ON THE cd-INDEX AND $\gamma$-VECTOR OF S*-SHELLABLE CW-SPHERES

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Abstract. We show that the $\gamma$-vector of the order complex of any polytope is the $f$-vector of a balanced simplicial complex. This is done by proving this statement for a subclass of Stanley’s S-shellable spheres which includes all polytopes. The proof shows that certain parts of the cd-index, when specializing $c = 1$ and considering the resulted polynomial in $d$, are the $f$-polynomials of simplicial complexes that can be colored with “few” colors. We conjecture that the cd-index of a regular CW-sphere is itself the flag $f$-vector of a colored simplicial complex in a certain sense.

1. Introduction

Let $P$ be an $(n-1)$-dimensional regular CW-sphere (that is, a regular CW-complex which is homeomorphic to an $(n-1)$-dimensional sphere). In face enumeration, one of the most important combinatorial invariants of $P$ is the cd-index. The cd-index $\Phi_P(c, d)$ of $P$ is a non-commutative polynomial in the variables $c$ and $d$ that encodes the flag $f$-vector of $P$. By the result of Stanley [St1] and Karu [Ka], it is known that the cd-index $\Phi_P(c, d)$ has non-negative integer coefficients. On the other hand, a characterization of the possible cd-indices for regular CW-spheres, or other related families, e.g. Gorenstein* posets, is still beyond reach. In this paper we take a step in this direction and establish some non-trivial upper bounds, as we detail now.

If we substitute 1 for $c$ in $\Phi_P(c, d)$, we obtain a polynomial of the form

$$\Phi_P(1, d) = \delta_0 + \delta_1 d + \cdots + \delta_{\lfloor \frac{n}{2} \rfloor} d^{\lfloor \frac{n}{2} \rfloor},$$

where $\lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$, such that each $\delta_i$ is a non-negative integer. In other words, $\delta_i$ is the sum of coefficients of monomials in $\Phi_P(c, d)$ for which $d$ appears $i$ times.

Let $\Delta$ be a (finite abstract) simplicial complex on the vertex set $V$. We say that $\Delta$ is $k$-colored if there is a map $c : V \to [k] = \{1, 2, \ldots, k\}$, called a $k$-coloring map of $\Delta$, such that if $\{x, y\}$ is an edge of $\Delta$ then $c(x) \neq c(y)$. Let $f_i(\Delta)$ denote the number of elements $F \in \Delta$ having cardinality $i + 1$, where $f_{-1}(\Delta) = 1$. The main result of this paper is the following.

Theorem 1.1. Let $P$ be an $(n-1)$-dimensional $S^*$-shellable regular CW-sphere, and let $\Phi_P(1, d) = \delta_0 + \delta_1 d + \cdots + \delta_{\lfloor \frac{n}{2} \rfloor} d^{\lfloor \frac{n}{2} \rfloor}$. Then there exists an $\lfloor \frac{n}{2} \rfloor$-colored simplicial complex $\Delta$ such that

$$\delta_i = f_{i-1}(\Delta) \quad \text{for} \quad i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor.$$
The precise definition of the S*-shellability is given in Section 2. The most important class of S*-shellable CW-spheres are the boundary complexes of polytopes. By the Kruskal-Katona Theorem (see e.g. [St2, II, Theorem 2.1]), the above theorem gives certain upper bound on $\delta_i$ in terms of $\delta_{i-1}$. Better upper bounds are given by Frankl-Füredi-Kalai theorem which characterizes the f-vectors of k-colored complexes [FFK].

The numbers $\delta_0, \delta_1, \delta_2, \ldots$ relate to the $\gamma$-vector (see Section 4 for the definition) of the barycentric subdivision (order complex) of $P$, namely the simplicial complex whose elements are the chains of nonempty cells in $P$ ordered by inclusion. Indeed, as an application of Theorem 1.1 we prove the following.

**Theorem 1.2.** Let $P$ be an $(n-1)$-dimensional S*-shellable regular CW-sphere and let $sd(P)$ be the barycentric subdivision of $P$. Then there exists an $\lfloor \frac{n}{2} \rfloor$-colored simplicial complex $\Gamma$ such that $\gamma_i(sd(P)) = f_{i-1}(\Gamma)$ for $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

Recall that an $(n-1)$-dimensional simplicial complex is said to be balanced if it is $n$-colored. If $P$ is the boundary complex of an arbitrary convex $n$-dimensional polytope, then $\delta_{\lfloor \frac{n}{2} \rfloor}(P) > 0$ and we conclude the following.

**Corollary 1.3.** Let $P$ be the boundary complex of an $n$-dimensional polytope. Then the $\gamma$-vector of $sd(P)$ is the $f$-vector of a balanced simplicial complex.

The above corollary supports the conjecture of Nevo and Petersen [NP, Conjecture 6.3] which states that the $\gamma$-vector of a flag homology sphere is the $f$-vector of a balanced simplicial complex. This conjecture was verified for the barycentric subdivision of simplicial homology spheres (in this case all the cells are simplices) in [NPT].

It would be natural to ask if the above theorems hold for all regular CW-spheres (or more generally, Gorenstein* posets). We conjecture a stronger statement on the cd-index, see Conjecture 4.3.

This paper is organized as follows: in Section 2 we recall some known results on the cd-index and define S*-shellability, in Section 3 we prove our main theorem, Theorem 1.1, in Section 4 we derive consequences for $\gamma$-vectors and present a conjecture on the cd-index, Conjecture 4.3.

2. cd-index of S*-shellable CW-spheres

In this section we recall some known results on the cd-index.

Let $P$ be a graded poset of rank $n+1$ with the minimal element $\hat{0}$ and the maximal element $\hat{1}$. Let $\rho$ denote the rank function of $P$. For $S \subseteq [n] = \{1, 2, \ldots, n\}$, a chain $\hat{0} = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{k+1} = \hat{1}$ of $P$ is called an $S$-flag if $\{\rho(\sigma_1), \ldots, \rho(\sigma_k)\} = S$. Let $f_S(P)$ be the number of $S$-flags of $P$. Define $h_S(P)$ by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T(P),$$

where $|X|$ denotes the cardinality of a finite set $X$. The vectors $(f_S(P) : S \subseteq [n])$ and $(h_S(P) : S \subseteq [n])$ are called the flag f-vector and flag h-vector of $P$ respectively.
Now we recall the definition of the cd-index. For \( S \subseteq [n] \), we define a non-commutative monomial \( u_S = u_1 u_2 \cdots u_n \) in variables \( a \) and \( b \) by \( u_i = a \) if \( i \notin S \) and \( u_i = b \) if \( i \in S \). Let
\[
\Psi_P(a, b) = \sum_{S \subseteq [n]} h_P(S) u_S.
\]

For a graded poset \( P \), let \( sd(P) \) be the order complex of \( P - \{0, 1\} \). Thus
\[
sd(P) = \{ \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \subset P - \{0, 1\} : \sigma_1 < \sigma_2 < \cdots < \sigma_k \}.
\]

We say that \( P \) is Gorenstein* if the simplicial complex \( sd(P) \) is a homology sphere. It is known that if \( P \) is Gorenstein* then \( \Psi_P(a, b) \) can be written as a polynomial \( \Phi_P(c, d) \) in \( c = a + b \) and \( d = ab + ba \) \([BK]\), and this non-commutative polynomial \( \Phi_P(c, d) \) is called the cd-index of \( P \). Moreover, by the celebrated results due to Stanley \([SL]\) (for convex polytopes) and Karu \([Ka]\) (for Gorenstein* posets), the coefficients of \( \Phi_P(c, d) \) are non-negative integers.

Next, we define S*-shellability of regular CW-spheres by slightly modifying the definition of S-shellability introduced by Stanley \([SL]\) Definition 2.1.

Let \( P \) be a regular CW-sphere (a regular CW-complex which is homeomorphic to a sphere) and \( F(P) \) its face poset. Then the order complex of \( F(P) \) is a triangulation of a sphere, so the poset \( F(P) \cup \{0, 1\} \) is Gorenstein*. We define the cd-index of \( P \) by \( \Phi_P(c, d) = \Phi_{F(P) \cup \{0, 1\}}(c, d) \). For any cell \( \sigma \) of \( P \), we write \( \overline{\sigma} \) for the closure of \( \sigma \). For an \((n-1)\)-dimensional regular CW-sphere \( P \), let \( \Sigma P \) be the suspension of \( P \), in other words, \( \Sigma P \) is the \( n \)-dimensional regular CW-sphere obtained from \( P \) by attaching two \( n \)-dimensional cells \( \tau_1 \) and \( \tau_2 \) such that \( \partial \overline{\tau_1} = \partial \overline{\tau_2} = P \). Also, for an \((n-1)\)-dimensional regular CW-ball \( P \) (a regular CW-complex which is homeomorphic to an \((n-1)\)-dimensional ball), let \( P' \) be the \((n-1)\)-dimensional regular CW-sphere which is obtained from \( P \) by adding an \((n-1)\)-dimensional cell \( \tau \) so that \( \partial \overline{\tau} = \partial P \).

**Definition 2.1.** Let \( P \) be an \((n-1)\)-dimensional regular CW-sphere. We say that \( P \) is S*-shellable if either \( P = \{\emptyset\} \) or there is an order \( \sigma_1, \sigma_2, \ldots, \sigma_r \) of the facets of \( P \) such that the following conditions hold.

(a) \( \partial \sigma_1 \) is S*-shellable.
(b) For \( 1 \leq i \leq r - 1 \), let
\[
\Omega_i = \overline{\sigma_1} \cup \overline{\sigma_2} \cup \cdots \cup \overline{\sigma_i}
\]
and for \( 2 \leq i \leq r - 1 \) let
\[
\Gamma_i = \left[ \partial \sigma_i \setminus (\partial \sigma_i \cap \Omega_{i-1}) \right].
\]

Then both \( \Omega_i \) and \( \Gamma_i \) are regular CW-balls of dimension \((n-1)\) and \((n-2)\) respectively, and \( \Gamma_i' \) is S*-shellable with the first facet of the shelling being the facet which is not in \( \Gamma_i \).

**Remark 2.2.** The difference between the above definition and Stanley’s S-shellability is that S-shellability only assume that \( P \) and \( \Gamma_i' \) are Eulerian and assume no conditions on \( \Omega_i \). However, S*-shellable regular CW-spheres are S-shellable, and the boundary complex of convex polytopes are S*-shellable by the line shelling \([BM]\). We leave the verification of this fact to the readers.
The next recursive formula is due to Stanley [St1].

**Lemma 2.3** (Stanley). With the same notation as in Definition 2.1, for $i = 1, 2, \ldots, r - 2$, one has

$$\Phi_{\Omega_i'}(c, d) = \Phi_{\Omega_i}(c, d) + \left\{ \Phi_{\partial\Omega_i'}(c, d) - \Phi_{\Sigma(\partial\Gamma_{i+1})}(c, d) \right\} c + \Phi_{\partial\Gamma_{i+1}}(c, d) d.$$

Since $\Omega_{r-1}' = P$ the above formula gives a way to compute the cd-index of $P$ recursively.

Next, we recall a result of Ehrenborg and Karu proving that the cd-index increases by taking subdivisions. Let $P$ and $Q$ be regular CW-complexes, and let $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ be a poset map. For a subcomplex $Q' = \sigma_1 \cup \cdots \cup \sigma_s \subset Q$, where each $\sigma_i$ is a cell of $Q$, we write $\phi^{-1}(Q') = \phi^{-1}(\sigma_1) \cup \cdots \cup \phi^{-1}(\sigma_s)$.

Following [EK] Definition 2.6, for $(n - 1)$-dimensional regular CW-spheres $P$ and $\hat{P}$, we say that $\hat{P}$ is a subdivision of $P$ if there is an order preserving surjective poset map $\phi : \mathcal{F}(\hat{P}) \to \mathcal{F}(P)$, satisfying that for any cell $\sigma$ of $P$, $\phi^{-1}(\sigma)$ is a homology ball having the same dimension as $\sigma$ and $\phi^{-1}(\partial\sigma) = \partial(\phi^{-1}(\sigma))$.

The following result was proved in [EK] Theorem 1.5.

**Lemma 2.4** (Ehrenborg-Karu). Let $P$ and $\hat{P}$ be $(n - 1)$-dimensional regular CW-spheres. If $\hat{P}$ is a subdivision of $P$ then one has a coefficientwise inequality $\Phi_{\hat{P}}(c, d) \geq \Phi_{P}(c, d)$

Back to $S^*$-shellable regular CW-spheres, with the same notation as in Definition 2.1, $\Omega_i'$ is a subdivision of $\Sigma(\partial\Omega_i)$ and $\partial\Omega_i$ is a subdivision of $\Sigma(\partial\Gamma_{i+1})$. Indeed, for the first statement, if $\tau_1$ and $\tau_2$ are the facets of $\Sigma(\partial\Omega_i)$ then define $\phi : \mathcal{F}(\Omega_i') \to \mathcal{F}(\Sigma(\partial\Omega_i))$ by

$$\phi(\sigma) = \begin{cases} 
\sigma, & \text{if } \sigma \in \partial\Omega_i, \\
\tau_1, & \text{if } \sigma \text{ is an interior face of } \Omega_i, \\
\tau_2, & \text{if } \sigma \notin \Omega_i.
\end{cases}$$

Similarly, for the second statement, if $\tau_1$ and $\tau_2$ are the facets of $\Sigma(\partial\Gamma_{i+1})$ then define $\phi : \mathcal{F}(\partial\Omega_i) \to \mathcal{F}(\Sigma(\partial\Gamma_{i+1}))$ by

$$\phi(\sigma) = \begin{cases} 
\sigma, & \text{if } \sigma \in \partial\Gamma_{i+1}, \\
\tau_1, & \text{if } \sigma \in \partial\Gamma_{i+1} \setminus \partial\Gamma_{i+1}, \\
\tau_2, & \text{otherwise}.
\end{cases}$$

Since $\Phi_{\Sigma P}(c, d) = \Phi_{P}(c, d) c$ for any regular CW-sphere $P$ (see [St1, Lemma 1.1]), Lemma 2.4 shows

**Lemma 2.5.** With the same notation as in Definition 2.1, for $i = 2, 3, \ldots, r - 2$, one has $\Phi_{\Omega_i'}(c, d) \geq \Phi_{\partial\Gamma_{i+1}}(c, d) c^2$.

3. **Proof of the main theorem**

In this section, we prove Theorem 1.1.

For a homogeneous cd-polynomial $\Phi$ (i.e., homogeneous polynomial of $\mathbb{Z}^n(c, d)$ with $\deg c = 1$ and $\deg d = 2$) of degree $n$, we define $\Phi_0, \Phi_2, \ldots, \Phi_n$ by

$$\Phi = \Phi_0 + \Phi_2 dc^{n-2} + \Phi_3 dc^{n-3} + \cdots + \Phi_{n-1} dc + \Phi_n d$$
where $\Phi_0 = \alpha c^n$ for some $\alpha \in \mathbb{Z}$ and each $\Phi_k$ is a $cd$-polynomial of degree $k - 2$ for $k \geq 2$. Also, we write $\Phi_{\leq k} = \Phi_0 + \Phi_2 dc^{n-2} + \cdots + \Phi_k dc^{n-k}$.

**Definition 3.1.**

- A vector $(\delta_0, \delta_1, \ldots, \delta_s) \in \mathbb{Z}^{s+1}$ is said to be $k$-FFK if there is a $k$-colored simplicial complex $\Delta$ such that $\delta_i = f_{i-1}(\Delta)$ for $i = 0, 1, \ldots, s$. ($\{\emptyset\}$ is a 0-colored simplicial complex.) A homogeneous $cd$-polynomial $\Phi = \Phi(c, d)$ is said to be $k$-FFK if, when we write $\Phi(1, d) = \delta_0 + \delta_1 d + \cdots + \delta_s d^s$, the vector $(\delta_0, \delta_1, \ldots, \delta_s)$ is $k$-FFK.
- A homogeneous $cd$-polynomial $\Phi$ of degree $n$ is said to be primitive if the coefficient of $c^n$ in $\Phi$ is 1.
- Let $\Phi$ be a homogeneous $cd$-polynomial. A primitive homogeneous $cd$-polynomial $\Psi$ is said to be $k$-good for $\Phi$ if $\Psi$ is $k$-FFK and $\Phi(1, d) \geq \Psi(1, d)$. Also, we say that a homogeneous $cd$-polynomial $\Psi$ is $k$-good for $\Phi$ if it is the sum of primitive homogeneous $cd$-polynomials that are $k$-good for $\Phi$.

We will use the following observation, which follows from [NPT, Lemma 3.1]:

**Lemma 3.2.** If $\Phi$ is a $k$-FFK homogeneous $cd$-polynomial of degree $n$, and if $\Psi'$ and $\Psi''$ are homogeneous $cd$-polynomials of degree $n'$ and $n''$ respectively, where $n', n'' \leq n - 2$, which are $k$-good for $\Phi$ then

$$\Phi + \Psi' dc^{n-n'-2} \quad \text{and} \quad \Phi + \Psi' dc^{n-n'-2} + \Psi'' dc^{n-n''-2}$$

are $(k + 1)$-FFK.

**Proof.** By Frankl-Füredi-Kalai theorem [FFK], for any $k$-colored simplicial complex $\Gamma$, there is the unique $k$-colored simplicial complex $\mathcal{C}(\Gamma)$, called a $k$-colored compressed complex, such that $f_i(\Gamma) = f_i(\mathcal{C}(\Gamma))$ for all $i$. Moreover, if $\Gamma'$ is a $k$-colored complex satisfying $f_i(\Gamma) \leq f_i(\Gamma')$ for all $i$, then one has $\mathcal{C}(\Gamma) \subseteq \mathcal{C}(\Gamma')$.

For a simplicial complex $\Gamma$, we write $f(\Gamma, d) = 1 + f_0(\Gamma)d + f_1(\Gamma)d^2 + \cdots$. There are $k$-colored complexes $\Delta, \Delta^{(1)}, \ldots, \Delta^{(m)}, \ldots, \Delta^{(s)}$ such that $f(\Delta, d) = \Phi(1, d)$, $\sum_{1 \leq i \leq m} f(\Delta^{(i)}, d) = \Psi'(1, d)$, $\sum_{m+1 \leq i \leq s} f(\Delta^{(i)}, d) = \Psi''(1, d)$ and each $\Delta^{(i)}$ is a subcomplex of $\Delta$. Let

$$\Gamma^{(i)} = \Delta \cup \left\{ \bigcup_{k=1}^i \left\{ F \cup \{v_k\} : F \in \Delta^{(k)} \right\} \right\},$$

where $v_1, \ldots, v_s$ are new vertices. Since each $\Delta^{(k)}$ is a subcomplex of $\Delta$, $\Gamma^{(i)}$ is a simplicial complex. Also, $f(\Gamma^{(m)}, d) = (\Phi + \Psi' dc^{n-n'-2})(1, d)$ and $f(\Gamma^{(s)}, d) = (\Phi + \Psi' dc^{n-n'-2} + \Psi'' dc^{n-n''-2})(1, d)$. We claim that each $\Gamma^{(i)}$ is $(k + 1)$-colored.

Let $V$ be the vertex set of $\Delta$ and $c : V \rightarrow [k]$ a $k$-coloring map of $\Delta$. Then the map $\hat{c} : V \cup \{v_1, \ldots, v_s\} \rightarrow [k + 1]$ defined by $\hat{c}(x) = c(x)$ if $x \in V$ and $\hat{c}(x) = k + 1$ if $x \notin V$ is a $(k + 1)$-coloring map of $\Gamma^{(i)}$. \qed

Let $P$ be an $(n - 1)$-dimensional $S^s$-shellable regular CW-sphere with the shelling $\sigma_1, \ldots, \sigma_r$. Keeping the notation in Definition 2.1 to simplify notations, we use the
following symbols.

\[
\Phi^{(i)} = \Phi^{(i)}(c, d) = \Phi_{\Omega_i}(c, d) \\
\Phi = \Phi_P(c, d) = \Phi^{(r-1)} \\
\Psi^{(i)} = \Phi_{\Gamma_{i+1}}(c, d) - \Phi_{\Sigma(\partial \Gamma_{i+1})}(c, d) \\
\Psi = \sum_{i=1}^{r-2} \Psi^{(i)} \\
\Pi = \Phi - \Phi^{(1)} .
\]

Thus Stanley’s recursive formula, Lemma 2.3, says

\[
\Phi^{(i+1)} = \Phi^{(i)} + \Psi^{(i)}c + \Phi_{\partial \Gamma_{i+1}}d
\]

and

\[
\Pi = \Psi c + \sum_{i=1}^{r-2} \Phi_{\partial \Gamma_{i+1}}(c, d)d .
\]

The last part of the following proposition is a restatement of Theorem 1.1.

**Proposition 3.3.** With notation as above, the following holds.

1. For \(2 \leq k \leq n\), \(\Psi^{(k)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}c\).
2. For \(2 \leq k \leq n\), \(\Pi_k\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi^{(i)}_{\leq k-2} + \Pi_{\leq k-2}\).
3. For \(2 \leq k \leq n\), \(\Phi_{\leq k}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\leq k-2}\).
4. For \(0 \leq k \leq n\), \(\Phi_{\leq k}\) is \(\lfloor \frac{k}{2} \rfloor\)-FFK. In particular, the cd-index of \(P\) is \(\lfloor \frac{n}{2} \rfloor\)-FFK.

**Proof.** The proof is by induction on dimension, where all statements clearly hold for \(n = 0, 1\). Suppose that all statements are true up to dimension \(n - 2\). To simplify notations, for a regular CW-sphere \(Q\), we write \(\Phi_Q = \Phi_Q(c, d)\).

**Proof of (1).** By applying the induction hypothesis to \(\Gamma_{i+1}'\) (use statement(2)), each \(\Psi^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \((\Phi_{\Sigma(\partial \Gamma_{i+1})})^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}\) and \((\Phi_{\Sigma(\partial \Gamma_{i+1})})^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi_{\Sigma(\partial \Gamma_{i+1})})^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}\). By Lemma 2.5,

\[
\Phi_{\Sigma(\partial \Gamma_{i+1})})^{(i)}c = \Phi_{\partial \Gamma_{i+1}}c^2 \leq \Phi_{\Omega_i'}^{(i)},
\]

thus \(\Psi^{(i)}\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}c\).

**Proof of (2).** By the definition of \(\Pi\),

\[
\Pi_k = \sum_{i=1}^{r-1} \Psi^{(i)}_k \text{ for } k < n
\]

and

\[
\Pi_n = \sum_{i=1}^{r-2} \Phi_{\partial \Gamma_{i+1}} .
\]

By (1), each \(\Psi^{(i)}_k\) is \(\lfloor \frac{k}{2} - 1 \rfloor\)-good for \(\Phi^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}c\). Then since

\[
\Phi^{(i)}_{\leq k-2} + \Psi^{(i)}_{\leq k-2}c \leq \Phi_{\leq k-2} = \Phi^{(1)}_{\leq k-2} + \Pi_{\leq k-2} ,
\]
Π_0 is \([k/2 - 1]\]-good for \(\Phi_{(k-2)}^{(i)} + \Pi_{k-2}^{(i)}\) for \(k < n\). Also, each \(\Phi_{\partial^i+1}\) is \([n/2 - 1]\]-FFK by the induction hypothesis (use (4)), and \(\Phi_{\partial^i+1}c^2 \leq \Phi^{(i)}\) by Lemma 2.5. The latter condition clearly says

\[
\Phi_{\partial^i+1}c^2 \leq \Phi_{\leq n-2}^{(i)} \leq \Phi_{\leq n-2} = \Phi_{\leq n-2} + \Pi_{\leq n-2}.
\]

Hence \(\Pi_n\) is \([n/2 - 1]\]-good for \(\Phi_{\leq n-2} + \Pi_{\leq n-2}\).

Proof of (3). Observe that since \(\Phi^{(1)} = \Phi_{\partial^2}, c\),

\[
\Phi_k = \Phi_{k}^{(1)} + \Psi_k \quad \text{for} \quad k < n
\]

and

\[
\Phi_n = \Pi_n.
\]

We already proved that \(\Phi_n = \Pi_n\) is \([n/2 - 1]\]-good for \(\Phi_{\leq n-2}\) in the proof of (2).

Suppose \(k < n\). Since \(\Phi^{(1)} = \Phi_{\partial^2}, c\) by the induction hypothesis (use (3)), \(\Phi^{(1)}_{\leq k-2}\) is \([k/2 - 1]\]-good for \(\Phi_{\leq k-2}\). Since \(\Phi^{(1)}_{\leq k-2} \leq \Phi_{\leq k-2}\) and since we already proved that \(\Psi_k = \Pi_k\) is \([k/2 - 1]\]-good for \(\Phi_{\leq k-2}\) in the proof of (2), \(\Phi_k\) is \([k/2 - 1]\]-good for \(\Phi_{\leq k-2}\).

Proof of (4). This statement easily follows from (3). For \(k = 0, 1\), the statement is obvious (as \(\Phi_{\leq 0} = \Phi_{\leq 1} = c^n\)). Suppose that \(\Phi_{\leq 2m+1}\) is \(m\)-FFK, where \(m \in \mathbb{Z}_{\geq 0}\). Then both \(\Phi_{2m+2}\) and \(\Phi_{2m+3}\) are \(m\)-good for \(\Phi_{\leq 2m+1}\) by (3), and therefore \(\Phi_{\leq 2m+2}\) and \(\Phi_{\leq 2m+3}\) are \((m + 1)\)-FFK by Lemma 3.2.

\[\Box\]

4. \(\gamma\)-vectors of polytopes and a conjecture on the cd-index

\(\gamma\)-vectors and the cd-index. Let \(\Delta\) be an \((n-1)\)-dimensional simplicial complex. Then the \(h\)-vector \(h(\Delta) = (h_0, h_1, \ldots, h_n)\) of \(\Delta\) is defined by the relation

\[
\sum_{i=0}^{n} h_i x^{n-i} = \sum_{i=0}^{n} f_{i-1}(\Delta)(x - 1)^{n-i}.
\]

If \(\Delta\) is a simplicial sphere (that is, a triangulation of a sphere), or more generally a homology sphere, then \(h_i = h_{n-i}\) for all \(i\) by the Dehn-Sommerville equations, and in this case the \(\gamma\)-vector \((\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})\) of \(\Delta\) is defined by the relation

\[
\sum_{i=0}^{n} h_i x^i = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1 + x)^{n-2i}.
\]

It was conjectured by Gal [Ga] that if \(\Delta\) is a flag homology sphere then its \(\gamma\)-vector is non-negative. Recently Nevo and Peterson [NP] further conjectured that the \(\gamma\)-vector of a flag homology sphere is the \(f\)-vector of a balanced simplicial complex. These conjectures are open in general, the latter conjecture was verified for barycentric subdivisions of simplicial homology spheres [NPT], and Gal’s conjecture is known to be true for barycentric subdivisions of regular CW-spheres by the following fact, combined with Karu’s result on the nonnegativity of the cd-index for Gorenstein* posets:

Let \(P\) be an \((n - 1)\)-dimensional regular CW-sphere. The barycentric subdivision \(sd(P)\) of \(P\) is the order complex of \(\mathcal{F}(P)\). Let \((h_0, h_1, \ldots, h_n)\) and \((\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})\) be the \(h\)-vector and \(\gamma\)-vector of \(sd(P)\), respectively. Then it is easy to see that
$$h_i = \sum_{S \subseteq [n], |S| = i} h_S(P).$$  Thus if \( \Phi_P(1, d) = \delta_0 + \delta_1 d + \delta_2 d^2 + \cdots + \delta_{\lceil \frac{n}{2} \rceil} d^{\lceil \frac{n}{2} \rceil} \), then for all \( i \geq 0 \),

$$\gamma_i = 2^i \delta_i.$$  

Since \( \delta_i \) is non-negative, we conclude that \( \gamma_i \) is also non-negative.

The next simple statement, combined with Theorem \ref{thm:1.1} proves Theorem \ref{thm:1.2}.

**Lemma 4.1.** With the same notation as above, if \((\delta_0, \delta_1, \ldots, \delta_{\lceil \frac{n}{2} \rceil})\) is k-FFK then \((\gamma_0, \gamma_1, \ldots, \gamma_{\lceil \frac{n}{2} \rceil})\) is also k-FFK.

**Proof.** Let \( \Delta \) be a \( k \)-colored simplicial complex on the vertex set \( V \) with \( f_{i-1}(\Delta) = \delta_i \) for all \( i \geq 0 \) and let \( c : V \to [k] \) be a \( k \)-coloring map of \( \Delta \). Consider a collection of subsets of \( W = \{ x_v : v \in V \} \cup \{ y_v : v \in V \} \)

$$\hat{\Delta} = \{ x_G \cup y_F \mid G \in \Delta, F \subseteq G \},$$

where \( x_H = \{ x_v : v \in H \} \) and \( y_H = \{ y_v : v \in H \} \) for any \( H \subseteq V \). Then \( \hat{\Delta} \) is a simplicial complex with \( f_{i-1}(\hat{\Delta}) = 2^i f_{i-1}(\Delta) = \gamma_i \) for all \( i \). The map \( \hat{c} : W \to [k], \hat{c}(x_v) = \hat{c}(y_v) = c(v) \), shows that \( \hat{\Delta} \) is \( k \)-colored. \( \square \)

**Proof of Corollary \ref{cor:3.3}**. By Theorem \ref{thm:1.2} in order to prove Corollary \ref{cor:3.3} it is enough to show that \( \delta_{\lceil \frac{n}{2} \rceil}(P) > 0 \) where \( P \) is the boundary complex of an \( n \)-polytope.

Billera and Ehrenborg showed that the cd-index of \( n \)-polytopes is minimized (coefficient-wise) by the \( n \)-simplex, denoted \( \sigma^n [BE] \). Thus, it is enough to verify that \( \delta_{\lceil \frac{n}{2} \rceil}(\sigma^n) > 0 \). It is known that all the cd-coefficients of \( \sigma^n \) are positive (e.g., by using the Ehrenborg-Readdy formula for the cd-index of a pyramid over a polytope [ER Theorem 5.2]). \( \square \)

**A conjecture on the cd-index.** It would be natural to ask if Theorems \ref{thm:1.1} and \ref{thm:1.2} hold for all regular CW-spheres (or all Gorenstein* posets). We phrase a conjecture on the the cd-index, that, if true, immediately implies Theorem \ref{thm:1.1} as well as the entire Proposition \ref{prop:3.3} (4).

For an arbitrary cd-monomial \( w = c^{s_0} d^{s_1} d \cdots d^{s_k} \) of degree \( n \) (where \( 0 \leq s_i \) for all \( i \) and \( s_0 + \cdots + s_k + 2k = n \)), let \( F_w \) be the following subset of \([n-1]\):

$$F_w = \{ s_0 + 1, s_0 + s_1 + 3, s_0 + s_1 + s_2 + 5, \ldots, s_0 + \cdots + s_{k-1} + 2k - 1 \}.$$  

Note that \( F_w \) contains no two consecutive numbers. For example, \( F_{e^n} = \emptyset \), \( F_{d^k} = \{ 1, 3, \ldots, 2k-1 \} \) and \( F_{cd^k} = \{ 2, 4, \ldots, 2k \} \). Let \( \mathcal{A} \) be the set of subsets of \([n-1]\) that have no two consecutive numbers, and let \( \mathcal{B} \) be the set of cd-monomials of degree \( n \). Then \( w \mapsto F_w \) is a bijection from \( \mathcal{B} \) to \( \mathcal{A} \) (as \( k = |F_w| \) and \( s_k = n-2k-s_{k-1}-\cdots-s_0 \) we see that the inverse map exists).

Let \( \Delta \) be a \( k \)-colored simplicial complex with the vertex set \( V \) and a \( k \)-coloring map \( c : V \to [k] \). For any subset \( S \subseteq [k] \), let \( f_S(\Delta) = |\{ F \in \Delta : c(F) = S \}| \). The vector \( (f_S(\Delta) : S \subseteq [k]) \) is called the flag f-vector of \( \Delta \). Note that the flag f-vector of a Gorenstein* poset \( P \) is equal to the flag f-vector of \( sd(P) \) by the coloring map defined by the rank function.

**Definition 4.2.** Let \( \Phi = \sum_w a_w w \) be a homogeneous cd-polynomial of degree \( n \) with \( w \) the cd-monomials and \( a_w \in \mathbb{Z} \). For \( S \subseteq [n-1] \), we define

$$\alpha_S(\Phi) = \begin{cases} a_w, & \text{if } S = F_w \text{ for some } w \in \mathcal{B} \\ 0, & \text{if } S \notin \mathcal{A}. \end{cases}$$
Conjecture 4.3. Let \( P \) be an \((n - 1)\)-dimensional regular CW-sphere (or more generally, Gorenstein* poset of rank \( n + 1 \)). Then there exists an \((n - 1)\)-colored simplicial complex \( \Delta \) such that \( f_S(\Delta) = \alpha_S(\Phi_P) \) for all \( S \subset [n - 1] \).

Thus the above conjecture states that the \( cd \)-index is itself the flag \( f \)-vector of a colored complex. If the above conjecture is true then \( \Phi_P(1, d) = 1 + f_0(\Delta) + d + \cdots + f_{\lfloor \frac{n}{2} \rfloor - 1}(\Delta) d^{\lfloor \frac{n}{2} \rfloor} \). Although \( \Delta \) is \((n - 1)\)-colored, this fact implies Theorem \[\[11\]. Indeed, since \( f_S(\Delta) = \alpha_S(\Phi_P) = 0 \) if \( S \) has consecutive numbers, if \( c : V \to [n - 1] \) is an \((n - 1)\)-coloring map of \( \Delta \) then the map \( \hat{c} : V \to [\lfloor \frac{n}{2} \rfloor] \) defined by \( \hat{c}(v) = \lfloor \frac{c(v) + 1}{2} \rfloor \) is an \([\frac{n}{2}]\)-coloring map of \( \Delta \).

The next result supports the conjecture in low dimension.

**Proposition 4.4.** Let \( P \) be a Gorenstein* poset of rank \( n + 1 \). For all \( i, j \in [n - 1] \),
\[
\alpha_{\{i\}}(\Phi_P)\alpha_{\{j\}}(\Phi_P) \geq \alpha_{\{i,j\}}(\Phi_P).
\]

**Proof.** Let \( (h_S(P) : S \subset [n]) \) be the flag \( h \)-vector of \( P \). Let \( \{i, i + j\} \subset [n - 1] \) with \( j \geq 2 \). What we must prove is \( \alpha_{\{i\}}(\Phi_P)\alpha_{\{i+j\}}(\Phi_P) \geq \alpha_{\{i,i+j\}}(\Phi_P) \).

Observe that
\[
\begin{align*}
\alpha_{\{i\}}(\Phi_P) & = \alpha_{\{i,i+j\}}(\Phi_P) + \alpha_{\{i\}}(\Phi_P) + \alpha_{\{i+j\}}(\Phi_P), \\
\alpha_{\{i+j\}}(\Phi_P) & = \alpha_{\{i\}}(\Phi_P) + \alpha_{\{i+j\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P), \\
\alpha_{\{i,j\}}(\Phi_P) & = \alpha_{\{i\}}(\Phi_P) + \alpha_{\{i,j\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P)
\end{align*}
\]
(as \( h_{\{i\} \cup \{i+j+1,...,n\}}(P) \) is the coefficient of \( b^i a^j b^{n-i-j} \) in \( \Psi_P(a, b) \), etc.). Since \( \alpha_{\emptyset} = 1 \), it is enough to prove that
\[
h_{\{i\}}(P)h_{\{i+j+1,...,n\}}(P) \geq h_{\{n-i-j\} \cup \{n-i+1,...,n\}}(P).
\]
It follows from \[\[St2\] III, Theorem 4.6\] that there is an \( n \)-colored simplicial complex \( \Delta \) with a coloring \( c : V \to [n] \) such that \( f_S(\Delta) = h_S(P) \) for all \( S \subset [n] \). Let
\[
\Delta_S = \{ F \in \Delta : c(F) = S \}
\]
for \( S \subset [n] \). Then it is clear that
\[
\Delta_{\{i\} \cup \{i+j+1,...,n\}} \subset \{ F \cup G : F \in \Delta_{\{i\}}, G \in \Delta_{\{i+j+1,...,n\}} \},
\]
which implies the desired inequality. \( \square \)

It is straightforward that the above proposition proves the next statement.

**Corollary 4.5.** Conjecture \[\[4.3\] holds for \( n \leq 5 \).

**Non-existence of \( d \)-polynomials.** For a Gorenstein* poset \( P \), we call \( \Phi_P(1, d) \) the \( d \)-polynomial of \( P \). It is a challenging problem to classify all possible \( d \)-polynomials of Gorenstein* posets, which give a complete characterization of all possible face vectors of Gorenstein* order complexes since knowing \( d \)-polynomials is equivalent to knowing \( \gamma \)-vectors. The problem is open even for the 3-dimensional case. To study this problem, by virtue of Theorem \[\[11\] it is natural to ask which FFK vector is realizable as the \( d \)-polynomial of a Gorenstein* poset. The next result shows that not all \([\frac{n}{2}]\)-FFK vectors are realizable as the \( d \)-polynomial of a Gorenstein* poset of rank \( n + 1 \).

First recall that the ordinal sum \( Q_1 + Q_2 \) of two disjoint posets \( Q_1 \) and \( Q_2 \) is the poset whose elements are the union of elements in \( Q_1 \) and \( Q_2 \) and whose relations
are those in \( Q_1 \) union those in \( Q_2 \) union all \( q_1 < q_2 \) where \( q_1 \in Q_1 \) and \( q_2 \in Q_2 \).

For Gorenstein* posets \( Q_1 \) and \( Q_2 \), the poset \( Q_1 \ast Q_2 = (Q_1 - \{ \hat{1} \}) + (Q_2 - \{ \hat{0} \}) \) is called the join of \( Q_1 \) and \( Q_2 \), and \( \Sigma Q_1 = Q_1 \ast B_2 \), where \( B_2 \) is a Boolean algebra of rank 2, is called the suspension of \( Q_1 \). By [ST] Lemma 1.1, \( \Phi_{Q_1 \ast Q_2}(c, d) = \Phi_{Q_1}(c, d) \cdot \Phi_{Q_2}(c, d) \).

**Proposition 4.6.** Let \( P \) be a Gorenstein* poset of rank 5, and let

\[
\Phi_P(c, d) = c^4 + \alpha_{\{1\}}c^2d + \alpha_{\{2\}}cdc + \alpha_{\{3\}}dc^2 + \alpha_{\{1,3\}}d^2
\]

be its cd-index. Suppose \( \alpha_{\{2\}} = 0 \). Then there are Gorenstein* posets \( P_1 \) and \( P_2 \) of rank 3 such that \( P = P_1 \ast P_2 \). In particular, \( \alpha_{\{1,3\}} = \alpha_{\{1\}}\alpha_{\{3\}} \).

**Proof.** Let \( r \) denote the rank function \( r: P \to \{0, 1, \ldots, 5\} \) \( (r(\hat{0}) = 0, r(\hat{1}) = 5) \). Let \( P_1 := \{ F \in P : r(F) \leq 2 \} \) and \( P_2 := \{ F \in P : r(F) \geq 3 \} \).

As \( P \) is Gorenstein*, to show that \( P = P_1 \cup P_2 \) it is enough to show that \( P_2 \cup \{ \hat{0} \} \) is Gorenstein* (as a Gorenstein* poset contains no proper subposet which is Gorenstein* of the same rank, and each interval \([F, \hat{1}]\) with \( r(F) = 2 \) in \( P \) is Gorenstein*).

For this, it is enough to show that any rank 4 element in \( P \) covers exactly two rank 3 elements in \( P \). Indeed, this guarantees that the dual poset to \( P \), denoted \( P^*_2 \), is the face poset of a union of CW 1-spheres, and as \( P \) is Gorenstein* so is its dual \( P^*_2 \), hence \( P^*_2 \) is Cohen-Macaulay since \( P^*_2 \) is a rank selected poset [ST2] III, Theorem 4.5], which implies that \( P^*_2 \) is the face poset of one CW 1-sphere, i.e. \( P_2 \cup \{ \hat{0} \} \) is Gorenstein*.

Let \( F \) be a rank 4 element of \( P \). Then \( P \) is a subdivision of \( \Sigma(0, F) \) (Recalling [EK] Definition 2.6], this is shown by the map \( \phi: P \to \Sigma(0, F) \), \( \phi(\sigma) = \sigma \) if \( \sigma < F \), \( \phi(\sigma) = \sigma_1 \) if \( \sigma \) and \( F \) are incomparable, and \( \phi(\sigma) = \sigma_2 \), where \( \sigma_1, \sigma_2 \) are the rank 4 elements in \( \Sigma(0, F) \). Thus, by Lemma 2.4, the coefficient of \( cdc \) in the cd-index of \( \Sigma(0, F) \) is zero, hence the coefficient of the monomial \( cd \) in the cd-index of \( [0, F] \) is zero.

This fact implies, when expanding the cd-index of \( [0, F] \) in terms of \( a, b \), that \( h_{\{3\}}([0, F]) \) equals the coefficient of \( c^3 \), namely \( h_{\{3\}}([0, F]) = 1 \). Switching to the flag f-vector of \( [0, F] \) we get \( f_{\{3\}}([0, F]) = h_{\{3\}}([0, F]) + h_{\{3\}}([0, F]) = 1 + 1 = 2 \). Thus, \( F \) covers exactly two rank 3 elements in \( P \).

**Example 4.7.** Consider the 2-FFK vector \((1, 6, 7)\). We claim that \( \Phi_P(1, d) \neq 1 + 6d + 7d^2 \) for all Gorenstein* poset \( P \) of rank 5. Indeed, if \( \Phi_P(1, d) = 1 + 6d + 7d^2 \), then \( \alpha_{\{1,3\}} = 7 \). Then \( \alpha_{\{1\}} + \alpha_{\{3\}} = 6 \) and \( \alpha_{\{2\}} = 0 \) by Proposition 4.3 which contradicts Proposition 4.6.

A similar argument shows that \((1, 2a, a^2 - 2)\), where \( a \geq 3 \), is 2-FFK, but not realizable as the d-polynomial of a Gorenstein* poset of rank 5.

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