Index theorems for holomorphic self-maps

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Introduction

The usual index theorems for holomorphic self-maps, like for instance the classical holomorphic Lefschetz theorem (see, e.g., [GH]), assume that the fixed-points set contains only isolated points. The aim of this paper, on the contrary, is to prove index theorems for holomorphic self-maps having a positive dimensional fixed-points set.

The origin of our interest in this problem lies in holomorphic dynamics. A main tool for the complete generalization to two complex variables of the classical Leau-Fatou flower theorem for maps tangent to the identity achieved in [A2] was an index theorem for holomorphic self-maps of a complex surface fixing pointwise a smooth complex curve $S$. This theorem (later generalized in [BT] to the case of a singular $S$) presented uncanny similarities with the Camacho-Sad index theorem for invariant leaves of a holomorphic foliation on a complex surface (see [CS]). So we started to investigate the reasons for these similarities; and this paper contains what we have found.

The main idea is that the simple fact of being pointwise fixed by a holomorphic self-map $f$ induces a lot of structure on a (possibly singular) subvariety $S$ of a complex manifold $M$. First of all, we shall introduce (in §3) a canonically defined holomorphic section $X_f$ of the bundle $T_M|_S \otimes (N_S^\ast)^{\otimes \nu_f}$, where $N_S$ is the normal bundle of $S$ in $M$ (here we are assuming $S$ smooth; however, we can also define $X_f$ as a section of a suitable sheaf even when $S$ is singular — see Remark 3.3 — but it turns out that only the behavior on the regular part of $S$ is relevant for our index theorems), and $\nu_f$ is a positive integer, the order of contact of $f$ with $S$, measuring how close $f$ is to being the identity in a neighborhood $S$ (see §1). Roughly speaking, the section $X_f$ describes the directions in which $S$ is pushed by $f$; see Proposition 8.1 for a more precise description of this phenomenon when $S$ is a hypersurface.

The canonical section $X_f$ can also be seen as a morphism from $N_S^{\otimes \nu_f}$ to $T_M|_S$; its image $\Xi_f$ is the canonical distribution. When $\Xi_f$ is contained in $T_S$ (we shall say that $f$ is tangential) and integrable (this happens for instance if $S$ is a hypersurface), then on $S$ we get a singular holomorphic
foliation induced by $f$ — and this is a first concrete connection between our discrete dynamical theory and the continuous dynamics studied in foliation theory. We stress, however, that we get a well-defined foliation on $S$ only, while in the continuous setting one usually assumes that $S$ is invariant under a foliation defined in a whole neighborhood of $S$. Thus even in the tangential codimension-one case our results will not be a direct consequence of foliation theory.

As we shall momentarily discuss, to get index theorems it is important to have a section of $TS \otimes (N^*_S)^{\otimes \nu_f}$ (as in the case when $f$ is tangential) instead of merely a section of $TM|_S \otimes (N^*_S)^{\otimes \nu_f}$. In Section 3, when $f$ is not tangential (which is a situation akin to dicriticality for foliations; see Propositions 1.4 and 8.1) we shall define other holomorphic sections $H_{\sigma,f}$ and $H^1_{\sigma,f}$ of $TS \otimes (N^*_S)^{\otimes \nu_f}$ which are as good as $X_f$ when $S$ satisfies a geometric condition which we call comfortably embedded in $M$, meaning, roughly speaking, that $S$ is a first-order approximation of the zero section of a vector bundle (see §2 for the precise definition, amounting to the vanishing of two sheaf cohomology classes — or, in other terms, to the triviality of two canonical extensions of $N_S$).

The canonical section is not the only object we are able to associate to $S$. Having a section $X$ of $TS \otimes F^*$, where $F$ is any vector bundle on $S$, is equivalent to having an $F^*$-valued derivation $X^#$ of the sheaf of holomorphic functions $\mathcal{O}_S$ (see §5). If $E$ is another vector bundle on $S$, a holomorphic action of $F$ on $E$ along $X$ is a $\mathbb{C}$-linear map $\hat{X}: \mathcal{E} \to \mathcal{F} \otimes \mathcal{E}$ (where $\mathcal{E}$ and $\mathcal{F}$ are the sheaves of germs of holomorphic sections of $E$ and $F$) satisfying $\hat{X}(gs) = X^#(g) \otimes s + g\hat{X}(s)$ for any $g \in \mathcal{O}_S$ and $s \in E$; this is a generalization of the notion of $(1,0)$-connection on $E$ (see Example 5.1).

In Section 5 we shall show that when $S$ is a hypersurface and $f$ is tangential (or $S$ is comfortably embedded in $M$) there is a natural way to define a holomorphic action of $N^*_S^{\otimes \nu_f}$ on $N_S$ along $X_f$ (or along $H_{\sigma,f}$ or $H^1_{\sigma,f}$). And this will allow us to bring into play the general theory developed by Lehmann and Suwa (see, e.g., [Su]) on a cohomological approach to index theorems. So, exactly as Lehmann and Suwa generalized, to any dimension, the Camacho-Sad index theorem, we are able to generalize the index theorems of [A2] and [BT] in the following form (see Theorem 6.2):

**Theorem 0.1.** Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f: M \to M$, $f \neq \text{id}_M$, be a holomorphic self-map of $M$ fixing pointwise $S$, and denote by $\text{Sing}(f)$ the zero set of $X_f$. Assume that

(a) $f$ is tangential to $S$, and then set $X = X_f$, or that

(b) $S^0 = S \setminus (\text{Sing}(S) \cup \text{Sing}(f))$ is comfortably embedded into $M$, and then set $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$. 
Assume moreover $X \not\equiv O$ (a condition always satisfied when $f$ is tangential), and denote by $\text{Sing}(X)$ the zero set of $X$. Let $\text{Sing}(S) \cup \text{Sing}(X) = \bigcup \Sigma_{\lambda}$ be the decomposition of $\text{Sing}(S) \cup \text{Sing}(X)$ in connected components. Finally, let $[S]$ be the line bundle on $M$ associated to the divisor $S$. Then there exist complex numbers $\text{Res}(X, S, \Sigma_{\lambda}) \in \mathbb{C}$ depending only on the local behavior of $X$ and $[S]$ near $\Sigma_{\lambda}$ such that

$$\sum_{\lambda} \text{Res}(X, S, \Sigma_{\lambda}) = \int_{S} c_{1}^{n-1}([S]),$$

where $c_{1}([S])$ is the first Chern class of $[S]$.

Furthermore, when $\Sigma_{\lambda}$ is an isolated point $\{x_{\lambda}\}$, we have explicit formulas for the computation of the residues $\text{Res}(X, S, \{x_{\lambda}\})$; see Theorem 6.5.

Since $X$ is a global section of $TS \otimes (N_{S}^{*})^{\otimes \nu_{f}}$, if $S$ is smooth and $X$ has only isolated zeroes it is well-known that the top Chern class $c_{n-1}(TS \otimes (N_{S}^{*})^{\otimes \nu_{f}})$ counts the zeroes of $X$. Our result shows that $c_{1}^{n-1}(N_{S})$ is related in a similar (but deeper) way to the zero set of $X$. See also Section 8 for examples of results one can obtain using both Chern classes together.

If the codimension of $S$ is greater than one, and $S$ is smooth, we can blow-up $M$ along $S$; then the exceptional divisor $E_{S}$ is a hypersurface, and we can apply to it the previous theorem. In this way we get (see Theorem 7.2):

**Theorem 0.2.** Let $S$ be a compact complex submanifold of codimension $1 < m < n$ in an $n$-dimensional complex manifold $M$. Let $f: M \to M$, $f \not\equiv \text{id}_{M}$, be a holomorphic self-map of $M$ fixing pointwise $S$, and assume that $f$ is tangential, or that $\nu_{f} > 1$ (or both). Let $\bigcup \Sigma_{\lambda}$ be the decomposition in connected components of the set of singular directions (see §7 for the definition) for $f$ in $E_{S}$. Then there exist complex numbers $\text{Res}(f, S, \Sigma_{\lambda}) \in \mathbb{C}$, depending only on the local behavior of $f$ and $S$ near $\Sigma_{\lambda}$, such that

$$\sum_{\lambda} \text{Res}(f, S, \Sigma_{\lambda}) = \int_{S} \pi_{*} c_{1}^{n-1}([E_{S}]),$$

where $\pi_{*}$ denotes integration along the fibers of the bundle $E_{S} \to S$.

Theorems 0.1 and 0.2 are only two of the index theorems we can derive using this approach. Indeed, we are also able to obtain versions for holomorphic self-maps of two other main index theorems of foliation theory, the Baum-Bott index theorem and the Lehmann-Suwa-Khanedani (or variation) index theorem: see Theorems 6.3, 6.4, 6.6, 7.3 and 7.4. In other words, it turns out that the existence of holomorphic actions of suitable complex vector bundles defined only on $S$ is an efficient tool to get index theorems, both in our setting and (under slightly different assumptions) in foliation theory; and this is another reason for the similarities noticed in [A2].
Finally, in Section 8 we shall present a couple of applications of our results to the discrete dynamics of holomorphic self-maps of complex surfaces, thus closing the circle and coming back to the arguments that originally inspired our work.

1. The order of contact

Let $M$ be an $n$-dimensional complex manifold, and $S \subseteq M$ an irreducible subvariety of codimension $m$. We shall denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on $M$, and by $\mathcal{I}_S$ the subsheaf of functions vanishing on $S$. With a slight abuse of notations, we shall use the same symbol to denote both a germ at $p$ and any representative defined in a neighborhood of $p$. We shall denote by $T_M$ the holomorphic tangent bundle of $M$, and by $\mathcal{T}_M$ the sheaf of germs of local holomorphic sections of $T_M$. Finally, we shall denote by $\text{End}(M, S)$ the set of (germs about $S$ of) holomorphic self-maps of $M$ fixing $S$ pointwise.

Let $f \in \text{End}(M, S)$ be given, $f \not\equiv \text{id}_M$, and take $p \in S$. For every $h \in \mathcal{O}_{M,p}$ the germ $h \circ f$ is well-defined, and we have $h \circ f - h \in \mathcal{I}_{S,p}$.

Definition 1.1. The $f$-order of vanishing at $p$ of $h \in \mathcal{O}_{M,p}$ is given by

$$\nu_f(h; p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}_{S,p}^\mu\},$$

and the order of contact $\nu_f(p)$ of $f$ at $p$ with $S$ by

$$\nu_f(p) = \min\{\nu_f(h; p) \mid h \in \mathcal{O}_{M,p}\}.$$

We shall momentarily prove that $\nu_f(p)$ does not depend on $p$.

Let $(z^1, \ldots, z^n)$ be local coordinates in a neighborhood of $p$. If $h$ is any holomorphic function defined in a neighborhood of $p$, the definition of order of contact yields the important relation

$$(1.1) \quad h \circ f - h = \sum_{j=1}^n (f^j - z^j) \frac{\partial h}{\partial z^j} \pmod{\mathcal{I}_{S,p}^{2\nu_f(p)}},$$

where $f^j = z^j \circ f$.

As a consequence we have

Lemma 1.1. (i) Let $(z^1, \ldots, z^n)$ be any set of local coordinates at $p \in S$. Then

$$\nu_f(p) = \min_{j=1, \ldots, n} \{\nu_f(z^j; p)\}.$$  

(ii) For any $h \in \mathcal{O}_{M,p}$ the function $p \mapsto \nu_f(h; p)$ is constant in a neighborhood of $p$.

(iii) The function $p \mapsto \nu_f(p)$ is constant.
Proof. (i) Clearly, \( \nu_f(p) \leq \min_{j=1, \ldots, n} \{ \nu_f(z^j; p) \} \). The opposite inequality follows from (1.1).

(ii) Let \( h \in \mathcal{O}_{M,p} \), and choose a set \( \{ \ell^1, \ldots, \ell^k \} \) of generators of \( I_{S,p} \). Then we can write

\[
(1.2) \quad h \circ f - h = \sum_{|I| = \nu_f(h; p)} \ell^I g_I,
\]

where \( I = (i_1, \ldots, i_k) \in \mathbb{N}^k \) is a \( k \)-multi-index, \( |I| = i_1 + \cdots + i_k \), \( \ell^I = (\ell^1)^{i_1} \cdots (\ell^k)^{i_k} \) and \( g_I \in \mathcal{O}_{M,p} \). Furthermore, there is a multi-index \( I_0 \) such that \( g_{I_0} \notin I_{S,p} \). By the coherence of the sheaf of ideals of \( S \), the relation (1.2) holds for the corresponding germs at all points \( q \in S \) in a neighborhood of \( p \). Furthermore, \( g_{I_0} \notin I_{S,q} \) for all \( q \in S \) close enough to \( p \). Putting these two observations together we get the assertion.

(iii) By (i) and (ii) we see that the function \( p \mapsto \nu_f(p) \) is locally constant and since \( S \) is connected, it is constant everywhere. \( \square \)

We shall then denote by \( \nu_f \) the order of contact of \( f \) with \( S \), computed at any point \( p \in S \).

As we shall see, it is important to compare the order of contact of \( f \) with the \( f \)-order of vanishing of germs in \( I_{S,p} \).

**Definition 1.2.** We say that \( f \) is tangential at \( p \) if

\[
\min \{ \nu_f(h; p) \mid h \in I_{S,p} \} > \nu_f.
\]

**Lemma 1.2.** Let \( \{ \ell^1, \ldots, \ell^k \} \) be a set of generators of \( I_{S,p} \). Then

\[
\nu_f(h; p) \geq \min \{ \nu_f(\ell^1; p), \ldots, \nu_f(\ell^k; p), \nu_f + 1 \}
\]

for all \( h \in I_{S,p} \). In particular, \( f \) is tangential at \( p \) if and only if

\[
\min \{ \nu_f(\ell^1; p), \ldots, \nu_f(\ell^k; p) \} > \nu_f.
\]

**Proof.** Let us write \( h = g_1 \ell^1 + \cdots + g_k \ell^k \) for suitable \( g_1, \ldots, g_k \in \mathcal{O}_{M,p} \). Then

\[
(1.2) \quad h \circ f - h = \sum_{j=1}^k [(g_j \circ f)(\ell^j \circ f - \ell^j) + (g_j \circ f - g_j)\ell^j],
\]

and the assertion follows. \( \square \)

**Corollary 1.3.** If \( f \) is tangential at one point \( p \in S \), then it is tangential at all points of \( S \).

**Proof.** The coherence of the sheaf of ideals of \( S \) implies that if \( \{ \ell^1, \ldots, \ell^k \} \) are generators of \( I_{S,p} \), then the corresponding germs are generators of \( I_{S,q} \) for
all \( q \in S \) close enough to \( p \). Then Lemmas 1.1.(ii) and 1.2 imply that both
the set of points where \( f \) is tangential and the set of points where \( f \) is not
tangential are open; hence the assertion follows because \( S \) is connected.

Of course, we shall then say that \( f \) is tangential along \( S \) if it is tangential
at any point of \( S \).

**Example 1.1.** Let \( p \) be a smooth point of \( S \), and choose local coordinates
\( z = (z^1, \ldots, z^n) \) defined in a neighborhood \( U \) of \( p \), centered at \( p \) and such that
\( S \cap U = \{z^1 = \cdots = z^m = 0\} \). We shall write \( z' = (z^1, \ldots, z^m) \) and \( z'' =
(z^{m+1}, \ldots, z^n) \), so that \( z'' \) yields local coordinates on \( S \). Take \( f \in \text{End}(M, S), \)
\( f \not\equiv \text{id}_M \); then in local coordinates the map \( f \) can be written as \((f^1, \ldots, f^n)\)
with

\[
f^j(z) = z^j + \sum_{h \geq 1} P^j_h(z', z''),
\]

where each \( P^j_h \) is a homogeneous polynomial of degree \( h \) in the variables \( z' \),
with coefficients depending holomorphically on \( z'' \). Then Lemma 1.1 yields

\[
\nu_f = \min\{h \geq 1 \mid \exists 1 \leq j \leq n : P^j_h \neq 0\}.
\]

Furthermore, \( \{z^1, \ldots, z^m\} \) is a set of generators of \( \mathcal{I}_{S,p} \); therefore by Lemma 1.2
the map \( f \) is tangential if and only if

\[
\min\{h \geq 1 \mid \exists 1 \leq j \leq m : P^j_h \neq 0\} > \min\{h \geq 1 \mid \exists m + 1 \leq j \leq n : P^j_h \neq 0\}.
\]

**Remark 1.1.** When \( S \) is smooth, the differential of \( f \) acts linearly on the
normal bundle \( N_S \) of \( S \) in \( M \). If \( S \) is a hypersurface, \( N_S \) is a line bundle, and
the action is multiplication by a holomorphic function \( b \); if \( S \) is compact, this
function is a constant. It is easy to check that in local coordinates chosen as in
the previous example the expression of the function \( b \) is exactly \( 1 + P^1_1(z)/z^1 \);
therefore we must have \( P^1_1(z) = (b_f - 1)z^1 \) for a suitable constant \( b_f \in \mathbb{C} \). In
particular, if \( b_f \neq 1 \) then necessarily \( \nu_f = 1 \) and \( f \) is not tangential along \( S \).

**Remark 1.2.** The number \( \mu \) introduced in \([\text{BT}, (2)]\) is, by Lemma 1.1, our
order of contact; therefore our notion of tangential is equivalent to the notion
of nondegeneracy defined in \([\text{BT}]\) when \( n = 2 \) and \( m = 1 \). On the other hand,
as already remarked in \([\text{BT}]\), a nondegenerate map in the sense defined in \([\text{A2}]\)
when \( n = 2 \), \( m = 1 \) and \( S \) is smooth is tangential if and only if \( b_f = 1 \) (which
was the case mainly considered in that paper).

**Example 1.2.** A particularly interesting example (actually, the one inspir-
ing this paper) of map \( f \in \text{End}(M, S) \) is obtained by blowing up a map tangent
to the identity. Let \( f_0 \) be a (germ of) holomorphic self-map of \( \mathbb{C}^n \) (or of any
complex \( n \)-manifold) fixing the origin (or any other point) and **tangent to the**
identity, that is, such that \(d(f_o)_O = \text{id}\). If \(\pi: M \to \mathbb{C}^n\) denotes the blow-up of the origin, let \(S = \pi^{-1}(O) \cong \mathbb{P}^{n-1}(\mathbb{C})\) be the exceptional divisor, and \(f \in \text{End}(M, S)\) the lifting of \(f_o\), that is, the unique holomorphic self-map of \(M\) such that \(f_o \circ \pi = \pi \circ f\) (see, e.g., \([A1]\) for details). If

\[f^j_o(w) = w^j + \sum_{h \geq 2} Q^j_h(w)\]

is the expansion of \(f^j_o\) in a series of homogeneous polynomials (for \(j = 1, \ldots, n\)), then in the canonical coordinates centered in \(p = [1 : 0 : \cdots : 0]\) the map \(f\) is given by

\[f^j(z) = \begin{cases} 
 z^1 + \sum_{h \geq 2} Q^1_h(1, z'', (z^1)^h) & \text{for } j = 1, \\
 z^j + \sum_{h \geq 2} \left[ Q^j_h(1, z'') - z^j Q^1_h(1, z'') \right] (z^1)^{h-1} \left/ \left(1 + \sum_{h \geq 2} Q^1_h(1, z'')(z^1)^{h-1}\right) \right. & \text{for } j = 2, \ldots, n,
\end{cases}\]

where \(z'' = (z^2, \ldots, z^n)\). Therefore \(b_f = 1\),

\[\nu_f(z^1; p) = \min\{h \geq 2 \mid Q^1_h(1, z'') \neq 0\},\]

and

\[\nu_f = \min\{\nu_f(z^1; p), \min\{h \geq 1 \mid \exists 2 \leq j \leq n : Q^j_{h+1}(1, z'') - z^j Q^1_{h+1}(1, z'') \neq 0\}\}.\]

Let \(\nu(f_o) \geq 2\) be the order of \(f_o\), that is, the minimum \(h\) such that \(Q^j_h \neq 0\) for some \(1 \leq j \leq n\). Clearly, \(\nu_f(z^1; p) \geq \nu(f_o)\) and \(\nu_f \geq \nu(f_o) - 1\). More precisely, if there is \(2 \leq j \leq n\) such that \(Q^j_{\nu(f_o)}(1, z'') \neq z^j Q^1_{\nu(f_o)}(1, z'')\), then \(\nu_f = \nu(f_o) - 1\) and \(f\) is tangential. If on the other hand we have \(Q^j_{\nu(f_o)}(1, z'') \equiv z^j Q^1_{\nu(f_o)}(1, z'')\) for all \(2 \leq j \leq n\), then necessarily \(Q^1_{\nu(f_o)}(1, z'') \neq 0\), \(\nu_f(z^1; p) = \nu(f_o) = \nu_f\), and \(f\) is not tangential.

Borrowing a term from continuous dynamics, we say that a map \(f_o\) tangent to the identity at the origin is dicritical if \(w^h Q^k_{\nu(f_o)}(w) \equiv w^k Q^h_{\nu(f_o)}(w)\) for all \(1 \leq h, k \leq n\). Then we have proved that:

**Proposition 1.4.** Let \(f_o \in \text{End}(\mathbb{C}^n, O)\) be a (germ of) holomorphic self-map of \(\mathbb{C}^n\) tangent to the identity at the origin, and let \(f \in \text{End}(M, S)\) be its blow-up. Then \(f\) is not tangential if and only if \(f_o\) is dicritical. Furthermore, \(\nu_f = \nu(f_o) - 1\) if \(f_o\) is not dicritical, and \(\nu_f = \nu(f_o)\) if \(f_o\) is dicritical.

In particular, most maps obtained with this procedure are tangential.
2. Comfortably embedded submanifolds

Up to now $S$ was any complex subvariety of the manifold $M$. However, some of the proofs in the following sections do not work in this generality; so this section is devoted to describe the kind of properties we shall (sometimes) need on $S$.

Let $E'$ and $E''$ be two vector bundles on the same manifold $S$. We recall (see, e.g., [Ati, §1]) that an extension of $E''$ by $E'$ is an exact sequence of vector bundles

$$O \to E' \xrightarrow{i} E \xrightarrow{\pi} E'' \to O.$$ 

Two extensions are equivalent if there is an isomorphism of exact sequences which is the identity on $E'$ and $E''$.

A splitting of an extension $O \to E' \xrightarrow{i} E \xrightarrow{\pi} E'' \to O$ is a morphism $\sigma : E'' \to E$ such that $\pi \circ \sigma = \text{id}_{E''}$. In particular, $E = \iota(E') \oplus \sigma(E'')$, and we shall say that the extension splits. We explicitly remark that an extension splits if and only if it is equivalent to the trivial extension $O \to E' \to E' \oplus E'' \to E'' \to O$.

Let $S$ now be a complex submanifold of a complex manifold $M$. We shall denote by $TM|_S$ the restriction to $S$ of the tangent bundle of $M$, and by $N_S = TM|_S/TS$ the normal bundle of $S$ into $M$. Furthermore, $T_{M,S}$ will be the sheaf of germs of holomorphic sections of $TM|_S$ (which is different from the restriction $T_M|_S$ to $S$ of the sheaf of holomorphic sections of $TM$), and $N_S$ the sheaf of germs of holomorphic sections of $N_S$.

**Definition 2.1.** Let $S$ be a complex submanifold of codimension $m$ in an $n$-dimensional complex manifold $M$. A chart $(U_\alpha, z_\alpha)$ of $M$ is adapted to $S$ if either $S \cap U_\alpha = \emptyset$ or $S \cap U_\alpha = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$, where $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$. In particular, $\{z_\alpha^1, \ldots, z_\alpha^n\}$ is a set of generators of $I_{S,p}$ for all $p \in S \cap U_\alpha$. An atlas $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$ of $M$ is adapted to $S$ if all charts in $\mathcal{U}$ are. If $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$ is adapted to $S$ we shall denote by $\mathcal{U}_S = \{(U_\alpha', z_\alpha')\}$ the atlas of $S$ given by $U_\alpha' = U_\alpha \cap S$ and $z_\alpha' = (z_\alpha^{m+1}, \ldots, z_\alpha^n)$, where we are clearly considering only the indices such that $U_\alpha \cap S \neq \emptyset$. If $(U_\alpha, z_\alpha)$ is a chart adapted to $S$, we shall denote by $\partial_{\alpha,r}$ the projection of $\partial/\partial z_\alpha^r|_{S \cap U_\alpha}$ in $N_S$, and by $\omega_\alpha^r$ the local section of $N_S^*$ induced by $dz_\alpha^r|_{S \cap U_\alpha}$; thus $\{\partial_{\alpha,1}, \ldots, \partial_{\alpha,m}\}$ and $\{\omega_\alpha^1, \ldots, \omega_\alpha^m\}$ are local frames for $N_S$ and $N_S^*$ respectively over $U_\alpha \cap S$, dual to each other.

From now on, every chart and atlas we consider on $M$ will be adapted to $S$.

**Remark 2.1.** We shall use the Einstein convention on the sum over repeated indices. Furthermore, indices like $j$, $h$, $k$ will run from 1 to $n$; indices like $r$, $s$, $t$, $u$, $v$ will run from 1 to $m$; and indices like $p$, $q$ will run from $m + 1$ to $n$. 

Definition 2.2. We shall say that $S$ splits into $M$ if the extension $O \to TS \to TM|_S \to NS \to O$ splits.

Example 2.1. It is well-known that if $S$ is a rational smooth curve with negative self-intersection in a surface $M$, then $S$ splits into $M$.

Proposition 2.1. Let $S$ be a complex submanifold of codimension $m$ in an $n$-dimensional complex manifold $M$. Then $S$ splits into $M$ if and only if there is an atlas $\mathcal{U} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ adapted to $S$ such that

$$\frac{\partial z^p_{\beta}}{\partial z^q_{\alpha}}|_S \equiv 0,$$

for all $r = 1, \ldots, m$, $p = m + 1, \ldots, n$ and indices $\alpha$ and $\beta$.

Proof. It is well known (see, e.g., [Ati, Prop. 2]) that there is a one-to-one correspondence between equivalence classes of extensions of $NS$ by $TS$ and the cohomology group $H^1(S, \text{Hom}(NS, TS))$, and an extension splits if and only if it corresponds to the zero cohomology class.

The class corresponding to the extension $O \to TS \to TM|_S \to NS \to O$ is the class $\delta(id_{NS})$, where $\delta: H^0(S, \text{Hom}(NS, NS)) \to H^1(S, \text{Hom}(NS, TS))$ is the connecting homomorphism in the long exact sequence of cohomology associated to the short exact sequence obtained by applying the functor $\text{Hom}(NS, \cdot)$ to the extension sequence. More precisely, if $\mathcal{U}$ is an atlas adapted to $S$, we get a local splitting morphism $\sigma_\alpha: N_{U_\alpha} \to TM|_{U_\alpha}$ by setting $\sigma_\alpha(\partial r, \alpha) = \partial/\partial z^r_{\alpha}$, and then the element of $H^1(\mathcal{U}_S, \text{Hom}(NS, TS))$ associated to the extension is $\{\sigma_\beta - \sigma_\alpha\}$. Now,

$$(\sigma_\beta - \sigma_\alpha)(\partial r, \alpha) = \frac{\partial z^s_{\beta}}{\partial z^q_{\alpha}}|_S \frac{\partial}{\partial z^r_{\beta}} - \frac{\partial}{\partial z^r_{\alpha}} = \frac{\partial z^s_{\beta}}{\partial z^q_{\alpha}} \frac{\partial z^p_{\beta}}{\partial z^q_{\beta}}|_S \frac{\partial}{\partial z^p_{\alpha}}.$$

So, if (2.1) holds, then $S$ splits into $M$. Conversely, assume that $S$ splits into $M$; then we can find an atlas $\mathcal{U}$ adapted to $S$ and a 0-cochain $\{c_\alpha\} \in H^0(\mathcal{U}_S, TS \otimes NS^*)$ such that

$$\frac{\partial z^s_{\beta}}{\partial z^r_{\alpha}} \frac{\partial z^p_{\alpha}}{\partial z^q_{\beta}}|_S = (c_\beta)^q_s \frac{\partial z^s_{\beta}}{\partial z^r_{\alpha}} \frac{\partial z^p_{\alpha}}{\partial z^q_{\beta}}|_S - (c_\alpha)^p_s$$

on $U_\alpha \cap U_\beta \cap S$. We claim that the coordinates

$$(2.3) \begin{cases} z^r_{\alpha} = z^r_{\alpha}, \\
z^p_{\alpha} = z^p_{\alpha} + (c_\alpha)^p_s (z^s_{\alpha})z^s_{\alpha} \end{cases}$$
satisfy (2.1) when restricted to suitable open sets \( \hat{U}_\alpha \subseteq U_\alpha \). Indeed, (2.2) yields

\[
\frac{\partial z^p_{\beta}}{\partial z^r_{\alpha}} \frac{\partial z^s_{\beta}}{\partial z^r_{\alpha}} + \frac{\partial z^p_{\beta}}{\partial z^s_{\alpha}} \frac{\partial z^q_{\beta}}{\partial z^s_{\alpha}} = \frac{\partial z^p_{\beta}}{\partial z^q_{\alpha}} (c^q_{\alpha})^r_{\beta} + R_1
\]

where \( R_1 \) denotes terms vanishing on \( S \), and we are done.

**Definition 2.3.** Assume that \( S \) splits into \( M \). An atlas \( U = \{(U_\alpha, z_\alpha)\} \) adapted to \( S \) and satisfying (2.1) will be called a splitting atlas for \( S \). It is easy to see that for any splitting morphism \( \sigma: N_S \rightarrow TM|_S \) there exists a splitting atlas \( U \) such that \( \sigma(\partial_{r, \alpha}) = \partial/\partial z^r_{\alpha} \) for all \( r = 1, \ldots, m \) and indices \( \alpha \); we shall say that \( U \) is adapted to \( \sigma \).

**Example 2.2.** A local holomorphic retraction of \( M \) onto \( S \) is a holomorphic retraction \( \rho: W \rightarrow S \), where \( W \) is a neighborhood of \( S \) in \( M \). It is clear that the existence of such a local holomorphic retraction implies that \( S \) splits into \( M \).

**Example 2.3.** Let \( \pi: M \rightarrow S \) be a rank \( m \) holomorphic vector bundle on \( S \). If we identify \( S \) with the zero section of the vector bundle, \( \pi \) becomes a (global) holomorphic retraction of \( M \) on \( S \). The charts given by the trivialization of the bundle clearly give a splitting atlas. Furthermore, if \((U_\alpha, z_\alpha)\) and \((U_\beta, z_\beta)\) are two such charts, we have \( z'_\alpha = \varphi_{\beta \alpha}(z'_\beta) \) and \( z'_\beta = a_{\beta \alpha}(z'_\alpha)z'_\alpha \), where \( a_{\beta \alpha} \) is an invertible matrix depending only on \( z'_\alpha \). In particular, we have

\[
\frac{\partial z^p_{\beta}}{\partial z^r_{\alpha}} \equiv 0 \quad \text{and} \quad \frac{\partial^2 z^r_{\beta}}{\partial z^s_{\alpha} \partial z^t_{\alpha}} \equiv 0
\]

for all \( r, s, t = 1, \ldots, m, p = m + 1, \ldots, n \) and indices \( \alpha \) and \( \beta \).

The previous example, compared with (2.1), suggests the following

**Definition 2.4.** Let \( S \) be a codimension \( m \) complex submanifold of an \( n \)-dimensional complex manifold \( M \). We say that \( S \) is comfortably embedded in \( M \) if \( S \) splits into \( M \) and there exists a splitting atlas \( U = \{(U_\alpha, z_\alpha)\} \) such that

\[
\left. \frac{\partial^2 z^r_{\beta}}{\partial z^s_{\alpha} \partial z^t_{\alpha}} \right|_S \equiv 0
\]

for all \( r, s, t = 1, \ldots, m \) and indices \( \alpha \) and \( \beta \).

An atlas satisfying the previous condition is said to be comfortable for \( S \). Roughly speaking, then, a comfortably embedded submanifold is like a first-order approximation of the zero section of a vector bundle.
Let us express condition (2.4) in a different way. If \((U_\alpha, z_\alpha)\) and \((U_\beta, z_\beta)\) are two charts about \(p \in S\) adapted to \(S\), we can write
\[
(2.5) \quad z_\beta^r = (a_{\beta \alpha})_s^r z_\alpha^s
\]
for suitable \((a_{\beta \alpha})_s^r \in \mathcal{O}_M, p\). The germs \((a_{\beta \alpha})_s^r\) (unless \(m = 1\)) are not uniquely determined by (2.5); indeed, all the other solutions of (2.5) are of the form \((a_{\beta \alpha})_s^r + e_s^r\), where the \(e_s^r\)'s are holomorphic and satisfy
\[
(2.6) \quad e_s^r z_\alpha^s \equiv 0.
\]
Differentiating with respect to \(z_\alpha^s\) we get
\[
(2.7) \quad e_t^r + \frac{\partial e_s^r}{\partial z_t^t} z_\alpha^s \equiv 0;
\]
in particular, \(e_t^r|_S \equiv 0\), and so the restriction of \((a_{\beta \alpha})_s^r\) to \(S\) is uniquely determined — and it indeed gives the 1-cocycle of the normal bundle \(N_S\) with respect to the atlas \(U_S\).

Differentiating (2.7) we obtain
\[
(2.8) \quad \frac{\partial e_t^r}{\partial z_s^s} + \frac{\partial e_s^r}{\partial z_t^t} + \frac{\partial^2 e_s^r}{\partial z_s^s \partial z_t^t} z_u^u \equiv 0;
\]
in particular,
\[
\left[ \frac{\partial e_t^r}{\partial z_s^s} + \frac{\partial e_s^r}{\partial z_t^t} \right]|_S \equiv 0,
\]
and so the restriction of
\[
\frac{\partial (a_{\beta \alpha})_s^r}{\partial z_\alpha^s} + \frac{\partial (a_{\beta \alpha})_s^r}{\partial z_\alpha^s}
\]
to \(S\) is uniquely determined for all \(r, s, t = 1, \ldots, m\).

With this notation, we have
\[
\frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_t^t} = \frac{\partial (a_{\beta \alpha})_s^r}{\partial z_\alpha^s} + \frac{\partial (a_{\beta \alpha})_s^r}{\partial z_\alpha^s} + \frac{\partial^2 (a_{\beta \alpha})_s^r}{\partial z_\alpha^s \partial z_t^t} z_u^u,
\]
therefore (2.4) is equivalent to requiring
\[
(2.9) \quad \left( \frac{\partial (a_{\beta \alpha})_s^r}{\partial z_\alpha^s} + \frac{\partial (a_{\beta \alpha})_s^r}{\partial z_\alpha^s} \right)|_S \equiv 0
\]
for all \(r, s, t = 1, \ldots, m\), and indices \(\alpha\) and \(\beta\).

**Example 2.4.** It is easy to check that the exceptional divisor \(S\) in Example 1.2 is comfortably embedded into the blow-up \(M\).

Then the main result of this section is
Theorem 2.2. Let $S$ be a codimension $m$ complex submanifold of an $n$-dimensional complex manifold $M$. Assume that $S$ splits into $M$, and let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a splitting atlas. Define a 1-cocycle $\{h_{\beta\alpha}\}$ of $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$ by setting

\[
(2.10) \quad h_{\beta\alpha} = \frac{1}{2} \frac{\partial \varepsilon_r^*}{\partial z_{\beta}^*} \frac{\partial^2 \varepsilon_r^*}{\partial z_{\beta}^* \partial z_{\alpha}^*} \left|_{S} \right. \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t \]

Then:

(i) $\{h_{\beta\alpha}\}$ defines an element $[h] \in H^1(S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes \mathcal{N}_S^*)$ independent of $\mathfrak{U}$;

(ii) $S$ is comfortably embedded in $M$ if and only if $[h] = 0$.

Proof. (i) Let us first prove that $\{h_{\beta\alpha}\}$ is a 1-cocycle with values in $\mathcal{N}_S^* \otimes \mathcal{N}_S \otimes \mathcal{N}_S^*$. We know that

\[
(a_{\alpha\beta})_u^r (a_{\beta\alpha})_u^s = \delta_u^r + \epsilon_u^r,
\]

where $\delta_u^r$ is Kronecker’s delta, and the $\epsilon_u^r$’s satisfy (2.6). Differentiating we get

\[
\frac{\partial (a_{\alpha\beta})_u^r (a_{\beta\alpha})_u^s}{\partial z_{\alpha}^s} + (a_{\alpha\beta})_u^s \frac{\partial (a_{\beta\alpha})_u^r}{\partial z_{\alpha}^s} = \frac{\partial \epsilon_u^r}{\partial z_{\alpha}^s};
\]

therefore (2.8) yields

\[
(a_{\beta\alpha})_s^u \frac{\partial (a_{\alpha\beta})_u^r}{\partial z_{\alpha}^s} \big|_S + (a_{\alpha\beta})_t^u \frac{\partial (a_{\beta\alpha})_u^r}{\partial z_{\alpha}^s} \big|_S = -(a_{\alpha\beta})_u^r \left( \frac{\partial (a_{\beta\alpha})_u^r}{\partial z_{\alpha}^s} + \frac{\partial (a_{\beta\alpha})_t^r}{\partial z_{\alpha}^s} \right) \big|_S.
\]

Hence

\[
h_{\beta\alpha} = \frac{1}{2} (a_{\alpha\beta})_u^r \left( \frac{\partial (a_{\beta\alpha})_u^r}{\partial z_{\alpha}^s} + \frac{\partial (a_{\beta\alpha})_t^r}{\partial z_{\alpha}^s} \right) \big|_S \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t
\]

\[
= \frac{1}{2} (a_{\alpha\beta})_u^r (a_{\alpha\beta})_t^s (a_{\beta\alpha})_s^t \times \left( \frac{\partial (a_{\beta\alpha})_u^r}{\partial z_{\alpha}^s} + (a_{\alpha\beta})_s^2 \frac{\partial (a_{\beta\alpha})_t^r}{\partial z_{\alpha}^s} \right) \big|_S \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t
\]

\[
= \frac{1}{2} \left( (a_{\alpha\beta})_s^t \frac{\partial (a_{\alpha\beta})_u^r}{\partial z_{\alpha}^s} + (a_{\alpha\beta})_t^s \frac{\partial (a_{\alpha\beta})_t^r}{\partial z_{\alpha}^s} \right) \big|_S \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t
\]

\[
= -h_{\beta\alpha},
\]

where in the second equality we used (2.1). Analogously one proves that $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$, and thus $\{h_{\beta\alpha}\}$ is a 1-cocycle as claimed.

Now we have to prove that the cohomology class $[h]$ is independent of the atlas $\mathfrak{U}$. Let $\tilde{\mathfrak{U}} = \{\tilde{U}_\alpha, \tilde{z}_\alpha\}$ be another splitting atlas; up to taking a common
refinement we can assume that $U_\alpha = \hat{U}_\alpha$ for all $\alpha$. Choose $(A_\alpha)_s^r \in \mathcal{O}(U_\alpha)$ so that $\hat{z}_\alpha^r = (A_\alpha)_s^r z_\alpha^r$; as usual, the restrictions to $S$ of $(A_\alpha)_s^r$ and of
\[
\frac{\partial (A_\alpha)_s^r}{\partial z_\alpha^t} + \frac{\partial (A_\alpha)_t^r}{\partial z_\alpha^s}
\]
are uniquely defined. Set, now,
\[
C_\alpha = \frac{1}{2} (A_\alpha^{-1})_u^u \left[ \frac{\partial (A_\alpha)_s^u}{\partial z_\alpha^t} + \frac{\partial (A_\alpha)_t^u}{\partial z_\alpha^s} \right] \left| \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t \right|_S
\]
then it is not difficult to check that
\[
h_\beta - \hat{h}_\beta = C_\beta - C_\alpha,
\]
where $\{\hat{h}_\beta\}$ is the 1-cocycle built using $\hat{U}$, and this means exactly that both $\{h_\beta\}$ and $\{\hat{h}_\beta\}$ determine the same cohomology class.

(ii) If $S$ is comfortably embedded, using a comfortable atlas we immediately see that $[h] = 0$. Conversely, assume that $[h] = 0$; therefore we can find a splitting atlas $\hat{U}$ and a 0-cochain $(c_\alpha)$ of $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$ such that $h_\beta = c_\alpha - c_\beta$. Writing
\[
c_\alpha = (c_\alpha)_s^r \partial_{\alpha,r} \otimes \omega_\alpha^s \otimes \omega_\alpha^t,
\]
with $(c_\alpha)_s^r$ symmetric in the lower indices, we define $\hat{z}_\alpha$ by setting
\[
\begin{aligned}
\left\{
\begin{array}{ll}
\hat{z}_\alpha^r = z_\alpha^r + (c_\alpha)_s^r (z_\alpha^s) z_\alpha^t & \text{for } r = 1, \ldots, m, \\
\hat{z}_\alpha^p = z_\alpha^p & \text{for } p = m+1, \ldots, n,
\end{array}
\right.
\end{aligned}
\]
on a suitable $\hat{U}_\alpha \subset U_\alpha$. Then $\hat{U} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ clearly is a splitting atlas; we claim that it is comfortable too. Indeed, by definition the functions
\[
(\hat{a}_\beta)_s^r = [\delta_\beta^r + (c_\beta)_s^r (a_\beta)_s^r] (a_\beta)_t^r \partial_{\beta,\alpha}(\hat{a}_\beta)_s^r d_s^{\alpha r}
\]
satisfy (2.5) for $\hat{U}$, where the $d_s^{\alpha r}$'s are such that $z_\alpha^u = d_s^{\alpha r} \hat{z}_\alpha^r$. Hence
\[
\left( \frac{\partial (\hat{a}_\beta)_s^r}{\partial z_\alpha^t} + \frac{\partial (\hat{a}_\beta)_t^r}{\partial z_\alpha^s} \right)|_S = 2(c_\beta)_s^r (a_\beta)_s^r (a_\beta)_t^r |_S + \left( \frac{\partial (a_\beta)_s^r}{\partial z_\alpha^t} + \frac{\partial (a_\beta)_t^r}{\partial z_\alpha^s} \right)|_S + (a_\beta)_s^r \left( \frac{\partial d_s^{\alpha r}}{\partial z_\alpha^t} + \frac{\partial d_s^{\alpha r}}{\partial z_\alpha^s} \right)|_S.
\]
Now, differentiating
\[
z_\alpha^u = d_v^u (t_\alpha^r + (c_\alpha)_r^s r_\alpha^s z_\alpha^s)
\]
we get
\[
\delta_\alpha^u = \frac{\partial d_v^u}{\partial z_\alpha^t} (t_\alpha^r + (c_\alpha)_r^s r_\alpha^s z_\alpha^s) + d_v^u (\delta_\alpha^u + 2(c_\alpha)_r^s z_\alpha^s)
\]
and
\[
0 = \left( \frac{\partial d_s^{\alpha r}}{\partial z_\alpha^t} + \frac{\partial d_s^{\alpha r}}{\partial z_\alpha^s} \right)|_S + 2(c_\alpha)_s^r d_s^{\alpha t}.
\]
Recalling that $h_\beta = c_\alpha - c_\beta$ we then see that $\hat{U}$ satisfies (2.9), and we are done. \qed
Remark 2.2. Since $N_S \otimes N_S^* \otimes N_S^* \cong \text{Hom}(N_S, \text{Hom}(N_S, N_S))$, the previous theorem asserts that to any submanifold $S$ splitting into $M$ we can canonically associate an extension

$$O \to \text{Hom}(N_S, N_S) \to E \to N_S \to O$$

of $N_S$ by $\text{Hom}(N_S, N_S)$, and $S$ is comfortably embedded in $M$ if and only if this extension splits. See also [ABT] for more details on comfortably embedded submanifolds.

3. The canonical sections

Our next aim is to associate to any $f \in \text{End}(M, S)$ different from the identity a section of a suitable vector bundle, indicating (very roughly speaking) how $f$ would move $S$ if it did not keep it fixed. To do so, in this section we still assume that $S$ is a smooth complex submanifold of a complex manifold $M$; however, in Remark 3.3 we shall describe the changes needed to avoid this assumption.

Given $f \in \text{End}(M, S)$, $f \neq \text{id}_M$, it is clear that $df|_{TS} = \text{id}$; therefore $df - \text{id}$ induces a map from $N_S$ to $TM|_S$, and thus a holomorphic section over $S$ of the bundle $TM|_S \otimes N^*_S$. If $(U, z)$ is a chart adapted to $S$, we can define germs $g^h_r$ for $h = 1, \ldots, n$ and $r = 1, \ldots, m$ by writing

$$z^h \circ f - z^h = z^1 g^h_1 + \cdots + z^m g^h_m.$$ 

It is easy to check that the germ of the section of $TM|_S \otimes N^*_S$ defined by $df - \text{id}$ is locally expressed by

$$g^h_r|_{U \cap S} \frac{\partial}{\partial z^h} \otimes \omega^r,$$

where we are again indicating by $\omega^r$ the germ of section of the conormal bundle induced by the 1-form $dz^r$ restricted to $S$.

A problem with this section is that it vanishes identically if (and only if) $\nu_f > 1$. The solution consists in expanding $f$ at a higher order.

Definition 3.1. Given a chart $(U, z)$ adapted to $S$, set $f^j = z^j \circ f$, and write

$$f^j - z^j = z^{r_1} \cdots z^{r_{\nu_f}} g^j_{r_1 \cdots r_{\nu_f}},$$

where the $g^j_{r_1 \cdots r_{\nu_f}}$'s are symmetric in $r_1, \ldots, r_{\nu_f}$ and do not all vanish restricted to $S$. Let us then define

$$X_f = g^h_{r_1 \cdots r_{\nu_f}} \frac{\partial}{\partial z^h} \otimes dz^{r_1} \otimes \cdots \otimes dz^{r_{\nu_f}}.$$ 

This is a local section of $TM \otimes (T^*M)^{\otimes \nu_f}$, defined in a neighborhood of a point of $S$; furthermore, when restricted to $S$, it induces a local section of $TM|_S \otimes (N^*_S)^{\otimes \nu_f}$. 
Remark 3.1. When \( m > 1 \) the \( g^j_{r_1 \ldots r_{\nu f}} \)'s are not uniquely determined by (3.1). Indeed, if \( e^j_{r_1 \ldots r_{\nu f}} \) are such that

\[
(3.3) \quad e^j_{r_1 \ldots r_{\nu f}} z^1 \cdots z^{r_{\nu f}} \equiv 0
\]

then \( g^j_{r_1 \ldots r_{\nu f}} + e^j_{r_1 \ldots r_{\nu f}} \) still satisfies (3.1). This means that the section (3.2) is not uniquely determined too; but, as we shall see, this will not be a problem. For instance, (3.3) implies that \( e^j_{r_1 \ldots r_{\nu f}} \in \mathcal{I}_S \); therefore \( X_f|_{U \cap S} \) is always uniquely determined — though \textit{a priori} it might depend on the chosen chart. On the other hand, when \( m = 1 \) both the \( g^j_{r_1 \ldots r_{\nu f}} \)'s and \( X_f \) are uniquely determined; this is one of the reasons making the codimension-one case simpler than the general case.

We have already remarked that when \( \nu_f = 1 \) the section \( X_f \) restricted to \( U \cap S \) coincides with the restriction of \( df - \text{id} \) to \( S \). Therefore when \( \nu_f = 1 \) the restriction of \( X_f \) to \( S \) gives a globally well-defined section. Actually, this holds for any \( \nu_f \geq 1 \):

**Proposition 3.1.** Let \( f \in \text{End}(M, S) \), \( f \neq \text{id}_M \). Then the restriction of \( X_f \) to \( S \) induces a global holomorphic section \( X_f \) of the bundle \( TM|_S \otimes (N^*_S)^{\otimes \nu_f} \).

**Proof.** Let \( (U, z) \) and \( (\hat{U}, \hat{z}) \) be two charts about \( p \in S \) adapted to \( S \). Then we can find holomorphic functions \( a_r^s \) such that

\[
(3.4) \quad \hat{z}^{r_1} a_{s_1}^r z^{s_1} = \partial \hat{z}^r \partial z^{s_1} \quad \text{(mod } \mathcal{I}_S \text{)}
\]

in particular,

\[
(3.5) \quad \frac{\partial \hat{z}^r}{\partial z^s} = a_r^s \quad \text{(mod } \mathcal{I}_S \text{)} \quad \text{and} \quad \frac{\partial \hat{z}^r}{\partial z^p} = 0 \quad \text{(mod } \mathcal{I}_S \text{)}.
\]

Now set \( f^j = z^j \circ f \), \( \hat{f}^j = \hat{z}^j \circ f \), and define \( g^j_{r_1 \ldots r_{\nu f}} \) and \( \hat{g}^j_{r_1 \ldots r_{\nu f}} \) using (3.1) with \((U, z)\) and \((\hat{U}, \hat{z})\) respectively. Then (3.4) and (1.1) yield

\[
\begin{align*}
a_s^{r_1} \cdots a_{s_{\nu f}}^{r_{\nu f}} g^j_{r_1 \ldots r_{\nu f}} z^{s_1} \cdots z^{s_{\nu f}} &= \hat{g}^j_{r_1 \ldots r_{\nu f}} \hat{z}^{r_1} \cdots \hat{z}^{r_{\nu f}} \quad \text{(mod } \mathcal{I}_S \text{)} \\
&= \hat{f}^j - \hat{z}^j = (f^h - z^h) \frac{\partial \hat{z}^j}{\partial z^h} + R_{2\nu_f} \\
&= g^h_{s_1 \ldots s_{\nu f}} \frac{\partial \hat{z}^j}{\partial z^h} z^{s_1} \cdots z^{s_{\nu f}} + R_{2\nu_f},
\end{align*}
\]

where the remainder terms \( R_{2\nu_f} \) belong to \( \mathcal{I}_S^{2\nu_f} \). Therefore we find

\[
(3.6) \quad a_s^{r_1} \cdots a_{s_{\nu f}}^{r_{\nu f}} \hat{g}^j_{r_1 \ldots r_{\nu f}} = \frac{\partial \hat{z}^j}{\partial z^h} g^h_{s_1 \ldots s_{\nu f}} \quad \text{(mod } \mathcal{I}_S \text{)}.
\]
Recalling (3.5) we then get
\[ \hat{g}_{r_1 \ldots r_{\nu f}} \frac{\partial}{\partial \hat{z}} \otimes d\hat{z}^{r_1} \otimes \cdots \otimes d\hat{z}^{r_{\nu f}} = \frac{\partial z^h}{\partial \hat{z}} \frac{\partial \hat{z}^{r_1}}{\partial z^{k_1}} \otimes \cdots \otimes \frac{\partial \hat{z}^{r_{\nu f}}}{\partial z^{k_{\nu f}}} \otimes d\hat{z}^k \otimes \cdots \otimes d\hat{z}^{r_{\nu f}} \] (mod \( I_S \))
\[ = a_{s_1}^r \cdots a_{s_{\nu f}}^r \hat{g}_{r_1 \ldots r_{\nu f}} \frac{\partial z^h}{\partial \hat{z}} \otimes d\hat{z}^{s_1} \otimes \cdots \otimes d\hat{z}^{s_{\nu f}} \] (mod \( I_S \)),
and we are done. \( \square \)

Remark 3.2. For later use, we explicitly notice that when \( m = 1 \) the germs \( a_s^r \) are uniquely determined, and (3.6) becomes
\[(3.7) (a_1^1)^{r_{1 \ldots 1}} \hat{g}_{1 \ldots 1} = \frac{\partial \hat{z}^j}{\partial z^h} g^h_{1 \ldots 1} \] (mod \( I_S^{\nu_{r_1}} \)).

Definition 3.2. Let \( f \in \text{End}(M, S), f \not\equiv \text{id}_M \). The canonical section \( X_f \in H^0(S, T_{M,S} \otimes (N^*_S)^{\otimes \nu_f}) \) associated to \( f \) is defined by setting
\[(3.8) X_f = g_{s_1 \ldots s_{\nu f}}^h \frac{\partial}{\partial z^{s_1}} \otimes \omega^{s_1} \otimes \cdots \otimes \omega^{s_{\nu f}} \]
in any chart adapted to \( S \). Since \( (N^*_S)^{\otimes \nu_f} = (N^*_S)^{\otimes \nu_{r_1}} \), we can also think of \( X_f \) as a holomorphic section of \( \text{Hom}(N^*_S^{\otimes \nu_{r_1}}, TM|_S) \), and introduce the canonical distribution \( \Xi_f = X_f(N^*_S^{\otimes \nu_{r_1}}) \subseteq TM|_S \).

In particular we can now justify the term “tangential” previously introduced:

Corollary 3.2. Let \( f \in \text{End}(M, S), f \not\equiv \text{id}_M \). Then \( f \) is tangential if and only if the canonical distribution is tangent to \( S \), that is if and only if \( \Xi_f \subseteq TS \).

Proof. This follows from Lemma 1.2. \( \square \)

Example 3.1. By the notation introduced in Example 1.2, if \( f \) is obtained by blowing up a map \( f_o \) tangent to the identity, then the canonical coordinates centered in \( p = [1 : 0 : \cdots : 0] \) are adapted to \( S \). In particular, if \( f_o \) is non-dicritical (that is, if \( f \) is tangential) then in a neighborhood of \( p \),
\[ X_f = \left[ Q_{\nu(f_o)}^q(1, z'') - z^q Q_{\nu(f_o)}^1(1, z'') \right] \frac{\partial}{\partial z^q} \otimes (\omega^1)^{\otimes (\nu(f_o) - 1)} \]
\[ = \hat{g}_{r_1 \ldots r_{\nu f}} \frac{\partial}{\partial \hat{z}} \otimes d\hat{z}^{r_1} \otimes \cdots \otimes d\hat{z}^{r_{\nu f}} \] (mod \( I_S \)),
where \( \hat{g}_{r_1 \ldots r_{\nu f}} \) is the symmetric \( \nu_{r_1} \)-fold tensor product of \( N^*_S \).

Remark 3.3. To be more precise, \( X_f \) is a section of the subsheaf \( T_{M,S} \otimes \text{Sym}^{\nu_{r_1}}(N^*_S) \), where \( \text{Sym}^{\nu_{r_1}}(N^*_S) \) is the symmetric \( \nu_{r_1} \)-fold tensor product of \( N^*_S \).
Now, the sheaf $N_S^*$ is isomorphic to $I_S/I_S^2$, and it is known that $\text{Sym}^{\nu'}I_S/I_S^{\nu'+1}$ is isomorphic to $I_S/I_S^{\nu'+1}$. This allows us to define $X_f$ as a global section of the coherent sheaf $T_{M,S} \otimes \text{Sym}^{\nu'}(I_S/I_S^2)$ even when $S$ is singular. Indeed, if $(U, z)$ is a local chart adapted to $S$, for $j = 1, \ldots, n$, the functions $f_j - z_j$ determine local sections $[f_j - z_j]$ of $I_S/I_S^{\nu'+1}$. But, since for any other chart $(\hat{U}, \hat{z})$,

$$\hat{f}_j - \hat{z}_j = (f^h - \hat{z}^h) \frac{\partial \hat{z}_j}{\partial z^h} + R_{2
u'},$$

then $\partial z^j \otimes [f_j - z_j]$ is a well-defined global section of $T_{M,S} \otimes \text{Sym}^{\nu'}(I_S/I_S^2)$ which coincides with $X_f$ when $S$ is smooth.

**Remark 3.4.** When $f$ is tangential and $\Xi_f$ is involutive as a sub-distribution of $TS$ — for instance when $m = 1$ — we thus get a holomorphic singular foliation on $S$ canonically associated to $f$. As already remarked in [Br], this possibly is the reason explaining the similarities discovered in [A2] between the local dynamics of holomorphic maps tangent to the identity and the dynamics of singular holomorphic foliations.

**Definition 3.3.** A point $p \in S$ is singular for $f$ if there exists $v \in (N_S)_p$, $v \neq 0$, such that $X_f(v \otimes \cdots \otimes v) = 0$. We shall denote by $\text{Sing}(f)$ the set of singular points of $f$.

In Section 7 it will become clear why we choose this definition for singular points. In Section 8 we shall describe a dynamical interpretation of $X_f$ at nonsingular points in the codimension-one case; see Proposition 8.1.

**Remark 3.5.** If $S$ is a hypersurface, the normal bundle is a line bundle. Therefore $\Xi_f$ is a 1-dimensional distribution, and the singular points of $f$ are the points where $\Xi_f$ vanishes. Recalling (3.8), we then see that $p \in \text{Sing}(f)$ if and only if $g_1^1(p) = \cdots = g_1^n(p) = 0$ for any adapted chart, and thus both the strictly fixed points of [A2] and the singular points of [BT], [Br] are singular in our case as well.

As we shall see later on, our index theorems will need a section of $TS \otimes (N_S^*)^{\otimes \nu'}$; so it will be natural to assume $f$ tangential. When $f$ is not tangential but $S$ splits in $M$ we can work too.

Let $O \to TS \xrightarrow{\iota} TM|_S \xrightarrow{\pi} N_S \to O$ be the usual extension. Then we can associate to any splitting morphism $\sigma: N_S \to TM|_S$ a morphism $\sigma': TM|_S \to TS$ such that $\sigma' \circ \iota = \text{id}_TS$, by $\sigma' = \iota^{-1} \circ (\sigma \circ \pi - \text{id}_TM|_S)$. Conversely, if there is a morphism $\sigma': TM|_S \to TS$ such that $\sigma' \circ \iota = \text{id}_TS$, we get a splitting morphism by setting $\sigma = (\pi|_{\text{Ker} \sigma'})^{-1}$. Then

**Definition 3.4.** Let $f \in \text{End}(M, S)$, $f \neq \text{id}_M$, and assume that $S$ splits in $M$. Choose a splitting morphism $\sigma: N_S \to TM|_S$ and let $\sigma': TM|_S \to TS$...
be the induced morphism. We set
\[ H_{\sigma,f} = (\sigma' \otimes \text{id}) \circ X_f \in H^0(S, \mathcal{T}_S \otimes (N^*_S)^{\otimes \nu_f}). \]

Since the differential of \( f \) induces a morphism from \( N_S \) into itself, we have a dual morphism \((df)^* : N^*_S \to N^*_S\). Then if \( \nu_f = 1 \) we also set
\[ H^1_{\sigma,f} = (\text{id} \otimes (df)^*) \circ H_{\sigma,f} \in H^0(S, \mathcal{T}_S \otimes N^*_S). \]

Remark 3.6. We defined \( H^1_{\sigma,f} \) only for \( \nu_f = 1 \) because when \( \nu_f > 1 \) one has \((df)^* = \text{id}\). On the other hand, when \( \nu_f = 1 \) one has \((df)^* = \text{id}\) if and only if \( f \) is tangential. Finally, we have \( X_f \equiv H_{\sigma,f} \) if and only if \( f \) is tangential, and \( H_{\sigma,f} \equiv O \) if and only if \( \Xi_f \subseteq \text{Im } \sigma = \text{Ker } \sigma' \).

Finally, if \((U, z)\) is a chart in an atlas adapted to the splitting \( \sigma \), locally we have
\[ H_{\sigma,f} = g^p_{s_1...s_{\nu_f}} |_S \frac{\partial}{\partial z^p} \otimes \omega^{s_1} \otimes \cdots \otimes \omega^{s_{\nu_f}}, \]
and, if \( \nu_f = 1 \),
\[ H^1_{\sigma,f} = (\delta^s_r + g^s_r) g^p_s |_S \frac{\partial}{\partial z^p} \otimes \omega^r. \]

4. Local extensions

As we have already remarked, while \( X_f \) is well-defined, its extension \( X_f \) in general is not. However, we shall now derive formulas showing how to control the ambiguities in the definition of \( X_f \), at least in the cases that interest us most.

In this section we assume \( m = 1 \), i.e., that \( S \) has codimension one in \( M \). To simplify notation we shall write \( g^j \) for \( g^j_{1...1} \) and \( a \) for \( a^1_{1...1} \). We shall also use the following notation:

- \( T_1 \) will denote any sum of terms of the form \( g^j \frac{\partial}{\partial z^p} \otimes dz^{h_1} \otimes \cdots \otimes dz^{h_{\nu_f}} \) with \( g \in \mathcal{I}_S \);
- \( R_k \) will denote any local section with coefficients in \( \mathcal{I}_S^k \).

For instance, if \((U, z)\) and \((\hat{U}, \hat{z})\) are two charts adapted to \( S \),
\begin{equation}
\frac{\partial}{\partial z^h} \otimes (dz^1)^{\otimes \nu_f} = a^\nu_f \frac{\partial z^k}{\partial z^h} \frac{\partial}{\partial z^k} \otimes (dz^1)^{\otimes \nu_f} \\
+ \frac{\partial z^1}{\partial z^h} a^{\nu_f-1}_{\nu_f} z^{1} \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{j_\ell}} \frac{\partial}{\partial z^1} \otimes dz^1 \otimes \cdots \otimes dz^{j_\ell} + T_1 + R_2,
\end{equation}
where
\[ T_1 = \frac{\partial z^p}{\partial \hat{z}^h} \nu_f \sum_{j=1}^{\nu_f} \frac{\partial a}{\partial z^{j_1}} \otimes dz^1 \otimes \cdots \otimes dz^{j_\nu_f} \otimes \cdots \otimes dz^1. \]

Assume now that \( f \) is tangential, and let \((U, z)\) be a chart adapted to \( S \). We know that \( f^1 - z^1 \in T^{\nu_f + 1}_S \), and thus we can write
\[ f^1 - z^1 = h^1 (z^1)^{\nu_f + 1}, \]
where \( h^1 \) is uniquely determined. Now, if \((\hat{U}, \hat{z})\) is another chart adapted to \( S \)
then
\[ a^{\nu_f + 1} \hat{h}^1 (z^1)^{\nu_f + 1} = \hat{f}^1 - \hat{z}^1 = (a \circ f) f^1 - a z^1 \]
\[ = a(f^1 - z^1) + (a \circ f - a) z^1 + (a \circ f - a)(f^1 - z^1) \]
\[ = a(f^1 - z^1) + \frac{\partial a}{\partial z^p} (f^p - z^p) z^1 + R_{\nu_f + 2} \]
\[ = \left[ a h^1 + \frac{\partial a}{\partial z^p} g^p \right] (z^1)^{\nu_f + 1} + R_{\nu_f + 2}. \]
Therefore
\[ (4.2) \quad a^{\nu_f + 1} \hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p + R_1. \]
Since \( g^1 = h^1 z^1 \) we then get
\[ (4.3) \quad a^{\nu_f} g^1 = a g^1 + \frac{\partial a}{\partial z^p} g^p z^1 + R_2, \]
which generalizes (3.6) when \( f \) is tangential and \( m = 1 \).

Putting (4.3), (3.6) and (4.1) into (3.2) we then get
\[ \text{Lemma 4.1.} \quad \text{Let } f \in \text{End}(M, S), \; f \neq \text{id}_M. \; \text{Assume that } f \text{ is tangential, and that } S \text{ has codimension 1. Let } (U, z) \text{ and } (\hat{U}, \hat{z}) \text{ be two charts about } p \in S \text{ adapted to } S, \text{ and let } \hat{X}_f, X_f \text{ be given by (3.2) in the respective coordinates. Then} \]
\[ \hat{X}_f = X_f + T_1 + R_2. \]

When \( S \) is comfortably embedded in \( M \) and of codimension one we shall also need nice local extensions of \( H_{\sigma, f} \) and \( H_{\sigma, f}^1 \), and to know how they behave under change of (comfortable) coordinates.

\[ \text{Definition 4.1.} \quad \text{Let } S \text{ be comfortably embedded in } M \text{ and of codimension 1, and take } f \in \text{End}(M, S), \; f \neq \text{id}_M. \; \text{Let } (U, z) \text{ be a chart in a comfortable atlas, and set } b^1(z) = g^1(O, z''); \text{ notice that } f \text{ is tangential if and only if } b^1 \equiv O. \; \text{Write } g^1 = b^1 + h^1 z^1 \text{ for a well-defined holomorphic function } h^1; \text{ then set} \]
\[ (4.4) \quad H_{\sigma, f} = h^1 z^1 \frac{\partial}{\partial z^1} \otimes (dz^1)^{\otimes \nu_f} + g^p \frac{\partial}{\partial z^p} \otimes (dz^1)^{\otimes \nu_f}, \]
and if \( \nu_f = 1 \) set
\[
\mathcal{H}^\nu_{\sigma,f} = h^1 z^1 \frac{\partial}{\partial z^1} \otimes dz^1 + g^p(1 + b^1) \frac{\partial}{\partial z^p} \otimes dz^1.
\]
Notice that \( \mathcal{H}_{\sigma,f} \) (respectively, \( \mathcal{H}^1_{\sigma,f} \)) restricted to \( S \) yields \( H_{\sigma,f} \) (respectively, \( H^1_{\sigma,f} \)).

**Proposition 4.2.** Let \( f \in \text{End}(M, S) \), \( f \neq \text{id}_M \). Assume that \( S \) is comfortably embedded in \( M \), and of codimension one. Fix a comfortable atlas \( \mathcal{U} \), and let \((U, z), (\hat{U}, \hat{z})\) be two charts in \( \mathcal{U} \) about \( p \in S \). Then if \( \nu_f = 1 \),
\[
\hat{\mathcal{H}}^1_{\sigma,f} = \mathcal{H}^1_{\sigma,f} + T_1 + R_2,
\]
while if \( \nu_f > 1 \),
\[
\hat{\mathcal{H}}_{\sigma,f} = \mathcal{H}_{\sigma,f} + T_1 + R_2,
\]
where \( T_1 = T_1^0 + T_1^1 \) with
\[
T_1^0 = \frac{1}{a} q^1 z^1 \sum_{k=1}^{1} \partial a \frac{\partial}{\partial z^k} \otimes dz^1 \otimes \cdots \otimes dz^p \otimes \cdots \otimes dz^1,
\]
\[
T_1^1 = -g^1 \frac{\partial q^1}{\partial \hat{z}^1} \frac{\partial}{\partial z^1} \otimes (dz^1)^{\nu_f}.
\]

**Proof.** First of all, from (3.7), \( a^{\nu_f} \hat{b}^1 = ab^1 \pmod{I_S^2} \). But since we are using a comfortable atlas we get
\[
\frac{\partial (a^{\nu_f} \hat{b}^1 - ab^1)}{\partial z^1} = (\nu_f a^{\nu_f-1} \hat{b}^1 - b^1) \frac{\partial a}{\partial z^1} + R_1 = R_1,
\]
and thus
\[
a^{\nu_f} \hat{b}^1 = ab^1 \pmod{I_S^2}.
\]
If \( \nu_f > 1 \) then by (3.7) and (4.8),
\[
a^{\nu_f} \hat{h}^1 \hat{z}^1 = (ah^1 + \frac{\partial a}{\partial z^p} g^p)z^1 \pmod{I'_S},
\]
which implies
\[
a^{\nu_f+1} \hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} g^p \pmod{I_S}.
\]
If \( \nu_f = 1 \), using (2.4) we can write
\[
\hat{b}^1 \hat{z}^1 + \hat{h}^1 (\hat{z}^1)^2 = f^1 - \hat{z}^1 = \frac{\partial \hat{z}^1}{\partial \hat{z}^j} (f^j - z^j) + \frac{1}{2} \frac{\partial^2 \hat{z}^1}{\partial z^h \partial z^k} (f^h - z^h)(f^k - z^k) + R_3
\]
\[
= ab^1 z^1 + \left[ ah^1 + \frac{\partial a}{\partial z^p} g^p (1 + b^1) \right] (z^1)^2 + R_3.
\]
and by (4.8),

\begin{equation}
(4.10) \quad a^2 \hat{h}^1 = ah^1 + \frac{\partial a}{\partial z^p} q^p (1 + b^1) \pmod{I_S}.
\end{equation}

So if we compute $\hat{\mathcal{H}}_{\sigma,f}$ for $\nu_f > 1$ (respectively, $\hat{\mathcal{H}}^1_{\sigma,f}$ for $\nu_f = 1$) using (3.7), (4.1) and (4.9) (respectively, (3.7), (4.1), (4.8) and (4.10)), we get the assertions.

\section{5. Holomorphic actions}

The index theorems to be discussed depend on actions of vector bundles. This concept was introduced by Baum and Bott in [BB], and later generalized in [CL], [LS], [LS2] and [Su]. Let us recall here the relevant definitions.

Let $S$ again be a submanifold of codimension $m$ in an $n$-dimensional complex manifold $M$, and let $\pi_E : E \rightarrow S$ be a holomorphic vector bundle on $S$. We shall denote by $\mathcal{F}$ the sheaf of germs of holomorphic sections of $F$, by $\mathcal{T}_S$ the sheaf of germs of holomorphic sections of $TS$, and by $\Omega^1_S$ (respectively, $\Omega^1_M$) the sheaf of holomorphic 1-forms on $S$ (respectively, on $M$).

A section $X$ of $\mathcal{T}_S \otimes \mathcal{F}^*$ (or, equivalently, a holomorphic section of $TS \otimes F^*$) can be interpreted as a morphism $X : \mathcal{F} \rightarrow \mathcal{T}_S$. Therefore it induces a derivation $X^\# : \mathcal{O}_S \rightarrow \mathcal{F}^*$ by setting

\begin{equation}
(5.1) \quad X^\#(g)(u) = X(u)(g)
\end{equation}

for any $p \in S$, $g \in \mathcal{O}_{S,p}$ and $u \in \mathcal{F}_p$. If $\{f_1^*, \ldots, f_k^*\}$ is a local frame for $F^*$ about $p$, and $X$ is locally given by $X = \sum_j v_j \otimes f_j^*$, then

\begin{equation}
(5.2) \quad X^\#(g) = \sum_j v_j(g) f_j^*.
\end{equation}

Notice that if $X^* : \Omega^1_S \rightarrow \mathcal{F}^*$ denotes the dual morphism of $X : \mathcal{F} \rightarrow \mathcal{T}_S$, by definition we have

\begin{equation}
X^*(\omega)(u) = \omega(X(u))
\end{equation}

for any $p \in S$, $\omega \in (\Omega^1_S)_p$ and $u \in \mathcal{F}_p$, and so

\begin{equation}
X^\#(g) = X^*(dg).
\end{equation}

\textit{Definition 5.1.} Let $\pi_E : E \rightarrow S$ be another holomorphic vector bundle on $S$, and denote by $\mathcal{E}$ the sheaf of germs of holomorphic sections of $E$. Let $X$ be a section of $\mathcal{T}_S \otimes \mathcal{F}^*$. A \textit{holomorphic action of $F$ on $E$ along $X$} (or an $X$-\textit{connection} on $E$) is a $\mathbb{C}$-linear map $\tilde{X} : \mathcal{E} \rightarrow \mathcal{F}^* \otimes \mathcal{E}$ such that

\begin{equation}
(5.3) \quad \tilde{X}(gs) = X^\#(g) \otimes s + g \tilde{X}(s)
\end{equation}

for any $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$. 

Example 5.1. If $F = TS$, and the section $X$ is the identity $id : TS \to TS$, then $X^\#(g) = dg$, and a holomorphic action of $TS$ on $E$ along $X$ is just a $(1,0)$-connection on $E$.

Definition 5.2. A point $p \in S$ is a singularity of a holomorphic section $X$ of $T_S \otimes F$ if the induced map $X_p : F_p \to T_p S$ is not injective. The set of singular points of $X$ will be denoted by $\text{Sing}(X)$, and we shall set $S^0 = S \setminus \text{Sing}(X)$ and $\Xi_X = X(F|_{S^0}) \subseteq TS^0$. Notice that $\Xi_X$ is a holomorphic subbundle of $TS^0$.

The canonical section previously introduced suggests the following definition:

Definition 5.3. A Camacho-Sad action on $S$ is a holomorphic action of $N_S \otimes \nu S$ on $N_S$ along a section $X$ of $T_S \otimes (N_S \otimes \nu S)^*$, for a suitable $\nu \geq 1$.

Remark 5.1. The rationale behind the name is the following: as we shall see, the index theorem in [A2] is induced by a holomorphic action of $N_S \otimes \nu S$ on $N_S$ along $X_f$ when $f$ is tangential, and this index theorem was inspired by the Camacho-Sad index theorem [CS].

Let us describe a way to get Camacho-Sad actions. Let $\pi : TM|_S \to NS$ be the canonical projection; we shall use the same symbol for all other projections naturally induced by it. Let $X$ be any global section of $T_S \otimes (N_S \otimes \nu S)^*$. Then we might try to define an action $\tilde{X} : NS \to (N_S \otimes \nu S)^* \otimes NS = \text{Hom}(N_S \otimes \nu S, NS)$ by setting

\begin{equation}
(5.4) \quad \tilde{X}(s)(u) = \pi([\mathcal{X} (\tilde{u}), s]|_S)
\end{equation}

for any $s \in NS$ and $u \in N_S \otimes \nu S$, where: $\tilde{s}$ is any element in $T_M|_S$ such that $\pi(\tilde{s}|_S) = s$; $\tilde{u}$ is any element in $T_M \otimes (\Omega^1_M)^{\otimes \nu}$ such that $\pi(\tilde{u}|_S) = u$; and $\mathcal{X}$ is a suitably chosen local section of $T_M \otimes (\Omega^1_M)^{\otimes \nu}$ that restricted to $S$ induces $X$.

Surprisingly enough, we can make this definition work in the cases interesting to us:

**Theorem 5.1.** Let $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$, be given. Assume that $S$ has codimension one in $M$ and that

(a) $f$ is tangential to $S$, or that
(b) $S$ is comfortably embedded into $M$.

Then we can use (5.4) to define a Camacho-Sad action on $S$ along $X_f$ in case (a), along $H_{\sigma, f}$ in case (b) when $\nu_f > 1$, and along $H_{\sigma, f}^1$ in case (b) when $\nu_f = 1$.

Proof. We shall denote by $X$ the section $X_f$, $H_{\sigma, f}$ or $H_{\sigma, f}^1$ depending on the case we are considering. Let $\mathcal{U}$ be an atlas adapted to $S$, comfortable and adapted to the splitting morphism $\sigma$ in case (b), and let $X$ be the local
extension of $X$ defined in a chart belonging to $\mathcal{U}$ by Definition 3.1 (respectively, Definition 4.1). We first prove that the right-hand side of (5.4) does not depend on the chart chosen. Take $(U, z), \tilde{(U, \tilde{z})} \in \mathcal{U}$ to be local charts about $p \in S$. Using Lemma 4.1 and Proposition 4.2 we get

$$[\tilde{X}(\tilde{u}), \tilde{s}] = [(X + T_1 + R_2)(\tilde{u}), \tilde{s}] = [X(\tilde{u}) + T_1 + R_2, \tilde{s}] = [X(\tilde{u}), \tilde{s}] + T_0 + R_1,$$

where $T_0$ represents a local section of $TM$ that restricted to $S$ is tangent to it. Thus

$$\pi([\tilde{X}(\tilde{u}), \tilde{s}]|_S) = \pi([X(\tilde{u}), \tilde{s}]|_S),$$

as desired.

We must now show that the right-hand side of (5.4) does not depend on the extensions of $s$ and $u$ chosen. If $s'$ and $\tilde{u}'$ are other extensions of $s$ and $u$ respectively, we have $(\tilde{s}' - \tilde{s})|_S = T_0$, while $(\tilde{u}' - \tilde{u})|_S$ is a sum of terms of the form $V_1 \otimes \cdots \otimes V_{\nu_f}$ with at least one $V_\ell$ tangent to $S$. Therefore $\mathcal{X}(\tilde{u}' - \tilde{u})|_S = O$ and

$$[\mathcal{X}(\tilde{u}'), \tilde{s}']|_S = [\mathcal{X}(\tilde{u}), \tilde{s}]|_S + [\mathcal{X}(\tilde{u}), \tilde{s}' - \tilde{s}]|_S + [\mathcal{X}(\tilde{u}' - \tilde{u}), \tilde{s}]|_S$$

so that $\pi([\mathcal{X}(\tilde{u}'), \tilde{s}']|_S) = \pi([\mathcal{X}(\tilde{u}), \tilde{s}]|_S)$, as wanted.

We are left to show that $\tilde{X}$ is actually an action. Take $g \in \mathcal{O}_S$, and let $\tilde{g} \in \mathcal{O}_M|_S$ be any extension. First of all,

$$\tilde{X}(s)(gu) = \pi([\mathcal{X}(\tilde{g}u), \tilde{s}]|_S) = g\tilde{X}(s)(u) - \tilde{s}(\tilde{g})|_S \pi(X(u)) = g\tilde{X}(s)(u),$$

and so $\tilde{X}(s)$ is a morphism. Finally, (5.1) yields

$$\mathcal{X}(\tilde{u})(\tilde{g})|_S = \mathcal{X}^\#(g)(u),$$

and so

$$\tilde{X}(gs)(u) = \pi([\mathcal{X}(\tilde{u}), \tilde{g}s]|_S) = g\tilde{X}(s)(u) + \mathcal{X}(\tilde{u})(\tilde{g})|_S s = g\tilde{X}(s)(u) + \mathcal{X}^\#(g)(u)s,$$

and we are done. \qed

Remark 5.2. If $\nu_f = 1$ and $f$ is not tangential then (5.4) with $\mathcal{X} = \mathcal{H}_{\sigma,f}$ does not define an action. This is the reason why we introduced the new section $\mathcal{H}^1_{\sigma,f}$ and its extension $\mathcal{H}^1_{\sigma,f}$.

Later it will be useful to have an expression of $\tilde{X}_f, \tilde{H}_{\sigma,f}$ and $\mathcal{H}^1_{\sigma,f}$ in local coordinates. Let then $(U, z)$ be a local chart belonging to a (comfortable, if necessary) atlas adapted to $S$, so that $\{\partial_1\}$ is a local frame for $N_S$, and $\{\omega^1_\sigma \otimes \partial_1\}$ is a local frame for $(N^*_S)^* \otimes N_S$. There is a holomorphic function $M_f$ such that

$$\tilde{X}_f(\partial_1)(\partial^1_1) = M_f \partial_1.$$
Now, recalling (3.2), we obtain
\[ \tilde{X}_f(\partial_1)(\partial^{\otimes \nu}_1) = \pi \left( \left[ X_f \left( \left( \frac{\partial}{\partial z^1} \right)^{\otimes \nu}_1 \right), \frac{\partial}{\partial z_1} \right] \big|_S \right) \]
\[ = \pi \left( \left[ g^1 \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z_1} \right] \big|_S \right) = - \frac{\partial g^1}{\partial z_1} \big|_S \partial_1, \]
and so
\[ (5.5) \quad M_f = - \frac{\partial g^1}{\partial z_1} \big|_S. \]
In particular, recalling that \( f \) is tangential we can write \( g^1 = z^1 h^1 \), and hence (5.5) yields
\[ (5.6) \quad M_f = - h^1|_S. \]
Similarly, if we write \( \tilde{H}_{\sigma,f}(\partial_1)(\partial^{\otimes \nu}_1) = M_{\sigma,f} \partial_1 \) and \( \tilde{H}^1_{\sigma,f}(\partial_1)(\partial_1) = M^1_{\sigma,f} \partial_1 \), we obtain
\[ (5.7) \quad M_{\sigma,f} = M^1_{\sigma,f} = - h^1|_S, \]
where \( h^1 \) is defined by \( f^1 - z^1 = b^1(z^1)^{\nu_1} + h^1(z^1)^{\nu_1 + 1}. \)

Following ideas originally due to Baum and Bott (see [BB]), we can also introduce a holomorphic action on the virtual bundle \( TS - N_S^{\otimes \nu} \). But let us first define what we mean by a holomorphic action on such a bundle.

**Definition 5.4.** Let \( S^0 \) be an open dense subset of a complex manifold \( S \), \( F \) a vector bundle on \( S \), \( X \in H^0(S, TS \otimes F^*) \), \( W \) a vector bundle over \( S^0 \) and \( \tilde{W} \) any extension of \( W \) over \( S \) in \( K \)-theory. Then we say that \( F \) acts **holomorphically on** \( \tilde{W} \) along \( X \) if \( F|_{S^0} \) acts holomorphically on \( W \) along \( X|_{S^0} \).

Let \( S \) be a codimension-one submanifold of \( M \) and take \( f \in \text{End}(M, S) \), \( f \not\equiv \text{id}_M \), as usual. If \( f \) is tangential set \( X = X_f \). If not, assume that \( S \) is comfortbly embedded in \( M \) and set \( X = H_{\sigma,f} \) or \( X = H^1_{\sigma,f} \) according to the value of \( \nu_f \); in this case, we shall also assume that \( X \not\equiv O \). Set \( S^0 = S \setminus \text{Sing}(X) \), and let \( Q_f = TS/X(N_X^{\otimes \nu_f}) \). The sheaf \( Q_f \) is a coherent analytic sheaf which is locally free over \( S^0 \). The associated vector bundle (over \( S^0 \)) is denoted by \( Q_f \) and it is called the **normal bundle of** \( f \). Then the virtual bundle \( TS - N_S^{\otimes \nu} \), represented by the sheaf \( Q_f \), is an extension (in the sense of \( K \)-theory) of \( Q_f \).

**Definition 5.5.** A **Baum-Bott action** on \( S \) is a holomorphic action of \( N_S^{\otimes \nu} \) on the virtual bundle \( TS - N_S^{\otimes \nu} \) along a section \( X \) of \( TS \otimes N_S^{\otimes \nu} \), for a suitable \( \nu \geq 1 \).
Theorem 5.2. Let $f \in \text{End}(M,S)$, $f \neq \text{id}_M$, be given. Assume that $S$ has codimension one in $M$, and that either $f$ is tangential to $S$ (and then set $X = X_f$) or $S$ is comfortably embedded into $M$ (and then set $X = H_{\sigma,f}$ or $X = H_{\sigma,f}^1$ according to the value of $\nu_f$). Assume moreover that $X \neq 0$. Then there exists a Baum-Bott action $\tilde{B}: Q_f \rightarrow (N^{\otimes \nu_f}_S)^* \otimes Q_f$ of $N^{\otimes \nu_f}_S$ on $TS - N^{\otimes \nu_f}_S$ along $X$ defined by

\begin{equation}
(5.8) \quad \tilde{B}(s)(u) = \pi_f([X(u), \tilde{s}])
\end{equation}

where $\pi_f: T_S \rightarrow Q_f$ is the natural projection, and $\tilde{s} \in T_S$ is any section such that $\pi_f(\tilde{s}) = s$.

Proof. If $\hat{s} \in T_S$ is another section such that $\pi_f(\hat{s}) = s$ we have $\hat{s} - \tilde{s} \in X(N^{\otimes \nu_f}_S)$; hence $\pi_f([X(u), \hat{s} - \tilde{s}]) = O$, and (5.8) does not depend on the choice of $\hat{s}$. Finally, one can easily check that $\tilde{B}$ is a holomorphic action on $S^0$.

Remark 5.3. Since $S$ has codimension one, $X: N^{\otimes \nu_f}_S \rightarrow TS$ yields a (possibly singular) holomorphic foliation on $S$, and the previous action coincides with the one used in [BB] for the case of foliations.

We can also define a third holomorphic action, on the virtual bundle $(TM|_S - N^{\otimes \nu_f}_S)|_{T_S}$ and set $N^{\otimes \nu_f}_S = S \setminus \text{Sing}(X_f)$, as before. Then the sheaf $W_f = T_{M,S}^1/X_f(N^{\otimes \nu_f}_S)$ is a coherent analytic sheaf, locally free over $S^0$, let $W_f = TM|_{S^0}/X_f(N^{\otimes \nu_f}_S|_{S^0})$ be the associated vector bundle over $S^0$. Then the virtual bundle $TM|_S - N^{\otimes \nu_f}_S$, represented by the sheaf $W_f$, is an extension (in the sense of $K$-theory) of $W_f$.

Definition 5.6. A Lehmann-Suwa action on $S$ is a holomorphic action of $N^{\otimes \nu}_S$ on $TM|_S - N^{\otimes \nu}_S$ along a section $X$ of $T_S \otimes N^{\otimes \nu}_S$, for a suitable $\nu \geq 1$.

Again, the name is chosen to honor the ones who first discovered the analogous action for holomorphic foliations in any dimension; see [LS], [LS2] (and [KS] for dimension two).

To present an example of such an action we first need a definition.

Definition 5.7. Let $S$ be a codimension-one, comfortably embedded submanifold of $M$, and choose a comfortable atlas $\mathcal{U}$ adapted to a splitting morphism $\sigma: N_S \rightarrow TM|_S$. If $v \in (N^{\otimes \nu}_S)_p$ and $(U, \varphi) \in \mathcal{U}$ is a chart about $p \in S$, we can write $v = \lambda(z^n)|_1^{\otimes \nu}$ for a suitable $\lambda \in \mathcal{O}(U \cap S)$. Then the local extension of $v$ along the fibers of $\sigma$ is the local section $\tilde{v} = \lambda(z^n)(\partial|_1^{\otimes \nu}) \in (TM|_S^{\otimes \nu})_p$.

If $(U, \tilde{z})$ is another chart in $\mathcal{U}$ about $p$, and $v \in (N^{\otimes \nu}_S)_p$, we can also write $v = \tilde{\lambda}|_1^{\otimes \nu}$, and we clearly have $\tilde{\lambda} = (a|_S)^{\nu} \lambda$. But since $S$ is comfortably
embedded in $M$ we also have
\[
\frac{\partial(\hat{\lambda} - a^\nu \lambda)}{\partial z^1} \bigg|_S = 0,
\]
and thus
\[
a^\nu \lambda = \hat{\lambda} + R_2.
\]
Therefore if $\hat{v}$ denotes the local extension of $v$ along the fibers of $\sigma$ in the chart $\hat{U}, \hat{\varphi}$ we have
\[
(5.9) \hat{v} = \hat{\lambda} \left( \frac{\partial}{\partial \hat{z}^1} \right)^{\otimes^\nu} = a^\nu \lambda \frac{\partial z^{h_1}}{\partial \hat{z}^1} \cdots \frac{\partial z^{h_\nu}}{\partial \hat{z}^1} \otimes \cdots \otimes \frac{\partial}{\partial z^{h_\nu}} + R_2 = \hat{v} + T_1 + R_2,
\]
where
\[
T_1 = a^\lambda \sum_{\ell=1}^\nu \frac{\partial z^{p_\ell}}{\partial z^1} \frac{\partial}{\partial z^1} \otimes \cdots \otimes \frac{\partial}{\partial z^{p_\ell}} \otimes \cdots \otimes \frac{\partial}{\partial z^1}.
\]
Hence:

**Theorem 5.3.** Let $f \in \text{End}(M, S)$, $f \neq \text{id}_M$, be given. Assume that $S$ is of codimension one and comfortably embedded in $M$, and that $f$ is tangential with $\nu_f > 1$. Let $\rho_f: T_{M,S} \rightarrow W_f$ be the natural projection. Then a Lehmann-Suwa action $\hat{V}: W_f \rightarrow (N_S^{\otimes \nu})^* \otimes W_f$ of $N_S^{\otimes \nu}$ on $TM|_S - N_S^{\otimes \nu}$ may be defined along $X_f$ by setting
\[
(5.10) \hat{V}(s)(v) = \rho_f([X_f(\hat{v}), \hat{s}]|_S),
\]
for $s \in W_f$ and $v \in N_S^{\otimes \nu}$, where $\hat{s}$ is any element in $T_{M|_S}$ such that $\rho_f(\hat{s}|_S) = s$, and $\hat{v} \in T_{M|_S}^{\hat{S}^{\otimes \nu}}$ is an extension of $v$ constant along the fibers of a splitting morphism $\sigma$.

**Proof.** Since $X_f(\hat{v})|_S \in T_S$ then clearly (5.10) does not depend on the extension $\hat{s}$ chosen. Using (5.9) and (4.7), since $f$ tangential implies $X_f = H_{\sigma,f}$ and $T_1^1 = R_2$, we have
\[
[X_f(\hat{v}), \hat{s}] = [(X_f + T_1^0 + R_2)(\hat{v} + T_1 + R_2), \hat{s}] = [X_f(\hat{v}), \hat{s}] + R_1,
\]
and therefore (5.10) does not depend on the comfortable coordinates chosen to define it. Finally, arguing as in Theorem 5.1 we can show that $\hat{V}$ actually is a holomorphic action, and we are done.

6. Index theorems for hypersurfaces

Let $S$ be a compact, globally irreducible, possibly singular hypersurface in a complex manifold $M$, and set $S' = S \setminus \text{Sing}(S)$. Given the following data:

(a) a line bundle $F$ over $S'$;
(b) a holomorphic section $X$ of $T S' \otimes F^*$;

(c) a vector bundle $E$ defined on $M$;

(d) a holomorphic action $\tilde{X}$ of $F|_{S'}$ on $E|_{S'}$ along $X$;

we can recover a partial connection (in the sense of Bott) on $E$ restricted to $S^0 = S' \setminus \text{Sing}(X)$ as follows: since, by definition of $S^0$, the dual map $X^*: \Xi_X \to F^*|_{S^0}$ is an isomorphism, we can define a partial connection (in the sense of Bott [Bo]) $D: \Xi_X \times H^0(S^0, E|_{S^0}) \to H^0(S^0, E|_{S^0})$ by setting

$$D_v(s) = (X^* \otimes \text{id})^{-1}(\tilde{X}(s))(v)$$

for $p \in S^0$, $v \in (\Xi_X)_p$, and $s \in H^0(S^0, E|_{S^0})$. Furthermore, we can always extend this partial connection $D$ to a $(1,0)$-connection on $E|_{S^0}$, for instance by using a partition of unity (see, e.g., [BB]). Any such connection (which is a $\Xi_X$-connection in the terminology of [Bo], [Su]) will be said to be induced by the holomorphic action $\tilde{X}$.

We can then apply the general theory developed by Lehmann and Suwa for foliations (see in particular Theorem 1’ and Proposition 4 of [LS], as well as [Su, Th. VI.4.8]) to get the following:

**Theorem 6.1.** Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$, and set $S' = S \setminus \text{Sing}(S)$. Let $F$ be a line bundle over $S'$ admitting an extension to $M$, and $X$ a holomorphic section of $T S' \otimes F^*$. Set $S^0 = S' \setminus \text{Sing}(X)$, and let $\text{Sing}(S) \cup \text{Sing}(X) = \bigcup \lambda \Sigma_\lambda$ be the decomposition of $\text{Sing}(S) \cup \text{Sing}(X)$ in connected components. Finally, let $E$ be a vector bundle defined on $M$. Then for any holomorphic action $\tilde{X}$ of $F|_{S'}$ on $E|_{S'}$ along $X$ and any homogeneous symmetric polynomial $\varphi$ of degree $n - 1$, there are complex numbers $\text{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of $\tilde{X}$ and $E$ near $\Sigma_\lambda$, such that

$$\sum \lambda \text{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) = \int_S \varphi(E),$$

where $\varphi(E)$ is the evaluation of $\varphi$ on the Chern classes of $E$.

Recalling the results of the previous section, we then get the following index theorem for holomorphic self-maps:

**Theorem 6.2.** Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \text{End}(M,S)$, $f \neq \text{id}_M$, be given. Assume that

(a) $f$ is tangential to $S$, and $X = X_f$, or that

(b) $S^0 = S \setminus (\text{Sing}(S) \cup \text{Sing}(f))$ is comfortably embedded into $M$, and $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$. 

Assume moreover $X \not\equiv O$. Let $\text{Sing}(S) \cup \text{Sing}(X) = \bigcup_{\lambda} \Sigma_{\lambda}$ be the decomposition of $\text{Sing}(S) \cup \text{Sing}(X)$ in connected components. Finally, let $[S]$ be the line bundle on $M$ associated to the divisor $S$. Then there exist complex numbers $\text{Res}(X, S, \Sigma_{\lambda}) \in \mathbb{C}$, depending only on the local behavior of $X$ and $[S]$ near $\Sigma_{\lambda}$, such that

$$\sum_{\lambda} \text{Res}(X, S, \Sigma_{\lambda}) = \int_{S} c^{n-1}_1([S]).$$

**Proof.** By Theorem 5.1 we have a Camacho-Sad action on $S$ along $X$ on $N_{S^0}$. Since $[S]$ is an extension to $M$ of $N_{S^0}$, we can apply Theorem 6.1. □

**Remark 6.1.** If $M$ has dimension two, and $S$ has at least one singularity or $X_f$ has at least one zero, then $S' \setminus \text{Sing}(f)$ is always comfortably embedded in $M$. Indeed, it is an open Riemann surface; so $H^1(S' \setminus \text{Sing}(f), \mathcal{F}) = \mathbb{O}$ for any coherent analytic sheaf $\mathcal{F}$, and the result follows from Proposition 2.1 and Theorem 2.2.

In a similar way, applying [Su, Th. IV.5.6], Theorem 5.3, and recalling that $\varphi(H - L) = \varphi(H \otimes L^*)$ for any vector bundle $H$, line bundle $L$ and homogeneous symmetric polynomial $\varphi$, we get

**Theorem 6.3.** Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$, be given. Assume that $S' = S \setminus \text{Sing}(S)$ is comfortably embedded into $M$, and that $f$ is tangential to $S$ with $\nu_f > 1$. Let $\text{Sing}(S) \cup \text{Sing}(X_f) = \bigcup_{\lambda} \Sigma_{\lambda}$ be the decomposition of $\text{Sing}(S) \cup \text{Sing}(X_f)$ in connected components. Finally, let $[S]$ be the line bundle on $M$ associated to the divisor $S$. Then for any homogeneous symmetric polynomial $\varphi$ of degree $n - 1$ there exist complex numbers $\text{Res}_{\varphi}(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_{\lambda}) \in \mathbb{C}$, depending only on the local behavior of $X_f$ and $TM|_S - [S]^{\otimes \nu_f}$ near $\Sigma_{\lambda}$, such that

$$\sum_{\lambda} \text{Res}_{\varphi}(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_{\lambda}) = \int_{S} \varphi(TM|_S \otimes ([S]^*)^{\otimes \nu_f}).$$

Finally, applying the Baum-Bott index theorem (see [Su, Th. III.7.6]) and Theorem 5.2 we get

**Theorem 6.4.** Let $S$ be a compact, globally irreducible, smooth complex hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$, be given. Assume that

(a) $f$ is tangential to $S$, and $X = X_f$, or that

(b) $S^0 = S \setminus \text{Sing}(f)$ is comfortably embedded into $M$, and $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$. 
Assume moreover \( X \neq O \). Let \( \text{Sing}(X) = \bigcup \Sigma_\lambda \) be the decomposition of \( \text{Sing}(X) \) in connected components. Finally, let \( [S] \) be the line bundle on \( M \) associated to the divisor \( S \). Then for any homogeneous symmetric polynomial \( \varphi \) of degree \( n - 1 \) there exist complex numbers \( \text{Res}_\varphi(X, TS - [S]^{\otimes \nu'}, \Sigma_\lambda) \in \mathbb{C} \), depending only on the local behavior of \( X \) and \( TS - [S]^{\otimes \nu'} \) near \( \Sigma_\lambda \), such that

\[
\sum_\lambda \text{Res}_\varphi(X, TS - [S]^{\otimes \nu'}, \Sigma_\lambda) = \int_S \varphi(TS \otimes ([S]^*)^{\otimes \nu'}).
\]

Thus, we have recovered three main index theorems of foliation theory in the setting of holomorphic self-maps fixing pointwise a hypersurface.

Clearly, these index theorems are as useful as the formulas for the computation of the residues are explicit; the rest of this section is devoted to deriving such formulas in many important cases.

Let us first describe the general way these residues are defined in Lehmann-Suwa theory. Assume the hypotheses of Theorem 6.1. Let \( \tilde{U}_0 \) be a tubular neighborhood of \( S^0 \) in \( M \), and denote by \( \rho: \tilde{U}_0 \to S^0 \) the associated retraction. Given any connection \( D \) on \( E|_{S^0} \) induced by the holomorphic action \( \tilde{X} \) of \( F \) along \( X \), set \( D^0 = \rho^*(D) \). Next, choose an open set \( \tilde{U}_\lambda \subset M \) such that \( \tilde{U}_\lambda \cap \left( \text{Sing}(S) \cup \text{Sing}(X) \right) = \Sigma_\lambda \), and a compact real \( 2n \)-dimensional manifold \( \tilde{R}_\lambda \subset \tilde{U}_\lambda \) with \( C^\infty \) boundary containing \( \Sigma_\lambda \) in its interior and such that \( \partial \tilde{R}_\lambda \) intersects \( S \) transversally. Let \( D^\lambda \) be any connection on \( E|_{\tilde{U}_\lambda} \), and denote by \( B(\varphi(D^0), \varphi(D^\lambda)) \) the Bott difference form of \( \varphi(D^0) \) and \( \varphi(D^\lambda) \) on \( \tilde{U}_0 \cap \tilde{U}_\lambda \). Then (see [LS] and [Su, Chap. IV])

\[
\text{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) = \int_{R_\lambda} \varphi(D^\lambda) - \int_{\partial R_\lambda} B(\varphi(D^0), \varphi(D^\lambda)),
\]

where \( R_\lambda = \tilde{R}_\lambda \cap S \). A similar formula holds for virtual vector bundles too; see again [Su, Chap. IV].

Remark 6.2. When \( \Sigma_\lambda = \{x_\lambda\} \) is an isolated singularity of \( S \), the second integral in (6.1) is taken on the link of \( x_\lambda \) in \( S \). In particular if \( S \) is not irreducible at \( x_k \) then the residue is the sum of several terms, one for each irreducible component of \( S \) at \( x_k \).

We now specialize (6.1) to our situation. Let us begin with the Camacho-Sad action: we shall compute the residues for connected components \( \Sigma_\lambda \) reduced to an isolated point \( x_\lambda \). Let again \([S]\) be the line bundle associated to the divisor \( S \), and choose an open set \( \tilde{U}_\lambda \subset M \) containing \( x_\lambda \) so that \( \tilde{U}_\lambda \cap \left( \text{Sing}(S) \cup \text{Sing}(X) \right) = \{x_\lambda\} \) and \([S]\) is trivial on \( \tilde{U}_\lambda \); take as \( D^\lambda \) the trivial connection for \([S]\) on \( W \) with respect to some frame. In particular, then, \( \varphi(D^\lambda) = O \) on \( R_\lambda \). By (6.1) the residue is then obtained simply by integrating
Let $\eta^j$ be a connection one-form of $D^j$, for $j = 0, \lambda$; with respect to a suitable frame for $[S]$ we can assume that $\eta^\lambda \equiv 0$. Let

$$\tilde{\eta} := t \eta^0 + (1 - t) \eta^\lambda = t \eta^0,$$

and let $\tilde{K} := d\tilde{\eta} + \tilde{\eta} \wedge \tilde{\eta} = d\tilde{\eta}$. From the very definition of the Bott difference form, it follows that

$$B(\varphi(D^0), \varphi(D^\lambda)) = \left(\frac{1}{2\pi i}\right)^n \int_0^1 \tilde{K}^{n-1}. $$

A straightforward computation shows that

$$\tilde{K}^{n-1} = (n-1)t^{n-2}dt \wedge \eta^0 \wedge d\eta^0 \wedge \cdots \wedge d\eta^0 + N,$$

where $N$ is a term not containing $dt$. Therefore

$$B(\varphi(D^0), \varphi(D^\lambda)) = \left(\frac{1}{2\pi i}\right)^n \eta^0 \wedge d\eta^0 \wedge \cdots \wedge d\eta^0. $$

Assume now that $x_\lambda \in \text{Sing}(X)$ and $S$ is smooth at $x_\lambda$. Up to shrinking $U_\lambda$ we may assume that $U_\lambda$ is the domain of a chart $z$ adapted to $S$ (and belonging to a comfortable atlas if necessary), so that $\{\partial_1\}$ is a local frame for $N_{S^0}$, and $\{dz^2, \ldots, dz^n\}$ is a local frame for $T^*S^0$. Then any connection $D$ induced by the Camacho-Sad action is locally represented by the $(1,0)$-form $\eta^0$ such that $D(\partial_1) = \eta^0 \otimes \partial_1$. To compute $\eta^0$, we first of all notice that $X = g^p \frac{\partial}{\partial z^p} \otimes (\omega^1)^{\otimes \nu_f}$, if $X = X_f$ or $X = H_{\sigma,f}$, and $X = (1 + b^1)g^p \frac{\partial}{\partial z^p} \otimes \omega^1$, if $X = H_{\sigma,f}^1$. Then, when $X$ is $X_f$ or $H_{\sigma,f}$,

$$(X^*)^{-1}(\omega^1)^{\otimes \nu_f} = \frac{1}{g^p} dz^p,$$

while when $X = H_{\sigma,f}^1$,

$$(X^*)^{-1}(\omega^1)^{\otimes \nu_f} = \frac{1}{(1 + b^1)g^p} dz^p.$$

Therefore, recalling formulas (5.6) and (5.7), we can choose $D$ so that when $X$ is $X_f$ or $H_{\sigma,f}$,

$$\eta^0 = (X^* \otimes \text{id})^{-1}(\tilde{X}(\partial_1)) = -\frac{h^1}{g^p} \bigg|_S dz^p,$$

while when $X = H_{\sigma,f}^1$,

$$\eta^0 = (X^* \otimes \text{id})^{-1}(\tilde{H}_{\sigma,f}^1(\partial_1)) = -\frac{h^1}{(1 + b^1)g^p} \bigg|_S dz^p.$$
Remark 6.3. When \( n = 2 \) and \( X = X_f \) we recover the connection form obtained in [Br]. The form \( \eta \) introduced in [A2], which is the opposite of \( \eta^0 \), is the connection form of the dual connection on \( N_\Sigma \), by [A2, (1.7)]. Since the definition of Chern class implicitly used in [A2] is the opposite of the one used in [Br] everything is coherent. Finally, when \( n = 2 \) and \( X = H^1_{\sigma,f} \) we have obtained the correct multiple of the form \( \eta \) introduced in [A2] when \( S \) was the smooth zero section of a line bundle (notice that \( 1 + b^1 \) is constant because \( S \) is compact, and that the form \( \eta \) of [A2] must be divided by \( b = 1 + b^1 \) to get a connection form).

Now we can take \( R_1 = \{ |g^p(x)| \leq \varepsilon \mid p = 2, \ldots, n \} \) for a suitable \( \varepsilon > 0 \) small enough. In particular, if we set \( \Gamma = \{ |g^p(x)| = \varepsilon \mid p = 2, \ldots, n \} \cap S \), oriented so that \( d\theta^2 \wedge \cdots \wedge d\theta^n > 0 \) where \( \theta^p = \arg(g^p) \), then arguing as in [L, §5] or [LS, §4] (see also [Su, pp.105–107]) we obtain

\[
\text{Res}(X, S, \{x_\lambda\}) = \left( -\frac{i}{2\pi} \right)^{n-1} \int_{\Gamma} \frac{\lambda^1}{g^2 \cdots g^n} \, dz^2 \wedge \cdots \wedge dz^n,
\]

when \( X = X_f \) or \( H_{\sigma,f} \), while when \( X = H^1_{\sigma,f} \) we have

\[
\text{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \left( -\frac{i}{2\pi} \right)^{n-1} \int_{\Gamma} \frac{\lambda^1}{(1 + b^1)^{n-1} g^2 \cdots g^n} \, dz^2 \wedge \cdots \wedge dz^n.
\]

Remark 6.4. For \( n = 2 \), formulas (6.5) and (6.6) give the indices defined in [A2]. Thus, if \( S \) is smooth, Theorem 6.2 implies the index theorem of [A2], because \( c_1([S]) = c_1(N_S) \). In an analogous way, Lehmann and Suwa (see [L], [LS], [LS2]) proved that the Camacho-Sad index theorem also is a consequence of Theorem 6.1.

When \( x_\lambda \) is an isolated singular point of \( S \) the computation of the residue is more complicated, because one cannot apply directly the results in [LS] as before, for in general there is no natural extension of \( \Xi_X \) and the Camacho-Sad action to \( \text{Sing}(S) \). However we are able to compute explicitly the index in this case too when \( n = 2 \), and when \( n > 2 \) and \( f \) is tangential with \( \nu_f > 1 \).

If \( n = 2 \) we can choose local coordinates \( \{(w^1, w^2)\} \) in \( \tilde{U}_\lambda \) so that \( S \cap \tilde{U}_\lambda = \{ l(w^1, w^2) = 0 \} \) for some holomorphic function \( l \), and \( dl \wedge dw^2 \neq 0 \) on \( S \cap \tilde{U}_\lambda \setminus \{x_\lambda\} \). In particular \( (l, w^2) \) are local coordinates adapted to \( S^0 \) near \( S \cap \tilde{U}_\lambda \setminus \{x_\lambda\} \) and \( \frac{\partial}{\partial \bar{w}^2} \) can be chosen as a local frame for \( N_{S^0} \) on \( \partial R_1 \).

Remark 6.5. When \( S^0 \) is comfortably embedded in \( M \) the chart \( (l, w^2) \) should belong to a comfortable atlas. Studying the proofs of Propositions 2.1 and of Theorem 2.2 one sees that this is possible up to replacing \( l \) by a function of the form \( \tilde{l} = (1 + c(w^2))l \), where \( c \) is a holomorphic function defined on \( S \cap \tilde{U}_\lambda \setminus \{x_\lambda\} \). Since to compute the residues we only need the behavior of \( l \) and
$w^2$ near $\partial R_1$, it is easy to check that using $\hat{l}$ or $l$ in the following computations yields the same results. So for the sake of simplicity we shall not distinguish between $l$ and $\hat{l}$ in the sequel.

Up to shrinking $\bar{U}_\lambda$, we can again assume that $[S]$ is trivial on $\bar{U}_\lambda$. The function $l$ is a local generator of $\mathcal{I}_S$ on $\bar{U}_\lambda$. Then the dual of $[l] \in \mathcal{I}_S/\mathcal{I}_S^2$, denoted by $s$, is a holomorphic frame of $[S]$ on $\bar{U}_\lambda$ which extends the holomorphic frame $\frac{\partial}{\partial \tau}$ of $N_S$ (see [Su, p.86]). In particular $s|_{\partial R_1} = \frac{\partial}{\partial \tau}$. We then choose $s$ on $[S]|_{\hat{U}_\lambda}$ the trivial connection with respect to $s$, so that $\eta^1 = O$. We are left with the computation of the form $\eta^0$ near $\partial R^1$. But if $X = X_f$ or $X = H_{\sigma,f}$ we can apply (6.3) to get

$$\eta^0|_{\partial R_1} = \left( \frac{l}{l \cdot (w^2 \circ f - w^2)} \right)_{\partial R_1} \frac{b^1 l}{l \cdot (w^2 \circ f - w^2)} \frac{\partial}{\partial R_1} dw^2,$$

where

$$b^1 = \left. \frac{l \circ f - l}{l \nu f} \right|_S$$

is identically zero when $f$ is tangential. On the other hand, when $X = H^1_{\sigma,f}$, applying (6.4) we get

$$\eta^0|_{\partial R_1} = \left( \frac{l \circ f - l}{l + (l \circ f - l)(w^2 \circ f - w^2)} \right)_{\partial R_1} \frac{b^1 l}{l + (l \circ f - l)(w^2 \circ f - w^2)} \frac{\partial}{\partial R_1} dw^2.$$  

Hence the residue is

$$(6.7) \quad \text{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \frac{(l \circ f - l) - b^1 l}{l + (l \circ f - l)(w^2 \circ f - w^2)} \left. \frac{\partial}{\partial R_1} \right|_S dw^2,$$

when $X = X_f$ or $X = H_{\sigma,f}$, while when $X = H^1_{\sigma,f}$,

$$(6.8) \quad \text{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \frac{(l \circ f - l) - b^1 l}{l + (l \circ f - l)(w^2 \circ f - w^2)} \left. \frac{\partial}{\partial R_1} \right|_S dw^2.$$

**Remark 6.6.** When $f$ is tangential we have $b^1 \equiv 0$; therefore the formula (6.7) gives the index defined in [BT], and Theorem 6.2 implies the index theorem of [BT].

When $n > 2$, $f$ is tangential and $\nu_f > 1$, we can define a local vector field $\tilde{v}_f$ which generates the Camacho-Sad action $\tilde{X}_f$ and compute explicitly the residue even at a singular point $x_\lambda$ of $S$. To see this, assume $(w^1, \ldots, w^n)$ are local coordinates in $\bar{U}_\lambda$ so that $S \cap \bar{U}_\lambda = \{l(w^1, \ldots, w^n) = 0\}$ for some holomorphic function $l$. Define the vector field $\tilde{v}_f$ on $\bar{U}_\lambda$ by

$$\tilde{v}_f = \frac{w^1 \circ f - w^1}{l \nu_f} \frac{\partial}{\partial w^1} + \ldots + \frac{w^n \circ f - w^n}{l \nu_f} \frac{\partial}{\partial w^n}.$$
We claim that the “holomorphic action” \( \theta_{\tilde{v}_f} \) in the sense of Bott [Bo] of \( \tilde{v}_f \) on \( N_{S^\lambda} \) as defined in [LS, p.177] coincides with our Camacho-Sad action, and thus we can apply [LS, Th. 1] to compute the residue. To prove this we consider \( W_1 = \{ x \in \tilde{U}_\lambda | \frac{\partial l}{\partial w^1}(x) \neq 0 \} \). On this open set we make the following change of coordinates:

\[
\begin{align*}
  z^1 &= l(w^1, \ldots, w^n), \\
  z^p &= w^p & \text{for} \ p = 2, \ldots, n.
\end{align*}
\]

The new coordinates \((z^1, \ldots, z^n)\) are adapted to \( S \) on \( W_1 \). If \( f^j = z^j + g^j(z^1)^{\nu_j} \) as usual, we have

\[(6.10)\]

\[w^p \circ f - w^p = g^p(z^1)^{\nu_j},\]

and

\[(6.11)\]

\[w^1 \circ f - w^1 = \frac{\partial w^1}{\partial z^1}(z^1)^{\nu_j} + R_{2w^1} = \left( \frac{\partial l}{\partial w^1} \right)^{-1} \left[ g^1 - \frac{\partial l}{\partial w^p} \right](z^1)^{\nu_j} + R_{2w^1}.
\]

Therefore, from (6.10) and (6.11), taking into account that \( \nu_j > 1 \), we get

\[(6.12)\]

\[
\tilde{v}_f = \left( \frac{w^1 \circ f - w^1}{(z^1)^{\nu_j}} + \frac{w^p \circ f - w^p}{(z^1)^{\nu_j}} \right) \frac{\partial l}{\partial z^1} + \frac{w^p \circ f - w^p}{(z^1)^{\nu_j}} \frac{\partial}{\partial z^1} \\
+ \frac{w^1 \circ f - w^1}{(z^1)^{\nu_j}} \frac{\partial}{\partial z^1} = X_f(\partial_1^{\otimes \nu_j}) + R_2,
\]

which gives the claim on \( W_1 \). Since the same holds on each \( W_j = \{ x \in \tilde{U}_\lambda | \frac{\partial l}{\partial w^1}(x) \neq 0 \} \), \( j = 1, \ldots, n \), and \( (\tilde{U}_\lambda \cap S) \setminus \{ x_\lambda \} = \bigcup_j W_j \), it follows that the Bott holomorphic action induced by \( \tilde{v}_f \) is the same as the Camacho-Sad action given by \( \tilde{X}_f \). Thus, if we choose — as we can — the coordinates \((w^1, \ldots, w^n)\) as in [LS, Th. 2], that is so that \( \{ l, (w^p \circ f - w^p)/l^{\nu_j} \} \) form a regular sequence at \( x_\lambda \), the residue is expressed by the formula after [LS, Th. 2].

Taking into account that, since \( f \) is tangential and by (6.13), the function \( l \) divides \( dl(\tilde{v}_f) \), we get

\[(6.13)\]

\[\text{Res}(X_f, S, \{ x_\lambda \}) = \left( \frac{-i}{2\pi \epsilon} \right)^{n-1} \int_\Gamma \frac{\left[ \sum_{j=1}^n \frac{\partial l}{\partial w^j}(w^j \circ f - w^j) \right]^{n-1}}{l^{n-1} \prod_{p=2}^n (w^p \circ f - w^p)} \, dw^2 \wedge \cdots \wedge dw^n,
\]

where this time

\[\Gamma = \left\{ w \in \tilde{U}_\lambda \left| \left| \frac{w^p \circ f - w^p}{l^{\nu_j}}(w) \right| = \epsilon, \ l(w) = 0 \right\} \right.,
\]

for a suitable \( 0 < \epsilon << 1 \), and \( \Gamma \) is oriented as usual (in particular \( \Gamma = (-1)^{\frac{\nu_j}{2}} R_{w^0} \) where \( R_{w^0} \) is the set defined in [LS, Th. 2]).

Note that for \( n = 2 \) we recover, when \( \nu_j > 1 \), formula (6.7). On the other hand, if \( x_\lambda \) is nonsingular for \( S \), then the previous argument with \( l = w^1 \) works for \( \nu_j = 1 \) as well, and we get formula (6.5).

Summing up, we have proved the following:
Theorem 6.5. Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$, be given. Assume that

(a) $f$ is tangential to $S$, and $X = X_f$, or that

(b) $S^0 = S \setminus (\text{Sing}(S) \cup \text{Sing}(f))$ is comfortably embedded into $M$, and $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.

Assume $X \not\equiv O$. Let $x_\lambda \in S$ be an isolated point of $\text{Sing}(S) \cup \text{Sing}(X)$. Then the number $\text{Res}(X, S, \{x_\lambda\}) \in \mathbb{C}$ introduced in Theorem 6.2 is given

(i) if $x_\lambda \in \text{Sing}(X) \cap (S \setminus \text{Sing}(S))$, and $f$ is tangential or $S^0$ is comfortably embedded in $M$ and $\nu_f > 1$, by

$$\text{Res}(X, S, \{x_\lambda\}) = \left(-\frac{i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{(h^1)^{n-1} dz^2 \wedge \cdots \wedge dz^n}{g^2 \cdots g^n};$$

(ii) if $x_\lambda \in \text{Sing}(X) \cap (S \setminus \text{Sing}(S))$, $S^0$ is comfortably embedded in $M$ and $\nu_f = 1$, by

$$\text{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \left(-\frac{i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{(h^1)^{n-1} dz^2 \wedge \cdots \wedge dz^n}{(1 + b^1)^{n-1}g^2 \cdots g^n};$$

(iii) if $n = 2$, $x_\lambda \in \text{Sing}(S)$, and $f$ is tangential or $S^0$ is comfortably embedded in $M$ and $\nu_f > 1$, by

$$\text{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \frac{(l \circ f - l - b^1l^{\nu_f})}{l \cdot (w^2 \circ f - w^2)^{\nu_f}} \left|_{S}\right. dw^2;$$

(iv) if $n = 2$, $x_\lambda \in \text{Sing}(S)$, $S^0$ is comfortably embedded in $M$ and $\nu_f = 1$, by

$$\text{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \frac{(l \circ f - l - b^1l)}{(l + (l \circ f - l))(w^2 \circ f - w^2)} \left|_{S}\right. dw^2;$$

(v) if $n > 2$, $x_\lambda \in \text{Sing}(S)$, $f$ is tangential and $\nu_f > 1$, by

$$\text{Res}(X_f, S, \{x_\lambda\}) = \left(-\frac{i}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{\sum_{j=1}^n \frac{\partial}{\partial w^j}(w^j \circ f - w^j)^{n-1}}{l^{n-1} \prod_{\nu=2}^{n-2}(w^\nu \circ f - w^\nu)} dw^2 \wedge \cdots \wedge dw^n.$$

Our next aim is to compute the residue for the Lehmann-Suwa action, at least for an isolated smooth point $x_\lambda \in \text{Sing}(X_f)$. Let $(W, w)$ be a local chart about $x_\lambda$ belonging to a comfortable atlas. Set $l = w^1$ and define $\tilde{v}_f$ as in (6.9). By (6.13) the Lehmann-Suwa action $\tilde{V}$ is given by the holomorphic action (in
the sense of Bott) of $\tilde{v}_f$ on $TM|_S - [S]^{\otimes \nu_f}$. Therefore we can apply [L], [LS] (see also [Su, Ths. IV.5.3, IV.5.6], and [Su, Remark IV.5.7]) to obtain

$$\text{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \text{Res}_\varphi(X_f, TM|_S, \{x_\lambda\}),$$

where $\text{Res}_\varphi(X_f, TM|_S, \{x_\lambda\})$ is the residue for the local Lie derivative action of $\tilde{v}_f$ on $TM|_S$ given by

$$\tilde{V}_I(s)(\tilde{v}_f) = [\tilde{v}_f, \tilde{s}]|_S,$$

where $s$ is a section of $TM|_S$ and $\tilde{s}$ is a local extension of $s$ constant along the fibers of $\sigma$.

We can write an expression of $\tilde{V}_I$ in local coordinates. Let $(U, z)$ be a local chart belonging to a comfortable atlas. Then $\{(\omega^1)^{\otimes \nu_f} \otimes \frac{\partial}{\partial z^1}|_S, \ldots, (\omega^1)^{\otimes \nu_f} \otimes \frac{\partial}{\partial z^n}|_S\}$ is a local frame for $(N_S^{\otimes \nu_f})^* \otimes TM|_S$. Thus there exist holomorphic functions $V^k_j$ (for $j, k = 1, \ldots, n$) so that

$$\tilde{V}_I(\frac{\partial}{\partial z^j})(\partial_1^{\otimes \nu_f}) = V^k_j \frac{\partial}{\partial z^k}.$$

Now, from (4.4) we get

$$\tilde{V}_I(\frac{\partial}{\partial z^j})(\partial_1^{\otimes \nu_f}) = \left[ X_f \left( (\frac{\partial}{\partial z^1})^{\otimes \nu_f} \right), \frac{\partial}{\partial z^j} \right]_S\bigg|_S = \left( h^1_1 z^1 \frac{\partial}{\partial z^1} + g^p_1 \frac{\partial}{\partial z^p} \right)_S = -h^1_1 |_S \delta^1_j \frac{\partial}{\partial z^1} - \frac{\partial g^p_1}{\partial z^j} |_S \frac{\partial}{\partial z^p},$$

and hence

$$(6.14) \quad V^1_1 = -h^1_1 |_S, \quad V^1_p \equiv 0, \quad V^p_j = -\frac{\partial g^p_1}{\partial z^j} |_S.$$

Therefore [Su, Th. IV.5.3] yields

**Theorem 6.6.** Let $S$ be a compact, globally irreducible, possibly singular hypersurface in an $n$-dimensional complex manifold $M$. Let $f \in \text{End}(M, S)$, $f \neq \text{id}_M$, be given. Assume that $S' = S \setminus \text{Sing}(S)$ is comfortably embedded into $M$, and that $f$ is tangential to $S$ with $\nu_f > 1$. Let $x_\lambda \in \text{Sing}(S) \cup \text{Sing}(X_f)$. Then for any homogeneous symmetric polynomial $\varphi$ of degree $n - 1$ the complex number

$$\text{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\})$$

introduced by Theorem 6.3 is given by

$$(6.15) \quad \text{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \int_\Gamma \varphi(V) dz^2 \wedge \cdots \wedge dz^n = \frac{g^2 \cdots g^n}{g(\nu_f - 1)}$$

where $V = (V^k_j)$ is the matrix given by (6.14) and $\Gamma$ is as in (6.5).
Remark 6.7. We adopt here the convention that if $V$ is an $n \times n$ matrix then $c_j(V)$ is the $j^{th}$ symmetric function of the eigenvalues $V$ multiplied by $(i/2\pi)^j$, and $\varphi(V) = \varphi(c_1(V), \ldots, c_{n-1}(V))$.

Finally, if $x_\lambda$ is an isolated point in $\text{Sing}(X)$, the complex numbers $\text{Res}_\varphi(X, TS - [S]^\otimes \nu_f, \{x_\lambda\})$ appearing in Theorem 6.4 can be computed exactly as in the foliation case using a Grothendieck residue with a formula very similar to (6.15); see [BB], [Su, Th. III.5.5].

7. Index theorems in higher codimension

Let $S \subset M$ be a complex submanifold of codimension $1 < m < n$ in a complex $n$-manifold $M$. A way to get index theorems for holomorphic self-maps of $M$ fixing pointwise $S$ is to blow-up $S$ and then apply the index theorems for hypersurfaces; this is what we plan to do in this section.

We shall denote by $\pi: M_S \to M$ the blow-up of $M$ along $S$, and by $E_S = \pi^{-1}(S)$ the exceptional divisor, which is a hypersurface in $M_S$ isomorphic to the projectivized normal bundle $\mathbb{P}(N_S)$. 

Remark 7.1. If $S$ is singular, the blow-up $M_S$ is in general singular too. So this approach works only for smooth submanifolds.

If $(U, z)$ is a chart adapted to $S$ centered in $p \in S$, in $M_S$ we have $m$ charts $(\tilde{U}_r, w_r)$ centered in $[\partial_1], \ldots, [\partial_m]$ respectively, where if $v \in N_{S,p}$, $v \neq 0$, we denote by $[v]$ its projection in $\mathbb{P}(N_S)$. The coordinates $z^j$ and $w^h_r$ are related by

$$z^j(w_r) = \begin{cases} w^j_r & \text{if } j = r, m + 1, \ldots, n, \\ w^r_r w^j_r & \text{if } j = 1, \ldots, r - 1, r + 1, \ldots, m. \end{cases}$$

Remark 7.2. We have $\tilde{U}_r \cap E_S = \{w^r_r = 0\}$, and thus $(\tilde{U}_r, w_r)$ is adapted to $E_S$ up to a permutation of the coordinates.

Now take $f \in \text{End}(M, S)$, $f \neq \text{id}_M$, and assume that $df$ acts as the identity on $N_S$ (this is automatic if $\nu_f > 1$, while if $\nu_f = 1$ it happens if and only if $f$ is tangential). Then we can lift $f$ to a unique map $\tilde{f} \in \text{End}(M_S, E_S)$, $\tilde{f} \neq \text{id}_{M_S}$, such that $f \circ \pi = \pi \circ \tilde{f}$ (see, e.g., [A1] for details). If $(U, z)$ is a chart adapted to $S$ and we set $f^j = z^j \circ f$ and $\tilde{f}^j_r = w_r^j \circ \tilde{f}$,

$$\tilde{f}^j_r(w_r) = \begin{cases} f^j_r(z(w_r)) & \text{if } j = r, m + 1, \ldots, n, \\ f^j_r(z(w_r)) & \text{if } j = r, m - 1, r + 1, \ldots, m. \end{cases}$$
If \( f \) is tangential we can find holomorphic functions \( h^r_{r_1 \ldots r_{r_f+1}} \) symmetric in the lower indices such that
\[
(7.2) \quad f^r - z^r = h^r_{r_1 \ldots r_{r_f+1}} z^{r_1} \cdots z^{r_{r_f+1}} + R_{r_f+2};
\]
as usual, only the restriction to \( S \) of each \( h^r_{r_1 \ldots r_{r_f+1}} \) is uniquely defined. Set then
\[
Y = h^r_{r_1 \ldots r_{r_f+1}}|_{S} \partial_r \otimes \omega^{r_1} \otimes \cdots \otimes \omega^{r_{r_f+1}};
\]
it is a local section of \( N_S \otimes (N^*_S)^{\otimes_1} \).

On the other hand, if \( f \) is not tangential set \( B = (\pi \otimes \text{id})_* \circ X_f \), where \( \pi : TM|_S \to N_S \) is the canonical projection. In this way we get a global section of \( N_S \otimes (N^*_S)^{\otimes_1} \), not identically zero if and only if \( f \) is not tangential, and given in local adapted coordinates by
\[
B = g^r_{r_1 \ldots r_{r_f}}|_{S} \partial_r \otimes \omega^{r_1} \otimes \cdots \otimes \omega^{r_{r_f}}.
\]

**Definition 7.1.** Take \( p \in S \). If \( f \) is tangential, a non-zero vector \( v \in (N_S)_p \) is a **singular direction** for \( f \) at \( p \) if \( X_f(v \otimes \cdots \otimes v) = 0 \) and \( Y(v \otimes \cdots \otimes v) \wedge v = O \). If \( f \) is not tangential, \( v \) is a **singular direction** if \( B(v \otimes \cdots \otimes v) \wedge v = O \).

**Remark 7.3.** The condition \( Y(v \otimes \cdots \otimes v) \wedge v = O \) is equivalent to requiring \( Y(v \otimes \cdots \otimes v) = \lambda v \) for some \( \lambda \in \mathbb{C} \).

Of course, in the tangential case we must check that this definition is well-posed, because the morphism \( Y \) depends on the local coordinates chosen to define it. First of all, if \( (U, z) \) is a chart adapted to \( S \) and centered at \( p \) then \( X_f(v \otimes \cdots \otimes v) = O \) when \( f \) is tangential means
\[
(7.3) \quad g^p_{r_1 \ldots r_{r_f}}(O) v^{r_1} \cdots v^{r_{r_f}} \frac{\partial}{\partial z^p} = O,
\]
where \( v = v^r \partial_r \). Now let \( (\hat{U}, \hat{z}) \) be another chart adapted to \( S \) centered in \( p \). Then we can find holomorphic functions \( a^r_s \) and \( \hat{a}^r_s \) such that \( \hat{z}^r = a^r_s z^s \) and \( z^r = \hat{a}^r_s \hat{z}^s \). Arguing as in the proof of (4.2) we get
\[
a^r_{s_1} \cdots a^r_{s_{r_f+1}} h^r_{r_1 \ldots r_{r_f+1}} = a^r_s h^s_{s_1 \ldots s_{r_f+1}} + \sum_{\ell=1}^{\nu_{r_f+1}} \frac{\partial a^r_s}{\partial z^p} g^p_{s_1 \ldots s_{r_f+1}} + R_1,
\]
where the index with the hat is missing from the list. Therefore
\[
\hat{Y} = Y + \hat{a}^r_s \sum_{\ell=1}^{\nu_{r_f+1}} \frac{\partial a^r_s}{\partial z^p} g^p_{s_1 \ldots s_{r_f+1}} \big|_{S} \partial_s \otimes \omega^{s_1} \otimes \cdots \otimes \omega^{s_{r_f+1}};
\]
in particular if \( X_f(v \otimes \cdots \otimes v) = O \) equation (7.3) yields
\[
\hat{Y}(v \otimes \cdots \otimes v) = Y(v \otimes \cdots \otimes v),
\]
and the notion of singular direction when \( f \) is tangential is well-defined.
Proposition 7.1. Let $S \subset M$ be a complex submanifold of codimension $1 < m < n$ of a complex $n$-manifold $M$, and take $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$, such that $df$ acts as the identity on $N_S$ (that is $f$ is tangential, or $\nu_f > 1$, or both). Denote by $\pi: M_S \to M$ the blow-up of $M$ along $S$ with exceptional divisor $E_S$, and let $\tilde{f} \in \text{End}(M_S, E_S)$ be the lifted map. Then:

(i) if $S$ is comfortably embedded in $M$ then $E_S$ is comfortably embedded in $M_S$, and the choice of a splitting morphism for $S$ induces a splitting morphism for $E_S$;

(ii) $d\tilde{f}$ acts as the identity on $N_{E_S}$;

(iii) $\tilde{f}$ is always tangential; furthermore $\nu_{\tilde{f}} = \nu_f$ if $f$ is tangential, $\nu_{\tilde{f}} = \nu_f - 1$ otherwise;

(iv) a direction $[v] \in E_S$ is a singular point for $\tilde{f}$ if and only if it is a singular direction for $f$.

Proof. (i) Let $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$ be a comfortable atlas adapted to $S$; we claim that $\tilde{\mathcal{U}} = \{(\tilde{U}_{\alpha,r}, w_{\alpha,r})\}$ is a comfortable atlas adapted to $E_S$ (and in particular determines a splitting morphism for $E_S$). Let us first prove that it is a splitting atlas, that is that

$$\frac{\partial w^j_{\alpha,r}}{\partial w^s_{\beta,r}} \bigg|_{E_S} \equiv 0$$

for every $r$, $s$, $j \neq s$ and indices $\alpha$ and $\beta$. We have

$$z^j_\beta = z^j_\beta|_S + \frac{\partial z^j_\beta}{\partial z^s_\alpha}|_S z^s_\alpha + \frac{1}{2} \frac{\partial^2 z^j_\beta}{\partial z^u_\alpha \partial z^v_\alpha}|_S z^u_\alpha z^v_\alpha + R_3.$$ 

Since $w^r_{\alpha,r} = z^r_\alpha$, if $j = p > m$ we immediately get

$$\frac{\partial w^p_{\beta,s}}{\partial w^r_{\alpha,r}} \bigg|_{E_S} = \frac{\partial z^p_\beta}{\partial z^r_\alpha}|_S \equiv 0,$$

because $\tilde{\mathcal{U}}$ is a splitting atlas. If $j = t \leq m$,

$$z^t_\beta = \frac{\partial z^t_\beta}{\partial z^s_\alpha}|_S z^s_\alpha + \frac{1}{2} \frac{\partial^2 z^t_\beta}{\partial z^u_\alpha \partial z^v_\alpha}|_S z^u_\alpha z^v_\alpha + R_3$$

$$= \left[ \frac{\partial z^t_\beta}{\partial z^s_\alpha}|_S + \sum_{u \neq r} \frac{\partial z^t_\beta}{\partial z^u_\alpha}|_S w^u_{\alpha,r} \right] w^r_{\alpha,r} + O((w^r_{\alpha,r})^3),$$

because $\tilde{\mathcal{U}}$ is a comfortable atlas. Therefore if $t \neq s$,

$$w^t_{\beta,s} = \frac{z^t_\beta}{z^s_\beta} = \frac{\partial z^t_\beta}{\partial z^s_\alpha}|_S + \sum_{u \neq r} \frac{\partial z^t_\beta}{\partial z^u_\alpha}|_S w^u_{\alpha,r} + O((w^r_{\alpha,r})^2)$$

$$= \left[ \frac{\partial z^t_\beta}{\partial z^s_\alpha}|_S + \sum_{u \neq r} \frac{\partial z^t_\beta}{\partial z^u_\alpha}|_S w^u_{\alpha,r} + O((w^r_{\alpha,r})^2) \right] w^r_{\alpha,r} + O((w^r_{\alpha,r})^3),$$

which gives the desired result.
Then

$$v \in (N_S)_{p}, v \neq O, \text{ and a chart } (U, z) \text{ adapted to } S \text{ centered in } p.$$ Then $v = v^s \partial_s$, with $v^s \neq 0$ for some $r$. Hence $[v] \in \hat{U}_r$ has coordinates

$$w^j([v]) = \begin{cases} 0 & \text{if } j = r, m + 1, \ldots, n, \\ v^j/v^r & \text{if } j = 1, \ldots, r - 1, r + 1, \ldots, m. \end{cases}$$

If $f$ is not tangential, then $[v]$ is a singular point for $\tilde{f}$ if and only if

$$[v^r g_{r_1 \ldots r_f}^{\ast} (O) - v^s g_{r_1 \ldots r_f}^{\ast} (O)] v^{r_1} \cdots v^{r_{r_f}} = 0$$

for all $s$, and thus if and only if $B(v \otimes \cdots \otimes v) \wedge v = O$, as claimed.
If \( f \) is tangential, writing \( f^a - z^a \) as in (7.2) we get
\[
\hat{f}_r^s(w_r) = w_r^s + (w_r^s)^{\nu_r} \left[ h_{r_1 \ldots r_{\nu_2+1}}^s(z(w_r)) - w_r^s h_{r_1 \ldots r_{\nu_2+1}}^r(z(w_r)) \right] w_{r_2}^{\nu_2} \cdots w_{r_1}^{\nu_1} + O((w_r^s)^{\nu_r+1})
\]
for \( s \neq r \), and then it is clear that \([v]\) is a singular point for \( \hat{f} \) if and only if \( v \) is a singular direction for \( f \).

We therefore get index theorems in any codimension:

**Theorem 7.2.** Let \( S \) be a compact complex submanifold of codimension \( 1 < m < n \) in an \( n \)-dimensional complex manifold \( M \). Let \( f \in \text{End}(M, S) \), \( f \not\equiv \text{id}_M \), be given, and assume that \( df \) acts as the identity on \( N_S \). Let \( \bigcup \Sigma_\lambda \) be the decomposition in connected components of the set of singular directions for \( f \) in \( \mathbb{P}(N_S) \). Then there exist complex numbers \( \text{Res}(f, S, \Sigma_\lambda) \in \mathbb{C} \), depending only on the local behavior of \( f \) and \( S \) near \( \Sigma_\lambda \), such that

\[
\sum_\lambda \text{Res}(f, S, \Sigma_\lambda) = \int_{E_S} c_1^{n-1}([E_S]) = \int_S \pi_* c_1^{n-1}([E_S]),
\]

where \( E_S \) is the exceptional divisor in the blow-up \( \pi : M_S \to M \) of \( M \) along \( S \), and \( \pi_* \) denotes the integration along the fibers of the bundle \( \pi|_{E_S} : E_S \to S \).

**Proof.** This follows immediately from Theorem 6.2, Proposition 7.1, and the projection formula (see, e.g., [Su, Prop. II.4.5]).

**Remark 7.4.** The restriction to \( E_S \) of the cohomology class \( c_1([E_S]) \) is the Chern class \( \zeta = c_1(T) \) of the tautological bundle \( T \) on the bundle \( \pi|_{E_S} : E_S \to S \) and it satisfies the relation

\[
\zeta^{n-m} - \pi|_{E_S} c_1(N_S) \zeta^{n-m-1} + \pi|_{E_S}^* c_2(N_S) \zeta^{n-m-2} + \cdots + (-1)^{n-m} \pi|_{E_S}^* c_{n-m}(N_S) = 0
\]

in the cohomology ring of the bundle (see, e.g., [GHI, pp. 606–608]). This formula can sometimes be used to compute \( \zeta \) in terms of the Chern classes of \( N_S \) and \( TM \) in specific examples.

**Theorem 7.3.** Let \( S \) be a compact complex submanifold of codimension \( 1 < m < n \) in an \( n \)-dimensional complex manifold \( M \). Let \( f \in \text{End}(M, S) \), \( f \not\equiv \text{id}_M \), be given, and set \( \nu = \nu_f \) if \( f \) is tangential, and \( \nu = \nu_f - 1 \) otherwise. Assume that \( S \) is comfortably embedded into \( M \), and that \( \nu > 1 \). Let \( \bigcup \Sigma_\lambda \) be the decomposition in connected components of the set of singular directions for \( f \) in \( \mathbb{P}(N_S) \). Finally, let \( \pi : M_S \to M \) be the blow-up of \( M \) along \( S \), with exceptional divisor \( E_S \). Then for any homogeneous symmetric polynomial \( \varphi \) of degree \( n - 1 \) there exist complex numbers \( \text{Res}_\varphi(f, TM|_{E_S} - N_{E_S}^{\otimes \nu}, \Sigma_\lambda) \in \mathbb{C} \),

depending only on the local behavior of $f$ and $T M|_{E_S} - N_{E_S}^{\otimes \mu}$ near $\Sigma_{\lambda}$, such that

$$\sum_{\lambda} \text{Res}_{\varphi}(f, T M|_{E_S} - N_{E_S}^{\otimes \mu}, \Sigma_{\lambda}) = \int_{S} \pi_* \varphi(T M|_{E_S} \otimes (N_{E_S}^{*})^{\otimes \mu}),$$

where $\pi_*$ denotes the integration along the fibers of the bundle $E_S \to S$.

**Proof.** This follows immediately from Theorem 6.3, Proposition 7.1, and the projection formula.

**Theorem 7.4.** Let $S$ be a compact complex submanifold of codimension $1 < m < n$ in an $n$-dimensional complex manifold $M$. Let $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$, be given, and assume that $df$ acts as the identity on $N_S$. Set $\nu = \nu_f$ if $f$ is tangential, and $\nu = \nu_f - 1$ otherwise. Let $\bigcup_{\lambda} \Sigma_{\lambda}$ be the decomposition in connected components of the set of singular directions for $f$ in $\mathbb{P}(N_S)$. Finally, let $\pi : M_S \to M$ be the blow-up of $M$ along $S$, with exceptional divisor $E_S$. Then for any homogeneous symmetric polynomial $\varphi$ of degree $n - 1$ there exist complex numbers $\text{Res}_{\varphi}(f, T E_S - N_{E_S}^{\otimes \mu}, \Sigma_{\lambda}) \in \mathbb{C}$, depending only on the local behavior of $f$ and $T E_S - N_{E_S}^{\otimes \mu}$ near $\Sigma_{\lambda}$, such that

$$\sum_{\lambda} \text{Res}_{\varphi}(f, T E_S - N_{E_S}^{\otimes \mu}, \Sigma_{\lambda}) = \int_{S} \pi_* \varphi(T E_S \otimes (N_{E_S}^{*})^{\otimes \mu}),$$

where $\pi_*$ denotes the integration along the fibers of the bundle $E_S \to S$.

**Proof.** This follows immediately from Theorem 6.4, Proposition 7.1, and the projection formula.

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**8. Applications to dynamics**

We conclude this paper with applications to the study of the dynamics of endomorphisms of complex manifolds, first recalling a definition from [A2]:

**Definition 8.1.** Let $f \in \text{End}(M, p)$ be a germ at $p \in M$ of a holomorphic self-map of a complex manifold $M$ fixing $p$. A parabolic curve for $f$ at $p$ is an injective holomorphic map $\varphi \colon \Delta \to M$ satisfying the following properties:

(i) $\Delta$ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial \Delta$;

(ii) $\varphi$ is continuous at the origin, and $\varphi(0) = p$;

(iii) $\varphi(\Delta)$ is invariant under $f$, and $(f|_{\varphi(\Delta)})^n \to p$ as $n \to \infty$.

Furthermore, we say that $\varphi$ is tangent to a direction $v \in T_p M$ at $p$ if for one (and hence any) chart $(U, z)$ centered at $p$ the direction of $z(\varphi(\zeta))$ converges to the direction $dz_p(v)$ as $\zeta \to 0$. 
Now we have the promised dynamical interpretation of $X_f$ at nonsingular points:

**Proposition 8.1.** Assume that $S$ has codimension one in $M$, and take $f \in \text{End}(M, S)$, $f \neq \text{id}_M$. Let $p \in S$ be a regular point of $X_f$, that is $X_f(p) \neq O$. Then

(i) If $f$ is tangential then no infinite orbit of $f$ can stay arbitrarily close to $p$. More precisely, there is a neighborhood $U$ of $p$ such that for every $q \in U$ there is $n_0 \in \mathbb{N}$ such that $f^{n_0}(q) \notin U$ or $f^{n_0}(q) \in S$.

(ii) If $\Xi_f(p)$ is transversal to $T_p S$ (so in particular $f$ is non-tangential) and $\nu_f > 1$ then there exists at least one parabolic curve for $f$ at $p$ tangent to $\Xi_f(p)$.

(iii) If $\Xi_f(p)$ is transversal to $T_p S$, $\nu_f = 1$, and $|b(p)| \neq 0$, 1 or $b(p) = \exp(2\pi i \theta)$ where $\theta$ satisfies the Bryuno condition (and $b$ is the function defined in Remark 1.1) then there exists an $f$-invariant one-dimensional holomorphic disk $\Delta$ passing through $p$ tangent to $\Xi_f(p)$ such that $f|\Delta$ is holomorphically conjugated to the multiplication by $b(p)$.

**Proof.** In local adapted coordinates centered at $p \in S$ we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z),$$

so that

$$\Xi_f(p) = \text{Span} \left( g^1(O) \frac{\partial}{\partial z^1} \bigg|_p + \cdots + g^n(O) \frac{\partial}{\partial z^n} \bigg|_p \right).$$

In case (i), we have $g^1 = z^1 h^1$ for a suitable holomorphic function $h^1$, and $g^{p_0}(O) \neq 0$ for some $2 \leq p_0 \leq n$, because $p$ is not singular. Therefore we can apply [AT, Prop. 2.1] (see also [A2, Prop. 2.1]), and the assertion follows.

Now, $\Xi_f(p)$ is transversal to $T_p S$ if and only if $g^1(O) \neq 0$. In case (ii) we can then write

$$f^j(z) = z^j + g^j(O)(z^1)^{\nu_f} + O(\|z\|^{{\nu_f}+1})$$

with $g^1(O) \neq 0$. Then $\Xi_f(p)$ is a non-degenerate characteristic direction of $f$ at $p$ in the sense of Hakim, and thus by [H1, 2] there exist at least $\nu_f - 1$ parabolic curves for $f$ at $p$ tangent to $\Xi_f(p)$.

If $\nu_f = 1$, it is easy to see that $b^1(O) = 1 + g^1(O)$, and $b^1(p) \neq 1$ because $\Xi_f(p)$ is transversal to $T_p S$. Therefore we can write

$$f^j(z) = \begin{cases} b^1(p) z^1 + O(\|z\|^2) & \text{if } j = 1, \\ z^j + g^j(O) z^1 + O(\|z\|^2) & \text{if } 2 \leq j \leq n, \end{cases}$$

and the assertion in case (iii) follows immediately from [Pö] (see also [N]). \[\square\]
In other words, $X_f$ essentially dictates the dynamical behavior of $f$ away from the singularities — or, from another point of view, we can say that the interesting dynamics is concentrated near the singularities of $X_f$.

**Remark 8.1.** If $\Xi_f(p)$ is transversal to $T_pS$, $\nu_f = 1$ and $b(p) = 0$ or $b(p) = \exp(2\pi i \theta)$ with $\theta$ irrational not satisfying the Bryuno condition, there might still be an $f$-invariant one-dimensional holomorphic disk passing through $p$ and tangent to $\Xi_f(p)$. On the other hand, if $b(p) = \exp(2\pi i \theta)$ is a $k$th root of unity, necessarily different from one, several things might happen. For instance, if $b(p) = -1$, up to a linear change of coordinates we can write

$$f^j(z) = \begin{cases} 
z^1 + z^1(-2 + (z^1)^{\mu_1} \hat{g}^1(z)) & \text{if } j = 1, \\
z^j + (z^1)^{\mu_j+1} \hat{g}^j(z) & \text{if } j = 2, \ldots, n, \end{cases}$$

for suitable $\mu_1, \ldots, \mu_n \in \mathbb{N}$ and holomorphic functions $\hat{g}^j$ not divisible by $z^1$ and such that $\hat{g}^j(O) = 0$ if $\mu_j = 0$. Then if $\mu_1 = 0$,

$$(f \circ f)^j(z) = \begin{cases} z^1 - z^1[\hat{g}^1(z) + \hat{g}^1(f(z)) - \hat{g}^1(f(z))\hat{g}(f(z))] & \text{if } j = 1, \\
z^j + (z^1)^{\mu_j+1}[\hat{g}^j(z) - (1 + \hat{g}^1(z))^{\mu_j+1}\hat{g}^j(f(z))] & \text{if } j = 2, \ldots, n. \end{cases}$$

So $\nu_{f \circ f} = 1$, $f \circ f$ is non-tangential but $p$ is singular for $f \circ f$. On the other hand, if $\mu_1 = 1$,

$$(f \circ f)^j(z) = \begin{cases} z^1 - (z^1)^2[\hat{g}^1(z) - \hat{g}^1(f(z)) + O(z^1)] & \text{if } j = 1, \\
z^j + (z^1)^{\mu_j+1}[\hat{g}^j(z) + (1)\hat{g}^j(f(z)) + O(z^1)] & \text{if } j = 2, \ldots, n. \end{cases}$$

Now if, for instance, $\mu_2 = 0$ we get $\nu_{f \circ f} = 1$, but $f \circ f$ is tangential and $p$ is singular for $f \circ f$. But if $\mu_2 = 2$ and $\mu_j \geq 2$ for $j \geq 3$ we get $\nu_{f \circ f} = 3$ and $p$ can be either singular or nonsingular for $f \circ f$.

**Remark 8.2.** If $\nu_f = 1$, $\Xi_f(p) \subset T_pS$ and $S$ is compact, necessarily $f$ is tangential, because $b \equiv 1$ and then $g^1(0, z'') \equiv 0$. If $S$ is not compact we might have an isolated point of tangency, and in that case we might have parabolic curves at $p$ not tangent to $\Xi_f(p)$. For instance, the methods of [A1] show that this happens for the map

$$f^j(z) = \begin{cases} z^1 + z^1(a z^2 + b z^3 + h_1(z'') + z^1 h_2(z)) & \text{if } j = 1, \\
z^2 + z^1(c + h_3(z)) & \text{if } j = 2, \\
z^3 + z^1 g^3(z) & \text{if } j = 3, \end{cases}$$

when $a, c \neq 0$.

Finally, we describe a couple of applications to endomorphisms of complex surfaces:
Corollary 8.2. Let $S$ be a smooth compact one-dimensional submanifold of a complex surface $M$, and take $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$. Assume that $f$ is tangential, or that $S \setminus \text{Sing}(f)$ is comfortably embedded in $M$, and let $X$ denote $X_f, H_{\sigma, f}$ or $H_{\sigma, f}^1$ as usual; assume moreover that $X \not\equiv O$. Then

(i) if $c_1(N_S) \neq 0$ then $\chi(S) - \nu_f c_1(N_S) > 0$;
(ii) if $c_1(N_S) > 0$ then $S$ is rational, $\nu_f = 1$ and $c_1(N_S) = 1$.

Proof. The well-known theorem about the localization of the top Chern class at the zeroes of a global section (see, e.g., [Su, Th. III.3.5]) yields

\begin{equation}
\sum_{x \in \text{Sing}(X)} N(X; x) = \chi(S) - \nu_f c_1(N_S),
\end{equation}

where $N(X; x)$ is the multiplicity of $x$ as a zero of $X$. Now, if $c_1(N_S) \neq 0$ then by Theorem 6.2 the set $\text{Sing}(X)$ is not empty. Therefore the sum in (8.1) must be strictly positive, and the assertions follow.

Definition 8.2. Let $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$. We say that a point $p \in S$ is \textit{weakly attractive} if there are infinite orbits arbitrarily close to $p$, that is, if for every neighborhood $U$ of $p$ there is $q \in U$ such that $f^n(q) \in U \setminus S$ for all $n \in \mathbb{N}$. In particular, this happens if there is an infinite orbit converging to $p$.

Then we can prove the following

Proposition 8.3. Let $S$ be a smooth compact one-dimensional submanifold of a complex surface $M$, and take $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$. If $f$ is tangential then there are at most $\chi(S) - \nu_f c_1(N_S)$ weakly attractive points for $f$ on $S$.

Proof. By (8.1) the sum of zeros of the section $X_f$ (counting multiplicity) is equal to $\chi(S) - \nu_f c_1(N_S)$. Thus the number of zeros (not counting multiplicity) is at most $\chi(S) - \nu_f c_1(N_S)$. The assertion then follows from Proposition 8.1.

Finally, the previous index theorems allow a classification of the smooth curves which are fixed by a holomorphic map and are dynamically trivial.

Theorem 8.4. Let $S$ be a smooth compact one-dimensional submanifold of a complex surface $M$, and take $f \in \text{End}(M, S)$, $f \not\equiv \text{id}_M$. Moreover assume that $\text{sp}(df_p) = \{1\}$ for some $p \in S$. If there are no weakly attractive points for $f$ on $S$ then only one of the following cases occurs:

(i) $\chi(S) = 2, c_1(N_S) = 0$, or
(ii) $\chi(S) = 2$, $c_1(N_S) = 1$, $\nu_f = 1$, or

(iii) $\chi(S) = 0$, $c_1(N_S) = 0$.

Proof. Since $N_S$ is a line bundle over a compact curve $S$, the action of $df$ on $N_S$ is given by multiplication by a constant, and since $df_p$ has only the eigenvalue 1 then this constant must be 1. If $f$ were nontangential then by Proposition 8.1.(ii) all but a finite number of points of $S$ would be weakly attractive. Therefore $f$ is tangential. By [A2, Cor. 3.1] (or [Br, Prop. 7.7]) if there is a point $q \in S$ so that $\text{Res}(X_f, N_S, p) \not\in \mathbb{Q}^+$ then $q$ is weakly attractive. Thus the sum of the residues is nonnegative and by Theorem 6.2 it follows that $c_1(N_S) \geq 0$. Thus (8.1) yields

$$
\chi(S) \geq \nu_f c_1(N_S) \geq 0.
$$

Therefore the only possible cases are $\chi(S) = 0, 2$. If $\chi(S) = 0$ then (8.2) implies $c_1(N_S) = 0$. Assume that $\chi(S) = 2$. Thus $c_1(N_S) = 0, 1, 2$. However if $c_1(N_S) = 1$ and $\nu_f = 2$ or if $c_1(N_S) = 2$ (and necessarily $\nu_f = 1$) then (8.1) would imply that $X_f$ has no zeroes, and thus $c_1(N_S) = 0$ by Theorem 6.2.

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