Chaotic motion of neutral and charged particles in the magnetized Ernst-Schwarzschild spacetime

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Neutral test particles around a Schwarzschild black hole immersed in an external uniform magnetic field have no interactions of electromagnetic forces, but their motions can be chaotic. This chaotic behavior is induced owing to the gravitational effect of the magnetic field leading to the nonintegrability of the magnetized Ernst-Schwarzschild spacetime geometry. In fact, chaos is strengthened typically with an increase of the energy or the magnetic field under appropriate circumstances. When these test particles have charges, the electromagnetic forces are included. As a result, the electromagnetic forces have an effect on strengthening or weakening the extent of chaos caused by the gravitational effect of the magnetic field.

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I. INTRODUCTION

Schwarzschild, Kerr and Kerr-Newman black hole spacetimes are highly nonlinear, but they have a sufficient number of isolating integrals to separate equations of motion and therefore are strictly integrable, regular. Only when perturbations are included and destroy the integrability of these spacetime geometries, may chaos be induced. Here the ‘chaos’ means that the dynamics of test particles in these spacetimes display exponentially sensitive dependence on initial conditions. In addition, the perturbations come from materials (such as disks, massive halos, shells and rings) surrounding the central gravitational bodies, spins of the particles or the central bodies, and so on. In fact, chaos can occur in static, axially symmetric relativistic core-shell systems with a monopolar core and an exterior shell of dipoles, quadrupoles and octopoles [1-3]. The chaotic motions of particles around black holes with discs or rings were also confirmed in [4-6]. Systems of compact binaries with one or two bodies spinning can be chaotic in some cases. For reference, see e.g. [7-13].

In addition to the above-mentioned materials and spins, magnetic or electromagnetic fields act as an additional perturbation to the spacetime geometries for inducing chaos. The magnetized Ernst-Schwarzschild metric [14] is a static, axially symmetric spacetime, and describes the motion of electrically charged and neutral test particles around a Schwarzschild black hole immersed in an external uniform magnetic field. Employing the methods of Poincaré surfaces of section and Lyapunov exponents, Karas & Vokrouhlický [15] showed that higher energy yields stronger chaotic behavior regardless of whether the particles in this spacetime geometry have charges or not. In fact, this chaotic behavior can be induced because no additional constant of motion exists and the equations of motion of test particles cannot be successfully separated and solved with an analytical method. Recently, Wang et al. [16] investigated the chaotic motion of a test scalar particle coupling to the Einstein tensor in Schwarzschild-Ernst black hole spacetime. When an external magnetic field whose four-vector potential have two nonzero components $A_t$ and $A_\phi$ is included near the Kerr black hole, the motion of charged particles in this axially symmetric spacetime geometry is nonintegrable and probably chaotic [17-19]. If the four-vector potential of a magnetic field has four nonzero components and breaks axial symmetry, the transition from regular to chaotic dynamics occurs more easily [20].

As claimed in [15], the gravitational effect of the magnetic field can induce chaos although neutral particles in the magnetized Ernst-Schwarzschild spacetime have no interactions of electromagnetic forces. It was also found in [15] that higher energy leads to stronger chaotic behavior. However, one question remains how the extent of chaos changes with an increase of the magnetic field. Another question remains whether the chaotic behavior caused by the gravity of the magnetic field is strengthened when electrically charged particles have interactions of electromagnetic forces. To address these questions, we employ a geometric numerical integration algorithm with preservation of geometric properties of the flows of the differential equations of motion of particles. Manifold correction schemes [21-25], symplectic integrators [26-29] and symmetric methods [30] are some of the geometric algorithms. Because the equations of motion are inseparable in the present problem, one of the explicit symmetric algorithms in extended phase space [31-35] is considered as a numerical integration tool. On the other hand, methods to determine the onset of chaos should be unambiguous declarations of chaos. That is, they do not depend on the spacetime coordinate system used in general relativity. They are called invariant chaotic indicators. The methods of poincaré sections, Lyapunov exponents of two nearby orbits with the proper time and distances [36,37], and fast Lyapunov indicators of two nearby orbits with the proper time and distances [37] are what we expect to use. In a word, the fundamental aim of this paper is to apply an appropriate geometric

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introduces an unambiguous indicators of chaos to truly
describe the effect of the electromagnetic forces on chaos
induced by the gravity of the magnetic field.

In what follows, we introduce the magnetized Ernst-
Schwarzschild spacetime and discuss the effective poten-
tial and stable circular orbits at the equatorial plane in
Section 2. Then, we survey the dynamics of generic or-
bits using analytical or numerical methods in Section 3.
Finally, the main results of this paper are concluded in
Section 4.

II. MAGNETIZED ERNST-SCHWARZSCHILD
SPACETIME

In this section, we introduce the evolution model of
charged particles in the magnetized Ernst-Schwarzschild
spacetime geometry and an electromagnetic field. Then,
the effective potential and innermost stable circular or-
bits at the equatorial plane are discussed.

A. Dynamical model of charged particles

Ernst [14] described the motion of neutral particles
around a non-rotating black hole in an external magnetic
field using the Ernst-Schwarzschild metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$
$$= \Lambda^2\left[\frac{2M - r}{r}dt^2 + \frac{rdr^2}{r - 2M} + r^2d\theta^2\right]$$
$$+ \frac{r^2}{\Lambda^2} \sin^2 \theta d\phi^2,$$

(1)

where $\Lambda = 1 + (1/4)B^2r^2 \sin^2 \theta$, $B$ is a magnetic field
parameter, and $M$ stands for the mass of the black hole.
The superscripts of $x$, $x^\alpha$ and $x^\beta$ with $\alpha, \beta \in \{0, 1, 2, 3\}$,
represent the components of Schwarzschild-like coordi-
nates $x = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$. This metric is an
invariant line element in the four-dimensional spacetime.
In fact, $ds^2 = -dt^2$, where $\tau$ stands for the proper time.
Here the constant of gravity $G$ and the speed of light $c$
between the geometric units $c = G = 1$. The spacetime has
even horizon at $r = 2M$ and no singularity outside the
horizon. It is the same as the Schwarzschild metric
in the two points, with the difference between them lies in that the Schwarzschild spacetime is asymp-
totically flat, but this Ernst-Schwarzschild spacetime is
not due to the magnetic field acting as a gravitational
effect. When the variables and parameters are given to
scale transformations $t \to tM$, $r \to rM$, $B \to B/M$ and
$\tau \to \tau M$, we obtain a dimensionless form of the metric,

which corresponds to the dimensionless Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\frac{dS}{d\tau}\right)^2 = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$$
$$= \frac{1}{2}\Lambda^2\left[-(1 - \frac{2}{r})t^2 + (1 - \frac{2}{r})^{-1}r^2 + r^2\dot{\theta}^2\right]$$
$$+ \frac{1}{2}\Lambda^{-2}r^2 \sin^2 \theta \dot{\phi}^2.$$  

(2)

Notice that $\mathcal{L}$ is identical to $-1/2$, $\mathcal{L} = -1/2$. We define the
covariant momentum

$$P_\alpha = \partial \mathcal{L}/\partial \dot{x}^\alpha = g_{\alpha\beta}\dot{x}^\beta.$$  

(3)

This Lagrangian is exactly equivalent to the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\dot{y}^{\alpha\beta}P_\alpha P_\beta.$$  

(4)

However, Lagrangian and Hamiltonian formulations at
the same post-Newtonian order are not exactly equiv-
alent and are approximately related in general [38-40].
The difference between them is mainly due to higher-
order post-Newtonian terms truncated.

On the other hand, Ernst [14] provided the non-zero components of the magnetic field

$$B_r = \Lambda^{-2}B \cos \theta,$$
$$B_\theta = -\Lambda^{-2}B(1 - \frac{2}{r})^{1/2} \sin \theta.$$  

(5)

The electromagnetic tensor $F_{\mu\nu}$ contains non-vanishing components [41]

$$F_{\theta\phi} = \frac{1}{2}B\Lambda^{-2}r^2 \sin 2\theta,$$
$$F_{r\phi} = B\Lambda^{-2}r^2 \sin \theta.$$  

(6)

Because the four-vector potential $A$ and the electromag-
netic tensor $F_{\mu\nu}$ satisfy the relation $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$,
we have the four-vector potential with a non-zero com-
ponent

$$A_\phi = \frac{B}{2\Lambda} \sin^2 \theta.$$  

(7)

The potential used in this paper is the same as that
used in [42] but is slightly different from those of [15]
and [16]. That is, $A_\phi = -Br^2 \sin^2 \theta/(2\Lambda)$ in [15] and
$A_\phi = -Br^2 \sin^2 \theta/\Lambda$ in [16]. Let $q$ be an electric charge
of the test particle with the dimensionless operation
$q \to qM$. The charged particle moves under the grav-
ities of the system (2) and the electromagnetic force, and
its covariant momentum obeys the relation

$$p_\alpha = P_\alpha + qA_\alpha.$$  

(8)

Based on Eqs. (4) and (8), the motion of the charged
particle subject to the influences of the gravities and the
electromagnetic force is restrained by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}g^{\alpha\beta}(p_\alpha - qA_\alpha)(p_\beta - qA_\beta).$$  

(9)
Hamilton’s canonical equations of motion are given as
\[ \frac{dx^a}{d\tau} = \frac{\partial H}{\partial p_a}, \quad \frac{dp_a}{d\tau} = -\frac{\partial H}{\partial x^a}. \] (10)

Both electromagnetic field and spacetime geometry are stationary and axial symmetric, i.e., the Hamiltonian $H$ does not explicitly depend on $t$ and $\phi$. Therefore, Eq. (10) implies that the system $H$ has constant specific energy $E$ and angular momentum $L$ as follows:

\[ p_t = P_t + qA_t = P_t + g_M, \]
\[ = -\Lambda^2(1 - \frac{2}{r})\dot{t} = -E, \]
\[ p_\phi = P_\phi + qA_\phi = g_\phi, \phi + qA_\phi \]
\[ = \Lambda^{-1}r^2\sin^2\theta\dot{\phi} + \frac{qB\sin^2\theta}{2\Lambda} \]
\[ = L. \] (11)

This constant angular momentum $L$ exists because the system (9) preserves axial symmetry. However, there is no constant angular momentum in the system of [20] for the lack of axial symmetry. In terms of the two constants of motion, the Hamiltonian $H$ is rewritten as

\[ H = \frac{1}{2} \frac{r^2 - 2}{r^2\Lambda^2} p_r^2 + \frac{p_\theta^2}{r^2\Lambda^2} + \frac{rE^2}{(2 - r)\Lambda^2} \]
\[ + \frac{\Lambda^2}{r^2\sin^2\theta}(L - \frac{qB\sin^2\theta}{2\Lambda}). \] (12)

The system $H$ has a four-dimensional phase space made of $(r, \theta, p_r, p_\theta)$. Its canonical equations of motion are

\[ \frac{dr}{d\tau} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{d\tau} = -\frac{\partial H}{\partial r}, \]
\[ \frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta}, \quad \frac{dp_\theta}{d\tau} = -\frac{\partial H}{\partial \theta}. \] (13)

Obviously, the Hamiltonian itself is an integral of motion. In fact, it is always identical to $-1/2$,

\[ H = -\frac{1}{2}. \] (14)

B. Effective potential and innermost stable circular orbits

Noting Eqs. (12) and (14), we have

\[ (1 - \frac{2}{r})^2 p_r^2 + r^2(1 - \frac{2}{r})p_\theta^2 = E^2 - V^2, \]
\[ V^2 = (1 - \frac{2}{r})[1 + \frac{\Lambda^2}{r^2}(L - \frac{qB\sin^2\theta}{2\Lambda})]\Lambda^2, \] (15)

where $V$ is the effective potential at the equatorial plane $\theta = \pi/2$. Fig. 1 plots the effective potentials for various values of the parameters $B$, $q$ and $L$. When $V$ has its local minimum, $V = 0$, that is, a circular orbit exists. In this case, the value of $r$ is the radius of the circular orbit. If $V > 0$, then this circular orbit is stable. Table 1 lists the radii of the stable circular orbits in Fig. 1. Especially for $V = 0$, the stable circular orbit obtained is the innermost stable circular orbit (ISCO). For example, $r = 6$ is the radius of ISCO of the system (12) with $B = 0$, equivalently, that of the Schwarzschild spacetime. Since the magnetic or electromagnetic field we consider is weak, i.e., $0 < |B| < 1$ and $0 < |qB| < 1$, the radii of ISCOs approach to 6 for the various values of the parameters $B$ and $q$, as shown in Fig. 2 and Table 1. In the following discussions, we apply analytical or numerical methods to study the dynamics of generic orbits.

III. INVESTIGATIONS OF ORBITAL DYNAMICS

We explore the dynamics of the system (12) according to three cases: $B = 0$, $B \neq 0$ with $q = 0$, and $B \neq 0$ with $q \neq 0$.

A. Case 1: $B = 0$

For the case of $B = 0$, the system (12) corresponds to the Schwarzschild spacetime

\[ H = \frac{1}{2} \left[ \frac{r^2 - 2}{r\Lambda^2} p_r^2 + \frac{p_\theta^2}{r^2\Lambda^2} + \frac{rE^2}{2 - r} \right] + \frac{L^2}{r^2\sin^2\theta}. \] (16)

Noting Eq. (14) and the Hamilton-Jacobi equation, we have

\[ -1 = \frac{r - 2}{r} \frac{\partial S}{\partial r} - \frac{1}{r^2} \frac{\partial S}{\partial \theta} + \frac{rE^2}{2 - r} \]
\[ + \frac{L^2}{r^2\sin^2\theta}. \] (17)

where $S$ is a generating function of the form

\[ S = \frac{1}{2}r^2 + L\phi + S_1(r) + S_2(\theta). \] (18)

It is clear that Eq. (17) has a separable form of the variables $r$ and $\theta$ and can be split into two equations

\[ K = r^2 - \left(1 - \frac{2}{r}\right)^{-1}E^2r^2 + \left(1 - \frac{2}{r}\right)r^2 \frac{\partial S_1}{\partial r}^2, \]
\[ K = -\frac{\partial S_2}{\partial \theta}^2 - \frac{L^2}{\sin^2\theta}. \] (19)

where $K$ is a constant. Considering that $\partial S_1/\partial r = p_r = r\dot{\phi}/(r - 2)$ and $\partial S_2/\partial \theta = p_\theta = r^2\dot{\theta}$, we modify Eq. (19) as

\[ K = r^2 + r^3(i^2 - E^2)/(r - 2), \] (20)
\[ K = -(r^2\dot{\phi})^2 - \frac{L^2}{\sin^2\theta}. \] (21)
Obviously, \((r, \theta, \dot{r}, \dot{\theta})\) is easily, analytically solved. In other words, the 4-dimensional Hamiltonian (16) (i.e. the Schwarzschild spacetime) is integrable due to the existence of the two constants of motion given by Eqs. (14) and (20) [or (21)]. All non-circular orbits of test particles in this spacetime are quasi-periodic and regular. Of course, all circular orbits should be strictly periodic.

\[ H = \frac{1}{2\lambda^2} \left( \frac{r - 2}{r} p_r^2 + \frac{p_{\theta}^2}{r^2} + \frac{rE^2}{2 - r} \right) + \frac{1}{2} \frac{\lambda^2 L^2}{r^2 \sin^2 \theta}. \]  

(22)

Similar to Eq. (17), the Hamilton-Jacobi equation is

\[ -\Lambda^2 = \frac{r - 2}{r} \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{rE^2}{2 - r} \]

\[ + \frac{\Lambda^2 L^2}{r^2 \sin^2 \theta}. \]  

(23)

This equation has no separable form of the variables \(r\) and \(\theta\) like Eq. (19). That means that the constant similar to Eq. (20) or (21) is no longer present. The system (22) holds only one constant (14) and therefore is nonintegrable. This can be understood easily from the physical point of view. Although the electromagnetic force is absent for the case of \(B \neq 0\) with \(q = 0\), the magnetic field acts as the gravitational effect and leads to the loss of the second constant in the system. Namely, the gravitational effect of the magnetic field included in the Schwarzschild spacetime destroys the integrability of this spacetime. In this sense, it is possible that chaos is hidden in the system (22). In the following demonstrations, we focus on the dynamics of order and chaos in this system using numerical techniques.

1. **Fourth-order explicit symmetric algorithms in extended phase space**

The conventional fourth-order Runge-Kutta method (RK4) is naturally suitable for explicitly solving the inseparable Hamiltonian (22). However, RK4 shows secular growth in the error of the Hamiltonian, as shown in Fig. 3(a). Instead, symplectic or symmetric methods concern the approximate preservation of constants of motion without drift in the errors. Usually implicit symplectic integrators are directly applied for such an inseparable Hamiltonian problem. A fourth-order implicit symplectic integrator with a symmetric composition of three second-order symplectic implicit midpoint methods (IS4) [27-29] exhibits good long term stability and error behavior in Fig. 3(b). In spite of this, the implicit method needs a lot of additional computational cost compared to the explicit algorithm RK4. This can be shown clearly in Table 3.

To save the computational cost, Pihajoki [31] doubled the phase space variables and constructed explicit leapfrog integration schemes of inseparable Hamiltonian systems. It is good to use two mixing maps on permutations of momenta to restrict the extended (new) variables to agree with the original (old) ones for given equal initial conditions. The permutations destroy the symplecticity of the algorithms in general. However, the methods are symmetric and therefore can still be similar to symplectic integrators that preserve the original Hamiltonian without secular growth in the error. In this sense, these methods are called extended phase space explicit symplectic-like (or symmetric) algorithms. Liu et al. [32] pointed out that it is better to replace the two permutations of momenta with the permutations of coordinates and the permutations of momenta (i.e. the sequent two permutations of coordinates and momenta). Unfortunately, the sequent two permutations may fail to work in some cases. Instead, the midpoint permutations between the old variables and their corresponding new variables are recommended in [33]. In particular, one of the advantages for the midpoint permutations lies in that the usual symplectic integration formulae can be directly applied to the extended phase space Hamiltonian but the methods of Pihajoki [31] and Liu et al. [32] can not. These extended phase space explicit symmetric integrators have been shown to have good performance in the conservation of the original Hamiltonian when they are used to solve some inseparable Hamiltonian problems [3,34,35,43].

Now let us consider the application of an extended phase space fourth-order explicit symmetric algorithm with the midpoint permutations to the inseparable Hamiltonian (22). Extending the 4-dimensional phase space variables \((r, \theta, p_r, p_\theta)\) to an 8-dimensional phase space variables \((\tilde{r}, \tilde{\theta}, \tilde{p}_r, \tilde{p}_\theta)\), we obtain a new Hamiltonian

\[ \tilde{H}(\tilde{r}, \tilde{\theta}, \tilde{p}_r, \tilde{p}_\theta) = H_1(\tilde{r}, \tilde{\theta}, \tilde{p}_r, \tilde{p}_\theta), \]  

(24)

where \(H_1 = H_2 = H\), and each of the four pairs \((r, \theta, p_r, p_\theta)\), \((\tilde{r}, \tilde{\theta}, \tilde{p}_r, \tilde{p}_\theta)\) is canonical each other. The old variables and their corresponding new variables have the same initial values: \(r_0 = \tilde{r}_0, \theta_0 = \tilde{\theta}_0, p_{r0} = \tilde{p}_r, p_{\theta0} = \tilde{p}_\theta\). It is clear that \(H_1\) and \(H_2\) are independently, analytically solvable. Thus \(\tilde{H}\) is a separable system with respect to the old variables and the new ones in the extended phase space although \(H\) is not to the old variables. Taking \(H_1\) as an operator for analytically solving the Hamiltonian \(H_1\) and \(H_2\) as another operator for analytically solving the Hamiltonian \(H_2\), we have the second-order explicit leapfrog algorithm

\[ S2(h) = H_2(h) H_1(h) H_2\left(\frac{h}{2}\right), \]  

(25)
where $h$ is a time step. A symmetric product of three leapfrogs yields the fourth-order method of Yoshida [44]

$$S4(h) = MS2(\lambda_1 h)S2(\lambda_2 h)S2(\lambda_1 h),$$

where $\lambda_1$ and $\lambda_2$ are time coefficients with the following expressions

$$\lambda_1 = \frac{1}{2 - 2^{1/3}}, \quad \lambda_2 = 1 - 2\lambda_1.$$

Additionally, $M$ is a mixing map on the midpoint permutations between the old and new variables: $(r + \tilde{r})/2 \rightarrow r$, $(\theta + \tilde{\theta})/2 \rightarrow \theta$, $(p_r + \tilde{p}_r)/2 \rightarrow p_r$, $(p_r + \tilde{p}_r)/2 \rightarrow \tilde{p}_r$, $(p_\theta + \tilde{p}_\theta)/2 \rightarrow p_\theta$, and $(p_\theta + \tilde{p}_\theta)/2 \rightarrow \tilde{p}_\theta$. As claimed above, this map is powerful to prevent from the divergence of the numerical solution $(r, \theta, \tilde{p}_r, \tilde{p}_\theta)$ for $H_1$ and the numerical solution $(\tilde{r}, \tilde{\theta}, p_r, p_\theta)$ for $H_2$ [or the original solution $(r, \theta, p_r, p_\theta)$ and the extended solution $(\tilde{r}, \tilde{\theta}, \tilde{p}_r, \tilde{p}_\theta)$] with time. As a result, the extended phase space explicit symmetric algorithm $S4$ in Fig. 3(c) gives excellent behavior in the error growth and conservation of the original Hamiltonian. In view of such good computational efficiency and accuracy, we employ the method $S4$ to investigate the dynamics of neutral particles.

2. Chaotic indicators

Since the system (22) has four dimensions and the integral (14), its structure in the original phase space can be described clearly with the aid of Poincaré section method. When the energy of the system (22) is not too high, e.g. $E = 0.99$ in Fig. 4(a), all phase orbits on the section at the plane $\theta = \pi/2$ are KAM tori and therefore are quasi-periodic, regular. When the magnetic field $B = 0.001$ is fixed but the energy slightly increases, e.g. $E = 0.9905$, one of the orbits is no longer a torus and has a small number of random distributed points in Fig. 4(b). This seems to show the chaoticity of this orbit. With a further increase of the energy, e.g. $E = 0.9915$, two orbits have a large number of random distributed points in Fig. 4(c). That seems to mean that the extent of chaos is typically strengthened.

In fact, the orbital dynamical feature of order and chaos depends on the rate of divergence of an orbit and its nearby orbit in the phase space. The rate can be measured precisely by the maximum Lyapunov exponent [36,37]

$$\lambda = \lim_{\tau \to \infty} \frac{1}{\tau} \ln \frac{d(\tau)}{d(0)}.$$  

Here, $d(\tau)$ is the proper distance between the two nearby orbits at the proper time $\tau$ and is defined by

$$d(\tau) = \sqrt{|g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta|}.$$  

The FLIs in Fig. 4(h), only one of the orbits in Fig. 4(b) behaves chaotic behavior. In particular, the FLIs of the two chaotic orbits in Fig. 4(i) grow faster than the FLI of the chaotic orbit in Fig. 4(h). This means that chaos in Fig. 4(c) is stronger than in Fig. 4(b).

The methods of Poincaré sections, Lyapunov exponents and FLIs provided the consistent result that chaos can occur in the magnetized Ernst-Schwarzschild spacetime although these particles without charges have no interaction of the electromagnetic force. This chaoticity is caused due to the magnetic field acting as the gravitational effect and destroying the integrability of the original spacetime. These methods also showed that chaos is strengthened typically with the energy increasing. In a similar way, we can confirm that the chaotic behavior occurs more easily when the magnetic field $B$ increases for a given energy (e.g. $E = 0.991$) in Fig. 5. This is because an increase of the magnetic field (or the energy) means
that of the gravitational effect from the magnetic field. The integrability of the original spacetime is typically destroyed so that there is no great difference between the gravitational effect of the magnetic field and the gravity of the black hole.

C. Case 3: $B \neq 0, q \neq 0$

For the case of $B \neq 0$ with $q \neq 0$, charged particles move in the electromagnetic field and spacetime geometry. Without doubt, the system (12) for the description of the motion of charged particles is nonintegrable. Thus, the onset of chaos is not unexpected.

The three types of parameter combinations in Fig. 5 are still considered, but the charge $q = 0.01$ is chosen in Fig. 6. Although the electromagnetic force is included, the phase space structures for each parameter combination in Fig. 6 are not explicitly different from those for the same parameter combination in Fig. 5. An increase of the magnetic field still leads to that of chaos. When the charge $q = -0.01$ is used instead of the charge $q = 0.01$, no typical changes of the phase space structures occur in Fig. 7.

Fixing the parameters $E = 0.9913$, $L = 3.8$ and $B = 0.001$, we find in Fig. 8 that a larger charge results in stronger chaos. If the fixed magnetic field $B = -0.001$ is used in Fig. 9, increasing the charge gives rise to decreasing the extent of chaos. These facts show that the electromagnetic forces in Figs. 8 and 9 exert different influences on chaos. Even if the electromagnetic forces are much small in Figs. 8(a) and 9(a), weak chaos is present. As mentioned above, this chaotic behavior is mainly due to the gravitational effect of the magnetic field. With the electromagnetic forces increasing, the onset of chaos is more likely to occur. In some cases, chaos is strengthened typically with the energy or the magnetic field increasing. When the charged particles move in the electromagnetic field and spacetime geometry, the electromagnetic forces play an important role in strengthening or suppressing the extent of chaos caused by the gravitational effect of the magnetic field.

IV. SUMMARY

In this work, we provide some insight into the dynamics of both electrically charged and neutral particles around the Schwarzschild black hole in the magnetic universe. The effective potentials and stable circular orbits at the equatorial plane are discussed. For weak magnetic or electromagnetic fields, the radii of the innermost stable circular orbits approach to that of the Schwarzschild spacetime. If these particles have no charges and the electromagnetic forces are absent, then the magnetic field acts as the gravitational effect and destroys the integrability of the original spacetime. Because of this, the occurrence of chaos is possible. In some cases, chaos is strengthened typically with the energy or the magnetic field increasing. When the charged particles move in the electromagnetic field and spacetime geometry, the electromagnetic forces play an important role in strengthening or suppressing the extent of chaos caused by the gravitational effect of the magnetic field.

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TABLE I: Radii $r$ of stable circular orbits at the equatorial plane in Fig. 1.

| $q$ | $B$ | $L$ | $r$  | $q$ | $B$ | $L$ | $r$  |
|-----|-----|-----|------|-----|-----|-----|------|
| 0   | 0.0 | 3.4641 | 10.7699999999 | 0 | 0 | 3.8643 | 10.7699998151 |
| 4.1 | 12.8999999999 | 4.1 | 12.87999976799 |

$\pm 10^{-3} \pm 10^{-3}$ 3.8543 10.66999981739 $\mp 10^{-3}$ 3.8356 10.49999982119
| 4.1 | 12.87999976799 |

$\pm 10^{-3} \pm 10^{-3}$ 3.8345 10.48999982141
| 4.1 | 12.86999976821 |

$\pm 10^{-3} \pm 10^{-3}$ 3.4637 5.999132540466 $\mp 10^{-3}$ 3.4643 5.999136706604
| 4.1 | 12.87999976799 |

$\pm 10^{-3} \pm 10^{-3}$ 3.4641 6 3.4643 5.999136705541
| 4.1 | 12.889990076777 |

TABLE II: Radii $r$ of the innermost stable circular orbits at the equatorial plane in Fig. 2.

| $q$ | $B$ | $L$ | $r$  | $q$ | $B$ | $L$ | $r$  |
|-----|-----|-----|------|-----|-----|-----|------|
| 0   | 0.0 | 5.999136705686 10^{-4} | 10^{-3} 5.999136705382 |
| $\pm 10^{-3} \pm 10^{-3}$ 5.999136677351 10^{-2} 5.99913647471 |
| $\pm 0.1 \pm 10^{-3}$ 5.999132244625 0.1 5.999132540466 |
| $\pm 1 \pm 10^{-3}$ 5.998705579809 1 5.998705579809 |
| $\pm 10^{-3} \pm 10^{-3}$ 5.919971855034 10^{-4} 5.919969263988 |
| $\pm 10^{-3} \pm 10^{-3}$ 5.919979831736 10^{-2} 5.919933924028 |
| $\pm 0.1 \pm 10^{-3}$ 5.919727012147 0.1 5.91970675229 |
| $\pm 1 \pm 10^{-3}$ 5.884025383442 5.884025383442 |

TABLE III: CPU times (seconds) for the three algorithms in Fig. 3.

| Algorithm | RK4 | IS4 | S4 |
|-----------|-----|-----|-----|
| CPU Time  | 10  | 50  | 27  |

FIG. 1: Effective potentials for several sets of parameter combinations.
FIG. 2: (color online) Radii $r$ of the innermost stable circular orbits varying with the parameters $B$ and $q$.

FIG. 3: Relative errors of the original Hamiltonian (22) for three integrators, including the conventional fourth-order explicit Runge-Kutta method RK4, the fourth-order implicit symplectic integrator IS4 and the extended phase space fourth-order explicit symmetric method S4. The time step is $h = 0.1$, and the parameters are $q = 0$, $B = 10^{-4}$, $E = 0.956$ and $L = 3.6$. The initial conditions are $r = 10$, $p_r = 0$, $\theta = \pi/2$, and the initial value of $p_\theta > 0$ is determined by Eq. (14).
FIG. 4: (color online) (a)-(c): Poincaré sections at the plane $\theta = \pi/2$ with $p_\theta > 0$ for three different values of the energy $E$ in the system (22). The other parameters are $B = 0.001$, $L = 3.6$ and $q = 0$, and the initial values $p_r = 0$ and $\theta = \pi/2$ are fixed. All orbits in panel (a) are regular KAM tori. There is a chaotic orbit with the initial value $r = 10$ in panel (b). Two orbits with the initial values $r = 10$ and $r = 55$ are chaotic in panel (c). (d)-(f): Lyapunov exponents $\lambda$ corresponding to the orbits in panels (a)-(c). (g)-(i): Fast Lyapunov indictors (FLIs) corresponding to the orbits in panels (a)-(c).

FIG. 5: (color online) Poincaré sections for three different values of the magnetic field $B$. The other parameters are $E = 0.991$, $L = 3.6$ and $q = 0$. 

The system (22). The other parameters are $B = 0.001$, $L = 3.6$ and $q = 0$, and the initial values $p_r = 0$ and $\theta = \pi/2$ are fixed. All orbits in panel (a) are regular KAM tori. There is a chaotic orbit with the initial value $r = 10$ in panel (b). Two orbits with the initial values $r = 10$ and $r = 55$ are chaotic in panel (c). (d)-(f): Lyapunov exponents $\lambda$ corresponding to the orbits in panels (a)-(c). (g)-(i): Fast Lyapunov indicators (FLIs) corresponding to the orbits in panels (a)-(c).
FIG. 6: (colour online) Same as Fig. 5 but the charge $q = 0.01$.

FIG. 7: (color online) Same as Fig. 6 but the charge $q = -0.01$.

FIG. 8: (color online) Poincaré sections for three different values of the charge $q$. The other parameters are $E = 0.9913$, $L = 3.8$ and $B = 0.001$.

FIG. 9: (color online) Same as Fig. 8 but the negative magnetic filed $B = -0.001$ is given.