Critical Trapped Surfaces Formation in the Collision of Ultrarelativistic Charges in \((A)dS\)

I.Ya. Aref’eva\(^1\), A.A. Bagrov\(^1\) and L.V. Joukovskaya\(^2\)

\(^1\)Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina st. 8, 119991, Moscow, Russia
\(^2\)Centre for Theoretical Cosmology, DAMTP, CMS, University of Cambridge, Wilberforce Road, CB3 0WA, Cambridge, United Kingdom

Abstract: We study the formation of marginally trapped surfaces in the head-on collision of two ultrarelativistic charges in \((A)dS\) space-time. The metric of ultrarelativistic charged particles in \((A)dS\) is obtained by boosting Reissner-Nordström \((A)dS\) space-time to the speed of light. We show that formation of trapped surfaces on the past light cone is only possible when charge is below certain critical - situation similar to the collision of two ultrarelativistic charges in Minkowski space-time. This critical value depends on the energy of colliding particles and the value of a cosmological constant. There is richer structure of critical domains in \(dS\) case. In this case already for chargeless particles there is a critical value of the cosmological constant only below which trapped surfaces formation is possible. Appearance of arbitrary small nonzero charge significantly changes the physical picture. Critical effect which has been observed in the neutral case does not take place more. If the value of the charge is not very large solution to the equation on trapped surface exists for any values of cosmological radius and energy density of shock waves. Increasing of the charge leads to decrease of the trapped surface area, and at some critical point the formation of trapped surfaces of the type mentioned above becomes impossible.

Keywords: Anti de Sitter, de Sitter, Trapped Surfaces, Collision of Shock Waves, Black Holes, \(AdS/CFT\).
Contents

1. Introduction 2

2. Set up
   2.1 Metric of an ultrarelativistic charge in the (A)dS background 4
   2.2 Two waves picture and trapped surface equations 7

3. Solution to Trapped Surface Equation in $AdS_5$ 10
   3.1 Critical charge 10
   3.2 Area of the trapped surface in $AdS_5$ 12

4. Solution to Trapped Surface Equation in $dS_4$ 14
   4.1 Critical charges 14
   4.2 Area of the trapped surface in $dS_4$ 16

5. Conclusion 18

A. Appendix 19
   A.1 Standard form of Lemma 19
   A.2 Modified form of Lemma 19
1. Introduction

Black holes are expected to form in collisions of ultrarelativistic particles with energies above the Planck scale \cite{1, 2, 3, 4, 5}. The Planck energy could be few TeV in the framework of TeV-gravity, where our space is a 3-brane situated in a large extra dimensional space and elementary particles are confined on the brane \cite{6}. In this scenario the black hole production in collisions of particles with the center-mass energy of a few TeV and their experimental signatures \cite{8, 9, 10, 11} at the LHC became the subject of intensive analytical and numerical investigations \cite{12, 13, 14, 15, 16}. We also note a discussion of the possible production of wormholes and others more exotic objects at the LHC \cite{17, 18, 19}. See \cite{20, 21} for a consideration of wormholes in astrophysics.

There are motivations to study similar processes in $AdS$ background. In the framework of $AdS/CFT$ correspondence \cite{22} the formation of trapped surfaces in $AdS_5$ is interesting because it is supposed to be dual to real four-dimensional formation of the quark-gluon plasma and its thermal equilibration \cite{23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34}. Existence of a critical impact parameter, beyond which the trapped surface does not exist has been considered as an indication for similar critical impact parameter in real collisions of heavy ions at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Lab. We would like to recall that applications of the $AdS/CFT$ correspondence to strongly coupled quark-gluon plasma has lead to many interesting results \cite{27, 35, 36, 37, 38}, but these results are related to description equilibrium or close to equilibrium quark-gluon plasma.

There are observational as well as theoretical reasons to consider collisions of ultrarelativistic particles in asymptotically $dS_4$ space-times. Well-known experimental fact is that our universe is expanding with constant acceleration which is well described by a small positive cosmological constant and this implies that the space-time is asymptotically $dS$. Furthermore processes of collisions of a ultrarelativistic matter could be also interesting in the context of the early Universe \cite{39, 40, 41, 42, 43, 44}. For the current status of black hole astrophysics see \cite{45}.

Let us note that trapped surface formation in the collision of ultra-relativistic particles in $dS$ space-time could be interesting also in the context of $dS/CFT$ correspondence \cite{46, 47, 48, 49, 50}.

It is known that in the flat background \cite{51, 52} charges of ultrarelativistic colliding particles (for a fixed impact parameter) decrease the trapped surface area and by increasing the charges one may reach a critical point at which no trapped surfaces can be formed. It is natural to ask a similar question about critical effects related to charges of ultrarelativistic particles propagating in $(A)dS$ background.

In this paper the formation of trapped surfaces in collisions of two charged shock waves in $(A)dS$ background is being considered. The first work in this direction in the $AdS/CFT$ context was the paper by Gubser, Pufu and Yarom \cite{23}, who consid-
ered central collisions of neutral ultrarelativistic particles and found the (marginally) trapped surface forming at the collision moment. The trapped surface area was then used as a lower bound estimation of the entropy produced in a heavy ion collision. In the limit of very large collision energy $E$ they found that the entropy grows as $E^{2/3}$.

Lin and Shuryak [29] have discussed matching heavy ion collisions to those of gravitational shock waves and have noted that the effective size of objects in gravitational collision grows with energy with an exponent ten times that in the real QCD. They have stressed that one cannot tune the scale of the cosmological constant $\Lambda$ or the impact parameter of a bulk colliding objects to the size of nucleus and tune it perhaps to the parton density. Non-zero impact parameter case is considered in [23, 29] and a natural critical phenomenon analogous to neutral shock wave collision in Minkowski background [53, 54, 55] has been found. Beyond certain impact parameter, the trapped surface disappears and black hole formation does not happen.

Other type of criticality has been found by Alvarez-Gaume, Gomez, Vera, Tavanfar and Vazquez-Mozo [24]. They have considered central collision of shock waves sourced by a matter distribution in the transverse space and have found critical phenomenon occurring where the shock wave reaches some diluteness limit and the formation of the marginally trapped surface is no longer possible.

In this paper we find a new critical phenomenon in $AdS$ background. Namely, in the $AdS$ case beyond certain charge, the trapped surface disappears and black hole formation does not happen. The value of the critical charge depends on the energy of colliding particles as well as the value of the cosmological constant. This phenomenon is analogous to the critical phenomenon found by Mann and Yoshino in the charged shock wave collision in Minkowski background [51]. We consider here head-on collisions only and no-head ones can be considered by straightforward generalization of the technics developed in [25] and one can expect that there exists the domain in $(Q, b)$-plane beyond which the trapped surface disappears (here $Q$ is charge and $b$ is impact parameter).

There is a richer structure of critical domains in the $dS$ case. In this case already in absence of a charge there is a critical value for the ratio of the shock wave energy density and the cosmological radius only below which trapped surfaces formation is possible [56]. We observe that small charge substantially violates this critical effect. If the charge is nonzero but enough small the trapped surface on the past light cone can be formed for given values of cosmological constant and energy density of shock wave even if in the neutral case for the same parameters the trapped surface can not be formed. Increasing of the charge decrease the area of formed trapped surface and it can be produced until the charge does not overcome critical value which depends on the energy of colliding particles and the value of the cosmological constant.

The paper is organized as follows. In Sect.2.1. we present the $(A)dS$ analog of the charged Aichelburg-Sexl shock wave which describes charged ultrarelativistic
particles in the (A)dS space time. This is a generalization of results of Hotta and Tanaka [62] to the case of charged particles. We note a problem that appears in the dS case and that is associated with the cosmological horizon of dS metric and can be solved by regularization procedure. Physical meaning of this regularization requires more detailed investigations. In Sect.2.2 we present the standard picture of two ultrarelativistic colliding particles and the trapped surface equation for the central collision in the the (A)dS case. Sect. 3 is devoted to description of the solutions to equation which defines a radius of trapped surface for the central collision in the $AdS_5$ case. The existence of the critical charge is being demonstrated. In Sect. 4 we discuss solutions to the trapped surface equation in the $dS_4$ case.

2. Set up

2.1 Metric of an ultrarelativistic charge in the (A)dS background

The metric of an ultrarelativistic charge in the flat background has been obtained [57, 58] by applying the Aichelburg-Sexl boost [59] to $D$-dimensional Reissner-Nordström space-time [60], see also [61].

We apply the same procedure to Reissner-Nordström-(anti)de Sitter black hole metric. Our calculations generalize results of Hotta and Tanaka [62], see also [57, 63, 64, 65, 58, 66, 67, 68, 69]. In a spherical static (Schwarzschild) coordinates the RN-(A)dS metric has the form (see for example, [70])

\[ ds^2 = -g(R)dT^2 + g(R)^{-1}dR^2 + R^2d\Omega_{D-2}^2, \]  

(2.1)

where

\[ g(R) = 1 - \frac{2M}{R^{D-3}} + \frac{Q^2}{R^{2(D-3)}} \pm \frac{\Lambda}{3}R^2, \]  

(2.2)

and the electromagnetic field

\[ A = A_TdT = \left( \sqrt{\frac{D-2}{2(D-3)}} \frac{Q}{R^{D-3}} + \Phi \right)dT, \]  

(2.3)

gives a solution to the E.O.M. for a (pure electric) gauge potential, here $\Phi$ is a constant and $\Lambda$ is a cosmological constant, $\Lambda/3 \equiv 1/a^2$.

Here and below ($-$) corresponds to $dS$ and ($+$) to $AdS$. $Q$ and $M$ are related to charge $q$ and mass $m$

\[ Q^2 = \frac{8\pi G_D q^2}{(D-2)(D-3)}, \]  

(2.4)

\[ M = \frac{8\pi G_D m}{(D-2)\Omega_{D-2}}. \]  

(2.5)
To get the metric of an ultrarelativistic charge in $(A)dS$ we take metric (2.1) and expand it on $M$ and $Q^2$. Thereafter we take the first order terms on these parameters

\[ds^2 = ds^2_{ds,AdS} + \left(\frac{2M}{R^{D-3}} - \frac{Q^2}{R^{2(D-3)}}\right)dT^2 + \frac{1}{(1 + \frac{R^2}{a^2})^2}\left(\frac{2M}{R^{D-3}} - \frac{Q^2}{R^{2(D-3)}}\right)dR^2 \quad (2.6)\]

We rewrite (2.6) in the plane coordinates which are satisfying relations

\[- Z_0^2 + Z_1^2 + Z_2^2 + ... + Z_{D-1}^2 + Z_D^2 = a^2, \quad (2.7)\]
\[- Z_0^2 + Z_1^2 + Z_2^2 + ... + Z_{D-1}^2 - Z_D^2 = -a^2, \quad (2.8)\]

Plane coordinates are related to $T$ and $R$ via formula

\[Z_0 = \sqrt{a^2 - R^2} \sinh T/a \quad (2.9)\]
\[Z_D = \pm \sqrt{a^2 - R^2} \cosh T/a \quad (2.10)\]

for $dS$ and

\[Z_0 \equiv \sqrt{a^2 + R^2} \sin(T/a), \quad (2.11)\]
\[Z_D \equiv \sqrt{a^2 + R^2} \cos(T/a), \quad (2.12)\]

for $AdS$ case and

\[Z_1 = R \cos \theta_1, \quad (2.13)\]
\[Z_2 = R \sin \theta_1 \cos \theta_2, ..., \quad (2.14)\]
\[Z_{D-1} = R \sin \theta_1 \sin \theta_2... \sin \theta_{D-2}, \quad (2.15)\]

for both cases. As a result we have

\[ds^2 = ds^2_0 + ds^2_p, \quad (2.16)\]

where $ds^2_0$ is $(A)dS$ metric and the perturbation $ds^2_p$ has the form

\[ds^2_p = G_{00}dZ_0^2 + G_{DD}dZ_D^2 + G_{0D}dZ_0dZ_D, \quad (2.17)\]

where

\[G_{MN} = \chi(Z_0^2, Z_D^2, M, Q) \cdot g_{MN}(Z_0^2, Z_D^2) \quad (2.18)\]

and nonzero components of $g_{MN}$ and the overall factor $\chi(Z_0^2, Z_D^2, M, Q)$ are given by

\[g_{00} = \pm a^2 Z_D^2 + Z_0^2 Z_D^2 + Z_1^4 + a^2 Z_0^2, \quad (2.19)\]
\[g_{DD} = a^2 Z_0^2 + Z_1^4 \pm Z_D^2 Z_0^2 + a^2 Z_0^2, \quad (2.20)\]
\[g_{0D} = -2(\pm 2a^2 + Z_0^2 \mp Z_D^2)Z_0 Z_D, \quad (2.21)\]
\[\chi = \frac{a^2}{(Z_D^2 + Z_0^2)^2}\left(\frac{2M}{(\pm a^2 + Z_0^2 + Z_D^2)^{D-1}} - \frac{Q^2}{(\pm a^2 + Z_0^2 + Z_D^2)^{D-2}}\right). \quad (2.22)\]
Performing a boost in the $Z_1$-direction

$$Z_0 = \gamma(Y_0 + vY_1), \quad \gamma \equiv (1 - v^2)^{-1/2},$$  \hfill (2.23)

$$Z_1 = \gamma(vY_0 + Y_1),$$  \hfill (2.24)

$$Z_2 = Y_2, \ldots Z_D = Y_D.$$  \hfill (2.25)

and rescaling

$$M = \bar{M}\sqrt{1 - v^2} \equiv \bar{M}/\gamma,$$  \hfill (2.26)

$$Q^2 = \bar{Q}^2\sqrt{1 - v^2} \equiv \bar{Q}^2/\gamma,$$  \hfill (2.27)

we put the first order deformation of the metric into the form

$$ds_p^2 = \gamma G_{00}(\gamma^2(Y_0 + vY_1)^2, Y_D^2) d(Y_0 + vY_1)^2$$
$$+ G_{DD}(\gamma^2(Y_0 + vY_1)^2, Y_D^2) dY_D dY_D$$
$$+ \frac{1}{\gamma} G_{DD}(\gamma^2(Y_0 + vY_1)^2, Y_D^2) dY_D^2.$$  \hfill (2.28)

To get the limit when $\gamma \to \infty$ in the case of $AdS$ like in absence of a charge we can apply the lemma of \[59, 62\] that is dealing with distributions:

$$\lim_{v \to 1} \gamma f(\gamma^2(Y_0 + vY_1)^2) = \delta(Y_0 + Y_1) \int f(x^2) dx$$  \hfill (2.29)

The case of $dS$ is more subtle. The problem is that in the $dS$ case components of $ds_p^2$ are not distributions for any value of $D$. Indeed, we need to consider the following expressions

$$\frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} f(\gamma^2 Y^2),$$  \hfill (2.30)

where $f$ is a smooth function.

It is well known \[77, 78\] that $\frac{1}{(Z^2 - \gamma^2 Y^2)^2}$ (as function of $Y$) is not a distribution for any $\gamma \neq 0$. To treat the expression (2.30) as a distribution one has to use a regularization. All natural regularizations have been studied \[77, 78\] and we would use one of them. We can construct modification of the formula (A.1) which operates with

$$\lim_{\gamma \to \infty} \left( \frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} f(\gamma^2 Y^2) \right)$$  \hfill (2.31)

and is suitable for the $dS$ case (see Appendix A.2):

$$\lim_{\gamma \to \infty} \left( \frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} f(\gamma^2 Y^2) \right) f(x^2) dx$$  \hfill (2.32)

According to mentioned formulas we have to select only terms proportional to $\gamma$ ($\gamma \to \infty$ when $v \to 1$). This prescription gives the following answer

$$ds^2 = ds_0^2 + F_{D,dS/AdS}(\bar{M}, \bar{Q}^2, Y_D) \delta(u) du^2,$$  \hfill (2.33)
where $u = Y_0 + Y_1$, and the shape function $F_{D,dS/AdS}(\bar{M}, \bar{Q}^2, Z)$ is given by the following formula

$$F_{D,dS/AdS}(\bar{M}, \bar{Q}^2, Z) = F_{D,dS/AdS}(\bar{M}, Z) - \frac{\bar{Q}^2}{2\bar{M}} F_{2D-3,dS/AdS}(\bar{M}, Z), \quad (2.34)$$

where

$$F_{D,dS/AdS}(\bar{M}, Z) = 2\bar{M} a^2 \int_{-\infty}^{\infty} \left[ \frac{(a^2(\pm Z^2 + x^2) + Z^2(x^2 \mp Z^2))}{(Z^2 \mp x^2)^2 \cdot (\pm a^2 + x^2 \mp Z^2)^{D-1 \over 2}} \right] dx. \quad (2.35)$$

Therefore, $F_{D,dS/AdS}(\bar{M}, \bar{Q}^2, Z_D)$ is a sum of the profile function for the chargeless shock wave in the same space-time dimension plus the profile function of $2D - 3$ dimensional chargeless shock wave multiplied by a ratio of the charge square to mass $\bar{M}$.

2.2 Two waves picture and trapped surface equations

In the previous subsection we presented gravitational properties of one ultrarelativistic charged particle traveling in $(A)dS$ space-time. The gravitational field of the particle is infinitely Lorentz-contracted and forms a shock wave. Except at the shock wave, the space-time is $(A)dS$. To deal with two colliding ultrarelativistic particles one deals with picture schematically shown in Fig.1. There are two shock waves and except at the shock waves the space-time is $(A)dS$ before the collision (i.e., regions I, II, and III). After the collision these two shocks nonlinearly interact with each other and the space-time within the future lightcone of the collision (i.e., region IV) becomes highly curved. It is unknown how to derive the metric in region IV even numerically. But it is possible to investigate the apparent horizon on the slice $u \leq 0 = v$ and $v \leq 0 = u$ and calculate the cross section for the apparent horizon formation $\sigma_{AH}$ in $AdS$ and $dS$ cases, see [23] and [56], respectively, similarly to the flat case [1, 13, 14].

In non asymptotically flat cases one has no general theorems that guarantee formation of a black hole if the marginally trapped surface is formed [71, 72]. However there is a common opinion that the existence of the marginally trapped surface can be used as an indication of a black hole formation and the area of the trapped surface can be used as a low bound to estimate the cross-section of the black hole formation [73, 74, 75, 76]. The problem of horizon formation in asymptotically non-flat spacetimes has been considered in [79].

This picture generalizes the picture for ultrarelativistic particles without charge and there is nothing special in the charged case except the different form of shape functions for the shock waves.

According to this picture the trapped surface is made up of two pieces, each of them is associated with one of two shock waves and one has to solve the boundary
Figure 1: Schematics picture of two colliding ultrarelativistic particles. $u$ and $v$ is a set of the light-cone coordinates

problem for two functions that define shapes of these two pieces of the trapped surface. For a central collision these two functions are reduced to one trapped surface function $\Psi$ and one deals with one equation. The form of this equation in the flat, AdS and dS spaces has been obtained by Eardley and Giddings [53], Gubser, Pufu and Yarom [23] and in [56], respectively. Corresponding trapped surface functions are related to solutions to the second order differential equation defining eigenfunction problem for the Beltrami-Laplace operators on the plane, Lobachevski space $\mathbb{H}^{D-2}$, and the sphere $S^{D-2}$ for the flat, AdS, and dS cases respectively. The corresponding equations in last two cases have the form

$$\left( \Delta_{S^{D-2}} \pm \frac{D-2}{a^2} \right) \left( \frac{\Psi - \kappa H}{1 \pm \rho^2/2a^2} \right) = 0, \quad (2.36)$$

$$H = \frac{1}{2} \left( 1 \pm \frac{\rho^2}{2a^2} \right) F, \quad (2.37)$$

where the upper sign corresponds to the dS case and the lower one to the AdS case; $\Psi$ is the trapped surface function and $H$ is related to the shape of shock wave. Note that this form of the trapped surface equation does not depend on a particular choice of the shape $F$ of shock wave. Here $\kappa = \theta(0)$ and we suppose that Heaviside function $\theta(0) = \frac{1}{2}$.

In the (A)dS case it is convenient to work with a chordal distance $q$, that is related to the plane coordinates $Z_D$ via

$$q_{AdS} = \frac{Z_D}{2a} - \frac{1}{2}, \quad (2.38)$$

in the AdS case and

$$q_{dS} = \frac{1}{2} - \frac{Z_D}{2a}, \quad (2.39)$$
in the $dS$ case.

Also we use here radial coordinate $\rho$ related to the chordal coordinates via

$$
\rho_{AdS} = a \sqrt{\frac{2 q_{AdS}}{1 + q_{AdS}}},
$$

(2.40)

and

$$
\rho_{dS} = a \sqrt{\frac{2 q_{dS}}{1 - q_{dS}}},
$$

(2.41)

The trapped surface function $\Psi$ for head-on collisions has to satisfy the following boundary conditions on a submanifold $\rho = \rho_0$:

$$
\Psi_{|\rho=\rho_0} = 0,
$$

(2.42)

$$
\partial_\rho \Psi_{|\rho=\rho_0} = -2.
$$

(2.43)

Existence of the trapped surface means the existence of a real solution to the following equation

$$
\frac{1}{4} (1 - \rho_0^2) F'(\rho_0) - \frac{\rho_0}{2a^2 + \rho_0^2} F(\rho_0) + \frac{\sqrt{2}}{2\kappa} = 0,
$$

(2.44)

for the $AdS$ case [23] and

$$
\frac{1}{4} (1 + \rho_0^2) F'(\rho_0) + \frac{\rho_0}{2a^2 - \rho_0^2} F(\rho_0) + \frac{\sqrt{2}}{2\kappa} = 0,
$$

(2.45)

for the $dS$ case [56], $\rho_0$ defines the radius of the corresponding trapped surface.

One can rewrite equations (2.44) and (2.45) in terms of $q$

$$
F'(q_0) - \frac{2}{1 + 2q_0} F(q_0) + \frac{2a}{\kappa \sqrt{q_0(1 + q_0)}} = 0
$$

(2.46)

for the $AdS$ case and

$$
F'(q_0) + \frac{2}{1 - 2q_0} F(q_0) + \frac{2a}{\kappa \sqrt{q_0(1 - q_0)}} = 0.
$$

(2.47)

for $dS$ case.

In [23] $\kappa = 1$ and in [56] $\kappa = 1/2$. Note that in [56] we used different normalization $F_{[56]} = \sqrt{2} F$.

These equations determine values of critical charges for the trapped surfaces formation. In the next two sections we are going to analyze particular cases of these equations.
3. Solution to Trapped Surface Equation in AdS

3.1 Critical charge

According to (2.34) the profile of the charged shock wave in the AdS$_5$ background has the form

$$F_{5,\text{AdS}}(\bar{M}, \bar{Q}^2, Z_5) = -3\frac{\pi \bar{M}}{a} \left( \frac{1 - \frac{2Z_5^2}{a^2}}{\sqrt{\frac{Z_5^2}{a^2}} - 1} + \frac{2Z_5}{a} \right) + \frac{5\pi \bar{Q}^2 12Z_5^2}{8a^3} - 3 - \frac{8Z_5^4}{a^3} + \sqrt{\frac{Z_5^2}{a^2}} - 1 + 8\frac{Z_5}{a} \left( \frac{Z_5^2}{a^2} - 1 \right)^{\frac{3}{2}},$$

or in terms of the chordal coordinate $q$

$$F_{5,\text{AdS}}(\bar{M}, \bar{Q}^2, Z_5(q)) = \frac{3\pi \bar{M}}{a} \left( \frac{8q^2 + 8q + 1}{2\sqrt{q(1+q)}} - 4(2q + 1)\sqrt{q(1+q)} \right) + \frac{5\pi \bar{Q}^2}{64a^3} \left( \frac{144q^2 + 16q - 1 + 128q^4 + 256q^3 - 64(2q + 1)q(1+q)\sqrt{q(1+q)}}{q(1+q)\sqrt{q(1+q)}} \right).$$

To find solutions to (2.46) for $F$ given by (3.2) we present the LHS of (2.46) for $F$ given by (3.2)

$$F_{5,\text{AdS}}(a, \bar{M}, \bar{Q}^2, q) = \frac{N_{5,\text{AdS}}(a, \bar{M}, \bar{Q}^2, q) - 2F_{5,\text{AdS}}(\bar{M}, \bar{Q}^2, q)}{1 + 2q} + \frac{2a}{\sqrt{q(1+q)}},$$

as

$$F_{5,\text{AdS}}(a, \bar{M}, \bar{Q}^2, q) = \frac{N_{5,\text{AdS}}(a, \bar{M}, \bar{Q}^2, q)}{D_{5,\text{AdS}}(a, q)}.$$ 

The numerator $N_{5,\text{AdS}}(a, \bar{M}, \bar{Q}^2, q)$ contains just one term with dependence on $\bar{Q}^2$. This dependence is linear with a positive coefficient

$$N_{5,\text{AdS}}(a, \bar{M}, \bar{Q}^2, q) = N_{5,\text{AdS}}(a, \bar{M}, q) + 15\frac{\pi}{a} \bar{Q}^2.$$ 

The denominator in (3.4) does not take infinite values. To find solutions to (2.46) for the shape function given by (3.2) we can draw the function

$$-N_{5,\text{AdS}}(a, \bar{M}, q) \equiv -(512a^3q^5 + 1280a^3q^4 + 96a^3q^3 - 96\bar{M}\pi aq^2 + 1024a^3q^3 - 96\bar{M}\pi aq + 256a^3q^2),$$

and see where this function is positive and where it can be equal to a given value $15\bar{Q}^2 \frac{\pi}{a}$. The maximum values of the function $-N_{5,\text{AdS}}(a, \bar{M}, q)$ define the critical values of $\bar{Q}^2$ for given $a$ and $\bar{M}$, see Fig. 2. In 3 we present the same pictures for different fixed values of $a$ and $\bar{M}$.
Figure 2: $-N_{5,\text{AdS}}(a, \bar{M}, q)$ as function of two variables $q$ and $a$, $\bar{M} = 1$. The yellow plane corresponds to a given value of $15\frac{\pi}{a} \bar{Q}^2$.

Figure 3: A. $-N_{5,\text{AdS}}(a, \bar{M}, q)$ as function of $q$ for different values of $a$ and different values of $\bar{M}$. Thick lines correspond to shock waves with higher energy. Magenta lines correspond to $a = 0.5$, red lines to $a = 1$, blue lines to $a = 2$ and green lines to $a = 3$. Critical values of the charge increase with increasing of the cosmological radius as well as the energy of shock waves. B. $-N_{5,\text{AdS}}(a, \bar{M}, q)$ as function of $q$ for different values of $a$ and $\bar{M} = 1$. The horizontal lines show the critical values of $\bar{Q}$ for the correspondent value of $a$. Magenta lines correspond to $a = 0.5$, red lines to $a = 1$, blue lines to $a = 2$ and green lines to $a = 3$ and critical values of the charge increase with increasing of the cosmological radius.
3.2 Area of the trapped surface in $AdS_5$

In the case of $AdS_5$ background the shape function $F(\rho)$ is

$$F_{5,AdS}(\rho) = \frac{3\sqrt{2} \pi \bar{M}}{a} \left( \frac{4a^4 + 12a^2 \rho^2 + \rho^4}{4a \rho (2a^2 - \rho^2)} - \sqrt{2} \frac{2a^2 + \rho^2}{2a^2 - \rho^2} \right)$$

$$- 5 \frac{\sqrt{2} \pi \bar{Q}^2}{a^3} \left( \frac{16a^8 - 160a^6 \rho^2 - 360a^4 \rho^4 - 40a^2 \rho^6 + \rho^8}{256a^2 \rho^3 (2a^2 - \rho^2)} + \frac{\sqrt{2} 2a^2 + \rho^2}{2} \frac{2}{2a^2 - \rho^2} \right)$$

(3.7)

And the equation on the radius of trapped surface (2.44) takes the form (here we introduce $x = \rho_0/a$)

$$\frac{(2 - x^2)^3}{x^2 (2 + x^2)} \left( \frac{\bar{M}}{a^2} - \frac{\bar{Q}^2}{a^4} \frac{5 (2 - x^2)^2}{64 x^2} \right) = \frac{16}{3\pi \kappa}.$$  

(3.8)

We solve this equation graphically, see Fig.4. We assume that value $\bar{M}/a^2$ is fixed, we variate parameter $\bar{Q}/a^2$ and analyze how the left hand side changes. On the Fig.4 the blue straight line represents the right hand side of (3.2), i.e. $16/3\pi \kappa$ (for $\kappa = 1/2$). The green line corresponds to $\bar{Q}^2/a^4 < Q^2_{cr}$. The magenta line represents the left hand side of equation (3.2) for $\bar{Q}^2/a^4 = Q^2_{cr}$. The red line represents the left hand side of equation (3.2) for $\bar{Q}^2/a^4 > Q^2_{cr}$.

**Figure 4:** A. The left hand side of equation (3.2) as function of $x$ for the fixed value of $\bar{M}/a^2 = 1$ and different values of $\bar{Q}^2/a^4$. The blue line corresponds to $\bar{Q} = 0$ and the straight blue line represents the right hand side of equation (3.2), i.e. $16/3\pi \kappa$ (for $\kappa = 1/2$). The green line corresponds to $\bar{Q}^2/a^4 < Q^2_{cr}$. The magenta line represents the left hand side of equation (3.2) for $\bar{Q}^2/a^4 = Q^2_{cr}$. The red line represents the left hand side of equation (3.2) for $\bar{Q}^2/a^4 > Q^2_{cr}$. B. The thick green line corresponds to shock wave with higher value of $\bar{M}/a^2 > 1$ and the same value of $\bar{Q}^2/a^4 < Q^2_{cr}$ as the fat green line. The thick red line corresponds to shock wave with higher value of $\bar{M}/a^2 > 1$ and the same value of $\bar{Q}^2/a^4 > Q^2_{cr}$ as the fat red line.
When $\bar{Q} = 0$ equation (3.8) has only one root (intersection of two blue lines in Fig.4A). A small charge leads to appearance of one new root of equation (3.8). This root is very close to zero and the initial ”chargeless” root moves to the left. These two roots correspond to two intersections of the blue straight line and the green line in Fig.4A). Further increasing of the charge makes for increasing the smallest root and decreasing the biggest one. At the critical value of $Q^2/a^4 = Q^2_{cr}$, $Q^2_{cr} \simeq 0.5$, two roots coincide (the magenta line in Fig.4A) and we have only one root of equation (3.8). For $\bar{Q}/a^4 > Q^2_{cr}$ equation (3.8) has no physical solution for given $\bar{M}/a^2$ and trapped surface can not be formed. Fig.4B shows that by increasing $\bar{M}/a^2$ we also increase $Q^2_{cr}$.

Suppose that $\bar{Q}/a^4 < Q^2_{cr}$. The area of the trapped surface is given by (for detailed derivation of similar formula in $dS$ see [56])

$$A_{AdS_5} = 2 \cdot VolS^2 \int_0^{\rho_0} \frac{2\sqrt{2}\rho^2}{(1 - \frac{\rho^2}{2a^2})^3} d\rho =$$

$$= 4\sqrt{2}a^3\pi \left( \frac{2\rho_0(2a^2 + \rho_0)}{(2a^2 - \rho_0)^2} - \sqrt{2}\arctanh\left(\frac{\sqrt{2}\rho_0}{2a}\right) \right).$$

To estimate the area of the trapped surface we use the biggest root of equation (3.8) because in the neutral limit this root tends to the ”chargeless” one.

Firstly we assume that $\bar{M} << a^2$. In this case a solution to (3.8) is given by

$$x^2 \approx \frac{3\pi}{8a^2} \left( 2\kappa - \frac{5\bar{Q}^2}{6\pi M^2} \right).$$

For small $x = \rho/a$ the area is

$$A_{AdS_5} \approx \frac{16\sqrt{2}\pi}{3} a^3 x^3 + O(x^5)$$

and we get

$$A_{AdS_5} \approx \frac{16\sqrt{2}\pi}{3} a^3 \left( \frac{3\pi}{8a^2} \left( 2\kappa - \frac{5\bar{Q}^2}{6\pi M^2} \right) \right)^{3/2}$$

Secondly we consider the case $\bar{M} >> a^2$. In this regime a solution to (3.8) is given by $x = \sqrt{2} - y$, where $y$ is small and satisfies the following equation

$$y^2 + \frac{3}{2\sqrt{2}}y^4 + \frac{7}{8}y^5 + \ldots \left( 1 - \frac{\bar{Q}^2}{\bar{M}a^2} \frac{5}{16} (y^2 + \ldots) \right) = \frac{4\sqrt{2}}{3\pi\kappa} \frac{a^2}{\bar{M}}.$$

Denoting the RHS of (3.13) by $\alpha^3 \equiv \frac{4\sqrt{2}}{3\pi\kappa} \frac{a^2}{\bar{M}}$ we find a perturbative solution

$$y = \alpha + p_1\alpha^2 + p_2\alpha^3 + \ldots$$
where the coefficients are given by

\[ p_1 = -\frac{\sqrt{2}}{4} \quad (3.15) \]

\[ p_2 = \frac{1}{12} + \frac{\bar{Q}^2}{Ma^2} \frac{5}{16} \cdot 3 \quad (3.16) \]

The area of the trapped surface is given by

\[ A_{AdS_5} \approx 4\sqrt{2}\pi a^3 \left( \frac{\sqrt{2}}{y^2} - \frac{1}{y} + ... \right) \]

\[ = 4\pi a^3 \left( \frac{3\pi \kappa \bar{M}}{2a^2} \right)^{2/3} \left( 1 - \frac{1}{24} (1 + \frac{5\bar{Q}^2}{Ma^2}) (\frac{4\sqrt{2} a^2}{3\kappa \pi \bar{M}})^{2/3} + ... \right) \quad (3.17) \]

and we see that the first term reproduces the Gubser, Pufu and Yarom answer \[23\] and the charge decreases the area of the trapped surface.

4. Solution to Trapped Surface Equation in $dS_4$

4.1 Critical charges

The profile of the charged shock wave in $dS_4$ in the chordal coordinate is given by

\[ F_{4,dS}(\bar{M}, \bar{Q}^2, q) = 8\bar{M} \left[ -1 + \frac{1 - 2q}{2} \ln \left( \frac{1 - q}{q} \right) \right] - \bar{Q}^2 3\pi \left( \frac{1}{4a} - \frac{8q}{\sqrt{q(1-q)}} \right) \quad (4.1) \]

To visualize the solutions of equation (2.47) for the shape function (4.1) one can use the same method as in Sect. 3. Let us introduce the function

\[ F_{4,dS}(a, \bar{M}, \bar{Q}^2, q) = F'_{4,dS}(a, \bar{M}, \bar{Q}^2, q) + \frac{2F_{4,dS}(a, \bar{M}, \bar{Q}^2, q)}{1 - 2q} + \frac{2a}{\sqrt{q(1-q)}}. \quad (4.2) \]

It can be presented as

\[ F_{4,dS}(a, \bar{M}, \bar{Q}^2, q) = \frac{N_{4,dS}(a, \bar{M}, \bar{Q}^2, q)}{D_{4,dS}(a, q)}, \quad (4.3) \]

where

\[ N_{4,dS}(a, \bar{M}, \bar{Q}^2, q) = N_{4,dS}(a, \bar{M}, q) + 3\pi \frac{\bar{Q}^2}{a^2}, \quad (4.4) \]

and

\[ N_{4,dS}(a, \bar{M}, q) = 32q^3 - 48q^2 + 16q - 32\frac{\bar{M}}{a} \sqrt{q(1-q)}. \quad (4.5) \]

To solve equation

\[ F_{4,dS}(a, \bar{M}, \bar{Q}^2, q) = 0 \quad (4.6) \]
one can draw the function $-N_{4,ds}(\tilde{M}/a, q)$ and see where it can be equal to $3\tilde{Q}^2/a^2\pi$.

The shape of $-N_{4,ds}(\tilde{M}/a, q)$ for a fixed value of $\tilde{M} = 1$ is presented in Fig.4. In Figure 4 A, we can see results obtained in [56], namely the existence of a critical value of $a = a_{cr,0}(\tilde{M})$, below which the trapped surface formation is impossible. Indeed, for $a$ small enough values of this function are positive and equation (4.2) has no solution. The charge makes this critical value of $a = a_{cr,Q}(\tilde{M})$ for the same energy less then the corresponding $a_{cr,0}(\tilde{M})$ for the chargeless case. (see Fig.4 B).

**Figure 5:** A. The multicolored surface represents $-N_{4,ds}(a, \tilde{M}, q)$ as function of two variables $q$ and $a$; $\tilde{M} = 1$. The intersection with the yellow plane shows the existence of solutions of equation (4.2) for $Q = 0$ for $a > a_{cr,0}$. For $0 < a < a_{cr,Q}$ there is no intersection of the yellow plane with the multicolored surface B. The intersection the multicolored surface with the green plane shows the existence of solutions of equation (4.2) for given $\tilde{Q}$ for $a < a_{cr,Q}$. We see that the critical value of the gravitational radius for two colliding chargeless shock waves is smaller as compare with the critical radius for charged shock waves with the same energy.

In Fig.4 the shape of function $-N_{4,ds}(a, \tilde{M}, q)$ as a functions of $q$ for different fixed values of $a$ and $\tilde{M} = 1$ is presented.

Fig.4 A. shows that in the case of $a < a_{cr,0}(\tilde{M})$, i.e. in the case when chargeless version of equation (4.1) has not solution, the presence of the charge drastically change the situation. Already a small charge produces a nonzero solution of the equation (4.6). By increasing the charge we reach a domain where charge effect dominates, i.e. the second term in the RHS of (4.1) dominates and equation (4.6) has no solutions.

Fig.4 B. shows that for $a > a_{cr,0}(\tilde{M})$ there is the maximum value of the function $-N_{4,ds}(a, \tilde{M}, q)$. This maximum for different $a$ defines critical values of $\tilde{Q}^2$, $Q_{max}^2 = Q_{max}^2(a, \tilde{M})$, above which there is no formation of the trapped surface. Similar maximum have curves representing the case of $a < a_{cr,0}$ (the green and magenta lines in Fig.4 B).
Figure 6: A. The shape of functions $-N_{4,\text{dS}}(a, M, q)$ as functions of $q$ for different fixed values of $a$, $\bar{M} = 1$. The red line presents the shape for $a$ equal to the critical value $a_{cr,0}$. The blue line presents the shape for $a > a_{cr,0}$, magenta and green lines correspond to $a < a_{cr,0}$; the dark green line and magenta lines have not nontrivial local minimum. B. Here is shown that all curves in the region $0 < q < 1$ have maxima that demonstrates the existence of the critical values of the charge above which there is no trapped surface formation.

4.2 Area of the trapped surface in $dS_4$

In $dS_4$ background the shape is

$$F_{4,\text{dS}}(\rho) = 4\bar{M} \left( -2 + \frac{2a^2 - \rho^2}{2a^2 + \rho^2} \ln \frac{2a^2}{\rho^2} \right) - \frac{3\sqrt{2}\pi\bar{Q}^2}{8a^2} \frac{4a^4 - 12a^2\rho^2 + \rho^4}{\rho(2a^2 + \rho^2)},$$

(4.7)

and equation (2.45) has the form:

$$\frac{(2 + x^2)^2}{x(2 - x^2)} \left( \frac{\bar{M}}{a} - \frac{\bar{Q}^2}{2a^2} \frac{3\sqrt{2}\pi}{32} \frac{2 + x^2}{x} \right) = \frac{\sqrt{2}}{2\kappa}. \quad (4.8)$$

For $\bar{Q}^2 = 0$ there are two possibilities. Whether $\bar{M} > a/4$ and we have not a solution to the equation on the radius of trapped surface, or $\bar{M} \leq a/4$ and two solution to this equation exist (see [56]). We take the smaller one because the larger is localized near spatial infinity. The graphical solution of this equation is presented in Fig.7. The LHS of (4.8) for $\bar{M} > a/4$ is drawn by the red line in Fig.7.A and for $\bar{M} < a/4$ by the magenta line in Fig.7.B. The RHS of (4.8) is presented by the green straight lines.

Let us consider graphically as well the influence of charge.

In Fig.7 different lines correspond to the left hand side of (4.8) for various values of charge $\bar{Q}$.

Let us consider first $a < a_{cr,0}$, Fig.7.A. Note that the function representing the case $\bar{Q} = 0$ has a positive singularity at $x = 0$ (the red curve). If $\bar{Q} \neq 0$ and is arbitrary small then the sign of the singularity at $x = 0$ changes and this fact leads
Figure 7: A. $a < a_{cr,0}$. The LHS of eq. (4.2) for $\bar{M}/a = 0.5 > \mu_{cr,0}$ and different values of $\bar{Q}^2/a^2$: $\bar{Q}^2/a^2 = 0$ (red line), $0 < \bar{Q}^2/a^2 < Q_{cr}^2 \approx 0.94$ (blue lines), and $\bar{Q}^2/a^2 > Q_{cr}^2$ (orange lines) The green straight line represents the value of the RHS of eq.(4.2). B. $a > a_{cr,0}$. LHS of eq.(4.2) for $\bar{M}/a = 0.1 < \mu_{cr,0}$ and different values of $\bar{Q}^2/a^2$ from $\bar{Q}^2/a^2 = 0$ (magenta line), $0 < \bar{Q}^2/a^2 < Q_{cr}^2 \approx 0.41$ (blue lines), and $\bar{Q}^2/a^2 > Q_{cr}^2$ (orange lines). The green line represents the value of the RHS of eq.(4.2)

to appearance of one root. Growth of the $\bar{Q}$ makes for growth of the root (the set of blue curves in Fig.7.A; lower curve corresponds to the bigger value of $\bar{Q}^2$).

Further increasing of $\bar{Q}^2$ causes the change of the sign of the second singularity at $x = \sqrt{2}$ and we have no more intersection points between the green line and the lines representing the left hand side of (4.8). The trapped surface cannot be formed as in the neutral case.

Slightly different behavior of roots can be observed in the case, when the trapped surface exists already for $\bar{Q} = 0$.

When equation (4.8) for $\bar{Q} = 0$ has two roots (intersection of the green and magenta lines), i.e. $a > a_{cr,0}$ (Fig.7.B), arbitrary small increasing of the charge leads to appearance of one new roots of (4.8) and to shift of the existent roots. Both of two new small roots are smaller than initial (intersection of green and the upper blue line). The third root near $x = \sqrt{2}$ seems nonphysical because it is localized near spatial infinity. At some critical value of $\bar{Q}$ (the second blue line) only one small root remains, and after that we have no physical solution to the equation for trapped surface radius. And if $\bar{Q}$ becomes much larger (brown curves with two negative singularities) there is no solution, even nonphysical.

We now assume that $\bar{Q}^2 \leq Q_{cr}^2$ and $\bar{M} \ll a$. In this case solution to (4.8) is very close to $x_0 = \sqrt{2 \frac{\bar{M}}{a}}$

$$\rho_0(\bar{M}/a, \bar{Q}^2/a^2) \approx 2\sqrt{2} \kappa \bar{M} (1 - \sqrt{2} \frac{3\pi}{64} \frac{\bar{Q}^2}{\kappa \bar{M}^2})$$

(4.9)
i.e. the presence of the charge decreases the radius of the formed trapped surface.

The area of the trapped surface is given by

$$A_{dS_4} = 2 \cdot Vol S^1 \int_0^{\rho_0} \frac{2\rho}{\left(1 + \frac{\rho^2}{2a^2}\right)^2} d\rho = 8\pi \frac{a^2 \rho_0^2}{2a^2 + \rho_0^2}$$  \hspace{1cm} (4.10)

For $Q^2 \leq Q_{cr}^2$ and $\bar{M} \ll a$

$$A_{dS_4} \approx 4\pi \rho_0^2 \left(1 - \frac{\rho_0^2}{2a^2}\right) \approx 32\pi \kappa^2 \bar{M}^2 \left(1 - \frac{4\kappa^2 \bar{M}^2}{a^2} - \frac{3\pi}{64} \frac{\bar{Q}^2}{\kappa \bar{M}^2}\right)$$  \hspace{1cm} (4.11)

5. Conclusion

We have presented the charged version of Aichelburg-Sexl metric in the \((A)dS\) case. We have calculated the area of the trapped surface produced in a head-on collision of two charged shock waves in \((A)dS\) background. We observe that the physical picture in \(AdS\) is similar to the flat case. Namely, there is a critical value of charge above which no marginally trapped surface on the past light cone is formed in the head-on collision. The phenomenon is analogous to the critical behavior found in flat space [51]. The value of critical charge in the \(AdS\) case depends both on the collision energy and the value of the cosmological constant.

It would be interesting to study non-head-on collisions of charged particles in \(AdS\) background and, in particular, to study an influence of charges on a possible elongation of the shape of the trapped surface in spherical coordinates.

We have found new interesting phenomena in the collision of shock waves in \(dS\) background. As has been already noticed in [56], there is a critical value of a cosmological radius below which there is no trapped surfaces formation. Non zero charge changes situation and formation of trapped surface becomes possible even if in the neutral case for given values of $\bar{M}$ and $a$ the ratio $\bar{M}/a$ is over critical point. However a large enough charge stops a formation of the trapped surface. It would be also interesting to study non-head-on collisions of charged particles in \(dS\) background and to find how a non-zero impact parameter influences on the trapped surface formation in the collision of charged particles.

Acknowledgments

I.A. is grateful to I.Volovich for fruitful discussions. I.A. and A.B. are supported in part by RFBR grant 08-01-00798, I.A. is supported also in part by grant NS-795.2008.1. L.J. acknowledges the support of the Centre for Theoretical Cosmology, in Cambridge.
A. Appendix

A.1 Standard form of Lemma

Lemma I. For an integrable function \( f \) takes place the identity
\[
\lim_{v \to 1} \gamma f (\gamma^2 (Y_0 + v Y_1)^2) = \delta(Y_0 + Y_1) \int f(x^2)dx
\]  
(A.1)

This one might be found in [62].

A.2 Modified form of Lemma

As has been mentioned in the text in the dS case we have to deal with expressions which require a regularization, in particular with expression (2.30). Let us first consider a simple example and prove the following

Lemma II. In the sense of distributions one has
\[
\lim_{\gamma \to \infty} \left( \frac{\gamma}{Z^2 - \gamma^2 Y^2} \right)_{\text{reg}} f(\gamma^2 Y^2) = \delta(Y) \int \left( \frac{1}{Z^2 - x^2} \right)_{\text{reg}} f(x^2)dx 
\]  
(A.2)

here we use the following regularization
\[
\left( \frac{\gamma}{Z^2 - \gamma^2 Y^2} \right)_{\text{reg}} f(\gamma^2 Y^2), g
\]  
(A.3)

Remark 1 To write regularization (A.3) we assume:
i) the following relation
\[
\int \frac{1}{Z^2 - x^2} f(x^2)g(\gamma^2 Y^2)dy \equiv \int \left( \frac{1}{Z^2 - x^2} \right)_{\text{reg}} f(x^2)g(x^2)dx
\]  
(A.4)
i) the regularization in the RHS of (A.4) in the following sense
\[
\int \left( \frac{1}{Z^2 - x^2} \right)_{\text{reg}} f(x^2)dx = \int \frac{f(x^2)}{2Z} (\frac{1}{Z-x})_{\text{reg}}dx + \int \frac{f(x^2)}{2Z} (\frac{1}{Z+x})_{\text{reg}}dx
\]  
(A.5)
i) the natural regularization for two terms in (A.5)
\[
\int (\frac{1}{Z-x})_{\text{reg}} f(x^2)dx = \int \frac{f(x^2)}{2Z} (\frac{1}{Z-x})_{\text{reg}}dx + \int \frac{f(x^2)}{2Z} (\frac{1}{Z+x})_{\text{reg}}dx
\]  
(A.6)
\[
\int (\frac{1}{Z+x})_{\text{reg}} f(x^2)dx + \int \frac{f(x^2)}{2Z} (\frac{1}{Z-x})_{\text{reg}}dx + \int \frac{f(x^2)}{2Z} (\frac{1}{Z+x})_{\text{reg}}dx
\]  
(A.7)
Remark 2 For simplicity we have made a cut-off in (A.7) and (A.6) as well in (A.3) at 1, but one can make it at an arbitrary dimensional parameter, say $\epsilon$.

Proof Taking the limit $\gamma \to \infty$ in the RHS of (A.3) we get

$$\lim_{\gamma \to \infty} \left( \frac{\gamma}{Z^2 - \gamma^2 Y^2} \right)_{reg} f(\gamma^2 Y^2), g$$

(A.8)

$$= \lim_{\gamma \to \infty} \left( \int_{|z-x|<1} \frac{1}{2Z} f(x^2) \frac{g(\gamma z) - g(\gamma z)}{Z-x} \, dx + \int_{|z-x|>1} \frac{1}{2Z} f(x^2)g(\gamma z) \, dx \right)$$

$$+ \lim_{\gamma \to \infty} \left( \int_{|z-x|<1} \frac{1}{2Z} f(x^2) \frac{g(\gamma z) - g(\gamma z)}{Z+x} \, dx + \int_{|z-x|>1} \frac{1}{2Z} f(x^2)g(\gamma z) \, dx \right)$$

$$= \int_{|z-x|<1} \frac{1}{2Z} f(x^2)g(0) \, dx + \int_{|z-x|>1} \frac{1}{2Z} f(x^2)g(0) \, dx$$

$$+ \int_{|z-x|<1} \frac{1}{2Z} f(x^2)g(\gamma z) \, dx + \int_{|z-x|>1} \frac{1}{2Z} f(x^2)g(\gamma z) \, dx$$

$$= g(0) \left( \int_{|z-x|<1} \frac{1}{2Z} f(x^2) \, dx + \int_{|z-x|>1} \frac{1}{2Z} f(x^2) \, dx \right)$$

$$+ g(0) \left( \int_{|z-x|<1} \frac{1}{2Z} f(x^2)g(\gamma z) \, dx + \int_{|z-x|>1} \frac{1}{2Z} f(x^2)g(\gamma z) \, dx \right)$$

$$= g(0) \left( \int \frac{f(x^2)}{2Z} \left( \frac{1}{Z-x} \right)_{reg} \, dx + \int \frac{f(x^2)}{2Z} \left( \frac{1}{Z+x} \right)_{reg} \, dx \right)$$

that is nothing but

$$g(0) \int f(x^2)(\frac{1}{Z^2 - x^2})_{reg} \, dx$$

(A.9)

that proves the Lemma II.

Let us prove the Lemma that is suitable for dS case.

Lemma III In the sense of distributions one has

$$\lim_{\gamma \to \infty} \left( \frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} \right)_{reg} f(\gamma^2 Y^2) = \delta(Y) \int \left( \frac{1}{(Z^2 - x^2)^2} \right)_{reg} f(x^2) \, dx$$

(A.10)

We use the following regularization

$$\left( \frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} \right)_{reg} f(\gamma^2 Y^2), g$$

(A.11)

$$= \int_{|z-x|<1} \frac{1}{2(Z^2 + x^2)} f(x^2)g(\gamma z) - f(Z^2)g(\gamma z) - \frac{\partial}{\partial Z} \left( f(Z^2)g(\gamma z) \right) (Z-x) \, dx$$

$$+ \int_{|z-x|>1} \frac{1}{2(Z^2 + x^2)} f(x^2)g(\gamma z) \, dx$$
Remark 3. To write this regularization we are motivated by the following relation

\[
\int \left( \frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} \right)_{\text{reg}} f(\gamma^2 Y^2) g(Y) \, dy \equiv \int \left( \frac{1}{(Z^2 - x^2)^2} \right)_{\text{reg}} f(x^2) g\left(\frac{Z}{\gamma}\right) \, dx \quad (A.12)
\]

Proof Using

\[
\frac{\partial}{\partial Z} \left( f(Z^2) g\left(\frac{Z}{\gamma}\right) \right) = 2Z f'(Z^2) g\left(\frac{Z}{\gamma}\right) + \frac{1}{\gamma} f(Z^2) g'\left(\frac{Z}{\gamma}\right) \quad (A.13)
\]

here

\[
f'(x) = \frac{\partial}{\partial x} f(x) \quad (A.14)
\]

and taking the limit \( \gamma \to \infty \) in the RHS of \((A.11)\) we get

\[
\lim_{\gamma \to \infty} \left( \frac{\gamma}{(Z^2 - \gamma^2 Y^2)^2} f(\gamma^2 Y^2) \right)_{\text{reg}} \cdot g
\]

\[
\quad = \lim_{\gamma \to \infty} \left( \int_{|Z-x|<1} \frac{1}{2(Z^2 + x^2)} f(x^2) g\left(\frac{Z}{\gamma}\right) - f(Z^2) g\left(\frac{Z}{\gamma}\right) - \frac{\partial}{\partial Z} \left( f(Z^2) g\left(\frac{Z}{\gamma}\right) \right) (x-Z) \, dx + \int_{|Z-x|>1} \frac{1}{2(Z^2 + x^2)} \left( Z - x \right) \right) \]

\[
\quad + \lim_{\gamma \to \infty} \left( \int_{|Z-x|<1} \frac{1}{2Z} f(x^2) g\left(\frac{Z}{\gamma}\right) - f(Z^2) g\left(\frac{Z}{\gamma}\right) - \frac{\partial}{\partial Z} \left( f(Z^2) g\left(\frac{Z}{\gamma}\right) \right) (Z+x) \, dx + \int_{|Z+x|>1} \frac{1}{2(Z^2 + x^2)} \left( Z + x \right) \right) \]

\[
\quad = g(0) \left( \int_{|Z-x|<1} \frac{1}{2(Z^2 + x^2)} f(x^2) - f(Z^2) - \left( \frac{\partial}{\partial Z} f(Z^2) \right) (Z-x) \, dx \right) \quad (A.16)
\]

\[
\quad + \int_{|Z-x|>1} \frac{1}{2(Z^2 + x^2)} f(x^2) \, dx
\]

\[
\quad + g(0) \left( \int_{|Z+x|<1} \frac{1}{2(Z^2 + x^2)} f(x^2) - f(Z^2) - \left( \frac{\partial}{\partial Z} f(Z^2) \right) (Z+x) \, dx \right) \quad (A.17)
\]

\[
\quad + \int_{|Z+x|>1} \frac{1}{2(Z^2 + x^2)} f(x^2) \, dx
\]
\[ g(0) \left( \int \frac{f(x^2)}{2(Z^2 + x^2)} \left( \frac{1}{(Z - x)^2} \right)_{\text{reg}} dx + \int \frac{f(x^2)}{2(Z^2 + x^2)} \left( \frac{1}{(Z + x)^2} \right)_{\text{reg}} dx \right) \]
\[ \equiv g(0) \int f(x^2) \left( \frac{1}{(Z^2 - x^2)^2} \right)_{\text{reg}} dx \]  
(A.18)

In the above calculations we use that the coefficient in front of the derivative of \( g \) goes to zero when \( \gamma \to \infty \).

**Remark 4.** One can replace ”1” in the domain of the integration in (A.11) by some dimensional parameter, say \( \epsilon \) and the regularization prescription which we use in fact means what we deal with the principle values of integral and just remove \( \epsilon \) neighborhood in integrals near points \( x = \pm Z_D \). This prescription give D-dimensional answers presented in [62].
References

[1] G. ’t Hooft Gravitational collapse and particle physics, Proceedings: Proton-antiproton Collider Physics, Aachen, 1986, pp. 669–688;
G. ’t Hooft, Graviton Dominance in Ultrahigh-Energy Scattering, Phys. Lett. B 198 (1987) 61;
G. ’t Hooft, On the factorization of universal poles in a theory of gravitating point particles, Nucl. Phys. B 304 (1988) 867.

[2] T. Dray and G. ’t Hooft, The Gravitational Shock Wave of a Massless Particle, Nucl. Phys. B253 (1985) 173.

[3] D. Amati, M. Ciafaloni and G. Veneziano, Superstring collisions at Planckian energies, Phys. Lett. B197 (1987) 81.
D. Amati, M. Ciafaloni and G. Veneziano, Classical and quantum gravity effects from Planckian energy superstring collisions, Int. J. Mod. Phys. A3 (1988) 1615.
D. Amati, M. Ciafaloni and G. Veneziano, Can space-time be probed below the string size? Phys. Lett. B216 (1989) 41.
D. Amati, M. Ciafaloni and G. Veneziano, Higher order gravitational deflection and soft bremsstrahlung in Planckian energy superstring collisions, Nucl. Phys. B347 (1990) 550.
D. Amati, M. Ciafaloni and G. Veneziano, Planckian scattering beyond the semiclassical approximation, Phys. Lett. B289 (1992) 87.

[4] P. D. D’Eath and P. N. Payne, Gravitational radiation in high speed black hole collisions. 1. Perturbation treatment of the axisymmetric speed of light collision, Phys. Rev. D46 (1992) 658674.
P. D. D’Eath and P. N. Payne, Gravitational radiation in high speed black hole collisions. 2. Reduction to two independent variables and calculation of the second order news function, Phys. Rev. D46 (1992) 675693.
P. D. D’Eath and P. N. Payne, Gravitational radiation in high speed black hole collisions. 3. Results and conclusions, Phys. Rev. D46 (1992) 694701.

[5] I. Ya. Aref’eva, K. Viswanathan and I. Volovich, Planckian-energy scattering, colliding plane gravitational waves and black hole creation, Nucl. Phys. B452 (1995) 346 [Erratum-ibid. B 462, 613 (1996)] hep-th/9412157
I. Ya. Aref’eva, K. S. Viswanathan and I. V. Volovich, On black hole creation in Planckian energy scattering, Int. J. Mod. Phys. D 5, 707 (1996), hep-th/9512170

[6] N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, The Hierarchy Problem and New Dimensions at a Millimeter, Phys.Lett. B429 (1998) 263, hep-ph/9803315;
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, New Dimensions at a Millimeter to a Fermi and Superstrings at a TeV, Phys.Lett. B436 (1998) 257, hep-ph/9804398.

[7] G.F. Giudice, R. Rattazzi and J.D. Wells, Quantum gravity and extra dimensions at high-energy colliders, Nucl. Phys. B 544 (1999) 3, hep-ph/9811291; G.F. Giudice, R.
Rattazzi and J.D. Wells, Transplanckian Collisions at the LHC and Beyond, Nucl.Phys. B630 (2002) 293, hep-ph/0112161.

[8] T. Banks and W. Fischler, A Model for High Energy Scattering in Quantum Gravity, hep-th/9906038

[9] I.Ya. Aref’eva, High-energy scattering in the brane world and black hole production, Part.Nucl.31 (2000) 169, hep-th/9910269

[10] S. Dimopoulos and G. Landsberg, Black Holes at the LHC, Phys.Rev.Lett. 87 (2001) 161602, hep-ph/0106295.

[11] S.B. Giddings and S. Thomas, High Energy Colliders as Black Hole Factories: The End of Short Distance Physics, Phys.Rev. D65 (2002) 056010, hep-ph/0106219

[12] D.M. Eardley and S. B. Giddings, Classical Black Hole Production in High-Energy Collisions, Phys.Rev. D66 (2002) 044011, gr-qc/0201034;

[13] A. Ringwald, H. Tu, Collider versus cosmic ray sensitivity to black hole production, Phys.Lett.B525 (2002) 135, hep-ph/0111042;
E.J. Ahn, M. Cavaglia and A.V. Olinto, Brane Factories, Phys.Lett. B551 (2003) 1, hep-th/0201042;
S. N. Solodukhin, Classical and quantum cross-section for black hole production in particle collisions, Phys.Lett. B533 (2002) 153; hep-ph/0201248.
E. Kohlprath and G. Veneziano, Black holes from high-energy beam-beam collisions, J. High Energy Phys 06 (2002) 057 [arXiv:gr-qc/0203093].

[14] H. Yoshino and Y. Nambu, Black hole formation in the grazing collision of high-energy particles, Phys. Rev. D67 (2003) 024009, gr-qc/0209003

[15] H. Yoshino and V. S. Rychkov, Improved analysis of black hole formation in high-energy particle collisions, Phys. Rev. D 71 (2005) 104028 ; hep-th/0503171.
M. Cavaglia, Black Hole and Brane Production in TEV Gravity: a Review, Int.J.Mod.Phys. A18 (2003) 1843, hep-ph/0210296;
P.Kanti, Black Holes In Theories With Large Extra Dimensions: a Review, Int.J.Mod.Phys. A19 (2004) 4899, hep-ph/0402168;
S. B. Giddings and V. S. Rychkov, Black holes from colliding wavepackets, Phys. Rev. D70 (2004) 104026, hep-th/0409131.
V. Cardoso, E. Berti and M. Cavaglia, What we (dont) know about black hole formation in high-energy collisions, Class.Quant.Grav.22:L61-R84,2005, hep-ph/0505125.
G.L. Landsberg, Black Holes at Future Colliders and Beyond, J.Phys.G32 (2006) R337, hep-ph/0607297;
D.M. Gingrich, Black Hole Cross Section at the LHC, Int. J. Mod. Phys. A 21 (2006) 6653, hep-ph/0609055;
H. Stoecker, Mini black holes in the first year of the LHC: Discovery through di-jet suppression, multiple mono-jet emission and ionizing tracks in ALICE, J.Phys.G32 (2006) S429;
B. Koch, M. Bleicher, H. Stoecker, Black Holes at LHC?, J.Phys.G34 (2007) S535, hep-ph/0702187;
N. Kaloper, J. Terning, How Black Holes Form in High Energy Collisions, arXiv:0705.0408.
M. Cavaglia et all., Signatures of black holes at the LHC, arXiv:0707.0317
P. Mende and L. Randall, Black Holes and Quantum Gravity at the LHC, arXiv:0708.3017
S. B. Giddings, High-energy black hole production, arXiv:0709.1107.

[16] U. Sperhake, V. Cardoso, F. Pretorius, E. Berti and J. A. Gonzalez, Phys. Rev. Lett. 101, 161101 (2008) [arXiv:0806.1738 [gr-qc]].
M. Shibata, H. Okawa and T. Yamamoto, Phys. Rev. D 78, 101501 (2008) [arXiv:0810.4735 [gr-qc]].
U. Sperhake, V. Cardoso, F. Pretorius, E. Berti, T. Hinderer and N. Yunes, Phys. Rev. Lett. 103, 131102 (2009) [arXiv:0907.1252 [gr-qc]].
M. W. Choptuik and F. Pretorius, arXiv:0908.1780 [gr-qc].

[17] I. Ya. Aref’eva and I. V. Volovich, Time Machine at the LHC, Int. J. Geom. Meth. Mod. Phys. 05, 641 (2008), arXiv:0710.2696

[18] A. Mironov, A. Morozov and T. N. Tomaras, If LHC is a Mini-Time-Machines Factory, Can We Notice?, arXiv:0710.3395

[19] P. Nicolini and E. Spallucci, Noncommutative geometry inspired wormholes and dirty black holes, arXiv:0902.4654.

[20] I. D. Novikov, N. S. Kardashev and A. A. Shatskiy, “The multicomponent Universe and the astrophysics of wormholes,” Phys. Usp. 50, 965 (2007) [Usp. Fiz. Nauk 177, 1017 (2007)].

[21] E.I. Novikova, I. D. Novikov, Homogeneous singularities inside collapsing wormholes, arXiv:0907.1936

[22] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[23] S. S. Gubser, S. S. Pufu and A. Yarom, Entropy production in collisions of gravitational shock waves and of heavy ions, Phys. Rev. D78 (2008) 066014, arXiv:0805.1551

[24] L. Alvarez-Gaume, C. Gomez, A. S. Vera, A. Tavanfar, and M. A. Vazquez-Mozo, Critical formation of trapped surfaces in the collision of gravitational shock waves, arXiv:0811.3969
[25] S. S. Gubser, S. S. Pufu and A. Yarom, *Off-center collisions in AdS with applications to multiplicity estimates in heavy-ion collisions*, arXiv:0902.4062 [hep-th]

[26] D. M. Hofman, J. M. Maldacena, “Conformal collider physics: Energy and charge correlations,” JHEP 0805 (2008) 012, arXiv:0803.1467 [hep-th]

[27] E. Shuryak, “Physics of Strongly coupled Quark-Gluon Plasma,” Prog. Part. Nucl. Phys. 62, 48 (2009) [arXiv:0807.3033 [hep-ph]].

[28] D. Grumiller, P. Romatschke, “On the collision of two shock waves in AdS5,” JHEP 0808 (2008) 027, arXiv:0803.3226 [hep-th]

[29] S. Lin and E. Shuryak, “Grazing Collisions of Gravitational Shock Waves and Entropy Production in Heavy Ion Collision,” arXiv:0902.1508 [hep-th].

[30] J. L. Albacete, Y. V. Kovchegov, A. Taliotis, “Asymmetric Collision of Two Shock Waves in AdS5,” JHEP 0905 (2009) 060, arXiv:0902.3046 [hep-th]

[31] W. A. Horowitz, Y. V. Kovchegov, “Shock Treatment: Heavy Quark Drag in a Novel AdS Geometry,” arXiv:0904.2536 [hep-th]

[32] P. M. Chesler, L. G. Yaffe, “Boost invariant flow, black hole formation, and far-from-equilibrium dynamics in N = 4 supersymmetric Yang-Mills theory,” arXiv:0906.4426 [hep-th]

[33] E. Avsar, E. Iancu, L. McLerran, D. N. Triantafyllopoulos, “Shockwaves and deep inelastic scattering within the gauge/gravity duality,” arXiv:0907.4604 [hep-th]

[34] G. T. Horowitz, M. M. Roberts, “Zero Temperature Limit of Holographic Superconductors,” arXiv:0908.3677 [hep-th]

[35] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B534 (1998) 202

[36] G. Policastro, D. T. Son and A. O. Starinets, “The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma,” Phys. Rev. Lett. 87 (2001) 081601.

[37] J. Casalderrey-Solana and D. Teaney, “Heavy quark diffusion in strongly coupled N = 4 Yang Mills,” hep-ph/0605199.

[38] S.-J. Sin and I. Zahed, “Holography of radiation and jet quenching,” Phys. Lett. B608 (2005) 265–273, hep-th/0407215.

H. Liu, K. Rajagopal, and U. A. Wiedemann, “Calculating the jet quenching parameter from AdS/CFT,” hep-ph/0605178.

C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz, and L. G. Yaffe, “Energy loss of a heavy quark moving through N = 4 supersymmetric Yang-Mills plasma,” hep-th/0605158.

S. S. Gubser, “Drag force in AdS/CFT,” Phys.Rev.D74:126005,2006, hep-th/0605182.
A. Buchel, ‘On jet quenching parameters in strongly coupled non-conformal gauge theories,” hep-th/0605178. hep-th/0605182.

S.-J. Sin and I. Zahed, “Ampere’s Law and Energy Loss in AdS/CFT Duality,” hep-ph/0606049. S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, Nonlinear Fluid Dynamics from Gravity, JHEP 02 (2008) 045, 0712.2456.

[39] D. R. Brill and S. A. Hayward, “Global structure of a black hole cosmos and its extremes,” Class. Quant. Grav. 11 (1994) 359 [arXiv: gr-qc/9304007].

[40] L. J. Romans, “Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory,” Nucl. Phys. B 383 (1992) 395 [arXiv: hep-th/9203018].

[41] S. F. Ross and R. B. Mann, “Cosmological production of charged black hole pairs,” Phys. Rev. D 52 (1995) 2254 [arXiv: gr-qc/9504015].

[42] R. Bousso, “Quantum global structure of de Sitter space” Phys. Rev. D 60 (1999) 063530 [arXiv: hep-th/9902183].

[43] J. D. E. Creighton and R. B. Mann, “Quasilocal thermodynamics of dilaton gravity coupled to gauge fields,” Phys. Rev. D 52 (1995) 4569 [arXiv: gr-qc/9505007].

[44] D. Kastor, J. Traschen, Cosmological multi-black-hole solutions, Phys. Rev. D, vol. 47, 12 (1993).

[45] R. Narayan, Black Holes in Astrophysics, New J. Phys. 7, 199 (2005), gr-qc/0506078

[46] C. M. Hull, “Timelike T-duality, de Sitter space, large N gauge theories and topological field theory,” JHEP 9807 (1998) 021 [arXiv: hep-th/9806146].

[47] C. M. Hull, “Duality and the signature of space-time,” JHEP 9811 (1998) 017 [arXiv: hep-th/9807127].

[48] V. Balasubramanian, P. Horava and D. Minic, “Deconstructing de Sitter,” JHEP 0105 (2001) 043 [arXiv: hep-th/0103171].

[49] E. Witten, “Quantum gravity in de Sitter space,” arXiv:hep-th/0106109.

[50] A. Strominger, “The dS/CFT correspondence,” JHEP 0110 (2001) 034 [arXiv: hep-th/0106113].

[51] H. Yoshino and R. B. Mann, Black hole formation in the head-on collision of ultrarelativistic charges, 0605131

[52] Y. Hatta, $e^+e^-$ annihilation to high energy scattering at weak and strong coupling, JHEP0811:057,2008, arXiv:0810.0889

[53] D. M. Eardley and S. B. Giddings, Phys. Rev. D 66, 044011 (2002) [arXiv:gr-qc/0201034].

[54] H. Yoshino and Y. Nambu, Phys. Rev. D 67, 024009 (2003) [arXiv:gr-qc/0209003].
[55] E. Kohlprath and G. Veneziano, JHEP **0206**, 057 (2002) [arXiv:gr-qc/0203093].

[56] I.Ya. Aref’eva, A.A. Bagrov, E.A. Guseva, *Critical Formation of Trapped Surfaces in the Collision of Non-expanding Gravitational Shock Waves in de Sitter Space-Time*, arXiv:0905.1087.
I.Ya. Aref’eva, A.A. Bagrov, *Trapped surfaces formation in collisions of non-expanding gravitational shock waves in AdS$_4$ spacetime*, Theor. Math. Phys., vol. 161, 3, 385-403 (2009)

[57] C. O. Lousto and N. Sanchez, *The Curved Shock Wave Space-Time Of Ultrarelativistic Charged Particles And Their Scattering*, Int. J. Mod. Phys. A **5**, 915 (1990), *Scattering Processes at the Planck Scale Authors*, gr-qc/9410041

[58] M. Ortaggio, “Ultrarelativistic boost of spinning and charged black rings,” arXiv:gr-qc/0601093.

[59] P. C. Aichelburg and R. U. Sexl, “On the gravitational field of a massless particle,” Gen. Rel. Grav. **2**, 303 (1971).

[60] R. C. Myers and M. J. Perry, “Black Holes In Higher Dimensional Space-Times,” Annals Phys. **172**, 304 (1986).

[61] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope, *The general Kerr-de Sitter metrics in all dimensions*, J. Geom. Phys. 53 (2005) 4973, hep-th/0404008.

[62] Hotta M., Tanaka M. *Shock wave geometry with non-vanishing cosmological constant*, Class. Quant. Grav. **10** (1993) 307.

[63] K. Sfetsos, On gravitational shock waves in curved space-times, Nucl. Phys. B436 (1995) 721746, hep-th/9408169. hep-th/9408169

[64] J. Podolsky and J. B. Griffiths, Impulsive waves in de Sitter and anti-de Sitter space-times generated by null particles with an arbitrary multipole structure, Class. Quant. Grav. 15 (1998) 453463, gr-qc/9710049.

[65] R. Emparan, Exact gravitational shockwaves and Planckian scattering on branes, Phys. Rev. D64 (2001) 024025, hep-th/0104009.

[66] Esposito G., Pettorino R., Scudellaro P., On boosted space-times with cosmological constantand their ultrarelativistic limit, arXiv:gr-qc/0606126v3.

[67] K. Kang and H. Nastase, Planckian scattering effects and black hole production in low M(PI) scenarios, Phys. Rev. D71 (2005) 124035, hep-th/0409099.

[68] K. Kang and H. Nastase, High energy QCD from Planckian scattering in AdS and the Froissart bound, Phys. Rev. D72 (2005) 106003, hep-th/0410173.

[69] H. Nastase, On high energy scattering inside gravitational backgrounds, hep-th/0410124.
[70] D. Astefanesei, R. B. Mann and E. Radu, *Reissner-Nordstrom-de Sitter black hole, planar coordinates and dS / CFT*, JHEP 0401:029, 2004, hep-th/0310273

Y. Brihaye, T. Delsate, *Charged-Rotating Black Holes in Higher-dimensional (A)DS-Gravity*, arXiv:0806.1583

[71] S. W. Hawking and R. Penrose, The Singularities of gravitational collapse and cosmology, Proc. Roy. Soc. Lond. A314 (1970) 529548.

[72] R. Penrose, The question of cosmic censorship, J. Astrophys. Astron. 20 (1999) 233248.

[73] G. W. Gibbons, Some comments on gravitational entropy and the inverse mean curvature flow, Class. Quant. Grav. 16 (1999) 16771687, hep-th/9809167.

[74] P. T. Chrusciel, E. Delay, G. J. Galloway, and R. Howard, The Area Theorem, Annales Henri Poincare 2 (2001) 109178, gr-qc/0001003.

[75] P. T. Chrusciel and W. Simon, Towards the classification of static vacuum spacetimes with negative cosmological constant, J. Math. Phys. 42 (2001) 17791817, gr-qc/0004032.

[76] S. Bhattacharyya et. al., Local Fluid Dynamical Entropy from Gravity, 0803.2526.

[77] I.M. Gelfand, G.E. Shilov, *Generalized functions*, vol. 1, 2, New York and London, Academic Press Inc, 1964

[78] V.S.Vladimirov, *Generalized functions in mathematical physics*, Moscow, 1976, (In Russian); English transl.(Mir, Moscow), 1979

[79] T. Chiba, K. Maeda, *Cosmic hoop conjecture?*, Phys. Rev. D, vol. 50, 8 (1994).

D. Ida, K. Nakao, M. Siino, S. Hatward, *Hoop conjecture for colliding black holes*, Phys. Rev. D 58 121501 (1998)