Invariant Cones in Lie Algebras and Positive Energy Representations and Contractions of Conformal Algebras

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Abstract.
We recall some important results, due to Kostant and others, about invariant convex cones in Lie algebras and positive energy representations. We apply these results to a study of positive energy representation of the conformal groups in $n$ dimensions, and we present a proof of the converse of a theorem attributed to I.E. Segal, which relates positive energy representations to positivity of the action of the generator of time translations for representations of the $n$-dimensional conformal group.

We also discuss related notions of deformation and contractions of Lie algebras and describe a deformation of the Poincaré subalgebra of the conformal algebra which generalizes the usual treatment. We consider the positive energy representations of the anti-deSitter subalgebras in the physically important four dimensional case, and apply this generalization to argue that the singelton representations cannot have nontrivial contractions to representations of the Poincaré algebra. We believe that our results represent a sharpening of the meaning of “kinematical confinement”, introduced by Flato, Fronsdal and their coworkers.

1. Invariant Convex Cones in Simple Lie Algebras
Let $V$ be a finite dimensional vector space over $\mathbb{R}$. A cone in $V$ is a closed convex subset stable under scalar multiplication by nonnegative real numbers i.e. $C$ is a cone if $C$ is closed and convex and such that

$$tC \subseteq C \forall \ t \geq 0.$$  

By convexity we mean [1]

$$\lambda \ v + \mu \ w \in C \text{ if } v, \ w \in C$$

where $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = 1$. A cone $C$ in a Lie algebra $\mathcal{G}$ is called invariant if it is invariant relative to the adjoint group i.e. if

$$\exp(\text{ad}X)C \subseteq C \forall X \in \mathcal{G}.$$  

A causal cone is one for which $C \cap -C = 0$.

The classification of invariant cones in Lie algebras was initiated by Kostant and Vinberg, and the classification of invariant cones in simple Lie algebras was accomplished by S.M. Paneitz in 1980 as part
of his Ph.D. thesis and then independently by G. I. Olshanskii [3]. Classification schemes for arbitrary Lie algebras are reported in [4]. A main theorem in the subject is the following theorem of Kostant [2].

**Theorem 1.1.** (Kostant)

Let $G$ be connected semisimple Lie group acting in a real finite dimensional vector space $V$. Let $K$ be a maximal compact subgroup of $G$. Then there exists a continuous $G$-invariant convex cone $C$ in $V$ satisfying $C \cap -C = 0$ if and only if $V$ has a nonzero $K$-invariant vector which lies in $\mathcal{H}$, the Lie algebra of $K$.

For simple Lie algebras there is a corollary of this theorem, namely: *there exists a non-trivial casual cone in $\mathcal{G} \iff G/K$ is Hermitian symmetric* [2]. A list of hermitian symmetric algebras may be found in [5]. It suffices for us to note that they include $so(2, n)$ ($n > 3$), $sp(n, \mathbb{R})$ and $su(p, q)$ ($p \geq q \geq 1$). Let $\mathcal{C}$ be the (one-dimensional) center of $\mathcal{H}$. Choose $Z \in \mathcal{C}$ suitably normalize. Note that for $so(2, n)$, $Z = X_0$ is the Einstein energy [6].

We let $C_{Min}$ be the closed cone generated by $\text{Ad}(G)Z$. It is a theorem that any (invariant) cone in $\mathcal{G}$ contains $C_{Min}$. Self-dual cones are those for which $C_{Min} = C_{Max} = (C_{Min})^*$. If $\mathcal{G} 
less\cong sl(n, \mathbb{R})$, then $C_{Min}$ is self-dual [2]. Since $so(2, 3) \cong sp(2, \mathbb{R})$ [5], we have for $\mathcal{G} = so(2, 3)$ the existence of a minimal, invariant self-dual cone in $\mathcal{G}$.

2. Lie Algebra Deformations and Contractions

Given $V$, a finite dimensional vector space of dimension $n$ over $\mathbb{R}$, let $\mathcal{M}_n$ be the space of all Lie structures on $V$ i.e. the space of all equivalence classes of bilinear mappings $\mu : V \times V \rightarrow V$ such that $\mu(x, y) = -\mu(y, x)$ and

$$\sum_{\text{cyclic} \{x, y, z\}} \mu(x, \mu(y, z)) = 0 \quad \text{Jacobi identity}$$

where two such maps are equivalent if they give isomorphic Lie algebras.

Let $\Lambda \in \mathbb{R}_+^n$, then a deformation of a given Lie algebra $\mathcal{G}$ over $\mathbb{R}$ is a continuous mapping $\psi : [0, \Lambda] \longrightarrow \mathcal{M}_n$ with $\mathcal{G} = (V, \psi(0))$ i.e. $\psi(0)$ is the Lie structure of $\mathcal{G}$, where continuous means continuous in the topology $\mathcal{M}_n$ inherits from the inclusion of the space of structure constants in $\mathbb{R}^{n^3}$ (i.e. in the Segal topology) [7], [8]. $\psi$ is a trivial deformation if $\psi(t)$ for all $t \in [0, \Lambda]$ is equivalent to $\psi(0)$.

The process of deformation of a Lie algebra may be viewed, at least in special circumstances, as the inverse of contraction of a Lie algebra, which idea goes back to Segal [9] and Inönü and Wigner [10] and formalized by Salzen [11]. We now turn to this notion of contraction of a Lie algebra. Let $\mathcal{G} = (V, [\cdot, \cdot])$ be an $n$-dimensional Lie algebra over $\mathbb{R}$ with an underlying $n$-dimensional vector space $V$ over $\mathbb{R}$ and a Lie bracket $[\cdot, \cdot]$. Consider a continuous family $(\phi_{\lambda})_{\lambda \in \mathbb{R}_+}$ of surjective mappings $\phi_{\lambda} \in \text{Hom}(V, W)$ where $W$ is another vector space of the same dimension as $V$. If the $\phi_{\lambda}$ are injective, we may define a new bracket on $V$ as follows:

$$[x, y]_{\lambda} = \phi_{\lambda}^{-1}[\phi_{\lambda}(x), \phi_{\lambda}(y)] \quad \forall \lambda \in \mathbb{R}_+^n, \quad \forall \ x, y \in V. \quad \text{(1)}$$

Set $\mathcal{G}_{\lambda} = (V, [\cdot, \cdot]_{\lambda})$ and assume that $\mathcal{G}_1 = \mathcal{G}$, so that $\phi_1$ defines an isomorphism of $\mathcal{G}$ onto its image. If

$$\lim_{\lambda \rightarrow 0} [x, y]_{\lambda} := [x, y]_0 \quad \text{(2)}$$

exists for all $x, y \in V$, then it defines a (possibly) new Lie algebra and we call this new Lie algebra, $\mathcal{G}_0 = (V, [\cdot, \cdot]_0)$, a contraction of the Lie algebra $\mathcal{G}$ or (simply) the *contracted of $\mathcal{G}$*. For the case when
$V = W$ and so $\phi_\lambda \in GL(V)$ our definition reduces to the usual definition given in the literature c.f. [8], [12], [13].

We now make use of the family of mappings $(\phi_\lambda)_{\lambda \in \mathbb{R}^+}$ to define contractions of representations of Lie algebras [14], [15]. We start with a given Lie algebra $\mathcal{G} = (W,[\cdot,\cdot])$ and an infinitesimally unitarizable representation $(d\pi(\mathcal{G}),\mathcal{H})$ of $\mathcal{G}$ defined on a fixed Hilbert space, $\mathcal{H}$. Given a continuous family $(\Pi_\lambda)_{\lambda \in \mathbb{R}^+}$ of closed invertible linear transformations of $\mathcal{H}$ with $\Pi_1 = \text{Id}$ (Id = the identity on $\mathcal{H}$) it is easy to see that the map

$$\mathcal{G}_\lambda = (V, [\cdot,\cdot]) \ni X \rightarrow d\pi_\lambda(X) = \Pi_\lambda^{-1} d\pi(\phi_\lambda(X)) \Pi_\lambda$$

(3)
defines a representation of $\mathcal{G}_\lambda$ on $\mathcal{H}$. In order to assure infinitesimal unitarizability of the contracted representation, the family $(\Pi_\lambda)_{\lambda \in \mathbb{R}^+}$ should be chosen so that $d\pi_\lambda(X)$ is skew-symmetric. To check the representation condition we have [15]:

$$[d\pi_\lambda(X), d\pi_\lambda(Y)] = \Pi_\lambda^{-1} [d\pi(\phi_\lambda(X)), d\pi_\lambda(\phi(Y))] \Pi_\lambda = \Pi_\lambda^{-1} d\pi([\phi_\lambda(X), \phi_\lambda(Y)]) \Pi_\lambda = \Pi_\lambda^{-1} d\pi([X, Y]) \Pi_\lambda = d\pi_\lambda([X, Y])$$

(4)

where in the last line we have made use of eqn. (1). We define the representation of the contracted Lie algebra, $\mathcal{G}_0$, as $d\pi_0(X) = \lim_{\lambda \rightarrow 0} d\pi_\lambda(X)$ provided this limit exists, and we call it the contracted of the representation $(d\pi(\mathcal{G}),\mathcal{H})$.

3. Positive Energy Representations of the Conformal Group in $n$ dimensions

Now we want to apply the above to obtain some physically useful results about representations of the conformal group in $n$ dimensions and some of its subgroups. Consider the quadratic form $Q(x)$ defined on $\mathbb{R}^{n+2}$ by

$$Q(x) = x_1^2 + x_0^2 - x_1^2 - x_2^2 - \ldots - x_n^2$$

(5)

where $x = (x_{-1},x_0,x_1,\ldots,x_n) \in \mathbb{R}^{n+2}$. Thus $\mathbb{R}^{n+2}$ equipped with the metric defined by $Q(x)$ is $n+2$ dimensional Minkowski space. We denote $\mathbb{R}^{n+2}$ dimensional Minkowski space by $M_{0,1}$. Let $G = SO_0(2,n)$ denote the connected component of the group of linear transformations of $\mathbb{R}^{n+2}$ preserving the symmetric bilinear form which is associated to $Q(x)$ by polarization. We shall call $G$ the $n$-dimensional conformal group, and we denote the universal cover of $G$ by $G^\ast = SO_0(2,n)^\ast$. Let $\mathcal{G}$ be the Lie algebra of $G$. $\mathcal{G}$ is identified with the set of all matrices $(a_{ij})$ ($-1 \leq i,j \leq n$) such that $a_{ii} = 0$ $(0 \leq i \leq n)$, $a_{ij} = -a_{ji}$ $(1 \leq i \leq j \leq n)$, $a_{0j} = a_{ij}$ $(1 \leq j \leq n)$, $a_{i-1j} = a_{ij-1}$ $(1 \leq j \leq n)$ and $a_{i-10} = -a_{0j-1}$. We define subalgebras $\mathcal{H}$, $\mathcal{G}^i$, $\mathcal{A}^\ast$, $\mathcal{N}_+$ and $\mathcal{N}_-$ as follows. Let $E_{ij}$ be the matrix such that the $(i,j)$ component is equal to 1 and the other components are all equal to 0. Let $L_{ij} = E_{ij} - E_{ji}$ $(1 \leq i \leq j \leq n)$, $L_{0i} = E_{0,i} + E_{i,0}$ $(1 \leq i \leq n)$, $L_{-1,i} = E_{i,-1} + E_{-1,i}$ $(1 \leq i \leq n)$ and $L_{-10} = E_{-1,0} - E_{0,-1}$. Let $\mathcal{H}$ be the subalgebra spanned by: $L_{ij}$ $(1 \leq i,j \leq n)$ and $L_{-1,0}$; $\mathcal{G}^i$ be the subalgebra spanned by: $L_{ij}$ $(1 \leq i,j \leq n-1)$, $L_{0i}$ $(1 \leq i \leq n-1)$, $L_{-1,i}$ $(1 \leq i \leq n-1)$ and $L_{-1,0}$; $\mathcal{A}^\ast$ be the subalgebra spanned by $L_{-1,n};$ and $\mathcal{N}_+$ $(\mathcal{N}_-)$ be the subalgebra spanned by $P_i = \frac{1}{2}(L_{n,i} + L_{-1,i})$ $(0 \leq i \leq n-1)$ (This is the Iwasawa-like decomposition of $G$ i.e. the map $H^\ast \times A \times N_+ \rightarrow G$ is an injective diffeomorphism onto an open, dense subset of $G$, where $H^\ast = SO_0(2,n-1)$) [16].

Consider the $n+1$ dimensional isotropic cone in $\mathbb{R}^{n+2}$ defined by

$$C = \{x \in \mathbb{R}^{n+2} | Q(x) = 0\}.$$  

(6)

Let $\mathbb{R}^{n+2}$ and $C^\ast$ be the sets of nonzero elements in $\mathbb{R}^{n+2}$ and $C$, respectively. Let $P \subset G$ be the stabilizer subgroup of $e = e_{-1} + e_n$ where $e_i = (0,0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^{n+2}$ (i.e. $e_i$ is the vector
Theorem 3.1. Theorem then follows from the observation that any cone in representation with positive energy if \( P \) where \( P \) is the \( n \)-dimensional Poincaré group. The orbit of \( e \) under \( G \) is \( C \). Hence \( C^* = G/P \).

Now we consider a representation \( \pi \) of \( SO(2, n)^- \) and \( d\pi \) the associated representation of \( so(2, n) \) which lifts to a representation of the enveloping algebra. For brevity, write \( X \) for \( d\pi(X) \) with \( X \in \mathfrak{g}(\mathfrak{g}) \), the enveloping algebra of the complexification \( \mathfrak{g}_F \) of \( \mathfrak{g} \). We let \( \mathcal{H} \) be a Cartan subalgebra of \( \mathfrak{g}_C \) and let \( \Delta \) be the root system of \((\mathfrak{g}, \mathcal{H})\). Define \( \mathcal{G}^\Delta = \{ X \in \mathfrak{g}| [h, X] = \lambda(h)X \ \forall \ h \in \mathcal{H} \text{ and } \lambda \in \Delta \} \). Set \( \mathcal{N} = \sum_{\lambda \in \Delta^+} \mathcal{G}^\lambda \) and \( \mathcal{N}^0 = \sum_{\lambda \in \Delta_-^0} \mathcal{G}^\lambda \). Let \( \mathfrak{v} \) be a \( \mathfrak{g}(\mathfrak{g}) \) module with action \( d\pi \) of \( \mathfrak{g}(\mathfrak{g}) \) on \( \mathfrak{v} \). Let: a) \( \mathfrak{v} = \sum_\lambda \mathfrak{v}_\lambda \) with \( \mathfrak{v}_\lambda = \{ v \in \mathfrak{V}| h = \lambda(H)v \}; \) b) \( \forall v_0 \in \mathfrak{V} \ni (i)X^+(X^-)v_0 = 0 \text{ with } X^+ \in \mathcal{N} \text{ and } X^- \in \mathcal{N}^0 \); (ii) \( H_0 \) \( v_0 = \lambda_0(H_0) \) \( v_0 \); (iii) \( \mathfrak{g}(\mathfrak{g}) \) \( v_0 = \mathfrak{v} \). If \( d\pi \) comes from a unitary representation of \( G^- \), then it is an infinitesimally unitarizable lowest (highest) weight representation with lowest (highest) weight \( \Lambda_0 \).

An irreducible, infinitesimally unitarizable representation \( d\pi \) of \( \mathfrak{g} \) on a Hilbert space \( \mathcal{H} \) is a representation with positive energy if

\[
C_\pi = \{ X \in \mathfrak{g} \mid \text{id}(X) \text{ is a nonnegative self adjoint operator} \}
\]

is a non-zero and proper cone in \( \mathfrak{g} \).

Theorem 3.1.

Let \( d\pi \) be an irreducible, infinitesimally unitarizable representation of \( \mathfrak{g} \) on \( \mathcal{H} \). Then \( d\pi \) is positive energy if and only if \( d\pi \) is a highest (lowest) weight representation.

To prove the theorem we refer the reader to [17] where it is shown that if \( \text{id}(X_0) \) a positive, self-adjoint operator then the representation \( (d\pi(\mathfrak{g}), \mathcal{H}) \) is a lowest (highest) weight representation. The theorem then follows from the observation that any cone in \( \mathfrak{g} = so(2, n) \) must contain \( C_{\text{Min}} \).

Next consider the 3 dimensional (simple) subalgebra (TDS) spanned by:

\[
\{ L_{-10} = -X_0, \ L_{0n}, \ L_{-1n} = -S \}
\]

\( X_0 \) is the Einstein energy, and \( S \) is the generator of scale.

Recall

\[
P_0 = i T_0 = \frac{i}{2} (L_{-10} + L_{0n})
\]

We have

\[
[S, X_0] = L_{0n}, \ [S, L_{0n}] = X_0, \ [X_0, L_{0n}] = S.
\]

Additionally

\[
L_{-10} (\alpha) = e^{\alpha S} (L_{-10}) e^{-\alpha S}, \ L_{-11} (\alpha) = [L_{0n}, L_{-10} (\alpha)], \ L_{\mu\nu} (\alpha) = L_{\mu\nu} (\mu, \nu = 0, 1, 2, \ldots, n - 1)
\]

These \( L_{\mu\nu} (\alpha) \)'s are a basis for a conjugate copy of \( so(2, n - 1) \), conjugate under \( so(2, n) \), and

\[
P_\mu = \frac{i}{2} \lim_{\alpha \to \infty} e^{-\alpha} L_{-1\mu} (\alpha), \ L_{\mu\nu}
\]

where \( P_\mu \) and \( L_{\mu\nu} \) are the generators of the \( n \)-dimensional Poincaré Lie algebra, \( \mathfrak{P} \).
Theorem 3.2. \( \psi \) is a positive energy representation of \( \text{so}(2,n) \) \( \iff \) \( \pi(P_0) = (d\pi(P_0)) \) is a positive, self-adjoint operator.

The proof in the \( \Leftarrow \) direction is due to Segal and is given in [17]. It uses the conformal inversion. Although the other direction (the converse of Segal’s theorem) seems to be known to experts in the field, to our knowledge an explicit proof of it, at least in the physics literature, seems to be lacking, and so we record it here: we observe that positivity of the Einstein energy implies

\[
(\psi, d\pi(iX_0(\alpha))\psi) = (\psi, id\pi(e^{\alpha S}X_0e^{-\alpha S})\psi) = (\psi', d\pi(iX_0)\psi') > 0 \quad \forall \quad \alpha > 0
\]  

(12)

where \( \psi' = e^{-\alpha S}\psi \). Now eqn. (9) implies

\[
-X_0(\alpha) = e^{\alpha S}(-X_0)e^{-\alpha S} = \{\cosh(\alpha)(-X_0) + \sinh(\alpha)L_{\text{out}}\} \to -2iP_0 e^{\alpha} \quad \text{as} \quad \alpha \to \infty.
\]  

(13)

The desired result is easily seen to be a consequence of these two equations.

4. A Deformation of the Poincaré Subalgebra and the Contracted of the Di and Rac

Denote by \( \mathcal{E}(\mathcal{G}) \) and \( \mathcal{E}(\mathcal{P}) \) be the universal enveloping algebras of \( \mathcal{G} = \text{so}(2,n-1) \) and \( \mathcal{P} \), respectively. We introduce the following element of \( \mathcal{E}(\mathcal{G}) \):

\[
Q_2 = \sum_{i=1}^{n-1} L_{ij} - \frac{1}{2} \sum_{i,j=1}^{n-1} L_{ij}^2 .
\]

Let \( \mathcal{K}(\mathcal{P}) \) be the skew field of \( \mathcal{P} \) and \( \mathcal{K}(\mathcal{G}_{\lambda}) \) be the skew field of \( \mathcal{G}_{\lambda} \) (see below for definition of \( \mathcal{G}_{\lambda} \)). Define a commutative algebraic extensions of \( \mathcal{K}(\mathcal{P}) \) by \( \mathcal{K}(\mathcal{P})^{\text{ext}} = \{a+bY \mid a, b \in \mathcal{K}(\mathcal{P}), [Y, a] = 0 \quad \forall \quad a \in \mathcal{K}(\mathcal{P}), Y^2 = P^2 \} \) where \( Y \) commutes with all elements of \( \mathcal{K}(\mathcal{P}) \) and \( P^2 = \sum_{k=0}^{n-1} P_k P_k \).

Now define a mapping \( \tau_\lambda \) from \( \mathcal{G}_{\lambda} \) to \( \mathcal{K}(\mathcal{P}) \) by

\[
\tau_\lambda(\hat{L}_{\mu\nu}) = L_{\mu\nu} , \quad \tau_\lambda(\hat{L}_{-1\mu}) = \frac{i\lambda}{2Y} [Q_2, P_\mu] + P_\mu .
\]  

(14)

The \( \lambda^{-1}\tau_\lambda(\hat{L}_{-1\mu}) \) and \( \tau_\lambda(\hat{L}_{\mu\nu}) \) satisfy the commutation relations of the generators of \( \mathcal{G} \). The \( \tau_\lambda(\hat{L}_{-1\mu}) \) and \( \tau_\lambda(\hat{L}_{\mu\nu}) \) are a basis for an isomorphic copy \( \mathcal{G}_{\lambda} \) of \( \mathcal{G} \).

For the remainder of the paper we take \( n = 4 \), but it seems clear that much of the followings holds for the higher dimensions [18] and even for other groups like \( \text{SL}(n, \mathbb{R}) \) [20], especially regarding contractions of the positive energy representations of the anti-deSitter group [15]. Let \( \tau_{\lambda=1}(\hat{Y}) = Y \), then \( \tau = \tau_{\lambda=1} \) and define a commutative algebraic extension of \( \mathcal{K}(\mathcal{G}_{\lambda}) \) by

\[
\mathcal{K}(\mathcal{G}_{\lambda})^{\text{ext}} = \left\{a + b \hat{Y} + c \hat{Y}^2 + d \hat{Y}^3 \mid a, b, c, d \in \mathcal{K}(\mathcal{G}_{\lambda}) \right\}
\]

where \( \hat{Y} \) commutes with all elements of \( \mathcal{K}(\mathcal{G}_{\lambda}) \). Then \( \tau_\lambda \) can be extended to a homomorphism of \( \mathcal{K}(\mathcal{G})^{\text{ext}} \) into \( \mathcal{K}(\mathcal{P})^{\text{ext}} \) in an obvious way, which, because of Lemma 4.2, is actually surjective in the case \( n = 4 \) which we now denote. Denote this extension also by \( \tau = \tau_{\lambda=1} \). Elements of \( \mathcal{K}(\mathcal{G})^{\text{ext}} \) have a tilde to keep them distinct from elements of \( \mathcal{K}(\mathcal{P})^{\text{ext}} \), and we introduce \( * \) structures on \( \mathcal{K}(\mathcal{P})^{\text{ext}} \) and \( \mathcal{K}(\mathcal{G})^{\text{ext}} \).

Lemma 4.1.

Let \( \tau(\mathcal{G}_{\lambda}) \) be the isomorphic copy of \( \mathcal{G} \) having basis elements \( L_{ij} \in \mathcal{G} \) and \( L_{-1\mu} \in \mathcal{K}(\mathcal{G})^{\text{ext}} \) defined by eqns. (14). Then (for \( \lambda = 1 \)) the following holds:

\[
C_2 = -Y^2 - \left[ \frac{W}{Y^2} + \frac{9}{4} \right] , \quad C_4 = \left[ Y^2 + \frac{1}{4} \right] \frac{W}{Y^2} .
\]  

(15)
Vectors is given in [21]. The results there are: let

Due to ker $\lambda$ in the just stated theorem, $\phi_\lambda$ must be a vector space isomorphism onto its image and thus we can define a new Lie bracket on $\mathcal{P}$ by:

Thus, if we let $\psi : I \subset \mathcal{R} \rightarrow \mathcal{M}_n$ be defined as $\psi(\lambda) = \lfloor \cdot \mid \cdot \rfloor_\lambda$ with $\lambda \in I$, then $\psi$ is a (non-trivial) deformation of $\mathcal{P}$, with $\psi(1)$ being the Lie structure of $\mathcal{G}$. Since $\tau_\lambda$ is an isomorphism, we can apply it in the other direction to obtain contractions of representations of $\mathcal{G}$.

A classification of the infinitesimally unitarizable, irreducible lowest weight representations of $\mathcal{G} = so(2, 3)$ using a modern representation theoretic approach involving Verma modules and singular vectors is given in [21]. The results there are: let $D(E_0, s_0)$ be a given such lowest weight representation with lowest weight $\Lambda \ni \Lambda(H_1) = s_0$ and $\Lambda(H_2) = E_0$. Then we have: i) $D(E_0, s_0) = D(\frac{1}{2}, 0)$ (Rac); ii) $D(E_0, s_0) = D(1, \frac{1}{2})$ (Di); iii) $D(E_0 > \frac{1}{2}, s_0 = 0)$; iv) $D(E_0 > 1, s_0 = \frac{1}{2})$; v) $D(E_0 \geq s_0 + 1, s_0 \geq 1)$. The massless representations are: $D(E_0) = s_0 + 1, s_0 \geq 1$.

All of the representations given here have non-trivial contractions to representations of $\mathcal{P}$, except for the case of the singlet representations (cf. [15] for details on the contractions of the massless representations in $n$ dimensions). For the singletons, i.e. the Di and the Rac (which is the minimal
representation [22] of $SO(2, 3)$, what goes wrong is that it is necessary that both $d\pi(\tilde{D})^{-1}$ and $d\pi(\tilde{D})^\dagger$ of Theorem 4.1 must exist on a suitable, dense domains in $\mathcal{H}$, in order that there exists a representation $d\tilde{\pi}$ of $\mathcal{G}$ on $D(1/2, 0) \subset \mathcal{H}$ (e.g. for the Rac). For both the $Di$ and $Rac$ is $\tilde{Y}^2 = -1, -\frac{1}{4}$ which implies not only that $d\pi(\tilde{P}_\mu)$ are not skew-symmetric operators, but, e.g. for the Rac: $d\pi(\tilde{D}) = 0$ (with $\tilde{Y} = -i$)! Thus the $d\tilde{\pi}(\tilde{P}_\mu)$ cannot exist in any sense as operators on the Hilbert space of the representation with $d\pi(\tau_1(\tilde{L}_{-1\mu}))$ being the action of $\tilde{L}_{-1\mu}$ in the Rac representation. Using this fact it is straightforward to see that the contracted of $d\pi(\mathcal{G}')$ where $d\pi$ is the $Di$ or Rac representation, which is induced by the isomorphism $\tau_\lambda$ cannot exist except trivially i.e. as the identity representation of $\mathcal{G}$.

5. References

[1] Eggleston 1958 Convexity (Cambridge: Cambridge Univ. Press)
[2] Paneitz S M 1981 Jour. Funct. Anal. 43 313-359
[3] Olshanski G I 1981 Funct. Anal. and Its Applications 15, 4, 53-66
[4] Hilgert J, Hoffman K H 1988 Semigroup Forum 37 241-252
[5] Helgason S 1978 Differential Geometry, Lie Groups and Symmetric Spaces’ (New York: Academic Press)
[6] Paneitz S M, Segal I E 1982 Jour. Funct. Anal., 47, 78-142; Paneitz S M 1983 Jour. Funct. Anal., 54, 18-112.
[7] Ritter W G 2003 Quantum Field Theory and the Space of All Lie Algebras Preprint: Harvard Univ., (arXiv: math-ph/0304031)
[8] Levy-Nahas M 1967 J. Math. Phys. 8 1211-1222 (This paper contains explicit information about the Segal topology i.e. the topology in the space $\mathcal{M}_\mu$ of structure constants.)
[9] Segal I E 1951 Duke Math. J. 18 221-265.
[10] In"on"u E, Wigner E P 1953 Proc. Nat. Acad. Sci. U S A 39 510-524.
[11] Saletan E 1961 J. Math. Phys. 2 1-21.
[12] L. "O hmus Ya H 1967 Constructions of Lie Groups, Volume 4 of Proceedings of the 1967 Summer School on Elementary Particle Physics held at Otep"a"a, Estonian SSR, August 1-13.
[13] Weimar-Woods E 2000 Reviews in Math. Phys. 12, 11, 1505-1529.
[14] Dooley A H, Rice J W 1985 Trans. Amer. Math. Soc. 289, 1, 185-202.
[15] Angelopoulos E, Laues M 1998 Reviews in Math. Phys. 10, 3, 271-299.
[16] Sekiguchi J 1980 Nagoya Math. Jour. 79 211-230.
[17] Mack G 1977 Comm. Math. Phys., 55 1-21.
[18] Moylan P 2004 Studies in Mathematical Physics Research Ch. 2, (New Jersey: Nova Science)
[19] Havlíček M, Moylan P 1993 Jour. Math. Phys. 34, 11, 5320-5332.
[20] Salom I, Šijacčki D 2009 arXiv:0906.2106v1 [math-ph] 11 Jun 2009
[21] V.K. Dobrev, E. Sezgin, Lecture Notes in Physics, 379, (Springer, 1990), 227.
[22] Kostant B 1990 The vanishing of scalar curvature and the minimal representation of $SO(4, 4)$ in Progr. Math. 92 (Boston: Birkhäuser), 85-124; Kobayashi T, Mano G 2008 The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$, arXiv: 0712.1769v2 [math.RT] 19 Jul 2008; Kobayashi, T. Orsted B 2003 Adv. Math. 180 486-595;
[23] Flato M, Fronsdal C 1980 Phys. Lett. B 97, 2, 236-240; Flato M, Fronsdal C, Sternheimer D 1999 Singleton Physics, arXiv:hep-th/9901043 v1 12 Jan 1999;