First, Second Quantization and Q-Deformed Harmonic Oscillator

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Abstract. Relations between the first, the second quantized representations and deform algebra are investigated. In the case of harmonic oscillator, the axiom of first quantization (the commutation relation between coordinate and momentum operators) and the axiom of second quantization (the commutation relation between creation and annihilation operators) are equivalent. We shown that in the case of $q$-deformed harmonic oscillator, a violence of the axiom of second quantization leads to a violence of the axiom of first quantization, and inverse. Using the coordinate representation, we study fine structures of the vacuum state wave function depend in the deformation parameter $q$. A comparison with fine structures of Cooper pair of superconductivity in the coordinate representation is also performed.

1. Introduction

In the last decade, studies of quantum algebras (or groups) have a great attention in modern physics and mathematics. They motivated by the passage from classical physical systems to quantum systems. Despite concept of algebras was suggested long before the discovery of quantum mechanics in the beginning of the last century, and an important achievement was mad by Heisenberg, the Heisenberg deformed algebra still rare applications in comparison with wave representation (Schrödinger wave equation, or the first quantization), and particle representation (occupation numbers approach, or second quantization) of quantum mechanics. Recently, quantum group and deformed Heisenberg algebras with $q$-deformed harmonic oscillator have been a subject of intensive investigation. This approach is found some applications in various branches of physics and chemistry \cite{1–7}. The method of $q$-deformed quantum mechanics was developed on the base of Heisenberg commutation relation (the Heisenberg algebra). The main parameter of this method is the deformation parameter $q$ usually considered to variety in the range $0 < q < 1$, and the models have been constructed that the behavior of studying objects reduce to theirs standard counterparts as $q \to 1$. In this work we investigate relations between the first, the second quantized representations of quantum mechanics and deform algebra. In the case of harmonic oscillator, the axiom of first quantization (the commutation relation between coordinate and momentum operators) and the axiom of second quantization (the commutation relation between creation and annihilation operators) are equivalent, so harmonic
oscillator is a good object for investigation the link between the two representations of quantum mechanics. We shown that in the case of $q$-deformed harmonic oscillator, a violence of the axiom of second quantization leads to a violence of the axiom of first quantization, and inverse. Using the coordinate representation, we study fine structures of the vacuum state wave function depend on the deformation parameter $q$. A comparison with fine structures of Cooper pair of superconductivity in the coordinate representation also is performed. In this work we use the atomic unit system with $\hbar = c = k_B = 1$.

2. Relation between the first and the second quantized representations of quantum mechanics

One of most important behaviors of nature is the wave-particle dualism. In the first quantization, classical particles turned to some waves (or fields in general). Example electrons can be expressed by wave function $\Psi_e (r,t)$, satisfied the wave equation Schrodinger. The regular habitants in first quantization are plane waves. Vacuum of the first quantization (the home land of wave habitants) is four-dimension flat Hilbert space, which is denoted by the ket-bra $|0\rangle = (r,t)$. The axioms of the first quantization are the commutation relations between coordinate and momentum $[\hat{x}, \hat{p}] = -i$. In the second quantization, waves (or fields) become particle (quasi-particles in general). Example light wave become photon. The regular habitants in second quantization are particles (quasi-particles). Vacuum of the second quantization (the home land of particle habitants) is the Fock space (or occupation number space), which is denoted by the round-bras $|0\rangle$. The axioms of the second quantization are the commutation relations between annihilation and creation operators (commutation for bosons and anti-commutation for fermions). Relations between the first and the second quantized representations of quantum mechanics are expressed in the table 1.
Table 1. Relations between the first and the second quantized representations of quantum mechanics.

Note here harmonic oscillator is a good bridge between the first and the second quantized representations of quantum mechanics.

3. Harmonic oscillator \((q = 1)\)

For simplicity we put the mass of oscillator is \(m = 1\), and take the case of one dimensional harmonic oscillator. In the first quantization representation the Hamiltonian of harmonic oscillator is

\[
H_0 = \frac{\omega}{2} \left( \hat{p}^2 + \hat{x}^2 \right),
\]

where \(\omega\) is the oscillation frequency, \(\hat{x}\) and \(\hat{p} = -i \frac{d}{dx}\) are the coordinate and momentum operators, which satisfied the commutative relation \([\hat{x}, \hat{p}] = i\). In the second quantization representation the Hamiltonian operator of harmonic oscillator is

\[
H_0 = \frac{\omega}{2} \left( a_0 a_0^+ + a_0^+ a_0 \right),
\]

where \(a_0^+\), \(a_0\) are creation and annihilation operators, which satisfied the commutation relation \([a_0, a_0^+] = a_0 a_0^+ - a_0^+ a_0 = 1\). Energy spectrum of harmonic oscillator has the form

\[
E_{0n} = \frac{\omega}{2} (2n + 1),
\]
where $n = 0, 1, 2, 3, \ldots$ are the integers. The creation and annihilation operators of harmonic oscillator can be expressed in terms of coordinate and momentum operators as

$$a_0 = \frac{\hat{x} + i\hat{p}_x}{\sqrt{2}},$$  \hspace{1cm} (4)

$$a_0^+ = \frac{\hat{x} - i\hat{p}_x}{\sqrt{2}}.$$  \hspace{1cm} (5)

From the definition of the annihilation operator $a_0|0\rangle$, the ground state wave function of harmonic oscillator $\Psi_0$ satisfies the equation

$$a_0|0\rangle = 0 = \left[\frac{\hat{x} + i\hat{p}_x}{\sqrt{2}}\right]\Psi_0 = \frac{1}{\sqrt{2}}(\hat{x} + \frac{d}{dx})\Psi_0.$$  \hspace{1cm} (6)

The solution of this equation gives us the ground state wave function of harmonic oscillator $\Psi_0$ in a Gaussian form

$$\Psi_0(x) = Ce^{-x^2/2},$$  \hspace{1cm} (7)

where $C$ is the normalization constant. The ground state wave function of harmonic oscillator $C\Psi_0$ with is plotted in the figure 1.

![Figure 1. The ground state wave function of harmonic oscillator $\Psi_0$ has a Gaussian form.](image)

Easily to check that

$$\langle 0|a_0^+a_0|0\rangle = 0,$$  \hspace{1cm} (8)

and

$$\langle 0|a_0^+a_0|0\rangle = \langle 0|\Psi_0^+\Psi_0|0\rangle = \langle 0|C^2e^{-x^2}|0\rangle = C^2\int_{-\infty}^{\infty} dx e^{-x^2} = C^2\Pi^{1/2} \neq 0,$$  \hspace{1cm} (9)

dependence, but $\langle 0|a_0^+a_0|0\rangle \neq 0$ so $|0\rangle \neq |0\rangle$. We note here the vacuums of the first and the second quantized representations of quantum mechanics are not the same. Introducing density operator $\rho_0 = \Psi_0^+\Psi_0$ and consider that the density fluctuation of vacuum in first quantization (Hilbert space) is corresponding the zero level oscillation $(n = \frac{1}{2})$ of harmonic oscillator in the second quantization Fock space (see Casimir effect)

$$\langle 0|\rho_0|0\rangle = \frac{1}{2},$$  \hspace{1cm} (10)

from that we can calculate the value of constant $C$

$$C = \left(2^{1/2}\Pi^{1/4}\right)^{-1}.$$  \hspace{1cm} (11)
4. \textit{q}-deformed harmonic oscillator

Creation and annihilation operators of \textit{q}-deformed harmonic oscillator satisfied the commutation relation

\[ [a, a^+] = aa^+ - qa^+a = 1, \]  

where \( q \) is deformation parameter taking values in \([0,1]\). Introducing the new deformation parameter \( \alpha \) taking values in \([0,\infty]\)

\[ \alpha = \sqrt{-\frac{\ln q}{2}}. \]  

In the second quantized representation, the Hamiltonian operator of \textit{q}-deformed harmonic oscillator is

\[ H = \frac{\omega}{2} (aa^+ + a^+a). \]  

Energy spectrum of \textit{q}-deformed harmonic oscillator has the form

\[ E_n = \frac{\omega}{2} ([n] + [n + 1]), \]  

where \([n]\) = \(\frac{1-q^n}{1-q}\) are the d-integer, they differ from natural numbers. The values of ratio \([n]/n\) depend on deformation parameter \( q \) are presented in the figure 2.

Figure 2. The values of ratio \([n]/n\) depend on deformation parameter \( q \).

This ratio tends to zero \( [n]/n \rightarrow 0 \) when \( q \rightarrow 0 \). The maximum value of \([n]_{\text{max}} \rightarrow \infty \) when \( q \rightarrow 1 \) and we return to case of standard harmonic oscillator. The creation and annihilation operators of \textit{q}-deformed harmonic oscillator can be expressed in terms of coordinate and momentum operators as

\[ a = \frac{\exp(-2i\alpha x) - \exp\left(i\alpha \frac{d}{dx}\right) \exp(-i\alpha x)}{\sqrt{1 - \exp(-2\alpha^2)}}, \]  

\[ a^+ = \frac{\exp(2i\alpha x) - \exp(i\alpha x) \exp\left(i\alpha \frac{d}{dx}\right)}{\sqrt{1 - \exp(-2\alpha^2)}}. \]  

Denote \( \varepsilon = 1 - q \), the energy spectrum of \textit{q}-deformed harmonic oscillator becomes quadratic

\[ E_n = \omega \left(n + \frac{1}{2} - \frac{n^2}{2} \varepsilon + O(\varepsilon^2)\right). \]
As above, the solution of the equation \( a(0) = 0 \)

\[
a(0) = \frac{\exp(-2i\alpha x) - \exp\left(i\alpha \frac{d}{dx}\right) \exp(-i\alpha x)}{\sqrt{1 - \exp(-2\alpha^2)}},
\]

(19)
gives us the ground state wave function \( \Psi_0 \) of \( q \)-deformed harmonic oscillator in the form

\[
\Psi_0(x) = C \exp\left( -\frac{x^2}{2} + 3\sqrt{2}i\alpha x \right),
\]

(20)

In the case of \( q \to 1, \alpha \to 0 \) we return to case of standard harmonic oscillator. Introduce density operator \( \rho = \Psi_0^\dagger \Psi_0 \) of \( q \)-deformed harmonic oscillator, because \( \Psi_0^\dagger \Psi_0 = \psi_0^\dagger \psi_0 \) therefore \( \rho = \rho_0 \), so that \( \langle 0 | \rho | 0 \rangle = \langle 0 | \rho_0 | 0 \rangle = \frac{1}{2} \). Values of density vacuum fluctuations are the same for both harmonic oscillator and \( q \)-deformed harmonic oscillator. Real part \( \text{Re}\Psi_0 \) and imaginary part \( \text{Im}\Psi_0 \) of ground state wave function \( \Psi_0 \) are presented in figure 3.

At \( q = 1 \), back to harmonic oscillator case with \( \text{Re}\Psi_0 \) has a Gaussian form, and \( \text{Im}\Psi_0 = 0 \).

5. Comparison with Cooper pairs in superconductivity

From condensed matter physics we have learn that the real part of wave function plays important role. In this part we try to explore the meaning of real part \( \text{Re}\Psi_0 \) of ground state wave function \( \Psi_0 \) of \( q \)-deformed harmonic oscillator. For comparison we take the well known case of Cooper pairs in superconductivity. In a long history of the BCS theory, the Cooper pair usually analyzed in the momentum-space. The first investigation Cooper pair in coordinate-space was done in the work [8], where was shown that this leads to a spherically symmetrical quasi-atomic wave function, with an identical onion-like layered structure for each of the electrons constituting the Cooper pair. The internal structure of Cooper pair wave function \( \Psi_C \) (also called the singlet pair function or the Gorkov function) is given by

\[
\Psi(r) \propto \sqrt{\frac{\cos(k_{FR} + \varepsilon r')}{\varepsilon^2 + 1}} \approx \cos(k_{FR}) \frac{\cos(\varepsilon r')}{\sqrt{\varepsilon^2 + 1}} = \cos(k_{FR}) K_0\left(\frac{r}{\pi\xi_0}\right),
\]

(21)
where \( k_F \) is the Fermi wave vector at the top of the Fermi sea, \( K_0 \) is the zero-order modified Bessel function with an asymptotic form that is similar to an exponential for large \( x \). The wave function \( \Psi_C \) of Cooper pair is plotted in the figure 4.

![Figure 4](image)

**Figure 4.** The wave function \( \Psi_C \) of Cooper pairs in superconductivity.

We can realize that the real part \( \text{Re}\Psi_0 \) of ground state wave function \( \Psi_0 \) of \( q \)-deformed harmonic oscillator (see figure 5) is very similar the wave function \( \Psi_C \) of Cooper pair.

![Figure 5](image)

**Figure 5.** The real part \( \text{Re}\Psi_0 \) of ground state wave function \( \Psi_0 \) of \( q \)-deformed harmonic oscillator \( \alpha = 10 \).

The similarity between \( \text{Re}\Psi_0 \) of \( q \)-deformed harmonic oscillator and the wave function \( \Psi_C \) of Cooper pair might be not accidental and will be a subject for further investigation.

6. Uncertainty relation for \( q \)-deformed harmonic oscillator

The coordinate-momentum uncertainty relation is an important character of quantum world. This uncertainty for \( q \)-deformed harmonic oscillator is

\[
\Delta x \Delta p = \frac{E}{\omega} = \frac{([n] + [n + 1])}{2}.
\]
For the case of harmonic oscillator \((\Delta x \Delta p)_0 = \frac{(2n+1)}{2}\). The ratio of \(K = \frac{\Delta x \Delta p}{(\Delta x \Delta p)_0}\) of uncertainty for q-deformed harmonic oscillator and uncertainty for harmonic oscillator is plotted in the figure 6.

**Figure 6.** The ratio of \(K = \frac{\Delta x \Delta p}{(\Delta x \Delta p)_0}\) of uncertainty for q-deformed harmonic oscillator and uncertainty for harmonic oscillator.

We note here uncertainty for q-deformed harmonic oscillator is less than the analogous value for standard harmonic oscillator for all \(n\), except \(n = 0\). At \(n = 0\), the coordinate-momentum uncertainty is minimum and given by \(1/2\).

### 7. Discussion

The main results of this work are the table 1 showing the relations between the first and the second quantized representations of quantum mechanics with an attention that harmonic oscillator is a good bridge between them, also the investigation deformation parameter \(q\) depending on the most important physical characters of q-deformed harmonic oscillator in explicit coordinate representations, such as annihilation and creation operators, q-deformed integer, the ground wave function, coordinate-momentum uncertainty relation. Introducing density operators, we explored the physical meaning of wave function of ground state of the two types oscillators with and absence deformation, and shown that the vacuums of first and second quantization are not the same but the values of vacuum density fluctuation or zero level oscillation are equal for both. We shown that the real part \(\text{Re}\Psi_0\) of ground state wave function \(\Psi_0\) of q-deformed harmonic oscillator is very similar the wave function \(\Psi_C\) of Cooper pair in superconductivity. This similarity might be not accidental and will be a subject for further investigation. We noted that uncertainty for q-deformed harmonic oscillator is less than the analogous value for standard harmonic oscillator for all \(n\), except \(n = 0\), which in contrast to standard quantum mechanics that uncertainty of harmonic oscillator is minimal value. The physical meaning of this also will be a subject for further investigation.

### References

[1] V. V. Eremin, A. A. Meldianov 2008 arXiv:0810.1967v1.
[2] S. Abe, C. Beck, E. G. D. Cohen 2007 Phys. Rev. E 76 031102.
[3] S. Abe 2009 Phys. Rev. E 79 041116.
[4] A. Lavagno, P. P. Swamy 2010 Physica. A 389 933.
[5] A. Algin, M. Senay 2012 Phys. Rev. E 85 041123.
[6] F. M. Andrade, E. O. Silva 2013 Phys. Lett. B 719 467.
[7] V. M. Tkachuck 2013 Phys. Rev. A 86 062112.
[8] A. M Kadin 2007 JSNM 20 285.