INTEGRATION IN THE GHP FORMALISM II: AN OPERATOR APPROACH FOR SPACETIMES WITH KILLING VECTORS, WITH APPLICATIONS TO TWISTING TYPE N SPACES.

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Held has proposed a coordinate- and gauge-free integration procedure within the GHP formalism built around four functionally independent zero-weighted scalars constructed from the spin coefficients and the Riemann tensor components. Unfortunately, a spacetime with Killing vectors (and hence cyclic coordinates in the metric, and in all quantities constructed from the metric) will be unable to supply the full quota of four scalars of this type. However, for such a spacetime additional scalars are supplied by the components of the Killing vectors; by using these alongside the spin coefficients and the Riemann tensor components we have the possibility of constructing the full quota of four functionally independent zero-weighted scalars, and of exploiting Held’s procedure.

As an illustration we investigate the vacuum Type N spaces admitting a Killing vector and a homothetic Killing vector. We show how the properties of separability, redundancy, decoupling and reduction of order can be understood and interpreted in a very general manner in our approach, and the advantages of being able to postpone the explicit coordinate choice to the very last step, when it can be used to simplify the residual ordinary differential equations. In a direct manner, we reduce the problem to a pair of ordinary differential operator ‘master equations’, making use of a new zero-weighted GHP operator. By first rewriting the master equations as a closed set of complex first order equations, we reduce the problem to one real third order operator differential equation for a complex function of a real variable — but with still the freedom to choose explicitly our fourth coordinate. It is then easy to see there are a whole class of coordinate choices where the problem reduces essentially to one real third order differential equation for a real function of a real variable. An alternative, more algorithmic approach, using a closed chain of real first order equations for real functions, reduces the problem to the same order, but in a more natural and much more concise form. It is also outlined how the various other third order differential equations, which have been derived previously in work on this problem, can be deduced from our master equations.
1. Introduction.
In a previous paper [1] we developed and illustrated a coordinate-dependent integration procedure in the GHP formalism [2], but, in spite of its advantages, especially in efficiency, over the analogous NU [3,4] approach in the NP formalism [5], we feel that it does not exploit the GHP formalism to its full potential. We also noted there an alternative coordinate-independent, operator approach advocated and developed by Held [6-10], and modified and illustrated recently in [11]. In this present paper we will develop this approach further — specifically for spacetimes admitting at least one Killing vector — and illustrate it by an application to the same problem as considered in the previous paper, [1] — the $G_2$ class of the twisting type N vacuum spaces (NT spaces). We emphasise however, that both papers are self-contained.

In the remainder of this section we will outline this GHP operator-integration procedure — first of all in general, and then specialised to the particular problem to be considered.

1.1. Outline of the integration procedure: the general case.
As discussed in detail in the introduction in [1], the GHP formalism consists of a complete system of three sets of equations: the GHP commutator equations, the GHP Ricci equations, and the GHP Bianchi equations. A procedure for integrating this system was summarised in five steps in [11]:
The first two steps are a coordinate-free integration procedure for the operator $\mathbb{P}$, exactly as proposed by Held, [6]; this $\rho$-integration process is a generalisation of the $r$-integration process in the NP formalism. The third step involves the application of the commutator equations to three complex quantities — two zero-weighted complex quantities which supply four functionally-independent zero-weighted real quantities and one weighted (by which we shall mean neither weight being zero, i.e. $s \neq 0 \neq t$ or equivalently $p \neq \pm q$, in the usual notations, [2]) complex quantity — so that the commutator equations are replaced by an alternative set of equations; we shall refer to these four real functionally-independent zero-weighted quantities as ‘coordinate candidates’ because, at the last two steps, they will usually be the obvious (but not necessarily always the most suitable) choice for the four coordinates, whereas the one weighted complex quantity will usually be transformed to unity by choice of gauge. We can refer to this new set of equations as the ‘GHP metric candidate equations’, but emphasise that although this new set of equations is constructed in a manner analogous to the NP metric equations in the NP-NU procedure, and to the
GHP metric equations in the earlier paper, [1] there is a fundamental difference; these coordinate candidates are chosen from within the system of equations, motivated only by considerations of their mathematical structure, whereas in [1] choices were imposed from the outside, motivated largely by physical and external geometric considerations. Another important difference is that many equations in this new set turn out to be identities, modulo the equations in the other sets; in fact we deliberately choose our coordinate candidates with the intention of achieving such structural simplifications.

After these three steps we will have reduced the complete system to a much smaller sufficient subsystem of differential (operator) equations; this subsystem will consist of (the equivalent of) six real tables for the action of the four GHP operators on each of the four (real) coordinate candidates and on the one complex weighted quantity, together with any residual differential equations from the original system of equations.

The last two steps involve the introduction of an explicit coordinate system and the adoption of a specific tetrad gauge; these choices will be made in such a manner that the reduced sufficient subsystem of differential equations become as simple and manageable as possible.

If we choose our four coordinate candidates to be our coordinates, then the four real tables for these quantities simply become the definitions of the four differential operators (equivalently, the tetrad vectors) in this choice of coordinate system; if we choose to use our gauge freedom to reduce the weighted complex quantity to unity, then the complex table for this quantity simply becomes the definitions of the badly behaved NP spin coefficients. Almost inevitably there will be residual differential equations, which we would hope would be in a reasonably simple form in the chosen coordinate system; however, this often may not be the case, and we may then wish to modify our final coordinate choice to make the differential equations more manageable.

The choice of the four coordinate candidates in step 3 is a crucial, and difficult step. Held [6] originally envisaged an ‘optimal situation’ where (i) six real quantities — more precisely, four functionally independent zero-weighted quantities and one complex weighted quantity, [11] — are suggested explicitly by the spin coefficients and Riemann tensor components, and (ii) the integration procedure, after the first three steps, yields only a complete involutive set of tables of the GHP operators on these six quantities. (In such a situation the problem would be essentially solved; since the differential operators — equivalently the tetrad components — could be written down directly from the tables with the four zero-weighted quantities as coordinates, and the complex weighted quantity gauged to unity.)
Unfortunately, so far, in any practical application of this method, it has not been possible to obtain, directly from the spin coefficients and Riemann tetrad components, the required four functionally independent zero-weighted quantities. In practice, less than the full quota of four zero-weighted quantities is usually supplied by the spin coefficients and Riemann tensor; at this stage it has been customary [12] to make a translation back to the NP formalism or to introduce the ‘missing coordinates’ from outside the formalism [6,7,10,13]. So, in fact, usually a compromise has had to be made at this stage, and the ideals of the coordinate-independent integration procedure in the GHP formalism as proposed by Held have had to be modified. However, we shall show in this paper that it is possible, even when the spin coefficients and Riemann tensor components fail to provide the full quota of zero-weighted quantities, to continue working in the GHP formalism along the lines proposed by Held.

We now know why, in practice, it has not been possible to obtain the full quota of four functionally independent zero-weighted quantities from the spin coefficients and Riemann tensor components: the investigations in GHP formalism which have been carried out were in specialised classes of spaces, which almost inevitably means that there exists at least one Killing vector. This in turn means the existence of at least one cyclic coordinate in the metric, and hence our inability to obtain four functionally independent quantities from quantities constructed from this metric (e.g. from spin coefficients and Riemann tensor components). However, recently, without going outside the GHP formalism, a means of overcoming this problem has been proposed and illustrated with an example in [11]. The space under consideration (implicitly) contained two Killing vectors, and it was found that the spin coefficients and Riemann tensor components only supplied two functionally independent scalars; the additional two coordinate candidates were obtained by taking zero-weighted ‘potentials’ for some of the spin coefficients and Riemann tensor components. The procedure for picking out such a potential relies on intuition rather than any standard procedure; we would prefer a more algorithmic method where various possible choices of coordinate candidates can be generated and tested directly, and the most suitable chosen.

In this paper we present such an alternative method for choosing coordinate candidates. Fortunately, in spaces containing Killing vectors, there is another source from which we may find additional coordinate candidates; if we include explicitly the Killing vector equations alongside the other field equations, then — since the Killing vector components are not constructed from the metric — we can look to the Killing vector components for additional
scalars, functionally independent of the metric. So we have found another possible way of satisfying the first condition of Held’s optimal situation — although to obtain the required six quantities we have to go beyond the spin coefficients and Riemann tensor components, which was Held’s original proposal. This is precisely the situation which occurs in the $G_2$ class of the NT spaces, and makes it an ideal application to illustrate this method. (We should emphasise that we are not claiming, in this paper, that the Killing vector components will always supply all the missing coordinate candidates; the relationship of zero-weighted GHP quantities to Killing and homothetic Killing vectors needs careful and detailed treatment, which will be given elsewhere. In this paper, we are simply pointing out the possibility of using these components, and giving an application where it is possible.)

It is emphasised that once the tables are found for the derivatives of our four coordinate candidates and for the one complex weighted quantity, then — alongside the Ricci, Bianchi and Killing equations — we have all the information from all the equations. We would hope that this complete system would reduce to satisfy the second condition of Held’s ‘optimal situation’ i.e. that only a complete involutive set of six tables remain. However, this is unlikely in practice; we can expect that in the tables some additional unknown functions will occur, which have to satisfy some residual differential equations from the Ricci, Bianchi and Killing sets. In such a situation the choice of the coordinate candidates as coordinates is not necessarily the best choice; an alternative choice may make the residual differential equations more manageable. At this stage we could of course write down the differential equations with the coordinate candidates as coordinates, and attempt to obtain further simplifications by explicit coordinate transformations, in the usual manner. However, we believe that there are still advantages to be gained by keeping within the GHP formalism, obtaining further simplification within that formalism, and finally allowing the structure of the suitably simplified GHP operator equations to suggest the optimum coordinate choices. Again, the application in this paper illustrates precisely how this procedure can be implemented.

1.2. Outline of the integration procedure: the $G_2$ case in the NT problem.

For the particular problem being considered in this paper the complete GHP system consisting of the three sets of equations — Bianchi, Ricci and commutator equations — has to be supplemented by three other sets of GHP equations: the Killing vector equations, the homothetic Killing vector equations, and the non-Abelian $G_2$ condition. The five steps
in the integration procedure will then be carried out for the whole system of six sets of equations. Before beginning, we will of course ensure that all the equations are specialised to vacuum Petrov Type $N$ spaces.

Held, in \[8,9\], has investigated some algebraically special vacuum spaces admitting Killing vectors, and in Section 2 and in the Appendix we clarify some differences in approach and notation between his work and our previous paper, [1]; in Section 3 we make use of the first part of Held’s work in \[8\] — the $\rho$–integration — and specialise the relevant calculations to Petrov type $N$ spaces, as well as extending to the $G_2$ case. This $\rho$–integration (corresponding to the first two steps of the integration procedure) is summarised at the beginning of Section 3, and we then also carry out the third step — applying the commutators to six appropriate quantities. The choice of these quantities is dictated by our wish to keep the calculations as simple and manageable, and to end up with as concise a system, as possible; however, we also try to keep in touch with the earlier method in [1], and to draw comparisons. Since the spaces under consideration have one Killing vector we will not be able to obtain all our coordinate candidates directly from the spin coefficients and Riemann tensor components; in fact none of our candidates is chosen in this manner: three candidates are constructed from the tetrad components of the Killing vector and homothetic Killing vector combined with the spin coefficients, while the fourth candidate is chosen as a potential for a combination of some of these quantities. Since we choose our six quantities as far as possible to suit the structure of the system of equations, and to give as simple a presentation as possible, when the tables are constructed for these six quantities we find considerable redundancy and simplifications once these equations are put alongside the Ricci, Bianchi and Killing equations. Unfortunately, we do not achieve Held’s optimal situation, since in addition to the six quantities, the six tables involve other functions, which are themselves subject to two residual Ricci equations.

The fourth and fifth steps of the integration procedure involve the choice of coordinates and gauge. In the last part of Section 3 we make the most straightforward choice — choosing the coordinate candidates as coordinates. We find that these choices do cause considerable structural simplification; in particular they enable the residual partial differential equations to separate into a pair of coupled ordinary differential equations for one unknown complex function of one real coordinate. However, these residual ordinary differential equations do not look very manageable, and, in particular, decoupling seems a problem.

In our work in Section 3 our primary concern is to ensure that we consider a sufficient
system of equations; up to this stage we are not so concerned with the most efficient presentation of the residual partial differential equations. However, we do keep in mind the need for subsequent simplification, and of course we are looking for the simplest options; and in particular we are hoping for candidates which permit separation. So although, in this case, we will ultimately choose coordinates different from the coordinate candidates, yet we realise that they will have to be closely related in order to share the separation property; specifically we will retain the first three coordinate candidates as coordinates, but we will allow the structure of the residual pair of ordinary operator differential equations to suggest our fourth coordinate.

The great advantage of our GHP approach is that we can get an overall picture of the structures of the equations, and this leads to insights not just into separability but also into redundancy, decoupling and order reduction in a manner which does not first require a precise coordinate choice; the final coordinate choice can then be made in an informed manner which best exploits the separability and decoupling properties, and seeks for maximum order reduction and/or simplification in the final differential equation. These advantages are demonstrated and illustrated in the remaining sections.

In Section 4 we go back to the residual pair of Ricci equations as given in the GHP formalism, and by modifying the GHP weighted operator $\tilde{P}'$ to a new zero-weighted operator $D$ we obtain a simpler version of these operator equations, where it is explicit that all the terms depend on only one real variable. We consider this version of these equations as our ‘master equations’. From these equations we are able to experiment with various different approaches to decoupling and reduction of order; in particular, we find one particularly structurally simple presentation of the master equations, as a set of first order ordinary differential operator equations in complex functions.

In Section 5, we show that by choosing our fourth coordinate from a class which exploits the structure of this simple version of the master equations, the decoupling problem is less complicated than in the previous coordinate choice in Section 3; by a little manipulation the problem reduces to one real third order differential operator equation for a complex function of a real variable. However, we still have the freedom to make our explicit coordinate choice, and by a suitable choice the problem easily reduces to one real third order differential equation for a real function of a real variable. We note that there is a class of such coordinates which reduce the problem to this order.

In Section 6 we present an alternative and more algorithmic approach to obtaining the
residual equation. By a systematic approach we construct, from our set of complex first order equations, a set of real first order equations in the form of a closed chain. These equations suggest both the coordinate and dependent variable in a natural way; the decoupling problem then disappears, and we obtain a comparatively concise form for the residual real third order differential equation.

The early work on the $G_2$ case of the NT problem by McIntosh [14] reduced the problem to a complicated third order complex differential equation for a complex function of a real variable, which he rearranged to a sixth order complex differential equation for the complex function alone — without its complex conjugate occurring explicitly. In recent years the problem has been reduced to essentially a single real ordinary differential equation of a real function of a real variable; the order of the final differential equation has gradually been lowered (at the expense of increasing non-linearity) and most recently there have been obtained a few (very complicated non-linear) third order real differential equations for a real unknown function, each highly dependent on its background formalism and particular coordinate choice [15-18]. The only known solution, the Hauser solution [19,20,21], is found to be a singular case. Each of these analyses began with a preferred coordinate system, and a lot of involved manipulation and complicated coordinate changes were needed in order to arrive at the final third order equation; it is not clear from each individual case whether alternative (and perhaps simpler) third (or lower) order equations could be obtained, nor is there any obvious links between the very different equations so far obtained. On the other hand, our method supplies us with a whole class of third order equations, and with the possibility of obtaining even more in a manner where we have some understanding of the structures involved, as well as some freedom to simplify these structures.

We highlight the insights we have obtained into the GHP integration procedure and into this particular application in the concluding section; we also outline how the various differential equations obtained previously by other approaches to this application can be deduced from our master equations.

2. Formalism and Notation.

As noted above, Held [8,9] has used the GHP formalism to investigate algebraically special vacuum spacetimes admitting Killing vectors, and we shall make direct use of some of his results in this work. Some points need to be clarified at the outset:
Firstly, Held, [6,8,9] slightly modifies three of the usual GHP operators $\mathbf{P}', \partial, \partial'$ to $\tilde{\mathbf{P}}', \tilde{\partial}, \tilde{\partial}'$; this makes no essential difference to the overall structures of the GHP formalism, but simplifies calculations by taking advantage of some properties of algebraically special vacuum metrics. In particular, it is easy to carry out a ‘coordinate-free $\rho$-integration’.

Secondly, as noted in Section 1, the set of Killing equations with which Held [8,9] works differ a little — being somewhat simpler — from the conventional set, and in particular from the set used in [12], and in the previous paper. We show in Appendix I the relationship between these two sets, and confirm that the two apparently different sets of equations in the two papers are equivalent for Killing vectors. We also generalise the set in [8,9] for the presence of homothetic Killing vectors, and show its equivalence to the analogous set in [12].

Thirdly, there is a question of notations, and this needs to be set out in detail to avoid confusion:

(i) In the previous paper, [1] we used the symbols $b, c, a$ for tetrad components of Killing vectors, and $\Sigma^o$ for the twist. Held uses the symbols $\xi_0, \xi_1, \xi_2, [9]$ or $\theta_0, \theta_1, \theta_2, [8]$ for Killing vector components, and $\Omega^o (= -2i\Sigma^o)$ for the twist. When we make use of Held’s results we shall translate them into the former symbols for consistency with the previous paper.

(ii) Quantities which are annihilated by the operator $\mathbf{P}$ are labelled in Held’s work with $^o$, e.g. $\mathbf{P}\eta^o = 0$; an equivalent notation has already been adopted in the previous paper.

(iii) Standard GHP usage, also followed by Held, uses the prime notation for half of the spin coefficients, and half of the GHP operators; in this paper we shall also use that notation. Although our previous paper uses the more familiar NP versions of the spin coefficients, since only $\kappa'(= -\nu)$ and $\rho'(= -\mu)$ occur explicitly in this paper there will be no difficulty in comparison.

(iv) The letter $P$ has a special usage in NP notation [3,4], and has been used extensively in the previous paper, [1]; the related $P^o$ plays an important role in Held’s work [8,9,10], and we retain it in this paper. If a comparison is being made we must take into account a factor of 2, $(P^o \sim 2P)$ as can be seen in the defining equations in each paper. We also point out that $\tilde{P}$ defined in [1] is real.
3. An operator-integration approach with coordinate candidates as coordinates.

3.1. Steps 1,2: The $\rho$–integration.

We specialise the $\rho$–integration results in [8] to the Petrov type N case with the substitutions

$$\Psi_2 = 0 = \Psi_3$$

(3.1)

giving:

The residual Bianchi and Ricci equations.

$$\rho' = \bar{\rho}\rho'^o$$

(3.2a)

$$\kappa' = \kappa'^o$$

(3.2b)

$$\Psi_4 = \rho\Psi_4^o$$

(3.3)

with

$$\Psi_4^o = -\tilde{\partial}'\kappa'^o$$

(3.4a)

$$\tilde{\partial}'\rho'^o = 2i\Sigma^o\kappa'^o$$

(3.4b)

$$\tilde{\partial}\kappa'^o = \tilde{\Phi}'\rho'^o$$

(3.4c)

$$\tilde{\partial}\tilde{\partial}'\Sigma^o = 2\Sigma^o\rho'^o$$

(3.4d)

and

$$\tilde{\Phi}\rho = \rho^2$$

$$\tilde{\partial}\rho = 0$$

$$\tilde{\partial}'\rho = -2i\rho^2\tilde{\partial}'\Sigma^o$$

(3.5)

The commutator equations.

$$[\tilde{\Phi}, \tilde{\partial}] = 0$$

(3.6a)

$$[\tilde{\Phi}, \tilde{\partial}'] = 0$$

(3.6b)

$$[\tilde{\Phi}, \tilde{\Phi}'] = 0$$

(3.6c)

$$[\tilde{\Phi}', \tilde{\partial}] = -\frac{\kappa'^o}{\rho}\tilde{\Phi} + q\kappa'^o$$

(3.6d)
\[
[\tilde{\mathbf{F}}', \tilde{\partial}'] = -\frac{\kappa'\rho}{\rho} \tilde{\mathbf{F}} + p\kappa'
\] (3.6e)

\[
[\tilde{\partial}, \tilde{\partial}'] = (\frac{\dot{\rho}'}{\rho} - \frac{\rho'}{\rho}) \tilde{\mathbf{F}} - 2i\Sigma'\tilde{\mathbf{F}} + p\rho'' - q\rho'
\] (3.6f)

(In addition to the Bianchi and Ricci equations given above there are the two equations

\[
2i\tilde{\mathbf{F}}'\Sigma' = -\dot{\rho}' + \rho'
\]

\[
\tilde{\partial}\Psi_4' = 0
\]

which follow from [8]. However, both are identically satisfied by virtue of the other Bianchi and Ricci equations, given above, and the commutators. We show this as follows: apply commutator (3.6f) to \(\Sigma'\) and use (3.4d) and its conjugate to get the first; apply commutator (3.6f) to \(\kappa'\), followed by (3.6e) applied to \(\rho'\) and then use appropriate equations from (3.4) together with the first equation to get the second.)

The equations for one Killing vector.

When the Killing vector \(\xi_1\) is given by

\[
\xi_1^\mu = a_1 l^\mu + b_1 n^\mu - c_1 m^\mu - \bar{c}_1 \bar{m}^\mu
\] (3.7)

the Killing equations can be \(\rho\)-integrated to obtain, [8]

\[
a_1 = -2i\bar{c}_1^o \bar{\partial}' \Sigma^o - \dot{\rho}' b_1^o + i\Sigma' \bar{\partial}' b_1^o + \frac{1}{2}(\rho + \frac{1}{\rho}) \bar{\partial}' b_1^o
\]

\[
b_1 = b_1^o
\]

\[
c_1 = c_1^o / \rho
\] (3.8)

with

\[
\tilde{\mathbf{F}}' b_1^o = -\frac{1}{2}(\tilde{\partial} c_1^o + \tilde{\partial}' c_1^o)
\] (3.9a)

\[
\tilde{\partial} b_1^o = -2i\Sigma^o c_1^o
\] (3.9b)

\[
\tilde{\partial}' c_1^o = 0 = \tilde{\partial}' c_1^o
\] (3.9c)

\[
b_1^o (\rho^\prime - \rho') - c_1^o \bar{\partial}' \rho' - c_1^o \bar{\partial} \rho' + 2\rho' \bar{\partial}' b_1^o = 0
\] (3.9d)

\[
b_1^o (\rho^\prime - \rho') + 2i\bar{c}_1^o \bar{\partial} \Sigma^o + 2i\bar{c}_1^o \bar{\partial}' \Sigma^o + i\Sigma^o (\bar{\partial} c_1^o + \bar{\partial}' c_1^o) = 0
\] (3.9e)

where the subscript \(1\) denotes quantities associated with the Killing vector \(\xi_1\).
The last equation was not explicitly displayed in [8] but is given in [9] where it is pointed out that it follows from the condition that $a_1$ is real.

However, when we examine this last equation we note that it is simply the commutator (3.6f) applied to $b_1^o$. In addition, when we apply the commutator (3.6d) to $b_1^o$ and differentiate the resulting equation by $\tilde{\partial}'$, after using (3.6f) a number of times together with some equations from (3.4), we obtain (3.9d). Hence the two equations (3.9d,e) are identities modulo the residual equations and the commutators, and so can be omitted.

The equations for one homothetic Killing vector.

When the homothetic Killing vector $\xi_2$ is given by

$$\xi_2^\mu = a_2 l^\mu + b_2 n^\mu - c_2 m^\mu - \bar{c}_2 \bar{m}^\mu$$  (3.10)

we use the set of equations (AI.2) in the first Appendix, and in the same manner as in [8] we can easily integrate to obtain

$$a_2 = -2ic_2^o \tilde{\partial}' \Sigma^o - \rho^\alpha b_2^o + i\Sigma^o \tilde{\Pi}' b_2^o + \frac{1}{2}(\frac{1}{\rho} + \frac{1}{\bar{\rho}})(\tilde{\Pi}' b_2^o - \phi)$$

$$b_2 = b_2^o$$  (3.11)

$$c_2 = c_2^o / \bar{\rho}$$

with

$$\tilde{\Pi}' b_2^o = -\frac{1}{2}(\tilde{\partial} c_2^o + \tilde{\partial}' c_2^o - \phi)$$  (3.12a)

$$\tilde{\partial} b_2^o = -2i\Sigma^o c_2^o$$  (3.12b)

$$\tilde{\Pi}' c_2^o = 0 = \tilde{\partial}' c_2^o$$  (3.12c)

$$b_2^o \tilde{\Pi}' \rho^\alpha - c_2^o \tilde{\partial}' \rho^\alpha - c_2^o \tilde{\partial} \rho^\alpha + 2\rho^\alpha \tilde{\Pi}' b_2^o = 0$$  (3.12d)

$$b_2^o (\rho^\alpha - \rho^\alpha) + 2ic_2^o \tilde{\partial}\Sigma^o + 2ic_2^o \tilde{\partial}' \Sigma^o + i\Sigma^o (\tilde{\partial} c_2^o + \tilde{\partial}' c_2^o - \phi) = 0$$  (3.12e)

where the subscript $2$ denotes quantities associated with the homothetic Killing vector $\xi_2$. We note that in this case also, the last two equations (3.12d,e) are identities modulo the residual equations and the commutators, and so can be omitted.

The non-Abelian $G_2$ condition.
We now consider the case for a non-Abelian $G_2$ of homothetic motions; by an appropriate choice of basis in the Lie algebra we get

$$[\xi_1, \xi_2] + 2\xi_1 = 0 \quad (3.13)$$

where $\xi_1(=\xi_1^{\mu}\nabla_\mu)$ and $\xi_2(=\xi_2^{\mu}\nabla_\mu)$ are the Lie derivative operators associated with the Killing vector $\xi_1$ and the homothetic Killing vector $\xi_2$ respectively. Equation (3.13) has to be satisfied when applied to an arbitrary (scalar) quantity or equivalently to four real functionally independent quantities. In the present context that means that it must be applied to four real zero-weighted functionally independent quantities, and for such zero-weighted scalars the operators can be written as

$$\xi_1(\equiv \xi_1^i \nabla_i) = a_1\hat{P} + b_1\hat{\bar{P}}' - c_1^o\bar{\partial} - \bar{c}_1^o\partial'$$

$$\xi_2(\equiv \xi_2^i \nabla_i) = a_2\hat{P} + b_2\hat{\bar{P}}' - c_2^o\bar{\partial} - \bar{c}_2^o\partial' \quad (3.14)$$

So for the case of Petrov type N vacuum spaces admitting one ordinary Killing vector $\xi_1$, and one homothetic Killing vector $\xi_2$, we would be required to solve the residual Ricci and Bianchi equations (3.4,5) together with the residual Killing equations (3.9a,b,c), the residual homothetic Killing equations (3.12a,b,c), the commutator equations (3.6) and non-Abelian condition (3.13).

### 3.2. Step 3: Choosing the coordinate candidates and applying the commutators.

Since the first two steps of the procedure set out in [11] have been carried out, the next step is to ensure that the commutator equations are completely satisfied; to ensure this we must apply them to four real, zero-weighted, functionally independent quantities and to one complex weighted quantity. We have just noted the similar requirements for (3.13) so it will be convenient to apply both sets of equations — (3.6) and (3.13) respectively — to the same four coordinate candidates. We would prefer to select those four coordinate candidates in a manner which would ensure that any additional constraint equations resulting from the commutator equations assume as simple a form as possible; but we also wish at this stage not to move too far from the coordinate choices of the first method. Fortunately, for the first three coordinate candidates, these two wishes coincide. For each of the four real and one complex quantities we will obtain a table for the action of the operators; we may also obtain some new differential equations when the commutators
and $G_2$ condition are used on each of these quantities, as well as some simplifications. We now set out these results systematically for each quantity in turn.

**Choice of the first two real (one complex) coordinate candidates, $\zeta$.**

Noting the simple equations for the $(0,2)$ quantities $c_1^0$ and $c_2^0$ we begin with the comparatively obvious choice for our first (complex zero-weighted) coordinate candidates

$$\zeta = i(c_2^0/c_1^0)$$  \hspace{1cm} (3.15)

so that

$$\tilde{P}\zeta = \tilde{P}'\zeta = \tilde{\partial}'\zeta = 0$$  \hspace{1cm} (3.16)

(We choose the factor $i$ in our definition of $\zeta$ in order to make easy direct comparison with [1], and also because this choice simplifies a later part of the calculation. We also omit $^o$ on $\zeta$, even though it is annihilated by $\tilde{P}$.)

When the commutators are applied to this zero-weighted quantity the only non-trivial results are,

$$\tilde{P}P^o = \tilde{P}'P^o = \tilde{\partial}'P^o = 0$$  \hspace{1cm} (3.17)

where $P^o$ is the $(0,-2)$ quantity,

$$P^o = \tilde{\partial}\zeta$$  \hspace{1cm} (3.18)

Noting the similarity between these equations and those for $c_1^0$, we can write

$$\tilde{P}(c_1^0P^o) = \tilde{P}'(c_1^0P^o) = \tilde{\partial}'(c_1^0P^o) = 0$$  \hspace{1cm} (3.19)

Since $(c_1^0P^o)$ is also a zero-weighted quantity comparison between its equations (3.19) and those for $\zeta$ show that

$$c_1^0P^o = 2g(\zeta)$$  \hspace{1cm} (3.20)

where $g$ is an arbitrary function of $\zeta$. (The factor 2 has been introduced to give correspondence with [1])

Turning next to equation (3.13), and applying (3.14) to $U$ we find

$$\xi_1\zeta = -c_1^0\tilde{\partial}\zeta = -2g(\zeta)$$  \hspace{1cm} (3.21)

$$\xi_2\zeta = -c_2^0\tilde{\partial}\zeta = ic_1^0\zeta\tilde{\partial}\zeta = 2i\zeta g(\zeta)$$

and so obtain from (3.13),

$$g(\zeta) = i$$  \hspace{1cm} (3.22)
or

\[ c_1^o = \frac{2i}{P_o} \quad \text{and} \quad c_2^o = \frac{2\zeta}{P_o} \quad (3.23) \]

where \( \zeta \) satisfies (3.16) and \( P^o \) satisfies (3.17).

In summary, we have a table (3.16,18) for the action of the operators on the two real coordinate candidates \( \zeta \); and the action of the commutators on these quantities yields the additional equations (3.17) — the table giving the action of most of the operators on the weighted quantity \( P^o \). The result of applying the condition (3.13) has been to obtain explicit expressions for the Killing vector components \( c_1^o, c_2^o \).

(Although this is not immediately apparent from the respective definitions, a little work confirms that \( \zeta \) as defined above essentially agrees with \( \zeta \) as defined in [1]. We could have chosen to define \( \zeta \) in this paper in a manner more closely analogous to [1] by introducing it as a potential for \( P^o \), which, in turn, could have been introduced as a potential for some of the spin coefficients. However, as we have emphasised before, we wish in this paper to illustrate the direct method of choosing coordinate candidates.)

**Choice of the third coordinate candidate, \( R \).**

A rearrangement of (3.5)

\[
\begin{align*}
\mathcal{P}\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)/|P^o| &= -2/|P^o| \\
\mathcal{P}'\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)/|P^o| &= -(\rho^o + \bar{\rho}^o)/|P^o| \\
\tilde{\partial}\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)/|P^o| &= -2i\frac{\tilde{\partial}\Sigma^o}{|P^o|} - \frac{\tilde{\partial}P^o}{2P^o}(\frac{1}{\rho} + \frac{1}{\bar{\rho}})/|P^o| \\
\tilde{\partial}'\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)/|P^o| &= 2i\frac{\tilde{\partial}'\Sigma^o}{|P^o|} - \frac{\tilde{\partial}'P^o}{2P^o}(\frac{1}{\rho} + \frac{1}{\bar{\rho}})/|P^o|
\end{align*}
\]  

suggests an obvious choice for our third coordinate candidate \( R \),

\[ R = -\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)/|P^o| \quad (3.25) \]

satisfying

\[
\begin{align*}
\mathcal{P}R &= 2/|P^o| \\
\mathcal{P}'R &= (\rho^o + \bar{\rho}^o)/|P^o| \\
\tilde{\partial}R &= 2i\frac{\tilde{\partial}\Sigma^o}{|P^o|} - \frac{\tilde{\partial}P^o}{2P^o} \\
\tilde{\partial}'R &= -2i\frac{\tilde{\partial}'\Sigma^o}{|P^o|} - \frac{\tilde{\partial}'P^o}{2P^o}
\end{align*}
\]  

(3.26)
From their respective differential equations given in tables (3.16) and (3.26), the three coordinate candidates \((\zeta + \bar{\zeta}), i(\zeta - \bar{\zeta}), R\) are easily seen to be functionally independent of each other; and when the commutators are applied to \(R\), the only two commutators (3.6d,f) which are not identically satisfied, give respectively,

\[
\kappa^o = -\bar{\Phi}' \left( \frac{\bar{\partial} P^o}{2P^o} \right) \tag{3.27a}
\]

\[
\rho^o = -\bar{\vartheta}' \left( \frac{\bar{\partial} P^o}{2P^o} \right) \tag{3.27b}
\]

Substituting these values into (3.4a) shows that

\[
\Psi^o_4 = \bar{\vartheta}' \bar{\Phi}' \left( \frac{\bar{\partial} P^o}{2P^o} \right) \tag{3.28}
\]

and also that (3.4c) is identically satisfied by virtue of (3.6d).

So the only remaining Ricci and Bianchi equations to be solved are (3.4b) and (3.4d) which now become

\[
\bar{\vartheta}' \bar{\partial} \left( \bar{\partial} \bar{P}^o \right) = 2i \Sigma^o \bar{\Phi}' \left( \frac{\bar{\partial} P^o}{P^o} \right) \tag{3.29}
\]

\[
\bar{\vartheta}' \bar{\partial} \Sigma^o = -\Sigma^o \bar{\vartheta}' \left( \frac{\bar{\partial} P^o}{P^o} \right)
\]

The first of these may be rearranged to

\[
\bar{\vartheta}' \bar{\partial} \left( \frac{\bar{\partial} P^o}{P^o^{3/2}} \right) = 0 \tag{3.30}
\]

There are also the inequalities

\[
\Sigma^o \neq 0 \tag{3.31a}
\]

\[
\bar{\vartheta}' \bar{\Phi}' \left( \frac{\bar{\partial} P^o}{2P^o} \right) \neq 0 \tag{3.31b}
\]

The remaining constraint equations from the Killing equations and from the homothetic Killing equations, after the substitutions (3.23), are

\[
\bar{\Phi}' b_1^o = i \left( \frac{\bar{\partial} P^o}{P^o^2} - \frac{\bar{\partial}' P^o}{P^o^2} \right)
\]

\[
\bar{\vartheta} b_1^o = -4 \Sigma^o / \bar{P}^o
\]

\[
\bar{\vartheta} b_2^o = -4i \Sigma^o / \bar{P}^o
\]

\[
\bar{\Phi}' b_2^o = -2 + \frac{1}{2} \varphi + \frac{\zeta \bar{\partial} P^o}{P^o^2} + \frac{\bar{\zeta} \bar{\partial}' P^o}{P^o^2}
\]

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Turning next to equation (3.13), and applying (3.14) to \( R \) we find

\[
\begin{align*}
\xi_1 R &= 0 \\
\xi_2 R &= (2 + \frac{\phi}{2})R
\end{align*}
\] (3.33)

and so we easily confirm that (3.13) is satisfied identically when applied to \( R \).

In summary, we have a table (3.26) for the action of the operators on the coordinate candidate \( R \); and the action of the commutators on \( R \) yields explicit expressions for some of the spin coefficients and \( \Psi_4^o \); some of the Killing equations are also simplified. The result of applying the condition (3.13) to \( R \) yields no new information.

**Choice of the fourth coordinate candidate, \( u \).**

We have noted in the introduction to this paper that we can introduce each of the coordinate candidates either directly as one of the (combination of) existing elements in the GHP formalism, or as a ‘potential’ for existing elements. Such potentials are chosen by a careful consideration of the structure of the equations, especially the commutators, paying particular attention to the appropriate weights, [11]; specifically the four derivatives \( \hat{P} \eta, \hat{P}' \eta, \partial \eta, \partial' \eta \) of a zero-weighted potential \( \eta \) are equated to (combinations of) some of the spin coefficients and Riemann tensor components in such a way that the commutators acting on \( \eta \) are identically satisfied. Since for zero-weighted \( \eta \),

\[
\nabla_\mu \eta = n_\mu \hat{P} \eta + l_\mu \hat{P}' \eta - \bar{m}_\mu \partial \eta - m_\mu \partial' \eta
\] (3.34)

clearly fixing its GHP derivatives determines \( \eta \) uniquely (up to an additive constant).

We can look for hints for possible potentials by rearranging the residual equations, paying particular attention to weights; for instance the first and last of (3.32) can be combined to give

\[
\hat{P}' \left( \frac{P^o |P^o| (b^o_2 - ib^o_1 \bar{\zeta})}{(\zeta + \bar{\zeta})} \right) = 2 \bar{\partial} |P^o| - \frac{(4 - \phi) P^o |P^o|}{2(\zeta + \bar{\zeta})}
\] (3.35)

i.e.

\[
\hat{P}' \left( \frac{P^o |P^o| (b^o_2 - ib^o_1 \bar{\zeta})}{(\zeta + \bar{\zeta})} \right) = 2 \bar{\partial} |P^o| - \frac{(4 - \phi) P^o |P^o|}{2(\zeta + \bar{\zeta})}
\] (3.36)

Therefore, the choice

\[
\hat{P} u = 0
\] (3.37a)
\[
\tilde{\mathcal{P}}' u = |P^o| (\bar{\zeta} + \zeta)^{-1 - m} \tag{3.37b}
\]
\[
\tilde{\partial} u = P^o |P^o| (b^o_2 - ib^o_1 \bar{\zeta})(\bar{\zeta} + \zeta)^{-2 - m}/2 \tag{3.37c}
\]

where \( u \) is a real zero-weighted quantity, guarantees that commutator (3.6a) is satisfied identically when applied to \( u \). (In fact it follows easily that (3.6f) is the only commutator not satisfied identically when applied to \( u \).)

We now need to check explicitly that the four coordinate candidates \( R, \zeta, \bar{\zeta}, u \), are functionally independent. We can write the four derivatives of each of these four real zero-weighted quantities in determinant form as

\[
\begin{vmatrix}
2/|P^o| & (\rho^o + \bar{\rho}^o)/|P^o| & (2i\tilde{\partial} \Sigma^o/|P^o|) - R\tilde{\partial}P^o/2P^o & (-2i\tilde{\partial}^\epsilon \Sigma^o/|P^o|) - R\tilde{\partial}^\epsilon \bar{P}^o/2\bar{P}^o \\
0 & 0 & P^o & 0 \\
0 & 0 & 0 & \bar{P}^o \\
0 & |P^o|(\bar{\zeta} + \zeta)^{-1 - m} & P^o\ell(\bar{\zeta} + \zeta)^{-1} & \bar{P}^o\bar{\ell}(\bar{\zeta} + \zeta)^{-1}
\end{vmatrix} \tag{3.38}
\]

where we have defined the function \( \ell(\zeta, \bar{\zeta}, u) \) by,

\[
\ell = \frac{(\zeta + \bar{\zeta})\tilde{\partial} u}{P^o} \tag{3.39}
\]

Hence, using (3.39), (3.6d) and (3.37b), we have

\[
\frac{\tilde{\partial}P^o}{P^o} = 2\frac{\tilde{\mathcal{P}}' \ell}{|P^o|} (\zeta + \bar{\zeta})^m + 2(1 + m)(\zeta + \bar{\zeta})^{-1} \tag{3.40}
\]

It is easy to confirm that this determinant is non-zero. (It can be seen that this definition for \( \ell \) agrees with the definition in the previous paper, [1].)

When we apply the remaining commutator (3.6f) to \( u \) we obtain

\[
2|P^o|\Sigma^o = i(\tilde{\partial} \tilde{\partial}' u - \tilde{\partial}' \tilde{\partial} u)(\bar{\zeta} + \zeta)^{1 + m} \tag{3.41}
\]

which becomes

\[
2|P^o|\Sigma^o = i\left( \bar{P}^o \tilde{\partial} \bar{\ell} - P^o \tilde{\partial}' \bar{\ell} + |P^o|^2(\bar{\zeta} + \zeta)^{-1}(\ell - \bar{\ell}) \right)(\bar{\zeta} + \zeta)^m \tag{3.42}
\]

We can find \( b^o_1, b^o_2 \) explicitly from (3.37c),

\[
b^o_1 = 2i \frac{(\ell - \bar{\ell})(\zeta + \bar{\zeta})^m}{|P^o|} \tag{3.43}
\]
\[ b_2^o = 2 \frac{(\zeta \ell + \bar{\zeta} \ell)(\zeta + \bar{\zeta})^m}{|P^o|} \] (3.44)

and when we substitute these values into the remaining Killing equations (3.32) we get

\[ \frac{\tilde{\partial} \ell}{P^o} = \ell \frac{\tilde{\partial}' \ell}{|P^o|} (\zeta + \bar{\zeta})^m \] (3.45a)

\[ \frac{\tilde{\partial}' \ell}{P^o} = \ell \frac{\tilde{\partial} \ell}{|P^o|} (\zeta + \bar{\zeta})^m \] (3.45b)

We can now use (3.39) and (3.37b) to rewrite (3.37c) as

\[ \frac{\tilde{\partial} u}{P^o} = \ell \frac{\tilde{\partial}' u}{|P^o|} (\zeta + \bar{\zeta})^m \] (3.46)

and a comparison with (3.45) shows that these equations give the very crucial simplification that \( \ell \) and \( u \) are functionally dependent.

When (3.45) are substituted into (3.42) we obtain

\[ 2 \frac{\Sigma^o}{|P^o|} = i \left( \frac{\ell \tilde{\partial}' \ell}{|P^o|} - \frac{\ell \tilde{\partial}' \ell}{|P^o|} + (\zeta + \bar{\zeta})^{-1-m}(\ell - \bar{\ell}) \right) (\zeta + \bar{\zeta})^{2m} \] (3.47)

The two residual Ricci and Bianchi equations (3.29) now become equations for \( \ell \) by the substitutions (3.40,47). We now seem to have too many equations — one real and one complex — for the complex unknown \( \ell \). However, we shall see that this is compensated for by the fact that these two equations are not independent; it will be easier to show this explicitly when we introduce our coordinates.

The inequalities (3.31) can also be rearranged with these substitutions.

Turning next to equation (3.13), and noting that

\[ \xi_1 u = 0 \]
\[ \xi_2 u = 0 \] (3.48)

we confirm that (3.13) is satisfied identically when applied to \( u \).

In summary, we have a table (3.37) for the action of the operators on the coordinate candidate \( u \); and the action of the commutators on \( u \) gives explicit expressions in terms of \( \ell \) for the twist \( \Sigma^o \) and for the Killing vector components \( b_1^o, b_2^o \), as well as the action of the remaining operator \( \tilde{\partial} \) on weighted \( P^o \). The remaining two Killing equations give
the important result that the complex function $\ell$ and the coordinate candidate $u$ are functionally dependent. The result of applying the condition (3.13) to $u$ yields no new information.

*Choice of one complex weighted quantity, $P^o$.*

When we apply the commutators to $P^o$ and all the earlier equations are taken into account, all the commutator equations acting on $P^o$ are identically satisfied.

*Summary of results in this subsection.*

Having confirmed that our four coordinate candidates are functionally independent, we now know that we have extracted all the information available from the commutators and the non-Abelian condition. The net result is that we have obtained explicit expressions for all the spin coefficients, Weyl tensor component and Killing vectors components, as well as six tables of twenty-four (real) equations — (3.16,18) and their complex conjugates, (3.24), (3.37a,b,39), (3.17,40) — for the action of the operators on the four coordinate candidates $R, u, \zeta, \bar{\zeta}$ and on the one complex weighted quantity, $P^o$. However, this set of tables is not self-contained; it contains explicit expressions in $\ell$, which is a function of $u$. The quantity $\ell$ must satisfy the two residual Ricci equations (3.29), subject to the inequalities (3.31), when the substitutions (3.40,47) are made.

The next obvious step is to use the coordinate candidates as coordinates.

### 3.3. Steps 4,5: Choice of coordinate candidates $R, \zeta, \bar{\zeta}, u$ as coordinates and choice of gauge.

We will now choose the coordinate candidates as coordinates $R, \zeta, \bar{\zeta}, u$, which means that their respective four tables of equations (3.16,18), (3.24) and (3.37) become identities, simply defining, in these coordinates, the GHP operators when acting on zero-weighted quantities as,

\[
\begin{align*}
\check{\Phi} & \equiv \check{\Phi}(R) \partial_R + \check{\Phi}(\zeta) \partial_\zeta + \check{\Phi}(\bar{\zeta}) \partial_{\bar{\zeta}} + \check{\Phi}(u) \partial_u = 2|P^o|^{-1}\partial_R \\
\check{\Phi}' & = (\rho^o + \bar{\rho}^o)|P^o|^{-1}\partial_R + |P^o|(\zeta + \bar{\zeta})^{-1-m}\partial_u \\
\check{\partial} & = \left(2i\frac{\check{\Omega}^o}{|P^o|} - R\frac{P^o\check{\Phi}'\ell}{|P^o|}(\zeta + \bar{\zeta})^m - RP^o(1+m)(\zeta + \bar{\zeta})^{-1}\right)\partial_R + P^o\partial_\zeta + P^o\ell(\zeta + \bar{\zeta})^{-1}\partial_u \\
\check{\partial}' & = \left(-2i\frac{\check{\Omega}'^o}{|P^o|} - R\frac{P^o\check{\Phi}'\bar{\ell}}{|P^o|}(\zeta + \bar{\zeta})^m - RP^o(1+m)(\zeta + \bar{\zeta})^{-1}\right)\partial_R + P^o\partial_\bar{\zeta} + P^o\bar{\ell}(\zeta + \bar{\zeta})^{-1}\partial_u
\end{align*}
\] (3.49)
The Killing operator becomes
\[ \xi_1 = -2i \partial_\zeta + 2i \partial_\bar{\zeta}, \] (3.50)
while the homothetic Killing operator becomes
\[ \xi_2 = -2\zeta \partial_\zeta - 2\bar{\zeta} \partial_{\bar{\zeta}} - 4mR \partial_R \] (3.51)

There is of course also the table of operator equations for the weighted \( P^o \), but this simply yields the badly behaved spin coefficients (if we want them) when we make our choice of gauge.

This coordinate choice exploits the separability property, and we find that the problem has been reduced to only two differential equations which can now be written out explicitly in these coordinates. Since \( \ell \) is a function of \( u \) only, i.e \( \ell \equiv \ell(u) \), we can replace \( \tilde{P}' \ell \) with
\[ \tilde{P}' \ell = \dot{\ell} |P^o| (\zeta + \bar{\zeta})^{-1-m} \] (3.52)
where \( \dot{\cdot} \) denotes differentiation with respect to \( u \) and we have used (3.37). So the only remaining equations to be solved are the two residual Ricci and Bianchi equations (3.29) for \( \ell \) which is a complex function of the real coordinate \( u \). When the substitutions (3.42,47) together with (3.58) are made into the equations (3.29) and the inequalities (3.31) we obtain
\[ \ddot{\ell} \ell + 3\dot{\ell} \ell' + \ell (2m + 1) \ell - 2\ell \ell' - 2\ell^2 - (4m + 2)\dot{\ell} - 2m(m + 1) = 0 \] (3.53)
\[ \{ \ell \ell^2 \ell' + \ell \ell' \left( (2m + 1) \ell + (2m - 3) \ell + 3\dot{\ell} \ell + \ddot{\ell} \right) + \dddot{\ell} \ell + (2m - 3)\ell \ell^2 \]
\[ + (2m + 1) \ell \ell \ell' + (4m - 2) \ddot{\ell} \ell - (4m^2 - 4m - 3) \ell \ell - (4m^2 - 4m - 2) \ell \}
\[ - \{ \text{c.c.} \}
\]
\[ = 0 \] (3.54)
\[ \dddot{\ell} - \dot{\ell} \ell + \ell - \ell \neq 0 \] (3.55)
\[ \ell \ell' + 3\dot{\ell} \ell + (2m + 1) \ell \neq 0 \] (3.56)

(These correspond to the equations and inequalities (111,112) in the previous paper [1], as well as to the equations and inequalities (25,26,28,29) originally given in [17], when the latter two sets of equations are specialised to \( f = \text{constant} \).)
We can substitute (3.53) (and its complex conjugate) into (3.54) to obtain a simpler version,

\[
\begin{align*}
\{ \bar{\ell}\ell (\dot{\ell} + 2m - 1)\dot{\ell} + \ddot{\ell} + (2m - 1)\dddot{\ell}^2 \\
+ (2m + 1)\dddot{\ell} + (4m^2 - 1)\dddot{\ell} - (4m - 2)\dddot{\ell} - (6m^2 - 2m - 2)\dddot{\ell}\}
- \{\text{c.c}\}
= 0
\end{align*}
\]

(3.57)

We now find explicitly the redundancy relation between this pair of equations to be

\[
\frac{d}{du}[\text{eq.}(3.57)] = (2m - 1 + \dot{\ell})[\text{eq.}(3.53)] - (2m - 1 + \dot{\ell})[\overline{\text{eq.}(3.53)}]
\]

(3.58)

The gauge choice does not affect the two residual differential equations, but putting \(P^0 = 1\) will cause some minor simplification in the form of the differential operators.

So we have succeeded in reproducing the results in the previous paper [1] by a method which was clearly influenced by the choice of the fourth coordinate \(u\) in that paper.

### 3.4 Summary of this section.

In Sections 3.1 and 3.2 we have succeeded in reducing the problem to essentially a pair of coupled ordinary differential equations for \(\ell\) by a method which was coordinate invariant. It may be argued that we have really introduced coordinates (the coordinate candidates) in all but name, so perhaps we should emphasise that our method is coordinate invariant in the sense that there is no background coordinate metric imposed, and that we do not have to adopt the coordinate candidates as our final coordinates. We emphasise again that it is structurally imperative that we apply the commutators to (the equivalent of) four functionally independent real scalars to ensure that we have a complete system of equations. Therefore, our prime concern was simply to establish a complete system and we have not been thinking of the suitability of these quantities in their optional additional role as coordinates; although of course we are always concerned with getting the complete system of equations in a reasonably concise and manageable form. Hence whether our coordinate candidates are the coordinates which give separation, decoupling and reduced order is of secondary importance, at this stage; it is of course preferable if it happens, as it makes subsequent work shorter.

Having obtained explicitly a complete system we then decided to tentatively adopt the coordinate candidates as our coordinates and to explore their usefulness. We have seen in Section 3.3 that this particular coordinate choice does give separation, and this is a bonus
for us; of course, we also want the equations to decouple and reduce to as low an order as possible. Although we could continue to work in the coordinates chosen in the last section we have pointed out before that we have the freedom to choose coordinates other than the coordinate candidates; therefore, we will now investigate these possibilities. From now on, having learned something of the structure of the residual differential equations still to be solved, motivation for preferring a particular choice will be its usefulness not just in separation, but also in enabling us to decouple the residual equations and to reduce the order of the equations.

4. The master equations, and alternative coordinates to coordinate candidates.

Although allowing the coordinate candidates to become the coordinates, as in the last section, is the most obvious choice, it is not the only one. Of course, in practice, we want to choose as coordinates those four quantities on which each of the four derivative operators yield simple expressions, since these expressions give us the explicit form of the operators and tetrad components in that coordinate system; as well we recognise that we will be left at the end of the operator-integration procedure with a residual set of differential equations still to be solved, and further progress towards their solution will depend on a suitable choice of coordinates. However, the particular choice made in the last section enabled us to obtain separation of the coordinates and reduce the problem to ordinary differential equations in the fourth coordinate $u$. Clearly we would wish any other coordinate choice to give us the same separation properties; so we will retain the first three coordinate candidates as coordinates, but postpone the explicit choice of the fourth coordinate. To make a skillful and informed choice of the fourth coordinate we need more understanding of the precise structure of the residual differential equations. In this section we will first get a better picture of this structure, which will motivate the eventual explicit choice for the fourth coordinate; so we will now continue on directly from the end of subsection 3.2 — where we had reduced the problem to solving the residual pair of Ricci equations (3.29), subject to the inequalities (3.31) — and develop further the equations (3.29) in the GHP operator notation.

4.1. The master equations and redundancy.

We have noted in Section 3.2 the crucial result that $\ell$ and $u$ are functionally dependent. Further, when we compare the differential equations (3.45) for $\ell$ with those for $u$, (3.37b, 3.46)
we note that
\[ \nabla_i \ell = \mathcal{D}(\ell) \nabla_i u \] (4.1)
where
\[ \mathcal{D}(\ell) = (\bar{\zeta} + \zeta)^{1+m} \frac{\tilde{\mathcal{P}}'}{|P_o|} \] (4.2)
so \( \ell \) and \( \mathcal{D}(\ell) \) are functionally dependent on \( u \) and hence on each other. The operator
\[ \mathcal{D} = \frac{(\bar{\zeta} + \zeta)^{1+m}}{|P_o|} \tilde{\mathcal{P}}' \] (4.3)
is simply a zero-weighted operator formed by scaling the (1,1)-weighted operator \( \tilde{\mathcal{P}}' \); this new operator also has the property that when it operates on a zero-weighted function of \( u \), it yields another zero-weighted function of \( u \).

We now wish to write out explicitly the two residual Ricci equations (3.29) in this notation, and so we first rewrite (3.40) and (3.47) as
\[ \frac{\partial P_o}{P_o^2} = 2(\mathcal{D} \ell + m + 1)(\zeta + \bar{\zeta})^{-1} \] (4.4)
\[ 2\Sigma^o = i(\ell D\bar{\ell} - \bar{\ell} D\ell + \ell - \bar{\ell})(\zeta + \bar{\zeta})^{m-1} \] (4.5)
The two residual Ricci and Bianchi equations now become
\[ \bar{\ell} D^3 \ell + 3\bar{\ell} D\ell D^2 \ell + \bar{\ell}(2m + 1)D^2 \ell - 2\ell D^2 \ell - 2(\mathcal{D} \ell)^2 - (4m + 2)\mathcal{D} \ell - 2m(m + 1) = 0 \] (4.6)
\[ \left\{ \bar{\ell} D^3 \ell + \bar{\ell} D^2 \ell ((2m + 1)\bar{\ell} + (2m - 3)\ell + 3\bar{\ell} D\ell + \ell D\bar{\ell}) + \bar{\ell} (\mathcal{D} \ell)^2 \bar{\ell} D\ell + (2m - 3)\bar{\ell} (\mathcal{D} \ell)^2 \\
+ (2m + 1)\bar{\ell} D\ell D\bar{\ell} + (4m - 2)\bar{\ell} D\ell - (4m^2 - 4m - 3)\ell D\bar{\ell} - (4m^2 - 4m - 2)\ell \right\} \\
- \left\{ \text{c.c.} \right\} = 0 \] (4.7)
and the inequalities become
\[ \bar{\ell} D\ell - \ell D\bar{\ell} + \bar{\ell} - \ell \neq 0 \] (4.8)
\[ \ell D^3 \ell + 3D^2 \ell D\ell + (2m + 1)D^2 \ell \neq 0 \] (4.9)
These two equations (4.6) and (4.7) are not independent, which can be seen as follows. The terms \( D^3 \ell, D^3 \bar{\ell} \) can be eliminated from the second equation using the first giving
\[ \left\{ \bar{\ell} D D^2 \ell + (2m - 1)D^2 \ell + \bar{\ell}(\mathcal{D} \ell)^2 \bar{\ell} D\ell + (2m - 1)\bar{\ell} (\mathcal{D} \ell)^2 \\
+ (2m + 1)\bar{\ell} D\ell D\bar{\ell} + (4m^2 - 1)\bar{\ell} D\ell - (4m - 2)\ell D\ell - (6m^2 - 2m - 2)\ell \right\} \\
- \left\{ \text{c.c.} \right\} = 0 \] (4.10)
and it is then found that a derivative of this third equation is related to the first by

$$D[(4.10)] = (2m - 1 + D\ell)[(4.6)] - (2m - 1 + D\ell)[(4.6)]$$

(4.11)

In the same way as above, inequality (4.9) can be simplified to

$$\ell D^2\ell + (D\ell)^2 + (2m + 1)D\ell + m(m + 1) \neq 0$$

(4.12)

The equations (4.6,10) and inequalities (4.8,12) for \(\ell\) carry essentially the same information as the equations (111,112) for \(\ell\) given in [1], but the correspondence is not so obvious; for instance we note the additional explicit unknown function \(f\) given in [1], and point out that in this paper there is also an additional unknown function, namely \(W\), which is implicitly built into the operator \(D\).

There are different possible ways to exploit the redundancy noted above. From one point of view, it means that we have essentially only two independent real equations, e.g. (4.10) and the real part (4.6); from another point of view we can think of the complex equation (4.6) as the main equation with (4.10) as a supplementary equation, which is essentially a special first integral of (a part of) (4.6). We could, at this stage, simply write \(D = W \frac{d}{dv}\) with \(v\) as coordinate, and attempt to decouple and simplify the two equations, in coordinate form, making use of the redundancy and coordinate freedom; we shall show in a subsequent paper how this can be done, continuing on from the equations (111,112) for \(\ell\) given in [1].

But in this paper we shall prefer to exploit the redundancy and simplify the equations (4.6,10) further, yet remain within the operator formalism; we shall then show that the final coordinate choice and decoupling follows in a very natural manner.

We shall consider the pair of equations (4.6,10) as our ‘master equations’, and will simplify them in different ways in the remainder of this paper, as well as pointing out how other approaches from these equations will enable us to retrieve the results of earlier workers. The equations and inequalities now have no terms which are functions of the three coordinates \(R, \zeta, \bar{\zeta}\), being functions only of the coordinate candidate \(u\). This of course means that when we introduce explicitly our fourth coordinate we will be dealing with a pair of ordinary differential equations in one real variable.

### 4.2. The master equations rewritten as a complex chain of first order equations.

We will now rewrite (4.6) and (4.10) in a more concise and manageable form. Defining

$$\lambda = \ell(D\ell + 2m - 1)$$

$$\Lambda = D\lambda + \frac{2\lambda}{\ell} + (m - 1)(m - 2)$$

(4.13)
we find that the differential equation (4.6) can be written in the simple form

\[ \mathcal{D}\Lambda = \frac{2\Lambda}{\bar{\ell}} \]  

(4.14)

while the differential equation (4.10) yields simply

\[ \lambda\bar{\Lambda} - (m - 1)(m - 2)(2\ell + \lambda) = \bar{\lambda}\Lambda - (m - 1)(m - 2)(2\bar{\ell} + \bar{\lambda}) \]  

(4.15)

(We can divide by \( \ell \) since a non-zero \( \ell \) is guaranteed by the inequalities.)

The inequality (4.8) remains in the same form, while the inequality (4.9) becomes very simply

\[ \mathcal{D}\Lambda \neq 0 \]  

(4.16)

So an equivalent presentation of these equations is,

\[ \mathcal{D}\Lambda = \frac{2\Lambda}{\bar{\ell}} \]
\[ \mathcal{D}\ell = 1 - 2m + \left( B + 2(m - 1)(m - 2)\ell \right)/\ell(\bar{\Lambda} - (m - 1)(m - 2)) \]  

(4.17)
\[ \mathcal{D}B = \Lambda\bar{\Lambda} - (m - 1)(m - 2)(\Lambda + \bar{\Lambda} - m(m + 1)) \]

where \( B = \lambda\bar{\Lambda} - (m - 1)(m - 2)(\lambda + 2\ell) \) is a real function.

(The denominator in the second equation is non-zero, since \( \Lambda \) cannot be constant, because of (4.16); nor can \( \ell \) be zero, as we have noted above.)

At first sight this set of first order (operator) differential equations — under the obvious substitutions of \( \ell \) from the first into the second, and \( B \) from the resulting equation into the third — appears to reduce to a real differential operator equation of third order for the complex function \( \Lambda \); however, a complication is that since \( B \) on the right hand side of the second equation is real, this equation also carries the implicit information

\[ \ell\bar{\Lambda}(\mathcal{D}\ell - 1 + 2m) - (m - 1)(m - 2)(\mathcal{D}\ell + 1 + 2m) \]
\[ = \bar{\ell}\Lambda(\mathcal{D}\bar{\ell} - 1 + 2m) - (m - 1)(m - 2)(\mathcal{D}\bar{\ell} + 1 + 2m) \]  

(4.18)

which complicates the deceptively simple structure in (4.17). Of course we also still need to introduce the fourth coordinate explicitly through the operator \( \mathcal{D} \), which will introduce another function; ultimately we will need to decouple the real and imaginary parts of \( \Lambda \).

We note that we have exploited the redundancy (4.11) to, in effect, replace the 7 real equations (4.13,14,15) in the 6 real unknowns \( \ell, \lambda, \Lambda \) by the 5 real equations (4.17) in the 5 real unknowns \( \ell, B, \Lambda \); in particular, it is emphasised that the introduction of the real function \( B \) means that the set of five equations (4.17) has no redundancy.
4.3. Introducing different coordinates to achieve decoupling.

The above set of equations reveal a structure built around the complex function $\Lambda$ which we know depends on only one variable. In Section 3 we chose the coordinate candidate $u$ as coordinate, and we could of course do the same for the alternative set of equations (4.17). In that coordinate the operator $D = \frac{d}{du}$, and substitution into (4.17) will lead to two real differential equations — of order three and two respectively — for the complex function $\Lambda$; although these are of a simpler structure than their counterparts obtained in Section 3, the difficulty of obtaining decoupling — after the separation of $\Lambda$ into real and imaginary parts — still remains.

However, we shall illustrate two approaches respectively in the next two sections, whereby we can obtain decoupling in a natural way.

With our additional freedom of choice for the fourth coordinate, there is the possibility — suggested both by the structure of the equations (4.6,10) and the absence of $u$ explicitly in these equations — to choose the fourth coordinate in terms of $\Lambda$. It would of course be very attractive if we could choose $\Lambda$ itself as the coordinate; unfortunately $\Lambda$ is a complex function of the real variable $u$ and so $\Lambda$ itself cannot play the role of the fourth coordinate; however, some real combination formed from $\Lambda$ (e.g. $\Re(\Lambda)$, $\arg(\Lambda)$, ...) would be a possible choice for the fourth coordinate. With such a choice the decoupling problem for complex $\Lambda$ disappears, since one part of $\Lambda$ is now the coordinate, and the other part is the real dependent unknown; there is also a reduction in order. (It might appear that we would also be left with a supplementary first order equation to solve for $u$ as a function of $\Lambda$, at the end of the analysis. In fact, as noted above, $u$ actually does not occur explicitly in any of the spin coefficients or Killing vector components and so it would not even be necessary to calculate $u$ explicitly.)

On the other hand, if we could rearrange our equations (4.17) into a closed chain of real equations in real functions then we could choose as our fourth coordinate the real function at the top of the chain.

5. A class of real third order differential equations for all values of $m$.

Let us label our fourth coordinate, $v$, so that we can write

$$D = W \frac{d}{dv} \quad (5.1)$$
where $W(= \frac{dv}{du})$ is a function of $v$. The second of the three equations in (4.17), with the first substituted, becomes

$$4\bar{\Lambda}^2 \left( \bar{\Lambda} - (m-1)(m-2) \right) \left( \ddot{\Lambda} + (\ln W)_v \dot{\Lambda} \right) + \dot{\Lambda}^2 \left( WB\dot{\Lambda} + 2(m-1)(m-2)(2m+3)\bar{\Lambda} - 2(1+2m)\bar{\Lambda}^2 \right) = 0$$

(5.2)

while the third can be written as

$$(WB)_v - WB(\ln W)_v = \Lambda\bar{\Lambda} - (m-1)(m-2)(\bar{\Lambda} + \Lambda - m(m+1))$$

(5.3)

where $\cdot$ denotes differentiation with respect to $v$.

From (5.2) we obtain

$$(\ln W)_v = \dot{\Lambda}^3 \beta + \dot{\Lambda}^3 \bar{\beta}$$

$$WB = -4(\bar{\Lambda} - (m-1)(m-2))\bar{\Lambda}^2 \dot{\Lambda} \beta - 4(\Lambda - (m-1)(m-2))\Lambda^2 \dot{\Lambda} \bar{\beta}$$

(5.4)

where

$$\beta = \left( 2\Lambda^3 \bar{\Lambda} - (m-1)(m-2)(2\Lambda^2 \bar{\Lambda} - (2m+3)\Lambda \bar{\Lambda}^2) - (1+2m)\Lambda^2 \bar{\Lambda}^2 \right)$$

$$/2\dot{\Lambda} \bar{\Lambda} \left( (\bar{\Lambda} - (m-1)(m-2))\Lambda^2 \dot{\Lambda}^2 - (\Lambda - (m-1)(m-2))\Lambda^2 \bar{\Lambda}^2 \right)$$

(5.5)

(We note that we can assume the denominator of $\beta$ is non-zero; otherwise we obtain a flat space solution — with inequality (4.16) violated — or the Hauser solution [19,20,21].)

Substituting the two equations (5.4) into (5.3) gives a real third order equation for the complex function $\Lambda$ of the real coordinate $v$,

$$\left( \Lambda - (m-1)(m-2) \right) \left( 4\Lambda^2 \bar{\Lambda} (\dot{\beta} - \dot{\Lambda}^3 \beta^2 - \dot{\Lambda}^3 \beta \bar{\beta}) + 2(2m+7)\Lambda \Lambda^2 \bar{\beta} \right)$$

$$+ \left( \bar{\Lambda} - (m-1)(m-2) \right) \left( 4\bar{\Lambda}^2 \bar{\Lambda} (\dot{\beta} - \dot{\Lambda}^3 \beta^2 - \dot{\Lambda}^3 \beta \bar{\beta}) + 2(2m+7)\bar{\Lambda} \bar{\Lambda}^2 \beta \right)$$

$$= -\Lambda\bar{\Lambda} + (m-1)(m-2)(\bar{\Lambda} + \Lambda - m(m+1))$$

(5.6)

But we have still the freedom to choose $v$; and providing we choose $v$ as a real function of $\Lambda$, and make an appropriate choice for the dependent variable — essentially choose it as a second, independent, real function of $\ell$ — then we can obtain the resulting equation which is of third order.

As an example, we put

$$\Lambda = v + iX$$

(5.7)
and write out the equation for the special case \( m = 1 \). When we substitute (5.7) into (5.5), for this case, we obtain
\[
\beta = (v + iX)^2 \left( 2i(v + iX)\dot{X} - 3(1 + i\dot{X})^2 \right) / 2(1 + \dot{X}^2) \left( (v - iX)^3(1 + i\dot{X})^2 - (v + iX)^3(1 - i\dot{X})^2 \right)
\]
\[
(5.8)
\]
where \( \dot{\cdot} \) denotes differentiation with respect to \( v \).

When this is in turn substituted into (5.6), which for \( m = 1 \) simplifies to
\[
(v + iX)^2(1 + i\dot{X}) \left( 2(v + iX)(\dot{\beta} - (1 + i\dot{X})^4 \beta^2 - (1 - i\dot{X})^4 \beta) + 9(1 + i\dot{X})\beta \right)
\]
\[
+ (v - iX)^2(1 - i\dot{X}) \left( 2(v - iX)(\dot{\beta} - (1 - i\dot{X})^4 \beta^2 - (1 + i\dot{X})^4 \beta) + 9(1 - i\dot{X})\beta \right)
\]
\[
= -(v^2 + X^2)/2
\]
\[
(5.9)
\]
Clearly there is no decoupling problem, and we have a real third order equation for the real function \( X \) of the real coordinate \( v \). Picking out the third order expression, we can write it explicitly as
\[
(\Lambda \bar{\Lambda} - \bar{\Lambda} \Lambda) = -2i\dot{X}
\]
\[
(5.10)
\]
We emphasise that once we have solved this single equation (5.9) — subject of course to the inequalities (4.8,16) — then the problem is essentially solved; once \( X \) is obtained we can write down \( \Lambda \) from (5.7), obtain \( \ln W \), from (5.4), and \( W \) by integration; hence we can obtain \( \ell \) from the first of (4.17) by substituting for \( W \) and \( \Lambda \).

So, we can then write down our table for the fourth coordinate \( v \),
\[
\hat{P} v = 0 \quad \hat{P}' v = \frac{dv}{du} \hat{P}' u = W|P^o|(\bar{\xi} + \zeta)^{-1-m}
\]
\[
\hat{\partial} v = \frac{dv}{du} \hat{\partial} u = W P^o \ell(\bar{\xi} + \zeta)^{-1}
\]
\[
(5.12)
\]
which gives us, in the coordinate system \( R, \zeta, \bar{\zeta}, v \), the GHP operators when acting on zero-weighted quantities as,
\[
\hat{P} = \hat{P}(R) \partial_R + \hat{P}(\zeta) \partial_\zeta + \hat{P}(\bar{\zeta}) \partial_{\bar{\zeta}} + \hat{P}(v) \partial_v = 2|P^o|^{-1} \partial_R
\]
\[
\hat{P}' = (\rho^o + \bar{\rho}^o)|P^o|^{-1} \partial_R + W|P^o|(\bar{\zeta} + \zeta)^{-1-m} \partial_v
\]
\[
\hat{\partial} = \left( \frac{2i}{|P^o|} (P^o \Sigma^\zeta + W P^o \ell(\bar{\xi} + \zeta)^{-1}\Sigma^\zeta_v) - RP^o(W\dot{\ell} + m + 1)(\zeta + \bar{\zeta})^{-1} \right) \partial_R + P^o \partial_\zeta + WP^o \ell(\bar{\xi} + \zeta)^{-1} \partial_v
\]
\[
(5.13)
\]
\[
\hat{\partial}' = \left( \frac{-2i}{|P^o|} (P^o \Sigma^\zeta + W P^o \ell(\bar{\xi} + \zeta)^{-1}\Sigma^\zeta_v) - RP^o(W\dot{\ell} + m + 1)(\zeta + \bar{\zeta})^{-1} \right) \partial_R + P^o \partial_\zeta + WP^o \ell(\bar{\xi} + \zeta)^{-1} \partial_v
\]
where \( \Sigma^o \) is given by (4.5) and \( \ell, W \) are obtained as described above. The Killing operators remain as in (3.50,51).

Finally, we emphasise that the reduction to a real third order equation of a real function is not only for the particular coordinate choice \( v = \Re(\Lambda) \); we could have chosen as coordinate any real function of \( \Lambda \) (e.g. \( \Im(\Lambda), \arg(\Lambda), |\Lambda|, \ldots \)) and with an appropriate choice of dependent variable, achieved an equation of the third order. So there is a whole class of real third order equations of a real unknown which can easily be found from (5.6). As noted above, the leading term in (5.6) has the form \((\bar{\Lambda} \Lambda - \Lambda \bar{\Lambda})\), and this raises the question whether one of these alternative coordinate choices would enable third order terms to cancel, and yield an equation of second order. For the special coordinate choices just mentioned it is easy to see that such reduction does not occur, and indeed we shall show in a subsequent paper that there is no possible way to choose our coordinate as a real function of \( \ell \) in order to achieve reduction of order. Of course this does not mean that there may not be some other choices of coordinate for which reduction of order can occur.

6. A concise third order equation for the case \( m = 1 \).

In this subsection we reduce the system of complex equations (4.17) to a real system, which form a closed chain of real first order (operator) differential equations. So as not to obscure the technique by details we consider the special case \( m = 1 \). Beginning with the one real equation from (4.17),

\[
\mathcal{D} B = C \tag{6.1}
\]

where

\[
B = \lambda \bar{\Lambda} \tag{6.2}
\]

\[
C = \Lambda \bar{\Lambda}
\]

we use the other equations in (4.16) to obtain the two real equations

\[
\mathcal{D} C = 2CE \tag{6.3}
\]

and

\[
\mathcal{D} E = -\frac{B}{\Lambda \ell^3} - \frac{B}{\Lambda \ell^3} + \frac{1}{\ell^2} + \frac{1}{\ell^2} \tag{6.4}
\]

where

\[
E = \left( \frac{1}{\ell} + \frac{1}{\ell} \right) \tag{6.5}
\]
with $B, C, E \neq 0$.

We could obtain the next equation in the chain by operating with $\mathcal{D}$ on the right hand side of (6.4); but the calculations are shorter if we introduce the complex function $F$ by

$$F = \ell^2 \bar{\Lambda}$$

and note that not only can we write out the right hand side of (6.4) in terms of real $B, C, E$ and complex $F$, but we also have, from (4.17), the very simple and useful result

$$\mathcal{D}F = 2B$$

Since $B$ is real we can put

$$A + ik = F = \ell^2 \bar{\Lambda}$$

where $A$ is a real function, but $k$ is a constant. So we have now obtained the following closed chain of real first order equations of real functions:

$$\mathcal{D}A = 2B$$
$$\mathcal{D}B = C$$
$$\mathcal{D}C = 2CE$$
$$\mathcal{D}E = -\frac{B}{A^2 + k^2} \left( AE \pm k \sqrt{E^2 - 4 \frac{C}{A^2 + k^2}} \right) + E^2 - 2\sqrt{C} \right)$$

Although we now have four real differential equations compared to the (equivalent of) five real differential equations in (4.17) we have not lost any information; the missing differential equation is simply $\mathcal{D}k = 0$; and by using only real functions all the information is explicit within the chain. Substituting the first three equations of (6.9) into the fourth one will clearly give a fourth order operator equation; however, an appropriate choice of coordinate will reduce the corresponding coordinate equation to third order. Let us choose $A$ as our fourth coordinate so that from the first equation the operator $\mathcal{D}$ is given by

$$\mathcal{D} = 2B \frac{d}{dA}$$

and substituting from the first three equations of (6.9) in the fourth equation gives a real third order differential equation for $B$, a real function of $A$, which can be presented in the comparatively concise form,

$$2\dot{S}S\ddot{S} + \dot{S}^2 \ddot{S} - 3S\ddot{S}^2 = -\frac{\dot{S}}{(A^2 + k^2)} \left( A\dot{S}S \pm k \sqrt{S^2 \ddot{S}^2 - 4\dot{S} \ddot{S}^2 \sqrt{\frac{\dot{S}}{A^2 + k^2}}} \right) - 2\ddot{S}^2 \sqrt{\frac{\dot{S}}{A^2 + k^2}}$$

(6.11)
where \( \dot{\cdot} \) denotes differentiation with respect to \( A \) and

\[
S = B^2
\]  
(6.12)

The constant parameter \( k \) can be absorbed by the following relabelling

\[
\tilde{A} = A/k, \quad \tilde{S} = S/k
\]  
(6.13)

giving

\[
2\ddot{\tilde{S}}\dddot{\tilde{S}} + \tilde{S}^2\dddot{\tilde{S}} - 3\ddot{\tilde{S}}^2 = -\frac{\dot{\tilde{S}}}{(\tilde{A}^2 + 1)}\left(\tilde{A}\dddot{\tilde{S}} + \sqrt{\dddot{\tilde{S}}^2\dddot{\tilde{S}} - 4\dddot{\tilde{S}}^2\sqrt{\frac{\ddot{\tilde{S}}}{\tilde{A}^2 + 1}}}\right) - 2\dddot{\tilde{S}}^2\sqrt{\frac{\dddot{\tilde{S}}}{\tilde{A}^2 + 1}}
\]  
(6.14)

where \( \dot{\cdot} \) now denotes differentiation with respect to the new coordinate \( \tilde{A} \).

It is clear that once \( \tilde{S} \) is obtained the problem is essentially solved: \( C, E \) can be obtained from (6.9) by differentiation, and can be combined algebraically to give \( \ell \),

\[
\frac{2\sqrt{k}}{\ell} = \frac{\sqrt{\dddot{\tilde{S}}}}{\dddot{\tilde{S}}^2} + \sqrt{\dddot{\tilde{S}}^2\dddot{\tilde{S}} - 4\dddot{\tilde{S}}^2\sqrt{\frac{\ddot{\tilde{S}}}{\tilde{A}^2 + 1}}}
\]  
(6.15)

The table of GHP operators in this coordinate system, \( R, \zeta, \bar{\zeta}, \tilde{A} \) is obtained from (5.10) where \( v \) is replaced by \( A \) and then by \( \tilde{A} \), and \( W \) is replaced by \( B \), and then by \( \tilde{S} \).

We emphasise that the inequalities (4.8,16) must also be satisfied; this constraint prevents us from making the simple choice \( k = 0 \), because it violates the first of these inequalities.

We could have chosen other coordinates e.g. \( B \) or \( C \) or combinations, and in the same manner obtained alternative third order equations.

Finally, we note the relative simplicity of the version of the final equation obtained here as compared to the version, for \( m = 1 \), in the last section.

7. Summary.

The original purpose of this paper was not primarily to try and make significant new progress in the NT problem; it was rather to set out in detail new insights into the GHP operator-integration approach — whose principles are applicable in very general contexts — and then to illustrate these insights by applying them to the NT problem. But, as well as
the NT problem providing an ideal laboratory for the demonstration of our GHP operator-integration approach, we have been able to obtain new insights into the NT problem itself — extending existing results and suggesting new approaches.

The major new insight regarding the GHP operator-integration approach is the important fact that, although spacetimes admitting Killing vectors do not immediately supply — from the spin coefficients and Riemann tensor components — the four functionally independent scalars which integration within the GHP formalism demands, the missing scalars may be supplied in a very simple and natural way — by also using the (tetrad components of the) Killing vectors. The relationship in the GHP formalism between Killing vectors and functionally independent scalars, as well as that between homothetic Killing vectors and separability, are very important topics which we have only touched on here; deeper implications for the GHP formalism will be developed in a separate paper.

An additional insight regarding the GHP operator-integration approach is that, even after we have reduced the problem to the residual ordinary differential operator equations, there are advantages to remaining within an operator formalism rather than immediately using coordinates explicitly. We have shown that, by constructing a closed chain of real first order ordinary differential operator equations, we can avoid the difficulties associated with decoupling, and can make the final coordinate choice in an efficient manner.

We chose the $G_2$ case of the NT problem as our illustration because of its non-trivial nature and because of the wide range of mathematical procedures necessary for its simplification. The intention was that the advantages and strengths of our method be thoroughly illustrated and tested. Much work has already gone into this particular problem, and many insights have already been gained — although often in a rather narrow manner specific to a particular formalism and coordinate system. In this paper, we have been able to understand the procedures of separability, redundancy, decoupling and reduction of order in a very general manner, in the context of the GHP formalism; we have also seen how it is possible to retain our coordinate choice to the very last step when it can be used to simplify the final equations in the most advantageous way. This we believe is the strength of this GHP operator approach; separability, redundancy even decoupling can be exploited, still retaining some coordinate freedom, and then at the very last stage when everything has been reduced to the decoupled residual ordinary differential operator equations this freedom can be exploited in such a manner as to present the final equations in the most reduced and/or manageable form.
Other workers, [15,16,17,18] who have reduced this problem to third order real differential equations have used very different formalisms, and special techniques, and there is no simple relationship between them. However, it is possible to obtain these very different equations from our master equations:

(i) The form of the residual real third order ordinary differential equation for \( h(\varphi) \) obtained by Ludwig and Yu [17] follows in a very concise manner from our master equations (4.6,10), by making the special choice of coordinate \( \varphi = \arg(\ell) \) and of independent variable \( h = \frac{d\varphi}{du}|\ell| \). We note, that unlike in the approach in Section 5, the particular coordinate choice is crucial; most other coordinate choices built on \( \ell \) give fourth order equations.

(ii) For arbitrary values of the parameter \( m \), Herlt [16] has first of all reduced the problem essentially to a real third order ordinary differential equation for a complex function \( g \), which he then transforms to a first order integro-differential equation for the real function \( \varphi(u) \) by defining the coordinate \( u \) by

\[
g = g, u e^{i\varphi(u)} \tag{7.1}
\]

Finally, he is able to obtain a real third order ordinary differential equation for a real function \( \chi(\varphi) \) defined by

\[
\chi(\varphi) = \int \sin \varphi \, du - 2\varphi(u) \tag{7.2}
\]

It is easy to see that Herlt’s function \( g \) corresponds essentially to our function \( \lambda \), and so we can retrieve his result by first rearranging our master equations (4.6,10), and then following his procedure. However, it is also possible to obtain other third order equations by constructing different coordinate choices around \( \lambda \) (equivalently Herlt’s \( g \)) than the choice (7.1).

For the special case \( m = 1 \) (corresponding to \( N=2 \), in the notation of [16]) Herlt has obtained a particularly concise form for the residual real third order differential equation, just as we have also, for this special case — in Section 6. Although there are obvious similarities in structure, some manipulation still needs to be carried out to show the direct equivalence of the two equations.

(iii) Chinea [15] — using an elegant approach with matrix valued differential forms — has reduced the NT problem with two commuting Killing vectors (i.e. \( m = 0 \)) to a single complex second order differential equation for a complex function. If we choose \( B \) as our coordinate, then after substituting the first and third into the second of equations (4.17) we
obtain a complex second order differential equation for arbitrary values of the parameter \( m \); for the special case \( m = 0 \) this equation is

\[
(\Lambda \ddot{\Lambda} - 2\Lambda - 2\dot{\Lambda})(2\dddot{\Lambda} + \dot{\Lambda}^2) = 2\dot{\Lambda}(2\dddot{\Lambda} - 2\dddot{\Lambda} + \dot{\Lambda}^2\dot{\Lambda}) - B(\Lambda \ddot{\Lambda} - 2\Lambda - 2\dot{\Lambda})^2/2\dot{\Lambda}^2
\]  

(7.3)

where \( \dot{\cdot} \) means differentiation with respect to \( B \). This equation is similar in structure to, but is not yet as neat as that of Chinea [15]. However, for non-commuting Killing vectors the two special cases \( m = 1, 2 \) have the remarkably simple forms, given respectively by

\[
\Lambda \dddot{\Lambda} + \dot{\Lambda} \ddot{\Lambda} = (2\dddot{\Lambda} - B\dot{\Lambda}\dot{\Lambda})\dot{\Lambda}\ddot{\Lambda}^2/4\dot{\Lambda}^2
\]  

(7.4)

\[
\Lambda \dddot{\Lambda} + \dot{\Lambda} \ddot{\Lambda} = (6\dddot{\Lambda} - B\dot{\Lambda}\dot{\Lambda})\dot{\Lambda}\ddot{\Lambda}^2/4\dot{\Lambda}^2
\]  

(7.5)

Chinea was able to transform his complex second order equation, by a number of coordinate changes, to a very complicated real third order ordinary differential equation for a real function of a real variable. We could do the same for the equations above (and also for the equation for arbitrary \( m \)), but the resultant equations seem more complicated than the versions obtained in the previous sections.

(iv) Finley et al. [18], beginning from the point of view of groups of point transformations, also reduce the problem to one real third order ordinary differential equation for a real function of a real variable \( q(y) \). They also define a function \( \Psi(y) \), which is second order in \( q \) and point out that the complicated third order equation can be rewritten as a comparatively simple, but interesting, first order expression for the real function \( \Psi(y) \) — spoiled only by one additional term explicit in \( q(y) \). McIntosh [23] has also found a similar interesting first order expression, again, unfortunately with one additional complicated term. As Finley et al. point out, these first order expressions have suggestive symmetries, which may lead to further significant simplifications. Essentially the idea being followed is to write the very complicated third order equation as a coupled pair of equations; in particular one of these equations should be simple enough to enable solutions for it to be found, yet complicated enough that the solutions are not trivial, and also that the second equation in the coupled pair should be significantly simplified.

In fact, our versions of the master equations in both (4.17) and (6.9) are in a form which enables us to experiment in precisely this manner. For instance, let us choose \( A \) as coordinate and rewrite the last equation in the set (6.9) as

\[
2X_{,A} - X^2 + \frac{AX}{A^2 + k^2} = Y
\]  

(7.6)
where we have relabelled $X = E/B$, and where $Y$ is given from (6.9) by

$$Y = \frac{XC}{2B^2} \pm \frac{k}{A^2 + k^2} \sqrt{X^2 - \frac{4}{B^2} \sqrt{\frac{C}{A^2 + k^2}} - \frac{2}{B^2} \sqrt{\frac{C}{A^2 + k^2}}} \quad (7.7)$$

From this equation we can easily find an expression for $X$ in terms of $B, C, Y$ and the coordinate $A$. If we differentiate this expression twice we get two new expressions respectively in $Y, A, Y, B, C,$ and $Y, A, Y, B, C$; by eliminating $B, C$ between these three expressions we are left with an expression for $X$ in terms of $Y, Y, A, Y$ and the coordinate $A$ i.e. an expression for $X$ which is second order in $Y$.

The structure of the left hand side of (7.6) is very similar to the interesting first order equations obtained respectively by Finley et al. and McIntosh.

(v) Finally we point out that the general third order equation which we obtained in Section 5 can also be obtained using the coordinate method of our earlier paper, [1]. By a suitable coordinate transformation, and appropriate combination of equations (111) in [1], we can obtain [5,6], after a lengthy calculation.

We shall show in a further paper the full details of how all of the existing equations, together with some simpler new ones, can be obtained from our master equations.

It is clear that there are many versions of a final third order equation, and so it is certainly possible that more manageable ones exist; there is even the possibility that a second order equation may still be found. We believe that the approach in this paper, which has the power to give us a unifying picture of such different approaches, supplies us with very powerful equipment for this ongoing task.

APPENDIX: SIMPLIFICATION OF GHP EQUATIONS IN INTRINSIC TETRADS.

A.1. Spacetimes admitting a Killing Vector.

We emphasise that in [12] the system of equations given was for the presence of a homothetic Killing vector, while in [8,9] only the presence of a Killing vector is considered.

In this subsection we are only considering Killing vectors, so when we refer to the equations in [12] we understand that we have made the substitution $\phi = 0$ which reduces the conformal Killing equations to the Killing equations.

In [8,9] the Killing equations consist of the seven complex and the two real equations (A.8a,b,c,9a,b,10a,b,13,14) while in [12] the Killing equations consist of the eight complex
and four real equations ((21), (21)', (21)*, (21)'*, (22), (22)', (22)*, (22)'*, (23), (23)',
(23)*, (23)'*).

The discrepancy in the number of equations is easily explained when we realize that (A13) in
[8,9] has simply been split in [12] into the two equations (22)' and (22) by the introduction
of an arbitrary real quantity \( P \); similarly (A14) in [8,9] has simply been split in [12] into
the two equations (22)* and (22)'* by the introduction of an arbitrary real quantity \( P^* \).

Finally in [12], (21)'* is simply the complex conjugate of (21)*; in [8,9] only (A.8b), the
counterpart to (21)* in [12], is given explicitly.

But there is a second apparent discrepancy. In [12] there occurs in (23) and (23)*((23)' and (23)*) an arbitrary complex quantity \( Q \); no such arbitrary quantities occur in
the counterpart equations (A9a,b) and (A10a,b) in [8,9]. However, it is easy to see that
when we apply the vacuum Type N restrictions with the particular tetrad gauge used in
[12] to the complete system of equations in [12], that it follows that

\[
Q = 0 = Q'
\]  

(AI.1)

(In fact, in [12] this calculation was carried out for the more general system — admitting
a homothetic Killing vector — resulting in (AI.1).

Therefore, the apparent discrepancy between the two systems of equations in [12] and [8,9]
respectively, in the presence of a Killing vector is resolved — for the particular class of
spacetimes under consideration in this paper.

However, we know from Held’s argument that such simplifications can be made in much
more general circumstances. This is a very fundamental property for spacetimes admitting
homothetic Killing vectors, and has very important consequences for the GHP formalism.
We will deal with this topic elsewhere; here we will just illustrate other more general cases
where this simplification can be made.

(i) For instance, suppose we choose \( l^i \) along the principal null direction of the Weyl tensor,
so that

\[
\Psi_0 = 0
\]

Then we can deduce that

\[
Q = 0
\]

provided that at least one of the \( \Psi_n \)'s (\( n = 0, ..., 4 \)) is nonzero, by an iterative argument
using in order (43), (44), (45), (44)', from [12]. (See [22] for a related type of discussion.)

Similarly we can choose $n^i$ along the principal null direction of the Weyl tensor, so that

$$\Psi_4 = 0$$

and this is compatible with

$$Q' = 0$$

(We note that the choices $\Psi_3 = 0 = \Psi_1$ also lead to the same result.)

(ii) For algebraically special spaces if we choose $l^i$ such that

$$\Psi_0 = 0 = \Psi_1$$

then, as in (i),

$$Q = 0$$

We can use the null rotation about $l^i$ to put

$$Q' = 0$$

In an analogous manner we could fix $l^i$ with respect to the Ricci tensor etc., [22].

A.2. Spacetimes admitting a Homothetic Killing Vector.

We now consider the set of equations in [12] in the presence of a homothetic Killing vector, and as noted above, it has been shown in [12] that (AI.1) follows when we apply the vacuum Type N restrictions with the particular tetrad gauge used in [12]. Therefore, for this special case, we can combine the 12 equations in [12] into the following 9 equations,

\[
\begin{align*}
\tilde{\mathbf{P}}' a &= -\kappa' \bar{c} - \bar{\kappa}' \bar{c} \\
\bar{\rho} \tilde{\mathbf{P}} a &= -\kappa' b + (\rho' - \rho') \bar{c} \\
\tilde{\mathbf{P}} b &= 0 \\
\bar{\rho} \tilde{\mathbf{P}} b &= (\rho - \bar{\rho}) \bar{c} \\
\tilde{\mathbf{P}} c &= -\bar{\rho} c \\
\tilde{\mathbf{P}}' c &= -\rho' c \\
\rho \tilde{\mathbf{P}}' c &= 0 \\
\tilde{\mathbf{P}}' b + \tilde{\mathbf{P}} a &= \phi \\
\rho \tilde{\mathbf{P}}'\bar{c} + \bar{\rho} \tilde{\mathbf{P}}c &= -\phi - (\rho' + \bar{\rho}') b - (\rho + \bar{\rho}) a
\end{align*}
\]  

(AI.2)
These equations are just the homothetic Killing vector generalisations of the analogous Killing vector equations (3.1)–(3.9) in [8], specialised to vacuum Type N spacetimes. We could have derived these equations for much more general classes of spacetimes, using a generalisation of the geometric argument used by Held in [8]; however, this would involve us in deeper questions than we wish to consider here, and the above equations are sufficient for our purposes.

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