Modeling the Evolution of Networks as Shrinking Structural Diversity

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September 22, 2020

Abstract

This article reviews and evaluates models of network evolution based on the notion of structural diversity. We show that diversity is an underlying theme of three principles of network evolution: the preferential attachment model, connectivity and link prediction. We show that in all three cases, a dominant trend towards shrinking diversity is apparent, both theoretically and empirically. In previous work, many kinds of different data have been modeled as networks: social structure, navigational structure, transport infrastructure, communication, etc. Almost all these types of networks are not static structures, but instead dynamic systems that change continuously. Thus, an important question concerns the trends observable in these networks and their interpretation in terms of existing network models. We show in this article that most numerical network characteristics follow statistically significant trends going either up or down, and that these trends can be predicted by considering the notion of diversity. Our work extends previous work observing a shrinking network diameter to measures such as the clustering coefficient, power-law exponent and random walk return probability, and justifies preferential attachment models and link prediction algorithms. We evaluate our hypothesis experimentally using a diverse collection of twenty-seven temporally evolving real-world network datasets.
1 Introduction

Networks provide an adequate and established model for studying a broad range of complex structures and processes. Common examples include friendships between people, Web pages connected by hyperlinks, interactions between users and content items, references between scientific works, communication and transport infrastructure. What these examples all have in common is their dynamics: Over time, all networks change. In some cases, new edges appear, such as in an email network when a new email is sent, or new nodes appear, such as in a scientific publication network in which a new author appears. An important question in the area of network analysis is thus: What are the patterns under which networks evolve? This question can and has been answered in many different ways, depending on the type of network and the network measure considered. In this work, we argue that the question can be answered by the notion of diversity. We note that many numerical network measures can be interpreted as a measure of structural diversity, and are observed in actual networks to evolve monotonically over time. The main observation that we make is that these numerical measures are as a general rule evolving in the direction that can be interpreted as shrinking diversity, regardless of the application area of the network. This statement is of course not an absolute one – it is not true that all network measures evolve consistently towards less diversity over time in all network types. Instead, we will show that this happens in a high number of cases, which is statistically and systematically larger than what would be expected by a random behavior.

The notion of diversity that we consider is entirely structural. In other words, we consider only the structure of a network, and not its content. As an example, we do not consider the change in the contents of emails written by people over time, but only who people are writing to. However, content can often be represented as a network, and in that case we do consider it. For instance, the bag of words model can be seen as a bipartite graph connecting documents with words. To make the notion of diversity more precise, we propose three principles of network evolution that fall under the umbrella term shrinking diversity:

- **Preferential attachment**: The notion that the richer get richer, i.e., that nodes with many neighbors tend to attract new links faster than other nodes is a well-established principle in modeling networks, and
constitutes the basis for the scale-free graph model of Barabási and Albert [3] and many other models. Measures of the equality of the degree distribution can be interpreted as measures of diversity and are predicted to increase under this model.

- **Increasing connectivity**: Many numerical network characteristics can be interpreted as a form of connectivity. In a highly connected network, less subgraphs with low connectivity to the rest of the network exist, and thus an increasing connectivity can be interpreted as shrinking diversity.

- **Link prediction algorithms**: The problem of link prediction is the task of predicting which edges will appear in a network, given the current network. For a given link prediction algorithm, we can identify numerical network measures that must increase or decrease over time if edges are added according to the predictions of the algorithm. A network that grows according to a link prediction algorithm will tend to consolidate its structure and not add any new structure, decreasing the diversity of the network.

The contribution of this work is thus to (1) review numerical network measures in light of the notion of structural diversity, (2) introduce a methodology for measuring whether a numerical network measure is significantly increasing or decreasing, (3) perform corresponding tests for a large set of numerical measures and network datasets from different areas, and (4) show that the three network evolution principles are valid, validating the shrinking diversity hypothesis. The article is structured as follows: Section 2 gives the definition of the concept of diversity, states the three principles of network evolution, and states our hypothesis of shrinking diversity. Section 3 introduces the eleven network measures we consider. Section 4 describes our systematic experiments on a collection of twenty-seven temporal network datasets in order to test our hypothesis. We conclude in Section 5. This article is partially based on previous conference papers by the author [41, 42].

## 2 Network Evolution and Diversity

Diversity is generally defined as the quality of a collection of things containing many *different* or *unlike* objects. In the context of networks, the diversity of
Table 1: The measures of diversity we study. The first column gives the aspect of a network that is covered by the measure. The second column describes in what case a network can be called diverse under that aspect.

| Aspect                | A network has shrinking diversity when                                      |
|-----------------------|-------------------------------------------------------------------------------|
| 3.1 Preferential attachment | Nodes with many neighbors acquire new neighbors faster than others        |
| 3.2 Connectivity      | Distinct parts of the network become better connected over time              |
| 3.3 Link prediction   | Its evolution follows link prediction functions                             |

a system can be understood as the diversity of opinions, topics, communities or any other entities represented by the network. For instance, in a movie recommender system, we understand that the community has more diversity when the movies being watched and rated are different from one user to another. On the other hand, a community in which most people watch the same fixed set of movies is not diverse. This notion of diversity is independent of the notion of size: A movie recommender community may have many users and include many movies, and still lack diversity, because most users have seen the same set of movies. Thus, diversity does not denote the size but the distinctness of the content. In the context of a network such as the user–movie graph, diversity is thus achieved when many users have seen different sets of movies. Equivalently, we can require that individual movies have been seen by different sets of users.

We investigate three concepts of network analysis that capture the notion of diversity:

- **Preferential attachment**: By the principle of preferential attachment, nodes with high degree receive new edges faster than nodes with small degree, implying that the inequality of the degree distribution increases over time. Thus, a network has diversity when all nodes have approximately the same number of neighbors, and has low diversity when some nodes have many neighbors and some nodes have very few neighbors.

- **Increasing connectivity**: A well-known result in network analysis is
that of a shrinking diameter, i.e., the observation that the diameter of real-world networks tends to decrease over time. This result can be generalized to other measures of connectivity, giving predictions of increasing connectivity, which can be interpreted as shrinking diversity, in the sense that a network that is well connected allows less room for individual communities, and thus is less diverse.

- **Link prediction functions**: A network has shrinking diversity when its evolution follows link prediction functions. A function that predicts the evolution of a network cannot predict new network structures (since they do not exist yet), but can only predict the strengthening of existing network structure. Thus, if a network evolves in accordance with a link prediction function, we interpret its diversity as shrinking. The numerical measures in this category are thus such that their temporal evolution is monotonous under various link prediction models.

Based on these three aspects of diversity, we derive individual numerical measures that we interpret in terms of diversity in Section 3. Table 1 gives an overview of the aspects and measures.

### 2.1 Related Work

The concept of *diversity* is broad and relates to many different areas of research, with many different definitions. In this subsection, we very briefly mention approaches to the concept that are not directly used in the rest of the work.

Changes in variables describing communities have been studied in organizational behavior studies [1]. An article by Hannan [22] analyses stability of organizational structures, in the continuous domain, and on the level of individual organizations. Another article [23] considers the inertia of organizations. A study about multiple organizations and their decline (which can be interpreted as less diversity) is given by [24].

Similarly, the competitive exclusion principle [25] can be interpreted as a decrease in diversity: the fact that multiple species competing in a single ecological niche does not represent a stable equilibrium, but will tend to the extinction of all but one species.

In sociology, inequalities in “human” distributions have been considered, as well as urban hierarchies, and how humans adapt, [26]. In this context, humans tend to move to an equilibrium, or evolve [27].
The concept of the diffusion of innovations by Rogers [56] is an early (pre-network) theory that explains the diffusion of an innovation. It can be seen as related to the concept of preferential attachment in that people are likely to connect to new, popular trends (innovations). In terms of diversity, the adoption of a single innovation by a population can be interpreted as the reduction of diversity, as it results in the complete population using one and the same technology.

Shrinking diversity is also explained in sociology by the theory of ingroups and outgroups [28, 60], in which people tend to favor persons from their own social groups, thus reinforcing existing social structures.

Also related are theories of sociological change [50].

The article [51] reviews multiple models of (among others) changes in network properties. The review covers various densities, Leskovec’s diameter, evolutionary-like approaches, whose outcome depends strongly on the community’s properties, and thus do not generalize. The work also considers communities that interact with each other, explaining the structural changes in each community by their interaction with other communities.

Homophily can also explain the dynamics of groups, i.e., that people tend to form groups with other persons that have similar attributes to themselves, resulting in clustered social networks [29].

Another related field of research is that of the dynamics of interorganizational ties [13, 58, 59], according to which organizational systems reach capacity in terms of the number of competitors that are in a field. When a field reaches capacity, the nature of the relationships changes from competitors helping and communicating with each other to more conventionally competitive behavior and turning to network ties that are more complementary, thus decreasing the diversity of the network.

Case studies that have observed shrinking diversity include the work by Bryant and Monge [8], as well as that by Brown and Ashman [7].

2.2 Evolution of Diversity Measures

Within the field of network analysis, an important aspect is that of network evolution, i.e., the understanding of the temporal changes in a network’s structure. A well-known example of a temporal trend in a numerical network measure is that of a shrinking diameter. By studying several large temporal networks, Leskovec et al. [45] discovered that the 90-percentile effective diameter (i.e., the average number of hops needed to reach 90% of
a network’s nodes) shrinks over time. This result leads to several follow-up questions: Since the diameter can be understood as a measure of connectivity, are other measures of connectivity shrinking too? Since the authors of that work described a network growth model to explain shrinking diameters, do other network growth models also lead to monotonous trends in network measures? In order to answer this and related questions, this article will study eleven numerical network measures. The hypothesis studied in this paper is as follows:

**Hypothesis** The structural diversity of most evolving networks is shrinking.

During the lifetime of a network, a given network measure will be subject to nontrivial fluctuations, and the temporal evolution of network measures is not always necessarily monotonous. For instance, the network density (i.e., the mean degree of nodes) has been shown to first grow very fast, then decline, and then end up growing slowly for the rest of a network’s lifetime [36]. As this example shows, it is important to distinguish between the behavior of a network measure of the network’s full lifetime, and the behavior of a network measure at one specific point in the lifetime of the network. Thus, we will test both cases in our experiments.

Since it is not possible to define in a general manner what most networks represent, we are only able to test this hypothesis on a large collection of networks that are available to us. Although these networks are from a diverse range of areas, we are aware of the bias inherent in any such study.

### 2.3 Definitions

Let $G = (V, E)$ be an undirected multigraph. We allow multiple edges between two vertices, and thus $E$ is a multiset. The degree $d(u)$ of a vertex $u$ is the number of incident edges to $u$, taking into account parallel edges.

Some networks are bipartite, i.e., their vertex set can be partitioned into two sets such that all edges connect one set with the other. An example is an interaction network between users and movies, in which each edge represents an interaction between a user and a movie. Most network measures apply to bipartite networks without problem. Out of the network measures covered in this article, the only exception is the clustering coefficient, which is well-defined and always zero for bipartite networks. We will therefore not consider the clustering coefficient for bipartite networks.

Let $A \in \mathbb{R}^{|V| \times |V|}$ be the symmetric adjacency matrix of $G$, containing the
multiplicity of the edges. Thus, \( A_{uv} \) equals the number of edges connecting \( u \) and \( v \). We also consider the diagonal degree matrix \( D \in \mathbb{R}^{|V| \times |V|} \) defined by \( D_{uu} = d(u) \). Finally, the matrix normalized adjacency matrix \( Z \) is defined as \( Z = D^{-1/2} A D^{-1/2} \).

# 3 Measures of Structural Network Diversity

We now review the eleven numerical measures of structural network diversity. All eleven measures are summarized in Table 2. In addition to these eleven measures of diversity, we will also explore the average degree \( d = 2|E|/|V| \) in our experiments, since it has been shown to evolve monotonically in the literature [36, 45].

For some measures of connectivity, monotonicity proofs exist, which show that when an edge is added to a connected network, the diversity cannot increase, but can only decrease or remain constant. These proofs are only valid for the connected case, i.e., when an edge is added to a connected network. The case of an added node is not considered, as it renders the graph disconnected. The proofs will be given along with the definitions of the measures. As a trivial example, the average degree \( d \) is strictly monotonous since adding an edge increases \( |E| \) but not \( |V| \).

## 3.1 Preferential Attachment

The temporal evolution of equality measures for the degree distribution can be predicted under the model of preferential attachment. Preferential attachment is a general principle of network growth which states that new edges will connect to a vertex with a probability that is proportional to the importance of that vertex [3]. The preferential attachment model thus predicts that nodes with high degree will receive more edges fast, and thus their degree will grow fast, while small degrees will only grow slowly. The result is a long-tailed degree distribution, in which most degrees are small, and few degrees are large. To measure the extent of this long-tailedness of the degree distribution, we use four different measures.

A related but different type of preferential attachment is preferential attachment on eigenvectors, and is equivalent to the spectral network evolution hypothesis [39], which is exploited in link prediction algorithms and is covered in Section 3.3 together with other link prediction methods.
Table 2: The eleven measures of diversity considered in this article. We also show the average degree as reference measure (first line).

| Aspect | Measure | Symbol | Range | Pr. | Mono. |
|--------|---------|--------|-------|-----|-------|
| 3.1.1  | Pref. att. | Gini coefficient | $G$ | $[0, 1]$ | ↗ | — |
| 3.1.2  | Pref. att. | Jain’s index | $J$ | $(0, 1]$ | ↘ | — |
| 3.1.3  | Pref. att. | Power-law exponent | $\gamma$ | $(1, \infty)$ | ↘ | — |
| 3.1.4  | Pref. att. | Relative edge distribution entropy | $H_{er}$ | $(0, \infty)$ | ↘ | — |
| 3.2.1  | Connect. | 90-percentile effective diameter | $\delta_{0.9}$ | $(0, \infty)$ | ↘ | ✓ |
| 3.2.2  | Connect. | Random walk return probability | $\vartheta_r(n)$ | $[1, \infty)$ | ↘ | — |
| 3.2.3  | Connect. | Relative controllability | $C_r$ | $(0, 1]$ | ↘ | ✓ |
| 3.2.4  | Connect. | Algebraic connectivity | $\alpha$ | $(0, \infty)$ | ↗ | ✓ |
| 3.3.1  | Link pred. | Clustering coefficient | $c$ | $[0, 1]$ | ↗ | — |
| 3.3.2  | Link pred. | Fractional rank | $\text{rank}_F$ | $[1, \infty)$ | ↘ | — |
| 3.3.3  | Link pred. | Eigenvalue power-law exponent | $\alpha$ | $(1, \infty)$ | ↗ | — |

**Pr.** Predicted trend according to the shrinking diversity hypothesis.

**Mono.** A monotonicity proof exists, showing that adding an edge to a connected network cannot increase the diversity according to the given measure.

In an undirected network, each edge is attached to two nodes. We can therefore consider each edge to belong to the two nodes that the edge connects. Thus, a network can be viewed as a distribution of half-edges over vertices. The number of edges owned by a vertex is then equal to the number of neighbors of that vertex, i.e., the degree. The sum of all degrees in the network thus equals twice the number of edges in the network.

The interpretation of the equality of degrees as diversity can be illustrated with a user–movie network. When all movies in that network have been seen by the same number of people, the diversity of the network is high. If instead a small number of movies have been seen by many people and most movies have been seen by only few people, then the network is not diverse. Thus, network diversity can be measured by inspecting how far the distribution of edges to nodes is away from an equitable distribution.

### 3.1.1 Gini Coefficient

The Gini coefficient is a measure of the inequality of a distribution, typically used in economics to measure the inequality of the income distribution in
a country. As a diversity measure for networks, we apply it to the degree distribution, as described in [41]. A measure of inequality can be interpreted to denote the opposite of diversity, since a network in which the distribution of edges is equal can be understood as having more diversity.

**Related Measures** The Hoover index (or Robin Hood index) is also a measure from economics equal to the relative amount of total income that must be redistributed for the distribution to become fully equal [12]. Another related measure is the balanced inequality ratio [41], which appears in statements of the form “X percent of the population own (100 – X) percent of assets.” For instance, the well-known 80–20 rule states in its original form that 80% of land area in Italy was owned by 20% of people, leading to a value of 0.2. All three measures are ultimately based on the Lorenz curve, and were found to correlate highly in our experiments, and thus only the Gini coefficient is investigated.

### 3.1.2 Jain’s Index

Another index of equality that can be applied to the degree distribution is the index of Jain [32]. This measure is used in computer networking to measure the fairness of resource allocation. For instance, it is used to process Internet packets from different sources equally. It is defined as

\[
J = \frac{\left(\sum_{u \in V} d(u)\right)^2}{n \cdot \sum_{u \in V} d(u)^2}.
\]

(1)

This index is maximally one for a completely equal distribution of degrees. The theoretical minimum of this index is \(1/|V|\) when all edges attach to a single node. The minimum realizable by a simple network is \(4(|V| - 1)/|V|^2\), i.e., slightly under \(4/|V|\). Thus, this index can be considered to be in range \((0, 1]\) for large networks.

### 3.1.3 Power-law Exponent

In the area of network analysis, the phrase *degree distribution* is mostly associated with the phrase *power law*. This is based on the observation that in many networks, the number of vertices with degree \(d\) is roughly proportional to the power \(d^{-\gamma}\), where the exponent \(\gamma > 2\) is a parameter, called the power-law exponent. Power-law degree distributions arise for instance in the
preferential attachment model of Barabási and Albert [3], giving a value of \( \gamma = 3 \) for the basic preferential attachment model.

Since not all degree distributions are precise power laws, the power law exponent is not strictly defined for all networks. Nonetheless, an estimation of the exponent is often used as a numerical network measure. In the experiments of this article, we use the method described in [52, Equation (5)] to estimate the power-law exponent, defining the power-law exponent \( \gamma \) as

\[
\gamma = 1 + n \left( \sum_{u \in V} \ln \frac{d(u)}{d_{\text{min}}} \right)^{-1},
\]

in which \( d_{\text{min}} \) is the minimum degree in the network. Note that this method returns values of \( \gamma \) in the range \((1, \infty)\), i.e., the values may be smaller or equal to two.

A misconception about the exponent \( \gamma \) is that the degrees are more unequal when its value is high. However, the opposite is true: The degrees are more unequal when \( \gamma \) is small [41, Fig. 6]. Thus, a shrinking diversity implies a shrinking value of \( \gamma \).

### 3.1.4 Relative Edge Distribution Entropy

The entropy is a measure used in thermodynamics to characterize the disorder of a physical system. In information theory, the entropy is a measure of the quantity of information. In a network, we can compute both the entropy of the edge distribution as well as the entropy of the degree distribution.

Given a probability distribution \( P(x) \) over a finite set \( x \in X \), the entropy \( H \) of \( P \) is defined as \( H(P) = \sum_{x \in X} -P(x) \ln P(x) \). The entropy can be interpreted as a measure of the uniformity of a distribution: It is zero when \( P(x_0) = 1 \) for some \( x_0 \), and reaches its maximal value of \( \ln(|X|) \) for the uniform distribution \( P(x) = 1/|X| \) for all \( x \). We apply the entropy to the distribution of edges over vertices to define the edge distribution entropy. In a graph \( G = (V, E) \), the edge distribution entropy is thus

\[
H_e = \sum_{u \in V} \frac{d(u)}{2|E|} \ln \frac{d(u)}{2|E|}.
\]

The entropy is nonnegative, and its maximal possible value is \( \ln |V| \), which is attained when all nodes \( u \in V \) have the same degree \( d(u) = 2|E|/|V| \).
Thus, the edge distribution entropy $H_e$ is a measure of equality. The edge distribution entropy is called the *entropy of degree sequence* (EDS) in [67].

Because the edge distribution entropy has a maximal value of $\ln |V|$, we may expect it to be highly correlated to the network’s size $|V|$ itself. Therefore, we normalize it by dividing by $\ln |V|$, resulting in the relative edge distribution entropy

$$H_{er} = \frac{1}{\ln |V|} \sum_{u \in V} \frac{d(u)}{2|E|} \ln \frac{d(u)}{2|E|}. \tag{3}$$

By construction, $H_{er}$ varies in the range $[0, 1]$, with zero denoting complete inequality and one denoting complete equality. A slightly different definition of the relative edge distribution entropy is called the *normalized entropy of degree sequence* (NEDS) in [67]. The relative edge distribution entropy $H_{er}$ is thus a measure of equality, and we expect it to shrink over time.

**Related Measures** The Theil index is an economic measure of inequality [61] often used to measure income inequality, and related to the entropy by $T_T = H_e/|V|$. Another related measure is the Atkinson index, which adds a parameter that can be used to define the relative importance of small and large contributions to the inequality measure [2].

**Entropy of Other Distributions** In addition to the edge distribution, the entropy can also be applied to the degree distribution [63]. This entropy is invariant under exchanges of the number of nodes having any different degree values $d_1$ and $d_2$, and thus two very different degree distributions could share the same entropy value. Thus, it should only be used under specific circumstances, such as the network having a power-law degree distribution, a problem which it shares with the power-law exponent $\gamma$.

### 3.2 Connectivity

The concept of connectivity characterizes a network whose nodes are easily reachable from other nodes. Different definitions of *easily reachable* lead to different measures of connectivity. For instance, counting the maximal number of edges needed leads to the diameter, and measuring how likely a random walk of $n$ steps returns to its starting node leads to the random walk return probability. A measure of connectivity can be interpreted as a measure
of diversity in the following way: When connectivity is high, any part of the network is easily reachable from any other part, and thus the network lacks diversity. On the other hand, a network with a low connectivity value has more subgraphs well-separated from the rest of the network, and thus more local diversity.

We study four measures of connectivity: the diameter, which corresponds to the maximal length of shortest paths in the network; the random walk return probability, which corresponds to the probability of a random walk to return to its starting node; the relative controllability, based on the number of nodes needed to control a full network; and the algebraic connectivity, based on a spectral clustering of the network.

3.2.1 Diameter

The diameter is a very common network measure that equals the longest shortest path in the network. It is typically used to describe a network as a small world [64]. A small-world network is one in which the diameter is small, and the clustering coefficient is high. The intuition behind pairing these two measures is to combine a measure of local coherence, the clustering coefficient, with a measure of the overall coherence, the diameter. The given reference shows that the diameter places each network on a continuum between two extremes. On one hand, a lattice graph has a high local coherence, and thus a high clustering coefficient, but a low global coherence, and thus a large diameter. It could be said that the lattice has a high diversity, since its parts are very far from each other, as measured by the typical distance of nodes. On the other hand, a random graph has a low clustering coefficient and a low diameter, denoting low diversity, due to the fact that every node is reachable in few hops from every node. This is consistent with the interpretation of the random graph as having low diversity, since all nodes are near to each other, and thus any local structure is lost. Therefore, the diameter of a network can be considered a measure of the network diversity.

To be precise, the diameter measures the largest distance between two nodes of a network. In practice, the diameter is susceptible to long branches connected to the rest of the network on just one end, and therefore a common variant is the 90-percentile effective diameter, defined as the number of steps needed to reach 90% of all nodes, counted over all nodes. We refer to this graph property as $\delta_{0.9}$, and will use it in the experiments of this article.

One important restriction of the diameter is that it can only be applied
to a connected network. For an unconnected network, some node pairs are not reachable from each other, and the diameter is undefined or infinite. Therefore, we always measure the diameter on a network’s largest connected component. In [4] and [45], the diameter is observed to shrink over time, implying that the diversity of the network is becoming less over time. Due to the high runtime complexity of computing the exact effective diameter, we estimate it by sampling vertices, and computing their distance to all other vertices.

**Monotonicity** Adding an edge to a connected network cannot increase the distance between nodes, and thus the diameter can only decrease or remain constant when such an edge is added. The diameter is thus monotonous with regard to adding an edge to a connected network. Note that this does not hold for unconnected networks. In the general case, adding an edge to an unconnected network may increase the diameter of the largest connected component.

### 3.2.2 Random Walk Return Probability

Another way to measure connectivity in a network is to consider random walks. A random walk is a process starting at a given node $u$ and proceeding along edges in a random manner. At each node $v$, the random walk chooses one of the neighbors of that node uniformly at random; i.e., with probability equal to $1/d(v)$. Random walks can be used to measure how well connected a network is. For instance, one can consider the probability of return to the starting node $u$ after $n$ steps. If it is low, then the network possesses low locality, which we interpret as a high connectivity and thus low diversity. If the probability of returning to the initial node is high, then we interpret that as a sign of low connectivity and thus high diversity.

The random walk return probability was introduced by Fay et al. [16] as a metric for comparing two graphs defined by

$$
\vartheta(n) = \sum_k (1 - \lambda_k[Z])^n = \sum_C \frac{1}{d(u_1)d(u_2)\ldots d(u_n)},
$$

where $C$ is the set of all cycles of size $n$ in the graph$^1$ and $d(u_i)$ is the degree of the $i^{th}$ node in a cycle.

$^1$The authors in [16] recommend a value of $n = 4$, which we use in this article.
To compute the random walk return probability $\vartheta(n)$, we thus need to compute all eigenvalues of the matrix $Z$, or alternatively to enumerate all $n$-cycles. Both operations are expensive, and cannot be achieved in practice on large datasets. Therefore, we use, as a proxy, the sum $\vartheta_r$ only over the $r$ dominant eigenvalues, i.e., those with the greatest distance to $\lambda = 1$. This is not an approximation to $\vartheta(n)$, but a value that varies in conjunction with it, both reproducing shifts in the overall distribution of eigenvalues in the range $[0, 2]$.

As a network evolves, a shrinking random walk return probability shows that this sum is shrinking and so the probability of taking a random walk of length $n$ and returning to the source node is in general shrinking in the network. Another way of expressing this is that the number of escape routes or non-cycles has increased. This in turn occurs when the community structure of networks becomes more blurred; random walks are more likely to jump away from the community where they started. Thus the lower the random walk return probability, the lower the diversity of a network.

### 3.2.3 Controllability

A less-known way to assess the structure of a network consists in measuring how well it can be controlled. For instance, assume that we want to influence opinions in a social network, but are only able to directly influence $k$ persons in the network, much less than the number of vertices $|V|$. Assuming that opinions will spread through the network, how big has $k$ to be in order for us to be able to influence all nodes in a network, in a way that any arbitrary opinion can be given to any node? A solution to this problem is given by Liu et al. [46], in which such driver nodes are identified and, surprisingly, they are not necessarily the nodes with highest degree. In fact, the authors of that article state that driver nodes tend to avoid the hubs of the network.

The resulting computational model uses differential equations to model diffusion and can be reduced to finding a maximal matching in the bipartite double cover of the network [46]. The maximal matching in a bipartite graph can be computed efficiently by exploiting Königs theorem, which states that perfect matchings and minimal vertex covers have equal size in bipartite graphs [6], and thus the corresponding integer program formulations are equivalent to their relaxations, implying that the two problems can be solved in polynomial time. In fact, a maximal matching in a bipartite graph can be found in runtime $O(|V|^{1/2}|E|)$, and thus can be computed efficiently.
even for large networks.

The number of driver nodes $C$ needed to control a graph $G = (V, E)$ equals $|V|$ minus the size of the maximal directed 2-matching in the network. A 2-matching is a set of edges such that each vertex is incident to at most two edges. A directed 2-matching is a set of directed edges, such that each vertex is incident to at most one ingoing and one outgoing edge. Here, we interpret an undirected graph as a directed graph where each edge corresponds to two directed edges:

$$|V| - C = \max_{M \subseteq V^2} |M|$$

s.t.  
$$|\{(v, w) \in M \mid v = u, \{v, w\} \in E\}| \leq 1 \quad \text{for all } u \in V,$$
$$|\{(v, w) \in M \mid w = u, \{v, w\} \in E\}| \leq 1 \quad \text{for all } u \in V.$$  

The result is the number $C$ of vertices needed to control a given network. A network that is hard to control (i.e., has a high value $C$) can be interpreted as having a low connectivity and thus a higher diversity. Thus, we expect $C$ to be a measure of the diversity of a network.

Since the number of nodes in a network is changing over time, the number of driver nodes and thus $C$ is dependent on the size of the network. Therefore, we use as a measure of diversity the relative controllability $C_r$ defined as

$$C_r = \frac{C}{|V|}. \quad (4)$$

**Monotonicity**  When an edge is added to a connected graph $G = (V, E)$, the size of a maximal matching in its bipartite double cover can only increase or remain constant, and therefore the number of driver nodes $C$ and the relative controllability $C_r$ can only decrease or stay constant, but not increase.

### 3.2.4 Algebraic Connectivity

In some graphs, removing a single edge can make the graph disconnected. These kinds of graphs have low connectivity. On the other hand, some graphs can only be made disconnected by removing a much larger number of edges. These kinds of graphs have a high connectivity. To measure these differences, the edge connectivity of a graph can be defined as the number of edges that have to be removed to make the graph disconnected [5]. This number is a characteristic number of the graph. However, it is not very expressive.
For instance, the edge connectivity can be reduced to a value of one by adding to the graph a new vertex and a new edge between that vertex and an already existing vertex. Instead, a more robust measure of connectivity is the algebraic connectivity, which is based on the Laplacian matrix \( L = D - A \). The matrix \( L \) is positive-semidefinite. All its eigenvalues are nonnegative, and its smallest eigenvalue is zero. Its second smallest eigenvalue is a measure of the network’s connectivity: When the network is disconnected, it is zero too. Otherwise, it is larger the harder it is to find small cuts dividing the network into two parts. We will denote the algebraic connectivity

\[
a = \lambda_2[L],
\]

where \( \lambda_2[L] \) is the second-smallest eigenvalue of \( L \). The algebraic connectivity was initially defined by Fiedler [17]. We compute the algebraic connectivity only in the largest connected component of a graph, as we do for the diameter. Otherwise, only the algebraic connectivity of the smallest connected component would be considered, since the eigenvalues of \( L \) are the eigenvalues of the Laplacian of each connected component.

**Monotonicity** Adding an edge between two vertices that are already connected indirectly can only increase the algebraic connectivity of a graph. This result can be deduced by considering the following interlacing theorem, stated e.g. in [66, p. 97]. This theorem states that when adding a positive-semidefinite matrix \( E \) of rank one to a given symmetric matrix \( X \) with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \), the new matrix \( \tilde{X} = X + E \) has eigenvalues \( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_n \) which interlace the eigenvalues of \( X \):

\[
\lambda_1 \leq \tilde{\lambda}_1 \leq \lambda_2 \leq \tilde{\lambda}_2 \leq \ldots \leq \lambda_n \leq \tilde{\lambda}_n
\]

In the case of the Laplacian matrix \( L \), adding an edge \( \{u, v\} \) adds to it the rank-one matrix \( xx^T \), where \( x \in \mathbb{R}^{|V|} \) is a vertex vector defined by \( x_u = +1 \), \( x_v = -1 \) and \( x_w = 0 \) otherwise. Note that the chosen orientation of the edge does not matter in this definition. Thus, a positive-definite matrix is added to \( L \), and the spectrum of \( L \) thus shifts up.

If \( G \) is already connected, its spectrum is \( \{0, \lambda_2, \ldots\} \) and after addition of the edge \( \{u, v\} \) it becomes \( \{0, \tilde{\lambda}_2, \ldots\} \), from which it follows that \( \tilde{\lambda}_2 \geq \lambda_2 \), i.e., the algebraic connectivity can only grow or remain constant, but not decrease.
3.3 Link Prediction Functions

The problem of link prediction in networks consists in predicting which edges will appear in an evolving network, given the current network. Many link prediction functions exist, each corresponding to different models of network growth. Link prediction functions make assumptions of regularity in a network. For instance, the triangle closing model is based on the assumption that triangles will form in a network. Thus, triangle closing predicts a growing regularity in a network. This is also true for other link prediction functions, as they tend to predict common structures, and therefore predict that common structures will be reinforced while uncommon structures will stay uncommon. Regularity can be interpreted as an aspect of diversity, in the sense that regularity indicates the lack of structural diversity. Thus, link prediction functions can be interpreted as predicting an increasing regularity and a shrinking diversity.

We consider three link prediction functions which lead to three measures of diversity: the clustering coefficient, the fractional rank and the eigenvalue power-law exponent.

3.3.1 Clustering Coefficient

The clustering coefficient measures the fraction of adjacent edge pairs that are completed by a third edge to form a triangle. The tendency of networks to form triangles represents one half of the small-world network model along with the network diameter [64], and leads to the simplest network growth models that goes beyond preferential attachment to take into account the shared neighborhood of two nodes [43]: triangle closing, i.e., the prediction that new edges will appear such that many triangles are formed. As a link prediction function, the resulting common neighbor count function is one of the simplest possible link prediction methods.

The clustering coefficient is the only measure we consider that only makes sense for unipartite networks. For bipartite networks, it is zero, because a bipartite network does not contain triangles.

In the triangle closing model, new edges are predicted to form new triangles and thus, the clustering coefficient is expected to increase, and thus the diversity to shrink.
3.3.2 Fractional Rank

An important class of link prediction functions are graph kernels. Graph kernels are positive-semidefinite functions of the adjacency matrix $A$ and can be used for modeling network growth [31]. Graph kernels as considered here have the property that they can be expressed in terms of the eigenvalue decomposition of the matrix $A$. Let $A = U\Lambda U^T$ be the eigenvalue decomposition of $A$, in which $U$ is an orthogonal matrix and $\Lambda$ a diagonal matrix. Then, a graph kernel $F$ can be expressed as a function of the form $F(A) = UF(\Lambda)U^T$, where $F(\Lambda)$ is given by applying a function $f$ to each eigenvalue separately such that $(F(\Lambda))_{kk} = f(\Lambda_{kk})$ [40]. Common graph kernels of this form have the property that the function $f$ is convex, i.e., they make large eigenvalues grow faster than smaller ones. Thus, large eigenvalues will tend to dominate smaller ones if a network evolves according to such graph kernels.

The two main graph kernels we study are the exponential kernel [35] $e^{\alpha A} = Ue^{\alpha \Lambda}U^T$ and the Neumann kernel [33] $(I - \alpha A)^{-1} = U(I - \alpha \Lambda)^{-1}U^T$. Both kernels take a positive parameter $\alpha$. For the Neumann kernel, this parameter must be smaller than the inverse spectral norm of $A$, i.e., smaller than the inverse of the largest absolute eigenvalue of $A$.

Let $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ be the ordered eigenvalues of $A$, i.e., the diagonal elements of $\Lambda$. From the convexity of the function $f$, it follows that the ratio $|\lambda_k|/|\lambda_1|$ and its square shrink during the application of a graph kernel, and therefore both the absolute and the squared eigenvalue sums are predicted to shrink if graph kernels are correct link prediction functions.

The sum of absolute or squared eigenvalues of the adjacency matrix can be derived as a fractional extension of the rank of a matrix. These measures are nonnegative and generalize the notion of matrix rank as follows. The ordinary matrix rank of $A$ can be written as $\text{rank}(A) = \sum_k[\lambda_k \neq 0]$, in which $[\lambda \neq 0] = 1$ when $\lambda \neq 0$ and $[\lambda \neq 0] = 0$ otherwise. We thus see that the matrix rank counts the number of nonzero eigenvalues. This is clearly a measure of the diversity of the network, but not a very good one, because very small eigenvalues contribute a value of one, although their contribution to the network is very small.

Therefore, we propose to compute a fractional rank in which each eigenvalue is counted in proportion to its size [42]. We start with the largest eigenvalue $\lambda_1$ and define its weight to be one. Then, each subsequent eigenvalue $\lambda_k$ is weighted as $(\lambda_k/\lambda_1)^2$. The sum of these values then gives the
network rank:

\[ \text{rank}_F = \sum_k \left( \frac{\lambda_k}{\lambda_1} \right)^2 \]  

(6)

We can rewrite this as the ratio of the Frobenius norm \( \|A\|_F \) and the spectral norm \( \|A\|_2 \) of \( A \):

\[ \text{rank}_F = \left( \sum_k \lambda_k^2 \right) / \lambda_1^2 = \frac{\|A\|_F^2}{\|A\|_2^2} = \frac{2|E|}{\lambda_1^2} \]

This is true because the spectral norm equals the largest absolute eigenvalue \( |\lambda_1| \), and the Frobenius norm equals the square root the the sum of squared eigenvalues of \( A \).\(^2\) We will call this number the fractional rank of \( G \). Note that because the squared Frobenius norm \( \|A\|_F^2 \) equals the sum of squared eigenvalues, we have \( \text{rank}_F(A) \geq 1 \). The fractional rank can be easily computed using the number of edges in the graph and the spectral norm, because \( \|A\|_F^2 = \sum_{i,j} A_{ij}^2 = 2|E| \). The spectral norm \( \|A\|_2 \) equals the largest absolute value and can be computed by power iteration.

**Preferential Attachment on Eigenvectors**  A shrinking fractional rank can also be explained by a modification of the preferential attachment model: The eigenvector centrality preferential attachment model, which states that the probability that an edge attaches to a vertex is proportional to that node’s eigenvector centrality. The eigenvector centrality is a centrality measure for nodes in a network, based on the eigenvalue decomposition of the network’s adjacency matrix. It is defined as the vertex’s entry in the adjacency matrix’s dominant eigenvector. This value is always nonnegative as a result of the Perron–Frobenius theorem.

When an unconnected node is added to a network, the fractional rank does not change. This follows directly from the fact that adding a zero row and column to a matrix will add an eigenvalue of zero to the spectrum. When an edge is added, the situation is more complex. In the case of the fractional rank \( \text{rank}_F \) of a graph \( G = (V, E) \) we can make the following derivation. Let \( \tilde{G} = (V, E \cup \{u, v\}) \) be the graph \( G \) to which the edge \( \{u, v\} \) has been added. Also, let \( \tilde{A} \) be its adjacency matrix. Then, the new largest

\(^2\) Note that \( \lambda_1 \geq 0 \) in our case, because the entries of \( A \) are nonnegative.
eigenvalue \( \tilde{\lambda}_1 \) can be estimated in the following way [9]. Let \( e_u \in \mathbb{R}^{|V|} \) be the vertex vector defined by \( (e_u)_v = 1 \) when \( u = v \) and \( (e_u)_v = 0 \) otherwise. Also, let \( A = U\Lambda U^T \) be the eigenvalue decomposition of \( A \). Then, the new adjacency matrix \( \tilde{A} \) can be written as \( \tilde{A} = A + e_u e_u^T + e_v e_v^T \). Now, assuming we want to write the new adjacency matrix as \( \tilde{A} = U\tilde{\Lambda}U^T \), we get \( \tilde{\Lambda} = \Lambda + U^T(e_u e_u^T + e_v e_v^T)U \). The matrix \( \tilde{\Lambda} \) defined in this way is not diagonal. However, in practice it is usually almost diagonal under the spectral network evolution model [39], and its largest diagonal value can be estimated as

\[
\tilde{\lambda}_1 = \lambda_1 + U_{u1}U_{v1} + U_{v1}U_{u1}
\]

The meaning of this expression is that approximately, by adding the edge \( \{u, v\} \), the dominant eigenvalue of the adjacency matrix \( A \) will grow by the double of the product of the entries \( u \) and \( v \) of the dominant eigenvector of \( A \). Plugging this result into the definition of the fractional rank, it follows that \( \text{rank}_F \) shrinks when

\[
\frac{2|E|}{\lambda_1} > \frac{2(|E| + 1)}{\lambda_1 + 2U_{u1}U_{v1}},
\]

or equivalently when \( U_{u1}U_{v1} > \lambda_1/2|E| \). In other words, the fractional rank shrinks when the values \( U_{u1} \) and \( U_{v1} \) are large enough. Remember that the dominant eigenvector \( U_{1} \) of \( A \) is nonnegative and can be interpreted as the eigenvector centrality of nodes in \( G \). Thus, the fractional rank shrinks when the product of the eigenvector centralities of the connected vertices are large enough. This can be understood as a form of preferential attachment: When new edges connect to central nodes, the fractional rank shrinks. The difference with the preferential attachment model is in the choice of the eigenvector centrality instead of the degree centrality.

**Spectral Growth** Another way to analyse the evolution of the fractional network rank is to look at models predicting the evolution of the largest eigenvalue \( \lambda_1 \). It follows from the definition of \( \text{rank}_F \) that the fractional network rank shrinks when the largest eigenvalue \( \lambda_1 \) grows faster than the square root of the number of edges \( |E| \). A corresponding model is given in [19], where the largest eigenvalue \( \lambda_1 \) grows as \(|V|^{1/4}\). According to [45], the number of edges \( |E| \) grows super-linearly in the number of vertices \( |V| \).
i.e., there is a constant $c > 1$ such that $|E| \sim |V|^c$. Plugging this into the definition of the fractional network rank, we get

$$\text{rank}_F = \frac{2|E|}{\lambda_1^2} \sim \frac{|V|^c}{\lambda_1^2} \sim \frac{|V|^c}{(|V|^{1/4})^2} = |V|^{c-1/2}.$$  

Thus, the fractional network rank will shrink when $c < 3/2$. Coincidentally, the constant $c$ has been reported to vary between 1.1 and 1.7. This is consistent with our experiments, in which the fractional rank does not always shrink, but only in most cases.

**Linear Spectral Evolution**  The spectral evolution model from [38] implies that the fractional rank decreases. In this model, it is assumed that over time, only the eigenvalues of the adjacency matrix $A$ grow, and that the eigenvectors of $A$ stay constant. Specifically, it predicts that the evolution of each eigenvalue is linear.

If spectral growth is extrapolated linearly into the future, the eigenvalue with the largest growth rate will overtake all others, and the network rank will decrease until it reaches the number of eigenvalues that have the same maximal growth rate. This explains a shrinking fractional rank in many networks, as a single eigenvalue becomes dominant.

### 3.3.3 Eigenvalue Power-law Exponent

Another way to measure the effect of graph kernels on growth of networks is given by the eigenvalue power-law exponent. The largest eigenvalues of a network’s adjacency matrix almost always follow a long tailed distribution. In other words, there are much more small eigenvalues than large eigenvalues. In [15], the distribution of the largest eigenvalues of adjacency matrix $A$ of the Internet topology network where observed to follow a power law $\lambda_k[A] \approx \lambda_1[A]/\alpha^k$ with $\alpha > 1$. Examples are shown in Figure 1.

As shown in the previous section, graph kernels predict that large eigenvalues grow faster than smaller eigenvalues. Thus, the exponent $\alpha$ of eigenvalue power laws is expected to grow and we expect the value $\alpha$ to grow when the diversity of a network is shrinking. We estimate the power-law exponent $\alpha$ by using the method described in [52, Equation (5)]. This is the same method we use to estimate the degree power-law exponent $\gamma$ as described in Section 3.1.3. The degree power-law exponent and the eigenvalue power-law exponent are derived to be related in [49] by the expression $\alpha = \gamma/2$. 

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Figure 1: Examples of eigenvalue power laws. The plots show the cumulated distribution of the eigenvalues of the adjacency matrix $A$, on a log-log scale.

This is explained in [18] by the observation that the largest eigenvalues of a scale-free graph follow the square roots of the largest degrees. Figure 2 shows a scatter plot of both exponents for all datasets. This plots shows no correlation between the two power-law exponents. Also, the fact that $\gamma$ is a measure of diversity while $\alpha$ is a measure of non-diversity is consistent with no such linear relationship.

4 Experiments

To validate the hypothesis of shrinking structural diversity, we use the Koblenz Network Collection (KONECT\(^3\)), which, at the time of the experiments, consisted of 187 network datasets, of which 72 had information about edge creation times. Out of these datasets, we use twenty-seven datasets for which we were able to compute all measures in reasonable time. Of these, thirteen are unipartite and fourteen bipartite. For directed and weighted networks, we ignore edge directions and edge weights. In all networks used, the edges are labeled by edge creation times. The full list of network datasets used is given in Table 3.

There are two ways in which the evolution of a network can be measured, which we both perform in our experiments:

**FULL:** The first type of measurement looks at the evolution of the complete network for times ranging from the network’s inception to the

\(^3\)http://konect.cc/ [37]
last added edge. In this type of experiment, the number of nodes $V$ in the network varies, as new nodes appear in the network. The drawback of this method is that a network is in general not connected, and some measures, such as the diameter, can only be computed for a connected network. Thus, a method is needed to study only the connected network.

**CONNECTED:** In the second type of measurement, the set of vertices $V_1$ is fixed at some time $t_1$, the largest connected component of nodes $\bar{V}_1$ is found, and then only the subnetworks consisting of the nodes in $\bar{V}_1$ are considered at later times $t > t_1$. This ensures that the network is connected, but restricts the experiments to times after the time $t_1$. This method is also used for evaluating the problem of link prediction, under the assumption that links connecting two disconnected components cannot be predicted sensibly. This method is then suitable because it ensures that each new edge connects two nodes already connected.

Let $\{G_i\}_{i=1}^N$ be a set of $N$ network datasets. Each network is split into one hundred timepoints $t = 1, \ldots, 100$, each containing $\lfloor |E|/100 \rfloor$ of the oldest of all edges $E$ in the network. Let $G_t^i$ be the network $G_i$ containing all edges up to time $t$. Then each measure $f$ is computed for all networks at all timepoints. Thus, $f(G_1^1), f(G_2^2), \ldots, f(G_{100}^{100})$ is the time series representing the evolution of the network measure $f$ for the Full variant.
Table 3: The list of twenty-seven network datasets used in this study.

| Network | Flags | $|V|$ | $|E|$ |
|---------|-------|-----|-----|
| ben     | B     | 167,525 | 1,164,576 |
| bfr     | B     | 30,997 | 201,727 |
| DG      | U M   | 30,308 | 87,627 |
| el      | B     | 149,904 | 1,837,141 |
| EL      | U M   | 8,297 | 107,071 |
| EN      | U M   | 87,273 | 1,148,072 |
| EP      | U M   | 131,828 | 841,372 |
| Fe      | B     | 75,360 | 1,266,753 |
| HA      | U     | 274 | 28,244 |
| HY      | U     | 113 | 20,818 |
| IF      | U     | 410 | 17,298 |
| M1      | B     | 2,625 | 100,000 |
| M2      | B     | 9,746 | 1,000,209 |
| Mt1     | MovieLens tag–movie B | 24,129 | 95,580 |
| Mui     | MovieLens user–movie B | 11,610 | 95,580 |
| Mut     | MovieLens user–tag B | 20,537 | 95,580 |
| nen     | Wikinews, English B | 173,772 | 901,416 |
| nfr     | Wikinews, French B | 26,546 | 193,618 |
| Ol      | Facebook friendships U | 63,731 | 1,545,686 |
| Ow      | Facebook wall posts U M | 63,891 | 876,993 |
| qen     | Wikiquotes, English B | 116,363 | 549,210 |
| RM      | Reality Mining U | 96 | 1,086,404 |
| SD      | Slashdot threads U M | 51,083 | 140,778 |
| SX      | Sexual escorts B | 16,730 | 50,632 |
| TO      | Internet topology U | 34,761 | 171,403 |
| UC      | UC Irvine messages U M | 1,899 | 59,835 |
| UF      | UC Irvine forum B | 1,421 | 33,720 |

U Unipartite network
B Bipartite network
M Network with multiple edges

For the CONNECTED variant, we chose the starting time to be $t_1 = 75$, i.e., three quarters of the available time range. This specific choice is arbitrary, and constitutes a trade-off between the requirement that the chosen time must not be too early, as otherwise the connected component is too small, and the requirement that the chosen time must not be too late, as otherwise the considered time range is too short.

To test whether a single network dataset has a shrinking diversity for a given diversity measure, we apply the Mann–Kendall test [47]. Given a series $(x_i)$, the Mann–Kendall test consists of applying a $t$-test to all pairwise differences $x_i - x_j$. We accept the hypothesis of a decreasing diversity for one network/measure combination when the $p$-value is below the threshold of $\alpha = 0.05$. The result of the individual Mann–Kendall tests for all net-
work/measure combinations are shown in Appendix I. For simplicity, we will describe the test procedure for measures of diversity such as the entropy. For measures that measure the opposite of diversity, the test is analogous. For each measure \( f \), we aggregate the results for all networks and test the hypothesis that the diversity measure is decreasing. The null hypothesis is thus that the measure \( f \) is not decreasing. Since the Mann–Kendall test is performed to a value of \( \alpha = 0.05 \), the probability of having \( k \) out of \( n \) successes for the measure \( f \) equals a binomial distribution with probability parameter \( \alpha \). Thus, the \( p \)-value for the null hypothesis that the measure is not decreasing equals \( p = \sum_{x=k}^{n} \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} \). We compute this \( p \)-value for each measure \( f \) and accept the hypothesis that the diversity measure \( f \) is decreasing when \( p < \alpha = 0.05 \).

The summarized results of the statistical tests are shown in Table 4.

4.1 Discussion

Out of the eleven measures of structural diversity, all but one show a temporal trend consistent with shrinking in either the FULL or CONNECTED case. Two measures show shrinking diversity in both cases: the power-law exponent and the diameter. Three measures are predicted to show shrinking diversity in the CONNECTED case mathematically, and also do so in the experiments.

Our experimental results thus show that for a large majority of structural network measures, a trend exists that can be interpreted as shrinking diversity. This leads us to conclude that the notion of structural diversity is a legitimate one, which explains in a more intuitive way the temporal evolution of different network measures. Thus, the shrinking diversity hypothesis gives an additional justification for models of preferential attachment, connectivity and link prediction. Detailed discussions follow.

Preferential Attachment Out of the four measures of degree equality or inequality, three show statistically significant trends consistent with shrinking diversity. The single exception is Jain’s index \( J \), which shows a trend consistent with increasing diversity. The preferential attachment model is thus validated by our experiments, up to the differing behavior of Jain’s index. For the fractional rank rank\(_F\), which can be interpreted as following from a process of preferential attachment on eigenvectors, we observe a shrinking trend in the connected case. The different behavior of Jain’s index is intriguing. On the face of it, we would be inclined to conclude that

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Table 4: Statistical significance tests for shrinking of diversity according to the eleven different measures. Statistically significant trends are shown as \textit{Up} and \textit{Down}. No statistically significant trend is denoted by a dash (—). Numbers in parentheses give the number of networks following the given trend according to the Mann–Kendall test, out of all 27 networks. (In cases without a trend, the number counts the networks following the predicted trend.)

| Measure | Observed trends | Predicted trends | Monotonicity |
|---------|-----------------|-----------------|--------------|
|         | Full            | Connected       |              |
| $d$     | (24) Up         | (27) Up         | Up           |
| $G$     | (24) Up         | (17) —          | Up           |
| $J$     | (23) Up         | (20) Up         | Down         |
| $\gamma$ | (21) Down      | (25) Down       | Down         |
| $H_{\alpha}$ | (19) Down   | (12) —          | Down         |
| $\delta_{0.9}$ | (18) Down | (26) Down       | Down         |
| $\vartheta_{r}(n)$ | (10) —       | (22) Down       | Down         |
| $C_{r}$  | (12) —          | (22) Down       | Down         |
| $a$     | (15) —          | (27) Up         | Up           |
| $c$     | (7) —           | (10) Up         | Up           |
| $\text{rank}_{F}$ | (13) —      | (19) Down       | Down         |
| $\alpha$ | (19) Up         | (23) Up         | Up           |

* For the clustering coefficient, the total number of networks is 13, since bipartite networks are excluded.

the preferential attachment hypothesis is not correct, according to our experiments with Jain’s index. However, the clear and consistent results for the Gini coefficient, power-law exponent and entropy lead us rather to conclude that Jain’s index is not a typical measure of diversity, and correlates negatively with other such measures.

**Connectivity** Of the three types of network measures studied in this article, the measures based on connectivity are the weakest in matters of shrinking diversity. For three of them, a proof exists that they must evolve according to a shrinking diversity in the connected case. Thus, the only nontrivial result is the shrinking diameter, which is observed even when taking new nodes into account, and the shrinking random walk return probability in the
The diameter is decreasing in general. We observe however that the diameter varies from one timepoint to the other sometimes as much as its overall trend. This is an indication that the diameter is not a robust measure. This is in opposition to reference [45], where a consistently shrinking effective diameter is reported for multiple networks.

The random walk return probability $\vartheta_r(n)$ is shrinking for most networks, in accordance with a shrinking diversity. This result also implies that the eigenvalues of the normalized Laplacian matrix $\mathbf{Z}$ move towards the value one, or equivalently, that the eigenvalues of the normalized adjacency matrix $\mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{Z}$ shrink.

The relative controllability $C_r$ is decreasing for almost all networks. This pattern is more consistent than for the three other connectivity measures. A decreasing relative number of driver nodes means that less and less vertices are necessary to be controlled in order to control the whole network. Thus, the diversity of the network is going down.

**Link Prediction** All three network measures based on link prediction evolve in a way consistent with shrinking diversity in the connected case, but only the power-law exponent does so in the unconnected case. This result confirms that link prediction methods can normally only be applied to connected networks, and that they do not give sensible results for unconnected nodes. For instance, a neighborhood-based link prediction method cannot predict a new edge connecting two disconnected components, since they always have zero neighbors in common.

The fractional rank is decreasing for the majority of networks, but by far not for all networks. This is an indication that common graph kernels such as the matrix exponential and the Neumann kernel are accurate link prediction functions. By interpreting each latent dimension of the eigenvalue decomposition of $\mathbf{A}$ as a community or topic (depending on the network) and the corresponding eigenvalue as the weight of the community or topic, implies that large communities or topics get larger over time, and go on to dominate smaller topics.

The clustering coefficient is the only measure considered that is only meaningful for unipartite networks, of which there are thirteen in our tests. Despite this reduced number, the clustering coefficient is increasing in ten of these networks, showing that the triangle closing model is correct in a
majority of networks.

5 Conclusion

The evolution of networks can indeed be understood in terms of a shrinking diversity, and common models of network evolution thus admit an interpretation in terms of diversity. The preferential attachment model is thus true because it predicts that new edges attach to popular nodes, decreasing the diversity of connections. The connectivity interpretation implies that connectivity is generally increasing over time in real-world networks, an observation in line with previous results. Finally, link prediction algorithms that are found to perform well in practice are justified as they presuppose shrinking diversity. These results justify the notion of structural diversity, and show that it is a primary driver in network evolution, and thus represents a basis for many temporal network analysis methods.

In terms of overall diversity, which may also include non-structural measures, our methods can however only give answer to the point that diversity can be represented equivalently as a network. While for instance the diversity of movies watched by the public is well represented by the structural diversity of the bipartite person–movie network, other types of diversity may not. The ubiquity of networks as a model however suggests that this is only rarely the case: many things whose diversity we are interested in such as opinions, languages, friendships, words, etc., can be represented as nodes in a network, and are thus amenable to our methodology. The positive results in our study should of course not be taken for natural; network evolution rules that defy the shrinking diversity hypothesis can of course not be ruled out by it, and may very well give particularly salient insight into processes at work in networks.

Acknowledgments We thank Sergej Sizov, Julia Perl, Damien Fay and Felix Schwagereit. The research leading to these results has received funding from the European Community’s Seventh Framework Programme under grant agreement n° 257859, ROBUST.
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6 Appendix I: Network Diversity Results

Figure 3 shows the evolution of all eleven network diversity measures and the average degree $d$ applied to our collection of network datasets. The plots show the FULL scenario, i.e., the evolution from $t = 1$ to $t = 100$ including all vertices and edges.
Figure 3: The evolution of all eleven diversity measures for all evaluated network datasets. The plots correspond to the Full scenario, i.e., time on the X axis goes from $t = 1$ to $t = 100$ and all vertices and edges are included. The color of the plot indicates whether the measure is evolving according to the prediction (green, $p$-value shown) or not (red, no $p$-value shown).