About interpolation of subspaces of rearrangement invariant spaces generated by Rademacher system

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Abstract

The Rademacher series in rearrangement invariant function spaces "closed" to the space \(L_\infty\) are considered. In terms of interpolation theory of operators a correspondence between such spaces and spaces of coefficients generated by them is stated. It is proved that this correspondence is one-to-one. Some examples and applications are presented.

1. Introduction

Let
\[ r_k(t) = \text{sign} \sin 2^{k-1} \pi t \quad (k = 1, 2, \ldots) \]
be the Rademacher functions on the segment \([0, 1]\). Define the linear operator
\[ Ta(t) = \sum_{k=1}^{\infty} a_k r_k(t) \quad \text{for} \quad a = (a_k)_{k=1}^{\infty} \in l_2. \tag{1} \]

It is well-known (see, for example, [1, p.340-342]) that by fixed \(a\) \(Ta(t)\) is an almost everywhere finite function on \([0, 1]\). Moreover, from Khintchine’s inequality it follows that
\[ ||Ta||_{L_p} \asymp ||a||_2 \quad \text{for} \quad 1 \leq p < \infty, \tag{2} \]
where \(||a||_p = (\sum_{k=1}^{\infty} |a_k|^p)^{1/p}\), as usual. The last means the existence of two-sided estimates with constants depending only on \(p\). Also, it can easily be checked that
\[ ||Ta||_{L_\infty} = ||a||_1. \tag{3} \]

A more detailed information on the behaviour of Rademacher series can be obtained by treating them in the framework of general rearrangement invariant spaces.

Recall that a Banach space \(X\) of measurable functions \(x = x(t)\) on \([0, 1]\) is said to be an rearrangement invariant space (r.i.s.) if the inequality \(x^*(t) \leq y^*(t)\) for \(t \in [0, 1]\) and \(y \in X\) imply \(x \in X\) and \(||x|| \leq ||y||\). Here and next \(z^*(t)\) is the nonincreasing rearrangement of a function \(|z(t)|\) concerning to Lebesgue measure denoted by meas \([2, p.83]\).

Important examples of r.i.s.’s are Marcinkiewicz and Orlicz spaces.

By \(\mathcal{P}\) will be denoted the cone of nonnegative increasing concave functions on the semiaxis \((0, \infty)\).

If \(\varphi \in \mathcal{P}\), then the Marcinkiewicz space \(M(\varphi)\) consists of all measurable functions \(x = x(t)\) such that
\[ ||x||_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_{\alpha}^{t} x^*(s) \, ds : 0 < t \leq 1 \right\} < \infty. \]
If $S(t)$ is a nonnegative convex continuous function on $[0, \infty)$, $S(0) = 0$, then the Orlicz space $L_S$ consists of all measurable functions $x = x(t)$ such that

$$||x||_S = \inf \left\{ u > 0 : \int_0^1 S\left(\frac{|x(t)|}{u}\right) dt \leq 1 \right\} < \infty.$$ 

In particular, if $S(t) = t^p$ ($1 \leq p < \infty$), then $L_S = L_p$.

For any r.i.s. $X$ on $[0, 1]$ we have $L_\infty \subset X \subset L_1$ [2, p.124]. By $X^0$ will be denoted the closure of $L_\infty$ in an r.i.s. $X$.

In problems discussed below, a special role is played by the Orlicz space $L_{N(t)}$ where $N(t) = \exp(t^2) - 1$ or, more precisely, by the space $G = L_{N_0}^0$. In the paper [3], V.A.Rodin and E.M.Semenov proved a theorem about the equivalence of Rademacher system to the standard basis in the space $l_2$.

**Theorem A.** Suppose that $X$ is an r.i.s. Then

$$||Ta||_X = \left\| \sum_{k=1}^\infty a_k r_k \right\|_X \asymp ||a||_2$$

if and only if $X \supset G$.

By Theorem A, the space $G$ is the minimal space among r.i.s.’s $X$ such that the Rademacher system is equivalent in $X$ to the standard basis of $l_2$.

In this paper, we consider problems related to the behaviour of Rademacher series in r.i.s.’s intermediate between $L_\infty$ and $G$. The main role is played here by concepts and methods of interpolation theory of operators.

For a Banach couple $(X_0, X_1), x \in X_0 + X_1$, and $t > 0$ we introduce the Peetre $K$-functional

$$K(t, x; X_0, X_1) = \inf \{ ||x_0||_{X_0} + t||x_1||_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \}.$$ 

Let $Y_0$ be a subspace of $X_0$ and $Y_1$ a subspace of $X_1$. A couple $(Y_0, Y_1)$ is called a $K$-subcouple of a couple $(X_0, X_1)$ if

$$K(t, y; Y_0, Y_1) \asymp K(t, y; X_0, X_1),$$

with constants independent of $y \in Y_0 + Y_1$ and $t > 0$.

In particular, if $Y_i = P(X_i)$, where $P$ is a linear projector bounded from $X_i$ into itself for $i = 0, 1$, then $(Y_0, Y_1)$ is a $K$-subcouple of $(X_0, X_1)$ (see [5] or [6, p.136]). At the same time, there are many examples of subcouples that are not $K$-subcouples (see [6, p.589], [7], and Remark 2 of this paper).

We shall consider the case: $X_0 = L_\infty$, $X_1 = G$, $Y_0 = T(l_1)$, and $Y_1 = T(l_2)$, where $T$ is given by (1). From (3) and Theorem A it follows that

$$K(t, Ta; T(l_1), T(l_2)) \asymp K(t, a; l_1, l_2).$$

In spite of the fact that $T(l_1)$ is uncomplemented in $L_\infty$ (see [8] or [9, p.134]) the following assertion holds.

**Theorem 1.** The couple $(T(l_1), T(l_2))$ is a $K$-subcouple of the couple $(L_\infty, G)$. In other words (see (4)),

$$K(t, Ta; L_\infty, G) \asymp K(t, a; l_1, l_2),$$

with constants independent of $a = (a_k)_{k=1}^\infty \in l_2$ and $t > 0$. 

We shall use in the proof of Theorem 1 an assertion about the distribution of
Rademacher sums. It was proved by S. Montgomery-Smith [10].

**Theorem B.** There exists a constant \( A \geq 1 \) such that for all \( a = (a_k)_{k=1}^\infty \in \ell_2 \) and \( t > 0 \)
\[
\text{meas}\left\{ s \in [0, 1] : \sum_{k=1}^\infty a_k r_k(s) > A^{-1} \varphi_a(t) \right\} \geq A^{-1} \exp(-At^2),
\]
where \( \varphi_a(t) = K(t, a; l_1, l_2) \).

We need now some definitions from interpolation theory of operators.

We say that a linear operator \( U \) is bounded from a Banach couple \( \vec{X} = (X_0, X_1) \) into a Banach couple \( \vec{Y} = (Y_0, Y_1) \) (in short, \( U : \vec{X} \to \vec{Y} \)) if \( U \) is defined on \( X_0 + X_1 \) and acts as bounded operator from \( X_i \) into \( Y_i \) for \( i = 0, 1 \).

Let \( \vec{X} = (X_0, X_1) \) be a Banach couple. A space \( X \) such that \( X_0 \cap X_1 \subset X \subset X_0 + X_1 \) is called an interpolation space between \( X_0 \) and \( X_1 \) if each linear operator \( U : \vec{X} \to \vec{X} \) is bounded from \( X \) into itself.

For every \( p \in [1, \infty] \), we shall denote by \( l_p(u_k), u_k \geq 0 \) \( k = 0, 1 \ldots \) the space of all two-sided sequences of real numbers \( a = (a_k)_{k=-\infty}^\infty \) such that the norm \( \|a\|_{l_p(u_k)} = \|(a_k u_k)\|_p \) is finite. Let \( E \) be a Banach lattice of two-sided sequences, \( (\min(1, 2^k))_{k=-\infty}^\infty \in E \). If \( (X_0, X_1) \) is a Banach couple, then the space of the real \( K \)-method of interpolation \( (X_0, X_1)^K_E \) consists of all \( x \in X_0 + X_1 \) such that
\[
\|x\| = \|(K(2^k x; X_0, X_1))_k\|_E < \infty.
\]
It is readily checked that the space \( (X_0, X_1)^K_E \) is an interpolation space between \( X_0 \) and \( X_1 \) (see, for example, [11, p.422]). In the special case \( E = l_p(2^{-k\theta}) \) \((0 < \theta < 1, 1 \leq p \leq \infty)\) we obtain the spaces \( (X_0, X_1)_{\theta,p} \) (for the detailed exposition of their properties see [4]).

A couple \( \vec{X} = (X_0, X_1) \) is said to be a \( K \)-monotone couple if for every \( x \in X_0 + X_1 \) and \( y \in X_0 + X_1 \) there exists a linear operator \( U : \vec{X} \to \vec{X} \) such that \( y = Ux \) whenever
\[
K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1) \quad \text{for all} \quad t > 0.
\]

As it is well-known (see, for example, [11, p.482]), any interpolation space \( X \) with respect to a \( K \)-monotone couple \( (X_0, X_1) \) is described by the real \( K \)-method. It means that for some \( E \)
\[
X = (X_0, X_1)^K_E.
\]

In particular, by the Sparr theorem [12] the couple \( (l_1, l_2) \) is a \( K \)-monotone couple. Therefore, if \( F \) is an interpolation space between \( l_1 \) and \( l_2 \), then there exists \( E \) such that
\[
F = (l_1, l_2)^K_E.
\]
Hence Theorem 1 allows to find a r.i.s. that contains Rademacher series with coefficients belonging to an arbitrary interpolation space between \( l_1 \) and \( l_2 \). In the paper [3], the similar result was obtained for sequence spaces satisfying more restrictive conditions (see Remark 3).

**Theorem 2.** Let \( F \) be an interpolation sequence space between \( l_1 \) and \( l_2 \) and \( F = (l_1, l_2)^K_E \). Then for the r.i.s. \( X = (L_\infty, G)^K_E \) we have
\[
\left\| \sum_{k=1}^\infty a_k r_k \right\|_X \approx ||a||_F,
\]
Proof of Theorem 1. It is known [2, p.164] that the Marcinkiewicz spaces is given by the formula

\[ H_u \]

Hence, \( X \)

\[ \text{where} \quad k \quad \text{and} \quad \phi \]

Since \( x \)

\[ N \]

Therefore,

\[ \text{Let r.i.s.'s} \quad X_0 \quad \text{and} \quad X_1 \quad \text{be two interpolation spaces between} \quad L_\infty \quad \text{and} \quad G. \]

If

\[ \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_0} \leq \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_1}, \]

then \( X_0 = X_1 \) and the norms of \( X_0 \) and \( X_1 \) are equivalent.

In the papers [13] and [13], the similar results were obtained by additional conditions with respect to spaces \( X_0 \) and \( X_1 \).

2. Proofs

Proof of Theorem 1. It is known [2, p.164] that the \( \mathcal{K} \) -functional of a couple of Marcinkiewicz spaces is given by the formula

\[ \mathcal{K}(t, x; M(\varphi_0), M(\varphi_1)) = \sup_{0<u \leq 1} \frac{\int_0^u x^*(s) \, ds}{\max(\varphi_0(u), \varphi_1(u)/t)} \]

If \( N(t) = \exp(t^2) - 1 \), then the Orlicz space \( L_N \) coincides with the Marcinkiewicz space \( M(\varphi_1) \), where \( \varphi_1(u) = u \log_2^{1/2}(2/u) \) [3]. In addition, \( L_\infty = M(\varphi_0) \) where \( \varphi_0(u) = u \). Therefore,

\[ \mathcal{K}(t, x; L_\infty, G) = \sup_{0<u \leq 1} \left\{ \frac{1}{u} \int_0^u x^*(s) \, ds \min(1, t\log_2^{1/2}(2/u)) \right\} \quad \text{for} \quad x \in G. \quad (7) \]

Since \( x^*(u) \leq 1/u \int_0^u x^*(s) \, ds \), then from (7) it follows

\[ \mathcal{K}(t, x; L_\infty, G) \geq \sup_{k=0,1,..} \left\{ x^*(2^{-k}) \min(1, t(k+1)^{-1/2}) \right\}. \]

Hence,

\[ \mathcal{K}(t, x; L_\infty, G) \geq x^*(2^{-k_t}) \quad \text{for} \quad t \geq 1, \quad \text{(8)} \]

where \( k_t = [t^2] - 1 \) ([z] is the integral part of a number z).

Let now \( a = (a_k)_{k=1}^\infty \in l_2 \) and \( x(t) = Ta(t) = \sum_{k=1}^\infty a_k r_k(t) \). By the Holmstedt formula [14],

\[ \varphi_a(t) \leq \sum_{k=1}^{[t^2]} a_k^* + t \left\{ \sum_{k=[t^2]+1}^\infty (a_k^*)^2 \right\}^{1/2} \leq B \varphi_a(t), \quad \text{(9)} \]

where \( \varphi_a(t) = \mathcal{K}(t, a; l_1, l_2) \), \( (a_k^*)_{k=1}^\infty \) is nonincreasing rearrangement of the sequence \( (|a_k|)_{k=1}^\infty \), and \( B > 0 \) is a constant independent of \( a = (a_k)_{k=1}^\infty \) and \( t > 0 \).

Assume, at first, that \( a \notin l_1 \). Then inequality (9) shows that

\[ \lim_{t \to d_1} \varphi_a(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \varphi_a(t) = \infty. \]

The function \( \varphi_a \) belongs to the class \( \mathcal{P} \) [4, p.55]. Therefore it maps the semiaxis \((0, \infty)\) onto \((0, \infty)\) one-to-one, and there exists the inverse function \( \varphi_a^{-1} \). By Theorem B, we have

\[ n_{|x|}(\tau) = \operatorname{meas}\{s \in [0, 1] : |x(s)| > \tau\} \geq \psi(\tau) \quad \text{for} \quad \tau > 0, \]

with constants independent of \( a = (a_k)_{k=1}^\infty \).
where \( \psi(\tau) = A^{-1} \exp\{-A[\varphi^{-1}_a(\tau A)]^2\} \). Passing to rearrangements we obtain
\[
x^*(s) \geq \psi^{-1}(s) \quad \text{for} \quad 0 < s < A^{-1}.
\] (10)

Obviously, by condition \( t \geq C_1 = C_1(A) = \sqrt{2 \log_2(2A)} \), it holds
\[
2^{-k_t/2} < A^{-1},
\] (11)
for \( k_t = \lfloor t^2 \rfloor - 1 \).

Hence (8) and (10) imply
\[
\mathcal{K}(t, x; L_\infty, G) \geq \psi^{-1}(2^{-k_t}).
\] (12)

Combining the definition of the function \( \psi \) with (11), we obtain
\[
\psi^{-1}(2^{-k_t}) = A^{-1} \varphi_a \left( A^{-1/2} \ln^{1/2}(A^{-1}2^{k_t}) \right) \geq A^{-1} \varphi_a \left( \sqrt{k_t \ln 2/(2A)} \right) \geq A^{-3/2} \sqrt{\ln 2/2} \varphi_a(\sqrt{k_t}) \geq A^{-3/2} \sqrt{\ln 2/2t^{-1}} \sqrt{k_t} \varphi_a(t).
\]

From the inequality \( t \geq C_1 \geq \sqrt{2} \) it follows
\[
\frac{\sqrt{k_t}}{t} \geq \frac{\sqrt{\lfloor t^2 \rfloor} - 1}{\sqrt{t^2} + 1} \geq 3^{-1/2}.
\]

Therefore, by (12), we have
\[
\mathcal{K}(t, x; L_\infty, G) \geq C_2 \varphi_a(t) \quad \text{for} \quad t \geq C_1,
\]
where \( C_2 = C_2(A) = \sqrt{\ln 2/6} A^{-3/2} \).

If now \( t \geq 1 \), then the concavity of the \( \mathcal{K} \)-functional and the previous inequality yield
\[
\mathcal{K}(t, x; L_\infty, G) \geq C_1^{-1} \mathcal{K}(tC_1, x; L_\infty, G) \geq \frac{C_2}{C_1} \varphi_a(C_1 t) \geq \frac{C_2}{C_1} \varphi_a(t).
\]

Using the inequalities \( ||a||_2 \leq ||a||_1 (a \in l_1) \) and \( ||x||_G \leq ||x||_\infty (x \in L_\infty) \), the definition of the \( \mathcal{K} \)-functional, and Theorem A, we obtain
\[
\mathcal{K}(t, x; L_\infty, G) = t ||x||_G \geq C_3 t ||a||_2 = C_3 \varphi_a(t) \quad \text{for} \quad 0 < t \leq 1.
\]

Thus,
\[
\mathcal{K}(t, a; l_1, l_2) \leq C \mathcal{K}(t, Ta; L_\infty, G),
\]
if \( C = \max(C_3^{-1}, C_1/C_2) \).

Suppose now \( a \in l_1 \). By (9), without loss of generality, we can assume that the function \( \varphi_a \) maps the semi-axis \( (0, \infty) \) onto the interval \( (0, ||a||_1) \) one-to-one. Hence we can define the mappings \( \varphi_a^{-1} : (0, ||a||_1) \rightarrow (0, \infty) \), \( \psi : (0, A^{-1}||a||_1) \rightarrow (0, A^{-1}) \), and \( \psi^{-1} : (0, A^{-1}) \rightarrow (0, A^{-1}||a||_1) \). Arguing as above, we get inequality (13).

The opposite inequality follows from Theorem A and relation (3). Indeed,
\[
\mathcal{K}(t, Ta; L_\infty, G) \leq \inf\{||Ta^0||_\infty + t||Ta^1||_G : a = a^0 + a^1, a^0 \in l_1, a^1 \in l_2\} \leq D \mathcal{K}(t, a; l_1, l_2).
\]
Proof of Theorem 2. It is sufficient to use Theorem 1 and the definition of the real $\mathcal{K}$ -method of interpolation.

For the proof of Theorem 3 we need some definitions and auxiliary assertions. These results also are of some independent interest.

Let $f(t)$ be a function defined on the interval $(0, l)$, where $l = 1$ or $l = \infty$. Then the dilation function of $f$ is defined as follows:

$$\mathcal{M}_f(t) = \sup \left\{ \frac{f(st)}{f(s)} : s, st \in (0, l) \right\}, \text{ if } t \in (0, l).$$

Since this function is semimultiplicative, then there exist numbers

$$\gamma_f = \lim_{t \to 0^+} \frac{\ln \mathcal{M}_f(t)}{\ln t} \text{ and } \delta_f = \lim_{t \to \infty} \frac{\ln \mathcal{M}_f(t)}{\ln t}.$$

A Banach couple $\vec{X} = (X_0, X_1)$ is called a partial retract of a couple $\vec{Y} = (Y_0, Y_1)$ if each element $x \in X_0 + X_1$ is orbitally equivalent to some element $y \in Y_0 + Y_1$. The last means that there exist linear operators $U : \vec{X} \to \vec{Y}$ and $V : \vec{Y} \to \vec{X}$ such that $Ux = y$ and $Vy = x$.

**Proposition 1.** Suppose $M(\varphi)$ is an Marcinkiewicz space on $[0, 1]$. If $\gamma_\varphi > 0$, then $\vec{X} = (L_\infty, M(\varphi))$ is a $\mathcal{K}$ -monotone couple.

**Proof.** It is sufficient to show that the couple $\vec{X}$ is a partial retract of the couple $\vec{Y} = (L_\infty, L_\infty(\tilde{\varphi}))$, where

$$||x||_{L_\infty(\tilde{\varphi})} = \sup_{0 < t \leq 1} \tilde{\varphi}(t)|x(t)|, \tilde{\varphi}(t) = t/\varphi(t).$$

Indeed, a partial retract of a $\mathcal{K}$ -monotone couple is a $\mathcal{K}$ -monotone couple [11, p.420], and by the Sparr theorem [12], $\vec{Y}$ is a $\mathcal{K}$ -monotone couple.

First note that the inclusion $L_\infty \subset M(\varphi)$ implies $L_\infty + M(\varphi) = M(\varphi)$. So, let $x \in M(\varphi)$. Without loss of generality [2, p.87], assume that $x(t) = x^*(t)$. Define the operator

$$U_1 y(t) = \sum_{k=1}^{\infty} 2^k \int_0^{2^{-k}} y(s) ds \chi_{(2^{-k}, 2^{-k+1})}(t) \text{ for } y \in M(\varphi).$$

The concavity of the function $\varphi$ and properties of the nonincreasing rearrangement imply

$$||U_1 y||_{L_\infty(\tilde{\varphi})} \leq 2 \sup_{k=1,2} (\varphi(2^{-k+1}))^{-1} \int_0^{2^{-k}} y^*(s) ds \leq 2||y||_{M(\varphi)},$$

Hence $U_1 : \vec{X} \to \vec{Y}$. Since $x(t)$ nonincreases, then $U_1 x(t) \geq x(t)$. Therefore the linear operator

$$U y(t) = \frac{x(t)}{U_1 x(t)} U_1 y(t)$$

is bounded from the couple $\vec{X}$ into the couple $\vec{Y}$. In addition, $U x(t) = x(t)$.

Take for $V$ the identity mapping, i.e., $V y(t) = y(t)$. Since $\gamma_f > 0$, then, by [2, p.156], we have

$$||V y||_{M(\varphi)} \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t)y^*(t) \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t)|y(t)| = C||y||_{L_\infty(\tilde{\varphi})}.$$
Therefore $V : \tilde{Y} \to \tilde{X}$ and $Vx = x$.

Thus an arbitrary element $x \in M(\varphi)$ is orbitally equivalent to itself as to element of the space $L_\infty + L_\infty(\tilde{\varphi})$. This completes the proof. \hfill \Box

**Corollary 1.** If $\gamma_\varphi > 0$, then $(L_\infty, M(\varphi)^0)$ is a $\mathcal{K}$-monotone couple.

**Proof.** Assume that $x$ and $y$ belong to the space $M(\varphi)^0$ and

$$\mathcal{K}(t, y; L_\infty, M(\varphi)^0) \leq \mathcal{K}(t, x; L_\infty, M(\varphi)^0) \quad \text{for } t > 0.$$ If $z \in M(\varphi)^0$, then

$$\mathcal{K}(t, z; L_\infty, M(\varphi)^0) = \mathcal{K}(t, z; L_\infty, M(\varphi)).$$

Therefore,

$$\mathcal{K}(t, y; L_\infty, M(\varphi)) \leq \mathcal{K}(t, x; L_\infty, M(\varphi)) \quad \text{for } t > 0.$$

Hence, by Proposition 1, there exists an operator $T : (L_\infty, M(\varphi)) \to (L_\infty, M(\varphi))$ such that $y = Tx$. It is readily seen that $M(\varphi)^0$ is an interpolation space concerning to the couple $(L_\infty, M(\varphi))$. Therefore $T : (L_\infty, M(\varphi)^0) \to (L_\infty, M(\varphi)^0)$.

We define now two subcones of the cone $\mathcal{P}$. Let us denote by $\mathcal{P}_0$ the set of all functions $f \in \mathcal{P}$ such that $\lim_{t \to 0^+} f(t) = \lim_{t \to \infty} f(t)/t = 0$. If $f \in \mathcal{P}$, then $0 \leq \gamma_f \leq \delta_f \leq 1$ [2, p.76]. Let $\mathcal{P}^{+-}$ be the set of all $f \in \mathcal{P}$ such that $0 < \gamma_f \leq \delta_f < 1$. It is obvious that $\mathcal{P}^{+-} \subset \mathcal{P}_0$.

A couple $(X_0, X_1)$ is called a $\mathcal{K}_0$-complete couple if for any function $f \in \mathcal{P}_0$ there exists an element $x \in X_0 + X_1$ such that

$$\mathcal{K}(t, x; X_0, X_1) \asymp f(t).$$

In other words, the set $\mathcal{K}(X_0 + X_1)$ of all $\mathcal{K}$-functionals of a $\mathcal{K}_0$-complete couple $(X_0, X_1)$ contains, up to equivalence, the whole of the subcone $\mathcal{P}_0$.

**Proposition 2.** The Banach couple $(L_1(0, \infty), L_2(0, \infty))$ is a $\mathcal{K}_0$-complete couple.

**Proof.** By the Holmstedt formula for functional spaces [14],

$$\mathcal{K}(t, x; L_1, L_2) \asymp \max \left\{ \int_0^t x(s) \, ds, \int_0^\infty (x(s))^2 \, ds \right\}^{1/2}$$

(14)

If $f \in \mathcal{P}_0$, then $g(t) = f(t^{1/2})$ belongs to $\mathcal{P}_0$, also. Let us denote $x(t) = g'(t)$. Then $x(t) = x^*(t)$ and

$$\int_0^t x(t) \, ds = g(t).$$

(15)

Assume that $f \in \mathcal{P}^{+-}$. If $\delta_f < 1$, then there exists $\varepsilon > 0$ such that for some $C > 0$

$$G(s) = f(s^{1/2}) \leq C(\sqrt{s}/t)^{1-\varepsilon} f(t^{1/2}), \quad \text{if } s \geq t.$$ Since $g \in \mathcal{P}_0$, then $g'(t) \leq g(t)/t$. Therefore for $t > 0$

$$\int_t^\infty (x(s))^2 \, ds \leq \int_t^\infty \frac{g^2(s)}{s^2} \, ds \leq C^2 \varepsilon^{-1} (f(t^{1/2}))^2 \int_t^\infty s^{-1-\varepsilon} \, ds = C^2 \varepsilon t^{-1} (g(t))^2.$$
Combining this with (14) and (15) we obtain
\[ K(t, x; L_1, L_2) \asymp g(t^2) = f(t). \]
Thus \( K(L_1 + L_2) \supset \mathcal{P}^+ \). Hence, in particular, the intersection \( K(X_0 + X_1) \cap \mathcal{P}^+ \) is not empty. Therefore, by [15, 4.5.7], \((L_1, L_2)\) is a \( K_0 \)-complete Banach couple. This completes the proof.

Let \( K(l_1 + l_2) \) be the set of all \( K \)-functionals corresponding to the couple \((l_1, l_2)\). By \( \mathcal{F} \) we denote the set of all functions \( f \in \mathcal{P} \) such that
\[
 f(t) = f(1)t \quad \text{for} \quad 0 < t \leq 1 \quad \text{and} \quad \lim_{t \to \infty} f(t)/t = 0.
\]

**Corollary 2.** *Up to equivalence,*
\[ K(l_1 + l_2) \supset \mathcal{F}. \]

**Proof.** It is well-known (see, for example, [4, p.142]) that for \( x \in L_1(0, \infty) + L_\infty(0, \infty) \) and \( u > 0 \)
\[
 K(u, x; L_1, L_\infty) = \int_0^u x^*(s) \, ds.
\]
(16)
In addition,
\[
 L_1 = (L_1, L_\infty)_{l^\infty}^K \quad \text{and} \quad L_2 = (L_1, L_\infty)_{l^{2(2-k)/2}}^K.
\]
The spaces \( l^\infty \) and \( l^{2(2-k)/2} \) are interpolation spaces concerning to the couple \( \vec{l}^\infty = (l^\infty, l^{2(2-k)}) \) [4]. Therefore, by the reiteration theorem (see [16] or [17]),
\[
 K(t, x; L_1, L_2) \asymp K(t, K(*, x; L_1, L_\infty); l^\infty, l^{2(2-k)/2}) \quad \text{for} \quad x \in L_1 + L_2
\]
(17)
Introduce the average operator:
\[
 Qx(t) = \sum_{k=1}^\infty \int_{k-1}^k x(s) \, ds \, \chi_{(k-1,k]}(t), \quad \text{if} \quad t > 0.
\]
From (16) it follows that
\[
 K(t, Qx^*; L_1, L_\infty) = K(t, x; L_1, L_\infty),
\]
for all positive integer \( t \). Both functions in the last equation are concave. Therefore,
\[
 K(t, Qx^*; L_1, L_\infty) \asymp K(t, x; L_1, L_\infty) \quad \text{for all} \quad t \geq 1.
\]
Hence (17) yields
\[
 K(t, Qx^*; L_1, L_2) \asymp K(t, x; L_1, L_2), \quad \text{if} \quad t \geq 1.
\]
(18)
Let now \( f \in \mathcal{F} \). Since \( \mathcal{F} \subset \mathcal{P}_0 \), then, by Proposition 2, there exists a function \( x \in L_1(0, \infty) + L_2(0, \infty) \) such that
\[
 K(t, x; L_1, L_2) \asymp f(t).
\]
(19)
Clearly, the operator $Q$ is a projector in the spaces $L_1$ and $L_2$ with norm 1. Moreover, $Q(L_1) = l_1$ and $Q(L_2) = l_2$. Hence, by the theorem about complemented sub-couples mentioned in Introduction (see [5] or [6, p.136]),

$$\mathcal{K}(t, Qx^*; L_1, L_2) \asymp \mathcal{K}(t, a; l_1, l_2) \text{ for } t > 0,$$

where $a = (\int_{t-1}^t x^*(s)ds)_{k=1}^\infty$.

Thus (18) and (19) imply

$$\mathcal{K}(t, a; l_1, l_2) \asymp f(t) \text{ for } t \geq 1.$$ 

The last relation also holds if $0 < t \leq 1$. Indeed, in this case

$$\mathcal{K}(t, a; l_1, l_2) = t||a||_2 = t\mathcal{K}(1, a; l_1, l_2) \asymp tf(1) = f(t).$$

This completes the proof. \qed

Proof of Theorem 3. As it was already mentioned in the proof of Theorem 1, the Orlicz space $L_N$, $N(t) = \exp(t^2) - 1$, coincides with the Marcinkiewicz space $M(\varphi_1)$, for $\varphi_1(u) = u \log^{1/2}(2/u)$. Since $\gamma_{\varphi_1} = 1$, then Corollary 1 implies that the couple $(L_\infty, G)$ is a $\mathcal{K}$-monotone couple. Hence,

$$X_0 = (L_\infty, G)_{E_0}^\mathcal{K} \text{ and } X_1 = (L_\infty, G)_{E_1}^\mathcal{K},$$

for some parameters of the real $\mathcal{K}$-method of interpolation $E_0$ and $E_1$. By Theorem 2,

$$\left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_i} \asymp ||(a_k)||_{F_i},$$

where $F_i = (l_1, l_2)_{E_i}^\mathcal{K}$ ($i = 0, 1$). So, by condition,

$$(l_1, l_2)_{E_0}^\mathcal{K} = (l_1, l_2)_{E_1}^\mathcal{K}. \quad (21)$$

The last means that the norms of spaces $E_0$ and $E_1$ are equivalent on the set $\mathcal{K}(l_1 + l_2)$. It is readily to check that this set coincides, up to the equivalence, with the set $\mathcal{K}(L_\infty + G)$ of all $\mathcal{K}$-functionals corresponding to the couple $(L_\infty, G)$. More precisely,

$$\mathcal{K}(l_1 + l_2) = \mathcal{K}(L_\infty + G) = \mathcal{F}. \quad (22)$$

In fact, by Theorem 1 and Corollary 1, $\mathcal{F} \subset \mathcal{K}(l_1 + l_2) \subset \mathcal{K}(L_\infty + G)$. On the other hand, since $L_\infty \subset G$ with the constant 1 and $L_\infty$ is dense in $G$, then $\mathcal{K}(L_\infty + G) \subset \mathcal{F}$ [11, p.386].

Let now $x \in X_0$. By (20), we have $(\mathcal{K}(2^k, x; L_\infty, G))_k \in X_0$. Using (22), we can find $a \in l_2$ such that

$$\mathcal{K}(2^k, a; l_1, l_2) \asymp \mathcal{K}(2^k, x; L_\infty, G),$$

for all positive integer $k$. Since a parameter of $\mathcal{K}$-method is a Banach lattice, then this implies $(\mathcal{K}(2^k, a; l_1, l_2))_k \in E_0$. Therefore, by (21), $(\mathcal{K}(2^k, a; l_1, l_2))_k \in E_1$, i.e., $(\mathcal{K}(2^k, x; L_\infty, G))_k \in E_1$ or $x \in X_1$. Thus $X_0 \subset X_1$. Arguing as above, we obtain the converse inclusion, and $X_0 = X_1$ as sets. Since $X_0$ and $X_1$ are Banach lattices, then their norms are equivalent.

This completes the proof. \qed
3. Final remarks and examples

**Remark 1.** Combining Theorems 1 — 3 with results obtained in the paper [18], we may also prove similar assertions for lacunary trigonometric series. Moreover, taking into account the main result of the paper [19], we may extend Theorems 1 — 3 to Sidon systems of characters of a compact Abelian group.

**Remark 2.** In Theorem 1, we cannot replace the space $G$ by $L_q$ with some $q < \infty$. Indeed, suppose that the couple $(T(l_1), T(l_2))$ is a $\mathcal{K}$ -subcouple of the couple $(L_\infty, L_q)$, i.e.,

$$
\mathcal{K}(t, a; l_1, l_2) \simeq \mathcal{K}(t, Ta; L_\infty, L_q).
$$

Let $E = l_p(2^{-\theta k})$, where $0 < \theta < 1$ and $p = q/\theta$. Applying the $\mathcal{K}$ -method of interpolation $(\cdot, \cdot)_E^\mathcal{K}$ to the couples $(l_1, l_2)$ and $(L_\infty, L_q)$ we obtain

$$
||Ta||_p \asymp ||a||_r,p = \left\{ \sum_{k=1}^{\infty} (a_k^*)^p k^{p/r-1} \right\}^{1/p}.
$$

Since $r = 2/(2 - \theta) < 2$ [4, p.142], then this contradicts with (2).

**Remark 3.** Clearly, a partial retract of a couple $\vec{Y} = (Y_0, Y_1)$ is also a $\mathcal{K}$ -subcouple of $\vec{Y}$. The opposite assertion is not true, in general (nevertheless, some interesting examples of $\mathcal{K}$ -subcouples and partial retracts simultaneously are given in the paper [20]). Indeed, by Theorem 1, the subcouple $(l_1, l_2)$ is a $\mathcal{K}$ -subcouple of the couple $(L_\infty, G)$. Assume that $(l_1, l_2)$ is a partial retract of this couple. Then (see the proof of Proposition 1) $(l_1, l_2)$ is a partial retract of the couple $(L_\infty, L_\infty(\log_2^{-1/2}(2/t)))$, as well. Therefore, by Lemma 1, from [21] and [4, p.142] it follows that

$$
[l_1, l_2]_\theta = (l_1, l_2)_{\theta, \infty} = l_{p, \infty},
$$

where $[l_1, l_2]_\theta$ is the space of the complex method of interpolation [4], $0 < \theta < 1$, and $p = 2/(2 - \theta)$. On the other hand, it is well-known [4, p.139] that

$$
[l_1, l_2]_\theta = l_p \quad \text{for} \quad p = \frac{2}{2 - \theta}.
$$

This contradiction shows that the couple $(l_1, l_2)$ is not a partial retract of the couple $(L_\infty, G)$.

Using Theorem 2, we can find coordinate sequence spaces of coefficients of Rademacher series from certain r.i.s.’s.

**Example 1.** Let $X$ be the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t \log_2 \log_2(16/t)$, $0 < t \leq 1$. Show that

$$
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\varphi)} \asymp ||a||_{l_1(\log)},
$$

where $l_1(\log)$ is the space of all sequences $a = (a_k)_{k=1}^{\infty}$ such that

$$
||a||_{l_1(\log)} = \sup_{k=1,2,..} \log_2^{-1}(2k) \sum_{i=1}^{k} a_i^*.
$$

(23)
is finite. Taking into account Theorem 2, it is sufficient to check that
\[
(l_1, l_2)_F^K = l_1(\log)
\]
and
\[
(L_\infty, G)_F^K = M(\varphi),
\]
for some parameter $F$ of the $K$-method of interpolation. More precisely, we shall prove that (25) and (26) are true for $F = l_\infty(u_k)$, where $u_k = 1/(k+1)$ ($k \geq 0$) and $u_k = 1$ ($k < 0$).

By the Holmstedt formula (9),
\[
\varphi_a(2^k) \leq \sum_{i=1}^{2^{2k}} a_i^* + 2^k \left[ \sum_{i=2^{2k}+1}^{\infty} (a_i^*)^2 \right]^{1/2} \leq B\varphi_a(2^k) \quad \text{for} \quad k = 0, 1, 2, \ldots,
\]
where, as before, $\varphi_a(t) = K(t, a; l_1, l_2)$. Without loss of generality, assume that $a_i = a_i^*$.

If $||a||_{l_1(\log)} = R < \infty$, then by (24),
\[
\sum_{i=1}^{2^{2k}} a_i^* \leq 2R(k+1).
\]

In particular, this implies $a_{2^{2k}} \leq 2^{-2k+1}R(k+1)$ for nonnegative integer $k$. Using the last inequality, we obtain
\[
\sum_{i=2^{2k}+1}^{\infty} a_i^2 \leq \sum_{j=k}^{\infty} \sum_{i=2^{2j}+1}^{2^{2(j+1)}} a_i^2 \leq \sum_{j=k}^{\infty} 2^{2j}a_{2^{2j}}^2 \leq 12R^2 \sum_{j=k}^{\infty} 2^{-2j}(j+1)^2 \leq 192R^2 \int_{k+1}^{\infty} x^2 2^{-2x} dx \leq 144R^2(k+1)^22^{-2k}.
\]

Hence the second term in (27) does not exceed $12R(k+1)$. Therefore, if $E = (l_1, l_2)_F^K$, then (28) implies
\[
||a||_E = \sup_{k=0,1,\ldots} \frac{\varphi_a(2^k)}{k+1} \leq 14||a||_{l_1(\log)}.
\]

Conversely, if $2^j + 1 \leq k \leq 2^{2(j+1)}$ for some $j = 0, 1, 2, \ldots$, then from (27) it follows
\[
\sum_{i=1}^{2^{2j}} a_i \leq B\varphi_a(2^{j+1}) \leq \sum_{i=1}^{2^{2(j+1)}} a_i \leq B||a||_E(j+2) \leq 2B\log_2(2k)||a||_E.
\]
Therefore $||a||_{l_1(\log)} \leq 2B||a||_E$ and (25) is proved.

We pass now to function spaces. At first, we introduce one more interpolation method which is, actually, a special case of the real method of interpolation.

For a function $\varphi \in P$ and an arbitrary Banach couple $(X_0, X_1)$ define generalized Marcinkiewicz space as follows:
\[
M_\varphi(X_0, X_1) = \left\{ x \in X_0 + X_1 : \sup_{t>0} \frac{K(t, x; X_0, X_1)}{\varphi(t)} < \infty \right\}.
\]
By equation (16), we have

\[ L_\infty = M_{\varphi_0}(L_1, L_\infty) \quad \text{and} \quad L_N = M_{\varphi_1}(L_1, L_\infty), \]

where these spaces are function spaces on the segment \([0, 1]\). Here \(\varphi_0(t) = \min(1, t)\), \(\varphi_1(t) = \min(1, t \log_2^{1/2}[\max(2, 2/t)])\), and \(N(t) = \exp(t^2) - 1\), as before. In addition, using similar notation, it is easy to check that

\[ (X_0, X_1)_{F}^c = M_\rho(X_0, X_1), \]

for an arbitrary Banach couple \((X_0, X_1)\) and \(\rho(t) = \log_2(4+t)\). Hence, by the reiteration theorem for generalized Marcinkiewicz spaces [11, p.428], we obtain

\[ (L_\infty, L_N)^c_F = M_\rho(M_{\varphi_0}(L_1, L_\infty), M_{\varphi_1}(L_1, L_\infty)) = M_{\varphi_0}(L_1, L_\infty) = M(\varphi), \]

where \(\varphi(t) = \varphi_0(t)\rho(\varphi_1(t) / \varphi_0(t))\). A simple calculation gives \(\varphi(t) \sim \varphi(t)\), if \(t > 0\). Thus,

\[ (L_\infty, L_N)^c_F = M(\varphi). \]

It is readily seen that \(K(t, x; L_\infty, G) = K(t, x; L_\infty, L_N)\), for all \(x \in G\). Therefore, for such \(x\) the norm \(\|x\|_{M(\varphi)}\) is equal to the norm \(\|x\|_Y\), where \(Y = (L_\infty, G)^c_F\). On the other hand, for \(x \in M(\varphi)\)

\[
\frac{1}{t \log_2^{1/2}(2/t)} \int_0^t x^*(s) ds \leq \|x\|_{M(\varphi)} \frac{\log_2 \log_2(16/t)}{\log_2^{1/2}(2/t)} \to 0 \quad \text{as} \quad t \to 0^+.\]

This implies that \(M(\varphi) \subset G\) [2, p.156].

Thus \(Y = M(\varphi)\) and therefore (26) is proved. Equivalence (23) follows now, as already stated, from (25) and (26).

**Remark 4.** Theorems 2 and 3 strengthen results of the papers [3] and [22], where similar assertions are obtained for sequence spaces \(F\) satisfying more restrictive conditions. For instance, we can readily show that the norm of the dilation operator

\[ \sigma_n a = (a_1, a_1, a_2, \ldots) \]

in the space \(l_1(\ln)\) (see Example 2) is equal to \(n\). Therefore condition (11) from [3] fails for this space and the theorems obtained in the papers [3] and [22] cannot be applied to it. Similarly, the Marcinkiewicz space \(M(\varphi)\) from Example 1 does not satisfy conditions of Theorem 8 of [3].

Using Theorems 2 and 3, we can derive certain interpolation relations.

**Example 2.** Let \(\varphi \in \mathcal{P}\) and \(1 \leq p < \infty\). Recall that the Lorentz space \(\Lambda_p(\varphi)\) consists of all measurable functions \(x = x(s)\) such that

\[ ||x||_{\varphi,p} = \left\{ \int_0^1 (x^*(s))^p d\varphi(s) \right\}^{1/p} < \infty. \]

In the paper [3], V.A.Rodin and E.M.Semenov proved that

\[ \left\| \sum_{k=1}^\infty a_k r_k \right\|_{\varphi,p} \asymp ||(a_k)||_{p}, \]
where $\varphi(s) = \log_2^{1-p}(2/s)$ and $1 < p < 2$. Moreover, the space $\Lambda_p(\varphi)$ is the unique r.i.s. having this property. Note that $l_p = (l_1, l_2)_{\theta,p}$, where $\theta = 2(p - 1)/p$ [4, p.142]. Therefore, by Theorem 2, we obtain

$$(L_\infty, G)_{\theta,p} = \Lambda_p(\varphi),$$

for the same $p$ and $\theta$. 
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