Deformations of Nonholonomic Two-plane Fields in Four Dimensions

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Abstract: An Engel structure is a maximally non-integrable field of two-planes tangent to a four-manifold. Any two Engel structures are locally diffeomorphic. We investigate the deformation space of Engel structures obtained by deforming certain canonical Engel structures on four-manifolds with boundary. When the manifold is $\mathbb{RP}^3 \times I$ where $I$ is a closed interval, we show that this deformation space contains a subspace corresponding to Zoll metrics on the two-sphere (metrics all of whose geodesics are closed) modulo ‘projective’ equivalence. The main tool is a construction of an Engel manifold from a three-dimensional contact manifold and a method of reversing this construction. These are special instances of Cartan’s method of prolongation and deprolongation. The double prolongation of a surface $X$ is an Engel manifold of the form $SX \times S^1$ where $SX$ denotes the unit tangent bundle to $X$. The $\mathbb{RP}^3 \times I$ example occurs in this way, since the unit tangent bundle of the two-sphere is $\mathbb{RP}^3$. Besides proving these new results, the article has the flavour of a review.

1 Introduction

The past three decades have seen the birth and flowering of the field of contact (and symplectic) topology ([11]). A key feature giving contact structures a topological, as opposed to purely geometric, nature is the Darboux theorem. This asserts that any two contact (or symplectic) structures of the same dimension are locally diffeomorphic. Hence contact structures have no local invariants. In the language of singularity theory, contact structures are stable germs. (See the next §, just preceding background theorem 2 for a definition of this term.) Besides contact fields, their even-dimensional analogues, and line fields, the only other stable germ of k-plane fields in n-dimensions occurs for 2-plane fields in
4-dimensions. The corresponding structures are called Engel structures \cite{23, 5}. They are not nearly as well investigated as contact structures.

A basic theorem in contact topology, referred to as Gray’s theorem, \cite{14}, asserts that any deformation of a global contact structure is diffeomorphic to the original. (See for example the notes of Eliashberg \cite{11} or the text of Bryant \cite{7}.) The analogous theorem is false in Engel geometry, as was observed by Gershkovich \cite{12}. The reason underlying this failure is simple. One can canonically associate to every Engel structure a certain line field (see §14). When the Engel planes are deformed so is the line field. But line fields on closed manifolds are well-known to be topologically unstable. Very recently, Alex Golubev \cite{13} has proved a version of Gray’s theorem in which the line field is fixed throughout the deformation.

The situation on manifolds with boundary is more subtle. If the Engel manifold is of the form $M \times I$, $I$ an interval, with the line field tangent to $I$ then any small deformation of the line field will yield a diffeomorphic line field. The purpose of this paper is to investigate deformations of certain canonical Engel manifolds of precisely this form. We prove that (A) any sufficiently small deformation of this structure can be embedded into a larger structure $M \times J$ of the same canonical form, but despite this fact, that (B) the typical deformation is not diffeomorphic to the original. We then relate the deformations of a canonical structure on $RP^3 \times I$ to the deformations of the round metric on the two-sphere through Zoll metrics.

There are two surprises in this paper. The first is that in a semi-local sense the moduli space of Engel deformations is equal to the quotient space of the space of (germs of) 2nd order scalar ODEs $y'' = f(x, y, y')$ by (germs of) diffeomorphisms of the $xy$ plane of dependent and independent variables. The second surprise is that in some sense Engel geometry is the product of three-dimensional contact geometry with that of the real projective line. More precisely, an Engel structure induces a transverse contact structure and tangential real projective structure on the four manifold. The tangential structure is tangent to the canonical line field. This second fact is known to a few experts (Bryant and Hsu \cite{6}, but we believe it is still worth emphasizing.

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2 Preliminaries.
2.1 Definitions

We begin rather generally. By a distribution we mean a smooth linear subbundle $D \subset TQ$ of the tangent bundle $TQ$ of a manifold. We will also call $D$ a k-plane field where $k$ is the rank of the subbundle. We may think of $D$ as a locally free sheaf of smooth vector fields on the manifold. Then we can use the notation $[D, D]$ for the sheaf generated by all Lie brackets $[X, Y]$ of sections $X, Y$ of $D$. Set $D^2 = [D, D]$ and continue to take Lie brackets, setting $D^{j+1} = D^j + [D, D^j]$. (One can check that, as sheaves $D^j \subset D^{j+1}$.)

Definition 1 An Engel field (or structure) on a four-manifold $Q$ is a rank 2 subbundle $D \subset TQ$ of the tangent bundle with the property that $D^2$ is a rank 3 distribution and $D^3$ is the entire tangent bundle.

Concretely, an Engel structure on a four-manifold is a two-plane field such that in a neighborhood of any point of the manifold we can find a local frame $X, Y$ for the field such that $X, Y, [X, Y], [X, [X, Y]]$ is a frame for the tangent bundle.

2.2 Prolongations

Let $E$ be a distribution of $k$-planes on a manifold $Q$. The projectivization $\text{IP}_E$ is a fiber bundle over $Q$ whose typical fiber $\text{IP}_E_q$ is the real projective space of dimension $k - 1$ which consists of the one-dimensional subspaces $\ell \subset E_q$. The projectivization inherits a canonical distribution defined by declaring that a curve $(q(t), \ell(t))$ in $\text{IP}_E$ is tangent to the distribution if and only if the derivative of the point of contact, $q(t) \in Q$, lies in the line $\ell(t)$. Alternatively, the distribution plane at the point $(q, \ell)$ is $d\pi_{(q, \ell)}^{-1}(\ell)$ where $\pi : \text{IP}_E \to Q$ is the projection and $d\pi$ its differential. We call $\text{IP}_E$ with this distribution the prolongation of $Q, E$. This is a special case of the of Cartan’s process of prolongation $\text{IP}_E$. See final section of [5], and other references to Cartan therein.

Instead of considering lines in $E$ we could consider rays in $E$. In this way we would obtain a sphere bundle $S(E)$ over $Q$, with a distribution defined in the same way as for $\text{IP}_E$. Its distribution is the pull-back of of the distribution just defined on $\text{IP}_E$ by the 2:1 cover $S(E) \to \text{IP}_E$. Either of the spaces $\text{IP}_E$ or $S(E)$, together with the distribution just defined will be called the prolongation of $E$. When we need to differentiate between them, we will call $S(E)$ the oriented prolongation. Observe that if $E$ has rank 2, then the process of prolonging increases the dimension of the space by 1, but keeps the rank of the distribution the same.

2.3 Examples

2.3.1 The basic Engel manifolds

Let $M$ be a contact three-manifold with contact structure $\xi \subset TM$. Then its prolongation $Q = \text{IP}_\xi$ and oriented prolongation $S\xi$ form the canonical examples
of Engel manifolds. We will refer to $S\xi$ as the basic examples. We compute in §2.3.3 that these are indeed Engel manifolds.

We will be mostly concerned with the case where $\xi$ is topologically trivial, which is to say it admits two independent global sections. In this case the prolongation $S\xi$ is equal to $Q \times S^1$ as a manifold.

### 2.3.2 The contact elements to a surface

Take $X$ to be a two-dimensional surface and $E$ to be its entire tangent space, $TX$. Then its prolongation $IPTX$ is called the space of contact elements to $X$. It forms a contact three-manifold and is one of the most basic examples of a contact manifold. If $X$ is oriented, then $STX$ can be canonically identified with $ST^*X$, the space of co-oriented contact elements. A metric on $X$ defines a map between the oriented prolongation $STX$ and the space of unit tangent vectors to $X$.

Combining this with the previous example, we see that the double prolongation of a surface $X$ forms an Engel manifold. This Engel manifold is diffeomorphic to $STX \times S^1$. One of these sections is the vertical field relative to the fibration $STX \to X$. Its flow rotates tangent lines to the surface. The other section depends on choosing a metric on $X$. Its integral curves correspond to the geodesics of this metric.

If we take $X = S^2$, the standard round two-sphere, then $STX \cong SO(3) \cong \mathbb{R}P^3$. So the double prolongation of the two-sphere yields an Engel structure on the Lie group $SO(3) \times S^1$. It is invariant under left translations in this group. The explicit identification $SO(3) \cong STS^2$ is given by $g \in SO(3) \mapsto (ge_3, ge_1) \in STX$ where $e_1, e_2, e_3$ are the standard basis for $\mathbb{R}^3$. Let $I, J, K$ be the standard basis for the Lie algebra of the rotation group, thought of as left-invariant vector fields. $K$ and $I$ span $\xi$, $K$ is the vertical vector field relative to the projection $STS^2 \to S^2$. The field $I$ represents geodesic flow. (The field $J$ represents a different realization of geodesic flow. See §4.1, especially the remarks following Theorem 4 and the review article of Arnold, concerning these two realizations of geodesic flow.) A global frame for the Engel structure is $\frac{\partial}{\partial \theta}$ and $\cos(\theta)K + \sin(\theta)I$ where $\theta$ is the angular coordinate on the $S^1$ factor.

### 2.3.3 Coordinates

According to the Darboux theorem, centered at any point of a contact three-manifold one can find coordinates $x, y, z$ such that in this neighborhood the contact distribution $\xi$ is described by the vanishing of the one-form $dz - ydx$. That is to say: $\xi_{(x,y,z)} = \{dz - ydx = 0\}$.

Now $dx$ and $dy$ form linear coordinates on each contact plane $\xi_{(x,y,z)}$. Thus almost any contact line $\ell \subset \xi_{(x,y,z)}$ is characterized by its slope, $w$ relative to these coordinates. (The only line not so covered is of course the vertical line $dx = 0$.) In other words: $\ell = \{dy = wdx\}$. Thus $w$ forms an affine coordinate
on the real projective line $IPξ_{x,y,z}$ and $x, y, z, w$ coordinatize the prolongation $IPξ$. It follows from the definition in §2.3.1 that the Engel distribution on $IPξ$ (or $Sξ$) is the two-plane field annihilated by the two one-forms $dz - ydx$ and $dy - wdx$. One calculates that the vector fields

$$W = \frac{\partial}{\partial w} \quad (1)$$

and

$$X = \frac{\partial}{\partial x} + w \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \quad (2)$$

frame this distribution.

### 2.4 Background Results

In this section we present background results on which we will be building, and which are necessary to understand the importance of Engel distributions.

**Background Theorem 1 (Engel normal form.)** Any two Engel manifolds are locally diffeomorphic. More precisely, in a neighborhood of any point of an Engel manifold there exist coordinates $x,y,z,w$ centered at that point such the distribution is spanned by the vector fields $W, X$ given by the equations (1) and (2) above.

This theorem is attributed to Engel. It is proved several times in several different ways in the works of Cartan. See [6], §2, and references therein. A modern proof of this theorem can be found in the text [3], Theorem II.5.1.

The conditions for a distribution to be Engel are open conditions, so that if we slightly perturb an Engel distribution (relatively to the Whitney topology) the resulting distribution is still Engel. In particular, it is locally diffeomorphic to the original. A geometric object is called locally stable (in the sense of singularity theory): if any sufficiently small deformation of the object is locally diffeomorphic to the original. Thus Engel distributions are locally stable. This is a property which they share with contact distributions. Unlike contact distributions, Engel distributions are not globally stable. **ONE OF THE MAIN AIMS OF THIS PAPER IS TO UNDERSTAND AND QUANTIFY THIS FAILURE OF GLOBAL STABILITY.**

In order to explain the importance of Engel distributions we will want another definition.

**Definition 2** A distribution is called regular if for each $j = 2, 3, \ldots$ the dimensions of the spaces $D^j(x)$ obtained by evaluating all of the vector fields in the sheaf $D^j$ at the point $x$ are constant.
Background Theorem 2 (stability theorem) Let \((k,n)\) be the rank of a distribution and the dimension of the space in which it lives. Then the only stable distributions occur when \((k,n) = (1,n), (n-1,n)\) or \((2,4)\). Any stable regular distribution with \((k,n) = (2,4)\) is an Engel distribution.

A proof of this can be found in Vershik and Gershkovich [23], [22], and also [18].

Background Theorem 3 (Gray’s theorem, [14]) Let \(\xi = \xi_0\) be a contact distribution and \(\xi_t\) a deformation of this distribution through contact distributions. Then there is a one-parameter family of diffeomorphisms \(\phi_t\) of the underlying manifold such that \(\phi_t^*\xi_t = \xi_0\)

Below, in lemma 10 of §9, we give a proof of a slight generalization of this theorem. As discussed above, this theorem asserts that contact distributions are globally stable.

Is the analogous theorem true for Engel manifolds? NO! The reason is that Engel manifolds inherit a canonical line field, as we will now describe.

2.5 The Characteristic line field

An Engel distribution \(\mathcal{D}\) determines a line field

\[ L \subset \mathcal{D}. \]

We will call it the characteristic line field. It is a central object in our investigations. \(L\) may be defined by the property

\[ [L, \mathcal{D}^2] \equiv 0 \mod \mathcal{D}^2. \]

In canonical local coordinates \(L\) is the span of the vector field \(W\) of eq(1) above. In the BASIC EXAMPLES the integral curves of \(L\) are the fibers of the fibration \(S\xi \to M\). In other words, they are obtained by spinning the contact line without moving the point of contact. The corresponding vector field will be written \(\partial/\partial \theta\).

Because \(L\) forms such a basic object it is worthwhile giving another description for it which accentuates its intrinsic aspect. Given any distribution \(\mathcal{D}\), the Lie bracket \([X,Y]\) of vector fields \(X, Y\) induces bilinear maps \(\mathcal{D}^j \times \mathcal{D}^k \to \mathcal{D}^{j+k}\). Set \(V_j = \mathcal{D}^j / \mathcal{D}^{j-1}\). Observe that \([X,fY] = f[X,Y] \mod f\). It follows that the Lie bracket induces maps \(V_j \otimes V_k \to V_{j+k}\) of a tensorial nature. In particular if the distribution is regular so that the \(V_j\) are themselves vector bundles then this induced bracket is a linear vector bundle map. In the Engel case \(V_2\) and \(V_3\) are both one-dimensional real vector bundles which we write (locally) as \(\mathbb{R}\).

Thus the Lie bracket induces a bilinear map \(\mathcal{D} \otimes \mathbb{R} \to \mathbb{R}\). Its one-dimensional kernel is \(L \otimes \mathbb{R}\). More concretely, if \(X\) and \(Y\) form a local frame for \(\mathcal{D}\) then \(L \subset \mathcal{D}\) is the kernel of the map \(v \mapsto [X,Y],v \mod (\mathcal{D}^2), v \in \mathcal{D}\).
REMARKS. The Engel line field enjoys several remarkable properties. Fix the endpoints of any sufficiently short integral curve $C$ of $L$. Consider the space of all unparameterized curves tangent to the Engel distribution and joining these particular points, and put the $C^1$-topology on this path space. Then $C$ is an isolated point in this path space. Bryant and Hsu \([6]\) call this property $C^1$-rigidity.

The second, related, property is that the curve $C$ is a minimizing subRiemannian (or Carnot-Caratheodory) geodesic, independent of choice of metric on the 2-plane fields $\mathcal{D}$. See \([19]\), \([20]\), \([17]\). We will not have the occasion to use these properties here though.

### 2.5.1 Application: Deformations on Closed manifolds.

Line fields are essentially vector fields. An important, basic fact in dynamical systems that there are vast classes of vector fields which are globally unstable: most nearby vector fields are not conjugate to them. Hence the following result should come as no surprise:

**Background Theorem 4 (Gershkovich, \([12]\))** A typical deformation of the canonical Engel structure on $S\xi$ or $IP(\xi)$ is not diffeomorphic to the canonical structure.

**Proof:** Suppose for simplicity that $\xi$ is parallelizable so that $S\xi = Q \times S^1$. The characteristic line field associated to the canonical Engel structure on $S\xi$ is spanned by the vertical vector field $\frac{\partial}{\partial \theta}$. All the orbits of this vector field are periodic. Fixing a value of $\theta \in S^1$ determines a global section for this line field and the corresponding Poincare return map is the identity. This is not a generic property for line fields! When the Engel field is deformed, so is its canonical line field. This deformation is rather arbitrary. In lemma 6 of §6 below we show that any contact map isotopic to the identity can be realized as the Poincare return map for some deformed Engel line field. QED

### 3 Global Engel Deformations

The last theorem asserts that Engel deformations are not globally stable. What can we save anything from the wreckage of global stability? Since Engel distributions are not globally stable, what is their deformation space?

**Question** [Courtesy of Viktor Ginzburg.] Suppose that the Engel line field $L_t$ of a deformation $\mathcal{D}_t$ of Engel structures remains constant. Is $\mathcal{D}_t$ then diffeomorphic to $\mathcal{D}_0$?

This question was answered very recently in the affirmatively by Alexander Golubev \([13]\). As a special case of his theorem we have:

**Background Theorem 5 (Golubev)** Let $\mathcal{D}_t$ be a deformation of the canonical Engel structure $\mathcal{D}_0$ on $Q = S\xi$ where $\xi$ is a parallelizable contact structure
on a three-manifold. Suppose that the Engel line fields \( L_t \) are independent of \( t: L_t = L_0 \). Then there is a diffeomorphism \( \phi_t : Q \to Q \) taking \( D_t \) to \( D_0 \).

We could provide a proof of Golubev’s theorem in this special case by slightly altering the lines of the proof of theorem 1 below. We choose not to do this in the interests of space and refer the interested reader to Golubev’s preprint.

We now come to the main subject of this article. Instead of fixing \( L \), we destroy its recurrence by cutting out a section of \( Q = M \times S^1 \) thus obtaining an Engel four-manifold (with boundary) of the form \( M \times I \), \( I \) an interval. The line field is tangent to the interval. If we perturb the field \((C^1-)\) slightly we will obtain a diffeomorphic line-field so that the answer stability question is no longer obvious. **The main question of the paper will be: to what extent does this ‘cutting’ restore the global stability of \( D \)?**

Henceforth we assume that \( \xi \subset TM \) is a parallelizable contact structure. This is the same as an oriented contact structure with a global non-vanishing section. We select two Legendrian direction fields \( V_0, V_1 : M \to S\xi \) which are never collinear. Together these define a domain

\[
\Omega = \Omega(V_0, V_1) \subset S\xi
\]

consisting of all directions \( V \) lying between them. To be more specific, let us use the same symbol \( V \) for a direction and for any vector whose positive span is this direction. Any direction \( V \) can be expanded

\[
V = V(\theta, m) = \cos \theta V_0(m) + \sin \theta V_1(m).
\]

uniquely in terms of the two fields \( V_0, V_1 \). The domain \( \Omega \) consists of those directions \( V \) for which \( 0 \leq \theta \leq \frac{\pi}{2} \).

**Definition 3** The domain \( \Omega = \Omega(V_0, V_1) \) just defined will be called the standard Engel domain determined by the pair \( V_0, V_1 \) of Legendrian direction fields.

The expression \((3)\) defines a global trivialization \( S\xi \cong M \times S^1 \). This trivialization maps \( \Omega \) diffeomorphically onto \( M \times [0, \frac{\pi}{2}] \). The image of the section \( V_0 \) corresponds to \( M \times 0 \) and will be called the bottom of the domain. Similarly, the image of \( V_1 \) will be called the top. The maps \( m \to V(\theta, m) \) defined by \((3)\) are Legendrian direction fields

\[
V(\theta) : M \to S\xi
\]

interpolating between \( V_0 \) and \( V_1 \). Their images will be denoted

\[
M_\theta = \text{image}(V(\theta))
\]

and correspond to \( M \times \{\theta\} \)’s.
By using our trivialization, we may also think of expression (3) as defining a vector field (or direction field) on $\Omega$ which is tangent to the foliation by the $M_\theta$s. Let $\frac{\partial}{\partial \theta}$ denote the vertical vector field on $S^1$ which is tangent to the $S^1$-factor of $M \times S^1$ under our trivialization. Then $V$ together with $\frac{\partial}{\partial \theta}$ span the Engel field. The Engel line field $L$ is spanned by $\frac{\partial}{\partial \theta}$.

**Question 1** Suppose that we deform the Engel distribution $D = D_0$ on a standard Engel domain $\Omega$ into a new Engel distribution $D_t$. Is $(\Omega, D_t)$ diffeomorphic to $(\Omega, D_0)$? If not, how can one describe the space of nontrivial deformations?

**Answer:**

**Theorem 1** For all sufficiently small $t$ there is an Engel isomorphism $\psi_t : (\Omega, D_t) \rightarrow (\Omega_t, D_0)$ between the standard Engel domain with varying Engel structure $D_t$ and a varying Engel domain $\Omega_t = \Omega(V_0, V_1)$ with standard Engel structure $D_0$ induced from the inclusion of $\Omega_t$ into $S\xi$. In other words, we can “straighten out” the Engel deformation at expense of varying the top and bottom of the domain.

If, during the course of deformation, the structure is constant in some neighborhoods of the top and bottom, then we may take $V_0 = V_0$ and $V_1 = \phi_t V_1$ for some contact diffeomorphism $\phi_t : (M, \xi) \rightarrow (M, \xi)$. In other words, the bottom remains unchanged and the top varies by a contact transformation. This transformation $\phi_t$ is canonically defined by the deformation and will be called the **bottom-to-top map**.

**Theorem 2** Let $\Omega_0 = \Omega(V_0, V_1)$ and $\Omega_1 = \Omega(W_0, W_1)$ be two standard Engel domains in $S\xi \cong M \times S^1$. Then there is an orientation preserving Engel diffeomorphism $\Omega_0 \rightarrow \Omega_1$ if and only if there is a contact diffeomorphism of $M, \xi$ which takes the pair $(V_0, V_1)$ of Legendrian direction fields to the pair $(W_0, W_1)$.

**Remark** When we say that the Engel diffeomorphism preserves orientation we mean that it preserves the orientation of both the manifold and the distribution. See §3.4 for more on this.

**Lemma 1 (Realization lemma)** Let $\phi_t : M \rightarrow M$ be any sufficiently $C^1$-small contact isotopy and $V_0, V_1$ any pair of everywhere independent Legendrian vector fields for $(M, \xi)$. Then there is a deformation $D_t$ of the canonical Engel structure on $\Omega(V_0, V_1) \subset S\xi$ for which $D_t = D_0$ near the top and bottom of the domain and such that the bottom-to-top map induced by the deformation is $\phi_t$.

**Remark** “Sufficiently small” in the realization lemma can be replaced by a certain ‘positivity’ condition concerning the sense of twisting of $\phi_t$. This positive twist property may prove to be important in studying questions regarding extensions and existence of Engel structures.
Combining the lemma with the previous two theorems, we have a complete translation of our Engel deformation problem into contact terms. Namely, the problem of understanding the Engel deformations of the Engel domain $\Omega(V_0, V_1)$ which are trivial near the top and bottom is equivalent to the problem of understanding certain deformations $(V_0t, V_1t)$ of the original Legendrian line field pair $(V_0, V_1)$ on $(M, \xi)$. The deformations are restricted to be of the form $V_0t = V_0$ and $V_1t = \phi^*_t V_1$ for some contact isotopy $\phi_t$. The deformation is considered trivial if we can find a contact isotopy $\psi_t$ which preserves $V_0$ (i.e. such that $\psi_1^* V_0 = V_0$) and such that $\psi_1^* V_1t = V_1$. Two such deformations $(V_0, V_1t)$ and $(V_0, \tilde{V}_1t)$ are equivalent if we can find a contact isotopy preserving $V_0$ and taking $V_{1t}$ to $V_{1t}$.

As we will see in the next section, such contact isotopies are very rare. Generically two Engel domains are not diffeomorphic and the moduli of such domains involved functional parameters.

4 The Appearance of Differential Equations

4.1 General 2nd Order Differential Equations

We have reduced our deformation question to a question in contact geometry concerning a pair of Legendrian direction fields. Our method of reduction was formalized by Cartan and is referred to as deprolongation in [6]. We will now see how a further deprolongation reduces this contact question to a question regarding 2nd order differential equations on surfaces.

Consider a pair $(V_0, V_1)$ of direction fields on a three-manifold $M^3$. Construct the local quotient of $M^3$ by the integral curves of $V_0$. By “local quotient” we mean that we restrict the field $V_0$ to a small enough neighborhood $U$ so that the quotient forms a smooth two-manifold $X^2$ with the quotient map $\pi : U \to X^2$ a submersion. For example, $U$ could be a flow-box for $V_0$. Now use the projection $\pi$ to push the integral curves of $V_1$ down to $X^2$. The result is a one-parameter family of curves passing through each point of $X^2$. If the pair spans a contact field in $M$ then the curves in $X$ passing through a given point will (microlocally) be in one-to-one correspondence with tangent directions through that point. In other words, as we move along an integral curve for $V_0$, the tangents of the projections of the $V_1$-curves also move, to first order. A family of curves parameterized by their tangents is nothing more than a 2nd order differential equation.

The discussion above paraphrases that of Arnold, [1], (especially p. 52-53) and Cartan, [9], (beginning on p. 25). We find it helpful to make this discussion more explicit.

Lemma 2 (Normal form theorem) Suppose we are given a of direction fields on a three-manifold whose span is a contact field. Then we can find,
in a neighborhood of any point, local coordinates \(x, y, p\) and vector fields \(V_0, V_1\) spanning the pair of direction fields such that:

\[
V_0 = \frac{\partial}{\partial p},
\]

\[
V_1 = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + f(x, y, p) \frac{\partial}{\partial p},
\]

where \(f\) is a function of \(x, y, z\) which can be chosen so as to vanish at the origin \((0, 0, 0) = 0\). We call this the normal form for the pair. The contact form annihilating \(V_0\) and \(V_1\) is \(dy - pdx\).

Our appendix contains a proof of this lemma. It follows from the lemma that any regular Legendrian curve transverse to \(V_0\) can be parameterized as \(x \mapsto (x, y(x), p(x))\) where \(p(x) = \frac{dy}{dx}\). Now we proceed with the construction above. Project the integral curves of the second line field \(V_1\) along the leaves of \(V_0\) onto this quotient space. According to the normal form for \(V_1\), the resulting curves must satisfy the 2nd order differential equation

\[
\frac{d^2 y}{dx^2} = f(x, y, p).
\]

To reverse this procedure, suppose we are given a system of 2nd order differential equations on a two-dimensional surface \(X\). We are interested in “parameterization independent” differential equations. The Euler-Lagrange equations for a Lagrangian \(L : TX \to \mathbb{R}\), homogeneous of degree one in the velocities, provide a class of example. This class of examples contains the geodesic equations for metrics on \(X\). In modern terms, 2nd order differential equations are usually described in terms of a spray. (Alternatively, they could be described as sections of the bundle \(J^2 X \to TX\) where \(J^2 X\) denotes the 2nd jet space for maps of the real line into \(X\). Recall that the tangent space \(TX\) is the 1st jet space for such maps.) In local coordinates \((x, y)\) such a system of second-order differential equations is written \(\frac{d^2 x}{dt^2} = f_1(x, y, \dot{x}, \dot{y}),\)

\(\frac{d^2 y}{dt^2} = f_2(x, y, \dot{x}, \dot{y})\). The parameterization-independence condition implies that \(f_1(x, y, \lambda \dot{x}, \lambda \dot{y}) = \lambda^2 f_1(x, y, \dot{x}, \dot{y})\) and that \(f_2(x, y, \lambda \dot{x}, \lambda \dot{y}) = \lambda^2 f_2(x, y, \dot{x}, \dot{y})\) for \(\lambda\) a positive number. However these are not sufficient conditions. (For example, all the solutions to a standard form for the geodesic equations, the form in which the Lagrangian is the kinetic energy, are parameterized by a constant multiple of arc-length, and so the equations are not parameter-independent.)

A more concise way to write down parameterization-independent 2nd order differential equations on the surface is the one described above. We take \(M = STX, \xi\) to be the standard contact form, and \(V_0\) to be the vertical direction field. (See §2.3.2.) Then a Legendrian direction field \(V_1\) which is transverse to \(V_0\) describes a 2nd order differential equation on \(X\) whose integral curves have, a priori, no distinguished parameterization. The Legendrian condition
insures that the integral curves of $V_1$ are “holonomic” – they are tangent to the appropriate contact elements. The fact that $V_1$ is a direction field and is defined on $STX$ rather than $TX$ implies that the integral curves have no distinguished parameterization. (However they do have a distinguished orientation. If we wanted unoriented curves then we would do better to work with line fields on $IPTX$.) The fact that $V_1$ is transverse to $V_0$ implies that for each tangent direction there is a corresponding integral curve.

Pick a particular contact element $\ell \subset T_pX$ to $X$. Choose coordinates $(x, y)$ centered at the point $p$ and such that the x-axis in these coordinates represents this contact element $\ell$. Any $C^1$ curve on $X$ whose initial contact element is close to $\ell$ can then be parameterized, near $p$, in the form of a graph: $y = y(x)$. If we use this parameterization then the system of parameterization-independent differential equations becomes a single 2nd order scalar differential equation as above, with $p = dy/dx$.

We call two parameter-independent differential equations on a surface projectively equivalent if there is a diffeomorphism of the surface which takes the solutions of one thought of as unparameterized curves to the solutions of the other.

Example: Central projection is a local projective equivalence between the geodesics on the round sphere and the geodesics on the flat plane.

Projective equivalence of differential equations boils down to contact equivalence between Legendrian line-field pairs. We summarize the above discussion as a background theorem.

**Background Theorem 6** A pair of direction fields $(V_0, V_1)$ on a three-manifold whose span is a contact field locally defines a parameter-independent 2nd order differential equation on a surface. And associated to every such equation is a pair of Legendrian direction fields. The two pairs of direction fields are locally diffeomorphic if and only if their corresponding equations are projectively equivalent.

We have thus reduced our Engel deformation question to one regarding deformations of 2nd order differential equations, a theory well-developed near the turn of the last century. (See Arnol’d (loc. cit.) and Cartan (loc.cit.).) The crucial fact for us is that the space of germs of parameter-independent 2nd order differential equations modulo projective equivalence is infinite-dimensional.

Consider now an Engel deformation $D_t$ of the standard Engel structure $D_0$ on the domain $\Omega(V_0, V_1)$ whose support is some small neighborhood $U$ of a point $(m, \theta)$ relative to our trivialization. It follows from the proof of theorem 1, that the direction fields $V_0, V_1$ described there equal $V_0, V_1$ except in the small neighborhood, $\pi(U)$ of the point $m \in M^3$. This deformation of fields in turn defines a deformation of parameterization-independent second order differential equations, with the support of the deformation being the quotient of $\pi(U)$ by the leaves of $V_0$. 

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Theorem 3  The bottom \( V_0 \), and top \( V_1 \) of the standard Engel domain \( \Omega(V_0, V_1) \), together with a point \( m \in M^3 \) define a germ of a parameter-independent second order differential equation which can be put into the form (4). Consider a point \( q \in \Omega \) with \( \pi(q) = m \), and the corresponding space of Engel deformations with support in a small neighborhood of \( q \). The germ (at \( q \)) of this space of Engel deformations of \( \Omega(V_0, V_1) \) modulo Engel isotopies is isomorphic to the space of germs of deformations of this 2nd order differential equation modulo projective equivalence. This latter space is infinite-dimensional.

Remark As can be seen from an inspection of the proof of theorem 1, this theorem is also true if we consider an apparently larger class of germs – namely germs along an integral curve of the Engel line field. In other words, consider deformations supported in arbitrarily small neighborhoods of the form \( \{m\} \times I \) in our trivialization, and impose the usual equivalence relation on such deformations in order to obtain germs of deformations along a leaf.

The point of the next subsection is to give a global version of these musings on differential equations.

4.2 Geodesics and Zoll Surfaces.

We refer the reader back to §2.3.2. Take \( M = STX \) to be the space of oriented contact elements for an oriented Riemannian surface \( X \) with metric \( g \). Take \( V_0 \) to be the vertical direction field for the fibration \( \pi : STX \to X \). Take \( V_1 \) to be the direction field corresponding to the geodesics on \( X \). The projections of the integral curves of \( V_1 \) along the \( V_0 \) curves are geodesics on \( X \) for the metric \( g \). The geodesics are to be thought of as unparameterized oriented curves. In this manner the Engel domain \( \Omega(V_0, V_1) \) encodes the ‘unparameterized geodesic flow’ for \( X \).

Recall that a ZOLL METRIC on the sphere is a metric all of whose geodesics are closed. Two Riemannian metrics are said to be PROJECTIVELY EQUIVALENT if there is a diffeomorphism which takes the geodesics of one to the geodesics of the other, where the geodesics are to be viewed as unparameterized curves. For example, as discussed earlier, central projection from the hemisphere to the plane establishes a local projective equivalence between the round sphere and the plane.

Theorem 4 Let \( g_t \) be a deformation of the round metric \( g_0 \) on the sphere through Zoll metrics. Let \( V_{1t} \) be the corresponding deformation of Legendrian direction fields on \( STS^2 \) which describe the (unparameterized) geodesics for \( g_t \).

Then

- (A) \( \Omega(V_0, V_1) \) can be realized as an Engel deformation of the standard domain \( \Omega(K, I) \subset SO(3) \times S^1 \cong STS^2 \times S^1 \) corresponding to a deformation which is trivial near the top and bottom.
• (B) the Engel deformation of (A) is trivial if and only if the metrics $g_t$ are isotopic to the round metric on a sphere $S^2(r_t)$ of radius $r_t$, and

• (C) two such ‘Zoll’ deformations are equivalent if and only if the corresponding metrics $g_t$ and $\tilde{g}_t$ are projectively equivalent.

Discussion. Theorem 4 asserts that the Engel deformation space for $\Omega(K,I)$ contains a copy of the “Zoll deformation space” consisting of (Zoll metrics)/(projective equivalence). To get a rough idea of this space, first consider the perhaps more well-known space of Zoll metrics modulo isometric equivalence which was investigated in the last few decades by Guillemin and others. (See [16], [24] and [4].) It is infinite-dimensional and Guillemin showed that its tangent space at the round metric consists of ‘one-quarter’ of the functions on the sphere. Projective equivalence is a coarser equivalence relation than isometric equivalence, so that we have a map (Zoll metrics)/(isometric equivalence) → (Zoll metrics)/(projective equivalence). The fiber of this map consists of the space of all metrics projectively equivalent to a given Zoll metric, modulo the equivalence relation induced by isometry. These fibers were investigated by Beltrami. (See the discussion in [10].) They are discrete unless the given metric has constant curvature in which case the fiber is one-dimensional with the single one-dimensional parameter corresponding to homotheties of the metric (or, what is the same, to the constant curvature). It follows that the space of Zoll metrics modulo projective equivalence is also an infinite dimensional space.

The real content of theorem 4 is assertion (A). Assertion (B) follows immediately from (A) and the discussion of the previous section concerning differential equations. Assertion (C) is classical and can be found in the book of Darboux [10].

In view of the realization lemma, (A) asserts that the geodesic flows $V_t$ for Zoll deformations can be written in the form $\phi_t^* V_1$ for some contact isotopy $\phi_t$. This should be compared to a result of Weinstein. [24], esp. the final remark and the proof of theorem 5.1. See also [10], Appendix B.) Weinstein’s theorem makes this same assertion, but for a different representation of geodesic flow. Weinstein used the Hamiltonian framework for geodesic flow. There the geodesic flow is defined as the Hamiltonian vector field for the Hamiltonian corresponding to kinetic energy. Any nonzero level set of this Hamiltonian can be identified with $ST^*X$. The resulting vector field is a Reeb vector field. It is transverse to the contact field. In contrast, for our theorem the geodesic ‘flow’ consists of the integral curve of a Legendrian direction field. There can be no contact automorphism taking the representation of Weinstein to the representation which we use (Darboux’s representation.) Geometrically, the difference between the two is that in Weinstein’s representation momenta represent tangent elements to wavefronts and as such are perpendicular to the direction of the geodesic’s motion, whereas in our (Darboux) representation momenta represent tangent elements to the geodesics themselves.
5  Contactifying an Engel manifold

We constructed our basic examples of Engel manifolds (§2.3.1) from three-dimensional contact manifolds. Here we reverse this procedure by dividing out by the orbits of the Engel line field. This is an instance of ‘deprolongation’ in Cartan’s sense. See the example titled ‘Prolongation and Deprolongation’ in the final section of [6], and other references to Cartan therein.

5.1  Contactification

Let \((Q, \mathcal{D})\) be an Engel manifold and let \(\mathcal{L}\) denote the foliation of \(Q\) by leaves of the canonical line field \(L \subset \mathcal{D}\). Let \(M = Q/\mathcal{L}\) be the quotient space. Its points are the leaves of the foliation. We will assume that it is a nice quotient, by which we mean that it is a manifold and that the quotient map \(\pi : Q \to M\) is a submersion.

The rank 3 distribution \(\mathcal{D}^2 = [\mathcal{D}, \mathcal{D}]\) is invariant under the flow along \(\mathcal{L}\). This is because the line field \(L\) is defined by the condition \([L, \mathcal{D}^2] \subset \mathcal{D}^2\). We can thus push \(\mathcal{D}^2\) down to the quotient obtaining a distribution \(\xi = \pi_* \mathcal{D}^2\). This quotient distribution has rank 2 because \(L\) is the kernel of the differential \(d\pi\).

**Lemma 3 (Contactification lemma)** If the quotient \(M = Q/\mathcal{L}\) is nice (as defined above) then it is a contact manifold with contact structure \(\xi = \pi_* \mathcal{D}^2\). We call it the contactification of \(Q\).

**Proof:** It only remains to prove that \(\xi\) is contact. This follows immediately from the fact that \([\mathcal{D}^2, \mathcal{D}^2]\) is the entire tangent space to \(Q\). Alternatively, we can use the local normal form (1), (2). In terms of those canonical coordinates \(x, y, z, w\), the leaves are the \(w\)-lines so that \(x, y, z\) coordinatize the quotient \(M\). The Engel distribution is defined locally by the Pfaffian system \(\theta_1 = \theta_2 = 0\) where \(\theta_1 = dz - ydx\) and \(\theta_2 = dy - wdx\). (See also the proof of the contactification theorem above.) The form \(\theta_1\) is the annihilator of \(\mathcal{D}^2\) and pushes down to a contact form on \(M\).

**Example:** Return to the basic example. The original contact manifold \(M\) can be identified with the quotient of \(Q\) by the one-dimensional foliation. The contact structure induced by this construction is the original contact structure.

5.1.1  Transverse Contact Structure

Let \(S \subset Q\) be a three-dimensional submanifold of our Engel manifold transverse to the line field \(L\). We think of \(S\) as a small disc. Then \(S\) may be identified with the quotient space \(U/\mathcal{L}\) where \(U\) is some (perhaps small) neighborhood of \(S\). For example, pick a vector field spanning \(L\) and then using the flow of this vector field to sweep out \(U\). \(S\) inherits a contact structure by identifying it with the quotient. This structure is simply the intersection of \(TS\) with \(\mathcal{D}^2\).
Let \( \tilde{S} \) be another contact slice, and suppose that there is a leaf \( \ell \) of \( L \) which intersects both \( S \) and \( \tilde{S} \). Choose a tube \( U \) around \( \ell \), that is to say, neighborhood \( U \) of \( \ell \) consisting of arcs of leaves. Now \( U \cap S \) and \( U \cap \tilde{S} \) are both realizations of the contactification of \( U \). The flow along the leaves in \( U \) induces a contactomorphism between these two realizations. This situation is summarized by saying that the foliation \( L \) has a transverse contact structure. This means that (A) any local transverse to a given leaf inherits a contact structure, and (B) these contact structures are consistent: the flow along \( L \) defines a map between two different local transverses to the same leaf which is a contact map.

### 5.2 Development Map

Let \((M, \xi)\) be the contactification of an Engel manifold \( Q \), as in §5.1.1, so that \( M = Q/L \). Although the distribution \( D^2 \) descends to a distribution on \( M \) (the contact distribution) the Engel field \( D \) itself does not descend. Define the map

\[
\Psi : Q \to IP^1\xi
\]

by mapping the point \( q \in Q \) to the image of \( D_q \) in \( \xi_{\pi(q)} = \pi_*D_q^2 \). In other words

\[
\Psi(q) = d\pi(q)(D(q)) \subset \xi(\pi(q)).
\]

Note that the image of \( D(q) \) under \( d\pi \) is indeed a one-dimensional subspace of \( \xi(m), m = \pi(q) \) because the kernel of \( d\pi \) is the line field \( L \). We call \( \Psi \) the development map for \( Q \).

Restricting \( \Psi \) to a leaf \( \ell = \pi^{-1}(m) \subset Q \), of the line field \( L \) we obtain a map

\[
\ell \to IP^1 = IP^1(\xi(m)).
\]

This map is monotone, i.e. its derivative is everywhere nonzero. This is merely the condition that that \([L, D] \neq D\), which must alway hold, since \( D^2 \) has rank 3.

This construction gives each leaf \( \ell \) a real projective structure. Let \( w \) denote an affine coordinate on the line \( IP^1 \). We may pull \( w \) back to the leaf thus obtaining a coordinate on part of the leaf. If \( \bar{w} \) is another affine coordinate, then it is related to \( w \) by a linear fractional transformation: \( \bar{w} = (aw + b)/(cw + d) \). Each leaf can be covered by such coordinates. As we vary from leaf to leaf the coefficients \( a, b, c, d \) change but in such a way that they are functions of coordinates \( x, y, z \) on the leaf space \( M \).

The fundamental theorem of projective geometry states that a projective map from one projective line to another is determined by its values on three points. It follows that if \( \Psi : Q \to Q \) is an Engel isomorphism, then the values of \( \Psi \) along any leaf \( \ell \) are completely determined by its values on any small interval of this leaf.

Let us summarize this discussion so far.
Lemma 4 Let $(M, \xi)$ be the contactification of an Engel manifold $Q$. Then there is a canonically defined Engel immersion,

$$\Psi : Q \to IP\xi,$$

which we call the development of $Q$. If $Q$ is closed then the development is a covering map. In any case, it maps the leaves of the canonical line field on $Q$ to the fibers of $IP\xi \to M$.

If $\phi$ and $\psi$ are two Engel immersions whose values agree at three distinct but sufficiently close points of a leaf of the canonical line field, then they are equal everywhere along this leaf.

5.2.1 Tangential Projective structure

The real projective structure on the leaves of $L$ is well-defined whether or not the global quotient $Q/L$ is nice. To construct it, use local slices to the leaf in order to define a local development map. The differential of the contact map intertwining the two slices defines a linear fractional map between the projectivizations of the contact planes at the leaf, and consequently a linear fractional transformation between the corresponding projective coordinates. Consequently the leaf has an atlas whose overlaps are linear fractional transformations. We summarize this by saying that $Q$ has a TANGENTIAL PROJECTIVE STRUCTURE.

5.2.2 Oriented Development

If $D$ and $Q$ are oriented then so are $L$ and $D^2$. (See [12] or [21].) It follows that $M$ and $\xi$ inherit natural orientations. In this case it makes sense to formulate an oriented version of the above proposition.

Proposition 1 If $M$ is the contactification of an oriented Engel manifold $Q$ (assumed “nice” as before) then there is a canonically defined oriented Engel immersion

$$\Psi : Q \to S\xi,$$

which we will again refer to as the DEVELOPMENT or ORIENTED DEVELOPMENT of $Q$.

5.2.3 Orientations

Suppose that the manifold $Q$ and the Engel field $D$ are both oriented. The action of Lie bracketing induces orientations in both $L$ and $D^2$. See [12] or [21] for details of this orientation construction. If $M$ is an oriented three manifold with oriented contact structure $\xi$ then $S\xi$ and its Engel distribution are both canonically oriented. Thus the line field is canonically oriented. Under this orientation of the characteristic line field positive motion corresponds to rotating
the contact directions in the counterclockwise sense. It follows that any Engel diffeomorphism preserving orientations and mapping one standard domain \( \Omega(V_0, V_1) \) to another, \( \Omega(W_0, W_1) \) must take bottom \( (V_0) \) to bottom \( (W_0) \) and top \( (V_1) \) to top \( (W_1) \). See theorem 1 above and the statement following it, where this was alluded to.

### 5.3 Contact slices

**Definition 4** A local cross-section or slice to an Engel manifold is an embedded three-dimensional sub-manifold which is transverse to the Engel line-field. A global cross-section or global slice is a local cross-section which intersects every leaf of the line-field.

**Example:** In the oriented basic example, \( Q = S\xi \), the submanifolds \( M_\theta \) are global slices.

**Lemma 5.** Let \( M \subset Q \) be a cross-section of the Engel manifold \( Q \). Then \( \xi = D^2 \cap TM \) defines a contact field on \( M \). The projectivization \( IP\xi \) of this contact field admits a global section over \( M \), namely \( Z = D \cap TM \). Any sufficiently small neighborhood of \( M \) in \( Q \) is diffeomorphic as an Engel manifold to a neighborhood of \( Z \subset IP\xi \) with its canonical Engel structure.

**Proof:** The slice condition implies that there is a neighborhood of \( U \) of \( M \) such that \( U/L_U \cong M \). The development map induces the Engel isomorphism of neighborhoods. QED

Suppose now that the line field \( L \) is oriented, that \( Q \) is a closed compact manifold, and that \( M \) is a global slice. Define a Poincare return map \( M \to M \) by mapping \( m \in M \) to the next intersection point of the leaf through \( m \) with \( M \). We are guaranteed that such a point exists because of the compactness of \( M \) and \( Q \) and the assumption that \( M \) is a global slice. This map can be realized as the flow of a vector field along \( L \) and this flow preserves \( D^2 \). It follows from this and the lemma 2 that the Poincare return map is a contact map of \( M \).

As a variation on this theme, suppose that \( M \) and \( M' \) are two connected global contact slices which do not intersect, and suppose that the characteristic line field is oriented. Flowing along the leaves of \( L \) now defines a contact map \( M \to M' \). The model example here is the domain \( \Omega(V_0, V_1) \subset S\xi \), with \( M \) and \( M' \) being \( M_0 \) and \( M_{\pi/2} \).

### 6 Proofs

**6.1 Proof of the first half of theorem 1.**

We apply the constructions of the last section to prove the assertions of the first paragraph of theorem 1.
Recall that the $M_\theta = \text{image}(V(\theta))$ are global slices for the unperturbed Engel structure. In terms of the isomorphism of $S\xi$ to $M \times S^1$ these slices are obtained by setting the $\theta$ coordinate equal to a constant. As contact manifolds, they are all isomorphic to $M$ with the section $V(\theta)$ providing the isomorphism.

Now consider a deformation $D^\epsilon$ of the canonical Engel structure having the property that $D^\epsilon(q) = D(q)$ for $q$ near the “top”, $M_{\pi/2}$ and the “bottom”, $M_0$ of $\Omega$. For small $\epsilon$, the $M_\theta$’s will still be global slices, and the canonical line field will be spanned by a vector field $W^\epsilon$ of the form

$$W^\epsilon = \frac{\partial}{\partial \theta} + W^\epsilon_1$$

where $W^\epsilon_1$ is tangent to the foliation by the $M_\theta$. The $M_\theta$’s have a 1-parameter family of contact structures, namely the intersections of the rank 3 distribution $(D^\epsilon)^2$ with their tangent spaces.

By the last lemma of the previous section, the time $t$ flow of the vector field $W^\epsilon$ defines contact diffeomorphisms between slices $M_\theta$ and $M_{\theta+t}$. Let us write $\phi^\epsilon_t$ for this map restricted to the bottom $M_0 = M$ of $\Omega$. Using the unperturbed identification (as the sections $V_\theta$) of each of the $M_t$ with $M$, we may think of $\phi^\epsilon_t$ as a map $M \to M$. For $t$ close to 0 or close to $\pi/2$ these maps $\phi^\epsilon_t$ are contact maps with respect to the original contact structure. This is because the original Engel structure has been left undeformed near the top and bottom of $\Omega$. It follows (see lemma 2) that the contactification of the perturbed Engel structure remains constant under perturbation and is contact diffeomorphic to the original structure $(M, \xi)$.

Let us describe this diffeomorphism explicitly. Let $L^\epsilon$ denote the foliation by curves defined by the perturbed line-field. Then we have that $\Omega/L^\epsilon \cong M$ as contact manifolds. The identification is obtained by mapping the leaf $L^\epsilon(q)$ through a point $q$ to the point where that leaf intersects the bottom $M_0 = M$, of $\Omega$. The projection map $\pi^\epsilon : \Omega \to M$, whose fibers are the leaves of the perturbed foliation $L^\epsilon$, can then be written as

$$\pi^\epsilon(m, \theta) = \phi^{-1}_0(m).$$

And the induced contact structure, $\pi^\epsilon_* (D^2)$, equals the original contact structure.

The intersection of the perturbed Engel field $D^\epsilon$ with the tangent space to the $M_\theta$ defines a line field, say $\ell^\epsilon$ which is canonically oriented. This line field is spanned by a vector field which together with $W^\epsilon$ spans $D^\epsilon$. The development map (see §5.2 and 5.3)

$$\Psi^\epsilon : (\Omega, D^\epsilon) \to (S\xi, D)$$

is now given by:

$$\Psi^\epsilon(q) = d\pi^\epsilon(q)(\ell^\epsilon(q)).$$

For $\theta$ close to 0 or $\pi/2$ the line field $\ell^\epsilon$ on $M_\theta$ is the one spanned by our original $V(\theta)$. 

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It follows from this discussion and the previous section that \( \Psi^\epsilon \) is an Engel embedding of the perturbed Engel manifold with boundary, \((\Omega, D^\epsilon')\) into the original Engel manifold \( S\xi \). Moreover the image of \( \Psi^\epsilon \) is \( \Omega(V_0, V_1') \) where

\[
V_1' = (\phi^\epsilon_{\pi/2})^* V_1.
\]

QED.

### 6.2 Engel automorphisms; Proof of the rest of Theorem 1

In order to prove the assertions in the second paragraph of theorem 1 we need to know more about Engel automorphisms. Let \( \phi : Q \to Q \) be an Engel automorphism: \( \phi^* D = D \). Then \( \phi \) preserves all the invariants of \( D \) so we have \( \phi^* D^2 = D^2 \) and \( \phi^* L = L \). Let \( U \) be any open set such that \( U/L \) is a nice quotient. For example, \( U \) might be a small tubular neighborhood of any local slice \( M \). Then \( \phi \) induces a contact automorphism of the local contactifications, that is, a map from the local quotient \( U/L \) to the local quotient \( \phi(U)/L \). It follows that the restriction of \( \phi \) to a local slice is also a contact automorphism, \( M \to \phi(M) = M' \).

**Lemma 6** The restriction of an Engel isomorphism to a local slice uniquely determines that isomorphism in a neighborhood of the slice. If the slice is global then this restriction uniquely determines the entire map.

**Proof** Let \( \phi \) and \( \psi \) be two such isomorphisms, and let \( M \) denote the slice. Then \( F = \psi^{-1} \circ \phi \) is an automorphism of \( X \) which is the identity on \( M \). We will first show that \( F \) is the identity in a neighborhood of \( M \). If \( m \in M \) then, using the construction in the BASIC EXAMPLE of \( \S 2 \) and lemma 2, we can find canonical Engel coordinates \( x, y, z, w \) such that near \( m \), the slice is defined by \( w = 0 \). The remaining functions \( x, y, z \) are coordinates on \( M \) so that \( F \) has the form: \( F(x, y, z, w) = (x, y, z, g(x, y, z, w)) \). Now the Engel distribution is defined by the Pfaffian system \( \theta_1 = \theta_2 = 0 \) where \( \theta_1 = dz - ydx \) and \( \theta_2 = dy - wdx \). For \( F \) to be an Engel automorphism we must have \( F^* \theta_2 = 0 \mod \{\theta_1, \theta_2\} \), i.e.

\[
dy - gdx = a\theta_1 + b\theta_2
\]

for some functions \( a, b \). Since no \( dz \)'s occur on the left hand side we have \( a = 0 \). Now write \( g = w + h \). We obtain the equation \( \theta_2 - hdx = b\theta_2 \). Considering this equation mod \( \theta_2 \), we obtain \( h = 0 \) so that \( g = w \) and \( F \) is the identity.

To prove the statement regarding global slices, first use the fact that global slices are local slices. It follows that the two maps agree in some neighborhood of the slice. Now use the second paragraph of lemma 4 which asserted that if two Engel automorphisms agree on three points of such a leaf, then they agree everywhere along that leaf.

QED
6.3 Proof of Theorem 2:

$M = M_0$ is a global slice for the Engel structure on $S\xi$. It follows from the lemma above and the preceding discussions that every Engel isomorphism from the one domain to the other is induced by a contact map $\psi : M \to M$. Such an isomorphism must take the boundary of the first domain to the boundary of the second. Being oriented, it must take the bottom boundary to the boundary and top boundary to top boundary. Hence it takes $V_0$ to $W_0$ and $V_1$ to $W_1$.

QED

6.4 Realization Lemma

Here we prove the realization lemma (lemma 1) regarding contact isotopies.

Proof: Let $X = X(m; t)$ be a time-dependent vector field on $M$ which generates the isotopy $\phi$ at time $t = \pi/2$. We may insist that $X(t, m) = 0$ for $t$ near zero and $t$ near $\pi/2$. The original Engel distribution $D$ is spanned by vector fields $V(m, \theta) = \cos(\theta)V_0(m) + \sin(\theta)V_1(m)$ and $\frac{\partial}{\partial m}$ where the contact structure $\xi$ on $M$ is spanned by the vector fields $V_0$ and $V_1$. Define a new Engel structure $E = \text{span}\{V, W\}$ by declaring it to be framed by the same vector field $V(\theta)$, parallel to the levels $M_\theta$ and by the vector field $W = \frac{\partial}{\partial \theta} + X(\theta, m)$ transverse to the $M_\theta$.

Set $U(\theta, m) = \frac{\partial}{\partial m} V$. Then $U = -\sin(\theta)V_0(m) + \cos(\theta)V_1(m)$ and for each fixed $\theta$, the fields $\{U(\theta, \cdot), V(\theta, \cdot)\}$ form a global frame for the contact distribution. One computes that $[W, V] = U + [X(\theta, \cdot), V(\theta, \cdot)]$. Since $X(\theta, \cdot)$ is an infinitesimal contact transformation we can expand $[X, V] = f(\theta, m)V + g(\theta, m)U$ where the contact structure $E$ on $M$ is spanned by the vector fields $V_0$ and $V_1$. Define a new Engel structure $E = \text{span}\{V, W\}$ by declaring it to be framed by the same vector field $V(\theta)$, parallel to the levels $M_\theta$ and by the vector field $W = \frac{\partial}{\partial \theta} + X(\theta, m)$ transverse to the $M_\theta$.

In order to describe the perturbed $D^2$, we need some more information regarding the contact generator $X$. Let $\alpha$ be a choice of contact one-form (By a conformal change of $\alpha$ we may assume that $\partial \alpha(V, U) = 1$.) Let $Z$ be its corresponding Reeb vector field, so that $\alpha(Z) = 1, d\alpha(Z, \cdot) = 0$. Then $X$ must have the form $hZ + X_h$ where $h$ is a function and where $X_h$ is tangent to the contact field and is determined by the relation $dh|_\xi + i_{X_h} d\alpha|_\xi = 0$. One computes that the perturbed $D^2$, which we will call $E^2$, is spanned by $\frac{\partial}{\partial \theta} + hZ$ and $\xi$ or by $W$ and $\xi$ and hence has rank 3. Since $[\xi, \xi]$ contains $Z$ we see that $[\xi, E^2]$ has rank 4 so that the distribution is indeed Engel. A straightforward calculation using the fact that $X$ is an infinitesimal contact automorphism shows that $W$ spans the new characteristic line field: $[W, E^2] \subset E^2$. Let $\phi_s$ denote the time $s$ flow of the time-dependent vector field $X$, starting at time 0. By construction, the time $s$ flow of $W$ is of the form $(m, 0) \mapsto (\phi_s(m), s)$, when applied to a point $(m, 0)$ at the ‘bottom level’ of $\Omega$. It follows that the bottom-to-top map is the desired contact map $\phi$. 

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QED

Remark on smallness of the contact automorphism

The smallness condition on $\phi$ is solely a consequence of the constraint $g \neq -1$ for its generator $X$. This constraint is equivalent to $g > -1$ because $g$ is continuous and is zero near the top and bottom of the domain. We attempt to give this last constraint a geometric meaning. The initial choice of frame $V_0, V_1$ determines an orientation $A = V_0 \wedge V_1$ for $\xi$. $V_0 \wedge V_1 = V \wedge U$. $V \wedge [X, V] = gV \wedge U$. Then $(V \wedge [X, V]) / A = g$ is a measure of the amount of ‘contact spinning’ induced by the infinitesimal contact automorphism $X$. The constraint $g > -1$ says that the contact transformation may not “spin” the contact planes “too much” in the negative direction. This remark may be useful in the future in understanding large deformations or obstruction to existence of of Engel structures. (Eliashberg, private communication.)

6.5 Proof of Theorem 4 on Zoll surfaces

Theorem 4 follows directly from the following lemma, combined with the realization lemma, and the discussion following the statement of the theorem.

Lemma 7 Let $g_t$ be a family of Zoll metrics on the two-sphere. Let $V_t$ be the corresponding Legendrian direction fields on $STS^2$ defining their geodesics. Then there is a contact diffeomorphism $\psi_t$ taking $V_t$ to $V_0$.

The proof of this lemma in turn follows fairly directly from part of the following theorem of Weinstein [24] (see also the appendix of [16]) and the following lemma. At first glance this theorem of Weinstein appears closely related to ours. However, as discussed after the statement of theorem 4, they are rather different theorems.

Background Theorem 7 (Weinstein) Let $g_t$ be a family of Zoll metrics on the two-sphere. Let $Z_t$ be the corresponding Hamiltonian vector fields defined on the energy levels $\Sigma_t = \{H_t = 1\}$ for kinetic energy. There exists a one-parameter family of contact maps $\beta_t : \Sigma_0 \rightarrow \Sigma_t$ such that $\beta_t^* Z_t = Z_0$.

Lemma 8 Let $\xi_t$ be a deformation of contact structures all of which contain a fixed Legendrian line field $L$. Then there is an isotopy $\phi_t$ with $\phi_t^* \xi_t = \xi_0$ and $\phi_t^* L = L$.

Proof of Lemma 8.

The metric $g_t$ induces a diffeomorphism $F_t$ between $STS^2$ and $\Sigma_t$. Namely, a tangent ray $\ell = \text{pos.span} \{ \frac{dx}{dt} \}$ attached at $x \in S^2$ is mapped to the unique covector $p = F_t(\ell)$ whose kernel $\ker(p)$ defines a line orthogonal to $\ell$ and normalized so that $p(\ell) >$ and so that the $g_t$ length of $p$ is 1. It is important to observe that $F_t$ is not a contactomorphism.
We claim that the pull-back \( F_t^*Z_t \) spans the direction field \( V_t \). To see this let \((x(s), p(s))\) denote a solution to the geodesic equation written in canonical variables and let \( \dot{x} \) be the tangent vector to the curve \( x(s) \) in \( S^2 \). It suffices to show that \( F_t(x, p) = (x, \text{pos.span}(\dot{x})) \). This follows directly from the the Legendre transformation relation \( \dot{x}^j = \Sigma g^{ij}p_i \). For the normal vector to a ‘hyperplane’ \( \{ p_i = 0 \} \) is given by raising the indices of \( p_i \). Consequently this normal vector equals the tangent vector to the curve. Moreover the condition \( H = \Sigma g_{ij}p_ip_j = \frac{1}{2} \) implies that \( p_i \dot{x}^i = 1 \) so that the orientations are as claimed.

Now use Weinstein’s theorem to find a diffeomorphism \( \beta_t \) of \( \Sigma_t \) such that \( \beta_t^*Z_t = Z_0 \). (We do not use the fact that \( \beta_t \) is contact.) Set \( f_t = F_t^{-1} \circ \beta_t \circ F_0 \) and check that we have \( f_t^*V_t = V_0 \).

Set \( \xi_t = f_t^*\xi \), a deformation of the contact structure \( \xi \). The pair \((\xi_t, V_0)\) fulfills the hypothesis of lemma 9. Thus there exists a contactomorphism \( \phi_t \) of \( \Sigma \) such that \( \phi_t^*\xi_t = \xi_0 \) and \( \phi_t^*V_0 = V_0 \). The isotopy \( \psi_t = f_t \circ \phi_t \) now satisfies the conclusions of lemma 8.

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**Proof of Lemma 9.**

This can be proved using the standard deformation method as in Moser’s proof of analogous results for volume forms and symplectic forms. We look for an isotopy \( \phi_t \) generated by a time-dependent vector field \( X_t \) which we must solve for. Let \( \theta_t \) denote contact forms for \( \xi_t \). These forms are well-defined up to a nonvanishing function \( f_t \) so we must solve

\[
\phi_t^*(f_t\theta_t) = 0.
\]

Differeniating this equation, using the assumption that \( X_t \) generates the flow, and factoring out a common prefactor of \( \phi_t^* \) yields:

\[
(L_X, f_t)\theta_t + f_t(di_X, \theta_t + i_X, d\theta_t) = -\dot{f}_t - f_t\dot{\theta}_t.
\]

Here dots denote \( d/dt \) and we have used Cartan’s magic formula \( L_X = di_X + i_X d \) for the Lie derivative of forms. We view this as a linear inhomogeneous equation to be solved for \( X_t, f_t \). If they can be solved then the flow \( \phi_t \) of \( X_t \) is the desired isotopy. The only assumption on the inhomogeneous terms is that \( \dot{\theta}(L) = 0 \).

To proceed we assume that \( X_t \subset \xi_t \). Then \( i_X, \theta_t = 0 \) so that \( di_X, \theta_t = 0 \). We now break these equations into their ‘horizontal’ \((\xi_t)\) and ‘vertical’ \((\text{Reeb})\) components by restricting both sides to \( \xi_t \) and by applying both sides to the Reeb vector field. The horizontal equations read:

\[
i_X, d\theta_t|_{\xi_t} = -\dot{\theta}_t|_{\xi_t}
\]

Since \( d\theta_t|_{\xi_t} \) is symplectic this equation has a unique solution \( X_t \in \xi_t \). Moreover the condition \( \dot{\theta}(L) = 0 \) implies that the solution \( X_t \) must lie in \( L \subset \xi_t \).

Using this \( X_t \) and the Reeb field we now write down the ‘vertical’ equations. Recall that the Reeb vector field \( R_t \) is defined by the conditions \( i_{R_t}, \theta_t = 1 \) and
\(i_{R_t}d\theta_t = 0\). Applying \(i_{R_t}\) to both sides of our deformation equation yields the equation

\[L_{X_t}f_t = -\dot{f}_t - f_t \dot{\theta}_t(R_t).\]

Set \(g(t, x) = \log(f_t(x))\). The equation for \(f\) is satisfied if \(g\) satisfies

\[\dot{g} = -L_{X_t}g - \dot{\theta}_t(R_t).\]

This last equation is a linear inhomogeneous equation for \(g_t(x)\) which can be solved by the method of variation of parameters. The solution is

\[g(t, x) = \left(\phi^*_{t} - t \int_{0}^{t} \phi^*_s h(s, \cdot) ds \right)(x),\]

where \(h(t, x) = \dot{\theta}_t(x)(R_t(x))\) and where \(\phi_t\) denotes the time \(t\) flow of the non-autonomous vector field \(X_t\), starting at time 0.

We have found a solution \((X_t, f_t)\) for our system. The flow \(\phi_t\) yields the desired isotopy. Observe that \(\phi_t^* L = L\) since \(X_t\) is everywhere tangent to \(L\).

QED

7 Appendix

Geometry of a pair of direction fields

Here we prove the lemma used in our final section. We restate the theorem with our previous coordinates \((x, y, p)\) replaced by \((x, z, y)\).

**Lemma 9** Let \(\ell_0, \ell_1\) be two oriented line fields on a three-manifold which span a contact field. Then in a neighborhood of any point of we can find coordinates \(x, y, z\) such that

\[\ell_0 = \text{span}\{\frac{\partial}{\partial y}\}\]

\[\ell_1 = \text{span}\{\frac{\partial}{\partial x} + y \frac{\partial}{\partial z} + f \frac{\partial}{\partial y}\}\]

In these coordinates the contact field is \(\{dz - ydx = 0\}\). If the line fields depend continuously on a parameter so do the coordinates and the function \(f\).

**Remark.** The corresponding lemma for a single line field \(\ell_0\) contained in a contact field is fairly well-known. It is the 3-dimensional case of the ‘Darboux’s theorem for Legendrian foliations’. See p. 72 of [3]. This lemma can be found in some form in the text on differential equations by Arnol’d.

**Proof:** The proof proceeds through a series of coordinate and frame changes. The vector field spanning \(\ell_0\) will be denoted by \(Y\) and the vector field spanning \(\ell_1\) will be denoted by \(X\). The coordinates will be denoted \(x, y, z\) and in each given step we will go through one or more coordinate
changes: \( \bar{x} = x(x, y, z), \bar{y} = y(x, y, z), \bar{z} = z(x, y, z) \) and perhaps a frame change: \( Y \rightarrow fY, X \rightarrow gX \) with \( f, g \) positive functions.

Step 1. Find coordinates \( x, y, z \) straightening \( \ell_0 \), so that in this neighborhood \( \ell_0 \) is spanned by \( \frac{\partial}{\partial y} \). Then \( \ell_1 \) is spanned by a vector field of the form \( f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z} \). Without loss of generality, we may assume \( f_1(0) > 0 \). (If \( f_1 < 0 \) we can change the coordinate \( x \) to \(-x\) to effect this change. If \( f_1(0) = 0 \) then it must be that \( f_3(0) \neq 0 \) or otherwise the two line-fields are not transverse. In this case we can switch the roles of \( x \) and \( y \).) Dividing by \( f_1 \) we have found coordinates with

\[
Y = \frac{\partial}{\partial y},
\]

\[
X = \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}
\]

By making a linear change of coordinates of the form

\[
\bar{x} = x
\]

\[
\bar{y} = y + c_1 x
\]

\[
\bar{z} = z + c_2 x
\]

we can cancel the constant term of \( f_2 \) and \( f_3 \). This is done by taking \( c_1 = -f_2(0), c_2 = -f_3(0) \), and using the chain rule. Reverting to the unbarred coordinates, we can thus impose the condition:

\[
f_2(0) = f_3(0) = 0
\]

Step 2. We calculate that

\[
[Y, X] = \frac{\partial f_3}{\partial y} \frac{\partial}{\partial z} + \frac{\partial f_2}{\partial y} \frac{\partial}{\partial y} + \frac{\partial f_2}{\partial y} \frac{\partial}{\partial y}
\]

The contact condition is that \( X, Y, [X, Y] \) are linearly independent everywhere, which means that \( \frac{\partial f_3}{\partial y} \neq 0 \). We may assume that \( \frac{\partial f_3}{\partial y} > 0 \) for if not, change the coordinate \( z \) to \(-z\) to insure this. Now define new coordinates

\[
\bar{x} = x
\]

\[
\bar{y} = f_3(x, y, z)
\]

\[
\bar{z} = z
\]

Then

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial f_3}{\partial x} \frac{\partial}{\partial \bar{y}}
\]

\[
\frac{\partial}{\partial y} = \frac{\partial f_3}{\partial y} \frac{\partial}{\partial \bar{y}}
\]

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\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial \bar{z}} + \frac{\partial f_3}{\partial z} \frac{\partial}{\partial y}
\]

It follows that in the new coordinates,

\[
Y = \frac{\partial f_3}{\partial y} \frac{\partial}{\partial y}
\]

\[
X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} + (f_3 + L_x f_2) \frac{\partial}{\partial y}.
\]

Now, divide \(Y\) by the positive function \(\frac{\partial f_3}{\partial y}\) and set \(f = (f_3 + L_x f_2)\) and revert to unbarred coordinates. We now have

\[
Y = \frac{\partial}{\partial y}
\]

\[
X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} + f \frac{\partial}{\partial y}
\]

where \(f\) is some arbitrary smooth function of \(x, y, z\).

Finally, we observe that if the line fields depend smoothly on a parameter, then all our coordinate changes and our final function \(f\) can be made to depend smoothly on this parameter as well.

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