Radiation of a Charge in a Waveguide with a Boundary between Two Dielectrics

T Yu Alekhina and A V Tyukhtin
Radiophysics Department, Physical Faculty of St. Petersburg State University,
1 Ulyanovskaya str., St. Petersburg, 198504, Russia

E-mail: tanya@niirf.spbu.ru, tyukhtin@bk.ru

Abstract. We analyze the electromagnetic field of a charged particle moving uniformly and crossing a boundary between two homogenous dielectrics in a metal cylindrical waveguide. We make an analysis of exact expressions for components of waveguide modes with the methods of the complex variable function theory. An original algorithm for computation of the field components is presented as well. Two instances are investigated in detail: the particle is flying from vacuum into dielectric and from dielectric into vacuum. In the first case, Cherenkov radiation (so-called 'wakefield') is partially compensated with the free field. We demonstrate areas where the waveguide modes practically coincide with ones in the regular waveguide and areas where the boundary influence is principal. For the second case, we demonstrate the possibility of generation of large quasi monochromatic radiation (Cherenkov-transition radiation) in the vacuum region at certain conditions.

1. Introduction
Investigation of a field of a particle bunch in a waveguide loaded with dielectric is important for the wakefield acceleration technique and for other problems in the accelerator physics. One of them consists in analysis of effect of the boundary on the wave field when the bunch flies into (or from) the dielectric structure. It should be noticed that a charge field structure in a regular waveguide loaded with dielectric was analyzed frequently (see, for example, [1, 2]). However, the case of an irregular waveguide with two different media has not been examined sufficiently, although such investigation is important as well. This especially concerns the study of the electromagnetic field (EMF) structure in contrast to the energetic characteristics analyzed in papers [3, 4]. Note that the case of a semi-infinite waveguide with the metal end was analyzed in papers [5, 6]. But this problem differs essentially from the one considered here because matching of fields in different parts of the waveguide is of great importance. Note as well that analogous problem with a boundary between vacuum and cold plasma was investigated in [7]. But the Cherenkov radiation (CR) is not generated in such situation; therefore it differs radically from the case under consideration.

In present paper we consider the electromagnetic field (EMF) generated by a point charge particle $q$ moving in the metal circular waveguide of radius $a$ along its axis ($z$-axis) through the interface ($z = 0$) between homogeneous isotropic dielectrics with permittivity $\varepsilon_1$ ($z < 0$) and $\varepsilon_2$ ($z > 0$) (figure 1). The media have no dispersion but small losses are taken into account. The charge moves uniformly with a velocity $\vec{V} = c\beta \vec{e}_z$ and intersects the boundary at the moment $t = 0$. 

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The analytical solution for this problem is found traditionally as an expansion into a series of eigenfunctions of the transversal operator \[3\]. This expression has two summands: \(E_{1,2}^q = E_{1,2}^q + E_{1,2}^b\).

The first one \((E_{1,2}^q)\) gives the field in the regular waveguide with homogeneous filling. V.L. Ginzburg [8] called it the ‘forced’ field. It contains CR if the charge velocity exceeds the Cherenkov threshold.

The second summand \((E_{1,2}^b)\) is connected with the influence of the boundary and gives so-called ‘free’ field. It includes transition radiation (TR).

We investigate the exact solution with analytical and computational methods. Analytical research is an asymptotic investigation using the steepest descent technique. Computations are based on original algorithm using certain transformation of the initial integration path. Such approaches were applied as well in some papers concerning both boundless homogenous media [9] and problems with interface between two media [10, 11] including the case of waveguide partially filled with a cold plasma [7].

Further we study two cases in detail: (I) flying from vacuum \((\varepsilon_1 = 1)\) into dielectric \((\varepsilon_2 > 1)\), (II) flying from dielectric \((\varepsilon_1 > 1)\) into vacuum \((\varepsilon_2 = 1)\).

2. The case of flying from vacuum into dielectric

The exact expressions for the forced and free fields can be easily derived from the ones for the general case of the boundary between arbitrary homogeneous isotropic media [6] if we take into account that \(\mu_1 = \mu_2 = 1\) and dispersion is negligible. The study of the forced field (which can contain CR) does not cause any difficulties [1]. So, we pay the main attention to research of the free field. Here we give expressions for longitudinal components of the free field:

\[
E_{1,2}^0 = \frac{2\pi \varepsilon_0^2 c^2 \beta}{\pi \varepsilon_1^2} \sum_{n=1}^{\infty} \chi_{0n}^2 J_1(\chi_{0n}^2/a) \frac{d\omega}{\omega} B_{n,1,2} \exp[-i(\omega t - k_{n,1,2} z)].
\]

\[
B_{n,1,2} = \frac{\left(\bar{\varepsilon}_{1,2} - \bar{\varepsilon}_{2,1}\right)}{\bar{g}(\omega)} \left[\sqrt{\omega^2 \varepsilon_{1,2} \varepsilon_{2,1}^2 + 1} - \sqrt{\omega^2 \varepsilon_{2,1}^2 - \omega_n^2}\right]
\]

\[
\bar{g}(\omega) = \left|\omega^2 \varepsilon_{1,2} - \omega_n^2\right|, \quad \bar{g}(\omega) = \varepsilon_1 \sqrt{\omega^2 \varepsilon_2 - \omega_n^2} + \varepsilon_2 \sqrt{\omega^2 \varepsilon_1 - \omega_n^2},
\]

where \(\omega_n = \chi_{0n} c/a\), \(\chi_{0n}\) is the \(n\)th zero of the Bessel function \((J_0(\chi_{0n}) = 0)\), \((\text{Im} k_{n,1,2} > 0)\).

In the analytical method, asymptotic expressions for the free field components of each mode can be obtained with the steepest descend technique [12]. We perform this analysis for the case of negligible losses taken into account for the determination of the location of the integrand singularities only. The first step in such research is to study the singularities of integrands in (1) in a complex plane \((\omega)\). One can show that integrands have the following singularities:

- four branch points \(\pm \bar{\omega}_n^{(1)} = \pm \omega_n - i\delta_1\) and \(\pm \bar{\omega}_n^{(2)} = \pm \omega_n / \sqrt{\varepsilon_2} - i\delta_2\);

- two poles on the imaginary axis \((\omega)\): \(\pm \omega_n^{(0)} = \pm i\beta \omega_n (1 - \beta^2)^{-1/2}\);
– two poles on the imaginary axis \( \pm \omega^{(2)}_{0n} = \pm i \beta \omega_n \left( 1 - \varepsilon \beta^2 \right)^{-1/2} \) if \( \beta < \varepsilon^{-1/2} \) or on the real axis \( \pm \omega^{(2)}_{0n} = \pm \beta \omega_n \left( \varepsilon \beta^2 - 1 \right)^{-1/2} - i \delta \) if \( \beta > \varepsilon^{-1/2} \).

Here \( \delta_1, \delta_2, \delta_3 \) are positive infinitesimal quantities. Therefore, all of the singularities take place below a real axis (figure 2 (a)).

Since the medium is passive it is necessary to guarantee exponential decrease of the free field with increase of \(|\nu|\). It means that \( \text{Re} \sqrt{\omega^2 - \omega_n^2} > 0 \) and \( \text{Re} \sqrt{\omega^2 \varepsilon \omega_n^2} > 0 \) for real values of \( \omega \). It is convenient to use these definitions of radicals for all complex values of \( \omega \) as well. Thus we define the branch cuts by the equations

\[
\text{Re} \sqrt{\omega^2 - \omega_n^2} = 0 , \quad \text{Re} \sqrt{\omega^2 \varepsilon \omega_n^2} = 0 .
\]

The integration path goes along the upper edge of the cuts (figure 2 (a)).

For obtaining asymptotic expressions, we use the steepest descend technique. At first, we make the following change of variables for the vacuum region:

\[
\omega = \omega_n \sinh \chi , \quad \sqrt{\omega^2 - \omega_n^2} = \omega_n \cosh \chi .
\]

This replacement removes the pair of branch points \( \pm \omega^{(1)}_{0n} = \pm \omega_n - i \delta \). The correspondence between quadrants of the complex plane (\( \omega \)) and areas in the plane (\( \chi \)) is presented in figure 2 (with Roman numerals). The initial integration path in the plane (\( \chi \)) is the contour \( \Gamma \). The branch point \( \omega^{(2)}_{0n} \) and poles \( \omega^{(1)}_{01}, \omega^{(1)}_{00} \) turn into \( \chi^{(2)}_{0n}, \chi^{(1)}_{01} \) and \( \chi^{(2)}_{00} \) respectively (figure 2). There are two saddle points on contour \( \Gamma : \chi_0 = \text{arch}(ct/R) \) and \( \chi_1 = \chi_0 + i \pi \), where \( R = \sqrt{(ct)^2 - z^2} \). The steepest descending paths (SDP) \( \Gamma^* \) consist of two branches (\( \Gamma_0^* \) and \( \Gamma_1^* \)).

The poles and the branch points can be crossed in the transformation of contour \( \Gamma \) into the SDP \( \Gamma^* \) passing through the saddle points \( \chi_{0,1} \), and the contributions from the corresponding singularities should be included in asymptotic expressions. So, if \( \text{Re} \chi^{(1)}_{01} < \text{Re} \chi^{(2)}_{0n} \) the poles \( \chi^{(2)}_{0n} \) can be crossed.
The contributions from the poles are evaluated by the residue theorem. They give so-called Cherenkov-transition radiation (CTR). But in this case there is no CTR in vacuum because of peculiarities of integrands.

A similar investigation can be made for the free field in the dielectric area. Here it is convenient to make another change of variables:

\[
\frac{ct}{a} = 5, \quad \frac{ct}{a} = 15, \quad \frac{ct}{a} = 30
\]

\[
\beta = 0.5, \quad \beta = 0.7, \quad \beta = 0.9
\]

\[
\gamma = 100
\]

**Figure 3.** The case of flying from vacuum into dielectric. Dependence of normalized longitudinal component \( \vec{E}_z = \vec{E}_0 \pi c^2 / 2q \omega_0^2 \) of the first mode of the whole field (continuous line 1) and the forced one (dashed line 2) on distance \( z/a \) for different dimensionless times \( ct/a \) and velocities \( \beta \) (or \( \gamma = \sqrt{1 - \beta^2} \)); \( \omega_0 = 2\pi \cdot 10\text{GHz} \), \( \epsilon_1 = 1 \), \( \epsilon_2 = 3 \), \( a = 5 \text{ mm} \). Sighting point is on the waveguide axis \( r = 0 \).

A similar investigation can be made for the free field in the dielectric area. Here it is convenient to make another change of variables:
\[ \omega = \omega_c e^{\frac{1}{2}}, \quad \sqrt{\omega^2 e_2 - \omega_n^2} = \omega_n e^{\frac{1}{2}}. \]  

In this case not only the poles but also the branch points can be crossed in transformation of \( \Gamma \) into \( \Gamma^* \). For the pole contributions one can obtain the following expression:

\[ E_{z2}^{\text{CTR}} = \frac{4q}{a^2 e_2} \sum_{n=1}^{\infty} \left[ J_0(z_n e^{1/2}) \cos \left( z_n (c t - z) a^{-1} e_2 \beta^2 - 1 \right)^{1/2} \right] \Theta(z_2 - z), \quad z_2 = c t / \beta e_2. \]  

\( E_{z2}^{\text{CTR}} \) is the exact formula for the forced field taken with an opposite sign. So, there is a compensation for the forced field with a part of free one in some domain from the boundary at \( z < z_2 \).

For numerical calculations, the exact integral representations (1) are used. The efficient algorithm developed is based on a certain transformation of the initial integration path in the complex plane \( \{ \omega \} \). Earlier, such an algorithm was used for the computation of the field in different dispersive unbounded or semi-bounded media [9, 10] and in a waveguide with semi-bounded cold plasma [6]. As it is shown in figure 2, the poles are located near the integration path \( \Gamma \). This leads to the rather abrupt behaviour of the integrands in (1). The numerical algorithm is adapted for overcoming this difficulty. We transform contour \( \Gamma \) into some new contour in the upper-half plane \( \{ \omega \} \) for \( c t - \sqrt{\epsilon_{1,2}} | z | < 0 \) (before the ‘wave front’ \( z_{f1,2} = c t / \sqrt{\epsilon_{1,2}} \) and into another contour in the lower-half plane \( \{ \omega \} \) for \( c t - \sqrt{\epsilon_{1,2}} | z | > 0 \) (behind the ‘wave front’). These new contours should bypass all of the singularities (the poles and the branch points) and then go parallel to SDP. Note that we can choose optimal parameters of contours for each computation.

Further we present the behaviour of the first mode of the longitudinal component \( E_z \) of the total field and the forced one in vacuum and dielectric for different velocities of \( \beta \) and at different moments \( c t / a \) (figure 3). If \( \beta < \beta_{C2} = 1 / \sqrt{e_2} \approx 0.56 \) the total field is TR everywhere except some small neighbourhood of the charge (figure 3, a, b, c). That is also true for all velocities in the vacuum area. If \( \beta > \beta_{C2} \) Cherenkov radiation is generated in dielectric, and the total field here is a combination of CR and TR. In the domain \( 0 < z < z_2 = c t / (\beta e_2) \) the forced wave field (so-called ‘wakefield’) is compensated by some part of the free field which is equal to the wakefield taken with an opposite sign. Note that the point \( z_2 \) is determined with the group velocity \( v_{g2} = c / \beta e_2 \) in a regular waveguide.

The compensation domain is large for the velocities closed to the Cherenkov threshold (figure 3, d, e, f), and it is reduced with increase in \( \beta \) (figure 3, g, h, i). As a result, there is the area where the wave field practically coincides with the wakefield in the regular waveguide (\( z_2 < z < c t / \beta \)). As well, there is the area where the boundary influence is principal (\( z > z_2 \)). Note that the first area is large if \( e_2 \) takes on a large value. This conclusion is important for the wakefield acceleration technique. It is explained the fact that limited dielectric structure can be applied for this technique, and the length of structure can be easily estimated.

3. The case of flying from dielectric into vacuum

Similar investigation was made for the case of flying from dielectric into vacuum. In this situation the contribution of poles can exist: this is so-called Cherenkov-transition radiation (CTR) as in the case of two semi-infinite media without metal wall [9–11]. One can show that this radiation in vacuum takes place at \( \beta_{C1} = e_1^{1/2} < \beta < \beta_{C1} = (e_1 - 1)^{1/2} \). The lower threshold \( \beta_{C1} \) is connected with condition of the Cherenkov radiation generation. Presence of the upper threshold \( \beta_{C1} \) is explained by total internal reflection of Cherenkov waves from the boundary (that takes place at \( \beta > \beta_{C1} \)). So, there is the reflected wave of CR in dielectric in the domain \( |z| < z_1 = c t / (\beta e_1) \):
\[ E_{z1}^{\text{CTR}} = \frac{4q}{a^2} \left( e_1 \beta^2 - 1 - \sqrt{1 - \beta^2 (e_1 - 1)} \right) \left( 1 + \sqrt{1 - \beta^2 (e_1 - 1)} \right)^{-1} \left( 1 + e_1 \sqrt{1 - \beta^2 (e_1 - 1)} \right)^{-1} \times \sum_{n=1}^{\infty} J_n(x_0 r/a) \nu_i^2(x_0 n) \cos \left[ x_0 n (ct \beta - \beta \left[ a \sqrt{e_1 \beta^2 - 1} \right] \Phi(z_1 - \beta) \right] \phi \]

(8)

In vacuum in the area \( z < z_3 = ct \sqrt{1 - \beta^2 (e_1 - 1)} / \beta \) there is the transmitted wave of CR:

![Graphs showing transmitted wave of CR](image)

**Figure 4.** The case of flying from dielectric into vacuum. The same as for figure 3 at \( e_1 = 3, \ e_2 = 1, \beta_{C1} \approx 0.56, \beta_{CTR} \approx 0.71.\)
So, CR can penetrate from dielectric into vacuum at velocities \( \varepsilon_1^{-1/2} < \beta < (\varepsilon_1 - 1)^{1/2} \).

The case of flying from dielectric into vacuum is presented in figure 4. If \( \beta < \beta_{c1} \), the total field is TR in dielectric and in vacuum everywhere except the charge vicinity (figure 4, a, b, c). If \( \beta > \beta_{c1} \) CR emerges in dielectric. The total wave field here consists of CR, reflected wave of CR (in the area \( z < z_1 \)) and TR. One can find that reflected wave of CR differs only a little from CR at \( \beta_{c1} < \beta < \beta_{cT1} \) (figures 4 (d-i)) and significantly at \( \beta > \beta_{cT1} \) (figure 4, j, k, l).

In vacuum, the total wave field consists of TR and transmitted wave of CR (CTR). CTR exists at \( \beta_{c1} < \beta < \beta_{cT1} \) in the area \( z < z_3 \), \( z_3 = ct \sqrt{1 - \beta^2 (\varepsilon_1 - 1)} / \beta \) (figure 4, d,e,f). At \( \beta > \beta_{cT1} \) this field decreases exponentially with the distance from the boundary. So, CR penetrates from dielectric into vacuum at some conditions. It should be noted that similar effect takes place at the boundary between vacuum and ‘left-handed medium’ [10] where the CTR has ‘reversed’ directivity. Note as well that there is some increase and concentration of TR near the “wave front” \( z_f = ct \) in vacuum for the case of the ultra-relativistic particle (figure 4, j, k, l).

3. Conclusion
Underline the main conclusions of the work. If the charge flies from vacuum into dielectric there is area where the wave field practically coincides with the wakefield in the regular waveguide and area where the boundary influence is principal. The first area is large if the dielectric permittivity takes on a large value. This conclusion is important for the wakefield acceleration technique. If the charge flies from dielectric into vacuum with certain velocity a large quasi monochromatic radiation can be generated in the vacuum region. This conclusion is of interest for development of new methods of generation of electromagnetic radiation.

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