Implications of mean field accretion disc theory for vorticity and magnetic field growth

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In addition to the scalar Shakura-Sunyaev $\alpha_{ss}$ turbulent viscosity transport term used in simple analytic accretion disc modeling, a pseudoscalar transport term also arises. The essence of this term can be captured even in simple models for which vertical averaging is interpreted as integration over a half-thickness and one separately studies each hemisphere. The additional term highlights a complementarity between mean field magnetic dynamo theory and accretion disc theory treated as a mean field theory. Such pseudoscalar terms have been studied, and can lead to large scale magnetic field and vorticity growth. Here it is shown that vorticity can grow even in the simplest azimuthal and half-height integrated disc model, for which mean quantities depend only on radius. The simplest vorticity growth solutions seem to have scales and vortex survival times consistent those required for facilitating planet formation. Also it is shown that when the magnetic back-reaction is included to lowest order, the pseudoscalar driving the magnetic field growth and that driving the vorticity growth will behave differently with respect to shearing and non-shearing flows: the former can reverse sign in the two cases, while the latter will have the same sign.

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1. Introduction

Analytic accretion disc theory and mean field magnetic dynamo theory are often thought of disjointly, but the two are intimately related in certain important senses and exploring this relation has some interesting consequences. In traditional $\alpha$ analytic accretion disc theory, (e.g. Shakura & Sunyaev 1973; Pringle 1981; Balbus-Hawley 1998) the Reynolds+Maxwell stress tensor representing a correlation of turbulent velocities, is replaced by $\alpha_{ss} c_s^2$, where $c_s$ is the sound speed. In the Navier-Stokes equation, this amounts to replacing the microphysical viscosity by a turbulent viscosity, $\nu = \alpha_{ss} c_s H$, where $H$ is the scale height. The flow variables then being solved for must be interpreted as mean fields, with gradient scales larger than those of the turbulent motions providing the viscosity. (Consequences of fluctuations for variability were studied in Balbus et al. 1994; Blackman 1998; 2000a). The continuity, radial momentum, angular momentum, and energy equations are solved, often with the assumption of axisymmetry, for radial solutions. The magnetic field is often added in as a pressure, rather than coupling in the magnetic induction equations (e.g. Narayan & Yi 1994; Narayan et al. 1998).

Indeed it is widely known that the $\alpha$ viscosity approach is incomplete. For more realistic discs, the dynamical evolution of the magnetic field must be coupled to the fluid equations. The problem is fully non-linear, and ultimately requires non-linear MHD turbulence simulations understanding magneto-shearing instabilities etc. (c.f. Balbus & Hawley 1991,1998, Brandenburg et al. 1995; Miller & Stone 2000). Nevertheless, the $\alpha$ viscosity approach provides a useful phenomenological framework for exploring solutions. This is where the relationship to mean field dynamo theory enters.

As traditional accretion disc theory ignores the induction equation, traditional kinematic mean field dynamo theory solves only the magnetic induction equation (e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Rädler 1999) taking the velocity that appears in that equation as given and not subject to the influence of the field. Thus traditional accretion disc theory and kinematic magnetic dynamo theory are complements of each other. Mean field dynamo theory in its analytic regime, kinematic or dynamic, is also subject to the complementary criticisms of accretion disc theory in that rigorously the fully dynamical set of equations have to be solved and the physics of MHD turbulence must be understood. For a sheared accretion disc, the problem of accretion and the problem of field growth are one fully coupled problem. Disc simulations with appropriate boundary conditions may show some evidence for mean field dynamo operation in a turbulent accretion disc (Brandenburg & Donner 1997), though higher magnetic Reynolds number simulations are needed. Note also that the mean field dynamo is not to be confused with turbulent amplification ($\equiv$ small scale dynamo). The former models an inverse cascade of magnetic helicity (e.g. Pouquet et
al. 1976), not simply field line stretching on the scale of the turbulent velocities. In addition, the mean field dynamo is often invoked to generate a mean magnetic flux, not just a mean field energy.

The formalism of mean field theory as applied to the induction equation leads to both a viscosity term that corresponds to the Shakura-Sunyaev $\alpha$ viscosity term, and an additional pseudoscalar term which can be similarly parameterized. As applied to the Navier-Stokes equation, the same formalism leads again to a Shakura-Sunyaev $\alpha_{ss}$ type viscosity term, and also to a pseudoscalar helicity term. The latter would vanish if the vertical averaging is interpreted as integration over the entire disc height, but does not vanish when the integration is taken over a half scale height (or less) and one considers each hemisphere independently. The pseudoscalar term has been recognized elsewhere (e.g. Moiseev et al 1983; Frisch et al. 1987; Khomenko et al 1991; Kitchatinov 1994ab; Tanga et al. 1996; Blackman & Chou 1997). Here it will be shown that it can be treated on par with the Shakura-Sunyaev viscosity. The pseudoscalar term can lead to vorticity growth in discs.

Enstrophy exhibits an inverse cascade in 2-D turbulence (Kraichnan & Montgomery 1980). The growth of vorticity in primarily 2-D rotating fluids has been seen in nature (e.g. Jupiter, c.f. Ingersoll 1990; Marcus 1993) as well as in simulation (Marcus 1990; McWilliams 1990) and experiment (e.g. Sommeria et al. 1988). Statistical mechanics approaches have been successful in modeling this (e.g. Chavanis & Sommeria 1998). Vorticity growth in sheared accretion discs is less well studied (reference added in proof: Godon & Livio 2000), even though vortex evolution has been studied (e.g. Adams & Watkins 1995; Godon & Livio 1999; Chavanis 2000). It will be later shown that an instability results in a sheared disc even when mean quantities are integrated to be only a function of radius in each hemisphere. Note here that the use of a pseudoscalar requires the turbulence to be 3-D even if the mean fields are taken to be 2-D.

In sections 2 and 3 the mean vorticity and mean magnetic field equations are derived with the back-reaction between the two included to lowest order, under the assumption that 1st order cross-correlations between fluctuating fields and velocities are small. In section 4, the resulting diffusion and pseudoscalar transport coefficients a parameterized to be proportional to the sound speed and, growing solutions for vorticity are derived. In section 5 the relation to vorticity growth for planet formation is discussed. Why the pseudoscalar growth coefficient for vorticity can have the opposite sign to that driving mean magnetic field growth is also discussed. Section 6 is the conclusion.
2. Derivation of Vorticity Equation

The Navier-Stokes equation with B-fields is given by

\[ \partial_t \mathbf{v} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla p/\rho - \nabla v^2/2 + \nu \nabla^2 \mathbf{v} + (\zeta + \frac{1}{3} \eta) \nabla (\nabla \cdot \mathbf{v}) + \mathbf{m} + \mathbf{F}(x, t) + \nabla \phi, \]

(1)

where \( \nu \) and \( \zeta \) are the constant viscosity and second viscosity and \( p \) is the pressure and \( \mathbf{m} \equiv (\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla \mathbf{B}^2)/4\pi \rho \). The \( \nabla \phi \) includes the potential force of gravity. The vorticity equation,

\[ \partial_t \omega = \nabla \times (\mathbf{v} \times \omega) + \nu \nabla^2 \omega + \nabla \rho \times \nabla p/\rho^2 + \nabla \times \mathbf{F}(x, t) + \nabla \times \mathbf{m}, \]

(2)

where \( \omega \equiv \nabla \times \mathbf{v} \), is obtained by taking the curl of (1). Hereafter we ignore the microphysical viscosities.

Now divide quantities such as \( \omega, \mathbf{v}, \) etc. into mean (indicated by an overbar or \( \langle \rangle \)) and fluctuating (indicated by prime) components \( \mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}' \) and \( \omega = \bar{\omega} + \omega' \) and respectively. The interest is in the application to a thin disc. I take the average to mean full integration over azimuth, vertical integration over the top half of the disc only, and local radial averaging, leaving mean quantities only a function of radius in each hemisphere, considering the top and bottom hemispheres of the disc separately. This is important because pseudoscalar averages would vanish if the average is taken over the whole disc.

The presence of \( \mathbf{F}(x, t) \) represents any function (e.g. forcing) not included in the other terms. It will drop out later. Note that ultimately, pseudoscalar generation results from a vertical density gradient combined with the shear and underlying rotation. Later I will parameterize the pseudoscalar in a similar way that Shakura & Sunyaev (1973) parameterize the viscosity.

It is assumed that derivatives with respect to \( x \) or \( t \) obey \( \partial_{x,x} \langle X_i X_j \rangle = \langle \partial_{x,x}(X_i X_j) \rangle \) and \( \langle \bar{X}_i X'_j \rangle = 0 \) (Reynolds relations (Rädler 1980)), where \( X_i = \bar{X}_i + X'_i \) are components of vector functions of \( x \) and \( t \). For the spatial mean, defined by \( \langle X_i(x, t) \rangle = |\zeta|^{-3} \int_{-L}^{+L} X_i(s, t) d^3s \), the relations hold when the averaging is taken over a large enough scale, such that \( l \ll |\zeta| \ll L \), where \( L \sim \bar{v}/\nabla \bar{v} \), and \( \ell \sim v'/\nabla v' \).

Subtracting the mean of (1) from itself, and assuming \( \nabla \phi' = 0 \), gives

\[ \partial_t \mathbf{v}' = \langle \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle - \mathbf{v}' \cdot \nabla \mathbf{v}' - \bar{\mathbf{v}} \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \bar{\mathbf{v}} - \nabla p'/\rho - \nabla (\mathbf{v}' \cdot \mathbf{v}) + \mathbf{F}'(x, t) + \mathbf{m}' \]

(3)

ignoring the viscosities. To proceed, (2) needs to be averaged. To simplify, assume that the density has only a mean time independent spatially varying quantity, and ignore its
fluctuating gradients. Then, since the means are only functions of the radial coordinate, the pressure term drops out of the vorticity. (The mean pressure thus satisfies the barotropic equation of state $\bar{p} = \bar{p}(\bar{\rho})$.) This gives

$$d_t\bar{\omega} = \nabla \times (v' \times \omega') + \nabla \times \bar{m} + \bar{\omega} \cdot \nabla \bar{v} - \bar{\omega} \nabla \cdot \bar{v},$$

(4)

where I neglected terms second order in time-varying mean quantities and and $d_t$ indicates working in the Local Standard of Rest (LSR) frame, i.e. that which moves with $\bar{v}$. I have also assumed $\mathbf{F} = \mathbf{F}'$. Subtracting (4) from (2) (ignoring the pressure term for reasons described above) gives

$$d_t\omega' = \omega' \cdot \nabla \bar{v} - \omega' \nabla \cdot \bar{v} + \bar{\omega} \cdot \nabla v' - \bar{\omega} \nabla \cdot v'$$

$$-v' \cdot \nabla \bar{\omega} + \omega' \cdot \nabla v' - \omega' \nabla \cdot v' - v' \cdot \nabla \omega' - \nabla \times (v' \times \omega') + \nabla \times m'$$

$$+ \nabla \times \mathbf{F}'(x, t).$$

(5)

For $\bar{\omega}$ to grow, the $\nabla \times$ terms in (4) must be non-vanishing.

Following previous work (c.f. Blackman & Chou 1997; Field et al. 1999), I expand the turbulent quantities on the right of Eqs. (4) to linear order in $\nabla \bar{v}$ using the equations for the fluctuating fields here it is assumed that correlations of zeroth order quantities can be functions of radius. To find the lowest order terms, assume weakly anisotropic turbulence: terms linear in the mean shear contribute, but their averaged $0^{th}$ order coefficients, are taken to be isotropic. These coefficients can be reflection asymmetric and radially dependent.

In order to “ignore” the terms which are products of one zeroth order turbulent quantity with one 1st order turbulent quantity (as I will do here) one assumes that these terms are small compared to the associated terms that involve products of one zeroth order quantity with one mean quantity. For example, one would assume that the 1st term on the right of (3) is greater than the sixth term on the right of (2). This weaker than the “usual” first order smoothing approximation in that the present requirement amounts to $\omega^{(1)} < \nabla \bar{v}$ and $v^{(1)} < \nabla$, rather than the usual $\omega' < \nabla \bar{v}$ or $v' < \nabla$. The latter two conditions are stricter as they apply to the total fluctuating quantities rather than only the anisotropic part.

An alternative potential justification for dropping the troubling terms that are products of one zeroth order turbulent quantity with one 1st order turbulent quantity, is that they themselves likely have positive and negative contributions which might nearly cancel: In a steady state, the turbulent quantities are balanced by input and cascade. This motivates a possible replacement of e.g. the sum of 6th 7th 8th and 9th terms by $(\chi - \xi)\omega^{(1)}$, where $\chi$ and $\xi$ are positive, and their difference represents the combination of growth and decay. (The 9th term is actually irrelevant as it vanishes when correlated with a fluctuating quantity, which is the only context in which it will enters.) Ignoring these terms would then amount
to the assumption that $\zeta - \xi$ is small. Such an approach avoids comparing each individual "offending" term to those dependent on the mean fields, since here the offending terms would cancel themselves.

Working in the LSR frame is also important to emphasize here. Although the turbulence in a sheared disc is more than weakly anisotropic, by working in the LSR frame, the anisotropy then manifests through terms like the first in (5) $\omega'(0) \cdot \nabla \bar{v}$ rather than through terms like $\bar{v} \cdot \nabla \omega'(0)$. To assess the implications, terms like $v' \cdot \nabla \bar{v}$ must be compared to terms like $v'(0) \cdot \nabla v'(0)$. For accretion discs whose turbulence is ultimately driven by a magneto-shearing instability, the ratio of these terms is of order $v'(0)/\Omega (v'(0)^2/l_T)$. But $v'(0)/l_T \sim \Omega$ for the magneto-shearing instability, so the anisotropy is of order 1 in the LSR frame, compared to $>> 1$ in the lab frame. Indeed, even in the LSR, this still means that the anisotropy should be considered to more than linear order, but because it is "only" of order 1 we expect qualitative similarities of the results to the expansion to all orders. Blackman (2000b) gave a restricted approach to treating this shear anisotropy to all orders.

Using the formal solutions for the turbulent field $\omega'(t) = \omega'(t = 0) + \int d\tau \omega'^{t}$, and using times appropriately chosen such that the correlation $\langle \omega'(t) \times \omega'(0) \rangle \approx 0$, we get

$$\langle \omega' \times \omega' \rangle^{(1)} = \langle \omega'(0)(t) \times \int_0^t d\tau \omega'^{(1)} \rangle + \langle \int_0^t d\tau \omega'^{(1)} \times \omega'(0)(t) \rangle,$$

(6)

The calculation of these averages requires Eqs. (3) and (5) for the time integrals, invoking the approximations discussed above. Using (3) also requires an expression for the pressure, which arises in (3) via the term

$$\overline{\rho^{-1} \omega'(0)} \times \int_0^t \nabla \rho^{(1)} dt',$$

(7)

where the $\overline{\rho}$ is pulled out under the assumption that it is time independent. Using isotropy, homogeneity, and Reynolds rules, Blackman and Chou (1997) showed that terms of the form (7) vanish in the derivation of the mean field equations to the first order considered. However in the present case, the averages of statistical correlations can be functions of radius. Thus several terms dropped from Blackman & Chou (1997) must be considered. Following that approach, the energy equation can be used to obtain

$$\nabla p^{(1)} = \nabla p^{(1)}(0) - \nabla \int (\rho^{(0)} \cdot \nabla \rho + \rho \cdot \nabla \rho^{(0)})$$

$$+ \gamma \rho^{(0)} \nabla \cdot \vec{v} + \gamma \rho \nabla \cdot \vec{v}' \)dt',$$

(8)

where $\gamma$ is the adiabatic index and we must now relax the $\nabla \cdot \bar{v}'$ constraint. To simplify the analysis, I have ignored any radiative contributions to higher than zeroth order quantities.
The pressure dependent contribution to $\langle \mathbf{v}' \times \mathbf{\omega}' \rangle^{(1)}$ can then be written
\begin{align*}
\langle \mathbf{\omega}'^{(0)} \times \int_0^t dt' \nabla p' dt' \rangle_k &= \epsilon_{ijk} \epsilon_{ims} \int_0^t dt' \int_0^t dt'' (\partial_j \bar{\nu}_l \partial_i p''(0) + \bar{\nu}_l \partial_j \partial_i p''(0) + \partial_j v_i''(0) \partial_l \bar{p} + v_l''(0) \partial_j \partial_l \bar{p} \\
&+ \gamma p''(0) \partial_j \bar{\nu}_l + \gamma \partial_j p''(0) \partial_l \bar{\nu}_l + \gamma \partial_j \bar{\nu}_l \partial_l v_i''(0) + \gamma \partial_j \bar{p} \partial_l v_i''(0)) (\partial_m v_s''(0) - \partial_s v_m''(0))
\end{align*}
\begin{equation}
= \frac{2\pi}{3} \int_0^t dt' \bar{p} \nabla \times \langle (\nabla \mathbf{v}'')^2(0) - \partial_k \nabla \mathbf{v}' \partial_i v_j''(0) - \mathbf{\omega} \langle \nabla p''(0) \cdot \nabla \times \mathbf{v}'''(0) \rangle \rangle
\end{equation}

where it is assumed that mean fields vary on time scales longer than the fluctuating fields, and $\nabla p''(1)(t = 0)$ is uncorrelated with or $\mathbf{\omega}'(t)$. The vanishing of terms to get to the last equality follows from careful application of isotropy (i.e. rank 2 and rank 3 averaged tensors of fluctuating 0th order quantities are proportional to $\delta_{ij}$ and $\epsilon_{ijk}$ respectively), but homogeneity of the 0th order turbulence has not been used (i.e. $\partial_i \langle X_j X_k \rangle^{(0)} = 0$) nor has $\nabla \cdot \mathbf{v}' = 0$ been used. Had homogeneity been used, then the first term on the right of the last equality would not have survived. Note however, that when put inside the curl of (4), the first two terms on the right of (9) vanish. This is because the curl of the pressure gradient vanishes, and all gradients are in the radial direction, so the cross product of gradients vanishes. Finally, the last term in (9) vanishes from use of Reynolds rules and isotropy, which is seen from using the chain rule with the gradient on $p''(0)$. The pressure does not seem to contribute to lowest order under the approximations used herein.

Collecting all of the above, crudely approximating time integrals by factors of the correlation time $\tau_c$ (Ruzmaikin et al. 1988), and freely employing Reynolds rules and incompressibility of fluctuating components,
\begin{equation}
\nabla \times \langle \mathbf{v}' \times \mathbf{\omega}' \rangle^{(1)} = \nabla \times \alpha_0 \mathbf{\omega} - \nabla \times (\beta_0 \nabla \times \mathbf{\omega})
\end{equation}

where it is assumed $\mathbf{F} = \mathbf{F}^{(0)}$, and the microphysical viscosity is ignored. The coefficients are
\begin{equation}
\begin{align*}
\alpha_0 &= (\tau_c/3) \langle \mathbf{\omega}^{(0)} \cdot \mathbf{v}^{(0)} \rangle \\
\beta_0 &= (\tau_c/3) \langle \mathbf{v}^{(0)} \cdot \mathbf{v}^{(0)} \rangle + \mathbf{b}^{(0)} \cdot \mathbf{b}^{(0)}
\end{align*}
\end{equation}

where $\mathbf{b}^{(0)} \equiv \mathbf{B}^{(0)} / 4\pi \bar{\rho}$. Upon substituting these into (10),
\begin{equation}
d_i \mathbf{\omega} = \nabla \times \alpha_0 \mathbf{\omega} - \nabla \times (\beta_0 \nabla \times \mathbf{\omega}) + \mathbf{\omega} \cdot \nabla \mathbf{v} - \mathbf{\omega} \nabla \cdot \mathbf{v}.
\end{equation}

This equation presumes that 1st order cross correlation terms vanish, that is
\begin{equation}
\langle \mathbf{b}^{(0)} \cdot \mathbf{v}^{(0)} \rangle = \langle \mathbf{\omega}^{(0)} \cdot \mathbf{b}'^{(0)} \rangle = \langle \mathbf{\omega}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle = 0.
\end{equation}

When this assumption is not made, then the mean magnetic field and mean vorticity field equations are coupled (Blackman & Chou 1997). Note that since the averaging is such that
the mean fields are only functions of radius, the penultimate term in (12) vanishes. The last term vanishes by using the continuity equation for a steady mean density (which has only a mean component) which gives \( \overline{\rho} \nabla \cdot \mathbf{v} = -\mathbf{v} \cdot \nabla \overline{\rho} \), and then recalling that we are working in the LSR frame. Thus

\[
d_t \boldsymbol{\omega} = \nabla \times \alpha_0 \boldsymbol{\omega} - \nabla \times (\beta_0 \nabla \times \boldsymbol{\omega})
\]

is the equation to be solved. Before proceeding to do so, I derive the magnetic field growth equation for comparison.

### 3. Derivation of Magnetic Field Equation

As mentioned above when cross correlations between velocity and magnetic field components are ignored, the mean field equations for the magnetic field and the vorticity field decouple for the case in which the gradient of the fluctuating components of the density are ignored. The induction equation for \( \mathbf{B} \) is then

\[
\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nu_M \nabla^2 \mathbf{B},
\]

Similarly, the equation for the mean \( \mathbf{B} \)-field, derived by averaging (15) is

\[
d_t \overline{\mathbf{B}} = \nabla \times \langle \mathbf{v}' \times \mathbf{B}' \rangle + \overline{\mathbf{B}} \cdot \nabla \overline{\mathbf{v}},
\]

where the resistivity has been ignored and \( \nabla \cdot \overline{\mathbf{v}} \) has been assumed. Subtracting (16) from (15) yields the equation for the fluctuating \( \mathbf{B} \)-field

\[
d_t \mathbf{B}' = \mathbf{B}' \cdot \nabla \overline{\mathbf{v}} - \mathbf{v} \cdot \nabla \mathbf{B}' + \mathbf{B} \cdot \nabla \mathbf{v}' - \overline{\mathbf{B}} \nabla \cdot \mathbf{v}' - \mathbf{v}' \cdot \nabla \overline{\mathbf{B}} + \mathbf{B}' \cdot \nabla \mathbf{v}' - \mathbf{B} \nabla \cdot \mathbf{v}' - \mathbf{v}' \cdot \nabla \mathbf{B}' - \nabla \times \langle \mathbf{v}' \times \mathbf{B}' \rangle
\]

again ignoring the resistivity, and assuming \( \nabla \cdot \overline{\mathbf{v}} = 0 \). Following the same procedure to first order in \( \overline{\mathbf{B}} \) and \( \overline{\mathbf{v}} \) for \( \langle \mathbf{v}' \times \mathbf{B}' \rangle \) in (16) that was followed above for \( \langle \mathbf{\omega}' \times \mathbf{v}' \rangle \) assuming (13) holds, the mean field induction equation becomes

\[
d_t \overline{\mathbf{B}} = \nabla \times \alpha_{m0} \overline{\mathbf{B}} - \nabla \times (\beta_{m0} \nabla \times \overline{\mathbf{B}}) + \overline{\mathbf{B}} \cdot \nabla \overline{\mathbf{v}},
\]

where the coefficients are

\[
\alpha_{m0} = (\tau_c/3)(2\langle \mathbf{B}'^{(0)} \cdot \nabla \times \mathbf{B}'^{(0)} \rangle/4\pi \overline{\rho} - \langle \mathbf{v}'^{(0)} \cdot \nabla \times \mathbf{v}'^{(0)} \rangle)
\]

\[
\beta_{m0} = (\tau_c/3)(2\langle \mathbf{B}'^{(0)} \cdot \mathbf{B}'^{(0)} \rangle/4\pi \overline{\rho} + \langle \mathbf{v}'^{(0)} \cdot \mathbf{v}'^{(0)} \rangle)
\]

and we have used \( \overline{\rho} = \overline{\rho} \).
4. Instability growth

Since the mean quantities are taken to be only a function of radius in each hemisphere, it is convenient to break up the vorticity equation into toroidal and poloidal components. Recalling that we are working in the LSR frame so that we can ignore terms that include factors of $\vec{v}$ without derivatives, the two resulting equations that we need to solve are then of the form

$$
d_t \omega_\phi = \alpha_0 \nabla \times \vec{w}_P + \nabla \alpha_0 \times \vec{w}_P + \beta_0 \nabla^2 \omega_\phi - \nabla \beta_0 \times (\nabla \times \omega_\phi)$$

and

$$
\partial_t \bar{v}_\phi = \alpha_0 \bar{w}_\phi + \beta_0 \nabla^2 \bar{v}_\phi,
$$

where the subscript $\phi$ indicates the toroidal component and $P$ the poloidal component.

Writing these equations for their components in cylindrical coordinates, and recalling that only the radial spatial derivatives of mean quantities contribute, gives

$$
\partial_t \bar{w}_\phi = -\alpha_0 \partial_r^2 \bar{v}_\phi - \frac{\alpha_0}{r} \partial_r \bar{v}_\phi + \frac{\alpha_0}{r^2} \bar{v}_\phi + \beta_0 \partial_r \bar{w}_\phi + \frac{\beta_0}{r} \partial_r \bar{w}_\phi - \frac{\beta_0}{r} \partial_r \partial_r \bar{w}_\phi
$$

and

$$
\partial_t \bar{v}_\phi = \alpha_0 \bar{w}_\phi + \beta_0 \partial_r^2 \bar{v}_\phi + \frac{\beta_0}{r} \partial_r \bar{v}_\phi - \frac{\beta_0}{r^2} \partial_r \bar{v}_\phi - \frac{\beta_0}{r^2} \partial_r \bar{w}_\phi.
$$

We look for solutions to (22) and (23) of the form

$$
\bar{w}_\phi = \bar{w}_{\phi0}(r)e^{ikr+nt} \quad \text{and} \quad \bar{v}_\phi = \bar{v}_{\phi0}(r)e^{ikr+nt},
$$

where $k_r$ is the radial wave number, $v_k$ is the Keplerian speed, $v_{\phi0}(r) = v_k(r) \propto r^{-1/2}$, and $\bar{w}_{\phi0}(r) \propto r^{-p}$. We can take $p = 3/2$ since all velocities will scale with the Keplerian speed, and the vorticity scales with the associated curl.

Since the correlation coefficients can also depend on $r$, this dependence must be addressed. From (??) it is evident that $\beta_0$ is the turbulent viscosity which can be parameterized in the Shakura-Sunyaev (Shakura & Sunyaev 1973) form, that is

$$
\beta_0 = \alpha_{ss} c_s h,
$$

where $c_s$ is the sound speed, $h$ is the disc half thickness, and $\alpha_{ss}$ is the Shakura-Sunyaev parameter. Similarly, $\alpha_0$ can be parameterized as

$$
\alpha_0 = q \alpha_{ss} c_s = q \beta_0 / h,
$$
where \( q \) satisfies \( -1 < q < 0 \) in the top half of the disc, and \( 0 < q < 1 \) in the bottom half. Since \( c_s \approx \frac{h v k}{r} \) for \( h \propto r, \beta_0 \propto r^{-1/2} \) and \( \alpha_0 \propto r^{-3/2} \). Taking into account the radial dependences, and plugging (24) into (22) and (23) gives

\[
\left[ n + \beta_0 k_r^2 - \frac{3 \beta_0}{2 r^2} + \frac{5 i k_r \beta_0}{2 r} \right] \tilde{\omega}_\phi - \left[ \frac{3 i k_r \alpha_0}{2 r} + \frac{3 \alpha_0}{2 r^2} + \alpha_0 k_r^2 \right] \tilde{v}_\phi = 0 \tag{27}
\]

and

\[- \alpha_0 \tilde{\omega}_\phi + \left[ n + \beta k_r^2 + \frac{3 \beta_0}{4 r^2} \right] \tilde{v}_\phi = 0. \tag{28}\]

Solving for \( n \) we have

\[n = -\beta_0 k_r^2 - \frac{5 i k_r \beta_0}{4 r} \pm \frac{1}{\sqrt{2}} \sqrt{\left[ \alpha_0^2 k_r^2 - \frac{25 k_r^2 \beta_0^2}{16 r^2} + \frac{3 i k_r \alpha_0^2}{2 r} \right]^2 + \left[ \frac{\alpha_0^2 k_r^2 - \frac{25 k_r^2 \beta_0^2}{16 r^2} + \frac{3 i k_r \alpha_0^2}{2 r}}{2} \right]^2} \tag{29}\]

To find the real part of \( n \), first write the bracketed term on the right of (29) as

\[(a + bi) = (c + di)^{1/2}, \tag{30}\]

where

\[c = \alpha_0^2 k_r^2 - \frac{25 k_r^2 \beta_0^2}{16 r^2} = \left( \frac{k_r \beta_0}{h} \right)^2 \left( q^2 - \frac{25 h^2}{16 r^2} \right), \tag{31}\]

and

\[d = 3 k_r \alpha_0^2 / 2 r = \frac{3 k_r q^2 \beta_0^2}{2 r h^2}, \tag{32}\]

and where (23) and (26) have been used. Note that \( a, b, c, d \) \((31), (32)\) are all real. Solving for \( a \) and \( b \) we get

\[a = \pm \frac{1}{\sqrt{2}} \left[ c + (c^2 + d^2)^{1/2} \right] \tag{33}\]

and

\[b = \pm \frac{1}{\sqrt{2}} \left[ -c + (c^2 + d^2)^{1/2} \right]. \tag{34}\]

Combining with (24) we get for the real part of \( n \):

\[\text{Re}(n) \approx -\beta_0 k_r^2 \pm \frac{1}{\sqrt{2}} \left( \frac{k_r \beta_0}{h} \right) \left[ q^2 - \frac{25 h^2}{16 r^2} + \left( q^4 + \left( \frac{25 h^2}{16 r^2} \right)^2 \right)^{1/2} \right] \tag{35}\]

where in calculating \( (c^2 + d^2) \), I have neglected terms of order \( q^2 k_r^2 / h^2 r^2 \) and \( q^4 / h^4 r^2 \) compared to a term of the order \( q^4 k_r^2 / h^4 \), on the assumption that the relevant growth modes have \( k >> 1/r \) (verified later).
For $q < 5h/4r$, only the negative term in (35) survives and we have only decaying solutions. For $q >> 5h/4r$ we have

$$Re(n) = \pm k_r \beta_0 q/h - k_r^2 \beta_0.$$  \hspace{1cm} (36)

Taking the positive sign, growth occurs for $k_r < q/h$ and the maximum growth rate occurs at $k_{\text{max}} = q/2h$ with maximum growth rate $q^2 \beta_0/4h^2$. The growth is strongly dependent on $q$.

A similar analysis shows that the imaginary part of $n$ is given by

$$Im(n) \simeq -\frac{5k_r \beta_0}{4r} \pm \frac{1}{\sqrt{2}} \left( \frac{k_r \beta_0}{h} \right) \left[ \frac{25h^2}{16r^2} - q^2 + \left( q^4 + \left( \frac{25h^2}{16r^2} \right)^2 \right)^{1/2} \right]^{1/2},$$ \hspace{1cm} (37)

for which only the first term on the right contributes when $q >> 5h/4r$. When $q << 5h/4r$ the positive solution gives zero imaginary part, whilst the negative solution gives $Im(n) = -5k_r \beta_0/2r$.

5. Discussion

5.1. Planet formation

The simple approach in the previous section turns out to provide vorticity growth with numbers comparable to what is required by planet formation.

Studies of planet formation have invoked the idea of trapping dust particles in vortices, to provide the required agglomeration of material (Barge & Sommeria 1995; Tanga et al 1996; Hodgson & Brandenburg 1998 Godon & Livio 1999; Bracco et al. 1999 Chavanis 2000) In all of the above references, anti-cyclonic vortices (vortices opposed to the underlying rotation) are the only ones of interest for planet formation. Consider a vortex centered in the disc at radius $r_0$. Dust particles entering the vortex from $r > r_0$ lose angular momentum and fall inward toward the vortex core, while those entering from $r < r_0$ gain angular momentum and move outward, also toward the radius of the vortex core. While there is a limited size range of dust particles that couple to the vortex, the result is migration of material to the center for suitable anti-cyclonic vortices.

To see that the discussion of the previous section leads to anti-cyclonic vortex growth, first note that ultimately, $\alpha_0$ results from a vertical density gradient and the Coriolis force (hidden in the averaging rather than explicitly considered), so $\alpha_0$ is positive in the northern hemisphere and negative in the southern hemisphere. Now the dominant growth term on
the right of (22) is the first term. That term is thus negative in the north and positive in the south. Thus the toroidal vorticity grows a negative contribution in the north and positive contribution in the south. This means that in (23), the toroidal velocity and thus the poloidal vorticity grow negative contributions (anti-cyclonic) in the north and also in the south.

If we consider the case for which $q \sim 1 \gg h/r$, then we have $k_{\text{max}} \sim 1/2h$ and $n_{\text{max}} = \beta/4h^2$ from the previous section. When the vortex reaches saturation, its characteristic evolution time would then be given by the oscillating part of the solution. From the $q \gg 5h/4r$ limit of (37) we have

$$\tau_v \sim 2rh/\beta_0 \sim \frac{2r}{\alpha_{ss}c_s} \sim \frac{2r}{h\alpha_{ss}\Omega},$$

(38)

where we have used $k \sim k_{\text{max}}$. For $\alpha_{ss} \sim 0.01$ and $h/r \sim 0.1$, this limit would give $\gtrsim 300$ orbital periods. This is consistent with what is required to form planets (c.f. Godon & Livio 1999).

The saturated energy density associated with the mean field vortex should be no greater than that associated with the turbulent energy density. Using the fact that the growth time for the magneto-shearing instability driving the growth of turbulence in accretion discs is of order the dominant eddy turnover time, and is equal to the rotation period $\Omega^{-1}$, we have for the viscosity $\beta_0 = \alpha_{ss}c_s h \sim v_T^2/\Omega$, where $v_T$ is the dominant turbulent speed. Thus $v_T \sim \alpha_{ss}^{1/2}c_s$ and a vortex with the associated velocity for a mode of $k_r \sim q/h$ would have

$$\omega \sim \alpha_{ss}^{1/2}c_s k_{\text{max}} \approx \alpha_{ss}^{1/2}\Omega |q| \sim 0.1\Omega,$$

(39)

for $q^2 \gg h^2/r^2$, and $\alpha_{ss} \sim 0.01$. This is typical of that employed by by Godon & Livio (1999).

Rewriting Godon & Livio’s (1999) allowed range in size of surviving sub-sonic vortices we have

$$\frac{\beta_0}{v_T} \leq k^{-1} \leq \frac{v_T}{\partial_r \Omega},$$

(40)

where the lower bound is a viscous length, the upper bound ensures that the vortex is not subject to destruction by shear, and we have used $v_T$ for the vortex speed. Using the relations for $v_T$ and $\beta_0$ above, we get the condition $\alpha_{ss}^{1/2}h \leq k^{-1} \leq \alpha_{ss}^{1/4}(rh)^{1/2}$. For $k \sim h^{-1}$ the lower inequality is always satisfied, and the upper inequality is satisfied for $h/r \leq \alpha^{1/2}$ which is consistent with the range allowed at least in FU Orionis for which $\alpha_{ss}$ could be even as large as 0.1 (Bell et al. 1995).

These values are encouraging but it is also important to understand why a vortex core appears at a particular radius. For this simple-minded approach, the radial dependence of
the growth rate could enter through any hidden dependences in $q$. One could extremize $(q - 5h/4r)$ to find the radius corresponding to the maximum. However, it is likely that the favorable radius of planet formation ultimately has much to do with the ambient dust concentration and size distribution as a function of radius in a stellar nebula (Barge & Sommeria 1995).

It is very important to note that the overly simplified approach herein does not locate the vortices on azimuth, as the azimuth is averaged over. Thus for a given radius, only the net vorticity is measured. The location of the vortex, or the number of the vortices is not determined.

5.2. Implications for accretion disc dynamos

The growth of mean vorticity and mean magnetic field require the presence of a pseudoscalar. It is interesting to compare the resulting pseudoscalars $\alpha_0$ and $\alpha_m$ by including the magnetic back-reaction to lowest order.

Note that $\alpha_0$ is proportional only to the negative of kinetic helicity whilst $\alpha_m$ represents the residual between this and a current helicity contribution. (The relation between this and the form in Pouquet et al. (1976) is discussed in Field et al. 1999). This can in principle be tested in simulations, though one must be careful not to use periodic boundary conditions over the simulation box half-thickness over which the vertical averaging is done. In addition, it was noted in Brandenburg & Donner (1997) and Brandenburg (1999) that $\alpha_m$ seems to have the opposite sign than expected from just the kinetic helicity part. It was suggested that the shear may play the role of flipping the sign of a rising loop of field (see Brandenburg & Donner 1997). Here we can go a step further and note that it is actually the current helicity term that will be affected by that sign: imagine a seed toroidal field from which a loop rises and twists less than 1/4 turn, in the direction opposite to the underlying rotation. This would have positive current helicity. But now as the Keplerian shear takes over, the loop rotates past a 1/4 turn, and the sign of the current helicity changes. By contrast, the sign of the vorticity growth coefficient $\alpha_0$ is the same for a Keplerian and non-shearing flow, since the kinetic helicity does not change sign. Thus in magneto-shearing simulations in which the turbulent magnetic energy slightly dominates the turbulent kinetic energy, the current helicity term may dominate $\alpha_m$. 
5.3. Implications for accretion disc modeling

The growth of vorticity as modeled herein may have implications for angular momentum transport in accretion discs. Indeed the presence of anti-cylconic vortices in the underlying flow transports angular momentum out of the ambient flow. Note however, that in the present treatment we have taken a base turbulent state that acts as a viscosity, so angular momentum is being transported also by the underlying turbulence. It is not entirely clear what the consequences of the vorticity growth would be for angular momentum transport, above and beyond their ability to agglomerate dust particles in the accretion flow. Such an additional effect of $\alpha_0$ needs to be calculated, since the pseudoscalar term is as naturally present as the viscosity term when integrating only over the half-thickness of the disc.

Note that the generation of Rossby wave vortices has been considered when there is no turbulence present and leads to outward transport of angular momentum independently of any magnetic field. (Lovelace et al. 1999; Li et al 1999).

Additional consequences of vortex generation for high energy accretion discs may include concentrating disc emission into collimated beams (Acosta-Pulido et al. 1990, Abramowicz 1992). X-ray iron line modeling could include the presence of such vortices to diagnose their presence. Also, Yoshizawa & Yokoi (1993) discussed how the interplay of magnetic field and vorticity can lead to generation of collimated large scale jets.

6. Conclusions

Standard axisymmetric Shakura-Sunyaev type turbulent accretion disc theory should always be regarded as mean field theory. When the vertical averaging is taken to be integration over 1/2 the disc scale height, the presence of a pseudoscalar transport coefficient in addition to the usual scalar diffusion term should survive. This pseudoscalar term allows vorticity growth in each hemisphere even when mean quantities are only a function of radii there. The simplest growth rates, survival times and spatial scales seem to be consistent with that required by planet formation studies. The simple pseudoscalar term driving vorticity growth differs from that which drives mean magnetic field growth in that the latter, unlike the former, could have an opposite sign for cases of a rigid rotator vs. a sheared rotator such as an accretion disc.

All of these points could be tested numerically, though one must be very careful about boundary conditions. Future analytic work can investigate the implications of including the pseudoscalar term in a generalization of the Shakura-Sunyaev/slim disc models. Azimuthal dependence must also be considered.
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