On the boundary complex of the $k$-Cauchy–Fueter complex

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Abstract

The $k$-Cauchy–Fueter complex, $k = 0, 1, \ldots$, in quaternionic analysis are the counterpart of the Dolbeault complex in the theory of several complex variables. In this paper, we construct explicitly boundary complexes of these complexes on boundaries of domains, corresponding to the tangential Cauchy–Riemann complex in complex analysis. They are only known boundary complexes outside of complex analysis that have interesting applications to the function theory. As an application, we establish the Hartogs–Bochner extension for $k$-regular functions, the quaternionic counterpart of holomorphic functions. These boundary complexes have a very simple form on a kind of quadratic hypersurfaces, which have the structure of right-type nilpotent Lie groups of step two. They allow us to introduce the quaternionic Monge–Ampère operator and open the door to investigate pluripotential theory on such groups. We also apply abstract duality theorem to boundary complexes to obtain the generalization of Malgrange’s vanishing theorem and the Hartogs–Bochner extension for $k$-CF functions, the quaternionic counterpart of CR functions, on this kind of groups.

Keywords Boundary complexes · The $k$-Cauchy–Fueter complex · The Hartogs–Bochner extension for $k$-regular functions · Right-type groups of step two · Abstract duality theorem · Malgrange’s vanishing theorem

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1 Introduction

The Cauchy–Riemann operator and the Dolbeault complex play central roles in the theory of several complex variables. In quaternionic analysis, we have a family of operators, the $k$-Cauchy–Fueter operator, acting on $\bigotimes^k \mathbb{C}^2$-valued functions, $k = 0, 1, \ldots$. This is because the group SU(2) of unit quaternionic numbers has a family of irreducible representations $\bigotimes^k \mathbb{C}^2$, while the group of unit complex numbers has only one irreducible representation space $\mathbb{C}$.

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The corresponding complexes are the \( k \)-Cauchy–Fueter complexes, which are already known explicitly (cf. \[ 1, 7, 11, 12, 15, 16, 40, 41 \] and references therein), and used to show several interesting properties of \( k \)-regular functions, the quaternionic counterpart of holomorphic functions. The 0-Cauchy–Fueter complex also has important applications to the quaternionic Monge–Ampère operator and quaternionic plurisubharmonic functions (cf. \[ 39, 45 \] and references therein). For a differential complex, a fundamental problem is to characterize domains on which the complex is exact, i.e. the Poincaré Lemma holds or its cohomology groups vanish. The Neumann problem associated to the \( k \)-Cauchy–Fueter complex on \( k \)-pseudoconvex domains was investigated in \[ 42 \]. It is expected that the nonhomogeneous \( k \)-Cauchy–Fueter equation is solvable if and only if the domain is \( k \)-pseudoconvex.

In the complex case, when the Dolbeault complex is restricted to a CR submanifold, ones obtain the tangential Cauchy–Riemann complex, which is a powerful tool to investigate holomorphic functions on domains and the Dolbeault complex. One way to study the \( k \)-Cauchy–Fueter complex and \( k \)-regular functions is to study its boundary complex. The theory of boundary complexes of general differential complexes began in 1970s by Andreotti, Hill, Lojasiewicz, Mackichan, and Nacinovich et al. (cf., e.g. \[ 5, 6, 28, 29 \] and references therein).

In this paper, we will write down explicitly the boundary complex of the \( k \)-Cauchy–Fueter complex on boundaries of domains, and apply it to establish the Hartogs–Bochner extension for \( k \)-regular functions, and construct the quaternionic Monge–Ampère operator on right-type nilpotent Lie groups of step two, corresponding to a kind of rigid quadratic hypersurfaces. On this kind of groups, we also apply abstract duality theorem to boundary complexes to obtain the generalization of Malgrange’s vanishing theorem and the Hartogs–Bochner extension for \( k \)-CF functions, the quaternionic counterpart of CR functions, under the momentum condition. They are only known boundary complexes outside of complex analysis that have interesting applications to the function theory.

### 1.1 The \( k \)-Cauchy–Fueter complex

Denote

\[
\mathcal{V}^{\sigma,\tau} := \bigodot^\sigma \mathbb{C}^2 \otimes \wedge^\tau \mathbb{C}^{2(n+1)},
\]

where \( \bigodot^\sigma \mathbb{C}^2 \) is the \( \sigma \)-th symmetric product of \( \mathbb{C}^2 \), and \( \wedge^\tau \mathbb{C}^{2(n+1)} \) is the \( \tau \)-th exterior product of \( \mathbb{C}^{2(n+1)} \). For fixed \( k = 0, 1, \ldots \), the \( k \)-Cauchy–Fueter complex on \( \mathbb{H}^{n+1} \) is given by

\[
0 \to \Gamma(D, \mathcal{V}_0) \xrightarrow{D_0} \cdots \to \Gamma(D, \mathcal{V}_j) \xrightarrow{D_j} \Gamma(D, \mathcal{V}_{j+1}) \to \cdots \to \Gamma(D, \mathcal{V}_{2n}) \to 0,
\]

(1.1)

for a domain \( D \) in \( \mathbb{H}^{n+1} \), where \( \Gamma(D, \mathcal{V}_j) \) is the space of smooth \( \mathcal{V}_j \)-valued functions with \( \mathcal{V}_j := \mathcal{V}^{\sigma_j,\tau_j} \) (see (2.7) for \( \sigma_j \) and \( \tau_j \)).

To write down operators in the complex (1.1), we need complex vector fields [40]

\[
(\nabla_{\hat{A}'}) := \begin{pmatrix}
\partial_1 + i\partial_2 & -\partial_3 - i\partial_4 \\
\partial_3 - i\partial_4 & \partial_1 - i\partial_2 \\
\vdots & \vdots \\
\partial_{4l+3} + i\partial_{4l+4} & \partial_{4l+1} - i\partial_{4l+2}
\end{pmatrix},
\]

(1.2)
where $\hat{A} = 0, \ldots, 2n + 1$, $A' = 0'$, $1'$, and $\partial_j = \frac{\partial}{\partial x_j}$. It is motivated by the embedding $\tau$ of quaternionic algebra $H$ into $gl(2, \mathbb{C})$:

$$
\tau(x_1 + x_2i + x_3j + x_4k) = \left( \begin{array}{c}
 x_1 + ix_2 \\
 x_3 - ix_4 \\
 x_3 - ix_4
\end{array} \right).
$$

The quaternionic structure of $\mathbb{H}^{n+1}$ is encoded in these vector fields. In the sequel, we identify $\mathbb{H}^{n+1}$ with the underlying space $\mathbb{R}^{4(n+1)}$. For a fixed basis $\{\omega^0, \ldots, \omega^{2n+1}\}$ of $\mathbb{C}^{2(n+1)}$, define two differential operators $d_{A'} : \Gamma(D, \land \tau C^{2(n+1)}) \to \Gamma(D, \land \tau C^{2(n+1)})$ by

$$
d_{A'} F := \sum_{A} \sum_{A=0}^{2n+1} \nabla_{\bar{A}} A_{A'} f_{A} \omega^A \land \omega^\hat{A},
$$

$(1.4)$

$A' = 0', 1'$, for $F = \sum_A f_{A} \omega^A \in \Gamma(D, \land \tau C^{2(n+1)})$, where $\omega^\hat{A} := \omega^{\hat{A}_1} \land \ldots \land \omega^{\hat{A}_r}$ for the multi-index $\hat{A} = \hat{A}_1 \cdots \hat{A}_r$. As $\partial$ and $\overline{\partial}$ in complex analysis, $d_{0'}$ and $d_{1'}$ introduced in [39] are a pair of anti-commutative operators behaving like exterior differentials:

$$
d_{0'}^2 = d_{1'}^2 = 0, \quad d_{0'} d_{1'} = -d_{1'} d_{0'}.
$$

They give a very useful expression of the quaternionic Monge–Ampère operator, and allow us to prove many important results in quaternionic pluripotential theory (cf. [38, 39, 45] and references therein).

By raising primed indices, we have operator $d^{A'} (2.4)$. It is convenient to identify $\otimes^\sigma \mathbb{C}^2$ with the space $\mathcal{P}_\sigma (\mathbb{C}^2)$ of homogeneous polynomials of degree $\sigma$ on $\mathbb{C}^2$ [25]. A $\mathcal{V}_j$-valued function $f$ can be viewed as a function in variables $x \in \mathbb{R}^{4(n+1)}$, $s^{A'} \in \mathbb{C}^2$ and Grassmannian variables $\omega^A$:

$$
\sum_{A',\hat{A}} f_{A'\hat{A}} (x) s^{A'} \omega^A
$$

where $s^{A'} := s^{A'_1} \cdots s^{A'_{n'}}$ for the multi-index $A' = A'_1 \cdots A'_{n'}$. Let $\partial_{A'} = \partial_{A'}/\partial_{x^t}$. Under this identification, differential operators in the $k$-Cauchy–Fueter complex have a very simple form:

$$
D_j = \begin{cases} 
\partial_{A'} d^{A'}, & \text{if } j = 0, \ldots, k - 1, \\
 d^{0'} d^{1'}, & \text{if } j = k, \\
 s_{A'} d^{A'}, & \text{if } j = k + 1, \ldots, 2n,
\end{cases}
$$

$(1.6)$

### 1.2 The boundary complex

Consider a domain

$$
D = \{ q = (q', q_{n+1}) \in \mathbb{H}^n \times \mathbb{H}; \varphi(q) > 0 \}.
$$

$(1.7)$

By rotation if necessary, we can assume the defining function $\varphi$ has the following form near the origin:

$$
\varphi(q) = \text{Re } q_{n+1} - \phi(q', \text{Im } q_{n+1}),
$$

$(1.8)$

where $\phi(q', \text{Im } q_{n+1}) = O((q', \text{Im } q_{n+1})^2)$. As the boundary version of operators $d^{A'}$, we introduce operators $\partial^{A'} : \Gamma(bD, \land \tau C^{2n}) \to \Gamma(bD, \land \tau + 1 C^{2n})$ by

$$
\partial^{A'} f = \sum_{A,A'} Z^{A'}_{A} f_{A} \omega^{A} \land \omega^{A'},
$$

$(1.9)$
for \( f = \sum_{|A|=\tau} f_A \omega^A \), where complex vector field \( Z_A' \) is tangential to the boundary: \( Z_A' \circ 0 = 0 \), for \( A = 0, \ldots, 2n - 1 \), \( A' = 0', 1' \) (cf. Sect. 2.2). Denote
\[
\gamma^{\sigma, \tau} := \mathcal{O}_\sigma \mathcal{C}^2 \otimes \Lambda^\tau \mathcal{C}^n.
\]

**Theorem 1.1** The boundary complex of the \( k \)-Cauchy–Fueter complex is the differential complex
\[
0 \rightarrow \Gamma (bD, \gamma_0) \xrightarrow{\mathcal{D}_0} \cdots \rightarrow \Gamma (bD, \gamma_j) \xrightarrow{\mathcal{D}_j} \Gamma (bD, \gamma_{j+1}) \rightarrow \cdots \rightarrow \Gamma (bD, \gamma_{2n}) \rightarrow 0,
\]
where \( \gamma_j := \gamma_j^{(1)} \oplus \gamma_j^{(2)} \) with
\[
\gamma_j^{(1)} := \gamma^{\sigma_j, \tau_j}, \quad \gamma_j^{(2)} := \gamma^{\sigma_{j+1}, \tau_j - 1}, \quad \text{if} \quad j \neq k,
\]
and \( \gamma_0^{(2)} = 0 \). For \( F = (F_1, F_2) \in \Gamma (bD, \gamma_0^{(1)}) \oplus \Gamma (bD, \gamma_0^{(2)}) \), \( \mathcal{D}_jF = (\mathcal{D}_j^{(1)}F, \mathcal{D}_j^{(2)}F) \) with
\[
\mathcal{D}_j^{(1)} = \begin{cases}
\partial_A \phi^A F_1 + \mathcal{E}_0 \wedge F_2, & \text{if} \quad j = 0, \ldots, k - 1, \\
\delta^0 \delta^1 F_1 - \mathcal{E}_0 \wedge (\mathcal{T}^0 F_1 + F_2), & \text{if} \quad j = k, \\
s_A^0 \phi^A F_1 + \mathcal{E}_0 \wedge F_2, & \text{if} \quad j = k + 1, \ldots, 2n - 2,
\end{cases}
\]
where \( \mathcal{E}_0 := -\delta^0 \delta^1 \circ 0 \).

Explicit formulae for \( \mathcal{D}_j \)'s are given by (3.28) and (4.15). Compared to \( d^A \) in (1.5), operators \( \partial^A \)'s usually do not behave like anti-commutative exterior differentials. We call a hypersurface \( bD \) right-type if \( \mathcal{E}_0 \) vanishes, because on such groups, operator \( \partial^A \)'s behave like that on the right quaternionic Heisenberg group (1.16). In this case, \( \mathcal{D}_j \) maps a \( \gamma_j^{(1)} \)-valued function to a \( \gamma_{j+1}^{(1)} \)-valued one by (1.12), i.e. we obtain a differential subcomplex:
\[
0 \rightarrow \Gamma (bD, \gamma_0^{(1)}) \xrightarrow{\mathcal{D}_0} \cdots \rightarrow \Gamma (bD, \gamma_j^{(1)}) \xrightarrow{\mathcal{D}_j} \cdots \xrightarrow{\mathcal{D}_{2n-2}} \Gamma (bD, \gamma_{2n-1}^{(1)}) \rightarrow 0
\]
with
\[
\mathcal{D}_j = \begin{cases}
\partial_A \phi^A, & \text{if} \quad j = 0, \ldots, k - 1, \\
\delta^0 \delta^1, & \text{if} \quad j = k, \\
s_A^0 \phi^A, & \text{if} \quad j = k + 1, \ldots, 2n - 2.
\end{cases}
\]

This subcomplex and its operators are very similar to the \( k \)-Cauchy–Fueter complex on \( \mathbb{H}^n+1 \).

### 1.3 The Hartogs–Bochner extension for \( k \)-regular functions

On a domain \( D \subset \mathbb{H}^{n+1} \), a function \( f \in \Gamma (D, \mathcal{O}_k^2) \) is called \( k \)-regular if \( \mathcal{D}_0f = 0 \). The space of all \( k \)-regular functions on a domain \( D \) is denoted by \( \partial_k(D) \). On a domain \( \Omega \) in \( bD \), a function \( F \in \Gamma (\Omega, \mathcal{O}_k^2) \) is called \( k \)-CF if \( \mathcal{D}_0F = 0 \). We have the following Hartogs–Bochner extension for \( k \)-regular functions.
Theorem 1.2 Let \( D \) be a bounded domain in \( \mathbb{H}^{n+1} \) (\( n \geq 1 \)) with smooth boundary such that \( \mathbb{H}^{n+1} \setminus D \) is connected. If \( f \) is a smooth \( k \)-CF function on \( bD \), then there exists \( \tilde{f} \in \mathcal{O}_k(D) \) smooth up to the boundary such that \( \tilde{f} = f \) on \( bD \).

This theorem for \( k = 1 \) was proved by Maggesi-Pertici-Tomassini [26] recently. They introduced the notion of admissible functions on the boundary, which coincides with the notion of 1-CF functions here, although it is written in a different form.

For a \( k \)-CF function \( f \) on \( bD \), the boundary complex allows us to construct a representative \( \tilde{f} \in H^{k}(D, \mathcal{O}_{k}^{C}\mathbb{C}^{2}) \) such that \( \tilde{f}|_{bD} = f \) and \( \mathcal{D}_{0}f \) is flat on \( bD \). Then by using the solution to the nonhomogeneous \( k \)-Cauchy–Fueter equation with compact support, we can construct the \( k \)-regular function \( \tilde{f} \).

1.4 Right-type groups and the quaternionic Monge–Ampère operator

As in the complex case, we call a domain \( D \) a rigid domain if it has a defining function of the following form:

\[
\varphi(q) = \text{Re} q_{n+1} - \phi(q'),
\]

i.e. \( \phi \) is independent of \( \text{Im} q_{n+1} \). If we take \( \phi(q') = \sum_{j,k=1}^{4n} S_{jk} x_j x_k \) for a real symmetric \( 4n \times 4n \)-matrix \( S \), the boundary is a rigid quadratic hypersurface. It has the structure of a nilpotent Lie group of step two, denoted by \( \mathcal{N}_{S} \), with the multiplication given by

\[
(x, t) \cdot (y, s) = \left( x + y, t_\beta + s_\beta + 2 \sum_{a,b=1}^{4n} B_{ab}^\beta x_a y_b \right),
\]

for \( x, y \in \mathbb{R}^{4n}, t, s \in \mathbb{R}^{3}, \beta = 1, 2, 3 \), where skew symmetric matrices \( B^\beta = (B_{ab}^\beta) \) are determined by \( S \) in (6.4). We call the associated group \( \mathcal{N}_{S} \) right-type if the quadratic hypersurface is. A simple characterization of right-type groups in terms of matrices \( B^\beta \) is given in Proposition 6.1. This kind of groups are abundant.

The right quaternionic Heisenberg group is \( \mathbb{H}^{n} \times \text{Im} \mathbb{H} \) with the multiplication given by

\[
(p, t) \cdot (q, s) = (p + q, t + s + 2\text{Im}(pq)),
\]

where \( p, q \in \mathbb{H}^{n}, t, s \in \text{Im} \mathbb{H} \). This group is right-type and the tangential \( k \)-Cauchy–Fueter complex constructed in [37] is a special case of the subcomplex (1.13). While the left quaternionic Heisenberg group is \( \mathbb{H}^{n} \times \text{Im} \mathbb{H} \) with the multiplication given by

\[
(p, t) \cdot (q, s) = (p + q, t + s + 2\text{Im}(\overline{pq})).
\]

This group is not right-type, but we have already constructed the tangential \( k \)-Cauchy–Fueter complex on this group by using the twistor method (cf. [43, Theorem 1.0.1]), which is different from the complex on the right one. To understand this difference, which is explained by Theorem 1.1, is one of main motivations of this paper.

On a right-type group, operators \( \partial^{A} \)'s behave like anti-commutative exterior differentials:

\[
\partial_{0'}^{2} = \partial_{1'}^{2} = 0, \quad \partial_{0'} \partial_{1'} = -\partial_{1'} \partial_{0'}.
\]

This kind of nice operators was first observed for \( (4n+1) \)-dimensional Heisenberg group in [45]. \( \partial^{A} \)'s allow us to introduce the quaternionic Monge–Ampère operator on a right-type group as

\[
\Delta u \wedge \ldots \wedge \Delta u, \quad \text{where} \quad \Delta u := \partial_{0'} \partial_{1'} u,
\]
and plurisubharmonicity of a function $u$ in terms of the positivity of the 2-current $\Delta u$ (2-form for a $C^2$-function). A direct application of (1.18) gives us a key identity for $\partial_0', \partial_1'$ and $\Delta$, by which we can show important Chern-Levine-Nirenberg type estimate in Theorem 6.1, and obtain the existence of the Monge–Ampère measure for a continuous plurisubharmonic function. This opens the door to investigate the quaternionic Monge–Ampère equation and pluripotential theory on right-type groups, generalizing the theory on the Heisenberg group [45].

1.5 The generalization of Malgrange’s vanishing theorem and the Hartogs–Bochner extension for $k$-CF functions

It is important to investigate the validity of the Poincaré lemma, i.e. vanishing of its cohomology groups, for boundary complexes in terms of the Levi-type forms, which were introduced in the study of the Neumann problem for the $k$-Cauchy–Fueter complex in [42], and the behavior of $k$-CF functions related to hypoellipticity, unique continuation, the maximum modulus principle, hypoanaliticity etc., as in the theory of CR functions and the tangential Cauchy–Riemann complex on CR manifolds (see e.g. [3, 4, 9, 10, 18–24, 28, 30, 33]). In this paper, we discuss the most simple case: right-type groups. We have subelliptic estimate.

**Proposition 1.1** Suppose that $N_S$ is a stratified right-type group and $K \Subset N_S$. Then there are positive constants $C_1, C_2$ only depending on $K$ such that

$$\|D_0 f\|_{0}^2 \geq C_1 \|f\|_{2}^2 - C_2 \|f\|_{0}^2,$$

for any $f \in C^\infty_0(K, \gamma_0)$.

Abstract duality theorem for a Fréchet–Schwartz space or the dual of a Fréchet–Schwartz space with topological homomorphisms can be applied to our case. Let $\hat{\mathcal{V}}^\bullet$ be the differential complex dual to the $k$-Cauchy–Fueter complex. We have the following generalization of Malgrange’s vanishing theorem.

**Theorem 1.3** On a right-type group $N_S$ satisfying condition (H), the homology groups $H_0(\mathcal{E}(N_S, \hat{\mathcal{V}}^\bullet))$ and $H_0(\mathcal{D}(N_S, \hat{\mathcal{V}}^\bullet))$ vanish.

In the step two case, a stratified group is exactly a group satisfying Hörmander’s condition, which is used to promise hypoellipticity of the SubLaplacian, while the assumption of condition (H) is a technique condition to promise unique continuation. We also prove the Hartogs–Bochner extension for $k$-CF functions under the momentum condition in Theorem 7.4. See [10, 23, 24, 28, 30] for Malgrange’s vanishing theorem and the Hartogs–Bochner extension for CR functions on CR manifolds.

This paper is organized as follows. In Sect. 2, we give basic notations, complex tangential vector fields $Z_{AA'}$, and recall the definition of the boundary complex of a general differential complex. In Sect. 3, we determine vector spaces of the boundary complex of the $k$-Cauchy–Fueter complex and induced operators for $j < k - 1$. For the case $j \geq k - 1$, it is done in Sect. 4. In Sect. 5, the Hartogs–Bochner extension theorem for $k$-regular functions is established. In Sect. 6, we prove the characterization of right-type groups in Proposition 6.1 and the Chern-Levine-Nirenberg type estimate. In Sect. 7, after establish subelliptic estimate, we apply abstract duality theorem to boundary complexes on right-type groups to obtain the generalization of Malgrange’s vanishing theorem and the Hartogs–Bochner extension for $k$-CF functions under the momentum condition. In the appendix, we show differential operators
(1.6) in the $k$-Cauchy–Fueter complex coincide with the usual form used before (e.g. in [40–42]).

## 2 Preliminaries

### 2.1 Notations

We adopt the following index notations:

\[
\hat{A}, \hat{B}, \hat{C}, \ldots \in \{0, 1, \ldots, 2n + 1\}, \quad A, B, C, \ldots \in \{0, 1, \ldots, 2n - 1\},
\]

\[A', B', C', D', \ldots \in \{0', 1'\}.\]

We will use the Einstein convention of taking summation for repeated indices. It is similar to lower or raise indices by a metric in differential geometry that we use for symmetrization. For example, we have

\[
\hat{f} A' = f_{B'}\hat{f}^{B' A'}, \quad f^A A' = f_{C'}. \quad (2.2)
\]

This is the same when an index is raised (or lowered) and then lowered (or raised) [31, 32]. Here $(\hat{e} A' B')$ is a volume element in $\mathbb{C}^2$. The contraction of an upper and a lower primed indices is invariant under the action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$ (cf. e.g. [41, 42]). These primed and unprimed indices are the generalization of Penrose’s two-spinor notations (cf. [31, 32]). By raising primed indices, we have

\[
\hat{\nabla}_A A':= \nabla_\hat{A} \hat{f} A' \omega^{\hat{A}} \land \omega_{\hat{A}}. \quad (2.4)
\]

### Proposition 2.1

(Proposition 2.2 in [39]) (1) $d^{A'} d^{B'} = 0$, i.e. (1.5) holds.

(2) For $F \in \Gamma(\land^7 \mathbb{C}^{2n})$, $G \in \Gamma(\land^5 \mathbb{C}^{2n})$, we have

\[
d^{A'} (F \land G) = d^{A'} F \land G + (-1)^{A'} F \land d^{A'} G, \quad A' = 0', 1'.
\]

**Proof** See (8.1) for symmetrization. For $F = F_A \omega^A$ with $|A| = \tau$,

\[
d^{A'} d^{B'} F = \nabla^A_{\hat{A}} \nabla^B_{\hat{B}} F_A \omega_{\hat{A}} \land \omega_{\hat{B}} \land \omega^A = -\nabla^B_{\hat{B}} \nabla^A_{\hat{A}} F_A \omega_{\hat{B}} \land \omega^A \land \omega^A = -d^{B'} d^{A'} F,
\]

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by $\nabla^A$’s commuting each other as differential operators of constant coefficients. The proof of (2) is the same as (3.2).

The Leibnitz law in Proposition 2.1 (2) will be frequently used. It is convenient to identify $\mathbb{O}^\sigma \mathbb{C}^2$ with the space $\mathcal{P}_\sigma (\mathbb{C}^2)$ of homogeneous polynomials of degree $\sigma$ on $\mathbb{C}^2$ [25]. $V_j$ is realized as $\mathcal{P}_{\sigma j}(\mathbb{C}^2) \otimes \wedge^{\rho_j} \mathbb{C}^{2(n+1)}$. Let $s^{0'}$, $s^1$ be coordinate functions of $\mathbb{C}^2$. We choose

$$S^a_\sigma := \frac{(s^{0'})^{\sigma-a} (s^1)^a}{(\sigma-a)!} a!,$$ 

$a = 0, \ldots, \sigma$, as a basis of $\mathcal{P}_\sigma (\mathbb{C}^2)$. The advantage of this basis is that

$$\partial_0 S^a_\sigma = S^a_{\sigma-1}, \quad \partial_1 S^a_\sigma = S^a_{\sigma-1}.$$

(2.5)

We write $s^{A'} := s^{A'_1} \cdots s^{A'_{\sigma}}$ for $A' = A'_1 \cdots A'_\sigma$ and set $|A'| = \sigma$ and $o(A')$ to be the number of $1'$ in $A'$.

For $j \geq k$, we also use $\tilde{S}^a_\sigma := (s^{0'})^{\sigma-a} (s^1)^a$, $a = 0, \ldots, \sigma$, as a basis of $\mathcal{P}_\sigma (\mathbb{C}^2)$. Here $s^{A'}_\sigma$’s are obtained by lowering primed indices. The advantage of this basis is that

$$s_0 \tilde{S}^a_\sigma = \tilde{S}^a_{\sigma+1}, \quad s_1 \tilde{S}^a_\sigma = \tilde{S}^a_{\sigma+1}.$$

(2.6)

Here and in the sequel, we use the convention $S^a_{\sigma-1} = 0 = S^{-1}_{\sigma-1}, \tilde{S}^a_{\sigma-1} = 0 = \tilde{S}^{-1}_{\sigma-1}$.

For fixed $k$, indices in $V_j := V^{\sigma_j, \tau_j}$ in (1.1) are given by

$$\sigma_j := \begin{cases} k - j, & \text{if } j = 0, \ldots, k, \\ j - k - 1, & \text{if } j = k + 1, \ldots, 2n + 1, \end{cases}$$

(2.7)

$$\tau_j := \begin{cases} j, & \text{if } j = 0, \ldots, k, \\ j + 1, & \text{if } j = k + 1, \ldots, 2n + 1. \end{cases}$$

Under this realization, we can easily see (1.1) is a complex, i.e. $D_{j+1} D_j = 0$, by

$$\partial_\sigma' \partial_B d^{A'} d^{B'} = 0, \quad s_{A'} s_{B'} d^{A'} d^{B'} = 0,$$

(2.8)

since $d^{A'} d^{B'}$ is skew-symmetric in $A'$, $B'$ by Proposition 2.1 (1), while $\partial_\sigma' \partial_B$ and $s_{A'} s_{B'}$ are both symmetric in $A'$, $B'$, and

$$d^{0'} d^{1'} \partial_\sigma' d^{A'} = 0, \quad s_{A'} d^{A'} d^{0'} d^{1'} = 0.$$

(2.9)

2.2 Complex tangential vector fields $Z_{AA'}$

Write $q_{l+1} := x_{4l+1} + i x_{4l+2} + j x_{4l+3} + k x_{4l+4}$, $l = 0, \ldots, n$. The Cauchy–Fueter operator on $\mathbb{H}^{n+1}$ is

$$\overline{D} q_{l+1} = \partial x_{4l+1} + i \partial x_{4l+2} + j \partial x_{4l+3} + k \partial x_{4l+4},$$

$l = 0, \ldots, n$. We have quaternionic tangential vector fields on the boundary:

$$\overline{Q}_l = \overline{D} q_{l} - \overline{D} q_{l} \cdot (\overline{D} q_{n+1} \theta)^{-1} \cdot \overline{D} q_{n+1}$$

(2.10)

$l = 1, \ldots, n$, since $\overline{Q}_l \theta = 0$ by definition.

The definition of the map $\tau$ in (1.3) can extended to a mapping from quaternionic $l \times m$-matrices to complex $2l \times 2m$-matrices by setting $\tau (a) := (\tau (a_{jk}))$ for a quaternionic matrix

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It also also direct to check that
\[ Z_{AA'} = \nabla_{(2n+\alpha(B'))^{C'}}. \] (2.11)

The vector fields \( Z_{AA'} \)'s are tangential to the boundary (1.8), i.e. \( Z_{AA'} \) = 0, since \( \overline{Q}_l \) = 0. If we write \( Q_l = X_{4l+1} + iX_{4l+2} + jX_{4l+3} + kX_{4l+4} \), then we have

\[
(Z_{AA'}) = \begin{pmatrix}
X_{4l+1} + iX_{4l+2} & -X_{4l+3} - iX_{4l+4} \\
X_{4l+3} - iX_{4l+4} & X_{4l+1} - iX_{4l+2} \\
\vdots & \vdots
\end{pmatrix}.
\] (2.12)

Denote \( N_{B'C'} := \nabla_{(2n+\alpha(B'))^{C'}}. \) Then,

\[
(N_{B'C'}) = \begin{pmatrix}
\nabla_{(2n)0'} & \nabla_{(2n)1'} \\
\nabla_{(2n+1)0'} & \nabla_{(2n+1)1'}
\end{pmatrix},
\]

and \( N_{\Omega} \) is the \( 2 \times 2 \)-matrix \( \tau(\overline{q}_{n+1} \Omega) \). By applying \( \tau \) to (2.10), \( Z_{AA'} \) can be written as

\[
Z_{AC'} = \nabla_{AC'} - n_{AB'} N_{B'C'},
\] (2.13)

with

\[
n_{AB'} = \nabla_{AD'} \cdot (N_{\Omega})^{-1}_{D'B'}.\] (2.14)

Then by raising primed indices (2.2), we get

\[
Z_{A'} = \nabla_{A'} - n_{AB'} N_{B'},
\] (2.15)

with

\[
(Z_{A'}) = \begin{pmatrix}
-X_{4l+3} - iX_{4l+4} & -X_{4l+1} - iX_{4l+2} \\
X_{4l+1} - iX_{4l+2} & -X_{4l+3} + iX_{4l+4} \\
\vdots & \vdots
\end{pmatrix}.
\] (2.16)

It also also direct to check that \( Z_{A'} \) = 0. The operator \( d_{A'} \) in (2.4) can be rewritten as

\[
d_{A'} f = \left\{ \nabla_{A'} f \omega^A + N_{C'} f_{A} \omega^{2n+\alpha(C')} \right\} \wedge \omega^A.
\] (2.17)

### 2.3 Definition of the boundary complex

Let us recall the definition of the boundary complex of a general differential complex (cf. e.g. [5, 6, 28]). Let \( D \) be a domain in \( \mathbb{R}^N \). Suppose that we have a differential complex on \( \mathbb{R}^N \):

\[
\Gamma(\mathbb{R}^N, E^{(0)}) \xrightarrow{A_0(x, \partial)} \Gamma(\mathbb{R}^N, E^{(1)}) \xrightarrow{A_1(x, \partial)} \Gamma(\mathbb{R}^N, E^{(2)}) \xrightarrow{A_2(x, \partial)} \ldots.
\]
We say \( u \in \Gamma(D, E^{(j)}) \) has zero Cauchy data on the boundary \( bD \) for \( A_j(x, \partial) \) if for any \( \psi \in \Gamma(U, E^{(j+1)}) \) compactly supported in \( U \), we have
\[
\int_D \langle A_j(x, \partial)u, \psi \rangle dV = \int_D \langle u, A_j^*(x, \partial)\psi \rangle dV, \tag{2.18}
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product of \( E^{(j)} \), and \( A_j^*(x, \partial) \) is the formal adjoint operator of \( A_j(x, \partial) \). Set
\[
\mathcal{J}_{A_j}(bD, U) := \{ u \in \Gamma(U, E^{(j)}); u \text{ has zero Cauchy data on } bD \text{ for } A_j(x, \partial) \},
\]
for any open set \( U \). Then \( U \rightarrow \mathcal{J}_{A_j}(bD, U) \) is a sheaf. We must have
\[
A_j(x, \partial)\mathcal{J}_{A_j}(bD, U) \subset \mathcal{J}_{A_{j+1}}(bD, U). \tag{2.19}
\]
This is because
\[
\int_D \langle A_{j+1}(x, \partial)A_j(x, \partial)u, \psi \rangle dV = 0 = \int_D \langle u, A_j^*(x, \partial)A_{j+1}^*(x, \partial)\psi \rangle dV = \int_D \langle A_j(x, \partial)u, A_{j+1}^*(x, \partial)\psi \rangle dV,
\]
for any \( u \in \mathcal{J}_{A_j}(bD, U) \) and compactly supported \( \psi \in \Gamma(U, E^{(j+1)}) \), by \( A_{j+1}A_j = 0, A_j^*A_{j+1}^* = 0 \). Setting
\[
Q^{(j)}(bD) = \frac{\Gamma(\mathbb{R}^N, E^{(j)})}{\mathcal{J}_{A_j}(bD, \mathbb{R}^N)},
\]
we obtain a quotient complex of the form
\[
\hat{Q}^{(0)}(bD) \xrightarrow{\hat{\alpha}_0} \hat{Q}^{(1)}(bD) \xrightarrow{\hat{\alpha}_1} \hat{Q}^{(2)}(bD) \xrightarrow{\hat{\alpha}_2} \cdots
\]
where \( \hat{\alpha}_j \) is induced by the differential operator \( A_j(x, \partial) \), but is not necessarily a differential operator (cf. [6, Section 6 (d)–(g)]).

Note that when the differential operator is of first order, (2.18) is satisfied if and only if
\[
\int_{bD} \langle A(x, \nu)u, \psi \rangle dS = 0
\]
by Stokes’ formula, where \( \nu \) is the unit vector outer normal to the boundary \( bD \). Since \( \psi \) is arbitrarily chosen, it is equivalent to
\[
A(x, \nu)u|_{bD} = 0
\]
by \( k \)-Cauchy–Fueter complex, \( \mathcal{D}_j \) in (1.6) with \( j \neq k \) is a differential operator of the first order, and so \( Q^{(j)}(bD) \cong \Gamma(bD, \ker \sigma^*_j(v)) \) by the exactness of its symbol sequence [41, Proposition 3.2]
\[
0 \rightarrow \mathcal{V}_0 \xrightarrow{\sigma_0(v)} \mathcal{V}_1 \xrightarrow{\sigma_1(v)} \mathcal{V}_2 \rightarrow \cdots \xrightarrow{\sigma_{2n}(v)} \mathcal{V}_{2n+1} \rightarrow 0 \tag{2.21}
\]
for any \( 0 \neq v \in \mathbb{R}^{4(n+1)} \), where \( \sigma_j(v) \) is the symbol of \( \mathcal{D}_j \) at the direction \( v \).
3 The boundary complex for $j < k - 1$

3.1 Vector spaces of the boundary complex

The first step is to identify $\mathcal{J}_j(U) := \mathcal{J}_{D_j}(bD, U)$ for the $k$-Cauchy–Fueter complex. To do so, we need the formula of $d^A$ in terms of vector fields tangential to $bD$. Here and in the sequel, we extend the definition (1.9) of $\partial^A$ to $\Gamma(U, \wedge^\tau \mathbb{C}^{2n+2}) \to \Gamma(U, \wedge^{\tau+1} \mathbb{C}^{2n+2})$ on the domain by

$$\partial^A F = Z^A \frac{\partial A}{\partial \bar{A}} \wedge \omega^A,$$

(3.1)

for $F = f_A \omega^A$. Operators $\partial^A$ also satisfy the Leibnitz law: if $|\bar{A}| = \tau$, and $G = g_B \omega^B$, we have

$$\partial^A (F \wedge G) = Z^A (f_A g_B) \wedge \omega^A \wedge \omega^B \nabla_{A \wedge B} \omega^B + \omega^A \wedge Z^A g_B \omega^A \wedge \omega^B$$

(3.2)

by $\omega^A \wedge \omega^B = (-1)^{\tau} \omega^A \wedge \omega^B$. Denote

$$\Omega^A := d^A \Omega, \quad \epsilon := -d^0 d^V \varrho$$

By definition,

$$d^V \Omega^V = -d^V \Omega^V = -\epsilon, \quad d^V \Omega^A = d^A d^A \Omega = 0.$$  

(3.3)

Proposition 3.1 We have

$$d^A = \Omega^A \wedge \partial_{4n+1} + d^B,$$

(3.4)

where $d^B_B := \partial^B f + \Omega^B \wedge T^B_B$ is the part only involving vector fields tangential to the boundary $bD$, and

$$T^B_B := R_{B'}^C \nabla^A_C - \delta^A_B \partial_{4n+1},$$

with $R_{B'C'} := \epsilon_{B' D'} (\mathbb{N}_{\bar{Q}})^{-1}_{D'C'}$.

Proof If we set $\Theta^C := \omega^{2n+\omega(C)} + n_{BC}, \omega^B$, then

$$\Theta^C = \omega^{2n+\omega(C)} + \nabla_{B\bar{D}} \varrho \cdot (\mathbb{N}_{\bar{Q}})^{-1}_{D'C'} \wedge \omega^B$$

(3.5)

$$= \left[ \omega^{2n+\omega(B')} N^{B'}_{D'} \varrho + \nabla_{B\bar{D}} \varrho \cdot \omega^B \right] (\mathbb{N}_{\bar{Q}})^{-1}_{D'C'} = d_{B D'} (\mathbb{N}_{\bar{Q}})^{-1}_{D'C'} = \Omega^{B'} R_{B'C'}$$

by (2.14), (2.17) and lowering index. So for $f = f_A \omega^A \in \Gamma(D, \wedge^\tau \mathbb{C}^{2n+2})$, we can write

$$d^A f = \left\{ Z^A f_A \omega^A + N^A_C f_A \omega^{2n+\omega(C)} + n_{AB} N^A_B f_A \omega^A \right\} \wedge \omega^A$$

(3.6)

$$= \partial^A f + \Theta^C \wedge N^A_C f$$

$$= \partial^A f + \Theta^C \wedge (N^A_C - N^A_{C'} \partial_{4n+1} f) + \Theta^C \nabla^A_C \varrho \wedge \partial_{4n+1} f$$

$$= \partial^A f + \Omega^B f + T^A_B f + \Omega^A \wedge \partial_{4n+1} f$$

by the formula (2.17) of $d^A$, the expression (2.15) of $Z^A_A$, (3.5) and using

$$R_{B'C'} \nabla^A_C \varrho = \epsilon_{B' D'} (\mathbb{N}_{\bar{Q}})^{-1}_{D'C'} (\mathbb{N}_{\bar{Q}})_{C'E} \epsilon_{E'A'} = \epsilon_{B' D'} \epsilon^D^A = \delta^A_B.$$
The proposition is proved. □

**Remark 3.1** (1) By definition, we have

$$ (N_{A'B'}\varrho(q)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(|q|), \quad \Omega^{A'} = \omega^{2n+o(A')} + O(|q|). \quad (3.7) $$

If the hypersurface (1.14) is rigid, then $\hat{\partial}_{q_{n+1}}\varrho = 1$. Thus $(N_{A'B'}\varrho)$ is the identity matrix, and so $R_{B'C'} = \varepsilon_{B'C'}$. Hence

$$ (T_{A'}^{A'}) = \begin{pmatrix} -i\partial_{4n+2} - \partial_{4n+3} + i\partial_{4n+4} \\ \partial_{4n+3} + i\partial_{4n+4} \end{pmatrix}. \quad (3.8) $$

(2) Vector fields $Z^{A'}_A$’s span the tangential space perpendicular to the quaternionic line of the normal vector, while $T_{B'}^{A'}$’s span 3-dim tangential space contained in the quaternionic line of the normal vector.

Up to a factor $i$, the symbol of the differential operator $\partial_A d^{A'}$ at the direction $\text{grad} \varrho$ is

$$ \mathbb{D} := \Omega^{A'} \wedge \partial_A, \quad (3.9) $$

because the symbol of $d^{A'}$ is easily seen to be $d^{A'} \varrho \wedge = \Omega^{A'} \wedge$ by definition if we replace $\partial_{x_j}$ by $\partial_{x_j} \varrho$. Set

$$ \mathbb{D} : = \frac{1}{2}(\Omega^{0'} \wedge \partial_0' - \Omega^{1'} \wedge \partial_1'), $$

$$ \Omega^{a'}_{\sigma_j} : = \mathbb{D} S^{a'}_{\sigma_j+1} = \frac{1}{2} \left( S^{a} a, \Omega^{0'} - S^{a'}_{\sigma_j-1} \Omega^{1'} \right), \quad (3.10) $$

where $a = 1, \ldots, \sigma_j$. When acting on $\mathcal{Y}^{\sigma, \tau} = \otimes^\sigma \mathbb{C}^2 \otimes \wedge^\tau \mathbb{C}^{2n+2}$ in (1.1), $\mathbb{D}$ and $\mathbb{D}$ are both linear transformations. $\mathcal{J}_j$ ’s are determined by the following proposition.

**Proposition 3.2** Let $k$ be fixed and $j = 0, \ldots, k - 1$. Then,

(1) $\mathcal{J}_j(U)$ consists of

$$ \mathbb{D} S^{a'}_{\sigma_j+1} \wedge f_a + \varrho f, \quad (3.11) $$

where $f_a \in \Gamma(U, \wedge^{j-1} \mathbb{C}^{2n+2}), a = 0, \ldots, \sigma_j + 1$ and $f \in \Gamma(U, \wedge^{j+1} \mathbb{C}^{2n+2})$.

(2) The $j$-th vector space of the boundary complex is $\mathcal{Y}_j = \mathcal{Y}_j^{(1)} \oplus \mathcal{Y}_j^{(2)}$ with

$$ \mathcal{Y}_j^{(1)} = \otimes^{\sigma_j} \mathbb{C}^2 \otimes \wedge^{\sigma_j+1} \mathbb{C}^{2n}, \quad \mathcal{Y}_j^{(2)} = \otimes^{\sigma_j+1} \mathbb{C}^2 \otimes \wedge^{\sigma_j+1} \mathbb{C}^{2n}, $$

and $\mathcal{Y}_0^{(2)} = \emptyset$. As $\Gamma(U, \mathbb{C})$-modules, $\Gamma(U, \mathcal{Y}_j^{(1)})$ and $\Gamma(U, \mathcal{Y}_j^{(2)})$ are spanned by

$$ S^{a'}_{\sigma_j} \omega^A, \quad \mathbb{D} \quad (3.12) $$

$$ a = 0, \ldots, \sigma_j, |A| = j, \quad \text{and} \quad \Omega^{b+1}_{\sigma_j} \wedge \omega^B, \quad (3.13) $$

$$ b = 0, \ldots, \sigma_j - 1 = \sigma_j+1, |B| = j - 1, \quad \text{respectively.} $$

**Proof** (1) Let $\langle \cdot, \cdot \rangle$ be the inner product of $\mathcal{V}_j$ by choosing $S^{a'}_{\sigma_j} \omega^A$ to be an orthonormal basis. For $f \in \Gamma(U, \mathcal{V}_j)$ and $\psi \in \Gamma(U, \mathcal{V}_{j+1})$ with compact support, by integration by part and
using Proposition 3.1, we get
\[
\int_D \left\{ \partial_A' dA' f, \psi \right\} dV = \int_D \left\{ \partial_A' \left( \Omega^{A'} \wedge \partial_{\mathcal{A}+1} + d_A' \right) f, \psi \right\} dV
\]
\[
= -\int_{bD} \left\{ \frac{dS}{\text{grad } \varrho} \right\} + \int_D \left\{ f, \left( \partial_A' dA' \right)^* \psi \right\} dV.
\]
where \(-\text{grad } \varrho / \text{grad } \varrho\) is the unit vector outer normal to \(bD\) with
\[
\text{grad } \varrho(x) = (-\varphi_{x1}(x), \ldots, -\varphi_{x4n+1}, 1, -\varphi_{x4n+2}, -\varphi_{x4n+3}, -\varphi_{x4n+4}).
\]
Since \(d_A'\) only involves vector fields tangential to the boundary \(bD\), there is no boundary term after integration by part. Thus, \(f \in \mathcal{J}_j(U)\) if and only if
\[
\mathbb{D} f|_{bD \cap U} = 0.
\]
It is direct to see that
\[
\mathbb{D} \mathbb{S}_{\sigma_j+1}^a = \Omega^{A'} \wedge \Omega^{B'} \partial_A' \partial_B' \mathbb{S}_{\sigma_j+1}^a = 0,
\]
since the term is skew-symmetric in superscripts \(A', B'\) and is symmetric in subscripts \(A', B'\). Thus, forms in (3.11) belongs to \(\mathcal{J}_j(U)\). On the other hand,
\[
\mathbb{D}(\Omega_{\sigma_j}^{B+1} \wedge \omega^B) = -\mathbb{S}_{\sigma_j-1}^b \Omega^{0'} \wedge \Omega^{1'} \wedge \omega^B \neq 0,
\]
for \(b = 0, \ldots, \sigma_j - 1\), by (2.5), and
\[
\mathbb{D}(\sigma_{\sigma_j}^b \omega^A) = \left( \mathbb{S}_{\sigma_j-1}^b \Omega^{0'} + \mathbb{S}_{\sigma_j-1}^{b-1} \Omega^{1'} \right) \wedge \omega^A.
\]
It is direct to see that terms in the right hand sides of (3.15)-(3.16) are linearly independent. So the linear combination of \(\Omega_{\sigma_j}^{B+1} \wedge \omega^B\) and \(\sigma_{\sigma_j}^b \omega^C\) with coefficients in \(\Gamma(U, \mathbb{C})\) can not be annihilated by \(\mathbb{D}\).

On the other hand, since \(\{\Omega^{0'}, \Omega^{1'}, \omega^0, \ldots, \omega^{2n-1}\}\) locally is also a basis of \(\mathbb{C}^{2(n+1)}\) by (3.7), we see that \(\Gamma(U, \mathbb{C}^2 \otimes \mathbb{C}^{2n+2})\) has a basis consisting of
\[
S_{\sigma_j}^a \omega^A, \quad S_{\sigma_j}^a \Omega^{A'} \wedge \omega^B, \quad S_{\sigma_j}^a \Omega^{0'} \wedge \Omega^{1'} \wedge \omega^C,
\]
where \(|A| = |B| + 1 = |C| + 2 = j, a = 0, 1, \ldots, \sigma_j\). But
\[
S_{\sigma_j}^a \Omega^{0'} \wedge \Omega^{1'} \wedge \omega^C \in \mathcal{J}_j(U),
\]
by (3.15), and
\[
S_{\sigma_j}^a \Omega^{A'} \wedge \omega^B = (-1)^{\alpha(A')} \mathbb{S}_{\sigma_j}^{\omega a+\alpha(A')} \wedge \omega^B, \quad \text{mod } \mathcal{J}_j(U).
\]
Thus \(\Gamma(U, \mathbb{C}^2 \otimes \mathbb{C}^{2n+2}) \mod \mathcal{J}_j(U)\) is spanned by terms in (3.12)-(3.13), and so \(\mathcal{J}_j(U)\) is spanned by terms in (3.11). (3.19) holds because
\[
S_{\sigma_j}^a \Omega^{0'} \wedge \omega^B = \Omega^{1'} \partial_1 S_{\sigma_j+1}^{a+1} \wedge \omega^B = -\Omega_{\sigma_j}^{a+1} \wedge \omega^B, \quad \text{mod } \mathcal{J}_j(U),
\]
for \(a = 0, \ldots, \sigma_j - 1\), and similarly, \(S_{\sigma_j}^a \Omega^{0'} \wedge \omega^B = \Omega_{\sigma_j}^a \wedge \omega^B \mod \mathcal{J}(U)\). The proposition is proved.

\textbf{Remark 3.2} The indices \(b\) in \(\Omega_{\sigma_j}^{B+1} \) (3.13) are only taken over \(0, \ldots, \sigma_j - 1\), because
\[
2 \Omega_{\sigma_j}^0 = S_{\sigma_j}^0 \Omega^{0'} = \mathbb{D} S_{\sigma_j+1}^0, \quad 2 \Omega_{\sigma_j}^{1+1} = -S_{\sigma_j}^1 \Omega^{1'} = -\mathbb{D} S_{\sigma_j+1}^1 \in \mathcal{J}_j(U).
\]
Lemma 3.1 For \( j = 0, 1, \cdots, k - 1 \),
\[
\partial_0 \Omega^{d+1}_{\sigma_j} = \Omega^{d+1}_{\sigma_{j-1}}, \quad \partial_1 \Omega^a_{\sigma_j} = \Omega^a_{\sigma_{j-1}}, \quad \partial_A d^A' \Omega^{d+1}_{\sigma_j} = \mathcal{E} S^a_{\sigma_j}.
\] (3.20)

Proof The first two identities follow from definition (3.10), while for the third one, we have
\[
\partial_A d^A' \Omega^{d+1}_{\sigma_j} = \frac{1}{2} \left( \partial_1 S^a_{\sigma_j} d^A' \Omega^0 - \partial_0 S^a_{\sigma_j} d^A' \Omega^1 \right) = \mathcal{E} S^a_{\sigma_j-1},
\]
by (3.3) and \( \sigma_{j+1} = \sigma_j - 1 \).

3.2 The induced action of operators \( D_j' \)'s on the boundary complex

By Proposition 3.2, an element of \( \Gamma(U, V_j) \) modulo \( J_j(U) \) has the form
\[
f_a S^a_{\sigma_j} + \Omega^{b+1}_{\sigma_j} \wedge G_b,
\]
for some \( f_a \in \Gamma(U, \wedge^j \mathbb{C}^n) \), \( G_b \in \Gamma(U, \wedge^{j-1} \mathbb{C}^n) \). We need to calculate the action of \( \partial_A d^A' \) on such elements modulo \( J_{j+1}(U) \). Here and in the sequel, we will use the convention that the summation of repeated indices \( a \) in \( S^a_{\sigma_j} \) is taken over \( a = 0, \ldots, \sigma_j \), while the summation of repeated indices \( b \) in \( \Omega^{b+1}_{\sigma_j} \) is taken over \( b = 0, \ldots, \sigma_j - 1 = \sigma_{j+1} \). We can write
\[
\mathcal{E} = \mathcal{E}_0 + \Omega^{A'} \wedge \mathcal{E}_A' + \mathcal{E}_{0V'} \wedge \Omega^0 \wedge \Omega^1
\] (3.21)
for some \( \mathcal{E}_0, \mathcal{E}_{0V'}, \mathcal{E}_{V'} \) and \( \mathcal{E}_{0V'} \) to be 2-, 1-, 1- and 0-forms only involving \( \omega^A \). Denote
\[
\mathcal{B}_c^{\pm} = \partial^A f_c \pm o(A') + \mathcal{E}_0 \wedge G_c,
\]
\[
\mathcal{C}_c^{\pm} = -\partial^A G_c \pm o(A') - \mathcal{E}^{A'} \wedge G_c \pm o(A') - T^{A'B'} f_c \pm o(A'B'),
\] (3.22)
where \( o(A'B') = o(A') + o(B') \). By raising indices,
\[
T^{00'} = T_{00}', \quad T^{01'} = -T_{00}', \quad T^{10'} = T_{10}', \quad T^{11'} = -T_{10}'.
\]

Proposition 3.3 If \( j = 0, 1, \cdots, k - 2 \), we have
\[
\partial_A d^A' \left[ f_a S^a_{\sigma_j} + \Omega^{b+1}_{\sigma_j} \wedge G_b \right] = \mathcal{C}_b S^b_{\sigma_{j+1}} + \Omega^{b+1}_{\sigma_{j+1}} \wedge \mathcal{B}_c^+ \mod J_{j+1}(U).
\]

Proof Note that
\[
\partial_A d^A' \Omega^{b+1}_{\sigma_j} \wedge G_b = S^b_{\sigma_{j+1}} \mathcal{E} \wedge G_b = \left[ \mathcal{E}_0 S^b_{\sigma_{j+1}} + \Omega^b_{\sigma_{j+1}} \wedge \mathcal{E}_{0V'} - \Omega^{b+1}_{\sigma_{j+1}} \wedge \mathcal{E}_{V'} \right]
\mod J_{j+1}(U).
\]
by using Lemma 3.1, (3.19) and (3.21). By using the formula (3.6) of \( d^A' \) and Lemma 3.1 again, we get
\[
\partial_A d^A' \left[ \Omega^{b+1}_{\sigma_j} \wedge G_b \right] = \partial_A d^A' \Omega^{b+1}_{\sigma_j} \wedge G_b - \partial_A \Omega^{b+1}_{\sigma_j} \wedge d^A' G_b
\]
\[
= S^b_{\sigma_{j+1}} \mathcal{E}_0 \wedge G_b - \Omega^b_{\sigma_{j+1}} \wedge \left( \partial^0 G_b + \mathcal{E}_{0V'} \wedge G_b \right)
\]
\[- \Omega^b_{\sigma_{j+1}} \wedge \left( \partial^1 G_b - \mathcal{E}_{0V'} \wedge G_b \right)
\]
\mod J_{j+1}(U),
since
\[
\Omega^b_{\sigma_{j+1}} \wedge \Omega^{A'} \wedge \cdots = 0 \mod J_{j+1}(U),
\] (3.23)
by (3.18). Now by relabeling and raising indices, we get
\[
\partial_A' d^{A'} \left[ \Omega^{b+1}_{\sigma j+1} \wedge G_b \right] = S^a_{\sigma j} \mathcal{E}_0 \wedge G_b - \Omega^{c+1}_{\sigma j+1} \wedge \left[ \mathcal{D}^{A'} G_{c+o(A')} + \mathcal{E}^{A'} \wedge G_{c+o(A')} \right] \mod J_{j+1}(U).
\]

On the other hand,
\[
\partial_A' \Omega^{A'} \wedge \partial_{4n+1} (f_a S^a_{\sigma j}) = \Box S^a_{\sigma j} \wedge \partial_{4n+1} f_a \in J_{j+1}(U),
\]
and so
\[
\partial_A' d^{A'} \left[ f_a S^a_{\sigma j} \right] = \partial_A' S^a_{\sigma j} \left( \Omega^{B'} f_a + \mathcal{T}^{A'}_{B'} f_a \right)
\]
\[
= S^{a-o(A')}_{\sigma j} \partial_A' f_a + S^a_{\sigma j-1} \left( \Omega^{0'} T^{0'}_{0'} f_a + \Omega^{1'} T^{0'}_{1'} f_a \right)
\]
\[
+ S^{a-1}_{\sigma j-1} \left( \Omega^{0'} T^{0'} f_a + \Omega^{1'} T^{0'}_{1'} f_a \right)
\]
\[
= S^{a}_{\sigma j+1} \partial_A' f_{a+o(A')} + \Omega^{a+1}_{\sigma j+1} \left( \left( T^{0'}_{0'} - T^{1'}_{1'} \right) f_a - \Omega^{a+1}_{\sigma j+1} T^{0'}_{1'} f_a + \Omega^{a-1}_{\sigma j+1} T^{0'}_{0'} f_a \right)
\]
\[
= S^{a}_{\sigma j+1} \partial_A' f_{a+o(A')} - \Omega^{a+1}_{\sigma j+1} \left[ \left( T^{0'}_{0'} + T^{1'}_{1'} \right) f_{c+1} + T^{0'}_{0'} f_c + T^{1'}_{1'} f_{c+2} \right] \mod J_{j+1}(U)
\]
by using (3.6), (3.19), Proposition 3.2 (1), and relabeling indices. Where \( c \) is taken over 0, \( \ldots, \sigma_{j+1} - 1 = \sigma_{j+2} \). The result follows from the sum of the above identities. \( \square \)

### 3.3 Proof of theorem 1.1 for \( j = 0, 1, \ldots, k - 1 \)

To write down operators \( \mathcal{D}_j \) in the boundary complex, define isomorphisms
\[
\Pi_j : \Gamma(U, V_j)/J_j(U) \longrightarrow \Gamma(bD \cap U, \gamma_j(1)) \oplus \Gamma(bD \cap U, \gamma_{j+1}),
\]
\[
f_a S^a_{\sigma j} + \Omega^{b+1}_{\sigma j} \wedge G_b \mapsto \left( f_a S^a_{\sigma j}, G_b S^b_{\sigma j+1} \right),
\]
where the first summation is taken over \( a = 0, \ldots, \sigma_j \), while the second one is taken over \( b = 0, \ldots, \sigma_{j+1} \).

The operator \( \mathcal{D}_j : \Gamma(bD, \gamma_j) \longrightarrow \Gamma(bD, \gamma_{j+1}) \) is given by
\[
\mathcal{D}_j = \Pi_{j+1} \circ \partial_A' d^{A'} \circ \Pi_j^{-1} \mod J_{j+1}(U).
\]

By Proposition 3.3, we get
\[
\mathcal{D}_j \left( f_a S^a_{\sigma j}, G_b S^b_{\sigma j+1} \right) = \Pi_{j+1} \left( \partial_A' d^{A'} \left[ f_a S^a_{\sigma j} + \Omega^{b+1}_{\sigma j} \wedge G_b \right] \right)
\]
\[
= \Pi_{j+1} \left( \mathcal{B}^{b} S^b_{\sigma j+1} + \Omega^{c+1}_{\sigma j+1} \wedge \mathcal{C}^{c+1} \right)
\]
\[
= \left( \mathcal{B}^{b} S^b_{\sigma j+1}, \mathcal{C}^{c} S^c_{\sigma j+2} \right),
\]
where \( c \) is taken over 0, \( \ldots, \sigma_{j+2} \). On the other hand, we have
\[
\mathcal{B}^{b} S^b_{\sigma j+1} = \partial_A' \mathcal{D}^{A'} \left( f_a S^a_{\sigma j} \right) + \mathcal{E}_0 \wedge \left( G_b S^b_{\sigma j+1} \right),
\]
\[
\mathcal{C}^{c} S^c_{\sigma j+2} = -\partial_A' \mathcal{D}^{A'} \left( G_b S^b_{\sigma j+1} \right) - \mathcal{E}^{A'} \wedge \partial_A' \left( G_b S^b_{\sigma j+1} \right) - \partial_A' \partial_B' \mathcal{T}^{A'B'} \left( f_a S^a_{\sigma j} \right)
\]
\[\square \text{ Springer}\]
by direct differentiation with respect to $A'$, and formulae (3.22) of $\mathcal{R}^+_b$ and $\mathcal{R}^+_c$. Thus, if taking $F_1 = f B S_{\sigma_j}^a$ and $F_2 = G B S_{\sigma_{j+1}}^b$, we see that $\mathcal{R}_j$ is given by

$$\mathcal{R}_j F = \left( \partial_A' \partial A' F_1 + \mathcal{E}_0 \land F_2, -\partial_A' (\partial A' + \mathcal{E} A') \land F_2 - \partial_A' \partial B' T A' B' F_1 \right).$$

(3.28)

4 The boundary complex for $j \geq k - 1$

4.1 Vector spaces of the boundary complex

For $j = k + 1, \ldots, 2n + 1$, let

$$\overline{\mathcal{D}} := s A' \Omega A' \land$$

which is the symbol of the differential operator $s A' d A'$ at the direction grad $\varrho$, and

$$\widetilde{\Omega}_{\sigma_j}^b := \frac{1}{2} \left( S_{\sigma_j}^{b-1} \Omega' - \overline{\Omega}_{\sigma_j}^b \Omega' \right).$$

(4.2)

where $b = 0, \ldots, \sigma_j + 1$. Unlike $\Omega_{\sigma_j}^a$, elements $\widetilde{\Omega}_{\sigma_j}^0 := -\frac{1}{2} S_{\sigma_j}^0 \Omega'$ and $\widetilde{\Omega}_{\sigma_j}^{a+1} := \frac{1}{2} S_{\sigma_j}^a \Omega'$ are not in ker $\overline{\mathcal{D}}$. It is direct to check that

Lemma 4.1

$$s_0 \widetilde{\Omega}_{\sigma_j}^b = \widetilde{\Omega}_{\sigma_{j+1}}^b,$$

$$s_1 \widetilde{\Omega}_{\sigma_j}^b = \widetilde{\Omega}_{\sigma_{j+1}}^{b+1},$$

$$s A' d A' \widetilde{\Omega}_{\sigma_j}^b = \mathcal{E} \widetilde{\Omega}_{\sigma_{j+1}}^b.$$  

(4.3)

Proposition 4.1 For fixed $k$, we have (1) $\mathcal{J}_{k+1}(U)$ consists of

$$\Omega' \land \Omega' \land f + \varrho F,$$

(4.4)

where $f \in \Gamma(U, \land^k \mathbb{C}^{2n})$ and $F \in \Gamma(U, \land^{k+2} \mathbb{C}^{2n+2})$.

(2) if $j = k + 2, \ldots, 2n - 1$, $\mathcal{J}_j(U)$ consists of

$$\overline{\mathcal{D}} S_{\sigma_j}^b \land f_b + \varrho F,$$

(4.5)

where $f_b \in \Gamma(U, \land^{j-1} \mathbb{C}^{2n+2})$, $F \in \Gamma(U, \land^j \mathbb{C}^{2n+2})$, and $b = 0, \ldots, \sigma_j - 1$.

(3) $\mathcal{J}_k(U)$ consists of

$$\Omega A' \land f A' + \Omega' \land \Omega' \land g + \varrho \left( \partial A' \partial A' - \mathcal{E}_0 \land g \right),$$

(4.6)

where $f A' \in \Gamma(U, \land^{k-1} \mathbb{C}^{2n})$ and $g \in \Gamma(U, \land^{k-2} \mathbb{C}^{2n})$. If $k = 1$, then $g = 0$.

Proof (1) For $f \in \Gamma(U, V_j)$ and $\psi \in \Gamma(U, V_{j+1})$ with compact support, we have

$$\int_D \left( s A' d A' f, \psi \right) d V = \int_D \left( s A' \left( \Omega A' \land \partial A' + d A' \right) f, \psi \right) d V$$

$$= \int_{b D} \left( \overline{\mathcal{D}} f, \psi \right) \left. \frac{d S}{\text{grad } \varrho} \right| + \int_D \left( f, \left( s A' d A' \right)^* \psi \right) d V,$$

where $(s A' d A' )^*$ is the formal adjoint of $s A' d A'$. Thus, $f \in \mathcal{J}_j(U)$ if and only if

$$\overline{\mathcal{D}} f \big|_{b D \cap U} = 0.$$  

(4.7)
When $j = k + 1$, we have $\sigma_{k+1} = 0$. The condition \((4.7)\) is equivalent to $\Omega' \land f = \Omega' \land f = 0$ on the boundary for $f \in \Gamma(U, \land^{k+2}\mathbb{C}^2)$. Thus, $J_{k+1}(U)$ consists of elements of the form \((4.4)\).

(2) For $j = k + 2, \ldots, 2n - 1$, we have $\sigma_j = 1, 2, \ldots$, and it is direct to see that

\[
\widetilde{\mathbf{D}} \mathbf{D} \mathbf{S}^b_{\sigma_j-1} = 0, \quad b = 0, \ldots, \sigma_j - 1,
\]

\[
\widetilde{\mathbf{D}} \Omega^c_{\sigma_j} = -\widetilde{\mathbf{S}}_{\sigma_j+1}^c \Omega'^\land \Omega' \neq 0, \quad c = 0, \ldots, \sigma_j + 1,
\]

by \((4.2)\) and $\widetilde{\mathbf{D}} = 0$ as in \((3.14)\). Thus $\widetilde{\mathbf{S}}_{\sigma_j-1} = \mathbf{S}_{\sigma_j} + \mathbf{S}_{\sigma_j+1}^\land \Omega' \in J_j(U)$. Note that

\[
\widetilde{\mathbf{S}}_{\sigma_j} \Omega'^\land \omega^b = (1)_{\theta(A')} \mathbf{S}_{\sigma_j} \Omega'^\land (\omega^b), \quad \text{mod } J_j(U),
\]

because

\[
\mathbf{S}_{\sigma_j} \Omega' = \mathbf{S}_{\sigma_j+1}^b, \quad \mathbf{S}_{\sigma_j} \Omega' = -\mathbf{S}_{\sigma_j} \mod J_j(U),
\]

by definition. Thus $J_j(U)$ consists of elements in \((4.5)\) as in the case $j < k$ in Proposition \(3.2\).

(3) Since $\partial_{n+1} \Omega'^\land = 0$, we have that for $f = f_A \omega^A \in \Gamma(U, \land^k\mathbb{C}^2)$,

\[
d'^\land f = \Omega'^\land \land \partial_{n+1} f + d'^\land \Omega'^\land = \Omega'^\land \land \partial_{n+1} f + d'^\land d'^\land f
\]

by using \((3.4)\) twice. Recall that there is no boundary term for $d'^\land f$ after integration by part. We get

\[
\int_D \left\{d'^\land f, \psi \right\} dV = \int_D \left\{f, (d'^\land d'^\land)^* \psi \right\} dV - \int_{bd} \left\{d'^\land (\Omega'^\land \land f) + \Omega'^\land \land d'^\land f, \psi \right\} \frac{dS}{\|\text{grad} \psi\|}.
\]

by integration by part twice. Thus $f \in J_k(U)$ if and only if

\[
d'^\land (\Omega'^\land \land f) \bigg|_{bd} = 0,
\]

\[
d'^\land (\Omega'^\land \land f) \bigg|_{bd} = 0,
\]

by definition of $d'^\land$. By the first equation, we can write

\[
f = \Omega'^\land \land f = \Omega'^\land \land f + \omega F + O(\psi^2)
\]

for some $f_A, g, F$ independent of $\psi$ and valued in $\land^{n}\mathbb{C}^2$. Then,

\[
d'^\land (\Omega'^\land \land f) \bigg|_{bd} = d'^\land (\Omega'^\land \land f + \omega \Omega'^\land \land F) \bigg|_{bd} = -\mathbf{E} \land \Omega'^\land \land f_0 - \Omega'^\land \land \mathbf{F} + \mathbf{E} \land \mathbf{F},
\]

by using the Leibnitz law and \((3.6)\). Similarly,

\[
\Omega'^\land \land d'^\land f \bigg|_{bd} = \Omega'^\land \land d'^\land f \bigg|_{bd} - \Omega'^\land \land \mathbf{F} \bigg|_{bd} = \Omega'^\land \land \mathbf{E} \land f_0 - \Omega'^\land \land \mathbf{F} + \mathbf{E} \land \mathbf{F}.
\]

\( \Box \) Springer
Their sum gives us that the second equation in (4.10) is equivalent to
\[ \Omega^0 \wedge \Omega^1 \wedge (F - \delta^A f_A + \mathcal{E} \wedge g) \bigg|_{bD} = 0, \]
i.e. \( F = \delta^A f_A - \mathcal{E}_0 \wedge g \mod \Omega^0 \), \( \Omega^1 \). The result follows. \( \square \)

By Proposition 4.1 and its proof, we know vector spaces of the boundary complex.

**Corollary 4.1** (1) for \( j = k + 1, \ldots, 2n - 1 \),
\[ \mathcal{Y}^{(1)}_j \equiv \mathcal{C} \tau_j \mathcal{C}^2 \wedge j+1 \mathcal{C}^2n, \quad \mathcal{Y}^{(2)}_j \equiv \mathcal{C} \tau_{j+1} \mathcal{C}^2 \wedge j \mathcal{C}^2n, \]
are spanned by
\[ \tilde{S}^a_{\sigma_j} \omega^B, \quad \tilde{\Omega}^b_{\sigma_j} \wedge \omega^C, \] (4.11)
respectively, where \( a = 0, \ldots, \sigma_j \), \( b = 0, \ldots, \sigma_{j+1} \), \( |B| = j + 1 \), \( |C| = j \).

(2) \( \mathcal{Y}^{(1)}_k \equiv \wedge^k \mathcal{C}^2n \) and \( \mathcal{Y}^{(2)}_k \equiv \wedge^k \mathcal{C}^2n \) are spanned by
\[ \omega^A, \quad \psi \omega^B, \] (4.12)
respectively, where \( |A| = |B| = k \).

4.2 The induced action of operators \( \mathcal{D}_j \)'s on the boundary complex

**Proposition 4.2** If \( j = k + 1, \ldots, 2n \), we have
\[ s_A d^A \left[ f_A \tilde{S}^a_{\sigma_j} + \tilde{\Omega}^b_{\sigma_j} \wedge G_b \right] = \mathcal{R}_b \tilde{S}^b_{\sigma_{j+1}} + \tilde{\Omega}^c_{\sigma_{j+1}} \wedge \mathcal{J}_c^- \mod \mathcal{J}_{j+1}(U); \]

**Proof** Note that
\[ s_A d^A \tilde{\Omega}^b_{\sigma_j} \wedge G_b = \tilde{S}^b_{\sigma_{j+1}} \mathcal{E} \wedge G_b = \tilde{S}^b_{\sigma_{j+1}} \left( \mathcal{E}_0 + \Omega^A \wedge \mathcal{E}_A \right) \wedge G_b \]
\[ = \mathcal{E}_0 \tilde{S}^b_{\sigma_{j+1}} + \tilde{\Omega}^{b+1}_{\sigma_{j+1}} \wedge \mathcal{E}_b - \tilde{\Omega}^b_{\sigma_{j+1}} \wedge \mathcal{E}_1 \wedge G_b \mod \mathcal{J}_{j+1}(U). \]

by (4.9). By using (4.3), we find that
\[ s_A d^A \left[ \tilde{\Omega}^b_{\sigma_j} \wedge G_b \right] \]
\[ = s_A d^A \tilde{\Omega}^b_{\sigma_j} \wedge G_b - s_A \tilde{\Omega}^b_{\sigma_j} \wedge d^A G_b \]
\[ = \tilde{S}^b_{\sigma_{j+1}} \mathcal{E}_0 \wedge G_b - \tilde{\Omega}^b_{\sigma_{j+1}} \left( \mathcal{E}_0 \wedge G_b \right) - \tilde{\Omega}^{b+1}_{\sigma_{j+1}} \wedge \mathcal{E}_0 \wedge G_b \]
\[ = \tilde{S}^b_{\sigma_{j+1}} \mathcal{E}_0 \wedge G_c - \tilde{\Omega}^b_{\sigma_{j+1}} \wedge \mathcal{E}_0 \wedge G_b \mod \mathcal{J}_{j+1}(U). \]

by \( \tilde{\Omega}^b_{\sigma_{j+1}} \wedge \Omega^A \wedge * \in \mathcal{J}_{j+1}(U) \) (cf. (4.8)).
As in (3.24), \( s_A^* \Omega^A \wedge \partial_{a+n+1} (f_a \tilde{S}_{\sigma_j}^a) = \tilde{B} \tilde{S}_{\sigma_j}^a \wedge \partial_{a+n+1} f_a \in \mathcal{J}_{j+1}(U) \). So we have

\[
\begin{align*}
  s_A^* d^A (f_a \tilde{S}_{\sigma_j}^a) \\
  &= s_A \tilde{S}_{\sigma_j}^a \left( \tilde{d}^A f_a + \Omega^B \wedge T^A_B f_a \right) \\
  &= \tilde{S}_{\sigma_{j+1}}^a \left( \tilde{d}^A f_a + \tilde{S}_{\sigma_{j+1}}^b \left( \tilde{\Omega}_0^0 \wedge T^0_0 + \Omega^1 \wedge T^0_1 \right) f_b \right) \\
  &\quad + \tilde{S}_{\sigma_{j+1}}^b \left( \tilde{\Omega}_0^0 \wedge T^1_0 + \Omega^1 \wedge T^1_1 \right) f_b \\
  &= \tilde{S}_{\sigma_{j+1}}^a \tilde{d}^A f_a - \tilde{\Omega}_{\sigma_{j+1}}^b \left[ \left( T^0_0 - T^1_1 \right) - \tilde{\Omega}_{\sigma_{j+1}}^b \wedge T^0_j + \tilde{\Omega}_{\sigma_{j+1}}^b \wedge T^1_j \right] f_b \\
  &= \tilde{S}_{\sigma_{j+1}}^a \tilde{d}^A f_a - \tilde{\Omega}_{\sigma_{j+1}}^b \left[ \left( T^0_0 + T^1_0 \right) f_{b-1} + T^0_0 f_b + T^1_1 f_{b-2} \right] \\
  &= \tilde{S}_{\sigma_{j+1}}^a \tilde{d}^A f_a - \tilde{\Omega}_{\sigma_{j+1}}^b T^A_B f_{b-o(A'B')} \mod \mathcal{J}_{j+1}(U),
\end{align*}
\]

by using (3.6), (4.9) and relabeling indices. The result follows from the sum of these two identities. \( \square \)

**Proposition 4.3** We have (1) if \( j = k - 1 \),

\[
\partial_A d^A [f_A S^1_{\sigma} + \Omega_1^1 \wedge G] = \partial^A f_{o(A')} + \varepsilon_0 \wedge G \\
+ \varepsilon \left[ \partial^0 \partial^1 G - \partial^A (\varepsilon_A' \wedge G) + \varepsilon_0 \wedge \left( T^{01}_{01} + \varepsilon_{01}' \right) G - \partial^B T^A_B f_{o(A')} \right] \mod \mathcal{J}_k(U)
\]

for \( f_a \in \Gamma(U, \wedge^{k-1} \mathbb{C}^{2n}) \) and \( G \in \Gamma(U, \wedge^{k-2} \mathbb{C}^{2n}) \). Here \( T^{01}_{01} = \frac{1}{2} (T^{01} - T^{10}) \) is skew-symmetrization;

(2) if \( j = k \),

\[
\partial^0 \partial^1 (f_0 + \varepsilon f_1) = \partial^0 \partial^1 f_0 - \varepsilon \wedge \left( T^{01} f_0 + f_1 \right) + \Omega^A \wedge \left( 2 T^0_{A1} \partial^1 f_0 - \partial^A f_1 \right)
\]

holds \( \mod \mathcal{J}_{k+1}(U) \) for \( f_0, f_1 \in \Gamma(U, \wedge^k \mathbb{C}^{2n}) \) independent of \( \varphi \).

**Proof** (1) Since \( \sigma_{k-1} = 1 \), \( \Omega_{o_{o_{k-1}}}^1 \) has only one element: \( \Omega_1^1 = \frac{1}{2} \left( s^0 \Omega^0 - s^0 \Omega^1 \right) \). Here we can assume \( f_a \) and \( G \) are independent of \( \varphi \), because \( \partial_A d^A (\varphi \omega) \) for \( \omega \in \Gamma(U, \wedge^{k-1} \mathbb{C}^{2n+2}) \) belongs to \( \mathcal{J}_k(U) \) automatically by (2.20). Thus

\[
\partial_A d^A [\Omega_1^1 \wedge G] = \varepsilon \wedge G - \partial_A \Omega_1^1 \wedge d^A G \\
= \varepsilon_0 \wedge G + \Omega^A \wedge \varepsilon_A' \wedge G + \Omega_0^1 \wedge \varepsilon_{01}' \wedge G \\
+ \frac{1}{2} \Omega^1 \wedge \partial^0 G - \frac{1}{2} \Omega^0 \wedge \partial^1 G + \frac{1}{2} \Omega^1 \wedge \Omega^0 \wedge \left( T^0_0 + T^1_1 \right) G \\
= \varepsilon_0 \wedge G + \varepsilon \left[ -\partial^A (\varepsilon_A' \wedge G) + \partial^0 \partial^1 G + \varepsilon_0 \wedge T^{01}_{01} G + \varepsilon_0 \wedge \varepsilon_{01}' G \right] \mod \mathcal{J}_k(U),
\]

by Proposition 4.1 (3). On the other hand, \( f_a S^1_{\sigma} = f_0 s^0 + f_1 s^1 \) by definition, and

\[
\begin{align*}
\partial_A d^A (f_a S^1_{\sigma}) &= \partial^0 f_0 + \partial^1 f_1 + \Omega^B \wedge \left( T^0_0 f_0 + T^1_1 f_1 \right) \\
&= \partial^A f_{o(A')} - \varepsilon \left( \partial^B T^0_0 f_0 + \partial^B T^1_1 f_1 \right) \mod \mathcal{J}_k(U).
\end{align*}
\]
Their sum gives us the result.

(2) We have

\[
d^0 d^{1'} (f_0 + \varrho f_1) = d^0' \left[ 0^{1'} f_0 + \Omega^{A'} \wedge T^{A'}_j f_0 + \Omega^{1'} \wedge f_1 + \varrho \left( 0^{1'} f_1 + \Omega^{A'} \wedge T^{1'}_j f_1 \right) \right]
\]

\[
= 0^0 0^{1'} f_0 + \Omega^{A'} \wedge T^{A'}_j 0^{1'} f_0 - \varepsilon \wedge T^{1'}_j f_0 - \Omega^{A'} \wedge 0^0' T^{1'}_j f_0
\]

\[
- \varepsilon \wedge f_1 + (\Omega^{0^0'} \wedge 0^{1'} - \Omega^{1'} \wedge 0^0') f_1, \quad \text{mod } J_{k+1}(U)
\]

(4.13)

by using (3.3), (3.6) and the characterization of \( J_{k+1}(U) \) in (4.4). The result follows. \( \square \)

### 4.3 The operator \( D_j \) in the boundary complex

For \( j = k + 1, \ldots, 2n \), define isomorphisms

\[
\Pi_j : \Gamma(U, V_j)/J_j(U) \to \Gamma(bD \cap U, \mathcal{V}_j^{(1)}) \oplus \Gamma(bD \cap U, \mathcal{V}_j^{(2)}),
\]

\[
\quad \quad f_a S_{\sigma_j}^a + \tilde{\Omega}_b^{j,j} \wedge G_b \mapsto \left( f_a S_{\sigma_j}^a, G_b \tilde{S}_{\sigma_j+1}^b \right),
\]

where the first summation is taken over \( a = 0, \ldots, \sigma_j \), while the second one is taken over \( c = 0, \ldots, \sigma_{j+1} \).

**Theorem 4.1** For \( F = (F_1, F_2) \in \Gamma(bD, \mathcal{V}_j) \), we have

\[
D_j F = \left( s_{A'} d^A F_1 + \varepsilon_0 \wedge F_2, - s_{A'} (d^A + \mathcal{E}^A \wedge) F_2 - s_{A'} s_{B'} T^{A'B'} F_1 \right), \quad j = k + 1, \ldots, 2n,
\]

\[
D_{k-1} F = \left( \delta_{A'} \wedge F_1 + \varepsilon_0 \wedge F_2, \delta^{0^0'} 0^{1'} F_2 - \delta^{0^0'} 0^{1'} \mathcal{E}_A \wedge \mathcal{F}_2 + \varepsilon_0 \wedge (\mathcal{T}^{0^0'} F_1 + \mathcal{E}_0') \mathcal{F}_2 \right.
\]

\[
- \delta_{A'} \wedge \mathcal{T}^{0^0'} F_1 \mathcal{F}_1 \right),
\]

\[
D_k F = \left( \delta^{0^0'} 0^{1'} F_1 - \varepsilon_0 \wedge (\mathcal{T}^{0^0'} F_1 + \mathcal{F}_2), - \delta_{A'} F_2 - \mathcal{E}_A \wedge (\mathcal{T}^{0^0'} F_1 + \mathcal{F}_2) + 2 \mathcal{T}^{0^0'} 0^{1'} F_1 \right).
\]

**Proof** (1) For \( j = k + 1, \ldots, 2n \), we have

\[
D_j \left( f_a S_{\sigma_j}^a, G_b \tilde{S}_{\sigma_j+1}^b \right) = \Pi_{j+1} \left( s_{A'} d^A \left[ f_a S_{\sigma_j}^a + \tilde{\Omega}_b^{j,j} \wedge G_b \right] \right)
\]

\[
= \Pi_{j+1} \left( \mathcal{R}_b \tilde{S}_{\sigma_j+1}^b + \tilde{\Omega}_c^{j,j} \wedge \mathcal{F}_c \right)
\]

\[
= \mathcal{R}_b \tilde{S}_{\sigma_j+1}^b + \mathcal{F}_c \tilde{S}_{\sigma_j+2}^c \mod J_{j+1},
\]

by Proposition 4.2, where

\[
\mathcal{R}_b \tilde{S}_{\sigma_j+1}^b = s_{A'} \mathcal{D}_A \left( f_a S_{\sigma_j}^a \right) + \varepsilon_0 \wedge \left( G_b \tilde{S}_{\sigma_j+1}^b \right),
\]

\[
\mathcal{F}_c \tilde{S}_{\sigma_j+2}^c = - s_{A'} \mathcal{D}_A \left( G_b \tilde{S}_{\sigma_j+1}^b \right) - \mathcal{E}_A \wedge s_{A'} \left( G_b \tilde{S}_{\sigma_j+1}^b \right) - s_{A'} s_{B'} T^{A'B'} \left( f_a S_{\sigma_j}^a \right),
\]

by multiplying \( s_{A'} \). Thus \( D_j \) is given by the first formula.

(2) Since \( \sigma_{k-1} = 1 \), let \( F_1 = f_a S_{\sigma_1}^a = f_0 s^{0^0'} + f_1 s^{1'} \) and \( G = F_2 \). We have

\[
\delta^A f_{o(A')} = \delta_{A'} d^A F_1,
\]

\[
\delta^B T^{A'}_{B'} f_{o(A')} = \delta_{A'} \mathcal{D}_B \mathcal{T}^{A'} F_1.
\]

The result follows Proposition 4.3 (1).

(3) Note that for \( j = k, F_1 = f_0 \) and \( F_2 = f_1 \). The result follows Proposition 4.3 (2). \( \square \)
Remark 4.1 \( \mathcal{D}_j \)'s for \( j > k + 1 \) in (4.15) are exactly operators for \( j < k - 1 \) in (3.28) with \( \partial_{A'} \) replaced by \( s_{A'} \).

(2) By comparing (3.27) with (4.16), we see that \( +o(A') \) in (3.22) comes from \( \partial_{A'} \), while \( -o(A') \) in (3.22) comes from multiplying \( s_{A'} \).

5 The Hartogs–Bochner extension of \( k \)-regular functions

Proposition 5.1 For fixed \( k \), if \( f \in \Gamma(bD, \mathbb{O}^k \mathbb{C}^2) \) is \( k \)-CF, then there exists a representative \( \hat{f} \in \Gamma(D, \mathbb{O}^k \mathbb{C}^2) \) such that \( \hat{f}|_{bD} = f \) and \( D_0 \hat{f} \) is flat on \( bD \).

Proof Denote also by \( f \) any fixed extensions of \( f \) to \( U \supset D \) as a \( C^\infty \) function.

(1) The case \( k > 1 \). Since \( f \) is \( k \)-CF, \( \partial_{A'}d^{A'}_k f \in \mathcal{J}_1(U) \) and so
\[
\partial_{A'}d^{A'}_k f = \mathbb{D}S^{a}_{\sigma_1} \cdot f_a + \varrho F,
\]
(5.1)
for some functions \( f_a \in \Gamma(U, \mathbb{O}) \), \( F \in \Gamma(U, \mathcal{V}_1) \), \( a = 0, \ldots, \sigma_1 \), by the characterization of \( \mathcal{J}_1(U) \) in Proposition 3.2. Note that \( \mathbb{D} \) given by (3.9) is globally defined since \( \Lambda_{A'} \)'s are. But
\[
\partial_{A'}d^{A'}_k \left( \varrho S^{a}_{\sigma_1} f_a \right) = \Lambda_{A'} \Lambda_{A'}S^{a}_{\sigma_1} \cdot f_a + O(\varrho) = \mathbb{D}S^{a}_{\sigma_1} \cdot f_a + O(\varrho).
\]
We see that
\[
\partial_{A'}d^{A'}_k \left( f - \varrho S^{a}_{\sigma_1} f_a \right) = \varrho F^{(1)},
\]
(5.2)
for some \( F^{(1)}_1 \in \Gamma(U, \mathcal{V}_1) \). As \( \partial_{A'}d^{A'}_k \) applied to the left hand side gives zero by (2.8), we get
\[
0 = \partial_{A'}d^{A'}_k(\varrho F^{(1)}_1) = \mathbb{D}F^{(1)}_1|_{bD} + O(\varrho),
\]
Namely, \( \mathbb{D}F^{(1)}_1|_{bD} = 0 \). Hence
\[
F^{(1)}_1 = \mathbb{D}S^{a}_{\sigma_1} \cdot F_a + \varrho F^{(2)}_1
\]
for some functions \( F_a \in \Gamma(U, \mathbb{O}) \) by Proposition 3.2 again, and so
\[
\partial_{A'}d^{A'}_k \left( f - \varrho S^{a}_{\sigma_1} f_a \right) = \varrho \mathbb{D}S^{a}_{\sigma_1} \cdot F_a + \varrho^2 F^{(2)}_1.
\]
(5.3)
Repeating this procedure, we get
\[
\partial_{A'}d^{A'}_k \left( f + \varrho f^{(1)} + \varrho^2 f^{(2)} + \cdots \right) \equiv O_{bD}^\infty
\]
with a formal power series in \( \varrho \) with coefficients \( C^\infty \) on \( bD \), where \( O_{bD}^\infty \) denotes functions vanishing of infinite order on \( bD \). By using the Whitney extension theorem (cf. [3, 4, 1, Proposition 22]), we get the conclusion.

(2) The case \( k = 0 \). Write \( f = u_0 + \varrho u_1 \). Since \( (u_0, u_1) \) is \( 0 \)-CF on \( bD, d^{0'}d^{1'}(u_0 + \varrho u_1) \in \mathcal{J}_1(U) \),
\[
d^{0'}d^{1'}(u_0 + \varrho u_1) = \varrho \alpha_1 + \Omega^{0'} \cdot \Omega^{1'} \cdot \beta_1
\]
for some \( \alpha_1 \in \Gamma(U, \lambda^2 \mathbb{C}^{2n+2}) \), \( \beta_1 \in \Gamma(U, \mathbb{O}) \), by the characterization of \( \mathcal{J}_1(U) \) for \( k = 0 \) in Proposition 4.1 (1). Then
\[
d^{0'}d^{1'} \left( u_0 + \varrho u_1 - \frac{1}{2} \varrho^2 \beta_1 \right) = \varrho \alpha'_1
\]
for some $\alpha'_1 \in \Gamma(U, \wedge^2 \mathbb{C}^{2n+2})$. As $d^{0'}$ and $d^{1'}$ applied to the left hand side give zero by (2.9), we get
\[
\Omega^{0'} \wedge \alpha'_1|_{bD} = 0, \quad \Omega^{1'} \wedge \alpha'_1|_{bD} = 0,
\]
and so
\[
\alpha'_1 = \Omega^{0'} \wedge \Omega^{1'} \wedge \beta_2 + \varrho \alpha_2
\]
for some $\alpha_2 \in \Gamma(U, \wedge^2 \mathbb{C}^{2n+2})$ and $\beta_2 \in \Gamma(U, \mathbb{C})$. Thus,
\[
d^{0'} d^{1'} \left( u_0 + \varrho u_1 - \frac{1}{2} \varrho^2 \beta_1 - \frac{1}{6} \varrho^3 \beta_2 + \cdots \right) = \varrho^2 \alpha'_2.
\]
Repeating this procedure, we get
\[
d^{0'} d^{1'} (u_0 + \varrho u_1 + \cdots) \equiv O_{bD}^\infty
\]
and get the conclusion as the case $k > 1$.

(3) The case $k = 1$. Since $f$ is 1-CF, $\partial A' d^{A'} f \in \mathcal{J}_1(U)$, and so
\[
\partial A' d^{A'} f = \Omega^{A'} f_{A'} + \varrho \partial^{A'} f_{A'} + O(\varrho^2)
\]
for some functions $f_{A'} \in \Gamma(U, \mathbb{C})$, by the characterization of $\mathcal{J}_1(U)$ for $k = 1$ in Proposition 4.1 (3). Since
\[
\partial A' d^{A'} \left( \varrho s^{B'} f_{B'} \right) = \Omega^{A'} f_{A'} + \varrho \left( \partial^{A'} f_{A'} + \Omega^{B'} + \left( T^{A'}_{B'} + d_{n+1} \right) f_{A'} \right) + O(\varrho^2)
\]
we see that
\[
\partial A' d^{A'} \left( f - \varrho s^{A'} f_{A'} \right) = \varrho \Omega^{A'} \cdot f_{A'}^{(1)} + \varrho^2 G + O(\varrho^2)
\]
for some $f_{A'}^{(1)} \in \Gamma(U, \mathbb{C})$ and $G \in \Gamma(U, \mathbb{C}^{2n+2})$ independent of $\varrho$. As $d^{0'} d^{1'}$ applied to the left hand side gives zero, we get
\[
\Omega^{1'} \wedge \Omega^{0'} \wedge \partial^{0'} f_{0'}^{(1)} = \Omega^{0'} \wedge \Omega^{1'} \wedge \partial^{1'} f_{1'}^{(1)} + 2 \Omega^{0'} \wedge \Omega^{1'} \wedge G = 0
\]
on the boundary $bD$, i.e.
\[
2G = \partial^{A'} f_{A'}^{(1)}, \mod \Omega^{A'}.
\]
Therefore
\[
\partial A' d^{A'} \left( f - \varrho s^{A'} f_{A'} \right) = \varrho \left[ \Omega^{A'} \cdot f_{A'}^{(1)} + \frac{\varrho}{2} \left( \partial^{A'} f_{A'}^{(1)} + \Omega^{A'} \cdot f_{A'}^{(2)} \right) + O(\varrho^2) \right],
\]
for some $f_{A'}^{(2)} \in \Gamma(U, \mathbb{C})$ independent of $\varrho$, and so
\[
\partial A' d^{A'} \left( f - \varrho s^{A'} f_{A'} - \frac{\varrho^2}{2} s^{A'} f_{A'}^{(1)} \right) = \varrho^2 \left[ \Omega^{A'} \cdot f_{A'}^{(2)} + O(\varrho) \right].
\]
for some $f_{A'}^{(2)} \in \Gamma(U, \mathbb{C})$. Repeating this procedure, we get the conclusion as above. \qed

**Remark 5.1** Such extension for CR functions was constructed by Andreotti-Hill [3, 4], while extension for pluriharmonic functions satisfying $\partial \bar{\partial}$-equation was constructed by Andreotti-Nacinovich in [6].

To show the Hartogs–Bochner extension, we need to solve nonhomogeneous $k$-Cauchy–Fueter equation $D_0 u = f$, for $f$ satisfying the compatibility condition $D_1 f = 0.$
Theorem 5.1 [40, Theorem 5.3] For \( f \in C_0(\mathbb{R}^{4n+4}, \mathcal{V}_1) \) such that \( \mathcal{D}_1 f = 0 \) in the sense of distributions, then there exists a function \( u \in \mathcal{C}_0(\mathbb{R}^{4n+4}, \mathcal{V}_0) \cap W^{1,2}(\mathbb{R}^{4n+4}, \mathcal{V}_0) \) \((u \in C_0(\mathbb{R}^{4n+4}, \mathcal{C}) \cap W^{2,2}(\mathbb{R}^{4n+4}, \mathcal{C}) \) if \( k = 0 \)\) satisfying \( \mathcal{D}_0 u = f \) and vanishing on the unbounded connected component of \( \mathbb{R}^{4n+4} \setminus \text{supp} f \).

Let us recall the construction of solutions in [40]. Consider the associated Hodge-Laplacian on \( \Gamma(D, \mathcal{V}_1) \):

\[
\square_1 = \mathcal{D}_0 \mathcal{D}_0^* + (D_1^* D_1)^2, \quad \square_1 = (\mathcal{D}_0 D_0^*)^2 + D_1^* D_1.
\]

if \( k = 0 \) and \( k = 1 \), respectively, and

\[
\square_1 = (\mathcal{D}_0 D_0^*)^2 + (D_1^* D_1)^2,
\]

if \( k \geq 2 \). They are all uniformly elliptic differential operators of 4-th order with constant coefficients. Its inverse \( \mathbf{G}_1 \) in \( L^2(\mathbb{R}^{4n+4}, \mathcal{V}_j) \) is a convolution operator with matrix kernel of \( C^\infty(\mathbb{R}^{4n+4} \setminus \{0\}) \) homogeneous functions of degree \(-4n\). The solution is given by \( \mathcal{D}_0^* \mathcal{D}_0 \mathcal{D}_0^* \mathbf{G}_1 f \) for \( k \geq 1 \), i.e.

\[
\mathcal{D}_0(\mathcal{D}_0^* \mathcal{D}_0 \mathcal{D}_0^* \mathbf{G}_1 f) = f.
\]

The solution is \( \mathcal{D}_0^* \mathbf{G}_1 f \) for \( k = 0 \).

Proof of Theorem 1.2 By Proposition 5.1, we can extend \( f \) to a smooth function \( \hat{f} \) on \( \overline{D} \) and extend \( \mathcal{D}_0 \hat{f} \) by 0 outside of \( \overline{D} \) to get a \( \mathcal{D}_1 \)-closed element \( F \in \Gamma(\mathbb{H}^{n+1}, \mathcal{V}_1) \) supported in \( \overline{D} \). Then, by Theorem 5.1, there exists \( H \in \mathcal{C}(\mathbb{H}^{n+1}, \mathcal{V}_0) \) with compact support such that \( F = \mathcal{D}_0 H \). Note that \( H \) is \( k \)-regular on \( \mathbb{H}^{n+1} \setminus \overline{D} \). Since a \( k \)-regular function is harmonic [40] (see this fact in (7.3) for right-type groups), by analytic continuation, \( H \) vanishes on the connected open set \( \mathbb{H}^{n+1} \setminus \overline{D} \). Then \( F = f - H \) gives us the required extension. \( \square \)

6 Right-type groups and the quaternionic Monge–Ampère operator

6.1 The nilpotent Lie groups of step two associated to rigid quadratic hypersurfaces

Consider rigid model domain \( D \) in (1.14) and the projection \( \pi : D \to \mathbb{H}^n \times \mathbb{H}_+ \) given by

\[
(q', q_{n+1}) \mapsto (q', q_{n+1} - \phi(q')),
\]

where \( q' \in \mathbb{H}^n \), which maps \( bD \) to \( \mathbb{H}^n \times \text{Im} \mathbb{H} \). The push forward vector field \( \pi_\ast \overline{\partial}_l \) is exactly \( \overline{\delta}_{ql+1} + \overline{\delta}_{ql+1} \phi \cdot \overline{\partial}_t \), where \( t = t_1 + t_2 \mathbf{j} + t_3 \mathbf{k} \in \text{Im} \mathbb{H} \) (cf. [37, Subsection 5.1]). Denote

\[
X_{4l+1} + iX_{4l+2} + jX_{4l+3} + kX_{4l+4} := \overline{\delta}_{ql+1} + \overline{\delta}_{ql+1} \phi \cdot \overline{\partial}_t,
\]

where \( l = 0, \ldots, n - 1 \). Then

\[
X_b = \overline{\partial}_b + 2 \sum_{\beta=1}^{4n} (\mathbf{S} \overline{\beta})_{ab} x_a \overline{\partial}_b
\]

by direct calculation [37, Proposition 5.1], and so

\[
[X_a, X_b] = 2 \sum_{\beta=1}^{3} B_{ab}^\beta \overline{\partial}_b,
\]
where

\[ B^\beta := \mathbb{I}^\beta + \mathbb{I}^\beta \mathbb{S} \]  \hspace{1cm} (6.4)

([37, Subsection 5.1] [44]). Here \( \mathbb{I}^\beta = \text{diag}(I^\beta, I^\beta, \ldots) \) with

\[
I^1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad I^2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad I^3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]

(6.5)

satisfying commutating relation of quaternions \((I^1)^2 = (I^2)^2 = (I^3)^2 = -I_4, I^1 I^2 = I^3\). So \( \text{span}_\mathbb{R} \{X_1, \ldots, X_{4n}, \partial_{\theta_1}, \partial_{\theta_2}, \partial_{\theta_3}\} \) is a nilpotent Lie algebra with center \( \text{span}_\mathbb{R} \{\partial_{\theta_1}, \partial_{\theta_2}, \partial_{\theta_3}\} \). The corresponding nilpotent Lie group of step two has the multiplication given by (1.15).

Recall that the Lie algebra \( \mathfrak{so}(4) \) of skew symmetric \( 4 \times 4 \) matrices has the decomposition

\[ \mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \]

with one \( \mathfrak{sp}(1) \) spanned by (6.5) and the other one spanned by

\[
J^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]

(6.6)

which also satisfy the quaternionic commutating relation. If we write \( \mathbb{S} \) as a block matrix \((\mathbb{S}^{lm})\) with \( \mathbb{S}^{lm} \) to be \( 4 \times 4 \) matrices, then

\[ B^\beta_{lm} = I^\beta \mathbb{S}^{lm} + \mathbb{S}^{lm} I^\beta, \]  \hspace{1cm} (6.7)

where \( l, m = 0, \ldots, n - 1 \), and \( B^\beta = \{B^\beta_{lm}\} \). The following is a direct characterization of right-type groups in terms of matrices \( B^\beta \).

**Proposition 6.1** The group \( N_S \) is right-type if and only if \( B^\beta_{lm} \in \text{span} \{J^1, J^2, J^3, I_4\} \) for each \( l, m, \beta \).

**Proof** Note that \( S^{\beta}_{ab} = \frac{1}{2} \partial_{\theta_{4l+a}} \partial_{\theta_{4m+b}} \phi = S_{(a+b)(4m+4l)}^{(ml)} = S_{ba}^{(ml)} \), i.e. \( S^{lm} = (S^{ml})^t \), which implies \( B^\beta_{lm} = - (B^\beta_{ml})^t \). Write \( E = d^\phi d^\psi \phi = E_{AB} \omega^A \wedge \omega^B \) and note that \( E = E_0 \) in the rigid case. Then by the expression (2.3) of \( \nabla_A^\psi \), we get

\[
E_{(2l)(2m)} = \frac{1}{2} \left( \nabla^\psi_{2l} \nabla^\psi_{2m} - \nabla^\psi_{2m} \nabla^\psi_{2l} \right) \phi \\
= \frac{1}{2} \left( \partial_{\theta_{4l+3}} + i \partial_{\theta_{4l+4}} \right) \left( \partial_{\theta_{4m+1}} + i \partial_{\theta_{4m+2}} \right) \phi - \frac{1}{2} \left( \partial_{\theta_{4l+1}} + i \partial_{\theta_{4l+2}} \right) \left( \partial_{\theta_{4m+3}} + i \partial_{\theta_{4m+4}} \right) \phi \\
= S^{lm}_{31} - S^{lm}_{32} + S^{lm}_{42} + S^{lm}_{42} - i \left( S^{lm}_{14} - S^{lm}_{41} + S^{lm}_{23} - S^{lm}_{32} \right),
\]

\[
E_{(2l+1)(2m+1)} = \frac{1}{2} \left( \nabla^\psi_{2l+1} \nabla^\psi_{2m+1} - \nabla^\psi_{2m+1} \nabla^\psi_{2l+1} \right) \phi \\
= \frac{1}{2} \left( \partial_{\theta_{4l+3}} + i \partial_{\theta_{4l+4}} \right) \left( \partial_{\theta_{4m+3}} - i \partial_{\theta_{4m+4}} \right) \phi + \frac{1}{2} \left( \partial_{\theta_{4l+1}} + i \partial_{\theta_{4l+2}} \right) \left( \partial_{\theta_{4m+1}} - i \partial_{\theta_{4m+2}} \right) \phi \\
= S^{lm}_{11} + S^{lm}_{22} + S^{lm}_{33} + S^{lm}_{44} + i \left( S^{lm}_{14} - S^{lm}_{41} - S^{lm}_{12} + S^{lm}_{21} \right),
\]

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by \( \nabla_{2l+1}' = -\nabla_{2l}' \), \( \nabla_{2m+1}' = \nabla_{2m}' \). Thus \( \mathcal{E} = 0 \) if and only if

\[
\begin{align*}
S_{11}^{(lm)} + S_{22}^{(lm)} + S_{33}^{(lm)} + S_{44}^{(lm)} &= 0, \\
S_{12}^{(lm)} - S_{21}^{(lm)} + S_{34}^{(lm)} - S_{43}^{(lm)} &= 0, \\
S_{13}^{(lm)} - S_{31}^{(lm)} - S_{24}^{(lm)} + S_{42}^{(lm)} &= 0, \\
S_{14}^{(lm)} - S_{41}^{(lm)} + S_{23}^{(lm)} - S_{32}^{(lm)} &= 0.
\end{align*}
\]  

(6.8)

By direct calculation, we have

\[
B_{lm}^1 := \begin{pmatrix} S_{21}^* - S_{12}^* & S_{22}^* + S_{11}^* & S_{23}^* + S_{14}^* & S_{24}^* - S_{13}^* \\
-S_{11}^* - S_{22}^* - S_{31}^* + S_{43}^* & -S_{13}^* + S_{24}^* - S_{14}^* + S_{32}^* \\
-S_{41}^* - S_{32}^* - S_{42}^* + S_{31}^* & -S_{43}^* + S_{34}^* - S_{44}^* + S_{33}^* \\
S_{31}^* + S_{42}^* + S_{32}^* + S_{41}^* & S_{33}^* + S_{44}^* + S_{34}^* + S_{43}^* \end{pmatrix}
\]  

(6.9)

where * = (lm). By (6.5)-(6.6), it is direct to see that (6.8) holds if and only if

\[
B_{lm}^1 := (S_{11}^* + S_{22}^*)J^1 + (S_{14}^* + S_{23}^*)J^2 + (S_{24}^* - S_{13}^*)J^3 + (S_{21}^* - S_{12}^*)I_4. \]  

(6.10)

Similarly, we have

\[
B_{lm}^2 := (S_{32}^* - S_{14}^*)J^1 + (S_{11}^* + S_{33}^*)J^2 + (S_{12}^* + S_{34}^*)J^3 + (S_{31}^* - S_{13}^*)I_4, \]  

\[
B_{lm}^3 := (S_{13}^* + S_{24}^*)J^1 + (S_{34}^* - S_{12}^*)J^2 + (S_{11}^* + S_{44}^*)J^3 + (S_{41}^* - S_{14}^*)I_4. \]  

(6.11)

For \( l = m \), coefficients of \( I_4 \) vanish since \( B_{ll}^1 \) is skew-symmetric. The Proposition is proved. \( \Box \)

The right quaternionic Heisenberg group (1.16) is associated to the rigid quadratic hypersurface (1.14) with \( \phi = \sum_{l=0}^{n-1} (-3x_{2l+1}^2 + x_{4l+2}^2 + x_{4l+3}^2 + x_{4l+4}^2) \) and \( B^\beta = \text{diag}(-J^\beta, -J^\beta, \ldots) \) [37]. This group is right-type by Proposition 6.1, and the tangential \( k \)-Cauchy–Fueter complex constructed in [37] is a special case of the subcomplex (1.13). Such a subcomplex was also constructed on the Heisenberg group in [34]. The left quaternionic Heisenberg group (1.17) is associated to the rigid quadratic hypersurface with \( \phi(q') = |q'|^2 \). The domain is the quaternionic Siegel upper half space [13, 14]. In this case, \( B^\beta = \text{diag}(J^\beta, I^\beta, \ldots) \). This group is not right-type by Proposition 6.1.

**Proposition 6.2** On rigid hypersurfaces,

\[
\nabla^{(A'} \nabla^{B')} = -\mathcal{E} \wedge T^{(A'B')}.
\]  

(6.12)

**Proof** For functions independent of \( x_{4n+1} \), we have

\[
d^{B'} d^{A'} = d^{B'} (\nabla^{A'} + \Omega^{C'} \wedge T^{A'}_{C'}). 
\]

\[
= \nabla^{B'} \nabla^{A'} + \Omega^{C'} \wedge T^{B'}_{C'} \nabla^{A'} - \varepsilon^{B'C'} \mathcal{E} \wedge T^{A'}_{C'} - \Omega^{C'} \wedge \nabla^{B'} T^{A'}_{C'} - \Omega^{D'} \wedge T^{B'}_{D'} T^{A'}_{C'},
\]

by using the the Leibnitz law (3.2). Hence \( d^{(A'} d^{B')} = 0 \) in Proposition 2.1 (1) gives us

\[
\nabla^{(A'} \nabla^{B')} = -\mathcal{E} \wedge T^{(A'B')} - \frac{1}{2} \Omega^{C'} \wedge ([T^{A'}_{C'}, \nabla^{B'}] + [T^{B'}_{C'}, \nabla^{A'}]) + \Omega^{C'} \wedge \Omega^{D'} \wedge T^{(A'B')}_{[D']T^{C'}_{C']}}.
\]

The result follows from \([T^{A'}_{C'}, \nabla^{B'}] = 0 = T^{(A'}_{[D']T^{B')}_{C']}}\), by expressions of \( T^{A'B'} \) in (3.8). \( \Box \)
Thus the anti-commutativity (1.18) of $\partial A^l$’s holds on right-type groups. It is equivalent to the following brackets between $Z_A^p$, generalizing the result for the right quaternionic Heisenberg group [37].

**Corollary 6.1** On the group $N\mathbb{S}$, $Z_A^{(A', B')}$ = $-E_{AB} T^{(A'B')}$. In particular, $Z_A^{(A' Z_B')}$ = 0 if the group is right-type.

**Proof** It follows from Proposition 6.2 by $\partial (A' \partial B')u = Z_A^{(A' Z_B') u} \omega^A \wedge \omega^B$ for a scalar function $u$. \hfill $\Box$

Although a $2n$-form is not an authentic differential form and we cannot integrate it, we can define a functional on $L^1(\Omega, \wedge^{2n} \mathbb{C}^2n)$ [39] [45]. For $F = f \Omega_{2n} \in L^1(\Omega, \wedge^{2n} \mathbb{C}^{2n})$, let

$$\int_{\Omega} F := \int_{\Omega} f dV, \quad (6.13)$$

where $dV$ is the Lebesgue measure on $\mathbb{R}^{4n+3}$, which is invariant on the group $N\mathbb{S}$, $\Omega_{2n} := \omega^0 \wedge \omega^1 \ldots \wedge \omega^{2n-1} \in \wedge_{\mathbb{R}+}^{2n} \mathbb{C}^2n$. Then $\beta_n = n! \Omega_{2n}$ for $\beta_n := \sum_{i=0}^{n-1} \omega^i \wedge \omega^{2i+1}$.

**Lemma 6.1** (Stokes-type formula) Let $\Omega$ be a bounded domain on the group $N\mathbb{S}$ with smooth boundary and defining function $\rho$ (i.e. $\rho = 0$ on $\partial \Omega$ and $\rho < 0$ in $\Omega$) such that $|\text{grad} \rho| = 1$. Assume that $T = \sum_A T_A \omega^A$ is a smooth $(2n - 1)$-form in $\Omega$, where $\omega^A = \omega^A |_{\Omega_{2n}}$. Then for $h \in C^1(\overline{\Omega})$, we have

$$\int_{\Omega} h \partial A^l T = - \int_{\Omega} \partial A^l h \wedge T + \sum_{A=0}^{2n-1} \int_{\partial \Omega} h T_A Z_A^l \rho dS, \quad (6.14)$$

where $dS$ denotes the surface measure of $\partial \Omega$. There is no boundary term if $h = 0$ on $\partial \Omega$.

**Proof** The proof is the same as that on the Heisenberg group [45]. Note that

$$\partial A^l (h T) = \sum_B Z_B^l (h T_A) \omega^B \wedge \omega^A = \sum_A Z_A^l (h T_A) \Omega_{2n}. \quad (6.15)$$

Then

$$\int_{\overline{\Omega}} \partial A^l (h T) = \int_{\overline{\Omega}} \sum_A Z_A^l (h T_A) dV = \int_{\partial \Omega} \sum_A h T_A Z_A^l \rho dS,$$

by definition (6.13) and integration by part,

$$\int_{\Omega} X_j f dV = \int_{\partial \Omega} f X_j \rho dS, \quad (6.15)$$

for $j = 1, \ldots, 4n$. (6.15) holds because coefficients of $\partial t_A$’s in $X_j$ are independent of $t$. (6.14) follows from the above formula and $\partial A^l (h T) = \partial A^l h \wedge T + h \partial A^l T$. \hfill $\Box$

**6.2 The quaternionic Monge–Ampère operator on right-type groups**

Recall that [39, 45] a $2p$-form $\omega$ is said to be **elementary strongly positive** if there exist linearly independent right $\mathbb{H}$-linear mappings $\eta_j : \mathbb{H}^n \to \mathbb{H}$, $j = 1, \ldots, p$, such that

$$\omega = \eta_1^* \omega^0 \wedge \eta_1^* \omega^1 \wedge \ldots \wedge \eta_p^* \omega^0 \wedge \eta_p^* \omega^1.$$
where \( \{ \tilde{w}^0, \tilde{w}^1 \} \) is a basis of \( \mathbb{C}^2 \) and \( \eta^p_\ast : \mathbb{C}^2 \to \mathbb{C}^{2n} \) is the induced \( \mathbb{C} \)-linear pulling back transformation of \( \eta_\ast \). It is called strongly positive if it belongs to the convex cone \( SP^2 \mathbb{C}^{2n} \) generated by elementary strongly positive \( 2p \)-elements. A \( 2p \)-element \( \omega \) is said to be positive if for any strongly positive element \( \eta \in SP^{2n-2p} \mathbb{C}^{2n} \), \( \omega \wedge \eta \) is positive. For \( \eta \) a domain \( \Omega \) in a right-type group \( \mathcal{N}_\mathbb{G} \), let \( \mathcal{D}_p^p(\Omega) = C^\infty_0(\Omega, \wedge^p \mathbb{C}^{2n}) \). An element of the dual space \( \{ \mathcal{D}_p^p(\Omega) \} \)’ is called a \( p \)-current. A \( 2p \)-current \( T \) is said to be positive if we have \( T(\eta) \geq 0 \) for any strongly positive form \( \eta \in \mathcal{D}_p^{2n-2p}(\Omega) \). Now for the \( p \)-current \( F \), we define a \( (p + 1) \)-current \( \partial_A' F \) as \( (\partial_A' F)(\eta) := -F(\partial_A' \eta) \), for any test \( (2n - p - 1) \)-form \( \eta, A' = 0, \cdot, 1 \). We say a current \( F \) is closed if \( \partial_0' F = \partial_1' F = 0 \).

A \([-\infty, \infty)\)-valued upper semicontinuous function on a right-type group \( \mathcal{N}_\mathbb{G} \) is said to be plurisubharmonic if it is \( L^1_{loc} \) and \( \Delta u \) is a closed positive 2-current. For a \( C^2 \) plurisubharmonic functions \( u, \Delta u \) is a closed strongly positive 2-form.

For positive \( (2n - 2p) \)-form \( T \) and an arbitrary compact subset \( K \), define \( \| T \|_K := \int_K T \wedge \beta_n^p \). For \( u_1, \ldots, u_n \in C^2 \) on a right-type group \( \mathcal{N}_\mathbb{G} \), we have

\[
\Delta u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n = \partial_0' (\partial_1' u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n)
= -\partial_1' (\partial_0' u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n)
= \partial_0' \partial_1' (u_1 \Delta u_2 \wedge \ldots \wedge \Delta u_n) = \Delta (u_1 \Delta u_2 \wedge \ldots \wedge \Delta u_n).
\]

It directly follows from \( \partial_A' \Delta = 0 \) by the anti-commutativity (1.18) of \( \partial_A' \)'s on this kind of groups.

**Theorem 6.1** Let \( \Omega \) be a domain in a right-type group \( \mathcal{N}_\mathbb{G} \). Let \( K \) and \( L \) be compact subsets of \( \Omega \) such that \( L \) is contained in the interior of \( K \). Then there exists a constant \( C \) depending only on \( K, L \) such that for any \( C^2 \) plurisubharmonic functions \( u_1, \ldots, u_p \) on \( \Omega \), we have

\[
\| \Delta u_1 \wedge \ldots \wedge \Delta u_p \|_L \leq C \prod_{i=1}^p \| u_i \|_{C^0(K)}.
\]

**Proof** The proof is similar to that on the Heisenberg group [45]. By definition, \( \Delta u_1 \wedge \ldots \wedge \Delta u_p \) is already closed and strongly positive. Since \( L \) is compact, there is a covering of \( L \) by a family of balls \( B_j' \subseteq B_j \subseteq K \). Let \( \chi \geq 0 \) be a smooth function equals to 1 on \( \overline{B_j'} \) with support in \( B_j \). We have

\[
\int_{B_j} \chi \Delta u_1 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p} = -\int_{B_j} \partial_0' \chi \wedge \partial_1' u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p}
= -\int_{B_j} u_1 \Delta \Delta \chi \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p}
= \int_{B_j} u_1 \Delta \chi \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p}
\]

(6.18)
by using the identity (6.16) and Stokes-type formula in Lemma 6.1. Then
\[
\|\Delta u_1 \wedge \ldots \wedge \Delta u_p\|_{L^\infty(B_j)} = \int_{L \setminus B_j} \Delta u_1 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p} \\
\leq \int_{B_j} \chi \Delta u_1 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p} \\
= \int_{B_j} u_1 \Delta \chi \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p} \\
\leq \frac{1}{\varepsilon} \|u_1\|_{L^\infty(K)} \|\Delta \chi\|_{L^\infty(K)} \int_{B_j} \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge \beta_n^{n-p+1}.
\]
for some \(\varepsilon > 0\), by using (6.18) and the positivity of \(-\varepsilon \Delta \chi + \|\Delta \chi\|_{L^\infty(K)} \beta_n\) for sufficiently small \(\varepsilon > 0\) (cf. [39, Lemma 3.3]). The result follows by repeating this procedure. \(\square\)

It is standard to yield the existence of Monge–Ampère measure (cf. e.g. [2] [44]) from the Chern-Levine-Nirenberg type estimate in Theorem 6.1. We omit details. Let \(\{u_j\}\) be a sequence of \(C^2\) plurisubharmonic functions converging to \(u\) uniformly on compact subsets of a domain \(\Omega\) in a right-type group \(N_\mathbb{S}\). Then \(u\) is a continuous plurisubharmonic function on \(\Omega\). Moreover, \((\Delta u_j)^n\) is a family of uniformly bounded measures on each compact subset \(K\) of \(\Omega\) and weakly converges to a non-negative measure on \(\Omega\). This measure depends only on \(u\) and not on the choice of an approximating sequence \(\{u_j\}\).

7 The generalization of Malgrange’s vanishing theorem and the Hartogs–Bochner extension for \(k\)-CF functions on right-type groups

7.1 Subelliptic estimate

For a nilpotent Lie group of step 2, its Lie algebra \(\mathfrak{g}\) has decomposition: \(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2\) satisfying \([\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_2\), \([\mathfrak{g}_2, \mathfrak{g}_2] = 0\). The group is called stratified if \([\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2\) (cf. [8]). Consider the condition \((H)\): for any \(\lambda \in \mathfrak{g}_2^* \setminus \{0\}\), the skew-symmetric bilinear form \(B_\lambda(Y, Y') = \langle \lambda, [Y, Y'] \rangle\), for \(Y, Y' \in \mathfrak{g}_1\), is non-degenerate.

Proof of Proposition 1.1 The estimate is proved in [37] for the right quaternionic Heisenberg group. The proof can be adapted to right-type groups since we have similar brackets by Corollary 6.1 and the associated Hodge-Laplacian operator.

Vector fields in (6.2) are not left invariant on \(N_\mathbb{S}\), but \(X_b = 2^a \partial_x + 2 \sum_{\beta=1}^3 \sum_{a=1}^{4n} B^{\beta}_{ab} X_a \partial_{\beta}\) is (cf. [37, Subsection 5.1]). Since they have the same brackets, we denote them also by \(X_b\)’s by abuse of notations. The SubLaplacian on the groups \(N_\mathbb{S}\) is \(\Delta_b := -\sum_{a=1}^{4n} X_a^2\).

By Theorem 1.1, \(f \in \Gamma(N_\mathbb{S}, \mathcal{G}_0)\) on a right-type group can be written as \(f = f_a S^a_k\) for scalar functions \(f_a\). Then \(\mathcal{D}_0 f = (\mathcal{D}_0^{(1)} f, \mathcal{D}_0^{(2)} f)\) with
\[
\mathcal{D}_0^{(1)}(f_a S^a_k) = \partial A f_a S^a_k = Z_a^Y f_a S^a_{k-1} \omega^A + Z_a^V f_a S^a_{k-1} \omega^A \\
= (Z_a^Y f_a + Z_a^V f_a) S^a_{k-1} \omega^A \\
= Z_a^V f_{a+\alpha(A')} S^b_{k-1} \omega^A. \tag{7.1}
\]
We introduce the inner products $\langle \cdot, \cdot \rangle$ of $\varphi_0$ and $\varphi_1^{(1)}$ by requiring $\{S_{k}^{\omega} \}$ and $\{S_{k-1}^{\omega} \}$ to be orthonormal bases, respectively. Then the operator formal adjoint to $\mathcal{D}_0^{(1)}$ is given by

$$\mathcal{D}_0^{(1)\ast} g = - \left. Z_{A}^{\omega} g_{A,0} S_{k}^{0} - Z_{A}^{\gamma} g_{A,k-1} S_{k}^{\gamma} \right|_{\mathcal{C}} - \sum_{a=1}^{k-1} Z_{A}^{B} g_{A,a-\alpha(b')} S_{k}^{a},$$

for $g = g_{A,b} S_{k-1}^{\omega} \in \mathcal{C}_{\mathcal{O}}(N_{3}, \varphi^{(1)}_{1})$, where $b$ is taken over $0, \ldots, k - 1.$ This is because

$$\int_{N_{3}} \langle \mathcal{D}_0^{(1)} f, g \rangle = \int_{N_{3}} Z_{A}^{\omega} f_{b+\alpha(b')} g_{A,b} dV = - \int_{N_{3}} f_{b+\alpha(b')} Z_{A}^{B} g_{A,b} dV$$

by relabeling indices and integration by part (6.15). The Hodge-Laplacian operator associated to $\mathcal{D}_0^{(1)}$ is

$$\mathcal{D}_0^{(1)\ast} \mathcal{D}_0^{(1)} f = - \left. Z_{A}^{\omega} Z_{A}^{\omega} f_{b} S_{k}^{0} - Z_{A}^{\gamma} Z_{A}^{\gamma} f_{k} S_{k}^{\gamma} \right|_{\mathcal{C}} - \sum_{a=1}^{k-1} \sum_{A=A',B'} Z_{A}^{B} Z_{A}^{A'} f_{a+\alpha(A')-\alpha(b')} S_{k}^{a}\quad \text{by (6.15)}.$$

Note that

$$\sum_{A=2l,2l+1} Z_{A}^{\omega} Z_{A}^{\omega} = X_{2l+3}^{2} + X_{2l+4}^{2} + i[X_{4l+3}, X_{4l+4}] + X_{2l+1}^{2} + X_{2l+2}^{2} - i[X_{4l+1}, X_{4l+2}]$$

$$= X_{2l+1}^{2} + X_{2l+2}^{2} + X_{2l+3}^{2} + X_{2l+4}^{2} = \sum_{A=2l,2l+1} Z_{A}^{\gamma} Z_{A}^{\gamma},$$

by (2.16), since

$$0 = 4Z_{2l}^{\omega} Z_{2l+1}^{\omega} = [Z_{2l}^{\omega}, Z_{2l+1}^{\omega}] + [Z_{2l}^{\gamma}, Z_{2l+1}^{\gamma}] = -i[X_{4l+3}, X_{4l+4}] + i[X_{4l+1}, X_{4l+2}].$$

by using Corollary 6.1. Similarly, we have and

$$\sum_{A=2l,2l+1} Z_{A}^{\omega} Z_{A}^{\gamma} = Z_{2l+1}^{\gamma} Z_{2l}^{\omega} - Z_{2l}^{\gamma} Z_{2l+1}^{\omega} = 2Z_{2l}^{\gamma} Z_{2l+1}^{\omega} = 0,$$

$$\sum_{A=2l,2l+1} Z_{A}^{\gamma} Z_{A}^{\omega} = -Z_{2l+1}^{\omega} Z_{2l}^{\gamma} + Z_{2l}^{\omega} Z_{2l+1}^{\omega} = 2Z_{2l}^{\omega} Z_{2l+1}^{\omega} = 0,$$

by $Z_{2l}^{\gamma} = -Z_{2l+1}^{\omega}, Z_{2l+1}^{\gamma} = Z_{2l}^{\omega}. So we get

$$\mathcal{D}_0^{(1)\ast} \mathcal{D}_0^{(1)} f = \Delta_{b} f_{0} S_{k}^{0} + 2 \sum_{a=1}^{k-1} \Delta_{b} f_{a} S_{k}^{a} + \Delta_{b} f_{k} S_{k}^{k}.$$

If identify $f \in \mathcal{C}(bD, \varphi_{0})$ with a vector $f = (f_{0}, \ldots, f_{k})^{t} \in \mathcal{C}^{k+1} \cong \mathcal{C}^{k} \otimes \mathcal{C}^{2}$, we have

$$\mathcal{D}_0^{(1)\ast} \mathcal{D}_0^{(1)} = \text{diag}(\Delta_{b}, 2\Delta_{b}, \ldots, 2\Delta_{b}, \Delta_{b}).$$
Now we have
\[
\left\| \mathcal{D}_0^{(1)} f \right\|^2_0 = \left\| \mathcal{D}_0^{(1)} f \right\|^2_0 + \left\| \mathcal{D}_0^{(2)} f \right\|^2_0 \geq \left\| \mathcal{D}_0^{(1)} f \right\|^2_0 \\
= \int_{\mathcal{N}_0} (\mathcal{D}_0^{(1)}, \mathcal{D}_0^{(1)} f, f) \geq - \int_{\mathcal{N}_0} \sum_{a=1}^{4n} (X_a^2 f, f) = \sum_{a=1}^{4n} \|X_a f\|^2_0.
\]

It is well known that the \( \frac{1}{2} \)-subelliptic estimate is satisfied when \( \{X_a, X_b\}'s \) span \( \{\partial_{t_1}, \partial_{t_2}, \partial_{t_3}\} \), i.e. it is stratified (cf. e.g. [17]).

If \( k = 0 \), note that
\[
d^0 d^1 u = Z_A^0 Z_B^1 \omega^A \wedge \omega^B = \sum_{l=0}^{2n-1} \Delta_l \omega^{2l} \wedge \omega^{2l+1} + \sum_{|A-B|\neq1} Z_A^0 Z_B^1 \omega^A \wedge \omega^B,
\]
with
\[
\Delta_l = Z_{2l}^0 Z_{2l+1}^1 - Z_{2l+1}^0 Z_{2l}^1 = X_{4l+1}^2 + X_{4l+2}^2 + X_{4l+3}^2 + X_{4l+4}^2
\]
\[
- \mathcal{I}[X_{4l+3}, X_{4l+4}] + \mathcal{I}[X_{4l+1}, X_{4l+2}]
\]
\[
= X_{4l+1}^2 + X_{4l+2}^2 + X_{4l+3}^2 + X_{4l+4}^2,
\]
by sing (2.16) and (7.2). So we have
\[
\left\| d^0 d^1 u \right\|^2_0 \geq \left\| \Delta_b u \right\|^2_0 \geq \frac{1}{2} \sum_{a=1}^{4n} \|X_a u\|^2_0 - \|u\|^2_0
\]
by the Cauchy–Schwarz inequality. The estimate follows similarly.

**Corollary 7.1** Suppose that \( \mathcal{N}_0 \) is stratified. For any \( s \in \mathbb{Z} \), if \( u \in \mathcal{E}'(\mathcal{N}_0, \mathcal{V}) \) satisfies \( \mathcal{D}_0 u \in W^s(\mathcal{N}_0, \mathcal{V}) \), we have \( u \in W^{s+\frac{1}{2}}(\mathcal{N}_0, \mathcal{V}) \). Moreover, if \( \text{supp}(u) \subset K \), there are constants \( c_s, K \geq 0 \) such that
\[
C_{s,K} \|u\|^2_s + \|\mathcal{D}_0 u\|^2_s \geq c_s,K \|u\|^2_{s+\frac{1}{2}}.
\]

Recall that a distribution in \( \mathcal{E}' \) always has compact support. This regularity follows from subelliptic estimate by the standard procedure (cf. [30]). We just mention that \( \mathcal{D}_0 u \in W^s(\mathcal{N}_0, \mathcal{V}) \) implies \( \Delta_b u \in W^{s-1}(\mathcal{N}_0, \mathcal{V}) \) and omit details (cf. [17] for a version of regularity for SubLaplacians).

### 7.2 Abstract duality theorem

A cohomological complex of topological vector spaces is a pair \( \mathbb{E}^\bullet, d \) where \( \mathbb{E}^\bullet = (E^q)_{q \in \mathbb{Z}} \) is a sequence of topological vector spaces and \( d = (d^q)_{q \in \mathbb{Z}} \) is a sequence of continuous linear maps \( d^q : E^q \to E^{q+1} \) satisfying \( d^{q+1} \circ d^q = 0 \). Its cohomology groups \( H^q(\mathbb{E}^\bullet) \) are the quotient spaces \( \ker d^q / \text{Im} \ d^{q-1} \), endowed with the quotient topology.

A homological complex of topological vector spaces is a pair \( \mathbb{E}_\bullet, d \) where \( \mathbb{E}_\bullet = (E_q)_{q \in \mathbb{Z}} \) is a sequence of topological vector spaces and \( d = (d_q)_{q \in \mathbb{Z}} \) is a sequence of continuous linear maps \( d_q : E_{q+1} \to E_q \) satisfying \( d_{q-1} \circ d_q = 0 \). Its homology groups \( H_q(\mathbb{E}_\bullet) \) are the quotient spaces \( \ker d_{q-1} / \text{Im} \ d_q \), endowed with the quotient topology. The dual complex of a cohomological complex \( \mathbb{E}^\bullet, d \) of topological vector spaces is the homological complex
(E'\_\ast, d') where E'_q = (E'_q)_{q \in \mathbb{Z}} with E'_q the strong dual of E_q and d' = (d'_q)_{q \in \mathbb{Z}} with d'_q the transpose map of d_q.

Recall that a Fréchet–Schwartz space is a topological vector space whose topology is defined by an increasing sequence of seminorms such that the unit ball with respect to the seminorm is relatively compact for the topology associated to the previous seminorm. A Fréchet–Schwartz space and the dual of a Fréchet–Schwartz space are both reflexive [23]. We need the following abstract duality theorem.

**Theorem 7.1** [23, Theorem 1.6] Let (E\_\ast, d) be a cohomological complex of Fréchet–Schwartz spaces or of dual of Fréchet–Schwartz spaces and let (E\_\ast, d) be its dual complex. For each q ∈ \mathbb{Z}, the following assertions are equivalent:

(i) \text{Im} d^q = \{ g ∈ E^{q+1} \mid g, f = 0 \text{ for any } f ∈ \ker d'_q \};

(ii) H^{q+1}(E\_\ast) is separated;

(iii) d^q is a topological homomorphism;

(iv) d'_q is a topological homomorphism;

(v) \hat{H}_q(E\_\ast) is separated;

(vi) \text{Im} d'_q = \{ f ∈ E'_q \mid \langle f, g \rangle = 0 \text{ for any } f ∈ \ker d^q \}.

A continuous linear map \psi between topological vector spaces L_1 and L_2 is called a **topological homomorphism** if for each open subset U ⊂ L_1, the image \psi(U) is an open subset of \psi(L_1). It is known that if L_1 is a Fréchet space, \psi is a topological homomorphism if and only if \psi(L_1) is closed [35, P. 77]. See e.g. [9, 19, 23] for applications of abstract duality theorem to \overrightarrow{\partial}- or \overleftarrow{\partial_b}-complex. We adapt their methods to the boundary complex of the k-Cauchy–Fueter complex.

For a complex vector space V, let \mathcal{E}(N, V) be the space of smooth V-valued functions with the topology of uniform convergence on compact sets of the functions and all their derivatives. Endowed with this topology \mathcal{E}(N, V) is a Fréchet–Schwartz space. Let \mathcal{D}(N, V) be the space of compactly supported elements of \mathcal{E}(N, V). For a compact subset K of N, let \mathcal{D}_K(N, V) the closed subspace of \mathcal{E}(N, V) with support in K endowed with the induced topology. Choose \{K_n\}_{n ∈ \mathbb{N}} an exhausting sequence of compact subsets of N. Then \mathcal{D}(N, V) = \bigcup_{n=1}^{\infty} \mathcal{D}_K(N, V). We put on \mathcal{D}(N, V) the strict inductive limit topology defined by the Fréchet–Schwartz spaces \mathcal{D}_K(N, V) [23]. Denote by \mathcal{E}'(N, V)' the dual of \mathcal{E}(N, V) and by \mathcal{D}'(N, V)' the dual of \mathcal{D}(N, V).

### 7.3 The generalization of Malgrange’s vanishing theorem and Hartogs–Bochner extension for k-CF functions

On a right-type group, we have the dual differential complex \overrightarrow{\partial}_j:

\[ 0 \leftarrow \Gamma(N, \overrightarrow{\partial}_0) \xleftarrow{\overrightarrow{\partial}_0} \cdots \leftarrow \Gamma(N, \overrightarrow{\partial}_j) \xleftarrow{\overrightarrow{\partial}_j} \cdots \xleftarrow{\overrightarrow{\partial}_{2n-2}} \Gamma(N, \overrightarrow{\partial}_{2n-1}) \xleftarrow{\overrightarrow{\partial}_{2n-1}} 0. \] (7.5)

where \overrightarrow{\partial}_j = \overrightarrow{\partial}_j^{(1)} ⊕ \overrightarrow{\partial}_j^{(2)} with

\[ \overrightarrow{\partial}_j^{(1)} := \gamma^{j, 2n−j}, \quad \overrightarrow{\partial}_j^{(2)} := \gamma^{j+1, 2n+1−j}, \text{ if } j ≠ k, \] (7.6)

and \overrightarrow{\partial}_0^{(2)} = \emptyset. Then, \mathcal{E}'(N, \overrightarrow{\partial}_j)' (respectively, \mathcal{D}'(N, \overrightarrow{\partial}_j)') can be identified with \mathcal{E}'(N, \overrightarrow{\partial}_j) (respectively, \mathcal{D}'(N, \overrightarrow{\partial}_j)).
The dual can be realized as follows (we only need $j = 0, 1$ here). For $F \in \mathcal{E}(N_{\mathbb{S}}, \otimes^j \mathbb{C}^2 \otimes \Lambda^{2n-1} \mathbb{C}^2)$, write it as $(F^A)$ with $F^A \in \mathcal{E}(N_{\mathbb{S}}, \Lambda^{2n-1} \mathbb{C}^2)$ (see the Appendix for this notation). It defines a functional on $\mathcal{D}(N_{\mathbb{S}}, \gamma_0)$ by

$$
\langle F, \varphi \rangle := \int F^A \varphi^A,
$$

(7.7)

for $\varphi \in \mathcal{D}(N_{\mathbb{S}}, \gamma_0)$. Similarly, for $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2) \in \mathcal{E}(M, \widehat{\gamma}_1)$, write $\mathbb{F} = (\mathbb{F}_1', \mathbb{F}_2')$ with $\mathbb{F}_1' \in \mathcal{E}(N_{\mathbb{S}}, \Lambda^{2n-1} \mathbb{C}^2)$, $\mathbb{F}_2' \in \mathcal{E}(N_{\mathbb{S}}, \Lambda^{2n-1} \mathbb{C}^2)$, $|A'| = \sigma_1 = k - 1$, $|B'| = \sigma_2 = k - 2$. It defines a functional on $\mathcal{D}(N_{\mathbb{S}}, \gamma_1)$ by

$$
\langle \mathbb{F}, \psi \rangle := \int \left( \mathbb{F}_1' \wedge \psi^1_A + \mathbb{F}_2' \psi^2_B \right),
$$

(7.8)

for any $\psi = (\psi^1_A, \psi^2_B) \in \mathcal{D}(N_{\mathbb{S}}, \gamma_1)$, since $\mathbb{F}_1' \wedge \psi^1_A + \mathbb{F}_2' \psi^2_B$ is an element of $\mathcal{D}(N_{\mathbb{S}}, \Lambda^{2n-1} \mathbb{C}^2)$.

Recall that on a right-type group, $\mathcal{D}_0 : \mathcal{E}(N_{\mathbb{S}}, \gamma_0) \to \mathcal{E}(N_{\mathbb{S}}, \gamma_1)$ is given by

$$
\mathcal{D}_0 f = \left( \partial_A \partial_A f, -\partial_A \partial_B T^{AB} f \right)
$$

for $f \in \mathcal{E}(N_{\mathbb{S}}, \gamma_0)$ by (3.28). Then for $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2) \in \mathcal{D}(N_{\mathbb{S}}, \widehat{\gamma}_1)$

$$
\langle \mathcal{D}_0 \mathbb{F}, f \rangle = \langle \mathbb{F}, \mathcal{D}_0 f \rangle = \int \mathbb{F}_1' \wedge \partial_A' f' A' A - \int \mathbb{F}_2' T^{AB} f A'B' B
$$

$$
= -\int \partial (\mathbb{F}_1') f' A' A + \int T(A'B') \mathbb{F}_2' f A'B' B
$$

by using Stokes-type formula in Lemma 6.1 and symmetrization. So we get

$$
\langle \mathcal{D}_0 \mathbb{F} \rangle A'B' B' = -\partial (\mathbb{F}_1') A'B' B + T(A'B') \mathbb{F}_2'.
$$

(7.9)

Mézvièr proved the following theorem for analytic hypoellipticity.

**Theorem 7.2** ([27, Theorem 0]) Let $P$ be a homogeneous left invariant differential operator on a nilpotent Lie group satisfying condition (H). Then $P$ is analytic hypoelliptic if and only if $P$ is $C^\infty$ hypoelliptic.

**Corollary 7.2** On a right-type group $N_{\mathbb{S}}$ satisfying condition (H), $\Delta_b$ is analytic hypoelliptic.

It follows from Theorem 7.2 and subelliptic estimate (7.4) in Corollary 7.1.

**Theorem 7.3** On a right-type group $N_{\mathbb{S}}$ satisfying condition (H), $\mathcal{D}_0 : \mathcal{E}'(N_{\mathbb{S}}, V_0) \to \mathcal{E}'(N_{\mathbb{S}}, V_1)$ and $\mathcal{D}_0 : \mathcal{D}(N_{\mathbb{S}}, V_0) \to \mathcal{D}(N_{\mathbb{S}}, V_1)$ have closed range.

**Proof** Let $\{ f_v \}$ be a sequence in $\mathcal{E}'(N_{\mathbb{S}}, \gamma_0)$ such that all $\mathcal{D}_0 f_v \to g$ in $\mathcal{E}'(N_{\mathbb{S}}, \gamma_0)$. Then $\mathcal{D}_0 f_v$ are all supported in a fixed compact subset $K$ for all $v$ and there is $s \in \mathbb{Z}$ such that $\mathcal{D}_0 f_v \in W^s(N_{\mathbb{S}}, \gamma_0)$, and $\mathcal{D}_0 f_v \to g$ in $W^s(N_{\mathbb{S}}, \gamma_1)$ (cf. [36]). Consequently, $\mathcal{D}_0 f_v \equiv 0$ outside of $K$, i.e., $f_v$ is $k$-CF on $N_{\mathbb{S}} \setminus K$. We can assume that $N_{\mathbb{S}} \setminus K$ has no compact connected component. Since $\Delta_b$ is analytic hypoelliptic by Corollary 7.2, and each component of a $k$-CF function annihilated by $\Delta_b$ by (7.3), $k$-CF functions $f_v|_{N_{\mathbb{S}} \setminus K}$ vanish on each connected component of $N_{\mathbb{S}} \setminus K$, and thus $\{ f_v \}$ are also supported in $K$.

This argument also implies that the estimate (7.4) in Corollary 7.1 holds with $C_{s,K} = 0$. If this is not true, there exist a sequence $h_v \in W^s(N_{\mathbb{S}}, \gamma_0)$ such that

$$
\| \mathcal{D}_0 h_v \|_s^2 < \frac{1}{v} \| h_v \|_{s + \frac{1}{2}}^2.
$$
By rescaling we can assume that $\|h_v\|_s = 1$ for each $v$. By (7.4)

$$C_{s,K} \geq \left( c_{s,K} - \frac{1}{v} \right) \|h_v\|_{s+\frac{1}{2}}^2.$$ 

Thus $\{h_v\}$ is bounded in the Sobolev space $W^{s+\frac{1}{2}}(N_G, \gamma_0)$. By the well known compactness of the inclusion $W^{s+\frac{1}{2}}(N_G, \gamma_0) \subset W^s(N_G, \gamma_0)$, there is a subsequence that converges to a function $h_\infty$ in $W^s(N_G, \gamma_0)$. We have

$$\|h_\infty\|_s = 1, \quad \mathcal{D}_0 h_\infty = 0.$$ 

Then $\Delta_b h_\infty = 0$ and $h_\infty$ is compactly supported. So $h_\infty \equiv 0$ by analytic continuation, which contradicts to $\|h_\infty\|_s = 1$.

By the estimate (7.4) in Corollary 7.1 with $C_{s,K} = 0$, we see that $\{f_v\}$ is uniformly bounded in $W^{s+\frac{1}{2}}(N_G, \gamma_0)$, and hence contains a subsequence which converges to a compactly supported weak solution $f \in W^s(N_G, \gamma_0)$ of $\mathcal{D}_0 f = g$. Namely, the image of $\mathcal{D}_0$ in $\mathcal{E}'(N_G, \gamma_1)$ is closed. The closedness of the image of $\mathcal{D}_0$ in $\mathcal{D}(N_G, \gamma_1)$ follows from the proved result for $\mathcal{E}'(N_G, \gamma_1)$ and the hypoellipticity.

**Proof of Theorem 1.3** By Theorem 7.3 and its proof, the sequences

$$0 \to \mathcal{D}(N_G, \gamma_0) \xrightarrow{\mathcal{D}_0} \mathcal{D}(N_G, \gamma_1), \quad (7.10)$$

$$0 \to \mathcal{E}'(N_G, \gamma_0) \xrightarrow{\mathcal{D}_0} \mathcal{E}'(N_G, \gamma_1),$$

both are exact and have closed ranges. Thus $\mathcal{D}_0$’s in (7.10) are topological homomorphisms [35, P. 77]. Now we can apply abstract duality theorem 7.1 (vi) to sequences in (7.10) to get exact sequences

$$0 \leftarrow \mathcal{D}'(N_G, \gamma_0) \xleftarrow{\mathcal{D}_0} \mathcal{D}'(N_G, \gamma_1), \quad (7.11)$$

$$0 \leftarrow \mathcal{E}(N_G, \gamma_0) \xleftarrow{\mathcal{D}_0} \mathcal{E}(N_G, \gamma_1),$$

i.e. $\mathcal{D}_0$’s are surjective. The result follows.

We have the following Hartogs–Bochner extension for $k$-CF functions.

**Theorem 7.4** Suppose that the right-type group $N_G$ satisfies condition (H) and $k > 0$. Let $\Omega$ be a domain of $N_G$ with smooth boundary such that $N_G \setminus \Omega$ connected, and let $\rho$ be a defining function (i.e. $\rho = 0$ on $\partial\Omega$ and $\rho < 0$ in $\Omega$) such that $|\text{grad } \rho| = 1$. Suppose that $f$ is the restriction to $\partial\Omega$ of a $C^2(\Omega, \gamma_0)$ function, with $\mathcal{D}_0 f$ vanishing to the second order on $\partial\Omega$, and satisfies the momentum condition

$$\int_{\partial\Omega} \left[ f_A^{A'}(G_1^A)_{A'} Z_A^{A'} \rho - f_A^{A'B'} T^{(A'B')} \rho G_2^{B'} \right] dS = 0 \quad (7.12)$$

for any $G \in \mathcal{E}'(N_G, \gamma_1)$. Then there exists a $k$-CF function $\tilde{f} \in C^2(\Omega, \gamma_0)$ such that $\tilde{f} = f$ on $\partial\Omega$.

**Proof** Extending $\mathcal{D}_0 f$ by 0 outside of $\overline{\Omega}$, we get a $\mathcal{D}$-closed continuous element $F \in \mathcal{E}'(N_G, \gamma_1)$ supported in $\overline{\Omega}$. Since $H_0(\mathcal{E}(N_G, \gamma_2))$ vanish by Theorem 1.3, so it is separated. Thus, we can apply abstract duality theorem 7.1 (i) to the second sequences in (7.10) and (7.11) to see that

$$\text{Im } \mathcal{D}_0 = \{ F \in \mathcal{E}(N_G, \gamma_1) | (F, G) = 0 \text{ for any } G \in \ker \mathcal{D}_0 \}.$$
In particular, if operators of first order given by \( j_\sigma \) by a 2\( \text{-} \)tuple \( \pi \) the group of permutations on \( \sigma \) letters. The symmetrization of indices is defined as
\[
(f_{A_1'...A_\sigma'}) = \frac{1}{\sigma!} \sum_{\pi \in S_\sigma} f_{A_\pi(1)...A_\pi(\sigma)}'.
\]
In particular, if \( (f_{A_1'...A_\sigma'}) \in \otimes^\sigma \mathbb{C}^2 \) is symmetric in \( A_2' \ldots A_\sigma' \), then we have
\[
f_{(A_1'...A_\sigma')} = \frac{1}{\sigma} \left( f_{A_1' A_2'...A_\sigma'} + \cdots + f_{A_1' A_2'...A_\sigma'} + \cdots + f_{A_\sigma' A_1'...A_{\sigma'-1}'} \right). \tag{8.2}
\]
For \( j = 0, 1, \ldots, k-1 \), we write an element of \( \Gamma(\mathbb{H}^{n+1}, V_j) \) as a tuple \( f = (f_{A'}) \) with symmetric indices \( A' = A_1' \ldots A_\sigma' \) and \( f_{A'} \in \Gamma(\mathbb{H}^{n+1}, \Lambda_j^0 \mathbb{C}^{2(n+1)}) \). \( D_j \) is a differential operators of first order given by
\[
(D_j f)_{A'} = \sum_{A' = 0, 1'} \overline{d}^{A'} f_{A'A'}; \tag{8.3}
\]
while for \( j = k + 1 \ldots, 2n + 1 \), we write an element of \( \Gamma(\mathbb{H}^{n+1}, V_j) \) as a tuple \( f = (f'A') \) with symmetric indices \( A' \) and \( f'A' \in \Gamma(\mathbb{H}^{n+1}, \Lambda_j^0 \mathbb{C}^{2(n+1)}) \). Then,
\[
(D_j f)^{A'A'} = d^{(A'} f^{A')}, \tag{8.4}
\]
where \( (\cdot, \cdot) \) is the symmetrization of indices.

Conflict of interest  The author declare that they have no conflict of interest.

8. Appendix

The \( \sigma \text{-th symmetric power} \otimes^\sigma \mathbb{C}^2 \) is a subspace of \( \otimes^\sigma \mathbb{C}^2 \), and an element of \( \otimes^\sigma \mathbb{C}^2 \) is given by a 2\( \text{-} \)tuple \( (f_A') \in \otimes^\sigma \mathbb{C}^2 \) with \( A' = A_1' \ldots A_\sigma' \) \( (A_1', \ldots, A_\sigma' = 0', 1') \) such that \( f_{A'} \) is invariant under permutations of subscripts, i.e.
\[
f_{A_1'...A_\sigma'} = f_{A_\pi(1)...A_\pi(\sigma)}'.
\]
Define isomorphisms
\[ \tilde{\Pi}_j : \Gamma(\mathbb{H}^{n+1}, V_j) \longrightarrow \Gamma(\mathbb{H}^{n+1}, \mathcal{P}_{\sigma_j}(\mathbb{C}^2) \otimes \wedge^j \mathbb{C}^{2n+2}). \] (8.5)

For \( j = 0, 1, \cdots, k \), the isomorphism \( \tilde{\Pi}_j \) is given by
\[ \tilde{\Pi}_j (f_{A'}) = f_a S^a_{\sigma_j}, \] (8.6)
where \( f_a := f_{A'} \) with \( o(A') = a \), and the summation is taken over \( a = 0, 1, \ldots, \sigma_j \). If \( j = k + 1, \cdots, 2n + 1 \), the isomorphism \( \tilde{\Pi}_j \) is given by
\[ \tilde{\Pi}_j (f_{A'}) = f_{A'} S^a_{\sigma_j} = f_{A'} \tilde{S}^a_{\sigma_j} \left( \frac{\sigma_j}{a} \right) \] (8.7)
where \( f^a = f_{A'} \) with \( o(A') = a \). Under this realization, operators \( \mathcal{D}_j \)'s in (1.6) in the \( k \)-Cauchy–Fueter complex is the same as (8.6)-(8.7) by the following proposition.

**Proposition 8.1** For \( f \in \Gamma(\mathbb{H}^{n+1}, V_j) \), we have
\[ \tilde{\Pi}_{j+1}(\mathcal{D}_j f) = \begin{cases} \partial_{A'}d^A \tilde{\Pi}_j (f), & \text{if } j = 0, \ldots, k - 1, \\ s_{A'}d^A \tilde{\Pi}_j (f), & \text{if } j = k + 1, \ldots, 2n + 1. \end{cases} \]

**Proof** Note that for \( j = 0, \ldots, k - 1 \),
\[ \partial_{A'}d^A \left( f_a S^a_{\sigma_j} \right) = d^0 f_a S^a_{\sigma_{j+1}} + d^1 f_a S^a_{\sigma_{j+1}} = \left( d^0 f_b + d^1 f_{b+1} \right) S^b_{\sigma_{j+1}}, \]
by (2.5), where \( b \) is taken over 0, 1, \ldots, \( \sigma_{j+1} = \sigma_j - 1 \). Apply the mapping \( \tilde{\Pi}^{-1}_{j+1} \) to get
\[ \left[ \tilde{\Pi}^{-1}_{j+1} \left( \partial_{A'}d^A \tilde{\Pi}_j (f) \right) \right]_{A'} = d^0 f_{0'}A' + d^1 f_{1'}A'. \]
For \( j = k + 1, \cdots, 2n + 1 \), noting that \( \sigma_{j+1} = \sigma_j + 1 \), we get
\[ s_{A'}d^A \left( f^a \tilde{S}^a_{\sigma_j} \left( \frac{\sigma_j}{a} \right) \right) = \left( d^0 f^a \tilde{S}^a_{\sigma_{j+1}} + d^1 f^a \tilde{S}^a_{\sigma_{j+1}} \right) \left( \frac{\sigma_j}{a} \right) = \left( \frac{\sigma_{j+1} - (a + 1)}{\sigma_{j+1}} d^0 f^a_{\sigma_{j+1}} + \frac{a + 1}{\sigma_{j+1}} d^1 f^a_{\sigma_{j+1}} \right) \tilde{S}^a_{\sigma_{j+1}} \left( \frac{\sigma_{j+1}}{a} \right). \]
Thus, for \( A'A' \) with \( |A'A'| = \sigma_{j+1} \) and \( o(A'A) = a + 1 \), we have
\[ \left[ \tilde{\Pi}^{-1}_{j+1} \left( s_{A'}d^A \tilde{\Pi}_j (f) \right) \right]_{A'A'} = \frac{\sigma_{j+1} - (a + 1)}{\sigma_{j+1}} d^0 f_{0'}a_{0'}a_{0'} \cdots a_{0'} + \frac{a + 1}{\sigma_{j+1}} d^1 f_{0'}a_{0'}a_{0'} \cdots a_{0'} \]
by (8.2). \( \square \)

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