Well-posedness of the Cauchy problem for the fractional power dissipative equations

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Abstract

This paper studies the Cauchy problem for the nonlinear fractional power dissipative equation

\[ u_t + (-\Delta)^\alpha u = F(u) \]

for initial data in the Lebesgue space \( L^r(\mathbb{R}^n) \) with \( r \geq r_d \triangleq nb/(2\alpha - d) \) or the homogeneous Besov space \( \dot{B}^{-\sigma}_{p,\infty}(\mathbb{R}^n) \) with \( \sigma = (2\alpha - d)/b - n/p \) and \( 1 \leq p \leq \infty \), where \( \alpha > 0 \), \( F(u) = f(u) \) or \( Q(D)f(u) \) with \( Q(D) \) being a homogeneous pseudo-differential operator of order \( d \in [0, 2\alpha) \) and \( f(u) \) is a function of \( u \) which behaves like \( |u|^b u \) with \( b > 0 \).

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1 Introduction

In this paper we study the Cauchy problem for the semi-linear fractional power dissipative equation

\[
\begin{align*}
&u_t + (-\Delta)^\alpha u = F(u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\
&u(0, x) = \varphi(x), \quad x \in \mathbb{R}^n
\end{align*}
\]

with the nonlinear term \( F(u) \) is equal either to \( f(u) \) or to \( Q(D)f(u) \), where \( Q(D) \) is a homogeneous pseudo-differential operator of order \( d \in [0, 2\alpha) \) with real number \( \alpha > 0 \) and \( f(u) \) is a function of \( u \) which behaves like \( |u|^b u \) or \( |u|^{b_1} u + |u|^{b_2} u \) with \( b > 0 \), \( b_1 > 0 \) and \( b_2 > 0 \). The evolution equation in (1.1) models several classical equations, for example,

(1) the semi-linear fractional power dissipative equation

\[ u_t + (-\Delta)^\alpha u = \pm \nu |u|^b u. \]
(2) the dissipative quasi-geostrophic (QG) equation
\[
\begin{aligned}
\begin{cases}
  \theta_t + u \cdot \nabla \theta + \kappa (-\Delta)^{\alpha} \theta = 0, \\
  u = (u_1, u_2) = \nabla^2 \psi, \quad (-\Delta)^{1/2} \psi = \theta,
\end{cases}
\end{aligned}
\]
\[(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.2}\]
where $1/2 < \alpha \leq 1$.

(3) the generalized Navier-Stokes equation
\[
\begin{aligned}
  u_t + (-\Delta)^{\alpha} u - (u \cdot \nabla) u + \nabla P = 0, \quad \nabla u = 0.
\end{aligned}
\]

(4) the generalized convection-diffusion equation
\[
\begin{aligned}
  u_t + (-\Delta)^{\alpha} u = a \cdot \nabla (|u|^b u), \quad a \in \mathbb{R}^n/\{0\}.
\end{aligned}
\]

(5) the Ginzburg-Landau equation
\[
\begin{aligned}
  u_t + a_1 \nabla^4 u + a_2 \nabla^2 u + a \nabla^2 u^3, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,
\end{aligned}
\]
where $a_1 > 0$, $a > 0$ and $a_2 \neq 0$.

The case $\alpha = 1$ for the problem (1.1) corresponds to the classical semi-linear heat equation and has been studied extensively (see, e.g. [7, 13, 16, 18, 20, 21, 24, 26]). Concerning the generalized Navier-Stokes equations, please refer to [3]. In the case when $\alpha$ is an integer, [14, 15] established the space-time estimates and well-posedness of strong solutions in Lebesgue spaces to the problem (1.1). For general $\alpha$, [16] studied the global well-posedness of solutions to (1.1) for small initial data in pseudomeasure spaces. In the case $1/2 < \alpha \leq 1$ (i.e. in the case of dissipative quasi-geostrophic equation (1.2)), well-posedness of solutions has been studied in, e.g. Lebesgue spaces [27], Sobolev spaces [28], Hölder spaces [30], Besov spaces [29, 31] and Triebel spaces [4].

In this paper we shall give a unified method to deal with the well-posedness of the Cauchy problem (1.1) for initial data in the Lebesgue space $L^r(\mathbb{R}^n)$ ($r \geq r_0 \triangleq nb/(2\alpha - d)$) or in the Besov space $B_{p, \infty}^{-\sigma}(\mathbb{R}^n)$ ($\sigma = (2\alpha - d)/b - n/p$ and $1 \leq p \leq \infty$), employing appropriate space-time spaces such as $C([0, \infty); L^r(\mathbb{R}^n)) \cap L^q([0, \infty); L^p(\mathbb{R}^n))$ or $C(I; L^r(\mathbb{R}^n)) \cap C_q(I; L^p(\mathbb{R}^n))$. In Section 2, we give a detailed analysis of the kernel function of the fractional power operator semigroup $S_\alpha(t) = e^{t(-\Delta)^{\alpha}}$. In particular, we derive the point-wise estimates of the kernel function of the semigroup $S_\alpha(t)$ by an invariant derivative technique (see Lemma 2.1 below) which leads to an equivalent characterization of the Besov space (see [31] for the special case $1/2 < \alpha \leq 1$ and $n = 2$). In Section 3 making use of the point-wise estimates of the kernel function obtained in Section 2 we establish the space-time estimates for the corresponding linear fractional power dissipative equation. Section 4 is devoted to the well-posedness in Lebesgue spaces of the Cauchy problem (1.1), using the space-time estimates established in Section 3 in conjunction with the Banach contraction mapping principle. In Section 5, we consider the fractional power dissipative equations with more general nonlinear terms. In particular, the interaction between two different nonlinear terms is discussed, and the local and small global well-posedness of solutions are established. Finally, in Section 6 we establish the well-posedness of solutions to the fractional power dissipative equation (1.1) for initial data in the Besov space $B_{p, \infty}^{-\sigma}(\mathbb{R}^n)$ or in the critical Lebesgue space $L^{nb/(2\alpha - d)}(\mathbb{R}^n)$ but with small norm in the Besov space $B_{p, \infty}^{-\sigma}(\mathbb{R}^n)$. Since the Besov space $B_{p, \infty}^{-\sigma}(\mathbb{R}^n)$ contains self-similar initial data in the sense that the initial data $\varphi(x)$ satisfies $\lambda^{-\frac{\alpha}{n}} \varphi(\lambda x) = \varphi(x)$ for any $\lambda > 0$, then our results in Section 6 implies the existence of global self-similar solutions to (1.1). Concerning the systematic scaling analysis of nonlinear parabolic equations please refer to Karch [10, 11].

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2 Analysis of the operator semigroup \( S_\alpha(t) \)

In this section we consider the linear semigroup \( S_\alpha(t) \triangleq e^{-t(-\triangle)^\alpha} \) generated by the following linear fractional power dissipative equation (2.1). We show that the kernel function of the operator semigroup \( S_\alpha(t) \) generates a bounded linear operator on \( L^p(\mathbb{R}^n) \) for \( p \in [1, \infty] \).

Consider the Cauchy problem for the homogeneous linear fractional power dissipative equation

\[
\begin{cases}
u_t + (-\triangle)^\alpha u = 0, & (t, x) \in [0, \infty) \times \mathbb{R}^n; \\
u(0) = \varphi(x), & x \in \mathbb{R}^n.
\end{cases}
\]

By the Fourier transform the solution of the problem (2.1) can be written as

\[
u(t, x) = \mathcal{F}^{-1}\left(e^{-t|\xi|^{2\alpha}} \mathcal{F}\nu(\xi)\right) = \mathcal{F}^{-1}e^{-t|\xi|^{2\alpha}} \ast \varphi(x) \triangleq K_t(x) \ast \varphi(x).
\]

Here \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier and inverse Fourier transforms, respectively, defined by

\[
\mathcal{F}(f) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx,
\]

\[
\mathcal{F}^{-1}(g) = \check{g}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) \, d\xi
\]

for any \( f, g \in \mathcal{S}' \), where \( \mathcal{S}' \) denotes the space of tempered distributions.

It is well known that for \( \alpha = 1 \) and \( \alpha = \frac{1}{2} \), \( K_t(x) \) is the Gaussian and Poisson kernel function, respectively, and their properties have been fully understood. In what follows we consider the general case \( \alpha \in (0, \infty) \). From (2.2) and Young’s inequality it is seen that, to guarantee the \( L^p \rightarrow L^p \) boundedness of the linear operator \( S_\alpha(t) \) one needs only that the kernel function \( K_t(x) \) is bounded on \( L^1 \). By scaling we have

\[
K_t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^{2\alpha}} \, d\xi
\]

\[
= (2\pi)^{-n/2} t^{-\frac{n}{2\alpha}} \int_{\mathbb{R}^n} e^{ixt/2\alpha} e^{-|\eta|^{2\alpha}} \, d\eta
\]

\[
\triangleq t^{-\frac{n}{2\alpha}} K\left(\frac{x}{t^{1/2\alpha}}\right).
\]

Thus it is enough to consider the kernel function

\[
K(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^{2\alpha}} \, d\xi.
\]

It is obvious that \( e^{-|\xi|^{2\alpha}} \in L^1(\mathbb{R}^n) \), so

\[
K(x) \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)
\]

and by the Riemann-Lebesgue theorem, \( \lim_{|x| \to \infty} K(x) = 0 \).

Similarly, we have \( |\xi|^{\nu} e^{-|\xi|^{2\alpha}} \in L^1(\mathbb{R}^n) \) and

\[
(-\triangle)^{\frac{\nu}{2}} K(x) \in L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)
\]

for \( \nu > 0 \), where \( C_0(\mathbb{R}^n) \) denotes the space of functions \( f \in C(\mathbb{R}^n) \) satisfying that \( \lim_{|x| \to \infty} f(x) = 0 \).

In the same way, we have \( \nabla K(x) \in L^\infty(\mathbb{R}^n) \cap C_0(\mathbb{R}^n) \) by the fact \( i\xi e^{-|\xi|^{\alpha}} \in \left(L^1(\mathbb{R}^n)\right)^n \).
Lemma 2.1. The kernel function \( K(x) \) has the following point-wise estimate
\[
|K(x)| \leq C(1 + |x|)^{-n-2\alpha}, \quad x \in \mathbb{R}^n
\]
for \( \alpha > 0 \). Consequently one has
\[
K \in L^p(\mathbb{R}^n), \quad K_t \in L^p(\mathbb{R}^n), \quad 0 < t < \infty
\]
for any \( 1 \leq p \leq \infty \).

Proof. Define the invariant derivative operator
\[
L(x, D) = \frac{x \cdot \nabla \xi}{i|x|^2}.
\]
Then we have
\[
L(x, D)e^{ix \cdot \xi} = e^{ix \cdot \xi}.
\]
The conjugate operator is
\[
L^*(x, D) = \frac{x \cdot \nabla \xi}{i|x|^2}.
\]
Thus we may write \( K(x) \) as
\[
K(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} L^* (e^{-|\xi|^{2\alpha}}) d\xi
\]
\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \rho(\frac{\xi}{\delta}) L^* (e^{-|\xi|^{2\alpha}}) d\xi
\]
\[
+ (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 - \rho(\frac{\xi}{\delta})) L^* (e^{-|\xi|^{2\alpha}}) d\xi \triangleq I + II,
\]
where \( \delta > 0 \) to be chosen later and \( \rho(\xi) \) is a \( C_c^\infty(\mathbb{R}^n) \)-function satisfying
\[
\rho(\xi) = \begin{cases}
1, & |\xi| \leq 1; \\
0, & |\xi| > 2.
\end{cases}
\]
It is clear that
\[
|I| \leq \frac{C}{|x|} \int_{|\xi| \leq 2\delta} |\xi|^{2\alpha-1} d\xi \leq C|x|^{-1} \delta^{2\alpha+n-1}.
\]
To estimate \( II \), take a sufficiently large natural number \( N > [2\alpha] + n \) and integrate by parts to obtain that
\[
|II| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{ix \cdot \xi} (L^*)^{N-1} (1 - \rho(\frac{\xi}{\delta})) L^* (e^{-|\xi|^{2\alpha}}) | d\xi
\]
\[
\leq C|x|^{-N} \int_{|\xi| \geq \delta} \sum_{k=1}^{N} |\xi|^{2\alpha - N} e^{-|\xi|^{2\alpha}} d\xi
\]
\[
+ C|x|^{-N} \sum_{k=1}^{N-1} C_k \delta^{-k} \int_{\delta \leq |\xi| \leq 2\delta} \sum_{l=1}^{N-k} C_l |\xi|^{2\alpha - (N-k)} e^{-|\xi|^{2\alpha}} d\xi
\]
\[
\leq C|x|^{-N} \int_{|\xi| \geq \delta} |\xi|^{2\alpha - N} e^{-|\xi|^{2\alpha}} d\xi + C|x|^{-N} \int_{|\xi| \geq \delta} |\xi|^{2\alpha - N} |\xi|^{2\alpha(N-1)} e^{-|\xi|^{2\alpha}} d\xi
\]
\[
+ C|x|^{-N} \sum_{k=1}^{N-1} \int_{\delta \leq |\xi| \leq 2\delta} \left( |\xi|^{2\alpha - N} e^{-|\xi|^{2\alpha}} + |\xi|^{2\alpha(N-k)} e^{-|\xi|^{2\alpha}} \right) d\xi,
\]
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In view of the facts that
\[ |\xi|^{2\alpha(N-1)}e^{-|\xi|^{2\alpha}} \leq C, \quad |\xi|^{2\alpha(N-k-1)}e^{-|\xi|^{2\alpha}} \leq C \]
for \( k = 1, 2, \cdots, N-1 \), it is found that \(|II|\) is dominated by
\[ C|x|^{-N}\left( \int_{|\xi|\geq \delta} |\xi|^{2\alpha-N} d\xi + \int_{\delta \leq |\xi| \leq 2\delta} \delta^{2\alpha-N} d\xi \right) \leq C|x|^{-N}\delta^{2\alpha-N+n}. \]
Thus it follows that
\[ |K(x)| \leq C|x|^{-1}\delta^{2\alpha+n-1} + C|x|^{-N}\delta^{2\alpha-N+n}. \]
Taking \( \delta = |x|^{-1} \) gives
\[ |K(x)| \leq C|x|^{-n-2\alpha}. \]
This together with the boundedness of \( K(x) \) (see (2.4)) completes the proof of the lemma. \( \square \)

We now take the \( \nu \)-th derivative of the kernel \( K(x) \) and have
\[ K^\nu(x) = (-\Delta)^{\nu/2} K(x), \quad K^\nu_t(x) = (-\Delta)^{\nu/2} K_t(x). \]
Then \( K^\nu(x) \) can be split up into
\[
K^\nu(x) = \left( 2\pi \right)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \rho(\xi/\delta) |\xi|^\nu e^{-|\xi|^{2\alpha}} d\xi \\
+ \left( 2\pi \right)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 - \rho(\xi/\delta)) |\xi|^\nu e^{-|\xi|^{2\alpha}} d\xi \\
\triangleq I + II.
\]
Clearly,
\[ |I| \leq C \int_{|\xi| \leq 2\delta} \delta^\nu d\xi \leq C\delta^{n+\nu}. \]
To estimate \( II \) we use the technique of invariant derivatives together with integration by parts to obtain that
\[ |II| \leq C \int_{\mathbb{R}^n} \left| e^{ix \cdot \xi} (L^*)^N ((1 - \rho(\xi/\delta)) |\xi|^\nu e^{-|\xi|^{2\alpha}}) \right| d\xi. \]
Arguing similarly as the proof of Lemma 2.1 we have
\[
|II| \leq C|x|^{-N} \left( \int_{|\xi|\geq \delta} |\xi|^{\nu-N} d\xi + \int_{\delta \leq |\xi| \leq 2\delta} \delta^{\nu-N} d\xi \right) \\
\leq C|x|^{-N}\delta^{\nu-N+n}.
\]
Taking \( \delta = |x|^{-1} \) leads to the estimate
\[ |K^\nu(x)| \leq C|x|^{-\nu-n}. \]
Thus we have the following lemma.
Lemma 2.2. The kernel function $K^\nu(x)$ has the following pointwise estimate
\[ |K^\nu(x)| \leq C(1 + |x|)^{-n-\nu}, \quad x \in \mathbb{R}^n \]
for $\nu > 0$. Consequently one has
\[ K^\nu \in L^p(\mathbb{R}^n), \quad K_t^\nu \in L^p(\mathbb{R}^n), \quad 0 < t < \infty \]
for any $1 \leq p \leq \infty$.

Remark 2.1. (i) Thank to $i\xi e^{-|\xi|^\alpha} \in (L^1(\mathbb{R}^n))^n$, one has by the same argument of Lemma 2.2 that
\[ |\nabla K(x)| \leq C(1 + |x|)^{-n-1}, \]
and
\[ \nabla K(x), \quad \nabla K_t(x) \in L^p(\mathbb{R}^n), \quad 0 < t < \infty \]
for any $1 \leq p \leq \infty$.

(ii) Similar to (2.3) the kernel function $K_t^\nu(x)$ satisfies the same scaling as follows:
\[ K_t^\nu(x) = t^{-\frac{n}{2\alpha}} t^{-\frac{n}{2\alpha}} K^\nu \left( \frac{x}{t^{1/2\alpha}} \right). \quad (2.5) \]

In Proposition 2.1 below we give another characterization of the negative index homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$, employing the pointwise estimate of the kernel $K(x)$ in Lemma 2.2 and the boundedness of the fractional power dissipative operator semigroup $S_\alpha(t)$ on the space $L^p(\mathbb{R}^n)$, where $s < 0$. The idea essentially comes from [19] (see also [17]). For the case $n = 2$ and $1/2 < \alpha \leq 1$ the reader is also referred to [31]. For completeness we give a proof of Proposition 2.1 here for any $n \in \mathbb{N}$ and $0 < \alpha < \infty$. We first recall the definition of homogeneous Besov spaces.

Choose a radial bump function $\hat{\psi}(\xi) \in C^\infty_c(\mathbb{R}^n)$ such that
\[ \hat{\psi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ \text{smooth}, & \text{if } 1 < |\xi| < 2, \text{ and } 0 \leq \hat{\psi}(\xi) \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases} \]
where $\hat{\psi}(\xi)$ denotes the Fourier transform of $\psi(x)$. Set $\hat{\phi}(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$ and let $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi), \ \xi \neq 0$ for $j \in \mathbb{Z}$, $\hat{\psi}_j(\xi) = \hat{\psi}(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Let $\Delta_j f = \phi_j * f$, $S_j f = \psi_j * f$. Then for any $f \in L^2(\mathbb{R}^n)$ we have the following Littlewood-Paley decomposition
\[ f(x) = \sum_{j=-\infty}^{\infty} \phi_j * f(x), \]
where the sum is taken in the $L^2(\mathbb{R}^n)$ sense.

The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by the dyadic decomposition as
\[ \dot{B}^s_{p,q} = \{ f \in Z'(\mathbb{R}^n) | \|f\|_{\dot{B}^s_{p,q}} < \infty \}, \]
where

\[ \|f\|_{\dot{B}^s_{p,q}} = \left\{ \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \|\phi_j * f\|_p^q \right) \right\}^{1/q}, \quad 1 \leq q < \infty \]

is the norm of \( \dot{B}^s_{p,q} \) and \( \mathcal{Z}'(\mathbb{R}^n) \) denotes the dual space of

\[ \mathcal{Z}(\mathbb{R}^n) = \{ f \in \mathcal{S}(\mathbb{R}^n) \mid D^\alpha \hat{f}(0) = 0, \text{ for any multi-index } \alpha \in \mathbb{N}^n \} \]

and can be identified by the quotient space \( \mathcal{S}'/\mathcal{P} \) with the polynomial \( \mathcal{P} \). See \[1\], \[13\] and \[25\] for details.

**Proposition 2.1.** Let \( 1 \leq p, q \leq \infty, s < 0 \) and assume that \( n \in \mathbb{N} \) and \( 0 < \alpha < \infty \). Then \( f \in \dot{B}^s_{p,q}(\mathbb{R}^n) \) if and only if

\[
\begin{cases}
\left( \int_0^\infty \left( t^{s/p} \|S_\alpha(t)f\|_p \right)^{q \alpha \frac{dt}{t}} \right)^{1/q} < \infty, & 1 \leq q < \infty, \\
\sup_{t>0} t^{s/p} \|S_\alpha(t)f\|_p, & q = \infty.
\end{cases}
\]

(2.6)

**Proof.** We only consider the case \( 1 \leq q < \infty \). The case \( q = \infty \) can be shown similarly. We first prove that

\[
\left( \sum_{j=-\infty}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q \right)^{1/q} \leq C \left( \int_0^\infty \left( t^{s/p} \|S_\alpha(t)f\|_p \right)^{q \alpha \frac{dt}{t}} \right)^{1/q}.
\]

In fact, let

\[
\Phi_j(x) = \mathcal{F}^{-1} \left( \tilde{\phi} \left( \frac{\xi}{2^j} \right) e^{2^{-j} \|\xi\|^{2\alpha}} \right)(x),
\]

\[
h_t(x) = \mathcal{F}^{-1} \left( e^{-\|\xi\|^{2\alpha}} \right)(x).
\]

Then by the definition of \( \Delta_j \) one has \( \Delta_j f = \Phi_j * h_{2^{-j}} * f(x) \).

By the Young inequality we get

\[ \|\Delta_j f\|_p \leq \|\Phi_j\|_1 \|h_{2^{-j}} * f\|_p \leq C \|h_{2^{-j}} * f\|_p, \]

where we have used the fact that

\[ \|\Phi_j(x)\|_1 = \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \left( e^{\|\xi\|^{2\alpha}} \tilde{\phi}(\xi) \right)(x)| dx < \infty. \]

A direct calculation shows that

\[
h_{2^{-j}} * f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-(2^{-j} \|\xi\|)^{2\alpha}} \hat{f}(\xi) e^{ix \cdot \xi} d\xi
\]

\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\|\xi\|^{2\alpha}(2^{-2\alpha} - t^{2\alpha})} e^{-(t \|\xi\|)^{2\alpha}} \hat{f}(\xi) e^{ix \cdot \xi} d\xi
\]

\[ = S_\alpha(2^{-2\alpha} - t^{2\alpha})(h_t * f)(x). \]
Thus it follows that
\[ \| h_{2^{-j}} \ast f \|_p \leq C \| h_t \ast f \|_p \] (2.7)
for any \( t \in [2^{-j-1}, 2^{-j}] \), which implies that
\[
\sum_{j=-\infty}^{\infty} 2^{sjq} \| \triangle_j f \|_q^q \leq C \sum_{j=-\infty}^{\infty} \int_{2^{-j-1}}^{2^{-j}} \left( t^{-s} \| h_{2^{-j}} \ast f \|_p \right) \frac{q \, dt}{t}
\leq C \int_0^{\infty} \left( t^{-s} \| h_t \ast f \|_p \right) \frac{q \, dt}{t}
\leq C \int_0^{\infty} \left( t^{-\frac{q}{2s}} \| S_\alpha(t) \ast f \|_p \right) \frac{q \, dt}{t},
\]
where we have used the fact that \( h_t \ast f(x) = S_\alpha(t^{2\alpha})f(x) \).

We now prove that
\[
\left( \int_0^{\infty} \left( t^{-\frac{q}{2s}} \| S_\alpha(t) \ast f \|_p \right) \frac{q \, dt}{t} \right)^{1/q} \leq C \left( \sum_{j=-\infty}^{\infty} 2^{sjq} \| \triangle_j f \|_q^q \right)^{1/q}.
\] (2.8)

In fact, for any \( j \in \mathbb{Z} \) one has the decomposition
\[ h_{2^{-j}} \ast f(x) = \sum_{k=-\infty}^{\infty} h_{2^{-j}} \ast \triangle_k f(x). \]

Arguing similarly as in deriving (2.7) one has
\[ \| h_t \ast f(x) \|_p \leq C \| h_{2^{-j}} \ast f(x) \|_p \]
for any \( t \in [2^{-j}, 2^{-j+1}] \). The left-hand side of the estimate (2.8) can be estimated as follows:
\[
\int_0^{\infty} \left( t^{-\frac{q}{2s}} \| S_\alpha(t) \ast f \|_p \right) \frac{q \, dt}{t} = 2^{s} \int_0^{\infty} \left( t^{-s} \| h_t \ast f \|_p \right) \frac{q \, dt}{t}
\leq C \sum_{j=-\infty}^{\infty} \int_{2^{-j-1}}^{2^{-j+1}} \left( 2^{js} \| h_{2^{-j}} \ast f \|_p \right) \frac{q \, dt}{t}
\leq C \sum_{j=-\infty}^{\infty} \left( 2^{js} \sum_{k=-\infty}^{\infty} \| h_{2^{-j}} \ast \triangle_k f \|_p \right) \frac{q \, dt}{t}. \] (2.9)

If we can show that
\[ \| h_{2^{-j}} \ast \triangle_k f \|_p \leq 2^{ks} \| \triangle_k f \|_p \] (2.10)
for any \( s < 0 \), then taking \( s_1 < s < s_0 < 0 \) we have by using the Minkowski inequality that the
right-hand side of (2.9) is bounded above by

\[
C \sum_{j=-\infty}^{\infty} \left( 2^{js} \sum_{k=-\infty}^{0} 2^{ks} \| \Delta_{k+j} f \|_p \right)^q + C \sum_{j=-\infty}^{\infty} \left( 2^{js} \sum_{k=1}^{\infty} 2^{ks} \| \Delta_{k+j} f \|_p \right)^q
\]

\[
\leq C \left( \sum_{j=-\infty}^{0} 2^{k(s_0-s)} \left( \sum_{j=-\infty}^{\infty} 2^{(k+j)s} \| \Delta_{k+j} f \|_p^q \right)^{1/q} \right)^q
\]

\[
+ C \left( \sum_{k=1}^{\infty} 2^{-k(s-s_1)} \left( \sum_{j=-\infty}^{\infty} 2^{(k+j)s} \| \Delta_{k+j} f \|_p^q \right)^{1/q} \right)^q
\]

\[
\leq C \sum_{j=-\infty}^{\infty} 2^{sj} \| \Delta_j f \|_p^q,
\]

which completes the proof of Proposition 2.1. We now prove the estimate (2.10). Note first that

\[
h_{2^{-j}} \ast \Delta_{k+j} f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-2^{-j|\xi|}2^n (\hat{\phi}(2^{-k-j}\xi) \hat{f}(\xi)) e^{ix \cdot \xi} d\xi
\]

\[
= \frac{2^{ks}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-2^{-j|\xi|}2^n \left( \frac{2^j}{|\xi|} \right) s} \left( \frac{|\xi|}{2^{k+j}} \right)^{s} \phi(2^{-k-j}\xi) \hat{f}(\xi) e^{ix \cdot \xi} d\xi,
\]

where \( \hat{\phi}(2^{-j}\xi) = \hat{\phi}(2^{-j+1}\xi) + \hat{\phi}(2^{-j}\xi) + \hat{\phi}(2^{-j-1}\xi). \) Since

\[
\left\| F^{-1} \left( e^{-2^{-j|\xi|}2^n \left( \frac{2^j}{|\xi|} \right) s} \right) \right\|_{L^1} = \left\| 2^{jn} F^{-1}(e^{-|\xi|^{2(n-s)}}(2^j x)) \right\|_{L^1} = \| F^{-1}(e^{-|\xi|^{2(n-s)}}) \|_{L^1} < \infty,
\]

\[
\left\| F^{-1} \left( \left( \frac{|\xi|}{2^{k+j}} \right)^{s} \phi(\frac{\xi}{2^{k+j}}) \right) \right\|_{L^1} = \| F^{-1}(|\xi|^s \phi(\xi)) \|_{L^1} < \infty,
\]

the estimate (2.10) follows easily from the Young inequality. \( \square \)

3. Space-time estimates for the linear equation

In this section we discuss the space-time estimates of solutions to the Cauchy problem of the linear fractional power dissipative equation

\[
\begin{cases}
\frac{du}{dt} + (-\triangle)^\alpha u = f(t, x), & (t, x) \in [0, \infty) \times \mathbb{R}^n; \\
u(0) = \varphi(x), & x \in \mathbb{R}^n.
\end{cases}
\] (3.1)

By Duhamel’s principle, the solution to the problem (3.1) can be written in the integral form as

\[
u(x, t) = S_\alpha(t) \varphi(x) + \int_0^t S_\alpha(t - \tau) f(\tau, x) d\tau \triangleq S_\alpha(t) \varphi(x) + (\mathcal{G} f)(t, x).
\] (3.2)

We first consider the space-time estimates for the homogeneous part of the solution \( u \) given in the integral form (3.2).
Lemma 3.1. Let \(1 \leq r \leq p \leq \infty\) and let \(\varphi \in L^r(\mathbb{R}^n)\). Then the homogeneous part of the solution \(u(t, \cdot)\) satisfies the estimates

\[
\| S_\alpha(t)\varphi(x) \|_p \leq C t^{-\frac{n}{2\alpha} \left( \frac{1}{r} - \frac{1}{p} \right)} \| \varphi \|_{L^r}, \tag{3.3}
\]

\[
\| (-\Delta)^{\nu/2} S_\alpha(t)\varphi(x) \|_p \leq C t^{-\frac{n}{2\alpha} \left( \frac{1}{r} - \frac{1}{p} \right)} \| \varphi \|_{L^r} \tag{3.4}
\]

for \(\alpha > 0\) and \(\nu > 0\).

Proof. It follows from the Young inequality combined with scaling property of the kernel \(K_t\).

To derive the space-time estimates of the homogeneous part of the solution \(u\) given in \(3.2\) we need to introduce the following definition on admissible triplets and generalized admissible triplets for the fractional power dissipative equation. For the corresponding definition for parabolic equations the reader is referred to [15, 14, 18].

Definition 3.1. The triplet \((q, p, r)\) is called an admissible triplet (for the fractional power dissipative equation) if

\[
\frac{1}{q} = \frac{n}{2\alpha} \left( \frac{1}{r} - \frac{1}{p} \right),
\]

where

\[
1 < r \leq p \begin{cases} \frac{nr}{n-2\alpha}, & \text{for } n > 2\alpha, \\ \infty, & \text{for } n \leq 2\alpha. \end{cases}
\]

Definition 3.2. The triplet \((q, p, r)\) is called a generalized admissible triplet (for the fractional power dissipative equation) if

\[
\frac{1}{q} = \frac{n}{2\alpha} \left( \frac{1}{r} - \frac{1}{p} \right),
\]

where

\[
1 < r \leq p \begin{cases} \frac{nr}{n-2r\alpha}, & \text{for } n > 2r\alpha, \\ \infty, & \text{for } n \leq 2r\alpha. \end{cases}
\]

Let \(B\) be a Banach space and let \(I = [0, T)\). We define the time-weighted space-time Banach space \(C_\sigma(I; B)\) and the corresponding homogeneous space \(\dot{C}_\sigma(I; B)\) as follows

\[
C_\sigma(I; B) = \{ f \in C(I; B) \mid \| f; C_\sigma(I; B) \| = \sup_{t \in I} \| t^{\frac{1}{p}} f \|_B < \infty \},
\]

\[
\dot{C}_\sigma(I; B) = \{ f \in C_\sigma(I; B) \mid \lim_{t \to 0^+} t^{\frac{1}{p}} \| f \|_B = 0 \}.
\]

In this paper the Banach space \(B\) is taken to be \(L^p(\mathbb{R}^n)\) with \(1 < p < \infty\).

With the above definitions we now have the following results on the space-time estimates for the homogeneous part of the solution \(u\) given in \(3.2\). These estimates can be proved by following 8 (see also 15). Here we give a proof for completeness.
Lemma 3.2. (i) Let \((q,p,r)\) be any admissible triplet and let \(\varphi \in L^r(\mathbb{R}^n)\). Then \(S_\alpha(t)\varphi \in L^q(I; L^p(\mathbb{R}^n)) \cap C_b(I; L^r(\mathbb{R}^n))\) with the estimate
\[
\|S_\alpha(t)\varphi(x)\|_{L^q(I; L^p)} \leq C\|\varphi\|_{L^r},
\]
for \(0 < T \leq \infty\), where \(C\) is a positive constant.

(ii) Let \((q,p,r)\) be any generalized admissible triplet. For any \(\varphi \in L^r(\mathbb{R}^n)\) we have \(S_\alpha(t)\varphi \in C_q(I; L^p(\mathbb{R}^n)) \cap C_b(I; L^r(\mathbb{R}^n))\) and
\[
\|S_\alpha(t)\varphi\|_{C_q(I; L^p)} \leq C\|\varphi\|_{L^r}.
\]

Hereafter, for a Banach space \(X\) we denote by \(C_b(I; X)\) the space of bounded continuous functions from \(I\) to \(X\).

Proof. The statement (ii) follows easily from Lemma 3.1. So we only need to prove (i). For the case \(p = r, q = \infty\), the estimate (3.5) is true from Lemma 3.1. We now consider the case \(p \neq r\).

Let
\[
U(t)\varphi = \|S_\alpha(t)\varphi\|_p.
\]
Then, and since \((q,p,r)\) is an admissible triplet, we deduce by Young’s inequality that
\[
U(t)\varphi \leq Ct^{-\frac{1}{q}}\|\varphi\|_{L^r}.
\]
It is easy to see that
\[
\mu\{t : |U(t)\varphi| > \tau\} \leq \mu\{t : Ct^{-\frac{1}{q}}\|\varphi\|_{L^r} > \tau\} = \mu\left\{ t : t < \left(\frac{C\|\varphi\|_{L^r}}{\tau}\right)^q \right\}
\leq \left(\frac{C\|\varphi\|_{L^r}}{\tau}\right)^q,
\]
which implies that \(U(t)\) is a weak type \((r,q)\) operator.

On the other hand, by Lemma 3.1, \(U(t)\) is sub-additive and satisfies that
\[
U(t)\varphi = \|S_\alpha(t)\varphi\|_p \leq C\|\varphi\|_p
\]
for \(r \leq p \leq \infty\), which means that \(U(t)\) is a \((p,\infty)\) operator. Since for any admissible triplet \((p,q,r)\) we can always find another admissible triplet \((p,q_1,r_1)\) such that
\[
q_1 < q < \infty, \quad r_1 < r < p
\]
and
\[
\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{\infty}, \quad \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{p},
\]
then the Marcinkiewicz interpolation theorem (see [13] or [23]) implies that \(U(t)\) is a strong \((r,q)\)-type operator. The estimate (3.5) thus follows, and the proof of Lemma 3.2 is complete.

We now derive the space-time estimates of the non-homogeneous part \(Gf\) of the solution \(u\) given in (3.2). For the case when \(\alpha\) is a positive integer, see also [13] [18].
Lemma 3.3. For $b > 0$ and $T > 0$ let $r_0 = nb/(2\alpha)$, $I = [0,T)$. Assume that $r \geq r_0 > 1$ and that $(q,p,r)$ is an admissible triplet satisfying that $p > b + 1$.

(i) If $f \in L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^\infty(\mathbb{R}^n))$, then

$$\|Gf\|_{L^\infty(I;L^r)} \leq CT^{1-\frac{nb}{2Dr}}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^p)}$$

for $p < r(1+b)$, and

$$\|Gf\|_{L^\infty(I;L^r)} \leq CT^{1-\frac{nb}{2Dr}}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^p)}\|f\|_{L_{\frac{q}{p-r}}^1(I;L_{\frac{q}{p-r}}^1)}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^\infty)}$$

for $p \geq r(1+b)$, where $\theta = \frac{p - r(b+1)}{(b+1)(p-r)}$.

(ii) If $f \in L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^\infty(\mathbb{R}^n))$, then

$$\|Gf\|_{L^q(I;L^p)} \leq CT^{1-\frac{nb}{2Dr}}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^p)}$$

for $p < r(1+b)$, and

$$\|Gf\|_{L^q(I;L^p)} \leq CT^{1-\frac{nb}{2Dr}}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^p)}\|f\|_{L_{\frac{q}{p-r}}^1(I;L_{\frac{q}{p-r}}^1)}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^\infty)}$$

for $p \geq r(1+b)$, where $\theta$ is the same as in (i).

Proof. We first prove (i). Consider first the case when $p < r(1+b)$. Using Young’s inequality one has

$$\|Gf\|_{L^\infty(I;L^r)} \leq C \int_0^t (t-s)^{-\frac{n(r+b)}{Dq} \frac{b+1}{r} \|f\|_{L^{\frac{q}{p-r}}} ds$$

$$\leq C \left( \int_0^t (t-s)^{-\frac{n(r+b)}{Dq} \frac{b+1}{r} \|f\|_{L^{\frac{q}{p-r}}} ds \right)^\frac{1}{\lambda} \|f\|_{L_{\frac{q}{p-r}}(I;L_{\frac{q}{p-r}}^p)}$$

$$\leq CT^{1-\frac{nb}{2Dr}}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^p)},$$

where $\frac{1}{\lambda} = 1 - \frac{b+1}{q}$ and $C = C(n,p,r,b)$ depends only on $n$, $p$, $r$, $b$.

For the case $p \geq r(1+b)$, by means of the Riesz interpolation theorem (see e.g. [23, 13] or [1]) and the Hölder inequality we have

$$\|Gf\|_{L^\infty(I;L^r)} \leq \int_0^t \|f(s,x)\|_{L^\infty(I;L^p)} ds\|f(s,x)\|_{L^\infty(I;L^p)}$$

$$\leq C \left( \int_0^t \|f(s,x)\|_{L^\infty(I;L^p)} ds\|f(s,x)\|_{L^\infty(I;L^p)} \right)\|f\|_{L_{\frac{q}{p-r}}(I;L_{\frac{q}{p-r}}^p)}$$

$$\leq CT^{1-\frac{(b+1)(1-\theta)}{q}}\|f\|_{L_{\frac{q}{p-r}}(I;L^p)}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^\infty)}$$

$$\leq CT^{1-\frac{nb}{2Dr}}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^p)}\|f\|_{L_{\frac{q}{p-r}}^1(I;L_{\frac{q}{p-r}}^1)}\|f\|_{L_{\frac{q}{p-r}}^\infty(I;L_{\frac{q}{p-r}}^\infty)},$$
where $\theta$ satisfies
$$
\frac{1}{r(b+1)} = \frac{\theta}{r} + \frac{1 - \theta}{p},
$$
the index of the Hölder inequality is
$$
1 = \left(\frac{1 + b}{1} + \frac{1 - \theta}{p}\right),
$$
and use has been made of the fact that
$$
1 - \frac{(b + 1)(1 - \theta)}{q} < 1 - \frac{b + 1}{q} + \frac{n(b + 1)}{2\alpha}
\left(\frac{1}{r(b+1)} - \frac{1}{p}\right)
= 1 - \frac{nb}{2\alpha r}.
$$

We now prove (ii). For the case $p < r(b+1)$ we have by Young’s inequality that
$$
\|Gf\|_{L^q(I;L^p)} \leq C \left\| \int_0^t (t-s)^{-\frac{\alpha}{2\alpha} \left(\frac{1}{r(b+1)} - \frac{1}{p}\right)} \|f(s, x)\|_{L^p} ds \right\|_{L^q}
\leq C \left( \int_0^T t^{-\frac{\alpha}{2\alpha} \left(\frac{1}{r(b+1)} - \frac{1}{p}\right)} \|f\|_{L^p} ds \right)^{\frac{1}{\alpha}}
\leq CT^{1 - \frac{nb}{2\alpha r}} \|f\|_{L^p(I;L^p)}^{\frac{1}{\alpha}} (I;L^p),
$$
where $1 + \frac{1}{q} = \frac{1 + b}{q} + \frac{1}{\chi}$.

For $p \geq r(b+1)$, arguing similarly as in the proof of (i) gives
$$
\|Gf\|_{L^q(I;L^p)} \leq C \left\| \int_0^t (t-s)^{-\frac{\alpha}{2\alpha} \left(\frac{1}{r(b+1)} - \frac{1}{p}\right)} \|f(s, x)\|_{L^p} ds \right\|_{L^q}
\leq C \left( \int_0^T t^{-\frac{\alpha}{2\alpha} \left(\frac{1}{r(b+1)} - \frac{1}{p}\right)} \|f\|_{L^p} ds \right)^{\frac{1}{\alpha}}
\leq CT^{1 - \frac{nb}{2\alpha r}} \|f\|_{L^p(I;L^p)}^{\frac{1}{\alpha}} (I;L^p),
$$
where $\theta$ and $\chi$ satisfy that
$$
\frac{1}{r(b+1)} = \frac{\theta}{r} + \frac{1 - \theta}{p},
1 + \frac{1}{q} = \frac{(b + 1)(1 - \theta)}{q} + \frac{1}{\chi},
$$
which is meaningful by the fact that $r < r(b+1) < p$.

Arguing similarly in the proof of Lemma 3.4, we can derive the estimates in the spaces $C_q(I;L^p(\mathbb{R}^n))$ and $C_b(I;L^p(\mathbb{R}^n))$ of the non-homogeneous term. In fact, for the case $p < r(b+1)$ (which implies $q > b + 1$), one has by Lemma 3.1 that
$$
\|Gf\|_{L^\infty(I;L^r)} \leq C \int_0^t (t-s)^{-\frac{\alpha}{2\alpha} \left(\frac{b+1}{p} - \frac{1}{r}\right)} \|f(s, x)\|_{L^\infty} ds
\leq C \int_0^t (t-s)^{-\frac{\alpha}{2\alpha} \left(\frac{b+1}{p} - \frac{1}{r}\right)} s^{-\frac{b+1}{q}} ds \|f\|_{C^\alpha(I;L^\infty)}
\leq CT^{1 - \frac{nb}{2\alpha r}} \|f\|_{C^\alpha(I;L^\infty)}^{\frac{1}{\alpha}} (I;L^\infty),
$$

13
where $C = C(n, p, r, b)$ depends only on $n$, $p$, $r$, $b$. Making use of the space-time estimates for the heat equation (cf. [15]) and Young’s inequality we get

$$
\|Gf\|_{C_q(t;L^p)} \leq C \sup_{t \in I} \int_0^t (t-s)^{-\frac{n}{2n} \left( \frac{b+1}{b} - \frac{1}{p} \right)} \|f(s, x)\|_{L^p} \, ds
$$

$$
\leq C \sup_{t \in I} \int_0^t (t-s)^{-\frac{n}{2n} \left( \frac{b+1}{b} - \frac{1}{q} \right)} \|f\|_{C_q(I;L^q)} \, ds
$$

$$
\leq C T^{1 - \frac{nb}{2n \alpha}} \|f\|_{C_q(I;L^q)}
$$

For the case $p \geq r(b+1)$, we use the Riesz interpolation theorem (see [23] [13] or [11]) to get, on noting the definition of the space $C_q(I;L^p(\mathbb{R}^n))$, that for any $0 < t \leq T$

$$
\|Gf\|_{L^\infty(I;L^r)} \leq C \int_0^t \|f(s, x)\|_{L^q} \left( \frac{b+1}{r(b+1)} \right) \, ds
$$

$$
= C \int_0^t \|f(s, x)\|_{L^q} \left( \frac{b+1}{r(b+1)} \right) \|f(s, x)\|_{L^p} \left( \frac{1}{r(b+1)} \right) \, ds
$$

$$
\leq C \|f\|_{L^q} \left( \frac{b+1}{r(b+1)} \right) \|f\|_{L^p} \left( \frac{1}{r(b+1)} \right) \int_0^t \|f\|_{C_q(I;L^p)} \left( \frac{1}{r(b+1)} \right) \, ds
$$

$$
\leq C \|f\|_{L^q} \left( \frac{b+1}{r(b+1)} \right) \|f\|_{L^p} \left( \frac{1}{r(b+1)} \right) \int_0^t \|f\|_{C_q(I;L^p)} \left( \frac{1}{r(b+1)} \right) \, ds
$$

$$
\leq C T^{1 - \frac{nb}{2n \alpha}} \|f\|_{C_q(I;L^p)}
$$

where $\theta$ satisfies $\frac{1}{r(b+1)} = \frac{\theta}{r} + \frac{1 - \theta}{p}$. To get the estimate in the time weighted space $C_q(I;L^p(\mathbb{R}^n))$ we make use of the Riesz interpolation theorem again and obtain that

$$
\|Gf\|_{C_q(I;L^p)} \leq C \sup_{t \in I} \int_0^t (t-s)^{-\frac{n}{2n} \left( \frac{b+1}{b} - \frac{1}{p} \right)} \|f(s, x)\|_{L^p} \left( \frac{b+1}{r(b+1)} \right) \, ds
$$

$$
\leq C \sup_{t \in I} \int_0^t (t-s)^{-\frac{n}{2n} \left( \frac{b+1}{b} - \frac{1}{q} \right)} \|f(s, x)\|_{L^q} \left( \frac{b+1}{r(b+1)} \right) \|f\|_{C_q(I;L^q)} \left( \frac{1}{r(b+1)} \right) \, ds
$$

$$
\leq C \sup_{t \in I} \int_0^t (t-s)^{-\frac{n}{2n} \left( \frac{b+1}{b} - \frac{1}{q} \right)} \|f\|_{C_q(I;L^q)} \left( \frac{1}{r(b+1)} \right) \, ds
$$

$$
\leq C T^{1 - \frac{nb}{2n \alpha}} \|f\|_{C_q(I;L^q)}
$$

where $\theta$ is the same as in (3.8). Thus we have obtained the following results.

**Lemma 3.4.** For $b > 0$ and $T > 0$, let $r_0 = nb/(2\alpha)$, $I = [0, T)$. Assume that $r \geq r_0 > 1$. Let $(q, p, r)$ be any generalized admissible triplet satisfying that $p > b+1$.

(i) If $f \in C_{\frac{b+1}{r(b+1)}}(I;L^\infty(\mathbb{R}^n))$, then

$$
\|Gf\|_{L^\infty(I;L^r)} \leq C T^{1 - \frac{nb}{2n \alpha}} \|f\|_{C_{\frac{b+1}{r(b+1)}}(I;L^\infty(\mathbb{R}^n))}
$$

for $p < r(b+1)$, and

$$
\|Gf\|_{L^\infty(I;L^r)} \leq C T^{1 - \frac{nb}{2n \alpha}} \|f\|_{C_{\frac{b+1}{r(b+1)}}(I;L^\infty(\mathbb{R}^n))}
$$

for $p \geq r(b+1)$.
for \( p \geq r(b+1) \), where \( \theta = \frac{p-r(b+1)}{(b+1)(p-r)} \).

(ii) If \( f \in C_{\infty}^{b+1}(I; L^{p+1}(\mathbb{R}^n)) \), then

\[
\| Gf \|_{C_p(I; L^p)} \leq CT^{1-\frac{nb}{2\alpha}} \| f \|_{C_{\infty}^{b+1}(I; L^{p+1})}
\]

for \( p < r(b+1) \), and

\[
\| Gf \|_{C_p(I; L^p)} \leq CT^{1-\frac{nb}{2\alpha}} \| f \|_{\frac{1}{b+1}} || f \|_{L^{\infty}(I; L^r)} \| f \|_{\frac{1}{b+1}} || f \|_{C_p(I; L^p)}^{(1-\theta)(b+1)}
\]

for \( p \geq r(b+1) \), where \( \theta \) is the same as in (i).

4 Well-posedness in Lebesgue spaces

In this section we consider the following Cauchy problem for the semi-linear fractional power dissipative equation

\[
\begin{align*}
\begin{cases}
u_t + (-\Delta)^\alpha u &= \pm|u|^b u, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n; \\
u(0) &= \varphi(x),
\end{cases}
\end{align*}
\]

(4.1)

We shall study the well-posedness of the Cauchy problem (4.1) for the initial data \( \varphi \in L^r(\mathbb{R}^n) \), \( r \geq r_0 = \frac{nb}{2\alpha} > 1 \). The corresponding integral equation is

\[
u(x, t) = S_\alpha(t)\varphi(x) + \int_0^t S_\alpha(t-\tau)f(\tau, x)d\tau = S_\alpha(t)\varphi(x) + Gf(u) \triangleq T(u),
\]

where \( f(u) = \pm|u|^b u \). The solution to the integral equation (4.2) is called a mild solution which, by the standard regularity effect, is regular for \( t > 0 \).

We first consider the solution to (4.1) (or equivalently (4.2)) in the space

\[
X(I) = C(I; L^r(\mathbb{R}^n)) \cap L^b(I; L^p(\mathbb{R}^n)),
\]

(4.3)

where \( I = [0, T) \) for \( T > 0 \). Using Lemmas 3.2 and 3.3 and applying the Banach contraction mapping principle to the integral operator \( T \), it is easy to establish the following theorems on the existence of local solutions or global small solutions to the problem (4.1). We omit the proof here for succinctness.

**Theorem 4.1.** Let \( 1 < r_0 = \frac{nb}{2\alpha} \leq r \) and let \( \varphi \in L^r(\mathbb{R}^n) \). Assume that \( (q, p, r) \) is an arbitrary admissible triplet.

(i) There exist \( T > 0 \) and a unique mild solution \( u \in X(I) \) to the problem (4.1), where \( T = T(\| \varphi \|_{L^r}) \) depends on the norm \( \| \varphi \|_{L^r} \) for \( r > r_0 \), and \( T = T(\varphi) \) depends on \( \varphi \) itself for the case \( r = r_0 \).

(ii) If \( r = r_0 \), then \( T = \infty \) provided that \( \| \varphi \|_{L^r} \) is sufficiently small. In other words, there exists a global small solution \( u \in C_0([0, \infty); L^b(\mathbb{R}^n)) \cap L^q([0, \infty); L^p(\mathbb{R}^n)) \).

(iii) Let \( [0, T^*) \) be the maximal existence interval of the solution \( u \) to the problem (4.1) (or equivalently (4.2)) such that \( u \in L^q([0, T^*); L^p(\mathbb{R}^n)) \cap C_0([0, T^*); L^r(\mathbb{R}^n)) \) for \( r > r_0 \). Then

\[
\| u(s) \|_{L^r} \geq \frac{C}{(T^* - s)^{\frac{2\alpha}{b}}}.
\]
We now consider the solution to (4.1) (or equivalently (4.2)) in the space

\[ Y(I) = C_0(I; L'(\mathbb{R}^n)) \cap \hat{C}_q(I; L^p(\mathbb{R}^n)), \]

where \( I = [0, T] \) for \( T > 0 \). Making use of Lemmas 3.2 and 3.4 together with the Banach contraction mapping principle to the integral equation (4.2) we can derive the following well-posedness results.

**Theorem 4.2.** Let \( 1 < r_0 = nb/(2\alpha) \leq r \) and let \( \varphi \in L'^{(\mathbb{R}^n)} \). Assume that \((q, p, r)\) is any generalized admissible triplet.

(i) There exist \( T > 0 \) and a unique mild solution \( u \in Y(I) \) to the problem (4.1), where \( T = T(\|\varphi\|_{L^r}) \) depends on the norm \( \|\varphi\|_{L^r} \) for the case \( r > r_0 \), and \( T = T(\varphi) \) depends on \( \varphi \) itself for the case \( r = r_0 \).

(ii) If \( r = r_0 \), the \( T = \infty \) provided that \( \|\varphi\|_{L^{r_0}} \) is sufficiently small. In other words, there exists a global small solution \( u \in C_0([0, \infty); L'((\mathbb{R}^n)) \cap \hat{C}_q([0, \infty); L^p(\mathbb{R}^n)) \).

(iii) Let \( I = [0, T^*] \) be the maximal existence interval of the solution \( u \) to the problem (4.2) such that \( u \in C_0(I; L'(\mathbb{R}^n)) \cap \hat{C}_q(I; L^p(\mathbb{R}^n)) \) for \( r > r_0 \). Then

\[ \|u(s)\|_{L^r} \geq \frac{C}{(T^* - s)^{\frac{1}{b} - \frac{1}{2\alpha}}} . \]

Our method is also valid for the case of convective nonlinear term, that is, the following Cauchy problem for the fractional power dissipative convective equation:

\[
\begin{align*}
& a \cdot \nabla (\varphi + (-\Delta)^a u) = (a \cdot \nabla)g(u), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n; \\
& u(0) = \varphi(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( b > 0, \alpha > 0 \) and \( a \in \mathbb{R}^n \) is a given \( n \)-dimensional vector. By Duhamel's principle the problem (4.3) is equivalent to the integral equation:

\[ u(t, x) = S_\alpha(t)\varphi(x) + \int_0^t S_\alpha(t - \tau)(a \cdot \nabla)g(u) d\tau \triangleq S_\alpha(t)\varphi(x) + \tilde{G}g(u), \quad (4.6) \]

where \( g(u) = \pm |u|^b u \).

Arguing similarly as in the proof of Lemmas 3.3 and 3.4 we have the following nonlinear estimates.

**Lemma 4.1.** For \( b > 0, \alpha > 1/2 \) and \( T > 0 \), let \( r_1 = \frac{nb}{2\alpha - 1} \) and \( I = [0, T] \). Assume that \( r \geq r_1 > 1 \). Let \((q, p, r)\) be an arbitrary admissible triplet satisfying that \( p > b + 1 \). If \( f \in L^{\frac{q'}{r+b}}(I; L^{\frac{p'}{r+b}}(\mathbb{R}^n)) \), then

\[
\|\tilde{G}f\|_{L^\infty(I;L^r)} + \|\tilde{G}f\|_{L^{q}(I;L^p)} \leq C T^{1 - \frac{1}{2\alpha}} \frac{nb}{2\alpha} \|f\|_{L^{\frac{q'}{r+b}}(I;L^{\frac{p'}{r+b}})}
\]

for \( p < r(1 + b) \), and

\[
\|\tilde{G}f\|_{L^\infty(I;L^r)} + \|\tilde{G}f\|_{L^{q}(I;L^p)} \leq C T^{1 - \frac{1}{2\alpha}} \frac{nb}{2\alpha} \|f\|_{L^{\frac{q'}{r+b}}(I;L^{\frac{p'}{r+b}})} \|f\|_{L^{\frac{1}{b+1}}(I;L^{\frac{1}{b+1}})} \leq \theta^{(b+1)} \|f\|_{L^{\frac{1}{b+1}}(I;L^{\frac{1}{b+1}})}
\]

for \( p \geq r(1 + b) \), where \( \theta = \frac{p - r(b + 1)}{(b + 1)(p - r)} \).
Lemma 4.2. For $b > 0$, $\alpha > 1/2$ and $T > 0$, let $r_1 = \frac{nb}{2\alpha - 1}$ and $I = [0, T)$. Assume that $r \geq r_1 > 1$. Let $(q, p, r)$ be an arbitrary generalized admissible triplet satisfying that $p > b + 1$. If $f \in C_{q,r}^+(I; L^{\frac{p}{p-1}}(\mathbb{R}^n))$, then

$$\|Gf\|_{L^\infty(I; L^p)} \leq \frac{C}{T^1 - \frac{1}{2\alpha}} \cdot \frac{b^\alpha}{r^\alpha} \cdot \|f\|_{C_q^+(I; L^p)}$$

for $p < r(1 + b)$, and

$$\|Gf\|_{L^\infty(I; L^p)} \leq \frac{C}{T^1 - \frac{1}{2\alpha}} \cdot \frac{b^\alpha}{r^\alpha} \cdot \|f\|_{L^\infty(I; L^p)} \cdot \|f\|_{C_q^+(I; L^p)}$$

for $p \geq r(b + 1)$, where $\theta = \frac{p - r(b + 1)}{(b + 1)(p - r)}$.

Using Lemmas 3.2 and 4.1 together with the Banach contraction mapping principle we can get the well-posedness in the space $X(I)$ defined by (4.3) of the Cauchy problem (4.5).

Theorem 4.3. Let $1 < r_1 = \frac{nb}{2\alpha - 1} \leq r$, $\varphi \in L^r(\mathbb{R}^n)$ and for $T > 0$ let $I = [0, T)$. Assume that $(q, p, r)$ is an arbitrary admissible triplet.

(i) There exist a $T > 0$ and a unique mild solution to the problem (4.5) such that $u \in X(I)$, where $T = T(\|\varphi\|_{L^r})$ depends on the norm $\|\varphi\|_{L^r}$ for the case $r > r_1$, or $T = T(\varphi)$ depends on $\varphi$ itself for the case $r = r_1$.

(ii) If $r = r_1$, then we can take $T = \infty$ provided that $\|\varphi\|_{L^r}$ is sufficiently small. In other words, there exists a global small solution $u \in C_0((0, \infty); L^r(\mathbb{R}^n)) \cap L^q((0, \infty); L^p(\mathbb{R}^n))$.

(iii) Let $I = [0, T^*)$ be the maximal existence interval of the solution $u$ to the problem (4.5) (or equivalently (4.7)) such that $u \in L^q(I; L^p(\mathbb{R}^n)) \cap C_b(I; L^r(\mathbb{R}^n))$ for $r > r_1$. Then

$$\|u(s)\|_{L^r} \geq \frac{C}{(T^* - s) \frac{1}{b} - \frac{1}{2\alpha} - \frac{b}{2\alpha}}.$$

Similarly, making use of Lemmas 3.2 and 4.2 and the Banach contraction mapping principle we can establish the following well-posedness in the space $Y(I)$ defined in (4.4) of the Cauchy problem (4.5).

Theorem 4.4. Let $1 < r_1 = \frac{nb}{2\alpha - 1} \leq r$, $\varphi \in L^r(\mathbb{R}^n)$ and for $T > 0$ let $I = [0, T)$. Assume that $(q, p, r)$ is an arbitrary generalized admissible triplet.

(i) There exist a $T > 0$ and a unique mild solution to the problem (4.5) such that $u \in Y(I)$, where $T = T(\|\varphi\|_{L^r})$ depends on the norm $\|\varphi\|_{L^r}$ for the case $r > r_1$ or $T = T(\varphi)$ depends on $\varphi$ itself for the case $r = r_1$.

(ii) If $r = r_1$, then $T = \infty$ provided that $\|\varphi\|_{L^r}$ is sufficiently small, that is, there exists a global small solution $u \in C_0((0, \infty); L^r(\mathbb{R}^n)) \cap C_b((0, \infty); L^p(\mathbb{R}^n))$.

(iii) Let $I = [0, T^*)$ be the maximal existence interval of the solution $u$ to the problem (4.5) such that $u \in C_0(I; L^r(\mathbb{R}^n)) \cap C_b(I; L^r(\mathbb{R}^n))$ for $r > r_1$. Then

$$\|u(s)\|_{L^r} \geq \frac{C}{(T^* - s) \frac{1}{b} - \frac{1}{2\alpha} - \frac{b}{2\alpha}}.$$
5 Fractional power dissipative equations with more general nonlinear terms

5.1 The case of more general nonlinear terms

In this subsection we study well-posedness in Lebesgue spaces for the case of more general nonlinear terms. In particular, we consider the following cases:

\begin{align}
\begin{cases}
  u_t + (-\Delta)^{\alpha}u = f_1(u) + f_2(u), & (t, x) \in [0, \infty) \times \mathbb{R}^n, \quad \alpha > 0, \\
  u(0, x) = \varphi(x), & x \in \mathbb{R}^n;
\end{cases}
\end{align}

\begin{align}
\begin{cases}
  u_t + (-\Delta)^{\alpha}u = f_1(u) + (\beta \cdot \nabla)f_2(u), & (t, x) \in [0, \infty) \times \mathbb{R}^n, \quad 2\alpha > 1, \\
  u(0, x) = \varphi(x), & x \in \mathbb{R}^n;
\end{cases}
\end{align}

and

\begin{align}
\begin{cases}
  u_t + (-\Delta)^{\alpha}u = f_1(u) + \nabla^2 f_2(u), & (t, x) \in [0, \infty) \times \mathbb{R}^n, \quad \alpha > 1, \\
  u(0, x) = \varphi(x), & x \in \mathbb{R}^n.
\end{cases}
\end{align}

Here \(f_1(u) = \pm|u|^{b_1}u\), \(f_2(u) = \pm|u|^{b_2}u\) and \(\beta \in \mathbb{R}^n\). Without loss of generality we assume \(b_1 > b_2 > 0\). Set \(r_0 = nb_1/(2\alpha)\) and let \((q,p,r)\) be an arbitrary admissible or generalized admissible triplet for \(r \geq r_0\). For \(T > 0\) let \(I = [0, T)\) and let

\begin{align*}
  X(I) &= C(I; L^q(I; L^p)) \cap L^q(I; L^p(\mathbb{R}^n)), \\
  Y(I) &= C(I; L^q(I; \mathbb{R}^n)) \cap C_q(I; L^p(\mathbb{R}^n)).
\end{align*}

Then, similarly to Lemmas 3.3 and 3.4 we have the following variant space-time estimates for the operator \(G\) (cf. (3.2) for its definition).

**Lemma 5.1.** Assume that \(r \geq r_0 > 1\). Let \((q,p,r)\) be an arbitrary admissible triplet satisfying that \(p > b_1 + 1\). If \(f_1 \in L^{q/(b_1+1)}(I; L^{p/(b_1+1)}(\mathbb{R}^n))\) and \(f_2 \in L^{q/(b_2+1)}(I; L^{p/(b_2+1)}(\mathbb{R}^n))\), then

\[
\|G(f_1 + f_2)\|_{L^q(I; L^p)} + \|G(f_1 + f_2)\|_{L^q(I; L^p)} \leq CT^{-\frac{nb_1}{2\alpha}}\|f_1\|_{L^{q/(b_1+1)}(I; L^{p/(b_1+1)})} + CT^{-\frac{nb_2}{2\alpha}}\|f_2\|_{L^{q/(b_2+1)}(I; L^{p/(b_2+1)})}
\]

for the case \(p < r(1 + b_2)\), and

\[
\|G(f_1 + f_2)\|_{L^q(I; L^p)} + \|G(f_1 + f_2)\|_{L^q(I; L^p)} \leq CT^{-\frac{nb_1}{2\alpha}}\|f_1\|_{L^{q/(b_1+1)}(I; L^{p/(b_1+1)})}^{\theta_1(b_1+1)}\|f_1\|_{L^{q/(b_1+1)}(I; L^{p/(b_1+1)})}^{\frac{1}{(1-\theta_1)(b_1+1)}} + CT^{-\frac{nb_2}{2\alpha}}\|f_2\|_{L^{q/(b_2+1)}(I; L^{p/(b_2+1)})}^{\theta_2(b_2+1)}\|f_2\|_{L^{q/(b_2+1)}(I; L^{p/(b_2+1)})}^{\frac{1}{(1-\theta_2)(b_2+1)}}
\]

for the case \(p \geq r(1 + b_1)\), where \(\theta_1 = \frac{p - r(b_1 + 1)}{(b_1 + 1)(p - r)}\) and \(\theta_2 = \frac{p - r(b_2 + 1)}{(b_2 + 1)(p - r)}.\) If \(r(1 + b_2) \leq p < r(1 + b_1)\), then

\[
\|G(f_1 + f_2)\|_{L^q(I; L^p)} + \|G(f_1 + f_2)\|_{L^q(I; L^p)} \leq CT^{-\frac{nb_1}{2\alpha}}\|f_1\|_{L^{q/(b_1+1)}(I; L^{p/(b_1+1)})}^{\frac{1}{b_1+1}} + CT^{-\frac{nb_2}{2\alpha}}\|f_2\|_{L^{q/(b_2+1)}(I; L^{p/(b_2+1)})}^{\frac{1}{b_2+1}}\|f_2\|_{L^{q/(b_2+1)}(I; L^{p/(b_2+1)})}^{(1-\theta_2)(b_2+1)}.
\]
Lemma 5.2. Assume that $r \geq r_0 > 1$. Let $(q,p,r)$ be an arbitrary generalized admissible triplet satisfying that $p > b_1 + 1$. If $f_1 \in C_{\frac{q}{q+1}}(I; L_{\frac{p}{p+1}}^q(\mathbb{R}^n))$ and $f_2 \in C_{\frac{q}{q+1}}(I; L_{\frac{p}{p+1}}^q(\mathbb{R}^n))$, then

$$\|G(f_1 + f_2)\|_{L_q(I; L_r)} + \|G(f_1 + f_2)\|_{C_q(I; L_p)} \leq CT^{1 - \frac{nb_1}{2r_0}} \|f_1\|_{C_{\frac{q}{q+1}}(I; L_{\frac{p}{p+1}}^q(\mathbb{R}^n))} + CT^{1 - \frac{nb_2}{2r_0}} \|f_2\|_{C_{\frac{q}{q+1}}(I; L_{\frac{p}{p+1}}^q(\mathbb{R}^n))}$$

for the case $p < r(1 + b_2)$, and

$$\|G(f_1 + f_2)\|_{L_q(I; L_r)} + \|G(f_1 + f_2)\|_{C_q(I; L_p)} \leq CT^{1 - \frac{nb_1}{2r_0}} \|f_1\|_{L_{\frac{p}{p+1}}^q(I; L_r)} \|\theta_1(b_1 + 1)\|_{C_q(I; L_p)} + CT^{1 - \frac{nb_2}{2r_0}} \|f_2\|_{L_{\frac{p}{p+1}}^q(I; L_r)} \|\theta_2(b_2 + 1)\|_{C_q(I; L_p)}$$

for the case $p \geq r(b_1 + 1)$, where $\theta_1$ and $\theta_2$ are the same as defined in Lemma 5.1. If $r(b_2 + 1) \leq p < r(b_1 + 1)$, then

$$\|G(f_1 + f_2)\|_{L_q(I; L_r)} + \|G(f_1 + f_2)\|_{C_q(I; L_p)} \leq CT^{1 - \frac{nb_1}{2r_0}} \|f_1\|_{L_{\frac{p}{p+1}}^q(I; L_r)} + CT^{1 - \frac{nb_2}{2r_0}} \|f_2\|_{L_{\frac{p}{p+1}}^q(I; L_r)}.$$

Using Lemmas 5.2 and 5.1 and the space $X(I)$ it is easy to prove the well-posedness of the Cauchy problem by the Banach contraction mapping principle.

Theorem 5.1. For $r \geq r_0 > 1$ let $\varphi \in L^r(\mathbb{R}^n)$. Let $(q,p,r)$ be an arbitrary admissible triplet.

(i) There exist a $T > 0$ and a unique mild solution to the problem [5.7] such that $u \in X(I)$, where $T = T(\|\varphi\|_{L^r})$ depends on the norm $\|\varphi\|_{L^r}$ for the case $r > r_0$ or $T = T(\varphi)$ depends on $\varphi$ itself for the case $r = r_0$.

(ii) Let $I = [0, T^*)$ be the maximal existence interval of the solution $u$ to the problem [5.7] such that $u \in L^q([0, T^*]; L^p(\mathbb{R}^n)) \cap C_b([0, T^*]; L^r(\mathbb{R}^n))$ for $r > r_0$. Then

$$\|u(s)\|_{L^r} \geq \frac{C}{(T^* - s)^{\frac{1}{b_1} - \frac{nb_1}{2r_0}}}.$$

Consequently, if $T^* < \infty$, then

$$\lim_{t \to T^*} \|u(t)\|_{L^r} = \infty.$$

Similarly, utilizing the space $Y(I)$ and Lemmas 3.2 and 5.2 in conjunction with the Banach contraction mapping principle we can establish the following well-posedness of the Cauchy problem 5.1

Theorem 5.2. For $r \geq r_0 > 1$ let $\varphi \in L^r(\mathbb{R}^n)$. Let $(q,p,r)$ be an arbitrary generalized admissible triplet.

(i) There exist a $T > 0$ and a unique mild solution to the problem [5.7] such that $u \in Y(I)$, where $T = T(\|\varphi\|_{L^r})$ depends on the norm $\|\varphi\|_{L^r}$ for the case $r > r_0$ or $T = T(\varphi)$ depends on $\varphi$ itself for the case $r = r_0$.  

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(ii) Let \( I = [0, T^* ) \) be the maximal existence interval of the solution \( u \) to the problem (5.1) such that \( u \in \mathcal{C}_q([0, T^* ) ; L^p(\mathbb{R}^n)) \cap C_b([0, T^* ) ; L^r(\mathbb{R}^n)) \) for \( r > r_0 \). Then
\[
\| u(s) \|_{L^r} \geq \frac{C}{(T^* - s)^{\frac{1}{q} - \frac{n}{2a}}}
\]
Consequently, if \( T^* < \infty \), then
\[
\lim_{t \to T^*} \| u(t) \|_{L^r} = \infty.
\]
Consider now the problems (5.2) and (5.3) with the convective effect or with higher-order derivative term, respectively, and for \( 0 \leq d < 2\alpha \) let \( r_d = nb_1/(2\alpha - d) \). Similarly to Lemmas 5.1 and 5.2 we have the following estimates of the nonlinear terms.

**Lemma 5.3.** Let \( r \geq r_d > 1 \) and let \((q, p, r)\) be an arbitrary admissible triplet satisfying that \( p > b_1 + 1 \). If \( f_1 \in L^{\frac{np}{n+q}}(I; L^{\frac{np}{b_1+1}}(\mathbb{R}^n)) \) and \( f_2 \in L^{\frac{np}{n+q}}(I; L^{\frac{np}{b_2+1}}(\mathbb{R}^n)) \), then
\[
\|G(f_1 + g)\|_{L^{\infty}(I;L^r)} + \|G(f_1 + g)\|_{L^q(I;L^p)} \\
\leq CT^{1-\frac{nb_1}{2\alpha}}\|f_1\|_{L^{\frac{np}{b_1+1}}(I;L^{\frac{np}{b_1+1}}(\mathbb{R}^n))} + CT^{1-\frac{d}{2\alpha}}\|f_2\|_{L^{\frac{np}{b_2+1}}(I;L^{\frac{np}{b_2+1}}(\mathbb{R}^n))}
\]
for \( p < r(1 + b_2) \), and
\[
\|G(f_1 + g)\|_{L^{\infty}(I;L^r)} + \|G(f_1 + g)\|_{L^q(I;L^p)} \\
\leq CT^{1-\frac{nb_1}{2\alpha}}\|f_1\|_{L^{\frac{np}{b_1+1}}(I;L^{\frac{np}{b_1+1}}(\mathbb{R}^n))} + CT^{1-\frac{d}{2\alpha}}\|f_2\|_{L^{\frac{np}{b_2+1}}(I;L^{\frac{np}{b_2+1}}(\mathbb{R}^n))}
\]
for \( p \geq r(b_1 + 1) \), where \( \theta_1 \) and \( \theta_2 \) are the same as in Lemma 5.1. If \( r(1 + b_2) \leq p < r(b_1 + 1) \), then
\[
\|G(f_1 + g)\|_{L^{\infty}(I;L^r)} + \|G(f_1 + g)\|_{L^q(I;L^p)} \\
\leq CT^{1-\frac{nb_1}{2\alpha}}\|f_1\|_{L^{\frac{np}{b_1+1}}(I;L^{\frac{np}{b_1+1}}(\mathbb{R}^n))} + CT^{1-\frac{d}{2\alpha}}\|f_2\|_{L^{\frac{np}{b_2+1}}(I;L^{\frac{np}{b_2+1}}(\mathbb{R}^n))}
\]
Here \( g = (\beta \cdot \nabla)f_2, d = 1 \) for the problem (5.2) or \( g = \nabla^2 f_2, d = 2 \) for the problem (5.3).

**Lemma 5.4.** Let \( r \geq r_d > 1 \) and let \((q, p, r)\) be an arbitrary generalized admissible triplet satisfying that \( p > b_1 + 1 \). If \( f_1 \in C_{\frac{q}{n+q}}(I; L^{\frac{pn}{b_1+n}}(\mathbb{R}^n)) \) and \( f_2 \in C_{\frac{q}{n+q}}(I; L^{\frac{pn}{b_2+n}}(\mathbb{R}^n)) \), then
\[
\|G(f_1 + g)\|_{L^\infty(I;L^r)} + \|G(f_1 + g)\|_{C^q(I;L^p)} \\
\leq CT^{1-\frac{nb_1}{2\alpha}}\|f_1\|_{C_{\frac{q}{n+q}}(I; L^{\frac{pn}{b_1+n}}(\mathbb{R}^n))} + CT^{1-\frac{d}{2\alpha}}\|f_2\|_{C_{\frac{q}{n+q}}(I; L^{\frac{pn}{b_2+n}}(\mathbb{R}^n))}
\]
for \( p < r(1 + b_2) \), and
\[
\|G(f_1 + g)\|_{L^\infty(I;L^r)} + \|G(f_1 + g)\|_{C^q(I;L^p)} \\
\leq CT^{1-\frac{nb_1}{2\alpha}}\|f_1\|_{C_{\frac{q}{n+q}}(I; L^{\frac{pn}{b_1+n}}(\mathbb{R}^n))} + CT^{1-\frac{d}{2\alpha}}\|f_2\|_{C_{\frac{q}{n+q}}(I; L^{\frac{pn}{b_2+n}}(\mathbb{R}^n))}
\]
for $p \geq r(b_1+1)$, where $\theta_1$ and $\theta_2$ are the same as in Lemma 5.1. If $r(b_2+1) \leq p < r(b_1+1)$, then

$$
\|G(f_1+g)\|_{L^\infty(I;L^r)} + \|G(f_1+g)\|_{C_q(I;L^p)}
\leq CT^{-\frac{n b}{2 + \alpha}} \|f_1\|_{C^\frac{1}{q} \Lambda (I;L^\frac{1}{r})} + CT^{-\frac{d}{2 + \alpha}} \|f_2\|_{L^\infty(I;L^r)} \|f_2\|_2^{\frac{1-\theta_2}{2 + \alpha}} \|C_q(I;L^p)\|^{(1-\theta_2)(b_2+1)}.
$$

Here $g$ and $d$ are the same as defined in Lemma 5.3.

Similarly as before, using Lemma 3.2 and Lemma 5.3 or 5.4 and the space $X(I)$ or $Y(I)$ we can establish the following results (Theorem 5.3 or 5.4, respectively) on well-posedness of the Cauchy problem (5.2) and (5.3).

**Theorem 5.3.** For $r \geq r_d = nb_1/(2\alpha - d) > 1$ with $0 \leq d < 2\alpha$ let $\varphi \in L^r(\mathbb{R}^n)$. Assume that $(q,p,r)$ is an arbitrary admissible triplet.

(i) There exist a $T > 0$ and a unique mild solution $u \in X(I)$ to the problem (5.2) or (5.3), where $T = T(\|\varphi\|_r)$ depends on the norm $\|\varphi\|_r$ for the case $r > r_d$ or $T = T(\varphi)$ depends on $\varphi$ itself for the case $r = r_d$.

(ii) Let $I = [0,T^*)$ be the maximal existence interval of the solution $u$ to the problem (5.2) or (5.3) such that $u \in L^q([0,T^*);L^p(\mathbb{R}^n)) \cap C_0([0,T^*);L^r(\mathbb{R}^n))$ for $r > r_d$. Then

$$
\|u(s)\|_{L^r} \geq \frac{C}{(T^* - s)^{\frac{1}{b_1} - \frac{d}{2 + \alpha} + \frac{n}{2\alpha}}},
$$

Consequently, if $T^* < \infty$, then

$$
\lim_{t \to T^*} \|u(t)\|_{L^r} = \infty.
$$

Here $d = 1$ in the case of (5.2) or $d = 2$ in the case of (5.3).

**Theorem 5.4.** For $r \geq r_d = nb_1/(2\alpha - d) > 1$ with $0 \leq d < 2\alpha$ let $\varphi \in L^r(\mathbb{R}^n)$. Assume that $(q,p,r)$ is an arbitrary generalized admissible triplet.

(i) There exist a $T > 0$ and a unique mild solution $u \in X(I)$ to the problem (5.2) or (5.3), where $T = T(\|\varphi\|_r)$ depends on the norm $\|\varphi\|_r$ for the case $r > r_d$ or $T = T(\varphi)$ depends on $\varphi$ itself for the case $r = r_d$.

(ii) Let $[0,T^*)$ be the maximal existence interval of the solution $u$ to (5.2) or (5.3) such that $u \in C_0([0,T^*);L^p(\mathbb{R}^n)) \cap C_0([0,T^*);L^r(\mathbb{R}^n))$ for $r > r_d$. Then

$$
\|u(s)\|_{L^r} \geq \frac{C}{(T^* - s)^{\frac{1}{b_1} - \frac{d}{2 + \alpha} + \frac{n}{2\alpha}}},
$$

Consequently, if $T^* < \infty$, then

$$
\lim_{t \to T^*} \|u(t)\|_{L^r} = \infty.
$$

Here $d = 1$ in the case of (5.2) or $d = 2$ in the case of (5.3).
The proof is broken down into the following three steps.

Step 1. Assume that the generalized admissible triplets \((q_j, p_j, r_j)\) \((j = i, 1)\) satisfy the conditions

\[
1 + b_j < r_j(1 + b_j), \quad j = 1, 2
\]

and

\[
\frac{r_1}{p_1} = \frac{r_2}{p_2}. \tag{5.5}
\]
For $I = [0, \infty)$ define the solution space as

$$Z(I) = \{ u \mid u \in C_b(I; L^{r_1} \cap L^{r_2}) \cap \hat{C}_{q_1}(I; L^{p_1}) \cap \hat{C}_{q_2}(I; L^{p_2}) \}$$

with the norm

$$\| u; Z(I) \| = \sum_{j=1}^{2} \sup_{t \in I} \frac{1}{t^j} \| u \|_{L^{p_j}} + \sum_{j=1}^{2} \sup_{t \in I} \| u \|_{L^{r_j}}.$$

The problem (1.1) can be written in the integral form as

$$u(x, t) = S_{\alpha}(t) \varphi(x) + \hat{G}(f_1(u) + f_2(u)) \triangleq T u,$$

where

$$\hat{G}(f_1(u) + f_2(u)) = \int_0^t S_{\alpha}(t - \tau) Q(D) [f_1(u(\tau, x)) + f_2(u(\tau, x))] d\tau.$$

Now consider the operator $T$ in the complete metric space

$$E(I) = \{ u \in Z(I) \mid \| u; Z(I) \| \leq \delta \}$$

with the metric

$$d(u, v) = \| u - v; Z(I) \|, \quad u, v \in E(I),$$

where $\delta > 0$ is a sufficiently small constant to be determined later. By Lemmas 3.1 and 3.2 one has

$$\| S_{\alpha}(t) \varphi; Z(I) \| \leq C(\| \varphi \|_{r_1} + \| \varphi \|_{r_2}). \quad (5.6)$$

By Lemma 5.4 and the Hölder inequality for $\frac{1}{p} = \frac{1}{p_1} + \frac{b_2}{p_2}$ we get

$$\| \hat{G}(f_1(u) + f_2(u)); C_{q_1}(I; L^{p_1}) \| \leq C\| u; C_{q_1}(I; L^{p_1}) \|^b_1 + \sup_{t \in I} \frac{1}{t^1} \int_0^t (t - \tau)^{-\frac{d}{2n} - \frac{b_2}{2p_2}} \| u \|^b_2 \| u \|_{p_2} d\tau$$

$$\leq C\| u; C_{q_1}(I; L^{p_1}) \|^b_1 + C \int_0^1 (1 - \tau)^{-\frac{d}{2n} - \frac{b_2}{2p_2} - \frac{1}{r_1} - \frac{b_1}{r_2}} d\tau \| u; C_{q_2}(I; L^{p_2}) \|^b_2 \| u; C_{q_1}(I; L^{p_1}) \|$$

$$\leq C \sum_{j=1}^{2} \| u; C_{q_j}(I; L^{p_j}) \|^b_1 \| u; C_{q_1}(I; L^{p_1}) \| \quad (5.7)$$

and

$$\| \hat{G}(f_1(u) + f_2(u)); C(I; L^{r_1}) \| \leq C \sup_{t \in I} \int_0^t (t - \tau)^{-\frac{d}{2n} - \frac{b_1}{2p_1} - \frac{1}{r_1}} \| u \|^1 \| u \|^b_1 d\tau + C \sup_{t \in I} \int_0^t (t - \tau)^{-\frac{d}{2n} - \frac{b_1}{2p_1} - \frac{1}{r_1}} \| u \|^b_2 \| u \|^b_1 d\tau$$

$$\leq C \sup_{t \in I} \int_0^t (t - \tau)^{-\frac{d}{2n} - \frac{b_1}{2p_1} - \frac{1}{r_1}} d\tau \| u; C_{q_1}(I; L^{p_1}) \|^{1+b_1}$$

$$+ C \sup_{t \in I} \int_0^t (t - \tau)^{-\frac{d}{2n} - \frac{b_2}{2p_1} + \frac{1}{r_1}} (t - \tau)^{-\frac{b_2}{r_2} - \frac{1}{r_1}} d\tau \| u; C_{q_2}(I; L^{p_2}) \|^b_2 \| u; C_{q_1}(I; L^{p_1}) \|$$

$$\leq C \sum_{j=1}^{2} \| u; C_{q_j}(I; L^{p_j}) \|^b_1 \| u; C_{q_1}(I; L^{p_1}) \|. \quad (5.8)$$
Similarly, we have
\[
\|\tilde{G}(f_1(u) + f_2(u)); C_{q_2}(I; L^{p_2})\| \leq C \sum_{j=1}^{2} \|u; C_{q_j}(I; L^{p_j})\|^b_j \|u; C_{q_2}(I; L^{p_2})\|, \quad (5.9)
\]
\[
\|\tilde{G}(f_1(u) + f_2(u)); C(I; L^{r_1})\| \leq C \sum_{j=1}^{2} \|u; C_{q_j}(I; L^{p_j})\|^b_j \|u; C_{q_2}(I; L^{p_2})\|. \quad (5.10)
\]
Combining the estimates (5.6)-(5.10) and choosing \(\delta > 0\) small enough we get
\[
\|T u; Z(I)\| \leq C \|\varphi; L^{r_1} \cap L^{r_2}\| + C \sum_{j=1}^{2} \|u; C_{q_j}(I; L^{p_j})\|^b_j \|u; C_{q_1}(I; L^{p_1})\| +
\]
\[
C \sum_{j=1}^{2} \|u; C_{q_j}(I; L^{p_j})\|^b_j \|u; C_{q_2}(I; L^{p_2})\|
\leq C \|\varphi; L^{r_1} \cap L^{r_2}\| + C \delta^{b_1+1} + C \delta^{b_2+1} < \delta \quad (5.11)
\]
provided that \(C \|\varphi; L^{r_1} \cap L^{r_2}\| < \frac{\delta}{2}\). Noting the definition of \(E(I)\) and the fact that
\[
d(T u, T v) \leq C \sum_{j=1}^{2} (\|u; C_{q_j}(I; L^{p_j})\|^b_j + \|v; C_{q_j}(I; L^{p_j})\|^b_j) d(u, v),
\]
one has \(d(T u, T v) \leq \frac{1}{2} d(u, v)\). Furthermore, from (5.11), and since
\[
\lim_{t \to 0^+} t^{\frac{1}{b_j}} \|S_\alpha(t)\varphi; L^{r_1} \cap L^{r_2}\| = 0,
\]
it follows that
\[
\lim_{t \to 0^+} t^{\frac{1}{b_j}} \|T u\|_{L^{p_j}} = 0, \quad j = 1, 2.
\]
Thus \(T\) is a contraction mapping from \(E(I)\) into itself so, by the Banach contraction mapping principle there exists a unique solution \(u \in E(I)\).

Step 2. We show that \(u \in C_q(I; L^p)\) for any generalized admissible triplet \((q, p, r_1)\) satisfying the condition (5.3). Without lost of generality we assume that the generalized admissible triplet \((q_j, p_j, r_j)\) satisfies the conditions (5.4) and (5.5) for \(j = 1, 2\). Arguing similarly as in deriving (5.7) we have
\[
\|u; C_q(I; L^p)\| \leq C \|\varphi\|_{L^\infty} + \sup_{\tau \in I} \int_{0}^{\tau} (t - \tau)^{-\frac{1}{2m} - \frac{ab}{2m(p_1)}} \|u\|_{L_1} \|u\|_{L^p} d\tau
\]
\[
\quad + \sup_{\tau \in I} \int_{0}^{\tau} (t - \tau)^{-\frac{1}{2m} - \frac{ab}{2m(p_2)}} \|u\|_{L_1} \|u\|_{L^p} d\tau
\leq C \|\varphi\|_{L^\infty} + C \|u; C_{q_1}(I; L^{p_1})\| \|u; C_{q_2}(I; L^{p_2})\|
\]
\[
+ C \|u; C_{q_2}(I; L^{p_2})\|^b_2 \|u; C_{q_1}(I; L^{p_1})\|
\leq C \|\varphi\|_{L^\infty} + C (\delta^{b_1} + \delta^{b_2}) \|u; C_q(I; L^p)\|,
\]

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which implies that for small $\delta > 0$

$$\|u; C_q(I; L^p)\| < \infty.$$  

Similar arguments as above give

$$\lim_{t \to 0^+} t^{\frac{1}{q}}\|u\|_{L^p} = 0.$$  

Thus, it is derived that $u \in \mathcal{C}_q(I; L^p)$ for any generalized admissible triplet $(q, p, r_1)$ satisfying the condition (5.4).

Step 3. Finally, if $(q, p, r_1)$ is a generalized admissible triplet satisfying $p \leq 1 + b_1$, then the result that $u \in \mathcal{C}_q(I; L^p)$ follows by interpolation between $C(I; L^p)$ and $\mathcal{C}_{\tilde{q}(\tilde{p}, r_1)}(I; L^p_{\tilde{p}})$ with the generalized admissible triplet $(\tilde{q}, \tilde{p}, r_2)$ satisfying the condition (5.4). The proof is thus complete.

Remark 5.2. One can easily see that there exist generalized admissible triplets $(q_j, p_j, r_j)$ $(j = 1, 2)$ satisfying condition (5.3). In fact, without loss of generality we assume $b_1 \geq b_2$, which implies $r_1 \geq r_2$. Since

$$\frac{1 + b_1}{r_1} < \frac{p_1}{r_1} \leq (1 + b_1), \quad \frac{1 + b_2}{r_2} < \frac{p_2}{r_2} \leq (1 + b_2)$$

and

$$\left(\frac{2\alpha - d}{n} \left(1 + \frac{1}{b_2}\right), 1 + b_2\right) \subset \left(\frac{2\alpha - d}{n} \left(1 + \frac{1}{b_1}\right), 1 + b_1\right), \quad \text{for } d \in [0, 2\alpha),$$

it is not difficult to choose generalized admissible triplets $(q_j, p_j, r_j)$ $(j = 1, 2)$ satisfying the condition (5.3). Moreover, since $r_j = \frac{nb_j}{2\alpha - d}$, we also have

$$\frac{b_1}{p_1} = \frac{b_2}{p_2}, \quad \text{and} \quad \frac{b_1}{q_1} = \frac{b_2}{q_2}.$$

6 Global well-posedness for high frequency data

In the previous sections we proved the well-posedness of the problem (4.1) for the initial data $\varphi \in L^r(\mathbb{R}^n)$ with $r \geq r_0 = nb/(2\alpha)$, that is, if the norm $\|\varphi\|_{L^{r_0}}$ of the initial data is small enough, then the solution $u$ exists globally. In this section we shall show that the solution $u$ exists globally if the norm $\|\varphi; B^{\frac{2\alpha - 2b}{2\alpha}}_{p, \infty}(\mathbb{R}^n)\|$ is small enough (in this case, the norm $\|\varphi\|_{L^{r_0}}$ may be large). To this end, let $I = [0, \infty)$ and let us introduce the following solution space

$$X(I) = C(I; L^{r_0}(\mathbb{R}^n)) \cap C_q(I; L^p(\mathbb{R}^n))$$

with the norm

$$\|u\|_{X(I)} = \sup_{t > 0} \|u(t)\|_{\dot{B}^{\frac{2\alpha - 2b}{2\alpha}}_{p, \infty}} + \sup_{t > 0} t^{\frac{1}{q}}\|u\|_{L^p},$$

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where \( r_0 = nb/(2\alpha) \leq p < r_0(b + 1), \) \( p > b + 1, \) \( \sigma = 2\alpha/b - n/p \geq 0 \) and \( \frac{1}{q} = \frac{n}{2\alpha} \left( \frac{1}{r_0} - \frac{1}{p} \right). \)

Consider the operator \( T, \) defined in (1.2), in the complete metric space

\[
X_\delta = \{ u(t) \in X(I) \mid \| u(t) \|_{X(I)} \leq 2\delta \}
\]

with the metric \( d(u, v) = \| u - v \|_{X(I)} \) for \( u, v \in X_\delta, \) where \( \delta \) is a small constant to be determined later. Using the equivalent characterization of Besov spaces (see Proposition 2.1) we have

\[
\| S_\alpha(t) \varphi \|_{X(I)} = \sup_{t > 0} \| S_\alpha(t) \varphi \|_{B_{p, \infty}^{\delta}} + \sup_{t > 0} t^{\frac{1}{q}} \| S_\alpha(t) \varphi \|_{L^p} \leq C \| \varphi \|_{B_{p, \infty}^{\delta}}. \tag{6.1}
\]

By Lemma 3.1 and the Sobolev embedding \( L^p(\mathbb{R}^n) \hookrightarrow B_{p, \infty}^{\delta}(\mathbb{R}^n), \) it is seen that

\[
\| Gf \|_{B_{p, \infty}^{\delta}} \leq \| Gf \|_{L^p} \\
\leq C \sup_{0 < t < \infty} \int_0^t (t - \tau)^{-\frac{n}{2\alpha} \left( \frac{k+1}{p} - \frac{2\alpha}{nb} \right)} \| f(\tau) \|_{L^p}^{b+1} d\tau \\
\leq C \sup_{0 < t < \infty} \int_0^t (t - \tau)^{-\frac{n}{2\alpha} \left( \frac{k+1}{p} - \frac{2\alpha}{nb} \right) - \frac{b+1}{q}} d\tau \| u \|_{C_q(t; L^p)}^{b+1} \\
\leq C \| u \|_{X(I)}^{b+1}, \tag{6.2}
\]

\[
\| Gf \|_{C_q(t; L^p)} \leq \sup_{0 < t < \infty} t^{\frac{1}{q}} \int_0^t (t - \tau)^{-\frac{n}{2\alpha} \left( \frac{k+1}{p} - \frac{1}{p} \right)} \| f(\tau) \|_{L^p}^{b+1} d\tau \\
\leq \sup_{0 < t < \infty} t^{\frac{1}{q}} \int_0^t (t - \tau)^{-\frac{n}{2\alpha} \left( \frac{k+1}{p} \right) - \frac{b+1}{q}} d\tau \| u \|_{C_q(t; L^p)}^{b+1} \\
\leq C \| u \|_{X(I)}^{b+1}. \tag{6.3}
\]

Combining (6.1)-(6.4) we have on noting the definition (1.2) of \( T \) that for \( u \in X_\delta \)

\[
\| T(u) \|_{X(I)} \leq C \| \varphi \|_{B_{p, \infty}^{\delta}} + C \| u \|_{X(I)}^{b+1} \leq C \| \varphi \|_{B_{p, \infty}^{\delta}} + C \delta^{b+1}.
\]

Thus, if we take \( \delta = C \| \varphi \|_{B_{p, \infty}^{\delta}} \) to be small enough, then \( T \) is a contraction mapping from \( X_\delta \) into itself. The Banach contraction mapping principle implies that \( T \) has a unique fixed point in \( u \in X_\delta \) or equivalently the problem (1.1) has a unique solution \( u \in X_\delta. \) Furthermore, one can verify that

\[
t^{\frac{1}{q}} \| G(u) \|_{L^p} \leq t^{\frac{1}{q}} \int_0^t (t - \tau)^{-\frac{n}{2\alpha} \left( \frac{k+1}{p} \right) - \frac{b+1}{q}} d\tau \left( \sup_{0 < \tau < t} \tau^{\frac{1}{q}} \| u(\tau) \|_{L^p} \right)^{b+1} \\
\leq C \delta^b \sup_{0 < \tau < t} \tau^{\frac{1}{q}} \| u(\tau) \|_{L^p},
\]

and

\[
\lim_{t \to 0} t^{\frac{1}{q}} \| S_\alpha(t) \varphi \|_{L^p} = 0.
\]

Consequently, it follows that

\[
\lim_{t \to 0} t^{\frac{1}{q}} \| u(t) \|_{L^p} = 0. \tag{6.5}
\]

Thus we arrive at
**Theorem 6.1.** Let \((q,p,r_0)\) be a generalized admissible triplet and let \(\sigma = 2\alpha/b - n/p\). Assume that \(\varphi \in L^{r_0}(\mathbb{R}^n)\). If \(\|\varphi\|_{\dot{B}_{p,\infty}^{-\sigma}}\) is small enough then the problem (4.1) has a unique mild solution \(u\) satisfying that

\[
 u(t, x) \in C \left( [0, \infty); L^{r_0}(\mathbb{R}^n) \right) \cap C_q \left( [0, \infty); L^p(\mathbb{R}^n) \right).
\]

Moreover, the solution \(u\) satisfies Remark 6.2, that is,

\[
 u(t, x) \in C \left( [0, \infty); L^{r_0}(\mathbb{R}^n) \right) \cap \dot{C}_q \left( [0, \infty); L^p(\mathbb{R}^n) \right).
\]

**Remark 6.1.** Theorem 6.1 indicates that the global solution \(u \in C \left( [0, \infty); L^{r_0}(\mathbb{R}^n) \right) \) exists provided that the initial data \(\varphi \in L^{r_0}(\mathbb{R}^n)\) and its norm in the Besov space \(\dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)\) is small enough. Note that the norm in \(L^{r_0}(\mathbb{R}^n)\) of \(\varphi\) may be arbitrarily large. For more details see the example in [2].

We may also consider the well-posedness in the Besov space \(\dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)\) of the problem (4.1). In doing so, we only need to use the solution space

\[
 X(I) = C_*(\mathbb{R}; \dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)) \cap C_q([0, \infty); L^p(\mathbb{R}^n))
\]

with the norm

\[
 \|u\|_{X(I)} = \sup_{t>0} \|u(t)\|_{\dot{B}_{p,\infty}^{-\sigma}} + \sup_{t>0} t^{\frac{\alpha}{p}} \|u\|_p,
\]

where \(I = [0, \infty)\), \(r_0 = nb/(2\alpha)\) \(\leq p < r_0(b + 1), \ p > b + 1, \ \sigma = 2\alpha/b - n/p \geq 0\) and \(\frac{1}{q} = \frac{n}{2\alpha(r_0 - \frac{1}{p})}\). Since the Besov space \(\dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)\) contains the self-similar initial data \(\varphi\), that is, \(\lambda^{\frac{\alpha}{\sigma}} C\varphi(\lambda x) = \varphi(x)\) for any \(\lambda > 0\), we also obtain self-similar solutions to the problem (4.1) in this case. By a similar argument as in the proof of Theorem 6.1 we have the following well-posedness in the homogeneous Besov space \(\dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)\) of the problem (4.1).

**Theorem 6.2.** Let \((q,p,r_0)\) be a generalized admissible triplet and let \(p \geq r_0, b + 1 < p < r_0(b + 1)\). Assume that \(\varphi \in \dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)\). Then, if \(\|\varphi\|_{\dot{B}_{p,\infty}^{-\sigma}}\) is sufficiently small then the problem (4.1) has a unique mild solution \(u\) satisfying that

\[
 u \in C_*(\mathbb{R}; \dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)) \cap C_q([0, \infty); L^p(\mathbb{R}^n)).
\]

Moreover, the solution \(u\) satisfies Remark 6.3, that is,

\[
 u \in C_*(\mathbb{R}; \dot{B}_{p,\infty}^{-\sigma}(\mathbb{R}^n)) \cap \dot{C}_q([0, \infty); L^p(\mathbb{R}^n)).
\]

**Remark 6.2.** For the problem (4.5) similar results to Theorems 6.1 and 6.2 hold if we take \(r_0 = nb/(2\alpha - 1)\) and \(\sigma = (2\alpha - 1)/b - n/p\).

**Remark 6.3.** (i) Consider the problem (1.1) with \(F(u) = -|u|^b u\), that is, the problem

\[
 \begin{cases}
 u_t + (-\Delta)^\alpha u = -|u|^b u, \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\
 u(0, x) = \varphi(x), \ x \in \mathbb{R}^n.
 \end{cases}
\]
It can be shown that the global solution to (6.6) exists under the condition $b < \frac{4}{\alpha/n}$. In fact, multiplying both sides of the first equation of (6.6) by $u$ and integrating the equation thus obtained over $\mathbb{R}^n$ gives
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}} u(t)\|_2^2 + \|u(t)\|^{b+2}_{b+2} = 0.
\]
Thus we have
\[
\|u\|_2 \leq C\|\varphi\|_2.
\]
Therefore, let $(q,p,2)$ be an admissible triplet and define the solution space
\[
X(I) = C(I; L^2(\mathbb{R}^n)) \cap L^q(I; L^p(\mathbb{R}^n))
\]
(or let $(q,p,2)$ be a generalized admissible triplet and define the solution space $Y(I) = C(I; L^2(\mathbb{R}^n)) \cap \mathcal{C}_q(I; L^p(\mathbb{R}^n))$, where $I = [0,T]$ for $T > 0$. Then by Theorem 4.1 or 4.2 the problem (6.6) has a unique local solution $u \in X(I)$ (or $u \in Y(I)$), where $T = T(\|\varphi\|_2)$ and $b < 4\alpha/n$. Picard’s method implies that the local solution can be extended to be a global one.

(ii) If $\alpha = 1$, then the restriction $b < 4\alpha/n$ on the nonlinear growth can be removed.

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