THE LIMITLESS FIRST INCOMPLETENESS THEOREM

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Abstract. This work is motivated from finding the limit of the applicability of the first incompleteness theorem (G1). A natural question is: can we find a minimal theory for which G1 holds? We examine the Turing degree structure of recursively enumerable (RE) theories for which G1 holds and the interpretation degree structure of RE theories weaker than the theory R with respect to interpretation for which G1 holds. We answer all questions that we posed in [2], and prove more results about them. It is known that there are no minimal essentially undecidable theories with respect to interpretation. We generalize this result and give some general characterizations which tell us under what conditions there are no minimal RE theories having some property with respect to interpretation.

1. Introduction

This work is motivated by the problem of finding the limit of the applicability of the first incompleteness theorem (G1). In this paper, we work with first-order theories with finite signature, and we equate a theory T with the set of theorems provable in T. Let T be a consistent recursively enumerable (RE) theory. To generalize G1 to weaker theories than PA with respect to (w.r.t. for short) interpretation, we introduce the notion “G1 holds for a theory T”.

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Definition 1.1 ([2]). We say that \( G_1 \) holds for an RE theory \( T \) if any consistent RE theory that interprets \( T \) is incomplete.

We say a theory \( T \) is essentially incomplete if any consistent RE extension of \( T \) in the same language is incomplete. The first incompleteness theorem tells us that \( \text{PA} \) is essentially incomplete. Motivated by finding the limit of the applicability of the first incompleteness theorem, we are interested in to what extent \( G_1 \) holds for theories weaker than \( \text{PA} \). The notion of essential incompleteness only refers to consistent RE extensions in the same language. To give a general formulation of \( G_1 \), in [2] we use the notion of interpretation such that we can compare different theories from distinctive languages.

We could define the notion “\( G_1 \) holds for a theory \( T \)” in different ways. For example, we could define this notion in a way that it is equivalent with \( T \) is recursively inseparable or \( T \) is effectively inseparable (for the definitions of these notions, we refer to Definition 2.9). This is a follow up paper of [2]. In this paper, we just follow the definition of “\( G_1 \) holds for a theory” in [2].

The theory of completeness/incompleteness is closely related to the theory of decidability/undecidability (see [19]). We say a theory \( T \) is essentially undecidable (EU) if any consistent RE extension of \( T \) in the same language is undecidable.

Proposition 1.2 ([19], see also [2]). For any consistent RE theory \( T \), \( G_1 \) holds for \( T \) iff \( T \) is essentially incomplete iff \( T \) is essentially undecidable.

It is well known that \( G_1 \) holds for Robinson Arithmetic \( \text{Q} \) and the theory \( \text{R} \) (see [19]).

In this paper, the notion of interpretation we use closely follows [22, 23]. An interpretation of a theory \( U \) in a theory \( V \) is based on a translation \( \tau \) of the \( U \)-language into the \( V \)-language. We will use the most inclusive notion to wit piecewise, multidimensional (with dimensions varying over pieces), relative, non-identity-preserving interpretability with parameters ([22]). A translation for the relational case commutes with the propositional connectives, and in some broad sense, it also commutes with the quantifiers but here there are a number of extra features. The translations could be more-dimensional in which a variable could be translated to an appropriate sequence of variables; we may have domain relativisation allowing the range of the translated quantifiers to be some...
domain definable in the $V$-language; the new domain may be built up from pieces of possibly different dimensions; we also allow parameters in an interpretation, and the translation may specify a parameter-domain; moreover, identity need not be translated to identity but can be translated to a congruence relation (23). We refer the reader for these definitions to 22, 23.

We say a theory $T$ is interpretable in a theory $S$ if there exists an interpretation of $T$ in $S$. Given theories $S$ and $T$, let $S \triangleleft T$ denote that $S$ is interpretable in $T$ (or $T$ interprets $S$); let $S \triangleright T$ denote that $T$ interprets $S$ but $S$ does not interpret $T$; we say $S$ and $T$ are mutually interpretable, denoted by $S \equiv I T$, if $S \triangleleft T$ and $T \triangleleft S$. In fact, $G1$ holds for many theories weaker than $\text{PA}$ w.r.t. interpretation.

In summary, we have the following picture (for definitions of these weak theories, we refer to [2]):

- $\text{Q} \triangleleft \text{I} \Sigma_0 + \exp \triangleleft \text{I} \Sigma_1 \triangleleft \text{I} \Sigma_2 \triangleleft \cdots \triangleleft \text{I} \Sigma_n \triangleleft \cdots \triangleleft \text{PA}$, and $G1$ holds for them.
- The theories $\text{Q}, \text{I} \Sigma_0, \text{I} \Sigma_0 + \Omega_1, \cdots, \text{I} \Sigma_0 + \Omega_n, \cdots, B \Sigma_1, B \Sigma_1 + \Omega_1, \cdots, B \Sigma_1 + \Omega_n, \cdots$ are all mutually interpretable, and $G1$ holds for them.
- Theories $\text{PA}^-, \text{Q}^+, \text{Q}^-, \text{TC}, \text{AS}, S_2^1$ and $\text{Q}$ are all mutually interpretable, and $G1$ holds for them.
- $\text{R} \not\triangleright \text{Q} \not\triangleleft \text{EA} \not\triangleleft \text{PRA} \not\triangleright \text{PA}$, and $G1$ holds for them.

A natural question is: can we find a minimal theory for which $G1$ holds? The answer of this question depends on our definition of minimality. Recall that we equate a theory with the set of theorems provable in it. If we define minimality as having the minimal number of axioms, then any finitely axiomatized essentially undecidable theory (e.g., Robinson Arithmetic $\text{Q}$) is a minimal RE theory for which $G1$ holds.

When we talk about minimality, we should specify the degree structure involved. Given two theories $U$ and $V$, let $U \leq_T V$ denote that $U$ is Turing reducible to $V$, and let $U <_T V$ denote that $U \leq_T V$ but $V \not\leq_T U$. In [2], we examine two degree structures that are respectively induced from Turing reducibility and interpretation: $\langle \text{D}_T, \leq_T \rangle$ and $\langle \text{D}_<, < \rangle$. We first introduce two key notions.

**Definition 1.3** ([2]).

This is only a selective picture to give readers a sense that there are many theories weaker than $\text{PA}$ for which $G1$ holds. For the full picture of the hierarchy of weak arithmetic theories, we refer to [9].
(1) Let $D_T = \{ S : S \text{ is RE, } S \lessdot_T R \text{ and } G_1 \text{ holds for } S \}$. 
(2) Let $D_I = \{ S : S \text{ is RE, } S \not\lessdot R \text{ and } G_1 \text{ holds for } S \}$. 

In [2], we show that there are no minimal RE theories w.r.t. Turing reducibility for which $G_1$ holds, and prove some results about the structure $⟨D_I, <⟩$. It is a question in [2] that whether $⟨D_I, <⟩$ has a minimal element? This question is answered negatively in [13] (c.f., Theorem 4.15). The following questions about $⟨D_I, <⟩$ are unanswered in [2]: 

**Question 1.4.**

(1) Can we show that for any Turing degree $0 < d \leq 0'$, there exists a theory $U \in D_I$ with Turing degree $d$?

(2) Are elements of $⟨D_I, <⟩$ comparable?

In this paper, we examine above questions and prove more facts about $⟨D_T, \leq_T⟩$ and $⟨D_I, <⟩$. The structure of this paper is as follows. In Section 2, we list definitions and facts we use in this paper. In Section 3, we examine and prove some facts about the structure $⟨D_T, \leq_T⟩$. In Section 4, we answer all questions about the structure $⟨D_I, <⟩$ in [2] and prove more results about $⟨D_I, <⟩$. It is proved in [13] that there are no minimal essentially undecidable theories with respect to interpretation. In Section 5, we generalize this result and give some general characterizations which tell us under what conditions there are no minimal RE theories having some property with respect to interpretation. In Section 6, we give some concluding remarks. In the Appendix, we give a brief overview of interpretation degree structures of three classes of theories in the literature: general RE theories, RE theories extending $PA$ and finitely axiomatized theories.

2. Preliminaries

In this paper, we always assume the *arithmetization* of the base theory. Given a sentence $φ$, let $⌜φ⌝$ denote the Gödel number of $φ$. Under arithmetization, we equate a set of sentences with the set of Gödel numbers of sentences.

Robinson Arithmetic $Q$ and the theory $R$ are both introduced in [19] by Tarski, Mostowski and Robinson as base axiomatic theories for investigating incompleteness and undecidability.
**Definition 2.1** (Robinson Arithmetic $Q$, [19]). Robinson Arithmetic $Q$ is defined in the language $\{0, S, +, \times\}$ with the following axioms:

1. $Q_1$: $\forall x \forall y(Sx = Sy \rightarrow x = y)$;
2. $Q_2$: $\forall x(Sx \neq 0)$;
3. $Q_3$: $\forall x(x \neq 0 \rightarrow \exists y(x = Sy))$;
4. $Q_4$: $\forall x \forall y(x + 0 = x)$;
5. $Q_5$: $\forall x \forall y(x + Sy = S(x + y))$;
6. $Q_6$: $\forall x(x \times 0 = 0)$;
7. $Q_7$: $\forall x \forall y(x \times Sy = x \times y + x)$.

Now, we introduce the theory $R$, which contains all key properties of arithmetic for the proof of $G_1$. The theory $R$ has the Turing degree $0'$.

**Definition 2.2** (The theory $R$, [19]). Let $R$ be the theory consisting of $Ax_1$-$Ax_5$ in the language $\{0, S, +, \times, \le\}$ where $n = S^n0$ for $n \in \omega$.

1. $Ax_1$: $m + n = m + n$;
2. $Ax_2$: $m \times n = m \times n$;
3. $Ax_3$: $mn \neq nm$ if $m \neq n$;
4. $Ax_4$: $\forall x(x \le n \rightarrow x = 0 \lor \cdots \lor x = n)$;
5. $Ax_5$: $\forall x(x \le \pi \lor \pi \le x)$.

Now, we introduce some basic notions in recursion theory.

**Definition 2.3** (Basic recursion theory, [15]).

1. Let $\langle \phi_e : e \in \omega \rangle$ be the list of all Turing programs, and $\langle W_e : e \in \omega \rangle$ be the list of all RE sets, where $W_e = \{x : \exists y T_1(e, x, y)\}$ and $T_1(z, x, y)$ is the Kleene predicate (cf.[11]).
2. Given $A, B \subseteq \omega$, we say that $A$ is *Turing incomparable* with $B$ if $A \not\le_T B$ and $B \not\le_T A$.
3. We say $S$ is a *minimal* RE theory w.r.t. Turing reducibility if there is no RE theory $V$ such that $V <_T S$.

Note that $x \in W_e$ if and only if for some $y$, the $e$-th Turing program with input $x$ yields an output in precisely $y$ steps.
Now, we introduce some notions about incompleteness and undecidability in the literature.

**Definition 2.4** (Incompleteness and undecidability, [19]).

1. We say a theory $T$ is *incomplete* if there exists a sentence in the language of $T$ which is neither provable nor refutable in $T$; otherwise, $T$ is *complete*.
2. A theory $T$ is *locally finitely satisfiable* if any finitely axiomatized sub-theory of $T$ has a finite model.
3. We say a theory $T$ is *undecidable* if the set of theorems of $T$ is not recursive; otherwise, $T$ is *decidable*.
4. We say a theory $T$ is *hereditarily undecidable* if any sub-theory $S$ of $T$ with the same language as $T$ is undecidable.

Now, we introduce some notions about interpretation and explain the meaning of minimality with respect to interpretation.

**Definition 2.5** (Interpretations).

- Given theories $S$ and $T$, we say that $S$ *tolerates* $T$ if $T$ is interpretable in some consistent extension of $S$ with the same language of $S$ (or equivalently, for some interpretation $\tau$, the theory $S + T^\tau$ is consistent).
- We say that a theory $S$ is *weaker* than a theory $T$ w.r.t. interpretation if $S \not\prec T$.
- Given theories $S$ and $T$, we say that $S$ is *incomparable* with $T$ w.r.t. interpretation if $S$ is not interpretable in $T$ and $T$ is not interpretable in $S$.
- We say $S$ is a *minimal* RE theory w.r.t. interpretation if there is no RE theory $T$ such that $T \not\prec S$.

The notion of interpretation provides us with a method to compare different theories in different languages. If $T$ is interpretable in $S$, then all sentences provable (refutable) in $T$ are mapped, by the interpretation function, to sentences provable (refutable) in $S$. The equivalence classes of theories, under the equivalence relation $\equiv_I$, are called the interpretation degrees.

The following two theorems are about important properties of the theory $R$ which we will use in this paper.

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3Note that $T^\tau$ should encompass the translations of the identity axioms and of the functionality axioms.
Theorem 2.6 (Visser, Theorem 6, [21]). For any RE theory $T$ with finite signature, $T$ is interpretable in the theory $R$ iff $T$ is locally finitely satisfiable.\footnote{In fact, if $T$ is locally finitely satisfiable, then $T$ is interpretable in $R$ via a one-piece one-dimensional parameter-free interpretation.}

Theorem 2.7 (Cobham, [22]). Any consistent RE theory that tolerates the theory $R$ is undecidable.

The following theorem provides us with a method for proving essential undecidability of a theory via interpretation.

Theorem 2.8 (Theorem 7, Corollary 2, [19]). Let $T_1$ and $T_2$ be two consistent theories with finite signature such that $T_1$ is interpretable in $T_2$. If $T_1$ is essentially undecidable, then $T_2$ is essentially undecidable.

Now, we introduce the notion of recursively inseparable theories and effectively inseparable theories.

Definition 2.9 (Inseparability, [17]). Let $T$ be a consistent RE theory, and $(A, B)$ be a pair of disjoint RE sets.

1. The pair $(T_P, T_R)$ is called the 	extit{nuclei} of the theory $T$, where $T_P$ is the set of Gödel numbers of sentences provable in $T$, and $T_R$ is the set of Gödel numbers of sentences refutable in $T$ (i.e., $T_P = \{ \varphi^* : T \vdash \phi \}$ and $T_R = \{ \varphi^* : T \vdash \neg \phi \}$).

2. We say $(A, B)$ is a recursively inseparable (RI) pair if there is no recursive set $X \subseteq \omega$ such that $A \subseteq X$ and $X \cap B = \emptyset$.

3. We say $T$ is recursively inseparable (RI) if $(T_P, T_R)$ is RI.

4. We say $(A, B)$ is effectively inseparable (EI) if there is a recursive function $f(x, y)$ such that for any $i$ and $j$, if $A \subseteq W_i$ and $B \subseteq W_j$ with $W_i \cap W_j = \emptyset$, then $f(i, j) \notin W_i \cup W_j$.

5. We say $T$ is effectively inseparable (EI) if $(T_P, T_R)$ is EI.

We list some properties of recursively inseparable theories and effectively inseparable theories we will use.

Fact 2.10. Let $S$ and $T$ be consistent RE theories.

1. If $S \subsetneq T$ and $S$ is RI(EI), then $T$ is RI(EI) \footnote{[17]}.\footnote{[17]}

2. The theory $R$ is EI. As a corollary, if $R \subsetneq T$, then $T$ is EI \footnote{[17]}.\footnote{[17]}.
(3) If \( T \) is EI, then \( T \) has Turing degree 0’ \([15]\).
(4) If \( T \) is EI, then \( T \) is RI, and if \( T \) is RI, then \( T \) is EU \([17]\).

**Definition 2.11 \([13, 2]\).** We introduce two natural operators on RE theories. The infimum \( U \oplus V \) is defined as follows: \( U \oplus V \) is a theory in the disjoint sum of the signatures of \( U \) and \( V \) plus a fresh 0-ary predicate symbol \( P \). The theory is axiomatised by all \( P \to \varphi \), where \( \varphi \) is a sentence in \( U \) plus \( \neg P \to \psi \), where \( \psi \) is a sentence in \( V \).

We can show that \( U \oplus V \) is the infimum of \( U \) and \( V \) in the interpretability ordering \(<\). This result works for all choices of our notion of interpretation (see \([23]\)).

3. The structure \( \langle D_T, \leq_T \rangle \)

In this section, we examine the structure \( \langle D_T, \leq_T \rangle \). Hanf shows that there is a finitely axiomatizable theory in each recursively enumerable tt-degree (see \([15]\)). Feferman shows in \([6]\) that if \( A \) is any recursively enumerable set, then there is a recursively axiomatizable theory \( T \) having the same Turing degree as \( A \). In \([16]\), Shoenfield improves Feferman’s result and shows that if \( A \) is a non-recursive RE set, then there is an essentially undecidable theory having the same Turing degree as \( A \).

**Theorem 3.1** (Shoenfield, \([16]\)). Suppose \( A \) is a non-recursive RE set. Then:

(1) there is a recursively inseparable pair \( \langle B, C \rangle \) such that \( A, B \) and \( C \) have the same Turing degree;

(2) there is a consistent axiomatizable theory \( T \) having one non-logical symbol which is essentially undecidable and has the same Turing degree as \( A \).

Thus, the structure \( \langle D_T, \leq_T \rangle \) is as complex as the Turing degree structure of RE sets. Janiczak’s theory \( J \) is introduced in \([10\ p.136]\), and is used in Shoenfield’s proof of Theorem 3.1 in \([16]\). The theory \( J \) has only one binary relation symbol \( E \) with the following axioms.

**J1:** \( E \) is an equivalence relation.

**J2:** There is at most one equivalence class of size precisely \( n \).
J3: There are at least \( n \) equivalence classes with at least \( n \) elements.\(^5\)

Let \( \Phi_n \) denote the sentence: there exists an equivalence class of size precisely \( n+1 \). Note that the \( \Phi_n \)'s are mutually independent over \( J \).

**Theorem 3.2** (Janiczak, [10]).

- \( J \) is decidable (see [10] Theorem 4).
- Over \( J \), every sentence is equivalent with a Boolean combination of the \( \Phi_n \)'s (see [10] Lemma 2), and this Boolean combination can be found explicitly from the given sentence.\(^6\)

**Theorem 3.3.** Suppose \( A \) is a non-recursive RE set. Then there is a partial recursive function \( g \) such that the following holds, where \( B_n = \{ x : g(x) = n \} \):

1. \((B_n, B_m)\) is a recursively inseparable pair for \( m \neq n \) and each \( B_n \) has the same Turing degree as \( A \);
2. The theory \( T(B_n, B_m) = J + \{ \Phi_i : i \in B_n \} + \{ \neg \Phi_i : i \in B_m \} \) for any \( m \neq n \) is recursively inseparable and has the same Turing degree as \( A \).

**Proof.** Our proof is inspired by Exercise 10-20(ii) in p.178 of [15]. Suppose \( A \) is a non-recursive RE set. Let \( f \) be the recursive function that enumerates \( A \) without repetitions. Define a function \( g \) as follows: \( g(\langle x, y \rangle) = n \) iff for some \( s \), \( f(s) = x \) and the program \( \phi_y \) with input \( \langle x, y \rangle \) yields \( n \) as output in \( < s \) steps.

Note that \( g \) is partial recursive. Define \( B_n = \{ x : g(x) = n \} \). Note that for any \( n \neq m \), \( B_n \) and \( B_m \) are disjoint RE sets.

We first show that for any \( m \neq n \), \((B_n, B_m)\) is a recursively inseparable pair and \( B_n \equiv_T B_m \equiv_T A \).

**Claim.** \( B_n \leq_T A \).

\(^5\)Our presentation of the theory \( J \) follows [13]. We include the axiom J3 to make the proof of the following fact in Theorem 3.2 more easy: over \( J \), every sentence is equivalent with a boolean combination of the \( \Phi_n \)'s.

\(^6\)This is a reformulation of Janiczak’s Lemma 2 in [10] in the context of J. Janiczak’s Lemma is proved by means of a method known as the elimination of quantifiers.
Proof. We test \langle x, y \rangle \in B_n as follows. If \( x \notin A \), then \( \langle x, y \rangle \notin B_n \). Suppose \( x \in A \). Take the unique \( s \) such that \( f(s) = x \). We can decide whether the program \( \phi_y \) with input \( \langle x, y \rangle \) yields \( n \) in \( < s \) steps. If yes, then \( \langle x, y \rangle \in B_n \); if no, then \( \langle x, y \rangle \notin B_n \). \hfill \Box

Claim. \( A \leq_T B_n \).

Proof. We test \( x \in A \) as follows. Suppose the index of the program with constant output value \( n \) is \( e_0 \), i.e. \( \phi_{e_0}(x) = n \) for all \( x \). If \( \langle x, e_0 \rangle \in B_n \), then \( x \in A \). Suppose \( \langle x, e_0 \rangle \notin B_n \), let \( w \) be the number of computation steps such that \( \phi_{e_0}(\langle x, e_0 \rangle) = n \). Decide whether \( f(s) = x \) and \( s \leq w \) for some \( s \). If yes, then \( x \in A \); if no, then \( x \notin A \). \hfill \Box

Claim. There is no recursive set \( X \) that separates \( B_n \) and \( B_m \) (i.e., \( B_n \subseteq X \) and \( X \cap B_m = \emptyset \)).

Proof. Suppose not, i.e., we can find a recursive function \( h \) such that \( \text{ran}(h) = \{m, n\} \), \( h[B_n] = \{m\} \) and \( h[B_m] = \{n\} \). Let \( h = \phi_{e_1} \).

Subclaim. If \( x \in A \), then the program \( \phi_{e_1} \) with input \( \langle x, e_1 \rangle \) yields output in \( \geq s \) steps where \( f(s) = x \).

Proof. Suppose not, i.e., \( x \in A \), but the program \( \phi_{e_1} \) with input \( \langle x, e_1 \rangle \) halts in less than \( s \) steps where \( f(s) = x \). Then by definition, we have \( \phi_{e_1}(\langle x, e_1 \rangle) = n \iff \phi_{e_1}(\langle x, e_1 \rangle) = m \), which leads to a contradiction. \hfill \Box

Let \( t \) be the recursive function such that \( t(x) = \) the number of steps to compute the value of \( \langle x, e_1 \rangle \) in the program \( \phi_{e_1} \). Then we have:

\[ x \in A \iff \exists s (s \leq t(x) \land f(s) = x). \]

Thus, \( A \) is recursive which leads to a contradiction. \hfill \Box

Thus, for any \( n \neq m \), \((B_n, B_m)\) is a RI pair and \( B_n \equiv_T B_m \equiv_T A \).

Define the theory \( T(B_n, B_m) = J + \{ \Phi_i : i \in B_n \} + \{ \neg \Phi_i : i \in B_m \} \) for any \( m \neq n \). Now we show that \( T(B_n, B_m) \) has the same Turing degree as \( A \).

Note that \( T(B_n, B_m) \) is a consistent RE theory. Since the \( \Phi_i \)'s are mutually independent over \( J \), we have \( \Phi_i \) is provable iff \( i \in B_n \), and \( \neg \Phi_i \) is provable iff \( i \in B_m \). Thus, \( B_n \) and
B_m are recursive in T_{(B_n,B_m)}. By Theorem 3.2, T_{(B_n,B_m)} is recursive in B_n and B_m. Since B_n \equiv_T B_m \equiv_T A, T_{(B_n,B_m)} \equiv_T A. Since (B_n,B_m) is a RI pair, by a standard argument, T_{(B_n,B_m)} is recursively inseparable. □

Theorem 3.3 improves Theorem 3.1 and shows that Shoenfield’s theory T in Theorem 3.1 is not unique. Since there are only countably many RE theories, Theorem 3.3 is the best result we can have.

Now, we list some key results about the Turing degree structure of RE sets in the literature.

**Fact 3.4 ([15]).**

1. The RE degrees are dense: for any RE sets A < T B, there is a RE set C such that A < T C < T B (Sacks, Theorem 4.1, [18]).
2. There are no minimal non-zero RE degrees (follows from (1)).
3. For any RE set 0 < T A < T 0', there exists an RE set B such that B is incomparable with A w.r.t. Turing reducibility. Moreover, an index for B can be found effectively from the index for A (Yates, Theorem 4.3, [18]).
4. Given RE sets A < T B, there is an infinite RE sequence of RE sets C_n such that A < T C_n < T B and C_n’s are incomparable w.r.t. Turing reducibility (Robinson, p. 147, [18]).
5. If a and b are RE degrees such that a < b, then any countably partially ordered set can be embedded in the RE degrees between a and b (Robinson, p. 147, [18]).

**Theorem 3.5.** (1) \( \langle D_T, \leq T \rangle \) is dense: for any A, B \in D_T such that A < T B, there exist countably many theories C \in D_T such that A < T C < T B.

2. \( \langle D_T, \leq T \rangle \) has no minimal elements.
3. For any theory A \in D_T, there exist countably many theories B \in D_T such that A < T B. Thus, \( \langle D_T, \leq T \rangle \) has no maximal element.
4. For any theory A \in D_T, there exist countably many theories B \in D_T such that B is incomparable with A w.r.t. Turing reducibility.
5. Given theories A, B \in D_T such that A < T B, there is an infinite RE sequence of theories C_n \in D_T such that A < T C_n < T B and C_n’s are incomparable w.r.t. Turing reducibility.
Given theories $A, B \in D_T$ such that $A <_T B$, any countably partially ordered set can be embedded in the structure $\langle D_T, \leq_T \rangle$ between $A$ and $B$.

$\langle D_T, \leq_T \rangle$ is a dense distributive lattice without endpoints.

Proof. (1): follows from Fact 3.4(1) and Theorem 3.3.
(2): follows from Fact 3.4(2).
(3): follows from Fact 3.4(1) and Theorem 3.3.
(4): follows from Fact 3.4(3) and Theorem 3.3.
(5): follows from Fact 3.4(4).
(6): follows from Fact 3.4(5).
(7): follows from (1)-(3).

4. The structure $\langle D_I, \triangleleft \rangle$

The interpretation degree structure of essentially incomplete RE theories weaker than the theory $R$ is much more complex. In this section, we answer all questions proposed in [2], and prove more results about $\langle D_I, \triangleleft \rangle$.

**Theorem 4.1** ([2]). For any recursively inseparable pair $(A, B)$, there exists a consistent RE theory $T_{(A, B)}$ such that $G_1$ holds for $T_{(A, B)}$ and $T_{(A, B)} \triangleleft R$.

Since by Theorem 3.3 there are countably many recursively inseparable pairs, by Theorem 4.1 the cardinality of $D_I$ is $\aleph_0$.

Theorem 4.1 is a reformulation of the proof of Theorem 1 in [16]. For proof details, we refer to [13].

**Theorem 4.2** (Shoenfield, [16]). Let $A$ be a RE set. Then we can effectively find a disjoint pair $(B, C)$ of RE sets such that:

1. $B, C \leq_T A$;
2. for any RE set $D$ that separates $B$ and $C$ (i.e., $B \subseteq D$ and $D \cap C = \emptyset$), we have $A \leq_T D$;
3. If $A$ is non-recursive, then $(B, C)$ is recursively inseparable.

There is no direct relation between the notion of interpretation and the notion of Turing reducibility. Given RE theories $U$ and $V$, $U \triangleleft V$ does not imply $U \leq_T V$, and
$U \leq_T V$ does not imply $U \vartriangleleft V$. Now, we introduce the notion of Turing persistence which establishes the relationship between $U \vartriangleleft V$ and $U \leq_T V$.

**Definition 4.3** ([13]). We say a RE theory $U$ is Turing persistent if for any consistent RE theory $V$, if $U \subseteq V$, then $U \leq_T V$.

Note that if $U$ is Turing persistent, then for any RE theory $V$, if $U \vartriangleleft V$, then $U \leq_T V$. Note that since any consistent RE theory interpreting $R$ is $EI$ and hence has the Turing degree $0'$, any consistent RE theory interpreting $R$ is Turing persistent.

Now we answer Question 1.4(1) in [2]: can we show that for any Turing degree $0 < d \leq 0'$, there is a theory $U$ such that $G_1$ holds for $U$, $U \not\vartriangleleft R$ and $U$ has Turing degree $d'$? Proposition 4.5 answers this question positively.

**Lemma 4.4.** Let $T$ be a consistent RE theory. If $T$ tolerates $R$, then $T$ is hereditarily undecidable.

*Proof.* Suppose $T$ tolerates $R$, but $T$ is not hereditarily undecidable. Then there exists a sub-theory $S$ of $T$ with the same language as $T$ such that $S$ is decidable. Since $T$ tolerates $R$, $S$ tolerates $R$. By Theorem 2.7, $S$ is undecidable, which leads to a contradiction. □

Proposition 4.5(1) is due to Shoenfield (see Theorem 4.9 in [13]). For completeness, we give full details of the properties of the theory $T_d$ in Proposition 4.5.

**Proposition 4.5.** Let $A$ be a RE set with Turing degree $d$. Then we can effectively find a consistent RE theory $T_d$ having one non-logical symbol such that

1. $T_d$ has the same Turing degree as $A$, and $T_d$ is Turing persistent (see Theorem 4.9 in [13]);
2. $T_d \not\vartriangleleft R$;
3. If $A$ is not recursive, then $T_d$ is $RI$.

*Proof.* Suppose $A$ is a RE set with Turing degree $d$, $(B,C)$ is the disjoint pair of RE sets effectively constructed from $A$ as in Theorem 4.2.

1. We show that $T_d$ has the same Turing degree as $A$. Define the theory $T_{(B,C)} = J + \{\Phi_n : n \in B\} + \{\neg\Phi_n : n \in C\}$. Let $T_d$ denote the theory $T_{(B,C)}$. Note that $T_d$ is
a consistent RE theory. Since the $\Phi_n$’s are mutually independent over $J$, we have $\Phi_n$ is provable iff $n \in B$, and $\neg \Phi_n$ is provable iff $n \in C$. Thus, $B$ and $C$ are recursive in $T_d$.

By Theorem 3.2, $T_d$ is recursive in $B$ and $C$. Since $B, C \leq_T A$ by Theorem 4.2, we have $T_d$ is recursive in $A$. Since $B$ separates $B$ and $C$, by Theorem 4.2 $A \leq_T B$. Thus, since $B$ is recursive in $T_d$, $T_d$ has the same Turing degree as $A$.

Now we show that $T_d$ is Turing persistent. Suppose $V$ is a consistent RE extension of $T_d$. Define $D = \{n : V \vdash \Phi_n\}$. Note that $B \subseteq D$ and $D \cap C = \emptyset$. By Theorem 4.2 $A \leq_T D$. Since $T_d$ has the same Turing degree as $A$, we have $T_d \equiv_T A \leq_T D \leq_T V$. Thus, $T_d$ is Turing persistent.

(2) We show that $T_d \not\preceq \mathbb{R}$. By Theorem 3.2 every sentence is equivalent with a Boolean combination of the $\Phi_n$’s over $J$. Since the $\Phi_n$’s are mutually independent over $J$, any finitely axiomatized sub-theory of $T_d$ has a finite model. Thus, $T_d$ is locally finitely satisfiable. Thus, by Theorem 2.6 we have $T_d \not\preceq \mathbb{R}$. We show that $\mathbb{R}$ is not interpretable in $T_d$. Note that the theory $J$ is a decidable sub-theory of $T_d$ in the same language. Since $T_d$ is not hereditarily undecidable, by Lemma 4.1 $\mathbb{R}$ is not interpretable in $T_d$. Thus $T_d \not\preceq \mathbb{R}$.

(3) We show that if $A$ is non-recursive, then $T_d$ is RI. By Theorem 4.2 $(B, C)$ is RI. We denote the theory $T_d$ by $T$. Define $g : n \mapsto \neg \Phi_n$. Note that $g$ is recursive, if $n \in B$, then $g(n) \in T_P$, and if $n \in C$, then $g(n) \in T_R$. We show that $T$ is RI. Suppose $T$ is not RI, i.e., there is a recursive set $X$ such that $T_P \subseteq X$ and $X \cap T_R = \emptyset$. Note that $B \subseteq g^{-1}[T_P] \subseteq g^{-1}[X]$ and $C \subseteq g^{-1}[T_R] \subseteq g^{-1}[X] = g^{-1}[X]$. Since $g$ and $X$ are recursive, $g^{-1}[X]$ is recursive. Thus, $g^{-1}[X]$ is a recursive set separating $B$ and $C$, which contradicts that $(B, C)$ is RI. Thus, $T_d$ is RI.

Finally, we remark that the construction of the theory $T_d$ is effective: there exists a recursive function $f$ such that if $A = W_e$, then there exists a RE theory $T$ with index $f(e)$ such that $T$ has the properties stated in Proposition 4.5. □

Corollary 4.6. There is no minimal recursively inseparable (RI) RE theory w.r.t. Turing reducibility.

Note that for Turing persistent theories, if they are comparable w.r.t. interpretation, then they are comparable w.r.t. Turing reducibility. From Proposition 4.5, given Turing
incomparable non-recursive RE sets, we can find incomparable RE theories in $D_I$ w.r.t. interpretation. Now we answer the following question in [2]: are elements of $\langle D_I, \triangleleft \rangle$ comparable? Theorem 4.7 answers this question negatively.

**Theorem 4.7.** Given RE sets $A <_T B$, there is a sequence of RE theories $\langle S_n : n \in \omega \rangle$ such that:

1. $S_n \in D_I$;
2. $A <_T S_n <_T B$;
3. $S_n$ is Turing persistent;
4. $S_n$ are incomparable w.r.t. interpretation.

**Proof.** By Fact 3.4(4), there exists a sequence of RE sets $\langle C_n : n \in \omega \rangle$ such that $A <_T C_n <_T B$ and $C_n$ are incomparable w.r.t. Turing reducibility. By Proposition 4.5, for each $n$, we can find a Turing persistent RE theory $S_n \in D_I$ with the same Turing degree as $C_n$. Thus, $A <_T S_n <_T B$ for each $n$. Since each $S_n$ is Turing persistent, and $C_n$’s are incomparable w.r.t. Turing reducibility, we have $S_n$’s are incomparable w.r.t. interpretation. □

As a corollary of Theorem 4.7, there are countably many incomparable elements of $\langle D_I, \triangleleft \rangle$. This answers Question 1.4(2).

**Proposition 4.8.** If $T \in D_I$, then $T$ is not finitely axiomatized.

**Proof.** Suppose $T \in D_I$, but $T$ is finitely axiomatized. Since $T \triangleleft R$, $T$ is locally finitely satisfiable. Since $T$ is finitely axiomatized, $T$ has a finite model, which contradicts the fact that $T$ is essentially undecidable. □

Proposition 4.8 shows that there is no finitely axiomatized theory interpretable in $R$ for which G1 holds.

**Lemma 4.9** (Lemma 4.8 in [2]; Theorem 2.2 in [13]). For RE theories $A$ and $B$, if G1 holds for both $A$ and $B$, then G1 holds for $A \oplus B$.

**Definition 4.10** ([22]). Given consistent RE theories $U$ and $V$, the supremum theory $U \otimes V$ is defined as follows. The signature of $U \otimes V$ is the disjoint union of the signatures of $U$ and $V$, plus two new unary predicates $\triangle_0$ and $\triangle_1$. The axioms of $U \otimes V$ are:
(1) \( P(x_0, \cdots, x_{n-1}) \rightarrow \bigwedge_{i<n} \Delta_0(x_i) \), if \( P \) is derived from the signature of \( U \);
(2) \( Q(y_0, \cdots, y_{m-1}) \rightarrow \bigwedge_{j<m} \Delta_0(x_j) \), if \( Q \) is derived from the signature of \( V \);
(3) the axioms of \( U \) relativized to \( \Delta_0 \);
(4) the axioms of \( V \) relativized to \( \Delta_1 \);
(5) \( \forall x (\Delta(x) \vee \Delta_1(x)) \);
(6) \( \forall x (\neg (\Delta_0(x) \land \Delta_1(x))) \).

We know that the interpretation degree structures of RE theories and essentially undecidable RE theories with the operators \( \oplus \) and \( \otimes \) are a distributive lattice.

**Theorem 4.11.** Let \( A \) be a RE theory with Turing degree less than \( \text{0}' \) for which \( G_1 \) holds. Then we can effectively find a Turing persistent theory \( S \), which is incomparable with \( A \) with respect to Turing degrees, such that \( G_1 \) holds for \( S \) and \( A \otimes S \not\equiv A \).

**Proof.** By Fact 3.4(3), we can effectively find a RE set \( C \) such that \( A \) is incomparable with \( C \) w.r.t. Turing reducibility. By Proposition 4.5, from \( C \) we can effectively find a Turing persistent theory \( S \) with the same Turing degree as \( C \) such that \( G_1 \) holds for \( S \). Let \( B = A \otimes S \). Suppose \( B \subset A \). Since \( S \subset B \subset A \) and \( S \) is Turing persistent, we have \( C \equiv_T S \leq_T A \), which leads to a contradiction. Thus, \( A \not\subset B \). \( \square \)

**Fact 4.12** (Theorem XVI, [15]). The set \( \{ e : W_e \text{ is recursive} \} \) is \( \Sigma^0_3 \)-complete.

**Theorem 4.13.** \( \{ e : G_1 \text{ holds for } W_e \} \) is \( \Pi^0_3 \)-complete.

**Proof.** Define \( U = \{ e : W_e \text{ is undecidable} \} \) and \( V = \{ e : G_1 \text{ holds for } W_e \} \). By Fact 4.12 \( U \) is \( \Pi^0_3 \)-complete. It is easy to check that \( V \) is \( \Pi^0_3 \).

**Claim.** There is a recursive function \( s \) such that for any \( n \in \omega \):
(1) \( n \in U \iff s(n) \in V \);
(2) \( W_n \equiv_T W_{s(n)} \).

**Proof.** Recall that the construction of the pair \( (B, C) \) from a given RE set \( A \) as in Theorem 4.2 is effective: there are recursive functions \( f \) and \( g \) such that if \( A = W_e \), then

\footnote{The first \( W_e \) in the definition of \( U \) stands for a set of numbers, and the second \( W_e \) in the definition of \( V \) stands for a theory as a set of theorems.}
Thus, there are no minimal elements of \( \langle \{ (1) \} \rangle \). Recall that the construction of \( T_{(B,C)} \) from a given disjoint pair \( (B,C) \) of RE sets in Proposition 4.5 is also effective: there is a recursive function \( h(x) \) such that if \( B = W_n \) and \( C = W_m \) and \( B \cap C = \emptyset \), then \( T_{(B,C)} = W_{h(n,m)} \) such that \( T_{(B,C)} \) has the same Turing degree as \( A \).

Define \( s(n) = h(f(n), g(n)) \). Note that \( s \) is recursive. Suppose \( A = W_e \) and \( (B,C) = (W_{f(e)}, W_{g(e)}) \) are constructed from \( A \) as in Theorem 4.2. Then \( s(e) \) is the index of the theory \( T_{(B,C)} \) constructed from \( (B,C) \). By Proposition 4.5, we have \( W_e \equiv_T W_{s(e)} \).

Now, we show that \( e \in U \) iff \( s(e) \in V \). Suppose \( e \in U \). Since \( W_e \) is non-recursive, \((W_{f(e)}, W_{g(e)}) \) is RE by Theorem 4.2. By Proposition 4.5, \( T_{(W_{f(e)}, W_{g(e)})} = W_{s(e)} \) is recursively inseparable and hence essentially undecidable. Thus, \( s(e) \in V \). Suppose \( e \notin U \). Since \( W_e \) is recursive and \( T_{(W_{f(e)}, W_{g(e)})} = W_{s(e)} \equiv_T W_e \), we have \( s(e) \notin V \). Thus \( e \in U \leftrightarrow s(e) \in V \). \( \square \)

Thus, \( V \) is \( \Pi^0_3 \)-complete. \( \square \)

**Remark.** Let \( U = \{ e : W_e \) is undecidable \( \} \) and \( V = \{ e : W_e \in D_1 \) and \( W_e \) is Turing persistent \( \} \). Let \( s \) be the recursive function defined in Theorem 4.14. Note that \( s(e) \) is the index of the theory \( T_{(B,C)} \) constructed in Proposition 4.5 where \( (B,C) \) is constructed from \( A = W_e \) as in Theorem 4.2. If \( e \in U \), by Proposition 4.5, \( s(e) \in V \). If \( e \notin U \), then \( W_{s(e)} \) is decidable. Thus, \( e \in U \leftrightarrow s(e) \in V \).

**Fact 4.14** ([15]).

1. \( \{ e : W_e \) is simple \( \} \) is \( \Pi^0_3 \)-complete (Exercise 14-31(i), p. 334, [15]).
2. \( \{ e : W_e \) is co-infinite \( \} \) is \( \Pi^0_3 \)-complete (Corollary XVI, p. 328, [15]).

As a corollary of Fact 4.14 and Theorem 4.13, there exists a recursive function \( f \) such that \( W_e \) is simple iff \( G_1 \) holds for \( W_{f(e)} \); and there exists a recursive function \( g \) such that \( W_e \) is co-infinite iff \( G_1 \) holds for \( W_{g(e)} \).

**Theorem 4.15** (Murwanashyuka, Pakhomov and Visser, Theorem 1.1 in [13]). *There are no minimal essentially undecidable recursively enumerable theories w.r.t. interpretation.*

As a corollary of Theorem 4.15, for any \( T \in D_1 \), there exists \( S \in D_1 \) such that \( S \nsubseteq T \). Thus, there are no minimal elements of \( \langle D_1, \prec \rangle \). This answers an question in [2].
5. Some generalizations

The paper [13] gives two proofs of Theorem 4.15. The first proof employs a direct
diagonalisation argument and the second proof uses some recursion theoretic result. In
this section, we generalize the two proofs of Theorem 4.15 in [13] as in Theorem 5.4 and
Theorem 5.5.

Now we first generalize the first proof of Theorem 4.15 in [13]. Recall the Janiczak
theory $J$ and that the sentence $\Phi_n$ says that there exists an equivalence class of size
precisely $n + 1$.

**Definition 5.1** ([13]). Given $X \subseteq \mathbb{N}$, we say that $W$ is a $J,X$-theory when
$W$ is axiomatised over $J$ by boolean combinations of the sentences $\Phi_n$ for
$n \in X$.

**Theorem 5.2** (Theorem 4.5 in [13]). Given any essentially undecidable theory $U$, we
can effectively find a recursive set $X$ (from an index of $U$) such that no consistent $J,X$-
theory interprets $U$. As a corollary, for any essentially undecidable theory $U$, we can
effectively find an essentially undecidable theory $W$ such that $U$ is not interpretable in
$W$.

From Theorem 5.2, for any essentially undecidable theory $U$, we can effectively find
an essentially undecidable theory $V$ such that $V \not\preceq U$.\footnote{Take the essentially undecidable theory $W$ from Theorem 5.2 such that $U$ is not interpretable in $W$. Let $V = U \oplus W$. By Lemma 4.9, $V$ is essentially undecidable. Since $U$ is not interpretable in $W$, we have $U$ is not interpretable in $V$. Thus, $V \not\preceq U$.}

**Lemma 5.3.** Suppose $(Y,Z)$ is an effectively inseparable pair of RE sets. Define the
theory $V = J + \{\Phi_n : n \in Y\} + \{\neg\Phi_n : n \in Z\}$. Then $V$ is EI.

**Proof.** We want to find a recursive function $h(i,j)$ such that if $V_P \subseteq W_i, V_R \subseteq W_j$ and
$W_i \cap W_j = \emptyset$, then $h(i,j) \notin W_i \cup W_j$.

Define the function $f : n \mapsto \Phi_n$. Clearly, $f$ is recursive. By s-m-n theorem, there is a
recursive function $g$ such that $f^{-1}[W_i] = W_g(i)$. Suppose $(Y,Z)$ is effectively inseparable
via the recursive function $t(i,j)$. Define $h(i,j) = f(t(g(i),g(j)))$. Clearly, $h$ is recursive.

Suppose $V_P \subseteq W_i, V_R \subseteq W_j$ and $W_i \cap W_j = \emptyset$. Note that $Y \subseteq f^{-1}[V_P] \subseteq f^{-1}[W_i] = W_g(i)$, and $Z \subseteq f^{-1}[V_R] \subseteq f^{-1}[W_j] = W_g(j)$. Note that $W_{g(i)} \cap W_{g(j)} = \emptyset$ since $W_i \cap W_j = \emptyset$.\footnote{Take the essentially undecidable theory $W$ from Theorem 5.2 such that $U$ is not interpretable in $W$. Let $V = U \oplus W$. By Lemma 4.9, $V$ is essentially undecidable. Since $U$ is not interpretable in $W$, we have $U$ is not interpretable in $V$. Thus, $V \not\preceq U$.}
∅. Since \((Y, Z)\) is EI via the function \(t\), we have \(t(g(i), g(j)) \notin W_{g(i)} \cup W_{g(j)}\). Then, \(h(i, j) \notin W_i \cup W_j\). Thus, \(V\) is EI. □

**Theorem 5.4.** Let \(P\) be some property of RE theories, and \(S, T\) be any consistent RE theories. Suppose the following conditions hold:

1. The operator \(\oplus\) is closed under the property \(P\): if \(S\) and \(T\) have the property \(P\), then \(S \oplus T\) also has the property \(P\);
2. If \(S\) has the property \(P\), then \(S\) is essentially undecidable;
3. If \(S\) is EI, then \(S\) has the property \(P\).

Then for any RE theory \(U\) with the property \(P\), we can effectively find a RE theory \(T\) with the property \(P\) such that \(T \nsubseteq U\). As a corollary, there are no minimal RE theories with the property \(P\) w.r.t. interpretation.

**Proof.** Let \(U\) be a consistent RE theory with the property \(P\). By Condition (2), Theorem 5.2 applies to RE theories with the property \(P\). Thus, we can effectively find a recursive set \(X\) (from an index of \(U\)) such that no consistent \(J, X\)-theory interprets \(U\). Let \(Y\) and \(Z\) be effectively inseparable RE sets which are subsets of \(X\). Define the theory \(V\) as \(V = J + \{\Phi_n : n \in Y\} + \{\neg \Phi_n : n \in Z\}\). By Lemma 5.3, \(V\) is EI. By Condition (3), \(V\) has the property \(P\). Let \(T = U \oplus V\). By condition (1), \(T\) has the property \(P\). Since no consistent \(J, X\)-theory interprets \(U\), \(U\) is not interpretable in \(V\). Thus, \(T \nsubseteq U\) since \(U\) is not interpretable in \(T\). □

Now we generalize the second proof of Theorem 4.15 in [13]. We say \(\subseteq\) is a partial pre-ordering on RE theories if \(\subseteq\) is reflexive and transitive: i.e., for any RE theory \(S, T\) and \(U\), we have \(S \subseteq S\), and if \(S \subseteq T\) and \(T \subseteq U\), then \(S \subseteq U\). Given any partial pre-ordering \(\subseteq\) on RE theories, we define \(S \equiv T \equiv S \subseteq T\) and \(T \subseteq S\). From this equivalence relation, we can introduce a degree structure generated from this equivalence relation. Let \(T \sqsubseteq S\) denote that \(T \subseteq S\) but \(S \subseteq T\) does not hold. We say \(S\) is a minimal RE theory w.r.t. the relation \(\subseteq\) if there is no RE theory \(T\) such that \(T \sqsubseteq S\).

Let \(P\) be any property of RE theories (e.g., \(EU\), \(EI\), etc). Let \(D_P\) be the collection of RE theories with the property \(P\). Theorem 5.5 provides us with a general way to
show that \( \langle D_P, \sqsubseteq \rangle \) has no minimal elements w.r.t. the degree structure induced from the relation \( \sqsubseteq \).

**Theorem 5.5.** Given a property \( P \) of RE theories, suppose \( \sqsubseteq \) is a partial pre-ordering on RE theories satisfying the following conditions for some \( n \in \omega \):

1. For any \( A, B \in D_P \), there is a lower bound \( A \boxplus B \) of \( A \) and \( B \) under \( \sqsubseteq \) such that \( A \boxplus B \in D_P \);
2. If \( A \in D_P \) and \( A \sqsubseteq B \), then \( B \in D_P \);
3. The relation \( A \sqsubseteq B \) is \( \Sigma^0_n \);
4. \( V = \{ e : W_e \text{ has the property } P \} \) is \( \Pi^0_n \)-complete.

Then \( \langle D_P, \sqsubseteq \rangle \) has no minimal element.

**Proof.** Suppose \( \langle D_P, \sqsubseteq \rangle \) has a minimal element \( T \). We show that for any RE theory \( A \in D_P \), we have \( T \sqsubseteq A \). Note that \( T \boxplus A \in D_P \). Since \( T \) is a minimal element of \( D_P \), and \( T \boxplus A \in D_P \) is a lower bound of \( T \) and \( A \) by condition (1), we have \( T \sqsubseteq T \boxplus A \).

Since \( T \boxplus A \sqsubseteq A \), we have \( T \sqsubseteq A \).

Note that by condition (2), \( e \in V \iff T \sqsubseteq W_e \). By condition (3), \( V \) is \( \Sigma^0_n \) which contradicts condition (4). \( \square \)

**Theorem 5.6.** Let \( P \) be some property of RE theories, and \( S, T \) be any consistent RE theories. Suppose the following conditions hold:

1. the operator \( \oplus \) is closed under the property \( P \): if \( S \) and \( T \) have the property \( P \), then \( S \oplus T \) also has the property \( P \);
2. For any essentially undecidable RE theory \( U \), we can effectively find (an index of) a RE theory \( V \) with the property \( P \) such that \( U \nvdash V \).
3. If a theory has the property \( P \), then it is essentially undecidable.

Then for any RE theory \( U \) with the property \( P \), we can effectively find a RE theory \( T \) with the property \( P \) such that \( T \not\sqsubset U \).

**Proof.** Let \( U \) be a RE theory with the property \( P \). By Condition (3), \( U \) is essentially undecidable. By Condition (2), we can effectively find (an index of) a RE theory \( V \) with

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\( \text{Here, } "W_e \text{ has the property } P" \) means that the RE theory with index \( e \) has the property \( P \).
the property $P$ such that $U \not\supset V$. Let $T = U \oplus V$. By Condition (1), $T$ has the property $P$. Since $U \not\supset V$, we have $T \not
i U$. □

Note that Theorem 5.4 and Theorem 5.6 tell us more information than Theorem 5.5. The proof of Theorem 5.5 is not effective: it shows that there are no minimal theories with the property $P$ w.r.t. interpretation, but it does not tell us that given a theory with the property $P$, we can effectively find a weaker theory with the property $P$ w.r.t. interpretation. But Theorem 5.4 and Theorem 5.6 also tell us that we can effectively find a weaker theory with the property $P$ w.r.t. interpretation, given a theory with the property $P$.

We can view the second proof of Theorem 4.15 in [13] as a special case of Theorem 5.5 in which $n = 3$, $P$ is the property of essential undecidability, $\sqsubseteq$ is the interpretation relation, and $A \boxplus B$ is $A \oplus B$. We can view the first proof of Theorem 4.15 in [13] as a special case of Theorem 5.4 and Theorem 5.6 where $P$ is the property of essential undecidability.

**Theorem 5.7** (Theorem 3.15, [4]). *There are no minimal effectively inseparable theories with respect to interpretability: for any effectively inseparable theory $T$, we can effectively find a theory which is effectively inseparable and strictly weaker than $T$ w.r.t. interpretation.*

We can view the proof of Theorem 3.15 in [4] as a special case of Theorem 5.4 and Theorem 5.6 where $P$ is the property of effective inseparability.

Before we discuss the application of Theorem 5.4, Theorem 5.5 and Theorem 5.6 to essentially hereditarily undecidable theories, we first introduce the notion of essentially hereditarily undecidable theories.

**Definition 5.8** (Essentially hereditarily undecidable theories, [19]). Let $T$ be a consistent RE theory.

1. We say $T$ is hereditarily undecidable (HU) if every sub-theory $S$ of $T$ over the same language is undecidable.

2. We say $T$ is essentially hereditarily undecidable (EHU) if any consistent RE extension of $T$ is HU.
Visser gave two proofs that there is no interpretability minimal essentially hereditarily undecidable RE theory: Theorem 41 and Theorem 44 in [23].

**Theorem 5.9 ([23]).**

1. For any essentially undecidable RE theory \( U \), we can effectively find (an index of) an essentially hereditarily undecidable RE theory \( V \) such that \( U \not\equiv V \) (Theorem 44 in [23]).
2. There is no interpretability minimal essentially hereditarily undecidable RE theory (Theorem 41 in [23]).

Visser proved in [23] that essentially hereditarily undecidable theories have the following properties:

1. If \( S \) and \( T \) are essentially hereditarily undecidable theories, then \( S \oplus T \) is also essentially hereditarily undecidable (Theorem 15, [23]);
2. Suppose \( U \) is consistent and essentially hereditarily undecidable and \( U \equiv V \). Then \( V \) is essentially hereditarily undecidable (Theorem 16, [23]).

It is easy to check that if \( T \) is essentially hereditarily undecidable, then \( T \) is essentially undecidable. From the argument of Theorem 41 in [23], we can show that \( \{ e : \mathcal{W}_e \text{ is essentially hereditarily undecidable} \} \) is \( \Pi^0_3 \)-complete. From these properties, it is easy to see that Theorem 5.5 and Theorem 5.6 apply to essentially hereditarily undecidable theories. We can view the proof of Theorem 44 in [23] as a special case of Theorem 5.6 where \( P \) is the property of essential hereditary undecidability. We can view the proof of Theorem 41 in [23] as a special case of Theorem 5.5 in which \( n = 3, P \) is the property of essential hereditary undecidability, \( \sqsubseteq \) is the interpretation relation, and \( A \sqcup B \) is \( A \oplus B \). However, Condition (3) of Theorem 5.4 does not apply to essentially hereditarily undecidable theories since effective inseparability does not imply essential hereditary undecidability (see Theorem 5.4 in [5]).

We conclude this section with some comments about Creative theories and Rosser theories. We first give some definitions. Let \( T \) be a consistent RE theory in the language of arithmetic or in a language admitting numerals, and \( (A, B) \) be a disjoint pair of RE sets. We say \( T \) is **Creative** if \( T_P \) is creative. We say \( (A, B) \) is **separable** in \( T \) if there is a
formula $\phi(x)$ with only one free variable such that if $n \in A$, then $T \vdash \phi(n)$, and if $n \in B$, then $T \vdash \neg \phi(n)$. We say $T$ is Rosser if any disjoint pair of RE sets is separable in $T$.

Note that $\{ e : W_e \text{ is creative} \}$ is $\Sigma^0_3$-complete (Exercise 14-3(iii), p. 334 in [15]). Thus, Condition (4) in Theorem 5.5 does not apply to Creative theories. Being Creative does not imply being essentially undecidable (see Theorem 4.12 in [5]). Thus, Condition (2) in Theorem 5.4 and Condition (3) in Theorem 5.6 does not apply to Creative theories.

For Rosser theories, we can show that if $S$ and $T$ are Rosser theories, then $S \oplus T$ is also Rosser as follows. Let $(A, B)$ be a disjoint pair of RE sets. Since $S$ is Rosser, there exists a formula $\phi(x)$ such that $n \in A \Rightarrow S \vdash \phi(n)$ and $n \in B \Rightarrow S \vdash \neg \phi(n)$. Since $T$ is Rosser, there exists a formula $\psi(x)$ such that $n \in A \Rightarrow T \vdash \psi(n)$ and $n \in B \Rightarrow T \vdash \neg \psi(n)$. Define $\varphi(x) := (P \rightarrow \phi(x)) \land (\neg P \rightarrow \psi(x))$. We can check that $n \in A \Rightarrow S \oplus T \vdash \varphi(n)$ and $n \in B \Rightarrow S \oplus T \vdash \neg \varphi(n)$. However, we do not know whether for any essentially undecidable RE theory $U$, we can effectively find (an index of) a Rosser theory $V$ such that $U \not\equiv V$. We did not explore this in this work.

6. Conclusion

We give some concluding remarks. The existence of a non-recursive RE set is essential to understand the incompleteness phenomenon. Given any non-recursive RE set $A$, we can effectively construct a RE theory with the same Turing degree as $A$ for which $G_1$ holds. In [8], Harvey Friedman proves $G_2$ for theories interpreting $I\Sigma_1$ based on the existence of a remarkable set which is equivalent to the existence of a non-recursive RE set. Whether there are minimal RE theories for which $G_1$ holds depends on the definition of minimality. The fact that there are no minimal RE theories for which $G_1$ holds w.r.t. Turing reducibility and interpretation shows that the incompleteness phenomenon is omnipresent, and there is no limit of $G_1$ w.r.t. Turing reducibility and interpretation.

Both [2] and this paper are about the limit of $G_1$. A natural question is: what is the limit of the second incompleteness theorem $G_2$ if any? Both mathematically and philosophically, $G_2$ is more problematic than $G_1$. In the case of $G_1$, we are mainly

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\footnote{For example, let $X$ be a creative set. Define $T := J + \{ \Phi_n : n \in X \}$. It is easy to check that $T$ is Creative, but $J + \{ \Phi_n | n \in \omega \}$ is a consistent complete decidable RE extension of $T$ (Theorem 4.12 in [5]).}
interested in the fact that *some* sentence is independent of the base theory. But in the case of \( G_2 \), we are also interested in the content of the consistency statement. We can say that \( G_1 \) is extensional in the sense that we can construct a concrete mathematical statement which is independent of the base theory without referring to arithmetization or provability predicate. However, \( G_2 \) is intensional and “whether the consistency of a theory \( T \) is provable in \( T \)” depends on many factors such as the way of formalization, the base theory we use, the way of coding, the way to express consistency, the provability predicate we use, etc. For the discussion of the intensionality of \( G_2 \), we refer to [3].

Let \( T \) be a consistent RE theory in the language of arithmetic or in a language admitting numerals via interpretation. We say \( G_2 \) holds for \( T \) if \( T \not\vdash \text{Con}(T) \).\(^{11}\) Pudlák shows that no RE theory \( T \) (in predicate logic) interprets \( Q + \text{Con}(T) \) (see [14]). As a corollary, \( G_2 \) holds for \( Q \). Authors in [1] proved that \( G_2 \) holds for \( Q \) using a totally different method.

**Theorem 6.1** (Theorem 1, [1]). Let \( L \) be any formal system with a recursive set of axioms, a finite number of finitary and recursive rules of inference including modus ponens and having \( A \rightarrow A \) as a theorem for all sentences \( A \). Then \( \text{Con}(L) \) is not provable in \( T_0 \), where \( T_0 \) is a weak arithmetic theory with the property that \( Q \subseteq T_0 \subseteq \text{PA} \).\(^{12}\)

I list the question whether \( G_2 \) holds for \( R \) (i.e. whether \( R \not\vdash \text{Con}(R) \) holds) as an open question in [3]. In fact, as a corollary of Theorem 6.1 since \( R \subseteq \text{PA} \), we have \( G_2 \) holds for the theory \( R \).

We know that \( G_1 \) holds for many RE theories \( T \) such that \( T \) has Turing degree less than \( 0' \) and \( T \not\subseteq R \). However, we do not know whether there exists a RE theory \( T \) such that \( G_2 \) holds for \( T \), \( T \) has Turing degree less than \( 0' \) and \( T \not\subseteq R \). We did not explore this question.

We conclude this paper with some informal comments. We know that given an essentially incomplete (effectively inseparable, essentially hereditarily undecidable) RE

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\(^{11}\)We assume that \( \text{Con}(T) \) is the canonical arithmetic formula expressing the consistency of the base theory \( T \) defined as \( \neg \text{Pr}_T(0 \neq 0) \) saying that \( 0 \neq 0 \) is not provable in \( T \) where the coding we use is the standard Gödel coding and the provability predicate we use satisfies Hilbert-Bernays-Löb derivability conditions.

\(^{12}\)For the definition of \( T_0 \), we refer to [1].
theory, we can effectively find another strictly weaker essentially incomplete (effectively inseparable, essentially hereditarily undecidable) theory w.r.t. interpretation. These facts reveal that the first incompleteness theorem is limitless, and provide us with more evidences from logic that the incompleteness phenomenon is ubiquitous (pervasive) in abstract formal theories. From the viewpoint of logic, in matters of incompleteness everything is always as bad as it can get via the magic tools from logic, and these negative results are just weird facts about an already strange phenomenon. However, if we only consider those “natural” mathematical theories, my personal view is that the theory $R$ is the weakest “natural” mathematical theory which is essentially incomplete even if we do not have a formal definition of naturalness in the literature.

7. Appendix

In this Appendix, we give a brief overview of interpretation degree structures of three classes of theories in the literature: general RE theories, RE theories extending $\text{PA}$ and finitely axiomatized theories. In this Appendix, we use the most inclusive notion of interpretation as explained in the introduction section.

Definition 7.1 (Distributive lattice, [12]).

1. A lattice is an algebraic structure $(L, \lor, \land)$, consisting of a set $L$ and two binary, commutative and associative operations $\lor$ and $\land$ on $L$ satisfying the following axiomatic identities for all elements $a, b \in L$ (sometimes called absorption laws):
   
   (A) $a \lor (a \land b) = a$;
   
   (B) $a \land (a \lor b) = a$.

2. A lattice is distributive if it satisfies $a \lor (b \land c) = (a \lor b) \land (a \lor c)$ for all $a, b, c \in L$.

We first examine the interpretation degree structure of general RE theories. The interpretation degree structure of RE theories extending $\text{PA}$ is well known in the literature. We have some equivalent characterizations of the interpretation relation between consistent RE extensions of $\text{PA}$. Given a theory $T$, define $T \upharpoonright k = \{n \in T : n \leq k\}$.

Fact 7.2 (Orey-Hájek Characterization, [12]). Suppose theories $S$ and $T$ are consistent RE extensions of $\text{PA}$. Then the following are equivalent:

1. $S \triangleleft T$;
Let \( \langle \mathcal{D}_{PA}, < \rangle \) denote the interpretation degree structure of consistent RE extensions of \( PA \). In fact, \( \langle \mathcal{D}_{PA}, <, \downarrow, \uparrow \rangle \) is a dense distributive lattice\(^{13}\), where

\[
A \downarrow B = T + \{ \text{Con}(A | k) \lor \text{Con}(B | k) : k \in \omega \}
\]

and

\[
A \uparrow B = T + \{ \text{Con}(A | k) \land \text{Con}(B | k) : k \in \omega \}.
\]

For more properties of \( \langle \mathcal{D}_{PA}, < \rangle \), we refer to [12].

The interpretation degree structure of finitely axiomatized theories, denoted by \( \langle \mathcal{D}_{\text{fin}}, < \rangle \), is studied by Harvey Friedman in [7]. Note that there are only \( \aleph_0 \) many interpretation degrees of finitely axiomatized theories. The structure \( \langle \mathcal{D}_{\text{fin}}, < \rangle \) forms a (reflexive) partial ordering with a minimum element \( \top \) and a maximum element \( \bot \), where \( \top \) is the equivalence class of all sentences with a finite model, and \( \bot \) is the equivalence class of all sentences with no models.

**Theorem 7.3** ([7]).

1. The structure \( \langle \mathcal{D}_{\text{fin}}, <, \oplus, \otimes \rangle \) is a distributive lattice.
2. The structure \( \langle \mathcal{D}_{\text{fin}}, < \rangle \) is dense, i.e., \( a \not< b \rightarrow (\exists c)(a \not< c \not< b) \).
3. For any \( a, b \in \mathcal{D}_{\text{fin}} \), if \( a \not< b \), then there exists an infinite sequence \( c_n \) such that \( a \not< c_n \not< b \) for each \( n \) and \( c_n \)'s are incomparable w.r.t. interpretation.

Thus, \( \langle \mathcal{D}_{\text{fin}}, <, \oplus, \otimes \rangle \) is a dense distributive lattice. The structure \( \langle \mathcal{D}_{\text{fin}}, < \rangle \) bears a rough resemblance to the Turing degree structure of RE sets.

\(^{13}\)We say that \( \langle \mathcal{D}_{PA}, <, \downarrow, \uparrow \rangle \) is dense if for any \( A, B \in \mathcal{D}_{PA} \) such that \( A \not< B \), there exists \( C \in \mathcal{D}_{PA} \) such that \( A \not< C \not< B \).
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