Finite sets in fibres of holomorphic maps

Michał Kwieciński* and Piotr Tworzewski
Uniwersytet Jagielloński, Kraków, Poland.

Abstract

We consider the following topological invariant of holomorphic maps: the maximal number of points of a special fibre that can be simultaneously approximated by points in one sequence of arbitrarily general fibres. Several results about this invariant and their applications describe the structure of holomorphic maps (notably non-equidimensional maps).

1 Introduction.

From the work of Thom [17], Fukuda [5] and Nakai [12], it follows that one cannot stratify arbitrary complex algebraic maps so as to have local topological triviality, such as in the case of Whitney stratified spaces. Indeed, an arbitrary complex map can have a locally infinite number of local topological types at points of the source space.

Thus, research on the topology of complex maps was mainly focused on maps satisfying Thom’s $a_f$ condition or similar conditions implying some kind of local topological triviality and for example leading the way to vanishing cycles (see e.g. [1], [6] and references therein for both classical and recent results on $a_f$ maps). Therefore, the topology of equidimensional maps seems to have received much greater attention than that of non-equidimensional ones. In particular, it seems to have been unknown, that if the generic fibre is discrete but there are special fibres of positive dimension, then there is a lower bound on the number of points in the generic fibre (Theorem 14), which has a simple form if the fibres of positive dimension are isolated (Theorem 15).

Not having topological constructibility in general, we can still get some insight into the topological structure of holomorphic maps. Using Hironaka’s flattening theorem [6] (and its local version by Hironaka, Lejeune-Jalabert and Teissier [8]), Sabbah proves that any map can be made into an $a_f$ map, after a base change by a blowup, thus giving a precise

*Supported by a Foundation for Polish Science (FNP) scholarship.
meaning to Thom’s “hidden blowups”. A recent result of Parusiński [13] states that the set of points at which a holomorphic map is not open is analytically constructible. In this paper we shall deal with the following natural problem concerning fibres of holomorphic maps which, to our best knowledge, has not been treated even for complex algebraic maps. Take $i$ points in a fibre of a holomorphic map $f$ and ask whether one can approximate them simultaneously by systems of $i$ points in arbitrarily general neighbouring fibres. Then ask for what maximal $i$ this is always possible and call that number $\phi(f)$. Our aim is to prove several theorems about $\phi(f)$ and give some applications of it. As will become clear from our results, $\phi(f)$ gives some idea of how general fibres converge to special fibres.

In particular we shall prove, that for maps to a locally irreducible space, $\phi(f)$ is infinite iff the map is open and on the other hand, if $\phi(f)$ is finite, then it is smaller than the dimension of the target space (Theorem 4). We also have similar results for maps to general spaces (Theorems 5 and 6). For maps to a smooth space, we shall obtain an effective formula for $\phi(f)$ in terms of dimensions of the loci where fibres have constant dimension (Theorem 11). As a consequence, we obtain a lower bound for the number of points in a generic discrete fibre of a holomorphic map, which also has fibres of small positive dimension (Theorems 14 and 15).

## 2 Statement of results.

We start by defining $\phi(f)$ precisely. For the sake of clarity, as above we break the definition up into two parts.

**Definition 1** Let $f : X \to Y$ be a holomorphic map of analytic spaces. Let $x_1, \ldots, x_i$ belong to one fibre $f^{-1}(y)$. We say that the sequence of points $x_1, \ldots, x_i$ can be approximated by general fibres iff for any boundary set (set with empty interior) $B \subset Y$, there exists a sequence $\{y_j\}$ in $Y - B$ such that $y_j \to y$ and sequences $\{x_{1j}\}, \ldots, \{x_{ij}\}$ such that for all $k$, $x_{kj} \in f^{-1}(y_j)$, for all $j$ and $x_{kj} \to x_k$, with $j \to \infty$.

**Definition 2** For a holomorphic map of analytic spaces $f : X \to Y$ define $\phi(f)$ as the supremum of all $i$, such that any sequence of $i$ points in (any) one fibre of $f$ can be approximated by general fibres (and as zero if no such $i$ exists).

In this paper analytic space means reduced complex analytic in the sense of Serre (cf. [11]). Analytic spaces shall always be considered with their transcendental topology (and not the Zariski topology). While no assumption will be made on the source space $X$, our results will depend on the different assumptions that we shall make on the target space $Y$. In particular, throughout the paper we assume that $Y$ is of finite dimension. Notice that the value of $\phi(f)$ will not change if in the definition we demand only that any sequence of pairwise different points in one fibre can be approximated. The arbitrary choice of the boundary set translates the intuitive notion of arbitrarily general fibre. It
will follow easily from our results that for proper maps “boundary” can be replaced by “nowhere dense analytic”, without changing $\phi(f)$. For (not necessarily proper) algebraic maps “boundary” can be replaced by “nowhere dense algebraic”.

The following examples illustrate the meaning of the number $\phi$. The value of $\phi$ is infinite for a locally trivial fibration and is zero for a closed (nontrivial) embedding in a complex manifold. For a blowup and more generally for any modification, $\phi$ is equal to 1. The example below, shows that different values of $\phi$ are possible.

**Example 3** Fix an integer $d \geq 1$. Consider $\mathbb{C}^d \times \mathbb{C}^d$ with variables $(y_1, \ldots, y_d, x_1, \ldots, x_d)$ and let $X$ be the hypersurface given by the equation $y_1x_1 + \cdots + y_dx_d = 0$. Let $f : X \to \mathbb{C}^d$ be the restriction of the first projection. Then it is easy to see that $\phi(f) = d - 1$.

The map in the above example is in fact the canonical projection of a spectrum of a symmetric algebra $[18]$ of a $\mathbb{C}[y_1, \ldots, y_d]$-module to the spectrum of $\mathbb{C}[y_1, \ldots, y_d]$. Using different terminology [4], one would say that it is the structural projection of a linear space (“vector bundle with singularities”) associated to a coherent sheaf on $\mathbb{C}^d$. In this particular case, an equivalent problem to that of bounding $\phi(f)$ has been studied in [10] and has produced a criterion for projectivity.

The main goal of this paper is to prove the following four theorems.

**Theorem 4** Let $f : X \to Y$ be a holomorphic map of analytic spaces, the space $Y$ being of dimension $d$ and locally irreducible. Then the following conditions are equivalent:

1. $\phi(f) = \infty$,
2. $\phi(f) \geq d$,
3. $f : X \to Y$ is an open map.

The above theorem says in particular that if $\phi(f) < \infty$, then then the number of points in a special fibre that can be approximated by points in a general fibre is small: $\phi(f) \leq d - 1$. Notice that this bound does not depend on the source space and the map, but only on the dimension of the target space.

This theorem can be generalized to the case of a non locally irreducible target in two ways. The first one is yet again a characterization of openness.

**Theorem 5** Let $f : X \to Y$ be a holomorphic map of analytic spaces. Let $d = \dim Y$ and let $\pi : \hat{Y} \to Y$ be the normalization of $Y$. Let $\hat{f} : \hat{Y} \times_Y X \to \hat{Y}$ be the canonical map. Then the following conditions are equivalent:

1. $\phi(\hat{f}) = \infty$,
2. $\phi(\hat{f}) \geq d$,
3. $f : X \to Y$ is an open map.
Remember that we are dealing with the transcendental topology, where the openness of a map does not have to agree with openness in the Zariski topology (consider the normalization of an irreducible curve with an ordinary double point). In fact, in the algebraic case, openness in the transcendental topology is equivalent to universal openness in the Zariski topology (see [13]).

Another generalization of theorem 4 requires us to recall some notions. Recall (cf. [11] p.295, [16] p.16) that for any holomorphic map \( f : Z \to Y \) of analytic spaces, the fibre dimension and the Remmert Rank of \( f \) at \( z \in Z \) are defined by

\[
\text{fbd}_z f = \dim_z f^{-1}(f(z)), \quad \rho_z f = \dim_z Z - \text{fbd}_z f.
\]

Recall also, that we have the inequality \( \rho_z f \leq \dim_{f(z)} Y \). As in [11], given a map \( f : Z \to Y \) and a subset \( V \subset Y \), we shall denote the two-sided restriction \( f|_{f^{-1}V} : f^{-1}V \to V \), by the symbol \( f^V \).

**Theorem 6** Let \( f : X \to Y \) be a holomorphic map of analytic spaces. Suppose that there is an integer \( D \) such that the sum of dimensions of irreducible components of any germ of \( Y \) is at most equal \( D \). Then the following conditions are equivalent:

1. \( \phi(f) = \infty \),
2. \( \phi(f) \geq D \),
3. for any \( y \in Y \), there is a neighbourhood \( V \) of \( y \) in \( Y \), an irreducible component \( V_1 \) of \( V \) passing through \( y \) and irreducible at \( y \), such that for any \( x \in f^{-1}(y) \) we have \( \rho_x f^V_1 = \dim_y V_1 \).

As we shall see from further results, the above theorem is a direct generalization of the locally irreducible case. Two things differ: \( f \) need not be an open map if \( \phi(f) \) is infinite and the value of \( \phi(f) \) can be greater than the dimension of \( Y \) even if \( \phi(f) \) is finite. The examples below illustrate these two phenomena respectively.

**Example 7** Consider \( \mathbb{C}^4 \) with variables \((y_1, y_2, y_3, y_4)\) and let \( Y \subset \mathbb{C}^4 \) be the “cross”, given by the equations \( y_1y_3 = y_1y_4 = y_2y_3 = y_2y_4 = 0 \). Consider \( Y \times \mathbb{C}^2 \) with additional variables \((x_1, x_2)\) and let \( X \subset Y \times \mathbb{C}^2 \) be given by the equation \( y_1x_1 + y_2x_2 = 0 \). Let \( f : X \to Y \) be the restriction of the first projection. Then \( f \) is not open, but \( \phi(f) = \infty \).

**Example 8** Fix a positive integer \( n \) and positive integers \( d_1, \ldots, d_n \). Let \( D = d_1 + \ldots + d_n \) and consider \( \mathbb{C}^D \) with variables \((y_{jk})\), \( j = 1, \ldots, n \) and \( k = 1, \ldots, d_j \). Let \( Y \) be the reduced subspace of \( \mathbb{C}^D \), defined by the monomial equations

\[
\{ y_{j_1k_1} \cdots y_{j_nk_n} = 0 \mid j_s \neq s \text{ and } k_s = 1, \ldots, d_{j_s}, \text{ for } s = 1, \ldots, n \}.
\]
Then it is obvious, that the germ of $Y$ at 0 has $n$ irreducible components, with respective dimensions $d_1, \ldots, d_n$. Now fix $j$ and consider $Y \times \mathbb{C}^{d_j}$, with additional variables $(x_1, \ldots, x_{d_j})$ on $\mathbb{C}^{d_j}$. Let $X_j$ be the subspace of $Y \times \mathbb{C}^{d_j}$, defined by the equation

$$y_1x_1 + \cdots + y_{d_j}x_{d_j} = 0.$$  

Let $f_j : X_j \to Y$ be the restriction of the natural projection. Now, define the space $X$ as the disjoint sum of the spaces $X_1, \ldots, X_n$. Consider the map $f : X \to Y$, which coincides with $f_j$ on each $X_j$. One can easily calculate that $\phi(f) = D - 1$.

Our final results will be an effective formula for $\phi(f)$, for maps to a smooth space and its consequences. Our invariant will be read off a partition of the source space which is well behaved with respect to the Remmert Rank.

**Definition 9** Let $f : X \to Y$ be a holomorphic map of analytic spaces. A countable partition $\{X_p\}_{p \in P}$ of $X$ is called a rank partition (for $f$) if for each $p \in P$:

1. $X_p$ is a nonempty irreducible locally analytic subset of $X$,
2. $f|_{X_p} : X_p \to Y$ has constant Remmert Rank.

Standard arguments in stratification theory provide us with the following proposition.

**Proposition 10** For any holomorphic map $f : X \to Y$, there exists a rank partition of $X$.

Actually it is always possible to find a locally finite rank partition which also has the property that the closure of each set $X_p$ is analytic. With a little more work one can prove that any holomorphic map has a rank stratification, i.e. a partition as above, which satisfies the boundary condition: for any $p, q$ if $\bar{X}_p \cap X_q \neq \emptyset$, then $\bar{X}_p \supset X_q$. However, a partition is more than enough for our results.

From a rank partition as above, we can read off some numerical data:

$r_p$ – the constant Remmert Rank of $f|_{X_p}$,

$k_p = \dim X_p$,

$h_p = \min\{\dim_x X : x \in X_p\}$.

These data alone allow us to evaluate our invariant. This is done in the theorem below.

**Theorem 11** Let $f : X \to Y$ be a holomorphic map of analytic spaces, the space $Y$ being smooth of pure dimension $d$. Let $\{X_p\}_{p \in P}$ be a rank partition of $X$. Then

$$\phi(f) = \inf\left\{\left[\frac{d - r_p - 1}{(k_p - r_p) - (h_p - d)}\right] : p \in P, \ k_p - r_p > h_p - d \right\},$$

where the square brackets indicate the integer part of a rational number.
The geometric meaning of the fraction in the above theorem is roughly the following: the denominator is the difference between the dimension of a special fibre and the dimension of the general fibre and the numerator is the codimension in $Y$ of the locus where that change in dimension occurs minus one.

Although, from our proof it will easily follow that $\phi(f)$ is smaller or equal to the infimum in the above theorem even if $Y$ is not smooth, equality no longer holds in the general case. This is shown in the following example.

**Example 12** Let $Y$ be the space of 2 by 2 complex matrices with vanishing determinant. Let $X$ be the subset of $Y \times \mathbb{P}^1\mathbb{C}$, consisting of all points $(A, (\lambda : \mu))$ satisfying the equation $A \binom{\lambda}{\mu} = 0$. Let $f : X \to Y$ be the restriction of the first projection. Then $\phi(f) = 1$, but the infimum in Theorem 11 is equal to 2.

In fact, a formula for $\phi(f)$, in the case when $Y$ is singular, would have to include more data than are used in the formula of theorem 11:

**Example 13** Embed the space $Y$ of the preceding example as the hypersurface of $\mathbb{C}^4$ satisfying $xy - zw = 0$. Define $X' \subset Y \times \mathbb{P}^1\mathbb{C}$, by the equation $x^2 + y\lambda\mu + (z - w)\mu^2 = 0$ and again, let $g : X' \to Y$ be the restriction of the first projection. The numerical data used in the formula in theorem 11 corresponding to $f$ and $g$ are the same, but $\phi(f) = 1$ and $\phi(g) = 2$.

Theorem 11 gives us a relationship between the dimensions of different fibres and the way they are attached to each other. From it one can deduce other information concerning the topology of holomorphic maps in more specific cases. For example, when dealing with a map $f$ whose generic fibres are discrete sets, it is obvious that if $f$ also has fibres of positive dimension, then $\phi(f)$ is not greater than the number of points in a generic fibre. In this situation, Theorem 11 provides the following lower bound for such a number (square brackets still denote the integer part).

**Theorem 14** Let $f : X \to Y$ be a holomorphic map of analytic spaces, the space $Y$ being smooth and both $X$ and $Y$ being of pure dimension $d$. Let $\{X_p\}_{p \in P}$ be a rank partition of $X$. Suppose that $f^{-1}(f(x))$ is a discrete set for $x$ belonging to some open dense subset of $X$ and that $f$ has at least one fibre of positive dimension. Then there exists an open dense subset $U$ of $X$, such that for all $x \in U$

$$\#f^{-1}(f(x)) \geq \inf \left\{ \left\lfloor \frac{d - r_p - 1}{k_p - r_p} \right\rfloor : p \in P, k_p > r_p \right\}.$$ 

For an isolated fibre of positive dimension this gives:

**Theorem 15** Let $f : X \to Y$ be a holomorphic map of analytic spaces, the space $Y$ being smooth and both $X$ and $Y$ being of pure dimension $d$. Let $y_0$ be a point in $Y$, such that
\[ \dim f^{-1}(y_0) = w_0 > 0 \text{ and } \dim f^{-1}(y) = 0, \text{ for } y \neq y_0. \] Then there exists an open dense set \( U \) of \( X \), such that for all \( x \in U \)

\[ \#f^{-1}(f(x)) \geq \left\lfloor \frac{d-1}{w_0} \right\rfloor. \]

The meaning of the above theorems is that if a map has discrete generic fibre, but also special fibres of small positive dimension along small sets, then there must be many points in the discrete generic fibre. An example of this situation is the universal homogeneous polynomial:

**Example 16** Consider \( \mathbb{C}^d \) with coordinates \( x_0, \ldots, x_{d-1} \) and \( \mathbb{P}^1 \mathbb{C} \) with homogeneous coordinates \((\lambda : \mu)\). Let \( X \) be the subspace of \( \mathbb{C}^d \times \mathbb{P}^1 \mathbb{C} \) defined by the equation

\[ x_0 \lambda^{d-1} + x_1 \lambda^{d-1} \mu + \cdots + x_{d-1} \mu^{d-1} \]

and let \( f : X \to \mathbb{C}^d \) be the restriction of the first projection. Then the fibre of \( f \) at \( 0 \) is of dimension one; all the other fibres are zero-dimensional and the generic fibre has \( d-1 \) points (its smallest possible cardinality by theorem 15).

Again, if \( Y \) is singular, then theorems 14 and 15 fail, as is seen from example 12. Further counterexamples are provided by "small contractions" (see [2]), which are the basis of the study of threefolds.

The invariant \( \phi(f) \) is much less well behaved in real geometry. For example, in theorem 6 the only implications that are true are: openness implies \( \phi(f) = \infty \) implies \( \phi(f) \geq d \). In particular one can find real algebraic maps to \( \mathbb{R}^2 \) with arbitrary value of \( \phi(f) \):

**Example 17** Fix a positive integer \( n \). Let \( \kappa_n = \tan \left( \frac{1}{2} \left( \frac{1}{n+1} + \frac{1}{n} \right) \pi \right) \). Define the the real algebraic subset \( X_n \subset \mathbb{R}^5 \) by the equation

\[ x_5^2 = (x_1 x_4 + x_2 x_3)(x_1 x_4 + x_2 x_3 - \kappa_n (x_1 x_3 - x_2 x_4)) \]

and let \( f_n : X_n \to \mathbb{R}^2 \) be the restriction of the orthogonal projection on the \((x_1, x_2)\) plane. It is easy to calculate that \( \phi(f_n) = n \).

### 3 Fibred powers and quasiopenness.

In the category of analytic spaces fibred products exist are isomorphic to the usual ones (see [9], p.200) after reduction.

**Definition 18** Let \( f : X \to Y \) be a holomorphic map of analytic spaces and \( i \geq 1 \). By the \( i \)-th fibred power of \( f \), we mean the pair \((X^{(i)}, f^{(i)})\) consisting of the space \( X^{(i)} = X \times_Y \ldots \times_Y X \) and the canonical map \( f^{(i)} : X^{(i)} \to Y \).
We shall use the same definition for fibred powers of continuous maps of topological spaces. The $i$-fold direct product of a space by itself will be denoted $X^i$. By definition, $X^0$ will be a point.

Since a point in $X^{(i)}$ is nothing else but a sequence of $i$ points in a fibre of $f$, we can easily obtain the following:

**Remark 19** For any $i \geq 1$, $\phi(f) \geq i$ iff $\phi(f^{(i)}) \geq 1$.

Hence, it is natural to determine what maps $f$ have $\phi(f) \geq 1$. We introduce the following notion.

**Definition 20** A map of topological spaces $f : Z \to Y$ is called quasiopen if for any subset $A \subset Z$ with nonempty interior in $Z$, its image $f(A)$ has nonempty interior in $Y$.

Any open map is quasiopen. The blowup $\mathbb{C}^2$ at the origin is an example of a quasiopen map which is not open. It is immediate that a map is quasiopen if and only if the image of any nonempty open set has nonempty interior. By elementary point-set topology one proves the following for first countable topological spaces.

**Remark 21** For a map of topological spaces $f : Z \to Y$ the following conditions are equivalent.

1. $f$ is quasiopen,
2. for any boundary set $B \subset Y$ its inverse image $f^{-1}(B)$ is a boundary set in $Z$,
3. $\phi(f) \geq 1$.

Thus, by the third equivalent condition, what we shall be looking at, will be the quasiopeness of fibred powers of holomorphic maps. The above two remarks easily imply the following

**Proposition 22** $\phi(f) = \sup \{\{0\} \cup \{i \geq 1 : f^{(i)} \text{ is quasiopen}\}\}$.

We must see more closely what quasiopeness means in the analytic case. The following proposition shows us that. We leave out its proof, which can be done by standard techniques of analytic geometry.

**Proposition 23** For a holomorphic map of analytic spaces $f : Z \to Y$, the following conditions are equivalent:

1. $f$ is quasiopen,
2. the restriction of $f$ to each irreducible component of $Z$ is quasiopen,
3. the image by $f$ of each irreducible component of $Z$ has nonempty interior in $Y$. 

8
4 The Remmert Rank.

In this section we have gathered some facts about the Remmert Rank which we shall need in the sequel. The usefulness of the Remmert Rank comes from the following well known theorem (see e.g. [1], p. 296).

**Remmert Rank Theorem** Let \( f : X \to Y \) be a holomorphic map of analytic spaces, the space \( X \) being of pure dimension. Suppose that \( \rho_x f = k \) for all \( x \in X \). Then every point of \( X \) has an arbitrarily small open neighbourhood, whose image is a locally analytic subset of \( Y \), of pure dimension \( k \).

We shall need mainly some results about the sets where the Remmert Rank takes on a different value from its generic value. The first of these are two remarks.

**Remark 24** Let \( f : W \to Y \) be a quasiopen holomorphic map to an analytic space of pure dimension \( d \). Let \( W_1 \) be an irreducible locally analytic subset of \( W \) such that \( \rho_z (f|_{W_1}) < d \) for all \( z \in W_1 \). Then \( \dim W_1 < \min \{ \dim_z W | z \in W_1 \} \).

**Proof of Remark 24.** If the conclusion of the remark were false, then \( W_1 \) would contain a nonempty open subset of \( W \). By the Remmert Rank Theorem this would contradict quasiopenness. Remark 24 immediately implies the next one.

**Remark 25** Let \( f : Z \to Y \) be a holomorphic map of analytic spaces, the space \( Y \) being irreducible of positive dimension. If \( f \) is quasiopen, then \( \text{fbd}_z f < \dim_z Z \) for any \( z \in Z \).

In the following sections we shall also make use of a lemma describing the ”critical values” with respect to the Remmert Rank.

**Lemma 26** (Sard theorem for the Remmert Rank.) Let \( f : X \to Y \) be a holomorphic map of analytic spaces, the space \( Y \) being irreducible of dimension \( d \). Then the set \( C(f) = f(\{ x \in X | \rho_x f < d \}) \) is a first category set.

**Proof of lemma 26.** Take a rank partition \( \{ X_p \}_{p \in P} \) for \( f \). The lemma will follow from the the Remmert Rank Theorem if we prove that the set \( C(f) \) is contained in the union of images of those sets \( X_p \) for which \( r_p < d \). (Notice that not all points \( x \in X \) with \( \rho_x f < d \) have to belong to some \( X_p \) with \( r_p < d \).) So, take \( y \in C(f) \). There exists a point \( x \) in the fibre \( f^{-1}(y) \) such that \( \rho_x f < d \). Let \( Z \) be a component of \( f^{-1}(y) \) passing through \( x \), of maximal dimension among such components. Then it is clear that \( \min \{ \dim_z X : z \in Z \} - \dim Z < d \). Then the family \( \{ Z \cap X_p \}_{p \in P} \) is an analytic partition of \( Z \) and therefore for some \( p, X_p \) contains a nonempty open subset of \( Z \). Then \( y \in f(X_p) \) and from the above inequality it follows that \( r_p < d \).
5 Maps to a locally irreducible space.

Theorem 4 is an immediate corollary of the theorem below and Proposition 22.

**Theorem 27** Let $f : X \to Y$ be a holomorphic map of analytic spaces, the space $Y$ being of dimension $d$ and locally irreducible. Then the following conditions are equivalent:

1. the maps $f^{(i)} : X^{(i)} \to Y$ are quasiopen for all $i = 1, 2, \ldots$,
2. the map $f^{(d)} : X^{(d)} \to Y$ is quasiopen,
3. the map $f^{(i)} : X^{(i)} \to Y$ is quasiopen for some $i \geq d$,
4. the map $f : X \to Y$ is open.

The above theorem provides an effective way of checking whether a given holomorphic map is open. Indeed, by condition 2 of theorem 27, one has to investigate the quasiopenness of the $d$-th fibred power of the map, which by condition 3 of proposition 23 can be tested just by looking at images of irreducible components. Thus, combined with primary decomposition algorithms ([3]), it provides algorithms for testing the openness of a map.

**Proof of Theorem 27.**

The space $Y$ being locally irreducible, its irreducible components are actually its connected components. Their dimensions are bounded from above by $d$. Therefore it is clear that in the proof of theorem 27 we can assume that $Y$ is actually irreducible. Our proof will be structured as follows. First, we observe that condition 1 implies condition 2 and condition 2 implies condition 3 in a trivial way. It is also fairly easy to see that condition 4 implies condition 1, when one notices that each map $f^{(i)} : X^{(i)} \to Y$ is actually open as the restriction of the open map $(f, \ldots, f) : X \times \ldots \times X \to Y \times \ldots \times Y$ to the inverse image of the diagonal in $Y \times \ldots \times Y$.

The hard part of the proof of Theorem 27 lies in showing that condition 3 implies condition 4, which we shall now do. We shall need the following lemma.

**Lemma 28** Let $f : X \to Y$ be a holomorphic map of analytic spaces, the space $Y$ being irreducible of dimension $d$. Suppose that $i \geq d$ and $f^{(i)} : X^{(i)} \to Y$ is quasiopen. Then $\rho_x f = d$ for every $x \in X$.

Notice that in the above lemma we do not need $Y$ to be locally irreducible.

**Proof of Lemma 28.** Fix $x_0 \in X$ and suppose that $\rho_{x_0} f = d - k$, $0 \leq k \leq d$. Let $m = \dim_{x_0} X$. Without loss of generality we can assume that $\dim X = m$. Let $C(f)$ be as in lemma 26. Observe that $\dim f^{-1}(y) \leq m - d$ and hence $\dim (f^{(i)})^{-1}(y) \leq i(m - d)$ for $y \notin C(f)$. 


Now, in $X^{(i)}$ consider the subset $A = (f^{(i)})^{-1}(C(f))$. Since, by lemma 26, $C(f)$ is a boundary set, therefore by remark 21, $A$ is a boundary set in $X^{(i)}$. Therefore we have
\[
\dim X^{(i)} = \sup \{ \dim_z X^{(i)} : z \notin A \} \leq i(m - d) + d.
\]
We can restrict our attention to the case $d \geq 1$. Set $z_0 = (x_0, \ldots, x_0) \in X^{(i)}$ and observe that $\text{fbd}_{x_0} f = m - d + k$ and so $\text{fbd}_{z_0} f^{(i)} = i(m - d + k)$. Since, by remark 23, $\text{fbd}_{z_0} f^{(i)} < \dim X^{(i)}$, we get $k < \frac{d}{i}$ and so $k = 0$. This completes the proof of the lemma.

Now we can conclude the proof of theorem 27. Take $x_0 \in X$. By lemma 28, $\rho_{x_0} f = d$. Let $X_1$ be an irreducible component of maximal dimension passing through $x_0$. Notice that also $\rho_x (f|_{X_1}) = d$, for any $x$ in a small neighbourhood $U$ of $x_0$ in $X_1$. Since $Y$ is locally irreducible, by the Remmert Rank Theorem $f|_{U}$ is open. Therefore, for any neighbourhood $V$ of $x_0$ in $X$, the image $f(V)$ contains $f(V \cap U)$ and hence is a neighbourhood of $f(x_0)$. Since this holds for any $x_0 \in X$, the map $f$ is open.

To conclude this section, remark that since condition 4 of theorem 27 implies openness of all maps $f^{(i)}$, $i = 1, 2, \ldots$, as an immediate corollary we obtain that for any $i \geq d$ the map $f^{(i)}$ is quasiopen iff it is open.

Notice that Theorem 27 and Lemma 28 combined provide an easy proof of Remmert’s Open Mapping Theorem.

6 Openness in the general case.

To prove Theorem 5, we first state and prove a purely topological proposition.

**Proposition 29** Let $f : X \to Y$ be a map of topological spaces. Let $\pi : \hat{Y} \to Y$ be a surjective, continuous map of topological spaces with the property that for any point $y \in Y$ and for any open neighbourhood $U$ of $\pi^{-1}(y)$ in $\hat{Y}$, $\pi(U)$ is a neighbourhood of $y$. Let $\hat{f} : \hat{Y} \times_Y X \to \hat{Y}$ be the canonical (base change) map. Then $f$ is open if and only if $\hat{f}$ is open.

**Proof of Proposition 29.** If $f$ is open, then $\hat{f}$ is open just by the continuity of $\pi$: embedding $\hat{Y} \times_Y X$ in $\hat{Y} \times X$ one verifies easily that for open sets $U \subset \hat{Y}$ and $V \subset X$, one has $\hat{f}((U \times V) \cap (\hat{Y} \times_Y X)) = U \cap \pi^{-1}(f(V))$ . Thus $\hat{f}$ is indeed open.

Now, suppose that $\hat{f}$ is open. For any point $x \in X$, taking a neighbourhood $V$ of $x$ in $X$, one observes that $f(V) = \pi(\hat{f}((\hat{Y} \times V) \cap (\hat{Y} \times_Y X)))$, and thus $f(V)$ is a neighbourhood of $y = f(x)$ by the openness of $\hat{f}$ and the properties of $\pi$. Hence $f$ is open. This ends the proof of proposition 29.
Remark 30 If $\pi : \hat{Y} \to Y$ is a closed, surjective, continuous map, then the condition imposed on $\pi$ in proposition 29 is satisfied. In particular this is the case when $Y$ is Hausdorff and first countable and $\pi$ is proper, surjective and continuous.

Proposition 29 and remark 30 easily imply the following (cf. also Lemma 1.5 in [13]).

Proposition 31 Let $f : X \to Y$ be a holomorphic map of analytic spaces. Let $\pi : \hat{Y} \to Y$ be the normalization of $Y$ and let $\hat{f} : \hat{Y} \times_Y X \to \hat{Y}$ be the canonical map. Then $f$ is open if and only if $\hat{f}$ is open.

Since the normalization of an analytic space is locally irreducible, we can apply theorem 4 to the map $\hat{f}$. Then, together with proposition 31 they imply theorem 5.

7 Quasiopen fibred powers in the general case.

This section is devoted to proving theorem 6. As before, it will follow easily from a theorem about the quasiopenness of fibred powers and Proposition 22.

Theorem 32 Let $f : X \to Y$ be a holomorphic map of analytic spaces. Suppose that there is an integer $D$ such that the sum of dimensions of irreducible components of any germ of $Y$ is at most equal to $D$. Then the following conditions are equivalent:

1. the maps $f^{(i)} : X^{(i)} \to Y$ are quasiopen for all $i = 1, 2, \ldots$,
2. the map $f^{(D)} : X^{(D)} \to Y$ is quasiopen,
3. the map $f^{(i)} : X^{(i)} \to Y$ is quasiopen for some $i \geq D$,
4. for any $y \in Y$, there is a neighbourhood $V$ of $y$ in $Y$, an irreducible component $V_1$ of $V$ passing through $y$ and irreducible at $y$, such that for any $x \in f^{-1}(y)$ we have $\rho_x f^{V_1} = \dim_y V_1$.

To prove theorem 32, we shall need the following lemma.

Lemma 33 Let $f : X \to Y$ be a holomorphic map of analytic spaces. Fix $y \in Y$ and suppose that $Y$ is locally irreducible at $y$. If $d = \dim_y Y$, then the following conditions are equivalent:

1. for all $x \in f^{-1}(y)$ we have $\rho_x f = d$,
2. for any $i = 1, 2, \ldots$, for any open set $U$ in $X^{(i)}$, with $y \in f^{(i)}(U)$, $f^{(i)}(U)$ has nonempty interior,

3. for any open set $U$ in $X^{(d)}$, with $y \in f^{(d)}(U)$, $f^{(d)}(U)$ has nonempty interior.

Proof of Lemma 33. To prove that condition 1 implies condition 2, first observe, that by the Remmert Rank Theorem condition 1 implies that for any $x \in f^{-1}(y)$, the image by $f$ of any neighbourhood of $x$ is a neighbourhood of $y$. Now take $U$ as in condition 2. The fact that $y \in f^{(i)}(U)$, implies that $U$ contains an element $z = (x_1, \ldots, x_i)$, with $f(x_1) = \ldots = f(x_i) = y$. Thus there are neighbourhoods $U_j$ of each $x_j$ in $X$, such that $U \supset X^{(i)} \cap (U_1 \times \cdots \times U_i)$ (here $X^{(i)}$ is embedded in $X^i$). Hence $f^{(i)}(U)$ contains the intersection of all the $f(U_j)$, which as we have observed is a neighbourhood of $y$. In particular it has nonempty interior, thus showing that condition 2 is fulfilled.

It is trivial that 2 implies 3. To prove that condition 3 implies condition 1, let $Z$ be the sum of those components of $X^{(d)}$ on which $f^{(d)}$ is quasiopen. Remark that condition 3 implies that the fibre $(f^{(d)})^{-1}(y)$ is contained in $Z$. Now we can copy the proof of lemma 28, taking $i = d$ and replacing $X^{(i)}$ by $Z$. We have thus ended the proof of lemma 33.

Proof of Theorem 24. Again, 1 implies 2 implies 3 in a trivial way. Now let us prove that 3 implies 4. Suppose that condition 4 is not fulfilled for a point $y$ in $Y$. Take a neighbourhood $V$ of $y$ in $Y$ such that all the irreducible components $V_1, \ldots, V_s$ of $V$ contain $y$ and are locally irreducible at $y$. Let $d_j = \dim V_j$. By our assumption, for each $j$, one can choose a point $x_j \in f^{-1}(y)$, such that $\rho_{x_j}f^{V_j} < d_j$. Embed canonically $(f^{-1}V_j)^{\{d_j\}} \subset X^{(d_j)}$. By lemma 33, for each $j$ there is an open subset $U_j$ of $X^{(d_j)}$, with $y \in f^{(d_j)}(U_j)$ and such that the set $(f^{V_j})^{\{d_j\}}(U_j \cap (f^{-1}V_j)^{\{d_j\}})$ has empty interior in $V_j$. In other words, the set $f^{(d_j)}(U_j) \cap V_j$ has empty interior in $V$.

Now, fix $i \geq D$ and embedding $X^{(i)} \subset X^{(d_1)} \times \cdots \times X^{(d_s)} \times X^{i-(d_1+\cdots+d_s)}$, let $U = X^{(i)} \cap (U_1 \times \cdots \times U_s \times X^{i-(d_1+\cdots+d_s)})$. Now we have $y \in f^{(d_j)}(U_j)$ for all $j$ and hence $U$ is a nonempty (open) set in $X^{(i)}$. Furthermore, for all $j$, the intersection $f^{(i)}(U) \cap V_j$ is contained in $f^{(d_j)}(U_j) \cap V_j$ and hence has empty interior. Therefore, $f^{(i)}(U)$ has empty interior in $V$. We have thus proved that $f^{(i)}$ is not quasiopen and so ended the proof of this implication.

Now we shall prove that 4 implies 1. Fix $i$ and take a nonempty open set $W$ in $X^{(i)}$. Choose $z \in W$ and $y = f(z)$. Take $V_1$ from condition 4 and apply lemma 33 to $f^{V_1}$, to find that $f^{(i)}(W) \cap V_1$ has nonempty interior. Hence $f^{(i)}(W)$ has nonempty interior. We have shown quasiopenness, ending the proof.

8 Maps to a smooth space.

This section is devoted to the proof of Theorem 11. First notice, that if the map $f$ itself is not quasiopen, then there exists $p$, with $h_p = k_p$ and $r_p < d$. Therefore, the formula in Theorem 11 produces 0 as it should. Hence, we can suppose that $f = f^{(1)}$ is
quasiopen in our proof. For convenience, in addition to the numerical data defined after
the statement of Proposition 10, we shall denote \( w_p = k_p - r_p = \text{fbd}_x(f|_{X_p}) \) for all \( x \in X_p \).
Given \( (p_1, \ldots, p_i) \in P^i \) we shall denote \( X_{p_1} \times_Y \cdots \times_Y X_{p_i} \) by \( X^{(p_1, \ldots, p_i)} \). We shall use the
expression of \( \phi(f) \) given in Proposition 24. The proof will be carried out in 8 steps.

**Proof of Theorem 11**

**Step 1** \( \dim X^{(p_1, \ldots, p_i)} \leq r_{p_j} + (w_{p_1} + \ldots + w_{p_i}) \) for \( j = 1, \ldots, i \).

Fix \( j \). The fibres of the natural map \( X^{(p_1, \ldots, p_i)} \to Y \) are of dimension \( w_{p_1} + \ldots + w_{p_i} \).
The image of a small neighbourhood of any point \( (x_1, \ldots, x_i) \in X^{(p_1, \ldots, p_i)} \) is contained
in the image of a small neighbourhood of \( x_j \in X_{p_j} \), which is of dimension \( r_{p_j} \) by the
Remmert Rank Theorem. Notice that \( \dim (X_p^{(i)}) \) such that \( \dim X^{(i)} = \text{dim} X^{(p_1, \ldots, p_i)} \).
Now take \( j \), such that \( w_{p_j} = \max\{w_{p_1}, \ldots, w_{p_i}\} \). By step 1 we obtain
\( \dim X^{(i)} \leq r_{p_j} + iw_{p_j} \) and hence \( \dim X^{(i)} \leq \sup\{r_p + iw_p : p \in P\} \). On the other hand
\( X^{(i)} \) contains all the \( (X_p^{(i)}) \) and so by step 2 we also have the converse inequality.

**Step 4** If \( f^{(i)} \) is quasiopen and \( h_p - d < w_p \) then \( r_p + iw_p < \dim X^{(i)} \).

Let \( W_1 \) be any irreducible component of maximal dimension of \( (X_p^{(i)}) \). By step 2, for
any \( z \in W_1 \) we have in particular \( r_z(f|_{W_1}) \leq r_p \) and by the assumption on \( p, r_p < d \). The
inequality now follows from Remark 24 and Step 2.

**Step 5** If \( f^{(i)} \) is quasiopen then \( \dim X^{(i)} = d + i(\dim X - d) \).

This follows from the formula in Step 3 in which we can eliminate certain indices \( p \), by Step 4.

**Step 6** If \( f^{(i)} \) is quasiopen and \( w_p > h_p - d \), then \( i \leq \frac{d - r_p - 1}{w_p - (h_p - d)} \). This implies one
inequality in Theorem 14.

Let \( W \) be the union of irreducible components of \( X \) with dimension not greater than
\( h_p \) minus the other components. Now we can apply step 4, with \( X \) replaced by \( W \) and
\( X_p \) replaced by \( X_p \cap W \). We obtain \( r_p + iw_p < \dim W^{(i)} \). The previous step gives us a
formula for \( \dim W^{(i)} \), which implies the inequality.
Step 7 Suppose that $f^{(i+1)}$ is not quasiopen. Choose $(p_1, \ldots, p_{i+1})$, such that $X^{(p_1, \ldots, p_{i+1})}$ contains a nonempty open subset $U$ of $X^{(i+1)}$, which is irreducible (as a locally analytic set) and whose image by $f^{(i+1)}$ has empty interior in $Y$. Then $\dim U \geq h_{p_1} + \cdots + h_{p_{i+1}} - \text{id}$.

Embed $X$ in a smooth complex space $M$ (if this can only be done locally, one can carry out a slightly more cumbersome proof using the same idea). For each $j = 1, \ldots, i+1$, let $Z_j$ be the union of irreducible components of $X$ of dimension not greater than $h_{p_j}$. Now $X^{(i+1)}$ is isomorphic to the subspace of the smooth space $M^{i+1} \times Y^{i+1}$, defined as the intersection of the graph of the product map $(f|_{Z_1}, \ldots, f|_{Z_{i+1}}) : Z_1 \times \cdots \times Z_{i+1} \to Y^{i+1}$ and the product space $M^{i+1} \times \Delta$, where $\Delta$ is the diagonal subspace of $Y^{i+1}$. The bound then follows directly from the estimate of the codimension of components of an intersection in a smooth space.

Step 8 If $f^{(i+1)}$ is not quasiopen, then for some $p \in P$, with $w_p > h_p - d$

\[ i \geq \left\lceil \frac{d - r_p - 1}{w_p - (h_p - d)} \right\rceil, \]

where square brackets denote the integer part of a rational number. This proves the remaining inequality in Theorem 11.

We have $\dim U \leq \dim X^{(p_1, \ldots, p_{i+1})}$ and hence by Step 1, $\dim U = r + (w_{p_1} + \cdots + w_{p_{i+1}})$, for some $r$ with $r \leq r_j$ for all $j = 1, \ldots, i+1$. Further, because the Remmert Rank of $f|_U$ is strictly smaller than $d$, we have $r < d$. Combining the above expression of $\dim U$ with the inequality from Step 7 and taking $j$ such that $w_p - h_{p_j} = \max\{w_{p_1} - h_{p_1}, \ldots, w_{p_{i+1}} - h_{p_{i+1}}\}$ one obtains $d - r \leq (i + 1)(d + w_{p_j} - h_{p_j})$. Take $p = p_j$. First, we see that $w_p > h_p - d$. Then, since $r \leq r_p$, we have $d - r_p - 1 < (i + 1)(d + w_p - h_p)$ and so

\[ i + 1 > \frac{d - r_p - 1}{w_p - (h_p - d)}, \]

from where the inequality follows automatically. Theorem 11 follows immediately from steps 6 and 8.

References

[1] J. Briançon, Ph. Maisonobe, M. Merle, Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom, Invent. Math. 117 (1994), 531–550.

[2] H. Clemens, J. Kollár, Sh. Mori, Higher dimensional complex geometry, Astérisque 166 (1988).
[3] D. Eisenbud, C. Huneke, W. Vasconcelos, Direct Methods for primary decomposition, Invent. Math. 110, 207-235 (1992).

[4] G. Fischer, Complex Analytic Geometry, Springer L.N. 538 (1976).

[5] T. Fukuda, Types topologiques des polynômes, Publ. Math. I.H.E.S. 46 (1976), 87–106.

[6] J. P. Henry, M. Merle, C. Sabbah, Sur la condition de Thom pour un morphisme analytique complexe, Ann. Scient. Éc. Norm. Sup., 4e série, t. 17 (1984), 227–268.

[7] H. Hironaka, Flattening theorem in complex analytic geometry, Amer. J. Math. 97 (1975), 199-265.

[8] H. Hironaka, M. Lejeune-Jalabert, B. Teissier, Platificateur local en géométrie analytique et aplatissement local, Astérisque 7-8 (1973), 441–446.

[9] L. Kaup, B. Kaup, Holomorphic Functions of Several Variables, de Gruyter Studies in Maths. 3, 1983.

[10] M. Kwieciński, Tensor powers of symmetric algebras, Communications in Algebra, 24 (1996), 793–801.

[11] S. Lojasiewicz, Introduction to Complex Analytic Geometry, Birkhäuser, Basel, 1991.

[12] I. Nakai, On topological types of polynomial mappings, Topology 23 (1984), 45–66.

[13] A. Parusiński, Constructibility of the set of points where a complex analytic morphism is open, Proc. AMS 117 (1993), 205–211.

[14] R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, Math. Ann. 133 (1957), 328–370.

[15] C. Sabbah, Morphismes analytiques sans éclatement et cycles évanescent, Astérisque 101-102 (1983), 286–319.

[16] W. Stoll, The Multiplicity of a Holomorphic Map, Inv. Math. 2 (1966), 15–58.

[17] R. Thom, La stabilité topologique des applications polynomiales, Enseign. Math. 8 (1962), 24-33.

[18] V.W. Vasconcelos, The Arithmetic of Blowup Algebras, LMS Lecture Note Series 195, Cambridge University Press 1994.

Uniwersytet Jagielloński,
Instytut Matematyki,
ul.Reymonta 4,
30-059 Kraków, Poland.

**e-mail:** kwiecins@im.uj.edu.pl, tworzews@im.uj.edu.pl