THE ROLE OF SYMMETRY AND DISSIPATION IN BIOLOCOMOTION

JAAP ELDERING AND HENRY O. JACOBS

ABSTRACT. In this paper we illustrate the potential role which relative limit cycles may play in biolocomotion. We do this by describing, in great detail, an elementary example of reduction of a lightly dissipative system modelling crawling-type locomotion. The symmetry group \((\mathbb{R})\) is the set of translations along a one-dimensional ground. Given a time-periodic perturbation, the system will admit a relative limit cycle whereupon each period is related to the previous by a shift along the ground. Generalization to a two-dimensional ground is described later in the paper with respect to the symmetry group \(SE(2)\). In this case the resulting limit cycles allow the body to turn and translate by a fixed angle with each period of the perturbation. These toy models identify how symmetry reduction and dissipation can conspire to create robust behavior in crawling, and possibly walking, locomotion.

1. INTRODUCTION

Traditionally, the role of symmetry in understanding biolocomotion has been studied in the extreme regimes where viscosity dominates or the system is purely conservative \([25, 27, 22, 23, 24, 19, 18, 16]\). This paper will expand some of these ideas to regimes in the middle, and explore how dissipation and symmetry can conspire to create biolocomotion through limit cycles. The notion of limit cycles is important in biolocomotion because simple periodic behavior is a defining characteristic of walking, running, swimming, flapping flight,... In order to construct realistic mathematical models that exhibit limit cycles, it is helpful to first identify some core mechanisms of limit cycle production. In this paper we present a simple phenomenological model for crawling-type locomotion. We combine techniques from geometric mechanics, hyperbolic stability and singular perturbation theory. Through this analysis, one can see how these techniques can be generalized to more sophisticated and realistic models.

1.1. Outline of the paper. We start with an overview of the background and motivation. Then, in section 3 we introduce our model of an (unactuated) crawler: a mass-spring system resting on the ground. We regularize the no-slip and no-penetration conditions imposed by the ground by ‘smearing them out’ over a small region around \(z = 0\) to smooth viscous friction and potential energies. What
results from this regularization is a constant dimension, lightly dissipative Lagrangian system. Having described the problem as an ODE on a space of constant dimension, we apply reduction and smooth dynamical systems theory in section 4. We observe that the system is invariant under isometric transformations along the ground and implement a reduction by symmetry of the model. Under mild regularity conditions (Assumptions 5) on the rest lengths of the springs and the spring and ground stiffness, we can use singular perturbation theory to find a robustly stable equilibrium for the reduced model (detailed proofs can be found in the Appendix). Next, under small time-periodic forcing (i.e. actuation of the crawler) this equilibrium persists as a limit cycle in the symmetry reduced model. The limit cycle in the reduced space corresponds to a relative limit cycle in the original phase space, (see Figure 1). A phase reconstruction formula gives the phase shift of the lifted, relative limit cycle; this phase shift corresponds to a stride length. Both relative periodicity and stability are characteristics of biolocomotion, and so we can consider the relatively periodic orbit as a model of crawling in this sense.

These results are summarized by the following theorem.

**Theorem 1** (main theorem). Let a simple crawler model be described by the Lagrange–d’Alembert equations (8). Under Assumptions 5 the symmetry reduced system has a stable rest state. For sufficiently small time-periodic forcings, this rest state persists as a stable limit cycle, which corresponds to a relative limit cycle in the unreduced system that models crawling.

It turns out that the phase shift of the relative limit cycle depends on the magnitude of the perturbation size to second order.

In section 5 we illustrate the theory with numerical experiments. We conclude with some remarks that the techniques are robust to small symmetry breaking, such as a non-flat ground or nearly periodic forcing and an outlook of how these ideas may be applied to other models of biolocomotion.

Most of the paper will be devoted to the case of a mass spring system in $\mathbb{R}^2$, where the ground is modeled by a one-dimensional space. The group symmetry is then $(\mathbb{R}, +)$. This setting allows a simple and explicit presentation of the ideas.
The mathematics for the three-dimensional case is virtually identical. In three dimensions the ground is given by a plane, and the symmetry is given by the set of rotations and translations in this plane. An illustration of the 3D setup is given in Figure 2, the details of this generalization can be found in section 6.

![Figure 2. An illustration of a 3D crawler moving on the plane.](image)

### 2. Background & motivation

Biomechanics in general requires knowledge from a range of fields. This particular paper draws upon previous research in geometric mechanics, stability theory, as well as inspiration from experimental and numerical observations.

#### 2.1. Geometric mechanics and locomotion

There is a long history of using geometric mechanics to study locomotion. Purcell’s three link swimmer [25] inspired Alfred Shapere and Frank Wilczek to interpret locomotion in Stokes flow as phase shift due to the curvature of a principal connection [27]. The simplicity of this perspective has proven useful in other dissipation dominated systems such as granular media [13]. More importantly, it was later found that a range of examples of locomotion fit within this geometric framework [23, 19]. In particular, many conservative systems could be analyzed in this way [22, 18, 16, 24].

Despite the success of the gauge theoretic picture of locomotion, the vast majority of examples of this perspective (perhaps all the examples, to the best of the authors’ knowledge) concern systems which are either conservative (i.e. Hamiltonian or Lagrangian), or friction dominated (i.e. where Newton’s law, $\ddot{q} \propto F$, is replaced by $\dot{q} \propto F$). There appear to be very few examples which invoke the gauge theoretic perspective of [27] in a regime which exhibits a mixture of viscous and inertial forces. Perhaps one reason for this is that the gauge theoretic picture does not translate without some alterations. In the case of mixed viscous and inertial effects, the use of a mechanical connection becomes non-physical, and one must
consider alternative routes to understanding phase-shifts. Nonetheless a similar framework to understanding locomotion is possible, and this paper pursues one such route.

2.2. **Contact problems.** The regularization we are going to pursue is in contrast to the hybrid systems approach, where transitions between different types of phase spaces are given by various transition maps. The hybrid systems approach expresses the non-constant nature of the dimension in contact problems explicitly, and has yielded a number of insights and useful models. For example, a hybrid systems formulation was introduced by McGeer [21], where the transition maps led to a dimension reduction; it was suggested that a limit cycle was approached passively. Since the work of [21], the notion of walking as a limit cycle has become more common, and more sophisticated analyses have lent further support to this idea [8, 9]. The most compelling arguments are the original videos of McGeer which accompany [21].

2.3. **Experimental observations.** On the biological side, ‘central pattern generators’ (CPGs) have been hypothesized as fundamental neural mechanism used in biolocomotion [11]. These CPGs are non-localized collections of neurons which produce rhythmic activity, and respond to various inputs which modulate these rhythms. Therefore the link between CPGs and limit cycle biolocomotion is one which links periodic activation of the controls to periodic motion of the body. This link is used in the creation of simple models which can be feasibly analyzed (see for example [9]).

2.4. **Theoretical significance.** Lastly, viewing biolocomotion as a limit cycle allows for great reductions in complexity. In particular, under weak assumptions, the existence of limit cycles in hybrid systems implies the existence of a reduced order model for the system as a whole. The recent paper [2] dealt with the singular nature of hybrid systems by using relatively weak assumptions on the transition maps to obtain regular dynamics on a subset which spans smoothly across the transition regions. Our paper can be seen as a *dual approach* to [2] in that we regularize the transitions maps themselves, using viscous friction forces [1, 17] and a smooth potential [26, 28]. A primary advantage of our approach is that we may invoke the more developed theory of smooth dynamical systems to prove the existence of a relative limit cycle. A similar regularization of ground contact is used in [30], which studies robustness and efficiency of a simple passive dynamic walking model actuated by CPGs.

Secondly, the ‘relative’ part of ‘relative limit cycle’ is not explicitly acknowledged in much of the biolocomotion literature. A notable exception is [10], where the conserved Noetherian momentum associated to an $S^1$ symmetry was used to create turning trajectories. We will be exploring a different, but related, application of symmetry reduction. In particular, dissipation prevents us from invoking Noether’s
theorem, while the conservation of momenta is not a crucial ingredient for creating relative limit cycles.

2.5. Approach. We will find that our model is invariant under $x$-translations. This symmetry permits us to do Lagrangian reduction \[3\]. The reduced system exhibits a stable fixed point which traces out a (trivial) limit cycle in time-augmented phase space. Because limit cycles are an instance of normally hyperbolic invariant manifolds, the persistence theorem \[7, 14\] permits us to assert the continued existence of diffeomorphic stable limit cycles when the system is disturbed by sufficiently small time-periodic oscillations. The resulting deformed limit cycle in the reduced phase space can be lifted to the unreduced phase space, wherein each period is related to the previous period by a constant shift in the $x$-direction. This construction can be seen as a terrestrial counterpart to research done on swimming wherein the phase space for Navier-Stokes fluid structure interaction exhibited an SE(3) symmetry which allows us to interpret swimming as a limit cycle on an appropriate quotient space \[15\].

2.6. Acknowledgments. The notion of realizing the no-slip condition as a limit of viscous friction was brought to the attention of H.J. by Dmitry V. Zenkov, while J.E. learned this from Hans Duistermaat. Sam Burden first enlightened H.J. on the role of limit cycles in model reduction for hybrid systems. We also thank Tony Bloch, Justin Seipel, and Ram Vasudevan for helpful conversations during the development of this paper. Finally, the initial stimulus to write this paper was given by Jair Koiller, who has been very supportive of our foray into biomechanics. Both authors were supported by the European Research Council Advanced Grant 267382 FCCA and H.J. also by the NSF grant CCF-1011944.

3. THE MODEL

The model can be broken into two distinct components: the crawler and the environment. The crawler consists of three masses connected by springs while the environment consists of the ground and a gravitational field. We will discuss the model of the crawler in empty space before we elaborate on how to include interactions with the environment.

3.1. A model of a crawler (in a vacuum). The crawler consists of three point particles of unit mass all connected by springs of stiffness $\kappa$, with light viscous damping $\nu$, see Figure 3. We describe the crawler as a Lagrangian mechanical system with additional forces to model the spring damping. The point particles move through $\mathbb{R}^2$ with positions $q_i = (x_i, z_i)$ and velocities $\dot{q}_i = (\dot{x}_i, \dot{z}_i)$ for $i = 1, 2, 3$. Thus the configuration space is $Q = \mathbb{R}^6$ with standard coordinates $q = (q_1, q_2, q_3) \equiv (x_1, z_1, x_2, z_2, x_3, z_3)$.

The kinetic energy is given by $T = \frac{1}{2} \| \dot{q} \|^2$ with the usual Euclidean metric. The potential energy from the springs, $U_{\text{shape}}$, is more easily expressed in other
coordinates: the spring lengths, i.e. the distances between the points \(q_i, q_j\). We therefore introduce three (local) coordinate functions

\[
\ell_k = \|q_i - q_j\| = \sqrt{(x_i - x_j)^2 + (z_i - z_j)^2}
\]

where \((i, j, k) \in S_3\) is a permutation of \(\{1, 2, 3\}\), that is, the spring length \(\ell_k\) is opposite mass \(k\) in the triangle, see Figure 3. The potential energy of the springs is now simply given by

\[
U_{\text{shape}} = \frac{\kappa_s}{2} \sum_{k=1}^{3} (\ell_k - \bar{\ell}_k)^2
\]

where \(\bar{\ell}_k\) denotes the rest length of spring \(k\).

\[\text{Figure 3. Depicted is a cartoon of our crawler.}\]

We define the viscous force of each spring by the one-form

\[
F_k = -\nu_s \dot{\ell}_k \, d\ell_k.
\]

In terms of the usual \(q_i = (x_i, z_i)\) coordinates these three forces \(F_k\) can be written as a sum of six force vectors \(F_{ij}\) describing the force exerted on particle \(i\) by the viscous friction of the spring connecting it to particle \(j\). We have

\[
F_{ij} = -\nu_s \frac{\langle \dot{q}_i - \dot{q}_j, q_i - q_j \rangle}{\|q_i - q_j\|^2} (q_i - q_j) = -\nu_s \frac{d\|q_i - q_j\|}{dt} \hat{n}_{ij}
\]

where \(\hat{n}_{ij} = \frac{q_i - q_j}{\|q_i - q_j\|}\) is the unit vector pointing from mass \(j\) to mass \(i\). The expression (4) constitutes the components of (3) with respect to the standard basis one-forms (\(dx_i, dz_i\)). More precisely, if we denote the components of \(F_{ij}\) by \(F_{ij}^x\)
and $F_{ij}^z$, then $F_{ij} = F_{ij}^x dx_i + F_{ij}^z dz_i$ is the one-form acting upon mass $i$, and we have $F_k = F_{ij} + F_{ji}$ for $(i, j, k) \in \mathcal{S}_3$. Thus, expression (3) conveniently captures the string damping force applied to the particles at both its endpoints. We see that the viscous friction forces oppose length change of the springs, exactly as expected. In any case, we can define the force $F_{\text{shape}} = \sum_{k=1}^3 F_k$.

Later in the paper we will make the rest lengths $\ell_k$ time-dependent as a means to indirectly control the actual lengths of the springs. Upon performing the substitution by functions $\ell_k(t)$, one should be careful about what kind of system is modeled by the resulting equations of motion. In our case, one could imagine that the viscous damping is realized through the addition of dashpots being placed in parallel to the springs.

3.2. A regularized model of the ground. The ground is described by the line $\{z = 0\}$ in $\mathbb{R}^2$. Ideally, the ground is impenetrable and imposes a no-slip condition, mathematically represented by the constraints
\begin{align}
z_i &\geq 0, \\
\dot{x}_i &= 0 \text{ if } z_i = 0
\end{align}
for $i = 1, 2, 3$, where equation (5) is the no-penetration condition and equation (6) is the no-slip condition. Both conditions present challenges of a singular nature because they abruptly ‘turn on’ at $z = 0$ and are inactive otherwise. It is precisely this ‘on/off’ character which we will regularize. To do this we will repeatedly make use of the differentiable function
\[
\chi(x) = \begin{cases} 
\frac{1}{2}x^2 & \text{if } x < 0, \\
0 & \text{else.}
\end{cases}
\]
to construct forces and potentials.

We approximate the no-penetration condition by considering a potential energy that grows rapidly for each $z_i < 0$ and is zero when $z_i \geq 0$ for all $i = 1, 2, 3$. Therefore, we define the potential energy $U_{\text{np}}: Q \rightarrow \mathbb{R}$ by
\[
U_{\text{np}}(q) = \kappa_{\text{np}} \sum_{i=1}^3 \chi(z_i).
\]

As the one-form $dx_i$ is independent of $dx_j$ when $i \neq j$ we see that $F_{ij} \neq -F_{ji}$ as one-forms.

Equations (2) and (4) (with the expression for $\ell_k$ substituted) show that the system is ill-defined when $q_i = q_j$ for some $i \neq j$. This is a set of codimension 2 which we shall stay away from in our analysis.

The function $\chi$ is of class $C^1$ only. However, this can be dealt with by applying a smoothing mollifier concentrated around 0. The width of the mollifier can be made arbitrary small, such that it does not overlap the fixed point to be found in Proposition 6 this prevents any possible circular dependencies in size estimates later on. Thus, without loss of generality we may assume that the system is smooth by viewing $\chi(\cdot)$ as a proxy for a smooth function with the same behavior away from 0.
This penalizes particles for falling through the floor and the penetration depth for a particle at rest can be controlled with $\kappa_{np}$. When $\kappa_{np}$ approaches infinity, the penetration depth goes to zero and our model approaches an exact model of a perfectly impenetrable ground. This can be viewed as modeling a one-sided holonomic constraint in the spirit of [26, 28].

The no-slip condition is similar to the no-penetration condition in that it is only active at $\{z = 0\}$. However, unlike the no-penetration condition, the no-slip condition is not derivable from a potential energy but instead can be viewed as a limit of viscous friction [1, 17]. In particular, consider the viscous force given by

$$F_{ns}(q, \dot{q}) = -\nu_{ns} \sum_{i=1}^{3} \chi'(z_i) \dot{x}_i \, dx_i.$$  

The force $F_{ns}$ dampens the horizontal motion of particles in a region around $\{z = 0\}$. Moreover, we can see that $F_{ns}$ is proportional to $dU_{np}$, the normal force exerted by the ground. This is consistent with standard (first-order) assumptions about the nature of slip-friction. As before, the coefficient $\nu_{ns}$ controls the amplitude of this force and when $\nu_{ns}$ goes to infinity we arrive at a no-slip condition.

Similarly, we dampen bouncing at the impact of a particle with the ground by including the viscous friction force

$$F_{db}(q, \dot{q}) = -\nu_{db} \sum_{i=1}^{3} \chi(z_i) \dot{z}_i \, dz_i.$$  

Finally, we incorporate gravity via the potential energy

$$U_g(q) = \sum_{i=1}^{3} z_i$$

which imposes the gravitational force $-dU_g(q) = -\sum_{i=1}^{3} dz_i$.

### 3.3. The full model.

Now that we have established the Lagrangian of the crawler, as well as the environmental forces imposed on it, we can finally provide the equations of motion. These equations of motion are obtained by adding the viscous forces, $F_{ns}$ and $F_{db}$, and the potential forces, $-dU_{np}$ and $-dU_g$, to the equations for the crawler in a vacuum. The equations of motion are the Lagrange–d’Alembert equations,

$$\ddot{q} = F_{\text{shape}}(q, \dot{q}) - dU_{\text{shape}}(q) - dU_{np}(q) + F_{ns}(q, \dot{q}) + F_{db}(q, \dot{q}) - dU_g(q).$$

If we define the total potential energy by $U: Q \rightarrow \mathbb{R}$ and the total force by $F: TQ \rightarrow T^*Q$, then the equations of motion are simply $\ddot{q} = F(q, \dot{q}) - dU(q)$. 
4. Analysis

In this section we prove the existence of a robustly stable equilibrium in a symmetry reduced phase space. To begin, we review the general process of reduction by symmetry before handling the specific case at hand. We reduce our system by an \( \mathbb{R} \)-symmetry to obtain a reduced vector field on the reduced phase space \( TQ/\mathbb{R} \). Subsequently, we prove the existence of a robustly stable equilibrium which can then be periodically perturbed to obtain a limit cycle. We reconstruct from it a relative limit cycle in the unreduced system. Finally, we provide some illustrative numerical results to support our claim that the reconstructed relative limit cycle typically has a non-trivial phase shift.

4.1. Reduction by symmetry in general. In this section we briefly describe the notion of reduction by symmetry. The constructions to be presented in this section may initially strike a reader as “needlessly abstract”. However, this abstraction rewards us with theorems which will allow us to skip many lengthy and mundane coordinate calculations in sections to follow.

Let \( G \) be a Lie group which acts freely and properly on a manifold \( M \). The orbit of \( x \in M \) is given by the set \( [x] := \{ g \cdot x \mid g \in G \} \), and the set of orbits is denoted \( [M] \). In this case \( [M] \) is a smooth manifold and the map \( \pi: x \in M \mapsto [x] \in [M] \) is a smooth surjection. We call the triple \((M, G, \pi)\) a principal \( G \)-bundle. We say that a vector field \( X \in \mathcal{X}(M) \) is \( G \)-invariant if \( X(g \cdot x) = g \cdot X(x) \) for all \( x \in M \) and \( g \in G \). The flow of a \( G \)-invariant vector field is also \( G \)-invariant in the sense that \( \Phi^X_t(g \cdot x) = g \cdot \Phi^X_t(x) \). Hence, group orbits are sent by \( \Phi^X_t \) to other group orbits. As a result there is a unique vector field \( \left[X\right] \in \mathcal{X}([M]) \) with flow \( \Phi^X_t \), such that the diagrams

\[
\begin{array}{ccc}
M & \xrightarrow{X} & TM \\
\downarrow \pi & & \downarrow T\pi \\
[M] & \xrightarrow{} & T[M]
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M & \xrightarrow{\Phi^X_t} & M \\
\downarrow \pi & & \downarrow \pi \\
[M] & \xrightarrow{\Phi^X_t} & [M]
\end{array}
\]

commute. We call \( [X] \) the reduced vector field of \( X \).

**Proposition 2.** Let \( x \in M \) be a fixed point of \( X \in \mathcal{X}(M) \). If \( X \) is \( G \)-invariant, then \( [x] \) is a fixed point of the reduced vector field \( [X] \in \mathcal{X}([M]) \) and the linearization of \([X]\) about \([x]\) is given by \( T_x[M] \rightarrow T_0(T_x[M]) \rightarrow T_x[M] \cdot (T_x\pi)^{-1}_{\text{right}} \), where \((T_x\pi)^{-1}_{\text{right}} \) is an arbitrary right-inverse to \( T_x\pi \).

**Lemma 3.** Assume the setup of Proposition 3. Then the kernel of \( T_x\pi \) is a subset of the kernel of \( T_x X : T_x M \rightarrow T_0(T_x M) \).

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\(^4\)We advise readers who are not well versed in differential geometry to imagine \( M = [M] \times G \) with the Lie group action \( g \cdot ([x], h) = ([x], gh) \) so that the quotient projection is simply \( \pi: ([x], h) \in M \mapsto [x] \in [M] \).
Proof. If \( \delta x \in T_xM \) is in the kernel of \( T_x\pi \) then it must be of the form \( \delta x = \frac{d}{dx}\big|_{x=0} g_x \cdot x \) for some curve \( g_x \in G \) which originates at \( g_0 = \text{id} \). We find
\[
T_x\Phi^X_t(\delta x) := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Phi^X_t(g_x \cdot x) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} g_x \cdot \Phi^X_t(x) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} g_x \cdot x = \delta x.
\]
Therefore, \( T_x\Phi^X_t \) is the identity on the subspace of \( T_xM \) tangent to a \( G \)-orbit.

Proof of Proposition \( 2 \). By the commutative diagrams above we observe that \( [x] = \pi(x) \in [M] \) is a fixed point of \( \Phi^X_t \) and \([X]\). As \( T_x\pi \) is surjective, we may define the formal inverse \( (T_x\pi)^{-1} : T_{\pi([x])}(\pi[M]) \rightarrow \frac{T_xM}{\ker(T_x\pi)} \). By Lemma \( 3 \), \( \ker(T_x\pi) \subset \ker(T_xX) \), so that \( T_0(T_x\pi) \cdot T_xX \cdot (T_x\pi)^{-1} \) is a well defined map from \( T_{\pi(x)}[M] \rightarrow T_0(T_{\pi(x)}[M]) \).
In other words \( T_{\pi(x)}[X] = T_0(T_x\pi) \cdot T_xX \cdot (T_x\pi)^{-1} \). We may now replace the formal inverse, \( T_x\pi^{-1} \), with an arbitrary right inverse, \((T_x\pi^{-1})_{\text{right}}\) to conclude the proof. \( \square \)

4.2. Reduction by translation symmetry. Consider the Abelian Lie group \( \mathbb{R} \) with the left action \( \rho^Q : \mathbb{R} \times Q \rightarrow Q \) given by translating the \( x \) coordinate of each particle. Explicitly, this action is given by
\[
\rho^Q(g)(x_1, z_1, x_2, z_2, x_3, z_3) := (x_1 + g, z_1, x_2 + g, z_2, x_3 + g, z_3) \quad \forall g \in \mathbb{R}.
\]
This action on \( Q \) can be lifted to \( TQ \). If we denote an element of \( TQ \) by \((q, u) = ((x_1, z_1, x_2, z_2, x_3, z_3), (v_1, w_1, v_2, w_2, v_3, w_3)) \) where \((v_i, w_i)\) denotes a velocity vector over \((x_i, z_i)\), then the action on \( TQ \), denoted by \( \rho^{TQ} \), is given explicitly by
\[
\rho^{TQ}(g) \cdot \begin{bmatrix} (x_1, z_1, x_2, z_2, x_3, z_3) \\ (v_1, w_1, v_2, w_2, v_3, w_3) \end{bmatrix} = \begin{bmatrix} (x_1 + g, z_1, x_2 + g, z_2, x_3 + g, z_3) \\ (v_1, w_1, v_2, w_2, v_3, w_3) \end{bmatrix}.
\]
Direct application of the definitions shows that this action is free and proper. The vector field \( X \in \mathfrak{X}(TQ) \) induced by \( \rho^{TQ} \) is invariant under this action, as can be explicitly verified from the equations. Thus there exists a unique vector field \([X] \in \mathfrak{X}(TQ/\mathbb{R})\) which is \( \pi^{TQ} \)-related to \( X \), with \( \pi^{TQ} : TQ \rightarrow TQ/\mathbb{R} \) the quotient map with respect to the action \( \rho^{TQ} \).

Just as the equations of motion are \( \mathbb{R} \)-invariant, quantities such as the potential energy are \( \mathbb{R} \)-invariant as well. This \( \mathbb{R} \)-invariance simply means that the system is unchanged by translations along the \( x \)-axis. As a result, there exists a unique reduced potential \( \tilde{U} : Q/\mathbb{R} \rightarrow \mathbb{R} \) such that \( U = \tilde{U} \circ \pi^Q \). The viscous friction forces can be expressed in terms of Rayleigh dissipation functions, and reduced forms can then easily be found for these. A quadratic function \( R : TQ \rightarrow \mathbb{R} \) is called a Rayleigh function for a friction force \( F \) when
\[
F(q, u) = -\frac{\partial R}{\partial u} : TQ \rightarrow T^*Q,
\]
where the fiber derivative in $TQ$ is taken \cite[definition 7.8.9]{20}. Then $R$ can be written in coordinates as

$$R(q, u) = \frac{1}{2} \nu_{ij}(q) u^i u^j. \tag{10}$$

In our case, the matrices $\nu_{ij}$ associated to the friction forces only depend on $z_i$, and $\rho^{TQ}$ acts trivially on the fibers of $TQ$, so there exist reduced Rayleigh functions that satisfy $R = \hat{R} \circ \pi^{TQ}$. Since the action of $\rho^{TQ}$ is trivial, we shall not distinguish between $R$ and $\hat{R}$.

4.3. Linearizations about equilibria. Let $q_* \in Q$ be such that $dU(q_*) = 0$. Then $(q_*, 0) \in TQ$ is an equilibrium point of the equations of motion \cite{8}. We can therefore consider the linearized equations over $(q_*, 0)$ by considering a local chart with coordinates $q = (q^1, \ldots, q^6)$. The induced coordinates on the tangent bundle will be denoted by $(q, u) = (q^1, \ldots, q^6, u^1, \ldots, u^6)$. It is a well-known result of the theory of linear oscillations, that the linearized system takes the form of a damped harmonic oscillator,

$$\frac{d}{dt} \begin{bmatrix} q \\ u \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -\nu \end{bmatrix} \begin{bmatrix} q \\ u \end{bmatrix}$$

where $K, \nu \in \mathbb{R}^{6 \times 6}$ are positive (semi-)definite matrices given by

$$K_{ij} := \frac{\partial^2 U}{\partial q^i \partial q^j}$$

and \cite{10}, respectively. In particular, $\nu$ is the local manifestation of the Rayleigh dissipation force, and $K$ is the linear approximation of the potential energy \cite{29}.

Without loss of generality we will let $q^6$ be a local fiber coordinate for the principal bundle projection $\pi: Q \to Q/\mathbb{R}$. For example, $q^6$ could measure the average $x$-coordinate of the three masses. In any case, the principal bundle projection is locally given by $p(q^1, \ldots, q^5, q^6) = (q^1, \ldots, q^5)$, where $p = [I_5 \quad 0]$. Moreover, the principal bundle projection on $TQ$ is locally given by $\pi^{TQ}(q^1, \ldots, q^6, u^1, \ldots, u^6) = (q^1, \ldots, q^5, u^1, \ldots, u^6)$. Note that $\pi^{TQ}$ drops the fiber coordinate $q^6$, but it does not drop the velocity $u^6$. Thus the linearization of $\pi^{TQ}$ at $(q_*, 0)$ is locally given by the matrix

$$T_{(q_*, 0)} \pi^{TQ} = \begin{bmatrix} p & 0 \\ 0 & I_6 \end{bmatrix} \tag{11}.$$ 

Under certain reasonable assumptions (see Assumption \cite{5} on page \cite{12}), the reduced potential energy $\hat{U}$ has a non-degenerate minimum which corresponds to the crawler resting motionless on the ground. In this case we can verify that the kernel of $K$ is $\text{span}(\frac{\partial}{\partial q^6})$. As a result, the kernel of the matrix in \cite{11} is also $\text{span}(\frac{\partial}{\partial q^6})$. Finally, $\text{span}(\frac{\partial}{\partial q^6})$ is also the kernel of $T_{(q_*, 0)} \pi^{TQ}$. Therefore, by Proposition \cite{2}, the linearization of the reduced system on $TQ/\mathbb{R}$ about the equilibrium
\[(\dot{q}_*, 0) = \pi^{TQ}(q_*, 0)\] is given by
\[
\frac{d}{dt} \begin{bmatrix} \dot{q} \\ u \end{bmatrix} = \begin{bmatrix} 0 & -K \cdot p^T \\ -\nu & -\nu \end{bmatrix} \begin{bmatrix} \dot{q} \\ u \end{bmatrix},
\]
where we have used the right inverse
\[
(T_{(q_*, 0)}\pi^{TQ})^{-1}_{\text{right}} = \begin{bmatrix} p^T & 0 \\ 0 & I_6 \end{bmatrix}.
\]

### 4.4. Stable equilibria.

It is easy to intuit the existence of a stable equilibrium which corresponds to a stationary crawler resting on the ground. Such a point in phase space would be merely a single element of an entire \(\mathbb{R}\)-orbit of equilibria obtained by translating the crawler along the \(x\)-direction. Therefore, these equilibria can only be marginally stable at best, as the vector field vanishes along the direction of this symmetry. However, it is possible that this \(\mathbb{R}\)-orbit projects to a (robustly) stable equilibrium in the reduced system (in the sense of Definition 4 below). We therefore turn to the reduced system and identify reasonably general conditions under which there exists a configuration \(\dot{q}_* \in Q/\mathbb{R}\) which is a non-degenerate minimum of \(\dot{U}\). Then we apply Proposition 8 to conclude that \((\dot{q}_*, 0) \in TQ/\mathbb{R}\) is a stable equilibrium. To be completely unambiguous about what we mean, let us define

**Definition 4 (Stable equilibrium).** Let \(\dot{x} = f(x)\) denote a dynamical system on a manifold \(M\). Then we call \(x_* \in M\) a robustly stable equilibrium if \(f(x_*) = 0\) and the spectrum of \(Df(x_*)\) lies strictly left of the imaginary axis.

This definition is to be seen in contrast to weaker notions such as marginal stability wherein eigenvalues may lie on the imaginary axis. In particular, a robustly stable equilibrium is a hyperbolic fixed point which (locally) attracts solution curves at an exponential rate.

To find a robustly stable equilibrium in our system, we make the following assumptions:

**Assumption 5.**

1. The rest lengths \(\bar{\ell}_1, \bar{\ell}_2\) and \(\bar{\ell}_3\) of the springs form a non-degenerate triangle, and
2. the spring and ground stiffness \(\kappa_s\) and \(\kappa_{np}\) are sufficiently large.

We shall formulate the precise results that lead towards the existence of a robustly stable equilibrium in the propositions below and indicate the ideas of the proofs; the details can be found in Appendix A.

**Proposition 6.** Under Assumptions 5 there exists a (local) minimum \(\dot{q}_* \in Q/\mathbb{R}\) of the reduced potential \(\dot{U}\). This minimum is non-degenerate in the sense that the Hessian, \(\dot{K}\), of \(\dot{U}\) at \(\dot{q}_*\) is positive definite.
For reasons which will be clear soon, we must have a guarantee that one mass of the equilibrium configuration has a larger $z$-coordinate than the others. Such a guarantee requires that the springs be sufficiently stiff to support the weight. This minimum spring stiffness, $\kappa_s$, will implicitly depend on how close to degeneracy the triangle of rest lengths is; this ensures that the actual lengths, $\ell_k$, of the energy-minimizing configuration form a non-degenerate triangle. The system is invariant under a relabeling of the masses, so without loss of generality we may also assume that $\bar{\ell}_3 \geq \max(\bar{\ell}_1, \bar{\ell}_2)$. The idea now is to search for a configuration where the masses 1 and 2 ‘rest on the ground’ and 3 has coordinate $z_3 > 0$ raised above the influence of the ground potential. We view this as a singular perturbation problem: when the stiffnesses $\kappa_s, \kappa_{np}$ are infinite, then the solution is trivially the rigid triangle with sides $\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3$, and resting on the ground $z = 0$. By rescaling, we turn it into a regular perturbation problem and apply the implicit function theorem to find a slightly perturbed stable configuration for large but finite $\kappa_s, \kappa_{np}$.

Secondly, the viscous friction is non-degenerate. To make this statement precise, recall that the Rayleigh dissipation function of a viscous force is given by (9), and that it is a quadratic function in the fiber variable $u$. As a preliminary result to proving hyperbolic attractivity of the fixed point in Proposition 8, we prove

**Proposition 7.** The total Rayleigh dissipation function $R(q,u)$ is positive definite on the fiber of $TQ/\mathbb{R}$ above $\hat{q}_*$ as found in Proposition 6.

The frictional forces are thus non-degenerate in the sense that the matrix $\nu$ associated to $R$ is positive definite. Together with the nature of the minimum $\hat{q}_*$ of $\hat{U}$, this provides all prerequisites for the following

**Proposition 8.** Let $\hat{q}_* \in Q/\mathbb{R}$ be a non-degenerate minimum of $\hat{U}$, that is, $d\hat{U}(\hat{q}_*) = 0$ and its Hessian $\hat{K}$ is positive definite. Then $\hat{q}_*, 0) \in TQ/\mathbb{R}$ is a robustly stable equilibrium for the reduced system.

The idea is that if no friction were present, then starting close to the stable equilibrium $(\hat{q}_*, 0)$ in phase space, the motion would be oscillatory. Since the friction force is non-degenerate by Proposition 7, the energy will decay asymptotically to zero sending the system to a standstill at $(\hat{q}_*, 0)$. We prove that this decay towards $(\hat{q}_*, 0)$ is exponential, even though the friction force does not act when the velocity is zero. Note that any $q \in Q$ such that $\pi^q(q) = \hat{q}_*$ produces an equilibrium $(q, 0) \in TQ$ for the unreduced system. However, any such $q$ is not a robustly stable equilibrium (it is only marginally stable).

4.5. **Time periodic perturbations.** Given a dynamical system $\dot{x} = f(x)$ on a manifold $M$ with a robustly stable equilibrium $x_* \in M$, one can embed the system into a time-periodic augmented phase space $S^1 \times M$ by using the vector field $(\dot{\theta}, \dot{x}) = (1, f(x))$. Then the trajectory $\gamma_0(\theta) = (\theta, x_*) \in S^1 \times M$ is a limit cycle for the system on $S^1 \times M$ which locally attracts at an exponential rate. The orbit $\Gamma_0 := S^1 \times \{x_*\}$ is a compact normally hyperbolic invariant submanifold, and so the
theorem on persistence of normally hyperbolic invariant manifolds \cite{7, 14} applies. Specifically, given a sufficiently small time-periodic perturbation \( f \mapsto f + \varepsilon g_\theta \), we can assert the existence of a persistent limit cycle, \( \gamma_\varepsilon \), in a neighborhood of \( \gamma_0 \) (see also ‘The Averaging Theorem’ in \cite{12}).

In the previous subsection, we found a robustly stable equilibrium in \( TQ/\mathbb{R} \). In this section, we will perturb this system by substituting time \( T \)-periodic lengths \( \bar{\ell}_k(t) \) for the constant rest lengths \( \bar{\ell}_k \). If these oscillations are small, we can expect to observe a \( T \)-periodic limit cycle, \( (\theta, \hat{\gamma}(\theta)) \), in the augmented phase space \( S^1 \times TQ/\mathbb{R} \). Thus \( \hat{\gamma} \) is a stable periodic trajectory of the original time-periodic system on \( TQ/\mathbb{R} \). However, if \( \gamma(t) \) is a trajectory in \( TQ \) which projects down to \( \hat{\gamma}(t) \in TQ/\mathbb{R} \), then it is generally not the case that \( \gamma(t) \) is periodic. In particular, any periodic trajectory \( \hat{\gamma} \subset TQ/\mathbb{R} \) is reduced from a trajectory \((q,u)(t)\) in \( TQ \) such that

\[
(13) \quad (q,u)(t+T) = \rho^{TQ}(\Delta x) \cdot (q,u)(t),
\]

where \( \Delta x \in \mathbb{R} \) is obtained from the \( u \)-component of \( \hat{\gamma}(t) \) via

\[
(14) \quad \Delta x = \int_0^T v_3(t) \, dt.
\]

The above integral may be viewed as a phase reconstruction formula with respect to the reduction by \( \mathbb{R} \)-symmetry. Trajectories which satisfy conditions such as (13) are known as relatively periodic orbits. A relatively periodic orbit \( \gamma(t) \) emanating from an initial condition \( \gamma(0) \in TQ \) will project down to a periodic orbit \( \hat{\gamma}(t) = \pi^{TQ}(\gamma(t)) \) in \( TQ/\mathbb{R} \). Conversely, an orbit \( \gamma(t) \) which projects down to a periodic orbit \( \hat{\gamma}(t) = \pi^{TQ}(\gamma(t)) \) in \( TQ/\mathbb{R} \) is necessarily a relatively periodic orbit in \( TQ \).

Moreover, if \( \hat{\gamma} \) is a stable limit cycle in \( TQ/\mathbb{R} \), then the relatively periodic orbits in \( TQ \) are stable as well. In this case the orbits in \( TQ \) are dubbed ‘relatively limit cycles’ in that they are relatively periodic and stable. For our system, the phase shift \( \Delta x \) corresponds to a ‘step’ and the stable limit cycle corresponds to the leg movement in a coordinate frame attached to the crawler. These observations combined suggest that we call the lifted trajectories in \( TQ \) crawling-like when \( \Delta x \neq 0 \) (and when \( \Delta x = 0 \) we could call it ‘rocking-like’). This completes the proof of all claims in our main theorem.

To compute the phase shift \( \Delta x \), we have to integrate (14) over the persistent limit cycle. There is in general no explicit formula for this cycle, since it depends implicitly on the perturbation. Fortunately, the present system is simple enough to be studied in computer simulations, see section 5. The simulations we carried out revealed that the phase shift \( \Delta x \) appears generically to be non-zero, but to

\footnote{To be more precise, the perturbation must be small in \( C^1 \) supremum norm. The Lagrange–d’Alembert vector field was already smooth (after application of a mollifier). Since we augmented the phase space with periodic time, these theorems also require the perturbation to be \( C^1 \) with respect to time. Note however that this can be relaxed to \( C^0 \) \cite[Remark 4.1]{6} and possibly integrable dependence on time.}
depend on the perturbation size to second order. A heuristic explanation for this result can be given by the fact that ‘making a step’ requires the combined variation of position and velocity of the masses, leading to a quadratic dependence on the perturbation size. The variation in position is needed to displace the crawler’s weight towards one leg and the variation in velocity to actually move the other leg. A more precise argument can be given as follows. We wish to prove that $\Delta x \in \mathcal{O}(\varepsilon^2)$ with $\varepsilon$ the perturbation size parameter. First of all, since the fixed point $(\hat{q}_*, 0) \in TQ/G$ is hyperbolic, the perturbation of the limit cycle will scale linearly with $\varepsilon$ as well; let us denote this periodic orbit by $(\hat{q}_*^\varepsilon(t), u^\varepsilon(t))$. We choose a complete set of local coordinates around $G \cdot \hat{q}_* \subset Q$, e.g. $(\ell_1, \ell_2, \ell_3, z_1, z_2, x_3)$, such that $x_3$ coordinatizes the orbits of the symmetry group $G = (\mathbb{R}, +)$. Hence, $x_3$ is a cyclic variable and the associated Lagrange–d’Alembert equation reads

$$\frac{d}{dt} \frac{\partial T}{\partial v_3} = -\frac{\partial R}{\partial v_3} = -p_{v_3}^* \cdot \nu(\hat{q}) \cdot u$$

where the right-hand side is the friction force and $p_{v_3}^*$ is projection onto the dual of the $v_3$ basis vector. The left-hand side is a total derivative, so its integral over a full period along $(\hat{q}_*^\varepsilon(t), u^\varepsilon(t))$ is zero. We study the right-hand side using Taylor expansion in powers of $\varepsilon$, denoted like

$$\hat{q}_*^\varepsilon(t) = \hat{q}_*^{(0)}(t) + \varepsilon \hat{q}_*^{(1)}(t) + \varepsilon^2 \hat{q}_*^{(2)}(t) + \mathcal{O}(\varepsilon^3)$$

$$u^\varepsilon(t) = u^{(0)}(t) + \varepsilon u^{(1)}(t) + \varepsilon^2 u^{(2)}(t) + \mathcal{O}(\varepsilon^3)$$

$$\Delta x^\varepsilon = \Delta x^{(0)} + \varepsilon \Delta x^{(1)} + \varepsilon^2 \Delta x^{(2)} + \mathcal{O}(\varepsilon^3).$$

Note that $\hat{q}_*^{(0)}(t) = \hat{q}_*$ and $u^{(0)}(t) = 0$, so the right-hand side of (15) expands to

$$-p_{v_3}^* \cdot \nu(\hat{q}(t)) \cdot u(t) = -p_{v_3}^* \cdot \nu(\hat{q}_*) \cdot (\varepsilon u^{(1)}(t) + \varepsilon^2 u^{(2)}(t)) - \varepsilon^2 p_{v_3}^* \cdot (D \nu(\hat{q}_*) \cdot \hat{q}_*^{(1)}(t)) \cdot u^{(1)}(t) + \mathcal{O}(\varepsilon^3).$$

We collect powers of $\varepsilon$ and integrate over a full period. For $\varepsilon^1$ this gives

$$0 = -p_{v_3}^* \cdot \nu(\hat{q}_*) \int_0^T u^{(1)}(t) \, dt.$$

In particular, since $\nu(\hat{q}_*)$ is positive definite and all velocity components except for $v_3$ are total derivatives of periodic position functions (hence integrate to zero), this implies that

$$\Delta x^{(1)} = \int_0^T v_3^{(1)}(t) \, dt = 0.$$

For $\varepsilon^2$ we find

$$0 = -p_{v_3}^* \int_0^T \left[ \nu(\hat{q}_*) \cdot (u^{(2)}(t) + (D \nu(\hat{q}_*) \cdot \hat{q}_*^{(1)}(t)) \cdot u^{(1)}(t)) \right] \, dt.$$
and with similar arguments this yields

$$\Delta x^{(2)} = -\frac{1}{\nu_{v_3}} \int_0^T \left( D\nu(\hat{q}_*) \cdot \dot{\hat{q}}^{(1)}(t) \right) \cdot \dot{u}^{(1)}(t) \, dt$$

where $\nu_{v_3} = \nu(\hat{q}_*)_{\text{span}(\frac{\partial}{\partial v_3})} > 0$ denotes the damping matrix $\nu(\hat{q}_*)$, interpreted as a quadratic form restricted to the space spanned by $v_3$. The expression (17) is zero when $D\nu(\hat{q}_*) \cdot \dot{\hat{q}}^{(1)}(t)$ is $L^2$-orthogonal to $\dot{v}^{(1)}(t)$, which is almost always false. Indeed, our numerical simulations indicate that $\Delta x^{(2)}$ is generically non-zero, and so it seems that the first contribution to $\Delta x$ is of order $\varepsilon^2$.

It is worth noting that our argument depends on the fact that the damping matrix $\nu$ depends non-trivially on the reduced configuration variables $\hat{q} \in \hat{Q}/G$. This can be viewed in contrast to [4], where damping induced self-recovery of a cyclic variable is studied, and hence a non-zero phase shift cannot occur. That setting assumes that $\nu$ does not depend on the other variables.

5. Numerical simulations

In this section we numerically compute trajectories to better understand this system. In particular we consider the time-dependent lengths

$$\tilde{\ell}_1(t) = 1 + \varepsilon \cos(\omega t)$$
$$\tilde{\ell}_2(t) = 1 - \varepsilon \sin\left(\omega\left(t - \frac{1}{2}\right)\right)$$
$$\tilde{\ell}_3(t) = 3 - \tilde{\ell}_1(t) - \tilde{\ell}_2(t)$$

where $\omega = 2\pi$ and we vary the amplitude $\varepsilon > 0$. Additionally we use the parameters: $\kappa_{np} = 10$, $\nu_{ns} = 10$, $\kappa_s = 10$, $\nu_{db} = 5$, and $\nu_s = 10$.

To test our theory we allow the system 10 seconds of inactivity (i.e. $\varepsilon = 0$) so that the system settles towards an equilibrium. Then, at $t = 10$ we set $\varepsilon = 0.5$. The system appears to converge to a relatively periodic orbit after a few periods, see Figure 4. This relatively periodic orbit exhibits a phase shift of $\Delta x = 0.046$, and so we observe a steady drift in the positive $x$-direction. We observe that both the $x$ and $z$ coordinates oscillate with angular frequencies of $2\pi$, as predicted by our analysis in Section 4.5, i.e. the period of the relative limit cycle is identical to the period of the perturbation. To further illustrate this relatively cyclic behavior we have plotted the locations of the masses over three time-periods in Figure 5 where one can clearly see how each period is identical to the previous period up to the constant shift $\Delta x = 0.046$. Finally, this value of $\Delta x$ was observed to be robust to small but randomly chosen changes in the initial conditions. This is in agreement with the theory that $\Delta x$ is ultimately a function of the time dependent lengths $\tilde{\ell}_k(t)$ only, implicitly defined through the phase reconstruction formula (14).

Although we do not have a proof that $\Delta x$ is generically non-zero, a few trial perturbations all yielded non-zero $\Delta x$ values. The simulations do support the claim that the first variation of $\Delta x$ with respect to the perturbation is zero, while
Figure 4. This plot depicts the $x$ coordinates (bold lines) and the $z$ coordinates (thin lines) of the three masses for a trajectory where $\varepsilon = 0.5$. The system is activated at $t = 10$.

Figure 5. Depicted are the trajectories of the masses in space over the final three periods plotted in figure 4. Above the trajectory of the bottom right mass we have indicated the phase shift of $\Delta x = 0.046$. 
the second variation is non-zero. In particular we have calculated trajectories for various $\varepsilon$'s, and computed the quantities

$$p_k = \frac{\log|\Delta x_k| - \log|\Delta x_{k-1}|}{\log(\varepsilon_k) - \log(\varepsilon_{k-1})},$$

to detect the scaling of $\Delta x$ with the perturbation size. If $\Delta x$ is proportional to $\varepsilon^2$ then we should find that $p_k \approx 2$. The results are summarized in Table 1.

| $\varepsilon$ | $\Delta x$ | $p$     |
|--------------|-----------|--------|
| 1            | 0.17870   | 1.9372 |
| 1/2          | 0.04666   | 1.9932 |
| 1/4          | 0.01172   | 1.9980 |
| 1/8          | 0.002934  | 1.9990 |
| 1/16         | 0.000734  | 2.0039 |
| 1/32         | 0.000183  | N/A    |

Table 1. The values of $\Delta x$ for various perturbation sizes $\varepsilon$.

6. Generalizing to 3D

We can easily generalize our model to three-dimensional space with $\text{SE}(2)$ acting as symmetries of the plane (see figure 2). The setup requires few changes, which we shall indicate here.

As a model of our crawler we take a tetrad of four masses described by coordinates $q_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ and $q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^{12}$. These masses are all connected by springs of lengths $\ell_{ij}$ where $i < j$ range over 1 to 6; we alternatively reference these lengths by $\ell_k$ and rest lengths by $\bar{\ell}_k$. We generalize the kinetic and potential energy, and friction forces from section 3 in the natural way, and we obtain, for example, for the ground friction force

$$F_{\text{ns}}(q, \dot{q}) = -\nu_{\text{ns}} \sum_{i=1}^{4} \chi'(z_i) \left( \dot{x}_i \, dx_i + \dot{y}_i \, dy_i \right).$$

By extending the horizontal directions to $\mathbb{R}^2$, this 3D model is naturally invariant under the isometry group $\text{SE}(2)$, acting on the planar part of $\mathbb{R}^3$; we denote this by the projection $p_{xy}: \mathbb{R}^3 \to \mathbb{R}^2$. Note that $\text{SE}(2)$ does not act freely on all of $\mathbb{R}^{12}$; if all masses $q_i$ have the same $(x, y)$ coordinates, then $q$ is fixed by the non-trivial stabilizer subgroup $\text{SE}(2)_{(x,y)}$ that fixes that point in the plane. We thus take the open subset $Q \subset \mathbb{R}^{12}$ with these configurations excluded and find:

**Proposition 9.** The action of $\text{SE}(2)$ on $Q$ is free and proper.

**Proof.** The action is free if $(\Phi, X, Y) \cdot q = q$ implies that $(\Phi, X, Y) \in \text{SE}(2)$ is the identity. Since the points $q_i$ project to at least two distinct points in the plane, this is well-known to hold.
To prove that the action is proper, we have to show that
\[ A: \text{SE}(2) \times Q \to Q \times Q : (g, q) \mapsto (g \cdot q, q) \]
is a proper continuous map, see [5, p. 53]. We shall do so by proving that \( A \) has a continuous inverse, defined on its image. Let \( (q', q) \in \text{Im}(A) \) and without loss of generality assume that \( p_{xy}(q_1) \neq p_{xy}(q_2) \). Then \( A^{-1}(q', q) = ((\Phi, X, Y), q) \) where \( (X, Y) = p_{xy}(q' - q_1) \) and \( \Phi = \angle(p_{xy}(q'_2 - q'_1), p_{xy}(q_2 - q_1)) \), and this angle depends continuously on the arguments, since these have non-zero lengths. \( \square \)

In assumption 5 we now assume that the tetrad formed by the four masses at rest lengths is non-degenerate, and that if the masses \( q_1, q_2, q_3 \) are at \( z = 0 \), then \( p_{xy} \) projects \( q_4 \) inside the triangle formed by \( q_1, q_2, q_3 \), while \( z_4 > 0 \). In other words, the tetrad would not tumble over. Similar to the 2D case, we choose local coordinates \( z_1, z_2, z_3, \{\ell_k\}_{1 \leq k \leq 6} \) for \( Q/\text{SE}(2) \). The reduced potential is still of the form (21):
\[
\hat{U} = \frac{\kappa_s}{2} \sum_{k=1}^{6} (\ell_k - \bar{\ell}_k)^2 + \sum_{i=1}^{3} \left( z_i + \kappa_{np} \chi(z_i) \right) + z_4(z_1, z_2, z_3, \ell_k).
\]

With our modified assumptions above, the proof of Proposition 6 remains essentially unchanged for the 3D case. To generalize the proof of Proposition 7 to 3D, we note that
\[
\ker(R_{db}) \cap \ker(R_{ns}) = \text{span} \left( \frac{\partial}{\partial x_4}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial z_4} \right),
\]
i.e. translations of the particle \( q_4 \), as in the 2D case. On the other hand,
\[
\ker(R_{\text{shape}}) = \mathfrak{se}(3) \cdot Q.
\]
By the same arguments as before, i.e. since \( q_4 \) is part of a rigid, space spanning tetrad, it follows that the intersection of the kernels is zero. These are all the steps required to show that the model generalizes to 3D. In particular, the Lagrangian and dissipative forces, as well as perturbations on the springs lengths are all invariant under the action of rotations and translations in the plane, so virtually the same reduction can be performed as for the 2D case. In order to implement the reduction, instead of using the coordinates \( (x_1, \ldots, x_4, y_1, \ldots, y_4, z_1, \ldots, z_4) \), it is more convenient e.g. to use the coordinates \((\ell, z, \theta, x, y)\) where \((x, y) = (x_1, y_1)\), \(\ell = (\ell_1, \ldots, \ell_6)\), \(z = (z_1, z_2, z_3)\), and \(\theta\) is the angle of the line which passes through \((x_1, y_1)\) and \((x_2, y_2)\) measured against the \(x\)-axis. In these coordinates the action of an element \((\Phi, X, Y) \in \text{SE}(2)\) is given by
\[
(\ell, z, \theta, x, y) \mapsto (\ell, z, \theta + \Phi, \cos(\Phi)x - \sin(\Phi)y + X, \sin(\Phi)x + \cos(\Phi)y + Y).
\]
These coordinates trivialize \( Q \cong M \times \text{SE}(2) \) as a principal \( \text{SE}(2) \) bundle in that the quotient projection \( \pi/\text{SE}(2) \) simply drops the last three coordinates, and the space \( Q/\text{SE}(2) \) is nine-dimensional with coordinates \((\ell_1, \ldots, \ell_6, z_1, z_2, z_3)\). More
Figure 6. Depicted are the trajectories in the $xy$-plane of the three masses on the ground for the 3D crawler. The zoom box shows the single cycles, while the overall trajectories are clearly seen to curve.

importantly, since SE(2) acts only on the fiber component of this trivialization, we can identify

$$TQ/\text{SE}(2) \cong TM \times \text{se}(2)$$

by left-trivialization of $T\text{SE}(2)$, with coordinates given by

$$(\ell, z, v_\ell, v_z, \omega, \xi_x, \xi_y).$$

Given a limit cycle of period $T$ in $TQ/\text{SE}(2)$ we can reconstruct the trajectory in $TQ$ by solving the ODE

$$(20) \quad \dot{g} = g \cdot (\omega(t), \xi_x(t), \xi_y(t))$$

with initial condition $g(0) = h := (\theta(0), x(0), y(0))$ to recover $g(T)$. Since $20$ is left-invariant, this is equivalent to $g(T) = h \cdot \Delta g$ with $\Delta g$ the solution at time $T$ of $20$ with initial condition $g(0) = e = (0, 0, 0)$. Remark that the phase shift $\Delta g = (\Phi, X, Y)$ acts from the right on the configuration space variables $(\theta, x, y)$. Thus, each period will be related to the previous by a rotation $\Phi$ around $(x_1, y_1)$ and a translation $(X, Y)$ relative to the crawler orientation $\theta$. A sample numerical simulation can be seen in figure 6.
7. Outlook & conclusion

In this paper we have shown that regularized models are capable of exhibiting behavior which resembles crawling, by constructing a model with a robust relative limit cycle. Such models are open to classical techniques in dynamical systems, and allow one to view crawling as a limit cycle in a reduced space, while the absolute motion manifests as a phase shift after reconstruction. These ideas are generic enough that it seems feasible to apply them to a range of other scenarios. Furthermore, the work suggests a number of follow-up questions to pursue:

(1) It would be interesting to investigate if the limit cycles in the regularized model persist under singular perturbation limits $\kappa_{np}, \nu_{ns} \to \infty$. Such an observation would help bridge the gap between this perspective and the hybrid systems approach.

(2) While the limit cycle in the paper is stable, the size of the stability basin is not addressed. Having a large stability basin is one method of achieving robustness, and so a lower bound for the radius of this basin would be useful to have.

(3) A non-flat ground breaks symmetry, but may still be addressed using normal hyperbolicity theory if the ground is still sufficiently close to flat. Similarly, small random or time dependent perturbations will only slightly perturb the relative limit cycle; in particular, the phase shift of each cycle is close to that without these perturbations.

Lastly, we would hope that at least a portion of these ideas would aide in studying stable walking models. In our model we constructed a crawling-like limit cycle as a small perturbation of an unactuated system and made use of the fact that stability along all of the limit cycle was preserved. This makes our model not directly applicable to walking, which is typically considered to be ‘statically unstable’ (e.g. in the inverted pendulum models, the walker collapses to the floor when the joints are not active). On the other hand, if one finds a model for walking with a limit cycle that is stable as a whole (that is, its Poincaré map is stable), then that cycle can be used as a starting point, and Lie theory can still be used to find a reduced description and a reconstruction formula for the phase shift. Furthermore, the resulting limit cycles would persists under small perturbations as described above.

Appendix A. Stability proofs

In this appendix we collect the detailed proofs for the statements in Section 4.4.

Proof of Proposition 6. To simplify the analysis we change to a (local) coordinate system for $Q/\mathbb{R}$ given by $(z_1, z_2, \ell_1, \ell_2, \ell_3)$. In these coordinates, and under the
assumption that \( z_3 > 0 \), the (reduced) potential energy takes the form

\[
\dot{U} = \frac{\kappa_s}{2} \sum_{k=1}^{3} (\ell_k - \bar{\ell}_k)^2 + z_1 + z_2 + z_3(z_1, z_2, \ell_i) + \kappa_{np} \chi(z_1) + \kappa_{np} \chi(z_2). 
\]

Note that \( z_3 \), the gravity potential of mass 3, depends on the variables in an intricate way which we shall not endeavor to make explicit. Thus we search for a solution \( \hat{q}_s \) of

\[
0 = d\dot{U} = \sum_{i=1,2} dz_i \left( 1 + \kappa_{np} \chi'(z_i) + \frac{\partial z_3}{\partial z_i} \right) + \sum_{i=1}^{3} d\ell_i \left( \kappa_s (\ell_i - \bar{\ell}_i) + \frac{\partial z_3}{\partial \ell_i} \right).
\]

We recover the solution \( \hat{q}_s \) by an implicit function argument. Let us define

\[
F((z_1, z_2, \ell_i), \varepsilon) = \begin{bmatrix} 1 + \kappa_{np} \chi'(z_1) + \frac{\partial z_3}{\partial z_1} \\ \frac{\partial z_3}{\partial z_2} \\ \ell_i - \bar{\ell}_i + \varepsilon \frac{\partial z_3}{\partial \ell_i} \end{bmatrix} \in \mathbb{R}^5 \text{ with } i = 1, 2, 3.
\]

A zero of \( F \) corresponds to a solution of \( (22) \) if we set the parameter \( \varepsilon = 1/\kappa_s \); we first search for a zero with \( \varepsilon = 0 \) though. That is, we consider the singular limit of infinite spring stiffness. This implies \( \ell_i = \bar{\ell}_i \). Note that when the ground potential \( \kappa_{np} \chi \) rises steeply enough, it follows by energy arguments that \( z_1 \approx z_2 \approx 0 \), so \( \ell_3 \) is oriented approximately horizontally. A simple geometric argument now shows that \( \frac{\partial z_3}{\partial z_i} > 0 \) for \( i = 1, 2 \): the \( x \) coordinate of mass 3 is located in between that of masses 1 and 2. Therefore, when either \( z_1 \) or \( z_2 \) are increased with all \( \ell_i = \bar{\ell}_i \) fixed, then the entire configuration must transform by a rigid rotation and translation wherein \( z_3 \) would increase as well. Hence, for \( i = 1, 2 \) we have that \( 1 + \frac{\partial z_3}{\partial z_i} > 0 \). Moreover, \( \chi'(z) \) is monotonically decreasing without bound from 0 as \( z \to -\infty \). It follows that there are unique values \( z_i < 0 \) such that the point \( \hat{q}_0 = (z_1, z_2, \bar{\ell}_1) \) solves \( F(\hat{q}_0, 0) = 0 \).

The derivative of \( F \) with respect to the variables \( (z_1, z_2, \ell_i) \) is found to be

\[
DF(\hat{q}_0, 0) = \begin{bmatrix} A + \kappa_{np} I_2 & B \\ 0 & I_3 \end{bmatrix},
\]

where \( B_{ij} = \frac{\partial^2 z_3}{\partial \ell_i \partial z_j} \) and \( A \) is the Hessian of \( (z_1, z_2) \mapsto z_3(z_1, z_2, \ell_i) \). Note that if \( \kappa_{np} \) is sufficiently large, then \( A + \kappa_{np} I_2 \) is positive definite. The eigenvalues \( \lambda \) of \( DF(\hat{q}_0, 0) \) are recovered from

\[
0 = \det(DF(\hat{q}_0, 0) - \lambda I_3) = (1 - \lambda)^3 \det(A + \kappa_{np} I_2 - \lambda I_2)
\]

and found to be all positive. In particular \( DF(\hat{q}_0, 0) \) is invertible and we can apply the implicit function theorem to conclude that there exists an \( \varepsilon_0 > 0 \) such that for
any $0 \leq \varepsilon < \varepsilon_0$ there exists a $\hat{q}_\varepsilon$ such that $F(\hat{q}_\varepsilon, \varepsilon) = 0$. Setting $\kappa_s = 1/\varepsilon$ will give that $\hat{q}_s = \hat{q}_\varepsilon$ is a solution for (22).

Before fixing $\varepsilon$, let us prove that the Hessian $\hat{K}$ of $\hat{U}$ at a candidate minimizer $\hat{q}_\varepsilon$ is positive definite. From the definition of the potential it follows that

$$
\hat{K} = \begin{bmatrix}
\kappa_{np} I_2 & 0 \\
0 & \kappa_s I_3
\end{bmatrix} + D^2 z_3,
$$

where $D^2 z_3$ is the Hessian of $z_3$ as a function of $z_1, z_2, \ell_1, \ell_2, \ell_3$. Note that the first term is positive definite and by choosing $\kappa_s$ and $\kappa_{np}$ sufficiently large, we can make it dominate the term $D^2 z_3$ such that $\hat{K}$ as a whole is positive definite. We finally choose $\varepsilon$ sufficiently small such that we obtain both that $\hat{q}_\varepsilon = \hat{q}_s$ is a minimizer of $\hat{U}$ and $\kappa_s = 1/\varepsilon$ is large enough that $\hat{K}$ is positive definite. \hfill \Box

To prove Proposition 7 we invoke the following Lemma.

**Lemma 10.** If $A_1, \ldots, A_n$ are positive semi-definite linear operators on a finite dimensional inner-product space $(V, \langle \cdot, \cdot \rangle)$ and $\bigcap_{k=1}^n \ker(A_k) = \{0\}$, then $A = \sum_{k=1}^n A_k$ is positive definite.

**Proof.** Clearly $A$ is positive semi-definite as a sum of semi-definite operators. We must prove that $A$ is definite. Assume $A$ is not definite so that there exists some non-zero $x \in V$ such that $\langle x, Ax \rangle = 0$. This latter equation can be written as $\sum_{k=1}^n \langle x, A_k x \rangle = 0$. By semi-definiteness of each $A_k$ this implies $\langle x, A_k x \rangle = 0$. This means that $A_k x = 0$ for each $k$. However the only such $x$ is 0. \hfill \Box

**Proof of Proposition 7** The Rayleigh dissipation function $R$ is a sum of three parts, each of which is a positive semi-definite operator. Specifically $R = R_{db} + R_{ns} + R_{shape}$, where $R_{db}$ is the Rayleigh function of $F_{db}$, and similarly for $R_{ns}$ and $R_{shape}$. As $F_{shape}$ acts only on the lengths between the particles, it is easy to see that it does not dampen rigid translations and rotations. In other words, the kernel of $F_{shape}$ is the (left) generator of the Lie algebra $\mathfrak{se}(2)$. Specifically, we find that the kernels (above $\hat{q}_\varepsilon$) are given by

$$
\ker(R_{db}) = \text{span}\left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_3} \right),
$$

$$
\ker(R_{ns}) = \text{span}\left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3} \right),
$$

$$
\ker(R_{shape}) = \text{span}\left( \sum_{i=1}^3 \frac{\partial}{\partial x_i}, \sum_{i=1}^3 \frac{\partial}{\partial z_i}, \sum_{i=1}^3 \left( z_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial z_i} \right) \right),
$$

where on the third line, the three vectors generate global translations in the $x$ and $z$ directions, and rotation about the origin, respectively, hence together span $\mathfrak{se}(2)$. We see that

$$
\ker(R_{db}) \cap \ker(R_{ns}) = \text{span}\left( \frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_3} \right).
$$
This only leaves translations of the third mass, but since the masses were assumed to form a non-degenerate triangle, the action of $\mathfrak{se}(2)$ cannot keep masses 1 and 2 fixed, while acting non-trivially on mass 3. Hence neither of $\frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_3}$ is in the kernel of $R_{\text{shape}}$. More precisely, assume that $\frac{\partial}{\partial x_3}$ is in the kernel of $R_{\text{shape}}$. Since it is obviously not in the kernel of the first two vectors spanning $R_{\text{shape}}$, it must hold that

$$\frac{\partial}{\partial x_3} = a \sum_{i=1}^{3} \frac{\partial}{\partial x_i} + b \sum_{i=1}^{3} \frac{\partial}{\partial z_i} + c \sum_{i=1}^{3} \left( z_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial z_i} \right)$$

with $c \neq 0$. This implies in particular that $0 = (b - cx_i) \frac{\partial}{\partial z_i}$, hence all $x_i$ must be equal (to $b/c$), which contradicts the assumption that the triangle is non-degenerate. The same argument can be made for $\frac{\partial}{\partial x_3}$ and so $\ker(R_{\text{dlb}}) \cap \ker(R_{\text{ms}}) \cap \ker(R_{\text{shape}}) = \{0\}$. By Lemma 10 it follows that $R$ is positive definite. □

**Proof of Proposition 8.** Firstly, $(\dot{q}_c, 0)$ is an equilibrium for the reduced system, and its linearization is given by Proposition 2. To assert that it is a robustly stable equilibrium, we consider its linearization (12),

$$\frac{d}{dt} \begin{bmatrix} \dot{q} \\ u \end{bmatrix} = A \begin{bmatrix} \dot{q} \\ u \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} 0 & p \\ -Kp^T & -\nu \end{bmatrix},$$

where $p = [I_5 \ 0]$ represents the principal bundle projection $\pi: Q \to Q/R$ in fiber-adapted coordinates. Recall that $K$ and $\nu$ are positive (semi-)definite matrices describing the linearized potential and friction forces, respectively. It follows from the definition $\dot{U} = U \circ p$ and $pp^T = I_5$, that $Kp^T = p^T K$.

Note that it is sufficient to prove that the linear flow satisfies $\|e^{At}\| \leq r < 1$ for some $t_0 > 0$, $r < 1$, and any choice of norm. From this it follows that the flow contracts exponentially for large $t$: write $t = nt_0 + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0, t_0)$, then we have

$$\|e^{At}\| = \|e^{A(n\tau_0 + \tau)}\| = \|e^{At_0}e^{A\tau}\| \leq \sup_{0 \leq \tau \leq t_0} \|e^{A\tau}\| \rho^n = Ce^{\rho t}$$

with $\rho = \frac{\log(r)}{t_0} < 0$ and $C = \sup_{0 \leq \tau \leq t_0} \|e^{A\tau}\| e^{-\rho \tau} < \infty$.

We choose the norm induced by the energy function

$$E_L(\dot{q}, u) = \frac{1}{2} \langle u, u \rangle + \frac{1}{2} \langle \dot{q}, \dot{K} \dot{q} \rangle$$

for the linear system (12), i.e. $E_L = \| \cdot \|^2$. This energy is a (non-strict) Lyapunov function in the sense that

$$\frac{dE_L}{dt} = \frac{\partial E_L}{\partial \dot{q}} \frac{d\dot{q}}{dt} + \frac{\partial E_L}{\partial u} \frac{du}{dt} = \langle \dot{K} \dot{q}, pu \rangle + \langle u, -Kp^T \dot{q} - \nu u \rangle = -\langle u, \nu u \rangle < 0$$

for all $u \neq 0$, since $\nu$ is positive definite. To prove that $\|e^{At_0}\| \leq r < 1$, let $\|(\dot{q}, u)\| = 1$ and note that since $E_L$ is non-increasing along solution curves, we can from now on restrict our analysis to the compact ball $B(0; 1) = E_L^{-1}([0, 1])$. 


The proof would be finished if $E_L$ were strictly decreasing, but this does not hold true for points $(\hat{q}, 0)$ in phase space. Instead, then, we have $\dot{u} = -p^T K \hat{q} \neq 0$, so after a short time interval, $u \neq 0$, and thus $E_L$ starts decreasing. Thus fixing a $t_0 > 0$, we find that $E_L$ strictly decreases along any solution curve over a time interval of length $t_0$, for all initial conditions $\|(\hat{q}, u)\| = 1$. By continuous dependence of a flow on initial parameters and compactness, it now follows that the decrease of $E_L$ is uniformly bounded away from zero, and hence we have $\|e^{A t_0}\| \leq r < 1$ for some $r < 1$. □

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**Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom**

*E-mail address:* j.eldering@imperial.ac.uk

**Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom**

*E-mail address:* hoj201@gmail.com