On Eigenvalues of Geometrically Finite Hyperbolic Manifolds with Infinite Volume

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April 4, 2020

Abstract

Let $M$ be an oriented geometrically finite hyperbolic manifold of dimension $n \geq 3$ with infinite volume. Then for all $k \geq 0$, we provide a lower bound on the $k$-th eigenvalue of the Laplacian operator acting on functions on $M$ by a constant and the $k$-th eigenvalue of some neighborhood $\tilde{M}$ of the thick part of the convex core.

1. Introduction

As analytic data of a manifold, the spectrum of the Laplacian operator acting on functions contains rich information about the geometry and topology of a manifold. For example, by results of Cheeger and Buser, with a lower bound on Ricci curvature, the first eigenvalue is equivalent to the Cheeger constant of a manifold, which depends on the geometry of the manifold. The rigidity and richness of hyperbolic geometry make the connection even more intriguing: by Canary [4], for infinite volume, topologically tame hyperbolic 3-manifolds, the bottom of the $L^2$-spectrum of $-\Delta$, denoted by $\lambda_0$, is zero if and only if the manifold is not geometrically finite. If $\lambda_0$ of a geometrically finite, infinite volume hyperbolic manifold is 1, by Canary and Burger [2], the manifold is either the topological interior of a handlebody or an $\mathbb{R}$-bundle over a closed surface, which is a very strong topological restriction.

Conversely, if we have data about the geometry of a manifold, we can convert it into data about its spectrum, like lower bounds on eigenvalues, dimensions of eigenspaces, existence of embedded eigenvalues, etc. Schoen [16] showed that the first nonzero eigenvalue of a closed hyperbolic manifold is bounded below by a constant multiple of the reciprocal of volume squared. White [18] showed that for $\epsilon$-thick closed hyperbolic 3-manifolds with upper bounds on the rank of their fundamental groups, the eigenvalues are roughly constant multiple of reciprocal of volume squared. For the finite-volume noncompact case, Randol [9] and Dodziuk [8] showed that the bottom of the spectrum have a similar lower bound by reciprocal of square volume. Theorem 2.12 of [12] establish the subtle issue of existence of discrete eigenvalues of finite-volume noncompact $n$-manifolds: there are finitely-many eigenvalues of finite multiplicity in $[0, ((n-1)/2)^2)$. There are examples where there are infinitely-many eigenvalues in $[((n-1)/2)^2, \infty)$, but they seem to occur quite rarely. Hamenstädt [11] relates the eigenvalues of an oriented noncompact finite-volume pinched negatively curved $n$-manifold to the eigenvalues of the compact, thick part of the manifold. In particular, for
every oriented finite volume Riemannian manifold $M$ of dimension $n \geq 3$ and curvature $\kappa \in [-b^2, -1]$ and for all $k \geq 0$, Hamenstädt proved that
$$\lambda_k(M) \geq \min \left\{ \frac{1}{3} \lambda_k(\widehat{M}_{thick}), \frac{(n-2)^2}{12} \right\}$$
where $\widehat{M}_{thick}$ denotes the thick part of $M$ under a suitable choice of Margulis constant, and $\lambda_k(M)$ denotes the $k$th discrete eigenvalue of the Laplacian operator acting on functions on $M$ counting with multiplicities. By [12], like in the finite-volume noncompact case, for the infinite-volume case, there are at most finitely-many eigenvalues of finite multiplicity in $[0, ((n - 1)/2)^2)$; but there are no eigenvalues in $[((n - 1)/2)^2, \infty)$. Based on the technique of [14], we generalize the lower bound result there to geometrically finite, hyperbolic $n$-manifolds with infinite volume:

**Theorem 1.1.** Let $M$ be an oriented geometrically finite hyperbolic manifold of dimension $n \geq 3$ with infinite volume, whose cusps are all of maximal rank. Denote by $\widetilde{M}$ a suitable closed thick neighborhood of the convex core of $M$, which will be chosen later, and $\lambda_k(M)$ the $k$th discrete eigenvalue of the Laplacian operator acting on functions on $M$. Then for all $k \geq 0$, we have
$$\lambda_k(M) \geq \min \left\{ \frac{1}{3} \lambda_k(\widetilde{M}), \frac{(n-2)^2}{12} \right\}$$
where the boundary condition on $\widetilde{M}$ is Neumann.

**Remark 1.** The intuition for the constant $\frac{(n-2)^2}{12}$ is as follows. The constant $\frac{(n-2)^2}{4}$ is closely related to the geometry of the Margulis tubes and cusps: it is the infimum of Rayleigh quotients for tubes and finite volume cusps, and the computation of the Rayleigh quotients follow directly from the metric. For the class of functions we are interested in, the $L^2$-norm of $f$ on $\widetilde{M}$ is at least $1/3$ of the $L^2$-norm of $f$ on $M$. The latter can be summarized as, for functions with small enough Rayleigh quotients, the mass is $\frac{1}{3}$-concentrated on $\widetilde{M}$.

**Acknowledgment**

The author would like to express a deep gratitude to his advisor Nathan Dunfield for continuing and unwavering support, countless helpful insights and comments through the years of his PhD study. The author thanks Pierre Albin, John D’Angelo, Anil Hirani, Rick Laugesen and Hadrian Quan for helpful discussions and emails. Special thanks to Rick Laugesen for a helpful argument in the proof of Theorem 3.5. This work is partially supported by NSF grant DMS-1811156.

**2. Definition and Backgrounds**

In this section we introduce several definitions and properties that will be used later. Let $M$ be an oriented geometrically finite hyperbolic manifold of dimension $n \geq 3$ with infinite volume, whose cusps are of maximal rank.
1. Rayleigh Quotient:

For a smooth square integrable function \( f \) on \( M \), the Rayleigh quotient of \( f \), denoted by \( \mathcal{R}(f) \), is

\[
\mathcal{R}(f) = \frac{\int_M \|\nabla f\|^2}{\int_M f^2}
\]

In the following sections we assume \( f \) is not identically zero.

2. Min-max Theorem for Self-Adjoint Operators

The Laplacian operator acting on functions on \( M \) is a self-adjoint operator by [12]. Min-max characterization of the eigenvalues of a possibly unbounded operator can be useful when we have some inequalities regarding eigenvalues or eigenfunctions. It will also be used in a critical way when we restrict eigenfunctions on \( M \) to its core \( \tilde{M} \). We only study discrete eigenvalues \( \lambda_{n-2}^2 \), and they are below the essential spectrum (see Theorem 2.12 of [12]). The following Min-max theorem [17] is used later:

**Theorem 2.1.** Let \( A \) be a self-adjoint operator, and let \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) be the eigenvalues of \( A \) below the essential spectrum. Then

\[
\lambda_n = \min_{\psi_1, \ldots, \psi_n} \max_{\psi \in \text{span}(\psi_1, \ldots, \psi_n)} \{ \langle \psi, A\psi \rangle : \|\psi\| = 1 \}.
\]

where \( \psi_1 \) is in the domain of \( A \). If we only have \( N \) eigenvalues, then we let \( \lambda_N := \inf \sigma_{\text{ess}}(A) \) (the bottom of the essential spectrum) for \( n > N \), and the above statement holds after replacing min-max with inf-sup. In our case for a normalized function, \( \langle f, \Delta f \rangle = \int_M \|\nabla f\|^2 = \mathcal{R}(f) \), the Rayleigh quotient.

3. Thick-thin Decomposition for Geometrically Finite Manifolds with Infinite Volume:

Let \( M \) be a geometrically finite hyperbolic manifold \( M \) of dimension \( n \geq 3 \) with infinite volume. Denote by \( \epsilon = \epsilon(n) > 0 \) the Margulis constant. \( M_{\text{thin}} \) denotes all points of \( M \) whose injectivity radius \( \text{inj}(x) \leq \epsilon \) while \( M_{\text{thick}} = \{ x | \text{inj}(x) \geq \epsilon \} \). In our case, \( M_{\text{thin}} \) is a finite union of Margulis tubes and cusps with finite volume. \( M_{\text{thick}} \) will be decomposed into two parts: \( \tilde{M} \) and neighborhoods of geometrically finite ends with infinite volume which grows exponentially.

4. Core \( \tilde{M} \) and Metric Structure in its Complement:

The descriptions and properties of metric on cusps and tubes are relatively standard. Notations and descriptions are from [11] and [10].

(a) Cusp

A cusp \( T \) is an unbounded component of \( M_{\text{thin}} \). It corresponds to the quotient of a horoball under the action of a parabolic subgroup \( \Gamma \). A cusp of maximal rank corresponds to a parabolic subgroup of rank \( n - 1 \). Such a cusp is naturally disjoint from the ends with infinite volume of \( M \), which is not true for intermediate rank cusps with infinite volume. The metric structure on \( T \) can be described as follows:
The subgroup $\Gamma$ stabilizes a horosphere $H$ in the universal cover $X$ of $M$. $T$ is diffeomorphic to $H/\Gamma \times [0, \infty)$ with the diffeomorphism mapping each ray $z \times [0, \infty)$ to a geodesic in $T$. Denote by $dx$ the volume element on $H/\Gamma$ of the restriction of the Riemannian metric on $M$; then the volume form $\omega$ on $T$ can be written in the form of $\omega = e^{-(n-1)t}dx \wedge dt$.

(b) Margulis Tube

A Margulis tube $T$ is a bounded tubular neighborhood of a closed geodesic $\gamma$ in $M$ of length $l < 2\epsilon$, where $\epsilon$ is the Margulis constant. The geodesic $\gamma$ is called the core geodesic of the tube. For the metric and computations, we use Fermi coordinates adapted to $\gamma$. We start by fixing a parametrization of $\gamma$ by arc length on the interval $[0, l)$. Let $\sigma$ be the standard angular coordinates on the fibers of the unit normal bundle $N(\gamma)$ of $\gamma$ in $M$ obtained by parallel transport of the fiber over $\gamma(0)$ (this unit normal bundle is an $S^{n-2}$-bundle over $\gamma$). Let $s$ be the length parameter of $\gamma$ and let $\rho \geq 0$ be the radial distance from $\gamma$. Via the normal exponential map, these functions define coordinates $(\sigma, s, \rho)$ for $T - \gamma$, defined on $N(\gamma) \times \gamma \times (0, \infty)$. In these coordinates, the maps $\rho \rightarrow (\sigma, s, \rho)$ are unit speed geodesics with starting point on $\gamma$ and initial velocity perpendicular to $\gamma'(\sigma)$. There exists a continuous function $R : N(\gamma) \rightarrow (0, \infty), (\sigma, s) \rightarrow R(\sigma, s)$ such that in these coordinates, we have $T = \{\rho \leq R\}$. The metric on $T - \gamma$ is of the form $h(\rho) + d\rho^2$ where $h(\rho)$ is a family of smooth metrics on the hypersurfaces $\rho = const$. Up to slightly adjusting the thick-thin decomposition and replacing $M_{thick}$ by its union with all Margulis tubes whose distance between the core geodesic and the boundary is $\leq 1$, we can assume for each Margulis tube the radial distance is uniformly bounded below by 1.

(c) A Neighborhood of An End with Infinite Volume

The convex core is the smallest convex submanifold $C(M)$ such that the inclusion map $C(M) \hookrightarrow M$ is a homotopy equivalence. From [1], $M - C(M)$ is an open subset of $M$ with finitely many components. The convex core $C(M)$ need not have smooth boundary and the boundary of an $r$-neighborhood of a convex subset in a Riemannian manifold is not necessarily smooth. The $r$-neighborhoods inside the geometrically finite ends are at least $C^1$ and we can enlarge them slightly to obtain neighborhoods with smooth boundary. The complement of the neighborhoods in the ends can be parameterized by $\mathbb{R}^+ \times \Omega$, where $\Omega$ is a finite union of hyperbolic surfaces. To construct the core $\tilde{M}$, we take enlarged neighborhoods of convex core in the ends, whose boundary are smooth and at least distance $r$ away from the boundary of the convex core, where $r$ is a fixed constant such that $r \geq \tanh^{-1} \left( \frac{(n-2)^2}{(n-1)} \right)$. We then remove all the cusps and Margulis tubes and perturb the boundary slightly to make it smooth, using ideas from [3]. What we have now is $\tilde{M}$, which intuitively is a neighborhood of the thick part intersecting with the convex core. We have chosen $r$ above so that some critical estimates on the eigenvalue will work.

5. Shell Estimates
Intuitively, for a cusp or tube, a shell is a subset whose points have small distance to the boundary of a cusp or tube, for example, less than or equal to 1. The Margulis constant $\epsilon(n)$ can be chosen so that the shell of each component of thin part is also contained in the thick part, and we will henceforth suppose that this is the case. It is equivalent to slightly enlarging the thin part. Shell estimates refer to a set of prescribed conditions on functions on shells, so that useful conclusions can be drawn, such as inequalities involving the functions or their gradient. Shell estimates appeared in the literature of discrete spectrum of hyperbolic or negatively curved manifolds for the first time in [9] as Lemma 2, which we record here for convenience. It is a prototype of an argument that we use several times later.

**Lemma 2.2.** For each $n$, there exists a constant $\delta > 0$ such that if $T$ is a thin component of $M$ with shell $S$, and if $f$ is a function defined on $T$ and satisfying:

(a) $\int_T |f|^2 = c > 0$

(b) $\int_S \|\nabla f\|^2 < \delta c$

(c) $\int_S |f|^2 < \delta c$

then $\int_T \|\nabla f\|^2 > (c/2)((n - 1)/2)^2$.

This lemma is used later in [9] to provide a lower bound for the first eigenvalue in the hyperbolic case. Shell estimates are also used in [11], in a critical way to provide a converse inequality to Theorem 1 for hyperbolic 3-manifold with finite volume. We will use the shell estimates later to facilitate the argument for some inequalities.

### 3. Bounding Small Eigenvalues from Below

Recall our goal is to understand the relationships between the eigenvalues of the manifold $M$ and eigenvalues of the core $\tilde{M}$. Compared to the core, its complement has concrete and simpler geometry for computations, which is not too different from warped products. It consists of Margulis tubes, finite volume cusp and neighborhoods of geometrically finite ends with infinite volume. Works about functional inequalities on the tubes and cusps have been done by Dodziuk and Randol in [9]. Then we use the inequalities to establish the relationship between eigenvalues of the manifold and those of the core. We start by considering components of geometrically finite ends. In general, the metric on the end, which is exponentially expanding, is only quasi-isometric to $\cosh^2 ts^2_{\partial C_1(M)} + dt^2$ where $ds^2_{\partial C_1(M)}$ is a hyperbolic metric on the boundary of 1-neighborhood of the convex core (see [5] and [14]). The eigenvalues under quasi-isometry has certain stability behavior, but depends on the constant of the quasi-isometry (see comments under Lemma 2.2 of [7]).

**Lemma 3.1.** Let $\overline{C_r(M)}$ denote the closure of the r-neighborhood of the convex core, $T$ a component of $M - \overline{C_r(M)}$, that is a neighborhood of a geometrically finite end with infinite volume. Let $f$ be a smooth function with compact support on $T$. Then
\[
\int_{T} \|\nabla f\|^2 \geq \frac{(n-1)^2(tanh\ r)^2}{4} \int_{T} f^2
\]

**Proof.** By Lemma 2.3 of [10], the infimum of Rayleigh quotients \(\mathcal{R}(f) = \int_{M} \|\nabla f\|^2 / \int_{M} f^2\) of all such functions on \(T\) is bounded below by \((n-1)^2(tanh\ r)^2/4\), which is equivalent to the desired inequality. \(\square\)

We can apply the above formula in our case, as in its proof a lower bound for the growth of metric in terms of distance to the boundary of convex core is provided, which is local. As long as we are distance \(r\) away from the boundary, the same estimate works.

Next we consider the thin parts. For Margulis tubes and finite volume cusps, we rewrite the concluding inequality of Lemma 2.3 of [11] as Lemma 3.2 to suit us later.

**Lemma 3.2.** Suppose \(T \subset M_{\text{thin}}\) is a Margulis tube or a cusp with boundary \(\partial T\), and \(f\) is a smooth function on \(T\) with \(\int_{T} f^2 \geq \int_{\partial T} f^2\); then
\[
\int_{T} \|\nabla f\|^2 \geq \frac{(n-2)^2}{4} \int_{T} f^2
\]

For a proof, see Lemma 2.3 of [11]. The argument is standard by Jacobian comparison under variation of curvatures and integration by parts.

The following inequality is taken from Lemma 1 of [9], which we omit the proof:

**Lemma 3.3.** Let \(T\) be a cusp corresponding to a parabolic subgroup \(\Gamma\) of maximal rank \(n-1\). Suppose \(f\) is a smooth function with compact support in the interior of \(T\). Then
\[
\int_{T} \|\nabla f\|^2 \geq \frac{(n-1)^2}{4} \int_{T} f^2
\]

The proof of the above lemma is a simple consequence of formula (3) on page 3 from [15]. We only need to show similar inequality is true in the \(t\) direction, and then integrate over the base \(H/\Gamma\).

The following proposition generalizes Lemma 2.4 of [11] with essentially the same technique. Here comes the choice of \(r\) that we mention earlier so that the estimates on the ends with infinite volume fit in with the estimates of the thin part from [11].

**Proposition 3.4.** \(f : M \rightarrow \mathbb{R}\) is a smooth, square integrable function with Raleigh quotient \(\mathcal{R}(f) < (n-2)^2/12\). Then
\[
\int_{\tilde{M}} f^2 \geq \frac{1}{3} \int_{M} f^2
\]
where \(r \geq \tanh^{-1} \frac{(n-2)^2}{(n-1)^2}\) (for example, when \(n = 3\), \(r = 1\) would suffice).
Proof. Let $M^*$ denote the disjoint union of all components of Margulis tubes, cusps and neighborhoods of ends with infinite volume, say, $M^* = \bigcup_{i=1}^{k} T_i$. The union is finite for a geometrically finite manifold. For each $T_i$ which belongs to the thin component, denote by $r_i$ the radial distance function to the boundary hypersurface, i.e., $r_i(x) =$ length of a radial arc connecting $x \in T$ to $\partial T$. For each $T_i$ which belongs to the ends with infinite volume, $r_i(x)$ denote the distance to $\partial \widetilde{M}$, the boundary of the core. In each $T_i$, we consider the shell consisting of points whose distance to $\partial T_i$ is less than or equal to 1. Denote by $A$ the finite disjoint union of all such shells. The thick core $\widetilde{M} = (M - \text{int}(M^*)) \cup A$. Moreover, by reordering, we may assume that there exists $p \leq k$ such that for $i \leq p$, there exists $s_i \leq 1$ such that

$$\int_{\{r_i=s_i\} \cap T_i} f^2 \leq \int_{\{r_i\geq s_i\} \cap T_i} f^2$$

and that for $i > p$, such an $s_i$ does not exist. Here $p$ could be zero, in which case we do not appeal to use Lemma 3.2. The volume element on the hypersurfaces $r_i = s_i$ is as in Lemma 2.3 of [11] and Lemma 2.3 of [10].

If $\int_{M^* - A} f^2 \leq \frac{2}{3} \int_M f^2$ we are done. Thus we assume $\sum_{i=1}^{k} \int_{T_i - A} f^2 = \int_{M^*} f^2 > \frac{2}{3} \int_M f^2$ (Recall that the shells are also part of the thick core $\widetilde{M}$). There are two cases. In the first one, $\sum_{i=1}^{p} \int_{T_i - A} f^2 \geq \frac{1}{3} \int_M f^2$. Suppose among the $p$ components, $p_1$ of them are tubes and cusps, $p-p_1$ of them are neighborhoods of ends infinite volume. Combining Lemma 3.1, Lemma 3.2 and Lemma 3.3 shows that

$$\int_M \| \nabla f \|^2 \geq \sum_{i=1}^{p} \int_{T_i \cap \{ r_i \geq s_i \}} \| \nabla f \|^2$$

$$\geq \frac{(n-2)^2}{4} \sum_{i=1}^{p_1} \int_{\{ r_i \geq s_i \} \cap T_i} f^2 + \frac{(n-1)^2}{4} \tanh r \sum_{i=p_1}^{p} \int_{\{ r_i \geq s_i \} \cap T_i} f^2$$

$$\geq \frac{(n-2)^2}{12} \int_M f^2$$

using the assumption on $r$. This contradicts our assumption on the Rayleigh quotient of $f$. In the second case, $\sum_{i=1}^{p} \int_{T_i - A} f^2 < \frac{1}{3} \int_M f^2$ implies that

$$\sum_{i=p+1}^{k} \int_{T_i - A} f^2 \geq \frac{1}{3} \int_M f^2$$

For each $i > p$, if we integrate the defining equation

$$\int_{\{ r_i = s_i \} \cap T_i} f^2 \geq \int_{\{ r_i \geq s_i \} \cap T_i} f^2$$

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over the shell $T_i \cap A = \{0 \leq r_i \leq 1\}$, we obtain

$$\int_0^1 ds \int_{\{r_i=s_i\}\cap T_i} f^2 = \int_{A\cap T_i} f^2 \geq \int_0^1 ds \int_{\{r_i\geq s_i\}\cap T_i} f^2 \geq \int_0^1 ds \int_{\{r_i\geq 1\}\cap T_i} f^2 = \int_{T_i-A} f^2$$

Summing over $i \geq p + 1$ and using inequality, we obtain

$$\int_{\bigcup_{i=p+1}^k T_i\cap A} f^2 \geq \sum_{i=p+1}^k \int_{T_i-A} f^2 \geq \frac{1}{3} \int_M f^2$$

As $A \subset \tilde{M}$, this contradicts the assumption on $f$. The lemma follows. \qed

Now we are ready to complete the proof of our main theorem, the argument is essentially the same as [11] for the finite volume case.

**Theorem 3.5.** Let $M$ be an oriented geometrically finite hyperbolic manifold $M$ of dimension $n \geq 3$ with infinite volume. Then for all $k \geq 0$, we have

$$\lambda_k(M) \geq \min \left\{ \frac{1}{3} \lambda_k(\tilde{M}), \frac{(n-2)^2}{12} \right\}$$

**Proof.** Let $M$ be an oriented geometrically finite hyperbolic manifold $M$ of dimension $n \geq 3$ with infinite volume. By Theorem 2.12 of [12], in the interval $[0, (n-2)^2/12)$, $M$ has at most finitely many eigenvalues with finite multiplicity. The idea of the proof is that the projection of the first $k$ eigenfunctions to some function space $\mathcal{H}(\tilde{M})$ is nondegenerate and span a $k$-dimensional subspace for $\mathcal{H}(\tilde{M})$. Applying Min-max theorem and proposition 3.4 gives the result.

Let $\mathcal{H}(M)$ (resp. $\mathcal{H}(\tilde{M})$) be the Sobolev space of square integrable functions with square integrable weak derivatives on $M$ (resp. $\tilde{M}$, $\tilde{M}$ has smooth boundary by construction). Here the weak derivative of a function $f$ on $\tilde{M}$ is a vector field $Y$ so that

$$\int_{\tilde{M}} <Y, X> = -\int_{\tilde{M}} f \text{div}(X)$$

for all smooth vector fields $X$ on $\tilde{M}$ with compact support in the interior of $\tilde{M}$. The class $\mathcal{H}(\tilde{M})$ contains all functions smooth in the interior of and up to the boundary of $\tilde{M}$ (see page 14 to 17 in [6]). Let $k > 0$ be such that $\lambda_k(M) < (n-2)^2/12$. We construct a $k$-dimensional linear subspace of the Hilbert space $\mathcal{H}(M)$ which correspond to the direct sum of the first $k$ eigenspaces. Denote by $E_j$ the eigenspace corresponding to $\lambda_j$ and $E = \bigoplus_{j=1}^k E_j$. If the $k$th eigenvalue is not simple (has multiplicity more than 1), we have to choose among all the eigenspaces corresponding to $\lambda_k$ but our choice will not affect later arguments.

Now to relate eigenvalues $\tilde{M}$ to $M$, we consider the projection/restriction map $\pi : \mathcal{H}(M) \to \mathcal{H}(\tilde{M})$. Since smooth functions are dense in $\mathcal{H}(M)$ and $\mathcal{H}(\tilde{M})$, $\pi$ is a one-Lipschitz linear map. Denote $\pi(E)$ by $W$. Now we use lemma to show that $\pi$ is non-degenerate, i.e.,
the dimension of $W$ is $k$, the same as that of $E$. Our first step is to show if there is nontrivial linear combination of eigenfunctions whose restrictions to the core is zero, then they must correspond to the same eigenvalue. We start with the case of two functions. To simplify notations, assume there is a nontrivial linear combination of $f_1$ and $f_2$ such that $c_1 f_1 + c_2 f_2 = 0$, where $f_i$ correspond to $\lambda_i$. Apply the Laplacian operator to the equation, we get $c_1 \lambda_1 f_1 + c_2 \lambda_2 f_2 = 0$. If $\lambda_1 \neq \lambda_2$ we have $f_1 = f_2 = 0$ on $\widetilde{M}$, contradicting Proposition 3.4. The general case follows from induction. Now if there is a nontrivial linear combination $f := c_1 f_1 + ... + c_k f_k = 0$ on the core, all $f_i$ must correspond to the same eigenvalue $\lambda_1 < (n−2)^2/12$. Thus $f$ is an eigenfunction corresponding to $\lambda_1$ whose restriction is zero. Contradiction.

We proceed by contradiction. Suppose there exists a normalized function $f \in E$, s.t. the restriction of $f$ to $\widetilde{M}$ vanishes. But functions in $E$ have Rayleigh quotient $< (n−1)^2/12$, therefore by lemma, $\int_{\widetilde{M}} f^2 \geq \frac{1}{3} \int_M f^2 \geq \frac{1}{3}$, which is impossible. It is also not possible for two distinct eigenfunctions on $M$ whose projections coincide on $\widetilde{M}$, as their difference satisfies the Laplacian equation on $\widetilde{M}$ and is identically zero.

As $\text{dim}(W) = k$ and $f \in E$ we have $R(f) \leq \lambda_k(M)$ by min-max characterization of eigenvalues. Note that both $f$ and $\pi(f)$ are smooth. Again, using

$$\int_{\widetilde{M}} f^2 \geq \frac{1}{3};$$

we have

$$\lambda_k(M) = \sup_{f_i \in E} R(f_i) = \sup_{f_i \in E} \int_M \| \nabla f_i \|^2 \geq \sup_{f_i \in E} \int_{\widetilde{M}} \| \nabla f_i \|^2$$

$$= \sup_{f_i \in E} R(f_i|_{\widetilde{M}}) \int_{\widetilde{M}} f^2 \geq \sup_{f_i \in E} \frac{1}{3} \| R(f_i|_{\widetilde{M}}) \| \geq \frac{1}{3} \lambda_k(\widetilde{M}).$$

The last inequality follows from min-max characterization of eigenvalues and the fact that $W$ is only some $k$-dimensional subspace, not necessarily the one spanned by the first $k$ eigenfunctions of $\widetilde{M}$.}

\[ \Box \]

4. Future Directions

In dimension three, using special property of Margulis tube, for finite volume oriented hyperbolic 3-manifold $M$, Hamenstadt [11] proves

$$\lambda_k(M) \leq c \log(\text{vol}(M_{\text{thin}}) + 2) \lambda_k(M_{\text{thick}})$$

for all $k \geq 1$ such that $\lambda_k(M_{\text{thick}}) < 1/96$.

It is natural to ask whether it is possible to provide an upper bound for the eigenvalues of infinite volume hyperbolic manifold in terms of the eigenvalues of the core. Moreover, many questions remain uninvestigated. What is a pattern for the distribution of the eigenvalues below essential spectrum of noncompact negatively curved hyperbolic manifolds? Do they
distribute relatively uniformly, or can they cluster around one point, like zero or the bottom of the essential spectrum? Since there are finitely many, is there an upper bound on the number of eigenvalues depending on the geometry/topology of the manifold? Moreover, recently, eigenvalues for differential forms, show up in an unexpected and surprising way as a bridge to connect Floer homology on 3-manifolds and hyperbolic geometry. See [13] and the reference therein. While monopole Floer homology is notoriously hard to compute directly, using eigenvalues of forms to mediate between geometry and Floer homology seems to be elegant, practical, and can see lots of potential use in the future. Unfortunately, such immediate and important applications of eigenvalues on forms have not seen the counterpart for eigenvalues for functions. It will be interesting to develop a stronger connection between such eigenvalues and the underlying hyperbolic geometry.

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