Using incomplete indefinite $LDL^T$ preconditioning for inexact interior point methods for linear programming

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Abstract

Most linear algebra kernels in interior point methods for linear programming require the solution of linear systems of equation with the matrix $N = A^T D^{-1} A$ (or $AD^{-1} A^T$), where $A$ denotes the constraint matrix of the linear program. This matrix $N$ arises from the reduced KKT system by block elimination. If the number of non-zeros in $N$ or in its Cholesky factorization $N = LL^T$ is very large, the computational cost and memory requirement to solve the linear systems of equations with $N$ may be prohibitively large.

In this work we implement an interior point method described by R. Freund and F. Jarre [1]. Forming the normal equation matrix $N$ is avoided altogether and we work with the reduced KKT system instead. We solve the linear systems for the Newton directions iteratively only to low accuracy using SQMR and an indefinite multilevel preconditioner (PARDISO [2,3]). Preliminary numerical results are encouraging.

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1. Introduction

We investigate an iterative alternative for the linear algebra kernel in interior point methods (IPM) for linear optimization. By that we understand that the linear systems of equations that arise in the course of the IPM are solved only to a certain (low) relative accuracy using an iterative solution method. A typical situation where the usual direct approach for solving those systems using a Cholesky factorization of the normal equations is not satisfying arises when the normal equations or, more often, its Cholesky factorization suffers a large amount of fill. So the setting we have in mind is the case where forming the normal equation matrix or its Cholesky factorization is not desired

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and thus we work exclusively with the augmented indefinite KKT system. We note in passing the computational cost of an matrix vector product (the only relevant cost in iterative linear systems solvers) is the same for the augmented system and the normal equation matrix. Further it can easily be seen that every preconditioner for the normal equation matrix is mathematically equivalent to one for the augmented KKT matrix (see, for example, [4]) but not vice versa.

A large number of publications on this subject and, more generally, inexact Newton methods for nonlinear systems of equations exist. We do not attempt to cite all the relevant ones in this context. We remark that we do not assume any knowledge of structure in the LP constraint matrix (in case such structure exists) or take advantage of heuristics that try to identify an optimal basis partitioning as in [4, 5].

Our work was prompted by promising numerical results obtained in non-convex optimization [6] and a certain eigenvalue problem [2] where an indefinite multilevel preconditioner is used that is part of the software package PARDISO. Further our work is based on the inexact IPM described by R. Freund and F. Jarre [1].

2. Statement of the problem

We consider the primal-dual linear programming pair
\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
\quad & \quad x \geq 0
\end{align*}
\]
\[
\begin{align*}
\max & \quad b^T y \\
\text{s.t.} & \quad A^T y + z = c \\
\quad & \quad z \geq 0.
\end{align*}
\]
For a given iterate \((x, y, z)\) and a centering parameter \(\mu\) the residuals for the corresponding first order optimality system are
\[
\begin{align*}
r_p &= Ax - b, \\
r_d &= A^T y + z - c, \\
r_c &= Xz - \mu e.
\end{align*}
\]
(For \(x \in \mathbb{R}^n\) the diagonal matrix \(\text{diag}(x)\) is denoted by \(X\).) Then the Newton direction \([d_y, d_x, d_z]^T\) at this particular step is implied by the solution of
\[
K \tilde{d} = \begin{bmatrix} 0 & A \\ A^T & X^{-1}Z \end{bmatrix} \begin{bmatrix} d_y \\ d_x \end{bmatrix} = - \begin{bmatrix} r_p \\ r_d - X^{-1}Zr_c \end{bmatrix} = \tilde{r},
\]
where \(d_x, d_z\) can be recovered from \(\tilde{d_x}\) by block elimination with \(X\) and \(Z\). Our goal is to solve (1) up to a certain accuracy \(\eta \in (0, 1)\), so that \(\|\tilde{r} - K\tilde{d}\| \leq \eta \|\tilde{r}\|\). Here \(\eta\) has to be chosen such that the norms of the residuals \(r_p^*\) and \(r_d^*\) of the next iterate are sufficiently smaller than the current ones (As pointed out in [1], \(\|r_c^*\| \approx 0\) can always be achieved.). The solution of (1) should require significantly less memory than the solution of the corresponding normal equation.

\(^1\)Another situation where the augmented system may be preferred is the presence of free variables in the primal LP, which are conceptually simpler to treat in the augmented system.

\(^2\)http://www.pardiso-project.org
We now describe very briefly the techniques in the PARDISO multilevel preconditioner; details can be found in [2]. The preconditioner is based on an incomplete multilevel indefinite $LDL^T$ factorization, where a specified bound $\kappa$ on the norm of $\|L^{-1}\|$ is guaranteed. In the course of the factorization, all pivots (1-by-1 or 2-by-2) that fail to satisfy the guaranteed bound on the norm of $L^{-1}$ are postponed and constitute the next level of the factorization after the Schur complement has been built. The dropping of entries happens both when forming the Schur complements or as a means to keep the resulting $L$ factor sparse. Both dropping tolerances can be controlled independently. The factorization is preceded by a reordering strategy that aims at reducing the resulting fill in the $L$ factor as well as increasing the (block) diagonal dominance of the reordered matrix.

The iterative solver that is built-in in PARDISO is based on the simplified QMR iteration (see [1] and references within). Part of this work is the discussion and comparison of various heuristics that control the parameters of the preconditioner in the course of the IPM iteration as well as the termination criteria for the QMR iteration.

4. A motivating example

As outlined in the introduction we focus on LP instances where the fill in the Cholesky factor of the normal equation matrix is large. A set of LP instances with that characteristic was provided to us by the Zuse Institute Berlin, from which the example in table 1 is taken.

We compare our implementation with the interior point method that comes with CPLEX (we used version 12.1). The linear algebra kernel in that method is based on the Cholesky factorization of the normal equations. In the example above the fill ratios shown refer to the quotient of the number of nonzeros in the Cholesky factor of the normal equation matrix and the number of nonzeros in $A$ (lower triangle of $AA^T$, respectively). Note that storing the Cholesky factor requires about 6GB of memory, while the preconditioner requires much less memory. The total solution time for CPLEX is about a factor of five larger than the total solution time of the inexact IPM.

In our work we present a detailed numerical comparison of the performance of our inexact IPM implementation with CPLEX on linear programming instances that suffer much fill in the Cholesky factor.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & CPLEX & inexact IPM \\
\hline
IPM iterations & 17 & 67 \\
fill ratio (w.r.t. $A$) & 210 & 4.5 (avg.) \\
fill ratio (w.r.t. $AA^T$) & 42 & n/a \\
solution time & $\sim 18,000s$ & $\sim 3,200s$ \\
\hline
\end{tabular}
\caption{Comparison for an LP instance with 248,127 columns, 386,294 rows and 3,254,730 non-zeros in $A$.}
\end{table}
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