Centralised Connectivity-Preserving Transformations by Rotation: 3 Musketeers for all Orthogonal Convex Shapes

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Programmable matter systems

Multi-Agent systems
Decentralised
Weak

Centralised variant - feasibility
Overview

• Model and problem definition
• Orthogonal convex shapes
• 6/7-Robot movement
• Transformation
Model and Problem definitions

• Set $S$ of $n$ agents in the shape $A$ occupying cells in a 2D grid
• Transformation of $A$ into shape $B$ in time $t$ by a series of configurations $C_0 \ldots C_t$ each reachable by a single move from a single node
• Centralised model to explore feasibility
• Rotation – movement of one node $90^\circ$ around another node
• Rot-Transformability – Rotation only
• RotC-Transformability – Rotation only, connectivity must be preserved
Rotation Only
Rotation Only

Rotation
Rotation Only
• Our focus – transforming orthogonal convex shapes into each other in the RotC setting
• Rotation only – simple operations are easier to implement in real-world systems, colour restrictions in square/triangular grids technically interesting
• Connectivity – programmable matter, not swarm robotics
• Certain orthogonal convex shapes cannot meaningly transform in RotC
• Therefore transformations are aided by seeds – nodes placed in empty cells neighbouring a shape to create a new shape to aid transformation
• We discard the seed at the end
Related Work

• Various programmable matter models developed e.g. [Dumitrescu, Pach, Symposium on Computational Geometry, 2004]

• Programmable materials developed e.g. [Rothemund, Nature, 2006]

Recent papers on the concept of seed-assisted transformations in programmable matter

• Universal transformation for Rot-Transformability for all unblocked shapes, introduction of RotC-Transformability and seeds, line folding with seeds, impossibility of 5-node line traversal and orthogonal convex idea [Michail et al., JCSS, 2019]

• Any pair of color-consistent nice shapes [Almethen, Michail, Potapov, TCS, 2020] A, B in O(n²) moves with a 4-seed [C, Michail, Potapov, ALGOSENSORS 2021] – not directly comparable with orthogonal convex

• Universal transformation with connectivity preservation using “leapfrog” and “monkey” movement and a 5-node seed [Akitaya et al., Algorithmica, 2021]
**Proposition.** A shape $S$ is a connected orthogonal convex shape iff its perimeter satisfies both the following properties:

- It is described by the regular expression

$$d_1(d_1 \mid d_2)^*d_2(d_2 \mid d_3)^*d_3(d_3 \mid d_4)^*d_4(d_4 \mid d_1)^*$$

under the additional constraint that $N_1 = N_3$ and $N_2 = N_4$.

- Its interior has no empty cell.
Orthogonal convex shapes

Orthogonal Convex

Non Orthogonal Convex
Rotation-only: Blocked Shapes

No move: 0-blocked (or blocked)

If connectivity must be preserved: $k$-blocked
Rotation-only: Blocked Shapes

No move: 0-blocked

If connectivity must be preserved: $k$-blocked
Rotation-only: Blocked Shapes

No move: 0-blocked

If connectivity must be preserved: \( k \)-blocked
Rotation-only: Blocked Shapes

No move: 0-blocked

If connectivity must be preserved: $k$-blocked
Main Results

• **Theorem 1.** For any orthogonal convex shape $S$, a 6-robot is capable of traversing the perimeter of $S$.

• **Theorem 2.** For any orthogonal convex shape $S$, a 7-robot is capable of traversing the perimeter of $S$.

• **Theorem 3.** Let $S$ and $S'$ be connected colour-consistent orthogonal convex shapes. Then there is a connected shape $M$ of 3 nodes (the 3 musketeers) and an attachment of $M$ to the bottom-most row of $S$, such that $S \cup M$ can reach the configuration $S'$ in $O(n^2)$ time steps.
6-robot movement

6 node block
Set $C$ of corner cases depending on height and width
Invariant – robot in new location with same structure
Solve for one quadrant, the rest follow by rotation
Cases

\[ h(C) = \left| y_u - y_d \right| \]
\[ w(C) = \left| x_r - x_l \right| \]

(a) The height 1 cases, with widths 1 and 2+.

(b) The height 2 case.

(c) The height 3+ case.

Fig. 8: The four basic corner scenarios of \( C \).
7-robot movement

• Represents a 6-robot carrying an extra node (the load)
• Mostly the same as 6-robot movement
• Key difference – two positions for the extra node
• Double the cases!
Any pair of color-consistent orthogonal convex shapes $A$, $B$ in $O(n^2)$ moves with a 3-seed.
Orthogonal-convex shape $A$

3 musketeers to 6-robot

Orthogonal-convex shape $A'$
The transformation process

We now have the main structure of our transformation process:
• Add a 3-node seed to create a 6-robot
• Move the 6-robot around the shape
• Remove a node according to a shape elimination sequence
• Move the resulting 7-robot
• Place the node according to a shape generation sequence
Shape $B$ + Canonical Shape + Diagonal line-with-leaves $D$
Algorithm 2 OConvexToDLL($S, M$)

**Input:** shape $S \cup M$, where $S$ is a connected orthogonal convex shape of $n$ nodes and $M$ is a 3-node seed on the cell perimeter of $S$, row elimination sequence $\sigma = (u_1, u_2, \ldots, u_n)$ of $S$, extended staircase generation sequence of $W \cup T = \sigma' = (u'_1, u'_2, \ldots, u'_n)$ which is colour-order preserving w.r.t. $\sigma$, shape elimination sequence $\sigma = (u_1, u_2, \ldots, u_{|T|})$ of $T$, shape generation sequence of $X = \sigma' = (u'_1, u'_2, \ldots, u'_{|T|})$ which is colour-order preserving w.r.t. $\sigma$

**Output:** shape $G = W \cup X \cup M$, where $G$ is a diagonal line-with-leaves and $M$ is a connected 3-node shape on the cell perimeter of $S$.

$R \leftarrow \text{GenerateRobot}(S, M)$

$\sigma \leftarrow \text{rowEliminationSequence}(S)$

$\sigma' \leftarrow \text{ExtendedStaircase}(\sigma)$

$W \cup T \leftarrow \text{OConvexToExtStaircase}(S, R, \sigma, \sigma')$

$\sigma \leftarrow \text{repsEliminationSequence}(W \cup T)$

$\sigma' \leftarrow \text{stairExtensionSequence}(W \cup T)$

$G \leftarrow \text{ExtStaircaseToDLL}(W \cup T, R, \sigma, \sigma')$

$\text{TerminateRobot}(G, R)$
**Lemma 15.** Let $S$ be a connected orthogonal convex shape. Then there is a connected shape $M$ of 3 nodes (the 3 musketeers) and an attachment of $M$ to the bottom-most row of $S$, such that $S \cup M$ can reach a configuration $S' \cup M'$ satisfying the following properties. $S' = S \setminus \{u_1, u_2, u_3\}$, where $\{u_1, u_2, u_3\}$ is the 3-prefix of a row elimination sequence $\sigma$ of $S$ starting from the bottom-most row of $S$. $M'$ is a 6-robot on the perimeter of $S'$. 
Proof Overview

**Lemma 2.** Every connected orthogonal convex shape $S$ has a row (and column) elimination sequence $\sigma$.

**Lemma 3.** Let $\sigma$ be a bicoloured sequence of nodes that fulfills all the following conditions:

- The set of the first two nodes in $\sigma$ is not single-coloured.
- The third node of $\sigma$ is black.
- $\sigma$ does not contain a single-coloured 3-sub-sequence.

Then there is an extended staircase generation sequence $\sigma' = (u'_1, u'_2, \ldots, u'_n)$ which is colour-order preserving with respect to $\sigma$.

**Lemma 4.** For any connected orthogonal convex shape $S$ of $n$ nodes, given a row elimination sequence $\sigma = (u_1, u_2, \ldots, u_n)$ of $S$ where the set of the first two nodes in $\sigma$ is not single-coloured and $u_3$ is black, there is an extended staircase generation sequence $\sigma' = (u'_1, u'_2, \ldots, u'_n)$ which is colour-order preserving w.r.t $\sigma$ and such that, for all $1 \leq i \leq |\sigma|$, $D_i = \{u'_1, u'_2, \ldots, u'_i\}$ is a connected orthogonal convex shape.
Proof Overview

Orthogonal-convex shape + 6-robot

Extended staircase

row elimination sequence
Proof Overview

Orthogonal-convex shape + 6-robot

Extended staircase

row elimination sequence

Diagonal line-with-leaves
Any orthogonal-convex shape

Proof Overview
Any orthogonal-convex shape

Proof Overview
Any orthogonal-convex shape
Any orthogonal-convex shape

**Theorem.** Let $S$ and $S'$ be connected color-consistent orthogonal convex shapes. Then there is a connected shape $M$ of 3 nodes (the 3 musketeers) and an attachment of $M$ to the bottom-most row of $S$, such that $S \cup M$ can reach the configuration $S'$ in $O(n^2)$ moves.
Theorem. Let $S$ and $S'$ be connected color-consistent orthogonal convex shapes. Then there is a connected shape $M$ of 3 nodes (the 3 musketeers) and an attachment of $M$ to the bottom-most row of $S$, such that $S \cup M$ can reach the configuration $S'$ in $O(n^2)$ moves.
Summary and open problems

• Seeds can aid the transformation of blocked shapes
• Rot-transformability is universal
• Minimal seed RotC-transformability for nice shapes
• Minimal seed transformations of orthogonal convex shapes
  ▪ Movement of 6/7-node robots around the perimeter - Theorem 1 and Theorem 2
  ▪ Transformation of orthogonal convex into other orthogonal convex by reversibility – Theorem 3

Open problems:
• Decentralising the execution
• Extending the class – universal transformation?
• Double spiral – example of problems of universal transformation

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Questions?