on the integrable highest weight modules of \(\mathfrak{su}(n)_1\), as well as on the truncated Hilbert spaces. The last fact could be understood from the fact that the truncated characters, i.e., the solution \(X_l\) of the recursion relation (5.3), are precisely the characters of the Haldane-Shastry spin chain of length \(L = nl\) which is known to have exact \(Y(\mathfrak{su}(n))\) symmetry (even for finite \(L\)). While \(Y(\mathfrak{so}(n))\) does not act on the integrable \(\mathfrak{so}(n)_1\) modules, the combinations of \(\mathfrak{so}(n)\) characters occurring in the recursion relations (5.6) and (5.21) are precisely the ‘minimal affinizations’ on which \(Y(\mathfrak{so}(n))\) does act (cf. \[59\]). Thus, even for \(\mathfrak{so}(n)\), the solution of the recursion relations and hence also the affine characters are actually virtual characters of \(Y(\mathfrak{so}(n))\). The interpretation in terms of quasi-particles is as follows \[21\]. The spinon representation of \(\mathfrak{so}(n)\) extends to a representation of \(Y(\mathfrak{so}(n))\) \[61\], thus we have an action of \(Y(\mathfrak{so}(n))\) on the one-spinon Fock space. This action extends to the multi-spinon Fock space by co-multiplication. However, this multi-spinon Fock space is bigger than just a direct sum of integrable modules of \(\mathfrak{so}(n)_1\). To get the integrable modules of \(\mathfrak{so}(n)_1\), subtractions are needed. While these subtractions occur in a \(Y(\mathfrak{so}(n))\) invariant manner (and therefore produce virtual \(Y(\mathfrak{so}(n))\) characters) the Yangian no longer acts on the resulting integrable module.

9. Physical applications and final remarks

Before closing this paper, we like to mention some applications of the formalism that we developed.

A particularly interesting application concerns the spinor quasi-particles for the \(\mathfrak{so}(5)_1\) WZW model. In a recent preprint \[25\], we have explained the relevance of these quasi-particles for the description of a particular model for strongly correlated electrons on a 2-leg ladder. The model describes itinerant electrons with kinetic (hopping) term and various interaction constants. By tuning some of these constants, one can

\*The explicit decomposition of the \(\mathfrak{so}(n)_1\) characters as virtual \(Y(\mathfrak{so}(n))\) characters has apparently been achieved in \[60\].
reach a situation where the model is in a critical $SO(5)$ superspin phase. The low-temperature dynamics are then described by the $so(5)_1$ WZW model. Inspecting the quantum numbers of the various primary fields, one finds that the $so(5)$ vector quasi-particles carry integer spin $S = 0, 1$ and charge $0$ or $\pm 2e$. In contrast, the spinor quasi-particles carry the quantum numbers of a single electron and can directly be probed in photo-emission type experiments. This then opens up the possibility that the non-abelian statistics of the $so(5)$ spinor quasi-particles can, in principle, be observed in experiments.

In a very recent preprint [62], Read and Rezayi have proposed an interesting new class of non-abelian quantum Hall states. The corresponding edge theories are conformal field theories, and the quasi-particle statistics, both in the bulk and at the edge, are conveniently studied using the formalism developed here and in [17]. [See [19] for the statistics of quasi-particles over the pfaffian quantum Hall states.] Some first successes of this approach have already been reported in [62].

For further applications to condensed matter systems, one wants to go beyond the level of equilibrium thermodynamics and apply the quasi-particle formalism to transport phenomena. Of great importance then are finite temperature form factors and Green’s functions for the quasi-particles of choice. While this subject is still being developed [63], it is clear that the fractional statistics carried by quasi-particles is expressed in some of these finite temperature characteristics.

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**A. Appendix: $q$-deformed tensor products**

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank $\ell$. Let $\Lambda_a$, $\alpha_a$ and $\alpha_a^\lor$ ($a = 1, \ldots, \ell$) be the fundamental weights, simple roots and simple co-roots, respectively,
normalized such that $\theta^2 = 2$ for the longest root $\theta$. Following the conventions of [56], for any pair of dominant integral weights $\lambda, \mu$, we define a polynomial $\tilde{K}_{\lambda\mu}(q)$ by

$$
\tilde{K}_{\lambda\mu}(q) = \sum_m q^{\tilde{c}(m)} \prod_{a=1}^{\ell} \prod_{i=1}^{\infty} \left[ \frac{P_i^a(m) + m_i^a}{m_i^a} \right],
$$

where the sum is taken over all nonnegative integers $m_i^a$ ($a = 1, \ldots, \ell, i = 1, 2, \ldots$), such that

$$
\mu - \lambda = \ell \sum_a \left( \sum_i \alpha_i^a \right) \alpha_a.
$$

Here

$$
P_i^a(m) = (\mu, \alpha_a^\vee) - \ell \sum_b \sum_{j=1}^\infty \Phi_{ij}^{ab} m_j^b,
$$

$$
\Phi_{ij}^{ab} = 2 \frac{(\alpha_a^a, \alpha_b^j)}{\alpha_a^a \alpha_b^j} \min(i\alpha_a^a, j\alpha_b^j),
$$

and $\tilde{c}(m)$ is the cocharge

$$
\tilde{c}(m) = \frac{1}{2} \sum_{a,b=1}^\ell \sum_{i,j=1}^\infty m_i^a \Phi_{ij}^{ab} m_j^b.
$$

We will refer to the polynomial $\tilde{K}_{\lambda\mu}(q)$ as the dual Kostka polynomial of the Lie algebra $\mathfrak{g}$.

For $\mathfrak{g} = \mathfrak{sl}(n)$, the dual Kostka polynomial $\tilde{K}_{\lambda\mu}(q)$ is related to the conventional Kostka polynomial (see, e.g., [64]) $K_{\lambda\mu}(q)$ by (cf. [35])

$$
\tilde{K}_{\lambda\mu}(q) = q^{n(\mu')} K_{\lambda\mu'}(q^{-1}),
$$

where $\mu'$ is the transposition of $\mu$ as a Young diagram, and

$$
n(\mu) = \sum_i (i - 1)\mu_i = \sum_i \left( \frac{\mu_i^2}{2} \right).
$$

Denote by $V_\lambda$ the irreducible highest weight module of $\mathfrak{g}$ with highest weight $\lambda$. Let $W_a$ denote the “minimal affinization” of $V_{\lambda_a}$, i.e., the minimal irreducible module
of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ (or Yangian $Y(\mathfrak{g})$) such that $V_{\Lambda_a} \subset W_a$. Let $n_a = (\mu, \alpha_a^\vee)$, i.e., $\mu = \sum n_a \Lambda_a$. From an analysis of the Bethe equations for a $\mathfrak{g}$ invariant spin chain (cf. [65, 59]) it follows that we have the following tensor product decomposition as $\mathfrak{g}$ modules

$$W_1^{n_1} \otimes W_2^{n_2} \otimes \ldots \otimes W_\ell^{n_\ell} \cong \sum_{\lambda} \widehat{K}_{\lambda \mu}(1) V_\lambda. \quad (A.7)$$

It is therefore natural to define the $q$-deformed tensor product as [56]

$$[W_1^{n_1} \otimes W_2^{n_2} \otimes \ldots \otimes W_\ell^{n_\ell}]_q \cong \sum_{\lambda} \widehat{K}_{\lambda \mu}(q) V_\lambda. \quad (A.8)$$

We associate character valued polynomials $M_\lambda(x_a, q)$ to the $q$-deformed tensor products (A.8) as follows. Let $\chi_\lambda$ denote the formal character of the finite dimensional irreducible $\mathfrak{g}$ representation $V_\lambda$ with dominant integral weight $\lambda$, i.e.,

$$\chi_\lambda = \sum_{\mu \in P(V_\lambda)} e^\mu, \quad (A.9)$$

where the sum runs over the weights $P(V_\lambda)$ of $V_\lambda$. The character valued polynomial $M_\lambda(x_a, q)$, where $\lambda$ is a dominant integral weight, is now defined as

$$M_\lambda(x_a, q) = \sum_{\mu} \left( \prod_a x_a^{(\mu, \alpha_a^\vee)} \right) \widehat{K}_{\mu \lambda}(q) \chi_\mu. \quad (A.10)$$

Because of eqn. (A.5), the polynomials $M_\lambda(x_a, q)$ at $x_a = 1$ are ‘dual’ to the polynomials $Q_\lambda'(q)$ defined by Milne [66] (see also [35]), i.e.,

$$Q_\lambda'(q) = \sum_{\mu} K_{\mu \lambda}(q) \chi_\mu. \quad (A.11)$$

Therefore we will sometimes refer to $M_\lambda(x_a, q)$ also as a (dual) Milne polynomial. Note that the polynomials $Q_\lambda'(q)$ are also sometimes referred to as ‘modified Hall-Littlewood’ polynomials.
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