What actually happens when you approach a gravitational singularity?∗

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Abstract

Roger Penrose’s 2020 Nobel Prize in Physics recognises that his identification of the concepts of “gravitational singularity” and an “incomplete, inextendible, null geodesic” is physically very important. The existence of an incomplete, inextendible, null geodesic doesn’t say much, however, if anything, about curvature divergence, nor is it a helpful definition for performing actual calculations. Physicists have long sought for a coordinate independent method of defining where a singularity is located, given an incomplete, inextendible, null geodesic, that also allows for standard analytic techniques to be implemented. In this essay we present a solution to this issue. It is now possible to give a concrete relationship between an incomplete, inextendible, null geodesic and a gravitational singularity, and to study any possible curvature divergence using standard techniques.

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Roger Penrose was awarded a 50% share of the 2020 Nobel Prize in Physics. The short form of Penrose’s contribution is cited as “for the discovery that black hole formation is a robust prediction of the general theory of relativity.” The longer form explicitly mentions Penrose’s 1965 paper that proved the first “modern” singularity theorem [1].

Readers of Penrose’s paper might be surprised to discover that he doesn’t actually prove the existence of a black hole. Instead Penrose presents a very general result that he uses as evidence for the stability of black hole solutions under perturbations, specifically the Schwarzschild and Kerr black holes. It was the very general nature of Penrose’s paper that changed the astronomical community’s thinking about the physical reality of black holes and, subsequently, justified his Nobel Prize. His paper doesn’t prove the existence of black holes, however.

A black hole is popularly imagined as a “place” where gravity becomes infinitely strong. There are a handful of different ways to mathematically express this. They all boil down to the idea that the curvature of spacetime diverges to infinity along some curve that is assumed to approach the singularity. Penrose’s paper says nothing about these ideas.

Penrose proved that, under very general and physically reasonable hypotheses, there exists an incomplete, inextendible, null geodesic. More recent singularity theorems prove the existence of incomplete, inextendible, timelike geodesics under similarly general hypotheses. The existence of an incomplete, inextendible, causal geodesic challenges our physical intuition of time. If I have experienced a finite amount of time, then I should always be able to experience a bit more time. Yet the singularity theorems say that this is not true.

Penrose implicitly assumes that the existence of a gravitational singularity is equivalent to the existence of an incomplete, inextendible, null curve. We agree with Penrose that a gravitational singularity is only a singularity if it can be approached in finite proper time. There are examples of spacetimes with incomplete, inextendible, timelike curves yet all causal geodesics are complete. Therefore, following Penrose (and ignoring issues associated with trapped causal curves), we take the existence of an incomplete, inextendible, causal curve as equivalent to the existence of a gravitational singularity.

This is an extremely general definition of gravitational singularity, but it covers all possible unphysical behaviour. Now that we have an, admittedly difficult to handle, definition of
gravitational singularity, the obvious question is “Does our physical intuition fit the mathematics?” Does curvature necessarily diverge along the incomplete, inextendible curves predicted by Penrose’s singularity theorem? This question is surprisingly deep, and has generated numerous branching research programs interested in various special cases.

We know a lot about what happens as an observer approaches singularities in certain parametrised families of spacetimes, but almost nothing about what happens on approach to a general singularity. This boils down to the difficulty of using our definition of a singularity in computations. This essay presents the evidence that something called, The Abstract Boundary, provides the tools necessary to perform curvature computations in general spacetimes starting only with the assumptions of Penrose’s singularity theorem.

The Kerr black hole solution \((m > a)\) is a clear example of how the analysis of a singularity usually proceeds. Presented in Boyer-Lindquist coordinates the spacetime is mapped to points in Euclidean space \(\mathbb{R}^4\) minus the origin. The origin is a point on the closure of the image of the coordinates, however. We will call such a point a boundary point. With respect to this chart any incomplete, inextendible curve ends at the origin. It therefore appears that all incomplete, inextendible curves have the same endpoint at the same gravitational singularity; at the “centre” of the black hole. The curvature scalar \(R_{abcd}R^{abcd}\) evaluated along incomplete, inextendible curves in the limit to the origin can be either finite or infinite. See Figure (1). We call such boundary points mixed.

Switching to Kerr-Schild coordinates presents a different picture. In these coordinates there is a 2-dimensional disc of boundary points. The endpoints of images of incomplete, inextendible, causal curves in Kerr-Schild coordinates are points on this disc. These curves, which in Boyer-Lindquist coordinates all appeared to have the same endpoint, could now have different endpoints. See Figure (2). It turns out that it is possible to analytically extend Kerr-Schild coordinates through the interior of the disc. Such boundary points are called regular as we can extend both the manifold and metric through them.

The boundary points, in Kerr-Schild coordinates, on the boundary of the disc are also endpoints of incomplete, inextendible curves. The curvature scalar \(R_{abcd}R^{abcd}\) diverges to infinity, however, along all such curves, at these boundary points. Thus these curves can be taken to correspond to “real” gravitational singularities. We call such singularities essential.
Figure 1: The curvature scalar $R_{abcd}R^{abcd}$ for the Kerr black hole in the $(r,\theta)$ plane of Boyer-Lindquist coordinates [2]. The left hand plot gives a 3D image, while the right hand plot presents the contours of the 3D image (darker colours represent larger values). We have truncated the value of $R_{abcd}R^{abcd}$ at $\pm 1000$. Note the dependence on $\theta$ for the value of the limit to $r = 0$.

Figure 2: A profile view of the disc of boundary points associated with Kerr-Schild coordinates [2]. The thick horizontal line between the two black dots is the set of regular boundary points through which the metric can be extended. The two black dots are the singular boundary points at which $R_{abcd}R^{abcd}$ diverges. The dashed and solid contours are lines of constant radial and one of the angular Boyer-Lindquist coordinates.
Note that in Boyer-Lindquist coordinates the internal structure of the singularity is muddled up with regular behaviour of the metric. Kerr-Schild coordinates present the structure of the singularity more cleanly. The physical behaviour is clearer. There are no mixed boundary points any more. This property of Kerr-Schild coordinates is used to justify the ring representation of the gravitational singularity as “more” physically reasonable than the point representation in Boyer-Lindquist coordinates.

The Kerr-Schild coordinates are therefore used to give a definition of a gravitational singularity that is useful for computations. The union of the manifold with the boundary points is a closed topological space. It is possible to perform computations “on the boundary” by taking limits on approach to boundary points. There is a classification of boundary points into singularities and regular points, and we know that singular boundary points and incomplete, inextendible, causal curves are related.

This definition is, however, dependent on the Kerr-Schild coordinates to define the boundary points. The identification of boundary points with the concept of gravitational singularity is chart dependent. This is a problem as any concept defined with respect to a manifold should be independent of coordinates. Moreover, it is not clear how to extend the example above to any Lorentzian manifold to which Penrose’s singularity theorem applies.

Our very recent paper [3] completes a body of work [4, 5, 6, 7, 8, 9, 10, 11] that presents a solution to all these issues—the Abstract Boundary. The Abstract Boundary is an algorithm that takes a spacetime and produces a compact, non-Hausdorff, locally Euclidean topological space into which the original spacetime is homeomorphically embedded. The boundary of this embedding, also called the Abstract Boundary, is denoted by $B(M)$.

Let $(\mathcal{M}, g)$ be a spacetime. Let $\Phi = \{\phi : \mathcal{M} \to \mathcal{N}\}$ be the set of all open embeddings of $\mathcal{M}$ into a manifold $\mathcal{N}$ of the same dimension. An element of $B(\mathcal{M})$ is an equivalence class of subsets of $\partial \phi(\mathcal{M})$ for all $\phi \in \Phi$. The equivalence relation is defined so that for each subset, $A \subset \partial \phi_1(\mathcal{M})$ and $B \subset \partial \phi_2(\mathcal{M})$, $[A] = [B] \in B(\mathcal{M})$ if the set of sequences $(a_i)_{i \in \mathbb{N}} \subset \mathcal{M}$ so that $\phi_1(a_i) \to a \in A$ and the set of sequences $(b_i)_{i \in \mathbb{N}} \subset \mathcal{M}$ so that $\phi_2(b_i) \to b \in B$ are the same. Full details of the construction can be found in [4].

In the Kerr solution, the origin in Boyer-Lindquist coordinates is equivalent to the disc in Kerr-Schild coordinates. See [8] for the creation of the necessary embedding from a chart.
Note that the construction of $\mathcal{B}(\mathcal{M})$ requires nothing more than $\mathcal{M}$ itself. Thus the Abstract Boundary can be applied to any pseudo-Riemannian manifold.

In this way, any method of studying a spacetime that involves attaching ideal points via a chart or embedding is contained in the Abstract Boundary. In particular, it contains, and thus generalises, both the classical coordinate dependent methods of studying singularities and Penrose’s conformal boundary [2] (see Figure (3)). Concrete examples of this can be found in [8].

![Figure 3: An illustration of Penrose’s conformal boundary for the Schwarzschild black hole [2]. The standard classification of boundary points, in terms of the conformal factor, agrees with the Abstract Boundary classification [8].](image)

Given a point $[A]$ in $\mathcal{B}(\mathcal{M})$ we can take any representative $B \in [A]$. This representative is a subset of the topological boundary of some embedding, $B \subset \partial \phi(\mathcal{M})$, for $\phi : \mathcal{M} \rightarrow \mathcal{N}$. One can therefore perform computations in the charts of $\mathcal{N}$ using the pushforward of $g$ by $\phi$. If these computations are invariant under the equivalence relation that defines $[A]$ then the results of the computation will be chart independent and well-defined. This equips the Abstract Boundary with a differential structure sufficient for computations of the sort performed with Kerr-Schild coordinates.
One of the miracles of the Abstract Boundary is that it is possible to generalise the classification of boundary points, as hinted at in the Kerr example. Thus points in the Abstract Boundary fall into one of a handful of classes, including “singularities”, “points at infinity” and “mixed points” \([4]\). The classification is constructed from the metric \(g\), the set \(\Phi\) and a set of curves (usually the set of all piecewise \(C^1\) causal curves).

Our recent result, the Endpoint Theorem \([3]\), shows that for any sequence \((x_i)_{i \in \mathbb{N}} \subset \mathcal{M}\) with no accumulation points, there exists an element \([x] \in \mathcal{B}(\mathcal{M})\) and an embedding \(\phi\) so that \(\phi(x_i) \to x\). This theorem establishes that \(\mathcal{M} \cup \mathcal{B}(\mathcal{M})\) is compact in the topology given in \([5, 6]\) or \([8]\) and thereby shows that \(\mathcal{B}(\mathcal{M})\) is large enough to capture all possible singular behaviour. Most importantly, it provides locations (the endpoints) for the gravitational singularities associated with the incomplete, inextendible, causal geodesics predicted by the singularity theorems.

The Abstract Boundary has its own singularity theorem \([7]\), which complements Penrose’s singularity theorem. The Abstract Boundary result shows that the incomplete, inextendible curves predicted by Penrose’s singularity theorem have actual singularities, according to the Abstract Boundary classification, as endpoints. This renders the Abstract Boundary an excellent mathematical representation of Penrose’s implicit definition of a gravitational singularity. Put another way, the Abstract Boundary is a mathematical encoding of our physical intuition.

Lastly, the papers \([10, 11]\) show that elements of \(\mathcal{B}(\mathcal{M})\) have good topological properties under continuous maps, and the paper \([9]\) shows that the classification of singularities of \(\mathcal{B}(\mathcal{M})\) is stable and therefore a physically reasonable definition.

The Abstract Boundary generalises the Kerr-Schild example to all Lorentzian manifolds. It also generalises a well respected alternative construction—Penrose’s conformal boundary. It solves the problem of how to convert Penrose’s definition of a gravitational singularity into a concept amenable to computation, by providing a location for the endpoint of the incomplete, inextendible, causal curve predicted by Penrose’s singularity theorem in \(\mathcal{B}(\mathcal{M})\).

With our semi-“completion” of the singularity theorems, one now has the framework required to proceed to investigate whether Penrose’s theorem actually predicts infinite curvature singularities like the ones found at the heart of black holes. This brings us much
closer to answering the fundamental question “What actually happens when you approach a gravitational singularity?”.

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