Pairwise coexistence of effects versus coexistence

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Abstract. The concept of coexistence of quantum mechanical effects is reviewed. We distinguish between coexistence and pairwise coexistence and give an example showing the non-equivalence the two concepts. A theorem on some closures of sets of pairwise coexistent effects is proved. Some proofs of the well-known fact that any set of pairwise coexistent (mutually compatible) sharp effects is coexistent are considered. In particular, a proof is presented that is based on the statement that every countably complete Boolean lattice of sharp effects is closed in some topology.

1. Introduction
One of the most crucial facts of quantum mechanics is the existence of observables that cannot be measured jointly. This issue has, from the early days of the theory up to the present, been investigated extensively (see, e.g., [1–3]). As is well known, the notion of an observable can be traced back to that of simple observables with only two outcomes, the effects. We review the concepts for the joint measurability of simple observables and present some results the author did not find elsewhere.

In quantum mechanics, an effect is described by and identified with a positive self-adjoint operator $A$ acting in a separable complex Hilbert space $\mathcal{H}$ satisfying $\|A\| \leq 1$, i.e., $0 \leq A \leq I$, $I$ being the unit operator. The set $\mathcal{E}(\mathcal{H})$ of all effects is a closed convex subset of the real Banach space $\mathcal{B}_s(\mathcal{H})$ of all bounded self-adjoint operators. It is well known that the extreme points of $\mathcal{E}(\mathcal{H})$ are the orthogonal projections $P$; an effect $P \in \partial_e \mathcal{E}(\mathcal{H})$, $\partial_e \mathcal{E}(\mathcal{H})$ denoting the extreme boundary of $\mathcal{E}(\mathcal{H})$, is called sharp whereas the other effects are called unsharp.\(^1\)

The states are given by the density operators $W$; $W$ is a positive self-adjoint trace-class operator of trace 1. The real vector space $\mathcal{T}_s(\mathcal{H})$ of all self-adjoint trace-class operators is, equipped with the trace norm, also a real Banach space. The set $\mathcal{S}(\mathcal{H})$ of all density operators is a closed convex subset of $\mathcal{T}_s(\mathcal{H})$ whose extreme points are the one-dimensional orthogonal projections $P_\psi := |\psi\rangle\langle\psi|$; a state $P_\psi \in \mathcal{S}(\mathcal{H})$ is called pure. The probability for the occurrence of an effect $A \in \mathcal{E}(\mathcal{H})$ (respectively, for 1 where the two outcomes of $A$ are 0 and 1) in a state $W \in \mathcal{S}(\mathcal{H})$ is given by $\text{tr} WA$.

According to the bilinear functional $\langle V, A \rangle \mapsto \text{tr} VA$ where $V \in \mathcal{T}_s(\mathcal{H})$ and $A \in \mathcal{B}_s(\mathcal{H})$ are arbitrary, $\mathcal{B}_s(\mathcal{H})$ is isometrically isomorphic to the dual space $\mathcal{T}_s(\mathcal{H})'$. Thus, the weak-* topology $\sigma(\mathcal{B}_s(\mathcal{H}), \mathcal{T}_s(\mathcal{H}))$ is defined. We call this topology briefly the $\sigma$-topology; it is weakest topology such that the elements of $\mathcal{T}_s(\mathcal{H})$, considered as linear functionals on $\mathcal{B}_s(\mathcal{H})$, are continuous. Since

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\(^1\) The terms “effect” and “coexistence” were introduced by Ludwig [4–7], whereas the terms “sharp” and “unsharp” for effects and observables were introduced by Busch. Ludwig calls a sharp effect a decision effect.
\[ \sigma(B_s(H), \mathcal{T}_s(H)) = \sigma(B_s(H), S(H)), \]

a neighborhood base of \( A \in B_s(H) \) is given by the open sets

\[
U(A; W_1, \ldots, W_n; \varepsilon) := \left\{ \tilde{A} \in B_s(H) \middle| |\text{tr} W_j \tilde{A} - \text{tr} W_j A| < \varepsilon \text{ for } j = 1, \ldots, n \right\}
\]

where \( W_j \in S(H) \) and \( \varepsilon > 0 \). Note that the \( \sigma \)-topology can be interpreted as the topology of the physical approximation of effects. Namely, an effect \( A \in \mathcal{E}(H) \) is physically approximated by \( \tilde{A} \in \mathcal{E}(H) \) if in many (but finitely many) states \( W_j \) the probabilities \( \text{tr} W_j \tilde{A} \) differ from \( \text{tr} W_j A \) by an amount less than a small \( \varepsilon > 0 \). This statement can be tested experimentally and can be characterized mathematically by \( \tilde{A} \in U(A; W_1, \ldots, W_n; \varepsilon) \).

The \( \sigma \)-topology is the topology on \( B_s(H) \) that is induced by the ultraweak operator topology on \( B_s(H) \), i.e., \( \sigma(B_s(H), \mathcal{T}_s(H)) = \sigma(B(H), \mathcal{T}(H)) \cap B_s(H), B(H) \) and \( \mathcal{T}(H) \) denoting the complex Banach spaces of all bounded, resp., trace-class operators. The weak operator topology on \( B_s(H) \) is just the topology \( \sigma(B_s(H), \partial_s \mathcal{E}(H)) \). On bounded sets the \( \sigma \)-topology and the weak operator topology coincide. This can be concluded by means of the neighborhoods (1) and the corresponding neighborhood base of the weak operator topology or, alternatively, by means of the following compactness argument. According to the Banach-Alaoglu theorem, the closed unit ball of \( B_s(H), B_{B_s(H)} := \{ A \in B_s(H) | ||A|| \leq 1 \} \), is \( \sigma \)-compact. The weak operator topology is Hausdorff and weaker than the \( \sigma \)-topology, hence, on \( B_{B_s(H)} \) both topologies must be equal.

The contemporary conception of an observable is that of a normalized effect-valued measure \( F \) on some measurable space \( (M, \Xi) \) where \( M \) is interpreted to be the value space of \( F \), i.e., a space encompassing the possible measuring values. In particular, in quantum mechanics \( F \) is a normalized positive-operator-valued measure on \( (M, \Xi) \), i.e., a map \( F : \Xi \to \mathcal{E}(H) \), \( B \mapsto F(B) \), satisfying \( F(\emptyset) = 0, F(M) = I \), and \( F(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} B_j \) where the measurable sets \( B_j \) are mutually disjoint and the sum converges in the \( \sigma \)-topology, for instance. The set \( R(F) := \{ F(B) \mid B \in \Xi \} \) is the range of \( F \). An observable \( E : \Xi \to \mathcal{E}(H) \) is called sharp if \( R(E) \subseteq \partial_s \mathcal{E}(H) \) and can be considered as a map \( E : \Xi \to \partial_s \mathcal{E}(H) \), otherwise an observable is called unsharp.

In Section 2, we recall the concept of the coexistence of a set of effects and define the pairwise coexistence of a set of effects. We show that a set of pairwise coexistent effects is not necessarily coexistent and prove some closure properties of pairwise coexistence. In Section 3 we give a proof of the well-known fact that two sharp effects are coexistent if and only if they commute; two coexistent sharp effects are also called compatible. Furthermore, following Gudder ([1], [8, pp. 53–129]), we use the concept of the compatant of a set of sharp effects to show that every set of mutually compatible sharp effects is contained in some complete Boolean sublattice of \( \partial_s \mathcal{E}(H) \).

It is well known that any set of mutually compatible sharp effects is contained in the range of a (sharp) observable. That is, according to the definition in Section 2, any set of pairwise coexistent sharp effects is coexistent, in contrast to the case of arbitrary effects. The statement of the penultimate sentence can quickly be proved as follows. Let \( S \) be a set of mutually compatible sharp effects. Then \( S \) is a set of commuting orthogonal projections of the separable Hilbert space \( H \) and, according to a theorem of von Neumann [9, 10], there exists a (bounded) self-adjoint operator \( A \) acting in \( H \) such that all \( P \in S \) are functions of \( A \), i.e.,

\[
P = f_P(A) = \int f_P \, dE = \int f_P(\lambda) \, E(d\lambda) = \int \lambda(E \circ f_P^{-1})(d\lambda)
\]

where \( f_P : \mathbb{R} \to \mathbb{R} \) is a bounded Borel-measurable function, \( E \) the spectral measure of \( A \), and \( (E \circ f_P^{-1})(B) = E(f_P^{-1}(B)), B \subseteq \mathbb{R} \) being a Borel set. From the representation of \( P \) by the last integral of (2) it follows that \( E \circ f_P^{-1} \) is the spectral measure of the projection \( P \) and therefore \( P = E(f_P^{-1}(\{1\})) \in R(E) \). Hence, every \( P \in S \) is in the range of the sharp observable \( E \) and
$S \subseteq R(E)$. Alternatively, this can be concluded from $\int f_P \, dE = P = P^2 = \int f_H \, dE$ which implies that $f_P = f_H^2$ $E$-almost everywhere. Thus, $f_P$ can be chosen to be the characteristic function $\chi_B$ of some Borel set $B$ which entails that $P = \int \chi_B \, dE = E(B) \in R(E)$.

Another proof of $S \subseteq R(E)$, as in the preceding paragraph, is due to Varadarajan given in the context of quantum logic (see [11]). This proof is based on a representation theorem of Loomis [12] which states that, given a countably complete Boolean lattice $L$ (Boolean $\sigma$-algebra), there exists a measurable space $(M, \Xi)$ and a $\sigma$-homomorphism of $\Xi$ onto $L$.

Since both the mentioned theorem of von Neumann, as well as the theorem of Loomis, are considerably deep and, moreover, the latter is beyond the scope of Hilbert space, it is worth presenting a proof of $S \subseteq R(E)$, as before, that uses only the theory of Hilbert space and bounded linear operators acting in Hilbert space and does not presuppose von Neumann’s theorem. This is performed in Sections 4 and 5. It turns out that even such a proof is quite involved so that, against the author’s original plan, further topics concerning the pairwise coexistence of effects cannot be included in this paper.

In Section 4 we prove several statements on countably complete Boolean sublattices of $\partial, \mathcal{E}(H)$ which prepare the proof of $S \subseteq R(E)$, $S$ as before. The central point is the corollary of Theorem 4 which establishes that every countably complete Boolean sublattice $L$ of $\partial, \mathcal{E}(H)$ is closed in the topology on $\partial, \mathcal{E}(H)$ that is induced by the $\sigma$-topology. This property of $L$ and its $\sigma$-separability entail that $L$ is countably generated, and this result enables the envisaged proof which is given in Section 5. Much of the subject matter of Sections 4 and 5 is a review.

2. Coexistence

According to Ludwig [2, 7], we call two effects $F, G \in \mathcal{E}(H)$ coexistent if there exist effects $F', G', H \in \mathcal{E}(H)$ such that

$$F = F' + H$$
$$G = G' + H$$

and $F' + G' + H \leq I$. The decomposition (3) is in general not unique. It is easy to see that two effects are coexistent if and only if they are contained in the range of some observable $F$, i.e., $F, G \in R(F) = F(\Xi)$. So two coexistent effects can be measured jointly by one measuring device. We notice the following simple statements:

(a) The trivial effects $0$ and $I$ are coexistent with every $F \in \mathcal{E}(H)$.
(b) The pair $(F, F)$, $F \in \mathcal{E}(H)$, is coexistent.
(c) Any $F, G \in \mathcal{E}(H)$ satisfying $F \leq G$ are coexistent.
(d) The effects $F$ and $G$ are coexistent if $F + G \in \mathcal{E}(H)$.
(e) Any $F, G \in \frac{1}{2} \mathcal{E}(H) = \frac{1}{2}[0, I] = \{A \in \mathcal{E}(H) \mid 0 \leq A \leq \frac{1}{2}I\}$ are coexistent.
(f) If $F, G \in \mathcal{E}(H)$ are coexistent, then $F$ and $I - G$ also.

Corresponding to the interpretation of the coexistence of effects as their joint measurability by one measuring device, a set $S$ is called coexistent if $S$ is contained in the range of some observable $F$, i.e., $S \subseteq R(F)$. We call $S \subseteq \mathcal{E}(H)$ pairwise coexistent if any two effects $F, G \in S$ are coexistent. Clearly, coexistence of set of effects implies pairwise coexistence, but the converse does in general not hold, as an example of a set of three effects shows. The following simple lemma prepares this example and is also useful for later considerations.

**Lemma 1.** If $A \in \mathcal{E}(H)$ and $P \in \partial, \mathcal{E}(H)$ are related by $A \leq P$, then $A = PA = AP$.

**Proof.** For $\chi \in R(P)^\perp$ where $R(P)$ denotes the range $PH$ of $P$, we have that

$$\|A^\frac{1}{2} \chi\|^2 = \langle \chi | A \chi \rangle \leq \langle \chi | P \chi \rangle = 0$$
and so $A\chi = 0$. For $\varphi \in R(P)$ it then follows, for all $\chi \in R(P) \perp$, that
\[ \langle \chi | A \varphi \rangle = \langle A \chi | \varphi \rangle = 0, \]
i.e., $A \varphi \in R(P)$. Hence, for any $\psi \in \mathcal{H}$, $\psi = \varphi + \chi$ with $\varphi \in R(P)$ and $\chi \in R(P) \perp$, we obtain
\[ A\psi = A\varphi = P A\varphi = P A\varphi + P A\chi = P A\psi. \]
Thus, $A = PA$ and $A = A^* = AP$. \hfill \Box

Now let $\varphi_1, \varphi_2 \in \mathcal{H}$ be two orthogonal unit vectors. Define the vectors
\[ \psi_1 := \varphi_1, \quad \psi_2 := \frac{1}{2} \sqrt{3} \varphi_1 + \frac{1}{2} \varphi_2, \quad \psi_3 := \frac{1}{2} \varphi_1 + \frac{1}{2} \sqrt{3} \varphi_2 \]
and the effects
\[ G_j := \frac{1}{2} P_{\psi_j} = \frac{1}{2} |\psi_j\rangle \langle \psi_j|, \]
$j = 1, 2, 3$. According to the above statement (e), the set $S := \{G_1, G_2, G_3\}$ is pairwise coexistent. We show that, however, $S$ is not coexistent. An easy calculation yields
\[ \langle \psi_2 | (G_1 + G_2 + G_3) \psi_2 \rangle = \frac{5}{4} > 1, \]
consequently, $G_1 + G_2 + G_3 \not\in I$. Assume $S$ were coexistent, i.e., there would exist an observable an observable $F$ and sets $B_j \in \Xi$ such that $G_j = F(B_j)$. We obtain that, for $j \neq k$,
\[ F(B_j \cap B_k) \leq F(B_j) = \frac{1}{2} P_{\psi_j}, \]
\[ F(B_j \cap B_k) \leq F(B_k) = \frac{1}{2} P_{\psi_k}. \]
So $2F(B_j \cap B_k) \leq P_{\psi_j}$, and by means of the lemma it follows that $F(B_j \cap B_k) = P_{\psi_j} F(B_j \cap B_k) = P_{\psi_j} F(B_j \cap B_k) P_{\psi_j} \leq \frac{1}{2} P_{\psi_j}$. Thus, $F(B_j \cap B_k) = \lambda_{jk} P_{\psi_j}$ with some $\lambda_{jk} \leq \frac{1}{2}$; analogously, $F(B_j \cap B_k) = \mu_{jk} P_{\psi_k}$ with some $\mu_{jk} \leq \frac{1}{2}$. In particular, we have that $\lambda_{jk} P_{\psi_j} = \mu_{jk} P_{\psi_k}$ which entails $\lambda_{jk} = \mu_{jk} = 0$ since $\psi_j$ and $\psi_k$ are linearly independent. Hence, $F(B_j \cap B_k) = 0$ for all $j \neq k$, in consequence, $F(B_1 \cap B_2 \cap B_3) = 0$ also. Therefore,
\[ F(B_1 \cup B_2 \cup B_3) = F(B_1) + F(B_2) + F(B_3) - F(B_1 \cap B_2) - F(B_1 \cap B_3) - F(B_2 \cap B_3) + F(B_1 \cap B_2 \cap B_3) \]
\[ = G_1 + G_2 + G_3 \]
\[ \not\in I \]
which contradicts $F(B_1 \cup B_2 \cup B_3) \leq I$. We have shown that pairwise coexistance does not imply coexistance. \footnote{2}

From $\|A\| = \sup_{\|\varphi\| \leq 1} |\langle \varphi | A \varphi \rangle|$ it follows that the unit ball $B_{\mathcal{B}(\mathcal{H})} = \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\| \leq 1\}$ is equal to the order-unit interval $[-I, I]$. Therefore, $\mathcal{E}(\mathcal{H}) = [0, I] = \frac{1}{2}([-I, I] + I) = \frac{1}{2}(B_{\mathcal{B}(\mathcal{H})} + I)$, and the $\sigma$-compactness of $B_{\mathcal{B}(\mathcal{H})}$ entails that of the convex set $\mathcal{E}(\mathcal{H})$. The latter implies that, according to the Krein-Milman theorem, $\mathcal{E}(\mathcal{H}) = \text{conv} \partial_\mathcal{E}(\mathcal{H})$ where the extreme boundary consists of the sharp effects. Moreover, even the stronger statement $\mathcal{E}(\mathcal{H}) = \partial_\mathcal{E}(\mathcal{H})$ is true, \footnote{2 This example was, without the details of the proof, already published in the author’s paper \cite[p. 12]{13}. A similar example is given in \cite[Theorem 1]{14}.}
provided that \( \dim \mathcal{H} = \infty \). From this we can further conclude that the norm-topological boundary \( \partial \mathcal{E}(\mathcal{H}) = \frac{1}{2}(S_{B_2(\mathcal{H})} + I) \), \( S_{B_2(\mathcal{H})} \) being the unit sphere \( S_{B_2(\mathcal{H})} := \{ A \in B_2(\mathcal{H}) \mid \| A \| = 1 \} \), is \( \sigma \)-dense in \( \mathcal{E}(\mathcal{H}) \); alternatively, \( \partial \mathcal{E}(\mathcal{H})^\sigma = \mathcal{E}(\mathcal{H}) \) follows from the fact that \( S_{B_2(\mathcal{H})} \) is \( \sigma \)-dense in \( B_{B_2(\mathcal{H})} \).

It is well known that, for a separable Banach space \( V \), the weak-* topology on the closed unit ball \( B_V \) of the dual space \( V' \), \( \sigma(V', V) \cap B_V \), is second countable (a straightforward proof is given in [13]). So w.r.t. \( \sigma(V', V) \cap B_V \), the topological concepts can be characterized using sequences. In particular, the weak-* compactness of \( B_V \) is equivalent to the weak-* sequential compactness. Since our Hilbert space is separable, the space \( \mathcal{T}(\mathcal{H}) \) of the self-adjoint operators is separable, and the above remarks apply to the topology on the unit ball \( B_{B_2(\mathcal{H})} \) induced by our \( \sigma \)-topology. In consequence, the topology \( \sigma \cap \mathcal{E}(\mathcal{H}) \) is second countable as well, and \( \mathcal{E}(\mathcal{H}) \) is sequentially \( \sigma \cap \mathcal{E}(\mathcal{H}) \)-compact and therefore also sequentially \( \sigma \)-compact.

The question arises what one can say about the convex hull and the \( \sigma \)-closure of a set \( S \) of pairwise coexistent effects. In particular, two effects \( F, G \in \mathcal{S}^\sigma \setminus S \) can, in the \( \sigma \)-topology, be approximated by two coexistent effects of \( S \). Since the \( \sigma \)-topology describes the physical approximation of effects by other ones, \( F \) and \( G \) should be coexistent. The following theorem confirms this expectation.

**Theorem 1.** The convex hull and the \( \sigma \)-closure of a set \( S \) of pairwise coexistent effects are pairwise coexistent. In particular, the smallest \( \sigma \)-closed convex set containing \( S \), \( \text{conv} \mathcal{S}^\sigma \), is pairwise coexistent.

**Proof.** The convex hull of \( S \) consists of the finite convex linear combinations of elements of \( S \). We have to show that any two effects

\[
F = \sum_{j=1}^{m} \alpha_j F_j \quad \text{and} \quad G = \sum_{k=1}^{n} \beta_k G_k
\]

where \( F_j, G_k \in S \), \( \alpha_j, \beta_k \geq 0 \), and \( \sum_{j=1}^{m} \alpha_j = \sum_{k=1}^{n} \beta_k = 1 \), are coexistent. The effects \( F \) and \( G \) can be written in the form

\[
F = \sum_{l=1}^{N} \delta_l \tilde{F}_l \quad \text{and} \quad G = \sum_{l=1}^{N} \delta_l \tilde{G}_l
\]

where \( \delta_l \geq 0 \), \( \sum_{l=1}^{N} \delta_l = 1 \), each \( \tilde{F}_l \) equals some \( F_j \), and each \( \tilde{G}_l \) equals some \( G_k \) (for more details of this part of the proof, see [13]). Now, from the coexistence of \( \tilde{F}_l \) and \( \tilde{G}_l \),

\[
\tilde{F}_1 = A_{l1} + A_{l0} \quad \text{and} \quad \tilde{G}_1 = A_{l2} + A_{l0},
\]

\( A_{l1}, A_{l2}, A_{l0} \in \mathcal{E}(\mathcal{H}) \), \( A_{l1} + A_{l2} + A_{l0} \leq I \), we obtain that

\[
F = \sum_{l=1}^{N} \delta_l A_{l1} + \sum_{l=1}^{N} \delta_l A_{l0} =: A_1 + A_0
\]

\[
G = \sum_{l=1}^{N} \delta_l A_{l2} + \sum_{l=1}^{N} \delta_l A_{l0} =: A_2 + A_0
\]

where \( A_1, A_2, A_0 \in \mathcal{E}(\mathcal{H}) \) and \( A_1 + A_2 + A_0 = \sum_{l=1}^{N} \delta_l (A_{l1} + A_{l2} + A_{l0}) \leq I \). Hence, \( F \) and \( G \) are coexistent.

Let \( F, G \in \mathcal{S}^\sigma \). Then there exist sequences \( \{ F_n \}_{n \in \mathbb{N}} \), \( \{ G_n \}_{n \in \mathbb{N}} \) in \( S \) and sequences \( \{ F'_n \}_{n \in \mathbb{N}} \), \( \{ G'_n \}_{n \in \mathbb{N}} \), and \( \{ H_n \}_{n \in \mathbb{N}} \) in \( \mathcal{E}(\mathcal{H}) \) such that

\[
F = \sigma\text{-lim}_{n \to \infty} F_n \quad \text{and} \quad G = \sigma\text{-lim}_{n \to \infty} G_n
\]

and

\[
F_n = F'_n + H_n \quad \text{and} \quad G_n = G'_n + H_n
\]
where \( F'_n + G'_n + H_n \leq I \). Due to the sequential \( \sigma \)-compactness of \( \mathcal{E}(\mathcal{H}) \), there exists a subsequence \( \{F'_n\}_n \in \mathbb{N} \) converging to some \( F' \in \mathcal{E}(\mathcal{H}) \) in the \( \sigma \)-topology. We write \( F'_n := F'_n \).

Further, there is a subsequence \( \{H_{n(j(k))}\}_{k \in \mathbb{N}} \) of \( \{H_{n(j)}\}_{j \in \mathbb{N}} \) such that \( H_{n(j(k))} \xrightarrow{\sigma} H \) as \( k \to \infty \); finally, there is a subsequence \( \{G'_{n(j(k))}\}_{k \in \mathbb{N}} \) of \( \{G'_{n(j(k))}\}_{k \in \mathbb{N}} \) with \( G'_{n(j(k))} \xrightarrow{\sigma} G' \) as \( l \to \infty \). Abbreviating \( n(j(k(l))) \) simply by \( n_l \), Eqs. (5) imply that

\[
\begin{align*}
F_{n_l} &= F'_l + H_{n_l} \\
G_{n_l} &= G'_l + H_{n_l} \\
\end{align*}
\]

where \( F'_l + G'_l + H_{n_l} \leq I \). By (4) and the limit behaviour of the considered subsequences as \( l \to \infty \), it follows from (6) that

\[
\begin{align*}
F &= F' + H \\
G &= G' + H \\
\end{align*}
\]

where \( F' + G' + H \leq I \). Hence, any two effects \( F, G \in \overline{S'} \) are coexistent, that is, \( \overline{S'} \) is pairwise coexistent.

An implication of Theorem 1 is that a set of pairwise coexistent effects or its convex hull cannot be \( \sigma \)-dense in \( \mathcal{E}(\mathcal{H}) \) (provided that \( \dim \mathcal{H} \geq 2 \)). In [13] this fact is proved differently and some physical consequences are discussed.

### 3. Compatibility

It is well known that the set \( \partial \mathcal{E}(\mathcal{H}) \) of the sharp effects is, with the order relation of \( \mathcal{B}_s(\mathcal{H}) \), a complete orthomodular orthocomplemented lattice. One can prove that the greatest lower and the least upper bound of \( P, Q \in \partial \mathcal{E}(\mathcal{H}) \) taken in \( \partial \mathcal{E}(\mathcal{H}) \) coincide with the corresponding bounds taken in \( \mathcal{E}(\mathcal{H}) \), but they do not exist when taken in \( \mathcal{B}_s(\mathcal{H}) \) unless \( P \) and \( Q \) are comparable. Furthermore, \( \mathcal{E}(\mathcal{H}) \) is not a lattice (for the lattice properties of \( \mathcal{E}(\mathcal{H}) \), see [16] and some references quoted therein). We use only the lattice structure of \( \partial \mathcal{E}(\mathcal{H}) \) and the fact—which we shall prove—that the infimum of an arbitrary subset of \( \partial \mathcal{E}(\mathcal{H}) \) taken in \( \partial \mathcal{E}(\mathcal{H}) \) is the same as taken in \( \mathcal{E}(\mathcal{H}) \).

We give a simple proof of the following known result (cf. [2]).

**Theorem 2.** Two sharp effects \( P, Q \in \partial \mathcal{E}(\mathcal{H}) \) are coexistent if and only if \( PQ = QP \). If \( P \) and \( Q \) are coexistent, then, in the representation

\[
\begin{align*}
P &= F_1 + H \\
Q &= F_2 + H \\
\end{align*}
\]

where \( F_1, F_2, H \in \mathcal{E}(\mathcal{H}) \) satisfying \( F_1 + F_2 + H \leq I \), \( F_1, F_2, \) and \( H \) are uniquely determined and given by the orthogonal projections \( F_1 = P(I - Q) = P \wedge Q^\perp \), \( F_2 = Q(I - P) = Q \wedge P^\perp \), and \( H = PQ = P \wedge Q \).

**Proof.** If \( PQ = QP \), then

\[
\begin{align*}
P &= P(I - Q) + PQ \\
Q &= Q(I - P) + PQ \\
\end{align*}
\]

and \( P(I - Q) + Q(I - P) + PQ = P + Q(I - P) \leq P + (I - P) = I \). Thus, \( P \) and \( Q \) are coexistent.

Conversely, assume (7) holds with \( F_1, F_2, H \in \mathcal{E}(\mathcal{H}) \) and \( F_1 + F_2 + F_3 \leq I \). From \( P = F_1 + H \) it follows that \( F_1 \leq P \) and from this, according to Lemma 1, \( F_1 = PF_1 \) and

\[
PF_1 = F_1 P.
\]
The relation $F_1 + F_2 + H = F_1 + Q \leq I$ implies $F_1 \leq I - Q$; therefore, analogously to (8), we have that $F_1(I - Q) = (I - Q)F_1$ and in consequence

$$QF_1 = F_1Q. \quad (9)$$

Furthermore, $H \leq Q$ and

$$QH = HQ. \quad (10)$$

Eqs. (9), (10), and $P = F_1 + H$ entail

$$PQ = PQ. \quad (11)$$

Since $F_1 + Q \leq I$, $(I - Q) - F_1 \geq 0$. By (8) and (11), it follows that $PF_1 \leq P(I - Q)$. Taking account of $F_1 = PF_1$ we obtain

$$F_1 \leq P(I - Q). \quad (12)$$

Moreover, $H \leq P, Q$ implies $H \leq P \wedge Q$ which is, by (11), equivalent to

$$H \leq PQ. \quad (13)$$

From (12) and (13) we conclude that

$$P = F_1 + H \leq P(I - Q) + PQ = P.$$

Hence,

$$F_1 + H = P(I - Q) + PQ,$$

or, equivalently,

$$F_1 - P(I - Q) = PQ - H. \quad (14)$$

By (12), $F_1 - P(I - Q) \leq 0$ whereas, by (13), $PQ - H \geq 0$. Thus, we obtain from (14) that

$$F_1 - P(I - Q) = PQ - H = 0;$$

consequently, $F_1 = P(I - Q), H = PQ$, and $F_2 = Q - H = Q(I - P)$. \hfill \Box$

Two coexistent sharp effects are often called compatible (commensurable in the terminology of Ludwig), and the compatibility of $P, Q \in \partial_e \mathcal{E}(\mathcal{H})$ is frequently denoted by $P \leftrightarrow Q$. The denotation $P \leftrightarrow S$ where $S \subseteq \partial_e \mathcal{E}(\mathcal{H})$ means that $P \leftrightarrow Q$ for all $Q \in S$. The compatibility of a set of sharp effects is understood in the pairwise sense, that is, $S \subseteq \partial_e \mathcal{E}(\mathcal{H})$ is compatible if $P \leftrightarrow Q$ for all $P, Q \in \partial_e \mathcal{E}(\mathcal{H})$ (in other words, $S$ is pairwise coexistent). We make use of these conventions.

It is well known that a set of sharp effects is compatible if and only if it is contained in a Boolean sublattice of $\partial_e \mathcal{E}(\mathcal{H})$. To prove an even stronger result, the concept of the compatant of a set of sharp effects is useful (cf. [1], [8, pp. 53–129]). The compatant $S^c$ of a set $S \in \partial_e \mathcal{E}(\mathcal{H})$ is the set of all sharp effects that are compatible with all effects of $S$, i.e., $S^c := \{Q \in \partial_e \mathcal{E}(\mathcal{H}) \mid Q \leftrightarrow S \}$. We notice the following simple statements:

(a) If $S_1 \subseteq S_2 \subseteq \partial_e \mathcal{E}(\mathcal{H})$, then $S_2^c \subseteq S_1^c$.
(b) For every $S \subseteq \partial_e \mathcal{E}(\mathcal{H})$, $S \subseteq S^{cc}$ where $S^{cc} := (S^c)^c$ is the bicompatant of $S$.
(c) For every $S \subseteq \partial_e \mathcal{E}(\mathcal{H})$, $S^c = S^{cc}$.
(d) A set $S$ of sharp effects is compatible if and only if $S \subseteq S^c$.
(e) If $S \subseteq \partial_e \mathcal{E}(\mathcal{H})$ is compatible, then $S^{cc}$ is compatible.
Statements (a), (b), and (d) are obvious. From (b) it follows that \( S^c \subseteq (S^c)^cc = S^{ccc} \), whereas (b) and (a) imply that \( S^{ccc} = (S^c)^c \subseteq S^c \), hence, (c) holds. If \( S \) is compatible, then, by (d), (a), and (c), \( S^{cc} \subseteq S = (S^c)^cc \). Using (d) again, we obtain from \( S^{cc} \subseteq S = (S^c)^cc \) that \( S^{cc} \) is compatible, thus proving (e). The next lemma is in particular necessary to prove some further statements about \( S^c \).

**Lemma 2.** Let \( P \in \partial_c \mathcal{E}(\mathcal{H}) \) and \( S \subseteq \partial_c \mathcal{E}(\mathcal{H}) \) nonempty. If \( P \leftrightarrow S \), then \( P \leftrightarrow \bigvee_{Q \in S} Q \) and \( P \leftrightarrow \bigwedge_{Q \in S} Q \), and the distributive laws

\[
P \land \bigvee_{Q \in S} Q = \bigvee_{Q \in S} P \land Q
\]

\[
P \lor \bigwedge_{Q \in S} Q = \bigwedge_{Q \in S} P \lor Q
\]

hold.

**Proof.** Since \( P \leftrightarrow Q \) for all \( Q \in S \), we have, by Theorem 2,

\[
\bigvee_{Q \in S} Q = \bigvee_{Q \in S} ((Q \land P^\perp) \lor (P \land Q)) = \left[ \bigvee_{Q \in S} (Q \land P^\perp) \right] \lor \left[ \bigvee_{Q \in S} (P \land Q) \right],
\]

(16)

and since \( \bigvee_{Q \in S} (P \land Q) \leq P \),

\[
P = \left[ P \land \left( \bigvee_{Q \in S} (P \land Q) \right)^\perp \right] \lor \left[ \bigvee_{Q \in S} (P \land Q) \right].
\]

(17)

Obviously, the two bracketed projection-valued expressions of (16) are orthogonal to each other, and the same is true for (17). Due to

\[
\bigvee_{Q \in S} (Q \land P^\perp) \leq P^\perp \leq P^\perp \lor \bigvee_{Q \in S} (P \land Q) = \left[ P \land \left( \bigvee_{Q \in S} (P \land Q) \right)^\perp \right]^\perp
\]

all the three different expressions inside the brackets of (16) and (17) are mutually orthogonal, so \( P \) and \( \bigvee_{Q \in S} Q \) commute. Hence, by Theorem 2, \( P \leftrightarrow \bigvee_{Q \in S} Q \). Now, from the uniqueness statement of Theorem (2) it follows that

\[
P \land \bigvee_{Q \in S} Q = \bigvee_{Q \in S} P \land Q.
\]

If \( P \leftrightarrow E, E \in \partial_c \mathcal{E}(\mathcal{H}), \) then \( P \leftrightarrow E^\perp = I - E \). Thus, \( P \leftrightarrow Q^\perp \) for all \( Q \in S \), and \( P \leftrightarrow \left( \bigvee_{Q \in S} Q^\perp \right)^\perp = \bigwedge_{Q \in S} Q \). Finally, \( P^\perp \leftrightarrow Q^\perp \) for all \( Q \in S \) so that

\[
P \lor \bigwedge_{Q \in S} Q = \left( P^\perp \land \bigvee_{Q \in S} Q^\perp \right)^\perp = \left( \bigvee_{Q \in S} P^\perp \land Q^\perp \right)^\perp = \bigwedge_{Q \in S} P \lor Q.
\]

\( \square \)
We recall that an orthocomplemented sublattice \( \mathcal{L} \) of \( \partial_e \mathcal{E}(\mathcal{H}) \) is Boolean if the distributive laws
\[
E \land (P \lor Q) = (E \land P) \lor (E \land Q) \\
E \lor (P \land Q) = (E \lor P) \land (E \lor Q),
\]
hold. As a consequence, any two \( P, Q \in \mathcal{L} \) are compatible, i.e., \( \mathcal{L} \) is a compatible set. In fact, because
\[
P = (P \land Q^\perp) \lor (P \land Q) \\
Q = (Q \land P^\perp) \lor (P \land Q)
\]
and \( (P \land Q^\perp) \), \( (Q \land P^\perp) \), and \( (P \land Q) \) are mutually orthogonal, \( P \) and \( Q \) commute, i.e., \( P \leftrightarrow Q \). If \( \mathcal{L} \) is complete, then, according to Lemma 2, the validity of the general distributive laws \( (15) \) in \( \mathcal{L} \) is implied. Thus, Eqs. \( (18) \) entails compatibility, and the completeness of \( \mathcal{L} \) and its compatibility yield Eqs. \( (15) \).

Now we can prove the following simple, but important theorem about compatants.

**Theorem 3.** Let \( S \subseteq \partial_e \mathcal{E}(\mathcal{H}) \).

(a) The compatant \( S^c \) is a complete orthocomplemented sublattice of \( \partial_e \mathcal{E}(\mathcal{H}) \). Moreover, \( S^c \) is closed in the topology \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \).

(b) If \( S \) is compatible, the bicompatant \( S^{cc} \) is a complete Boolean sublattice of \( \partial_e \mathcal{E}(\mathcal{H}) \) which, moreover, is \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \)-closed.

**Proof.** The first statement of (a) is a consequence of \( 0, I \in S^c \), \( Q \in S^c \Rightarrow Q^\perp \in S^c \), and Lemma 2. To prove the second statement of (a), recall that the topology \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \) is second countable, which implies the second countability of \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \). Consequently, if \( E \in \overline{S^c \cap \partial_e \mathcal{E}(\mathcal{H})} \), then there exists a sequence \( \{Q_n\}_{n \in \mathbb{N}} \) in \( S^c \) such that \( Q_n \xrightarrow{n \to \infty} E \) as \( n \to \infty \). Taking account of \( Q_n \leftrightarrow P, P \in S \), we obtain, since the product of operators of \( B_s(\mathcal{H}) \) is sequentially \( \sigma \)-continuous separately in each factor,
\[
PE = P\sigma \lim_{n \to \infty} Q_n = \sigma \lim_{n \to \infty} PQ_n = \sigma \lim_{n \to \infty} Q_nP = (\sigma \lim_{n \to \infty} Q_n)P = EP.
\]
Hence, \( E \in S^c \), and \( S^c = \overline{S^c \cap \partial_e \mathcal{E}(\mathcal{H})} \).

Concerning (b), we only have to prove that in the complete orthocomplemented sublattice \( S^{cc} \) the distributive laws are satisfied. We already remarked that \( S^{cc} \) is compatible if \( S \) is compatible. Thus, by Lemma 2, \( S^{cc} \) is distributive.

**Corollary.** If a set \( S \) of sharp effects is contained in a Boolean sublattice \( \mathcal{L} \) of \( \partial_e \mathcal{E}(\mathcal{H}) \), then \( S \) is compatible. Conversely, any compatible set \( S \) of sharp effects is contained in a complete Boolean sublattice of \( \partial_e \mathcal{E}(\mathcal{H}) \).

**Proof.** The first statement of the corollary is clear since \( \mathcal{L} \) is compatible as shown above. The second follows from \( S \subseteq S^{cc} \) and part (b) of the theorem since \( S \) is compatible.

In Section 2 we noticed that \( \partial_e \mathcal{E}(\mathcal{H}) \) is \( \sigma \)-dense in \( \mathcal{E}(\mathcal{H}) \). So one cannot expect that the compatant of \( S \) is \( \sigma \)-closed in \( \mathcal{E}(\mathcal{H}) \) or, equivalently, in \( B_s(\mathcal{H}) \); \( S^c \) is closed in the \( \sigma \)-topology on \( \partial_e \mathcal{E}(\mathcal{H}) \).
4. Some crucial properties of countably complete Boolean sublattices

According to Lemma 2, the bicompatant of a compatible set is a $\sigma \cap \partial_e \mathcal{E} (\mathcal{H})$-closed complete Boolean sublattice of $\partial_e \mathcal{E} (\mathcal{H})$. We are going to prove that every countably complete Boolean sublattice of $\partial_e \mathcal{E} (\mathcal{H})$ is $\sigma \cap \partial_e \mathcal{E} (\mathcal{H})$-closed; we understand a countably complete lattice to be a partially ordered set in which the greatest lower and the least upper bound of every countable subset exist. To prove the indicated property of countably complete Boolean sublattices of $\partial_e \mathcal{E} (\mathcal{H})$, we need the following two lemmata.

**Lemma 3.** Let $\{P_n\}_{n \in \mathbb{N}}$ be a sequence of mutually compatible sharp effects. Then

$$\bigvee_{n \in \mathbb{N}} P_n = \bigvee_{n \in \mathbb{N}} Q_n$$

where $\{Q_n\}_{n \in \mathbb{N}}$ is a sequence in $\partial_e \mathcal{E} (\mathcal{H})$ whose members are mutually orthogonal.

*Proof.* Define $Q_1 := P_1$ and

$$Q_n := P_n \land \left( \bigvee_{j=1}^{n-1} P_j \right)^\perp$$

for $n \geq 2$. It follows that, for $m < n$,

$$Q_n \leq \left( \bigvee_{j=1}^{n-1} P_j \right)^\perp = \bigwedge_{j=1}^{n-1} P_j^\perp \leq P_m^\perp \lor \bigvee_{j=1}^{m-1} P_j = \left( P_m \land \left( \bigvee_{j=1}^{m-1} P_j \right)^\perp \right)^\perp = Q_m^\perp;$$

therefore, the projections $Q_n$ are mutually orthogonal. By induction, we now show that

$$\bigvee_{j=1}^n P_j = \bigvee_{j=1}^n Q_j. \quad (20)$$

The case $n = 1$ is clear. Assume (20) holds for $n = k$. Then, for $n = k + 1$,

$$\bigvee_{j=1}^{k+1} Q_j = \left( \bigvee_{j=1}^k Q_j \right) \lor Q_{k+1} = \left( \bigvee_{j=1}^k P_j \right) \lor \left( P_{k+1} \land \left( \bigvee_{j=1}^k P_j \right)^\perp \right)$$

$$= \left( \bigvee_{j=1}^{k+1} P_j \right) \land \left( \left( \bigvee_{j=1}^k P_j \right) \lor \left( \bigvee_{j=1}^k P_j \right)^\perp \right)$$

$$= \bigvee_{j=1}^{k+1} P_j$$

where we used one of the distributive laws, which, by Lemma 2, is possible since $\bigvee_{j=1}^k P_j \leftrightarrow \left( \bigvee_{j=1}^k P_j \right)^\perp$ and, again by Lemma 2, $\bigvee_{j=1}^k P_j \leftrightarrow P_{k+1}$. Thus, (20) has been proved.

---

A countably complete Boolean lattice is often called a Boolean $\sigma$-algebra. The author would like to call a countably complete lattice $\sigma$-complete; but this is misleading because we could understand a $\sigma$-complete subset of $\mathcal{B}_\sigma (\mathcal{H})$ to be set that is complete w.r.t. the uniform structure induced by our $\sigma$-topology.
Eq. (19) implies \( Q_n \leq P_n \), consequently,
\[
Q_n \leq \bigvee_{n \in \mathbb{N}} P_n. \tag{21}
\]

Let \( E \in \partial_\varepsilon \mathcal{E}(\mathcal{H}) \) be any upper bound of all \( Q_n \). Then, by (20), \( E \geq \bigvee_{j=1}^n Q_j = \bigvee_{j=1}^n P_j \) holds for all \( n \in \mathbb{N} \). Hence, \( E \geq P_n \) for all \( n \in \mathbb{N} \), which entails that \( E \geq \bigvee_{n \in \mathbb{N}} P_n \). From this and (21) we conclude that \( \bigvee_{n \in \mathbb{N}} Q_n = \bigvee_{n \in \mathbb{N}} P_n \). \( \square \)

The first statement of the next lemma is of general functional analytic character.

**Lemma 4.**

(a) Let \( (\mathcal{V}, \| \cdot \|) \) be a separable Banach space and \( (\mathcal{V}', \| \cdot \|) \) its dual. Then there exists a second norm \( \| \cdot \|_\sigma \) on \( \mathcal{V}' \) such that \( \| \cdot \|_\sigma \)-topology on the \( \| \cdot \| \)-closed unit ball \( B_{\mathcal{V}'} \) coincides with the weak*- topology \( \sigma(\mathcal{V}', \mathcal{V}) \) on \( B_{\mathcal{V}'} \), that is, the topology on \( B_{\mathcal{V}'} \) induced by the norm \( \| \cdot \|_\sigma \) equals \( \sigma(\mathcal{V}', \mathcal{V}) \cap B_{\mathcal{V}'} \).

(b) In the case \( \langle \mathcal{V}, \mathcal{V}' \rangle = \langle \mathcal{H}, B_0(\mathcal{H}) \rangle \) the norm \( \| \cdot \|_\sigma \) can be chosen such that, in addition,
\[
\| A_1 \|_\sigma \leq \| A_2 \|_\sigma
\]
holds for \( A_1, A_2 \in B_0(\mathcal{H}) \) whenever \( 0 \leq A_1 \leq A_2 \).

**Proof.** Choose a countable dense subset \( D := \{ v_1, v_2, \ldots \} \) of the \( \| \cdot \| \)-closed unit ball \( B_{\mathcal{V}} \) of \( \mathcal{V} \) being \( \| \cdot \| \)-dense in \( B_{\mathcal{V}} \) and define
\[
\| \ell \|_\sigma := \sum_{j=1}^{\infty} \lambda_j |\ell(v_j)| \quad \tag{22}
\]
where \( \ell \in \mathcal{V}' \) and \( \lambda_j > 0 \), \( \sum_{j=1}^{\infty} \lambda_j = 1 \). It is easily shown that \( D \) separates \( \mathcal{V}' \) and \( \| \cdot \|_\sigma \) is a norm on \( \mathcal{V}' \). Moreover, the weak topology \( \sigma(\mathcal{V}', D) \) is Hausdorff and coincides on \( B_{\mathcal{V}'} \) with the \( \| \cdot \|_\sigma \)-topology, as we are going to show. To that end, observe that the sets
\[
U(\ell; v_1, \ldots, v_n; \varepsilon) := \{ \ell \in \mathcal{V}' \mid |\ell(v_j) - \ell(v_j)| < \varepsilon, j = 1, \ldots, n \},
\]
n \( \in \mathbb{N} \), \( \varepsilon > 0 \), constitute a neighborhood base of \( \ell \in \mathcal{V}' \) (compare to Eq. (1)).

Let \( \ell_1, \ell_2 \in \mathcal{V}' \) with \( \ell_1 \neq \ell_2 \). Since \( D \) separates \( \mathcal{V}' \), there exists a vector \( v_N \in D \) such that \( \ell_1(v_N) \neq \ell_2(v_N) \). Let \( |\ell_1(v_N) - \ell_2(v_N)| =: \varepsilon \) and consider the neighborhoods
\[
U(\ell_k; v_N; \frac{\varepsilon}{2}) := \left\{ \ell \in \mathcal{V}' \mid |\ell(v_N) - \ell_k(v_N)| < \frac{\varepsilon}{2} \right\}
\]
of \( \ell_1 \) and \( \ell_2 \), respectively, \( k = 1, 2 \). For \( \tilde{\ell} \in U(\ell_1; v_N; \frac{\varepsilon}{2}) \) we have that
\[
|\tilde{\ell}(v_N) - \ell_1(v_N)| = |(\tilde{\ell}(v_N) - \ell_1(v_N)) - (\ell_2(v_N) - \ell_1(v_N))| \geq |\tilde{\ell}(v_N) - \ell_1(v_N)| - |\ell_2(v_N) - \ell_1(v_N)| = \varepsilon - |\tilde{\ell}(v_N) - \ell_1(v_N)| > \frac{\varepsilon}{2}.
\]
In consequence, \( \tilde{\ell} \notin U(\ell_2; v_N; \frac{\varepsilon}{2}) \); so \( U(\ell_1; v_N; \frac{\varepsilon}{2}) \) and \( U(\ell_2; v_N; \frac{\varepsilon}{2}) \) are disjoint, and \( \sigma(\mathcal{V}', D) \) is Hausdorff.
Using (22), we obtain from $\|\tilde{\ell} - \ell\|_\sigma < \varepsilon$ that $|(\tilde{\ell} - \ell)(v_j)| < \varepsilon$ for all $j \in \mathbb{N}$. Therefore, $\tilde{\ell} \in U(\ell; v_1, \ldots, v_n; \varepsilon)$ for all $n \in \mathbb{N}$. Hence, on $\mathcal{V}^\prime$ and in particular on $B_{\mathcal{V}^\prime}, \sigma(\mathcal{V}^\prime, D)$ is weaker than the $\|\cdot\|_\sigma$-topology. On $B_{\mathcal{V}^\prime}$, the converse is also true. Namely, let $\ell, \tilde{\ell} \in B_{\mathcal{V}^\prime}$, $\varepsilon > 0$, $\sum_{j=n+1}^\infty \lambda_j < \frac{\varepsilon}{2}$, and $\tilde{\ell} \in U(\ell; v_1, \ldots, v_n; \frac{\varepsilon}{2})$. It follows that

$$\|\tilde{\ell} - \ell\|_\sigma = \sum_{j=1}^n \lambda_j |(\tilde{\ell} - \ell)(v_j)| + \sum_{j=n+1}^\infty \lambda_j |(\tilde{\ell} - \ell)(v_j)| < \sum_{j=1}^n \lambda_j \frac{\varepsilon}{2} + \sum_{j=n+1}^\infty \lambda_j \|\tilde{\ell} - \ell\|_\sigma < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \cdot 2 = \varepsilon,$$

in the estimation of the infinite sums $|(\tilde{\ell} - \ell)(v_j)| \leq \|\tilde{\ell} - \ell\| \leq 2$ has been used where $\|\tilde{\ell} - \ell\|_\sigma$ is the usual norm of $\tilde{\ell} - \ell$ and $\|v_j\|, \|\tilde{\ell}\|, \|\ell\| \leq 1$ has been taken into account. Thus, $\tilde{\ell} \in U(\ell; v_1, \ldots, v_n; \frac{\varepsilon}{2}) \cap B_{\mathcal{V}^\prime}$ implies $\tilde{\ell} \in \{\tilde{\ell} \in B_\mathcal{S}(\mathcal{H}) | \|\tilde{\ell} - \ell\|_\sigma < \varepsilon\} \cap B_{\mathcal{V}^\prime}$, and on $B_{\mathcal{V}^\prime}$ the $\|\cdot\|_\sigma$-topology and $\sigma(\mathcal{V}^\prime, D)$ are equal.

Now, the topology $\sigma(\mathcal{V}^\prime, D)$ is Hausdorff and weaker than $\sigma(\mathcal{V}, \mathcal{V})$, and $\sigma(\mathcal{V}, \mathcal{V}) \cap B_{\mathcal{V}^\prime}$ is compact. Hence, $\sigma(\mathcal{V}^\prime, D) \cap B_{\mathcal{V}^\prime} = \sigma(\mathcal{V}^\prime, \mathcal{V}) \cap B_{\mathcal{V}^\prime}$. Thus, on $B_{\mathcal{V}^\prime}$ the $\|\cdot\|_\sigma$-topology and $\sigma(\mathcal{V}^\prime, \mathcal{V})$ coincide.

In the case $\langle \mathcal{V}, \mathcal{V}^\prime \rangle = \langle \mathcal{T}_n(\mathcal{H}), B_\mathcal{S}(\mathcal{H}) \rangle$, Eq. (22) reads

$$\|A\|_\sigma = \sum_{j=1}^\infty \lambda_j |\text{tr} V_j A|$$

where $A \in B_\mathcal{S}(\mathcal{H})$ and the $V_j \in \mathcal{T}_n(\mathcal{H})$ form, w.r.t. the trace norm, a dense subset of the unit ball of $\mathcal{T}_n(\mathcal{H})$. We can replace the set $D = \{V_1, V_2, \ldots\}$ by a set $\tilde{D} := \{W_1, W_2, \ldots\}$ of density operators being dense in the set $\mathcal{S}(\mathcal{H})$ of all density operators. Since $\tilde{D}$ separates $B_\mathcal{S}(\mathcal{H})$, $\sigma(B_\mathcal{S}(\mathcal{H}), \tilde{D})$ is Hausdorff, and since $\|W_j\| = 1$, the $\|\cdot\|_\sigma$-topology corresponding to $\tilde{D}$ coincides on $B_{\mathcal{S}(\mathcal{H})}$ with $\sigma(B_\mathcal{S}(\mathcal{H}), \tilde{D})$. Again, the Hausdorff topology $\sigma(B_\mathcal{S}(\mathcal{H}), \tilde{D}) \cap B_{\mathcal{S}(\mathcal{H})}$ is weaker than the compact topology $\sigma(B_\mathcal{S}(\mathcal{H}), \mathcal{T}_n(\mathcal{H})) \cap B_{\mathcal{S}(\mathcal{H})}$, so they must be equal; thus, on $B_{\mathcal{S}(\mathcal{H})}$ the $\|\cdot\|_\sigma$-topology corresponding to $\tilde{D}$ and our $\sigma$-topology coincide. Finally, from

$$\|A\|_\sigma = \sum_{j=1}^\infty \lambda_j |\text{tr} W_j A|$$

and $0 \leq A_1 \leq A_2$ it follows that

$$\|A_1\|_\sigma = \sum_{j=1}^\infty \lambda_j |\text{tr} W_j A_1| \leq \sum_{j=1}^\infty \lambda_j |\text{tr} W_j A_2| = \|A_2\|_\sigma.$$  

The following important theorem and its corollary can be proved on the basis of the preceding two lemmata (cf. [2, pp. 95–96]).

**Theorem 4.** Let $\{P_j\}_{j \in \mathbb{N}}$ be a sequence of compatible sharp effects converging to $P \in \partial_\mathcal{E}(\mathcal{H})$ in the $\sigma$-topology.

(a) If $Q \in \partial_\mathcal{E}(\mathcal{H})$ is compatible with all $P_j$, then $Q \leftrightarrow P$. The limit $P$ is compatible with all $P_j$. 

12
(b) There exists a subsequence \( \{P_{jk}\}_{k \in \mathbb{N}} \) such that

\[
P = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} P_{jk}.
\]

Proof. From \( P_j Q = Q P_j \) for all \( j \in \mathbb{N} \), \( P_j \xrightarrow{\sigma} P \) as \( j \to \infty \), and the sequential \( \sigma \)-continuity of the product of operators of \( B_0(\mathcal{H}) \) separately in each factor, it follows that \( P Q = Q P \). So the first statement of (a) has been shown. Setting \( Q = P_k \), \( P_k \leftrightarrow P_j \) implies \( P_k \leftrightarrow P \).

We are now going to prove statement (b). From \( P_j \xrightarrow{\sigma} P \) as \( j \to \infty \) we obtain that

\[
P_j(I - P) \xrightarrow{\sigma} P(I - P) = 0.
\]

Using a norm according to part (b) of Lemma 4, we have that

\[
\|P_j(I - P)\|_{\sigma} \to 0
\]
as \( j \to \infty \). We can choose a subsequence \( \{P_{jk}\}_{k \in \mathbb{N}} \) of \( \{P_j\}_{j \in \mathbb{N}} \) such that

\[
\|P_{jk}(I - P)\|_{\sigma} < \frac{1}{2^k}.
\]

Observe that, for every \( n \in \mathbb{N} \), the infinite sum \( \sum_{k=n}^{\infty} P_{jk}(I - P) \) exists in the \( \sigma \)-topology because \( \sum_{k=n}^{\infty} P_{jk}(I - P) \) exists in the \( \sigma \)-topology because

\[
\sum_{k=n}^{\infty} \|P_{jk}(I - P)\|_{\sigma} < \sum_{k=n}^{\infty} \frac{1}{2^k} \leq 1.
\]

Since \( \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} P_{jk} \) is a decreasing sequence of orthogonal projections, it is \( \sigma \)-convergent and its limit is equal to \( \tilde{P} := \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} P_{jk} \in \partial_c \mathcal{E}(\mathcal{H}) \). From \( \bigvee_{k=n}^{\infty} P_{jk} \geq P_n \) and \( P_{jk} \xrightarrow{\sigma} P \) it then follows that \( \tilde{P} \geq P \); hence,

\[
\bigvee_{k=n}^{\infty} P_{jk} \geq \tilde{P} \geq P.
\]

Because the sharp effects \( P_{jk} \) are compatible, Lemma 3 implies that

\[
\bigvee_{k=n}^{\infty} P_{jk} = \bigvee_{k=n}^{\infty} Q_{jk} = \sum_{k=n}^{\infty} Q_{jk}
\]

where the \( Q_{jk} \) are mutually orthogonal. From (24) and (25) we obtain that

\[
\left( \bigvee_{k=n}^{\infty} P_{jk} \right) - P = \left( \bigvee_{k=n}^{\infty} P_{jk} \right)(I - P) = \sum_{k=n}^{\infty} Q_{jk}(I - P).
\]

Choosing \( Q_{jk} \) according to (19), we have that \( P_l \leftrightarrow Q_{jk} \) for all \( k, l \in \mathbb{N} \). Consequently, \( Q_{jk} \leftrightarrow P \) and \( P_{jk} P = PP_{jk} \); furthermore, \( P_{jk} \leftrightarrow P \) and \( P_{jk} P = PP_{jk} \). Thus, \( Q_{jk} \leq P_{jk} \) implies \( Q_{jk}(I - P) \leq P_{jk}(I - P) \). Now, from (26) it follows that

\[
\left( \bigvee_{k=n}^{\infty} P_{jk} \right) - P = \sum_{k=n}^{\infty} Q_{jk}(I - P) \leq \sum_{k=n}^{\infty} P_{jk}(I - P)
\]

where the last sum exists in the \( \sigma \)-topology, as we already remarked. Finally, from (27) and (23) we conclude that

\[
\left\| \left( \bigvee_{k=n}^{\infty} P_{jk} \right) - P \right\|_{\sigma} \leq \sum_{k=n}^{\infty} \|P_{jk}(I - P)\|_{\sigma} \leq \sum_{k=n}^{\infty} \|P_{jk}(I - P)\|_{\sigma} < \sum_{k=n}^{\infty} \frac{1}{2^k} \to 0
\]
as \( n \to \infty \). Hence, \( P = \sigma-\lim_{n \to \infty} \bigvee_{k=n}^{\infty} P_{jk} = \tilde{P} = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} P_{jk} \). \( \square \)
Corollary. Every countably complete Boolean sublattice \( \mathcal{L} \) of \( \partial_e \mathcal{E}(\mathcal{H}) \) is \( \sigma \)-closed in \( \partial_e \mathcal{E}(\mathcal{H}) \), i.e., closed in the topology \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \).

Proof. Let \( P \in \overline{\mathcal{L}^{\sigma \cap \partial_e \mathcal{E}(\mathcal{H})}} \). Then, as a consequence of the second countability of \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \), there exists a sequence \( \{P_j\}_{j \in \mathbb{N}} \) in \( \mathcal{L} \) converging to \( P \) in the \( \sigma \)-topology. Recall that \( \mathcal{L} \) is a compatible set. Then, according to the theorem, there is a subsequence \( \{P_{j_k}\}_{j \in \mathbb{N}} \) such that \( P = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} P_{j_k} \). Hence, \( P \in \mathcal{L} \), and \( \mathcal{L} \) is \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \)-closed.

For our purpose, the theorem is important because of its corollary. The latter enables a short proof of the next lemma on which the lemma afterwards is based.

**Lemma 5.** Every countably complete Boolean sublattice \( \mathcal{L} \) of \( \partial_e \mathcal{E}(\mathcal{H}) \) is countably generated, i.e., there exists a countable set \( \{P_1, P_2, \ldots\} \subseteq \mathcal{L} \) such that \( \mathcal{L} \) is the smallest countably complete Boolean sublattice of \( \partial_e \mathcal{E}(\mathcal{H}) \) containing \( \{P_1, P_2, \ldots\} \).

Proof. Since the topology \( \sigma \cap \mathcal{L} \) on \( \mathcal{L} \) is second countable as well, \( \mathcal{L} \) is separable in this topology. That is, there exists a countable set \( \{P_1, P_2, \ldots\} \subseteq \mathcal{L} \) being \( \sigma \cap \mathcal{L} \)-dense in \( \mathcal{L} \). In consequence, \( \{P_1, P_2, \ldots\} \) is also \( \sigma \cap \partial_e \mathcal{E}(\mathcal{H}) \)-dense in \( \mathcal{L} \) or, equivalently, \( \overline{\{P_1, P_2, \ldots\}}^{\sigma \cap \partial_e \mathcal{E}(\mathcal{H})} = \overline{\mathcal{L}^{\sigma \cap \partial_e \mathcal{E}(\mathcal{H})}} \).

Thus, by the corollary, \( \mathcal{L} = \overline{\{P_1, P_2, \ldots\}}^{\sigma \cap \partial_e \mathcal{E}(\mathcal{H})} \). Let \( \tilde{\mathcal{L}} \) be the smallest countably complete Boolean sublattice containing \( \{P_1, P_2, \ldots\} \). Clearly, \( \tilde{\mathcal{L}} \subseteq \mathcal{L} \). Again, by the corollary, \( \tilde{\mathcal{L}} = \overline{\mathcal{L}^{\sigma \cap \partial_e \mathcal{E}(\mathcal{H})}} \supseteq \overline{\{P_1, P_2, \ldots\}}^{\sigma \cap \partial_e \mathcal{E}(\mathcal{H})} = \mathcal{L} \). Hence, \( \mathcal{L} = \tilde{\mathcal{L}} \).

The final lemma of this section is the basis of our conclusions in the next section. The author took the idea of its proof from [10, pp. 268–270], and [18, pp. 64–65]; however, the idea obviously originates from [19, proof of Satz 10].

**Lemma 6.** Let \( \mathcal{L} \) be a countably complete Boolean sublattice of \( \partial_e \mathcal{E}(\mathcal{H}) \). Then there exists an increasing family \( \{E_\lambda\}_{\lambda \in \mathbb{R}} \) of orthogonal projections that is \( \sigma \)-continuous from the right, satisfies \( E_\lambda = 0 \) for \( \lambda < 0 \), \( E_\lambda = I \) for \( \lambda \geq I \), and generates \( \mathcal{L} \).

Proof. Choose a set \( \{P_1, P_2, \ldots\} \subseteq \mathcal{L} \) that generates \( \mathcal{L} \). The sharp effects \( P_1 \) are mutually compatible. We consider the case of an infinite set \( \{P_1, P_2, \ldots\} \); if the generating set is finite (i.e., if \( \mathcal{L} \) is finite), the following argumentation can easily be modified. By induction, we construct the following increasing sequence of increasing families of orthogonal projections:

(i) \( 0 \leq P_1 \leq I \).

(ii) \( 0 \leq P_1 \wedge P_2 \leq P_1 \leq P_1 \vee P_2 \leq I \) or, what is the same, \( 0 \leq P_1 P_2 \leq P_1 \leq P_1 + P_2(I - P_1) \leq I \).

(iii) Rename the family of (ii) as \( 0 \leq Q_1 \leq Q_2 \leq Q_3 \leq I \) and define

\[
0 \leq Q_1 \wedge P_3 \leq Q_1 \leq Q_1 \vee (Q_2 \wedge P_3) \leq Q_2 \leq Q_2 \vee (Q_3 \wedge P_3) \leq Q_3 \leq Q_3 \vee P_3 \leq I
\]

or

\[
0 \leq Q_1 P_3 \leq Q_3 \leq Q_1 + (Q_2 - Q_1)P_3 \leq Q_2 \leq Q_2 + (Q_3 - Q_2)P_3 \leq Q_3 \leq Q_3 + (I - Q_3)P_3 \leq I.
\]

(vi) In general, if the \( n \)th family is \( 0 \leq E_1 \leq E_2 \leq \ldots \leq E_{2^{n-1}} \leq I \), define

\[
0 \leq E_1 \wedge P_{n+1} \leq E_1 \leq E_1 \vee (E_2 \wedge P_{n+1}) \leq E_2 \leq E_2 \vee (E_3 \wedge P_{n+1}) \leq E_3 \leq \ldots \\
\leq E_{2^{n-1}} \leq E_{2^{n-1}} \vee P_{n+1} \leq I.
\]

4 It should be remarked that, due to the separability of the Hilbert space \( \mathcal{H} \), every countably complete Boolean lattice is already complete. We do not use this fact; it is proved in [17, pp. 303–304], where the well-ordering principle is used.
Each of these families $0, E_1, \ldots, E_{2^n-1}, I$ where $n \in \mathbb{N}$ contains the preceding one as a subfamily. The members of each such family are mutually compatible with each other and with all $P_1, P_2, \ldots$. Moreover, the Boolean lattice $L_n$ generated by the family $0, E_1, \ldots, E_{2^n-1}, I$ contains $P_1, \ldots, P_n$. This is true for $n = 1$. Assume $L_n$ contains $P_1, \ldots, P_n$. Since $L_n \subseteq L_{n+1}$, $P_1, \ldots, P_n$ also belong to $L_{n+1}$, and we only have to show that $P_{n+1} \in L_{n+1}$. From

$$ (E_1 \land P_{n+1}) \lor ((E_1 \lor (E_2 \land P_{n+1})) \lor E_1) = E_2 \land P_{n+1} $$

it follows that $E_2 \land P_{n+1} \in L_{n+1}$, and

$$ (E_2 \land P_{n+1}) \lor ((E_2 \lor (E_3 \land P_{n+1})) \lor E_2) = E_3 \land P_{n+1} $$

then implies $E_3 \land P_{n+1} \in L_{n+1}$. Continuing this way, we obtain $E_{2^n-1} \land P_{n+1} \in L_{n+1}$ and finally

$$ (E_{2^n-1} \land P_{n+1}) \lor ((E_{2^n-1} \lor P_{n+1}) \lor E_{2^n-1}) = P_{n+1}. $$

Thus, $P_{n+1} \in L_{n+1}$.

As a consequence of $P_1, \ldots, P_n \in L_n$, the Boolean sublattice $\tilde{L}_n$ generated by $P_1, \ldots, P_n$ is contained in $L_n$. By the construction of the family $0, E_1, \ldots, E_{2^n-1}, I$, we have that $0, E_1, \ldots, E_{2^n-1}, I \in \tilde{L}_n$ which entails $L_n \subseteq \tilde{L}_n$. Hence, $L_n = \tilde{L}_n$.

Let $S$ be the set of all members of all families $0, E_1, \ldots, E_{2^n-1}, I$, $n \in \mathbb{N}$. Because the projections of each family are increasing and the families themselves increase, $S$ is a countable totally ordered set of orthogonal projections. Let $L(S)$ be the Boolean sublattice generated by $S$ and $L_c(S)$ the countably complete Boolean sublattice generated by $S$. From $L(S) \supseteq L_n$ it follows that $L(S) \supseteq \bigcup_{n \in \mathbb{N}} L_n$. Since $L_n \subseteq L_{n+1}$, $\bigcup_{n \in \mathbb{N}} L_n$ itself is a Boolean lattice containing $S$, so $L(S) \subseteq \bigcup_{n \in \mathbb{N}} L_n$. Hence,

$$ L(S) = \bigcup_{n \in \mathbb{N}} L_n = \bigcup_{n \in \mathbb{N}} \tilde{L}_n. \quad (28) $$

The original countably complete Boolean sublattice $L$ is generated by $\{P_1, P_2, \ldots\}$. Consequently, $L \supseteq \tilde{L}_n$ and $L \supseteq \bigcup_{n \in \mathbb{N}} \tilde{L}_n$; further, by (28), $L \supseteq L(S)$ and in particular $L \supseteq S$. Therefore,

$$ L \supseteq L_c(S) \supseteq L(S) \supseteq \{P_1, P_2, \ldots\} \quad (29) $$

where the latter inclusion is a consequence of $\tilde{L}_n \supseteq \{P_1, \ldots, P_n\}$ and (28) again. Now, from (29) we obtain $L \supseteq L_c(S) \supseteq \{P_1, P_2, \ldots\}$ which implies

$$ L \supseteq L_c(S) \supseteq L. $$

Thus, $L = L_c(S)$.

We have proved that $S$ is a countable totally ordered set of orthogonal projections generating $L$. Now, to avoid confusion, we rename the families $0, E_1, \ldots, E_{2^n-1}, I$ as $0, Q_1, \ldots, Q_{2^n-1}, I$ and use them to construct the family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ occurring in the statement of the lemma. Consider the interval $[0, 1]$ of real numbers and define $E_\lambda = P_1$ for $\lambda \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Next define

$$ E_\lambda := \begin{cases} 
Q_1 & \text{if } \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
Q_2 & \text{if } \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
Q_3 & \text{if } \lambda \in \left[\frac{7}{9}, \frac{8}{9}\right].
\end{cases} $$

where $Q_1, Q_2, Q_3$ are the projections of step (ii) above and $Q_2 = P_1$. The considered intervals are just the open intervals that occur in the first two steps of the construction of the Cantor
set. If the values of $E \lambda$ on the open intervals $I_1, \ldots, I_{2n-1}$ occurring in the $n$-th step of the construction of the Cantor set are defined to be $Q_1, \ldots, Q_{2n-1}$, respectively, then the respective values of $E \lambda$ on the open intervals $J_1, \ldots, J_{2n+1}$ of the step $n+1$ are to be $Q_1 \land P_{n+1} \leq Q_1 \lor (Q_2 \land P_{n+1}) \leq Q_2 \lor (Q_3 \land P_{n+1}) \leq Q_3 \land \ldots \leq Q_{2n-1} \lor P_{n+1}$.

So $E \lambda$ is consistently defined for all $\lambda \in [0, 1) \setminus C$ where $C$ is the Cantor set. Defining $E_\lambda := 0$ for $\lambda < 0$ and $E_\lambda := I$ for $\lambda \geq 1$, $\{E_\lambda\}_\lambda \in [R(C) \cup \{1\}]$ is an increasing family of orthogonal projections whose members constitute the set $S$.

To define $E_\lambda$ on the Cantor set, let $\lambda \in C$, $\lambda \neq 1$. Since $[0, 1) \setminus C$ is dense in $[0, 1]$, there exist a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ in $[0, 1) \setminus C$ satisfying $\lambda < \lambda_{n+1} \leq \lambda_n$ for all $n$ and converging to $\lambda$. For every such sequence, the decreasing sequence $\{E_{\lambda_n}\}_{n \in \mathbb{N}}$ is $\sigma$-convergent. Given two different decreasing sequences in $[0, 1) \setminus C$ converging to $\lambda$ from the right, one can construct a third such sequence containing subsequences the first two ones. From this it follows that $\lim_{n \to \infty} E_{\lambda_n}$ does not depend on the chosen sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ and $E_{\lambda} := \lim_{n \to \infty} E_{\lambda_n}$ for $\lambda \in C \setminus \{0, 1\}$ is well defined.

Since $\{E_\lambda\}_\lambda$ increases on $(\mathbb{R} \setminus C) \cup \{1\}$ and $E_{\lambda} = \lim_{n \to \infty} E_{\lambda_n} = \bigwedge_{n \in \mathbb{N}} E_{\lambda_n} = \bigwedge_{\mu > \lambda} E_{\mu}$ is valid for all $\lambda \in C \setminus \{1\}$, $\{E_\lambda\}_\lambda$ is increasing on $\mathbb{R}$. It remains to show its $\sigma$-continuity from the right. This is trivially the case for $\lambda < 0$ and $\lambda \geq 1$. It is also true for $\lambda \in [0, 1) \setminus C$ because then $\lambda$ belongs to one of the open boundary intervals of $C$ and $E_\lambda$ is constant on such an interval. We still have to consider $\lambda \in C \setminus \{1\}$.

Let $\lambda \in C \setminus \{1\}$ and $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence used in the definition of $E_\lambda$, i.e., $\lambda_n \in [0, 1) \setminus C$, $\lambda < \lambda_n \leq \lambda_{n+1}$ for all $n \in \mathbb{N}$ and $\lambda_n \to \lambda$ as $n \to \infty$. We then have $E_\lambda = \lim_{n \to \infty} E_{\lambda_n}$ and, for all states $W \in \mathcal{S}(\mathcal{H})$, $\text{tr} WE_{\lambda_n} \to \text{tr} WE_\lambda$. Now consider an arbitrary sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converging to $\lambda$ from the right, i.e., $\mu_n < \lambda_n$, and $\mu_n \to \lambda$ as $n \to \infty$. Given $W \in \mathcal{S}(\mathcal{H})$ and $\varepsilon > 0$, choose $n_0$ such that $\text{tr} WE_{\lambda_n} \leq \text{tr} WE_{\lambda_{n_0}} < \text{tr} WE_\lambda + \varepsilon$ and $N \in \mathbb{N}$ such that $\lambda < \mu_n < \lambda_{n_0}$ for all $n \geq N$. It follows that $E_{\lambda_n} \leq E_{\mu_n} \leq E_{\lambda_{n_0}}$ and

$$\text{tr} WE_{\lambda_n} \leq \text{tr} WE_{\mu_n} \leq \text{tr} WE_{\lambda_{n_0}} < \text{tr} WE_\lambda + \varepsilon$$

hold for all $n \geq N$. Hence, $\text{tr} WE_{\mu_n} \to \text{tr} WE_\lambda$ for all $W \in \mathcal{S}(\mathcal{H})$ as $n \to \infty$, and in consequence $E_\lambda = E_{\lim_{n \to \infty} E_{\mu_n}}$. Thus, for all $\lambda \in C \setminus \{1\}$, $\{E_\lambda\}_\lambda$ is $\sigma$-continuous from the right.

Summarizing, $\{E_\lambda\}_\lambda$ is increasing, $\sigma$-continuous from the right, and equal to 0 and $I$ for $\lambda < 0$ and $\lambda \geq 1$, respectively. Moreover, since $\{E_\lambda | \lambda \in (\mathbb{R} \setminus C) \cup \{1\}\} = S$, $\{E_\lambda\}_\lambda$ generates $\mathcal{L}$.

5. Sharp observables and countably complete Boolean sublattices

An increasing family $\{E_\lambda\}_\lambda$ of orthonormal projections being $\sigma$-continuous from the right and fulfilling $\lim_{\lambda \to \infty} E_\lambda = 0$ and $\lim_{\lambda \to \infty} E_\lambda = I$ is usually called a spectral family. So the family of Lemma 6 is a particular spectral family. Given any spectral family $\{E_\lambda\}_\lambda$, it is well known that there is a unique spectral measure $E$ on the Borel sets $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ (in physical terms, a sharp observable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying

$$E(|\lambda, \mu|) = E_\mu - E_\lambda$$

for all $\lambda, \mu \in \mathbb{R}$, $\lambda < \mu$. To present a self-contained treatise, we prove this statement.

Let $\{E_\lambda\}_\lambda$ be a spectral family and $\varphi \in \mathcal{H}$. The function $\lambda \mapsto <\varphi|E_\lambda \varphi>$ is, except for normalization, a cumulative distribution function, and there is exactly one (finite) measure $m_\varphi$ on $\mathcal{B}(\mathbb{R})$ such that

$$m_\varphi(|\lambda, \mu|) = <\varphi|E_\mu \varphi> - <\varphi|E_\lambda \varphi> = <\varphi|E(|\lambda, \mu|)\varphi>$$

(31)
where $E(\lambda, \mu) := E_\mu - E_\lambda$, $\lambda < \mu$ (see, e.g., [20]). Defining the complex measures

$$m_{\varphi\psi} := \frac{1}{4} \sum_{j=0}^{3} i^j m_{\psi+i^j\varphi}$$

we obtain, by polarization,

$$m_{\varphi\psi}(\lambda, \mu) = \langle \varphi | E(\lambda, \mu) | \psi \rangle.$$  

It follows that

$$m_{\varphi,\psi+\psi}(\lambda, \mu) = m_{\varphi\psi_1}(\lambda, \mu) + m_{\varphi,\psi_2}(\lambda, \mu).$$

Since the measures $m_{\varphi,\psi+\psi}$ and $m_{\varphi\psi_1} + m_{\varphi,\psi_2}$ coincide on all half-open intervals, they must be equal. Therefore, adding some analogous arguments, $(\varphi, \psi) \mapsto m_{\varphi\psi}$ is linear in $\psi$ and conjugate linear in $\varphi$.

By means of the limit properties of $\{ E_\lambda \}_{\lambda \in \mathbb{R}}$ for $\lambda \to \pm \infty$ and the continuity properties of a measure, Eq. (31) implies that $m_{\varphi}(\mathbb{R}) = \| \varphi \|^2$. Thus, it follows from (32) that, for any Borel set $B \in \mathcal{B}(\mathbb{R})$,

$$|m_{\varphi\psi}(B)| \leq \frac{1}{4} \sum_{j=0}^{3} m_{\psi+i^j\varphi}(B) \leq \frac{1}{4} \sum_{j=0}^{3} m_{\psi+i^j\varphi}(\mathbb{R}) = \frac{1}{4} \sum_{j=0}^{3} \| \psi + i^j \varphi \|^2 \leq (\| \varphi \| + \| \psi \|)^2.$$

For unit vectors $\varphi, \psi \in \mathcal{H}$, we have $|m_{\varphi\psi}(B)| \leq 4$; by linearity, this entails that $|m_{\varphi\psi}(B)| \leq 4\| \varphi \||\psi\|$ for arbitrary vectors $\varphi, \psi \in \mathcal{H}$. Hence, for every $B$, $(\varphi, \psi) \mapsto m_{\varphi\psi}$ is a bounded sesquilinear functional and there exists a bounded linear operator $E(B)$ such that

$$m_{\varphi\psi}(B) = \langle \varphi | E(B) | \psi \rangle.$$  

Since $0 \leq \langle \varphi | E(B) | \varphi \rangle = m_{\varphi}(B) \leq m_{\varphi}(\mathbb{R}) = \| \varphi \|^2$ we obtain that $E(B) = E(B)^*$ and $0 \leq E(B) \leq I$, i.e., $E(B) \in \mathcal{E}(\mathcal{H})$.

We still have to show that, for all Borel sets $B \in \mathcal{B}(\mathbb{R})$, $E(B)$ is a projection. To do this, we use a representation of the measure $m_{\varphi}$ that is related to the construction of $m_{\varphi}$ by means of the cumulative distribution function $\lambda \mapsto \langle \varphi | E_{\lambda}\varphi \rangle$. In fact, we have, for all $B \in \mathcal{B}(\mathbb{R})$,

$$m_{\varphi}(B) = m_{\varphi}^*(B) = \inf_{j=1}^{\infty} \left( \langle \varphi | E_{\lambda_j}\varphi \rangle - \langle \varphi | E_{\lambda_j}\varphi \rangle \right) = \inf_{j=1}^{\infty} \left( \langle \varphi | E(\lambda_j, \mu_j) \varphi \rangle \right),$$

where $m_{\varphi}^*$ is the outer measure occurring in the construction of $m_{\varphi}$ and the infimum is taken over all sequences of half-open intervals $[\lambda_j, \mu_j]$ with $B \subseteq \bigcup_{j=1}^{\infty} [\lambda_j, \mu_j]$. Defining $D_1 := \lambda_1, \mu_1$ and $D_j := [\lambda_j, \mu_j] \setminus \bigcup_{k=1}^{j-1} [\lambda_k, \mu_k]$ for $j \geq 2$, we observe that $\bigcup_{j=1}^{\infty} [\lambda_j, \mu_j] = \bigcup_{j=1}^{\infty} D_j$ and $D_j \cap D_k = \emptyset$ for $j \neq k$ (note the analogy with Lemma 3 and Eq. (19)). Further, each $D_j$ is the union of finitely many disjoint half-open intervals. Hence, there is a sequence of pairwise disjoint intervals $[a_k, b_k]$ so that $B \subseteq \bigcup_{j=1}^{\infty} [\lambda_j, \mu_j] = \bigcup_{k=1}^{\infty} a_k, b_k$ and $\sum_{k=1}^{\infty} (\langle \varphi | E_{b_k}\varphi \rangle - \langle \varphi | E_{a_k}\varphi \rangle) \leq \sum_{j=1}^{\infty} (\langle \varphi | E_{\lambda_j}\varphi \rangle - \langle \varphi | E_{\lambda_j}\varphi \rangle)$. In consequence, it suffices to take the infimum in (34) over all sequences of disjoint intervals $[a_k, b_k]$ with $B \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k]$.

For the disjoint pairwise intervals $[a_k, b_k]$ the projections $E([a_k, b_k])$ are mutually orthogonal, and we obtain from (33) and (34) that

$$\langle \varphi | E(B) | \varphi \rangle = \inf_{j=1}^{\infty} \langle \varphi | E([a_k, b_k]) | \varphi \rangle = \inf_{j=1}^{\infty} \left( \langle \varphi \left( \sum_{j=1}^{\infty} E([a_k, b_k]) \right) \varphi \right),$$

where $E([a_k, b_k]) := E_{b_k} - E_{a_k}$.
where the latter sum converges in the σ-topology, for instance. Let \( S \) be the set of all orthogonal projections \( \sum_{j=1}^{\infty} E([a_k, b_k]) \) occurring in (35). So we can write
\[
\langle \varphi | E(B) | \varphi \rangle = \inf_{P \in S} \langle \varphi | P | \varphi \rangle. \tag{36}
\]

It follows that \( E(B) \leq P \) for all \( P \in S \) and thus \( E(B) \leq \inf_{P \in S} P \). But note that this infimum must be taken in \( \mathcal{E}(\mathcal{H}) \), provided it exists. Fortunately, the infimum of a subset of \( \partial \varepsilon \mathcal{E}(\mathcal{H}) \) taken in \( \mathcal{E}(\mathcal{H}) \) is the same as the infimum taken in \( \partial \varepsilon \mathcal{E}(\mathcal{H}) \), as we shall prove below. So \( \inf_{P \in S} P = \bigwedge_{P \in S} P \) exists and belongs to \( \partial \varepsilon \mathcal{E}(\mathcal{H}) \). Now, from \( \bigwedge_{P \in S} P \leq P \) for all \( P \in S \) and (36) we conclude that
\[
\left\langle \varphi \left| \left( \bigwedge_{P \in S} P \right) \right| \varphi \right\rangle \leq \inf_{P \in S} \langle \varphi | P | \varphi \rangle = \langle \varphi | E(B) | \varphi \rangle.
\]

Hence, \( \bigwedge_{P \in S} P \leq E(B) \). Thus, \( E(B) = \bigwedge_{P \in S} P \), and \( E(B) \) is an orthogonal projection.

Let \( S \subseteq \partial \varepsilon \mathcal{E}(\mathcal{H}) \) and \( \bigwedge_{P \in S} P \) be the greatest lower bound of \( S \) taken in \( \partial \varepsilon \mathcal{E}(\mathcal{H}) \). It could be that the infimum of \( S \) taken in \( \mathcal{E}(\mathcal{H}) \) does not exist or is greater than \( \bigwedge_{P \in S} P \). However, we now show that, for \( A \in \mathcal{E}(\mathcal{H}) \), \( A \leq P \) for all \( P \in S \) implies \( A \leq \bigwedge_{P \in S} P \). Then \( \bigwedge_{P \in S} P \) is also the greatest lower bound of \( S \) taken in \( \mathcal{E}(\mathcal{H}) \).

For \( E := \bigwedge_{P \in S} P \) we have that \( R(E) = \bigcap_{P \in S} R(P) \) and
\[
R(E)^\perp = \left( \bigcap_{P \in S} R(P) \right)^\perp = \bigvee_{P \in S} R(P)^\perp = \text{span} \bigcup_{P \in S} R(P)^\perp, \tag{37}
\]
where \( R(E) \) and \( R(P) \) denote the ranges of \( E \) and \( P \), respectively, and \( \bigvee_{P \in S} R(P)^\perp \) is the smallest closed subspace of \( \mathcal{H} \) containing all the subspaces \( R(P)^\perp \). Let \( A \in \mathcal{E}(\mathcal{H}) \) and suppose \( A \leq P \) for all \( P \in S \). Considering any \( P \in S \), it follows that, for \( \varphi \in R(E) \subseteq R(P) \),
\[
\langle \varphi | A \varphi \rangle \leq \langle \varphi | P \varphi \rangle = \langle \varphi | \varphi \rangle = \langle \varphi | E \varphi \rangle \tag{38}
\]
and, for \( \chi \in R(P)^\perp \),
\[
\| A^{1/2} \chi \|^2 = \langle \chi | A \chi \rangle = \langle \chi | P \chi \rangle = 0.
\]

Thus, \( A \chi = 0 \) for \( \chi \) belonging to any subspace \( R(P)^\perp \), and, by (37), \( A \chi = 0 \) for \( \chi \in R(E)^\perp \). Hence, decomposing \( \psi \in \mathcal{H} \) according to \( \psi = \varphi + \chi \), \( \varphi \in R(E) \), \( \chi \in R(E)^\perp \), and taking account of (38), we obtain that
\[
\langle \psi | A \psi \rangle = \langle \psi | A \varphi \rangle = \langle \varphi | A \varphi \rangle \leq \langle \varphi | E \varphi \rangle = \langle \psi | E \psi \rangle
\]
for every \( \varphi \in \mathcal{H} \).

Summarizing, we have constructed a map \( E : B(\mathbb{R}) \to \partial \varepsilon \mathcal{E}(\mathcal{H}) \), \( B \mapsto E(B) \), such that \( E([\lambda, \mu]) = E_\mu - E_\lambda \), \( \lambda < \mu \), and for every \( \varphi \in \mathcal{H} \)
\[
B \mapsto \langle \varphi | E(B) | \varphi \rangle
\]
is a measure, namely, \( m_\varphi \). From \( m_\varphi(\emptyset) = 0 \) and \( m_\varphi(\mathbb{R}) = \| \varphi \|^2 \) it follows that \( E(\emptyset) = 0 \) and \( E(\mathbb{R}) = I \), and the σ-additivity of \( m_\varphi \) implies that \( E(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} E(B_j) \) where \( B_j \cap B_k = \emptyset \) for \( j \neq k \) and the sum converges in the σ-topology, for instance. That is, \( E \) is a spectral measure satisfying (30). Two spectral measures satisfying (30) for a given spectral family must be the same since every measure \( m_\varphi \) is uniquely determined by its values on the half-open intervals and all the \( m_\varphi \) determine \( E \) uniquely according to \( m_\varphi(B) = \langle \varphi | E(B) | \varphi \rangle \).

Finally, \( E(B_1 \cup B_2) = E(B_1) + E(B_2) \) for \( B_1 \cap B_2 = \emptyset \) entails \( E(B_1)E(B_2) = 0 \) because all \( E(B) \) are orthogonal projections.

The following lemma states important properties of any sharp observable.
Lemma 7. Let $E$ be a sharp observable on any measurable space $(M, \Xi)$.

(a) The range $R(E) = E(\Xi) = \{E(B) | B \in \Xi\}$ is a countably complete Boolean sublattice of $\partial_e \mathcal{E}(\mathcal{H})$. Moreover, $E$ satisfies (i) $E(\emptyset) = 0$, $E(M) = I$, (ii) $E(CB) = E(B)^\perp$ where $CB$ denotes the complement of $B$, and (iii) $E(\bigcup_{j=1}^{\infty} B_j) = \bigvee_{j=1}^{\infty} E(B_j)$ and $E(\bigcap_{j=1}^{\infty} B_j) = \bigwedge_{j=1}^{\infty} E(B_j)$ where $\{B_j\}_{j \in \mathbb{N}}$ is any sequence of measurable sets. That is, $E : \mathcal{B}(\mathbb{R}) \to \partial_e \mathcal{E}(\mathcal{H})$ is a $\sigma$-homomorphism.

(b) If the $\sigma$-algebra $\Xi$ is generated by $\Gamma \subseteq \Xi$, then $R(E)$ is generated by $E(\Gamma) = \{E(B) | B \in \Gamma\}$.

Proof. Clearly, $E(\emptyset) = 0 \in R(E)$ and $E(M) = I \in R(E)$. By the additivity of $E$, we have the orthogonal decompositions

$$
E(B_1) = E(B_1 \setminus B_2) + E(B_1 \cap B_2)
$$

$$
E(B_2) = E(B_2 \setminus B_1) + E(B_1 \cap B_2),
$$

where $B_1, B_2 \in \Xi$. So any two sharp effects of $R(E)$ are compatible and

$$
E(B_1)E(B_2) = E(B_1 \cap B_2) = E(B_2)E(B_1).
$$

This implies that $E(B_1) \wedge E(B_2) = E(B_1 \cap B_2) \in R(E)$ and further, by induction,

$$
\bigwedge_{j=1}^{n} E(B_j) = E\left(\bigcap_{j=1}^{n} B_j\right) \in R(E),
$$

where $B_1, \ldots, B_n \in \Xi$. To show that, for a sequence of sets $B_j \in \Xi$, the equality

$$
\bigwedge_{j=1}^{\infty} E(B_j) = E\left(\bigcap_{j=1}^{\infty} B_j\right) \in R(E)
$$

holds, observe that $E(\bigcup_{j=1}^{\infty} B_j) \leq E(B_j)$ is valid for every $B_j$, and let $P \in \partial_e \mathcal{E}(\mathcal{H})$ with $P \leq E(B_j)$ for all $B_j$. Then, by (40), $P \leq \bigwedge_{j=1}^{n} E(B_j) = E\left(\bigcap_{j=1}^{n} B_j\right)$. Using one of the continuity properties of the measure $B \mapsto \langle \varphi | E(B) \varphi \rangle$, $\varphi \in \mathcal{H}$, we obtain that

$$
\langle \varphi | P \varphi \rangle \leq \left\langle \varphi \left| \, E\left(\bigcap_{j=1}^{n} B_j\right) \, \varphi \right. \right\rangle \rightarrow \left\langle \varphi \left| \, E\left(\bigcap_{j=1}^{\infty} B_j\right) \, \varphi \right. \right\rangle
$$

as $n \to \infty$. Therefore, $P \leq E(\bigcap_{j=1}^{\infty} B_j)$ and thus $\bigwedge_{j=1}^{\infty} E(B_j) = E(\bigcap_{j=1}^{\infty} B_j)$.

From (39) it follows that

$$
E(B_1) \lor E(B_2) = \left( E(B_1 \setminus B_2) \lor E(B_1 \cap B_2) \right) \lor \left( E(B_2 \setminus B_1) \lor E(B_1 \cap B_2) \right)
$$

$$
= E(B_1 \setminus B_2) \lor E(B_2 \setminus B_1) \lor E(B_1 \cap B_2)
$$

$$
= E(B_1 \setminus B_2) + E(B_2 \setminus B_1) + E(B_1 \cap B_2)
$$

$$
= E(B_1 \cup B_2)
$$

and further

$$
\bigvee_{j=1}^{n} E(B_j) = E\left(\bigcup_{j=1}^{n} B_j\right) \in R(E).
$$
Since $E(CB) = I - E(B)$ where $B \in \Xi$, 
\[ E(B)^\perp = E(CB) \in R(E). \]  
(43)

According to (40), (42), and (43), $R(E)$ is an orthocomplemented sublattice of $\partial_e \mathcal{E}(\mathcal{H})$, and because of (41) this sublattice is countably complete. Namely, 
\[ \bigvee_{j=1}^{\infty} E(B_j) = \left( \bigwedge_{j=1}^{\infty} E(B_j)^\perp \right) = \left( E \left( \bigcap_{j=1}^{\infty} CB_j \right) \right)^\perp = E \left( \bigcup_{j=1}^{\infty} B_j \right), \] 
so 
\[ \bigvee_{j=1}^{\infty} E(B_j) \in R(E) \] 
also holds.

Finally, according to Lemma 2 the sublattice $R(E)$ is Boolean because its elements are mutually compatible.

To prove part (b) of the lemma, let $\mathcal{L}(E(\Gamma))$ be the countably complete Boolean sublattice of $\partial_e \mathcal{E}(\mathcal{H})$ generated by $E(\Gamma)$. Since $E(\Gamma) \subseteq E(\Xi) = R(E)$ and $R(E)$ is a countably complete sublattice, we have that 
\[ \mathcal{L}(E(\Gamma)) \subseteq R(E). \]  
(44)

To show the converse, let $\mathcal{L}$ be any countably complete Boolean sublattice contained in $R(E)$. Then $E^{-1}(\mathcal{L}) = \{ B \in \Xi \mid E(B) \in \mathcal{L} \}$ is a $\sigma$-subalgebra of $\Xi$. If $B \in \Gamma$, then $E(B) \in \mathcal{L}(E(\Gamma))$ and so $B \in E^{-1}(\mathcal{L}(E(\Gamma)))$. Hence, 
\[ \Gamma \subseteq E^{-1}(\mathcal{L}(E(\Gamma))) \subseteq \Xi. \] 
In virtue of the preceding remark, $E^{-1}(\mathcal{L}(E(\Gamma)))$ is a $\sigma$-subalgebra of $\Xi$. Since $\Gamma$ generates $\Xi$, we obtain $\Xi = E^{-1}(\mathcal{L}(E(\Gamma)))$. Consequently, 
\[ R(E) = E(\Xi) = E[E^{-1}(\mathcal{L}(E(\Gamma)))] \subseteq \mathcal{L}(E(\Gamma)) \] 
and thus, by (44), $R(E) = \mathcal{L}(E(\Gamma))$. \hfill \square

After these extensive preparations, we are now able to prove the following main theorem which asserts the converse of the first statement of part (a) of the preceding lemma.

**Theorem 5.** Let $\mathcal{L}$ be a countably complete Boolean sublattice of $\partial_e \mathcal{E}(\mathcal{H})$. Then there exists a sharp observable $E$ on the Borel space $(\mathbb{R}, B(\mathbb{R}))$ (in other words, a spectral measure) such that $\mathcal{L} = R(E)$.

**Proof.** According to Lemma 6, there exists a spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ that generates $\mathcal{L}$. The family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines uniquely a spectral measure $E : B(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$ satisfying $E(\lambda, \mu) = E_\mu - E_\lambda$. One of the continuity properties of the measures $B \mapsto E_\lambda(B)$, $\varphi \in \mathcal{H}$, and $\sigma\text{-}\lim_{\lambda \rightarrow -\infty} E_\lambda = 0$ imply that $\langle \varphi | E(\lambda, \mu) \varphi \rangle = \langle \varphi | E_\mu \varphi \rangle$ which entails $E(\lambda, \mu) = E_\lambda$ for all $\lambda \in \mathbb{R}$.

The $\sigma$-algebra $B(\mathbb{R})$ is generated by the half-open intervals $\lambda, \mu$; since $\lambda, \mu = (-\infty, \mu] \setminus (-\infty, \lambda]$ and every interval $]-\infty, \lambda]$ can be represented as a countable union of intervals of the form $\lambda, \mu$, $B(\mathbb{R})$ is generated by the intervals $]-\infty, \lambda]$, $\lambda \in \mathbb{R}$, as well. Then, by part (b) of Lemma 7, $R(E)$ is generated by 
\[ \{ E(]-\infty, \lambda]) \mid \lambda \in \mathbb{R} \} = \{ E_\lambda \mid \lambda \in \mathbb{R} \}. \] 
Hence, $R(E) = \mathcal{L}$. \hfill \square

20
Corollary. Any set of mutually compatible (i.e., pairwise coexistent) sharp effects is contained in the range of a sharp observable on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) and is therefore coexistent.

Proof. An arbitrary set \(S \subseteq \partial_e \mathcal{E}(\mathcal{H})\) is contained in its bicompatant \(S^{cc}\). If \(S\) is compatible, then \(S^{cc}\) is compatible also. According to Theorem 3, \(S^{cc}\) is a countably complete Boolean sublattice of \(\partial_e \mathcal{E}(\mathcal{H})\). Taking account of the last theorem, we obtain that

\[
S \subseteq S^{cc} = R(E)
\]

where \(E\) is some sharp observable on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\).

Within the framework of Hilbert space and bounded linear operators acting in Hilbert space, we proved that, for sharp effects, the concepts of pairwise coexistence and coexistence are equivalent.

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