UNIFORM GAP IN LYAPUNOV EXPONENTS AND DOMINATED SPLITTING FOR LINEAR COCYCLES

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Abstract. Given a linear cocycle over an ergodic homeomorphism on a compact metric space, we show that the existence of a uniform gap between the \( p \)–th and \((p+1)\)–th Lyapunov Exponent on a \( C^0 \)–neighbourhood implies the existence of a Dominated splitting of index \( p \).

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1. Introduction

The theory of characteristic exponents was founded by the end of 19th century with the study of stability of ordinary differential equations by Lyapunov [13]. It was the works of Furstenberg and Kesten [9] and Oseledets [18] who brought the concept into the field of Ergodic Theory. Lyapunov Exponents appear naturally on the field of Smooth Dynamical Systems by the works of Pesin [20] and Spectral Theory [8].

Let \( X \) be a compact metric space and \( f \in \text{Hom}(X) \). Let \( A : X \to \mathcal{G} \) be a continuous function where \( \mathcal{G} \) is a group of matrices, either \( \text{GL}(d, \mathbb{R}) \) or \( \text{SL}(d, \mathbb{R}) \). A linear cocycle

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is a map $F_A : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ given by $(x, v) \mapsto (f(x), A(x)v)$. The Lyapunov exponent related to $x$ and $v \neq 0$ is given by the limit

$$\lim_{n \to \infty} \frac{1}{n} \|A^n(x)v\| = \gamma(x, v).$$

One of the main properties of Lyapunov exponents for linear cocycles is the existence of an invariant splitting. This splitting, which is called the Oseledet splitting defined in a total probability set (a set of full measure for any $f$-invariant measure) and is related to each exponent. Every vector in the splitting grows exponentially in a rate given by the respective Lyapunov exponent. Such splitting is a measurable function on $X$, and provides the asymptotic dynamical information of the action of the cocycle. Therefore, understanding the behaviour of Lyapunov exponents, the Oseledets splitting and how they behave has become a broad and active field of study.

Let $E$ be a vector bundle over a topological space $X$ and $F_A : E \to E$ a linear cocycle induced by a map $A : X \to G$. An invariant splitting $E = E \oplus F$ is called hyperbolic if vectors in $E$ are uniformly expanded and vectors in $F$ are uniformly contracted by the action of $A$ on fibers. Hyperbolicity is a very strong condition, therefore weaker forms have been widely studied in the literature, one of those forms is domination. A cocycle $A$ has a Dominated splitting if there exists an $A$-invariant splitting $E \oplus F$ where all vectors in $E$ are more expanded than vectors in $F$ by the action of $A$.

The notion of Dominated splitting appeared and played an important role in the works of Mañé [14], Pliss [21] and Liao [12] on Smale’s Stability Conjecture. The term “Dominated splitting” was introduced by Mañé in [16]. The concept itself was independently used in the theory of Ordinary Differential Equations by the Russian School under the name of exponential separation (see [19] for further details). Dominated splitting continue to be an important subject that appears frequently in Dynamical Systems [22] and Control Theory [7]. Despite being defined for smooth dynamical systems, the notion of Dominated splitting extends naturally to Linear Cocycles.

It is natural to ask when the Oseledets splitting is a Dominated splitting. Due to the properties of Dominated splittings this implies that the Oseledets splitting, that is defined measurably, admits a continuous extension. The main response to this in the $C^0$ topology was given by Bochi and Viana in [5] which can be summarized by the following dichotomy: If $A_0$ is a continuity point for the Lyapunov Spectrum then the Oseledets splitting is either dominated or trivial. Note that this means that the entire splitting is dominated. We remark that continuity of the Lyapunov Spectrum is $C^0$-generic, meaning that this dichotomy is present in a residual set in the space of linear cocycles.

In this paper, we explore further the techniques used by Bochi and Viana to give an answer to the following weaker question.
**Question.** What conditions on the Lyapunov exponents imply the existence of a Dominated splitting?

Our Main Theorem states that, in the presence of a uniform gap between the $p$-th and the $(p+1)$-th Lyapunov exponent in a $C^0$ neighbourhood of a cocycle, there exists a Dominated splitting $E \oplus F$ of index $p$. More precise definitions are given in section 2.

We say that the Lyapunov spectrum $\lambda_1(A) \geq \ldots \geq \lambda_d(A)$ of a cocycle $A \in C(X, G)$ has the uniform $p-$gap property if there exists a $C^0$-neighbourhood $U_A$ and $\beta > 0$ such that for every $B \in U_A$ it holds that

$$\lambda_p(B) - \lambda_{p+1}(B) \geq \beta.$$ 

The number $\beta$ is the $p-$gap between exponents.

Our main result states that this property ensures the existence of a dominated splitting of index $p$.

**Main Theorem.** Let $X$ be an infinite compact metric space and $\mu$ a Borel regular measure. Let $f \in \text{Hom}_\mu(X)$ such that $\mu$ is an ergodic measure and let $A \in C(X, G)$ be a Linear Cocycle over $f$ such that its Lyapunov spectrum has the uniform $p-$gap property. Then there exists a Dominated splitting of index $p$ on $\text{supp(}\mu\text{)}$.

For convenience, we will develop the document as if the measure $\mu$ is fully supported. However, all proofs and statements are correct by restricting to $\text{supp(}\mu\text{)}$ when the measure is not fully supported.

This Dominated splitting is given by the Oseledets splitting, where it is defined, as follows: Let $\gamma_1(A) > \ldots > \gamma_{k(A)}(A)$ be the Lyapunov exponents counted without multiplicity and let $s \in \{1, \ldots, k(A)\}$ be given by $\gamma_s(A) = \lambda_p(A)$ and $\gamma_{s+1}(A) = \lambda_{p+1}(A)$. Denote by

$$E = \bigoplus_{i=1}^{s} E_i, \quad F = \bigoplus_{i=s+1}^{k(A)} E_i.$$ 

The Main Theorem implies the existence of a continuous extension of the splitting $E \oplus F$ over all of $X$ and such extension is dominated.

Our main result keeps in the same line as other similar results that link the existence of uniform gaps among the Lyapunov spectrum with the existence of dominated splittings. A very similar conclusion as our Main Theorem has been recently announced by Kassel and Potrie in [10]. The authors define uniform $p-$gap as a gap between $p$-th and $(p+1)$-th exponents of a cocycle $A$ among all invariant measures over a subshift of finite type. They include a discussion and further references for similar results with this alternative notion of uniform gap.
Salgado in [23] showed that, provided the existence of a continuous invariant splitting, a uniform gap of Lyapunov exponents in a full probability set is equivalent to Dominated splitting. Araujo and Salgado have other relevant work related to dominated splittings and linear cocycles, and exterior powers and dominated splittings [1, 2].

We now state some consequences of the Main Theorem involving continuity of Lyapunov exponents. We say that a cocycle $A \in C(X, G)$ is a point of continuity of the $p$-th Lyapunov exponent $\lambda_p$ if it is a point of continuity of the function $\text{Lyap}_p : C(X, G) \to \mathbb{R}$ that maps $A \mapsto \lambda_p(A)$. That is, for every sequence of cocycles $A_n$ such that $A_n \to A$ then $\lambda_p(A_n) \to \lambda_p(A)$.

**Corollary 1.** Let $A \in C(X, \text{SL}(3, \mathbb{R}))$ be a point of continuity of the 2nd Lyapunov exponent. Then, either the whole Oseledet splitting is dominated or $\lambda_2 = 0$.

Since continuity of consecutive Lyapunov exponents imply the presence of uniform gaps, the Main Theorem gives an alternate route to show Bochi-Viana Theorem above, in the context of Linear Cocycles and ergodic measures. In what follows we explain this briefly.

Note that if a cocycle $A \in C(X, G)$ is a point of continuity of the $p$-th and $(p+1)$-th Lyapunov exponent and it holds that $\lambda_p(A) > \lambda_{p+1}(A)$ it follows that, taking a small $\varepsilon > 0$, there exists a $C^0$-neighbourhood of $A$ such that there is a uniform gap between the $p$-th and $(p+1)$-th Lyapunov exponents. The following corollary follows:

**Corollary 2.** Let $A \in C(X, G)$ be a point of continuity of the $p$-th and $(p+1)$-th Lyapunov exponent such that $\lambda_p(A) > \lambda_{p+1}(A)$. Then there exists a Dominated splitting of index $p$ for $A$.

Moreover, if the cocycle is a point of continuity of the whole Lyapunov Spectrum we get this known result which was first proved in [5].

**Corollary 3** (Bochi-Viana Corollary 1 [5]). Let $A \in C(X, G)$ be a point of continuity of the Lyapunov Spectrum. Then either all exponents are equal at almost every point, or there exists a dominated splitting which coincides with the Oseledets splitting almost everywhere.

Note that in [5] there is a different scope, focusing mainly on perturbations on volume-preserving diffeomorphisms, not necessarily ergodic, and then applying those techniques on linear cocycles take values not only on GL$(d, \mathbb{R})$ or SL$(d, \mathbb{R})$ but on any submanifold of GL$(d, \mathbb{R})$ fulfilling an accessibility condition. Therefore the original proof of Corollary 3 had to circumvent the use of specific Lebesgue measure tools, such as Vitali’s Covering Lemma, which are not valid for any measure on a compact
topological space. By using our Main Theorem we give an alternate path that avoids this by using topological properties of the base space $X$, such as Urysohn’s Lemma.

For the proof we maintain the original path taken by Bochi and Viana. The contributions in this article lie on the improvement of intermediate lemmas (basically optimizing quantifiers), and combining them with a new idea (considering a cocycle that minimizes the sum of exponents).

We now give an outline of the path we take to prove the main theorem. Suppose the Oseledets splitting, grouped in the spaces $E \oplus F$ as shown in eq (1) is not $m-$dominated for a large $m$, that is

$$ \|A^m(x)\|_F \|((A^m(x))^{-1}\| > \frac{1}{2}; $$

then, if $m > m_0$ for some $m_0$ that depends on the how large the norm of $A$ is, we can make small perturbations by composing with rotations and hyperbolic matrices along the orbit of $x$ that moves a vector in $E(x)$ into a vector of $F(f^m x)$.

In fact, for $\mu$-almost every point $x$ and any $n$ sufficiently large we find an $\ell \approx n/2$ and $y \in f^{\ell}(x)$ that fulfills (2). This way, by composing with rotations and hyperbolic matrices, we can get a new cocycle $\tilde{B}$ near $A$ such that, coincides with $A$ along the orbit of $x$ for $\approx n/2$ steps. The new cocycle changes a vector from $E$ into a vector of $F$ and coincides again with $A$ along the rest of the orbit ($\approx n/2$ steps again). It follows that, if we compare the norm of the $p-$th exterior cocycles we see we lose some expansion. While $\|\wedge^p A^n(x)\| \approx \exp(n(\lambda_1 + \cdots + \lambda_p))$ we notice that

$$ \|\wedge^p B^n(x)\| \leq \exp \left( n(\lambda_1 + \cdots + \lambda_{p-1} + \frac{\lambda_p + \lambda_{p+1}}{2}) \right). $$

We repeat this local process on a large measure set and using a Kakutani Rokhlin argument we can get a cocycle $\tilde{B}$ arbitrarily close to $A$ such that the sum of the first $p$ Lyapunov exponents drops by approximately half of the gap.

As we stated above, this perturbation can be done in an open set as long as there is lack of $m-$domination. So if we have a cocycle $A$ with the uniform $p-$gap property and such that $E_A \oplus F_A$ is not dominated, then we can do a similar perturbation to any $A'$ in a neighbourhood of $A$ and drop the sum of the first $p$ Lyapunov exponents. This eventually contradicts the boundness of exponents.

The paper is organised as follows. In section 2 we give the basic definitions, theorems and properties used in the paper.

In Section 3 we introduce the space interchanging sequences which are the main tool for perturbations. We show the conditions for existence on neighbourhoods and state a result estimating their norm when applied to the Oseledets splitting.
In Section 4 the space interchanging sequences are used to create perturbations of cocycles, then following a Kakutani-Rokhlin towers argument we show such perturbations can be globalised to obtain cocycles with lower Lyapunov exponents.

In Section 5 gives a proof for the Main Theorem by arriving to a contradiction using the perturbations defined in the previous sections and explain how to derive corollary 1.

In Section 6 we conclude by posing some questions that derive from the Main Theorem.

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2. Basic definitions and preliminaries

Let $X$ be an infinite compact metric space endowed with $\mu$ a probability measure. We say $\mu$ is Borel regular if for any measurable set $M$ it holds that

$$
\mu(M) = \inf\{\mu(O) : M \subset O : O \text{ is an open set}\}
$$

$$
\mu(M) = \sup\{\mu(C) : C \subset M : C \text{ is a closed set}\}.
$$

Since $X$ is a compact space, every closed set is compact, so the measure $\mu$ is trivially tight

$$
\mu(M) = \sup\{\mu(K) : K \subset M : K \text{ is a compact set}\}.
$$

We suppose $\mu$ is fully supported this is equivalent to $\mu$ being strictly positive, that is, every non empty open set $U \subset X$ holds that $\mu(U) > 0$.

We note that these hypotheses on the measure are satisfied on many important examples. Such as compact Riemannian manifolds when $\mu$ is the measure induced by a volume form. Or a sample space $\{1, \ldots, N\}^\mathbb{Z}$ when $\mu$ is the Bernoulli measure.

Let $f : X \to X$, we say that $f$ preserves the measure $\mu$ if for every measurable set $M \subset X$ it holds that $\mu(f^{-1}(M)) = \mu(M)$. We denote by $\text{Hom}_\mu(X)$ the set of homeomorphisms that preserve the measure $\mu$. The measure $\mu$ is ergodic if for every measurable set $M$ such that $f(M) = M$ then $\mu(M) = 0$ or 1. Let $f$ be a homeomorphism, then fully supported ergodic measures are aperiodical that is the set of periodic points $\text{Per}(f)$ holds that $\mu(\text{Per}(f)) = 0$. 

2.1. Linear cocycles. Let $f \in \text{Hom}_\mu(X)$. Let $\mathcal{G}$ denote either of the matrices groups $\text{GL}(d, \mathbb{R})$ or $\text{SL}(d, \mathbb{R})$. Given a continuous map $A : X \to \mathcal{G}$ we associate the linear cocycle on $X \times \mathbb{R}^d$

$$F_A : X \times \mathbb{R}^d \to X \times \mathbb{R}^d, \quad F_A(x, v) = (f(x), A(x)v).$$

Since we keep the base $f$ of the cocycle fixed, as an abuse of notation we identify the linear cocycle $F_A$ with the function $A : X \to \mathcal{G}$. We denote the space of continuous functions from $X$ to $\mathcal{G}$, and the space of linear cocycles, as $C(X, \mathcal{G})$.

For cocycles we use the following notation for iterations

$$F^n_A(x, v) = (f^n(x), A^n(x)v).$$

Observe that then $A$ satisfies the cocycle property: $A^{n+m}(x) = A^n(f^m(x))A^m(x)$ for every $n, m \in \mathbb{Z}$.

For any $M \in \mathcal{G}$ we define the operator norm as

$$\|M\| := \max\{|Mv|, \ v \in \mathbb{R}^d, \ |v| = 1\}.$$ 

This induces the $C^0$ or uniform norm on $C(X, \mathcal{G})$ as follows: given $A \in C(X, \mathcal{G})$,

$$\|A\| = \max_{x \in X}\{\|A(x)\|\}.$$ 

The continuity of the function $A$ implies that $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$. We say that $\gamma$ is a Lyapunov exponent for the cocycle at the point $x$ if there exists $v \in \mathbb{R}^d$ different than zero such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)v\| = \gamma,$$

in this case we say that $v$ is a Lyapunov vector for $x$.

A point $x \in X$ is regular if every vector $v \in \mathbb{R}^d$ different than zero is a Lyapunov vector. We shall denote the set of regular points for the cocycle $A$ as $\mathcal{R}(A)$. The following theorem (see [24] for details) guarantees that, far from being empty, the set $\mathcal{R}(A)$ has full probability, this means that it has full measure for any $f$—invariant probability. We denote the Grassmanian of $\mathbb{R}^d$ by $\mathcal{G}(d)$.

**Theorem 2.1** (Oseledeets). For $\mu$ almost every $x \in X$, there exist functions $k = k(x)$, $\gamma_1(A, x) > \gamma_2(A, x) > \cdots > \gamma_k(A, x)$ together with a direct sum splitting $\mathbb{R}^d = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x)$ that verify

1. These functions are $f$—invariant: $k(f(x)) = k(x)$ and $\gamma_i(f(x)) = \gamma_i(x)$.
2. The functions $x \mapsto k(x)$, $x \mapsto \gamma_i(x)$ and $x \mapsto E_i(x)$ (taking values in $\mathbb{N}$, $\mathbb{R}$ and $\mathcal{G}(d)$ respectively) are measurable.
3. The splitting is invariant under the action of the cocycle, this means that $A(x)E_i(x) = E_i(f(x))$. 

(4) It holds that, for every non-null \( v \in E_i(x) \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \| A^n(x)v \| = \gamma_i(x).
\]

(5) For any disjoint sum of the spaces within the splitting
\[
\lim_{n \to \pm \infty} \frac{1}{n} \log |\sin \angle(\bigoplus_{i \in I} E_i(f^n(x)), \bigoplus_{j \in J} E_j(f^n(x)))| = 0 \text{ whenever } I \cap J = \emptyset.
\]

The splitting \( \mathbb{R}^d = E_1(x) \oplus \cdots \oplus E_k(x) \) is called \textit{Oseledets splitting}.

We define the \textit{multiplicity} of the exponent \( \gamma_i(x) \) as the dimension of \( E_i(x) \) and we shall denote the Lyapunov exponents, counted with multiplicity, by \( \lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x) \). This is the \textit{Lyapunov Spectrum} of the cocycle.

The Oseledets Theorem is a refinement of the following theorem, which finds the greater and lower Lyapunov exponents by means of the norm of the cocycle.

\textbf{Theorem 2.2 (Furstenberg-Kesten).} The limit
\[
\gamma_1(x) = \lim_{n \to \infty} \frac{1}{n} \log \| A^n(x) \|
\]
exists at \( \mu \)-almost every point. Moreover, \( \gamma_1 \in L^1(\mu) \) and
\[
\int \gamma_1(x) d\mu = \lim_{n \to \infty} \frac{1}{n} \int \log \| A^n(x) \| d\mu.
\]

When the measure \( \mu \) is ergodic for \( f \) then all the invariant functions in Theorem 2.1 are constant. We can overlook the dependency of Lyapunov exponents on \( x \) and just associate them to the cocycle \( A \).

Let \( \ell \in \{1, \ldots, d\} \), any linear map \( L \in \mathcal{G} \) induces a linear map \( \wedge^\ell L \) on the exterior power \( \bigwedge^\ell \mathbb{R}^d \) that maps every decomposable vector \( v_1 \wedge \cdots \wedge v_\ell \) into \( L(v_1) \wedge \cdots \wedge L(v_\ell) \). Thus in a straightforward manner any cocycle \( A \in \mathcal{C}(X, \mathcal{G}) \) induces a cocycle \( \wedge^\ell A \) on the Exterior Power \( \bigwedge^\ell \mathbb{R}^d \).

The Exterior Power \( \bigwedge^\ell \mathbb{R}^d \) is endowed with a norm, this norm on every decomposable vector \( \| v_1 \wedge \cdots \wedge v_\ell \| \) coincides with the volume of the \( \ell \)-dimensional parallelepiped defined by \( v_1, \ldots, v_\ell \). The operator norm for linear maps and the \( C^0 \) norm are defined analogously to their \( \mathbb{R}^d \) counterparts.

For a cocycle \( A \in \mathcal{C}(X, \mathcal{G}) \) with a Lyapunov Spectrum given by \( \lambda_1(A) \geq \cdots \geq \lambda_d(A) \), the cocycle \( \wedge^\ell A \) has Lyapunov exponents, counted with multiplicity, \( \lambda_{i_1}(A) + \cdots + \lambda_{i_\ell}(A) \) with \( 1 \leq i_1 < \cdots < i_\ell \leq d \). In particular, \( \lambda_1^\wedge A = \lambda_1(A) + \cdots + \lambda_\ell(A) \) and \( \lambda_2^\wedge A = \lambda_1(A) + \cdots + \lambda_{\ell - 1}(A) + \lambda_{\ell + 1}(A) \). See [24, 3] for further details.

2.2. \textbf{Dominated splittings}. Let \( \Lambda \subset X \) be an \( f \)-invariant subset, suppose that for every \( x \in \Lambda \) there exists a splitting in non-null spaces \( E(x) \oplus F(x) \), with constant dimensions such that \( \mathbb{R}^d = E(x) \oplus F(x) \), this defines a splitting of the bundle \( T_\Lambda = E \oplus F \). The \textit{index} of the splitting is \( \dim(E(x)) \).
Suppose the splitting is invariant under the cocycle $A$, this means: $A(E(x)) = E(f(x))$ and $A(F(x)) = F(f(x))$ for every $x \in \Lambda$.

**Definition 2.3.** Given $m \in \mathbb{N}$, we say that $T\Lambda = E \oplus F$ is an $m$–dominated splitting if for every $x \in \Lambda$,

$$\frac{\|A^m(x)|_{F(x)}\|}{\text{m}(A^m(x)|_{E(x)})} \leq \frac{1}{2}$$

where $\text{m}$ denote the induced lower bound matrix norm $\text{m}(L) = \|L^{-1}\|^{-1}$.

We say that $T\Lambda = E \oplus F$ is a dominated splitting if it is $m$–dominated for some $m \in \mathbb{N}$. In such case we will write $E \succ F$.

Moreover, if we have a splitting $T\Lambda = E_1 \oplus E_2 \oplus E_3$, we say it is a dominated splitting if $E_1 \succ E_2 \succ E_3$ or, equivalently, $E_1 \succ (E_2 \oplus E_3)$ and $(E_1 \oplus E_2) \succ E_3$. This can be generalised, in an analogous manner, to any number (at most $d$) of subspaces.

We list some basic properties of dominated splittings, for details and proofs see [6].

- **Transversality.** If $T\Lambda = E \oplus F$ is a dominated splitting, then the angle $\angle(E(x), F(x))$ is bounded away from zero for all $x \in \Lambda$.
- **Uniqueness.** If $T\Lambda = E^1 \oplus F^1$ and $T\Lambda = E^2 \oplus F^2$ are two dominated splittings with equal index then $E^1 = E^2$ and $F^1 = F^2$.
- **Continuity.** A dominated splitting $T\Lambda = E \oplus F$ is continuous and extends continuously to a dominated splitting over the closure of $\Lambda$.
- **Persistence under perturbations.** Let $\Lambda_A$ an invariant set where $T\Lambda_A$ admits a dominated splitting, then there exists a neighbourhood $U$ of $\Lambda_A$ and a $C^0$ neighbourhood $\mathcal{U}$ of $A \in C(X, G)$ such that for every $B \in \mathcal{U}$ and every compact invariant set $\Lambda_B \subset U$ holds that $T\Lambda_B$ admits a dominated splitting.

We say then that an invariant splitting $E \oplus F$ is not $m$–dominated at $y \in X$ if:

$$\frac{\|A^m(y)|_{F(y)}\|}{\text{m}(A^m(y)|_{E(y)})} > \frac{1}{2}$$

(3)

And $E \oplus F$ is eventually not $m$–dominated at $x \in X$, if there exists $n \in \mathbb{Z}$ such that the splitting is not $m$–dominated at $f^n(x)$.

Given $m \in \mathbb{N}$, let $\mathcal{D}_{E \oplus F}(A, m)$ be the set of points $x$ such that the splitting $E \oplus F$ is $m$–dominated along the orbit of $x$. The set $\mathcal{D}_{E \oplus F}(A, m)$ is closed and invariant, therefore for any $m \in \mathbb{N}$, if $\mu$ is an ergodic measure, $\mu(\mathcal{D}_{E \oplus F}(A, m))$ is either 0 or 1.

For such splitting, we say a cocycle $A$ is not $m$–dominated when the set $X \setminus \mathcal{D}_{E \oplus F}(A, m)$ has full measure. In conclusion, if $\mu$ is ergodic then either $E \oplus F$ is $m$–dominated or $\mu$-almost every point is eventually not $m$–dominated.

**2.3. Other classical theorems.** In this subsection we state two Classical Theorems that will be used. We give references for the interested reader.
A classical theorem in Ergodic Theory is Birkhoff’s Theorem. Given a measurable set \( E \subset X \) and a point \( x \in X \), we define the \textit{mean sojourn time} of \( x \) to \( E \) as the number

\[
\tau(E, x) = \lim_{n \to \infty} \frac{1}{n} \# \{ j \in \{0, \ldots, n-1\} \text{ such that } f^j(x) \in E \}.
\]

Note that, if exists, then \( \tau(E, f(x)) = \tau(E, x) \). This number though is not defined in general. Birkhoff’s Theorem states that it is defined almost everywhere when \( f \) preserves the measure \( \mu \).

\textbf{Theorem 2.4} (Birkhoff). Let \( f : X \to X \) be a measurable transformation and \( \mu \) be a probability measure invariant under \( f \). Given any measurable set \( E \subset X \) the mean sojourn time \( \tau(E, x) \) exists \( \mu \)-almost every point \( x \in X \). Moreover, it holds that \( \tau(E, x) \) is integrable over \( \mu \) and

\[
\int \tau(E, x) d\mu(x) = \mu(E).
\]

From Birkhoff Theorem it follows that if the measure \( \mu \) is ergodic then for \( \mu \)-almost every \( x \in X \) it holds that \( \tau(E, x) = \mu(E) \). See [25, 15] for details.

A topological space \( X \) is \textit{normal} if for every disjoint pair of closed sets, \( C \) and \( D \), there are disjoint open sets \( U \) and \( V \) such that \( C \subset U \) and \( D \subset V \). Every metric space is normal. The following is a characterization by continuous functions of normal spaces. See [11, 17] for details.

\textbf{Theorem 2.5} (Urysohn’s Lemma). A topological space \( X \) is normal if and only if for any pair of disjoint closed sets \( C, D \subset X \) there exists a continuous function \( h : X \to [0, 1] \) such that \( h(x) = 0 \) for every \( x \in C \) and \( h(x) = 1 \) for every \( x \in D \).

\section{3. Space interchanging sequences}

The following statement is the first ingredient for creating perturbations. It states that given two directions that do not dominate the other we can get, for long enough orbits, a sequence of nearby matrices that change one direction for the other.

\textbf{Proposition 3.1.} Let \( M > 0 \) and \( \varepsilon \in (0,1) \). There exists a positive integer \( m_0 = m_0(M, \varepsilon) \in \mathbb{N} \) such that if \( m \geq m_0 \) then the following property holds:

Let \( y \in X \) be a nonperiodic point and \( A \in C(X, \mathcal{G}) \) such that \( \|A\| < M \) and there exists two nonzero, non colinear vectors \( v_0 \) and \( w_0 \) such that

\[
\|A^m(y)v_0\| < 2\|A^m(y)w_0\|.
\]

Then there exists a sequence \( \{L_0, \ldots, L_{m-1}\} \) of length \( m \), such that \( \|A(f^j(y)) - L_j\| < \varepsilon \) for \( 0 \leq j \leq m-1 \) and

\[
L_{m-1} \ldots L_0(v_0) \in \text{span}(A^m(y)w_0).
\]
Proof. Let $A \in C(X, G)$ such that $\|A\| < M$ as in the hypotheses of the Proposition. For every $j \geq 0$ denote by $E_j = A^j(y)(\text{span}(v_0))$, $F_j = A^j(y)(\text{span}(w_0))$ and write $\alpha_\varepsilon = \cos^{-1}(\sqrt{1 - \varepsilon/4})$. We will give an explicit definition for $m_0 = m_0(M, \varepsilon)$ later in the proof. Suppose it is already defined and $m \geq m_0$. We consider two cases.

First case: Suppose there exists $l \in \{0, 1, \ldots, m - 1\}$ such that

$$\angle(E_l, F_l) = \alpha < \alpha_\varepsilon.$$  

Take unit vectors $u_E \in E_l$ and $u_F \in F_l$ such that $\angle(u_E, u_F) = \alpha$. Let $V = E_l \oplus F_l$ and $W = V^\perp$. Let $\hat{R} : V \to V$ be the rotation of angle $\alpha$ such that $\hat{R}(u_E) = u_F$. Then $\|\hat{R} - 1\| = 2 - 2 \cos(\alpha) < \varepsilon$. Let $R : \mathbb{R}^d \to \mathbb{R}^d$ be such that $R|_V = \hat{R}$ and $R|_W = I_{d-2}$.

For each $j \neq l$ we define the sequence $L_j$ as $L_j = A(f^j(y))$ and $L_l = R$. By construction this sequence maps $v = A^{-1}(f^j(y))(u_E)$ to $w = A^{m-1}(f^j(y))(u_F)$. Then, the sequence $\{L_0, \ldots, L_{m-1}\}$ satisfies $\|A(f^j(y)) - L_j\| < \varepsilon$ for $0 \leq j \leq m - 1$ and $L_{m-1} \ldots L_0(v_0) \in \text{span}(A^m(y)w_0)$. The proposition follows.

Second case: In what follows we assume that $\angle(E_j, F_j) \geq \alpha_\varepsilon$ for every $j = 0, \ldots, m - 1$. From the hypotheses there exist two unit vectors $v_0 \in E_0$ and $w_0 \in F_0$ such that $\|A^m(y)(v_0)\| < 2\|A^m(y)(w_0)\|$.

We now describe the main idea for constructing the sequence $\{L_0, \ldots, L_{m-1}\}$ and defining $m_0$. First we rotate $v_0$ a tiny angle in direction to $w_0$, let $\hat{v}_0$ be this new vector. Then along the orbit we make hyperbolic perturbations contracting in the direction of $v_j = A^j(y)(v_0)$ and expanding in the direction of $w_j = A^j(y)(w_0)$. We continue doing so until $\hat{v}_0$, the image of $v_0$ under these perturbations, has an angle small enough with $F_j$ and finish with another rotation. We define $m_0 \in \mathbb{N}$ big enough that allows us to do this for any $m \geq m_0$.

To simplify notation we consider for every $j$ the basis of $\mathbb{R}^d$ given by the sets $B_j = \{v_j, w_j, e_j^1, \ldots, e_j^{d-2}\}$ where $e_j^i$ determine an orthonormal basis for $\text{span}(v_j, w_j)^\perp$.

Since for every $j$ we have that $\angle(v_j, w_j) \geq \alpha_\varepsilon$, it follows that the norms of the change of basis matrices from the canonical $\mathbb{R}^d$ basis to $B_j$, namely $M_j$ and $M^{-1}_j$, are bounded. Let $C_\varepsilon = \max\{\|M_j\|, \|M^{-1}_j\|, 0 \leq j \leq m - 1\}$.

Let $\varepsilon_1$ and $\alpha_1$ be such that

$$0 < \varepsilon_1 < \frac{\varepsilon}{MC_\varepsilon}, \quad \tan(\alpha_1) = \frac{\tan(\alpha_\varepsilon)}{1 + \varepsilon_1}.$$  

We denote by $H_{j, \varepsilon_1} : \mathbb{R}^d \to \mathbb{R}^d$ the linear transformations such that in the bases $B_j$ are associated to the matrix

$$[H_{j, \varepsilon_1}]_{B_j} = \begin{bmatrix} 1/(1 + \varepsilon_1) & 0 & 0 \\ 0 & 1 + \varepsilon_1 & 0 \\ 0 & 0 & I_{d-2} \end{bmatrix}.$$
We now define the perturbations $L_j$. Denote $Y = \text{span}(v_0, w_0)$ and let $\hat{R}_0 : Y \to Y$ be the rotation of angle $\alpha_1$ in direction from $v_0$ to $w_0$ and let $R_0 : \mathbb{R}^d \to \mathbb{R}^d$ be such that $R_0|_Y = \hat{R}_0$ and $R_0|_{Y^\perp} = I_{d-2}$. This defined in the canonical basis of $\mathbb{R}^d$. We define $L_0 = A(y)R_0$.

For $0 < j < m - 1$ define $L_j = A(f^j(y))H_{j, \varepsilon}$. Note that

\[
\|A(f^j(y)) - L_j\| \leq \|A\|\|I_d - H_{j, \varepsilon}\| \leq \|A\|\|M_j^{-1}\|\|I_d - [H_{j, \varepsilon}]_{8j}\|\|M_j\| \leq MC\varepsilon^2 \varepsilon_1 < \varepsilon.
\]

Note that if we denote $\hat{L}_j = L_j \circ \ldots \circ L_0$ it holds that

\[
\hat{v}_j = \hat{L}_j(v_0) = \cos(\alpha_1)(1 + \varepsilon_1)^{-j}v_j + \sin(\alpha_1)(1 + \varepsilon_1)^j w_j.
\]

Let $\alpha_{2,j} = \angle(\hat{v}_j, w_j)$ and denote $Y_j = \text{span}(v_j, w_j)$. Note that if $R_{2,j} : Y_j \to Y_j$ is a rotation of angle $\alpha_{2,j}$ in direction from $v_j$ to $w_j$ it holds that

\[
R_{2,j}(\hat{v}_j) = \left(\cos(\alpha_{2,j})\cos(\alpha_1)(1 + \varepsilon_1)^{-j} - \sin(\alpha_{2,j})\sin(\alpha_1)(1 + \varepsilon_1)^j\right)v_j
\]

\[+ \left(\sin(\alpha_{2,j})\cos(\alpha_1)(1 + \varepsilon_1)^{-j} + \cos(\alpha_{2,j})\sin(\alpha_1)(1 + \varepsilon_1)^j\right)w_j.
\]

Observe that $R_{2,j}(\hat{v}_j)$ is parallel to $w_j$ if and only if

\[
\cos(\alpha_{2,j})\cos(\alpha_1)(1 + \varepsilon_1)^{-j} - \sin(\alpha_{2,j})\sin(\alpha_1)(1 + \varepsilon_1)^j = 0.
\]

Therefore it suffices to determine $m_0 \in \mathbb{N}$ such that for every $m > m_0$ it holds that $\alpha_2 = \alpha_{2,m-2} < \alpha_\varepsilon$ and $R_{2,m-2}(\hat{v}_{m-2})$ is parallel to $w_{m-2}$. Note that from (4) the second condition translates to

\[
\frac{1}{\tan(\alpha_2)}\left(\frac{1}{(1 + \varepsilon_1)^2}\right)^{m-2} = \tan(\alpha_1).
\]

Note that $\alpha_2 < \alpha_\varepsilon$ if and only if

\[
\left(\frac{1}{1 + \varepsilon_1}\right)^{2m-4} < \tan(\alpha_1)\tan(\alpha_\varepsilon).
\]

Recall from $\alpha_1$ is defined such that $\tan(\alpha_1) = \tan(\alpha_\varepsilon)/(1 + \varepsilon_1)$. Therefore equation (5) implies

\[
\left(\frac{1}{1 + \varepsilon_1}\right)^{2m-5} < \tan^2(\alpha_\varepsilon) = \frac{1 - (1 - \varepsilon/4)}{(1 - \varepsilon/4)}.
\]

The right hand equality follows from the definition of $\alpha_\varepsilon = \cos^{-1}(\sqrt{1 - \varepsilon/4})$. This last inequality is equivalent to,

\[
1 - \frac{\varepsilon}{4} < \frac{(1 + \varepsilon_1)^{2m-5}}{1 + (1 + \varepsilon_1)^{2m-5}}.
\]
As $\varepsilon_1 > 0$ and $\varepsilon \in (0, 1)$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$, the inequality (6) above is true. It follows that for such $m$ then $\alpha_2 = \alpha_{2,m-2} < \alpha_\varepsilon$.

In sum, by setting $\alpha_2 = \alpha_{2,m-2}$ satisfying the equation (4) above. We define $L_{m-1} = A(f^m(y))R_{\alpha_2}$ being $R_{\alpha_2}$ defined as $R_{\alpha_{2,m-2}}$ when restricted to the subspace $Y_{m-2} = \text{span}(v_{m-2},w_{m-2})$ and $R_{\alpha_2}|_{Y^\perp}$ as the identity. This finishes the definition of the sequence $\{L_0, \ldots, L_{m-1}\}$. Let $v = v_0$ and the proposition now follows. \qed

**Definition 3.2.** A sequence $\{L_0, \ldots, L_{m-1}\}$ such that for two different directions $v_0, w_0$ it holds that $L_{m-1} \ldots L_0(v_0) \in A^{m-1}(y)(\text{span}(w_0))$ and $\|A(f^j(y))-L_j\| < \varepsilon$, as in the previous Proposition, will be called an *interchanging $\varepsilon$-sequence over $y$ of length $m$.*

Recall that $\mathcal{R}(A)$ is the set of the regular points For $x \in \mathcal{R}(A)$, let $s \in \{1, \ldots, k(A)\}$ be given by $\gamma_s(A) = \lambda_p(A)$ and $\gamma_{s+1}(A) = \lambda_{p+1}(A)$, denote by $E(x) = \bigoplus_{i=1}^{s} E_i(x)$, and $F(x) = \bigoplus_{i=s+1}^{k(A)} E_i(x)$. Then the bundle sum $E \oplus F$ defines an splitting of index $p$.

If $A$ is not $m-$dominated then up to a measure zero set we can assume then that the set $\mathcal{R} = \mathcal{R}(A)$ of regular points have eventually no $m$-dominated splitting and no periodic points. Recall that by these assumptions every $x \in \mathcal{R}$ has an iterate $y = f^n(x)$, for some $n \in \mathbb{Z}$ that holds the non $m$–domination condition (3)

$$\frac{\|A^m(y)|_{F(y)}\|}{m(A^m(y)|_{E(y)})} > \frac{1}{2}.$$ 

The Non $m$–domination condition ensures that there are non periodic points $y \in X$ such that $\|A^m(y)(v)\| < 2\|A^m(y)(w)\|$ for non zero vectors $v \in F$ and $w \in E$ so we are in the conditions of Proposition 3.1 and we have interchanging $\varepsilon$–sequences.

For any cocycle $A$ we define $\Lambda_i(A) = \lambda_1(A) + \lambda_2(A) + \ldots + \lambda_i(A)$ the sum of the first $i$–th Lyapunov exponents.

The following lemma states that if $E \oplus F$ is not dominated, it is possible to find interchanging $\varepsilon$–sequences with an apropriate norm. Recall that every Linear transformation $L : \mathbb{R}^d \to \mathbb{R}^d$ induces a Linear transformation on the exterior powers $\wedge^i L : \wedge^i \mathbb{R}^d \to \wedge^i \mathbb{R}^d$ that maps every decomposable vector $v_1 \wedge \ldots \wedge v_j$ into $L(v_1) \wedge \ldots \wedge L(v_j)$.

The lemma is a slight modification of [5, Proposition 4.2], but the proof in [5] also works in our setting. We will include a proof for the reader’s convenience.

**Lemma 3.3.** Let $M > 0$, $\varepsilon \in (0, 1)$, $\delta > 0$. Then for every $m \in \mathbb{N}$ such that $m \geq m_0(M, \varepsilon)$ and $A \in C(X, G)$ such that $\|A\| < M$ and the Oseledets splitting of index $p$ $E \oplus F$ is not an $m$–dominated splitting of index $p$. Then there exists a measurable function $N_A : \mathcal{R} \to \mathbb{N}$ with the following properties: For almost every $x \in \mathcal{R}$ and every $n \geq N_A(x)$ there exists an interchanging $\varepsilon$-sequence $\{\hat{L}^{(x,n)}_0, \ldots, \hat{L}^{(x,n)}_{n-1}\}$.
over $x$ of length $n$ such that
\[
\frac{1}{n} \log \| \wedge^p (\hat{L}_{n-1}^{(x,n)}, \ldots, \hat{L}_0^{(x,n)}) \| \leq \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \delta.
\]

**Remark.** The function $N_A$ depends on the cocycle $A$, $\varepsilon$, $\delta$ and $m$.

Before giving a proof we need to give some preliminaries. First, we introduce the following norm. We represent a linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ by the following associated matrix

\[
T = \begin{bmatrix}
T^{++} & T^{+-} \\
T^{-+} & T^{--}
\end{bmatrix}
\]

with respect to two splittings $\mathbb{R}^d = E_0 \oplus F_0$ and $\mathbb{R}^d = E_1 \oplus F_1$. We then define the norm
\[
\|T\|_{\text{max}} = \max \{ \|T^{++}\|, \|T^{+-}\|, \|T^{-+}\|, \|T^{--}\| \}.
\]

The relation between this norm and the usual operator norm is established in the following lemma

**Lemma 3.4.** Let $\theta_0 = \angle(E_0, F_0)$ and $\theta_1 = \angle(E_1, F_1)$. Then:

1. $\|T\| \leq 4(\sin \theta_0)^{-1}\|T\|_{\text{max}}$;
2. $\|T\|_{\text{max}} \leq (\sin \theta_1)^{-1}\|T\|$.

**Proof.** Let $v = v_0 + u_0 \in E_0 \oplus F_0$. From the Sine Theorem it follows that $\|v_0\| \leq \|v\|/\sin \theta_0$ and the same inequality holds for $\|u_0\|$. Thus,
\[
\|Tv\| \leq \|T^{++}v_0\| + \|T^{+-}u_0\| + \|T^{-+}v_0\| + \|T^{--}u_0\| \leq 4\|T\|_{\text{max}}\|v\|/\sin \theta_0.
\]

This proves the first inequality.

For the second inequality let $v_0 \in E_0$. Its image splits as $Tv_0 = T^{++}v_0 + T^{-+}v_0 \in E_1 \oplus F_1$. Once again from Sine Theorem it follows that,
\[
\|T^{++}v_0\| \leq \|Tv_0\| (\sin \theta_1)^{-1} \leq \|T\|\|v_0\| (\sin \theta_1)^{-1}
\]
for $* = +$ and $* = -$. Together with the corresponding estimate for $T^{*-}u_0$, $u_0 \in F_0$ the second inequality follows. $\square$

We also need the following recurrence result from [4, Lemma 3.12]:

**Lemma 3.5.** Let $f : X \to X$ be a measurable homeomorphism. Let $C \subset X$ be a measurable set with $\mu(C) > 0$ and let $\Gamma = \bigcup_{n \in \mathbb{Z}} f^n(C)$. Fix any $\gamma > 0$. Then there exists a measurable function $N_0 : \Gamma \to \mathbb{N}$ such that for almost every $x \in \Gamma$, and for all $n \geq N_0(x)$ and $t \in (0, 1)$, there exists $\ell \in \{0, 1, \ldots, n\}$ such that $t - \gamma \leq \ell/n \leq t + \gamma$ and $f^\ell(x) \in C$.

We are now ready to give a proof for Lemma 3.3
Proof of Lemma 3.3. Fix $M$, $\varepsilon$ and $\delta$, and $A$ such that $\|A\| < M$. Following [5] we divide the proof in two parts for the sake of clarity.

Part 1: Definition of $N_A(\cdot)$ and the sequence $\hat{L}_{(x,n)}^\perp$.

Since $m \in \mathbb{N}$ is greater than $m_0$ the conclusion of Proposition 3.1 holds true. Let $C \subset \mathcal{R}$ be the set of points such that the nondomination condition holds (see equation (3)). Since every point $x \in \mathcal{R}$ has no dominated splitting, we can rewrite $\mathcal{R}$ as follows:

$$\mathcal{R} = \bigcup_{n \in \mathbb{Z}} f^n(C).$$

Let $\lambda_1(A), 1 \leq i \leq \binom{d}{p}$, denote the Lyapunov exponents of the cocycle $\lambda^n(A)$ over $f$, in nonincreasing order. Recall that

$$\lambda_1^n(A) = \lambda_1(A) + \ldots + \lambda_{p-1}(A) + \lambda_p(A),$$

$$\lambda_2^n(A) = \lambda_1(A) + \ldots + \lambda_{p-1}(A) + \lambda_{p+1}(A).$$

Since $\lambda_p(A) > \lambda_{p+1}(A)$ then $\lambda_1^n(A) > \lambda_2^n(A)$, this gives us a splitting $\lambda^n \mathbb{R}^d = V \oplus H$ where $V$ denotes the Oseledets subspace associated to the upper exponent and $H$ the sum of all other Oseledets subspaces. Note that $V$ is one-dimensional.

For $x \in \mathcal{R}$ the Oseledets’ Theorem 2.1 gives $Q(x) \in \mathbb{N}$ such that for every $n \geq Q(x)$ we have:

1. $\frac{1}{n} \log \frac{\|\lambda^n A^n(x)v\|}{\|v\|} < \lambda_1^n(A) + \delta$ for every $v \in V \setminus \{0\}$;
2. $\frac{1}{n} \log \frac{\|\lambda^n A^n(x)w\|}{\|w\|} < \lambda_2^n(A) + \delta$ for every $w \in H \setminus \{0\}$;
3. $\frac{1}{n} \log \sin \angle(V(f^n x), H(f^n x)) > -\delta$.

For $q \in \mathbb{N}$, let $B_q = \{x \in \mathcal{R}; Q(x) \leq q\}$. Then $B_q$ form a nondecreasing sequence and their union is $\mathcal{R}$. Let $q$ be the first positive integer such that $\mu(C \cap f^{-m}(B_q)) > 0$, from ergodicity the set

$$C_q = \bigcup_{n \in \mathbb{Z}} f^n(C \cap f^{-m}(B_q))$$

is equal to $\mathcal{R}$ almost everywhere. We are defining the function $N_B$ in the set $C_q$.

Let $c$ be a strict upper bound for $\log \|\lambda^n A\|$ and $\gamma = \min\{c^{-1}\delta, 1/10\}$. Using the sets (9) and Lemma 3.5, we find a measurable function $N_0 : C_q \to \mathbb{N}$ such that for almost every $x \in C_q$, every $n \geq N_0(x)$ and every $t \in (0,1)$ there is $\ell \in \{0,1,\ldots,n\}$ with $|\ell/n - t| < \gamma$ and $f^\ell(x) \in C \cap f^{-m}(B_q)$. We define $N_B(x)$ for $x \in C_q$ as the least integer such that

$$N_B(x) \geq \max\{N_0(x), 10Q(x), m\gamma^{-1}, \delta^{-1} \log[4/\sin \angle(E_x, F_x)]\}.$$
Fix a point $x \in C_q$ and $n \geq N(x)$. We construct the interchanging $\varepsilon$-sequence $\{\dot{L}_j^{(x,n)}\}$. As $n \geq N_0$, there exists $\ell \in \mathbb{N}$ such that
\[
\frac{\ell - 1}{n} < \gamma \quad \text{and} \quad y = f^\ell(x) \in C \cap f^{-m}(B_q).
\]
Since $y \in C$, where the nondomination condition holds, Proposition 3.1 gives an interchanging $\varepsilon$-sequence $\{L_0, \ldots, L_{m-1}\}$ such that there are nonzero vectors $v_0 \in E_y$ and $w_0 \in F_{f^m(y)}$ for which
\[
L_{m-1} \ldots L_0(v_0) = w_0.
\]
We form the interchanging $\varepsilon$-sequence $\{\dot{L}_0^{(x,n)}, \ldots, \dot{L}_{n-1}^{(x,n)}\}$ of length $n$ by concatenating
\[
\{A(f^i(x)); \; 0 \leq i < \ell\}, \{L_0, \ldots, L_{m-1}\}, \{A(f^i(x)); \; \ell + m \leq i < n\}.
\]
By construction this is an interchanging $\varepsilon$-sequence over $x$ of length $n$.

Part 2. Estimation of $\|\wedge^p (\dot{L}_{n-1}^{(x,n)} \ldots \dot{L}_0^{(x,n)})\|$.

To simplify notation, write $\wedge^p (\dot{L}_{n-1}^{(x,n)} \ldots \dot{L}_0^{(x,n)}) = A_1 L A_0$, with $A_0 = \wedge^p A^{\ell-1}(x)$, $A_1 = \wedge^p A^{n-\ell-m}(f^{\ell+m}(x))$ and $L = \wedge^p L_{m-1} \ldots L_0$.

Claim 1. The map $L : \wedge^p(\mathbb{R}^d) \rightarrow \wedge^p(\mathbb{R}^d)$ satisfies $L(V(y)) \subset H(f^m(y))$.

Proof of Claim 1. Let $x \in \mathcal{R}$ and consider a basis $\{e_1(x), \ldots, e_d(x)\}$ of $\mathbb{R}^d$ such that $e_i(x) \in E_j(x)$ for $\dim E_1(x) + \cdots + \dim E_{j-1}(x) < i \leq \dim E_1(x) + \cdots + \dim E_j(x)$.

The $V(x)$ is generated by $e_1(z) \wedge \cdots \wedge e_p(x)$ and $H(x)$ is generated by the vectors $e_{i_1}(x) \wedge \cdots \wedge e_{i_p}(x)$ with $1 \leq i_1 < \cdots < i_p \leq d$, $i_p > p$. Notice as well that $\{e_1(x), \ldots, e_p(x)\}$ and $\{e_{p+1}(x), \ldots, e_d(x)\}$ are bases for $E(x)$ and $F(x)$ respectively.

Consider the vectors $v_0 \in E(y)$ and $w_0 = L(v_0) \in F(f^m y)$ where $L = L_{m-1} \ldots L_0$. There is $j \in \{1, \ldots, p\}$ such that
\[
\{v_0, e_1(y), \ldots, e_{j-1}(y), e_j+1(y), \ldots, e_p(y)\}
\]
is a basis for $E(y)$. Therefore $V(y)$ is generated by the vector
\[
v_0 \wedge e_1(y) \wedge \cdots \wedge e_{j-1}(y) \wedge e_{j+1}(y) \wedge \cdots \wedge e_p(y),
\]
whose image by $L = \wedge^p(L)$ is
\[
w_0 \wedge L(e_1(y)) \wedge \cdots \wedge L(e_{j-1}(y)) \wedge L(e_{j+1}(y)) \wedge \cdots \wedge L(e_p(y)).
\]
As $w_0$ is a linear combination of vectors $e_{p+1}(f^m(y)), \ldots, e_d(f^m(y))$ and each $L(e_i(y)$ is a linear combination of vectors $e_1(f^m(y)), \ldots, e_d(f^m(y))$ we must ensure that $e_1(f^m(y)) \wedge \cdots \wedge e_p(f^m(y))$ does not appear. This shows the vector (11) belongs to $H(f^m(y))$ and the claim follows. \qed
This claim provides us the key observation that \( \mathcal{L}(V(f^t x)) \subset H((f^t x)_i) \). Following the blocks notations introduced at equation (7), the associated matrices to these cocycles have the form:

\[
\mathcal{A}_i = \begin{bmatrix} \mathcal{A}_i^{++} & 0 \\ 0 & \mathcal{A}_i^{--} \end{bmatrix}, \quad i = 0, 1, \quad \text{and} \quad \mathcal{L} = \begin{bmatrix} 0 & \mathcal{L}^{++} \\ \mathcal{L}^{--} & \mathcal{L}^{+-} \end{bmatrix}
\]

\( \mathcal{A}_i^{+-} = 0 = \mathcal{A}_i^{--} \) since the sub-bundles \( V \) and \( H \) are \( \wedge^p A \)-invariant, and \( \mathcal{L}^{++} = 0 \) due to Claim 1. Then

\[
\hat{L}_{n-1} \ldots \hat{L}_0 = \begin{bmatrix} 0 & \mathcal{A}_1^{++} \mathcal{L}^{+-} \mathcal{A}_0^{--} \\ \mathcal{A}_1^{+-} \mathcal{L}^{--} \mathcal{A}_0^{++} & \mathcal{A}_1^{--} \mathcal{L}^{+-} \mathcal{A}_0^{++} \end{bmatrix}.
\]

Claim 2. For \( i = 0, 1, x \in C_q \setminus C_{q-1} \) and \( n \geq N(x) \),

\[
\log \| A_0^{++} \| < \frac{1}{2} n(\lambda_1^{\wedge p}(A) + 5\delta) \quad \text{and} \quad \log \| A_0^{--} \| < \frac{1}{2} n(\lambda_2^{\wedge p}(A) + 5\delta).
\]

Proof of Claim 2. Since \( \ell > (\frac{1}{2} - \gamma)n > \frac{1}{10} n \geq \frac{1}{10} N(x) \geq Q(x) \) we have that

\[
\log \| A_0^{++} \| = \log \| \wedge^p (A^t(x)) \|_{V(x)} = \ell(\lambda_1^{\wedge p}(A) + \delta),
\]

\[
\log \| A_0^{--} \| = \log \| \wedge^p (A^t(x)) \|_{H(x)} = \ell(\lambda_2^{\wedge p}(A) + \delta).
\]

Let \( \lambda \) be either \( \lambda_1^{\wedge p}(A) \) or \( \lambda_2^{\wedge p}(A) \). Using \( \gamma \lambda < \gamma c \leq \delta \) and \( \gamma < 1 \), we find

\[
\ell(\lambda + \delta) < n(\frac{1}{2} + \gamma)(\lambda + \delta) < n(\frac{1}{2} \lambda + \frac{1}{2} \delta + \delta + \delta) = \frac{1}{2} n(\lambda + 5\delta).
\]

This proves the case \( i = 0 \).

On the other hand, for \( i = 1 \) we have \( n - \ell - m > n(\frac{1}{2} - \gamma) - n\gamma > \frac{1}{10} n \geq Q(x) \geq q \). Also \( f^t(x) \in f^{-m}(B_q) \) and so \( Q(f^t+m(x)) \leq q \). Therefore

\[
\log \| A_1^{++} \| = \log \| \wedge^p (B^{n-\ell-m}(f^{t+m}x)) \|_{V(f^{t+m}x)} < (n - \ell - m)(\lambda_1^{\wedge p}(A) + \delta),
\]

\[
\log \| A_1^{--} \| = \log \| \wedge^p (A^{n-\ell-m}(f^{t+m}x)) \|_{H(f^{t+m}x)} < (n - \ell - m)(\lambda_2^{\wedge p}(A) + \delta).
\]

As before, \( (n - \ell - m)(\lambda + \delta) < n(\frac{1}{2} + \gamma)(\lambda + \delta) \leq \frac{1}{2} n(\lambda + 5\delta) \) and this concludes the proof of the claim. \( \square \)

Claim 3. \( \log \| \mathcal{L} \|_{\text{max}} < 2n\delta. \)

Proof of Claim 3. Since \( \{ L_0, \ldots, L_{m-1} \} \) is an interchanging \( \varepsilon \)-sequence, each \( L_j \) is close to the value of \( A \) at some point. We may then assume that \( \log \| L_j \| < c \). In particular, \( \log \| \mathcal{L} \| < mc \leq nc\gamma \leq n\delta \). We have that \( \ell + m \geq n(\frac{1}{2} - \gamma) \geq \frac{1}{10} n \geq Q(x) \).

So \( \log \left[ 1/ \sin \mathcal{L}(V_{f^{t+m}x}, H_{f^{t+m}x}) \right] < \delta \) and, by Lemma 3.4 part 2, \( \log \| \mathcal{L} \|_{\text{max}} < 2n\delta. \) \( \square \)
To conclude, we use the previous claims to bound each of the blocks in (12):

\[
\log \| A_{1^-} \mathcal{L}^- A_{0^-} \| < \frac{1}{2} n (\lambda_1^{\lambda_p}(A) + \lambda_2^{\lambda_p}(A) + 14\delta),
\]

\[
\log \| A_{1^-} \mathcal{L}^+ A_{0^+} \| < \frac{1}{2} n (\lambda_1^{\lambda_p}(A) + \lambda_2^{\lambda_p}(A) + 14\delta),
\]

\[
\log \| A_{1^-} \mathcal{L}^- A_{0^-} \| < \frac{1}{2} n (2\lambda_2^{\lambda_p}(A) + 14\delta).
\]

The last expression is smaller than the first two, it follows that

\[
\log \| \hat{L}_{n-1} \ldots \hat{L}_0 \|_{\max} < n \left( \frac{\lambda_1^{\lambda_p}(A) + \lambda_2^{\lambda_p}(A)}{2} + 7\delta \right).
\]

Therefore, by the first inequality of Lemma 3.4 and

\[
\log \left[ \frac{4}{\sin \angle (V(x), H(x))} \right] < n\delta,
\]

\[
\log \| \hat{L}_{n-1} \ldots \hat{L}_0 \| < n \left( \frac{\lambda_1^{\lambda_p}(A) + \lambda_2^{\lambda_p}(A)}{2} + 8\delta \right).
\]

Replacing \( \delta \) by \( \delta/8 \) this concludes the proof of Lemma 3.3. \( \square \)

4. Lowering the main Lyapunov exponent

In the last section we established that under non \( m \)-domination for a large \( m \) we can find a sufficiently long interchanging \( \varepsilon \)-sequence that lowers the norm along the orbit of a non periodic point. Our aim is using these sequences to construct nearby cocycles such that the main Lyapunov exponent is lower.

The main ingredient is the following perturbation result. Recall that \( \mathcal{R} \) contains no periodic points.

**Proposition 4.1.** Let \( x \in \mathcal{R}, M > 0, \varepsilon \in (0,1) \) and \( \delta > 0 \). Assume that the cocycle \( A \in C(X, \mathcal{G}) \) such that \( \| A \| < M \) is not \( m \)-dominated for some \( m > m_0(M, \varepsilon) \) and let \( n \geq N_A(x) \). Then there exists an open set \( U_x \) containing \( x \) such that for every open set \( V \subset U_x \) and any compact set \( K \subset V \), there exists a cocycle \( B \in C(X, \mathcal{G}) \) such that:

1. \( \| A - B \| < \varepsilon \).
2. \( B \) is constant within \( f^i(K), \forall j = 0, 1, \ldots, n - 1 \).
3. \( B = A \) in \( X \setminus \bigcup_{i=0}^{n-1} f^i(V) \).
4. For every \( y \in K \) it holds that

\[
\frac{1}{n} \log \| \wedge^p B^n(y) \| < \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \delta.
\]

**Proof.** Since \( x \) is a nonperiodic point, there exists an open set \( U \) containing \( x \) such that \( f^i(U) \cap f^j(U) = \emptyset \) for every \( i \neq j \) and \( i, j \in \{0, \ldots, n - 1\} \). Let \( \delta_{A,u} \) be given by the uniform continuity of the cocycle \( A \) for \( \varepsilon/2 \). Let \( U_x \) be the connected component of \( U \cap \left( \bigcap_{i=0}^{n-1} f^{-i}(B(f^i(x), \delta_{A,u})) \right) \) that contains the point \( x \).
Let \( V \subset U_x \) be an open set and let \( K \subset V \) be a compact set. Since \( K \) and \( \partial V \) are compact sets it holds that \( \rho = d(K, \partial V) \) is a positive number. We consider the following set

\[ V_0 = \{ y \in V, \ d(y, K) < \rho/2 \}. \]

The set \( V_0 \) is open and it holds that \( K \subset V_0 \subset V \). Since \( f \) is a homeomorphism we have that \( \hat{V} = \bigcup_{i=0}^{n-1} f^i(V_0) \) is an open set and \( \hat{K} = \bigcup_{i=0}^{n-1} f^i(K) \) is a compact set.

By Urysohn’s Lemma 2.5, there exists a continuous function \( h : X \rightarrow \mathbb{R} \) such that \( h|_{V_0} \equiv 0 \) and \( h|_{\hat{K}} \equiv 1 \). By virtue of Lemma 3.3, using \( \delta \) and \( \varepsilon/2 \) we can find an interchanging \( \varepsilon/2 \)-sequence of length \( n \) over \( x \) namely \( \{\hat{L}_i(x,n)\}_{i=0}^{n-1} \).

To define the cocycle \( B \) we consider two different cases. First, consider \( \mathcal{G} = \text{GL}(d, \mathbb{R}) \). We can define the cocycle \( B \in C(X, \mathcal{G}) \) as:

\[
B(y) = \begin{cases} 
A(y) & \text{if } y \in X \setminus \bigcup_{i=0}^{n-1} f^i(U) \\
(1 - h(y)) A(y) + h(y) \hat{L}_i(x,n) & \text{if } y \in f^i(U), \ i \in \{0, \ldots, n-1\}.
\end{cases}
\]

By construction \( B \) satisfies items 2 to 4.

In order to prove our proposition we need to calculate \( \|A - B\| \). It follows from the definition of the cocycle \( B \) that given \( y \in X \), if \( y \in X \setminus \hat{V} \) then

\[ \|A(y) - B(y)\| = 0. \]

Otherwise, for \( i \in \{0, \ldots, n-1\} \) and \( y \in f^i(U) \cap \hat{V} \),

\[
\|A(y) - B(y)\| = h(y) \|\hat{L}_i(x,n) - A(y)\| \leq \|\hat{L}_i(x,n) - A(f^i(x))\| + \|A(f^i(x)) - A(y)\| < \varepsilon
\]

where the last inequality follows from the fact that the distance \( d(y, f^i(x)) < \delta_{A,u} \) and this shows the proposition in the first case.

For the case \( \mathcal{G} = \text{SL}(d, \mathbb{R}) \) we proceed similarly. Note that \( \text{SL}(d, \mathbb{R}) \) is not convex so the construction for \( B \) does not work. We fix this as follows.

Denote \( \mathbf{B}(A(f^i(x)), \varepsilon/2) \subset \text{SL}(d, \mathbb{R}) \) the ball of radius \( \varepsilon/2 \) and center \( A(f^i(x)) \) for \( i \in \{0, \ldots, n-1\} \). Since \( \text{SL}(d, \mathbb{R}) \) is locally contractible, we can find a continuous function \( \gamma \in \text{Cont}(\bar{V} \times [0, 1], \bigcup_{i=0}^{n-1} \mathbf{B}(A(f^i(x)), \varepsilon/2)) \) in the closure of \( \hat{V} \times [0, 1] \). Such that

\[ \gamma(y, 0) = A(y), \quad \gamma(y, 1) = \hat{L}_i(x,n), \text{ for } y \in f^i(V_0). \]

We proceed to define the cocycle \( B \in C(X, \mathcal{G}) \) as:

\[
B(y) = \begin{cases} 
A(y) & \text{if } y \in X \setminus \hat{V} \\
\gamma(y, h(y)) & \text{if } y \in \hat{V}.
\end{cases}
\]
Similarly to the GL\((d, \mathbb{R})\) case, \(B\) satisfies items 2 to 4 by construction. The calculation of \(\|A - B\|\) is similar as well, from the definition of the cocycle \(B\) that given \(y \in X\), if \(y \in X \setminus \hat{V}\) then
\[
\|A(y) - B(y)\| = 0.
\]

Otherwise, for \(i \in \{0, \ldots, n - 1\}\) and \(y \in f^i(U) \cap \hat{V}\),
\[
\|A(y) - B(y)\| = \|\gamma(y, h(y)) - A(y)\|
\leq \|\gamma(y, h(y)) - A(f^i(x))\| + \|A(f^i(x)) - A(y)\| < \varepsilon
\]
where the last inequality follows, again, from the fact that the distance \(d(y, f^i(x)) < \delta_{A,u}\) showing the proposition for the second case.

The current objective is to use perturbations as in last proposition to obtain a cocycle with a lower Lyapunov exponent. The following construction will be useful.

**Definition 4.2.** Let \(f \in \text{Hom}_\mu(X)\). A *tower* is a pair of measurable sets \((T, T_b)\) such that there is a positive integer \(n\), called the *height* of the tower, such that the sets \(T_b, f(T_b), \ldots, f^{n-1}(T_b)\) are pairwise disjoint and their union is \(T\). The set \(T_b\) is called the *base* of the tower. We call the sets \(f^i(T_b)\) the \(i\)-th *floor* of the tower.

A *castle* is a pair of measurable sets \((Q, Q_b)\) such that there exists a finite or countable family of pairwise disjoint towers \((T^i, T^i_b)\) such that \(Q = \bigcup_i T^i\) and \(Q_b = \bigcup_i T^i_b\). We call \(Q_b\) the *base* of the castle.

Given a positive measure set \(G \subset X\), consider the return time map \(\tau : G \to \mathbb{N}\) defined by \(\tau(x) = \min\{n \geq 1; f^n(x) \in G\}\). Denoting \(G_n = \tau^{-1}(n)\), then \((T_n, G_n)\) is a tower, where \(T_n = G_n \cup f(G_n) \cup \ldots \cup f^{n-1}(G_n)\). Consider the castle \((Q, G)\), with base \(G\), given by the union of the towers \(T_n\). This is called the *Kakutani-Rokhlin Castle* with base \(G\). Note that \(Q = \bigcup_{n \in \mathbb{Z}} f^n(G)\) modulo a zero measure set. In particular it holds that \(Q\) is an invariant set.

The following Lemma assures that if we have a measurable set such that its boundary measures zero, then we can assume up to a zero measure set that its Kakutani-Rokhlin castle has towers with open bases by substituting \(G_i\) with \(\text{int}(G_i)\).

**Lemma 4.3.** Let \(G\) be a measurable set such that \(\mu(\partial G) = 0\). Consider the Kakutani-Rokhlin castle \((Q, G)\) and its towers \((T_i, G_i)\). Then it holds that \(\mu(\partial G_i) = 0\) for every \(i \in \mathbb{N}\).

**Proof.** We will show this by induction in the basis of the Kakutani-Rokhlin towers. Note that, by definition, it holds that \(G_1 = G \cap f^{-1}(G)\). It follows that
\[
\partial G_1 = (\partial G \cap f^{-1}(G)) \cup (G \cap f^{-1}(\partial G))
\]
By hypothesis \(\mu(\partial G) = 0\), due to \(\mu\) being an \(f\)-invariant measure it follows that \(\mu(f^{-1}(\partial G)) = 0\). Thus \(\mu(\partial G_1) = 0\).
Now, suppose the conclusion of the Lemma is true for $G_n$. Note that the set

$$G_{n+1} = (G \cap f^{-(n+1)}(G)) \setminus G_n.$$  

Then

$$\partial G_{n+1} = \partial (G \cap f^{-(n+1)}(G)) \setminus G_n \cup (\partial G_n \cap G \cap f^{-(n+1)}G).$$

We conclude that $\mu(\partial (G \cap f^{-(n+1)}(G))) = 0$ analogously as the $\partial G_1$ case. And $\mu(\partial G_n \cap G \cap f^{-(n+1)}G) = 0$ since $\mu(\partial G_n) = 0$. It follows that $\mu(\partial G_{n+1}) = 0$ which concludes the proof. □

Our main tool for creating sets with zero-measure boundary are open balls. This is justified by the following remark.

**Remark 4.4.** Given any point $x \in X$ there are uncountably infinitely many radii of balls $B(x, r)$ such that $\mu(\partial B(x, r)) = 0$.

This follows immediately from the fact that the set $S = \{ \partial B(x, r) : 0 < r < 1 \}$ is an uncountable set and it is composed of disjoint sets. As the measure $\mu$ is finite it follows that $\mu(\partial B(x, r)) = 0$ except, possibly, in countably many radii. In what follows we will use this fact to construct sets with zero-measure boundaries.

We are ready to globalize the perturbation in the last proposition in order to get a cocycle with a lower first Lyapunov exponent.

**Theorem 4.5.** Let $\varepsilon \in (0, 1)$, $\delta > 0$ and $M > 0$. Let $A \in C(X, G)$ such that $\|A\| < M$ and assume that the cocycle $A$ is not $m-$dominated for some $m \geq m_0(\|A\|, \varepsilon)$. Then there exists a cocycle $B \in C(X, G)$ such that:

1. $\|A - B\| < \varepsilon$,
2. $\Lambda_p(B) < \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \delta$.

**Proof.** Fix $\varepsilon > 0$ and $\delta > 0$. In order to prove the theorem, we will make many perturbations as in Proposition 4.1 within a large set. For the sake of clarity we divide the proof in two parts.

**Part 1: Construction of the cocycle $B$.**

First we note that, as $\|A\| < M$, then $\|A^{p^*} A\| < M^p$. Therefore the norm of cocycles in $U(A^{p^*} A, \varepsilon)$ is bounded by $c = M^p + \varepsilon$. In what follows we denote

$$\varepsilon_1 = \frac{\delta}{6 \log c}.$$  

For every positive integer $N$ we consider the set

$$N_N = \{ y \in \mathcal{R} \text{ such that } N_A(y) \leq N \}.$$  

These sets form a non decreasing sequence and up to a zero measure set their union is the set $X$. We take $N_0 \in \mathbb{N}$ such that

$$\mu(N_{N_0}) \geq 1 - \frac{9}{10} \varepsilon_1.$$
Let $x \in \mathcal{R}$, as it is a nonperiodic point there exists an open set $G$ containing $x$ such that $f^i(G) \cap f^j(G) = \emptyset$ for $i \neq j$, and $i, j \in \{0, \ldots, 10N_0 - 1\}$. Note that we can assume that the boundary $\partial G$ is a zero measure set by taking $G$ from open balls as in Remark 4.4 if necessary.

We consider the Kakutani-Rokhlin castle of basis $G$. From Lemma 4.3 it follows that we can assume the bases $\{G_{i,j}\}_{i=0}^{\infty}$ of the Kakutani-Rokhlin towers to be open sets with a zero measure boundary.

Since the measure $\mu$ is a probability measure, it follows that there exists $m$ such that the towers $\{(T_i,G_i)\}_{i=0}^{\infty}$ hold that $\mu(\bigcup_{i=m}^{\infty} T_i) < \varepsilon_1$.

We aim to partition each remaining tower in smaller subtowers of convenient size. To do this fix $i \in \{N_0, \ldots, m - 1\}$ and let $\delta_{A,u}$ be given by the uniform continuity of the cocycle $A$ for $\varepsilon/2$. For every $x \in G_i$ consider $V_x$ the connected component containing $x$ of $\bigcap_{j=0}^{m-1} f^{-j}(B(f^j(x), \delta_{A,u}))$. Since $V_x$ is open, we can choose a ball $B(x, r_x) \subset V_x$ such that $\mu(\partial B(x, r_x)) = 0$ as in Remark 4.4.

As the balls $B(x, r_x)$ cover $\partial G_i$, there exists a finite subcover $B_1, \ldots, B_{n(i)}$. We denote $G_{i,1} = B_1 \cap G_i$. And inductively we define

$$G_{i,j} = (B_j \setminus \bigcup_{l=1}^{j-1} G_{i,l}) \cap G_i$$

for $j \in \{2, \ldots, n(i)\}$. This creates a partition of the tower $(T_i, G_i)$ in smaller disjoint towers $(T_{i,j}, G_{i,j})$ with the same number of floors. Note that, by construction, $\mu(\partial G_{i,j}) = 0$.

For every tower $T_{i,j}$ consider

$$k(i,j) = \min\{k \in \{0, \ldots, i - N_0\}, \text{ such that } f^k(G_{i,j}) \cap \mathcal{N}_{N_0} \neq \emptyset\},$$

note that the towers where $k(i,j)$ is not well defined are either entirely contained in $(\mathcal{N}_{N_0})^c$ or the intersection is in the last $N_0$ floors. We denote the set where $k(i,j)$ is not well defined and $(\mathcal{N}_{N_0})^c$ by $\mathcal{K}$.

Note that, if $x \in f^{k(i,j)}(G_{i,j}) \cap \mathcal{N}_{N_0}$, from Proposition 4.1 we get an open set $U_x$. The open set $f^{k(i,j)}(G_{i,j})$ is a subset of $U_x$. This follows from the construction of $G_{i,j}$ which mimics the construction of $U_x$ but uses higher towers, making it a smaller set. Recall that $\mu(f^{k(i,j)}(G_{i,j})) = \mu(G_{i,j})$ thus, we denote by $K_{i,j}$ a compact subset of $f^{k(i,j)}(G_{i,j})$ such that

$$\mu(K_{i,j}) \geq (1 - \varepsilon_1) \mu(G_{i,j}).$$

Using Proposition 4.1 we construct cocycles $B_{i,j}$ using $\varepsilon$, $\delta/2$, and setting the number $n = i - k(i,j)$, the open sets $f^{k(i,j)}(G_{i,j})$ and the compact sets $K_{i,j}$. We denote $\hat{T}_{i,j} = T_{i,j} \setminus \bigcup_{k=0}^{k(i,j)-1} f^k(G_{i,j})$. As sketched by figure 1
We define the cocycle $B$ by parts as follows. Let $y \in X$,

$$B(y) = \begin{cases} B_{i,j}(y) & \text{if } y \in \hat{T}_{i,j} \\ A(y) & \text{everywhere else.} \end{cases}$$

From the definition it follows that $\|A - B\| < \varepsilon$.

**Part 2: Estimation of the sum of the $p$ first Lyapunov exponents $\Lambda_p(B)$.

Recall that $\Lambda_p(B) = \lambda_1^{\wedge p}(B)$, that is, the main Lyapunov exponent of the Exterior Cocycle $\wedge^p B$. In order to estimate such main Lyapunov exponent, let

$$y \in K = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n(i)} K_{i,j}.$$

Since all $K_{i,j}$ are disjoint, let $n(y) = i - k(i,j)$ that is, $n(y)$ is the height of the tower $\hat{T}_{i,j}$. Let $\tau_1$ be the first return time of $y$ to $K$.

Denoting by $y_2 = f^{\tau_1}(y)$, we denote analogously by $n(y_2) = i - k(i,j)$ such that $y_2 \in K_{i,j}$, the height of the tower $\hat{T}_{i,j}$, and $\tau_2$ the first return time to $K$. Denote then by $y_3 = f^{\tau_2}(y_2)$ and so on. For consistency in notation denote $y$ by $y_1$. 
Fix $N \in \mathbb{N}$, we can write $N = \tau_1 + \tau_2 + \ldots + \tau_k + \hat{N}$, for some $k \in \mathbb{N}$ such that $\hat{N} = N - \sum_{i=1}^{k} \tau_i$ is lower than $\tau_{k+1}$. It holds that, for the exterior cocyle $\wedge^p(B)$,

$$\wedge^p(B^N)(y) = \wedge^p(B^\hat{N})(y_k) \wedge^p(B^{\tau_k-n(y_k)}(f^{n(y_k)})(y_k)) \wedge^p(B^{n(y_k)})(y_k) \ldots \wedge^p(B^{\tau_1-n(y_1)}(f^{n(y_1)})(y_1)) \wedge^p(B^{n(y_1)})(y_1).$$

In particular, it follows that the norm for $\wedge^p B^N(y)$ holds that

$$\| \wedge^p B^N(y) \| \leq \| \wedge^p B^\hat{N}(y_k) \| \prod_{i=1}^{k} \| \wedge^p B^{\tau_i-n(y_i)}(f^{n(y_i)})(y_i) \| \| \wedge^p B^{n(y_i)}(y_i) \|.$$

Recall that $c$ is an upper bound of the norm of cocycles in $\mathcal{U}(\wedge^p A, \varepsilon)$, then if $n_N = \hat{N} + \sum_{i}^{k} (\tau_i - n(y_i))$ it follows that

$$\| \wedge^p B^N(y) \| \leq \| \wedge^p B^{n(y_1)}(y_1) \| \| \wedge^p B^{n(y_2)}(y_2) \| \ldots \| \wedge^p B^{n(y_k)}(y_k) \| e^{n_N}.$$

Now we calculate,

$$\frac{1}{N} \log \| \wedge^p B^j(y) \| \leq \frac{1}{N} \left( \log \| \wedge^p B^{n(y_1)}(y_1) \| + \ldots + \log \| \wedge^p B^{n(y_k)}(y_k) \| \right) + \frac{n_N}{N} \log c.$$

From Proposition 4.1 it follows that, for $i \in \{1, \ldots, k\}$,

$$\frac{1}{n(y_i)} \log \| \wedge^p B^{n(y_i)}(y_i) \| \leq \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \frac{\delta}{2},$$

thus

$$\frac{1}{N} \log \| \wedge^p B^N(y) \|$$

$$\leq \frac{1}{N} \left( \frac{n(y_1)}{n(y)} \log \| \wedge^p B^{n(y_1)}(y_1) \| + \ldots + \frac{n(y_k)}{n(y_k)} \log \| \wedge^p B^{n(y_k)}(y_k) \| \right) + \frac{n_N}{N} \log c$$

$$\leq \frac{\sum_{i=1}^{k} n(y_i)}{N} \left( \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \frac{\delta}{2} \right) + \frac{n_N}{N} \log c$$

$$\leq \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \frac{\delta}{2} + \frac{n_N}{N} \log c.$$

Since $\mu(K) > 0$ it follows that almost every $y \in K$ is a regular point for the cocyle $B$, thus the limit $\lim_{N \to \infty} \frac{1}{N} \log \| \wedge^p B^N(y) \|$ exists for $\mu$—almost every $y$ and it holds that

$$\Lambda_1(B) = \lim_{N \to \infty} \frac{1}{N} \log \| \wedge^p B^N(y) \| \leq \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \frac{\delta}{2} + \lim_{N \to \infty} \frac{n_N}{N} \log c.$$
The last limit is the mean sojourn time of $y$ at the complement of the towers with base $K_{i,j}$ within the castle $Q$. Since $\mu$ is ergodic it follows from Birkhoff’s Theorem 2.4 that this limit is equal to the measure of the complement of these towers.

For the purpose of calculating this measure, recall that we can divide this complement in three not necessarily disjoint sets. The first one is the towers of the castle $(Q, G_{i})$ with at least $m$ floors, that is $\cup_{i=m}^{\infty} T_{i}$. Recall that $m$ was chosen such that $\mu(\cup_{i=m}^{\infty} T_{i}) < \varepsilon_{1}$.

The second one is the set $K$. Recall that it contains the towers of the castle such that all except the last $N_{0}$ floors are entirely contained in the complement of $N_{N_{0}}$, recall that the minimum height of a tower is $10N_{0}$ floors. It follows that the measure is $10/9\mu((N_{N_{0}})^{c}) < \varepsilon_{1}$ since $\mu(N_{N_{0}}) > 1 - 9/10\varepsilon_{1}$.

The third set is

$$\mathcal{T} = \bigcup_{n(i) \geq k(i,j)} \bigcup_{i=N_{0}}^{m-1} \bigcup_{j=1}^{n(i)} \bigcup_{\ell=0}^{n(i)-k(i,j)} f^{\ell} \left( f^{k(i,j)} G_{i,j} \setminus K_{i,j} \right)$$

recall the relation $\mu(K_{i,j}) \geq (1 - \varepsilon_{1}) \mu(G_{i,j})$. This implies that the measure of $\mathcal{T}$ is lower than $\varepsilon_{1}$.

We conclude from Birkhoff Theorem that:

$$\lim_{N \to \infty} \frac{n_{N}}{N} \leq \mu(\mathcal{K} \cup \mathcal{T} \cup (\cup_{i=m}^{\infty} T_{i})) < 3\varepsilon_{1} = \frac{\delta}{2 \log c}.$$

Thus we deduce that

$$\Lambda_{p}(B) \leq \frac{\Lambda_{p-1}(A) + \Lambda_{p+1}(A)}{2} + \delta,$$

the theorem now follows. \hfill \Box

5. Proof of Main Theorem

Before proving the main theorem, we say that $\alpha$ is the limit inferior of the $p$–sum of the first Lyapunov exponents as the cocycle $B$ tends to $A$ when

$$\alpha = \liminf_{B \to A} \Lambda_{p}(B) = \sup (\inf \{\Lambda_{p}(B) : 0 < \|B - A\| < \varepsilon\}).$$

It is also worthy of note that, when a cocycle $A$ has no dominated splitting of index $p$ then, for any sequence of cocycles $A_{n} \to A$, it holds that either $A_{n}$ possesses no dominated splitting of index $p$ or such dominated splitting is weak (i.e: the $m$ for the $m$–domination gets arbitrarily large), in particular, it is greater than $m_{0}(\|A\| + 1, \varepsilon)$ for any small $\varepsilon$. This is a simple consequence of the continuity of $m$–dominated splittings. See [6] for further details.
Proof of main theorem. Suppose the cocycle $A$ holds the uniform $p$-gap property and it also has no dominated splitting of index $p$. From the uniform gap property it holds that there exists a neighbourhood $U$ of $A$ and $\beta > 0$ such that for every $B \in U$ then $\lambda_p(B) - \lambda_{p+1}(B) > \beta$.

Let $\alpha = \liminf_{B \to A} \Lambda_p(B)$. We can assume, by taking a smaller neighbourhood if necessary, that every cocycle $B$ in $U$ holds that $\Lambda_p(B) > \alpha - \beta/4$. Let $\varepsilon \in (0,1)$ be such that the $C^0$ ball $B(A,\varepsilon)$ is entirely contained in $U$.

Let $A'$ such that $\|A - A'\| < \varepsilon/2$. By the observation above $A'$ is not $m$-dominated for some $m \geq m_0(\|A\| + 1, \varepsilon/2)$, $m_0$ defined as in Proposition 3.1. $A'$ can be assumed to hold that $\Lambda_p(A') \in [\alpha - \beta/5, \alpha + \beta/5]$.

From Theorem 4.5, taking $\varepsilon/2$ and $\delta < \beta/20$, there exists a cocycle $B$ such that $\|B - A'\| < \varepsilon/2$ and $\Lambda_p(B) < \Lambda_p(A') - \beta/2 + \delta$. The right side of the inequation is true since we can rewrite

$$\frac{\Lambda_{p-1}(A') + \Lambda_{p+1}(A')}{2} = \Lambda_{p-1}(A') + \frac{\lambda_p(A') + \lambda_{p+1}(A')}{2} = \Lambda_p(A') + \frac{\lambda_{p+1}(A') - \lambda_p(A')}{2} < \Lambda_p(A') - \frac{\beta}{2}.$$ 

The cocycle $B$ is in the neighbourhood $U$ thus it follows that

$$\alpha + (\beta/5 - \beta/2) + \delta > \Lambda_p(B) > \alpha - \beta/4$$

which is a contradiction. The theorem now follows.

We can now give a proof of the corollary 1.

Proof of Corollary 1. : Let $A \in C(X,\text{SL}(3,\mathbb{R}))$ such that $A$ is a continuity point of the second Lyapunov exponent.

If $A$ is a point of continuity of the first Lyapunov exponent then the whole Oseledet splitting is dominated. Note that a splitting of index 1 implies the whole splitting is dominated as well. So now we suppose there is no dominated splittings.

To show $\lambda_2(A) = 0$ suppose that $\lambda_2(A) < 0$. Write $\varepsilon_0 = -\lambda_2(A)/2$, from continuity we can find $C^0$ neighbourhood $U$ small enough we can assume every cocycle $B \in U$ holds that $\lambda_2(B) < \lambda_2(A) + \varepsilon_0 < 0$.

As $A$ is not dominated, it follows from Main theorem that we can find in $U$ a cocycle $B$ such that $|\lambda_1(B) - \lambda_2(B)| < \varepsilon_0$. This implies $\lambda_1(B) < 0$ and thus $\lambda_1(B) + \lambda_2(B) + \lambda_3(B) < 0$ which contradicts $B \in \text{SL}(3,\mathbb{R})$.

We can make an analogous contradiction if we suppose $\lambda_2(A) > 0$. This finishes the proof.
6. SOME QUESTIONS

In this section we pose some questions that arise after the proof of the theorem. The first question is about consequences of our Main Theorem. Suppose a cocycle \( A \in C(X, G) \) is such that the Lyapunov spectrum holds \( \lambda_p(A) > \lambda_{p+1}(A) \) and the splitting \( E \oplus F \) of index \( p \) given by the Oseledet splitting is not dominated. It follows from our Main Theorem that for every \( C^0 \)-neighbourhood \( U \) of \( A \) and given any \( \varepsilon > 0 \) we can find a cocycle \( B \in U \) such that \( 0 \leq \lambda_p(B) - \lambda_{p+1}(B) < \varepsilon \), that is, we can get the exponents arbitrarily close.

**Question 1.** Does there always exist a cocycle \( B \in U \) such that \( \lambda_p(B) = \lambda_{p+1}(B) \)?

Note that we have the \( C^0 \)-generic dichotomy of either the Oseledets splitting being dominated or every Lyapunov exponents being equal. However, this fact alone is not enough to answer this question. Some more work may be needed to reach a conclusion.

The following question is about relaxing the hypothesis on the measure \( \mu \), which we suppose to be ergodic.

**Question 2.** Is the Main Theorem valid if the measure \( \mu \) is not ergodic?

This question evidently requires some adjustments on the definition of uniform \( p \)-gap property among the Lyapunov Exponents since they are no longer constant. We could conjecture the uniform gap has to exist on average, that is, it should exist among the integrated Lyapunov exponents. As given in a cocycle \( A \) there exists a \( C^0 \)-neighbourhood \( U \) and \( \beta > 0 \) such that for every cocycle \( B \in U \) it holds that

\[
\int_X [\lambda_p(B, x) - \lambda_{p+1}(B, x)] d\mu(x) > \beta.
\]

However, note that this question as is seems to have a negative answer\(^1\). A sketch of the argument follows: let \( X \) and \( f \) be as in the Main Theorem. Suppose there exists \( \Lambda_1 \) and \( \Lambda_2 \) two invariant sets such that there exists ergodic borel regular probability measures \( \mu_i \) supported on respective \( \Lambda_i \). We construct a cocycle \( A \in \text{GL}(d, \mathbb{R}) \) such that \( A|_{\Lambda_1} \) has a uniform \( p \)-gap property and \( A|_{\Lambda_2} \) is such that \( \lambda_p(A|_{\Lambda_2}) = \lambda_{p+1}(A|_{\Lambda_2}) \) and admits no dominated splitting of index \( p \).

Consider the measure \( \mu = \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2 \). This is a non-ergodic \( f \)-invariant borel regular probability measure. Note that the cocycle \( A \) holds the uniform \( p \)-gap property as defined above. However, the cocycle \( A \) does not admit a dominated splitting of index \( p \) on \( \text{supp} (\mu) \).

A reformulation of Question 2 that may be of interest is the following.

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\(^1\)Thanks to Mauricio Poletti for pointing this out and providing the sketch of the argument that is reproduced here.
**Question 2’.** Under the hypotheses of the Main Theorem, but the measure $\mu$ is not ergodic, does there exist an invariant set $\Lambda \subset X$ such that $\mu(\Lambda) > 0$ and $A$ admits a dominated splitting of index $p$?

Another question is the validity of main theorem on cocycles taking values on other matrices groups.

**Question 3.** Is the Main Theorem valid on cocycles taking values on other subgroups of $\text{GL}(d, \mathbb{R})$?

In the proof of Proposition 3.1 we used the fact of cocycle taking values in either in $\text{GL}(d, \mathbb{R})$ or $\text{SL}(d, \mathbb{R})$ (actually, that we could make perturbations by either hyperbolic special matrices or rotations). This restricts all the constructions afterwards. So this question deals with making the conclusion of the Proposition true on other groups of matrices.

In [5] the authors find conditions for the existence of space interchanging sequences for a fixed cocycle. This cocycle takes values on any submanifold of $\text{GL}(d, \mathbb{R})$ that fulfills an accessibility condition. This condition is fulfilled by any subgroup of $\text{GL}(d, \mathbb{R})$ that acts transitively on $\mathbb{P}\mathbb{R}^d$. One way of answering our question is verifying if the conclusions of Proposition 3.1 are true in cocycles taking values on manifolds that satisfy such accessibility condition.

As a final question let $M$ be a Riemannian manifold, $\mu$ the induced volume measure and $f \in \text{Diff}^1_{\mu}(X)$ be a volume preserving diffeomorphism. As in question 2, we say that its Lyapunov spectrum has the uniform $p-$gap property if there exists a $C^1$-neighbourhood $U_f$ and $\beta > 0$ such that for every $g \in U_f$ it holds that

$$\int_M \left[ \lambda_p(g, x) - \lambda_{p+1}(g, x) \right] d\mu(x) > \beta.$$  

We have an analogous statement for our Main theorem in this context.

**Question 4.** Does the uniform $p$-gap property imply the existence of a dominated splitting for $f$?

As a derived question, restrict the previous question to ergodic diffeomorphisms.

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