Reflection matrices for the $U_q[s spo(2n|2m)]$ vertex model

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Abstract

We propose a classification of the solutions of the graded reflection equations to the $U_q[s spo(2n|2m)]$ vertex model. We find twelve distinct classes of reflection matrices such that four of them are diagonal. In the non-diagonal matrices the number of free parameters depending on the number of bosonic ($2n$) and fermionic ($2m$) degrees of freedom while in the diagonal ones we find solutions with at most one free parameter.

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1 Introduction

The present work is the third of three papers devoted to the classification of the integrable reflection K-matrices for the vertex models associated with the $U_q[sl(r|2m)]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[spo(2n|2m)]$ Lie superalgebras. In the first paper [1] we have considered the $U_q[sl(r|2m)]$ vertex model and in the second paper [2] the $U_q[osp(r|2m)^{(1)}]$ vertex model ones. In this paper we have presented the general set of regular solutions of the graded reflection equation for the $U_q[spo(2n|2m)]$ vertex model.

It is well-known that the Lie superalgebra $osp(2m|2n)$ and $spo(2n|2m)$ are naturally isomorphic: From the classical Lie superalgebras[3], we have the notation $D(m|n) = osp(2m|2n) = spo(2n|2m)$. Thus, the transition from $osp(2m|2n)^{(1)}$ to $spo(2n|2m)$ amounts to shift of the gradation of the vector representation[4]. But, we will work with the natural gradations in order to avoid any shift of gradations. Moreover, for trivial values of $m$ and $n$, a Lie superalgebra coincides with a Lie algebra: $D(m|0) = D_n^{(1)}$, $D(0|n) = C_n^{(1)}$. Therefore, by considering the $U_q[spo(2n|2m)]$ vertex model we are completing the study of the integrable $K$-matrices for the graded version of the vertex models associated with the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, and $A_{2n-1}^{(2)}$ Lie algebras.

Our findings can be summarized into four classes of diagonal solutions and eight classes of non-diagonal ones. This paper is organized as follows. In the next section we present the $R$-matrix of the $U_q[spo(2n|2m)]$ vertex model in terms of standard Weyl matrices. In the section 3 we present four classes of diagonal solutions. In the section 4 we present eight classes of non-diagonal solutions what we hope to be the most general set of $K$-matrices for the vertex model here considered. Concluding remarks are discussed in the section 5, and in the appendix A we present solutions associated with the $U_q[spo(2|2)]$ case.

2 The $U_q[spo(2n|2m)]$ vertex model

The $U_q[spo(2n|2m)]$ invariant $R$-matrices are given by [3]

$$
R(x) = \sum_{i=1}^{N} (-1)^{p_i} a_i(x) \hat{e}_{ii} \otimes \hat{e}_{ii} + b(x) \sum_{i,j=1}^{N} \hat{e}_{ii} \otimes \hat{e}_{jj} + c(x) \sum_{i,j=1}^{N} (-1)^{p_i p_j} \hat{e}_{ji} \otimes \hat{e}_{ij} + \sum_{i,j=1}^{N} (-1)^{p_i p_j} d_{i,j}(x) \hat{e}_{ij} \otimes \hat{e}_{i'j'}
$$

(1)
where $N = 2n + 2m$ is the dimension of the graded space with $2n$ bosonic and $2m$ fermionic degrees of freedom. Here $i' = N + 1 - i$ corresponds to the conjugated index of $i$ and $\hat{e}_{ij}$ refers to a usual $N \times N$ Weyl matrix with only one non-null entry with value 1 at the row $i$ and column $j$.

In what follows we shall adopt the grading structure

$$p_i = \begin{cases} 0 & \text{for } i = 1, \ldots, m \text{ and } i = 2n + m + 1, \ldots, N, \\ 1 & \text{for } i = m + 1, \ldots, 2n + m \end{cases},$$

(2)

and the corresponding Boltzmann weights $a_i(x)$, $b(x)$, $c(x)$, $\bar{c}(x)$ and $d_{ij}(x)$ are then given by

$$a_i(x) = (x - \zeta)(x^{(1 - p_i)} - q^2x^{p_i}), \quad b(x) = q(x - 1)(x - \zeta),$$

$$c(x) = (1 - q^2)(x - \zeta), \quad \bar{c}(x) = x(1 - q^2)(x - \zeta)$$

(3)

and

$$d_{i,j}(x) = \begin{cases} q(x - 1)(x - \zeta) + x(q^2 - 1)(\zeta - 1), & (i = j = j') \\
(x - 1)[(x - \zeta)(-1)^{p_i}q^{2p_i} + x(q^2 - 1)], & (i = j \neq j') \\
(q^2 - 1)[\zeta(x - 1)\frac{\theta_{i,j'}}{\theta_{j,j'}} - \delta_{i,j'}(x - \zeta)], & (i < j) \\
(q^2 - 1)x[(x - 1)\frac{\theta_{i,j'}}{\theta_{j,j'}} - \delta_{i,j'}(x - \zeta)], & (i > j) \end{cases}$$

(4)

where $\zeta = q^{2n-2m+2}$. The remaining variables $\theta_i$ and $t_i$ depend strongly on the grading structure considered and they are determined by the relations

$$\theta_i = \begin{cases} (-1)^{\frac{p_i}{2}}, & 1 \leq i \leq \frac{N}{2}, \\
(-1)^{\frac{p_i}{2}}, & \frac{N}{2} + 1 \leq i \leq N \end{cases}$$

(5)

$$t_i = \begin{cases} i - \left[ \frac{1}{2} + p_i - 2\sum_{j=1}^{\frac{N}{2}} p_j \right], & 1 \leq i \leq \frac{N}{2}, \\
i + \left[ \frac{1}{2} + p_i - 2\sum_{j=\frac{N}{2}+1}^{i} p_j \right], & \frac{N}{2} + 1 \leq i \leq N \end{cases}$$

(6)

The $R$-matrix (1) satisfies important symmetry relations, besides the standard properties of regularity and unitarity, namely

$$\text{PT – Symmetry : } R_{21}(x) = R_{12}^{x_{t_1},x_{t_2}}(x)$$

$$\text{Crossing Symmetry : } R_{12}(x) = \frac{\rho(x)}{\rho(x^{-1}\eta^{-1})}V_1R_{12}^{x_{\eta^{-1}},x_{-1}}(x^{-1}\eta^{-1})V_1^{-1},$$

(7)

where the symbol $st_k$ stands for the supertransposition operation in the space with index $k$. In its turn $\rho(x)$ is an appropriate normalization function given by $\rho(x) = q(x - 1)(x - \zeta)$ and the crossing
parameter is $\eta = \zeta^{-1}$. The corresponding crossing matrix $V$ is an anti-diagonal matrix with the following non-null entries $V_{ii'}$:

$$
V_{ii'} = \begin{cases} 
(-1)^{i-p_{1}} 2^{i}, & i = 1, \\
(-1)^{i-p_{i}} q^{F^{(i)}_{1}}, & 1 < i < N + \frac{1}{2}, \\
(-1)^{i-p_{i}} q^{F^{(i)}_{2}}, & N + \frac{1}{2} < i \leq N.
\end{cases}
$$

(8)

where

$$
F^{(i)}_{1} = i - 1 - p_{1} - p_{i} - 2 \sum_{j=1}^{i-1} p_{j}, \quad F^{(i)}_{2} = i - p_{1} - p_{i} - 2 \sum_{j=2, j \neq i}^{i-1} p_{j}.
$$

(9)

The construction of integrable models with open boundaries was largely impulsed by Sklyanin’s pioneer work [6]. In Sklyanin’s approach the construction of such models are based on solutions of the so-called reflection equations [7] for a given integrable bulk system. The reflection equations determine the boundary conditions compatible with the bulk integrability and it reads

$$
R_{21}(x/y)K_{2}^{+}(x)R_{12}(xy)K_{1}^{-}(y) = K_{1}^{-}(y)R_{21}(xy)K_{2}^{+}(x)R_{12}(x/y),
$$

(10)

where the tensor products appearing in (10) should be understood in the graded sense. The matrix $K^{-}(x)$ describes the reflection at one of the ends of an open chain while a similar equation should also hold for a matrix $K^{+}(x)$ describing the reflection at the opposite boundary. As discussed above, the $U_q[spo(2n|2m)]$ $R$-matrix satisfies important symmetry relations such as the PT-symmetry and crossing symmetry. When these properties are fulfilled one can follow the scheme devised in [8, 9] and the matrix $K^{-}(x)$ is obtained by solving the Eq. (10) while the matrix $K^{+}(x)$ can be obtained from the isomorphism $K^{-}(x) \mapsto K^{+}(x)^{st} = K^{-}(x^{-1}\eta^{-1})V^{st}V$.

The purpose of this work is to investigate the general families of regular solutions of the graded reflection equation (10). Regular solutions mean that the $K$-matrices have the general form

$$
K^{-}(x) = \sum_{i,j=1}^{N} k_{i,j}(x) \hat{e}_{ij},
$$

(11)

such that the condition $k_{i,j}(1) = \delta_{ij}$ holds for all matrix elements.

The direct substitution of (11) and the $U_q[spo(2n|2m)]$ $R$-matrix (11) in the graded reflection equation (10), leave us with a system of $N^4$ functional equations for the entries $k_{i,j}(x)$. In order to solve these equations we shall make use of the derivative method. Thus, by differentiating the equation (11) with respect to $y$ and setting $y = 1$, we obtain a set of algebraic equations for the matrix elements $k_{i,j}(x)$. Although we obtain a large number of equations only a few of them are actually independent and a direct inspection of those equations, in the lines described in [10], allows us to find the branches of regular solutions. In what follows we shall present our findings for the regular solutions of the reflection equation associated with the $U_q[spo(2n|2m)]$ vertex model.
The diagonal solutions of the graded reflection equation (10) is characterized by a $K$-matrix of the form

$$K^{-}(x) = \sum_{i=1}^{N} k_{i,i}(x) \hat{e}_{ii}. \quad (12)$$

with the entries $k_{i,j}(x)$ related with $k_{1,1}(x)$ in a general form given by

$$k_{i,i}(x) = \frac{(\beta_{i,i} - \beta_{1,1})(x-1) + 2}{(\beta_{i,i} - \beta_{1,1})(x^{-1} - 1) + 2} k_{1,1}(x) \quad (13)$$

for $i = 2, \ldots, N-1$ and

$$k_{N,N}(x) = \frac{(\beta_{N,N} - \beta_{1,1})(x-1) + 2}{(\beta_{N,N} - \beta_{1,1})(x^{-1} - 1) + 2} k_{N-1,N-1}(x) \quad (14)$$

The parameters $\beta_{i,i} = \frac{d}{dx}[k_{i,i}(x)]_{x=1}$ are fixed in order to give us four families of diagonal $K$-matrices such that half of them with one free parameter $\beta$.

$$k_{1,1}(x) = \frac{\beta(x^{-1} - 1) + 2}{\beta(x-1) + 2}, \quad k_{22}(x) = \cdots = k_{N-1,N-1}(x) = 1, \quad k_{N,N}(x) = \frac{\beta(1-x)q^2\zeta - 2}{\beta(x - q^2\zeta) - 2x}. \quad (15)$$

and

$$k_{1,1}(x) = \cdots = k_{n+m,n+m}(x) = 1, \quad k_{n+m+1,m+m+1}(x) = \cdots = k_{N,N}(x) = \frac{\beta(x-1) + 2}{\beta(x^{-1} - 1) + 2}. \quad (16)$$

while the other half without free parameters:

$$k_{1,1}(x) = \cdots = k_{p-1,p-1}(x) = 1, \quad k_{p,p}(x) = \cdots = k_{N+1-p,N+1-p}(x) = \frac{x + \epsilon q^{2p-3} \sqrt{\zeta}}{x^{-1} + \epsilon q^{2p-3} \sqrt{\zeta}}, \quad k_{N+2-p,N+2-p}(x) = \cdots = k_{N,N}(x) = x^2. \quad (17)$$

where the discrete label $p$ assumes values in the interval $2 \leq p \leq m + 1$ and

$$k_{1,1}(x) = \cdots = k_{p-1,p-1}(x) = 1, \quad k_{p,p}(x) = \cdots = k_{N+1-p,N+1-p}(x) = \frac{x + \epsilon q^{4m+1-2p} \sqrt{\zeta}}{x^{-1} + \epsilon q^{4m+1-2p} \sqrt{\zeta}}, \quad k_{N+2-p,N+2-p}(x) = \cdots = k_{N,N}(x) = x^2. \quad (18)$$

for the discrete label $p$ with values in the interval $m + 2 \leq p \leq n + m$.

Here and in what follows, $\epsilon$ is a discrete parameter assuming the values $\pm 1$. We also notice that the case $p = 2$ is limit of the solution of the solution (15) and we have $(n + m)$ diagonal solutions.
4 Non-Diagonal K-matrix Solutions

Analyzing the reflection matrix equation (10) we can see that the non-diagonal matrix elements $k_{i,j}(x)$ have the form

\[
k_{i,j}(x) = \begin{cases} 
\beta_{i,j}xG(x), & i > j' \\
\beta_{i,j}xG(x), & i > j' \\
\beta_{i,j}G(x)H_f(x), & i = j' \text{ (fermionic)} \\
\beta_{i,j}G(x)H_b(x), & i = j' \text{ (bosonic)} 
\end{cases}
\]

(19)

Here and in what follows $G(x)$ is an arbitrary function satisfying the regular condition $k_{i,j}(1) = \delta_{ij}$ and

\[
\beta_{i,j} = \frac{d}{dx}[k_{i,j}(x)]_{x=1}, \quad H_f(x) = \frac{x - \epsilon q \sqrt{\zeta}}{1 - \epsilon q \sqrt{\zeta}}, \quad H_b(x) = \frac{qx + \epsilon \sqrt{\zeta}}{q + \epsilon \sqrt{\zeta}}.
\]

(20)

Moreover, the diagonal entries $k_{i,i}(x)$ satisfy defined recurrence relations which depend on the bosonic and fermionic degree of freedom. However, for the $U_q[spo(2n|2m)]$ model we have found $K$-matrix solutions with both degree of freedom only for the cases ($n \neq 1, m = 1$) and ($n = 1, m \neq 1$).

4.1 K-matrices with fermionic degree of freedom

Here we shall focus on the non-diagonal solutions of the graded reflection equation (10) with $k_{i,j}(x) = 0$ for the bosonic degree of freedom i.e., for $i \neq j = m + 1, \ldots, m + 2n$. We have found three classes of non-diagonal solutions that we refer in what follows as solutions of type $P_1$ to type $P_3$:

4.1.1 Solution $P_1$

The solution of type $P_1$ is valid only for the $U_q[spo(2n|2)]$ models with $n \geq 1$ and the $K$-matrix has the following block structure

\[
K^- = \begin{pmatrix} 
k_{1,1} & \mathbb{O}_{1 \times 2n} & k_{1,N} \\
\mathbb{O}_{2n \times 1} & \mathbb{K}_1 & \mathbb{O}_{2n \times 1} \\
k_{N,1} & \mathbb{O}_{1 \times 2n} & k_{N,N} 
\end{pmatrix},
\]

(21)

where $\mathbb{O}_{a \times b}$ is a $a \times b$ null matrix and

\[
\mathbb{K}_1(x) = \frac{x^2 - \zeta}{1 - \zeta} \mathbb{I}_{2n \times 2n}.
\]

(22)
Here and in what follows $I_{2n \times 2n}$ denotes a $2n \times 2n$ identity matrix and the remaining non-null entries are given by

\[ k_{1,1}(x) = 1, \quad k_{1,N}(x) = \frac{1}{2} \beta_{1,N}(x^2 - 1), \quad k_{N,1}(x) = \frac{2 \zeta}{(1 - \zeta)^2} \beta_{1,N}, \quad k_{N,N}(x) = x^2. \]

where $\beta_{1,N}$ is the free parameter. We remark here that this solution for $n = 1$ consist of a particular case of the three parameter solution given in the appendix for the $U_q[spo(2|2)]$ vertex model.

### 4.1.2 Solution $\mathcal{P}_2$

The $U_q[spo(2n|4)]$ vertex models admit the solution $\mathcal{P}_2$ whose corresponding $K$-matrix has the following structure

\[
K = \begin{pmatrix}
    k_{1,1} & k_{1,2} & k_{1,N-1} & k_{1,N} \\
    k_{2,1} & k_{2,2} & k_{2,N-1} & k_{2,N} \\
    \mathbb{K}_2 & \mathbb{O}_{2n \times 2} & \mathbb{K}_2 & \mathbb{O}_{2n \times 2} \\
    k_{N-1,1} & k_{N-1,2} & k_{N-1,N-1} & k_{N-1,N}
\end{pmatrix},
\]

where $\mathbb{K}_2 = k_{3,3}(x)I_{2n \times 2n}$. The non-diagonal entries can be written as

\[
k_{2,1}(x) = \beta_{1,2} G_1(x), \quad k_{2,1}(x) = \beta_{1,2} G_1(x),
k_{1,N-1}(x) = \beta_{1,N-1} G_1(x), \quad k_{N-1,1}(x) = q^2 \zeta \beta_{1,N} G_1(x),
k_{2,N-1}(x) = -\frac{\beta_{2,1,1} \beta_{1,N}}{\beta_{2,1}} G_1(x) H(x), \quad k_{N-1,2}(x) = -q^2 \zeta \beta_{2,1} \beta_{1,N} G_1(x) H(x),
k_{2,N}(x) = -\Gamma_n \frac{\beta_{1,N}}{\beta_{1,2}} x G_2(x), \quad k_{N,2}(x) = -q^2 \zeta \Gamma_n \beta_{2,1,1} \beta_{1,N} x G_2(x),
k_{N-1,N}(x) = -q^2 \zeta \Gamma_n \frac{\beta_{1,N}}{\beta_{1,N-1}} x G_2(x), \quad k_{N,N-1}(x) = -q^2 \zeta \Gamma_n \beta_{2,1} \beta_{1,N} x G_2(x),
k_{N,1}(x) = q^2 \zeta \frac{\beta_{2,1} \beta_{1,N}}{\beta_{1,N-1}} G_1(x) H(x), \quad k_{1,N}(x) = \beta_{1,N} G_1(x) H(x),
\]

and the auxiliary functions are given by

\[
G_1(x) = \left[ -q^2 \zeta \Gamma_n \frac{\beta_{1,N}}{\beta_{2,1} \beta_{1,N-1}} + \frac{x - q^2 \zeta}{x - 1} \right] \frac{x - 1}{x^2 - q^2 \zeta} \frac{k_{1,N}(x)}{\beta_{1,N}},
\]

\[
G_2(x) = \left[ -\frac{1}{\Gamma_n} \frac{\beta_{1,N}}{\beta_{1,2} \beta_{1,N-1}} + \frac{x - q^2 \zeta}{x - 1} \right] \frac{x - 1}{x^2 - q^2 \zeta} \frac{k_{1,N}(x)}{\beta_{1,N}},
\]

\[
H(x) = \frac{\beta_{1,2} \beta_{1,N-1} (x^2 - q^2 \zeta) - q^2 \zeta \Gamma_n \beta_{1,N} (x - 1)}{\beta_{1,2} \beta_{1,N-1} (x - q^2 \zeta) - q^2 \zeta \Gamma_n \beta_{1,N} (x - 1)}
\]

(26)
and

$$\Gamma_n = \frac{\beta_{1,1} \beta_{1,N}}{\beta_{1,N-1}} - \frac{2}{1 - q^2 \zeta}$$

(27)

With respect to the diagonal matrix elements, we have the following expressions

$$k_{11}(x) = \left\{ \frac{(x - q^2 \zeta) \beta_{2,1} \beta_{1,N}}{\beta_{1,N-1}} + \frac{\beta_{1,2} \beta_{1,N-1}}{\beta_{1,N}} \right\} + \frac{q^2 \zeta \Gamma_n (1 - x q^2 \zeta) \beta_{1,N}}{\beta_{1,2} \beta_{1,N-1}} + \frac{2(1 + q^2 \zeta)^2 x - 4q^2 \zeta (x^2 + 1)}{(1 - q^2 \zeta) (x - 1)} \frac{k_{1,N}(x)}{\beta_{1,N}}$$

(28)

for the recurrence relation

$$k_{2,2}(x) = k_{1,1}(x) + (\beta_{2,2} - \beta_{1,1}) G_1(x),$$

$$k_{3,3}(x) = k_{1,1}(x) + (\beta_{3,3} - \beta_{1,1}) G(x) + \Delta_1(x),$$

$$k_{N-1,N-1}(x) = k_{3,3}(x) + (\beta_{N-1,N-1} - \beta_{3,3}) x G_2(x) + \Delta_2(x),$$

$$k_{N,N}(x) = k_{N-1,N-1}(x) + (\beta_{N,N} - \beta_{N-1,N-1}) x G_2(x).$$

(29)

where

$$\Delta_1(x) = \frac{\beta_{2,1}}{\beta_{1,N-1}} \left( x + q^2 \zeta \Gamma_n \frac{\beta_{1,N}}{\beta_{1,2} \beta_{1,N-1}} \right) \left( \frac{x - 1}{x^2 - q^2 \zeta} \right) k_{1,N}(x)$$

$$\Delta_2(x) = -\frac{q^2 \zeta}{\Gamma_n} \frac{\beta_{1,N}}{\beta_{1,N-1}} \Delta_1(x).$$

(30)

The diagonal entries [29] depend on the variables \(\beta_{n,a}\) which are related to the four free parameters \(\beta_{1,2}, \beta_{2,1}, \beta_{1,N-1}\) and \(\beta_{1,N}\) through the expressions

$$\beta_{2,2} = \beta_{1,1} + \frac{\beta_{1,2} \beta_{1,N-1}}{\beta_{1,N}} - \Gamma_n,$$

$$\beta_{3,3} = \beta_{1,1} + \frac{\beta_{1,2} \beta_{1,N-1}}{\beta_{1,N}} + \frac{2}{1 - q^2 \zeta},$$

$$\beta_{N-1,N-1} = \beta_{1,1} + 2 + \frac{\beta_{1,2} \beta_{1,N-1}}{\beta_{1,N}} - q^2 \zeta \Gamma_n,$$

$$\beta_{N,N} = \beta_{1,1} + 2 + \frac{\beta_{1,2} \beta_{1,N-1}}{\beta_{1,N}} - q^2 \zeta \Gamma_n \frac{\beta_{1,N}}{\beta_{1,2} \beta_{1,N-1}}.$$ 

(31)
4.1.3 Solution \( \mathcal{P}_3 \)

This class of solution is valid for all \( U_q[spo(2n|2m)] \) vertex models with \( m \geq 3 \) and the corresponding \( K \)-matrix possess the following general form

\[
K^- = \begin{pmatrix}
k_{1,1} & \cdots & k_{1,m} & \cdots & k_{1,2n+m+1} & \cdots & k_{1,N} \\
\vdots & \ddots & \vdots & \Omega_{m \times 2n} & \vdots & \ddots & \vdots \\
k_{m,1} & \cdots & k_{m,m} & \cdots & k_{m,2n+m+1} & \cdots & k_{m,N} \\
\Omega_{2n \times m} & K_3 & \Omega_{2n \times m} & & \cdots & \ddots & \vdots \\
k_{2n+m+1,1} & \cdots & k_{2n+m+1,m} & \cdots & k_{2n+m+1,2n+m+1} & \cdots & k_{2n+m+1,N} \\
\vdots & \ddots & \vdots & \Omega_{m \times 2n} & \vdots & \ddots & \vdots \\
k_{N,1} & \cdots & k_{N,m} & \cdots & k_{N,2n+m+1} & \cdots & k_{N,N}
\end{pmatrix},
\]

where \( K_3 \) is a diagonal matrix given by

\[
K_3(x) = k_{m+1,m+1}(x) \mathbb{I}_{2n \times 2n}.
\]

With respect to the elements of the last column, we have the following expression

\[
k_{i,N}(x) = -\frac{\epsilon}{\sqrt{\zeta}} q^{t_1-t_1} \beta_{1,i} x G(x)
\]

\[
i = 2, \ldots, m \text{ and } i = 2n + m + 1, \ldots, N - 1,
\]

In their turn the entries of the first column are mainly given by

\[
k_{i,1}(x) = q^{t_1-t_j} \frac{\beta_{2,1} \beta_{1,i}}{\beta_{1,N-1}} G(x)
\]

\[
i = 3, \ldots, m \text{ and } i = 2n + m + 1, \ldots, N - 1.
\]

In the last row we have

\[
k_{N,j}(x) = -\frac{\epsilon}{\sqrt{\zeta}} q^{t_{N-1}} \frac{\beta_{2,1} \beta_{1,j}}{\beta_{1,N-1}} x G(x)
\]

\[
j = 2, \ldots, m \text{ and } j = 2n + m + 1, \ldots, N - 1,
\]

while the elements of the first row are \( k_{1,j}(x) = \beta_{1,j} G(x) \) for \( j = 2, \ldots, m \) and \( j = 2n + m + 1, \ldots, N - 1 \).

Concerning the elements of the secondary diagonal, they are given by

\[
k_{i,i'}(x) = q^{t_1-t_{i'}} \left( \frac{1 - \epsilon q \sqrt{\zeta}}{q + 1} \right)^2 \beta_{1,i'}^2 G(x) H_f(x)
\]

\[
i = 2, \ldots, m \text{, } i \neq i' \text{ and } i = 2n + m + 1, \ldots, N - 1,
\]
while the remaining entries $k_{1,N}(x)$ and $k_{N,1}(x)$ are determined by the following expressions

\[
   k_{1,N}(x) = \beta_{1,N} G(x) H_f(x)
   \]

\[
   k_{N,1}(x) = q^{N-1} G(x) \overline{\beta_{1,N}^2} H_f(x)
   \]

(38)

where $H_f(x)$ is given by (20).

The remaining matrix elements $k_{i,j}(x)$ with $i \neq j$ are then

\[
   k_{i,j}(x) = \begin{cases} 
   -\frac{\epsilon}{\sqrt{\zeta}} q^{n-i} \left( \frac{1 - \epsilon q \sqrt{\zeta}}{q + 1} \right) \frac{\beta_{1,i} \beta_{1,j}}{\beta_{1,N}} G(x), & i < j \quad 2 \leq i, j \leq N - 1 \\
   \frac{1}{\zeta} q^{n-i} \left( \frac{1 - \epsilon q \sqrt{\zeta}}{q + 1} \right) \frac{\beta_{1,i} \beta_{1,j}}{\beta_{1,N}} x G(x), & i > j \quad 2 \leq i, j \leq N - 1
   \end{cases}
   \]

(39)

and

\[
   k_{2,1}(x) = \frac{2(-1)^m (1 - \epsilon q \sqrt{\zeta}) q^{m-2}}{(1 + \epsilon \sqrt{\zeta})(\epsilon q^{m-1} \sqrt{\zeta} + (-1)^m)(\epsilon q^{m} \sqrt{\zeta} + (-1)^m) \beta_{1,N}^{-1}} G(x),
   \]

\[
   k_{1,m}(x) = \frac{2\epsilon q^{m-1}(q + 1)^2}{(1 - \epsilon q \sqrt{\zeta})(1 + \epsilon q \sqrt{\zeta})(\epsilon q^{m-1} \sqrt{\zeta} + (-1)^m)(\epsilon q^{m} \sqrt{\zeta} + (-1)^m) \beta_{1,m}} G(x),
   \]

(40)

and the parameters $\beta_{1,j}$ are constrained by the relation

\[
   \beta_{1,j} = -\beta_{1,j+1} \frac{\beta_{1,N-j}}{\beta_{1,N+1-j}} \quad j = 2, \ldots, m - 1.
   \]

(41)

With regard to the diagonal matrix elements, they are given by

\[
   k_{i,i}(x) = \begin{cases} 
   k_{1,1}(x) + (\beta_{i,i} - \beta_{1,1}) G(x), & 2 \leq i \leq m \\
   k_{1,1}(x) + (\beta_{i+1,m+1} - \beta_{1,1}) G(x) + \Delta(x), & i = m + 1 \\
   k_{m+1,m+1}(x) + (\beta_{2n+m+2n+m+1} - \beta_{1,m+1}) x G(x) - \epsilon q^{2n+1} \sqrt{\zeta} \Delta(x), & i = 2n + m + 1 \\
   k_{i-1,i-1}(x) + (\beta_{i,i} - \beta_{1,i-1}) x G(x), & 2n + m + 2 \leq i \leq N
   \end{cases}
   \]

(42)

The last term of the recurrence relation (42) is identified with

\[
   k_{N,N}(x) = x^2 k_{1,1}(x)
   \]

(43)

to find

\[
   k_{1,1}(x) = \left[ \frac{2x}{x^2 - 1} - \frac{\beta_{m+1,m+1} - \beta_{1,1}}{x + 1} \right] G(x) + \frac{\Delta(x)}{x^2 - 1}
   \]

(44)

where

\[
   \Delta(x) = \frac{2(x - 1) G(x)}{(1 + \epsilon \sqrt{\zeta})(\epsilon q^{m-1} \sqrt{\zeta} + (-1)^m)(\epsilon q^{m} \sqrt{\zeta} + (-1)^m)}
   \]

(45)

In their turn the diagonal parameters $\beta_{i,i}$ are fixed by the relations

\[
   \beta_{i,i} = \begin{cases} 
   \beta_{1,1} + \Lambda_m \sum_{k=0}^{i-2} (-\frac{1}{q})^k, & 2 \leq i \leq m \\
   \beta_{1,1} + 2 - \epsilon \sqrt{\zeta} \Lambda_m \sum_{k=0}^{N-1-i} (-q)^k, & 2n + m + 1 \leq i \leq N - 1 \\
   \beta_{1,1} + 2, & i = N
   \end{cases}
   \]

(46)
\[ \beta_{m+1,m+1} = \beta_{1,1} + \Lambda_m \left( \frac{q}{q+1} + \frac{(-1)^m}{q^{m-2}(q+1)^2} \frac{1 + \epsilon \sqrt{\zeta}}{\epsilon \sqrt{\zeta}} \right), \]  

(47)

with

\[ \Lambda_m = \frac{2(-1)^m q^{m-2}(q+1)^2 \epsilon \sqrt{\zeta}}{(1 + \epsilon \sqrt{\zeta})(\epsilon q^{m-1} \sqrt{\zeta} + (-1)^m)(\epsilon q^m \sqrt{\zeta} + (-1)^m)}. \]  

(48)

The class of solution \( \mathcal{P}_3 \) has a total amount of \( m \) free parameters namely \( \beta_{1,2n+m+1}, \ldots, \beta_{1,N} \).

Here we note that we can take the limit \( m = 2 \) in the class \( \mathcal{P}_3 \) to get a two parameter solution which is a particular case of the three parameter solution of the class \( \mathcal{P}_2 \). This complete the set of solutions with the fermionic degree of freedom.

### 4.2 K-matrices with bosonic degree of freedom

Here we consider \( k_{i,j}(x) = 0 \), for \( i \neq j = \{1, \ldots, m\} \) and \( i \neq j = \{2n + m + 1, \ldots, N\} \) in order to get \( K \)-matrix solutions with only bosonic degree of freedom. We have found three classes of non-diagonal solutions that we refer in what follows as solutions of type \( \mathcal{P}_4 \) to type \( \mathcal{P}_6 \):

#### 4.2.1 Solution \( \mathcal{P}_4 \):

This family of solutions is valid only for the \( U_q[spo(2|2m)] \) vertex model with \( m \geq 1 \) and the corresponding \( K \)-matrix has the following block diagonal structure

\[
K^- = \begin{pmatrix}
  k_{1,1} I_{m \times m} & 0_{m \times 2} & 0_{m \times m} \\
  0_{2 \times m} & k_{m+1,m+1} & k_{m+1,m+2} \\
  0_{m \times m} & k_{m+2,m+1} & k_{m+2,m+2} \\
  k_{N,N} & 0_{m \times 2} & I_{m \times m}
\end{pmatrix}
\]  

(49)

The non-null entries are given by

\[
k_{1,1}(x) = 1, \\
k_{m+1,m+1}(x) = x \frac{\alpha[q^2 \zeta^{-1} + 1]m - 1](x - 1) + 2}{\alpha[xq^2 \zeta^{-1} - 1](x - 1) + 2x}, \\
k_{m+1,m+2}(x) = \beta \frac{x(x^2 - 1)}{\alpha[xq^2 \zeta^{-1} - 1](x - 1) + 2x}, \\
k_{m+2,m+1}(x) = -\beta^2 \frac{x^2 \zeta^{-1}(x^2 - 1)}{\alpha[xq^2 \zeta^{-1} - 1](x - 1) + 2x}, \\
k_{m+2,m+2}(x) = -x^2 \frac{\alpha[q^2 \zeta^{-1} + 1](x - 1) - 2x}{\alpha[xq^2 \zeta^{-1} - 1](x - 1) + 2x}, \\
k_{N,N}(x) = x^2.
\]  

(50)

where \( \alpha = \beta_{m+1,m+1} \) and \( \beta = \beta_{m+1,m+2} \) are the two free parameters.
### 4.2.2 Solution $\mathcal{P}_5$:

The solution $\mathcal{P}_5$ is admitted for the $U_q[spo(4|2m)]$ models with the following $K$-matrix

$$
K^- = \begin{pmatrix}
  k_{1,1} I_{m \times m} & O_{m \times 4} & O_{m \times m} \\
  k_{m+1,m+1} k_{m+1,m+2} & k_{m+1,m+3} & k_{m+1,m+4} \\
  k_{m+2,m+1} & k_{m+2,m+2} & k_{m+2,m+3} & k_{m+2,m+4} \\
  k_{m+3,m+1} & k_{m+3,m+2} & k_{m+3,m+3} & k_{m+3,m+4} \\
  k_{m+4,m+1} & k_{m+4,m+2} & k_{m+4,m+3} & k_{m+4,m+4} \\
  O_{4 \times m} & O_{m \times 4} & k_{N,N} I_{m \times m}
\end{pmatrix}
$$

The non-diagonal elements are all grouped in the $4 \times 4$ central block matrix. With respect to this central block, the entries of the secondary diagonal are given by

$$
\begin{align*}
k_{m+1,m+4}(x) &= \beta_{m+1,m+4} G(x) H_b(x) \\
k_{m+2,m+3}(x) &= eq^{-2} \frac{\beta_{m+1,m+3}}{\beta_{m+1,m+2}} \Gamma_m G(x) H_b(x), \\
k_{m+3,m+2}(x) &= -eq^{-2} \frac{\beta_{m+1,m+2}}{\beta_{m+1,m+3}} \Gamma_m G(x) H_b(x), \\
k_{m+4,m+1}(x) &= -q^{-2} \frac{2}{\beta_{m+1,m+4}} \Gamma_m G(x) H_b(x),
\end{align*}
$$

where we recall $H_b(x) = \frac{qx + \epsilon \sqrt{x}}{q + \epsilon \sqrt{x}}$ and $G(x)$ arbitrary. The remaining non-diagonal elements can be written as

$$
\begin{align*}
k_{m+1,m+2}(x) &= \beta_{m+1,m+2} G(x) & k_{m+2,m+1}(x) &= -eq^{-2} \frac{\beta_{m+1,m+3}}{\beta_{m+1,m+4}} \Gamma_m G(x), \\
k_{m+1,m+3}(x) &= \beta_{m+1,m+3} G(x) & k_{m+3,m+1}(x) &= eq^{-2} \frac{\beta_{m+1,m+2}}{\beta_{m+1,m+4}} \Gamma_m G(x), \\
k_{m+2,m+4}(x) &= eq^{-2} \beta_{m+1,m+3} x G(x) & k_{m+4,m+2}(x) &= q^{-2} \beta_{m+1,m+2} \Gamma_m x G(x), \\
k_{m+3,m+4}(x) &= -eq^{-2} \beta_{m+1,m+2} x G(x) & k_{m+4,m+3}(x) &= q^{-2} \beta_{m+1,m+3} \Gamma_m x G(x),
\end{align*}
$$

where

$$
\Gamma_m = \frac{\beta_{m+1,m+2} \beta_{m+1,m+3}}{\beta_{m+1,m+4}} - \frac{2}{(q^{m-2} + \epsilon)(q^m + \epsilon)}. 
$$

In their turn the diagonal entries are given by the following expressions

$$
\begin{align*}
k_{1,1}(x) &= \frac{2(q^{m-2} + \epsilon)^2}{(q^{m-2} + \epsilon)(q^m + \epsilon)(q^{2m-4} - 1)(x-1)} - \frac{\beta_{m+1,m+2} \beta_{m+1,m+3}}{\beta_{m+1,m+4}} \left( \frac{q^m + \epsilon}{(q^{m-2} + \epsilon)x(x+1)} \right) G(x)
\end{align*}
$$

(55)
with the recurrence relation

\[
\begin{align*}
k_{m+1,m+1}(x) &= k_{1,1}(x) + (\beta_{m+1,m+1} - \beta_{1,1})G(x) + \Delta_1(x), \\
k_{m+2,m+2}(x) &= k_{m+1,m+1}(x) + (\beta_{m+2,m+2} - \beta_{m+1,m+1})G(x), \\
k_{m+3,m+3}(x) &= k_{m+2,m+2}(x) + \Delta_2(x), \\
k_{m+4,m+4}(x) &= k_{m+3,m+3}(x) + (\beta_{m+4,m+4} - \beta_{m+3,m+3})xG(x), \\
k_{N,N}(x) &= x^2k_{1,1}(x),
\end{align*}
\]  

(56)

where

\[
\begin{align*}
\Delta_1(x) &= \frac{\epsilon \Gamma_m}{q^{m-2} + \epsilon} \frac{(x - 1)G(x)}{x}, \\
\Delta_2(x) &= -\epsilon q^{m-2}(q^2 + 1) \Gamma_m \left( \frac{q^{m-2}x + \epsilon}{q^{m-2} + \epsilon} \right) G(x).
\end{align*}
\]  

(57)

The variables \(\beta_{i,j}\) are given in terms of the free parameters \(\beta_{m+1,m+2}, \beta_{m+1,m+3}\) and \(\beta_{m+1,m+4}\) through the relations

\[
\begin{align*}
\beta_{m+1,m+1} &= \beta_{1,1} - \Gamma_m \\
\beta_{m+2,m+2} &= \beta_{1,1} + \frac{2}{(q^{m-2} + \epsilon)(q^m + \epsilon)} + \epsilon q^{m-2} \frac{\beta_{m+1,m+2}\beta_{m+1,m+3}}{\beta_{m+1,m+4}}, \\
\beta_{m+3,m+3} &= \beta_{1,1} + 2 - \frac{2q^{2m-2}}{(q^{m-2} + \epsilon)(q^m + \epsilon)} - \epsilon q^{m-2} \frac{\beta_{m+1,m+2}\beta_{m+1,m+3}}{\beta_{m+1,m+4}}, \\
\beta_{m+4,m+4} &= \beta_{1,1} + 2 - \frac{2q^{2m-2}}{(q^{m-2} + \epsilon)(q^m + \epsilon)} + q^{2m-2} \frac{\beta_{m+1,m+2}\beta_{m+1,m+3}}{\beta_{m+1,m+4}}.
\end{align*}
\]  

(58)

We remark here that the form of this class of solution differs from general form \([19]\) by the number of free parameters.

4.2.3 Solution \(\mathcal{P}_6\):

The vertex model \(U_q[spo(2n|2m)]\) admits the solution \(\mathcal{P}_6\) for \(n \geq 3\), whose \(K\)-matrix has the following block structure

\[
K^- = \begin{pmatrix}
k_{1,1} & \mathbb{O}_{m \times 2n} & \mathbb{O}_{m \times m} \\
\mathbb{O}_{m \times 2n} & k_{m+1,m+1} & \cdots & k_{m+1,2n+m} \\
\mathbb{O}_{2n \times m} & \vdots & \ddots & \vdots \\
\mathbb{O}_{m \times m} & \mathbb{O}_{m \times 2n} & \cdots & k_{2n+m,m+1} \\
\mathbb{O}_{m \times m} & \mathbb{O}_{2n \times m} & \cdots & k_{N,N} & \mathbb{O}_{m \times m}
\end{pmatrix}
\]  

(60)

The central block matrix cluster all non-diagonal elements different from zero. Concerning
that central block, we have the following expressions determining entries of the borders,

\[
k_{i,2n+m}(x) = \frac{\epsilon}{\sqrt{\zeta}} \frac{\theta_{i}q^{i}}{\theta_{m+1}q^{m+1}} \beta_{m+1,i}xG(x), \quad i = m + 2, \ldots, 2n + m - 1
\]

\[
k_{2n+m,j}(x) = \frac{\epsilon}{\sqrt{\zeta}} \frac{\theta_{2n+m}q^{2n+m}}{\theta_{m+2}q^{m+2}} \beta_{m+2,m+1,i}xG(x), \quad j = m + 2, \ldots, 2n + m - 1
\]

\[
k_{i,m+1}(x) = \frac{\theta_{i}q^{i}}{\theta_{m+2}q^{m+2}} \beta_{m+2,m+1}xG(x), \quad i = m + 3, \ldots, 2n + m - 1
\]

\[
k_{m+1,j}(x) = \beta_{m+1,j}xG(x). \quad j = m + 2, \ldots, 2n + m - 1
\]

The entries of the secondary diagonal are given by

\[
k_{i,i'}(x) = \begin{cases} 
\beta_{m+1,2n+m}G(x)H_{b}(x), & \text{if } i = m + 1 \\
-q^{2(m-1)}\beta_{m+1,i}q^{i+m+1}\left(\frac{q + \epsilon \sqrt{\zeta}}{q + 1}\right)^{2}\beta_{m+1,i}G(x)H_{b}(x), & \text{if } i = m + 2, \ldots, 2n + m - 1 \\
\theta_{2n+m-1}q^{2n+m-1}\beta_{m+2,m+1}G(x)H_{b}(x), & \text{if } i = 2n + m
\end{cases}
\]

and the remaining non-diagonal elements are determined by the expression

\[
k_{i,j}(x) = \begin{cases} 
\epsilon \frac{\theta_{i}q^{i}}{\sqrt{\zeta}} \frac{\theta_{i}q^{i}}{\theta_{m+1}q^{m+1}} \left(\frac{q + \epsilon \sqrt{\zeta}}{q + 1}\right) \beta_{m+1,i}xG(x), & \text{if } i < j', \ m + 1 < i, j < 2n + m \\
\frac{1}{\theta_{m+1}q^{m+1}} \left(\frac{q + \epsilon \sqrt{\zeta}}{q + 1}\right) \beta_{m+1,i}xG(x), & \text{if } i > j', \ m + 1 < i, j < 2n + m
\end{cases}
\]

and

\[
k_{m+1,m+n}(x) = -\frac{2e(-1)^{n}\sqrt{\zeta}(1 + q)^{2}}{(1 - \epsilon \sqrt{\zeta})(q + \epsilon \sqrt{\zeta})} \frac{1}{(q^{m} - (-1)^{n}e)(q^{m-1} - (-1)^{n}e)} \beta_{m+1,2n+m}G(x),
\]

\[
k_{m+2,m+1}(x) = \frac{2q(q + \epsilon \sqrt{\zeta})}{\sqrt{\zeta}(1 - \epsilon \sqrt{\zeta})(q^{m} - (-1)^{n}e)(q^{m-1} - (-1)^{n}e)} \beta_{m+1,2n+m}G(x).
\]

In their turn the diagonal entries \(k_{i,i}(x)\) are given by

\[
k_{1,1}(x) = \frac{x - \epsilon \sqrt{\zeta}}{1 - \epsilon \sqrt{\zeta}} \frac{xq^{m} + (-1)^{n}e}{q^{m} + (-1)^{n}e} \frac{xq^{m-1} - (-1)^{n}e}{q^{m-1} - (-1)^{n}e} \frac{2G(x)}{x(x^{2} - 1)}
\]

\[
k_{i,i}(x) = \begin{cases} 
k_{1,1}(x) + \Gamma_{n,m}(x), & \text{if } i = m + 1 \\
k_{m+1,m+1}(x) + (\beta_{i,i} - \beta_{m+1,m+1})G(x), & \text{if } i = m + 2, \ldots, m + n \\
k_{n+m,n+m+1}(x) + (\beta_{n+m,n+m+1} - \beta_{n+m,n+m+1})xG(x) + \Delta_{n,m}(x), & \text{if } i = n + m + 1 \\
k_{n+m+1,n+m+1}(x) + (\beta_{i,i} - \beta_{n+m+1,n+m+1})xG(x), & \text{if } i = n + m + 2, \ldots, 2n + m \\
x^{2}k_{1,1}(x), & \text{if } i = N
\end{cases}
\]

where

\[
\Gamma_{n,m}(x) = \frac{2(qx + \epsilon \sqrt{\zeta})}{(1 - \epsilon \sqrt{\zeta})(q^{m} + (-1)^{n}e)(q^{m-1} - (-1)^{n}e)} \frac{G(x)}{x}.
\]
and
\[
\Delta_{n,m}(x) = -\frac{2(-1)^n q^{n-1} (q^2 + 1)}{(1 - \epsilon \sqrt{\zeta})(q^n + (-1)^n \epsilon)(q^{n-1} - (-1)^n \epsilon)(x - 1)G(x)}
\]

The diagonal parameters are then fixed by the relations
\[
\beta_{i,i} = \begin{cases} 
\beta_{1,1} = \frac{q + \epsilon \sqrt{\zeta}}{(1 + q)^2} Q_{n,m}, & i = m + 1 \\
\beta_{m+1,m+1} = Q_{n,m} \sum_{k=0}^{i-m-2} (-q)^k, & i = m + 2, \ldots, n + m \\
\beta_{n+n,n+n} = (1 + q^2)^{-1} \sum_{k=0}^{i-n-m-2} (-q)^k Q_{n,m}, & i = n + m + 1 \\
\beta_{n+n+1,n+n+1} = (1 + q^2)^{-1} \sum_{k=0}^{i-n-m-1} (-q)^k Q_{n,m}, & i = n + m + 2, \ldots, n + m
\end{cases}
\]

(68)

and the auxiliary parameter \( Q_{n,m} \) is given by
\[
Q_{n,m} = -\frac{2(1 + q)^2}{(1 - \epsilon \sqrt{\zeta})(q^n + (-1)^n \epsilon)(q^{n-1} - (-1)^n \epsilon)}.
\]

(69)

Besides the above relations the following constraints should also holds
\[
\beta_{m+1,m+j} = -\beta_{m+1,j+m+1} \frac{\beta_{m+1,2n+m-j}}{\beta_{m+1,2n+m+1-j}}, & j = 2, \ldots, n - 1
\]

(70)

and \( \beta_{m+1,m+n+1}, \ldots, \beta_{m+1,2n+m} \) are regarded as the \( n \) free parameters. The case \( n = 2 \) is a particular solution of the three parameter class \( P_6 \).

4.3 Complete K-matrices

The complete K-matrices are solutions with all entries different from zero. This kind of solution will be present only in two class: the models with two bosonic degree of freedom, \( U_q[spo(2|2m)] \) and those models with only two fermionic degree of freedom \( U_q[spo(2n|2)] \).

4.3.1 Solution \( P_7 \):

The series of solutions \( P_7 \) is valid for the \( U_q[spo(2|2m)] \) model and the corresponding K-matrix also possess all entries different from zero. In the first and last columns, the matrix elements are mainly given by
\[
k_{i,1}(x) = \frac{\theta_i q^{i_1} \beta_{2,1} \beta_{1,\nu}}{\theta_2 q^{i_2} \beta_{1,N-1}} G(x) & i = 3, \ldots, N - 1
\]

\[
k_{i,N}(x) = -\frac{\epsilon_m \theta_i q^{i_1}}{\sqrt{\zeta} \theta_1 q^{i_1}} \beta_{1,\nu} x G(x) & i = 2, \ldots, N - 1
\]

(71)

while the ones in the first and last rows are respectively
\[
k_{1,j}(x) = \beta_{1,j} G(x) & j = 2, \ldots, N - 1
\]

\[
k_{N,j}(x) = -\frac{\epsilon_m \theta_N q^{i_N} \beta_{2,1} \beta_{1,\nu}}{\sqrt{\zeta} \theta_2 q^{i_2} \beta_{1,N-1}} x G(x) & j = 2, \ldots, N - 1
\]

(72)
In the secondary diagonal we have the following expression determining the matrix elements

\[
k_{i,i'}(x) = \begin{cases} 
\beta_{1,N} G(x) H_f(x), & i = 1 \\
\frac{\theta_1 q^{i_1}}{\theta_{i'} q^{i_{i'}}} \left( \frac{1 - \epsilon_m q \sqrt{q}}{q + 1} \right) \frac{\beta_{1,i'}^2}{\beta_{1,N}} G(x) H_f(x), & i \neq \{1, m + 1, m + 2, N\} \\
\frac{\theta_1 q^{i_1}}{\theta_{i'} q^{i_{i'}}} \left( \frac{1 - \epsilon_m q \sqrt{q}}{q + 1} \right) \frac{\beta_{1,i'}^2}{\beta_{1,N}} G(x) H_f(x), & i = \{m + 1, m + 2\} \\
\frac{\theta_{N-1} q^{N-i-1}}{\theta_{2} q^{i_2}} \frac{\beta_{1,N} \beta_{2,1}}{\beta_{1,N-1}} G(x) H_f(x), & i = N.
\end{cases}
\]  

(73)

recalling that

\[
H_b(x) = \frac{q x + \epsilon_m \sqrt{q}}{q + \epsilon_m \sqrt{q}} \quad \text{and} \quad H_f(x) = \frac{x - \epsilon_m q \sqrt{q}}{1 - \epsilon_m q \sqrt{q}}.
\]  

(74)

In their turn the other non-diagonal entries satisfy the relation

\[
k_{i,j}(x) = \begin{cases} 
-\frac{\epsilon_m}{\sqrt{q}} \frac{\theta_{i} q^{i_1}}{\sqrt{q}} \left( \frac{1 - \epsilon_m q \sqrt{q}}{q + 1} \right) \frac{\beta_{1,i'} \beta_{1,j}}{\beta_{1,N}} G(x), & i < j', 2 < i, j < N - 1 \\
\frac{\theta_{1} q^{i_1}}{\theta_{i'} q^{i_{i'}}} \left( \frac{1 - \epsilon_m q \sqrt{q}}{q + 1} \right) \frac{\beta_{1,i'} \beta_{1,j}}{\beta_{1,N}} x G(x), & i > j', 2 < i, j < N - 1
\end{cases}
\]  

(75)

and

\[
\begin{align*}
k_{1,m}(x) &= \frac{i(q + 1)}{q - 1} \frac{\beta_{1,m+1} \beta_{1,m+2}}{\beta_{1,m+3}} G(x), \\
k_{1,m+1}(x) &= (-1)^{\delta_{m,2}} \frac{i \sqrt{q}(q + \epsilon_m)}{(1 + \epsilon_m \sqrt{q})(1 + q \sqrt{q})} \frac{\beta_{1,N}}{\beta_{1,m+2}} G(x), \\
k_{2,1}(x) &= (-1)^{\delta_{m,2}} \frac{\epsilon_m}{q \sqrt{q}} \frac{\beta_{1,N-1}}{\beta_{1,N-2}} G(x).
\end{align*}
\]  

(76)

and the parameters \(\beta_{1,j}\) are required to satisfy the recurrence relation

\[
\beta_{1,j} = -\frac{\beta_{1,j+1} \beta_{1,N-j}}{\beta_{1,N-j+1}} \quad j = 2, \ldots, m - 1.
\]  

(77)

Considering now the diagonal entries, they are given by

\[
k_{i,i}(x) = \begin{cases} 
k_{1,1}(x) + (\beta_{1,1} - \beta_{1,1}) G(x), & i = 2, \ldots, m + 1 \\
k_{1,1}(x) + (\beta_{m+2,m+2} - \beta_{1,1}) x G(x) + \Delta(x), & i = m + 2 \\
k_{m+2,m+2}(x) + (\beta_{i} - \beta_{m+2,m+2}) x G(x), & i = m + 3, \ldots, N - 1 \\
\alpha^2 k_{1,1}(x), & i = N
\end{cases}
\]  

(78)

where

\[
k_{1,1}(x) = \frac{G(x)}{x - 1} \quad \text{and} \quad \Delta(x) = (1 - x) G(x).
\]
The class of solutions $P$ contains only non-null entries. The border elements are mainly given by the following expressions

$$
\beta_{i,i} = \begin{cases} 
\beta_{1,1} + \frac{(q + 1)}{q(1 + \epsilon_m \sqrt{\zeta})} \sum_{k=0}^{i-2} (-\frac{1}{q})^k, & i = 2, \ldots, m \\
\beta_{m,m} - \frac{2\epsilon_m \sqrt{\zeta}}{q(q + 1)(1 + \epsilon_m \sqrt{\zeta})}, & i = m + 1 \\
\beta_{m+1,m+1} + \frac{1}{q(q + 1)(1 + \epsilon_m \sqrt{\zeta})}, & i = m + 2 \\
\beta_{m+2,m+2} - \frac{2q^2}{(q + 1)(1 + \epsilon_m \sqrt{\zeta})}, & i = m + 3 \\
\beta_{m+3,m+3} + \frac{(q + 1)}{1 + \epsilon_m \sqrt{\zeta}} \sum_{k=0}^{i-m-4} (-\frac{1}{q})^k, & i = m + 4, \ldots, N 
\end{cases}
$$

This solution has altogether $m + 1$ free parameters, namely $\beta_{1, m+2}, \ldots, \beta_{1, N}$. This complete solution depend on the parity of $m$ through the relation $\epsilon_m = (-1)^m$.

Here we notice that the case $m = 2$ is also special because $\beta_{1, m} = \beta_{1, 2}$ and we have an exchange of sign for $k_{2, 1}(x)$ and $k_{1, 3}(x)$ as indicate in equation (79).

### 4.3.2 Solution $P_8$:

The class of solutions $P_8$ is valid for the vertex model $U_q[spo(2n|2)]$ and the corresponding $K$-matrix contains only non-null entries. The border elements are mainly given by the following expressions

$$
k_{i,N}(x) = -\frac{\epsilon_n}{\sqrt{\zeta}} \theta_1 q^{i_1} \theta_1 q^{i_1} \beta_{1, i} x G(x) & \text{for } i = 2, \ldots, N - 1 \\
k_{i,1}(x) = \theta_1 q^{i_1} \beta_{2, 1} \beta_{1, i} x G(x) & \text{for } i = 3, \ldots, N - 1 \\
k_{N,j}(x) = -\frac{\epsilon_n}{\sqrt{\zeta}} \theta_2 q^{2q^2} \beta_{2, 1} \beta_{1, j} x G(x) & \text{for } j = 2, \ldots, N - 1 \\
k_{1,j}(x) = \beta_{1, j} G(x) & \text{for } j = 2, \ldots, N - 1.
$$

The secondary diagonal is constituted by elements $k_{i, i'}(x)$ given by

$$
k_{i, i'}(x) = \begin{cases} 
\beta_{1, N} G(x) H_f(x), & i = 1 \\
\frac{\theta_1 q^{i_1}}{\theta_1 q^{i_1}} \left( \frac{1 - \epsilon_n q \sqrt{\zeta}}{q - 1} \right) \frac{q + \epsilon_n \sqrt{\zeta}}{q + 1} \beta_{1, i'}^2 G(x) H_b(x), & i \neq \{1, N\} \\
\frac{\theta_2 q^{2q^2}}{\theta_2 q^{2q^2}} \beta_{2, 1} \beta_{1, i} G(x) H_f(x), & i = N 
\end{cases}
$$

where the functions $H_b(x)$ and $H_f(x)$ were already given in (74). The remaining non-diagonal entries are determined by the expression

$$
k_{i, j}(x) = \begin{cases} 
\frac{\epsilon_n}{\sqrt{\zeta}} \theta_1 q^{i_1} \left( \frac{1 - \epsilon_n q \sqrt{\zeta}}{q - 1} \right) \beta_{1, i} \beta_{1, j} G(x), & i < j', 2 < i, j < N - 1 \\
\frac{1}{\sqrt{\zeta}} \theta_1 q^{i_1} \left( \frac{1 - \epsilon_n q \sqrt{\zeta}}{q - 1} \right) \beta_{1, i}^2 \beta_{1, j} G(x), & i > j', 2 < i, j < N - 1 
\end{cases}
$$
and
\[ k_{1,n+1}(x) = \frac{-i\sqrt{\xi}(q+1)}{(q\sqrt{\xi} - \epsilon_n)(\sqrt{\xi} - \epsilon_n)} \beta_{1,n} G(x) \]
\[ k_{2,1}(x) = \frac{-i\epsilon_n(q\sqrt{\xi} - \epsilon_n)}{(\sqrt{\xi} - \epsilon_n)(q - 1)} \beta_{1,N} G(x). \]

and the parameters \( \beta_{1,j} \) are required to satisfy
\[ \beta_{1,j} = -\frac{\beta_{1,j+1}\beta_{1,N-j}}{\beta_{1,N+1-j}} \quad j = 2, \ldots, n. \] (83)

With respect to the diagonal entries, they are given by
\[ k_{i,i}(x) = \begin{cases} 
   k_{1,1}(x) + (\beta_{i,i} - \beta_{1,1})G(x), & i = 2, \ldots, n + 1 \\
   k_{n+1,n+1}(x) + (\beta_{n+2,n+2} - \beta_{n+1,n+1})xG(x) + \Delta(x), & i = n + 2 \\
   k_{n+2,n+2}(x) + (\beta_{i,i} - \beta_{n+2,n+2})xG(x), & i = n + 3, \ldots, N - 1 \\
   x^2 k_{1,1}(x), & i = N
\end{cases} \] (84)

where
\[ k_{1,1}(x) = \frac{G(x)}{x - 1} \quad \text{and} \quad \Delta(x) = \epsilon_n \left( \frac{\sqrt{\xi} + q}{\sqrt{\xi} + \epsilon_n q} \right) \left( \frac{q^2 + \epsilon_n}{\sqrt{\xi} - \epsilon_n} \right) \left( \frac{x - 1}{q - 1} \right) G(x). \]

The parameters \( \beta_{i,i} \) are determined by the expressions
\[ \beta_{i,i} = \begin{cases} 
   (q - 1)^n + \frac{(-1)^{n+i}q^{i-1} + q^{i-2} - (-1)^i(q - 1)}{(q - 1)(q\sqrt{\xi} - \epsilon_n)}, & i = 2, \ldots, n + 1 \\
   \frac{(\sqrt{\xi} + q)(q^2 + \epsilon_n)}{q(q - 1)(\sqrt{\xi} - \epsilon_n)}, & i = n + 2 \\
   \frac{\epsilon_n q(q + 1)^2}{(q - 1)(\sqrt{\xi} - \epsilon_n)} \sum_{k=0}^{i-n-3} (-q)^k, & i = n + 3, \ldots, N - 1 \\
   \beta_{1,1} + 2, & i = N
\end{cases} \] (85)

The variables \( \beta_{1,n+2}, \ldots, \beta_{1,N} \) give us a total amount of \( n + 1 \) free parameters. This complete solution depend on the parity of \( n \) through the relation \( \epsilon_n = (-1)^n \).

5 Concluding Remarks

In this work we have presented the general set of regular solutions of the graded reflection equation for the \( U_q[osp(2n|2m)] \) vertex model. Our findings can be summarized into four classes of diagonal solutions and eight classes of non-diagonal ones. Although the \( R \) matrix of the \( U_q[osp(2m|2n)^{(1)}] \) vertex model is isomorphic to the \( R \) matrix of the \( U_q[spo(2n|2m)] \) vertex model, we have find additional different \( K \)-matrices from those obtained by the exchange of the degree of freedom in these models (see, for example, the diagonal solutions of the \( U_q[spo(2|2)] \) vertex model).
We expect the results presented here to motivate further developments on the subject of integrable open boundaries for vertex models based on $q$-deformed Lie superalgebras. In particular, the classification of the solutions of the graded reflection equation for others $q$-deformed Lie superalgebras, for instance, the $U_q[osp(2n|2m)^{(2)}]$ Lie superalgebra, which we hope to report on a future work.

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Appendix A: The $U_q[spo(2|2)]$ case

The set of $K$-matrices associated with the $U_q[spo(2|2)]$ vertex model includes both diagonal and non-diagonal solutions. The four solutions intrinsically diagonal three contain only one free parameter $\beta$ and they are given by

$$K^-(x) = \operatorname{Diag}(\frac{2x - \beta(x - 1)}{2x + \beta(x - 1)}, 1, 1, \frac{2x + \beta x(q^4 - x)}{2x + \beta(q^4 - x)}) ,$$

$$K^-(x) = \operatorname{Diag}(1, 1, \frac{2x + \beta x(x - 1)}{2x - \beta(x - 1)}, \frac{2x + \beta x(x - 1)}{2x - \beta(x - 1)}) ,$$

$$K^-(x) = \operatorname{Diag}(1, \frac{2x + \beta x(x - 1)}{2x - \beta(x - 1)}, x^2, \frac{2x + \beta x(x - 1)}{2x - \beta(x - 1)} x^2)$$  \hspace{1cm} (B.1)$$

and one without free parameters

$$K^-(x) = \operatorname{Diag}(1, x \frac{q^2 + \epsilon x}{xq^2 + \epsilon}, x \frac{q^2 + \epsilon x}{xq^2 + \epsilon}, x^2).$$  \hspace{1cm} (B.2)$$

We have also found the following non-diagonal solutions

$$K^-(x) = \begin{pmatrix} 1 - \frac{(x^{-1} - \beta^2 + \alpha)(x-1)}{2\beta} & 0 & 0 & \frac{1}{2} \beta_{23} (x^2 - 1) \\ 0 & 1 + \frac{(\beta^2 - \alpha)(x-1)}{2\beta} & 0 & 0 \\ 0 & 0 & x^2 - \frac{(\beta^2 - \alpha)x(x-1)}{2\beta} & 0 \\ \frac{\alpha}{2\beta_{23}} (x^2 - 1) & 0 & 0 & x^2 + \frac{(\beta^2 + \alpha)x(x-1)}{2\beta} \end{pmatrix}$$  \hspace{1cm} (B.3)$$

containing three free parameters $\alpha, \beta$ and $\beta_{23}$, and another solution with one free parameter

$$K^-(x) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \beta_{1,4} (x^2 - 1) \\ 0 & \frac{x^2 - x^2}{q^2-1} & 0 & 0 \\ 0 & 0 & \frac{q^2 - x^2}{q^2-1} & 0 \\ 0 & \frac{2}{\beta_{1,4}} \frac{q^2}{(q^2-1)^2} (x^2 - 1) & 0 & x^2 \end{pmatrix}.$$  \hspace{1cm} (B.4)$$

We don’t find any complete solution for this model.