RESTRICITION ON TYPES OF COHERENT STATES
DUE TO GAUGE SYMMETRY

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From the viewpoint of the formulation of the SU(2) coherent states and their path
integrals labeled by a full set of Euler angles (φ, θ, ψ) which we developed in the previous
paper, we study the relations between gauge symmetries of Lagrangians and allowed
quantum states. We investigate permissible types of fiducial vectors in the full quantum
dynamics in terms of SU(2) coherent states. We propose a general framework for a
Lagrangian having a certain gauge symmetry with respect to one of the Euler angles ψ.
We find that when a Lagrangian has the gauge symmetry fiducial vectors are so restricted
that they belong to the eigenstates of $\hat{S}_3$ or to the orbits of them under the action of
the SU(2); And the strength of a fictitious monopole, which appears in the Lagrangian,
is a multiple of $\frac{1}{2}$. In this case Dirac strings are permitted. One exceptional case exists
when the fictitious monopole charge disappears. The reasoning here does not work for a
Lagrangian without the gauge symmetry. The relations between formulations and results
of the preceding work by Stone that has piloted us and those by ours are also discussed.

Keywords: Gauge symmetry; SU(2) coherent state path integral; fiducial vector;
monopole.

1. Introduction

Symmetry is one of the basic principles that penetrate all of physics: from classical
to quantum physics, from statistical or condensed matter physics to particle physics.
And from relativity to gauge field theory. Thus we see a wide range of symmetries:
from external (space-time) to internal ones; And from discrete to continuous ones.
This is partly because geometry is an indispensable element to describing physics. And a natural algebraic language to express geometrical
symmetry is group theory.

Classical mechanics is widely known to have close relations between symmetries
and Lagrangians or Hamiltonians. Since quantum mechanics and field theory are,
in some respects, modeled and devised after classical mechanics, we see that not
a few methods and notions, including symmetry, in quantum mechanics and field

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theory resemble those in classical mechanics. And thus symmetry plays a crucial role also in quantum physics. Or we can interpret that although “the physical world is quantum mechanical” the quantum feature are somewhat transmitted to the classical world; And through the latter we may try to grasp the former. Of course since we are not able to capture all Nature by classical analogies, there are sometimes discrepancies between quantum symmetries and classical ones.

Now, one of the typical mathematical tools that relate classical states with quantum ones is coherent states (CS). It was originally devised by Schrödinger as the states having classical “particle” nature. The state exhibits a wave packet whose center moves along with the classical trajectory with minimum uncertainty; Thus it shows classical nature. The original CS which is called canonical CS is, in the light of quantum optics, generated by displacing, or driving, the vacuum, i.e. the zero photon state. Later CS have been developed in a wide variety of directions. Viewed from a general framework, we may take up the following three subjects among the evolutions. First, CS have been extended to wider classes. A systematic way to broaden CS is constructing CS in terms of unitary irreducible representations of Lie groups due to Perelomov. In the approach, CS is defined by operating a unitary operator related to a physical system being considered on a “fiducial vector (FV)”, which we denote $|\Psi_0\rangle$. From this point of view, for the canonical CS the unitary operator is a displacement operator and a FV is the ground state or vacuum. Similarly, the spin CS can be constructed by operating a rotation operator on a FV; The FV is conventionally taken as $|s, s\rangle$ or $|s, -s\rangle$: the highest or lowest eigenvectors of $S_3$. We may perform the procedures to other Lie groups, which automatically produces CS for the corresponding Lie groups. Schur’s lemma coming from irreducibility always ensures the overcompleteness of CS. Second, since CS enjoy overcomplete relations, “coherent state path integrals (CSPI)”, i.e., path integrals (PI) via CS, have been developed. Such CSPI have been pushing the method of PI forward strongly. And besides CSPI turned out to be closely related to geometric phases; In fact it is remarkable that geometric phases follow from the topological terms of phase space PI or CSPI naturally. Third, following the fruits of the above two developments, CS and CSPI with arbitrary FV have been explored. In the case CS are obtained by operating a unitary operator on an arbitrary FV: For the canonical CS a FV is arbitrary superpositions of the Fock number states; And for the spin CS arbitrary superpositions of $|s, m\rangle$: a general eigenvector of $S_3$. We found that, as mentioned in Ref. the canonical CS evolving from arbitrary FV turned out to be an arbitrary superposition of displaced number states having no classical analogues. Similarly, we may regard spin CS evolving from a general FV other than the conventional one as quantum states without classical analogues. It is true that CS with the conventional FV are closest to classical states and have useful properties. However, recent technologies enable us to prepare quantum states which have no classical analogues; The typical one is the squeezed states of light.

\*We all realize that the very methods of quantization also fall within such examples.
We certainly regard the evolutions plausible since experimental developments due to high technologies have often created opportunities to reconsider Nature. In this respect we may take CS evolving from a generic FV as the mathematical tools, or a sort of new language, for describing non-classical quantum states. And thus what we have done in Refs. [25 and 26] is constructing new quantum states and investigating the dynamics: i.e., the time evolutions of the quantum states. We can interpret the attempts as extending both CS due to Schrödinger-Klauder-Glauber-Perelomov and PI due to Dirac-Feynman-Klauder-Kuratsuji-Suzuki.

In the previous paper [26], hereafter referred to as I, we have developed a basic formulation of the SU(2), i.e. spin, CS based on arbitrary FV and of their PI. The CS and CSPI are labeled by a full set of three Euler angles \( \Omega \equiv (\phi, \theta, \psi) \). Since the present paper flows directly out of I, we will look back the previous results concisely. In I we found out that the Lagrangian in the action appeared in the PI expression were composed of two parts: The topological term related to geometric phases and the dynamical one originating from a Hamiltonian. And the former is again split into two parts: One is the monopole type part which is the generalization of that of Balachandran et al.\(^{28,29,30}\) and the other represents the effect of entanglements between neighboring components of a FV. Such interweaving components of a FV appear in the dynamical term as well. The monopole is fictitious in that it does not represent a real physical monopole having a magnetic charge; Instead it stems from the topological or geometric phase terms. However, mathematical descriptions seem quite common to both real and fictitious monopoles. And we have confirmed the PI form by demonstrating from discrete PI to continuous ones. Moreover, it has been proved that the generic spin CSPI contract to the general canonical CSPI in the high spin limit.

At first sight it seems that we are free to choose FV; There are no restrictions on FV and we may take an arbitrary FV. However, when a Lagrangian varies at most a total derivative under a certain gauge transformation and possesses a sort of semiclassical symmetry, a full quantum state with an arbitrary FV does not always preserve the related symmetry. In the case, when a FV \(|\Psi_0\rangle\) cannot be reached from \(|s, m\rangle\) via \(\hat{R}(\Omega)\), we find some strange feature: Semiclassical orbits do not always represent exact quantal evolutions. It is Stone who first observed that there does exist one of the central problems at the point viewed from a general framework of CS with arbitrary FV. He raised the problem in a paper\(^{31}\) commenting on the precursory version of I\(^{32}\). Moreover, he went further enough to propose a criterion under which the CS capture full quantal evolutions. According to that, an arbitrary FV is not always realized and that there may be restrictions on FV so that quantum evolutions are consistent with the semiclassical ones which has an original symmetry. Actually, the FV have to be identical with \(|s, m\rangle\) or on the orbits of \(|s, m\rangle\) under the action of \(\hat{R}(\Omega)\). And, as Stone precisely pointed out, the problem is deeply related to the charge quantization of monopoles.

In this article we consider the problem posed in Ref. [31] from the general framework of spin CSPI developed in I. As mentioned earlier, we have demonstrated
the process of going from the discrete PI to the continuous PI in \( \mathbf{I} \). We have also showed that the spin CSPI contract to the canonical CSPI. So the PI expressions in \( \mathbf{I} \) are quite all right and are not responsible for not bringing the restriction on FV. Then one might wonder from where the restriction comes. We will approach the riddle in the light of the “gauge symmetry” associated with the invariance of Lagrangian appeared in the spin CSPI in \( \mathbf{I} \). So the present article is also concerned with “symmetry”. It will be one of the early attempts that relate CS and CSPI with symmetry, especially gauge symmetry.

The plan of the paper is as follows. First, we look back the spin(SU(2))CSPI based on arbitrary FV in \( \mathbf{I} \). Next, using the formulation of the spin CSPI, we discuss general properties of a Lagrangian in the light of gauge symmetries in \( \mathbf{3} \). Next, in \( \mathbf{4} \) we demonstrate, using simple examples, the relations between types of FV and semiclassical as well as full quantal dynamics. We then look over several real examples of Lagrangians in order to see the relations between Hamiltonians, FV and gauge symmetries of the whole Lagrangians. Main results are presented as theorems and proved in \( \mathbf{5} \). Theorem 1 gives the central result concerning the restriction on FV in the full quantum picture. We find that gauge symmetries bring restrictions on FV and thus on the form of CS. We look into the situation much deeper by investigating the generator of the symmetry transformation in Theorem 2. Next, we revisit the gauge symmetries in the light of a new kind of isotropy subgroups due to Ref. 31 in \( \mathbf{6} \). And we see the correspondence between the approach and that in \( \mathbf{3} \) and \( \mathbf{4} \). Finally we summarize the results in \( \mathbf{7} \). There we also discuss the related topics that we view in a future prospect.

2. General SU(2) Coherent State Path Integrals

Let us recall the results in \( \mathbf{I} \). First, we define the spin(SU(2))CS, \( |\Omega\rangle \), evolving from an arbitrary FV \( |\Psi_0\rangle \) as:

\[
|\Omega\rangle \equiv |\phi, \theta, \psi\rangle = \hat{R}(\Omega)|\Psi_0\rangle = \exp(-i\phi\hat{S}_3)\exp(-i\theta\hat{S}_2)\exp(-i\psi\hat{S}_3)|\Psi_0\rangle.
\]

(1)

The FV \( |\Psi_0\rangle \) is represented by:

\[
|\Psi_0\rangle = \sum_{m=-s}^{s} c_m|m\rangle \quad \text{with} \quad \sum_{m=-s}^{s} |c_m|^2 = 1.
\]

(2)

Hereafter \( |m\rangle \) stands for \( |s, m\rangle \). From (1) and (2) we obtain

\[
|\Omega\rangle = \sum_{m=-s}^{s} c_m|\Omega, m\rangle \quad \text{with} \quad |\Omega, m\rangle \equiv \hat{R}(\Omega)|m\rangle.
\]

(3)

See (I - 19)\(^b\) for the explicit form of \( |\Omega, m\rangle \) which we do not need in the present paper. We called \( |\Omega, m\rangle \) the “rotated spin number state” in \( \mathbf{I} \) where we saw that it corresponded to the “displaced number state”\(^c\) in the general canonical CS.\(^{25}\)

\(^b\) Eq. (1 - 19) denotes Eq. (1) in \( \mathbf{I} \).

\(^c\) See, e.g., Ref. 33 and references therein.
Then the quantum time evolution of a physical system with a Hamiltonian $\hat{H}(\hat{S}_+, \hat{S}_-, \hat{S}_3; t)$ in terms of $|\Omega\rangle$ is given by the propagator:

$$K(\Omega_f, t_f; \Omega_i, t_i) = \int \exp\{(i/\hbar)S[\Omega(t)]\} D[\Omega(t)],$$

where

$$S[\Omega(t)] = \int_{t_i}^{t_f} \left[ \{\Omega| i\hbar \frac{\partial}{\partial t}|\Omega\rangle - H(\Omega, t) \} dt \equiv \int_{t_i}^{t_f} L(\Omega, \dot{\Omega}, t) dt,$$

with

$$H(\Omega, t) \equiv \langle \Omega | \hat{H} | \Omega \rangle.$$ (6)

The explicit form of the Lagrangian yields:

$$L(\Omega, \dot{\Omega}, t) = \hbar \left[ A_0(\{c_m\})(\dot{\phi} \cos \theta + \dot{\psi}) + A_3(\Omega, \dot{\Omega}; \{c_m\}) \right] - H(\Omega, t),$$

where

$$A_3(\Omega, \dot{\Omega}; \{c_m\}) \equiv -A_1(\psi; \{c_m\}) \dot{\phi} \sin \theta + A_4(\psi; \{c_m\}) \dot{\theta}.$$ (8)

The following expressions include what $A_0$, $A_1$ and $A_4$ mean:

$$\begin{align*}
A_0(\{c_m\}) &= \sum_{m=-s}^{s} m|c_m|^2, \\
A_1(\psi; \{c_m\}) &= (1/2) \sum_{m=-s+1}^{s} f(s, m)[c_m^* c_{m-1} \exp(i\psi) + c_m c_{m-1}^* \exp(-i\psi)] \\
A_2(\Omega; \{c_m\}) &= (1/2) \sum_{m=-s+1}^{s} f(s, m) \exp(i\phi) \{ (1 + \cos \theta) \exp(i\psi)c_{m}^* c_{m-1} - (1 - \cos \theta) \exp(-i\psi)c_{m-1}^* c_{m} \} \\
A_4(\psi; \{c_m\}) &= [(1/(2i)) \sum_{m=-s+1}^{s} f(s, m)[c_m^* c_{m-1} \exp(i\psi) - c_m c_{m-1}^* \exp(-i\psi)] \\
f(s, m) &= [(s + m)(s - m + 1)]^{1/2}.
\end{align*}$$ (9)

The term with the square brackets in the Lagrangian (7):

$$A_0(\{c_m\})(\dot{\phi} \cos \theta + \dot{\psi}) + A_3(\Omega, \dot{\Omega}; \{c_m\}),$$

stemming from $\langle \Omega | (\partial/\partial t)|\Omega\rangle$, may be called the “topological term” that is related to the geometric phases.

For operators $S \equiv (\hat{S}_+, \hat{S}_-, \hat{S}_3)$ that constitute $\hat{H}$, we have

$$\begin{align*}
\langle \Omega | \hat{S}_3 | \Omega \rangle &= A_0(\{c_m\}) \cos \theta - A_1(\psi; \{c_m\}) \sin \theta \\
\langle \Omega | \hat{S}_+ | \Omega \rangle &= A_0(\{c_m\}) \sin \theta \exp(i\phi) + A_2(\Omega; \{c_m\}) \\
\langle \Omega | \hat{S}_- | \Omega \rangle &= A_0(\{c_m\}) \sin \theta \exp(-i\phi) + A_4(\psi; \{c_m\}) \sin \theta \exp(-i\phi),
\end{align*}$$

where $A_i (i = 0, 1, 2)$ and $f(s, m)$ have already been given in (9) and $\hat{S}_\pm = \hat{S}_1 \pm i \hat{S}_2$.

Variational equations associated with the Lagrangian (7) are:

$$\begin{align*}
\hbar \{ [A_0(\{c_m\}) \sin \theta + A_3(\Omega, \dot{\Omega}; \{c_m\}) \cos \theta] \dot{\phi} + A_1(\psi; \{c_m\}) \dot{\psi} \} &= -\partial H/\partial \theta \\
\hbar \{ [A_0(\{c_m\}) \sin \theta + A_3(\Omega, \dot{\Omega}; \{c_m\}) \cos \theta] \dot{\theta} - [A_4(\psi; \{c_m\}) \sin \theta] \dot{\psi} \} &= \partial H/\partial \phi \\
\hbar \{ [A_4(\psi; \{c_m\}) \sin \theta] \dot{\phi} + A_1(\psi; \{c_m\}) \dot{\theta} \} &= \partial H/\partial \psi,
\end{align*}$$

(12)
3. Gauge Symmetry of Lagrangian

Following the results in §2 we now consider the relations between semiclassical time evolutions of $|\Omega\rangle$ and the properties of symmetries that Lagrangians possess.

We see from (7)-(9) that the $\psi$-variable does not take effect in $A_i$, $(i = 1, 2, 4)$ provided no neighboring $\{c_m\}$ exists for any $c_m$; Then $A_3$-terms vanishes and the topological term (10) takes the form:

$$A_0(\{c_m\})(\dot{\phi}\cos\theta + \dot{\psi}).$$

The form of (13) is a generalization of that in Ref. 29. Taking $s = \frac{1}{2}, c_{1/2} = 1$ in (10) yields the latter form. For such a FV, if we assume that $\hat{H}$ is linear in $S$, we see from (11) that $H(\Omega)$ does not depend on $\psi$. Consequently the form of the variational equations (12) becomes the same as that for the usual spin CSPI evolving from $|\Psi_0\rangle = |s\rangle$, or $|-s\rangle$ with $\pm s$ replaced with $A_0$. And thus we can choose any $\psi$ as far as semiclassical dynamics is concerned. In what follows we will investigate a little deeper such FV and $\hat{H}$ as above-mentioned that yield (13) and leave semiclassical dynamics invariant.

From the viewpoint of the symmetry of Lagrangian, we grasp the situation as follows. To begin with, in the present case we have

$$L(\Omega, \dot{\Omega}, t) = \hbar A_0(\{c_m\})(\dot{\phi}\cos\theta + \dot{\psi}) - H(\Omega, t).$$

Under the “gauge $\psi$-transformation”:

$$\hat{R}(\Omega) \rightarrow \hat{R}(\Omega) \cdot \exp(-i\hat{S}_3\dot{\psi}') = \hat{R}(\Omega'), \quad (\Omega' \equiv (\phi, \theta, \psi + \dot{\psi}')),$$

which moves the $\psi$-Euler angle, a ket vector $|\Omega\rangle$ changes as:

$$|\Omega\rangle \equiv \hat{R}(\Omega)|\Psi_0\rangle \rightarrow \hat{R}(\Omega)\exp(-i\hat{S}_3\dot{\psi}')|\Psi_0\rangle \equiv |\Omega'\rangle.$$ 

And besides we assume

$$H(\Omega) = H(\Omega').$$

Then the Lagrangian (14) behaves as:

$$L \rightarrow L + \hbar A_0\dot{\psi}'.$$ 

Hence we see that the Lagrangian (14) has what is called “weak invariance” in Refs. 28 and 29. The Lagrangian varies only by a total derivative term under a certain transformation. 28, 29, 30 Let us call it the “weak gauge symmetry” or the “weak gauge $\psi$-symmetry” here so as to stress the effect of the gauge $\psi$-transformation (15). We state them once more in the following definition:

**Definition 1.** If a Lagrangian is transformed according to (18) under (13), we say that the Lagrangian possesses the weak gauge symmetry or the weak gauge $\psi$-symmetry.
So far we have concentrated on the semiclassical time evolutions. We see that a FV with certain conditions meet the symmetry of semiclassical motions. However, if we consider the full quantum dynamics conformable to the $\psi$-invariance, more stringent conditions are required of FV. Moreover, we will find that the conditions on FV, i.e. those on $\{c_m\}$, for the symmetry of Hamiltonians or of the whole Lagrangians are changed when we take up Hamiltonians that are quadratic or higher in $S_\pm$. We will look into the situations more deeply using real sample Lagrangians in the next § 4.

4. Sample Lagrangians

In this section we will discuss the relation between FV, Hamiltonians and weak symmetry of Lagrangians above mentioned by demonstrating several concrete examples. The problem is closely related to semiclassical versus full quantum time evolutions first pointed by Stone. 31

First, we investigate a few simple examples to see how FV relate to symmetries and what semiclassical and full quantum time evolutions look like in § 4.1. This may help one to grasp various wider examples collected in Table 1 in the following § 4.2. The examples in § 4.1 are also included in Table 1.

4.1. Simple illustrations

In this subsection we illustrate, using simple examples of a Hamiltonian and FV, the relations between semiclassical paths and exact quantal time evolutions.

Before going into the examples, let us recollect the general theory of CS. 16 Then we find that CS are determined in connection with FV. Let $G$ be a Lie group of our concern. Consider the case in which there exists a subgroup of $G$, say $H$, that leaves a FV, $|\Psi_0\rangle$, invariant:

$$H|\Psi_0\rangle = \exp(i\alpha)|\Psi_0\rangle \quad (\alpha : \text{phase}) \quad (19)$$

Such a subgroup $H$ is called the “isotropy subgroup” or a “stabilizer”. Then CS is actually defined on the coset space $G/H$. 3 If there is no isotropy subgroups, we may regard $H = \{1\}$. Then $G/H$ is $G$ itself. Hence we see that an isotropy subgroup depends upon the way in which we choose a FV, $|\Psi_0\rangle$.

Consider the present SU(2) CS case: $G = SU(2)$. If a FV is $|\Psi_0\rangle = |m\rangle$, then $H = U(1)$ and the spin CS is actually determined on the coset space $G/H = SU(2)/U(1)$ which corresponds to a Bloch sphere $S^2$; And thus $|\Omega\rangle \rightarrow |\theta, \phi\rangle$. This includes what we always do in constructing the conventional spin CS with $|\Psi_0\rangle = |s\rangle$ or $|-s\rangle$; And we may call the FV “standard”. The corresponding CS fall within what is called the “informative” CS in Ref. 31. We see that CS is invariant under the

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3 Usually $G$ and $H$ are frequently employed in place of $G$ and $H$. In order that one does not confuse them with the Hamiltonians $\hat{H}$, $H$ and the generator $G$ in the following sections we use calligraphic $G$ and $H$ to express general groups here.
transformation $\mathcal{H} = \{ \exp(-i\psi S_3) \}$. The case is illustrated in §4.1.1 below. Next, if we take a FV other than $|\Psi_0\rangle = |m\rangle$, then $\mathcal{H}$ is not $U(1)$ no more but $\{1\}$. Hence $\mathcal{G}/\mathcal{H} = \mathcal{G} = SU(2) \simeq S^3$ is specified by a full set of three Euler angles: $\Omega = (\phi, \theta, \psi)$. For this time $\exp(-i\psi S_3)|\Psi_0\rangle \neq \exp(i\alpha)|\Psi_0\rangle$ ($\alpha$ : phase). However, there are cases in which the “little group”\(^{[7]}\) that leaves semiclassical states invariant exists. The group turns out to be $\{ \exp(-i\psi' G) \}$ where $G$ is the generator of the weak symmetry transformation (15); The explicit form will be given later in §5.2. See §4.1.2 for a sample FV. The case has something to do with another type of isotropy subgroups proposed by Stone\(^{[31]}\); And we will investigate the case from a slightly different viewpoint in §6. The case without even the weak symmetry is treated in §4.1.3.

Now, we will treat concrete examples below. The examples, which are discussed by Stone\(^{[31]}\), are simplified versions of Ref.\(^{[32]}\). Take a spin in a constant magnetic field $B = (0, 0, B)$. The Hamiltonian is:

$$\hat{H} = -\mu B S_3. \quad (20)$$

In what follows we will illustrate, taking three typical types of FV, the relations between Hamiltonians, FV, the $\psi$-symmetry of Lagrangians. We also demonstrate how semiclassical motions concern the full quantum ones for the FV.

4.1.1. Standard FV

First, let a FV $|\Psi_0\rangle = |m\rangle$. It is rather a standard case where $\mathcal{H} = U(1)$. We have $A_0 = m$ and the $A_3$-term is absent; And thus the topological term has the weak $\psi$-symmetry as we saw it in §3. Besides since the $A_1$- and $A_2$-term vanish,

$$H(\Omega) = -m \mu B \cos \theta \quad (21)$$

contains no $\psi$-variables and is symmetric under (15). Consequently, the Lagrangian:

$$L = m (\dot{\phi} \cos \theta + \dot{\psi}) + m \mu B \cos \theta \quad (22)$$

possesses the weak $\psi$-gauge symmetry (18). The above situation is summarized in the column (i) in Table 1. See the following §4.2.

The variation equations (12) are:

$$\dot{\phi} = -(\mu B / \hbar), \quad \dot{\theta} = 0. \quad (23)$$

The third equation in (12), describing the behavior of $\dot{\psi}$, is automatically satisfied. It means that any $\psi$ works well as far as the semiclassical motions are concerned. From (23) we obtain, setting the initial state as $\Omega(t=0) \equiv \Omega_0 = (0, \theta_0, \psi_0)$:

$$\phi = -(\mu B / \hbar)t, \quad \theta = \theta_0, \quad \psi = \psi(t) \quad (\psi(t): \text{arbitrary}) \quad (24)$$

\(^{[7]}\) We take “$-i\psi$” after our Euler angle convention.

\(^{[31]}\) We take a minus sign for the convenience of later arguments.
Therefore the semiclassical motion is given by:

$$|\Omega_{SC}(t)\rangle = \exp\{-i[(\mu B)/\hbar]\hat{S}_3 t\} \cdot \exp(-i\theta_0 \hat{S}_2) \cdot \exp[-i\psi(t)]|m\rangle. \tag{25}$$

On the other hand, we see that the full quantal time evolution is also described in terms of spin CS with the Euler angle $\Omega_{FQ}(t)$:

$$|\Omega_{FQ}(t)\rangle = \exp\{-i[(\mu B)/\hbar]\hat{S}_3 t\}|\Omega(0)\rangle = \exp\{-i[(\mu B)/\hbar]\hat{S}_3 t\} \cdot \exp(-i\theta_0 \hat{S}_2) \cdot \exp(-i\psi_0)|m\rangle. \tag{26}$$

For the present FV, i.e., $|m\rangle$, each factor, $\exp[-i\psi(t)\hat{S}_3]$ in (25) and $\exp(-i\psi_0\hat{S}_3)$ in (26), when acting on $|m\rangle$, yields a trivial phase factor respectively; And thus two states (25) and (26) belong to the same ray living on $SU(2)/U(1) \simeq S^2$; Consequently, both semiclassical and genuine quantum time evolutions coincide. We may interpret that the $\psi$-gauge symmetry is preserved also in the full quantum dynamics.

Next, let us see the relation between the full quantum propagator and the semiclassical one. For an arbitrary final state $|\Omega_f\rangle$ at $t = t_f$ they are given by $\langle \Omega_f|\Omega_{FQ}(t)\rangle$ and $\langle \Omega_f|\Omega_{SC}(t)\rangle$ respectively. Then it is clear that from (25) and (26) the relation between two propagators are:

$$\langle \Omega_f|\Omega_{FQ}(t)\rangle = \langle \Omega_f|\Omega_{SC}(t)\rangle \cdot \exp\{i\ m \ [\psi(t) - \psi_0]\}. \tag{27}$$

The semiclassical propagator obeys the full quantal one up to $\psi$-gauge dependence. What we see here is one of the concrete examples of the dynamics for informative CS. Notice that for a round trip in which the final $\psi$ differs from $\psi_0$ by $2\pi$ or $4\pi$ two propagators fully agree since $m$ is an integer or a half integer.

### 4.1.2. Non-standard FV1

Second, we put FV as: $|\Psi_0\rangle = (\sqrt{2/3}, 0, \sqrt{1/3})^T \notin \hat{R}(\Omega)|m\rangle \ (m = 1, 0, -1)$. As the previous FV only $A_0$-term survives among $A_i \ (i = 0, \cdots, 4)$ since the terms involving $c_m c_{m-1}$ and its complex conjugate vanish. The Lagrangian reads:

$$L = \frac{1}{3}(\dot{\phi}\cos\theta + \dot{\psi}) + \frac{1}{3}\mu B \cos\theta. \tag{28}$$

It behaves like (18), showing the weak $\psi$-gauge symmetry. See the column (ii) in Table 1 in the following §4.2. The variation equations are the same as (23) and the solution gives:

$$|\Omega_{SC}(t)\rangle = \exp\{-[(\mu B)/\hbar]\hat{S}_3 t\} \cdot \exp(-i\theta_0 \hat{S}_2) \cdot \begin{pmatrix} \exp[-i\psi(t)]\sqrt{2/3} & 0 \\ \exp[i\psi(t)]\sqrt{1/3} & \exp[-i\psi(t)]\sqrt{1/3} \end{pmatrix}, \tag{29}$$

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\* One may confirm that $|\Psi_0\rangle \notin \hat{R}(\Omega)|m\rangle$ holds actually with the aid of the explicit form of $\hat{R}(\Omega)$. See, e.g., Messiah and references cited in I.
Since the full quantum time evolution operator is clearly the same as that in (26),
the corresponding state is given by:

$$|\Omega_{FQ}(t)\rangle = \exp\{-i(\mu B)/\hbar \hat{S}_3 t\} \cdot \exp(-i\theta_0 \hat{S}_2) \cdot \exp\left(\frac{i}{\sqrt{1/3}}\right) \cdot \exp\left(\frac{-i\psi_0}{\sqrt{2/3}}\right).$$  (30)

We realize that this time each factor, \(\exp\{-i\psi(t) \hat{S}_3\}\) in (29) and \(\exp\{-i\psi_0 \hat{S}_3\}\) in (30), when acting on the present FV yields a non-trivial phase factor that changes the quantum states basically. If we have \(\psi(t) \equiv \psi_0\) for all \(t\), two evolutions coincide. However, there seems no a priori reason to set \(\psi(t) \equiv \psi_0\); For we know that the Lagrangian (7) has the \(\psi\)-gauge symmetry. And thus we conclude that the semi-classical time evolution does not agree with the genuine quantum time evolution for this non-trivial FV as pointed out by Stone. \(31\) Notice that the relation between full quantum propagators and the semiclassical one is no longer simple as (27), but we have instead:

$$\langle \Omega_f | \Omega_{FQ}(t) \rangle = \langle \Omega_f | \hat{R}(\Omega_{SC}) | \tilde{\Psi}_0 \rangle \quad \text{with} \quad | \tilde{\Psi}_0 \rangle \equiv \sum_{m=-1}^{1} c_m \exp[i m(\psi(t) - \psi_0)] |m\rangle. \quad (31)$$

### 4.1.3. Non-standard FV2

Third, we consider the FV: \(|\Psi_0\rangle = (\sqrt{1/2}, \sqrt{1/6}, \sqrt{1/3})^T \not\in \hat{R}(\Omega)|m\rangle \quad (m = 1, 0, -1). \) See the column (v) in Table 4 in the following §4.2 and one will find the both \(A_3\)-term and \(H\) have different properties from two above-mentioned FV cases, thus yielding the Lagrangian without the \(\psi\)-symmetry. We have the Lagrangian:

$$L = \frac{1}{6} \hbar \{[\hat{\phi} \cos \theta + \psi] + \tilde{c} [\hat{\theta} \sin \psi - \hat{\phi} \sin \theta \cos \psi]\} - \frac{1}{6} \mu B [\cos \theta - \tilde{c} \sin \theta] \cos \psi, \quad (32)$$

where \(\tilde{c} \equiv 2 + \sqrt{6}\). The variation equations \(12\) give:

$$\begin{align*}
\hbar \{[\sin \theta + \tilde{c} \cos \theta \cos \psi] \hat{\phi} + \tilde{c} \hat{\psi} \cos \psi\} &= -\mu B (\sin \theta + \tilde{c} \cos \theta \cos \psi), \\
\hbar \{[\sin \theta + \tilde{c} \cos \theta \cos \psi] \hat{\theta} - (\tilde{c} \cos \theta \sin \psi) \hat{\psi}\} &= 0, \\
\hbar (\hat{\phi} \sin \theta \sin \psi + \hat{\theta} \cos \psi) &= -\mu B \sin \theta \sin \psi.
\end{align*}$$  (33)

One can verify that \(33\) yields a solution:

$$\hat{\phi} = -(\mu B/\hbar)t, \quad \hat{\theta} = \theta_0, \quad \hat{\psi} = \psi_0 \quad (\psi_0: \text{const.}), \quad (34)$$

which provides the same time evolution of the state as that for the exact quantum dynamics. For we know that the time development operator acting on a FV for the latter is the same as that in (26) or (30).
4.2. Various examples of Lagrangians

When a Hamiltonian $\hat{H}$ includes terms that are quadratic or higher in $\hat{S}_\pm$, $\psi$-dependence of $H$ will be changed from that for $\hat{H}$ linear in $\hat{S}_i (i = \pm, 3)$ even for the same FV. And thus the gauge $\psi$-symmetry of the whole Lagrangian will be also modified. We have to be careful that generally in CSPI the combination of a Hamiltonian and a FV together determines whether the corresponding Lagrangian possesses the weak gauge $\psi$-symmetry or not. This is a crucial difference from Refs. 28–30. In the present subsection we take up two typical Hamiltonians that are composed of $\hat{S}$; They are the NMR and NQR types of Hamiltonians given below:

\[
\hat{H}_{\text{NMR}} = -\mu B \cdot S \quad \text{(NMR type)}
\]

and

\[
\hat{H}_{\text{NQR}} = \omega_Q (B \cdot S)^2 \quad \text{(MQR type)}.
\]

We can obtain $H$ in (35) as $H_{\text{NMR}} \equiv \langle \Omega | \hat{H}_{\text{NMR}} | \Omega \rangle$; We may have the explicit form with the aid of (11). Similarly $H_{\text{NQR}}$ is evaluated with the aid of (11–A.8). For the present purpose, however, we do not have to know the explicit forms of $H_{\text{NMR}}$ and $H_{\text{NQR}}$; All that we need to grasp is the following: $H_{\text{NMR}}$ is independent of $\psi$ and hence invariant under (15) if and only if no nearest neighboring $\{c_m\}$ exists for any $c_m$; And the condition for $H_{\text{NQR}}$ to be invariant under (15) is that both no nearest neighboring $\{c_m\}$ and no next nearest neighboring $\{c_m\}$ in a given FV exist for any $c_m$.

We demonstrate various examples of combinations of Hamiltonians and FV, including those illustrated in §4.1, in Table 1 below; And we indicate the semiclassical weak gauge $\psi$-symmetry of their topological term, $H$ and the whole Lagrangian. As we put in §4.1 all the examples that meet the semiclassical symmetry do not always preserve the symmetry in full quantum dynamics. We will proceed to the problem in the next section.

In Table 1 notations like $(\sqrt{2/3}, 0, \sqrt{1/3})^T \equiv (\frac{2}{3})^{1/2}|1\rangle + (\frac{1}{3})^{1/2}|-1\rangle$, for instance, are used. And Hamiltonians (35) and (36) are referred to as NMR and NQR respectively. Notice that in case (i) the spin actually meets $s \geq 1$ for NQR type Hamiltonians.

5. Restriction on FV due to Weak Gauge Symmetry

We have looked over various examples of FV, Hamiltonians, Lagrangians and the weak symmetries in §4. Some Lagrangians meet the “semiclassical” weak symmetry; And some do not. Do the “full quantum” states realize the symmetry that the former Lagrangians possess? We know from §4.1 that the answer is not affirmative. So there may be a kind of restriction on quantum states. This falls within the problem that Stone took up more than a decade ago. It is the problem of realizable FV in CS.
Table 1. Examples of Lagrangians and weak gauge symmetry

|    | (i) | (ii) | (iii) | (iv) | (v)  | (vi) | (vii) |
|----|-----|------|-------|------|------|------|-------|
| spin | arbitrary | 1    | 1     | 1    | 1    | 3/2  | 3/2   |
| $\hat{H}$ | NMR | NMR  | NQR   | NMR  | NMR  | NMR  | NQR   |
| FV $|\Psi_0\rangle$ | $|m\rangle$ | $\left(\begin{array}{ccc} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{array}\right)$ | $\left(\begin{array}{ccc} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{array}\right)$ | $\left(\begin{array}{ccc} \sqrt{1/3} \\ 0 \\ \sqrt{1/3} \end{array}\right)$ | $\left(\begin{array}{ccc} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{array}\right)$ | $\left(\begin{array}{ccc} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{array}\right)$ | $\left(\begin{array}{ccc} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{array}\right)$ |
| $A_0$ | $m$ | 1/3  | 1/3   | 0    | 1/6  | 1/2  | 1/2   |
| $A_3$-term | absent | absent | absent | present | present | absent | absent |
| weak symmetry of the topological term | Yes | Yes | Yes | No | No | Yes | Yes |
| symmetry of $H$ | Yes | Yes | No | No | No | Yes | Yes |
| total weak symmetry of $L$ | Yes | Yes | No | No | No | Yes | Yes |

In this section we treat the problem of the restriction on types of spin CS, i.e. on those of the FV, in the full quantum picture when a spin CS Lagrangian has the weak semiclassical gauge $\psi$-symmetry in the light of the formalism in §2 and §3. This means that we impose such restrictions on FV in spin CS that reflect the weak semiclassical gauge symmetry. It provides one answer to the mystery on spin CSPI posed by Stone in Ref. 31. Or what we will perform is to see the results in Ref. 31 from a different point of view; from the viewpoint of the gauge symmetry of the action in spin CSPI. First, the general theorem, Theorem 1 is given and proved in §5.1. Second, in §5.2 we investigate the generator of the symmetry transformation, which yields Theorem 2. And then we revisit Theorem 1 via Theorem 2. It may help us to understand what is going on concretely.

5.1. General results

We consider a class of Lagrangians which we have formulated in §8 and illustrated in §4. We treat a Lagrangian with the weak gauge $\psi$-symmetry in which $A_3 = 0$ as well as $H(\Omega)$ is invariant under the transformation. Then the whole Lagrangian changes at most a total derivative. Among the examples in the preceding §4.2 (i), (ii), (vi) and (vii) meet the condition; See the Table 1. In the cases the following
theorem holds. And thus actually only the FV in (i) survives; FV in the form of (ii), (vi) or (vii) are ruled out. The results agree with those indicated by Stone from the viewpoint of two types of isotropy subgroups associated with semiclassical and full quantum dynamics. For the cases (iii), (iv) and (v) the theorem will not tell anything.

**Theorem 1.** If a Lagrangian associated with SU(2) CS has the weak gauge symmetry related to $\psi$-variable, the fiducial vector belongs to $|m\rangle$ or to the orbit of $|m\rangle$ under the action of $\hat{R}(\Omega)$ and the Dirac condition holds for $A_0$.

Before going into the following proof, we briefly mention in which direction we will proceed. Doing so is adequate for the purpose.

Since we concentrate on spin degrees of freedom, spin CS and FV, our Lagrangian differs from Refs. 28–30. The method by Aitchison in Ref. 30, however, applies also to ours; And thus we mainly proceed along Ref. 30 in the following with suitable changes; we particularly observe how a state vector $|\Omega\rangle$ behaves. That is our strategy. Let us start the proof now.

**Proof:** Since we assume that the gauge $\psi$-symmetry holds for a Lagrangian, the Lagrangian takes the form of (14). And under the transformation (15) a ket vector $|\Omega\rangle$ and the Lagrangian change in the manners of (16) and (18) respectively.

Now, denote $\hat{G}$ such an operator that makes a transformation (16) on $|\Omega\rangle$:

$$\exp(-i\hat{G}\psi')|\Omega\rangle = \exp(-i\hat{G}\psi')\hat{R}(\Omega)|\Psi_0\rangle = \hat{R}(\Omega)\exp(-i\hat{S}_3\psi')|\Psi_0\rangle. \tag{37}$$

This means that $\hat{G}$ is the generator of (16); And also of (18) since we have assumed the FV and $\hat{H}$ meet the conditions on weak symmetry. In terms of $\hat{G}$, the change of $L$ becomes:

$$L \rightarrow L + \hbar \langle \Omega|\hat{G}|\Omega\rangle \psi', \tag{38}$$

where we have used the expression of $L$ in (5). Then we obtain from (18) and (38)

$$\langle \Omega|\hat{G}|\Omega\rangle = A_0, \tag{39}$$

which is considered as a condition on semiclassical symmetry.

From (37) we have the following relation between operators:

$$\exp(-i\hat{G}\psi')\hat{R}(\Omega) = \hat{R}(\Omega)\exp(-i\hat{S}_3\psi'). \tag{40}$$

For an infinitesimal transformation we have

$$\hat{G}\hat{R}(\Omega) = \hat{R}(\Omega)\hat{S}_3. \tag{41}$$

The explicit form of $\hat{G}$, which we do not need in the present context, is given in §5.2. Notice that (41) is not identical with the corresponding expression in Ref. 30. The latter is $[\hat{G}, \hat{R}(\Omega)] = \hat{R}(\Omega)\hat{S}_3$.

Let us impose the condition on a full quantum state so that the dynamics is conformable to semiclassical $\psi$-symmetry. The definition of $\hat{G}$ in (37), the consequent weak symmetry (18) and Noether’s theorem tell us that the state vector is
subject to a certain restriction in the sense of Dirac. And thus we require a sort of Gauss’ law:

\[ \hat{G}|\Omega\rangle = A_0|\Omega\rangle. \]  

(42)

Note that (42) is different from (39). It is clear that (42) is more stringent than (39). A set of \{|\Omega\rangle\} that satisfies (42) also meets (39). However, the reverse does not always hold. We will return to the relation between (39) and (42) in §6.

As the finite form of (42) we have

\[ \exp(-i\hat{G}\psi')|\Omega\rangle = \exp(-iA_0\psi')|\Omega\rangle. \]  

(43)

On the other hand, with the aid of (40),

\[ \exp(-i\hat{G}\psi')|\Omega\rangle = \hat{R}(\Omega)\exp(-i\hat{S}_3\psi')|\Psi_0\rangle = \sum_{m=-s}^{s} c_m \exp(-im\psi')|\Omega, m\rangle. \]  

(44)

Combining (43) with (44) we obtain

\[ \exp(-iA_0\psi') \sum_{m=-s}^{s} c_m |\Omega, m\rangle = \sum_{m=-s}^{s} c_m \exp(-im\psi')|\Omega, m\rangle. \]  

(45)

Only a few FV meet the condition (45); And indeed it is possible if and only if

\[ |\Psi_0\rangle = |m\rangle. \]  

(46)

The FV has only one nonzero component: The state vector becomes

\[ |\Omega\rangle = |\Omega, m\rangle, \]  

(47)

which is what we call “rotated spin number states” in I. From this we see

\[ A_0 = m. \]  

(48)

Since \( m = \cdots, -\frac{1}{2}, 0, \frac{1}{2}, \cdots \), Eq. (45) indicates that the Dirac condition holds for \( A_0 \).

Let us put one more point about a FV. Decompose \( \hat{R}(\Omega) = \hat{R}(\Omega')\hat{R}(\Omega'') \) and we will obtain

\[ |\Omega\rangle = |\Omega'\rangle = \hat{R}(\Omega')|\Psi'_0\rangle \]  

with \( |\Psi'_0\rangle \equiv \hat{R}(\Omega'')|m\rangle \).

(49)

From this viewpoint we may interpret \( |\Psi'_0\rangle \) as a FV. Hence, from (46) and (49), we see that the FV coincides with \( |m\rangle \), one of the eigenstates of \( \hat{S}_3 \), or rides on the orbit of \( |m\rangle \) under the SU(2) rotations.

The above Theorem implies the strange feature of CSPI and FV: i.e., the Lagrangians, which depend upon the kinds of FV as shown in Table I in turn restrict them. The restriction condition determines the forms of FV. As a result, the FV belongs to \( |\Psi_0\rangle = |m\rangle \) \( (m = s, s - 1, \cdots, -s) \), or can be reached from \( |m\rangle \) by \( \hat{R}(\Omega) \). Mathematically, they are on orbits of \( |m\rangle \) under the action of the
SU(2) group. And the Dirac condition is permitted. This gives an answer to the problem posed in Ref. 31 in the light of our spin CSPI formalism.

Notice that the reverse statement does not hold: The Dirac condition does not always imply $|\Psi_0\rangle = |m\rangle$. This is apparent since a FV $|\Psi_0\rangle = (\frac{3}{2}, 0, 0, (\frac{1}{2}))^T$ in (vi) and (vii) in Table 1 yields $A_0 = \frac{1}{2}$.

However, there is clearly an exceptional case: i.e. $A_0 = 0$ case in which the FV may have several nonzero components.

Next, we revisit (45) from another viewpoint; For a spin $s = 0, \frac{1}{2}, \frac{3}{2}, \cdots$ case, $m = 0, \pm 1, \pm 2, \cdots$. Putting $\psi' = 2\pi$ in (45) we have $\exp(2\pi i A_0) = 1$, thus yielding $A_0 = n$ ($n : \text{integer}$). It is consistent with (48). For a spin half-integer case, putting $\psi' = 4\pi$ in (45), we have $\exp(4\pi i A_0) = 1$; And thus we obtain $A_0 = \frac{1}{2}n$ ($n : \text{integer}$). Since $m = \pm \frac{1}{2}, \pm \frac{3}{2}, \cdots$ in this case, the result agrees with (48). We have already made a similar argument about semiclassical and full quantum propagators after (27). Note that for a spin $\frac{1}{2}$ case, it is always possible to write any FV in the form of $|\Psi_0\rangle = R(\Omega_0)|\frac{1}{2}\rangle$ using a suitable Euler angle $\Omega_0$: For spin $\frac{1}{2}$ any FV can be reached from $|\frac{1}{2}\rangle$ or $|\frac{1}{2}\rangle$. We are able to describe any two-state system in terms of SU(2) CS. So the condition (49) holds automatically. What we have seen here is a CS version of magnetic charge quantization which has been known for a long time. 30

Now, it is widely known that the Dirac condition is related to Dirac strings. A Dirac string, extending from the origin to half infinity in the $(\phi, \theta)$-space, corresponds to choosing $\psi = \phi$ or $\psi = -\phi$ “section” in the topological term (13). In the present case it is surely possible to prepare the above $\psi$ since the gauge $\psi$-transformation promises the freedom. And the freedom, as we saw, comes from selecting the special type of FV: $|\Psi_0\rangle = |m\rangle$. Hence, looking from the present spin CSPI formalism, whether Dirac strings are permissible or not depends upon the types of FV.

One may observe that conventional arguments about the Dirac condition often imply that the particle interacting with a pole has spin 0 or $\frac{1}{2}$, which falls within the type of FV that meets Theorem 1. For a generic spin $s$, however, we have wider possibilities on FV. And thus we may expect so also on magnetic charge. It is, of course, an open question whether we can apply our discussion to real magnetic monopoles. However, the approach presented here may provide us with a fine view of real monopoles since real and fictitious monopoles enjoy common mathematical descriptions.

5.2. Explicit form of the generator $\hat{G}$

In the preceding subsection we used the operator $\hat{G}$ satisfying (37) or (40). Although we have not needed its explicit form, a natural question arises: what does it look like? We find the explicit form of the generator $\hat{G}$ easily in Theorem 2 below. This may be a by-product of Theorem 1. Conversely, however, the form brings us to Theorem 1 via another route again. Hence Theorem 2 helps us to understand Theorem 1.
Theorem 2. For an operator $\hat{G}$ to satisfy (37) or (40) it is necessary and sufficient that $\hat{G}$ is expressed as

$$\hat{G} = \hat{R}(\Omega)\hat{S}_3\hat{R}^+(\Omega) = S \cdot n,$$

(50)

where $n \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi)$.

Proof: Let us see the necessity first. From (40) we have

$$\exp(-i\hat{G}\psi') = \hat{R}(\Omega)\exp(-i\hat{S}_3\psi')\hat{R}^+(\Omega) = \exp[-i\hat{R}(\Omega)\hat{S}_3\hat{R}^+(\Omega)\psi'],$$

(51)

which, with the help of (I-A.7), results in (50). Next, that (50) is sufficient is obvious since we only need to cross the last equality in (51) in reverse order.

We see that the generator $\hat{G}$ is nothing but a sort of generalized “Hopf map”. The usual Hopf map is given by $\hat{R}(\Omega)\hat{S}_3\hat{R}^+(\Omega) = \sigma \cdot n.$ It is often referred to as the quantity which indicates the gauge $\psi$-symmetry; And it is also exactly what is taken as the tool of “gauge fixing” in Refs. 28 and 29.

Next, we revisit Theorem 1 via Theorem 2. Now, $\hat{S}_3$ is a $(2s+1) \times (2s+1)$ matrix whose eigenvectors are completely given by

$$\hat{S}_3|m\rangle = m|m\rangle.$$

(52)

Then, operating a non-singular matrix $\hat{R}(\Omega)$ on both sides of (52), we have, employing (50),

$$\hat{G}|\Omega, m\rangle = m|\Omega, m\rangle,$$

(53)

which specifies all the $(2s+1)$ eigenvectors of $\hat{G}$ thoroughly. Comparing (53) with (42), we realize that $|\Omega\rangle$ must coincide with one of the $\{|\Omega, m\rangle\}$. This leads us again to (47), from which we consequently obtain (40) and (48). We thus confirm that (40) and (48) hold again.

Next, we put two additional comments. First, the form of (50) can be obtained also by the infinitesimal relation (41). Since (41) is independent of $s$, we may try $2 \times 2$ matrices:

$$\hat{G} \equiv \alpha_+\hat{S}_+ + \alpha_-\hat{S}_- + \alpha_3\hat{S}_3 = \left(\begin{array}{cc} \frac{1}{2}\alpha_3 & \alpha_+ \\ \alpha_- & -\frac{1}{2}\alpha_3 \end{array}\right)$$

(54)

and $\hat{R}^{(1/2)}(\Omega)$; See (I-A.1) for the expression of $\hat{R}^{(1/2)}(\Omega)$. In this manner we obtain (50) again. It is clear that the direct evaluation of $\exp(-i\psi'\hat{G})$ using (51) leads to (40) as well; See Refs. 38 and 39 for such matrix calculations. Second, we point out that $A_0$ in (9), (39) and (53) are mutually consistent.
6. Another View of Semiclassical Motions

In Ref. 31 Stone characterized CS, FV and the consistency between semiclassical and full quantum dynamics by a slightly different way from ours. In order to see what they looked like, he introduced the isotropy subgroup $H_0$ for a given Lie group $G$ that stabilized the expectation values of the Lie group generators in the state of a FV in addition to the usual isotropy subgroup $H$ in (19). He showed that semiclassical orbits lived on $G/H_0$, whereas the full quantum dynamics was governed by $G/H$.

Let us concentrate on $H_0$ here. For the present spin CS case it reads:

$$H_0 = \{ h \in G | \langle h|\hat{S}_i|h\rangle = \langle \Psi_0|\hat{S}_i|\Psi_0\rangle \}, \quad (|h\rangle \equiv h|\Psi_0\rangle, \ i = \pm, 3). \quad (55)$$

Notice that $|0\rangle$ is used in place of $|\Psi_0\rangle$ and $H_{|0\rangle}$ instead of $H$ in Ref. 31. We take latter notations to keep harmony with the expressions in the preceding sections. One may feel that the relation between the description by $H_0$ and that by $H$ resembles the connection between (39) and (42). One describes the semiclassical dynamics; And the other is related to full quantum time evolution.

We now revisit the framework of Ref. 31 from our point of view in § 2 – § 5. Let us take up three representative types of FV which are numbered (i), (ii), (iv) in Table 1. Then we obtain the following results due to Ref. 31 illustrated in Table 2:

| Table 2. Two types of isotropy subgroups and FV |
|---|---|---|
| (i) | (ii) | (iv) |
| FV | $|m\rangle$ | $\begin{pmatrix} \sqrt{2/3} \\ 0 \\ \sqrt{1/3} \end{pmatrix}$ | $\begin{pmatrix} \sqrt{1/3} \\ \sqrt{1/3} \end{pmatrix}$ |
| $H$ | $\{ \exp(-i\psi'\hat{S}_3) \}$ | $\{ 1 \}$ | $\{ 1 \}$ |
| $H_0$ | $\{ \exp(-i\psi'\hat{S}_3) \}$ | $\{ \exp(-i\psi'\hat{S}_3) \}$ | $\{ 1 \}$ |

In cases (i) and (ii) we have $H_0 = \{ h \} = \{ \exp(-i\psi'\hat{S}_3) \}$. It is clear that the effect of $H_0$ in the Lagrangians amounts to that of (15) which features the weak $\psi$-symmetry; In fact we have used the convention and notation “$-\psi'$” so that one can see the accordance easily. Contrary, we have $H_0 = \{ 1 \}$ in the case (iv); And the case has nothing to do with the weak $\psi$-symmetry. Hence we are going to investigate (i) and (ii) with the aid of (55) instead of (15). Consider the topological term first. For a FV with parameter dependence we have a more appropriate interpretation of (I-35)
as:

\[ \hat{R}^+(\Omega) \frac{\partial}{\partial t} \hat{R}(\Omega) = \hat{R}^+(\Omega) \left( \frac{\partial}{\partial t} \hat{R}(\Omega) \right) + \frac{\partial}{\partial t} \]

\[ = -i(\dot{\phi} \cos \theta + \dot{\psi}) \hat{S}_3 + \frac{1}{2}(i\dot{\phi} \sin \theta - \dot{\theta}) \exp(i\psi) \hat{S}_+ \]

\[ + \frac{1}{2}(i\dot{\phi} \sin \theta + \dot{\theta}) \exp(-i\psi) \hat{S}_- + \frac{\partial}{\partial t}. \] (56)

Since (56) is linear in \( \hat{S}_i \), we observe that under the operation \( H_0 \) on \( |\Psi_0\rangle \) the following relation holds:

\[ \langle \Psi_0 | \hat{R}^+(\Omega)(\partial/\partial t) \hat{R}(\Omega) | \Psi_0 \rangle \rightarrow \langle h | \hat{R}^+(\Omega)(\partial/\partial t) \hat{R}(\Omega) | h \rangle \]

\[ = \langle \Psi_0 | \hat{R}^+(\Omega)(\partial/\partial t) \hat{R}(\Omega) | \Psi_0 \rangle - i\dot{\psi}', \] (57)

where we have used (55). Hence, the topological term \( \langle \Omega | (\partial/\partial t) | \Omega \rangle \) is weakly symmetric under \( H_0 \). The result may be obvious; For it is clear that (55) corresponds to (15). Besides, assuming \( \hat{H} = \hat{H}_{\text{NMR}} \) in (i), \( \hat{H} \) is also linear in \( \hat{S}_i \) in the present cases (i) and (ii) in Tables 1 and 2. And thus \( \langle \Omega | \hat{H} | \Omega \rangle \) is invariant under \( H_0 \). Then the whole Lagrangians possesses the weak \( \psi \)-symmetry. So we have confirmed that the condition (55) reproduces one about weak \( \psi \)-gauge symmetry for a Lagrangian discussed in §3.

If one tries to express the complete symmetry of Hamiltonians that have such higher order products of \( \hat{S}_i \) as \( \hat{H}_{\text{NQR}} \) in (36), more stringent conditions like

\[ H_0 = \{ h \in G | \langle h | \hat{S}_{i_1} \cdots \hat{S}_{i_\ell} | h \rangle = \langle \Psi_0 | \hat{S}_{i_1} \cdots \hat{S}_{i_\ell} | \Psi_0 \rangle \} \quad (i_1, \ldots, i_\ell = \pm, 3) \] (58)

may be required. Here \( \hat{S}_{i_1} \cdots \hat{S}_{i_\ell} \) is an arbitrary \( \ell \) product of \( \hat{S}_\pm \) or \( \hat{S}_3 \) that appears in the Hamiltonian. For example, we know \( \hat{H}_{\text{NQR}} \) gives \( \ell = 2 \); And augmented with (58) we see that the description due to \( H_0 \) brings a criteria for the weak gauge symmetry which works on all the cases in Table 1.

We have thus looked over two types of descriptions on CS, FV and semiclassical evolutions: one uses the shift of the \( \psi \) variable in \( \hat{R}(\Omega) \) as (15) and the other, i.e., (55) and (58), employs a transformed FV. They equally describe semiclassical dynamics well. And no matter what description we choose, both Ref. [3] and Theorem 1 in §5 tell us that if we have the gauge \( \psi \)-symmetry, we are led to a standard FV and informative CS in the full quantum dynamics.

7. Discussion

We have studied Lagrangians having a weak gauge symmetry in the light of spin CSPI with a general FV. We have set a condition on a state vector in order that the full quantum description keeps the semiclassical symmetry. This gives the restriction on FV. The types of CS get limited; It is mandatory that the spin CS ride on the orbits of \( |m\rangle \) under the action of \( \hat{R}(\Omega) \) in the full quantum dynamics. And the fictitious monopole charge \( A_0 \) is so quantized as to bring the Dirac condition.
Otherwise, the $\psi$-variable becomes “anomalous”. Of course in natural sciences it is Nature who gives a final decision. However, we expect that the results are all right in the case. Concerning the matter, we find that the results agree with those due to Stone,\cite{31} who first posed the problem in a general framework of CS and FV.

Notice that the rule to determine $A_0$, i.e. the monopole charge or strength, is not built in the CSPI a priori, but we impose it from physical demands — gauge $\psi$-symmetry. This is the way in which spin CSPI with general FV bring the Dirac condition for fictitious monopole charges. Remember that the monopole charge quantization condition is not derived by the Schrödinger equation itself, but by boundary or topological conditions also in the usual wave function formalisms.\cite{40,41,42}

Now, let us see a future prospect. First, in the present paper, we have treated Lagrangians with the $\psi$-gauge symmetry. If we consider a Lagrangian without the symmetry, the situation looks rather different. See, for example, (iii), (iv) and (v) in Table 1. In the cases, we could not impose a subsidiary condition, i.e. Gauss’ law, which produces the quantization of a fictitious monopole charge $A_0$ anymore. Theorem 1 does not bring us any information for the case. And then what would happen to a FV, $|\Psi_0\rangle$, and $A_0$? The problem, as well as physical applications of the present case, seems to be so intriguing for future investigations.

Second, it is clear that the present formalism may be extended to wider CSPI cases; Among them SU(1, 1) and SU(3) cases sound most probable candidates. The latter case is, of course, related to QCD. What do the restrictions on FV give physical systems described in terms of these CS? Do they bring new information other than the previous ones?\cite{28,29,30} In this respect it might be as well to remember that Stone\cite{31} actually discussed the restriction on FV for CS constructed from wider Lie groups.

Third, we give subsidiary comments on CS and FV: One might observe a close formal analogy between CS with general FV and the ground states of many body systems or vacua in field theory.\cite{67,68} At least mathematical apparatus, a Lie group $G$ and its coset space $G/H$, are common to both of the cases.\cite{43} FV look like ground states or vacua. Of course, in the many body systems the ground states themselves are expressed in terms of CS. However, such ground states are also absorbed into arbitrary FV. Moreover, we may be able to prepare room for dealing symmetries associated with higher energy levels than vacua with the aid of the present arbitrary FV formalism. It is an open question as to whether there are some deeper implications behind the formal resemblance. In addition we want to indicate one more point on CS and FV. It is on the definitions of two types of isotropy subgroups $\mathcal{H}_0$ and $\mathcal{H}$ in §6. They correspond to the relation between (39) and (42). It seems natural to feel that they remind us of subsidiary conditions in covariant quantization of photons due to Gupta-Bleuler.\cite{44} In that case the expectation value of the operator describing the Lorentz condition for a state vector is more crucial than the effect of the operation on the vector itself to establish the connection between quantized photon field and classical electrodynamics. And a certain gauge transformation leaves the expectation value invariant. This is also the case for the
above-mentioned vacua in field theory; There again the expectation value of a field operator plays a central role. We have not yet known whether there exists any deeper meaning of the analogies or not either.

Finally, we know that condensed matter systems have possibilities to simulate monopoles and gauge field theories. Condensed matter physics makes it possible to examine such concepts in laboratories. The problems of CS and gauge symmetry discussed here will be of interest for a variety of realms of physics including condensed matter physics; For symmetry is one of the basic principles that penetrate all of physics.

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