Binary response model with many weak instruments*

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Abstract

This paper considers an endogenous binary response model with many weak instruments. We in the current study employ a control function approach and a regularization scheme to obtain better estimation results for the endogenous binary response model in the presence of many weak instruments. Two consistent and asymptotically normally distributed estimators are provided, each of which is called a regularized conditional maximum likelihood estimator (RCMLE) and a regularized nonlinear least square estimator (RNLSE) respectively. Monte Carlo simulations show that the proposed estimators outperform the existing estimators when there are many weak instruments. We use the proposed estimation method to examine the effect of family income on college completion.

JEL codes: C31, C35

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1 Introduction

The conventional two-stage least square (TSLS) estimation method has been employed in empirical studies in economics and other fields of social science, although the dependent variable is binary (see, e.g., Miguel, Satyanath, and Sergenti, 2004, Norton and Han, 2008, Nunn and Qian, 2014, and Bastian and Michelmore, 2018). This approach is not recommended from a theoretical view. Nonetheless, it has been popularly employed not only because of the ease of interpretation, but also because of the fact that some essential statistical properties of estimators for nonlinear binary response models have only been studied in limited settings. For example, in contrast with the growing body of work on instrumental variable estimators for linear models in the presence of many instruments, statistical properties of estimators for binary response models in such a scenario have not been discussed yet. Moreover, only a little is known about them when there are weak or nearly weak instruments, see e.g., Magnusson (2010) and Dufour and Wilde (2018).

This paper aims to make the endogenous binary response model be better applied in practice; in particular, we focus on a valid estimation and inference method for the endogenous binary response model when the dimension of the instrumental variable (or sometimes called the number of instruments) is very large. In principle, using a large number of instruments is helpful for improving estimation efficiency. However, in spite of this advantage, this is not always recommended in finite samples due to the so-called many instruments bias (see Bekker, 1994). Thus, in practice, employing a way to reduce this bias is essential to make use of them. A possible candidate would be using only a few of instruments selected by a certain criterion, see, e.g., Donald and Newey (2001), Belloni et al. (2012), and Caner and Zhang (2014). However, in the current paper, we consider a situation where each element of a large number of instruments is allowed to be nearly weak (see Newey and Windmeijer, 2009), and this consideration makes the consistency results of the aforementioned variable selection procedures be no longer valid even in linear models. Therefore, to handle the scenario with “many weak instruments”, this study adopts a regularization scheme that is similar to those considered by Hausman, Lewis, Menzel, and Newey (2011), Carrasco (2012), Hansen and Kozbur (2014), Carrasco and Tchuente (2016), and Han (2020). Our regularization strategy is preferred to variable selection procedures for a couple of reasons to be detailed in Section 2.1.

Alongside the regularization scheme, we consider estimators similar to the two-stage conditional maximum likelihood estimator (2SCMLE) proposed by Rivers and Vuong (1988) which is one of the most popular estimators for endogenous binary response models (see, e.g., Wooldridge, 2010). Specifically, as detailed in Section 3, we use the first-stage residual, obtained using a regularization method, as an additional regres-
sor in the second stage. Then, we define two estimators, which are respectively called the regularized conditional maximum likelihood estimator (RCMLE) and regularized nonlinear least square estimator (RNLSE). Their asymptotic properties are studied under a parametric assumption similar to that in Rivers and Vuong (1988); although the parametric approach may not be satisfactory from a theoretical standpoint, it is (i) preferred in empirical studies compared to semi- and non-parametric methods and (ii) helpful for understanding how the regularization method improves estimation results in the presence of many weak instruments.

In our setting, instrumental variables are allowed to be nearly weak; such instruments are common in empirical studies using binary response models. For example, Miguel et al. (2004), Norton and Han (2008), Nunn and Qian (2014), Frijters, Johnston, Shah, and Shields (2009) and Bastian and Michelmore (2018) report small values of the first-stage F test statistic that is known to be closely related to the weakness of instruments in the linear model (Staiger and Stock, 1997; Stock and Yogo, 2005). Although this statistic is not a perfect measure of instrument strength in binary response models (e.g., Frazier, Renault, Zhang, and Zhao, 2020), such small values of F statistic are suggestive that their instrumental variables might be weak. However, in spite of its practical importance, only a few studies concern the issue of (nearly) weak instruments in endogenous binary response models. Frazier et al. (2020) study a way to test the weakness of instruments using a distorted J test under a parametric assumption. Andrews and Cheng (2014) investigate asymptotic properties of the generalized method of moment (GMM) estimator under weak identification and apply their estimation approach to an endogenous probit model. However, even in these articles, the case with many weak instruments has not been studied.

The discussions made in this paper are technically different from those in the previous literature on many weak instruments. In particular, we allow the instrumental variable to be function-valued; examples of the function-valued random variables include, but are not limited to, the age-specific fertility rate (Florens and Van Bellegem, 2015), skill-specific share of immigrants (Seong and Seo, 2022), and lagged cumulative intraday stock return trajectory (Chen et al., 2022). Such a functional instrumental variable has received recent attention in econometrics, but it has been used mostly in the linear model, see, e.g., Carrasco (2012), Florens and Van Bellegem (2015), Carrasco and Tchuente (2015, 2016), Benatia et al. (2017), Chen et al. (2022), and Seong and Seo (2022). However, in contrast with the cases considered therein, our moment condition is nonlinear and not additively separable from the error term. These issues require the use of a control function to adequately deal with the issue of endogeneity. Consequently, the limiting distributions of our estimators turn out to depend on the limiting distribution of the first-stage estimator which is given
by an operator acting on a possibly infinite-dimensional space in our setting (see Section 3.2). This in turn complicates our asymptotic analysis and makes it technically different from that of (i) estimators for linear models (e.g., Hansen and Kozbur, 2014; Carrasco, 2012; Carrasco and Tchuente, 2016; Benatia et al., 2017) and (ii) GMM estimators associated with an additively separable second stage (e.g., Newey and Windmeijer, 2009; Hausman et al., 2011; Andrews and Cheng, 2014; Han, 2020). Given this, the current paper uses an approach to asymptotic analysis that is differentiated from the previous papers.

This paper is relevant to practitioners who need to control for endogeneity using many weak instruments when the outcome is binary. We revisit Bastian and Michelmore (2018) and use our estimators to study the effect of family income in childhood and adolescent years on college completion in young adulthood. Following the authors, we first use three measures of Earned Income Tax Credit (EITC) exposure to address the endogeneity of family income and then employ interactions of the EITC measures with 47 state-of-residence dummies as additional instruments. This example shows that, in the presence of many instruments, Rivers and Vuong’s (1988) 2SCMLE is biased toward the probit estimator while the estimates obtained using our method are not so.

The paper is organized as follows. In Section 2, we discuss the model of interest. Section 3 proposes our estimators and presents their asymptotic properties. Section 4 provides Monte Carlo simulation results. The empirical application is given in Section 5. Section 6 concludes. All proofs of the results in this paper are provided in the Online Supplement. We include an additional simulation result and an empirical application in the same supplement.

2 The Model

We consider the following model with independent and identically distributed (iid) data.

\[ y_i = 1\{Y_{2i}'\beta_0 \geq u_i\}, \]

\[ Y_{2i} = \Pi_n Z(x_i, t) + V_i, \]

where \(1\{A\}\) is the indicator function that takes 1 if \(A\) is true and 0 otherwise. The response variable \(y_i\) is a scalar. The \(d_e\)-dimensional explanatory variable \(Y_{2i}\) consists of endogenous and exogenous variables; if a row of \(Y_{2i}\) is exogenous, the corresponding row of \(V_i\) is equal to zero. The variable \(x_i\) is \(d_x\)-dimensional and satisfies the exogeneity condition, i.e., \(E[u_i|x_i] = 0\) and \(E[V_i|x_i] = 0\).

The model above is similar to the standard endogenous binary response model considered by, e.g., Rivers and Vuong (1988), Rothe (2009), and Blundell and Powell (2004), other than that a general class
of random variables, such as a function-valued random variable, can be considered as $Z_i(= Z_i(x_i, t))$. The instrument $Z_i$ is a realized function of a $d_x$-dimensional exogenous variable $x_i$ and a deterministic index $t$. While the index is crucial for discussing technical details of $Z_i$ in the general case, we will temporarily set it aside to focus on the core aspects of the model. The mathematical details on the index $t$ will be explained in Section 3.2.1. Instead, by using illustrative examples, we note that our setup can accommodate various types of instruments suitable for practical use. First, if $x_i = (x_{1i}, \ldots, x_{d_i})$, then $Z_i$ may be given by $x_i$, a standard $d_x$-dimensional vector of different instruments (see, e.g., Examples 1 and 2). If $x_i$ is a scalar, we may consider $(1, x_i, x_i^2, \ldots, x_i^K)'$ as $Z_i$ and, if so, our first-stage estimator can be understood as a nonparametric estimator of $E[Y_{2i}|x_i]$ in some cases, see Carrasco (2012, Section 2.3) and Antoine and Lavergne (2014). Lastly, as a more general case, the realized instrument $Z_i$ itself can be a function-valued random variable discussed in the recent articles such as Carrasco (2012), Florens and Van Bellegem (2015) and Benatia et al. (2017). Examples of function-valued random variables include (but are not limited to) the density of $x_i$, continuum of moments of $x_i$ (e.g., Carrasco, 2012), rainfall growth curve in Example 3 and skill-specific share of immigrants (see, e.g., Seong and Seo, 2022).

As is often assumed in the literature, including the articles mentioned above, we assume $u_i|(x_i, V_i) \sim_{iid} (-V_i'\psi_0, \sigma_i^2)$. This assumption is helpful for addressing the issue of endogeneity of $Y_{2i}$. Under the assumption, (2.1) can be written as follows.

$$y_i = 1\{Y_{2i}'\beta_0 + V_i'\psi_0 \geq \eta_i\}, \quad (2.3)$$

where $\eta_i = u_i + V_i'\psi_0$ and $E[\eta_i|x_i, V_i] = 0$. (2.3) shows that the issue of endogeneity can be resolved once the first-stage error $V_i$ is used as an additional regressor in (2.1). In this regard, the variable $V_i$ is often called the control function in the literature. However, in practice, the control function is not observable and has to be replaced by its estimate. Rivers and Vuong (1988) suggested replacing $V_i$ with the residuals from the first-stage linear regression, say $\hat{V}_i$, and estimating the parameters in (2.3) by maximizing the conditional likelihood of $y_i|(x_i, \hat{V}_i)$ under a parametric assumption. This approach becomes one of the most popular estimation methods for endogenous binary response models and was extended by Rothe (2009) to the semiparametric case.

Although the aforementioned estimators have good asymptotic properties in general, it is not the case in our setting. This is because, to ensure good asymptotic properties of the estimators, the difference between $\hat{V}_i$ and $V_i$ should be asymptotically negligible. However, in practice, $Z_i$ may have a nearly singular covari-
ance, see, e.g., Examples 1 and 2. If so, the first-stage estimator may be asymptotically biased, resulting in \( \hat{V}_i \) being asymptotically different from \( V_i \). Another concern on the use of Rivers and Vuong’s (1988) approach is that the asymptotic distribution of their 2SCMLE depends on the asymptotic distribution of the first-stage estimator. In the current paper, the first-stage parameter and its estimator are represented by linear transformations from a possibly infinite dimensional Hilbert space to \( \mathbb{R}^{d_e} \). Therefore, the asymptotic distribution cannot be obtained as in Rivers and Vuong (1988), which prevents a naive application of their approach.

Below, we provide practical examples where the use of Rivers and Vuong’s (1988) approach may not be appropriate due to a nearly singular covariance of \( Z_i \), even with a considerably large sample size. In the examples, we use \( W_i \) and \( \tilde{Z}_i \) to denote the included and excluded instrumental variables for the ease of explanation. The vectors \( Y_{2i} \), and \( Z_i \) in (2.1) and (2.2) reduce to the vector of an endogenous variable and \( W_i \) and the vector of \( W_i \) and \( \tilde{Z}_i \) respectively. In addition, we use the condition number (the ratio of the largest to smallest eigenvalues) of a covariance matrix as a measure of how close the matrix is to being singular.

**Example 1.** Bastian and Michelmore (2018) studied the impact of family income in childhood and adolescent years on academic achievement in early adulthood using the linear probability model. However, from a theoretical perspective, their result may be improved by using a binary response model; this is because (i) the linear model often indicates that the predicted probability is outside the interval \([0, 1]\), (ii) the marginal effect of an explanatory variable on the probability of outcome occurring is constant, regardless of the value of an explanatory variable, and (iii) the standard errors of the coefficient estimates in the linear model should be corrected for heteroscedasticity and thus estimation results are less efficient. Moreover, when endogeneity is present, theoretical results developed for the linear model do not always remain valid for the binary response model, see, e.g., Li et al. (2022) and Frazier et al. (2020). Hence, as an alternative, one may consider the following.

\[
y_i = 1\{\beta_0 + I_i'\beta_1 + W_{1i}'\beta_2 + W_{2i,t}'\beta_3 \geq u_i\}, \quad I_i = \pi_0 + \tilde{Z}_i'\pi_1 + W_{1i}'\pi_2 + W_{2i,t}'\pi_3 + V_i,
\]

where \( y_i \) is 1 if individual \( i \) is a college graduate and 0 otherwise. \( I_i = (I_i,(0-5), I_i,(6-12), I_i,(13-18))' \) and each \( I_i,(a) \) is the family income for individual \( i \) at each age interval \( a \). \( W_{1i} \) and \( W_{2i} \) are the vectors of exogenous variables detailed in Section 5. As instruments, the authors suggested using measures of EITC exposure of individual \( i \) in three age intervals 0-5, 6-12, and 13-18, i.e., \( \tilde{Z}_i = (\text{EITC}_{i,(0-5)}, \text{EITC}_{i,(6-12)}\text{EITC}_{i,(13-18)})' \). We, in Section 5, use interactions of these EITC measures with state-of-residence dummies as additional instruments. This approach is expected to improve estimation efficiency by reducing the variance of the
estimators. Moreover, it will enable us to account for heterogeneous effects of EITC exposure on family income that may be influenced by the state of residence. Thus, it will mitigate a potential bias from factors such as state-specific costs of living. The maximum number of instruments used in the analysis is 144, and the sample size is 2,654. In this example, the largest and smallest eigenvalues of the sample covariance are around 1.94 and $9.35 \times 10^{-18}$. Hence, in spite of the considerably large sample size, the sample covariance of instruments has a very large condition number, which suggests that the matrix is nearly singular.

**Example 2.** Angrist and Krueger (1991) is a popular example of using many instruments and has been discussed in many econometric articles, see, e.g., Donald and Newey (2001), Carrasco (2012) and Hansen and Kozbur (2014), to name a few. Suppose that, instead of the effect of educational achievement on earnings that was studied by Angrist and Krueger (1991), we are interested in the effect of educational achievement on the probability of being employed. To study this, let $y_i = 1$ if individual $i$ is employed for a full year and 0 otherwise. Then, the model can be written as follows.

$$y_i = 1\{\beta_0 + \text{Educ}_i \beta_1 + W'_i \beta_2 \geq u_i\}, \quad \text{Educ}_i = \tilde{Z}_i' \Pi_1 + W'_i \Pi_2 + V_i,$$

where Educ$_i$ is the years of schooling of individual $i$. $W_i$ consists of a constant, 9 year-of-birth dummies (YoB), and 50 state-of-birth dummies (SoB). As in Angrist and Krueger (1991), let $\tilde{Z}_i$ be the 180-dimensional vector of 3 quarter-of-birth dummies and their interactions with YoB and SoB. The condition number of the covariance of $(\tilde{Z}_i', W_i')'$ is around 4941, even when standardized. This covariance would be nearly singular even with the large sample size of 329,509 individuals.

### 2.1 Regularization with many weak instruments

We in this paper use regularization to resolve the aforementioned issues related to many, and nearly weak, instruments. There are various regularization methods available in the literature. A popular approach is reducing the number of instruments by (i) choosing the optimal number of instruments that minimizes a model selection criterion (e.g., Donald and Newey, 2001) or (ii) using a variable selection procedure such as Lasso (e.g., Belloni et al., 2012) or Elastic-net (e.g., Caner and Zhang, 2014). This approach works particularly well if the signal from instruments is (i) concentrated on a few of them (called the sparsity condition) and (ii) strong enough to consistently select a set of instruments with non-zero coefficients in the first stage. However, in spite of its popularity, this is not always the best approach to use many instruments. A concern related to this approach is that the sparsity condition may not hold when the variables of interest are highly correlated with each other. In such a case, there would be a model selection mistake in which
irrelevant instruments are included; the selection mistake is not desirable since it can make the resulting estimators less efficient. Hence, it would be better to avoid using the variable selection procedure if such a selection mistake is likely to occur. In this regard, given that (i) we consider the situation with many (nearly) weak instruments and (ii) the variable selection procedures tend to select no instrument or almost randomly choose a set of instruments in such a scenario (see, e.g., Belloni et al., 2012), the variable selection procedure may not be appropriate in our setting. Moreover, as in Examples 1 and 2, if instruments are given by a set of dummies, it is not advisable to choose a few of them; this is because (i) a selected set of instruments will depend on how the variables are encoded and (ii) the categorical variables are not likely to be represented by only a few of them.

Therefore, this study adopts a relatively intuitive and conservative strategy of regularizing the sample covariance of instruments. Our regularization scheme, which will be formally introduced in Section 3, is preferred to the foregoing variable selection procedures for a couple of reasons. Firstly, we do not need the sparsity condition, and, secondly, our approach is free from the aforementioned concern on the selection mistake. In addition, from a practical perspective, valid instruments are likely to be (nearly) weak, and if so, the consistency results of the variable selection procedures are no longer valid. On the other hand, it is shown that asymptotic properties of the estimators obtained by regularizing the covariance of instruments remain valid even in such a circumstance, see, e.g., Hausman et al. (2011), Hansen and Kozbur (2014), and Carrasco and Tchuente (2015) to name a few. This regularization method has been widely employed especially in the context of linear models (see, e.g., Carrasco, 2012; Hansen and Kozbur, 2014; Carrasco and Tchuente, 2016), but its application to the binary response model has not been discussed yet.

3 Estimator and Asymptotic Properties

To illustrate our methodology, we first consider the simple case $Z_i \in \mathbb{R}^{d_z}$ for $d_e \leq d_z < \infty$; even in this case, regularization is needed if the covariance of $Z_i$ is nearly singular (e.g., Examples 1 and 2). In Section 3.2, we extend the discussion to the case where $Z_i$ is function-valued. In that case, not only is regularization needed to deal with the singularity of the covariance, but also a novel asymptotic analysis is needed to establish the asymptotic properties of our estimators. In the following, $Z_i$ is assumed to be centered without loss of generality.
3.1 Simple case: a finite-dimensional instrumental variable

3.1.1 Estimator

We let $K$ and $K_n$ denote the covariance and the sample covariance of $Z_i$, i.e., $K = \mathbb{E}[Z_i Z_i']$ and $K_n = n^{-1} \sum_{i=1}^{n} Z_i Z_i'$. In Section 3.2, we modify these definitions to handle function-valued $Z_i$. We first consider the following moment condition associated with (2.2).

$$
\Pi_n K = \Pi_n \mathbb{E} \left[ Z_i Z_i' \right] = \mathbb{E} \left[ Y_2 Z_i' \right],
$$

(3.1)

In many econometric studies, to solve (3.1), it has been assumed that the covariance $K$ is invertible. However, this assumption is not likely to hold when the dimension of $Z_i$ is large, see, e.g., Examples 1 and 2. Hence, instead of the inverse of $K$, we consider its regularized inverse. In particular, we here focus on Tikhonov regularization (in Section 3.2, we allow various regularization methods satisfying certain conditions). Specifically, given a regularization parameter $\alpha > 0$, let $K^{-1}_\alpha$ denote the regularized inverse of $K$ that is given by

$$
K^{-1}_\alpha = K (K^2 + \alpha I_d)^{-1}.
$$

The regularized solution to (3.1) (denoted $\Pi_\alpha$) and its sample counterpart (denoted $\hat{\Pi}_\alpha$) are given by

$$
\Pi_\alpha = \mathbb{E}[Y_2 Z_i'] K^{-1}_\alpha \quad \text{and} \quad \hat{\Pi}_\alpha = (n^{-1} \sum_{i=1}^{n} Y_2 Z_i') K^{-1}_n = (n^{-1} \sum_{i=1}^{n} Y_2 Z_i') K_n (K_n^2 + \alpha I_{d_z})^{-1},
$$

(3.2)

where $I_{d_z}$ is the identity matrix of dimension $d_z$. Using $\hat{\Pi}_\alpha$ in (3.2), we compute the predicted value of $Y_2$, denoted $\hat{\gamma}_{i,\alpha}$, and the first-stage residual, denoted $\hat{V}_{i,\alpha}$, i.e., $\hat{\gamma}_{i,\alpha} = \hat{\Pi}_\alpha Z_i$ and $\hat{V}_{i,\alpha} = Y_2 - \hat{\Pi}_\alpha Z_i$. These will be used as regressors in the second stage for mathematical convenience.

We further assume that $u_i | (x_i, V_i) \sim \text{iid } \mathcal{N}(-V_i' \psi_0, \sigma^2_\eta)$. Here, what enables us to apply the control function approach is not the conditional normality of $u_i$ but the fact that the endogeneity of $Y_2$ affects the second stage only through the conditional expectation of $u_i$. Thus, the normality assumption is only employed to facilitate our discussions, and it may be replaced by another distributional assumption. We next normalize the conditional variance $\sigma^2_\eta$ to 1 to achieve the identification of the second-stage parameters, see, e.g., Manski (1988). Moreover, let $g_i = (\gamma_i', V_i')' = ((\Pi_n Z_i)', (Y_2 - \Pi_n Z_i)')'$, $\bar{g}_{i,\alpha} = (\gamma_{i,\alpha}', \hat{V}_{i,\alpha}')'$ and $\theta = (\beta', \beta' + \psi')'$. In contrast with $\bar{g}_{i,\alpha}$, the vector $g_i$ is not defined with the regularized inverse. The RCMLE (denoted $\hat{\theta}_M$) and the RNLSE (denoted $\hat{\theta}_N$) are the estimators of the true coefficients $\theta_{0M}$ and $\theta_{0N}$.
which are the solutions to the following objective functions. Specifically, for \( j \in \{ M, N \} \)

\[
\hat{\theta}_j := \arg \max_{\theta \in \Theta} Q_{jn}(\theta, \hat{g}_i, \alpha) \quad \text{and} \quad \theta_{0j} := \arg \max_{\theta \in \Theta} Q_j(\theta, g_i),
\]

(3.3)

where \( Q_{jn}(\theta, \hat{g}_i, \alpha) := \frac{1}{n} \sum_{i=1}^{n} m_j(\theta, \hat{g}_i, \alpha) \) and \( Q_j(\theta, g_i) := \mathbb{E}[m_j(\theta, g_i)] \),

(3.4)

and the function \( m_j(\theta, g_i) \) is \( y_i \log \Phi(g_i^t \theta) + (1 - y_i) \log(1 - \Phi(g_i^t \theta)) \) (resp. \( \frac{1}{2} (y_i - \Phi(g_i^t \theta))^2 \)) if \( j = M \) (resp. \( j = N \)), and \( \mathbb{E}[\cdot] \) in the above is the expectation taken with respect to the distribution of \( g_i \) for sample size \( n \), but we suppress the index \( n \) unless it is necessary. Under the identification conditions in Section 3.1.2, we have \( \theta_{0M} = \theta_{0N} = \theta_0 \).

**Remark 1.** \( \mathbb{E}[y_i | x_i, V_{i,\alpha}] \) is in general different from \( \mathbb{E}[y_i | x_i, V_i] \). Specifically, under the aforementioned assumptions,

\[
\mathbb{E}[y_i | x_i, V_{i,\alpha}] = \Phi(\gamma_{i,\alpha}^t \beta_0 + V_{i,\alpha}^t (\beta_0 + \psi_0) - \psi_0 \Pi_n (I_{d_z} - K^{-1}\alpha \Lambda) Z_i),
\]

(3.5)

where \( \gamma_{i,\alpha} = \Pi_\alpha Z_i \), and \( V_{i,\alpha} = Y_{2i} - \gamma_{i,\alpha} \). (3.5) shows that, if \( \alpha > 0 \) is fixed, the conditional expectation depends not only on the regularized error \( V_{i,\alpha} \) but also on a bias term given by \( \Pi_n (I_{d_z} - K^{-1}\alpha \Lambda) Z_i \). Therefore, to eliminate an effect of this bias term on the asymptotic properties of our estimators, \( \alpha \) is assumed to shrink to zero as \( n \to \infty \) throughout the paper.

**Remark 2.** We in this paper apply the regularization approach to the conditional maximum likelihood and nonlinear least square estimation methods. These are chosen because of their ease of implementation and popularity in empirical research. It is also possible to use other estimators in Rivers and Vuong (1988), such as the instrumental variable probit and the limited information maximum likelihood, in a similar manner. However, this will not be further discussed here.

### 3.1.2 Asymptotic Properties

This section presents the asymptotic properties of our estimators. In this section, \( \langle a_1, a_2 \rangle \) (resp. \( ||a_1|| \)) for any \( a_1, a_2 \in \mathbb{R}^n \) is understood as the usual Euclidean inner product (resp. norm). Moreover, we let \( \| \cdot \|_{\text{HS}} \) denote \( \text{tr}(A^tA)^{1/2} \) for a matrix \( A \). We use a.s. and a.s.n. to denote almost surely and almost surely for \( n \) large enough.

**Assumption 1.** For \( \Pi_n \) satisfying (3.1), there exists a bounded matrix \( \Pi_0 : \mathbb{R}^{d_z} \to \mathbb{R}^{d_e} \) such that \( \Pi_n = \Lambda_n \Pi_0 / \sqrt{n} \) where \( \Lambda_n = \widetilde{\Lambda} \text{diag}(\mu_{1n}, \ldots, \mu_{d En}) \). The matrix \( \widetilde{\Lambda} : \mathbb{R}^{d_e} \to \mathbb{R}^{d_e} \) is bounded, and the smallest eigenvalue of \( \widetilde{\Lambda} \Lambda \) is bounded away from zero. For each \( j \), either \( \mu_{jn} = \sqrt{n} \) (strong) or \( \mu_{jn} / \sqrt{n} \to 0 \) (weak).
Moreover, \( \mu_{m,n} = \min_{1 \leq j \leq d} \mu_{jn} \to \infty \) as \( n \to \infty \).

We, hereafter, let \( f_i = \Pi_0 Z_i \). In Section 3.2, we allow the instrument \( Z_i \) to be function-valued. In that case, \( f_i \) can be viewed as the (infeasible) optimal instrument that summarizes all the signals from the infinite-dimensional random variable \( Z_i \). Assumption 1 is standard in the literature on many weak instruments and is related to the weakness of \( f_i \), see, e.g., Chao and Swanson (2005), Newey and Windmeijer (2009), Hansen and Kozbur (2014), and Carrasco and Tchuente (2016) to mention only a few. This condition essentially rules out weak instruments in the sense of Staiger and Stock (1997). Specifically, under the assumption, each row of \( f_i \) could be either (nearly) weak or strong, depending on the value of \( \mu_{jn} \). If \( \mu_{jn} = \sqrt{n} \) for all \( j \), then every element of \( f_i \) is strong and regularization will play a role of reducing the many instruments bias discussed in Bekker (1994). The above assumption is somewhat related to the nearly weak moment condition in Antoine and Renault (2009). We complement the article by exploring the use of possibly infinite-dimensional \( Z_i \) in the binary response model.

**Assumption 2.**

(i) \( \{y_i, Y_{2i}, x_i\}_{i=1}^n \) is iid, \( \mathbb{E}[V_i|x_i] = 0 \), and \( u_i|(f_i, V_i) \sim \text{iid } \mathcal{N}(-\psi_0'V_i, 1) \); (ii) The minimum eigenvalue of \( \mathbb{E}[f_i f_i'] \) is bounded away from zero; (iii) \( \mathbb{E}[\|V_i\|^2|x_i] < \infty \); (iv) \( \mathbb{E}[\|Z_i\|^2] < \infty \).

**Assumption 3.** The parameter space of \( \theta \), denoted \( \Theta \), is compact. \( \theta_0 \) is the unique point satisfying Assumption 2.(i) and the model (2.1), and is in the interior of \( \Theta \).

Assumption 2 formalizes the requirement for the control function approach; the endogeneity of \( Y_{2i} \) arises only through \( \mathbb{E}[u_i|f_i, V_i] \). In addition, we normalize the conditional variance of \( u_i|(f_i, V_i) \) to 1 (see Manski, 1988). A different normalization scheme, such as assuming the norm of \( \theta_0 \) to 1, could have been chosen, but, in the current study, normalizing the conditional variance is preferred in that it makes the distribution of \( u_i|(f_i, V_i) \) free from nuisance parameters. Assumption 3 implies that there exists a unique \( \theta_0 \in \Theta \) satisfying \( \mathbb{E}[y_i|Y_{2i}, V_i] = \Phi(g_0'\theta_0) \). Thus, \( \theta_{0M} = \theta_{0N} = \theta_0 \).

In the sequel, we provide a condition on \( \Pi_0 \). In the condition below, \( \text{ran } K \) is the range of \( K \).

**Assumption 4.** Let \( \pi_{\ell,0} \) be the \( \ell \)th row of \( \Pi_0 \). Then, for \( \ell = 1, \ldots, d, \pi_{\ell,0} \in \text{ran } K \).

As detailed in Section 3.2, a primitive condition for Assumption 4 is that \( (\pi'_{\ell,0} \varphi_j)^2 \) decreases at a rate slower than the decaying rate of the eigenvalues of \( K \), where \( \{\varphi_j\}_{j \geq 1} \) denotes the eigenvectors of \( K \). This assumption is needed to identify \( \Pi_0 \) without assuming the invertibility of \( K \), see Carrasco et al. (2007) and Benatia et al. (2017). Furthermore, it plays an important role in dealing with the bias term discussed in Remark 1.
The theorem below presents the consistency of $\hat{\theta}_j$ under the aforementioned assumptions.

**Theorem 1.** Suppose that $\alpha \to 0$ and $\mu_{m,n}^2\alpha^{1/2} \to \infty$ as $n \to \infty$, and Assumptions 1 to 4 are satisfied. Then, $\hat{\theta}_j - \theta_0 \xrightarrow{p} 0$ as $n \to \infty$ for $j \in \{M, N\}$.

The conditions on $\alpha$ and $\mu_{m,n}$ in Theorem 1 are related to the weakness of the optimal instrument $f_i$. If $\alpha = O(n^{-a})$ and $\mu_{m,n} = O(n^\ell)$ for some $0 < a$ and $0 < \ell \leq 1/2$, then the condition on $\mu_{m,n}^2\alpha^{1/2}$ implies that $0 < a \leq 2$. This is a necessary condition for $\alpha$ to ensure the consistency of $\hat{\theta}_j$. A similar condition can be found in Carrasco and Tchuente (2016, Proposition 2) and Hansen and Kozbur (2014, Assumption 2). In particular, in Hansen and Kozbur (2014), the inverse of the number of instruments plays a similar role of the regularization parameter in the current paper. We note that the convergence rate of $\alpha$ allows us to control for the bias associated with the nearly weak signal of $f_i$ and the use of a regularized inverse at the same time.

We next discuss the asymptotic distribution of $\hat{\theta}_j$. As in Rivers and Vuong (1988), we use the estimated control function to adequately address endogeneity underlying the model. This results in the limiting distribution of our estimators depending on that of the first-stage estimator. However, in our setting, the limiting distribution of the first-stage estimator cannot be straightforwardly obtained using the arguments in Rivers and Vuong (1988), because the covariance of $Z_i$ is allowed to be nearly singular. Although a similar scenario has been studied in the linear model (Carrasco, 2012; Hansen and Kozbur, 2014; Carrasco and Tchuente, 2015, 2016), the asymptotic distributions in these articles do not directly depend on the limiting distribution of their first-stage estimator. Thus, their asymptotic approaches cannot be applied in our context. Moreover, the use of a regularized inverse in the first-stage estimation also requires us to address the regularization bias in Remark 1. To deal with these issues and to encompass the general case in Section 3.2, we employ an asymptotic approach in Chen et al. (2003), which is substantially differentiated from that in Rivers and Vuong (1988). To this end, the following assumption is employed.

**Assumption 5.** (i) $\mathbb{E}[\|Z_i\|^4] < \infty$; (ii) $\mathbb{E}[\|V_i\|^4|x_i] < \infty$ and $\mathbb{E}[V_iV_i'|x_i]$ is positive definite; (iii) For all $\theta \in \Theta$, $\sup_i \Phi(g_i'|\theta)$ is bounded away from 0 and 1 a.s.n.; (iv) There exists $c > 0$ such that $\sup_i \|f_i\| \leq c$ a.s.n. and $\sup_i \mathbb{E}[\|V_i\|^2|x_i] \leq c$ a.s.

Assumptions 5.(i) and 5.(ii) are moment conditions for $Z_i$ and $V_i$. Assumptions 5.(iii) and 5.(iv) are needed to bound a particular quantity in our proof, but these can be relaxed by using a trimming term in the objective function as in Rothe (2009). Thus, Assumption 5 is not restrictive in practice.

We now discuss the asymptotic normality result. In the theorem below, $S_n^{-1}$ is a matrix given by
for the condition on in Antoine and Renault (2009) and Han and Renault (2020); these articles show that GMM estimators are mainly required to eliminate a possible effect of asymptotic bias of the first stage estimator on the limiting testing. Theorem 2 enables us to conduct inference on Remark 3. linear transformation of the asymptotic covariance of Rivers and Vuong’s (1988) 2SCMLE. If there exist non-zero 2 chi-square random variable with 2 degrees of freedom. Remark 4. If there exist non-zero 2_n and s_*, \in \mathbb{R}^{2d_e}\{0\} such that r_n^{-1/2}S_n^{-1}s \to s_* for some s_0 \in \mathbb{R}^{2d_e}\{0\}, then \(s_0'W_0s) \to N(0,1). This property allows us to conduct inference on s_0, see Newey and Windmeijer (2009, Theorem 3). For example, we can study the endogeneity of Y_2; by testing \(H_0 : \psi = 0\). Given that \(\mu_{m,n}A_n^{-1} \to A^{-1}\) and \(A^{-1}\ell_{d_e} \neq 0\) for the \(d_e\)-dimensional vector of ones \(\ell_{d_e}\), the endogeneity of \(Y_2\) can be tested by setting \(s = (-s_0', s_0')\), see Section 5.

3.2 General case: a functional instrumental variable

We now extend the discussion in Section 3.1 to the general case.
3.2.1 Details on the instrumental variable $Z(x_i, t)$

Before discussing our estimators, we present the details on $Z_i$ in the general case. Our definition of $Z_i$ is similar to that in Carrasco (2012) and Carrasco and Tchuente (2015, 2016). Specifically, let $t$ be an element of a set $\mathcal{C}$ and let $\tau$ denote a positive measure on $\mathcal{C}$ whose support is identical to $\mathcal{C}$. Moreover, let $\mathcal{H}$ be the space of square integrable functions defined on $\mathcal{C}$ equipped with the inner product $\langle h_1, h_2 \rangle = \int_{\mathcal{C}} h_1(t)h_2(t)\tau(t)dt$ and its induced norm $\|h_1\| = \langle h_1, h_1 \rangle^{1/2}$ for $h_1, h_2 \in \mathcal{H}$. Then, $\mathcal{H}$ is a separable Hilbert space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space and let the instrumental variable $Z_i$ be a measurable (with respect to the usual Borel $\sigma$-field) map from $\Omega$ to $\mathcal{H}$. Under the definition, $Z_i$ can be either a vector- or a function-valued instrumental variable.

The following example and Section C in the Online Supplement illustrate how function-valued instrument $Z_i$ can be used in binary response models.

**Example 3.** Miguel et al. (2004) studied the effect of economic growth on the occurrence of civil war in sub-Saharan Africa. The authors suggested addressing the potential endogeneity of economic growth using annual rainfall growth from the viewpoint that (i) rainfall variations are exogenous and (ii) a large volume of the economy in sub-Saharan Africa relies on agriculture. But, agricultural production may be better predicted if the instrument is not simply given by the annual *average* but given by a *function* of rainfall variations over an entire year, because of the importance of daily rainfall variations in agricultural productivity. To concretely illustrate this idea, let $x_{is}(t) = 1 - \ddot{x}_{is}(t)/\ddot{x}_{is-1}(t)$ where $\ddot{x}_{is}(t)$ is the daily rainfall in country $i$ on the $t$th day of year $s$. For each $i$ and $s$, we can obtain the rainfall growth curve, denoted $Z_{is}$, by smoothing $\{x_{is}(t)\}_{t=1}^{365}$ (see Horváth and Kokoszka, 2012 for a similar example). Then, one may consider the following model.

$$\text{conflict}_{is} = 1\{\beta_0 + \text{growth}_{is}\beta + W_{is}'\gamma \geq u_{is}\}, \quad \text{growth}_{is} = \Pi_n Z_{is} + W_{is}'\pi_2 + V_{is}.$$ 

The variable $\text{conflict}_{is}$ is 1 if country $i$ experiences a civil conflict in year $s$ and 0 otherwise. $\text{growth}_{is}$ us economic growth in country $i$ in year $s$. $W_{is}$ is a set of exogenous variables. In this case, the first-stage parameter $\Pi_n$ will be represented by a linear functional from $\mathcal{H}$ to $\mathbb{R}$.

The covariance of $Z_i$, denoted $\mathcal{K}$, and its sample counterpart, denoted $\mathcal{K}_n$, are given by

$$\mathcal{K} = \mathbb{E}[Z_i \otimes Z_i] \quad \text{and} \quad \mathcal{K}_n = n^{-1} \sum_{i=1}^{n} Z_i \otimes Z_i,$$

where $\otimes$ signifies the tensor product in $\mathcal{H}$ satisfying that $h_1 \otimes h_2(\cdot) = \langle h_1, \cdot \rangle h_2$ for any $h_1, h_2 \in \mathcal{H}$. The
Tensor product is a natural generalization of the outer product in Euclidean spaces, and thus, $K$ (resp. $K_n$) can also be understood as a generalization of the covariance (resp. sample covariance) matrix. Lastly, we note that $K$ (resp. $K_n$) allows the spectral decomposition with respect to its eigenvalues and eigenfunctions, denoted $\{\kappa_j, \varphi_j\}_{j \geq 1}$ (resp. $\{\hat{\kappa}_j, \hat{\varphi}_j\}_{j \geq 1}$). Assume that the eigenvalues are in descending order. Given the consistency of $K_n$, $\hat{\kappa}_j$ and $\hat{\varphi}_j$ are consistent estimators of $\kappa_j$ and $\varphi_j$, see Bosq (2000, Lemmas 4.2, 4.3).

### 3.2.2 Estimator

As in Section 3.1, we consider the following moment condition that is associated with (2.2).

$$\Pi_n \mathbb{E} [Z_i \otimes Z_i] = \mathbb{E} [Z_i \otimes Y_{2i}].$$

(3.6) is a generalization of (3.1) to the case where $Z_i$ is possibly infinite dimensional. As discussed by Carrasco et al. (2007), the unique solution to (3.6) exists only when $K = \mathbb{E} [Z_i \otimes Z_i]$ is bijective and its inverse is continuous. These conditions do not hold if $Z_i$ is a random variable taking values in the Hilbert space of square integrable functions. Thus, as in Section 3.1, we use a regularized inverse of $K$, denoted $K_{\alpha}^{-1}$, to obtain a solution to (3.6). The regularized inverse considered here satisfies

$$\lim_{\alpha \to 0} K_{\alpha}^{-1} = h \text{ for any } h \in \mathcal{H},$$

and has the following representation:

$$K_{\alpha}^{-1} = \sum_{j=1}^{\infty} \kappa_j^{-1} q(\kappa_j, \alpha) \varphi_j \otimes \varphi_j.$$

(3.7)

The conditions on $q(\kappa_j, \alpha)$ are detailed in Assumption 8. This representation includes various types of regularization schemes, such as (i) Tikhonov ($q(\kappa_j, \alpha) = \frac{\kappa_j^2}{(\kappa_j^2 + \alpha)}$), (ii) ridge ($q(\kappa_j, \alpha) = \frac{\kappa_j}{(\kappa_j + \alpha)}$), and (iii) spectral cut-off ($q(\kappa_j, \alpha) = 1 \{\kappa_j^2 \geq \alpha\}$). The sample counterpart of $K_{\alpha}^{-1}$, denoted $K_{n\alpha}^{-1}$, is similarly defined by replacing $\{\kappa_j, \varphi_j\}_{j \geq 1}$ in (3.7) with $\{\hat{\kappa}_j, \hat{\varphi}_j\}_{j \geq 1}$, i.e., $K_{n\alpha}^{-1} = \sum_{j=1}^{\infty} \hat{\kappa}_j^{-1} q(\hat{\kappa}_j, \alpha) \hat{\varphi}_j \otimes \hat{\varphi}_j$. Then, for $\alpha > 0$, the solution to (3.6) (denoted $\Pi_{\alpha}$) and its sample counterpart (denoted $\hat{\Pi}_{\alpha}$) are given as follows.

$$\Pi_{\alpha} = \mathbb{E}[Z_i \otimes Y_{2i}] K_{\alpha}^{-1} = \Pi_n K K_{\alpha}^{-1} \quad \text{and} \quad \hat{\Pi}_{\alpha} = (n^{-1} \sum_{i=1}^{n} Z_i \otimes Y_{2i}) K_{n\alpha}^{-1}.$$

(3.8)

Although the operators $\Pi_{\alpha}$ and $\hat{\Pi}_{\alpha}$ in (3.8) act on a Hilbert space of infinite dimension, the predicted value of $Y_{2i}$ and the first-stage residual, which are computed from (3.8), are still $d_e$-dimensional. Therefore, we can obtain our estimators by solving (3.3) as described in Section 3.1.

### 3.2.3 Asymptotic Properties

In the following, $A^*$ denotes the adjoint of a linear operator $A : \mathcal{H} \to \mathbb{R}^{d_e}$.
Assumption 6. (i) Assumption 1 holds for a bounded linear operator $\Pi_0 : \mathcal{H} \rightarrow \mathbb{R}^{d_x}$. (ii) For $\{\pi_{\ell,0} = \Pi_0^\ell e_{\ell}\}_{\ell=1}^{d_x}$, there is $\rho \geq 1$ such that \(\sum_{j=1}^{\infty} (\varphi_j, \pi_{\ell,0})^2 / \kappa_j^{2\rho} < \infty\).

Assumption 7. (i) Assumption 2 holds; (ii) Assumption 3 holds.

Assumption 8. $K_{-1}^{-1}$ allows the representation in (3.7). For $\kappa \geq 0$ and $\alpha > 0$, the function $q(\kappa, \alpha)$ in (3.7) satisfies $q(\kappa, \alpha) \in [0,1]$, $\alpha \kappa^{-1} q(\kappa, \alpha) \leq c$ and $\sup_{\kappa} \kappa^{2\tilde{\rho}} (q(\kappa, \alpha) - 1) \leq c \alpha^{\min\{\tilde{\rho},1\}}$ for some $\tilde{\rho} > 0$ and $c > 0$. Moreover, $\lim_{\alpha \rightarrow 0} q(\kappa, \alpha) = 1$ if $\kappa > 0$.

Assumptions 6.(i) and 7 are equivalent to Assumptions 1, 2 and 3 except that the first-stage parameter and the norm of $Z_i$ are adapted to the case where $Z_i$ is a $\mathcal{H}$-valued random variable. Assumption 6.(ii), which is similar to Assumption 4, is needed to identify $\Pi_0$ without the injectivity of $K$, see Carrasco et al. (2007) and Benatia et al. (2017). As briefly mentioned in Section 3.1, this assumption imposes a restriction on the relative decaying rate of $K$’s eigenvalues with respect to its Fourier coefficients $\{(\langle \pi_{\ell,0}, \varphi_j \rangle)_{j \geq 1}\}$, which is crucial for obtaining the main results of this study. Specifically, it enables us to prove that $n^{-1} \sum_{i=1}^{n} \|\hat{g}_{i,\alpha} - g_i\|^2 = o_p(1)$, which plays an important role in the derivation of our main findings.

Assumption 8 describes the conditions on the regularized inverse $K_{-1}^{-1}$. Many common regularization methods, such as Tikhonov or spectral cut-off, satisfy the conditions (see, e.g., Carrasco et al. 2007, Section 3.3). Hence, from a practical perspective, Assumption 8 is not restrictive.

The following theorem shows the consistency of $\hat{\theta}_j$ even when $Z_i$ is a functional variable.

Theorem 3. Suppose that $\alpha \rightarrow 0$ and $\mu_{m,n,\alpha}^2 \rightarrow \infty$ as $n \rightarrow \infty$, and Assumptions 6 to 8 are satisfied. Then, $\hat{\theta}_j - \theta_0 \xrightarrow{P} 0$ as $n \rightarrow \infty$ for $j \in \{M, N\}$.

The conditions on $\mu_{m,n}$ and $\alpha$ in Theorem 3 are different from those in Theorem 1, which is mainly due to Assumption 8. In fact, if we choose Tikhonov, spectral cut-off, or Landweber Fridman regularization methods, then $q(\kappa, \alpha) \leq c \kappa / \alpha^{-1/2}$ for a constant $c > 0$ and the condition on $\mu_{m,n,\alpha}^2$ in Theorem 3 can be replaced by $\mu_{m,n,\alpha}^{1/2} \rightarrow \infty$ which is equivalent to that in Theorem 1.

We next discuss the asymptotic distributions of the proposed estimators. As discussed in Section 3.1, their asymptotic distributions cannot be obtained straightforwardly from the previous works of Rivers and Vuong (1988), Carrasco (2012), and Carrasco and Tchuente (2016, 2015); this is not only because of some distinctive properties of our model which are described in the previous section, but also because of the fact that $\hat{\gamma}_{i,\alpha}$ and $\hat{V}_{i,\alpha}$, the arguments of the second-stage objective function, now depend on an infinite-dimensional parameter estimate, $\hat{\Pi}_\alpha$. This calls for an approach to asymptotic analysis that is different from
those in the foregoing articles. In fact, our case is similar to that considered by Chen et al. (2003) in which the criterion function contains an infinite-dimensional parameter estimate. Chen et al. (2003) discuss primitive conditions for the asymptotic normality result of an estimator that is obtained in such a circumstance, and we here make use of their methodology to establish the asymptotic normality results of our estimators.

To this end, we need a few additional conditions, one of which is the following condition on the integrability of the covering number of $\mathcal{H}$ with respect to $\|\cdot\|$ (denoted $\mathcal{N}(\epsilon, \mathcal{H}, \|\cdot\|)$), i.e.,

$$
\mathcal{N}(\epsilon, \mathcal{H}, \|\cdot\|) = \inf\{n \in \mathbb{N} : \mathcal{H} \subset \bigcup_{k=1}^{n} B_k(\epsilon)\}
$$

where $B_k(\epsilon) = \{h \in \mathcal{H} : \|h - h_k\| < \epsilon, h_k \in \mathcal{H}\}$.

**Assumption 9.** $\mathcal{H}$ satisfies $\int_{0}^{\infty} \sqrt{\log \mathcal{N}(\epsilon, \mathcal{H}, \|\cdot\|)} d\epsilon < \infty$.

In Chen et al. (2003, Condition 3.3), the covering number of the space of an infinite-dimensional parameter is required to satisfy the integrability condition similar to that in Assumption 9. In our setting, such a parameter space is given by the space of bounded linear operators from $\mathcal{H}$ to $\mathbb{R}^{d_e}$. Thus, it is not trivial to verify the condition in Chen et al. (2003). In contrast, the requirement of Assumption 9 is only for the domain of our infinite-dimensional parameter, which is easy to verify. This assumption is likely to hold in many practical settings; for example, the condition holds if $Z_i$ is finite dimensional as in Section 3.1 or if $\mathcal{H}$ is a Sobolev or Hölder space, see, e.g., van der Vaart and Wellner (1996). In the current paper, the restriction on the domain of $\Pi_0$ is sufficient, because it is a finite rank operator (see Lemma 3 in the Online Supplement).

**Assumption 10.** (i) Assumption 5 holds; (ii) For $\kappa \geq 0$ and $\alpha > 0$, $q(\kappa, \alpha)$ in (3.7) satisfies $\alpha^{1/2} \kappa^{-1} q(\kappa, \alpha) \leq c$ for some constant $c > 0$; (iii) Assumption 6.(ii) holds with $\rho \geq 3/2$.

Assumption 10 is needed to deal with the bias term mentioned in Remark 1. Assumption 10.(ii) implies that not all regularization methods are applicable in our setting. For example, ridge regularization may fail to satisfy the condition since its conservative upper bound of $\kappa^{-1} q(\kappa, \alpha)$ is $O(\alpha^{-1})$. In contrast, the other aforementioned methods, such as Tikhonov and spectral cut-off, satisfy the condition. Hence, in spite of its popularity, ridge regularization is less preferred in this paper.

Assumption 10.(iii) can be relaxed if the range of $\mathcal{V}_j^*$, denoted $\text{ran } \mathcal{V}_j^*$, is a subset of the range of $\mathcal{K}$, denoted $\text{ran } \mathcal{K}$, where $\mathcal{V}_j^*$ is the adjoint of $\mathcal{V}_j$ that is defined by $\mathbb{E}[\tilde{n}_{2j}(\theta_0, g_i) Z_i \otimes g_0]$. In Remark 6 in the Online Supplement, we show that the closure of $\text{ran } \mathcal{V}_j^*$, denoted $\text{cl}(\text{ran } \mathcal{V}_j^*)$, is always a subset of the closure of $\text{ran } \mathcal{K}$, denoted $\text{cl}(\text{ran } \mathcal{K})$. If $Z_i$ is finite dimensional or can be represented by a finite number of basis functions, then $\text{cl}(\text{ran } \mathcal{V}_j^*) = \text{ran } \mathcal{V}_j^*$ (resp. $\text{cl}(\text{ran } \mathcal{K}) = \text{ran } \mathcal{K}$), and in that case Assumption 10.(iii) will be satisfied. Hence, we expect, as Remark 6 in the Online Supplement suggests, Assumption 10.(iii) is
not restrictive in practice.

We now present the asymptotic normality result. In contrast with the case in Theorem 2, \( C_j \) here is a linear operator from \( \mathcal{H} \) to \( \mathbb{R}^{d_e} \) satisfying \( K^{1/2} C_j^* = \mathcal{V}_s^* \). The operator \( C_j \) is unique and bounded even when \( Z_i \) is a functional variable taking values in \( \mathcal{H} \) (see Baker 1973, Theorem 1).

**Theorem 4.** Suppose that \( \alpha \to 0, \mu_{m,n}^2 \to \infty \) and \( \sqrt{n} \alpha \to 0 \) as \( n \to \infty \), and Assumptions 6 to 10 are satisfied. Then, \( \mathcal{W}_{ij}^{-1/2} \mathcal{S}_n^{ij} \left( \hat{\theta}_j - \theta_0 \right) \xrightarrow{d} \mathcal{N} \left( 0, \mathcal{I}_{2d_e} \right) \) for \( j \in \{ M, N \} \), where \( \mathcal{W}_{ij} = (\mathcal{S}_n^{-1} \Gamma_{1,0j} \mathcal{S}_n^{-1})^{-1} \mathcal{J}_j (\mathcal{S}_n^{-1} \Gamma_{1,0j} \mathcal{S}_n^{-1})^{-1} \).

The asymptotic covariance \( \mathcal{W}_{ij} \) is nonsingular irrespective of the dimension of \( Z_i \). Thus, we can conduct inference on \( \theta_0 \) and \( \varsigma' \theta_0 \) as discussed in Remarks 3 and 4.

### 3.3 Estimation of Asymptotic Variances and Average Structural Functions

A consistent estimator of \( \mathcal{W}_{ij} \) in Theorems 2 and 4 is essential to conduct inference on \( \theta_0 \). As noted by Chen et al. (2003), a bootstrap method may be used for this purpose, but it may be computationally demanding, especially when the dimension of \( Z_i \) is very large. In this section, we provide a simple and practical method for computing a consistent estimator of \( \mathcal{W}_{ij} \).

In the following, \( \hat{\Gamma}_{1,j} = n^{-1} \sum_{i=1}^n \hat{m}_{2j} (\hat{\theta}_j, \hat{g}_{i,\alpha}) \hat{g}_{i,\alpha} \) and \( \hat{J}_{1,j} = n^{-1} \sum_{i=1}^n \hat{m}_{1j} (\hat{\theta}_j, \hat{g}_{i,\alpha}) \hat{g}_{i,\alpha} \) where \( m_{2m} (\theta, g_i) = \hat{m}_{2M} (\theta, g_i) \) and \( m_{1M} (\theta, g_i) = \left( y_i - \Phi (g_i') \right)^2 \phi^2 (g_i') \). Lemmas 6 and 7 in Appendix D.2 in the Online Supplement show that \( \| \mathcal{S}_n^{-1} \hat{\Gamma}_{1,j} - \Gamma_{1,0j} \|_{\text{HS}} \) and \( \| \mathcal{S}_n^{-1} \hat{J}_{1,j} \mathcal{S}_n^{-1} - \mathcal{J}_{1,j} \|_{\text{HS}} \) are \( o_p(1) \). Moreover, let \( \hat{J}_{2,j} = \hat{\sigma}_{j}^2 \hat{V}_{jn} \mathcal{K}_n^{-1} \mathcal{K}_n \mathcal{K}_n^{-1} \mathcal{V}_{jn}^* \) where \( \hat{\sigma}_{j}^2 = n^{-1} \sum_{i=1}^n (1) \hat{\theta}_j \hat{g}_{i,\alpha} \) and \( \hat{V}_{jn} \). Lemma 8 shows that \( \mathcal{S}_n^{-1} \hat{\mathcal{J}}_{2,j} \mathcal{S}_n^{-1} \) is a consistent estimator of \( \mathcal{J}_{2,j} \) under the conditions therein. Thus, \( \mathcal{S}_n^{-1} \hat{\mathcal{J}}_{2,j} \mathcal{S}_n^{-1} \) converges to \( \mathcal{J}_{2,j} \) in probability where \( \hat{\mathcal{J}}_{j} = \hat{\mathcal{J}}_{1,j} + \hat{\mathcal{J}}_{2,j} \). The theorem below summarizes Lemmas 6–8 in the Online Supplement.

**Theorem 5.** Let \( \hat{\mathcal{W}}_{j} = \hat{\Gamma}_{1,j} \hat{J}_{1,j} \) and suppose that the conditions in Theorem 4 hold. Then, for each \( j \), \( \| \mathcal{S}_n \hat{\mathcal{W}}_{j} \mathcal{S}_n - \mathcal{W}_{0j} \|_{\text{HS}} = o_p(1) \).

**Remark 5.** We note that interest often lies in how the outcome probability changes according to a change in \( Y_2 \). In practice, this is often measured by two estimates, the average structural function (ASF) and the average partial effect (APE), each of which is given by

\[
\text{ASF}(y_2) := \frac{1}{n} \sum_{i=1}^n \Phi (y_i' \hat{\beta} + \hat{V}_{i,\alpha}' \hat{\psi}) \quad \text{and} \quad \text{APE}(y_2) := \frac{1}{n} \sum_{i=1}^n \phi (y_i' \hat{\beta} + \hat{V}_{i,\alpha}' \hat{\psi}) \hat{\beta},
\]

see Wooldridge (2010), Blundell and Powell (2004) and Rothe (2009). The ASF (resp. APE) is a consistent estimator of \( \mathbb{E}_V [\mathbb{P}(y = 1 | Y_2 = y_2, V)] \) (resp. \( \mathbb{E}_V [(\partial \mathbb{P}(y = 1 | Y_2, V)/\partial Y_2)_{Y_2 = y_2}] \)).
3.4 Mean Squared Error and the Choice of $\alpha$

A practical challenge in implementing our estimation procedure is selecting the regularization parameter. A theoretically and empirically grounded approach would be to choose $\alpha$ in such a way as to minimize the conditional mean squared error (MSE) defined by $E[\|\hat{\theta}_j - \theta_0\|^2|x]$, as in Donald and Newey (2001) and Carrasco (2012). However, our analysis of the MSE is more complicated than theirs because of (i) the second-stage nonlinearity and (ii) the use of the control function estimates whose asymptotic properties rely on a possibly infinite-dimensional random element. In particular, we cannot deal with the linearization error without further conditions. Thus, to reduce the complexity, we here focus on the conditional MSE of $\bar{\theta}$ defined by the solution to the following:

$$M(\theta_0, \mathcal{G}(\pi_n)) + \Gamma_{1n}(\theta_0, \mathcal{G}(\pi_n))(\bar{\theta} - \theta_0) + \Gamma_{2n}(\theta_0, \mathcal{G}(\pi_n))(\hat{\pi}_n - \pi_n) = 0,$$

where the subscription $j$ for denoting the RCMLE and the RNLSE is omitted for the moment to reduce the notational clutter. Under the employed conditions, the asymptotic distribution of $\theta$ is equivalent with that of $\hat{\theta}_j$ (see Chen et al. 2003, p.1605) and thus the discussion on $\theta$ would be enough here for the purpose of discussing a way to choose the regularization parameter.

The proposition below summarizes the upper bound of $\theta$’s conditional MSE; we can provide further details if $\mu_{jn}$’s are known. While the following result may be considered conservative, it can still be useful in practice due to two reasons: (i) the unknown nature of $\{\mu_{jn}\}_{j=1}^{2d_e}$ and (ii) the fact that the rate below represents the outcome when at least one strong instrument is present.

**Proposition 1.** Suppose that the conditions in Theorem 4 hold. Furthermore, assume that (i) $E[\tilde{m}_{20,i}^k|x]$,

$E[\tilde{m}_{20,i}^k V_i|x]$ and $E[\tilde{m}_{20,i}^k V_i V'_i|x]$ are all constant across $i$ and bounded for $k = 1, 2$ a.s., where $\tilde{m}_{20,i} = E[\tilde{m}_{20}(\theta_0, g_i)|x, V]$. Then, $E[\|S_n'(\bar{\theta} - \theta_0)\|^2|x] = O_P(n^{-1}\alpha^{-1/2} + \alpha^2)$.

The control function approach makes the MSE of $\bar{\theta}$ be a certain transformation of $\hat{\Pi}_n$’s MSE and makes it be minimized where the conditional MSE of $\hat{\Pi}_n$ is minimized. Due to that, the above convergence rate is similar to that in the functional linear model studied by Benatia et al. (2017, Proposition 2) who suggested choosing $\alpha$ in a way to minimize the conditional MSE. However, their approach may not be preferred in our setup, because the optimal $\alpha$ minimizing the MSE does not ensure the asymptotic normality of our estimators. As mentioned before, our asymptotic normality results require a fast convergence of $\hat{\Pi}_n$’s bias even at the cost of a large variance; if we choose $\alpha$ as in Benatia et al. (2017), the limiting distribution of $\hat{\theta}_j$ will depend on the regularization bias of the first-stage estimator, which is not desired in our setup. Hence,
a slower convergence of $\bar{\theta}$ (and that of $\hat{\theta}_j$), resulting from not choosing the optimal rate of $\alpha$, should be understood as a cost to implement inference with a possibly infinite-dimensional instrumental variable.

The optimal $\alpha$ minimizing the conditional MSE in Proposition 1 shrinks at the rate of $n^{-2/5}$. Although this rate is not recommended, the above criterion still gives us a practical guideline to select the regularization parameter. For example, to ensure the asymptotic normality of our estimators, one may set the parameter $\alpha$ such that

$$c_g \left( \frac{h^T \hat{P}(\mathcal{P}_n \alpha - I)^2 \hat{f}_h}{n} + \sigma_{v_h}^2 \text{tr}(\hat{P}_n^2) \right),$$

for a deterministic $h \in \mathbb{R}^{d_e}$. The scalars $\sigma_{v_h}^2$ and $c_g$ respectively given by $\mathbb{E}[(V_i^T h)^2|x]$ and the largest eigenvalue of $n^{-1} \sum_i \hat{m}_{20,i}^2 g_{0i} d_{0i}$. The operator $\mathcal{P}_n \alpha$ is $\sum_{j=1}^n q(\hat{\kappa}_j, \alpha) \kappa_j^{-1} \mathcal{T}_n \varphi_j \otimes (\mathcal{T}_n \varphi_j)$ and $\hat{f}$ is $n \times d_e$ matrix given by $(\hat{\Pi}_n Z_1, \ldots, \hat{\Pi}_n Z_n)'$. The term in the parenthesis is similar to the conditional MSE in Carrasco (2012) and its estimators are available there. We use (3.9) to choose $\alpha$ in the sequel.

4 Simulation Study

We now study the finite sample performance of our estimators via Monte Carlo simulations. In Sections 4.1 and 4.2, the standard case with finite-dimensional $Z_i$ will be considered. The general case with function-valued $Z_i$ will be separately discussed in Section B in the Online Supplement.

4.1 Experiment 1: Gaussian Instrument

We consider the following data generating process (DGP).

$$y_i = 1\{y_{2i}/\beta_1 + z_{1i}/\beta_2 \geq u_i\},$$

$$y_{2i} = Z_i' \pi + v_i,$$

where $\beta_1 = 1, \beta_2 = -1, \rho = 0.6$, and $\sigma^2_\pi = 1/(1-\rho^2)$. Thus, $\eta_i = u_i + \rho \sigma_\pi^{-1} \sigma_1 v_i \sim \text{iid } \mathcal{N}(0, 1)$. The variable $z_{1i}$ is the first element of $Z_i \in \mathbb{R}^K$, and $Z_{1i} \sim \text{iid } \mathcal{N}(0, \Sigma_Z)$ where $\Sigma_Z = [\sigma^2_{Z_1} \rho_{Z_1Z_2}]$, $\sigma^2_{Z_2} = 0.5$ and $\rho_z = 0.7$. We let $\sigma^2_\pi = 1 - \rho' \Sigma_Z \pi$ so that the unconditional variance of $y_{2i}$ becomes 1.

We set $\pi = e_{\ell_1} \pi = e_{\ell_1} (\lfloor \ell_1+sK \rfloor, 0_{K-\lfloor sK \rfloor})'$ where $\lfloor \cdot \rfloor$ is the floor function and $\ell_1$ is the $\ell_1$-dimensional vector of ones. The parameter $s$ determines how sparse the first-stage coefficient $\pi$ is, and we consider sparse ($s=0.2$) and dense ($s=0.8$) cases. We here consider the case $K = 50$. The values of $c_*$ are chosen to have specific values for the concentration parameter, $\mu^2 = n \pi' \Sigma_Z \pi/(1 - \pi' \Sigma_Z \pi)$, as in Belloni et al. (2012). We consider two cases: $\mu^2 = 30$ and $\mu^2 = 60$. Under the design, the infeasible first-stage F test statistic,
given by $\mu^2/[sK]$, ranges from 0.375 to 6. The concentration parameter may not be the perfect measure of instrument strength for binary response models. However, given that this measure is suggestive to the weakness of instruments in linear models, this setup might be enough to examine whether our methodology improves estimation results in a scenario where the conventional TSLS estimator is not likely to perform well. Moreover, later in this section, we will study the performance of our estimators with different values of $\mu^2$, $s$ and $K$. There we will show that Rivers and Vuong’s (1988) estimator tends to be more biased as the concentration parameter decreases.

We consider two RCMLEs computed with different regularization methods: the TRCMLE (Tikhonov) and SCRCMLE (spectral cut-off).¹ Their regularization parameters are chosen to the value that minimizes the criterion in (3.9) with the parameter space of $\alpha$ being 25 equally spaced points between $c_an^{-0.6}$ and $c_an^{-0.6}$.² The constant $c_a$ is set to $\|\Sigma_Z\|_{\max}\{0.1, 1/\delta\}$ for the TRCMLE and $\|\Sigma_Z\|^2_{\max}\{0.1, 1/\delta\}$ for the SCRCMLE, where $\delta$ is the first-stage F test. This is designed to have a larger regularization parameter when a smaller concentration parameter is given. We compare the performance of our estimators with four alternatives: the 2SCMLE proposed by Rivers and Vuong (1988), Inf.2SCMLE that will be detailed shortly, naive probit estimator, and Tikhonov-regularized TSLS estimator (TTSLS). Similar to the TSLS estimator, the 2SCMLE is expected to be biased as the number of instruments increases. To separate this bias from the bias arising from a weak signal of instruments, we compute the 2SCMLE using only those instruments with non-zero first stage coefficients. This modified estimator is referred to the Inf.2SCMLE. If there is no distortion from the weak signal, the infeasible 2SCMLE is likely to perform well when $\mu^2 = 30$ and $s = 0.3$. On the other hand, if this estimator is not affected by the many instruments bias, it is expected to perform well when $\mu^2 = 60$ and $s = 0.8$. The TTSLS is similar to the estimators proposed by Carrasco (2012) and Carrasco and Tchuente (2015, 2016) other than the fact that it uses the first-stage residual as an additional regressor in the second stage.³ This estimator is considered because of the popularity of the linear probability model in empirical studies. As the true parameter of the TTSLS is different from that of all the other estimators, we evaluate its performance at $E[\phi(g_i^r)]\beta_1$ so that a reasonable comparison can be made (see Chapter 14 of Cameron and Trivedi, 2005).

For each estimator, we compute the median bias (Med.Bias) and median absolute deviation (MAD). We

¹The results from the RNLSE are similar to those from the RCMLE and thus are omitted.
²For computational convenience, we use the RNLSE and compute Mallows $C_p$ in Carrasco (2012) to estimate (3.9). The deterministic vector $h$ is set to $\iota_d$.
³The variance is computed with the heteroscedasticity-robust covariance estimator $HC_1$ in MacKinnon and White (1985) and we use the regularization parameter chosen for the TRCMLE.
Table 1: Simulation results (Gaussian instruments)

| n  | s  | \( \mu^2 = 30 \)       | \( \mu^2 = 60 \)       |
|----|----|-------------------------|-------------------------|
|    |    | Med.Bias | MAD | RP | Med.Bias | MAD | RP |
| 200| 0.2| TCRMLE  | 0.006 | 0.287 | 0.045 | 0.042 | 0.221 | 0.058 |
|    |    | SCRCMLE | -0.047 | 0.243 | 0.040 | -0.039 | 0.189 | 0.050 |
|    |    | Inf.2SCMLE | -0.221 | 0.196 | 0.134 | -0.128 | 0.166 | 0.096 |
|    |    | 2SCMLE | -0.559 | 0.121 | 0.840 | -0.461 | 0.113 | 0.729 |
|    |    | Probit | -0.729 | 0.068 | 0.788 | -0.712 | 0.068 | 0.839 |
|    |    | TTSLS | 0.078 | 0.273 | 0.110 | 0.073 | 0.204 | 0.136 |
| 400| 0.2| TCRMLE | -0.035 | 0.235 | 0.045 | 0.011 | 0.187 | 0.042 |
|    |    | SCRCMLE | -0.059 | 0.209 | 0.040 | -0.053 | 0.168 | 0.049 |
|    |    | Inf.2SCMLE | -0.209 | 0.182 | 0.130 | -0.128 | 0.152 | 0.095 |
|    |    | 2SCMLE | -0.526 | 0.117 | 0.845 | -0.419 | 0.108 | 0.721 |
|    |    | Probit | -0.738 | 0.046 | 0.968 | -0.730 | 0.046 | 0.975 |
|    |    | TTSLS | 0.029 | 0.233 | 0.072 | 0.048 | 0.177 | 0.091 |
| 0.8| 0.2| TCRMLE | -0.024 | 0.221 | 0.046 | 0.010 | 0.164 | 0.046 |
|    |    | SCRCMLE | -0.020 | 0.203 | 0.034 | -0.017 | 0.151 | 0.050 |
|    |    | Inf.2SCMLE | -0.450 | 0.119 | 0.662 | -0.325 | 0.108 | 0.496 |
|    |    | 2SCMLE | -0.489 | 0.113 | 0.794 | -0.371 | 0.104 | 0.640 |
|    |    | Probit | -0.729 | 0.048 | 0.977 | -0.711 | 0.049 | 0.989 |
|    |    | TTSLS | 0.025 | 0.216 | 0.078 | 0.036 | 0.157 | 0.094 |

Notes: The simulation results based on 2,000 replications are reported. Each cell reports the median bias (Med.Bias), median absolute deviation (MAD), and rejection probability at 5% significance level (RP).

also test \( H_0 : \beta_1 = 1 \) at 5% significance level as explained in Remark 4. The rejection rate is reported in columns labeled “RP” in Table 1. Figure 1 reports the estimated ASFs.

In Table 1, the TCRMLE and SCRCMLE outperform the 2SCMLE and Inf.2SCMLE. Specifically, the 2SCMLE is severely biased and does not have reasonable rejection rate. On the other hand, our estimators have smaller median bias and correct size for most cases in Table 1. It seems to be interesting that the SCRCMLE outperforms the TCRMLE in the dense design, though the difference diminishes with larger sample size. The superior performance of the SCRCMLE in the small sample size may be due to the relative stability of the regularized inverse when the signal is dense. Similar results can be found in Seong and Seo (2022).

We observe from Table 1 that the Inf.2SCMLE does not perform well when the signal is either dense (\( s = 0.8 \)) or weak (\( \mu^2 = 30 \)). The distortion in the dense design may be related to the many instruments bias; the estimator uses 40 instruments when the signal is dense. Meanwhile, the distortion when \( \mu^2 = 30 \) suggests that Rivers and Vuong’s (1988) 2SCMLE is not reliable when the concentration parameter is small.
This is consistent with the results in Figure 1. In the figure, theASFs of the 2SCMLE shift toward those of the probit estimator as $\mu^2$ gets smaller.

In Table 1, the TTSLS and TRCMLE produce similar estimation results. However, the TTSLS exhibits rejection rates above the nominal level for all cases in the table. Furthermore, as reported in Figure 1, the TTSLS does not provide good estimates of the ASFs, particularly at the boundaries of $y_2$, whereas the TRCMLE yields estimates close to the true values even at the boundaries.

Lastly, we examine the performance of the aforementioned estimators with different values of $K$, $\mu^2$ and $s$. The DGP is identical to the above, and we focus on three estimators: the TRCMLE, 2SCMLE, and probit estimator. In the first experiment, we change $K$ from 10 to 100 while fixing $s$ and $\mu^2$ to 0.5 and 50, respectively. In the next, we set $K = 50$ and $\mu^2 = 50$ and then change $s$ from 0.1 to 1. Lastly, we set $K = 50$ and $s = 0.5$ and change $\mu^2$ from 25 to 250. The sample size is 200.

The estimation results are reported in Figure 2. The TRCMLE shows the smallest median bias and appears to have correct size for all cases considered in Figure 2. On the other hand, as $K$ increases, the 2SCMLE and probit estimator tend to perform similarly; this may not be surprising, since, in the linear model, it is known that the TSLS estimator shifts substantially toward the OLS estimator as the number of instruments increases. As the value of $\mu^2$ decreases, these two estimators also produce similar performance, which suggests that Rivers and Vuong’s (1988) estimator may be unreliable when the concentration parameter is small.

4.2 Factor Model

Table 2: Simulation results (Factor model, $\tilde{K} = 100$, $K = 5$)

| $n$   | $\mu^2 = 30$                  | $\mu^2 = 60$                  |
|-------|------------------------------|------------------------------|
|       | Med.Bias | MAD | RP | Med.Bias | MAD | RP |
| 200   | TRCMLE   | -0.085 | 0.235 | 0.086 | -0.055 | 0.190 | 0.073 |
|       | SCRCMLE  | -0.015 | 0.241 | 0.038 | -0.012 | 0.187 | 0.039 |
|       | Inf.2SCMLE | -0.062 | 0.228 | 0.059 | -0.027 | 0.184 | 0.045 |
|       | 2SCMLE   | -0.637 | 0.099 | 0.968 | -0.576 | 0.097 | 0.945 |
|       | Probit    | -0.715 | 0.071 | 0.792 | -0.693 | 0.073 | 0.841 |
|       | TTSLS     | -0.093 | 0.220 | 0.121 | -0.064 | 0.169 | 0.122 |
| 400   | TRCMLE   | -0.080 | 0.216 | 0.062 | -0.036 | 0.165 | 0.060 |
|       | SCRCMLE  | -0.044 | 0.220 | 0.042 | -0.030 | 0.164 | 0.051 |
|       | Inf.2SCMLE | -0.080 | 0.212 | 0.058 | -0.037 | 0.164 | 0.055 |
|       | 2SCMLE   | -0.625 | 0.092 | 0.989 | -0.544 | 0.090 | 0.973 |
|       | Probit    | -0.738 | 0.049 | 0.953 | -0.725 | 0.050 | 0.971 |
|       | TTSLS     | -0.074 | 0.216 | 0.102 | -0.042 | 0.159 | 0.097 |

Notes: The simulation results based on 2,000 replications are reported. Each cell reports the median bias (Med.Bias), median absolute deviation (MAD), and rejection probability at 5% significance level (RP).
We now assume that only \( \tilde{Z}_i \in \mathbb{R}^{\tilde{K}} \) can be observed instead of the true instrument \( Z_i \in \mathbb{R}^K \). Let \( M \) be a \( \tilde{K} \times K \) matrix consisting of elements that are randomly drawn from \( \text{Unif}[-1, 1] \). This matrix is fixed across simulations, and let \( \tilde{Z}_i = M Z_i + \tilde{V}_i \) where \( \tilde{V}_i \sim \text{iid } \mathcal{N}(0, \tilde{\sigma}^2 I_{\tilde{K}}) \) and \( \tilde{\sigma} = 0.3 \). The measurement error \( \tilde{V}_i \) is independent of \( u_i \) and \( V_i \). The parameters are set as follows; \( K = 5, \tilde{K} = 100, s = 1, \rho_z = 0, \) and \( \sigma_z = 1 \). We consider two different values of \( \mu^2 \): 30 and 60.

The Inf.2SCMLE is computed using the true instrumental variables, while the others are computed using \( \tilde{Z}_i \). The simulation results are reported in Table 2 (the estimated ASFs are similar to those in Figure 1 and are reported in the Online Supplement). The results are similar to those in Section 4.1; our estimators have considerably small median bias and reasonable rejection rates. We note that SCRCMLE tends to have a larger bias but a smaller MAD when \( n = 400 \), which may be a result of a bias-variance tradeoff. As the Inf.2SCMLE is computed using the true instrumental variable, this estimator performs considerably well. On the other hand, the 2SCMLE has substantially large median bias and poor size control, which is consistent with the results in Table 1.

5 Empirical Example: Bastian and Michelmore (2018)

The EITC, which becomes the largest cash-transfer program in the U.S., is designed to increase labor supply and benefit low-income families with children. Recent studies on the EITC have shown that the program has a considerable impact on its recipients (see Meyer and Rosenbaum, 2001; Eissa and Hoynes, 2004) and their children (see Dahl and Lochner, 2012). In particular, Bastian and Michelmore (2018) provided empirical evidence that an increase in EITC exposure has a positive impact on family earnings, which leads to an increase in children’s long-term academic achievement. We in this section extend their work with taking into consideration the following issues. First, the first-stage F statistics reported in Bastian and Michelmore (2018, Table 5) are between 3.2 and 5.1, which suggests that their instrumental variables, measures of EITC exposure, might be weak. Moreover, as mentioned in Example 1, the results from the linear probability model could be improved by using the nonlinear binary response model from a theoretical perspective.

We complement Bastian and Michelmore’s (2018) results as follows. First, we employ the binary response model and use our estimators. Second, we utilize additional instruments to increase estimation efficiency and account for heterogeneous effects of EITC exposure on family income depending on the state of residence. Lastly, we examine the weakness of the suggested instruments in the context of the binary response model by using the test proposed by Frazier et al. (2020).
We use the data of Bastian and Michelmore (2018) from the 1968-2013 waves of the Panel Study of Income Dynamics (PSID). Please refer to Bastian and Michelmore (2018) for details on the data. Our model of interest is similar to theirs and is given as follows,

\[ y_i = \mathbb{1}\{\beta_0 + I_i'\beta_1 + W_{1i}'\beta_2 + W_{2i}'\beta_3 \geq u_i\}, \quad I_i = \pi_0 + \tilde{Z}_i'\pi_1 + W_{1i}'\pi_2 + W_{2i}'\pi_3 + V_i, \quad (5.1) \]

where \( I_i = (I_i,(0-5), I_i,(6-12), I_i,(13-18))' \) and each \( I_i,(a) \) is the family income of individual \( i \) at each age interval \( a \). The outcome variable \( y_i \) is 1 if individual \( i \) is a college graduate and 0 otherwise.\(^4\) \( W_{1i} \) is an 11-dimensional vector of personal characteristics: age, age square, the number of siblings at age 18, and indicators for black, Hispanic, female, ever-married parents, and whether the individual’s mother and father completed high school or are at least some college-educated. \( W_{2i} \), which is measured at age 18, is state-by-year economic indicators: per capita GDP, the unemployment rate, the top marginal income tax rate, the minimum wage, maximum welfare benefits, spending on higher education, and tax revenue. All the continuous variables are standardized.

We consider two sets of instruments. In Model 1, we follow Bastian and Michelmore (2018) and let \( \tilde{Z}_i \) be \( (\text{EITC}_{i,(0-5)}, \text{EITC}_{i,(6-12)}, \text{EITC}_{i,(13-18)})' \), where \( \text{EITC}_{i,(a)} \) is the standardized measure of EITC exposure of individual \( i \) at each age interval \( a \).\(^5\) As EITC exposure is measured in childhood and adolescent years, this could affect children’s long-term educational attainment only through \( I_i \), see Bastian and Michelmore (2018) for more discussions on the validity of these instruments. In Model 2, we use interactions of the measures of EITC exposure with 47 state-of-residence dummies (observed in the sample) as additional instruments. The state-specific cost of living can produce heterogeneous effects of EITC transfers on the family income. Therefore, including these additional instruments will allow us to account for a possible bias in the first stage. The number of instruments in Model 2 is 144. The sample size is 2,654.

Before estimating the model, we apply Frazier et al.’s (2020) distorted J test to see if our instruments are weak in the sense of Staiger and Stock (1997). This is needed to ensure that the conditions in Theorem 2 are satisfied. However, Frazier et al.’s (2020) test is not designed for the case where (i) the (sample) covariance of instruments is nearly singular and (ii) multiple endogenous variables are present. Hence, we apply the test only in Model 1. In addition, the test is implemented separately for each age interval \( a \) after replacing \( I_i \) in (5.1) with \( I_i,(a) \). The computed distorted J tests are respectively 37.15, 21.81, and 42.76 for each of the

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\(^4\) College graduation is assessed when individuals reach the age of 26.

\(^5\) The measures of EITC exposure and family income are in thousands of 2013 dollars and discounted at a 3% annual rate from age 18. A detailed description can be found in Bastian and Michelmore (2018).
age intervals 0-5, 6-12, and 13-18, and the critical value at 5% significance level is around 9.48. Therefore, the distorted J tests suggest that our model is not weakly identified in the sense of Staiger and Stock (1997).

We now apply our estimation procedure and compare the results with those from existing estimators. We consider the TRCMLE and SCRCMLE, and choose their regularization parameter as in Section 4, while the parameter set of \( \alpha \) is set to 25 equally spaced points between \( c_{\alpha}n^{-0.6} \times 10^{-4} \) and \( c_{\alpha}n^{-0.6} \times 10^{-1} \) with \( c_{\alpha} \) being \( \| \hat{\Sigma}_{Z} \| \) for the TRCMLE and \( \| \hat{\Sigma}_{Z} \|^{2} \) for the SCRCMLE. We also report estimation results from Rivers and Vuong’s (1988) 2SCMLE and the probit estimator. In addition to them, we consider another estimator similar to the post-Lasso estimator proposed by Belloni et al. (2012). This estimator is considered due to its popularity in empirical research. To compute this estimator, we first select a set of instruments with non-zero first-stage coefficients using the Lasso procedure. Then, we refit the first stage using the selected set of instruments and compute the first-stage fitted values and residuals. Lastly, a linear model, in which the fitted values and residuals are used as regressors, is estimated. We report the results in the column labeled “Lasso” in Table 3.

Table 3: Effects of family earnings on college completion

| Variables | TRCMLE | SCRCMLE | 2SCMLE | Lasso | Probit |
|-----------|--------|---------|--------|-------|--------|
| Model 1 (\( \dim(\hat{Z}_{i}) = 3 \)) | | | | | |
| \( I_{i,(0-5)} \) | 1.5237*** | 1.5032*** | 1.5032*** | 0.2720*** | 0.1396*** |
| (0.5502) | (0.5429) | (0.5429) | (0.0646) | (0.0484) |
| \( I_{i,(6-12)} \) | -2.9299*** | -2.8910*** | -2.8910*** | -0.4324*** | 0.2022*** |
| (1.1011) | (1.0809) | (1.0809) | (0.1217) | (0.0699) |
| \( I_{i,(13-18)} \) | 2.2176** | 2.2188** | 2.2188** | 0.3332*** | 0.0530 |
| (0.9601) | (0.9477) | (0.9477) | (0.0968) | (0.0502) |
| Exogeneity Test | 10.7426*** | 10.6553*** | 10.6553*** | 24.9366*** | |
| Model 2 (\( \dim(\hat{Z}_{i}) = 144 \)) | | | | | |
| \( I_{i,(0-5)} \) | 1.5519*** | 1.5330*** | 0.4107** | 0.1750*** | 0.1396*** |
| (0.5346) | (0.5343) | (0.1867) | (0.0435) | (0.0484) |
| \( I_{i,(6-12)} \) | -2.7763*** | -2.7998*** | -0.3722 | -0.2894*** | 0.2022*** |
| (0.9543) | (1.0032) | (0.2738) | (0.0805) | (0.0699) |
| \( I_{i,(13-18)} \) | 2.1080*** | 2.1979*** | 0.5260** | 0.2807*** | 0.0530 |
| (0.8114) | (0.8531) | (0.2138) | (0.0700) | (0.0502) |
| Exogeneity Test | 14.4770*** | 12.9709*** | 10.0550** | 29.7836*** | |

Notes: Standard errors of coefficient estimates are reported in parentheses. Significance *** \( p < 0.01 \), ** \( p < 0.05 \), * \( p < 0.1 \). \( \alpha \) is chosen to 0.00404 (TRCMLE) and 4.038 \times 10^{-6} \ (SCRCMLE) in Model 1 and 0.00411 (TRCMLE) and 3.429 \times 10^{-3} \ (SCRCMLE) in Model 2.

Table 3 shows that, in Model 1, there is no big difference in \( \beta_{1} \) estimates obtained using our and Rivers and Vuong’s (1988) approaches. However, the estimation results become substantially different in Model 2.

\(^6\)To conduct the test, we set \( \delta_{n}, \alpha_{i} \) and \( b_{i} \) in Frazier et al. (2020) to \( \bar{\psi}/ \log(\log(n)) \), \( (I_{i,(\alpha)}, W_{1i}', W_{2i}', \hat{Z}_{i}', 0_{k}') \), and \( (0_{k}', W_{1i}', W_{2i}', \hat{Z}_{i}') \) where \( k = \dim(W_{1i}) + \dim(W_{2i}) + \dim(\hat{Z}_{i}) + 1 = 23 \).

\(^7\)The results from the RNLSE are similar to those from the RCML and thus are omitted.
Specifically, in contrast to our estimators that report similar estimates of $\beta_1$ in Models 1 and 2, the 2SCMLE tends to shift toward the probit estimator in Model 2. This is consistent with the findings in Section 4 and shows that our estimators are not subject to the many instruments bias. Moreover, all the instrumental variable estimators report smaller standard errors in Model 2. This implies that the large set of instruments in Model 2 includes additional information that is useful for predicting the family income.

Similar results can be found in Figure 3, which report the estimated probability of college completion depending on the family income in each age interval. For the ease of explanation, let $\mu_a$ (resp. $\sigma_a$) denote the sample mean (resp. standard deviation) of $I_i(a)$. Then, Figure 3 shows how the probability varies as the family income $I_i(a)$ changes from $\mu_a - \sigma_a$ to $\mu_a + \sigma_a$ when the other variables are fixed to their medians. In the figure, there is no significant change in the probabilities of college completion computed with the TRCMLE in both models. On the other hand, the probability computed with the 2SCMLE substantially shifts toward that of the probit estimator in Model 2. Similar results can be found in APE estimates reported in the Online Supplement.

We now focus on the estimates computed using the Lasso procedure. As mentioned in Section 4, the parameters in linear probability models are different from those in nonlinear binary response models. Therefore, to make a reasonable comparison between the coefficient estimates, we compute the APEs at the sample mean of $Y_{2i}$ and report them in Table 7 in Appendix F in the Online Supplement. However, even in the table, the Lasso estimates tend to be biased toward the probit estimates in Model 2. This estimation result may be explained by the reason mentioned in Section 2.1 and a slower convergence rate of the eigenvalues of the sample covariance of $Z_i$.

We conduct the exogeneity test as explained in Remark 4, and the results are reported in Table 3. The $p$-values are computed from $\chi^2(3)$ distribution. The results indicate the presence of endogeneity at 1% significance level regardless of the number of instruments or estimation methods. This explains the difference in coefficient estimates between the probit estimator and the others.

The estimation results in Table 3 are similar to those in Bastian and Michelmore (2018); an increase in family income between age 13 and 18 has a positive impact on college completion in their early adulthood. The authors find a similar positive effect, but their estimates, reported in column 3 of their Table 6, are not significant. This discrepancy may come from differences between our and their estimation strategies; in addition to the different models of interest, we do not use the large number of fixed effects in Bastian and Michelmore (2018) to avoid having many explanatory variables in the second stage. Meanwhile, Table 3
reports a positive (resp. negative) impact of an increase in family income between birth and age 5 (resp. between age 6 and 12) on college completion. Bastian and Michelmore (2018) also report mixed signs of coefficients when the outcome is given by high school completion or employment status. This phenomenon might be caused by cross-age correlations in EITC exposure.

6 Conclusion

This paper proposes an estimation procedure to resolve the issue of many (nearly) weak instruments in endogenous binary response models. The proposed estimators are easy to compute and have good asymptotic properties. A Monte Carlo study suggests that our estimators outperform the existing estimators for endogenous binary response models when there are many (nearly) weak instruments. We apply the estimators to study the effect of family income in childhood and adolescent years on college completion.

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Figure 1: Estimated average structural functions (Gaussian instruments)

Notes: The simulation results based on 2,000 replications are reported. $K = 50$ and $n = 200$. 
Figure 2: Simulation results for different parameter values

(a) Median Bias

(b) Median Absolute Deviation

(c) Rejection Probability at 5% significance level

Notes: The simulation results based on 2,000 replications are reported. The default values of $K$, $s$ and $\mu^2$ are 50, $\lfloor 0.5 \rfloor$, and 50. The dashed lines indicate the nominal size 0.05.

Figure 3: Estimated average structural functions

Notes: Each figure is the estimated average structural function (ASF) that reports the probability of college completion according to family earnings in the age interval 0-5 (left), 6-12 (middle), and 13-18 (right).
Appendix A: Additional remark

**Remark 6.** Let \( \ker \mathcal{K} \) and \( (\ker \mathcal{K})^\perp \) denote the kernel of \( \mathcal{K} \) and its orthogonal complement space. Then, because \( (\ker \mathcal{K})^\perp = \text{cl}(\text{ran } \mathcal{K}) \), for any \( h \in \ker \mathcal{K} \),

\[
\langle V_j h, V_j h \rangle = \| E[\dot{m}^2_j(\theta_0, g_i) g_{0i}(Z_i, h)] \|^2 \leq E[\| \dot{m}^2_j(\theta_0, g_i) g_{0i}(Z_i, h) \|^2] = 0,
\]

where the inequality follows from the Cauchy-Schwarz inequality. The last equality is a consequence of Assumption 7 and the definition of \( \ker \mathcal{K} \). Thus, \( \ker \mathcal{K} \subset \ker V_j \), which implies \( (\ker V_j)^\perp = \text{cl}(\text{ran } V_j^\ast) \subset \text{cl}(\text{ran } \mathcal{K}) = (\ker \mathcal{K})^\perp \). \( \square \)

Appendix B: Additional simulation study with function-valued instruments

We now concern a scenario where the functional form of \( \mathbb{E}[y_{2i}|x_i] \) is unknown and, to estimate it, the continuum of moments is used as \( Z_i \). Specifically, we replace the first stage in (4.1) in Section 4.1 by

\[
Y_{2i} = \pi z_{1i} + \pi f(z_{2i}) + v_i,
\]

where \( z_{1i} \sim_{\text{iid}} \mathcal{N}(0, 1) \), \( z_{2i} \sim_{\text{iid}} \text{Unif}[0, 1] \) and \( f(\cdot) \) is set to the probability density function of the beta distribution with shape parameters 2 and 5. The constant \( \pi \) is computed as before with replacing \( \Sigma_Z \) by its estimates calculated with 1,000 samples of \((z_{1i}, f(z_{2i}))\).

Our estimators and the TTSLS are computed using the continuum of moments \( Z_i = \exp(tz_{2i}) \) as an instrument where \( t \) takes values in 100 equally spaced points between -5 and 5. The weight \( \tau(\cdot) \) in Section 3.2.1 is given by the standard normal density function. Then, the estimators are computed using the eigenanalysis approach as in Carrasco (2012). The Inf.2SCMLE is calculated with the realized values of \( f(z_{2i}) \), and the 2SCMLE, requiring the linear first stage, is omitted in this example. The concentration parameter is computed using the true values of \( f(z_{2i}) \) and is specified to 60 or 180. However, since we use the continuum of moments as \( Z_i \) instead of the true values of \( f(z_{2i}) \), the signal from \( Z_i \) may be weaker than what the concentration parameter suggests.

Table 4 summarizes the simulation results. For all the cases considered in the table, the SCRCMLE performs as good as the Inf.SCRCMLE that is computed using the realized values of \( f(z_{2i}) \). Meanwhile, the two Tikhonov regularized estimators report a relatively large median bias and MAD when \( \mu^2 = 60 \). This may be related to the fact that (i) the eigenvalues of the sample covariance of \( Z_i \) shrink at a relatively fast rate in this example and (ii) Tikhonov regularized estimators have a relatively slow convergence rate (see, e.g., Benatia et al. 2017, Remark 3.2). Hence, for this particular example, spectral cut-off regularization may be preferred.

Appendix C: Additional empirical illustration

Miguel, Satyanath, and Sergenti (2004) studied the effect of economic growth on the occurrence of civil war in sub-Saharan Africa by instrumenting the economic growth to the annual rainfall growth rate. As briefly mentioned in Example 3, the model may provide better insight into the relationship if the annual variations of the rainfall growth curve are utilized as an instrument, rather than its average. We leave this

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Since \( z_{i2} \) takes values on a compact interval, this function is bounded and square integrable.
Table 4: Simulation results (Functional instrument of the continuum of moments)

| n   | TRCMLE | SCRCMLE | Inf.2SCMLE | Probit | TTSLS |
|-----|--------|---------|------------|--------|-------|
| 200 | 0.166  | -0.000  | 0.035      | -0.753 | 0.177 |
|     | 0.371  | 0.296   | 0.239      | 0.074  | 0.361 |
|     | 0.026  | 0.050   | 0.037      | 0.633  | 0.118 |
|     | 0.223  | 0.197   | 0.181      | 0.079  | 0.209 |
|     | 0.053  | 0.065   | 0.043      | 0.605  | 0.102 |
| 400 | 0.101  | -0.025  | -0.001     | -0.754 | 0.115 |
|     | 0.333  | 0.277   | 0.218      | 0.051  | 0.331 |
|     | 0.024  | 0.038   | 0.040      | 0.918  | 0.088 |
|     | 0.185  | 0.168   | 0.143      | 0.052  | 0.177 |
|     | 0.045  | 0.056   | 0.047      | 0.896  | 0.078 |

Notes: The simulation results based on 2,000 replications are reported. Each cell reports the median bias (Med.Bias), median absolute deviation (MAD), and rejection probability at 5% significance level (RP).

question for future study given the current data availability. Instead, in this section, we use the continuum of moments of the annual rainfall growth as an instrument to see how the function-valued instrumental variable can be used in practice.

The model we consider is given by:

\[
\text{conflict}_{it} = 1\{\beta_0 + \text{growth}_{it-1}\beta + W_{it}'\gamma \geq u_{it}\}, \quad \text{growth}_{it-1} = \prod_{it}Z_{it-1} + W_{it}'\pi_2 + V_{it}. \quad (C.1)
\]

The dependent variable indicates if there was a civil conflict resulting in at least 25 deaths, and \(\text{growth}_{it-1}\) measures annual economic growth in country \(i\) at time \(s-1\). Miguel et al. (2004) found that the lagged economic growth plays an important role in the occurrence of civil conflicts associated with 25-death threshold. The vector \(W_{it}\) contains the explanatory variables: the log of GDP per capita in 1979, a lagged measure of democracy, ethnolinguistic fractionalization, religious fractionalization and the indicator on if country \(i\) exports oil. We refer the readers to Miguel et al. (2004) for detailed information on these variables and the data.

To compute our regularized estimators, the instrumental variable \(Z_{it-1}\) is set to the continuum of moments of rainfall growth rates of country \(i\) at time \(s-1\), i.e., \(Z_{it} = \exp(-tx_{it-1})\) where \(x_{it-1}\) is the lagged rainfall growth that is used as an instrument in Miguel et al. (2004). As in our simulation setup in Section B, the weight function is given by the standard normal density and the function-valued random variable is measured on 100 equally spaced points between -5 and 5. As before, all continuous exogenous variables are standardized, and \(Z_{i}\) is centered before analysis.

We compare estimation results of the TRCMLE and SCRCMLE, computed with the above function-valued instruments, with the naive probit estimator and Rivers and Vuong’s (1988) estimator calculated using \(x_{it-1}\) as an instrument. The regularization parameter for our estimators are obtained as in Section 5.

Table 5 summarizes estimation results. The exogeneity testing results indicate the presence of an endogeneity problem, which explains the significant difference between the estimates obtained from probit and other instrumental variable estimation approaches. All instrumental variable estimators suggest that economic growth has a greater impact on the occurrence of civil conflicts, although none of them are statistically significant. Similar results are obtained even when the dependent variable is specified to civil conflicts associated with deaths greater than 1,000. The insignificant results may be attributed to the failure to address
country-specific fixed effects that are not included in our model. As mentioned in Section 5, the fixed effects are not included to mitigate a possible issue related to the curse of dimensionality in the second stage. Despite the insignificance, the table provides valuable insight on our estimators. For example, the similar effects are not included to mitigate a possible issue related to the curse of dimensionality in the second stage.

Appendix D: Proofs of the results in Sections 3.2 and 3.3

D.1 Appendix D.1: Proofs

Since Theorems 1 and 2 are special cases of Theorems 3 and 4, we only prove Theorems 3 and 4. Beforehand, to facilitate the following discussions, we introduce a few notations. For any $a \in \mathbb{R}$, $\tilde{\phi}(a)$ denotes the inverse Mill’s ratio given by $\tilde{\phi}(a) = \phi(a)/\Phi(a)$. In addition, for a linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$, $\|A\|_{\text{op}}$ denotes its operator norm that is given by $\|A\|_{\text{op}} = \sup_{\|h\| \leq 1} \|Ah\|$. The Hilbert-Schmidt norm of $A$ is denoted by $\|A\|_{\text{HS}}$, i.e., $\|A\|_{\text{HS}} = \left(\sum_{j=1}^{\infty} \langle A^*A\zeta_j, \zeta_j \rangle \right)^{1/2}$ where $\{\zeta_j\}_{j \geq 1}$ is a set of orthonormal basis of $\mathcal{H}$. Note that $\|Ah\| \leq \|A\|_{\text{op}} \|h\|$ and $\|A\|_{\text{op}} \leq \|A\|_{\text{HS}}$.

Proof of Theorem 3. Since the compactness of $\Theta$ and the identification condition of $\theta_0$ are given in Assumption 3, we focus on the proof of the uniform convergence between $Q_{jH}(\cdot)$ and $Q_{j}(\cdot)$. To this end, let $\tilde{Q}_{jH}(\theta) \equiv \tilde{Q}_{jH}(\theta, g_i) = \frac{1}{n} \sum_{i=1}^{n} m_j(\theta, g_i)$ that is an auxiliary objective function associated with $Q_{j}(\cdot)$. Then, by the dominated convergence theorem and Assumptions 2 and 3, we have $\sup_{\theta} |\tilde{Q}_{jH}(\theta) - Q_{j}(\theta)| = o_p(1)$ for $j \in \{M, N\}$. Thus, by the triangular inequality, the consistency of $\tilde{\theta}_j$ can be established by proving the uniform convergence between $\tilde{Q}_{jH}(\cdot)$ and $Q_{jH}(\cdot)$.

For $j = N$, due to the boundedness of $\Phi(\cdot)$ and $\phi(\cdot)$, we have $\sup_{\theta \in \Theta} |\tilde{Q}_{Nn}(\theta, g_i) - Q_{Nn}(\theta, \hat{g}_{i,\alpha})| \leq c \frac{1}{n} \sum_{i=1}^{n} \|S_{n}^{-1}(\hat{g}_{i,\alpha} - g_i)\|$ for a positive constant $c$. Analogously, using the Lipschitz continuity of $\tilde{\phi}(\cdot)$, mean-value theorem, and triangular inequality, we find that $\sup_{\theta \in \Theta} |\tilde{Q}_{Mn}(\theta, g_i) - Q_{Mn}(\theta, \hat{g}_{i,\alpha})| \leq c \frac{1}{n} \sum_{i=1}^{n} \|S_{n}^{-1}(\hat{g}_{i,\alpha} - g_i)\|^2$.

Table 5: Estimation results for Section C

| Variable       | TRCMLE  | SCRCMLE | 2SCMLe  | Probit   |
|----------------|---------|---------|---------|----------|
| growth_{i-1}  | -0.5136 | -0.4930 | -0.6056 | -0.0335  |
|                | (0.4046)| (0.3967)| (0.4609)| (0.0522) |

Exogeneity Testing

Table 5: Estimation results for Section C

| Variable       | TRCMLE  | SCRCMLE | 2SCMLe  | Probit   |
|----------------|---------|---------|---------|----------|
|                | 1.4317***| 1.3647***| 1.5605***|          |

Notes: Each cell reports point estimates of $\beta$ in (C.1). The standard errors are reported in parentheses. The sample size is 743. The regularization parameters of the TRCMLE and SCRCMLE are respectively $1.413e^{-6}$ and $1.543e^{-5}$.
First, by using the spectral decomposition of $\mathcal{K}_{na}^{-1}$, we have the following.
\[
\frac{1}{n} \sum_{i=1}^{n} \left\| \Pi_0(\mathcal{K}_n \mathcal{K}_{na}^{-1} - I) Z_i \right\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left( (q(\tilde{\kappa}, \alpha) - 1) (Z_i, \tilde{\varphi}_j) \Pi_0 \tilde{\varphi}_j \right)^2 \\
= \sum_{j,k=1}^{d_o} (q(\tilde{\kappa}, \alpha) - 1) (q(\tilde{\kappa}, \alpha) - 1) \langle \tilde{\varphi}_k, \mathcal{K}_n \tilde{\varphi}_j \rangle \langle \Pi_0 \tilde{\varphi}_j, \Pi_0 \tilde{\varphi}_k \rangle \\
= \sum_{\ell=1}^{d_o} \sum_{j=1}^{\infty} \hat{\kappa}_j (q(\tilde{\kappa}, \alpha) - 1)^2 (\tilde{\varphi}_j, \pi_{\ell,0})^2 \leq \sup_j \hat{\kappa}_j^{2(\rho + 1)} (1 - q(\tilde{\kappa}, \alpha))^2 \sum_{\ell=1}^{d_o} \sum_{j=1}^{\infty} \frac{\langle \tilde{\varphi}_j, \pi_{\ell,0} \rangle^2}{\hat{\kappa}_j^{2(\rho)}} \tag{D.1}
\]
where the equalities follow from the definition of $\mathcal{K}_n$, $\mathcal{K}_{na}^{-1}$ and the orthogonality of $\{ \tilde{\varphi}_j \}_{j \geq 1}$ across $j$. The last term in (D.1) is bounded above by $O_p(\alpha^{(2\rho + 1)/2})$ because of Assumptions 6.(ii) and 8.9

Next, let $\mathcal{K}_n^{1/2} = \sum_{j=1}^{\infty} \mathcal{K}_j^{1/2} \tilde{\varphi}_j \otimes \tilde{\varphi}_j$ and $v_{i\ell}$ be the $\ell$th row of $V_i$. Then, the following inequality holds by the definition of $\| \cdot \|_{HS}$. Assumptions 6.(i) and 8, and Theorem 2.7 of Bosq (2000).
\[
\frac{1}{n} \sum_{j=1}^{n} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathcal{K}_n^{-1} Z_i \otimes \Lambda_n^{-1} V_i) Z_j \right\|^2 = \left\| \Lambda_n^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \otimes V_i \right) \mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \right\|_{HS}^2 \\
\leq \left\| \Lambda_n^{-1} \right\|_{op}^2 \left\| \mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \right\|_{op} \sum_{\ell=1}^{d_o} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i v_{i\ell} \right\|^2 = O_p(1/\mu_{m,n}^2), \tag{D.2}
\]
where the last line is obtained from the fact that $\sup_j \alpha q^2 (\hat{\kappa}_j, \alpha)/\hat{\kappa}_j \leq \sup_j \alpha q (\hat{\kappa}_j, \alpha)/\hat{\kappa}_j = O_p(1)$ due to Assumption 8.10

Therefore, from (D.1) and (D.2), we find the following.
\[
\frac{1}{n} \sum_{i=1}^{n} \left\| \mathcal{S}_n^{-1}(\hat{g}_i, \alpha - g_i) \right\|^2 \leq \frac{4}{\sqrt{n}} \sum_{i=1}^{n} \left\| \Pi_0 \mathcal{K}_n \mathcal{K}_{na}^{-1} Z_i - \Pi_0 Z_i \right\|^2 + \frac{4}{\sqrt{n}} \sum_{i=1}^{n} \left\| \frac{1}{\sqrt{n}} \sum_{j=1}^{\infty} \langle \mathcal{K}_{na}^{-1} Z_i, Z_j \rangle \Lambda_n^{-1} V_i \right\|^2 \\
= O_p(\alpha^{-2(\rho + 1)}) + O_p(1/\mu_{m,n}^2),
\]
from which it can be deduced that $\sup_{\theta \in \Theta} |Q_{jn}(\theta, \hat{g}_i, \alpha) - Q_{jn}(\theta, g_i)| = o_p(1)$. □

**Proof of Theorem 4.** Theorem 4 is obtained by applying Theorem 2 of Chen et al. (2003) to $\mathcal{S}_n^{-1} \partial \mathcal{Q}_1(\theta, G_t(\Pi)) / \partial \theta$. Lemmas 3-5 verify the primitive conditions in Chen et al. (2003). □

**Proof of Theorem 5.** Theorem 5 is a consequence of Lemmas 6-8 and Slutsky’s theorem. □

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9To be specific, Carrasco (2012) shows that $\sup_j \kappa_j^{2(\rho + 1)/2} (1 - q(\tilde{\kappa}_j, \alpha))^2$ is bounded above by $O_p(\alpha^{(2\rho + 1)/2})$ for spectral cut-off and $O_p(\alpha^{(2\rho + 1)/4})$ for Tikhonov regularization, and therefore,
\[
\sup_j \kappa_j^{2(\rho + 1)/2} (1 - q(\tilde{\kappa}_j, \alpha))^2 \leq (\sup_j \kappa_j^{2(\rho + 1)/2} (1 - q(\tilde{\kappa}_j, \alpha))^2 ^2 \leq O_p(\alpha^{min((2\rho + 1)/2)}).
\]

For ridge regularization, we have $\sup_j \kappa_j^{2\rho} (1 - q(\tilde{\kappa}_j, \alpha))^2 \leq \sup_j \kappa_j^{2\rho} \frac{\kappa_j^{2\rho}}{\kappa_j^{2\rho + 2}} \leq O_p(\alpha^{min(2\rho)})$, where the first inequality follows from the non-negativity of $\tilde{\kappa}_j$ and $\alpha$. The second is from the fact that $\kappa^\rho/(\kappa + \alpha^2)$ is strictly increasing in $\kappa$ if $\rho > 1$, and is maximized at $\kappa = \alpha^2/(1 - \rho)$ if $\rho \in (0, 1]$. Hence, the last term of (D.1) is bounded by $O_p(\alpha^\rho)$ where its rate of convergence could depend on the regularization scheme.

10We here note that (D.2) is bounded above by $O_p(1/(\mu_{m,n}^2 \sqrt{n}))$ under Assumption 10 for a later use.
D.2 Appendix D.2: Lemmas

We hereafter let $\mathcal{H}^r$ denote the Cartesian product of $r$ copies of $\mathcal{H}$ equipped with the inner product $\langle h_1, h_2 \rangle = \sum_{j=1}^{r} \langle h_{1j}, h_{2j} \rangle$ and its induced norm $\|h_1\| = (\sum_{j=1}^{r} \langle h_{1j}, h_{1j} \rangle)^{1/2}$ for $h_1 = (h_{11}, \ldots, h_{1r})'$ and $h_2 = (h_{21}, \ldots, h_{2r})'$. Moreover, $T_i$ denotes the operator from $\mathcal{H}^{de}$ to $\mathbb{R}^{de}$ given by

$$T_i = \begin{bmatrix} \langle \cdot, Z_i \rangle & 0 & 0 & \ldots & 0 \\ 0 & \langle \cdot, Z_i \rangle & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \langle \cdot, Z_i \rangle \end{bmatrix}.$$ 

Then, for $\Pi : \mathcal{H} \to \mathbb{R}^{de}$ and $\pi = (\Pi^* e_1, \ldots, \Pi^* e_N)' \in \mathcal{H}^{de}$, we have $\Pi Z_i = T_i \pi$. We let $G_i(\pi) = ((T_i \pi)', (Y_{2i} - T_i \pi)' )' = ((\Pi Z_i)', (Y_{2i} - \Pi Z_i)' )'$ so that $g_i$ and $\hat{g}_i$ can be understood as the function $G_i(\cdot)$ evaluated at $\pi_n = (\Pi^* e_1, \ldots, \Pi^* e_N)'$ and $\hat{\pi}_\alpha = (\hat{\Pi}_\alpha^* e_1, \ldots, \hat{\Pi}_\alpha^* e_N)'$, respectively.

Note that Condition 2.1 in Chen et al. (2003) holds by the definition of $\hat{\theta}_j$. Lemmas 1 and 2 verify Conditions (2.2)-(2.4) in Chen et al. (2003) that are adapted to our setting. It can be shown that their Theorem 2 still holds under these alternative conditions, by slightly modifying the proof given in Chen et al. (2003).

In the following, $M_j(\theta, g_i) = \partial Q_j(\theta, g_i)/\partial \theta$. Lastly, we provide the proofs of the following lemmas only for $j = N$, since, under Assumption 5.(iii), the proofs for the RCMLE can be obtained in a similar manner. Thus, we omit the details.

**Lemma 1.** Suppose that the conditions in Theorem 4 hold, and let $\Theta_\delta = \{ \theta \in \Theta : \|\theta - \theta_0\| \leq \delta \}$. Then the following holds for $j \in \{ M, N \}$.

(1.1) The second derivative $\Gamma_{1,j}(\theta, g_i)$ in $\theta$ of $M_j(\theta, g_i)$ exists for $\theta \in \Theta_\delta$, and is continuous at $\theta = \theta_0$.

Let $\Gamma_{1,0,j} \equiv \Gamma_{1,j}(\theta_0, g_i)$ and then $S_n^{-1} \Gamma_{1,0,j} S_n^{-1}$ is nonsingular.

(1.2) Let $L_{\delta_n} = \{ \pi \in \mathcal{H}^{de} : \|\pi - \pi_n\| \leq \mu_{m,n} \delta_n/\sqrt{n} \}$ for some $\delta_n > 0$ and $\pi_n = (\Pi^* e_1, \ldots, \Pi^* e_N)$. For $\theta \in \Theta_\delta$, the Fréchet derivative $\Gamma_{2,j}(\theta, G_i(\pi_n)) : \mathcal{H}^{de} \to \mathbb{R}^{2de}$ of $M_j(\theta, G_i(\pi_n))$ in $\pi = (\pi_1, \ldots, \pi_{de})' \in \mathcal{H}^{de}$ at $\pi_n$ exists. $\Gamma_{2,0,j} \equiv \Gamma_{2,j}(\theta_0, G_i(\pi_n))$. For all $(\theta, \pi) \in \Theta_\delta \times L_{\delta_n}$ with a positive sequence $\delta_n = o(1)$,

$$\|S_n^{-1} \Gamma_{2,j}(\theta, g_i) - \Gamma_{2,j}(\theta_0, g_i)\| - \pi_n\| \leq o(1) \delta_n,$$

$$\|S_n^{-1} (M_j(\theta, G_i(\hat{\pi}_\alpha)) - M_j(\theta, G_i(\pi_n)) - \Gamma_{2,j}(\theta, G_i(\pi_n)) (\hat{\pi}_\alpha - \pi_n))\| \leq O(\mu_{m,n}^{-1} \sqrt{n}) \| \hat{\Pi}_\alpha - \Pi_n \| H_5^{1/2}.$$ 

**Proof of Lemma 1.** $\Gamma_{1,j}(\theta, g_i) = \mathbb{E}[\hat{m}_{2j}(\theta, g_i) g_i g_i'] + \mathbb{E}[\hat{m}_{2j}(\theta, g_i)] g_i g_i'$ where

$$\hat{m}_{2M}(\theta, g_i) = \left( \frac{y_i - \Phi(g_i' \theta)}{\Phi(g_i' \theta)(1 - \Phi(g_i' \theta))} \phi(g_i' \theta) \right)^2, \quad \hat{m}_{2M}(\theta, g_i) = \frac{y_i - \Phi(g_i' \theta)}{\Phi(g_i' \theta)(1 - \Phi(g_i' \theta))} \phi'(g_i' \theta),$$

$$\hat{m}_{2N}(\theta, g_i) = -\phi^2(g_i' \theta), \quad \hat{m}_{2N}(\theta, g_i) = (y_i - \Phi(g_i' \theta)) \phi'(g_i' \theta).$$

By the law of iterated expectation, the second term of $\Gamma_{1,0,j}$, $\mathbb{E}[\hat{m}_{2j}(\theta_0, g_i) g_i g_i']$, is 0.

First, (1.1) follows from the continuity of $\hat{m}_{2j}(\cdot)$ and $\hat{m}_{2j}(\cdot)$ with respect to $\theta$, Assumptions 2 and 5, and the boundedness of $\phi(\cdot)$. 

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To prove (1.2.1), we note that the Fréchet derivative of $\mathcal{T}_lh$ in $h$ is $\mathcal{T}_l$ for any $h \in \mathcal{H}^{d_e}$. Hence, by the chain rule, the Fréchet derivative of $\partial \mathcal{M}_j(\theta, \mathcal{G}_i(\pi))$ at $\pi_n$ exists, and we have

\[
\mathcal{S}_n^{-1}\Gamma_{2, N}(\theta, g_i)(\pi - \pi_n) = \mathbb{E}[\tilde{m}_2 N(\theta, g_i)g_0\psi T_1(\pi - \pi_n)] + \mathbb{E}[\tilde{m}_2 N(\theta, g_i)g_0\psi T_1(\pi - \pi_n)]
\]

where $\tilde{S}_n^{-1} = \mathcal{S}_n^{-1}(\mathcal{I}_{d_e}, \mathcal{I}_{d_e})$. For $(\theta, \pi) \in \Theta_{\delta_n} \times \mathbb{L}_{\delta_n}$, we only prove that the first term in the RHS of (D.3) satisfies $\|\mathbb{E}[\tilde{m}_2 N(\theta, g_i)g_0\psi T_1(\pi - \pi_n)] + \mathbb{E}[\tilde{m}_2 N(\theta, g_i)g_0\psi T_1(\pi - \pi_n)]\| = o_p(1)\delta_n$, since the proofs for the remaining terms are similar. Thus, we omit the details. Note that

\[
\|\mathbb{E}[\tilde{m}_2 N(\theta, g_i)g_0\psi T_1(\pi - \pi_n)] + \mathbb{E}[\tilde{m}_2 N(\theta, g_i)g_0\psi T_1(\pi - \pi_n)]\| 
\]

where the first and second inequalities follow respectively from the triangular inequality and the Lipschitz continuity of $\phi^2(\cdot)$ and the boundedness of $\phi(\cdot)$ and $\phi'(\cdot)$. Then, the last line is deduced from Assumption 5 and the definitions of $\Theta_{\delta_n}$ and $\mathbb{L}_{\delta_n}$.

We then focus on (1.2.2). Given that $\mathcal{M}_j(\theta, \mathcal{G}_i(\pi))$ is Fréchet differentiable in $\pi$, it is Gâteaux differentiable and, by the continuity of Gâteaux derivative and the mean-value theorem, we have

\[
\|\mathcal{S}_n^{-1}(\mathcal{M}_j(\theta, \mathcal{G}_i(\pi)) - \mathcal{M}_j(\theta, \mathcal{G}_i(\pi_n)) - \Gamma_{2,j}(\theta, \mathcal{G}_i(\pi_n))(\pi - \pi_n))\| 
\]

for $\pi \in \mathbb{L}_{\delta_n}$, see Drábek and Milota (2007, Theorem 3.2.7). Using the representation of $\Gamma_{2,j}$ in (D.3), the RHS of (D.5) can be decomposed into three terms each of which is associated with the terms in the RHS of (D.3). Then, by using similar arguments in proving (D.4) and the fact that $\theta'(\mathcal{G}_i(\pi_n + t(\pi - \pi_n)) - \mathcal{G}_i(\pi_n)) = t\theta'(\mathcal{I}_{d_e}, \mathcal{I}_{d_e})\mathcal{T}_l(\pi - \pi_n)$,

\[
(D.5) \leq O(1)(\mathbb{E}[\|\mathcal{T}_l(\pi - \pi_n)\|^2\|g_0\|] + \|\mathcal{S}_n^{-1}\|_{op}\mathbb{E}[\|\mathcal{T}_l(\pi - \pi_n)\|^2])
\]

Then, the desired result is obtained because $\mathbb{E}[\|\mathcal{T}_l(\pi - \pi_n)\|^2] = \sum_{\ell=1}^{2d_e} \mathbb{E}[(Z_\ell, \pi - \pi_n)^2] = \sum_{\ell=1}^{2d_e} \mathbb{E}[\mathcal{K}(\Pi - \Pi_n)*^2]$ and $\|\mathcal{S}_n^{-1}\|_{op} = O(\sqrt{n}p_{\alpha,m,n})$, and $\mathbb{E}[\|g_0\|\|x\|] \leq c$ for a generic constant $c$.

In the lemma below, we will show that the term appearing in (1.2.2) is bounded above by $o_p(n^{-1/2})$. This result combined with Lemma (1.2.2) imply Condition (2.4) in Chen et al. (2003).

**Lemma 2.** Suppose that the conditions in Theorem 4 hold. Then, $\|\mathcal{K}^{1/2}(\mathcal{K}^{-1}_\alpha \mathcal{K}_n - \mathcal{I})\|_{\mathcal{H}}^2 = o_p(n^{-1/2})$ for all $\ell = 1, \ldots, d_e$ and $O(\mu_{m,n}^{-1}n^{1/2})\|\hat{\Pi}_\alpha - \Pi_n\|^{1/2} = o_p(n^{-1/2})$.

**Proof.** Note that

\[
\|\mathcal{K}^{1/2}(\mathcal{K}^{-1}_\alpha \mathcal{K}_n - \mathcal{I})\|_{\mathcal{H}}^2 \leq 2\|\mathcal{K}^{1/2}(\mathcal{K}^{-1}_\alpha \mathcal{K}_n - \mathcal{I})\|_{\mathcal{H}}^2 + 2\|\mathcal{K}^{1/2}(\mathcal{K}^{-1}_\alpha \mathcal{K}_n - \mathcal{K}^{-1}_\alpha \mathcal{K})\|_{\mathcal{H}}^2.
\]

(1.2.2)
Moreover, we have
\[ \|K^{1/2}(\kappa^{-1} - I)\pi\|_2^2 \leq (\sup_j \kappa_j^{\varphi_j^2}(q(\kappa_j, \alpha) - 1))^2 \sum_{j=1}^{\infty} \kappa_j^{-2\rho}(\pi_{\ell,0}, \varphi_j)^2 = O_p(\alpha^2), \] (D.7)
by Assumptions 6.(ii), 8 and 10. Given that \( \alpha \sqrt{n} \to 0 \), (D.7) = \( o_p(n^{-1}) \). We next focus on the second summand in (D.6). To this end, let \( \kappa_\alpha \) denote the generalized inverse of \( \kappa_\alpha^{-1} \) such that \( \kappa_\alpha = \sum_j q(\kappa_j, \alpha) > 0 \), \( \kappa_j^{-1} \varphi_j \varphi_j \), and \( \kappa_{na} \) is similarly defined with \( \kappa_j, \varphi_j \). Then, we have
\[ \|K^{-\rho}\pi\| = O(1), \quad \|K^{1/2}\kappa_{na}^{-1}\|_{op} = O_p(1) \quad \text{and} \quad \|K_{na}^{-1/2}K^{-1/2}\|_{op} = O_p(1). \] (D.8)
Moreover, we have
\[
K^{1/2}(\kappa^{-1} - I)\pi_{\ell,0} = K^{1/2}(\kappa_{na}^{-1} - \kappa_\alpha^{-1})\kappa_\alpha\pi_{\ell,0} + K^{1/2}\kappa_{na}^{-1}(I - \kappa_\alpha^{-1})\pi_{\ell,0}
\]
\[
= K^{1/2}\kappa_{na}^{-1}(\kappa - \kappa + \kappa - \kappa + \kappa - \kappa)\kappa_\alpha\pi_{\ell,0} + K^{1/2}\kappa_{na}^{-1}(I - \kappa_\alpha^{-1})\pi_{\ell,0}
\]
\[
= K^{1/2}\kappa_{na}^{-1}(\kappa_{na} - \kappa_{na})\kappa_\alpha\pi_{\ell,0} + K^{1/2}\kappa_{na}^{-1}(\kappa - \kappa)\kappa_\alpha\pi_{\ell,0} + K^{1/2}\kappa_{na}^{-1}(I - \kappa_\alpha^{-1})\pi_{\ell,0}.
\]
Using (D.8), the arguments similar to those in showing the above and the condition on \( \rho \), we also have
\[ \|K^{1/2}\kappa_{na}^{-1}(\kappa_{na} - \kappa_{na})\kappa_\alpha\pi_{\ell,0}\| = O_p(n^{-1}). \]
Lastly, by using \( \|\kappa_{na} - \kappa\|_{op} = O_p(n^{-1/2}) \), we find that
\[ \|K^{1/2}\kappa_{na}^{-1}(\kappa_{na} - \kappa)(I - \kappa_{\alpha}^{-1})\pi_{\ell,0}\| \leq O_p(1)\|\kappa_{na} - \kappa\|_{op}^2 \|I - \kappa_{\alpha}^{-1}\|_{op} = o_p(n^{-1}). \]
from which it can be deduced that \( K^{1/2}(\kappa_{na} - \kappa_{\alpha}^{-1})\pi_{\ell,0}\| = o_p(n^{-1}). \) This concludes the first part of the lemma.

We then move to the second part.
\[ \|(\hat{\Pi}_n - \Pi_n)K^{1/2}\|_{HS}^2 \leq 2\|\Pi_n(\kappa_{na}^{-1} - I)\|_{HS}^2 + 2\|K^{1/2}\kappa_{na}^{-1}\sum_{i=1}^{n} V_i \otimes Z_i\|_{HS}^2 \] (D.9)
where the first summand is bounded above by \( o_p(n^{-1}) \) as shown above. Because of Bosq (2000, Theorem 2.7) and Hörmann and Kokoszka (2010, Theorem 3.1), \( \|n^{-1/2}\sum_{i=1}^{n} V_i \otimes Z_i\|_{HS}^2 = O_p(n^{-1}). \) Combining this with the fact that \( \|K^{1/2}\kappa_{na}^{-1}\|_{op} = O_p(\alpha^{-1/2}) \), the second term in (D.9) is bounded above by \( O_p(n^{-1/2}) \). Thus, the desired result is obtained due to \( \mu_{n, \alpha}^2 \to \infty \).

Lemmas 3 and 4 verify Condition (2.5) in Chen et al. (2003) is fulfilled by using their Theorem 3.

**Lemma 3.** Suppose that Assumption 9 holds. Then, \( \int_{0}^\infty \sqrt{\log \mathcal{N}(\epsilon, \mathcal{H}^{de}, \|\cdot\|_p)} d\epsilon < \infty. \)

**Proof of Lemma 3.** Let \( \mathcal{N} = \mathcal{N}(\epsilon_1, \mathcal{H}, \|\cdot\|) \) and \( \{B_k(\epsilon_1)\}_{k=1}^{\infty} \) be the collection of \( \epsilon_1 \)-balls centered at \( \{h_k\}_{k=1}^{\infty} \) satisfying \( \mathcal{H} \subset \bigcup_{k=1}^{\infty} B_k(\epsilon_1) \). We also let \( \mathcal{D} \) be the set of \( d_e \)-tuples over \( \{h_k\}_{k=1}^{\infty} \). Then, for \( \epsilon = d_e^{1/2} \), we can define the collection of \( \epsilon \)-balls, denoted \( \{D_k(\epsilon)\}_{k=1}^{\infty} \), each of which is centered at an element of \( \mathcal{D} \). Note that, for any \( \pi \in \mathcal{H}^{de} \) and \( \ell = 1, \ldots, d_e \), there exists \( h^*_\ell \in \{h_k\}_{k=1}^{\infty} \) such that \( \|e^\ell \pi - h^*_\ell\| \leq \epsilon_1 \) by the definition of \( \{h_k\}_{k=1}^{\infty} \). Therefore, for any \( \pi \in \mathcal{H}^{de} \), there always exists some \( h^* = (h^*_1, \ldots, h^*_d) \) such that \( \|\pi - h^*\| \leq (d_e \max_{1 \leq \ell \leq d_e} \|e^\ell \pi - h^*_\ell\|)^{1/2} \leq \epsilon \). Hence, \( \mathcal{H}^{de} \subset \bigcup_{k=1}^{\infty} D_k(\epsilon) \), and thus, \( \int_{0}^\infty \sqrt{\log \mathcal{N}(\epsilon, \mathcal{H}^{de}, \|\cdot\|)} d\epsilon \leq \int_{0}^\infty \sqrt{d_e \log \mathcal{N}} d\epsilon_1 < \infty \) by Assumption 9.

\[ \square \]
Lemma 4. Suppose that the conditions in Theorem 4 hold. Let \( \tilde{m}_{1,j,\ell}(\theta, g_i) = e'_j \partial m_j(\theta, g_i)/\partial \theta \). Then, \( \tilde{m}_{1,j,\ell}(\theta, g_i) \) satisfies Conditions (3.1) and (3.2) of Chen et al. (2003) for \( j \in \{ M, N \} \) and \( \ell = 1, \ldots, 2d_e \).

Proof of Lemma 4. We will focus on the proof for the case \( j = N \) and the first \( d_e \ell \)'s, since those for the remainder and the RCMLE can be obtained in a similar manner.

For a linear operator \( \Pi_1 : \mathcal{H} \to \mathbb{R}^{d_e} \) (resp. \( \Pi_2 : \mathcal{H} \to \mathbb{R}^{d_e} \)), let \( \pi_{1,\ell} = \Pi_1^* e_\ell \) (resp. \( \pi_{2,\ell} = \Pi_2^* e_\ell \)) and, for each \( s = 1, 2 \), let \( \Phi_s = \Phi(G_s^*(\pi_s)\theta_s) \) and \( \phi_s = \phi(G_s^*(\pi_s)\theta_s) \). By the triangular inequality, we have

\[
|\tilde{m}_{1,N,\ell}(\theta_1, G_1^*(\pi_1)) - \tilde{m}_{1,N,\ell}(\theta_2, G_1^*(\pi_2))| = |(\Phi_1 - \Phi_2)\phi_1||\langle \pi_{1,\ell}, Z_i \rangle| + |(y_i - \Phi_2)(\phi_1 - \phi_2)||\langle \pi_{1,\ell}, Z_i \rangle| + |(y_i - \Phi_2)\phi_2||\langle \pi_{1,\ell} - \pi_{2,\ell}, Z_i \rangle|.
\]

We show the first two terms in (D.10) satisfy Condition (3.1) in Chen et al. (2003) because of the Lipschitz continuities of \( \Phi(\cdot) \) and \( \phi(\cdot) \) and the Cauchy-Schwarz inequality; specifically, we have

\[
|(\Phi_1 - \Phi_2)\phi_1||\langle \pi_{1,\ell}, Z_i \rangle| \leq (\phi(0))^2||\langle \theta_1 - \theta_2||\pi_{1,\ell}|| + \parallel \pi_1 - \pi_2\parallel\parallel \theta_2\parallel||\langle \pi_{1,\ell}, Z_i \rangle|,
\]

and

\[
|(y_i - \Phi_1)(\phi_1 - \phi_2)||\langle \pi_{1,\ell}, Z_i \rangle| \leq 2|\phi'(1)||\langle \theta_1 - \theta_2||\pi_{1,\ell}|| + \parallel \pi_1 - \pi_2\parallel\parallel \theta_2\parallel||\langle \pi_{1,\ell}, Z_i \rangle|.
\]

Hence, both (D.11) and (D.12) are bounded by \( c b_j(\pi_i)||\theta_1 - \theta_2|| + \parallel \pi_1 - \pi_2\parallel \) for some constant \( c > 0 \) and \( b_j(\pi_i) = ||Z_i||^2 \) for \( j = 1, \ldots, d_e \), and thus Condition (3.1) in Chen et al. (2003) is fulfilled.

The last term in (D.10) satisfies Condition (3.2) in Chen et al. (2003), since

\[
E\left[\sup_{(\theta, \Pi) : ||\theta_1 - \theta_2|| < \epsilon, ||\pi_1 - \pi_2|| < \epsilon} \left|\left|(y_i - \Phi_2)\phi_2\right|^2||\langle \pi_{1,\ell} - \pi_{2,\ell}, Z_i \rangle\right|^2\right]^{1/2}\]

\[
\leq \left(E\left[\sup_{(\theta, \Pi) : ||\theta_1 - \theta_2|| < \epsilon, ||\pi_1 - \pi_2|| < \epsilon} \phi^2(0)||\pi_1 - \pi_2||^2 ||Z_i||^2\right]\right)^{1/2} \leq \epsilon \phi^2(0) \left(E[||Z_i||^2]\right)^{1/2},
\]

where the inequality follows from the Cauchy-Schwarz inequality. \( \square \)

Lemma 5 proves Condition (2.6) in Chen et al. (2003).

Lemma 5. Suppose that the conditions for Theorem 4 hold. Then, for \( j \in \{ M, N \} \), as \( n \to \infty \)

\[
\sqrt{n} J_j^{-1/2} S_n^{-1}(M_{jn}(\theta_0, g_i) + \Gamma_{2,0j}(\hat{n} - \pi_n)) \overset{d}{\to} \mathcal{N}(0, I_{2d_e}).
\]

Proof of Lemma 5. We have

\[
\sqrt{n} S_n^{-1}\Gamma_{2,0j}(\hat{n} - \pi_n) = V_j \kappa_n^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i^* \psi_0 Z_i + \sqrt{n} V_j(\kappa_n^{-1} \kappa_n - \mathcal{I}) \Pi_n^* \psi_0,
\]

\[
= V_j \kappa_n^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i^* \psi_0 Z_i + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i^* \psi_0 V_j(\kappa_n^{-1} - \kappa_n^{-1}) Z_i + \sqrt{n} V_j(\kappa_n^{-1} \kappa_n - \mathcal{I}) \Pi_n^* \psi_0,
\]

where \( \sum_{i=1}^{n} V_i^* \psi_0 Z_i/\sqrt{n} = O_p(1) \) by Bosq (2000, Theorem 2.7). The proof consists of three steps. In the first step, we show the last term in (D.13) is \( o_p(1) \). Then, in Step 2, we show the second term in (D.13) is \( o_p(1) \). In the last step, we derive the limit of the first term in (D.13).
Step 1: As discussed in Section 3.2.3, there exists a bounded linear operator $C_j$ such that $V_j = C_j K^{1/2}$, see Baker (1973, Theorem 1), and thus, by Lemma 2, we have

$$
\|V_j(K^{-1}_{na}K_n - \mathcal{I})\sigma_{\psi_0}\|^2 \leq \sum_{\ell=1}^{d_\ell} \|C_j\|_{op}^2 \|K^{1/2}(K^{-1}_{na}K_n - \mathcal{I})\sigma_{\psi_0}\|^2 \|\psi_0\|^2 = o_p(n^{-1}).
$$

Step 2: We then show that $\|V_j(K^{-1}_{na} - K^{-1}_a)n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 Z_i\| = o_p(1)$. Note that $V_j(K^{-1}_{na} - K^{-1}_a)$ can be decomposed into the sum of $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ each of which is defined by

$$
\tilde{A}_1 = C_j K^{1/2}K^{-1}_a (K_n - K)K^{-1}_{na}, \quad \tilde{A}_2 = C_j K^{1/2}K^{-1}_a (K_n - K)K^{-1}_{na}, \quad \tilde{A}_3 = C_j K^{1/2}K^{-1}_a (K_n - K)K^{-1}_{na},
$$

where the subscript $j$ on $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ denoting the RCMLE and RNLS is omitted for the moment for the ease of explanation. We first consider $n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_1 Z_i$. Since $\tilde{A}_1 : \mathcal{H} \rightarrow \mathbb{R}^{2d_e}$, this quantity is $2d_e$-dimensional random vector. Specifically, due to the independence across $i$, $\mathbb{E}[n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_1 Z_i|x] = 0$ and

$$
\mathbb{E}[\|n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_1 Z_i\|^2|x] = \sigma_{\psi_0}^2 \sum_{i=1}^{n} \|\tilde{A}_1 Z_i\|^2 = \sigma_{\psi_0}^2 \sum_{i=1}^{2d_e} \left(n^{-1} \sum_{i=1}^{n} \langle Z_i, \tilde{A}_1 e_i \rangle^2 \right)
$$

$$
= \sigma_{\psi_0}^2 \sum_{i=1}^{2d_e} \|\mathbb{E}[\|K^{-1/2}_{na}(K_n - K)K^{-1}_{na}\|^2] = O_p((n\alpha)^{-1}),
$$

where the last bound follows from the facts that (i) $C_j = \mathbb{E}[m_{ij}(\theta_0, g_i) g_0 g_0']^{1/2}\mathcal{C}_j$ with $\|\mathcal{C}_j\|_{op}$ and $\|\mathbb{E}[m_{ij}(\theta_0, g_i) g_0 g_0']\|_{HS}$ being bounded above by 1 and a generic constant $c$ respectively (see, e.g., Baker (1973) and Assumption 5), (ii) $C_j$ is Hilbert-Schmidt due to (i), and (iii) $\|K_n - K\|_{op} = O_p(n^{-1/2})$ and $\|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{op} = O_p(n^{-1/4})$ (Assumption 8), and (iv) $\|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{op} = O_p((n\alpha)^{-1})$. Therefore, by applying the Markov’s inequality, we have $n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_1 Z_i = o_p(1)$. We then consider $n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_2 Z_i$. Note that $\mathbb{E}[n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_2 Z_i|x] = 0$ and

$$
\mathbb{E}[\|n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_2 Z_i\|^2|x] = \sigma_{\psi_0}^2 \sum_{i=1}^{n} \|\tilde{A}_2 Z_i\|^2 = \sigma_{\psi_0}^2 \|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{HS}^2.
$$

(D.14)

Thus, it suffices to show that $\|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{HS} = o_p(1)$, which is given below.

$$
\|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|^2_{HS} \leq \|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|^2_{op} = O(1), \quad \|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{HS}^2 = O(1).
$$

(D.15)

The second inequality is obtained from the Cauchy-Schwarz inequality, the commutative properties between $K$ and $K^{-1}_a$ and the fact that $\|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{op} = O(1)$ and $\|K^{1/2}_{na}(K_n - K)K^{-1}_{na}\|_{HS}^2 = (K^{-1}_a - \mathcal{I})K^{1/2}_{na}(K_n - K)K^{-1}_{na}/(K^{-1}_a - \mathcal{I})K^{1/2}_{na}.$ The last equality is from Assumption 8 and the arguments similar to those in Carrasco (2012, p.394). Because of (D.15) and the Markov’s inequality, we conclude that $n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_3 Z_i = o_p(1)$ and, by applying similar arguments in proving (D.15), $n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 \tilde{A}_3 Z_i = o_p(1)$. Thus, we have $\|V_j(K^{-1}_{na} - K^{-1}_a)n^{-1/2}\sum_{i=1}^{n} V_i'\psi_0 Z_i\| = o_p(1)$.

Step 3: Let $\mathcal{J}_2, j = \sigma_{\psi_0}^2 V_j K^{1/2}K^{-1}_a V_j^*$ and $\mathcal{J}_j = \mathcal{J}_{1,j} + \mathcal{J}_{2,j}$. Then, the following holds by the results in Steps 1 and 2, the central limit theorem (Bosq 2000, Theorem 2.7), continuous mapping theorem,
and orthogonality between $m_{ij}(\theta_0, g_i)g_{0i}$ and $V_i^t\psi_0 V_j K^{-1}Z_i$.

$$\sqrt{n}J^{-1/2}_{\alpha}S_{n}^{-1}(M_{j\alpha}(\theta_0, g_i) + \Gamma_2(\theta_0, g)(\pi - \pi_n))$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(m_{ij}(\theta_0, g_i)g_{0i} + V_i^t\psi_0 V_j K_{\alpha}^{-1}Z_i) + o_p(1) \overset{d}{\rightarrow} N(0, \mathcal{I}_{2d_e}).$$ (D.16)

The remaining part is to show that $\|J_{\alpha} - J\|_{\text{HS}} = \|J_{2,\alpha} - J_{2,\beta}\|_{\text{HS}} = o_p(1)$. This follows from Theorem 1 of Baker (1973) and the commutative property between $K^{1/2}, K$ and $K_{\alpha}^{-1}$; specifically, we have

$$\|J_{2,\alpha} - J_{2,\beta}\|_{\text{HS}} \leq \|C_{\beta}\|_{\text{HS}}\|K^{2}K_{\alpha}^{-2} - \mathcal{I}\|_{\text{op}} = o(1),$$ (D.17)

where $\mathcal{P}(\ker K)^\perp$ is the projection on $(\ker K)^\perp$. The result follows from Assumption 8.

**Lemma 6.** Suppose that the conditions in Theorem 4 hold. Then, $\|S_{n}^{-1}(\hat{\Gamma}_{1,j} - \Gamma_{1,0j})S_{n}^{-1}\|_{\text{HS}} = o_p(1)$.

**Proof of Lemma 6.**

$$\|S_{n}^{-1}(\hat{\Gamma}_{1,j} - \Gamma_{1,0j})S_{n}^{-1}\|_{\text{HS}} \leq \frac{1}{n}\sum_{i=1}^{n} \phi^2(\tilde{g}_{i,\alpha})S_{n}^{-1}(\tilde{g}_{i,\alpha}g_{i,\alpha} - g_{i,\alpha}g_{i,\alpha}^t)S_{n}^{-1}\|_{\text{HS}}$$

$$+ 2\phi(0)\frac{1}{n}\sum_{i=1}^{n}(\phi(\tilde{g}_{i,\alpha})\phi(\tilde{g}_{i,\alpha}^t))S_{n}^{-1}g_{i,\alpha}g_{i,\alpha}^tS_{n}^{-1}\|_{\text{HS}} + \frac{1}{n}\sum_{i=1}^{n}(\phi^2(\tilde{g}_{i,\alpha}))S_{n}^{-1}g_{i,\alpha}g_{i,\alpha}^tS_{n}^{-1}\|_{\text{HS}}.$$

The first term in the RHS satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\phi^2(\tilde{g}_{i,\alpha})S_{n}^{-1}(\tilde{g}_{i,\alpha}g_{i,\alpha} - g_{i,\alpha}g_{i,\alpha}^t)S_{n}^{-1}\|_{\text{HS}}$$

$$\leq \phi^2(0)\left(n^{-1}\sum_{i=1}^{n}\|S_{n}^{-1}(\tilde{g}_{i,\alpha} - g_{i,\alpha})\|^2 + 2(n^{-1}\sum_{i=1}^{n}\|g_{0i}\|^2)^21/2(n^{-1}\sum_{i=1}^{n}\|S_{n}^{-1}(\tilde{g}_{i,\alpha} - g_{i,\alpha})\|^2)^{1/2}\right),$$

(D.18)

where the inequality follows from the triangular inequality, Hölder’s inequality, and the boundedness of $\phi(\cdot)$. The bounds in (D.1) and (D.2) give that (D.18) = $o_p(1)$. By using similar arguments and the Lipschitz continuity of $\phi(\cdot)$, we also have

$$n^{-1}\sum_{i=1}^{n}\|\phi^2(\tilde{g}_{i,\alpha})S_{n}^{-1}g_{i,\alpha}g_{i,\alpha}^tS_{n}^{-1}\|_{\text{HS}} \leq O_p(1)(n^{-1}\sum_{i=1}^{n}\|g_{0i}\|^2)^{1/2} = o_p(1).$$ (D.19)

Lastly, we find that $E[\phi^2(g_{0i})S_{n}^{-1}g_{i,\alpha}g_{i,\alpha}^tS_{n}^{-1}] = \Gamma_{1,0N}^N$ and $E[\|\phi^2(g_{0i})S_{n}^{-1}g_{i,\alpha}g_{i,\alpha}^tS_{n}^{-1}\|^2_{\text{HS}}] = O(1)$ by Assumption 10.(i), and thus,

$$\frac{1}{n}\sum_{i=1}^{n}S_{n}^{-1}(\phi^2(g_{0i})g_{i,\alpha}g_{i,\alpha} - \Gamma_{1,0N}^N)S_{n}^{-1}\|_{\text{HS}} = o_p(1)$$

(D.20)

by the Markov’s inequality. Hence, Lemma 6 follows from (D.18)-(D.20).

**Lemma 7.** Let $\hat{\mathcal{J}}_{1,j} = \frac{1}{n}\sum_{i=1}^{n}m_{ij}(\hat{\theta}_j, \hat{g}_{i,\alpha})g_{i,\alpha}g_{i,\alpha}^t$. Suppose that the conditions for Theorem 4 hold. Then, $\|S_{n}^{-1}\hat{\mathcal{J}}_{1,j}S_{n}^{-1} - \mathcal{J}_{1,j}\|_{\text{HS}} = o_p(1)$.

The proof of Lemma 7 is similar to that of Lemma 6. Thus, the details are omitted.

**Lemma 8.** Suppose that the conditions for Theorem 4 hold. Then, $\|S_{n}^{-1}\hat{\mathcal{J}}_{2,j}S_{n}^{-1} - \mathcal{J}_{2,0j}\|_{\text{HS}} = o_p(1)$. 

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Proof. Let \( \hat{\sigma}_{\psi_0}^2 = \frac{1}{n} \sum_{i=1}^n (\psi_0'(V_i))^2 \). Under Assumptions 7 and 10, we have \( \mathbb{E}[|\psi_0'(V_i)|^2] = O(1) \), and thus, \( |\hat{\sigma}_{\psi_0}^2 - \sigma_{\psi_0}^2| = o_p(1) \). Theorem 4 implies that \( \|\hat{\theta} - \theta_0\| = O_p(\mu_{m,n}^{-1}) \). Hence,

\[
|\hat{\sigma}_{\psi_0}^2 - \sigma_{\psi_0}^2| \leq \frac{1}{n} \sum_{i=1}^n |(\psi_0 - \hat{\psi}_j)V_i V_i'(\psi_0 - \hat{\psi}_j)| + \frac{1}{n} \sum_{i=1}^n \|\hat{\psi}_j(V_i - \hat{\psi}_i,\alpha)(V_i - \hat{\psi}_i,\alpha)\| \hat{\psi}_j| \\
\leq \|\hat{\psi}_j - \psi_0\| \left( \frac{1}{n} \sum_{i=1}^n \|V_i\|^2 + \frac{1}{n} \sum_{i=1}^n \|V_i - \hat{\psi}_i,\alpha\|^2 \right) \|\psi_0\|^2 + \|\hat{\psi}_j - \psi_0\|^2 \tag{D.21}
\]

where the inequalities follow from the triangular inequality and the Cauchy-Schawrz inequality. Note that, because of (D.1), (D.2) and Theorems 3 and 4, (D.21) is bounded above by \( o_p(1) \).

Analogously, let \( \mathcal{V}_{jn} = \frac{1}{n} \sum_{i=1}^n m_{2j}(\theta_0, g_i)Z_i \otimes g_{0i} \). It satisfies \( \mathbb{E}[|m_{2j}(\theta_0, g_i)Z_i \otimes g_{0i}|^2_{HS}] = O(1) \) by Assumption 5 and thus \( \|\mathcal{V}_{jn} - \mathcal{V}_{j}\|_{HS} = O_p(n^{-1/2}) \) (see, e.g., Hörmann and Kokoszka, 2010, Theorem 3.1). Hence, we have

\[
\|\mathcal{V}_j \mathcal{K}_\alpha^{-1} \mathcal{K}_\alpha^{-1} \mathcal{V}_j^* - \mathcal{V}_{jn} \mathcal{K}_\alpha^{-1} \mathcal{K}_\alpha^{-1} \mathcal{V}_{jn}^*\|_{HS} \\
\leq 2\|\mathcal{C}_j \mathcal{K}_\alpha^{-1/2} \mathcal{K}_\alpha^{-1} \mathcal{C}_j^*\|_{op} \|\mathcal{V}_j - \mathcal{V}_{jn}\|_{HS} + \|\mathcal{K}_\alpha^{-1} \mathcal{K}_\alpha^{-1}\|_{op} \|\mathcal{V}_j - \mathcal{V}_{jn}\|_{HS} = o_p(1/\alpha^{1/2}), \tag{D.22}
\]

by Assumption 10 and Theorem 1 in Baker (1973). The above is \( o_p(1) \) due to \( \mu_{m,n,\alpha}^2 \rightarrow \infty \).

It remains to show that \( \|\mathcal{V}_{jn} \mathcal{K}_\alpha^{-1} \mathcal{K}_\alpha^{-1} \mathcal{V}_j^* - \hat{\mathcal{V}}_{jn} \mathcal{K}_\alpha^{-1} \mathcal{K}_\alpha^{-1} \hat{\mathcal{V}}_{jn}^*\|_{HS} = o_p(1) \). To this end, we let \( \tilde{D}_{j1,\ell} = n^{-1} \sum_{i=1}^n m_{2j}(\theta_0, g_{0i})Z_i(Z_i, (\hat{\Pi}_{\alpha} - \Pi_{\alpha})^* S_n^{-1/2} \varepsilon_\ell), \tilde{D}_{j2,\ell} = n^{-1} \sum_{i=1}^n (m_{2j}(\hat{\theta}_j, \hat{g}_i, \alpha) - m_{2j}(\theta_0, g_{0i}))Z_i(S_n^{-1/2} \varepsilon_\ell) \) and \( \tilde{D}_{j3,\ell} = n^{-1} \sum_{i=1}^n (m_{2j}(\hat{\theta}_j, \hat{g}_i, \alpha) - m_{2j}(\theta_0, g_{0i}))Z_i(Z_i, (\hat{\Pi}_{\alpha} - \Pi_{\alpha})^* S_n^{-1/2} \varepsilon_\ell) \). Then, we have

\[
\hat{\mathcal{V}}_{jn}^* - \mathcal{V}_{jn}^* = \sum_{\ell=1}^{d_\ell} (\tilde{D}_{j1,\ell} + \tilde{D}_{j2,\ell} + \tilde{D}_{j3,\ell}) e_\ell', \tag{D.23}
\]

and thus

\[
\|\mathcal{K}_\alpha^{-1/2} (\hat{\mathcal{V}}_{jn}^* - \mathcal{V}_{jn}^*)\|_{HS}^2 \leq 3 \sum_{\ell=1}^{d_\ell} (\|\mathcal{K}_\alpha^{-1/2} \tilde{D}_{j1,\ell}\|_{HS}^2 + \|\mathcal{K}_\alpha^{-1/2} \tilde{D}_{j2,\ell}\|_{HS}^2 + \|\mathcal{K}_\alpha^{-1/2} \tilde{D}_{j3,\ell}\|_{HS}^2). \tag{D.24}
\]

In the following we will show each term appearing in the RHS of (D.24) is \( o_p(1) \). For each \( \ell \), \( \tilde{D}_{j1,\ell} \) satisfies that

\[
\tilde{D}_{j1,\ell} = n^{-1} \sum_{i=1}^n m_{2j}(\theta_0, g_{0i})Z_i \otimes Z_i((\hat{\Pi}_{\alpha} - \Pi_{\alpha})^* S_n^{-1/2} \varepsilon_\ell) \\
= (\mathbb{E}[m_{2j}(\theta_0, g_{0i})Z_i \otimes Z_i] + O_p(n^{-1/2}))o_p(1), \tag{D.25}
\]

where the last line is a consequence of the central limit theorem in Bosq (2000, Theorem 2.7), the arguments in proving (D.1)-(D.2) and Baker (1973, Theorem 1). Due to \( \mathbb{E}[m_{2j}(\theta_0, g_{0i})Z_i \otimes Z_i] = \mathcal{K}_{1/2}^{1/2} \tilde{D}_{1j} \) for a bounded linear operator \( \tilde{D}_{1j} \) (Baker 1973, Theorem 1) and \( n^{-1/2} \alpha^{-1/2} = o(1) \), (D.25) implies that

\[
\|\mathcal{K}_\alpha^{-1/2} \tilde{D}_{j1,\ell}\|_{HS} = o_p(1). \tag{D.26}
\]

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Moreover, the mean-value theorem gives that
\[
\tilde{D}_{j,\ell} = n^{-1} \sum_{i=1}^{n} \tilde{m}^{(1)}_{2i} g_0 e_i \ell Z_i g_i(\hat{\theta}_j - \theta_0) + n^{-1} \sum_{i=1}^{n} \tilde{m}^{(1)}_{2i} g_0 e_i \ell Z_i(\hat{g}_i, - g_0) \hat{\theta}_j
\]
\[
= \mathbb{E}[\tilde{m}^{(1)}_{2j,0} g_0 e_i \ell Z_i g_i(\hat{\theta}_j - \theta_0)] + \mathbb{E}[\tilde{m}^{(1)}_{2j,0} g_0 e_i \ell Z_i \otimes Z_i](\mathcal{S}_n^{-1}(\hat{\Pi}_n - \Pi_n))^* S'_n \hat{\theta}_j + O_p(n^{-1/2})
\]
\[
= \mathcal{K}^{1/2} \tilde{D}_{j,\ell} O_p(\mu_{m,n}^{-1}) + \mathcal{K}^{1/2} \tilde{D}_{j,\ell} o_p(1) + O_p(n^{-1/2}),
\]  
(D.27)
where \(\tilde{m}^{(1)}_{2j}\) denotes the first derivative of \(\tilde{m}_{2j}(\cdot, \cdot)\) evaluated at \((\hat{\theta}_j, \hat{g}_i) \in [\hat{\theta}_j, \theta_0] \times [\hat{g}_i, g_0]\) and \(\tilde{m}^{(1)}_{2j,0} = \tilde{m}^{(1)}_{2j}(\theta_0, g_0) + o(1)\) and with some bounded linear operators \(\tilde{D}_{j,\ell}\) and \(\tilde{D}_{j,\ell}\) (Baker 1973; Theorem 1).
Therefore, we have
\[
\|\mathcal{K}_n^{-1/2} \tilde{D}_{j,\ell} \|_{\text{HS}} = o_p(1).
\] (D.28)
Before moving to the last term, \(\|\mathcal{S}_n^{-1}(\hat{\Pi}_n - \Pi_n)\|_{\text{op}}\) is \(o_p(1)\) and the rate of \(n^{-1} \sum_{i=1}^{n} \|\mathcal{S}_n^{-1}(\hat{g}_i, - g_i)\|^2\) is corrected to \(O_p(\alpha^2) + O_p(1/(\mu_{m,n}^{2}n^{-1/2})) = O_p(\alpha^2)\) under the employed conditions. Then, by using the Lipschitz continuity of \(\tilde{m}_{2j}(\cdot, \cdot)\) and the Cauchy-Schwarz inequality, we find that
\[
\|\mathcal{K}_n^{-1/2} \tilde{D}_{j,\ell} \|_{\text{HS}} \leq O_p(\alpha^{-1/4}) \|\hat{\theta}_j - \theta_0\| n^{-1} \sum_{i=1}^{n} \|\mathcal{S}_n^{-1}(\hat{g}_i, - g_i)\| \|\hat{g}_i, - g_i\| \|Z_i\|
\]
\[
+ O_p(\alpha^{-1/4}) \|\hat{\Pi}_n - \Pi_n\| n^{-1} \sum_{i=1}^{n} \|\mathcal{S}_n^{-1}(\hat{g}_i, - g_i)\| \|Z_i\| = o_p(1),
\] (D.29)
because of \(\mu_{m,n}^{2} \to \infty\). (D.24), (D.26), (D.28) and (D.29) give that
\[
\|\mathcal{K}_n^{-1/2} (\hat{V}_j - V_j)\|_{\text{HS}}^2 = o_p(1).
\] (D.30)
To conclude the proof, note that
\[
\|\hat{V}_j \mathcal{K}_n^{-1} \mathcal{K}_n \mathcal{K}_n^{-1} \hat{V}_j - \mathcal{V}_j \mathcal{K}_n^{-1} \mathcal{K}_n \mathcal{K}_n^{-1} \mathcal{V}_j \|_{\text{HS}}
\]
\[
\leq 2\|\mathcal{K}_n^{-1/2} (\hat{V}_j - V_j)\|_{\text{HS}} \|\mathcal{K}_n^{-1/2} \mathcal{K}_n \mathcal{K}_n^{-1} \mathcal{V}_j\|_{\text{HS}} + \|\mathcal{K}_n^{-1/2} (\hat{V}_j - V_j)\|_{\text{HS}} \|\mathcal{K}_n^{-1/2} \mathcal{K}_n \mathcal{K}_n^{-1/2} \|_{\text{op}}
\]
\[
+ \|\mathcal{V}_j \|\mathcal{K}_n^{-1} \mathcal{K}_n \mathcal{K}_n^{-1} \mathcal{K}_n^{-1} \mathcal{K}_n^{-1} \|_{\text{op}} \|\mathcal{V}_j \|_{\text{HS}}
\]
\[
\leq O_p(1) + \|\mathcal{C}_j \|_{\text{HS}} \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1} \mathcal{K}_n^{-1} \mathcal{K}_n^{-1} \|_{\text{op}} \|\mathcal{V}_j \|_{\text{op}}
\]
\[
\leq O_p(1) + O_p(1) \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}} \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}}
\]
\[
+ O_p(1) \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}} \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}}
\]
\[
\leq O_p(1) + O_p(1) \left( \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}} + \|\mathcal{K}_n^{-1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}} \right) \leq o_p(1),
\] (D.31)
where the first two inequalities are respectively follow from the triangular inequality and (D.30) and the boundedness of \(\|\mathcal{K}_n^{-1/2} \mathcal{V}_j\|_{\text{HS}}\) due to Baker (1973, Theorem 1). The remaining inequalities are obtained from \(\|\mathcal{K}_n\|_{\text{op}} = O_p(\alpha^{-1/2})\), \(\|\mathcal{K}_n - \mathcal{K}\|_{\text{HS}} = O_p(n^{-1/2})\), \(\|\mathcal{K}_n^{1/2} \mathcal{K}_n^{-1/2} \|_{\text{op}} \leq 1\) and the fact that
\[
\|A^* A - B^* B\|_{\text{op}} \leq \|A - B\|_{\text{op}} (\|A\|_{\text{op}} + \|B\|_{\text{op}}) + \|A - B\|_{\text{op}}^2.
\]
The last bound can be obtained by using arguments similar to those in the proof of Lemma 3.(i) in Carrasco (2012).
Appendix E: Proofs of the results in Section 3.4

We employ the approach in Donald and Newey (2001) to prove Proposition 1. For the subsequent discussions, we let \( \sum_i \) denote \( \sum_{i=1}^n \) whenever it is convenient. In addition, to simplify discussions, we let \( m_{1,0}, \tilde{m}_{2i,0} \) and \( \tilde{m}_{2i,0} \) respectively denote \( m_{ij}(\theta_0, G_i(\pi_n)) \), \( \tilde{m}_{j2}(\theta_0, G_i(\pi_n)) \) and \( \tilde{m}_{j2}(\theta_0, G_i(\pi_n)) \), regardless of \( j \in \{M, N\} \). The notation \( \tilde{m}_{2i,0} \) will be used to indicate \( \mathbb{E}[\tilde{m}_{2i,0}|\mathcal{G}] \) where \( \mathcal{G} \) is the \( \sigma \)-field generated by \( \{(x_i, V_i)_{i=1}^n\} \) and \( \tilde{\mathbb{M}}_{2,0} = \text{diag}(\tilde{m}_{21,0}, \ldots, \tilde{m}_{2n,0}) \). In addition, \( m_{1,0} \) and \( g_{0L} \) are \( n \times 1 \) vectors respectively given by \( (m_{11,0}, \ldots, m_{1n,0})' \) and \( (g_{01,\ell}, \ldots, g_{0n,\ell})' \), where \( g_{0i,\ell} \) is the \( \ell \)th row of \( \{g_{0i}\}_{i=1}^n \). For a matrix \( A \), its maximum eigenvalue and \( (i, j) \)th element will be respectively denoted by \( \lambda_{\text{max}}(A) \) and \( [A]_{ij} \).

We hereby introduce the operations \( T_n : \mathcal{H} \rightarrow \mathbb{R}^n \) and \( T_n^* : \mathbb{R}^n \rightarrow \mathcal{H} \) such that

\[
T_n h = (\langle Z_1, h \rangle, \ldots, \langle Z_n, h \rangle)', \quad T_n^* \nu = n^{-1} \sum_{i=1}^n \nu_i Z_i,
\]

for any \( h \in \mathcal{H} \) and \( \nu \in \mathbb{R}^n \). Then, the operations satisfy that \( T_n \hat{\varphi}_j = \hat{\varphi}_j^{1/2} \hat{\varphi}_j \) and \( T_n^* \varphi_j = \hat{\varphi}_j^{1/2} \varphi_j \) for \( j = 1, \ldots, n \). For notational simplicity, we use \( \hat{q}_j \) to indicate \( q, \alpha \) and, by using the notation, let \( P_{\alpha} = \sum_j \hat{q}_j \hat{\kappa}_j^{-1}(T_n \hat{\varphi}_j) \otimes (T_n \varphi_j) = \sum_j \hat{q}_j \hat{\varphi}_j \otimes \varphi_j = T_n \kappa_{\alpha}^{-1} T_n^* \) with the tensor product satisfying that \( (\hat{\varphi}_j \otimes \varphi_j) \nu = \langle \hat{\varphi}_j, \nu \rangle_n \hat{\varphi}_j = (\hat{\varphi}_j' \nu / n) \hat{\varphi}_j \) for any \( \nu \in \mathbb{R}^n \). To simplify the notation, we use \( v_{\ell, \nu}, v_{\psi, 0} \) and \( \pi_{\psi, 0} \) to denote \( [V]_{\ell, \nu}, V' \psi_0 \) and \( \psi_0 \pi_{\psi, 0} \) respectively. The following property will be used sometimes to facilitate discussions: for any \( h \in \mathcal{H} \),

\[
T_n (\kappa_{\alpha}^{-1} \kappa_n - I) = (P_{\alpha} - I) T_n h.
\]

Lastly, we let

\[
\Delta_1 = \sum_{\ell, \ell' = 1}^{2d_e} \frac{(T_n \pi_{\psi, 0})'(P_{\alpha} - I) \tilde{\mathbb{M}}_{2,0} g_{0,\ell} g_{0,\ell'}' \tilde{\mathbb{M}}_{2,0} (P_{\alpha} - I) T_n \pi_{\psi, 0}}{n^2} e_{\ell, \ell'} e_{\ell, \ell'},
\]

and

\[
\Delta_2 = \sum_{\ell = 1}^{2d_e} \frac{(T_n \pi_{\psi, 0})'(P_{\alpha} - I)^2 T_n \pi_{\psi, 0}}{n},
\]

Note that

\[
\mathbb{E}[\text{tr}(\Delta_1)|x] \leq \Delta_2 \sum_{\ell = 1}^{2d_e} [\mathbb{E}[\tilde{m}_{20, \ell} x | x f_{\ell, \ell'}^2 (1_{\ell \leq d_e}) + \mathbb{E}[\tilde{m}_{20, \ell} v_{\ell, \nu} v_{\ell, \nu'} | x] 1_{\ell > d_e}] = O_p(\Delta_2).
\]

Lemma 9. Suppose that the conditions in Proposition 1 hold. Then, the following holds for \( \ell, \ell' = 1, \ldots, 2d_e \).

(a) \( \sum_i [P_{\alpha}]_{ii} = O_p(\alpha^{-1/2}) \), \( \sum_i [P_{\alpha}^2]_{ii} = O_p(\sum_i [P_{\alpha}]_{ii}) \) and \( \sum_i [P_{\alpha}]_{ii}^2 = O_p(\sum_i P_{ii}) \).

(b) \( \Delta_2 = O_p(\alpha^2) \).

(c) \( \mathbb{E}[(g_{0,\ell} \tilde{\mathbb{M}}_{2,0} P_{\alpha} v_{\psi, 0})(g_{0,\ell'} \tilde{\mathbb{M}}_{2,0} P_{\alpha} v_{\psi, 0})|x] = \sigma_{\ell, \ell'}^2 \sum_j \sum_{i \neq j} [P_{\alpha}]_{ij} [P_{\alpha}]_{ji} \mathbb{E}[\tilde{m}_{20, \ell} g_{0, i, \ell'} | x] \mathbb{E}[\tilde{m}_{20, s} g_{0, s, \ell'} | x] + O_p(\alpha^{-1}) = O_p(n \alpha^{-1/2}) \).

(d) \( \mathbb{E}[(T_n \pi_{\psi, 0})'(P_{\alpha} - I) \tilde{\mathbb{M}}_{2,0} g_{0,\ell} g_{0,\ell'}' \tilde{\mathbb{M}}_{2,0} P_{\alpha} v_{\psi, 0}/n^2 | x] = O_p(\Delta_2^{1/2}/(n \alpha^{-1/4})) \).
(e) \( n^{-1}\mathbb{E}[g_0' m_1 m_0' [0] [x] = O_p(n^{-1}) \) and \( \mathbb{E}[g_0' m_1 m_0' (e, Irr \pi_1) T_0 \pi_0 [x] = O_p(n^{1/2}) \).

(f) Let \( n \times 1 \) vector consisting of an iid random variable \( \xi_i, \ldots, \xi_n \) satisfying \( \mathbb{E}[\xi_i | G] = 0 \), \( \lambda_{max}(\mathbb{E}[\xi_i G]) \leq \sigma \xi_i \) for some positive constant \( \sigma \). Then, \( \xi_i' P_{0,n} v_\ell = O_p(\alpha^{-1/4}) \) and \( \mathbb{E}[(\xi_i P_{0,n} v_\ell)(\xi_i' P_{0,n} v_\ell)][x] = O_p(\alpha^{-1/2}) \) and \( n^{-1/2} \xi_i' (P_{0,n} - \mathcal{I}) T_0 \pi_0 [x] = O_p(\Delta_{2}^{-1/2}) \).

(g) For \( \xi_i \) satisfying the conditions in (f), \( \sum_{i,j} \xi_i v_{ij} \pi_j v_{j\ell} = O_p(\alpha^{-1/4}) \) for \( \ell, \ell' = 1, \ldots, d_x \).

(h) \( \mathbb{E}[g_0' m_1 m_0' (P_{0,n} \pi_0 | x)] \), \( \mathbb{E}[g_0' m_1 m_0' (P_{0,n} - I) T_0 \pi_0 [x] \) and \( \mathbb{E}[m_1' m_0 g_0' m_0' (P_{0,n} - I) T_0 \pi_0 [x] \) are all 0.

Proof. (a): \( \sum_{i} [P_{0,n}]_{ii} = \sum_{i,j=1}^{n} \bar{g}_{ij} (\bar{\omega}_{ij}, e_i) n \bar{\omega}_{ij} e_i = \sum_{j} \bar{g}_{ij} \left( n^{-1} \bar{\omega}_{ij} \bar{\omega}_{ij} \right) = \sum_{j} \bar{g}_{ij}, \) due to the orthonormality of \( \bar{\omega}_{ij} \). Because \( ||Z_i||^2 \) is finite, \( \sum_{j} \bar{g}_{ij} < \infty \), from which we have \( \sum_{j} \bar{g}_{ij} \leq \max_j (\bar{g}_{ij} \bar{g}_{ij}^{-1}) \sum_j \bar{g}_{ij} = O_p(\alpha^{-1/2}) \) due to Assumption 8. The second part is given from the fact that \( P_{0,n}^2 = \sum_j \bar{g}_{ij}^2 \bar{\omega}_{ij} \otimes \bar{\omega}_{ij} \) and \( \bar{g}_{ij} \in [0, 1] \). Lastly, \( \sum_i [P_{0,n}]_{ii} = \sum_i e_i' P_{0,n} e_i e_i' P_{0,n} e_i \leq \sum_i [P_{0,n}]_{ii} \leq \sum_i \bar{g}_{ij}, \) due to \( \lambda_{max}(e_i e_i') = 1 \) and \( (\bar{\omega}_{ij} e_i) ^2 / n \leq 1 \).

(b) The following is deduced from arguments similar in (D.1) and Assumption 8.

\[
\Delta_2 = \sum_{\ell=1}^{d_x} \sum_{j=1}^{n} (\bar{g}_{ij} - 1)^2 (\bar{\omega}_{ij}, T_0 \pi_0 | x) = \sup_{\ell} \frac{\bar{g}_{j\ell}}{\bar{\omega}_{ij} \pi_0}^2 = O_p(\alpha^2).
\]

(c): \( g_0' \bar{m}_{2,0} P_{0,n} \pi_0 = \sum_{i,j} \bar{m}_{20,j}[P_{0,n}]_{ij} V_j' \pi_0 = \sum_{j} \bar{m}_{20,j}[P_{0,n}]_{jj} g_0[j] V_j' \pi_0 \) and \( \sum_{j} V_j' \pi_0 L_{-j, \ell} = h_{1,\ell} + h_{2,\ell} \) with \( L_{-j, \ell} \) being \( \sum_{i \neq j} \bar{m}_{20,i}[P_{0,n}]_{ji} g_0[j] V_j' \pi_0 \) and \( V_j' \pi_0 \).

We first consider \( \mathbb{E}[h_{1,\ell} h_{1,\ell} | x] \) as follows:

\[
\mathbb{E}[h_{1,j} h_{1,\ell} | x] = \sum_{j,k} \mathbb{E}[\bar{m}_{20,j}[P_{0,n}]_{jj} g_0[j] V_j' \pi_0 \bar{m}_{20,k}[P_{0,n}]_{kk} g_0[k] V_k' \pi_0 | x]
\]

\[
\begin{cases}
\sum_{j,k} \mathbb{E}[\bar{m}_{20,j} V_j' \pi_0 \bar{m}_{20,k} V_k' \pi_0 | x] \quad \text{if } 1 \leq \ell, \ell' \leq d_x, \\
\sum_{j,k} \mathbb{E}[\bar{m}_{20,j} V_j' \pi_0 \bar{m}_{20,k} V_k' \pi_0 | x] \quad \text{if } 1 \leq \ell \leq d_x < \ell' \leq 2d_x, \\
\sum_{j,k} \mathbb{E}[\bar{m}_{20,j} V_j' \pi_0 \bar{m}_{20,k} V_k' \pi_0 | x] \quad \text{if } 2d_x < \ell, \ell' \leq n
\end{cases}
\]

where the terms in the RHS are all bounded above by \( O(1) \sum_{j,k} [P_{0,n}]_{jj} [P_{0,n}]_{kk} = O_p(\alpha^{-1}) \).

We then focus on \( \mathbb{E}[h_{2,j} h_{2,\ell} | x] \) that can be decomposed into \( \sum_j \mathbb{E}[(V_j' \pi_0)^2 L_{-j,\ell} L_{-j,\ell} | x] \) and \( \sum_{j \neq k} \mathbb{E}[(V_j' \pi_0)(V_k' \pi_0)L_{-j,\ell} L_{-j,\ell} | x] \).

We will show that

1. \( \sum_j \mathbb{E}[(V_j' \pi_0)^2 L_{-j,\ell} L_{-j,\ell} | x] = O_p(n \alpha^{-1/2}), \)
2. \( \sum_{j \neq k} \mathbb{E}[(V_j' \pi_0)(V_k' \pi_0)L_{-j,\ell} L_{-j,\ell} | x] = O_p(\alpha^{-2}), \)

from which the desired result will be given.

Proof of (c)i:

\[
\sum_j \mathbb{E}[(V_j' \pi_0)^2 L_{-j,\ell} L_{-j,\ell} | x] = O_p(n \alpha^{-1/2}).
\]
where $\sigma^2_{\psi_0} = \mathbb{E}[(V'_j \psi_0)^2 | x]$, $h_{21, \ell \ell'} = \sum_j \sum_{i \neq j} [P_{\alpha}]_{ij} [P_{\alpha}]_{ji} \mathbb{E}[\tilde{m}^2_{\alpha, i, j, \ell \ell'} | x]$ and $h_{22, \ell \ell'} = \sum_j \sum_{i \neq j} \sum_{s \neq i, j} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} \mathbb{E}[\tilde{m}^2_{\alpha, i, j, s, \ell \ell'} | x]$. If $\ell, \ell' \leq d_e$, the term $h_{21, \ell \ell'}$ satisfies

$$h_{21, \ell \ell'} \leq \mathbb{E}[\tilde{m}^2_{\alpha, i, j, \ell \ell'} | x] \sum_j \left( \sum_{i \neq j} f_{i, \ell} [P_{\alpha}]_{ij} \right)^{1/2} \left( \sum_{i \neq j} f_{i, \ell'} [P_{\alpha}]_{ij} \right)^{1/2} \leq O_p \left( \sum_j [P^2_{\alpha}]_{jj} \right), \tag{E.4}$$

where the second inequality follows from the boundedness of $f_i$ and the fact that $\sum_{i \neq j} [P_{\alpha}]^2_{ij} \leq \sum_i [P_{\alpha}]_{ji} [P_{\alpha}]_{ij} = [P^2_{\alpha}]_{jj}$. Similarly, if $1 \leq \ell \leq d_e < \ell' \leq 2d_e$,

$$h_{21, \ell \ell'} = \mathbb{E}[\tilde{m}^2_{\alpha, i, j, \ell \ell'} | x] \sum_j \sum_{i \neq j} [P_{\alpha}]_{ij} [P_{\alpha}]_{ji} f_{i, \ell} \leq O(1) \sum_j [P^2_{\alpha}]_{jj} = O_p(\alpha^{-1/2}). \tag{E.5}$$

Lastly, if $d_e < \ell, \ell' \leq 2d_e$, $h_{21, \ell \ell'}$ satisfies that

$$h_{21, \ell \ell'} = \mathbb{E}[\tilde{m}^2_{\alpha, i, j, \ell \ell'} | x] \sum_j \sum_{i \neq j} [P_{\alpha}]_{ij} [P_{\alpha}]_{ji} \leq O_p \left( \sum_i [P^2_{\alpha}]_{ii} \right). \tag{E.6}$$

From (E.4) to (E.6), $h_{21, \ell \ell'} = O(\alpha^{-1/2})$.

Meanwhile, $h_{22, \ell \ell'}$ in (E.3) is given as follows.

$$h_{22, \ell \ell'} = \begin{cases} 
\mathbb{E}[\tilde{m}_{20, j}^2 \sum_j \sum_{i \neq j} \sum_{s \neq i, j} f_{i, s} f_{i, \ell} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} ] & \text{if } 1 \leq \ell, \ell' \leq d_e, \\
\mathbb{E}[\tilde{m}_{20, j}^2 \sum_j \sum_{i \neq j} \sum_{s \neq i, j} f_{i, s} f_{i, \ell} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} ] & \text{if } 1 \leq \ell, \ell' \leq d_e < \ell', \ell' \leq 2d_e, \\
\mathbb{E}[\tilde{m}_{20, j}^2 \sum_j \sum_{i \neq j} \sum_{s \neq i, j} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} ] & \text{if } 2d_e \leq \ell, \ell' \leq n \end{cases} \tag{E.7}$$

The term appearing in the last line is bounded above as follows,

$$\sum_j \sum_{i \neq j} \sum_{s \neq i, j} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} \leq \sum_j \left( \sum_{i \neq j} [P_{\alpha}]^2_{ij} \right)^2 + O_p(\alpha^{-1/2}) \leq n \sum_{i, j} [P_{\alpha}]^2_{ij} + O_p(\alpha^{-1/2}) = O_p(n\alpha^{-1/2}). \tag{E.8}$$

By using similar arguments and the boundedness of $f_i$, we find that $\sum_j \sum_{i \neq j} \sum_{s \neq i, j} f_{i, s} f_{i, \ell} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} = O_p(n\alpha^{-1/2})$. We then focus on the second line in (E.7). Because $|a - b| \leq |a| + |b|$ and $\sum_i f_{i, \ell} ([P^2_{\alpha}]_{ii} - [P_{\alpha}]_{ii}) = O_p(\alpha^{-1/2})$, we have

$$\left| \sum_j \sum_{i \neq j} \sum_{s \neq i, j} f_{i, \ell} [P_{\alpha}]_{ij} [P_{\alpha}]_{js} \right| \leq \left| \sum_j \sum_{i \neq j} f_{i, \ell} [P_{\alpha}]_{ij} \left( \sum_j [P_{\alpha}]_{js} \right) + O_p(\alpha^{-1/2}) \right|$$

$$\leq \sum_j \left( \sum_{i \neq j} f_{i, \ell}^2 [P_{\alpha}]^2_{ij} \right)^{1/2} \left( \sum_j [P_{\alpha}]^2_{js} \right)^{1/2} + O_p(\alpha^{-1/2})$$

$$\leq O_p(n) \sum_{i, j} [P_{\alpha}]^2_{ij} + O_p(\alpha^{-1/2}) = O_p(n\alpha^{-1/2}), \tag{E.9}$$

where the last two inequalities follow from Hölder’s inequality and the boundedness of $f_i$. From (E.8) to (E.9), we find that $h_{22, \ell \ell'} = O_p(n\alpha^{-1/2})$ for all $\ell, \ell'$, and thus, by combining with the bound of $h_{21, \ell \ell'}$, (E.3) is bounded above by $O_p(n\alpha^{-1/2})$.

**Proof of (c).ii:** Let $L_{-i, -j, \ell \ell'}$ being defined without $i$th and $j$th elements. Because $\mathbb{E}[V_j | x] = 0$,

$$\sum_j \sum_{s \neq j} \mathbb{E}[V'_j \psi_0 V'_s \psi_0 L_{-s, j, \ell \ell'} | x] = 0 \quad \text{and} \quad \sum_j \sum_{s \neq j} \mathbb{E}[V'_j \psi_0 V'_s \psi_0 L_{-s, j, \ell \ell'} | x] = 0.$$
From the above, we find that
\[
\sum_j \sum_{s \neq j} \mathbb{E}[V_j' \psi_0 V_s' \psi_0 L_{j,s} L_{-s,j} | x] = \sum_j \sum_{s \neq j} \mathbb{E}[V_j' \psi_0 \bar{m}_{20,j} | \mathcal{P}_{\mathcal{N}_0}]_{jj} \mathbb{E}[V_s' \psi_0 \bar{m}_{20,s} | \mathcal{P}_{\mathcal{N}_0}]_{ss} g_{0s} \tilde{\ell} | x]
\]
\[
= \begin{cases}
\sum_j \sum_{s \neq j} \mathbb{E}[V_j' \psi_0 \bar{m}_{20,j} | \mathcal{P}_{\mathcal{N}_0}]_{jj} \mathbb{E}[V_s' \psi_0 \bar{m}_{20,s} | \mathcal{P}_{\mathcal{N}_0}]_{ss} & \text{if } 1 \leq \ell, \ell' \leq d_e,
\sum_j \sum_{s \neq j} \mathbb{E}[V_j' \psi_0 \bar{m}_{20,j} | \mathcal{P}_{\mathcal{N}_0}]_{jj} \mathbb{E}[V_s' \psi_0 \bar{m}_{20,s} | \mathcal{P}_{\mathcal{N}_0}]_{ss} & \text{if } 1 \leq \ell \leq d_e < \ell' \leq 2d_e,
\sum_j \sum_{s \neq j} \mathbb{E}[V_j' \psi_0 \bar{m}_{20,j} | \mathcal{P}_{\mathcal{N}_0}]_{jj} \mathbb{E}[V_s' \psi_0 \bar{m}_{20,s} | \mathcal{P}_{\mathcal{N}_0}]_{ss} & \text{if } 2d_e < \ell, \ell' \leq n,
\end{cases}
\]

where \( \bar{\ell} = \ell - d_e \) and \( \bar{\ell}' = \ell' - d_e \). Given the boundedness of \( \bar{m}_{20,j} \) and \( \mathbb{E}[V_j V_j' | x] \), the terms appearing in the RHS are all bounded above by \( \sum_j \sum_{s \neq j} \mathbb{E}[V_j' \psi_0 | x] \mathbb{E}[V_s' \psi_0 | x] \mathcal{P}_{\mathcal{N}_0} = O_p(\alpha^{-1}) \), where the desired result is obtained.

(d): Let \( \gamma_i, \psi_0 \) and \( \mathcal{Q}_{\mathcal{N}_0} \) denote \( \{T_n \pi_{\psi_0,i} | \mathcal{I} \} \) and \( \mathcal{P}_{\mathcal{N}_0} = \mathcal{I} \) respectively. Then,
\[
\mathbb{E}\left[ (T_n \pi_{\psi_0})^T (\mathcal{P}_{\mathcal{N}_0} - \mathcal{I}) M_{20,j} g_{00} g_{0'} M_{20,j} \mathcal{P}_{\mathcal{N}_0} \psi_0 | x \right]
= \sum_{i,j} \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} \mathbb{E}[\mathcal{P}_{\mathcal{N}_0}]_{jj} t_{1j, \ell' t} + 2 \sum_{i,j} \sum_{k \neq j} \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} \mathbb{E}[\mathcal{P}_{\mathcal{N}_0}]_{jk} t_{2j, \ell' t} t_{3k, \ell'},
\]
where \( t_{1j, \ell'} = \mathbb{E}[\bar{m}_{20,j} g_{00}, g_{0'} V_j' \psi_0 | x] \), \( t_{2j, \ell'} = \mathbb{E}[\bar{m}_{20,j} g_{00}, g_{0'} V_j' \psi_0 | x] \) and \( t_{3j, \ell'} = \mathbb{E}[\bar{m}_{20,j} g_{00}, g_{0'} | x] \). The quantities are bounded regardless of \( \ell \) and \( \ell' \) under the employed assumptions. Hence, we have
\[
\sum_{i,j} \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} \mathbb{E}[\mathcal{P}_{\mathcal{N}_0}]_{jj} t_{1j, \ell' t} \leq \left( \sum_j [\mathcal{P}_{\mathcal{N}_0}]_{jj}^2 t_{1j, \ell' t}^2 \right)^{1/2} \left( \sum_i \sum_{\ell} \gamma_i, \psi_0 \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} [\mathcal{Q}_{\mathcal{N}_0}]_{\ell j} \right)^{1/2}
\leq O_p(\alpha^{-1/4}) \sum_{i,\ell} \sum_{\ell} \gamma_i, \psi_0 \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} \leq 2 = O_p(\alpha^{-1/4} n^{1/2} \Delta_2^{1/2}),
\]
where the last bound follows from the definition of \( \Delta_2 \). Lastly, we have
\[
\sum_{i,j} \sum_{k \neq j} \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} \mathbb{E}[\mathcal{P}_{\mathcal{N}_0}]_{jk} t_{2j, \ell' t} t_{3k, \ell'} \leq \left( \sum_j [\mathcal{P}_{\mathcal{N}_0}]_{jk} t_{2j, \ell' t} t_{3k, \ell'}^2 \right)^{1/2} \left( \sum_i \gamma_i, \psi_0 [\mathcal{Q}_{\mathcal{N}_0}]_{ij} \right)^{1/2}
\leq \left( \sum_j [\mathcal{P}_{\mathcal{N}_0}]_{jk} t_{2j, \ell' t} \sum_{k \neq j} t_{3k, \ell'}^2 \right)^{1/2} (n \Delta_2)^{1/2}
\leq (O(n) \sum_{j} [\mathcal{P}_{\mathcal{N}_0}]_{jk}^2)^{1/2} (n \Delta_2)^{1/2} \leq O_p(\alpha^{-1/4} n^{1/2} \Delta_2^{1/2}).
\]

(e): The first part follows from the facts that \( \mathbb{E}[m_{1,0} m_{1,0}' \mathcal{G}] = \text{diag}(\mathbb{E}[m_{2,0}^2 \mathcal{G}], \ldots, \mathbb{E}[m_{2,0}^2 \mathcal{G}]) \) and \( m_{1,0}^2 \) is almost surely bounded for all \( n \). The second is deduced from the independence across \( i \); specifically,
\[
\mathbb{E}[g_{0i, \ell} m_{1,0} m_{1,0}' \mathcal{P}_n \psi_0' | x] = \sum_i \mathbb{E}[m_{1,0}^2 \mathcal{G}] V_{\ell} g_{0i, \ell} | x] [\mathcal{P}_n]_{ii} = O_p(\alpha^{-1/2}).
\]
To obtain the last part, let \( \mathcal{T}_4, \ell \) be the \( n \times 1 \) vector consisting of \( \{E[g_{0i, \ell} \mathbb{E}[m_{1,0} m_{1,0}^2 \mathcal{G}] | x] \}_{i=1}^n \). Then,
\[
\mathbb{E}[g_{0i, \ell} m_{1,0} m_{1,0}' (\mathcal{P}_n - \mathcal{I}) T_n \pi_{00} | x] = \sum_{i,j,k} \mathbb{E}[g_{0i, \ell} \mathbb{E}[m_{1,0} m_{1,0}^2 \mathcal{G}] | x] [\mathcal{Q}_{\mathcal{N}_0}]_{jk} \langle Z_k, \pi_{00} \rangle
\leq (\mathcal{T}_4, \ell) (\mathcal{T}_4, \ell)' (\mathcal{P}_n - \mathcal{I})^2 T_n \pi_{00}^{1/2} = O_p(n \Delta_2^{1/2}).
\]

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(f): The first part follows from the Markov’s inequality and the fact that $E[\xi'P_{na}v_i|x] = 0$ and
\[
E[(\xi'P_{na}v_i)(\xi'P_{na}v_{i'})|x] = E[v'P_{na}E[\xi'|\mathcal{G}]P_{na}v_{i'}|x] \leq c_\xi tr(P_{na}E[v_i v_{i'}|x]P_{na}) = O_p(\alpha^{-1/2}),
\]
for $\ell, \ell' = 1, \ldots, d_e$. Analogously, the second part is obtained from $E[\xi'(P_{na} - I)T_n\pi_0|x] = 0$ and
\[
E[n^{-1}(\xi'(P_{na} - I)T_n\pi_0)^2|x] = n^{-1}(T_n\pi_0)^2(P_{na} - I)E[\xi'|x](P_{na} - I)T_n\pi_0 \leq c_\xi \Delta_2,
\]
since $E[\xi'|x] = \text{diag}(E[\xi_2|^2|x], \ldots, E[\xi_n|^2|x])$ which is bounded above by $c_\xi I_n$.

(g): Due to the law of iterated expectation, $E[\sum_{i,j} \xi_i v_i [P_{na}]_{ij} v_j|x] = 0$ for all $\ell$ and $\ell'$. Let $\tilde{L}_{-i, \ell'} = \sum_{j \neq i} [P_{na}]_{ij} v_j|x \ell'$ and decompose $\sum_{i,j} \xi_i v_i [P_{na}]_{ij} v_j|x$ into $\sum_{i} \xi_i v_i [P_{na}]_{ii}$ and $\sum_{i} \xi_i v_i \tilde{L}_{-i, \ell'}$. Then, we have
\[
E[(\sum_{i} \xi_i v_i [P_{na}]_{ii})^2|x] = \sum_{i} E[\xi_i^2 v_i^2 [P_{na}]_{ii}|x] \leq c_\xi E[||V_i||^4|x] \sum_{i} [P_{na}]_{ii}^2 = O_p(\alpha^{-1/2}).
\]
Because $\sum_{i,j} E[\xi_i^2 v_i^2 [P_{na}]_{ii}|x] \leq c_\xi E[\xi_i^2 v_i^2|x]^2 \sum_{i,j} [P_{na}]_{ii}^2 = O(1) \sum_{i}[P_{na}]_{ii}$ and $E[\xi_i^2 v_i^2|x] = 0$,
\[
E[(\sum_{i} \xi_i v_i \tilde{L}_{-i, \ell'})^2|x] = E[\sum_{i} \sum_{j \neq i} \xi_i v_i [P_{na}]_{ij} \tilde{L}_{-i, \ell'} \tilde{L}_{-j, -\ell'}|x] + O_p(\alpha^{-1/2})
\]
\[
= \sum_{i} \sum_{j \neq i} E[\xi_i^2 v_i [P_{na}]_{ii}|x] E[\xi_j^2 v_j [P_{na}]_{jj}|x] + O_p(\alpha^{-1/2}) = O_p(\alpha^{-1/2}).
\]
Hence, the desired result is given by applying the Markov’s inequality.

(h) can be shown by using the law of iterated expectation and the fact that $E[m_{1i,0}\mathcal{G}] = 0$.

\[
\square
\]

**Proof of Proposition 1**

We start from the following linearization:

\[
0 = \mathcal{M}(\bar{\theta}, \bar{g}_i) = \mathcal{M}(\theta_0, \mathcal{G}(\pi_n)) + \Gamma_n(\theta_0, \mathcal{G}(\pi_n))(\bar{\theta} - \theta_0) + \Gamma_2n(\theta_0, \mathcal{G}(\pi_n))(\bar{\pi}_n - \pi_n),
\]
From the above, $S_n'(\bar{\theta} - \theta)$ allows the following representation:

\[
S_n'(\bar{\theta} - \theta) = \tilde{H}^{-1}h,
\]
where
\[
\tilde{H} = n^{-1} \sum_{i=1}^{n} \tilde{m}_{2i,0}g_0\psi_0' + n^{-1} \sum_{i=1}^{n} (\bar{m}_{2i,0} - \bar{m}_{2i,0})g_0\psi_0' + n^{-1} \sum_{i=1}^{n} \tilde{m}_{2j,0}g_0\psi_0'
\]
\[
= \tilde{H}_{1} + \tilde{H}_{2} = H,
\]
and the numerator is given by
\[
h = n^{-1} \sum_{i=1}^{n} \psi_0'(\bar{\Pi}_{n,\alpha} - \Pi_n)Z_i \tilde{m}_{2i,0}g_0 + n^{-1} \sum_{i=1}^{n} m_{1i,0}g_0 + n^{-1} S_n^{-1}(\bar{\Pi}_{n,\alpha} - \Pi_n) \sum_{i=1}^{n} \tilde{m}_{1i,0}Z_i + h,
\]
\[
= h + \tilde{h}_1 + \tilde{h}_2 = h.
\]
with $Z_h = n^{-1} \sum_{i=1}^{n} \psi_0'(\bar{\Pi}_{n,\alpha} - \Pi_n)Z_i (\bar{m}_{2i,0} - \bar{m}_{2i,0} + \tilde{m}_{2i,0})g_0$. In the following, $\varrho_{na} = \Delta_2 + O_p(n^{-1}\alpha^{-1/2})$

Note that the first $d_e$ rows of $Z_h$ allow the representation such that $n^{-1} \xi'(P_{na} - I)T_n\pi_0 + n^{-1} \xi'P_{na}v_0$,
with \(\xi\) being the \(n \times 1\) vector whose \(i\)th element is given by \((\tilde{m}_{2i,0} + \tilde{m}_{2i,0} - \tilde{m}_{2i,0})g_{0,\ell}\). Given the boundedness of \(r_i\), the quantity \(\xi\) satisfies the conditions in 9.(f). For the remaining rows, \(Z_h\) satisfies 9.(g) and the second part of 9.(f). Thus, we have

\[
Z_h = o_p(\varrho_{n\alpha}).
\]

We then consider \(T_2^h\), which allows the following representation:

\[
T_2^h \equiv \sum_{\ell=1}^{d_e} n^{-1} m_{1,0}' \left( P_{n\alpha} - I \right) T_n \mathfrak{v}_0 e_\ell + n^{-1/2} \mu_{\ell}^{-1} m_{1,0}' P_{n\alpha} v_\ell e_\ell
\]

\[
\quad + \sum_{\ell=d_e+1}^{2d_e} n^{-1} m_{1,0}' \left( P_{n\alpha} - I \right) T_n \mathfrak{v}_n e_\ell + n^{-1} m_{1,0}' P_{n\alpha} v_\ell e_\ell.
\]

From 9.(f), we find that \(z_{1}^h, z_{2}^h\) and \(z_{3}^h\) are respectively \(o_p(\varrho_{n\alpha})\) under the employed assumptions. Hence, \(T_2^h = \sum_{\ell=1}^{d_e} t_\ell h e_\ell + o_p(\varrho_{n\alpha})\).

Before moving on, we note that \(h\) allows the following representation:

\[
n^{-1} G_0^i \mathfrak{M}_{20,n} \left( k_{n\alpha}^{-1} - I \right) \mathfrak{v}_0 + n^{-1} G_0^i \mathfrak{M}_{20,n} \left( k_{n\alpha}^{-1} - I \right) \mathfrak{v}_n + \Delta = 0.
\]

(a) \(\mathbb{E}(h|\mathfrak{G}) = 0\).

(b) \(\max_{\ell} \mathbb{E}(h^\ell h^\ell |\mathfrak{G}) = o_p(n^{-1}).\)

(c) \(\mathbb{E}(h^\ell h^\ell |\mathfrak{G}) = 0\).

(d) \(\sum_{\ell,\ell'=1}^{2d_e} \mathbb{E}(h^\ell h^{\ell'} |\mathfrak{G}) = o_p(n^{-1} \mu_{\ell}^{-1/2} \alpha^{-1/2}).\)

(e) \(\|h\|_H^2\) and \(\|h\|_H^2\) are all \(o_p(\varrho_{n\alpha})\). \(\|h\|_H^2\) is \(O_p(n^{-1} \mu_{\ell}^{-1/2} \alpha^{-1/4})\) for \(k = 1, 2\). Moreover, \(\|h\|_H^2\) is \(O_p(n^{-1})\) for \(k = 1, 2\).

(f) \(\mathbb{E}(hh' h^{-1} T_k^h |\mathfrak{G}) = 0\) for \(k = 1, 2\).

Let \(\hat{\alpha}(\alpha) = (h + T_1^h + \sum_{\ell=1}^{d_e} t_\ell h e_\ell)(h + T_1^h + \sum_{\ell=1}^{d_e} t_\ell h e_\ell)' - \sum_{k=1}^{2d_e} (hh' h^{-1} T_k^h + T_k^h h^{-1} hh').\) Then, from (a) to (f), we find that

\[
\mathbb{E}(\hat{\alpha}(\alpha) |\mathfrak{G}) = \mathbb{E}(hh' h^{-1} |\mathfrak{G}) + o_p(\varrho_{n\alpha})
\]

\[
= n^{-2} \sigma_{v_0}^2 \sum_{\ell,\ell' = 1}^{2d_e} (h_{21,\ell'} h_{22,\ell'}) e_\ell e_\ell' + \Delta_1 + O_p(\Delta_2^{1/2} / (n\alpha^{-1/4})).
\]

\[
= n^{-2} \sigma_{v_0}^2 \sum_{i,j} \sum_{i \neq j} \sum_{s \neq i,j} [P_{n\alpha}]_{ij} [P_{n\alpha}]_{js} \mathbb{E}(\tilde{m}_{20,i} g_{0i,\ell} |\mathfrak{G}) \mathbb{E}(\tilde{m}_{20,s} g_{0s,\ell'} |\mathfrak{G}) e_\ell e_\ell' + \Delta_1 + o_p(\varrho_{n\alpha})
\]

\[
= O_p(n^{-1} \alpha^{-1/2}) + \Delta_1 + o_p(\varrho_{n\alpha}),
\]

where the second line follows from the definition of \(\Delta_1\) and Lemma 9.(g). The remaining rows are obtained from Lemmas 9.(c). This and (E.2) conclude the proof.
Appendix F: Additional Tables and Figures

Table 6: Simulation results (Factor model with $\bar{K} = 30$)

| $n$ | $\mu^2 = 30$ | $\mu^2 = 60$ |
|-----|---------------|---------------|
|     | Med.Bias      | MAD           | RP  | Med.Bias      | MAD           | RP  |
| 200 | TRCMLE        | -0.056        | 0.236| 0.052         | -0.025        | 0.189 | 0.050 |
|     | SCRCMLE       | -0.070        | 0.231| 0.056         | -0.047        | 0.182 | 0.058 |
|     | Inf.2SCMLE    | -0.073        | 0.227| 0.058         | -0.034        | 0.181 | 0.049 |
|     | 2SCMLE        | -0.431        | 0.154| 0.476         | -0.319        | 0.142 | 0.351 |
|     | Probit        | -0.726        | 0.069| 0.781         | -0.703        | 0.071 | 0.840 |
|     | TTSLS         | -0.050        | 0.226| 0.080         | -0.028        | 0.165 | 0.086 |
| 400 | TRCMLE        | -0.073        | 0.231| 0.056         | -0.028        | 0.170 | 0.052 |
|     | SCRCMLE       | -0.089        | 0.219| 0.059         | -0.045        | 0.171 | 0.055 |
|     | Inf.2SCMLE    | -0.094        | 0.221| 0.064         | -0.046        | 0.169 | 0.059 |
|     | 2SCMLE        | -0.434        | 0.142| 0.517         | -0.310        | 0.127 | 0.376 |
|     | Probit        | -0.736        | 0.050| 0.953         | -0.724        | 0.050 | 0.975 |
|     | TTSLS         | -0.076        | 0.222| 0.080         | -0.031        | 0.164 | 0.084 |

Note: The simulation results based on 2,000 replications are reported. Each cell reports the median bias (Med.Bias), median absolute deviation (MAD), and rejection probability at 5% significance level (RP).

Figure 4: Estimated average structural functions (Factor model in Section 4)

Note: The estimated average structural functions based on 2,000 replications. The sample size $n$ is 200.
Table 7: Average partial effect estimates

| Variables | TRCMLE | SCRCMLE | 2SCMLE | Lasso  | Probit |
|-----------|--------|---------|--------|--------|--------|
| Model 1   |        |         |        |        |        |
| \( I,_{(0-5)} \) | 0.3322 | 0.3277  | 0.3277 | 0.2720 | 0.0285 |
| \( I,_{(6-12)} \) | -0.6388 | -0.6303 | -0.6303 | -0.4324 | 0.0441 |
| \( I,_{(13-18)} \) | 0.4835  | 0.4837  | 0.4837 | 0.3332 | 0.0116 |
| Model 2   |        |         |        |        |        |
| \( I,_{(0-5)} \) | 0.3372  | 0.3328  | 0.0901 | 0.1750 | 0.0285 |
| \( I,_{(6-12)} \) | -0.6032 | -0.6078 | -0.0816 | -0.2894 | 0.0441 |
| \( I,_{(13-18)} \) | 0.4580  | 0.4772  | 0.1153 | 0.2807 | 0.0116 |

Note: Each cell reports point estimates of the average partial effect computed with the TRCMLE, SCRCMLE, 2SCMLE, probit estimator, and Lasso estimator. The sample size is 2,654.

Figure 5: Estimated average partial effect of family income on the probability of college completion

Note: Each figure is the estimated average partial effect of family income on the probability of college completion according to the standardized family income in the age interval 0-5 (left), 6-12 (middle), and 13-18 (right).

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