L-functions of symmetric powers of Kloosterman sums

(unit root L-functions and p-adic estimates)

C. Douglas Haessig*

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Abstract

The L-function of symmetric powers of classical Kloosterman sums is a polynomial whose degree is now known, as well as the complex absolute values of the roots. In this paper, we provide estimates for the p-adic absolute values of these roots. Our method is indirect. We first develop a Dwork-type p-adic cohomology theory for the two-variable unit root L-function associated to the Kloosterman family, and study p-adic estimates of the eigenvalues of Frobenius. A continuity argument then provides the desired p-adic estimates.

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1 Introduction

Let $F_q$ be a finite field with $q = p^a$ elements, $p \geq 5$. Associated to each $\bar{t} \in \mathbb{F}_q^\ast$, define the Kloosterman sum

$$Kl_{\bar{t},m} := \sum_{\bar{x} \in \mathbb{F}_q^m} \psi \circ \text{Tr}_{F_q/F_q^{\bar{t}}} \left( \bar{x} + \frac{\bar{t}}{\bar{x}} \right) \quad m = 1, 2, 3, \ldots,$$

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where \( \deg(t) := [\mathbb{F}_q(t) : \mathbb{F}_q] \), \( q_t := q^{\deg(t)} \), and \( \psi \) is a fixed non-trivial additive character on \( \mathbb{F}_q \). It is well-known that the associated \( L \)-function is quadratic:

\[
L(Kl_t, T) := \exp \left( \sum_{m \geq 1} Kl_{t,m} \frac{T^m}{m} \right) = (1 - \pi_0(t)T)(1 - \pi_1(t)T),
\]

with roots satisfying \( \pi_0(t)\pi_1(t) = q_t \), \( |\pi_0(t)|_\mathbb{C} = |\pi_1(t)|_\mathbb{C} = \sqrt{q_t} \), and \( \pi_0(t) \) is a \( p \)-adic 1-unit, meaning \( |1 - \pi_0(t)|_p < 1 \). Let \( \kappa \in \mathbb{Z}_p \).

Let \( k \) be a positive integer, and define the \( k \)-th \textit{symmetric power} \( L \)-function of the Kloosterman family by

\[
L(Sym^k Kl, T) := \prod_{i \in [\mathbb{Z}_p]} \prod_{m=0}^k \frac{1}{1 - \pi_0(t)^k - m \pi_1(t)^m T^{\deg(t)}}.
\]

Robba [14] gave the first \( p \)-adic study of this \( L \)-function, and showed among other things that it is a polynomial defined over \( \mathbb{Z} \), and gave a conjectural formula for its degree. Using \( \ell \)-adic techniques, Fu-Wan [8] proved this conjecture as a special case of their study of the \( n \)-variable Kloosterman sum. In that paper, motivated by a conjecture of Gouvêa-Mazur, they asked whether there exists a uniform quadratic lower bound for the Newton polygon of \( L(Sym^k Kl, T) \) independent of \( k \). Our first main result of the paper affirmatively answers their question:

**Theorem 1.1.** For every positive integer \( k \), writing \( L(Sym^k Kl, T) = \sum c_m T^m \), then

\[
\ord_q c_m \geq \left( 1 - \frac{1}{p-1} \right) m(m-1).
\]

A uniform quadratic lower bound is known [16] for the Legendre family of elliptic curves \( E_t : x^2 = x_1(x_1 - 1)(x_1 - t) \), specifically the \( L \)-function of the \( k \)-th symmetric product of the first relative \( \ell \)-adic cohomology. This \( L \)-function, defined analogously to \( L(Sym^k Kl, T) \), equals the nontrivial part of the Hecke polynomial associated to the \( p \)-th Hecke operator \( T_{k+2}(p) \) for level 2 acting on cusp forms of weight \( k + 2 \) (see [1]). For the Kloosterman family, recent work of Yun [20], based on conjectures of Evans [3], gives an automorphic interpretation for \( L(Sym^k Kl, T) \) when \( k \) is small. It has also been shown by Fu-Wan [9] that \( L(Sym^k Kl, T) \) is geometric (or motivic) in nature, meaning it equals the local factor at \( p \) of the zeta function of a (virtual) scheme of finite type over \( \mathbb{Z} \).

Our motivation for the following study comes from a related but different direction. In [6], Dwork first defined the unit root \( L \)-function of the Legendre family of elliptic curves essentially as follows. Setting \( X := \mathbb{A}^1 \setminus \{0, 1, H(t) = 0\} \), where \( H(t) \) is the Hasse polynomial, the map \( f : E_t \mapsto t \in X \) gives a family of ordinary elliptic curves whose first relative \( p \)-adic étale cohomology \( R^1 f_* \mathbb{Z}_p \) gives a continuous rank one representation \( \rho_E : \pi_1^{\text{arith}}(X) \to \text{GL}_1(\mathbb{Z}_p) \). This has the property that, for a closed point \( t \in |X/\mathbb{F}_q| \), the image of the geometric Frobenius \( \text{Frob}_t \) is \( \rho_E(\text{Frob}_t) = \pi_0(t) \), where \( \pi_0(t) \) is the unique \( p \)-adic unit root of the zeta function of the fiber \( E_t \). For \( k \) a positive integer, define the unit root \( L \)-function

\[
L(\rho_E^\otimes k, T) := \prod_{t \in |X/\mathbb{F}_q|} \frac{1}{1 - \pi_0(t)^k T^{\deg(t)}}.
\]

For every \( k \), meromorphy was shown by Dwork in [6]. It also has a \( p \)-adic modular interpretation. Define the Fredholm determinant \( D(k, T) := \det(I - U_p T \mid M_k) \), where \( M_k \) is the space of overconvergent \( p \)-adic modular forms of level 2 and
weight $k$, and $U_p$ is the Atkin operator. Then there is the relation:

$$L(\rho_E \otimes k, T) = \frac{D(k+2,T)}{D(k,pT)}.$$  \hspace{1cm} (2)

This allows one to obtain results about modular forms from results about the unit root $L$-function. See \cite{15} for a detailed exposition; see also \cite{16}. Their special values $L(\rho_E \otimes k, 1)$ are also related to geometric Iwasawa theory as discussed in \cite{4}.

In this paper, we study the analogous unit root $L$-function attached to the Kloosterman family. Let $\kappa \in \mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the $p$-adic integers. The Kloosterman unit root $L$-function is defined by

$$L_{\text{unit}}(Kl, \kappa, T) := \prod_{\bar{t} \in |G_m/F_q|} \frac{1}{1 - \pi_0(\bar{t})^\kappa T^\deg(t)}.$$  \hspace{1cm} (3)

At the moment, we are unaware of a relation similar to (2).

Unit root $L$-functions are not expected to have a typical cohomological description outside the closed unit disk (see \cite{7}). However, as we demonstrate in this paper, an associated $L$-functions will: define the infinite $\kappa$-symmetric power $L$-function

$$L(Sym^\infty \kappa Kl, T) := \prod_{\bar{t} \in |G_m/F_q|} \prod_{m \geq 0} \frac{1}{1 - \pi_0(\bar{t})^\kappa - m \pi_1(\bar{t})^m T^\deg(t)},$$

and observe that we have the relation

$$L_{\text{unit}}(Kl, \kappa, T) = \frac{L(Sym^\infty \kappa Kl, T)}{L(Sym^\infty, \kappa - 2 Kl, qT)}.$$  \hspace{1cm} (4)

The infinite $\kappa$-symmetric power $L$-function was defined intrinsically by Wan \cite{17} in the proof of Dwork’s conjecture; see \cite{10} for details.

Our second main result is the development of a $p$-adic cohomology theory (Section 3) for the infinite $\kappa$-symmetric power $L$-function. This may be used to meromorphically describe the $L$-function:

$$L(Sym^\infty \kappa Kl, T) = \frac{\det(1 - \tilde{\beta}_\kappa T | H^1)}{\det(1 - q^{\tilde{\beta}_\kappa} T | H^0)},$$

where $\tilde{\beta}_\kappa$ is a completely continuous operator defined on $p$-adic spaces $H^1$ and $H^0$. As unit root $L$-functions are not typically rational functions, $H^1$ is expected to be infinite dimensional in general. For the Kloosterman family, we will show $H^0 = 0$ when $\kappa \neq 0$, and of dimension one when $\kappa = 0$, and when $\kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0}$ then $H^1$ is infinite dimensional. It is interesting that the case $\kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0}$ is the easiest to handle, whereas we are unable to compute cohomology when $\kappa \in \mathbb{Z}_{\geq 0}$.

Studying the action of Frobenius on cohomology leads to our third main result:

**Theorem 1.2.** For every $\kappa \in \mathbb{Z}_p$, $L(Sym^\infty \kappa Kl, T)$ is $p$-adic entire with $T = 1$ a root. Furthermore, writing $L(Sym^\infty \kappa Kl, T) = \sum_{m \geq 0} c_m T^m \in 1 + T \mathbb{Z}_p[[T]]$, then $ord_q c_m$ satisfies (4).

The proof of Theorem 1.1 now follows from the following identity:

$$L(Sym^k Kl, T) = \frac{L(Sym^\infty \kappa Kl, T)}{L(Sym^\infty, -(k+2) Kl, q^{k+1}T)}.$$  \hspace{1cm} (5)

Some heuristic calculations suggest that there is a chance the lower bound in Theorem 1.2 (and thus Theorem 1.1)
may be improved to simply \(m(m - 1)\), but we have been unable to prove this, and it may be that it fails for some \(m\).

We conjecture that the zeros and poles of \(L(Sym^\infty, \kappa)Kl, T)\) are all simple except for possibly finitely many, and that adjoining the collection of zeros and poles to \(\mathbb{Q}_p\) produces a finite extension field of \(\mathbb{Q}_p\) (the so-called \(p\)-adic Riemann hypothesis).

The above provides a cohomological description of the unit root \(L\)-function of the form:

\[
L_{\text{unit}}(Kl, \kappa, T) = \frac{\det(1 - \beta_\kappa T | H^1)}{\det(1 - q\beta_{\kappa-2} T | H^1)} \quad (\text{when } \kappa \neq 0 \text{ or } 2),
\]

with root \(T = 1\) and pole \(T = 1/q\). It is unclear whether there are any cancellations among the remaining zeros and poles, however, we expect that there are few if any. We note that while Artin’s conjecture does not carry over to geometric \(p\)-adic representations, it does for infinite symmetric powers over curves. See [19] for more details.

The \(p\)-adic cohomology theory developed here is of de Rham type, and may be seen as an extension of Dwork’s \(p\)-adic cohomology theory. We thus expect techniques from Dwork’s classical theory may be carried over to this theory. For example, a dual theory seems possible, perhaps giving rise to possible symmetry? It would be of great interest if this were the case. Another example is studying the variation of a family of unit root \(L\)-functions. In a future article joint with Steven Sperber, we examine how the unit roots of a family (of unit root \(L\)-functions) vary with respect to the parameter.

While we have restricted our study to the case of the one-variable Kloosterman family, the techniques developed here extend to the \(n\)-variable family as well as generalizations such as those studied in [12] and [13]. We hope to say more about their cohomology in a future article.

2 Relative Bessel cohomology

In this section we recall Dwork’s “Bessel cohomology” construction [5 Section 2] but modified according to [13]. Fix a prime \(p \geq 5\), and fix \(\pi \in \mathbb{Q}_p\) satisfying \(\pi^{p-1} = -p\). Set \(\Omega := \mathbb{Q}_p(\pi)\). Let \(b, b' \in \mathbb{R}_{>0}\) and \(\rho \in \mathbb{R}\). For \(u \in \mathbb{Z}\), set

\[
m(u) := \begin{cases} 
-u & \text{if } u \leq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Define the coefficient spaces

\[
L(b'; \rho) := \left\{ \sum_{n \geq 0} A(n) t^n \mid A(n) \in \Omega, \text{ord}_p A(n) \geq 2b' n + \rho \right\}
\]

\[
L(b') := \bigcup_{\rho \in \mathbb{R}} L(b'; \rho),
\]

and for \(q = p^a\) with \(a \geq 0\), define the \(L(b')\)-module

\[
K_q(b', b; \rho) := \left\{ \sum_{n \geq 0, u \in \mathbb{Z}} A(n, u) t^n \cdot q^{m(u)} x^u \mid A(n, u) \in \Omega, \text{ord}_p A(n, u) \geq 2b' n + b|u| + \rho \right\}
\]

\[
K_q(b', b) := \bigcup_{\rho \in \mathbb{R}} K_q(b', b; \rho).
\]
Lastly, define

$$V_q(b', b; \rho) := \left( \Omega[[t]] + \Omega[[t]] \cdot \frac{t^q}{x} \right) \cap K_q(b', b; \rho)$$

$$V_q(b', b) := \bigcup_{\rho \in \mathbb{R}} V_q(b', b).$$

When \( a = 0 \), we will write simply \( K(b', b) \) for \( K_1(b', b) \) and \( V(b', b) \) for \( V_1(b', b) \). Define the relative boundary operator

$$D_t q := x \frac{\partial}{\partial x} + \pi \left( t - \frac{t^q}{x} \right)$$

$$= e^{-\pi(x+t^q/x)} \circ x \frac{\partial}{\partial x} \circ e^{\pi(x+t^q/x)},$$

acting on \( K_q(b', b) \). This satisfies:

**Theorem 2.1** (Theorem 2.1 of [5]). Let \( b > \frac{1}{p - 1} \) and \( b \geq b' \). Set \( \varepsilon := b - \frac{1}{p - 1} \). Then \( \ker D_t q = 0 \) and

$$K_q(b'/q, b; \varepsilon) = V_q(b'/q, b; \varepsilon) \oplus D_t q K_q(b'/q, b; \varepsilon).$$

**Proof.** This is [5, Theorem 2.1] but modified slightly to suit the spaces defined above. Dwork does not explicitly point out that \( \ker D_t q = 0 \), however, this follows immediately from [5, Lemma 2.5].

Consequently, we may identify the first cohomology group \( H_1 t q(b'/q, b) := K_q(b'/q, b)/D_t q K_q(b'/q, b) \) with \( V_q(b'/q, b) \), a free \( \mathbb{L}(b'/q) \)-module of rank two with basis \( \{ 1, \pi t^q/x \} \).

**Relative Frobenius.** We now study the relative Frobenius \( \alpha_a \), which is defined as follows. First, define Dwork’s splitting function

$$\theta(z) := \exp \pi(z - z^p) = \sum_{i \geq 0} \theta_i z^i,$$

where it is well-known that \( \text{ord}_p \theta_i \geq (p - 1)i/p^2 \). Next, for each \( m \geq 1 \), define

$$F(t, x) := \theta(x) \theta(t/x)$$

$$F_m(t, x) := \prod_{i=0}^{m-1} F(t^p, x^p).$$

Define the Cartier operator \( \psi_x : \sum A_u x^u \mapsto \sum A_{pu} x^u \), and observe that \( \psi_x : K(b', b; 0) \rightarrow K(b', pb; 0) \). Set

$$\alpha_1(t) := \psi_x \circ F(t, x),$$

and

$$\alpha_m(t) := \psi_x^m \circ F_m(t, x)$$

$$= \alpha_1(t^{p^{m-1}}) \circ \cdots \circ \alpha_1(t^p) \circ \alpha_1(t).$$

Since \( F(t, x) \in K(\tilde{b}/p, \tilde{b}/p; 0) \), where \( \tilde{b} := (p - 1)/p \), we see that \( F(t^p, x^p) \in K(\tilde{b}/p^{p+1}, \tilde{b}/p^{p+1}; 0) \), and thus, for \( b, b' \leq \tilde{b} \),
and since \( \psi_a : t^{p^m} u \rightarrow t^{p^m} u \), we have \( \alpha_m(t) : K(b', b; 0) \rightarrow K_{b'}(b') b; 0) \). As

\[
p^m D_{\tilde{t}^{p^m}} \circ \alpha_m = \alpha_m \circ D_t,
\]

for \( \tilde{b} \geq b > \frac{1}{p-1} \) and \( b' \leq b \), we see that \( \alpha_m \) induces a map on relative cohomology \( \bar{\alpha}_m(t) : H^1_{b'}(b') \rightarrow H^1_{b'}(b') b \).

**Theorem 2.2.** Let \( \tilde{b} \geq b > \frac{1}{p-1} \) and \( b \geq b' \). With \( \{1, \pi t / x\} \) and \( \{1, \pi t^{p^m} / x\} \) as bases of \( H^1_{b'}(b) \) and \( H^1_{b'}(b\tilde{p}, b) \), respectively, the relative Frobenius \( \bar{\alpha}_m \) satisfies

\[
\bar{\alpha}_m(t)(1) = A_{m,1}(t) + A_{m,3}(t) \frac{\pi t^{p^m}}{x} \quad \text{and} \quad \bar{\alpha}_m(t) \left( \frac{\pi t}{x} \right) = A_{m,2}(t) + A_{m,4}(t) \frac{\pi t^{p^m}}{x},
\]

where

\[
A_{m,1} \in L(\tilde{b}/p; 0) \quad A_{m,3} \in L(b/p; \tilde{b} - \frac{1}{p-1})
\]

\[
A_{m,2} \in L(b/p; \frac{1}{p-1} - \frac{\tilde{b}}{p}) \quad A_{m,4} \in L(b/p; \tilde{b} - \frac{\tilde{b}}{p})
\]

Furthermore,

\[
A_{m,1}(0) = 1 \quad A_{m,2}(0) = 0 \quad A_{m,3}(0) \neq 0 \quad A_{m,4}(0) = p^m.
\]

### 3 Sym\(^{\infty, \kappa}\)-cohomology

Let \( \kappa \in \mathbb{Z}_p \). Define the falling factorial \( \kappa^m \) as follows. For \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \) and \( m \geq 0 \), the definition is conventional:

\[
\kappa^m := \kappa(\kappa - 1) \cdots (\kappa - m + 1),
\]

where \( \kappa^1 := 1 \). For \( \kappa = k \in \mathbb{Z}_{\geq 0} \), define

\[
\kappa^m_k := \begin{cases} 1 & \text{if } m = 0 \\ k(k - 1) \cdots (k - m + 1) & \text{if } 0 < m \leq k \\ k(k - 1) \cdots 2 \cdot 1 \cdot \hat{0} \cdot (-1) \cdot (-2) \cdots (k - m + 1) & \text{if } m \geq k + 1, \end{cases}
\]

where \( \hat{0} \) means it is counted in the \( m \) terms of the product, but omitted from the actual product. For example, \( \kappa^{m+1}_k = k \kappa^{m+1} = k! \) and \( \kappa^{m+2} = k! \cdot (-1); \) and \( \hat{0}^1 := 1, \hat{1}^0 := 0 = 1, \) and \( \hat{0}^2 := -1 \).

**Spaces.** Let \( b > \frac{1}{p-1} \) and \( b \geq b' \). Set \( \varepsilon := b - \frac{1}{p-1} \). Denote by \( S_0 \) the subring of power series in \( \Omega[[t, w]] \) which converge on the closed unit polydisk \( |t|_p \leq 1 \) and \( |w|_p \leq 1 \). In order to obtain an integral cohomology theory, we will weight the basis by \( w^{(m)} := \kappa^m w^m \). A more natural choice of basis (due to the definition of \( \bar{\alpha}_m \), below) is \( (\kappa^m) w^m \); however, this leads to non-integral cohomology theory, which in turn makes certain \( p \)-adic estimates difficult. Define the subspaces of
\[ S(b', \varepsilon; \rho) := \left\{ \sum_{n,m \geq 0} A(n,m)t^n w^{(m)} \mid A(n,m) \in \Omega, \operatorname{ord}_\rho A(n,m) \geq 2b'n + \varepsilon m + \rho \right\} \]

\[ S(b', \varepsilon) := \bigcup_{\rho \in \mathbb{R}} S(b', b; \rho). \]

**Boundary map** \( \partial \). With

\[
\partial := t \frac{\partial}{\partial t} + \frac{\pi t}{x} = e^{-\pi(x+t/x)} \circ t \frac{\partial}{\partial t} \circ e^{\pi(x+t/x)},
\]

observe that \( \partial \) and \( D_t \) commute, and thus \( \partial \) acts on relative cohomology \( H^1_t \) as follows. Since

\[
\partial(1) = \frac{\pi t}{x} \quad \text{and} \quad \partial\left( \frac{\pi t}{x} \right) = \pi^2 t,
\]

we see that on \( H^1_t \) with basis \( \{1, \pi t/x\} \), the map \( \partial \) takes the matrix form

\[
\partial = t \frac{d}{dt} + H, \quad \text{where} \quad H := \begin{pmatrix} 0 & 1 \\ \pi^2 t & 0 \end{pmatrix}
\]

acts on row vectors. It is convenient to rewrite (7) in terms of \( H \). Writing the basis vectors 1 and \( \pi t/x \) in row vector form \( (1, 0) \) and \( (0, 1) \), then (7) is equivalent to \( (1, 0)H = (0, 1) \) and \( (0, 1)H = (\pi^2 t, 0) \). We will abuse notation and write these as

\[
H(1) = \pi t/x \quad \text{and} \quad H(\pi t/x) = \pi^2 t.
\]

To motivate the following definition, consider the following. Intuitively, we will think of \( S_0 \) as the \( \kappa \)-symmetric power of the space \( H^1_t \) with basis \( \{1, \pi t/x\} \), where we intuitively set \( w = \pi t/x \) and \( 1 = 1 \). Thus, \( w^m \) should be viewed as \( 1^{\kappa-m}w^m \).

Rewrite (9) as \( H(1) = w \) and \( H(w) = \pi^2 t \). This means we should expect an extension of \( \partial \) to the \( \kappa \)-symmetric power \( S_0 \) of the form:

\[
\partial_\kappa(w^m) = " \partial(1^{\kappa-m}w^m) " = (\kappa-m)\partial(1)w^m + mw^{m-1}\partial(w) = (\kappa-m)H(1)w^m + mw^{m-1}H(w) = (\kappa-m)w^{m+1} + mw^{m-1}.
\]

Thus, we should have:

\[
\partial_\kappa(t^n w^m) := nt^n w^m + (\kappa-m)t^{n+1} w^{m-1}.
\]

While this will be the case, we instead use the weighted basis \( \{w^{(m)}\} \) to define the boundary map \( \partial_\kappa : S(b', \varepsilon) \to S(b', \varepsilon) \) as follows. Let \( m \geq 0 \). For \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \), define

\[
\partial_\kappa(t^n w^{(m)}) := nt^n w^{(m)} + t^n w^{(m+1)} + m(\kappa-m+1)\pi^2 t^{n+1}w^{(m-1)}.
\]

(10)
Note that $\partial_\kappa S(b', \varepsilon; 0) \subset S(b', \varepsilon; 0)$ if $b \geq b'$. For $\kappa = k \in \mathbb{Z}_{\geq 0}$, define $\partial_\kappa$ as follows. For $0 \leq m \leq k - 1$ or $m \geq k + 2$, then $\partial_\kappa(t^n w^{(m)})$ is defined by (10). When $m = k$ or $m = k + 1$, define

$$\partial_\kappa(t^n w^{(k)}) := nt^n w^{(k)} + k\pi^2 t^{n+1} w^{(k-1)}$$
$$\partial_\kappa(t^n w^{(k+1)}) := nt^n w^{(k+1)} + t^n w^{(k+2)} + (k + 1)\pi^2 t^{n+1} w^{(k)}.$$ 

Define the cohomology spaces

$$H^0_\kappa(S(b', \varepsilon)) := \ker(\partial_\kappa) \quad \text{and} \quad H^1_\kappa(S(b', \varepsilon)) := S(b', \varepsilon)/\partial_\kappa S(b', \varepsilon).$$

It is convenient to define

$$\mathcal{L}_{\kappa, H}(w^m) := (\kappa - m)H(1)w^m + mw^{m-1}H(w) = (\kappa - m)w^{m+1} + m\pi^2 tw^{m-1}. \quad (11)$$

Thus, $\partial_\kappa$ on $S(b', \varepsilon)$ takes the form

$$\partial_\kappa = t \frac{d}{dt} + \mathcal{L}_{\kappa, H},$$

an infinite differential system, where the matrix of $\mathcal{L}_{\kappa, H}$, acting on row vectors, takes the following form. For $\kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0}$:

$$\begin{pmatrix}
0 & 1 \\
\kappa \pi^2 t & 0 & 1 \\
2(\kappa - 1)\pi^2 t & 0 & 1 \\
3(\kappa - 2)\pi^2 t & 0 & \ddots \\
\ddots & \ddots & \ddots \\
(k - 1)\cdot 2\pi^2 t & 0 & 1 \\
k \cdot 1 \cdot \pi^2 t & 0 & 0 \\
(k + 1) \cdot \pi^2 t & 0 & 1 \\
(k + 2) \cdot (-1)\pi^2 t & 0 & 1 \\
\ddots & \ddots & \ddots 
\end{pmatrix}.$$ 

For $\kappa = k \in \mathbb{Z}_{> 0}$, the matrix takes the form

$$\begin{pmatrix}
0 & 1 \\
1 \cdot k \pi^2 t & 0 & 1 \\
2 \cdot (k - 1) \pi^2 t & 0 & 1 \\
3 \cdot (k - 2) \pi^2 t & 0 & 1 \\
\ddots & \ddots & \ddots \\
(k - 1) \cdot 2\pi^2 t & 0 & 1 \\
k \cdot 1 \cdot \pi^2 t & 0 & 0 \\
(k + 1) \cdot \pi^2 t & 0 & 1 \\
(k + 2) \cdot (-1)\pi^2 t & 0 & 1 \\
\ddots & \ddots & \ddots 
\end{pmatrix}.$$ 

Note
where 0 means it is omitted. Note the zero in the \((k + 2)\) column. For \(\kappa = 0\), we have

\[
\text{matrix of } \mathcal{L}_{0,H} = \begin{pmatrix}
0 & 0 & 0 \\
2 \cdot 0 \pi^2 t & 0 & 1 \\
2 \cdot (-1) \pi^2 t & 0 & 1 \\
\ddots & \ddots & \ddots 
\end{pmatrix}
\]

**Theorem 3.1.** Let \(\kappa \in \mathbb{Z}_p\). Then

\[
H^0_{\kappa} (S(b', \varepsilon)) = \begin{cases} 
0 & \text{if } \kappa \in \mathbb{Z}_p \setminus \{0\} \\
\Omega & \text{if } \kappa = 0.
\end{cases}
\]

**Proof.** We first suppose \(\kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0}\). Let \(\xi \in S(b', \varepsilon)\) such that \(\partial_\kappa \xi = 0\). Writing \(\xi = \sum_{n \geq 0} \xi_n w^{(n)}\), then \(\partial_\kappa \xi = 0\) takes the form

\[
t \xi_0' + \kappa \pi^2 t \xi_1 = 0 \\
t \xi_1' + \xi_0 + 2(\kappa - 1) \pi^2 t \xi_2 = 0 \\
t \xi_2' + \xi_1 + 3(\kappa - 2) \pi^2 t \xi_3 = 0 \\
\vdots & \vdots & \vdots \\
t \xi_k' + \xi_{k-1} + (k + (k - 1)) \pi^2 t \xi_{k+2} = 0
\]

(12)

We will show \(t^m \mid \xi_n\) for every \(m \geq 1\) and \(n \geq 0\) using induction. Observe that the second equation of (12) implies \(t \mid \xi_0\), and the third equation implies \(t \mid \xi_1\), and so forth: \(t \mid \xi_n\) for \(n \geq 0\). Next, suppose \(t^m \mid \xi_n\) for every \(n \geq 0\). The first equation of (12) then implies that \(t^{m+1} \mid \xi_0\). Using this, the second equation of (12) shows \(t^{m+1} \mid \xi_1\). Continuing, we see that \(t^{m+1} \mid \xi_n\) for every \(n \geq 0\). Hence, \(\xi = 0\) as desired.

Suppose now that \(\kappa \in \mathbb{Z}_{\geq 1}\). For convenience, set \(k := \kappa\). We first observe that \(\partial_\kappa \xi = 0\) breaks up into two systems of the form

\[
t \xi_0' + k \pi^2 t \xi_1 = 0 \\
t \xi_1' + \xi_0 + 2(k - 1) \pi^2 t \xi_2 = 0 \\
t \xi_2' + \xi_1 + 3(k - 2) \pi^2 t \xi_3 = 0 \\
\vdots & \vdots & \vdots \\
t \xi_k' + \xi_{k-1} + (k + 1) \pi^2 t \xi_{k+2} = 0
\]

(13)

and

\[
t \xi_{k+2} + (k + 2) \cdot (-1) \pi^2 t \xi_{k+3} = 0 \\
t \xi_{k+3} + \xi_{k+2} + (k + 3) \cdot (-2) \pi^2 t \xi_{k+4} = 0 \\
t \xi_{k+4} + \xi_{k+3} + (k + 4) \cdot (-3) \pi^2 t \xi_{k+5} = 0 \\
\vdots & \vdots & \vdots
\]

(14)
As (14) is of a form essentially identical to (12), a similar argument shows \( \xi_m = 0 \) for every \( m \geq k + 2 \). Thus, (13) takes the form of the (finite dimensional) differential system \( t \frac{d}{dt} + H_k \), where \( H_k \) is the \((k + 1) \times (k + 1)\) matrix

\[
H_k = \begin{pmatrix}
0 & 1 & & \\
\kappa^{-2}t & 0 & 1 & \\
2(k - 1)\kappa^{-2}t & 0 & 1 & \\
3(k - 2)\kappa^{-2}t & \ddots & \ddots & 1 \\
\kappa^{-2}t & 0 & & \\
\end{pmatrix}
\]

This is precisely the \( k \)-th symmetric power of the differential system (8). Using a deep result of Dwork’s, Robba [14, p.202] shows this system has no overconvergent solutions.

Lastly, set \( \kappa = 0 \), then \( \partial_\kappa \xi = 0 \) takes the form

\[
\begin{align*}
t\xi'_0 & + \pi^2 t\xi_1 = 0 \\
t\xi'_1 & + 2(-1)^2 t\xi_2 = 0 \\
t\xi'_2 + \xi_1 & + 3(-2)^2 t\xi_3 = 0 \\
t\xi'_3 + \xi_2 & + 4(-3)^2 t\xi_4 = 0 \\
\vdots & \vdots & \vdots \\
\end{align*}
\]

Ignoring the first equation, this system is of a similar form to (12), and so a similar argument shows \( \xi_m = 0 \) for every \( m \geq 1 \). Consequently, the first equation now becomes \( \xi'_0 = 0 \), and so \( \xi_0 \) is a constant, finishing the proof.

3.1 \( H^1 \) with \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \)

Throughout this section we will assume \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \). We may naturally view \( L(b') \) as a subspace of \( S(b', \varepsilon) \) by sending \( \xi \mapsto \xi \), the series with no \( w^{(m)} \) terms. For \( \rho \in \mathbb{R} \), define the spaces

\[
\begin{align*}
V(b', \varepsilon; \rho) & := L(b'; \rho) \cap S(b', \varepsilon; \rho) \\
V(b', \varepsilon) & := \bigcup_{\rho \in \mathbb{R}} V(b', \varepsilon; \rho).
\end{align*}
\]

**Lemma 3.2.** Let \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \). Then \( V(b', \varepsilon) \cap \partial_\kappa S(b', \varepsilon) = \{0\} \).

**Proof.** Let \( \eta \in V(b', \varepsilon) \cap \partial_\kappa S(b', \varepsilon) \). Write \( \eta = \sum_{n \geq 0} a_n t^n \), and let \( \xi \in S(b', \varepsilon; \rho) \) such that \( \partial_\kappa \xi = \eta \). Write \( \xi = \sum_{n,m \geq 0} A(n, m) t^n w^{(m)} \) with \( \text{ord}_p A(n, m) \geq 2b'n + \varepsilon m + \rho \) for some \( \rho \in \mathbb{R} \). Set \( \xi_m := \sum_{n \geq 0} A(n, m) t^n \). Using this, \( \xi = \sum_{m \geq 0} \xi_m w^{(m)} \). Using the \( \{w^{(m)}\}_{m \geq 0} \) as a basis of \( S(b', \varepsilon) \) as an \( L(b') \)-module, \( \xi \) takes the vector form \((\xi_0, \xi_1, \ldots)\),
and $\eta$ takes the form $(\sum a_nt^n, 0, 0, \ldots)$. Writing $\partial_a = t\frac{d}{dt} + L_{\kappa,H}$ then $\partial_a \xi = \eta$ is equivalent to the system

$$
\begin{align*}
\sum_{n \geq 0} a_n t^n &= t\xi_0' + \kappa \pi^2 t\xi_1 \\
0 &= t\xi_1' + \xi_0 + 2(\kappa - 1)\pi^2 t\xi_2 \\
0 &= t\xi_2' + \xi_1 + 3(\kappa - 2)\pi^2 t\xi_3 \\
&\quad \vdots
\end{align*}
$$

We will show by induction that $t^n \mid \eta$ for every $n$. Observe that the right-hand side of the first equation of the system is divisible by $t$, and thus $a_0 = 0$, or equivalently, $t$ divides $\eta$. Similarly, the second equation shows $t$ divides $\xi_0$. Continuing, we get that $t$ divides every $\xi_i$ for all $i \geq 0$.

Next, as $\kappa \pi^2 t\xi_1$ in the first equation is now divisible by $t^2$, the coefficient of $t$ in $\xi_0$ must equal $a_1$. Using this, from the second equation and the fact that $2(\kappa - 1)\pi^2 t\xi_2$ is divisible by $t^2$, the coefficient of $t$ of $\xi_1$ must equal $-a_1$. The same argument using the third equation shows the coefficient of $t$ in $\xi_2$ equals $a_1$. Continuing the argument, we must have $\xi_m = (-1)^m a_1 t + O(t^2)$ for every $m \geq 0$, and so $A(1,m) = (-1)^m a_1$. As $\text{ord}_p A(1,m) \geq 2b' + \varepsilon m + \rho$ for every $m$, it must be that $a_1 = 0$.

Assume now that $t^n$ divides $\eta$ and every $\xi_m$. We proceed with an identical argument as above to show $a_n = 0$. As $\kappa \pi^2 t\xi_1$ in the first equation is now divisible by $t^{n+1}$, the coefficient of $t^n$ in $\xi_0$ must equal $(1/n)a_n$. Using this, from the second equation and the fact that $2(\kappa - 1)\pi^2 t\xi_2$ is divisible by $t^{n+1}$, the coefficient of $t^n$ of $\xi_1$ must equal $-(1/n^2)a_n$. The same argument using the third equation shows the coefficient of $t^n$ in $\xi_2$ equals $(1/n^3)a_n$. Continuing the argument, we must have $\xi_m = (-1)^m (1/n^m+1)a_n t^n + O(t^{n+1})$ for every $m \geq 0$. This means $\text{ord}_p (\frac{1}{n^m} a_n) \geq 2b'n + \varepsilon m + \rho$ for every $m$, which is impossible unless $a_n = 0$. Thus, $\eta = 0$.

**Lemma 3.3.** Let $b > \frac{1}{p-1}$ and $b \geq b'$, and set $\varepsilon := b - \frac{1}{p-1}$. Then

$$S(b', \varepsilon; 0) = V(b', \varepsilon; 0) \oplus L_{\kappa,H} S(b', \varepsilon; \varepsilon).$$

(16)

**Proof.** We first observe that the right-hand side of (16) is contained in the left. We now show the reverse direction. First, observe that

$$w^{(m+1)} = -m(\kappa - m + 1)\pi^2 t w^{(m-1)} + L_{\kappa,H} w^{(m)}.$$

Using this recursively, it follows that:

- for $m \geq 0$: $w^{(2m+1)} = L_{\kappa,H} \sum_{l=0}^{m} \zeta_{2m+1,l} \pi^2 t^l w^{(2(m-l))}$
- for $m \geq 1$: $w^{(2m)} = \eta_{2m,0} \pi^2 m t^m + L_{\kappa,H} \sum_{l=0}^{m-1} \zeta_{2m,l} \pi^2 t^l w^{(2(m-l)-1)}$,

where $\eta_0$ and $\zeta_l$ are elements in $\mathbb{Z}_p$. The result follows easily from this. For future reference we record:

$$
\eta_{2m,0} = 2^{2m}(\kappa/2)^m (1/2)_m \\
\zeta_{2m,l} := 2^{2l} (m - \frac{1}{2}) (-\frac{\kappa}{2} + m + 1)_l \\
\zeta_{2m+1,l} := 2^{2l} (m)(\frac{\kappa + 1}{2} + m)_l.
$$
Theorem 3.4. Let \( b > \frac{1}{p-1} \) and \( b \geq b' \), and set \( \varepsilon := b - \frac{1}{p-1} \). Then
\[
S(b', \varepsilon; 0) = V(b', \varepsilon; 0) \oplus \partial_{\varepsilon} S(b', \varepsilon; \varepsilon).
\] (17)

Proof. First, observe that the righthand side of (17) is contained in the left. Let \( \xi \in S(b', \varepsilon; 0) \). By Lemma 3.3 there exists \( \eta_0 \in V(b', \varepsilon; 0) \) and \( \zeta_0 \in S(b', \varepsilon; \varepsilon) \) such that \( \xi = \eta_0 + \mathcal{L}_{\varepsilon} \mathcal{H} \zeta_0 \). Setting \( \xi_1 := -t \mathcal{L}_{\varepsilon} \mathcal{H} \zeta_0 \), then \( \xi = \eta_0 + \partial_{\varepsilon} \zeta_0 + \xi_1 \). As \( \xi_1 \in S(b', \varepsilon; \varepsilon) \), we may repeat this process to obtain \( \xi = \sum_{m \geq 0} \eta_m + \partial_{\varepsilon} \sum_{m \geq 0} \zeta_m \), with \( \eta_m \in S(b', \varepsilon; m) \) and \( \zeta_m \in S(b', \varepsilon; m + 1) \).

3.2 Frobenius

We first work on the fibers. Fix \( \bar{t} \in \bar{F}_q \) and let \( \hat{t} \) be its Teichmüller lift. Set \( d(\bar{t}) := [F_q(\bar{t}) : F_q] \) and define \( q_t := q^{d(\bar{t})} \).

Define
\[
\alpha_t := \psi_{x}^{\omega d(\bar{t})} \circ F_{u(\bar{t})}(\hat{t}, x).
\]

For \( \frac{p-1}{p} \geq b > \frac{1}{p-1} \), observe that \( \alpha_t \) is an endomorphism of \( K_{\bar{t}}(b) \), where \( K_{\bar{t}}(b) \) denotes the space obtained from \( K(b', b) \) by specializing \( t = \bar{t} \). Dwork’s trace formula states
\[
(q_t^m - 1)^n \text{Tr}(\alpha_t^m | K_{\bar{t}}(b)) = \sum_{x \in \bar{F}_q} \Psi \circ \text{Tr}_{K_{\bar{t}}(b)/F_q}(x + \bar{t})
\]
or equivalently
\[
L(K_{\bar{t}}, T) = \frac{\det(1 - \alpha_t T | K_{\bar{t}}(b))}{\det(1 - q_t \alpha_t T | K_{\bar{t}}(b))}.
\]

By Theorem 2.2 the operator \( D_t := x \frac{d}{dx} + \pi \left( x - \frac{1}{2} \right) \) acts on the space \( K_{\bar{t}}(b) \) such that the associated cohomology satisfies \( H^0(K_{\bar{t}}(b)) := \ker(D_t) = 0 \) and \( H^1(K_{\bar{t}}(b)) := K_{\bar{t}}(b)/D_t K_{\bar{t}}(b) \cong \Omega(\bar{t}) + \Omega(\bar{t}) \frac{d}{dx} \). Further, the Frobenius \( \alpha_t \) induces a map \( \bar{\alpha}_t \) on cohomology satisfying
\[
L(K_{\bar{t}}, T) = \det(1 - \bar{\alpha}_t T | H^1(K_{\bar{t}}(b)))
\]
\[
= (1 - \pi_0(\bar{t}) T)(1 - \pi_1(\bar{t}) T),
\]
where properties of the roots \( \pi_i(\bar{t}) \) were described in the introduction.

Infinite symmetric powers on the family. Let \( \frac{p-1}{p} \geq b > \frac{1}{p-1} \) and \( b \geq b' \), and set \( \varepsilon := b - \frac{1}{p-1} \). By Theorem 2.2 \( \bar{\alpha}_a(1) \in L(b'/q; 0) + L(b'/q; \varepsilon) \frac{d}{dx} \). Furthermore, \( \bar{\alpha}_a(1) = 1 + \eta + \zeta \frac{d}{dx} \), with \( \eta \in L(b'/q; 0) \), \( t | \eta \), and \( \zeta \in L(b'/q; \varepsilon) \). Define \( \Upsilon_t : H_{t}^{u}(b'/q, b) \to \Omega[[t, w]] \) by sending \( \zeta + \xi \frac{d}{dx} \mapsto \zeta + \xi w \). Thus, for \( \tau \in \mathbb{Z}_p \), \( (\Upsilon_t \circ \bar{\alpha}_a(1))^{\tau} \) is a well-defined element of \( \Omega[[t, w]] \). Define \( [\bar{\alpha}_a]_{\varepsilon} : S(b', \varepsilon) \to S(b'/q, \varepsilon) \) by linearly extending over \( \Omega[[t]] \) the action
\[
[\bar{\alpha}_a]_{\varepsilon}(w^{(m)}) := \kappa_{\varepsilon} \cdot (\Upsilon_t \circ \bar{\alpha}_a(1))^{\varepsilon}(\Upsilon_t \circ \bar{\alpha}_a(1))^{\varepsilon} \cdot \pi_1(\bar{t})^{m}.
\]

Due to the weighted basis \( \{w^{(m)}\} \), it is not immediate that this is well-defined, and in fact may not be without further conditions on \( b \) and \( b' \). Recall, \( \tilde{b} = (p - 1)/p \).
Lemma 3.5. Suppose \( \hat{b} - \frac{1}{p-r} > b > \frac{1}{p-r} \) and \( b \geq \hat{b} \). Then \([\tilde{\alpha}_a]_\kappa : S(b', \varepsilon) \to S(b'/q, \varepsilon)\) is well-defined.

Proof. By Theorem 3.2 and setting \( A_{a,3} := A_{a,3}/A_{a,1} \),

\[
[\tilde{\alpha}_a](w^{(m)}) = \kappa^m(A_{a,1} + A_{a,3}w)^{\kappa - m}(A_{a,2} + A_{a,4}w)^m
\]

By Lemma 3.5, \( \tilde{\alpha} \). Define the Cartier operator and so its \( p \) which is the local factor in the Euler product of \( t \) acting on \( \kappa \) acting on \( \kappa \) by specializing \( t = \hat{t} \). As a consequence of Theorem 3.2 observe that \( \tilde{\alpha}(1) = 1 + \eta \xi + \xi \hat{\eta} \) for some elements \( \eta, \xi \in \Omega(\hat{t}) \) satisfying \( |\eta|_p < 1 \) and \( |\xi|_p < 1 \). Define \( \Upsilon_{\xi t} : H^1(K_t(b)) \to S_t(\varepsilon) \) by \( \xi + \xi \xi \eta \). For any \( \tau \in \mathbb{Z}_p \), \( (\Upsilon_{\xi t} \circ \alpha_t(1))\tau \) is a well-defined element of \( \Omega(\hat{t})[w] \). Define \([\tilde{\alpha}_t]_\kappa \). By Lemma 3.3 \([\tilde{\alpha}_t]_\kappa \). By Lemma 3.3 \([\tilde{\alpha}_t]_\kappa \) is a well-defined endomorphism of \( S_t(\varepsilon) \) when \( \hat{b} - \frac{1}{p-r} \geq b \). The main purpose for working on the fibers is that, by an argument similar to [10] Corollary 2.4, part 2, we have

\[
\det(1 - [\tilde{\alpha}_t]_\kappa T | S_t(\varepsilon)) = \prod_{m=0}^{\infty} \left(1 - \tau_0(\hat{t})^{\kappa - m} \pi_1(\hat{t})^m T\right),
\]

which is the local factor in the Euler product of \( L(Sym^{n,\infty} Kt, T) \).

Dwork trace formula. Define the Cartier operator \( \psi_t : S(b'/p, \varepsilon) \to S(b', \varepsilon) \) by

\[
\psi_t : \sum_{n,m \geq 0} A(n, m) t^n w^{(m)} \mapsto \sum_{n,m \geq 0} A(pm, m) t^n w^{(m)}.
\]
Define
\[ \beta_\kappa := \psi_\kappa \circ [\bar{\alpha}]_\kappa : S(b', \varepsilon) \to S(b', \varepsilon), \]
a completely continuous operator.

**Proposition 3.6.** \( L(\text{Sym}^{\infty, \kappa} Kl, T) = \det(1 - \beta_\kappa T \mid S(b', \varepsilon))^\delta \) where \( \delta \) sends any function \( g(T) \) to \( g(T)/g(qT) \).

**Proof.** Let \( B_\kappa(t) \) be the infinite dimensional matrix of \([\bar{\alpha}]_\kappa\) with respect to the basis \( B := \{ w^{(m)} : m \geq 0 \} \). Write \( B_\kappa(t) = \sum_{n \geq 0} b_n t^n \), where \( b_n \) is an infinite matrix with entries in \( \mathbb{C}_p \). Define \( F_{B_\kappa} := (b_{qn-m})_{(n,m)} \) where \( n, m \geq 0 \), and we set \( b_{qn-m} := 0 \) if \( qn - m < 0 \), the zero matrix. As described prior to \([12, \text{Lemma 2.3}]\), the matrix of \( \beta_\kappa \) with respect to \( B \) is \( F_{B_\kappa} \). By \([13, \text{Lemma 4.1}]\), the Dwork trace formula gives
\[
(q^m - 1)Tr(\beta_\kappa^m) = (q^m - 1)Tr(F_{B_\kappa}^m)
= \sum_{\ell \in F_{B_\kappa}, \ell \in \text{Teich}(\hat{\kappa})} Tr(B_\kappa(\hat{\kappa}^{m-1}) \cdots B_\kappa(\hat{\kappa}) B_\kappa(\hat{\kappa}))
= \sum_{\ell \in F_{B_\kappa}, \ell \in \text{Teich}(\hat{\kappa})} Tr([\hat{\alpha}]^m_{\hat{\kappa}} \mid S_{\ell}(\varepsilon))
\]
It now follows from \([13]\) that
\[
L(\text{Sym}^{\infty, \kappa} Kl, T) = \det(1 - \beta_\kappa T \mid S(b', \varepsilon))^\delta.
\]
(See the argument succeeding \([10, \text{Equation 8}]\) for details.) \( \square \)

**Lemma 3.7.** \( q\partial_\kappa \circ \beta_\kappa = \beta_\kappa \circ \partial_\kappa \).

We will prove this lemma in the next section. Assuming this, then \( \beta_\kappa \) induces maps \( \tilde{\beta}_\kappa : H^1(S(b', \varepsilon)) \to H^1(S(b', \varepsilon)) \) and \( \tilde{\beta}_\kappa : H^0(S(b', \varepsilon)) \to H^0(S(b', \varepsilon)) \) with the property:
\[
L(\text{Sym}^{\infty, \kappa} Kl, T) = \frac{\det(1 - \tilde{\beta}_\kappa T \mid H^1(S(b', \varepsilon)))}{\det(1 - q\tilde{\beta}_\kappa T \mid H^0(S(b', \varepsilon)))}.
\]

**Theorem 3.8.** If \( \kappa \in \mathbb{Z}_p \) then \( L(\text{Sym}^{\infty, \kappa} Kl, T) \) is an entire function. When \( \kappa = 0 \), then
\[
L(\text{Sym}^{\infty, 0} Kl, T) = \frac{\det(1 - \tilde{\beta}_0 T \mid H^1(S(b', \varepsilon)))}{1 - qT},
\]
which is still an entire function (even though it doesn’t look like it).

**Proof.** If \( \kappa \neq 0 \), then the first statement follows immediately from Theorem 3.4. Suppose now that \( \kappa = 0 \). In this case, the constant 1 is a basis for \( H^0 \) with trivial action of Frobenius:
\[
q\bar{\alpha}_0(1) = q\psi^\kappa_0 \circ [\bar{\alpha}]_0(1) = q\psi^\kappa_0(\bar{\alpha}_0(1))^0 = q.
\]
This proves the first part of the theorem. To show entireness, since the unit root \( L \)-function with \( \kappa = 0 \) takes the form \( L_{\text{unit}}(0, T) = (1 - T)/(1 - qT) \), we see that
\[
L(\text{Sym}^{\infty, 0} Kl, T) = (1 - T)\frac{L(\text{Sym}^{\infty, -2} Kl, qT)}{1 - qT}.
\]
We will see in the next proposition that \( L(\text{Sym}^{\infty-2} K_l, T) \) has a root at \( T = 1 \), which proves the entireness for \( \kappa = 0 \).

**Theorem 3.9.** For every \( \kappa \in \mathbb{Z}_p \), \( T = 1 \) is a root of \( L(\text{Sym}^{\infty-k} K_l, T) \).

**Proof.** While one may use the cohomology above to prove this, we will use a different argument. From [14, Theorem B] \( T = 1 \) is a root of the \( k \)-th symmetric power \( L\)-function \( L(\text{Sym}^k K_l, T) \) for every positive integer \( k \). The result now follows by continuity: let \( \{k_m\} \) be any sequence of positive integers which tend to infinity and \( k_m \to \kappa \) \( p \)-adically, then

\[
\lim_{m \to \infty} L(\text{Sym}^{k_m} K_l, T) = L(\text{Sym}^{\infty-k} K_l, T).
\]

\[\square\]

### 3.3 Proof of Lemma 3.7

The result follows from a limit using finite symmetric powers. Let \( k \) be a positive integer. Define the map

\[
\text{Sym}^k \tilde{a}_m : \text{Sym}^k_{l, (\psi^{'}, b')} H^1_{l, \phi} (b', b) \to \text{Sym}^k_{l, (\psi^{'}, p^m)} H^1_{l, \phi} (b' / p^m, b)
\]

Define the length\((w^{(m)}) := m\). For \( \xi \in S(b', \varepsilon) \), define length\((\xi) \) as the supremum of the lengths of the individual terms in the series defining \( \xi \). In most cases, the length of \( \xi \) will be infinite. Set \( w_0 := 1 \) and \( w_1 := \frac{\pi t}{x} \), so that \( \{w_0, w_1\} \) is a basis of \( H^1_{l, \phi} (b', b) \), and \( \{w_0^m w_1^n : 0 \leq m \leq k\} \) is a basis of \( \text{Sym}^k_{l, (\psi^{'}, \phi)} (b', b) \) over \( L(b') \). As the basis is finite, \( \{w_0^m w_1^n : 0 \leq m \leq k\} \) is also a basis of \( \text{Sym}^k_{l, (\psi^{'}, \phi)} H^1_{l, \phi} (b', b) \) over \( L(b') \), where \( w_1^m := k^m w_1^m \). Define \( S^{(k)}(b', \varepsilon) := \{ \xi \in S(b', \varepsilon) \mid \text{length}(\xi) \leq k \} \). We may identify \( S^{(k)}(b', \varepsilon) \) with \( \text{Sym}^k H^1_{l, \phi} (b', b) \) using the map

\[
w^{(m)} \mapsto w_0^m w_1^{(m)}.
\]

Let \( k_n \) be a sequence of positive integers tending to infinity such that \( p \)-adically \( k_n \to \kappa \). For each \( n \in \mathbb{Z}_{\geq 0} \), define the approximation map \( [\tilde{a}_n]_{(\kappa, n)} : S(b', \varepsilon) \to S(b'/q, \varepsilon) \) by

\[
[\tilde{a}_n]_{(\kappa, n)}(w^{(m)}) := \begin{cases} [\tilde{a}_n]_{k_n}(w^{(m)}) & \text{if } m \leq k_n \\ 0 & \text{otherwise.} \end{cases}
\]

Using the identification with the \( k_n \)-symmetric power, we have

\[
[\tilde{a}_n]_{(\kappa, n)}(w^{(m)}) = ((\psi^{'}, \phi) \circ \tilde{a}_n (1))^{k_n - m} (\psi^{'}, \phi) \frac{\pi t}{x}^m
\]

\[
\cong (\text{Sym}^{k_n} \tilde{a}_n) (w_0^{k_n - m} w_1^{(m)}),
\]

and thus \( [\tilde{a}_n]_{(\kappa, n)} \cong \text{Sym}^{k_n} \tilde{a}_a \).

Next, an analogous argument to that in [10, Lemma 2.2] demonstrates that \( \lim_{n \to \infty} [\tilde{a}_n]_{(\kappa, n)} = [\tilde{a}_\kappa]_\kappa \) as maps from \( S(b', \varepsilon) \to S(b'/q, \varepsilon) \). Consequently, if we define \( \beta_{(\kappa, n)} := \psi^{'n} \circ [\tilde{a}_n]_{(\kappa, n)} \) then as operators on \( S(b', \varepsilon) \),

\[
\lim_{n \to \infty} \beta_{(\kappa, n)} = \beta_\kappa.
\]

(19)
Lastly, define \( \partial_{(k;n)} \) on \( S(b', \varepsilon) \) as follows. For \( 0 \leq m \leq k_n - 1 \),

\[
0 \leq m \leq k_n - 1 : \quad \partial_{(k;n)}(t^r w^{(m)}) := rt^r w^{(m)} + t^r w^{(m+1)} + m(k_n - m + 1)\pi^2 t^{r+1} w^{(m-1)},
\]

\[
m = k_n : \quad \partial_{(k;n)}(t^r w^{(k)}) := rt^r w^{(k)} + k_n\pi^2 t^{r+1} w^{(k-1)},
\]

\[
m > k_n : \quad \partial_{(k;n)}(t^r w^{(k)}) := 0.
\]

Similarly, define \( \tilde{\partial}_{(k;n)} \) on \( \text{Sym}^k_{L(b')}H^1_L(b', b) \) as follows. For \( (\xi_1, \ldots, \xi_{k_n}) \in H^1_L(b', b)_{p^{k_n}}, \) define

\[
\tilde{\partial}_{(k;n)}(\xi_1 \cdots \xi_{k_n}) := \sum_{i=1}^{k_n} \xi_1 \cdots \hat{\xi}_i \cdots \xi_{k_n} \partial(\xi_i),
\]

where \( \partial \) was defined by (3). Then \( \partial_{(k;n)} \equiv \tilde{\partial}_{(k;n)} \) through the identification of the spaces \( S^{(k_n)}(b', \varepsilon) \) and \( \text{Sym}^k_{L(b')}H^1_L(b', b) \).

Further, again using the identification,

\[
q \partial_{(k;n)} \circ \beta_{(k;n)} \equiv \beta_{(k;n)} \circ \partial_{(k;n)}.
\] (20)

Lastly, observe that with the topology of coefficient-wise convergence on \( S(b', \varepsilon) \), \( \partial_{(k;n)} \rightarrow \partial_k \), which follows by considering the case \( 0 \leq m \leq k_n - 1 \):

\[
|\partial_{(k;n)}(t^r w^{(m)}) - \partial_k(t^r w^{(m)})| = |m(k_n - k)\pi^2 t^{r+1} w^{(m-1)}|,
\]

which tends to zero as \( n \rightarrow \infty \). Lemma 3 now follows by taking the limit of (20).

### 4 \( p \)-adic estimates

In order to obtain the best possible \( p \)-adic estimates, we will modify the splitting function used above. This affects the spaces involved as follows. We will assume throughout this section that \( k \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \).

Let \( \pi \in \mathbb{Q}_p \) be a root of \( \sum_{i=0}^{\infty} \frac{\pi^i}{p^i} \) with \( \text{ord}_p(\pi) = \frac{1}{p-1} \), and set \( \Omega := \mathbb{Q}_p(\pi) \) with \( R \) as its ring of integers. Let \( E(t) := \exp \left( \sum_{i=0}^{\infty} \frac{\pi^i}{p^i} t^i \right) \) be the Artin-Hasse exponential. Define \( \pi_l := \sum_{i=0}^{l} \frac{\pi^i}{p^i} \) and note that \( \text{ord}_p(\pi_l) \geq \frac{l+1}{p-1} - l - 1 \).

Dwork’s (infinite) splitting function is defined as \( \theta_{\infty}(t) := E(\pi t) = \sum_{i=0}^{\infty} \lambda_i t^i \). Observe that each coefficient satisfies \( \text{ord}_p(\lambda_i) \geq \frac{l}{p-1} \) since the Artin-Hasse exponential has \( p \)-adic integral coefficients.

With \( b, b' > 0 \) and \( \rho \in \mathbb{R} \), define the spaces (with \( q = p^a \), \( a \geq 0 \))

\[
L(b': \rho) := \left\{ \sum_{n=0}^{\infty} A(n)t^n \mid A(n) \in \Omega, \text{ord}_p A(n) \geq 2b'n + \rho \right\}
\]

\[
L(b') := \bigcup_{\rho \in \mathbb{R}} L(b': \rho)
\]

\[
K_q(b', b; \rho) := \left\{ \sum_{n, u \geq 0, u \in \mathbb{Z}} A(n, u)t^n \cdot t^{\text{ord}_p(u)}x^u \mid A(n, u) \in \Omega, \text{ord}_p A(n, u) \geq 2b'n + b|u| + \rho \right\}
\]

\[
K_q(b', b) := \bigcup_{\rho \in \mathbb{R}} K_q(b', b; \rho).
\]

We will denote \( K_1(b', b) \) by \( K(b', b) \). Notice that \( f(t, x) := x + \frac{t}{x} \in K(b', b; -b) \).
Define
\[ F_\infty(t, x) := \theta_\infty(x) \frac{t}{x} \in R[[t, x]] \]
\[ F_{\infty, a}(t, x) := \prod_{i=0}^{a-1} F_\infty(t^{p^i}, x^{p^i}) \in R[[t, x]]. \]

Set \( \hat{b} := p/(p - 1) \). Observe that
\[ F_\infty(t, x) \in K(b'/p, b/p; 0) \quad \text{and} \quad F_{\infty, a}(t, x) \in K(b'/q, b/q; 0) \]
for all real numbers \( \hat{b} \geq b \geq b' \). Next, we define a function \( G(t, x) \) such that
\[ F_\infty(t, x) = \frac{G(t, x)}{G(t^{p^i}, x^{p^i})}. \]

Using this equation recursively, we see that \( G(t, x) \) must be defined by
\[ G(t, x) := \prod_{j=0}^{\infty} F_\infty(t^{p^j}, x^{p^j}) \in R[[t, x]]. \]

Set \( f_x := x^{\frac{\hat{b}}{p}} f(t, x) \). Define the boundary operator on \( K(b', b) \) by
\[ \hat{D}_t := \frac{1}{G(t, x)} \circ x \frac{\partial}{\partial x} \circ G(t, x) = x \frac{\partial}{\partial x} + W_1(t, x) \]
where
\[ W_1(t, x) := \sum_{j=0}^{\infty} \pi_j p^j f_x(t^{p^j}, x^{p^j}). \]

As \( W_1 \in K(\hat{b}, \hat{b}; -1) \) and acts by multiplication, \( \hat{D}_t \) is an endomorphism of \( K(b', b) \). Define the relative cohomology spaces
\[ H_0^d(K(b', b)) := \ker(\hat{D}_t \mid K(b', b)) \quad \text{and} \quad H_1^d(b', b) := K(b', b)/\hat{D}_t K(b', b). \]
For technical reasons, it will be useful to define the following. Write
\[ f_x(t^{p^i}, x^{p^i}) = f_x(t, x)^{p^i} + ph(t, x), \]
where \( h \) has \( p \)-adic integral coefficients. Using this, we may write
\[ W_1 = \pi f_x Q_1 + R_1, \]
where \( R_1, Q_1 \in K(\hat{b}, \hat{b}; 0) \) and \( Q_1 \) is a \( 1 \)-unit.

**Theorem 4.1.** Let \( \hat{b} \geq b > \frac{1}{p-1} \) and \( b \geq b' \), and set \( \varepsilon := b - \frac{1}{p-1} \). Then \( \ker \hat{D}_{\varepsilon} = 0 \) and
\[ K_\varepsilon(b'/q, b; 0) = V_\varepsilon(b'/q, b; 0) \oplus \hat{D}_{\varepsilon} K_\varepsilon(b'/q, b; \varepsilon). \]
Proof. The result follows from [3 Lemma 2.1], and a similar argument to [11 Section 3.3].

With \( f_t(t, x) := \frac{\partial}{\partial t} f(t, x) \), define

\[
\hat{\partial} := \frac{1}{G(t, x)} \circ t \frac{\partial}{\partial t} \circ G(t, x)
\]

\[
= t \frac{\partial}{\partial t} + W_2(t, x)
\]

where

\[
W_2(t, x) := \sum_{j=0}^{\infty} \pi_j p^j f_t(t^j, x^j) = \sum_{j=0}^{\infty} \pi_j p^j \left( \frac{t}{x} \right)^j
\]

Since \( W_2(t, x) \in K(\hat{b}, \hat{b}; -1) \), \( \hat{\partial} \) is an endomorphism of \( K(b', b) \). Also, \( \hat{\partial} \) commutes with \( \hat{D}(t) \) as endomorphisms of \( K(b', b) \), it induces an operator on relative cohomology

\[
\hat{\partial} : H_1^1(b', b) \to H_1^1(b', b).
\]

**Lemma 4.2.** On \( H_1^1(b', b) \) with basis \( \{1, \frac{x}{t}\} \), if we write the boundary operator in matrix form, \( \hat{\partial} = t \frac{\partial}{\partial t} + \hat{H} \), then there exists \( \eta_0 \) a 1-unit in \( L(b'; 0) \) and a matrix \( \hat{R} \) such that

\[
\hat{H} = \eta_0 H + \hat{R}
\]

and

\[
\hat{R} = \begin{pmatrix}
\hat{R}_{00} & \hat{R}_{01} \\
\hat{R}_{10} & \hat{R}_{11}
\end{pmatrix}
\]

with

\[
\hat{R}_{00} \in L(b'; 0) \quad \hat{R}_{01} \in L(b'; \varepsilon) \quad \hat{R}_{10} \in L(b'; -\varepsilon) \quad \hat{R}_{11} \in L(b'; 0).
\]

Proof. We first consider \( \hat{\partial}(1) = W_2 \). Write \( W_2 = \pi f_t Q_2 \) and note that \( Q_2 \in K(\hat{b}, \hat{b}; 0) \) is a 1-unit. From [3 Lemma 2.1], there exists a 1-unit \( \eta_0 \in L(b'; 0) \) and \( h \in K(b', b; \varepsilon) \) such that \( Q_2 = \eta_0 + \pi f_t h \). Then

\[
W_2 = \pi f_t Q_2
\]

\[
= \pi f_t \eta_0 + \pi f_t (\pi f_t h)
\]

\[
= \pi f_t \eta_0 + (W_1 - R_1) Q_1^{-1} (\pi f_t h)
\]

\[
= \pi f_t \eta_0 + \hat{D}_t (\zeta_1) - \xi_1
\]

where \( \zeta_1 \in K(b', b; 0) \) and \( \xi_1 \in K(b', b; 0) \). By Theorem [11] write \( \xi_1 = \xi_1^{(0)} + \frac{\pi t}{x} \zeta_1 + \hat{D}_t (\zeta_2) \) where \( \xi_1^{(0)} \in L(b'; 0) \), \( \xi_1^{(1)} \in L(b'; \varepsilon) \), and \( \zeta_2 \in K(b', b; \varepsilon) \). Then

\[
\hat{\partial}(1) = \eta_0 \frac{\pi t}{x} + \xi_1^{(0)} + \xi_1^{(1)} \frac{\pi t}{x} + \hat{D}_t (\zeta_1 + \zeta_2).
\]

Hence, \( \hat{R}_{00} = \xi_1^{(0)} \in L(b'; 0) \) and \( \hat{R}_{01} = \xi_1^{(1)} \in L(b'; \varepsilon) \) as desired.
We now compute \( \hat{\mathcal{H}}(\pi t) \). Write

\[
\hat{\partial} \left( \frac{\pi t}{x} \right) = \frac{\pi t}{x} + W_2 \frac{\pi t}{x}
= \frac{\pi t}{x} + \frac{\pi f_2^2 \pi t}{x} \]
\[
= \frac{\pi t}{x} + \frac{\pi f_2(\eta_0 + \pi f_2 h) \pi t}{x}
= \frac{\pi t}{x} + \eta_0 \left( \frac{\pi t}{x} \right)^2 + \pi f_2(\pi f_2 h) \frac{\pi t}{x}.
\]

Now,

\[
\hat{D}_1 \left( \frac{\pi t}{x} \right) = -\frac{\pi t}{x} + W_1 \frac{\pi t}{x}
= -\frac{\pi t}{x} + (\pi f_2 Q_1 + R_1) \frac{\pi t}{x}
= -\frac{\pi t}{x} + \pi^2 t - \left( \frac{\pi t}{x} \right)^2 + \pi f_2 \xi_1 + \xi_2
\]

where \( \xi_1 := (Q_1 - 1) \frac{\pi t}{x} \) and \( \xi_2 := R_1 \frac{\pi t}{x} \) are elements in \( K(b', b; -\varepsilon) \). By Theorem 4.1,

\[
\pi f_2 \xi_1 + \xi_2 = \tilde{\eta}_0 + \tilde{\eta}_1 \pi t + \hat{D}_1(\tilde{\zeta})
\]

for some \( \tilde{\eta}_0 \in L(b'; -\varepsilon) \), \( \tilde{\eta}_1 \in L(b'; 0) \), and \( \tilde{\zeta} \in K(b', b; 0) \). Next, setting \( \tilde{\xi} := \pi f_2 h \frac{\pi t}{x} \in K(b', b; -\varepsilon) \), then

\[
\pi f_2(\pi f_2 h \frac{\pi t}{x}) = \pi f_2 \tilde{\xi} = \hat{D}_1(\tilde{\zeta}) + \xi_2 + \xi_3 \pi t
\]

where \( \xi_3 \in K(b', b; -\varepsilon) \), \( \xi_2 \in L(b'; -\varepsilon) \), and \( \xi_3 \in L(b'; 0) \). Consequently,

\[
\hat{\partial}(\pi t) = \frac{\pi t}{x} + \eta_0 \left( \frac{\pi t}{x} \right)^2 + \pi f_2(\pi f_2 h \frac{\pi t}{x})
= \frac{\pi t}{x} + \eta_0 \left( \frac{\pi t}{x} + \pi^2 t + \tilde{\eta}_0 + \tilde{\eta}_1 \pi t + \hat{D}_1(\tilde{\zeta}) \right) + \hat{D}_1(\tilde{\zeta}) + \xi_2 + \xi_3 \pi t
\]

where

\[
\hat{R}_{10} := \eta_0 \tilde{\eta}_0 + \xi_2 \in L(b'; -\varepsilon)
\]
\[
\hat{R}_{11} := 1 - \eta_0 + \eta_0 \tilde{\eta}_1 + \xi_3 \in L(b'; 0).
\]

---

Corollary 4.3. \( \mathcal{L}_{\kappa, \tilde{R}}(b'; \varepsilon; 0) \subset \mathcal{L}_{\kappa, \tilde{R}}(b'; \varepsilon; 0) \), where \( \mathcal{L}_{\kappa, \tilde{R}} \) is defined similarly to \( \mathcal{L} \).

Lemma 4.4. \( S(b'; \varepsilon; 0) = V(b'; \varepsilon; 0) + \mathcal{L}_{\kappa, \tilde{R}}(b'; \varepsilon; \varepsilon) \).

Proof. Let \( \xi \in S(b'; \varepsilon; 0) \). By Lemma 4.3, there exists \( \eta \in V(b'; \varepsilon; 0) \) and \( \zeta \in S(b'; \varepsilon; \varepsilon) \) such that \( \xi = \eta + \mathcal{L}_{\kappa, \tilde{R}}(\zeta) \). Write \( \mathcal{L}_{\kappa, \tilde{R}} = (\mathcal{L}_{\kappa, \tilde{R}} - \mathcal{L}_{\kappa, \tilde{R}})\eta_0^{-1} \). Then

\[
\xi = \eta + (\mathcal{L}_{\kappa, \tilde{R}} - \mathcal{L}_{\kappa, \tilde{R}})\eta_0^{-1} \zeta = \eta + \mathcal{L}_{\kappa, \tilde{R}}(\eta_0^{-1} \zeta) - \mathcal{L}_{\kappa, \tilde{R}}(\eta_0^{-1} \zeta).
\]
By Lemma \[\ref{lemma:12}\] since \(\eta_0^{-1}\zeta \in S(b',\varepsilon;\varepsilon)\), \(\mathcal{L}_{\kappa,\tilde{R}}(\eta_0^{-1}\zeta) \in S(b',\varepsilon;\varepsilon)\). We may now repeat this procedure with \(\mathcal{L}_{\kappa,\tilde{R}}(\eta_0^{-1}\zeta)\). This shows the lefthand side is contained in the right.

To prove the other direction, let \(\zeta \in S(b',\varepsilon;\varepsilon)\). Then \(\mathcal{L}_{\kappa,\tilde{R}}(\zeta) = \eta_0 \mathcal{L}_{\kappa,H}(\zeta) + \mathcal{L}_{\kappa,\tilde{R}}(\zeta)\). From Lemma \[\ref{lemma:30}\] \(\mathcal{L}_{\kappa,H}(\zeta) \in S(b',\varepsilon;0)\), and Lemma \[\ref{lemma:12}\] gives \(\mathcal{L}_{\kappa,\tilde{R}}(\zeta) \in S(b',\varepsilon;\varepsilon)\). This proves the result.

**Lemma 4.5.** Suppose \(\kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0}\). Then \(V(b',\varepsilon) \cap \mathcal{L}_{\kappa,H} S(b',\varepsilon) = \{0\}\).

**Proof.** Let \(\xi = \sum_{m \geq 0} \xi_m w^{(m)} \in S(b',\varepsilon)\) and \(\eta \in V(b',\varepsilon)\) be such that \(\mathcal{L}_{\kappa,H} \xi = \eta\). This is equivalent to the system of equations

\[
\begin{align*}
\eta &= + \kappa \pi^2 t \xi_1 \\
0 &= \xi_0 + 2(\kappa - 1) \pi^2 t \xi_2 \\
0 &= \xi_1 + 3(\kappa - 2) \pi^2 t \xi_3 \\
0 &= \xi_2 + 4(\kappa - 3) \pi^2 t \xi_4 \\
&\vdots
\end{align*}
\]

Observe that the second equation shows \(t \mid \xi_0\), and the fourth equation shows \(t \mid \xi_2\), and hence \(t^2 \mid \xi_0\). Repeating this shows \(t^m \mid \xi_0\) for any \(m \geq 1\). Hence, \(\xi_0 = 0\), and thus \(\xi_{2m} = 0\) for \(m \geq 1\). A similar argument using the odd rows shows \(\xi_{2m+1} = 0\) for \(m \geq 0\).

**Theorem 4.6.** \(S(b',\varepsilon;0) = V(b',\varepsilon;0) \oplus \tilde{\partial}_\kappa S(b',\varepsilon;\varepsilon)\) and \(\ker \tilde{\partial}_\kappa = 0\).

**Proof.** First note that the righthand side is contain in the left. Next, let \(\xi \in S(b',\varepsilon;0)\) and set \(E := t \frac{d}{dt}\). By Lemma \[\ref{lemma:13}\] there exists \(\eta \in V(b',\varepsilon;0)\) and \(\zeta \in S(b',\varepsilon;\varepsilon)\) such that

\[
\xi = \eta + \mathcal{L}_{\kappa,H}(\zeta) = \eta + \tilde{\partial}_\kappa(\zeta) - E(\zeta).
\]

As \(E(\zeta) \in S(b',\varepsilon;\varepsilon)\), we may repeat this procedure to obtain \(S(b',\varepsilon;0) = V(b',\varepsilon;0) + \tilde{\partial}_\kappa S(b',\varepsilon;\varepsilon)\).

We now show directness. Let \(\eta \in V(b',\varepsilon) \cap \tilde{\partial}_\kappa S(b',\varepsilon)\). Let \(\zeta \in S(b',\varepsilon)\) be such that \(\tilde{\partial}_\kappa \zeta = \eta\). If \(\zeta \neq 0\), then there exists \(c \in \mathbb{R}\) such that \(\zeta \in S(b',\varepsilon;c)\) but \(\zeta \notin S(b',\varepsilon;c+\varepsilon)\). Now,

\[
\eta = \tilde{\partial}_\kappa \zeta = E \zeta + \eta_0 \mathcal{L}_{\kappa,H} \zeta + \mathcal{L}_{\kappa,\tilde{R}} \zeta.
\]

Setting \(\xi_1 := E \zeta + \mathcal{L}_{\kappa,\tilde{R}} \zeta \in S(b',\varepsilon;\varepsilon)\), then there exists \(\eta_1 \in V(b',\varepsilon;c)\) and \(\zeta_1 \in S(b',\varepsilon;c+\varepsilon)\) such that

\[
\xi_1 = \eta_1 + \tilde{\partial}_\kappa \zeta_1 = \eta_1 + E_1 \zeta_1 + \eta_0 \mathcal{L}_{\kappa,H}(\zeta_1) + \mathcal{L}_{\kappa,\tilde{R}}(\zeta_1).
\]

Set \(\xi_2 := E(\zeta_1) + \mathcal{L}_{\kappa,\tilde{R}} \zeta_1 \in S(b',\varepsilon;c+\varepsilon)\). Iterating this procedure, we obtain

\[
\eta = \sum_{i=1}^{\infty} \eta_i + \eta_0 \mathcal{L}_{\kappa,H}(\zeta) + \sum_{i=1}^{\infty} \zeta_i,
\]

which we rewrite as

\[
\mathcal{L}_{\kappa,H}(\zeta) + \sum_{i \geq 1} \zeta_i = (\eta - \sum_{i \geq 1} \eta_i) \eta_0^{-1}.
\]
By Lemma 3.5 we must have \( \zeta = -\sum_{i \geq 1} \zeta_i \in S(b', \varepsilon; c + \varepsilon) \), which contradicts our choice of \( c \). Observe that setting \( \eta = 0 \) shows \( \partial_\varepsilon = 0 \).

### 4.1 Frobenius

Set

\[
\alpha_{\infty,1}(t) := \psi_x \circ F_\infty(t, x),
\]

and

\[
\alpha_{\infty,m}(t) := \psi_x^m \circ F_{\infty,m}(t, x) = \alpha_{\infty,1}(t^{p^{m-1}}) \circ \ldots \circ \alpha_{\infty,1}(t) \circ \alpha_{\infty,1}(t).
\]

Since \( F_{\infty}(t, x) \in K(\hat{b}/b, \hat{b}/p; 0) \), where \( \hat{b} := p/(p-1) \), we see that \( \alpha_{\infty,m}(t) : K(\hat{b}/b; 0) \to K(\hat{b}/p^{m}; b, p) \). As

\[
p^m \hat{D}_{p^m} \circ \alpha_{\infty, m} = \alpha_{\infty, m} \circ \hat{D}_t,
\]

we see that \( \alpha_{\infty, m} \) induces a map on relative cohomology \( \alpha_{\infty, m}(t) : H^1_t(\hat{b}/b) \to H^1_t(\hat{b}/p^m, b) \).

**Theorem 4.7.** Set \( \hat{b} := p/(p-1) \). With \( \{1, \pi t/x\} \) and \( \{1, \pi t^{p^m}/x\} \) as bases of \( H^1_t(\hat{b}, \hat{b}) \) and \( H^1_t(\hat{b}/p^m, \hat{b}) \), respectively, the relative Frobenius \( \bar{\alpha}_{\infty, m} \) satisfies

\[
\bar{\alpha}_{\infty, m}(1) = A_{m,1}(t) + A_{m,3}(t) \frac{\pi t^{p^m}}{x} \quad \text{and} \quad \bar{\alpha}_{\infty, m}(\frac{\pi t}{x}) = A_{m,2}(t) + A_{m,4}(t) \frac{\pi t^{p^m}}{x},
\]

where

\[
A_{m,1} \in L(\hat{b}/p^m; 0) \quad \quad A_{m,3} \in L(\hat{b}/p^m; \hat{b} - \frac{1}{p-1})
\]

\[
A_{m,2} \in L(\hat{b}/p^m; 1 - \frac{\hat{b}}{p}) \quad \quad A_{m,4} \in L(\hat{b}/p^m; \hat{b} - \frac{\hat{b}}{p}) \quad (21)
\]

and \( A_{m,1}(0) = 1 \).

**Proof.** This follows from an analogous argument to [2] Section 3 or [11] Section 3.

Define \( [\bar{\alpha}_{\infty, a}]_\varepsilon : S(b', \varepsilon) \to \Omega[[t, u]] \) by linearly extending over \( L(b') \) the action

\[
[\bar{\alpha}_{\infty, a}]_\varepsilon (w^{(m)}) := \bar{\alpha}_{\infty, a} \cdot (\bar{\gamma} \circ \bar{\alpha}_{\infty, a}(1))^{k-1} \cdot (\bar{\gamma} \circ \bar{\alpha}_{\infty, a} \frac{\pi t}{x})^m.
\]

Just as in Lemma 5.3, \( [\bar{\alpha}_{\infty, a}]_\varepsilon \) is an endomorphism of \( S(b', \varepsilon) \) when \( \hat{b} - \frac{1}{p-1} \geq b > \frac{1}{p-1} \) and \( b \geq b' \). Define

\[
\beta_{\varepsilon, \kappa} := \psi_\varepsilon \circ [\bar{\alpha}_{\infty, a}]_\varepsilon : S(b', \varepsilon) \to S(b', \varepsilon).
\]

An analogous result to Lemma 5.7 shows \( \hat{\partial}_\varepsilon \circ \beta_{\varepsilon, \kappa} = \beta_{\varepsilon, \kappa} \circ \hat{\partial}_\varepsilon \), and so \( \beta_{\varepsilon, \kappa} \) induces maps on cohomology \( \beta_{\varepsilon, \kappa} : H^p(S(b', \varepsilon)) \to H^p(S(b', \varepsilon)) \) and \( \beta_{\varepsilon, \kappa} : H^1(S(b', \varepsilon)) \to H^1(S(b', \varepsilon)) \). Combining this with the Dwork trace formula (see
Proposition 3.6, and Theorem 4.6 for \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \),

\[
L(Sym_{\infty-k}^{\infty} Kl, T) = \det(1 - \bar{\beta}_{\infty-k} T | H^1(S(b', \varepsilon))).
\]

**Theorem 4.8.** Let \( \kappa \in \mathbb{Z}_p \). Writing \( L(Sym_{\infty-k}^{\infty} Kl, T) = \sum_{m=0}^{\infty} c_m T^m \), then for every \( m \geq 0 \),

\[
\ord_q c_m \geq \left(1 - \frac{1}{p^m - 1}\right)m(m - 1).
\]

**Proof.** We will prove this assuming \( \kappa \in \mathbb{Z}_p \setminus \mathbb{Z}_{\geq 0} \). As this set is dense in \( \mathbb{Z}_p \), the result will follow by continuity in \( \kappa \) of \( L(Sym_{\infty-k}^{\infty}, T) \).

With \( \beta_{\infty-k,1} := \psi_t \circ [\bar{\alpha}_{\infty,k}] : S(b', \varepsilon) \to S(b', \varepsilon) \), we have the relation

\[
\beta_{\infty-k,1} = \psi_t \circ [\bar{\alpha}_{\infty,k}](t) \circ \cdots \circ \psi_t \circ [\bar{\alpha}_{\infty,k}](t)
\]

\[
= \psi_t^a \circ [\bar{\alpha}_{\infty,1}(t^{p^{a-1}})] \circ \cdots \circ [\bar{\alpha}_{\infty,1}(t^p)] \circ [\bar{\alpha}_{\infty,1}(t)]
\]

\[
= \psi_t^a \circ [\bar{\alpha}_{\infty,1}(t^{p^{a-1}}) \circ \cdots \circ \bar{\alpha}_{\infty,1}(t^p) \circ \bar{\alpha}_{\infty,1}(t)]
\]

\[
= \beta_{\infty-k,a}
\]

where we have used an argument similar to [10] Corollary 2.4 for the third equality. Hence, on cohomology, \( \bar{\beta}_{\infty-k,a} = \beta_{\infty-k,a} \).

Now,

\[
det(1 - \bar{\beta}_{\infty-k,a} T^a | H^1(S(b', \varepsilon))) = det(1 - \bar{\beta}_{\infty-k,1} T^a | H^1(S(b', \varepsilon))) = \prod_{\zeta^n = 1} det(1 - \zeta \bar{\beta}_{\infty-k,1} T | H^1(S(b', \varepsilon))).
\]

Counting multiplicities, let \( m_i \) denote the number of reciprocal roots of \( det(1 - \bar{\beta}_{\infty-k,1} T | H^1(S(b', \varepsilon))) \) which have slope \( s_i \); note, we say \( \lambda \in \mathbb{C}_p \) has slope \( s_i \) if \( \ord_p(\lambda) = \lambda \). Then, from (22), \( det(1 - \bar{\beta}_{\infty-k,a} T | H^1(S(b', \varepsilon))) \) has \( m_i \) reciprocal roots of slope \( s_i/a \), or alternatively, it has \( m_i \) reciprocal roots of \( q \)-adic slope \( s_i \).

Set \( \hat{b} = \hat{b} - \frac{1}{p^m} \) and \( \varepsilon := b - \frac{1}{p^m} \). Note, \( \pi^{2n} \in V(\hat{b}/p; 0) \subset S(\hat{b}/p; \varepsilon; 0) \). As \( [\bar{\alpha}_{\infty,k}] \) is well-defined on this space, we have \( \beta_{\infty-k,1}(\pi^{2n} \varepsilon) \in S(\hat{b}, \varepsilon; 0) \). In order to reduce in cohomology, we view \( \beta_{\infty-k,1}(\pi^{2n} \varepsilon) \in S(\hat{b} - \frac{1}{p^m}, \varepsilon; 0) \). Hence, \( \beta_{\infty-k,1}(\pi^{2n} \varepsilon) = \sum_{m \geq 0} B(m, n) \pi^{2m} \in \sum_{m \geq 0} B(m, n) \pi^{-2m} \). Writing \( \sum_{m \geq 0} B(m, n) \pi^{2m} = \sum_{m \geq 0} B(m, n) \pi^{-2m} \cdot \pi^{2m} \), we see that \( \ord_p B(m, n) \pi^{-2m} \geq 2(\hat{b} - 1/(p - 1)m - 2m/(p - 1) = 2m(1 - \frac{1}{p^m - 1}) \). The result now follows from the previous paragraph.

**References**

[1] Alan Adolphson, *A p-adic theory of Hecke polynomials*, Duke Math. J. 43 (1976), no. 1, 115–145.

[2] Alan Adolphson and Steven Sperber, *Exponential Sums and Newton Polyhedra: Cohomology and Estimates*, Annals of Math. 130 (1989), no. 2, 367–406.
[3] H. Timothy Choi and Ronald Evans, *Congruences for sums of powers of Kloosterman sums*, Int. J. Number Theory 3 (2007), no. 1, 105–117. MR 2310495 (2008d:11090)

[4] Richard Crew, *L-functions of p-adic characters and geometric Iwasawa theory*, Invent. Math. 88 (1987), no. 2, 395–403. MR 880957 (89g:11049)

[5] B. Dwork, *Bessel functions as p-adic functions of the argument*, Duke Math. J. 41 (1974), 711–738. MR 0387281 (52 #8124)

[6] Bernard Dwork, *On Hecke polynomials*, Inventiones math. 12 (1971), 249–256.

[7] Matthew Emerton and Mark Kisin, *Unit L-functions and a conjecture of Katz*, Ann. of Math. (2) 153 (2001), no. 2, 329–354. MR 1829753 (2002k:11100)

[8] Lei Fu and Daqing Wan, *L-functions for symmetric products of Kloosterman sums*, J. Reine Angew. Math. 589 (2005), 79 – 103.

[9] _, *L-functions of symmetric products of the Kloosterman sheaf over Z*, Math. Ann. 342 (2008), no. 2, 387–404. MR 2425148 (2009i:14022)

[10] C. Douglas Haessig, *Meromorphy of the rank one unit root L-function revisited*, Finite Fields Appl. 30 (2014), 191–202. MR 3249829

[11] C. Douglas Haessig and Antonio Rojas-León, *L-functions of symmetric powers of the generalized Airy family of exponential sums*, Int. J. Number Theory 7 (2011), no. 8, 2019–2064. MR 2873140 (2012k:11114)

[12] C. Douglas Haessig and Steven Sperber, *L-functions associated with families of toric exponential sums*, J. Number Theory 144 (2014), 422–473. MR 3239170

[13] C. Douglas Haessig and Steven Sperber, *Families of generalized Kloosterman sums*, Trans. Amer. Math. Soc. (2015).

[14] Philippe Robba, *Symmetric powers of the p-adic Bessel equation*, J. Reine Angew. Math. 366 (1986), 194 – 220.

[15] Daqing Wan, *Meromorphic continuation of L-functions of p-adic representations*, Ann. of Math. (2) 143 (1996), no. 3, 469–498.

[16] _, *Dimension variation of classical and p-adic modular forms*, Invent. Math. 133 (1998), 469–498.

[17] _, *Dwork’s conjecture on unit root zeta functions*, Ann. Math. 150 (1999), 867–927.

[18] _, *A quick introduction to Dwork’s conjecture*, Contemporary mathematics 245 (1999), 147–163.

[19] _, *L-functions of function fields*, Number theory, Ser. Number Theory Appl., vol. 2, World Sci. Publ., Hackensack, NJ, 2007, pp. 237–241. MR 2364844 (2009a:11186)

[20] Zhiwei Yun, *Galois representations attached to moments of Kloosterman sums and conjectures of Evans*, Compos. Math. 151 (2015), no. 1, 68–120, Appendix B by Christelle Vincent. MR 3305309