Double Bicrosssum of Braided Lie algebras

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Abstract

The condition for double bicrosssum to be a braided Lie bialgebra is given. The result generalizes quantum double, bicrosssum, bicrosscosum, bisum. The quantum double of braided Lie bialgebras is constructed. The relation between double crosssum of Lie algebras and double crossproduct of Hopf algebras is given.

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0 Introduction

Lie bialgebras and quasitriangular Lie bialgebras were introduced by Drinfeld in his remarkable article [6]. In the sequel, many important examples of Lie bialgebras were found. For example, Witt algebra and Virasoro algebra in [19] and [23]; parabolic Lie bialgebra in [17]; quaternionic Lie bialgebra in [8], and so on. Low dimensional coboundary (or triangular ) Lie bialgebras are classified in [24]. Lie bialgebras play an important role in quantum groups and Lie groups (see [16, 5]). Majid studied the bism of Lie bialgebras in [15] and [17]. Grabowski studied the double bosonisation of Lie bialgebras [9].

Braided tensor categories become more and more important. In mathematical physics, Drinfeld and Jambo found the solutions of Yang-Baxter equations by means of braided tensor categories; In string theory, Yi-Zhi Huang and Liang Kong gave the relation between Open-string vertex algebras and braided tensor categories (see [11, 12]); In topology, there exits a tight contact between braided tensor categories and the invariants of links and 3-dimensional manifolds (see [4, 10, 14, 20]).

In this paper, we follows Majid’s work in papers [15, 17] and generalize his conclusion in two ways. On the one hand, we give the condition for double bicrosssum to become a Lie bialgebra, which generalizes quantum double, bicrosssum, bicrosscosum and bisum. We
build the relation between the bisum of Lie algebras and the biproduct of Hopf algebras. On the other hand, our conclusions hold in braided tensor categories, so our work will supply a useful tool for conformal fields, vector algebras and 3-manifolds.

Preliminaries

We begin with recalling additive category (see [7, Page 148]).

**Definition 0.1** A category $\mathcal{C}$ is called an additive category, if the following conditions are satisfied for any $U, V, W \in \mathcal{C}$:

(Ad1) $\text{Hom}_\mathcal{C}(U, V)$ is an additive group and $\text{Hom}_\mathcal{C}(V, U) \times \text{Hom}_\mathcal{C}(W, V) \to \text{Hom}_\mathcal{C}(W, U)$ is bilinear;
(Ad2) $\mathcal{C}$ has a zero object 0;
(Ad3) There exists the direct sum for any finite objects in $\mathcal{C}$.

**Example 0.2** (i) If $H$ is a Hopf algebra with an invertible antipode, then the Yetter-Drinfeld category $H\mathcal{YD}$ is an additive category;
(ii) The braided tensor category, which is studied in [11], is an additive category.

Otherwise, the most of important examples in [4] are additive braided tensor categories.

In this paper, unless otherwise stated, $(\mathcal{C}, \mathcal{C})$ is an additive braided tensor category with braiding $\mathcal{C}$.

If $\mathcal{E}$ is a family of objects in $\mathcal{C}$ and $C_{U,V} = (C_{V,U})^{-1}$ for any $U, V \in \mathcal{E}$, then the braiding $\mathcal{C}$ is said to be symmetric on $\mathcal{E}$. If $\mathcal{C}$ is symmetric on $L$ and $L$ has a left duality $L^*$, then, by [20], $\mathcal{C}$ is symmetric on the set $\{L, L^*\}$.

Let $L, U, V, A$ and $H$ be objects in $\mathcal{C}$. $\mathcal{C}$ is symmetric on $L, U$ and $V$, respectively; $\mathcal{C}$ is symmetric on set $\{A, H\}$; $\alpha : H \otimes A \to A$, $\beta : H \otimes A \to H$, $\phi : A \to H \otimes A$ and $\psi : H \to H \otimes A$ are morphisms in $\mathcal{C}$. We work on the five objects above throughout this paper.

**Definition 0.3** ([13]) $(L, [ , ])$ is called a braided Lie algebra in $\mathcal{C}$, if $[ , ] : L \otimes L \to L$ is a morphism in $\mathcal{C}$, and the following conditions are satisfied:

(L1) $\mathcal{C}$-anti- symmetry: $[ , ](\text{id} + C) = 0$,
(L2) $\mathcal{C}$-Jacobi identity: $[ , ]([ , ] \otimes \text{id})(\text{id} + C_{12}C_{23} + C_{23}C_{12}) = 0$, where $C_{12} = C_{L,L} \otimes \text{id}_L$ and $C_{23} = \text{id}_L \otimes C_{L,L}$.

If $(\Lambda, \beta)$ is a coquasitriangular Hopf algebra, then the category $\mathcal{C} = ^\Lambda\mathcal{M}$ of all left $\Lambda$-comodules is a symmetric braided tensor category. $\beta$-Lie algebras in [2] and [3], as well as $\epsilon$-Lie algebras in [21], are braided Lie algebras in $\mathcal{C}$.
Definition 0.4 \((L, \delta)\) is called a braided Lie coalgebra in \(C\), if \(\delta : L \to L \otimes LC\) is a morphism in \(C\) and the following conditions are satisfied:

1. **(CL1) C-co-anti-symmetry:** \((id + C)\delta = 0\),
2. **(CL2) C-co-Jacobi identity:** \((id + C_{12}C_{23} + C_{23}C_{12})(\delta \otimes id)\delta = 0\).

In the definition above, (L1), (L2), (CL1) and (CL2) are also called braided anti-symmetry, braided Jacobi identity, braided anti-co-symmetry, braided co-Jacobi identity.

Example 0.5 \(^1\) Assume that \(U\) is an object in \(C\) with a left duality \(U^*\) and \(X = U \otimes U^*, \ Y = U^* \otimes U\). Define the multiplication and unit in \(X\) as follows:

\[m_X = id_U \otimes ev_U \otimes id_{U^*}, \ \eta_X = coev_U.\]

It is clear that \((X, m, \eta)\) is an associative algebra in \(C\). Therefore, \(X\) becomes a braided Lie algebra under the following bracket operation:

\[\{ , \} = m_X - m_X C_{X,X}.\]

Dually, define the comultiplication and counit in \(Y\) as follows:

\[\Delta_Y = id_{U^*} \otimes coev_U \otimes id_U, \ \epsilon_Y = ev_U.\]

It is clear that \((Y, \Delta, \epsilon)\) is a braided coalgebra in \(C\). Therefore, \(Y\) is a braided Lie coalgebra under the following operation:

\[\delta = \Delta_X - C_{Y,Y} \Delta_Y.\]

Definition 0.6 Assume that \((H, [ , ])\) is a braided Lie algebra in \(C\) and \((H, \delta)\) is a braided Lie coalgebra in \(C\). If the following condition is satisfied:

\[\delta [ , ] = ((( , ) \otimes id_H)(id_H \otimes \delta)(id_H \otimes id_H - C_{H,H}) + (id_H \otimes ( , ))(C_{H,H} \otimes id_H)(id_H \otimes \delta))(id_H \otimes id_H - C_{H,H}),\]

then \((H, [ , ], \delta)\) is called a braided Lie bialgebra in \(C\).

In order to denote operations for convenience, we use braiding diagrams. Let

\[\{ , \} = \begin{array}{c}
H \\
H
\end{array}, \quad \delta = \begin{array}{c}
H \\
H
\end{array}, \quad \alpha = \begin{array}{c}
H \\
A
\end{array}, \quad \beta = \begin{array}{c}
A \\
H
\end{array}, \quad \phi = \begin{array}{c}
A \\
H
\end{array}, \quad \psi = \begin{array}{c}
A \\
A
\end{array}.
\]

In particular, when \(C\) is symmetric on \(\{U, V\}\), \(C_{U,V}\) is denoted by the following diagram:

\[C_{U,V} = \begin{array}{c}
U \\
V
\end{array}.\]

When \(U\) has a left duality \(U^*\), denote the evaluation \(ev_U = d_U\) and coevaluation \(coev_U = b_U\) by the following diagrams:

\[ev_U = \begin{array}{c}
U^* \\
U
\end{array}, \quad coev_U = \begin{array}{c}
U^* \\
U
\end{array}.\]
we have

\[(L1): \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array}
= - \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array},
\quad
(L2): \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array},
\end{array}
\]

\[(CL1): \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array}
= - \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array},
\quad
(CL2): \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} = 0 .
\]

\[(LB): \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} .
\]

Obviously, if \((L1)\) holds, then \((L2)\) is equivalent to anyone of the following two equations:

\[\begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} = 0 ,
\quad
\begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
  L \\
  L \\
  L \\
  L
\end{array}
\end{array} .
\]

If \(H\) is a braided Lie algebra and \(\alpha([,] \otimes \text{id}) = \alpha(\text{id} \otimes \alpha)(C_{H,H} \otimes \text{id})\), then \((A, \alpha)\) is called a left \(H\)-module. If \((\delta \otimes \text{id})\phi = (\text{id} \otimes \phi)\phi - (C_{H,H} \otimes \text{id})(\text{id} \otimes \phi)\phi\), then \((A, \phi)\) is called a left \(H\)-comodule. If \(H\) and \(A\) are braided Lie algebra, and \(\alpha(\text{id} \otimes [,]) = [ ,](\alpha \otimes \text{id}) + [,](\text{id} \otimes \alpha)(C_{H,A} \otimes \text{id})\), then \((A, [,], \alpha)\) is called an \(H\)-module Lie algebra. Assume that \(H\) is a braided Lie coalgebra and \(L\) is a braided Lie algebra; if \(L\)
is a left $H$-comodule and the bracket operation of $L$ is a morphism of $H$-comodules, i.e. 
$\phi[.] = (\text{id} \otimes [.]) (\phi \otimes \text{id}) + (\text{id} \otimes [.]) (C_{A,H} \otimes \text{id})(\text{id} \otimes \phi)$, then $L$ is called a left $H$-comodule braided Lie algebra.

Assume that $H$ is a braided Lie bialgebra, $(A, \alpha)$ is a left $H$-module and $(A, \phi)$ is a left $H$-comodule. If the following condition is satisfied:

\[(YD) : \quad \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array} = \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array} + \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array} + \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array},
\]

then $A$ is called a left-left- Yetter-Drinfeld module over $H$, or (left $H$- )YD module. Denote by $^{H}YD(C)$ the category of all left $H$-YD modules. Dually, assume that $H$ is a braided Lie bialgebra, $(H, \beta)$ is a right $A$-module and $(H, \psi)$ is a right $A$-comodule. If the following condition is satisfied:

\[(YD) : \quad \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array} = \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array} + \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array} + \begin{array}{c}
H & A \\
\downarrow & \downarrow \\
H & A
\end{array},
\]

then $H$ is called right-right-Yetter-Drinfeld module over $A$, or (right $A$-)YD module. Denote by $YD(C)^{A}$ the category of all right $A$-YD modules.

Assume that $H$ is a braided Lie bialgebra in $C$, $A$ is a left $H$-YD module and the bracket operation and co-bracket operation of $A$ are $H$-module morphism and $H$-comodule morphism. If the following condition is satisfied:

\[(SLB) : \quad \begin{array}{c}
A & A \\
\downarrow & \downarrow \\
A & A
\end{array} = \begin{array}{c}
A & A \\
\downarrow & \downarrow \\
A & A
\end{array} + \begin{array}{c}
A & A \\
\downarrow & \downarrow \\
A & A
\end{array} - \begin{array}{c}
A & A \\
\downarrow & \downarrow \\
A & A
\end{array} - \begin{array}{c}
A & A \\
\downarrow & \downarrow \\
A & A
\end{array} - \begin{array}{c}
A & A \\
\downarrow & \downarrow \\
A & A
\end{array},
\]
then $A$ is called a (left $H$-) infinitesimal braided Lie bialgebra, where $SC$ is called an infinitesimal braiding (see [17]) and is defined as follows:

$$SC = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A

\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}.
\end{array}$$

Similarly, we have right infinitesimal braided Lie bialgebra.

### 1 The double bicrosssum of braided Lie algebras

In this section we give the conditions for the double bicrosssum of braided Lie algebras to become a braided Lie bialgebra.

Let $D := A \oplus H$ as objects in $\mathcal{C}$; define the bracket and co-bracket operations of $D$ as follows:

$$[\ ]_D = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}.
\end{array}
$$

$$\delta_D = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}.
\end{array}
$$

Write $A^\phi_\alpha \bowtie^\psi_\beta H = (D, [\ ]_D, \delta_D)$. If one of $\alpha, \beta, \phi, \psi$ is zero, we can omit the zero morphism in $A^\phi_\alpha \bowtie^\psi_\beta H$. For example, when $\phi = 0, \psi = 0$, we denote $A^\phi_\alpha \bowtie^\psi_\beta H$ by $A^\alpha \bowtie_\beta H$; when $\alpha = 0, \beta = 0$, we denote $A^\phi_\alpha \bowtie^\psi_\beta H$ by $A^\phi \bowtie^\psi H$; Similarly, we have $A^\phi \bowtie^\psi H$, $A^\alpha \bowtie^\psi_\beta H$, $A^\phi \bowtie H$, $A^\alpha \bowtie^\psi H$. We call $A^\phi_\alpha \bowtie^\psi_\beta H$ the double bicrosssum; $A^\phi \bowtie^\psi H$ bicrosssum; $A^\phi \bowtie H$ bicrosscosum; $A^\alpha \bowtie^\psi_\beta H$ and $A^\phi \bowtie^\psi_\beta H$ bisum; $A^\phi \bowtie H$ and $A^\phi \bowtie^\psi H$ bicross sum; $A^\phi \bowtie^\psi H$ and $A^\phi \bowtie^\psi H$ semi-direct sum, or the smash sum; $A^\phi \bowtie H$ and $A^\phi \bowtie^\psi H$ semi-direct co-sum. We also use the following notations: $A^\phi_\alpha \bowtie^\psi_\beta H = A \bowtie_\beta H$; $A^\phi_\alpha \bowtie_\beta H = A \bowtie H$; $A^\phi \bowtie^\psi H = A \bowtie H$; $A^\alpha \bowtie^\psi \bowtie H = A \bowtie H$; $A^\phi \bowtie^\psi H = A \bowtie H$ and $A^\phi \bowtie^\psi H = A \bowtie H$. 

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\[ A^\phi \bowtie_{\beta} H = A \boxtimes H \quad \text{and} \quad A_\alpha \bowtie^\phi H = A \boxtimes H; \quad A_\alpha \bowtie H = A \bowtie H \quad \text{and} \quad A \bowtie_{\beta} H = A \bowtie H. \]

For conveniency, write \[ [ \ ]_D = A \bowtie H \quad \text{and} \quad \delta_D = A \bowtie H. \]

**Definition 1.1** ([16, Def.8.3.1]) Assume that \( A \) and \( H \) are braided Lie algebras. If \((A, \alpha)\) is a left \( H \)-module, \((H, \beta)\) is a right \( A \)-module, and the following \((M1)\) and \((M2)\) hold, then \((A, H, \alpha, \beta)\) (or \((A, H)\)) is called a matched pair of braided Lie algebras:

\[
(M1) : \quad H A A = H A A + H A A - H A A. \\
(M2) : \quad H H A = H H A + H H A - H H A. 
\]

**Proposition 1.2** ([16, Pro.8.3.2]) If \((A, H)\) is a matched pair of braided Lie algebras, then double cross sum \( A \bowtie H \) is a braided Lie algebra in \( \mathcal{C} \).

**Proof.** Obviously, \((L1)\) hold. Now we show \((L2)\). By Definition,

\[
H A A = H A A + H A A - H A A, \quad A H A = - A H A - A H A. 
\]

Therefore, it follows from \((I)\) and the conditions of matched pair that

\[
H A A = H A A + H A A - H A A. \\
\]

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Write the six terms of the right hand side above as (1), (2), · · · , (6), respectively.

\[(1) + (3) + (5) = \frac{H}{A} \frac{A}{A} \frac{A}{A} + \frac{H}{A} \frac{A}{A} \frac{A}{A} + \frac{H}{A} \frac{A}{A} \frac{A}{A} = \frac{H}{A} \frac{A}{A} \frac{A}{A}.\]

Using the braided anti- symmetry of \(A\), we have

\[\frac{H}{A} \frac{A}{A} \frac{A}{A} - \frac{H}{A} \frac{A}{A} \frac{A}{A} = -\frac{H}{A} \frac{A}{A} \frac{A}{A} - \frac{H}{A} \frac{A}{A} \frac{A}{A} = -\frac{H}{A} \frac{A}{A} \frac{A}{A}.\]

Therefore

\[\frac{H}{A} \frac{A}{A} \frac{A}{A} = \frac{H}{A} \frac{A}{A} \frac{A}{A} \frac{H}{A} \frac{A}{A} \frac{A}{A}.\]

Consequently, (L2) holds on \(H \otimes A \otimes A\). Similarly, we can show that (L2) holds for other cases. □

In the sequel, we immediately obtain: if \((A, \alpha)\) is an \(H\)-module braided Lie algebra, then semi-direct sum \(A \rtimes H\) is a braided Lie algebra.

**Definition 1.3** Assume that \(A\) and \(H\) are braided Lie coalgebras. If \((A, \phi)\) is a left \(H\)-comodule, \((H, \psi)\) is a right \(A\)-comodule and the following (CM1) and (CM2) hold, then \((A, H, \phi, \psi)\) (or \((A, H)\)) is called a matched pair of braided Lie coalgebras.

\[(CM1): \quad \frac{H}{A} \frac{A}{A} \frac{A}{A} = \frac{H}{A} \frac{A}{A} \frac{A}{A} + \frac{H}{A} \frac{A}{A} \frac{A}{A} + \frac{H}{A} \frac{A}{A} \frac{A}{A} - \frac{H}{A} \frac{A}{A} \frac{A}{A};\]

\[(CM2): \quad \frac{H}{H} \frac{A}{H} \frac{A}{H} = \frac{H}{H} \frac{A}{H} \frac{A}{H} + \frac{H}{H} \frac{A}{H} \frac{A}{H} + \frac{H}{H} \frac{A}{H} \frac{A}{H} - \frac{H}{H} \frac{A}{H} \frac{A}{H}.\]
Proposition 1.4 If \((A, H)\) is a matched pair of braided Lie coalgebras, then double cross co-sum \(A \triangleright H\) is a braided Lie coalgebra.

Proof. We only check the braided co-Jacobi identity. By definition,

\[
A \triangleright H = A \triangleright AA + HAA (1) + A \triangleright AA + HAA (2) - A \triangleright HAA (3) + A \triangleright HAA (4) + HAA (5)
\]

\[
- A \triangleright HAA (6) - A \triangleright HAA (7) - A \triangleright HAA (8) + A \triangleright HAA (9).
\]

So

\[
A \triangleright A = A \triangleright AA + HAA (10) + A \triangleright AA + HAA (11) - A \triangleright AAA (12) + A \triangleright AAA (13) + AAA (14)
\]

\[
- A \triangleright HAA (15) - A \triangleright HAA (16) - A \triangleright HHA (17) + A \triangleright HHA (18) \quad \text{and}
\]

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It follows from the braided Jacobi identity of $A$ that $(1) + (10) + (19) = 0$; Since $A$ is a left $H$-comodule, $(13) + (9) − (26) = 0$, $(4) − (17) + (27) = 0$, $(22) − (8) + (18) = 0$; By the conditions of matched pair of braided Lie coalgebras, $−(16) + (2) − (21) + (5) − (24) = 0$, $−(25) + (11) − (3) + (14) − (6) = 0$, $−(7) + (20) − (12) + (23) − (15) = 0$. Therefore, (CL2) holds on $A$. Similarly, (CL2) holds on $H$. □

In the sequel, we immediately obtain: if $(A, φ)$ is an $H$-comodule braided Lie coalgebra, then the semi-direct sum $A ▶ ▼ H$ is a braided Lie coalgebra.

**Definition 1.5** Assume that $A$ and $H$ are braided Lie algebras and braided Lie coalgebras, and (SLB) holds on $A ⊗ A$ and $H ⊗ H$. If the following (B1)- (B5) hold, then $(A, H, α, β, φ, ψ)$ (or $(A, H)$) is called a double matched pair:

\[
(B1): \quad \begin{array}{c}
\text{H} \\ A
\end{array} \begin{array}{c}
\text{A} \\ \text{A}
\end{array} = \begin{array}{c}
\text{H} \\ \text{A}
\end{array} \begin{array}{c}
\text{A} \\ \text{A}
\end{array} + \begin{array}{c}
\text{H} \\ \text{A}
\end{array} \begin{array}{c}
\text{A} \\ \text{A}
\end{array} - \begin{array}{c}
\text{H} \\ \text{A}
\end{array} \begin{array}{c}
\text{A} \\ \text{A}
\end{array} ;
\]

\[
(B2): \quad \begin{array}{c}
\text{H} \\ H
\end{array} \begin{array}{c}
\text{A} \\ H
\end{array} = \begin{array}{c}
\text{H} \\ H
\end{array} \begin{array}{c}
\text{A} \\ H
\end{array} + \begin{array}{c}
\text{H} \\ H
\end{array} \begin{array}{c}
\text{A} \\ H
\end{array} - \begin{array}{c}
\text{H} \\ H
\end{array} \begin{array}{c}
\text{A} \\ H
\end{array} ;
\]
Theorem 1.6 If \((A, H)\) is matched pairs of braided Lie algebras and braided Lie coalgebras in \(C\), and a double matched pair, then \(A \bowtie H\) is a braided Lie bialgebra in \(C\).

Proof. By Proposition 1.2 and Proposition 1.4, \(A \bowtie H\) is a braided Lie algebra and braided Lie coalgebra. Therefore, we have to check (LB). By definition,

\[
H A \overset{(1)}{=} H A + H A + H A - H A .
\]

Using (B1), (B2) and (B5) on the right hand side above, we have

\[
H A \overset{(1)}{=} H A + H A + H A - H A .
\]
We have to show the above is equal to

\[(g) + (h) - (i) - (j)\]
Thus (LB) holds on $H \otimes A$. Similarly, it follows from (B1), (B2) and (B5) that (LB) holds on $A \otimes H$.

Next we investigate the case of (LB) on $A \otimes A$: 

$$(k)_{\text{SLB}} = (1) - (2) + (3) - (4)$$

where
By definition, \((n) = (1) - (5) - (12) - (14); (o) = (3) + (10) + (8) + (16); -(p) = -(2) + (7) + (11) - (13); -(q) = -(4) + (9) - (6) - (15).\) Therefore (LB) holds on \(A \otimes A\). Similarly, (LB) holds on \(H \otimes H\). Consequently, (LB) holds on \((A \bowtie H) \otimes (A \bowtie H)\). \(\square\)

In the sequel, we have

**Corollary 1.7** ([16, Proposition 8.3.4]) Assume that \(A\) and \(H\) are braided Lie bialgebras; \((A,H)\) is a matched pair of braided Lie algebras; \(A\) is a left \(H\)-module braided Lie coalgebra; \(H\) is a right \(A\)-module braided Lie coalgebra. If \((\text{id}_H \otimes \alpha)(\delta_H \otimes \text{id}_A) + (\beta \otimes \text{id}_A)(\text{id}_H \otimes \delta_A) = 0\) holds, then \(A \bowtie H\) becomes a braided Lie bialgebra.
Corollary 1.8 Assume that $A$ and $H$ are braided Lie bialgebras; $(A,H)$ is a matched pair of braided Lie coalgebra; $A$ is a left $H$-comodule braided Lie algebra; $H$ is a right $A$-comodule braided Lie algebra. If $([,]H \otimes \text{id}_A)(\text{id}_H \otimes \phi) + (\text{id}_H \otimes [,]_A)(\psi \otimes \text{id}_A) = 0$ holds, then $A◮ H$ becomes a braided Lie bialgebra.

Corollary 1.9 Assume that $A$ and $H$ are braided Lie bialgebras; $A$ is a left $H$-comodule braided Lie coalgebra; $H$ is a right $A$-module braided Lie algebra. If $(B2)$ and $(B3)$ hold, and $(\beta \otimes \text{id})(\text{id} \otimes \delta_A) + ([,]H \otimes \text{id})(\text{id} \otimes \phi) = 0$ holds, then $A◮ H$ becomes a braided Lie bialgebra.

Corollary 1.10 ([16, Proposition 8.3.5]) Assume that $A$ and $H$ are braided Lie bialgebras; $A$ is a left $H$-module braided Lie algebra; $H$ is a right $A$-comodule braided Lie coalgebra. If $(B1)$ and $(B4)$ hold, and $(\text{id} \otimes \alpha)(\delta_H \otimes \text{id}) + ([,] \otimes \text{id})(\text{id} \otimes \phi) = 0$ holds, then $A\triangleright H$ becomes a braided Lie bialgebra.

Corollary 1.11 ([17, Theorem 3.7]) If $H$ is a braided Lie bialgebra in $C$ and $A$ is a left $H$-infinitesimal braided Lie bialgebra, then bisum $A \triangleright H$ is a braided Lie bialgebra in $C$.

Corollary 1.12 If $A$ is a braided Lie bialgebra in $C$ and $H$ is a right $A$-infinitesimal braided Lie bialgebra, then bisum $A \triangleright H$ is a braided Lie bialgebra in $C$.

Corollary 1.13 (i) Assume that $A$ and $H$ are braided Lie bialgebras in $C$. If $A$ is an $H$-module braided Lie algebra and an $H$-module braided Lie coalgebra, and $(\text{id} \otimes \alpha)(\delta_H \otimes \text{id}) = 0$ holds, then $A \triangleright H$ is a braided Lie bialgebra in $C$.

(ii) Assume that $A$ and $H$ are braided Lie bialgebras in $C$. If $A$ is an $H$-comodule braided Lie algebra and $H$-comodule braided Lie coalgebra, and $([,] \otimes \text{id})(\text{id} \otimes \phi) = 0$ holds, then $A \triangleright H$ is a braided Lie bialgebra in $C$.

Obviously, for any two braided Lie bialgebra $A$ and $H$, if set $\alpha = 0$, $\beta = 0$, $\phi = 0$ and $\psi = 0$, then $A_\alpha \triangleright H_\beta$ becomes a braided Lie bialgebras. In this case, denote $A_\alpha \triangleright H_\beta$ by $A \oplus H$. Therefore, the direct sum $A \oplus H$ of $A$ and $H$ is a braided Lie bialgebra. Using this, we can construct many examples of braided Lie bialgebras.

Example 1.14 Under Example 0.5, assume that $U$ has a left duality $U^*$ in $C$. Set $X = U \otimes U^*$ and $Y = U^* \otimes U$. Define the bracket operation and co-bracket operation of $X \oplus Y$ as follows: $[ , ]_X = m_X - m_X C_{X,X}$, $\delta_X = 0$, $[ , ]_Y = 0$, $\delta_X = \Delta_Y - C_{Y,Y} \Delta_Y$. It is clear that $X \oplus Y$ is a braided Lie bialgebra under the following operations: $[ , ]_X \oplus Y = [ , ]_X$, $\Delta_{X \oplus Y} = \delta_Y$. 
2 The construction of quantum doubles

In this section we construct the quantum double of braided Lie algebras.

Assume that \( H \) is a braided Lie bialgebra in \( C \) and \( I \) is the unit object in \( C \). If there exists a morphism \( R : I \to H \otimes H \) satisfying the following (COB), then \( H \) is called a coboundary braided Lie bialgebra.

\[(COB) : \quad H \overset{\bullet}{\otimes} H = H \overset{\bullet}{\otimes} R + H \otimes R \overset{\bullet}{\otimes} H . \]

If \((H, R)\) is a coboundary braided Lie bialgebra and \( R \) satisfies the following (CYBE), then \((H, R)\) is called quasitriangular braided Lie bialgebra:

\[(CYBE) : \quad H \overset{\bullet}{\otimes} R \overset{\bullet}{\otimes} H + H \otimes R \overset{\bullet}{\otimes} H + H \otimes R \overset{\bullet}{\otimes} H = 0 . \]

The equation above is called the classical Yang-Baxter equation. If (COB) holds, then (CYBE) is equivalent to the following (I) and (II), respectively.

\[(I) : \quad H \overset{\bullet}{\otimes} R \overset{\bullet}{\otimes} H = R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H , \]

\[(II) : \quad H \otimes R \overset{\bullet}{\otimes} H = R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H . \]

In fact, by the braided anti- symmetry of \( H \) and (COB),

\[H \overset{\bullet}{\otimes} H = R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H + R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H = R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H + R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H = - R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H - R \overset{\bullet}{\otimes} H \overset{\bullet}{\otimes} H . \]
Thus (CYBE) and (I) are equivalent. Similarly, (CYBE) and (II) are equivalent.

Assume that $H$ is a braided Lie bialgebra in $C$ with a left duality $H^*$. Define the bracket operation and co-bracket operation in $H^*$ as follows:

$$H^* \triangleright H^* = H^*$$

It is clear that $H^*$ is a braided Lie bialgebra in $C$ (see [16, Proposition 8.1.2]). Obviously, $H^{*\text{op}}$ is a braided Lie bialgebra, where the co-bracket operations of $H^{*\text{op}}$ and $H^*$ are the same, but their bracket operation have different sign. Similarly, $H^{*\text{cop}}$ is also a braided Lie bialgebra.

**Theorem 2.1** Assume that $H$ is a braided Lie bialgebra with a left duality $H^*$.

(i) ([16, Pro. 8.2.1]) Let $A = H^{*\text{op}}$,

Then $(A \triangleright H, R)$ is a quasitriangular braided Lie bialgebra.

(ii) Let $A = H^{*\text{cop}}$,

Then $(A \triangleright H, R)$ is a quasitriangular braided Lie bialgebra.
Then \((A \bowtie H, R)\) is a quasitriangular braided Lie bialgebra. \(H^{*\text{cop}} \bowtie H\) is called the quantum double of \(H\), written as \(D(H)\).

**Proof.** (i) the left hand side of \((M2) = \)

- \(H H H^{*\text{cop}} = \)
- \((1)+\)
- \(H H H^{*\text{cop}} = \)
- \((2)-\)
- \(H H H^{*\text{cop}} = \)
- \((3)-\)
- \(H H H^{*\text{cop}} = \)
- \((4),\)

the right hand side of \((M2) = \)

- \(H H H^{*\text{cop}} = \)
- \((5)+\)
- \(H H H^{*\text{cop}} = \)
- \((6)+\)
- \(H H H^{*\text{cop}} = \)
- \((7)-\)
- \(H H H^{*\text{cop}} = \)
- \((8).\)

It is clear that \((1) = (5), (2) = -(8), -(3) = (6), (7) = -(4). Therefore \((M2)\) holds.

Similarly, we can check that conditions in Corollary 1.7 hold, i.e. \(H^{*\text{cop}} \bowtie H\) is a braided Lie bialgebra. See

\[\begin{align*}
    A \bowtie H & \quad = \quad A \bowtie H \\
    A \bowtie H & \quad = \quad A \bowtie H \\
    A \bowtie H & \quad = \quad A \bowtie H \\
\end{align*}\]

and

\[\begin{align*}
    A \bowtie H & \quad = \quad A \bowtie H \\
    A \bowtie H & \quad = \quad A \bowtie H \\
\end{align*}\]

\footnote{Majid [16, Pro.8.2.1] called \(H^{*\text{cop}} \bowtie H\) the quantum double. Considering the quantum double of Hopf algebras, we do not use this definition in this paper.}
Therefore, we complete the proof of the first part. The second part can be proved similarly. □

**Corollary 2.2** Assume that \((H, R)\) is a quasitriangular braided Lie bialgebra and \((A, \alpha)\) is an \(H\)-module braided Lie algebra and \(H\)-module braided Lie coagebra. If we define

\[
\phi = \begin{array}{c}
\begin{array}{c}
A \\
H \\
A
\end{array}
\end{array}
\]

and \((SLB)\) holds on \(A \otimes A\), then \(A \triangleright H\) becomes a braided Lie bialgebra.

**Proof.** We first check that \((A, \phi)\) is an \(H\)-comodule. Indeed, by the definition above and the quasitriangular conditions,
Next we check the Yetter-Drinfeld module condition (YD):

\[
\begin{align*}
&H \otimes A \xrightarrow{r} H \otimes A = (1) + (2) + (3).
\end{align*}
\]

Since \( A \) is a left \( H \)-comodule, we have

\[
\text{the left hand side } = (a) + (b).
\]

For the right hand side,

\[
\begin{align*}
(3) &= - \quad (4) - \quad (5),
\end{align*}
\]

where \((-4) = (a), -5 + 1 = 0, (2) = (b)\). Therefore (\( YD \)) holds. Using Corollary 1.11, we complete the proof. \( \square \)

Dually, we shall give the definition of co-quasitriangular braided Lie bialgebras. Assume that \( B \) is a braided Lie bialgebra. If there exists a morphism \( r : H \otimes H \to I \) satisfying the following condition (Bo), then \((H, r)\) is called a coboundary Lie bialgebra. Furthermore, if \( r \) satisfies the following condition (CCYBE), then \((H, r)\) is called a co-quasitriangular braided Lie bialgebra:

\[
\begin{align*}
(Bo) : \quad H H \xrightarrow{r} H H &= H H + (H H) r. \quad (1) + (2) + (3).
\end{align*}
\]
The equation above is called the classical Co-Yang-Baxter equation.

If (Bo) holds, then (CCYBE) is equivalent to anyone of the following (I) and (II):

\[
\begin{array}{c}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\end{array}
\end{array}
\begin{array}{c}
= 0
\end{array}
\]

(I),

\[
\begin{array}{c}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\begin{array}{c}
H \\
H
\end{array}
\end{array}
\end{array}
\begin{array}{c}
= H
\end{array}
\]

(II).

**Corollary 2.3** Assume that \((H, r)\) is a co-quasitriangular braided Lie bialgebra; \((A, \phi)\) is an \(H\)-comodule braided Lie algebra and an \(H\)-comodule braided Lie coalgebra. If we define

\[
\alpha = \begin{array}{c}
\begin{array}{c}
H \\
A
\end{array}
\begin{array}{c}
A \\
A
\end{array}
\end{array}
\]

and (SLB) holds on \(A \otimes A\), then \(A \bowtie H\) becomes a braided Lie bialgebra.

**Corollary 3.3** and **Corollary 2.3** are called the bosonisation theorems.

We easily obtain: if \(H\) has a left duality \(H^*\), then \((H, R)\) is a quasitriangular braided Lie bialgebra if and only if \((H^*, R^*)\) is a co-quasitriangular braided Lie bialgebra. Here \(R^*\) can be denoted by the following diagram:

\[
R^* = \begin{array}{c}
\begin{array}{c}
H^* \\
H^*
\end{array}
\begin{array}{c}
R^*
\end{array}
\end{array}
\]

Otherwise, if \(U\) and \(V\) have the left duality \(U^*\) and \(V^*\), then \(U \oplus V\) has a left duality \(U^* \oplus V^*\). In fact, let \(D\) denote \(U \oplus V\). Define \(d_U = d_U(\pi_U \otimes \pi_U)\), \(b_U = (\iota_U \otimes \iota_U) b_U\), \(d_D = d_U + d_V\), \(b_D = b_U + b_V\), where \(\pi_U\) and \(\pi_U^*\) are the canonical projections from \(U \oplus U^*\) to \(U\) and to \(U^*\), respectively; \(\iota_U\) and \(\iota_U^*\) are canonical injection from \(U\) to \(U \oplus U^*\) and from \(U^*\) to \(U \oplus U^*\), respectively. It is clear that \(d_D\) and \(b_D\) is evaluation and coevaluation, respectively. Therefore, in Example 1.14, \(X \oplus Y\) has a left duality, so \(X \oplus Y\) has the quantum double,
Example 2.4 ([1] Lemma 3.4)] Assume that $\Gamma$ is a finite commutative group; $g_i \in \Gamma$, $\chi_i \in \hat{\Gamma}$, where $\hat{\Gamma}$ is the character group of $\Gamma$; $\chi_i(g_j)\chi_j(g_i) = 1$, $1 \leq N_i$, where $N_i$ is the order of $\chi_i(g_i)$, $1 \leq i, j \leq \theta$. Let $\mathcal{R}(g_i, \chi_i; 1 \leq i \leq \theta)$ be the algebra generated by the set $\{x_i \mid 1 \leq i \leq \theta\}$ with defining relations
\[ x_i^{N_i} = 0, \quad x_i x_j = \chi_j(g_i) x_j x_i \quad \text{where} \quad 1 \leq i, j, l \leq \theta. \] Define the coalgebra operations and $k\Gamma$-(co)module in $\mathcal{R}(g_i, \chi_i; 1 \leq i \leq \theta)$ as follows;
\[ \Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \]
\[ \delta^-(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)x_i. \]
It is clear that $\mathcal{R}(g_i, \chi_i; 1 \leq i \leq \theta)$ is a quantum commutative braided Hopf algebra in $k\Gamma\mathcal{YD}$, called a quantum linear space in $k\Gamma\mathcal{YD}$. Let $\mathcal{C} := k\Gamma\mathcal{YD}$ and $U := \mathcal{R}(g_i, \chi_i; 1 \leq i \leq \theta)$ as braided Hopf algebras; $A = H =: \mathcal{R}(g_i, \chi_i; 1 \leq i \leq \theta)$ as vector spaces. Define $\delta_A = 0, [\ , ]_H = 0, [\ ]_A = m_U - m_U C_{U,U}, \quad \delta_H = \Delta_U - C_{U,U}\Delta_U$. It is clear that $A$ and $H$ are braided bialgebras in $\mathcal{C}$ (see [28, Theorem 1.3]). So is $A \oplus H$. Since $A$ and $H$ are finite dimensional, they have left dualities in $k\Gamma\mathcal{YD}$ (see [27, Proposition 1.0.17] ), so $A \oplus H$ has a left duality in $k\Gamma\mathcal{YD}$, i.e. $A \oplus H$ has a quantum double in $k\Gamma\mathcal{YD}$.

3 The universal enveloping algebra of double cross sum of Lie bialgebras

In this section we show that the universal enveloping algebra of double cross sum of Lie bialgebras is isomorphic to the double cross product of their universal enveloping algebras.

Throughout this section, we work over ordinary vector spaces. That is, $\mathcal{C}$ is the category $\text{Vect}_k$ of ordinary vector spaces (see [27]). Let $i_H : H \to U(H)$ denote the canonical injection. One can see [13, Definition IX.2.1] and [25] about the matched pair of bialgebras.

**Lemma 3.1** Assume that $A$ and $H$ are two Lie algebras. If $(U(A), U(H), \alpha', \beta')$ is a matched pair and $\text{Im}(\alpha'(i_H \otimes i_A)) \subseteq A$ and $\text{Im}(\beta'(i_H \otimes i_A)) \subseteq H$, then there exist $\alpha$ and $\beta$ such that $(A, H, \alpha, \beta)$ becomes a matched pair of Lie algebras with $\alpha'(i_H \otimes i_A) = i_A \alpha$ and $\beta'(i_H \otimes i_A) = i_H \beta$.

**Proof.** Define $\alpha(x \otimes a) = \alpha'(x \otimes a) = x \triangleright a$ and $\beta(x \otimes a) = \beta'(x \otimes a) = x \triangleleft a$ for any $x \in H$, $a \in A$. Since $(U(A) \times U(H))$ is a matched pair of bialgebras, we have $x \triangleright (ab) = (x \triangleright a)((1 \triangleleft 1) \triangleright b) + (1 \triangleright a)((x \triangleleft 1) \triangleright b) + (1 \triangleright 1)((x \triangleleft a) \triangleright b) = (x \triangleright a)b + a(x \triangleright b) + (x \triangleleft a) \triangleright b$ and $x \triangleright (ba) = (x \triangleright b)a + b(x \triangleright a) + (x \triangleleft b) \triangleright a$ for any $x \in H$, $a, b \in A$. By subtraction of
two equations above, $x ⊲ [a, b] = [x ⊲ a, b] + [a, x ⊲ b] + (x ◼a) ▷ b - (x ◼ b) ▷ a$, so (M1) holds. Similarly, we can obtain (M2): $[x, y] ◼ a = [x, y ◼ a] + [x ◼ a, y] + x ◼ (y ▷ a) - y ◼ (x ▷ a)$.

Consequently, $(A, H)$ is a matched pair of Lie algebras. □

**Theorem 3.2** Assume that $A$ and $H$ are Lie algebras. If $(U(A), U(H), α', β')$ is a matched pair of bialgebras and $\text{Im}(α'(i_H ⊗ i_A)) ⊆ A$, $\text{Im}(β'(i_H ⊗ i_A)) ⊆ H$, then there exist a Hopf algebra isomorphism:

$$U(A ∅ H) ≅ U(A) ∅ U(H).$$

**Proof.** Assume that $(a, x), (b, y) ∈ A ∅ H$ with $a, b ∈ A$, $x, y ∈ H$. By the definition of $A ∅ H$, $[(a, x), (b, y)] = ([a, b] + x ▷ b - y ▷ a, [x, y] + x ◼ b - y ◼ a)$. Define $f : A ∅ H → U(A) ∅ U(H)$, $f(a, x) = i_A(a) ⊙ 1 + 1 ⊙ i_H(x) = a ⊙ 1 + 1 ⊙ x$. Then $f([(a, x), (b, y)]) = [a, b] ⊙ 1 + x ▷ b ⊙ 1 - y ▷ a ⊙ 1 + 1 ⊙ [x, y] + 1 ⊙ x ◼ b - 1 ⊙ y ◼ a)$. By the definition of double cross product (see [23]), $(a ⊙ x) ◼ b ⊙ y = (a ∅ x)(b ∅ y) = a(x(1) ▷ b(1)) ⊙ (x(2) ◼ b(2))y$, where $\cdot$ denotes the multiplication of double cross product. See

$$[f(a, x), f(b, y)]$$

$$= (a ⊙ 1 + 1 ⊙ x) ∅ (b ⊙ 1 + 1 ⊙ y) - (b ⊙ 1 + 1 ⊙ y) ∅ (a ⊙ 1 + 1 ⊙ x)$$

$$= ab ⊙ 1 + a ⊙ y + b ⊙ x + x ▷ b ⊙ 1 + 1 ⊙ x ◼ b + 1 ⊙ xy$$

$$- (ba ⊙ 1 + b ⊙ x + a ⊙ y + y ▷ a ⊙ 1 + 1 ⊙ y ◼ a + 1 ⊙ yx)$$

$$= (ab - ba) ⊙ 1 + x ▷ b ⊙ 1 - y ▷ a ⊙ 1$$

$$+ 1 ⊙ (xy - yx) + 1 ⊙ x ◼ b - 1 ⊙ y ◼ a).$$

Thus $f([(a, x), (b, y)]) = [f(a, x), f(b, y)]$, which implies that $f$ is a Lie algebra homomorphism from $A ∅ H$ to $U(A) ∅ U(H)$. Using the universal property of universal enveloping algebras, we know that $f$ can become an algebra homomorphism from $U(A ∅ H)$ to $U(A) ∅ U(H)$.

$f$ also is a coalgebra homomorphism. Indeed, for any $(a, x) ∈ A ∅ H$, since

$$Δf(a, x) = Δ(a ⊙ 1 + 1 ⊙ x) = a ⊙ 1 ⊙ 1 ⊙ 1$$

$$+ 1 ⊙ 1 ⊙ a ⊙ 1 + 1 ⊙ x ⊙ 1 ⊙ 1 + 1 ⊙ 1 ⊙ 1 ⊙ x$$

and

$$(f ⊗ f)Δ(a, x) = (f ⊗ f)((a, x) ⊙ 1 + 1 ⊙ (a, x))$$

$$= (f ⊗ f)(a ⊙ 1 + x ⊙ 1 + 1 ⊙ a + 1 ⊙ x)$$

$$= f(a) ⊙ f(1) + f(x) ⊙ f(1) + f(1) ⊙ f(a) + f(1) ⊙ f(x),$$

we have $Δf(a, x) = (f ⊗ f)Δ(a, x)$. Considering that $f$ and $Δ$ are algebra homomorphisms, we have that $f$ is a coalgebra homomorphism.
Using two Lie algebra homomorphisms $\iota_A : A \to A \bowtie H$, $a \to (a,0)$ and $\iota_H : H \to A \bowtie H$, $x \to (0,x)$, we can obtain two algebra homomorphisms $j_1 : U(A) \to U(A \bowtie H)$ and $j_2 : U(H) \to U(A \bowtie H)$, i.e. for any $a \in A$, $x \in H$,

$$j_1(a) = i_{A \bowtie H}(a, 0), \quad j_2(x) = i_{A \bowtie H}(0, x).$$

Define a map $j : U(A) \bowtie U(H) \to U(A \bowtie H)$ by sending $j(b \otimes y) = j_1(b)j_2(y)$

For any $a \in A$, $x \in H$, since $jf(a, x) = j(a \otimes 1) + j(1 \otimes x) = (a, 0) + (0, x) = (a, x)$, we know that $f$ is injective on $A \bowtie H$. By [22, Lemma 11.0.1], $f$ is injective on $U(A \bowtie H)$. Obviously, $f$ is surjective. Consequently $f$ is a Hopf algebra isomorphism. □

**Corollary 3.3** If $A$ and $H$ are Lie algebras, then there exists a Hopf algebra isomorphism:

$$U(A \oplus H) \cong U(A) \otimes U(H).$$

**Proof.** Let $\alpha'$ and $\beta'$ be trivial actions, i.e. $\alpha' = (\epsilon_{U(H)} \otimes id_{U(A)})$ and $\beta' = (id_{U(H)} \otimes \epsilon_{U(A)})$. Obviously, $(U(A), U(H), \alpha', \beta')$ is a matched pair of bialgebras with $\alpha = 0$ and $\beta = 0$. Applying Theorem 3.2 we complete the proof. □

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