DIRECTED POLYMERS IN A RANDOM ENVIRONMENT WITH A DEFECT LINE

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ABSTRACT. We study the depinning transition of the 1 + 1 dimensional directed polymer in a random environment with a defect line. The random environment consists of i.i.d. normal potential values assigned to each site of \( \mathbb{Z}^2 \); sites on the positive axis have the potential enhanced by a deterministic value \( u \). We show that for small inverse temperature \( \beta \) the quenched and annealed free energies differ significantly at most in a small neighborhood (of size of order \( \beta \)) of the annealed critical point \( u^a_c = 0 \).

1. INTRODUCTION.

1.1. Physical Motivation. The directed polymer in a random environment (DPRE) models a one-dimensional object interacting with disorder. The 1 + 1 dimensional version of the model first appeared in the physics literature in [19] as a model for the interface in two-dimensional Ising models with random exchange interaction. Since then it has been used in models of various growth phenomena: formation of magnetic domains in spin-glasses [19], vortex lines in superconductors [26], turbulence in viscous incompressible fluids (Burger turbulence) [5], roughness of crack interfaces [18], and the KPZ equation [24].

A related problem is the competition between extended and point defects as reflected in pinning phenomena, arising for example in the context of high-temperature superconductors [4, 11]. On a lattice this can be described by a random potential, typically i.i.d. at each lattice site, representing the point defects, with an additional fixed potential \( u \) added for sites along some line, representing the extended defect. The polymer must choose between roughly following the extended defect, or finding the best path(s) through the point defects. As \( u \) is decreased, one expects a depinning transition at some critical \( u_c \) where the polymer ceases to follow the extended defect.

In the (nonrigorous) physics literature, there have been disagreeing predictions as to whether \( u_c = 0 \). Kardar [23] examined this problem numerically and found that \( u_c > 0 \) for the 1 + 1 dimensional DPRE with defect line. On the other hand, Tang and Lyuksyutov in [27] argued that the same model satisfies \( u_c = 0 \), and claimed that \( u_c > 0 \) only above 1 + 1 dimensions. Their conclusion was supported by Balents and Kardar [2], numerically and via a functional renormalization group analysis, and later by Hwa and Natterman [20] in another renormalization group analysis. It is hoped that a mathematically rigorous analysis can eventually resolve the question.

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Mathematical Formulation of the Problem. The DPRE in 1 + d dimensions is formulated as follows. Let $P_D$ be the distribution of the symmetric simple random walk (SSRW) $S = \{S_j, j \geq 0\}$ on $\mathbb{Z}^d$ with initial distribution $\nu$, and let $E_\nu$ be the corresponding expectation. We write $P_x, E_x$ when $\nu = \delta_x$, and $P, E$ for $P_0, E_0$. The polymer configuration is represented by the path $\{(j, S_j)\}_{j=1}^n$ in $\mathbb{N} \times \mathbb{Z}^d$. The random environment, or bulk disorder, is given by i.i.d. random variables $V = \{v(i, x) : i \geq 1, x \in \mathbb{Z}^d\}$ with law denoted $Q$ satisfying $E^Q[e^{\beta v(i, x)}] < \infty$ for all $\beta \in \mathbb{R}$. Throughout, we assume that the disorder is Gaussian with zero mean and unit variance. The Hamiltonian for paths $s$ is

$$H_N(s) = \sum_{j=1}^N v(j, s_j),$$

and the quenched polymer measure $\mu_{\beta, q}^N$ is defined in the usual Boltzmann-Gibbs way:

$$(1.1) \quad \frac{d\mu_{\beta, q}^N}{dP}(s) = \frac{1}{Z_{\beta, q}^N} e^{\beta H_N(s)},$$

where $\beta > 0$ is the inverse temperature and $Z_{\beta, q}^N = E_0[e^{\beta H_N(S)}]$ is the quenched partition function.

The first rigorous mathematical work on directed polymers in 1 + d dimensions was done by Imbrie and Spencer [22], proving that in dimension $d \geq 3$ with small enough $\beta$, the end point of the polymer scales as $n^{1/2}$, i.e. the polymer is diffusive. Bolthausen [3] considered the nonnegative martingale $W_n^{\beta, q} = Z_n^{\beta, q} / E^Q[Z_n^{\beta, q}]$ and observed that the almost sure limit $W_\infty = \lim_{n \to \infty} W_n^{\beta, q}$ is subject to a dichotomy: there are only two possibilities for the positivity of the limit, $Q(W_\infty > 0) = 1$ (known as weak disorder) or $Q(W_\infty = 0) = 1$ (known as strong disorder), because the event $\{W_\infty = 0\}$ is a tail event. Bolthausen also improved the result of Imbrie and Spencer to a central limit theorem for the end point of the walk, which means that in $d \geq 3$ entropy dominates at high enough temperature, in that the polymer behaves almost as if the disorder were absent. Comets and Yoshida [8] showed that there exists a critical value $\beta_c = \beta_c(d, v) \in [0, \infty]$ with $\beta_c = 0$, for $d = 1, 2$ and $0 < \beta_c \leq \infty$ for $d \geq 3$, such that $Q(W_\infty > 0) = 1$ if $\beta \in \{0\} \cup (0, \beta_c)$ and $Q(W_\infty = 0) = 1$ if $\beta > \beta_c$. In particular, for the present 1 + 1 dimensional case, disorder is always strong. See [1] for a survey.

There has been substantial investigation of pinning models in which disorder is present only in the defect line $\{0\} \times \mathbb{N}$; see ([15] [16] and [28]) for surveys. In such models (which we call pinning models with defect-line potential), the energy gains from pinning compete only with the entropy loss inherent in the class of pinned paths. Here, by contrast, we enhance the potential in the DPRE by a fixed amount $u$ at each site of the defect line, so that energy gains from the enhancement for pinned paths also compete with the possibility of better energy gains from the potential $v(i, x)$ along depinned paths compared to pinned ones. Specifically, we define the Hamiltonian and the quenched polymer measure by

$$(1.2) \quad H_N^u(s) = \sum_{j=1}^N (v(j, s_j) + u 1_{s_j = 0}) = H_N(s) + u L_N(s),$$
\( \frac{d\mu_{N}^{\beta,u,q}}{dP}(s) = \frac{1}{Z_{N}^{\beta,u,q}} e^{\beta H_{N}(s)}, \)

where

\[ L_{N}(s) = \sum_{j=1}^{N} 1_{s_{j}=0}, \quad Z_{N}^{\beta,u,q} = \mathbb{E}_{0} \left[ e^{\beta H_{N}(s)} \right] \]

are the local time and the quenched partition function, respectively. Here \( P \) is the distribution of the SSRW with \( S_{0} = 0 \).

In general for a partition function \( Z \), the restriction to a set \( \Omega \) of SSRW paths will be denoted \( Z(\Omega) \); we add a subscript \( \nu \) when the SSRW has initial distribution \( \nu \), and include \( V \) as an argument of \( Z \) when we wish to emphasize the dependence on the disorder configuration \( V \).

Thus for example,

\[ Z_{N,\nu}^{\beta,u,q}(\Omega, V) := \mathbb{E}_{\nu} \left( e^{\beta H_{N}(s) 1_{\Omega}(S)} \right). \]

When \( \nu = \delta_{x} \) we write \( x \) in place of \( \nu \).

Our first result is on the existence of the quenched free energy of the model:

**Theorem 1.1.** For every \( \beta > 0 \) and \( u \in \mathbb{R} \),

\[ f^{q}(\beta, u) = \lim_{N \to \infty} \frac{1}{N} \log Z_{N}^{\beta,u,q} = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{Q}[\log Z_{N}^{\beta,u,q}] \]

exists \( Q \)-a.s. and in \( Q \)-mean.

The annealed polymer measure \( \mu_{N}^{\beta u} \) is obtained by taking the expected value over the disorder of the quenched Boltzmann-Gibbs weight, yielding

\[ \frac{d\mu_{N}^{\beta u}}{dP}(s) = \frac{1}{Z_{N}^{\beta u}} e^{\beta u L_{N}(s)+\beta^{2} N/2}, \]

where

\[ Z_{N}^{\gamma} = \mathbb{E}_{0}(e^{\gamma L_{N}(S)}), \quad Z_{N}^{\beta u} = Z_{N}^{\beta u} e^{\beta^{2} N/2} = \mathbb{E}_{0}(e^{\beta u L_{N}(S)+\beta^{2} N/2}) \]

is the annealed partition function. Note that \( \mu_{N}^{\beta u} \) depends only on the product \( \beta u \). Letting

\[ F(\gamma) = \lim_{N \to \infty} \frac{1}{N} \log Z_{N}^{\gamma}, \]

the annealed free energy is

\[ f^{a}(\beta, u) = \lim_{N \to \infty} \frac{1}{N} \log Z_{N}^{\beta u} = F(\beta u) + \frac{\beta^{2}}{2}. \]

Here \( F(\beta u) \) is the free energy of the pinning model with homogeneous defect-line potential, that is, with disorder \( v \equiv 0 \).

The quenched and annealed critical points are

\[ u_{c}^{q}(\beta) = \inf\{u : f^{q}(\beta, u) > f^{q}(\beta, 0)\}, \quad u_{c}^{a}(\beta) = \inf\{u : f^{a}(\beta, u) > f^{a}(\beta, 0)\}. \]

Note that \( f^{a}(\beta, 0) = \beta^{2}/2 \) so \( \beta u_{c}^{a}(\beta) \) does not depend on \( \beta \). In fact, it is standard (see [15]) that in the present situation \( u_{c}^{q}(\beta) = 0 \) for all \( \beta \) because the overlap is infinite, that is, the random walk on \( \mathbb{Z}^{2} \) with distribution \( P \times P \) is recurrent. When \( u > u_{c}^{q}(\beta) \) the quenched polymer is said to be pinned. Note also that \( f^{q}(\beta, u) \leq f^{a}(\beta, u) \) by Jensen’s inequality.
As mentioned above, physicists have differed on the question of whether $u^q_c(\beta) = 0$. One approach which at least provides a bound for $u^q_c(\beta)$ is to find a value $\Delta_0(\beta)$ such that for $u > \Delta_0(\beta)$, the quenched and annealed free energies are approximately the same; in particular this means the quenched free energy is strictly greater than $\beta^2/2$ and thus also strictly greater than $f^q(\beta, 0)$, meaning that $u > u^q_c(\beta)$. We thereby obtain that $u^q_c(\beta) \leq \Delta_0(\beta)$. This is the approach taken in [1] for the pinning model with defect-line potential; in the case where the underlying process is 1-dimensional SSRW one has $\Delta_0(\beta)$ of order at most $e^{-K/\beta^2}$ for some constant $K$, for small $\beta$. Here our main result has a similar form, but with bound $\Delta_0(\beta)$ of order $\beta$. This larger size of $\Delta_0(\beta)$ is rooted in the larger overlap present in the DPRE—overlap is counted throughout the bulk of $\mathbb{Z}^2$, as opposed to just on the axis. We do not know whether $\Delta_0(\beta)$ of order $\beta$ is optimal. At any rate, the theorem says in effect that the disorder alters the free energy significantly at most for $u$ in a neighborhood of size $O(\beta)$ of the annealed critical point $u^a_c(\beta) = 0$.

**Theorem 1.2.** Consider the 1 + 1 dimensional DPRE with defect line, with Hamiltonian as in (1.2). Suppose that the disorder variables $V = \{v(i, x) : i \geq 1, x \in \mathbb{Z}^d\}$ are i.i.d. standard normal. Then given $0 < \epsilon < 1$, there exists a $K = K(\epsilon)$ as follows. Provided that $\beta$ and $\beta u$ are sufficiently small and $u \geq K\beta$, we have

$$f^a(\beta, u) \geq f^q(\beta, u) \geq \frac{\beta^2}{2} + (1 - \epsilon)F(\beta, u).$$

Further, for small $\beta$,

$$0 \leq u^q_c(\beta) \leq K(\epsilon) \beta.$$  

As noted above, physicists have disagreed about whether in fact $u^q_c(\beta) = 0$.

In the following sections, the $K'_i$s are universal constants, except where they depend on a parameter, which is shown in parentheses.

2. Proof of Theorem 1.1 Existence of the Free Energy

In this section, we will first prove the existence of the free energy for the constrained model and then by using concentration of measure property of Gaussian processes, we relate the free energy of the constrained model with the free energy of the unconstrained model. We frequently follow [6] and [7].

2.1. The Constrained Model. In the constrained model (quenched or annealed), we restrict to paths ending at $s_N = 0$. To distinguish from the main model we append a superscript of $c$ to the polymer measure, partition function, etc., so for example

$$Z^\beta_{N\beta, u, q, c} = E_0 \left[ e^{\beta H^u_N(S)} 1\{s_N = 0\} \right]$$

is the constrained quenched partition function.

Due to periodicity of SSRW, we assume that $N, M$ are even integers for this section; and for notational convenience, we suppress the $\beta, u, q$, writing $Z_N(x)$ or $Z_N(x, V)$ for $Z^\beta_{N\beta, u, q}$. and $Z_N$ for $Z^\beta_{N\beta, u, q}$, and defining

$$Z_N(x, y) = Z_N(x, y; V) := E_x \left[ e^{\sum_{j=1}^N \beta(v(j, S_j) + u_1 S_j = 0)} 1_{s_N = y} \right].$$
where $P_x$ is the SSRW measure when $S_0 = x$. Let $\theta_{n,y}$ be the space-time shift operator on the environment $V$:

$$\theta_{n,y}(k, x) = \nu(k + n, x + y).$$

From the Markov property of SSRW, we have

$$Z_{2N}(0, 0; V) \geq Z_N(0, x; V)Z_N(x, 0; \theta_{N,0} V)$$

for all $x$, which after taking logs and expectations yields

$$E^Q[\log Z_N(0, x; V)] \leq \frac{1}{2} E^Q[\log Z_{2N}(0, 0; V)]$$

for all $x$.

Similarly we obtain

$$E^Q[\log Z_{N+M}(0, 0; V)] \geq E^Q[\log Z_N(0, 0; V)] + E^Q[\log Z_M(0, 0; V)].$$

This superadditivity establishes the existence of the limit

$$\lim_{N \to \infty} \frac{1}{N} E^Q[\log Z_N(0, 0; V)] = \sup_{N \geq 1} \frac{1}{N} E^Q[\log Z_N(0, 0; V)].$$

We have using Jensen’s inequality that

$$\frac{1}{N} E^Q[\log Z_N(0, 0; V)] \leq \frac{1}{N} \log E^Q[Z_N(0, 0; V)] \leq \frac{\beta^2 + \beta u}{2}.$$

It then follows from the subadditive ergodic theorem (see [12]) that the constrained free energy exists and $Q$-a.s. constant:

$$f_{q,c}(\beta, u) = \lim_{N \to \infty} \frac{1}{N} \log Z_N(0, 0; V) = \lim_{N \to \infty} \frac{1}{N} E^Q[\log Z_N(0, 0; V)].$$

The non-randomness (a.s.) of $f_{q,c}(\beta, u)$ is called the self-averaging property of the quenched free energy.

### 2.2. The Unconstrained Model

To prove the existence of the free energy of the unconstrained model we will use the following Gaussian concentration result.

**Proposition 2.1.** Let $G : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function with constant $A$, that is,

$$|G(x) - G(y)| \leq A||x - y||, \quad x, y \in \mathbb{R}^N,$$

where $|| \cdot ||$ denotes the Euclidean norm on $\mathbb{R}^N$. Then, if $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ is a vector of i.i.d. standard normal random variables, we have

$$P(|G(\xi) - E(G(\xi))| > t) \leq \exp\left(-\frac{t^2}{2A^2}\right)$$

for all $t > 0$.

For the proof of the above proposition, see [21].

We consider the set $\Lambda_N = \{(i, x) : 1 \leq i \leq N, |x| \leq i, x - i \text{ even}\}$. For $s$ an SSRW path of length $N$, define $a^s \in \mathbb{R}^A$ by

$$a^s(i, x) = 1_{s_i=x}.$$

We define a function $G : \mathbb{R}^A \to \mathbb{R}$ by

$$G(z) = \log E_0 \left[ e^{\sum_{i=1}^{N} \beta(s_i, S_{i-1} + u 1_{S_{i-1}})} 1_{S_N=x} \right] = \log E_0 \left[ e^{\beta(a^s \cdot z + \sum_{i=1}^{N} u 1_{S_{i-1}})} 1_{S_N=x} \right].$$
Since

\[ |a^S \cdot z - a^S \cdot z'| \leq ||a^S|| \cdot ||z - z'|| \leq \sqrt{N} ||z - z'||, \]

\( G \) is a Lipschitz function with constant at most \( \beta \sqrt{N} \). Using Proposition 2.1, we have

\[
\begin{align*}
E_Q \left[ e^{\frac{1}{\sqrt{N}} \log Z_N(0,x) - E_Q \log Z_N(0,x)} \right] &= \int_0^\infty Q(|\log Z_N(0, x) - E_Q \log Z_N(0, x)| \geq \sqrt{N} \log t) dt \\
&\leq 1 + \int_1^\infty e^{\frac{-(\log t)^2}{2\beta^2}} dt \\
&= K_1(\beta) < \infty
\end{align*}
\]

for all \((N, x) \in \Lambda_N\). Similarly, with the factor \(1_{S_N=x}\) removed from \(G\) we obtain

\[
\sum_{N=1}^\infty Q(|\log Z_N - E_Q \log Z_N| \geq \lambda N) < \infty, \quad \lambda > 0.
\]

We now compare free and constrained partition functions \(E_Q[\log Z_N]\) and \(E_Q[\log Z_{2N}(0,0)]\).

For \(\gamma \in (0,1)\), we have

\[
E_Q[\log Z_N] \leq \frac{1}{\gamma} \log E_Q[Z_N^\gamma] \\
= \frac{1}{\gamma} \log E_Q \left[ \left( \sum_{x:(N,x) \in \Lambda_N} Z_N(0,x) \right)^\gamma \right] \\
\leq \frac{1}{\gamma} \log E_Q \left[ \sum_{x:(N,x) \in \Lambda_N} Z_N(0,x)^\gamma \right] \\
= \frac{1}{\gamma} \log E_Q \left[ \sum_{x:(N,x) \in \Lambda_N} e^{\gamma(\log Z_N(0,x) - E_Q[\log Z_N(0,x)])} e^{\gamma E_Q[\log Z_N(0,x)]]} \right] \\
\leq \frac{1}{\gamma} \log \left( e^{\frac{1}{2}E_Q[\log Z_{2N}(0,0)]} \left( \sum_{x:(N,x) \in \Lambda_N} E_Q \left[ e^{\gamma(\log Z_N(0,x) - E_Q[\log Z_N(0,x)])} \right] \right) \right) \\
\leq \frac{1}{2} E_Q[\log Z_{2N}(0,0)] + \frac{1}{\gamma} \log (K_1(2N + 1)).
\]

In the third and fourth inequalities, we used (2.2) and (2.5), respectively. Taking \(\gamma = \frac{1}{\sqrt{N}}\), we have

\[
E_Q[\log Z_N(0,0)] \leq E_Q[\log Z_N] \leq \sqrt{N} \log (K_1(2N + 1)) + \frac{1}{2} E_Q[\log Z_{2N}(0,0)];
\]

and therefore

\[
\lim_{N \to \infty} \frac{1}{N} E_Q[\log Z_N(0,0)] = \lim_{N \to \infty} \frac{1}{N} E_Q[\log Z_N].
\]

With (2.6) and Borel-Cantelli, this establishes equality of the free energies in the original and constrained models.
3. Proof of Theorem 1.2

3.1. Proof Outline. We take a block length $N$ which is a multiple (of order $\epsilon^{-2}$) of the annealed correlation length, so that the associated finite-volume annealed free energy is large. We use the second moment method to show that on scale $N$, the quenched partition function is with high probability within a constant of the annealed one; here the condition $u \geq K(\epsilon)\beta$ allows necessary control of the overlap. This remains true if we restrict the partition functions to a set $\Omega_N$ of paths which stay inside an $N \times 4\sqrt{N}$ box centered on the axis, and end within $\sqrt{N}/4$ of the axis. Having paths end close to the axis facilitates concatenating a large number $L$ of the boxes together to make a length-$LN$ corridor in such a way that the corresponding partition function is approximately the product of the $L$ single-box partition functions.

Certain boxes in this corridor, though, may have very small values for the associated quenched partition function, making this product of single-box partition functions unacceptably small relative to the annealed one. This requires re-routing the corridor through off-axis boxes in places, to avoid “bad” on-axis boxes; bad off-axis boxes must also be avoided in this process. The result is a dependent percolation problem on coarse-grained scale; one needs an infinite directed path of “good” boxes, with most of these boxes being on-axis, where the extra potential $u$ is relevant. We use results of [13], [17] and [25] to establish the existence of such a path. The restriction of the quenched partition function to length-$LN$ paths following the corresponding (non-coarse-grained) corridor then provides a lower bound for the full quenched partition function at length $LN$, and taking a limit as $L \to \infty$ yields the desired result.

3.2. Further Preliminaries. Recall that $F(\gamma)$ denotes the free energy of the homogeneous (or annealed) model with defect-line potential. As observed in (15, equation (2.7)), $\gamma + \log E_0(e^{\gamma L_n})$ is subadditive in $n$ for all $\gamma \geq 0$. It follows that

$$E_0(e^{\gamma LN}) \geq e^{-\gamma}e^{NF(\gamma)}.$$  

The following is essentially the same as (15, equation (2.22)).

Lemma 3.1. There exist $K_2, K_3 > 0$ such that

$$\forall j \geq 1, \gamma > 0, \quad E_0(e^{\gamma L_j M}) \leq K_2 j e^{K_3 j},$$

where $M = 1/F(\gamma)$ is the correlation length.

For the proof of the following see [15] or [16].

Proposition 3.2. The free energy $F(\gamma)$ has the following properties:

a) $F(\gamma)$ is 0 on $(-\infty, 0]$ and strictly increasing and positive on $(0, \infty)$.

b) for some $K_4 > 0$, $F(\gamma) \sim K_4 \gamma^2$, as $\gamma \to 0^+$.

For any $x \in \mathbb{Z}$, $\gamma \geq 0$, conditioning on the hitting time of 0 yields

$$E_x e^{\gamma LN} \leq E_0 e^{\gamma (LN+1)}.$$  

For $k > 1$, conditioning on $S_{(k-1)N}$, applying (3.2) and iterating we obtain

$$E_x e^{\gamma Ln} \leq \left( E_0 e^{\gamma (LN+1)} \right)^k.$$
For SSRW paths \(s^1, s^2\), define the overlap

\[
B_N(s^1, s^2) = \sum_{i=1}^{N} 1_{s^1_i = s^2_i}
\]

For independent copies \(S^1, S^2\) of the Markov chain \(S\), \((S^1, S^2)\) is also a Markov chain, so as a special case of (3.2),

\[
E \otimes^2 e^{\gamma B_N} \leq E(0,0) e^{\gamma(B_N+1)},
\]

and as a special case of (3.3), for \(k \geq 1, \gamma \geq 0\), and \(x, x' \in \mathbb{Z}\), we have

\[
E \otimes^2 e^{\gamma B_k N} \leq \left(\frac{E(0,0) e^{\gamma(B_N+1)}}{k}\right)^k.
\]

The following is a straightforward consequence of Donsker’s invariance principle.

**Lemma 3.3.** For one dimensional SSRW, we have

\[
A_{\text{forward}} := \liminf_{N \to \infty} \inf_{|x| \leq \frac{\sqrt{N}}{4}} P_x \left( \max_{1 \leq i \leq N} |S_i| \leq 2\sqrt{N}, |S_N| \leq \frac{\sqrt{N}}{4} \right) > 0,
\]

\[
A_{\text{up}} := \liminf_{N \to \infty} \inf_{|x| \leq \frac{\sqrt{N}}{4}} P_x \left( \max_{1 \leq i \leq N} |S_i| \leq 2\sqrt{N}, |S_N - \sqrt{N}| \leq \frac{\sqrt{N}}{4} \right) > 0,
\]

\[
A_{\text{down}} := \liminf_{N \to \infty} \inf_{|x| \leq \frac{\sqrt{N}}{4}} P_x \left( \max_{1 \leq i \leq N} |S_i| \leq 2\sqrt{N}, |S_N + \sqrt{N}| \leq \frac{\sqrt{N}}{4} \right) > 0.
\]

The proof of the following is due to S.R.S. Varadhan [29].

**Lemma 3.4.** There exists a constant \(0 < \epsilon_0 < 1\), such that for \(\gamma > 0\), for all sufficiently large \(N\) and \(|x| \leq \frac{\sqrt{N}}{4}\),

\[
E_x \left( e^{\gamma L N} 1_{\Omega_N} \right) \geq \epsilon_0 E_x \left( e^{\gamma L N} \right),
\]

where

\[
\Omega_N = \{ s : \max_{1 \leq i \leq N} |s_i| \leq 2\sqrt{N}, |s_N| \leq \frac{\sqrt{N}}{4} \}.
\]

**Proof.** We define a polymer measure on the space of SSRW paths:

\[
\mu^\gamma_{N,x}(A) := \frac{E_x[e^{\gamma L N} 1_A]}{E_x[e^{\gamma L N}]}.
\]

Let \(W(n, x) = E_x[e^{\gamma L n}]\).
Under $\mu_{N,x}^{\gamma}(\cdot)$ we have a non-stationary Markov process with transition probabilities from $z$ to $y = z \pm 1$ at time $k < N$ given by

$$
\pi(z, y, k, N, \gamma) = \frac{E_x[e^{\gamma L_N}1_{S_k = z}1_{S_{k+1} = y}]}{E_x[e^{\gamma L_N}1_{S_k = z}]} = \frac{E_x[e^{\gamma L_k}1_{S_k = z}]E_x[e^{\gamma L_N-k}1_{S_1 = y}]}{E_x[e^{\gamma L_k}1_{S_k = z}]} = \frac{1}{2} \frac{e^{\gamma \delta_0(y)}E_x[e^{\gamma L_{N-k-1}}]}{E_x[e^{\gamma L_{N-k}}]} = \frac{e^{\gamma \delta_0(y)}W(N-k-1, y)}{2W(N-k, z)}.
$$

(3.7)

For all $z$,

$$
W(N-k, z) = \frac{1}{2} e^{\gamma \delta_0(z+1)}W(N-k-1, z+1) + \frac{1}{2} e^{\gamma \delta_0(z-1)}W(N-k-1, z-1)
$$

while for $z \geq 1$ we have monotonicity in $z$:

$$
W(N-k-1, z+1) \leq W(N-k-1, z) \text{ and } W(N-k-1, 1) \leq e^{\gamma}W(N-k-1, 0),
$$

and similarly for $z \leq -1$,

$$
W(N-k-1, z) \leq W(N-k-1, z+1) \text{ and } W(N-k-1, -1) \leq e^{\gamma}W(N-k-1, 0).
$$

Therefore for $z \geq 1$, the second term on the right in (3.8) is the larger one, and by (3.7) we thus have

$$
\pi(z, z-1, k, N, \gamma) \geq \frac{1}{2},
$$

while for $z \leq -1$, similarly,

$$
\pi(z, z+1, k, N, \gamma) \geq \frac{1}{2}.
$$

Hence, the $\mu_{N,x}^{\gamma}$ chain can be coupled to the $P_x$ chain (i.e. SSRW) in a such a way that the $\mu_{N,x}^{\gamma}$ chain is always smaller or equal in magnitude. Therefore

$$
\mu_{N,x}^{\gamma}(\Omega_N) \geq P_x(\Omega_N),
$$

and the result then follows from Lemma 3.3.

Let

$$
\tau_x = \inf\{n \geq 1 : S_n = x\}.
$$

**Lemma 3.5.** Let $0 < \epsilon < 1$ be given. Then, for sufficiently large $N$ and $|x| \leq \sqrt{N}$,

$$
E_x\left(e^{\beta u L_N}\right) \geq \frac{1}{2} P(\xi \geq \frac{1}{4\sqrt{\epsilon}})e^{(1-\epsilon)NF(\beta u)},
$$

where $\xi$ denotes a standard normal random variable.
Proof. For a given $0 < \epsilon < 1$, there exists an $N_0 = N_0(\epsilon)$ such that for all $N \geq N_0$ and for $0 < x \leq \frac{\sqrt{N}}{4}$,

$$P_x(\tau_0 \leq \epsilon N) = P_0(\tau_x \leq \epsilon N) \geq P_0(S_{\epsilon N} \geq \frac{\sqrt{N}}{4}) \geq \frac{1}{2} P(\xi \geq \frac{1}{4\sqrt{\epsilon}}).$$  \hspace{1cm} (3.9)$$

The right side of (3.9) is also a lower bound for the left side for $\frac{\sqrt{N}}{4} \leq x < 0$ by symmetry, and for $x = 0$ after increasing $N_0$ if necessary. Therefore, for sufficiently large $N$ and $|x| \leq \frac{\sqrt{N}}{4}$, using (3.1) and (3.9),

$$E_x(e^{\beta u L_N}) \geq \sum_{k=x}^{\epsilon N} e^{\beta u} E_0(e^{\beta u L_{N-k}}) P_x(\tau_0 = k) \geq \sum_{k=x}^{\epsilon N} e^{(1-\epsilon)NF(\beta u)} P_x(\tau_0 = k) = e^{(1-\epsilon)NF(\beta u)} P_x(\tau_0 \leq \epsilon N) \geq \frac{1}{2} P(\xi \geq \frac{1}{4\sqrt{\epsilon}}) e^{(1-\epsilon)NF(\beta u)}.$$

□

We need information about the excursion length distribution of $(p, q)$-walks. First, a definition:

**Definition 3.6.** A $(p, q)$-walk is a random walk in which the steps $X_i$ have distribution $P(X_1 = b) = P(X_1 = -b) = p/2 \in (0, 1/2)$ and $P(X_1 = 0) = q > 0$, where $p + q = 1$ and $b$ is a positive integer.

Let $\bar{S}_N = S^1_N - S^2_N$, where $S^1_N, S^2_N$ are independent SSRWs. Then $(\bar{S}_N)_{N \geq 1}$ is a $(1/2, 1/2)$-walk with $b = 2$, and $B_N(S^1, S^2) = L_N(\bar{S})$.

For the proof of the following, see [15] and [14].

**Proposition 3.7.** For any $(p, q)$-walk, $p \in (0, 1)$, we have

$$P(\tau_0 = n) \sim \sqrt{\frac{p}{2\pi}} n^{-3/2} \text{ as } n \to \infty.$$  

For $(1, 0)$-walk,

$$P(\tau_0 = 2n) \sim \sqrt{\frac{1}{4\pi}} n^{-3/2} \text{ as } n \to \infty.$$  

**Proposition 3.8.** Let $0 < a < 1$ be given. Then there exists a constant $K_5 = K_5(a) > 0$ such that for sufficiently small $\beta$ and $\beta u$, and $u \geq K_5\beta$, for $M = 1/F(\beta u)$ we have

$$E_{(0,0)}^{\otimes 2} \left( e^{2\beta^2(B_M(S^1, S^2)+1)} - 1 \right) \leq a.$$
Proof. Let $E_i$ denote the length of the $i^{th}$ excursion of $\vec{S} = S^1 - S^2$ from 0 (that is, the time from the $(i - 1)$st to the $i$th visit to 0.) Then

$$P(B_M + 1 > k) \leq P(\max_{1 \leq i \leq k} E_i \leq M) = (1 - P(E_1 > M))^k$$

for all $k \geq 1$.

By Proposition 3.7 $P(E_1 > M) \sim (\pi M)^{-1/2}$ as $M \to \infty$, that is as $\beta u \to 0$, so for sufficiently small $\beta u > 0$,

$$P(B_M + 1 > k) \leq \left(1 - \frac{1}{\sqrt{2\pi M}}\right)^k$$

for all $k \geq 1$.

Therefore $B_M + 1$ is stochastically dominated by a geometric random variable with parameter

$$p_M = (2\pi M)^{-1/2} \geq \frac{\beta u \sqrt{K}}{3}$$

for $M$ sufficiently large, or equivalently $\beta u$ sufficiently small, where the inequality follows from Proposition 3.2(b). Therefore we have for $\beta u$ small,

$$E_{(0,0)}^{\otimes 2} \left( e^{2\beta^2 (B_M(S^1,S^2) + 1)} - 1 \right) \leq \frac{p_M e^{2\beta^2}}{1 - (1 - p_M) e^{2\beta^2}} - 1,$$

provided that

$$p_M > 1 - e^{-2\beta^2}.$$  

To bound (3.12) by the given $a$, we need

$$p_M \geq \frac{a + 1}{a}(1 - e^{-2\beta^2}),$$

so by (3.11) it suffices for (3.13) and (3.14) that $a \geq K_5(a)\beta$. \hfill \Box

3.3. The Coarse Grained Lattice $\mathbb{L}_{CG}$. In this section, we introduce a coarse grained lattice

$$\mathbb{L}_{CG} := \{(I, J) \in \mathbb{Z}^2 : I \geq 0, 0 \leq J \leq I\}.$$  

Note this is really a “half lattice” since we only consider $J \geq 0$.

Recall that the annealed correlation length is $M = 1/F(\beta u)$. Let $N = k_0 M$, with $k_0$ to be specified. For notational convenience we assume that $N$ and $\sqrt{N}$ are integers. We use capital letters $(I, J)$ for a site in the coarse grained lattice which corresponds to the vertical window

$$R(I, J) := \{(k, l) \in \mathbb{Z}^2 : k = IN, (J - \frac{1}{4})\sqrt{N} \leq l \leq (J + \frac{1}{4})\sqrt{N}\}$$

in the original lattice $\mathbb{Z}^2$.

The box starting from the window $R(I, J)$ is the following region in $\mathbb{Z}^2$:

$$B(I, J) := [IN, (I + 1)N] \times [(J - 2)\sqrt{N}, (J + 2)\sqrt{N}].$$

We say that there is a link between sites $(I, J)$ and $(I + 1, L)$ if $|L - J| \leq 1$. The link is down, forward or up according as $L = J - 1, J$ or $J + 1$. A path $\Gamma = \Gamma_{(I,J)\rightarrow(K,L)}$ from site $(I, J)$ to site $(K, L)$ in $\mathbb{L}_{CG}$ is a sequence of sites $(I, J) = (I_0, J_0), (I_1, J_1), \ldots, (I_N, J_N) = (K, L)$ such that there is a link between $(I_i, J_i)$ and $(I_{i+1}, J_{i+1})$ for all $i < N$. $\Gamma(I_i)$ will denote the second coordinate $J_i$ of the unique site $(I_i, J_i)$ in the path $\Gamma$. We will use the alternate notation $\Gamma_{(I,J)}$.
for $\Gamma_{(0,0) \to (I,J)}$. Given paths $\Gamma^1, \Gamma^2$ from some $(I, J)$ to $(K, L)$, we say that $\Gamma^1$ is closer to the x-axis than $\Gamma^2$ if

$$\Gamma^1(I_i) \leq \Gamma^2(I_i) \text{ for each } I \leq I_i \leq K.$$ 

Suppose each site $(I, J) \in \mathbb{L}_{CG}$ is designated as open or closed. We than say a path $\Gamma_{(I,J) \to (K,L)}$ is

(i) open if its all sites are open;
(ii) maximal if it has the maximum number of open sites among all paths from site $(I, J)$ to site $(K, L)$;
(iii) optimal if it is the maximal path which is closest to the x-axis.

$\Gamma_{(I,J)}^\infty$ denotes a generic infinite open path from the site $(I, J)$. There is exactly one optimal path for given sites $(I, J)$ and $(K, L)$ and we denote it by $\Gamma_{(I,J) \to (K,L)}^{opt}$.

When an infinite open path from a site $(I, J)$ exists, the one which is closest to the x-axis among all such paths is called the infinite good path from the site $(I, J)$, and we denote it by $\Gamma_{(I,J)}^{G,\infty}$. $\Gamma_{(I,J)}^{G,\infty}$ denotes the infinite good path from the site $(0,0)$, when it exists. For $0 \leq I \leq K$, $\Gamma_{I \to K}$ will denote the segment of the path $\Gamma_{(I,J)}^{G,\infty}$ between the sites with first coordinates $I$ and $K$. Note that if the site $(I_0, J_0)$ is on the infinite good path from $(0,0)$, then

$$\Gamma_{(0,0) \to (I_0, J_0)}^{opt} = \Gamma_{(I,J)}^{G,\infty}_{(0,0) \to (I_0, J_0)}.$$ 

Given a path $\Gamma = \Gamma_{(0,0) \to (I,J)} = \{(L, J_L) : L \leq I\}$ in $\mathbb{L}_{CG}$, we identify a subset $\Omega_{I,J}$ of the SSRW paths of length $IN$ in the following way:

$$\Omega_{I,J} := \Omega_{I,J}(\Gamma) := \left\{ s = \{(n, s_n)\}_{n \leq IN} : s_0 = 0, s_{LN} \in R(L, J_L) \forall L \leq I, s \subset \bigcup_{L < I} B(L, J_L) \right\}.$$ 

When $\Gamma_{(I,J)}^{G,\infty} = \{(L, J_L^G) : L \geq 0\}$ exists, for $0 \leq I \leq K$ we define

$$\Omega_{I \to K}^{G,\infty} := \left\{ s = \{(n, s_n)\}_{1 \leq n \leq KN} : s_{LN} \in R(L, J_L^G) \forall L \leq K, s \subset \bigcup_{L \leq K} B(L, J_L^G) \right\}.$$ 

We define quenched probability measures on the windows $R(I, J)$, using SSRW paths associated to the optimal coarse-grained path to that window, as follows: for $I \geq 1$ and $x \in R(I, J)$, let

$$\nu_{(I,J)}^q(x) := \frac{Z_{1N}^{\beta,u,q} \left( \Omega_{I,J}^{\Gamma_{(0,0) \to (I,J)}^{opt}} \cap \{s_{1N} = x\} \right)}{Z_{1N}^{\beta,u,q} \left( \Omega_{I,J}^{\Gamma_{(0,0) \to (I,J)}^{opt}} \right)}, \quad x \in R(I, J),$$

and let $\nu_{(0,0)}^q := \delta_0$. The measure

$$\tilde{\nu}_{(I,J)}^q(x) = \nu_{(I,J)}^q((IN, JN) + x), \quad x \in R(0,0),$$

is the translate of $\nu_{(I,J)}^q$ to $R(0,0)$.

Define the following sets of SSRW paths, corresponding to up, forward and down links in a coarse-grained path:

$$\Omega_{N}^{up} := \{(s_0, \cdots, s_N) : \left| s_0 \right| \leq \frac{\sqrt{N}}{4}, \left| s_N - \sqrt{N} \right| \leq \frac{\sqrt{N}}{4}, \left| s_i \right| \leq 2\sqrt{N}, 1 \leq i \leq N\},$$

$$\Omega_{N}^{forward} := \{(s_0, \cdots, s_N) : \left| s_0 \right| \leq \frac{\sqrt{N}}{4}, \left| s_N \right| \leq \frac{\sqrt{N}}{4}, \left| s_i \right| \leq 2\sqrt{N}, 1 \leq i \leq N\}.$$
and

\[ \Omega_N^{\text{down}} := \{(s_0, \cdots, s_N) : |s_0| \leq \frac{\sqrt{N}}{4}, |s_N + \sqrt{N}| \leq \frac{\sqrt{N}}{4}, |s_i| \leq 2\sqrt{N}, 1 \leq i \leq N\}. \]

Note that the up, forward and down sets of SSRW paths start at the window \( R(I, J) \), stay in the box \( B(I, J) \), and end at the window \( R(I + 1, J + l) \), with \( l = +1, 0, -1 \), respectively.

Of particular interest are the link partition functions

\[ Z_{N, \nu(I,J)}^{\beta, u, q} (\Omega_N^g, \theta_{I, J}(V)), \quad g = \text{up, forward, down}, \]

corresponding to SSRW paths in the box \( B(I, J) \) from the window \( R(I, J) \) to \( R(I + 1, J + l) \), with \( l = 1, 0, -1 \) according to the value of \( g \). When \( J = 0 \) and \( g = \text{forward} \), we refer to the link or partition function as \textit{on-axis}, otherwise it is \textit{off-axis}.

### 3.4. Open and Closed Sites in the Coarse Grained Lattice.

Define the filtrations

\[ \mathcal{F}_I := \sigma(\{v(i, x) : 1 \leq i \leq IN, x \in \mathbb{Z}\}), \quad I \geq 1, \]

and note that the measures \( \nu(I,J) \) are \( \mathcal{F}_I \)-measurable for all \( J \geq 0 \). One expects on-axis link partition functions to be larger than off-axis ones in general, and we will specify constants \( U_{\text{on}} \geq U_{\text{off}} \) which will serve as lower bounds for these partition functions, satisfying

\[ U_{\text{on}} \leq \frac{1}{2} E^Q \left( Z_{N, \nu(I,J)}^{\beta, u, q} (\Omega_N^\text{forward}, \theta_{I, J}(V)) \mid \mathcal{F}_I \right) Q \text{ - a.s. for each } I \geq 0, \]

and for \( I > 0, J \leq I \) and \( g = \text{forward, up, down}, \)

\[ U_{\text{off}} \leq \frac{1}{2} E^Q \left( Z_{N, \nu(I,J)}^{\beta, u, q} (\Omega_N^g, \theta_{I, J}(V)) \mid \mathcal{F}_I \right) Q \text{ - a.s.} \]

For \( I \geq 1 \), by Lemma 3.35 and 6.1 for sufficiently small \( \beta u \), Q-a.s.

\[
E^Q \left( Z_{N, \nu(I,J)}^{\beta, u, q} (\Omega_N^\text{forward}, \theta_{I, J}(V)) \mid \mathcal{F}_I \right) = \sum_{x \in R(0,0)} \nu(I,J)(x) E^Q \left( E_x \left[ e^{\beta \sum_{k=1}^N \left( v(IN+k,S_k)+u1_{S_k=0} \right)} 1_{\Omega_N^\text{forward}} \right] \right)
= \sum_{x \in R(0,0)} \nu(I,J)(x) e^{\beta^2 N} E_x \left[ e^{\sum_{k=1}^N \beta u1_{S_k=0} 1_{\Omega_N^\text{forward}}} \right]
\geq \sum_{x \in R(0,0)} \nu(I,J)(x) e^{\beta^2 N} \frac{\epsilon_0}{2} P(\xi \geq \frac{1}{4\sqrt{\epsilon}} e^{(1-\epsilon)N(F(\beta u))})
\geq \frac{\epsilon_0}{2} P(\xi \geq \frac{1}{4\sqrt{\epsilon}} e^{(1-\epsilon)F(\beta u)N}).
\]

Hence we define

\[ U_{\text{on}} := \frac{\epsilon_0}{4} P(\xi \geq \frac{1}{4\sqrt{\epsilon}} e^{(1-\epsilon)F(\beta u)N}). \]
For sufficiently small $\beta u > 0$, for all $I \geq 0, J \geq 1$ and for $g = \text{forward}, \text{up}, \text{down}$, by Lemma 3.3 we have $Q$-a.s.

$$E^Q\left(Z_{N,\tilde{\nu}_{(I,J)}^q}^{\beta,u,q}(\Omega_N^g,\theta_{IN,JN}(V))|\mathcal{F}_I\right) \geq E^Q\left(Z_{N,\tilde{\nu}_{(I,J)}^q}^{\beta,0,q}(\Omega_N^g,\theta_{IN,JN}(V))|\mathcal{F}_I\right) \geq e^{\frac{\beta^2}{2}N} \sum_{x \in \mathbb{R}(0,0)} \tilde{\nu}_{(I,J)}^q(x) P_x(\Omega_N^g) \geq \frac{1}{2} e^{\frac{\beta^2}{2}N} \min(A_{\text{forward}}, A_{\text{up}}, A_{\text{down}}).$$

Hence we define

$$U_{\text{off}} := \frac{1}{4} e^{\frac{\beta^2}{2}N} \min(A_{\text{forward}}, A_{\text{up}}, A_{\text{down}}).$$

We can then define open sites inductively on $I$. The site $(0,0)$ is called open if

$$Z_{N,\tilde{\nu}_{(I,0)}^q}(\Omega_{\text{up}}^g,\theta_{IN,0}(V)) \geq U_{\text{off}}$$

and the site $(I,J), 0 < J \leq I$, is open if

$$Z_{N,\tilde{\nu}_{(I,J)}^q}(\Omega_{\text{up}}^g,\theta_{IN,JN}(V)) \geq U_{\text{off}}, \ g = \text{up}, \text{forward}, \text{down},$$

otherwise $(I,J)$ is closed. Note the inductive definition is necessary because the previously defined open/closed values determine the optimal path from $(0,0)$ to $(I,J)$, which determines $\tilde{\nu}_{(I,J)}^q$. Let $X_{(I,J)} = 1_{\{(I,J) \text{ is open}\}}$.

### 3.5 Second Moment Method and Probability of an Open Site

We will use the second moment method to show the probability of a closed site is small. In general, for $Y$ a random variable with finite mean and variance, and $\theta, \epsilon \in (0,1)$, by Chebychev’s Inequality we have

$$P((1 - \theta)EY \leq Y \leq (1 + \theta)EY) \geq 1 - \epsilon,$$

provided that

$$\frac{\text{Var}(Y)}{(EY)^2} \leq \theta^2 \epsilon. \tag{3.19}$$

Hence for a site $(I,0)$ on the $x$-axis, applying (3.19) and (3.20) with $\theta = 1/2$ we see that, $Q$-a.s.,

$$Q(X_{(I,0)} = 1|\mathcal{F}_I) \geq 1 - \epsilon, \tag{3.21}$$

provided

$$\frac{\text{Var}Q\left(Z_{N,\tilde{\nu}_{(I,0)}^q}(\Omega_N^g,\theta_{IN,0}(V)) \mid \mathcal{F}_I\right)}{(E^Q\left(Z_{N,\tilde{\nu}_{(I,0)}^q}(\Omega_N^g,\theta_{IN,0}(V)) \mid \mathcal{F}_I\right))^2} \leq \frac{\epsilon}{8}, \ g = \text{forward}, \text{up}. \tag{3.22}$$
Similarly, for $(I, J)$ with $J \geq 1$, we see that, $Q$-a.s.,

\[(3.23)\quad Q(X_{I,J} = 1 | F_I) \geq 1 - \epsilon,\]

provided

\[(3.24)\quad \frac{\text{Var}_Q \left( Z_{N, N, \nu}^{\beta, u, q} (\Omega^g_{N}, \theta_{IN, JN}(V)) \mid F_I \right)}{\left( E_Q \left( Z_{N, N, \nu}^{\beta, u, q} (\Omega^g_{N}, \theta_{IN, JN}(V)) \mid F_I \right) \right)^2} \leq \frac{\epsilon}{12}, \quad g = \text{up, forward, down}.\]

For SSRW paths $s^1$ and $s^2$, since the environment is Gaussian we have

\[(3.25)\quad E^Q \left( e^{\beta H_N(S^1) + \beta u L_N(S^1)} e^{\beta H_N(S^2) + \beta u L_N(S^2)} \right) = e^{\beta u L_N(S^1)} e^{\beta u L_N(S^2)} e^{\beta^2 B_N(S^1, S^2) e^{\beta^2 N}}.\]

Recall $N = k_0 M$. Using \[(3.3), (3.4)\], the Cauchy-Schwartz inequality and the fact that $(t-1)^2 \leq t^2 - 1$ for $t \geq 1$, for all $(I, J)$ we get $Q$-a.s.

\[(3.26)\quad E^Q \left( e^{\beta H_N(S^1) + \beta u L_N(S^1)} e^{\beta H_N(S^2) + \beta u L_N(S^2)} \right) = e^{\beta u L_N(S^1)} e^{\beta u L_N(S^2)} e^{\beta^2 B_N(S^1, S^2) e^{\beta^2 N}}.\]

For the denominator, by Lemma \[3.3\] for some $K_6 > 0$, $Q$-a.s.

\[
\begin{align*}
E^Q \left( Z_{N, N, \nu}^{\beta, u, q} (\Omega^g_{N}, \theta_{IN, JN}(V)) \mid F_I \right) & = \sum_{x \in R(I,J)} \tilde{\nu}^q_{(I,J)}(x) E^Q \left( e^{\beta \sum_{k=1}^{N} \left( v(IN + k, S_k) + u S_k = 0 \right)} \Omega^g_N \right) \\
& \geq e^{-\frac{N}{2} \beta^2 \nu} \sum_{x \in R(I,J)} \tilde{\nu}^q_{(I,J)}(x) P_x (\Omega^g_N) \\
& \geq e^{-\frac{N}{2} \beta^2 \nu} K_6.
\end{align*}
\]

By Proposition \[3.2\] we have $M = M(\beta u) \leq 5M(2\beta u)$ for small $\beta u$. Therefore by Lemma \[3.1\] for $K_2, K_3$ from that lemma,

\[
E_0 e^{2\beta u(L_{M+1})} \leq 6K_2 e^{5K_3} =: K_7.
\]
Combining this with (3.26) and (3.27) we obtain that the left side of (3.24) is bounded by

\[(3.28) \quad K_6^{-2} K_7 k_0 \left( \left( E_{0,0}^{\leq 2} \left[ e^{2\beta^2 (S_1^1, S_2^1) + 1} \right] + 1 \right)^{k_0} - 1 \right)^{1/2}.
\]

Hence for our given \(0 < \epsilon < 1\), we wish to apply Proposition 3.8 with

\[(3.29) \quad a = \left( \frac{K_6^4 \epsilon^2}{12^2 K_7^{2k_0}} + 1 \right)^{1/k_0} - 1;
\]

since \(0 < K_6 < 1\) and \(K_7 > 1\), we indeed have \(a < 1\) as needed. We thereby obtain from (3.28) that, provided \(u \geq K_5(a)\beta\), the left side of (3.24) (and also of (3.22)) is bounded by \(\epsilon/12\). Thus (3.21) and (3.23) hold, for \(\beta\) and \(\beta u\) small.

3.6. Lipschitz Percolation. Lipschitz percolation, the existence of open Lipschitz surfaces, was first introduced and studied in [13] and [17]. In this section, we briefly summarize and adapt some of their results for dimension \(d = 2\), to use in our context.

The independent site percolation model in \(Z^2\) is obtained by independently designating each site \(x \in Z^2\) open with probability \(p\), otherwise closed. The corresponding probability measure on the sample space \(\Omega = \{0, 1\}^{Z^2}\) will be denoted by \(P_p\), and expectation by \(E_p\).

Let \(Z^+ = \{0, 1, 2, 3, \ldots\}\). A function \(L : Z \to Z^+_0\) is called Lipschitz if for all \(x, y \in Z\) with \(|x - y| = 1\), we have \(|L(x) - L(y)| \leq 1\). \(L\) is called open if for each \(x \in Z\), the site \((x, L(x))\) is open.

**Remark 3.9.** In [13] and [17], it was assumed that \(L \geq 1\), but here it is more convenient to consider \(L(\cdot) \geq 0\), which of course does not change the results.

Let \(A_{Lip}\) be the event that there exists an open Lipschitz function \(L : Z \to Z^+_0\). Since \(A_{Lip}\) is invariant under horizontal translation, we have \(P_p(A_{Lip}) = 0\) or 1. Since \(A_{Lip}\) is also an increasing event, there exists a \(p_L \in [0, 1]\) such that

\[P_p(A_{Lip}) = \begin{cases} 
0 & \text{if } p < p_L, \\
1 & \text{if } p > p_L.
\end{cases}
\]

It was proved in [13] that \(0 < p_L < 1\). For any family \(F\) of Lipschitz functions, the lowest function

\[\bar{L}(x) = \inf\{L(x) : L \in F\}\]

is also Lipschitz. Hence if there exists an open Lipschitz function, then there exists a lowest open Lipschitz function, and it will be again denoted by \(L\). From [13], \((L(x) : x \in Z)\) is stationary and ergodic.

Let \(D\) be the set of all \(x \in Z\) for which \(L(x) > 0\). Let \(D_0\) be the connected component of 0 in \(D\), where connectedness is via adjacency in \(Z\). We define \(D_0 = \emptyset\) if \(0 \notin D\).

**Theorem 3.10.** ([13],[17]) Let \(L\) be the lowest open Lipschitz function. For \(p > p_L\), there exists \(\alpha = \alpha(p) > 0\) such that

\[P_p(L(0) > n) \leq e^{-\alpha(n+1)}, \quad n > 0.
\]

There exists \(p'_L < 1\) such that for \(p > p'_L\)

\[\exp (-\lambda n) \leq P_p(|D_0| > n) \leq \exp (-\gamma n), \quad n \geq 1,
\]

where \(\lambda = \lambda(p)\) and \(\gamma = \gamma(p)\) are positive and finite.
Remark 3.11. By Theorem 3.10 if the random field $X$ stochastically dominates independent site percolation of a sufficiently high density, then with positive probability there exists an infinite good path starting from $(0,0)$ in $\mathbb{L}_{CG}$.

By Theorem 3.10 for $p > p_L^*$ and $n \geq 1$ we have

$$1 - \mathbb{P}_p(\mathcal{L}(0) = \mathcal{L}(1) = 0) \leq \mathbb{P}_p(|D_0| > n) + \mathbb{P}_p((i,0) \text{ is closed for some } i \in (-n,n))$$

(3.30)

$$\leq e^{-\gamma(p)n} + (2n - 1)(1 - p).$$

We may assume $\gamma(p)$ is nondecreasing in $p$. Then given $\epsilon > 0$, we can first apply (3.30) with $p = p_L^*$, and choose $n$ large enough so $e^{-\gamma(p_L^*)n} < \epsilon/2$. Then for $p$ sufficiently close to 1, both terms on the right side of (3.30) are bounded by $\epsilon/2$, so by the ergodic theorem,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} 1(\mathcal{L}(i-1) = \mathcal{L}(i) = 0) = \mathbb{P}_p(\mathcal{L}(0) = \mathcal{L}(1) = 0) > 1 - \epsilon, \quad \mathbb{P}_p \text{-a.s.}$$

(3.31)

3.7. Stochastic Domination. To obtain the domination referenced in Remark 3.11 we will need the following result of Liggett, Schonmann and Stacey [25].

Theorem 3.12. Let $(X_s)_{s \in \mathbb{Z}}$ be a collection of 0-1 valued $k$-dependent random variables, and suppose that there exists a $p \in (0,1)$ such that for each $s \in \mathbb{Z}$

$$\mathbb{P}(X_s = 1) \geq p.$$  

Then if

$$p > p_{SD}(k) = 1 - \frac{k^k}{(k + 1)^{k+1}},$$

then $(X_s)_{s \in \mathbb{Z}}$ is dominated from below by a product random field with density $0 < \rho(p) < 1$. Furthermore, $\rho(p) \to 1$ as $p \to 1$.

Fix $\epsilon > 0$ and choose $p < 1$ so that an open Lipschitz function exists a.s. and (3.33) holds. Then choose $\eta$ with $\rho(1 - \eta) > p$. For fixed $I \geq 1$, the boxes $B(I, J), B(I, J')$ are disjoint for $|J - J'| > 4$, so conditionally on $\mathcal{F}_I$, $(X(I, J) : 0 \leq J \leq I)$ is a 4-dependent collection of random variables. From (3.21) and (3.23), for sufficiently small $\beta u > 0$ and $\beta > 0$ with $u \geq K(\eta)\beta$,

$$Q(X(I,J) = 1|\mathcal{F}_I) \geq 1 - \eta \quad Q \text{-a.s.} \text{ for each } I \geq 1, J \geq 0.$$  

(3.32)

We can apply Theorem 3.12 inductively on $I$ to see that there exists a collection of i.i.d. 0-1 valued random variables $\{Y(I,J) : (I,J) \in \mathbb{L}_{CG}\}$ with $Q(Y(I,J) = 1) = \rho(1 - \eta)$ and

$$Q(X(I,J) \geq Y(I,J)|\mathcal{F}_I) = 1 \quad Q \text{-a.s.}$$

and therefore also unconditionally, $X(I,J) \geq Y(I,J)$ a.s. It follows that the configuration $\{X(I,J) : (I,J) \in \mathbb{L}_{CG}\}$ also a.s. has a lowest open Lipschitz function $L = \mathcal{L}_X$ which satisfies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} 1(\mathcal{L}(i-1) = \mathcal{L}(i) = 0) > 1 - \epsilon, \quad \mathbb{P}_p \text{-a.s.},$$

(3.33)

by (3.33). With positive probability we have $\mathcal{L}_X(0) = 0$, in which case $\mathcal{L}_X = \Gamma^{G,\infty}$ is the infinite good path from $(0,0)$. 

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3.8. **Final Steps.** Let

\[ R_L := \sum_{I=1}^{L} 1_{\{\Gamma^G,\infty(I-1)=\Gamma^G,\infty(I)=0\}}. \]

By ergodicity and \((3.33)\), the limit

\[ \alpha = \alpha(\beta u) := \lim_{L \to \infty} \frac{R_L}{L} > 1 - \epsilon \]

exists. Recall that

\[ U_{\text{off}} = \Theta_1 e^{\beta^2 N}, \quad U_{\text{on}} = \Theta_2 e^{(\beta^2 + (1-\epsilon)F(\beta u))N}, \]

where \(\Theta_1 < 1\) is a constant and

\[ \Theta_2 = \Theta_2(\epsilon) = \frac{\epsilon_0}{4} P(\xi \geq \frac{1}{4\sqrt{\epsilon}}) \sim \frac{\epsilon_0 \sqrt{\epsilon}}{\sqrt{2\pi}} e^{-1/32\epsilon} \text{ as } \epsilon \to 0. \]

Define \(\Theta_3 = - (\alpha \log \Theta_2 + (1-\alpha) \log \Theta_1) > 0\). For some \(K_9 > 0\) we have

\[ \Theta_3 \leq \frac{K_9}{\epsilon}, \quad \epsilon \in (0, 1). \]

For \(L \geq 1\) we have

\[ \frac{1}{LN} \log Z_{\text{off}}^{\beta,u,q} \geq \frac{1}{LN} \log Z_{\text{off}}^{\beta,u,q}(\Omega^G,\infty) \]

and when an infinite good path from \((0,0)\) exists, using \((3.15)\),

\[ Z_{\text{off}}^{\beta,u,q}(\Omega^G,\infty) \]

where \(Z_{\text{off}}^{\beta,u,q} := 1\). Note that \((3.15)\) also guarantees that the measures \(\tilde{\nu}_{(I-1,\Gamma^G,\infty(I-1))}\) on the right side of \((3.37)\) are the ones used in the definition of open/closed coarse-grained sites.

Let \(p_{0,\infty} > 0\) be the probability that there is an infinite good path from \((0,0)\) in the configuration \(X\). When such a path exists, by \((3.37)\) we have for all \(L \geq 1\)

\[ Z_{\text{off}}^{\beta,u,q}(\Omega^G,\infty) \geq U_{\text{on}}^{L} U_{\text{off}}^{L-R_L} \]

Therefore

\[ Q \left( \frac{1}{LN} \log Z_{\text{off}}^{\beta,u,q} \geq \frac{1}{LN} \log U_{\text{on}}^{R_L} U_{\text{off}}^{L-R_L} \text{ for all } L \geq 1 \right) \geq p_{0,\infty}. \]

Since the quenched free energy is self-averaging, recalling \(N = k_0 M = k_0 / F(\beta u)\) and \(f^a(\beta, u) = F(\beta u) + \beta^2 / 2\), using \((3.36)\) we get

\[ f^q(\beta, u) \geq \frac{1}{N} \log U_{\text{on}} + (1 - \alpha) \frac{1}{N} \log U_{\text{off}} \]

\[ = \alpha \left( (1 - \epsilon)F(\beta u) + \frac{\beta^2}{2} \right) - \frac{1}{N} \Theta_3 + (1 - \alpha) \frac{\beta^2}{2} \]

\[ \geq \frac{\beta^2}{2} + \alpha (1 - \epsilon)F(\beta u) - \frac{K_9}{k_0 \epsilon} F(\beta u). \]
By choosing $k_0 = \lfloor K_9 \epsilon^{-2} + 1 \rfloor$, we make the third term on the right side of (3.38) greater than $-\epsilon F(\beta u)$. This and (3.34) show that

\begin{equation}
\beta^2 + (1 - 3\epsilon)F(\beta u) > \frac{\beta^2}{2} = f^a(\beta, 0) \geq f^q(\beta, 0),
\end{equation}

proving (1.6) and (1.7).

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