A Lawson-type exponential integrator for the Korteweg–de Vries equation

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Abstract

We propose an explicit numerical method for the periodic Korteweg–de Vries equation. Our method is based on a Lawson-type exponential integrator for time integration and the Rusanov scheme for Burgers’ nonlinearity. We prove first-order convergence in both space and time under a mild Courant–Friedrichs–Lewy condition $\tau = O(h)$, where $\tau$ and $h$ represent the time step and mesh size, respectively, for solutions in the Sobolev space $H^3((−\pi, \pi))$. Numerical examples illustrating our convergence result are given.

Keywords. exponential integrators; Lawson methods; Korteweg–de Vries equation; error estimates; Rusanov scheme

1 Introduction

Consider the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \Omega = (−\pi, \pi), \quad t > 0, \quad u(x, 0) = u_0(x), \quad x \in \Omega,$$

where we impose periodic boundary conditions for practical implementation. The KdV equation is a generic model for the study of weakly nonlinear long waves. It describes the propagation of shallow water waves in a channel [19] and is widely applied in science and engineering, such as in plasma physics where it gives rise to ion acoustic solitons [6] and in geophysical fluid dynamics where it describes long waves in shallow seas and deep oceans [25, 26]. The KdV equation is also relevant for studying the interaction between nonlinearity and dispersion.

For the well-posedness of the periodic KdV equation, we refer to [3, 9]. It was shown that the equation is globally well-posed for $H^s$ initial data with $s$ being a non-negative integer. For its numerical solution, various methods have been proposed and analyzed in the literature, such as finite difference methods (FDM) [10, 15, 32, 34], finite element methods [1, 2, 35], Fourier spectral methods [4, 24, 27, 29], splitting methods [13, 14, 17] and Petrov–Galerkin methods.

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for the KdV equation with nonperiodic boundary condition \[22, 23, 30\]. Numerical methods for the Kadomtsev–Petviashvili equation, which is a two-dimensional generalization of the KdV equation, were considered in \[7, 18\].

For finite difference methods, linear stability has been analyzed in \[8, 32, 34\]. The explicit leap-frog scheme \[32\] and the Lax–Friedrichs scheme \[34\] require both the rather severe stability condition \(\tau = O(h^3)\), where \(\tau\) and \(h\) represent the discretization parameters in time and space, respectively. To weaken the stability restriction, some implicit FDM were proposed in \[8, 32\]. Recently, the Lax–Friedrichs scheme with an implicit dispersion was proved to converge uniformly to the solution of the KdV equation for initial data in \(H^3\) under the stability condition \(\tau = O(h^{3/2})\) for both the decaying case on the full line and the periodic case. However, no convergence rate was obtained. Very recently, for the \(\theta\)-right winded FDM which applies the Rusanov scheme for the hyperbolic flux term and a 4-points \(\theta\)-scheme for the dispersive term, first-order convergence in space was proved under a hyperbolic Courant–Friedrichs–Lewy (CFL) condition \(\tau = O(h)\) when \(\theta \geq \frac{1}{2}\) and under an Airy Courant–Friedrichs–Lewy condition \(\tau = O(h^3)\) for solutions in \(H^6(\mathbb{R})\) \[5\].

On the other hand, the numerical approximation by Fourier spectral/pseudospectral methods has been studied by many authors \[21, 24\]. Maday and Quarteroni \[24\] showed that for solutions in \(H^r\), the error of Fourier spectral method is order of \(O(h^{r-1})\) in the \(L^2\) norm while the error of pseudospectral method is order of \(O(h^{r-2})\) in the \(H^1\) norm. The corresponding \(L^2\) estimate for the Fourier pseudospectral method was established in \[21\] with the aid of artificial viscosity to avoid the nonlinear instability caused by the aliasing error. More specifically, first-order convergence in time was shown in \[21\] for the fully discrete pseudospectral method under the stability condition \(\tau = O(h^3)\) and \(\tau = O(h^2)\) for explicit and implicit discretization methods of the nonlinear term, respectively. For the rigorous analysis of splitting methods or exponential time integrator methods, we refer to \[13, 16\] or \[12\], respectively.

In the present paper, we propose a Fourier pseudospectral method based on a Lawson-type exponential integrator which integrates the linear part exactly and the Rusanov scheme for Burgers’ nonlinearity with an added artificial viscosity. The main result of the paper is given in Theorem 2.1. There, first-order convergence in both space and time is shown under a mild CFL condition \(\tau = O(h)\).

The rest of this paper is organized as follows. In Section 2, we present the necessary notation, the numerical scheme and the main convergence result. Section 3 is devoted to the details of the error analysis. Numerical results are reported in Section 4 to illustrate our error bounds.

Throughout the paper, \(C\) represents a generic constant independent of the discretization parameters and the exact solution \(u\).

## 2 The Fourier pseudospectral method

We adopt the standard Sobolev spaces and denote by \(\| \cdot \|\) and \((\cdot, \cdot)\) the norm and inner product in \(L^2(\Omega)\), respectively. For \(m \in \mathbb{N}\), we denote by \(H^m_0(\Omega)\) the \(H^m\) functions on the one-dimensional torus \(\Omega = (-\pi, \pi)\). In particular, these functions have derivatives up to
order \( m - 1 \) that are all \( 2\pi \)-periodic. The space is equipped with the standard norm \( \| \cdot \|_m \)
and semi-norm \( | \cdot |_m \).

Let \( \tau = \Delta t > 0 \) be the time step size and denote the temporal grid points by \( t_k := k\tau \) for
\( k = 0, 1, 2, \ldots \). Given a mesh size \( h := 2\pi/(2N + 1) \) with \( N \) being a positive integer, let
\[
x_j := -\pi + jh, \quad j = 0, 1, \ldots, 2N,
\]
be the spatial grid points in \([-\pi, \pi)\). Denote
\[
X_N := \text{span}\left\{ e^{ikx} : |k| \leq N \right\}, \quad \tilde{X}_N := \left\{ v = \sum_{k=-N}^{N} v_k e^{ikx} \in \mathbb{R} \right\} \subseteq X_N,
\]
\[
Y_N := \left\{ v = (v_0, v_1, \ldots, v_{2N}) \in \mathbb{C}^{2N+1} \right\}, \quad \tilde{Y}_N = Y_N \cap \mathbb{R}^{2N+1}.
\]

For any \( u, v \in C(\Omega) \), define the following discrete inner product and norm by
\[
\langle u, v \rangle_N = h \sum_{j=0}^{2N} u(x_j)v(x_j), \quad \| u \|_N = \langle u, u \rangle_N^{1/2}.
\]

For a periodic function \( v(x) \) and a vector \( v \in Y_N \), let \( P_N : L^2(\Omega) \to X_N \) be the standard orthogonal projection operator, and \( I_N : C(\Omega) \to X_N \) or \( I_N : Y_N \to X_N \) be the interpolation operator \([31]\), i.e.,
\[
(P_N v, \varphi) = (v, \varphi), \quad \text{for all } \varphi \in X_N;
\]
\[
(I_N v)(x_j) = v(x_j), \quad \text{or } (I_N v)(x_j) = v_j, \quad j = 0, \ldots, 2N.
\]

More specifically, \( P_N v \) and \( I_N v \) can be written as
\[
(P_N v)(x) = \sum_{l=-N}^{N} \hat{v}_l e^{ilx}, \quad (I_N v)(x) = \sum_{l=-N}^{N} \tilde{v}_l e^{ilx},
\]
where \( \hat{v}_l \) and \( \tilde{v}_l \) are the Fourier and discrete Fourier coefficients, respectively, defined as
\[
\hat{v}_l = \frac{1}{2\pi} \int_{0}^{2\pi} v(x) e^{-ilx} dx, \quad \tilde{v}_l = \frac{1}{2N+1} \sum_{j=0}^{2N} v_j e^{-ilx_j}, \quad l = -N, \ldots, N.
\]

It was proved in \([31]\) that for any \( u, v \in C(\Omega) \),
\[
\langle u, v \rangle_N = (I_N u, I_N v), \quad \| u \|_N = \| I_N u \|.
\]

The semi-discrete pseudospectral method for (1.1) consists in finding \( u_N \) in \( \tilde{X}_N \) such that
\[
\partial_t u_N(x, t) + \partial_x^3 u_N(x, t) + \frac{1}{2} I_N \left( (u_N^2)_x \right)(x, t) = 0, \quad x \in \Omega = (-\pi, \pi), \quad t > 0,
\]
\[
u_N(x, 0) = I_N(u_0)(x), \quad x \in \overline{\Omega}.
\]
Thus, by Duhamel’s formula, we have
\[ u_N(t_n + \tau) = e^{-\tau \partial_x^3} u_N(t_n) - \frac{1}{2} \int_0^\tau e^{-(\tau-s)\partial_x^3} I_N \left( (u_N^2)_x(t_n + s) \right) ds. \]

By applying the approximation \( u_N(t_n + s) \approx u_N(t_n) \) and the first-order Lawson method [11, 20], we get a first-order approximation as
\[ u_N(t_n + \tau) \approx e^{-\tau \partial_x^3} u_N(t_n) - \frac{\tau}{2} e^{-\tau \partial_x^3} I_N \left( (u_N^2)_x(t_n) \right). \]

(2.3)

To ensure the stability, we apply the Rusanov scheme [5, 33] for Burgers’ nonlinearity, which consists of a centered hyperbolic flux and an added artificial viscosity. The scheme then reads as
\[ u_{n+1}^N = e^{-\tau \partial_x^3} u_n^N - \frac{\tau}{2} e^{-\tau \partial_x^3} I_N \left( \delta_x^0 (u_n^N)^2 \right) + \frac{c\tau h}{2} e^{-\tau \partial_x^3} \delta_x^2 u_n^N, \quad n \geq 0, \]
\[ u_0^N = I_N(u_0), \]

(2.4)

where the constant \( c \) is the so-called Rusanov coefficient, which has to satisfy a certain condition (cf. (3.28)). Moreover, we have used the notation
\[ \delta_x^0 v(x) = \frac{v(x+h) - v(x-h)}{2h}, \quad \delta_x^2 v(x) = \frac{v(x+h) - 2v(x) + v(x-h)}{h^2}, \]

where \( v(x) = v(x \pm 2\pi) \). Similarly, for a vector \( v \in Y_N \), define the standard finite difference operators as
\[ \delta_x^0 v_j = \frac{v_{j+1} - v_{j-1}}{2h}, \quad \delta_x^2 v_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}, \quad \delta_x^+ v_j = \frac{v_{j+1} - v_j}{h}, \quad j = 0, 1, \ldots, 2N, \]

with \( v_{j \pm (2N+1)} = v_j \) when necessary.

We are now in the position to present the main result of the paper.

**Theorem 2.1** Assume that the solution of (1.1) satisfies \( u \in C(0, T; H^3_{\alpha}(\Omega)) \) and let \( c_0 > 0 \) be given by condition (3.28). Then, for \( c \geq c_0 \), there exists \( h_0 > 0 \) such that for all \( h \leq h_0 \) and \( \tau \leq h/c \), the error of scheme (2.4) satisfies
\[ \|u_n^N - u(t_n)\| \leq M(\tau + h), \quad n\tau \leq T. \]

(2.5)

Here both of the constants \( M \) and \( h_0 \) depend on \( T, c \) and \( \|u\|_{L^{\infty}(0,T;H^3_{\alpha}(\Omega))} \) (cf. (3.29) and (3.16)).

### 3 Error estimate

The purpose of this section is to prove Theorem 2.1.
3.1 Some lemmas

We recall three lemmas from the literature and then prove an additional lemma. All the lemmas are used in the proof of Theorem 2.1.

Lemma 3.1 [31]. For any \( u \in H^m_p(\Omega) \) and \( 0 \leq \mu \leq m \),

\[
\| P_N u - u \|_\mu \leq C h^{m-\mu} |u|_m, \quad \| P_N u \|_m \leq C \| u \|_m.
\]

(3.1)

In addition, if \( m > \frac{1}{2} \), then

\[
\| P_N u - u \|_\mu \leq C h^{m-\mu} |u|_m, \quad \| P_N u \|_m \leq C \| u \|_m.
\]

(3.2)

Lemma 3.2 (Nikolski’s Inequality) [31]. For any \( u \in X_N \) and \( 1 \leq p \leq q \leq \infty \),

\[
\| u \|_{L^q} \leq \left( \frac{N p_0 + 1}{2 \pi} \right)^{\frac{1}{p} - \frac{1}{q}} \| u \|_{L^p},
\]

where \( p_0 \) is the smallest even integer \( \geq p \). In particular, for \( p = 2 \) and \( q = \infty \), we have

\[
\| u \|_{L^\infty} \leq h^{-1/2} \| u \|.
\]

(3.3)

Lemma 3.3 (Bernstein’s Inequality) [31]. For any \( u \in X_N \) and \( 0 \leq \mu \leq m \),

\[
\| u \|_m \leq C h^{\mu-m} \| u \|_\mu.
\]

(3.4)

Lemma 3.4 For \( a = (a_0, a_1, \ldots, a_{2N}) \), \( b = (b_0, b_1, \ldots, b_{2N}) \in \tilde{Y}_N \), we have

\[
\langle a, \delta_x^2 b \rangle_N = -\langle \delta_x^+ a, \delta_x^+ b \rangle_N,
\]

(3.5)

and

\[
\sum_{j=0}^{2N} a_j a_{j+1} \delta_x^+ a_j = -\frac{h^2}{3} \sum_{j=0}^{2N} (\delta_x^+ a_j)^3, \quad \sum_{j=0}^{2N} a_j a_{j+1} \delta_x^0 a_j = -\frac{4h^2}{3} \sum_{j=0}^{2N} (\delta_x^0 a_j)^3,
\]

(3.6)

\[
\sum_{j=0}^{2N} \delta_x^2 a_j \delta_x^0 (ab)_j = -\frac{1}{h^2} \sum_{j=0}^{2N} a_j a_{j+1} \delta_x^+ b_j + \frac{1}{h^2} \sum_{j=0}^{2N} a_j a_{j+1} \delta_x^0 b_j.
\]

(3.7)

Proof. The identity (3.5) is the discrete version of the integration by parts formula:

\[
\langle a, \delta_x^2 b \rangle_N = \frac{1}{h} \sum_{j=0}^{2N} a_j (b_{j+1} - 2b_j + b_{j-1}) = \frac{1}{h} \sum_{j=0}^{2N} a_j (b_{j+1} - b_j) - \frac{1}{h} \sum_{j=0}^{2N} a_j (b_j - b_{j-1})
\]

\[
= \sum_{j=0}^{2N} a_j \delta_x^+ b_j - \sum_{j=0}^{2N} a_{j+1} \delta_x^+ b_j = -h \sum_{j=0}^{2N} (\delta_x^+ a_j)(\delta_x^+ b_j) = -\langle \delta_x^+ a, \delta_x^+ b \rangle_N.
\]
By definition, we have
\[
\sum_{j=0}^{2N} a_j a_{j+1} \delta^+_x a_j = \sum_{j=0}^{2N} \frac{a_j a_{j+1}^2 - a_j^2 a_{j+1}}{h} = \frac{-2N \sum_{j=0}^{2N} a_j a_{j+1}^2 - 3a_j a_{j+1}^2 + 3a_j a_{j+1}^2 - a_j^3}{3h} = -\frac{h^2}{3} \sum_{j=0}^{2N} (\delta^+_x a_j)^3,
\]
which completes the proof of (3.6). For (3.7), a straightforward calculation shows that
\[
\sum_{j=0}^{2N} a_{j-1} a_{j+1} \delta^0_x a_j = \sum_{j=0}^{2N} \frac{a_{j+1}^2 a_j - a_{j+1} a_{j-1}^2}{2h} = \frac{-4}{3} \sum_{j=0}^{2N} \frac{a_{j+1}^2 - 3a_{j+1} a_j - 3a_{j+1} a_{j-1}^2 - a_{j-1}^3}{8h} = -\frac{4h^2}{3} \sum_{j=0}^{2N} (\delta^0_x a_j)^3,
\]
which completes the proof of Lemma 3.3. \(\square\)

### 3.2 Local error analysis

We introduce the local truncation error \(\xi^{n+1}\) as defect
\[
\xi^{n+1} = u(t_{n+1}) - e^{-\tau \partial_x^2} u(t_n) + \frac{\tau}{2} e^{-\tau \partial_x^2} \left[ \delta_x^0 (u(t_n))^2 - ch \delta_x^2 u(t_n) \right], \quad n \geq 0. \tag{3.8}
\]
The local error can be bounded as follows.

**Lemma 3.5** For \(u \in C(0, T; H^3_p(\Omega))\), we have
\[
\|\xi^{n+1}\| \leq M_3 \tau^2 + M_2 \tau h,
\]
where \(M_3\) and \(M_2\) depend on \(\|u\|_{L^\infty(0, T; H^3(\Omega))}\) and \(\|u\|_{L^\infty(0, T; H^2(\Omega))}\), respectively.
Proof. We first recall
\[ u(t_{n+1}) = e^{-\tau \partial_y^2} u(t_n) - \frac{1}{2} \int_0^\tau e^{-(\tau-s)\partial_y^2} (u^2)_x(t_n+s) \, ds \]
and that \( e^{i \partial_y^2} \) is a linear isometry for all \( t \in \mathbb{R} \). This yields that
\[
\| \xi^{n+1} \| \leq \frac{1}{2} \left\| \int_0^\tau e^{-(\tau-s)\partial_y^2} (u^2)_x(t_n+s) - e^{-\tau \partial_y^2} (u^2)_x(t_n) \, ds \right\|
\]
\[
+ \frac{\tau}{2} \left\| e^{-\tau \partial_y^2} [(u^2)_x(t_n) - \delta_x^0 (u(t_n)^2)] \right\| + \frac{ct \gamma}{2} \left\| e^{-\tau \partial_y^2} \delta_x^2 u(t_n) \right\|
\]
\[
= \frac{\tau}{2} \left\| (u^2)_x(t_n) - \delta_x^0 (u(t_n)^2) \right\| + \frac{ct \gamma}{2} \left\| \delta_x^2 u(t_n) \right\|
\]
\[
=: I_1 + I_2 + I_3.
\]
For the first part we get
\[
I_1 = \left\| \int_0^\tau (\tau - s) e^{-(\tau-s)\partial_y^2} \left[ \delta_x^2 (uu_x) + \partial_s (uu_x) \right] (t_n+s) \, ds \right\|
\]
\[
= \left\| \int_0^\tau (\tau - s) e^{-(\tau-s)\partial_y^2} \left[ 3u_x \delta_x^3 u + 3(\delta_x^2 u)^2 - u^2 \delta_x^2 u - 2uu_x^2 \right] (t_n+s) \, ds \right\|
\]
\[
\leq C \tau^2 \sup_{0 \leq t \leq T} \| u \|_3 \left[ \| u_x \|_{L^\infty} + \| \partial_x^2 u \|_{L^\infty} + \| u \|_{H^2} + \| u \|_{L^\infty} \| u_x \|_{L^2} \right]
\]
\[
\leq C \tau^2 \sup_{0 \leq t \leq T} \left( \| u \|_3^2 + \| u \|_3^3 \right),
\]
where we employed equation (11) and the Sobolev imbedding theorem \( H^3(\Omega) \hookrightarrow W^{2,\infty}(\Omega) \). Further, using Taylor expansion and Hölder’s inequality, we have
\[
I_2 = \frac{\tau}{4h} \left\| \int_0^h (h-y) \left[ \delta_x^2 (u^2)_x (\cdot + y, t_n) - \delta_x^2 (u^2)_x (\cdot - y, t_n) \right] \, dy \right\|
\]
\[
\leq \frac{\tau h^{1/2}}{4} \left( \int_0^\pi \int_0^h \left[ \delta_x^2 (u^2)_x (\cdot + y, t_n) - \delta_x^2 (u^2)_x (\cdot - y, t_n) \right]^2 \, dy \, dx \right)^{1/2}
\]
\[
\leq \tau h \| \delta_x^2 (u^2)_x (t_n) \| \leq 2\tau h (\| u_x (t_n) \|_{L^\infty} \| u(t_n) \|_1 + \| u(t_n) \|_{L^\infty} \| u(t_n) \|_2)
\]
\[
\leq C \tau h \| u(t_n) \|_2 \leq C \tau h \sup_{0 \leq t \leq T} \| u \|_2 .
\]
A similar calculation shows that
\[
I_3 = \frac{ct \gamma}{2h} \left\| \int_0^h (h-y) \left[ \delta_x^2 (u^2)_x (\cdot + y, t_n) + \delta_x^2 (u^2)_x (\cdot - y, t_n) \right] \, dy \right\|
\]
\[
\leq c \tau h \sup_{0 \leq t \leq T} \| u \|_2^2,
\]
which completes the proof. \( \square \)
3.3 Proof of Theorem 2.1.

Proof. Denote \( \omega_n^t = P_N(u(t_n)) \) and \( \eta^n = u_n^t - \omega_n^t \in \tilde{Y}_N \). In view of (3.11) and the triangle inequality, it is sufficient to show

\[
\| \eta^n \| \leq M(\tau + h),
\]

where \( M \) is independent of \( \tau \) and \( h \) for \( 0 \leq n\tau \leq T \).

The proof is given by induction. For \( n = 0 \), it is obvious by using Lemma 3.1

\[
\| \eta^0 \| = \| I_N(u_0) - P_N(u_0) \| \leq C h \| u_0 \|_1.
\]

Suppose the claim is true for \( n = 0, 1, \ldots, k \). We prove that \( \| \eta^{k+1} \| \leq M(\tau + h) \). Subtracting (2.4) from the projection of (3.8) in \( X_N \) and noticing that the operator \( P_N \) commutes with \( e^{-\tau \partial_3^2} \), we get for \( n = 0, 1, \ldots, k \),

\[
\eta^{n+1} = e^{-\tau \partial_3^2} \eta^n - \frac{\tau}{2} e^{-\tau \partial_3^2} \left[ I_N \delta_x^0 (u_n^t)^2 - P_N \delta_x^0 (u_n^t)^2 \right] + \frac{c \tau h}{2} e^{-\tau \partial_3^2} \delta_x^2 \eta^n - P_N (\xi^{n+1})
\]

\[
e^{-\tau \partial_3^2} \left[ \eta^n + \frac{c \tau h}{2} \delta_x^2 \eta^n - \frac{\tau}{2} I_N \delta_x^0 (u_n^t)^2 - (\omega_n^t)^2 + \zeta^{n+1} \right], \tag{3.11}
\]

where

\[
\zeta^{n+1} = \frac{\tau}{2} \left[ P_N \delta_x^0 (u(t_n)^2) - I_N \delta_x^0 (\omega_n^t)^2 \right] - e^{\tau \partial_3^2} P_N (\xi^{n+1}).
\]

It follows from Lemma 3.1 (2.1) and Lemma 3.5 that

\[
\| \zeta^{n+1} \| \leq \frac{\tau}{2} \| P_N \delta_x^0 (u(t_n)^2) - I_N \delta_x^0 (u(t_n)^2) \| + \frac{\tau}{2} \| I_N \delta_x^0 (u(t_n)^2) - (\omega_n^t)^2) \| + \| P_N (\xi^{n+1}) \|
\]

\[
\leq C h \| \delta_x^0 (u(t_n)^2) \|_1 + \tau \| \delta_x^0 (u(t_n)^2) - (\omega_n^t)^2) \|_1 + M_3 (\tau^2 + \tau h)
\]

\[
\leq C h \| u(t_n)^2 \|_1 + \| u(t_n)^2 - (\omega_n^t)^2) \|_1 + M_3 (\tau^2 + \tau h)
\]

\[
\leq C h \| u(t_n)^2 \|_1 + C h \| u(t_n) + \omega_n^t \|_1 + M_3 (\tau^2 + \tau h)
\]

\[
\leq C h \| u(t_n)^2 \|_1 + C h \| u(t_n) \|_1 + M_3 (\tau^2 + \tau h) \leq M_3 \tau^2 + \tau h, \tag{3.12}
\]

where \( M_3 \) depends on \( \| u \|_{L^\infty(0,T;H_0^2(\Omega))} \). Here for the third inequality we used the properties

\[
\| \delta_x^0 v \|^2 = \frac{1}{4 h^2} \int_{-\pi}^{\pi} (v(x + h) - v(x - h))^2 dx = \frac{1}{4 h^2} \int_{-\pi}^{\pi} \left( \int_{-h}^{h} v'(x + y) dy \right)^2 dx 
\]

\[
\leq \frac{1}{2 h} \int_{-\pi}^{\pi} \int_{-h}^{h} (v'(x + y))^2 dy dx = |v|^2_1,
\]

\[
\| \delta_x^0 v \|_N^2 = \frac{1}{4 h} \sum_{j=0}^{2N} (v(x_j + h) - v(x_j - h))^2 = \frac{1}{4 h} \sum_{j=0}^{2N} \left( \int_{-h}^{h} v'(x_j + y) dy \right)^2 
\]

\[
\leq \frac{1}{2} \sum_{j=0}^{2N} \int_{-h}^{h} (v'(x_j + y))^2 dy = |v|^2_1,
\]
and the well-known bilinear estimate \( \|fg\|_1 \leq C\|f\|_1\|g\|_1 \). For simplicity of notation, we denote \( u_j^n = u_j^n(x_j), \omega_j^0 = \omega_j^0(x_j), \eta_j^n = \eta_j^n(x_j) \) and \( c_j^{n+1} = \zeta_j^{n+1}(x_j) \). Recall that \( u_j^n, \omega_j^0, \eta_j^n, \zeta_j^{n+1} \in \mathbb{R} \) by definition. Applying (3.11), Young's inequality, (2.1) and (3.5), we obtain

\[
\|\eta^{n+1}\|^2 = \|\eta^n + \frac{\tau}{2} I_N \delta_x^0(u^n) - \frac{\tau}{2} I_N \delta_x^0(\omega^n) + \zeta^{n+1}\|^2 \\
= \|\eta^n + \zeta^{n+1}\|^2 + \frac{\tau^2}{4} \|c_0 \delta_x^2 \eta^n - I_N \delta_x^0((\eta^n)^2) - 2I_N \delta_x^0(\eta^n \omega^n)\|^2 \\
+ \tau \langle \eta^n + \zeta^{n+1}, c_0 \delta_x^2 \eta^n - I_N \delta_x^0((\eta^n)^2) - 2I_N \delta_x^0(\eta^n \omega^n)\rangle \\
\leq (1 + \tau)\|\eta^n\|^2 + (1 + \frac{1}{\tau})\|\zeta^{n+1}\|^2 + \frac{\tau^2}{4} \|c_0 \delta_x^2 \eta^n - \delta_x^0((\eta^n)^2)\|^2 + 2\delta_x^0(\eta^n \omega^n)\|_N \\
+ \tau \langle \eta^n + \zeta^{n+1}, c_0 \delta_x^2 \eta^n - \delta_x^0((\eta^n)^2)\rangle \\
= (1 + \tau)\|\eta^n\|^2 + (1 + \frac{1}{\tau})\|\zeta^{n+1}\|^2 + \frac{\tau^2}{4} \|c_0 \delta_x^2 \eta^n\|^2_N + \frac{\tau^2}{4} \||\eta^n\|^2_N \\
+ \tau^2 \|\delta_x^0(\eta^n, \omega^n)\|^2_N - \frac{\tau^2}{2} \langle \delta_x^0 \eta^n, \delta_x^0((\eta^n)^2)\rangle_N - \tau^2 \langle \delta_x^0 \eta^n, \delta_x^0(\eta^n \omega^n)\rangle_N \\
+ \tau^2 \langle \delta_x^0(\eta^n), \delta_x^0(\omega^n)\rangle_N - \tau \langle \zeta^{n+1}, \delta_x^0((\eta^n)^2)\rangle_N \\
- 2\tau \langle \eta^n, \delta_x^2(\eta^n \omega^n)\rangle_N - 2\tau \langle \zeta^{n+1}, \delta_x^0(\eta^n \omega^n)\rangle_N .
\]

By definition and the periodic boundary condition, we have

\[
\frac{\tau^2 h^2}{4} \|\delta_x^2 \eta^n\|^2_N = \frac{\tau^2 h^2}{4} \sum_{j=0}^{2N} (6(\eta_j^n)^2 - 8\eta_j^n \eta_{j+1}^n + 2\eta_{j+1}^n \eta_{j-1}^n) \\
= \frac{\tau^2 h^2}{4} \sum_{j=0}^{2N} (4(\eta_j^n)^2 - 8\eta_j^n \eta_{j+1}^n + 4(\eta_{j+1}^n)^2 - (\eta_{j+1}^n)^2 + 2\eta_{j+1}^n \eta_{j-1}^n - (\eta_{j-1}^n)^2) \\
= \tau^2 \left( \|\delta_x^0 \eta^n\|^2_N - \|\delta_x^0 \eta^n\|^2_N \right) .
\]

Moreover, it follows from (3.3), by induction \( \|\eta^n\| \leq M(\tau + h) \) and the assumption \( \tau \leq h/c \) that

\[
\|\eta^n\|_{L^\infty} \leq h^{-1/2}\|\eta^n\| \leq Mh^{-1/2}(\tau + h) \leq M(1 + 1/c)h^{1/2} \leq 1,
\]

whenever

\[
h \leq h_0 = M^{-2}(1 + 1/c)^{-2} .
\]

Thus

\[
\frac{\tau^2}{4} \|\delta_x^0((\eta^n)^2)\|^2_N = \frac{\tau^2 h}{4} \sum_{j=0}^{2N} (\delta_x^0 \eta_j^n)^2(\eta_{j+1}^n + \eta_{j-1}^n)^2 \leq \tau^2 \|\eta^n\|^2_{L^\infty} \|\delta_x^0 \eta^n\|^2_N \leq \tau^2 \|\delta_x^0 \eta^n\|^2_N .
\]

In view of the Sobolev inequality and (3.1), we have

\[
\|\omega^n\|_{L^\infty} \leq C\|\omega^n\|_1 \leq C\|u(t_n)\|_1 \leq M_1, \quad \|\partial_x \omega^n\|_{L^\infty} \leq C\|\omega^n\|_2 \leq C\|u(t_n)\|_2 \leq M_2 .
\]
where $M_1$ and $M_2$ depend on $\|u\|_{L^\infty(0,T;H^1_0(\Omega))}$ and $\|u\|_{L^\infty(0,T;H^1(\Omega))}$, respectively. This yields

$$
\tau^2 \| \delta_x^0(\eta^n, \omega^n) \|_N^2 = \tau^2 h \sum_{j=0}^{2N} (\eta^0_j + \delta_x^0 \omega^n_j + \omega^n_{j-1} \delta_x^0 \eta^n_j)^2
$$

$$
\leq 2\tau^2 h \sum_{j=0}^{2N} ((\eta^n_{j+1})^2(\delta_x^0 \omega^n_j)^2 + (\omega^n_{j-1})^2(\delta_x^0 \eta^n_j)^2)
$$

$$
\leq 2\tau^2 \| \partial_x \omega^n_N \|_{L^\infty} \| \eta^n \|_N^2 + 2\tau^2 \| \omega^n_N \|_{L^\infty} \| \delta_x^0 \eta^n \|_N^2
$$

$$
\leq 2\tau^2 \left( M_2^2 \| \eta^n \|_N^2 + M_1^4 \| \delta_x^0 \eta^n \|_N^2 \right). \quad (3.19)
$$

Applying (3.17) and (3.15), we obtain

$$
-\frac{c\tau^2 h}{2} \langle \delta_x^0 \eta^n, \delta_x(\eta^n)^2 \rangle_N = \frac{2}{c} \sum_{j=0}^{2N} \eta^n_j \eta^n_{j+1} \delta_x^0 \eta^n_j - \frac{c\tau^2}{2} \sum_{j=0}^{2N} \eta^n_{j-1} \eta^n_{j+1} \delta_x^0 \eta^n_j
$$

$$
= \frac{-c\tau^2 h}{6} \sum_{j=0}^{2N} \delta_x^0 \eta^n_j \leq \frac{-c\tau^2 h}{3} \sum_{j=0}^{2N} \delta_x^0 \eta^n_j \leq \frac{c\tau^2}{3} \left( \| \delta_x^0 \eta^n \|_N^2 \right). \quad (3.20)
$$

Similarly, using (3.7), (3.18) and the assumption $c\tau \leq h$ yields

$$
-c\tau^2 h \langle \delta_x^0 \eta^n, \delta_x(\eta^n \omega^n) \rangle_N = c\tau^2 \sum_{j=0}^{2N} \eta^n_j \eta^n_{j+1} \delta_x \omega^n_j - c\tau^2 \sum_{j=0}^{2N} \eta^n_{j-1} \eta^n_{j+1} \delta_x \omega^n_j
$$

$$
\leq 2\tau \| \partial_x \omega^n_N \|_{L^\infty} \| \eta^n \|_N \leq 2M_2 \| \eta^n \|_N. \quad (3.21)
$$

Some tedious calculations give

$$
\tau^2 \langle \delta_x^0((\eta^n)^2), \delta_x^0(\eta^n \omega^n_N) \rangle_N = \tau^2 h \sum_{j=0}^{2N} (\delta_x^0 \eta^n_j)(\eta^n_{j+1} + \eta^n_{j-1})(\omega^n_{j+1} \delta_x^0 \eta^n_j + \eta^n_{j-1} \delta_x^0 \omega^n_j)
$$

$$
= \tau^2 h \sum_{j=0}^{2N} (\delta_x^0 \eta^n_j)^2 \omega^n_{j+1} + \eta^n_{j+1} + \eta^n_{j-1}) + \tau^2 h \sum_{j=0}^{2N} \eta^n_{j-1} (\eta^n_{j+1} + \eta^n_{j-1}) (\delta_x^0 \eta^n_j)(\delta_x^0 \omega^n_j)
$$

$$
\leq 2\tau^2 \| \omega^n_N \|_{L^\infty} \| \eta^n \|_{L^\infty} \| \delta_x^0 \eta^n \|_N^2 + \frac{\tau}{h} \| \eta^n \|_{L^\infty} \| \partial_x \omega^n_N \|_{L^\infty} h \sum_{j=0}^{2N} \eta^n_{j-1} (|\eta^n_{j+1} + |\eta^n_{j-1}|)
$$

$$
\leq 2\tau^2 \| \omega^n_N \|_{L^\infty} \| \eta^n \|_{L^\infty} \| \delta_x^0 \eta^n \|_N^2 + 2\tau \| \eta^n \|_{L^\infty} \| \partial_x \omega^n_N \|_{L^\infty} \| \eta^n \|_N^2
$$

$$
\leq 2M_1 \tau^2 \| \delta_x^0 \eta^n \|_N^2 + \frac{2M_2 \tau}{c} \| \eta^n \|_N^2. \quad (3.22)
$$
Applying Young’s inequality, we have

$$c^2 h \langle \zeta^{n+1}, \delta^2 \eta^n \rangle_N = c^2 \sum_{j=0}^{2N} \zeta^{n+1}_j \left( \eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n \right)$$

$$\leq 2c \sum_{j=0}^{2N} (\zeta^{n+1}_j)^2 + \frac{c^2}{8} \sum_{j=0}^{2N} (\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n)^2$$

$$\leq 2c \sum_{j=0}^{2N} (\zeta^{n+1}_j)^2 + 2c^2 \sum_{j=0}^{2N} (\eta_j^n)^2 \leq \frac{2}{\tau} \|\zeta^{n+1}\|^2 + 2\tau \|\eta^n\|^2. \quad (3.23)$$

Furthermore, a straightforward calculation yields that

$$-\tau \langle \eta^n, \delta^0 (\eta^n)^2 \rangle_N = -\frac{T}{2} \sum_{j=0}^{2N} (\eta^0_j)^2 - (\eta^n_j)^2 = -\frac{T}{2} \sum_{j=0}^{2N} \left[ (\eta_{j+1}^n (\eta_{j+1}^n)^2 - (\eta_{j-1}^n)^2 \right]$$

$$= \frac{T}{6} \sum_{j=0}^{2N} \left[ (\eta_{j+1}^n)^3 - 3\eta_{j+1}^n (\eta_{j+1}^n)^2 + (\eta_{j-1}^n)^2 \right]$$

$$- 2\tau \langle \eta^n, \delta^0 (\eta^n \omega_N^n) \rangle_N = -\tau h \sum_{j=0}^{2N} \eta_j^n \eta_{j+1} \delta^x_j \omega_j^n \leq \tau \|\partial_x \omega_N^n\| \|\eta^n\|^2 \leq M_2 \tau \|\eta^n\|^2. \quad (3.25)$$

Similarly, one derives that

$$-\tau \langle \zeta^{n+1}, \delta^0 (\eta^n)^2 \rangle_N = -\frac{T}{2} \sum_{j=0}^{2N} \zeta^{n+1}_j \left[ (\eta_{j+1}^n)^2 - (\eta_{j-1}^n)^2 \right] + 2\eta_j^n \omega_j^n - 2\eta_{j-1}^n \omega_{j-1}^n$$

$$\leq 2 \sum_{j=0}^{2N} (\zeta^{n+1}_j)^2 + \frac{T^2}{4} \|\eta^n\|_2^2 \sum_{j=0}^{2N} (\eta_{j+1}^n - \eta_{j-1}^n)^2 + \frac{T^2}{4} \sum_{j=0}^{2N} (\eta_{j+1}^n \omega_{j+1}^n - \eta_{j-1}^n \omega_{j-1}^n)^2$$

$$\leq \frac{T}{h} \left[ \frac{2}{\tau} \|\zeta^{n+1}\|^2 + \tau (\|\eta^n\|_2^2 + \|\omega_N^n\|_2^2) \|\eta^n\|^2 \right] \leq \frac{1}{c} \frac{2}{\tau} \|\zeta^{n+1}\|^2 + \tau (1 + M_1^2) \|\eta^n\|^2. \quad (3.26)$$

Combining (3.13) and (3.14)- (3.26), we obtain that

$$\|\eta^{n+1}\|^2 \leq \left( 1 + A \tau \right) \|\eta^n\|^2 + \left( 1 + \frac{3}{\tau} + \frac{2}{c \tau} \right) \|\zeta^{n+1}\|^2 + B \tau^2 \|\delta^0 \eta^n\|^2$$

$$+ \tau h \sum_{j=0}^{2N} \left( h - c \tau \right) \left( \frac{h^2}{6} \delta^x_j \eta_j^n - c \right) \left( \delta^x_j \eta_j^n \right)^2, \quad (3.27)$$

where

$$A = 3 + 3M_2 + 2\tau M_2^2 + \frac{1}{c} (1 + 2M_2 + M_1^2),$$

$$B = 1 + \frac{2c}{3} - c^2 + 2M_1 + 2M_2^2.$$
Applying Young’s inequality, we get
\[ B \leq 2 + \frac{1}{9}c^2 - c^2 + 2M_1 + 2M_1^2 \leq 2(M_1 + 1)^2 - \frac{8}{9}c^2, \]
which implies that \( B \leq 0 \) whenever
\[ c \geq c_0 = \frac{3}{2}(1 + M_1), \tag{3.28} \]
where \( M_1 \) is given by (3.18) depending on \( \|u\|_{L^\infty(0,T;H^1_0(0))} \). Furthermore, in view of (3.15),
\[ \frac{h}{6}\delta_x \eta_j^n - c \leq \frac{1}{3}\|\eta^n\|_{L^\infty} - c < 0, \]
and the CFL condition \( ct \leq h \), this together with (3.12) yields that for \( n = 0, \ldots, k \),
\[
\|\eta^{n+1}\|^2 \leq (1 + \tau C(M_2, c))\|\eta^n\|^2 + \frac{C(c)}{\tau}\|\zeta^{n+1}\|^2 \\
\leq (1 + \tau C(M_2, c))\|\eta^n\|^2 + \tau C(M_3, c)(\tau + h)^2,
\]
where \( C(c, d) \) indicates that \( C \) depends on \( c \) and \( d \). Hence
\[
\|\eta^{k+1}\|^2 \leq e^{\tau C(M_2, c)}\|\eta^k\|^2 + \tau C(M_3, c)(\tau + h)^2 \\
\leq e^{2\tau C(M_2, c)}\|\eta^{k-1}\|^2 + \tau C(M_3, c)(\tau + h)^2(1 + e^{\tau C(M_2, c)}) \\
\leq \ldots \\
\leq e^{(k+1)\tau C(M_2, c)}\|\eta^0\|^2 + \tau C(M_3, c)(\tau + h)^2(1 + e^{\tau C(M_2, c)} + \ldots + e^{k\tau C(M_2, c)}) \\
\leq e^{\tau C(M_2, c)} [\|\eta^0\|^2 + \frac{C(M_3, c)}{C(M_2, c)}(\tau + h)^2] \\
\leq e^{\tau C(M_2, c)}C(M_3, c)(\tau + h)^2,
\]
which gives the error (3.29) for \( n = k + 1 \) by setting
\[ M = C(T, M_3, c) = e^{TC(M_2, c)/2}C^{1/2}(M_3, c). \tag{3.29} \]
This concludes the proof.

4 Numerical experiments

In this section, we present some numerical experiments to illustrate our analytic convergence rate given in Theorem 2.1.

Example 1. The well-known solitary-wave solution of the KdV equation (1.1) is given by
\[ u(x, t) = 12\lambda \text{sech}^2(\sqrt{\lambda}(x - 4\lambda t - a)), \quad a \in \mathbb{R}, \quad \lambda > 0. \tag{4.1} \]
It represents a single bump moving to the right with speed \( 4\lambda \). Here we choose \( \lambda = 1/4 \) and the torus \( \Omega = (-30, 30) \) which is large enough such that the periodic boundary conditions
Figure 1: Numerical simulation for the solitary-wave solution (4.1). (a) The error of the first-order scheme (2.4) at $T = 2$ for various choices of $h$ and $c$. The time step size $\tau$ satisfies $\tau = h/c$. The broken line has slope one. (b) The numerical solution at $T = 10, 20$ was obtained by the scheme (2.4) with $h = 1/200$ and $\tau = h/4$.

do not introduce significant errors, i.e., the soliton is far enough away from the boundary for the considered time interval.

Example 2. The initial data of the KdV equation (1.1) is now chosen as

$$u_0(x) = 3 \text{sech}^2(2x) \sin(x), \quad x \in [-\pi, \pi].$$  \hspace{1cm} (4.2)

The initial data and the numerical solution for $T = 3$ with $c = 3$, $h = \pi/2^{11}$ and $\tau = h/\pi$ are displayed in Figure 2 (b), where the reference solution is obtained by the second-order exponential integrator of [12] with $\tau = 10^{-6}$ and $h = \pi/2^{15}$. The error of the scheme (2.4) with $c = 3$ and $\tau = h/\pi$ is shown in Figure 2 (a). The graph clearly shows first-order convergence of the scheme (2.4).
Figure 2: Numerical simulation for the initial value \((4.2)\). (a) The error of the first-order scheme \((2.4)\) at \(T = 3\) for various choices of \(h\) with \(c = 3\) and \(\tau = h/\pi\). The broken line has slope one. (b) The numerical solution at \(T = 3\) was computed with the scheme \((2.4)\) using \(h = \pi/2^{11}\) and \(\tau = h/\pi\).

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