Quintessence and CMB

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Abstract

A particular kind of quintessence is considered, with equation of motion $p_Q/\rho_Q = -1$, corresponding to a cosmological term with time-dependence $\Lambda(t) = \Lambda(t_0)(R(t_0)/R(t))^P$ which we examine initially for $0 \leq P < 3$. Energy conservation is imposed, as is consistency with big-bang nucleosynthesis, and the range of allowed $P$ is thereby much restricted to $0 \leq P < 0.2$. The position of the first Doppler peak is computed analytically and the result combined with analysis of high-Z supernovae to find how values of $\Omega_m$ and $\Omega_\Lambda$ depend on $P$. 
Our knowledge of the universe has changed dramatically even in the last five years. Five years ago the best guess, inspired partially by inflation, for the makeup of the present cosmological energy density was $\Omega_m = 1$ and $\Omega_\Lambda = 0$. However, the recent experimental data on the cosmic background radiation and the high-$Z$ ($Z =$ red shift) supernovae strongly suggest that both guesses were wrong. Firstly $\Omega_m \simeq 0.3 \pm 0.1$. Second, and more surprisingly, $\Omega_\Lambda \simeq 0.7 \pm 0.2$. The value of $\Omega_\Lambda$ is especially unexpected for two reasons: it is non-zero and it is $\geq 120$ orders of magnitude below its “natural” value.

The fact that the present values of $\Omega_m$ and $\Omega_\Lambda$ are of comparable order of magnitude is a “cosmic coincidence” if $\Lambda$ in the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G N T_{\mu\nu} + \Lambda g_{\mu\nu}$$

is constant. Extrapolate the present values of $\Omega_m$ and $\Omega_\Lambda$ back, say, to redshift $Z = 100$. Suppose for simplicity that the universe is flat $\Omega_C = 0$ and that the present cosmic parameter values are $\Omega_m = 0.300...$ exactly and $\Omega_\Lambda = 0.700...$ exactly. Then since $\rho_m \propto R(t)^{-3}$ (we can safely neglect radiation), we find that $\Omega_m \simeq 0.9999..$ and $\Omega_\Lambda \simeq 0.0000..$ at $Z = 100$. At earlier times the ratio $\Omega_\Lambda/\Omega_m$ becomes infinitesimal. There is nothing to exclude these values but it does introduce a second “flatness” problem because, although we can argue for $\Omega_m + \Omega_\Lambda = 1$ from inflation, the comparability of the present values of $\Omega_m$ and $\Omega_\Lambda$ cries out for explanation.

In the present paper we shall consider a specific model of quintessence. In its context we shall investigate the position of the first Doppler peak in the Cosmic Microwave Background (CMB) analysis using results published by two of us with Rohm earlier [1]. Other works on the study of CMB include [2–5]. We shall explain some subtleties of the derivation given in [1] that have been raised since its publication mainly because the formula works far better than its expected order-of-magnitude accuracy. Data on the CMB have been provided recently in [6–13] and especially in [14].
The combination of the information about the first Doppler peak and the complementary analysis of the deceleration parameter derived from observations of the high-red-shift supernovae [15,16] leads to fairly precise values for the cosmic parameters $\Omega_m$ and $\Omega_\Lambda$. We shall therefore also investigate the effect of quintessence on the values of these parameters.

In [1], by studying the geodesics in the post-recombination period a formula was arrived at for the position of the first Doppler peak, $l_1$. For example, in the case of a flat universe with $\Omega_C = 0$ and $\Omega_M + \Omega_\Lambda = 1$ and for a conventional cosmological constant:

$$l_1 = \pi \left( \frac{R_t}{R_0} \right) \left[ \Omega_M \left( \frac{R_0}{R_t} \right)^3 + \Omega_\Lambda \right]^{1/2} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\Omega_M w^3 + \Omega_\Lambda}}$$

(1)

If $\Omega_C < 0$ the formula becomes

$$l_1 = \frac{\pi}{\sqrt{-\Omega_C}} \left( \frac{R_t}{R_0} \right) \left[ \Omega_M \left( \frac{R_0}{R_t} \right)^3 + \Omega_\Lambda + \Omega_C \left( \frac{R_0}{R_t} \right)^2 \right]^{1/2} \sin \left( \sqrt{-\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\Omega_M w^3 + \Omega_\Lambda}} \right)$$

(2)

For the third possibility of a closed universe with $\Omega_C > 0$ the formula is:

$$l_1 = \frac{\pi}{\sqrt{\Omega_C}} \left( \frac{R_t}{R_0} \right) \left[ \Omega_M \left( \frac{R_0}{R_t} \right)^3 + \Omega_\Lambda + 5\Omega_C \left( \frac{R_0}{R_t} \right)^2 \right]^{1/2} \sinh \left( \sqrt{\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\Omega_M w^3 + \Omega_\Lambda}} \right)$$

(3)

The use of these formulas gives iso-$l_1$ lines on a $\Omega_M - \Omega_\Lambda$ plot in $25 \sim 50\%$ agreement with the corresponding results found from computer code. On the insensitivity of $l_1$ to other variables, see [17,18]. The derivation of these formulas was given in [1]. Here we add some more details.

The formula for $l_1$ was derived from the relation $l_1 = \pi / \Delta \theta$ where $\Delta \theta$ is the angle subtended by the horizon at the end of the recombination transition. Let us consider the Legendre integral transform which has as integrand a product of two factors, one is the temperature autocorrelation function of the cosmic background radiation and the other factor is a Legendre polynomial of degree $l$. The issue is what is the lowest integer $l$ for which the two factors
reinforce to create the doppler peak? For small \( l \) there is no reinforcement because the horizon at recombination subtends a small angle about one degree and the CBR fluctuations average to zero in the integral of the Legendre transform. At large \( l \) the Legendre polynomial itself fluctuates with almost equispaced nodes and antinodes. The node-antinode spacing over which the Legendre polynomial varies from zero to a local maximum in magnitude is, in terms of angle, on average \( \pi \) divided by \( l \). When this angle coincides with the angle subtended by the last-scattering horizon, the fluctuations of the two integrand factors are, for the first time with increasing \( l \), synchronized and reinforce (constructive interference) and the corresponding partial wave coefficient is larger than for slightly smaller or slightly larger \( l \). This explains the occurrence of \( \pi \) in the equation for the \( l_1 \) value of the first doppler peak written as \( l_1 = \pi / \Delta \theta \).

Another detail concerns the use of the photon horizon as opposed to the acoustic horizon. If we examine the evolution of the recombination transition given in \([19]\) the degree of ionization is 99% at 5,000K (redshift \( Z = 1,850 \)) falling to 1% at 3,000K (\( Z = 1,100 \)). These times represent the beginning and end of the recombination transition. For matter domination \( R \sim t^{2/3} \) so in cosmic time the start of recombination is at \( t = 1.7 \times 10^5y \) (\( Z = 1,850 \)) and the end is at \( t = 3.8 \times 10^5y \) (\( Z = 1,100 \)). The photons we detect are from \( Z = 1,100 \). The baryon-photon plasma has fluctuations at \( Z = 1,850 \) with size the acoustic horizon \( 1.7 \times 10^5y / \sqrt{3} = 1.0 \times 10^5y \). Between \( Z = 1,850 \) and \( Z = 1,100 \) the transparency increases as does the size of the fluctuation. The sound speed is gradually replaced by the light speed in the fluctuation evolution. One can see quantitatively that during the cosmic time \( 2 \times 10^5y \) of the recombination transition the fluctuation can grow by the required amount if one uses a speed between sound and light. The agreement of the formula for \( l_1 \) with experiment is at the 10% level which shows phenomenologically that the fluctuation does grow (approximately by \( \sqrt{3} \)) during the recombination transition and that is why there is no \( \sqrt{3} \) in its numerator. Although the formula started out as an order-of-magnitude estimate the fact that it works far better gives insight about the physics of
recombination and cosmic background radiation.

To introduce our quintessence model as a time-dependent cosmological term, we start from the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \Lambda(t) g_{\mu\nu} + 8\pi G T_{\mu\nu} = 8\pi G T_{\mu\nu}$$

(4)

where $\Lambda(t)$ depends on time as will be specified later and $T_{\mu\nu} = \text{diag}(\rho, -p, -p, -p)$. Using the Robertson-Walker metric, the ‘00’ component of Eq.(4) is

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G \rho}{3} + \frac{1}{3} \Lambda$$

(5)

while the ‘ii’ component is

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p + \Lambda$$

(6)

Energy-momentum conservation follows from Eqs.(5,6) because of the Bianchi identity $D^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}) = D^\mu (\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}) = D^\mu T_{\mu\nu} = 0$.

Note that the separation of $T_{\mu\nu}$ into two terms, one involving $\Lambda(t)$, as in Eq(4), is not meaningful except in a phenomenological sense because of energy conservation.

In the present cosmic era, denoted by the subscript ‘0’, Eqs.(5,6) become respectively:

$$\frac{8\pi G}{3} \rho_0 = H_0^2 + \frac{k}{R_0^2} - \frac{1}{3} \Lambda_0$$

(7)

$$-8\pi G p_0 = -2q_0 H_0^2 + H_0^2 + \frac{k}{R_0^2} - \Lambda_0$$

(8)

where we have used $q_0 = -\frac{\ddot{R}_0}{R_0 H_0^2}$ and $H_0 = \frac{\dot{R}_0}{R_0}$.

For the present era, $p_0 \ll \rho_0$ for cold matter and then Eq.(3) becomes:

$$q_0 = \frac{1}{2} \Omega_M - \Omega_\Lambda$$

(9)

where $\Omega_M = \frac{8\pi G \rho_0}{3 H_0^2}$ and $\Omega_\Lambda = \frac{\Lambda_0}{3 H_0^2}$.
Now we can introduce the form of \( \Lambda(t) \) we shall assume by writing

\[
\Lambda(t) = bR(t)^{-P}
\]

(10)

where \( b \) is a constant and the exponent \( P \) we shall study for the range \( 0 \leq P < 3 \). This motivates the introduction of the new variables

\[
\tilde{\Omega}_M = \Omega_M - \frac{P}{3-P}\Omega_\Lambda, \quad \tilde{\Omega}_\Lambda = \frac{3}{3-P}\Omega_\Lambda
\]

(11)

It is unnecessary to redefine \( \Omega_C \) because \( \tilde{\Omega}_M + \tilde{\Omega}_\Lambda = \Omega_M + \Omega_\Lambda \). The case \( P = 2 \) was proposed, at least for late cosmological epochs, in \[20\].

The equations for the first Doppler peak incorporating the possibility of non-zero \( P \) are found to be the following modifications of Eqs.(1,2,3). For \( \Omega_C = 0 \)

\[
l_1 = \pi \left( \frac{R_t}{R_0} \right)^3 \left[ \tilde{\Omega}_M \left( \frac{R_0}{R_t} \right)^3 + \tilde{\Omega}_\Lambda \left( \frac{R_0}{R_t} \right)^P + \Omega_C \left( \frac{R_0}{R_t} \right)^2 \right]^{1/2} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P}}
\]

(12)

If \( \Omega_C < 0 \) the formula becomes

\[
l_1 = \frac{\pi}{\sqrt{-\Omega_C}} \left( \frac{R_t}{R_0} \right)^3 \left[ \tilde{\Omega}_M \left( \frac{R_0}{R_t} \right)^3 + \tilde{\Omega}_\Lambda \left( \frac{R_0}{R_t} \right)^P + \Omega_C \left( \frac{R_0}{R_t} \right)^2 \right]^{1/2} \times \]

\[
\times \sin \left( \sqrt{-\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P + \Omega_C w^2}} \right)
\]

(13)

For the third possibility of a closed universe with \( \Omega_C > 0 \) the formula is:

\[
l_1 = \frac{\pi}{\sqrt{\Omega_C}} \left( \frac{R_t}{R_0} \right)^3 \left[ \tilde{\Omega}_M \left( \frac{R_0}{R_t} \right)^3 + \tilde{\Omega}_\Lambda \left( \frac{R_0}{R_t} \right)^P + \Omega_C \left( \frac{R_0}{R_t} \right)^2 \right]^{1/2} \times \]

\[
\times \sinh \left( \sqrt{\Omega_C} \int_1^{\frac{R_0}{R_t}} \frac{dw}{\sqrt{\tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P + \Omega_C w^2}} \right)
\]

(14)

The dependence of \( l_1 \) on \( P \) is illustrated for constant \( \Omega_M = 0.3 \) in Fig. 1(a), and for the flat case \( \Omega_C = 0 \) in Fig. 1(b). For illustration we have varied \( 0 \leq P < 3 \) but as will
become clear later in the paper (see Fig 3 below) only the much more restricted range
$0 \leq P < 0.2$ is possible for a fully consistent cosmology when one considers evolution since
the nucleosynthesis era.

We have introduced $P$ as a parameter which is real and with $0 \leq P < 3$. For $P \to 0$ we
regain the standard cosmological model. But now we must investigate other restrictions
already necessary for $P$ before precision cosmological measurements restrict its range even
further.

Only for certain $P$ is it possible to extrapolate the cosmology consistently for all $0 < w =
(R_0/R) < \infty$. For example, in the flat case $\Omega_C = 0$ which our universe seems to approximate
[14], the formula for the expansion rate is

$$\frac{1}{H_0^2} \left( \frac{\dot{R}}{R} \right)^2 = \tilde{\Omega}_M w^3 + \tilde{\Omega}_\Lambda w^P$$  (15)

This is consistent as a cosmology only if the right-hand side has no zero for a real positive
$w = \hat{w}$. The root $\hat{w}$ is

$$\hat{w} = \left( \frac{3(1 - \Omega_M)}{P - 3\Omega_M} \right)^{1/P}$$  (16)

If $0 < \Omega_M < 1$, consistency requires that $P < 3\Omega_M$.

In the more general case of $\Omega_C \neq 0$ the allowed regions of the $\Omega_M - \Omega_\Lambda$ plot for $P = 0, 1, 2$
are displayed in Fig. 2.

We see from Eq.(16) that if we do violate $P < 3\Omega_M$ for the flat case then there is a $\hat{w} > 0$
where the cosmology undergoes a bounce, with $\dot{R} = 0$ and $\ddot{R}$ changing sign. This necessarily
arises because of the imposition of $D^\mu T_{\mu\nu} = 0$ for energy conservation. For this example it
occurs in the past for \( \dot{w} > 1 \). The consistency of big bang cosmology back to the time of nucleosynthesis implies that our universe has not bounced for any \( 1 < \dot{w} < 10^9 \). It is also possible to construct cosmologies where the bounce occurs in the future! Rewriting Eq. (16) in terms of \( \Omega_\Lambda \):

\[
\dot{w} = \left( \frac{3\Omega_\Lambda}{3\Omega_\Lambda - (3 - P)} \right)^{1/3}
\]

If \( P < 3 \), then any \( \Omega_\Lambda < 0 \) will lead to a solution with \( 0 < \dot{w} < 1 \) corresponding to a bounce in the future. If \( P > 3 \) the condition for a future bounce is \( \Omega_\Lambda < -\left( \frac{P-3}{3} \right) \). What this means is that for the flat case \( \Omega_C = 0 \) with quintessence \( P > 0 \) it is possible for the future cosmology to be qualitatively similar to a non-quintessence closed universe where \( \dot{R} = 0 \) at a finite future time with a subsequent big crunch.

Another constraint on the cosmological model is provided by nucleosynthesis which requires that the rate of expansion for very large \( w \) does not differ too much from that of the standard model.

The expansion rate for \( P = 0 \) coincides for large \( w \) with that of the standard model so it is sufficient to study the ratio:

\[
\frac{(\dot{R}/R)_P^2/(\dot{R}/R)_P^2}{(3\Omega_M - P)/((3 - P)\Omega_M)} \xrightarrow{w \to \infty} \frac{(4\Omega_R - P)/((4 - P)\Omega_R)}{(3\Omega_M - P)/((3 - P)\Omega_M)}
\]

where the first limit is for matter-domination and the second is for radiation-domination (the subscript R refers to radiation).

The overall change in the expansion rate at the BBN era is therefore

\[
\frac{(\dot{R}/R)_P^2/(\dot{R}/R)_P^2}{(3\Omega_M - P)/((3 - P)\Omega_M) \times (4\Omega_R^{\text{trans}} - P)/((4 - P)\Omega_R^{\text{trans}})} \xrightarrow{w \to \infty}
\]

where the superscript "trans" refers to the transition from radiation domination to matter domination. Putting in the values \( \Omega_M = 0.3 \) and \( \Omega_R^{\text{trans}} = 0.5 \) leads to \( P < 0.2 \) in order that
the acceleration rate at BBN be within 15% of its value in the standard model, equivalent
to the contribution to the expansion rate at BBN of one chiral neutrino flavor.

Thus the constraints of avoiding a bounce ($\dot{R} = 0$) in the past, and then requiring consistency
with BBN leads to $0 < P < 0.2$.

We may now ask how this restricted range of $P$ can effect the extraction of cosmic
parameters from observations. This demands an accuracy which has fortunately begun
to be attained with the Boomerang data [14]. If we choose $l_1 = 197$ and vary $P$ as
$P = 0, 0.05, 0.10, 0.15, 0.20$ we find in the enlarged view of Fig 3 that the variation in the
parameters $\Omega_M$ and $\Omega_\Lambda$ can be as large as $\pm 3\%$. To guide the eye we have added the line
for deceleration parameter $q_0 = -0.5$ as suggested by [16,17]. In the next decade, inspired
by the success of Boomerang (the first paper of true precision cosmology) surely the sum
($\Omega_M + \Omega_\Lambda$) will be examined at much better than $\pm 1\%$ accuracy, and so variation of the
exponent of $P$ will provide a useful parametrization of the quintessence alternative to the
standard cosmological model with constant $\Lambda$.

Clearly, from the point of view of inflationary cosmology, the precise vanishing of $\Omega_C = 0$
is a crucial test and its confirmation will be facilitated by comparison models such as the
present one.
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Figure 1.
Dependence of $l_1$ on $P$ for (a) fixed $\Omega_M = 0.3$; (b) fixed $\Omega_C = 0$.

Figure 2.
Regions of the $\Omega_M - \Omega_\Lambda$ plot where there is a future bounce (small dot lattice), no bounce (unshaded) and a past bounce (large dot lattice) for (a) $P = 0$; (b) $P = 1$; and (c) $P = 2$.

Figure 3.
Enlarged view of $\Omega_M - \Omega_\Lambda$ plot to exhibit sensitivity to $0 \leq P \leq 0.2$.
Contours are (right to left) $P = 0, 0.05, 0.10, 0.15, 0.20$. 
$\Omega_\Lambda = 0.5$

$\Omega_\Lambda = 0.6$

$\Omega_\Lambda = 0.7$

$\Omega_\Lambda = 0.8$

Figure 1.(a)
Figure 1. (b)
Figure 2. (a)
Figure 2. (b)
Figure 2. (c)

\[ \Omega_\Lambda = 1 - \Omega_M \]

\[ \Omega_\Lambda = \frac{1}{2} \Omega_M \]

\[ \Omega_\Lambda = \frac{1}{2} \Omega_M - \frac{1}{2} \]
\( \Omega_\Lambda = \frac{1}{2} \Omega_m + \frac{1}{2} \)

Figure 3.