COMPATIBILITY WITH CAP-PRODUCTS IN TSYGAN'S FORMALITY AND HOMOLOGICAL DUFLO ISOMORPHISM

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Abstract. In this paper we prove, with details and in full generality, that the isomorphism induced on tangent homology by the Shoikhet-Tsygan formality $L_\infty$-quasi-isomorphism for Hochschild chains is compatible with cap-products. This is a homological analog of the compatibility with cup-products of the isomorphism induced on tangent cohomology by Kontsevich formality $L_\infty$-quasi-isomorphism for Hochschild cochains.

As in the cohomological situation our proof relies on a homotopy argument involving a variant of Kontsevich eye. In particular we clarify the rôle played by the I-cube introduced in [4].

Since we treat here the case of a most possibly general Maurer-Cartan element, not forced to be a bidifferential operator, then we take this opportunity to recall the natural algebraic structures on the pair of Hochschild cochain and chain complexes of an $A_\infty$-algebra. In particular we prove that they naturally inherit the structure of an $A_\infty$-algebra with an $A_\infty$-(bi)module.

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Introduction

Given a (possibly curved) $A_\infty$-algebra $(A, \gamma_0, \gamma_1, \gamma_2, \ldots)$, it is known that its Hochschild cochain complex $C^\bullet(A, A)$ is naturally a (non curved) $A_\infty$-algebra, with structure maps $d_{\gamma, k}$ being defined thanks to the famous brace operations [17] introduced by Gerstenhaber and Voronov:

(1) $d_{\gamma, 1}(P) := \sum_i (\gamma_i \{ P \} \mp P \{ \gamma_i \})$ and $d_{\gamma, k}(P_1, \ldots, P_k) := \sum_i \gamma_i \{ P_1, \ldots, P_k \}$ ($k \geq 2$).

This statement can be reformulated and proved using $B_\infty$-algebras [19] and twisting procedure for them with respect to Maurer-Cartan elements, following Getzler and Jones. Namely, given a $B_\infty$-algebra $(B, d, m)$ and a Maurer-Cartan element (shortly MCE) $\gamma$, i.e. a degree 1 element in $B$ satisfying the Maurer-Cartan equation

$d(\gamma) + m(\gamma, \gamma) = d_1(\gamma) + m_{1, 1}(\gamma, \gamma) = 0$,

then there is a new $B_\infty$-algebra $(B, d_\gamma, m)$ with

$d_\gamma := d + m(\gamma \otimes \bullet) - m(\bullet \otimes \gamma)$.

The result of this paper were mainly obtained when D.C. was working in ETH (on leave of absence from Université Lyon 1). His research was fully supported by the European Union thanks to a Marie Curie Intra-European Fellowship (contract number MEIF-CT-2007-042212).
For any graded vector space $A$ the brace operations on $B = \text{End}(A)$, with
$$\text{End}(A) := \bigoplus_{n \geq 0} \text{Hom}(A^\otimes n, A)[1 - n],$$
define a $B_\infty$-algebra structure on $B$ such that a MCE $\gamma$ tantamounts to a curved $A_\infty$-algebra structure on $A$, and the structure maps $d_{\gamma, k}$ of $d_\gamma$ are precisely given by (1). From this formalism it is easy to see that two homotopy equivalent $A_\infty$-algebra structures on $A$ induce homotopy equivalent $A_\infty$-algebra structures on $B = \text{End}(A)$. All this is recalled in the first Section of the paper.

The first aim of the present paper is to develop a similar machinery for Hochschild chains of a curved $A_\infty$-algebra $(A, \gamma_0, \gamma_1, \gamma_2, \ldots)$. Unfortunately, things do not appear to go as easily as in the case of cochains. We can nevertheless prove that there is an $A_\infty$-bimodule structure on the Hochschild chain complex $C_{\gamma}(A, A)$ (with reversed grading), over the $A_\infty$-algebra $C^*(A, A)$. To do so, we prove that there are two distinct left $B_\infty$-actions of $\text{End}(A)$ on
$$A^\otimes: = \bigoplus_{n \geq 0} A \otimes A^\otimes n.$$Then, to any (curved) $A_\infty$-algebra structure on $A$, we define the $A_\infty$-bimodule structure on $A^\otimes A = C_{\gamma}(A, A)$, as usual, as the adjoint action of the MCE $\gamma$. Being easier to say than to do, the above claim requires some work, and to introduce new notions such as $B_\infty$-(bi)modules. This is the subject of Section 2, which can be viewed as the explanation of the sketch of a construction of Tamarkin–Tsygan [25] regarding dualities between Hochschild cochain and chain complex.

The above constructions extend to the following setting: a smooth real manifold $X$ and a commutative and unital differential graded algebra (shortly, DGA) $(m, d_m)$ splitting as $m = n \oplus R$, with $n$ a (pro)nilpotent ideal and $R$ a unital subalgebra concentrated in degree 0. Then the complex of $m$-valued polydifferential operators $D^m_{\text{poly}}(X)$, is naturally a $B_\infty$-algebra in which MCEs are deformations of the DGA $C^\infty(X, m)$ as an $A_\infty$-algebra over $(m, d_m)$.

Similarly, the complex of $m$-valued Hochschild chains $C^\text{poly,m}(X)$ with reversed grading (see e.g. [12] for a precise definition) naturally carries two distinct left $B_\infty$-actions of $D^m_{\text{poly}}(X)$, for which any MCE in $D^m_{\text{poly}}(X)$ induces an $A_\infty$-bimodule structure on $C^\text{poly,m}(X)$.

Then we recall from [20] that there exists an $L_\infty$-quasi-isomorphism $\mathcal{U}$ from the differential graded Lie algebra (shortly, DGLA) $T^m_{\text{poly}}(X)$ of $m$-valued polyvector fields to the DGLA $D^m_{\text{poly}}(X)$. Therefore, given a MCE $\gamma$ in $T^m_{\text{poly}}(X)$ one obtains a chain map
$$\mathcal{U}_{\gamma, 1} : \left(T^m_{\text{poly}}(X), d_m + [\gamma, -]\right) \longrightarrow \left(D^m_{\text{poly}}(X), d_m + [\tilde{\gamma}, -]\right),$$where (below $\mu$ denotes the standard commutative product on $C^\infty(X, m)$)
$$\tilde{\gamma} := \mu + \sum_{n \geq 1} \frac{1}{n!} \mathcal{U}_n(\gamma, \ldots, \gamma).$$Moreover, Kontsevich claimed and sketchily proved in [20, Section 8] (see [3, 21] for detailed proofs in particular cases) that $\mathcal{U}_{\gamma, 1}$ is compatible with cup-products in the sense that it induces an algebra isomorphism
$$H^*(T^m_{\text{poly}}(X), d_m + [\gamma, -]) \longrightarrow H^*(D^m_{\text{poly}}(X), d_m + [\tilde{\gamma}, -]).$$Analogously, we recall from [24] that there exists an $L_\infty$-quasi-isomorphism $\mathcal{S}$ from the differential graded Lie module (shortly, DGLM) $A^m(X)$ of $m$-valued differential forms (with reversed grading) to the DGLM $C^\text{poly,m}(X)$.

Therefore, given a MCE $\gamma$ as above one obtains a quasi-isomorphism
$$\mathcal{S}_{\gamma, 0} : \left(C^\text{poly,m}(X), d_m + L_{\gamma}\right) \longrightarrow \left(A^m(X), d_m + L_\gamma\right).$$The second aim of the paper is to prove the following Theorem, which is a very natural generalization of [4]:

**Theorem A.** The quasi-isomorphism $\mathcal{S}_{\gamma, 0}$ is compatible with cap-products in the sense that it induces an isomorphism of $H^*(T^m_{\text{poly}}(X), d_m + [\gamma, -])$-modules
$$H^*(C^\text{poly,m}(X), d_m + L_{\gamma}) \longrightarrow H^*(A^m(X), d_m + L_\gamma).$$

---

1In the following, we assume that $m$ is bounded below as a graded vector space: $m^k = \{0\}$ for $k < 0$. Moreover, tensor products with $m$ have to be understood as completed tensor products with respect to the $n$-adic topology.
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We thank Alberto Cattaneo, who raised the question of generalizing our results in a context where all the boundary faces of the I-cube could contribute, for his interest in our work. We also thank Giovanni Felder for many interesting discussions on this project. We finally thank Vasily Dolgushev for a particularly enlightening remark about the globalisation of the compatibility between cup products, and Dmitry Tamarkin for his kind help on a tricky point in Section 2.

Remark. A. Cattaneo pointed our attention on this possible generalization of our previous work, in which we made use of the geometry of the I-cube. In [4] and previous versions of the present paper we made use of the geometry of the I-cube to prove that a homotopy exists for the cap-products (following Kontsevich’s original idea [20] for cup-products), but in the simpler cases (namely, when $\gamma$ is at most a bivector) we have considered, three boundary faces did not contribute. They actually do in the general context we consider in the present paper; nevertheless, we give here a cleaner proof, in which the I-cube is, at the end, not strictly needed.

The proof of Theorem A requires several steps. As it is now usual in deformation quantization we first prove the desired result in the local situation $X = \mathbb{R}^d$. For this purpose we recall, in Section 3, the construction of Kontsevich’s [20] and Shoikhet’s [24] local formality maps. The main ingredients of both constructions are appropriate compactified configuration spaces and integrals of angle forms over them. We detail in particular two remarkable compactified configuration spaces which are of some use for the compatibility between cup and cap products: Kontsevich’s eye and the I-cube.

We then review quickly in Section 4 the proof of the compatibility between cup products in our very general framework for $X = \mathbb{R}^d$. The main argument, of homotopical nature, was sketched by Kontsevich in [20], later clarified by Manchon and Torossian in [21] in the framework of deformation quantization, and finally adapted to the case of Q-manifolds in [3]. The globalisation of the compatibility between cup products was first seriously considered in [5], and is addressed in Section 8.

The proof of the compatibility between cap products for $X = \mathbb{R}^d$ occupies Section 5, and is based on a homotopy argument very similar to the one of Section 4. Contrarily to what we first guessed in [4], the I-cube is not strictly needed for the proof, but definitely gives insight to understand how things work. Again, the question of globalisation is pushed-forward to the final Section of the paper.

Before going through the globalisation of the previous results, we discuss three special cases of interest and an application. Namely, the cases of interest, detailed in Section 6, are the following ones. The first case is when $m = \mathbb{R}[[\hbar]]$; it corresponds to Shoikhet’s conjecture [24], which is originally motivated by deformation quantization, and is proved in [4]. The second one is when the MCE $\gamma$ is of polyvector degree at most 1; then one can prove that so is its image $U(\gamma)$, which can be interpreted in terms of a Fedosov connection and its Weyl curvature on a deformed algebra, following the terminology of [10]. Finally, the third case of interest is when the MCE $\gamma$ is precisely a vector field; we are able to compute explicitly the quasi-isomorphisms $U_{\gamma,1}$ and $S_{\gamma,0}$ by means of a rooted Todd class $j(\gamma)$, following [5] (see also [3]).

In Section 7 we present an application of the third case described in the preceding Section. Shortly speaking, we prove a (co)homological analogon of the so-called Duflo isomorphism. Here the rooted Todd class $j(\gamma)$ is precisely the Duflo element that is used to modify the Poincaré-Birkhoff-Witt isomorphism. We remind the reader that the use of the compatibility between cup products to prove the Duflo isomorphism goes back to Kontsevich’s seminal paper [20], where its cohomological extension was claimed (and of which one can find a complete proof in [22]). In [4] we proved a version of the Duflo isomorphism on coinvariants, and the result presented in this Section ends the story by extending it to homology.

The final Section of the paper is devoted to the proof of the main Theorem A. It is basically obtained by means of now standard globalisation methods. These methods were introduced by Fedosov [15] for the deformation quantization of symplectic manifolds, generalized by Cattaneo-Felder-Tomassini [10] to the case of Poisson manifolds, and finally adapted (and popularized) by Dolgushev [11, 12] to the context of the formality (both for cochains and chains). Our presentation follows closely [12] and is quite sketchy, focusing essentially on the main specific points for the compatibility. We end the Section in explaining how the approach of Cattaneo-Felder-Tomassini is contained in this description.

Remark. We finally mention that our main result can be obtained as a consequence of a very recent preprint [13] of Dolgushev-Tamarkin-Tsygan, where they prove the formality of the homotopy calculus algebra of Hochschild (co)chains. Their proof is more conceptual and does not require to check compatibility with cup and cap products, as both are part of the generating operations of the coloured operad calc of calculus algebras. Nevertheless, our approach seems to have the advantage of being, to some extent, computable (see e.g. Subsection 6.3).

Acknowledgements. We thank Alberto Cattaneo, who raised the question of generalizing our results in a context where all the boundary faces of the I-cube could contribute, for his interest in our work. We also thank Giovanni Felder for many interesting discussions on this project. We finally thank Vasily Dolgushev for a particularly enlightening remark about the globalisation of the compatibility between cup products, and Dmitry Tamarkin for his kind help on a tricky point in Section 2.
Notation

Unless otherwise specified, we work over a field \( k \) of characteristic zero: algebras, modules etc ... are over \( k \).

A graded vector space means a \( \mathbb{Z} \)-graded \( k \)-vector space. The category of graded vector spaces is symmetric monoidal with non-trivial commutativity isomorphism \( \sigma \) being given by the Koszul sign rule:

\[
\sigma_{VW} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|}w \otimes v.
\]

Dealing with graded vector spaces and morphisms between them, this rule is always tacitly assumed.

If \((\mathcal{O}, d\mathcal{O}, m\mathcal{O})\) is a (possibly colored) DG operad, then a \( \mathcal{O} \)-algebra up to homotopy is the data of a DG vector space \((V, d)\) together with a DG linear map \( \rho : (\mathcal{O}, d\mathcal{O}) \to (\operatorname{End}(V), [d_V, -]) \) such that \( \rho \circ m\mathcal{O} \) is homotopic to \( m_{\operatorname{End}(V)} \circ (\rho \otimes \rho) \). In particular, any \( \mathcal{O}_\infty \)-algebra is a \( \mathcal{O} \)-algebra up to homotopy, while the converse is false.

Exemplarily, an associative algebra up to homotopy is a DG vector space \((A, d_A)\) together with a product \( m_A : A \otimes A \to A \) such that \([d_A, m_A] = 0\) and \((m_A \otimes \operatorname{id}) \circ m_A \) is homotopic to \((\operatorname{id} \otimes m_A) \circ m_A\).

The homotopy is not considered as a part of the structure. The obvious notion of a morphism of \( \mathcal{O} \)-algebra up to homotopy can then be guessed by the reader. Exemplarily, a morphism of associative algebras up to homotopy is a graded linear map \( f : A \to B \) such that \( f \circ d_A = d_B \circ f \), and \( f \circ m_A \) is homotopic to \( m_B \circ (f \otimes f) \).

1. \( B_\infty \)-structure on Hochschild cochains

In this Section, we discuss in its generality the \( B_\infty \)-algebra structure on \( \operatorname{End}(E) \), for a graded vector space \( E \), and the twisting procedure that allows one to deduce from this the \( B_\infty \)-structure on the Hochschild cochain complex of an \((A_\infty)\)-algebra \( A \). It has been first exploited by Getzler–Jones [19] and Gerstenhaber–Voronov [17], to which we refer for more details and for complete proofs; we will nonetheless write down explicitly certain formulæ and some arguments, which will be helpful for upcoming computations.

1.1. \( B_\infty \)-algebras and twistings. We consider a graded vector space \( V \): for a homogeneous element \( v \) in \( V \), we denote by \( |v| \) its degree. The cofree, coassociative coalgebra with counit cogenerated by \( V \) is the tensor coalgebra \( T(V) = \bigoplus_{n \geq 0} V^\otimes n \), \( V^\otimes 0 = k \), with the natural coproduct, resp. counit,

\[
\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n), \quad \text{resp.}
\]

\[
\varepsilon(v_1 \otimes \cdots \otimes v_n) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1, \end{cases}
\]

where \( v_0 = v_{n+1} = 1 \), 1 being the unit of the ground field \( k \).

**Definition 1.1.** \( V \) is called a \( B_\infty \)-algebra, if there exist linear maps

\[
d : T(V) \to T(V) \quad \text{and} \quad m : T(V) \otimes T(V) \to T(V),
\]

such that the 6-tuple \((T(V), m, \Delta, \eta, \varepsilon, d)\), where \( \eta \) is the natural unit, is a DG bialgebra.

In other words, \( m \) is an associative product of degree 0 on \( T(V) \) and a morphism of coalgebras, i.e. the following identities hold true

\[
m \circ (m \otimes 1) = m \circ (1 \otimes m), \quad \Delta \circ m = (m \otimes m) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes \Delta),
\]

where \( \tau \) denotes the standard braiding in the category of graded vector spaces.

Further, \( d \) is a linear operator on \( T(V) \) of degree 1, which squares to 0, and which is simultaneously a derivation w.r.t. \( m \) and a coderivation w.r.t. \( \Delta \): more explicitly,

\[
d^2 = 0, \quad d \circ m = m \circ (d \otimes 1 + 1 \otimes d), \quad \Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta,
\]

tacitly assuming Koszul’s sign rule.

The fact that \( m \) is a morphism of coalgebras, resp. \( d \) is a coderivation of degree 1, implies that \( m, \) resp. \( d \), is uniquely specified by its components

\[
m_{p,q} : V^\otimes p \otimes V^\otimes q \to V, \quad \text{resp.} \quad d_p : V^\otimes p \to V.
\]
For the sake of clarity of upcoming computations, we need to write down explicitly the product \( m \) and the differential \( d \) in terms of its components: namely, \( m \), resp. \( d \), is determined via the formulæ

\[
(2) \quad m(v_1 \otimes \cdots \otimes v_p \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_q) = \sum_{i=1}^{p+q} \sum_{\mu, \nu \in P} \sigma(\mu, \nu) m_{\mu, \nu}(v_1 \otimes \cdots \otimes v_p \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_q), \quad \text{resp.}
\]

\[
(3) \quad d(v_1 \otimes \cdots \otimes v_p) = \sum_{i=1}^{p} \sum_{j=0}^{p-1} (-1)^{j+1} d_i v_1 \otimes \cdots \otimes d_i (v_{j+1} \otimes \cdots \otimes v_{j+1}) \otimes \cdots \otimes v_p.
\]

Formula (2) needs some explanations, also for later computations.

For positive integers \( l, p \), we define

\[
\mathcal{P}_l(p) = \left\{ (\mu_1, \ldots, \mu_l) \in \mathbb{Z}^l : \mu_i \geq 0, |\mu| = \sum_{i=1}^l \mu_i = p \right\},
\]

the set of (generalized) partitions of \( p \) into \( l \) subsets; we observe that the entries of a generalized partition \( \mu \) are \textbf{not} ordered. Furthermore, for positive integers \( l, p \) and \( q \), the pairing \( \vee \) between \( \mathcal{P}_l(p) \) and \( \mathcal{P}_l(q) \) is defined via

\[
(\mu \vee \nu)_i = \begin{cases} 
1, & \mu_i, \nu_i \geq 1 \\
\mu_i, & \mu_i \geq 1, \nu_i = 0 \\
\nu_i, & \mu_i = 0, \nu_i \geq 1 \\
0, & \mu_i = \nu_i = 0.
\end{cases}
\]

For non-negative integers \( l, p \) and \( q \), such that \( 1 \leq l \leq p + q \), a pair \((\mu, \nu)\) in \( \mathcal{P}_l(p) \times \mathcal{P}_l(q) \), such that \(|\mu \vee \nu| = l\), determines a linear map \( m_{\mu, \nu} \) from \( V^\otimes p \otimes V^\otimes q \) to \( V^\otimes l \), via

\[
m_{\mu, \nu}(v_1 \otimes \cdots \otimes \bar{v}_1 \otimes \cdots) = \prod_{i=1}^{l} m_{\mu_i, \nu_i} \left( (v_{l+1-\sum_{j=1}^{i-1} \mu_j} \otimes \cdots v_{l+1-\sum_{j=1}^{i-1} \mu_j} \otimes (\bar{v}_{l+1-\sum_{j=1}^{i-1} \nu_j} \otimes \cdots \bar{v}_{l+1-\sum_{j=1}^{i-1} \nu_j}) \right),
\]

where the factors in the tensor product are ordered from 1 to \( l \) from the left to the right: if either \( \mu_i = 0 \) or \( \nu_i = 0 \), for some \( i = 1, \ldots, l \), then we set \( m_{\mu_i, \nu_i} = \text{id} \); if both indices are 0, we set \( m_{0,0} = 0 \). Finally, the sign \( \sigma(\mu, \nu) \) is determined by Koszul’s sign rule.

For later purposes, it is useful to write down explicitly the associativity condition for the product \( m \) in terms of its components:

\[
\sum_{l=1}^{p+q} \sum_{\mu, \nu \in P} \sigma(\mu, \nu) m_{l, \tau}(m_{\mu, \nu}(v_1 \otimes \cdots \otimes v_p \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_q) \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_r) = \sum_{l=1}^{q+r} \sum_{\nu, \pi \in P} \sigma(\nu, \pi) m_{l, \tau}(v_1 \otimes \cdots \otimes v_p \otimes (m_{\nu, \pi}(\bar{v}_1 \otimes \cdots \otimes \bar{v}_q \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_r)),
\]

Remark 1.2. Writing down explicitly the previous families of identities for a few simple cases, we find that, if \( V \) is a \( B_\infty \)-algebra, the binary operations

\[
[v_1, v_2] := m_{1,1}(v_1 \otimes v_2) - (-1)^{|v_1||v_2|} m_{1,1}(v_2 \otimes v_1) \quad \text{and} \quad v_1 \cup v_2 := d_2(v_1 \otimes v_2),
\]

together with the differential \( d_3 \), endow \( V[1] \) with the structure of a Gerstenhaber algebra up to homotopy, with homotopies expressible via \( d_3 \) (for the associativity of \( \cup \)), \( m_{1,2} \) (for the Leibniz rule between \( \cup \) and \( [\cdot, \cdot] \)), \( m_{2,1} \) (for the Jacobi identity of \( [\cdot, \cdot] \)), and \( m_{1,1} \) (for the commutativity of \( \cup \)).

Let now \( V \) be a \( B_\infty \)-algebra in the sense of Definition 1.1.

Definition 1.3. A Maurer–Cartan element \( \gamma \) for the \( B_\infty \)-algebra \( V \) is an element of \( V \) of degree 1, which obeys the Maurer–Cartan equation

\[
d \gamma + m(\gamma, \gamma) = 0.
\]

Since \( \gamma \) belongs to \( V \), it is obviously primitive in the bialgebra \( T(V) \); further, (5) simplifies to

\[
d_1 \gamma + m_{1,1}(\gamma \otimes \gamma) = 0.
\]

The MC equation (5) for the MC element \( \gamma \), together with the primitivity of \( \gamma \), implies that the map

\[
d_\gamma = d + m(\gamma \otimes \cdot) - m(\cdot \otimes \gamma),
\]
tacitly assuming Koszul’s sign rule, defines a twisted $B_\infty$-structure on $V$, i.e. $(T(V), m, \Delta, \eta, \varepsilon, d_\gamma)$ is a DG bialgebra.

1.2. $B_\infty$-algebra structure on the Hochschild cochain complex. We consider a graded vector space $E$: to it, we associate

$$V = \text{End}(E) := \bigoplus_{n \geq 0} \text{Hom}(E^\otimes n, E)[1 - n].$$

**Remark 1.4.** Regarding the grading on $V$, we use the following notation: since $E$ is graded, then any tensor power of $E$ is also naturally graded, as well as $\text{Hom}(E^\otimes n, E)$. The degree referring to this grading will be denoted by $|\cdot|$. Further, if $P$ is an element of $\text{Hom}(E^\otimes p, E)$ of degree $|P|$, then its degree in $V$ is called its (shifted) total degree and is denoted by $|P|$. We thus have

$$|P| = p - 1 + |P|. $$

A $B_\infty$-algebra structure on $V$ has been constructed explicitly by Getzler–Jones [19] and Gerstenhaber–Voronov [17]: we review here its construction and some of its main features.

The differential $d$ on $T(V)$ is the trivial one; the multiplication $m$ on $T(V)$ is defined by components $m_{p,q}$, which are non-trivial precisely when $p \leq 1$, with no restrictions on $q$: the unit axiom for $m$ forces $m_{0,q}$ to be equal to the identity map, while $m_{1,q}$ is defined via

$$m_{1,q}(P, Q_1, \ldots, Q_q)(e_1, \ldots, e_n) = P\{Q_1, \ldots, Q_q\}(e_1, \ldots, e_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_q \leq n} (-1)^{\sum_{k=1}^q |Q_k| (i_k - 1 + \sum_{j=1}^{k-1} |e_j|)} P(e_1, \ldots, Q_1(e_{i_1}, \ldots), \ldots, Q_q(e_{i_q}, \ldots), \ldots, e_n),$$

with the previous grading conventions; in (7), $n = p + \sum_{a=1}^q (q_a - 1), 1 \leq i_1, i_k + q_k \leq i_{k+1}, k = 1, \ldots, q - 1, i_q + q_q - 1 \leq n$. It is not difficult to prove that the brace operations (7) have (total) degree 0.

It is useful to have a pictorial representation of certain operations: we depict an operator $P$ of with $p$ inputs (and one output) as a corolla with $p$ leaves, going from the bottom to the top. Thus, the component $m_{1,q}$ has the following graphical representation:

![Pictorial representation of the component $m_{1,q}$](image)

The conditions for $V$ to be a $B_\infty$-algebra reduce to the associativity condition (4), which simplifies in the present situation to

$$\left(P\{Q_1, \ldots, Q_q\}\right)\{R_1, \ldots, R_r\} = \sum_{1 \leq i_1 \leq \cdots \leq i_q \leq r} (-1)^{\sum_{a=1}^q |Q_a| (i_a - 1 + \sum_{b=1}^{a-1} |R_b|)} P\{R_1, \ldots, R_{i_1}-1, Q_1\{R_{i_1}, \ldots\}, \ldots, R_{i_q}-1, Q_q\{R_{i_q}, \ldots\}, \ldots, R_r\},$$

We recall from Remark 1.2 that we have a bracket of total degree 0 on $V$:

$$[P_1, P_2] = P_1\{P_2\} - (-1)^{|P_1||P_2|} P_2\{P_1\}.$$ 

For $q = r = 1$, Condition (8) simplifies to

$$\left(P_1\{P_2\}\right)\{P_3\} = P_1\{P_2\{P_3\}\} + P_1\{P_2, P_3\} + (-1)^{|P_2||P_3|} P_1\{P_3, P_2\}.$$ 

Whence the bracket satisfies the Jacobi identity\(^2\), and thus $V$ is a DGLA with trivial differential.

Since the differential $d$ is trivial, a MCE $\gamma$ satisfies the identity

$$\gamma\{\gamma\} = \frac{1}{2}[\gamma, \gamma] = 0.$$

\(^2\)This can be recovered from the fact (see Remark 1.2) that $m_{2,1}$, which is the homotopy for the Jacobi identity in a general $B_\infty$-algebra, vanishes here.
Example 1.5. Now we consider a (possibly curved) $A_{\infty}$-algebra $(A, \gamma_0, \gamma_1, \gamma_2, \ldots)$. By abuse of terminology, we may say that

$$\gamma := \gamma_0 + \gamma_1 + \gamma_2 + \cdots \in \prod_{n \geq 0} \text{Hom}(A^{\otimes n}, A)^{1-n}$$

is a MCE (in the present situation the Maurer-Cartan equation (5) makes sense since each of its homogeneous component is a finite sum).

Therefore the twisting procedure of Subsection 1.1 applies and one obtains a new $B_{\infty}$-structure on $\text{End}(A)$, such that

$$d_{\gamma,1}(P) = [\gamma, P]$$

is (up to a sign) the standard Hochschild coboundary operator for cochains of an $A_{\infty}$-algebra, and

$$P_1 \cup_\gamma P_2 := d_{\gamma,2}(P_1 \otimes P_2) = \gamma\{P_1, P_2\}$$

defines a product which is associative up to homotopy$^3$. More precisely, if $p_i$ is the number of entries of $P_i$, then, for $p_1 + p_2 \leq p$, we have, by construction,

$$P_1 \cup_\gamma P_2(a_1, \ldots, a_p) = \sum_{1 \leq j_1, j_2 \leq r, p_1 \leq j_1 + p_2 - j_2 \leq p} (-1)^{\sum_{i=1}^{2} |P_i(p_{j_i} - 1 + \sum_{k=1}^{i-1} |a_k|)} \gamma_{p_1 - p_2}(a_1, \ldots, P_1(a_{j_1}, \ldots), \ldots, P_2(a_{j_2}, \ldots), a_p).$$

Example 1.6. As a special case, if $A$ has the structure of a graded algebra, the associative product $\mu$ on $A$ is a MCE and the differential $d_{\mu}$ has only two components: $(-1)^{p-1}d_{\mu,1}$, resp. $(-1)^{p_1(p_2-1)}d_{\mu,2}$, is the standard Hochschild differential, resp. the standard product, on the Hochschild cochain complex of the algebra $A$.

1.3. A more general example of a $B_{\infty}$-algebra structure. We may consider more generally a commutative DG algebra $(m, d_m)$ as in the introduction. Its differential extends to a differential $d_m := \text{id} \otimes d_m$ on $V := \text{End}(E) \otimes m$ of total degree 1, which further extends to a (co)differential on $T(V)$, which we denote, by abuse of notations, by the same symbol.

The brace operations defined on $\text{End}(E)$ naturally extend to $V$ in the following way:

$$P \otimes m\{Q_1 \otimes n_1, \ldots, Q_r \otimes n_r\} := (-1)^rP\{Q_1, \ldots, Q_r\} \otimes mn_1 \cdots n_r,$$

where the sign $(-1)^r$ is determined by the appropriate braiding (with corresponding Koszul’s sign rule).

Then, the construction of the previous Subsection can be repeated verbatim, except that we have the additional non zero structure map $d_1 := d_m$.

Therefore the Maurer-Cartan equation reads

$$d_m(\gamma) + \gamma\{\gamma\} = 0$$

and makes sense for a generalized element

$$\gamma = \gamma_0 + \gamma_1 + \gamma_2 + \cdots \in \prod_{n \geq 0} (\text{Hom}(E^{\otimes n}, E) \otimes m)^{1-n}.$$

Such MCEs are in bijection with $(m, d_m)$-$A_{\infty}$-algebra structures on $E \otimes m$.

We implicitly make use of this $B_{\infty}$-structures in Sections 4 and 5 below (see also the Introduction above).

2. $B_{\infty}$-structures on Hochschild chains

In this Section, we discuss two $B_{\infty}$-module structures on $E \otimes E$, for a graded vector space $E$, and the twisting procedure that allows one to deduce from these two distinct left $B_{\infty}$-module structures on the Hochschild chain complex of an $(A_{\infty})$-algebra $A$. We believe this clarifies and makes more explicit a construction roughly sketched by Tamarkin–Tsygan in [25].

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$^3$Recall that it is actually commutative up to homotopy.
2.1. $B_\infty$-bimodules. We assume the graded vector space $V$ to be a $B_\infty$-algebra; we borrow the main notations from Subsection 1.1. Let $W$ be another graded vector space.

**Definition 2.1.** A $B_\infty$-bimodule structure on $W$ (over $V$) is a $B_\infty$-algebra structure on $V \oplus W[-1]$ such that

- $V$ is a $B_\infty$-subalgebra,
- all components of structure maps involving $W$ more than once are zero.

**Remark 2.2.** To $W$, we may associate the bi-comodule $W$ cogenerated by $W$, namely, $W = T(V) \otimes W \otimes T(V)$, with left-, resp. right-, coaction $\Delta_L$, resp. $\Delta_R$, defined via

$$\Delta_L = \Delta \otimes 1, \text{ resp. } \Delta_R = 1 \otimes \Delta.$$ 

A $B_\infty$-bimodule structure on $W$ w.r.t. the $B_\infty$-algebra structure on $V$ tantamounts to the data of linear maps $b : W \rightarrow W$, $m_L : T(V) \otimes W \rightarrow W$, $m_R : W \otimes T(V) \rightarrow W$, such that the 6-tuple $(W, b, m_L, \Delta_L, m_R, \Delta_R)$ is a DG bi-(co)module over the DG bialgebra $(T(V), m, \Delta, \eta, \varepsilon, d)$.

More precisely, $m_L$ (resp. $m_R$) defines a left (resp. right) action of $T(V)$ on $W$, which is required to be a morphism of bi-comodules; moreover, the left and right actions are required to commute, $b$ is required to square to 0, and to be a bi-(co)derivation of the bi-(co)module structure on $W$.

From this, we see that there is an obvious notion of left (resp. right) $B_\infty$-module.

Compatibility of $b$, $m_L$ and $m_R$ with coalgebra and comodule structures implies that they are uniquely determined by their structure maps (i.e. their evaluation on homogeneous components, composed with the standard projection $W \rightarrow W$):

$$b_{p,q} : V^\otimes p \otimes W \otimes V^\otimes q \rightarrow W,$$

$$m_L^{p,q,r} : V^\otimes p \otimes V^\otimes q \otimes W \otimes V^\otimes r \rightarrow W,$$

$$m_R^{p,q,r} : V^\otimes p \otimes W \otimes V^\otimes q \otimes V^\otimes r \rightarrow W.$$

Exemplarily, we write down the condition for $m_L$ to be a left action w.r.t. $m$ in terms of their respective components:

$$\sum_{l=1}^{p+q} \sum_{(\mu,\nu) \in \mathcal{P}_l(p) \times \mathcal{P}_l(q)} \sigma(\mu,\nu) m_L^{l,r,s} (m_{\mu,\nu}(v_1 \otimes \cdots \otimes v_p \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_r \otimes w \otimes \hat{v}_1 \otimes \cdots \otimes \hat{v}_s)) =$$

$$\sum_{l=1}^{q+r+s+1} \sum_{l=1}^{p} \sum_{(\mu_1,\nu_1) \in \mathcal{P}_l(q) \times \mathcal{P}_l(r+s+1)} \sigma(\mu_1,\nu_1) m_L^{p,q,r} (m_{\mu_1,\nu_1}(\bar{v}_1 \otimes \cdots \otimes \bar{v}_1 \otimes w \otimes \hat{v}_1 \otimes \cdots \otimes \hat{v}_s),$$

the notations are obvious generalizations of those introduced in Subsection 1.1. Finally, we consider a MC element $\gamma$ for the $B_\infty$-algebra $V$ as in Definition 1.3, Subsection 1.1: if $W$ is a $B_\infty$-bimodule as in Definition 2.1, then $\gamma$ determines a twisted differential $b_\gamma$ on $W$ via

$$b_\gamma = b + m_L(\gamma \otimes \bullet) - m_R(\bullet \otimes \gamma),$$

tacitly using Koszul’s sign rule, and the 6-tuple $(W, b_\gamma, m_L, \Delta_L, m_R, \Delta_R)$ again defines a $B_\infty$-bimodule structure on $W$.

**Example 2.3.** Let $V = \oplus_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded $B_\infty$-algebra, whose structure maps are degree preserving. Then $V_0$ is obviously a $B_\infty$-algebra w.r.t. the restriction of $m$. If we assume that $V_k = \{0\}$ when $k < -1$, then $V_{-1}[-1]$ is a $B_\infty$-bimodule over $V_0$, with left, resp. right, action $m_L$, resp. $m_R$, whose components are given by

$$m_L^{p,q,r} = m_{p,q+1+r}, \quad m_R^{p,q,r} = m_{p+1+q,r}.$$ 

It is clear e.g. that (9) follows immediately from (4), and analogous arguments imply the claim. Moreover, if $\gamma$ is a MCE in $V_0$, then the $B_\infty$-bimodule structure induced by the twisted ($\mathbb{Z}$-graded) $B_\infty$-algebra $(V, d_\gamma, m)$ on $V_{-1}[-1]$ obviously coincides with the twisted $B_\infty$-bimodule structure $(b_\gamma, m_L, m_R)$ on it.

2.2. Left $B_\infty$-module structures on the Hochschild chain complex. Let $E$ be a graded vector space, to which we associate

$$E \otimes E := \bigoplus_{n \geq 0} E \otimes E^\otimes n[n].$$
We then define $F := E \oplus E^*$ with the following additional $\mathbb{Z}$-grading: $E$, resp. $E^*$, has $\mathbb{Z}$-degree 0, resp. $-1$. Therefore, $V := \text{End}(F)$, becomes a $\mathbb{Z}$-graded $B_\infty$-algebra that satisfies the condition of Example 2.3. Explicitly,

$$V_0 = \text{End}(E) \oplus \bigoplus_{p,q \geq 0} \text{Hom}(E^{\otimes p} \otimes E^* \otimes E^{\otimes q}, E)^*[-p - q] \quad \text{and} \quad V_{-1} = \bigoplus_{n \geq 0} \text{Hom}(E^{\otimes n}, E^*)[1 - n].$$

In particular, $V_{-1}[-1]$ is canonically isomorphic to $(E \otimes E)^*$: explicitly, the identification is given by

$$\langle P(e_1, \ldots, e_n), e_0 \rangle = (-1)^{\sum_{i=1}^n |e_i|} \langle \overline{P}, (e_0 \cdots | e_n) \rangle, \quad P \in V_{-1}, \quad (e_0 \cdots | e_n) \in E^{\otimes n}.$$

Moreover, there is an inclusion $P \mapsto \overline{P}$

$$\text{Hom}(E^{\otimes n+1}, E) \hookrightarrow \bigoplus_{p+q=n} \text{Hom}(E^{\otimes p} \otimes E \otimes E^{\otimes q}, E),$$

explicitly given by the formula

$$\overline{P}(e_{i_1}, \ldots, e_{i_p}, e_{i_1+1}, \ldots, e_{i_p}) := (-1)^{\sum_{i=1}^p |e_i|} \langle \overline{P}, (e_0 \cdots | e_n) \rangle, \quad P \in V_{-1}, \quad (e_0 \cdots | e_n) \in E^{\otimes n}.$$

We observe that cyclic permutations enter into the game explicitly at this step. In turn, Formula (10) induces an inclusion

$$\text{End}(E) \hookrightarrow \text{End}(E) \oplus \bigoplus_{p,q \geq 0} \text{Hom}(E^{\otimes p} \otimes E^* \otimes E^{\otimes q}, E^*)[-p - q] \subset V_0, \quad P \mapsto P + \overline{P}.$$

Obviously, the identity morphism preserves the $B_\infty$-algebra structure. On the other hand, we may compute, using Formula (10), the inclusion $P\{Q_1, \ldots, Q_q\}$, for $P, Q_i, i = 1, \ldots, q$, general elements of $\text{End}(E)$, and we get

$$P\{Q_1, \ldots, Q_q\} = \sum_{i=1}^q P\{Q_1, \ldots, Q_i, Q_{i+1}, \ldots, Q_q\} + \sum_{i=1}^q Q_i \{P\{Q_1, \ldots, Q_{i-1}, Q_i, Q_{i+1}, \ldots, Q_q\}\}.$$

The two terms on the right-hand side of Identity (12) need some explanations. The cyclic permutations of the elements $Q_i, i = 1, \ldots, q$, appear evidently because of Formula (10): $P\{Q_i, \ldots, Q_j \}$, resp. $Q_i \{P\{Q_1, \ldots, Q_{i-1}, Q_i, Q_{i+1}, \ldots, Q_q\}\}$, acts non-trivially precisely on those terms, where the argument labelled by $e_0$ is placed between $Q_{i-1}$ and $Q_i$, resp. as an argument of $Q_i$. Finally, we observe that we have omitted the signs in Identity (12): these are easily obtained by Koszul's sign rule w.r.t. total degree.

In the special case where we consider only $P$ and $Q$, the defect of Inclusion (11) to be a $B_\infty$-algebra morphism can be characterized in a nice way, namely

$$\iota(P) \{ \iota(Q) \} - \iota(P\{Q\}) = [\overline{P}, \overline{Q}].$$

Since the inclusion (11) is NOT a $B_\infty$-algebra morphism, then we do NOT obtain a $B_\infty$-bimodule structure on $E \otimes E$ over $\text{End}(E)$. Nevertheless, as we will now explain, we will get two distinct left $B_\infty$-module structure, which we now explicitly describe.

The only non-trivial structure maps of the right $B_\infty$-module structure $m_R$ on $V_{-1}[-1]$ over $V_0$ are

$$m_{R,0,0}^p(P, Q_1, \ldots, Q_q) = P\{Q_1, \ldots, Q_q\},$$

for $P \in V_{-1}$ and $Q_1, \ldots, Q_q \in \text{End}(E) \subset V_0$. In particular, we can see that the induced left $B_\infty$-bimodule structure $m_{L,2}$ on $E \otimes E$, over $\text{End}(E)$, has only non-trivial structure maps $m_{L,2}^{p,0,0}$ given as follows: for any $c = (e_0 \cdots | e_m) \in E^{\otimes m}$,

$$m_{L,2}^{p,0,0}(P_1, \ldots, P_p, e) = \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq m} (-1)^{\sum_{j=1}^p \{i_{a_j-1}+\sum_{d=0}^{a_{j-1}-1} |e_d|\}} (e_0 \cdots | P_1(e_{i_1}, \ldots) \cdots | P_p(e_{i_r}, \ldots) \cdots | e_m),$$

where the summation is over indices $i_1, \ldots, i_p$, such that $1 \leq i_1, i_k + p_k \leq i_{k+1}, k = 1, \ldots, r - 1, i_p + p_p - 1 \leq m$ (if $p = 0$, $m_{L,2}^{0,0,0}$ is the identity map). The fact that summation is over all indices $1 \leq i_1$ is a consequence of the duality between $V_{-1}$ and $(E \otimes E)^*$, which highlights the special element $e_0$ in a given chain.

As for the brace operations, we have a pictorial representation for those structure maps:

\footnote{We have an induced left $B_\infty$-bimodule structure since that, for $P, Q_1, \ldots, Q_q$ as above, $P\{\iota(Q_1), \ldots, \iota(Q_q)\} = P\{Q_1, \ldots, Q_q\}$.}
Then we observe that, even if $P$ maps $m$-algebra structure on $A$, the only non-trivial structure maps of the left $B_\infty$-module structure $m_L$ on $V_{-1}[-1]$ are

$$m_{L,1}^{q,r}(P,Q_1,\ldots, Q_q,S,R_1,\ldots, R_r) = P(Q_1,\ldots, Q_q,S,R_1,\ldots, R_r),$$

where $P \in \text{Hom}(E^{\otimes k} \otimes E^{\otimes l} \otimes E^{\otimes \ell}, E^*)$, $Q_a$ and $R_b$, $a = 1,\ldots, q$, $b = 1,\ldots, r$ elements of $\text{End}(E)$ and $S \in V_{-1}$. In particular, we get an induced left $B_\infty$-bimodule structure $m_{L,1}$ on $E \otimes E$, over $\text{End}(E)$, with only non-trivial structure maps $m_{L,1}^{q,r}$, via

$$\langle \mathcal{P} (R_1,\ldots, R_r, S, Q_1,\ldots, Q_q) \rangle, c = \langle S, m_{L,1}^{q,r}(P,Q_1,\ldots, Q_q,c,R_1,\ldots, R_r) \rangle,$$

where $S$, resp. $c$, is a general element of $V_{-1}$, resp. $E \otimes E$, such that the previous expression makes sense.

The brace identities (8), Subsection 1.2, together with Identity (12), imply that the previous formula yields a left $B_\infty$-action: still, we observe that two dualizations are hidden in the previous formula, the first one in the inclusion $P \hookrightarrow \mathcal{P}$, the second one between $V_{-1}$ and $(E \otimes E)^*$.

For $P$, $Q_a$, $R_b$ as before in $\text{End}(E)$, and $c = (e_0|\cdots|e_m)$ in $E \otimes E$, we have the explicit form

$$m_{L,1}^{q,r}(P,Q_1,\ldots, Q_q,c,R_1,\ldots, R_r)) =$$

$$\sum_{i \leq j_i \leq -\sum_{i=1}^{r} |c_i| + \sum_{i=1}^{m} |Q_a|} (-1)^{f_1} (P(e_i,\cdots, Q_1(e_{j_1},\ldots), Q_q(e_{j_q},\ldots),\cdots, e_0, R_1(e_{k_1},\ldots),\cdots, R_r(e_{k_r},\ldots),\cdots) |e_n| \cdots |e_{n-1}|),$$

and the indices in the summation satisfy $j_i + q_i \leq j_{i+1}$, $i = 1,\ldots, q - 1$, $j_q + q_q - 1 \leq m$, and $k_i + r_i \leq k_{i+1}$, $i = 1,\ldots, r - 1$, $k_r + r_r - 1 \leq n - 1$.

Now we consider a (possibly curved) $A_\infty$-algebra on $(A, \gamma_0, \gamma_1, \gamma_2, \ldots)$. We allow ourselves the same abuse of language as in Example 1.5 and consider the formal sum $\gamma = \gamma_0 + \gamma_1 + \gamma_2 + \cdots$ as a MCE of the $B_\infty$-algebra $\text{End}(A)$. Then we observe that, even if $\iota$ is NOT a $B_\infty$-algebra morphism,

$$\iota(\gamma) = \sum_{n \geq 0} \gamma_n + \sum_{p,q \geq 0} \gamma_{p,q},$$

defines a MCE (again, by abuse of notation) in $\text{End}(A \oplus A^*)$. Namely, $\gamma_{p,q}$ $(p,q \geq 0)$ are the structure maps of the natural $A_\infty$-module structure on $A^*$.5

We may then apply the twisting procedure sketched at the end of Subsection 1.1 to $\text{End}(A \oplus A^*)$ w.r.t. the MCE $\iota(\gamma)$. Following the same lines of reasoning as above, we get the following

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5To prove that it is truly a MCE, we simply recall that, if one has an $A_\infty$-algebra $B$ together with an $A_\infty$-bimodule $M$, then by definition $B \otimes M$ is an $A_\infty$-algebra.
Theorem 2.4. $A \otimes A$ has two distinct left $B_\infty$-module structures over $(\text{End}(A), d_\gamma, m)$, given by $m_{L,i}$, $i = 1, 2$, and the formula

$$b_\gamma := m_{L,1}(\gamma \otimes \bullet) - m_{L,2}(\gamma \otimes \bullet),$$

specifies a degree 1 operator, which squares to 0 (i.e. an $A_\infty$-module structure on $A \otimes A$ over $\text{End}(A)$).

We only observe that the twisting procedure, as in Subsection 1.1, cannot be applied verbatim in the present situation because of Identity (12). Still, the same identity implies that $b_\gamma$ squares to 0, as can be verified by a direct computation.

Remark 2.5. It follows directly from Subsection 1.3 that this construction generalizes to the situation where we tensorize $\text{End}(E)$ and $E \otimes E$ by a commutative DG algebra $(m, d_m)$ as in the introduction.

2.3. $T$-algebra structure on Hochschild (co)homology. First, we observe that from the very definition of a $B_\infty$-bimodule, and according to the fact that any $B_\infty$-algebra is a Gerstenhaber algebra up to homotopy (see Remark 1.2), we have the following:

For any $B_\infty$-algebra $V$ together with a $B_\infty$-bimodule $W$, the pair $(V[1], W)$ naturally inherits the structure of Gerstenhaber algebra and module up to homotopy.

The crucial point is that, on the Hochschild chain complex of an $A_\infty$-algebra we only have two left $B_\infty$-module structures, having the same differential, but NOT satisfying the axioms of a $B_\infty$-bimodule (see the previous subsection). We therefore do NOT have the structure of a Gerstenhaber module up to homotopy. Nevertheless, we prove below that we have something close to a Gerstenhaber module; namely, a $T$-algebra.

We first recall the definition of a $T$-algebra (or precalculus following the recent terminology of [13]) from [25].

Definition 2.6. A $T$-algebra is a pair $(V, W)$ of a Gerstenhaber algebra $(V, \cup, [, ])$ and a graded vector space $W$ together with

- an action $\cap$ of the GA $(V, \cup)$, turning $(W, \cap)$ into a GM,
- an action $L$ of the GLA $(V[-1], [, ])$, turning $(W, L)$ into a GLM,

such that the following identities hold true for any $v_1, v_2 \in V$ and $w \in W$:

$$(v_1 \cap w) = [v_1, v_2] \cap w + (-1)^{|v_1||v_2|}v_2 \cap (L_{v_1}w),$$

$$(v_1 \cup v_2)w = L_{v_1}(v_2 \cap w) + (-1)^{|v_1|}v_1 \cap (L_{v_2}w).$$

We may say, by abuse of language, that $W$ is a $T$-module over the Gerstenhaber algebra $V$.

Remark 2.7. It is worth mentioning that Identity (16) in Definition 2.6 can be re-written as

$$(v_1 \cup v_2)w = (-1)^{|v_1||v_2|}v_2 \cap (L_{v_1}w) + (-1)^{|v_1|}v_1 \cap (L_{v_2}w) - (-1)^{|v_1|}v_1 \cap w,$$

and we therefore see that a $T$-module is almost a Gerstenhaber module, the default being given by the last term in the r.h.s. of (17). The modified Identity (17) will be particularly useful in the upcoming computations.

We now prove that $A \otimes A$ is a $T$-module up to homotopy over $\text{End}(A)$.

We first recall that $\text{End}(A)$ is a graded Lie algebra with bracket $[P, Q] := P\{Q\} - (-1)^{|P||Q|}Q\{P\}$. We observe that the same is true for $\text{End}(A \otimes A^*)$. This allows us to define a right Lie action of $\text{End}(A)$ onto $V_{-1}$. Namely, for $P \in \text{End}(A)$ and $Q \in V_{-1}$, we set

$$R_P(Q) := [Q, \iota(P)] = Q\{P\} - (-1)^{|P||Q|}T_1(Q)\{P\}.$$

We now prove that the previous formula defines a right Lie action. First of all, we evaluate explicitly $R_P_1(R_P_2Q)$, using the brace relations (8), Subsection 1.2:

$$R_{P_1}(R_{P_2}Q) = Q\{P_2, P_1\} + Q\{P_2\{P_1\}\} + (-1)^{|P_1||P_2|}Q\{P_1, P_2\} - (-1)^{|Q||P_2|}T_2\{Q, P_1\} - (-1)^{|Q||P_1|}T_2\{Q, P_2\} - (-1)^{|Q||P_1||P_2|}T_1\{P_2, Q\} - (-1)^{|Q||P_1||P_2|}T_1\{P_2, Q\}.$$

A similar expression is obtained evaluating $R_{P_2}(R_{P_1}Q)$: summing up the two terms with the correct signs, we find

$$[R_{P_1}, R_{P_2}](Q) = Q\{P_2\{P_1\}\} - (-1)^{|P_1||P_2|}Q\{P_1, P_2\} - (-1)^{|Q||P_2|}T_2\{Q, P_1\} - (-1)^{|Q||P_1|}T_2\{Q, P_2\} - (-1)^{|Q||P_1||P_2|}T_1\{P_2, Q\} + (-1)^{|Q||P_1||P_2|}T_1\{P_2, Q\} + (-1)^{|Q||P_1||P_2|}T_1\{P_2, Q\}.$$
Obviously, the first two terms on the right hand-side sum up to \(Q\{P_2, P_1\}\). On the other hand, we consider \([P_2, P_1]\{Q\}: explicitly, in virtue of Formula (10), Subsection 2.2,
\[
P_2\{P_1\}\{Q\} = \overline{P_2}\{P_1, Q\} + (-1)^{||Q||}P_2\{Q, P_1\} + (-1)^{||P_1||}P_2\{P_2\{P_1\}\{Q\}\},
\]
and a similar formula holds true for \(P_2\{P_1\}\{Q\}\) with obvious due changes. This yields \([R_{P_1}, R_{P_2}]\{Q\} = R_{\{P_2, P_1\}}(Q)\).

Dually, \(R\) defines a (left) graded Lie module structure on \(A \otimes A\), which we denote by \(L\).

We have not yet considered the \(A_{\infty}\)-algebra structure on \(A\), i.e. the MCE \(\gamma\). Since we have a graded Lie algebra together with a graded Lie module, then we can twist them by the MCE \(\gamma\) and obtain a DGLA \((\widehat{\text{End}}(A), [\gamma, ], [, \cdot, \cdot])\) together with a DGLM \((A \otimes A, L, \circ)\).

Further, we recall that we have a product \(\cup_{\gamma} := \cup_{\gamma, 2}\) which makes \(\widehat{\text{End}}(A)[1]\) into a Gerstenhaber algebra up to homotopy. Analogously, we define an action (from the right) of \(\text{End}(A)\) on \(A \otimes A\), which we again denote by \(\cup_{\gamma}\).

Moreover, the differential \(d_{\gamma, 1} = [\gamma, ]\), resp. \(b_{\gamma, 0, 0} = L\), is by definition compatible with the product \(\cup_{\gamma}\), resp. the action \(\cup_{\gamma}\).

Remark 2.8. A similar formula, where we switch \(P\) and \(Q\), defines accordingly a right action of \(\widehat{\text{End}}(A)\) on \(A \otimes A\), which we will also denote by \(\cup_{\gamma}\): one may think that this would lead to some confusion, but that both actions commute in the graded sense, since \(\cup\) commutes up to homotopy, thus, later on, we will not distinguish between left and right action from the notational point of view.

Proposition 2.9. \((A \otimes A, L, \circ, \cup_{\gamma})\) is a \(T\)-module up to homotopy over the Gerstenhaber algebra up to homotopy \((\widehat{\text{End}}(A)[1], d_{\gamma, 1}, [, \cdot, \cdot], \cup_{\gamma})\).

Proof. By the above arguments and computations, it remains to prove the homotopical versions of Identities (15) and (16) in Definition 2.6: in particular, we observe that we will prove the homotopical version of the modified Identity (17).

We will only write down the explicit homotopy formulae with signs: the computations leading to their proof make use of the brace identities (8), Subsection 1.2, and of Identity (12), Subsection 2.2, since \(L\) and \(\circ\) can be described explicitly in terms of the brace operations on \(V_{-1}\), and Identity (12), Subsection 2.2 measures the failure of the two left \(B_{\infty}\)-actions \(m_{L, i}, i = 1, 2\), of being compatible.

Explicitly, we have the homotopy formulae,
\[
L_{P_i} (P_2 \cup_{\gamma} c) = [P_1, P_2] \cup_{\gamma} c + (-1)^{||P_1||+||P_2||-1}P_2 \cup_{\gamma} L_{P_1} c + (-1)^{||P_1||}L_{\gamma} (m_{L, 1}^{1, 0}(P_1, P_2, c) - \nu_{L, 1}^{1, 0}(d_{\gamma, 1}P_1, P_2, c)) - (-1)^{||P_1||+||P_2||}m_{L, 1}^{1, 0}(P_1, P_2, L_{\gamma} c),
\]
and
\[
L_{P_1 \cup_{\gamma} P_2} c + (-1)^{||P_1||+||P_2||-1}L_{P_1 \cup_{\gamma} P_2} c = \\
= (-1)^{||P_1||}P_1 \cup_{\gamma} L_{P_2} c + (-1)^{||P_1||+||P_2||-1}L_{P_2} c \cup_{\gamma} P_1 + \\
+ (-1)^{||P_1||+||P_2||-1}P_2 \cup_{\gamma} L_{P_1} c + (-1)^{||P_1||+||P_2||-1}L_{P_1} c \cup_{\gamma} P_2 + \\
+ (-1)^{||P_1||}L_{P_2} c \cup_{\gamma} (P_1, P_2) + (-1)^{||P_1||+||P_2||+||c||-1}L_{P_1} c \cup_{\gamma} (P_1, P_2) + \\
+ (-1)^{||P_2||}L_{\gamma} (m_{L, 2}^{0, 0}(P_1, P_2, c) - m_{L, 2}^{0, 0}(d_{\gamma, 1}P_1, P_2, c)) - (-1)^{||P_1||}m_{L, 2}^{0, 0}(P_1, d_{\gamma, 1}P_2, c) - \\
- (-1)^{||P_1||+||P_2||}m_{L, 2}^{0, 0}(P_1, P_2, L_{\gamma} c) + (-1)^{||P_1||}L_{\gamma} (m_{L, 2}^{0, 0}(P_2, P_1, c)) - m_{L, 2}^{0, 0}(d_{\gamma, 1}P_2, P_1, c) - \\
- (-1)^{||P_1||+||P_2||}m_{L, 2}^{0, 0}(P_2, d_{\gamma, 1}P_1, c) - (-1)^{||P_1||+||P_2||}m_{L, 2}^{0, 0}(P_2, P_1, L_{\gamma} c),
\]
for \(P_i, i = 1, 2\), in \(\text{End}(A)\) and \(c\) in \(A \otimes A\).

We observe that there is a homotopy formula, similar to the first one we have written down, for the right action \(\cup_{\gamma}\): this explains the appearance of many terms in the second homotopy formula. We observe that the graded anti-commutators in the second homotopy formula sum up in the corresponding cohomology, whence the second homotopy formula restricts on cohomology to (17). \(\square\)
In this Section we discuss in some details compactifications of configuration spaces of i) points in the complex upper-half plane $\mathcal{H}$ and on the real axis $\mathbb{R}$, and ii) points in the interior of the punctured unit disk $D^\times$ and on the unit circle $S^1$.

We will focus our attention on $C_2.0 \cong D_{1.1}$ and on its boundary stratification: it will play a central rôle in the proof of both compatibilities with cup and cap products. We will also take a better look at the compactified configuration space $C_{2.1} \cong D^{+1.2}$: though it is not crucial in the forthcoming proofs, its boundary stratification leads to a better understanding of the homotopy formula for the compatibility between cap products, see e.g. [4].

3. Configuration spaces and integral weights

3.1. Configuration spaces and their compactifications. In this Subsection we recall compactifications of configuration spaces of points in the complex upper-half plane $\mathcal{H}$ and on the real axis $\mathbb{R}$, and of points in the interior of the punctured unit disk $D^\times$ and on the unit circle $S^1$.

3.1.1. Configuration spaces $C^+_{A,B}$ and $C_A$. We consider a finite set $A$ and a finite (totally) ordered set $B$. We define the open configuration space $C^+_{A,B}$ as

$$C^+_{A,B} := \{(p,q) \in \mathcal{H}^A \times \mathbb{R}^B | p(a) \neq p(a') \text{ if } a \neq a', \quad q(b) < q(b') \text{ if } b < b'\} / G_2,$$

where $G_2$ is the semidirect product $\mathbb{R}^+ \ltimes \mathbb{C}$, which acts diagonally on $\mathcal{H}^A \times \mathbb{R}^B$ via

$$(\lambda, \mu)(p,q) = (\lambda p + \mu, \lambda q + \mu) \quad (\lambda \in \mathbb{R}^+, \mu \in \mathbb{C}).$$

The action of the 2-dimensional Lie group $G_2$ on such $n + m$-tuples is free, precisely when $2|A| + |B| - 2 \geq 0$: in this case, $C^+_{A,B}$ is a smooth real manifold of dimension $2|A| + |B| - 2$.

The configuration space $C_A$ is defined as

$$C_A := \{p \in \mathbb{C}^A | p(a) \neq p(a') \text{ if } a \neq a'\} / G_3,$$

where $G_3$ is the semidirect product $\mathbb{R}^+ \ltimes \mathbb{C}$, which acts diagonally on $\mathbb{C}^A$ via

$$(\lambda, \mu)p = \lambda p + \mu \quad (\lambda \in \mathbb{R}^+, \mu \in \mathbb{C}).$$

The action of $G_3$, which is a real Lie group of dimension 3, is free precisely when $2|A| - 3 \geq 0$, in which case $C_A$ is a smooth real manifold of dimension $2|A| - 3$.

Finally, we observe that the spaces $C^+_{A,B}$ and $C_A$ are orientable, see e.g. [1] for a complete discussion of orientations of such configuration spaces.

The configuration spaces $C^+_{A,B}$, resp. $C_A$, admit compactifications à la Fulton–MacPherson, obtained by successive real blow-ups: we will not discuss here the construction of their compactifications $C^+_\infty A,B$, $C_A$, which are smooth manifolds with corners, referring to [20], [21], [3] for more details, but we focus mainly on their stratification, in particular on the boundary strata of codimension 1 of $C^+_\infty A,B$.

Namely, the compactified configuration space $C^+_\infty A,B$ is a stratified space, and its boundary strata of codimension 1 look like as follows:

i) there is a subset $A_1$ of $A$, resp. an ordered subset $B_1$ of successive elements of $B$, such that

$$\partial A_1, B_1 C^+_\infty A,B \cong C^+_A, B_1 \times C^+_\infty A_1, B \setminus B_1 \cup \{\ast\};$$

intuitively, this corresponds to the situation, where points in $\mathcal{H}$, labelled by $A_1$, and successive points in $\mathbb{R}$ labelled by $B_1$, collapse to a single point labelled by $\ast$ in $\mathbb{R}$. Obviously, we must have $2|A_1| + |B_1| - 2 \geq 0$ and $2(|A| - |A_1|) + (|B| - |B_1| + 1) - 2 \geq 0$.

ii) there is a subset $A_1$ of $A$, such that

$$\partial A, C^+_\infty A,B \cong C_A \times C^+_\infty A_1 \setminus \{\ast\}.B;$$

this corresponds to the situation, where points in $\mathcal{H}$, labelled by $A_1$, collapse together to a single point $\ast$ in $\mathcal{H}$, labelled by $\ast$. Again, we must have $2|A_1| - 3 \geq 0$ and $2(|A| - |A_1| + 1) + |B| - 2 \geq 0$.

3.1.2. Configuration spaces $D^+_{A,B}$ and $D_A$. We consider a finite set $A$ and a finite, cyclically ordered set $B$. We define the open configuration space $D^+_{A,B}$ as

$$\{(p,q) \in (D^\times)^A \times (S^1)^B | p(a) \neq p(a'), a \neq a', \quad q(b_1) < q(b_2) < \cdots < q(b_1), b_1 < b_2 < \cdots < b_1\} / S^1,$$

where $D^\times$ denotes the punctured unit disk. Here the group $S^1$ acts on $D^+_{A,B}$ by rotations: the action is free, precisely when $2|A| + |B| - 1 \geq 0$, in which case $D^+_{A,B}$ is a smooth real manifold of dimension $2|A| + |B| - 1$. 
We also consider the configuration space
\[ D_A = \{ p \in (\mathbb{C}^\times)^A \mid p(a) \neq p(a'), a \neq a' \} / \mathbb{R}^+, \]
where \( \mathbb{R}^+ \) acts by rescaling. It is obviously a smooth real manifold of dimension \( 2|A| - 1 \), when \( 2|A| - 1 \geq 0 \).

We also observe that, analogously to the configuration spaces \( C_{A,B}^+ \) and \( C_A \) of the previous Subsection, \( D_{A,B}^+ \) and \( D_A \) are orientable: it follows along the same lines of the discussion in \cite{1}. Moreover, they also admit natural compactifications obtained by successive real blow-ups.

The boundary strata of codimension 1 of \( D_{A,B}^+ \) are given in the following list:

i) there is a subset \( A_1 \) of \( A \), such that
\[ \partial A_1 \circ D_{A,B}^+ \cong D_{A_1} \times D_{A \setminus A_1,B}^+. \]
Intuitively, this corresponds to the situation, where points in \( D^\times \) labelled by \( A_1 \) tend together to the origin.

Clearly, we must have \( 2|A_1| - 1 \geq 0 \) and \( 2(|A| - |A_1|) + |B| - 1 \geq 0 \).

ii) There is a subset \( A_1 \) of \( A \), such that
\[ \partial A_1 \circ D_{A,B}^+ \cong C_{A_1} \times D_{A \setminus A_1,B \cup \{\ast\},B}. \]
This corresponds to the situation, where points in \( D^\times \) labelled by \( A_1 \) collapse together to a point in \( D^\times \), labelled by \( \ast \). We must impose \( 2|A_1| - 3 \geq 0 \) and \( 2(|A| - |A_1|) + |B| - 1 \geq 0 \).

iii) Finally, there is a subset \( A_1 \) of \( A \) and an ordered subset \( B_1 \) of successive elements of \( B \), such that
\[ \partial A_1 \circ D_{A,B}^+ \cong C_{A_1,B_1} \times D_{A \setminus A_1,B \setminus B_1 \cup \{\ast\}}. \]
which describes the situation, where points in \( D^\times \) labelled by \( A_1 \) and successive points labelled by \( B_1 \) collapse together to a point in \( S^1 \), labelled by \( \ast \). We have to impose \( 2|A_1| + |B_1| - 2 \geq 0 \) and \( 2(|A| - |A_1|) + (|B| - |B_1| - 1) \geq 0 \).

3.1.3. An identification. Considering the special case \( A = [n] \) and \( B = [m] \), \( m \geq 1 \), we may use the action of \( S^1 \) to construct a section of \( D_{A,B}^+ \) by fixing the first point in \( S^1 \) to 1. This section is diffeomorphic, by means of the Möbius transformation
\[ \psi : \mathcal{H} \cup \mathbb{R} \rightarrow D \cup S^1 \setminus \{1\}; \quad z \mapsto \frac{z - \frac{1}{\ast}}{z + 1}, \]
where \( D \) is the unit disk, to a smooth section of \( C_{n+1,m-1}^+ \), given by fixing e.g. the first point in the complex upper half-plane \( \mathcal{H} \) to \( i \) by means of the action of \( G_2 \).

Then, the compactified configuration space \( D_{n,m}^+ \) can be identified with \( C_{n+1,m-1}^+ \), and we observe that the cyclic order of the \( m \) points in \( S^1 \) translates naturally into an order of the \( m - 1 \) points on the real axis \( \mathbb{R} \).

We point out that in certain situation it is better to use the compactified configuration spaces \( D_{n,m}^+ \) instead of the equivalent \( C_{n+1,m-1}^+ \), because i) a cyclic order is visualized in an easier way on \( S^1 \) and ii) we need two special points (the origin and the first point in \( S^1 \)), which are also better visualized in the punctured disk \( D \) with boundary.

We further consider the manifold \( D_n \), for \( n \geq 1 \) and notice the identification \( D_n \cong C_{n+1}^+ \): to be more precise, by means of complex translation, we may put e.g. the first point in \( C_{n+1} \) at the origin, and using rescalings, one can put the remaining points in the punctured unit disk with boundary. Analogously as before, the compactification \( D_n \) of \( D_n \) can be identified with \( C_{n+1} \).

More generally, this identification remains possible for arbitrary \( A,B \), after the choice of distinguished elements \( \bullet \in A \) and \( \circ \in B \). We consequently identify the codimension 1 boundary strata of \( D_{A,B}^+ \) with those of \( C_{A\cup\{\bullet\},B\setminus\{\circ\}}^+ \) (higher codimension can be worked out along the same lines very easily):

i) the situation (20) where points labelled by \( A_1 \) collapse together to the origin corresponds to the situation (19), where points labelled by \( A_1 \cup \{\bullet\} \) collapse together to a single point in \( \mathcal{H} \), which takes the rôle of the marked point \( \bullet \).

ii) The situation (21), where points labelled by \( A_1 \) collapse to a single point in \( D \), corresponds to the situation (19), where points labelled by \( A_1 \) collapse together to a single point in \( \mathcal{H} \), which will not be the new marked point \( \bullet \).

iii) The situation (22), where points labelled by \( A_1 \cup B_1 \), with \( \circ \notin B_1 \), collapse to a single point in \( S^1 \) corresponds to the situation (18), where points labelled by \( A_1 \cup B_1 \) collapse to a single point in \( \mathbb{R} \), which will not be the new marked point \( \circ \).

iv) Finally, the situation (22), where points labelled by \( A_1 \cup B_1 \), with \( \circ \in B_1 \), collapse to a single point in \( S^1 \) corresponds to the situation (18), where points labelled by the set \( (A \setminus A_1 \cup \{\bullet\}) \cup (B \setminus B_1) \) collapse to a single point in \( \mathbb{R} \), which will be the new marked point \( \circ \).
3.2. Two remarkable compactified configuration spaces. We describe two remarkable compactified configuration spaces: the eye and the I-cube.

3.2.1. The eye. We now describe explicitly the compactified configuration space $C_{2,0}$, known as Kontsevich's eye. Here is a picture of it, with all boundary strata of codimension 1, labelled by Greek letter, and codimension 2, labelled by Latin letters, which we will describe shortly afterwards:

![Kontsevich's eye](image)

We first describe the boundary strata of codimension 1.

1) The stratum labelled by $\alpha$ corresponds to $C_2 = S^1$: intuitively, it describes the situation, where the two points collapse to a single point in $\mathcal{H}$;
2) the stratum labelled by $\beta$ corresponds to $C_{1,1} \cong [0,1]$: it describes the situation, where the first point goes to the real axis;
3) the stratum labelled by $\gamma$ corresponds to $C_{1,1} \cong [0,1]$: it describes the situation, where the second point goes to the real axis.

As already observed, we have the identification $C_{2,0} \cong D_{1,1}$, and we can thus reinterpret its codimension 1 boundary strata as follows:

1) the stratum labelled by $\alpha$ corresponds to $D_1 = S^1$, which describes the situation, where the point in $D^\times$ goes to the origin;
2) the stratum labelled by $\beta$ corresponds to $C_{1,1} \cong [0,1]$, which describes the situation, where the point in $D^\times$ and the point on $S^1$ collapse together in $S^1$;
3) the stratum labelled $\gamma$ corresponds to $D_{0,2}^+ \cong [0,1]$, which describes the situation, the point in $D^\times$ goes to $S^1$, without collapsing to the single point of $S^1$.

We use the following pictorial representation for the latter situations, where we consider Kontsevich’s eye as $D_{1,1}$:

![Boundary strata of codimension 1 of $D_{1,1}$](image)

Finally, the two boundary strata of codimension 2 are each one a copy of $C_{0,2}^+ = \{\text{pt}\}$. The situations they describe can be depicted as follows:

![Boundary strata of codimension 2 of $D_{1,1}$](image)

3.2.2. The I-cube. We now describe shortly the compactified configuration space $C_{2,1} \cong D_{1,2}^+$, which will be called the I-cube: in particular, we are interested in its boundary strata of codimension 1 and 2. As in Subsubsection 3.2.1, we use Greek letters, resp. Latin letters, for labelling boundary strata of codimension 1, resp. 2.

Pictorially, the I-cube looks like as follows:
Its boundary stratification consists of 9 strata of codimension 1, 20 strata of codimension 2 and 12 strata of codimension 3.

**Boundary strata of codimension 1.** We illustrate explicitly the boundary strata of codimension 1: again, before describing them mathematically, it is better to depict them:

![Boundary strata of the I-cube of codimension 1](image)

The strata labelled by $\alpha$ and $\beta$ are both described by $C_{0,2}^+ \times D_{1,1}$, depending on the cyclic order of the two points in $S^1$: since $C_{0,2}^+$ is 0-dimensional, the strata $\alpha$ and $\beta$ are two copies of Kontsevich’s eye $D_{1,1}$.

As for the strata labelled by $\gamma$ and $\delta$, they are both described by $C_{1,1} \times D_{0,2}^+$: both $C_{1,1}$ and $D_{0,2}^+$ correspond to closed intervals, whence $\gamma$ and $\delta$ are topologically two squares.

The strata labelled by $\varepsilon$ and $\theta$ correspond both to $C_{1,2}^+ \times D_{0,1}$, depending on the cyclic order of points in $S^1$: recalling the results of Subsection 2.1, $\varepsilon$ and $\theta$ are topologically two copies of the hexagon (this will be also clearer after the description of the boundary strata of codimension 2 of the I-cube).

On the other hand, the strata labelled by $\eta$ and $\zeta$ are both described by $C_{1,0} \times D_{0,3}^+$, depending on the cyclic order on $S^1$: again, since $C_{1,0}$ is 0-dimensional, an inspection of $D_{0,3}^+$ shows that $\eta$ and $\zeta$ are topologically two copies of the hexagon (again, we deserve a careful explanation, when dealing with boundary strata of codimension 2 of the I-cube).

Finally, the stratum labelled by $\xi$ corresponds to $D_1 \times D_{0,2}^+$: since $D_1 = S^1$ and $D_{0,2}^+$ is a closed interval, topologically $\xi$ looks like a cylinder.

The above picture describes the boundary strata of codimension 1 of $D_{1,2}^+$: using the prescriptions of Subsubsection 3.1.3, it is then easy to identify these boundary strata with the corresponding boundary strata of codimension 1 of $C_{2,1}$.

**Boundary strata of codimension 2.** We discuss now some relevant boundary strata of the I-cube of codimension 2: we first illustrate all of them pictorially as follows, referring to the picture of the I-cube for the notations:
For our purposes, we need only describe explicitly the boundary strata labelled by $e, f, h, j, p, q$ and $o$: in fact, these describe certain boundary components of a particular imbedding of the plane square into the I-cube, which will be useful later on.

The strata labelled by $e$ and $f$ are described as $D_1 \times C_0^{+2} \times D_{0,1}$: since $C_0^{+2}$ and $D_{0,1}$ are both 0-dimensional, while $D_1 = S^1$, they can be identified with the pupils of the two brave new eyes.

The strata labelled by $h, j$ and $p$ correspond all to $C_{1,0} \times C_0^{+3} \times D_{0,1}$, hence they correspond all to a closed interval, the only difference depending on the cyclic order on $S^1$ and on the corresponding order on $\mathbb{R}$.

As for the stratum labelled by $o$, it is described as $C_{1,0} \times C_0^{+2} \times D_{0,2}$, which is also topologically, by previous arguments, a closed interval.

Finally, the stratum labelled by $q$ is $C_{1,1} \times C_0^{+2} \times D_{0,1}$: once again, it is topologically a closed interval.

It is left as an exercise to identify the boundary strata of $D_{1,2}^+$ with the boundary strata of $C_{2,1}$, according to the prescriptions of Subsubsection 3.1.3.

3.3. Integral weights associated to graphs. In this Subsection we recall Kontsevich’s, resp. Shoikhet’s, angle forms and the corresponding weights, resp. modified weights, associated to graphs.

3.3.1. Angle forms. We first need to specify a smooth 1-form on the configuration space $C_{2,0}$. For any two distinct points $p, q$ in $H \sqcup \mathbb{R}$, we define

$$\varphi(p, q) = \frac{1}{2\pi} \arg \left( \frac{q - p}{q - p} \right).$$

The real number $\varphi(p, q)$ represents the (normalized) angle from the geodesic from $p$ to the point $\infty$ on the positive imaginary axis to the geodesic between $p$ and $q$ w.r.t. the hyperbolic metric of $H \sqcup \mathbb{R}$, measured in counterclockwise direction. It is therefore defined up to a constant, and thus $\omega := d\varphi$ is a well-defined 1-form.

**Lemma 3.1.** The 1-form $\omega$ extends to a smooth 1-form on Kontsevich’s eye $C_{2,0}$, with the following properties:

a) the restriction of $\omega$ to the boundary stratum $C_2 = S^1$ equals the total derivative of the (normalized) angle measured in counterclockwise direction from the positive imaginary axis;

b) the restriction of $\omega$ to $C_{1,1}$, where the first point goes to the real axis, vanishes. \hfill \Box

We then define a smooth 1-form on the configuration space $C_{3,0}$. For any three pairwise distinct points $p, q, r$ in $H \sqcup \mathbb{R}$, we define the modified Kontsevich angle function

$$\varphi_D(p, q, r) = \varphi(q, r) - \varphi(q, p),$$

and set $\omega_D = d\varphi_D$.

**Lemma 3.2.** The 1-form $\omega_D$ extends to a smooth 1-form on $C_{3,0}$, with the following properties:

a) its restriction to $C_{2,1}$, when the second point approaches the real axis, vanishes;

b) its restriction to $C_{2,0} \times C_{1,1}$, when the first and second point collapse together to the real axis, equals $-\pi^1 \omega$;

c) its restriction to $C_{2} \times C_{2,0}$, when the first and second point collapse together in the upper half-plane, equals $\pi_2^2 \omega - \pi_1^1 \omega$;
d) its restriction to $C_{2,0} \times C_{1,1}$ (resp. $C_2 \times C_{2,0}$), when the first and third point collapse together to the real axis (resp. in the upper half-plane), vanishes;

e) its restriction to $C_{2,0} \times C_{1,1}$, when the second and third point collapse together to the real axis, equals $\pi_1^* \omega$.

Using the prescriptions of Subsubsection 3.1.3, it is easy to identify the boundary strata of $C_{2,0}$ in Lemma 3.1 and the boundary strata of $C_{3,0}$ in Lemma 3.2 with the corresponding boundary strata of $D_{1,1}$ and $D_{2,1}$.

In particular, here is a useful pictorial description of the angle function (23):

![Figure 11 - The modified Kontsevich’s angle functions $\varphi_D$](image)

3.3.2. Integral weights associated to graphs. We consider, for given positive integers $n$ and $m$, directed graphs $\Gamma$ with $m + n$ vertices labelled by the set $\mathcal{V}(\Gamma) = \{1, \ldots, n, \bar{1}, \ldots, \bar{m}\}$. Here, “directed” means that each edge of $\Gamma$ carries an orientation. Additionally, the graphs we consider are required to have no loop (a loop is an edge beginning and ending at the same vertex). To any edge $e = (i, j)$ of such a directed graph $\Gamma$, we associate the smooth 1-form $\omega_e := \pi_e^* \omega$ on $C_{n,m}^+$, where $\pi_e : C_{n,m}^+ \to C_{2,0}$ is the smooth map given by

$$
[ (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_m) ] \mapsto [ (z_i, z_j) ].
$$

Then, to any directed graph $\Gamma$ without loop, and denoting by $\mathcal{E}(\Gamma)$ the set of its edges, we associate a differential form

$$
\omega_\Gamma := \bigwedge_{e \in \mathcal{E}(\Gamma)} \omega_e
$$
on the (compactified) configuration space $C_{n,m}^+$.

Remark 3.3. We observe that, a priori, it is necessary to choose an ordering of the edges of $\Gamma$ since $\omega_\Gamma$ is a product of 1-forms: two different orderings of the edges of $\Gamma$ simply differ by a sign. This sign ambiguity precisely coincide (and thus cancel) with the one appearing in the definition of $B_\Gamma$ (see next Section).

We recall that $C_{n,m}^+$ is orientable and its orientation specifies an orientation for any boundary stratum thereof.

**Definition 3.4.** The weight $W_\Gamma$ of the directed graph $\Gamma$ is

$$
W_\Gamma := \int_{C_{n,m}^+} \omega_\Gamma.
$$

Observe that the weight (25) truly exists since it is an integral of a smooth differential form over a compact manifold with corners.

In the same way, we define a modified weight associated to a graph without loop $\Gamma$ with $m + n + 1$ vertices labelled by $\mathcal{V}(\Gamma) := \{0, \ldots, n, \bar{1}, \ldots, \bar{m}\}$. To any edge $e = (i, j) \in \mathcal{E}(\Gamma)$, we associate a smooth 1-form $\omega_{D,e}$ on $D_{n,m}^+$ by the following rules:

- if neither $i$ nor $j$ lies in $\{0, \bar{1}\}$, then $\omega_{D,e} := \pi_{(0,i,j)}^* \omega_D$, where

$$
\pi_{(0,i,j)} : D_{n,m}^+ \cong C_{n+1,m-1}^+ \to C_{3,0}
$$

$$
[ (z_0, \ldots, z_n, \bar{z}_\bar{1}, \ldots, \bar{z}_\bar{m}) ] \mapsto [ (z_0, z_i, z_j) ];
$$

- if $i = 0$ and $j \neq \bar{1}$, then $\omega_{D,e} := \pi_{(i,j)}^* \omega_D$, where

$$
\pi_{(i,j)} : D_{n,m}^+ \cong C_{n+1,m-1}^+ \to C_{2,0}
$$

$$
[ (z_0, \ldots, z_n, \bar{z}_i, \ldots, \bar{z}_m) ] \mapsto [ (z_i, z_j) ];
$$

- if $i \neq 0$ and $j \neq \bar{1}$, then $\omega_{D,e} := \pi_{(i,j)}^* \omega_D$, where

$$
\pi_{(i,j)} : D_{n,m}^+ \cong C_{n+1,m-1}^+ \to C_{1,0}
$$

$$
[ (z_0, \ldots, z_n, \bar{z}_i, \ldots, \bar{z}_m) ] \mapsto [ (z_i, z_j) ];
$$
• if $j = \overline{1}$ and $i \neq 0$, then $\omega_{D,e} := p^*_{(i,j)}\omega$, where

$$p_{(i,j)} : D^+_{n,m} \rightarrow D_{1,1} \cong C_{2,0}$$

$$[ (z_1, \ldots, z_m, \overline{1}, \ldots, \overline{m}) ] \mapsto [ (z_i, z_j) ];$$

• if $i = \overline{1}$ or $j = 0$ or $(i, j) = (0, \overline{1})$, then $\omega_{D,e} = 0$.

Then, as above,

$$\omega_{D,\Gamma} := \bigwedge_{e \in E(\Gamma)} \omega_{D,e}$$

defines a differential form on $D^+_{n,m}$.

**Definition 3.5.** The modified weight $W_{D,\Gamma}$ of the directed graph $\Gamma$ is

$$W_{D,\Gamma} := \int_{D^+_{n,m}} \omega_{D,\Gamma}.$$  

3.4. Explicit formulæ for Kontsevich’s and Tsygan’s formality morphisms. We quickly review the construction of i) Kontsevich’s $L_\infty$-quasi-isomorphism $U$, and ii) Shoikhet’s $L_\infty$-quasi-isomorphism $S$ respectively.

3.4.1. The $L_\infty$-quasi-isomorphism $U$. For any pair of non-negative integers $(n, m)$, a **K-admissible graph** $\Gamma$ of type $(n, m)$ is by definition a directed graph without loops and with labels obeying the following requirements:

i) the set of vertices $V(\Gamma)$ is given by $\{ 1, \ldots, n, \overline{1}, \ldots, \overline{m} \}$; vertices labelled by $\{ 1, \ldots, n \}$, resp. $\{ \overline{1}, \ldots, \overline{m} \}$, are called vertices of the first, resp. second, type;

ii) every edge in $E(\Gamma)$ starts at some vertex of the first type and there is at most one edge between any two distinct vertices of $\Gamma$.

For a given vertex $v$ of $\Gamma$, we denote by $\text{star}(v)$ the subset of $E(\Gamma)$ of edges starting at $v$: then, we assume that, for any vertex of the first type $v$ of $\Gamma$, the elements of $\text{star}(v)$ are labelled as $\{ e^1_v, \ldots, e^{\text{star}(v)}_v \}$. By definition, the **valence** of a vertex $v$ is the cardinality of the star of $v$. The set of K-admissible graphs of type $(n, m)$ is denoted by $G^K_{n,m}$.

Remark 3.6. In the following we will use integral weights associated to graphs introduced in the previous Section. We can restrict ourselves safely to K-admissible graphs such that $|E(\Gamma)| = 2n + m - 2$ as one can easily see that the weights vanish in any other situation.

Finally, we define the $n$-th structure map $U_n$ of Kontsevich’s $L_\infty$-quasi-isomorphism by

$$U_n := \sum_{m \geq 0} \sum_{\Gamma \in G^K_{n,m}} W_{T^\Gamma} : T^\Gamma_{\text{poly}}(V) \rightarrow D^\Gamma_{\text{poly}}(V)[1 - n],$$

where $U_n(\alpha_1, \ldots, \alpha_n)$ is a $m$-polydifferential operator naturally associated to the graph $\Gamma$ and polyvector fields $\alpha_1, \ldots, \alpha_n$, as defined in [20] (see also [8, Appendix A.8]).

**Theorem 3.7** (Kontsevich). The Taylor components (28) combine to an $L_\infty$-quasi-isomorphism

$$U : T^\Gamma_{\text{poly}}(V) \rightarrow D^\Gamma_{\text{poly}}(V)$$

of $L_\infty$-algebras, whose first order Taylor component reduces to the Hochschild–Kostant–Rosenberg quasi-isomorphism in cohomology.

The complete proof of Theorem 3.7 is given in [20]: the main argument of the proof relies on a clever use of Stokes’ Theorem to derive quadratic identities for the weights (24) of (28), which in turn imply the quadratic identities for (28), corresponding to the fact that $U$ is an $L_\infty$-morphism.

3.4.2. The $L_\infty$-quasi-isomorphism $S$. The construction of $S$ is similar, in principle, to the construction sketched in the previous Subsection, but presents certain subtleties, which we need to discuss also for later purposes.

An **S-admissible graph** of type $(n, m)$ is a directed labelled graph $\Gamma$ without loops and such that:

i) the set of vertices $V(\Gamma)$ is given by $\{ 0, \ldots, n, \overline{1}, \ldots, \overline{m} \}$; vertices labelled by $\{ 1, \ldots, n \}$, resp. $\{ \overline{1}, \ldots, \overline{m} \}$, are called vertices of the first, resp. second, type;

ii) every edge in $E(\Gamma)$ starts at some vertex of the first type and there is at most one edge between any two distinct vertices of $\Gamma$;

iii) there is no edge ending at the special vertex $0$.

The set of S-admissible graphs of type $(n, m)$ is denoted by $G^S_{n,m}$. 
Remark 3.8. As above we can restrict ourselves safely to $S$-admissible graphs such that $|\mathcal{E}(\Gamma)| = 2n + m - 1$, as the modified weights vanish in any other situation.

We now consider an $S$-admissible graph in $\mathcal{G}^S_{n,m}$, such that $|\text{star}(0)| = l$. To $n$ polyvector fields $\{\gamma_1, \ldots, \gamma_n\}$ on $V$, such that $|\text{star}(k)| = |\gamma_k| + 1$, $k = 1, \ldots, n$, and to a Hochschild chain $c = (a_0|a_1| \cdots |a_{m-1})$ of degree $-m + 1$, we associate an $l$-form on $V$ (whose actual degree is $-l$, following the grading in [24]) defined via

\begin{equation}
(\alpha, S_\Gamma(\gamma_1, \ldots, \gamma_n; c)) := U_\Gamma(\alpha, \gamma_1, \ldots, \gamma_n)(a_0, \ldots, a_{m-1}).
\end{equation}

The graded vector space $C^\text{poly}(\cdot)$ with Hochschild differential and DGLA action $L$ over the DGLA $D^\text{poly}(\cdot)$ has a structure of a DGM: if we compose $L$ with the $L^\infty$-quasi-isomorphism $U$, $C^\text{poly}(\cdot)$ becomes an $L^\infty$-module over the DGLA $T^\text{poly}(\cdot)$. The $n$-th Taylor component $S_n$ of $S$ from the $L^\infty$-module $C^\text{poly}(\cdot)$ to the $L^\infty$-module $\Omega^\bullet(\cdot)$ (actually, this is a true DGM, with trivial differential and action $L$ given by the Lie derivative w.r.t. polyvector fields) over $T^\text{poly}(\cdot)$ is given by

\begin{equation}
S_n := \sum_{m \geq 1} \sum_{\Gamma \in \mathcal{G}^S_{n,m}} W_{D,\Gamma} S_\Gamma : \bigwedge^n T^\text{poly}(\cdot) \otimes C^\text{poly}(\cdot) \rightarrow \Omega^\bullet(\cdot)[-n].
\end{equation}

Theorem 3.9 (Shoikhet). The Taylor components (30) combine to an $L^\infty$-quasi-isomorphism

\begin{equation}
S : C^\text{poly}(\cdot) \rightarrow \Omega^\bullet(\cdot)
\end{equation}

of $L^\infty$-modules over $T^\cdot(\cdot)$, whose $0$-th order Taylor component reduces to the Hochschild–Kostant–Rosenberg quasi-isomorphism in homology.

We refer to [24] for a complete proof of Theorem 3.9: the proof of the quadratic identities satisfied by the weights (27) of (30) can be found in [24], and relies again on a clever use of Stokes’ Theorem.

4. The compatibility between cup products

We borrow the notation from Sections 1, 2 and 3: in particular, $(m,d_m)$ is as in the introduction, and accordingly we consider the twisted DGLAs $T^m_{\text{poly}}(\cdot)$, $D^m_{\text{poly}}(\cdot)$ with corresponding new gradings, products, differentials etc.

We further consider a general MCE $\gamma$ in $T^\text{poly}(\cdot)$

\begin{equation}
\gamma = \gamma_{-1} + \gamma_0 + \gamma_1 + \gamma_2 + \cdots,
\end{equation}

where the suffix refers to the polyvector degree, which satisfies the MC equation

\begin{equation}
d_m \gamma + \frac{1}{2}[\gamma, \gamma] = 0.
\end{equation}

We denote by $U(\gamma)$ its image w.r.t. the (extension of the) $L^\infty$-quasi-isomorphism $U$ of Theorem 3.7, Subsubsection 3.4.1, i.e.

\begin{equation}
U(\gamma) = \sum_{n \geq 1} \frac{1}{n!} U_n(\gamma, \ldots, \gamma).
\end{equation}

It is a MCE in $D^m_{\text{poly}}(\cdot)$, with infinitely many components of different polydifferential operator degree.

4.1. The homotopy argument for the cup product. We consider Kontsevich’s eye $C_{2,0}$, and a smooth curve $\ell$ therein, with starting point $\ell(0)$ on the pupil $C_2$ and final point $\ell(1)$ in any one of the boundary strata of codimension 2, $C^+_{0,2}$, and such that $\ell(t)$ in $C_{2,0}$, for $t$ in $(0, 1)$, e.g.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{The curve $\ell$ in $C_{2,0}$}
\end{figure}
More generally, for any pair of non-negative integers \((n, m)\), such that \(n \geq 2\) (hence, automatically, \(2n + m - 2 \geq 0\)), we consider the subset \(Z_{n,m}^+\) of \(C_{n,m}^+\), which consists of those configurations, whose projection onto \(C_{2,0}\) through \(\pi_{1,2}\) is in \(\ell\).

Subsets of the form \(Z_{n,m}^+\) were introduced in [20], and they were analyzed more carefully in [21]: they are actually submanifolds with corners of \(C_{n,m}^+\) of codimension 1, and they inherit an orientation from the orientation of \(\ell\) and of the spaces \(C_{n,m}^+\) themselves, as shown in [21].

Another important feature of \(Z_{n,m}^+\) is the characterization of its boundary: for our purposes, we are interested only in its boundary strata of codimension 1, which are of the following type:

i) Configurations of \(C_{n,m}^+\), such that their projection onto \(C_{2,0}\) is \(\ell(0)\): this boundary stratum is denoted by \(Z_{n,m,0}^+\): explicitly, a general component \(Z\) of \(Z_{n,m,0}^+\) splits as
\[
Z \cong C_A^+ \times C_{n-|A|+1,m}^+,
\]
where \(A\) is a subset of \([n]\) with \(2 \leq |A| \leq n\), which contains the points labelled by 1 and 2, and \(C_A^+\) denotes the subset of \(C_A\), such that the projection onto \(C_2 \cong S^1\) (corresponding to the points 1 and 2) is a fixed point of \(S^1\).

ii) Configurations of \(C_{n,m}^+\), such that their projection onto \(C_{2,0}\) is \(\ell(1)\): this boundary stratum is denoted by \(Z_{n,m,1}^+\). A general component \(Z\) of \(Z_{n,m,1}^+\) splits as
\[
Z \cong C_{A_1,B_1}^+ \times C_{A_2,B_2}^+ \times C_{A_3,B_3}^+,
\]
where \(1 \leq |A_1| \leq n\), \(0 \leq |B_1| \leq m\), \(1 \leq |A_2| \leq n\), \(0 \leq |B_2| \leq m\), \(0 \leq |A_3| \leq n\), \(2 \leq |B_3| \leq m\), \(1 \in A_1\) and \(2 \in A_2\).

iii) Non-trivial intersections of boundary strata of codimension 1 of \(C_{n,m}^+\) with the interior \(Z_{n,m}^0\) of \(Z_{n,m}^+\): this we denote by \(Z_{n,m}^{+0}\). We observe that, in this case, the first and second point of \(C_{n,m}^+\) are distinct and lie on the curve \(\ell\). The explicit form of a general component \(Z\) of \(Z_{n,m}^{+0}\) will be described explicitly later on.

Pictorially, configurations of points in the boundary strata of \(Z_{n,m}^+\) of type i) and ii) look like as follows:

![Figure 13 - Typical configurations in the boundary strata of \(Z_{n,m}^+\) of type i) and ii)\]

For a MCE \(\gamma\) as in (31) and any two \(m\)-valued polyvector fields \(\alpha, \beta\) on \(V\), we will construct a bilinear map \(\mathcal{H}_{\gamma}^{U}\) from \(T^m_{\text{poly}}(V) \otimes T^m_{\text{poly}}(V)\) to \(D^m_{\text{poly}}(V)\) such that the following identity holds true,
\[
\mathcal{U}_\gamma (\alpha \cup \beta) - \mathcal{U}_\gamma (\alpha) \cup \mathcal{U}_\gamma (\beta) = \mathcal{H}_{\gamma}^{U} (\langle d_m \alpha + [\gamma, \alpha], \beta \rangle + (-1)^{|\alpha|} \mathcal{H}_{\gamma}^{U} (\langle d_m \beta + [\gamma, \beta] \rangle + \langle d_m + d_H + L_H(\gamma) \rangle \mathcal{H}_{\gamma}^{U} (\alpha, \beta)).
\]

(32)

First of all, for a non-negative integer \(m\), we define
\[
\mathcal{H}_{\gamma}^{U, m} (\alpha, \beta) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in G^K_{n+2, m}} \mathcal{W}_\Gamma \mathcal{U}_\Gamma (\alpha, \beta, \gamma, \ldots, \gamma),
\]
where now the weight \(\mathcal{W}_\Gamma\) of an admissible graph \(\Gamma\) in \(G^K_{n+2, m}\) is
\[
\mathcal{W}_\Gamma = \int_{Z_{n+2, m}^+} \omega_\Gamma,
\]
with the same notations as above. Finally, we set
\[
\mathcal{H}_{\gamma}^{U} (\alpha, \beta) = \sum_{m \geq 0} \mathcal{H}_{\gamma}^{U, m} (\alpha, \beta).
\]
We want to reinterpret (32) in terms of equalities between weights: we observe that the \( m \)-valued polyvector fields \( \alpha \) and \( \beta \) are put at the first and second vertex of the first type of any graph \( \Gamma \) appearing in the morphisms \( \mathcal{U}_n \).

We first observe that both expressions on the left-hand side of (32) can be re-written as

\[
\mathcal{U}_n(\alpha \cup \beta) = \sum_{i \geq 0} \frac{1}{i!} \mathcal{U}_{i+2}^0(\alpha, \beta, \gamma_1, \ldots, \gamma_i),
\]

\[
\mathcal{U}_n(\alpha) \cup \mathcal{U}_n(\beta) = \sum_{i \geq 0} \frac{1}{i!} \mathcal{U}_{i+2}^1(\alpha, \beta, \gamma_1, \ldots, \gamma_i),
\]

where the morphisms \( \mathcal{U}_{i+2}^0, i = 0, 1, n \geq 0 \), are defined as in (28), the only difference being that the weights (25) have been replaced by

\[
W_i^\gamma = \int_{Z_{n+2,m}^+} \omega_\Gamma, i = 0, 1,
\]

for any graph \( \Gamma \) in \( \mathcal{G}_{n+2,m}^K \). We refer to [3, 21] for an explicit proof of (33) and (34), which we omit here.

Stokes' Theorem implies the following identity between weights:

\[
W_1^\gamma - W_0^\gamma = \overline{W}_\Gamma := \int_{Z_{n+2,m}^+} \omega_\Gamma,
\]

for any graph \( \Gamma \) in \( \mathcal{G}_{n+2,m}^K \). Hence, the proof of (32) is equivalent to evaluating explicitly the weights on the right-hand side of the previous identity.

We inspect more carefully the weights \( \overline{W}_\Gamma \), for a general graph \( \Gamma \) in \( \mathcal{G}_{n+2,m}^K \). For this purpose, we consider \( Z_{n+2,m}^+ \) as was observed earlier, \( Z_{n+2,m}^+ \) is the intersection of boundary strata of codimension 1 of \( C_{n+2,m}^+ \) with the interior of \( Z_{n+2,m}^+ \). Hence, the possible strata \( Z \) of \( Z_{n+2,m}^+ \) are of the following form, recalling that in \( Z_{n+2,m}^+ \), the first two points in \( \mathcal{H} \) remain distinct and lie on the curve \( \gamma \):

\( i \) there is a subset \( A \) of \( [n + 2] \), which contains at most one of the first two points of \( C_{n+2,m}^+ \), such that

\[
Z \cong (C_A \times C_{n-|A|+3,m}) \cap \mathcal{Z}_{n+2,m}^+ = C_A \times \mathcal{Z}_{n-|A|+3,m}^+,
\]

where \( 2 \leq |A| \leq n + 1 \).

\( ii \) There are a subset \( A \) of \( [n + 2] \) and a subset \( B \) of \( [m] \) of successive integers, such that \( A \) does not contain neither the first nor the second point of \( C_{n+2,m}^+ \), such that

\[
Z \cong (C_{A,B}^+ \times C_{n-|A|+2,m-|B|+1}) \cap \mathcal{Z}_{n+2,m}^+ = C_{A,B}^+ \times \mathcal{Z}_{n-|A|+2,m-|B|+1}^+,
\]

where \( 0 \leq |A| \leq n \) and \( 0 \leq |B| \leq m \).

\( iii \) There is a subset \( A \) of \( [n + 2] \), which contains both the first two points of \( C_{n+2,m}^+ \), such that

\[
Z \cong (C_{A,B}^+ \times C_{n-|A|+2,m-|B|+1}) \cap \mathcal{Z}_{n+2,m}^+ = C_{A,B}^+ \times \mathcal{Z}_{n-|A|+2,m-|B|+1}^+,
\]

where \( 2 \leq |A| \leq n + 2 \) and \( 0 \leq |B| \leq m \).

We consider now the restriction of weights to strata of type \( i \), \( ii \) and \( iii \) of \( Z_{n+2,m}^+ \).

4.1.1. Strata of type \( i \). Graphically, a typical configuration of points in a general component of the boundary stratum \( Z_{n,m}^+ \) of type \( i \) looks like as follows:

![Figure 14 - A typical configuration in a general component of the boundary stratum of Z_{n,m}^+ of type i)](image)
We have two subcases of \(i\), namely, when \(i_1\) exactly one of the first two points is in \(A\), or \(i_2\) neither of them is in \(A\). Any weight (25) splits as
\[
\int_Z \omega_\Gamma = \int_{C_A} \omega_{\Gamma A} \int_{Z_{n-|A|+3,m}} \omega_{\Gamma A} ,
\]
where \(\Gamma_A\), resp. \(\Gamma^A\), denotes the subgraph of \(\Gamma\), whose vertices are labelled by \(A\) and whose edges have both endpoints in \(A\), resp. obtained by contracting the subgraph \(\Gamma_A\) to a single vertex.

By Kontsevich’s Lemma 9.1, Appendix 9, the first integral on the right-hand side does not vanish, only if \(|A| = 2\) and \(\Gamma_A\) consists of a single edge connecting the two points in \(A\), in which case, by Lemma 3.1, it equals 1. Therefore, the graph \(\Gamma^A\) is in \(G_{n+1,m}^K\); the weighted sum of polydifferential operators associated to subgraphs \(\Gamma^A\) corresponds either to the action of \(\alpha\) or \(\beta\) on \(\gamma\), in case \(i_1\), or, in case \(i_2\), to the action of \(\gamma\) on itself w.r.t. the Schouten–Nijenhuis brackets.

The sum over all possible admissible graphs \(\Gamma\), whose splitting as above is non-trivial, of the corresponding weights and polydifferential operators yields the first two terms of (32), up to \(d_\alpha\), and polydifferential operators containing the Schouten–Nijenhuis brackets of \(\gamma\) with itself.

4.1.2. *Strata of type \(ii\).* For a better understanding, here is the graphical representation of a typical configuration of points in a general component of the boundary stratum \(Z_{n,m}^+\) of type \(ii\):

![Figure 15 - A typical configuration in a general component of \(Z_{n,m}^+\) of type \(ii\)](image)

We consider the case \(ii\): for any graph \(\Gamma\) in \(G_{n+2,m}^K\), the weight \(\tilde{W}_\Gamma\) restricted to a component \(Z\) of \(Z_{n+2,m}\) splits as
\[
\int_Z \omega_\Gamma = \int_{\tilde{Z}_{n-|A|+2,m-|B|+1}} \left( \int_{C_{A,B}} \omega_{\Gamma A,B} \right) \omega_{\Gamma A,B} ,
\]
with notations similar to those in Subsubsection 4.1.1.

By Lemma 3.1, there can be no outgoing edge from \(\Gamma_{A,B}\); thus, the subgraph \(\Gamma_{A,B}\) is in \(G_{|A|,|B|}^K\), and so is \(\Gamma^{A,B}\), and we have the splitting
\[
\int_Z \omega_\Gamma = \int_{C_{A,B}} \omega_{\Gamma A,B} \int_{\tilde{Z}_{n-|A|+2,m-|B|+1}} \omega_{\Gamma A,B} .
\]

Therefore, the sum over all possible admissible graphs \(\Gamma\) (whose splitting as above is non-trivial) of the corresponding weights and polydifferential operators yields \(H_C^C(\alpha, \beta)\{\tilde{\gamma}\}\), recalling the brace identities and \(\tilde{\gamma} := \mu + U(\gamma)\). We observe that the standard multiplication \(\mu\) appears, when \(A_3 = \emptyset\) and \(B_3 = \{1, 2\}\).

4.1.3. *Strata of type \(iii\).* Here is a pictorial representation of a typical configuration of points in a general component of the boundary stratum of \(Z_{n,m}^+\) of type \(iii\):

![Figure 16 - A typical configuration in a general component of \(Z_{n,m}^+\) of type \(iii\)](image)
For an admissible graph in $G_{n+2,m}$, the weight $\tilde{W}_\Gamma$ restricted to $Z$ splits as

$$\int_Z \omega_\Gamma = \int_{C^+_{n-|A|+2,m-|B|+1}} \left( \int_{Z_{A,B}} \omega_{\Gamma_{A,B}} \right) \omega^{\Gamma_{A,B}},$$

with the same notations as before.

By Lemma 3.1, again, there are no outgoing edges from $\Gamma_{A,B}$, hence both $\Gamma_{A,B}$ and $\Gamma^{A,B}$ are admissible, and in fact we have the splitting

$$\int_Z \omega_\Gamma = \int_{C^+_{n-|A|+2,m-|B|+1}} \omega_{\Gamma_{A,B}} \int_{Z_{A,B}} \omega_{\Gamma^{A,B}}.$$

Therefore, summing up over all possible admissible graphs $\Gamma$, with non-trivial splitting as above, of the corresponding weights and polydifferential operators gives $\tilde{\gamma} \{ \mathcal{H}^S_d(\alpha, \beta) \}$; we observe, once again, that the standard multiplication $\mu$ appears, when $A = [n+2]$ and $|B| = m - 1$.

Finally, since $d_n$ is a differential, the MC equation for $\gamma$ permits to re-insert it in (32), using the Leibniz rule.

### 5. The compatibility between cap products

We now come to the proof of the compatibility between cap product in the case of $X = \mathbb{R}^d$. Borrowing the notation from Sections 4 and 5, we construct a linear operator

$$\mathcal{H}^S_\gamma : T^m_{\text{poly}}(V) \otimes C^{\text{poly},m}(V) \rightarrow \Omega^m(V),$$

(with abuse of notations from Section 4), which is required to satisfy the following homotopy property:

$$\mathcal{S}_\gamma(\mathcal{U}_\gamma(\alpha) \cap \mathcal{S}_\gamma(c)) - \alpha \cap \mathcal{S}_\gamma(c) = (d_m + L_\gamma)(\mathcal{H}^S_d(\alpha, c)) + \mathcal{H}^S_d(d_m\alpha + [\gamma, \alpha], c) + (-1)^{|\alpha|}h^S_d(\alpha, d_m c + b\mathcal{H}^S_\gamma + L_\mathcal{H}^S_\gamma c),$$

(35)

for a MCE $\gamma$ in $T^m_{\text{poly}}(V)$ as in (31), Section 4, a general $m$-valued polyvector field $\alpha$ and a general Hochschild chain $c$ on $A$ with values in $A$, $A$ being the graded algebra of $m$-valued functions on $V$. As usual, we define $\mathcal{H}^S_d$ in terms of graphs and weights associated to them. Namely, with $\alpha$ and $c = (a_0|\cdots|a_m)$ as above $(m \geq 0)$, we have

$$\mathcal{H}^S_d(\alpha, c) := \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in G_{n+1,m+1}} \bigcirc_{\text{n times}} \tilde{W}_{D,\Gamma} \mathcal{S}_\Gamma(\alpha, \gamma, \gamma, \cdots, c),$$

(36)

where $\tilde{W}_{D,\Gamma}$ are certain weights defined as suitable integrals.

#### 5.1. A curve on Kontsevich’s eye $D_{1,1}$ and related configuration spaces

We consider the curve $\ell$ on Kontsevich’s eye $D_{1,1}$, with initial point $\ell(0)$ on $\alpha$, and final point $\ell(1) = b$, which corresponds to the following imbedding of the open unit interval into the open configuration space $D_{1,1}$:

![Figure 17 - The curve $\ell$ in $D_{1,1}$](image)

For the labelling of all boundary strata of the brave new eye, we refer to Subsubsection 3.2.1. We also observe that, under the identification $D_{1,1} = C_{2,0}$, the curve $\ell$ corresponds to the curve on Kontsevich’s Eye of Subsection 4.1.

We consider the subset $Y_{n,m}^+$ of $D_{n,m}^+$ for $n \geq 1$ and $m \geq 1$, consisting of those configurations, whose projection onto $D_{1,1}$ (which extends the natural projection from $D_{n,m}$ onto $D_{1,1}$, onto the first point in $D$ and the first point in $S^1$) belongs to the curve $\ell$. Pictorially,
The dashed line represents the curve, along which the first point in $D$ moves, which we have denoted by $\circ$, connecting it with the first point in $S^1$ (w.r.t. the cyclic order).

The set $Y_{n,m}^+$, for $n \geq 1$, is a smooth submanifold with corners of $D_{n,m}^+$, of codimension 1; it inherits an orientation from the orientation of the curve and of $D_{n,m}^+$.

We need to characterize explicitly the boundary strata of $Y_{n,m}^+$ of codimension 1: this is analogous to what was done in Subsection 4.1, whence the strata consist of

i) configurations in $D_{n,m}^+$, whose projection onto $D_{1,1}$ is $\ell(0)$ (the corresponding strata are denoted collectively by $Y_{n,1,0}^+$);

ii) configurations in $D_{n,m}^+$, whose projection onto $D_{1,1}$ is $\ell(1)$ (the corresponding strata are denoted collectively by $Y_{n,m,1}^+$);

iii) the intersection of boundary strata of codimension 1 of $D_{n,m}^+$ with the interior $\circ Y_{n,m}^+$ of $Y_{n,m}^+$ (the corresponding strata are denoted collectively by $Y_{n,m}^+$).

It is clear that all such boundary strata are submanifolds with corners of $D_{n,m}^+$ of codimension 2.

In the forthcoming Subsections, we prove that $\alpha \cap S_\gamma(c)$, $S_\gamma(U_\gamma(\alpha) \cap c)$, and the r.h.s. of (35) can be expressed via a formula similar to (36), involving new weights $W_{0,\Gamma}^0$, $W_{1,\Gamma}^1$, and $W_{1,\Gamma}$, where, for any S-admissible graph $\Gamma \in \mathcal{G}_n^+$,

$$W_{i,\Gamma}^i := \int_{Y_{n,m,i}^+} \omega_{D,\Gamma}^i \quad i = 0, 1,$$

$$W_{0,\Gamma}^0 = \int_{\circ Y_{n,m}^+} \omega_{D,\Gamma}^0.$$

5.2. A formula for $\alpha \cap S_\gamma(c)$. We first consider the boundary strata $Y_{n,m,0}^+$: it is not difficult to verify that a general component $Z$ of $Y_{n,m,0}^+$ has the form

$$Z \cong D_A^0 \times D_{n-|A|,m}^+,$$

where $A$, $|A| \geq 1$, contains at least the first point in $D$, and $D_A^0$ is a smooth submanifold of $D_A$ of codimension 1, whose elements are configurations in $D_A$, whose projection onto the first point is fixed (since $\ell(0)$ represents in fact a point in $D$ which approaches the origin along a fixed direction in $S^1$).

Graphically,

**Figure 19 - A typical configuration in $Y_{n,m,0}^+$**

**Lemma 5.1.** For any admissible graph $\Gamma$ in $\mathcal{G}_n^+$, the weight $W_{0,\Gamma}^0$ vanishes if the vertex 1 has at least one incoming edge. Otherwise, the identity

$$W_{0,\Gamma}^0 = W_{D,\Gamma_0}$$

holds true, where $\Gamma_0 \in \mathcal{G}_n^+$ is obtained from $\Gamma$ by collapsing the vertices 0 and 1.
Proof. By the above characterization of the components $Z$ of $Y_{n+1,m,0}^+$, for a general admissible graph $\Gamma$ as in the assumptions of Lemma 5.1, we have the following factorization

$$W^0_{D,\Gamma}|_Z = \int_{D^+_{n-|A|+1,m+1}} \left( \int_{D^0_A} \omega_{D,\Gamma_A} \right) \omega_{D,\Gamma^A},$$

where $Z$ is a general component of $Y_{n,m,0}^+$ and where $\Gamma_A$ is the subgraph of $\Gamma$, whose vertices are labelled by $A$ and whose edges have at least one endpoint in $A$, and $\Gamma^A$ is obtained from $\Gamma$ by contracting $\Gamma_A$ to the single vertex labelled by $0$.

We focus our attention on the inner integral in the previous factorization.

We first observe that, if there is exactly one edge from 0 to 1, then, by means of Lemma 3.1, a), the integrand vanishes, since $\omega|_{D^1}=c_2$ is the derivative of a constant angle, hence it is trivial.

Further, using Lemma 3.1, a), and Lemma 3.2, c) and f) (and the characterization of the restriction of $\omega_D$ to the boundary stratum $C_3 \times C_{1,0}$ of $C_{3,0}$) to explicitly evaluate the integrand $\omega_{D,\Gamma_A}$ on $\mathcal{D}^0_A$, and by dimensional reasons, all factors in $\omega_{D,\Gamma^A}$ which live on $\mathcal{D}^0_A$ are products of Kontsevich's angle function on $C_2$. We can therefore apply the arguments of the proof of Kontsevich's Lemma 9.1, Appendix 9, whence the only possibly non-trivial integrals appear, when $|A|=1$, i.e. $\mathcal{D}^0_A$ consists of a single point on $S^1$.

Since there are no edges connecting 0 with 1, when $|A| = 1$, there can be some edge from 1 to some other vertices (of first or second type), or some edge with endpoint 1.

We write star(1) = $\{e_1, \ldots, e_i\}$. In the first case, by Lemma 3.2, c), the integrand $\omega_{D,\Gamma_A}$ is a product of the form

$$\omega_{D,\Gamma_A}|_{\mathcal{D}^1_A} = \int_{k=1}^p (\omega_{D,e_1} + \omega|_{C_2}).$$

Again, since Kontsevich’s angle function is constant by construction, $\omega|_{C_2}$ vanishes, the inner integration over a 0-dimensional point may be then discarded, and the only non-trivial factor surviving integration is $\int_{k=1}^p \omega_{D,e_k}$. The previous product can be re-inserted into the remaining integrand $\omega_{D,\Gamma^A}$, and, denoting by $\Gamma_0$ this new graph, the claim follows.

In the second case, Lemma 3.2, d), immediately implies the claim. \(\square\)

Proposition 5.2. For $\gamma$, $\alpha$ and $c$ as above, the following identity holds true:

$$(37) \quad \alpha \cap S_\gamma(c) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}^S_{n+1,m+1}} W^0_{D,\Gamma} S_{\Gamma}(\alpha, \gamma, \ldots, \gamma; c).$$

(Sketch of proof). On the one hand, the l.h.s. of (37) can be re-written as

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma_0 \in \mathcal{G}^S_{n,m+1}} W_{D,\Gamma_0} \iota_\alpha S_{\Gamma_0}(\gamma, \ldots, \gamma; c).$$

On the other hand, we consider the r.h.s. of (37): by Lemma 5.1, the weights $W^0_{D,\Gamma}$ are non-trivial only for those admissible diagrams $\Gamma$, whose vertex labelled by 1 has no incoming edges, in which case the sum simplifies to

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma_0 \in \mathcal{G}^S_{n,m+1}} \sum_{\Gamma_0 \prec \gamma} W_{D,\Gamma_0} S_{\Gamma}(\alpha, \gamma, \ldots, \gamma; c).$$

In the previous expression, the third summation is exactly over those admissible graphs, whose vertex 1 has no incoming edges and whose contraction of the vertices 0 and 1 is the admissible graph $\Gamma_0$.

Finally, we observe that for any fixed graph $\Gamma_0 \in \mathcal{G}^S_{n,m+1}$,

$$\iota_\alpha S_{\Gamma_0}(\gamma, \ldots, \gamma; c) = \sum_{\Gamma_0 \prec \gamma} S_{\Gamma}(\alpha, \gamma, \ldots, \gamma; c),$$

which ends the proof of the proposition. \(\square\)

5.3. A formula for $S_\gamma(\mathcal{U}_\gamma(\alpha) \cap c)$. A general component $Z$ of the boundary stratum $Y_{n,m,1}^+$ has the explicit form

$$(38) \quad Z \cong C^+_{A_1,B_1} \times C^+_{A_2,B_2} \times D^+_A, B_3,$$

where $A_i$, $i = 1, 2, 3$, are disjoint subsets of $[n]$, with $1 \leq |A_1| \leq n$, $0 \leq |A_2| \leq n$, $0 \leq |A_3| \leq n$, with $n = |A_1| + |A_2| + |A_3|$, and $B_i$, $i = 1, 2, 3$, are disjoint ordered subsets of $[m]$ of successive elements, such that $1 \leq |B_1| \leq m$, $2 \leq |B_2| \leq m$, $1 \leq |B_3| \leq m$, and $m = |B_1| + |B_2| + |B_3|$.
Here is a pictorial representation of a typical component \( Z \) of \( Y_{n,m,1}^+ \):

![Figure 20 - A typical configuration in \( Y_{n,m,1}^+ \)](image)

We consider a component \( Z \) of \( Y_{n+1,m+1}^+ \) as in (38), and, for an S-admissible graph \( \Gamma \) in \( G_{n+1,m+1}^S \) as before, we denote by

1. \( \Gamma_1^Z \) the subgraph of \( \Gamma \), whose vertices are labelled by \( A_1 \cup B_1 \);
2. \( \Gamma_2^Z \) the graph with vertices labelled by \( A_2 \cup (B_2 \sqcup \{\ast\}) \), which is the quotient of the subgraph \( \widehat{\Gamma} \) (with vertices labelled by \( (A_1 \sqcup A_2) \cup (B_1 \sqcup B_2) \)) by \( \Gamma_1^Z \), and \( \ast \) corresponds to the contraction of \( \Gamma_1^Z \);
3. \( \Gamma_3^Z \) the graph with vertices labelled by \( (A_3 \sqcup \{0\}) \sqcup (B_3 \sqcup \{1\}) \), which is the quotient of \( \Gamma \) by \( \widehat{\Gamma} \).

**Lemma 5.3.** For a general component \( Z \) of \( Y_{n+1,m+1}^+ \) as in (38) and a general S-admissible graph \( \Gamma \) in \( G_{n+1,m+1}^S \), the identity

\[
\int_Z \omega_{D,\Gamma} = W_{1,\Gamma}^Z W_{2,\Gamma}^Z W_{3,\Gamma}^Z
\]

holds true.

**Proof.** First of all, if there is an edge e.g. from \( A_1 \) to \( A_2 \), we may apply Lemma 3.2, a) or d) to show that the corresponding contribution vanishes; the same argument implies the claim in all other cases. We observe that this also implies that \( \Gamma_i, i = 1, 2, 3 \), is admissible.

Hence, we have the factorization

\[
W_{D,\Gamma}^1|_{C_{A_1,B_1}^+,C_{A_2,B_2}^+,C_{A_3,B_3}^+} = \int_{C_{A_1,B_1}^+} \omega_{D,\Gamma_1} \int_{C_{A_2,B_2}^+} \omega_{D,\Gamma_2} \int_{C_{A_3,B_3}^+} \omega_{D,\Gamma_3}.
\]

Finally, we use Lemma 3.2, c), to reduce the first two factors in the previous factorization to usual Kontsevich’s weights (25): in fact, we have

\[
\omega_{D,\Gamma_i}|_{C_{A_i,B_i}^+} = \omega_{T_i}, \quad i = 1, 2,
\]

and the claim follows directly from the definition of (25).

Hence, Lemma 5.3 implies, more generally, the following factorization property:

\[
W_{D,\Gamma}^1 = \sum_Z W_{1,\Gamma}^Z W_{2,\Gamma}^Z W_{3,\Gamma}^Z,
\]

where \( Z \) runs over components of the type (38) of \( Y_{n+1,m+1,1}^+ \).

**Proposition 5.4.** For \( \gamma, \alpha \) and \( c \) as above, the following identity holds true:

\[
S_\gamma(U_\alpha(\gamma) \cap c) = \sum_{n \geq 0} \sum_{\Gamma \in G_2^S_{n+1,m+1}} W_{D,\Gamma}^1 S_\Gamma(\alpha, \gamma, \ldots, \gamma; c).
\]

**Sketch of proof.** We consider the left-hand side of (41): it can be re-written as

\[
\sum_{n_1,n_2,n_3 \geq 0} \frac{1}{n_1!n_2!n_3!} \sum_{\Gamma_1 \in G_{n_1+1,m+1}^S} \sum_{\Gamma_2 \in G_{n_2+2,m+2}^S} \sum_{\Gamma_3 \in G_{n_3+3,m+3}^S} W_{\Gamma_1} W_{\Gamma_2} W_{D,\Gamma_3} \sum_{0 \leq k \leq p \leq m_1+m_2+m_3} W_{U_\Gamma(\gamma, \ldots, \gamma)_{n_3}}(a_{p+1}, \ldots, a_k, U_{\Gamma_1}(\alpha, \gamma, \ldots, \gamma)_{n_1}(a_{k+1}, \ldots, a_{k+m_1}, \ldots, a_{n_1})| \ldots | a_p).
\]
As for the right-hand side of (37), we apply Lemma 5.3:
\[
\sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in G_{n+1,m+1}^{D}} \sum_{Z} W_{\Gamma_{1}^{Z}} W_{\Gamma_{2}^{Z}} W_{D,\Gamma_{3}^{Z}} S_{\Gamma}(\alpha, \gamma, \ldots, \gamma; c),
\]
where, for an admissible graph $\Gamma$, $Z$ runs over all possible decompositions of $\Gamma$ into three admissible graphs as above. Finally, for any triple $(\Gamma_{1}, \Gamma_{2}, \Gamma_{3})$ and any component $Z$ of $\mathcal{Y}_{n+1,m+1}^{1}$ as above, one can show that\(^6\)
\[
S_{\Gamma_{3}}(\gamma, \ldots, \gamma; (U_{\Gamma_{2}}(\gamma, \ldots, \gamma) (a_{p+1}, \ldots, a_{k}, U_{\Gamma_{1}}(\alpha, \gamma, \ldots, \gamma) (a_{k+1}, \ldots, a_{k+m_{1}}), \ldots, a_{l})) \cdots a_{p}) = \sum_{\Gamma \in G_{n+1,m+1}^{D}, \Gamma_{2} \Gamma_{1}^{Z}} S_{\Gamma}(\alpha, \gamma, \ldots, \gamma; c).
\]
We observe that the component $Z$ determines the indices $k, l, p$. To finish the proof of the Proposition, it remains to compute the number of elements in the sum of the r.h.s. of the last identity: we let the reader check that it is precisely $\frac{n!}{n_{1}! n_{2}! n_{3}!}$.

\[\square\]

**Remark 5.5.** In the case $m \geq 1$, the projection $\mathcal{D}_{n+1,m+1}^{+} \to \mathcal{D}_{1,1}^{+}$ factors through $\mathcal{D}_{n+1,m+1}^{+} \to \mathcal{D}_{1,2}^{+} \to \mathcal{D}_{1,1}^{+}$. Moreover, the inverse image of $\ell(1)$ through the last projection consists of the union of the following components of the I-cube (see Figure 8): $\ell, q, p, o$ and $h$. The detailed contribution of the inverse image of each of these components through $\mathcal{D}_{n+1,m+1}^{+} \to \mathcal{D}_{1,2}^{+}$ is as follows: $j)$ $k > 0$ and $p \neq m$, $q)$ $k = 0$, $m_{1} \neq 0$ and $p \neq m$, $p)$ $m_{1} = 0$ and $m_{2} \neq 0$, $o)$ $m_{1} = m_{2} = 0$ and $p \neq m$, $h)$ $p = m$ and $m_{1} = m_{2} = 0$.

### 5.4. The homotopy formula.
Summarizing the results of Propositions 5.2 and 5.4, we may write
\[
\mathcal{S}_{\gamma}(\mathcal{U}_{\gamma}(\alpha) \cap c) - \alpha \cap \mathcal{S}_{\gamma}(c) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in G_{n+1,m+1}^{D}} (W_{\Gamma_{1}^{Z}} - W_{\Gamma_{2}^{Z}}) S_{\Gamma}(\alpha, \gamma, \ldots, \gamma; c).
\]
Then, for any S-admissible graph $\Gamma \in G_{n,m}^{S}$, we have
\[
0 = \int_{\gamma_{n,m}^{+}} d\omega_{D,\Gamma} = \int_{\partial \gamma_{n,m}^{+}} \omega_{D,\Gamma} = W_{\Gamma_{1}^{Z}} - W_{\Gamma_{2}^{Z}} - W_{D,\Gamma},
\]
where the second equality is a consequence of Stokes' Theorem. It thus remains to analyze the contributions coming from integration over components of $Y_{n,m}^{+}$, which are of three types: 

i) there is a subset $A$ of $[n]$ which may or may not contain the vertex 1, such that
\[
Z \cong (\mathcal{C}_{A} \times \mathcal{D}_{n-|A|+1,m}^{+}) \cap \gamma_{n,m}^{+} \cong \mathcal{C}_{A} \times \gamma_{n-|A|+1,m}^{+}
\]
and $2 \leq |A| \leq n$.

ii) There is a subset $A$ of $[n]$, which does not contain the vertex 1, such that
\[
Z \cong (\mathcal{D}_{A} \times \mathcal{D}_{n-|A|,m}^{+}) \cap \gamma_{n,m}^{+} \cong \mathcal{D}_{A} \times \gamma_{n-|A|,m}^{+}.
\]

iii) There is a subset $A$ of $[n]$, which does not contain the vertex 1, and an ordered subset $B$ of successive elements in $[m]$, such that
\[
Z \cong (\mathcal{C}_{A,B}^{+} \times \mathcal{D}_{n-|A|,m-|B|+1}^{+}) \cap \gamma_{n,m}^{+} \cong \mathcal{C}_{A,B}^{+} \times \gamma_{n-|A|,m-|B|+1}^{+}.
\]
Namely, such components are intersections of boundary strata of codimension 1 of $\mathcal{D}_{n,m}^{+}$ with the interior of $\pi^{-1}(\ell(0,1])$. Hence, a configuration in $Y_{n,m}^{+}$ is such that the first point in $D_{\gamma}$ approaches neither the origin nor $S^{1}$.

For any S-admissible graph $\Gamma \in G_{n,m}^{S}$ and any component $Z$ of $Y_{n,m}^{+}$, we write
\[
W_{D,\Gamma}^{Z} := \int_{Z} \omega_{D,\Gamma}.
\]

\[^6\]Here we assume that $n_{1} + n_{2} + n_{3} = n$ and $m_{1} + m_{2} + m_{3} = m + 1$. 

5.4.1. **Contribution of components of type i**. We consider components of type i of \(Y_{n,m}^+\): to get a better understanding of them, here is a pictorial representation of possible configurations in two distinct general components

![Figure 21 - Two distinct typical configurations of type i) in \(Y_{n,m}^+\)](image)

**Lemma 5.6.** For an \(S\)-admissible graph \(\Gamma \in \mathcal{G}_{n+1,m+1}^S\) and a component \(Z\) of \(Y_{n+1,m+1}^+\) of type i) as before, the weight \(W^Z_{D,\Gamma}\) is non-trivial only if \(|A| = 2\) and there is exactly one edge connecting the vertices labelled by \(A\), in which case we have

\[
W^Z_{D,\Gamma} = \hat{W}_{\Gamma^A} := \int_{\mathcal{Y}_{n,m+1}^A} \omega_{D,\Gamma^A},
\]

where \(\Gamma^A\) is obtained from \(\Gamma\) by contracting the vertices labelled by \(A\) and eliminating the edge between them.

**Proof.** First of all, we have the following factorization of (42):

\[
W^Z_{D,\Gamma} = \int_{\mathcal{Y}_{n-|A|+1,m+1}^A} \left( \int_{\mathcal{C}_A} \omega_{D,\Gamma^A} \right) \omega_{D,\Gamma^A},
\]

where \(\Gamma_A\), resp. \(\Gamma^A\), is the subgraph of \(\Gamma\), whose vertices are labelled by \(A\) and whose edges have at least one endpoint in \(A\), resp. the graph obtained from \(\Gamma\) by contracting \(\Gamma_A\) to a single vertex.

Using Lemma 3.2, \(f\), we re-write the first factor in the previous factorization as

\[
\int_{\mathcal{C}_A} \omega_{\Gamma^A},
\]

if \(|E_{\Gamma_A}| \geq 2\), whence, by Kontsevich's Lemma 9.1, we conclude that it vanishes.

Thus, \(\Gamma_A\) can have at most one edge: dimensional reasons imply that \(|A| = 2\). The corresponding integral does not vanish iff \(\Gamma_A\) consists of a single edge connecting the two vertices labelled by \(A\): by Lemma 3.2, \(f\), the contribution of such an integral is 1 (since the integral over \(C_2 = S^1\) of the piece \(-\pi^2\omega\) of the restriction of \(\omega_D\) vanishes, as \(\pi^2\omega\) is not on \(S^1\)). It is also clear that, in this case, \(\Gamma^A\) is an admissible graph in \(\mathcal{G}_{n-1,m+1}^S\). \(\Box\)

**Proposition 5.7.** For \(\gamma, \alpha\) and \(c\) as above, the following identity holds true:

\[
\sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}_{n+1,m+1}^S} \sum_Z W^Z_{D,\Gamma} S_T(\alpha, \ldots, \gamma; c) =
\]

\[
= \frac{1}{2} \sum_{n \geq 2} \frac{1}{(n-2)!} \sum_{\Gamma \in \mathcal{G}_{n,m+1}^S} \tilde{W}_{D,\Gamma} S_T(\alpha, \gamma, \ldots, \gamma; c) +
\]

\[
+ \sum_{n \geq 1} \frac{1}{(n-1)!} \sum_{\Gamma \in \mathcal{G}_{n-1,m+1}^S} \tilde{W}_{D,\Gamma} S_T(\gamma, \alpha, \gamma, \ldots, \gamma; c),
\]

where \(Z\) runs over components of type i) of \(Y_{n+1,m+1}^+\).

**Sketch of proof.** By Lemma 5.6, the only components \(Z\) of type i) yielding non-trivial weights are those of the form \(\mathcal{C}_A \times \mathcal{Y}_{n,m+1}^+\), where \(|A| = 2\): thus, the left-hand side of (43) can be re-written as

\[
\sum_{n \geq 0} \frac{1}{n!} \sum_A \sum_{\Gamma^A \in \mathcal{G}_{n,m+1}^S} \sum_{\Gamma \in \mathcal{G}_{n+1,m+1}^S} \tilde{W}_{D,\Gamma^A} S_T(\alpha, \ldots, \gamma; c),
\]

where the notation \(\Gamma_A \times \Gamma\) means that \(\Gamma^A\) is obtained from \(\Gamma\) by collapsing the vertices of \(\Gamma\) labelled by \(A\) and the only edge between them; the second sum is over all \(A \subset [n]\) such that \(|A| = 2\) (i.e. over the above components \(Z\) of type i)).
Terms involving $[\gamma, \alpha]$, resp. $\frac{1}{2}[\gamma, \gamma]$, correspond to components $Z$ for which $A$ contains 1, resp. does not contain 1.

Now, combining the MC equation for $\gamma$ with Leibniz’s rule, we get

$$(n - 1)S_T(\alpha, [\gamma, \gamma], \gamma, \ldots, \gamma; c) = d_m\left(S_T(\alpha, [\gamma, \gamma], \gamma, \ldots, \gamma; c)\right) + S_T\left(d_m(\alpha), [\gamma, \gamma], \gamma, \ldots, \gamma; c\right) \pm S_T\left(\alpha, [\gamma, \gamma], \gamma, \ldots, \gamma; d_m(c)\right),$$

which implies, thanks to Proposition 5.7, that

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in G_{n+1,m+1}} \sum_{Z_n \in Z} W_D^Z S_T(\alpha, [\gamma, \ldots, \gamma]; c) = d_m(H_1^S(\alpha, c)) + H_1^S(d_1(\alpha, c)) \pm H_1^S(\alpha, d_m(c)), \quad (44)$$

5.4.2. Contribution of components of type ii). We begin with a pictorial representation of a general configuration in a possible component $Z$ of type ii) of $Y_{+}^{n,m}$:

![Diagram](https://via.placeholder.com/150)

**Figure 22 - A typical configurations of type ii) in $Y_{+}^{n,m}$**

**Lemma 5.8.** For an $S$-admissible graph $\Gamma \in G_{n+1,m+1}^S$ and a component $Z$ of $Y_{+}^{n+1,m+1}$ of type ii) as before, the weight $W_{D,\Gamma}^Z$ is non-trivial only if $|A| = 1$ and if either a) there is exactly one edge from 0 to the vertex labelled by $k$, or b) there is no edge from 0 to $k$ and there is at least one edge with starting point labelled by $k$. We then have

$$W_{D,\Gamma}^Z = W_{D,\Gamma,A}^0, \text{ in case a)},$$

$$W_{D,\Gamma}^Z = \sum_{e_{v_A}^l \in \text{star}(v_A)} \pm W_{D,\Gamma,A,l}^0, \text{ in case b),}$$

where $\Gamma_A$, resp. $\Gamma_{+A,l}$, is obtained from $\Gamma$ by contracting 0 and the vertex labelled by $A$ and eliminating the edge from 0 to this vertex, resp. eliminating the edge $e_{v_A}^l$.

**Proof.** The decomposition of the component $Z$ implies the factorization of the corresponding weight,

$$W_{D,\Gamma}^{Z,ii} = \int_{\gamma_{n-|A|+1,m+1}}^{\gamma_{+}} \left(\int_{D_A} \omega_{D,\Gamma_A}\right) \omega_{D,\Gamma_A},$$

where now $\Gamma_A$ denotes the subgraph of $\Gamma$, whose vertices are labelled by $A$ and whose edges have at least one vertex in $A$, and $\Gamma_A$ is obtained from $\Gamma$ by contracting $\Gamma_A$ to a single vertex.

We focus on the first factor: first of all, we use Lemma 3.1, a), and Lemma 3.2, c) and d) (and again the restriction of $\omega_D$ to $C_3 \times C_{1,0}$) to evaluate the restriction of $\omega_{D,\Gamma_A}$ on $D_A$. All factors of this restriction, which live on $D_A$, are pull-backs of Kontsevich’s angle form $\omega$ to $C_2$: hence, we may apply Kontsevich’s Lemma 9.1 to conclude that all such contributions vanish unless $|A| = 1$.

If $|A| = 1$ (the corresponding vertex is denoted by $v_k$), and there is at least one edge, whose endpoint is $v_k$, Lemma 3.2, d), implies immediately that the corresponding integral vanishes.

We assume now $\text{star}(v_A) = \{e_{v_A}^1, \ldots, e_{v_A}^P\}$. If there is exactly one edge from 0 to $v_A$, Lemma 3.2, c), implies that $\omega_{D,\Gamma_A}$ is a product of the form (forgetting about signs)

$$\omega_{D,\Gamma_A}|_{D_A} = \omega|_{c_2} \wedge \bigwedge_{k=1}^P (\omega_{D,e_{v_A}^k} + \omega|_{c_2}) = \omega|_{c_2} \wedge \bigwedge_{k=1}^P \omega_{D,e_{v_A}^k},$$

by the antisymmetry of the wedge product. The inner integration over $S^1$ produces hence a factor 1, and the form $\bigwedge_{k=1}^P \omega_{D,e_{v_A}^k}$ can be inserted into the outer factor $\omega_{\Gamma_A}$. 
If there is no edge from 0 to \( v_A \), by Lemma 3.2, \( \omega_{D,R_A} \) is a product (again forgetting about signs)

\[
\omega_D, R_A|_{D_A} = \prod_{k=1}^{p} (\omega_{D,e^i_k} + \omega_{c_2}) = \prod_{k=1}^{p} (-1)^{k-1} \omega_{c_2} \land \bigwedge_{j \neq k} \omega_{D,e^j_k},
\]

again by the antisymmetry of the wedge product, since we need the factor \( \omega_{c_2} \) because of the integral. The inner integration over \( S^1 \) produces hence a factor 1, and the forms \( \bigwedge_{j \neq k} \omega_{D,e^j_k} \), \( k = 1, \ldots, p \) can be inserted into the outer factor \( \omega_{R_A} \), and the claim follows. □

**Proposition 5.9.** For \( \gamma, \alpha \) and \( c \) as above, the following identity holds true:

\[
(45) \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}_{n+1,m+1}} \sum_Z W^Z_{D,R}(\alpha, \gamma, \ldots, \gamma; c) = L_\gamma(H_\gamma(\alpha, c)),
\]

where \( Z \) runs over components of type ii) of \( Y^{+}_{n+1,m+1} \).

**Proof.** Using Lemma 5.8, we re-write the left-hand side of (45) as

\[
\sum_{n \geq 0} \frac{1}{n!} \sum_{k=2}^{n+1} \sum_{r \in \mathcal{G}_{n,m+1}} \sum_{r^k \in \mathcal{G}_{n,m+1}} \omega_{D,R}^k S_\Gamma(\alpha, \gamma, \ldots, \gamma; c) + \sum_{n \geq 0} \frac{1}{n!} \sum_{k=2}^{n+1} \sum_{c_k \in \text{star}(k)} \sum_{r \in \mathcal{G}_{n,m+1}} \sum_{r^k \in \mathcal{G}_{n,m+1}} \pm \omega_{D,R}^k \cdot S_\Gamma(\alpha, \gamma, \ldots, \gamma; c),
\]

borrowing notations from Lemma 5.8. On the other hand, recalling the homotopy formula

\[
L_\gamma = d \circ \iota_\gamma \pm \iota_\gamma \circ d,
\]

and using slight modifications of the arguments of Subsection 4.2 and 4.3 of [21] and in the proof of Proposition 5.2, we have

\[
L_\gamma \left( S_{\Gamma_\alpha}(\alpha, \gamma, \ldots, \gamma; c) \right) = \sum_{k=2}^{n+1} \sum_{r \in \mathcal{G}_{n+1,m+1}} \sum_{r^k \in \mathcal{G}_{n+1,m+1}} \pm S_\Gamma(\alpha, \gamma, \ldots, \gamma; c) + \sum_{k=2}^{n+1} \sum_{c_k \in \text{star}(k)} \sum_{r \in \mathcal{G}_{n+1,m+1}} \sum_{r^k \in \mathcal{G}_{n+1,m+1}} \pm S_\Gamma(\alpha, \gamma, \ldots, \gamma; c),
\]

whence the claim follows. More precisely, the first, resp. second, term on the right-hand side of the previous equality corresponds to the composition \( d \circ \iota_\gamma \), resp. \( \iota_\gamma \circ d \), by an explicit evaluation of the contraction operation and by means of Leibniz’s rule for \( d \). □

**5.4.3. Contribution of components of type iii).** We discuss weights associated to admissible graphs and to components of type iii) of \( Y^{+}_{n,m} \): before entering into the discussion, a pictorial representation of the two distinct possible configurations in such components could be helpful:

![Figure 23 - Two possible configurations in \( Y^{+}_{n,m} \) of type iii)](image)
Lemma 5.10. For an S-admissible graph $\Gamma \in \mathcal{G}_{n+1, m+1}^S$ and a component $Z$ of $Y_{n+1, m+1}^+$ of type iii) as before, the weight $W_{D, \Gamma}^Z$ vanishes unless there are no edges connecting $A$ to its complement. In this case we have

$$W_{D, \Gamma}^Z = W_{\Gamma_{A,B}} W_{\Gamma^{A,B}}^z,$$

where $\Gamma_{A,B}$ denotes the subgraph of $\Gamma$ whose vertices are labelled by $A \sqcup B$ and the graph $\Gamma^{A,B}$ is obtained from $\Gamma$ by contracting $\Gamma_{A,B}$ to a single vertex of the second type.

Proof. The above weight vanishes, if there is at least one edge connecting a vertex labelled by $A$ to a vertex not labelled by $A$ in virtue of Lemma 3.2, a), similarly to the first step in the proof of Lemma 5.3: this forces, by the way, $\Gamma_A$ and $\Gamma_Z$ to be both admissible, since all stars of $\Gamma_A$ and $\Gamma_Z$ belong to $E_{\Gamma_A}$ and $E_{\Gamma_Z}$ respectively.

Additionally, the following factorization of the above weight holds true:

$$W_{D, \Gamma}^Z = \int_{\gamma=0}^{\gamma=2} \omega_{D, \Gamma_{A,B}} \int_{n=1}^{n=1} \omega_{D, \Gamma^{A,B}}.$$

Finally, we use Lemma 3.2, c), to prove that the integrand $\omega_{D, \Gamma_{A,B}}$ equals in fact $\omega_{\Gamma_{A,B}}$, whence the last claim follows from the previous factorization and from the definition of (25).

Proposition 5.11. For $\gamma$, $\alpha$ and $c$ as above, the following identity holds true:

$$(46) \quad \sum_{n \geq 0} \frac{1}{n!} \sum_{\gamma \in \mathcal{G}_{n+1, m+1}^S} \sum_{Z} W_{D, \Gamma}(\alpha, \gamma; \ldots; \gamma; c) = \mathcal{H}_{\mu}^S(\alpha, L_{\mu+\mu}(\gamma)(c)),$$

where $Z$ runs over components of type iii) of $Y_{n+1, m+1}^+$, and $\mu$ denotes the standard multiplication in $A$.

Proof. We use Lemma 5.10 to re-write the left-hand side of (46):

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\gamma \in \mathcal{G}_{n+1, m+1}^S} \sum_{A,B} W_{\Gamma_{A,B}} \mathcal{W}_{D, \Gamma^{A,B}} S_\Gamma(\alpha, \gamma; \ldots; \gamma; c).$$

We fix $n, m \geq 0$, $0 \leq k \leq n$, $0 \leq l \leq m + 1$, and we observe that for pair $(\Gamma_1, \Gamma_2)$ of admissible graphs $\Gamma_1 \in \mathcal{G}_{n,l}^K$ and $\Gamma_2 \in \mathcal{G}_{n-k+1, m-l+2}^S$, one finds

$$\sum_{|A|=k, |B|=l} \sum_{\gamma \in \mathcal{G}_{n+1, m+1}^S} S_\Gamma(\alpha, \gamma; \ldots; \gamma; c) = S_{\Gamma_2}(\alpha, \gamma; \ldots; \gamma; L_{\mu+\mu}(\gamma)(c)).$$

We finally observe that the graph $\Gamma_1$, which consists of only two vertices of the second type, yields exactly the multiplication $\mu$.

This ends the proof of the Proposition (we leave to the reader the check of the consistency of the combinatorial coefficients).

5.4.4. End of the proof of the compatibility between cap products in the case $X = \mathbb{R}^d$. It follows from identities (44), (45) and (46) that

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\gamma \in \mathcal{G}_{n+1, m+1}^S} W_{D, \Gamma} S_\Gamma(\alpha, \gamma; \ldots; \gamma; c)$$

is precisely equal to the r.h.s. of the homotopy equation (35), whence the result follows.

6. SOME SPECIAL CASES OF INTEREST

In this Section we discuss some interesting special cases.

i) We first give the recipe for proving Shoikhet’s conjecture [4, 24] starting from the main result of this paper; we want to point out that the proof of Shoikhet’s conjecture was the starting point of the investigations that have led us to the present paper, thanks to a stimulating question from A.S. Cattaneo.

ii) We further discuss the case of a MCE of polyvector degree less or equal than 1, in view of a forthcoming application towards globalisation of the results of [21] and of the present paper in the framework of deformation quantization (we refer to Section 8 for more details).

iii) We finally consider the special case of a MCE $\gamma$ of polyvector degree 0: in this case, we may compute explicitly both quasi-isomorphisms $\mathcal{L}_h$ and $S_\gamma$ using results of [5]: this is an important computational result, which will play a fundamental rôle in the proof of Calderaru’s conjecture [5–7].
6.1. The result of [21] and Shoikhet’s conjecture. We consider the special case \( m = \mathbb{R}[\hbar] \), viewed as a DGA concentrated in degree 0 and trivial differential (\( n = h\mathbb{R}[\hbar] \) being the pronilpotent ideal).

For \( V = \mathbb{R}^d \), we consider the DGLA of \( h \)-formal polyvector fields \( T_{\text{poly}}(V)[\hbar] \), resp. of \( h \)-formal polydifferential operators \( D_{\text{poly}}(V)[\hbar] \), and the corresponding DGM \( \Omega(V)^{-\bullet} [\hbar] \), resp. the Hochschild chain complex \( C^\bullet_{\text{poly}}[\hbar] \). All algebraic structures are extended \( h \)-linearly and tensor products are completed w.r.t. the \( h \)-adic topology.

We consider a MCE \( \gamma \) of \( h T_{\text{poly}}(V)[\hbar] = T^n_{\text{poly}}(V) \) of the form
\[
\gamma = \pi_h = h\pi_1 + \cdots, \quad \pi_i \in T^1_{\text{poly}}(V), \quad i \geq 1.
\]
Its image \( \mathcal{U}(\gamma) \) w.r.t. the \( L_\infty \)-quasi-isomorphism \( \mathcal{U} \) is an element of degree 1.

Since \( d_1 = [\mu, \cdot] \), where \( \mu \) is the standard product on \( A \), the MC equation for \( \mathcal{U}(\gamma) \) is equivalent to the fact that \( \mu + \mathcal{U}(\gamma) \) defines an \( h \)-linear associative product, which we denote by \( \ast \), on \( C^\infty(V)[\hbar] \).

We now consider the Gerstenhaber algebras (up to homotopy) \( (T_{\text{poly}}(V)[\hbar], [\cdot, \cdot, \cdot], \cup) \) and \( (D_{\text{poly}}(V)[\hbar], \cup, \cup, \cup) \); for the latter, we have \( d_1 + \mathcal{U}(\gamma) = d_{\mathcal{U}, \gamma} \), where \( d_{\mathcal{U}, \gamma} \) is the Hochschild differential on the Hochschild cochain complex of \( \mathcal{U}(\gamma) \) and Shoikhet’s conjecture.

We then consider the Gerstenhaber algebras (up to homotopy) \( (T_{\text{poly}}(V)[\hbar], [\cdot, \cdot, \cdot], \cup) \) and \( (D_{\text{poly}}(V)[\hbar], \cup, \cup, \cup) \); for the latter, we have \( d_{\mathcal{U}, \gamma} \), where \( \mathcal{U}(\gamma) \) is the Hochschild differential on the Hochschild cochain complex of \( C^\infty(V)[\hbar] \), endowed with the deformed product \( \ast \), with values in itself. Furthermore, the product \( \cup \) of degree 1 takes the form
\[
(D_1 \cup D_2)(a_1, \ldots, a_n) = D_1(a_1, \ldots, a_{D_1+1}) \ast D_2(a_{D_1+2}, \ldots, a_n), \quad D_i \in D_{\text{poly}}(V)[\hbar], \quad a_i \in A,
\]
and \( n = |D_1| + |D_2| + 2 \) (up to a sign depending on \( D_i \)).

On the other hand, we also consider the \( T \)-module, resp. \( T \)-module up to homotopy, \( (\Omega^{-\bullet}(V)[\hbar], L_\gamma, L, \cap) \) and \( (C^\bullet_{\text{poly}}(V)[\hbar], b_\mathcal{U}, L, \cap) \): in the latter, we have \( b_\mathcal{U} = d_{\mathcal{U}, \gamma} \), where \( b_\mathcal{U} \) is the Hochschild differential on the Hochschild cochain complex of \( C^\infty(V)[\hbar] \).

6.2. The case of a MCE of polyvector degree at most 1. We then consider a MCE of \( T^n_{\text{poly}}(V) \) of polyvector degree at most 1, i.e.
\[
\gamma = \gamma_1 + \gamma_0 + \gamma_0,
\]
where i) \( \gamma_1 \) is a degree 2, \( n \)-valued function on \( V \), ii) \( \gamma_0 \) is an \( n \)-valued vector field of degree 1 on \( V \), and iii) \( \gamma_1 \) is a degree 0, \( n \)-valued bivector field on \( V \).

The image \( \mathcal{U}(\gamma) \) of a MCE as in (48) satisfies the Maurer–Cartan equation in \( D^n_{\text{poly}}(V) \): since \( \gamma \) is the sum of three types of \( n \)-valued polyvector fields on \( V \), the degree requirement of the classical morphisms \( \mathcal{U}_n \) and the (graded) anticommutativity of the wedge product on \( m \)-valued polyvector fields implies the decomposition of \( \mathcal{U}(\gamma) \)
\[
\mathcal{U}(\gamma) = \sum_{n \geq 1} \frac{1}{n!} \mathcal{U}_n(\gamma_1, \ldots, \gamma_n) + \sum_{n \geq 0} \frac{1}{n!} \mathcal{U}_{n+1}(\gamma_0, \gamma_1, \ldots, \gamma_n) + \sum_{n \geq 0} \frac{1}{n!} \mathcal{U}_{n+2}(\gamma_0, \gamma_0, \gamma_1, \ldots, \gamma_n).
\]

For the sake of simplicity, we write from now on \( B \), resp. \( Q \), resp. \( F \), for the first term, resp. second term, resp. sum of the third and fourth term, in (49).

The graded commutative product on \( A = C^\infty(V) \otimes m \) defines a 1-cocycle \( \mu \) of \( D^n_{\text{poly}}(V) \). We may then consider the \( m \)-valued bidifferential operator \( \mu + B \) of degree 0 and the linear operator \( \tilde{Q} = d_m + Q \) of degree 1 on \( m \)-valued functions on \( V \). Accordingly, the Maurer–Cartan equation for \( \mathcal{U}(\gamma) \) is equivalent to
i) \( \mu + B \) defines an \( m \)-linear associative product \(*\) of degree 0 on \( A \).

ii) \( \bar{Q} \) is a derivation of degree 1 of \((A, *)\); its square equals

\[ \bar{Q}^2 = -[F, *], \]

where \([ \cdot , \cdot \] \) denotes the graded commutator w.r.t. the product \(*\).

iii) The \( m \)-valued function \( F \) of degree 2, which, by the previous equation, can be viewed as a sort of “curvature” of the “connection” \( \bar{Q} \), satisfies the Bianchi identity, i.e. it is annihilated by \( \bar{Q} \): \( \bar{Q}(F) = 0 \).

In other words, \( A \) equipped with the product \(*\), the derivation \( \bar{Q} \) and the element \( F \), is a curved DGA in the framework of [10], \( \bar{Q} \) is a Weyl connection on \( A \) with Weyl curvature \( F \), see Section 8.

For a MCE \( \gamma \) as in (48), we consider the \( T \)-algebra \( \left( T^m_{\text{poly}}(V), d_m + [\gamma, \cdot, \cdot, \cdot], \cup \right) \),

We also consider the Gerstenhaber algebra up to homotopy \( \left( D^m_{\text{poly}}(V), d_m + d_H + [\tilde{\omega}(\gamma), \cdot, \cdot, \cdot], \cup \right) \), where

\[ (D_1 \cup \ast D_2)(a_1, \ldots, a_n) = D_1(a_1, \ldots, a_{|D_1|+1}) \ast D_2(a_{|D_1|+2}, \ldots, a_n), \quad n = |D_1| + |D_2| + 2, \]

for \( D_i, i = 1, 2 \), general elements of \( D^m_{\text{poly}}(V) \), and \( a_j, j = 1, \ldots, |D_1| + |D_2| + 2 \), general elements of \( A \). Additionally, we have \( d_m + d_H + \tilde{\omega}(\gamma), \cdot, \cdot, \cdot \) \( \ast \) \( [\tilde{Q}, \cdot, \cdot, \cdot] \) \( \ast \) \( [F, \cdot, \cdot, \cdot] \), where \( d_{H,*} \) denotes the Hochschild differential w.r.t. the product \(*\). We further consider the \( T \)-module up to homotopy \( \left( C^m_{\text{poly}}(V), d_m + b_H + L_{\tilde{\omega}(\gamma)}, \cup \right) \), where

\[ D \cap \ast (a_0 a_1 \cdots a_n) = (a_0 \ast D(a_1, \ldots, a_{|D|+1}) a_{|D|+2} \cdots a_n), \]

for \( D \), resp. \( c \), a general element of \( D^m_{\text{poly}}(V) \), resp. \( C^m_{\text{poly}}(V) \). Furthermore, we have \( d_m + b_H + L_{\tilde{\omega}(\gamma)} = b_{H,*} + L_{\tilde{Q}} + L_F \).

**Theorem 6.2.** For any MCE \( \gamma \) as in (31), \( U_\gamma \) and \( S_\gamma \) are quasi-isomorphisms of Gerstenhaber algebras and \( T \)-modules up to homotopy respectively, fitting into the following commutative diagram:

\[ \begin{array}{ccc} 
T^m_{\text{poly}}(V), d_m + [\gamma, \cdot, \cdot, \cdot], \cup & \xrightarrow{U_\gamma} & D^m_{\text{poly}}(V), d_m + d_H + [\tilde{\omega}(\gamma), \cdot, \cdot, \cdot], \cup \\
(\Omega^m(V), d_m + L_{\tilde{\omega}(\gamma)}, \cup) & \xrightarrow{S_\gamma} & C^m_{\text{poly}}(V), d_m + b_H + L_{\tilde{\omega}(\gamma)}, \cup + L_F, \cup 
\end{array} \]

### 6.3. Explicit computation of the tangent quasi-isomorphisms

Borrowing notation from Subsection 6.2, we consider the case of a MCE \( \gamma = \gamma_0 \) concentrated in polyvector degree 0.

We consider now the morphism \( S_{\gamma} \): for a general Hochschild chain \( c = (a_0) \cdots (a_m) \) of (Hochschild) degree \(-m\), \( m \geq 0 \), we have

\[ S_{\gamma}(c) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}^n_{n,m+1}} W_{D,T} \cdot \mathcal{S}_{\Gamma}(\gamma, \ldots, \gamma, c), \]

with the notations from Subsection 3.2.

We first observe that the valence of any vertex of the first type of an admissible graph \( \Gamma \) as in (52) is 1, since it associated to a copy of the vector field \( \gamma \). Thus, we sum only over those admissible graphs \( \Gamma \) with univalent vertices of the first type.

Dimensional reasons imply that the weight (27) of an admissible graph \( \Gamma \) in \( \mathcal{G}^S_{n,m+1} \) is non-trivial, only if \( 2n + m \) equals the degree of the integrand, which is in this case \( n + l \), where \( l \) is the number of edges starting from the vertex 0, whence \( l = n + m \). Since such an admissible graph has exactly \( n + m + 1 \) vertices (of the first and second type), and since there are neither multiple edges nor loops by assumption, there is exactly one edge joining the vertex 0 to all vertices except one, namely the first vertex of the second type w.r.t. the cyclic order: this is because the integrand \( \omega_{D,\Gamma} \) vanishes, if star(0) contains an edge \( e \) joining 0 to the first vertex in \( S^1 \) w.r.t. the cyclic order, by the constructions of Subsubsection 3.3.2.

Since \( m \geq 0 \), we use the section of \( D_{n,m+1}^+ \) which, by means of the Möbius transformations \( \psi \) is diffeomorphic to a section of \( C_{n+1,m}^+ \), see also Subsubsection 3.1.3 (we observe that the origin 0 of the disk is mapped to \( i \), while the point 1 is mapped to the half-circle at infinity in the complex upper half-plane \( \mathcal{H} \)).

Recalling (23), Subsubsection 3.2.1, the weight (27) of an admissible graph \( \Gamma \) in \( \mathcal{G}^S_{n,m+1} \) is mapped to a weight of type (25), the only difference is that the factors of \( \omega_{D,\Gamma} \) are mapped to \( i \) usual forms \( \omega_\ast \), whenever \( e \) is an edge from 0 to some vertex (of the first and of the second type), and the “new” \( e \) is now an edge from \( i \) to the image w.r.t. \( \psi \) of the endpoint, and \( ii \) differences between \( \omega_\ast \) and \( \omega_{(e)} \), if \( e \) is an edge between two vertices (of the first and second
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We have used dashed arrows to denote forms on $D_{n,m+1}^+$ as in (27), in the graph on the left-hand side, while we have used black, resp. dashed, arrows to denote forms on $C_{n+1,m}^+$, as in (25), resp. differences of such forms.

From now on, when considering weights (27) of admissible graphs in $G_{n,m+1}^S$ as above, we implicitly assume that we are considering them on $C_{n+1,m}^+$ by means of the previous correspondence.

Lemma 6.3. If an admissible graph $\Gamma$ in $G_{n,m+1}^S$ as above has a vertex of the first type of valence 1, which is the endpoint of exactly one edge, then its weight vanishes.

Proof. It is more instructive to give a graphical proof (which will be also useful later on)

On the right-hand side of both equalities above, we may apply Lemmata 9.2 and 9.3, Appendix; we also notice that, by dimensional reasons, whenever there is a 1-valent vertex, the corresponding weight vanishes (since we integrate a 1-form over a 2-dimensional space, a subset of the complex upper half-plane $\mathcal{H}$).

We consider now an admissible graph $\Gamma$ in $G_{n,m+1}^S$, satisfying the above dimensional non-triviality condition. We consider a vertex $v_1$ of the first type: it is the endpoint of an edge starting at i, and exactly one edge departs from it. Moreover, the edge $e_1$ starting at $v_1$ must connect it to a different vertex of the first type: if not, $e_1$ connects $v_1$ to a vertex of the second type. By Lemma 6.3, there must be an edge $e_2$ from a vertex $v_2$ of the second type with endpoint $v_1$; again by Lemma 6.3, there must be an edge $e_3$ from a vertex $v_3$ of the second type with endpoint $v_2$, and so on, until we arrive at the vertex $v_n$, which is necessarily as in Lemma 6.3 by dimensional reasons, whence the weight vanishes.

By the very same procedure, we find that all admissible graphs appearing in (52) and having possibly non-trivial weights must be as in Figure 24 on the left-hand side, i.e. they must be wheeled trees (with dashed and black directed edges). Using the first graphical identity in the proof of Lemma 6.3, we may replace all dashed arrows by black ones: in fact,
Any wheeled tree with dashed edges, by repeatedly applying this computation, can be written as the sum of the same wheeled tree and other graphs with only black edges; except the wheeled tree with only black edges, each of the remaining graphs has at least one vertex of the first type as in Lemma 9.2 or 9.3, whence they vanish.

We denote by $T_{n,m}^K \subset \mathcal{G}_{n,m}^K$ the set of wheeled trees as above, which we view as admissible graphs as in Subsubsection 3.1: then, by the previous computations, we may re-write (52) as

$$S_j(c) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\Gamma \in T_{n,m+1}^K} W_T S_{\Gamma}(\gamma_j, \ldots, \gamma_j, c).$$

We consider a wheeled tree $\Gamma$ in $T_{n,m}^K$: by the same arguments as in [20], Paragraph 8.3.3.1, if $\Gamma$ contains at least a wheel with an odd number of vertices, then its weight vanishes. Thus, in (53), we may sum only w.r.t. even integers $n$.

From now on, we may follow the arguments of Subsection 10.1 of [5] to evaluate (53): of course, there are some sign modifications to keep into account, but the end result turns out to be the same (since $n$ is even and since only wheels with an even number of vertices appear on the above sum). First, we write

$$\gamma = \gamma_0 \otimes m_0, \quad \gamma_0 \in T_{\text{poly}}^0(V), \quad m_0 \in m_1,$$

and, following the notations of [5], we introduce the $m$-valued, matrix-valued 1-form $\Xi$ via

$$\Xi = \Xi_0 \otimes m_0 = (\partial_1 \partial_0^1 \gamma^j_k dx_k) \otimes m_0.$$

Then, following almost verbatim the computations of Subsection 10.1 of [5], we find

$$S_\gamma(c) = \det \sqrt{\frac{\Xi}{e^\frac{\Xi}{2} - e^{-\frac{\Xi}{2}}} \land HKR(c)} = j(\gamma) \land HKR(c),$$

The right-hand side of (54) needs some explanations. First of all, we have improperly written a determinant: in fact, it should be denoted by the more appropriate notation Ber, which represents the super-determinant, or Berezinian.

In fact, $\Xi$ represents a 1-form on $V$ with values in $m$-valued matrices: $m$-valued matrices form a GA, hence usual trace and determinant have to be replaced by their super-analogues.

Further, the square root of the quotient in the Berezinian has to be interpreted as a power series. More precisely, we have

$$\frac{1}{2} \log \left( e^{\frac{\Xi}{2}} - e^{-\frac{\Xi}{2}} \right) = \sum_{l \geq 0} \beta_l x^l,$$

and the coefficients $\beta_l$ are called modified Bernoulli numbers.

Aside from some sign differences, which, as already remarked, do not cause changes in the main arguments, the only point we want to stress is that, by the above arguments, the weights in (53) are the same weights examined in [5], whose computation has been performed, using different approaches, in [27, 28].

Summarizing all the computations so far, we have the following

**Theorem 6.4.** For a MCE of $T_{\text{poly}}^m(V)$ of polyvector degree 0, the following identity holds true:

$$S_\gamma(c) = j(\gamma) \land HKR(c), \quad c \in C_{\text{poly}, m}^0(V),$$

where HKR is the Hochschild–Kostant–Rosenberg quasi-isomorphism in homology, and $j(\gamma)$ is the rooted Todd class analogon appearing in the main result of [5].

7. Application: (co)homological Duflo isomorphism

We consider a finite dimensional Lie algebra $\mathfrak{g}$ over a field $k$ of characteristic zero.

7.1. Statement of the result. We recall the definition of the (modified) Duflo element

$$J := \det \left( e^{\text{ad}/2} - e^{-\text{ad}/2} \right) \in \hat{S}(\mathfrak{g}^*)^\mathfrak{g}.$$

We also remind the reader that the completed algebra $\hat{S}(\mathfrak{g}^*)$ naturally acts on $S(\mathfrak{g})$:

$$\xi^k \cdot x^n := \frac{n!}{(n-k)!} \xi(x)^k x^{n-k} \quad (x \in \mathfrak{g}, \xi \in \mathfrak{g}^*, k > 0, n > 0).$$

The following result (proved in [20, 22]) is a cohomological extension of the original Duflo isomorphism [14].
Theorem 7.1 (Cohomological Duflo isomorphism). The morphism of $\mathfrak{g}$-modules

$$D := \text{sym} \circ (J^{1/2} : S(\mathfrak{g}) \to U(\mathfrak{g}))$$

induces an algebra isomorphism

$$H^\bullet (\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} H^\bullet (\mathfrak{g}, U(\mathfrak{g}))$$

at the level of Chevalley-Eilenberg cohomology.

We now observe that, if $A$ is an algebra on which $\mathfrak{g}$ acts by derivations, the Chevalley–Eilenberg Lie algebra homology $H_{-\bullet}(\mathfrak{g}, A)$ is equipped with an $H^\bullet (\mathfrak{g}, A)$-module structure in the following way: on the level of the complexes, for any Chevalley-Eilenberg cochain $\alpha = \xi \otimes a$, resp. chain $c = x \otimes a'$, one defines

$$\alpha(c) = \iota_\xi(x) \otimes aa',$$

where $\iota$ denotes the usual contraction operation\(^7\). In what follows we will prove the following homological version of the Duflo isomorphism.

Theorem 7.2 (Homological Duflo isomorphism). The morphism $D$ induces an isomorphism of $H^\bullet (\mathfrak{g}, S(\mathfrak{g}))$-modules

$$H_{-\bullet}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} H_{-\bullet}(\mathfrak{g}, U(\mathfrak{g}))$$

at the level of Chevalley-Eilenberg homology.

Considering the degree zero (co)homology, one obtains

**Corollary 7.3.** $D$ restricts to an isomorphism of algebras $S(\mathfrak{g})^0 \xrightarrow{\sim} U(\mathfrak{g})^0 = \mathcal{Z}(U(\mathfrak{g}))$ on invariants, and induces an isomorphism of $S(\mathfrak{g})^0$-modules $S(\mathfrak{g})_0 \xrightarrow{\sim} U(\mathfrak{g})_0 = \mathcal{A}(U(\mathfrak{g}))$ on coinvariants.

Here $\mathcal{Z}(B)$ denotes the center of an algebra $B$, and $\mathcal{A}(B) = B/[B, B]$ its abelianization.

### 7.2. Proof of the results.

7.2.1. **Proof of Theorem 7.1.** In this Subsubsection we follow closely [3, Subsection 5.2].

Let us consider the superspace $V = \Pi \mathfrak{g}$; then, the GA $A$ of superfunctions on $V$ is $A = \wedge^\bullet (\mathfrak{g}^*)$. Therefore the Chevalley-Eilenberg differential $d_C$ defines a cohomological vector field $\gamma$ on $V$ (i.e. a degree one derivation squaring to 0)\(^8\). In other words, $\gamma$ is a MCE in $T^\bullet_{\text{poly}}(V)$.

On the one hand, $T^\bullet_{\text{poly}}(V)$ is naturally isomorphic to $\wedge^\bullet (\mathfrak{g}^*) \otimes S(\mathfrak{g})$ and, under this identification, $[\gamma, \ ]$ precisely gives the coboundary operator $d_C$ of the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with values in $S(\mathfrak{g})$. On the other hand, $(D^\bullet_{\text{poly}}(V), d_H + [\gamma, \ ])$ identifies with the complex $CC^\bullet(A, d_C)$ of Hochschild cochains of the DGA $(A, d_C)$ with values in itself.

Now we observe that we have a quasi-isomorphism of DGAs

$$\ell : (D^\bullet_{\text{poly}}(V), d_H + [\gamma, \ ]) \to C^\bullet(\mathfrak{g}, U(\mathfrak{g})),$$

where $C^\bullet(\mathfrak{g}, U(\mathfrak{g}))$ denotes the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with values in $U(\mathfrak{g})$, given by the following composition of maps

$$D^\bullet_{\text{poly}}(V) = \wedge^\bullet (\mathfrak{g}^*) \otimes T\left( \wedge^\bullet (\mathfrak{g}) \right) \to \wedge^\bullet (\mathfrak{g}^*) \otimes T(\mathfrak{g}) \to \wedge^\bullet (\mathfrak{g}^*) \otimes U(\mathfrak{g}) = C^\bullet(\mathfrak{g}, U(\mathfrak{g})).$$

This is a manifestation of the fact that the quadratic DGA $(\wedge^\bullet (\mathfrak{g}^*), d_C)$ and the quadratic-linear algebra $U(\mathfrak{g})$ are related by a Koszul-type duality (see e.g. [23]). Moreover, the following diagram of quasi-isomorphisms of complexes commutes:

$$\begin{array}{ccc}
T^\bullet_{\text{poly}}(V), [\gamma, \ ] & \xrightarrow{\text{HKR}} & D^\bullet_{\text{poly}}(V), [\gamma, \ ] \\
\parallel & & \downarrow \ell \\
C^\bullet(\mathfrak{g}, S(\mathfrak{g})) & \xrightarrow{\text{sym}} & C^\bullet(\mathfrak{g}, U(\mathfrak{g})),
\end{array}$$

Finally, we recall (see e.g. [3]) that $H_{\mathfrak{g}} = \text{HKR} \circ \iota_{\mathfrak{g}(\gamma)}$, and that one has the following

**Lemma 7.4 ([3], Lemma 5.6).** Under the obvious identification $IIV = \mathfrak{g}$, the supermatrix valued 1-form $\Xi$, restricted to $\mathfrak{g}$, which we implicitly identify with the space of vector fields on $V$ with constant coefficients, satisfies $\Xi = \text{ad}$. \(^7\)

\(^7\)In fact, this defines an actual DG-module structure on the level of the complexes.

\(^8\)This is abstract nonsense.
Therefore, since
\[ j(\gamma) := \det \sqrt{\frac{e^{\Xi/2} - e^{-\Xi/2}}{2}}, \]
we have the following commutative diagram of algebra isomorphisms
\[
\begin{array}{ccc}
H^\bullet\left( T^\bullet_{\text{poly}}(V), [\gamma, \cdot] \right) & \xrightarrow{U,} & H^\bullet\left( D^\bullet_{\text{poly}}(V), d_H + [\gamma, \cdot] \right) \\
\downarrow & & \downarrow \epsilon \\
H^\bullet(g, S(g)) & \xrightarrow{\mathcal{D}} & H^\bullet(g, U(g)).
\end{array}
\]
Hence Theorem 7.1 follows. \qed

7.2.2. Proof of Theorem 7.2. We keep the same notations as in the previous paragraph.

First of all, we observe that \( \Omega^\bullet(V) \) is naturally isomorphic to \( \wedge^\bullet(g^*) \otimes S(g^*) \) and that, under this identification, \( L_{\gamma} \) precisely gives the coboundary operator of the Chevalley-Eilenberg cochain complex of \( g \) with values in \( S(g^*) \).

Then, \( (C^\bullet_{\text{poly}}(V), b_H + L_{\gamma}) \) identifies with the complex \( CC_{-\bullet}(A, d_C) \) of Hochschild chains (with reversed grading) of the DGA \( (A, d_C) \) with values in itself.

Applying Theorem 6.4 to the present situation, we have \( S_\gamma = j(\gamma) \wedge \text{HKR} \). We observe that, in order for this map to be well-defined, we need to consider completed versions \( \tilde{\Omega}^\bullet(V) = \wedge^\bullet(g^*) \otimes \tilde{S}(g^*) \) and \( \tilde{C}^\bullet_{\text{poly}}(V) = \wedge^\bullet(g^*) \otimes \tilde{T}(\wedge^\bullet(g^*)) \) of the spaces involved in the formality for chains.

Now, we recall that we have a quasi-isomorphism of complexes
\[
\lambda : C^\bullet(g, U(g)^*) \longrightarrow \left( \tilde{C}^\bullet_{\text{poly}}(V), b + L_{\gamma} \right)
\]
given by the following composition of maps
\[
C^\bullet(g, U(g)^*) = \wedge^\bullet(g^*) \otimes U(g)^* \longrightarrow \wedge^\bullet(g^*) \otimes T(g)^* = \wedge^\bullet(g^*) \otimes \tilde{T}(g^*) \hookrightarrow \wedge^\bullet(g^*) \otimes \tilde{T}(\wedge^\bullet(g^*)),
\]
which induces an isomorphism of \( H^\bullet(g, U(g)^*) \)-modules on cohomology.

Moreover, the following diagram of quasi-isomorphisms of complexes commutes:
\[
\begin{array}{ccc}
\left( \tilde{\Omega}(V), L_{\gamma} \right) & \xrightarrow{\text{HKR}} & \left( \tilde{C}^\bullet_{\text{poly}}(V), b + L_{\gamma} \right) \\
\downarrow \text{sym}^* & & \downarrow \lambda \\
C^\bullet(g, \tilde{S}(g^*)) & \xrightarrow{\lambda} & C^\bullet(g, U(g)^*),
\end{array}
\]
We observe that, for any \( g \)-module \( M \), the Chevalley-Eilenberg cochain complex \( C^\bullet(g, M^*) \) is naturally isomorphic to the dual of the Chevalley-Eilenberg chain complex \( C_{-\bullet}(g, M) \) with reversed grading. Moreover, a direct computation shows that

Lemma 7.5. For any \( \omega \in \tilde{\Omega}^\bullet(V) = C^\bullet(g, \tilde{S}(g^*)) \) and any \( c \in C_{-\bullet}(g, S(g)) \),

i) \( \langle j(\gamma) \wedge \omega, c \rangle = \langle \omega, j(\gamma) \cdot c \rangle \); 
ii) for any \( \alpha \in T^\bullet_{\text{poly}}(V) = C^\bullet(g, S(g)) \), \( \langle i_\alpha \omega, c \rangle = \langle \omega, \alpha(c) \rangle \).

Therefore the transpose of \( S_\gamma \) induces an isomorphism of \( H^\bullet(g, S(g)^*) \)-modules
\[
H_{-\bullet}(g, S(g)) \longrightarrow H_{-\bullet}(g, U(g))
\]
which is precisely \( \mathcal{D} \), whence the proof of Theorem 7.2. \qed

Remark 7.6. As we already mentioned, there is a duality between the DGA \( (A, d_C) \) and the quadratic-linear algebra \( U(g) \): in [4], we give a more direct proof of Corollary 7.3 in the same spirit of Kontsevich’s approach to the original Duflo isomorphism [20], which does not make use of the aforementioned duality.
7.3. Why we can work over $\mathbb{Q}$. In this Subsection we explain why Theorem 6.4, Theorem 7.1, Theorem 7.2 and Corollary 7.3 are valid over any field of zero characteristic.

First of all, we observe that we have been able to compute explicitly $\mathcal{U}$ (Section 9 of [3], see also [5]) and $\mathcal{S}$, (Section 6 of the present paper), and both have rational coefficients.

Then to prove that the mentioned results remain true over $\mathbb{Q}$ (and thus over any field of zero characteristic) we have to find homotopies with rational coefficients.

Finally, we observe that both homotopy equations (32) and (35) are linear w.r.t. the weights of graphs appearing in the homotopy operator $\mathcal{H}$, respectively.

To conclude, we have a real solution of a system of linear equations with rational coefficients. Therefore a rational solution exists. □

8. Proof of the main result

The main goal of this final Section is to globalize to a general manifold $X$ the local results obtained above in the paper. The globalisation procedure is based on [12] (see also [10]).

8.1. Fedosov resolutions and the globalisation procedure. We consider a general $d$-dimensional manifold $X$. According to Section 3, [12], we consider the algebra $B = C^\infty(X)$ of smooth functions on $X$; we associate to $X$ the DGLAs $(T_{\text{poly}}(X), 0, [\cdot, \cdot])$ of polyvector fields on $X$ with trivial differential and Schouten–Nijenhuis bracket, and $(D_{\text{poly}}(X), d, [\cdot, \cdot])$ of polydifferential operators on $X$, which is viewed as the subcomplex of the Hochschild cochain complex of $B$, consisting of cochains, which are smooth differential operators w.r.t. any of their arguments, and with induced Hochschild differential and Gerstenhaber bracket. Both DGLAs are graded w.r.t. the shifted degree.

Furthermore, we have corresponding DGMs, $(\Omega(X), 0, L)$, with trivial differential and action $L$ by polyvector fields given by Lie derivative, and $(C_{\text{poly}}(X), b_H, L)$, where $C_{\text{poly}}(X)$ has been defined in [12]: it is defined as a suitable completion of the Hochschild cochain complex of $C^\infty(X)$ with negative grading, with the Hochschild differential $b_H$ and the action $L$. We observe that both DGMs are negatively graded.

We quote (without proof) from [12] the main result towards the globalisation of Kontsevich’s and Tsygan’s formality $L_\infty$-quasi-isomorphism.

**Theorem 8.1.** For a given $d$-dimensional manifold $X$, there exist DGLAs $\mathfrak{g}_i^X$, resp. DGMs $\mathfrak{M}_i^X$, $i = 1, 2$, and $L_\infty$-quasi-isomorphisms $\mathcal{U}_X$ and $\mathcal{E}_X$, which fit into the following commutative diagram of DGLAs and DGMs:

\[
\begin{array}{ccc}
T_{\text{poly}}(X) & \xrightarrow{\mathfrak{g}_1^X} & D_{\text{poly}}(X) \\
\Omega(X) & \xrightarrow{\mathfrak{M}_1^X} & C_{\text{poly}}(X) \\
\end{array}
\]

where the vertical arrows denote DGLA-actions, and the hooked arrows denote quasi-isomorphisms.

Here we briefly discuss the construction of the DGLAs $\mathfrak{g}_i^X$ and corresponding DGMs $\mathfrak{M}_i^X$, $i = 1, 2$, which are called Fedosov resolutions of the corresponding DGLAs and DGMs, which have been defined above. Using the notations of [12], we have

\[
\begin{align*}
\mathfrak{g}_1^X &= \Omega(X, T_{\text{poly}}), & \mathfrak{g}_2^X &= \Omega(X, D_{\text{poly}}), \\
\mathfrak{M}_1^X &= \Omega(X, \mathcal{E}), & \mathfrak{M}_2^X &= \Omega(X, C_{\text{poly}}),
\end{align*}
\]

where $T_{\text{poly}}$, resp. $D_{\text{poly}}$, denotes the bundle over $X$, whose global sections are formal polyvector fields, resp. formal polydifferential operators, w.r.t. fiber coordinates of $TX$, and $\mathcal{E}$, resp. $C_{\text{poly}}$, denotes the bundle, whose global sections are formal differential forms, resp. formal Hochschild chains, w.r.t. fiber coordinates of $TX$. $T_{\text{poly}}$ and $D_{\text{poly}}$ are bundles of DGLAs, the former with trivial differential and $B$-linear Schouten–Nijenhuis bracket w.r.t. the formal coordinates of $TX$, the latter with Hochschild differential and Gerstenhaber bracket induced from the DGLA-structure on the Hochschild cochain complex of the algebra $F = \mathbb{R}[y_1, \ldots, y_d]$; similarly, $\mathcal{E}$ and $C_{\text{poly}}$ are bundles of DGMs over $T_{\text{poly}}$ and $D_{\text{poly}}$ respectively, the former with trivial differential and Lie derivative w.r.t. the fiber coordinates of $TX$, the latter with Hochschild differential and action $L$ induced from the DGM-structure on the Hochschild cochain complex of $F$.

Obviously, the De Rham differential $d$ on $X$ defines, by linear extension, a differential (which we denote by the same symbol) on the DGLAs $\mathfrak{g}_i^X$ and on the DGMs $\mathfrak{M}_i^X$, $i = 1, 2$, which is compatible with all aforementioned algebraic structures; it is then clear that all DGLAs and DGMs considered so far are naturally bi-graded.

A very important tool in the proof of Theorem 8.1 is the Fedosov connection on $\mathfrak{g}_i^X$ and $\mathfrak{M}_i^X$, $i = 1, 2$: it is customary to denote it as $D = d + \omega$, where $\omega$ decomposes as $\omega = A + \tilde{\omega}$, where $A$ is the connection 1-form of a torsion-free connection $\nabla = d + A$ on $TX$ and $\tilde{\omega}$ is an element of $\Omega^1(X, T_{\text{poly}})$. The Fedosov connection is flat, i.e.
\[ D^2 = 0, \text{ or, equivalently, } d\omega + \frac{1}{2}[\omega, \omega] = 0; \text{ we observe that, since } \nabla \text{ is torsion-free, then } \nabla \text{ extends to a derivation of degree 1 on } \mathfrak{g}_1^X \text{ and } \mathfrak{M}_1^X, \ i = 1, 2. \text{ Since } D \text{ is flat and is compatible with all algebraic structures, we may consider the cohomology of } \mathfrak{g}_1^X \text{ and } \mathfrak{M}_1^X, \ i = 1, 2 \text{ w.r.t. } D: \text{ it turns out that all cohomologies are concentrated in degree 0, and that we have isomorphisms of DGLAs and DGMs}
\]
\[
\begin{align*}
H^0(\Omega(X, T_{\text{poly}}), D) & \cong T_{\text{poly}}(X), & H^0(\Omega(X, D_{\text{poly}}), D) & \cong D_{\text{poly}}(X), \\
H^0(\Omega(X, E), D) & \cong \Omega(X), & H^0(\Omega(X, C_{\text{poly}}), D) & \cong C_{\text{poly}}(X).
\end{align*}
\]
(57)

We finally briefly present the construction of the \( L_\infty \)-quasi-isomorphisms \( \mathcal{U}_X \) and \( \mathcal{S}_X \).

We pick a local chart \( U \) of \( X \), and we set \((m, d_m) = (\Omega(U), d)\), \( d \) being the De Rham differential: we observe that \( \Omega(U) \) is commutative (in the graded sense), and has a decomposition into a nilpotent part \( n = \Omega^{\geq 1}(U) \) and a commutative algebra (concentrated in degree 0), which contains a unit annihilated by \( d \).

On the other hand, Kontsevich’s and Tsygan’s \( L_\infty \)-quasi-isomorphisms \( \mathcal{U} \) and \( \mathcal{S} \) extend to \( L_\infty \)-quasi-isomorphisms on formal polyvector fields, polydifferential operators, differential forms and Hochschild chains on \( V_{\text{formal}} \) and \( \mathfrak{g}_1^X \).

Similarly, the \( L_\infty \)-quasi-isomorphism \( \mathcal{U}_X \) and \( \mathcal{S}_X \) are constructed by gluing the local \( L_\infty \)-morphisms \( \mathcal{U} \) and \( \mathcal{S} \), obtained by twisting the natural extensions of \( \mathcal{U} \) and \( \mathcal{S} \) w.r.t. the MCE \( \mathcal{U}|_U \); we only observe that the additional properties of \( \mathcal{U} \) and \( \mathcal{S} \), proved in [12, 20], imply that a change of local charts does not affect \( \mathcal{U}_X \) and \( \mathcal{S}_X \), whence it follows that they glue together.

8.2. End of the proof of Theorem 8.1. Let \((m, d_m)\) be a commutative DGA as in the introduction, and consider a MCE \( \gamma \) in \( T_{\text{poly}}^m(X) \). First of all, we observe that the commutative diagram (56) can be extended by \( m \)-linearity, and we denote by \( \mathcal{U} \) (resp. \( \mathcal{S} \)) the \( L_\infty \)-quasi-isomorphism obtained by means of a quasi-inverse \( \mathcal{I} \), resp. \( \mathcal{J} \) (both not unique), of the right-most, resp. left-most, hooked arrow of the first, resp. second, line of (56).

Then, the quasi-isomorphism of DGLAs \( T_{\text{poly}}^m(X) \to \mathfrak{g}_1^{X, m} \) produces, out of \( \gamma \), a MCE \( \gamma_1 \) in \( \mathfrak{g}_1^{X, n} \), which satisfies by construction, and by means of (57), \( D(\gamma_1) = 0 = d_m(\gamma_1) + \frac{1}{2}[\gamma_1, \gamma_1] \).

Similarly, the \( L_\infty \)-quasi-isomorphism \( \mathcal{U}_X \) produces, out of \( \gamma_1 \), a MCE \( \gamma_2 \) in \( \mathfrak{g}_2^{X, n} \), which itself gives rise to a MCE \( \gamma' \) obtained by means of the quasi-inverse \( \mathcal{I} \) of \( D_{\text{poly}}^m(X) \to \mathfrak{g}_2^{X, m} \).

By construction, \( \gamma' = \mathcal{U}(\gamma) \), and \( \mathcal{U}_\gamma \) can be computed as the composed \( L_\infty \)-morphism
\[
(58)
\]
It follows from [2,12] that hooked arrows in (56) preserve all algebraic structures (namely, \( B_\infty \)-structures). Moreover, the compatibility between cup products from Section 4 leads us to the following

Lemma 8.2. The first Taylor component \( \mathcal{U}_X^{(1)} \gamma_1 \) of \( \mathcal{U}_X \gamma_1 \) induces an isomorphism of Gerstenhaber algebras
\[
H^\bullet(\mathfrak{g}_1^{X, m}) \to H^\bullet(\mathfrak{g}_2^{X, m}).
\]

Proof. We pick up a local chart \( U \) of \( X \) and write \( D = d + \omega|_U \). Then, we set \( \gamma_1 = (\omega + \gamma_1)|_U \): \( \gamma_1 \) is a MCE of \( \mathfrak{g}_1^{U, m} \).

We now introduce the DG algebra \( \bar{m} := \Omega(U) \otimes m \) and observe that
- it is of the form \( \bar{m} = C_\infty(U) \oplus \bar{n} \), with \( \bar{n} \) (pro)nilpotent;
- \( \mathfrak{g}_1^{U, m} \cong T_{\text{poly}}(V_{\text{formal}}) \) and \( \mathfrak{g}_2^{U, m} \cong D_{\text{poly}}(V_{\text{formal}}) \);
- \( \gamma_1 \) lies in the (pro)nilpotent part.

We thus obtain a MCE \( \mathcal{U}(\gamma_1) \) in \( \mathfrak{g}_2^{U, m} \), which decomposes as \( \mathcal{U}(\gamma_1) = (\omega + \gamma_2)|_U \) by the properties of \( \mathcal{U}_U \). We are therefore in the framework to which the compatibility between cup-products applies: the homotopy identity (32) holds true (for \( \gamma_1 \) and \( \bar{m} \)).
Finally, by construction we have $\mathcal{U}^c_{\gamma} = \Omega_{\mathcal{U}_{\gamma}}$. It thus remains to prove that the homotopy operators, which are well-defined locally, glue together to a globally well-defined operator. This follows directly from the arguments of [5, Lemma 10.1.1].

Let us now prove that the first Taylor component of $I_{\gamma_2}$ induces an isomorphism of Gerstenhaber algebras between $H^\bullet \left( g_{2,\gamma_2}^X \right)$ and $H^\bullet \left( D_{\text{poly}}^m(X)_{\gamma'} \right)$. Observe that we can view $\gamma'$ as a MCE in $g_2^S$, since we have an inclusion of $B_\infty$-algebras $D_{\text{poly}}^m(X) \hookrightarrow g_2^S$. Moreover, since this inclusion is a quasi-isomorphism having $I$ as a quasi-inverse then $\gamma_2$ and $\gamma'$ are gauge equivalent in $g_2^S$. As a consequence the $B_\infty$-algebras $g_{2,\gamma_2}^X$ and $g_{2,\gamma'}^X$ are isomorphic. In the end, we have the following commutative diagram of isomorphisms

$$
\begin{array}{ccc}
H^\bullet \left( g_{2,\gamma_2}^X \right) & \longrightarrow & H^\bullet \left( g_{2,\gamma'}^X \right) \\
\downarrow I_{\gamma_2,1} & & \downarrow \\
H^\bullet \left( D_{\text{poly}}^m(X)_{\gamma'} \right),
\end{array}
$$

two of them being isomorphisms of Gerstenhaber algebras; so the third (i.e. $I_{\gamma_2}$) is.

We have therefore proved that the first Taylor component $\mathcal{U}_{\gamma_1,1}$ of $\mathcal{U}_\gamma$ induces an isomorphism of (Gerstenhaber) algebras $H^\bullet \left( T_{\text{poly}}^m(X)_{\gamma} \right) \longrightarrow H^\bullet \left( D_{\text{poly}}^m(X)_{\mathcal{U}(\gamma)} \right)$.

As for $\mathcal{U}_\gamma$, the quasi-isomorphism $\mathcal{S}_{\gamma,0}$ can be decomposed as follows:

$$
(59) \quad C_{\text{poly}}^m(X)_{\gamma'} \longrightarrow M_{2,\gamma_2}^X \longrightarrow M_{2,\gamma_2}^X \longrightarrow M_{1,\gamma_2}^X \longrightarrow \Omega^m(X)_{\gamma}.
$$

Finally, Theorem A is a consequence of the following

**Proposition 8.3.** Sequences (58) and (59) of quasi-isomorphisms fit into a commutative diagram of Gerstenhaber algebras and $T$-modules

$$
\begin{array}{ccccccc}
H^\bullet \left( T_{\text{poly}}^m(X)_{\gamma} \right) & \longrightarrow & H^\bullet \left( g_{1,\gamma_1}^X \right) & \longrightarrow & H^\bullet \left( g_{2,\gamma_2}^X \right) & \longrightarrow & H^\bullet \left( D_{\text{poly}}^m(X)_{\gamma'} \right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^\bullet \left( \Omega^m(X)_{\gamma} \right) & \longrightarrow & H^\bullet \left( M_{1,\gamma_1}^X \right) & \longrightarrow & H^\bullet \left( M_{2,\gamma_2}^X \right) & \longrightarrow & H^\bullet \left( C_{\text{poly}}^m(X)_{\gamma'} \right),
\end{array}
$$

where the vertical arrows denote $T$-module structures and horizontal arrows isomorphisms of Gerstenhaber algebras and $T$-modules.

**Proof.** We only have to prove that $\mathcal{S}_{X,\gamma_1}^0$ induces an isomorphism of $T$-modules on cohomology.

The proof goes along the same lines of arguments as in the proof of Lemma 8.2. It therefore remains to prove that the homotopy operator for the compatibility between cap products satisfies the appropriate property so that gluing is allowed. Namely, we have the following

**Lemma 8.4.** Let $\Gamma$ be an $S$-admissible graph in $G_{n+1,m+1}^S$, $n \geq 1$, $\gamma_i$ general elements of $T_{\text{poly}}(V_{\text{formal}})$, such that, for some $j = 2, \ldots, n$, $\gamma_j$ is a linear vector field, $c$ a general element of $C_{\text{poly}}$ of Hochschild degree $-m$. Then

$$
\mathcal{W}_{D,\Gamma} \mathcal{S}_\Gamma(\gamma_1, \gamma_2, \ldots, \gamma_n, c) = 0.
$$

**Proof.** We may assume without loss of generality that $\gamma_2$ is linear.

The first point of the first type in $J_{n+1,m+1}^+$, which moves along a fixed trajectory in the punctured unit disk, corresponds to $\gamma_1$; we observe that any other point in the punctured unit disk $D_0^+$ can move freely.

It is clear that $\mathcal{S}_\Gamma(\gamma_1, \gamma_2, \ldots, c)$ does not trivially vanish, only if the vertex of $\Gamma$ corresponding to the linear vector field $\gamma_2$ has at most an incoming edge and exactly one outgoing edge: then, Lemma 6.3 yields the vanishing of the corresponding integral weight $\mathcal{W}_{D,\Gamma}^\circ$. □

Now, the homotopy operator for the compatibility between cap products is globally well-defined, because a change of local charts of $X$ changes the local expression for the Fedosov connection by a local 1-form with values in linear vector fields: the previous identity implies that such a change does not in fact contribute, whence the Proposition follows. □
8.3. Relation with deformation quantization and the work of Cattaneo-Felder-Tomassini. We consider
again the special case when \( m = \mathbb{R}[\hbar] \), and \( \gamma \) is a MCE of the form
\[
\gamma = \pi_1 + \hbar \pi_2 + \mathcal{O}(\hbar^2) \in \hbar T^1_{\text{poly}}(X)[\hbar].
\]
In particular \( \pi_1 \) defines a Poisson structure on \( X \). Below we assume the reader is familiar with the subject of
defformation quantization.

Borrowing the notation from the proof of Lemma 8.2 in Subsection 8.2, we see that w.r.t. a local chart \( U \) of \( X \),
\( \tilde{\gamma}_1 = (\omega + \gamma_1)|_U \) has polyvector degree less or equal to 1, and \( D(\gamma_1) = 0 \).

According to the computations of Subsection 6.2, \( U(\tilde{\gamma}_1) \) induces the following structure on \( A^U := \mathcal{O}_{\text{formal}} \otimes \mathcal{O}^0 = C^\infty(U, \mathcal{O}_{\text{formal}})[\hbar] \): i) a \( C^\infty(U)[\hbar] \)-linear associative product \( \ast \) on \( A^U \), ii) a Fedosov connection \( \tilde{Q} \) on \( (A^U, \ast) \) and
iii) the Weyl curvature \( F \) of \( \tilde{Q} \), in the terminology of Subsection 4.2, [10].

By inspecting (49) and recalling that \( \pi_1 = \mathcal{O}(\hbar) \), it follows that i) the Fedosov connection \( \tilde{Q} \) satisfies \( \tilde{Q} = D + \mathcal{O}(\hbar) \),
where \( D \) is the previously introduced Fedosov connection on \( \mathfrak{g}^X \) and ii) the Weyl curvature \( F \) of \( \tilde{Q} \) satisfies \( F = \mathcal{O}(\hbar) \).

We recall now that \( D \) is flat and that the corresponding cohomology on \( \mathfrak{g}^X[\hbar] \) is concentrated in degree 0: then, the assumptions of Lemmata 4.5 and 4.6 in [10] are satisfied, whence we may recover\(^9\) the fact that the MCE in \( \mathfrak{g}^X[\hbar] \) corresponding to the triple \( (\ast, \tilde{Q}, F) \) is gauge-equivalent to the MCE corresponding to the product \( \ast \) and the underformed connection \( D \), with zero curvature.

This last fact is of crucial use in [10] to prove that one has an isomorphism of algebras between the algebra of
Casimir functions for the formal Poisson structure \( \pi_1 \) and the center of the corresponding quantized algebra, which
is precisely the degree zero part of the compatibility between cap products on cohomology.

Let us simply recall that the quantized algebra is constructed as the subspace of \( D \)-flat sections in \( A^X \), which is
isomorphic to \( C^\infty(X)[\hbar] \), equipped with the associative product \( \ast \).

9. Appendix

In this Appendix, we quote two main technical Lemmata from [20], which are used in many computations throughout
the paper.

First of all, we consider the compactified configuration space \( C_n \), with \( n \geq 3 \); further, for any two distinct indices
\( 1 \leq i \neq j \leq n \), there is natural projection \( \pi_{ij} \) from \( C_n \) onto \( C_2 \), and we denote by \( \omega_{ij} \) the pull-back of \( \omega|_{C_2} \) w.r.t. the
projection \( \pi_{ij} \) (see Lemma 3.1, Subsection 3.1).

**Lemma 9.1.** For a positive integer \( n \geq 3 \), the integral
\[
\int_{C_n} \bigwedge_{\alpha = 1}^{2n-3} \omega_{i_\alpha j_\alpha}
\]
vanishes, for any distinct indices \( 1 \leq i_\alpha \neq j_\alpha \leq n \), \( \alpha = 1, \ldots, 2n - 3 \).

For a proof of Lemma 9.1, we refer to [20], Subsection 6.6; in [9] one can find an alternative proof to the original one
of Kontsevich. Lemma 9.1 is often used in Subsection 4.1, Subsection 5.2 and related Subsubsections, and in
Subsection 5.4 and related Subsubsections.

We now consider the 1-form \( \omega \) on Kontsevich’s eye \( C_{2,0} \) as in Subsection 3.1.

**Lemma 9.2.** The integral
\[
\int_{C_{2,0}} \omega_{12} \wedge \omega_{21} = \int_{\mathcal{H} \setminus \{z_0\}} \omega(z_0, z) \wedge \omega(z, z_0)
\]
vanishes, where \( \omega_{12} \), resp. \( \omega_{21} \), denotes the usual form \( \omega \) on \( C_{2,0} \), resp. the usual form \( \omega \), but with the arguments
permuted (we observe that Kontsevich’s angle function is not symmetric w.r.t. its arguments, hence the vanishing of
the integral is non-trivial). In the second expression, \( z_0 \) is some fixed point in the complex upper half-plane \( \mathcal{H} \).

**Lemma 9.3.** If \( z_1, z_2 \) are either two distinct points in the complex upper half-plane \( \mathcal{H} \) or if \( z_1 \) is in the complex
upper half-plane \( \mathcal{H} \) and \( z_2 \) on the real axis \( \mathbb{R} \), the integral
\[
\int_{\mathcal{H} \setminus \{z_1, z_2\}} \omega(z_1, z) \wedge \omega(z, z_2)
\]
vanishes, where we integrate w.r.t. \( z \).

For a proof of both Lemmas 9.2 and 9.3, we refer to [20], Lemmata 7.3, 7.4 and 7.5, or to [3].

\(^9\)As a particular case of our claim, in Subsection 8.2, that \( \gamma' \) and \( \gamma_2 \) are gauge equivalent in \( \mathfrak{g}^{X,m}_2 \).
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