ON THE SUBMETRIZABILITY NUMBER AND $i$-WEIGHT OF QUASI-UNIFORM SPACES AND PARATOPOLOGICAL GROUPS

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Abstract. We derive many upper bounds on the submetrizability number and $i$-weight of paratopological groups and topological monoids with open shifts. In particular, we prove that each first countable Hausdorff paratopological group is submetrizable thus answering a problem of Arhangel’skii posed in 2002. Also we construct an example of a zero-dimensional (and hence regular) Hausdorff paratopological abelian group $G$ with countable pseudocharacter which is not submetrizable. In fact, all results on the $i$-weight and submetrizability are derived from more general results concerning normally quasi-uniformizable and bi-quasi-uniformizable spaces.

Introduction

This paper was motivated by the following problem of Arhangel’skii [1, 3.11] (also repeated by Tkachenko in his survey [24, 2.1]): Does every first countable Hausdorff paratopological group admit a weaker metrizable topology? A surprisingly simple answer to this problem was given by the authors in [4]. We just observed that each Hausdorff paratopological group $G$ carries a natural uniformity generated by the base consisting of entourages \( \{(x, y) \in G \times G : y \in U^{-1}xU \cap U^{-1}yU \} \) where $U$ runs over open neighborhoods of the unit $e$ in $G$. In [4] this uniformity was called the quasi-Roelecke uniformity on $G$ and denoted by $Q$. If $G$ is first-countable, then the quasi-Roelecke uniformity $Q$ is metrizable, which implies that the space $G$ is submetrizable. Moreover, if the quasi-Roelecke uniformity $Q$ is $\omega$-bounded, then the topology generated by the uniformity $Q$ is metrizable and separable, which implies that $G$ has countable $i$-weight, i.e., admits a continuous injective map onto a metrizable separable space.

In fact, for the submetrizability of $G$ it suffices to require the countability of the pseudocharacter $\psi(Q)$ of $Q$, i.e., the existence of a countable subfamily $\mathcal{U} \subset Q$ such that $\bigcap\mathcal{U} = \Delta_X$. So, the aim of the paper is to detect paratopological groups $G$ whose quasi-Roelecke uniformity $Q$ has countable pseudocharacter. For this we shall find some upper bounds on the pseudocharacter $\psi(Q)$. These bounds will give us upper bounds on the submetrizability number $sm(G)$ and the $i$-weight $iw(G)$ of a paratopological group $G$. In fact, the obtained upper bounds on $sm(G)$ and $iw(G)$ have uniform nature and depends on the properties of the two canonical quasi-uniformities $L$ and $R$ on $G$ called the left and right quasi-uniformities of $G$. These quasi-uniformities are studied in Sections 5 and 6. In Sections 3 and 4 we study properties of topological spaces whose topology is generated by two quasi-uniformities which are compatible in some sense (more precisely, are $\pm$-subcommuting or normally $\pm$-subcommuting). In Section 3 we prove that any two normally $\pm$-subcommuting quasi-uniformities are normal in the sense of [4]. This motivates the study of topological spaces whose topology is generated by a normal quasi-uniformity. For such spaces we obtain some upper bounds on the $i$-weight, which is done in Section 4. Section 1 has preliminary character. It contains the necessary information of topological spaces, quasi-uniform spaces, and their cardinal characteristics. In Section 7 we present two counterexamples to some natural conjectures concerning submetrizable paratopological groups.

1. Preliminaries

In this section we collect known information on topological spaces, quasi-uniformities, and their cardinal characteristics. For a set $X$ by $|X|$ we denote its cardinality. By $\omega$ we denote the set of all finite ordinals and by $\mathbb{N} = \omega \setminus \{0\}$ the set of natural numbers.

For a cardinal $\kappa$ by $\log(\kappa)$ we denote the smallest cardinal $\lambda$ such that $2^\lambda \geq \kappa$.

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1.1. Topological spaces and their cardinal characteristics. For a subset $A$ of a topological space $X$ by $\overline{A}$ and $A^\circ$ and $\overline{A}$ we denote the closure, interior and interior of the closure of the set $A$ in $X$, respectively.

A family $\mathcal{N}$ of subsets of a topological space $X$ is called a network of the topology of $X$ if each open set $U \subset X$ can be written as the union $\bigcup \mathcal{U}$ of some subfamily $\mathcal{U} \subset \mathcal{N}$. If each set $N \in \mathcal{N}$ is open in $X$, then $\mathcal{N}$ is a base of the topology of $X$.

A subset $D$ of a topological space $X$ is called strongly discrete if each point $x \in D$ has a neighborhood $U_x \subset X$ such that the family $(U_x)_{x \in D}$ is discrete in the sense that each point $z \in X$ has a neighborhood that meets at most one set $U_x$, $x \in D$. It is easy to see that each strongly discrete subset of (a $T_1$-space) $X$ is discrete (and closed) in $X$. A topological space $X$ is called (strongly) $\sigma$-discrete if $X$ can be written as the countable union $X = \bigcup_{n \in \omega} X_n$ of (strongly) discrete subsets of $X$.

A topological space $X$ is called

- Hausdorff if any two distinct points $x, y \in X$ have disjoint open neighborhoods $O_x \ni x$ and $O_y \ni y$;
- collectively Hausdorff if each closed discrete subset of $X$ is strongly discrete in $X$;
- functionally Hausdorff if for any two distinct points $x, y \in X$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$;
- regular if for any point $x \in X$ and a neighborhood $O_x \subset X$ there is a neighborhood $V_x \subset X$ of $x$ such that $\overline{V}_x \subset O_x$;
- completely regular if for any point $x \in X$ and a neighborhood $O_x \subset X$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f^{-1}(0, 1) \subset O_x$;
- quasi-regular if each non-empty open set $U \subset X$ contains the closure $\overline{V}$ of another non-empty open set $V \subset X$;
- submetrizable if $X$ admits a continuous metric (or equivalently, admits a continuous injective map into a metrizable space).

It is clear that each submetrizable space is functionally Hausdorff.

In Section 2 we shall need the following property of strongly $\sigma$-discrete spaces.

Proposition 1.1. Each strongly $\sigma$-discrete Tychonoff space $X$ is zero-dimensional and submetrizable. Moreover, $X$ admits an injective continuous map into the Cantor cube $\{0, 1\}^\kappa$ of weight $\kappa = \log(|X|)$.

Proof. The proposition trivially holds if $X$ is discrete. So, we assume that $X$ is not discrete and hence infinite. Write $X$ as the countable union $X = \bigcup_{n \in \omega} X_n$ of pairwise strongly discrete non-empty subsets $X_n$ of $X$. Let $\beta X$ be the Stone-Čech compactification of $X$. Using the strong discreteness of each $X_n$, we can extend each continuous bounded function $f : X_n \to \mathbb{R}$ to a continuous bounded function on $X$. This implies that the closure $\overline{X}_n$ of $X_n$ in $\beta X$ is homeomorphic to the Stone-Cech compactification $\beta X_n$ of the discrete space $X_n$ and hence has covering dimension dim($\beta X_n$) = 0 (see [10] 3.6.7 and 7.1.17]). By the Countable Sum Theorem [10] 3.1.8 for covering dimension in normal spaces, the $\sigma$-compact (and hence normal) space $Z = \bigcup_{n \in \omega} \overline{X}_n$ has covering dimension dim($Z$) = 0, which implies that its subspace $X = \bigcup_{n \in \omega} X_n$ is zero-dimensional.

Now we prove that $X$ is submetrizable. For every $n \in \omega$ and every $x \in X_n$ we can choose a closed-and-open neighborhood $U_x \subset X$ of $x$ such that $U_x \cap \bigcup_{k < n} X_k = \emptyset$ and the indexed family $(U_x)_{x \in X_n}$ is discrete in $X$. Then the union $\bigcup_{x \in X_n} U_x$ is a closed-and-open subset in $X$ and the function $d_n : X \times X \to \{0, 1\}$ defined by

$$d_n(x, y) = \begin{cases} 0, & \text{if } x, y \in U_x \text{ for some } x \in X_n \text{ or } x, y \not\in \bigcup_{z \in X_n} U_z, \\ 1, & \text{otherwise}, \end{cases}$$

is a continuous pseudometric on $X$. Consequently, the function $d = \max_{n \in \omega} \frac{1}{\kappa} d_n$ is a continuous metric on $X$, which implies that $X$ is submetrizable.

It follows that the space $X$ admits a continuous injective map into the countable product $\prod_{n \in \omega} D_n$ of discrete spaces $D_n$ of cardinality $|D_n| = 1 + |X_n| \leq |X|$. By definition of the cardinal $\kappa = \log(|X|)$, every discrete space $D_n$, $n \in \omega$, admits an injective (and necessarily continuous) map into the Cantor cube $\{0, 1\}^\kappa$. Then $\prod_{n \in \omega} D_n$ and hence $X$ also admits a continuous injective map into $\{0, 1\}^\kappa$. $\square$

For a cover $\mathcal{U}$ of a set $X$ and a subset $A \subset X$ we put $\mathcal{S}^0(A; \mathcal{U}) = A$ and $\mathcal{S}^{n+1}(A; \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap \mathcal{S}^n(A; \mathcal{U}) \neq \emptyset \}$ for $n \geq 0$.

1.2. Cardinal characteristics of topological spaces, I. For a topological space $X$ let

- $nw(X) = \min\{|N| : \mathcal{N} \text{ is a network of the topology of } X\}$ be the network weight of $X$;
- $d(X) = \min\{|A| : A \subset X, \overline{A} = X\}$ be the density of $X$;
- $hd(X) = \sup\{ d(Y) : Y \subset X \}$ the hereditary density of $X$;
The cardinal characteristics \( nw, d, s, e, c, l \) are well-known in General Topology (see [9], [13]) whereas \( \bar{l}, \bar{l}^* \) are relatively new and notations for these cardinal characteristics are not fixed yet. For example, the weak Lindelöf number \( \bar{l} \) often is denoted by \( wL \), but in [13] §3 it is denoted by \( wc \) and called the weak covering number. According to [21], the weak extent \( l^* \) can be called the star cardinality. Spaces with countable weak extent are called star-Lindelöf in [20] and strongly star-Lindelöf in [8]. Observe that \( e \leq de \) and \( e(X) = de(X) \) for any T\(_1\)-space \( X \).

The relations between the above cardinal invariants are described in the following version of Hodel’s diagram [13]. In this diagram an arrow \( f \to g \) (resp \( f \rightarrow g \)) indicates that \( f(X) \leq g(X) \) for any (T\(_1\)) space \( X \).

\[
\begin{array}{cccccccc}
l^* & \quad \Rightarrow & de & \quad \Rightarrow & l & \quad \Rightarrow & hd & \quad \Rightarrow \\
dc & \quad \Rightarrow & e & \quad \Rightarrow & s & \quad \Rightarrow & nw & \quad \Rightarrow \\
\bar{l} & \quad \Rightarrow & c & \quad \Rightarrow & d & \quad \Rightarrow & hd \\
\end{array}
\]

In fact, the cardinal characteristics \( d, l, \bar{l}, l^* \) are initial representatives of the hierarchy of cardinal characteristics \( l^{n} \) and \( \bar{l}^{n} \), \( n \in \mathbb{N} \), describing star-covering properties of topological spaces (see the survey paper [20] of Matveev for more information on this subject).

For a topological space \( X \) and an integer number \( n \geq 0 \) let

- \( l^{n}(X) \) be the smallest cardinal \( \kappa \) such that for every open cover \( \mathcal{U} \) of \( X \) there is a subset \( A \subset X \) of cardinality \( |A| \leq \kappa \) such that \( \mathcal{S}^{n}(A; \mathcal{U}) = X \);
- \( \bar{l}^{n}(X) \) be the smallest cardinal \( \kappa \) such that for every open cover \( \mathcal{U} \) of \( X \) there is a subset \( A \subset X \) of cardinality \( |A| \leq \kappa \) such that \( \mathcal{S}^{n}(A; \mathcal{U}) \) is dense in \( X \);
- \( l^{n+\frac{1}{2}}(X) \) be the smallest cardinal \( \kappa \) such that every open cover \( \mathcal{U} \) of \( X \) contains a subfamily \( \mathcal{V} \subset \mathcal{U} \) of cardinality \( |\mathcal{V}| \leq \kappa \) such that \( \mathcal{S}^{n}(\bigcup \mathcal{V}; \mathcal{U}) = X \);
- \( \bar{l}^{n+\frac{1}{2}}(X) \) be the smallest cardinal \( \kappa \) such that every open cover \( \mathcal{U} \) of \( X \) contains a subfamily \( \mathcal{V} \subset \mathcal{U} \) of cardinality \( |\mathcal{V}| \leq \kappa \) such that \( \mathcal{S}^{n}(\bigcup \mathcal{V}; \mathcal{U}) \) is dense in \( X \);
- \( l^{\omega}(X) = \min_{n \in \mathbb{N}} l^{n}(X) \) and \( \bar{l}^{\omega} = \min_{n \in \mathbb{N}} \bar{l}^{n}(X) \).

Observe that \( l^{0} = |.|, \bar{l}^{0} = d, l^{\frac{1}{2}} = l, \bar{l}^{\frac{1}{2}} = \bar{l}, \) and \( l^{1} = l^* \).

In [7] the cardinal characteristics \( l^{n} \) and \( \bar{l}^{n} \) are denoted by \( st_{n}l \) and \( st_{n}\bar{l} \), respectively. In [8] spaces \( X \) with countable \( l^{n+\frac{1}{2}}(X) \) and \( \bar{l}^{n}(X) \) are called \( n\)-star-Lindelöf and strongly \( n\)-star Lindelöf, respectively.

The following diagram describes provable inequalities between cardinal characteristics \( l^{n}, \bar{l}^{n}, l^{n+\frac{1}{2}}, \) and \( \bar{l}^{n+\frac{1}{2}} \) for \( n \in \mathbb{N} \). For two cardinal characteristics \( f,g \) an arrow \( f \rightarrow g \) indicates that \( f(X) \leq g(X) \) for any topological space \( X \).

\[
\begin{array}{cccccccc}
l^{\omega} & \quad \Rightarrow & l^{(n+1)} & \quad \Rightarrow & l^{(n+\frac{1}{2})} & \quad \Rightarrow & l^{n} & \quad \Rightarrow & l^{1} & \quad \Rightarrow & de & \quad \Rightarrow & l^{\frac{1}{2}} & \quad \Rightarrow & hd \\
\bar{l}^{\omega} & \quad \Rightarrow & \bar{l}^{(n+\frac{1}{2})} & \quad \Rightarrow & \bar{l}^{n} & \quad \Rightarrow & \bar{l}^{(n-\frac{1}{2})} & \quad \Rightarrow & \bar{l}^{\frac{1}{2}} & \quad \Rightarrow & c & \quad \Rightarrow & \bar{l}^{0} & \quad \Rightarrow & d & \quad \Rightarrow \\
\end{array}
\]
The unique non-trivial inequalities $l^{*1} \leq dc$ and $\bar{l}^{*1} \leq dc$ in this diagram follow from the next proposition whose proof can be found in [5].

**Proposition 1.2.** Any topological space $X$ has $l^{*1}(X) \leq dc(X)$ and $\bar{l}^{*1}(X) \leq dc(X)$.

For quasi-regular spaces many star-covering properties are equivalent. Let us recall that a topological space $X$ is called **quasi-regular** if each non-empty open set $U \subset X$ contains the closure $\overline{\overline{U}}$ of another non-empty open set $V$ in $X$. The following proposition was proved in [9] (and for regular spaces in [8]).

**Proposition 1.3.** Let $X$ be a quasi-regular space. Then

1. $dc(X) = \bar{l}^{*1}(X) = l^{*\omega}(X)$.
2. If $X$ is normal, then $dc(X) = \bar{l}^{*1}(X)$.
3. If $X$ is perfectly normal, then $dc(X) = c(X) = \bar{l}^{*\frac{1}{2}}(X)$.
4. If $X$ is collectively Hausdorff, then $dc(X) = de(X) = l^{*1}(X)$.
5. If $X$ is paracompact, then $dc(X) = l(X)$.
6. If $X$ is perfectly paracompact, then $dc(X) = hd(X)$.

Proposition 1.3 implies that for quasi-regular spaces the diagram describing the relations between the cardinal characteristics simplifies to the following form.

![Diagram](image)

Next, we consider some local cardinal characteristics of topological spaces. Let $X$ be a topological space, $x$ be a point of $X$, and $N_x$ be the family of all open neighborhoods of $x$ in $X$.

- The **character** $\chi_x(X)$ of $X$ at $x$ is the smallest cardinality of a neighborhood base at $x$.
- The **pseudocharacter** $\psi_x(X)$ of $X$ at $x$ is the smallest cardinality of a subfamily $\mathcal{U} \subset N_x$ such that $\bigcap \mathcal{U} = N_x$.
- The **closed pseudocharacter** $\overline{\psi}_x(X)$ of $X$ at $x$ is the smallest cardinality of a subfamily $\mathcal{U} \subset N_x$ such that $\bigcap_{U \in \mathcal{U}} \overline{U} = \bigcap_{V \in N_x} \overline{V}$.

It is easy to see that for any point $x$ of a Hausdorff topological space $X$ we get

$$\psi_x(X) \leq \overline{\psi}_x(X) \leq \chi_x(X).$$

The cardinals

$$\chi(X) = \sup_{x \in X} \chi_x(X), \quad \psi(X) = \sup_{x \in X} \psi_x(X), \quad \text{and} \quad \overline{\psi}(X) = \sup_{x \in X} \overline{\psi}_x(X)$$

are called the **character**, the **pseudocharacter**, and the **closed pseudocharacter** of $X$, respectively. It follows that

$$\psi(X) \leq \overline{\psi}(X) \leq \chi(X)$$

for any Hausdorff topological space $X$.

The (closed) pseudocharacter is upper bounded by the (closed) diagonal number defined as follows. Let $X$ be a Hausdorff topological space. By $\Delta_x = \{(x,y) \in X \times X : x = y\}$ we denote the **diagonal** of the square $X \times X$.

- The **diagonal number** $\Delta(X)$ of $X$ is the smallest cardinality of a family $\mathcal{U}$ of open subsets of $X \times X$ such that $\bigcap \mathcal{U} = \Delta_x$.
- The **closed diagonal number** $\overline{\Delta}(X)$ of $X$ is the smallest cardinality of a family $\mathcal{U}$ of open subsets of $X \times X$ such that $\bigcap_{U \in \mathcal{U}} \overline{U} = \Delta_x$.

It is easy to see that $\psi(X) \leq \Delta(X) \leq \overline{\Delta}(X)$ and $\overline{\psi}(X) \leq \overline{\Delta}(X)$ for any Hausdorff space $X$.

Following [12] [2.1] we say that a space $X$ has **regular $G_\delta$-diagonal** if $\Delta(X) \leq \omega$ (resp. $\overline{\Delta}(X) \leq \omega$).

The (closed) diagonal number of a functionally Hausdorff space $X$ is upper bounded by

- the **submetrizability number** $sm(X)$ of $X$, defined as the smallest number of continuous pseudometrics which separate points of $X$, and

$$sm(X) = \sup \{ |\mathcal{F}| : \mathcal{F} \text{ is a family of continuous pseudometrics on } X \}.$$
the \textit{i-weight} $iw(X)$ of $X$, defined as the smallest number of continuous real-valued functions that separate points of $X$.

The following diagram describes relations between these cardinal characteristics. In this diagram for two cardinal characteristics $f,g$ an arrow $f \to g$ indicates that $f(X) \leq g(X)$ for any functionally Hausdorff topological space $X$. 

![Diagram](image)

The unique non-trivial inequality $iw \leq sm \cdot \log dc$ in this diagram is proved in the following proposition.

**Proposition 1.4.** Each infinite functionally Hausdorff space $X$ has 

$$iw(X) \cdot \omega = sm(X) \cdot \log(dc(X)) \quad \text{and} \quad |X| \leq dc(X)^{\omega \cdot sm(X)} \leq 2^{\omega \cdot iu(X)}.$$ 

Proof. The inequality $sm(X) \cdot \log(dc(X)) \leq iw(X) \cdot \omega$ follows from the inequalities $sm(X) \leq iw(X)$ and $dc(X) \leq |X| \leq [0, 1]^{iu(X)} = 2^{iu(X)}$, the latter of which implies $\log(dc(X)) \leq \log(2^{iu(X)}) = iw(X) \cdot \omega$.

Now we prove the inequalities $iw(X) \cdot \omega \leq sm(X) \cdot \log(dc(X))$ and $|X| \leq dc(X)^{\omega \cdot sm(X)}$. The definition of the submetrizability number implies that $X$ admits a continuous injective map $f : X \to \prod_{\alpha \in sm(X)} M_\alpha$ into the Tychonoff product of $sm(X)$ many metric spaces $M_\alpha$. We lose no generality assuming that each metric space $M_\alpha$ is a continuous image of $X$ and hence $d(M_\alpha) = dc(M_\alpha) \leq dc(X)$ and $|M_\alpha| \leq d(M_\alpha)^\omega$. Then

$$|X| \leq \prod_{\alpha \in sm(X)} |M_\alpha| \leq \prod_{\alpha \in sm(X)} d(M_\alpha)^\omega \leq \prod_{\alpha \in sm(X)} dc(X)^\omega = dc(X)^{\omega \cdot sm(X)}.$$ 

By [H 4.4.9], for every $\alpha \in sm(X)$ the metric space $M_\alpha$ admits a topological embedding into the countable power $H^\kappa_\omega$ of the hedgehog $H_\kappa = \{ (x_i)_{i \in \kappa} : \{ i \in \kappa : x_i \neq 0 \} \leq 1 \}$ with $\kappa = dc(X) \geq d(M_\alpha)$ many spines. The hedgehog $H_\kappa$ can be thought as a cone over a discrete space $D$ of cardinality $\kappa$. The discrete space $D$ admits an injective continuous map into the Tychonoff cube $[0, 1]^{\log(\kappa)}$. Consequently, $H_\kappa$ admits an injective continuous map into the cone over the Tychonoff cube $[0, 1]^{\log(\kappa)}$, which implies that $iw(H_\kappa) \leq \log(\kappa) = \log(dc(X))$ and $iw(H^\kappa_\omega) \leq \log(dc(X)) \cdot \omega = \log(dc(X))$. Then $iw(X) \leq sm(X) \cdot iw(H^\kappa_\omega) \leq sm(X) \cdot \log(dc(X))$. This completes the proof of the equality $iw(X) \cdot \omega = sm(X) \cdot \log(dc(X))$.

To complete the proof of the proposition, observe that 

$$|X| \leq dc(X)^{\omega \cdot sm(X)} \leq (2^{\log(dc(X))})^{\omega \cdot sm(X)} = 2^{\omega \cdot \log(dc(X)) \cdot \omega \cdot sm(X)} = 2^{\omega \cdot iu(X)}.$$ 

\[\blacksquare\]

1.3. **Pre-uniform spaces and their cardinal characteristics.** By an \textit{entourage} on a set $X$ we understand any subset $U \subset X \times X$ containing the diagonal $\Delta_X = \{(x, y) \in X \times X : x = y\}$ of $X \times X$. For an entourage $U$ on $X$, point $x \in X$ and subset $A \subset X$ let $B(x; U) = \{ y \in X : (x, y) \in U \}$ be the \textit{U-ball} centered at $x$, and $B(A; U) = \bigcup_{a \in A} B(a; U)$ be the \textit{U-neighborhood} of $A$ in $X$.

Now we define some operations on entourages. For two entourages $U, V$ on $X$ let 

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$$

be the \textit{inverse} entourage and 

$$UV = \{(x, z) \in X \times X : \exists y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}$$

be the \textit{composition} of $U$ and $V$. It is easy to see that $(UV)^{-1} = V^{-1}U^{-1}$. For every entourage $U$ on $X$ define its powers $U^n$, $n \in \mathbb{Z}$, by the formula: $U^0 = \Delta_X$ and $U^{n+1} = U^n U$, $U^{-n} = U^{-n} U^{-1}$ for $n \in \omega$. Define also the \textit{alternating powers} $U^{+n}$ and $U^{-n}$ of $U$ by the recursive formulas: $U^{\pm 0} = U^{\mp 0} = \Delta_X$, and $U^{\pm (n+1)} = U U^{\mp n}$, $U^{\mp (n+1)} = U^{-1} U^{\pm n}$ for $n \geq 0$. If $U$ is an entourage on a topological space $X$, then put $U = \bigcup_{x \in X} B(x; U)$ be the closure of $U$ in the product $X_d \times X$ where $X_d$ is the set $X$ endowed with the discrete topology.

The following lemma proved in [L] shows that the alternating power $U^{\pm n}$ on an entourage $U$ is equivalent to taking the star with respect to the cover $U = \{ B(x; U) : x \in X \}$.

**Lemma 1.5.** For any entourage $U$ on a set $X$ and a point $x \in X$ we get $B(x; U^{-1} U) = St(x; U)$ where $U = \{ B(x; U) : x \in X \}$. Consequently, $B(x; U^{\pm n}) = B(x; (U^{-1} U)^n) = St^n(x; U)$ for every $n \in \mathbb{N}$.

A family $\mathcal{U}$ of entourages on a set $X$ is called a \textit{uniformity} on $X$ if it satisfies the following four axioms: 

(1) for any $U \in \mathcal{U}$, every entourage $V \subset X \times X$ containing $U$ belongs to $\mathcal{U}$;
(U2) for any entourages \( U, V \in \mathcal{U} \) there is an entourage \( W \in \mathcal{U} \) such that \( W \subset U \cap V \);

(U3) for any entourage \( U \in \mathcal{U} \) there is an entourage \( V \in \mathcal{U} \) such that \( VV \subset U \);

(U4) for any entourage \( U \in \mathcal{U} \) there is an entourage \( V \in \mathcal{U} \) such that \( V \subset U^{-1} \).

A family \( \mathcal{U} \) of entourages on \( X \) is called a quasi-uniformity (resp. pre-uniformity) on \( X \) if it satisfies the axioms (U1)–(U3) (resp. (U1)–(U2)). So, each uniformity is a quasi-uniformity and each quasi-uniformity is a pre-uniformity. Observe that a pre-uniformity is just a filter of entourages on \( X \).

A subfamily \( \mathcal{B} \subset \mathcal{U} \) is called a base of a pre-uniformity \( \mathcal{U} \) on \( X \) if each entourage \( U \in \mathcal{U} \) contains some entourage \( B \in \mathcal{B} \). Each base of a preuniformity satisfies the axiom (U2). Conversely, each family \( \mathcal{B} \) of entourages on \( X \) satisfying the axiom (U2) is a base of a unique pre-uniformity \( \langle \mathcal{B} \rangle \) consisting of entourages \( U \subset X \times X \) containing some entourage \( B \in \mathcal{B} \). If the base \( \mathcal{B} \) satisfies the axiom (U3) (and (U4)), then the pre-uniformity \( \langle \mathcal{B} \rangle \) is a quasi-uniformity (and a uniformity).

Next we define some operations over preuniformities. Given two preuniformities \( \mathcal{U}, \mathcal{V} \) on a set \( X \) put
\[
\mathcal{U}^{-1} = \{ U^{-1} : U \in \mathcal{U} \}, \quad \mathcal{U} \cap \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \}, \quad \mathcal{U} \cup \mathcal{V} = \{ U \cup V : U \in \mathcal{U}, V \in \mathcal{V} \}
\]
and let \( \mathcal{U} \mathcal{V} \) be the pre-uniformity generated by the base \( \{ UV : U \in \mathcal{U}, V \in \mathcal{V} \} \). For every \( n \in \omega \) let \( \mathcal{U}^{\pm n}, \mathcal{U}^{\mp n}, \mathcal{U}^{\wedge n}, \mathcal{U}^{\vee n} \) be the pre-uniformities generated by the bases \( \{ U^{\pm n} : U \in \mathcal{U} \}, \{ U^{\mp n} : U \in \mathcal{U} \}, \{ U^{\wedge n} \cap U^{\mp n} : U \in \mathcal{U} \}, \{ U^{\wedge n} \setminus U^{\mp n} : U \in \mathcal{U} \} \), respectively. Observe that \( \mathcal{U}^{\wedge n} = \mathcal{U}^{\pm n} \wedge \mathcal{U}^{\mp n} \) and \( \mathcal{U}^{\vee n} = \mathcal{U}^{\pm n} \vee \mathcal{U}^{\mp n} \). For a pre-uniformity \( \mathcal{U} \) on a topological space \( X \) let \( \mathcal{U} \) be the pre-uniformity generated by the base \( \{ U : U \in \mathcal{U} \} \).

The pre-uniformities \( \mathcal{U}^{\pm n}, \mathcal{U}^{\mp n}, \mathcal{U}^{\wedge n}, \mathcal{U}^{\vee n} \) feet into the following diagram (in which an arrow \( \mathcal{V} \to \mathcal{W} \) indicates that \( \mathcal{V} \subset \mathcal{W} \)):

\[
\begin{array}{ccc}
\mathcal{U}^{\pm n} & \mathcal{U}^{\mp n} & \mathcal{U}^{\wedge n} \\
\mathcal{U}^{\vee(n+1)} \downarrow & & \mathcal{U}^{\vee n} \downarrow \mathcal{U}^{\vee(n-1)} \\
\mathcal{U}^{\wedge n} \uparrow & & \mathcal{U}^{\mp n}
\end{array}
\]

We shall say that a preuniformity \( \mathcal{U} \) on \( X \) is
- \( \pm n \)-separated if \( \bigcap \mathcal{U}^{\pm n} = \Delta_X \);
- \( \mp n \)-separated if \( \bigcap \mathcal{U}^{\mp n} = \Delta_X \);
- \( n \)-separated if \( \mathcal{U} \) is both \( \pm n \)-separated and \( \mp n \)-separated.

Observe that for an odd number \( n \) a pre-uniformity \( \mathcal{U} \) is \( n \)-separated if and only if it is \( \pm n \)-separated if and only if it is \( \mp n \)-separated (this follows from the equality \( (U^{\pm n})^{-1} = U^{\mp n} \) holding for every entourage \( U \)).

This equivalence does not hold for even \( n \).

**Example 1.6.** For every \( m \in \mathbb{N} \) consider the entourage \( U_m = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : y \in \{ x \} \cup \{ x + m, \infty \} \} \) on the half-line \( \mathbb{R}_+ = [0, \infty) \). The family \( \{ U_m \}_{m \in \mathbb{N}} \) is a base of a quasi-uniformity \( \mathcal{U} \) on \( \mathbb{R}_+ \) which is \( \mp 2 \)-separated but not \( \pm 2 \)-separated.

Each preuniformity \( \mathcal{U} \) on a set \( X \) generates a topology \( \tau_\mathcal{U} \) consisting of all subsets \( W \subset X \) such that for each point \( x \in W \) there is an entourage \( U \in \mathcal{U} \) with \( B(x; U) \subset W \). This topology \( \tau_\mathcal{U} \) will be referred to as the topology generated by the pre-uniformity \( \mathcal{U} \). If \( \mathcal{U} \) is a quasi-uniformity, then for each point \( x \in X \) the family of balls \( \{ B(x; U) : U \in \mathcal{U} \} \) is a neighborhood base of the topology \( \tau_\mathcal{U} \) at \( x \). This implies that for a quasi-uniformity \( \mathcal{U} \) on a set \( X \) the topology \( \tau_\mathcal{U} \) is Hausdorff if and only if for any distinct points \( x, y \in X \) there is an entourage \( U \in \mathcal{U} \) such that \( B(x; U) \cap B(y; U) = \emptyset \) if and only if \( \bigcap U^{-1} = \Delta_X \) if and only if the quasi-uniformity \( \mathcal{U} \) is \( \pm 2 \)-separated. It is known (see [16] or [17]) that the topology of each topological space \( X \) is generated by a suitable quasi-uniformity (in particular, the Pervin quasi-uniformity, generated by the subbase consisting of the entourages \( (U \times U) \cup ((X \setminus U) \times X) \) where \( U \) runs over open sets in \( X \)).

Now we consider some cardinal characteristics of pre-uniformities. Let \( \mathcal{U} \) be a pre-uniformity on a topological space \( X \).

- The *boundedness number* \( \ell(\mathcal{U}) \) of \( \mathcal{U} \) is defined as the smallest cardinal \( \kappa \) such that for any entourage \( U \in \mathcal{U} \) there is a subset \( A \subset X \) of cardinality \( |A| \leq \kappa \) such that \( B(A; U) = X \);

- the *weak boundedness number* \( \ell(\mathcal{U}) \) of \( \mathcal{U} \) is defined as the smallest cardinal \( \kappa \) such that for any entourage \( U \in \mathcal{U} \) there is a subset \( A \subset X \) of cardinality \( |A| \leq \kappa \) such that \( B(A; U) \) is dense in \( X \);

- the *character* \( \chi(\mathcal{U}) \) of \( \mathcal{U} \) is the smallest cardinality of a subfamily \( \mathcal{V} \subset \mathcal{U} \) such that each entourage \( U \in \mathcal{U} \) contains some entourage \( V \in \mathcal{V} \);
the pseudocharacter $\psi(U)$ of $U$ is the smallest cardinality of a subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcap \mathcal{V} = \bigcap \mathcal{U}$;

- the closed pseudocharacter $\overline{\psi}(U)$ of $U$ is the smallest cardinality of a subfamily $\mathcal{V} \subset \mathcal{U}$ such that for every $x \in X$ we get $\bigcap_{V \in \mathcal{V}} B(x;V) = \bigcap_{U \in \mathcal{U}} B(x;U)$ (so, $\overline{\psi}(U) = \psi(U)$);

- the local pseudocharacter $\psi(U)$ of $U$ is the smallest cardinal $\kappa$ such that for every $x \in X$ there is a subfamily $\mathcal{V}_x \subset \mathcal{U}$ of cardinality $|\mathcal{V}_x| \leq \kappa$ such that $\bigcap_{V \in \mathcal{V}_x} B(x;V) = \bigcap_{U \in \mathcal{U}} B(x;U)$.

For any Hausdorff topological space $X$ and a quasi-uniformity $U$ generating the topology of $X$ we get the inequalities $\psi(X) = \psi(U) \leq \overline{\psi}(U) \leq \psi(U)$ and $\chi(X) \leq \chi(U)$, which fit into the following diagram (in which an arrow $a \rightarrow b$ indicates that $a \leq b$).

$$
\begin{align*}
\psi(X) & \rightarrow \overline{\psi}(X) \rightarrow \chi(X) \\
\psi(U) & \rightarrow \overline{\psi}(U) \rightarrow \chi(U)
\end{align*}
$$

The boundedness number $\ell(U)$ combined with the pseudocharacter $\psi^2(U)$ can be used to produce a simple upper bound on the cardinality of a $\exists 2$-separated pre-uniform space (cf. [4, 4.3]).

**Proposition 1.7.** Any set $X$ has cardinality $|X| \leq \ell(U)^{\psi^2(U)}$ for any $\exists 2$-separated pre-uniformity $U$ on a set $X$.

**Proof.** The pre-uniformity $U^{\psi^2}$, being separated, contains a subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| = \psi(U^{\psi^2})$ such that $\bigcap_{V \in \mathcal{V}} V^{-1}V = \Delta X$. By the definition of the boundedness number $\ell(U)$, for every entourage $V \in \mathcal{V}$ there is a subset $L_V \subset X$ of cardinality $|L_V| \leq \ell(U)$ such that $X = B(L_V;V)$. For every $x \in X$ choose a function $f_x \in \prod_{V \in \mathcal{V}} L_V$ assigning to every entourage $V \in \mathcal{V}$ a point $f_x(V) \in L_V$ such that $x \in B(f_x(V);V)$. We claim that for any distinct points $x, y \in X$ the functions $f_x, f_y$ are distinct. Indeed, the choice of the family $V$ yields an entourage $V \in \mathcal{V}$ such that $(x, y) \notin V^{-1}V$. Then $f_x(V) \neq f_y(V)$ and hence $f_x \neq f_y$. This implies that

$$|X| \leq \prod_{V \in \mathcal{V}} |L_V| \leq |\chi(X)|^{\mathcal{V}|} = \ell(U)^{\psi(U^{\psi^2})}. $$

\[ \square \]

Following [4] we define a quasi-uniformity $U$ on a topological space $X$ to be normal if for any subset $A \subset X$ and entourage $U \in \mathcal{U}$ we get $\overline{A} \subset \overline{B(A;U)}$. A topological space $X$ is called normally quasi-uniformizable if the topology of $X$ is generated by a normal quasi-uniformity. Normally quasi-uniformizable spaces possess the following important normality-type property proved in [4].

**Theorem 1.8.** Let $X$ be a topological space and $U$ be a normal quasi-uniformity generating the topology of $X$. Then for every subset $A \subset X$ and entourage $U \in \mathcal{U}$ there exists a continuous function $f : X \rightarrow [0, 1]$ such that $A \subset f^{-1}(0)$ and $f([0, 1)) \subset \overline{B(A;U)}$.

1.4. **Cardinal characteristics of topological spaces.** Let $X$ be a topological space. An entourage $U$ on $X$ is called a neighborhood assignment if for every $x \in X$ the $U$-ball $B(x;U)$ is a neighborhood of $x$. The family $\mathcal{U}_X$ of all neighborhood assignments on a topological space $X$ is a pre-uniformity called the universal pre-uniformity on $X$. It contains any pre-uniformity generating the topology of $X$ and is equal to the union of all pre-uniformities generating the topology of $X$.

The universal pre-uniformity $\mathcal{U}_X$ contains

- the universal quasi-uniformity $q\mathcal{U}_X = \bigcup \{ U \subset \mathcal{U}_X : U$ is a quasi-uniformity on $X \}$, and

- the universal uniformity $\mathcal{U}_X = \bigcup \{ U \subset \mathcal{U}_X : U$ is a uniformity on $X \}$

of $X$. It is clear that $\mathcal{U}_X \subset q\mathcal{U}_X \subset \mathcal{U}_X$. The interplay between the universal pre-uniformities $\mathcal{U}_X$, $q\mathcal{U}_X$ and $\mathcal{U}_X$ are studied in [5].

Since the topology of any topological space is generated by a quasi-uniformity, the universal quasi-uniformity $q\mathcal{U}_X$ generates the topology of $X$. In contrast, the universal uniformity $\mathcal{U}_X$ generates the topology of $X$ if and only if the space $X$ is completely regular.

Cardinal characteristics of the pre-uniformities $\mathcal{U}_X$, $q\mathcal{U}_X$ and $\mathcal{U}_X$ or their alternating powers can be considered as cardinal characteristics of the topological space $X$. In particular, for a Hausdorff space $X$ we have the equalities:

$$
\chi(X) = \chi(\mathcal{U}_X), \quad \psi(X) = \psi(\mathcal{U}_X), \quad \overline{\psi}(X) = \overline{\psi}(\mathcal{U}_X), \quad \Delta(X) = \psi(\mathcal{U}_X^{\psi^2}).
$$
The last equality follows from Lemma [1.3]. On the other hand, the boundedness number $\ell(p\mathcal{U}_X)$ of $p\mathcal{U}_X$ coincides with the Lindelöf number $l(X)$ of $X$.

Observe that for the universal pre-uniformity $p\mathcal{U}_X$ on a Hausdorff topological space $X$ the upper bound $|X| \leq \ell(p\mathcal{U}_X)^{\psi(p\mathcal{U}_X^{\infty})}$ proved in Proposition [1.7] turns into the known upper bound $|X| \leq l(X)^{\Delta(X)}$.

Having in mind the equality $l(X) = \ell(p\mathcal{U}_X)$, for every $n \in \mathbb{N}$ let us define the following cardinal characteristics:

\[
\ell^\pm(X) := \ell(p\mathcal{U}_X\pm n), \quad \ell^\mp(X) := \ell(p\mathcal{U}_X^\mp n), \quad \ell^n(X) := \ell(p\mathcal{U}_X^n), \quad \ell^{\infty}(X) := \ell(p\mathcal{U}_X^{\infty}),
\]

\[
q\ell^\pm(X) := \ell(qp\mathcal{U}_X\pm n), \quad q\ell^n(X) := \ell(qp\mathcal{U}_X^n), \quad q\ell^{\infty}(X) := \ell(qp\mathcal{U}_X^{\infty}).
\]

Let also

\[
\ell^\omega(X) = \min_{n \in \mathbb{N}} q\ell^n(X), \quad q\ell^\omega(X) = \min_{n \in \mathbb{N}} q\ell^n(X), \quad \text{and} \quad u\ell(X) = \ell(\mathcal{U}_X).
\]

Observe that $u\ell(X) = \ell(\mathcal{U}_X^\pm) = \ell(\mathcal{U}_X^n) = \ell(\mathcal{U}_X^{\infty})$ for every $n \in \mathbb{N}$ (this follows from the equality $\mathcal{U}_X = \mathcal{U}_X^\pm = \mathcal{U}_X^n$ holding for every $n \in \mathbb{N}$).

The above cardinal characteristics were introduced and studied in [5].

Some inequalities between the cardinal characteristics $\ell^\pm, \ell^\mp, \ell^n, q\ell^n, q\ell^{\infty}, \ell^{\infty}$, and $u\ell$ are described in the following diagram in which an arrow $a \to b$ indicates that $a(X) \leq b(X)$ for any topological space $X$.

It turns out that the cardinal invariants $l^\pm, l^\mp, \bar{l}^n, \tilde{l}^n$, and $\tilde{l}^n\frac{1}{2}$ can be expressed via the cardinal invariants $\ell^\pm, \ell^\mp, \ell^n, \ell^\infty$ for a suitable number $m$. The following proposition is proved in [5] (or can be easily derived from the definitions).

**Proposition 1.9.** For every $n \in \omega$ we have the equalities:

\[
l^n = \ell^{\pm 2n}, \quad \bar{l}^n = \ell^{\mp 2n}, \quad l^n\frac{1}{2} = \ell^{\pm (2n+1)}, \quad \tilde{l}^n\frac{1}{2} = \ell^{\pm (2n+1)}.
\]

The following proposition (proved in [5]) describes the relation of the cardinal invariants $\ell^\pm, \ell^\mp$ to classical cardinal invariants.

**Proposition 1.10.** Let $X$ be a topological space. Then

1. $\ell^{\infty}(X) \leq s(X) \leq q\ell^{\infty}(X) \leq \ell^{\infty}(X) \leq nw(X)$;
2. $e(X) \leq de(X) \leq q\ell^{\pm 1}(X) \leq \ell^{\pm 1}(X) = l(X)$;
3. $c(X) \leq q\ell^{\mp 1}(X) \leq \ell^{\mp 1}(X) \leq d(X)$;
4. If $X$ is quasi-regular, then $\ell^{\pm 3}(X) = \ell^{\pm 3}(X) = \ell^{\infty}(X) = d(X)$;
5. If $X$ is completely regular, then $q\ell^{\pm 3}(X) = q\ell^{\infty}(X) = u\ell(X) = d(X)$.

Taking into account Propositions [1.3, 1.9, and 1.10] we see that for quasi-regular spaces the cardinal characteristics $\ell^\pm, \ell^\mp, \ell^n, \ell^\infty$ relate to other cardinal characteristics of topological spaces as follows.
The space $X$ is uniformizable if and only if $\alpha \in X$ for some generating the topology of $X$. In this section we apply Theorem 1.8 to derive some upper bounds on the $i$-weight of a normally quasi-uniformizable space.

Proposition 2.1. Let $X$ be a topological space whose topology is generated by a normal quasi-uniformity $\mathcal{U}$. The space $X$ has $i$-weight $iw(X) \leq \kappa$ for some cardinal $\kappa$ if there exists a family of subsets $\{A_\alpha\}_{\alpha \in \kappa}$ of $X$ and a family of entourages $\{U_\alpha\}_{\alpha \in \kappa} \subset \mathcal{U}$ such that for any distinct points $x, y \in X$ there is $\alpha \in \kappa$ such that $x \in A_\alpha$ and $y \notin B(A_\alpha; U_\alpha)$.

Proof. For every $\alpha \in \kappa$ apply Theorem 1.8 to construct a continuous map $f_\alpha : X \to [0,1]$ such that $f_\alpha(A_\alpha) \subset \{0\}$ and $f_\alpha^{-1}([0,1)) \subset B(A_\alpha; U_\alpha)$. It follows that the family of continuous maps $\{f_\alpha\}_{\alpha \in \kappa}$ separates points of $X$. So, $iw(X) \leq \kappa$. 

This proposition will be used to prove:

Theorem 2.2. A Hausdorff space $X$ has $i$-weight $iw(X) \leq \psi(A^{-1}\mathcal{U}) \cdot \ell(A)$ for any normal quasi-uniformity $\mathcal{U}$ generating the topology of $X$ and any pre-uniformity $\mathcal{A}$ on $X$ such that $\bigcap A^{-1}\mathcal{U} \neq \emptyset$.

Proof. If the cardinal $\psi(A^{-1}\mathcal{U})$ is finite, then $\psi(A^{-1}\mathcal{U}) = 1$, which implies that $A^{-1}A = A = U$ for some $A \in \mathcal{A}$ and $U \in \mathcal{U}$. In this case $\ell(A) = \ell(A)$ and hence $iw(X) \leq \ell(A)$.

So, we assume that the cardinal $\kappa = \psi(A^{-1}\mathcal{U})$ is infinite. Since $\bigcap A^{-1}\mathcal{U} = \Delta_X$, we can choose subfamilies $(A_\alpha)_{\alpha \in \kappa} \subset \mathcal{A}$ and $(U_\alpha)_{\alpha \in \kappa} \subset \mathcal{U}$ such that $\bigcap_{\alpha \in \kappa} B(x, A_\alpha^{-1}A_\alpha U_\alpha) = \{x\}$ for every $x \in X$. For every $\alpha \leq \kappa$ choose a subset $Z_\alpha \subset X$ of cardinality $\ell(A)$ such that $X = B(Z_\alpha; A_\alpha)$. Consider the family of sets $Z = \bigcup_{\alpha \in \kappa} \{B(z; A_\alpha) : z \in Z_\alpha\}$. We claim that for any distinct points $x, y \in X$ there is a set $Z \in Z$ and ordinal $\alpha \in \kappa$ such that $x \in Z$ and $y \notin B(Z; A_\alpha)$.

By the choice of the families $(A_\alpha)$, $(U_\alpha)$, for the points $x, y$ there is an index $\alpha \in \kappa$ such that $y \notin B(x; A_\alpha^{-1}A_\alpha U_\alpha)$. Since $X = B(Z_\alpha; A_\alpha)$, we can find a point $z \in Z_\alpha$ such that $x \in B(z; A_\alpha)$ and hence...
$z \in B(x; A^{-1}_\alpha)$. We claim that the set $Z = B(z; A_\alpha) \in \mathcal{Z}$ has the required properties: $x \in Z$ and $y \notin \overline{B(Z; U_{\alpha})}$. To derive a contradiction, assume that $y \in B(Z; U_{\alpha})$, which implies
\[ y \in B(Z; U_{\alpha}) = B(B(z; A_\alpha); U_{\alpha}) = B(z; A_\alpha U_{\alpha}) \subset B(B(x; A^{-1}_\alpha); A_\alpha U_{\alpha}) = B(x; A^{-1}_\alpha A_\alpha U_{\alpha}). \]
But this contradicts the choice of the index $\alpha$.

This contradiction allows us to apply Proposition 2.1 and conclude that
\[ iw(X) \leq |Z| \cdot \kappa \leq \sum_{\alpha \in \kappa} |Z_\alpha| \cdot \kappa \leq \kappa^2 \cdot \ell(A) = \overline{\psi(A^{-1} A U)} \cdot \ell(A). \]

\[ \square \]

Applying Theorem 2.2 to some concrete pre-uniformities $A$, we get the following corollary.

**Corollary 2.3.** Let $X$ be a functionally Hausdorff space and $U$ be a normal quasi-uniformity generating the topology of $X$. If for some $n \in \mathbb{N}$ the quasi-uniformity $U$ is
1. $\pm(4n - 2)$-separated, then $iw(X) \leq \overline{\psi(U^{\pm(4n - 3)})} \cdot \ell(U^{\ell(2n - 1)}) \leq \chi(U) \cdot q \ell^{\ell(2n - 1)}(X)$;
2. $\mp(4n - 1)$-separated, then $iw(X) \leq \overline{\psi(U^\mp(4n - 2)}) \cdot \ell(U^{\ell(2n - 1)}) \leq \chi(U) \cdot q \ell^{\ell(2n - 1)}(X)$;
3. $\pm(4n)$-separated, then $iw(X) \leq \overline{\psi(U^{\pm(4n - 1)})} \cdot \ell(U^{\ell(2n)}) \leq \chi(U) \cdot q \ell^{\ell(2n)}(X)$;
4. $\mp(4n + 1)$-separated, then $iw(X) \leq \overline{\psi(U^\mp(4n - 2)}) \cdot \ell(U^\mp(2n)) \leq \chi(U) \cdot q \ell^{\ell(2n)}(X)$.

**Proof.** 1. If $U$ is $\pm(4n - 2)$-separated, then for the pre-uniformity $A = U^{\pm(2n - 1)} \cup U^{\ell(2n - 1)}$ we get
\[ A^{-1} A U \subset U^{\pm(2n - 1)} U^{\pm(2n - 1)} U = U^{\pm(4n - 3)} U = U^{\pm(4n - 3)} \]
and hence $\bigcap A^{-1} A U = \bigcap A^{-1} A U^{\ell(2n - 1)} = \bigcap U^{\pm(4n - 2)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $A = U^{\ell(2n - 1)}$, we get
\[ iw(X) \leq \overline{\psi(U^{\pm(4n - 3)})} \cdot \ell(U^{\ell(2n - 1)}) \leq \chi(U) \cdot q \ell^{\ell(2n - 1)}(X). \]

2. If $U$ is $\mp(4n - 1)$-separated, then for the pre-uniformity $A = U^{\pm(2n - 1)}$ we get
\[ A^{-1} A U = U^{\mp(2n - 1)} U^{\pm(2n - 1)} U = U^{\mp(4n - 2)} U = U^{\mp(4n - 2)} \]
and hence $\bigcap A^{-1} A U = \bigcap A^{-1} A U^{\pm(2n - 1)} = \bigcap U^{\mp(4n - 1)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $A = U^{\mp(2n - 1)}$, we get
\[ iw(X) \leq \overline{\psi(U^{\mp(4n - 2)})} \cdot \ell(U^{\pm(2n - 1)}) \leq \chi(U) \cdot q \ell^{\ell(2n - 1)}(X). \]

3. If $U$ is $\pm(4n)$-separated, then for the pre-uniformity $A = U^{\ell(2n)}$ we get
\[ A^{-1} A U \subset U^{\pm(2n)} U^{\ell(2n)} U = U^{\pm(4n - 1)} U = U^{\pm(4n - 1)} \]
and hence $\bigcap A^{-1} A U = \bigcap A^{-1} A U^{\pm(2n)} = \bigcap U^{\pm(4n)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $A = U^{\ell(2n)}$, we get
\[ iw(X) \leq \overline{\psi(U^{\pm(4n - 1)})} \cdot \ell(U^{\pm(2n)} \cup U^{\ell(2n)}) \leq \chi(U) \cdot q \ell^{\ell(2n)}(X). \]

4. If $U$ is $\mp(4n + 1)$-separated, then for the pre-uniformity $A = U^\mp(2n)$ we get
\[ A^{-1} A U = U^{\mp(2n)} U^{\mp(2n)} U = U^{\mp(4n)} \]
and hence $\bigcap A^{-1} A U = \bigcap A^{-1} A U^{\mp(2n)} = \bigcap U^{\mp(4n + 1)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $A = U^{\mp(2n)}$, we get
\[ iw(X) \leq \overline{\psi(U^{\mp(4n - 2)})} \cdot \ell(U^{\mp(2n)}) \leq \chi(U) \cdot q \ell^{\ell(2n)}(X). \]

\[ \square \]

**Corollary 2.4.** If $X$ is a Hausdorff space and $U$ is a normal quasi-uniformity generating the topology of $X$, then the space $X$ has $i$-weight $iw(X) \leq \overline{\psi(U)} \cdot \ell(U \cup U^{-1}) \leq \chi(U) \cdot \ell(U^{\ell(1)})$. Moreover, if the quasi-uniformity $U$ is
1. $\mp 3$-separated, then $iw(X) \leq \overline{\psi(U^3)} \cdot \ell(U) \leq \chi(U) \cdot q \ell^{\ell(2)}(X)$;
2. $\pm 4$-separated, then $iw(X) \leq \overline{\psi(U^{\pm 3})} \cdot \ell(U^{\ell(2)}) \leq \chi(U) \cdot q \ell^{\ell(2)}(X)$.
Consequently, the space $X$ is functionally Hausdorff.

Motivated by Proposition 3.1, let us introduce the following

**Definition 3.2.** Two quasi-uniformities $L$ and $R$ on a set $X$ are called

- **commuting** if $LR = RL$;
- **$\pm$-subcommuting** if $LR^{-1} \subset R^{-1}L$ and $RL^{-1} \subset L^{-1}R$;

A topological space $X$ is defined to be **bi- quasi-uniformizable** if the topology of $X$ is generated by two $\pm$-subcommuting quasi-uniformities.

**Theorem 3.3.** For any $\pm$-subcommuting quasi-uniformities $L, R$ generating the topology $\tau$ of a topological space $X$ the pre-uniformity $Q = LR^{-1} \vee RL^{-1}$ is a uniformity generating a completely regular topology $\tau_Q$, weaker than the topology $\tau$ of $X$. If the space $X$ is Hausdorff, then the topology $\tau_Q$ generated by the uniformity $Q$ is Tychonoff, the space $X$ is functionally Hausdorff and has submetrizability number

$$sm(X) \leq \psi(Q) \cdot \log(\ell(Q)) \leq \chi(L) \cdot \chi(R) \cdot \log(dc(X)).$$

and $i$-weight

$$iw(X) \leq \psi(Q) \cdot \log(\ell(X)) \leq \chi(L) \cdot \chi(R) \cdot \log(dc(X)).$$

**Proof.** By Proposition 3.1, the pre-uniformity $Q$ is a quasi-uniformity. Since $Q^{-1} = Q$, it is a uniformity. Then the topology $\tau_Q$ generated by the uniformity $Q$ is Tychonoff (see [9, 8.1.13]) Since $Q \subset L$, the topology $\tau_Q$ is weaker than the topology $\tau_L = \tau$ of the space $X$.

Now assume that the topology $\tau$ is Hausdorff. In this case for any distinct points $x, y \in X$ we can find entourages $L \in L$ and $R \in R$ such that $B(x; L) \cap B(y; R) = \emptyset$. Then $y \notin B(x; LR^{-1})$ and hence $(y, x) \notin \bigcap Q$, which means that the uniformity $Q$ is separated and the topology $\tau_Q$ generated by $Q$ is Tychonoff. Consequently, the space $X$ is functionally Hausdorff.

To show that $sm(X) \leq \psi(Q)$, fix a subfamily $V \subset Q$ of cardinality $|V| = \psi(Q)$ such that $\bigcap V = \Delta_X$. By [9, 8.1.11], for every entourage $V \in Q$ there exists a continuous pseudometric $d_V$ on $X$ such that the entourage $[d_V]_{<1} = \{(x, y) \in X \times X : d_V(x, y) < 1\}$ is contained in $V$. Then the family of pseudometrics $D = \{d_V\}_{V \in Q}$ separates points of $X$, which implies that $sm(X) \leq |D| \leq |V| = \psi(Q)$.

Taking into account that the topological weight of a metric space is equal to its boundedness number, which does not exceed the discrete cellularity, and applying Proposition 3.1, we conclude that

$$iw(X) \leq \psi(Q) \cdot \log(\ell(X)) \leq \chi(L) \cdot \chi(R) \cdot \log(dc(X)).$$

$\square$
Theorem 3.3 implies:

**Corollary 3.4.** Each Hausdorff bi-quasi-uniformizable topological space is functionally Hausdorff.

We do not know if this corollary can be reversed.

**Problem 3.5.** Is each functionally Hausdorff space bi-quasi-uniformizable?

**Proposition 3.6.** Let \( L, R \) be two \( \pm \)-subcommuting quasi-uniformities generating the same Hausdorff topology on \( X \). If the quasi-uniformities \( L^{-1}, R^{-1} \) generate the same topology on \( X \), then the quasi-uniformities \( L \) and \( R \) are 3-separated.

**Proof.** Given two distinct points \( x, y \in X \) we shall find an entourage \( R \in R \) such that \( (x, y) \notin R^{-1}RR^{-1} \). Since the topology generated by the quasi-uniformities \( L \) and \( R \) on \( X \) is Hausdorff, there are two entourages \( L \in L \) and \( R \in R \) such that \( B(x; R) \cap B(y; L) = \emptyset \) and hence \( (x, y) \notin RL^{-1}L^{-1} \). Replacing \( R \) by a smaller entourage, we can additionally assume that \( B(y; R) \subset B(y; L) \). Then \( B(x; R) \cap B(y; RL) = \emptyset \) and hence \( y \notin B(x; RL^{-1}R^{-1}) \).

Since the quasi-uniformities \( L \) and \( R \) are \( \pm \)-subcommuting, for the entourages \( L \) and \( R \) there are entourages \( \bar{L} \in L \) and \( \bar{R} \in R \) such that \( \bar{L}^{-1} \bar{R} \subset RL^{-1} \). Since quasi-uniformities \( L^{-1} \) and \( R^{-1} \) generate the same topology on \( X \), for the entourage \( \bar{L}^{-1} \) there is an entourage \( \bar{R} \in R \) such that \( B(x; \bar{R}^{-1}) \subset B(x; \bar{L}^{-1}) \). Then for the entourage \( R = \bar{R} \cap R \) we get \( B(x; R^{-1}RR^{-1}) \subset B(x; R^{-1}R^{-1}) \subset B(x; RL^{-1}R^{-1}) \subset B(x; RL^{-1}R^{-1}) \) and hence \( y \notin B(x; R^{-1}RR^{-1}) \). So, \( \bigcap \bar{R}^{-1}R^{-1} = \Delta_X \) and after inversion, \( \bigcap R^{-1}R = \Delta_X \), which means that the quasi-uniformity \( R \) is 3-separated. By analogy we can prove that the quasi-uniformity \( L \) is 3-separated. \( \square \)

### 4. Normally bi-quasi-uniformizable spaces

Observe that for two quasi-uniformities \( L, R \) on a set \( X \) the inclusion \( LR^{-1} \subset R^{-1}L \) is equivalent to the existence for every entourages \( L \in L \) and \( R \in R \) two entourages \( \bar{L} \in L \) and \( \bar{R} \in R \) such that \( \bar{R}^{-1} \bar{L} \subset LR^{-1} \). Changing the order of quantifiers in this property we obtain the following notion.

**Definition 4.1.** A topological space \( X \) is called normally bi-quasi-uniformizable if its topology is generated by quasi-uniformities \( L \) and \( R \) satisfying the following properties:

- \( \forall L \in L \exists \bar{L} \in L \forall R \in R \exists \bar{R} \in R \) such that \( \bar{R}^{-1} \bar{L} \subset LR^{-1} \) and \( \bar{L}^{-1} \bar{R} \subset RL^{-1} \);  
- \( \forall R \in R \exists \bar{R} \in R \forall L \in L \exists \bar{L} \in L \) such that \( \bar{L}^{-1} \bar{R} \subset RL^{-1} \) and \( \bar{R}^{-1} \bar{L} \subset RL^{-1} \).

In this case we shall say that the quasi-uniformities \( L \) and \( R \) are normally \( \pm \)-subcommuting.

By analogy we can introduce normally commuting quasi-uniformities.

**Definition 4.2.** Two quasi-uniformities \( L \) and \( R \) on a set \( X \) are defined to be normally commuting if it satisfy the following two conditions:

- \( \forall L \in L \exists \bar{L} \in L \forall R \in R \exists \bar{R} \in R \) such that \( \bar{R} \bar{L} \subset LR \) and \( \bar{L} \bar{R} \subset RL \);  
- \( \forall R \in R \exists \bar{R} \in R \forall L \in L \exists \bar{L} \in L \) such that \( \bar{L} \bar{R} \subset RL \) and \( \bar{R} \bar{L} \subset RL \).

**Proposition 4.3.** Any two normally \( \pm \)-subcommuting quasi-uniformities \( L, R \) generating the same topology on a set \( X \) are normal. Consequently, each normally bi-quasi-uniformizable topological space is normally quasi-uniformizable.

**Proof.** To show that \( L \) is normal, fix a subset \( A \subset X \) and entourage \( L \in L \). Since \( L \) and \( R \) are normally \( \pm \)-subcommuting, for the entourage \( L \) there exists an entourage \( \bar{L} \in L \) such that for every entourage \( R \in R \) there is an entourage \( \bar{R} \in R \) with \( \bar{L}^{-1} \bar{R} \subset RL^{-1} \). We claim that \( B(A; L) \subset B(A; \bar{L}) \). Given any point \( x \in B(A; \bar{L}) \), we need to show that \( x \in B(A; \bar{L}) \). Since \( O(x) \subset X \), find an entourage \( R \in R \) such that \( B(x; R) \subset O(x) \). By the choice of the entourage \( \bar{L} \), for the entourage \( R \) there is an entourage \( \bar{R} \in R \) such that \( \bar{L}^{-1} \bar{R} \subset RL^{-1} \). It follows from \( x \in B(A; \bar{L}) \) that \( B(x; \bar{L}^{-1}) \cap A \neq \emptyset \) and hence \( \emptyset \neq B(x; \bar{L}^{-1}R) \cap A \subset B(x; RL^{-1}) \cap A \). Then \( \emptyset \neq B(x; R) \cap B(A; L) \subset O(x) \cap B(A; L) \), which means \( x \in B(A; L) \). So, \( B(A; \bar{L}) \subset B(A; L) \) and hence \( A \subset B(A; \bar{L}) \circ B(A; L) \), which means that \( L \) is normal. By analogy we can prove the normality of the quasi-uniformity \( R \). \( \square \)

**Theorem 4.4.** If \( L \) and \( R \) are two normally \( \pm \)-subcommuting quasi-uniformities generating the topology of a Hausdorff topological space \( X \), then the quasi-uniformities \( LR^{-1} \) and \( R^{-1}L \) are 1-separated and have pseudocharacter

\[
(1) \quad \psi(LR^{-1}) = \psi(R^{-1}L) \leq \psi(LL^{-1}) \cdot \ell(L^{-1}) \leq \psi(LL^{-1}) \cdot q(\ell^{-1})(X);
\]
(2) \( \psi(LR^{-1}) = \psi(RL^{-1}) \leq \psi(L^{-1}L) \cdot \ell(L) \leq \psi(L^{-1}L) \cdot q \ell^{\pm 2}(X) \) if \( L^{-1}, R^{-1} \) are normally \( \pm \)-subcommuting and generate the same topology on \( X \).

(3) \( \psi(LR^{-1}) = \psi(RL^{-1}) \leq \psi(LL^{-1}L) \cdot \ell(LL^{-1}L) \leq \psi(L^{-1}L) \cdot q \ell^{\pm 2}(X) \) if the quasi-uniformities \( L \) and \( R \) are normally commuting and \( \bigcap \LL^{-1}L = \Delta_X \).

(4) \( \psi(LR^{-1}) = \psi(RL^{-1}) \leq \psi(A^{-1}A) \cdot \ell(A) \cdot \ell^{\pm 2}(X) \) for any pre-uniformity \( A \) on \( X \) such that \( \bigcap A^{-1}A = \Delta_X \).

**Proof.** First we show that the quasi-uniformities \( LR^{-1} \) and \( RL^{-1} \) are 1-separated. Since the topology of \( X \) is Hausdorff, for any distinct points \( x, y \in X \) we can find two disjoint open sets \( O_x \ni x \) and \( O_y \ni y \). Taking into account that the quasi-uniformities \( L \) and \( R \) generate the topology of \( X \), we can find two entourages \( L \in \mathcal{L} \) and \( R \in \mathcal{R} \) such that \( B(x; L) \subset O_x \) and \( B(y; R) \subset O_y \). Then \( B(x; L) \cap B(y; R) = \emptyset \) and hence \( y \notin B(x; LR^{-1}) \) and \( x \notin B(y; RL^{-1}) \), which implies that \( \bigcap LR^{-1} = \Delta_X = \bigcap RL^{-1} \). So, the quasi-uniformities \( LR^{-1} \) and \( RL^{-1} \) are 1-separated. Taking into account that \( LR^{-1} = RL^{-1} \) we conclude that \( \psi(LR^{-1}) = \psi(RL^{-1}) \).

1. Now we shall prove the inequality \( \psi(LR^{-1}) \leq \psi(LL^{-1}L) \cdot \ell(L^{-1}) \). Fix a family of entourages \( \Lambda \subset \mathcal{L} \) of cardinality \( |\Lambda| \leq \psi(LL^{-1}L) \) such that \( \bigcap_{L \in \Lambda} LL^{-1}L = \Delta_X \). Replacing every \( L \in \Lambda \) by a smaller entourage, we can assume that \( \bigcap_{L \in \Lambda} (LL)(LL)^{-1} = \Delta_X \).

Since the quasi-uniformities \( L \) and \( R \) are normally \( \pm \)-subcommuting, for the entourage \( L \in \mathcal{L} \) there exists an entourage \( \tilde{L} \in \mathcal{L} \) such that for any entourage \( R \in \mathcal{R} \) there exists an entourage \( \tilde{R} \in \mathcal{R} \) such that \( \tilde{L}^{-1}\tilde{R} \subset RL^{-1} \). Replacing \( L \) by \( L \cap \tilde{L} \), we can assume that \( L \subset \tilde{L} \). For the entourage \( L \) choose a subset \( Z_L \subset X \) of cardinality \( |Z_L| \leq \ell(L^{-1}) \) such that \( X = B(Z_L; \tilde{L}^{-1}) \). For every \( z \in Z_L \) choose an entourage \( R_z \in \mathcal{R} \) such that \( B(z; R_z) \subset B(z; L) \). By the choice of \( \tilde{L} \), for the entourage \( R_z \) there exists an entourage \( \tilde{R}_z \in \mathcal{R} \) such that \( \tilde{L}^{-1}\tilde{R}_z \subset R_z L^{-1} \). Consider the family

\[
\mathcal{P} = \bigcup_{L \in \Lambda} \{(L, \tilde{R}_z) : z \in Z_L \} \subset \mathcal{L} \times \mathcal{R}.
\]

We claim that for any distinct points \( x, y \in X \) there is a pair \( (L, \tilde{R}_z) \in \mathcal{P} \) such that \( B(x; L) \cap B(y; \tilde{R}_z) = \emptyset \).

By the choice of the family \( \Lambda \), there is an entourage \( L \in \Lambda \) such that \( x \notin B(y; LL^{-1}L^{-1}) \). Since \( y \in X = B(Z_L; \tilde{L}^{-1}) \), there exists a point \( z \in Z_L \) such that \( y \in B(z; \tilde{L}^{-1}) \) and hence \( z \in B(y; \tilde{L}) \). We claim that the pair \( (L, \tilde{R}_z) \in \mathcal{P} \) has the desired property: \( B(x; L) \cap B(y; \tilde{R}_z) = \emptyset \). Assuming that \( B(x; L) \cap B(y; \tilde{R}_z) \neq \emptyset \), we would conclude that

\[
x \in B(y; \tilde{R}_z L^{-1}) \subset B(z; \tilde{L}^{-1}\tilde{R}_z L^{-1}) \subset B(z; R_z L^{-1} L^{-1}) \subset B(z; LL^{-1}L^{-1}) \subset B(y; \tilde{L}LL^{-1} L^{-1}) \subset B(y; LLL^{-1} L^{-1})
\]

which contradicts the choice of \( L \). So \( B(x; L) \cap B(y; \tilde{R}_z) = \emptyset \), which is equivalent to \( y \notin B(x; L\tilde{R}_z) \). Then

\[
\psi(LR^{-1}) \leq |\mathcal{P}| \leq \sum_{L \in \Lambda} |Z_L| \leq |\Lambda| \cdot \ell(L^{-1}) \leq \psi(LL^{-1}L) \cdot \ell(L^{-1}L).
\]

2. If the quasi-uniformities \( \mathcal{L}^{-1} \) and \( \mathcal{R}^{-1} \) are normally \( \pm \)-subcommuting and generate the same topology on \( X \), then by Proposition 3.4 this topology is Hausdorff, which allows us to apply the first item to the quasi-uniformities \( \mathcal{L}^{-1}, \mathcal{R}^{-1} \) and obtain the upper bound \( \psi(L^{-1}R) \leq \psi(L^{-1}L) \cdot \ell(L) \). The \( \pm \)-subcommutativity of \( \mathcal{L}^{-1} \) and \( \mathcal{R}^{-1} \) implies that \( \psi(R(L^{-1}) \leq \psi(L^{-1}R) \). So,

\[
\psi(LR^{-1}) = \psi(RL^{-1}) \leq \psi(LL^{-1}L) \leq \psi(L^{-1}L) \cdot \ell(L) \leq \psi(L^{-1}L) \cdot q \ell^{\pm 2}(X).
\]

3. Next, assuming that the quasi-uniformities \( \mathcal{L} \) and \( \mathcal{R} \) are normally commuting and \( \bigcap \LL^{-1}L = \Delta_X \), we prove the inequality \( \psi(R(L^{-1}) = \psi(LR^{-1}) \leq \psi(LL^{-1}L) \cdot \ell(LL^{-1}L) \cdot \ell)L^{-1}L \). Fix a subfamily \( \Lambda \subset \mathcal{L} \) of cardinality \( |\Lambda| = \psi(LL^{-1}L) \) such that \( \bigcap_{L \in \Lambda} LL^{-1}L = \Delta_X \). Replacing every entourage \( L \in \Lambda \) by a smaller entourage, we can assume that \( \bigcap_{L \in \Lambda} L^2L^{-1}L^{-1}L = \Delta_X \).

Since the quasi-uniformities \( \mathcal{L} \) and \( \mathcal{R} \) are normally commuting and normally \( \pm \)-subcommuting, for every entourage \( L \in \Lambda \) there exists an entourage \( \tilde{L} \in \mathcal{L} \), \( \tilde{L} \subset L \), such that for every entourage \( R \in \mathcal{R} \) there exists an entourage \( \tilde{R} \in \mathcal{R} \) such that \( \tilde{L} \tilde{R} \subset RL \) and \( \tilde{L}^{-1}\tilde{R} \subset RL^{-1} \).

By the definition of the boundedness number \( \ell(LL^{-1}L^{-1}L) \), for every \( L \in \Lambda \) there exists a subset \( A_L \subset X \) of cardinality \( |A_L| \leq \ell(LL^{-1}L^{-1}L) \) such that \( X = B(A_L; L\tilde{L}^{-1}L^{-1}L) \).

For every point \( a \in A_L \) choose an entourage \( R_a \in \mathcal{R} \) such that \( B(a; R_a) \subset B(a; L) \). By the choice of \( \tilde{L} \) for the entourage \( R_a \) there exists an entourage \( \tilde{R}_a \in \mathcal{L} \) such that \( \tilde{L}\tilde{R}_a \subset R_a L \), and for the entourage \( \tilde{R}_a \in \mathcal{R} \) there
is an entourage $\tilde{R}_a \in R$ such that $\tilde{L}^{-1}\tilde{R}_a \subset \tilde{R}_a L^{-1}$. Consider the family of pairs

$$ \mathcal{P} = \bigcup_{L \in \mathcal{A}} \{(L, \tilde{R}_a) : a \in A_L\} \subset \mathcal{L} \times R. $$

We claim that for any distinct points $x, y \in X$ there exists a pair $(L, R) \in \mathcal{P}$ such that $B(x; L) \cap B(y; R) = \emptyset$.

Given two distinct points $x, y \in X$, find an entourage $L \in \mathcal{A}$ such that $(x, y) \notin L^{2L^{-3}}L$.

Since $y \in X = B(L; \tilde{L} L^{-1} \cap \tilde{L}^{-1}\tilde{L})$, we can find a point $a \in A_L$ such that $y \in B(a; \tilde{L} L^{-1} \cap \tilde{L}^{-1}\tilde{L})$ and hence $y \in B(a; \tilde{L} L^{-1})$ and $a \in B(y; \tilde{L}^{-1}\tilde{L}) \subset B(y; L^{-1}L)$. We claim that $B(x; L) \cap B(y; \tilde{R}_a) = \emptyset$. To derive a contradiction, assume that $B(x; L) \cap B(y; \tilde{R}_a) \neq \emptyset$. Observe that

$$ B(y; \tilde{R}_a) \subset B(a; \tilde{L} L^{-1}) \subset B(a; R L^{-1}\tilde{L}) \subset B(y; L^{-1}L). $$

Then $\emptyset \neq B(x; L) \cap B(y; \tilde{R}_a) \subset B(x; L) \cap B(y; L^{-1}L L^{-1}L)$ implies $y \notin B(x; L^{2L^{-3}}L)$, which contradicts the choice of the entourage $L$. This contradiction shows that $B(x; L) \cap B(y; \tilde{R}_a) = \emptyset$ and hence

$$ \psi(\mathcal{R}^{-1}) = \psi(\mathcal{L} \mathcal{L}^{-1}) \leq |\mathcal{P}| \leq \sum_{L \in \mathcal{A}} |A_L| \leq \psi(\mathcal{A}^{-1} \mathcal{A}) \cdot \ell(\mathcal{A}) \cdot \ell^{2}(X). $$

4. Finally we prove that $\psi(\mathcal{R}^{-1}) = \psi(\mathcal{L} \mathcal{L}^{-1}) \leq \|\mathcal{P}\| \leq \sum_{L \in \mathcal{A}} |A_L| \leq \psi(\mathcal{A}^{-1} \mathcal{A}) \cdot \ell(\mathcal{A}) \cdot \ell^{2}(X)$ for any pre-uniformity $A$ on $X$ such that $\bigcap_{L \in \mathcal{A}} A_L = \Delta_X$. If $\psi(\mathcal{A}^{-1} \mathcal{A}) = 1$, which implies that $A^{-1} AL = \Delta_X = A = L$ for some $A \in \mathcal{A}$ and $L \in \mathcal{L}$. In this case $\ell(\mathcal{A}) = |X|$ and the topological space $X$ is discrete. Then for every point $x \in X$ we can choose an entourage $R_x \in R$ such that $B(x; R_x) = \{x\}$. Then $\bigcap_{x \in X} R_x \mathcal{L}^{-1} = \bigcup_{x \in X} R_x = \Delta_X$ and hence $\psi(\mathcal{R}^{-1}) \leq |X| \leq \ell(\mathcal{A}) \leq \psi(\mathcal{A}^{-1} \mathcal{A}) \cdot \ell(\mathcal{A}) \cdot \ell^{2}(X)$.

So, we assume that the cardinal $\kappa = \psi(\mathcal{A}^{-1} \mathcal{A})$ is infinite. Since $\bigcap_{L \in \mathcal{A}} A_L = \Delta_X$, we can choose subfamilies $(A_{\alpha})_{\alpha \in \kappa} \subset A$ and $(L_{\alpha})_{\alpha \in \kappa} \subset \mathcal{L}$ such that $\bigcap_{\alpha \in \kappa} B(x; A_{\alpha}^{-1}A_{\alpha} \mathcal{L}^{-1}_{\alpha}) = \{x\}$ for every $x \in X$.

For every $\alpha \in \kappa$ consider the entourage $A_{\alpha} \in A$ and find a subset $Z_{\alpha} \subset X$ of cardinality $|Z_{\alpha}| \leq \ell(\mathcal{A})$ such that $X = B(Z_{\alpha}; A_{\alpha})$. Since the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally $\pm$-subcommuting, for the entourage $L_{\alpha}$ there is an entourage $\tilde{L}_{\alpha}$ such that for every $R \in \mathcal{R}$ there is $\tilde{R}_a \in \mathcal{R}$ such that $\tilde{L}_\alpha^{-1} \tilde{R}_a \subset R \mathcal{L}^{-1}_{\alpha}$.

Now fix any point $z \in Z_{\alpha}$. The normality of the quasi-uniformity $\mathcal{L}$ (proved in Proposition 4.3) guarantees that $B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha}) \subset B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$. Put $W_{\alpha,z} = B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$. For every point $y \in X \setminus W_{\alpha,z}$ choose an entourage $R_{\alpha,y} \in \mathcal{R}$ such that $B(y; R_{\alpha,y}) \cap B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha}) = \emptyset$ and hence $B(y; R_{\alpha,y} \mathcal{L}^{-1}_{\alpha}) \cap B(z; A_{\alpha} L_{\alpha}) = \emptyset$. For every $y \in X \setminus B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$ we can replace $R_{\alpha,y}$ by a smaller entourage and assume additionally that $B(y; R_{\alpha,y})$ is disjoint with $B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$.

By the choice of the entourage $L_{\alpha}$ for every $y \in X \setminus W_{\alpha,z}$ there is an entourage $\tilde{R}_y \in \mathcal{R}$ such that $\tilde{R}_y \subset R_{\alpha,y}$ and $\tilde{R}_y \subset R_{\alpha,z} \subset R_{\alpha,y} \mathcal{L}^{-1}_{\alpha}$. For every $y \in W_{\alpha,z}$ choose an entourage $\tilde{R}_y \in \mathcal{R}$ such that $B(y; \tilde{R}_y) \subset W_{\alpha,z}$. Now consider the neighborhood assignment $V = \bigcup_{y \in X} (y) \times B(y; R_y \cap L_{\alpha})$. By the definition of $\ell^{2}(X)$, there exists a subset $A_{\alpha} \subset X$ of cardinality $|A_{\alpha}| \leq \ell^{2}(X)$ such that $X = B(Z_{\alpha}; V V^{-1})$.

Consider the family $\mathcal{P} = \bigcup_{\alpha \in \kappa} \bigcup_{y \in A_{\alpha}} \{(L_{\alpha}, \tilde{R}_a) : a \in A_L\} \subset \mathcal{L} \times R$. We claim that for any distinct points $x, y \in X$ there is a pair $(L, R) \in \mathcal{P}$ such that $B(x; L) \cap B(y; R) = \emptyset$.

Indeed, for the points $x, y \in X$ we can find an ordinal $\alpha \in \kappa$ such that $y \notin B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$. Since $X = B(Z_{\alpha}; A_{\alpha})$, there is a point $z \in Z_{\alpha}$ such that $x \in B(z; A_{\alpha})$. Then $y \notin B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$ and hence $B(y; \tilde{R}_a) \subset B(y; R_{\alpha,y}) \mathcal{L}^{-1}_{\alpha}$ is disjoint with $B(z; A_{\alpha} \mathcal{L}^{-1}_{\alpha})$.

Since $y \in X = B(a; V V^{-1})$, there is a point $a \in A_{\alpha}$ such that $y \in B(a; V V^{-1})$, which implies that $\emptyset \neq B(y; V \cap B(a; V) = B(y; R_y \cap L_{\alpha}) \cap B(a; \tilde{R}_a \cap L_{\alpha})$ and hence $y \in B(a; \tilde{R}_a \cap L_{\alpha})$. Since $B(y; \tilde{R}_a) \cap B(a; \tilde{R}_a \cap L_{\alpha}) = \emptyset$ and $B(a; R_y \cap L_{\alpha}) = \emptyset$. Now observe that the $\tilde{R}_a$-ball $B(y; \tilde{R}_a) \subset B(a; V V^{-1} \tilde{R}_a) \subset B(a; R_y \cap L_{\alpha})$ is disjoint with the $L_{\alpha}$-ball $B(x; L_{\alpha}) \subset B(z; A_{\alpha} L_{\alpha})$.

The family $\mathcal{P}$ witnesses that

$$ \psi(\mathcal{R}^{-1}) = \psi(\mathcal{R} \mathcal{L}^{-1}) \leq |\mathcal{P}| \leq \psi(\mathcal{A}^{-1} \mathcal{A}) \cdot \ell(\mathcal{A}) \cdot \ell^{2}(X). $$

Taking into account that $\psi(\mathcal{R} \mathcal{L}^{-1} \cap \mathcal{L} \mathcal{L}^{-1}) \leq \psi(\mathcal{R}^{-1})$, and applying Theorem 4.4 we obtain:

**Theorem 4.5.** Let $X$ be a Hausdorff topological space and $\mathcal{L}, \mathcal{R}$ be two normally $\pm$-subcommuting quasi-uniformities generating the topology of $X$. Then the uniformity $\mathcal{Q} = \mathcal{L} \mathcal{L}^{-1} \cap \mathcal{R} R^{-1}$ has pseudocharacter:

1. $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}) \cdot \ell(\mathcal{L} \mathcal{L}^{-1}) \cdot \ell^{2}(X);$
(2) $\psi(Q) \leq \psi(\mathcal{L}\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{+2}) \cdot q\ell^{+1}(X)$.

Moreover, if the quasi-uniformity $\mathcal{L}$ is

(3) $\exists$ 3-separated, then $\psi(Q) \leq \overline{\psi}(\mathcal{L}^{-1} \mathcal{L}) \cdot (\mathcal{L} \cdot \ell(\mathcal{L}) \leq \overline{\psi}(\mathcal{L}^{+2}) \cdot \ell^{+1}(X)$.

(4) $\pm4$-separated, then $\psi(Q) \leq \overline{\psi}(\mathcal{L}^{-1} \mathcal{L}) \cdot (\ell(\mathcal{L}^{-1}) \lor \mathcal{L}^{-1} \mathcal{L}) \cdot \ell^{+2}(X) \leq \overline{\psi}(\mathcal{L}^{+3}) \cdot \ell^{+2}(X)$;

(5) $\exists$ 5-separated, then $\psi(Q) \leq \overline{\psi}(\mathcal{L}^{-1} \mathcal{L}^{-1} \mathcal{L}) \cdot (\mathcal{L}^{-1} \mathcal{L}) \cdot \ell^{+3}(X) \leq \overline{\psi}(\mathcal{L}^{+4}) \cdot q\ell^{+2}(X) \cdot \ell^{+2}(X)$;

(6) $\pm6$-separated, then $\psi(Q) \leq \overline{\psi}(\mathcal{L}^{-1} \mathcal{L}^{-1} \mathcal{L}) \cdot \ell^{+2}(X) = \overline{\psi}(\mathcal{L}^{+5}) \cdot \ell^{+2}(X)$.

If the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally commuting and 3-separated, then

(7) $\psi(Q) \leq \psi(\mathcal{L}^{-1} \mathcal{L}) \cdot (\mathcal{L}^{-1} \lor \mathcal{L}) \leq \psi(\mathcal{L}^{+3}) \cdot q\ell^{+2}(X)$.

If the quasi-uniformities $\mathcal{L}^{-1}$, $\mathcal{R}^{-1}$ are normally $\pm$-subcommuting and generate the same topology on $X$, then

(8) $\psi(Q) \leq \psi(\mathcal{L}^{-1} \mathcal{L}) \cdot (\ell(\mathcal{L}) \leq \psi(\mathcal{L}^{+2}) \cdot \ell^{+1}(X)$ and

(9) $\psi(Q) \leq \psi(\mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L}) \cdot (\ell(\mathcal{L}) \leq \psi(\mathcal{L}^{+2}) \cdot q\ell^{+1}(X) \cdot q\ell^{+1}(X)$.

Proof. 1. The first inequality follows from Theorem 4.3(4) applied to the pre-uniformity $\mathcal{A} = \mathcal{U} \lor \mathcal{U}^{-1}$.

2. The second item follows from Theorem 4.3(4).

3-6. The items (3)–(6) follow from Theorem 4.3(4) applied to the pre-uniformities $\mathcal{L}$, $\mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L}$, and $\mathcal{L} \mathcal{L}^{-1}$, respectively.

7. The seventh item follows from Theorem 4.3(3).

8.9. Assume that the quasi-uniformities $\mathcal{L}^{-1}$, $\mathcal{R}^{-1}$ are normally $\pm$-subcommuting and generate the same topology on $X$. The inequalities $\psi(Q) \leq \psi(\mathcal{L}^{-1} \mathcal{L}) \cdot (\ell(\mathcal{L}) \leq \psi(\mathcal{L}^{+2}) \cdot q\ell^{+1}(X)$ follow from Theorem 4.3(2).

To prove that $\psi(Q) \leq \psi(\mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L}) \cdot (\ell(\mathcal{L}) \leq \psi(\mathcal{L}^{+2}) \cdot \ell^{+1}(X)$, fix a subset $A \subset \mathcal{L}$ of cardinality $|A| = \psi(\mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L})$ such that $\bigcap_{A \in \mathcal{L}} \mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L} \neq \emptyset$. Replacing every $L \in A$ by a smaller entourage, we can assume that $\bigcap_{L \in A} L^{-2} \lor L^{-2} \subset \Delta_X$. Since the quasi-uniformities $\mathcal{L}$, $\mathcal{R}$ are normally $\pm$-subcommuting and the quasi-uniformities $\mathcal{L}^{-1}$, $\mathcal{R}^{-1}$ are normally $\pm$-subcommuting, for every $L \in A$ there exists an entourage $\tilde{L} \subset \mathcal{L}$ with $\tilde{L} \subset L$ such that for every $R \in \mathcal{R}$ there is $\tilde{R} \subset \mathcal{R}$ such that $\tilde{L}^{-1} \tilde{L} \subset R^{-1} \mathcal{L}$ and $\tilde{R}^{-1} \subset R^{-1} L^{-1}$.

For every $L \in A$ fix a subset $Z_L \subset X$ of cardinality $|Z_L| \leq (L)(\mathcal{L}^{-1}) \subset \Delta_X$. Given any distinct points $x, y$ find an entourage $L \in A$ such that $(x, y) \notin L^{-2} \lor L^{-2} L^{-2}$ and hence $(x, y) \notin L^{-2} L^{-2}$ or $(x, y) \notin L^{-2} L^{-1}$. If $(x, y) \notin L^{-2} L^{-2}$, then $B(y; L^{2}) \cap B(x; L^{2}) = \emptyset$. Since $y \in X = B(Z_L; L)$, there is $z \in Z_L$ such that $y \in B(\tilde{L}; L)$, then $z \in B(y; L^{2})$ and the $L$-ball $B(z; LL) \subset B(y; LL)$ does not intersect $B(x; L^{2})$, which implies $B(z; LL^{2}) \cap B(x; L) = \emptyset$. Observe that $B(y; \tilde{R}_z) \subset B(z; \tilde{L}^{-1} \tilde{R}_z) \subset B(z; R_z^{1} \mathcal{L}^{-1} \subset B(z; LL^{-1})$ and hence $B(y; \tilde{R}_z) \cap B(x; L) \subset B(z; LL^{-1}) \cap B(x; L) = \emptyset$. So, $(x, y) \notin \tilde{L}^{-1} \tilde{R}_z^{-1}$ and hence $(x, y) \notin \tilde{L}^{-1} \tilde{R}_z^{-1}$.

If $(x, y) \notin L^{-2} L^{-2}$, then $B(y; L^{-2}) \cap B(x; L^{-2}) = \emptyset$. Since $y \in X = B(Z_L; L)$, there is $z \in Z_L$ such that $y \in B(z; \tilde{L})$. Then $z \in B(y; L^{-2}) \subset B(y; L^{-1})$ and the $L^{-2}$-ball $B(z; L^{-1}) \subset B(y; L^{-2})$ does not intersect $B(x; L^{-2})$, which implies $B(z; L^{-1} L) \cap B(x; L^{-1}) = \emptyset$. Observe that $B(y; \tilde{R}_z^{-1}) \subset B(z; \tilde{L}^{-1} \tilde{R}_z^{-1}) \subset B(z; R_z^{-1} L^{-1} \subset B(z; L^{-1} L) and hence $B(y; \tilde{R}_z^{-1}) \cap B(x; L^{-1}) \subset B(z; L^{-1} L) \cap B(x; L^{-1}) = \emptyset$. So, $(x, y) \notin \tilde{L}^{-1} \tilde{R}_z$. Since $\tilde{R}_z \tilde{L}^{-1} \subset L^{-1} \tilde{L}_z$, we get also $(x, y) \notin \tilde{R}_z \tilde{L}^{-1}$.

This completes the proof of the equality $\bigcap_{L \in A} \mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L} = \Delta_X$, which implies the desired inequality

$$\psi(Q) \leq |P| \leq \sum_{L \in A} |Z_L| \leq \psi(\mathcal{L}^{-1} \lor \mathcal{L}^{-1} \mathcal{L}) \cdot (\ell(\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}) - \ell(\mathcal{L}^{-1})$$.

\[ \square \]

In Section 6 we shall need the following upper bound on the local pseudocharacters $\psi(\mathcal{L}^{-1})$ and $\psi(\mathcal{R}^{-1})$ of normally $\pm$-subcommuting quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$.

**Proposition 4.6.** If the topology of a Hausdorff space $X$ is generated by two normally $\pm$-subcommuting quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$, then $\psi(\mathcal{L}^{-1}) \leq \overline{\psi}(X) \cdot \ell(\mathcal{L}^{-1})$ and $\psi(\mathcal{R}^{-1}) \leq \overline{\psi}(X) \cdot \ell(\mathcal{L}^{-1})$.  

Proof. First we prove that $\hat{\psi}(LL^{-1}) \leq \hat{\psi}(X) \cdot \ell^{\leq 2}(X)$. Fix any point $x \in X$. Since the topology of $X$ is generated by the quasi-uniformity $\mathcal{R}$, we can fix a subfamily $\mathcal{R}_x \subseteq \mathcal{R}$ of cardinality $|\mathcal{R}_x| \leq \overline{\psi}(X) \leq \hat{\psi}(X)$ such that $\bigcap_{\mathcal{R}_x} B(x; RR\mathcal{R}) = \{x\}$.

By the normality of the quasi-uniformity $\mathcal{R}$, for every $R \in \mathcal{R}_x$ we get $B(x; RR) \subseteq B(x; RR\mathcal{R})$. Then for every point $z \in X \setminus B(x; RR\mathcal{R})$ we can find an entourage $L_z \in \mathcal{L}$ such that $B(z; L_z \cap L_z) \cap B(x; RR) = \emptyset$. For every point $z \in B(x; RR)$ choose an entourage $L_z \in \mathcal{L}$ such that $B(z; L_z \cap L_z) \subseteq B(x; RR)$. Since the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally $\pm$-subcommuting, for the entourage $R \in \mathcal{R}$ there is an entourage $\tilde{R} \in \mathcal{R}$ such that for every entourage $L \in \mathcal{L}$ there is an entourage $\tilde{L} \in \mathcal{L}$ such that $\tilde{R} \cap L \subseteq \mathcal{L}$. In particular, for every $z \in \mathcal{L}$ there is an entourage $\tilde{L}_z \in \mathcal{L}$ such that $\tilde{R} \cap L \subseteq \mathcal{L}$. Replacing $\tilde{L}_z$ by a smaller entourage we can assume that $\tilde{L}_z \subseteq L_z$ and $B(z; \tilde{L}_z) \subseteq B(x; R)$.

By the definition of $\ell^{\leq 2}(X)$, for the neighborhood assignment $N_R = \bigcup_{z \in X} \{z\} \times B(z; \tilde{L}_z \cap \tilde{R})$ there is a subset $Z_R \subseteq X$ of cardinality $|Z_R| \leq \ell^{\leq 2}(X)$ such that $X = B(Z_R; N_R N_R^{-1})$.

We claim that the subfamily $\mathcal{L}' = \bigcap_{\mathcal{L} \subseteq \mathcal{L}} B(x; LL^{-1}) = \{x\}$. Given any point $y \in X \setminus \{x\}$, find an entourage $R \in \mathcal{R}_x$ such that $y \notin B(x; RR\mathcal{R})$. Since $y \in X = B(Z_R; N_R N_R^{-1})$, there is a point $z \in Z_R$ such that $y \notin B(z; N_R N_R^{-1})$ and hence $B(y; L_y \cap \tilde{R}) \cap B(z; L_z \cap \tilde{R}) = B(y; N_R \cap B(z; N_R) \neq \emptyset$ and $y \in B(z; \tilde{L}_z \cap \tilde{R})$. Since $y \notin B(x; RR\mathcal{R})$, the choice of the entourages $L_y, L_z$ implies that $z \notin B(x; RR\mathcal{R})$. We claim that $B(y; \tilde{L}_z) \cap B(x; \tilde{L}_z) = \emptyset$. To derive a contradiction, assume that $B(y; \tilde{L}_z) \cap B(x; \tilde{L}_z) \neq \emptyset$. Then

$$\emptyset \neq B(y; \tilde{L}_z) \cap B(x; \tilde{L}_z) \subseteq B(z; \tilde{L}_z \cap \tilde{R}) \cap B(x; R) \cap B(z; \tilde{L}_z \cap \tilde{R})$$

and hence $B(z; L_z \cap \tilde{L}_z \cap \tilde{R}) \neq \emptyset$, which contradicts the choice of the entourage $L_z$. This contradiction completes the proof of the inequality $\hat{\psi}(LL^{-1}) \leq \hat{\psi}(X) \cdot \ell^{\leq 2}(X)$.

By analogy (or changing $\mathcal{L}$ and $\mathcal{R}$ by their places) we can prove that $\hat{\psi}(RR^{-1}) \leq \hat{\psi}(X) \cdot \ell^{\leq 2}(X)$. $\square$

5. QUASI-UNIFORMITIES ON TOPOLOGICAL MONOIDS

A topological monoid is a topological semigroup $X$ possessing a (necessarily unique) two-sided unit $e \in X$. We shall say that a topological monoid $S$ has open shifts if for any elements $a, b \in X$ the two-sided shift $s_{a,b} : X \to X$, $s_{a,b} : x \mapsto axb$, is an open map.

A typical example of a topological monoid with open shifts is a paratopological group, i.e., a group endowed with a topology making the group operation $G \times G \to G$, $(x, y) \mapsto xy$, continuous.

The closed half-line $[0, \infty)$ endowed the Sorgenfrey topology (generated by the base $B = \{[a, b) : 0 \leq a < b < \infty\}$) and the operation of addition of real numbers is a topological monoid with open shifts, which is not a (paratopological) group.

Each topological monoid $X$ carries five natural quasi-uniformities:

- the left quasi-uniformity $\mathcal{L}$, generated by the base $\{(x, y) \in X \times X : y \in xU\} : U \in \mathcal{N}_e$,
- the right quasi-uniformity $\mathcal{R}$, generated by the base $\{(x, y) \in X \times X : y \in xu\} : U \in \mathcal{N}_e$,
- the two-sided quasi-uniformity $\mathcal{L} \vee \mathcal{R}$, generated by the base $\{(x, y) \in X \times X : y \in xu \cap xU\} : U \in \mathcal{N}_e$,
- the Roelcke quasi-uniformity $\mathcal{RL} = \mathcal{L} \vee \mathcal{R}$, generated by the base $\{(x, y) \in X \times X : y \in xu \cap xU\} : U \in \mathcal{N}_e$,

and

- the quasi-Roelcke uniformity $\mathcal{Q} = \mathcal{RL}^{-1} \cap \mathcal{RL}^{-2}$, generated by the base $\{(x, y) \in X \times X : xu \cap yU \neq \emptyset \neq Uy \cap xu\} : U \in \mathcal{N}_e$.

Here by $\mathcal{N}_e$ we denote the family of all open neighborhoods of the unit $e$ in $X$. The quasi-uniformities $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L} \vee \mathcal{R}$, and $\mathcal{RL}$ are well-known in the theory of topological and paratopological groups (see [22, Ch.2], [2, §1.8]). The quasi-Roelcke uniformity was recently introduced in [4]. It should be mentioned that on topological groups the quasi-Roelcke uniformity coincides with the Roelcke (quasi-)uniformity. The following diagram describes the relation between these five quasi-uniformities (an arrow $U \to V$ in the diagram indicates that $U \subseteq V$).

```
\begin{tikzpicture}
    \node (L) at (0,0) {$\mathcal{L}$};
    \node (R) at (1,0) {$\mathcal{R}$};
    \node (Q) at (0.5,-1) {$\mathcal{RL}$};
    \node (LOR) at (0.5,0.5) {$\mathcal{L} \vee \mathcal{R}$};

    \draw[->] (L) -- (LOR);
    \draw[->] (LOR) -- (Q);
    \draw[->] (Q) -- (R);
    \draw[->] (L) -- (Q);
    \draw[->] (R) -- (Q);
\end{tikzpicture}
```
If a topological monoid $X$ has open shifts, then the quasi-uniformities $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L} \lor \mathcal{R}$ and $\mathcal{RL}$ generate the original topology of $X$ (see [15, 18]) whereas the quasi-Roelcke uniformity $\mathcal{Q}$ generates a topology $\tau_\mathcal{Q}$, which is (in general, strictly) weaker than the topology $\tau$ of $X$. If $X$ is a paratopological group, then the topology $\tau_\mathcal{Q}$ on $G$ coincides with the joint $\tau_2 \lor (\tau^{-1})_2$ of the second oscillator topologies considered by the authors in [3]. The topology $\tau_\mathcal{Q}$ turns the paratopological group into a quasi-topological group, i.e., a group endowed with a topology in which the inversion and all shifts are continuous (see Proposition 6.3).

**Proposition 5.1.** On each topological monoid $X$ with open shifts the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally commuting, normally $\pm$-subcommuting, and normal. The topology of $X$ is Hausdorff if and only if the quasi-Roelcke uniformity $\mathcal{Q} = \mathcal{L}\mathcal{R}^{-1} \lor \mathcal{RL}^{-1}$ on $X$ is separated.

**Proof.** To see that the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally commuting, fix any entourage $L \in \mathcal{L}$ and find a neighborhood $U \subset G$ of the unit $e$ such that $L = \{(x, y) \in X \times X : y \in xU\} \subset L$. Given any entourage $R \in \mathcal{R}$, find a neighborhood $V \subset G$ of the unit $e$ such that $R = \{(x, y) \in X \times X : y \in Vx\} \subset R$. Then

$$\tilde{L} \tilde{R} = \{(x, y) \in X \times X : \exists z \in X \text{ such that } (x, z) \in \tilde{L} \text{ and } (z, y) \in \tilde{R}\} = \{(x, y) \in X \times X : \exists z \in xU \text{ and } y \in Vz\} = \{(x, y) \in X \times X : y \in (Vx)U\} = \tilde{R} \tilde{L} \subset RL \cap LR.$$

This implies that the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally commuting.

Next, we prove that $L^{-1} \tilde{R} \subset \tilde{R} L^{-1} \subset RL^{-1}$. Given any pair $(x, y) \in \tilde{L}^{-1} \tilde{R}$, find a point $z \in X$ such that $(x, z) \in \tilde{L}^{-1}$ and $(z, y) \in \tilde{R}$. Then $x \in zU$ and $y \in Vz$. So, we can find points $u \in U$ and $v \in V$ such that $x = zu$ and $y = vz$. Multiplying $x = zu$ by $v$, we get $vx = vz u = yu$ and hence $(x, vx) \in \tilde{R}$ and $(y, vx) = (y, yu) \in \tilde{L}$, which implies that $(x, y) \in \tilde{R} \tilde{L}^{-1} \subset RL^{-1}$. So, $\tilde{L}^{-1} \tilde{R} \subset \tilde{R} L^{-1} \subset RL^{-1}$. By analogy we can prove that $\tilde{R}^{-1} \tilde{L} \subset L^{-1} \tilde{R} \subset LR^{-1}$.

By Proposition 3.1, the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$, being normally $\pm$-subcommuting, are normal.

If $X$ is Hausdorff, then for any distinct points $x, y \in X$ we can find a neighborhood $U \subset X$ of the unit $e$ such that $Ux \cap yU = \emptyset$. Then for the entourages $L = \{(x, y) \in X : y \in xU\} \in \mathcal{L}$ and $R = \{(x, y) \in X \times X : y \in Ux\}$ we get $y \notin B(x; RL^{-1}) \supset B(x; RL^{-1} \cap LR^{-1})$. This means that $\Delta_X \subset \mathcal{Q}$ and the quasi-Roelcke uniformity $\mathcal{Q}$ is separated.

Now assume that the quasi-Roelcke uniformity $\mathcal{Q}$ is separated. Given two distinct points $x, y \in X$, find two entourages $L \in \mathcal{L}$ and $R \in \mathcal{R}$ such that $(x, y) \notin LR^{-1} \cap RL^{-1}$ and hence $(x, y) \notin LR^{-1}$ or $(x, y) \notin RL^{-1}$.

For the entourages $L, R$, find a neighborhood $U \subset X$ of $e$ such that $(x, y) \in X \times X : y \in xU \subset L$ and $(x, y) \in X \times X : y \in Ux \subset R$. If $(x, y) \notin LR^{-1}$, then $xU \cap yU = \emptyset$. If $(x, y) \notin RL^{-1}$, then $Ux \cap yU = \emptyset$. In both cases the points $x, y$ have disjoint neighborhoods in $X$, which means that $X$ is Hausdorff. $\square$

Proposition 5.1 and Theorem 3.1 imply:

**Theorem 5.2.** Each Hausdorff topological monoid $X$ with open shifts is functionally Hausdorff and has submetrizability number $sm(X) \leq \psi(\mathcal{Q}) \leq \chi(X)$ and $i$-weight $iw(X) \leq \psi(\mathcal{Q}) \cdot \log(\ell(\mathcal{Q})) \leq \chi(X) \cdot \log(d(\mathcal{Q})).$

Observe that for a paratopological group $G$ the quasi-Roelcke uniformity $\mathcal{Q}$ generates the topology of $G$ if and only if $G$ is a topological group.

**Problem 5.3.** Study properties of topological monoids $S$ with open shifts whose topology is generated by the quasi-Roelcke uniformity $\mathcal{Q}$.

6. **The Submetrizability Number and $i$-Weight of Paratopological Groups**

In this section we apply the results of the preceding sections to paratopological groups, i.e., groups $G$ endowed with a topology making the group operation $G \times G \to G$, $(x, y) \mapsto xy$, continuous. It is easy to see that the inversion map $G \to G$, $x \mapsto x^{-1}$, is a uniform homeomorphism of the quasi-uniform spaces $(G, L^{-1})$ and $(G, R)$ and also a uniform homeomorphism of the quasi-uniform spaces $(G, \mathcal{R}^{-1})$ and $(G, \mathcal{L})$. This observation combined with Propositions 3.6 and 6.1 implies:

**Proposition 6.1.** On each paratopological group $G$

1. the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are normally commuting, normally $\pm$-subcommuting, and normal;
2. the quasi-uniformities $L^{-1}$ and $R^{-1}$ are normally commuting, normally $\pm$-subcommuting, and generate the same topology on $G$. 
If the topology of $G$ is Hausdorff, then the quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are 3-separated and the quasi-Roelcke uniformity $Q = LR^{-1} \vee RL^{-1}$ is separated.

Next, we prove that a paratopological group endowed with the quasi-Roelcke uniformity is a uniform quasi-topological group.

**Definition 6.2.** A uniform quasi-topological group is a group $G$ endowed with a uniformity $U$ such that the inversion $G \to G, x \mapsto x^{-1}$, is uniformly continuous and for every $a,b \in G$ the shifts $s_{a,b} : G \to G, s_{a,b} : x \mapsto axb$, is uniformly continuous.

**Proposition 6.3.** Any paratopological group $G$ endowed with the quasi-Roelcke uniformity $Q = LR^{-1} \vee RL^{-1}$ is a uniform quasi-topological group.

**Proof.** Observe that for any neighborhood $V \in \mathcal{N}_c$ and points $x,y \in G$ the inclusion $y \in V_xV^{-1} \cap V^{-1}xV$ is equivalent to $y^{-1} \in Vx^{-1}V^{-1} \cap V^{-1}x^{-1}V$, which implies that the inversion map $G \to G, x \mapsto x^{-1}$, is uniformly continuous.

Next, we show that for every $a,b \in G$ the shift $s_{a,b} : G \to G, s_{a,b} : x \mapsto axb$, is uniformly continuous. Fix any neighborhood $V \in \mathcal{N}_c$ of $e$. By the continuity of the shifts on $G$, there exists a neighborhood $U \subset V$ of $e$ such that $aU \subset Va,Ub \subset bV, Ua^{-1} \subset a^{-1}V$, and $b^{-1}U \subset bV^{-1}$. Inverting the two latter inclusions, we get $aU^{-1} \subset V^{-1}a$ and $U^{-1}b \subset bV^{-1}$. Then for any points $x,y \in G$ with $y \in U^{-1}xU \cap UxU^{-1}$, we get $yab \in aU^{-1}xUb \cap aUxU^{-1}b \subset V^{-1}axbV \cap VaxbV^{-1}$, which means that the shift $s_{a,b}$ is uniformly continuous.

The following theorem is a partial case of Theorem 5.2.

**Theorem 6.4.** Each Hausdorff paratopological group $G$ is functionally Hausdorff and has submetrizability number $sm(G) \leq \psi(Q) \leq \chi(G)$ and $i$-weight $iw(G) \leq \psi(Q) \cdot \log(\ell(Q)) \leq \chi(G) \cdot \log(\ell(G))$.

In light of this theorem it is important to have upper bound on the pseudocharacter $\psi(Q)$ of the quasi-Roelcke uniformity. Such upper bounds are given in the following theorem, which unifies or generalizes the results of [23] and [19].

**Theorem 6.5.** For any Hausdorff paratopological group $G$ its quasi-Roelcke uniformity $Q = LR^{-1} \vee RL^{-1}$ has pseudocharacter

1. $\psi(Q) \leq \min\{\psi(L^{-1}) \cdot \ell(L^{-1}), \psi(L^{-1}L) \cdot \ell(L)\} \leq \psi(G) \cdot \ell^{\pm 2}(G) \cdot \min\{\ell(L), \ell(L^{-1})\} \leq \psi(G) \cdot \ell^{\pm 2}(G) \cdot \min\{q^{\pm 1}(G), q^{\pm 1}(G)\}$;
2. $\psi(Q) \leq \psi(L^{-1} \vee L^{-1}L) \cdot \ell(L^{-1}) \cdot \ell(L) \leq \psi(L^{\pm 2}) \cdot q^{\pm 1}(G) \cdot q^{\pm 1}(G)$;
3. $\psi(Q) \leq \psi(L^{-1}L^{-1} \vee L^{-1}L) \leq \psi(L^{\pm 3}) \cdot q^{\pm 2}(G)$.

Moreover, if the quasi-uniformity is

4. $\mp 4$-separated, then $\psi(Q) \leq \psi(L^{-1}L^{-1}L^{-1}) \cdot \ell^{\pm 2}(X) \leq \psi(L^{\pm 4}) \cdot q^{\pm 2}(X) \cdot \ell^{\pm 2}(G)$;
5. $\pm 6$-separated, then $\psi(Q) \leq \psi(L^{-1}L^{-1}L^{-1}) \cdot \ell^{\pm 2}(G) \leq \psi(L^{\pm 5}) \cdot \ell^{\pm 2}(G)$.

**Proof.** 1. The inequality $\psi(Q) \leq \psi(L^{-1}) \cdot \ell(L^{-1})$ follows from Theorem 4.6(2), which also implies $\psi(Q) \leq \psi(R^{-1}) \cdot \ell(R^{-1}) \leq \psi(L^{-1}L) \cdot \ell(L)$. By Proposition 4.6, $\psi(L^{-1}L^{-1}) = \psi(L^{-1}) \leq \psi(Q) \cdot q^{\pm 2}(G)$ and $\psi(L^{-1}L) = \psi(R^{-1}) = \psi(Q) \cdot \ell^{\pm 2}(G)$, which implies

$$\min\{\psi(L^{-1}L^{-1}) \cdot \ell(L^{-1}), \psi(L^{-1}L^{-1}L^{-1}) \cdot \ell(L^{-1})\} \leq \psi(Q) \cdot \ell^{\pm 2}(G) \cdot \min\{\ell(L), \ell(L^{-1})\}.$$

2, 3. The upper bounds from the second and third items follow from Theorem 4.6(9,7) and Proposition 6.1.

4. Assume that the quasi-uniformity $\mathcal{L}$ is $\mp 4$-separated. Then we can choose a subfamily $U \subset \mathcal{N}_c$ of cardinality $|U| = \psi(L^{-1}L^{-1}L^{-1})$ such that $\bigcap_{U \in \mathcal{U}} U^{-1}UU^{-1}U = \{e\}$. Replacing every $U$ by a smaller neighborhood of $e$, we can assume that $\bigcap_{U \in \mathcal{U}} U^{-1}UU^{-1}U = \{e\}$. Since $U^{-1}UU^{-1}U \subset U^{-1}(U^{-1}UU^{-1}U)$, we conclude that $\bigcap_{U \in \mathcal{U}} U^{-1}UU^{-1}U = \{e\}$ and $\psi(L^{-1}L^{-1}L^{-1}) \leq |U| = \psi(L^{-1}L^{-1}L^{-1})$. Applying Theorem 4.6(4) to the pre-uniformity $\mathcal{A} = \mathcal{L}^{-1}$, we get the upper bound

$$\psi(Q) \leq \psi(A^{-1}AU) \cdot \ell(A) \cdot \ell^{\pm 2}(G) = \psi(L^{-1}L^{-1}L^{-1}L^{-1}) \cdot \ell(L^{-1}L^{-1}) \cdot \ell^{\pm 2}(G) = \psi(L^{-1}LL^{-1}L^{-1}) \cdot \ell(L^{-1}L^{-1}L^{-1}) \cdot \ell(L^{-1}L^{-1}L^{-1}) \cdot \ell^{\pm 2}(G).$$

5. The fifth item follows from Theorem 4.6(6).
7. Two counterexamples

In this section we construct two examples of paratopological groups that have some rather unexpected properties.

7.1. A paratopological group with countable pseudocharacter which is not submetrizable. In Theorem 6.3.1 we proved that for each Hausdorff paratopological group $G$ its quasi-Roelcke uniformity has pseudocharacter $\psi(Q) \leq \varphi(G) \cdot \ell(\mathcal{L}) \cdot \min\{\ell(\mathcal{L}), \ell(\mathcal{L}^{-1})\}$. It is natural to ask if this upper bound can be improved to $\psi(Q) \leq \psi(G)$. In this section we show that this inequality is not true in general. Namely, we present an example of a zero-dimensional (and hence) Hausdorff abelian paratopological group which has countable pseudocharacter but is not submetrizable. Some properties of this group can be proved only under Martin Axiom [27], whose topological equivalent says that each countabably cellular compact Hausdorff space is $\kappa$-Baire for every cardinal $\kappa < \mathfrak{c}$. We say that a topological space $X$ is $\kappa$-Baire if for any family $\mathcal{U}$ consisting of $\kappa$ many open dense subsets of $X$ the intersection $\bigcap \mathcal{U}$ is dense in $X$. Under Martin’s Axiom for $\sigma$-centered posets, each separable compact Hausdorff space is $\kappa$-Baire for every cardinal $\kappa < \mathfrak{c}$. This implies that under Martin’s Axiom (for $\sigma$-centered posets) the space $\mathbb{Z}^\kappa$ endowed with the Tychonoff product topology is $\kappa$-Baire for every cardinal $\kappa < \mathfrak{c}$. Here $\mathfrak{c}$ stands for the cardinality of continuum. In the statement (4) of the following theorem by $\mathfrak{d}$ we denote the cofinality the partially ordered set $(\mathbb{N}^\kappa, \leq)$. It is known [26] that $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$ and $\mathfrak{d} = \mathfrak{c}$ under Martin’s Axiom (for countable posets).

Let $\kappa$ be an uncountable cardinal. On the group $\mathbb{Z}^\kappa$ of all functions $g : \kappa \to \mathbb{Z}$ consider the shift-invariant topology $\tau_\gamma$ whose neighborhood base at the zero function $e : \kappa \to \mathbb{Z}$ consists of the sets

$$W_{F, m} = \{g \in \mathbb{Z}^\kappa : g|F = 0, \ g(\kappa) \subset \{0\} \cup [m, \infty)\}$$

where $m \in \mathbb{N}$ and $F$ runs over finite subsets of $\kappa$. The group $\mathbb{Z}^\kappa$ endowed with the topology $\tau_\gamma$ is a paratopological group, denoted by $\mathbb{Z}^\kappa$. Since the group $\mathbb{Z}^\kappa$ is abelian, the fours standard uniformities of $\mathbb{Z}^\kappa$ coincide (i.e., $\mathcal{L} = \mathcal{R} = \mathcal{L} \vee \mathcal{R} = \mathcal{R} \mathcal{L}$) whereas the quasi-Roelcke uniformity $Q$ coincides with the pre-uniformities $\mathcal{L} \mathcal{L}^{-1}$ and $\mathcal{R} \mathcal{R}^{-1}$.

**Theorem 7.1.** For any uncountable cardinal $\kappa$ the paratopological group $G = \mathbb{Z}^\kappa$ has the following properties:

1. $G$ is a zero-dimensional (and hence regular) Hausdorff abelian paratopological group;
2. the topology on $G$ induced by the quasi-Roelcke uniformity $Q$ coincides with the Tychonoff product topology $\tau$ on $\mathbb{Z}^\kappa$;
3. $\psi(Q) = \chi(G) = \kappa$ but $\psi(G) = \varphi(G) = \omega$;
4. $\ell(Q) = \omega$ but $\ell(\mathcal{L}) \geq \mathfrak{d} > \omega$;
5. $c(G) \geq \kappa$ but $dc(G) = \omega$;
6. $iw(G) \cdot \omega = sm(G) \cdot \omega \geq \log(2^\kappa)$.
7. If $2^\kappa > \mathfrak{c}$, then $G$ is not submetrizable.
8. If the space $\mathbb{Z}^\kappa$ is $\kappa$-Baire, then $G$ fails to have $G_\delta$-diagonal and hence is not submetrizable.

**Proof.** 1. It is clear that the topology $\tau_\gamma$ on $\mathbb{Z}^\kappa$ is stronger than the Tychonoff product topology $\tau$ on $\mathbb{Z}^\kappa$. This implies that the paratopological group $G = \mathbb{Z}^\kappa$ is Hausdorff. Observing that each basic neighborhood $W_{F, m}$ of the zero function $e \in \mathbb{Z}^\kappa$ is $\tau_\gamma$-closed, we conclude that it is $\tau_\gamma$-closed, which implies that the space $\mathbb{Z}^\kappa$ is zero-dimensional and hence regular.

2. Observe that for every basic neighborhood $W_{F, m}$ of zero, the set $W_{F, m} - W_{F, m}$ coincides with the basic neighborhood $W_F = \{g \in \mathbb{Z}^\kappa : g|F = 0\}$ of zero in the Tychonoff product topology $\tau$. This implies that $\tau$ coincides with the topology induced by the quasi-Roelcke uniformity $Q$.

3. The equality $\chi(G) = \kappa = \psi(Q)$ easily follows from the definition of the topology $\tau_\gamma$ and the fact that the quasi-Roelcke uniformity $Q$ generates the Tychonoff product topology on $\mathbb{Z}^\kappa$. To see that $\psi(G) = \varphi(G) = \omega$, observe that $\bigcap_{m \in \mathbb{N}} W_{0, m} = \{e\}$.

4. To see that $\ell(Q) = \omega$, take any basic open neighborhood $W_{F, m}$ of zero in the group $G$ and observe that $Z^F = \{g \in \mathbb{Z}^\kappa : g|\kappa \setminus F = 0\}$ is a countable subgroup of $G$ such that $G = Z^F + (W_{F, m} - W_{F, m})$, which implies that $\ell(Q) \leq \omega$. On the other hand, the boundedness number $\ell(\mathcal{L})$ of the left quasi-uniformity on the paratopological group $\mathbb{Z}^\kappa$ is equal to the cofinality of the partially ordered set $(\mathbb{N}^\kappa, \leq)$ which is not smaller that $\mathfrak{d}$, the cofinality of the partially ordered set $(\mathbb{N}^\kappa, \leq)$.

5. For every $x \in \kappa$ denote by $\delta_x : \kappa \to \{0, 1\} \subset \mathbb{Z}$ the characteristic function of the singleton $\{x\}$ and let $U_x = \delta_x + W_{\{x\}, 2}$ be a basic neighborhood of $\delta_x$. We claim that for any distinct points $x, y \in \kappa$ the sets $U_x$ and $U_y$ are disjoint. To derive a contradiction, assume that $U_x \cap U_y$ contains some function $f \in \mathbb{Z}^\kappa$. The inclusion
$f \in U_x$ implies that $f(x) = \delta_x(x) = 1$. On the other hand, if $f \in U_y$ implies $f(x) \in \{\delta_y(x)\} \cup \{\delta_y(x) + 2, \infty\} = \{0\} \cup [2, \infty) \neq \emptyset$. So, the closed-and-open sets $U_x$, $x \in \kappa$, are pairwise disjoint and hence $|c(G)| \geq |\{U_x\}_{x \in \kappa}| = \kappa$.

By Proposition 11.10 $dc(G) = \ell^{\omega+1}(G)$. So, it suffices to prove that $\ell^{\omega+1}(G) = \omega$. Given a neighborhood assignment $V$ on $G$, we need to find a countable subset $C \subset G$ such that $B(C; AVV^{-1}VV^{-1}) = G$. Using Zorn’s Lemma, find a maximal subset $C \subset G$ such that $B(x; VV^{-1}) \cap B(y; VV^{-1}) = \emptyset$ for any distinct points $x, y \in C$. By the maximality of $C$, for every $x \in G$ there is a point $c \in C$ such that $B(c; V V^{-1}) \cap B(x; V V^{-1}) \neq \emptyset$, which implies $x \in B(C; V V^{-1} V V^{-1})$ and hence $X = B(C; V V^{-1} V V^{-1})$. It remains to prove that the set $C$ is countable. To derive a contradiction, assume that $C$ is uncountable. For every $x \in G$ find a finite subset $F_x \subset \kappa$ and a positive number $m_x \in \mathbb{N}$ such that $x + W_{F_x, m_x} \subset B(x; V)$. By the $\Delta$-system Lemma 16.1, the uncountable set $C$ contains an uncountable subset $D \subset C$ such that the family $(F_x)_{x \in D}$ is a $\Delta$-system with kernel $K$, which means that $F_x \cap F_y = K$ for any distinct points $x, y \in D$. For every $n \in \mathbb{N}$ and $f \in Z^K$ consider the subset $D_{n,f} = \{x \in D : f(K) = \kappa, m_x \leq n, \sup_{\alpha \in F_x} |x(\alpha)| \leq n\}$ of $D$ and observe that $D = \bigcup_{n \in \mathbb{N}} \bigcup_{f \in Z^K} D_{n,f}$. By the Pigeonhole Principle, for some $n \in \mathbb{N}$ and $f \in Z^K$ the set $D_{n,f}$ is countable. Consider the clopen subset $Z^K(f) = \{x \in Z^\kappa : x[K] = f\}$ of $Z^\kappa$. Since $Z^K(f)$ is a Baire space, for some $m \in \mathbb{N}$ the set $X_m = \{x \in Z^\kappa(f) : m_x = m\}$ is not nowhere dense in $Z^\kappa(f)$. Consequently, there is a finite subset $K \subset \kappa$ containing $K$ and a function $f_0 : K \rightarrow Z$ such that the set $X_m \cap Z^\kappa(f_0)$ is dense in $Z^\kappa(f) = \{x \in Z^\kappa : x[K] = f\}$. Since the family $(F_x \setminus K)_{x \in D}$ is disjoint, the set $\{x \in D : (F_x \setminus K) \cap K = \emptyset\}$ is finite, so we can find two functions $x, y \in D_{n,f}$ such that $(F_x \cup F_y) \cap K = K$. Put $K = F_x \cup F_y \cup K$ and choose any function $f : K \rightarrow Z$ such that $f(K) = f_0$ and $f(\alpha) = x(\alpha) - n - m$ for any $\alpha \in K \setminus K$. The function $f$ determines a non-empty open subset $Z^\kappa(f) = \{x \in Z^\kappa : x[K] = f\}$, which contains some function $z \in X_m$ (by the density of $X_m \cap Z^\kappa(f)$ in $Z^\kappa(f)$). Choose a function $z \in Z^\kappa(f)$ such that $\exists f_x = z[F_x]$ and $\exists (\alpha) \geq \max\{m + z(\alpha), m_x + x(\alpha)\}$ for every $\alpha \in K \setminus F_x$. Then $\exists (z + W_{F_x, m}) \cap (x + W_{F_x, m_x}) \subset B(z; V) \cap B(x; V)$, which implies $z \in B(x; V V^{-1})$.

By analogy we can prove that $z \in B(y; V V^{-1})$. So, $B(x; V V^{-1}) \cap B(y; V V^{-1}) \neq \emptyset$, which contradicts the choice of the set $C \ni x, y$. This contradiction shows that $C$ is countable and hence $dc(G) = \ell^{\omega+1}(G) = \omega$.

6. By Proposition 14 iw$(G) \cdot \omega = sm(G) \cdot \log(dc(G)) = \sm(G) \cdot \omega$. On the other hand, $2^\kappa = |G| \leq |\{0, 1\}^{iw(G)}| = |\{0, 1\}^{iw(G)}| \omega$ implies that $\log(2^\kappa) \leq iw(G) \cdot \omega$.

7. If $\theta > \kappa$, then $sm(G) \cdot \omega \geq \log^*(\theta) \geq \log^{(\omega)}(\theta^+) > \omega$, which implies that $sm(G) > \omega$ and hence $G$ is not submetrizable.

8. Suppose that the space $Z^\kappa$ is $\kappa$-Baier. Assuming that the space $G$ is $\kappa^\kappa$ has $G_\delta$-diagonal, we can apply Theorem 2.2 in [12] and find a countable family $(U_n)_{n \in \omega}$ open covers of $G$, which separates the points of $G$ in the sense that for every distinct points $f, g \in G$ there is $n \in \mathbb{N}$ such that no set $U \in U_n$ contains both points $f$ and $g$. Since the space $G$ is zero-dimensional, we can assume that each set $U \in \bigcup_{n \in \omega} U_n$ is closed-and-open in $G$. Put $U_0 = \{G\}$.

We shall construct an increasing sequence $(F_n)_{n \in \omega}$ of finite subsets and a sequence $f_n \in Z^{F_n}$, $n \in \omega$, of functions such that for every $n \in \omega$ the clopen set $Z^\kappa(f_n) = \{f \in Z^\kappa : f|F_n = f_n\}$ is contained in $U_n \cap Z^\kappa(f_{n-1})$ for some set $U_n \in U_n$.

We start the inductive construction letting $F_0 = \emptyset$ and $f_0 : \emptyset \to Z$ be the unique function. Then $Z^\kappa(f_0) = Z^\kappa \in U_0$. Assume that for some $n \in \omega$ we have defined a finite set $F_{n-1} \subset \kappa$ and a function $f_{n-1} \in Z^{F_{n-1}}$ such that $Z^\kappa(f_{n-1}) \subset U_{n-1}$ for some $U_{n-1} \in U_{n-1}$.

The $F$ be the family of all triples $(F, f, m)$ where $F$ is a finite subset of $\kappa$ containing $F_{n-1}$, $f : F \to Z$ is a function extending the function $f_{n-1}$ and $m \in \mathbb{N}$ is a positive integer. Observe that $|F| = \kappa$. For every function $g \in \kappa^\kappa$ choose a closed-and-open subset $U_g \in U_n$ containing $g$ and choose a finite subset $F_g \subset \kappa$ containing $F_{n-1}$ and a number $m_g$ such that $g + W_{F_g, m_g} \subset U_g$. For every triple $(F, f, m) \in F$ consider the subset $Z(F, f, m) = \{g \in Z^\kappa : (F_g, g|F_g, m_g) = (F, f, m)\}$ and observe that $Z^\kappa(f_{n-1}) = \bigcup_{(F, f, m) \in F} Z(F, f, m)$. Since the space $Z^\kappa(f_{n-1})$ is $\kappa$-Baier, there is a triple $(F, f, m) \in F$ such that the set $Z(F, f, m)$ is not nowhere dense in $Z^\kappa(f_{n-1})$. Consequently we can find a finite set $F_n \subset \kappa$ and a function $f_n \in Z^{F_n}$ such that for the basic open set $Z^\kappa(f_n) = \{g \in Z^\kappa : g|F_n = f_n\}$ the intersection $Z^\kappa(f_n) \cap Z(F, f, m)$ is dense in $Z^\kappa(f_n)$. It follows that $F_n \supset F \supset F_{n-1}$ and $f_n|F = f$. Choose any point $g \in Z(F, f, m) \cap Z^\kappa(f_n)$. We claim that $Z^\kappa(f_n) \subset U_g \in U$. Assuming that $Z^\kappa(f_n) \not\subset U_g$, choose a function $h \in Z^\kappa(f_n) \setminus U_g$ and find a basic neighborhood $h + W_{F_h, m} \subset Z^\kappa(f_n) \setminus U_g$ of $h$. It follows from the inclusion $h + W_{F_h, m} \subset Z^\kappa(f_n)$ that $E \supset F_n \supset F$ and $h|F_n = f_n$. Then $f = f_n|F = f$. Choose a function $h : \kappa \to Z$ such that $h|E = h|E$ and $h(x) = \max\{g(x) + m, h(x) + 1\}$ for every $x \in \kappa \setminus E$. Then $h \in (h + W_{F_h, m}) \subset (Z^\kappa(f_n) \setminus U_g) \cup U_g = \emptyset$, which is a desired contradiction completing the inductive step.

After completing the inductive construction, consider the countable set $F_\omega = \bigcup_{n \in \omega} F_n$ and the function $f_\omega : F_\omega \to Z$ such that $f_\omega|F_n = f_n$ for all $n \in \omega$. Since the complement $\kappa \setminus F_\omega$ is not empty, the “cube”
Corollary 7.2. For every cardinal \( \kappa \geq \kappa \) the paratopological group \( \uparrow \mathbb{Z}^\kappa \) has countable pseudocharacter but fails to be submetrizable.

It is known \([27]\) that under Martin’s Axiom the space \( \mathbb{Z}^\kappa \) is \( \kappa \)-Baire for every cardinal \( \kappa < \kappa \). This fact combined with Theorem 7.6 implies the following MA-improvement of Corollary 7.2.

Corollary 7.3. Under Martin’s Axiom, for any uncountable cardinal \( \kappa \) the paratopological group \( \uparrow \mathbb{Z}^\kappa \) has countable pseudocharacter but fails to be submetrizable.

Problem 7.4. Can the space \( \uparrow \mathbb{Z}^\omega \) be submetrizable in some model of ZFC?

In Theorem 6.4 we proved that the paratopological group \( G = \uparrow \mathbb{Z}^\omega \) has \( d(G) \geq c(G) \geq \kappa \) and \( dc(G) = \omega \). By Propositions 1.3 and 1.10 \( \ell^{\omega_1}(G) = \ell^{\omega_1}(G) = dc(G) = \omega \). It would be interesting to know the values of some other cardinal characteristics of \( G \), intermediate between \( dc(G) \) and \( c(G) \).

Problem 7.5. For the paratopological group \( G = \uparrow \mathbb{Z}^\kappa \) calculate the values of cardinal characteristics \( \ell^{\pm n}(G) \), \( \ell^{\pm n}(G) \), \( \ell^{\pm n}(G) \) for all \( n \in \mathbb{N} \).

7.2. A submetrizable paratopological group whose quasi-Roecke uniformity has uncountable pseudocharacter. By Theorem 6.4 each Hausdorff paratopological group \( G \) has submetrizability number \( sm(G) \leq \psi(\mathbb{Q}) \). This inequality can be strict as shown by an example constructed in this subsection.

Given an uncountable cardinal \( \kappa \) in the paratopological group \( \uparrow \mathbb{Z}^\kappa \) consider the subgroup \( H = \{ f \in \uparrow \mathbb{Z}^\kappa : |\text{supp}(f)| < \omega \} \) consisting of functions \( f : \kappa \to \mathbb{Z} \) that have finite support \( \text{supp}(f) = \{ \alpha \in \kappa : f(\alpha) \neq 0 \} \). A neighborhood base of \( H \) at zero consists of the sets

\[ W_{F,m} = \{ h \in H : h \neq 0, h(\kappa) \in \{0\} \cup [m, \infty) \} \]

where \( F \) runs over finite subsets of \( \kappa \) and \( m \in \mathbb{N} \).

Theorem 7.6. For any uncountable cardinal \( \kappa \) the paratopological group \( H \) has the following properties:

1. \( H \) is a zero-dimensional (and hence regular) Hausdorff abelian paratopological group;
2. \( H \) is strongly \( \sigma \)-discrete and submetrizable;
3. \( iw(H) \cdot \omega = \log(\kappa) \);
4. \( \psi(H) = \chi(H) = \kappa \) but \( \psi(H) = \overline{\psi(H)} = \omega \);
5. \( \ell(Q) = \omega \) but \( \ell(L) = dc(H) = \kappa \).

Proof. The items (1), (2), (5) follow (or can be proved by analogy with) the corresponding items of Theorem 7.1.

(2)–(3): To see that the space \( H \) is strongly \( \sigma \)-discrete, write \( H = \bigcup_{n,m \in \mathbb{N}} H_{n,m} \) where \( H_{n,m} = \{ h \in \uparrow \mathbb{Z}^\kappa : |\text{supp}(h)| = n, \| h \| \leq m \} \) and \( \| h \| = \sup_{\alpha \in \kappa} |h(\alpha)| \). We claim that each set \( H_{n,m} \) is strongly discrete in \( \uparrow \mathbb{Z}^\kappa \).

To each function \( h \in H_{n,m} \) assign the neighborhood \( U_{h} = h + W_{\text{supp}(h),m+1} \). Given any two distinct functions \( g, h \in H_{n,m} \), we shall prove that \( U_{g} \cap U_{h} = \emptyset \). Assuming that \( U_{g} \cap U_{h} \) contains some function \( f \in H \), we would conclude that \( |\text{supp}(g)| = |\text{supp}(g)| \) and \( |\text{supp}(h)| = |\text{supp}(h)| \). So, \( g|\text{supp}(g)| = h|\text{supp}(h)| \) and \( g \neq h \) implies \( \text{supp}(g) \neq \text{supp}(h) \). Since \( |\text{supp}(g)| = |\text{supp}(h)| = n \), there is \( \alpha \in \text{supp}(g) \) \( \text{supp}(h) \) such that \( g(\alpha) \neq 0 = h(\alpha) \). Then \( f(\alpha) \in \{ g(\alpha) \} \cap \{ m+1, \infty \} = \emptyset \), which is a contradiction showing that the indexed family \( (U_{h})_{h \in H_{n,m}} \) is disjoint.

To show that this family \( (U_{h})_{h \in H_{n,m}} \) is discrete, for every function \( g \in H \setminus \bigcup_{h \in H_{n,m}} U_{h} \) consider its neighborhood \( U_{g} = g + W_{\text{supp}(g),m+1} \). We claim that \( U_{g} \cap U_{h} = \emptyset \) for every \( h \in H_{n,m} \). Assume conversely that for some \( h \in H_{n,m} \) the intersection \( U_{g} \cap U_{h} \) contains a function \( f \in H \). Then \( f|\text{supp}(g)| = g|\text{supp}(g)| \) and \( f|\text{supp}(h)| = h|\text{supp}(h)| \). If \( \text{supp}(h) \setminus \text{supp}(g) \neq \emptyset \), then we can find \( \alpha \in \text{supp}(h) \setminus \text{supp}(g) \) and conclude that \( f(\alpha) = h(\alpha) \neq 0 = g(\alpha) \) and hence \( f(\alpha) \in \{ g(\alpha) \} \cap \{ m+1, \infty \} = \emptyset \), which is a contradiction. So, \( \text{supp}(g) \setminus \text{supp}(h) \) and \( \text{supp}(h) \setminus \text{supp}(g) \). It follows from \( g \notin U_{h} \) that for some \( \alpha \in \kappa \setminus \text{supp}(h) \) we get \( g(\alpha) \notin \{0\} \cup \{m+1, \infty \} \). Then \( \alpha \in \text{supp}(g) \) and \( f(\alpha) = g(\alpha) \notin \{ m+1, \infty \} \). On the other hand, the inclusion \( f \in U_{h} \) and \( f(\alpha) = 0 = h(\alpha) \) implies \( f(\alpha) \in \{ m+1, \infty \} \). This contradiction completes the proof of the equality \( U_{g} \cap U_{h} = \emptyset \), which shows that the family \( (U_{h})_{h \in H_{n,m}} \) is discrete in \( H \) and the set \( H_{n,m} \) is strongly discrete in \( H \). Then the space \( H = \bigcup_{n,m \in \mathbb{N}} H_{n,m} \) is strongly \( \sigma \)-discrete. By Proposition 1.1 it is submetrizable and has \( i \)-weight \( iw(H) \cdot \omega = \log(|H|) = \log(\kappa) \).

- \( Z^\kappa(f_\omega) = \{ g \in Z^\kappa : g|Z_\omega = f_\omega \} \) contains two distinct functions \( f, g \). By the choice of the family \( (U_n)_{n \in \omega} \) there is a number \( n \in \omega \) such that no set \( U \in \mathcal{U} \) contains both points \( f \) and \( g \). On the other hand, by the inductive construction, \( f, g \in Z^\kappa(f_\omega) \subset Z^\kappa(f_n) \subset U_n \) for some set \( U_n \in \mathcal{U} \), which is a desired contradiction completing the proof of the theorem.

- \( \mathcal{U} \cdot h \subset U \cdot h \) for all \( \mathcal{U} \in \mathcal{U} \). We claim that each set \( U_{h} \in H_{n,m} \) is strongly \( \kappa \)-discrete. By Proposition 1.1 it is submetrizable and has \( i \)-weight \( iw(H) \cdot \omega = \log(|H|) = \log(\kappa) \).
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