A NEW CLASS OF PATH EQUATIONS IN AP-GEOMETRY

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Abstract

In the present work, it is shown that, the application of the Bazanski approach to Lagrangians, written in AP-geometry and including the basic vector of the space, gives rise to a new class of path equations. The general equation representing this class contains four extra terms, whose vanishing reduces this equation to the geodesic one. If the basic vector of the AP-geometry is considered as playing the role of the electromagnetic potential, as done in a previous work, then the second term (of the extra terms) will represent Lorentz force while the fourth term gives a direct effect of the electromagnetic potential on the motion of the charged particle. This last term may give rise to an effect similar to the Aharanov-Bohm effect. It is to be considered that all extra terms will vanish if the space-time used is torsion-less.

1 Introduction

In a previous attempt [1], to explore paths in AP-geometry that are analogous to the geodesic of Riemannian geometry using the Bazanski approach, it has been shown that there are three such paths in which a torsion term appears. The vanishing of this term would reduce these equations to the geodesic one. This term is found to have a jumping coefficient from one equation to the other. One of the authors of the present work [2] has generalized this set of equations and suggested that the torsion term represents a type of interaction between the torsion of the background gravitational field and the quantum spin of the moving particle. The generalized path equation mentioned above has been used, in its linearized form, to interpret [3] qualitatively and quantitatively the discrepancy in the COW-experiment. Also, the same equation has been used [4] to explain the time delay of massless spinning particles received from SN1987A. It is to be considered that the generalized path equation can be used to describe trajectories of spinning particle in any gravity theory (including General Relativity) if it is written in the AP-geometry.

In the context of the philosophy of geometerization of physics, it is well known that paths in an appropriate geometry are used to represent trajectories of test particles in the background field described completely using this geometry. For example, in constructing his theory of general relativity (GR), Einstein has used the paths of Riemannian geometry, the geodesic and the null-geodesic, to represent the trajectories of scalar test particles and of photons, respectively. Most of the achievements of GR come from the use of these paths in studying the motion of such particles. The path equations, in Riemannian geometry, are usually derived by imposing an action principle on a Lagrangian function of form (cf. [5]),

\[ L \overset{\text{def}}{=} g_{\mu\nu}U^\mu U^\nu \]  

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where $g_{\mu\nu}$ is the metric tensor and $U^\alpha$ is a vector tangent to the path satisfying the conditions $U^\alpha U_\alpha = 1$ for geodesic and $U^\alpha U_\alpha = 0$ for null-geodesic. The resulting path equation can be written in the form (cf. [6])

$$\frac{dU^\mu}{dS} + \left\{ \mu_{\alpha\beta} \right\} U^\alpha U^\beta = 0$$

where $\left\{ \mu_{\alpha\beta} \right\}$ is the Christoffel symbol of the second kind. This is a well known method for deriving the path equations in Riemannian geometry.

In the literature, there is a less known method for deriving path equations in Riemannian Geometry. This method has been suggested by Bazanski [7], in which an action principle is imposed on a Lagrangian of the form,

$$L_B \overset{\text{def}}{=} g_{\mu\nu} U^\mu D\Psi^\nu$$

where $\Psi^\nu$ is the deviation vector and

$$\frac{D\Psi^\nu}{DS} \overset{\text{def}}{=} \Psi^\nu;^\alpha U^\alpha.$$  

The semicolon (;) denotes covariant differentiation using Christoffel symbol. Bazanski was able\textsuperscript{3} to derive the path equation (1.2) by varying the Lagrangian (1.3) w.r.t. the deviation vector $\Psi^\nu$. It is clear that the two Lagrangian (1.1) and (1.3) give identical results, concerning paths in Riemannian geometry.

As stated above, using this approach, Wanas et al. [1] have obtained a set of three path equations, which can be written in the compact form,

$$\frac{dU^\mu}{d\tau} + \left\{ \mu_{\alpha\beta} \right\} U^\alpha U^\beta = -a \Lambda_{(\alpha\beta)} U^\alpha U^\beta,$$

where ($\tau$) is a parameter varying along the path and $\Lambda_{(\alpha\beta)}^{\mu}$ is the torsion tensor of the AP-space. The parameter ($a$) takes the values $0, \frac{1}{2}, 1$ to reproduce the set of path equations mentioned above. Wanas [2] has generalized the set (1.5) and suggested that the torsion term in the generalized path equation may give rise an interaction between the torsion of space-time and the quantum spin of moving test particle. The jumping torsion coefficient, in the path equations of the AP-geometry, may give a strong evidence that this geometry is naturally quantized (i.e. without performing any known quantization schemes). Wanas and Kahil [8] have shown that the quantum properties discovered in AP-geometry, can be found in other non-symmetric geometries (i.e. geometries with non-vanishing torsion). It is to be considered that the path equations obtained, in Riemannian geometry or in AP-geometry, are used to represent trajectories of neutral test particles in a pure gravitational field.

The appearance of the torsion term, with jumping coefficients, is a direct consequence of applying the Bazanski approach in AP-geometry. The appearance of the jumping

\textsuperscript{3}He was also able to derive the equation of geodesic deviation, using the same Lagrangian (1.3), by performing variation w.r.t. the tangent $U^\mu$. 


coefficients has been tempting to attribute quantum properties (in the sense of Planck’s quantization) to the suggested spin-torsion interaction. It is the aim of the present work to explore the consequences of using the Bazanski approach to derive path equations in the presence of both gravity and electromagnetism. In this framework, we are going to suggest a Lagrangian containing the basic vector \(4\) of the AP-geometry. This may throw some light on other physical interactions, if any. The paper is organized as follows. In section 2 we give a brief summary of the AP-relations needed in the present work. In section 3 we derive the new class of path equations. The physical meaning of the geometric terms, appeared in the new class, is given section 4. The work is discussed and concluded in section 5.

2 Summary of Some Relations in AP-Geometry

AP-space is an \(n\)-dimensional manifold, each point of which is labelled by a set of \(n\)-independent variables \(x^\nu, (\nu = 1, 2, 3, ..., n)\). The structure of this space is defined completely by a set of \(n\)-contravariant vector fields \(\lambda_i^\mu\) where \(i(= 1, 2, 3, ..., n)\) denotes the vector number, and \(\mu(= 1, 2, 3, ..., n)\) denotes the coordinate component. The normalized cofactor \(\lambda_{\mu}^i\) of the vectors \(\lambda_i^\mu\), in the determinant \(\mid \mid \lambda_i^\mu \mid \mid\), is defined such that (cf.[10])

\[
\lambda_i^\mu \lambda_j^\nu = \delta_{ij}^\nu, \quad (2.1)
\]

\[
\lambda_i^\mu \lambda_i^\nu = \delta_{\mu\nu}. \quad (2.2)
\]

Using these vectors, the following second order symmetric tensors are defined:

\[
g^{\mu\nu} \overset{\text{def}}{=} \lambda_i^\mu \lambda_i^\nu, \quad (2.3)
\]

\[
g_{\mu\nu} \overset{\text{def}}{=} \lambda_i^\mu \lambda_i^\nu. \quad (2.4)
\]

Consequently,

\[
g^{\mu\alpha} g_{\nu\alpha} = \delta_{\nu}^\mu. \quad (2.5)
\]

The tensor \(g_{\mu\nu}\) can be used to play the role of the metric tensor, of Riemannian space, associated with AP-space, when needed. Consequently, using (2.3) and the derivatives (2.4), one can define Christoffel symbols and covariant derivatives using this symbol, in the usual manner. The following third order tensor, the contortion tensor, can be defined as,

\[
\gamma_{\mu\nu}^\alpha \overset{\text{def}}{=} \lambda_i^\alpha \lambda_i^\mu \lambda_i^\nu, \quad (2.6)
\]

which is non-symmetric in its last two indices \(\mu, \nu\). It can be shown that \(\gamma_{\mu\nu}^\alpha\) is skew-symmetric in its first two indices.

\footnote{This vector has been shown to represent the electromagnetic generalized potential in the context of the generalized field theory [9].}
It is well known that the addition of any third order tensor to an affine connection gives another connection, thus the object defined by,
\[
\Gamma_{\alpha,\mu\nu}^{\alpha} \equiv \left\{ \frac{\alpha}{\mu\nu} \right\} + \gamma_{\alpha,\mu\nu}^{\alpha},
\] (2.7)
is a non-symmetric connection defined in AP-geometry. So, in addition to the covariant derivative, defined using Christoffel symbol, one can define three more tensor derivatives using (2.7). For example, if \( A^\mu \) is an arbitrary contravariant vector, we can define the following tensor derivatives for this vector,
\[
A_{\cdot |\nu}^{\mu} \equiv A^\mu + A^\alpha \Gamma_{\cdot \alpha\nu}^{\mu}
\] (2.8)
\[
\tilde{A}_{\cdot |\nu}^{\mu} \equiv A^\mu + A^\alpha \tilde{\Gamma}_{\cdot \alpha\nu}^{\mu}
\] (2.9)
\[
A_{\cdot \mu |\nu} \equiv A_{\mu |\nu} + A^\alpha \Gamma_{\cdot \alpha\nu}^\mu
\] (2.10)
where the comma (,) denotes ordinary partial derivative, the stroke and (+) sign denotes tensor derivative using affine connection (2.7), the stroke and the (-) sign denotes the tensor derivative using the dual connection,
\[
\tilde{\Gamma}_{\cdot \mu\nu}^{\alpha} \equiv \Gamma_{\cdot \mu\nu}^{\alpha}
\] (2.11)
while the stroke without signs characterizes tensor derivatives using the symmetric connection,
\[
\Gamma_{\cdot (\mu\nu)}^{\alpha} \equiv \frac{1}{2}(\Gamma_{\cdot \mu\nu}^{\alpha} + \Gamma_{\cdot \nu\mu}^{\alpha}),
\] (2.12)
or using (2.7), we can write
\[
\Gamma_{\cdot (\mu\nu)}^{\alpha} \equiv \left\{ \frac{\alpha}{\mu\nu} \right\} + \frac{1}{2}\Delta_{\cdot \mu\nu}^{\alpha}
\] (2.13)
where
\[
\Delta_{\cdot \mu\nu}^{\alpha} \equiv \gamma_{\cdot \mu\nu}^{\alpha} + \gamma_{\cdot \nu\mu}^{\alpha}.
\] (2.14)
As a consequence of using the connection (2.7), it is easy to show that,
\[
\lambda_{\cdot \mu |\nu}^{\alpha} = 0
\] (2.15)
which is usually known, in the literature, as the AP-condition. The solution of equation (2.15) will give an alternative definition for the non-symmetric connection,
\[
\Gamma_{\cdot \mu\nu}^{\alpha} \equiv \lambda_{\cdot \mu\nu}^{\alpha}
\] (2.16)
Since \( \Gamma_{\cdot \mu\nu}^{\alpha} \) is non-symmetric, then one can define the torsion tensor of AP-geometry as,
\[
\Lambda_{\cdot \mu\nu}^{\alpha} \equiv \Gamma_{\cdot \mu\nu}^{\alpha} - \Gamma_{\cdot \nu\mu}^{\alpha}
\] (2.17a)
or using (2.7),

\[
\Lambda_{\mu \nu} \overset{\text{def}}{=} \gamma_{\mu \nu}^\alpha - \gamma_{\nu \mu}^\alpha ,
\]

(2.17b)

which is a skew symmetric tensor in the last two indices. As a direct result of using (2.15) and definition (2.4), one can obtain

\[
g_{\mu \nu} = 0
\]

(2.18)

which can be written in the form

\[
g_{\mu \nu, \sigma} = g_{\mu \alpha} \Gamma_{\alpha \nu}^\sigma + g_{\nu \alpha} \Gamma_{\alpha \mu}^\sigma .
\]

(2.19)

Also, recall that Christoffel symbol is defined as a result of a metricity condition,

\[
g_{\mu \nu; \sigma} = 0 ,
\]

(2.20)

which gives,

\[
g_{\mu \nu, \sigma} = g_{\mu \alpha} \left\{ \alpha \right\}_{\nu \sigma} + g_{\nu \alpha} \left\{ \alpha \right\}_{\mu \sigma} .
\]

(2.21)

In view of (2.18) and (2.20) it is clear that the operation of raising or lowering tensor indices commutes with covariant differentiation using Christoffel symbol and with tensor differentiation using (2.7). Contracting (2.17b) one can obtain the following covariant vector

\[
C_{\mu} \overset{\text{def}}{=} \Lambda_{\mu \alpha} = \gamma_{\mu \alpha}^\alpha
\]

(2.22)

which is called the basic vector (cf. [11])

3 The New Class of Path Equation

In the present work we are going to construct a new class of path equations, motivated by the following points:

1. In case of Riemannian geometry if we add a term containing a vector field \(A^\mu\) to the Lagrangian (1.1) we get,

\[
L_1 \overset{\text{def}}{=} g_{\mu \nu}(\dot{U}^\mu + \beta A^\mu)\dot{U}^\nu
\]

(3.1)

where \(\beta\) is a converting parameter. The application of an action principle to this Lagrangian will give the path equation (cf. [12])

\[
\frac{d\dot{U}^\mu}{d\hat{s}} + \left\{ \mu \right\}_{\alpha \beta} \dot{U}^\alpha \dot{U}^\beta = -\frac{1}{2} \beta f_{\nu \rho} \dot{U}^\nu ,
\]

(3.2)

where \(\hat{s}\) is the evolution parameter of the path, \(\dot{U}^\mu\) is the tangent of the resulting path and

\[
f_{\mu \nu} \overset{\text{def}}{=} (A_{\mu \nu} - A_{\nu \mu}) .
\]

(3.3)
This is what is done to get the equation of motion of a charged particle (mass $m$, charge $e$) in a combined gravitational and electromagnetic field, in the context of Einstein-Maxwell’s theory. In this case, $A_\mu$ represents the electromagnetic potential, $\beta = \frac{e}{mc}$, $f_{\mu\nu}$ is the electromagnetic field strength tensor and $c$ is the speed of light. The term on R.H.S. of (3.2) represents Lorentz force. In the context of the philosophy of geometerization, this approach is subject to the objection that the vector field $A^\mu$ is not part of the geometric structure.

In the context of the Generalized Field Theory (GFT), constructed in the AP-geometry by Mikhail and Wanas [9], the vector $C_\mu$ defined by (2.22), represents the electromagnetic generalized potential (using certain system of units). The skew-symmetric part of the field equations of this theory can be written as,

$$F_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu}, \quad (3.4)$$

where $F_{\mu\nu}$ is a second order skew-symmetric tensor, defined in the AP-geometry [9], playing the role of the electromagnetic field strength tensor. Now $C_\mu$ has the physical meaning mentioned above and is a part of the geometric structure. The addition of this vector to the Lagrangian (1.1), in the sense given by (3.1), will give rise to a path equation, in the AP-geometry, similar to (3.2). This supports the physical meaning attributed to $C_\mu$ and $F_{\mu\nu}$ in the framework of GFT. But this is not our main goal. This goal will be clarified in the next point.

2. Recently, some quantum (jumping) properties have been discovered in geometries with non-vanishing torsion [1], [2] and [13]. It is shown that these properties are closely related to affine connections defined in the geometry. The use of Lagrangian functions of the form given by (1.1) or (3.1), in constructing path equations, will not give rise to such properties since it is independent of the affine connection (cf. (1.2) and (3.2)). The Lagrangian which may give rise to such properties should be of the Bazanski’s type (1.3). This is obvious, as it contains, in its structure, the derivative of the deviation vector which depends on the affine connection used. The use of such Lagrangian may give rise to quantized objects and may give rise to new interactions (eg. (1.5)). Our main goal is to modify (3.1), in AP-geometry, to include the affine connection. This may throw some light on the quantized quantities related to the geometric objects $C_\mu$ and $F_{\mu\nu}$ and will help one to discover new interactions, if any.

In a similar way, used to modify the Lagrangian function (1.1) to become in the form (1.3), we are going to modify (3.1) , to be written as,

$$L^* \overset{\text{def}}{=} g_{\mu\nu} (U^{*\mu} + \dot{C}^\mu) \frac{D\Psi^\nu}{Ds^*} \quad (3.5)$$

where $U^{*\mu}$ is the tangent to the path and $s^*$ is its evolution parameter. For dimensional consideration, concerning the Lagrangian (3.5), we have redefined the electromagnetic potential to be $\dot{C}^\mu \overset{\text{def}}{=} l C^\mu$, where $l$ is a dimensional parameter.

Now the derivative of the deviation vector in (3.5) can be evaluated, in the AP-geometry, using one of the connections related to (2.16). The first connection is (2.16) itself, the second is its dual (2.11) and the third is its symmetric part (2.13).

In the following subsections we are going to derive the path equations resulting from the use of these connections.
3.1 The First Path Equation (using $\Gamma_{\mu\nu}^{\alpha}$)

Using the non-symmetric connection given by (2.7) or (2.16), the Lagrangian (3.5) can be written in the form:

$$L^+ \overset{\text{def}}{=} g_{\mu\nu}(V^\mu + \hat{C}^\mu) \frac{D\xi^\nu}{Ds^+}$$  \hspace{1cm} (3.6)

where $V^\mu$ is the tangent of the resulting path, $\xi^\nu$ is the vector giving the deviation from this path and $s^+$ is the evolution parameter along the path. The derivative of the deviation vector is given by,

$$\frac{D\xi^\nu}{Ds^+} \overset{\text{def}}{=} \dot{\xi}^\nu + \xi^\alpha \Gamma^\nu_{\alpha\beta} V^\beta$$  \hspace{1cm} (3.7)

where,

$$\dot{\xi}^\nu \overset{\text{def}}{=} \frac{d\xi^\nu}{ds^+}.$$  \hspace{1cm} (3.8)

Now, we have

$$\frac{\partial L^+}{\partial \dot{\xi}^\sigma} = g_{\mu\sigma}(V^\mu + \hat{C}^\mu).$$

Then, making use of (2.19) we can write

$$\frac{d}{ds^+} \frac{\partial L^+}{\partial \dot{\xi}^\sigma} = g_{\mu\sigma} \frac{d}{ds^+}(V^\mu + \hat{C}^\mu) + [g_{\lambda\sigma} \Gamma^\lambda_{\mu\rho} + g_{\mu\lambda} \Gamma^\lambda_{\sigma\rho}](V^\mu + \hat{C}^\mu)V^\rho.$$  \hspace{1cm} (3.9)

Also from (3.6) we can write,

$$\frac{\partial L^+}{\partial \xi^\sigma} = g_{\mu\sigma}(V^\mu + \hat{C}^\mu) \Gamma^\nu_{\sigma\beta} V^\beta.$$  \hspace{1cm} (3.10)

Substituting (3.9) and (3.10) into the Euler-Lagrange equation,

$$\frac{d}{ds^+} \frac{\partial L^+}{\partial \dot{\xi}^\sigma} - \frac{\partial L^+}{\partial \xi^\sigma} = 0,$$

we get, after necessary reductions, the path equation

$$\frac{dV^\mu}{ds^+} + \left\{ \begin{array}{c} \mu \\ (\alpha\beta) \end{array} \right\} V^\alpha V^\beta = -\Lambda_{(\alpha\beta).} \overset{\text{def}}{=} V^\alpha V^\beta - \hat{F}_\mu V^\nu - g^{\mu\alpha} \hat{C}^\nu_{\alpha\beta} V^\nu - \Lambda_{\alpha\beta.} \overset{\text{def}}{=} \hat{C}^\alpha V^\beta, \hspace{1cm} (3.11)$$

where

$$\hat{F}_\mu \overset{\text{def}}{=} \hat{C}_{\mu,\nu} - \hat{C}_{\mu,\nu}.$$  \hspace{1cm} (3.9)

3.2 The Second Path Equation (using $\Gamma_{\mu\nu}^{\alpha}$)

We use in this subsection the symmetric connection given by (2.13) to evaluate the Lagrangian (3.5), which can be written as,

$$L^o \overset{\text{def}}{=} g_{\mu\nu}(W^\mu + \hat{C}^\mu) \frac{D\xi^\nu}{Ds^o}$$  \hspace{1cm} (3.12)
where $W^{\mu}$ is the tangent to the resulting path, $\zeta^{\nu}$ is the deviation vector and $s^{o}$ is the parameter varying along the path. Similar to subsection 3.1 we can write the definitions

$$
\frac{D\zeta^{\nu}}{Ds^{o}} \overset{\text{def}}{=} \zeta^{\nu} + \zeta^\alpha \Gamma_{\alpha \beta}^{\nu} W^\beta \tag{3.13}
$$

and

$$
\dot{\zeta}^{\nu} \overset{\text{def}}{=} \frac{d\zeta^{\nu}}{ds^{o}}.
$$

Evaluating the variational derivatives of the Lagrangian (3.12) and substituting in the Euler-Lagrange equation, as done in section 3.1, we get after some, relatively long but straightforward, calculations the second path equation

$$
\frac{dW^{\mu}}{ds^{o}} + \left\{ \frac{\mu}{\alpha \beta} \right\} W^{\alpha} W^{\beta} = -\frac{1}{2} \Lambda_{(\alpha \beta)}^{\mu} W^{\alpha} W^{\beta} - \hat{F}_{\nu}^{\mu} W^{\nu} - g^{\mu \delta} \hat{C}_{\nu}^{\alpha \beta} W^{\nu} - \frac{1}{2} \Lambda_{\alpha \beta}^{\mu} \hat{C}^{\alpha} W^{\beta}. \tag{3.14}
$$

### 3.3 The Third Path Equation (using $\tilde{\Gamma}_{\alpha \mu \nu}$)

For this path equation, the Lagrangian (3.5) can be written in the form,

$$
L^{-} \overset{\text{def}}{=} g_{\mu \nu} (J^{\mu} + \hat{C}^{\mu}) \frac{D\eta^{\nu}}{Ds^{-}} \tag{3.15}
$$

where $J^{\mu}$ is the tangent to the path, $\eta^{\nu}$ is the deviation vector and $s^{-}$ is the evolution parameter characterizing the path. Similar to the definitions given in the previous subsections we write,

$$
\frac{D\eta^{\nu}}{Ds^{-}} \overset{\text{def}}{=} \dot{\eta}^{\nu} + \eta^{\alpha} \tilde{\Gamma}_{\alpha \beta}^{\nu} J^{\beta}, \tag{3.16}
$$

where,

$$
\dot{\eta}^{\nu} \overset{\text{def}}{=} \frac{d\eta}{ds^{-}}. \tag{3.17}
$$

Performing necessary variational calculations, as done in the previous subsections and substituting in the Euler-Lagrange equation we get, after some rearrangements, the third equation

$$
\frac{dJ^{\mu}}{ds^{-}} + \left\{ \frac{\mu}{\alpha \beta} \right\} J^{\alpha} J^{\beta} = -\hat{F}_{\nu}^{\mu} J^{\nu} - g^{\mu \delta} \hat{C}_{\nu}^{\alpha \beta} J^{\nu} - \frac{1}{2} \Lambda_{\alpha \beta}^{\mu} \hat{C}^{\alpha} J^{\beta}. \tag{3.18}
$$

### 4 Physical Meaning of the Geometric Terms

The set of path equations (3.11), (3.14) and (3.18) comprises a new class of path equations which can be written in the general form

$$
\frac{dZ^{\mu}}{d\tau} + a_{1} \left\{ \frac{\mu}{\alpha \beta} \right\} Z^{\alpha} Z^{\beta} = -a_{2} \Lambda_{(\alpha \beta)}^{\mu} Z^{\alpha} Z^{\beta} - a_{3} \hat{F}_{\nu}^{\mu} Z^{\nu} - a_{4} g^{\mu \delta} \hat{C}_{\nu}^{\alpha \beta} Z^{\nu} - a_{5} \Lambda_{\alpha \beta}^{\mu} \hat{C}^{\alpha} Z^{\beta}, \tag{4.1}
$$

where $Z^{\mu}$ is the tangent of the resulting path, $\tau$ is the evolution parameter of the path, $a_{1}, a_{2}, a_{3}, a_{4},$ and $a_{5}$ are numerical parameters whose values are listed in the following table.
Table 1: Values of the parameters of (4.1) corresponding to connection used

| Connection used | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | Equation |
|-----------------|-------|-------|-------|-------|-------|----------|
| $\Gamma^\alpha_{\mu\nu}$ | 1     | 1     | 1     | 1     | 1     | (3.11)   |
| $\Gamma^\alpha_{(\mu\nu)}$ | 1     | $\frac{1}{2}$ | 1     | 1     | $\frac{1}{2}$ | (3.14)   |
| $\tilde{\Gamma}^\alpha_{\mu\nu}$ | 1     | 0     | 1     | 1     | 0     | (3.18)   |

In the context of the scheme of geometerization, if the general equation (4.1) is used to study trajectories of charged test particles, we can attribute the following physical meanings to some of its terms:

1- The term whose coefficient is $(a_1)$, $\left\{^\mu_{\alpha\nu}\right\}Z^\alpha Z^\nu$, represents, as usual, the effects of gravitation on the motion of the particle.

2- The term whose coefficient is $(a_2)$, $\Lambda^\mu_{(\alpha\nu)}Z^\alpha Z^\nu$, is suggested [2] to represent a type of interaction between the torsion of space-time and quantum spin of the moving particle. This interaction will affect the motion of the particle. This term is quantized as clear from the values of $(a_2)$ in Table 1. It is shown that there are some experimental [3] and observational [4] evidences for the existence of this interaction.

3- The term whose coefficient is $(a_3)$, $\hat{F}^\mu_{\alpha\beta}$, represents the effect of the electromagnetic field on the motion of a charged particle, the Lorentz force, as obvious from the comparison with the R.H.S. of (3.2). The coefficient of this term, as clear from Table 1, does not vary from equation to another. So, this effect is not quantized.

4- The term whose coefficient is $(a_5)$, $\Lambda^\mu_{\alpha\nu}\hat{C}^\alpha Z^\nu$, represents an interaction between the electromagnetic potential and the torsion of space-time. The term is already quantized, as clear from Table 1. This term represents a direct effect of the electromagnetic potential on the motion of a charged particle similar to the Aharonov-Bohm (AB) effect [14], [15], with one main difference, that is the influence of the torsion on this term. This will be discussed in the following section.

5 Discussion and Concluding Remarks

In the present work, we derived a new class of path equation, in AP-geometry, using Bazanski method. All terms appearing in this class are parts of the geometric structure used. The general equation, representing this class can be used, qualitatively, to explore different interactions affecting trajectories of charged particles in a combined gravitational and electromagnetic field.

If the terms representing the electromagnetic effects on the trajectory, $\hat{F}_{\mu\nu}$ and $\hat{C}_\alpha$, are switched off, then the new class will reduce to the old set of path equations [1] with spin-torsion term. If further, we neglect this term we get the ordinary geodesic equation.
Two of the terms, on the R.H.S. of the general equation (4.1), are naturally quantized in Planck’s sense (terms with jumping coefficients). One of these terms is the spin-torsion term with the coefficient ($a_2$) and the other is the term giving rise to AB-type effect (the term with coefficient ($a_5$), see Table 1). All other terms, including the Lorentz force term, of this equation are not quantized, in the above mentioned sense. It is clear from this equation that the appearance of the quantum properties in this equation is closely connected to the explicit appearance of the torsion in the terms concerned. Since Riemannian geometry is a torsion free geometry, such quantum properties did not appear in any field theory constructed in this geometry.

We would like to point out that we are not claiming that we are doing Quantum Mechanics or Quantum Field Theories. Actually, we are dealing with a property, in AP-geometry that some terms of the equations have jumping coefficients, which reflects some quantum features in the sense of Planck’s quantization. It is well known, in the literature, that the AB-effect is a quantum phenomenon which is impossible to be accounted for using classical electrodynamics. The scheme followed in present work is a pure geometric scheme, which is considered by many authors as a classical scheme. Now, if the term, whose coefficient is $a_5$, is interpreted to give rise to AB-effect, then one can draw the following conclusion: Either the AB-effect is a classical phenomenon and can be accounted for using a classical scheme, or the type of geometry used, the AP-geometry, admits some quantum properties.

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