Bose-Einstein correlations in multiple particle production from 1959 to 1989

K.Zalewski *
M.Smoluchowski Institute of Physics
Jagellonian University, Cracow†
and
Institute of Nuclear Physics, Cracow

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Abstract

The evolution of the experimental knowledge and of the theoretical ideas about Bose-Einstein correlation in multiple particle production processes, during the first thirty years from their discovery, is reviewed.

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Bose-Einstein correlations.

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†Address: Reymonta 4, 30 059 Krakow, Poland, e-mail: zalewski@th.if.uj.edu.pl
1 Introduction

The Bose-Einstein correlations in multiple particle production processes were discovered almost fifty years ago [87], [88]. Since then many old results got forgotten. For instance now most people believe that what is being observed in multiple particle production processes is the Hanbury Brown and Twiss effect, in spite of the very clear proof to the contrary given by Kopylov and Podgoretsky [114] (cf. [119] for a more recent brief discussion of this point). Concepts are renamed. E.g. what is now called Bose-Einstein or HBT correlations used to be the GGLP effect. Results are rediscovered and priorities are ascribed at random. It seems, therefore, useful to summarize and review critically the old results. We chose the first thirty years, which corresponds to little more than two hundred papers. Even so, this is a vast subject and I will be grateful for comments and remarks, which could help to
improve the text. After 1989 the explosion in the number of publications on
the subject has been such that now the number of papers can be estimated
to be over one thousand. Around 1990 the first big review papers devoted to
Bose-Einstein correlations appeared [134], [123], [48]. For obvious reasons,
however, they concentrated on the problems which the authors considered
topical at the time of writing and did not have the hindsight we have now.

This review is chronological, but not historical. A historical review would
have to include all the wrong and/or sterile ideas which influenced the re-
search workers in the field. No effort in this direction has been made.

When reviewing the work of many people, done over a long period of time,
the choice of notation becomes a problem. We have chosen a simple, uniform
notation without trying to keep track of the notations of the various authors.
Our notation will be introduced when necessary, but here we summarize the
main points. The four-momentum of a particle will be denoted 
\( p \) or \( p' \) and
its positions in space-time \( x \) or \( x' \). If the particle is on its mass shell, the
time component of \( p \) is \( E = \sqrt{p^2 + m^2} \). For a pair of particles we introduce

\[
K = \frac{1}{2}(p_1 + p_2), \quad K' = \frac{1}{2}(p'_1 + p'_2), \quad q = p_1 - p_2, \quad q' = p'_1 - p'_2, \quad (1)
\]

\[
q^2 = -Q^2 \quad (2)
\]

\[
X = \frac{1}{2}(x_1 + x_2), \quad X' = \frac{1}{2}(x'_1 + x'_2), \quad Y = x_1 - x_2, \quad Y' = x'_1 - x'_2. \quad (3)
\]

with the obvious notation \( \sqrt{Q^2} = Q \). For single particles

\[
K_1 = \frac{1}{2}(p_1 + p'_1), \quad q'_1 = p_1 - p'_1, \quad X_1 = \frac{1}{2}(x_1 + x'_1), \quad Y_1 = x_1 - x'_1. \quad (4)
\]

The subscript is changed from 1 to 2, if the position in space-time and the
four-momentum of the particle are \( x_2, p_2 \). Sometimes, when only one particle
is considered and there is no risk of confusion with the previous formulae,
the subscripts will be dropped.

Let us note for further reference two useful identities

\[
p_1 x_1 + p_2 x_2 = 2KX - \frac{1}{2}qY, \quad (5)
\]

\[
p_1 x_1 - p_2 x_2 = KY + qX. \quad (6)
\]

and their analogues for the primed and subscripted parameters as well as for
the space vectors.
We will use pseudo density matrices \( \tilde{\rho}(p_1, \ldots, p_n; p'_1, \ldots, p'_n) \). Their diagonal elements yield the momentum distributions, but the relations are model dependent and in general more complicated than for the standard density matrices. The pseudo density matrices are hardly ever normalized to unity. Moreover, in the GGLP model discussed in the next chapter their diagonal elements have to be integrated over phase space, with energy momentum conservation imposed, in order to give the momentum distributions (see text).

The radius \( R \) of the interaction region means different things in different papers. For a Gaussian distribution of sources

\[
\rho(r) = \sqrt{\frac{\alpha}{\pi R^2}} e^{-\alpha r^2/R^2}
\]

\( \sqrt{\langle r^2 \rangle} = \sqrt{3/2} \alpha R \), but both \( \alpha = 1 \) and \( \alpha = 1/2 \) are being used. For sources distributed on the surface of a sphere \( \sqrt{\langle r^2 \rangle} = R \), while if the sources are distributed uniformly over this sphere \( \sqrt{\langle r^2 \rangle} = \sqrt{3/5} R \). One should also keep in mind that the average of the square of the distance between two points is \( 2 \langle r^2 \rangle \), while e.g. \( \sqrt{\langle x^2 \rangle} = \sqrt{1/3} \sqrt{\langle r^2 \rangle} \). These ambiguities have been discussed and illustrated by many examples from the literature in ref. [36].

In the next section we review the two papers which created the field. Then we present the subsequent results divided in ten years periods. Finally in the last section we summarize the highlights of this development.

## 2 Beginnings

### 2.1 Discovery

Bose-Einstein correlations in multiple particle production were discovered accidentally [89] as a byproduct of an unsuccessful attempt to find the \( \rho \) meson, which had just been predicted by Frazer and Fulco [82]. The group of the Goldhabers at Berkeley was studying annihilations of antiprotons of momentum 1.05 GeV/c in a propane bubble chamber. Since the predicted \( \rho \) meson had isospin one, they hoped to find a bump in the invariant mass distribution of the unlike-sign pairs \( (\pi^+\pi^-) \), and no bump in the mass distributions of the like-sign pairs \( (\pi^\pm\pi^\pm) \). They analyzed the \( 2\pi^+2\pi^- \) and \( 3\pi^+3\pi^- \) semi-inclusive channels\(^1\) [87], but instead of finding the expected bumps discovered that the distributions of the angle between pion momenta, as measured in the \( p\bar{p} \) cms frame, strongly depended on the pion charges. For like-sign pion

\(^1\)i.e. channels with an unspecified number of \( \pi^0 \)-s
pairs the distributions in $\cos \theta_{\pi\pi}$ were almost flat, while for unlike-sign pairs large opening angles were more probable as expected from momentum conservation. In fact, the distributions for the unlike-sign pions were even more peaked than required by phase space, so that the combined distributions of $\cos \theta_{\pi\pi}$ for all the pion pairs at given multiplicity agreed within errors with calculations from the statistical model with Lorentz invariant phase space. In order to characterize the distributions of the opening angle, the group introduced for each distribution a parameter which was being very popular for many years,

$$\gamma = \frac{n_>}{n_<},$$

where $n_>$ ($n_<$) is the number of pion pairs with the c.m.s. opening angle greater (smaller) than $90^\circ$. Thus, the experimental result was that $\gamma_{\text{like}} < \gamma_{\text{unlike}}$ with the value of $\gamma$ calculated from Lorentz invariant phase space falling in between.

### 2.2 The GGLP paper

It took over a month and the help of an eminent theorist to understand that what was being observed were correlations due to Bose-Einstein statistics [89]. The resulting paper written by the Goldhabers, Lee and Pais [88] (further quoted GGLP) became a standard reference. According to the SPIRES data base, which omits citations previous to year 1973, this paper has been quoted over 300 times. For many years the effect was being referred to as the GGLP effect. The realization that an analogous effect had been discovered earlier in astronomy by Hanbury Brown and Twiss [98] and applied to the determination of the angular radii of stars came only much later [94], [89].

The starting point for GGLP was the statistical model with Lorentz invariant phase space [145]. According to this model the distribution of the particle momenta for a final state produced in a $p\bar{p}$ annihilation, containing $n$ pions and nothing else, is given by the formula

$$E_{p_1} \cdots E_{p_n} \frac{dN}{d^3p_1 \cdots d^3p_n} = C(n_+, n_0, n_-) \frac{a^n}{n!} \delta^4(P - \sum_{i=1}^{n} p_i).$$

Here $a$ is a constant, $P$ is the initial total momentum four-vector and the momentum four-vectors of the final particles are $(E_{p_i}, \mathbf{p}_i), i = 1, \ldots, n$. The factor $C$ contains a normalization constant and an isospin factor dependent on the numbers of positive, neutral and negative pions $(n_+, n_0, n_-)$. It will not be discussed here any further. Today it is well known that prescription
is very unrealistic, but at the time of GGLP, when beam energies were low and for comparison with experiment the formula was integrated over all the degrees of freedom except at most one, agreement with data was usually reasonably good.

The point made by GGLP was that formula \( (9) \) contains no correlations between the electric charges of the pions and their momenta. Consequently, the distribution of opening angles for pion pairs, as predicted by the model, does not depend on the charge of the pair, which is in violent contradiction with the data \[87\]. They proposed to replace the factor \( 1/n! \) by a product of three factors, one for pions of each charge. In view of future generalizations we will present their argument using operators and matrices instead of the wave functions they used. By analogy with the formulae for the density matrices in the momentum representation, we are looking for a replacement

\[
1/n! \rightarrow \prod_k \tilde{\rho}_k(p_1, \ldots, p_{n_k}; p_1, \ldots, p_{n_k}),
\]

where the product is over the pion charges \( k = +, 0, - \) and the tildes are there to remind that \( \tilde{\rho}_k \) do not have all the properties of textbook density matrices. We will call the matrices \( \tilde{\rho} \) pseudo density matrices. This includes a variety of matrices similar to, but not identical with, the density matrices.

Let us define a single particle density operator

\[ \hat{\rho} = \int d^3x |x\rangle \rho(x) \langle x| \]

with the usual normalization condition

\[ \int d^3x \rho(x) = 1. \]

Function \( \rho(x) \) must be real, because operator \( \hat{\rho} \) is hermitian. This single particle state is an incoherent superposition of pure states, each corresponding to a given production point \( x \). GGLP considered two weight functions \( \rho(x) \). Following the statistical model they put \( \rho(x) = \text{const} \) within a spherical volume of radius \( R_S \) and zero outside. They also introduced a "Gaussian-shaped volume" meaning that \( \rho(x) \sim \exp[-x^2/2\lambda] \). Comparing the matrix elements \( \tilde{\rho}(p_1, p_2; p_1, p_2) \) calculated within their model (see further) for the two choices of \( \rho(x) \) they found that the results were similar, with the closest agreement for

\[ R_S = 2.15\sqrt{\lambda}. \]
Therefore, they chose the Gaussian as it leads to simpler calculations. The density matrix in momentum representation, which corresponds to (11), is
\[
\rho(p; p') \equiv \langle p | \hat{\rho} | p' \rangle \sim \int d^3x \rho(x) e^{-iqx} \equiv \langle e^{iqx} \rangle.
\] (14)

The corresponding momentum distribution can be obtained by putting \( q = 0 \) and is constant, as was to be expected from the uncertainty principle. Let us note that \( \rho(p; p') \) is a single particle density matrix, but not necessarily the one describing the system being considered.

Let us define an unsymmetrized \( n \)-particle density matrix as a product of the single particle density matrices (14)
\[
\rho^U(p_1, \ldots, p_n, p'_1, \ldots, p'_n) = \prod_{i=1}^{n} \rho(p_i; p'_i).
\] (15)

Taking the diagonal elements and imposing energy momentum conservation one could "derive" in this way the statistical model (9). GGLP, however, proposed to symmetrize for each charge of pions. For \( k = +, 0, - \) their assumption, written in a more general form (16), is
\[
\tilde{\rho}_k(p_1, \ldots, p_{n_k}, p'_1, \ldots, p'_{n_k}) = \frac{1}{n_k!} \sum_{P,Q} \rho^U_P(p_1, \ldots, p_{n_k}, p'_1, \ldots, p'_{n_k}),
\] (16)

where the summation extends over the \( n_k! \) permutations \( P \) of the momenta \( p_i \) and over the \( n_k! \) permutations \( Q \) of the momenta \( p'_i \). This formula can be rewritten as
\[
\tilde{\rho}_k(p_1, \ldots, p_{n_k}, p'_1, \ldots, p'_{n_k}) = \sum_{Q} \tilde{\rho}_k^U(p_1, \ldots, p_{n_k}, p'_1, \ldots, p'_{n_k}),
\] (17)

GGLP considered no more than two pions of the same sign, because they assumed in the calculations that every event with more than one \( \pi^0 \) contains exactly two \( \pi^n \)-s. Moreover, they needed only the distributions of momenta i.e. the diagonal elements of the pseudo density matrices. Thus, their recipe was as follows. For \( n_k = 0, 1 \) there is no dependence of the factor \( \tilde{\rho}_k \) on the momenta of the pions. For \( n_k = 2 \)
\[
\tilde{\rho}_k(p_1, p_2; p'_1, p'_2) = \tilde{\rho}(p_1; p_1)\tilde{\rho}(p_2; p_2) + |\tilde{\rho}(p_1; p_2)|^2,
\] (18)
where the hermiticity of the single particle pseudo density matrix has been used. Substituting expression (14) one finds:

$$\tilde{\rho}_k(p_1, p_2; p_1, p_2) = 1 + \left| \int d^3 x \rho(x) e^{i(p_2 - p_1) \cdot x} \right|^2,$$  

(19)

which for a Gaussian $\rho(x)$ gives

$$\tilde{\rho}_k(p_1, p_2; p_1, p_2) = 1 + e^{-\lambda(p_1 - p_2)^2}.$$  

(20)

As seen from the derivation, the momenta in the exponent are three-momenta. GGLP, however, replaced them by four-momenta, in order to simplify further the calculations. This makes the formula Lorentz invariant and does not change much their numerical results. Incidentally, it improved agreement with experiment. Let us follow GGLP and introduce their ”correlation function”

$$\psi(p_1, p_2) = 1 + e^{-\lambda Q^2},$$  

(21)

where $Q^2 = -(p_1 - p_2)^2$. This formula has later become very popular. Let us stress, however, that in the GGLP model it was a factor in the integrand of the phase space integral, while later it was being used without further integrations. For the channels $2\pi^+2\pi^-, 2\pi^+2\pi^-\pi^0$ and $2\pi^+2\pi^-2\pi^0$ the GGLP proposal amounts to replacing in formula (9) the factor $1/n!$ by

$$\psi(p_1, p_2)\psi(p_3, p_4)\tilde{\psi}(p_5, p_6),$$  

(22)

where $p_1$ and $p_2$ are the four-momenta of the positive pions, $p_3$ and $p_4$ are the four-momenta of the negative pions and $\tilde{\psi}(p_5, p_6) = 1$ unless $n_0 = 2$. For $n_0 = 2$: $\tilde{\psi}(p_5, p_6) = \psi(p_5, p_6)$, where $p_5$ and $p_6$ are the four-momenta of the two neutral pions. The results, after averaging with suitable weights over $n_0$, were found to be in good qualitative agreement with the corresponding experimental results from [87]. Also the radius of the interaction region $R_s$ came out between 0.7 fm and 1 fm which is the expected order of magnitude. Quantitatively, however, $\gamma^{unlike}$ was predicted too small and $\gamma^{like}$ too large whatever the choice of $R_s$. Thus the model underestimated the difference between the like- and unlike-sign pion pairs.

Besides their calculations GGLP made some important general remarks. They realized that ”an adequate model should at the same time give a reasonable account of all combined aspects of the annihilation process” and that, therefore, one should look for other evidence for or against the model. They predicted that for any given exclusive channel the effects of Bose-Einstein statistics should decrease with increasing energy. They worried about the
multiplicity distributions, because the volume \( \frac{4}{3} \pi R_s^3 \), which they found, was significantly smaller than the volume which in the statistical model governs the multiplicity distribution. They stressed that the study of Bose-Einstein correlations may give valuable information about the reaction mechanism, but that ”results of this study should not be construed to imply that dynamical effects (such as, for example, \( \pi-\pi \) interactions) are definitely negligible”. In fact they suspected that the ”\( \pi \) isobars”, or in today’s language resonances, may be important.

3 First decade (1961 — 1970)

In the sixties the study of Bose-Einstein correlations was gradually gaining popularity. The conceptual framework was still that of the GGLP paper, though an isolated attempt to study three- and four-body correlations \(^7\) should be noted. Usually, people plotted the distributions of \( \cos \theta_{\pi\pi} \), calculated the parameters \( \gamma \), or some other parameters which could be expressed in terms of the parameters \( \gamma \), and supported, or very rarely criticized, the implications of the GGLP paper. The effect found in \(^8\) was confirmed for other \( pp \) interactions. A compilation from 1967 \(^6\) lists \( \gamma^u, \gamma^l \) pairs for 25 different (exclusive) channels and/or energies. The effect was found in \( \pi^+p \), \( \pi^-p \), \( pp \), and \( K^+p \) \(^8\) interactions. The effective attraction in momentum space was found for \( \pi^0\pi^0 \) pairs \(^2\), but not for \( K^+\pi^\pm \) pairs \(^8\). The results for the \( \pi^0 \)-s and the \( K^+\pi^+ \) pairs eliminated the possibility \(^\) that the effect for \( \pi^\pm\pi^\pm \) pairs was due to the fact that these pairs are ”exotic” i.e. unable to form resonances, and not to the fact that they consist of identical bosons. A related possibility was that the effect is due to resonances. Here the results were not so clear (cf. eg. the review \(^1\) and references given there), however, cutting out the resonances did not change much the measured \( \gamma \)-s \(^8\). Moreover, a particularly strong effect was observed for \( \pi^+p \) interactions at 8 GeV/c with nine charged pions and a proton or ten charged pions and a neutron in the final state, where the energy per particle is too low for a significant production of resonances \(^2\). Thus Bose-Einstein correlations remained as the only plausible explanation.

Some qualitative implications of the GGLP model got confirmed. A compilation of the data for annihilations showed \(^6\) that with increasing energy the effect, as measured by the parameter

\[
C = \frac{\gamma^u - \gamma^l}{(\gamma^u + 1)(\gamma^l + 1)},
\]

(23)
decreases rapidly. A rough fit was

$$C = \frac{73.9}{n_c^{1.64}(n_0 + 1)^{1.54}}e^{-\frac{\sqrt{s}}{0.132(n_c+n_0)}}. \tag{24}$$

where $n_c$ and $n_0$ denote the numbers of charged and neutral pions respectively. The centre of mass energy $\sqrt{s}$ is here expressed in GeV. The uncertainty of the normalizing factor 73.9 was about fifty percent, while the uncertainties of the other three parameters were at the ten per cent level.

For events with many particles in the final state it was possible to show that the effect occurs in every event and not only in some special ones \cite{76}. The effect is enhanced when one chooses pairs of pions with similar values of momenta $|\mathbf{p}_i|$ i.e. with similar energies \cite{37}.

Quantitatively, however, the model did not work. The conclusion in \cite{67} was that "The model of Goldhaber et al. is in violent disagreement with the data". The main argument was that at low energy per particle the values predicted for $C$ are much too small. Moreover, assuming a constant interaction radius, $R = 1.3$ fm, the authors found that for the exclusive process $\bar{p}p \to 3\pi^+3\pi^-$ the model grossly underestimates the energy dependence of parameter $C$. Incidentally, \cite{67} seems to have been the first calculation with three identical particles in the final state. The statistical model contains no peripherality and consequently with increasing energy per particle its predictions for the single particle distributions become completely wrong. The GGLP model did nothing to correct that, but the point was that also its predictions for the correlations were wrong. One tried to improve the starting point by replacing the statistical model by the uncorrelated jet model, which includes peripherality, or by the CLA model, which included moreover some multiperipheral correlations, but this did not help \cite{68}. The situation was summarized at the 1-st multiparticle conference in Paris \cite{150}. To put it short: the GGLP model was good enough to convince people that the effect is due to Bose-Einstein statistics, but not good enough to make quantitative predictions.

## 4 Progress in the seventies

### 4.1 The approaches of Kopylov and Podgoretskii, and of Shuryak

In 1971 an important series of papers by Kopylov, Podgoretskii and collaborators (further quoted KP) begun to appear \cite{94}. The approach of KP differed significantly from that of GGLP. There was a major difference in
motivation. The purpose of GGLP was to explain the charge dependence of the distributions of the opening angles for pion pairs. They noticed that the radius of the interaction region, obtained as a byproduct, was of the expected order of magnitude, though probably too small, but this remark was not followed. According to a detailed review \[48\] only two experimental papers previous to 1976 gave values for the radius of the interaction region (\[67\] and \[73\]). Looking up these papers, one sees that in both cases the radii were given only to convince the reader that the authors worked hard before recognizing that they are unable to fit the data. KP were familiar with the work of Hanbury Brown and Twiss \[98\] and their purpose was to use quantum interference, for identical fermions as well as for identical bosons, to get information about the lifetimes and extensions of various sources. They considered a variety of sources \[112\]: resonances of particles and of nuclei, highly excited nuclei where both lifetimes and shapes were of interest and interaction regions in multiple particle production processes. KP did not compare their results with experimental data.

The GGLP approach did not include time. It corresponds to a picture where all the pions are emitted instantaneously and simultaneously. Then, when it exactly happened is irrelevant. The sources of KP were time dependent and extended in time. They considered it to be the main difference between the two approaches \[24\] and claimed to have ”a more correct theoretical approach” \[112\]. Another difference was technical, but of great practical importance. GGLP had reasonably simple formulae for the integrands of the phase space integrals. The integration over phase space, however, remained to be done. In practice this limited their calculations of the physical distributions to exclusive channels with no more than six particles. KP assumed that the constraints of energy and momentum conservation have little effect on the distributions of the momentum difference \((q_0, \mathbf{q}) = (E_{p_1} - E_{p_2}, \mathbf{p}_1 - \mathbf{p}_2)\) in the region of small \(|\mathbf{q}|\). Thus, their results could be directly compared with the measured cross-sections. This eliminated the necessity of evaluating the phase space integrals and made the predictions applicable also to inclusive processes. Obviously, it was a reasonable assumption only for sufficiently high multiplicities for exclusive processes, or for sufficiently high energies for inclusive processes. The KP results for multiple particle production were sketched in \[111\] and then described in more detail in \[112\]. In ref. \[113\] the theory was reformulated using a more powerful formalism borrowed from optics and some generalizations were given. A short summary, including important new ideas, was given in \[109\]. Reference \[114\] explained the relation between the work of GGLP and that of Hanbury Brown and Twiss.

Both GGLP and KP considered an incoherent superposition of pure pion states produced by sources excited at some common time, say \(t = 0\), at vari-
ous points of space \( \mathbf{x} \) and emitting one pion per source. The GGLP sources, however, emit the pions instantly and without changing their locations. The KP sources are much more general. The amplitude of a pion emitted by a source was assumed to be the solution of the Klein-Gordon equation

\[
\left( \nabla^2 - \frac{\partial^2}{\partial t^2} - m^2 \right) A(\mathbf{x}) = -4\pi J(\mathbf{x}),
\]  

(25)

where \( m \) is the pion mass, \( \mathbf{x} \) is the four-vector \( \mathbf{x}, t \) and \( J(\mathbf{x}) \) is the (classical) source. Further there will be many sources contributing incoherently, but let us start with just one. The equation simplifies when the Fourier transforms in time:

\[
A(\mathbf{x}; \varepsilon) = \int_{-\infty}^{\infty} dt A(\mathbf{x}) e^{i\varepsilon t}; \quad J(\mathbf{x}; \varepsilon) = \int_{-\infty}^{\infty} dt J(\mathbf{x}) e^{i\varepsilon t};
\]  

(26)

are introduced. Note that, according to the general rules of quantum mechanics, \( \varepsilon \) can be interpreted as the energy of the pion. The transforms satisfy the equation

\[
\left( \nabla^2 - \mathbf{p}^2 \right) A(\mathbf{x}; \varepsilon) = -4\pi J(\mathbf{x}; \varepsilon),
\]  

(27)

where \( \mathbf{p}^2 = \varepsilon^2 - m^2 \) and, therefore, it is the square of the momentum of the pion. The solution of this equation is well known and described in detail in [113], but we will need here only a special case. Let us assume that all the source functions \( J(x) \) are negligible outside a small region of space \( V \) and that the detector is at a large distance \( r_D \) from the centre of region \( V \). Large means here much larger than the diameter of region \( V \). Then, in the vicinity of the detector, the (approximate) solution is

\[
A(\mathbf{x}; \varepsilon) = \tilde{A}(p) \frac{\mathbf{e}^{i|\mathbf{p}|r_D}}{r_D}; \quad \tilde{A}(p) = \int d^4x J(x) e^{ipx};
\]  

(28)

where \( p \) is the pion energy-momentum four-vector \( (\mathbf{p}, E_p) \) with the momentum \( \mathbf{p} \) oriented from the centre of region \( V \) to the detector and \( |\mathbf{p}| = \sqrt{\mathbf{p}^2} \). Note that all the dependence on the source is contained in the factor \( \tilde{A}(p) \). The other factors are irrelevant from the point of view of symmetrization and will be further omitted. Thus in general one source, say the one labelled \( i \), contributes to the single particle pseudo density matrix a term \( \tilde{A}_i(p) \tilde{A}^*_i(p') \).

In general this product depends on some parameters and has to be averaged over them. The result should be summed over all the sources. Note that the contributions from all the space-time points where \( J(x) \) is significantly different from zero are summed coherently (adding amplitudes) in the formula
above, while the averaging and the summation over sources correspond to incoherent (adding probabilities) superpositions. Finally

\[ \tilde{\rho}(p, p') = \sum_i \langle \tilde{A}_i(p) \tilde{A}_i^*(p') \rangle. \] (29)

Formulæ of this kind, which are more general than the KP model, were introduced by Shuryak [141, 142]. Shuryak’s results can be derived starting from the density operator

\[ \hat{\rho}(t) = \int_{-\infty}^{t} dt_0 \int_{-\infty}^{t} dt_0' e^{-iH(t-t_0)}|J(t_0)\rangle\langle J(t_0)| e^{iH(t-t_0')} \]. (30)

Actually it is a pseudo density operator, because its trace does not have to be equal one. Further we will take \( t \to \infty \). When \( J(x) = \langle x|J(t)\rangle \) is a classical current this leads to the KP theory. \( J(x) \), however, can be any single particle production amplitude (cf. e.g. [139]). This formula, as it stands, is for one coherent source. In order to make a realistic model it is necessary to sum over the sources and, sometimes, to average over some parameters of the sources. Including that, introducing, after the first and before the last factor of the integrand, unit operators built from the eigenstates of the position operator and converting to a pseudo density matrix element in the momentum representation we find

\[ \tilde{\rho}(p, p') = \int d^4x \int d^4x' \langle p|e^{-iH(t-t_0)}|x\rangle \sum_i \langle J_i(x) J_i^*(x') \rangle \langle x'|e^{iH(t-t_0')}|p'\rangle. \] (31)

Evaluating the matrix elements of the time-evolution operators and omitting the uninteresting factor \( \exp[it(E_{p'} - E_p)] \) we get

\[ \tilde{\rho}(p, p') = \int d^4x \int d^4x' \left( \sum_i J_i(x) J_i^*(x') \right) e^{i(px - p'x')}, \] (32)

or equivalently in other variables

\[ \tilde{\rho}(p_1, p'_1) = \int d^4X_1 \int d^4Y_1 \sum_i \left( J_i(X_1 + \frac{Y_1}{2}) J_i^*(X_1 - \frac{Y_1}{2}) \right) e^{i(q_1X_1 + K_1Y_1)}, \] (33)

This formula is the starting point for much modern work. KP proposed

\[ J(x) = \tau^{-1/2} \delta(x - x_0 - vt)e^{-iEt - \frac{1}{2\tau} \theta(t)} \]. (34)
This corresponds to a classical point source created at time $t = 0$ at point $x = x_0$ and moving from that point on with constant velocity $v$. The source has energy $E$ and decays emitting a pion according to an exponential decay law with average life time $\tau$. Thus the parameters $x_0, v, E, \tau$ depend on the source label $i$. The integrations necessary to get $\tilde{A}(p)$ are in this case trivial and one finds using (29)

$$\tilde{\rho}(p, p') = \sum_i \left\langle \frac{\tau e^{-i q \cdot x_0}}{\left(\frac{1}{2} + i \tau (E - \varepsilon + p \cdot v)\right) \left(\frac{1}{2} - i \tau (E - \varepsilon' + p' \cdot v)\right)} \right\rangle. \quad (35)$$

Here KP assumed that the averaging should be done over the source energy $E$ and that this can be reduced to an integration over $E$ from minus to plus infinity. The integral is easily done using the method of residues. The sources are labelled by the points $x_0$ of their creation. Thus, the summation over $i$ reduces to an integration over space with some weight $\rho(x_0)$. Finally, omitting an irrelevant factor $2\pi$,

$$\tilde{\rho}(p, p') = \int d^3 x_0 \rho(x_0) e^{-i q \cdot x_0} \left. \frac{1}{1 - i \tau (q_0 - q \cdot v)} \right|_{q_0 = q}.$$ \quad (36)

Usually one puts $v = 0$ so that

$$|\tilde{\rho}(p, p')|^2 = \left| \int d^3 x_0 \rho(x_0) e^{-i q \cdot x_0} \right|^2. \quad (37)$$

The symmetrization is done as in GGLP. Thus for the two-body correlations formula (18) remains valid. Since $\tilde{\rho}(p, p) = 1$, the first (non interference) term in (18) remains equal one. The second (interference) term is modified by the factor $(1 + \tau^2 q_0^2)^{-1}$. From the experimental point of view this is a beautiful formula. From the dependence of the correlation function on the vector $q$ at fixed $q_0$ one can determine the space distribution of the sources just like in the GGLP model. Studying the dependence on $q_0$ at fixed $q$, however, one can additionally measure the life time of the source.

An obvious objection to formula (36) is that a density matrix depending on the difference of the momenta only is grossly unrealistic. GGLP did not have this difficulty, because they kept only the states allowed by energy-momentum conservation and the necessary projection introduces a dependence on $p_1 + p_2$. KP proposed the following way out. Suppose that the incoherent sources are not point-like in space, but smeared. Thus

$$\hat{\rho} = \int d^4 x_s e^{-i H_0 (t-t_s)} |\psi_s\rangle \rho(x_s) \langle \psi_s| e^{+i H_0 (t-t_s)}. \quad (38)$$
Here the four-vector $x_s$ labels the sources. It gives the space and time position of the source at the moment when the source got created, or of any other point in space-time which defines unambiguously the source. Function $\rho(x_s)$ gives the distribution in space-time of the incoherent sources, thus, for any function $f(x_s)$ the average over the sources is

$$\langle f(x_s) \rangle \equiv \int d^4x_s \rho(x_s)f(x_s). \quad (39)$$

$H_0$ is the free particle Hamiltonian. The exponentials give the time dependence of the state vectors $|\psi_s\rangle$ and their dependence on the time component of $x_s$. The state vectors $|\psi_s\rangle$ are equivalent in the sense that

$$\langle x | \psi_s \rangle = \psi(x - x_s). \quad (40)$$

Thus all the incoherent sources are related by rigid shifts in space-time. It is convenient to use a picture where the density matrix does not depend on the time $t$. This can be achieved either by using the time dependent momentum eigenstates $e^{-iH_0t}|p\rangle$, or equivalently by using the interaction picture, where each state vector gets an additional factor $e^{+iH_0t}$, which cancels the $t$-dependent factors both in the density operator and in the time dependent momentum eigenstates. Let us define the function

$$A(p) = \int d^3x e^{-ipx}\psi(x). \quad (41)$$

Then, taking the matrix element of the density operator and dropping some constant factors,

$$\tilde{\rho}(p_1, p_2) = A(p_1)A^*(p_2) \langle e^{iqx_s} \rangle. \quad (42)$$

This formula has several interesting features. The diagonal elements, which yield the single particle momentum distribution are

$$\tilde{\rho}(p, p) = |A(p)|^2. \quad (43)$$

Thus, any single particle distribution can be reproduced perfectly. Function $A(p)$ is related to the properties of a single source defined by the state vector $|\psi\rangle$, but tells us nothing about the distribution of sources in space-time. On the other hand, for the correlation functions defined in the next section the factors $A(p)$ cancel and the formulae are as if the sources were point-like. Therefore, the correlation function depends only on $q$ and not on the vectors $p_1, p_2$ separately. This is similar to the situation in the GGLP model. Note, however, the change in the interpretation of the dependence of function $\rho(x_s)$.
on the space vector \( x_s \). Now it gives the distribution of the positions of the sources. Since the sources are smeared in space, the actual region, where the pions are produced is larger than would follow, if the function \( \rho(x_s) \) were interpreted as the GGLP function \( \rho(\mathbf{x}) \) was. What is more, the function of four variables \( \rho(x_s) \) cannot be determined unambiguously from measurements of the three-momenta. Thus unavoidable model dependence comes in.

Another less important, but much publicized, difference between KP and GGLP was in the assumption about the space distribution of sources. KP pointed out that the interaction region is likely to be opaque and that, consequently, only the sources on its surface can contribute. They assumed that the sources are created on the surface of a sphere of radius \( R \), but that their radiation satisfies Lambert’s law taken to mean that this surface can be replaced by a circular disc of radius \( R \) perpendicular to the direction from the interaction region to the detector\(^2\). Thus for the integral in (19) they got

\[
\frac{1}{\pi R^2} \int_0^R dr \int_0^{2\pi} e^{i q_T r \cos \theta} = \frac{2 J_1(q_T R)}{q_T R}.
\]

Here \( q_T \) is the length of the component of \( q \) in the plane of the disc and \( J_1 \) is a Bessel function. The integral is most simply done by expanding the exponent in the integrand and integrating term by term. Cocconi\(^63\) proposed to replace this expression by a Gaussian, according to

\[
\left( \frac{2 J_1(x)}{x} \right)^2 \to e^{- \frac{x^2}{4}}.
\]

Actually, for not too large values of the argument, the right hand side is a very good approximation to the more complicated expression on the left-hand side. This short reference does not give justice to the importance of reference \( ^63 \). Cocconi derived independently many of the KP results. He proposed alternative interpretations. E.g. in his picture \( c \tau \) was the thickness of the “photosphere” i.e. of the spherical crust from which the pions were emitted. Last not least, he contributed much in discussions with experimentalists, so that ref. \( ^63 \) has got more citations than any of the KP papers and even sometimes the whole approach was referred to as the Kopylov, Podgoretskii Cocconi model.

\(^2\)This is a hand-waving argument. An exact evaluation of the corresponding integral can be found in ref. \( ^{115} \). It yields the result (44) in the limit \( c \tau \gg R \). This condition was clearly stated in refs \( ^{109}, ^{112} \). Unfortunately, it is hardly ever realized in practice. Another case when the formula is valid is for \( q_\parallel = 0 \), where \( q_\parallel \) is the component of vector \( q \) along the direction of the total momentum of the pair.
For the distribution in the difference in momenta KP got [109]

\[
\frac{d^3\sigma}{d^2q_T dq_0} \sim 1 + \left[ \frac{2J_1(q_T R)}{q_T R} \right]^2 \frac{1}{q_0^2 \tau^2 + 1}.
\]

(46)

Kopylov [109] rewrote this in the form

\[
\frac{d^3\sigma}{d^2q_T dq_0} = \left( \frac{d^3\sigma}{d^2q_T dq_0} \right)_{off} \left( 1 + \frac{I(q_T R)}{1 + q_0^2 \tau^2} \right).
\]

(47)

Note that this formula would not follow, if the second factor on the right-hand side depended on \( \mathbf{p}_1 + \mathbf{p}_2 \), because the two-particle pseudo density matrix yields \( d^6\sigma/dp^6 \) and after integration over \( \mathbf{p}_1 + \mathbf{p}_2 \) factorization would not hold any more. Function \( I(q_T R) \) can be obtained by comparison with the previous formula, or from some other model. The important distinction is between the cross-section on the left-hand side, which is the physical one containing all the interference effects, and that on the right-hand-side, which is calculated with the interference effects turned off. In the GGLP and KP models this would be up to a constant factor \( \rho(\mathbf{p}_1, \mathbf{p}_1) \rho(\mathbf{p}_2, \mathbf{p}_2) \), but the formula is more general than that. Then the "off" cross-section, which was soon renamed by experimentalists "background", should be taken from experiment. The question is how? GGLP used for comparison with like-sign pion pairs the unlike-sign pion pairs. These have certainly no Bose-Einstein correlations, but they have other unwanted correlations due e.g. to the formation of resonances in the \( \pi^+\pi^- \) system. Kopylov [109] proposed to take like sign pairs, but with each pion taken from a different event\(^3\). This background got later the name "mixed" and became quite popular, though it eliminates too many correlations e.g. the correlations due to energy and momentum conservation or to the production of jets.

The relation between the work of GGLP and that of Hanbury Brown and Twiss was clarified in ref. [114]. The authors developed a more general model. It introduces two scales of time, or energy: the life time of the emitting source, or equivalently the width of the energy distribution of the produced particles, \( \tau \sim 1/\Gamma \) and the resolution of the detector in time, or equivalently in energy, \( \Delta t \sim 1/\gamma \). In astronomy the life time of the excited atomic state, which emits the radiation, is much shorter than the time resolution of the detector. Therefore, one can neglect \( \tau \) compared to \( \Delta t \). In particle physics, on the other hand, the width in energy of the source is much larger than the energy resolution of the detector. Therefore, the opposite limit is justified: one can neglect \( \gamma \) as compared to \( \Gamma \). These two limits correspond respectively to the models of Hanbury Brown and Twiss and to that of GGLP. They

\(^3\)According to ref. [110] this was an idea of Podgoretskii.
are completely different limits. E.g. the HBT results can be derived from classical physics without ever mentioning quantum physics. Nevertheless, the acronym GGLP got in the seventies gradually replaced by the acronym HBT. The analysis from [114] implies that the motivation must have been sociological rather than physical. In this review we use the neutral BEC, which is also quite popular.

4.2 Correlation functions

Since the study of various correlations was a popular subject in the seventies, it soon became common to express formulae like (47) in terms of two-particle correlation functions, cf. e.g. [44], defined by

\[ C(p_1, p_2) = \frac{N(p_1, p_2)}{\bar{N}(p_1)\bar{N}(p_2)}. \] (48)

Here

\[ N(p) = \frac{1}{\sigma} \frac{d^3\sigma}{dp^3}; \quad N(p_1, p_2) = \frac{1}{\sigma} \frac{d^6\sigma}{dp_1^3dp_2^3}; \] (49)

are respectively the single particle and the two-particle distributions of the particles studied, \( \sigma \) is the integrated single particle cross-section, and \( d^3p \) means either \( dp_xdp_ydp_z \) or \( dp_xdp_ydp_z/E_p \). \( \bar{N} \) is a normalization constant.

Theorists usually either put it equal one, or interpret it as the ratio of the normalization of the denominator to the normalization of the numerator. In the latter case, for inclusive processes they put \( \bar{N} = \frac{\langle n \rangle^2}{\langle n(n-1) \rangle} \). The resulting confusion has been reviewed in ref. [128]. Experimentalists more often use this factor to make the correlation function tend to one with increasing \( Q^2 \).

The correlation function is a nice, well-defined object, but for extracting information about the production region it is often "extremely impractical" as put later by experimentalists [148]. Consider for example a process, where the final state consists of two narrow jets back to back. Let the orientation of the axis of this pencil-like structure have an isotropic distribution.

Then function \( N(p_1, p_2) \) has strong maxima for \( \theta_{\pi\pi} = 0 \) and \( \theta_{\pi\pi} = \pi \), while function \( N(p) \) is spherically symmetric. The correlation function also has a strong maximum at \( \theta_{\pi\pi} = 0 \) (and another one at \( \theta_{\pi\pi} = \pi \)), but this tells us little about Bose-Einstein correlations. On the other hand, if for each event the coordinate system is rotated so that the axis of the jets is along the \( z \) axis, also function \( N(p) \) acquires maxima at \( \theta_{\pi\pi} = 0 \) and \( \theta_{\pi\pi} = \pi \). The maxima in the denominator of (48) largely cancel the dynamical part of the maxima in the numerator and Bose-Einstein correlations can be studied,
though usually some additional selections to improve the analysis are introduced (c.f. e.g. [4]). Other examples, where the correlation function (48) is in practice useless for gaining information about the interaction region, have been described in ref. [95]. Formally, the way out is to define the correlation function $C_0(p_1, p_2)$ for the background, i.e. for the fictitious case considered by KP: the same process, but with the Bose-Einstein correlations switched off. Then the reduced correlation function

$$C(p_1, p_2)/C_0(p_1, p_2) \equiv 1 + R(p_1, p_2).$$

is a good replacement for the correlation function (48). In the following we will mostly discuss the function $R(p_1, p_2)$ defined by this formula. It corresponds to the interference term in formula (47) and should tend to zero, when $Q^2$ becomes so large that no more Bose-Einstein correlations are expected. When the background two-particle distribution is a product of the standard inclusive single particle distributions, $C_0(p_1, p_2) = 1$ and $1 + R(p_1, p_2)$ reduces to the correlation function $C(p_1, p_2)$. This is the case for the mixed background, while usually it is not the case for the background from unlike sign pion pairs. It is also not the case when the inclusive single particle distribution is modified by some selection or transformation like in the two-jet model mentioned above. Since it is usual to call $1 + R(p_1, p_2)$ a correlation function, one could argue as follows. The uncorrected experimental correlation function $C_{\text{uncorr}}(p_1, p_2)$ should be corrected for the unwanted correlations by dividing it by $C_0(p_1, p_2)$. Then $1 + R(p_1, p_2)$ is this corrected correlation function. In the seventies theorists usually put $C_0 = 1$ and experimentalists chose $C_0$ constant and such as to make $R(p_1, p_2)$ tend to zero with increasing $Q^2$. Later the sophistication was greatly increased, but the best way of choosing $C_0(p_1, p_2)$ is even today controversial.

4.3 Further experimental results

On the experimental side some more qualitative implications of the GGLP picture got confirmed. The effect was found in nucleus-nucleus collisions [83] and for $K^0_sK^0_s$ pairs [61]. It was shown [77] that within the experimental errors the GGLP effect could be interpreted as a reflection of the difference in the distributions of the masses of pion pairs $m_{\pi\pi}$ for pairs of like-sign and unlike-sign pions. Actually GGLP proposed that the weight factor, which equals one for unlike-sign pairs, should depend on $Q^2 = -(p_1 - p_2)^2$ for like-sign pairs. Since, however, $Q^2 = m_{\pi\pi}^2 - 4m_{\pi}^2$, the two statements are equivalent. Three- and four-body correlations were studied [47], [49], [118]. They were found strong, but at that time could be fully explained as reflections of
the GGLP two-body correlations. Gradually the groups begun to determine
the radii and life times of the interaction regions as suggested by KP. Re-
sults were obtained for $K^+p$ scattering [72], [91], [92], for $\pi^\pm p$ scattering [22],
[39], [70], for $\bar{p}p$ [18], [41], [64], [122] and $\bar{p}n$ [50] scattering , for $pp$ scattering
[41], [74], [78], for $p$-nucleus (Be, Ti, W) [11] and $\pi^- - C$ [23] scattering
as well as for nucleus-nucleus scattering [83] ($Ar + BaI_2$ and $Ar + Pb_3O_4$),
[24] ($C + Ta$). The background distribution was usually identified with the
distribution for unlike sign pairs. Only for scattering on nuclei the mixed
background proposed by Kopylov was found more convenient, because using
it one could use the negatively charged particles only and thus, avoid the
contamination by protons. Sometimes results for two different backgrounds
were compared in order to show that the uncertainty about the background
distribution does not affect too much the results. The normalization of the
background distribution with respect to the distribution of like-sign pairs was
chosen so that the numbers of pairs in both samples coincided, or so that the
distributions became identical for large values of $Q^2$. Note that these
two normalizations are not equivalent. There were various other sources of
uncertainty. Statistics was poor and the results depended on the binning
chosen. The fits, as judged by the $\chi^2$ test, were often poor. The maps of
$\chi^2(R, \tau)$ exhibited large valleys with secondary minima. Formula (47) is not
covariant. Some groups used it in the overall cms, others preferred the cms
of the charged pions. On that were superposed the ordinary experimental
uncertainties e.g. particle misidentification and limited momentum resolution.
The theory was not very robust either. E.g. when the authors of ref
[122] studying $\bar{p}p$ annihilations got $R \approx c\tau \approx 3$fm and, not very surprisingly,
considered this result too large, they replaced the assumption that $v = 0$
by the assumption that there are two sources moving with velocities $v$ and
$-v$ respectively, added some phenomenological assumptions about $v$ and got
from the same data $c\tau \approx 0.7$fm and $R \approx 0.7 - 0.8$fm. Another degree of free-
dom, which was not exploited by experimentalists however, was to assume
that there is a scatter of the time when the sources got excited. This gives
another factor dependent on $q_0$ [112].

In spite of these uncertainties the results were not unreasonable. Results
from $K^0\bar{K}^0$ pairs [64] were roughly consistent with those from $\pi^\pm \pi^\pm$ pairs.
For hadron-hadron scattering $R$ was usually about one fermi. For nucleus-
nucleus scattering it was somewhat bigger, perhaps $3 - 4$ fm. In both cases
$c\tau$ was about one fm, perhaps a bit less. According to most papers, though
not to all, $c\tau < R$ and the interaction region had the shape of a pancake
rather than that of a cigar. Within errors, no systematic dependence of
$R$ and/or $\tau$ on energy and/or the kind of hadrons involved was seen. The
collision energies ranged up to $\sqrt{s} = 52.3$ GeV [74], which would have been

20
impossible to study using the old parameters $\gamma$ which hardly depend on the 
charges at so high energy per particle [67].

The KP parameterization was not the only one used. In particular Biswas 
et al. in ref. [44] found

$$C(p_1, p_2) = 1 + (0.80 \pm 0.10)e^{-(11.2^{\pm2.4})Q^2}. \quad (51)$$

According to this result $R(p, p) = 0.8 \pm 0.1$. This was surprising, as ac-
cording to both GGLP and KP, as well as to a more general argument model-
elled on quantum optics [44], $R(p, p) = 1$. The experimental evidence for
$R(p, p) < 1$ was not yet compelling. Moreover, experimental factors like 
particle misidentification and finite momentum resolution could also reduce
$R(p, p)$. Nevertheless, the problem attracted attention.

### 4.4 Further theoretical ideas

One way to obtain $R(p, p) < 1$ is to reject the assumption that the pion 
sources associated with space-time points are incoherent. When all the 
sources act coherently, they can be replaced by one common source. Then 
there is no need to symmetrize and consequently $R(p, p) = 0$ - there is no 
attraction between the momenta of identical bosons. Fowler and Weiner 
[80], [81] pointed out that for a partially coherent source one can obtain 
$0 < R(p, p) < 1$. A surprising conclusion was that in order to explain the 
small reduction observed by [44] one needs about 50% of coherence. In a 
more quantitative model proposed in [45] the coherent sources were respon-
sible for 70% of the pion production. A related strategy was to use a more 
sophisticated model for multiple particle production, where the symmetry 
with respect to exchanges of identical pions is built in from the outset. Work 
along such lines was done by Giovannini and Veneziano [86], but unfortu-
nately it gave the prediction $R(p, p) = 0$ for multiple particle production in 
e$^+e^-$ annihilations, which soon got disproved by experiment.

Implications of the fact that some of the interfering pions were decay 
products of resonances have been considered since the beginning of the KP 
series of papers [94], [108]. In the mid seventies, however, it was established 
experimentally that for multiple particle production in hadron-hadron scat-
tering the fraction of prompt, i.e produced directly and not from decaying 
resonances, pions in multiple particle production processes is small. Typical 
estimates were between 10% and 20% [98]. This suggested that interference 
effects could be used to study resonances, as foreseen earlier [108], but also 
that studies of the Bose-Einstein correlations, which did not take resonances 
into account, might be unrealistic [93], [108]. In particular Grassberger [93]
produced a realistic model including resonances and worked out many of its predictions. He found that the GGLP and KP models require important revisions. Since there is a time lag between the production of the prompt pions and the production of the pions from resonance decays, the interference between pions of these two origins should produce narrow peaks which should be added to the more smooth functions $R(q^2)$ obtained neglecting resonances. This gives $R(p, p) > 1$, but the peaks corresponding to the long lived resonances, in particular to the $\eta$ and $\omega$, are so narrow that they are missed under the experimental conditions and one expects that effectively in practice $R(p, p) < 1$. Another argument in favor of the narrow peaks is that the resonances fly away before decaying which makes the region where the pions are created larger. It should be kept in mind, however, that if the resonances did not move at all, the narrow peaks would still be there because of the time lag. Since the resonance fractions depend on momenta, the correlation functions $C(p_1, p_2)$, should depend not only on the momentum difference $q$, but also on the sum of momenta i.e. on $K$. The pions from resonance decays, especially from the long-lived resonances, are produced in a region, where the density of particles is low, which makes doubtful the claim that pion production happens only on the surface of the production region. Finally, from the explicit formulae it is seen that the correlation functions depend on the orientation of the difference of momenta $q$. For $q$ parallel to the beam axis the peak in $Q^2$ is narrower than that for $q$ perpendicular to the beam. This means that in this model the production region is cigar-shaped. A proposal [93] with a great future was to choose for study, besides the longitudinal (with respect to the beam) component of $q$, its transverse component parallel to the transverse momentum of the pion pair, later called the out component, and the transverse component perpendicular to the transverse momentum of the pion pair, later called the side component. In the examples worked out in the paper $R_{\text{side}}$ is always the smallest. $R_{\text{out}}$ for $p = 0$ must be equal $R_{\text{side}}$ from symmetry, but with increasing $|p_T|$ it increases and is already larger than $R_{\text{long}}$ at $|p_T| = 300$ MeV/c. $R_{\text{long}}$ decreases significantly when $|p_T|$ increases from 100 MeV to 300 MeV. At the time the experimental data were not good enough to check these predictions. The differences between the new picture and the GGLP-KP one were so large that Grassberger concluded: "If they [the peaks] are not found, this would have serious implications for resonance cross-sections. If they are found, however, the currently observed $\pi^-\pi^-$ correlations cannot be the particle physics equivalent of the Hanbury Brown-Twiss effect".

Another influential contribution was the paper by Yano and Koonin [147]. They were interested in heavy ion collisions and their starting point was a
classical single particle distribution

\[ D(x, p) = \frac{E}{\sigma} \frac{d\sigma}{d^3p} (\pi^2 r_0^3 \tau)^{-1} \exp \left[-a(x \cdot P)^2 + b x^2 \right], \]

(52)

where \( P \) is the total initial four-momentum. The physical origin of this formula is simple. The measured momentum distribution is multiplied by a guessed distribution in space-time and the product is interpreted as the phase-space distribution. Since the source function \( D(x, p) \) depends on both coordinates and momenta it is, of course, unsuitable for substitution into the Klein-Gordon equation. The parameters \( a \) and \( b \) can be rewritten as

\[ a = s^{-1} \left(r_0^{-2} + \tau^{-2}\right); \quad b = r_0^{-2}. \]

(53)

In the overall cms frame \( P = (\sqrt{s}, 0) \) and formula (52) becomes

\[ D(x, p) = \frac{E}{\sigma} \frac{d\sigma}{d^3p} (\pi^2 r_0^3 \tau)^{-1} \exp \left[-\frac{t^2}{\tau^2} - \frac{x^2}{r_0^2} \right]. \]

(54)

The corresponding two-body distribution is

\[ \frac{E_1 E_2}{\sigma} \frac{d\sigma_e}{d^3p_1 d^3p_2} = \int d^4x_1 d^4x_2 D(x_1, P_1) D(x_2, P_2) |\phi^e_{p_1 p_2}(x_1, x_2)|^2, \]

(55)

where \( \phi^e_{p_1 p_2}(x_1, x_2) \) is the symmetrized two-body wave function. In the GGLP and KP models symmetrized plane waves had been used. Yano and Koonin considered also Coulomb distorted waves and Coulomb distorted waves including strong \( \pi^-\pi^- \) interactions in the \( I = 2, L = 0, 2 \) states. The Coulomb and strong corrections turned out to be marginal (below 0.1% in the interesting region of \( q \)), so finally only the plane waves were used. These give

\[ R(p_1, p_2) = \exp \left[-\frac{a}{2} r_0^2 \tau^2 (q \cdot P)^2 + \frac{1}{2b} q^2 \right] \]

(56)

and in the overall cms system

\[ R(p_1, p_2) = \exp \left[-\frac{1}{2} q_T^2 \tau^2 - \frac{1}{2} q^2 r_0^2 \right]. \]

(57)

This formula, or formula (54), gives the physical interpretation of the parameters \( \tau \) and \( r_0 \). As compared to the KP formalism it suggests a change from the parameter \( q_T \) to \( q^2 \). Note also the change, with respect to the GGLP formula (21), of the sign in the term proportional to \( q_T^2 \) in the exponent.
Let us consider now the quantum-mechanical interpretation of the YK formulae. Using the plane wave approximation for function $\phi_s$ and the fact that function $D(x,p)$ is real, we get by comparison with (18)

$$|\rho(p_1, p_2)|^2 = \Re \int d^4x_1 D(x_1, p_1) e^{iqx_1} \int d^4x_2 D(x_2, p_2) e^{-iqx_2}. \quad (58)$$

This cannot be generally true, since the right hand side does not even have to be positive. Following YK we assume further, however, that

$$D(x, p) = |A(p)|^2 \rho(x), \quad (59)$$

where $\rho(x)$ is real non-negative. Then the $\Re$ is unnecessary and one can put

$$\rho(p_1, p_2) = A(p_1)A^*(p_2) \langle e^{iqx} \rangle, \quad (60)$$

which formally coincides with (42) and thus, is acceptable as a formula consistent with quantum mechanics. The factorization of the $D(x, p)$ function assumed in (59) means that there are no $p - x$ correlations. This is a reasonable assumption \[101\] for the low energy heavy ion collisions considered in \[147\], but not in general.

When there is no factorization, the YK integrands contain the non-factorizable analogues of $\rho(x_1)|A(p_1)|^2$ and $\rho(x_2)|A(p_2)|^2$ respectively, while in quantum mechanics the integrands contain the non factorizable analogue of $\rho(x)A(p_1)A^*(p_2)$ and its complex conjugate. This can give similar results only under very special assumptions (cf. \[130\]). In order to see how the YK approach can break down when there are momentum-position correlations, let us consider the following example\footnote{This is a simplified version of the models described in refs \[131\], \[127\]. See also \[65\].}. Replace in formula (58) the normalized Gaussian by $\delta^4(x - \lambda p)$, where $\lambda$ is a constant. Then the integrals in formula (58) can be done and one finds

$$|\rho(p_1, p_2)|^2 = \rho(p_1, p_1)\rho(p_2, p_2) \cos (\lambda Q^2), \quad (61)$$

which is evidently wrong since the right-hand side is not positive definite. For some more realistic models where the YK method fails because of the $x - p$ correlations and for its derivation using the "smoothness assumption" see \[138\].

### 4.5 The GKW paper

A detailed discussion of particle correlations in heavy ion collisions was given by Gyulassy, Kauffmann and Wilson (further quoted GKW) \[96\]. This is the...
second most quoted paper (after GGLP [33]) on Bose-Einstein correlations in multiple particle production processes. The starting point of GKW was the Klein-Gordon equation with a classical current on the right-hand side (27), but with $A$ interpreted as a Heisenberg pion field. The normalized solution, which had been well known, is the coherent state

$$|A⟩ = e^{-\pi/2} \exp \left( i \int d^3p \tilde{J}(p)a^\dagger(p) \right) |0⟩,$$

where $a^\dagger(p)$ is the creation operator for a pion with momentum $p$,

$$\tilde{J}(p) = \int d^4x \frac{e^{iE_p (t-ix) \cdot x}}{(2E_p(2\pi)^3)^{1/2}} J(x),$$

and

$$\overline{\pi} = \int d^3p |\tilde{J}(p)|^2.\quad (64)$$

The use of coherent states to describe Bose-Einstein correlations is standard in optics and had been applied to multiple hadron production processes [34], but GKW pushed the analysis much further than their predecessors.

Let us denote by $|J⟩$ the coherent state corresponding to the current $J(x)$. Since as is well known

$$a^\dagger(p)|J⟩ = i\tilde{J}(p)|J⟩,$$

one easily checks that in state $|J⟩$ the multiplicity distribution is Poissonian with average multiplicity $\overline{\pi}$. For a given multiplicity $m$ the momentum distribution is

$$N(p_1, \ldots, p_m) = \prod_{j=1}^m |\tilde{J}(p_j)|^2,$$

and consequently $R(p,p) = 0$ - there is no GGLP effect. Now the authors introduce $N$ source currents related by space-time translations. The density matrix is assumed to be diagonal in $N$ and the probability distribution $P(N)$ is part of the theoretical input. Thus the overall source current for given $N$ is

$$J(x) = \frac{1}{\sqrt{N}} \sum_{j=1}^N J_\pi(x - x_j),$$

25
or equivalently

\[ \tilde{J}(p) = \tilde{J}_\pi(p) \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i\omega_j t - ip \cdot x}, \]  

(68)

where \( \tilde{J}_\pi(p) \) is the Fourier transform of \( J_\pi(x) \). The factor \( 1/\sqrt{N} \) was not used by GKW. It makes, however, the limits \( N \to \infty \) easier to see (cf. e.g. [31]). The probability distribution in space-time for the points \( x_j \) is given by some function \( \rho(x) \). It is convenient to define its Fourier transform

\[ \rho(p) \equiv \rho^*(p) = \int d^4x \rho(x)e^{ipx}. \]  

(69)

Note the crucial difference with respect to the KP approach: it is not assumed that the constituent sources \( J(x - x_j) \) are incoherent. Thus the solution (62) can be used with the new current.

Now the single-particle momentum distribution is

\[ N(p) = |\tilde{J}_\pi(p)|^2 \left( 1 + \langle N + 1 \rangle_{\rho(p)} \right), \]  

(70)

where \( \langle \cdots \rangle \) means averaging over the probability distribution \( P(N) \). One of the important points of the GKW paper is the observation that since realistic functions \( \rho(p) \) decrease rapidly with the increase of any of the components of \( p \), they used as a guide formula (54), and since \( E_p \geq m_\pi \) the second term on the right hand side can be neglected. Another argument for the same conclusion is that function \( \rho(p) \) is small, when \( |p| \) exceeds \( 1/R \), where \( R \) is the radius of the region where the pions are produced, while typical values of \( |p| \) are of the order of \( m_\pi \) or more. Therefore, in the large source limit, which is a reasonable picture of heavy ion collisions, the second term is negligible. It is a correction term which is calculable, can and should be studied, but as a reasonable approximation one can omit it. Then the result is that on the average \( N \) sources contribute just \( N \) times as much as one source, as if the sources were incoherent. In the two-body distribution there are six kinds of terms, but after simplifying as in the previous case only two kinds survive: those which do not contain \( \rho(\cdots) \) and those which contain \( \rho(q) \), where all the components of the argument can be made small. One gets

\[ R(p_1, p_2) = \langle N - 1 \rangle_{\rho(q)} |\rho(q)|^2. \]  

(71)

For large values of \( \langle N \rangle \) the first factor on the right-hand side is approximately equal one and the GGLP result is recovered. Since similar results hold for any \( m \)-pion distribution, one can assume, as a mathematical shortcut, that
the constituent sources are incoherent. GKW called this approximation the chaotic field limit and pointed out that this limit, though not the finite $N$ corrections to it, had been found from essentially the same model in ref. [34]. Because of their careful handling of $N$, however, GKW were able to derive the incoherence of the sources as a reasonable approximation, while in ref. [34] it was introduced as an assumption. GKW realized that the interaction region depends on the impact parameter of the collision, which they consider to be well-defined in a collision. Thus, they recommended either to use data corresponding to a narrow range of impact parameters, or to average incoherently over the impact parameters.

GKW estimated that resonance production is not very important in heavy ion collisions. Therefore, in order to explain the fact that $R(p, p) \neq 1$ they introduced partial coherence [80]. Generalizing the proposal of Fowler and Weiner they introduced the degree of coherence for momentum $p$ by the formula

$$D(p) = \frac{n_0(p)}{n_0(p) + n_{ch}(p)}, \quad (72)$$

where $n_0$ refers to the pions produced coherently and $n_{ch}$ to the pions produced incoherently (chaotically). This function should be obtained from the relation

$$R(p, p) = 1 - D^2(p). \quad (73)$$

The degree of coherence must be known before conclusions about the geometry of the source can be extracted from the data, because\(^5\) e.g.

$$C(p_1, p_2) = 1 + (1 - D(p_1))(1 - D(p_2))\rho^2(q) + 2\sqrt{D(p_1)D(p_2)(1 - D(p_1))(1 - D(p_2))}\rho(q). \quad (74)$$

Only for $D \equiv 0$ one recovers the result [18]. The origin of the coherent component was ascribed to some collective effects in the colliding nuclei. Thus it would be specific to heavy ion collisions.

The most influential part of the GKW paper was their analysis of the final state interactions, i.e. of the interactions taking place after the pions had been produced. These interactions can affect the pion momentum distribution. Final state interactions were considered before [93], [147], but always with the conclusion that they are unimportant. GKW proposed a

\(^5\)Assuming that $\rho(p)$, $\tilde{J}_0(p)$ and $\tilde{J}_{ch}(p)$ are real.
model for these interactions and when it later became used by experimentalists the corrections were often found significant. There are two kinds of final state interactions. The pions move in a single particle potential including the Coulomb potential from the positive charge of the nuclear cores. Moreover, the pions interact with each other. Here one can consider the Coulomb interactions between the charged pions and their strong interactions \[147\].

Taking into account the single particle potential, say \(V(x)\), is comparatively easy, because it does not affect the independent particle approximation. One assumes that the potential \(V(x)\) does not produce particle pairs and does not support bound states. Then it is enough to replace everywhere the pion plane waves, which are solutions to the free particle Klein-Gordon equation, by the corresponding solutions of the Klein-Gordon equation with the potential \(V(x)\). We will denote these solutions \(\psi_p(x)\), where the subscript \(p\) means that this function corresponds to the plane wave describing a free pion with momentum \(p\), i.e. that for \(x^2 \to \infty\) it tends to this plane wave. With the additional assumption that the range of the single particle potential \(V\) is much larger than \(m_\pi^{-1}\), one finds the single particle density matrix

\[
\rho_V(p, p') = \int d^4x \rho(x) \psi_p(x) \psi_{p'}^*(x).
\]

From this density matrix the correlation function is built as usual, using formula \[153\] to construct the necessary elements of the two-body pseudo density matrix. The density matrix may look similar to the density matrix from the KP model with extended sources \[38\], but the physical interpretation is very different. In the KP model the functions \(\langle x | \psi(x_0) \rangle\) can be normalized to one and describe the shape of a source with a finite space extension. Here the functions \(\psi_p(x)\) are almost plane waves extending over all space. They are only slightly deformed in a finite region of space due to the potential \(V(x)\).

The effect of coherence is calculated as before, but using the distorted functions \(\rho_V(p, p'), n_{ov}(p)\) and \(n_{chV}(p)\). The distorted functions are calculated with the plane waves replaced by the functions \(\psi_p(x)\). In practice this part of the final state interaction does not affect much the distribution of the momentum difference \(q\), because both bosons are pushed in the same way by the potential.

The two-pion interactions spoil the independent particle approximation and, therefore, are much more difficult to include. GKW considered various approximations. They pointed out that the distorted wave Born approximation would be the best tool, but it is rather complicated. The main effect is to change the wave function of the pion pair from a symmetrized product of plane waves to a two-body Coulomb wave function. Using the nonrelativistic Schrödinger equation, factoring out the motion of the pair as a whole and
neglecting in the relative coordinate space the changes of the wave function in a region of radius $O(R)$ where $R$ is the radius of the interaction region around the point $\mathbf{x}_1 - \mathbf{x}_2 = 0$, one finds \cite{96} that the pseudo density matrix element $\tilde{\rho}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_1, \mathbf{p}_2)$ gets multiplied by the “well-known Gamov factor”

$$G(\mathbf{p}_1, \mathbf{p}_2) = \frac{2\pi \eta}{\epsilon^{2\pi\eta} - 1}; \quad \eta = \frac{\alpha m}{|\mathbf{p}_1 - \mathbf{p}_2|},$$

(76)

where $\alpha$ is the fine structure constant. According to the theory the measured two-particle distribution contains this factor. Thus, the suggestion is that experimentalists could correct their results for Coulomb interactions by dividing the measured functions $\rho(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_1, \mathbf{p}_2)$ by the corresponding Gamov factor.

Let us look at the numbers. The Bohr radius for a pair of identical particles of mass $m$ and unit charge is

$$a_B = \frac{2}{\alpha m},$$

(77)

where $\alpha = 1/137$. This is 1.96 MeV$^{-1}$ or 386 fm for charged pions and 0.56 MeV$^{-1}$ or 110 fm for charged kaons. Since the Bohr radii are much larger than the interactions radii $R$, which do not exceed several fermis, it had long been believed that the Coulomb final state interactions can be safely neglected. From the formula for the Gamov factor it is seen, however, that the dimensionless parameter is not $R/a_B$, but $1/(qa_B)$, which can be large when $q$ is small. Moreover

$$2\pi \eta = \frac{4\pi}{qa_B} \equiv \frac{c}{q},$$

(78)

The coefficient $c$ is 25 MeV for pions and 7 MeV for kaons. It is enhanced by the $4\pi$ factor.

5 Refinements and criticism in the eighties

5.1 New experimental results and problems

In the eighties the Bose-Einstein correlations were studied in further processes. They were found in $\mu p$ \cite{27} and $\nu D$ \cite{13} collisions and studied in detail in $e^+e^-$ annihilations \cite{4}, \cite{14}, \cite{15}, \cite{30}, \cite{104} and $\gamma\gamma$ scattering \cite{104}. The radii of the hadron production regions in these processes, as compared to hadronic interactions, were found somewhat smaller, as expected. The values of $R(p,p)$, however, were not reduced. This contradicted the prediction from some models (cf. e.g. \cite{86}) that $e^+e^-$ annihilations should be much
more coherent than hadronic interactions and consequently exhibit little or no GGLP effect. In fact, it made the interpretation of $\lambda$ as the coherence parameter doubtful. Correlations between charged kaons have been found and studied [6]. In general, there was a spectacular improvement in the quality of the data.

As the experimental data improved the uncertainties and ambiguities in the analysis of Bose-Einstein correlations became apparent. In ref. [71] the important result [44] that $R(p,p) \neq 1$ was confirmed. The authors used a formula similar to the KP formula (47)

$$\frac{N_L}{N_{bcg}} = \text{Const} \left(1 + \lambda \frac{I^2(qTR)}{1 + (q_0\tau)^2}\right), \quad (79)$$

where $I(x) = 2J_1(x)/x$. The introduction of the factor $\lambda$ into the interference term significantly improves the fits to experimental data and was soon followed by most groups. Note that in such parameterizations $R(p,p) = \lambda$. Where we write in the following $R(p,p)$, many writers would have written $\lambda$. The interpretation of this coefficient, however, remained doubtful. Experimental uncertainties, coherence and resonance production were all invoked. The left-hand side is here the ratio of the distribution for pairs of like sign pions to the corresponding background distribution of pion pairs. The strongest result in ref. [71] was that for pairs of like sign pions produced in $\pi^+p$ interactions at 16 GeV/c incident momentum: $\lambda = 0.49 \pm 0.02$. Statistically this is about 25 standard deviations from one, but changing the background from pairs of unlike charge pions to another background, which looked at least as reasonable though later it was criticized [1], the authors obtained $\lambda = 0.88 \pm 0.02$. Introducing the Gamov factor, which was not done in ref. [71], further significantly increases the fitted values of the parameter $\lambda$ [38], [61], [148]. Incidentally, introducing the Gamov factor makes the fits worse. In a particularly bad case [148] the confidence level dropped from 65% to 0.1%. This explains why not all the groups were willing to include this correction factor. The uncertainty in $\lambda$ was particularly striking for $\bar{p}p$ annihilations at rest [71]. With the background of unlike sign pion pairs, $\lambda = 1.20 \pm 0.08$ was the biggest among the three $\lambda$-s for the reactions considered in the paper. With the other background it became 0.63 $\pm$ 0.05 - the smallest of the three. For $Q^2$ decreasing towards zero function $R(p_1,p_2)$ was found to increase faster than the Gaussian fit suggested. This was described either, following [121], by including in the fit another Gaussian with a larger value of $R$ [8], [9], [25], [31], [116], or by using exponentials in $Q$ instead of Gaussians [4], [116], [123]. Taking into account this additional rise again increased the value of $\lambda$. The uncertainty in $\lambda$ is seen to be large and mainly
systematic. It was not clear how to reduce it or even how to estimate it. This has made progress in the understanding of the origin of this factor difficult. The key problem is a good choice of the background distribution.

Let us note some new ideas concerning the evaluation of the background distributions. In order to eliminate the effects of resonances and their reflections, the measured distributions for like meson pairs and those for the background were divided by the corresponding Monte-Carlo distributions [14], [15], [27]. A factor \((c_0 + c_2Q^2)\) [1], [14], [27], [58], [104], \((c_0 + c_1Q)\) [4] (with constant \(c_i\)-s), or similar [15], [31], were introduced into \(C_0(p_1, p_2)\) to keep the correlation function close to one in the region, where Bose-Einstein correlations are no more expected. The groups using mixed backgrounds realized that, in order to get a good mixed background, members of the pairs of events supplying the background pairs must be similar. Depending on the process this could mean similar multiplicity (cf. e.g. [1], [38], [39]), rotating one of the events to make the sphericity axes coincide (cf. e.g. [4]) etc. The mixed distribution was sometimes [4], [148] improved as follows. From (48) one finds

\[
N(p_1) = N' \int d^3p_2 \frac{N(p_1, p_2)}{1 + R(p_1, p_2)}, \tag{80}
\]

where \(N'\) is a suitable normalizing factor. In first approximation \(R(p_1, p_2) = 0\) and this is just the single particle distribution as used to build the ordinary mixed background. One can do better, however, by substituting into the integrand the function \(R(p_1, p_2)\) calculated in the first approximation. After a few iterations one gets a single particle distribution which satisfies the equation above. This, however, is a small correction. Note that the product of such single particle distributions still does not contain the two-body non-Bose-Einstein correlations between identical particles, which should be there.

Since the results for the life-time \(\tau\) were also usually found with large uncertainties, most of the discussion concerned the radii \(R\) of the interaction regions. As a crude characteristics one could say that within the large uncertainties all these radii are similar except for the much bigger radii found in heavy ion collisions. Nevertheless, it was possible to find some interesting regularities. Indications were found [6], [64] that when identical kaons are used, the radii come out smaller than the corresponding radii obtained when identical pions are used. This could mean a decrease of the effective (measured) radius with increasing mass of the identical particles.

The increase of \(R\) with increasing particle multiplicity or particle density in pseudorapidity was at first somewhat confusing. For hadron-hadron collisions, when increasing the particle multiplicity by selection or by rising the energy, an increase of \(R\) was visible, but only at energies \(\sqrt{s} > 30 GeV\) [5],
For $e^+e^-$ collisions at energies $\sqrt{s} < 30$ GeV no increase was seen. For heavy ion collisions increasing the mass number $A$, or reducing the collision parameter gave an increase of $R$ even for energies of the order of $1$ GeV \cite{3, 10, 40, 124}. What counts here is the mass number of the projectile, provided the target is heavy enough \cite{3}. This has been nicely illustrated in ref. \cite{69}, where for $pp$ and $p + Xe$ scattering the radii are the same within errors, while the particle density in rapidity changes by a factor of two. Refs. \cite{32, 102} reported for $^{16}O + Au$ collisions a strong increase of $R_T$ (almost by a factor of two) for rapidities of the pion pairs close to the cms rapidity, but this was not confirmed. From a thorough compilation of data for heavy ion collisions it was concluded \cite{35} that the measured radius $R$ increases linearly with the cubic root of the mass number of the lighter of the two colliding nuclei. A fit gave $R = 1.21 A_p^{\frac{1}{3}}$. Some evidence for the decrease of the measured radius with increasing momentum \cite{25} and transverse momentum \cite{10} of the pion pair was also reported.

Another interesting observation was that for $e^+e^-$ \cite{15, 30, 104} and hadron-hadron \cite{7, 8, 58} interactions, $R(p_1, p_2)$ could be well approximated by a function of $Q^2$ only – in first approximation by a Gaussian as proposed by GGLP. This has in particular two important implications. Contrary to the Ansatz made for heavy ion collisions in ref. \cite{147}, for $e^+e^-$ annihilations the enhancement due to Bose-Einstein correlations is large for $Q^2$ small even if $q_0^2$ and $q^2$ taken separately are large. The other conclusion is that the distribution of $q$ in the rest frame of the pair is spherically symmetric. In the KP formula the increase of either $q_0^2$ or $q_T^2$ reduces $R(p_1, p_2)$ like in \cite{147}, but because of the absence of the dependence on $q_\parallel$ (of the component of $q$ along the direction of $p_1 + p_2$) the fits are much better than in the YK case. The reason is that \cite{15, 104}

$$e^{-\frac{\mu^2}{4} q_\parallel^2 + \frac{\gamma^2 \mu^2}{4 (\gamma^2+1)} q_0^2} \approx 1,$$

where $\gamma$ is the Lorentz factor for the transformation from the cms to the rest frame of the pion pair. Introducing this factor into the KP formula for the correlation function is approximately equivalent to a change from $q_T^2$ to $q^2$ accompanied by the introduction of a term, which increases with increasing $q_0$. Thus the KP formula can be rewritten so that it becomes similar to the GGLP formula.

### 5.2 String model

For $e^+e^-$ annihilations, where the final state is believed to evolve from a single string, the effects of Bose-Einstein correlations were calculated \cite{51, 17}
in the framework of a slightly extended \cite{17}, \cite{28} string model. The string model is closely related to the realistic and highly successful LUND model \cite{16}. Therefore, its predictions concerning the distribution and properties of particle sources must be taken seriously. Already the qualitative results were striking. There are strong correlations between the position of the source and the momentum spectrum of the particles it emits. As a result, the volume determined from the study of Bose-Einstein correlations is not the volume of the interaction region. Only particles with similar momenta contribute to the GGLP effect. Therefore, the measured volume is the volume of the region, where pions with similar momenta are being produced and not the total interaction volume. This has little effect on the transverse dimensions, but the measured longitudinal dimension becomes almost energy independent, while the true length of the interaction region is believed to grow roughly like $\sqrt{s}$. In order to introduce $\mathbf{x} - \mathbf{p}$ correlations into the KP sources \cite{113} it is enough to remove the assumption that the sources are equivalent \cite{51}, but to find a quantitative relation between the string model and the KP model generalized in this way is not easy. The attempt in ref. \cite{53} does not look very convincing. Thus the geometrical interpretation of the measured volume is somewhat doubtful. The one-dimensional string model, which is more constrained than the three-dimensional version, suggests that the space-time distribution of sources should depend mainly on the variable $\tau^2 = t^2 - z^2$. A natural extension to three dimension is to assume that this distribution depends on $\tau^2 = t^2 - r^2$, which according to the standard method of analysis corresponds to a correlation function depending only on the variable $Q^2$ \cite{51}. This is supported by experiment. The model suggests also that the distribution in $Q$ is more peaked at small $Q$ and has a longer tail at large $Q$ than a Gaussian. At the quantitative level the comparison of the model with experiment is difficult and requires the use of Monte Carlo programs and additional assumptions e.g. about resonance production. Nevertheless, good agreement with the data from ref. \cite{4} was obtained \cite{17}, \cite{52}, though after an interesting detour. At first it seemed \cite{17} that the introduction of long lived resonances ruins the agreement. Then it was pointed out \cite{52} that the amount of $\eta'$ (0.41 per event) commonly assumed at that time was only a guess and probably a very bad one. Neglecting $\eta'$ production good agreement with experiment was obtained. Later experiment confirmed that the production of $\eta'$ in $e^+e^-$ annihilations is indeed small.
5.3 Wigner functions, emission functions and covariant currents

It has always been a problem how to put together the available information about the space-time structure of the interaction region and about the momentum distribution of the produced particles. Usually in quantum mechanics one works in the momentum representation or in the coordinate representation, but not in both simultaneously. KP suggested to consider the space-time distribution of the points labelling the incoherent sources and a wave function in the momentum representation, the same for each source [113]. Yano and Koonin [147] introduced a distribution $D(x, p)$ without bothering about its interpretation in quantum mechanics. Pratt [135] proposed to use for the sources the Wigner function (cf. [100] and references quoted there). The Wigner function $W(X_1, K_1)$ can be defined by either of the two equivalent formulae:

$$W(X_1, K_1) = \int \frac{d^3q_1}{(2\pi)^3} \rho(K_1 + q_1/2, K_1 - q_1/2, t_1) e^{i q_1 \cdot X_1}$$  \hspace{1cm} (82)$$

$$W(X_1, K_1) = \int \frac{d^3Y_1}{(2\pi)^3} \rho(X_1 + Y_1/2, X_1 - Y_1/2, t_1) e^{-i Y_1 \cdot K_1}.$$  \hspace{1cm} (83)$$

Inverting the first equation one finds

$$\rho(p_1, p_1', t_1) = \int d^3X e^{-i q_1 \cdot X} W(X_1, K_1).$$  \hspace{1cm} (84)$$

where in the exponent stands the Lorentz invariant product of four-vectors. Further the subscripts $I$ will be skipped.

Now come the characteristic assumptions of this approach: the sources are labelled by their freeze out time $t_0$, and the summation over the sources, assumed incoherent, is reduced to an integration, with a suitable weight factor, over the freeze out time $t_0$. Since for any source $\rho_I(p, p', t)$ does not depend on the time $t$ for $t \geq t_0$, where $t_0$ is the freeze out time of this source, one can put $t = t_0$ on the right hand side. Then the formula for the full
single particle density matrix at any time $t$ larger than the latest freeze out time, reads

$$\rho(p_1, p'_1) = \int d^4X e^{i q_1 \cdot X} S(X, K_1),$$

(86)

where the matrix element on the left hand side is time independent and the time component $X_0$ on the right hand side is understood as the freeze-out time $t_0$ of the source labelled $t_0$. As seen from this derivation function $S$, which will be further referred to as the emission function\(^6\), can be understood as a product of the Wigner function for the particles produced by the source $t_0$ and of the density factor necessary to convert the summation over the sources into the integration over $t_0$. The point is, however, that besides this emission function there is an infinity of other emission functions which, when substituted into this formula, yield the same density matrix and therefore are equally acceptable. In particular a small source with a large life-time can give the same correlation function as a large source with a short life-time\([136]\). The statement\([135]\) that $S(x, p)$ "can be identified as the probability of emitting a pion of momentum $p$ from space-time point $x$" suggests a way of thinking about the source function which has become very popular. What can be proved, however, is much less: the Wigner function $W(x, p)$ when integrated over space $(d^3x)$ gives the momentum distribution at time $t$ and when integrated over momenta $(d^3p)$ gives the distribution in space-time\([100]\). Nevertheless, it soon became customary to discuss the source function using Pratt’s picture.

When sources freezing out at different times contribute coherently the relation of the emission function to Wigner functions becomes rather tenuous\([151]\). In this case it seems better to relate the emission function to the history of the sources rather than to some states of the system\([139], [62]\). Let us define the emission function in space-time by the formula (c.f. formula (29))

$$\tilde{S}(X_1, Y_1) = \sum_i \langle J_i(x_1) J_i^*(x'_1) \rangle .$$

(87)

Then according to Shuryak’s formula\([33]\)

$$\tilde{\rho}(p_1, p'_1) = \int \tilde{S}(X, Y) e^{i K_1 Y + i q_1 X} d^4X d^4Y,$$

(88)

\(^6\)Also known as Wigner function, pseudo Wigner function, source function etc. The name emission function was used in ref.\([136]\).
Formula (86) is reproduced for

\[ S(X_1, K_1) = \int \tilde{S}(X_1, Y) e^{iK_1 Y} d^4 Y. \]  

(89)

This is an unambiguous definition of the emission function. The emission function is real, because complex conjugation is equivalent to a change of the sign of \( Y \) and this can be compensated by a change of the integration variables from \( Y \) to \(-Y\). Note, however, that in order to calculate this emission function it is not enough to know the state of the system at some time, or even to know the state of the system at every time. Phase relations among the states generated at different times are also important. The quantity being averaged depends on two times, and not on one as would be case for state vectors, density matrices or Wigner functions.

Using the formula for the single particle density matrix and formula (18) one finds

\[ C(p_1, p_2) = 1 + \frac{\left| \int d^4 X S(X, K) e^{i qx} \right|^2}{\int d^4 X S(X, p_1) \int d^4 X' S(X', p_2)}. \]  

(90)

Often it is considered an acceptable approximation to replace each of the momenta in the denominator by \( K \). Then the formula takes a simpler form

\[ C(p_1, p_2) = 1 + \left| \frac{\int d^4 x S(x, K) e^{i qx}}{\int d^4 x S(x, K)} \right|^2. \]  

(91)

The second term on the right hand side is the function \( R(p_1, p_2) \).

An alternative approach has been worked out by Gyulassy and collaborators [96], [106], [97], [129], [130]. It is an extension of the labelled wave packets method of KP [113]. Suppose that each source (wave packet) is labelled by a space-time point \( x_s \) and a momentum \( p_s \). The energy corresponding to this momentum is obtained from the condition \( p_s^2 = m^2 \). Then one assumes that

\[ \rho(p_1, p_2) = \langle e^{i q x_s} \rho_{p_s}(p_1, p_2) \rangle, \]  

(92)

where the averaging is a summation over the sources \( s \). The new point, as compared to the KP formula [12], is that the single source density matrix on the right-hand side depends on \( p_s \). A particularly important special case is when

\[ \rho_{p_s}(p_1, p_2) = j(\frac{p_s p_1}{m}) j^*(\frac{p_s p_2}{m}). \]  

(93)
Interpreting \( p_s \) as the average momentum of the pions emitted by the source this means that each source looks the same in its rest frame, while for KP each source looked the same in the overall frame of the event (except for a space-time translation). Also the condition \( p_s^2 = m^2 \) means that the labels correspond to the emitted pions rather than to a heavy source. Further, it is assumed that the averaging over the sources can be expressed as an integration with weight function \( D(x_s, p_s) \). Thus

\[
\tilde{\rho}(p_1, p_2) = \int d^4 x_s \int d^3 p_s D(x_s, p_s) \rho_{p_s}(p_1, p_2) = \\
\int d^4 x_s \int d^3 p_s D(x_s, p_s) j\left(\frac{p_s p_1}{m}\right) j^*(\frac{p_s p_2}{m})
\]

This version of the model is known as the covariant current formalism. A more general model, where the single source density matrix does not factorize, i.e. does not correspond to a pure state, and the second equality does not hold, was considered in [130]. The authors proposed

\[
\rho_{p_s}(p_1, p_2) = N \exp \left[ \frac{(p_s - (p_1 + p_2)/2)^2}{2\Delta p^2} + \frac{(p_1 - p_2)^2 \Delta x^2}{2} \right],
\]

where \( N \) is the normalization constant, while \( \Delta x^2 \) and \( \Delta p^2 \) are constants, variances of \( x \) and \( p \) respectively, constrained by the Heisenberg uncertainty relation \( \Delta x^2 \Delta p^2 \geq 1/4 \). This density matrix factorizes if and only if the terms proportional to \( p_1 p_2 \) in the exponent cancel, or equivalently when the product \( \Delta x^2 \Delta p^2 \) has the smallest value allowed by the Heisenberg uncertainty principle, i.e. for minimal wave packets [130]. The factors are then functions of \( (p_s - p_i)^2 = 2m^2 + 2p_s p_i, \ i = 1, 2 \) and the density matrix can be written in the form (93)

Both approaches described here use simultaneously momenta and coordinates. Their interpretations, however, are different. In the Wigner function, or Shuryak, approach \( X \) and \( K \) are arithmetical averages of the arguments of \( \tilde{\rho}(x, x') \) and \( \tilde{\rho}(p, p') \) respectively. In the wave packets approach they are labels of the sources, which may, but do not have to, be the centres of the wave packets. Single particle energy is no problem since the particles are on mass shell. Time in the wave packet formalism is just one more label. In the Wigner function approach its meaning depends on the details of the model.
5.4 Difficulties with the final state interactions

Final state interactions were further studied. It had been known (cf. Section 4.5) that the two-body Coulomb interactions are controlled by the dimensionless parameter $\xi_c = R/a_B$ [136], where $a_B$ is the Bohr radius and $R$ is the radius of the interaction region. For pion pairs $a_B \approx 386$ fm. Consequently, $\xi_c \approx 0.01$ for heavy ion collisions and much less for the other cases. Therefore, it had been believed for some time that Coulomb correction cannot be important. This conclusion is not justified, because the factor $R$ in the small parameter $\xi_c$ enters as an estimate of $1/|q|$. In the very small $|q|$ region this is a gross underestimate. Moreover, as seen from the Gamov factor, there are large numerical coefficients. The leading correction in the small $\xi_c$ limit is given by the Gamov factor [96]. This correction is significant and spoils the agreement with experiment in the small $Q^2$ region (cf. e.g. [85]).

Pratt [136] has used the GGLP model with a Gaussian distribution of sources and with the symmetrized product of plane waves replaced by a symmetrized Coulomb wave functions. This method of taking into account Coulomb interactions had been used before [107], [120], but mainly in the study of the Fermi-Dirac correlations among nucleons. Pratt found that including the next term of the expansion in powers of $\eta$, besides the Gamov factor, changes the estimated interaction radius by about 20% for $R$ as small as a few fermi.

In order to describe the two-body strong interaction among identical charged pions it is enough to consider the $s$-wave, $I = 2$ phase shift [144]. The observed phase shift corresponds to repulsion, as one would expect for an exotic two pion system where no resonances can be formed. The key observation [54] is that since the interactions are short range (about 0.2 fm) they are negligible for pions produced at a distance of one fermi, or so, from each other. Consequently, this correction for strong interactions (not to be confused with the corrections for resonance production) is negligible for the study of Bose-Einstein correlation in heavy ion collisions. For $e^+e^-$ annihilations, on the other hand, this correlation can be important. According to a rough estimate [54] it reduces the factor $\lambda$ by at least 30%. Moreover, in agreement with experiment, it produces a minimum of the correlation function at $Q \approx 0.6$ GeV, where the correlation function drops below one. In spite of this success the quantitative applicability of this model, where only an isolated pion pair is considered, is doubtful [55]. For $\pi^+\pi^-$ pairs there is an attractive strong interaction due to the $I = 0$ s-wave phase shift. Therefore, three-body $\pi^+\pi^-\pi^+$ interactions give an effective attraction in the $\pi^+\pi^+$ system which, according to rough estimates, could easily overcompensate the repulsion due to the $I = 2$ phase shift [55]. A more careful analysis including
resonance production \cite{56} suggests that the strong interactions must somehow cancel, because otherwise, the fully corrected $\lambda$ would come out negative, which does not make sense. Another important remark \cite{54} is that experimentally, at low $Q^2$, the backgrounds formed from unlike sign pions are not very different from the mixed backgrounds, where by construction the strong interaction among members of pairs is absent. This supports the point of view that final state interactions are less important than naively expected. A study of two-body correlations for pairs of unlike sign pions could bring here interesting additional information. Bowler's final conclusion for the strong final state interactions is \cite{57} "I think there is good reason to believe that in multiple production the effects of final state interactions among all pairs largely CANCEL and only the propagation and decay of resonances modifies the underlying source structure". An exact analysis of such many body strong interaction is, however, "impossibly complicated" \cite{56}. Bowler concludes also \cite{55} that "the chaoticity and range parameters extracted from experiment have no simple interpretation and should be regarded as purely descriptive".

5.5 Early models with $x-p$ correlations

The improved formalism has been used to build models more realistic than the early attempts of GGLP and KP. A common feature of these models is that they include $x-p$ correlations. Thus Pratt \cite{135} considered

$$S(x,p) = \delta(r-R)\delta(t) \exp \left[ -\gamma \frac{E_p - \mathbf{v} \cdot \mathbf{p}}{T} \right], \quad (96)$$

where the temperature $T$, velocity $v$ and Lorentz factor $\gamma = 1/\sqrt{1-v^2}$ are constant. The pions are ejected from a sphere of radius $R$ and tend to go radially outwards. Thus, contrary to the KP picture, there are $x-p$ correlations. Evaluating $R(p_1,p_2)$ for this source function one finds \cite{135} that the effective (i.e. measured) radius of the interaction region is:

$$R_{\text{eff}}(K) = R[(y \tanh y)^{-1} - (\sinh y)^2]^{1/2}; \quad y = \frac{|K| \gamma v}{T}. \quad (97)$$

The expression in the square bracket equals $\frac{2}{3}$ for $|K| = 0$ and decreases towards zero with increasing $|K|$. However, in this model we know that the true radius of the interaction region is $R$ for all $K$. Let us stress the implications of this important observation. One should study the Bose-Einstein correlations at given $|K|$. A $|K|$ dependence of the deduced radius of the

\footnote{And in fact latter brought.}
interaction range implies the presence of $x - p$ correlations. Wrong assumptions about the $x - p$ correlations lead to wrong determinations of the radius of the interaction region. For instance in Pratt’s model averaging over $|K|$ and interpreting the results as if there were no $x - p$ correlations one obtains a radius below $\sqrt{2/3}$ of the correct (input) value.

A more realistic model with $x - p$ correlations, inspired by Bjorken’s inside-outside cascade, was given in ref. [106] in the framework of the covariant current approach. The authors chose

$$D(x, p) = \frac{1}{\pi R_T^2} \delta(t - \tau_0 \cosh y)(\delta(z - \tau_0 \sinh y)e^{-x^2/R_T^2}, \quad (98)$$

$$j\left(\frac{p_sp_p}{m}\right) = \sqrt{a} \exp\left[-\frac{p_sp_p}{2mT}\right], \quad (99)$$

where $a,T,R_T$ and $\tau_0$ are constants. This corresponds to a distribution of sources which is uniform in rapidity and concentrated at the longitudinal proper time $\tau_0 \equiv \sqrt{t^2 - z^2}$. Each source in its rest frame produces pions according to a ”pseudothermal model” which is a modification of the Boltzmann distribution chosen so as to make the subsequent integrations possible to perform analytically. The corresponding single particle pseudo density matrix is [106], [130]

$$\rho(p_1, p_2) = aK_0(\sqrt{u})e^{-u^2/R_T^2}, \quad (100)$$

$$u = \left[\frac{m_{1T} + m_{2T}}{2T} - i\tau_0(m_{1T} - m_{2T})\right]^2 +$$

$$2\left(\frac{1}{4T^2 + \tau_0^2}\right)m_{1T}m_{2T}[\cosh(y_1 - y_2) - 1], \quad (101)$$

where $m_{iT}$ are the transverse masses, $y_i$ the rapidities of the two particles and $K_0$ is the modified Bessel function. It is interesting that this model, extended by the inclusion of resonances which requires the use of a Monte Carlo program, explains the data of the NA35 experiment [32], which had been previously explained [43] by assuming the presence of a quark-gluon plasma. Resonance production is the most important correction, because other corrections like a spread in the freeze-out proper time $\tau_0$ or a nonuniform distribution in rapidity tend to cancel [129]. This analysis can be generalized and made more rigorous [130]. Let us also note attempts to use Monte Carlo programs to get more realistic distributions of the sources [101], [66].

A rather different approach to Bose-Einstein correlations in hadron and heavy ion collisions was developed by Makhlin, Sinyukov and collaborators
Their starting point was Landau’s hydrodynamic model. According to this model the production of hadrons in high energy hadronic interactions proceeds in three stages. First the content of the interaction region thermalizes. This happens very fast. Then the fluid expands and cools according to the laws of hydrodynamics. Whenever the temperature of an element of the fluid drops to a critical temperature \( T = T_{cr} \), its content gets converted into hadrons which have the thermal equilibrium distribution corresponding to temperature \( T_{cr} \). This hadronization of a fluid element is also assumed to be a very rapid process. Since at every space point the temperature is a function of time, the space-time points where \( T(x, t) = T_{cr} \) form a three-dimensional hypersurface \( \Sigma_{cr} \) in space-time and all the hadrons are produced there. In order to get predictions one should in principle assume some initial conditions i.e. the distribution at the beginning of the hydrodynamic stage, solve the hydrodynamical equations with these initial conditions and find the hypersurface \( \Sigma_{cr} \) as well as the velocity distribution of the fluid on it. This would be a very difficult task, but there are simple estimates. E.g. one can define \( \Sigma_{cr} \) by the equation \( t^2 - z^2 = \tau^2 \), where \( \tau \) is a constant and assume that the fluid velocity is parallel to the \( z \) axis and at space-time point \( x, t \) equal \( z/t \). The authors suggest that by selecting events with a given heights of the central plateau in rapidity one can approximately fix \( \tau \). There is also a more complicated estimate due to Landau. When transverse expansion is included, parts of the hypersurface \( \Sigma_{cr} \) become time like which complicates the formalism, but can be dealt with \[143\]. The hydrodynamic approach got very popular later, though usually it just means introducing collective velocities of the sources.

5.6 General proposals

Some results more general than specific models have also been given. For heavy ion collisions the plausible assumption that the transverse profile of the interaction region is given by the overlap of the two nuclear distributions explains easily two experimental facts. For a given pair of ions the average transverse radius of the interaction region \( R_T \) increases with decreasing impact parameter \[33\] and for central collisions \( R_T \) is given by the radius of the smaller of the two colliding ions \[60\].

In ref. \[117\] the authors, generalizing earlier results from \[132\], proved that when \( \rho(p_1, p_2) \) integrated over \( p_1 + p_2 \) can be written in the form \( \rho(a^2 q_T^2 + q_z^2) \), where \( a \) is a constant, the distribution of the angle \( \theta \) between
vector $\mathbf{q}$ and the axis of the event $z$ is

$$\phi(\cos \theta) = \frac{a^2}{2[a^2 + (1 - a^2) \cos^2 \theta]^{3/2}}. \quad (102)$$

This formula, which follows by simple integration, can be useful to distinguish, cigar shaped ($a^2 < 1$) from pancake shaped ($a^2 > 1$) sources. Pratt [137] and Bertsch [42] reintroduced for the difference of momenta $\mathbf{q}$ the coordinate system which had been used some ten years before by Grassberger [93]. The notation and terminology is from Bertsch [42]: $q_L$ for the longitudinal component along the event axis, $q_o$ for the outward component parallel to $K_T$ and $q_s$ for the sideward component perpendicular to both the event axis and to $K$. His formula

$$R(p_1, p_2) = e^{-\left[\frac{1}{2}q_z^2 + \frac{1}{2}q_y^2 + \frac{1}{2}R_L^2 q^2\right]}$$

soon became very popular. The main point of both authors was that, if the time interval when the pions are produced is long - this could be due to a slowly hadronizing quark-gluon plasma but other possibilities were also considered - then one expects $R_o \gg R_s$, because the pions produced late behave like pions produced far along the out direction. Let us finally note a diagrammatic classification of the various terms occurring in the theory of multiparticle Bose-Einstein correlation with coherence included [46]. For an attempt to estimate the effect of multibody Bose-Einstein correlations on the two particle correlation function by direct calculation of the permanents see [149].

Podgoretsky [133] [134] studied the dependence of the parameters $R$ and $\tau$ on the boosts along the event axis. Let us denote the event axis by $z$ and the axes in the out and side directions by $x$ and $y$ respectively. Let us further assume that in some reference frame the distribution of sources is given by a formula similar to the Yano-Koonin formula (54):

$$\rho(x) = \frac{1}{4\pi^2 R^2 RT} \exp \left[ -\frac{x^2 + y^2}{2R^2} - \frac{z^2}{2R^2} - \frac{\tau^2}{2T^2} \right]. \quad (104)$$

Then, in the reference frame moving with respect to this frame with velocity $\beta$ (Lorentz factor $\gamma$) parallel to the $z$ axis, the distribution of $\mathbf{q}$ is [133]

$$W \sim 1 + \exp \left[ -q_x^2 R^2 - q_y^2 R^2 - \gamma^2(q_z(1 - \beta \bar{u}_z) - q_x \beta \bar{u}_x)^2 R^2 - \gamma^2(q_z(\beta - u_z) - q_x u_x)^2 T^2 \right]. \quad (105)$$

---

8Actually in the paper the coefficient of $q_L^2$ reads $\frac{1}{2}R_L^{-2}$, but this seems to be a misprint.
Here $u$ is the velocity of the pion pair $K/K_0$ and the identity

$$q_0 = u \cdot K$$

has been used. Yano and Koonin [147] got a similar result, but instead of using explicitly the Lorentz transformation they expressed the exponent in terms of invariants, see formula (102). Podgoretsky noticed that whenever there is a frame where all the pion sources are at rest and where their distribution is invariant under the change of orientation of the event axis, the parameters $R_L$ and $\tau$ must be extremal in this frame. Thus, it could be possible to identify the frame, where all the sources are at rest. Early estimates [20], [21] favored the quark frame, but later the much more detailed analysis of [2] demonstrated, that there is no clear minimum of $R_L$ as a function of $p_p/p_{\pi^+}$ and consequently no privileged frame, where all the sources are at rest. On the other hand the parameterization (105) turned out to be very useful.

### 6 Conclusions

In his review talk entitled *The GGLP effect alias Bose-Einstein effect alias HBT effect* at a conference in Marburg in 1990 [90] G. Goldhaber said: What is clear is that we have been working on this effect for thirty years. What is not as clear is that we have come much closer to a precise understanding of the effect. Let us discuss this statement from the present point of view.

There is no doubt that GGLP was a great paper. It correctly identified the Bose-Einstein correlations as the reason for the small $Q^2$ enhancement for the like sign pion pairs in $\bar{p}p$ interactions. Moreover, the authors proposed the invariant Gaussian parameterization, noticed the possibility of getting from BEC information about the interaction region, pointed out that a joint analysis of many experimental distributions should be made and mentioned the possible importance of resonances.

The most important message: the possibility of measuring the interaction region, however, passed unnoticed. Only after the work of Kopylov, Podgoretsky, and Cocconi in the early seventies it became gradually accepted that BEC are more interesting as a tool to study the interaction region than as a peculiarity of the distribution of the relative momentum of identical pions. This was certainly a breakthrough. Another breakthrough happened

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9 Amusingly, in the special case of $\bar{p}N$ interactions at low energy there is now evidence [84], [19], [26] that the GGLP effect is due to resonances more than to BEC between directly produced pions.
on the technical side. The GGLP method of modifying the integrand of the integral over phase space was applicable only to low multiplicity exclusive channels, while the method, pioneered by Kopylov and Podgoretsky, of modifying the correlation function could be applied at any (sufficiently high) multiplicity and also to inclusive distributions (at sufficiently high energy).

The relation between the properties of the interaction region and the data on interparticle correlations turned out to be much more complicated than anticipated. Final state interactions, resonances, experimental resolution and particle identification are all important factors. Even, however, when all these technical problems are solved, important sources of systematic error remain. Once it is recognized that the hadrons are not all produced instantly and simultaneously, the problem arises: how to reconstruct a four-dimensional, space-time distribution of sources from the measured three-dimensional distribution of momenta. Moreover, BEC can be only observed among particles with similar momenta. Particles with similar momenta, according to many models, are produced in a homogeneity region, which is a fraction of the total interaction region. Thus, what one measures is the size of the homogeneity region and not that of the total hadronization region. In other words, it is not possible to draw conclusions from BEC without previous knowledge about the position- (in space-time) momentum correlations of the produced hadrons.

The conclusion is that attempts to get information about the interaction region from a model independent analysis of the data seem hopeless. It is necessary to start with a model and then, within the model, it may be possible to use the data on BEC to get information about the interaction region. This information may be crucial, like in the case of hydrodynamic models, or of marginal interest like in string models.

To summarize: the idea to study BEC as a way of learning about the interaction region has certainly made their study much more interesting. Many problems not anticipated by GGLP have been identified and some of them have been solved. A reliable and non controversial method of getting quantitative information about the interaction region from BEC is still, however, not available, as was strikingly illustrated by the failure of the predictions made for RHIC. If this is meant by the precise understanding of the effect Goldhaber’s position is still defendable.

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