On the equational complexity of RRA

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Abstract
We prove that the equational complexity function for the variety of representable relation algebras is bounded below by a log-log function.

1 Introduction
Let RRA denote the class of representable relation algebras. RRA is definable by equations [10], but not by finitely many [8]. Indeed, any equational basis must contain equations containing arbitrarily many variables [5]. It is an open question whether RRA is definable by first-order formulas using some bounded number of variables—see [3], page 625.

A weak representation of a relation algebra is an isomorphism to an RRA that doesn’t necessarily preserve the operations + and − but does preserve ∩. Let wRRA denote the class of weakly representable relation algebras. wRRA is not finitely based [4], and RRA is not finitely based over wRRA [1]. It was recently shown that wRRA is a variety [9]. Since RRA has no finite-variable equational basis it must be the case that at least one of the following holds:

(i) wRRA has no finite-variable equational basis;

(ii) there is no finite-variable equational basis that defines RRA over wRRA.

It would be interesting to know which of these hold. The author submits this to the reader as an open problem.

All of these results speak to the “bad behavior” of RRA. In this note, we want to focus on a related question for finite algebras: given a finite A ∈ RA, how much of the equational theory of RRA do we have to verify in A before we know that A ∈ RRA?
2 Definitions

We take the following definition from [7]:

**Definition 1.** The *length* of an equation is the total number of operation symbols and variables appearing in the equation. For a variety $V$ of finite signature, the *equational complexity* of $V$ is defined to be a function $\beta_V$ such that for a positive integer $m$, $\beta_V(m)$ is the least integer $N$ such that for any algebra $A$ of the similarity class of $V$ with $|A| \leq m$, $A \in V$ iff $A$ satisfies all equations true in $V$ of length at most $N$.

For example, the length of $(x + y) \cdot z = x \cdot z + y \cdot z$ is 12. We note that for a variety $V$ of finite signature, $\beta_V$ always exists. To see this, fix $m$, and consider the collection of algebras in the similarity class of $V$ of size of most $m$ that are not in $V$. For each algebra in the collection, take the shortest equation that witnesses the algebra’s non-membership in $V$. Let $\ell$ be the length of the longest such shortest equation. Then $\ell + 1$ is an upper bound for $\beta_V(m)$.

Throughout the rest of this paper, let $V = \text{RRA}$. In [6], Roger Lyndon gave a general construction of relation algebras from projective geometries. We are interested in the algebras that come from finite projective lines, and we will use them to find a lower bound on $\beta_V$. We give a definition here that is equivalent to the one Lyndon gave.

Let $E_{n+1}$ be a finite integral relation algebra with $n$ symmetric diversity atoms $a_1, \ldots, a_n$ and one identity atom $1'$. Composition on the atoms is defined thus:

$$a_i; a_i = 1' + a_i \quad \text{and} \quad a_i; a_j = a_i + a_j + 1' \quad \text{for } i \neq j$$

Lyndon proved that $E_{n+1}$ is representable iff there exists a projective plane of order $n - 1$. Bruck and Ryser proved in [2] that there is no projective plane of order $2 \cdot 3^{2n+1}$; hence, $E_{2 \cdot 3^{2n+1} + 2}$ is non-representable. However, every proper subalgebra $A$ of $E_{n+1}$ embeds into $E_{p+1}$ for any prime $p > n$, and hence is representable. Jónsson used this fact in [5] to give a proof that $\text{RRA}$ has no $k$-variable basis for $k < \omega$. This implies that $\beta_V(m)$ is not bounded above.

3 The lower bound

The computation of this lower bound follows the proof of Lemma 6 in [7]. Consider $E_{2 \cdot 3^{2n+1} + 2}$: since there is no projective plane of order $2 \cdot 3^{2n+1}$, $E_{2 \cdot 3^{2n+1} + 2} \notin \text{RRA}$. Therefore, there is some equation $\varepsilon$ such that $\text{RRA} \models \varepsilon$ but $E_{2 \cdot 3^{2n+1} + 2} \not\models \varepsilon$.

We recall that every proper subalgebra of $E_{2 \cdot 3^{2n+1} + 2}$ is representable. Consider the number of distinct variables in $\varepsilon$, and suppose that it is no more than $k = \log_2 3 \cdot (2n + 1)$. Then take $b_1, \ldots, b_k \in E_{2 \cdot 3^{2n+1} + 2}$. The subalgebra generated by $b_1, \ldots, b_k$ is the boolean subalgebra generated by $1', b_1, \ldots, b_k$. This subalgebra is no larger than $2^{2^{k+1}}$, and thus is proper, since

$$2^{2^{\log_2 3 \cdot (2n + 1) + 1}} < 2^{2 \cdot 3^{2n+1} + 2} = |E_{2 \cdot 3^{2n+1} + 2}|$$
Thus we can conclude that $\varepsilon$ contains more than $\log_2 3 \cdot (2n + 1)$ variables, since any equation of fewer variables true in all representable relation algebras would have to be satisfied by the representable subalgebra of $E_{2,3^{2n+1}+2}$ generated by $b_1, \ldots, b_k$. Now consider the length of $\varepsilon$: since $\varepsilon$ contains $k$ distinct variables, it must contain at least $k - 2$ binary operation symbols, hence its length is at least $2k - 2$. This gives us that

$$2 \log_2 3 \cdot (2n + 1) - 2 < \beta_V \left(2^{2 \cdot 3^{2n+1}+2}\right) \tag{\star}$$

Now choose $m \in \mathbb{Z}^+$, with $m \geq 2^8$. Then there is some $n \in \mathbb{Z}$ so that

$$2^{2 \cdot 3^{2n+1}+2} \leq m \leq 2^{2 \cdot 3^{2n+1}+2}$$

Then $m \leq 2^{2 \cdot 3^{2n+1}+2}$ gives us that

$$\frac{1}{2} \log_3 \left(\frac{1}{2} \log_2 (m) - 1\right) - \frac{3}{2} \leq n \tag{\star \star}$$

Let $f(n) = 2 \log_2 3 \cdot (2n + 1) - 2$. We apply $f$ to both sides of $(\star \star)$, which (since $f$ is increasing) yields

$$2 \log_2 3 \cdot (\log_3 \left(\frac{1}{2} \log_2 (m) - 1\right) - 2) - 2 \leq 2 \log_2 3 \cdot (2n + 1) - 2$$

$$< \beta_V \left(2^{2 \cdot 3^{2n+1}+2}\right) \text{ by (\star)}$$

$$\leq \beta_V(m),$$

where the last line follows from the monotonicity of $\beta_V$.

Therefore $\beta_V(m) > 2 \log_2 3 \cdot (\log_3 \left(\frac{1}{2} \log_2 (m) - 1\right) - 2) - 2$ for all $m \geq 2^8$.

Since the size of a finite relation algebra is always a power of 2, we can make some aesthetic changes. Let $M$ be the number of atoms of a finite algebra $A$, and let $\beta_\ast_V$ be the equational complexity function that takes as input the number of atoms of an algebra (rather than the cardinality). Then we get

$$\beta_\ast_V(M) > 2 \log_2 3 \cdot [\log_3 (M/2 - 1) - 2] - 2$$

### 4 Conclusion

Since the language of RA has finite signature, $\beta_V$ is always finite. In [7], tools are given for finding upper bounds for locally finite varieties. For the variety RRA, the derivation of an upper bound may prove more difficult. The author submits this as another open problem.
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