Thermoelectric transport through a quantum nanoelectromechanical system and its backaction

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We present a comprehensive study of thermoelectric transport properties of a quantum nanoelectromechanical system (NEMS) described by a single-electron-transistor (SET) coupled to a quantum nanomechanical resonator (NR). The effects of a quantum NR on the electronic current are investigated with special emphasis on how the SET-NR coupling strength plays a role in such a NEMS. We find that the SET-NR coupling is not only able to suppress or enhance the thermoelectric current but can also switch its direction. The effect of the NR on the thermoelectric coefficients of the SET are studied and we find that even a small SET-NR coupling could dramatically suppress the figure of merits ZT. Lastly, we investigate the backaction of electronic current on the NR and possible routes of heating or cooling the NR are discussed. We find that by appropriately tuning the gate voltage the backaction can be eliminated, which could find possible applications to enhance the sensitivity of detection devices.

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I. INTRODUCTION

Nanomechanical resonators (NRs) have been in the limelight because of their possible applications in ultra-sensitive detection1,2 and quantum-controlled devices3,4,5. Recent advances in high-frequency NR fabrication and cooling technology have made it possible to achieve quantum behaved resonators6,7, which opens up the possibility for mechanical systems to be coherently coupled to the electronic ones to form a quantum nanoelectromechanical system (NEMS)8,9,10. Such NEMSs have shown a wide range of applications such as charge probing11, coherent sensing12,13, and electron shuttling14. In such applications an important parameter of the NEMS is the coupling strength between the NR and the electronic system. Traditionally, the coupling strength is assumed to be weak15-16, hence the effect of the NR on the electronic system is often regarded as a perturbation. However, recent experiments have demonstrated that the coupling between the NR and electronic system can be strong17,18 and even be tailored19,20. The influence of the NR on the electronic system in this strong coupling regime is a fascinating facet which has not yet been fully explored theoretically under a quantum mechanical description.

An important class of the NEMSs is an NR coupled to a single-electron-transistor (SET-NR system), which has been widely investigated both theoretically15,16,19,21 and experimentally2,12,13,18. In most cases, the NR is treated classically under the condition that the resonant frequency is much smaller than the electron tunnelling frequency. The transport properties of such classical SET-NR systems have been extensively studied, including the calculation of current, current noise15,20, and dynamics of the NR16,21,22. Among these investigations an interesting perspective is to understand how the NR affects the electronic current21,24, and hence facilitates the analysis of the mechanically tuned electronic signals from the detectors, or even helps in the design of mechanically controlled electronic nano-devices. Previous studies have shown that the NR causes changes to electronic current near the resonant frequency22 and this phenomenon is widely used in experiments18,25 for various applications. However, these works focus on the influence of NR on the voltage biased current. The influence of a quantum NR on thermoelectric current, where the electronic current is induced by a temperature bias, is largely ignored. Furthermore, when the resonant frequency of NR is comparable or even larger to the electron tunnelling frequency, the SET-NR coupling exhibits quantum behaviour and the coherences between the NR and the SET become important20. Theoretical investigations taking the coherence between NR and SET are quite limited. In this work we study transport properties of a complete quantum mechanical SET-NR system and find intriguing effects due to the influence of the NR on thermoelectric current. In particular, the NR is not only able to adjust the magnitude of the current as previously discovered, but also able to tune its direction which is impossible in voltage biased mode.

Another important aspect to fully understand the transport properties is the backaction on the NR caused by the passage of electronic current. Experimentally the
backaction has been constructively employed to cool the NR to the quantum regime\textsuperscript{2,19,26}. However, for most applications, like displacement detection, the backaction is not preferred since it generates noise which in turn reduces the sensitivity of the detector\textsuperscript{3,27}. For voltage biased current backaction is generally not avoidable and hence it imposes a fundamental limit to measurement sensitivity. However, in this work we find that by using thermoelectric current it is possible to fully eliminate the backaction.

In the following sections we present in detail how the thermoelectric current varies with the SET-NR coupling strength. We then demonstrate a complete picture of how a quantum NR affects the thermoelectric coefficients of the SET including the Seebeck coefficients and figure of merit. We then go on to study the backaction of the electronic system on the NR and the role of SET-NR coupling played in the backaction. Our comprehensive study allows us to fully understand the thermoelectric transport properties of the SET-NR system in the quantum regime and provides a better understanding on how to eliminate the backaction.

II. PHYSICAL MODEL AND FORMALISM

We consider a SET-NR system where a SET is capacitively coupled to an NR as schematically shown in Fig. 1. The full setup consists of a SET-NR island coupled to two electronic leads with temperature $T_L$ ($T_R$) and chemical potential $\mu_L$ ($\mu_R$) for the left (right) lead. A background environment with temperature $T_E$ acts on the NR.

The energy $E_N$ of the NR island will be in the strict Coulomb blockade regime such that only a single electron can tunnel through the SET when an excess charge is tunnelling into the island. This is sensitive to the separation distance between the SET and the NR. Hence we can expand the energy $E_N$ of the NR given by\textsuperscript{28}

$$E_N = E_C(N - N_g)^2,$$

where $E_C = e^2/(2C_S)$ is the charging energy of the island. When the capacitance is so small that the charging energy dominates over the thermal fluctuations, the SET-NR island will be in the strict Coulomb blockade regime.

The energy $\varepsilon$ above depends on the displacement of the NR via $E_C$ and $N_g$. In experimental set-ups the displacement $x$ is much smaller than the separation distance between the SET and the NR. Hence we can expand the energy to linear order in $x$ as $\varepsilon = \varepsilon_0 + \lambda x$, where $\varepsilon_0$ is a constant evaluated at equilibrium separation when there are $N$ excess charges. The coefficient $\lambda$ is given by

$$\lambda = -\frac{eC_g'}{C_S}[C_S V_g + (N - N_g + 1/2)e].$$

Here $C_g'$ is the derivative of $C_g$ with respect to $x$ which is assumed to be a constant. Equation (4) is consistent with different forms obtained in the literature\textsuperscript{11,18,25} in the sense that the coefficient $\lambda$ above can be physically interpreted as the extra force acting on the NR by the SET when an excess charge is tunnelling into the island. It also characterizes the coupling strength between the SET and NR and it is possible to be tuned by adjusting the oscillation mode of the NR\textsuperscript{2}. The constant $\varepsilon_0$ is proportional to the gate voltage $V_g$ and it can also be adjusted experimentally. Hence in the following discussion we will use $\varepsilon_0$ instead of $V_g$ to describe the gate properties.

By writing the Hamiltonian in the basis of excess charges we can quantize the SET-NR island to obtain\textsuperscript{28}

$$H_S = (\varepsilon_0 + \lambda x)|N+1\rangle\langle N+1| + H_{NR},$$

where $x$ is a quantum mechanical position operator. The Hilbert space of the excess charges is spanned by $|N\rangle$ and $|N+1\rangle$ with fixed $N$. The NR Hamiltonian $H_{NR}$ is the standard Harmonic oscillator

$$H_{NR} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2,$$
where $\omega_0$ is the fundamental frequency of the NR and $m$ is its mass. The SET-NR island is connected to two electronic leads which act as a source and a drain. Since the experimental set-up cannot be isolated from dissipative effects acting on the NR, we subject the NR to an external environment which is kept at temperature $T_E$. Therefore, the total Hamiltonian reads

$$H_{tot} = H_S + H_L + H_R + H_E + H_T + H_{SE},$$

where the lead Hamiltonians $H_{L,R}$ are modelled as an infinite set of free-fermions given by

$$H_{L,R} = \sum_{k \in L,R} \epsilon_k c_k^\dagger c_k.$$  \hspace{1cm} (8)

Above $c_k^\dagger$ and $c_k$ are the fermionic creation and annihilation operators respectively. The environment Hamiltonian $H_E$ describes dissipative effects and is assumed to be a phononic heat bath given by

$$H_E = \sum_{n=1}^{\infty} \frac{p_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2q_n^2.$$  \hspace{1cm} (9)

The coupling of the system to this background environment is via the linear position coupling, $H_{SE} = -\sum_{n=1}^{\infty} C_n q_n x / \sqrt{2\omega_0}$. The tunnelling Hamiltonian $H_T$ connects the SET and the electrodes which can be written as

$$H_T = \sum_{k \in L,R} V_k \left( c_k^\dagger |N\rangle \langle N+1| + c_k |N+1\rangle \langle N| \right),$$  \hspace{1cm} (10)

where the first term describes the tunnelling of a charge from the island to the lead while the second term expresses the reverse process.

Given the total Hamiltonian we can now describe the state of the SET-NR island via a reduced density matrix $\rho$ by tracing over the lead and environment degrees of freedom. Since the system Hamiltonian contains nonlinear interactions the nonequilibrium transport problem is not solvable exactly. Common approximations include: treating the NR at equilibrium 32,33, ignoring the coherences between the SET and the NR 22,23,34,35, and regarding the SET-NR coupling as a perturbation 36,37. Since we would like to study the system in the regime of strong SET-NR coupling we resort to the alternative of treating the system-lead and system-environment coupling as weak while the nonlinearity in SET-NR coupling is treated accurately. Specially we do not use the commonly used canonical transformation which transform the nonlinearity of the system to the system-lead coupling, making the perturbative parameter an abstract quantity. Within this weak system-lead coupling approximation we employ the standard techniques of the theory of open quantum systems to write a quantum master equation of the Bloch-Redfield type given by 38,39.

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_S,\rho] - \frac{1}{\hbar^2} \sum_{\alpha,\beta} \delta_{ij} \rho_{ij}(\tau - t)\rho C^{\alpha\beta}(t - \tau) + h.c. + d\rho/dt = 0,$$  \hspace{1cm} (11)

where $h.c.$ stands for Hermitian conjugate and $S = \{ |N\rangle \langle N+1|, |N+1\rangle \langle N|, x / \sqrt{2\omega_0} \}$ is a vector with each component denoting a particular system operator coupled to the leads and the environment. The corresponding operator of the leads or environment can also be expressed in a vector form and is given by $B = \{ \sum_{k \in L,R} V_k c_k^\dagger, \sum_{k \in L,R} V_k c_k, -\sum_{n=1}^{\infty} C_n q_n \}$. Subsequently the correlation functions used in Eq. (11) are defined as $C^{\alpha\beta}(t) = \langle B^\alpha(t)B^\beta(0) \rangle$, where the superscript denotes a particular component of the B-vector. Detailed calculations of the correlation functions are presented in the appendix. Operators with time arguments indicate free evolutions with respect to the non-interacting Hamiltonian $H_0 = H_S + H_L + H_R + H_E$.

In order to solve Eq. (11) we numerically diagonalize the system Hamiltonian $H_S$ so that the SET-NR coupling is treated exactly. In particular we do not make the common assumption that the NR is at equilibrium 32,33, which would be true only if the SET and NR are weakly coupled. Thus, in the energy eigenbasis of the system Hamiltonian $H_S$ the quantum master equation reads

$$\frac{d\rho_{nm}}{dt} = -\frac{i}{\hbar} \Delta_{nm}\rho_{nm} + R_{nm}^{ij} \rho_{ij},$$  \hspace{1cm} (12)

where

$$R_{nm}^{ij} = \frac{1}{\hbar} \sum_{\alpha,\beta} \left\{ \sum_{l} S_{ni}^\alpha S_{jm}^\beta W_{ij} S_{nl}^{\alpha\beta} - \delta_{jm} \sum_{l} S_{nl}^{\alpha\beta} W_{il}^{\alpha\beta} \right\} + h.c.$$  \hspace{1cm} (13)

The relaxation coefficients are given by

$$W_{ij}^{\alpha\beta} = \int_{-\infty}^{t} d\tau e^{i\Delta_{ij}(\tau - t)/\hbar} C^{\alpha\beta}(t - \tau),$$  \hspace{1cm} (14)

where $\Delta_{ij} = E_i - E_j$ are the energy spacing of the system Hamiltonian.

Since we are interested in the steady-state thermoelectric transport properties we solve the above quantum master equation in the steady state by setting $d\rho/dt = 0$ and taking the limit $t \rightarrow 0$. In order to evaluate the currents at the lowest order we require only the 0-th order reduced density matrix, which is obtained by solving $R_{nm}^{ii}(0) = 0$ along with the normalization condition $\text{Tr}(\rho(0)) = 1$. The off-diagonal elements satisfy $\rho_{ij}(0) = 0 \forall i \neq j$. The price to pay in order to use only the 0-th order reduced density matrix is that we then require the information about the system-lead coupling in the current operator. This obstacle can be overcome if we follow the techniques of Thingna et al. in refs. 41,42 to obtain the reduced definition of the current operator. This reduced current operator could then be combined with the 0-th order reduced density matrix $\rho(0)$ to obtain the average currents at the lowest order of system-lead coupling. Thus, in order to obtain the reduced definition of the current operators we begin with the standard
definition \( \Delta \) of the electron and heat current operators out of the left lead as

\[
I_e = -e \frac{dN_L}{dt} \\
= \frac{i}{\hbar} \sum_{k \in L} V_k \left( c_k^\dagger |N\rangle \langle N+1| - c_k |N+1\rangle \langle N| \right), \tag{15}
\]

\[
I_h = -\frac{d(H_L - \mu_L N_L)}{dt} \\
= \frac{i}{\hbar} \sum_{k \in L} V_k (\varepsilon_k - \mu_L) \left( c_k^\dagger |N\rangle \langle N+1| - c_k |N+1\rangle \langle N| \right), \tag{16}
\]

where \( \mu_L \) represents the chemical potential of the left lead and \( N_L = \sum_{k \in L} c_k^\dagger c_k \) is the left lead electron number operator.

Instead of treating the electron and heat current separately we use a unified notation,

\[
I_{e(h)} = \frac{i}{\hbar} \sum_{\alpha=1,2} S_\alpha \otimes B_{e(h)}^\alpha,
\]

to treat both currents on the same footing. Above the operator-vectors, \( B_e = \{ e \sum_{k \in L} V_k c_k^\dagger - e \sum_{k \in L} V_k c_k \} \) and \( B_h = \{ \sum_{k \in L} (\varepsilon_k - \mu_L) V_k c_k^\dagger - i \sum_{k \in L} (\varepsilon_k - \mu_L) V_k c_k \}, \) contain information about the left lead. Following the derivation of refs. \[41\] \[42\] we obtain the lowest order of the reduced steady-state current operators as

\[
I^r_{e(h)} = \frac{i}{\hbar} \sum_{\alpha,\beta} \int_{-\infty}^0 d\tau S_\alpha S_\beta^\dagger \langle \alpha^\beta \rangle_{e(h)}(-\tau) + h.c., \tag{18}
\]

where \( \langle \alpha^\beta \rangle_{e(h)}(\tau) = \langle B^\alpha(\tau) B_{e(h)}^\beta(0) \rangle \) are the correlation functions between the lead operators occurring in the current operator definition of Eq. \[17\] and the tunnelling Hamiltonian \( H_T \). The actual calculations of these correlation functions are further discussed in the appendix. Expressing the reduced current operators in the energy eigenbasis of \( H_S \) we obtain

\[
(I^r_{e(h)})_{ij} = \frac{i}{\hbar} \sum_{\alpha,\beta,k} \left[ S_\alpha S_\beta^\dagger W_{e(h)}^{\alpha\beta} (\Delta_{kj}) + c.c. \right], \tag{19}
\]

where \( c.c. \) denotes complex conjugate and the relaxation coefficients are different from that used in the quantum master equation and are given by

\[
W_{e(h)}^{\alpha\beta}(\Delta_{kj}) = \int_{-\infty}^0 d\tau e^{i\Delta_{kj}\tau/\hbar} C_{e(h)}^{\alpha\beta}(-\tau). \tag{20}
\]

Now since the reduced current operators are known, the average steady-state currents can be easily calculated using the 0-th order reduced density matrix as

\[
I_{e(h)} = \text{Tr} \left( \rho^{(0)} I^r_{e(h)} \right).
\]

To explore the thermoelectric properties of the SET-NR system, we also need to calculate the transport coefficients in the linear response regime. In this regime the relation between heat and electron current can be expressed in a matrix form as

\[
\begin{pmatrix} I_e \\ I_h \end{pmatrix} = \begin{pmatrix} L_0 & L_1 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} \Delta \mu \\ -T \end{pmatrix}. \tag{21}
\]

Here, the off-diagonal elements \( L_1 \) are the same due to the Onsager reciprocal relations. Thus, we can express the transport coefficients in terms of the above matrix coefficients as

\[
G_e = e L_0, \tag{22}
\]

\[
S = \frac{L_1}{G_e T}, \tag{23}
\]

\[
\kappa = \frac{L_2}{T} - G_e S^2 T, \tag{24}
\]

\[
ZT = \frac{G_e S^2 T}{\kappa}, \tag{25}
\]

where \( G_e \) is the electronic conductance, \( \kappa \) is the thermal conductance, \( S \) is the Seebeck coefficient, and \( ZT \) is the thermoelectric figure of merit which determines the efficiency of the device to convert (waste) thermal energy into (useful) electrical current.

### III. RESULTS AND DISCUSSION

#### A. Thermoelectric transport

We first study the effect of gate voltage, governed by \( \varepsilon_0 \), and SET-NR coupling strength \( \lambda \) on the thermoelectric current as shown in Fig. \[2\]. From now onwards, we measure all quantities in units of \( \hbar \omega_0 \) and choose the
parameters such that the thermal energy \( k_B T \), chemical potential bias \( \epsilon \Delta V \), and energy scale of the SET-NR island are comparable to \( \hbar \omega_0 \). This choice of parameters allows us to work in the quantum regime and is experimentally realizable since it is possible to create NRs with \( \omega_0 \) up to a few Gigahertz\(^\ref{ga}\), which would correspond to the background temperature of tens of millikelvin\(^\ref{gb}\) and a SET energy scale in micro-\(eV\) range. From Fig.\(\ref{fig:thermo}\) we clearly see that when \( \lambda = 0 \) the thermoelectric current changes direction when \( \varepsilon_0 \) transverses across the Fermi-level of the leads. This can be easily understood from the fact that the dominant charge carriers of the SET are altered by varying the gate voltage\(^\ref{gc}\). Interestingly, we can also see that the thermoelectric current changes sign with the SET-NR coupling strength. This particularly happens for \( \varepsilon_0 \) larger than the Fermi-level of the leads. For example if we focus on \( \varepsilon_0 = \hbar \omega_0 \) we find that at around \( \lambda \approx 1.2 \hbar \omega_0 \) the current changes sign from positive to negative. The red dotted line on the contour plot helps locate the boundary of the sign change. This phenomenon is particularly interesting because it clearly demonstrates that the mechanical motion of the SET-NR structure is able to influence the dominant charge carriers inside the SET. One could understand this change of dominant charge carriers from electrons to holes in terms of a polaron shift process. When the quantized vibration of the NR couples to the SET it forms a polaron, which in turn shifts the energy spacing \( \varepsilon_0 \) downwards by an amount \( \lambda^2/(\hbar \omega_0) \). Thus the stronger the SET-NR coupling strength \( \lambda \) the larger would be the energy shift. Once the polaron shift becomes large enough to shift the energy from above the Fermi-level of the leads to below, we see a change in the dominant carrier type of the SET. In term of the magnitude of the current, in most regimes the SET-NR coupling will decrease the current. However, in some domains we also find that SET-NR coupling can be used to increase the current. For example, when \( \varepsilon_0 = -\hbar \omega_0 \) the magnitude of current increases to a maximum at \( \lambda = 1.5 \hbar \omega_0 \) and then decreases again as a function of \( \lambda \). The enhancement effect is even more pronounced in the low temperature regime. This phenomenon clearly exhibits the potential capability to build mechanical tunable quantum NEMS such that the mechanical system can either suppress, enhance or even change the direction of electronic current.

Next we comprehensively study the thermoelectric transport properties of the SET-NR island in the linear response regime by plotting all the transport coefficients, namely the electronic conductance \( G_e \), thermal conductance \( \kappa \), Seebeck coefficient \( S \), and the figure of merit \( ZT \), in Fig.\(\ref{fig:thermo}\). Here we again clearly observe the polaron shift in the electronic conductance [Fig.\(\ref{fig:thermo}\) (a)] and Seebeck coefficient [Fig.\(\ref{fig:thermo}\) (c)] plots. The maximum electronic conductance [Fig.\(\ref{fig:thermo}\) (a)] and zero Seebeck coefficient [blue dashed line in Fig.\(\ref{fig:thermo}\) (c)] shift towards the righthand side for increasing \( \lambda \). In the calculation of thermal conductance, since in our model the electrodes do not contain phonons, the entire contribution to the thermal conductance is from the electrons. We approximate the phonon contribution of the leads as a small constant added to the thermal conductance. This assumption is valid because the phonon contribution to thermal conductance is generally small in nano-junctions connecting to metallic leads\(^\ref{gd}\). From Fig.\(\ref{fig:thermo}\) (b) we observe that in the regime of small \( \lambda \) the thermal conductance approaches zero. This is because when the SET-NR coupling strength approaches zero, there is only one channel for the electron to tunnel through the SET island. In this case, since only the electron carries energy, the heat flow is only possible when there is some electron current. As a result, the thermal conductance is always zero. Subsequently \( ZT \) will be large due to the small thermal conductance. Therefore, materials with restricted tunnelling channels, or delta-shaped transport distribution have been suggested as potentially good thermoelectric materials\(^\ref{ge}\). However, here we find that the SET-NR coupling can open up extra channels for tunnelling so that the thermal conductance will increase quickly in presence of a quantum NR. From Fig.\(\ref{fig:thermo}\) (d) we can see that \( ZT \) decreases dramatically with the SET-NR coupling strength. These results are explicitly shown in Fig.\(\ref{fig:thermo}\) (a). Here we observe that \( ZT \) decays rapidly (even in the log scale) for both positive and negative \( \varepsilon_0 \). This would cause a severe hindrance to experiments based on constrained tunnelling to enhance \( ZT \), since small (but finite) vibrational couplings are unavoidable in such systems. In this figure the position of the dip in the curve for \( \varepsilon = 4\hbar \omega_0 \) corresponds to the van-
lishing Seebeck coefficient $S$. In other words this is the position at which the energy spacing $\varepsilon_0$ is located exactly at the Fermi-level of the leads, taking the polaron shift into account. Thus the effect of a decreasing $\varepsilon_0$ is to shift the dip towards left-hand side and eventually it vanishes for negative $\varepsilon_0$, in the process causing $\log(ZT)$ to become negative for $\varepsilon_0 = 0$. Even though at small values of SET-NR coupling strength $\lambda$, $\log(ZT)$ rapidly decreases, it saturates for large values of $\lambda$ to $\approx ZT = 1$ and is no longer sensitive to the gate voltage. This saturation behaviour is due to the fact that when the strength of the nonlinearity is quite high, the electronic and thermal conductance are always increasing or decreasing simultaneously as we can see from Figs. 3(a) and Figs. 3(b).

B. Backaction on the NR

The backaction of the electronic current on the NR states is of primary interest due to its effects on device sustainability or measurement sensitivity. The backaction can be strong in some cases, but for detector applications a small backaction is preferred to stabilize the NR leading to accurate measurements. It is therefore of immense interest to build devices in which the mechanical motion can tune the electronic properties effectively while the mechanical system itself is well sustained and not affected by the backaction of the electronic current. In order to study this backaction we investigate the effects on NR vibrational states due to the electronic current passing through the SET-NR island. If the NR is weakly coupled to the SET it will equilibrate to the background temperature. Therefore one would expect that the expectation value of the energy level of the NR, $n_{eq}$, obeys the Boson-Einstein distribution $n_{eq} = \left[ e^{\beta_E \varepsilon} - 1 \right]^{-1}$ with the equilibrium temperature $k_B T_E = \beta_E^{-1}$. This assumption completely fails in the strong SET-NR coupling regime since the charges on the SET will affect the vibrational state of the NR. Hence we generalize the distribution of the NR to capture the strong SET-NR coupling effects.

In order to do this we first obtain the canonical density matrix of the entire SET-NR island which equilibrates to the background environment given by $\rho \propto e^{-\beta_E H_S}$. Only after that we trace over the SET degrees of freedom to get the reduced density matrix of the NR. Using this approach, the distribution of the energy levels of the NR will no longer be a Bose-Einstein distribution but will be exactly given by

$$n_{eq} = \frac{1}{e^{\beta_E h\omega_0} - 1} + \frac{\lambda^2/\beta_E}{e^{\beta_E (\varepsilon_0 - \lambda^2/\beta_E h\omega_0)} + 1}.$$  

(26)

Clearly the first term accounts for the Bose-Einstein distribution, whereas the second term captures the effect of strong SET-NR coupling strength. This term can be understood physically as the formation of a polaron with energy $\lambda^2/(\hbar \omega_0)$. The polaron only exists when an extra charge is present on the island and the polaron energy serves as the chemical potential to that charge. Hence, on average the extra energy applied on the NR, due to the finite SET-NR coupling, follows a Fermi-Dirac distribution multiplied by the polaron energy as given by the second term in Eq. (26). This second term can dominate over the first term in the strong SET-NR coupling regime or in the low temperature regime. In these regimes if the charging energy ($\varepsilon_0$) is smaller than the polaron energy then an increase in temperature would cause the average NR excitation number ($n_{eq}$) to decrease due to the reduced probability of the polaron formation.

In the nonequilibrium transport regime Eq. (26) is no longer valid because the passage of electronic current will disturb the energy distribution of NR. However, the energy distribution of the NR can be obtained numerically with the help of the 0-th order reduced density matrix as, $n_{eq} = \text{Tr}(\rho^{(0)} a^\dagger a)$, where $a$ and $a^\dagger$ are creation and annihilation operators of the NR. This nonequilibrium energy distribution allows us to study the effects of the current on the NR energy distribution, i.e., the backaction, using a distribution difference defined as $\Delta n = n_{eq} - n_{eq}$. Figure 5 shows the contour plot of $\Delta n$ as function of $\varepsilon_0$ and $\lambda$ under both voltage and temperature bias. The red regions represent heating whereas the blue region represents cooling. The yellow regions represents the parameter regime where the backaction is not present as $\Delta n = 0$. As we can see for small values of $\lambda$ the backaction almost vanishes in both cases, which is expected because in this situation the electronic system and the vibrational system are almost decoupled so the NR equilibrates to its background environment. Interesting effects appear in the intermediate to strong SET-NR coupling regime. Figure 5(a) shows $\Delta n$ under the voltage bias condition and we find in this case $\Delta n$ is always positive. However, for temperature bias situation as shown in Fig. 5(b) we find both positive and negative $\Delta n$, which indicates that it is even possible to cool the NR. Importantly, we find that for every $\lambda$ there are two corresponding values of $\varepsilon_0$ where $\Delta n$ vanishes [dashed blue lines in Fig. 5(b)], which
We have investigated the thermoelectric transport properties of a SET coupled to a quantum NR and importantly its dependence on the SET-NR coupling strength. We demonstrated that a quantum NR is capable of suppressing, enhancing or even changing the direction of thermoelectric current of the SET. This is because the NR and the electron form a polaron which can effectively shift the charging energy of the SET. The charging energy controls the dominant carrier type (holes or electrons) and thus leads to a change in the direction of thermoelectric current.

Furthermore, we have shown that even a small SET-NR coupling can dramatically suppress $ZT$ because a finite SET-NR coupling strength will introduce extra tunneling channels for electrons which will greatly enhance the thermal conductance and thus reduce $ZT$. On the other hand, in the strong SET-NR coupling regime we find that $ZT$ saturates and becomes insensitive to the gate voltage.

The backaction of the electronic current on the NR is also examined and we have observed that cooling of the NR is possible for electronic current under a temperature bias (thermoelectric current), whereas in the standard voltage bias regime only heating of the NR is possible. Importantly we also find that it is possible to eliminate the backaction in the parameter regimes where one can effectively generate the thermoelectric current. As a result, we propose that thermoelectric currents can be a possible solution for detector applications, where one needs an electronic current passing through the SET without immensely affecting the NR.

IV. SUMMARY

In this appendix we give explicit formulas for the correlation functions used in our calculations. The information of the electrodes, environment and their coupling to the SET-NR island can be summarized in the spectral density $\Gamma(\omega)$ (for the electronic leads) and $J(\omega)$ (for the environment) given by

$$\Gamma_\alpha(\omega) = 2\pi \sum_{k \in \alpha} |V_k|^2 \delta(\omega - \epsilon_k), \quad \alpha = L, R \quad (A.1)$$

$$J(\omega) = \pi \sum_{n=1}^{\infty} \frac{e_n}{2m_\hbar \omega_n} \delta(\omega - \omega_n). \quad (A.2)$$

In the thermodynamic limit of the leads and the environment the spectral densities become a continuous function and in this work we choose them to be of the form

$$\Gamma_\alpha(\omega) = \frac{\eta_\alpha}{1 + (\omega/\omega_D)^2}, \quad \alpha = L, R \quad (A.3)$$

$$J(\omega) = \frac{\eta_\omega}{1 + (\omega/\omega_D)^2}, \quad (A.4)$$

where $\eta_L, \eta_R$ (units of energy), $\eta_\omega$ (units of $\hbar$) represent the square of the system-lead and system-environment
coupling strength and \( \varepsilon_D \) (omegaD) represents a cut-off energy (frequency) of the Lorentz-Drude form to avoid ultraviolet divergences.\(^{20}\)

Since the vector-operator \( B \) used in the quantum master equation is given by
\[
B = \sum_{k \in L,R} V_k c_k^\dagger \sum_{k \in L,R} V_k c_k - \sum_{n=1} C_n \eta_n
\]
the non-vanishing correlation functions are given by \( C^{12}(t) = \langle B^1(t) B^2(0) \rangle \), \( C^{21}(t) = \langle B^2(t) B^1(0) \rangle \) and \( C^{33}(t) = \langle B^3(t) B^3(0) \rangle \). Using the above definition of the spectral densities these correlation functions can be expressed as
\[
C^{12}(t) = \sum_{\alpha = L,R} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Gamma_\alpha(\epsilon) f_\alpha(\epsilon)e^{i\epsilon t/\hbar}, \quad (A.5)
\]
\[
C^{21}(t) = \sum_{\alpha = L,R} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Gamma_\alpha(\epsilon)(1 - f_\alpha(\epsilon))e^{-i\epsilon t/\hbar}, \quad (A.6)
\]
\[
C^{33}(t) = \int \frac{d\omega}{\pi} J(\omega)n(\omega)e^{i\omega t}, \quad (A.7)
\]

where \( f_\alpha(\epsilon) = \left[ e^{\beta_\alpha(\epsilon - \mu_\alpha)} + 1 \right]^{-1} \) is the Fermi-Dirac distribution of \( \alpha \)-th lead, \( n(\omega) = \left[ e^{\beta_\omega h\omega} - 1 \right]^{-1} \) is the Bose-Einstein distribution and we have assumed that the left and right leads are uncorrelated. In order to obtain the correlation function of the environment we have assumed \( J(-\omega) = -J(\omega) \), which is true for ohmic spectral density chosen in this work. Above \( \beta_\omega = k_B T_\omega^{-1} \) and \( \mu_\alpha \) correspond to the inverse temperature and chemical potential of the leads and the environment with appropriate subscripts \( \alpha \).

In the electronic current operator the correlation functions are only related to the left-lead since we are calculating the current flowing out of the left-lead. Thus the non-zero correlation functions are given by
\[
C^{12}_c(t) = -e \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Gamma_L(\epsilon) f_L(\epsilon)e^{i\epsilon t/\hbar}, \quad (A.8)
\]
\[
C^{21}_c(t) = e \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Gamma_L(\epsilon)(1 - f_L(\epsilon))e^{-i\epsilon t/\hbar}. \quad (A.9)
\]

The non-zero correlation functions in the heat current operator are scaled by a factor of energy and can be expressed as
\[
C^{12}_h(t) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} (\epsilon - \mu_L) \Gamma_L(\epsilon) f_L(\epsilon)e^{i\epsilon t/\hbar}, \quad (A.10)
\]
\[
C^{21}_h(t) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} (\epsilon - \mu_L) \Gamma_L(\epsilon)(1 - f_L(\epsilon))e^{-i\epsilon t/\hbar}. \quad (A.11)
\]

For the spectral densities chosen in this work, Eqs. \([A.3]\) and \([A.4]\), the correlation functions can be explicitly evaluated in terms of the Matsubara summations. Thus, the correlation functions used in the quantum master equation are given by
\[
C^{12}(t) = F_L(t) + F_R(t), \quad (A.12)
\]
\[
C^{21}(t) = \frac{(\eta_L + \eta_R)\varepsilon_D}{2} e^{-\varepsilon_D t/\hbar} - C^{12}(-t). \quad (A.13)
\]

When \( t > 0 \), the function \( F_\alpha(t) \) reads
\[
F_\alpha(t) = \sum_{l = 1,3,5, \ldots} \frac{1}{\beta_\alpha} \frac{i\eta_\alpha \varepsilon_D^{2}}{\left( \mu_\alpha - i\eta_\alpha \varepsilon_D^{2} \right) + \varepsilon_D^{2}} e^{-(\varepsilon_D^{2} + i\eta_\alpha \varepsilon_D^{2}) t/\hbar} + \frac{\eta_\alpha \varepsilon_D^{2}}{2 \left( e^{-\beta_\alpha(\mu_\alpha - i\eta_\alpha \varepsilon_D^{2})} + 1 \right)} e^{-\varepsilon_D t/\hbar}. \quad (A.14)
\]

In case of \( t < 0 \) the function \( F_\alpha(t) \) can be obtained using \( F_\alpha(t) = F_\alpha(-t) \). Here \( \varepsilon_D^{2} = \pi l / \beta_\alpha \) is the Matsubara frequency. The correlation functions used in the current operators are
\[
C^{12}_c(t) = -e F_L(t), \quad (A.15)
\]
\[
C^{21}_c(t) = \frac{e(\eta_L + \eta_R)\varepsilon_D}{2} e^{-\varepsilon_D t/\hbar} + C^{12}_c(-t), \quad (A.16)
\]
\[
C^{12}_h(t) = -H_L(t), \quad (A.17)
\]
\[
C^{21}_h(t) = \frac{\eta_L \varepsilon_D(\varepsilon_D - \mu_L)}{2} e^{-\varepsilon_D t/\hbar} + C^{12}_h(-t), \quad (A.18)
\]

where
\[
H_\alpha(t) = \sum_{l = 1,3,5, \ldots} \frac{1}{\beta_\alpha} \frac{i\eta_\alpha (\varepsilon_D^{2} - \mu_\alpha)}{\left( \mu_\alpha - i\eta_\alpha \varepsilon_D^{2} \right) + \varepsilon_D^{2}} e^{-(\varepsilon_D^{2} + i\eta_\alpha \varepsilon_D^{2}) t/\hbar} + \frac{\eta_\alpha \varepsilon_D^{2}}{2 \left( e^{-\beta_\alpha(\mu_\alpha + i\eta_\alpha \varepsilon_D^{2})} + 1 \right)} e^{-\varepsilon_D t/\hbar}, \quad (A.19)
\]

for \( t > 0 \). Similarly we can use the relation \( H_\alpha(t) = H_\alpha(-t) \) to obtain \( \mathcal{H}_\alpha(t) \) when \( t < 0 \).
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