Quantitative homogenization with relatively soft inclusions and interior estimates

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Abstract

We establish large-scale interior Lipschitz estimates for solutions to systems of linear elasticity with rapidly oscillating periodic coefficients and Dirichlet boundary conditions in domains with periodically placed inclusions of size $O(\varepsilon)$ and magnitude $\delta$ by establishing $H^1$-convergence rates for such solutions. The interior estimates at the macroscopic scale are derived directly without the use of compactness via a Campanato-type scheme presented by S. Armstrong and C.K. Smart and that was adapted for uniformly elliptic equations in by Armstrong and Z. Shen.

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1 Introduction

The purpose of this paper is to establish large-scale interior Lipschitz estimates for solutions to systems of linear elasticity with $\varepsilon$-periodic coefficients in domains with periodically placed inclusions of size $O(\varepsilon)$ and magnitude $\delta$ and to establish $H^1$-convergence rates in periodic homogenization. To be precise, let $\omega \subseteq \mathbb{R}^d$ be an unbounded domain with 1-periodic structure, i.e., if $1_+$ denotes the characteristic function of $\omega$, then $1_+$ is a 1-periodic function in the sense that

$$1_+(y) = 1_+(z + y) \quad \text{for } y \in \mathbb{R}^d, z \in \mathbb{Z}^d. \quad (1.1)$$

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Let \(1_\cdot\) denote the characteristic function of \(\mathbb{R}^d \setminus \omega\), and note it also satisfies (1.1). For \(\varepsilon > 0, 0 \leq \delta \leq 1\), we consider the operator

\[
L_{\varepsilon, \delta} = -\text{div}(k_\delta^\varepsilon \tilde{A}(x) \nabla) = -\frac{\partial}{\partial x_i} \left( k_\delta^\varepsilon a_{ij}^\alpha \left( \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right),
\]

for \(x \in \mathbb{R}^d\), where \(A^\varepsilon(x) = A(x/\varepsilon), A(y) = \{a_{ij}^{\alpha\beta}(y)\}_{1 \leq i,j,\alpha,\beta \leq d}\) for \(y \in \mathbb{R}^d, d \geq 2\), \(k_\delta^\varepsilon = k_\delta(\cdot/\varepsilon)\), and

\[
k_\delta(y) = 1_+ (y) + \delta 1_-(y).
\]

The specific case \(\delta = 0\) is discussed in [23]. Naturally, \(k_\delta\) is 1-periodic. We assume the coefficient matrix \(A(y)\) is real, measurable, and satisfies the elasticity conditions

\[
a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{ij}^{i\beta}(y),
\]

\[
\kappa_1 |\xi|^2 \leq a_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \kappa_2 |\xi|^2,
\]

for a.e \(y \in \mathbb{R}^d\) and any symmetric matrix \(\xi = \{\xi_i^\alpha\}_{1 \leq i,\alpha \leq d}\), where \(\kappa_1, \kappa_2 > 0\). We also assume \(A\) is 1-periodic in the sense of (1.1), i.e.,

\[
A(y) = A(y + z) \quad \text{for } y \in \mathbb{R}^d, z \in \mathbb{Z}^d.
\]

The coefficient matrix of the systems of linear elasticity describes the relation between the stress and strain a material experiences during relatively small elastic deformations. Consequently, the elasticity conditions (1.3), (1.4), and \(\delta\) should be regarded as physical parameters of the system, whereas \(\varepsilon\) and (1.5) are clearly geometric characteristics of the system.

Let \(\Omega\) be a bounded domain. In this paper, we consider the Dirichlet boundary value problem given by

\[
\begin{cases}
L_{\varepsilon, \delta}(u_{\varepsilon, \delta}) = 0 & \text{in } \Omega, \\
u_{\varepsilon, \delta} = f & \text{on } \partial \Omega.
\end{cases}
\]

We say \(u_{\varepsilon, \delta}\) is a weak solution to (1.6) provided

\[
\int_{\Omega} k_\delta^\varepsilon a_{ij} a_{ij}^\alpha \frac{\partial u_{\varepsilon, \delta}}{\partial x_j} \frac{\partial u^\alpha}{\partial x_i} = 0 \quad \text{for any } w = \{w^\alpha\}_\alpha \in H^1_0(\Omega; \mathbb{R}^d)
\]

and \(u_{\varepsilon, \delta} - f \in H^1_0(\Omega; \mathbb{R}^d)\). Note when \(\delta = 0\), Neumann boundary conditions on the perforations are implied. The boundary value problem (1.6) models relatively small elastic deformations of composite material.
materials reinforced with soft inclusions and subject to zero external body forces [10, 19, 22]. In particular, soft inclusions are comparatively “weaker” than the cementing matrix $\omega$, but their embedding can be otherwise advantageous. For example, a material’s compressive strength can be indirectly proportional with the increasing volume of soft inclusions but the thermal inertia and energy efficiency may be directly proportional [15].

For each $\delta \in (0, 1]$, the existence and uniqueness of a weak solution $u_{\varepsilon, \delta} \in H^1(\Omega; \mathbb{R}^d)$ to (1.6) for $f \in H^{1/2}(\partial \Omega; \mathbb{R}^d)$ follows easily from the Lax-Milgram theorem and Korn’s inequality. For $\delta = 0$, the existence and unqieness follows from Lax-Milgram and Korn’s inequality for perforated domains [9, 22]. It should be noted that the solution $u_{\varepsilon, \delta}$ is not bounded uniformly in $H^1(\Omega; \mathbb{R}^d)$. Indeed, if $L_{\varepsilon, \delta}(u_{\varepsilon, \delta}) = 0$ in $\Omega$ and $u_{\varepsilon, \delta} = f$ on $\partial \Omega$, then one may deduce by energy estimates

$$\|k^\varepsilon_{\delta} u_{\varepsilon, \delta}\|_{L^2(\Omega)} + \|k^\varepsilon_{\delta} \nabla u_{\varepsilon, \delta}\|_{L^2(\Omega)} \leq C \|f\|_{H^{1/2}(\partial \Omega)},$$

where $C$ depends on $\kappa_1, \kappa_2$.

One of the main results of this paper is the following theorem. We emphasize that no smoothness assumptions are required on the coefficients $A$, only the elasticity conditions (1.3), (1.4), and the periodicity condition (1.5).

**Theorem 1.1.** Suppose $A$ satisfies (1.3), (1.4), and (1.5). Let $u_{\varepsilon, \delta}$ denote a weak solution to $L_{\varepsilon, \delta}(u_{\varepsilon, \delta}) = 0$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$. For $0 < r < R$, there exists a constant $C$ depending on $d$, $\omega$, $\kappa_1$, and $\kappa_2$ such that

$$\left( \frac{1}{B(x_0, r)} \int_{B(x_0, r)} |k^\varepsilon_{\delta} \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2} \leq C \left( \frac{1}{B(x_0, R)} \int_{B(x_0, R)} |k^\varepsilon_{\delta} \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2}$$

for $0 \leq \delta \leq 1$.

The scale-invariant estimate in Theorem 1.1 should be regarded as a Lipschitz estimate at the large scale, e.g., $1 \leq r/\varepsilon$, and it is proved in Section 4. Indeed, if Theorem 1.1 were to hold also for $0 < r < \varepsilon$, then by letting $r \to 0$ we would have

$$|k^\varepsilon_{\delta} \nabla u_{\varepsilon, \delta}(x_0)| \leq C \left( \frac{1}{B(x_0, R)} \int_{B(x_0, R)} |k^\varepsilon_{\delta} \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2}$$

for all $x_0$ in some compact subset of $\Omega$. In particular, we would have a Lipschitz estimate independent of $\delta$ for $u_{\varepsilon, \delta}$ in the connected substrate.
\( \omega \) and a Lipschitz estimate for \( u_{\epsilon, \delta} \) in the inclusions \( \mathbb{R}^d \setminus \omega \) with explicit knowledge of the effect of the parameter \( \delta \). Unfortunately, (1.9) does not hold without more assumptions on the smoothness of the coefficients \( A \) and the domain \( \omega \). That is, the periodicity assumptions (1.1), (1.5) and elasticity conditions (1.3), (1.4) alone contribute to the large-scale average behavior of the solution.

Under additional assumptions that \( A \) is Hölder continuous and the domain \( \omega \) has a sufficiently regular boundary, an interior Lipschitz estimate at the microscopic scale for solutions to (1.6) follows from local \( C^{1, \alpha} \)-estimates for the operator \( L_{1, \delta} \). This follows from a layer potential argument of Escauriaza, Fabes, and Verchota where nontangential estimates were obtained for single equation interface problems [14]. Yeh modified this same argument to obtain local \( W^{1, p} \)-estimates and Hölder estimates for (1.6) in the case of single equations with diagonal coefficients [25, 26]. The necessary modifications for our setting is discussed in Appendix A.

Nevertheless, if \( A \) is \( \alpha \)-Hölder continuous, i.e., there exists a \( \alpha \in (0, 1) \) with

\[
|A(x) - A(y)| \leq C|x - y|^\alpha \quad \text{for } x, y \in \mathbb{R}^d
\]  

(1.8)

for some constant \( C \) uniform in \( x \) and \( y \), then the following corollary holds.

**Corollary 1.2.** Suppose \( A \) satisfies (1.3), (1.4), (1.5), and (1.8) for some \( \alpha \in (0, 1) \). Suppose \( \omega \) is an unbounded \( C^{1, \alpha} \)-domain. Let \( u_{\epsilon, \delta} \) denote a weak solution to \( L_{\epsilon, \delta}(u_{\epsilon, \delta}) = 0 \) in \( B(x_0, R) \) for some \( x_0 \in \mathbb{R}^d \) and \( R > 0 \). Then for \( 0 \leq \delta \leq 1 \),

\[
\|k_{\delta}^\epsilon \nabla u_{\epsilon, \delta}\|_{L^\infty(B(x_0, R/3))} \leq C \left( \frac{1}{B(x_0, R)} \int_{B(x_0, R)} |k_{\delta}^\epsilon \nabla u_{\epsilon, \delta}|^2 \right)^{1/2}
\]  

(1.9)

some constant \( C \) independent of \( \epsilon \) and \( \delta \).

Interior Lipschitz estimates for the case \( \delta = 1 \) were first obtained indirectly through the method of compactness by Avellaneda and Lin [5]. The celebrated method of compactness has been applied in other settings [17, 25, 26]. For example, uniform Hölder estimates for a single elliptic equation with diagonal coefficients in the case \( \delta = \epsilon \) were obtained indirectly by Yeh with this method [25]. The method of compactness is essentially “proof by contradiction” and relies on qualitative convergence, which for (1.6) can be ambiguous and complicated.
Interior Lipschitz estimates for the case $\delta = 1$ were obtained directly by Shen [24] through a general scheme developed by Armstrong and Smart [4] for establishing large-scale Lipschitz estimates for local minimizers of convex integral functionals arising in homogenization. The method was adapted for divergence form elliptic equations with almost-periodic coefficients by Armstrong and Shen [3]. The same estimates were directly proved for the case $\delta = 0$ by the author of this paper using the general scheme [23]. Essentially, in this paper we establish sub-optimal quantitative convergence rates for solutions to (1.6) and use the same scheme.

Hueristically, the scheme is a Campanato-type iteration verifying that on mesoscopic scales the solution $u_{\varepsilon, \delta}$ is “flatter.” If $P_1$ denotes the space of affine functions in $\mathbb{R}^d$ and $H_{\varepsilon, \delta}(r)$ defined by

$$H_{\varepsilon, \delta}(r) = \frac{1}{r} \left( \inf_{p \in P_1} \int_{B(r)} |k_{\varepsilon, \delta}(u_{\varepsilon, \delta} - p)|^2 \right)^{1/2}$$

quantifies a weighted $L^2$-“flatness” of the solution in some ball $B(r)$ with radius $r$, then we show there exists a $\theta \in (0, 1)$ such that

$$H_{\varepsilon, \delta}(\theta r) \leq C H_{\varepsilon, \delta}(r) + \text{error},$$

(1.10)

where the “error” term is controllable whenever $\varepsilon \leq r$ and the constant $0 \leq C < 1$ indicates an improvement in “flatness.” Indeed, (1.10) follows from the fact that $u_{\varepsilon, \delta}$—at least in the connected substrate—can be well-approximated in $L^2$ by a solution to a constant coefficient system. It is known from classical $C^2$ estimates that solutions to constant coefficient systems satisfy (1.10) with no error. In contrast to compactness methods, showing (1.10) relies on tractable $L^2$-convergence rates of $u_{\varepsilon, \delta}$, which we will see follows from new results regarding quantitative homogenization in $H^1$. These sub-optimal $H^1$-convergence rates are stated in Theorem 1.3 and proved in Section 3.

For fixed $\delta \geq 0$, the estimate

$$\|u_{\varepsilon, \delta} - u_{0, \delta} - \varepsilon \chi_\delta^\varepsilon K_\varepsilon^2 (\nabla u_{0, \delta}) \eta_\varepsilon\|_{H^1(\Omega)} \leq C \varepsilon^{1/2} \|u_{0, \delta}\|_{H^1(\partial \Omega)},$$

(1.11)

is known, where $u_{0, \delta} \in H^1(\Omega; \mathbb{R}^d)$ denotes the weak solution of the boundary value problem for the homogenized system corresponding to (1.6), $\chi_\delta = \{\chi_{j, \delta}^\beta\}_{1 \leq j, \beta \leq d} \in H^1_{\text{per}}(\mathbb{R}^d; \mathbb{R}^d)$ denotes the matrix of correctors associated with the coefficients $k_\delta A$ (see (2.8)), $K_\varepsilon$ denotes the smoothing operator at scale $\varepsilon$ defined by (2.1), and $\eta_\varepsilon \in C_0^\infty(\Omega)$ be
the cut-off function defined by (3.1). However, the explicit dependence of $C$ on the parameter $\delta$ is not known. The estimate (1.11) was proved by the author of this paper in [23] when $\delta = 0$. For $\delta = 1$, the estimate was proved by Shen in [24]. The following theorem is therefore also a main result of this paper, as it holds for any $0 \leq \delta \leq 1$ and the constant $C$ is completely independent of the parameter $\delta$.

**Theorem 1.3.** Let $\Omega$ be a bounded Lipschitz domain and $\omega$ be an unbounded Lipschitz domain with 1-periodic structure. Suppose $A$ is real, measurable, and satisfies (1.3), (1.4), and (1.5). Let $u_{\varepsilon, \delta}$ denote a weak solution to (1.6) for $0 \leq \delta \leq 1$. There exists a constant $C$ depending on $\kappa_1$, $\kappa_2$, $d$, $\Omega$, and $\omega$ and a $\mu > 0$ depending on $\kappa_1$, $\kappa_2$, $d$, and $\Omega$ such that

$$
\|k_{\delta}^\varepsilon r_{\varepsilon, \delta}\|_{L^2(\Omega)} + \|k_{\delta}^\varepsilon \nabla r_{\varepsilon, \delta}\|_{L^2(\Omega)} \leq C \varepsilon^\mu \|f\|_{H^1(\partial\Omega)},
$$

where

$$
r_{\varepsilon, \delta} = u_{\varepsilon, \delta} - u_{0, \delta} - \varepsilon \chi_{\delta}^\varepsilon K^2_{\varepsilon}((\nabla u_{0, \delta})\eta_\varepsilon).
$$

Theorem 1.3 is particularly new for small yet positive $\delta$. Indeed, if $0 < \delta_0 \leq \delta \leq 1$, then estimate (1.12) follows from work in [24] where the constant $C$ depends somehow on $\delta_0$. With regards to the regularity estimates, the theorem essentially establishes the estimate

$$
\left( \int_{B(r)} (k_{\delta}^\varepsilon (u_{\varepsilon, \delta} - v))^2 \right)^{1/2} \lesssim \left( \frac{\varepsilon}{r} \right)^\mu
$$

for some $v$ satisfying a constant coefficient system. The established $C^2$ estimates for $v$ are used to give (1.10). The “error” term of (1.10) is on the order of the RHS of (1.14), and so to carry out the scheme it is important that $\mu > 0$. The typical iteration of (1.10) depends on the smallness of the RHS of (1.14) and is written in full detail in [3, 4]. We use a generalization of the iteration process provided by Shen [24] (see Lemma 4.6).

The paper is structured in the following way. In Section 2, we establish more notation and recall various preliminary results from other works. The convergence rate presented in Theorem 1.3 is proved in Section 3. In Section 4, we prove the interior Lipschitz estimates at the macroscopic scale, i.e., Theorem 1.1. In Appendix A, we argue the local interior Lipschitz estimates at the microscopic scale by applying
the argument of [14]. It should be noted throughout that \( C \) is a harmless constant that may be change from line to line. At no point does \( C \) depend on \( \varepsilon \) or \( \delta \).

## 2 Preliminaries

Fix \( \varphi \in C_0^\infty(B(0,1)) \) with \( \varphi \geq 0 \) and \( \int_{\mathbb{R}^d} \varphi = 1 \). Define

\[
K_\varepsilon g(x) = \int_{\mathbb{R}^d} g(x-y) \varphi_\varepsilon(y) \, dy, \quad g \in L^2(\mathbb{R}^d),
\]

where \( \varphi_\varepsilon(y) = \varepsilon^{-d} \varphi(y/\varepsilon) \). Note \( K_\varepsilon \) is a continuous map from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d) \). A proof for the following two lemmas is readily available in [24], and so we do not present either here. For any function \( g \), set \( g^\varepsilon(\cdot) = g(\cdot/\varepsilon) \).

**Lemma 2.1.** Let \( g \in H^1(\mathbb{R}^d) \). Then

\[
\|g - K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon\|\nabla g\|_{L^2(\mathbb{R}^d)},
\]

where \( C \) depends only on \( d \).

**Lemma 2.2.** Let \( h \in L^2_{\text{loc}}(\mathbb{R}^d) \) be a 1-periodic function. Then for any \( g \in L^2(\mathbb{R}^d) \),

\[
\|h^\varepsilon K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq C\|h\|_{L^2(Q)} \|g\|_{L^2(\mathbb{R}^d)},
\]

where \( Q = [0,1)^d \) and \( C \) depends on \( d \) and \( \Omega \).

A proof of Lemma 2.3 can be found in [22].

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain. For any \( g \in H^1(\Omega) \),

\[
\|g\|_{L^2(\Omega_r)} \leq C r^{1/2} \|g\|_{H^1(\Omega)},
\]

where \( \Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < r \} \) and \( C \) depends only on \( d \).

A proof of Lemma 2.4 can be found in [19].

**Lemma 2.4.** Suppose \( B = \{b_{ij}^{\alpha\beta}\}_{1 \leq i,j,\alpha,\beta \leq d} \) is 1-periodic and satisfies

\[
b_{ij}^{\alpha\beta} \in L^2_{\text{loc}}(\mathbb{R}^d) \text{ with } \frac{\partial}{\partial y_j} b_{ij}^{\alpha\beta} = 0, \quad \text{and} \quad \int_Q b_{ij}^{\alpha\beta} = 0.
\]
There exists \( \pi = \{\pi^{\alpha\beta}_{kij}\}_{1 \leq i,j,k,\alpha,\beta \leq d} \) with \( \pi^{\alpha\beta}_{kij} \in H^1_{\text{loc}}(\mathbb{R}^d) \) that is 1-periodic and satisfies
\[
\frac{\partial}{\partial y_k} \pi^{\alpha\beta}_{kij} = b^{\alpha\beta}_{ij} \quad \text{and} \quad \pi^{\alpha\beta}_{kij} = -\pi^{\alpha\beta}_{ikj}.
\]

If \( \delta > 0 \), it can be shown that the weak solution to (1.6) converges weakly in \( H^1(\Omega; \mathbb{R}^d) \) and consequently strongly in \( L^2(\Omega; \mathbb{R}^d) \) as \( \varepsilon \to 0 \) to some \( u_{0,\delta} \), which is a solution of a constant-coefficient equation in the domain \( \Omega \) (see [7, 19, 22] and references therein). Indeed, the following theorem is well-known.

**Theorem 2.5.** Suppose \( \Omega \) is a bounded Lipschitz domain and that \( A \) satisfies (1.3), (1.4), and (1.5). Let \( u_{\varepsilon,\delta} \) satisfy \( L_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \) in \( \Omega \), and \( u_{\varepsilon,\delta} = f \) on \( \partial \Omega \) for some fixed \( \delta > 0 \). Then there exists a \( u_{0,\delta} \in H^1(\Omega; \mathbb{R}^d) \) such that
\[
u_{\varepsilon,\delta} \rightharpoonup u_{0,\delta} \quad \text{weakly in} \quad H^1(\Omega; \mathbb{R}^d).
\]
Consequently, \( u_{\varepsilon,\delta} \to u_{0,\delta} \) strongly in \( L^2(\Omega; \mathbb{R}^d) \).

For a proof of the previous theorem, see [9, Section 10.3] and notice \( k_\delta A^\varepsilon \) is uniformly elliptic in \( \mathbb{R}^d \) for \( \delta > 0 \). The function \( u_{0,\delta} \) is called the homogenized solution and the boundary value problem it solves is the homogenized system corresponding to (1.6).

Theorem 2.6 is a typical result in the study of periodically perforated domains, i.e., the case when \( \delta = 0 \). For a proof of the following theorem, consult the work of Acerbi, Piat, Dal Maso, and Percivale [1]. Let \( \Omega_\varepsilon = \Omega \cap \varepsilon \omega \) and for \( p \in (1, \infty) \) let \( W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \) denote the closure in \( W^{1,p}(\Omega_\varepsilon; \mathbb{R}^d) \) of \( C^\infty(\mathbb{R}^d; \mathbb{R}^d) \) function vanishing on \( \Gamma_\varepsilon := \partial \Omega_\varepsilon \cap \partial \Omega \).

**Theorem 2.6.** Let \( \Omega \) and \( \Omega_0 \) be bounded Lipschitz domains with \( \Omega \subset \Omega_0 \) and \( \text{dist}(\partial \Omega_0, \Omega) > 1 \). Let \( p \in (1, \infty) \). For \( \varepsilon \) small enough, there exists a linear extension operator \( P_\varepsilon : W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \to W^{1,p}_0(\Omega_0; \mathbb{R}^d) \) such that
\[
P_\varepsilon w = w \quad \text{a.e. in} \quad \Omega_\varepsilon \quad \text{(2.2)}
\]
\[
\|P_\varepsilon w\|_{L^p(\Omega_0)} \leq C_1 \|w\|_{L^p(\Omega_\varepsilon)}, \quad \text{(2.3)}
\]
\[
\|\nabla P_\varepsilon w\|_{L^p(\Omega_0)} \leq C_2 \|\nabla w\|_{L^p(\Omega_\varepsilon)}, \quad \text{(2.4)}
\]
for some constants \( C_1, C_2 \) depending on \( \Omega, \omega, d, \) and \( p \).
If $\delta = 0$, then it is difficult to qualitatively discuss the convergence of $u_{e,0}$, although in this case it is discussed in [2, 7, 9, 10] and many others. Quantitatively, however, we have the estimate (1.11). A stronger estimate for this case is proved in [23]. The homogenized system of elasticity corresponding to (1.6) in the case $\delta = 0$ and of which $u_{0,0}$ is a solution is given by

\[
\begin{cases}
L_{0,0}(u_{0,0}) = 0 \quad \text{in } \Omega \\
u_{0,0} = f \quad \text{on } \partial \Omega,
\end{cases}
\]

where $L_{0,0} = -\text{div}(A_0 \nabla)$, $A_0 = \{a_{ij,0}^{\alpha\beta}\}_{1 \leq i,j,\alpha,\beta \leq d}$ denotes a constant matrix given by

\[
\hat{a}_{ij,0}^{\alpha\beta} = \int_Q k_0 a_{ik}^{\alpha\gamma} \frac{\partial X_{j,0}^{\gamma\beta}}{\partial y_k},
\]

and $X_{j,0}^{\beta} = \{X_{j,0}^{\gamma\beta}\}_{1 \leq \gamma \leq d}$ denotes the solution to the following variational problem

\[
\begin{cases}
\int_Q k_0 a_{ik}^{\alpha\gamma} \frac{\partial X_{j,0}^{\gamma\beta}}{\partial y_k} \frac{\partial \phi^\alpha}{\partial y_i} dy = 0, \quad \text{for any } \phi = \{\phi^\alpha\}_\alpha \in H^1_{\text{per}}(Q;\mathbb{R}^d) \\
\chi_{j,0}^{\beta} := X_{j,0}^{\beta} - y_j e^\beta \text{ is 1-periodic,} \quad \int_Q k_0 \chi_{j,0}^{\beta} = 0,
\end{cases}
\]

where $e^\beta \in \mathbb{R}^d$ has a 1 in the $\beta$th position and 0 in the remaining positions and $H^1_{\text{per}}(Q;\mathbb{R}^d)$ denotes the closure of $C_\text{per}^\infty(\mathbb{R}^d;\mathbb{R}^d)$ functions in the $H^1(Q;\mathbb{R}^d)$ norm. For details on the existence of solutions to (2.7), see [22]. The functions $\chi_{j,0}^{\beta}$ are referred to as the first-order correctors for the system (1.6) with $\delta = 0$. The coefficients $\hat{A}_0$ are known to be uniformly elliptic. Indeed, we have the following lemma. For a proof, see [22].

**Lemma 2.7.** Suppose $A$ satisfies (1.3), (1.4), and (1.5). If $X_{j,0}^{\beta} = \{X_{j,0}^{\gamma\beta}\}$ are defined by (2.7), then $\hat{A}_0 = \{a_{ij,0}^{\alpha\beta}\}$ defined by (2.6) satisfies

\[
\hat{a}_{ij,0}^{\alpha\beta} = \hat{a}_{ij,0}^{i\beta} = \hat{a}_{ij,0}^{\alpha\beta}
\]

for $1 \leq i, j, \alpha, \beta \leq d$ and

\[
\hat{\kappa}_1 |\xi|^2 \leq \hat{a}_{ij,0}^{\alpha\beta} \xi_i \xi_j \leq \hat{\kappa}_2 |\xi|^2
\]

for some $\hat{\kappa}_1, \hat{\kappa}_2 > 0$ and any symmetric matrix $\xi = \{\xi_{i,0}\}_{i,\alpha}$. 


For $\delta \geq 0$, let $\chi_{j,\delta}^\beta = \{\chi_{\gamma,\delta}^\beta\}_{1 \leq \gamma \leq d}$ denote the solution to the following variational problem

$$
\begin{cases}
\int_Q k_\delta a_{ik}^{\alpha\gamma} \frac{\partial X_{j,\delta}^\beta}{\partial y_k} \frac{\partial \phi^\alpha}{\partial y_i} \, dy = 0 & \text{for any } \phi \in H^1_\text{per}(Q; \mathbb{R}^d) \\
\chi_{j,\delta}^\beta := X_{j,\delta}^\beta - y_j e^\beta \text{ is } 1\text{-periodic, } \int_Q \chi_{j,\delta}^\beta = 0,
\end{cases}
$$

(2.8)

which coincides with (2.7) if $\delta = 0$. To show the existence and uniqueness of the solutions $\chi_{j,\delta}^\beta$, apply the Lax-Milgram theorem to the space $H^1_\text{per}(Q; \mathbb{R}^d)$. As a consequence, with the appropriate choice of test functions, one can obtain the bound

$$
\|k_\delta \chi_{j,\delta}^\beta\|_{L^2(Q)} + \|k_\delta \nabla \chi_{j,\delta}^\beta\|_{L^2(Q)} \leq C
$$

for some constant $C$ depending on $\kappa_1$, $\kappa_2$, and $\omega$.

Define the constant matrix $\tilde{A}_\delta = \{a_{ij,\delta}^{\alpha\beta}\}$ by

$$
\tilde{a}_{ij,\delta}^{\alpha\beta} = \int_Q k_\delta a_{ik}^{\alpha\gamma} \frac{\partial X_{j,\delta}^\beta}{\partial x_k} \, dy,
$$

(2.9)

where $X_{j,\delta}^\beta$ is defined in (2.8). The constant matrix $\tilde{A}_\delta$ is uniformly elliptic uniformly in $\delta$. For details, see Section 3. Let $u_{0,\delta}$ denote a solution to the homogenized boundary value problem corresponding to (1.6) with $\delta \geq 0$, i.e., $u_{0,\delta}$ satisfies

$$
\begin{cases}
\mathcal{L}_{0,\delta}(u_{0,\delta}) = 0 & \text{in } \Omega \\
u_{0,\delta} = f & \text{on } \partial \Omega,
\end{cases}
$$

(2.10)

where $\mathcal{L}_{0,\delta} = -\text{div}(\tilde{A}_\delta \nabla)$ and $\tilde{A}_\delta$ is defined by (2.9).

Throughout, it is assumed that any two connected components of $\mathbb{R}^d \setminus \omega$ are separated by some positive distance. Specifically, if $\mathbb{R}^d \setminus \omega = \bigcup_{k=1}^\infty H_k$, where $H_k$ is simply connected and bounded for each $k$, then there exists a constant $g^\omega$ so that

$$
0 < g^\omega \leq \inf_{k_1 \neq k_2} \left\{ \inf_{x_{k_1} \in H_{k_1}} \inf_{x_{k_2} \in H_{k_2}} |x_{k_1} - x_{k_2}| \right\}.
$$

(2.11)

It should be noted that $\|\nabla u_{1,0}\|_{L^\infty}$ grows uncontrollably as $g^\omega \to 0$. For more details regarding this and explicit results, see [13].
3 Homogenization with Soft Inclusions

In this section, we quantitatively discuss the convergence of solutions to (1.6) as $\varepsilon, \delta \to 0$ by proving Theorem 1.1. In Subsection 3.1, we discuss the ellipticity of $\hat{A}_\delta$ which is shown to be uniform in $\delta$. In Subsection 3.2, we provide the proof of Theorem 1.1.

3.1 Ellipticity of $\hat{A}_\delta$

If $A$ satisfies (1.3) and (1.4), then $\hat{A}_\delta$ defined by (2.6) satisfies conditions (1.3) and (1.4) but with possibly different constants $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ depending on $\kappa_1$ and $\kappa_2$ but not $\delta$. In particular, we have the following lemma.

Lemma 3.1. Let $\hat{A}_\delta$ be defined by (2.6) for $0 \leq \delta \leq 1$. Then

$$
\hat{a}_{ij,\delta}^{\alpha\beta}(y) = \hat{a}_{ji,\delta}^{\beta\alpha}(y) = \hat{a}_{ij,\delta}^{\alpha\beta}(y)
$$

$$
\tilde{\kappa}_1|\xi|^2 \leq a_{ij,\delta}^{\alpha\beta}(y)\xi^i\xi_j \leq \tilde{\kappa}_2|\xi|^2
$$

for any symmetric matrix $\xi = \{\xi^i\}$, where $\tilde{\kappa}_1, \tilde{\kappa}_2 > 0$ depend on $\kappa_1, \kappa_2$, and $|Q \cap \omega|$.

Lemma 3.1 follows from the following two lemmas. The first discusses the convergence of $\chi_{ij,\delta}^\beta$ in the connected substrate for each $1 \leq j, \beta \leq d$ as $\delta \to 0$, and the second discusses the convergence of $\hat{A}_\delta$ to $\hat{A}_0$ as $\delta \to 0$. As $\hat{A}_0$ is known to be uniformly elliptic (see Lemma 2.7), we obtain Lemma 3.1.

Lemma 3.2. If $X_0 = \{X_{j,\delta}^\beta\}_{1 \leq j, \beta \leq d}$, $X_\delta = \{X_{j,\delta}^\beta\}_{1 \leq j, \beta \leq d}$ are defined by (2.7) and (2.8), respectively, then for $\delta > 0$ we have the following estimates:

(i) $\|I_+ \nabla (X_0 - X_\delta)\|_{L^2(Q)} \leq C_1 \delta^{1/2}$,

(ii) $\|I_- \nabla X_\delta\|_{L^2(Q)} \leq C_2 \delta^{-1/4}$,

where $C_1, C_2$ depend on $\kappa_1$ and $\kappa_2$.

Proof. Let $\tilde{\chi}_{j,0}^\beta = P\chi_{j,0}^\beta \in H^1(Q; \mathbb{R}^d)$ be a periodic extension of $\chi_{j,0}^\beta$ for each $1 \leq j, \beta \leq d$, where $P$ is the bounded linear extension operator given in [22, Lemma 4.1]. Let

$$
\tilde{X}_{j,\delta}^\beta(y) = y_j e^\beta + \tilde{\chi}_{j,0}^\beta(y).
$$

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Recall that \(1_{\widetilde{X}_0}\) satisfies (2.7) and \(X_\delta\) satisfies (2.8), and so for any \(\phi \in H^1_{\text{per}}(Q; \mathbb{R}^d)\) we have

\[
\int_Q k_\delta A \nabla (\widetilde{X}_0 - X_\delta) \cdot \nabla \phi = \delta \int_Q 1_- A \nabla \widetilde{X}_0 \cdot \nabla \phi
\]

Note \(\widetilde{X}_0 - X_\delta = \chi_0 - \chi_\delta \in H^1_{\text{per}}(Q; \mathbb{R}^d)\), and so by the ellipticity of \(A\) and Cauchy-Schwarz,

\[
\int_Q k_\delta |\nabla (\widetilde{X}_0 - X_\delta)|^2 \leq C \int_Q k_\delta A \nabla (\widetilde{X}_0 - X_\delta) \cdot \nabla (\widetilde{X}_0 - X_\delta)
\]

\[
= C \delta \int_Q 1_- \nabla \widetilde{X}_0 \cdot \nabla (\widetilde{X}_0 - X_\delta)
\]

\[
= C_1 \delta \int_Q 1_+ |\nabla X_0|^2 + \delta \int_Q 1_- |\nabla (\widetilde{X}_0 - X_\delta)|^2,
\]

where \(C_1\) only depends on \(\kappa_1\) and \(\kappa_2\). This gives (i). For (ii), note

\[
\delta \int_Q 1_- A \nabla X_\delta \cdot \nabla X_\delta = - \int_Q 1_+ A \nabla (X_0 - X_\delta) \cdot \nabla X_\delta
\]

\[
\leq C \delta^{1/2} \|1_+ \nabla X_0\|_{L^2(Q)} \|1_+ \nabla X_\delta\|_{L^2(Q)},
\]

where \(C\) only depends on \(\kappa_2\). By (i),

\[
\delta \int_Q 1_- |\nabla X_\delta|^2 \leq C \delta^{1/2} \|1_+ \nabla X_0\|_{L^2(Q)}^2,
\]

where \(C\) depends on \(\kappa_1, \kappa_2\), which gives (ii).

Lemma 3.3. If \(\hat{\mathbf{A}}_0\) and \(\hat{\mathbf{A}}_\delta\) are defined by (2.6) and (2.9), then

\[
|Q \cap \omega| \hat{\mathbf{A}}_0 - \hat{\mathbf{A}}_\delta \leq C \delta^{1/2} \|1_+ \nabla X_0\|_{L^2(Q)},
\]

where \(C\) depends on \(\kappa_1\) and \(\kappa_2\).

Proof. Note

\[
|Q \cap \omega| \hat{\mathbf{A}} - \hat{\mathbf{A}}_\delta = \int_Q 1_+ A \nabla (X_0 - X_\delta) - \delta \int_Q 1_- \nabla X_\delta,
\]

from which the desired estimate follows by Lemma 3.2. \(\square\)
3.2 Convergence Rates

Let $K_\varepsilon$ be defined as in Section 2. Let $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfy

\[
\begin{cases}
0 \leq \eta_\varepsilon(x) \leq 1 & \text{for } x \in \Omega, \\
\text{supp}(\eta_\varepsilon) \subset \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq 3\varepsilon \}, \\
\eta_\varepsilon = 1 & \text{on } \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq 4\varepsilon \}, \\
|\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}.
\end{cases}
\]

(3.1)

Let $\Gamma_\varepsilon = \partial \Omega \cap \varepsilon \omega$, and let $H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$ denote the closure in $H^1(\Omega; \mathbb{R}^d)$ of $C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ functions vanishing on $\Gamma_\varepsilon$.

Lemma 3.4. Let $r_{\varepsilon,\delta} = u_{\varepsilon,\delta} - u_{0,\delta} - \varepsilon \chi_\delta K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)$. Then

\[
\int_\Omega k_\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w = \int_\Omega (\eta_\varepsilon - 1) k_\varepsilon A^\varepsilon \nabla [u_{\varepsilon,\delta} - u_{0,\delta}] \cdot \nabla w + \int_\Omega k_\varepsilon A^\varepsilon \nabla [u_{\varepsilon,\delta} - u_{0,\delta}] \cdot [w \nabla \eta_\varepsilon]
\]

\[
+ \int_\Omega \left[ \tilde{A}_\varepsilon - k_\varepsilon A^\varepsilon \right] \left[ \nabla u_{0,\delta} - K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \right] \cdot \nabla w
\]

\[
- \int_\Omega \left[ \tilde{A}_\varepsilon - k_\varepsilon A^\varepsilon \nabla \chi_\delta \right] K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w
\]

\[
- \varepsilon \int_\Omega k_\varepsilon A^\varepsilon \chi_\delta \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w
\]

for any $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$.

Proof. Since $u_{\varepsilon,\delta}$ and $u_{0,\delta}$ solve (1.6) and (2.10), respectively,

\[
\int_\Omega k_\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla[w \eta_\varepsilon] = \int_\Omega \tilde{A}_\varepsilon \nabla u_{0,\delta} \cdot \nabla[w \eta_\varepsilon] = 0
\]

for any $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$, where $\eta_\varepsilon$ denotes the cut-off function defined by (3.1). Hence,

\[
\int_\Omega k_\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w
\]

\[
= \int_\Omega k_\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w - \int_\Omega k_\varepsilon A^\varepsilon \nabla u_{0,\delta} \cdot \nabla w
\]

\[
- \int_\Omega k_\varepsilon A^\varepsilon \nabla [\varepsilon \chi_\delta K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)] \cdot \nabla w
\]

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This proves the lemma.

and

\[
\int_{\Omega} \tilde{\Delta}_\delta \nabla u_{0,\delta} \cdot \nabla w = \int_{\Omega} (1 - \eta_\delta) \tilde{\Delta}_\delta \nabla u_{0,\delta} \cdot \nabla w + \int_{\Omega} \tilde{\Delta}_\delta \nabla u_{0,\delta} \cdot [w \nabla \eta_\delta].
\]

This proves the lemma.

\[\Box\]

**Lemma 3.5.** For \(w \in H^1(\Omega, \Gamma_{\tilde{\delta}}; \mathbb{R}^d)\),

\[
\left| \int_{\Omega} k_{\delta}^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w \right| \leq C \left\{ \| \nabla u_{0,\delta} \|_{L^2(\Omega)} + \| (\nabla u_{0,\delta}) \eta_\varepsilon - K_\varepsilon (\nabla u_{0,\delta}) \eta_\varepsilon \|_{L^2(\Omega)} \right. \\
+ \varepsilon \| K_\varepsilon (\nabla^2 u_{0,\delta}) \eta_\varepsilon \|_{L^2(\Omega)} + \| k_{\delta}^\varepsilon \nabla u_{\varepsilon,\delta} \|_{L^2(\Omega)} \} \| \nabla w \|_{L^2(\Omega)}
\]

**Proof.** By Lemma 3.4,

\[
\int_{\Omega} k_{\delta}^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w = I_1 + I_2 + I_3 + I_4 + I_5,
\]

(3.2)
where

\[ I_1 = \int_{\Omega} (\eta_\varepsilon - 1) k_\delta^\varepsilon A^\varepsilon \nabla [u_{\varepsilon,\delta} - u_{0,\delta}] \cdot \nabla w \]
\[ I_2 = \int_{\Omega} k_\delta^\varepsilon A^\varepsilon \nabla [u_{\varepsilon,\delta} - u_{0,\delta}] \cdot [w \nabla \eta_\varepsilon] \]
\[ I_3 = \int_{\Omega} \left[ \hat{A}_\delta - k_\delta^\varepsilon A^\varepsilon \right] [\nabla u_{0,\delta} - K_\varepsilon^2 ((\nabla u_{0,\delta}) \eta_\varepsilon)] \cdot \nabla w \]
\[ I_4 = -\int_{\Omega} \left[ \hat{A}_\delta - k_\delta^\varepsilon A^\varepsilon \nabla \chi_\delta^\varepsilon \right] K_\varepsilon^2 ((\nabla u_{0,\delta}) \eta_\varepsilon) \cdot \nabla w \]
\[ I_5 = -\varepsilon \int_{\Omega} k_\delta^\varepsilon A^\varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2 ((\nabla u_{0,\delta}) \eta_\varepsilon) \cdot \nabla w \]

and \( w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d) \). Since \( \text{supp}(1 - \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon} \), where

\[ \mathcal{O}_{4\varepsilon} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < 4\varepsilon \}, \]

by Cauchy-Schwarz, (3.1), and (1.4) we have

\[ |I_1| \leq C \left\{ \| \nabla u_0 \|_{L^2(\mathcal{O}_{4\varepsilon})} + \| k_\delta^\varepsilon \nabla u_{\varepsilon,\delta} \|_{L^2(\mathcal{O}_{4\varepsilon})} \right\} \| \nabla w \|_{L^2(\Omega)}. \] (3.3)

Similarly, as \( \text{supp}((\nabla \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon} \), Cauchy-Schwarz, [23, Lemma 3.4], and (3.1) imply

\[ |I_2| \leq C \left\{ \| \nabla u_0 \|_{L^2(\mathcal{O}_{4\varepsilon})} + \| k_\delta^\varepsilon \nabla u_{\varepsilon,\delta} \|_{L^2(\mathcal{O}_{4\varepsilon})} \right\} \| \nabla w \|_{L^2(\Omega)}. \] (3.4)

Using (3.1) again,

\[ \| \nabla u_0 - K_\varepsilon^2 ((\nabla u_0) \eta_\varepsilon) \|_{L^2(\Omega)} \]
\[ \leq \| (1 - \eta_\varepsilon) \nabla u_0 \|_{L^2(\Omega)} + \| (\nabla u_0) \eta_\varepsilon - K_\varepsilon^2 ((\nabla u_0) \eta_\varepsilon) \|_{L^2(\Omega)} \]
\[ + \| K_\varepsilon ((\nabla u_0) \eta_\varepsilon - K_\varepsilon^2 ((\nabla u_0) \eta_\varepsilon)) \|_{L^2(\Omega)} \]
\[ \leq \| \nabla u_0 \|_{L^2(\mathcal{O}_{4\varepsilon})} + C \| (\nabla u_0) \eta_\varepsilon - K_\varepsilon^2 ((\nabla u_0) \eta_\varepsilon) \|_{L^2(\Omega)}. \]

Therefore, by Cauchy-Schwarz,

\[ |I_3| \leq C \| \nabla u_0 - K_\varepsilon^2 ((\nabla u_0) \eta_\varepsilon) \|_{L^2(\Omega)} \| \nabla w \|_{L^2(\Omega)} \]
\[ \leq C \left\{ \| \nabla u_0 \|_{L^2(\mathcal{O}_{4\varepsilon})} \right\} \| \nabla w \|_{L^2(\Omega)}. \] (3.5)
Set $B_\delta = \tilde{A}_\delta - k_\delta A \nabla X_\delta$. By (2.8) and (2.9), $B_\delta = \{b_{ij,\delta}^{\alpha\beta}\}$ satisfies the assumptions of Lemma 2.4. Therefore, there exists $\pi_\delta = \{\pi_{kij,\delta}^{\alpha\beta}\}$ that is 1-periodic with

$$
\frac{\partial}{\partial y_k} \pi_{kij,\delta}^{\alpha\beta} = b_{ij,\delta}^{\alpha\beta} \quad \text{and} \quad \pi_{kij,\delta}^{\alpha\beta} = -\pi_{ikj,\delta}^{\alpha\beta},
$$

where

$$
b_{ij,\delta}^{\alpha\beta} = \tilde{\alpha}_{ij,\delta}^{\alpha\beta} - k_\delta^2 a_{k,\delta}^{\gamma\alpha} \frac{\partial}{\partial y_k} X_\delta^{\gamma\beta}.
$$

Moreover, $\|\pi_{kij,\delta}^{\alpha\beta}\|_{H^1(Q)} \leq C$ for some constant $C$ depending on $\kappa_1$, $\kappa_2$, but not $\delta$ given Lemma 3.1. Hence, integrating by parts gives

$$
\int_{\Omega} b_{ij,\delta}^{\alpha\beta} K_\varepsilon^2 \left( \frac{\partial u_{0,\delta}^{\beta}}{\partial x_j} \right) \frac{\partial w^{\alpha}}{\partial x_i} = -\varepsilon \int_{\Omega} \pi_{kij,\delta}^{\alpha\beta} \frac{\partial}{\partial x_k} \left[ K_\varepsilon^2 \left( \frac{\partial u_{0,\delta}^{\beta}}{\partial x_j} \right) \frac{\partial w^{\alpha}}{\partial x_i} \right]
$$

$$
= -\varepsilon \int_{\Omega} \pi_{kij,\delta}^{\alpha\beta} \frac{\partial}{\partial x_k} \left[ K_\varepsilon^2 \left( \frac{\partial u_{0,\delta}^{\beta}}{\partial x_j} \right) \frac{\partial w^{\alpha}}{\partial x_i} \right],
$$

since

$$
\int_{\Omega} \pi_{kij,\delta}^{\alpha\beta} K_\varepsilon^2 \left( \frac{\partial u_{0,\delta}^{\beta}}{\partial x_j} \right) \frac{\partial^2 w^{\alpha}}{\partial x_k \partial x_i} = 0
$$

due to the antisymmetry of $\pi_\delta$. Thus, by Lemma 2.2, and (3.1),

$$
|I_4| \leq C \varepsilon \|\pi_\delta^\varepsilon \nabla K_\varepsilon^2 ((\nabla u_{0,\delta}) \eta_\varepsilon)\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}
$$

$$
\leq C \left\{ \|\nabla u_{0,\delta}\|_{L^2(\Omega)} + \varepsilon \|K_\varepsilon ((\nabla^2 u_{0,\delta}) \eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|\nabla w\|_{L^2(\Omega)}. \tag{3.6}
$$

Finally, by Lemma 2.2 and (3.1),

$$
|I_5| \leq C \left\{ \|\nabla u_{0,\delta}\|_{L^2(\Omega)} + \varepsilon \|K_\varepsilon ((\nabla^2 u_{0,\delta}) \eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|\nabla w\|_{L^2(\Omega)} \tag{3.7}
$$

The desired estimate follows from (3.3)–(3.7).

**Lemma 3.6.** For $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$,

$$
\left| \int_{\Omega} k_\delta^\varepsilon A \nabla r_{\varepsilon,\delta} \cdot \nabla w \right| \leq C \varepsilon \|f\|_{H^1(\delta \Omega)} \|\nabla w\|_{L^2(\Omega)},
$$

where $\mu > 0$ depends on $d$, $\kappa_1$, and $\kappa_2$.  

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Proof. Recall that $u_0, \delta$ satisfies $L_0, \delta(u_0, \delta) = 0$ in $\Omega$, and so it follows from estimates for solutions in Lipschitz domains to constant-coefficient systems that

$$\| (\nabla u_{0, \delta})^* \|_{L^2(\partial \Omega)} \leq C \| f \|_{H^1(\partial \Omega)},$$  \hspace{1cm} (3.8)

where $(\nabla u_{0, \delta})^*$ denotes the nontangential maximal function of $\nabla u_{0, \delta}$ (see [12]). By the coarea formula,

$$\| \nabla u_{0, \delta} \|_{L^2(\Omega)} \leq C \varepsilon^{1/2} \| (\nabla u_{0, \delta})^* \|_{L^2(\partial \Omega)} \leq C \varepsilon^{1/2} \| f \|_{H^1(\partial \Omega)}. \hspace{1cm} (3.9)$$

Notice that if $u_{0, \delta}$ solves (2.10), then $L_0, \delta(\nabla u_{0, \delta}) = 0$ in $\Omega$, and so we may use an interior Lipschitz estimate for $L_0, \delta$. That is,

$$|\nabla^2 u_{0, \delta}(x)| \leq \frac{C}{\rho(x)} \left( \int_{B(x, \rho(x)/8)} |\nabla u_{0, \delta}|^2 \right)^{1/2}, \hspace{1cm} (3.10)$$

where $\rho(x) = \text{dist}(x, \partial \Omega)$. In particular,

$$\| (\nabla^2 u_{0, \delta}) \eta_\varepsilon \|_{L^2(\Omega)} \leq \left( \int_{\Omega \setminus \sigma_{3\varepsilon}} |\nabla^2 u_{0, \delta}|^2 \right)^{1/2} \leq C \left( \int_{\Omega \setminus \sigma_{3\varepsilon}} \int_{B(x, \rho(x)/8)} \left| \frac{\nabla u_{0, \delta}(y)}{\rho(x)} \right|^2 \frac{dy}{\rho(x)} dx \right)^{1/2} \leq C \left( \int_{\Omega \setminus \sigma_{3\varepsilon}} \int_{B(x, \rho(x)/8)} |\nabla u_{0, \delta}(y)|^2 dy dS(x) dt \right)^{1/2} \leq C \| (\nabla u_{0, \delta})^* \|_{L^2(\partial \Omega)} \left( \int_{3\varepsilon}^{C_0} t^{-2} dt \right)^{1/2} + C_1 \| \nabla u_{0, \delta} \|_{L^2(\Omega)} \leq C \left\{ \varepsilon^{-1/2} \| f \|_{H^1(\partial \Omega)} + \| f \|_{H^{1/2}(\partial \Omega)} \right\} \leq C_0 \varepsilon^{-1/2} \| f \|_{H^1(\partial \Omega)}, \hspace{1cm} (3.11)$$

where $C_0$ is a constant depending on $\Omega$, and we have used (3.1), (3.8) (3.9), the coarea formula, energy estimates, and (3.10). Hence,

$$\varepsilon \| K_\varepsilon((\nabla^2 u_{0, \delta}) \eta_\varepsilon) \|_{L^2(\Omega)} \leq C \varepsilon^{1/2} \| f \|_{H^1(\partial \Omega)} . \hspace{1cm} (3.12)$$
By Lemma 2.1,
\[
\| (\nabla u_{0,\delta} )_{\eta} - K_\varepsilon ((\nabla u_{0,\delta} )_{\eta} ) \|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \| f \|_{H^1(\partial\Omega)},
\]
(3.13)
where the last inequality follows from (3.12) and (3.1).

Finally, we establish a $W^{1,p}$-estimate for some $p > 2$ for $u_{\varepsilon,\delta}$ uniform in $\varepsilon$ and $\delta$ by establishing a reverse Hölder inequality. Indeed, if there exists a $p > 2$ so that
\[
\left( \int_{\Omega} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^p \right)^{1/p} \leq C \| f \|_{H^1(\partial\Omega)}^2,
\]
then Hölder’s inequality implies
\[
\int_{\mathcal{O}_{\delta\varepsilon}} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2 \leq C\varepsilon^{(p-2)/p} \| f \|_{H^1(\Omega)}^2.
\]
(3.14)
The existence of such a $p$ follows from the Lemma 3.7. Equations (3.9), (3.12), (3.13), and (3.14) give the desired result.

**Lemma 3.7.** There exists a $p_0 > 2$ such that
\[
\left( \int_{\Omega} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^{p_0} \right)^{1/p_0} \leq C \| f \|_{H^1(\partial\Omega)}
\]
for some constant $C$ depending on $\kappa_1$, $\kappa_2$, $d$, $p_0$, and $\Omega$.

**Proof.** The desired estimate essentially follows from Cacciopoli’s inequality, the Poincaré-Sobolev inequality, and the self-improving property of reverse Hölder inequalities. We prove an interior estimate, and the boundary estimate follows with an analogous proof.

Take $B(x_0,2r) \subset \Omega$, and note that Cacciopoli’s inequality (see Lemma 4.1) implies
\[
\left( \int_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2 \right)^{1/2} \leq C \left( \int_{B(x_0,2r)} |k_\delta^\varepsilon u_{\varepsilon,\delta}|^2 \right)^{1/2}
\]
\[
\leq \frac{C}{r} \left( \int_{B(x_0,2r)} |k_\delta^\varepsilon u_{\varepsilon,\delta}|^2 \right)^{1/2}
\]
\[
\leq \frac{C}{r} \left\{ \delta \left( \int_{B(x_0,2r)} |u_{\varepsilon,\delta}|^2 \right)^{1/2} + \left( \int_{B(x_0,2r)} |P_\varepsilon (1_+ u_{\varepsilon,\delta})|^2 \right)^{1/2} \right\},
\]

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which is invariant if we subtract a constant vector from $u_{\varepsilon, \delta}$. If we subtract the average value of $u_{\varepsilon, \delta}$ over the ball $B(x_0, 2r)$, then by the Poincaré-Sobolev inequality
\[
\left( \frac{1}{\| B(x_0, r) \|} \left| \int_{B(x_0, r)} k_{\delta}^\varepsilon \nabla u_{\varepsilon, \delta} \right|^2 \right)^{1/2} \\
\leq \delta \left( \frac{1}{\| B(x_0, 2r) \|} \left| \nabla u_{\varepsilon, \delta} \right|^s \right)^{1/s} + \frac{C}{r} \left( \frac{1}{\| B(x_0, 2r) \|} \left| P_{\varepsilon} (1_{\mathbb{L}} u_{\varepsilon, \delta}) \right|^2 \right)^{1/2},
\]
where $s = 2d/(d + 2)$. Similarly, by subtracting another constant we can show
\[
\left( \frac{1}{\| B(x_0, r) \|} \left| \int_{B(x_0, r)} k_{\delta}^\varepsilon \nabla u_{\varepsilon, \delta} \right|^2 \right)^{1/2} \\
\leq \delta \left( \frac{1}{\| B(x_0, 2r) \|} \left| \nabla u_{\varepsilon, \delta} \right|^s \right)^{1/s} + \left( \frac{1}{\| B(x_0, 2r) \|} \left| \nabla P_{\varepsilon} (1_{\mathbb{L}} u_{\varepsilon, \delta}) \right|^s \right)^{1/s},
\]
which by Lemma 2.6 shows
\[
\left( \frac{1}{\| B(x_0, r) \|} \int_{B(x_0, r)} w^q \right)^{1/q} \leq C \frac{1}{\| B(x_0, 2r) \|} \int_{B(x_0, 2r)} w,
\]
where $w = |k_{\delta}^\varepsilon \nabla u_{\varepsilon, \delta}|^s$ and $q = 2/s$. By the self-improving property of reverse Hölder inequalities (see [16, Chapter V, Proposition 1.1]),
\[
\left( \frac{1}{\| B(x_0, r) \|} \int_{B(x_0, r)} w^t \right)^{1/t} \leq C \left( \frac{1}{\| B(x_0, 2r) \|} \int_{B(x_0, 2r)} w^q \right)^{1/q},
\]
for any $t \in [q, q + \nu)$ for some $\nu > 0$ depending on $\kappa_1, \kappa_2$, and $d$. That is,
\[
\left( \frac{1}{\| B(x_0, r) \|} \int_{B(x_0, r)} |k_{\delta}^\varepsilon \nabla u_{\varepsilon, \delta}|^p \right)^{1/p} \leq C \left( \frac{1}{\| B(x_0, 2r) \|} \int_{B(x_0, 2r)} |k_{\delta}^\varepsilon \nabla u_{\varepsilon, \delta}|^2 \right)^{1/2} \tag{3.15}
\]
for any $p \in [2, 2 + \nu)$ and any $B(x_0, 2r) \subset \Omega$.

We may show a similar estimate for any ball $B(x_0, 2r)$ with $x_0 \in \partial \Omega$. That is, if $F = f$ on $\partial \Omega$ and $F \in H^{3/2}(\Omega)$, then the continuous injection $H^{3/2}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ for any $q \geq 2d/(d - 1)$ gives the estimate
\[
\left( \frac{1}{\| B(x_0, r) \|} \int_{B(x_0, r) \cap \Omega} |k_{\delta}^\varepsilon \nabla u_{\varepsilon, \delta}|^p \right)^{1/p}
\]
\[ \leq C \left( \int_{B(x_0,2r)^c \cap \Omega} |k_\delta^e \nabla u_{e,\delta}|^2 \right)^{1/2} + \left( \int_{B(x_0,2r)^c \cap \Omega} |\nabla F|^q \right)^{1/q}. \] (3.16)

Patching together inequalities (3.15) and (3.16) gives the desired estimate for some \( p_0 > 2 \).

**Proof of Theorem 1.3.** Note \( \delta r_{\varepsilon,\delta} \in H^1_0(\Omega; \mathbb{R}^d) \subset H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d) \), and so by Lemmas 3.6 and (1.4),

\[
\| \delta e(r_{\varepsilon,\delta}) \|_{L^2(\Omega)}^2 \leq C \delta \int_{\Omega} k_\delta^e A^e \nabla r_{\varepsilon,\delta} \cdot \nabla r_{\varepsilon,\delta} \\
\leq C \varepsilon^\mu \| f \|_{H^1(\partial\Omega)} \| \delta \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)},
\]

where \( e(r_{\varepsilon,\delta}) \) denotes the symmetric part of \( \nabla r_{\varepsilon,\delta} \). Korn’s first inequality then implies

\[
\| \delta \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)} \leq C \varepsilon^\mu \| f \|_{H^1(\partial\Omega)}.
\] (3.17)

Note also \( P_\varepsilon(1^e_+ r_{\varepsilon,\delta}) \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d) \), and so by Lemmas 3.6 and 2.6,

\[
\| 1^e_+ e(P_\varepsilon(1^e_+ r_{\varepsilon,\delta})) \|_{L^2(\Omega)}^2 \\
\leq C \int_{\Omega} k_\delta A^e \nabla r_{\varepsilon,\delta} \cdot \nabla P_\varepsilon(1^e_+ r_{\varepsilon,\delta}) - \delta \int_{\Omega} 1^e_+ A^e \nabla r_{\varepsilon,\delta} \cdot \nabla P_\varepsilon(1^e_+ r_{\varepsilon,\delta}) \\
\leq C \varepsilon^\mu \| f \|_{H^1(\partial\Omega)} \| 1^e_+ \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)},
\]

where we’ve used (3.17). Korn’s first inequality for periodically perforated domains then implies

\[
\| 1^e_+ \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)}^2 \leq C \| 1^e_+ e(P_\varepsilon(1^e_+ r_{\varepsilon,\delta})) \|_{L^2(\Omega)}^2 \leq C \varepsilon^\mu \| f \|_{H^1(\partial\Omega)} \| 1^e_+ \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)},
\]

and so

\[
\| 1^e_+ \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)} \leq C \varepsilon^\mu \| f \|_{H^1(\partial\Omega)}. \] (3.18)

Equations (3.17) and (3.18) give the desired estimate. Indeed,

\[
\| k_\delta \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)} \leq \| 1^e_+ \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)} + \| \delta \nabla r_{\varepsilon,\delta} \|_{L^2(\Omega)} \leq C \varepsilon^\mu \| f \|_{H^1(\partial\Omega)}.
\]
4 Interior Estimates at the large scale

In this section, we discuss \textit{a priori} interior estimates for the boundary value problem (1.6) at the macroscopic scale by proving Theorem 1.1. By macroscopic, we refer to the case when \(\varepsilon/r \leq 1\). Throughout this section, let \(B(r) \equiv B(x_0, r)\) denote the ball of radius \(r > 0\) centered at some \(x_0 \in \mathbb{R}^d\).

The following lemma is essentially Cacciopoli’s inequality for the operator \(L_{\varepsilon, \delta} \) defined by (1.2). The proof is similar to a proof of the classical Cacciopoli’s inequality, but nevertheless we present a proof for completeness.

\textbf{Lemma 4.1.} Suppose \(L_{\varepsilon, \delta}(u_{\varepsilon, \delta}) = 0\) in \(B(2r)\) for some \(r > 0\). Then
\[
\left( \frac{1}{|B(r)|} \int_{B(r)} |k_{\delta} A u_{\varepsilon, \delta}|^2 \right)^{1/2} \leq \frac{C}{r} \left( \frac{1}{|B(2r)|} \int_{B(2r)} |k_{\delta} u_{\varepsilon, \delta}|^2 \right)^{1/2},
\]
where \(C\) depends only on \(\kappa_1\) and \(\kappa_2\).

\textit{Proof.} By rescaling we may assume \(r = 1\), i.e., set \(U(x) = u_{\varepsilon, \delta}(rx)\) and note \(U\) satisfies \(L_{\varepsilon/r, \delta}(U) = 0\) in \(B(2)\). Let \(\zeta \in C_0^\infty(B(2))\). Then
\[
0 = \int_{B(2)} k_{\delta} A u \cdot \nabla (u \zeta^2) \geq \kappa_1 \int_{B(2)} k_{\delta} |\nabla u|^2 \zeta^2 - 2 \int_{B(2)} (u \zeta) k_{\delta} A u \cdot \nabla \zeta, \tag{4.1}
\]
where \(u \equiv u_{\varepsilon, \delta}\). Equation (4.1), \(\delta \leq 1\), and Cauchy-Schwarz imply
\[
\int_{B(2)} 1/\gamma |\delta \nabla u|^2 \zeta^2 \leq \frac{C_1}{\gamma} \int_{B(2)} k_{\delta} |\nabla u|^2 \zeta^2 + \gamma C_2 \int_{B(2)} k_{\delta} u^2 |\nabla \zeta|^2
\]
for any \(\gamma > 0\). Similarly, equation (4.1) and Cauchy-Schwarz give
\[
\int_{B(2)} 1/\gamma' |\nabla u|^2 \zeta^2 \leq \frac{C_1}{\gamma'} \int_{B(2)} k_{\delta} |\nabla u|^2 \zeta^2 + \gamma' C_2 \int_{B(2)} k_{\delta} u^2 |\nabla \zeta|^2
\]
for any \(\gamma' > 0\). Choosing \(\gamma, \gamma'\) large enough gives
\[
\int_{B(2)} |k_{\delta} \nabla u|^2 \zeta^2 \leq C \int_{B(2)} |k_{\delta} u|^2 |\nabla \zeta|^2
\]
for some constant \(C\) depending on \(\kappa_1\) and \(\kappa_2\). Choose \(\zeta\) so that \(\zeta \equiv 1\) in \(B(1)\) and \(|\nabla \zeta| \leq C\). The desired inequality follows. \(\Box\)
Lemma 4.2. Suppose $\mathcal{L}_{\epsilon, \delta}(u_{\epsilon, \delta}) = 0$ in $B(3r)$. There exists $v \in H^1(B(r); \mathbb{R}^d)$ satisfying $\mathcal{L}_{0, \delta}(v) = 0$ in $B(r)$ and

$$
\left( \int_{B(r)} |k_\delta^\epsilon(u_{\epsilon, \delta} - v)|^2 \right)^{1/2} \leq C \left( \frac{\epsilon}{r} \right)^\mu \left( \int_{B(3r)} |k_\delta^\epsilon u_{\epsilon, \delta}|^2 \right)^{1/2},
$$

where $C$ depends on $\kappa_1$, $\kappa_2$, and $d$ and $\mu > 0$.

Proof. First we prove the lemma for $r = 1$. By Lemma 4.1 and estimate (2.4) in Theorem 2.6 of Section 2,

$$
\left( \int_{B(3/2)} |
abla P_\epsilon(1_+^\epsilon u)|^2 \right)^{1/2} + \left( \int_{B(3/2)} |\delta \nabla u|^2 \right)^{1/2} \leq C \left( \int_{B(3)} |k_\delta^\epsilon u|^2 \right)^{1/2},
$$

where $u \equiv u_{\epsilon, \delta}$. Specifically, there exists a $t \in [1, 5/4]$ such that

$$
\|P_\epsilon(1_+^\epsilon u)\|_{H^1(\partial B(t))} + \delta \|u\|_{H^1(\partial B(t))} \leq C \|k_\delta^\epsilon u\|_{L^2(B(3))}.
$$

(4.2)

Let $v$ denote the weak solution to the Dirichlet problem $\mathcal{L}_{0, \delta}(v) = 0$ in $B(t)$ and $v = P_\epsilon(1_+^\epsilon u)$ on $\partial B(t)$. Note $v = u = P_\epsilon(1_+^\epsilon u)$ on $\partial B(t) \cap \epsilon \omega$.

By Theorem 1.3,

$$
\|k_\delta^\epsilon(u - v)\|_{L^2(B(1))} \leq C \epsilon^\mu \|P_\epsilon(1_+^\epsilon u)\|_{H^1(\partial B(t))} + \delta \|\nabla P_\epsilon(1_+^\epsilon u) - \nabla u\|_{L^2(B(t))},
$$

(4.3)

since

$$
\|k_\delta^\epsilon \chi_\delta^\epsilon \kappa_\epsilon^2(\nabla v)\eta_\epsilon\|_{L^2(B(t))} \leq C \|\nabla v\|_{L^2(B(t))} \leq C \|P_\epsilon(1_+^\epsilon u)\|_{H^1(\partial B(t))},
$$

where we’ve used notation consistent with Theorem 1.3.

By Lemma 4.1,

$$
\int_{B(t)} \left| k_\delta^\epsilon \nabla w \right|^2 \leq \int_{B(t)} \left| \nabla P_\epsilon(1_+^\epsilon u) \right|^2 + C \int_{B(2t)} |k_\delta^\epsilon w|^2,
$$

(4.4)

where $w = P_\epsilon(1_+^\epsilon u) - u$. Equation (4.4) follows from the fact that $\mathcal{L}_{\epsilon, \delta}(w) = \mathcal{L}_{\epsilon, \delta}(P_\epsilon(1_+^\epsilon u))$ in $B(3)$ and $t \in [1, 5/4]$. Note by Lemma 2.6, $w = 0$ a.e. in $B(3) \cap \epsilon \omega$. Hence, Poincaré’s inequality gives

$$
\int_{B(2t)} |k_\delta^\epsilon w|^2 = \delta^2 \int_{B(2t)} 1_+^\epsilon |w|^2 \leq \epsilon^2 \int_{B(3)} |k_\delta^\epsilon \nabla w|^2.
$$

(4.5)
Indeed, set \( W(x) = w(\varepsilon x) \), and let \( \{H_k\}_{k=1}^{N(\varepsilon)} \) denote the bounded, connected components of \( \mathbb{R}^d \setminus \omega \) with \( \varepsilon H_k \cap B(2t) \neq \emptyset \). Then \( W = 0 \) on \( \partial H_k \) for each \( k \), and so

\[
\int_{B(2t)} |w|^2 \leq \sum_{k=1}^{N} \int_{H_k} |W|^2 \leq C \varepsilon^2 \sum_{k=1}^{N} \int_{\varepsilon H_k} |\nabla w|^2 \leq C \varepsilon^2 \int_{B(3)} |\nabla w|^2,
\]

where \( C \) is independent of \( \varepsilon \) since \( \omega \) is periodic. Lemma 2.6 together with (4.2), (4.3) and (4.5) give the estimate for \( r = 1 \).

Now we prove the estimate for arbitrary \( r > 0 \). To this end, let \( U(x) = u(rx) \), and note \( \mathcal{L}_{\varepsilon/r,\delta}(U) = 0 \) in \( B(3) \). By the above, there exists a \( V \in H^1(B(1); \mathbb{R}^d) \) satisfying \( \mathcal{L}_{0,\delta}(V) = 0 \) in \( B(1) \) and

\[
\left( \frac{1}{B(1)} \int |k_{\delta/r}^\varepsilon (U - V)|^2 \right)^{1/2} \leq C \left( \frac{\varepsilon}{r} \right) \left( \frac{1}{B(3)} \int |k_{\delta}^\varepsilon U|^2 \right)^{1/2},
\]

The change of variables \( rx \mapsto x \) gives the desired estimate.

**Lemma 4.3.** Suppose \( \mathcal{L}_{0,\delta}(v) = 0 \) in \( B(2r) \). Then for \( r \geq \varepsilon \),

\[
\left( \frac{1}{B(r)} \int |v|^2 \right)^{1/2} \leq C \left( \frac{1}{B(2r)} \int |k_{\delta}^\varepsilon v|^2 \right)^{1/2}
\]

for a constant \( C \) depending on \( \omega, \kappa_1, \kappa_2, \) and \( d \).

**Proof.** See [23] for a proof when \( \delta = 0 \). The case \( \delta > 0 \) follows similiarly given Lemma 3.1.

For \( w \in L^2_{\text{loc}}(B(r); \mathbb{R}^d) \), \( \delta \geq 0 \), and \( \varepsilon, r > 0 \), set

\[
H_{\varepsilon,\delta}(r; w) = \frac{1}{r} \inf_{M \in \mathbb{R}^{d\times d}} \left( \frac{1}{B(r)} \int |k_{\delta}^\varepsilon (w - Mx - q)|^2 \right)^{1/2}.
\]

**Lemma 4.4.** Suppose \( v \) satisfies \( \mathcal{L}_{0,\delta}(v) = 0 \) in \( B(1) \). For any \( r \in [\varepsilon, 1] \) and \( \theta \in (0, 1/4) \),

\[
H_{\varepsilon,\delta}(\theta r; v) \leq C \theta H_{\varepsilon,\delta}(r; v)
\]

for some constant \( C \) depending on \( d, \kappa_1, \kappa_2, \) and \( \omega \).
Proof. It follows from interior $C^2$-estimates for elasticity systems with constant coefficients that for any $\theta \in (0, 1/4)$,

$$H_{\varepsilon, \delta}(\theta r; v) \leq H_{\varepsilon, 1}(\theta r; v) \leq C_1 \theta H_{\varepsilon, 1}(r/2; v),$$

where $C_1$ a constant depending on $d, \kappa_1, \kappa_2$. By Lemma 4.3, we have the desired estimate. □

Lemma 4.5. Suppose $\mathcal{L}_{\varepsilon, \delta}(u_{\varepsilon, \delta}) = 0$ in $B(1)$. For any $\varepsilon \leq r \leq 1/3$,

$$H_{\varepsilon, \delta}(\theta r; u) \leq C_1 \theta H_{\varepsilon, \delta}(r; u) + \frac{C_2}{r} \left( \varepsilon \right) \mu \inf_{q \in \mathbb{R}^d} \left( \int_{B(r)} |k_{\delta}^\varepsilon(u - q)|^2 \right)^{1/2},$$

where $u \equiv u_{\varepsilon, \delta}$, $\theta \in (0, 1/4)$, and $\mu > 0$.

Proof. Fix $r \geq \varepsilon$, and let $v \equiv v_r$ denote the function given by Lemma 4.2. We have

$$H_{\varepsilon, \delta}(\theta r; u) \leq \frac{1}{\theta r} \left( \int_{B(\theta r)} |k_{\delta}^\varepsilon(u - v)|^2 \right)^{1/2} + H_{\varepsilon, \delta}(\theta r; v)$$

$$\leq \frac{C}{r} \left( \int_{B(r)} |k_{\delta}^\varepsilon(u - v)|^2 \right)^{1/2} + C_1 \theta H_{\varepsilon, \delta}(r; v)$$

$$\leq \frac{C}{r} \left( \int_{B(r)} |k_{\delta}^\varepsilon(u - v)|^2 \right)^{1/2} + C_1 \theta H_{\varepsilon, \delta}(r; u),$$

where we’ve used Lemma 4.4. By Lemma 4.2,

$$H_{\varepsilon, \delta}(\theta r; u) \leq \frac{C_2}{r} \left( \varepsilon \right) \mu \left( \int_{B(3r)} |k_{\delta}^\varepsilon u|^2 \right)^{1/2} + C_1 \theta H_{\varepsilon, \delta}(r; u). \quad (4.8)$$

Since (4.8) remains invariant if we subtract a constant from $u$, the desired estimate follows. □

Lemma 4.6. Let $H(r)$ and $h(r)$ be two nonnegative continuous functions on the interval $[0, 1]$. Let $0 < \varepsilon < 1/6$. Suppose that there exists a constant $C_0$ with

$$\begin{cases}
\max_{r \leq t \leq 3r} H(t) \leq C_0 H(3r), \\
\max_{r \leq t, s \leq 3r} |h(t) - h(s)| \leq C_0 H(3r),
\end{cases} \quad (4.9)$$
for any $r \in [\varepsilon, 1/3]$. We further assume
\[ H(\theta r) \leq \frac{1}{2} H(r) + C_0 \left( \frac{\varepsilon}{r} \right)^\mu \{H(3r) + h(3r)\} \]  
for any $r \in [\varepsilon, 1/3]$ and some $\mu > 0$, where $\theta \in (0, 1/4)$. Then
\[ \max_{\varepsilon \leq r \leq 1} \{H(r) + h(r)\} \leq C \{H(1) + h(1)\}, \]
where $C$ depends on $C_0$ and $\theta$.

**Proof.** See [24, Lemma 8.5].

**Proof of Theorem 1.1.** By rescaling, we may assume $R = 1$. We assume $\varepsilon \in (0, 1/6)$, and we let $H(r) \equiv H_{\varepsilon, \delta}(r; u)$, where $u \equiv u_{\varepsilon, \delta}$ and $H_{\varepsilon, \delta}(r; u)$ is defined above by (4.7). Let $h(r) = r^{-1}|M_r|$, where $M_r \in \mathbb{R}^{d \times d}$ satisfies
\[ H(r) = \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B(r)} |k_\delta(u - M_r x - q)|^2 \right)^{1/2}. \]

Note there exists a constant $C$ independent of $r$ so that
\[ H(t) \leq CH(3r), \quad t \in [r, 3r]. \]  
(4.11)

Suppose $s, t \in [r, 3r]$. We have
\[ |h(t) - h(s)| \leq C \inf_{q \in \mathbb{R}^d} \left( \int_{B(r)} k_\delta(M_t - M_s)x - q|^2 \right)^{1/2} \]
\[ \leq C \inf_{t \in \mathbb{R}^d} \left( \int_{B(t)} k_\delta|u - M_t x - q|^2 \right)^{1/2} \]
\[ + C \inf_{s \in \mathbb{R}^d} \left( \int_{B(s)} k_\delta|u - M_s x - q|^2 \right)^{1/2} \]
\[ \leq CH(3r), \]
where we’ve used (4.11) for the last inequality. Specifically,
\[ \max_{r \leq t, s \leq 3r} |h(t) - h(s)| \leq CH(3r). \]
(4.12)

Clearly
\[ \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B(3r)} k_\delta(u - q)|^2 \right)^{1/2} \leq H(3r) + h(3r), \]
and so Lemma 4.5 implies
\[
H(\theta r) \leq \frac{1}{2} H(r) + C \left( \frac{\varepsilon}{r} \right)^{\mu} \{ H(3r) + h(3r) \} \quad (4.13)
\]
for any \( r \in [\varepsilon, 1/3] \) and some \( \theta \in (0, 1/4) \). Note equations (4.11), (4.12), and (4.13) show that \( H(r) \) and \( h(r) \) satisfy the assumptions of Lemma 4.6. Consequently,
\[
\left( \int_{B(r)} |k^\varepsilon \nabla u|^2 \right)^{1/2} \leq C \inf_{r \in \mathbb{R}^d} \left( \int_{B(3r)} |k^\varepsilon (u - q)|^2 \right)^{1/2} \leq C \{ H(3r) + h(3r) \} \leq C \{ H(1) + h(1) \} \leq C \left( \int_{B(1)} |k^\varepsilon u|^2 \right)^{1/2} \quad (4.14)
\]
Since (4.14) remains invariant if we subtract a constant from \( u \), the desired estimate in Theorem 1.1 follows from Poincaré’s inequality.

\[\square\]

**A  Interior estimates at the small-scale**

In this appendix, we discuss combining the large-scale estimate Theorem 1.1 with \( C^{1,\alpha} \) estimates for interface problems to derive interior estimates at both the macroscopic and microscopic scale. In particular, we show Corollary 1.2. First, we prove the following lemma.

**Lemma A.1.** Suppose \( A \) satisfies (1.3), (1.4), (1.5), and is \( \alpha \)-Hölder continuous for some \( \alpha \in (0, 1) \), i.e., \( A \) satisfies (1.8). Suppose \( \omega \) is an unbounded \( C^{1,\alpha} \) domain. Let \( u_{1,\delta} \) denote a weak solution to \( \mathcal{L}_{1,\delta}(u_{\delta}) = 0 \) in \( B(x_0, 1) \) for some \( x_0 \in \mathbb{R}^d \). Then
\[
\| \nabla u_{\delta} \|_{C^{0,\alpha}(B(x_0, r) \cap \omega)} + \delta \| \nabla u_{\delta} \|_{C^{0,\alpha}(B(x_0, r) \backslash \omega)} \leq C \| k_{\delta} \nabla u_{\delta} \|_{L^2(B(x_0, 1))},
\]
for a constant \( C \) independent of \( \delta \) and \( 0 < r \leq 1/3 \). In particular,
\[
\| k_{\delta} \nabla u_{\delta} \|_{L^\infty(B(x_0, r))} \leq C \| k_{\delta} \nabla u_{\delta} \|_{L^2(B(x_0, 1))}
\]
for \( 0 < r \leq 1/3 \).

Lemma A.1 was proved for scalar equations with diagonal coefficients in smooth domains in \([14, 25, 26]\). Lemma A.1 continues to hold
for elliptic systems with coefficients and domains satisfying the given assumptions. Together, Lemma A.1 and Theorem 1.1 give interior Lipschitz estimates for $L_{\epsilon,\delta}$ at every scale.

Let $\Gamma(\cdot, x)$ denote the matrix-valued fundamental solution associated with $L_{1,1}$ in $\mathbb{R}^d$. That is, $\Gamma(\cdot, x) = \{\Gamma^\alpha\beta(\cdot, x)\}_{1 \leq \alpha, \beta \leq d}$ satisfies

$$f^\beta(x) = \int_{\mathbb{R}^d} a_{ij}^\alpha(\xi) \frac{\partial \Gamma^\gamma\beta}{\partial x_j}(\xi, x) \frac{\partial f^\alpha}{\partial x_i}(\xi) d\sigma(\xi)$$

for $f = \{f^\beta\}_{1 \leq \beta \leq d} \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d)$. Indeed, if $A$ is VMO, i.e.,

$$\sup_{x \in \mathbb{R}^d} \int_{B(x, r)} A(y) - \int_{B(x, r)} A \left| dy \right. \rightarrow 0 \quad \text{as} \quad R \rightarrow 0^+ \quad (A.1)$$

then $\Gamma(\cdot, x) \in W^{1,1}_\text{loc}(\mathbb{R}^d \setminus \{x\}; \mathbb{R}^{d \times d})$ exists uniquely for each $x \in \mathbb{R}^d$ (see work of Hofmann and Kim [18] for $d \geq 3$ and work of Brown, Kim, and Taylor [6] for $d = 2$). If $A$ satisfies (1.8), then $A$ satisfies (A.1).

For a bounded, simply-connected domain $H$ and $g \in L^2(\partial H; \mathbb{R}^d)$, the single-layer potential $Sg = \{(Sg)^\alpha\}_{1 \leq \alpha \leq d}$ is given by

$$(Sg)^\alpha(x) = \int_{\partial H} \Gamma^{\alpha\beta}(x, \xi) g^\beta(\xi) d\sigma(\xi), \quad x \in \mathbb{R}^d \setminus \partial H \quad (A.2)$$

and the double-layer potential $Dg = \{(Dg)^\alpha\}_{1 \leq \alpha \leq d}$ is given by

$$(Dg)^\alpha(x) = \int_{\partial H} n_i(\xi) a_{ij}^\alpha(\xi) \frac{\partial \Gamma^{\beta\gamma}}{\partial x_j}(\xi, x) g^\gamma(\xi) d\sigma(\xi), \quad x \in \mathbb{R}^d \setminus \partial H \quad (A.3)$$

where $n(\xi) = \{n_i(\xi)\}_{1 \leq i \leq d}$ denotes the unit vector outward normal to $H$ at $\xi \in \partial H$.

It is known (see [20, Theorem 4.6]) that if $g \in L^2(\partial H; \mathbb{R}^d)$, then

$$Dg^\pm = \pm \frac{1}{2}g + Kg \quad \text{on} \quad \partial H, \quad (A.4)$$

where $K$ is given by

$$Kg(x) = \text{p.v.} \int_{\partial H} n_i(\xi) a_{ij}^\alpha(\xi) \frac{\partial \Gamma^{\beta\gamma}}{\partial x_j}(\xi, x) g^\gamma(\xi) d\sigma(\xi), \quad x \in \partial H$$

and

$$Dg^\pm(x) = \lim_{h \to 0^+} Dg(x \pm hn), \quad 27$$
i.e., $Dg^+$ and $Dg^-$ denote the traces of $Dg$ on $\partial H$ from the exterior of $\Pi$ and the interior of $H$, respectively. In particular, $w = Dg$ satisfies $L_{1,1}(w) = 0$ in $H$ and $w = (-\frac{1}{2} + \mathcal{K})g$ on $\partial H$. It is also known (see [20, Lemma 5.7]) that if $H$ is Lipschitz and $A$ satisfies (1.3), (1.4), and (1.8), then

$$\frac{1}{2} + \mathcal{K} : L^2(\partial H; \mathbb{R}^d) \rightarrow L^2(\partial H; \mathbb{R}^d)$$

is bounded and continuously invertible. For single equations, this follows from the compactness of $\mathcal{K}$ and Fredholm theory (see the argument of Yeh in [25, Lemma 3.2]). For systems with variable coefficients, the operator $\mathcal{K}$ is not compact (see the work of Kenig and Shen [20] for an alternative proof of invertibility on $L^2$). The following lemma, however, is more or less known.

**Lemma A.2.** Suppose $A$ satisfies (1.3), (1.4), (1.5) and is $\alpha$-Hölder continuous, i.e., satisfies (1.8), for some $\alpha \in (0, 1)$. Suppose $H$ is a bounded $C^{1,\alpha}$ domain. The operators

$$S : C^{0,\alpha}(\partial H; \mathbb{R}^d) \mapsto C^{1,\alpha}(\partial H; \mathbb{R}^d)$$

and

$$D : C^{1,\alpha}(\partial H; \mathbb{R}^d) \mapsto C^{1,\alpha}(\partial H; \mathbb{R}^d)$$

defined by (A.2) and (A.3), respectively, are bounded.

From (A.4), we have the jump relations

$$g = w^+ - w^- \quad \text{and} \quad \left( \frac{\partial w}{\partial n} \right)^+ = \left( \frac{\partial w}{\partial n} \right)^-,$$

(A.6)

where $w = Dg$ and $\partial w/\partial n = n \cdot \nabla w$ denotes the normal derivative of $w$.

The following lemma essentially follows from the jump relations (A.6) and regularity problems for the exterior Neumann and interior Dirichlet problems.

**Lemma A.3.** There exists a constant $C$ depending on $\kappa_1, \kappa_2$ and $H$ such that

$$\|g\|_{1,\alpha} \leq C \left\| \left( -\frac{1}{2} + \mathcal{K} \right) g \right\|_{1,\alpha}
$$

for any $g \in C^{1,\alpha}(\partial H; \mathbb{R}^d)$, where $\| \cdot \|_{1,\alpha} = \| \cdot \|_{C^{1,\alpha}(\partial H)}$. 

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As mentioned in Section 2, any two connected components of \( \mathbb{R}^d \setminus \omega \) are separated by some positive distance \( g^\omega \). If \( \mathbb{R}^d \setminus \omega = \bigcup_{k=1}^{\infty} H_k \), write \( H_k^* \) to denote the set

\[
H_k^* = \{ x \in \mathbb{R}^d : \text{dist}(x, H_k) \leq g^\omega/4 \}.
\]

To prove Lemma A.1, it suffices to show the result holds in each \( H_k^* \).

Indeed, if \( A \) satisfies (1.8), the boundedness of \( \nabla u_{1,\delta} \) in the interior of \( \omega \) follows from classical results regarding elliptic systems with Hölder continuous coefficients.

**Lemma A.4.** Suppose \( A \) satisfies (1.3), (1.4), and is \( \alpha \)-Hölder continuous for some \( \alpha \in (0,1) \). Suppose \( \omega \) is an unbounded \( C^{1,\alpha} \) domain. If \( L_{1,\delta}(u_{1,\delta}) = \text{div}(f) \) in \( H_k^* \) and \( u_{1,\delta} = 0 \) on \( \partial H_k^* \), then

\[
\| k_\delta \nabla u_{1,\delta} \|_{C^{0,\alpha}(H_k^*)} \leq C \left\{ \| k_\delta \nabla u_{1,\delta} \|_{L^2(H_k^*)} + \| k_{\delta^{-1}} f \|_{C^{0,\alpha}(H_k^*)} \right\},
\]

where \( C \) depends only on \( \| A \|_{C^{\alpha}}, \alpha, \omega, \kappa_1, \kappa_2 \).

**Proof.** Note that if \( \delta_0 \leq \delta \leq 1 \), then the result follows from general results regarding divergence form elliptic equations with \( \alpha \)-Hölder continuous coefficients in \( C^{1,\alpha} \) domains. Hence, we may assume \( 0 \leq \delta \leq \delta_0 \) for some \( \delta_0 \) to be determined.

Let \( u_1 \) satisfy the boundary value problem

\[
\begin{cases}
-\text{div}(\delta^2 A \nabla u_1) = \text{div}(f) & \text{in } H_k \\
-\text{div}(A \nabla u_1) = \text{div}(f) & \text{in } H_k^* \setminus H_k \\
u_1 = 0 & \text{on } \partial H_k \cup \partial H_k^*
\end{cases}
\]

By \( C^{1,\alpha} \) estimates for elliptic systems with \( \alpha \)-Hölder continuous coefficients in \( C^{1,\alpha} \) domains (see [8, Chapter 9, Theorem 2.7]), we have

\[
\| k_\delta \nabla u_1 \|_{C^{0,\alpha}(H_k^*)} \leq C \| k_{\delta^{-1}} f \|_{C^{0,\alpha}(H_k^*)}.
\]

Set \( u_2 = u - u_1 \), where \( u \equiv u_{1,\delta} \). Note then \( u_2 \) satisfies the equation and jump conditions

\[
\begin{cases}
-\text{div}(k_{\delta^2} A \nabla u_2) = 0 & \text{in } H_k^{*} \\
[k_\delta^2 A \nabla u_2]_{\partial H_k} \cdot n = -[k_\delta^2 A \nabla u_1]_{\partial H_k} \cdot n \\
[u_2]_{\partial H_k} = 0, \\
u_2 = 0 & \text{on } \partial H_k^{*}
\end{cases}
\]

(A.7)

(A.8)
where $|g|_{\partial H_k} = g^+ - g^-$, $g^\pm = \lim_{t \to 0^+} g(\cdot \pm tn)$, and $n$ denotes the unit vector outward normal to $H_k$. Hence, for $x \in H_k$,

$$u_2(x) = -\int_{\partial H_k} \frac{\partial \Gamma}{\partial n_A}(x,y) u(y) \, d\sigma(y) + \Gamma(x,y) \frac{\partial u_2}{\partial n_A}(y) \, d\sigma(y), \quad (A.9)$$

where $\partial g/\partial n_A = A\nabla g \cdot n$. For $x \in H_k^* \setminus H_k$,

$$u_2(x) = \int_{\partial H_k^*} \Gamma(x,y) \frac{\partial u_2}{\partial n_A^*}(y) \, d\sigma(y)$$

$$-\int_{\partial H_k^*} \Gamma(x,y) \frac{\partial u_2}{\partial n_A^*}(y) \, d\sigma(y) + \frac{\partial \Gamma}{\partial n_A^*}(x,y) u(y) \, d\sigma(y), \quad (A.10)$$

where $n^*$ denotes the unit vector outward normal to $H_k^*$. Then (A.9) and (A.10) imply

$$u(x) = \left\{ \frac{1}{2} u(x) - Du(x) \right\} + \int_{\partial H_k} \Gamma(x,y) \frac{\partial u_2}{\partial n_A}(y) \, d\sigma(y), \quad (A.11)$$

and

$$u(x) = \int_{\partial H_k^*} \Gamma(x,y) \frac{\partial u_2}{\partial n_A^*}(y) \, d\sigma(y)$$

$$-\int_{\partial H_k^*} \Gamma(x,y) \frac{\partial u_2}{\partial n_A^*}(y) \, d\sigma(y) - \left\{ -\frac{1}{2} u(x) - Du(x) \right\}, \quad (A.12)$$

for $x \in \partial H_k$ (see [21, Chapter 7]), where $D \equiv D_{\partial H_k}$. Equations (A.8), (A.11), and (A.12) then imply

$$\left\{ \frac{1}{2} + D \right\} u(x) = S \left( \frac{\partial u_2}{\partial n_A^*} \right)(x)$$

$$\left\{ \frac{1}{2} - D \right\} u(x) = \int_{\partial H_k^*} \Gamma(x,y) \frac{\partial u_2}{\partial n_A^*}(y) \, d\sigma(y) - S \left( \frac{\partial u_2}{\partial n_A^*} \right)(x),$$

where $S \equiv S_{\partial H_k}$. Finally, by (A.8),

$$\left[ 1 - 2 \left( \frac{1 - \delta^2}{1 + \delta^2} \right) D \right] u(x)$$

$$= \frac{2}{1 + \delta^2} \left\{ \int_{\partial H_k} \Gamma(x,y) \frac{\partial u_2}{\partial n_A^*}(y) \, d\sigma(y) + S \left( |k_{\delta^2}A\nabla u_1|_{\partial H_k} \cdot n \right) \right\}.$$
for $0 \leq \delta \leq \delta_0$, where $C$ is some constant independent of $\delta$. Indeed, it is sufficient to take $\delta_0$ so that
\[ 4C \left( \frac{\delta^2}{1 + \delta^2} \right) < 1 \quad \text{for } \delta \leq \delta_0, \]
where $C$ is a constant depending only on the operator norm of $D$, which is finite by Lemma (A.2). By (A.8) and (A.7),
\[ \| k_\delta \nabla u \|_{C^{0,\alpha}(H^*_\delta)} \leq C \| k_{\delta^{-1}} f \|_{C^{0,\alpha}(H^*_\delta)}. \quad (A.13) \]
Equations (A.7) and (A.13) give the desired estimate.

**Proof of Corollary 1.2.** By rescaling, we may assume $R = 1$. To prove the desired estimate, assume $\varepsilon \in (0, 1/9)$. Indeed, if $\varepsilon \geq 1/9$, then (1.9) follows from Theorem 1.1. From Lemma A.1, Theorem 1.1, and a “blow-up argument” (see the proof of Lemma 4.2), we deduce
\[ \| k_\varepsilon \nabla u \|_{L^\infty(B(y,\varepsilon))} \leq C \left( \int_{B(y,3\varepsilon)} |k_\varepsilon \nabla u| \right)^{1/2} \]
\[ \leq C \left( \int_{B(x_0,1)} |k_\varepsilon \nabla u| \right)^{1/2} \]
for any $y \in B(x_0,1/3)$. The desired estimate follows by covering $B(x_0,1/3)$ with balls $B(y,\varepsilon)$. \qed

**References**

[1] E. Acerbi, V. Chaido Piat, G. Dal Maso, and D. Percivale. An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Anal.*, 18:481–496, 1992.

[2] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23:1482–1518, 1992.

[3] S.N. Armstrong and J.C. Mourrat. Lipschitz regularity for elliptic equations with random coefficients. *Arch. Ration. Mech. Anal.*, 219:255–348, 2016.

[4] S.N. Armstrong and C.K. Smart. Quantitative stochastic homogenization of convex integral functionals. *Ann. Sci. Éc. Norm. Supér.*, 49:423–481, 2016.
[5] M. Avellaneda and F. Lin. Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, 40:803–847, 1987.

[6] R. Brown, S. Kim, and J. Taylor. The green function for elliptic systems in two dimensions. *Comm. Partial Differential Equations*, 38:1574–1600, 2013.

[7] G. A. Chechkin, A. L. Piatnitski, and A. S. Shamaev. *Homogenization Methods and Applications*. American Mathematical Society, Providence, Rhode Island, 2000.

[8] Ya-Zhe Chen and Lan-Cheng Wu. *Second Order Elliptic Equations and Elliptic Systems*. American Mathematical Society, Providence, Rhode Island.

[9] D. Cioranescu and P. Donato. *An Introduction to Homogenization*. Oxford University Press, New York City, 1999.

[10] D. Cioranescu and J. Saint Jean Paulin. Homogenization in open sets with holes. *J. Math. Anal. Appl.*, 71:590–607, 1979.

[11] J. Conway. *A Course in Functional Analysis*. Springer New York, New York, New York, 1990.

[12] B. Dahlberg, C. Kenig, and G. Verchota. Boundary value problems for the system of elastostatics in lipschitz domains. *Duke Math. J.*, 57:795–818, 1988.

[13] B. Yin E. S. Bao, Y. Y. Li. Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. *Comm. in PDEs*, 35:1982–2006, 2010.

[14] L. Escauriaza, E. B. Fabes, and G. Verchota. On a regularity theorem for weak solutions to transmission problems with internal lipschitz boundaries. *Proc. Amer. Math. Soc.*, 115:1069–1076, 1992.

[15] G. Falzone et al. The influences of soft and stiff inclusions on the mechanical properties of cementious composites. *Cement and Concrete Composites*, 71:153–165, 2016.

[16] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*. Princeton University Press, New Jersey.

[17] S. Gu and Z. Shen. Homogenization of stokes systems and uniform regularity estimates. *SIAM J. Math. Anal.*, 47:4025–4057, 2015.
[18] S. Hofmann and S. Kim. The green function estimates for strongly elliptic systems of second order. *Manuscripta Math.*, 124:139–172, 2007.

[19] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, Berlin.

[20] Carlos E. Kenig and Zhongwei Shen. Layer potential methods for elliptic homogenization problems. *Comm. Pure Appl. Math.*, 64:1–44, 2011.

[21] William McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, New York, New York.

[22] O.A. Oleinik, A.S. Shamaev, and G.A. Yosifian. *Mathematical Problems in Elasticity and Homogenization*. Elsevier Science Publishers, Amsterdam, Netherlands, 1992.

[23] B.C. Russell. Homogenization in perforated domains and interior lipschitz estimates. *J. Diff. Eqs.*, 263:3396–3418, 2017.

[24] Z. Shen. Boundary estimates in elliptic homogenization. *Anal. PDE*, 10:653–694, 2017.

[25] L.M. Yeh. Elliptic equations in highly heterogeneous porous media. *Math. Methods Appl.*, 33:198–223, 2010.

[26] L.M. Yeh. *$L^p$ gradient estimate for elliptic equations with high-contrast conductivities in $\mathbb{R}^n$*. *Math. Methods Appl.*, 33:198–223, 2010.

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