Neutrino Mass, Dark Energy, and the Linear Growth Factor

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(Dated: March 28, 2008)

We study the degeneracies between neutrino mass and dark energy as they manifest themselves in cosmological observations. In contradiction to a popular formula in the literature, the suppression of the matter power spectrum caused by massive neutrinos is not just a function of the ratio of neutrino to total mass densities $f_\nu = \Omega_\nu/\Omega_m$, but also each of the densities independently. We also present a fitting formula for the logarithmic growth factor of perturbations in a flat universe, $f(z; k; f_\nu, w, \Omega_{DE}) \approx [1 - A(k)\Omega_{DE}f_\nu + B(k)f_\nu^2 - C(k)f_\nu^3]\Omega_m(z)$, where $\alpha$ depends on the dark energy equation of state parameter $w$. We then discuss cosmological probes where the $f$ factor directly appears: peculiar velocities, redshift distortion and the Integrated Sachs-Wolfe effect. We also modify the approximation of Eisenstein & Hu (1999) for the power spectrum of fluctuations in the presence of massive neutrinos and provide a revised code\textsuperscript{1}.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
            & \textsuperscript{\nu} mass & dark energy \\
\hline
Geometry   & $\times$                  & $\checkmark$       \\
Linear Power Spectrum $P(k, z = 0)$ & $\checkmark$    & $\times$                  \\
Linear Growth Function $\delta(z)$ & $\checkmark$ & $\checkmark$ \\
\hline
\end{tabular}
\caption{The ability of observational tests based on geometry, matter power spectrum and linear growth to probe neutrino mass and dark energy. In the text we discuss the sensitivity of $P(k, z)$ to neutrino mass.}
\end{table}

I. INTRODUCTION

The latest results from the WMAP satellite\textsuperscript{1} confirm the success of the $\Lambda$CDM model, where $\sim 75\%$ of the mass-energy density is in the form of dark energy, and matter, most of it in the form of Cold Dark Matter (CDM) making up the remaining 25\%. Neutrinos with masses on the eV scale or below will be a hot component of the dark matter and will free-stream out of overdensities and thus wipe out small-scale structures. This fact makes it possible to use observations of the clustering of matter in the universe to put upper bounds on the neutrino masses. A thorough review of the subject is found in \cite{3}.

Present cosmological neutrino mass limits make use of the suppression effect of the neutrino free-streaming at a fixed, given redshift. As our ability to map out the mass distribution at different epochs of the cosmic history improves, by doing, e.g., weak lensing tomography, we will gain sensitivity by in addition using the effect of massive neutrinos on the growth rate of density fluctuations. One key issue which then arises is possible degeneracies between neutrino masses and cosmological parameters. In this paper we focus on the degeneracy between the dark energy equation of state and the neutrino masses. We show that the combined effect of neutrinos and dark energy can be parametrized in a simple manner, and that the degeneracy can be broken by mapping out the large-scale structure over a reasonably wide range of redshifts. We also explore the effect of massive neutrinos on the matter power spectrum $P(k)$, and we modify the formula given by Eisenstein & Hu \cite{4}, such that it would be valid for realistic properties of massive neutrinos. The dependencies of three important ingredients of the universe on the equation of state parameter $w$ and the neutrino density $\Omega_\nu$ are shown in Table \textsuperscript{1}.

The outline of the paper is as follows. In Section II we contrast common approximations to the power spectrum of fluctuations with the exact results from CAMB, and we provide a new modified approximation. In Section III we provide a new fitting formula for the linear theory growth of perturbations in the presence of massive neutrinos. In section IV we discuss the parameter degeneracy between dark energy parameters and neutrino masses, and we illustrate how it manifests itself in peculiar velocities and the Integrated Sachs-Wolfe effect. Our conclusions are summarized in Section V.

II. MASSIVE NEUTRINOS AND THE MATTER POWER SPECTRUM

Most cosmological neutrino mass limits make use of the matter power spectrum $P(k)$ in some guise, although it is also possible to obtain a limit from cosmic microwave background data alone, see \cite{5, 6, 7}.

\textsuperscript{1}http://www.star.ucl.ac.uk/~lahav/nu_matter_power.f
A useful way to consider the effect of neutrino mass on the power spectrum is to consider the following quantity at a given redshift:

\[
\frac{\Delta P(k)}{P(k)} = \frac{P(k; f_\nu) - P(k; f_\nu = 0)}{P(k; f_\nu = 0)}.
\]

(1)

A common heuristic explanation for the role of the matter power spectrum in deriving neutrino mass limits is the approximate expression \(\frac{\Delta P(k)}{P(k)} \approx -8f_\nu\), describing the suppression of small-scale power caused by neutrino free-streaming, a result valid for \(f_\nu \ll 1\), and first given in [5]. See [2, 9] for derivations. In Figure (1), we compare this approximation to the matter power spectrum using CAMB [10], but also to the Eisenstein & Hu approximation [4], to our modification to this E&H approximation and to the numerical solution given by equation (6), for \(f_\nu = \frac{\Omega_\nu}{\Omega_m} = 0.04\) and \(f_\nu = 0.16\). An approximation for \(P(k, z)\) is given in [2]

\[
P(k, z) = \begin{cases}
\left(\frac{g(z)}{1 + zg(0)}\right)^2 P(k, 0) & \text{for } aH < k < k_{nr}, \\
\left(\frac{g(z)}{1 + zg(0)}\right)^{2p} P(k, 0) & \text{for } k \gg k_{nr},
\end{cases}
\]

(2)

where \(p \approx 1 - 3/5 f_\nu\) based on equation (7) below. We give a better approximation to this factor in Section III B.

The non-relativistic scale, \(k_{nr}\), is given by

\[
k_{nr} \approx 0.018\Omega_m^{1/2} \left(\frac{m}{1\,\text{eV}}\right)^{1/2} h\,\text{Mpc}^{-1}.
\]

(3)

For another approximation valid for Mixed Dark Matter see [12].

As a rule of thumb, the present-epoch matter power spectrum is considered to be in the linear regime for comoving wavenumbers \(k < 0.10 - 0.15 h\,\text{Mpc}^{-1}\), and we see from Figure (1) that \(\Delta P/P\) obtained from CAMB tends to a constant only for \(k > 1.5\,\text{Mpc}^{-1}\), which is well into the non-linear regime of structure formation. This fact is also evident from Figures (12) and (13) in [2]. Thus, \(\Delta P \approx -8f_\nu\) should only be used as a heuristic to the effect of massive neutrinos on the power spectrum since it’s not valid at the length scales where the matter power spectrum can be said to be in the linear regime. Furthermore, we note that, as expected, it works well only for very small \(f_\nu\) whereas for large neutrino masses this approximation breaks down. Moreover, Figure (1) also shows that for small neutrino masses the E&H approximation breaks down. However, by modifying the master transfer function used by E&H, we managed to minimise the error between CAMB and the E&H approximation for \(f_\nu = 0.04\), from 20% to only 3% for \(0.02 < k < 0.15 h\,\text{Mpc}^{-1}\). This modification works much better for small neutrino masses than its predecessor used to, as well as for very massive neutrinos. For \(0.15 \leq \Omega_m \leq 0.8, \Omega_b/\Omega_m \leq 0.3, f_\nu \leq 0.3, z = 0\) and \(N_\nu = 3\) the accuracy of the fitting formula is quite high. Note that the formula works equally well for \(\Omega_\nu = 0\). Our revised code can be downloaded from [http://zuserver2.star.ucl.ac.uk/~lahav/matter_power.f](http://zuserver2.star.ucl.ac.uk/~lahav/matter_power.f).

We also explore the effect of the dark energy equation of state \((w)\) on the matter power spectrum. For \(0.15 \leq \Omega_m \leq 0.8, \Omega_b/\Omega_m \leq 0.3, f_\nu \leq 0.3, z = 0\) and \(N_\nu = 3\) the accuracy of the fitting formula is quite high. Note that the formula works equally well for \(\Omega_\nu = 0\). Our revised code can be downloaded from [http://zuserver2.star.ucl.ac.uk/~lahav/matter_power.f](http://zuserver2.star.ucl.ac.uk/~lahav/matter_power.f).

Accuracy of CAMB is set to 0.3 %.
spectrum is controlled by the fractional contribution of neutrinos to the total mass density in the Universe, i.e. $f_\nu = \frac{\Omega_\nu}{\Omega_m}$. The scale where the suppression of power from neutrino free-streaming sets in is controlled by the comoving Hubble radius at the time when the neutrinos became non-relativistic, corresponding to a comoving wavenumber

$$k_{fs} = 0.10\Omega_m h \sqrt{f_\nu}. \quad (4)$$

Note the obvious degeneracy with $\Omega_m h$. This parameter, in models with negligible baryon density, sets the scale of the Hubble radius at matter-radiation equality, and so it determines the scale at which the matter power spectrum bends over in the case of massless neutrinos. For realistic baryon densities, there is an additional dependence on $\Omega_\nu$. There is a statement, sometimes found in the literature, that the power spectrum depends only on $f_\nu$ and not the independent values of the neutrino and matter densities. This is only true in the case where $\Omega_m h$ and $\Omega_\nu$ are held fixed as $f_\nu$ varies, and only for $k > 0.8 \ h \ Mpc^{-1}$, since the neutrino free-streaming also enters this picture. However, this can be altered by normalising $k$ by $k_{fs}$ (free-streaming scale). As shown in Figure 2, $\frac{\Delta P}{P}$ is a function of the ratio $f_\nu$ and $k_{fs}$. Moreover, we see that $(\Delta P/P)/f_\nu$ tends to -8 only for small neutrino masses whereas it goes to -4.5 for $f_\nu = 0.2$.

**FIG. 2:** The dependence of $\Delta P/P$ at $z = 0$ on $\Omega_m$ and $\Omega_\nu$, illustrating, via the scaling by $k_{fs}$, equation (4), that just the ratio $f_\nu = \frac{\Omega_\nu}{\Omega_m}$ is insufficient to fully parametrize $\Delta P/P$. From top to bottom: $f_\nu = 0.2$ (dashed line), 2 models with $f_\nu = 0.1(\Omega_\nu=0.02, \Omega_m=0.2)$ (dash-dotted line), $f_\nu = 0.03$ (dotted line). We see that the ratio $\frac{\Delta P}{f_\nu}$ tends to a constant only for $k > 50k_{fs}$, but not to a universal constant.

Exploring the E&H approximation to CAMB in Figure [3], we observe that their formula provides best results for only 1 massive neutrino and 2 massless neutrinos with large $f_\nu$, whereas there is considerably less power on small scales for 3 massive neutrinos. However with our modified formula, this effect is altered and the approximation is now valid for 3 massive neutrinos of any mass.

**FIG. 3:** $\Delta P/P$ at $z = 0$ for 1 massive, 2 massless neutrinos (top panel) and 3 massive neutrinos (bottom panel), using CAMB (full line), E&H (dotted line), and our modified fitting formula (dashed line). In all cases $f_\nu = 0.04, \Omega_m = 0.25, \Omega_\nu = 0.04$.

### III. LINEAR GROWTH: AN ANALYTICAL APPROXIMATION

The equation for linear evolution of density perturbations is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_0 \delta, \quad (5)$$

where $\delta = \delta \rho_m / \rho_m$, and $\rho_m$ and $\delta \rho_m$ is the density and the overdensity of matter, respectively. Light, massive neutrinos inhibit structure formation on small scales because they free-stream out of the dark matter potential wells. Roughly, this can be taken into account by multiplying the driving term on the right-hand side of equation (5) by a factor $\frac{\Omega_{cdm}}{\Omega_{cdm} + \Omega_\nu}$, $\frac{\Omega_{cdm}}{\Omega_m} = 1 - f_\nu, f_\nu = \frac{\Omega_\nu}{\Omega_m}$. This equation is valid for $k > 0.2 \ h \ Mpc^{-1}$.

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_0 (1 - f_\nu) \delta. \quad (6)$$

#### A. Einstein De Sitter Universe

For an Einstein de Sitter universe, the solution to equation (6) is given by [2, 13, 14]
where \( p \approx 1 - \frac{1}{3} f_\nu \). Some authors (e.g. [14]), have used this relation to estimate crudely the suppression of the power spectrum due to massive neutrinos \(^2\) to be \( \frac{\Delta^2}{\delta^2} \approx -8 f_\nu \). However, as we showed in the previous section this is a poor approximation. Our Figure 4 shows the limitation of this approach as it is only valid for very large scales \( k > 0.5 \, h \, \text{Mpc}^{-1} \) and small neutrino masses \( f_\nu < 0.05 \). More importantly, we have shown in Figure 2 that the suppression is not just a function of the ratio \( f_\nu \), but of the matter and neutrino densities separately.

**B. A Dominated Universe**

We now evaluate the linear growth of perturbation, equation (10), for a Universe with a cosmological constant. On length scales below the present horizon dark energy does not cluster, and so it only affects \( H = \dot{a}/a \). For a flat cosmology with a dark energy component with constant equation of state \( p = w \rho \), the Hubble parameter as a function of redshift \( z \) is given by

\[
H = \dot{a} = H_0 \sqrt{\Omega_m (1 + z)^3 + (1 - \Omega_m)(1 + z)^{3(1+w)}},
\]

where \( H_0 = 100 h \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \) is the present value of the Hubble parameter, parametrized by the dimensionless Hubble parameter \( h \), and \( \Omega_m \) is the present value of the matter density in units of the critical density that gives a spatially flat universe. The linear growth factor \( f \) is defined by \(^2\)

\[
f \equiv \frac{d \ln \delta}{d \ln a}, \tag{11}
\]

\[^2\] The qualitative derivation for the suppression of \( P(k) \) goes as follows. From equation (7), the growth from matter-radiation equality epoch \( a_{eq} \) to the present \( a_0 \) is

\[
\frac{\delta(a_0)}{\delta(a_{eq})} = (1 + z_{eq})(1 + z_{eq})^{-\frac{2}{3}} f_\nu = (1 + z_{eq}) e^{-\frac{2}{3} f_\nu \ln(1 + z_{eq})}. \tag{8}
\]

The power spectrum \( P(k) \) is the variance of the fluctuations \( \delta \) in Fourier space, so massive neutrinos suppress it by the same factor as it suppresses \( \delta^2 \), i.e.

\[
P(k,f_\nu) = \frac{P(k,f_\nu = 0)}{P(k,f_\nu = 0)} \approx \frac{6}{5} f_\nu \ln(1 + z_{eq}). \tag{9}
\]

Conceptually this derivation contrasts two scenarios (with and without massive neutrinos) which yield at the present epoch the same amplitude of fluctuations. It also assumes that \( \Omega_m \) is the same for both scenarios, hence \( (1 + z_{eq}) \approx 23900 \Omega_m h^2 \). For the concordance model \( \Omega_m h^2 = 0.175 \). This gives for the RHS of equation (9), \( \frac{\Delta^2}{\delta^2} \approx -9.6 f_\nu \). Different coefficients may be obtained by taking into account whether the neutrinos became non-relativistic before or after matter-radiation equality.

\[
\frac{df}{d \ln a} = -f^2 \left[ \frac{1}{2} - \frac{3}{2} w(1 - \Omega_m(z)) \right] f + \frac{3}{2} \Omega_m(z)(1 - f_\nu), \tag{12}
\]

FIG. 4: Density fluctuation \( \delta(z) \) on comoving \( 8h^{-1}\text{Mpc} \) scale, normalized to CMB (top panel) and normalized to the value at \( z = 0 \) (middle panel). The models shown in the figure have \( (w, f_\nu) \) equal to \((-1,0)\) (solid line), \((-1,0.04)\) (dotted line), \((-0.8,0)\) (dashed line), and \((-0.8,0.04)\) (dash-dotted line). In the case of \( w = -0.8 \), we included dark energy perturbations according to CMBFAST. We have assumed a spatially flat universe, adiabatic fluctuations, and fixed the matter density \( \Omega_m = 0.25 \), baryon density \( \Omega_b = 0.04 \), the Hubble constant \( h = 0.7 \), scalar spectral index \( n_s = 1 \), and the optical depth to reionization \( \tau = 0 \). The lower plot shows the density fluctuation \( \delta(z) \) as a function of \( f_\nu \) at \( z = 0 \) (solid line), \( z = 1 \) (dotted line), and \( z = 2 \) (dashed line).
where $a = (1 + z)^{-1}$ is the scale factor of the universe, and $\Omega_m(z) = H_0^2\Omega_{m0}(1 + z)^3/H^2(z)$ is the time dependent density parameter of matter. The Runge-Kutta integration of the set of equations (11), (12) simultaneously gives the growth factor and the logarithmic derivative of the growth factor. Equations (11), (12) are valid even when $w$ evolves with time.

Equation (13) is a simplified description of the effect of massive neutrinos on the growth of structures, but its solution is in good agreement with the results of detailed calculations with e.g. CAMB [16]. We have checked this by comparing $\ln \delta$ found by solving (12) to the value obtained from CAMB (expressed there as $\sigma_8$, the root-mean-square mass fluctuation in spheres of radius $8 \, h^{-1} \text{Mpc}$). We note that the difference between the exact result and the solution of equation (12) is at most 7.5%.

Since the simple prescription employed above to describe the effect of neutrino masses on linear growth seems to work well over a range of parameters and cosmic epochs, we can motivate a simple analytical approximation to the linear growth factor $f$. In matter-dominated cosmologies this quantity is well approximated by $f \approx \Omega_m^\alpha(z)$ [17, 18]. An analytical approximation is also available for models with a cosmological constant, see e.g. [19]. For models without massive neutrinos but with a more general dark energy component, Wang and Steinhardt [20] found that $f = \Omega_m^\alpha(z)$, with

$$\alpha = \alpha_0 + \alpha_1 [1 - \Omega_m(z)]$$

(13)

where

$$\alpha_0 = \frac{3}{5 - \frac{3w}{\alpha}}$$

(14)

and

$$\alpha_1 = \frac{3}{125} \frac{(1 - w)(1 - 3w/2)}{(1 - 6w/5)^3}$$

(15)

for a constant $w$. We should note that $\alpha$ is a very weak function of redshift for $z > 1$.

To include the effect of massive neutrinos in the approximation for $f$, we note that for massive neutrinos the growth is not only redshift dependent but scale dependent as well. By using CAMB, we calculated $P(k, z)$ over a range of values for $f_\nu$, and varying $w$ and $\Omega_m$ in the proximity of $w = -1$ and $\Omega_m = 0.25$, we find that a reasonable fit for a flat universe, $\Omega_m + \Omega_{DE} = 1$, is given by

$$f(z, k; f_\nu, w, \Omega_{DE}) \approx \mu(k, f_\nu, \Omega_{DE})\Omega_m^\alpha(z),$$

(16)

where

$$\mu(k, f_\nu, \Omega_{DE}) = 1 - A(k)\Omega_{DE}f_\nu + B(k)f_\nu^2 - C(k)f_\nu^3.$$  

(17)

Numerical values for $A, B, C$ as a function of $k$ are given in Table II. They vary over the redshift range $z < 10$ up to 1%. Specifically for $8 \, h^{-1} \text{Mpc}$ we recorded the $\sigma_8$ given by CMBFAST, and the resulting approximation is given as:

$$f(z; f_\nu, w, \Omega_{DE}) \approx (1 - 0.8\Omega_{DE}f_\nu + 3.9f_\nu^2 - 9.8f_\nu^3)\Omega_m^\alpha(z).$$

(18)

This formula is valid for $f_\nu \leq 0.15$, and its accuracy is quite high for $w = -1, z = 1$, see Figure 4. Performance at $w = -0.5$ or at $z = 10$ is at most 2% worse than the $w = -1, z = 1$ case.

As expected $\mu = 1$ on very large scales where $k < k_{nr}$. On very small scales, where $k \gg k_{nr}$, we would expect that $f \approx [\Omega_m(1 - f_\nu)]^{\alpha_0}$ based on equation (10) and (14), i.e.

$$\mu = (1 - f_\nu)^{\alpha_0}.$$  

(19)

Another approximation on these small scales follows from equation (2), and by [21]. However we note that this is based on the solution equation (7), which is only valid for an Einstein-de Sitter universe. We find that by solving the set of equations (11), (12) for a $\Lambda$ dominated universe, $\mu = 1 - 0.619f_\nu$, for $z = 1, \Omega_m = 0.25, w = -1$, compared to $\mu = p = 1 - 0.600f_\nu$ of equations (2), (7). We emphasize again this is only valid on very small scales, and for intermediate scales one should use equation (17).

### IV. THE $w$-$f_\nu$ DEGENERACY

Hannestad [22] pointed out that cosmological neutrino mass limits are considerably weakened if one allows for $w < -1$ (a case which is peculiar, as in this case the density will increase with the expansion of the universe). In his analysis, the parameter combination $\Omega_m h^2$ instead of $f_\nu$ was varied, and the explanation of the degeneracy he gave was as follows: since $f_\nu = \Omega_m h^2$ determines the small-scale suppression of the matter power spectrum, one can compensate for a larger $\Omega_m$ by increasing $\Omega_m h^2$. If one assumes a fixed $w = -1$ in the analysis, the Hubble diagram from supernovae Type Ia rule out values of $\Omega_m h^2$ much larger than 0.3. However, if one allows for $w < -1$, then the supernova data are compatible with considerably larger values of $\Omega_m h^2$, and hence a higher value of

| $k$ (scales) | A($k$) | B($k$) | C($k$) | $f$ |
|-------------|--------|--------|--------|-----|
| 0.001       | 0      | 0      | 0      | 0.825265 |
| 0.01        | 0.132  | 1.62   | 7.13   | 0.824272 |
| 0.05        | 0.613  | 5.59   | 21.13  | 0.815508 |
| 0.07        | 0.733  | 6.0    | 21.45  | 0.811596 |
| 0.1         | 0.786  | 5.09   | 15.5   | 0.806680 |
| 0.5         | 0.813  | 0.803  | -0.844 | 0.789606 |
surveys and does not affect the shape of the matter

does not cluster on the scales probed by galaxy redshift
time, the degeneracy is almost completely broken when using
the WMAP 3-year data as opposed to the first year [7].

Based on the considerations in the previous subsections
of this paper we would expect there to be a further and
more direct degeneracy between \(f_\nu\) and \(w\): the same lin-
ear growth rate can result from a range of combinations
of values of \(w\) and \(f_\nu\). In physical terms, this degeneracy
can be understood as follows: neutrino free-streaming
will supress growth of structure on small scales. How-
ever, decreasing \(w\) (for fixed \(\Omega_m\)) prolongs the era of
structure formation, and reduces the value of the Hubble
parameter in the matter-dominated phase. The Hubble
parameter acts as a ‘friction term’ in equation (18), so re-
ducing it will enhance the linear growth factor, see equa-
tion (18). To sum up, one can at least partly compensate
for increasing \(f_\nu\) by decreasing \(w\). This degeneracy would
be relevant if one were to use the growth of structure
to constrain \(f_\nu\) and \(w\). In Figure (4) we illustrate this
by plotting the root-mean-square mass fluctuation am-
plitude \(\delta(z)\) as a function of redshift, for four different
combinations of \(w\) and \(f_\nu\). The middle panel shows the
situation when if only the growth rate is measured (nor-
malized at \(z = 0\)): in that case distinguishing between
the four cases will require very accurate measurements.
The situation is better when the absolute values of the
fluctuations are measured, as shown in the top panel of
Figure (4). This can be understood by noting that the
absolute values depend, in the case when we normalize
to the CMB on large scales, on \(w\), but is only weakly
dependent on \(f_\nu\). The reason for this is that dark energy
fluctuations are relevant on scales of the size of the hori-
zon, and dominate integrated Sachs-Wolfe effect on the
very largest scales. In contrast, the main effect of neutrions
on the CMB is on smaller scales through a small shift
in the position of the peaks and a slight enhancement of
their amplitude. Thus, if the large-scale normalization
is combined with a measurement of the growth rate, the
degeneracy between \(w\) and \(f_\nu\) may to a large extent be
broken. Figure (6) also shows the degeneracy of \(f_\nu\) and \(w\)
through the linear growth factor \(f\).

As indicated in Table I there is an interplay in the
roles of neutrino mass and the DE equation of state,
when estimated observationally, which depend on geo-
metry, the growth rate of perturbations and the shape of
the power spectrum. We discuss briefly two probes where
results from this paper, in particular the growth rate \(f\)
in equation (18), could help in understanding degeneracy.

The first is peculiar velocities (see e.g. [2] for review).
The rms bulk flow is predicted in linear theory as:

\[
\langle v^2(R_\ast) \rangle = (2\pi^2)^{-1} H_0^3 \int dk f^2 P(k) W_G^2(k R_\ast)
\]

where \(W_G(k R_\ast)\) is a window function, e.g.
\(W(k R_\ast) = \exp(-k^2 R_\ast^2/2)\) for a Gaussian sphere of radius \(R_\ast\). The

\(\Omega_{\nu 0}\) can be accommodated by a given value of \(f_\nu\). The
degeneracy described above between \(\Omega_{\nu 0}\) and \(w\) is indirect.
We also emphasize again and again that the suppression
is not just a function of \(f_\nu\).

The degeneracy is most relevant when the shape of
the matter power spectrum \(P(k)\) at \(z = 0\) (or another
fixed redshift) is used to constrain the neutrino mass
(i.e. the bias between the galaxy distribution and total
mass distribution is assumed constant and marginalized
over). Dark energy with a constant equation of state
does not cluster on the scales probed by galaxy redshift
surveys and does not affect the shape of the matter

\begin{align*}
\Omega_{DE} & \equiv \Omega_m (1 + z) \left( 1 + \frac{1}{3} \Omega_m \right) \\
\Omega_{DE} & = 0.7, \quad \Omega_\nu = 0.3, \quad \Omega_b = 0.04, \quad h = 0.7.
\end{align*}
monic amplitudes are (e.g. [26, 27]): small angle approximation the predicted spherical har-
correlation of the CMB with galaxy samples [25]. In the
the Integrated Sachs Wolfe effect derived from cross-
tortion. 

FIG. 7: The product $D^2 (f - 1) P$ in equation (21), using equation (18) on $8h^{-1}$Mpc scale, at $\ell = 50$ (lower set of curves) and at $\ell = 100$ (upper set of curves) for spatially flat universe models with $\Omega_m = 0.3$, $\Omega_b = 0.04$, $h = 0.7$, $n_s = 1.0$, and $w = -1$, $\Omega_{\nu} = 0.0$ (full lines), $w = -1$, $\Omega_{\nu} = 0.01$ (dotted lines), $w = -0.8$, $\Omega_{\nu} = 0$ (dashed lines).

velocity field at low redshift is insensitive to geometry. The power spectrum $P(k)$ depends on the neutrino mass, but not on dark energy. Massive neutrinos would suppress bulk flows [24]. However in [24] any dependence of $f$ on neutrino mass and dark energy was ignored. Our equation (18) shows such dependence, which would result in extra suppression of the bulk flows and in some degeneracy with $w$. This is also relevant on redshift distortion.

A second cosmological probe which depends on $f$ is the Integrated Sachs Wolfe effect derived from cross-correlation of the CMB with galaxy samples [27]. In the small angle approximation the predicted spherical harmonic amplitudes are (e.g. [26, 27]):

$$C_{T \ell}(\ell) = \frac{-3b_g H_0^2 \Omega_m \Omega_{\nu}}{c^4 (\ell + 1/2)^2} \int dr \Theta(r) D^2 H(f-1) P \left( \frac{\ell + 1/2}{r} \right) \tag{21}$$

where $\Theta(r)$ is a radial selection function for the galaxy survey, $b_g$ is the galaxy biasing factor, and $D(t)$ is the linear theory growth function. The approximate relations for $f$, $D$ and $P(k)$ are useful to explore the degeneracies between neutrino mass and dark energy. As a simple illustration, we show in Figure 7 the product $D^2 (f - 1) P$ in the integrand above. We see that the selection function must extend to higher redshifts to distinguish between neutrino masses and $w \neq -1$. See [28] for further analysis on the ISW effect in the presence of massive neutrinos.

V. CONCLUSIONS

We have demonstrated that the popular heuristic formula for the linear theory suppression of the matter fluctuations by free-streaming neutrinos, $\Delta P(k)/P(k) \approx -8f_\nu$, is valid only on very small scales ($k > 0.8h$/Mpc). However, it is not of practical use as this is in the strongly non-linear regime of matter clustering. We also provide a modified code for the E&H fitting formula. The linear growth factor in models with both massive neutrinos and a dark energy component with equation of state parameter $w =$constant, has been calculated numerically, and we have provided a fitting formula justified by simple arguments. Furthermore, we have explained the indirect degeneracy between $w$ and $m_\nu$ found in [22], and argued that there is a further, more direct degeneracy between $w$ and $f_\nu$ when the linear growth rate of density perturbations is considered. This degeneracy can, however, be lifted by measurements of the absolute scale of the mass fluctuations, for example from the CMB.

Acknowledgments

AK acknowledges support by the University of London Perren Studentship. The work of OE is supported by the Research Council of Norway, project numbers 159637 and 162830. OL acknowledges support by PPARC Senior Research Fellowship. Special thanks to Filipppe Abdalla and Ole Host, for fruitful discussions.

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