In this paper (as in previous ones) we identify and investigate polynomials $p^{(n)}(x)$ featuring at least one additional parameter $\nu$ besides their argument $x$ and the integer $n$ identifying their degree. They are orthogonal (provided the parameters they generally feature fit into appropriate ranges) as much as they are defined via standard three-term linear recursion relations; and they are interesting inasmuch as they obey a second linear recursion relation involving shifts of the parameter $\nu$ and of their degree $n$, and as a consequence, for special values of the parameter $\nu$, also remarkable factorizations, often having a Diophantine connotation. The main focus of this paper is to relate our previous machinery to the standard approach to discrete integrability, and to identify classes of polynomials featuring these remarkable properties.

Keywords: Discrete integrability; recursion relations; orthogonal polynomials; Diophantine factorizations; Askey polynomial classification.
after reviewing these properties, we found that most of the named polynomials belonging
to the Askey scheme [6] (in some cases, up to minor modifications) could be fitted — for
appropriate assignments of their parameters — into this machinery and thereby shown to
possess these properties (although generally the factorization formulae we obtained were
applicable for parameters falling outside the ranges for which the standard orthogonality
property holds). In this paper we continue to focus on classes of orthogonal polynomials to
which our machinery [4] is applicable. These monic polynomials are again defined by the
standard three-term recursion relation
\[ p_n^{(r)}(x) = (x + a_n^{(r)}) p_{n-1}^{(r)}(x) + b_n^{(r)} p_{n-1}^{(r)}(x) \] (1a)
with the “initial” assignments
\[ p_{-1}^{(r)}(x) = 0, \quad p_0^{(r)}(x) = 1, \] (1b)
clearly entailing
\[ p_1^{(r)}(x) = x + a_0^{(r)}, \quad p_2^{(r)}(x) = (x + a_1^{(r)})(x + a_0^{(r)}) + b_1^{(r)} \] (1c)
and so on.

**Notation:** Here and hereafter the index \( n \) (as well as analogous indices such as \( m, \ell \); see
below) is generally an arbitrary nonnegative integer — unless otherwise explicitly indicated:
note that this implies that (1a) is not required to hold for \( n = -1 \), when clearly it would
contradict (1b), and that (1b) entails that, in all formulae, the polynomials \( p_\ell^{(r)}(x) \) should
be set to zero whenever \( \ell \) is negative. Of course \( a_n^{(r)}, b_n^{(r)} \) are given functions of the index
\( n \) and of the parameter \( \nu \). The polynomials \( p_n^{(r)}(x) \), as well as the parameters \( a_n^{(r)}, b_n^{(r)} \),
might also depend on other parameters besides \( \nu \) (indeed they often do, see below); but
the parameter \( \nu \) plays a crucial role, and the classes of orthogonal polynomials featuring
remarkable factorizations are associated with special values of this parameter (generally
simply related to the order \( n \) of these polynomials). Some of the formulae written below
might require a special interpretation for \( n = 0 \), and note that hereafter the value \( b_0^{(r)} \) of
the coefficient \( b_n^{(r)} \) at \( n = 0 \) should play no role (see (1a) and (1b)).

In the following we will also employ, whenever convenient, the quantities \( A_n^{(r)} \) and \( B_n^{(r)} \)
related to \( a_n^{(r)} \) and \( b_n^{(r)} \) by the simple relations
\[ a_n^{(r)} = A_n^{(r)} - A_{n-1}^{(r)}, \quad b_n^{(r)} = \frac{B_n^{(r)}}{B_{n-1}^{(r)}} \] (2a)
entailing of course
\[ A_n^{(r)} = A_0^{(r)} + \sum_{m=0}^{n-1} a_m^{(r)}, \quad B_n^{(r)} = B_0^{(r)} \prod_{m=1}^{n} [(-1)^m b_m^{(r)}]. \] (2b)

Here and hereafter we use the standard convention according to which sums are set to zero,
and products are set to unity, when their lower limits exceed their upper limits; this is
consistent with the validity of these formulae for \( n = 0 \).
Let us now recall tersely our previous findings [4]. Assume that there exist quantities $A_n^{(\nu)}$ and $\omega^{(\nu)}$ satisfying the nonlinear recursion relation
\begin{align*}
(A_n^{(\nu)} - A_{n-1}^{(\nu-1)}) (A_{n-1}^{(\nu)} - A_{n-2}^{(\nu-1)} + \omega^{(\nu)})
&= (A_n^{(\nu-1)} - A_{n-2}^{(\nu-2)}) (A_{n-1}^{(\nu-1)} - A_{n-2}^{(\nu-2)} + \omega^{(\nu-1)}) \quad (3a)
\end{align*}
with the “initial” condition
\begin{equation}
A_0^{(\nu)} = 0 \quad (3b)
\end{equation}
(note that this initial condition guarantees the validity of (3a) for $n = 0$, and thereby eliminates the need to assign $A_1^{(\nu)}$). Then (see [4, Proposition 2.1]), provided the coefficients $a_n^{(\nu)}$ are defined in terms of these quantities by the first of the relations (2a) and the coefficients $b_n^{(\nu)}$ are defined as follows,
\begin{equation}
b_n^{(\nu)} = (A_n^{(\nu)} - A_{n-1}^{(\nu-1)}) (A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \omega^{(\nu)}), \quad n = 1, 2, \ldots, \quad (3c)
\end{equation}
the polynomials $P_n^{(\nu)}(x)$ identified by the corresponding recursion relation (1) satisfy the following additional three-term recursion relation (involving a shift both in the order $n$ of the polynomials and in the parameter $\nu$):
\begin{equation}
\begin{split}
P_n^{(\nu)}(x) = P_{n-1}^{(\nu-1)}(x) + g_n^{(\nu)} P_{n-1}^{(\nu-1)}(x),
\end{split} \quad (4a)
\end{equation}
with
\begin{equation}
g_n^{(\nu)} = A_n^{(\nu)} - A_{n-1}^{(\nu-1)}, \quad n = 1, 2, \ldots \quad (4b)
\end{equation}
As a consequence there hold for some of these polynomials (characterized by special assignments of the parameter $\nu$, generally simply related to the degree $n$ of the polynomial) remarkable Diophantine factorizations (see [4] and below). Note that, via (3b), the formulas (3c) respectively (4b) — if assumed valid also for $n = 0$ — entail the vanishing of $b_0^{(\nu)}$ respectively $g_0^{(\nu)}$, namely the “initial” conditions
\begin{equation}
b_0^{(\nu)} = 0, \quad g_0^{(\nu)} = 0. \quad (5)
\end{equation}
Let us moreover recall that conditions — equivalent to (3) and (4b) but characterizing directly the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$, and $g_n^{(\nu)}$, hence being also sufficient for the validity of the second recursion relation (4a) — read as follows (see Appendix B of [2], as well as [4]):
\begin{equation}
a_n^{(\nu)} = a_n^{(\nu)} - a_{n-1}^{(\nu)} = a_n^{(\nu)} - a_{n-1}^{(\nu-1)}, \quad (6a)
\end{equation}
\begin{equation}
b_n^{(\nu)} = b_n^{(\nu)} - b_{n-1}^{(\nu-1)} = b_n^{(\nu)} - b_{n-1}^{(\nu-1)}, \quad (6b)
\end{equation}
with
\begin{equation}
g_n^{(\nu)} = \frac{b_n^{(\nu)} - b_{n-1}^{(\nu-1)}}{a_n^{(\nu)} - a_{n-1}^{(\nu-1)}} \quad (6c)
\end{equation}
and the “initial” conditions (5). It is indeed plain that (6a) is implied by the first (2a) and (4b), that (6b) corresponds to (3a) via (3c) and (4b), and the diligent reader will also verify that (6c) corresponds as well to (3a) via the first relation (2a) with (3c) and (4b).
Let us mention in passing that we investigated — by trial and error techniques, but somewhat more systematically than we had previously done [4] — solutions of the non-linear system (3); we found several results, but all of them eventually yielded polynomials belonging to the Askey scheme [6] (possibly up to rescaling and shifts of their arguments); we have therefore decided not to report these findings.

In this paper we firstly investigate, in Sec. 2, the connection of the machinery developed in previous papers (see in particular [4]) with standard approaches to discrete integrability. In this manner we show how some of our previous findings can be fitted in that context, and how they can be extended: in particular we find a new nontrivial class of nonlinear integrable equations satisfied by the single function $A^{(1)}_n$ of the two discrete variables $n$ and $\nu$ (see below (39) and (46)). Then, in Sec. 3, we report new factorization formulae applicable to polynomials satisfying two recursion relations, such as those yielded by the treatment of the preceding Sec. 2. We then focus, in Sec. 4, on the identification — via trial-and-error searches — of classes of orthogonal polynomials to which the extension of our approach (see Sec. 2), including in some cases the Diophantine factorizations it yields (see Sec. 3), is applicable. We thereby again end up with polynomials belonging to the Askey scheme [6]; and occasionally we thereby obtain for these polynomials results — recursion relations and Diophantine factorizations — that are not reported in standard compilations (although presumably they could also be obtained by other approaches, such as the connection of these polynomials with the hypergeometric function). Some developments are confined to appendices to avoid interruptions in the flow of the presentation.

Although this paper reports findings obviously belonging to a continued research line [1–4], its presentation is self-contained, while also minimizing repetitions. So we tersely reviewed — see above — only those previous findings that are necessary and sufficient for the comprehension of the results obtained in this paper, which can therefore be understood without having read the preceding papers of this series [1–4] (although this oversight is not recommended).

Finally — also to take account of remarks by Referees — let us underline that the main results reported in this paper — as indeed made clear by its title and abstract — are the connection of our approach to discrete integrability (which has yielded the identification of new integrable discrete nonlinear evolution equations, see below (39) and (46)) and the identification of classes of “named” polynomials satisfying remarkable properties, such as Diophantine factorizations. A tool to obtain these results are a second type of recursion relations, playing — together with the more standard, three term ones satisfied by orthogonal polynomials — an analogous role to a Lax pair underlying the property of integrability.

It is certainly the case that the additional recursion relations we utilize could be obtained, as pointed out by Referees, by different techniques than those we employ to get them, for instance via the Geronimus [7] and Christoffel transforms [8]; and let us re-emphasize the obvious fact that all the Diophantine factorizations we identified could be — after they have been discovered — also demonstrated by different techniques, such as the connection with hypergeometric functions of the classes of polynomials we consider. It is common knowledge that mathematical results — especially in the field of special functions — can be arrived at by different routes; but the identification of new routes is generally considered a worthwhile achievement; and the first identification of a finding deserves special recognition, even if it can be later shown that the same result can be arrived at by alternative approaches.
2. The Connection of our Approach with Standard “Discrete Integrability” Treatments

As tersely surveyed above, our approach (see for instance [4]) focused on the identification — and on the remarkable properties, including in particular Diophantine factorizations — of classes of (orthogonal) polynomials \(p^{(\nu)}(x)\) satisfying both the linear three-term recursion relation (1a) — involving (only) shifts in the index \(x\) playing the role of the operators \(\hat{E}^{(\pm)}_{\pm}\) associated to them. Hence they rather naturally fit within the context in which polynomials also play a key role, see for instance [14–17] and references quoted in these papers; but none of them appears to coincide with our treatment, see below. We are of course aware of various previous treatments in the “discrete integrability” context in which polynomials also play a key role, see for instance [10–13] and references quoted there. It seems therefore appropriate that we also review our treatment in such a context; the special feature we shall of course have to keep in mind is the requirement that these two linear recursion relations be compatible entails that the coefficients \(a^{(\nu)}_n\), \(b^{(\nu)}_n\) respectively \(g^{(\nu)}_n\) featured by them satisfy certain conditions, which can be reduced [4] to the nonlinear relations (3) satisfied by the quantities \(A^{(\nu)}_n\) and \(\omega^{(\nu)}\) (in terms of which the quantities \(a^{(\nu)}_n\), \(b^{(\nu)}_n\) respectively \(g^{(\nu)}_n\) are easily retrieved via the first (2a), via (3c) respectively via (4b)). This entails that these nonlinear relations, (3), can be categorized as discrete integrable equations, inasmuch as the two linear recursion relations (1a) and (4a) play the role of a Lax pair associated to them. Hence they rather naturally fit within that major development in the investigation of integrable discrete systems that occurred over the last few decades: see [9] and many subsequent papers and some books, for instance [10–13] and references quoted there. It seems therefore appropriate that we also review our treatment in such a context; the special feature we shall of course have to keep in mind is the requirement that the functions \(p^{(\nu)}_n(x)\) be monic polynomials of degree \(n\), as entailed by (1).

We start by reinterpreting our basic recursion, (1a), as a discrete spectral problem (with \(x\) playing the role of eigenvalue and \(p\) that of eigenfunction, see below),

\[
L_{\nu} = x_{\nu},
\]

via the convenient introduction of the following self-evident notation:

\[
p \equiv p^{(\nu)}(x), \quad a \equiv a^{(\nu)}_n, \quad b \equiv b^{(\nu)}_n, \quad L = \hat{E}_+ - a\hat{I} - b\hat{E}_-,
\]

where the operators \(\hat{E}_\pm\), here and hereafter, are the “raising” and “lowering” operators acting on the index \(n\), while \(\hat{I}\) is the identity operator:

\[
\hat{E}_\pm f^{(\nu)}_n = f^{(\nu)}_{n \pm 1}, \quad \hat{I} f^{(\nu)}_n = f^{(\nu)}_n,
\]

and more generally

\[
\hat{E}_k f^{(\nu)}_n = f^{(\nu)}_{n + k},
\]

with \(k\) an arbitrary integer, positive or negative (and of course \(\hat{E}_0 = \hat{I}\)). Here \(f \equiv f^{(\nu)}_n\) indicates a generic quantity depending on the index \(n\) and on the parameter \(\nu\) (and possibly on the variable \(x\) and on additional parameters). Likewise we introduce the “raising” and “lowering” operators \(\hat{E}^{(\pm)}_{\pm}\) acting on the parameter \(\nu\):

\[
\hat{E}^{(\pm)} f^{(\nu)}_n = f^{(\nu \pm 1)}_n.
\]
Here and hereafter, for notational transparency, we equip with a superimposed hat the mathematical symbols denoting operators acting via shifts on the index \( n \) or on the parameter \( \nu \) (note that they do not act on the polynomial variable \( x \), playing the role of “eigenvalue” in the discrete spectral problem (7)).

We then associate, to the eigenvalue problem (7), a second linear (recursion) relation reading

\[ \hat{E}^{(+)} p = \hat{H} p, \]  

with the operator \( \hat{H} \) acting on the index \( n \), and depending on the indices \( n \) and possibly on the parameter \( \nu \), in a manner still to be determined. The introduction of this relation is suggested by (4a), to which (with \( \nu \) replaced by \( \nu + 1 \)) it clearly reduces for the special assignment

\[ \hat{H} = \hat{I} + h^{(\nu+1)} \hat{E} \ldots. \]  

(10)

The fact that the coefficient of the identity operator \( \hat{I} \) in the right-hand side of this formula is unity is of course required by the property of the polynomials \( p^{(\nu)}(z) \) to be monic, see (1).

Before proceeding let us also introduce the following convenient short-hand notation:

\[ f^{(\pm)} \equiv f^{(\nu\pm 1)}_n, \quad f^{(\nu)} \equiv f^{(\nu)}_{n\pm 1}, \]  

(11a)

applicable to any quantity depending on the index \( n \) and on the parameter \( \nu \). The following obvious operator identities are then useful (see below):

\[ \hat{E}^{\pm}_n f = f^{(\pm)}_n \hat{E}^{\pm} \ldots, \quad \hat{E}^{(\pm)}_n f = f^{(\pm)} \hat{E}^{(\pm)} \ldots, \]  

\[ \hat{E}^{(\pm)} \hat{E}^{(\pm)} = \hat{E}^{(\pm) \hat{E}^{(\pm)} = \hat{I} \ldots. \]  

(11b)

We now report several propositions, the proofs of which are relegated to Appendix A in order to avoid interrupting the flow of our presentation. Let us emphasize that our treatment here is quite standard, see for instance [11] (with the discrete time \( t \) replaced by our parameter \( \nu \)) and, for the case of continuous time, [18].

**Proposition 2.1.** The eigenvalue equation (7) and the recursion relation (9) are compatible if and only if

\[ \hat{L}^{(+)} \hat{H} - \hat{H} \hat{L} = 0 \]  

(12a)

where of course (consistently with the notation (11a) and the identities (11b))

\[ \hat{L}^{(+)} = \hat{E}_+ - \hat{E}_- = \hat{E}^{(+) \hat{E}^{(-)} = \hat{E}^{(-) \hat{E}^{(+) = \hat{I} \ldots. \]  

(12b)

Note that every \( \hat{H} \) reading as follows,

\[ \hat{H} = \sum_{r=0}^{M} h^{(\nu)}_r \hat{E}_{k-r}, \quad k \in \mathbb{Z}, \quad M \in \mathbb{N} \]  

(13a)

with

\[ h^{(\nu)}_r \equiv h_{\nu}^{(r)} \]  

(13b)
and k an arbitrarily assigned integer (negative or positive), satisfies (12), provided the coefficients $h^{[\nu]}_{n+1}[\nu] \equiv 0$, satisfy the following equations:

$$
\begin{align*}
&h^{[m+1]}_{n+1}[\nu] - h^{[m+1]}_{n}[\nu] - a^{[m+1]}_{n+1}h^{[m]}_{n}[\nu] + a^{[m]}_{n+1}h^{[m]}_{n}[\nu] \\
&- b^{[m]}_{n+1}h^{[m-1]}_{n}[\nu] + b^{[m]}_{n+1}h^{[m-1]}_{n}[\nu] = 0, \quad m = -1, 0, 1, \ldots, M + 1,
\end{align*}
$$

where $M$ is an arbitrary positive integer, and we assume $h^{[\nu]}$ to vanish beyond the boundaries (in the parameter $n$), i.e.

$$
\begin{align*}
h^{[\nu]} &= 0 \quad \text{if } r < 0 \quad \text{or} \quad r > M.
\end{align*}
$$

This system of algebraic nonlinear equations, (14), clearly features $M + 3$ equations in the $M + 3$ unknowns $a, b, h^{[\nu]}$ with $r = 0, 1, \ldots, M$ (of course with $a, b, h^{[\nu]}$ being functions of $n$ and $\nu$ as explicitly indicated above and below), and it is “bi-triangular” in the following sense: when displayed in more explicit form (starting from the two boundaries), these equations of motion read as follows:

$$
\begin{align*}
h^{[M]}_{n+1}[\nu] - h^{[0]}_{n}[\nu] &= 0, \\
h^{[2]}_{n+1}[\nu] - h^{[1]}_{n}[\nu] - a^{[1]}_{n+1}h^{[0]}_{n}[\nu] + a^{[0]}_{n+1}h^{[0]}_{n}[\nu] &= 0, \\
h^{[4]}_{n+1}[\nu] - h^{[3]}_{n}[\nu] - a^{[3]}_{n+1}h^{[2]}_{n}[\nu] + a^{[2]}_{n+1}h^{[2]}_{n}[\nu] &= 0, \\
h^{[6]}_{n+1}[\nu] - h^{[5]}_{n}[\nu] - a^{[5]}_{n+1}h^{[4]}_{n}[\nu] + a^{[4]}_{n+1}h^{[4]}_{n}[\nu] &= 0, \\
h^{[8]}_{n+1}[\nu] - h^{[7]}_{n}[\nu] - a^{[7]}_{n+1}h^{[6]}_{n}[\nu] + a^{[6]}_{n+1}h^{[6]}_{n}[\nu] &= 0, \\
&\vdots
\end{align*}
$$

Hence to solve this system one can start from (15a) yielding $h^{[0]}$ as an arbitrary function $h^{[0]}(\nu)$ (independent of $n$):

$$
h^{[0]} = \tilde{h}^{[0]}(\nu).
$$

Next (15b) determines (easily) $h^{[1]}$ in terms of $a$:

$$
h^{[1]} \equiv a^{[1]}_{n}[\nu] = \tilde{h}^{[1]}[\nu] + \sum_{\ell=0}^{n-1} a^{[\nu]}_{\ell+1} - a^{[\nu]}_{\ell+2},
$$

with $\tilde{h}^{[1]}[\nu]$ another arbitrary function of $\nu$ only (independent of $n$). Next (15c) determines (easily) $h^{[2]}$ in terms of $a$ and $b$:

$$
h^{[2]} \equiv h^{[2]}_{n}[\nu] = \tilde{h}^{[2]}_{n}[\nu] + \sum_{\ell=0}^{n-1} a^{[\nu]}_{\ell+1}h^{[1]}_{\ell}[\nu] - a^{[\nu]}_{\ell+2}h^{[1]}_{\ell}[\nu] + b^{[\nu]}_{\ell+1}h^{[0]}_{\ell}[\nu] - b^{[\nu]}_{\ell+2}h^{[0]}_{\ell}[\nu],
$$

with $\tilde{h}^{[2]}_{n}[\nu]$ another arbitrary function of $\nu$ only (independent of $n$).
with $\hat{h}^{(2)(\nu)}$ another arbitrary function of $\nu$ only (independent of $n$). And so on up to (15d) that determines (easily) $\hat{h}^{(M)}$ in terms of $a$ and $b$. Finally the system of two, highly nonlinear, algebraic equations (15e), (15f) determine — at least in principle — the two functions $a$ and $b$.

One could also proceed in reverse order, starting from (15f) to obtain (albeit not so easily) the function $\hat{h}^{[M]}$ in terms of $b$, then using (15e) to obtain $\hat{h}^{[M-1]}$ in terms of $a$ and $b$, and so on.

Remark 2.1. Since we are focussing on polynomial eigenfunctions of (7a), and we moreover require these polynomials to be monic, the only acceptable versions of the operator $\hat{H}$ in (9) are the following subclass of (13):

$$\hat{H} = \hat{I} + \sum_{r=1}^{M} \hat{h}^{[r]} \hat{E}, \quad M \in \mathbb{N}. \quad (17)$$

Such operators could in principle be obtained, for every value of $M$, following the procedure we just described; but we also describe now a more global — and, in the integrability context, perhaps more standard — procedure (see, for instance, [11,18]), based on several propositions which lead to the introduction of a recursion operator allowing to express in more compact form the entire hierarchy of relevant nonlinear equations.

Proposition 2.2. Suppose that there exist an operator $\hat{H}$ such that

$$\hat{L}^{(+)} \hat{H} - \hat{H} \hat{L} = W \hat{I} + w \hat{E}, \quad (18)$$

where $W \equiv W^{(\nu)}_a$ and $w \equiv w^{(\nu)}_a$ are now assumed to be known (and to be independent of $x$; they shall of course depend on $a^{(\nu)}$ and $b^{(\nu)}$, or equivalently on $A^{(\nu)}$ and $B^{(\nu)}$). Then we can construct another operator, say $\hat{H}'$, such that

$$\hat{L}^{(+)} \hat{H}' - \hat{H}' \hat{L} = W' \hat{I} + w' \hat{E}, \quad (19)$$

where $\hat{H}'$ is given by the formula

$$\hat{H}' = \hat{H} + Q \hat{I} + q \hat{E}, \quad (20a)$$

with $Q'$ and $q'$ determined by the relations

$$Q' - Q'_\nu = W, \quad (20b)$$

$$b \cdot q' - b^{(+)} q'_\nu = b \cdot w, \quad (20c)$$

while $W'$ and $w'$ are given by the formulæ

$$W' = -aW + w + q'_\nu - q' + (a - a^{(+)} Q'), \quad (21a)$$

$$w' = -bW - a \cdot w + (a - a^{(+)} q') + b Q' - b^{(+)} Q'_\nu, \quad (21b)$$

where of course $Q'$ and $q'$ are determined in terms of $W, w$ and $b$ by (20b) and (20c).

These formulæ are instrumental to set up a kind of bootstrap mechanism suitable to generate by iteration a sequence of solutions of the key compatibility relation (12a). To this
Proposition 2.3. There hold the two formulae 
\[ (20b) \] and 
\[ (20c). \]

If one makes the assignment 
\[ \tag{22a} \]
then there holds the relation 
\[ \tag{22b} \]
is a short-hand version of (21), again with \( Q' \) and \( q' \) determined in terms of \( W, w \) and \( b \) by (20b) and (20c).

Proposition 2.4. If one makes the assignment (see the notation (22a) and (22b))
\[ \tag{25} \]
then there holds the relation 
\[ \tag{26a} \]
with 
\[ \tag{26b} \]
Here the parameters \( \epsilon^{[j(n)]} \) and \( \varphi^{[k(u)]} \) are independent of \( n \) (and \( s \)), but otherwise arbitrary (restrictions on them shall be introduced below).
Proposition 2.5. For any nonnegative integer $K$ the second operator appearing in the right-hand side of \((25)\) has the following structure:
\[
\left( \sum_{k=0}^{K} \left[ \hat{\mathcal{B}}^{[k]} \hat{\rho}^{[k]} \right] \right) \hat{B}_+^{(+)} \hat{E}_- = \sum_{k=0}^{K} \left[ \hat{\mathcal{B}}^{[k]} \hat{E}_-^{k+1} \right] \equiv \hat{\mathcal{B}} \hat{E}_- + \sum_{k=1}^{K} \left[ \hat{\mathcal{B}}^{[k]} \hat{E}_-^{k+1} \right],
\]
(27)

where the quantities $\hat{\mathcal{B}}^{[k]}$ depend now on $K$, $n$ and $\nu$ (in addition of course to $k$ : $\hat{\mathcal{B}}^{[k]} \equiv \hat{\mathcal{B}}_{K,k}$), and (in the second line) $\hat{\mathcal{B}}$ is clearly a short-hand notation for $\hat{\mathcal{B}}^{[0]} \equiv \hat{\mathcal{B}}_{K,0}$.

Note in particular that the raising operator $\hat{E}_+$ does not appear in the right-hand side of this formula: indeed the operator \((27)\) always lowers the index $n$ (unless it yields an identically vanishing result). In the following we shall not be interested in the specific form of the coefficients $\hat{\mathcal{B}}^{[k]}$, but only in the property demonstrated by the structure of the right-hand side of \((27)\).

Proposition 2.6. For any nonnegative integer $J$ the first operator appearing in the right-hand side of \((25)\) has the following structure:
\[
\left( \sum_{j=0}^{J} \left[ \hat{\mathcal{C}}^{[j]} \hat{\rho}^{[j]} \right] \right) \hat{I} = (\hat{E}_+)^J + \sum_{j=1}^{J-1} \left[ \hat{\mathcal{C}}^{[j]} \hat{E}_-^j \right] + \sum_{j=1}^{J} \left[ \hat{\mathcal{C}}^{[j]} \hat{E}_-^{j+1} \right],
\]
(28)

where the quantities $\hat{\mathcal{C}}^{[j]}$ and $\sigma^{[j]}$ depend on $J$, $n$ and $\nu$ (in addition of course to $j$).

Proposition 2.7. Within the class \((25)\), only the subclass
\[
\hat{H} = \hat{I} + \left( \sum_{k=0}^{K} \left[ \hat{B}_+^{[k]} \hat{\rho}^{[k]} \right] \right) \frac{\hat{B}_+^{(+)} \hat{E}_-}{\hat{B}_- \hat{E}_-}
\]
(29)

(corresponding to $J = 0$, $\hat{\mathcal{C}}^{[0]} = 0$) is consistent, via the second recursion relation \((9)\), with the property of the polynomials $p_n^{(\nu)}(x)$ to be monic, implied by the first recursion relation \((1)\) defining them.

Proposition 2.8. If the quantities $A \equiv A_{h,w}^{(\nu)}$, $B \equiv B_{h,w}^{(\nu)}$ satisfy the spinor system
\[
\begin{pmatrix}
    a - a^{(+)} \\
    b - b^{(+)}
\end{pmatrix} + \left( \sum_{k=0}^{K} \left[ \hat{C}_+^{[k]} \hat{\rho}^{[k]} \right] \right) \begin{pmatrix}
    B_+^{(+)} / B - B^{(+)} / B_+ \\
    (a - a^{(+)}) / (B_+^{(+)} / B_+ - B^{(+)} / B_+)
\end{pmatrix} = 0,
\]
(30)

then there holds the second recursion \((9)\) with $\hat{H}$ given by \((29)\), hence reading as follows:
\[
p_n^{(\nu+1)}(x) = \left[ I + \left( \sum_{k=0}^{K} \left[ \hat{C}_+^{[k]} \hat{\rho}^{[k]} \right] \right) \frac{B_+^{(+)} / B - B^{(+)} / B_+}{B_+^{(+)} / B_+ - B^{(+)} / B_+} \right] p_n^{(\nu)}(x).
\]
(31)

Let us re-emphasize that, for notational convenience, we employed throughout a mixed notation, using the quantities $a \equiv a_{h,w}^{(\nu)}$, $b \equiv b_{h,w}^{(\nu)}$ as well as $A \equiv A_{h,w}^{(\nu)}$, $B \equiv B_{h,w}^{(\nu)}$: let us recall in this connection that the relation among the quantities $A_{h,w}^{(\nu)}$, $B_{h,w}^{(\nu)}$ and the quantities $a^{(\nu)}$, $b^{(\nu)}$ is specified by \((2)\) (and see also \((3c)\)), while the monic polynomials $p_n^{(\nu)}(x)$ are defined by the latter quantities via the basic three-term recursion relation \((1)\). In the following subsections
we investigate the classes of these polynomials which are defined by the three-term recursion (1) with coefficients satisfying the relation (30), so that the corresponding polynomials also satisfy the second recursion relation (31). We shall of course limit our consideration to the simpler cases, corresponding to the simpler assignments of the arbitrary coefficients $b_k^{(v)}$ in (30) and (31).

2.1. $K = 0, \bar{e}^{(v)} = 0$

This is a quite trivial case. Indeed with this assignment (30) yields $a = a^{(+)}$ and $b = b^{(+)}$, entailing $a_n^{(v)} = a_n$ and $b_n^{(v)} = b_n$ (both independent of $v$). Hence the class of polynomials defined by (1) is independent of $v$, $p_k^{(v)}(x) = p_n(x)$ and the second recursion, as yielded by (31), becomes trivial, $p_k^{(v+1)}(x) = p_n^{(v)}(x) = p_n(x)$.

2.2. $K = 0, \bar{e}^{(v)} = \bar{e}^{(v)}$

With this assignment the second recursion (31) reads

$$p_n^{(v+1)}(x) = p_n^{(v)}(x) + \bar{e}^{(v)} \frac{b_n^{(v+1)}}{b_n^{(v)}} p_n^{(v)}(x),$$

hence it coincides with (4a) (with $v$ replaced by $v+1$) if one sets

$$\bar{e}^{(v+1)} = \bar{e}^{(v)} \frac{b_n^{(v+1)}}{b_n^{(v)}}.$$  

This entails

$$g_n^{(v)} \frac{b_n^{(v+1)}}{b_n^{(v+1)}} = \frac{b_n^{(v+1)}}{b_n^{(v+1)}} \frac{b_n^{(v-1)}}{b_n^{(v-1)}} = \frac{b_n^{(v+1)}}{b_n^{(v+1)}} \frac{b_n^{(v-1)}}{b_n^{(v-1)}},$$

where the last step is justified by the second (2a). Clearly this relation coincides with (6b).

Moreover, with this assignment the spinor formula (30) yields the two relations

$$a_n^{(v+1)} - a_n^{(v)} = \bar{e}^{(v)} \left( \frac{b_n^{(v+1)}}{b_n^{(v)}} - \frac{b_n^{(v+1)}}{b_n^{(v)}} \right),$$

$$b_n^{(v+1)} - b_n^{(v+1)} = \bar{e}^{(v)} (a_n^{(v+1)} - a_n^{(v+1)}) \frac{b_n^{(v+1)}}{b_n^{(v+1)}}.$$  

The first of these, via (33), becomes

$$a_n^{(v+1)} - a_n^{(v)} = g_n^{(v+1)} - g_n^{(v)};$$

which coincides with (6a) (with $v$ replaced by $v+1$); and the second, again via (33), becomes

$$b_n^{(v+1)} - b_n^{(v+1)} = g_n^{(v+1)} (a_n^{(v+1)} - a_n^{(v)}),$$

which coincides with (6c) (again, with $v$ replaced by $v+1$). It is thus seen that this assignment reproduces the results of [4], as reported above (see Sec. 1, in particular (4a) and (6)).
2.3. \( K = 1, \varepsilon^{(0)(\nu)} = 0, \varepsilon^{(1)(\nu)} \neq 0 \)

As shown in Appendix B, with this assignment the second recursion (31) reads

\[
p_n^{(\nu+1)}(x) = p_n^{(\nu)}(x) + \varepsilon^{(1)(\nu)} \left[ \frac{\phi_n^{(\nu+1)}}{B_{n-2}^{(\nu+1)}} (A^{(\nu)}_{n-1} - A^{(\nu+1)}_{n+1}) + \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu+1)}} p_{n-1}^{(\nu)}(x) \right],
\]

(37)

and the conditions to be satisfied by the coefficients defining the polynomials \( p_n^{(\nu)}(x) \) via the basic recursion relation (1) read

\[
A_n - A - A_{n-1}^{(+)} + A^{(+)} = \varepsilon^{(0)(\nu)} \left[ \frac{B_n^{(+)}}{B_{n-1}^{(+)}} (A^{(+)} - A) - \frac{B^{(+)}_{n-1}}{B_{n-2}^{(+)}} (A_{n-1}^{(+)} - A_{n-2}) \right],
\]

(38a)

\[
\frac{B^{(+)}_{n-1}}{B_{n-2}^{(+)}} = \varepsilon^{(0)(\nu)} \left[ \frac{B_n^{(+)}}{B_{n-1}^{(+)}} (A - A_{n-1} + A^{(+)} - A_{n+1}^{(+)}) \right. \\
\left. - \frac{B^{(+)}_{n-1}}{B_{n-2}^{(+)}} (A_{n-1}^{(+)} - A_{n-2}) \right].
\]

(38b)

As shown in Appendix B, this system of two (nonlinear discrete) equations, (38), for the two dependent variables \( A_n^{(\nu)} \) and \( B_n^{(\nu)} \), can be reformulated as the following single equation for the quantity \( A_n^{(\nu)} \):

\[
(A_n^{(+)} - A_{n-1})(A_{n+1}^{(+)} - A_{n-1}^{(+)} + A^{(+)} - A + \phi^{(\nu)}) + (A_{n+1}^{(+)} + A^{(+)} - A_{n-1}^{(+)} - A_{n-1}^{(+)} + A^{(+)} - A + \phi^{(\nu+1)})
\]

\[
\cdot (A_{n+1}^{(+)} + A^{(+)} - A_{n+1}^{(+)} + A^{(+)} - A_{n+1}^{(+)} + A^{(+)} - A + \phi^{(\nu+1)})
\]

\[
\cdot (A_{n+1}^{(+)} + A^{(+)} - A_{n+1}^{(+)} + A^{(+)} - A_{n+1}^{(+)} + A^{(+)} - A + \phi^{(\nu+1)})
\]

\[
\cdot (A_{n+1}^{(+)} + A^{(+)} - A_{n+1}^{(+)} + A^{(+)} - A_{n+1}^{(+)} + A^{(+)} - A + \phi^{(\nu+1)}).
\]

(39)

Here \( \phi^{(\nu)} \) and \( \psi^{(\nu)} \) are independent of \( n \) (and of course of \( x \)), but can depend arbitrarily on \( \nu \).

Let us now point out that the second recursion relation (37) holds trivially for \( n = 0 \) since \( p_n^{(\nu)} \) vanishes identically for negative \( m \), while for \( n = 1 \), via (1b), (1c) and (2b), it entails the following formula determining \( \varepsilon^{(1)(\nu)} \) in terms of “initial” values of the dependent variables \( A \) and \( B \) (recall (2a)):

\[
\varepsilon^{(1)(\nu)} = \varepsilon^{(0)(\nu)} \left[ \frac{B_0^{(\nu+1)} (A_0^{(\nu+1)} - A_1^{(\nu+1)})}{B_0^{(\nu+1)} (A_0^{(\nu+1)} + A_1^{(\nu+1)} - A_0^{(\nu+1)})} \right].
\]

(40)

Finally let us note that, via (1a) and (2), the second recursion relation (37) can now be reformulated as follows:

\[
p_n^{(\nu+1)}(x) = (1 - \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu+1)}}) p_n^{(\nu)}(x) + \varepsilon^{(1)(\nu)} \left( \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu+1)}} (x + A^{(\nu)} - A_{n+1}^{(\nu+1)}) p_{n-1}^{(\nu)}(x) \right).
\]

(41a)
Via (1a), that of course entails analogical to those detailed in the preceding Sec. 2.3 and especially in the related Appendix.

The computations to arrive at these findings are somewhat more cumbersome yet quite

hence, via the formula (135a) of Appendix B, it can also be rewritten in the following form (from which the parameter \( \tilde{\psi}^{(v)} \) disappeared):

\[
p^{(v+1)}_n(x) = \left( \frac{1 + \frac{A_0^{(v+1)} - A_0^{(v)} + \phi^{(v)}}{A_0^{(v+1)} - A_0^{(v)}}}{\frac{A_0^{(v+1)} - A_0^{(v)}}{A_0^{(v+1)} - A_0^{(v)}}} \right) p^{(v)}_n(x) + \frac{A_0^{(v+1)} - A_0^{(v)} + \phi^{(v)}}{A_0^{(v+1)} - A_0^{(v)}} p^{(v)}_n(x) - x p^{(v)}_{n-1}(x) + (A_0^{(v+1)} - A_0^{(v)}) p^{(v)}_{n-1}(x).
\]

(41b)

Via (1a), that of course entails

\[
p^{(v-1)}_n(x) - x p^{(v-1)}_{n-1}(x) = a^{(v-1)}_{n-1} P^{(v-1)}_{n-1}(x),
\]

(42)

as well as (2a) and the formula (140a) of Appendix B, another avatar of this formula, (41b), reads as follows:

\[
p^{(v)}_n(x) = p^{(v-1)}_n(x) + G^{(v)}_n P^{(v-1)}_{n-1}(x) + \tilde{G}^{(v)}_n P^{(v-1)}_{n-2}(x),
\]

(43a)

\[
G^{(v)}_n = (A_0^{(v)} - A_0^{(v-1)} + \phi^{(v-1)}),
\]

(43b)

\[
\tilde{G}^{(v)}_n = \phi^{(v-1)} + \frac{A_0^{(v)} - A_0^{(v-1)} + \phi^{(v-1)} + A_0^{(v-1)} - A_0^{(v-2)}}{(A_0^{(v-1)} + A_0^{(v-2)} - A_0^{(v)} - A_0^{(v-1)}) (A_0^{(v-1)} - A_0^{(v)}) (A_0^{(v-1)} + A_0^{(v-1)} - A_0^{(v-2)} - \phi^{(v-1)})}
\]

(43c)

\[2.4. \ K = 1, \ \tilde{\psi}^{(v)}(\nu) \neq 0, \ \tilde{\psi}^{(1)}(\nu) \neq 0\]

The findings reported in this subsection encompass those of the previous two Subsec. 2.2 respectively 2.3 (and of course reduce to them if one sets \( \tilde{\psi}^{(1)}(\nu) = 0 \) respectively \( \tilde{\psi}^{(v)}(\nu) = 0 \).

The second recursion (31) now takes again (after replacing \( \nu \) with \( \nu - 1 \)) the form (43a),

\[
p^{(v)}(x) = p^{(v-1)}(x) + G^{(v)}_n P^{(v-1)}_{n-1}(x) + \tilde{G}^{(v)}_n P^{(v-1)}_{n-2}(x),
\]

(44a)

but now with the following definition of the two quantities \( G^{(v)}_n \) and \( \tilde{G}^{(v)}_n \):

\[
G^{(v)}_n = \frac{\tilde{\psi}^{(v-1)}(\nu-1) A_0^{(v)} - A_0^{(v)} + \phi^{(v)} - A_0^{(v-1)}}{B^{(v-1)}_{n-1}}
\]

(44b)

\[
\tilde{G}^{(v)}_n = \frac{\tilde{\psi}^{(1)}(\nu-1) B^{(v)}_{n-1}}{B^{(v-1)}_{n-1}}
\]

(44c)
while the conditions to be satisfied by the coefficients defining the polynomials \( p^{(i)}(x) \) via the basic recursion relation (1) now read

\[
A_+ - A - A_{n+1}^{(i+1)} + A_{n+1}^{(i)} = \tilde{c}^{(0)}(n) \frac{B^{(i+1)} - B^{(i)}}{B_-},
\]

\[
+ \tilde{c}^{(1)}(n) \frac{B^{(i+1)} (A_{n+1}^{(i+1)} - A) - B^{(i)} (A_{n+1}^{(i)} - A_-)}{B_- (A - A_- + A^{(i)} - A)}.
\]

(45a)

\[
\frac{B^{(i+1)}}{B_-} = \tilde{c}^{(0)}(n) \frac{B^{(i)} (A - A_- + A^{(i)} - A_-)}{B_- (A - A_- + A^{(i)} - A)}
\]

\[
+ \tilde{c}^{(1)}(n) \frac{B^{(i+1)} (A_{n+1}^{(i+1)} - A_{n+1}^{(i)})}{B_- (A_{n+1}^{(i+1)} - A_{n+1}^{(i)} - A_-)}
\]

+ \frac{B^{(i+1)}}{B_-} - \frac{B^{(i)}}{B_-}

(45b)

This system of two (nonlinear discrete) equations, (38), for the two dependent variables \( A_{n+1}^{(i)} \) and \( B_{n+1}^{(i)} \), can be reformulated as the following (notationally compactified) single equation for the quantity \( \bar{A}_{n+1}^{(i)} \):

\[
\tilde{C}_{n+1}^{(i+1)} \tilde{c}_{n+1}^{(i)} \bar{A}_{n+1}^{(i)} = \tilde{c}_{n-1}^{(i)} \tilde{C}_{n+1}^{(i+1)} \bar{c}_{n+1}^{(i+1)} \bar{A}_{n+1}^{(i+1)},
\]

(46a)

where

\[
\tilde{C}_{n+1}^{(i+1)} \equiv \tilde{c}_{n+1}^{(i)} + \tilde{c}_{n+1}^{(i+1)} \bar{A}_{n+1}^{(i)} - A_{n+1}^{(i+1)},
\]

(46b)

\[
\tilde{C}_{n+1}^{(i)} \equiv \tilde{c}_{n+1}^{(0)} + \tilde{c}_{n+1}^{(i)} (A_{n+1}^{(i+1)} - A_{n+1}^{(i)}),
\]

(46c)

\[
\tilde{c}_{n+1}^{(0)} \equiv (A_{n+1}^{(i+1)} - A_{n+1}^{(i)}) [\tilde{c}_{n+1}^{(0)} - \tilde{c}_{n+1}^{(i)} (A_{n+1}^{(i+1)} - A_{n+1}^{(i)})] + \tilde{\varphi}_{n+1}^{(i)}
\]

(46d)

\[
\bar{A}_{n+1}^{(i)} \equiv A_{n+1}^{(i+1)} - A_{n+1}^{(i)} + \bar{F}_{n+1}^{(i)} + \bar{A}_{n+1}^{(i)} \equiv A_{n+1}^{(i+1)} - A_{n+1}^{(i)} + \bar{F}_{n+1}^{(i)}
\]

(46e)

Here \( F_{n+1}^{(i)} \) and \( \varphi_{n+1}^{(i)} \) are two arbitrary functions of \( i \) only (i.e. independent of the index \( n \)). On the other hand via (1) and (2) the initial conditions entail

\[
\bar{F}_{n+1}^{(i)} = A_{n+1}^{(i)} - A_{n+1}^{(i+1)}
\]

(47)

Let us also report the expression of the coefficient \( \tilde{b}_{n+1}^{(i)} \) (see (1)) in terms of these quantities:

\[
\tilde{b}_{n+1}^{(i)} = (\tilde{c}_{n+1}^{(0)} + \tilde{c}_{n+1}^{(i)} A_{n+1}^{(i)} - \tilde{c}_{n+1}^{(i)} A_{n+1}^{(i+1)} (-A_{n+1}^{(i+1)} + A_{n+1}^{(i)} - F_{n+1}^{(i)}))
\]

\[
\cdot [-\tilde{c}_{n+1}^{(i+1)} A_{n+1}^{(i)} + \tilde{c}_{n+1}^{(i+1)} F_{n+1}^{(i)} A_{n+1}^{(i+1)} - \tilde{c}_{n+1}^{(i)} F_{n+1}^{(i)} A_{n+1}^{(i)} + \tilde{c}_{n+1}^{(i)} (A_{n+1}^{(i)} A_{n+1}^{(i)} A_{n+1}^{(i)} 2)
\]

\[
- 2 \tilde{c}_{n+1}^{(i+1)} A_{n+1}^{(i)} A_{n+1}^{(i)} + \tilde{c}_{n+1}^{(i)} (A_{n+1}^{(i+1)} 2) - \tilde{c}_{n+1}^{(i)}]
\]

\[
\cdot [(\tilde{c}_{n+1}^{(0)} + \tilde{c}_{n+1}^{(i)} A_{n+1}^{(i-1)} - \tilde{c}_{n+1}^{(i)} A_{n+1}^{(i+1)} - \tilde{c}_{n+1}^{(i)} A_{n+1}^{(i)} + \tilde{c}_{n+1}^{(i)} F_{n+1}^{(i)} A_{n+1}^{(i)} A_{n+1}^{(i)} A_{n+1}^{(i)} A_{n+1}^{(i)} 2]
\]

\[
- 2 \tilde{c}_{n+1}^{(i+1)} A_{n+1}^{(i)} A_{n+1}^{(i)} + \tilde{c}_{n+1}^{(i)} (A_{n+1}^{(i+1)} 2) - \tilde{c}_{n+1}^{(i)}]^{-1}.
\]

(48)
The simultaneous validity for a class of polynomials $p_n^{(\nu)}$ is given by (2a), hence it is sufficiently simple not to require explicit display for the present case. But we do display the second recurrence:

$$p_{n+1}^{(\nu)} = \frac{\tilde{c}(\nu|x) A_n^{(\nu)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu)} - \tilde{c}(\nu|x) A_n^{(\nu+1)} - \tilde{c}(\nu|x) A_{n+1}^{(\nu+1)} + \tilde{c}(\nu|x) A_n^{(\nu)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu)} - \tilde{c}(\nu|x) A_n^{(\nu+1)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu+1)}}{\tilde{c}(\nu|x) A_n^{(\nu)} - \tilde{c}(\nu|x) A_{n+1}^{(\nu)} + \tilde{c}(\nu|x)}$$

(49)

3. Factorizations

The corresponding expression of the coefficients $a_m^{(\nu)}$ is given by (2a), hence it is sufficiently simple not to require explicit display for the present case. But we do display the second recurrence:

$$p_{n+1}^{(\nu)} = \frac{\tilde{c}(\nu|x) A_n^{(\nu)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu)} - \tilde{c}(\nu|x) A_n^{(\nu+1)} - \tilde{c}(\nu|x) A_{n+1}^{(\nu+1)} + \tilde{c}(\nu|x) A_n^{(\nu)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu)} - \tilde{c}(\nu|x) A_n^{(\nu+1)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu+1)}}{\tilde{c}(\nu|x) A_n^{(\nu)} - \tilde{c}(\nu|x) A_{n+1}^{(\nu)} + \tilde{c}(\nu|x)}$$

(49)

The corresponding expression of the coefficients $a_m^{(\nu)}$ is given by (2a), hence it is sufficiently simple not to require explicit display for the present case. But we do display the second recurrence:

$$p_{n+1}^{(\nu)} = \frac{\tilde{c}(\nu|x) A_n^{(\nu)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu)} - \tilde{c}(\nu|x) A_n^{(\nu+1)} - \tilde{c}(\nu|x) A_{n+1}^{(\nu+1)} + \tilde{c}(\nu|x) A_n^{(\nu)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu)} - \tilde{c}(\nu|x) A_n^{(\nu+1)} + \tilde{c}(\nu|x) A_{n+1}^{(\nu+1)}}{\tilde{c}(\nu|x) A_n^{(\nu)} - \tilde{c}(\nu|x) A_{n+1}^{(\nu)} + \tilde{c}(\nu|x)}$$

(49)

3. Factorizations

The simultaneous validity for a class of polynomials $p_n^{(\nu)}(x)$ of two recursion relations involving shifts in the degree $n$ of the polynomials and in their parameter $\nu$ allows to identify subclasses of these polynomials — characterized by appropriate restrictions on the coefficients defining them (see (1)) — for which there hold remarkably neat factorizations. Such results were indeed the first motivation of our investigation and are reported in previous papers of this series, see for instance [4] were results implied by the simultaneous validity of (1) and the second recursion obtained in Sec. 2.3, see (43); the proofs of these results are relegated to Appendix C. Note that the same version of the second recursion relation, see (43) respectively (44a), has been obtained in Secs. 2.3 respectively 2.4, although of course the corresponding nonlinear conditions on the quantities $A_n^{(\nu)}$ are different in the two cases, see (39) respectively (40).

Proposition 3.1. If for some value of the parameter $\mu$ and for all positive integer values of $n$ there holds the condition

$$b_{n+1}^{(\nu+1)} + G_n^{(\nu)} = 0,$$

(50)

with $G_n^{(\nu)}$ defined by (43c), then for the corresponding polynomials there holds the complete factorization

$$p_n^{(\nu)}(x) = \prod_{m=1}^{n} (x - x_m^{(\nu+1)}),$$

(51a)

with

$$x_m^{(\nu+1)} = -(a_{m-1}^{(\nu)} + G_m^{(\nu)})$$

(51b)

where of course $G_m^{(\nu)}$ is defined by (43b).

Proposition 3.2. If for some value of the parameter $\mu$ and for all positive integer values of $n$ there hold the conditions

$$b_{n+1}^{(2n+1)} + G_n^{(2n-1+\nu)} + G_n^{(2n+\nu)} + G_n^{(2n+\nu)} + G_n^{(2n-1+\nu)} = 0,$$

(52a)
factorization

where of course $G^{(v)}_m$ defined — as the case may be — by (43b) respectively (43c) or by (44b) respectively (44c), then for the corresponding polynomials there holds the complete factorization

$$p_m^{(2m+\nu)}(x) = \prod_{m=1}^{\nu} \left( x - x_m^{(2m+\nu)} \right),$$

with

$$x_m^{(2m)} = -\left( n_{m-1} + G^{(v-1)}_m + G^{(v)}_m \right),$$

where of course $G^{(v)}_m$ is defined by (43b) or (44b), as the case may be.

Proposition 3.3. If for some value of the parameter $\mu$ and for all positive integer values of $n$ there hold the conditions

$$\begin{align*}
al_{n-1}^{(n+\mu)} + a_{n-1}^{(n+\mu)} G^{(n+\mu)}_{n-1} - a_{n-2}^{(n+\mu)} G^{(n+\mu)}_{n-2} &= 0, \\
b_{n-1}^{(n+\mu)} + b_{n-1}^{(n+\mu)} G^{(n+\mu)}_{n-1} - b_{n-2}^{(n+\mu)} G^{(n+\mu)}_{n-2} &= 0, \\
g_{n-1}^{(n+\mu)} + g_{n-1}^{(n+\mu)} G^{(n+\mu)}_{n-1} - g_{n-2}^{(n+\mu)} G^{(n+\mu)}_{n-2} &= 0,
\end{align*}$$

with $G^{(v)}_m$ respectively $\tilde{G}^{(v)}_m$ defined — as the case may be — by (43b) respectively (43c) or by (44b) respectively (44c), then for the corresponding polynomials there holds the complete factorization

$$p_m^{(3m+\nu)}(x) = \prod_{m=1}^{\nu} \left( x - x_m^{(3m+\nu)} \right),$$

with

$$x_m^{(3m)} = -\left( a_{m-1}^{(v)} + G^{(v)}_{m-1} \right),$$

where of course $G^{(v)}_m$ is defined by (43b) or (44b), as the case may be.

4. Classes of Orthogonal Polynomials Identified by Solutions of the Nonlinear Relations (46)

In this section various solutions are reported of the nonlinear relations (46) satisfied by the quantities $A^{(v)}_n$. These solutions are obtained by a trial and error procedure: ansätze (involving several free parameters), which specify the dependence of these quantities on $n$ and on $\nu$, are required to satisfy (46). Whenever a solution $A^{(v)}_n$ of (46) is obtained in this manner, its implications — based on the findings described above — for the corresponding polynomials $p_n^{(v)}(x)$ are reported, as well as the identification of these polynomials with named polynomials whenever this is possible.
In the following the new parameters introduced — for which various notations are used — are understood to be arbitrary numbers (unless otherwise explicitly stated), and their relations to the parameters introduced above are detailed whenever appropriate.

4.1. Polynomial case

In this subsection attention is restricted to quantities \( A_\nu^n \) depending polynomially on \( n \) and \( \nu \). For practical reasons only polynomials of degree less or equal to 3 are treated.

4.1.1. Quadratic case

The relevant solution reads

\[ A_\nu^n = 2^n \nu - n \rho - n^2 + u_0 + u_1 \nu, \quad (56a) \]

with

\[ \xi^{(0)} = \frac{c_0 (h_0 + h_1 \nu)}{h_0}, \quad \xi^{(1)} = h_0 + h_1 \nu; \quad (56b) \]

\[ \varphi^{(\nu)} = -w_1, \quad \varphi^{(\nu)} = -\frac{1}{4} (h_0 u_1 + c_0)^2 (h_0 + h_1 \nu). \quad (56c) \]

The corresponding coefficients \( a_\nu^n \) and \( b_\nu^n \) are

\[ a_\nu^n = -2n + 2\nu - \rho - 1, \quad (57a) \]

\[ b_\nu^n = \frac{1}{2} \left( -4h_0 \nu + c_0 + 2h_0 \rho + 2h_0 n - h_0 u_1 \right). \quad (57b) \]

The corresponding polynomials \( p_\nu^n(x) \) satisfy the second recurrence relation

\[ (\tau^{(\nu)} + 2h_0 n)p_\nu^{(n+1)}(x) = \tau^{(\nu)} p_\nu^n(x) + n(\tau^{(\nu)} + h_0 u_1 + c_0 + 2h_0 x)p_{\nu-1}^{(n)}(x), \quad (58a) \]

where

\[ \tau^{(\nu)} = -(2h_0 - 2h_0 \rho + 4h_0 \nu + h_0 u_1 - c_0). \quad (58b) \]

Via (44a)–(44c), this recursion can be reformulated as follows:

\[ p_\nu^{(n+1)} = p_\nu^{(n)} + G_n^{(\nu)} p_{\nu-1}^{(n)} + \tilde{G}_n^{(\nu)} p_{\nu-2}^{(n)}, \quad (59a) \]

with

\[ G_n^{(\nu)} = G_n = 2n, \quad \tilde{G}_n^{(\nu)} = \tilde{G}_n^{(\nu)} = n(n - 1). \quad (59b) \]

Note that the parameter \( h_1 \) plays no role in this last formulae, (57)–(59), as well as the simple form of the coefficients of this recursion relation, which turn out to be independent of \( \nu \). And it is easily seen that these polynomials coincide, up to a translation, with the (generalized) Laguerre polynomials:

\[ p_\nu^n(x) = (-1)^n n! L_n^{(\alpha)}(y), \quad (60a) \]
with
\[ y = x + \frac{(c_0 - b_0) y_1}{2b_0}, \quad \alpha = \frac{c_0 - b_0 y_1 - 4b_0 \nu + 2b_0 \rho}{2b_0} \] (60b)

And (58a) becomes the well-known relation
\[ (n + \alpha - 1) L_n^{(\nu - 1)}(y) = (\alpha - 1) L_{n+1}^{(\nu - 1)}(y) - (y + \alpha - 1) L_{n-1}^{(\nu)}(y). \] (61)

4.1.2. Cubic case
A solution, cubic in \( y \) and \( \nu \), of the relations (46) is
\[
A_n^{(\nu)} = \frac{2}{3} y^3 + \left( -\rho + 2\nu + \frac{3}{2} - \tau \right) y^2 + \frac{2(\tau - 1 + \rho) \nu - \frac{5}{6} + \tau + \rho - \tau \rho}{n} n
+ \tilde{\sigma} \nu - 2\nu^2 - \frac{8}{3} y^3 + \omega, \tag{62a}
\]
with
\[
\tilde{c}^{(0)(\nu)} = ch, \quad \tilde{c}^{(1)(\nu)} = h, \tag{62b}
\]
\[
\tilde{F}^{(\nu)} = -\tilde{\sigma} + 12\nu + \frac{14}{3} + 8\nu^2, \tag{62c}
\]
\[
\tilde{\varphi}^{(\nu)} = -\frac{(-3\nu + 24\nu + 24\nu^2 - 3\nu)(-3\nu + 23 + 48\nu + 24\nu^2 - 3\nu)}{36}. \tag{62d}
\]
The corresponding coefficients \( a_m^{(\nu)} \) and \( b_n^{(\nu)} \) are:
\[
a_m^{(\nu)} = -2n^2 + (1 - 2\nu + 4\nu - 2\rho)n + 2(\tau + \rho) \nu - \tau \rho, \tag{63a}
\]
\[
b_n^{(\nu)} = \frac{n(n + \tau + \rho - 1)}{6} \left[ 6n^2 + (-24\nu + 6(\tau + \rho) - 2)n + 24\nu^2 - 12(\tau - \rho - 2)\nu + 6\tau \rho - 6\tau - 6\rho - 3\tau + 5 \right]. \tag{63b}
\]
where \( \sigma = \tilde{\sigma} - c \). (63c)

and the corresponding polynomials \( p_k^{(\nu)}(x) \) satisfy the following second recursion relation:
\[
[6n(n - 4 + \rho + \tau - 4\nu) + \eta^{(\nu)}] p_n^{(\nu+1)}
= -([24\nu + 18]n - \eta^{(\nu)} p_n^{(\nu)} - n(n + \tau + \rho - 1)(24\nu + 18)n - \eta^{(\nu)}
+ 3\sigma - 6\nu - 24\nu^2 - 5 - 24\nu^2 p_{n-1}^{(\nu)}]. \tag{64a}
\]
where
\[ \eta^{(\nu)} = 24\nu^2 - 12(-4 + \rho + \tau)\nu + 6\tau \rho - 12\tau - 12\rho + 23 - 3\sigma. \tag{64b} \]

Via (44a)–(44c), this recursion can be reformulated as follows:
\[ p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)} p_{n-1}^{(\nu)} + G_n^{(\nu)} p_{n-2}^{(\nu)}. \tag{65a} \]
Polynomials Defined by Three-Term Recursion Relations

with

\[ G_n^{(\nu)} = 2n(n + \tau + \rho - 1), \quad (65b) \]
\[ \tilde{G}_n^{(\nu)} = n(n - 1)(n - 2 + \tau + \rho)(n + \tau + \rho - 1). \quad (65c) \]

And it is easily seen that the normalized Continuous Dual Hahn polynomials \( p_n^{(\nu)}(x; \rho, \sigma, \tau, c) \) defined by the standard recurrence (1) with \( a_n^{(\nu)} \) and \( b_n^{(\nu)} \) defined by (63) and the assignments:

\[ y = x - \frac{1 + 3\tau}{6}, \quad (66a) \]
\[ \alpha = \frac{3\tau + 3\rho - \sqrt{3} \sqrt{(3\tau^2 - 6\tau\rho + 3\rho^2 + 6\rho + 2)}}{6}, \quad (66b) \]
\[ \beta = \frac{3\tau + 3\rho + \sqrt{3} \sqrt{(3\tau^2 - 6\tau\rho + 3\rho^2 + 6\rho + 2)}}{6}, \quad (66c) \]
\[ \gamma = -2\nu. \quad (66d) \]

Hence the Continuous Dual Hahn polynomials \( p_n^{(\nu)}(y; \alpha, \beta, \gamma) \) satisfy a second recurrence relation, see (64).

Let us also recall that these polynomials \( p_n^{(\nu)}(y; \alpha, \beta, \gamma) \) are invariant under permutations of the 3 parameters \( \alpha, \beta, \gamma \).

**Factorizations.** When \( \nu = n + \mu \), \( (67a) \)

with

\[ \mu = \frac{1}{2} \left( \tau + \rho - \frac{1}{2} \right), \quad (67b) \]

the condition (50) is satisfied provided

\[ \sigma = -\tau - \rho + 2\tau + \frac{1}{6} \quad (67c) \]

Then for the corresponding polynomials \( p_n^{(n + \mu)}(x) \), there holds the complete factorization (51), with the zeros \( x_n \) depending quadratically on \( n \), namely

\[ x_n = -4n^2 + (-4\tau - 4\rho + 6)n + 5/2\tau - \tau^2 - 2 + 5/2\rho - \rho^2 - \tau \rho. \quad (68) \]

This entails, via (66) with (67c), the following complete Diophantine factorization for the normalized Continuous Dual Hahn polynomials:

\[ P_n^{(\nu)} \left( x - \frac{1}{4}, \frac{1}{2} + \frac{1}{2} \tau - \rho, \frac{1}{2}, \rho + \tau - \frac{1}{2}, -2n - \tau - 3 \right) = \prod_{m=1}^n \left| x - x_m \right|, \quad (69) \]

with the zeros \( x_m \) defined of course by (68).
4.2. Rational case

In this subsection attention is restricted to quantities $A_n^{(\nu)}$ depending rationally (and, for practical reasons, rather simply) on $n$ and $\nu$.

4.2.1. Two cases with linear numerator and denominator

We begin with two cases featuring a rational solution of the relations (46) with both numerator and denominator linear in $n$ and $\nu$. The first reads as follows:

$$A_n^{(\nu)} = -\frac{n\delta}{2n - 2\nu + \delta}.$$  \hspace{1cm} (70a)

with

$$\bar{c}^{(0)\nu} = 0, \quad \bar{c}^{(1)\nu} = h, \quad F^{(\nu)} = 0, \quad \varphi^{(\nu)} = G.$$  \hspace{1cm} (70b)

The corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are

$$a_n^{(\nu)} = \frac{\delta(2\nu - \delta)}{2(n + 2 - 2\nu + \delta) \nu(n\nu + \nu + 1)}$$ \hspace{1cm} (71a)

$$b_n^{(\nu)} = -\frac{(n - 2\nu + \delta)(n)(2n - 2\nu + \delta - 1)}{h(2n - 2\nu + \delta)^2(2n - 2\nu + \delta + 1)(2n - 2\nu + \delta - 1)}.$$ \hspace{1cm} (71b)

The corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the following second recursion relation:

$$(n - 2\nu + \delta - 1)(2n - 2\nu + \delta)p_n^{(\nu+1)} = (2n - 2\nu + \delta)(2n - 2\nu + \delta - 1)p_n^{(\nu)} - n(n - 2\nu + \delta + \delta)p_{n-1}^{(\nu)}$$ \hspace{1cm} (72)

or equivalently, via (44a)–(44c),

$$p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)} p_{n-1}^{(\nu)} + G_n^{(\nu)} p_{n-2}^{(\nu)}.$$ \hspace{1cm} (73a)

with

$$G_n^{(\nu)} = -\frac{2n\delta}{(2n - 2\nu + \delta)(2n + 2 - 2\nu + \delta)},$$ \hspace{1cm} (73b)

$$\tilde{G}_n^{(\nu)} = \frac{n(n - 1)(\nu^2 + G(2n - 2\nu + \delta))}{n(2n - 2\nu + \delta)(2n - 2\nu + \delta - 1)(2n - 2\nu + \delta + 1)}.$$ \hspace{1cm} (73c)

And it is easily seen that these polynomials coincide, up to a rescaling of the argument, with the standard Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$:

$$p_n^{(\nu)}(x) = \frac{2^n n!}{(n + \alpha + \beta + 1)} P_n^{(\alpha,\beta)}(y).$$ \hspace{1cm} (74a)
where

\[ y = \sqrt{\frac{Gk}{G}} z, \quad (74b) \]

\[ \alpha = \frac{Gk - 2G\nu - \delta \sqrt{Gk}}{2G}, \quad (74c) \]

\[ \beta = \frac{Gk - 2G\nu + \delta \sqrt{Gk}}{2G} \quad (74d) \]

**Factorizations.** When

\[ \nu = n + \mu \quad (75a) \]

with

\[ \mu = \frac{(G \pm \sqrt{Gk})\delta}{2G} \quad (75b) \]

the condition (50) is satisfied. Then for the corresponding polynomials \( p_n^{(n+\mu)}(x) \), there holds the complete factorization (51), which however merely entails the well known fact that the Jacobi polynomial \( P_n^{(\nu,-\nu)}(x) \) is proportional to \( (x - 1)^n \).

A second rational solution of the relations (46) with both numerator and denominator linear in \( n \) and \( \nu \) reads as follows:

\[ A_n^{(\nu)} = \frac{2\nu(k + 1) - \delta}{2n - 2\nu + \delta} + u_0 + u_1 \nu, \quad (76a) \]

with

\[ \tilde{c}^{(\nu)} = 2b(k + 1) + c, \quad \tilde{c}^{(1\nu)} = \frac{2b(k + 1) + c}{2(k + 1) + u_1}, \quad (76b) \]

\[ F^{(\nu)} = -u_1, \quad \tilde{c}^{(\nu)} = \frac{(k + 1 + u_1)^2 (2h k + 2h + c)}{2k + u_1 + 2} \quad (76c) \]

The corresponding coefficients \( a_n^{(\nu)} \) and \( b_n^{(\nu)} \) are

\[ a_n^{(\nu)} = -\frac{(2k\nu + 2\nu - \delta)(2\nu - \delta)}{[2n + 2 - 2\nu + \delta][2n - 2\nu + \delta]}, \quad (77) \]

\[ b_n^{(\nu)} = \frac{(\delta + n - 2\nu)(2nk + 2n - 4k\nu + k\delta - 4\nu + 2\nu)n(k\delta + 2nk + 2n)}{(2n - 2\nu + \delta)^2(-2\nu + 2n + \delta - 1)(-2\nu + \delta + 2n + 1)} \quad (78) \]

The corresponding polynomials \( p_n^{(\nu)}(x) \) satisfy the following second recursion relation:

\[ p_n^{(\nu+1)} = -\frac{(4\nu + 4k\nu - 2\delta + 2k + 2 - k\delta)(-2\nu + 2n + \delta - 1)}{(n - 1 - 2\nu + \delta)[2nk + 2n + k\delta - 2k - 2 + 2\delta - 4\nu - 4\nu]} p_n^{(\nu)} + \frac{n(k\delta + 2nk + 2n)}{(2nk + 2n + k\delta - 2k - 2 + 2\delta - 4\nu - 4\nu)} x(2n - 2\nu + \delta) + 2k(n - 3\nu + \delta - 1) + 2n - 6\nu + 3\delta - 2}{(n - 1 - 2\nu + \delta)(2n - 2\nu + \delta)} p_{n-1}^{(\nu)} \quad (79) \]
or equivalently, via (44a)–(44c),
\[ p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)} p_{n-1}^{(\nu)} + G_n^{(\nu)} p_{n-2}^{(\nu)} \]  
(80a)

with
\[ G_n^{(\nu)} = \frac{2n(2n + 2k + \delta)}{(2n - 2\nu + 2 + \delta)(2n - 2\nu + \delta)} \]  
(80b)
\[ G_n^{(\nu)} = \frac{n(n-1)(2n + 2k + \delta)(2n - 2 + 2k(n-1) + k\delta)}{(2n - 2\nu + \delta)^2(2n - 2\nu + \delta - 1)(2n - 2\nu + \delta + 1)} \]  
(80c)

Note that the parameters \( h, u \) do not appear in these formulae.

Again, it is easily seen that these polynomials coincide, up to a rescaling of the argument, with the Jacobi polynomials \( P_n^{(\alpha, \beta)}(z) \):
\[ p_n^{(\nu)}(x) = \frac{2^n n!}{(n+\alpha+\beta+1)_n} P_n^{(\alpha, \beta)}(y), \]  
(81a)
\[ y = \frac{1}{2(\epsilon+1)} x, \]  
(81b)
\[ \alpha = \delta - 2\nu - \frac{k\delta}{2(\epsilon+1)}, \quad \beta = \frac{k\delta}{2(\epsilon+1)}. \]  
(81c)

4.2.2. A case with quadratic numerator and linear denominator

A rational solution of Eq. (46), with quadratic numerator and linear denominator in \( n \) and \( \nu \), reads as follows:
\[ A_n^{(\nu)} = \frac{q_\nu^2 + \nu(q_2 + q_1) + q_1 n - q_\nu^2 w + (-q_0 w + q_1) \nu + q_0}{1 + w(n - \nu)} \]  
(82a)

with
\[ \tilde{c}^{(\nu)} = h(2q_2 + q_\nu w), \quad \tilde{c}^{(1)} = h, \]  
(82b)
\[ p^{(\nu)} = -q_1, \quad \tilde{c}^{(\nu)} = -h \frac{q_2^2}{w^2}, \]  
(82c)

The corresponding expressions of the coefficients \( a_n^{(\nu)} \) and \( b_n^{(\nu)} \), see (1), read as follows:
\[ a_n^{(\nu)} = \frac{\text{Num} a}{(1 - w n + w) (-1 - w n + w + w \nu)}, \]  
(83a)
\[ \text{Num} a = q_\nu^2 w + (-2q_2 w + q_2(w+2)) n + \nu^2 w + (-q_2 - w q_1 - q_2 w + q_2^2) \nu + q_2 + q_1 - q_0 w; \]  
(83b)
\[ b_n^{(\nu)} = \frac{\text{Num} h}{\text{Den} b}, \]  
(84a)
with
\[ G_n^{(\nu)} = \frac{\text{Num} \tilde{G}}{\text{Den} \tilde{G}}, \]
\[ \tilde{G}_n = \frac{n(q_0w^2 - wq_1 + q_2)}{(-1 - wn + w(\nu - 1))(-1 - wn + w(\nu - 1))}. \]
\[ \text{Num} \tilde{G} = n(n - 1)|wq_4(\nu - 1) - q_4 + w^4q_2q_0 + q_2^2 - 2w^3q_2q_3 - 2w^2q_2q_1 + w^3q_3^2 - wq_4(n - 1)|
\[ \cdot (-wq_4(\nu - 1) + q_4 + w^4q_2q_0 + q_2^2 - 2w^3q_2q_3 - 2w^2q_2q_1 + w^3q_3^2 - wq_4(n - 1))|w(n(n - 1) + 2w(\nu - 1) - 2)|, \]
\[ \text{Den} \tilde{G} = (q_0w^2 - wq_1 + q_2)^2[-wn + 2w(\nu - 1) - 2 + w]
\[ \cdot [-1 - wn + w(\nu - 1)]^{-2}[-2w(n(n - 1) + 2w(\nu - 1) + w - 2]
\[ \cdot [-2w(n - 1) + 2w(\nu - 1) - w - 2]. \]

This class of polynomials features 6 arbitrary parameters — namely \( q_0, q_2, q_3, q_4, w, \nu \) — but it can nevertheless be reduced, via an appropriate translation and rescaling of the argument of the polynomials, to the class of Jacobi polynomials \( P_n^{(\alpha,\beta)}(y) \):

\[ p_n^{(\nu)}(x) = P_n^{(\alpha,\beta)}(z), \]
\[ \alpha = \frac{1}{w - \nu} + \frac{(w^2q_0 + q_2 - wq_3)}{wq_4}, \]
\[ \beta = \frac{1}{w - \nu} - \frac{(w^2q_0 + q_2 - wq_3)}{wq_4}, \]
\[ z = -\frac{(wx + q_2)(w^2q_0 + q_2 - wq_3)}{q_4}. \]

We are grateful to a referee for pointing out this fact.
Factorizations. When

\[ \nu = n + \mu \]  

with

\[ \mu = \frac{q_2^2 h + q_4 + hw^3 q_4^2 + 2w^2 q_2 ph - 2hw^3 q_1 - 2wpq_2 h + hw^2 q_3^2}{w q_1}. \]  

then

\[ p_n^{(n+\nu)}(x) = \left( x + \frac{w^2 q_2 ph - wpq_2 h + q_2^2 h + q_1}{w(q_0 w^2 - w q_1 + q_2) h} \right)^n. \]  

This finding reproduces, via the identification (87), the following well-known property of Jacobi polynomials:

\[ P^{(−n,β)}_n(x) = (x−1)^n. \]  

4.2.3. Two cases with quartic numerator and linear denominator

A first rational solution of Eq. (46), with quartic numerator and linear denominator in \( n \) and \( \nu \), reads as follows:

\[ A_n^{(\nu)} = \frac{\text{Num}}{6(2n−2\nu−2+\sigma)h}, \]  

\[ \text{Num} = -2hw^4 + 4h(2\nu − \sigma + 2)w^3 + b(2\sigma - 3\nu + 9\sigma - 6\sigma - 10)\nu^2 \]
\[ + \left( -32h^3 - 24hw^3 + 6h + 2c + 2h(\rho - \sigma) \right)\nu - h(5\sigma - 6\rho + 6\sigma - 12\rho - 4) \]
\[ + 32hw^4 - 8h(2\sigma - 7)\nu^3 + 4(7h - 3\sigma - 3c)\nu^2 \]
\[ + 2[(\sigma - 2)(3c - h) - 6q_0h]\nu + 6q_0h(\sigma - 2). \]  

where

\[ \sigma = b + c + d, \quad \rho = be + bd + cd, \quad \tau = bcd, \quad \gamma = b^2 + c^2 + d^2. \]  

\[ \tilde{\phi}(\nu) = C, \quad \tilde{\phi}(\nu) = h, \]  

\[ \tilde{\phi}(\nu) = \frac{5h - C + 12h^2 + 8h^2}{h}. \]  

\[ \tilde{\phi}(\nu) = \frac{-(C + 4h + 8h\nu + 4h\nu^2)(C + h + 4h\nu + 4h\nu^2)}{h}. \]  

These polynomials coincide with the Wilson polynomials \( W_n(x; a, b, c, d) \) (see [6]) with

\[ a = -2\nu: \]

\[ p_n^{(\nu)}(x) = \frac{(-1)^n}{(a + a + e - 1)}W_n(x; -2\nu, b, c, d). \]  

(91)
Hence one finds, for the Wilson polynomials $W_n(x; a, b, c, d)$, the following recurrence relation:

$$W_n(x; a, b, c, d) = \frac{\text{Num} \, W_n(x; a, b, c, d) + \text{Num} \, W_{n-1}(x; a, b, c, d)}{\text{Den} \, W_{n-1}(x; a, b, c, d)},$$  \hspace{0.5cm} (92a)

$$\text{Num} = (-2n + 3 - a - \sigma)((-2a + 3)n^2 - (2a - 3)(a + \sigma - 3)n - (a + b - 2)(a + c - 2)(a + d - 2)),$$  \hspace{0.5cm} (92b)

$$\text{Den} = (n - 2 + a + b)(n - 2 + a + d)(n - 2 + a + c)(n - 1 + a + \sigma),$$  \hspace{0.5cm} (92c)

$$\text{Num} = (n - 1 + b + c)(n - 1 + b + d)(n - 1 + c + d)n \cdot \left[ -x(2n - 2 + a + \sigma) + (-2a + 3)n^2 + (-2a \sigma + 3a + 13a - 11 - 4 \sigma^2)n + p(2 - a) + 10a^2 - 2a^3 + 5(2 - \sigma) - \tau + 6a \sigma - 2a^2 \sigma - 17a \right] ,$$  \hspace{0.5cm} (92d)

$$\text{Den} = (n - 2 + a + b)(n - 2 + a + c)(n - 2 + a + d) \cdot (2n - 2 + a + \sigma)(n - 2 + a + \sigma),$$  \hspace{0.5cm} (92e)

with $\sigma, \rho$ and $\tau$ defined as above, see (90c). Analogous recurrence relations involving the other parameters can of course be obtained from the symmetry of the Wilson polynomials in their 4 parameters $a, b, c, d$. We have not found such relations in the standard compilations.

A second rational solution of the equation (46), with quartic numerator and linear denominator in $n$ and $\nu$, reads as follows:

$$A_n^{(c)} = \frac{\text{Num}}{6h(2n - 2 \nu + \sigma + \eta + \delta - 2)},$$  \hspace{0.5cm} (93a)

$$\text{Num} = -2h n^4 + (8h \nu - 4h(\delta - 2 + \eta + \sigma)) n^3 + (-6h^2 \nu + 6h(2\nu - 3 \delta + 2\sigma + \nu) - h(10 + 6h \eta - 9 \delta + 6h \sigma + 6\delta \sigma - 9\eta - 9\sigma)) n^2 + \left[ -8h^3 \nu + 6h(-1 - \delta - 2\sigma - \eta) \nu^2 + (-12h \delta - 12c + 6h \sigma \eta + 6h + 12h \eta \sigma + 6h \sigma - 12h \sigma^2 + 6h \delta \sigma - 12h \eta ) \nu - h(6h \delta \sigma + 5\sigma - 6\delta \sigma - 6h \eta + 5\delta + 4 - 12g \nu + 5\eta - 6\eta \nu) n + 8h^4 \nu - 4h(\delta + 4\sigma + \eta - 5) \nu^4 + (6h \sigma \eta + 16h - 30h \sigma - 6h \eta - 12c + 18h \sigma^2 + 6h \delta \sigma - 6h \delta \nu) \nu^2 + (-12c - 6h \sigma \eta^2 - 6h \delta \eta - 2h \eta - 6h \delta \sigma + 6\sigma - 12g \eta h + 6h \delta \sigma - 2h \delta + 6\eta h + 4h - 14h \sigma + 18h \sigma^2 + 6h \sigma \eta + 6c \delta) \nu + 6g \delta \nu (\delta - 2 + \eta + \sigma) \right],$$  \hspace{0.5cm} (93b)

with

$$\beta^{(0)} = e, \hspace{0.5cm} \beta^{(1)} = h, \hspace{0.5cm} (93c)$$

$$F^{(c)} = \frac{-2h - h \sigma^2 + c + 2h \sigma - 4h \nu + 2h \sigma \nu - 2h \nu^2}{h},$$  \hspace{0.5cm} (93d)
The corresponding coefficients \( a_n^{(v)} \) and \( b_n^{(v)} \) are

\[
a_n^{(v)} = \frac{\text{Num}_a}{(2n - 2\nu + \sigma + \eta + \delta - 2)}
\]

\[
\text{Num}_a = 2n^4 + (-8\nu - 4 + 4\sigma + 4\eta + 4\delta)n^3 + (10\nu^2 - (-10\nu - 12\delta + 10 - 12\eta)\nu
- 5\eta - 5\sigma + 2\delta^2 - 5\delta + 2\nu^2 + 6\eta\delta + 2 + 6\sigma + 2\delta\sigma + 6\delta\eta)n^2
\]

\[
\times [-4\nu^3 + (10\nu + 6\sigma - 6 + 10\delta)\nu^2 + (8\eta + 8\delta - 4\delta^2 - 10\delta\sigma - 2\sigma^2
- 4\eta^2 + 6\sigma - 12\eta - 10\sigma - 2\nu^2 + (2\nu\sigma + 2\sigma\sigma - \sigma - \delta - \eta + 2\delta\eta)
\times (\sigma - 1 + \delta + \eta)]n + (-2\eta - 2\nu + 4\sigma + 4\eta + 4\delta)^2
\]

\[
+ (2\nu\sigma + 2\sigma\sigma - \sigma - 2\nu^2 - 2\delta^2\eta - \eta^2 + 4\delta\eta - \delta^2\sigma - \delta^2\eta + 6\sigma\delta\sigma)\nu + \sigma\delta\eta(\delta - 2 + \eta + \sigma),
\]

\[
\text{Den}_b = (n + \eta + \delta - 1)n(n - 1 - 2\nu + \sigma)(\delta - 2 + \sigma + n + \eta - 2\nu)
- 4\nu^2 + (4\sigma - 6 + 4\eta - 8\nu + 4\delta)\nu + (4\delta - 4\eta + 6 - 4\sigma)\nu
+ 2\delta\sigma - 3\delta + 4\delta\eta + 2\nu^2 - 3\eta - 2 - 3\sigma + 2\nu^2
\]

\[
+ 4\nu^2 + (4\nu - 10 - 4\delta - 8\nu + 4\delta)\nu + 4\delta\sigma - (4\delta - 4\eta + 10 - 4\sigma)\nu
+ 2\delta\sigma - 5\nu + 4\delta\eta + 2\nu^2 - 5\eta - 5 + 2\sigma + 2\sigma^2],
\]

\[
\text{Den}_b = 16(2n - 2\nu + \sigma + \eta + \delta - 2)^2(\delta - 2 + \sigma + 3 + 2n + \eta)
- (\delta - 1 + \sigma - 2\nu + \eta + 2n).
\]

The corresponding polynomials \( p_n^{(v)}(x) \) satisfy the following second recursion relation:

\[
p_n^{(v+1)} = \frac{\text{Num}_1}{\text{Den}_1} p_n^{(v)} + \frac{\text{Num}_2}{\text{Den}_2} p_{n-1}^{(v)},
\]

\[
\text{Num}_1 = (2\nu + 2 - \sigma)(2n - \delta - 2\nu + \sigma - 3 + \eta),
\]

\[
\text{Den}_1 = (-2 + n - 2\nu + \sigma)(\delta - 3 + \eta - 2\nu + n + \sigma),
\]

\[
\text{Num}_2 = (n + \eta + \delta - 1)n[(-2 - 2\nu + \sigma)n^2 + 6\nu^2 + (12 - 2\eta - 2\delta - 6\sigma)\nu - 2\sigma
- 6\sigma + 2\nu + \eta\delta + 2x^2 + 6 + \delta\sigma - 2\delta\eta n - 4x^3 + (6\sigma + 3\delta - 12 + 3\eta)\nu^2
+ (-3\delta - 12 - 4\sigma^2 - 2x + 12\sigma + 6\eta + 6\delta - 3\sigma - 2\delta\eta)\nu
\]

\[
\times (-2 - 2\nu + \sigma - x + 6\nu + 6\delta + 3\sigma - 2\delta\eta)\nu + (6\sigma + 3\eta - 12 + 3\nu)\nu
\]

\[
\times (6\sigma + 3\eta - 12 + 3\nu)\nu + (6\sigma + 3\delta - 12 + 3\eta)\nu^2
\]

\[
+ (-3\delta - 12 - 4\sigma^2 - 2x + 12\sigma + 6\eta + 6\delta - 3\sigma - 2\delta\eta)\nu
\]

\[
+ (\delta - 2 + \eta + \sigma - 2\nu + \sigma - 3 - 2\nu + \sigma)(\eta - 2 + \nu + \sigma - 3 + \nu).
\]
of the nonlinear equation (46) for these two cases are different.

Case 1.

\[
\begin{align*}
\nu &= n + \mu, \\
\mu &= \frac{1}{2} \delta + \frac{1}{2} \eta - \frac{5}{4} \sigma + \frac{1}{4} \tau,
\end{align*}
\]

with

\[
\tau = \sqrt{4 \delta^2 - 8 \delta \eta + 4 \eta^2 + 1 - 4 \sigma^2}.
\]
It can then be verified that there holds the condition (50), hence for the corresponding polynomials \( p^{(n+\mu)}_n(x) \) there holds the complete factorization (51a), i.e.

\[
p^{(n+\mu)}_n(x) = \prod_{m=1}^n (x - x_m),
\]

(99a)

with

\[
x_m = \frac{1}{4(1 - \tau)} \{ -4(1 - \tau) m^2 - 2[2\delta + 2\eta - 4 + \tau](1 - \tau) m - 6(1 - \tau) - 4\eta^2
- 4\eta^2 \delta - 4\delta^2 + 10\delta^2 + 5\eta(1 - \tau) + 5\delta(1 - \tau) + 10\eta\delta + 2\eta^2\tau + 2\delta^2\tau
+ 4\eta^3 - 10\eta^2 + 4\delta^3 - 4\delta^2 + 8\eta^2 \}
\]

(99b)

and of course \( \mu \) given by (98b).

Case 2.

\[
\nu = n + \mu,
\]

(100a)

with

\[
\mu = \frac{1}{2} \delta + \frac{1}{2} \eta - \frac{5}{4} + \frac{1}{2} \tau - \frac{1}{4} \tau
\]

(100b)

and \( \tau \) defined as above, see (98c). It can then be verified that there holds again the condition (50), hence for the corresponding polynomials \( p^{(n+\mu)}_n(x) \) there holds the complete factorization (51a), i.e. again (99a) but now with

\[
x_m = \frac{1}{4(1 - \tau)} \{ -4(1 - \tau) m^2 - 2[2\delta + 2\eta - 4 + \tau](1 - \tau) m
- 2 - 4\eta^3 + 3\delta - 16\delta^2 + 5\eta + 2\tau - 8\eta^2 + 6\eta + 6\delta^2
+ 4\delta^2 + 4\eta^2 - 3\eta\tau + 4\eta^2\delta + 4\delta^2 + 2\eta^2\tau + 2\delta^2\tau - 4\delta^3 - 3\delta\tau \}
\]

(100c)

and of course \( \mu \) given by (100b).

Two additional complete factorizations of type (99a) obtain for

\[
\nu = n + \mu
\]

(101a)

with

\[
\mu = \delta - 1
\]

(101b)

or

\[
\mu = \delta - \frac{3}{2}
\]

(101c)

in both cases with the same zeros

\[
x_m = -(m + \delta - 1)^2.
\]

(102)

And two more factorizations obtain from these two by exchanging the roles of the two parameters \( \eta \) and \( \delta \), since the polynomials in question are invariant under this exchange.
Remark. In the special cases
\[ \sigma = \pm \frac{1}{2} \]  
these polynomials reduce to the Wilson polynomials \( W(x; a, b, c, d) \). The identification of the coefficients is given by the following simple rules:

\[ a = \alpha, \quad b = \beta, \quad c = \eta, \quad d = \delta \]  

and

\[ \alpha = -\nu, \quad \beta = -\nu + \sigma = \alpha + \sigma. \]  

Note however that these relations entail
\[ b = a + \sigma, \]  

hence only a subclass of the Wilson polynomials is obtained.

5. Special Solutions of the Nonlinear Equation (46)

In this section we report for completeness some special (indeed, rather trivial) solutions of the nonlinear equation (46).

The first such solution reads as follows:

\[ A_{\nu}^{(\nu)} = W(n) + Q(\nu), \]  

with \( W(n) \) and \( Q(\nu) \) arbitrary functions of their arguments,

\[ F^{(\nu)} = Q(\nu) - Q(\nu + 1) \]

and \( \tilde{\varphi}^{(\nu)}, \tilde{\varphi}_{\nu}, \tilde{\varphi}_{\nu}^{(\nu)} \) also arbitrary. The corresponding coefficient \( a_{\nu}^{(\nu)} \) is independent of \( \nu \),

\[ a_{\nu}^{(\nu)} = W(n + 1) - W(n), \]  

and the corresponding coefficient \( b_{\nu}^{(\nu)} \) vanishes:

\[ b_{\nu}^{(\nu)} = 0. \]  

Another simple solution of the nonlinear equation (46) reads

\[ A_{\nu}^{(\nu)} = f(n - 2\nu), \]  

with \( f(z) \) an arbitrary function of its argument,

\[ F^{(\nu)} = f(-2\nu) - f(-2 - 2\nu) \]  

and \( \tilde{\varphi}^{(\nu)} = 0, \tilde{\varphi}_{\nu}, \tilde{\varphi}_{\nu}^{(\nu)} \) also arbitrary. The corresponding coefficient \( a_{\nu}^{(\nu)} \) reads

\[ a = f(n - 2\nu + 1) - f(n - 2\nu), \]  

and again the corresponding coefficient \( b_{\nu}^{(\nu)} \) vanishes, see (106b).

A third simple solution of the nonlinear equation (46) reads

\[ A_{\nu}^{(\nu)} = f(n - \nu) = f(z), \]  

with \( f(z) \) an arbitrary function of its argument,
again with \( f(z) \) an arbitrary function of its argument,
\[
\hat{\phi}^{(\nu)} = 0, \quad (109b)
\]
\[
\hat{F}^{(\nu)} = f(-\nu) - f(-\nu - 1), \quad (109c)
\]
and \( \hat{\psi}^{(\nu)}, \hat{\delta}^{(\nu)} \) also arbitrary. The corresponding coefficient \( a_{\nu}^{(\nu)} \) reads
\[
a_{\nu}^{(\nu)} = f(n - \nu + 1) - f(n - \nu), \quad (110)
\]
while once more \( b_{\nu}^{(\nu)} \) vanishes, see (106b).

Note that in all these cases the vanishing of the coefficient \( b_{\nu}^{(\nu)} \) entails that the basic three-term recurrence relation (1) becomes a two-term recursion and the polynomials yielded by it therefore factorize as follows:
\[
p^{(\nu)}(x) = \prod_{k=1}^{n} (x + a_{k-1}^{(\nu)}). \quad (111)
\]

6. Outlook

We plan to pursue this line of research in various directions, including the possibility to take as point of departure three-term recursion relations (satisfied by polynomials) more general than (1) and the investigation of differential equations satisfied by the new class of polynomials we have identified. It will also be of interest to apply to the new integrable discrete equations introduced above — such as (46) — the techniques introduced by van der Kamp and Quispel [19, 20] and already applied by them to some of our previous findings.

Appendix A

Proof of Proposition 2.1. Clearly (8), (7) and (9) entail
\[
\hat{E}^{(+)} \hat{L} p = \hat{L}^{(+)} \hat{E}^{(+)} p = \hat{L}^{(+)} \hat{H} p. \quad (112)
\]
Now note that, via (7a), (9) and again (7a), and using the fact that the number \( x \) "commutes" with the operators \( \hat{E}^{(+)} \) and \( \hat{H} \), one gets
\[
\hat{E}^{(+)} \hat{L} p = \hat{E}^{(+)} x p = x \hat{E}^{(+)} p = x \hat{H} p = \hat{H} x p = \hat{H} \hat{L} p; \quad (113a)
\]
hence, via (112),
\[
(\hat{L}^{(+)} \hat{H} - \hat{H} \hat{L}) p = 0. \quad (113b)
\]
Clearly this last formula is implied by (12a), and since it must hold for the polynomials \( p^{(\nu)} \) with \( n \) an arbitrary nonnegative integer, it implies (12a).

Proof of Proposition 2.2. Via (20a) we get
\[
\hat{L}^{(+)} \hat{H} - \hat{H} \hat{L} = (\hat{L}^{(+)} \hat{H} - \hat{H} \hat{L}) \hat{L} + \hat{L}^{(+)} (Q' \hat{I} + q' \hat{E} \ldots) - (Q' \hat{I} + q' \hat{E} \ldots) \hat{L}, \quad (114a)
\]
hence, via (18),
\[
\hat{L}^{(+)} \hat{H} - \hat{H} \hat{L} = (\hat{W} \hat{I} + w \hat{E} \ldots) \hat{L} + \hat{L}^{(+)} (Q' \hat{I} + q' \hat{E} \ldots) - (Q' \hat{I} + q' \hat{E} \ldots) \hat{L}, \quad (114b)
\]
hence, via (12b) and (7c),
\[ L^{(+)} H' - H' L = (E_\pm - a^{(+)} I - b^{(+)} E_\pm)(Q' I + q' E_-) \]
\[ + [(W - Q') I + (w - q') E_-] (E_\pm - a I - b E_-), \]
(114c)
hence, using (11b),
\[ L^{(+)} H' - H' L = (Q'_\pm - Q' + W) E_\pm - [b^{(+)} q'_- + (w - q') b_-] E_- \]
\[ - [a^{(+)} Q' - q'_- - w + q' + (W - Q') a] I \]
\[ + [-a^{(+)} q' - b^{(+)} Q'_- - (W - Q') b - (w - q') a_-] E_- . \]
(114d)
Comparing this expression with (19) we immediately get:
\[ Q'_\pm - Q' + W = 0, \]
(115a)
which coincides with (20b) and determines \( Q' \) in terms of \( W \);
\[ b^{(+)} q'_- + (w - q') b_- = 0, \]
(115b)
which coincides with (20c) and determines \( q'_- \) in terms of \( w \) and \( b_- \);
\[ W' = -a^{(+)} Q' + q'_+ + w - q' - (W - Q') a, \]
(116a)
which coincides with (21a) and determines \( W' \) in terms of \( W \) and \( w \) as well as \( Q' \) and \( q'_- \),
themselves given by (20b) and (20c) in terms of \( W \) and \( w \); and finally
\[ w' = -a^{(+)} q' - b^{(+)} Q'_- - (W - Q') b - (w - q') a_- . \]
(116b)
which coincides with (21b) and determines \( w' \) in terms of \( W \) and \( w \) as well as \( Q' \) and \( q'_- \),
themselves given by (20b) and (20c) in terms of \( W \) and \( w \).

**Proof of Proposition 2.3.** The first formula, (23), is an immediate consequence of the definitions of \( L \) and \( L^{(+)} \); see (7c) and (12b). The second formula, (24), is as well easily verified by using these definitions and the definition of \( B \), see (2).

**Proof of Proposition 2.4.** It is an immediate consequence of Propositions 2.2 and 2.3: note that the independence of the coefficients \( c^{(2)}(n) \) and \( \tilde{c}^{(2)}(n) \) from \( n \) is of course required in order that these coefficients “commute” with the operators \( L \) and \( L^{(+)} \) which only act on the index \( n \), see their definitions (7c) and (12b).

**Proof of Proposition 2.5.** The proof is by induction. Clearly (27) holds for \( K = 0 \) when \( \rho \equiv \rho^{(0)} \equiv \tilde{c}^{(0)}(n) B^{(+)} / B_- \). To show that, if it holds at \( K \), it also holds at \( K + 1 \), we must show that, if
\[ \tilde{H} = \rho E_- + \sum_{k=1}^{K} [\rho^{(k)} E_-]^{k+1}, \]
(117)
then \( \tilde{H}' = \tilde{H} \tilde{H} \) has an analogous structure. The first step to arrive at \( \tilde{H}' \) is the formula (19), which via this ansatz (117) yields (after some standard steps using (7c), (12b) and (11b))
\[ W = \rho_+ - \rho, \]
(118)
Hence via (20b) we get

\[ Q' - Q'_+ = \rho_+ - \rho \]  \hspace{1cm} (119a)

yielding

\[ Q' = -\rho, \]  \hspace{1cm} (119b)

where, without loss of generality, we omitted to add an \( n \)-independent arbitrary quantity (since this is taken care of by lower terms in the iteration; we shall do so other times in the following, without repeating every time this justification). Hence via (20a)

\[ \hat{H}' = \left( \rho \hat{E}_+ + \sum_{k=1}^{\infty} [\rho^{B_{(k)}^{(1)}}] (\hat{E}_+ - a\hat{I} - b\hat{E}_-) - \rho \hat{I} + q' \hat{E}_- \right), \]  \hspace{1cm} (120)

and it is easily seen that the right-hand side of this expression contains no terms proportional to \( \hat{E}_- \) nor, thanks to a neat cancellation, a term proportional to \( \hat{I} \), but only terms proportional to \( (\hat{E}_-)^p \) with \( p \) a positive integer; thereby confirming that \( \hat{H}' \) has the same structure as \( \hat{H} \), see (117).

**Proof of Proposition 2.6.** Since this proof is analogous to the preceding one, we merely outline it. Let

\[ \hat{H}' = \hat{R} \hat{I}, \]  \hspace{1cm} (121)

then from (20b), (23) and (20b)

\[ Q'_+ - Q'_- = a^{(1)} - a = -[(A_+ - A_+^{(1)}) - (A - A^{(1)})] \]  \hspace{1cm} (122a)

hence (again, up to an \( n \)-independent quantity we set to zero)

\[ Q' = -(A - A^{(1)}) \]  \hspace{1cm} (122b)

hence, via (20a),

\[ \hat{H}' = \hat{E}_+ - (A_+ - A^{(1)}) \hat{I} + (q' - b) \hat{E}_-. \]  \hspace{1cm} (123)

The result then easily follows by further iterations. \( \square \)

**Proof of Proposition 2.7.** This proposition is an immediate consequence of the previous two Propositions 2.5 and 2.6, and of the monic character of the polynomials \( p_n^{(1)}(x) \) implied by the three-term recursion relation (1) defining them. \( \square \)

**Proof of Proposition 2.8.** This proposition is an immediate consequence of the previous propositions, see in particular Propositions 2.1, 2.4 and 2.7. \( \square \)

**Appendix B**

In this appendix we justify findings reported in Sec. 2.3.

Firstly the derivation of (37). The assignment under consideration implies, via (31),

\[ p_{n+1}^{(1)}(x) = p_{n}^{(1)}(x) + q_n^{(1)} \hat{B}_{(1)}^{(1)} \hat{E}_- p_n^{(1)}(x). \]  \hspace{1cm} (124)
Let us therefore evaluate the operator $\hat{\mathcal{R}}(B(B_+)/B_+ \hat{E}_-) \hat{E}_+$ appearing in the right-hand side of this formula. To this end we write (see (22a))

$$H' = \hat{\mathcal{R}} \frac{B(B_+)/B_+ \hat{E}_-}{B_+} \hat{E}_+ = \hat{\mathcal{R}} \hat{H}$$

(125a)

with

$$\hat{H} = \frac{B(B_+)/B_+ \hat{E}_-}{B_+} \hat{E}_+$$

(125b)

Hence (see (18))

$$\hat{L}' B(B_+)/B_+ \hat{E}_- - B(B_+)/B_+ \hat{E}_+ \hat{E}_+ = W' + w \hat{E}_-,$$

(126)

and (see (20))

$$\hat{H}' = \frac{B(B_+)/B_+ \hat{E}_-}{B_+} \hat{E}_+ + \hat{Q}' \hat{I} + q' \hat{E}_-$$

(127a)

with

$$\hat{Q}' \hat{Q}'_+ = W',$$

(127b)

$$b_- q' - b^{(+)q}'_+ = b_-' w,$$

(127c)

where $W$ and $w$ are now defined by (126), hence they read (see (24))

$$W = \frac{B(B_+)/B_+ \hat{E}_+}{B_+} - \frac{B(B_+)/B_+ \hat{E}_-}{B_+},$$

(128a)

$$w = \frac{B(B_+)/B_+ (a_+ - a^{(+)})}{B_+}.$$  

(128b)

Hence, as clearly implied by (127b) with (128a),

$$Q' = \frac{B(B_+)/B_+ \hat{E}_+}{B_+},$$

(129)

while (127c) with (128b) yield

$$b_- q' - b^{(+)q}'_+ = b_-' \frac{B(B_+)/B_+ (a_+ - a^{(+)})}{B_+},$$

(130a)

hence (via (2a))

$$\frac{B}{B^{(+)q}_+} q' - \frac{B}{B^{(+)q}_-} q'_+ = A - A^{(+)q}_+ - (A_+ - A^{(+)q}_+),$$

(130b)

clearly entailing

$$\frac{B}{B^{(+)q}_+} q' = A - A^{(+)q}_+,$$

(130c)
hence
\[ q' = \frac{B^+}{B^-} (A - A^+). \]  \tag{130d}

It is thus seen (from (127a), (129), (130d) and (7c)) that
\[ \hat{H}' = \frac{B^+}{B^-} (\hat{E}_+ - a \hat{I} - b \hat{E}_-) - \frac{B^+}{B^-} (A - A^+) \hat{E}_- \]  \tag{131a}

hence (via (11b) and the second (2a))
\[ \hat{H}' = \frac{B^+}{B^-} (A_+ - A^{(+)\dagger}) \hat{E}_+ + \frac{B^+}{B^-} \hat{E}_+ \hat{E}_-. \]  \tag{131b}

The insertion of this expression in place of the operator \( \hat{H} \) in the right-hand side of (9) yields the second recursion relation (37), which is thereby proven.

Next, let us obtain the conditions required for the validity of the results we just got. They are provided by (30), which, with the assignment under consideration here, reads
\[ \left(a - a^{(+)\dagger}\right) + 2^{(1)(\nu)} \left(\frac{B^+}{B_-} (A - A^{(+)\dagger}) - \frac{B^+}{B_-} (A_+ - A^{(+)\dagger})\right) = 0, \]  \tag{132a}

hence (see (22b) with (21) and (20b), (20c))
\[ a - a^{(+)\dagger} + 2^{(1)(\nu)} [-aW + w + q' - q' + (a - a^{(+)\dagger})Q'] = 0, \]  \tag{132b}
\[ b - b^{(+)\dagger} + 2^{(1)(\nu)} [-bW - aW + (a - a^{(+)\dagger}) q' + bQ' - b^{(+)\dagger} Q'] = 0, \]  \tag{132c}

where \( W \) and \( w \) are given by (128) and \( Q' \) and \( q' \) are given by (129) and (130d). Hence these two equations read
\[ a - a^{(+)\dagger} + 2^{(1)(\nu)} \left[ \frac{B^+}{B_-} (A - A^{(+)\dagger}) - \frac{B^+}{B_-} (A_+ - A^{(+)\dagger}) \right] = 0, \]  \tag{133a}
\[ b - b^{(+)\dagger} + 2^{(1)(\nu)} \left[ \frac{B^+}{B_-} (a_+ - a^{(+)\dagger}) (A_+ - A^{(+)\dagger}) + \frac{B^+}{B_-} (A_+ - A^{(+)\dagger}) \right] = 0, \]  \tag{133b}

and via (2a) they coincide with the two equations (38), which are thereby proven.

Finally let us derive (39) from (38). Firstly we note that (38) can be rewritten as follows,
\[ A_+ - A^{(+)\dagger} + 2^{(1)(\nu)} \frac{B^+}{B_-} (A - A^{(+)\dagger}) = A - A^{(+)\dagger} + 2^{(1)(\nu)} \frac{B^+}{B_-} (A_+ - A^{(+)\dagger}), \]  \tag{134a}

hence it clearly entails
\[ A_+ - A^{(+)\dagger} + 2^{(1)(\nu)} \frac{B^+}{B_-} (A - A^{(+)\dagger}) = \phi^{(\nu)}, \]  \tag{134b}
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yielding
\[ \tilde{\varphi}_n^{(\nu)} B_n^{(\nu)} = \frac{\varphi^{(\nu)} - A_+ + A_+^{(\nu)}}{A - A_+^{(\nu)}} \]  

\[ \text{(135a)} \]

hence as well (by replacing \( n \) with \( n - 1 \) respectively \( n - 2 \))
\[ \tilde{\varphi}_n^{(\nu)} B_n^{(\nu)} = \frac{\varphi^{(\nu)} - A + A_+^{(\nu)}}{A - A_+^{(\nu)}} \]  

\[ \text{(135b)} \]
\[ \tilde{\varphi}_n^{(\nu)} B_n^{(\nu)} = \frac{\varphi^{(\nu)} - A_+ + A_+^{(\nu)}}{A_+ - A_+^{(\nu)}} \]  

\[ \text{(135c)} \]

By cross multiplying the last two equations we get
\[ \frac{\varphi^{(\nu)} - A_+ + A_+^{(\nu)}}{A_+ - A_+^{(\nu)}} B_+^{(\nu)} = \frac{\varphi^{(\nu)} - A + A_+^{(\nu)}}{A - A_+^{(\nu)}} B_+^{(\nu)} \]  

\[ \text{(136a)} \]

hence
\[ B_+^{(\nu)} = \frac{(A_+ - A_+^{(\nu)}) \varphi^{(\nu)} - A_+ + A_+^{(\nu)}}{(A_+ - A_+^{(\nu)}) \varphi^{(\nu)} - A_+ + A_+^{(\nu)} B_+^{(\nu)}} \]  

\[ \text{(136b)} \]

As for (38b), it can be rewritten as follows:
\[ \tilde{\varphi}_n^{(\nu)} B_n^{(\nu)}(A - A_+ + A_+^{(\nu)} - A_+^{(\nu)})(A_+ - A_+^{(\nu)}) = B_+^{(\nu)} \left(1 - \tilde{\varphi}_n^{(\nu)} B_n^{(\nu)} \right) \]  

\[ \text{(137a)} \]

hence (see (135))
\[ \frac{\varphi^{(\nu)} - A + A_+^{(\nu)}}{A - A_+^{(\nu)}} B_+^{(\nu)} = \frac{B_+^{(\nu)} A_+ - A_+^{(\nu)} + A_+ - A_+^{(\nu)} - \varphi^{(\nu)}}{A_+ - A_+^{(\nu)}} \]  

\[ \text{(137b)} \]

hence (see (136b))
\[ A + A_+^{(\nu)} - A_+ = \frac{B_+^{(\nu)} A_+ - A_+^{(\nu)} + A_+ - A_+^{(\nu)} - \varphi^{(\nu)}}{B_+^{(\nu)} (\varphi^{(\nu)} - A + A_+^{(\nu)})} \]  

\[ - \frac{B_+^{(\nu)} A_+ - A_+^{(\nu)} + A_+ - A_+^{(\nu)} - \varphi^{(\nu)}}{B_+^{(\nu)} (\varphi^{(\nu)} - A_+ + A_+^{(\nu)})} \]  

\[ \text{(137c)} \]
We now introduce the quantity \( C \equiv c_n^{(\nu)} \) by setting

\[
\left( \frac{B}{B_{-}} \right) A - A_{+}^{(\nu)} + A - A_{+}^{(\nu)} - \phi^{(\nu)} = \frac{\phi^{(\nu)} - A + A_{+}^{(\nu)}}{A + A_{-} - A_{+}^{(\nu)} - \phi^{(\nu)}} C, \tag{138a}
\]

entailing of course (by replacing \( n \) with \( n - 1 \))

\[
\left( \frac{B}{B_{-}} \right) A_{-} - A_{+}^{(\nu)} + A - A_{-}^{(\nu)} - \phi^{(\nu)} = \frac{\phi^{(\nu)} - A_{-} + A_{+}^{(\nu)}}{A_{-} + A_{-} - A_{+}^{(\nu)} - \phi^{(\nu)}} C. \tag{138b}
\]

And by inserting the last two expressions in the preceding one we get

\[
(A - A_{+}^{(\nu)})(A - A_{+}^{(\nu)} - \phi^{(\nu)}) - (A_{-} - A_{+}^{(\nu)})(A_{-} - A_{+}^{(\nu)} - \phi^{(\nu)}) = C - C_{-}, \tag{139a}
\]

yielding

\[
C = (A - A_{+}^{(\nu)})(A - A_{+}^{(\nu)} - \phi^{(\nu)}) + \psi^{(\nu)}. \tag{139b}
\]

Hence, from (138a),

\[
\frac{B}{B_{-}} = \frac{(A - A_{+}^{(\nu)})(\phi^{(\nu)} - A + A_{+}^{(\nu)})}{A + A_{-} - A_{+}^{(\nu)} - \phi^{(\nu)}} C(A - A_{+}^{(\nu)})(A - A_{+}^{(\nu)} - \phi^{(\nu)}) + \psi^{(\nu)} \tag{140a}
\]

implying (by replacing \( \nu \) with \( \nu + 1 \) and \( n \) with \( n + 1 \))

\[
\frac{B_{+}}{B_{+}^{(\nu)}} = \frac{(A_{+}^{(\nu)} - A_{+}^{(\nu+1)})(\phi^{(\nu+1)} - A_{+}^{(\nu+1)})}{(A_{+}^{(\nu)} + A_{+}^{(\nu+1)} - A_{+}^{(\nu+1)} + A_{+}^{(\nu+1)} - \phi^{(\nu+1)})}
\]

\[
\frac{(A_{+}^{(\nu)} - A_{+}^{(\nu+1)} + A_{+}^{(\nu)} - A_{+}^{(\nu+1)} + \phi^{(\nu+1)})}{(A_{+}^{(\nu)} - A_{+}^{(\nu+1)} + A_{+}^{(\nu)} - A_{+}^{(\nu+1)} + \phi^{(\nu+1)})}. \tag{140b}
\]

Finally we use the identity

\[
\frac{B_{+}}{B} = \left( \frac{B_{+}^{(\nu)}}{B^{(\nu)}} \right) \left( \frac{B_{+}}{B_{-}} \right)^{-1} \tag{141}
\]

to get (from the last two formulae and (135b))

\[
\phi^{(\nu)} \frac{B_{+}^{(\nu)}}{B} = \frac{(A_{+}^{(\nu)} - A_{+}^{(\nu+1)})(\phi^{(\nu+1)} - A_{+}^{(\nu+1)} + A_{+}^{(\nu+1)})}{(A_{-} - A_{+}^{(\nu)})(A_{-} - A_{+}^{(\nu)})}
\]

\[
\frac{(A_{+}^{(\nu)} - A_{+}^{(\nu+1)})(A_{+}^{(\nu)} - A_{+}^{(\nu+1)} - \phi^{(\nu+1)}) + \psi^{(\nu+1)}}{(A_{-} - A_{+}^{(\nu)})(A_{-} - A_{+}^{(\nu)} + \phi^{(\nu)}) + \psi^{(\nu)}}. \]
Appendix C
In this Appendix C we prove the first two propositions reported in Sec. 3, and we indicate how the third one can be analogously proven.

Proof of Proposition 3.1. This proof is quite easy. By using (1a) (with $n$ replaced by $n-1$ and $\nu$ replaced by $\nu-1$) to replace the first term in the right-hand side of the second recursion (43a) we get

$$p_n^{(\nu)}(x) = (x + G_n^{(\nu)} + G_n^{(\nu-1)})p_{n-1}(x) + (G_n^{(\nu-1)} + G_n^{(\nu-2)})p_{n-2}(x).$$

(143a)

hence the condition (50) entails (for $\nu = n + \mu$)

$$p_n^{(n+\mu)}(x) = (x + G_n^{(n-1+\mu)} + G_n^{(n+\mu)})p_{n-1}(x),$$

(143b)

and clearly these entail the factorization formula (51), which is thereby proven.

Proof of Proposition 3.2. This proof is analogous to the previous one, albeit a bit longer. We must first iterate the second recursion (43a) (or, equivalently, (44a), as the case may be), by using this same relation to decrease by one unit the parameter $\nu$ in the right-hand side of this formula, obtaining thereby:

$$p_n^{(\nu)}(x) = (x + G_n^{(\nu-1)} + G_n^{(\nu-2)})p_{n-2}(x) + (G_n^{(\nu-1)} + G_n^{(\nu-2)})p_{n-3}(x).$$

(144a)

Next, we replace the first term in the right-hand side by using the basic recursion relation (1a) (with $\nu$ replaced by $\nu-2$ and $n$ by $n-1$), getting thereby

$$p_n^{(\nu)}(x) = (x + G_n^{(\nu-2)} + G_n^{(\nu-1)} + G_n^{(\nu-2)})p_{n-2}(x) + (G_n^{(\nu-2)} + G_n^{(\nu-2)} + G_n^{(\nu-2)})p_{n-3}(x) + G_n^{(\nu-2)}G_n^{(\nu-3)}p_{n-4}(x).$$

(144b)

Hence, by setting $\nu = 2n + \mu$, this formula reads

$$p_n^{(2n+\mu)}(x) = (x + G_n^{(2n-2+\mu)} + G_n^{(2n-1+\mu)} + G_n^{(2n+\mu)})p_{n-2}(x) + (G_n^{(2n-2+\mu)} + G_n^{(2n-1+\mu)} + G_n^{(2n+\mu)} + G_n^{(2n-1+\mu)})p_{n-3}(x) + G_n^{(2n+\mu)}G_n^{(2n-1+\mu)}p_{n-4}(x).$$

(144c)
yielding, when the 3 relations (52) hold,
\[ p_n^{(2n+\mu)}(x) = (x+q_n^{(2n-2+\mu)}) + G_n^{2n-1+\mu} + G_n^{(2n+\mu)} p_{n-1}^{(2n-2+\mu)}(x), \]  

(144d)

from which the factorization (55) immediately follows.

The proof of Proposition 3.3 is analogous to the proof of Proposition 3.2, except that one must first iterate once the recursion (1a) rather than (43a) (or, equivalently, (44a)).

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