Z-CATEGORIES I

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Abstract. This paper is the first in a series of two papers, Z-Categories I and Z-Categories II, which develop the notion of Z-category, the natural bi-infinite analog to strict \(\omega\)-categories, and show that the \((\infty, 1)\)-category of spectra relates to the \((\infty, 1)\)-category of homotopy coherent Z-categories as the pointed groupoids.

In this work we provide a 2-categorical treatment of the combinatorial spectra of [Kan] and argue that this description is a simplicial avatar of the abiding notion of homotopy coherent Z-category. We then develop the theory of limits in the 2-category of categories with arities of [BergerMelliesWeber] to provide a cellular category which is to Z-categories as \(\Delta\) is to 1-categories or \(\Theta_n\) is to \(n\)-categories. In an appendix we provide a generalization of the spectrification functors of 20th century stable homotopy theory in the language of category-weighted limits.

Preface

This paper is the first in a series of two papers, Z-Categories I and Z-Categories II, which develop the notion of Z-category, the natural bi-infinite analog to strict \(\omega\)-categories, and show that the \((\infty, 1)\)-category of spectra lives inside the \((\infty, 1)\)-category of pointed homotopy coherent Z-categories as the full sub-\((\infty, 1)\)-category of Z-groupoids. That theorem is carried out in the language of Cisinski model categories. The work is split as it is for the first is a work of (nearly formal) 2-category theory, and the second is almost entirely restricted to homotopical concerns.

In a sense these two works aims to synthesize the 1963 combinatorial model of spectra of [Kan], with the 1983 homotopy hypothesis of [Grothendieck], by way of Australian style 2-category theory and the 2-category of categories arities of [BergerMelliesWeber], in the context of the 2006 theory of Cisinski model categories of [Cisinkil] interpreted as presentations of categories of homotopy coherent models of limit theories.

The work is aimed to be readable both by topologists and category theorists and, as such, many elementary notions which are well known to one group, but much less well so to the other, are treated somewhat explicitly. We beg the reader’s patience.

Introduction

We begin with a long(er) and semi-technical introduction. This introduction should be read with the foreknowledge that all will be explained in more detail in the body of the text.

Part I: A 2-Categorical Treatment of Kan’s “Semisimplicial Spectra”

Kan’s Suspension. The work of [Kan] begins with a description of the reduced suspension which is adapted to the geometry/combinatorics of the simplex category. Kan describes,

\[1\text{See Cisinkil or Z-categories II Part II for a survey of the relevant results translated from the French.}\]
albeit not in these modern terms, the reduced suspension of pointed simplicial sets as the left Kan extension

\[ \Sigma_K = \text{Lan}_{\Delta^+} \left( \begin{array}{cc} \Delta & \to & \hat{\Delta} \\ [n] & \to & \Delta[n+1]_+ / \Delta[n]_+ \lor [0]_+ \end{array} \right) \]

which assigns to every freely pointed \( n \)-simplex a “fattened” circle whose cross section shares the geometry of the \( n \)-simplex. We will note that this functor is a quotient of (a post-composition) of the endofunctor

\[ K : \Delta \to \Delta \]

\[ [n] \to [n+1] \]

The implied factoring will become important when we understand categories of sequential spectrum objects and \( \Omega \)-spectrum objects as particular 2-categorical limits.

**Sequential spectra and \( \Omega \)-spectra as oplax-limits and pseudo-limits respectively.**

Now, in a 1-category there is a unique notion of the cone over a diagram. However, in passing to 2-categories, we immediately come upon three further notions of cone over a 1-diagram. Indeed, in a 2-category, we have in the strict cones an avatar of the 1-categorical notion of cone. But we also have pseudo-cones in which the role played by identity 2-cells in commuting triangles is instead played by a coherent choice of invertible 2-cells. More, we have and oplax and lax cones wherein those 2-cells remain coherent, i.e. functorial in the 1-cells of the diagram, but for which the hypothesis of invertibility is dropped (see Figures 1, 2, 3, and 4). The oplax and lax conventions are then but the two choices for the directions of the comparison 2-cells. Importantly, just as the data of a cone in a 1-category constitutes a natural transformation from a constant functor into the diagram, in a 2-category, the various notions of cones, pseudo-cones, oplax-cones, and lax-cones constitute pseudo-natural transformations, oplax-natural transformations, and lax-natural transformation\(^2\) from a constant functor into the diagram.

Now, it is no big leap to go from Adams spectra to sequential spectrum objects valued in a category \( \mathbf{A} \) with respect to an adjunction \( L \dashv R \). Indeed a sequential spectrum object is comprised of an \( \mathbb{N} \)-indexed set of objects of the category \( \mathbf{A} \), \( (X_i)_{i \in \mathbb{N}} \), together with a set of maps \( (X_i \xrightarrow{\varphi_i} R(X_{i+1}))_{i \in \mathbb{N}} \). As we’ll see however, these are but objects of the oplax limit

\[ \text{oplaxlim} \left\{ \cdots \to \mathbf{A} \to \mathbf{A} \right\} \]

in the 2-category \( \text{Cat} \) (or \( \text{CAT} \) etc. depending on size) where the data of those structure maps \( (\varphi_i)_{i \in \mathbb{N}} \) are the 2-cells of the associated classified cone, and \( \Omega \)-spectrum objects are but objects of the pseudo-limit over that same diagram. Indeed, when the functor \( R \) is an iso-fibration, the pseudo-limit and conical limit coincide, whence \( \Omega \)-spectrum objects are in fact but objects of the strict limit in all cases we concern ourselves with here.

So called coordinate-free spectra (see [ElmendorfKrizMandellMay]) too admit this description after replacing the diagram

\[ \cdots \to \mathbf{A} \to \mathbf{A} \to \mathbf{A} \]

\(^2\)Importantly, these four distinct notions of natural transformation define four distinct enrichments of a category of categories in a category of categories.
with the diagram

\[
\begin{array}{ccc}
\text{FinSub}(\mathcal{U})^\text{op} & \xrightarrow{\text{Cat}} & V \\
& \xrightarrow{S_*} & \Omega^{W-V}
\end{array}
\]

where the category $\text{FinSub}(\mathcal{U})$ is the category of sub-vector spaces of a particular choice of $\omega$-dimensional vector space $\mathcal{U} \xrightarrow{\sim} \mathbb{R}^\omega$.

Returning to Kan’s particular choice of suspension functor, and its factoring we will get a natural transformation between right adjoint functors

$K^* \Rightarrow \Omega_K : \hat{\triangle} \xrightarrow{} \hat{\triangle}$

where $\Omega_K$ is the functor right adjoint to Kan’s $\Sigma_K$. These 2-cells will then assemble to define an oplax natural transformation

\[
\cdots \xrightarrow{\Omega_K} \hat{\triangle} \xrightarrow{\Omega_K} \hat{\triangle} \xrightarrow{\Omega_K} \hat{\triangle} \\
\cdots \xrightarrow{id} \hat{\triangle} \xrightarrow{id} \hat{\triangle} \xrightarrow{id} \hat{\triangle}
\]

whence a functor between oplax limits

\[
\text{oplaxlim} \left\{ \cdots \xrightarrow{\Omega_K} \hat{\triangle} \xrightarrow{\Omega_K} \hat{\triangle} \right\} \rightarrow \text{oplaxlim} \left\{ \cdots \xrightarrow{K^*_*} \hat{\triangle} \xrightarrow{K^*_*} \hat{\triangle} \right\}
\]

But since $K^*_*$ is induced by an endomorphism of $\triangle$ we will show how may rewrite the category on the right-hand side in a more elementary fashion.
Factorization through $\big( (\_ \,)^{\text{op}}, \text{Set}_* \big) : \text{Cat}^{\text{coop}} \to \text{CAT}$ and the collage construction.

Indeed, we will observe that the diagram

$$
\cdots \to \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \xrightarrow{K^*} \hat{\triangle}
$$

factors through the 2-functor

$$
\big( (\_ \,)^{\text{op}}, \text{Set}_* \big) : \text{Cat}^{\text{coop}} \to \text{CAT}
$$

which takes a small category $\mathcal{A}$ and returns the large category of $\text{Set}_*$-enriched functors $\hat{\mathcal{A}} \to [\mathcal{A}^{\text{op}}, \text{Set}_*]$ from $\mathcal{A}^{\text{op}}$ to $\text{Set}_*$ where $\mathcal{A}$ is the free $\text{Set}_*$-enriched category on $\mathcal{A}$. That 2-functor however sends 2-categorical colimits to 2-categorical limits.

As such, we will find that we have equivalences

$$
\lim \left\{ \cdots \leftarrow \triangle \xleftarrow{K} \triangle \xleftarrow{K} \triangle \right\} \sim \lim \left\{ \cdots \to \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \right\}
$$

and

$$
\text{oplaxlim} \left\{ \cdots \leftarrow \triangle \xleftarrow{K} \triangle \xleftarrow{K} \triangle \right\} \sim \text{oplaxlim} \left\{ \cdots \to \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \right\}
$$

Then, invoking the collage construction, it will be easy to describe the oplax colimit

$$
\text{oplaxlim} \left\{ \cdots \leftarrow \triangle \xleftarrow{K} \triangle \xleftarrow{K} \triangle \right\}
$$

as the small category on the set of objects

$$
\left\{ ([n], -k) \, | \, n \in \mathbb{N}, -k \in \mathbb{Z}_{\leq 0} \right\} = \coprod_{-m \in \mathbb{Z}_{\leq 0}} \text{Ob} (\triangle)
$$

with $\text{Hom}$-sets defined by the expression

$$
\triangle_{\text{coll}} (([n], -m), ([\ell], -(m+k))) = \triangle ([n+k], [\ell])
$$

with the composition laws coming from the right hand side of that equality. The universal property of the strict colimit $\lim \left\{ \cdots \leftarrow \triangle \xleftarrow{K} \triangle \xleftarrow{K} \triangle \right\}$ is similarly enjoyed by $\triangle_{\text{st}}$, the category with set of objects

$$
\text{Ob} (\triangle_{\text{st}}) = \{ [z] \, | \, z \in \mathbb{Z} \}
$$

with $\text{Hom}$-sets generated by maps

$$
\left\{ d^i : [m-1] \to [m] \, | \, i \in \mathbb{N} \right\}
$$

and

$$
\left\{ s^j : [m+1] \to [m] \, | \, j \in \mathbb{N} \right\}
$$

subject to (un-bounded) simplicial identities. The obvious functor $\triangle_{\text{coll}} \to \triangle_{\text{st}}$ then induces an adjoint triple

$$
\triangle_{\text{coll}} \quad \perp \quad \triangle_{\text{st}}
$$

with the pullback functor in the middle recovering the inclusion (note the direction of the arrow)

$$
\text{oplaxlim} \left\{ \cdots \to \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \right\} \leftarrow \lim \left\{ \cdots \to \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \right\}
$$

More, the oplax limit

$$
\text{oplaxlim} \left\{ \cdots \to \hat{\triangle} \xrightarrow{\Omega K} \hat{\triangle} \xrightarrow{\Omega K} \hat{\triangle} \right\} \to \text{oplaxlim} \left\{ \cdots \to \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \xrightarrow{K^*} \hat{\triangle} \right\}
$$
too arises as a pullback along a pointed functor
\[ \Delta_{coll} \longrightarrow \text{plaxlim} \{ \cdots \to \widehat{\Delta} \xrightarrow{\Omega_K} \widehat{\Delta} \xrightarrow{\Omega_K} \widehat{\Delta} \} \]
which we will construct explicitly.

**Double duty and slick tricks.** Kan defines the category of semisimplicial spectra to be the full subcategory of \( \widetilde{\Delta}_{st}^\bullet \) subtended by colimits of pointed presheaves of the form
\[ \Delta_{st}^{|i|/d^{>n}=s} \]
which is a stable \([z]-simplex with something much like the geometry of an \( n \)-simplex, as its set of stable \([z-1],[z-2],\ldots,[z-n] \) simplices which are not the basepoint mirror the combinatorics of the usual \([n]-simplex. We call this subcategory \( \text{LocFin}(\Delta_{st}) \).

We will see that part of the adjoint triple induced by the functor above factors through \( \text{LocFin}(\Delta_{st}) \) as a co-reflective subcategory of \( \widetilde{\Delta}_{st}^\bullet \).

\[ \widetilde{\Delta}_{st}^\bullet \xrightarrow{\perp} \text{LocFin}(\Delta_{st}) \xleftarrow{\text{plaxlim}} \{ \cdots \xrightarrow{\Omega_K} \widehat{\Delta} \} \]

We will also correct a small error made in [Kan] - a related one in made [ChenKrizPultr] - which has it that the right hand adjunction above is a reflective subcategory - this is not the case and we provide an obvious counter-example. Instead, if we define \( \text{LocSph}(\Delta_{st}) \) to be the full subcategory of \( \widetilde{\Delta}_{st}^\bullet \) on pointed presheaves which are colimits of objects of the form
\[ S^{(z-n)}[n] = \Delta_{st}^{|i|/(d_0d_0^\ldots d_{n-1}d_{n}=d^{>n}=s)} \]
which are quotients of \( \Delta_{st}^{|i|/d^{>n}=s} \) which me may think as a \((z-n)\)-sphere with an \([n]\)-simplex worth of nontrivial faces\(^2\), then we then have a sequence of adjunctions

\[ \widetilde{\Delta}_{st}^\bullet \xrightarrow{\perp} \text{LocFin}(\Delta_{st}) \xleftarrow{\text{plaxlim}} \{ \cdots \xrightarrow{\Omega_K} \widehat{\Delta} \} \]

where the rightmost adjunction is a reflective subcategory and the left two adjunctions are co-reflective subcategories. As we will see however in developing model category structures on these categories the error becomes irrelevant as the data lost in the passage from \( \text{LocFin}(\Delta_{st}) \) to \( \text{LocSph}(\Delta_{st}) \) are coherently contractible.

**Homotopy-coherent models of...?** Kan goes on describe a horn-filling condition for mapping spaces of pointed presheaves \( \Delta_{st}^\bullet \longrightarrow \text{Set}_\bullet \) in \( \text{LocFin}(\widetilde{\Delta}_{st}^\bullet) \). In [Brown] this is replaced with an easier description. Brown defines the set of stable horns
\[ \Lambda^i_B = \{ \Lambda^i_B |n-k| \to S^{(-k)}[n] \mid n \in \mathbb{N}, k \in \mathbb{N}, 0 \leq i \leq k \} \]
where the stable horns \( \Lambda^i_B |n-k| \) are defined much as the simplicial horns \( \Lambda^i |n| \) of \( \Delta |n| \) are. Indeed, in [Brown] it was proven that this set comprises a generating set of acyclic cofibrations for a (left transfer of) a pointed Cisinski model category\(^3\) which was then proven, in [BousfieldFriedlander], to be Quillen equivalent to the Bousfield-Friedlander model structure on sequential spectra valued in pointed simplicial sets. The promised irrelevance of the

\(^3\)thinking in this way a banana is an arrow with \(\sim 5\) faces

\(^4\)again of course not in this much later language.
distinction between $\text{LocFin} (\Delta_{st})$ and $\text{LocSph} (\Delta_{st})$ is that the adjunction between them is a Quillen equivalence.

But Cisinski model categories are presentations of $(\infty, 1)$-categories of homotopy coherent models for limit theories. Consider, for example, Joyal’s model structure for quasi-categories. That model structure is the minimal Cisinski model structure on the category of simplicial sets in which the spine inclusions are trivial cofibrations, and the generating set $\Lambda^\infty (V)$ of anodyne extensions for that model structure identify the fibrant objects as those which are homotopy orthogonal to the spine inclusions. But strict orthogonality of a simplicial set with respect to the set of spine inclusions is equivalent to the so-called Segal condition on a simplicial set $X$,

$$X ([n]) \xrightarrow{\sim} X ([1]) \times_{X([0])} X ([1]) \times_{X([0])} \cdots \times_{X([0])} X ([1])$$

It is thus that we identify quasi-categories as providing a presentation of the $(\infty, 1)$-category of homotopy coherent 1-categories.

Returning to the topic of Kan’s model then one is wont to ask: what essentially algebraic structure are Kan spectra to be understood as homotopy coherent models of?

A Globular perspective on Kan’s suspension functor. In a sense the purely simplicial geometry obscures something of Kan’s suspension. Quasi-categories are as economical as they are precisely because the directed higher cells are obscure - everything is a composition datum. Thinking globularly however, or for that matter complicially, we see that since the simplex category $\Delta$ is a full subcategory of the category $\text{Cat}$ of small categories we may think of Kan’s suspension as taking each simplex, thought of as a 1-category, and returning the 2-category with a single object and that original 1-category as its unique non-trivial $\text{Hom}$-category. In this view, Kan’s suspension is but the simplicial avatar of the endofunctor

$$\Sigma_\omega : \text{Str}-\omega-\text{Cat}_* \longrightarrow \text{Str}-\omega-\text{Cat}_*$$

which takes an $\omega$-category and returns the $\omega$-category with a single object and that original $\omega$-category as its unique non-trivial $\text{Hom}$-$\omega$-category. Letting $\Omega_\omega$ denote the functor right adjoint to $\Sigma_\omega$, it is clear that the diagram which begat sequential spectra should then be understood an avatar of the diagram

$$\cdots \longrightarrow \text{Str}-\omega-\text{Cat}_* \xrightarrow{\Omega_\omega} \text{Str}-\omega-\text{Cat}_* \xrightarrow{\Omega_\omega} \text{Str}-\omega-\text{Cat}_*$$

and the diagram

$$\cdots \longrightarrow \widehat{\Delta}_* \xrightarrow{K^*} \widehat{\Delta}_* \xrightarrow{K^*} \widehat{\Delta}_*$$

should be considered as but the pointing of an avatar of the diagram

$$\cdots \longrightarrow \text{Str-\omega-Cat} \xrightarrow{S^*} \text{Str-\omega-Cat} \xrightarrow{S^*} \text{Str-\omega-Cat}$$

where $S^*$ is the functor right adjoint to the functor $S : \text{Str-\omega-Cat} \longrightarrow \text{Str-\omega-Cat}$ which send an $\omega$-category $X$ to the $\omega$-category with two objects 0 and 1 with $\text{Hom}(0, 1) = X$ and all other $\text{Hom}$-categories trivial.

5For those already familiar with Berger’s wreath notation, we might simply say the $\omega$-category $[1]: (X)$.
**Z-categories.** What then, we ask, is an object of the following limit?

\[
\lim \left\{ \cdots \to \text{Str}-\omega\text{-Cat} \xrightarrow{S^*} \text{Str}-\omega\text{-Cat} \xrightarrow{S^*} \text{Str}-\omega\text{-Cat} \right\}
\]

To answer this we consider that \(\text{Str}-\omega\text{-Cat}\) is an Eilenberg-Moore category of algebras for a monad on the category of globular sets. Indeed we recall that there exists a monad \(T\) on globular sets and an isomorphism \(\text{Str}-\omega\text{-Cat} \xrightarrow{\sim} \hat{G}^T\). Then, just as with the functor

\[
K^* : \hat{\Delta} \to \hat{\Delta}
\]

the functor

\[
S^* : \hat{G} \to \hat{G}
\]

which gave rise to

\[
S^* : \text{Str}-\omega\text{-Cat} \to \text{Str}-\omega\text{-Cat}
\]

is of the form \([S^{op}, \text{Set}]\) where \(S : G \to G\) is the obvious endomorphism of the globe category which sends the \(n\)-globe \(\pi\) to the \((n + 1)\)-globe \(\overline{n + 1}\).

The taking of the Eilenberg-Moore category for a monad \(T\) is the taking of a lax limit over the underlying endofunctor as a diagram in \(\text{CAT}\). As such, since this operation commutes with limits, we are granted the existence of an isomorphism

\[
\lim \left\{ \cdots \to \text{Str}-\omega\text{-Cat} \xrightarrow{S^*} \text{Str}-\omega\text{-Cat} \xrightarrow{S^*} \text{Str}-\omega\text{-Cat} \right\} \sim \lim \left\{ \cdots \xleftarrow{K} G \xleftarrow{\hat{G}} G \xleftarrow{\hat{G}} \right\}
\]

for a monad \(\lim T\). More, just as the universal property of the colimit

\[
\lim \left\{ \cdots \xleftarrow{K} \Delta \xleftarrow{\hat{K}} \Delta \xleftarrow{\hat{K}} \Delta \right\}
\]

is enjoyed by \(\Delta_{st}\) whose objects are in bijection with \(Z\), with maps being generated by \(\omega\)-many face and degeneracy maps, the universal property of

\[
\lim \left\{ \cdots \xleftarrow{\hat{S}} G \xleftarrow{\hat{G}} G \xleftarrow{\hat{G}} \right\}
\]

is enjoyed by the category \(G_Z\) of integer, as opposed to natural number, indexed globes.

\[
G_Z = \left\{ \cdots \xrightarrow{s} \to \overline{1} \xrightarrow{s} \overline{0} \xrightarrow{s} \overline{1} \xrightarrow{s} \cdots \mid s \circ t = s \circ s \quad t \circ t = t \circ s \right\}
\]

We refer to presheaves on \(G_Z\) as \(Z\)-globular sets. Then, to borrow a phrase from [Weiner](p. 52):

“If this thing \(\hat{G}_Z\) is to have a name, let that name be **Z-categories** (simony or sorcery)."

We’ll denote the category of such by \(Z\text{-Cat}\).

**Part II: Limits in the 2-category of categories with arities**

To complete the strict version of the story, we must identify a category \(A\) and a regulus\(^6\) \(R\) in \(\hat{A}\) which are to \(Z\)-categories as the simplex category \(\Delta\) and the spine inclusions are to small 1-categories. To clarify precisely what this means and to find such a category and

\(^6\)In [Kelly] we find the term regulus used to identify the set of morphisms which carve out orthogonal subcategories - we will use the term here, honoring and appreciating these “small rulers”.
regulus we appeal to Berger, Mellies and Weber's 2-category of categories with arities and Bourke and Garner's synthesis of this with Day's notion of density presentation.

In \text{BergerMelliesWeber} those authors develop the 2-category of categories with arities \( \text{CwA} \) - a sage choice of a 2-category of small dense subcategories. That work then uses this 2-category to organize numerous characterizations of various algebras on presheaf categories as instances of a more general theory. In particular, the same apparatus is seen to characterize 1, 2, \ldots, \( n \), \ldots, \( \omega \)-categories as reflective subcategories of \( \hat{\Delta}, \hat{\Theta}_2, \ldots, \hat{\Theta}_n, \ldots, \hat{\Theta} \) respectively satisfying an "abstract nerve criterion", where the \( \Theta_k \) are subcategories of Joyal’s category \( \Theta \). Bourke and Garner go on to note that in many cases, including those just mentioned, by way of Day’s notion of a density presentation, we may in fact easily extract a regulus which carves out precisely the subcategory of those presheaves satisfying the criterion of the "abstract nerve theorem". In the cases mentioned the canonical regulus is the set of spine inclusions which, together with a little slight of hand, implies that the inner horns carve out the same subcategories (this recovers the earlier result of \text{Berger1}).

What we do in this paper is to show that the obvious 2-functor \( \text{CwA} \rightarrow \text{CAT}^2 \) which sends a full-and-faithful right adjoint to the underlying functor reflects weighted limits. From this it follows (see Example 13.3) that

\[
\lim \left\{ \cdots \rightarrow \text{Str-}\omega-\text{Cat} \xrightarrow{S^*} \text{Str-}\omega-\text{Cat} \xrightarrow{S^*} \text{Str-}\omega-\text{Cat} \right\}
\]

is a reflective subcategory of \( \lim \left\{ \cdots \leftarrow \Theta \xleftarrow{S} \Theta \xleftarrow{S} \Theta \right\} \) and more that it is carved out by the regulus whose homotopical and groupoidal analogue carved out Kan’s semisimplicial spectra in \( \hat{\Delta}^{st} \).

\textit{Remark 0.1.} Since, in this work, we do not actually need to perform any explicit computations with the categories

\[
\Theta, \quad \text{oplaxlim} \left\{ \cdots \leftarrow \Theta \xleftarrow{S} \Theta \xleftarrow{S} \Theta \right\} \quad \text{or} \quad \lim \left\{ \cdots \leftarrow \Theta \xleftarrow{S} \Theta \xleftarrow{S} \Theta \right\}
\]

we leave explicit descriptions of those categories to the second paper.

\textbf{Appendix A - Generalized spectrification.} All four 2-categorical limits over 1-diagrams are but instances of the still more general enriched categorical notion of weighted limits. If limits of diagrams of categories are compatible collections of objects, then category-weighted limits are compatible collections of diagrams valued in the constituent categories of a diagram of categories. It is in this language that we identify the usual left adjoint to the inclusion of \( \Omega \)-spectrum objects into sequential spectrum objects as an instance of a rather more general reflection of weighted limits over a 1-diagram into conical limits over that same diagram.

\textbf{Appendix B - Oplax limits, oplax weights.} This second appendix explains why the weight \( (J \downarrow \_ \_ \_)_\text{op} : J \rightarrow \text{Cat} \) computes oplax limits as weighted limits. This is well known, but not in general to topologists\(^7\), as such we include it for the author’s edification and for the interested reader’s.

\textbf{Appendices C&D - Technical lemmata.} In a third appendix we assemble some technical lemmata which would clutter the exposition were they in the body of the text.

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\(^7\)A much larger library of such examples is given in \text{BourkeGarner}.

\(^8\)Consider, for example, the author.
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Part 1. A 2-Categorical Treatment of Kan’s “Semisimplicial Spectra”

In this part we use some 2-categorical notions to give a description of the “semisimplicial spectra” of [Kan]. Along the way, in Remark 1.4, we correct a subtle error of [Kan] and another related one found in [ChenKrizPultr].

1. Kan’s Suspension: The Kan decalage and its associated suspension and loop-space functors

Definition 1.1. Denote by $K$ the functor

\[
\begin{array}{ccc}
\Delta & \xrightarrow{K} & \Delta \\
[n] & \xrightarrow{[n+1]} & d^i \\
{s^j} & \xrightarrow{d^i} & s^j
\end{array}
\]

and observe that there are natural transformations

\[
\text{id}_\Delta \xRightarrow{\alpha} K \xLeftarrow{\beta} [0]
\]

whose components at $[n] \in \Delta$ correspond to the maps $d^{i+1} : [n] \rightarrow [n+1]$ and $\{n+1\} : [0] \rightarrow [n+1]$. These data comprise the Kan decalage (see [CisinskiMaltsiniotis]) and we illustrate them in Figure 5. Kan’s small suspension functor and its associated loop-space functor (see [Kan]) are given by taking the quotient of $K$ by $\alpha$ and $\beta$.

![Figure 5. The Kan decalage illustrated: the case of [2]](attachment:image.png)
Let the functor $\Sigma_K : \widehat{\Delta}_\bullet \to \widehat{\Delta}_\bullet$ be the left Kan extension

$$\Sigma_K := \text{Lan}_\Delta \left( \begin{array}{c} \Delta \rightarrow \widehat{\Delta}_\bullet \\ [n] \rightarrow \Delta[n+1]/\Delta[n] \vee \Delta[0] \end{array} \right)$$

with $\Delta$ being the pointed Yoneda embedding $\Delta : \Delta_\bullet \to \widehat{\Delta}_\bullet$ and let $\Omega_K$ denote its right adjoint.

Now, as proven in [Kan], we have homeomorphisms

$$|\Sigma_K X| \sim \rightarrow |\Sigma X|$$

for all pointed simplicial sets $X$ where $|\_| : \widehat{\Delta}_\bullet \to \text{Top}_\bullet$ is the usual geometric realization functor and $\Sigma$ is the usual reduced suspension. More, the unit $\eta : \text{id}_{\widehat{\Delta}_\bullet} \Rightarrow \Omega_K \Sigma_K$ of the adjunction $\Sigma_K \dashv \Omega_K$ is invertible.

**Proposition 1.2.** (Proposition 3 [ChenKrizPultr]) The unit $\eta : \text{id}_{\widehat{\Delta}_\bullet} \Rightarrow \Omega_K \Sigma_K$ of the adjunction $\Sigma_K \dashv \Omega_K$ is invertible.

**Proof.** The proof given in [ChenKrizPultr] is correct modulo the detail we’ll explore carefully in Remark 1.4. □

**Remark 1.3.** The invertibility of the unit is an important distinction between this adjunction and the associated one in pointed spaces. For any set $E$ we have that both the pointed simplicial set

$$\Sigma_K \left( \bigvee_{e \in E} \Delta[0]_+ \right)$$

and the pointed space

$$\Sigma \left( \text{Disc}(E)_+ \right)$$

are wedges of $E$-many circles. In $\text{Top}_\bullet$ however we may compose these circles whereas in $\widehat{\Delta}_\bullet$ we cannot compose them without first taking a fibrant replacement for $\Sigma_K \left( \bigvee_{e \in E} \Delta[0]_+ \right)$ in the pointed Kan (a.k.a. test) model structure.

**Remark 1.4.** In [ChenKrizPultr] we find a functor $\Omega : \widehat{\Delta}_\bullet \to \widehat{\Delta}_\bullet$ defined as follows.

$$\Omega : \widehat{\Delta}_\bullet \longrightarrow \widehat{\Delta}_\bullet$$

$$X \longmapsto \Omega(X) : \Delta^{op} \longrightarrow \text{Set}_\bullet$$

$$[n] \longmapsto \left\{ [n+1] \xrightarrow{x} X \mid d_{n+1}(x) = \bullet \right\}$$

Those authors then go on to describe the functor left adjoint to $\Omega$ as follows.

Let $\Delta_0$ in $\Delta$ be wide subcategory thereof generated by the objects and solid arrows of the following diagram (the dotted arrows are present only to guide the eye, they are not generators of $\Delta_0$)

$$[0] \quad [1] \quad [2] \quad \ldots$$

or Joyal’s model structure for that matter, if one is so inclined.
Let \( \iota : \Delta_0 \to \Delta \) be the name of the inclusion and see that \( \iota \) induces an adjoint triple
\[
\iota_\ast \dashv \iota^\ast \dashv \iota_! \nabla
\]

Let \( \Sigma_0 : \hat{\Delta}_n \to \hat{\Delta}_{n+1} \) be the functor defined as follows.

\[
\Sigma_0 : \hat{\Delta}_n \to \hat{\Delta}_{n+1} \\
X \xleftarrow{\iota_\ast} \Sigma_0 (X) : \Delta_0^\text{op} \to \text{Set} \\
[n+1] \xrightarrow{\iota_!} X ([n]) \\
[0] \xrightarrow{\iota^\ast} \{\bullet\}
\]

It’s not hard to see that we have a natural isomorphism of functors \( \Sigma_K \to \iota_\ast \Sigma_0 \), indeed the co-continuity of both functors provides that it suffices to check the claim on simplices and it can be checked by hand that \( \Sigma_K (\Delta [n]_{+}) \) agrees with \( \iota_\ast \Sigma_0 (\Delta [n]_{+}) \).

The functor \( \Omega \) of [ChenKrizPultr] however is not right adjoint to the functor \( \iota_\ast \Sigma_0 \) as is claimed in [ChenKrizPultr]. We will construct a counter-example. Consider the pointed simplicial set \( X \) defined by the following push-out.

\[
\begin{array}{ccc}
\Delta [0]_{+} & \xrightarrow{d^1_+} & \Delta [1]_{+} \\
\downarrow & & \downarrow \\
\bullet & \to & X
\end{array}
\]

By construction the unique non-degenerate 1-cell of \( X \), call it \( x \in X ([1]) \), has as its 1st face the base-point \( \bullet \). Thus we compute

\[
\hat{\Delta}_n (\Delta [0]_{+} , \Omega (X)) = \Omega (X) ([0]) \\
= \{ \Delta [1]_{+} \xrightarrow{\nu} X \mid d^1_+ (y) = \bullet \} \\
= \{ x, \bullet \}
\]

However, we find that

\[
\hat{\Delta}_n (\iota_\ast \Sigma_0 ([0]), X) = \{ \bullet \}
\]
as the only 1-cell of \( X \) whose 1st and 0th faces are both \( \bullet \) is the degenerate one at the base-point.

2. Sequential spectra as oplax limits, \( \Omega \)-spectra as pseudo-limits

Set \( X : \mathbf{Z}_{\leq 0} \to \text{CAT} \) to be the diagram which sends each \(-n\) to \( \hat{\Delta}_n \) and each \(- (n+1) \to -n\) to \( \Omega_K \).

\[
\begin{array}{ccc}
\cdots & \xrightarrow{\Omega_K} & \hat{\Delta}_n \\
& \xrightarrow{\Omega_K} & \hat{\Delta}_n
\end{array}
\]

As we stated in the introduction, \( \lim \) is \( \Omega \)-Sp \( (\hat{\Delta}_n , \Sigma_K \to \Omega_K) \) and \( \lim \) is \( \text{Sp} \left( \hat{\Delta}_n , \Sigma_K \to \Omega_K \right) \), the usual categories of \( \Omega \)-spectrum and spectrum objects respectively (this is treated in detail
in Example [A.3]). Then, for familiar reasons (or alternatively our weighted limit treatment of spectrification, Theorem [A.1]) we get a spectrification adjunction

\[ \lim X \rightarrow \operatorname{oplaxlim} X \]

We recall that the category \( \hat{\Delta} \) admits the pointed test model structure Quillen equivalent to pointed spaces with the classical model structure. Therefore, seeing as \( \operatorname{oplaxlim} X \) is equivalent to the category of sequential spectrum objects valued in \( \hat{\Delta} \) with respect to the adjunction \( \Sigma K \dashv \Omega K \), we may equip \( \operatorname{oplaxlim} X \) with Hovey’s stable model structure.

Then, since we have homeomorphisms \( |\Sigma K X| \sim \Sigma |X| \) for all pointed simplicial sets \( X \), it follows that Hovey’s stable model structure on \( \operatorname{oplaxlim} X \) is Quillen equivalent to the Bousfield-Friedlander model structure on sequential spectra, whence it presents the \((\infty, 1)\)-category of spectra. Kan’s observation, as we will elaborate below in the next section, is that by having chosen a particular presentation of the suspension, i.e. \( \Sigma K \), the category \( \operatorname{oplaxlim} X \) admits an elegant description as a subcategory of a category of pointed presheaves.

3. FACTORIZATION THROUGH \( [\_, \text{Set}_\bullet]_\bullet : \text{Cat}^{\text{coop}} \rightarrow \text{CAT and the Collage Construction} \)

Let

\[ \hat{K} : Z_{\leq 0} \rightarrow \text{CAT} \]

be the diagram which sends each \( -n \) to \( \hat{\Delta} \) and each \( -(n + 1) \rightarrow -n \) to the functor

\[ K^\bullet : \hat{\Delta} \rightarrow \hat{\Delta} \]

\[ X \rightarrow \hat{\Delta} \left( \Delta K (\_ + , X) \right) \]

We observe that this functor factors through \( \text{Cat}^{\text{coop}} \) as follows

\[ \begin{array}{ccc}
Z_{\leq 0} & \xrightarrow{\sim} & \hat{\Delta} \\
\downarrow & & \downarrow \hat{K} \\
Z^{\text{coop}}_{\geq 0} & \xrightarrow{\sim} & \hat{\Delta} \\
\downarrow K^{\text{coop}} & & \downarrow \text{[\_, Set\_]} \\
\text{Cat}^{\text{coop}} & & \\
\end{array} \]

where \( K : Z_{\geq 0} \rightarrow \text{Cat} \) is the functor which sends each \( n \) to \( \Delta \) and each \( n \rightarrow (n + 1) \) to \( K \).

Since the functor \( [\_, \text{Set}_\bullet] : \text{Cat}^{\text{coop}} \rightarrow \text{CAT} \) sends conical, pseudo, lax and oplax colimit to conical, pseudo, lax and oplax limits respectively it follows that we have further isomorphisms

\[ \begin{array}{ccc}
\lim \hat{K} & \sim & \lim K^\bullet \\
\downarrow & & \downarrow \\
\text{pslim} \hat{K} & \sim & \text{pslim} K^\bullet \\
\downarrow & & \downarrow \\
\text{oplaxlim} \hat{K} & \sim & \text{oplaxlim} K^\bullet \\
\end{array} \]
What’s more, since $K^*$ is an iso-fibration, it follows that $\widehat{K} : Z_{\leq 0} \to \text{CAT}$ is fibrant in the injective-canonical model structure, so we have isomorphisms

$$\lim \widehat{K} \sim \pslim \widehat{K} \sim \pslim \lim \sim \lim \widehat{K}.$$  

The categories $\text{oplaxlim} K$ and $\lim K$ admit explicit descriptions. Recall that the collage construction presents the oplax colimits of an arrow in $\text{Prof}$. Iterating this construction gives us a presentation of the category $\text{oplax} K$. Indeed, set $\triangle_{\text{coll}}$ to be the category generated by

$$\coprod_{m \in Z_{\leq 0}} \triangle$$

together with

$$\triangle_{\text{coll}} (([n], -m), ([\ell], -(m + k))) = \triangle ([n + k], [\ell])$$

and see that $\triangle_{\text{coll}}$ enjoys the universal property of $\text{oplaxlim} K$.

**Remark 3.1.** The category $\triangle_{\text{coll}}$ defined above can also be described as being generated by the diagram

```
\[\begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & 2 & 2 \\
  \vdots & 1 & 1 \\
  \vdots & 0 & 0 \\
  \vdots & \vdots & \vdots \\
\end{array}\]
```

whose vertical maps are the usual face and degeneracy maps, subject to the usual identities, and whose diagonal maps act so that the squares

\[
\begin{array}{ccc}
  [n + 2] & \to & [n + 2] \\
  d^i & \downarrow & s^j \\
  [n + 1] & \to & [n + 1] \\
  d^i & \downarrow & s^j \\
  [n] & \to & [n] \\
\end{array}
\]

\[
\begin{array}{ccc}
  [n + 2] & \to & [n + 2] \\
  s^j & \downarrow & s^j \\
  [n + 1] & \to & [n + 1] \\
  d^i & \downarrow & d^i \\
  [n] & \to & [n] \\
\end{array}
\]

\[\text{as it did in for } X \text{ in the previous example.}\]
The universal property of the strict colimit $\lim \rightarrow K$ is similarly enjoyed by $\Delta_{st}$, the category whose objects are in bijection with the set $\mathbb{Z}$,

$$\text{Ob} (\Delta_{st}) = \{ [z] | z \in \mathbb{Z} \}$$

with Hom-sets generated by maps

$$\{ d^i : [m - 1] \to [m] | i \in \mathbb{N} \}$$

and

$$\{ s^j : [m + 1] \to [m] | j \in \mathbb{N} \}$$

subject to (un-bounded) simplicial identities. One gets to the category $\lim \rightarrow K$ from the category $\text{oplaxlim} \rightarrow K$ by identifying objects along the diagonal maps of $\Delta_{coll}$. Indeed, this defines a functor $\rho : \Delta_{coll} \to \Delta_{st}$, whence an adjoint triple

$$\Delta_{coll} \overset{\rho_!}{\longrightarrow} \Delta_{st} \overset{\rho_*}{\longrightarrow} \Delta_{coll}$$

of categories of pointed presheaves.

Now, since $\Sigma_K$ is a quotient of $K_\bullet = \text{Lan}_\ast (\ast,K)$, we have oplax squares

$$\hat{\Delta} \overset{\Delta_{coll} \ast}{\longrightarrow} \hat{\Delta} \overset{id}{\downarrow} \hat{\Delta} \overset{\Sigma_K}{\longleftarrow} \hat{\Delta}$$

whence, taking mates, we have oplax squares

$$\hat{\Delta} \overset{\Delta_{coll} \ast}{\longrightarrow} \hat{\Delta} \overset{\Omega_K}{\downarrow} \hat{\Delta} \overset{id}{\longleftarrow} \hat{\Delta}$$

These squares can be pasted together providing an oplax natural transformation $X \overset{\text{oplax}}{\longrightarrow} \hat{K}$, whence we have a functor $\text{oplaxlim} X \to \text{oplaxlim} \hat{K}$. But, since $\text{oplaxlim} \hat{K}$ is a category of pointed presheaves, this map admits a more explicit description. Indeed that functor is a restricted Yoneda embedding, or nerve.
Observe that, for each \(-n \in \mathbb{Z}_{\leq 0}\), there is a pseudo-cone

where the instance of \(\mathbb{k}_n\) targets the \(-n^{th}\) copy of \(\triangle_n\).

These pseudo-cones define maps

\[
\Phi_{[-n, \infty)} : \triangle_n \rightarrow \text{pslim} X \xrightarrow{\sim} \text{lim} X
\]

**Remark 3.2.** In the perhaps more familiar notation of sequential spectrum objects the maps are described by the formula

\[
\left\{ \begin{array}{ll}
\Phi_{[-n, \infty)} (\triangle [r])_{-i} = \\
\Sigma_K (i-n) \triangle [r]_+ & i \geq n
\end{array} \right.
\]

We observe then that the diagrams
where the 2-cells are de-suspensions and suspensions of the quotient $K_\bullet \Rightarrow \Sigma_K$, induce an oplax co-cone

$$\text{lim} X \rightarrow \text{pilim} X \rightarrow \text{oplaxlim} X$$

whence a map

$$\Phi : \text{oplaxlim} X \rightarrow \text{pilim} X \rightarrow \text{lim} X$$

We may of course compose $\Phi$ with the reflective subcategory

$$\text{sp}_X \dashv \Delta_X : \text{lim} X \rightleftharpoons \text{oplaxlim} X$$
That composition $\Delta_X \Phi : \text{oplaxlim} K \longrightarrow \text{oplaxlim} X$ induces an adjoint triple.

As a note before moving on: we observe that $\Delta_X \Phi^* : \text{oplaxlim} K \longrightarrow \text{oplaxlim} X$ recovers the oplax limit $\text{oplaxlim} X$. Composing the adjunction $\Delta_X \Phi^* - \dashv \Delta_X \Phi!$ above with the adjunction $\rho^* - \dashv \rho_!$ provides a third adjunction

4. Double Duty and Slick Tricks

We will be concerned with two interesting and important subcategories of $\Delta_{st}$. Both are full subcategories on colimits of objects of specific forms.

**Definition 4.1.** For each $n \in \mathbb{N}$ and $z \in \mathbb{Z}$, define the pointed presheaf $\Delta_{st}[z]/d^{>n} =^*$ by the following pushout in $\Delta_{st}$.

Similarly, for each $n \in \mathbb{N}$ and $z \in \mathbb{Z}$, let

$S^{(z-n)} [n] = \Delta_{st}[z]/d^n.d^{n-1}...d^1.d^0 = d^{>n} =^*$

be defined by the following pushout in $\Delta_{st}$.

We define $\text{LocFin} (\Delta_{st})$ to be the full subcategory of $\Delta_{st}$ on objects which are colimits of objects of the form $\Delta_{st}[z]/d^{>n} =^*$ and we define $\text{LocSph} (\Delta_{st})$ to be the full subcategory of $\Delta_{st}$ on colimits of objects of the form $S^{(z-n)} [n]$.

**Remark 4.2.** As we discussed in the introduction, the pointed presheaf $\Delta_{st}[z]/d^{>n} =^*$ is a stable $[z]$-simplex with precisely $n$-many codimension 1 faces and the pointed presheaf $S^{(z-n)} [n]$ is a stable $[z]$-simplex which is, in essence, an $[n]$-simplex worth of $(z-n)$-spheres.
It is immediate, as observed in [ChenKrizPultin], that $\text{LocFin}(\Delta_{st})$ is a coreflective subcategory of $\hat{\Delta}_{st}$ with the right adjoint functor picking out, for each presheaf, the sub-object of cells of the form $\Delta_{st}^{[i]}/d^{\geq n}$. The same argument provides that $\text{LocSph}(\Delta_{st})$ is coreflective in $\text{LocFin}(\Delta_{st})$.

$$\hat{\Delta}_{st} \xhookrightarrow{\perp} \text{LocFin}(\Delta_{st}) \xleftarrow{\perp} \text{LocSph}(\Delta_{st}) \xleftarrow{\perp} \text{Set}_{\ast}$$

Now, as we will show, every presheaf $Z : \Delta_{st}^{\text{op}} \to \text{Set}_{\ast}$ which is $\rho_{\ast} : (\Delta_X \Phi)^{\ast}$ of something satisfies an important and interesting property - it is a colimit of pointed presheaves $S^{(z-n)} [n]$. Every object of the category $\operatorname{op laxlim} X$ is canonically the colimit of objects of the form

$$\text{in}_X \Phi (\Delta_{\text{coll}} ([n], -k))$$

as any simplex in any index gives rise to such. It then follows that, since $\rho_{\ast} : (\Delta_X \Phi)^{\ast}$ preserves colimits, every $Z : \Delta_{st}^{\text{op}} \to \text{Set}_{\ast}$ which is $\rho_{\ast} : (\Delta_X \Phi)^{\ast}$ of something is a colimit of objects of the form

$$\rho_{\ast} : (\text{in}_X \Phi)^{\ast} \circ \text{in}_X \Phi (\Delta_{\text{coll}} ([n], -k))$$

As it turns out however, these are precisely the stable simplicial sets $S^{(z-n)} [n]$.

**Lemma 4.3.** For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\rho_{\ast} : (\Delta_X \Phi)^{\ast} \circ (\Delta_X \Phi) (\Delta_{\text{coll}} ([n], -k)) = S^{-k} [n]$$

**Proof.** Consider the object $(\Delta_X \Phi)^{\ast} \circ (\Delta_X \Phi) (\Delta_{\text{coll}} ([n], -k))$ in the sequential notation.

$$(\Delta_X \Phi)^{\ast} \circ (\Delta_X \Phi) (\Delta_{\text{coll}} ([n], -k))_{-\ell} = \begin{cases} \bullet & \ell < k \\
\Sigma_{K}^{(\ell-k)} [n]_{+} & \ell \geq k \end{cases}$$

We’ve thus the promised (in the introduction) composition of adjunctions

$$\hat{\Delta}_{st} \xhookrightarrow{\perp} \text{LocFin}(\Delta_{st}) \xleftarrow{\perp} \text{LocSph}(\Delta_{st}) \xleftarrow{\perp} \text{Set}_{\ast}$$

**Example 4.4.** Now, Kan claims that the composite adjunction

$$\text{LocFin}(\Delta_{st}) \xleftarrow{\perp} \text{LocSph}(\Delta_{st}) \xleftarrow{\perp} \text{Set}_{\ast}$$

is a reflective subcategory, by asserting the co-unit to be a natural isomorphism, whence asserting the right adjoint functor to be full-and-faithful. It’s not hard to show however that $\Delta_{st}^{[i]}/d^{\geq n} \ast$ is in $\text{LocFin}(\Delta_{st})$ but not in $\text{LocSph}(\Delta_{st})$, so the component of co-unit of the adjunction at the object $\Delta_{st}^{[i]}/d^{\geq n} \ast$ is not the identity.

The right hand adjunction however is, by Kan’s observation, a reflective subcategory.

**Lemma 4.5.** The co-unit of the adjunction

$$\text{LocSph}(\Delta_{st}) \xleftarrow{\perp} \text{Set}_{\ast} \xleftarrow{\perp} \text{LocFin}(\Delta_{st})$$

is a natural isomorphism, whence $\text{LocSph}(\Delta_{st}) \to \text{Set}_{\ast}$ is full-and-faithful.
Proof. This can be shown by direct computation. Suppose \( Z : \Delta^\text{op}_{\text{st}} \to \text{Set}_* \) to be of \( \text{LocSph}(\Delta_{\text{st}}) \). Then the pointed simplicial sets \( \text{KPs}(Z) \) may be described as follows

\[
\text{KPs}(Z) : \Delta^\text{op}_{\text{st}} \to \text{Set}_ *
\]

\[
[n] \to \Delta_{\text{st}*}(S^{-i}[n], Z)
\]

so it is clear enough that the co-unit is a natural isomorphism.

\( \square \)

Remark 4.6. For the above to really be as clear as we assert, it is important to note that there are quotient relationships between the spheres \( S^{-k}[n] \). For example \( S^1[1] \to S^2[0] \) is a quotient.

As it turns out however, in the next section, we’ll show that this subtle error of Kan’s, repeated in a slightly different guise in \[\text{ChenKrizPultr}\], ends up irrelevant once we go from categories to model categories.

5. Homotopy-coherent models of...?

In 1973 \[\text{Brown}\] synthesized Kan’s treatment of spectra with the then new notion of model categories (really with the notion of categories of fibrant objects also developed in \[\text{Brown}\]).

Definition 5.1. (\[\text{Brown}\]) Let \( z \in \mathbb{Z}, n \in \mathbb{N} \) and \( 0 \leq i \leq n \) be given and set

\[
\Lambda_{\text{Brown}} = \left\{ \Delta_{\text{st}*}(z)[i]/d^{>n} \to \Delta_{\text{st}*}[i]/d^{>n} \mid z \in \mathbb{Z}, n \in \mathbb{N}, 0 \leq i \leq n \right\}
\]

and

\[
\partial_{\text{Brown}} = \left\{ \partial \Delta_{\text{st}*}(z)[i]/d^{>n} \to \Delta_{\text{st}*}[i]/d^{>n} \mid z \in \mathbb{Z}, n \in \mathbb{N} \right\}
\]

comprise a set of generating acyclic cofibrations and generating co-fibration for a model structure on \( \text{LocFin}(\Delta_{\text{st}}) \). More:

- This model structure is the left transfer of a pointed Cisinski model structure on \( \Delta_{\text{st}*} \);
- The adjunction

\[
\text{LocFin}(\Delta_{\text{st}}) \xrightarrow{\text{KSp}} \text{op laxlim} \xleftarrow{\text{KPs}}
\]

where the right hand side is given the stable model structure is a Quillen equivalence.

Proof. See Theorem 5 and Proposition 7 of \[\text{Brown}\].

This model structure can further be left transported to \( \text{LocSph}(\Delta_{\text{st}}) \).
Definition 5.3. Let \( z \in \mathbb{Z} \), \( n \in \mathbb{N} \), and \( 0 \leq i \leq n \) be given and set
\[
\Lambda_{S}^{i[z]/d^{>n}} \hookrightarrow S^{(z-n)}[n]
\]
to be defined by way of the epi-mono factorization
\[
\begin{array}{ccc}
\Lambda_{B}^{i[z]/d^{>n}} & \rightarrow & \Delta_{st}^{i[z]/d^{>n}} \\
\text{epi} & & \text{mono} \\
\Lambda_{S}^{i[z]/d^{>n}} & \hookrightarrow & S^{(z-n)}[n]
\end{array}
\]
Let \( \Lambda_{\text{Spherical}} \) be the following set of monomorphisms.
\[
\Lambda_{\text{Spherical}} = \left\{ \Lambda_{S}^{i[z]/d^{>n}} \hookrightarrow S^{(z-n)}[n] \mid z \in \mathbb{Z}, n \in \mathbb{N}, 0 \leq i \leq n \right\}
\]
Similarly we define
\[
\partial S^{(z-n)[n]} \hookrightarrow S^{(z-n)}[n]
\]
by the epi-mono factorization
\[
\begin{array}{ccc}
\partial \Delta_{st}^{i[z]/d^{>n}} & \rightarrow & \Delta_{st}^{i[z]/d^{>n}} \\
\text{epi} & & \text{mono} \\
\partial S^{(z-n)}[n] & \hookrightarrow & S^{(z-n)}[n]
\end{array}
\]
and we define the set \( \partial_{\text{Spherical}} \) as follows.
\[
\partial_{\text{Spherical}} = \left\{ \partial S^{(z-n)}[n] \hookrightarrow S^{(z-n)}[n] \mid z \in \mathbb{Z}, n \in \mathbb{N} \right\}
\]
Corollary 5.4. The sets \( \Lambda_{\text{Spherical}} \) and \( \partial_{\text{Spherical}} \) comprise a set of generating acyclic cofibrations and generating cofibrations for the left transport of the model category on \( \text{LocFin}(\Delta_{st}) \) of Theorem 5.2 along the adjunction
\[
\text{LocFin}(\Delta_{st}) \rightleftharpoons \text{LocSph}(\Delta_{st})
\]
Moreover the the Quillen adjunction adjunction is a Quillen equivalence.

Proof. To prove that the model structure generated by \( \Lambda_{\text{Spherical}} \) and \( \partial_{\text{Spherical}} \) coincides with the left transferred model structure is to prove, for \( f : X \rightarrow Y \) in \( \text{LocSph}(\Delta_{st}) \), that \( f \in \text{Cell}(\Lambda_{\text{Spherical}}) \) if and only if \( f \in \text{Cell}(\Lambda_{\text{Brown}}) \) and likewise for \( \partial_{\text{Spherical}} \) and \( \partial_{\text{Brown}} \).
Since the inclusion functor is left adjoint, and both \( \Lambda_{\text{Spherical}} \) and \( \partial_{\text{Spherical}} \) lie in \( \text{Cell}(\Lambda_{\text{Brown}}) \) and \( \text{Cell}(\partial_{\text{Brown}}) \) respectively, the “if direction” is clear. Conversely, since membership in \( \text{LocSph}(\Delta_{st}) \) is defined simplex-wise it follows by a simple factorization argument that any \( f \) from \( \text{LocSph}(\Delta_{st}) \) which lies in \( \text{Cell}(\Lambda_{\text{Brown}}) \) and \( \text{Cell}(\partial_{\text{Brown}}) \) actually lies in \( \text{Cell}(\Lambda_{\text{Spherical}}) \).
or $\text{Cell}(\partial_{\text{Spherical}})$ respectively. Indeed, it’s clear that we have factorizations as below.

\[
\begin{align*}
\Lambda^i B \triangleleft & \st[z]/d^{>n} = * \quad \partial \st[z]/d^{>n} = * \\
\Lambda^i S^{(z-k)}[k], \partial S^{(z-k)}[k] & \longrightarrow \st[z]/d^{>n} = * \\
S^{(z-k)}[k] & \text{ for different values of } z \text{ and } n.
\end{align*}
\]

That the Quillen adjunction is in fact a Quillen equivalence follows from the determination of weak-equivalences by their inducement of isomorphisms on stable homotopy groups, which are determined as homotopy classed of maps from objects in $\text{LocSph}(\st)$ - consider that

\[
\pi_z(\_ ) = [S^z[0], \_] .
\]

\[
\square
\]

Having found model categories for spectra which admit interpretation as homotopy coherent models of a limit theory, both the model structure on $\text{LocSph}(\st)$ and $\text{LocFin}(\st)$ permit this, we are wont to ask - what is that limit theory a description of?

6. A Globular Perspective on Kan’s Suspension Functor

Kan’s model structure is intuitively interpreted as describing homotopy coherent groupoids. However, in this interpretation one usually thinks of every simplex merely as composition datum - unless one is accustomed to thinks in terms of higher categorical nerves and oriented simplices. As such it may not be clear how to relate the notions of composition embodied by lifting against

\[
\Lambda^i S^{(z-k)}[k], \partial S^{(z-k)}[k] \longrightarrow \st[z]/d^{>n} = *
\]

for different values of $z$ and $n$. We contend that making the globular content more explicit makes things a bit more clear.

Since the simplex category $\Delta$ is a full subcategory of the category $\text{Cat}$ of small categories we may think of Kan’s suspension as taking each simplex, thought of as a 1-category, and returning the 2-category with a single object and that original simplex as its unique $\text{Hom}$-category. In this view, Kan’s suspension is but the simplicial avatar of the endo-functor

\[
\Sigma_\omega : \text{Str}-\omega-\text{Cat}_* \longrightarrow \text{Str}-\omega-\text{Cat}_*
\]

which takes an $\omega$-category and returns the $\omega$-category with a single object and that original $\omega$-category as its unique $\text{Hom}$-$\omega$-category.

**Remark 6.1.** Leaving simplicial sets and going to globular ones is not of course the only way to see this. Indeed we could, and probably should, think of Kan’s suspension $\Sigma_K$ as an endo-functor on strict $\omega$-categories as complicial sets, which sends an $[n]$-simplex to a thick $[n+1]$-simplex $\Sigma_K[n+1]$. Had this author known of that material prior to developing this work, he might well have gone in that direction instead.
Remark 6.2. Importantly, this perspective is compatible with the interpretation of Kan’s model structure as providing a description of homotopy coherent \( \omega \)-groupoids, as a composition datum for \( n \)-many 1-cells passes to composition datum for \( n \)-many 2-cells.

Let \( \Omega_\omega \) denote the functor right adjoint to \( \Sigma_\omega \). The diagram which begat sequential spectra should then be understood an avatar of the diagram

\[
\cdots \to \text{Str-\( \omega \)-Cat}_\bullet \overset{\Omega_\omega}{\to} \text{Str-\( \omega \)-Cat}_\bullet \overset{\Omega_\omega}{\to} \text{Str-\( \omega \)-Cat}_\bullet
\]

and the diagram

\[
\cdots \to \Delta_\bullet \overset{K^\ast}{\to} \Delta_\bullet \overset{K^\ast}{\to} \Delta_\bullet
\]

should be considered as but the pointing of an avatar of the diagram

\[
\cdots \to \text{Str-\( \omega \)-Cat} \overset{S^\ast}{\to} \text{Str-\( \omega \)-Cat} \overset{S^\ast}{\to} \text{Str-\( \omega \)-Cat}
\]

where \( S^\ast \) is the functor right adjoint to the functor \( S : \text{Str-\( \omega \)-Cat} \to \text{Str-\( \omega \)-Cat} \) which sends an \( \omega \)-category \( X \) to the \( \omega \)-category with two objects 0 and 1 with \( \text{Hom}(0, 1) = X \) and all other \( \text{Hom} \)-categories empty.\(^{11}\)

7. \textit{Z-categories}

7.1. \textbf{Strict \( \omega \)-categories and Berger’s Cellular Nerve.}

Definition 7.1. Let \( G \) denote the \textbf{globe category}, the category

\[
\begin{align*}
\begin{array}{c}
0 \quad s \\
\bowtie \quad \Upsilon \\
\cdots \quad \cdots
\end{array}
\end{align*}
\quad \begin{array}{c}
s \circ t = s \circ s \\
t \circ t = t \circ s
\end{array}
\]

Let the category of \textbf{globular sets} be the presheaf category \( \hat{G} \) and let \( T : \hat{G} \to \hat{G} \) be the monad whose algebras are strict-\( \omega \)-categories.

It was first proved in [Berger1] that strict \( \omega \)-categories are a reflective, indeed orthogonal, subcategory of the category of cellular sets \( \Theta \). We repeat the treatment of the category \( \Theta \) developed first in [Berger2], however our presentation borrows more from [CisinskiMaltsiniotis].

7.1.1. \textit{Segal’s category} \( \Gamma \). Segal’s category \( \Gamma \) is a skeleton - in the sense of a small category of chosen representatives for each isomorphism class - for the opposite category of the category of finite pointed sets.

Definition 7.2. Let \( \Gamma \), \textbf{Segal’s gamma category}, be the category specified thus: let

\[
\text{Ob}(\Gamma) = \{ (k) = \{1, \ldots, k\} | k \geq 1 \} \cup \{ (0) = \emptyset \},
\]

and let \( (\langle n \rangle, \langle m \rangle) \) be defined by the expression

\[
\Gamma ((\langle n \rangle, \langle m \rangle)) = \{ \varphi : \langle n \rangle \to \text{Sub}_{\text{Set}}(\langle m \rangle) \} \forall i \neq j \in \langle m \rangle, \varphi(i) \cap \varphi(j) = \emptyset
\]

where, for any category \( A \) and object \( a \) thereof, \( \text{Sub}_A(a) \) is the category of subobjects of \( a \). Define the composition of morphisms in \( \Gamma \) by setting

\[
\langle \ell \rangle \varphi \to \langle m \rangle \sigma \to \langle n \rangle
\]

to be the map

\[
\sigma \circ \varphi : i \mapsto \bigcup_{j \in \varphi(i)} \sigma(j).
\]

\(^{11}\)For those already familiar with Berger’s wreath notation, we might simply say the \( \omega \)-category \([1];(X)\).
Remark 7.3. The equivalence of categories between $\Gamma$ and $\text{FinSet}^{\text{op}}$ is a 0-truncated analogue of the Grothendieck construction - a map of finite pointed sets is replaced with the data of the fibres it parameterizes.

7.1.2. Berger’s Categorical wreath product.

Definition 7.4. Let $A$ and $B$ be small categories. Given a functor $G : B \longrightarrow \Gamma$, we define $B \int_G A = B \int A$ (with the second notation suppressing the functor $G$ when the meaning is clear) be the category the objects of which are

$$b; (a_1, \ldots, a_m)$$

where $b$ is an object of $B$, $G (b) = \langle m \rangle$, and $(a_1, \ldots, a_m)$ describes a function $G (b) \longrightarrow \text{Ob} (A)$. The morphisms of $B \int A$, denoted

$$g; f : b; (a_i)_{i \in G(b)} \longrightarrow d; (c_i)_{i \in G(d)}$$

are constituted of a morphism

$$g : b \longrightarrow d$$

of $B$ and a morphism of $\hat{A}$,

$$f = \left( (f_{ji} : a_i \rightarrow c_j)_{j \in G(g)(i)} \right)_{i \in G(b)} : \prod_{i \in G(b)} A^{a_i} \longrightarrow \prod_{j \in G(g)(i)} A^{c_j}$$

The composition

$$b; (a_i)_{i \in G(b)} \xrightarrow{g; f} d; (c_i)_{i \in G(d)} \xrightarrow{r; q} \ell; (k_i)_{i \in G(\ell)}$$

is denoted $r \circ g; q \circ f$ where the meaning of $r \circ g$ is clear and

$$q \circ f = \left( (q_{jk} \circ f_{ki})_{j \in G(g(r)(\ell)(i))} \right)_{i \in G(b)}$$

with the varying values for $k \in G (d)$ in the expression being those unique $k$ in $G (g) (i)$ such that $j \in G (r) (k)$.

Remark 7.5. We’ll also use the more explicit notation $g; (f_1, f_2, \ldots, f_n)$ where doing so simplifies the exposition.

Example 7.6. Define the functor $F : \triangle \longrightarrow \Gamma$ by setting

$$F ([n]) = \langle n \rangle$$

and setting for each $\varphi : [m] \longrightarrow [n]$,

$$F (\varphi) : \langle m \rangle \longrightarrow \langle n \rangle$$

to be the function

$$F (\varphi) : \langle m \rangle \longrightarrow \text{Sub}_{\text{Set}} (\langle n \rangle)$$

given thus:

$$F (\varphi) (i) = \{ j | \varphi (i - 1) < j \leq \varphi (i) \}$$
7.1.3. The Categories $\Theta$ and $\Theta_n$.

**Definition 7.7.** Let
\[
\gamma : \Delta \to \Delta \int \Delta
\]
be the obvious functor extending the assignment $\gamma([n]) = [n]; ([0], \ldots, [0])$. We define the categories $\Theta_n$ to be the $n$-fold wreath product of $\Delta$ with itself, i.e. we set
\[
\Theta_n = \Delta \int \left( \cdots \int \Delta \right)
\]
We set $\Theta$ to be the conical colimit\footnote{An elementary argument about the projective-canonical and Reedy-canonical model structures on $\text{CAT}(\mathbb{N}, \text{Cat})$ provides that this conical colimit enjoys the universal property of the pseudo-colimit.}
\[
\lim \left\{ \Delta \xrightarrow{\gamma} \Delta \int \Delta \xrightarrow{\text{id}} \cdots \right\}
\]

**Remark 7.8.** It should also be noted that
\[
\Theta \xrightarrow{\gamma} \Delta \int \Theta \xrightarrow{\gamma} \Delta \int \Delta \int \Theta \xrightarrow{\gamma} \cdots
\]
so we may denote cells - where cells are the objects of $\Theta$ - in many compatible ways. For example for any $T$ a cell of $\Theta$ we may also write $T = [n]; (T_1, \ldots, T_n)$ for some unique $n \in \mathbb{N}$ and unique $T_1, \ldots, T_n$ cells of $\Theta$.

7.1.4. Reedy Structures on $\Theta_n$ and $\Theta$. It is important to note that the categories $\Theta$ and $\Theta_n$ are Reedy categories. This fact was first proved in \cite{Berger1} and again by more general means in \cite{BergnerRezk}.

**Definition 7.9.** Inductively, we define
\[
\lambda_{\Theta_{n+1}} : \text{Ob} (\Theta_{n+1}) \to \mathbb{N}
\]
by setting
\[
\lambda_{\Theta_{n+1}} ([m]; (T_1, \ldots, T_m)) = m + \sum_{i=1}^{m} \lambda_{\Theta_n} (T_i)
\]
with $\lambda_{\Theta_1} = \lambda_{\Delta}$. The direct (wide) subcategories $\Theta_n^+, \Theta^+$ are subtended by the morphisms $f : S \to T$ which pass to monomorphisms of presheaves under $\mathcal{K}$. Similarly the inverse (wide) subcategories $\Theta_n^-, \Theta^-$ are subtended by the morphisms of $\Theta_n$ (resp. $\Theta$) which pass to epimorphisms under $\mathcal{K}$.

**Lemma 7.10.** (Lemma 2.4(a) of \cite{Berger1}) The datum $(\Theta_n, \Theta_n^+, \Theta_n^-, \lambda_n)$ (resp. $(\Theta, \Theta^+, \Theta^-, \lambda)$) comprises a Reedy structure.
7.1.5. Inner horns and $\omega$-categories, outer horns and $\omega$-groupoids.

**Definition 7.11.** Recall that a codimension 1 face $d : [m] \to [m+1]$ of is inner if it preserves top and bottom elements and outer if it does not. Inductively, we may then define **inner faces** of $\Theta_n$ as those maps

$$f; (g) : [m]; (S_1, \ldots, S_m) \to [n]; (T_1, \ldots, T_n)$$

for which:

- $f$ is inner for $\Delta$;
- and each component $g_{ji} : S_i \to T_j$ of $g$ is inner for $\Theta_{n-1}$.

Given a face $\kappa : S \to T$, let $\Lambda^\kappa [T] \subset \Theta [T]$ be the colimit

$$\varprojlim_{\Theta \downarrow [T]\to \Theta \downarrow [T]} \Theta [S] \cdot \kappa.$$

Then the canonical inclusion $\Lambda^\kappa [T] \to \Theta [T]$ is to be called an an **inner** (resp. outer) horn of $\Theta [T]$ if $\kappa$ is an inner (resp. outer) face of $T$. Let

$$\Lambda_{\text{InnerBerger}} = \{ \Lambda^\kappa [T] \to \Theta [T] | \kappa \in \text{InnerFace}(T), T \in \text{Ob}(\Theta) \}$$

and let

$$\Lambda_{\text{Berger}} = \{ \Lambda^\kappa [T] \to \Theta [T] | \kappa \in \text{Face}(T), T \in \text{Ob}(\Theta) \}.$$

**Proposition 7.12.** ([Berger1]) An $n$-cellular set (resp. cellular set) is the nerve of a strict $n$-category (resp. $\omega$-category) if and only if it is right orthogonal to the regulus $\Lambda_{\text{InnerBerger}}$. Similarly, an $n$-cellular set (resp. cellular set) is the nerve of a strict $n$-groupoid (resp. strict $\omega$-groupoid) precisely if it is right orthogonal to the regulus $\Lambda_{\text{Berger}}$.

7.2. $Z$-categories.

**Definition 7.13.** Denote by $S$ the functor

$$S : G \to G$$

This functor induces $S^* : \hat{G} \to \hat{G}$ which “pulls down” $n$-cells as $(n-1)$-cells and forgets the preexisting 0-cells. This operation preserves $T$-algebras; $S^*$ induces a functor $S^* : \hat{G}^T \to \hat{G}^T$.

What then we ask is an object of the limit

$$\varprojlim \{ \cdots \to \text{Str-$\omega$-Cat} \xrightarrow{S^*} \text{Str-$\omega$-Cat} \xrightarrow{S^*} \text{Str-$\omega$-Cat} \}$$

which we consider as the limit of the diagram $S^* : Z_{\leq 0} \to \text{CAT}$ which sends each $-n$ to $\hat{G}^T$ and each $-(n+1) \to -n$ to $S^*$.

$$\cdots \xrightarrow{S^*} \hat{G}^T \xrightarrow{S^*} \hat{G}^T$$

Now, Eilenberg-Moore objects for a monad $(T, \mu, \eta)$ in a 2-category $A$ (here $\text{CAT}$) are but lax-limits over the diagram $[T] : 2 \to A$, which classifies the underlying endo-functor. Then, since the taking of limits commutes with the taking of limits, we find that there exists some monad $\varprojlim T$ for which we have an isomorphism

$$\varprojlim (\hat{G}^T) \sim \varprojlim \hat{G}^{\varprojlim T}.$$
More, we have an isomorphism

\[ \lim \tilde{G} \sim \tilde{G}_Z \]

with \( G_Z \) defined as follows.

**Definition 7.14.** Let \( G_Z \) be the category

\[ \langle \cdots \xrightarrow{s} \xrightarrow{t} \xrightarrow{s} \xrightarrow{t} \xrightarrow{s} \cdots \mid s \circ t = s \circ s \quad t \circ t = t \circ s \rangle \]

Using the generalized spectrification of Theorem \[A.1\] we observe that the monad \( \lim T \), which we’ll denote now by \( T_Z \), is but \( \text{sp} \circ \text{oplaxlim} \) where \( \text{sp} \) is the left adjoint to the inclusion

\[ \lim \{ \cdots S \xrightarrow{\tilde{G}^T} S \xrightarrow{\tilde{G}^T} \} \longrightarrow \text{oplaxlim} \{ \cdots S \xrightarrow{\tilde{G}^T} S \xrightarrow{\tilde{G}^T} \} \]

**Definition 7.15.** Let the category \( \tilde{G}_Z^{T_Z} \) be called the 1-category of strict-\( Z \)-categories.

**Part 2. Limits in the 2-category of Categories with Arities**

In Proposition \[7.12\] above we recalled Berger’s characterization of strict \( \omega \)-categories and strict \( \omega \)-groupoids as cellular sets satisfying horn-filling conditions. In this second part we develop theory enough to allow us to extend this result to characterize \( Z \)-categories as an orthogonal subcategory of a category of \( Z \)-cellular sets. We do this by developing the theory of limits in Berger, Mellies and Weber’s 2-category with arities and Bourke and Garner’s synthesis of that work with Day’s density presentations.

**8. Basic definitions: Variations on the 2-category \text{CwA}**

In \[BergerMelliesWeber\] we find the following 2-category defined.

**Definition 8.1.** Let \( \text{CwA} \), the 2-category of categories with arities, be the 2-category with:

- **CWA0:** 0-cells are locally small categories \( E \) together with a dense, small, full subcategory \( A \xrightarrow{I} E \), denoted \( (A, E, I) \);
- **CWA1:** 1-cells \( (A, E, I) \longrightarrow (B, F, J) \), so called arities-respecting functors, are functors \( K : E \longrightarrow F \) such that the squares

\[ \begin{array}{ccc}
E & \xrightarrow{\mathcal{N}_I} & \hat{A} \\
\downarrow K & & \downarrow \mathcal{P} \\
F & \xrightarrow{\mathcal{N}_J} & \hat{B}
\end{array} \]
pseudo-commute, where \( \overline{K} \) is (and will remain by convention - at least most of the time) the following left Kan extension

\[
\begin{array}{cccccc}
A & \xrightarrow{I} & E & \xrightarrow{\mathcal{N}_I} & \widehat{A} \\
\downarrow & & \downarrow & & \downarrow \\
K & \downarrow & \hat{E} & \xrightarrow{\hat{\mathcal{E}}} & \widehat{B} \\
\end{array}
\]

\( \overline{K} = \text{Lan}_{\hat{A}} (\mathcal{N}_J \circ K \circ I) \)

**CWA2:** 2-cells \( \alpha : K \Rightarrow L : (A, E) \to (B, F) \) are (arbitrary) natural transformations \( \alpha : K \Rightarrow L : E \to F \).

Perhaps unsurprisingly we can, in a sense, switch the roles of \( K \) and \( \overline{K} \) in the definition of the 1 and 2-cells of \( \text{CwA} \) - from \( K \) as datum and \( \overline{K} \) as part of a condition to the reverse.

**Lemma 8.2.** Given categories with arities \((A, E, I)\) and \((B, F, J)\), and a co-continuous functor \( \overline{K} : \hat{A} \to \hat{B} \), if the square

\[
\begin{array}{cccccc}
E & \xrightarrow{\mathcal{N}_I} & \widehat{A} \\
\downarrow & & \downarrow \\
F & \xrightarrow{\mathcal{N}_J} & \widehat{B} \\
\end{array}
\]

pseudo-commutes, then \( \overline{K} = \text{Lan}_{\hat{K}} (\mathcal{N}_J \circ G \circ I) \).

**Proof.** Since \( \overline{K} \) is co-continuous and \( \hat{A} \) is a presheaf category, whence it is the free completion of \( A \) under colimits, it follows that \( \overline{K} = \text{Lan}_{\hat{K}} (K) \) for some \( K : A \to \hat{B} \). More, since

\[ \text{Lan}_{\hat{K}} (K) = \text{Lan}_{\mathcal{N}_I} (\text{Lan}_{I} (K)) \]

and the square pseudo-commutes by hypothesis we have

\[ \mathcal{N}_J \circ G = \text{Lan}_{\mathcal{N}_I} (\text{Lan}_{I} (K)) \circ \mathcal{N}_I \]

which, since \( \mathcal{N}_I \) is full-and-faithful, provides that

\[ \mathcal{N}_J \circ G = \text{Lan}_I (K) \]

which in turn provides, by pre-composition, the equality

\[ \mathcal{N}_J \circ G \circ I = \text{Lan}_I (K) \circ I \]

and lastly, by the full-and-faithfulness of \( I \), the equality

\[ \mathcal{N}_J \circ G \circ I = K \]

In light of the lemma above, we provide an alternative description of \( \text{CwA} \).

**Corollary 8.3.** The 2-category defined by the axioms **CWA0** from in Definition 8.1 with the further axioms
CWA1*: 1-cells \((A, E, I) \to (B, F, J)\) are co-continuous functors \(\overline{K} : \hat{A} \to \hat{B}\) which pseudo-factor as in

\[
\begin{array}{c}
E \\
\downarrow \ \\
\downarrow \ \\
F \\
\end{array} \quad \begin{array}{c}
\hat{A} \\
\hat{B} \\
\end{array}
\]

and we may replace CWA2 in that definition with

CWA2*: 2-cells \(\overline{K} \Rightarrow \overline{L} : (A, E, I) \to (B, F, J)\) of CwA are natural transformations of the underlying co-continuous functors.

comprise a 2-category CwA′ which is bi-equivalent to CwA 2-category.

Remark 8.4. We will never envoke CwA′ and CwA as distinct again.

Example 8.5. In the category of globular sets, \(\hat{G}\), we may identify the subcategory of linear pasting diagrams of globes: let \(P \to \hat{G}\) denote the full subcategory of \(\hat{G}\) subtended by objects of the form

\[
\lim \begin{array}{l}
\begin{array}{c}
\pi_0 \\
\pi_1 \\
\ldots \\
\pi_{\ell-1} \\
\pi_{\ell} \\
\end{array} \\
\begin{array}{c}
\pi_0 \to \pi_1 \\
\pi_1 \to \pi_{\ell-1} \\
\pi_{\ell-1} \to \pi_{\ell} \\
\end{array}
\end{array}
\]

for lists of non-negative integers,

\[n_0, m_1, n_1, \ldots, n_{\ell-1}, m_{\ell-1}, n_{\ell}\]

with each \(m_i \leq n_{i-1}, n_i\). Since the subcategory \(P \to \hat{G}\) factors the Yoneda embedding it is dense, whence \((P, \hat{G}, I)\) is a 0-cell of CwA.

9. Monads with Arities

Definition 9.1. A monad \(T\) on \(E\) has arities \(A\) if the endofunctor \(T\) is arities-respecting in the sense of CWA1.

Remark 9.2. Those concerned with the fact that no criteria are imposed on the 2-cells of the monad \(T\) may appeal to Lemma C.1.

Lemma 9.3. Given a monad \(T \cup E\) and category with arities \((A, E, I)\), \(T\) has arities \(A\) precisely if, setting \(\overline{T} = \text{Lan}_{\mathcal{N}_I}(\mathcal{N}_I \circ T)\), we have that \(\hat{A}_T \sim \hat{A}_{\overline{T}}\).

Proof. Suppose \(T \cup (A, E, I)\). Since \(\overline{T}\) is co-continuous \(\hat{A}_T = \hat{A}_{\overline{T}}\) and then, since \(\mathcal{N}_I \circ T = \overline{T} \circ \mathcal{N}_I\), \(A_T = A_{\overline{T}}\). Conversely, suppose \(\hat{A}_T = \hat{A}_{\overline{T}}\) with \(\overline{T} = \text{Lan}_{\mathcal{N}_I}(\mathcal{N}_I \circ T)\). Since \(\hat{A}_T\) is a presheaf topos precisely if \(\overline{T}\) is co-continuous it follows that \(\overline{T} = \text{Lan}_{\mathcal{N}_I}(\overline{T} \circ \mathcal{N}_I)\). But then

\[
\overline{T} = \text{Lan}_{\mathcal{N}_I}(\overline{T} \circ \mathcal{N}_I) \\
= \text{Lan}_{\mathcal{N}_I}(\overline{T} \circ \mathcal{N}_I \circ I) \\
= \text{Lan}_{\mathcal{N}_I}(\mathcal{N}_I \circ T) \circ \mathcal{N}_I \circ I \\
= \text{Lan}_{\mathcal{N}_I}(\mathcal{N}_I \circ T \circ I)
\]

which, according CWA1, is to say that \(T\) preserves arities \(A\). \(\square\)
Remark 9.4. Of the most useful and interesting aspects of the 2-category CwA is that not only does it have Eilenberg-Moore objects for all monads, but that the Eilenberg-Moore object of a monad is computable by way of the bijective-on-objects full-and-faithful factorization system. We will see this arising perhaps a bit less mysteriously from a limit reflection result later on.

Example 9.5. The monad T on \( \hat{G} \) whose algebras are strict-\( \omega \)-categories is an arities preserving monad for \( P \to G \). Moreover \( \Theta \to \text{Str-}\omega\text{-Cat} \) is the Eilenberg-Moore object for that monad in CwA.

10. FROM ARITIES TO SKETCHES

As described in [BourkeGarner] the notion of density presentation, due to Day, can be used to sharpen the abstract nerve theorem of Berger, Mellies and Weber.

Definition 10.1. A set of colimits \( \Psi \) in a category \( C \) is a set of diagrams
\[
\{ D_j : I_j \to C \}_{j \in J}
\]
for which the colimit \( \lim D_i \) exists for all \( j \in J \).

Given a full-and-faithful functor \( K : C \to A \) and a diagram \( D : I \to C \) with colimit \( \lim D \), we say that the colimit of \( D \) is \( K \)-absolute if the nerve \( \mathcal{N}_K : A \to \hat{C} \) preserves that colimit, i.e. if the canonical comparison map \( \lim \mathcal{N}_K D \to \mathcal{N}_K \left( \lim D \right) \) is an isomorphism.

Given a set of colimits \( \Psi \) in a category \( C \) and a full-and-faithful functor \( K : C \to A \), if \( A \) is the closure of \( C \) under a set of \( K \)-absolute colimits then \( \Psi \) is said to be a density presentation for \( K \).

Example 10.2. The set of wide spans
\[
\left\{ \begin{array}{c}
\overrightarrow{n_0} \\
\overrightarrow{m_1}
\end{array} \xleftarrow{\begin{array}{c}
t^0 \cdots t^m_1 s^1 \cdots s^m_1
\end{array}} \begin{array}{c}
\overrightarrow{n_1} \\
\overrightarrow{m_1}
\end{array} \xleftarrow{\begin{array}{c}
t^1 \cdots t^{m_1} s^1 \cdots s^{m_1}
\end{array}} \begin{array}{c}
\vdots
\end{array} \xleftarrow{\begin{array}{c}
t^\ell \cdots t^m_\ell s^1 \cdots s^m_\ell
\end{array}} \begin{array}{c}
\overrightarrow{n_\ell} \\
\overrightarrow{m_\ell}
\end{array}
\right\}
\]
indexed by the set of tuples of integers
\[
\{ (n_0, m_1, \ldots, m_{\ell-1}, n_\ell) | \ell \in \mathbb{N}, \ n_0 \geq m_1 \geq n_1 \geq \cdots \geq n_{\ell-1} \geq m_{\ell-1} \geq n_\ell \in \mathbb{N} \}
\]
constitute a density presentation for the functor \( G \to P \) which induced the functor \( P \to \hat{G} \).

Proposition 10.3. Given a density presentation
\[
\Psi = \{ D_\ell : I_\ell \to A \}_{\ell \in L}
\]
for a full-and-faithful functor \( C \to A \), if \( M \) is a monad with arities for \( J \) as a 0-cell of CwA, so we have a diagram
\[
\begin{array}{c}
A_M \\
\downarrow F_M
\end{array} \xleftarrow{\begin{array}{c}
J_M
\end{array}} \begin{array}{c}
\hat{C}_M \\
\downarrow F_M
\end{array} \xleftarrow{\begin{array}{c}
\mathcal{A}_J M
\end{array}} \begin{array}{c}
\hat{A}_M \\
\downarrow F_M
\end{array} \xleftarrow{\begin{array}{c}
\mathcal{A}_J
\end{array}} \begin{array}{c}
\hat{A}
\end{array}
\]
\[
C \to A \xleftarrow{\begin{array}{c}
J
\end{array}} \begin{array}{c}
\hat{C} \\
\downarrow \mathcal{A}_J
\end{array} \to \hat{A}
\]
then a presheaf $X : A_m^{op} \to \text{Set}$ is the nerve of an $M$-algebra if and only if $X$ is right orthogonal to the set of monomorphisms

$$V_\Psi = \left\{ \lim_{\to} \mathcal{N}_{J_m} F^M JD_\ell \to \mathcal{N}_{J_m} \lim_{\to} F^M JD_\ell \right\}$$

**Proof.** This follows from Theorem 36 of [BourkeGarner].

**Remark 10.4.** In the case where the nerves of the colimits $\mathcal{N}_{J_m} \lim_{\to} F^M D_\ell$ are representable by $T_\ell$ we will denote the colimit $\lim_{\to} \mathcal{N}_{J_m} F^M D_\ell$ by $V[T_\ell]$ for reasons which will become clear.

**Example 10.5.** Recall that the set of wide spans

$$\Psi = \left\{ \begin{array}{c}
\begin{array}{ccccccc}
\eta_0 & \to & m_1 & \to & \cdots & \to & m_{\ell-1} & \to & m_\ell \\
\quad \downarrow & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\epsilon_{0-m_1} & \to & s_{n_1-m_1} & \to & \cdots & \to & s_{n_{\ell-1}-m_{\ell-1}} & \to & m_{\ell-1}
\end{array}
\end{array} \right\}$$

constitute a density presentation for the inclusion $G \hookrightarrow P$. The set $V_\Psi$ is the set of spine inclusions

$$V = \{ V[T] \hookrightarrow \Theta[T] \}$$

It follows then, from the proposition above, that a cellular set $X$ is the nerve of a strict $\omega$-category precisely if it is right orthogonal to this set $V$. Right orthogonality with respect to the spines is of course the familiar Segal condition.

We can use this proposition this to provide a more abstract proof of Berger’s cellular nerve criterion in terms of inner horns.

**Example 10.6.** Let $\kappa : S \to T$ be a hyperface of a cell $T$ of $\Theta$. We recall that $\Lambda^\kappa [T]$ is the union in $\Theta[T]$ of all hyperfaces of $T$ which are not the hyperface $\kappa$. Each presheaf $\Lambda^\kappa [T]$ is therefore a co-equalizer. Let $F_{(\kappa,T)}$ be the set of hyperfaces of $T$ which are not the hyperface $\kappa$ and let $I_{(\kappa,T)}$ be the set of representables including into intersections if hyperfaces in $F_{(\kappa,T)}$.

**Remark 10.7.** The strange wording regarding the definition of $I_{(\kappa,T)}$ here is chosen to accommodate the fact that hyperfaces in $\Theta$ may intersect as coproducts of representables and not always representables as in the case of $\Delta$. Consider for example the intersection

$$[1] ; d^0 \cap [1] ; d^1 \to [1] ; [1]$$

which computes to the coproduct of representable presheaves $\Theta[0] \coprod \Theta[0]$.

Given any hyperface $\kappa : S \to T$ of cell $T$ of $\Theta$ it follows that the canonical map

$$\lim_{\to} \left\{ \prod_{R' \in I_{(\kappa,T)}} \Theta[R'] \Rightarrow \prod_{R \in F_{(\kappa,T)}} \Theta[R] \right\} \to \Theta[T]$$

is an isomorphism.

The set of all of these pairs of parallel arrows comprise a density presentation, call it $\Upsilon$, for the identity $\Theta \to \Theta$. We will use this density presentation to provide an alternative proof of Berger’s cellular nerve theorem as follows. Let $\overline{T}$ be the obvious left Kan extension of the monad $T \circ \hat{G}$ whose algebras are strict $\omega$-categories. Observe that $\Theta \to \hat{\Theta[\overline{T}]}$ is the Eilenberg-Moore object for the arities preserving monad $\overline{T}$. Thus Proposition 10.3 applies and a cellular set is the nerve of a strict $\omega$-category precisely if it is right orthogonal to $V_\Upsilon$ and the set $V_\Upsilon$ is the set $A_{inner}$ by design.
11. Cat-enriched Relative Adjunctions between $\text{Cat} \leftrightarrow \text{CAT}$ and $\text{Prof}$, and $\text{CwA}$

11.1. Cat-enriched relative adjunctions. In this short subsection we recall the definition of Cat-enriched relative adjunctions and recall the associated limit and colimit preserving properties of relative adjoint functors.

**Definition 11.1.** Given a (not necessarily commuting) triangle of 2-functors

$$
\begin{array}{ccc}
A & \xrightarrow{G} & B \\
\downarrow{J} & & \downarrow{D} \\
C & \xleftarrow{D} & B
\end{array}
$$

if there exists some natural equivalence of categories

$$
C(J(\_), D(\_)) \xrightarrow{\sim} D(G(\_), \_)
$$

we say that $G$ is left adjoint to $D$ relative to $J$.

**Lemma 11.2.** Given $G$ left adjoint to $D$ relative to $J$ as in the definition above:

- $G$ preserves all weighted colimits that $J$ does; and
- $D$ preserves all weighted limits if $J$ is dense.$^{13}$

**Proof.** Suppose $X : J \rightarrow A$ to be a 2-functor and $W : J \rightarrow \text{Cat}$ to be a weight. Then, letting $\lim^W X$ denote the $W$-weighted colimit of $X$, we see that if $J$ preserves $\lim^W X$ then, for each $n \in B$ we’ve the following sequence of equivalences of categories.

$$
\begin{align*}
B \left( G \left( \lim^W X \right), n \right) & \xrightarrow{\sim} C \left( J \left( \lim^W X \right), D(n) \right) \\
& \xrightarrow{\sim} C \left( \lim^W JX, D(n) \right) \\
& \xrightarrow{\sim} [J, \text{Cat}] (W, C(J \circ X(\_), D(n))) \\
& \xrightarrow{\sim} [J, \text{Cat}] (W, B(G \circ X(\_), n)) \\
& \xrightarrow{\sim} B \left( \lim^W GX, n \right)
\end{align*}
$$

Since the equivalences are natural in $n$ and vary over all of $B$, this provides an isomorphism

$$
G \left( \lim^W X \right) \xrightarrow{\sim} \lim^W GX
$$

The similar argument for the weighted limit preservation arrives at equivalences of categories

$$
C \left( J(a), D \left( \lim^W X \right) \right) \xrightarrow{\sim} C \left( J(a), \lim^W DX \right)
$$

which are natural in $a \in A$. In the context of our assumption that $J$ is dense, it follows that

$$
D \left( \lim^W X \right) \xrightarrow{\sim} \lim^W DX.
$$

$^{13}$As observed in footnote 13 of [Ulmer], this is certainly not necessary, but is at least sufficient.
11.2. Cat $\leftrightarrow$ CAT-relative adjunctions.

**Lemma 11.3.** The functor $\text{Prof} \rightarrow \text{CAT}$ is $\text{Cat} \leftrightarrow \text{CAT}$-right adjoint to $\text{Cat} \rightarrow \text{Prof}$.

\[
\begin{array}{c}
\text{Cat} \\
\downarrow \\
\text{CAT} \\
\downarrow \\
\text{Prof}
\end{array}
\]

*Proof.* See that we’ve a natural isomorphism of Hom-categories

\[
\text{Prof}(\tilde{A}, \tilde{B}) \sim \text{CAT}(A, B)
\]

\[\square\]

**Lemma 11.4.** The functor

\[
\begin{array}{c}
\text{Cat} \\
\downarrow \\
\text{CwA}
\end{array}
\]

is left $\text{Cat} \leftrightarrow \text{CAT}$ adjoint to the functor

\[
\begin{array}{c}
\text{CwA} \\
\downarrow \\
\text{CAT} \\
\downarrow \\
(E, A, I) \\
\downarrow \\
E
\end{array}
\]

*Proof.* See that

\[
\text{CwA}((B, B, \text{id}_B), (A, E, I)) \sim \text{CAT}(B, E)
\]

as the categories (left side) of pseudo-commuting squares (with right vertical map co-continuous) and (right side) arrows are equivalent.

\[\square\]

In fact, there is a morphism of $\text{Cat} \leftrightarrow \text{CAT}$ adjunctions between these two.\[14\]

\[\begin{array}{c}
\text{Cat} \\
\downarrow \\
\text{CwA} \\
\downarrow \\
\text{Prof} \\
\downarrow \\
\text{CAT}
\end{array}
\]

**Remark 11.5.** As consequences of these lemmata then, we find that:

\[14\]Indeed, someone more sophisticated than this author will smell Yoneda structures lurking behind the scenes as the non-identity 2-cells are the Yoneda embedding and the restricted Yoneda embedding respectively.
• $\mathbf{Cat} \to \mathbf{Prof}$ and $\mathbf{Cat} \to \mathbf{CwA}$ preserves all colimits which the inclusion $\mathbf{Cat} \to \mathbf{CAT}$ does; and

• since $\mathbf{Cat} \to \mathbf{CAT}$ is dense, the functors $\mathbf{Prof} \to \mathbf{CAT}$ and $\mathbf{CwA} \to \mathbf{CAT}$ preserves all limits.

12. A 3-Oriented, a Comma Category, and the Limit Reflection of $\mathbf{CwA} \to \mathbf{CAT}^2$

Now, it’s pleasing to the eye to observe that the relative adjunctions of the previous section assemble into a 3-oriental suggestive of a Yoneda structure. More, it’s probable that some rumination upon this will yield a formal theory of $\mathbf{CwA}$’s in an sufficiently endowed 3-category. However, our concern is more pedestrian.

The unit 2-cell

\[
\begin{array}{c}
\mathbf{Cat} \\
\downarrow \ \\
\mathbf{Prof} \\
\downarrow \\
\mathbf{CAT}
\end{array}
\]

induces a 2-functor $\mathbf{CwA} \to \mathbf{CAT} \downarrow (\mathbf{Prof} \to \mathbf{CAT})$, and it’s not hard to see that this 2-functor is full-and-faithful - consider our second definition for $\mathbf{CwA}$. Full-and-faithful functors however reflect all (weighted) limits and colimits; by way of this fact, and the limit preservation properties of Remark 11.5, we develop a criterion for the existence of some weighted limits in $\mathbf{CwA}$.

**Proposition 12.1.** The 2-functor

\[\mathbf{CwA} \to \mathbf{CAT}^2\]

reflects all limits.

**Proof.** As we’ve already observed, the 2-functor

\[\mathbf{CwA} \to \mathbf{CAT} \downarrow (\mathbf{Prof} \to \mathbf{CAT})\]

reflects limits as it is a full-sub-2-category. As such, we will show that

\[\mathbf{CAT} \downarrow (\mathbf{Prof} \to \mathbf{CAT}) \to \mathbf{CAT}^2\]

creates limits, whence the composition $\mathbf{CwA} \to \mathbf{CAT}^2$ will reflect limits as well.

For the proof that $\mathbf{CAT} \downarrow (\mathbf{Prof} \to \mathbf{CAT}) \to \mathbf{CAT}^2$ creates limits we will apply Corollary D.3 and for that it suffices to show that $\mathbf{Prof} \to \mathbf{CAT}$ creates limits. For the preservation part it is enough to recall that $\mathbf{Prof} \to \mathbf{CAT}$ is right adjoint relative to a dense functor. For the reflectivity we observe that $\mathbf{Prof} \to \mathbf{CAT}$ factors as

\[\mathbf{Prof} \to \mathbf{CoCompCAT} \to \mathbf{CAT}\]

and the first leg is a full sub-2-category, whence it reflects all limits, and

\[\mathbf{CoCompCAT} \to \mathbf{CAT}\]

creates all limits as it is monadic.\footnote{\textsuperscript{15}for the small presheaves monad}
Corollary 12.2. Let

\[(A, E, I): J \rightarrow \text{CwA} \]

be a 2-diagram and let \(W: J \rightarrow \text{Cat} \) be a weight. Then the weighted limit \(\lim^{W} (A, E, I) \) in \(\text{CwA} \) exists if:

- the weighted limit \(\lim^{W} A \) exists in \(\text{Prof} \);
- the weighed limit \(\lim^{W} A^{f} \) taken in \(\text{CAT}^{2} \) is induced by a full subcategory \(\lim^{W} A \rightarrow \lim^{W} E \)

and in such case it is

\[
\left( \lim_{\text{Prof}}^{W} A, \lim_{\text{CAT}}^{W} E, \lim^{W} A \rightarrow \lim^{W} E \right)
\]

Proof. By the limit reflection proposition above it suffices to show that, under the hypotheses of this corollary, the limit \(\lim^{W} A^{f} \) lies in the image of \(\text{CwA} \). To that end, see that if the weighted limit \(\lim^{W} A \) exists in \(\text{Prof} \), \(\lim^{W} A^{f} \) in \(\text{CAT}^{2} \) will be in the image of \(\text{CwA} \) if \(\lim^{W} A^{f} \) is the nerve associated to a subcategory - density follows from the fact that full-and-faithful functors are closed under limits, being, as they are, a right orthogonality class. \(\square\)

13. Examples

Example 13.1. Eilenberg-Moore Objects of Monads with Arities - Suppose that \(T \odot (A, E, I) \) is a monad in \(\text{CwA} \). It follows from Corollary 12.2 and Lemma 9.3 that \((A_{T}, E^{T}, I^{T}) \) is the Eilenberg-Moore object for \(T \) if \(A_{T} \rightarrow E^{T} \) is a full subcategory, but this much is clear.

Remark 13.2. The promised connection between this limit reflection result and the bijective-on-objects full-and-faithful factorization system comes from the fact that Kleisli objects in \(\text{Prof} \) are bijective on objects functors \(A \rightarrow A_{T} \).

Example 13.3. A cellular description of strict-\(Z\)-categories - In this example we will identify useful arities for the category

\[
\lim \left\{ \cdots \rightarrow S^{*} \rightarrow \text{Str-\(\omega\)-Cat} \rightarrow \text{Str-\(\omega\)-Cat} \right\} \sim \hat{G}_{Z}^{Tz}
\]

of \(Z\)-categories.

Now, the functor \(S: G \rightarrow G \) which induced \(S^{*} \) also induces \(S_{i}: \hat{G} \rightarrow \hat{G} \) which restricts along \(P \rightarrow G \) to \(S_{P} = S|_{P}: P \rightarrow P \). These functors fit into the the following commuting diagram.

\[
\begin{array}{ccc}
G & \xrightarrow{J} & P & \xrightarrow{I} & \hat{G} \\
S \downarrow & & \downarrow & & \downarrow S_{i} \\
G & \xrightarrow{J} & P & \xrightarrow{I} & \hat{G}
\end{array}
\]
It follows from that commutativity that the right square in
\[
\begin{array}{c}
P \xrightarrow{I} \tilde{G} \xrightarrow{\mathcal{N}_I} \hat{P} \\
S_P \\
P \xrightarrow{I} \tilde{G} \xrightarrow{\mathcal{N}_I} \hat{P}
\end{array}
\]
pseudo-commutes, whence it comprises a 1-cell in \( \text{CwA} \), as for any \( X \in \tilde{G} \),
\[
\mathcal{N}_I S^* X = \tilde{G} (I, S^* X) \\
\sim \tilde{G} (S_I \circ I, X) \\
\sim \tilde{G} (I \circ S_P) \\
\sim \hat{P} (S_P, \mathcal{N}_I X) \\
= S_P^* \mathcal{N}_I X
\]
and the left square pseudo-commutes as \( S_I \) is full-and-faithful\(^{16}\) so the unit of the adjunction \( S_I \dashv S^* \) is invertible.

We may then assemble a much larger pseudo-commuting diagram.
\[
\begin{array}{c}
P \xrightarrow{I} \tilde{G} \xrightarrow{\mathcal{N}_I} \hat{P} \\
P \xrightarrow{I} \tilde{G} \xrightarrow{\mathcal{N}_I} \hat{P} \\
P \xrightarrow{I} \tilde{G} \xrightarrow{\mathcal{N}_I} \hat{P} \\
\vdots \quad \vdots \quad \vdots
\end{array}
\]
We note that the columns of \( \tilde{G} \)’s and \( \hat{P} \)’s are comprised of iso-fibrations, so the conical limits over them compute pseudo-limits. The left-hand column of commuting squares induces a map
\[
I_Z : \lim_{\longrightarrow} \{ \cdots \leftarrow P \leftarrow P \} \longrightarrow \lim_{\longrightarrow} \{ \cdots \rightarrow \tilde{G} \rightarrow \tilde{G} \}
\]
and the nerve associated to this map
\[
\mathcal{N}_{I_Z} : \tilde{G}_Z \rightarrow \hat{P}_Z
\]
is the pseudo-limit in \( \text{CAT}^2 \) of the horizontal parts of the right-hand column. By Theorem \[12.2] to show that \( (P_Z, \tilde{G}_Z, I_Z) \) is the conical limit in \( \text{CwA} \) of
\[
\cdots \rightarrow (P, \tilde{G}, I) \rightarrow (P, \tilde{G}, I)
\]
\(^{16}\)It is important here that we have used the \textit{irreflexive} globe category and not the reflexive globe category. This is an instance of the principle/phenomenon which undergirds Makkai’s FOLDS or Henry’s related work on the language of a model category. The point is that inverse categories provide good “theories of sorts”, whereas other categories do not.
it then suffices to show that
\[ I_Z : P_Z \to \widehat{G}_Z \]
is a full subcategory.

Using the \( \mathbb{Z}^{\text{op}} \) indexing for the copies of \( P \) in \( P_Z \) and the \( \mathbb{Z}_{\leq 0} \) indexing for the copies of \( \widehat{G} \) in \( \varprojlim \widehat{G} \) we may write the map \( I_Z \) explicitly as
\[
I_Z : P_Z \to \varprojlim \{ \cdots \to \widehat{G} \to \widehat{G} \}
\]

Since \( N_I \) is a full subcategory, it follows that \( I_Z \) is injective on objects and faithful, but since \( S_P \) is also full-and-faithful, it follows that \( I_Z \) is full as well. Composing \( I_Z \) with the equivalence \( \varprojlim \{ \cdots \to \widehat{G} \to \widehat{G} \} \sim \widehat{G}_Z \) permits to identify \( I_Z \) as the full subcategory of \( \widehat{G}_Z \) on presheaves of the form
\[
\varprojlim \left\{ \begin{array}{c}
\begin{array}{cccc}
n_0 & n_1 & \cdots & n_{\ell-1} & n_{\ell} \\
m_1 & m_2 & \ddots & m_{\ell-1} & m_{\ell}
\end{array}
\end{array} \right\}
\]
for integers \( n_0, m_1, n_1, \ldots, n_{\ell-1}, m_{\ell-1}, n_{\ell} \) with each \( m_i \leq n_i-1, n_i; \) \( P_Z \) is the full subcategory of \( \widehat{G}_Z \) of finite \( \mathbb{Z} \)-globular pasting diagrams. These wide spans moreover comprise a density presentation \( \Psi_Z \) for \( G_Z \to P_Z \).

Lastly, since limits commute with limits, whence limits over towers commute with the taking of Eilenberg-Moore objects, it follows that
\[
\left( \Theta_Z := \varprojlim \{ \cdots \leftarrow \Theta \leftarrow \Theta \}, \widehat{G}_Z^{T_Z}, I_Z^T \right) \]
is the Eilenberg-Moore object for \( T_Z \) as a monad in \( \text{CwA} \) on \( (P_Z, \widehat{G}_Z, I_Z) \). Thus, by Proposition 10.3 and a repeat of our argument from Example 10.6 it follows that a \( \mathbb{Z} \)-cellular set is the nerve of a strict-\( \mathbb{Z} \)-category if and only if it is:

* right orthogonal to the \( \mathbb{Z} \)-cellular analog of the spines; or
* right orthogonal to the \( \mathbb{Z} \)-cellular analog of the inner horns.

**Remark 13.4.** In a sequel to this work, we will reprove a result of [LessardThesis], and we will show how the homotopy coherent version of this cellular presentation of \( \mathbb{Z} \)-categories provides a model of Spectra as pointed \( \mathbb{Z} \)-groupoids.

**Appendix**

**Appendix A. Generalized Spectrification**

Recall that in the 2-category \( \text{Cat} \), given:

* a 2-category \( J \);
* a weight 2-functor \( W : J \to \text{Cat} \); and
* a 2-diagram of categories \( X : J \to \text{Cat} \).
the Hom-category of natural transformations and modifications is determined by the Hom-categories $\text{Cat}(W_i, X_j)$ by way of the end formula

$$[J, \text{Cat}] (W, X) \xrightarrow{\sim} \int_{j \in J} \text{Cat}(W_j, X_j)$$

We may then be wont to ask: which properties or structures involving the values taken by the functor $\text{Cat}(W(_\_), X(_\_))$ are sufficiently (and appropriately) coherent so that they induce similar properties or structures on the weighted limit? In particular, if for some objects $i$ and $j$ of $J$, we have that $X_j$ has all $W_i$ colimits then we have adjunctions

$$\left\{ \begin{array}{c}
\lim^{(i,j)} \\
\Delta^{(i,j)}
\end{array} \right\} \xrightarrow{\perp} \left\{ \begin{array}{c}
W_i, X_j \\
\perp
\end{array} \right\}$$

$(i,j) \in \text{Ob}(J)^2$

When do these object-wise adjunctions beget adjunctions in $[J, \text{Cat}]$? Taking a hint from classical treatments of stable homotopy theory we develop an easy criterion and show, by explicit example, that the spectrification functors of stable homotopy theory are recovered from this 2-categorical treatment.

### A.1. A Criterion for small conical limits in CAT to be reflective in weighted-limits in CAT: generalized spectrification.

**Theorem A.1.** Let $J$ be a 1-category and let $X : J \rightarrow \text{CAT}$ be a diagram of categories. Suppose further that, for all $f : j \rightarrow k$ in $J$:

- the diagram $X : J \rightarrow \text{CAT}$ is fibrant in the injective-canonical model structure on $[J, \text{CAT}]$;
- the category $X_k$ admits all $W_j$ and $W_k$-colimits;
- the functor $X_f$ preserves $W_j$ colimits; and
- the functor $W_f$ is final.

Then, the functor

$$[J, \text{CAT}] (W, X) \xrightarrow{\Delta} [J, \text{CAT}] (\bullet, X)$$

admits a left adjoint, which we will call $\text{sp}$.

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc}
\text{Ps} (J, \text{CAT}) (W, X) & \xrightarrow{\Delta'} & \text{Ps} (J, \text{CAT}) (\bullet, X) \\
\downarrow I_W & & \downarrow I \\
[J, \text{CAT}] (W, X) & \xrightarrow{\Delta} & [J, \text{CAT}] (\bullet, X)
\end{array}$$

where $I$ and $I_W$ are the obvious inclusions. Since $I_W$ is full-and-faithful, if $I_W \Delta$ admits a left adjoint functor $L$, then $LI_W$ is left adjoint to $\Delta$. But $I_W \Delta = \Delta' I$, so to produce a left adjoint to $\Delta$ it suffices to produce one for $\Delta' I$. Since $X$ is fibrant however, and $[J, \text{CAT}] (\bullet, X)$ enjoys the universal property of the conical limit $\lim X$, $I$ is an adjoint equivalence, whence to produce a left adjoint to $\Delta' I$ it is enough to produce a left adjoint for $\Delta'$. It is just such an adjoint we will now describe.
Recall that for any weight $W$, the $\text{Hom}$-category $[J, \text{CAT}] (W, X)$, may be computed as the equalizer

$$[J, \text{CAT}] (W, X) \xrightarrow{\sim} \varprojlim \left\{ \prod_{i \in J} [W_i, X_i] \xrightarrow{\rho} \prod_{j \in J} \prod_{k \in J} J(j, k), [W_j, X_k] \right\}$$

where

$$\rho_{j, k} : [W_j, X_j] \longrightarrow [J(j, k), [W_j, X_k]]$$

$$[W_{j(-)}, X_k] : J(j, k) \longrightarrow [[W_j, X_j], [W_j, X_k]]$$

and like-wise

$$\lambda_{j, k} : [W_k, X_k] \longrightarrow [J(j, k), [W_j, X_k]]$$

$$[W(-), X_k] : J(j, k) \longrightarrow [[W_k, X_k], [W_j, X_k]]$$

More, since $J$ is a 1-category, we have an isomorphism

$$\prod_{j \in J} \prod_{k \in J} [J(j, k), [W_j, X_k]] \xrightarrow{\sim} \prod_{j \in J} \prod_{k \in J} \prod_{f \in J(j, k)} [W_j, X_k]$$

and thus the map $\rho$ may be made explicit as a product.

$$\left( ([W_j, X_f])_{f \in J(j, k)} \right)_{k \in J} : \prod_{j \in J} \left( [W_j, X_j] \longrightarrow \prod_{k \in J} \prod_{f \in J(j, k)} [W_j, X_k] \right)$$

Similarly $\lambda$ may be made explicit as the product.

$$\left( ([W_f, X_k])_{f \in J(j, k)} \right)_{j \in J} : \prod_{k \in J} \left( [W_k, X_k] \longrightarrow \prod_{j \in J} \prod_{f \in J(j, k)} [W_j, X_k] \right)$$

The pseudo-limit of that same diagram computes the $\text{Hom}$-category $\text{Ps}(W, X)$.

Thus, thinking of the taking of pseudo-equalizers as a functor

$$\text{pslim} : \text{Ps}(\partial \overline{1}, \text{CAT}) \longrightarrow \text{Cat}$$

we see that $\Delta'$ is $\text{pslim}$ of the pseudo-natural transformations of functors $\partial \overline{1} \longrightarrow \text{CAT}$ which comprises the diagram

\[
\begin{array}{ccc}
\prod_{i \in J} [W_i, X_i] & \xrightarrow{\rho} & \prod_{j \in J} \prod_{k \in J} \prod_{f \in J(j, k)} [W_j, X_k] \\
\prod \Delta^{(i, j)} & & \prod \prod \Delta^{(j, k)} \\
\prod_{i \in J} X_i & \xrightarrow{id} & \prod_{j \in J} \prod_{k \in J} \prod_{f \in J(j, k)} X_k \\
\end{array}
\]

As such, it suffices to find left adjoints $L^{(i, j)}$

$$[W_i, X_j] \xrightarrow{\perp_{\Delta^{(i, j)}}} X_j$$
for each pair of objects $i$ and $j$ of $J$ such that the squares - solid and dashed, and solid and dotted respectively - in the assembled diagram

\[
\begin{array}{c}
\prod_{i \in J} [W_i, X_i] \\
\prod_{i \in J} \prod_{k \in J} \prod_{f \in J(j,k)} [W_f, X_k] \\
\prod_{i \in J} X_i \\
\prod_{i \in J} X_i
\end{array}
\]

pseudo-commute.

Our hypotheses will imply that the family

\[\{ L^{(i,j)} = \lim^{(i,j)} : [W_i, X_j] \rightarrow X_j \}_{(i,j) \in \text{Ob}(J)^2}\]

satisfies this criterion. Since $W_f$ is final for all morphisms $f : j \rightarrow k$ of $J$, for any $\phi \in \text{Ob}([W_k, X_k])$ we have the following chase.

Thus, the square

\[
\begin{array}{c}
\prod_{i \in J} [W_i, X_i] \\
\prod_{i \in J} \prod_{k \in J} \prod_{f \in J(j,k)} [W_f, X_k] \\
\prod_{i \in J} X_i \\
\prod_{i \in J} X_i
\end{array}
\]
pseudo-commutes. Similarly, since $X_f$ preserves $W_j$ colimits for all $f : j \to k$ of $J$, we have for each $\phi \in \text{Ob}([W_j, X_j])$ the following chase.

Thus, the square

Thus, the square

pseudo-commutes. Then, defining functor $sp'$ as the pseudo-limit $\text{pslim} \left( \prod \lim^{(j,k)} \right)$, we have that $sp'$ is left adjoint to $\Delta' = \text{pslim} \left( \prod \Delta^{(i,i)} \prod \lim^{(j,k)} \right)$. In light of our earlier observations, we have $sp = sp' I_W$ is left adjoint to $\Delta$ and the theorem is proved. □

**Example A.2.** To make clear the necessity of the condition that $X$ is injective-canonical fibrant, consider the following example. Let $J = G_{\leq 1}$ (the ur-pair-of-parallel-arrow) and let $W = X$ be the $J$-diagram

where $\bullet$ is the terminal category and $I$ is the ur-isomorphism. Since $s$ and $t$ are equivalences of categories and $\bullet$ is co-complete it follows that all hypotheses of the theorem, save fibrancy, are satisfied. We see however that there does not exist any adjunction between $[J, \text{CAT}](\bullet, X)$ and $[J, \text{CAT}](W, X)$ as the former is the empty category whereas the latter is a terminal category.

To see that the fibrancy hypothesis excludes this example we observe that, since $J$ is both direct and inverse, it follows that the Reedy-canonical model structure on $[J, \text{CAT}]$ (picking $J^- = J$) and the injective-canonical model structure coincide. Thus $X$ is fibrant precisely if the matching maps

$$X (\overline{0}) \to X (1) \times X (1)$$
and

\[ X(T) \rightarrow \bullet \]

are iso-fibrations. While the second of those maps is an iso-fibration, as all maps into the terminal category are iso-fibrations, the first map, which evaluates to

\[ \bullet \xrightarrow{(s,t)} I \times I \]

is not an iso-fibration as \((s,t)\), as an object of \(I \times I\), is isomorphic therein to all three other objects of \(I \times I\) and none of those isomorphisms lift.

A.2. Examples: Spectrification Three Ways. We now attend to the promised examples.

**Example A.3. Sequential spectra** - Let \(S\) denote the usual category of spaces and let \(S_\bullet\) denote the category of pointed spaces. Let \(J = \mathbb{Z}_{\leq 0}\), let \(W = (\mathbb{Z}_{\leq 0} \downarrow -)^{\text{op}} : J \rightarrow \text{Cat}\), and let \(X : \mathbb{Z}_{\leq 0} \rightarrow \text{CAT}\) be the functor sending each non-positive integer to \(S_\bullet\) and each \(-(n + 1) \rightarrow n\) to \(\Omega : S_\bullet \rightarrow S_\bullet\). As we will show, these data satisfy the hypotheses of Theorem A.1.

First, since \(\mathbb{Z}_{\leq 0}\) is an inverse category, it follows that the fibrant objects of the injective-canonical model structure are but sequences of iso-fibrations - for any inverse category \(A\) and model category \(C\), the fibrant objects are diagrams of fibrations between fibrant objects and all categories are fibrant. Then since \(\Omega\) is an iso-fibration\(^\text{17}\), it follows that \(X\) is injective-canonical fibrant. Second, \(S_\bullet\) is co-complete so it has all \(W\)-colimits. Third, as \(W\)-final, it follows that \(W\) is a diagram whose morphisms are all final functors. Lastly, the functor \(\Omega = \text{Hom}(S^1, \_\_\_\_)\) preserves sequential colimits; given any \(G : \mathbb{Z}_{\geq 0} \rightarrow S_\bullet\), since any map \(S^1 \rightarrow \text{lim}G\) factors through some \(G(n) \rightarrow \text{lim}G\) it follows that \(S_\bullet(S^1, \text{lim}G) \rightarrow \text{lim}S_\bullet(S^1, G(\_\_\_\_\_))\). Theorem A.1 then provides that there is an adjunction

\[
\begin{array}{c}
\left[\mathbb{Z}_{\leq n}, \text{CAT}\right](W, X) \\
\downarrow \\
\left[\mathbb{Z}_{\leq 0}, \text{CAT}\right](\bullet, X)
\end{array}
\]

What we will now show is that this adjunction recovers \(\Omega\)-spectra as a reflective subcategory of sequential spectra.

A sequential spectrum object in \(S_\bullet\) is comprised of:

- an \(\mathbb{N}\) indexed family of pointed spaces \((X_n)_{n \in \mathbb{N}}\); together with
- morphisms \((\phi_n : X_{n} \rightarrow \Omega X_{-(n+1)})_{n \in \mathbb{N}}\) where \(\Omega = S_\bullet(S^1, \_\_\_\_)\), the usual loop-space functor.

The data of these objects is readily seen equivalent to that of oplax cones over \(X : \mathbb{Z}_{\leq 0} \rightarrow \text{CAT}\), so the usual category of sequential spectra is here recovered as \(\text{op}lax\text{lim}X\). As we will show, the weight \(W\) is in fact the oplax-weight for \(\mathbb{Z}_{\leq 0}\), meaning that those oplax cones are equivalent to natural transformations

\[ \Xi : W \Rightarrow X : \mathbb{Z}_{\leq 0} \rightarrow \text{CAT} \]

Observe that the component of some natural transformation \(\Xi\)

\[ \Xi : W \Rightarrow X : \mathbb{Z}_{\leq 0} \rightarrow \text{CAT} \]

\text{17}there are of course numerous arguments for this, for example \(\text{Top}_\bullet \rightarrow \text{Set}_\bullet\) is conservative, and non-identity isomorphisms of sets are just renamings, so we may rename the elements of \(X\) as \(Y\)' in such a way that \(\Omega Y = Y\).
at the object \(-n\) of \(\mathbb{Z}_{\leq 0}\), is but a diagram
\[
\cdots \leftarrow X_{-(n+2)} \leftarrow X_{-(n+1)} \leftarrow X_{-n}
\]
in \(S_\bullet\). Consider then that for each \(f : -m \to -n\) of \(\mathbb{Z}_{\leq 0}\), the naturality criterion
\[
\begin{align*}
(Z_{\leq 0} \downarrow -m)^{\text{op}} &\to S_\bullet, \\
(\mathbb{Z}_{\leq 0} \downarrow -n)^{\text{op}} &\to S_\bullet
\end{align*}
\]
\[
\begin{align*}
(-k \to -m) &\to X_{-m} \\
(-k - m - n) &\to X_{-n} = \Omega^{n-m} (X_{-k})
\end{align*}
\]
implies the identities
\[
\{X_{-n} = \Omega^{n-m} (X_{-k})\}_{-k \to -m \to -n \in \mathbb{Z}_{\leq 0}^2}
\]
so our natural transformations are equivalent to oplax cones as follows: the data of a natural transformation \((Z_{\leq 0} \downarrow -)\)^{\text{op}} \to X is given in the solid arrows and equalities in the diagram below, and the data of an oplax cone is given by the objects along the diagonal together with the dashed arrows.

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\Omega^2 (X_{-2}) & \Omega (X_{-1}) & \Omega (X_{-1}) & \cdots & \cdots \\
X_{-2} & X_{-1} & X_{0} & \cdots & \cdots \\
\end{array}
\]

Now, taking a very strict definition\(^{18}\), the category of sequential \(\Omega\)-spectra is the subcategory of sequential spectra on spectrum objects for which the constituent families of structure maps
\[
(\phi_{-n} : X_{-n} \to \Omega X_{-(n+1)})_{n \in \mathbb{N}}
\]
are families of homeomorphisms. In terms of our functor \(X : \mathbb{Z}_{\leq 0} \to \text{CAT}\) this is but \(\text{pslim}X\) inside of \(\text{oplaxlim}X\). But \(X\) is injective-canonical fibrant, so \(\lim X \sim \text{pslim}X\). Now, it’s not hard to see that the diagram
\[
\begin{array}{cccccc}
\text{oplaxlim}X & \leftarrow \text{pslim}X & \leftarrow \lim X \\
\downarrow & \downarrow & \downarrow \\
[W, X] \downarrow \Delta & [\mathbb{Z}_{\leq 0}, \text{CAT}] & \text{pslim}X & \text{pslim}X
\end{array}
\]
commutes, whence the left adjoint to \(\Delta\) is the left adjoint to the inclusion of sequential \(\Omega\)-spectra into sequential spectra, which is to say that this left adjoint is spectrification.

Lastly, this 2-categorical treatment of spectrification also applies to the more sophisticated notion of coordinate-free spectra found in \([\text{ElmendorfKrizMandellMay}]\).

\(^{18}\)our reason here for picking the strict definition is that this works aims to synthesize stabilization and strict notions of algebra. It seems likely however that the treatment of the notion should hold up well in an \((\infty, 2)\)-categorical treatment - a topic of future work.
Example A.4. Coordinate-free spectra - keeping $S$ and $S_*$ as above and fix some $\mathcal{U} \to R^\infty$ (as vector spaces). Let $\text{FinSub}(\mathcal{U})$ be the category of finite dimensional sub-vector-spaces of $\mathcal{U}$. For each $V \in \text{FinSub}(\mathcal{U})$, denote by $S^V$ the one-point compactification of $V$ and for each $V \subset W \in \text{FinSub}(\mathcal{U})$, let $W - V$ denote the orthogonal complement of $V$ in $W$. A $\mathcal{U}$-coordinate free spectrum object $X$ of $S_*$ is comprised of:

- for each $V \in \text{FinSub}(U)$, a pointed space $X_V$; and
- for each $V \to W$ of $\text{FinSub}(U)$, a morphism $\sigma_{V,W} : X_V \to \Omega^{W-V}(X_W)$ where $\Omega^{W-V} = S_*(S^{W-V}, -)$.

Denote the category of such by $\text{CFSp}(\mathcal{U}, S_*, \Omega^{W-V})$.

A coordinate-free spectrum is, as in the previous example, equivalent to an oplax cone over a $1$-diagram in $\text{Cat}$ - here the diagram is as follows.

$$\Omega_{\text{CFS}} : \text{FinSub}(\mathcal{U})^{\text{op}} \to \text{Cat}$$

$$\begin{array}{c}
V \\
\downarrow \Phi \\
S_*
\end{array}$$

$$\begin{array}{c}
V \subset W \\
\downarrow \Phi \\
\Omega^{W-V}
\end{array}$$

In the case of coordinate-free spectra, the meaning of $\Omega$-spectra is essentially the same as with sequential spectra: a coordinate free $\Omega$-spectrum has all constituent morphisms $\sigma_{V,W} : X_V \to \Omega^{W-V}(X_W)$ being homeomorphisms. We see again that

$$\Omega\text{-CFSp}(U, S_*, S^{(-)}) \to \text{CFSp}(U, S_*, S^{(-)})$$

commutes with the so labeled functors being equivalences of categories. As before, by Theorem A.1

$$\Omega\text{-CFSp}(U, S_*, S^{(-)}) \to \text{CFSp}(U, S_*, S^{(-)})$$

will have a left adjoint as:

- $\Omega_{\text{CFS}}$ is injective-canonical fibrant in $[\text{FinSub}(\mathcal{U})^{\text{op}}, \text{CAT}]$;
- Observe that picking $\text{FinSub}(\mathcal{U})^{\text{op}} = (\text{FinSub}(\mathcal{U})^{\text{op}})^-$ defines a Reedy structure on $\text{FinSub}(\mathcal{U})^{\text{op}}$ such that the injective-canonical and Reedy-canonical model structures on $[\text{FinSub}(\mathcal{U})^{\text{op}}, \text{CAT}]$ coincide. Thus to check fibrancy of $\Omega_{\text{CFS}}$ is to check that, for every $W \in \text{FinSub}(\mathcal{U})^{\text{op}}$, the map

$$S_* \to \lim_{W \leftarrow V : W \in \text{FinSub}(\mathcal{U})^{\text{op}}} S_*$$

is an iso-fibration. But since every square in $\text{FinSub}(\mathcal{U})^{\text{op}}$ commutes, that limit is just the product

$$\prod_{V : \text{codim-1}(W)} S_*$$

over all co-dimension 1 subspaces of $W$ and the map is the one induced by that universal property by the family of maps $(\Omega)_{V : \text{codim-1}(W)}$. Indeed this is an iso-fibration as $\Omega$ is an iso-fibration.

- the category $S_*$ admits all small colimits;
Appendix B. Oplax limits and the Oplax weight $(J \downarrow -)^{\text{op}} : J \to \text{Cat}$

The material of this section is well known to the 2-categorical cognoscenti\textsuperscript{19}. We include it here for completeness, for notation, and for ease of reference.

B.1. Oplax natural transformations and the universal property of the oplax limit.

**Definition B.1.** Given 2-categories $J$ and $D$, and two 2-functors $X, Y : J \to D$, an oplax-natural transformation $\alpha : X \xrightarrow{\text{oplax}} Y$ is comprised of:

- a set of 1-cells $\{a_j : X_j \to Y_j\}_{j \in \text{Ob}(J)}$; and
- a $\text{Mor}(J)$ indexed family of 2-cells

\[
\begin{array}{c}
\begin{array}{ccc}
X_i & \xrightarrow{X_f} & X_j \\
\downarrow a_i & \searrow \alpha_f & \downarrow a_j \\
Y_i & \xrightarrow{Y_f} & Y_j \\
\end{array}
\end{array}
\]

such that, for every $i \xrightarrow{g} j \xrightarrow{f} k$ of $J$, we have an equality of 2-cells

\[
\begin{array}{c}
\begin{array}{ccc}
X_i & \xrightarrow{X_g} & X_j \\
\downarrow a_i & \searrow \alpha_g & \downarrow a_j \\
Y_i & \xrightarrow{Y_g} & Y_j \\
\end{array} = \begin{array}{ccc}
X_i & \xrightarrow{X_f} & X_k \\
\downarrow a_k & \searrow \alpha_f & \downarrow a_j \\
Y_i & \xrightarrow{Y_f} & Y_k \\
\end{array}
\end{array}
\]

We define $\text{Oplax}(X, Y)$ to be the 1-category with:

- oplax-natural transformations $\alpha : X \xrightarrow{\text{oplax}} Y$ as 0-cells; and
- modifications $c : \alpha \Rightarrow \beta$ as 1-cells.

Having recalled the notion of oplax-natural transformation, we may recall the notion of the oplax limit.

**Definition B.2.** For 2-categories $J$ and $D$ and a 2-functor $X : J \to D$, then the oplax limit of the 2-diagram $X$, denoted $\xrightarrow{\text{oplax}} \text{Oplaxlim} \ X$, is comprised of:

- an object $\xleftarrow{\text{oplax}} \text{Oplaxlim} \ X$ of $D$;

\textsuperscript{19}read: “not the author until the writing of this article”

\textsuperscript{20}in the usual abuse of notation
and an oplax cone into $X$, i.e.

- projection maps $\{ \tilde{pr}_i : \text{oplax} \lim X \rightarrow X_i \}_{i \in \text{Ob}(J)}$; and
- 2-cells $\{ \pi_f : \tilde{pr}_i \Rightarrow f \circ \tilde{pr}_j \}_{f : i \rightarrow j \in \text{Mor}(J)}$

with:

- for all $i \xrightarrow{f} j \xrightarrow{g} k$ of $J$, the diagram of 2-cells commuting, i.e. $(g \pi_f) \circ \pi_g = \pi_{gof}$

enjoying the universal property:

- post-composition with the oplax-cone $((\tilde{pr}_i), (\pi_f))$ induces an equivalence of categories.

$$D \left( A, \text{oplax} \lim X \right) \xrightarrow{\sim} \text{Oplax} \left( J, C \right) \left( \Delta A, X \right)$$

**Remark B.3.** It’s worth noting that the suggestive (of the structure of a module) criterion $(g \pi_f) \circ \pi_g = \pi_{gof}$ in the definition of an oplax cone can be born out (see [Lack]).

**B.2. The Oplax weight** $(J \downarrow \_)^{\text{op}} : J \rightarrow \text{Cat}$.

**Definition B.4.** Given a 2-category $J$, call the 2-functor $(J \downarrow \_)^{\text{op}} : J \rightarrow \text{Cat}$

the **oplax weight**.

As one would hope, given the name, the oplax weight $(J \downarrow \_)^{\text{op}}$ computes oplax limits. See that, for a given 2-functor $X : J \rightarrow \text{Cat}$, we’ve an oplax cone

$$\Delta [J, \text{Cat}] ((J \downarrow \_)^{\text{op}}, X) \xrightarrow{\text{oplax}} X$$

comprised of 1-cells

$$\left\{ \Delta [J, \text{Cat}] ((J \downarrow \_)^{\text{op}}, X) \xrightarrow{\alpha_j (\text{id}_j : j \rightarrow j)} X_j \right\}$$

and 2-cells

$$\Delta [J, \text{Cat}] ((J \downarrow \_)^{\text{op}}, X) \xrightarrow{\Delta [J, \text{Cat}] ((J \downarrow \_)^{\text{op}}, X)}$$

whose component at some $\alpha : (J \downarrow \_)^{\text{op}} \Rightarrow X$ is given by 1-cells

$$\alpha : (J \downarrow \_)^{\text{op}} \Rightarrow X \xrightarrow{\alpha_k (\text{id}_k)} X$$

and

$$\alpha_j (\text{id}_j) \xrightarrow{\alpha_j (\text{id}_j)} X_f \circ \alpha_j (\text{id}_j) = \alpha_k (j \rightarrow k)$$
where by $\alpha_k(f \to k)$ we mean the application of $\alpha_k$ to the morphism of $(J \downarrow k)^{\text{op}}$ corresponding to the following triangle.

$$
\begin{array}{ccc}
j & \rightarrow & k \\
\downarrow f & \searrow & \nearrow \downarrow f \\
& k & \\
\end{array}
$$

The oplax cone thus described endows $[J, \text{Cat}]((J \downarrow _)^{\text{op}}, X)$ with the universal property of the oplax limit over $X$.

What remains a bit opaque however is how, precisely, we translate between oplax-natural transformations $\alpha : X \xrightarrow{\text{oplax}} Y$ and functors

$$
\text{oplaxlim} \alpha : [J, \text{Cat}]((J \downarrow _)^{\text{op}}, X) \to [J, \text{Cat}]((J \downarrow _)^{\text{op}}, Y)
$$

Every $\beta : (J \downarrow _)^{\text{op}} \Rightarrow X$ is comprised of a natural family of functors

$$
\{ \beta^j : (J \downarrow j)^{\text{op}} \to X_j \}_{j \in \text{Ob}(J)}
$$

We will describe $\text{oplaxlim} \alpha$ by describing its action on the objects of $[J, \text{Cat}]((J \downarrow _)^{\text{op}}, X)$, the natural transformations $\beta : (J \downarrow _)^{\text{op}} \Rightarrow X$, component-wise. We set $\text{oplaxlim} \alpha(\beta)$ to the natural transformation comprised of the family of functors $\{\bar{\alpha}(\beta^j)\}_{j \in \text{Ob}(J)}$

- whose actions on objects are given

$$
\bar{\alpha}(\beta^j) : (J \downarrow j)^{\text{op}} \to Y_j
$$

$$(p \to j) \quad \mapsto \quad Y_q \circ \alpha_p \circ \beta^p(\text{id}_p)
$$

and

- which sends morphisms $(k \to j) : (p \to q \to j) \to (\ell \to q \to \ell \to j)$ corresponding to triangles

$$
\begin{array}{ccc}
\ell & \to & k \\
q \searrow & \downarrow & \nearrow p \\
& j & \\
\end{array}
$$

to the compositions

$$
\begin{array}{c}
Y_q \alpha_p \beta^p(\text{id}_p) \xrightarrow{Y_q \alpha_p \beta^p(\text{id}_p)} Y_q \alpha_p \beta^p(\ell \to p) \\
Y_q \alpha_p \beta^p(\ell \to p) \xrightarrow{Y_q \alpha_k \beta^\ell(\text{id}_\ell)} Y_q Y_k \alpha_\ell \beta^\ell(\text{id}_\ell) \\
Y_q Y_k \alpha_\ell \beta^\ell(\text{id}_\ell) \xrightarrow{Y_q Y_k \alpha_\ell \beta^\ell(\text{id}_\ell)}
\end{array}
$$

\[21\]The idiosyncrasy of using super-script here will make something a little more natural in our examples section, so those irked by the choice should rest assured that the author has not chosen this notation just to be a pest - and is instead a pest with a mission.
Remark B.5. Those carefully type-checking the composition above may find the following 2-diagram useful.

\[
\begin{array}{c}
(J \downarrow \ell)^{\mathsf{op}} \xrightarrow{(J \downarrow k)^{\mathsf{op}}} (J \downarrow j)^{\mathsf{op}} \xrightarrow{(J \downarrow q)^{\mathsf{op}}} (J \downarrow p)^{\mathsf{op}} \\
\downarrow \beta^{\ell} \quad \quad \downarrow \beta^j \quad \quad \downarrow \beta^p \\
X_{\ell} \xrightarrow{X_k} X_{j} \xrightarrow{X_q} X_{p} \\
\downarrow \alpha_{\ell} \quad \quad \downarrow \alpha_j \quad \quad \downarrow \alpha_p \\
Y_{\ell} \xrightarrow{Y_k} Y_{j} \xrightarrow{Y_q} Y_{p} \\
\end{array}
\]

APPENDIX C. EXTENSIONS OF MONADS

Lemma C.1. Given categories $A$ and $B$, with $B$ co-complete, and a functor $N : A \rightarrow B$, then the composition
\[
\text{Lan}_N (N \circ \_ : [A, A] \xrightarrow{[A, N]} [A, B] \xrightarrow{\text{Lan}_N(\_)} [B, B])
\]
is monoidal. If moreover:
- $N$ is full-and-faithful and dense; and
- $\text{Lan}_N (N \circ \_ : [A, A] \rightarrow [B, B]_{\mathsf{cc}} \hookrightarrow [B, B])$;
then $\text{Lan}_N (N \circ \_)$ is strong monoidal.

Proof. The first part is well known.\(^{22}\) The second part is a weakening of a Math-overflow post by di-Liberti (https://mathoverflow.net/questions/321909/extending-monads-along-dense-functors).

Since $N$ is dense if and only if $\text{Lan}_N (N) = \text{id}_A$, thus for dense $N$, $\text{Lan}_N (N \circ \_)$ strictly preserves the unit. Since:
- $N$ is full-and-faithful, $N \circ F = \text{Lan}_N (N \circ F) \circ N$; and
- since $\text{Lan}_N (N \circ F)$ is co-continuous for all $F : A \rightarrow A$, $\text{Lan}_N (N \circ F)$ preserves left Kan extensions, so for any $F \circ G$ in $[A, A]$ we have that $\text{Lan}_N (N \circ F) \circ \text{Lan}_N (N \circ G) = \text{Lan}_N (\text{Lan}_N (N \circ F) \circ N \circ G)$;
thus
\[
\text{Lan}_N (N \circ F \circ G) = \text{Lan}_N (\text{Lan}_N (N \circ F) \circ N \circ G) = \text{Lan}_N (N \circ F) \circ \text{Lan}_N (N \circ G).
\]

Proof. although this author with his woeful preparation hadn’t thought about it much before this work was compiled in the context of fear, paranoia, and incipient unemployment.\(\square\)
Lemma D.1. Let \((V, \otimes, 1)\) be a symmetric monoidal category and let \(A, B,\) and \(C\) be \(V\)-categories powered in \(V\). Then, given a comma category

\[
\begin{array}{ccc}
F & \downarrow & G \\
\downarrow \text{pr}_1 & & \downarrow \text{pr}_2 \\
A & \rightarrow & B \\
\downarrow L & & \downarrow R \\
C & \rightarrow & \text{pr}_2 \\
\end{array}
\]

where \(G\) is continuous and \(C\) is complete\(^23\) it follows that, if for some diagram

\[
(a_*, f_* : L(a_*) \rightarrow R(b_*), b_*) : 1 \rightarrow L \downarrow R
\]

and some weight

\[
W : 1 \rightarrow V
\]

the weighted limits \(\lim^W a_*\) and \(\lim^W b_*\) exist in \(A\) and \(B\) respectively, then

\[
\left(\lim^W a_*, \left(\tilde{f}_*\right), \lim^W b_*\right)
\]

enjoys the universal property of the weighted limit \(\lim^W (a_*, f_*, b_*)\) - with \(\left(\tilde{f}_*\right)\) induced from the universal property\(^24\) enjoyed by \(\lim^W R(b_*) = R\left(\lim^W b_*\right)\) in \(C\).

Proof. We’ve assumed \(A, B,\) and \(C\) to be \(V\)-powered. It therefore suffices to consider conical limits.

Now, the cone

\[
L(\alpha) : L\left(\lim a_*\right) \Rightarrow L(a_*) : 1 \rightarrow C
\]

and the natural transformation

\[
f_* : L(a_*) \Rightarrow R(b_*)
\]

compose to cone \(\tilde{f}_* := f_* \circ \alpha : L\left(\lim a_*\right) \Rightarrow R(b_*)\). This cone induces, by the universal property of \(\lim R(b_*)\) enjoyed by \(R\left(\lim b_*\right)\) as \(R\) is continuous, a morphism in \(C\)

\[
\left(\tilde{f}_*\right) : L\left(\lim a_*\right) \rightarrow R\left(\lim b_*\right)
\]

whence \(\left(\lim a_*, \left(\tilde{f}_*\right), \lim b_*\right)\) is an object of \(L \downarrow R\). That \(\left(\lim a_*, \left(\tilde{f}_*\right), \lim b_*\right)\) enjoys the universal property of the limit in \(L \downarrow R\) follows immediately from the that of \(\lim a_*\) and \(\lim b_*\) in \(A\) and \(B\) respectively. \(\square\)

\(^{23}\)this can surely be weakened, but to do so in the statement of the lemma would make it more onerous to read than this author is willing to write.

\(^{24}\)in a fashion made more explicit in the proof
Corollary D.2. Let $(\mathcal{V}, \otimes, 1)$ be a symmetric monoidal category and let $\mathcal{B}$, and $\mathcal{C}$ be $\mathcal{V}$-categories powered in $\mathcal{V}$, with $\mathcal{C}$ complete. Then, given a comma category

$$
\begin{array}{c}
\mathcal{C} \\
\Downarrow R
\end{array}
\begin{array}{c}
\mathcal{B} \\
\Downarrow R
\end{array}
\begin{array}{c}
\mathcal{C} \\
\Downarrow R
\end{array}
$$

where $R : \mathcal{B} \to \mathcal{C}$ preserves all weighted limits, then $\mathcal{C} \downarrow R \to \mathcal{C} \times \mathcal{B}$ creates all weighted limits.

Proof. Again, by powering, it suffices to consider conical limits, and since $R$ preserves all weighted limits, Lemma D.1 applies and we have a formula for conical limits in $\mathcal{C} \downarrow R$. □

Preservation: from the formula we have that, when $\lim b_\bullet$ exists in $\mathcal{B}$,

$$
\lim (c_\bullet, f_\bullet : c_\bullet \to R(b_\bullet), b_\bullet) = \left( \lim c_\bullet, \left( \tilde{f}_\bullet \right), \lim b_\bullet \right)
$$

It is immediate then that $\mathcal{C} \downarrow R \to \mathcal{C} \times \mathcal{B}$ preserves limits.

Reflection: since limits and colimits in products of categories are computed factor-wise, if a cone

$$
(c, f : c \to R(b), b) \Rightarrow (c_\bullet, f_\bullet : c_\bullet \to R(b_\bullet), b_\bullet)
$$

in $\mathcal{C} \downarrow R$ passes to a limiting cone

$$
(c, b) \Rightarrow (c_\bullet, b_\bullet)
$$

in $\mathcal{C} \times \mathcal{B}$, then $c \Rightarrow c_\bullet$ and $b \Rightarrow b_\bullet$ are limiting. But then, since $R$ preserves limits, it follows from the formula that $(c, f : c \to R(b), b) \Rightarrow (c_\bullet, f_\bullet : c_\bullet \to R(b_\bullet), b_\bullet)$ is limiting.

Corollary D.3. Let $(\mathcal{V}, \otimes, 1)$ be a symmetric monoidal category and let $\mathcal{B}$, and $\mathcal{C}$ be $\mathcal{V}$-categories powered in $\mathcal{V}$, with $\mathcal{C}$ complete. Then, given a comma category

$$
\begin{array}{c}
\mathcal{C} \\
\Downarrow R
\end{array}
\begin{array}{c}
\mathcal{B} \\
\Downarrow R
\end{array}
\begin{array}{c}
\mathcal{C} \\
\Downarrow R
\end{array}
$$

where $R : \mathcal{B} \to \mathcal{C}$ creates all weighted limits, the functor $\mathcal{C} \downarrow R \to \mathcal{C}^2$ creates all weighted limits.

Proof. See that the square

$$
\begin{array}{ccc}
\mathcal{C} \downarrow R & \to & \mathcal{C} \times \mathcal{B} \\
\downarrow & & \downarrow \text{id}_\mathcal{C} \times R \\
\mathcal{C}^2 & \to & \mathcal{C} \times \mathcal{C}
\end{array}
$$
is a pullback. Since $C \downarrow R \longrightarrow C \times B$ creates all limits by Corollary [D.2], and $\text{res}$ creates all limits, to prove $C \downarrow R \longrightarrow C^2$ creates all limits, it suffices that $\text{id}_C \times R$ creates all limits, which it does as it creates the limits that $R$ does - all of them. □

**References**

[Ara2] Ara, Dimitri “Higher quasi-categories vs higher Rezk spaces” Journal of K-Theory 14(3) (2014), 701-749

[Berger1] Berger, Clemens “A Cellular Nerve for Higher Categories” Advances in Mathematics 169, 118–175 (2002)

[Berger2] - “Iterated wreath product of the simplex category and iterated loop spaces” Adv. Math. 213 (2007), 230-270.

[BergerMelliesWeber] Berger, Clemens, P.A. Mellies, M. Weber “Monads with arities and their associated theories” J. Pure Appl. Algebra 216 (2012), 2029-2048

[BergnerRezk] Bergner, Julie and Charles Rezk “Reedy categories and the $\Theta$-construction” Math. Z. 274 (2013), no. 1-2, 499–514.

[Borceux] Borceux, F. (1994). “Handbook of Categorical Algebra” (Encyclopedia of Mathematics and its Applications). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511525865

[BourkeGarner] Bourke, John and Richard Garner "Monads and theories" Advances in Mathematics 351 (2019), p.1024–1071

[BousfieldFriedlander] Bousfield, A. K. and E. M. Friedlander. “Homotopy theory of $\Gamma$-spaces, spectra, and bisimplicial sets” Springer Lecture Notes in Math., Vol. 658, Springer, Berlin, 1978, pp. 80-130.

[Brown] Brown, Kenneth S. “Abstract Homotopy Theory and Generalized Sheaf Cohomology” Transactions of the American Mathematical Society, Vol. 186 (1973), 419-458 (jstor:1996573).

[ChenKrizPultr] Chen, Ruian Igor Kriz and Ales Pultr “Kan’s Combinatorial Spectra and their Sheaves Revisisted” to appear in Theory and Applications of Categories

[Cisinski1] Cisinski, Denis-Charles “Les préfaisceaux comme modèles des types d’homotopie” Astérisque 308 (2006), xxiv+392 pp.

[CisinskiMaltsiniotis] Cisinski, Denis-Charles and Georges Maltsiniotis "La catégorie $\Theta$ de Joyal est une catégorie test". J. Pure Appl. Algebra, 215, pp. 962-982 (2011).

[Elmendorf] Elmendorf, Anthony “The Grassmanian Geometry of spectra” Journal of Pure and Applied Algebra Volume 54, Issue 1, October 1988, Pages 37-94

[ElmendorfKrizMandellMay] A. D. Elmendorf , I. Kriz , M.A. Mandell , J.P. May "Rings, Modules, and Algebras in Stable Homotopy Theory" 1995, American Mathematical Society Surveys and Monographs, American Mathematical Society Providence, R.I.

[Grothendieck] Grothendieck, A. “Pursuing stacks” 1983, manuscrit. À paraître dans la série Documents Mathématiques, Soc. Math. France.

[Hovey] “Spectra and Symmetric Spectra in General Model Categories" Journal of Pure and Applied Algebra, Volume 165, Issue 1, 23 November 2001, Pages 63-127

[Joyal] Joyal, Andre “Disks, duality and Theta-categories” https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf

[Kan] Kan, Daniel “Semisimplicial spectra” Illinois Journal of Mathematics, Volume 7, Issue 3 (1963), 463-478.

[Kelly] Kelly, G. Maxwell “The Basic Concepts of Enriched Category Theory.” Reprints in Theory and Applications of Categories [electronic only]. 2005.

[Lack] Lack, S. “Limits for Lax Morphisms” Appl Categor Struct 13, 189–203 (2005). https://doi.org/10.1007/s10485-005-2958-5

[LessardThesis] Lessard, Paul Roy “Spectra as Locally Finite Pointed $\mathbb{Z}$-groupoids” University of Colorado at Boulder. August 2019: 169 pages; 22589944.

[Lessard2] - “$\mathbb{Z}$-Categories II” zREF
[Ulmer] Ulmer, Friedrich. “Properties of dense and relative adjoint functors.” Journal of Algebra 8 (1968): 77-95.

[Weiner] Wiener, Norbert. 1990. God and Golem, Inc.: a comment on certain points where cybernetics impinges on religion. Cambridge, Mass: Massachusetts Institute of Technology Press.