THE EIGHTH MOMENT OF THE FAMILY OF
$\Gamma_1(q)$-AUTOMORPHIC $L$-FUNCTIONS

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ABSTRACT. We prove a Lindelöf on average bound for the eighth moment of a family of $L$-functions attached to automorphic forms on $GL(2)$, the first time this has been accomplished. Previously, such a bound had been proven for the sixth moment for our family by Djanković [4] and for a similar family by Young [11]. Our proof rests on a new approach which overcomes the lack of perfect orthogonality in the family initially observed by Iwaniec and Li [7].

1. Introduction

One particularly prominent field of study in analytic number theory is the size and distribution of values of $L$-functions. These are studied not only for its own sake, but also because good understanding of these properties frequently imply great results about arithmetic objects attached to these $L$-functions. The larger program here is to understand $L$-functions in general, and thereby understand many dissimilar arithmetic objects via the same route. However, it turns out that this is quite challenging, and different families of $L$-functions require very different methods, despite their superficial similarities.

In this paper, we study a family of $L$-functions attached to automorphic forms on $GL(2)$. In particular, we shall derive an upper bound for the eighth moment of this family, which is of the same quality as the Lindelöf hypothesis on average. Previously, Djankovic [4] had derived the same upper bound for the sixth moment of this family. A comparable family had been studied by M. Young [11] and Luo [8]. Essentially, they study the family of $L$-functions attached to Hecke Maass forms with Laplace eigenvalue $\lambda_j = 1/4 + t_j^2$, averaged over all $t_j \in [T, 2T]$. Young [11] then derived an upper bound for the sixth moment of this family of the same quality as the Lindelöf hypothesis on average. However, the best upper bound for the eighth moment due to [8] exceeds the Lindelöf quality bound by $T^{1/2}$. Thus, our work is the first time a Lindelöf on average bound has been achieved for the eighth moment of a $GL(2)$ family.

Typically upper bounds of this type are closely connected to large sieve type bounds. As noted by other authors, such bounds are more subtle for many $GL(2)$ families as compared to the classical $GL(1)$ family with Dirichlet characters. For our family, a large sieve was developed by Iwaniec and Xiaqing Li [7]. However, their large sieve indicates that the family is not perfectly orthogonal - in particular, certain terms are
larger than expected if the coefficients are chosen to look like certain Bessel functions twisted by Kloosterman sums. It is for this structural reason that Djanković was able to develop good upper bounds for the sixth moment, and his methods fail in the case of the eighth moment.

Now, we will be more precise. Let $S_k(\Gamma_0(q), \chi)$ be the space of cusp forms of weight $k \geq 2$ for the group $\Gamma_0(q)$ and the nebentypus character $\chi \pmod{q}$, where

$$
\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \ c \equiv 0 \pmod{q} \right\}.
$$

Also, let $S_k(\Gamma_1(q))$ be the space of holomorphic cusp forms for the group

$$
\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \ c \equiv 0 \pmod{q}, \ a \equiv d \equiv 1 \pmod{q} \right\}.
$$

Note that $S_k(\Gamma_1(q))$ is a Hilbert space with the Petersson’s inner product

$$
<f, g> = \int_{\Gamma_1(q) \backslash \mathbb{H}} f(z) \overline{g}(z) y^{k-2} \, dx \, dy,
$$

and

$$
S_k(\Gamma_1(q)) = \bigoplus_{\chi \pmod{q}} S_k(\Gamma_0(q), \chi).
$$

Let $\mathcal{H}_\chi \subset S_k(\Gamma_0(q), \chi)$ be an orthogonal basis of $S_k(\Gamma_0(q), \chi)$ consisting of Hecke cusp forms, normalized so that the first Fourier coefficient is 1. For each $f \in \mathcal{H}_\chi$, we let $L(f, s)$ be the $L$-function associated to $f$, defined for $\text{Re} \ (s) > 1$ as

$$
(1.1) \quad L(f, s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1},
$$

where $\{\lambda_f(n)\}$ are the Hecke eigenvalues of $f$. With our normalization, $\lambda_f(1) = 1$. In general, the Hecke eigenvalues satisfy the Hecke relation

$$
(1.2) \quad \lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right),
$$

for all $m, n \geq 1$. We define the completed $L$-function as

$$
(1.3) \quad \Lambda(f, s) = \left(\frac{q}{4\pi^2}\right)^{\frac{s}{2}} \Gamma\left(s + \frac{k-1}{2}\right)L(f, s),
$$

which satisfies the functional equation

$$
(1.4) \quad \Lambda(f, s) = i^k \eta_f \Lambda(\bar{f}, 1 - s),
$$

where $|\eta_f| = 1$ when $f$ is a newform.

Suppose for each $f \in \mathcal{H}_\chi$, we have an associated number $\alpha_f$. Then we define the harmonic average of $\alpha_f$ over $\mathcal{H}_\chi$ to be

$$
\sum_{f \in \mathcal{H}_\chi} \alpha_f = \frac{\Gamma(k - 1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_\chi} \frac{\alpha_f}{\|f\|^2}.
$$

We shall be interested in moments of the form
\[ M_k(q) := \frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{\chi(-1)=(-1)^k} h \left| L(f, 1/2) \right|^{2k}. \]

We note that the size of the family is around size \( q^2 \) while the conductor is around size \( q \). This should be compared with the family previously mentioned in the work of [11] and Luo [8] where the family is around size \( T^2 \) with conductor around size \( T \). \(^1\) For prime level, \( \eta_f \) can be expressed in terms of Gauss sums, and in particular we expect \( \eta_f \) to equidistribute on the circle as \( f \) varies over an orthogonal basis of \( S_k(\Gamma_1(q)) \). Thus, we expect our family of \( L \)-functions to be unitary.

As mentioned previously, Djanković [4] studied the sixth moment of this family and obtained the following upper bound, which is consistent with the Lindelöf hypothesis.

**Theorem 1.1 (Djanković).** Let \( q \) be a prime number and \( k \geq 3 \) an odd integer. Then we have as \( q \to \infty \)

\[ M_3(q) \ll q^\epsilon \]

for any \( \epsilon > 0 \). Note that the implied constant depends on \( \epsilon \).

In recent work [3], the authors were able to derive an asymptotic with a power saving for the sixth moment. In this paper, we prove the following Lindelöf upper bound for the eighth moment.

**Theorem 1.2.** Let \( q \) be a prime number and \( k \geq 3 \) an odd integer. Then we have as \( q \to \infty \)

\[ M_4(q) \ll q^\epsilon \]

for any \( \epsilon > 0 \). Note that the implied constant depends on \( \epsilon \).

Our methods may extend to help prove an asymptotic on a comparable eighth moment, perhaps involving an extra small average over the vertical line. We hope to return to this in the future.

### 2. Approximate Functional Equation and Initial Setup

The first step is to express \( \left| L(f, 1/2) \right|^8 \) in terms of an approximate functional equation. To this end, we introduce the following notations and lemmas.

The \( k \)-divisor function is defined by

\[ \sigma_k(n) = \sum_{n_1 n_2 \ldots n_k = n} 1. \]  

When \( k = 2 \), we write \( \sigma(n) \) instead of \( \sigma_2(n) \). \( \sigma \) is a multiplicative function, but is not completely multiplicative. The following Lemma records the well known multiplicative relation for \( \sigma \).

\(^1\)The mechanism is somewhat different however - this family exhibits a certain conductor dropping phenomenon while ours has increased size.
Lemma 2.1. We have
\[ \sigma(n_1n_2) = \sum_{d|(n_1,n_2)} \mu(d)\sigma\left(\frac{n_1}{d}\right)\sigma\left(\frac{n_2}{d}\right). \]

Let
\[ \mathcal{A}(b,c;d,j) := \mu(b)\mu(c)\sigma\left(\frac{d|b,c}{b}\right)\sigma\left(\frac{d|b,c}{c}\right)\sigma(j). \]

Now we write \( L^4(f,s) \) in terms of its Dirichlet series.

Lemma 2.2. Let \( L(f,s) \) be an \( L \)-function in \( \mathcal{H}_\chi \). For \( \Re(s) > 1 \), we have
\[
L^4(f,s) = \sum_{j \geq 1} \frac{\chi(j)\sigma(j)}{j^{2s}} \sum_{d \geq 1} \frac{\chi(d)}{d^{2s}} \sum_{n_1,n_2 \geq 1} \frac{\lambda_f(n_1)\lambda_f(n_2)\sigma(n_1)\sigma(n_2)}{n_1^s n_2^s} \sum_{n_1n_2 = u} \sigma(dn_1)\sigma(dn_2)
\]
\[
= \sum_{b,c,d,j \geq 1} \frac{\chi(j b,c)d}{(bc)^s[b,c]^{2s}d^{2s}j^{2s}} \sum_{n_1,n_2 \geq 1} \frac{\lambda_f(bcn)\sigma_4(n)}{n^s}.
\]

Proof. From the Hecke relation (1.2), we have
\[
L^2(f,s) = \sum_{j \geq 1} \frac{\chi(j)}{j^{2s}} \sum_{n \geq 1} \frac{\lambda_f(n)\sigma(n)}{n^s}.
\]

Then by the Hecke relation, we obtain
\[
L^4(f,s) = \sum_{j \geq 1} \frac{\chi(j)\sigma(j)}{j^{2s}} \sum_{n_1,n_2 \geq 1} \frac{\lambda_f(n_1)\lambda_f(n_2)\sigma(n_1)\sigma(n_2)}{n_1^s n_2^s} \sum_{n_1n_2 = u} \sigma(dn_1)\sigma(dn_2).
\]

From Lemma 2.1, we have
\[
\sum_{d \geq 1} \frac{\chi(d)}{d^{2s}} \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \sum_{n_1n_2 = n} \sigma(dn_1)\sigma(dn_2)
\]
\[
= \sum_{d \geq 1} \frac{\chi(d)}{d^{2s}} \sum_{n_1,n_2 \geq 1} \frac{\lambda_f(n_1n_2)}{n_1^s n_2^s} \sum_{b|(n_1,d), \ c|n_2,d} \mu(b)\mu(c)\sigma\left(\frac{n_1}{b}\right)\sigma\left(\frac{d}{b}\right)\sigma\left(\frac{n_2}{c}\right)\sigma\left(\frac{d}{c}\right)
\]
\[
= \sum_{b,c,d \geq 1} \mu(b)\mu(c) \sum_{d \geq 1} \frac{\chi(d)}{d^{2s}} \sum_{n_1,n_2 \geq 1} \frac{\lambda_f(n_1n_2)}{n_1^s n_2^s} \sigma\left(\frac{n_1}{b}\right)\sigma\left(\frac{d}{b}\right)\sigma\left(\frac{n_2}{c}\right)\sigma\left(\frac{d}{c}\right)
\]

By changing \( n_1 \) to \( bn_1, n_2 \) to \( cn_2 \) and \( d \) to \( d|b,c \), we derive the lemma. \( \square \)

Let \( H(s) \) be an even holomorphic function bounded in the region \( |\Re(s)| < 3 \) with \( H(0) = 1 \). We now prove an approximate functional equation for \( L^4(f,1/2) \).
Lemma 2.3. For any $\xi > 0$, let

$$V(\xi) = \frac{1}{2\pi i} \int_{(1)} H^4(s) \frac{\Gamma^4\left(\frac{k}{2} + s\right)}{\Gamma^4\left(\frac{k}{2}\right)} \xi^{-s} \frac{ds}{s}.$$

Then

$$(2.3)\quad L\left(f, \frac{1}{2}\right)^4 = \sum_{b,c,d,j,n} \sum_{\chi(bcn)\sigma_4(n)} \chi(j[b,c]d) \phi(b,c,d,j) \lambda_f(bcn) \sigma_4(n) \frac{V\left(\frac{(2\pi)^4(j[b,c]d)^2bcn}{q^2}\right)}{j[b,c]d\sqrt{bcn}}$$

$$= \lambda_f(bcn) \sigma_4(n) \frac{V\left(\frac{(2\pi)^4(j[b,c]d)^2bcn}{q^2}\right)}{j[b,c]d\sqrt{bcn}}.$$

Proof. We start by writing

$$J(f) = \frac{1}{2\pi i} \int_{(2)} \Lambda\left(f, \frac{1}{2} + s\right)^4 H^4(s) \frac{ds}{s},$$

where we recall that $\Lambda(f,s)$ is defined in (1.3). Moving the contour integral to $(-2)$ and using the functional equation (1.4) and $H(-s) = H(s)$, we obtain that

$$J(f) = \Lambda\left(f, \frac{1}{2}\right)^4 + \frac{1}{2\pi i} \int_{(-2)} \Lambda\left(f, \frac{1}{2} + s\right)^4 H^4(s) \frac{ds}{s}$$

$$= \Lambda\left(f, \frac{1}{2}\right)^4 - \frac{1}{2\pi i} \int_{(2)} \Lambda\left(f, \frac{1}{2} - s\right)^4 H^4(-s) \frac{ds}{s}$$

$$= \Lambda\left(f, \frac{1}{2}\right)^4 - (i^k \eta_f)^4 J(\bar{f}).$$

By (1.3) and Lemma 2.2, we have

$$J(f) = \left(\frac{\eta_{1/2}}{4\pi^2}\right) \Gamma^4\left(\frac{k}{2}\right) \sum_{b,c,d,j,n \geq 1} \chi(j[b,c]d) \phi(b,c,d,j) \lambda_f(bcn) \sigma_4(n) \frac{V\left(\frac{(2\pi)^4(j[b,c]d)^2bcn}{q^2}\right)}{j[b,c]d\sqrt{bcn}},$$

and a similar expression holds for $J(\bar{f})$. Using the expression for $\Lambda(f,1/2)$ from Equation (1.3) completes the proof of the Lemma.

By Stirling’s formula, for any fixed $A > 0$ and real number $x > 0$

$$V(x) \ll \frac{1}{(x+1)^4}$$

and

$$V(x) = 1 + O(x^4) \quad \text{for} \quad x \rightarrow 0.$$
and let \( \Psi_1(x) = \Psi(x) \left( \frac{x}{q^{2+\epsilon}} \right) \). Hence \( \Psi_1(x) \) has the same properties as \( \Psi(x) \) for all \( X \leq q^{2+\epsilon} \). By Lemma 2.3, it is enough to bound
\[
\frac{2}{\phi(q)} \sum_{\substack{\chi \text{ mod } q \text{ } \chi(-1) = (-1)^k}} \sum_{f \in \mathcal{H}_\chi} |T_f(X)|^2,
\]
where
\[
T_f(X) = \sum_{b,c,d,j,n \geq 1 \atop j \mid [b,c]d \leq \sqrt{X}} \frac{\lambda_f(bcn)\sigma_4(n)}{j[b,c]d} \sqrt{bcn} \Psi_1 \left( \frac{(2\pi)^4(j[b,c]d)^2bcn}{X} \right).
\]
Applying Cauchy-Schwarz inequality and writing \([b,c] = bc/(b,c)\), we obtain that
\[
|T_f(X)|^2 \leq \sum_{b,c,d,j,n \geq 1 \atop j \mid [b,c]d \leq \sqrt{X}} \frac{(b,c)\sigma_2^2(j)\sigma_2^2(d/b,c)\sigma_2^2(d/c)}{j dbc} \sum_{b,c,d,j,n \geq 1 \atop j \mid [b,c]d \leq \sqrt{X}} \frac{(b,c)}{j dbc} \times \left| \sum_{n \geq 1} \lambda_f(bcn)\sigma_4(n) \Psi_1 \left( \frac{(2\pi)^4(j[b,c]d)^2bcn}{X} \right) \right|^2.
\]
Since
\[
\sum_{b,c,d,j,n \geq 1 \atop j \mid [b,c]d \leq \sqrt{X}} \frac{(b,c)\sigma_2^2(j)\sigma_2^2(d/b,c)\sigma_2^2(d/c)}{j dbc} \ll X^\epsilon \ll q^\epsilon,
\]
and
\[
\sum_{b,c,d,j,n \geq 1 \atop j \mid [b,c]d \leq \sqrt{X}} \frac{(b,c)}{j dbc} \ll q^\epsilon,
\]
to prove Theorem 1.2 it suffices to show the following Proposition.

**Proposition 2.4.** Let \( b, c, n \) be defined as above, and
\[
a_{bcn} = \frac{\sigma_4(n)}{\sqrt{bcn}} \Psi_1 \left( \frac{bcn}{Y} \right),
\]
where \( Y = \frac{X}{16\pi^4(j[b,c]d)^2} \). Then we have
\[
\frac{2}{\phi(q)} \sum_{\substack{\chi \text{ mod } q \text{ } \chi(-1) = (-1)^k}} \sum_{f \in \mathcal{H}_\chi} \left| \sum_{n \geq 1} a_{bcn} \lambda_f(bcn) \right|^2 \ll q^\epsilon.
\]

Note that the implied constant depends on \( \epsilon \).
3. Preliminary lemmas

To deal with the summation in Proposition 2.4, we first apply the large sieve developed by Iwaniec and Xiaoqing Li [7] which we record below.

Lemma 3.1 (Asymptotic large sieve). Let $q$ be a prime number, $N \geq q$, $T = Nq^{-1}$ and $1 \leq H \leq T$. Then for any complex vectors $\alpha = (a_n)$ with $N < n \leq 2N$ we have

$$\frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{f \in \mathcal{H}} \left| \sum_{n \geq 1} a_n \lambda_f(n) \right|^2 = \frac{1}{d} \sum_{1 \leq l \leq T} \left( \frac{2\pi}{t} \right)^2 \sum_{1 \leq h \leq H} |P_{ht}(\alpha)|^2 + O \left( N^\epsilon \left( \frac{N}{q^2} + \sqrt{\frac{N}{qH}} \right) \right),$$

with any $\epsilon > 0$,

$$P_{ht}(\alpha) = \sum_n a_n S(h\bar{q}, n; t) J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{hn}{q}} \right),$$

$S(m, n; c)$ is the Kloosterman sum defined by

$$S(m, n; c) = \sum_{a \equiv 1 \mod c} e \left( \frac{am + \bar{a}n}{c} \right),$$

and $J_{k-1}(x)$ is the $J$-Bessel function of order $k - 1$. Moreover, the implied constant depends on $k$ and $\epsilon$.

Next, we provide some useful properties of Bessel functions.

Lemma 3.2. We have

$$J_{k-1}(2\pi x) = \frac{1}{2\sqrt{x}} \left\{ W(2\pi x) e \left( x - \frac{k}{4} + \frac{1}{8} \right) + W(2\pi x) e \left( -x + \frac{k}{4} - \frac{1}{8} \right) \right\},$$

where $W(j)(x) \ll_{j, k} x^{-j}$. Moreover,

$$J_{k-1}(2x) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{x^{2\ell+k-1}}{\ell!(\ell+k-1)!}.$$  

These results are standard and we refer the reader to [10] for these claims. Other than $J$-Bessel functions, we will also need to use some properties of a Bessel function of the second kind $Y_0(x)$ and a modified Bessel function of the second kind $K_0(x)$. These appear due to an application of Voronoi summation with coefficients $\tau(n)$, which we will see later. We express $Y_0(x)$ and $K_0(x)$ in three different forms. The first and second forms are useful for large $x$ and small $x$ respectively, and the third expression is helpful in separating variables.

Lemma 3.3. Let $Y_0(x)$ be the Bessel function of the second kind of order 0, and $K_0(x)$ is the modified Bessel function of the second kind of order 0. Then

$$Y_0(2\pi x) = \frac{1}{\pi \sqrt{x}} \text{Im} \left( W(2\pi x) e \left( x - \frac{1}{8} \right) \right),$$
and

\[(3.4) \quad K_0(x) = \left( \frac{1}{2\pi} \right)^{1/2} e^{-x} W_1(x),\]

where \(W^{(j)}(x) \ll_{j,k} x^{-j}\) and \(W_1^{(j)}(x) \ll_{j} x^{-j}\). This \(W\)-function is the same as the one in Lemma 3.2. Moreover,

\[(3.5) \quad Y_0(x) = \frac{2}{\pi} \left( \ln \left( \frac{x}{2} \right) + \gamma \right) \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2} + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} H_k \frac{x^{2k}}{4^k (k!)^2},\]

and

\[(3.6) \quad K_0(x) = -\left( \ln \left( \frac{x}{2} \right) + \gamma \right) \sum_{k=0}^{\infty} \frac{x^{2k}}{4^k (k!)^2} + \sum_{k=1}^{\infty} H_k \frac{x^{2k}}{4^k (k!)^2},\]

where \(\gamma\) is the Euler’s constant and \(H_k\) is a harmonic number, defined by

\[H_k = \sum_{m=1}^{k} \frac{1}{m}.\]

Let \(0 < \sigma < 1\). Then for some constants \(\kappa\) and \(\kappa_1\)

\[(3.7) \quad Y_0(x) = \frac{1}{2\pi i} \int_{(-\sigma)} \gamma(s)x^{-s} ds + \frac{2}{\pi} \ln x + \kappa,\]

and

\[(3.8) \quad K_0(x) = \frac{1}{2\pi i} \int_{(-\sigma)} \gamma_1(s)x^{-s} ds - \ln x + \kappa_1\]

for some \(\gamma(s)\) and \(\gamma_1(s)\) satisfying \(\int_{(-\sigma)} |\gamma(s)| |ds| \ll 1\) and \(\int_{(-\sigma)} |\gamma_1(s)| |ds| \ll 1\).

**Proof.** The result of (3.3) and (3.4) can be found on p.206 in [10], and Equation (3.5) and (3.6) are given in Section 9 of [1]. We are left to prove Equation (3.7) and (3.8). From the Mellin transform in Equation 17 of Section 6.8 [5], we obtain that

\[Y_0(x) = \frac{1}{2\pi i} \int_{(\sigma)} \gamma(s)x^{-s} ds; \quad 0 < \sigma < 3/2\]

where

\[\gamma(s) = -2^{s-1} \pi^{-1} \Gamma^2 \left( \frac{s}{2} \right) \cos \left( \frac{s\pi}{2} \right).\]

The integral representation above is only conditionally convergent, where the limit is taken for \(-T < \text{Im} s < T\) and letting \(T\) go to infinity. By a standard argument, we may shift the contour of integration to \(-\sigma\) where \(0 < \sigma < 1\). In so doing, we pick up the residue of \(\gamma(s)x^{-s}\) at \(s = 0\) which is \(\frac{2}{\pi} \ln x - \frac{2}{\pi} \ln 2 + \frac{2\gamma}{\pi}\) using the Laurent expansion of \(\Gamma(s)\) at \(s = 0\). Thus

\[Y_0(x) = \frac{1}{2\pi i} \int_{(-\sigma)} \gamma(s)x^{-s} ds + \frac{2}{\pi} \ln x - \frac{2}{\pi} \ln 2 + \frac{2\gamma}{\pi},\]
where $\gamma$ is the Euler’s constant. By Stirling’s formula, the integral above is now absolutely convergent. In particular,

$$|\gamma(s)| \ll |t|^{-\sigma-1},$$

for $s = -\sigma + it$ and $|t| > 1$. This proves Equation (3.7).

Finally, Equation (3.8) follows from the Mellin transform in Equation 26 of Section 6.8 [5], which is

$$K_0(x) = \frac{1}{2\pi i} \int_{(\sigma)} \gamma_1(s)x^{-s} \, ds; \quad \sigma > 0$$

where $\gamma_1(s) = 2^{s-2}\Gamma^2\left(\frac{s}{2}\right)$. We then shift the contour of integration to $-\sigma$ where $0 < \sigma < 1$ and pick up the residue of $\gamma_1(s)x^{-s}$ at $s = 0$, which is $\ln 2 - \ln x - \gamma$. This implies Equation (3.8). □

Our summation will involve divisor functions, and we will apply Voronoi summation formula (e.g. see Theorem 4.10 in [6]).

**Lemma 3.4** (Voronoi summation formula). Suppose $g(x)$ is smooth and compactly supported on $\mathbb{R}^+$. $Y_0$ and $K_0$ are Bessel functions defined as in Lemma 3.3. Let $ad \equiv 1 \pmod{c}$. Then

$$\sum_{n=1}^{\infty} \sigma(n) e\left(\frac{an}{c}\right) g(n) = \frac{1}{c} \int_0^\infty (\log x + 2\gamma - 2 \log c) g(x) \, dx$$

$$- \frac{2\pi}{c} \sum_{\ell=1}^{\infty} \sigma(\ell) e\left(-\frac{d\ell}{c}\right) \int_0^\infty Y_0\left(\frac{4\pi}{c}\sqrt{\ell x}\right) g(x) \, dx$$

$$+ \frac{4}{c} \sum_{\ell=1}^{\infty} \sigma(\ell) e\left(\frac{d\ell}{c}\right) \int_0^\infty K_0\left(\frac{4\pi}{c}\sqrt{\ell x}\right) g(x) \, dx.$$  

Finally, we will eventually reduce our bound to applications of the following large sieve inequality involving $GL(1)$ harmonics as stated in Exercise 5, Chapter 7 in [6].

**Lemma 3.5.** For any complex numbers $\alpha_m, \beta_n$, we have

$$\sum_{q \leq Q} \sum_{a \pmod{q}}^{*} \left| \sum_{m \leq M, n \leq N}^{\ast} \alpha_m \beta_n e\left(\frac{am\bar{n}}{q}\right) \right|^2 \leq (Q^2 + MN)\|\alpha\|^2\|\beta\|^2.$$
4. First step toward the proof of Proposition 2.4

By the asymptotic large sieve in Lemma 3.1, we obtain that

\[
\frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{f \in \mathcal{H}_q} \left| \sum_{n \geq 1} \lambda_f(bcn) a_{bcn} \right|^2 = \frac{1}{q} \sum_{1 \leq t \leq T} \left( \frac{T}{t} \right)^2 \sum_{1 \leq h \leq H} |P_{ht}(\alpha)|^2 + O \left( q^2 \left( \frac{Y}{q^2} + \sqrt{\frac{Y}{qH}} \right) \right) \|\alpha\|^2
\]

where \( T = Y/q \) and \( 1 \leq H \leq T \). We choose \( H = Y/q \) so that the error term is small. We note that since \( Y \leq X \leq q^2 + \epsilon \),

\[
\|\alpha\|^2 \leq \sum_{Y < bcn < 2Y} \frac{\sigma_4(n)}{bcn} \ll \frac{1}{bc} \left( \log \frac{Y}{bc} \right)^{16} \ll q^\epsilon.
\]

Throughout the paper, \( \epsilon \) denotes an arbitrary small positive constant that may vary from term to term, but \( \epsilon_1 \) will be a small fixed constant, which can be chosen later.

We divide \( h \) and \( t \) into dyadic intervals, and it is enough to consider

\[
\frac{1}{qT^2} \sum_{t \sim T_1} \sum_{(t,q)=1} \sum_{h} \Psi \left( \frac{h}{H_1} \right) |P_{ht}(\alpha)|^2,
\]

where we remind the reader that \( \Psi_1 \) denotes a compactly supported smooth function and where we write \( a \sim A \) as shorthand for \( A < a \leq 2A \). We know that \( \sigma_4(n) = \sum_{n_1,n_2=n} \sigma(n_1)\sigma(n_2) \), and we can divide the sum over \( n_1 \) into dyadic intervals. Therefore we write

\[
P_{ht}(\alpha) = \sum_n \frac{\sigma_4(n)}{\sqrt{bcn}} \Psi_1 \left( \frac{bcn}{Y} \right) S(hq, bcn; t) J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{bcn}{q}} \right)
\]

\[
= \sum_{N} \sum^*_{r \mod t} e \left( \frac{hq \bar{r}}{t} \right) \sum_{n_1} \frac{\sigma_2(n_1)}{\sqrt{bcn_1}} \Psi \left( \frac{n_1}{N} \right) \mathcal{N}(n_1; r, t, h, bc),
\]

where

\[
\mathcal{N}(n_1; r, t, h, bc) := \sum_{n_2} \frac{\sigma_2(n_2)}{\sqrt{n_2}} J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{hbcn_1n_2}{q}} \right) \Psi_1 \left( \frac{bcn_1n_2}{Y} \right) e \left( \frac{rbcn_1n_2}{t} \right),
\]

and \( \sum^*_{r \mod t} \) denotes a sum over \( 1 \leq r \leq t \) such that \( (r,t) = 1 \). Since \( n_1n_2 \ll Y \), by symmetry, we may without loss of generality assume that

\[
N_1 \ll \sqrt{\frac{Y}{bc}}.
\]
Thus it is enough to consider
\[
\mathcal{F}(T_1, H_1; q) := \sum_{t \sim T_1} \sum_{r \pmod{t}} \sum_{h} \Psi \left( \frac{h}{H_1} \right) \left| \sum_{n_1} e \left( \frac{\ell h}{t} \right) \sigma_2(n_1) \sqrt{bcn_1} \Psi \left( \frac{n_1}{N} \right) \right|^2.
\]

Let
\[
\frac{bcn_1}{\ell} = \frac{m}{\eta},
\]
where \((m, \eta) = 1\). Applying Voronoi summation formula in Lemma 3.4 to \(N(n_1; r, t, h, bc)\), we obtain that
\[
N(n_1; r, t, h, bc) = R_1 + R_2 + R_3,
\]
where
\[
R_1 := \frac{1}{\ell} \int_0^\infty \left( \log x + 2\gamma - 2\log \eta \right) \frac{1}{\sqrt{x}} J_{k-1} \left( \frac{4\pi}{t} \sqrt{\ell x} \right) \Psi \left( \frac{bcn_1 x}{Y} \right) dx;
\]
\[
R_2 := -\frac{2\pi}{\ell} \sum_{\ell=1}^\infty \sigma_2(\ell) e \left( \frac{-\ell m r \eta}{\ell} \right) \int_0^\infty Y_0 \left( \frac{4\pi}{\eta} \sqrt{\ell x} \right) \frac{1}{\sqrt{x}} J_{k-1} \left( \frac{4\pi}{t} \sqrt{\ell x} \right) \Psi \left( \frac{bcn_1 x}{Y} \right) dx;
\]
\[
R_3 := \frac{4}{\ell} \sum_{\ell=1}^\infty \sigma_2(\ell) e \left( \frac{\ell m r \eta}{\ell} \right) \int_0^\infty K_0 \left( \frac{4\pi}{\eta} \sqrt{\ell x} \right) \frac{1}{\sqrt{x}} J_{k-1} \left( \frac{4\pi}{t} \sqrt{\ell x} \right) \Psi \left( \frac{bcn_1 x}{Y} \right) dx.
\]

Note here that \(m \equiv 1 \pmod{\eta}\) and \(r \equiv 1 \pmod{\eta}\). Let
\[
\mathcal{F}_i(T_1, H_1; q) = \sum_{t \sim T_1} \sum_{r \pmod{t}} \sum_{h} \Psi \left( \frac{h}{H_1} \right) \left| \sum_{n_1} e \left( \frac{\ell h}{t} \right) \sigma_2(n_1) \sqrt{bcn_1} \Psi \left( \frac{n_1}{N} \right) R_i \right|^2
\]
for \(i = 1, 2, 3\). Since \(|a| + |b| + |c| \leq 3|a|^2 + |b|^2 + |c|^2\), we have that
\[
\mathcal{F}(T_1, H_1; q) \ll \mathcal{F}_1(T_1, H_1; q) + \mathcal{F}_2(T_1, H_1; q) + \mathcal{F}_3(T_1, H_1; q).
\]

Hence Proposition 2.4 will follow from the following Lemma.

**Lemma 4.1.** Let \(0 < T_1 \leq T\) and \(0 < H_1 \leq H\), where \(T = H = Y/q\). Then for \(i = 1, 2, 3\),
\[
\mathcal{F}_i(T_1, H_1; q) \ll q^\epsilon,
\]
where the implied constant depends on \(\epsilon\).
5. Bounding $\mathcal{F}_1(T_1, H_1; q)$

Firstly, $R_1$ defined in (4.2) does not depend on variable $r$. The sum over $r$ is then the Ramanujan's sum, which is

$$\sum_{r \,(\text{mod} \, t)}^* e \left( \frac{hqr}{t} \right) = \mu \left( \frac{t}{(t,h)} \right) \frac{\phi(t)}{\phi \left( \frac{t}{(t,h)} \right)} \ll (t,h).$$

We change the variable from $x \rightarrow \frac{x}{bcn_1}$ inside the integral in (4.2) and use Equation (3.1). Thus

$$R_1 = \frac{1}{\sqrt{bcn_1 \eta}} \int_0^\infty \left( \log \frac{x}{bcn_1} + 2\gamma - 2 \log \eta \right) \frac{1}{\sqrt{x}} J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{hx}{q}} \right) \Psi_1 \left( \frac{x}{Y} \right) dx \ll q \epsilon \left( \frac{bcn_1}{t} \right) \sqrt{H_1 Y} \frac{1}{T_1},$$

where $\Psi_2(x)$ is a smooth compactly supported function which may be expressed in terms of $W(x)$ and $\Psi_1(x)$. If $T_1 \ll \frac{1}{q^1} \sqrt{\frac{H_1 Y}{q}}$, then we can integrate by parts many times and obtain that

$$\mathcal{F}_1(T_1, H_1; q) \ll q^{-100},$$

which can be ignored. Otherwise, we bound the integral in Equation (5.1) trivially, use $T_1 \gg \frac{1}{q^1} \sqrt{\frac{H_1 Y}{q}}$ and derive that

$$\mathcal{F}_1(T_1, H_1; q) \ll q^\epsilon \frac{\sqrt{H_1 Y}}{T_1 \sqrt{q}} \ll q^{\epsilon + \epsilon_1}.$$ 

Upon choosing $\epsilon_1$ sufficiently small, we conclude the proof of Lemma 4.1 for $i = 1$. 
6. Bounding $\mathcal{F}_2(T_1, H_1; q)$

Since $m\bar{m} \equiv 1 \pmod{t}$, $m\bar{m} \equiv 1 \pmod{\eta}$ when $\eta | t$. Then after the change of variables $n_1 \to n$ and $x \to xY/(bcn)$, we write

$$
\sum_{r \pmod{t}}^* e\left(\frac{h\bar{q}r}{t}\right) \sum_n \frac{\sigma_2(n)}{\sqrt{bcn}} R_2
$$

$$
= -2\pi \frac{\sqrt{Y}}{bc} \sum_{r \pmod{t}}^* e\left(\frac{h\bar{q}r}{t}\right) \sum_n \frac{\sigma_2(n)}{n\eta} \Psi\left(\frac{n}{N}\right) \sum_{\ell=1}^\infty \sigma_2(\ell) e\left(\frac{-\bar{m}\bar{r}\ell}{\eta}\right)
$$

$$
\times \int_0^\infty Y_0\left(\frac{4\pi}{\eta} \sqrt{\frac{\ell xY}{bcn}}\right) J_{k-1}\left(\frac{4\pi}{t} \sqrt{\frac{hxY}{q}}\right) \Psi_3(x) \, dx,
$$

where $\Psi_3(x) = \frac{\Psi_1(x)}{\sqrt{x}}$, and $R_2$ is defined in (4.3). Hence

$$
\mathcal{F}_2(T_1, H_1; q) \ll \frac{Y}{qT_1^2(bc)^2} \sum_{t \sim T_1} \sum_h \Psi\left(\frac{h}{H_1}\right) \left| \sum_{r \pmod{t}}^* e\left(\frac{h\bar{q}r}{t}\right) \sum_n \frac{\sigma_2(n)}{n\eta} \Psi\left(\frac{n}{N}\right) \sum_{\ell=1}^\infty \sigma_2(\ell) e\left(\frac{-\bar{m}\bar{r}\ell}{\eta}\right) \right|
$$

$$
\times \left| \int_0^\infty Y_0\left(\frac{4\pi}{\eta} \sqrt{\frac{\ell xY}{bcn}}\right) J_{k-1}\left(\frac{4\pi}{t} \sqrt{\frac{hxY}{q}}\right) \Psi_3(x) \, dx \right|^2
$$

Recall that $\eta = \frac{t}{(bcn,t)}$. Writing $d = (t, bc)$ we obtain that

$$
\mathcal{F}_2(T_1, H_1; q) \ll \frac{Y}{qT_1^2(bc)^2} \sum_{d \mid (bc)} \sum_{t \sim T_1/d} \sum_h \Psi\left(\frac{h}{H_1}\right) \left| \sum_{r \pmod{td}}^* e\left(\frac{h\bar{q}r}{td}\right) \sum_n \frac{\sigma_2(n)(n,t)}{nt} \Psi\left(\frac{n}{N}\right) \right|
$$

$$
\times \left| \sum_{\ell=1}^\infty \sigma_2(\ell) e\left(\frac{-\bar{m}\bar{r}\ell}{t(t,n)}\right) \int_0^\infty Y_0\left(\frac{4\pi(t,n)}{t} \sqrt{\frac{\ell xY}{bcn}}\right) J_{k-1}\left(\frac{4\pi}{td} \sqrt{\frac{hxY}{q}}\right) \Psi_3(x) \, dx \right|^2
$$
Next, we remove the greatest common divisors \((t, n)\) and \((t, \ell)\) to facilitate an eventual application of the large sieve inequality as in Lemma 3.5. Thus

\[
\mathcal{F}_2(T_1, H_1; q) \ll \frac{Y}{q T_1^4 (bc)^2} \sum_{d | bc} d^2 \sum_{g \leq T/d} \sum_{g_1 \leq \frac{T}{d^2}} \sum_{r \equiv (t,dgg_1) \pmod{tdg_1}} \sum_{h} \Psi \left( \frac{h}{H_1} \right) \sum_{r' \equiv (t,dgg_1) \pmod{tdg_1}} e \left( \frac{h q r}{tdg g_1} \right)
\]

\[
\times \sum_{n, n' \equiv (n, t) \pmod{bc}} \sigma_2(n g) \sigma_2(n' g) \psi \left( \frac{n g}{N} \right) \psi \left( \frac{n' g}{N} \right)
\]

\[
\times \sum_{\ell, \ell' \equiv (\ell, t) \pmod{bc}} \sigma_2(\ell g_1) e \left( -\frac{bc \ell'}{d} n \right) \sigma_2(\ell' g_1) e \left( -\frac{bc \ell'}{d} n \right) \int_{0}^{\infty} Y_0 \left( \frac{4 \pi}{t} \sqrt{\ell x Y} \right) \psi \left( \frac{4 \pi}{t} \sqrt{\ell' x' Y} \right)
\]

\[
\times Y_0 \left( \frac{4 \pi}{t} \sqrt{\ell x' Y} \right) \psi \left( \frac{4 \pi}{t} \sqrt{\ell' x' Y} \right) dx \, dx'
\]

We divide into 3 cases, depending on the size of \(T_1\) with respect to \(\sqrt{H_1 Y q}\), which are

1. \(T_1 \gg \sqrt{H_1 Y q}\).
2. \(T_1 \ll \sqrt{ \frac{H_1 Y q}{q^2} \psi(x)}\).
3. \(T_1 \ll \frac{1}{\psi(x)}\).

Case 1: \(T_1 \gg \sqrt{H_1 Y q}\). For this case, we prove the following Lemma.

**Lemma 6.1.** Let \(T_1 \gg \sqrt{H_1 Y q}\). Then

\[
\mathcal{F}_2(T_1, H_1; q) \ll q^\epsilon,
\]

where the implied constant depends on \(\epsilon\).

We square the absolute value in Equation (6.1) and obtain that

\[
\mathcal{F}_2(T_1, H_1; q) \ll \frac{Y}{q T_1^4 (bc)^2} \sum_{d | bc} d^2 \sum_{g \leq T/d} \sum_{g_1 \leq \frac{T}{d^2}} \sum_{r \equiv (t,dgg_1) \pmod{tdg_1}} \sum_{r' \equiv (t,dgg_1) \pmod{tdg_1}} e \left( \frac{h q r}{tdg g_1} \right)
\]

\[
\times \sum_{n, n' \equiv (n, t) \pmod{bc}} \sigma_2(n g) \sigma_2(n' g) \psi \left( \frac{n g}{N} \right) \psi \left( \frac{n' g}{N} \right)
\]

\[
\times \sum_{\ell, \ell' \equiv (\ell, t) \pmod{bc}} \sigma_2(\ell g_1) e \left( -\frac{bc \ell'}{d} n \right) \sigma_2(\ell' g_1) e \left( -\frac{bc \ell'}{d} n \right) \int_{0}^{\infty} Y_0 \left( \frac{4 \pi}{t} \sqrt{\ell x Y} \right) \psi \left( \frac{4 \pi}{t} \sqrt{\ell' x' Y} \right)
\]

\[
\times Y_0 \left( \frac{4 \pi}{t} \sqrt{\ell x' Y} \right) \psi \left( \frac{4 \pi}{t} \sqrt{\ell' x' Y} \right) dx \, dx'
\]
where

\[ S(H_1, x, x', t) := \sum_h \Psi\left(\frac{h}{H_1}\right) J_{k-1}\left(\frac{4\pi}{tdgg_1} \sqrt{\frac{hx}{q}}\right) J_{k-1}\left(\frac{4\pi}{tdgg_1} \sqrt{\frac{hx'}{q}}\right) e^{\left(\frac{h\bar{q}(\bar{r} - \bar{r}')}{tdgg_1}\right)}. \]

Since \( \frac{1}{T_1} \sqrt{\frac{H_1Y}{q}} \ll 1 \), we can treat the Bessel function \( J_{k-1}(x) \) as a smooth function. Using Equation (3.2) for \( J_{k-1}(x) \), we obtain that \( S(H_1, x, x', t) \) is

\[
\sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha!(\alpha + k - 1)!} \left(\frac{4\pi}{tdgg_1} \sqrt{\frac{xYH_1}{q}}\right)^{2\alpha+k-1} \left(\frac{4\pi}{tdgg_1} \sqrt{\frac{x'YH_1}{q}}\right)^{2\alpha+k-1} \sum_h \mathcal{F}_\alpha\left(\frac{h}{H_1}\right) e^{\left(\frac{h\bar{q}(\bar{r} - \bar{r}')}{tdgg_1}\right)},
\]

where

\[
\mathcal{F}_\alpha\left(\frac{h}{H_1}\right) := \Psi\left(\frac{h}{H_1}\right) \left(\frac{h}{H_1}\right)^{2\alpha+k-1}.
\]

Note that \( \mathcal{F}_\alpha(x) \) is compactly supported and smooth. Applying Poisson summation to the sum over \( h \), we have that

\[
\sum_h \mathcal{F}_\alpha\left(\frac{h}{H_1}\right) e^{\left(\frac{h\bar{q}(\bar{r} - \bar{r}')}{{\it tdgg}_1}\right)} = H_1 \sum_{\beta \equiv -\bar{q}(\bar{r} - \bar{r}') \pmod{{\it tdgg}_1}} \int_{\mathbb{R}} \mathcal{F}_\alpha(y) e^{\left(-\frac{\beta H_1y}{{\it tdgg}_1}\right)} dy.
\]

When \( |\beta| \gg \frac{T_1}{H_1} q^\epsilon \), we do integration by parts many times to see that the contribution from these terms is negligible. Hence we focus only on those terms for which \( |\beta| \ll \frac{T_1}{H_1} q^\epsilon \).
Thus bounding the terms inside the sum over $\beta$ trivially, we have

\begin{align}
\mathcal{F}_2(T_1, H_1; q) &\lesssim \frac{YH_1 q^\epsilon}{q T_1^4 (bc)^2} \sum_{d|bc} d^2 \sum_{g \leq T/d} g \leq \frac{T}{n} \sum_{\beta \leq T_1/q} \sum_{n \geq 0} \frac{1}{\alpha!(\alpha + k - 1)!} \left( \frac{1}{T_1 \sqrt{YH_1 q}} \right)^{4\alpha + 2k - 2} \\
&\times \sum_{n} \sum_{\ell, \ell'} \left| \sum_{n, n'} \sigma_2(n g) \sigma_2(n' g', \beta q, \tau_{dgg}) \frac{\Psi(n/N)}{n} \frac{\Psi(n'/N)}{n'} \right| \\
&\times \left| \int_0^\infty \int_0^\infty Y_0 \left( \frac{4\pi}{t} \sqrt{\frac{\ell x Y}{bc n g g_1}} \right) Y_0 \left( \frac{4\pi}{t} \sqrt{\frac{\ell' x' Y}{bc n' g g_1}} \right) \Psi_4(x) \Psi_4(x') \, dx \, dx' \right| \\
&\lesssim \frac{YH_1 q^\epsilon}{q T_1^4 (bc)^2} \sum_{d|bc} d^2 \sum_{g \leq T/d} g \leq \frac{T}{n} \sum_{\beta \leq T_1/q} \sum_{n \geq 0} \frac{1}{\alpha!(\alpha + k - 1)!} \left( \frac{1}{T_1 \sqrt{YH_1 q}} \right)^{4\alpha + 2k - 2} \\
&\times \sum_{n} \sum_{\ell, \ell'} \left( \mathcal{A} + \mathcal{A}' \right),
\end{align}

where $\Psi_4(x) := \Psi_4(x, \alpha) = x^{\alpha + k/2 - 1} \Psi_3(x)$ is also compactly supported and smooth and moreover,

\begin{align}
\mathcal{A} &:= \left| \sum_{n, \ell} \frac{\sigma_2(n g)}{n} \frac{\Psi(n g/N)}{n} \sum_{\ell, \ell'} \sigma_2(\ell g_1) e \left( -\frac{bc d n r \ell}{t} \right) \int_0^\infty Y_0 \left( \frac{4\pi}{t} \sqrt{\frac{\ell x Y}{bc n g g_1}} \right) \Psi_4(x) \, dx \right|^2, \\
\mathcal{A}' &:= \left| \sum_{n, \ell} \frac{\sigma_2(n g)}{n} \frac{\Psi(n g/N)}{n} \sum_{\ell, \ell'} \sigma_2(\ell g_1) e \left( -\frac{bc}{t} n(\bar{r} + \beta q) \ell}{t} \right) \int_0^\infty Y_0 \left( \frac{4\pi}{t} \sqrt{\frac{\ell x Y}{bc n g g_1}} \right) \Psi_4(x) \, dx \right|^2.
\end{align}

By a change of variables, we see that both

\begin{align}
\sum_{n} \sum_{\ell, \ell'} \left( \mathcal{A} + \mathcal{A}' \right) \rangle
\end{align}
and
\[
\sum_{t \sim \frac{T}{\text{ggq}}} \sum_{r \equiv \frac{t}{\text{gqggq}}} \sum_{(n,t)=1}^{\ast} A',
\]
are bounded by
\[
dggq \sum_{t \sim \frac{T}{\text{ggq}}} \sum_{r \equiv \frac{t}{\text{gqggq}}} \sum_{(n,t)=1}^{\ast} \frac{\sigma_2(nq)}{n} \Psi \left( \frac{nq}{N} \right) \sum_{(t)\ell=1} \sigma_2(\ell g_1) e \left( \frac{\bar{\nu}r\ell}{t} \right) \int_0^\infty Y_0 \left( \frac{4\pi t}{\sqrt{\text{bcngg}_1}} \right) |\Psi_4(x) dx|^2,
\]
independently of \( \beta \).

Therefore it is enough to consider
\[(6.5)\]
\[
G_b(T_1, H_1; q) := \frac{Y q^\epsilon}{q T_1^3 (bc)^2} \sum_{d | bc} d^3 \sum_{g \leq T/d} g \sum_{g_1 \leq T/q} g_1 \sum_{a=0}^\infty \frac{1}{\alpha! (\alpha + k - 1)!} \left( \frac{1}{T_1} \right) \left[ \frac{\sqrt{Y H_1}}{q} \right]^{4\alpha + 2k - 2} \times \sum_{t \sim \frac{T}{\text{ggq}}} \sum_{r \equiv \frac{t}{\text{gqggq}}} \sum_{(n,t)=1}^{\ast} \frac{\sigma_2(nq)}{n} \Psi \left( \frac{nq}{N} \right) \sum_{(t)\ell=1} \sigma_2(\ell g_1) e \left( \frac{\bar{\nu}r\ell}{t} \right) \int_0^\infty Y_0 \left( \frac{4\pi t}{\sqrt{\text{bcngg}_1}} \right) |\Psi_4(x) dx|^2
\]
and show that
\[
G_b(T_1, H_1; q) \ll q^\epsilon.
\]
To prove this bound, we consider 3 cases depending on the range of \( \ell \) and write
\[
G_b(T_1, H_1; q) \ll G_{b,1}(T_1, H_1; q) + G_{b,2}(T_1, H_1; q),
\]
where \( G_{b,i}(T_1, H_1; q) \) has the same expression as \( G_b(T_1, H_1; q) \) but restricting the size of \( \ell \) to \( \ell \gg q^{\epsilon} \frac{T_{bc,N}^2}{d^2 g_1 Y} \) and \( \ell \ll q^{\epsilon} \frac{T_{bc,N}^2}{d^2 g_1 Y} \) respectively.

**Case 1.1:** \( \ell \gg q^{\epsilon} \frac{T_{bc,N}^2}{d^2 g_1 Y} \). In this case, we use the expression for \( Y_0(x) \) in Equation (3.3) and consider
\[
\sum_{(n,t)=1}^{\ast} \frac{\sigma_2(nq)}{n} \Psi \left( \frac{nq}{N} \right) \sum_{\ell \gg q^{\epsilon} \frac{T_{bc,N}^2}{d^2 g_1 Y}} \sigma_2(\ell g_1) e \left( \frac{\bar{\nu}r\ell}{t} \right) \left( t \sqrt{\frac{\text{bcngg}_1}{Y}} \right)^{1/2} \int_0^\infty e \left( \pm \frac{2}{t} \sqrt{\frac{\ell xY}{\text{bcngg}_1}} \right) \Psi_4(x) x^{1/4} dx.
\]
We can then integrate by parts many times with respect to \( x \) and obtain that \( G_{b,1}(T_1, H_1; q) \ll q^{-100} \).
Case 1.2: \( \ell \ll q^{\frac{1}{4}} \frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y} \). Using the integral expression for \( Y_{0} \) in Equation (3.7), we have for \( 0 < \sigma < 1 \)

\[
Y_{0}\left(\frac{4\pi}{t} \sqrt{\frac{\ell x Y}{b c n g_{1}}}\right) = \int_{(\sigma)}^{\infty} \gamma(s) \left(\frac{4\pi}{t} \sqrt{\frac{\ell x Y}{b c n g_{1}}}\right)^{-s} ds + \frac{2}{\pi} \ln \left(\frac{4\pi}{t} \sqrt{\frac{x Y}{b c n g_{1}}}\right) + \frac{1}{\pi} \log \ell - \frac{1}{\pi} \log n + \kappa.
\]

For brevity, we deal only with the integral term, the other terms being slightly easier. The sum over \( t \) from this integral of \( G_{2}(T_{1}, H_{1}; q) \) can be written as

\[
\sum_{t \sim T_{1}} \sum_{r \pmod{t}}^{*} \left[ \int_{0}^{\infty} \Psi_{3}(x) \int_{(\sigma)}^{\infty} \gamma(s) \left(\frac{tdg_{1}}{4\pi T_{1} \sqrt{x}}\right)^{s} \sum_{(n,t)=1}^{\infty} \frac{\sigma_{2}(ng)}{n} \left(\frac{n}{N/g}\right)^{s/2} \Psi\left(\frac{ng}{N}\right) \right. \times \sum_{\ell \ll q^{\frac{1}{4}} \frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y}}^{*} \sigma_{2}(\ell g_{1}) \left(\frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y} \right)^{s/2} e\left(\frac{\overline{n}r\ell}{t}\right) ds \ dx \right]^{2}.
\]

By Cauchy-Schwarz’s inequality and the large sieve inequality in Lemma 3.5, we obtain that the above is bounded by

\[
\ll \int_{0}^{\infty} |\Psi_{3}(x)| \int_{(\sigma)}^{\infty} |\gamma(s)| \sum_{t \sim T_{1}} \sum_{r \pmod{t}}^{*} \left[ \int_{0}^{\infty} \frac{\sigma_{2}(ng)}{n} \left(\frac{n}{N/g}\right)^{s/2} \Psi\left(\frac{ng}{N}\right) \right. \times \sum_{\ell \ll q^{\frac{1}{4}} \frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y}}^{*} \sigma_{2}(\ell g_{1}) \left(\frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y} \right)^{s/2} e\left(\frac{\overline{n}r\ell}{t}\right) ds \ dx \right]^{2}.
\]

\[
\ll q^{\epsilon} \left(\frac{T_{1}^{2}}{d^{2}g^{2}g_{1}^{2}} + \frac{N}{g} \frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y} \right) \left(\frac{g}{N}\right) \left(\frac{T_{1}^{2}bcN}{d^{2}g^{2}g_{1}Y} \right) \ll q^{\epsilon} \frac{T_{1}^{4}bc}{d^{4}g^{4}g_{1}^{2}Y}.
\]

Therefore,

\[
G_{2}(T_{1}, H_{1}; q) \ll \frac{Yq^{\epsilon}}{qT_{1}^{3}(bc)^{2}} \sum_{d \mid bc} d^{3} \sum_{g \leq \frac{T_{1}}{4}} g \sum_{g_{1} \leq \frac{T_{1}}{2g}} g_{1} \frac{T_{1}^{4}bc}{d^{4}g^{4}g_{1}^{2}Y} \ll q^{\epsilon}.
\]

From Case 1.1 - 1.2, we deduce that \( G_{2}(T_{1}, H_{1}; q) \ll q^{\epsilon} \) and conclude that

\( F_{2}(T_{1}, H_{1}; q) \ll q^{\epsilon} \), as desired.

Case 2: \( \frac{1}{q^{\frac{1}{4}}} \sqrt{\frac{H_{1}Y}{q}} \ll T_{1} \ll \sqrt{\frac{H_{1}Y}{q}} \). For this case, we will show that

**Lemma 6.2.** Let \( \frac{1}{q^{\frac{1}{4}}} \sqrt{\frac{H_{1}Y}{q}} \ll T_{1} \ll \sqrt{\frac{H_{1}Y}{q}} \). Then

\( F_{2}(T_{1}, H_{1}; q) \ll q^{\epsilon} \),
where the implied constant depends on $\epsilon$.

We first consider the sum over $h$ in Equation (6.2). Using Equation (3.1) to express $J_{k-1}(x)$, we have

$$S(H_1, x, x', t) = S_1(H_1, x, x', t) + S_2(H_1, x, x', t) + S_3(H_1, x, x', t) + S_4(H_1, x, x', t),$$

where each $S_i(H_1, x, x', t)$ is of the form

$$c_k t d g g_1 \sqrt{\frac{q}{H_1 Y}} \sum_h \Psi_5 \left( \frac{h}{H_1} \right) e \left( \pm \frac{2}{t d g g_1} \frac{h x Y}{q} \right) e \left( \pm \frac{h q T_1}{t d g g_1} \right),$$

for some choice of the sign $\pm$, some constant $c_k$ and where $\Psi_5(x)$ is a product of $\Psi(x)/\sqrt{x}$ and $W$ or $\bar{W}$. In particular, $\Psi_5(x)$ is smooth and compactly supported. Similar to Case 1, we apply Poisson summation formula to the sum over $h$ and derive that

$$S_i(H_1, x, x', t) = c_k t d g g_1 \sqrt{\frac{q}{H_1 Y}} \sum_h \Psi_5 \left( \frac{h}{H_1} \right) e \left( \pm \frac{2}{t d g g_1} \frac{h x Y}{q} \right) e \left( \pm \frac{h q T_1}{t d g g_1} \right),$$

(6.7)

If $|\beta| \gg q^3 \sqrt{\frac{Y}{H_1 q}}$, then $|\beta| \gg q^3 \frac{T_1}{H_1}$ since $T_1 \ll \sqrt{\frac{H_1 Y}{q}}$. We do integration by parts many times and obtain that the contribution from these terms is $\ll q^{-100}$. So we focus only terms from $|\beta| \ll q^3 \sqrt{\frac{Y}{H_1 q}}$. Similar to Case 1, the contribution from these terms can be bounded by

$$G_m(T_1, H_1; q) := \frac{q^3 Y H_1}{q T_1 (bc)^2} T_1 \sqrt{\frac{q}{H_1 Y}} \sqrt{\frac{Y}{H_1 q}} \int_{\mathbb{R}} |\Psi_5(z)| \sum_{d | b \ell} \frac{d}{d} \sum_{g \leq \frac{T_1}{d g g}} g \sum_{g_1 \leq \frac{T_1}{d g g_1}} g_1 \sum_{r \leq \frac{T_1}{d g g_1}} r \sum^* \sum_n \frac{\sigma_2(n g)}{n} \Psi \left( \frac{n g}{N} \right) \sum_{\ell \geq 1} \sigma_2(\ell g_1) e \left( \frac{\tilde{m} r \ell}{t} \right) \int_0^\infty Y_0 \left( \frac{4 \pi}{t} \sqrt{\frac{\ell Y x}{b c n g g_1}} \right)$$

$$\times \left| \sum_n \frac{\sigma_2(n g)}{n} \Psi \left( \frac{n g}{N} \right) \sum_{\ell \geq 1} \sigma_2(\ell g_1) e \left( \frac{\tilde{m} r \ell}{t} \right) \right| dx \right|^2 dz,$$

where $\Psi_6(x) = \Psi_3(x)/\sqrt{x}$. Note the factor in front of integral can be simplified to $\frac{q^3 Y}{q T_1 (bc)^2}$. In order to show that

$$G_m(T_1, H_1; q) \ll q^3,$$

we separate into 2 cases depending on the range of $\ell$. We write

(6.9) $$G_m(T_1, H_1; q) \ll G_{m,1}(T_1, H_1; q) + G_{m,2}(T_1, H_1; q),$$
where $\mathcal{G}_{m,1}(T_1, H_1; q)$ has the same expression as $\mathcal{G}_m(T_1, H_1; q)$ but restricting the size of $\ell$ to $\ell \gg q^{\epsilon_1} \frac{H_1}{q d^2 g^2 y_{11}}$ and $\ell \ll q^{\epsilon_1} \frac{H_1}{q d^2 g^2 y_{11}}$.

**Case 2.1.** $\ell \gg q^{\epsilon_1} \frac{H_1}{q d^2 g^2 y_{11}}$. We use the expression for $Y_0(x)$ in Equation (3.3) and write the integral inside the absolute value of Equation (6.8) as

\[
(6.10) \left( t \sqrt{\frac{\text{bcng} g_1}{\ell Y}} \right)^{1/2} \int_0^\infty \frac{1}{x^{1/4}} e \left( \pm \frac{2}{t} \sqrt{\frac{\ell Y x}{\text{bcng} g_1}} \right) e \left( \pm \frac{2}{tdg_1} \sqrt{\frac{z x Y H_1}{q}} \right) \Psi_0(x) \, dx.
\]

From the fact that $T_1 \ll \sqrt{\frac{H_1 Y}{q}}$, $\frac{H_1 Nbc}{q d^2 g^2 y_{11}} \gg T^2 bcN_{11}, H_{11}^2 Y$, and so $\ell \gg q^{\epsilon_1} T^2 bcN_{11}, H_{11}^2 Y$. We then integrate by parts many times and obtain that $\mathcal{G}_{m,1}(T_1, H_1; q) \ll q^{-100}$.

**Case 2.2:** $\ell \ll q^{\epsilon_1} \frac{N H_1 bc}{q d^2 g^2 y_{11}}$. Since $\frac{1}{q^{\epsilon_1}} \sqrt{\frac{H_1 Y}{q}} \ll T_1 \ll \sqrt{\frac{H_1 Y}{q}}$, then

\[
(6.11) \ell \ll q^{\epsilon_1} \frac{N H_1 bc}{q d^2 g^2 y_{11}} \ll q^{\epsilon_1} T^2 bcN_{11}, H_{11}^2 Y.
\]

We write $Y_0(x)$ using the integral representation in Equation (6.6). By the condition on $\ell$, we can treat $\mathcal{G}_{m,2}(T_1, H_1; q)$ in the same manner as Case 1.2 and obtain that $\mathcal{G}_{m,2}(T_1, H_1; q) \ll q^{\epsilon}$.

From Case 2.1-2.2 and Equation (6.9), we conclude that $\mathcal{G}_m(T_1, H_1; q) \ll q^{\epsilon}$ and this prove Lemma 6.2.

**Case 3:** $T_1 \ll \frac{1}{q^{\epsilon_1}} \sqrt{\frac{H_1 Y}{q}}$. For this case, we show that

**Lemma 6.3.** Let $T_1 \ll \frac{1}{q^{\epsilon_1}} \sqrt{\frac{H_1 Y}{q}}$. Then

\[
\mathcal{F}_2(T_1, H_1; q) \ll q^{\epsilon},
\]

where the implied constant depends on $\epsilon$.

We now divide the range for $\ell$ into 2 ranges, which are $\frac{1}{q^{\epsilon_1}} \frac{H_1 Nbc}{q d^2 g^2 y_{11}} \ll \ell < 3 \frac{H_1 Nbc}{q d^2 g^2 y_{11}}$ and the rest.

For the second range, we write the Bessel functions $J_{k-1}(x)$ and $Y_0(x)$ as in Equations (3.1) and (3.3), respectively. Then, the integral in Equation (6.1) is of the form

\[
\left( t \sqrt{\frac{\text{bcng} g_1}{\ell Y}} \right)^{1/2} \left( tdg_1 \sqrt{\frac{q}{z Y H_1}} \right)^{1/2} \int_0^\infty \frac{1}{x^{1/2}} e \left( \pm \frac{2}{t} \sqrt{\frac{\ell Y x}{\text{bcng} g_1}} \right) e \left( \pm \frac{2}{tdg_1} \sqrt{\frac{z x Y H_1}{q}} \right) \Psi_7(x) \, dx,
\]

where $\Psi_7(x)$ is a product of $\Psi_3(x)$ and $W$ or $\overline{W}$. Since $T_1 \ll \frac{1}{q^{\epsilon_1}} \sqrt{\frac{H_1 Y}{q}}$,

\[
\frac{2}{tdg_1} \sqrt{\frac{z x Y H_1}{q}} \gg q^{\epsilon_1}.
\]
If both terms inside the exponential function has the same sign then we can integrate by parts many times and obtain that the contribution from these terms is negligible. However, if the signs are different, then the size of the derivative of the phase inside the exponential is

$$\mathcal{E}(x) = \frac{1}{t} \sqrt{x} \sqrt{\frac{N}{bcngg_1}} - \frac{1}{tdgg_1 \sqrt{x}} \sqrt{\frac{zYH_1}{q}}.$$ 

If $\ell \geq 3\frac{H_1 Nbc}{q \sqrt{3} g^a g_1}$, then

$$\mathcal{E}(x) = \frac{1}{t} \sqrt{x} \sqrt{\frac{N}{bcngg_1}} - \frac{1}{tdgg_1 \sqrt{x}} \sqrt{\frac{zYH_1}{q}} \geq \left( \sqrt{3} \sqrt{\frac{N}{n}} - \sqrt{2} \right) \frac{1}{tdgg_1 \sqrt{x}} \sqrt{\frac{H_1 Y}{q}} \gg q^{1}.$$ 

Similarly, if $\ell \leq \frac{1}{4} \frac{H_1 Nbc}{q \sqrt{2} g^a g_1}$, then

$$\mathcal{E}(x) = \frac{1}{tdgg_1 \sqrt{x}} \sqrt{\frac{zYH_1}{q}} - \frac{1}{t} \sqrt{x} \sqrt{\frac{N}{bcngg_1}} \geq \left( 1 - \frac{1}{2} \sqrt{\frac{N}{n}} \right) \frac{1}{tdgg_1 \sqrt{x}} \sqrt{\frac{H_1 Y}{q}} \gg q^{1}.$$ 

Note moreover that $\mathcal{E}(x) \asymp \mathcal{E}^{(k)}(x)$ for any $k \geq 0$. We then do integration by parts many times and derive that the contribution from these terms is bounded by $q^{-100}$.

Now we can focus on terms from $\frac{1}{4} \frac{H_1 Nbc}{q \sqrt{2} g^a g_1} < \ell < \frac{3}{4} \frac{H_1 Nbc}{q \sqrt{2} g^a g_1}$. After squaring out Equation (6.1), we obtain that the contribution from these terms has the same expression as Equation (6.2), with the additional restriction $\frac{1}{4} \frac{H_1 Nbc}{q \sqrt{2} g^a g_1} < \ell, \ell' < \frac{3}{4} \frac{H_1 Nbc}{q \sqrt{2} g^a g_1}$.

Next, we treat the sum over $h$ similarly to the beginning of Case 2 and obtain the same expression as in Equation (6.3). Moreover, by the same argument as Case 2, the contribution from $|\beta| \gg q' \sqrt{\frac{Y}{H_{1q}}}$ is negligible. Next, we use Equation (6.3) for $Y_0(x)$ when $|\beta| \ll q' \sqrt{\frac{Y}{H_{1q}}}$. Hence, to bound $\mathcal{F}_2(T_1, H_1; q)$, it is sufficient to bound

$$\mathcal{G}_a(T_1, H_1; q) := \frac{\sqrt{H_1}}{T_1 \sqrt{n} \sqrt{bcngg_1}} \sum_{|\beta| \ll q' \sqrt{\frac{Y}{H_{1q}}}} \sum_{d|bc} \sum_{d|g} \sum_{g \leq T/d} \sum_{g_1} \sum_{g_1} \sum_{g_1} \sum_{t} \sum_{t} \sum_{r (mod tdgg_1)}$$

$$\times \sum_{n,n' (mod \ell, \ell')} \sigma_2(n) \Psi \left( \frac{ng}{N} \right) \sigma_2(n'g) \Psi \left( \frac{n'g}{N} \right) \frac{\sigma_2(\ell g_1)}{\ell^{1/4}} \frac{\sigma_2(\ell' g_1)}{\ell'^{1/4}} e \left( -\frac{\ell g_1}{t} \right)$$

$$\times \frac{\ell \beta q}{t^{1/4}} e \left( -\frac{\ell g_1}{t} \right) J(n, n', \ell, \ell'),$$
where

\[
\mathcal{J}(n, n', \ell, \ell') := \int_0^\infty \Psi_5(z) \int_0^\infty \int_0^\infty \Psi_7(x) \Psi_7(x') e\left(\pm \frac{2}{t} \sqrt{\frac{\ell x Y}{b c n g g_1}} \pm \frac{2}{t} \sqrt{\frac{\ell' x' Y}{b c n' g g_1}}\right) \\
\times e\left(\pm \frac{2}{t} \sqrt{\frac{z x H_1 Y}{q}} \pm \frac{2}{t} \sqrt{\frac{z' x' H_1 Y}{q}}\right) e\left(-\frac{\beta H_1 z}{t d g g_1}\right) dx \, dx' \, dz,
\]

and \(\Psi_7(x)\) is the product between \(\frac{\Psi_3(x)}{\sqrt{x}}\) and \(W\) or \(\overline{W}\). Due to the range of \(T_1\) and \(\ell, \ell'\), we have that

\[
\frac{2}{t d g g_1} \sqrt{\frac{z x H_1 Y}{q}}, \quad \frac{2}{t d g g_1} \sqrt{\frac{z' x' H_1 Y}{q}} \gg q^t,
\]

and

\[
\frac{2}{t} \sqrt{\frac{\ell x Y}{b c n g g_1}}, \quad \frac{2}{t} \sqrt{\frac{\ell' x' Y}{b c n' g g_1}} \gg \frac{1}{t d g g_1} \sqrt{\frac{H_1 Y}{q}} \gg q^t.
\]

If the signs in front of \(\frac{2}{t d g g_1} \sqrt{\frac{z x H_1 Y}{q}}\) and \(\frac{2}{t \sqrt{\frac{\ell x Y}{b c n g g_1}}}\) in \(\mathcal{J}(n, n', \ell, \ell')\) are the same, we can integrate by part many times with respect to \(x\) and show that the contribution from these terms is negligible. The same is true for the signs in front of \(\frac{2}{t d g g_1} \sqrt{\frac{z' x' H_1 Y}{q}}\) and \(\frac{2}{t \sqrt{\frac{\ell' x' Y}{b c n' g g_1}}}\). This motivates us to define

\[
\mathcal{J}_1(n, n', \ell, \ell') := \int_0^\infty \Psi_5(z) \int_0^\infty \int_0^\infty e\left(-\frac{2}{t} \sqrt{\frac{\ell x Y}{b c n g g_1}} - \frac{2}{t} \sqrt{\frac{\ell' x' Y}{b c n' g g_1}}\right) \\
\times e\left(\frac{2}{t d g g_1} \sqrt{\frac{z x H_1 Y}{q}} + \frac{2}{t d g g_1} \sqrt{\frac{z' x' H_1 Y}{q}}\right) e\left(-\frac{\beta H_1 z}{t d g g_1}\right) \Psi_7(x) \Psi_7(x') \, dx \, dx' \, dz,
\]

and

\[
\mathcal{J}_2(n, n', \ell, \ell') := \int_0^\infty \Psi_5(z) \int_0^\infty \int_0^\infty e\left(-\frac{2}{t} \sqrt{\frac{\ell x Y}{b c n g g_1}} + \frac{2}{t} \sqrt{\frac{\ell' x' Y}{b c n' g g_1}}\right) \\
\times e\left(\frac{2}{t d g g_1} \sqrt{\frac{z x H_1 Y}{q}} - \frac{2}{t d g g_1} \sqrt{\frac{z' x' H_1 Y}{q}}\right) e\left(-\frac{\beta H_1 z}{t d g g_1}\right) \Psi_7(x) \Psi_7(x') \, dx \, dx' \, dz.
\]

By symmetry it is enough to consider only \(\mathcal{J}_1\) and \(\mathcal{J}_2\). Further, let \(G_{s,i}(T_1, H_1; q)\) be the same expression as \(G(T_1, H_1; q)\) but replacing \(\mathcal{J}(T_1, H_1; q)\) by \(\mathcal{J}_i(T_1, H_1; q)\) for \(i = 1, 2\). It now suffices to show that for \(i = 1, 2\),

\[
G_{s,i}(T_1, H_1; q) \ll q^t.
\]
Calculation of $\mathcal{G}_{s,1}(T_1, H_1; q)$. By the change of variables $\sqrt{x} = u - \sqrt{x'}$, (6.12)
\[
\mathcal{J}_1(n, n', \ell, \ell') = \int_0^\infty \Psi_5(z) \int_0^\infty \int_0^\infty e \left(-\frac{2u}{t} \sqrt{\frac{\ell Y}{bcng_1}} + 2 \frac{x'Y}{t \cdot b c g_1} \left(\sqrt{\frac{\ell'}{n}} - \sqrt{\frac{\ell}{n'}}\right)\right) \times e \left(\frac{2u}{tdgg_1} \sqrt{\frac{z H_1 Y}{q}} - \frac{\beta H_1 z}{tdgg_1}\right) 2(u - \sqrt{x'}) \Psi_7 \left((u - \sqrt{x'}^2)\right) \Psi_7(x') du \, dx' \, dz.
\]

Note that the integrand vanishes when $u$ is outside the range $1 \leq u - \sqrt{x'} \leq \sqrt{2}$ because of the support of $\Psi_7(x)$. We consider the derivative with respect to $z$ of the expression inside the exponential, which is
\[
\mathcal{D}_\beta(z) = \frac{u}{tdgg_1} \sqrt{\frac{H_1 Y}{q z}} - \frac{\beta H_1 z}{tdgg_1}.
\]
If $|\mathcal{D}_\beta(z)| \geq q^{\epsilon_1/2}$, we can do integration by parts many times and obtain that the contribution from these terms is negligible. Hence it suffices to consider $u \in \mathcal{I}_{z,\beta}$, where $\mathcal{I}_{z,\beta}$ is the interval
\[
\left(\sqrt{z} \beta \sqrt{\frac{q H_1 Y}{Y}} - 2T_1 \sqrt{\frac{q}{H_1 Y}} \sqrt{z q^{\epsilon_1/2}}, \sqrt{z} \beta \sqrt{\frac{q H_1 Y}{Y}} + 2T_1 \sqrt{\frac{q}{H_1 Y}} \sqrt{z q^{\epsilon_1/2}}\right).
\]
Note that the length of $\mathcal{I}_{z,\beta}$ is
(6.13)
\[
|\mathcal{I}_{z,\beta}| \asymp T_1 \sqrt{\frac{q}{H_1 Y}} q^{\epsilon_1/2}.
\]

Let $R = \frac{T_1}{q^{\epsilon_1/4}} \sqrt{\frac{q}{H_1 Y}}$. If $|\ell' n - \ell n'| \gg q^{\epsilon_1/2} \frac{N H_1 bc}{g \cdot q d^2 g_1^2 g_1} R$, then
\[
|\sqrt{\frac{\ell}{n}} - \sqrt{\frac{\ell'}{n'}}| \asymp \left|\frac{\ell}{n} - \frac{\ell'}{n'}\right| \sqrt{\frac{N q d^2 g_1^2 g_1}{g \cdot N H_1 bc}} \gg q^{\epsilon_1/2} \frac{H_1 bc}{q d^2 g_1^2 g_1} R \sqrt{\frac{N q d^2 g_1^2 g_1}{g \cdot N H_1 bc}} \gg q^{\epsilon_1/4} T_1 \sqrt{\frac{bc}{d^2 g_1 Y}} \asymp q^{\epsilon_1/4} \frac{t}{\sqrt{bcgg_1}},
\]
for $t \sim \frac{T_1}{tdgg_1}$. Thus, integration by parts many times with respect to $x'$ shows that the contribution from these terms is negligible. So we now assume
(6.14)
\[
|\ell' n - \ell n'| \ll q^{\epsilon_1/2} \frac{N \cdot N H_1 bc}{g \cdot q d^2 g_1^2 g_1} R.
\]
In order to deal with the condition in (6.14), we divide the intervals for $\ell, \ell'$ into
divides of length $R \frac{N H_1}{q d^2 q' g_1}$ and $n, n'$ into intervals of length $R \frac{N}{g}$. Trivially, we need
$
\sim \frac{1}{R_{\ell}}$
such tuples $(I_n, I_n', I_n, I_n')$ to cover the whole range. However, for fixed $I_n, I_n, I_n'$, where $n, n', I, I'$ are the left-ended points of intervals $I_n, I_n', I_n, I_n'$, respectively, we may assume by (6.14) that

$$
\frac{\ell' n - \ell n'}{\ell} \ll q^{\epsilon/2} \frac{N}{g} R,
$$

for some $n \in I_n, n' \in I_n', \ell \in I_n$ and $\ell' \in I_n'$. However, due to our restriction on the
length of the intervals, this implies that in fact (6.15) is also satisfied by $n, n', I$ and $I'$. Thus, there are $O(q^{\epsilon/2})$ choices for $I_n'$. Hence there are only $O\left(\frac{q^{\epsilon/2}}{R^3}\right)$ relevant four
tuples $(I_n, I_n', I_n, I_n')$ with endpoints satisfying (6.14), and we obtain that

$$
G_{n,1}(T_1, H_1; q) \ll \frac{\sqrt{H_1}}{T_1} \frac{1}{\sqrt{q(bc)^{3/2}}} \sum_{0 \leq \beta \leq q} \int_{I_{\beta}}^{\infty} |\Psi(z)| \int_{I_{\beta}}^{\infty} \left|\left(u - \sqrt{x'}\right) \Psi\left((u - \sqrt{x'})^2\right)\right| 
\times \sum_{d|bc} \frac{\sqrt{g}}{d} \sum_{g \leq \ell' / d} g^{-1/2} \sum_{g_1 \leq \frac{\ell'}{d g'}} \sum_{(I_n, I_n, I_n', I_n')} \sum_{I_{\beta}}^{\text{rel}} S(I_n, I_n', I_n, I_n') \Psi_7(x') \, du \, dx' \, dz,
$$

where $\sum_{\text{rel}}$ is the sum over the relevant tuples with all points satisfying (6.15), and

$$
S(I_n, I_n', I_n, I_n') = \sum_{t \equiv \frac{1}{d g_1}}^{1} \sum_{(t, n') = 1}^{\ast} \psi_{\text{rel}}(\frac{n g}{N}) \psi_{\text{rel}}(\frac{n' g}{N})
\times \sum_{n \in I_n, n' \in I_n'} \sum_{(m n', t) = 1}^{\ast} \frac{\sigma_2(n g)}{n^{3/4}} \frac{\sigma_2(n' g)}{n'^{3/4}} e\left(\frac{n f}{t}\right)
\times \sum_{t \in I_n, t' \in I_n'} \sum_{(t, t') = 1} \frac{\sigma_2(\ell g_1)}{\ell^{1/4}} e\left(\frac{-\beta \ell}{d^2 n f} \frac{\ell}{t}\right) \frac{\sigma_2(\ell' g_1)}{\ell'^{1/4}} e\left(\frac{-\beta \ell'}{d^2 n' f} \frac{\ell'}{t}\right)
\times e\left(\frac{-2 u}{t} \sqrt{\frac{\ell Y}{bc n g_1}} + \frac{2}{t} \sqrt{\frac{x' Y}{bc g_1}} \left(\sqrt{\frac{\ell}{n}} - \sqrt{\frac{\ell'}{n'}}\right)\right).
$$

Further

$$
|S(I_n, I_n', I_n, I_n')| \leq |S_1(I_n, I_n)| + |S_2(I_n', I_n')|,
$$

where $S_1$ and $S_2$ are defined as follows:
where

\[ S_1 := S_1(I_n, I_t) = \sum_{t \sim \frac{t_1}{dgg_1}} \sum_{r \left( \text{mod} \ t dgg_1 \right)} r (\mod t dgg_1) e^{\left( -\frac{be}{t} n \ell \right)} e \left( -\frac{2u}{t} \sqrt{\frac{Y}{b cgg_1}} + 2 \frac{t}{b cgg_1} \sqrt{\frac{x'}{n'}} \right)^2 \]

and

\[ S_2 := S_2(I_n', I_t') = \sum_{t' \sim \frac{t_1}{dgg_1}} \sum_{r' \left( \text{mod} \ t' dgg_1 \right)} r' (\mod t' dgg_1) e^{\left( -\frac{be}{t'} n' \ell' \right)} e \left( -\frac{2u}{t'} \sqrt{\frac{Y}{b cgg_1}} + 2 \frac{t'}{b cgg_1} \sqrt{\frac{x'}{n'}} \right)^2 \]

By a change of variables in \( r' \), we have

\[ S_1(I_n, I_t) \leq dgg_1 \sum_{t \sim \frac{t_1}{dgg_1}} \sum_{r \left( \text{mod} \ t \right)} e \left( -\frac{2u}{t} \sqrt{\frac{Y}{b cgg_1}} + 2 \frac{t}{b cgg_1} \sqrt{\frac{x}{n}} \right) e \left( \frac{r n \ell}{t} \right)^2 \]

(6.17)

\[ \sum_{n \in I_n} \sum_{(n, t)} \frac{\sigma_2(n g)}{n^{3/4}} \psi \left( \frac{ng}{N} \right) \sum_{\ell \in I_t} \frac{\sigma_2(\ell g)}{\ell^{1/4}} = \sum_{n' \in I_n'} \sum_{(n', t')} \frac{\sigma_2(n' g)}{n'^{3/4}} \psi \left( \frac{n' g}{N} \right) \sum_{\ell' \in I_t'} \frac{\sigma_2(\ell' g)}{\ell'^{1/4}} \]

Similarly, we have

\[ S_2(I_{n'}, I_{t'}) \leq dgg_1 \sum_{t' \sim \frac{t_1}{dgg_1}} \sum_{r' \left( \text{mod} \ t' \right)} e \left( -\frac{2u}{t'} \sqrt{\frac{Y}{b cgg_1}} + 2 \frac{t'}{b cgg_1} \sqrt{\frac{x'}{n'}} \right) e \left( \frac{r' n' \ell'}{t'} \right)^2 \]

(6.18)

\[ \sum_{n' \in I_n'} \sum_{(n', t')} \frac{\sigma_2(n' g)}{n'^{3/4}} \psi \left( \frac{n' g}{N} \right) \sum_{\ell' \in I_t'} \frac{\sigma_2(\ell' g)}{\ell'^{1/4}} \]

Hence

\[ G_{s,1}(T_1, H_1; q) \ll \mathcal{P}_1(T_1, H_1; q) + \mathcal{P}_2(T_1, H_1; q), \]

where \( \mathcal{P}_i(T_1, H_1; q) \) has the same expression as the right hand side of \( (6.16) \) but we replace \( S(I_n, I_n', I_t, I_t') \) by the right hand side of \( S_i \) in \( (6.17) \) and \( (6.18) \). Hence to show that \( G_{s,1}(T_1, H_1; q) \ll q^e \), it is enough to prove the following.
Lemma 6.4. Let all notations be defined as above. Then for \( i = 1, 2 \), we have

\[
S_i \ll q^i \left( \frac{T_i^2}{dgg_1} + R_i^2 \frac{N NH_{1bc}}{g qdg} \right) R_i^2 \sqrt{\frac{H_{1bc}}{qd^2g_1}}.
\]

From this we deduce that

\[
P_i(T_1, H_1; q) \ll q^\epsilon.
\]

Proof. We start with bounding \( S_2(\mathcal{I}_n, \mathcal{I}_{\ell'}) \). We will eventually apply the large sieve Lemma 3.5 in order to do this, we wish to separate \( \ell' \) from \( n' \). We recall that \( n, n', l, l' \) are the left-ended points of intervals \( \mathcal{I}_n, \mathcal{I}_{n'}, \mathcal{I}_l, \mathcal{I}_{\ell'} \). Let

\[
f(x) = e(x).
\]

We will write a Taylor polynomial for \( e\left(-\frac{2}{t} \sqrt{\frac{xY}{bcn'gg_1}} \right) \) around \( x = -\frac{2}{t} \sqrt{\frac{xY}{bcn'gg_1}} \).

Hence

\[
e\left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} f^{(\alpha)} \left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) \left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) \left(\sqrt{\frac{\ell'}{n'}} - \sqrt{\frac{v}{n'}}\right) ^\alpha.
\]

Moreover since \( |\ell' - \ell'| \ll R \frac{NH_{1bc}}{qd^2g_1} \) and \( |n' - n'| \ll R \frac{N}{g} \), we have

\[
f^{(\alpha)} \left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) \left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) \left(\sqrt{\frac{\ell'}{n'}} - \sqrt{\frac{v}{n'}}\right) ^\alpha
\]

\[
\ll \alpha \left(\frac{1}{t} \sqrt{\frac{Y}{bcg_1}} \right) \sum_{j=0}^{\alpha} \left(\begin{array}{c} \alpha \\ j \end{array}\right) \frac{\sqrt{\ell'} - \sqrt{v}}{\sqrt{n'}} ^j \left(\sqrt{\frac{1}{\sqrt{n'}}} - \frac{1}{\sqrt{n'}} \right) ^{\alpha-j}
\]

\[
\ll \alpha \left(\frac{1}{t} \sqrt{\frac{Y}{bcg_1}} \right) \sum_{j=0}^{\alpha} \left(\begin{array}{c} \alpha \\ j \end{array}\right) \frac{\ell' - \ell'}{\sqrt{n'(\sqrt{\ell'} + \sqrt{v'})} ^j} \left(\sqrt{\frac{1}{\sqrt{n'}}} - \frac{n' - n'}{\sqrt{n'n'(\sqrt{n'} + \sqrt{v'})} ^j} \right) ^{\alpha-j}
\]

\[
\ll \alpha R^\alpha \left(\frac{1}{t} \sqrt{\frac{Y}{bcg_1}} \right) \left(\frac{NH_{1bc}}{qd^2g_1} \right) ^\alpha \left(\frac{1}{N/g} \right) ^\alpha \ll \alpha \left(\frac{R}{T_1} \sqrt{\frac{H_1Y}{q}} \right) ^\alpha \ll q^{-\alpha \epsilon_1/4}.
\]

Choosing \( B \) such that \( q^{-B\epsilon_1/4} \ll q^{-100} \), we obtain that

\[
f\left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) = \sum_{0 \leq \alpha \leq B} \frac{1}{\alpha!} f^{(\alpha)} \left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) \left(-\frac{2}{t} \sqrt{\frac{\ell'xY}{bcn'gg_1}} \right) \left(\sqrt{\frac{\ell'}{n'}} - \sqrt{\frac{v}{n'}}\right) ^\alpha + O(q^{-100}).
\]
Therefore,

\[ S_2(\mathcal{I}_n', \mathcal{I}_r) \ll dg_{1} q^e \sum_{0 \leq \alpha \leq B} \left( \frac{dg_{1}}{T_1} \right)^{2\alpha} \sum_{0 \leq j \leq \alpha} \sum_{r \text{ (mod } t)} \sum_{r' \in I_{n', (n', t)} = 1} \frac{\sigma_2(n'g)}{n'^{3/4}(\sqrt{n'})^j} \left( \frac{1}{\sqrt{n'}} - \frac{1}{\sqrt{n''}} \right)^{\alpha-j} \times \Psi \left( \frac{n'g}{N} \right) \sum_{\ell' \in I_{\ell'}} \left( \frac{(\sqrt{\ell'} - \sqrt{\ell})^j}{t} \right)^2 + O(q^{-50}). \]

Then by large sieve inequality in Lemma 3.5 and the same computation as Equation (6.13), we have

\[ S_2(\mathcal{I}_n', \mathcal{I}_r) \ll dg_{1} q^e \sum_{0 \leq \alpha \leq B} \left( \frac{dg_{1}}{T_1} \right)^{2\alpha} \left( \frac{T_1^2}{d^2g^2g_1^2} + R^2 \frac{N H_1 b c}{g q d^2g_2g_1} \right) \times \sum_{n' \in I_{n'}} \frac{1}{(\sqrt{n'})^{2j}} \left( \frac{1}{\sqrt{n'}} - \frac{1}{\sqrt{n''}} \right)^{2\alpha-2j} \sum_{\ell' \in I_{\ell'}} \left( \frac{(\sqrt{\ell'} - \sqrt{\ell})^2j}{t} \right)^2 \leq q^e \left( \frac{T_1^2}{d^2g^2g_1^2} + R^2 \frac{N H_1 b c}{g q d^2g_2g_1} \right) R^2 \sqrt{H_1 b c} \sqrt{q d^2g_1}. \]

The calculation of \( S_1(\mathcal{I}_n, \mathcal{I}_r) \) proceeds similarly. Here we write Taylor polynomials for \( e \left( \left( -\frac{2u}{t} + \frac{t}{n} \frac{xY_{bcg_1}}{\sqrt{y}} + \frac{t}{n} \frac{x^2Y_{bcg_1}}{\sqrt{y}} \right) \sqrt{\frac{1}{n}} \right) \) around \( x = \left( -\frac{2u}{t} + \frac{t}{n} \frac{xY_{bcg_1}}{\sqrt{y}} + \frac{t}{n} \frac{x^2Y_{bcg_1}}{\sqrt{y}} \right) \sqrt{\frac{1}{n}} \) instead, and we derive that \( S_1(\mathcal{I}_n, \mathcal{I}_r) \) can be bounded by the same quantity.

Next we bound \( P_1(T_1, H_1; q) \). First, the length of \( \mathcal{I}_{\alpha, \beta} \) in Equation (6.13) and the fact that the number of relevant tuples is \( O \left( \frac{q^{1/2}}{R^3} \right) \) yields

\[ P_1(T_1, H_1; q) \ll \frac{q \sqrt{H_1}}{T_1 \sqrt{q} (bc)^{3/2}} \sum_{0 \leq \beta < q^e} \sqrt{\frac{1}{n}} \int_0^\infty |\Psi_5(z)| \int_0^\infty \int_{\mathcal{I}_{\alpha, \beta}} |u - \sqrt{x'}||\Psi_7 \left( (u - \sqrt{x'})^2 \right) \Psi_7(x')| \times \sum_{d \mid b \mid g_{1}^{1/2}} \sum_{g \leq \sqrt{T_1/d}} \sum_{g_1 \leq \sqrt{T_1/d}} \sum_{(\mathcal{I}_{n}, \mathcal{I}_{n'}, \mathcal{I}_{r}) = 1} \left( \frac{T_1^2}{dg_{1}} + R^2 \frac{N H_1 b c}{g q d^2g_2g_1} \right) R^2 \sqrt{\frac{H_1 b c}{q d^2g_1}} \\
\ll q^{e+\epsilon_1} \frac{\sqrt{H_1}}{T_1 \sqrt{q}} \frac{\sqrt{Y_{H_1}} T_1}{H_1 q} \frac{q}{H_{1} Y_{R}} \left( \frac{T_1^2}{q} + \frac{R^2 N^2 H_1}{q} \right) \sqrt{\frac{H_1}{q}} \ll q^e, \]

where we recall that \( R = \frac{T_1}{q^{1/2}} \sqrt{H_{1} Y_{R}} \), \( N \ll \sqrt{Y} \), \( H_1 \ll q \), \( Y \leq q^{2+\epsilon} \), and we choosing sufficiently small \( \epsilon_1 \). This completes the lemma.
Calculation of $G_{s,2}(T_1, H_1; q)$. By changing variable and letting $u = \sqrt{x} - \sqrt{x'}$, we have that

$$J_2(n, n', \ell, \ell') = \int_0^\infty \Psi_5(z) \int_0^\infty \int_0^\infty e\left( -\frac{2u}{t} \sqrt{\frac{\ell Y}{bcngg_1}} - \frac{2}{t} \sqrt{x'Y'_{bcgg_1}} \left( \sqrt{\frac{\ell}{n}} - \sqrt{\frac{\ell'}{n'}} \right) \right)$$

$$\times e\left( \frac{2u}{tdgg_1} \sqrt{\frac{z H_1 Y}{q}} - \frac{\beta H_1 z}{tdgg_1} \right) 2(u + \sqrt{x'}) \Psi_7((u + \sqrt{x'})^2) \Psi_7(x') \, du \, dx' \, dz.$$  

Note that the function vanishes when $u$ is outside the range $1 \leq u + \sqrt{x'} \leq \sqrt{2}$ because of the support of $\Psi_7(x)$. Further, $J_2$ has a similar expression to $J_1$ in Equation (6.12). In fact, we can use the same arguments as bounding $G_{s,1}(T_1, H_1; q)$ to conclude that

$$G_{s,2}(T_1, H_1; q) \ll q'.$$

From the calculation of both $G_{s,i}(T_1, H_1; q)$, we arrive at Lemma 6.3. From Lemma 6.1 - 6.3, we obtain Lemma 4.1 for $i = 2$.

7. Bounding $F_3(T_1, H_1; q)$

In this section, we sketch how to prove Lemma 4.1 for $i = 3$. We recall that $F_3(T_1, H_1; q)$ is defined in Equation (4.5). This is the same expression as for $F_2$ with $Y_0$ replaced by $K_0$. Thus, the proof proceeds along the same lines, but we use (3.4) in place of (3.3) and (3.8) in place of (3.7). Here, the proofs of Case 1 and Case 2 are similar. The proof for Case 3 is significantly easier because the expression for $K_0$ in (3.4) is a rapidly decreasing smooth function with small derivatives, as opposed to the expression for $Y_0(2\pi x)$ in (3.3) which involves the phase $e(x - 1/8)$. Thus, in Case 3, we can simply integrate by parts many times to show that the contribution in that range is negligible.

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