Big Bang Nucleosynthesis constraints on Barrow entropy

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We use Big Bang Nucleosynthesis (BBN) data in order to impose constraints on the exponent of Barrow entropy. The latter is an extended entropy relation arising from the incorporation of quantum-gravitational effects on the black-hole structure, parameterized effectively by the new parameter $\Delta$. When considered in a cosmological framework and under the light of the gravity-thermodynamics conjecture, Barrow entropy leads to modified cosmological scenarios whose Friedmann equations contain extra terms. We perform a detailed analysis of the BBN era and we calculate the deviation of the freeze-out temperature comparing to the result of standard cosmology. We use the observationally determined bound on $|\delta T/T_f|$ in order to extract the upper bound on $\Delta$. As we find, the Barrow exponent should be inside the bound $\Delta \lesssim 1.4 \times 10^{-4}$ in order not to spoil the BBN epoch, which shows that the deformation from standard Bekenstein-Hawking expression should be small as expected.

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I. INTRODUCTION

Recently it was shown that quantum-gravitational effects may introduce deformations on the black hole surface, which, although complex and dynamical, as a first approximation can effectively and coarse-grained be described by a fractal structure. As a result, the black hole entropy will deviate from the standard Bekenstein-Hawking one, given by the expression [1]

$$S_B = \left( \frac{A}{A_0} \right)^{1+\Delta/2},$$

with $A$ the standard black-hole area and $A_0$ the Planck area. Hence, the quantum-gravitational deformation is quantified through the exponent $\Delta$, lying between the extreme values $\Delta = 0$, which corresponds to the standard Bekenstein-Hawking entropy, and $\Delta = 1$, which corresponds to the most intricate and deformed structure.

Later on, new developments have appeared in the literature aiming to test the performance of the above Barrow entropy in the cosmological framework. The validity and the constraints imposed by the generalized second law of thermodynamics, including the matter-energy content and the horizon entropy, were investigated in [2]. Additionally, the Barrow holographic dark energy model was proposed in [3] and has been tested against the latest cosmological data in [4, 5] where it was found that it describes very efficiently the late accelerated expansion of the universe, having additionally the correct asymptotic behavior [6]. Finally, Barrow entropy has been studied in the black-hole context in [7–12].

In [13] Barrow entropy was applied in the framework of “gravity-thermodynamics” conjecture [14–16], according to which the first law of thermodynamics can be applied on the universe apparent horizon. As a result, one obtains a modified cosmology, with extra terms in the Friedmann equations depending on the new exponent $\Delta$, which disappear in the case $\Delta = 0$, i.e when Barrow entropy becomes the standard Bekenstein-Hawking one. Although this construction is very efficient in describing the late-time universe, one should carefully examine whether the aforementioned extra terms are sufficiently small in order not to spoil the early-time behavior and in particular the Big Bang Nucleosynthesis (BBN) epoch.

In the current article we address the above crucial issue concerning the behavior of modified cosmology through Barrow entropy in the early universe. Since it is known that a given cosmological model is considered viable if and only if it satisfies the appropriate conditions imposed by BBN, we can impose the BBN observational requirements in order to extract constraints on the exponent $\Delta$ of Barrow entropy. The structure of the article is as follows: In Section II we present the modified Friedmann equations that arise from the “gravity-thermodynamics” application of Barrow entropy. In Section III we perform the investigation of the BBN epoch in such a cosmological...
scenario, and we extract the constraints on the Barrow exponent. Finally, Section IV contains a summary of our results.

II. MODIFIED COSMOLOGY THROUGH BARROW HORIZON ENTROPY

In this section we briefly review the construction of modified Friedmann equations through the application of “gravity-thermodynamics” conjecture using Barrow entropy [13]. We consider a Friedmann-Robertson-Walker (FRW) metric

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \]

(2)

with \( a(t) \) the scale factor, and where \( k = 0, +1, -1 \) corresponds to flat, close and open geometry respectively. Moreover, we assume that the universe is filled with matter and radiation perfect fluids.

Let us start by presenting the above procedure in the usual case of general relativity. The gravity-thermodynamics conjecture states that the first law can be applied on the universe horizon considered as a thermodynamical system separated by a causality barrier [14–16], with the standard choice being the apparent one [17–19]

\[ r_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \]

(3)

with \( H = \frac{\dot{a}}{a} \) the Hubble parameter and dots denoting time-derivatives. For the horizon temperature one uses the corresponding black hole expression, but with the apparent horizon replacing the black-hole one [20], namely

\[ T_h = \frac{1}{2\pi r_A}. \]

(4)

The apparent horizon entropy will be given by the black-hole one in a similar way, namely from Bekenstein-Hawking relation \( S = A/(4G) \), with \( A = 4\pi r_A^2 \) its area and \( G \) the gravitational constant (in units where \( h = k_B = c = 1 \)):

\[ S_h = \frac{1}{4G} A. \]

(5)

Hence, incorporating the energy flow through the horizon \( \delta Q = -dE = A(\rho_m + p_m + \rho_r + p_r)Hr_A dt \) with \( \rho_i \) and \( p_i \) the energy density and pressure of the conserved matter and radiation fluids, and using the first law of thermodynamics \( -dE = TdS \) as well as expressions (4),(5), we can finally extract the Friedmann equations [18]

\[ -4\pi G(\rho_m + p_m + \rho_r + p_r) = \dot{H} - \frac{k}{a^2}, \]

(6)

\[ \frac{8\pi G}{3}(\rho_m + \rho_r) = H^2 + \frac{k}{a^2} - \frac{\Lambda}{3}, \]

(7)

where the cosmological constant arises as an integration constant. Note that in the above steps, and in order to avoid non-equilibrium thermodynamics, one applies the usual equilibrium assumption, namely that the universe fluids have the same temperature with the horizon [16–18, 21, 22].

The gravity-thermodynamics conjecture has been widely and efficiently applied in many modified theories of gravity, as long as one uses the modified entropy relation corresponding to each theory [23–31]. Knowing the above, one can apply the gravity-thermodynamics conjecture in the case of Barrow entropy. Hence, using (1) instead of (5) one finally results to [13]

\[ \frac{(4\pi)^{(1-\Delta/2)}2^{(1+\Delta/2)}A_0^{(1+\Delta/2)}(\rho_m + p_m + \rho_r + p_r)}{H^2 + \frac{k}{a^2}} = \frac{\dot{H} - \frac{k}{a^2}}{\Delta/2}, \]

(8)

and through integration to

\[ \frac{2 + \Delta}{2 - \Delta} \left( \frac{H^2 + \frac{k}{a^2}}{a^2} \right)^{1-\Delta/2} = \frac{(4\pi)^{(1-\Delta/2)}2^{(1+\Delta/2)}A_0^{(1+\Delta/2)}}{6} \]

\[ + \frac{C}{3} A_0^{(1+\Delta/2)}, \]

(9)

where \( C \) is the integration constant. As we observe, we have resulted to modified Friedmann equations which include extra terms comparing to general relativity. Restricting for simplicity in the flat case \( k = 0 \), we can re-write them as

\[ H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_{DE}) \]

(10)

\[ \dot{H} = -4\pi G(\rho_m + p_m + \rho_r + p_r + \rho_{DE} + p_{DE}), \]

(11)

where we have defined the energy density and pressure of the effective dark energy sector as

\[ \rho_{DE} = \frac{3}{8\pi G} \left\{ \frac{\Lambda}{3} + H^2 \left[ 1 - \beta (1 + \frac{\Delta}{2}) H^{-\Delta} \right] \right\}, \]

(12)

\[ p_{DE} = -\frac{1}{8\pi G} \left\{ \frac{\Lambda}{3} + 2H \left[ 1 - \beta (1 + \frac{\Delta}{2}) H^{-\Delta} \right] + 3H^2 \left[ 1 - \frac{\beta (2 + \Delta)}{2} H^{-\Delta} \right] \right\}, \]

(13)

with \( \beta \equiv \frac{4(4\pi)^{\Delta/2}G}{A_0^{2\Delta/2}} \) a parameter with dimensions \( [L^{-\Delta}] \)

and \( \Lambda \equiv 4CG(4\pi)^{\Delta/2} \) a parameter with dimensions \( [L^{-2}] \)

(in units where \( h = k_B = c = 1 \)). As expected, for \( \Delta = 0 \) (which implies that \( \beta = 1 \)) the above modified Friedmann equations reduce to \( \Lambda CDM \) scenario.

Finally, note that applying the first Friedmann equation (10) at present time one obtains

\[ \Lambda = \frac{3}{2 - \Delta} \frac{H_0^2(2-\Delta)}{2} - 3H_0^2(\Omega_{m0} + \Omega_{r0}), \]

(14)

which is a convenient expression relating \( \Lambda, \Delta \) and \( \beta \) with the present values of the matter and radiation density parameters \( \Omega_{m0}, \Omega_{r0}, \) as well as with the present value of the Hubble parameter \( H_0 \).
III. BIG BANG NUCLEOSYNTHESIS
CONSTRAINTS ON BARROW EXPONENT $\Delta$

In this section we examine the Big Bang Nucleosynthesis (BBN) in the framework of modified cosmology through spacetime thermodynamics with Barrow entropy. Since BBN occurs in the radiation era we focus on the energy density of relativistic particles which is given by $\rho_r = \frac{\pi^2}{30} g_* T^4$, where the effective number of degrees of freedom is $g_* \sim 10$ and $T$ is the temperature (the details of BBN are provided in the Appendix). The neutron abundance is calculated using the protons-neutron conversion rate, i.e.

$$\lambda_{pn}(T) = \lambda_{(n+\bar{\nu}_e\rightarrow p+e^-)} + \lambda_{(n+e^+\rightarrow p+\bar{\nu}_e)} + \lambda_{(n-p+e^-+\bar{\nu}_e)}$$

(15)

and its inverse $\lambda_{np}(T)$, and the total rate is therefore

$$\lambda_{tot}(T) = \lambda_{np}(T) + \lambda_{pn}(T).$$

(16)

From (16) one can result to (see (A.22) in the Appendix)

$$\lambda_{tot}(T) = 4 A T^3 (4! T^2 + 2 \times 3! QT + 2! Q^2),$$

(17)

with $Q = m_n - m_p = 1.29 \times 10^{-3}$GeV the neutron-proton mass difference and $A = 1.02 \times 10^{-11}$GeV$^{-4}$. Concerning the primordial mass fraction of $^4$He, we can estimate it by using [32]

$$Y_p = \frac{2x(t_f)}{1 + x(t_f)},$$

(18)

with $\lambda = e^{-(t_n - t_f)/\tau}$, where $t_f$ is the freeze-out time of the weak interactions, $t_n$ the corresponding freeze-out time of nucleosynthesis, $\tau$ the neutron mean lifetime (A.19), and with $x(t_f) = e^{-Q/T(t_f)}$ the neutron-to-proton equilibrium ratio. The function $\lambda(t_f)$ accounts for the fraction of neutrons that decay into protons inside the time interval $t \in [t_f, t_n]$.

In any modified cosmological model one results with extra contributions in the Friedmann equations. The BBN happens at the radiation dominated era, and according to observations these extra contributions need to be small comparing to the radiation sector in the concordance cosmological model, i.e. the Standard Model radiation in the framework of General Relativity. Hence, the first Friedmann equation becomes approximately

$$H^2 = \frac{8\pi G}{3} \rho_r = H_{GR}^2,$$

(19)

and thus the scale factor evolves as $a \sim t^{1/2}$, with $t$ the cosmic time. The relation between temperature and time is therefore given by

$$\frac{1}{T} \simeq \left(\frac{32\pi^3 g_*}{90}\right)^{1/2} \frac{T^2}{M_P} \quad \text{(or } T(t) \simeq (t/\text{sec})^{-1/2}\text{MeV}),$$

(20)

which leads to

$$H \approx \left(\frac{4\pi^3 g_*}{45}\right)^{1/2} \frac{T^2}{M_P},$$

(21)

with $M_P = (8\pi G)^{-1} = 1.22 \times 10^{19}$ GeV the Planck mass.

Now, if the interaction rate $\lambda_{tot}(T)$ given in (17) is $\frac{1}{T} \ll \lambda_{tot}(T)$, i.e. if the expansion time is much smaller than the interaction time then as usual we can safely consider that all processes are in thermal equilibrium [32, 33]. On the other hand, if $\frac{1}{T} \gg \lambda_{tot}(T)$ then particles decouple since they do not have the necessary time intervals to interact. Hence, the temperature at which particles decouple, namely the freeze-out temperature $T_f$, corresponds to the equality time, i.e. when $H = \lambda_{tot}(T)$.

Since $H \approx \left(\frac{4\pi^3 g_*}{45 M_P^2 T^2}\right)^{1/6}$ while $\lambda_{tot}(T) \approx q T^5$, with $q = 4.44! \approx 9.8 \times 10^{-10}$GeV$^{-4}$, the above equality provides the freeze-out temperature as

$$T_f = \left(\frac{4\pi^3 g_*}{45 M_P^2 q^2}\right)^{1/6} \approx 0.0006 \text{ GeV}. $$

(22)

In the case of modified cosmology, the Hubble function $H$ will deviate from $H_{GR}$, and hence the freeze-out temperature $T_f$ will also present a deviation $\delta T_f$ from the GR result (21). This will in turn induce a deviation of the fractional mass $Y_p$, given by

$$\delta Y_p = Y_p \left[\left(1 - \frac{Y_p}{2\lambda}\right) \ln \left(\frac{2\lambda}{Y_p} - 1\right) - \frac{2t_f}{\tau} \frac{\delta T_f}{T_f}\right],$$

(23)

where we have imposed $\delta T(t_n) = 0$ due to the fact that $T_n$ is fixed by the deuterium binding energy [34–37]. The observational estimations of the mass fraction $Y_p$ of baryons converted to $^4$He during the BBN epoch are [38–44]

$$Y_p = 0.2476, \quad |\delta Y_p| < 10^{-4}. $$

(24)

Hence, inserting these into (22) we extract the upper bound on $\frac{\delta T_f}{T_f}$, i.e.

$$\left|\frac{\delta T_f}{T_f}\right| < 4.7 \times 10^{-4},$$

(25)

which is the allowed deviation from standard cosmology.

In the scenario at hand, the Barrow-entropy-related effective dark energy $\rho_{DE}$ given in (12) is in principle present at the BBN times too. Hence, this should be small comparing to $\rho_r$, and thus it can be treated as a perturbation, while the matter sector can be neglected as usual. Therefore, the Hubble function arises from (10) as

$$H = H_{GR} V\left[1 + \frac{\rho_{DE}}{\rho_r}\right] = H_{GR} + \delta H,$$

(26)

and hence

$$\delta H = \left(\sqrt{1 + \frac{\rho_{DE}}{\rho_r}} - 1\right) H_{GR}. $$

(27)

Thus, this deviation $\delta H$ from standard $H_{GR}$ will lead to $\delta T_f$, and since as we mentioned $H_{GR} = \lambda_{tot} \approx q T^5$, we find

$$\left(\sqrt{1 + \frac{\rho_{DE}}{\rho_r}} - 1\right) H_{GR} = 5q T^3 \delta T_f, $$

(28)
which since \( \rho_{DE} \ll \rho_r \) becomes

\[
\frac{\delta T_f}{T_f} \simeq \frac{\rho_{DE}}{\rho_r} \frac{H_{GR}}{10qT_f^3}.
\] (28)

We have now all the information to proceed to the investigation of the BBN bounds on the parameter \( \Delta \) of Barrow entropy. These constraints will be extracted using (28) and (12). Additionally, we will use the numerical values

\[
\Omega_{m0} = 0.3, \quad \Omega_{r0} = 0.000092, \quad H_0 = 1.4 \times 10^{-42} \text{GeV}.
\] (29)

Inserting (12) into (28), and eliminating \( \Lambda \) using (14), we find

\[
\left| \frac{\delta T_f}{T_f} \right| \simeq \left[ g_\ast \pi^2 T_f^4 (\Delta - 2) \right]^{-1} \left[ \frac{3}{2} M_p^2 H_0^2 M_p^2 \pi^2 (2 + \Delta) + H_0^2 (\Delta - 2) \Omega_{m0} \right. \\
+ q^2 T_f^{10} \left[ 2 - \Delta - 2 \pi^2 M_p^2 \pi^2 (qT_f^3) = (2 + \Delta) \right] \right].
\] (30)

Since all constants are known, if we insert the above expression into (24) we obtain the BBN bounds on Barrow \( \Delta \). As expected for \( \Delta = 0 \) we obtain \( \delta T_f/T_f = 0 \).

In Fig. 1 we depict \( \delta T_f/T_f \) from (30) vs \( \Delta \) (red curve), as well as the upper bound from (24). As we can see, constraints from BBN require \( \Delta \lesssim 1.4 \times 10^{-4} \).

IV. CONCLUSIONS

In this work we have used Big Bang Nucleosynthesis analysis and data in order to impose constraints on the exponent \( \Delta \) of Barrow entropy. The latter is an extended entropy relation arising from the incorporation of quantum-gravitational effects on the black-hole structure, parameterized effectively by the new parameter \( \Delta \). When considered in a cosmological framework and under the light of the gravity-thermodynamics conjecture, Barrow entropy leads to modified cosmological scenarios whose Friedmann equations contain extra terms. This construction is very efficient in describing the late-time universe, nevertheless one should examine whether the involved extra terms are sufficiently small in order not to spoil the early-time behavior and in particular BBN epoch.

We performed a detailed analysis of the BBN era in the above new cosmological scenarios and we calculated the deviation of the freeze-out temperature comparing to the result of standard cosmology, brought about by Barrow entropy exponent \( \Delta \). Hence, we used the observationally determined bound on \( \left| \frac{\delta T_f}{T_f} \right| \) in order to extract the upper bound on \( \Delta \). As we showed, the Barrow exponent should be inside the bound \( \Delta \lesssim 1.4 \times 10^{-4} \) in order not to spoil the BBN epoch. As expected the latter result shows that the deformation from standard Bekenstein-Hawking expression should be small.

It would be interesting to investigate the case where the complexity and dynamicality of quantum-gravitationally deformed horizon structure would be incorporated through a Barrow exponent that depends on time and scale, as it has already been done with Tsalis entropy exponent [45]. This construction could leave more freedom in the deviation of Barrow entropy from Bekenstein-Hawking one. However, such a detailed study lies beyond the scope of the present work and it is left for a future project.

Appendix: Big Bang Nucleosynthesis

In this Appendix we review briefly the Big Bang Nucleosynthesis features [32, 33]. The energy density of relativistic particles \( (T \gg m, \mu) \) where \( \mu \) is the chemical potential) filling up the early Universe is \( \rho = \frac{g_\ast}{(2\pi)^3} \int E \bar{n}(E/T) d^3 p = \frac{g_\ast^2}{3} g T^4 \), with \( g_\ast \) denoting the degeneracy factors for particle species \( (g_\gamma = 2, \ g_e = 4, \ g_\mu = 2) \), and \( g = g_\gamma + g_e + g_\mu = \frac{g_\gamma^2}{2} (g_\gamma = g_e + 3g_\mu = 10) \) are the effective degrees of freedom (one assumes implicitly that muon and tau neutrinos have a small mass comparing to the effective temperature, and that other massless species are not present).

The primordial \(^4\)He in the early Universe was formed at a temperature \( T \approx 100 \text{ MeV} \). The number and energy densities were formed by photons and relativistic leptons (electron, positron and neutrinos), while rapid collisions
were forcing all these particles to be in thermal equilibrium. In particular, protons and neutrons were maintained in thermal equilibrium through their interactions with leptons:

\begin{align}
\nu_e + n &\leftrightarrow p + e^- \quad \text{(A.1)} \\
e^+ + n &\leftrightarrow p + \bar{\nu}_e \quad \text{(A.2)} \\
n &\leftrightarrow p + e^- + \bar{\nu}_e \quad \text{(A.3)}
\end{align}

One can calculate the neutron abundance through the conversion rate of protons to neutrons, denoted by \(\lambda_{np}(T)\), and its inverse rate denoted by \(\lambda_{pn}(T)\). Hence, at suitably high temperature the weak interaction rates read as

\[\lambda_{tot}(T) = \lambda_{np}(T) + \lambda_{pn}(T).\]  

(A.4)

Now, \(\lambda_{np}\) is given by the sum of the rates corresponding to the processes (A.1)-(A.3), i.e.

\[\lambda_{np} = \lambda_{(n+\nu_e\to p+e^-)} + \lambda_{(n+e^+\to p+\bar{\nu}_e)} + \lambda_{(n\to p+e^-+\bar{\nu}_e)},\]  

(A.5)

while the rate \(\lambda_{np}\) is obtained from \(\lambda_{pn}(T) = e^{-Q/T}\lambda_{pn}(T)\), where \(Q = m_n - m_p = 1.29 \times 10^{-3}\) GeV is the neutron-proton mass difference.

During the freeze-out regime one can assume that [33]: (i) The particles temperatures are the same, namely \(T_{\nu} = T_e = T_q = T\), (ii) the temperature \(T\) is lower from the typical energies \(E\) that enter into the integrals that appear in the expressions for the rates (and thus one can use the Boltzmann distribution \(n \approx e^{-E/T}\) instead of the Fermi-Dirac one), (iii) \(m_e \ll E_e, E_\nu\). i.e. the electron mass \(m_e\) can be neglected comparing to the electron and neutrino energies.

From the above we conclude that the interaction rate of the process (A.1) is [32, 33]

\[d\lambda_{(n+\nu_e\to p+e^-)} = d\mu (2\pi)^4 |\langle M \rangle|^2 W,\]  

(A.6)

with

\[d\mu = \frac{d^3p_\nu}{(2\pi)^32E_\nu} \frac{d^3p_e}{(2\pi)^32E_e} \frac{d^3p_p}{(2\pi)^32E_p},\]  

(A.7)

\[M = \left(\frac{g_w}{8MW}\right)^2 [\bar{u}_e\Gamma^{\mu}u_n][\bar{\nu}_e\Sigma_\mu\nu_{\nu_e}],\]  

(A.8)

\[\Omega^\mu \equiv \gamma^\mu (c\nu_e - cA\gamma^5),\]  

(A.9)

\[\Sigma^\mu \equiv \gamma^\mu (1 - \gamma^5),\]  

(A.10)

\[W = \delta^{(4)}(P)n(E_{\nu_e})(1 - n(E_e)),\]  

(A.11)

\[P \equiv p_n + p_{\nu_e} - p_p - p_e.\]  

(A.12)

Note that in (A.8) we have made use of the condition \(q^2 \ll M_W^2\), with \(M_W\) the \(W\) vector gauge boson mass, and where \(q^\mu = p_{\nu_e}^\mu - p_p^\mu\) is the momentum transferred. From (A.6) we obtain

\[\lambda_{(n+\nu_e\to p+e^-)} = A T^5 I_y,\]  

(A.13)

with \(A = \frac{\lambda_{(n+\nu_e\to p+e^-)}}{2\pi^2} \approx 1.02 \times 10^{-11}\) GeV\(^{-4}\) [33], and with

\[I_y = \int_y^{\infty} (\epsilon - Q')^2 \sqrt{\epsilon^2 - y^2} \epsilon (1 - n(\epsilon)) d\epsilon,\]  

(A.14)

having defined

\[y \equiv \frac{m_\nu}{T}, \quad Q' = \frac{Q}{T}.\]  

(A.15)

Repeating the above calculation steps for the process (A.2) we acquire

\[\lambda_{(e^+\nu_e\to p+\bar{\nu}_e)} = A T^5 J_y,\]  

(A.16)

where

\[J_y = \int_y^{\infty} (\epsilon + Q')^2 \sqrt{\epsilon^2 - y^2} \epsilon (1 - n(\epsilon + Q')) d\epsilon,\]  

(A.17)

and finally we extract

\[\lambda_{(e^+\nu_e\to p+\bar{\nu}_e)} = A T^3 (4!T^2 + 2 \times 3!QT + 2!Q^2).\]  

(A.18)

Concerning the neutron decay (A.3) one has

\[\tau = \lambda_{(n\to p+e^-+\bar{\nu}_e)}^{-1} \approx 887\text{sec}.\]  

(A.19)

As a result, in the incorporation of the process (A.5) we can safely neglect the neutron decay, namely the neutron can be handled as a stable particle during the Big Bang Nucleosynthesis.

In summary, the aforementioned approximations (i)-(iii) result to [33]

\[\lambda_{(e^+\nu_e\to p+\bar{\nu}_e)} = \lambda_{(n+\nu_e\to p+e^-)}.\]  

(A.20)

Hence, substituting (A.20) into (A.5) and then into (A.4), leads to the expression for \(\lambda_{tot}(T)\) as

\[\lambda_{tot}(T) \simeq 2\lambda_{np} = 4\lambda_{(e^+\nu_e\to p+\bar{\nu}_e)};\]  

(A.21)

and hence inserting (A.18) results to

\[\lambda_{tot}(T) = A T^3 (4!T^2 + 2 \times 3!QT + 2!Q^2).\]  

(A.22)

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