ON THE BLOW-UP CRITERION OF NAVIER-STOKES EQUATION
ASSOCIATED WITH THE WEINSTEIN OPERATOR

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Abstract. In this paper we give Navier-Stokes system associated with the Weinstein operator
(NSW) (see Eq. (3.4)), We study the existence and uniqueness of solutions to equations (NSW)
in $L^p(\mathbb{R}^{d+1})$, $2\alpha + d + 2 < p \leq \infty$, and we proved some properties of the maximal solution of equation. If the maximum time $T^*$ is finite, we establish that the growth of $\|u(t)\|_{L^p}$ is at least of the order of $(T^* - t)^{-\frac{2\alpha}{p-2}}$ for all $t$ in $[0, T^*]$, also we give some blow-up results.

Keywords: blow-up criterion, critical spaces, Navier-Stokes equations, Weinstein transform, integral transform

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1. Introduction

Most physical systems are modeled by nonlinear partial differential equations, for example the incompressible Navier-Stokes equations in the whole $\mathbb{R}^3$ space, which proposed by H. Navier and G. Stokes in 1845, to describe the evolution of a viscous fluid:

\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p \\
\text{div} u = 0 \\
u_t = 0
\end{cases}
\]

Here the parameter $\nu > 0$ is the viscosity of fluid and $u(t, x)$ denotes the velocity field of a fluid at time $t$ and at position $x$, subjected to a pressure $p(t, x)$.

In 1934, J. Leray (see [3]) gave (without proof) an blow-up result of the Navier-Stokes solution in Lebesgue spaces $L^p(\mathbb{R}^3)$ for $3 < p \leq \infty$

\[
\|u(t)\|_{L^p} \geq C (T^* - t)^{-\frac{3}{p-3}}
\]

which was proved by Y. Giga in 1986 (see [10]). Moreover, several authors have been interested in this problem in Sobolev spaces $H^s(\mathbb{R}^3)$, $s > 1/2$ J. Benamneur in 2010 (see [11]), showed that for $s > 5/2$ the blow-up result index depends in an increasing way on the regularity index:

\[
\|u(t)\|_{H^s} \geq C (T^* - t)^{-s/3}
\]

using $\text{div} u = 0$, we obtain:

\[
\begin{align*}
u_t &= 0 \\
u_t &= \nabla u = \text{div}(u \otimes u)
\end{align*}
\]
thus the Navier-Stokes system is written in the form:

\[
\begin{align*}
(NS) & \quad \left\{ \begin{array}{l}
\partial_t u + \text{div}(u \otimes u) - \nu \Delta u = -\nabla p \\
d\text{div } u & = 0 \\
u_t = 0 & = u_0
\end{array} \right.
\end{align*}
\]

and \( u \otimes v = (u_1v, u_2v, u_3v), \ u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3). \)

The Weinstein operator \( \Delta_{W}^{\alpha,d} \) has several applications in pure and applied Mathematics especially in Fluid Mechanics (\[5\]). The harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem (cf. \[6\][7]). In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. This transform is called the Weinstein transform, and several authors have been interested in spaces related to this operator, in \[8\] the authors introduced the Sobolev space associated with a Weinstein operator \( H_{S,\alpha}^{\beta} (\mathbb{R}^{d+1}) \) and investigated their properties, and in \[9\] the authors introduced the Sobolev refine the inequality between the homogeneous Weinstein-Besov spaces \( \mathcal{B}^{s,\beta}_{p,q} (\mathbb{R}^{d+1}) \) and many more, such as the homogenous Weinstein-Riesz spaces \( \mathcal{R}_{\beta}^{s} (L^{p}_{q} (\mathbb{R}^{d+1})) \) and the generalized Lorentz spaces.

This work is devoted to define and we study the Navier-Stokes equations associated with the Weinstein operator (NSW) in the whole \( \mathbb{R}^{d+1} \) space.

This paper is organized as follows: In section 2 we recall some elements of harmonic analysis associated with the Weinstein operator, which will be needed in the sequel. In section 3 We study the existence and uniqueness of the solution the (NSW) equations in \( L^{p}_{\alpha} (\mathbb{R}^{d+1}) \), \( 2\alpha + d + 2 < p < \infty \), and also we give some blow-up results.

2. Harmonic analysis associated with the Weinstein-Laplace operator

Notations. In what follows, we need the following notations:
- \( \mathbb{R}^{d+1} = \mathbb{R}^d \times ]0, \infty[\) .
- \( x = (x_1, ..., x_d, x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}^{d+1} \)
- \( |x| = \sqrt{x_1^2 + x_2^2 + ... + x_{d+1}^2} \).
- \( \mathcal{C}_{0} (\mathbb{R}^{d+1}) \), the space of continuous functions on \( \mathbb{R}^{d+1} \), even with respect to the last variable.
- \( \mathcal{C}_{c} (\mathbb{R}^{d+1}) \), the space of continuous functions on \( \mathbb{R}^{d+1} \) with compact support, even with respect to the last variable.
- \( \mathcal{C}_{0}^{p} (\mathbb{R}^{d+1}) \), the space of functions of class \( C^{p} \) on \( \mathbb{R}^{d+1} \), even with respect to the last variable.
- \( \mathcal{S}_{c} (\mathbb{R}^{d+1}) \), the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^{d+1} \), even with respect to the last variable.
- \( \mathcal{S}_{c} (\mathbb{R}^{d+1}) \), the space of \( C^{\infty} \)-functions on \( \mathbb{R}^{d+1} \) which are of compact support, even with respect to the last variable.
- \( L^{p}_{\alpha} (\mathbb{R}^{d+1}) \), \( 1 \leq p \leq +\infty \), the space of measurable functions on \( \mathbb{R}^{d+1} \) such that

\[
\|f\|_{\alpha,p} = \left[ \int_{\mathbb{R}^{d+1}} |f(x)|^p d\mu_{\alpha,d}(x) \right]^{\frac{1}{p}} < +\infty, \text{ if } 1 \leq p < +\infty,
\]

\[
\|f\|_{\alpha,\infty} = \text{ess sup}_{x \in \mathbb{R}^{d+1}} |f(x)| < +\infty,
\]
where $\mu_{\alpha,d}$ is the measure defined on $\mathbb{R}^{d+1}_+$ by

$$d\mu_{\alpha,d}(x) = \frac{x^{2\alpha+1}}{(2\pi)^{\frac{d}{2}} \Gamma(\alpha+1)} dx,$$

and $dx$ is the Lebesgue measure on $\mathbb{R}^{d+1}$.

- $\mathcal{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Weinstein operator developed in [12].

The Weinstein operator $\Delta_{W}^{\alpha,d}$ is defined on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times [0, +\infty[$, by:

$$\Delta_{W}^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \; \alpha > -\frac{1}{2},$$

where $\Delta_d$ is the Laplacian for the $d$ first variables and $L_\alpha$ is the Bessel operator for the last variable defined on $[0, +\infty[$ by:

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}} = \frac{1}{x_{d+1}^{2\alpha+1}} \frac{\partial}{\partial x_{d+1}} \left( x_{d+1}^{2\alpha+1} \frac{\partial u}{\partial x_{d+1}} \right).$$

The Weinstein operator $\Delta_{W}^{\alpha,d}$, mostly referred to as the Laplace-Bessel differential operator is now known as an important operator in analysis. The relevant harmonic analysis associated with the Bessel differential operator $L_\alpha$ goes back to S. Bochner, J. Delsarte, B.M. Levitan and has been studied by many other authors such as J. L"{o}fstr"{o}m and J. peetre [18], I. Kipriyanov [14], K. Trim"{e}che [15], I.A. Aliev and B. Youssef [16].

Let us begin by the following result, which gives the eigenfunction $\Psi_{\lambda}^{\alpha,d}$ of the Weinstein operator $\Delta_{W}^{\alpha,d}$.

**Proposition 2.1.** For all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the system

$$\begin{cases}
\frac{\partial^2 u}{\partial x_j^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}} = -\lambda_j^2 u(x), & \text{if } 1 \leq j \leq d \\
L_\alpha u(x) = -\lambda_{d+1}^2 u(x), & \text{if } j = d+1 \\
u(0) = 1, & \text{if } j = d+1 \\
\frac{\partial u}{\partial x_j}(0) = -i\lambda_j, & \text{if } 1 \leq j \leq d.
\end{cases}$$

has a unique solution $\Psi_{\lambda}^{\alpha,d}(\cdot)$ given by:

$$\forall z \in \mathbb{C}^{d+1}, \; \Psi_{\lambda}^{\alpha,d}(z) = e^{-i\langle z', \lambda' \rangle} j_\alpha(\lambda_{d+1} z_{d+1}),$$

where $z = (z', x_{d+1})$, $z' = (z_1, z_2, \ldots, z_d)$ and $j_\alpha$ is the normalized Bessel function of index $\alpha$, defined by

$$\forall \xi \in \mathbb{C}, \; j_\alpha(\xi) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left( \frac{\xi}{2} \right)^{2n}.$$

**Proposition 2.2.**

i) For all $\lambda$, $z \in \mathbb{C}^{d+1}$ and $t \in \mathbb{R}$, we have

$$\Psi_d^{\alpha}(\lambda, 0) = 1, \; \Psi_d^{\alpha}(\lambda, z) = \Psi_d^{\alpha}(\lambda, tz) \text{ and } \Psi_d^{\alpha}(\lambda, tz) = \Psi_d^{\alpha}(t\lambda, z).$$

ii) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}^{d+1}_+$ and $z \in \mathbb{C}^{d+1}$, we have

$$|D_z^n \Psi_d^{\alpha}(x, z)| \leq \|x||^{|\nu|} \exp(||x|| \| \text{Im } z||),$$

(2.5)
where $D^\nu_z = \frac{\partial^\nu}{\partial z_1 \cdots \partial z_{d+1}}$ and $|\nu| = \nu_1 + \cdots + \nu_{d+1}$. In particular

$$\forall x, y \in \mathbb{R}^{d+1}, \ |\Psi_{\alpha,d}^\nu(x,y)| \leq 1. \quad (2.6)$$

**Definition 2.1.** The Weinstein transform is given for $f \in L_\alpha^1(\mathbb{R}^{d+1}_+)$ by

$$\forall \lambda \in \mathbb{R}^{d+1}_+, \ \mathcal{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}^{d+1}_+} f(x)\Psi_{\alpha,d}^\nu(x,\lambda)d\mu_{\alpha,d}(x). \quad (2.7)$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}^{d+1}_+$ given by the relation (2.1).

The Weinstein transform, referred to as the Fourier-Bessel transform, has been investigated by I. Kipriyanov [14], I.A. Aliev [1] and others (see [2, 12, 17] and [4]).

Using the properties of the classical Fourier transform on $\mathbb{R}^d$ and of the Bessel transform, one can easily see the following relation, which will play an important role in the sequel.

**Example 2.1.** Let $E_s, s > 0$, be the function defined by

$$\forall x \in \mathbb{R}^{d+1}, \ E_s(x) = e^{-s|x|^2}.$$  

Then the Weinstein transform $\mathcal{F}_W^{\alpha,d}$ of $E_s$ is given by:

$$\forall \lambda \in \mathbb{R}^{d+1}_+, \ \mathcal{F}_W^{\alpha,d}(E_s)(\lambda) = \frac{1}{(2s)^{\alpha+d+1}}e^{-|\lambda|^2/2s}. \quad (2.8)$$

Some basic properties of the transform $\mathcal{F}_W^{\alpha,d}$ are summarized in the following results. For the proofs, we refer to [12, 17].

**Proposition 2.3.** (see [12, 17])

i) For all $f \in L_\alpha^1(\mathbb{R}^{d+1}_+)$, we have

$$\|\mathcal{F}_W^{\alpha,d}(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}. \quad (2.9)$$

ii) For all $f \in L_\alpha^1(\mathbb{R}^{d+1}_+)$ and $\Delta_W^{\alpha,d} f \in L_\alpha^1(\mathbb{R}^{d+1}_+)$ we have

$$\mathcal{F}_W^{\alpha,d}(\Delta_W^{\alpha,d} f)(x) = -|x|^2\mathcal{F}_W^{\alpha,d}(f)(x) \quad (2.10)$$

**Theorem 2.1.** (see [12, 17])

i) The Weinstein transform $\mathcal{F}_W^{\alpha,d}$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^{d+1})$ onto itself and from $\mathcal{D}(\mathbb{R}^{d+1})$ onto $\mathcal{H}_+(\mathbb{C}^{d+1})$.

ii) Let $f \in \mathcal{S}(\mathbb{R}^{d+1})$. The inverse transform $(\mathcal{F}_W^{\alpha,d})^{-1}$ is given by

$$\forall x \in \mathbb{R}^{d+1}_+, \ (\mathcal{F}_W^{\alpha,d})^{-1}(f)(x) = \mathcal{F}_W^{\alpha,d}(f)(-x). \quad (2.11)$$

iii) Let $f \in L_\alpha^1(\mathbb{R}^{d+1}_+)$. If $\mathcal{F}_W^{\alpha,d}(f) \in L_\alpha^1(\mathbb{R}^{d+1}_+)$, then we have

$$f(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_W^{\alpha,d}(f)(y)\Psi_{\alpha,d}^\nu(-x,y)d\mu_{\alpha,d}(y), \ a.e \ x \in \mathbb{R}^{d+1}_+. \quad (2.12)$$

**Theorem 2.2.** (see [12, 17])

i) For all $f, g \in \mathcal{S}(\mathbb{R}^{d+1})$, we have the following Parseval formula

$$\int_{\mathbb{R}^{d+1}_+} f(x)g(x)d\mu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_W^{\alpha,d}(f)(\lambda)\mathcal{F}_W^{\alpha,d}(g)(\lambda)d\mu_{\alpha,d}(\lambda). \quad (2.13)$$
ii) (Plancherel formula).
For all \( f \in S(\mathbb{R}^{d+1}) \), we have:
\[
\int_{\mathbb{R}^{d+1}} |f(x)|^2 \, d\mu_{\alpha,d}(x) = \int_{\mathbb{R}^d} \left| \mathcal{F}_W^{\alpha,d}(f)(\lambda) \right|^2 \, d\mu_{\alpha,d}(\lambda).
\] (2.14)

iii) (Plancherel Theorem):
The transform \( \mathcal{F}_W^{\alpha,d} \) extends uniquely to an isometric isomorphism on \( L^2_\alpha(\mathbb{R}^{d+1}) \).

The following example is a consequence of the relation (2.14).

Example 2.2. Let \( q_t, \ t > 0 \), be the function defined by
\[
\forall x \in \mathbb{R}^{d+1}, \ q_t(x) = \frac{1}{(2t)^{\alpha+\frac{d}{2}+1}} e^{-\frac{4\pi x^2}{t}}.
\] (2.15)

Then the inverse transform \( \left( \mathcal{F}_W^{\alpha,d} \right)^{-1} \) of \( q_t \) is given by
\[
\forall x \in \mathbb{R}^{d+1}, \ \left( \mathcal{F}_W^{\alpha,d} \right)^{-1}(q_t)(x) = \mathcal{F}_W^{\alpha,d}(q_t)(x) = e^{-t\|x\|^2}.
\] (2.16)

Definition 2.3. The Weinstein convolution product of \( f, g \in \mathcal{S}(\mathbb{R}^{d+1}) \) is given by:
\[
\forall x \in \mathbb{R}^{d+1}, \ f \ast_W g(x) = \int_{\mathbb{R}^{d+1}} T_x f(y) g(y) \, d\mu_{\alpha,d}(y).
\] (2.19)
Proposition 2.5. (see [12, 17])

i) Let \( p, q, r \in [1, +\infty] \) such that \( \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \). Then for all \( f \in L^p_{\alpha}(\mathbb{R}^{d+1}_+) \) and \( g \in L^q_{\alpha}(\mathbb{R}^{d+1}_+) \), the function \( f *_{W} g \in L^r_{\alpha}(\mathbb{R}^{d+1}_+) \) and we have

\[
\|f *_{W} g\|_{\alpha,r} \leq \|f\|_{\alpha,p} \|g\|_{\alpha,q}.
\]

(2.20)

ii) For all \( f, g \in L^1_{\alpha}(\mathbb{R}^{d+1}_+) \), (resp. \( \mathcal{X}_{\alpha}(\mathbb{R}^{d+1}) \)), \( f *_{W} g \in L^1_{\alpha}(\mathbb{R}^{d+1}_+) \) (resp. \( \mathcal{X}_{\alpha}(\mathbb{R}^{d+1}) \)) and we have

\[
\mathcal{F}^{\alpha,d}_{W}(f *_{W} g) = \mathcal{F}^{\alpha,d}_{W}(f) \mathcal{F}^{\alpha,d}_{W}(g).
\]

(2.21)

3. Navier-Stokes equation associated with the Weinstein operator

In this section, we collect some notations and definitions that will be used later

- \( \Delta^{\alpha,d}_{W} f := (\Delta^{\alpha,d}_{W} f_1, \Delta^{\alpha,d}_{W} f_2, ..., \Delta^{\alpha,d}_{W} f_{d+1}) \): Weinstein-Laplace of \( f \) if \( f = (f_1, f_2, ..., f_{d+1}) \)

- \( \mathcal{F}^{\alpha,d}_{W}(f) := (\mathcal{F}^{\alpha,d}_{W} f_1, \mathcal{F}^{\alpha,d}_{W} f_2, ..., \mathcal{F}^{\alpha,d}_{W} f_{d+1}) \): Weinstein transform of \( f \).

- \( \left[ \mathbb{L}^p_{\alpha} \left( \mathbb{R}^{d+1}_+ \right) \right]^{d+1} = \mathbb{L}^p_{\alpha} \left( \mathbb{R}^{d+1}_+ \right) \times \ldots \times \mathbb{L}^p_{\alpha} \left( \mathbb{R}^{d+1}_+ \right) \)

- \( C \left( x \in [0, T[ \right. \mathbb{L}^p_{\alpha} \left( \mathbb{R}^{d+1}_+ \right) \right]^{d+1} \) : Space of continuous functions of \( [0, T[ \) in \( \mathbb{L}^p_{\alpha} \left( \mathbb{R}^{d+1}_+ \right) \)

- \( u_j \otimes v = (u_j v_1, ..., u_j v_{d+1}) \), \( u = (u_1, ..., u_{d+1}) \) and \( v = (v_1, ..., v_{d+1}) \)

- \( u \otimes v = (u_1 \otimes v, ..., u_{d+1} \otimes v) \)

Definitions 3.1. Let \( f \) is a function verified, \( f \in L^1_{\alpha}(\mathbb{R}^{d+1}_+) \) and \( \lambda \mapsto \lambda_j \mathcal{F}^{\alpha,d}_{W}(f)(\lambda) \in L^1_{\alpha}(\mathbb{R}^{d+1}_+) \), \( j = 1, 2, ..., d + 1 \). The Weinstein gradient \( \nabla^{\alpha,d}_{W}(f)(x) \) in \( x = (x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1}_+ \), defined by

\[
\nabla^{\alpha,d}_{W} f(x) := i \left( \mathcal{F}^{\alpha,d}_{W} \right)^{-1} \begin{pmatrix}
\lambda, \mathcal{F}^{\alpha,d}_{W} f(\lambda) \\
\end{pmatrix}(x)
\]

(3.1)

\[
= i \begin{pmatrix}
\left( \mathcal{F}^{\alpha,d}_{W} \right)^{-1} \left( \lambda_1 \mathcal{F}^{\alpha,d}_{W} f(\lambda) \right)(x) \\
\left( \mathcal{F}^{\alpha,d}_{W} \right)^{-1} \left( \lambda_2 \mathcal{F}^{\alpha,d}_{W} f(\lambda) \right)(x) \\
\vdots \\
\left( \mathcal{F}^{\alpha,d}_{W} \right)^{-1} \left( \lambda_{d+1} \mathcal{F}^{\alpha,d}_{W} f(\lambda) \right)(x) \\
\end{pmatrix}
\]

(3.2)

Let \( f = (f_1, f_2, ..., f_{d+1}) \) is a function verified, \( f \in \left( \mathbb{L}^1_{\alpha}(\mathbb{R}^{d+1}_+) \right)^{d+1} \) and \( \lambda \mapsto \lambda_j \mathcal{F}^{\alpha,d}_{W}(f_j)(\lambda) \in L^1_{\alpha}(\mathbb{R}^{d+1}_+) \), \( j = 1, 2, ..., d + 1 \). The Weinstein divergence \( \text{div}^{\alpha,d}_{W}(f) \) defined by

\[
\text{div}^{\alpha,d}_{W}(f)(x) := i \sum_{j=1}^{d+1} \left( \mathcal{F}^{\alpha,d}_{W} \right)^{-1} \left( \lambda_j \mathcal{F}^{\alpha,d}_{W} f_j(\lambda) \right)(x)
\]

(3.3)

Remarks 3.1. i) By the relation (2.17) we have

\[
\text{div}^{\alpha,d}_{W} \nabla^{\alpha,d}_{W} = \Delta^{\alpha,d}_{W}
\]

ii) If \( \alpha = -\frac{1}{2} \) we have

\[
\text{div}^{\alpha,d}_{W} = \text{div}
\]

\[
\nabla^{\alpha,d}_{W} = \nabla
\]

iii)

\[
\text{div}^{\alpha,d}_{W}(u \otimes v) = \left( \text{div}^{\alpha,d}_{W}(u_1 \otimes v), ..., \text{div}^{\alpha,d}_{W}(u_{d+1} \otimes v) \right),
\]

\[
= i \sum_{k=1}^{d+1} \left( \mathcal{F}^{\alpha,d}_{W} \right)^{-1} \left( \lambda_k \mathcal{F}^{\alpha,d}_{W}(v_k \otimes u)(\lambda) \right)
\]
so

\[
\mathcal{F}_W^{\alpha,d} \left( \text{div}^\alpha_W (u \otimes v) \right) (\lambda) = i \sum_{k=1}^{d+1} \lambda_k \mathcal{F}_W^{\alpha,d} (v_k \otimes u)(\lambda)
\]

We consider in the rest of this article that the incompressible Navier- Stokes-Weinstein system is given by:

\[
\begin{align*}
\partial_t u - \nu \Delta^{\alpha,d} W u + \text{div}^\alpha_W (u \otimes u) &= -\nabla^{\alpha,d} W p, \quad \text{in} \quad \mathbb{R}_+^+ \times \mathbb{R}_+^{d+1} \\
\text{div}^\alpha_W u &= 0 \quad \text{in} \quad \mathbb{R}_+^+ \times \mathbb{R}_+^{d+1} \\
u u(0) &= u^0 \quad \text{in} \quad \mathbb{R}_+^{d+1}
\end{align*}
\]

(3.4)

In the case \( \alpha = -\frac{1}{2} \), Navier- Stokes-Weinstein system \((NSW)\) reduces to the classical Navier-Stokes system \((NS)\) (see [1,1]).

with:
- \( \nu \) is the fluid viscosity.
- \( u = u(t, x) = (u_1, \ldots, u_{d+1}) : \mathbb{R}_+ \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) is the fluid velocity field,
- \( p = p(t, x) : \mathbb{R}_+ \times \mathbb{R}^{d+1} \to \mathbb{R} \) is the fluid pressure.
- \( u \) and \( p \) are the two unknowns of the system.

**Remark 3.1.** If \( u^0 \) is regular, applying the divergence operator to the Navier-Stokes-Weinstein equation, we can express the \( p \) pressure as a function of the fluid velocity \( u \)

\[
\text{div}^\alpha_W \left( \text{div}^\alpha_W (u \otimes u) \right) = -\Delta^{\alpha,d} W p
\]

So

\[
p = \left( -\Delta^{\alpha,d} W \right)^{-1} \left( - \sum_{k,j=1}^{d+1} \mathcal{F}_W^{\alpha,d} \left( \lambda_k \lambda_j \mathcal{F}_W^{\alpha,d} (u_j u_k)(\lambda) \right) \right)
\]

Moreover, under the same conditions, Duhamel’s formula implies

\[
u(t, x) = e^{\nu t} \Delta^{\alpha,d} W u^0 - \int_0^t e^{\nu(t-s)} \Delta^{\alpha,d} W \left( \text{div}^\alpha_W (u \otimes u) + \nabla^{\alpha,d} W p \right) ds
\]

then

\[
u(t, x) = e^{\nu t} \Delta^{\alpha,d} W u^0 - \int_0^t e^{\nu(t-s)} \Delta^{\alpha,d} W \mathbb{P} \left( \text{div}^\alpha_W (u \otimes u) \right) ds
\]

with:
- \( e^{\nu t} \Delta^{\alpha,d} W u = q_{\nu t} * W u = \frac{1}{(2 \pi)^{d+1}} e^{-|\xi|^2} * W u = \mathcal{F}_W^{\alpha,d} \left( e^{-\nu|\xi|^2} \mathcal{F}_W^{\alpha,d} u \right) \)
- \( \mathbb{P} \) designates the Leray projector defined by:

\[
\mathbb{P} = I - \nabla^{\alpha,d} W \left( \Delta^{\alpha,d} W \right)^{-1} \text{div}^{\alpha,d} W
\]

The operator \( \mathcal{F}_W^{\alpha,d} (\mathbb{P}_{i,j}) \) is a matrix and the coefficient \( \mathcal{F}_W^{\alpha,d} (\mathbb{P}_{i,j}) \) defined as follows:

\[
\mathcal{F}_W^{\alpha,d} (\mathbb{P}_{i,j}) = \delta_{i,j} - \frac{\xi_i \xi_k}{|\xi|^2} = \begin{cases} 
1 - \frac{\xi^2}{|\xi|^2} & \text{if } i = j \\
\frac{-\xi_i \xi_j}{|\xi|^2} & \text{if not}
\end{cases}
\]
3.1. Existence and uniqueness of solution for the (NSW) system in $L^p_\alpha(\mathbb{R}^{d+1})$ spaces.

Theorem 3.1. Let $2\alpha + d + 2 < p \leq \infty$, fixed and $u^0 \in (L^\infty_\alpha(\mathbb{R}^{d+1}))^{d+1}$ such that $\text{div}^{\alpha,d}(u) = 0$. Then they exist $T^* > 0$ and a unique solution $u$ for the Navier-Stokes-Weinstein system in $C \left([0,T^*], (L^p_\alpha(\mathbb{R}^{d+1}))^{d+1}\right)$

Proof. For $R > 0$ and $T > 0$ we pose

$$B_{R,T} = \left\{ u \in C \left([0,T], (L^p_\alpha(\mathbb{R}^{d+1}))^{d+1}\right) \mid \|u\|_{L^\infty([0,T],L^p_\alpha)} \leq R \right\}$$

We consider the following application

$$\phi : B_{R,T} \rightarrow C \left([0,T], (L^p_\alpha(\mathbb{R}^{d+1}))^{d+1}\right)$$

$$u \mapsto e^{\nu t\Delta^{\alpha,d}_W} u^0 - \int_0^t e^{\nu(t-\tau)\Delta^{\alpha,d}_W} P \left(\text{div}^{\alpha,d}_W(u \otimes v)\right) d\sigma$$

we pose

$$\begin{cases}
L_0(t) = e^{\nu t\Delta^{\alpha,d}_W} u^0 \\
B(u,v)(t) = \int_0^t e^{\nu(t-\tau)\Delta^{\alpha,d}_W} P \left(\text{div}^{\alpha,d}_W(u \otimes v)\right) d\sigma
\end{cases}$$ (3.5)

We want to apply Fixed Point Theorem, so we look for a good choice of $R$ and $T$ we’re going to do some research for the first condition on $R$ and $T$ such that $\phi(B_{R,T}) \subset B_{R,T}$. we have,

$$\left\|L_0(t)\right\|_{L^p_\alpha} = \left\|e^{\nu t\Delta^{\alpha,d}_W} u^0\right\|_{L^p_\alpha} = \left\|q_{\nu t} * u^0\right\|_{L^p_\alpha} \leq \left\|q_{\nu t}\right\|_{L^1_\alpha} \left\|u^0\right\|_{L^p_\alpha} \leq \left\|u^0\right\|_{L^p_\alpha}$$ (3.6)

Furthermore,

$$\left\|B(u,v)(t)\right\|_{L^p_\alpha} = \left\|\int_0^t e^{\nu(t-\tau)\Delta^{\alpha,d}_W} P \left(\text{div}^{\alpha,d}_W(u \otimes v)\right) d\sigma\right\|_{L^p_\alpha}$$

$$\leq \int_0^t \left\|e^{\nu(t-\tau)\Delta^{\alpha,d}_W} P \left(\text{div}^{\alpha,d}_W(u \otimes v)\right)\right\|_{L^p_\alpha} d\sigma$$

Using the inequality of Young with

$$1 + \frac{1}{p} = \frac{2}{p} + \frac{p-1}{p}$$

we obtain,

$$\left\|e^{\nu(t-\tau)\Delta^{\alpha,d}_W} P \left(\text{div}^{\alpha,d}_W(u \otimes v)\right)\right\|_{L^p_\alpha} \leq \sum_{j=1}^{d+1} \left\|\left(F^{\alpha,d}_W\right)^{-1} \left(\text{div}^{\alpha,d}_W(u \otimes v)\right)\right\|_{L^p_\alpha}$$

$$\leq \sum_{j=1}^{d+1} \left\|\left(F^{\alpha,d}_W\right)^{-1} \left(\text{div}^{\alpha,d}_W(u \otimes v)\right)\right\|_{L^p_\alpha}$$ (3.7)

The inequality of Young implies

$$\left\|e^{\nu(t-\tau)\Delta^{\alpha,d}_W} P \left(\text{div}^{\alpha,d}_W(u \otimes v)\right)\right\|_{L^p_\alpha} \leq \left\|\left(F^{\alpha,d}_W\right)^{-1} \left(\text{div}^{\alpha,d}_W(u \otimes v)\right)\right\|_{L^p_\alpha} \leq \frac{\nu t}{R^2} \left\|u \otimes v\right\|_{L^p_\alpha}^2$$ (3.8)
with $M(\xi) = \frac{1}{|\xi|^2} (\delta_{ij}|\xi|^2 - \xi_i \xi_j)_{1 \leq i, j \leq d+1}$ we have,
\[
\left\| (F_W^\alpha,d)^{-1} \left( e^{-\nu(t-\sigma)|\xi|^2} i \xi_j \frac{\xi_j}{|\xi|^2} \right) \right\|_{L_\alpha^p}^{\frac{1}{2}} \\
\leq \left\| \int \frac{\nu(t-\sigma)}{2} \int \Psi_d^\alpha (\xi, -x)e^{-\nu(t-\sigma)|\xi|^2} i \xi_j \frac{\xi_j}{|\xi|^2} d\mu_{\alpha,d}(\xi) \right\|_{L_\alpha^p}^{\frac{1}{2}} \\
\leq \left( \nu(t-\sigma) \right)^{\frac{2n+4d+3}{2}} \left\| \left( F_W^\alpha \right)^{-1} \left( e^{-|\xi|^2} \eta_j \eta_j \frac{\eta_j}{|\eta|^2} \right) \left( \nu(t-\sigma) \right)^{-1/2} \right\|_{L_\alpha^p}^{\frac{1}{2}} \\
\leq (\nu(t-\sigma))^{-\frac{2n+4d+3}{2p}} \left\| \left( F_W^\alpha \right)^{-1} \left( e^{-|\xi|^2} \eta_j \eta_j \frac{\eta_j}{|\eta|^2} \right) \right\|_{L_\alpha^p}^{\frac{1}{2}}
\]

Similarly, we have
\[
\left\| (F_W^\alpha,d)^{-1} \left( i \xi_j e^{-\nu(t-\sigma)|\xi|^2} \left( 1 - \frac{\xi_j^2}{|\xi|^2} \right) \right) \right\|_{L_\alpha^p}^{\frac{1}{2}} \\
\leq (\nu(t-\sigma))^{-\frac{2n+4d+2}{2p}} \left\| \left( F_W^\alpha,d \right)^{-1} \left( i \xi_j e^{-|\xi|^2} \left( 1 - \frac{\eta_j^2}{|\eta|^2} \right) \right) \right\|_{L_\alpha^p}^{\frac{1}{2}}
\]

Let $f, g$ such that
\[
f(\eta) = i \eta_j e^{-|\eta|^2} \left( 1 - \frac{\eta_j^2}{|\eta|^2} \right) \quad \text{and} \quad g(\eta) = e^{-|\eta|^2} i \eta_j \eta_j \frac{\eta_j}{|\eta|^2}
\]

Then
\[
\|B(u, v)\|_{L_p^\alpha} \leq R^2 \int_0^T (\nu(t-\sigma))^{-\frac{2n+4d+2}{2p}} \left( \left\| (F_W^\alpha,d)^{-1} (f) \right\|_{L_\alpha^p}^{\frac{1}{2}} + \left\| (F_W^\alpha,d)^{-1} (g) \right\|_{L_\alpha^p}^{\frac{1}{2}} \right) d\sigma \\
\leq R^2 \int_0^T (\nu(t-\sigma))^{-\frac{2n+4d+2}{2p}} \left( \left\| (F_W^\alpha,d)^{-1} (f) \right\|_{L_\alpha^p}^{\frac{1}{2}} + \left\| (F_W^\alpha,d)^{-1} (g) \right\|_{L_\alpha^p}^{\frac{1}{2}} \right) d\sigma \\
\leq CR^2 T^{-\frac{2n+4d+2}{2p}}
\]

with $C = \frac{\nu^{-\frac{2n+4d+2}{2p}}}{\nu^{-\frac{2n+4d+2}{2p}}}$

then we choose $R > \|u^0\|_{L_p^\alpha}$ and $T > 0$ such as
\[
\|\varphi(u)\|_{L_p^\alpha} \leq \|u^0\|_{L_p^\alpha} + CR^2 T^{-\frac{2n+4d+2}{2p}} \leq R \quad (3.9)
\]

From where, $\varphi (B_{R,T}) \subset B_{R,T}$

we're going to do some research for the second condition on $R$ and $T$ where $\varphi$ is contracting: we have
\[
\|\varphi(u) - \varphi(v)\|_{L_p^\alpha} \leq \|B(u, u)(t) - B(v, v)(t)\|_{L_p^\alpha} \\
\leq \|B(u - v, u)(t) + B(v, u - v)(t)\|_{L_p^\alpha} \\
\leq CT^{-\frac{2n+4d+2}{2p}} \left( \sup_{0 \leq t \leq T} \|u(t, x)\|_{L_p^\alpha} + \sup_{0 \leq t \leq T} \|v(t, x)\|_{L_p^\alpha} \right) \\
\times \sup_{0 \leq t \leq T} \|(u - v)(t, x)\|_{L_p^\alpha} \\
\leq 2CRT^{-\frac{2n+4d+2}{2p}} \sup_{0 \leq t \leq T} \|(u - v)(t, x)\|_{L_p^\alpha}
\]
We choose $T$ and $R$ as
\[ 2CR^{\frac{p-2a-d-3}{2p}} < 1/2 \]  \hspace{1cm} (3.10)

So, according to the inequalities (3.9) and (3.10), the theorem of the fixed point implies the existence of a unique solution of the system \((NSW)\) \(u\) in \( C \left( [0, T], \left( L^p_\alpha (\mathbb{R}^{d+1}) \right)^{d+1} \right) \)

\[ \blacksquare \]

**Remark 3.2.** We can choose \( T = \frac{C_{p, \alpha, d}}{\| u^0 \|^2_{L^p_{\alpha}} } \) : depends only on the norm of \( u^0 \)

Indeed, we take \( R = 2 \| u^0 \|_{L^p_{\alpha}} \) and using the inequalities (3.9) and (3.10), we can deduce
\[ T^{\frac{2-2\alpha-d-2}{2p}} = \frac{C_0}{\| u^0 \|^2_{L^p_{\alpha}} } \]

then
\[ T = \frac{C_{p, \alpha, d}}{\| u^0 \|^2_{L^p_{\alpha}} } \]

with \( C_{p, \alpha, d} = C_0^{\frac{2}{2p-2\alpha-d-2}} \).

**Proposition 3.1.** Let \( \nu > 0 \), \( 2\alpha + d + 2 < p \leq \infty \) and \( u \in C \left( [0, T^*]; \left( L^p_\alpha (\mathbb{R}^{d+1}) \right)^{d+1} \right) \) the solution of \((NSW)\) such as \( u \notin C \left( [0, T^*]; \left( L^p_\alpha (\mathbb{R}^{d+1}) \right)^{d+1} \right) \) with \( T^* < \infty \). Then
\[ \lim_{t \to T^*} \| u(t) \|_{L^p_{\alpha}} = \infty \]

Let \( T_1 = T = \frac{C_{p, \alpha, d}}{\| u^0 \|^2_{L^p_{\alpha}} } \), then according to Theorem 3.1, there is a unique solution \( u \in C \left( [0, T_1], \left( L^p_\alpha (\mathbb{R}^{d+1}) \right)^{d+1} \right) \) of \((NSW)\).

We consider the following system:

\begin{equation}
\text{(NSW)} \begin{cases}
\partial_t v - \nu \Delta_W^{\alpha, d} v + \text{div}_W^{\alpha, d} (v \otimes v) = -\nabla_W^{\alpha, d} p, \\
\text{div}_W^{\alpha, d} v = 0, \\
v(0) = u(T_1)
\end{cases}
\end{equation}

(3.11)

So, there is a unique solution for \((NSW)\) \( v \in C \left( [0, T_1]; \left( L^p_\alpha (\mathbb{R}^{d+1}) \right)^{d+1} \right) \)

\[ T_2 = \frac{C_{p, \alpha, d}}{\| u(0) \|^2_{L^p_{\alpha}}} = \frac{C_{p, \alpha, d}}{\| u(T_1) \|^2_{L^p_{\alpha}}} \]

So, by uniqueness of solution, we have
\[ u(t) = v(t - T_1), \quad \forall t \in [T_1, T_2] \]

moreover, \( u \in C \left( [0, T_1 + T_2]; \left( L^p_\alpha (\mathbb{R}^{d+1}) \right)^{d+1} \right) \) then \( T_1 + T_2 < T^* \)

We can then construct a series \( T_1, T_2, \ldots, T_n \) with
\[ T_k = \frac{C_{p, \alpha, d}}{\| u(T_1 + T_2 + \ldots + T_{k-1}) \|^2_{L^p_{\alpha}}} \quad \forall 2 \leq k \leq n \]

We now consider the following system:

\begin{equation}
\text{(NSW}_n) \begin{cases}
\partial_t w - \nu \Delta_W^{\alpha, d} w + \text{div}_W^{\alpha, d} (w \otimes w) = -\nabla_W^{\alpha, d} p, \\
\text{div}_W^{\alpha, d} w = 0, \\
w(0) = u(T_1 + T_2 + \ldots + T_n)
\end{cases}
\end{equation}

(3.12)
By uniqueness of the solution,

\[ u(t) = w \left( t - (T_1 + T_2 + \ldots + T_n) \right), \quad \forall T_1 + T_2 + \ldots + T_n \leq t \leq T_1 + T_2 + \ldots + T_{n+1} \]

So

\[ T_1 + T_2 + \ldots + T_{n+1} < T^* \]

then \( \sum_{k=1}^{n} T_k \) converges, which implies

\[ \sum_{n=1}^{\infty} \frac{C_{\alpha,d}}{\| u (\sum_{k=1}^{n} T_k) \|_{L^p_{\alpha}}} < \infty \]

from where,

\[ \lim_{m \to \infty} \sum_{n=m+1}^{\infty} \frac{C_{\alpha,d}}{\| u (\sum_{k=1}^{n} T_k) \|_{L^p_{\alpha}}} = 0 \]

so

\[ \lim_{n \to \infty} \frac{C_{\alpha,d}}{\| u (\sum_{k=1}^{n} T_k) \|_{L^p_{\alpha}}} = 0 \]

So he comes,

\[ \lim_{n \to \infty} \left\| u \left( \sum_{k=1}^{n} T_k \right) \right\|_{L^p_{\alpha}} = +\infty \]

As a result, we can deduce

\[ \sum_{k=1}^{\infty} T_k = T^* \quad \text{and} \quad \limsup_{t \to T^-} \left\| u \left( \sum_{k=1}^{n} T_k \right) \right\|_{L^p_{\alpha}} = +\infty \]

3.2. Blow-up result in \( L^p_{\alpha}(\mathbb{R}^{d+1}) \), \( 2\alpha + d + 2 < p \leq \infty \).

**Theorem 3.2.** Let \( \nu > 0 \), \( 2\alpha + d + 2 < p \leq \infty \) and \( u \in C \left( [0,T^*], L^p (\mathbb{R}^{d+1}) \right) \) a maximum solution of (NSW) such that \( T^* \leq \infty \), then there is a constant \( C > 0 \) such that

\[ \| u(t) \|_{L^p_{\alpha}} \geq \frac{C}{(T^* - t)^{\frac{2p}{2\alpha+d-2}}}, \quad \forall t \in [0,T^*] \]

**Proof.** Let \( T \in [0,T^*] \) define by

\[ T = \sup \left\{ t \in [0,T^*]; \sup_{0 \leq z \leq t} \| u(z) \|_{L^p} < 2 \| u^0 \|_{L^p_{\alpha}} \right\} \]

By combining proposition 2.2.2 and the continuity of \( t \mapsto \| u(t) \|_{L^p_{\alpha}} \), we obtain

\[ \| u(T) \|_{L^p_{\alpha}} = 2 \| u^0 \|_{L^p_{\alpha}} \]

For \( t \in [0,T] \) we have

\[ \| u(t) \|_{L^p_{\alpha}} \leq \| u^0 \|_{L^p_{\alpha}} + 4CT \left( \frac{2p}{2\alpha+d-2} \right) \| u^0 \|_{L^p_{\alpha}}^2 \]

In particular, for \( t = T \)

\[ \| u^0 \|_{L^p_{\alpha}} \leq 4CT \left( \frac{2p}{2\alpha+d-2} \right) \| u^0 \|_{L^p_{\alpha}}^2 \]

so

\[ 1 \leq 4CT \left( \frac{2p}{2\alpha+d-2} \right) \| u^0 \|_{L^p_{\alpha}} \]

Which give

\[ 1 \leq 4CT^* \left( \frac{2p}{2\alpha+d-2} \right) \| u^0 \|_{L^p_{\alpha}} \]

(3.13)
Let $t_0 \in [0, T^*[$

$$\begin{array}{l}
(\text{NSW}_0) \quad \left\{ 
\begin{array}{l}
\partial_t v - \nu \Delta_W^{\alpha,d} v + \text{div}_W^{\alpha,d}(v \otimes v) = -\nabla_W^{\alpha,d} p, \\
\text{div}_W^{\alpha,d} v = 0 \\
v(0) = u(t_0)
\end{array}
\right.
\end{array}$$

According to Theorem 3.1, there is a single maximum solution $v \in C \left([0, A^*], (L^p(\mathbb{R}^{d+1}))^{d+1}\right)$.

Now $t \mapsto u(t + t_0)$ is a solution on $[0, T^* - t_0[$, then

$$v(t) = u(t + t_0) \text{ and } A^* = T^* - t_0.$$ 

By applying (3.13) we obtain,

$$1 \leq 4C(T^* - t)^{\alpha - \frac{2p}{2p - \alpha - d}} \|v(0)\|_{L^p_{\alpha}}$$

By replacing the initial instant with any instant $t_0$, we deduce,

$$\|u(t_0)\|_{L^p_\alpha} \geq \frac{1}{4C (T^* - t_0)^{\alpha - \frac{2p}{2p - \alpha - d}}}$$

**Question:** What are the conditions on the soboleve space $H_0^{s,\alpha}(\mathbb{R}^{d+1}_+)$ (see §) to define solutions of the Navier-Stokes equations associated with the Weinstein operator?

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