Scalar-isoscalar states in the large-$N_c$ Regge approach

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Scalar-isoscalar states ($J^{PC} = 0^{++}$) are investigated within the large-$N_c$ Regge approach. We elaborate on the consequences of including the lightest $f_0(600)$ scalar-isoscalar state into such an analysis, where the position of $f_0(600)$ fits very well into the pattern of the radial Regge trajectory. Furthermore, we point out that the pion and nucleon spin-0 gravitational form factors, recently measured on the lattice, provide valuable information on the low-mass spectrum of the scalar-isoscalar states on the basis of the scalar-meson dominance in the spin-0 channel. Through the fits to these data we find $m_\sigma = 450 - 600$ MeV. We compare the predictions of various fits and methods. An analysis of the QCD condensates in the two-point correlators provides further constraints on the parameters of the scalar-isoscalar sector. We find that a simple two-state model suggests a meson nature of $f_0(600)$, and a glueball nature of $f_0(980)$, which naturally explains the ratios of various coupling constants. Finally, we note that the fine-tuned condition of the vanishing dimension-2 condensate in the Regge approach with infinitely many scalar-isoscalar states yields a reasonable value for the mass of the lightest glueball state.

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I. INTRODUCTION

The history and status of the $\sigma$-meson has been quite vacillating (for reviews see e.g.\textsuperscript{1} 1\textsuperscript{3} and references therein). A scalar-isoscalar state with a mass of $\sim 500$ MeV was originally proposed in the fifties\textsuperscript{4}, as an ingredient of the nucleon-nucleon force providing saturation and binding in nuclei. Along the years, there has always been some arbitrariness in the “effective” or “fictitious” $\sigma$ meson mass and the coupling constant to the nucleon, partly due to the lack of other sources of information. For instance, in the very successful Charge Dependent (CD) Bonn NN-potential\textsuperscript{5}, any partial wave $2S + 1 L J$-channel is fitted with a different scalar-isoscalar meson mass and coupling. The $\sigma$-meson was also introduced as the chiral partner of the pion to account for spontaneous breaking of the chiral symmetry\textsuperscript{6}. The lack of confidence in its existence motivated taking its mass to infinity, yielding the non-linear sigma model\textsuperscript{7}, which is the modern starting point for the Chiral Perturbation Theory\textsuperscript{8}.

During the last decade, the situation has steadily changed, and the $\sigma$-meson has been finally resurrected\textsuperscript{9}, culminating with the inclusion of the $0^{++}$ resonance in the Particle Data Group review (PDG)\textsuperscript{10} as the $f_0(600)$ state, seen as a $\pi\pi$ resonance. It has widespread values for the mass, $400 - 1200$ MeV, and for the width, $600 - 1200$ MeV\textsuperscript{11}. A rigorous definition of the $\sigma$ as a $\pi\pi$ resonance requires that it be a pole of the $\pi\pi$ scattering amplitude in the $(J, T) = (0, 0)$ channel in the second Riemann sheet in the Mandelstam variable $s$. Within such a framework, the uncertainties have recently been narrowed sharpened with a benchmark determination based on the Roy equations with constraints from the chiral symmetry\textsuperscript{12}, yielding the value\textsuperscript{1}\textsuperscript{1}

$$m_\sigma - i\Gamma_\sigma/2 = 441^{+16}_{-8} - i272^{+9}_{-12} \text{ MeV}.$$ (1)

The analysis of Ref.\textsuperscript{13} yields a result with somewhat higher $m_\sigma$, $473 \pm 6 \pm 11 - i257 \pm 5 \pm 2$ MeV (with the errors statistical and systematic, respectively), while the unitarized Chiral Perturbation Theory (χPT) gives a bit lower value of the mass, $401^{+12}_{-16} - i277^{+23}_{-26} \text{ MeV}\textsuperscript{14}$. Nevertheless, various determinations of the pole mass agree with the values\textsuperscript{11} within the uncertainties. These accurate determinations make somewhat tricky the original question on what $\sigma$ mass should be used \textit{a priori} within a meson-exchange picture, due to the very large width of the resonance. Moreover, the determinations mentioned above do not imply necessarily the standard assignment of the linear sigma-model where one takes $(\sigma, \pi)$ as chiral partners in the $(\frac{1}{2}, 0)$ representation of the chiral $SU(2)_R \otimes SU(2)_L$ group. \textit{A priori}, the $\sigma$ state

\textsuperscript{1} We use this definition for the pole mass in the $\sqrt{s}$ variable. A better one is $s_\sigma = m_\sigma^2 - i\Gamma_\sigma m_\sigma$, which coincides with the previous one in the narrow resonance limit.
could also belong to the $(0,0)$ representation. Admittedly, the heated debate on the nature of the $\sigma$ meson is not completely over, with the tetraquark $^{14}$ or glueball interpretations $^{17}$ considered (see, e.g., Ref. $^{1}$ and numerous references therein).

Our goal in the present paper is to point out that the large-$N_c$ Regge models may provide valuable insight into this problem. Clearly, the numbers in Eq. $^{11}$ represent the values for $N_c = 3$. We recall that in the large-$N_c$ limit QCD can be mapped onto a theory of infinitely many stable and non-interacting mesons and glueballs $^{18,19}$. In the standard large-$N_c$ counting, meson and glueball masses are both $O(N_c^0)$ whereas their widths are $O(N_c^{-1})$ and $O(N_c^{-2})$, respectively. However, there is also a large-$N_c$-suppressed mass shift which makes a literal use of Eq. $^{11}$ somewhat questionable. Actually, lattice calculations have shown that QCD$_\infty$ is generally not too far from QCD at $N_c = 3$ (see, e.g., Ref. $^{21}$ for an early investigation). The studies of the $\pi\pi$ scattering, based on a large-$N_c$ scaling of the $N_c = 3$ parameters and unitarized via the Inverse Amplitude Method (IAM) applied to the $\chi$PT amplitudes, provide an $N_c$ dependence of resonances which, regarding the $\sigma$ state, depends on the details of the scheme used. While the one-loop coupled channel approach $^{22}$ yields a very wide range of $m_\sigma$ and a large width (in an apparent contradiction with the standard large-$N_c$ counting $^{18,19}$ when extrapolated to $N_c \gg 10$), the two-loop approach $^{23}$ produces a large shift (by a factor of 2) of $m_\sigma$ when going from $N_c = 3$ to $N_c = \infty$ (the corresponding mass shift is small in the case of the $\rho$ meson). One should note the large uncertainties of the two-loop IAM, documented in Ref. $^{24}$. Along similar large-$N_c$ counting, the $K$-matrix approach $^{25}$ gives $m_\sigma \sim 2\sqrt{3}f_\pi = O(\sqrt{N_c})$. The Bethe-Salpeter approach $^{26}$ yields an estimate $m_\sigma \approx 500$ MeV $^{27}$ at $N_c \to \infty$. A more refined analysis using the large-$N_c$ consistency conditions between the unitarization and resonance saturation suggests $m_\sigma - m_\sigma = O(N_c^{-1})^{14}$.

The large-$N_c$ analysis of the NN force was carried out in Refs. $^{28,29}$ at the quark level and in Ref. $^{30}$ at the hadronic level, providing a justification of the meson-exchange picture. Moreover, it indicates what mesons are the leading ones, namely the infinite tower of $\sigma$, $\pi$, $\rho$, $\omega$, $A_1$, etc., states. Meson widths enter as relative $1/N_c^2$ corrections to the potential, on equal footing with many other effects (the spin-orbit force, relativistic dynamics, or the inclusion of other mesons), independently on how large the $N_c$ width is in the real $N_c = 3$ world. In this regard the large-$N_c$ analysis may provide a reliable determination of the asymptotic value of the $\sigma$ mass at $N_c \to \infty$. A fit to the $^1S_0$ leading-$N_c$ potential yields $m_\sigma = 501(20)$ MeV $^{27}$. Moreover, it was shown in Ref. $^{31}$ within the large-$N_c$ framework, that the correlated $2\pi$ exchange corresponds to a Yukawa potential where the corresponding Yukawa mass differs by a $1/N_c$ correction to the pole mass or equivalently to the Breit-Wigner mass (the difference between both masses is $O(N_c^{-2})^{32}$). Finally, we should also note that using an alternative large-$N_c$ counting $^{33,34}$, the scalar meson behaves as a $qqq$ state $^{35}$ and its width is $O(N_c^{-2})$.

In the present work we take into account an infinite number of scalar-isoscalar states. In order to gather information on their coupling to hadrons and the vacuum, we use a Regge model for the radial trajectories and study correlation functions between the energy-momentum tensor. Surprisingly, we find in Section $^{11}$ that quite naturally all the scalar-isoscalar states can be described by a single radial Regge trajectory with half the standard slope. The mass of the $\sigma$ state can be deduced from this trajectory as the mass of the lowest state. As a consequence, there seems to be no obvious difference between mesons and glueballs, as far as the spectrum is concerned. Despite this surprising ordering with a sort of a radial quantum number, we should warn the reader that states of different nature may not necessarily be characterized by a single quantum number and other interpretations are considered in the literature. For example, $f_0(980)$ is often regarded as $qqqq$, $f_0(1370)$ as mainly $\bar{u}u + \bar{d}d$, $f_0(1500)$ as mainly a glueball, and $f_0(1710)$ dominantly as $ss$. In such a case, there is no trajectory revealing the sigma meson mass, and the observed regularity would be accidental.

Existing lattice studies of hadronic matrix elements of the energy-momentum tensor, the so-called gravitational form factors of the pion and nucleon, provide a very valuable complementary information on the $\sigma$ properties, with the large-$N_c$ interpretation behind (Section $^{IV}$). Unfortunately, the data are too noisy as to pin down the coupling of the excited scalar-isoscalar states to the energy-momentum tensor. Nevertheless, useful information confirming the mass estimates for the $\sigma$-meson can be extracted (Sect. $^{IV}$). It is observed from the lattice data that the $\sigma$ mass grows with the value of the light current quark mass in a natural way.

It is interesting to discern the nature of the $\sigma$ state from an analysis of a truncated spectrum. The minimum number of states, allowed by certain sum rules and low energy theorems, is just two. In Section $^{V}$ we undertake such an analysis, which suggests that $f_0(600)$ is a $q\bar{q}$ meson, while $f_0(980)$ is a glueball. Finally, in Section $^{VI}$ we address the rather fundamental issue of the existence of dimension-2 condensates within the context of the Operator Product Expansion of QCD. Such objects appear naturally in the Regge approach and could not vanish if only a finite number of states were kept. Actually, we show that the dimension-2 condensate may vanish when infinitely many states are considered, in which case a
We get, after excluding the lowest $n = 0$ state and using the 8 remaining PDG states, the central parameter values $a = 1.35 \text{ GeV}^2$ and $m_{\sigma} = 530 \text{ MeV}$, with $\chi^2/\text{DOF} = \chi^2/(8 - 2) = 9.9$.

A non-trivial issue, not addressed in the analyses of Refs. [37, 38], is the determination of the parameters in the Regge fits. Actually, the large value of $\chi^2/\text{DOF} \sim 10 \gg 1$ found above prevents a reliable error determination of the model parameters $a$ and $m_{\sigma}$ and calls for corrections from different sources. A common way to overcome this difficulty is to rescale the weights in the $\chi^2$ variable. In our case we simply replace $\Delta M_{f,n} \rightarrow 3 \Delta M_{f,n}$, such that $\chi^2/\text{DOF} \sim 1$. In this case we get

$$a = 1.35(5) \text{ GeV}^2, \quad m_{\sigma} = 530(95) \text{ MeV} \quad \text{(Fit I)}.$$  

In the usual approaches (see, e.g., Refs. [38, 41]) a difference between the bare and physical pole masses, based on the $K$-matrix or the Breit-Wigner formula, is made. This generally introduces some model dependence and thus there may be systematic uncertainties in the masses related to the precise definition of the resonance parameters.

As we can see from Eq. (4), the fit is quite acceptable and implies the value of the string tension $\sqrt{\sigma} = 463(9) \text{ MeV}$. We note that the phenomenological value is $\sqrt{\sigma} = 420 \text{ MeV}$ from a global fit to the excited meson spectrum [37].

Formula (2) is equivalent to two parallel radial Regge trajectories with the standard slope

$$M_{S,-}(n)^2 = a n + m_{\sigma}^2,$$

$$M_{S,+}(n)^2 = a n + m_{\sigma}^2 + \frac{a}{2}, \quad (5)$$

From here we get the mass-splitting formula

$$M_{S,+}(n)^2 - M_{S,-}(n)^2 = \frac{a}{2} = \pi \sigma, \quad (6)$$

and, in particular,

$$2(m_{f_0(980)}^2 - m_{\sigma}^2) = a = 2 \pi \sigma, \quad (7)$$

which works extremely well for the value $m_{f_0} = 980 \text{ MeV}$ and $m_{\sigma} = 530 \text{ MeV}$, yielding 1.36 GeV$^2$ for the l.h.s. vs. 1.35 GeV$^2$ for the r.h.s.

Given the fact that all states fit quite naturally into the pattern, we see no obvious way how glueball states could be singled out solely on the basis of belonging to one of the trajectories [40]. Also note that a glueball corresponds to a bound state in gluodynamics, or equivalently QCD for infinitely heavy quarks (zero active flavors), while the states obtained here correspond to the limit of light quarks (two or three active flavors).
As is well known [18, 19], at large $N_c$ mesons and glueballs are stable states with fixed mass, $M_{S,n} \sim N_c^0$, and small decay widths $\Gamma_{S,n} \sim 1/N_c$ and $\sim 1/N_c^2$, respectively. Moreover, a general feature of the observed mesonic spectrum is that while the masses grow with $n$, the widths stay constant. Therefore, one would naively expect the large-$N_c$ approximation to work better for the higher masses of the excited states. One might incorporate the widths by using a particular model for the spectral density, such as the Breit-Wigner parametrization. Rather than using such a parametrization for the finite width, it may be better to take the widths themselves as an uncertainty on the value of the mass up to $O(1/N_c)$. This is clearly an upper bound on the uncertainty of the resonance position. Actually, the PDG resonance values are often quoted as the Breit-Wigner values, which, depending on the assumed background, changes from process to process and in general leads to some model dependence. However, a rigorous description of resonances as quantum mechanical states requires to determine them as poles of the meson-meson scattering amplitudes in the second Riemann sheet. Then their position in the complex plane becomes model-independent. For the lowest $0^{++}$ state the shift between the Breit-Wigner and the pole values is rather large, $m_\sigma = 800$ MeV vs. $m_\sigma = 556$ MeV, so the difference is compatible with the corresponding half-width. In view of this discussion we fit Eq. (2) by minimizing

$$\chi^2 = \sum_n \left( \frac{M_{f,n} - M_S(n)}{\Gamma_{f,n}/2} \right)^2 ,$$

where we have chosen the inverse of the half-width squared as the weight. The result of this procedure, called Fit II, is

$$a = 1.31(12) \text{ GeV}^2, \ m_\sigma = 556(127) \text{ MeV},$$

$$\chi^2/\text{DOF} = 0.12 \quad \text{(Fit II)}.$$

The correlation plot for the $a - m_\sigma$ parameters is presented in Fig. 2 for fits I (solid lines) and II (dashed lines). We notice the compatibility of the two methods.

We remark that the large-$N_c$-motivated analysis of the NN-scattering in the $^1S_0$-channel [27, 11] yields a quite similar value for the lowest scalar mass, $m_\sigma = 500(20)$ MeV.

Although our fitting strategy was rather crude, the resulting Regge-like spectrum of the scalar-isoscalar states turns out to be very reasonable, given the simplicity of the mass formula of Eq. (2). The crucial aspect of the foregoing analysis is that the $f_0(600)$ resonance corresponds to the lightest state in the radially-excited spectrum of the Regge-like scalar-isoscalar states. This is an important point, since scalar glueballs [42, 43] and scalar-isoscalar mesons [44, 45] have been studied within the AdS/CFT framework, however neglecting the possible role of the lightest scalar in any of these approaches. The soft-wall version has been reviewed in Ref. [46, 47].

In Sect. IV we will provide a mounting evidence on the coupling of the lightest state to the gravitational form factor of both the pion and the nucleon. This is relevant, as it shows that the $\sigma$ state should be explicitly considered in the evaluation of the correlation functions, where it plays an essential dynamical role.

III. FORMALISM FOR THE EFFECTIVE SCALAR FIELDS

In this Section we review the basic formalism used in the following parts of the paper. A particular attention is paid to the $N_c$-scaling of coupling constants of the meson and glueball scalar-isoscalar states.

| Resonance | $M$ [MeV] | $\Gamma$ [MeV] | $n$ | $M$ (Fit I) | $M$ (Fit II) |
|-----------|-----------|---------------|----|------------|-------------|
| $f_0(600)$ | 400 – 1200 | 500 – 1000 | 0 | 530 | 556 |
| $f_0(980)$ | 980(10) | 70(30) | 1 | 976 | 983 |
| $f_0(1370)$ | 1350(150) | 400(100) | 2 | 1275 | 1274 |
| $f_0(1500)$ | 1505(6) | 109(7) | 3 | 1516 | 1510 |
| $f_0(1710)$ | 1724(7) | 137(8) | 4 | 1724 | 1714 |
| $f_0(2020)$ | 1992(16) | 442(60) | 5 | 1909 | 1896 |
| $f_0(2100)$ | 2103(8) | 209(19) | 6 | 2078 | 2062 |
| $f_0(2200)$ | 2189(13) | 238(50) | 7 | 2234 | 2215 |
| $f_0(2330)$ | 2321(30) | 223(30) | 8 | 2380 | 2359 |

TABLE I: PDG values of resonance parameters [36], compared to the fits to the radial Regge spectrum of the $0^{++}$ scalar-isoscalar states, $M_n^2 = \frac{1}{2} m^2 + \frac{1}{2} m_n^2$. In fits I and II the uncertainties are taken as the error in the mass, or as the one half of the resonance width, respectively. In both cases the lowest $n = 0$ state, corresponding to the $f_0(600)$ resonance, is excluded from the fit.

FIG. 2: The $\Delta \chi^2 = 2.3$ and 4.6 contours, corresponding to the 68% and 90% confidence levels in the $m_\sigma - a$ plane, for Fit I (solid lines) and Fit II (dashed lines). The optimum values are indicated with the dots, filled for Fit I, and open for Fit II.
A. Interpolating fields

The first non-trivial question is to look for the interpolating QCD operator which triggers the $J^{PC} = 0^{++}$ isoscalar states from the vacuum. On the lattice this has been done in a variety of ways, mainly involving quark operators (such as $\bar{q}q$, see also [48]). A clear candidate, often used for the gluonium, is given by the trace of the energy-momentum tensor, $\Theta^{\mu\nu}$, which satisfies the trace anomaly equation [49],

$$\partial^\mu D_\mu = \Theta^{\mu}_\mu \equiv \Theta$$  \hspace{1cm} (10)

Here $\beta(\alpha) = \frac{\mu^2 d\alpha}{d\mu^2}$ denotes the beta function, $\alpha$ is the running coupling constant, $\gamma_m(\alpha) = d\log m/d\log \mu^2$ is the anomalous dimension of the current quark mass $m$, the symbol $D_\mu$ denotes the dilatation current, and $G^{\mu\nu}_a$ is the field strength tensor of the gluon field. The operator $\Theta$, besides having the $J^{PC} = 0^{++}$ quantum numbers, is renormal-invariant. In addition, in the chiral limit of massless quarks, $m_q \to 0$, it is $SU_R(2) \otimes SU_L(2)$ chirally invariant. This is different from the usual $\bar{q}q$ interpolating field, which is not chirally invariant and mixes under the chiral transformations with the $i\bar{q}\gamma_5 q$ operator. Actually, under the operation $q \to q\gamma_5$, the operator $\Theta$ is chirally even while $\bar{q}q$ is chirally odd.

In perturbation theory one has

$$\beta(\alpha) = -\alpha \left[ \beta_0 \left( \frac{\alpha}{4\pi} \right) + \beta_1 \left( \frac{\alpha}{4\pi} \right)^2 + \ldots \right],$$

$$\gamma_m(\alpha) = \frac{\alpha}{4\pi} + \ldots, \hspace{1cm} (11)$$

where

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f,$$

$$\beta_1 = \frac{34}{3} N_c^2 - \frac{13}{3} N_f N_c + \frac{N_f^2}{N_c}, \hspace{1cm} (12)$$

and $N_f$ denotes the number of active flavors. We recall that in the large-$N_c$ limit $\alpha \sim 1/N_c$, and $G^2 \sim (N_c^2 - 1)$, i.e., is proportional to the number of gluons. Hence, in Eq. (10) there is a contribution scaling as $N_c^2$ and flavor-independent, as well as another contribution, subleading in $N_c$ and scaling as $N_c^2 N_f$ (the higher orders in, for instance, the $\overline{MS}$ scheme, generate the $N_f^2$ terms which are dependent on the renormalization scheme). In the chiral limit we only have the gluonic operator contribution to Eq. (10), however, still some information on the quark degrees of freedom remains via the $\beta$-function. Thus, in the chiral limit we may distinguish between the gluonic and quark contributions by the large-$N_c$ and $N_f$ scaling behavior,

$$\Theta_g \sim N_c^2, \hspace{1cm} \Theta_q \sim N_c N_f. \hspace{1cm} (13)$$

B. Two-point correlations

For a scalar-isoscalar particle $|n\rangle$ we have a non-vanishing matrix element to the vacuum for the dilatation current,

$$\langle 0 | D^\mu | n \rangle = i q^\mu f_n, \hspace{1cm} (14)$$

and hence

$$\langle 0 | \partial^\mu D_\mu | n \rangle = \langle 0 | \Theta | n \rangle = m_n^2 f_n \hspace{1cm} (15)$$

for the on-shell particles. From now on the mass of the scalar-isoscalar states is denoted as $m_n$. The two-point correlation function of the energy-momentum tensor reads

$$\Pi^{\mu\nu\alpha\beta} (q) = i \int d^4 x e^{iqx} \langle 0 | T \{ \Theta^{\mu\nu}(x) \Theta^{\alpha\beta}(0) \} | 0 \rangle. \hspace{1cm} (16)$$

The energy momentum tensor is conserved, therefore

$$\langle 0 | \Theta^{\mu\nu} | n \rangle = \frac{1}{3} f_n (g^{\mu\nu} q^2 - q^\mu q^\nu), \hspace{1cm} (17)$$

where the factor of 1/3 complies with Eq. (15). After inserting a complete set of states we find

$$\Pi^{\mu\nu\alpha\beta} (q) = \frac{1}{9} (g^{\mu\nu} q^2 - q^\mu q^\nu)(g^{\alpha\beta} q^2 - q^\alpha q^\beta) \Pi_{\Theta\Theta} (q), \hspace{1cm} (18)$$

with

$$\Pi_{\Theta\Theta} (q) = i \int d^4 x e^{iqx} \langle 0 | T \{ \Theta(x) \Theta(0) \} | 0 \rangle$$

$$= \sum_n \frac{f_n^2 q^4}{m_n^2 q^2 + c.t.,} \hspace{1cm} (19)$$

where c.t. stands for the counterterms. According to the $N_c$ counting rules, the leading part of $\Pi_{\Theta\Theta}$ scales as $N_c^2$, and the next-to-leading part as $N_c N_f$.

The sum in Eq. (19) runs over all $J^{PC} = 0^{++}$ isoscalar states, i.e., both mesons ($m$) and glueballs ($g$). The difference is, however, in the scaling of the coupling constants $f_n$ with $N_c$. If all states are contributing and no cancellations occur, then the glueball must have a larger coupling, $f_n^2 \sim N_c^2$, whereas the $\bar{q}q$-meson has $f_n^2 \sim N_c N_f$:

$$f_n \sim N_c \hspace{1cm} \text{(glueball),}$$

$$f_n \sim \sqrt{N_c} \hspace{1cm} \text{(meson).} \hspace{1cm} (20)$$

The Operator Product Expansion (OPE) in QCD yields [50–54]

$$\Pi(q^2) = q^4 \left[ C_0 \log q^2 + \sum_n \frac{C_{2n}}{q^{2n}} \right], \hspace{1cm} (21)$$

as well as other contributions.
where
\[ C_0 = -\frac{1}{2\pi^2}(N_c^2 - 1) \left( \frac{\beta(\alpha)}{\alpha} \right)^2. \] (22)

In the conventional treatment insisting on the presence of the local gauge-invariant operators only, the sum over \( n \) starts with \( n = 2 \), while the admission of the dimension-2 condensate includes the \( n = 1 \) term as well. In Sect. VI we will analyze this issue further.

We have imposed the condition to have a well behaved limit for the on-shell particles we may use the equations of motion, \((-\partial^\mu \partial_\mu - m_n^2)\varphi_n = 0\), to obtain
\[ \Theta(x) = -\sum_n f_n m_n^2 \varphi_n(x) + O(\varphi^2). \] (31)

For the off-shell particles we need to introduce the improved energy momentum tensor \[\tilde{C}^\mu_\nu(x)\] which has a better high energy behavior, but the final result is the same.

The above properties may be described with a suitable extension of the effective dilatonic Lagrangeans \[\mathcal{L}_{\text{eff}}(x)\] of the form
\[ \mathcal{L}(x) = \sum_n \left[ \frac{1}{2} \partial^\mu \Phi_n \partial_\mu \Phi_n - V(\Phi_n) \right]. \] (32)

Here, the potential
\[ V(\Phi_n) = \frac{m_n^2}{8 f_n^2} \Phi_n^4 \left[ \log \left( \frac{\Phi_n}{f_n} \right)^2 - \frac{1}{2} \right] \] (33)
fulfills the conditions that the minimum is at \( \langle \Phi_n \rangle = f_n \) and the mass is \( V''(f_n) = m_n^2 \). The physical field \( \varphi_n \) is defined by \( \Phi_n = f_n + \varphi_n \). The value at the minimum is
\[ \epsilon_v = \sum_n V(f_n) = -\frac{1}{16} \sum_n m_n^2 f_n^2, \] (34)
which corresponds to the vacuum energy density \[\epsilon_v\].

The coupling to the pion can be considered by working in the non-linear representation and with the interaction Lagrangean
\[ \mathcal{L}_\pi = \frac{f_\pi^2}{4} \sum_n g_{n\pi\pi} \Phi_n^2 \langle \partial^\mu U^\dagger \partial_\mu U \rangle, \] (35)

where \( U(x) = e^{i\pi \cdot \pi / f_\pi} \) is the non-linearly transforming pion field and \( \langle \cdot \rangle \) represents the trace in the isospin space.

In the vacuum, \( \Phi_n = f_n \), we get
\[ \mathcal{L}_{\pi}^{(0)} = \frac{f_\pi^2}{4} \sum_n g_{n\pi\pi} f_n \langle \partial^\mu U^\dagger \partial_\mu U \rangle, \] (36)

which leads to the condition
\[ \sum_n g_{n\pi\pi} f_n = 1. \] (37)

C. Effective Lagrangeans

The results of the previous subsections can be directly translated into the language of the effective Lagrangeans. In particular, we may write the PCDC (Partial Conservation of the Dilaton Current) equation,
\[ D^\mu(x) = \sum_n f_n \partial^\mu \varphi_n(x) + O(\varphi^2), \] (30)

and the naive positive combination of Eq. (28) may in fact turn into a vanishing or even negative contribution.
Since $\mathcal{L}_\pi \sim N_c$ and $f_\pi \sim \sqrt{N_c}$, it follows (under the assumption that all states are contributing in the large-$N_c$ limit and no cancellations occur) that $g_{n\pi\pi}f_n \sim N_c^0$, or, due to Eq. (20),

\[
\begin{align*}
    g_{n\pi\pi} &\sim 1/N_c, & \text{(glueball)}, \\
    g_{n\pi\pi} &\sim 1/\sqrt{N_c}, & \text{(meson)}.
\end{align*}
\]

A construction of the chiral Lagrangean in the linear realization is presented in Appendix B.

The scalar fields $\Phi_n$ are invariant under chiral transformations.² Lagrangean (35) is nothing else but a generalization of the dilaton Lagrangean of Refs. [58, 60] for the case of many dilatons. We treat here all scalars on equal footing, as suggested by the fact that they all fit into the same radial Regge trajectory. The tree-level decay width of the scalar $n$ into two pions is given by the expression

\[
\Gamma(\Phi_n \to \pi\pi) = \frac{32}{3}g_{n\pi\pi}^2 m_n^2 \left[ 1 - \frac{2m_n^2}{m_n^2} \right]^2 \sqrt{m_n^2 - 4m_\pi^2} \sim \frac{32}{3}g_{n\pi\pi}^2 m_n^3. 
\]

Because of the scaling (38),

\[
\Gamma_{n\pi\pi} \sim 1/N_c^2 \quad \text{(glueball)}, \\
\Gamma_{n\pi\pi} \sim 1/N_c \quad \text{(meson)},
\]

which displays the weaker coupling of the glueballs to the pion compared to the mesons.

The Lagrangean describing the interaction of the nucleon with the scalar-isoscalar particles is

\[
\mathcal{L}_N = \bar{N} \left( i\partial - \sum_n g_{nNN}\Phi_n - \frac{g_A}{f_\pi} u^i \gamma^i \phi u \right) N, 
\]

with $u^2 = U^5 = e^{i\gamma^5 \pi^5/2}$. The nucleon field $N$ transforms non-linearly, but we can return to the linear realization by undoing the chiral transformation, $\Psi = uN$, which effectively replaces the nucleon mass $M_N$ by the combination $\sum_n g_{nNN}\Phi_n$. This yields the generalized Goldberger-Treiman relation for the scalar-isoscalar channel,

\[
M_N = \sum_n g_{nNN}f_n, 
\]

derived by Carruthers in the early seventies.² Again, under the assumption that all states contribute to (42) and no cancellations occur, we find

\[
\begin{align*}
    g_{nNN} &\sim N_c^0, & \text{(glueball)}, \\
    g_{nNN} &\sim \sqrt{N_c}, & \text{(meson)}.
\end{align*}
\]

Lagrangean (11) fulfills also the standard Goldberger-Treiman relation, $g_A M_N = f_\pi g_{\pi NN}$. In addition, it is scale-invariant, such that as a consequence that $D^\mu$ is the Noether current generating dilatations we have the identity [53, 61]

\[
\partial^\mu D_\mu = \Theta = \sum_n \left[ 4V(\Phi_n) - \Phi_n V'(\Phi_n) \right] 
\]

\[
= -\sum_n \frac{m_n^2}{4f_\pi^2} \Phi_n^4. 
\]

Upon subtraction of the vacuum part and using that $\Phi_n = f_n + \varphi_n$ it follows that

\[
\langle N|\Theta(0)|N \rangle = -\sum_n m_n^2 f_n \langle N|\varphi_n|N \rangle 
\]

\[
= \sum_n f_n g_{nNN}, 
\]

which vanishes at the threshold $q = 0$, in agreement with the Weinberg-Tomozawa result. According to Eqs. (38) and (40), the meson contribution scales as $N_c^0$ and the glueball contribution as $1/N_c$, hence is relatively suppressed.

Finally, the nucleon-nucleon central potential stemming from the exchange of the $J^{PC} = 0^{++}$ isoscalar particles reads in the large-$N_c$ limit

\[
V_C(|q|) = \sum_n \frac{g_{nNN}^2 q^2}{q^2 + m_n^2}. 
\]

Passing to the coordinate space we obtain the standard Yukawa potential,

\[
V_C(r) = -\sum_n \frac{g_{nNN}^2 e^{-m_n r}}{4\pi r}. 
\]

The large-$N_c$ counting rules [40] yield $V_C \sim N_c$ from the meson exchange, while the glueball exchange produces a subleading contribution $\sim N_c^0$. Other tensorial components of the NN potential scale as $N_c^2$ or $\sim 1/N_c$ [24, 30].

The $N_c$ scaling of various quantities is collected for convenience in Table III.

IV. GRAVITATIONAL FORM FACTORS

In this section we use the lattice data for the gravitational form factors of the pion and nucleon to place further bounds on the mass of the $\sigma$ meson. Our analysis is based on the fact that the spin-0 channel is dominated by the scalar-isoscalar mesons.
TABLE II: $N_c$-scaling of various quantities.

| quantity | glueball | q$^q$ meson |
|----------|----------|-------------|
| $m_0$    | 1        | 1           |
| $f_0$    | $N_c$    | $\sqrt{N_c}$|
| $\Gamma_{\pi\pi}$ | $1/N^2_\pi$ | $1/N_c$ |
| $g_{n\pi\pi}$ | $1/N_c$ | $1/\sqrt{N_c}$ |
| $g_{nNN}$ | 1        | $\sqrt{N_c}$ |

A. Pion

The gravitational form factors of the pion are defined via the matrix element of the energy-momentum tensor between the pion states,

$$\langle \pi^b(p') | \Theta^{\mu\nu}(0) | \pi^a(p) \rangle = \frac{1}{2} g^{ab} \left[ (g^{\mu\nu} q^2 - q^\mu q^\nu) \Theta_1(q^2) + 4 P^\mu P^\nu \Theta_2(q^2) \right],$$

where $P = \frac{1}{2} (p' + p)$, $q = p' - p$, and $a, b$ are the isospin indices. The trace part of this form factor corresponds to the coupling of the spin-0, while the traceless part to spin-2 states. In the present analysis we are interested in the spin-0 gravitational form factor is given by

$$\langle \pi^b(p') | \Theta(0) | \pi^a(p) \rangle = \delta^{ab} \Theta_1(q^2),$$

where

$$\Theta_1(q^2) = \frac{3}{2} q^2 \Theta_1(q^2) + \frac{1}{2} (4m^2_{\pi} - q^2) \Theta_2(q^2).$$

Due to chiral symmetry constraints, this form factor satisfies

$$\Theta_1(q^2) = q^2 + O(p^4).$$

Two low energy theorems follow:

$$\Theta_1(0) = 0,$$

$$\Theta_1'(0) = 1.$$  \hspace{1cm} (53)

These conditions can be achieved by assuming a derivative coupling

$$\langle \pi^a(p') | \Phi_n | \pi^b(p) \rangle = \delta^{ab} \frac{g_{n\pi\pi} q^2}{m^2_n - q^2},$$

written as

$$\Theta_1(q^2) = \sum_n \frac{g_{n\pi\pi} f_n q^2 m^2_n}{m^2_n - q^2},$$

and thus

$$\sum_n f_n g_{n\pi\pi} = 1,$$

exactly as Eq. (57). Since according to Table II we have the scaling $f_n g_{n\pi\pi} \sim N^2_c$, both mesons and glueballs may contribute to relation (57) on equal footing.

At large momenta the form factor behaves as

$$\Theta_1(q^2) \rightarrow - \sum_n g_{n\pi\pi} m^2_n f_n - \frac{\sum_n g_{n\pi\pi} m^4_n f_n}{q^2} + \ldots$$

If we require the constant term to vanish, we get

$$\sum_n f_n g_{n\pi\pi} m^2_n = 0.$$  \hspace{1cm} (59)

Clearly, cancellations are necessary and we need at least two contributing states to satisfy the sum rule (59). This high-energy relation plays a similar role in the scale-invariance considerations to the second Weinberg sum rule in the chiral framework.

The only determination of the pion gravitational form factor to date is the lattice calculation of Brommel et al. \cite{63, 64}. The data go up to $t = -4$ GeV$^2$, however with substantial errors. The quark contribution to the energy momentum tensor, considered in \cite{63, 64}, is

$$\Theta^{\mu\nu}_q(x) = \frac{1}{4} \bar{q}(x) \left[ \gamma^\mu i\gamma^5 \gamma^\nu + \gamma^\nu i\gamma^5 \gamma^\mu \right] q(x),$$

from which a decomposition similar to Eq. (49) and the corresponding spin-0 component Eq. (51) follow. The data points are shown in Fig. 3. Following the standard notation for the moments of the pion GPD’s \cite{65} we introduce

$$A_{20}(t) = \frac{1}{2} \Theta_1^q(t), \quad A_{22}(t) = -\frac{1}{2} \Theta_2^q(t),$$

FIG. 3: The $\Delta \chi^2 = 2.3$ and 4.6 contours, corresponding to the 68% and 90% confidence levels, in the $m_\pi - (x)_{u+d}$ plane. The fit is obtained from the lattice data for the pion spin-0 gravitational form factor, as described in the text.
where the symbols $\Theta_q^t$ denote the quark parts of the gravitational form factors of Eq. (19). A monopole fit has been undertaken, yielding $m_\pi = 0.89(27)(9)$ MeV and $A_{22}(0) = -0.076(5)$ (cf. Table C.8 on p. 105 of Ref. [64]). By using several parameterizations it has been noted that the precise form of the function cannot be pinned down from the data unambiguously due to large uncertainties in $A_{22}(t)$. In the following we will use the low-energy theorem

$$A_{22}(0) = -\frac{1}{4}A_{20}(0),$$

which provides a relatively accurate fixing of $A_{22}$ at the origin and greatly helps the regression analysis.

In terms of the form factors related to the Generalized Parton Distributions (GPD) of Ref. [63, 64], the quark contribution to the gravitational form factor reads

$$\Theta_q^t(t) = -4t^3A_{22}^t(t),$$

Due to the multiplicative QCD evolution one has

$$\Theta_q^t(t, \mu) = \langle x \rangle_q^t(\mu)\Theta_q^t(t),$$

where the (valence) quark momentum fraction depends on the renormalization scale $\mu$. Its leading-order perturbative evolution reads

$$R = \frac{\langle x \rangle_q^t(\mu)}{\langle x \rangle_q^t(\mu_0)} = \frac{\alpha(\mu)}{\alpha(\mu_0)}\gamma_1^{(0)}/(2\beta_0),$$

where the anomalous dimension is $\gamma_1^{(0)}/(2\beta_0) = 32/81$ for $N_F = N_c = 3$. The QCD running coupling constant is equal to

$$\alpha(\mu) = \frac{4\pi}{\beta_0 \log(\mu^2/A_{QCD}^2)},$$

where we take $A_{QCD} = 226$ MeV for $N_c = N_f = 3$.

Phenomenologically, it is known from the Durham group analysis [60], based mainly on the E615 Drell-Yan data [67] and the model assumption that the sea quarks carry $10 - 20\%$ fraction of the momentum, that $\langle x \rangle_{u+d}^\pi = 0.47(2)$ at the scale $\mu = 2$ GeV. The analysis of the Dortmund group [68], based on the assumption that the momentum fraction carried by the valence quarks in the pion coincides with that in the nucleon, yields $\langle x \rangle_{u+d}^\pi = 0.4$ at $\mu = 2$ GeV. The lattice data at the lattice spacing $a_{lat} = 0.1$fm as well as a recent chiral quark model calculation [69] support this view.

We recall that the large-$N_c$ analysis implies sums of monopoles in the form factors. The largest available momentum transfer, $t = -4$ GeV$^2$, obtained in Ref. [64], suggests that some information on the contribution of the excited states might be extracted. Therefore, following the approach already used for the electromagnetic pion form factor in Ref. [70] (see also [71]), we have attempted a Regge-like fit,

$$\Theta_\pi(t) = tf_b(t),$$

including infinitely many states, of the form

$$f_b(t) = \frac{B(b - 1, \frac{M^2 - t}{m^2})}{B(b - 1, \frac{M^2 - t}{a^2})},$$

with $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ denoting the Euler Beta function. The function [65] fulfills the normalization condition

$$f_b(0) = 1.$$  

For $x \gg y$ one has $B(x, y) \sim \Gamma(y)x^{-y}$, hence in the asymptotic region of $M^2 - t \gg (b - 1)a$ we find

$$f_b(t) \sim \frac{\Gamma\left(\frac{M^2}{a^2} + b - 1\right)}{\Gamma\left(\frac{M^2}{a^2}\right)} \left(\frac{a}{M^2 - t}\right)^{b-1}.$$  

The result for $a = 1.31$ is

$$\langle x \rangle_{u+d}^\pi = 0.52(2), \quad m_\sigma = 495^{+250}_{-135} \text{ MeV}, \quad b = 2.24^{+1.56}_{-0.55}.$$  

As we can see, the result is fully compatible with the monopole ($b = 2$) and at present the large errors in the lattice data wash out any insight from the excited scalar spectrum, despite the large momenta$^3$.

Thus, we restrict ourselves to a simple monopole fit

$$\Theta_q^t(t) = \langle x \rangle_q^t(\mu_0)\frac{t m_{\pi}^2}{m_{\pi}^2 - t},$$

which yields $\chi^2/\text{DOF} \sim 2.4$. Actually the large $\chi^2$ is due to incompatible values of nearby points. In such a situation, in order to obtain reliable estimate of the model parameters, we rescale the errors by a factor of 1.5 to make them mutually compatible. Moreover, we enforce the low energy theorem, Eq. (63) as a constraint – a possibility not directly considered in Ref. [63, 64]. As a result we get $\chi^2/\text{DOF} \simeq 1$ (after the mentioned rescaling of the data errors) and the optimum values

$$\langle x \rangle_{u+d}^\pi = 0.52(2), \quad m_\sigma = 445(32) \text{ MeV}. $$

In Fig. 3 we present the corresponding correlation ellipse. The gravitational form factor $\Theta_b(t)$ at the optimum values of the parameters [72] is presented in Fig. 4.

**B. Nucleon**

The scalar component of the nucleon gravitational form factor is

$$\langle N(p')|\Theta(0)|N(p)\rangle = \bar{u}(p')u(p)\Theta_N(q^2),$$

$^3$ We note that a similar fit [71] to the vector form factor using $a = 1.2(1)$ GeV$^2$ yields $m_\rho = 775(15)$ MeV and $b = 2.14(7)$, which can be distinguished from a simple monopole fit at the two-standard-deviation level.
where \( u(p) \) and \( u(p') \) are nucleon Dirac spinors. In the meson-dominance approximation it can be written as

\[
\Theta_N(q^2) = \sum_n g_n N \frac{f_n m_n^2}{m_n^2 - q^2}.
\]

(75)

The nucleon mass is given by Eq. (42).

The quark contributions to the GPD’s of the nucleon have been determined on the lattice [72, 73] for masses as small as twice the physical pion mass. The decomposition corresponding to the energy-momentum tensor reads

\[
\langle p' | \Theta^q_{\mu\nu} | p \rangle = \bar{u}(p') \left[ A_{20}^{q}(t) \frac{\gamma_\mu P_\nu + \gamma_\nu P_\mu}{2} + B_{20}^{q}(t) \frac{i(P_\mu \sigma_\nu + P_\nu \sigma_\mu)\Delta^e}{4M_N} + C_{20}^{q}(t) \frac{\Delta_\mu \Delta_\nu - g_{\mu\nu}\Delta^2}{M_N} \right] u(p),
\]

(76)

where \( \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \) (the Bjorken-Drell notation), the scalar functions are moments of the GPD’s, the momentum transfer is denoted as \( \Delta = p' - p \), and the average nucleon momentum is \( P = (p' + p)/2 \). Taking the trace and applying the Gordon identity, \( 2M_N \bar{u}(p') \gamma_\mu u(p) = \bar{u}(p')(i\sigma^{\mu\nu} \Delta_\nu + 2P_\mu)u(p) \), as well as the Dirac equation, \( (\slashed{p} - M_N)u(p) = 0 \) and \( \bar{u}(p')(\gamma^\mu - M_N) = 0 \), we obtain the following expression for the spin-0 gravitational form factor of the nucleon:

\[
\Theta_N^q(t) = M_N \left[ A_{20}^{q}(t) + \frac{t}{4M_M} B_{20}^{q}(t) - \frac{3t}{M_N^2} C_{20}^{q}(t) \right].
\]

(77)

Due to the multiplicative character of the QCD evolution for the conserved energy-momentum tensor operator, one has

\[
\sum_q \Theta_N^q(t) = \sum_q \langle x \rangle_q \Theta_N(t) \equiv \langle x \rangle_{u+d} \Theta_N(t).
\]

(78)

The combination (77) can be directly constructed from the results of the LHPC Collaboration [73, where the data points for the \( A_{20}, B_{20}, \) and \( C_{20} \) form factors are provided for six values the pion mass: \( m_\pi = 757, 681, 595, 495, 356, \) and 352 MeV. The values of the corresponding nucleon masses are \( M_N = 1565, 1489, 1379, 1292, 1216, \) and 1158 MeV. We add the statistical errors in quadrature. The result for the two lowest pion masses is shown in Fig. 6. We note that while the \( A_{20}(t) \) function is determined to a better than 10% accuracy, the large errors in \( \Theta_N^q \) originate from the less accurate \( B_{20}(t) \) and
$C_{20}(t)$ form factors.

Similarly to the pion case we have tried to deduce some information from the high-energy region, $t \sim -1.2$ GeV$^2$, by attempting the Regge form

$$\Theta_N(t) = M_N \frac{B(b - 1, \frac{M_N^2 - t}{M_N^2})}{B(b - 1, \frac{M_N^2}{M_N^2})}. \quad (79)$$

Unfortunately, again the large errors in the data do not allow to fix the $b$ parameter unambiguously, and it is compatible with $b = 2$. Thus we stick to the simple monopole form,

$$\Theta_N^b(t) = M_N \langle x \rangle_{u+d}^N \frac{m_\pi^2 - t}{m_\pi^2 - m_\pi^2_{\text{phys}}}. \quad (80)$$

Since most of the lattice data are accumulated at pion masses significantly above the physical value, extrapolation to the physical point is necessary. At low values of the pion mass and $t$ the chiral corrections are expected to be significant.\footnote{Experience with the vector form factor shows that the low momentum region is quite sensitive to chiral corrections, whereas the intermediate energy region is better described by the large-$N_c$ dynamics.} Not entering these intricacies, we assume, following many lattice studies, the simple dependence of $m_\sigma$ on $m_\pi$:

$$m_\sigma^2(m_\pi) = m_\sigma^2 + c \left( m_\pi^2 - m_\pi^2_{\text{phys}} \right). \quad (81)$$

Hence $m_\sigma$ (without an argument) denotes the value at the physical point. Similarly,

$$\langle x \rangle_{u+d}^N(m_\pi) = \langle x \rangle_{u+d}^N + d \left( m_\pi^2 - m_\pi^2_{\text{phys}} \right). \quad (82)$$

We use simultaneously the form factors\footnote{Experience with the vector form factor shows that the low momentum region is quite sensitive to chiral corrections, whereas the intermediate energy region is better described by the large-$N_c$ dynamics.} for the six pion masses from LHPC\cite{72}. There are several ways one can perform the fit, all leading to similar but not identical results. Our procedure is as follows: We first normalize the form factors to unity at $m_\pi = 0$, by dividing a data set at a particular $m_\pi$ by the value of the point at $t = 0$. That way we get rid of the dependence on $\langle x \rangle_{u+d}^N$ and can carry out a two-parameter fit, with the parameters of Eq. (83). The $\chi^2$ method yields

$$m_\sigma = 550^{+180}_{-200}\text{MeV}, \quad c = 0.95^{+0.90}_{-0.75}; \quad (83)$$

with $\chi^2/\text{DOF} = 5.2/(87 - 2)$. The resulting correlation plot is shown in Fig. 6.

One may perform separately the fit to the normalized form factors at each value of $m_\sigma$. An example for the two lowest pion masses, 356 and 352 MeV, is shown in Fig. 6. The band indicates the uncertainty in $m_\sigma$ obtained with the $\chi^2$ method. The result of the procedure at all values of $m_\sigma$ is shown in Fig. 7, together with the best fit curve with parameters (83). The point at the physical value of $m_\sigma$ indicates the resulting extrapolation for the physical value of $m_\sigma$, Eq. (83).

Taking the values of the unnormalized form factors at $t = 0$ we may compute the momentum fraction carried by the quarks as a function of $m_\pi$. The result of the $\chi^2$ fit is

$$\langle x \rangle_{u+d}^N = 0.447(14), \quad d = -0.13(3). \quad (84)$$

In the case of the lattice calculations of the QCDSF Collaboration\cite{72}, five points have been provided at values ranging between $0 \leq -t \leq 3$ GeV$^2$ for the pion masses $m_\pi = 930, 760, \text{and } 550$ MeV, then extrapolated to the physical pion mass. The corresponding nucleon masses are $M_N = 1724, 1519, \text{and } 1308$ MeV. The $A$, $B$, and $C$ form factors have been parametrized in Ref. 72.
by a dipole form with the mass of \( M = 1.1(2) \) GeV.\(^5\)

Note that in the presence of a heavy mass of a tensor meson the data for the spin-0 gravitational form factor become noisy at large values of the momentum transfer \( t \) due to the relative smallness of the signal and the propagation of errors in the formula for \( \Theta_N^q \), Eq. (77).

For this reason we proceed as follows: with the provided dipole fits we reconstruct the data points for the spin-0 gravitational form factor using Eq. (77) at the quoted values of \( t \), adding the statistical errors in quadrature. The resulting \( \Theta_N^q(t) \) becomes negative within errors at \(-t > 1.5 \) GeV\(^2\), precluding accurate fitting with the monopole form. Thus we determine the mass from the slope of the linearly-extrapolated form factor at \( t = 0 \), according to the formula \[ 1/m_\sigma^2 = \frac{4}{3} \Theta_N^q(0)/\Theta_N^q(0). \]

Figure 8 shows the outcome of this procedure for the available pion masses as well as for the data set resulting from an extrapolation to the physical point made in Ref. [72].

The result of the fit of the form (81) yields
\[
m_\sigma = 600^{+80}_{-100} \, \text{MeV}, \quad c = 0.8(2), \tag{85}\]

or \( m_\sigma^2 = 0.35 \) GeV\(^2\) + 0.8\( m_\sigma^2 \). These values are consistent within the uncertainties with the formula of the LHPC collaboration, \( m_\sigma^2 = 0.387 \) GeV\(^2\) + 0.91\( m_\sigma^2 \).

Similarly, fitting to the form factors at the origin, we obtain
\[
\langle x \rangle_{u+d} = 0.55(3), \quad d = 0.07(4). \tag{86}\]

### C. Summary of the \( \sigma \) mass

The various estimates of the mass of the lightest scalar-isoscalar state, \( m_\sigma \), are summarized in Fig. 9. We note that the values are compatible within the uncertainties. We remark that the definition of the resonance mass via the pole position on the second Riemann sheet of the scattering amplitude or via the Breit-Wigner function may differ significantly, even by a factor of two. However, the difference occurs at the level \( 1/N_c \) [32].

The previous results are based on extrapolation to physical pion masses, a subject which has received much attention in recent years, as chiral logs are expected to play a role. Comparison of Figs. 4 and 8 provides a coherent picture that the scalar mass grows with the current quark mass. Actually, by analyzing the heavy pion limit, a distinction between glueballs and mesons could be made. Indeed, glueballs are existing states in pure gluodynamics which corresponds to the limit of extremely heavy quarks. The lightest \( 0^{++} \) glueball in that limit has a mass of \( M_{0^{++}} = 1.7 \) GeV, and this would correspond to a physical state whose mass becomes a fixed number for infinitely heavy quarks. Clearly, states which become \( q\bar{q} \) states should decouple since they have masses scaling as \( 2m_q \) at large \( m_q \). The data are just too scarce and noisy to make a conclusive statement in that regard. From this viewpoint, lattice data in the heavy-pion regime would be most welcome.

### V. Calculations with a Finite Number of States

In this Section we return to the derived sum rules in an attempt to assess the nature of the scalar-isoscalar states. The sum rules [34, 42, 57, 59] are:
\[
\sum_n f_n g_n \pi \pi = 1, \tag{87}
\]
\[
\sum_n f_n g_n \pi \pi m_n^2 = 0, \]
\[
\sum_n f_n g_n NN = M_N, \]
\[
-\frac{1}{16} \sum_n f_n^2 m_n^2 = \epsilon_v. \]
In addition, the dimension-2 object of Eq. (87) with a finite number of states requires the presence of at least two scalar-isoscalar states. The presently-available lattice data do not allow to extend the analysis of Sect. IV to account for two or more states. Nevertheless, one can perform a qualitative analysis based solely on Eq. (87).

When we saturate the sum rules (87) with two states, we get

\[ \sum_n f_n^2 = C_2. \]  

(88)

It is clear that the satisfaction of the second sum rule (87) with a finite number of states requires the presence of at least two scalar-isoscalar states. The presently-available lattice data do not allow to extend the analysis of Sect. IV to account for two or more states. Nevertheless, one can perform a qualitative analysis based solely on Eq. (87).

When we saturate the sum rules (87) with two states, we get

\[ 1 = g_{\sigma\pi\pi} f_\sigma + g_{f\pi\pi} f_f, \]  

(89)

\[ 0 = g_{\sigma\pi\pi} f_\sigma m_\sigma^2 + g_{f\pi\pi} f_f m_f^2, \]  

(90)

\[ M_N = g_{\sigma NN} f_\sigma + g_{f N NN} f_f, \]  

(91)

\[ \epsilon_v = -\frac{1}{16} (f_\sigma^2 m_\sigma^2 + f_f^2 m_f^2), \]  

(92)

\[ C_2 = f_\sigma^2 + f_f^2. \]  

(93)

Let us identify \( \sigma \) with \( f_0(600) \), the state \( f \) with \( f_0(980) \), and treat \( m_\sigma \) as a variable. From the decay widths \( \Gamma_{\sigma \to \pi\pi} = 540(20) \text{ MeV} \) [12] and \( \Gamma_{f_0 \to \pi\pi} = 70(30) \text{ MeV} \) we infer (up to the sign) with the help of Eq. (39) the \( m_\sigma \)-dependent values of the coupling constants \( g_{\sigma\pi\pi} \) and \( g_{f\pi\pi} \). Then, equations (89) are used to find the constants \( f_\sigma \) and \( f_f \). We choose the convention where both of these constants are positive. Next, assuming \( g_{\sigma NN} = 9.5(5) \), we obtain \( g_{f N NN} \) from Eq. (91). Finally, from Eq. (92) we compute the vacuum energy density. The results are displayed in Figs. 10, 11 and 12.

The most important qualitative feature can be inferred from the analysis of the ratios of coupling constants. Figure 10 shows the ratios of the constants \( f_\sigma/f_f \), \( g_{f\pi\pi}/g_{\sigma\pi\pi} \), and \( g_{f N NN}/g_{f N NN} \). All these ratios are of the order \( 1/\sqrt{N_c} \approx 0.6 \) for the whole PDG range of \( m_\sigma \). According to the scaling of Table II this supports the view that \( \sigma \) is a \( q\bar{q} \) meson, while \( f_0(980) \) is a glueball. With this assignment the results of Fig. 10 emerge naturally. Certainly, the conclusion relies on the assumption of having just two states dominating the sum rules. Yet, it is appealing in its simplicity. In the following section we will see that if infinitely many states are considered this conclusion would not necessarily follow when a further sum rule (the vanishing of a dimension-2 condensate) is imposed. Note that in the two-state model the dimension-2 object, \( C_2 \), is necessarily positive, see Eq. (93).

Figure 11 shows the dependence of the constants \( f_\sigma \) and \( f_f \) on \( m_\sigma \). For comparison, we also plot the pion decay constant \( f_\pi = 93 \text{ MeV} \). We note that \( f_\sigma \) is of the order \( f_\pi \) near \( m_\sigma \approx 500 \text{ MeV} \). As discussed in Appendix B, \( f_f \) and \( f_\sigma \) need not be equal even if we postulate the scalar to be the chiral partner of the pion.

Finally, we examine the last sum rule (92), concerning the energy density of the vacuum due to the gluon condensate. According to Ref. [62], we have (for three active flavors)

\[ \epsilon_v = -\frac{9}{32} \frac{\alpha}{\pi} G^2 = -(224^{+35}_{-70} \text{ MeV})^4. \]  

(94)

We note that the two-state model favors the lower values of \( m_\sigma \), up to 550 MeV, where the sum rule (92) is satisfied within the values (34).

VI. DIMENSION TWO CONDENSATES IN THE SCALAR SECTOR

In the previous section we have explored phenomenological consequences of saturating the sum rules in the large-\( N_c \) limit with a finite number of states (two states). Such an approach is justified for observables where the coupling constants to higher-mass states are sharply suppressed, and we are essentially left with the dominance of the low-mass states. This is, however, not always the
case. For instance, as discussed in Sect. IIIB proper matching of the two-point functions to QCD necessarily requires infinitely many intermediate states present in the correlator. Only that way the leading logarithmic behavior \( \alpha_s \) can be reproduced. Similar studies have been carried out in the past for other correlators in the Regge approach [77, 85]. Here we focus on the \( \Pi_{\Theta \Theta} \) correlator defined in Eq. (16).

In QCD, keeping the leading terms in \( \alpha_s \), this object has the explicit twist expansion of the form [50–54]

\[
\Pi_{\Theta \Theta} = -\frac{\alpha_s^2(n_c^2 - 1)\beta_0^2}{32\pi^4}Q^4\log\frac{Q^2}{\mu^2} + C_2 Q^2 + \frac{\alpha_s^2}{16\pi}\left(\frac{\alpha}{\pi}G^2\right) + \ldots
\]  

(95)

(note a factor of 16 difference between our definition [106] and Ref. [54], carrying over to the definition of \( C_2 \)).

The dimension-2 gluon condensate, originally proposed by Celenza and Shakin [50], appears as an elusive gauge-invariant non-perturbative and non-local operator which generates the lowest \( 1/Q^2 \) power corrections in the twist expansion. Its dynamical origin remains unclear [51, 52], despite certain evidence provided by the instanton model [53], phenomenological QCD sum rules reanalyses [90], phenomenological studies of the \( \tau \) decay data [91], further theoretical considerations [92–95], quark-model calculations [96, 97], or lattice simulations [98, 99] and their relevance for the confinement-deconfinement phase transition [100]. Phenomenologically, upper bounds for the dimension-2 operators coming from the \( e^+e^- \) data were analyzed in [100, 101], yielding \( 16C_2 \leq 0.1 \text{ GeV}^2 \sim (0.3 \text{ GeV})^2 \). The possible appearance of the dimension-2 condensates in Regge-like models can be traced in Ref. [102]; it was also briefly discussed in Ref. [83]. A more quantitative analysis was carried out by us in Refs. [82, 84].

Very recently, the presence of the dimension-2 objects has been linked to the truncation of the full perturbative series [102], indicating duality between the resummed perturbation theory and the presence of the lowest power corrections. This is supported by the correlation between perturbative and non-perturbative contributions found in Ref. [102].

As we have already discussed in Sect. IIIB in the \( \Theta-\Theta \) correlator the dimension-2 object, \( C_2 = \sum f_n^2 \), appears naturally. Clearly, if there is a finite number of states, the sum is necessarily positive and hence non-vanishing. This is in an open contradiction to the Operator-Product-Expansion philosophy, where only dimension-4 and higher local gauge-invariant operators are allowed. Therefore, the only scenario where the dimension-2 object can possibly vanish is by assuming an infinite number of resonances, such that the sum diverges requiring regularization. In this case the asymptotic condition, Eq. (23), holds. As usual in the Regge models, we assume \( f_n \) to be independent on \( n \) for all values of \( n \), not only asymptotically, as requested by Eq. (2).

The basic object of our study is the Regge sum (in this section \( M \) denotes the lowest glueball mass)

\[
\Pi_{\Theta \Theta} = \sum_{n=0}^{\infty} \frac{Q^4 f_n^2}{m_n^2 + Q^2} = \sum_{n=0}^{\infty} \frac{Q^4 f_n^2}{M^2 + \alpha n + Q^2},
\]  

(96)

which can be evaluated using the formula (see Appendix A)

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{M^2 + \alpha n + Q^2} - \frac{1}{M^2 + \alpha n} \right] = \frac{1}{\alpha} \left[ \Psi(M^2/\alpha) - \Psi(M^2 + Q^2/\alpha) \right],
\]  

(97)

where

\[
\Psi(z) = \Gamma'(z)/\Gamma(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} + \mathcal{O}(z^{-6}),
\]  

(98)

is the polygamma function.

We note that the infinite Regge sum generates properly the \( Q^4 \log Q^2 \) term. We assume that we have one family of states contributing to the Regge sum at the leading-\( N_c \) order. The matching with the leading term of the OPE expansion gives immediately

\[
C_0 = \frac{f^2}{\alpha} = \frac{\alpha_s^2(n_c^2 - 1)\beta_0^2}{32\pi^4},
\]  

(99)

We read off that \( f \sim N_c \), in accordance with the fact that the glueballs provide the saturation at the leading-\( N_c \) level. Explicitly,

\[
f = \frac{11\sqrt{4\alpha N_c^2}}{12\sqrt{\pi}},
\]  

(100)

for \( \alpha = 0.2, a = 1.31 \text{GeV}^2 \), and \( N_c = 3 \) gives \( f \sim 135 \text{ MeV} \), a reasonable value. Note that taking the
full slope \( a = 1.31 \text{ GeV} \) is equivalent to taking every second state in the scalar-isoscalar trajectory, hence we are implicitly assuming that every second state on the trajectory is a glueball, alternating with \( q\bar{q} \) scalar-isoscalar mesons. This alternation is suggested by the two parallel trajectories, Eq. \([5]\).

For the case of the dimension-2 condensate the \( \zeta \)-function regularization \([84]\) (see Appendix \([A]\)) applied on the spectrum \( M_n^2 = a n + M^2 \) yields

\[
C_2 = \sum f_n^2 = f^2 \left( \frac{1}{2} - \frac{M^2}{a} \right) \tag{101}
\]

The condition for a vanishing dimension-2 condensate is thus equivalent to requesting that the mass of lowest state be given by the equality

\[
M^2 = \frac{1}{2}a, \tag{102}
\]

which for \( a = 1.31(12) \text{ GeV} \) from Eq. \([9]\) means \( M = 810(40) \text{ MeV} \). The value is not far from the mass of \( f_0(980) \), it is also in the PDG range for \( m_a \) (cf. Fig. \([9]\)) \footnote{As described in \([82]\), the shift from the pole resonance to the Breit-Wigner resonance is \( O(N_c^{-2}) \) suppressed, suggesting that the leading \( N_c \) values lies in between.}

Thus, on the basis of a simplest Regge model considered here it is not possible to sort out which state is a glueball, and which one is a \( q\bar{q} \) meson, if the dimension-2 condensate were indeed to vanish.

The condition for the vanishing \( C_2 \) is identical to that for the vector or axial flavor channels with \( C_{2,V} = \sum f_n^2 \) and \( C_{2,A} = \sum f_{n,A}^2 \), respectively \([82]\). Note that in that case the first Weinberg sum rule requires a non-vanishing dimension-2 object \( f_\pi^2 = C_{2,A} - C_{2,V} \). This shows that both dimension-2 operators cannot vanish simultaneously, despite the fact that they cannot be represented by local and gauge invariant operators.

One may extend the Regge model by modifying the parameters of the lowest states on the trajectory, for instance the \( f \) constants. It is known from similar studies that such modifications cause strong effects \([82]\). Also, the \( 1/N_c \) effects may play a role. Nevertheless, we see that the fine-tuning needed to cancel the dimension-2 object leads to a quite natural condition for the lowest mass of the Regge family.

For the dimension-4 condensate the matching of Eq. \([96]\) to Eq. \([95]\) leads to the large-\( N_c \) identity (see Appendix \([A]\))

\[
\frac{121 N_c^2 \alpha}{144 \pi} \left( \frac{\alpha}{\pi} G^2 \right) = -\frac{1}{2} \sum f_n^2 M_n^2 = \frac{f^2}{2} \left( \frac{M^4}{a} + \frac{a}{6} - M^2 \right) \tag{104}
\]

The condition for the gluon condensate to be positive is therefore \( a > (3 + \sqrt{3}) M^2 \) or \( a < (3 - \sqrt{3}) M^2 \). Combining Eq. \([103]\) and \([104]\) yields

\[
\left( \frac{\alpha}{\pi} G^2 \right) = \frac{\alpha N_c^2 \left( a^2 - 6aM^2 + 6M^4 \right)}{24\pi^3}. \tag{104}
\]

This quantity is plotted against the mass \( M \) in Fig. \([13]\). We note that we have two windows for the values of \( M \), above 1.3 GeV and below 0.45 GeV, which match the predictions of the considered Regge model to the QCD estimate of Ref. \([74]\) giving \( \left( \frac{\alpha}{\pi} G^2 \right) = 0.009(7) \text{ GeV}^4 \). Alternative determinations yield about twice larger values, \( \left( \frac{\alpha}{\pi} G^2 \right) = 0.022(3) \text{ GeV}^4 \) \([107]\), or even negative values, \( \left( \frac{\alpha}{\pi} G^2 \right) = -0.005(1) \text{ GeV}^4 \) \([108]\). In our model the gluon condensate is related to the lightest glueball mass \( M \) according to Eq. \([103]\). Modifications of the Regge model along the lines of Ref. \([82, 84]\) may modify or lift these constraints, making it possible for the approach to be fully consistent with the requirements of the Operator Product Expansion.

\section{VII. CONCLUSIONS}

We have presented a wide-scope analysis of the \( \sigma \)-state in the large-\( N_c \) Regge framework. Our main points are the following:

1. The \( \sigma \) state fits very well to the scalar-isoscalar Regge trajectory including all the known states from the PDG review. From the viewpoint of the spectrum, there seems to be no particular distinction among the scalar-isoscalar states. The glueball or meson nature has to be sought based on the \( N_c \) scaling of the couplings of the \( 0^{++} \) states to the vacuum or to hadrons.
2. The mass of the lowest 0++ state (the \( \sigma \)) can also be extracted from the analysis of gravitational form factors of the pion and nucleon, available from the lattice calculations. Extracted masses are compatible with the benchmark determinations within the expected accuracy of the large-\( N_c \) framework, showing the consistency of the approach and the proximity of the large-\( N_c \) limit to the real world in the scalar-isoscalar sector.

3. The constraints of chiral symmetry and the high-energy behavior of the energy-momentum correlator imposes a set of sum rules involving various coupling constants. Saturation of these sum rules with just two states suggest that \( f_0(600) \) is a meson and \( f_0(980) \) is a glueball. Such an assignment, specific to the simple two-state model, results in natural \( \epsilon \)-scaling of the ratios of the coupling constants of these states.

4. Correlation functions of the energy-momentum tensor are saturated by (infinitely many) scalar-iso scalar states. The analysis of the Operator Product Expansion shows that there appears the dimension-2 condensate, whose vanishing requires a fine tuning of parameters. Then, in the simplest Regge model the resulting mass of the lowest glueball state is \( \sim 800 \) MeV.

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**Appendix A: The \( \zeta \)-regularization**

In the large \( N_c \) limit only tree-level diagrams survive, however, with infinitely many resonances. Thus, the infinite sums may need to be regularized. In the dimensional regularization the coupling of the resonance to the current acquires an additional dimension \( f_n \rightarrow f_n \mu^{\epsilon} \) with \( \epsilon = d - 4 \). By choosing as a natural scale \( \mu = m_n \), one gets \( f_n^2 \rightarrow f_n^2 m_n^{2\epsilon} \), which means

\[
\sum_n f_n^2 m_n^{2\epsilon} \equiv \lim_{s \rightarrow k} \sum_n f_n^2 m_n^{2s}. \tag{A1}
\]

This is completely equivalent to formally expanding the two-point correlator at large \( Q^2 \) and reinterpreting the coefficients within the zeta-function regularization \([84]\). In our case the mass formula is \( m_n^2 = an + b^2 \), hence one has to consider the sums of the form

\[
\sum_n (n + \alpha)^{2s} \equiv \lim_{s \rightarrow k} \sum_n (n + \alpha)^{2s} = \zeta(\epsilon - 2s, \alpha) \tag{A2}
\]

where \( \zeta(\alpha) \) is the generalized Riemann zeta function. Then, we arrive at the explicit formulas

\[
\sum_{n=0}^{\infty} (n + \alpha)^0 = \frac{1}{2} - \alpha, \tag{A3}
\]

\[
\sum_{n=0}^{\infty} (n + \alpha)^1 = \frac{1}{2} \left( -\alpha^2 + \alpha - \frac{1}{6} \right), \tag{A4}
\]

**Appendix B: The linear Lagrangean for one scalar**

For only one scalar-iso scalar state, Lagrangean \([35]\) with condition \([37]\) reads

\[
\mathcal{L}_{\Phi \Phi} = \frac{f_\sigma^2}{4f_\sigma^2} \Phi^2 (\partial^\mu U^\dagger \partial_\mu U) + \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - V(\Phi). \tag{B1}
\]

Introducing the Cartesian decomposition

\[
\Sigma = \frac{f_\sigma}{f_\pi} \Phi U = \sigma + i \vec{\pi} \cdot \vec{\pi}, \tag{B2}
\]

one gets an extension of the linear sigma model with the so-called \( A \)-term \([109]\), allowed in the linear realization of the chiral symmetry:

\[
\mathcal{L} = \frac{1}{2} [\partial^\mu \sigma \partial_\mu \sigma + \partial^\mu \vec{\pi} \cdot \partial_\mu \vec{\pi}] + \frac{A^2}{2} [\sigma \partial^\mu \sigma + \vec{\pi} \cdot \partial_\mu \vec{\pi}]^2 - V(\sigma^2 + \vec{\pi}^2). \tag{B3}
\]

where in the present case

\[
A^2 \sigma^2_{\text{vac}} = \left( \frac{f_\sigma}{f_\pi} - 1 \right). \tag{B4}
\]

Note that \( \sigma \) and \( \vec{\pi} \) are chiral partners.

Imposing the condition \( f_\sigma = f_\pi \) removes the \( A \) term. Note, however, that the identity \( f_\sigma = f_\pi \) is not a consequence of any symmetry or physical requirement. With this condition the Lagrangean reads

\[
\mathcal{L} = \frac{1}{2} [\partial^\mu \sigma \partial_\mu \sigma + \partial^\mu \vec{\pi} \cdot \partial_\mu \vec{\pi}] - \frac{m_\pi^2}{8f_\pi^2} (\sigma^2 + \vec{\pi}^2)^2 \left[ \log \left( \frac{\sigma^2 + \vec{\pi}^2}{f_\pi^2} \right) - \frac{1}{2} \right]. \tag{B5}
\]

Note the resemblance to the bosonic part of the dilated chiral quark model of Ref. \([110]\). The main difference between potential \([B5]\) and the standard Mexican hat potential,

\[
V(\sigma, \vec{\pi}) = \frac{m_\pi^2}{8f_\pi^2} (\sigma^2 + \vec{\pi}^2 - f_\pi^2)^2, \tag{B6}
\]

is the scale behaviour of the latter, a different vacuum energy density (factor of two difference), and modified couplings containing more than one scalar. The \( \sigma \pi \pi \) vertex is the same on shell, thus yields the same decay width of the \( \sigma \) into two pions.
