Differentially Private Temporal Difference Learning with Stochastic Nonconvex-Strongly-Concave Optimization

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Abstract

Temporal difference (TD) learning is a widely used method to evaluate policies in reinforcement learning. While many TD learning methods have been developed in recent years, little attention has been paid to preserving privacy and most of the existing approaches might face the concerns of data privacy from users. To enable complex representational abilities of policies, in this paper, we consider preserving privacy in TD learning with nonlinear value function approximation. This is challenging because such a nonlinear problem is usually studied in the formulation of stochastic nonconvex-strongly-concave optimization to gain finite-sample analysis, which would require simultaneously preserving the privacy on primal and dual sides. To this end, we employ a momentum-based stochastic gradient descent ascent to achieve a single-timescale algorithm, and achieve a good trade-off between meaningful privacy and utility guarantees of both the primal and dual sides by perturbing the gradients on both sides using well-calibrated Gaussian noises. As a result, our DPTD algorithm could provide \((\epsilon, \delta)\)-differential privacy (DP) guarantee for the sensitive information encoded in transitions and retain the original power of TD learning, with the utility upper bounded by \(\tilde{O}\left(\left(\frac{d \log(1/\delta)}{n \epsilon}\right)^{3/8}\right)\), where \(n\) is the trajectory length and \(d\) is the dimension. Extensive experiments conducted in OpenAI Gym show the advantages of our proposed algorithm.

1 Introduction

Reinforcement learning (RL) has shown great success in a series of scenarios such as robot control tasks, planning tasks and games [22, 12, 24]. However, despite their superior empirical performance, most of these works do not consider privacy concerns regarding user data and many applications of RL algorithms are hindered due to data leakage [2]. As a motivating example, in medical research, users’ treatment records should remain confidential while RL policies are trained upon them. Without considering the data privacy, previous works have shown that the user historical information can be inferred by recursively interacting with the released policies [35].

Policy evaluation (PE), which aims to approximate a value function, is an essential step in many RL algorithms. For instance, in actor-critic [36], the resulting value function could be used to estimate the expected return of the states for a given policy, which can be further used in a policy improvement step. The first algorithm for PE achieving differential privacy (DP) is proposed by [2], which originates from Monte-Carlo methods. However, Monte-Carlo methods need a full trajectory before updating the estimation, which might be impractical when the task incurs a long trajectory for an episode.

Another classical PE method is the temporal difference (TD) learning [27], which allows incremental updates without using full trajectory information. To enable TD learning to approximate the value function well in large or continuous state space, function approximation is employed. A large amount of works [25, 26, 4, 23] focus on the analysis of TD learning with linear function approximation. To make the TD learning more effective in many RL tasks where the value function is more complex and can not be simply approximated by linear functions, Maei \textit{et al.} [17] build up the first framework for the analysis of TD learning with nonlinear value function approximation and a great number of advances [29, 21, 34] have been made for the effectiveness of

\textsuperscript{1}The tilde in this paper hides the log factor.
nonlinear TD learning. Though the effectiveness has been extensively studied, the importance of privacy in TD learning has long been ignored.

In this paper, we propose the first differentially private temporal difference (DPTD) learning algorithm to preserve privacy in TD learning with nonlinear value function approximation in the formulation of stochastic nonconvex-strongly-concave optimization. To analyze the sensitivity and achieve DP in nonlinear TD learning, we consider perturbing the gradients on both the primal and dual sides by injecting noise to the primal and dual sides simultaneously. However, different from canonical tasks of preserving privacy in stochastic gradient descent, devising such noises in the formulation of nonlinear TD learning is more challenging since the noises on the primal side will also suppress the convergence of the dual side and vice versa. We overcome this challenge by employing the momentum-based stochastic gradient descent ascent to achieve a single-timescale algorithm, which enables us to update the parameters of both the primal and dual sides with the learning rates of the same order. In this way, it is possible to preserve the privacy of both the primal and dual sides using noises with the same variances to avoid the large privacy cost. Finally, we perturb the gradients on primal and dual sides using Gaussian noises with the same and carefully chosen variances to efficiently preserve the privacy and make a good trade-off between the privacy and utility guarantees.

In summary, we make the following contributions.

- We propose the first TD learning method that achieves DP with nonlinear function approximation, named DPTD. We prove that our algorithm could protect the single state transition with $(\epsilon, \delta)$-DP guarantee.
- We prove that the utility of our algorithm is upper bounded by $\tilde{O}\left(\frac{d \log(1/\delta)}{\epsilon n^2}\right)$, where the tilde hides the log factor.
- We conduct extensive experiments in OpenAI Gym environments. The experimental results show clear improvements against previous approaches.

Notations Throughout this paper, we use $\| \cdot \|$ to denote the $\ell_2$ norm of the vectors and $(x, y)$ to denote the concatenation of two vectors $x$ and $y$. For a given set $\mathcal{X}$, let $\mathcal{P}_\mathcal{X}(\cdot)$ be the projection to the set $\mathcal{X}$. We denote $[n] = \{1, \cdots, n\}$ for $n \in \mathbb{N}^+$. Let $\tau$ represent a trajectory and $\xi_i = (s_i, a_i, s'_i)$ be the $i$-th state transition in a given trajectory.

2 Related Work

In this section, we present the works for studying TD learning and the recent advances in achieving DP in RL.

Temporal Difference Learning Policy evaluation (PE), which approximates the value function of a given policy, is a fundamental part of RL. One of the most widely used policy evaluation methods is temporal difference (TD) learning which is first proposed in [27] and aims to solve PE by minimizing the Bellman error. While most of the existing works focus on analyzing the convergence rate of TD learning with linear value function approximation [25, 26, 4, 23], nonlinear function approximation might be more preferable which can tackle the complex learning objectives in some complex tasks better. The most notable example might be using neural networks with nonlinearities to approximate the value functions. Maei et al. [17] present the first framework for TD learning with smooth nonlinear value functions. Wai et al. [29] reformulate the nonlinear TD learning as a primal-dual finite-sum optimization problem via Fenchel’s duality, where the primal side is nonconvex and the dual side is strongly-concave, and propose a TD learning method with variance reduction technique in the offline setting. Further, Qiu et al. [21] propose primal-dual online TD algorithms based on the variance reduction technique in the online setting.

Differential Privacy and Applications Differential privacy (DP) is first formally introduced by [8] which aims to provide rigorous privacy-preserving guarantee of the systems. In recent years, privacy-preserving machine learning algorithms have been extensively studied in empirical risk minimization (ERM) [33], deep learning (DL) [1] and RL [2].

We briefly discuss the DP RL algorithms. The first DP RL algorithm for PE is presented in [2], motivated by the protection of user records in medical research. However, their methods originate from Monte-Carlo methods, which require at least one full trajectory for updating the value function approximation once. Private
Q-learning algorithm is given by [30], achieving DP by protecting the reward function. Lebensold et al. [14] focus on how actor-critic methods perform when initialized with a privatized first-visit Monte-Carlo estimate in [2]. Vietri et al. [28] establish both the PAC and regret utility guarantees of an optimism-based private RL algorithm for episodic tabular MDPs.

3 Preliminaries

Before we formally present our algorithm, we first introduce PE, some definitions in DP and necessary assumptions. 

3.1 Policy Evaluation

In RL, a discounted Markov decision process (MDP) is denoted by a tuple $\mathcal{M} = (S, A, P, R, \gamma)$, where $S$ is the state space, $A$ is the action space, $P(\cdot|s,a)$ is the transition probability kernel, $R: S \times A \rightarrow \mathbb{R}$ is the reward function, and $\gamma$ is the discount factor. A policy $\pi$ takes state $s \in S$ as an input and gives a distribution over actions $A$.

We consider the PE problem, where the value function is learned for a policy. For a given policy $\pi$, the corresponding reward function is defined as $R(\pi) = \mathbb{E}_{s \sim \mathcal{D}} [R(s,a)]$ and the induced transition matrix is $P^{\pi}(s,s') = \int_A \pi(a|s)P(s,a,s')da$. The value function is defined as $V^{\pi}: S \rightarrow \mathbb{R}$ representing the long term expected discounted reward under the policy $\pi$, which is formally defined as

$$V^{\pi}(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t R^\pi(s_t) \mid s_0 = s, \pi \right].$$

To simplify the notations, we use $R^\pi, V^{\pi}$ through stacking up $R^\pi(s), V^{\pi}(s)$ for all $s$. By definition, $V^{\pi}$ satisfies the Bellman equation

$$V^{\pi} = R^{\pi} + \gamma P^{\pi} V^{\pi}.$$

Since the true value function is intractable, it is common to proceed the policy evaluation by minimizing the mean squared Bellman error (MSBE). We assume there exists a stationary distribution $\mu^{\pi}$ of the Markov chain induced by policy $\pi$. Let $D = \text{Diag}(\{\mu^{\pi}(s)\}_{s \in S})$. Then the MSBE could be formulated as

$$\text{MSBE} = \frac{1}{2} \| V^{\pi} - R^{\pi} - \gamma P^{\pi} V^{\pi} \|^2_D,$$

When $S$ is large or infinite, it is inefficient or even unrealistic to access $V^{\pi}$ through a tabular form and thus the function approximation is needed. In practice, however, we can not directly optimize the above objective as the approximated value functions usually lie in subspaces [24]. Thus a projection step is needed. We assume $V^{\pi}$ is parameterized by some parameter $\theta \in \mathbb{R}^d$ where $d$ is the dimension [24]. In the case where linear function approximation is used, i.e., $V^{\pi} = \Phi \theta$ with $\Phi \in \mathbb{R}^{|S| \times d}$ as the feature matrix, the projection $\Pi = \Phi(\Phi^\top D \Phi)^{-1} \Phi^\top D$ is well defined and well studied. For twice-differentiable nonlinear function approximation, [29] propose a general projected Bellman error (MSPBE) as follows

$$\text{MSPBE} = \frac{1}{2} \mathbb{E} \left[ \delta(s) \Psi(s)^\top \right] G^{-1}_\theta \mathbb{E} \left[ \delta(s) \Psi(s) \right], \quad (1)$$

where $V^{\pi}_{\theta}$ denotes the value function under policy $\pi$ parameterized by $\theta$, $\Psi(s) = \nabla_\theta V^{\pi}_{\theta}(s)$ is the gradient evaluated at state $s$, $G_\theta = \mathbb{E}_s \left[ \Psi(s) \Psi(s)^\top \right] \in \mathbb{R}^{d \times d}$, $\delta(s) = R^{\pi}(s) + \gamma P^{\pi} V^{\pi}_{\theta}(s') - V^{\pi}_{\theta}(s)$ is the TD error and the expectation is taken over $s \in S, a \sim \pi(\cdot|s), s' \sim P(s,a)$. Via the Fenchel’s duality that $\frac{1}{2} \| x \|^2_A - \frac{1}{2} y^\top A y, \text{the MSPBE minimization problem has a primal-dual formulation as}$

$$\min_{\theta \in \Theta} \text{MSPBE}(\theta)$$

$$= \min_{\theta \in \Theta} \max_{\omega \in \Omega} \left\{ L(\theta, \omega) := \mathbb{E}_{s,a,s'} \left[ \ell(\theta, \omega; s, a, s') \right] \right\}, \quad (2)$$

where

$$\ell(\theta, \omega; s, a, s') := \langle \delta(s) \Psi(s), \omega \rangle - \frac{1}{2} \omega^\top \left[ \Psi(s) \Psi(s)^\top \right] \omega.$$
More generally, let \( f(\theta, \omega; \xi) = \ell(\theta, \omega; s, a, s') \) and \( F(\theta, \omega) := \mathbb{E}_{\xi \sim \Xi} [f(\theta, \omega; \xi)] \). Then the original mini-max problem in Eq. (2) is transformed into the following form
\[
\min_{\theta \in \Theta} \text{MSPBE}(\theta) = \min_{\theta \in \Theta} \max_{\omega \in \Omega} F(\theta, \omega).
\] (3)

The difficulty of solving the above objective largely arises from the fact that it may be nonconvex in \( \Theta \) but concave in \( \Omega \). Like previous works, we need the following assumptions which are common in the field of nonconvex-strongly-concave primal-dual optimization [16, 21, 29] and DP ERM problem [33, 31, 32].

The first assumption guarantees the existence of a solution, which hence ensures the feasibility of the problem [16, 21].

**Assumption 3.1 (Existence of solutions).** The solution \( \theta^* = \arg\min_{\theta \in \Theta} \text{MSPBE}(\theta) \) exists. Let \( J(\theta) := \max_{\omega \in \Omega} F(\theta, \omega) \). We assume \( J(\theta^*) > -\infty \).

The next assumption is about continuity of the gradient, which holds when the parametric family of functions has bounded, smooth gradient and Hessian [21, 29]. Furthermore, this assumption implies that \( F(\theta, \cdot) \) and \( F(\cdot, \omega) \) are both \( L_F \)-Lipschitz smooth.

**Assumption 3.2 (Lipschitz continuity of \( \nabla F \)).** There exists some constant \( L_F > 0 \) such that for any \( \theta, \theta' \in \Theta \), \( \omega, \omega' \in \Omega \), the gradient \( \nabla F(\theta, \omega) = (\nabla_{\theta} F(\theta, \omega), \nabla_{\omega} F(\theta, \omega)) \) satisfies
\[
\| \nabla F(\theta, \omega) - \nabla F(\theta', \omega') \| \leq L_F \| (\theta, \omega) - (\theta', \omega') \|.
\]

The third assumption upper bounds the stochastic gradient, which is critical for bounding the sensitivity in the analysis of DP [33, 31].

**Assumption 3.3 (Stochastic G-Lipschitz).** For any \( \xi, (\omega, \theta) \) and \( (\omega', \theta') \), the stochastic function \( f \) satisfies
\[
\| f(\omega, \theta; \xi) - f(\omega', \theta'; \xi) \| \leq G \| (\omega, \theta) - (\omega', \theta') \|.
\]

The fourth assumption restricts the feasible sets of the parameter to be convex, which is common in TD learning [21, 23, 4].

**Assumption 3.4 (Convex sets).** The feasible sets \( \Theta \) for the primal variable \( \theta \) and \( \Omega \) for the dual variable \( \omega \) are closed convex sets.

The next assumption guarantees the existence and uniqueness of the solution \( \omega^* = \max_{\omega \in \Omega} F(\theta, \omega) \), for any fixed \( \theta \in \Theta \). It holds when \( G_\theta \) defined in Eq. (1) is positive definite [21, 29].

**Assumption 3.5 (Strong concavity).** For any given \( \theta \in \Theta \), the function \( F(\theta, \cdot) \) is \( \mu \)-strongly concave, i.e., \( \forall \omega, \omega' \in \Omega, F(\theta, \cdot) \) is concave and \( \| \nabla_{\omega} F(\theta, \omega) - \nabla_{\omega} F(\theta, \omega') \| \geq \mu \| \omega - \omega' \| \).

The last assumption assumes data is i.i.d. Though this assumption might be impractical for DP under state-action-state in Definition 3.3 since data points in a single trajectory might be correlated, it may hold more naturally with DP under trajectory in Definition 3.4. Moreover, this is standard in DP-relevant analysis [31, 33, 30].

**Assumption 3.6 (Sampling i.i.d. data).** For a given dataset \( S \), data points in \( S \) are independent and identical distributed (i.i.d.). Further, the algorithm samples the data points uniformly.

### 3.2 Differential Privacy

Two datasets \( X \) and \( X' \) are neighboring if they only differ in one data point. Then the DP is defined as follows.

**Definition 3.1 (\((\epsilon, \delta)\)-DP [8]).** A randomized mechanism \( \mathcal{M} : \mathcal{X} \to \mathcal{Y} \) satisfies \((\epsilon, \delta)\)-differential privacy if for any two neighbouring inputs \( X, X' \subseteq \mathcal{X} \) and any subset of outputs \( Y \subseteq \mathcal{Y} \), it holds that
\[
\mathbb{P}(\mathcal{M}(X) \in Y) \leq e^\epsilon \mathbb{P}(\mathcal{M}(X') \in Y) + \delta.
\]

To achieve \((\epsilon, \delta)\)-DP, we consider using Gaussian mechanism [7] which adds a \( d \)-dimensional Gaussian noise \( u \sim N(0, \sigma^2 I_d) \) to the output at time \( t \). The magnitude of the noise variance depends on the \( \ell_2 \)-sensitivity of the query function, which is formally defined in Definition B.1.

To analyze the mechanism of a sequence of randomized mechanisms more effectively, Rényi differential privacy (RDP) is proposed in [18] based on the Rényi divergence, which is a natural relaxation of DP.
Definition 3.2 ((α, ρ)-RDP [18]). A randomized mechanism \( M : \mathcal{X} \rightarrow \mathcal{Y} \) satisfies \((\alpha, \rho)\)-Rényi differential privacy if for any two neighbouring inputs \( X, X' \subseteq \mathcal{X} \) and any subset of outputs \( Y \subseteq \mathcal{Y} \), it holds that

\[
D_{\alpha}(M(X)||M(X')) := \frac{\log \mathbb{E}(M(X)/M(X'))^\alpha}{(\alpha - 1)} \leq \rho.
\]

When the RL algorithm is deployed online in applications such as recommender systems, sensitive user information is often encoded through experiences, i.e., the state-action-state triples. Our goal to protect the sensitive information in RL is realized by making the state-action-state triple approximately indistinguishable for attackers, which leads to our specification of neighboring datasets. This definition is applicable to our approach and other pure online RL algorithms.

For notational convenience, we use \( \xi_i = (s_i, a_i, s'_i) \) and \( \hat{\xi}_i = (\hat{s}_i, \hat{a}_i, \hat{s}'_i) \) to denote the state-action-state triples.

Definition 3.3 (DP under state-action-state). Let \( S = \{\xi_i\}_{i=1}^n \) and \( \hat{S} = \{\hat{\xi}_i\}_{i=1}^n \) be two trajectories of the same length. \( S \) and \( \hat{S} \) are neighbouring if there exists a unique \( i \in [n] \) such that \( \xi_i \neq \hat{\xi}_i \). If a randomized mechanism \( M \) is \((\epsilon, \delta)\)-DP under this definition of neighbourhood, this mechanism is \((\epsilon, \delta)\)-DP under state-action-state.

When the RL algorithm is deployed offline, the above definition may be insufficient to provide privacy guarantee since the setting where one trajectory composes a dataset is no longer feasible. In the case where the dataset is composed of multiple trajectories, we introduce a more general definition of DP under trajectory that allows at most one trajectory to differ in neighbouring datasets.

Definition 3.4 (DP under trajectory). Let \( S = \{\tau_i\}_{i=1}^m \) and \( \hat{S} = \{\hat{\tau}_i\}_{i=1}^m \) be two datasets consisting of \( m \) trajectories where \( \tau_i = \{\xi_{j_{i,j}}\}_{j=1}^{|\tau_i|} \) with \( |\tau_i| \leq n \) and \( \hat{\tau}_i = \{\hat{\xi}_{j_{i,j}}\}_{j=1}^{|\tau_i|} \) with \( |\tau_i| \leq n \). \( S \) and \( \hat{S} \) are neighbouring if there exists a unique \( i \in [m] \) such that \( \tau_i \neq \hat{\tau}_i \). If a randomized mechanism \( M \) is \((\epsilon, \delta)\)-DP under this definition of neighbourhood, this mechanism is \((\epsilon, \delta)\)-DP under trajectory.

4 Algorithm

We now present our algorithm, differentially private temporal difference learning (DPTD), detailed in Algorithm 1.

DPTD takes the adaptive step size \( \nu_t \), and the constant parameters \( \kappa, \eta, \alpha, \beta \) as the input. These constant parameters are used to adjust the step sizes when updating the primal and dual variables with the momentum-based gradient estimators. At each iteration, DPTD performs stochastic gradient descent and ascent of \( \theta_t \) and \( \omega_t \) respectively and then projects the updates to the feasible sets \( \Theta \) and \( \Omega \) (line 2). Then DPTD obtains \( \theta_{t+1} \) by taking a step from \( \theta_t \) to \( \tilde{\theta}_{t+1} \) with step size \( \nu_t \) and obtains \( \omega_{t+1} \) in the similar way (line 3). Then DPTD computes the stochastic momentum-based gradient estimator \( p'_{t+1} \) and \( d'_{t+1} \) (line 4), which are perturbed via the Gaussian noises with moderate variances to achieve DP (line 5).

One of the main technical challenges lie in controlling privacy noises for primal and dual sides simultaneously. The common two-timescale framework implies an imbalance of privacy noises on the two sides, hence leading to an inefficient convergence rate and an unnecessarily large privacy cost. To overcome this challenge, we employ a single-timescale framework via the momentum-based stochastic gradient descent ascent [21], which despite being more complicated to analyze the simultaneous descent dynamics, achieves desirable utility and privacy guarantees. The other key challenge is the choice of the variance of the Gaussian noises \( \sigma_{t+1} \), which is detailed in Section 5.

5 Theoretical Results

In this section, we provide the main theoretical results of privacy and utility with DP under state-action-state. The presentation and discussions with DP under trajectory is deferred to Appendix A.

5.1 Privacy Analysis

Theorem 5.1 provides a privacy guarantee in terms of DP under state-action-state for Algorithm 1.
Algorithm 1 Differentially Private Temporal Difference Learning

\textbf{Input}: $\nu_t > 0$, $\kappa > 0$, $\eta > 0$, $\alpha > 0$, $\beta > 0$, $\theta_0 \in \Theta$, $\omega_0 \in \Omega$.

\textbf{Initialize}:

\[ u^p_0 \sim N(0, \sigma^2_0 I_d), \quad p_0 = \nabla_{\theta} f(\theta_0, \omega_0; \xi_0) + u^p_0, \]
\[ u^d_0 \sim N(0, \sigma^2_0 I_d), \quad d_0 = \nabla_{\omega} f(\theta_0, \omega_0; \xi_0) + u^d_0. \]

1: \textbf{for} $t = 1, 2, \ldots$ \textbf{do}
2: \quad Perform stochastic gradient descent and ascent and project the updates to the feasible sets:
3: \quad \bar{\theta}_{t+1} = \mathcal{P}_\Theta (\theta_t - \kappa p_t), \quad \bar{\omega}_{t+1} = \mathcal{P}_\Omega (\omega_t + \eta d_t).
4: \quad Update the primal variable $\theta_{t+1}$ and dual variable $\omega_{t+1}$:
5: \quad $\theta_{t+1} = \theta_t + \nu_t (\bar{\theta}_{t+1} - \theta_t), \quad \omega_{t+1} = \omega_t + \nu_t (\bar{\omega}_{t+1} - \omega_t)$.
6: \quad Compute the momentum-based gradient estimator on primal side $p'_{t+1}$ and on dual side $d'_{t+1}$:
7: \quad $p'_{t+1} = (1 - \alpha \nu_t) p_t + \alpha \nu_t \nabla_{\theta} f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1})$, \quad $d'_{t+1} = (1 - \beta \nu_t) d_t + \beta \nu_t \nabla_{\omega} f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1})$.
8: \quad Draw the Gaussian noises with variance $\sigma_{t+1}$:
9: \quad $u^p_{t+1} \sim N(0, \sigma^2_{t+1} I_d), \quad u^d_{t+1} \sim N(0, \sigma^2_{t+1} I_d)$,
10: \quad and release the differentially private gradient estimator $p_{t+1}$, $d_{t+1}$:
11: \quad $p_{t+1} = p'_{t+1} + u^p_{t+1}, \quad d_{t+1} = d'_{t+1} + u^d_{t+1}$.

7: \textbf{Output}: $(\bar{\theta}, \bar{\omega})$ sampled uniformly at random from $\{(\theta_t, \omega_t)\}_{t=0}^{T-1}$.

\textbf{Theorem 5.1} (Privacy under state-action-state). \textit{Consider the DP defined in Definition 3.3. Under Assumption 3.3, 3.4, 3.6, given the total number of iterations $T$, for any $\delta > 0$ and the privacy budget $\epsilon$, Algorithm 1 satisfies $(\epsilon, \delta)$-DP under state-action-state with the variance}

\[ \sigma_t^2 = \frac{14G^2 T \alpha'}{n^2} \left( \frac{\log(1/\delta)}{\alpha - 1} \right) + \frac{14G^2 T \alpha'}{n^2 \beta \epsilon}, \quad \forall t \geq 0, \]

\text{where } \sigma^2 = \frac{\sigma_t^2}{1 + \sigma_t^2} \geq 0.7, \quad \alpha' = \frac{\log(1/\delta)}{(1-\beta')\beta'} + 1 \leq 2\sigma^2 \log(\frac{n}{\sigma^2(1+\sigma^2)})/3 + 1 \quad \text{and } \beta' \in (0, 1).

\textbf{Proof Sketch of Theorem 5.1.} Consider the randomized mechanisms on primal side induced by the update rule of the gradient estimator in Algorithm 1

\[ \mathcal{M}^p_t = \begin{cases} \nabla_{\theta} f(\theta_0, \omega_0; \xi_0) + u^p_0 & t = 0 \\ (1 - \alpha \nu_{t-1}) p_{t-1} + \alpha \nu_{t-1} \nabla_{\theta} f(\theta_t, \omega_t; \xi_t) + u^p_t & t > 0. \end{cases} \]

We show $\mathcal{M}^p_t$ satisfies RDP and the privacy guarantee of DP could be transformed from privacy guarantee of RDP using Lemma B.2. Notice that $\mathcal{M}^p_t$ is the composition of a series of randomized mechanisms $(\mathcal{G}^p_t, \ldots, \mathcal{G}^p_1)$. 

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It remains to show that $G^p_t$ achieves RDP so as to show $M^p_t$ achieves RDP by Lemma B.1. To this end, in the case when $t = 0$, we first consider the Gaussian mechanism $G^p_0 = \sum_{t=0}^{n-1} \nabla \theta f(\theta, \omega_0; \xi_t) + u^p_0$ which takes the whole trajectory $\tau$ as the input instead of one state transition $\xi_0$ of $\tau$. Gaussian mechanism $G^p_0$ consists of the Gaussian noise $u^p_0$ and the query $\tilde{G}^p_0(S) = \sum_{t=0}^{n-1} \nabla \theta f(\theta, \omega_0; \xi_t)$ whose $\ell_2$-sensitivity could be shown to satisfy $\Delta^p_0 \leq 2G$. Thus $G^p_T$ and $G^p_0$ satisfy RDP by Lemma B.3 if the variance of Gaussian noise $u^p_0$ takes the value as suggested in Theorem 5.1. In the similar manner, we can prove that $G^p_t$ satisfies RDP for the case $t > 0$. The proof sketch of the randomized mechanisms on the dual side is similar to that of the primal side. □

## 5.2 Utility Analysis

We first introduce the utility metric to measure the nonconvex-strongly-concave optimization of TD learning and then present the utility analysis of our algorithm.

**Utility Metric** To simultaneously measure the convergence on the primal and dual sides of our algorithm, we adopt the following metric to measure the utility and similar metrics are also adopted in the previous works [29, 11, 21], which is

$$\mathcal{M}(\theta_t, \omega_t) := \kappa^{-1} \| \hat{\theta}_{t+1} - \theta_t \| + \| \nabla \theta F(\theta_t, \omega_t) - p_t \| + L_F \| \omega_t - \omega^*(\theta_t) \| .$$  \hspace{1cm} (4)

The first two terms of RHS in Eq. (4) are used to measure the convergence of the primal variable $\theta$. If the first two terms $\kappa^{-1} \| \hat{\theta}_{t+1} - \theta_t \| + \| \nabla \theta F(\theta_t, \omega_t) - p_t \| \approx 0$, then $\nabla \theta F(\theta_t, \omega_t) \approx p_t$ and $\hat{\theta}_{t+1} \approx \theta_t$, which further indicates that $\theta_{t+1} = P_{\theta}(\theta_t - \kappa p_t) \approx P_{\theta}(\theta_t - \kappa \nabla \theta F(\theta_t, \omega_t)) \approx \theta_t$ due to the update rules in Algorithm 1. In this circumstance, $\theta_t$ will be a stationary point if $\nabla \theta F(\theta_t, \omega_t) = 0$ and a local minimizer on the boundary of $\Theta$ otherwise. In either situation, $\theta_t$ could be considered convergent in constrained nonconvex optimization [11, 21]. The convergence of $\omega_t$ to the optimal maximizer $\omega^*(\theta_t)$ is measured by the third term of RHS in Eq. (4).

Under this metric, we present the utility under state-action-state achieved by our algorithm in the following theorem, whose proof is deferred to C, with the specified Gaussian noises in Theorem 5.1.

**Theorem 5.2 (Utility under state-action-state).** Under Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, if we set the parameters $\alpha = \beta = 3, 0 < \eta \leq \mu/(4L^2_{\theta}), 0 < \kappa \leq \eta \mu^2/(9L^2_{\theta}), \nu_t = 1/4(t+b) \frac{\delta}{\Delta} \geq \max\{2kL^2_{\theta}/\mu^2, 3\}$ and choose the number of iterations $T = \sqrt{\frac{C\nu}{\eta \mu \log(1/\delta)}}$ where $C$ is a constant, then with the Gaussian noises in Theorem 5.1, the output of Algorithm 1 satisfies

$$\mathbb{E} \| \mathcal{M}(\tilde{\theta}, \tilde{\omega}) \| \leq \tilde{O} \left( \frac{(d \log(1/\delta))^{\frac{1}{2}}}{(n\epsilon)^{\frac{1}{4}}} \right).$$

Moreover, the total gradient complexity of Algorithm 1 is $2(T + 1) = O \left( \frac{n\epsilon}{\sqrt{d \log(1/\delta)}} \right).$
Figure 1: Compare DPTD with different algorithms (DPGLD, DPSRM, SGD, TD) on Cart Pole (a), Acrobot (b) and Atari 2600 Pong (c). Figure (a), (b) and (c) show the value of objective function versus the number of epochs. Each epoch has 5 finite trajectories. The shadow denotes 1-std. The learning curves are averaged over 10 random seeds and are generated without smoothing.

6 Experiments

To validate the effectiveness of our algorithm, we conduct comprehensive empirical evaluations and present the experiment results in this section.

6.1 Setting

We justify our proposed algorithms empirically through classical control tasks: Cart Pole [3], Acrobot [9] and Atari 2600 Pong in OpenAI Gym [5] environments. All the algorithms are evaluated with data generated from Sarsa for Cart Pole and Acrobot and DQN for Atari. To ensure that the generated trajectories are of good quality, we sample 5 trajectories for each environment.

6.2 Baselines

Since our algorithm is the first differentially private temporal difference method, we have no relevant TD algorithms which can also achieve DP to compare. Thus, we evaluate DPTD against several baseline methods in the DP ERM literature including differentially private gradient Langevin dynamics (DPGLD) [32], and differentially private stochastic recursive momentum (DPSRM) [33]. To study the utility where there is no need to achieve DP and thus no need to inject noises, we also include the non-private TD and stochastic gradient descent (SGD) [10, 19] as our baselines, which are not injected by any noise and thus there are no privacy guarantees of them. Though DPGLD, DPSRM, and SGD are designed for solving nonconvex optimization problems instead of nonconvex-strongly-concave primal-dual optimization problems, for a fair comparison, we also implement these baselines in the primal-dual form for comparing their performance with DPTD. Specifically, at each iteration, these algorithms are implemented to take a gradient descent step to minimize the objective function (i.e., Eq. (2)) on the primal side and simultaneously take a gradient ascent step to maximize the objective function on the dual side. The value functions of all the algorithms are parameterized by a two-layer fully-connected neural network with 50 hidden neurons and ELU activation function [6]. Other implementation details are deferred to Appendix G.

6.3 Results and Analysis

We report the experiment results in terms of utility in Figure 1, where the y-axis indicates the value of $\mathcal{L}(\theta, \omega)$ in Eq. (2) in the optimization process. The following conclusions are drawn in order. First, we observe that DPSRM, DPGLD and SGD can not converge well in all the three tasks, even though the gradients in SGD are not perturbed, since these methods are not able to leverage the property of the primal-dual optimization problem inherently. In particular, one can see that the performance of DPSRM, DPGLD and SGD degrades heavily in Figure 1 (c), perhaps due to the high-dimensional state space and the increasing complexity of the policies in the Atari task. Furthermore, DPTD converges faster in three tasks compared to DPSRM, DPGLD and SGD, which shows that DPTD has a better utility. Finally, TD without injected by any noises has the best utility compared
to all the other methods in three tasks, whose values of Eq. (2) converge to 0 rapidly. This is reasonable since TD is a non-private version of our algorithm.

Furthermore, to study the impact of different privacy budgets on convergence, we conduct experiments to show the utility of DPTD with varying $\epsilon$ and report the experiment results in Figure 2, deferred to Appendix H. One can see that as the privacy parameter $\epsilon$ decreases from 100.0 to 0.1, the variance of Gaussian noises increases and the performance of DPTD begins to degrade, matching our theoretical analysis.

7 Conclusions

In this paper, we make the first step to develop an efficient algorithm for differentially private primal-dual temporal difference (TD) learning, which protects the critical state transitions in reinforcement learning (RL) so as to make two neighboring trajectories indistinguishable and simultaneously achieve fast convergence rate. We also show that our algorithm can achieve differential privacy (DP) with a bounded utility under the case where the full trajectory needs to be protected. The privacy guarantee and the utility guarantee of our algorithm are validated by both the rigorous theoretical analysis and comprehensive experiments conducted in three OpenAI Gym environments. In our future work, we are interested in how to simultaneously achieve DP and keep a fast convergence rate of TD learning with nonlinear smooth function approximation under Markovian sampling.

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A Discussion on Privacy and Utility under Trajectory

A.1 Privacy and Utility Analysis

While our algorithm runs online with the utility and the privacy under state-action-state as discussed before, a more general definition is required for the offline setting where multiple trajectories are presented in one dataset, which motivates us to consider DP under trajectory in Definition 3.4.

Another motivation of DP under trajectory is about the assumption, including two aspects. From the first aspect, making data points in one trajectory independent and identically distributed (i.e., Assumption 3.6) cannot be easily satisfied in practice, caused by the property of the Markov chain. However, sampling different trajectories independently under the identical distribution can be achieved for less dependency between trajectories. From the second aspect, constraining the stochastic gradients (i.e., Assumption 3.3) is not necessary for DP under trajectory and this assumption can be replaced by a weaker one, shown below.

The following assumption gives a weaker version of Assumption 3.3, helping bound the averaged stochastic gradients. This assumption is a necessary but not sufficient condition for Assumption 3.3.

**Assumption A.1 (Averaged G-Lipschitz).** Given a full trajectory $\tau$ with length $|\tau| \leq n$, $\forall \theta \in \Theta$ and $\forall \omega \in \Omega$, 
\[
\left\| \sum_{i=0}^{\left\lfloor |\tau|-1 \right\rfloor} \nabla_{\theta} f(\theta, \omega; \xi_i) \right\| \leq nG , \quad \left\| \sum_{i=0}^{\left\lfloor |\tau|-1 \right\rfloor} \nabla_{\omega} f(\theta, \omega; \xi_i) \right\| \leq nG ,
\]
holds for some $G > 0$.

Armed with the above assumption, the theorems providing the privacy and the utility guarantee under trajectory are given as follows.

**Theorem A.1 (Privacy under trajectory).** Consider the DP defined in Definition 3.4. Under Assumption 3.4, 3.6, A.1, given the total number of iterations $T$, for any $\delta > 0$ and the privacy budget $\epsilon$, Algorithm 1 satisfies $(\epsilon, \delta)$-DP under trajectory with the variance
\[
\sigma_t^2 = \frac{14n^2G^2T\alpha'}{m^2 \left( \epsilon - \frac{\log(1/\delta)}{\alpha' - 1} \right)} , \quad \forall t \geq 0 ,
\]
where $\sigma^2 = \frac{\sigma_t^2}{4n^2G^2} \geq 0.7$, $\alpha' = \frac{\log(1/\delta)}{(1-\beta')\epsilon} + 1 \leq 2\sigma^2 \log \left( \frac{n}{\sigma'(1+\sigma^2)} \right)^{3} + 1 , \beta' \in (0, 1)$, $n$ is the maximum trajectory length and $m$ is the number of trajectories.

Theorem A.1 gives the privacy guarantee of Algorithm 1 under trajectory. One can see that the variance of Gaussian noises under trajectory grows as $n$ increases. It is reasonable since in DP under trajectory it requires to protect the privacy of two trajectories which have $n$ different state-action-state triples in the worst-case scenario.

The utility under trajectory of our algorithm is presented in the following theorem.

**Theorem A.2 (Utility under trajectory).** Under Assumption 3.1, 3.2, 3.4, 3.5, 3.6, A.1, if we set the parameters $\alpha = \beta = 3, 0 < \eta \leq \mu/(4L^2_P), 0 < \kappa \leq \eta \nu^2/(9L^2_P), \nu_i = 1/4(t+b)^{1/2}$ with $b \geq \max \{(2\kappa L^2_P/\mu)^2, 3\}$ and choose the number of iterations $T = Cn^\frac{Cme}{n\sqrt{d\log(1/\delta)}}$ where $C$ is a constant, then with the Gaussian variance in Theorem A.1, the output of Algorithm 1 satisfies the following
\[
\mathbb{E} \left\| \mathcal{M}(\bar{\theta}, \bar{\omega}) \right\| \leq \tilde{O} \left( \frac{n^{2}(d\log(1/\delta))^{1/2}}{(me)^{1/2}} \right).
\]
Moreover, the total gradient complexity of Algorithm 1 is $2(T + 1) = \mathcal{O} \left( \frac{me}{n\sqrt{d\log(1/\delta)}} \right)$.

Compared to the utility upper bound under state-action-state $\tilde{O} \left( \frac{(d\log(1/\delta))^{1/2}}{(n\sigma)^{1/2}} \right)$, the utility upper bound under trajectory is worse by a factor of $\mathcal{O} \left( \frac{n^{2}}{(me)^{1/2}} \right)$ since larger dependence on $n$ of variance of Gaussian noises is needed to protect the privacy under trajectory than under state-action-state. However, the utility under trajectory will still be acceptable if $m$ is larger than $n$, which is possible in practice due to the sample inefficiency of RL algorithms [13].
A.2 Comparisons with DP over Initial Visitation Estimate

Balle et al. [2] consider preserving the privacy in the definition of DP over initial visitation estimate, where they strictly restrict that two different trajectories can only differ in one state transition in two neighboring datasets. However, DP under trajectory in Definition 3.4 allows that two different trajectories can differ in at most \( n \) transitions in two neighboring datasets. Furthermore, they aim to preserve privacy in PE with linear function approximation, while we consider preserving privacy in PE with nonlinear function approximation.

B Proof of Theorem 5.1

The formal definition of \( \ell_2 \)-sensitivity is given as follows.

**Definition B.1** (\( \ell_2 \)-sensitivity [7]). The \( \ell_2 \)-sensitivity \( \Delta(g) \) of a function \( g \) is defined as \( \Delta(g) = \sup_{X,X'} \| g(X) - g(X') \| \), for any two neighboring datasets \( X \subseteq X' \subseteq X' \).

Before proving Theorem 5.1, we first present the following auxiliary lemmas. Lemma B.1 shows that the mechanism satisfies RDP if this mechanism is a composition of a series of mechanisms which satisfy RDP.

**Lemma B.1** ([18]). If \( k \) randomized mechanisms \( \mathcal{M}_i : X \to Y \) for \( i \in [k] \), satisfy \((\alpha, \rho_i)\)-RDP, then their composition \((\mathcal{M}_1(X), \cdots, \mathcal{M}_k(X))\) satisfies \((\alpha, \sum_{i=1}^k \rho_i)\)-RDP for \( X \subseteq X \). Moreover, the input of \( i \)-th mechanism can base on the outputs of previous \((i - 1)\) mechanisms.

Based on Lemma B.2, one can establish a DP privacy guarantee of one mechanism by leveraging the privacy guarantee in terms of RDP.

**Lemma B.2** ([18]). If a randomized mechanism \( \mathcal{M} : X \to Y \) satisfies \((\alpha, \rho)\)-RDP, then \( \mathcal{M} \) satisfies \((\rho + \log(1/\delta))/(\alpha - 1), \delta\)-DP for all \( \delta \in (0, 1) \).

In the online setting, it is unrealistic to access all the samples in a dataset via one query. For instance, the agent needs to update the approximation of the value function after the agent experiences a new state-action-state pair in TD learning. If a mechanism works under the samples that are subsampled from the whole dataset instead of the whole dataset, this mechanism is considered to use subsampling. Lemma B.3 can transform the RDP privacy guarantee for a mechanism without subsampling to the RDP privacy guarantee for the mechanism using uniform subsampling.

**Lemma B.3** ([33]). Given a function \( q : S^n \to \mathbb{R} \), then Gaussian Mechanism \( \mathcal{M} = q(S) + u \), where \( u \sim N(0, \sigma^2I) \), satisfies \((\alpha, \alpha \Delta^2(q)/(2\sigma^2))\)-RDP. In addition, if we apply the mechanism \( \mathcal{M} \) to a subset of samples using uniform sampling without replacement with sampling rate \( \tau \), \( \mathcal{M} \) satisfies \((\alpha, 3.5 \tau^2 \Delta^2(q)/\sigma^2)\)-RDP given \( \sigma^2 = \sigma^2/\Delta^2(q) \geq 0.7, \alpha \leq 2\sigma^2 \log(1/\tau \alpha(1 + \sigma^2))/3 + 1 \).

In our main proof of privacy guarantee, we will first prove that our Algorithm 1 satisfies RDP based on Lemma B.3 and Lemma B.1. Then we show that Algorithm 1 satisfies DP using Lemma B.2.

**Proof of Theorem 5.1.** Let \( \mathcal{M}^p_t \) and \( \mathcal{M}^d_t \) be the privacy protection mechanisms on primal side and dual side respectively at the \( t \)-th iteration constructed by the update rules in in Algorithm 1, i.e.,

\[
\mathcal{M}^p_t = \begin{cases} 
(1 - \alpha \nu_{t-1}) p_{t-1} + \alpha \nu_{t-1} \nabla_\theta f(\theta_t, \omega_t; \xi_t) + u^p_t, & t > 0 \\
\nabla_\theta f(\theta_0, \omega; \xi_0) + u^p_0, & t = 0
\end{cases}
\]

and

\[
\mathcal{M}^d_t = \begin{cases} 
(1 - \beta \nu_{t-1}) d_{t-1} + \beta \nu_{t-1} \nabla_\omega f(\theta_t, \omega_t; \xi_t) + u^d_t, & t > 0 \\
\nabla_\omega f(\theta_0, \omega; \xi_0) + u^d_0, & t = 0
\end{cases}
\]

We first show the mechanism on primal side \( \mathcal{M}^p_t \) satisfies the privacy guarantee for \( t = 0, 1, 2, \cdots, T - 1 \).

**Case (a).** If \( t = 0 \), we have

\[
\mathcal{M}^p_0 = \nabla_\theta f(\theta_0, \omega_0; \xi_0) + u^p_0.
\]

Therefore, we first consider the following Gaussian mechanism

\[
\mathcal{G}^p_0 = \nabla_\theta f(\theta_0, \omega_0; \xi_0) + u^p_0,
\]

\[
\mathcal{M}^d_T = \nabla_\omega f(\theta_T, \omega_T; \xi_T) + u^d_T.
\]

Therefore, we first consider the following Gaussian mechanism

\[
\mathcal{G}^d_T = \nabla_\omega f(\theta_T, \omega_T; \xi_T) + u^d_T.
\]

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where \( u_0^p \sim N(0, \sigma_0^2 I_d) \), \( u_0^d \sim N(0, \sigma_d^2 I_d) \). Note that \( G_0^p \) is based on the subsampling. Hence we will first consider the mechanisms without subsampling and get the final RDP by using Lemma B.3. Specifically, we consider the following Gaussian mechanism without subsampling

\[
G_0^p = \sum_{i=0}^{n-1} \nabla f(\theta_0, \omega_0; \xi_i) + u_0^p.
\]

**Sensitivity.** Consider the query on the trajectory \( \tau \) in \( S \) as follows

\[
\hat{q}_0^p(S) = \sum_{i=0}^{n-1} \nabla f(\theta_0, \omega_0; \xi_i).
\]

Similarly, we can get \( \hat{q}_0^p(S') \) where \( S' \) is one of \( S \)'s neighbouring datasets as defined in Definition 3.3. Thus, we have

\[
\hat{q}_0^p(S) - \hat{q}_0^p(S') = \nabla f(\theta_0, \omega_0; \xi_i) - \nabla f(\theta_0, \omega_0; \hat{\xi}_i).
\]

Then Assumption 3.3 implies that

\[
\hat{\Delta}_0^p = \| \nabla f(\theta_0, \omega_0; \xi_i) - \nabla f(\theta_0, \omega_0; \hat{\xi}_i) \| \leq 2G.
\]

**Privacy guarantee of \( G_0^p \).** By Lemma B.3, if the Gaussian noise \( u_0^p \) has the following variance

\[
\sigma_0^2 = \frac{14G^2 T \alpha'}{n^2 \left( \epsilon - \frac{\log(1/\delta)}{\alpha'} \right)},
\]

where \( \sigma^2 = \frac{\sigma_d^2}{4G^2} \geq 0.7 \) and \( \alpha' \leq 2\sigma^2 \log(\frac{n}{\alpha'(1+\sigma^2)})/3 + 1 \), then \( G_0^p \) will satisfy \( (\alpha', \frac{14G^2 \alpha^2}{n^2\sigma_d^2}) \)-RDP.

**Case (b).** If \( t > 0 \), we have

\[
M_t^p = (1 - \alpha u_{t-1}) p_{t-1} + \alpha u_{t-1} \nabla f(\theta_t, \omega_t; \xi_i) + u_t^p,
\]

which suggests us considering the following Gaussian mechanism

\[
G_t^p = \alpha u_{t-1} \nabla f(\theta_t, \omega_t; \xi_i) + u_t^p.
\]

Since the mechanism \( G_t^p \) uses subsampling, we first consider the following mechanism on the whole dataset without subsampling

\[
\tilde{G}_t^p(S) = \alpha u_{t-1} \sum_{i=0}^{n-1} \nabla f(\theta_t, \omega_t; \xi_i) + u_t^p.
\]

**Sensitivity.** Consider the following query without subsampling on the whole dataset

\[
\tilde{q}_t^p = \alpha u_{t-1} \sum_{i=0}^{n-1} \nabla f(\theta_t, \omega_t; \xi_i).
\]

Similarly, we can get \( \tilde{q}_t^p(S') \). Thus, we have

\[
\tilde{q}_t^p(S) - \tilde{q}_t^p(S') = \alpha u_{t-1} \left( \nabla f(\theta_t, \omega_t; \xi_i) - \nabla f(\theta_t, \omega_t; \hat{\xi}_i) \right).
\]

Then we can obtain the \( \ell_2 \)-sensitivity of the query \( \tilde{q}_t^p \) as follows

\[
\tilde{\Delta}_t^p = \| \alpha u_{t-1} (\nabla f(\theta_t, \omega_t; \xi_i) - \nabla f(\theta_t, \omega_t; \hat{\xi}_i)) \| \leq 2\alpha u_{t-1} G \leq 2G,
\]

where the first inequality comes from Assumption 3.3 and \( \alpha u_{t-1} \leq 1 \).
Updating rules shown in Algorithm 1, we have

\[ \sigma_t^2 = \frac{14G^2T \alpha'}{n^2 \left( \epsilon - \log(1/\delta) \right)}, \]  

(7)

where \( \sigma^2 = \frac{\sigma^2}{G^2} \geq 0.7, \alpha' = \frac{\log(1/\delta)}{(1-\alpha')^2} + 1 \leq 2\sigma^2 \log \left( \frac{n}{\alpha'(1+\sigma^2)} \right)/3 + 1 \) and \( \beta' \in (0, 1) \), then the mechanism \( G_t^{\alpha'} \) will satisfy \( \left( \alpha', \frac{14\alpha'G^2}{n^2\sigma_t^2} \right) \)-RDP.

Privacy guarantee of \( M_t^{\alpha'} \). By the definition of \( M_t^{\alpha'} \) in Eq. (5), \( M_t^{\alpha'} \) is composed of several Gaussian mechanisms, i.e., \( M_t^{\alpha'} = (G_0^{\alpha'}, \ldots, G_n^{\alpha'}) \). Then Lemma B.1 implies that \( M_t^{\alpha'} \) satisfies \( \left( \alpha', \sum_{i=0}^{T} \frac{14\alpha'G^2}{n^2\sigma_i^2} \right) \)-RDP. Thus the output on the primal side satisfies \( \left( \alpha', \sum_{i=0}^{T} \frac{14\alpha'G^2}{n^2\sigma_i^2} \right) \)-RDP. Finally, by using Lemma B.2, we transform RDP to DP and thus the output satisfies

\[ \left( \sum_{i=0}^{T} \left( \frac{14\alpha'G^2}{n^2\sigma_i^2} \right) + \frac{\log(1/\delta)}{\alpha' - 1}, \delta \right) \)-DP.

Substituting the value of \( \sigma_t \) in Eq. (7) simplifies the above result to \( (\epsilon, \delta) \)-DP. The proof of privacy guarantee of the dual side is similar to that of the primal side and is omitted here. \( \square \)

C Proof of Theorem 5.2

In this section, we provide the proof of Theorem 5.2, which gives the utility of Algorithm 1 with \((\epsilon, \delta)\)-DP under state-action-state-to. To this end, we first introduce the following lemmas.

Lemma C.1 ([15]). Under Assumptions 3.2, 3.5, the mapping \( \omega^*(\theta) = \arg\max_{\omega \in \Theta} F(\theta, \omega) \) is Lipschitz continuous, which is

\[ \| \omega^*(\theta) - \omega^*(\theta') \| \leq L_{\omega} \| \theta - \theta' \|, \quad \forall \theta, \theta' \in \Theta \]

where the Lipschitz constant is \( L_{\omega} = \frac{K\mu}{\mu} \).

Lemma C.2 ([21]). Under Assumptions 3.2, 3.4, 3.5, letting \( 0 < \kappa \nu_t \leq \mu/(16L_F^2) \) and \( \nu_t \leq 1 \), with the updating rules shown in Algorithm 1, we have

\[ J(\theta_{t+1}) - J(\theta_t) \leq -\frac{3\eta_t}{4\kappa} \| \tilde{\theta}_{t+1} - \theta_t \|^2 + 2L_F^2\kappa \nu_t \| \omega_t - \omega^*(\theta_t) \|^2 + 4\kappa \nu_t \| \nabla F(\theta_t, \omega_t) - p_t \|^2, \]

where \( J(\theta) = \max_{\omega \in \Omega} F(\theta, \omega) \) and \( \omega^*(\theta) := \arg\max_{\omega \in \Omega} F(\theta, \omega) \).

Lemma C.3 ([21]). Under Assumptions 3.2, 3.4, 3.5, letting \( 0 < \nu_t \leq 1/8 \) and \( 0 < \eta \leq (4L_F)^{-1} \), with the updating rules shown in Algorithm 1, we have

\[ \| \omega_{t+1} - \omega^*(\theta_t) \|^2 \leq \left( 1 - \frac{\nu_t \mu}{2} \right) \| \omega_t - \omega^*(\theta_t) \|^2 - \frac{3\nu_t}{4} \| \tilde{\omega}_{t+1} - \omega_t \|^2 + \frac{4\eta_t}{\mu} \| \nabla F(\theta_t, \omega_t) - d_t \|^2, \]

where \( \omega^*(\theta_t) = \arg\max_{\omega \in \Omega} F(\theta_t, \omega) \).

Lemma C.4 ([21]). Under Assumptions 3.2, 3.4, 3.5, letting \( 0 < \nu_t \leq 1/8 \) and \( 0 < \eta \leq (4L_F)^{-1} \), with the updating rules shown in Algorithm 1, we have

\[ \| \omega_{t+1} - \omega^*(\theta_{t+1}) \|^2 \leq \left( 1 - \frac{\mu \nu_t}{4} \right) \| \omega_t - \omega^*(\theta_t) \|^2 - \frac{3\nu_t}{4} \| \tilde{\omega}_{t+1} - \omega_t \|^2 + \frac{75\eta_t \mu}{16\mu} \| d_t - \nabla F(\theta_t, \omega_t) \|^2 \]

\[ + \frac{75L_F^2 \nu_t}{16\mu \eta} \| \tilde{\theta}_{t+1} - \theta_t \|^2, \]

where \( \omega^*(\theta_t) = \arg\max_{\omega \in \Omega} F(\theta_t, \omega) \) and \( \omega^*(\theta_{t+1}) = \arg\max_{\omega \in \Omega} F(\theta_{t+1}, \omega) \).
Lemma C.5 (Bounded variance). Under Assumption 3.3, the variance of the stochastic gradient $\nabla f(\theta, \omega; \xi) = (\nabla_\theta f(\theta, \omega; \xi), \nabla_\omega f(\theta, \omega; \xi))$ is bounded as $E_{\xi \sim \zeta}||\nabla f(\theta, \omega; \xi) - \nabla F(\theta, \omega)||^2 \leq \sigma^2$, where $\sigma^2 = 2G^2$.

Lemma C.5 shows that the stochastic gradient $\nabla f(\theta, \omega; \xi)$ is bounded by a constant $\sigma^2$, related to the property of function $f$ and $F$. Armed with Lemma C.5, the following lemma further bounds the variances of the gradient estimators on the primal side and the dual side and the detailed proof is deferred to Appendix F.

Lemma C.6 (With bounded variance). Under Assumptions 3.2, 3.4, 3.5, letting $0 < \nu_t \leq (8\alpha)^{-1}$ and $0 < \eta \leq (4L_F)^{-1}$, with the updating rules shown in Algorithm 1, we have

\[
E \|\nabla_\theta F(\theta_{t+1}, \omega_{t+1}) - p_{t+1}\|^2 \leq (1 - \alpha \nu_t) E \|\nabla_\theta F(\theta_t, \omega_t) - p_t\|^2 + \frac{9\nu_t L_F^2}{8\alpha} E \left( \|\bar{\theta}_{t+1} - \theta_t\|^2 + \|\bar{\omega}_{t+1} - \omega_t\|^2 \right) \\
+ \alpha^2 \nu_t^2 \sigma^2 + d\sigma_t^2,
\]

and

\[
E \|\nabla_\omega F(\theta_{t+1}, \omega_{t+1}) - d_{t+1}\|^2 \leq (1 - \alpha \nu_t) E \|\nabla_\omega F(\theta_t, \omega_t) - d_t\|^2 + \frac{9\nu_t L_F^2}{8\alpha} E \left( \|\bar{\theta}_{t+1} - \theta_t\|^2 + \|\bar{\omega}_{t+1} - \omega_t\|^2 \right) \\
+ \alpha^2 \nu_t^2 \sigma^2 + d\sigma_t^2,
\]

where $\sigma^2 = 2G^2$.

Proof of Theorem 5.2. Recall the step size is chosen as $\nu_t = \frac{a}{(t+\delta)^2}$ with $a = \frac{1}{16}$ in Theorem 5.2. By Assumption 3.2 and Assumption 3.5, it is clear that $L_F \geq \mu$. The parameter $\eta$ and $\nu_t$ in Theorem 5.2 could be further bounded as

\[
\eta \leq \frac{\mu}{4L_F} \leq \frac{1}{4L_F}
\]

and

\[
\nu_t \leq \frac{a}{b\delta^2} \leq \min \left\{ \frac{1}{2\delta}, \frac{\mu}{16\kappa L_F^2} \right\}
\]

Thus, with such parameter settings, we are able to apply Lemmas C.2, C.3 and C.4 in the following proof. By Lemma C.2, we have

\[
J(\theta_{t+1}) - J(\theta_t) \leq -\frac{3\nu_t}{4\kappa} \|\bar{\theta}_{t+1} - \theta_t\|^2 + 2L_F^2 \kappa \nu_t \|\omega_t - \omega(\theta_t)^\ast\|^2 \\
+ 4\kappa \nu_t \|\nabla_\theta F(\theta_t, \omega_t) - p_t\|^2.
\]

Taking expectation on both sides shows that

\[
E [J(\theta_{t+1}) - J(\theta_t)] \leq -\frac{3\nu_t}{4\kappa} E \|\bar{\theta}_{t+1} - \theta_t\|^2 + 2L_F^2 \kappa \nu_t E \|\omega_t - \omega(\theta_t)^\ast\|^2 + 4\kappa \nu_t E \|\nabla_\theta F(\theta_t, \omega_t) - p_t\|^2.
\]

In the above inequality, the LHS will be a telescoping sum if we sum over $t$ from $t = 0$ to $T - 1$. And then we can move the first term on the RHS to the LHS, which will give us an upper bound for the summation of $E \|\bar{\theta}_{t+1} - \theta_t\|^2$. Thus, to get the final bound, we need to get the upper bound of another two terms on the RHS of Eq. (8), i.e., $E \|\omega_t - \omega(\theta_t)^\ast\|^2$ and $E \|\nabla_\theta F(\theta_t, \omega_t - p_t)\|^2$. Furthermore, Lemma C.4 shows that

\[
E \|\omega_{t+1} - \omega(\theta_{t+1})\|^2 \leq \left( 1 - \frac{\mu \nu_t}{4} \right) E \|\omega_t - \omega(\theta_t)^\ast\|^2 - \frac{3\nu_t}{4} E \|\bar{\omega}_{t+1} - \omega_t\|^2 \\
+ \frac{75\nu_t \kappa L_F^2}{16\mu \eta} E \|d_t - \nabla F(\theta_t, \omega_t)\|^2 + \frac{75L_F^2 \kappa \nu_t}{16\mu \eta} E \|\bar{\theta}_{t+1} - \theta_t\|^2.
\]

Multiplying both sides of the above inequality by $10L_F^2 \kappa / (\mu \eta)$ leads to

\[
\frac{10L_F^2 \kappa}{\mu \eta} E \|\omega_{t+1} - \omega(\theta_{t+1})\|^2 \leq \frac{10L_F^2 \kappa}{\mu \eta} \left( 1 - \frac{\mu \nu_t}{4} \right) E \|\omega_t - \omega(\theta_t)^\ast\|^2 - \frac{15L_F^2 \kappa \nu_t}{2\mu \eta} E \|\bar{\omega}_{t+1} - \omega_t\|^2 \\
+ \frac{375L_F^2 \kappa \nu_t}{8\mu^2} E \|d_t - \nabla F(\theta_t, \omega_t)\|^2 + \frac{375L_F^2 \kappa \nu_t}{8\mu^2 \eta^2} E \|\bar{\theta}_{t+1} - \theta_t\|^2.
\]
Rearranging the terms shows that
\[
\frac{10L^2_{F,K}}{\mu \eta} \left( \mathbb{E} \| \omega_{t+1} - \omega^* (\theta_{t+1}) \|^2 - \mathbb{E} \| \omega_t - \omega^* (\theta_t) \|^2 \right)
\leq - \frac{5L^2_{F,K} \eta}{2 \mu} \mathbb{E} \| \omega_t - \omega^* (\theta_t) \|^2 - \frac{15L^2_{F,K} \nu_t}{2 \mu \eta} \mathbb{E} \| \bar{\omega}_{t+1} - \omega_t \|^2
+ \frac{375L^2_{F,K} \nu_t}{8 \mu^2} \mathbb{E} \| d_t - \nabla F (\theta_t, \omega_t) \|^2 + \frac{375L^2_{F,K} \nu_t}{8 \mu^2} \mathbb{E} \| \bar{\omega}_{t+1} - \bar{\omega}_t \|^2.
\] (9)

Then we define
\[
P_t := J (\theta_t) - J^* + \frac{10L^2_{F,K}}{\mu \eta} \| \omega_t - \omega^* (\theta_t) \|^2, \quad \forall t \geq 0
\]
where \(J^* > - \infty\) is the minimal value of \( J \). Thus we have \( J(\theta) - J^* > 0, \forall \theta \in \Theta \). Taking both Eq. (8) and Eq. (9) into consideration, we have
\[
\mathbb{E} [P_{t+1} - P_t] \leq - \left( \frac{3 \nu_t}{4 \kappa} - \frac{375L^2_{F,K} \nu_t}{8 \mu^2 \eta^2} \right) \mathbb{E} \| \bar{\omega}_{t+1} - \bar{\omega}_t \|^2 + \frac{15L^2_{F,K} \nu_t}{2 \mu \eta} \mathbb{E} \| \bar{\omega}_{t+1} - \omega_t \|^2
+ \frac{375L^2_{F,K} \nu_t}{8 \mu^2} \mathbb{E} \| d_t - \nabla F (\theta_t, \omega_t) \|^2 + 4 \nu_t \kappa \mathbb{E} \| p_t - \nabla \theta F (\theta_t, \omega_t) \|^2 - \frac{L^2_{F,K} \nu_t}{2} \mathbb{E} \| \omega_t - \omega^* (\theta_t) \|^2.
\]

We can simplify the coefficient \(- \left( \frac{3 \nu_t}{4 \kappa} - \frac{375L^2_{F,K} \nu_t}{8 \mu^2 \eta^2} \right)\) in the above inequality. First, by the parameter setting in Theorem 5.2, we have \(0 < \kappa \leq \eta \mu^2 / (9L^2_{F})\), which gives us \(\eta \geq 9L^2_{F} \kappa / \mu^2\) and further \(\eta^2 \geq 81L^2_{F} \kappa^2 / \mu^4\). Second, by Lemma C.1, we have \(L_{\omega} = L_{F} / \mu\). Thus
\[
\frac{375L^2_{F,K} \nu_t}{8 \mu^2 \eta^2} = \frac{375L^2_{F,K} \nu_t}{8 \mu^2 \eta^2} \leq \frac{125 \nu_t}{216 \kappa},
\]
and the coefficient is bounded by
\[
\left( \frac{3 \nu_t}{4 \kappa} - \frac{375L^2_{F,K} \nu_t}{8 \mu^2 \eta^2} \right) \leq - \frac{3 \nu_t}{216 \kappa} \leq - \frac{\nu_t}{8 \kappa},
\]
which implies that
\[
\mathbb{E} [P_{t+1} - P_t] \leq - \frac{\nu_t}{8 \kappa} \mathbb{E} \| \bar{\omega}_{t+1} - \bar{\omega}_t \|^2 + \frac{15L^2_{F,K} \nu_t}{2 \mu \eta} \mathbb{E} \| \bar{\omega}_{t+1} - \omega_t \|^2 - \frac{L^2_{F,K} \nu_t}{2} \mathbb{E} \| \omega_t - \omega^* (\theta_t) \|^2
+ \frac{375L^2_{F,K} \nu_t}{8 \mu^2} \mathbb{E} \| d_t - \nabla F (\theta_t, \omega_t) \|^2 + 4 \nu_t \kappa \mathbb{E} \| p_t - \nabla \theta F (\theta_t, \omega_t) \|^2.
\] (10)

The LHS of Eq. (10) will be a telescoping sum if we task summation from \( t = 0 \) to \( T - 1 \). And then we can move \( \mathbb{E} \| \bar{\omega}_{t+1} - \bar{\omega}_t \|^2 \) and \( \mathbb{E} \| \omega_t - \omega^* (\theta_t) \|^2 \) from the RHS to the LHS, which will help us bound the two terms. Thus, we expect to upper bound \( \mathbb{E} \| d_t - \nabla \theta F (\theta_t, \omega_t) \|^2 \) and \( \mathbb{E} \| p_t - \nabla \theta F (\theta_t, \omega_t) \|^2 \). By Lemma C.6, we have
\[
\mathbb{E} \| \nabla \theta F (\theta_{t+1}, \omega_{t+1}) - p_{t+1} \|^2 \leq (1 - \alpha \nu_t) \mathbb{E} \| \nabla \theta F (\theta_t, \omega_t) - p_t \|^2 + \frac{9 \nu_t L^2_{F}}{8 \alpha} \mathbb{E} \left( \| \bar{\omega}_{t+1} - \bar{\omega}_t \|^2 + \| \bar{\omega}_{t+1} - \omega_t \|^2 \right)
+ \alpha^2 \nu_t^2 \sigma^2 + d \sigma^2_{t+1},
\] (11)
and
\[
\mathbb{E} \| \nabla \theta F (\theta_{t+1}, \omega_{t+1}) - d_{t+1} \|^2 \leq (1 - \alpha \nu_t) \mathbb{E} \| \nabla \theta F (\theta_t, \omega_t) - d_t \|^2 + \frac{9 \nu_t L^2_{F}}{8 \alpha} \mathbb{E} \left( \| \bar{\omega}_{t+1} - \bar{\omega}_t \|^2 + \| \bar{\omega}_{t+1} - \omega_t \|^2 \right)
+ \alpha^2 \nu_t^2 \sigma^2 + d \sigma^2_{t+1}.
\] (12)

We define a Lyapunov function to enable a telescoping summation, which is for \( \forall t \geq 0 \),
\[
Q_t := P_t + \frac{2 \kappa}{\mu \eta} \| \nabla \theta F (\theta_t, \omega_t) - p_t \|^2 + \frac{2 \kappa}{\mu \eta} \| \nabla \theta F (\theta_t, \omega_t) - d_t \|^2.
\]
Multiplying both side of Eq. (11) and Eq. (12) by $2\kappa/(\mu \eta)$ and combining with Eq. (10), we have

$$
\mathbb{E}[Q_{t+1} - Q_t] \leq - \left( \frac{\nu_t}{8\kappa} - \frac{3\kappa \nu_t L_F^2}{2\mu \eta} \right) \mathbb{E} \left\| \tilde{\theta}_{t+1} - \theta_t \right\|^2 - \frac{L_F^2 \kappa \nu_t}{2\mu \eta} \mathbb{E} \left\| \omega_t - \omega^* (\theta_t) \right\|^2 + \frac{4d\kappa \sigma^2_{t+1}}{\mu \eta}
$$

$$
- \frac{6L_F^2 \kappa \nu_t}{\mu \eta} \mathbb{E} \left\| \tilde{\omega}_{t+1} - \omega_t \right\|^2 + \frac{36\sigma^2 \nu_t^2 \kappa}{\mu \eta} \mathbb{E} \left\| \nabla \theta F (\theta_t, \omega_t) - p_t \right\|^2
$$

$$
- \left( \frac{12\nu_t \kappa}{\mu \eta} - \frac{375L_F^2 \kappa \nu_t}{8\mu^2} \right) \mathbb{E} \left\| \nabla \omega F (\theta_t, \omega_t) - d_t \right\|^2.
$$

By the parameter setting in Theorem 5.2, we have $0 < \eta \leq \mu / (4L_F^2)$ and $0 < \kappa \leq \eta \mu^2 / (9L_F^2)$, or $\eta \geq 9L_F^2 \kappa / \mu^2$. Moreover, by our assumption on $L_F$-smoothness and $\mu$-strongly concavity, we have $L_F \geq \mu > 0$.

We now simply the coefficients in the above display. First, for $- \left( \frac{\nu_t}{8\kappa} - \frac{3\kappa \nu_t L_F^2}{2\mu \eta} \right)$, we have

$$
- \left( \frac{\nu_t}{8\kappa} - \frac{3\kappa \nu_t L_F^2}{2\mu \eta} \right) \leq - \frac{\nu_t}{8\kappa} + \frac{\nu_t \mu}{6},
$$

and we remove $\mu$ by the following inequality

$$
\mu \kappa \leq \frac{\eta \mu^3}{9L_F^2} \leq \frac{\eta \mu}{9} \leq \frac{\mu^2}{36L_F} \leq \frac{1}{36}.
$$

Thus we obtain

$$
- \left( \frac{\nu_t}{8\kappa} - \frac{3\kappa \nu_t L_F^2}{2\mu \eta} \right) \leq - \frac{\nu_t}{8\kappa} + \frac{\nu_t}{216\kappa} \leq - \frac{\nu_t}{16\kappa}.
$$

Second, for $- \left( \frac{12\nu_t \kappa}{\mu \eta} - 4\kappa \nu_t \right)$, notice that

$$
\mu \eta \leq \frac{\mu^2}{4L_F^2} \leq \frac{1}{4},
$$

and hence

$$
- \left( \frac{12\nu_t \kappa}{\mu \eta} - 4\kappa \nu_t \right) \leq - \frac{12\nu_t \kappa}{\mu \eta} + \frac{\kappa \nu_t}{\mu \eta} \leq - \frac{4\nu_t \kappa}{\mu \eta}.
$$

Third, for $- \left( \frac{12\nu_t \kappa}{\mu \eta} - \frac{375L_F^2 \kappa \nu_t}{8\mu^2} \right)$, we obtain the following simplified result

$$
- \left( \frac{12\nu_t \kappa}{\mu \eta} - \frac{375L_F^2 \kappa \nu_t}{8\mu^2} \right) \leq - \frac{\nu_t \kappa}{\mu} \cdot \frac{9L_F^2}{8\mu} < 0.
$$

Then plugging the three simplified coefficients into the inequality and omitting the terms with negative coefficients leads to

$$
\mathbb{E} [Q_{t+1} - Q_t] \leq - \frac{\nu_t}{16\kappa} \mathbb{E} \left\| \tilde{\theta}_{t+1} - \theta_t \right\|^2 - \frac{4 \nu_t \kappa L_F^2}{\mu \eta} \mathbb{E} \left\| p_t - \nabla \theta F (\theta_t, \omega_t) \right\|^2 - \frac{36\sigma^2 \nu_t^2 \kappa}{\mu \eta} + \frac{4d\kappa \sigma^2_{t+1}}{\mu \eta},
$$

(13)

Taking summation on both sides of Eq. (13) from $t = 0$ to $T - 1$ and rearranging shows that

$$
\sum_{t=0}^{T-1} \frac{\nu_t \kappa}{16} \left( \frac{1}{\kappa^2} \mathbb{E} \left\| \tilde{\theta}_{t+1} - \theta_t \right\|^2 + \frac{64}{\mu \eta} \mathbb{E} \left\| p_t - \nabla \theta F (\theta_t, \omega_t) \right\|^2 + 8L_F^2 \mathbb{E} \left\| \omega_t - \omega^* (\theta_t) \right\|^2 \right)
$$

$$
\leq \frac{36\sigma^2 \kappa \sum_{t=0}^{T-1} \nu_t^2 + 4\kappa d \sum_{t=1}^{T} \sigma_t^2}{\mu \eta} + \mathbb{E} [Q_0 - Q_T]
$$

$$
\leq \frac{36\sigma^2 \kappa \sum_{t=0}^{T-1} \nu_t^2 + 4\kappa d \sum_{t=1}^{T} \sigma_t^2}{\mu \eta} + Q_0,
$$

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where the last inequality comes from $Q_t \geq 0, \forall t \geq 0$ and $Q_0$ is determined in the initialization.

Since $\nu_t$ is set to $\frac{a}{(t+b)^{\frac{1}{2}}}$, we can use the fact $\nu_t \geq \nu_T$ for any $0 \leq t \leq T$ to upper bound the LHS of the above inequality. Since $\frac{\mu \eta}{4} \leq \frac{1}{2}$, combining $\nu_t \geq \nu_T$ and $\frac{1}{\mu \eta} \geq 1$ shows that

$$\frac{\nu_T \kappa}{16} \sum_{t=0}^{T-1} \left( \frac{1}{\kappa^2} \mathbb{E} \left[ \left\| \bar{\theta}_{t+1} - \theta_t \right\|^2 + \mathbb{E} \left\| p_t - \nabla_{\theta} F(\theta_t, \omega_t) \right\|^2 + L_F^2 \mathbb{E} \left\| \omega_t - \omega^*(\theta_t) \right\|^2 \right) \right) \leq \sum_{t=0}^{T-1} \frac{\nu_T \kappa}{16} \left( \frac{1}{\kappa^2} \mathbb{E} \left[ \left\| \bar{\theta}_{t+1} - \theta_t \right\|^2 + \frac{64}{\mu \eta} \mathbb{E} \left\| p_t - \nabla_{\theta} F(\theta_t, \omega_t) \right\|^2 + 8L_F^2 \mathbb{E} \left\| \omega_t - \omega^*(\theta_t) \right\|^2 \right) \right) \leq \frac{36\sigma^2}{\kappa \nu_T^2} \sum_{t=0}^{T-1} \frac{1}{\nu_t^2} + \frac{4kd \sum_{t=1}^{T-1} \sigma_t^2}{\mu \eta} + Q_0.$$

Rearranging the above display leads to

$$\frac{1}{T} \sum_{t=0}^{T-1} \left( \frac{1}{\kappa^2} \mathbb{E} \left[ \left\| \bar{\theta}_{t+1} - \theta_t \right\|^2 + \mathbb{E} \left\| p_t - \nabla_{\theta} F(\theta_t, \omega_t) \right\|^2 + L_F^2 \mathbb{E} \left\| \omega_t - \omega^*(\theta_t) \right\|^2 \right) \right) \leq \frac{576\sigma^2}{\kappa \nu_T^2} \sum_{t=0}^{T-1} \frac{1}{\nu_t^2} + \frac{64d \sum_{t=1}^{T} \sigma_t^2}{\mu \eta} + \frac{16Q_0}{\kappa \nu_T^2}.$$

Applying Jensen’s inequality to the LHS of Eq. (14) shows that

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left( \frac{1}{\kappa^2} \left\| \bar{\theta}_{t+1} - \theta_t \right\|^2 + \mathbb{E} \left\| p_t - \nabla_{\theta} F(\theta_t, \omega_t) \right\|^2 + L_F^2 \mathbb{E} \left\| \omega_t - \omega^*(\theta_t) \right\|^2 \right) \leq \frac{3}{T} \sum_{t=0}^{T-1} \mathbb{E} \left( \frac{1}{\kappa^2} \left\| \bar{\theta}_{t+1} - \theta_t \right\|^2 + \mathbb{E} \left\| p_t - \nabla_{\theta} F(\theta_t, \omega_t) \right\|^2 + L_F^2 \mathbb{E} \left\| \omega_t - \omega^*(\theta_t) \right\|^2 \right)^{1/2}.$$

The RHS of Eq. (14) could be bounded by of $\nu_t$ and $\sigma_t$. By the Gaussian noise set in Theorem 5.1, we have

$$\sigma_t^2 = \frac{14G^2T\alpha'}{n^2\beta'\epsilon}, \forall t \geq 0,$$

where $\alpha' = \frac{\log(1/\delta)}{(1-\beta')\epsilon} + 1$ and $\beta' \in (0, 1)$. Recall that $\nu_t = \frac{1}{4(t+b)^{\frac{1}{2}}}$. Then we obtain

$$\frac{576\sigma^2}{\kappa \nu_T^2} \sum_{t=0}^{T-1} \frac{1}{\nu_t^2} + \frac{64d \sum_{t=1}^{T-1} \sigma_t^2}{\mu \eta} \leq \frac{144\sigma^2}{\mu \eta T} (T + b)^{1/2} \log(T + b) + \frac{3584G^2dT(T + b)^{1/2}}{\beta' \eta \mu^2 \epsilon} \left( \log \left( \frac{T}{T + b} \right) + 1 \right) \leq \frac{144\sigma^2}{\mu \eta T} (T + b)^{1/2} \log(T + b) + \frac{3584G^2dT(T + b)^{1/2}}{\beta' \eta \mu^2 \epsilon} \left( \log \left( \frac{T}{T + b} \right) + 1 \right),$$

where the last inequality is due to $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for $x \geq 0, y \geq 0.$
Now combining Eq. (14), Eq. (15) and Eq. (16), we have

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left( \frac{1}{\kappa} \left\| \tilde{\theta}_{t+1} - \theta_t \right\| + \| p_t - \nabla_\theta F (\theta_t, \omega_t) \| + L_F \| \omega_t - \omega^* (\theta_t) \| \right)
\leq \sqrt{3} \left( \frac{576 \sigma^2 \sum_{t=0}^{T-1} \nu_t^2 + 64d \sum_{t=1}^{T} \sigma^2}{\mu T \nu_T} + \frac{16Q_0}{\mu \nu_T} \right)^{\frac{1}{2}} + 60\sqrt{3}Gd^2 \left( T^\frac{4}{3} + b^\frac{1}{2}T^\frac{1}{3} \right) \left( \frac{\log \left( \frac{1}{\delta} \right)}{(1-\beta')\frac{1}{2}} \right)^{\frac{1}{2}} + \frac{1}{\eta \mu^{\frac{1}{2}} \beta' \frac{1}{2}} \sqrt{\frac{8}{3}} \sum_{t=1}^{T} \sigma_t \log \left( \frac{1}{\delta} \right)
\leq \sqrt{3} \left( \frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta} \right)^{\frac{1}{2}} \left( \frac{1}{T^\frac{1}{2}} + \frac{b^\frac{1}{2}}{T^\frac{1}{2}} \right) \log \left( T + b \right) + \frac{120\sqrt{3}Gd^2 \left( T^\frac{4}{3} + b^\frac{1}{2}T^\frac{1}{3} \right) \log \left( \frac{1}{\delta} \right)}{\eta \mu^{\frac{1}{2}} \beta' \frac{1}{2}} (1 - \beta') \frac{1}{2} \sqrt{\frac{8}{3}} \sum_{t=1}^{T} \sigma_t \log \left( \frac{1}{\delta} \right),
(17)
$$

where the second inequality is due to $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for $x \geq 0, y \geq 0$. Simplifying Eq. (17) by replacing the constant coefficients with $C_1$ leads to

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left( \frac{1}{\kappa} \left\| \tilde{\theta}_{t+1} - \theta_t \right\| + \| p_t - \nabla_\theta F (\theta_t, \omega_t) \| + L_F \| \omega_t - \omega^* (\theta_t) \| \right)
\leq \sqrt{3} \left( \frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta} \right)^{\frac{1}{2}} \left( \frac{1}{T^\frac{1}{2}} + \frac{b^\frac{1}{2}}{T^\frac{1}{2}} \right) \log \left( T + b \right) + \frac{120\sqrt{3}Gd^2 \left( T^\frac{4}{3} + b^\frac{1}{2}T^\frac{1}{3} \right) \log \left( \frac{1}{\delta} \right)}{\eta \mu^{\frac{1}{2}} \beta' \frac{1}{2}} \sqrt{\frac{8}{3}} \sum_{t=1}^{T} \sigma_t \log \left( \frac{1}{\delta} \right)
= C_1 \left( \frac{1}{T^\frac{1}{2}} + \frac{b^\frac{1}{2}}{T^\frac{1}{2}} \right) \log \left( T + b \right) + \frac{C_2 \left( T^\frac{4}{3} + b^\frac{1}{2}T^\frac{1}{3} \right) d^\frac{1}{2} \log \left( \frac{1}{\delta} \right)}{\eta \mu^{\frac{1}{2}} \beta' \frac{1}{2} \sqrt{\frac{8}{3}} \sum_{t=1}^{T} \sigma_t \log \left( \frac{1}{\delta} \right)},
(18)
$$

where $C_1 = \sqrt{3} \left( \frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta} \right)^{\frac{1}{2}}$, $C_2 = b^\frac{1}{2}C_1$, $C_3 = \frac{120\sqrt{3}G}{(\beta'(1-\beta')\eta \mu^{\frac{1}{2}} \beta' \frac{1}{2}) \sqrt{\frac{8}{3}} \sum_{t=1}^{T} \sigma_t \log \left( \frac{1}{\delta} \right)}$, $C_4 = b^\frac{1}{2}C_3$.

By the parameter setting in Theorem 5.2, with $T = \frac{C_5 \mu \eta \sqrt{\log(1/\delta)}}{d}$ where $C_5$ is a constant, we can obtain the final bound is

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left( \frac{1}{\kappa} \left\| \tilde{\theta}_{t+1} - \theta_t \right\| + \| p_t - \nabla_\theta F (\theta_t, \omega_t) \| + L_F \| \omega_t - \omega^* (\theta_t) \| \right)
\leq \frac{\sqrt{d \log \left( \frac{1}{\delta} \right)}}{\sqrt{C_5 \mu \eta \sqrt{\log(1/\delta)}}} \left( \log \left( b + \frac{C_5 \mu \eta \sqrt{\log(1/\delta)}}{\sqrt{d \log \left( \frac{1}{\delta} \right)}} \right) + C_2 \frac{\sqrt{\log \left( \frac{1}{\delta} \right)}}{C_5 \mu \eta \sqrt{\log(1/\delta)}} \right)
+ C_1 C_5 \frac{\sqrt{d \log \left( \frac{1}{\delta} \right)}}{\sqrt{d \log(1/\delta)}} + C_3 C_5^\frac{1}{4} \sqrt{\eta \mu \log \left( \frac{1}{\delta} \right)}.
$$

If we hide the factor $\sqrt{\log \left( b + \frac{C_5 \mu \eta \sqrt{\log(1/\delta)}}{\sqrt{d \log(1/\delta)}} \right)}$, i.e., $\sqrt{\log(b+T)}$, we obtain

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left( \frac{1}{\kappa} \left\| \tilde{\theta}_{t+1} - \theta_t \right\| + \| p_t - \nabla_\theta F (\theta_t, \omega_t) \| + L_F \| \omega_t - \omega^* (\theta_t) \| \right) \leq \hat{O} \left( \frac{d \log(1/\delta)}{(n \epsilon) \frac{1}{2}} \right).
$$

**Gradient Complexity.** The gradient complexity is equal to $2(T+1) = \mathcal{O} \left( \frac{ne}{\sqrt{d \log(1/\delta)}} \right)$ since Algorithm 1 computes gradients for both the primal side and the dual side.

\[\square\]

### D Proof of Theorem A.1

In this section, we provide the proof of Theorem A.1.
Proof of Theorem A.1. According to the update rules in Algorithm 1, our mechanisms are constructed as

\[
\mathcal{M}_t^p = \begin{cases} 
(1 - \alpha \nu_{t-1})p_{t-1} + \alpha \nu_{t-1} \nabla \theta f(\theta_t, \omega_t; \xi_t) + u^p_t, & t > 0 \\
\nabla \theta f(\theta_0, \omega_0; \xi_0) + u_0^p, & t = 0,
\end{cases}
\]

(19)

and

\[
\mathcal{M}_t^d = \begin{cases} 
(1 - \beta \nu_{t-1})d_{t-1} + \beta \nu_{t-1} \nabla \omega f(\theta_t, \xi_t) + u^d_t, & t > 0 \\
\nabla \omega f(\theta_0, \xi_0) + u_0^d, & t = 0.
\end{cases}
\]

(20)

We aim to show the privacy guarantee of \(\mathcal{M}_t^p\) and \(\mathcal{M}_t^d\) for \(t = 0, \ldots, T-1\). We prove the privacy guarantee of the mechanism on the primal side (i.e., \(\mathcal{M}_t^p\)) and the proof of the privacy guarantee of the mechanism on the dual side (i.e., \(\mathcal{M}_t^d\)) follows similarly. Similar to the proof of 5.1, we start from the case when \(t = 0\) and then discuss the case when \(t > 0\).

Case (a). If \(t = 0\), we consider the following Gaussian mechanism

\[
G_0^p = \nabla \theta f(\theta_0, \omega_0; \xi_0) + u_0^p,
\]

where \(u_0^p \sim N(0, \sigma^2 I_d)\). To provide the privacy guarantee of the above mechanism, we first prove the privacy guarantee of the following Gaussian mechanisms \(\tilde{G}_0^p\) without subsampling, which means we have the access to the full dataset. Denote by \(\tau_i\) the \(i\)-th trajectory in dataset \(S\) and by \(\xi_{i,j}\) the \(j\)-th triple in \(\tau_i\). Specifically, \(\tilde{G}_0^p\) is constructed as

\[
\tilde{G}_0^p = \sum_{i=0}^{m-1} \sum_{j=0}^{\lfloor |\tau_i| - 1 \rfloor} \nabla \theta f(\theta_0, \omega_0; \xi_{i,j}) + u_0^p.
\]

Sensitivity. Consider the query on the dataset \(S\) as follows

\[
\tilde{\theta}_0^p(S) = \sum_{i=0}^{m-1} \sum_{j=0}^{\lfloor |\tau_i| - 1 \rfloor} \nabla \theta f(\theta_0, \omega_0; \xi_{i,j}) .
\]

Similarly we can get \(\tilde{\theta}_0^p(\hat{S})\). Then the \(\ell_2\)-sensitivity could be bounded as follows

\[
\tilde{\Delta}_t^p = \left\| \tilde{\theta}_0^p(S) - \tilde{\theta}_0^p(\hat{S}) \right\| \\
\leq \sum_{j=0}^{\lfloor |\tau_i| - 1 \rfloor} \left\| \nabla \theta f(\theta_0, \omega_0; \xi_{i,j}) \right\| + \left\| \nabla \theta f(\theta_0, \omega_0; \hat{\xi}_{i,j}) \right\| \\
\leq 2nG
\]

The last inequality is because of Assumption A.1. Similarly, we obtain \(\tilde{\Delta}_t^d \leq 2nG\).

Privacy guarantee of \(\tilde{G}_0^p\). By Lemma B.3, if the Gaussian noise \(u_0^p\) has the following variance

\[
\sigma_t^2 = \frac{14n^2G^2T\alpha'}{m^2 \left( \epsilon - \frac{\log(1/\delta)}{\alpha'-1} \right)} ,
\]

where \(\sigma_t^2 = \frac{\sigma^2}{m}, \quad 0.7, \quad \alpha' = \frac{\log(1/\delta)}{1 - \beta' n} + 1 \leq 2\sigma^2 \log\left( \frac{n}{\alpha(1 + \sigma^2)} \right)/3 + 1 \quad \text{and} \quad \beta' \in (0, 1), \) then our mechanism \(\tilde{G}_0^p\) will satisfy \((\alpha', \frac{14\alpha'n^2G^2}{m^2\sigma^2})\)-RDP.

Case (b). The sensitivity is bounded the same as in the Case (a). Then one can see that Gaussian mechanism \(\tilde{G}_0^p\) is bounded as

\[
\tilde{\Delta}_t^p \leq 2\alpha \nu_{t-1}nG \leq 2nG
\]
with the probability $P_E$.

It is clear that the mechanisms in Case (a) and Case (b) are able to satisfy the same RDP under the same Gaussian noise since they have the same upper bound of the sensitivity.

**Privacy guarantee of $\mathcal{M}_t^\ell$.** Due to the definition of $\mathcal{M}_t^\ell$ in Eq. (19), $\mathcal{M}_t^\ell$ is composed of several Gaussian mechanisms, i.e., $\mathcal{M}_t^\ell = (\mathcal{G}_0^\ell, \ldots, \mathcal{G}_t^\ell)$. Then Lemma B.1 implies that $\mathcal{M}_t^\ell$ and the output on the primal side satisfies $(\alpha', \sum_{i=0}^{T} \frac{14\alpha'^2_i \gamma^2_i}{m^2 \sigma_i^2})$-RDP. Applying Lemma B.2 shows that the output satisfies $\left(\sum_{i=0}^{T} \frac{(14\alpha_i' \gamma^2_i)}{m^2 \sigma_i^2} + \frac{\log(1/\delta)}{\alpha_i' - 1}, \delta\right)$-DP. Substituting the value of $\sigma_i$ simplifies it as $(\epsilon, \delta)$-DP under trajectory which concludes the proof. □

# E Proof of Theorem A.2

In this section, we provide the proof of Theorem A.2, which gives the utility of Algorithm 1 when achieving $(\epsilon, \delta)$-DP under trajectory.

**Proof of Theorem A.2.** The main proof is similar and the difference lies in that we inject different Gaussian noises and the variance of gradient is $\sigma_i^2$. The variance of the Gaussian noise is

$$\sigma_i^2 = \frac{14n^2G^2T\alpha_i'}{m^2(\epsilon - \log(1/\delta))}, \forall t \geq 0,$$

Thus, we start from rebounding the LHS of Eq. (16) as follows

$$\begin{align*}
576\sigma_t^2 \sum_{t=0}^{T-1} \nu_t^2 + 64d \sum_{t=1}^{T} \sigma_t^2 + 16Q_0 \frac{144\sigma^2}{\mu \eta T} (T + b) \frac{T}{\nu_t^2} & \leq \frac{144\sigma^2}{\mu \eta T} (T + b) \frac{T}{\nu_t^2} \log(T + b) \\
& + \frac{3584n^2G^2dT(T + b) \frac{T}{\nu_t^2} \left(\frac{\log(\frac{1}{\epsilon})}{(1 - \beta')^2} + 1\right)}{\beta' \eta \mu m^2 \epsilon} \\
& + \frac{64Q_0(T + b)^{1/2}}{\kappa T} \\
& \leq \left(\frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta}\right) \left(\frac{1}{\sqrt{T}} + \frac{\sqrt{b}}{T}\right) \log(T + b) \\
& + \frac{3584n^2G^2dT(T + b) \frac{T}{\nu_t^2} \left(\frac{\log(\frac{1}{\epsilon})}{(1 - \beta')^2} + 1\right)}{\beta' \eta \mu m^2 \epsilon} , \quad (21)
\end{align*}$$

where the last inequality is due to $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ for $x \geq 0, y \geq 0$.

Combining Eq. (14), Eq. (15) and Eq. (21) shows that

$$\begin{align*}
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left(\frac{1}{\kappa} \left\| \theta_{t+1} - \theta_t \right\| + \left\| p_t - \nabla_0 F(\theta_t, \omega_t) \right\| + L_F \left\| \omega_t - \omega^* (\theta_t) \right\|\right) \\
\leq \sqrt{3} \left(\frac{576\sigma_t^2 \sum_{t=0}^{T-1} \nu_t^2 + 64d \sum_{t=1}^{T} \sigma_t^2 + 16Q_0}{\mu \eta T^2} \right)^{1/2} \\
\leq \sqrt{3} \left(\frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta}\right)^{1/2} \left(\frac{1}{\sqrt{T}} + \frac{bT^2}{T^2}\right) \log(T + b) + \frac{60\sqrt{3}nGd^2(T^{\frac{1}{2}} + b^2T^{\frac{3}{2}}) \left(\frac{\log(\frac{1}{\epsilon})}{(1 - \beta')^2} + 1\right)}{\beta' \eta \mu m \epsilon^{1/2}} \\
\leq \sqrt{3} \left(\frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta}\right)^{1/2} \left(\frac{1}{\sqrt{T}} + \frac{bT^2}{T^2}\right) \log(T + b) + \frac{120\sqrt{3}nGd^2(T^{\frac{1}{2}} + b^2T^{\frac{3}{2}}) \log(\frac{1}{\epsilon})}{\eta \mu \beta' \left(1 - \beta'\right)^2 m \epsilon^{1/2}},
\end{align*}$$

where the second inequality is due to $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ for $x \geq 0, y \geq 0$. 

}
Simplifying Eq. (17) by replacing the constant coefficients with $C_t$ leads to

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| \tilde{\theta}_{t+1} - \theta_t \right\| + \left\| p_t - \nabla \theta \mathbb{E} \left[ F (\theta_t, \omega_t) \right] + L_F \left\| \omega_t - \omega^* (\theta_t) \right\| \right] 
\leq \sqrt{3} \left( \frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta} \right)^{\frac{1}{2}} \left( \frac{1}{T^\frac{3}{4}} + \frac{b_1^2}{T^2} \right) \log^2 \left( T + b \right) + \frac{120 \sqrt{3} \eta \Gamma (\frac{T^\frac{3}{4} + b_1^2 T^2) \log^2 (\frac{1}{\epsilon})}{\eta^2 \mu^2 \beta^2 (1 - \beta')^2 \eta m c}
\leq C_1 \log^2 (T + b) + C_2 \log^2 (T + b) + \left( C_3 T^\frac{3}{4} + C_4 T^2 \right) \log^2 \left( \frac{1}{\epsilon} \right)
\leq \tilde{O} \left( \frac{1}{T^\frac{3}{4}} \right) + \mathcal{O} \left( \frac{T^\frac{3}{4} \log^2 (1/\delta)}{m c} \right),
$$

where $C_1 = \sqrt{3} \left( \frac{64Q_0}{\kappa} + \frac{144\sigma^2}{\mu \eta} \right)^{\frac{1}{2}}$, $C_2 = b_1^2 C_1$, $C_3 = \frac{120 \sqrt{3} \eta \Gamma}{\eta^2 \mu^2 \beta^2 (1 - \beta')^2 \eta m c}$, $C_4 = b_1^2 C_3$.

By the parameter setting in Theorem A.2, if we set $T$ as follows

$$
T = \frac{C_5 m c}{n \sqrt{d \log(1/\delta)}},
$$

and hide the factor $\sqrt{\log(b + T)}$, the same as in Theorem 5.2, we will obtain

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left\| \mathcal{R}_t \right\| \leq \tilde{O} \left( \frac{n^\frac{1}{4} (d \log(1/\delta))^{\frac{1}{2}}}{m c} \right).
$$

**Gradient Complexity.** The gradient complexity is equal to $2(T + 1) = \mathcal{O} \left( \frac{m c}{n \sqrt{d \log(1/\delta)}} \right)$ since Algorithm 1 computes gradients for both the primal side and the dual side.

\[\square\]

## F Proof of Technical Lemmas

In this section, we give the detailed proof of several technical lemmas.

### F.1 Proof of Lemma C.5

**Lemma C.5** [Bounded Variance] Under Assumption 3.3, the variance of the stochastic gradient $\nabla f (\theta, \omega; \xi) = (\nabla f (\theta, \omega; \xi), \nabla f (\theta, \omega; \xi))$ is bounded as $\mathbb{E}_{\xi \sim \Xi} \left[ \nabla f (\theta, \omega; \xi) - \nabla F (\theta, \omega) \right]^2 \leq \sigma^2$, where $\sigma^2 = 4G^2$.

**Proof.** By Assumption 3.3, we have that for any $\theta \in \Theta$, $\omega \in \Omega$, $\left\| \nabla f (\theta, \omega; \xi) \right\| \leq G$ and $\left\| \nabla F (\theta, \omega) \right\| \leq G$. We start directly from the LHS,

$$
\mathbb{E}_{\xi \sim \Xi} \left[ \nabla f (\theta, \omega; \xi) - \nabla F (\theta, \omega) \right]^2 = \mathbb{E}_{\xi \sim \Xi} \left[ \left\| \nabla f (\theta, \omega; \xi) \right\|^2 + \left\| \nabla F (\theta, \omega) \right\|^2 - 2 \nabla f (\theta, \omega; \xi) \nabla F (\theta, \omega) \right]
\leq 2G^2 - 2 \mathbb{E}_{\xi \sim \Xi} \left[ \nabla f (\theta, \omega; \xi) \nabla F (\theta, \omega) \right]
= 2G^2 - 2 \mathbb{E}_{\xi \sim \Xi} \left[ \nabla f (\theta, \omega; \xi) \right] \nabla F (\theta, \omega)
\leq 2G^2,
$$

where $\sigma^2 = 2G^2$. Similarly, $\mathbb{E}_{\xi \sim \Xi} \left[ \nabla F (\theta, \omega; \xi) - \nabla F (\theta, \omega) \right]^2 \leq 2G^2$ also holds. Thus, if we set $\sigma^2 = 2G^2$, we have $\mathbb{E}_{\xi \sim \Xi} \left[ \nabla f (\theta, \omega; \xi) - \nabla F (\theta, \omega) \right]^2 \leq \sigma^2$.

\[\square\]

### F.2 Proof of Lemma C.6

**Lemma C.6** [With Bounded Variance]

Under Assumptions 3.2, 3.4, 3.5, letting $0 < \nu_t \leq (8\alpha)^{-1}$ and $0 < \eta \leq (4L_F)^{-1}$, with the updating rules shown in Algorithm 1, we have

$$
\mathbb{E} \left\| \nabla \theta F (\theta_{t+1}, \omega_{t+1}) - p_{t+1} \right\|^2 \leq (1 - \alpha \nu_t) \mathbb{E} \left\| \nabla \theta F (\theta_t, \omega_t) - p_t \right\|^2 + \frac{9\nu_t L_F^2}{8\alpha} \mathbb{E} \left\| \bar{\theta}_{t+1} - \theta_t \right\|^2 + \frac{9\nu_t L_F^2}{8\alpha} \mathbb{E} \left\| \bar{\omega}_{t+1} - \omega_t \right\|^2 + \alpha^2 \nu_t^2 \sigma^2 + d\sigma_{t+1}^2, \tag{22}
$$
Proof. We first show the detailed proof for Eq. (22) in the lemma, and for the proof of Eq. (23) we will only give a proof sketch since the proof of the two inequalities is similar.

We start from the LHS of Eq. (22). Decompose the term $\nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_{t+1}$ as follows

$$\nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_{t+1} = \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - (1 - \alpha \nu_t) p_t - u^p_{t+1} - \alpha \nu \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1})$$

$$= (1 - \alpha \nu_t) \left[ \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t \right] - u^p_{t+1} + \alpha \nu_t \left[ \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) \right],$$

where we use the updating rule $p_{t+1} = (1 - \alpha \nu_t)p_t + \nu \nu_t \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) + u^p_{t+1}$ in Algorithm 1.

Taking expectation of the square of the norm on both sides leads to

$$\mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_{t+1} \right\|^2$$

$$= (1 - \alpha \nu_t)^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t \right\|^2 + \alpha^2 \nu_t^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) \right\|^2$$

$$+ (2 \alpha - 2 \alpha^2 \nu_t) \nu_t \mathbb{E} \left\langle \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t, \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) \right\rangle$$

$$+ \mathbb{E} \left\| u^p_{t+1} \right\|^2 + 2 \mathbb{E} \left\langle (1 - \alpha \nu_t)p_t + \nu \nu_t \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}), u^p_{t+1} \right\rangle.$$

(24)

It remains to simplify the RHS of Eq. (24). By the Gaussian noise we define in Algorithm 1, we have

$$\mathbb{E} \left\| u^p_{t+1} \right\|^2 = d\sigma^2_{t+1},$$

(25)

and

$$\mathbb{E} \left\langle (1 - \alpha \nu_t)p_t + \nu \nu_t \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}), u^p_{t+1} \right\rangle = 0,$$

(26)

where the second equality is because $\mathbb{E}[u^p_{t+1}] = 0$.

By the tower rule of conditional expectation, one can see that

$$\mathbb{E} \left[ \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_{t+1} \right] = \mathbb{E} \left[ \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t \right]$$

$$- \mathbb{E} \left[ \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) \right] = 0.$$

(27)

Combining Eq. (25), Eq. (26), Eq. (27), we simplify Eq. (24) as

$$\mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_{t+1} \right\|^2 = (1 - \alpha \nu_t)^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t \right\|^2 + d\sigma^2_{t+1}$$

$$+ \alpha^2 \nu_t^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) \right\|^2.$$

(28)

Now we bound the first term $(1 - \alpha \nu_t)^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t \right\|^2$ in Eq. (28) as follows

$$(1 - \alpha \nu_t)^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - p_t \right\|^2$$

$$= (1 - \alpha \nu_t)^2 \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta F(\theta_t, \omega_t) \right\|^2$$

$$\leq (1 - \alpha \nu_t)^2 \left( 1 + \frac{1}{\alpha \nu_t} \right) \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta F(\theta_t, \omega_t) \right\|^2$$

$$+ (1 - \alpha \nu_t)^2 (1 + \alpha \nu_t) \mathbb{E} \left\| \nabla \theta F(\theta_t, \omega_t) - p_t \right\|^2$$

$$\leq \frac{9}{8 \alpha \nu_t} \mathbb{E} \left\| \nabla \theta F(\theta_{t+1}, \omega_{t+1}) - \nabla \theta F(\theta, \omega_t) \right\|^2 + (1 - \alpha \nu_t) \mathbb{E} \left\| \nabla \theta F(\theta_t, \omega_t) - p_t \right\|^2,$$

(29)

where the first inequality is by Young’s inequality $\|x + y\|^2 \leq (1 + \lambda)\|x\|^2 + (1 + \lambda^{-1})\|y\|^2$ with $\lambda = \alpha \nu_t$, and the second inequality is due to the condition $0 < \nu_t \leq (8 \alpha)^{-1}$ and then

$$(1 - \alpha \nu_t) \left( 1 + \frac{1}{\alpha \nu_t} \right) \leq 1 + \frac{1}{\alpha \nu_t} \leq \frac{9}{8 \alpha \nu_t},$$

and

$$(1 - \alpha \nu_t)^2 (1 + \alpha \nu_t) = 1 - \alpha \nu_t - \alpha^2 \nu_t^2 + \alpha^3 \nu_t^3 \leq 1 - \alpha \nu_t.$$
Furthermore, Assumption 3.2 implies that $\nabla_{\theta} F(\theta, \omega)$ is Lipschitz continuous. Recall that the update rule in Algorithm 1 is $\theta_{t+1} = \theta_{t} + \nu_{t}(\bar{\theta}_{t} - \theta_{t})$ and $\omega_{t+1} = \omega_{t} + \nu_{t}(\bar{\omega}_{t} - \omega_{t})$, which further leads to

$$
\frac{9}{8\alpha\nu_{t}}\mathbb{E}\left\| \nabla_{\theta} F(\theta_{t+1}, \omega_{t+1}) - \nabla_{\theta} F(\theta_{t}, \omega_{t}) \right\|^2 \leq \frac{9L_{F}^{2}}{8\alpha\nu_{t}}\left(\|\theta_{t+1} - \theta_{t}\|^2 + \|\omega_{t+1} - \omega_{t}\|^2\right) \\
\leq \frac{9L_{F}^{2}\nu_{t}}{8\alpha}\left(\|\bar{\theta}_{t+1} - \theta_{t}\|^2 + \|\bar{\omega}_{t+1} - \omega_{t}\|^2\right) \tag{30}
$$

Combining Eq. (29) and Eq. (30) leads to

$$
(1 - \alpha\nu_{t})^2 \mathbb{E}\left\| \nabla_{\theta} F(\theta_{t+1}, \omega_{t+1}) - p_{t} \right\|^2 \leq \frac{9L_{F}^{2}\nu_{t}}{8\alpha}\|\bar{\theta}_{t+1} - \theta_{t}\|^2 + \frac{9\nu_{t}L_{F}^{2}}{8\alpha}\|\bar{\omega}_{t+1} - \omega_{t}\|^2 \\
+ (1 - \alpha\nu_{t})\mathbb{E}\|\nabla_{\theta} F(\theta_{t}, \omega_{t}) - p_{t}\|^2 \tag{31}
$$

which upper bounds the first term of the RHS in Eq. (28).

For the third term of the RHS in Eq. (28), Lemma C.5 implies that

$$
\alpha^2\nu_{t}^{2}\mathbb{E}\left\| \nabla_{\theta} F(\theta_{t+1}, \omega_{t+1}) - \nabla_{\theta} f(\theta_{t+1}, \omega_{t+1}; \xi_{t+1}) \right\|^2 \leq \alpha^2\nu_{t}^{2}\sigma^{2} \tag{32}
$$

Eq. (22) follows from combining Eq. (28), Eq. (31) and Eq. (32). The proof of Eq. (23) is similar to that of Eq. (22) and is omitted here.\[\square\]

**G Implementation Details**

The parameters of all the algorithms are introduced as follows. The value function is parameterized by a two-layer fully-connected neural network with 50 hidden neurons and ELU activation function [6]. The discount factor $\gamma$ is set to 0.95 as in [29]. We set the feasible sets as $\Theta = [-1, 1]^{d}$ and $\Omega = [-1, 1]^{d}$ where $d$ is the dimension of the neural network’s parameters. For DPTD and TD, we set $\alpha = 3$, $\beta = 3$, $\kappa = 2$, $\eta = 2$, $\nu_{t} = \frac{1}{(t+3)^{1/2}}$ as suggested in Theorem 5.2. The step sizes of DPGLD and DPSRM are also taken as the suggested theoretical values in their original papers. For SGD, the step size is maintained in the same order with other algorithms, ranging from $10^{-3}$ to $10^{-4}$. We implement all the algorithms in PyTorch 1.5.1 [20] with Ubuntu 18.04 and an NVIDIA GTX 2080Ti GPU.
Additional Experiments

We conduct experiments to show the impact of different privacy budgets on the convergence of DPTD. Specifically, we plot the utilities of DPTD under $\epsilon = 100.0$, $\epsilon = 10.0$, $\epsilon = 1.0$ and $\epsilon = 0.1$ respectively. As $\epsilon$ increases, DPTD will have more privacy budgets and the primal and dual gradients of DPTD will be perturbed by Gaussian noises with smaller variance. Therefore, DPTD under larger $\epsilon$ will have better utility, which is validated in Figure 2, where DPTD under $\epsilon = 100.0$ has the best utility in all three tasks. Furthermore, the utility of DPTD will degrade as $\epsilon$ decreases and DPTD under $\epsilon = 0.1$ has the worst utility.

Figure 2: Utilities of DPTD under different privacy parameter $\epsilon$, including $\epsilon = 0.1$, 1.0, 10.0, 100.0. Each epoch has 5 finite trajectories. The shadow denotes 1-std. The learning curves are averaged over 10 random seeds and are generated without smoothing.