The Frobenius map, rank 2 vector bundles and Kummer’s quartic surface in characteristic 2 and 3

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November 18, 2018

1 Introduction

Let $X$ be a smooth projective curve of genus 2 defined over an algebraically closed field $k$ of characteristic $p > 0$. The moduli space $M_X$ of semi-stable rank 2 vector bundles with fixed trivial determinant is isomorphic to the linear system $|2\Theta| \cong \mathbb{P}^3$ over $\text{Pic}^1(X)$ and the $k$-linear relative Frobenius map $F : X \to X_1$ induces by pull-back a rational map (the Verschiebung)

$$
\begin{array}{ccc}
M_{X_1} & \xrightarrow{V} & M_X \\
D \downarrow & & \downarrow D \\
|2\Theta_1| & \xrightarrow{\tilde{V}} & |2\Theta|
\end{array}
$$

The vertical maps $D$ are isomorphisms and the Verschiebung $V : E \to F^*E$ coincides via $D$ with a rational map $\tilde{V}$ given by polynomial equations of degree $p$ (Proposition 1.2). The Kummer surfaces $\text{Kum}_X$ and $\text{Kum}_{X_1}$ are canonically contained in the linear systems $|2\Theta|$ and $|2\Theta_1|$ and coincide with the semi-stable boundary of the moduli spaces $M_X$ and $M_{X_1}$. Moreover $\tilde{V}$ maps $\text{Kum}_{X_1}$ onto $\text{Kum}_X$.

Our interest in the diagram (1.1) comes from questions related to the action of the Frobenius map on vector bundles like e.g. surjectivity of $V$, density of Frobenius-stable bundles, loci of Frobenius-destabilized bundles (see [LP]). These questions are largely open when the rank of the bundles, the genus of the curve or the characteristic of the field are arbitrary. In [LP] we made use of the exceptional isomorphism $D : M_X \to |2\Theta|$ in the genus 2, rank 2 case and determined the equations of $\tilde{V}$ when $X$ is an ordinary curve and $p = 2$, which allowed us to answer the above mentioned questions. In this paper we obtain the equations of $\tilde{V}$ in two more cases:

(1) $p = 2$ and $X$ non-ordinary with Hasse-Witt invariant equal to 1,

(2) $p = 3$ and any $X$.

In case (1) we consider a family $\mathcal{X}$ of genus 2 curves parametrized by a discrete valuation ring with ordinary generic fibre $X_\eta$ and special fibre isomorphic to $X$. We obtain the equations of $\tilde{V}$ for $X$ (Theorem 5.1) by specializing the quadrics $P_{ij}$ defining the Verschiebung $V_\eta : M_{X_{1,\eta}} \to M_{X_\eta}$ associated to $\mathcal{X}_\eta$ (section 5). In order to determine the limit of the $P_{ij}$’s we use the explicit formulae (Proposition 3.1) of the coefficients of the $P_{ij}$’s, which coincide with the coefficients of Kummer’s quartic surface $\text{Kum}_{X_\eta}$, in terms of the coefficients of an affine equation of the ordinary curve $X_\eta$. As in the ordinary case we easily deduce from the equations of $\tilde{V}$ a full description of the Verschiebung $V$ (Proposition 5.4).
In case (2) we show that the cubic equations of $\tilde{V}$ are given by the polar equations of a Kummer surface $S \subset |2\Theta_1|$ (Theorem 6.1). Moreover $S$ is isomorphic to $\text{Kum}_X$ and the 16 nodes of $S$ correspond to the 16 base points of $\tilde{V}$. We deduce that $V$ is surjective and of degree 11 (Corollary 6.4).

In the appendix we show that over any smooth curve $X$ of genus $g \geq 2$ defined over an algebraically closed field of characteristic $p > 0$ and for any integer $r \geq 2$, there exist Frobenius-destabilized bundles of rank $r$, i.e., semi-stable bundles $E$ such that $F^*E$ is not semi-stable (Theorem 7.4).

We thank M. Raynaud for helpful discussions.

2 Preliminaries on genus 2 curves in characteristic 2

We consider a smooth ordinary curve $X$ of genus 2 defined over a field $K$ of characteristic 2. We denote by $\overline{K}$ its algebraic closure. Let $k(X)$ be the function field of $X$ and $\omega_X$ the canonical bundle over $X$.

2.1 Weierstrass points

After taking a finite extension of $K$ and applying an automorphism of $\mathbb{P}^1_K$ we can assume that the three Weierstrass points of $X$ are 0, 1 and $\infty$. We consider the birational Abel-Jacobi map

$$AJ : S^2X \rightarrow JX, \quad P_1 + P_2 \mapsto \mathcal{O}_X(P_1 + P_2) \otimes \omega_X^{-1}. \quad (2.1)$$

We observe that the three nonzero elements $[0], [1], [\infty]$ of $JX[2]$ are $K$-rational and are given by

$$[0] := AJ(1 + \infty), \quad [1] := AJ(0 + \infty), \quad [\infty] := AJ(0 + 1). \quad (2.2)$$

For later use we mention that the sheaf of locally exact differential forms (see [R] section 4) equals

$$B = \mathcal{O}_X(0 + 1 + \infty) \otimes \omega_X^{-1}.$$ We recall that $B$ is a theta-characteristic of $X$, i.e., $B^{\otimes 2} \cong \omega_X$.

2.2 Level 2 structure

A level 2 structure is an isomorphism $\psi : JX[2] \xrightarrow{\sim} \mathbb{P}^2_2$. Note that two level 2 structures differ by an automorphism of $\mathbb{P}^2_2$, i.e., an element of $\text{GL}(2, \mathbb{F}_2) = \mathfrak{S}_2$, where $\mathfrak{S}_2$ is the symmetric group. It is well-known that a level 2 structure $\psi$ is equivalent to an ordering of the Weierstrass points of $X$. We refer e.g. to [DO] page 141 for the characteristic zero case, which can easily be adapted to the characteristic two case. Because of the choices involved in the degeneration of $X$ to a non-ordinary curve (section 5.1), we consider the ordering 1, $\infty$, 0 of the Weierstrass points. With the notation of (2.3) the corresponding level 2 structure $\psi$ is given by

$$\psi([0]) = (1, 0), \quad \psi([1]) = (0, 1), \quad \psi([\infty]) = (1, 1). \quad (2.3)$$

A level 2 structure $\psi$ allows us to construct ([LP] section 2) the theta basis $\{X_g\}_{g \in \mathbb{F}_2^*}$ of the space $H^0(JX, 2\Theta)$. We denote the four sections by $X_B, X_0, X_1, X_\infty$ and introduce the rational functions $Z_\bullet \in k(JX)$ defined by

$$Z_0 = \frac{X_0}{X_B}, \quad Z_1 = \frac{X_1}{X_B}, \quad Z_\infty = \frac{X_\infty}{X_B}. \quad (2.4)$$

We recall that $X_B$ is the theta function with associated zero divisor $2\Theta_B$, where

$$\Theta_B := \{L \in JX \mid h^0(X, L \otimes B) \geq 1\}.$$
2.3 Birational models

Let \( X \) be an ordinary smooth curve of genus 2 defined over \( \overline{K} \) and \( \psi \) a level 2 structure. It follows from [L] page 28 and [B] Proposition 1.4 that the pair \( (X, \psi) \) is uniquely represented by an affine equation of the form

\[
y^2 + x(x+1)y = x(x+1)(ax^3 + (a+b)x^2 + cx + c),
\]

with \( a, b, c \in \overline{K} \). Moreover if \( X \) is defined over \( K \), the coefficients \( a, b, c \) lie in a finite extension of \( K \). The next lemma is an immediate consequence of [B] Proposition 1.5.

2.1. Lemma. The curve \( X \) defined by the equation (2.5) is smooth if and only if \( abc \neq 0 \).

Let \( \tilde{M}_3 \) denote the moduli space parametrizing pairs \( (X, \psi) \) of smooth ordinary genus 2 curves \( X \) defined over \( \overline{K} \) equipped with a level 2 structure. It follows from the previous remark that \( \tilde{M}_3 \) is an affine variety,

\[
\tilde{M}_3 = \text{Spec} \, \overline{K}[a, b, c, \frac{1}{abc}].
\]

Fixing the curve \( X \) the symmetric group \( S_3 \) acts naturally on the level 2 structures \( \psi \). It can be shown that this \( S_3 \)-action on \( \tilde{M}_3 \) coincides with the permutation action of \( S_3 \) on the coefficients \( a, b, c \).

2.4 Normal form

We introduce the rational function \( Y \in k(X) \) defined by \( Y = \frac{y}{x(x+1)} \). Then (2.5) becomes

\[
Y^2 + Y = R(x), \quad \text{with} \quad R(x) = \frac{ax^3 + (a+b)x^2 + cx + c}{x(x+1)}.
\]

We also observe that, given a polynomial \( S(x) \), the involution \( i_S : \mathbb{A}^2_k \rightarrow \mathbb{A}^2_k, (x, y) \mapsto (x, y+x(x+1)S(x)) \) transforms the equation (2.5) of \( X \) into \( Y^2 + Y = R + S^2 + S \).

2.5 Kummer’s quartic equation

For a pair \( (X, \psi) \) it has been shown in [LP] Proposition 4.1 that there exist constants \( \lambda_0, \lambda_1, \lambda_\infty \in \overline{K} \) such that the following equality holds in \( k(JX) \),

\[
\lambda_0^2(Z_0^2 + Z_1^2Z_\infty^2) + \lambda_1^2(Z_1^2 + Z_0^2Z_\infty^2) + \lambda_\infty^2(Z_\infty^2 + Z_0^2Z_1^2) + \lambda_0\lambda_1\lambda_\infty Z_0Z_1Z_\infty = 0.
\]

The constants \( \lambda_0, \lambda_1, \lambda_\infty \) are related via \( \psi \) (2.3) to the \( \{\lambda_g\}_{g \in \mathbb{F}_2^2} \) used in [LP] Proposition 4.1 as follows: \( \lambda_0 = \frac{\lambda_{g_0}}{\lambda_{g_0}}, \lambda_1 = \frac{\lambda_{g_1}}{\lambda_{g_0}}, \lambda_\infty = \frac{\lambda_{g_\infty}}{\lambda_{g_0}} \).

3 The coefficients \( \lambda_\ast \) of Kummer’s quartic surface \( \text{Kum}_X \) for ordinary \( X \)

3.1. Proposition. Given a curve \( X \) with a level 2 structure \( \psi \) represented by an affine equation (2.5) with \( a, b, c \in K \). Then the coefficients of the equation (2.7) of its Kummer surface \( \text{Kum}_X \) are

\[
\lambda_0^2 = \frac{1}{ab}, \quad \lambda_1^2 = \frac{1}{ac}, \quad \lambda_\infty^2 = \frac{1}{bc}.
\]
Let \( \{x_i\} \) be the dual basis of the theta basis \( \{X_g\}_{g \in \mathbb{P}^2} \) of \(|2\Theta| = \mathbb{P}^3\). Then the homogeneous equation of \( \text{Kum}_X \) is
\[
c(x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2) + b(x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2) + a(x_{00}^2x_{11}^2 + x_{01}^2x_{10}^2) + x_{00}x_{01}x_{10}x_{11} = 0.
\]

The idea of the proof is to consider the pull-back of the rational functions \( Z, \) \( \text{(2.4)} \) by the Abel-Jacobi map \( \text{(2.1)} \) to the symmetric product \( S^2X \) and to do the computations in the function field \( k(S^2X) \leftrightarrow k(X \times X). \) Since \( X \) is given by the equation \( \text{(2.3)} \), we have natural coordinates \( x_1, y_1 \) and \( x_2, y_2 \) on \( X \times X. \) For notational convenience, we also use \( Y_i = \frac{y_i}{x_i(x_i+1)} \) for \( i = 1, 2. \) We use two lemmas.

### 3.2. Lemma

Suppose that there exist polynomials \( A, B \in K[x_1, x_2] \), which satisfy
\[
(Y_1 + Y_2)A(x_1, x_2) = B(x_1, x_2).
\]
Then \( A = B = 0. \)

**Proof.** Squaring the relation \( \text{(3.1)} \) and using \( \text{(2.6)} \) leads to the equation
\[
(Y_1 + Y_2)A^2 + (R(x_1) + R(x_2))A^2 + B^2 = 0.
\]
Applying again \( \text{(3.2)} \), the first term transforms into \( AB. \) Clearing denominators, we arrive at a polynomial equation, which only holds if \( A = B = 0; \) e.g. take the total degree of \( A \) and \( B. \) \( \Box \)

### 3.3. Lemma

The pull-back by the Abel-Jacobi map \( AJ \) of the rational function \( Z_\infty \in k(JX) \) equals
\[
AJ^*(Z_\infty) = \alpha_\infty \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2}, \quad \text{with } P(x_1, x_2) = \frac{(x_1 + x_2)^2}{x_1x_2(x_1 + 1)(x_2 + 1)}
\]

Similar we have
\[
AJ^*(Z_0) = \alpha_0 \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2}, \quad AJ^*(Z_1) = \alpha_1 \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2},
\]
for some nonzero constants \( \alpha_0, \alpha_1, \alpha_\infty \in K. \)

**Proof.** The first equality follows immediately from Theorem 2 [AG] and the other two from Proposition 5 [AG]. \( \Box \)

**Proof of Proposition 3.7.** We write \( Q = \frac{P(x_1, x_2)}{(Y_1 + Y_2)^2}. \) Using Lemma 3.3 the pull-back to \( S^2X \) of the equation \( \text{(2.7)} \) equals
\[
\lambda_0^2 \left[ \alpha_0^2a_1^2x_2^3Q^2 + \alpha_0^2\alpha_\infty^2(x_1 + 1)^2(x_2 + 1)^2Q^4 \right] + \lambda_1^2 \left[ \alpha_1^2(x_1 + 1)^2(x_2 + 1)^2Q^2 + \alpha_0^2\alpha_\infty^2x_1^2x_2^2Q^4 \right] + \lambda_\infty^2 \left[ \alpha_\infty^2Q^2 + \alpha_0^2\alpha_1^2x_1^2x_2^2(x_1 + 1)^2(x_2 + 1)^2Q^4 \right] + \lambda_0\lambda_1\lambda_\infty \left[ \alpha_0\alpha_1\alpha_\infty x_1x_2(x_1 + 1)(x_2 + 1)Q^3 \right] = 0
\]
We divide by \( Q^2 \) and multiply by \( (Y_1 + Y_2)^4, \)
\[
(Y_1 + Y_2)^4 \left[ \lambda_0^2\alpha_0^2x_1^2x_2^2 + \lambda_1^2\alpha_1^2(x_1 + 1)^2(x_2 + 1)^2 + \lambda_\infty^2\alpha_\infty^2 \right] + (Y_1 + Y_2)^2 \left[ \lambda_0\lambda_1\lambda_\infty\alpha_0\alpha_1\alpha_\infty P_1x_1x_2(x_1 + 1)(x_2 + 1) + S \right] = 0.
\]
where \( S \) is the sum of the remaining terms (not containing \( Y_1, Y_2). \) After taking the square root (note that the entire expression is a square of a polynomial in the \( x_i \)’s and \( Y_i \)’s), applying \( \text{(2.6)} \) and Lemma 3.2, we obtain that the coefficients of \( (Y_1 + Y_2)^2 \) and \( (Y_1 + Y_2)^4 \) are the same, i.e.,
\[
\lambda_0^2\alpha_0^2x_1^2x_2^2 + \lambda_1^2\alpha_1^2(x_1 + 1)^2(x_2 + 1)^2 + \lambda_\infty^2\alpha_\infty^2 = \lambda_0\lambda_1\lambda_\infty\alpha_0\alpha_1\alpha_\infty P_1x_1x_2(x_1 + 1)(x_2 + 1).
\]
An easy computation shows that this equality only holds if
\[ \lambda_0 a_0 = \lambda_1 a_1 = \lambda_\infty a_\infty = 1. \]

Now we replace the \( \alpha \)'s and the sum \( S \) by their expressions in the square root of the equation \((3.2)\)
\[
(P \left[ \frac{\lambda_0}{\lambda_1 \lambda_\infty} (x_1 + 1)(x_2 + 1) + \frac{\lambda_1}{\lambda_0 \lambda_\infty} x_1 x_2 + \frac{\lambda_\infty}{\lambda_0 \lambda_1} x_1 x_2 (x_1 + 1)(x_2 + 1) \right] = 0. \]

We introduce the constants \( \mu_0, \mu_1, \mu_\infty \in K \) defined by \( \mu_\infty = \frac{\lambda_\infty}{\lambda_0 \lambda_1}, \mu_0 = \frac{\lambda_0}{\lambda_1 \lambda_\infty}, \mu_1 = \frac{\lambda_1}{\lambda_0 \lambda_\infty} \). The previous equality becomes after replacing \( P \) by its expression and dividing by \((x_1 + x_2)\)
\[
\left[ Y_1^2 + Y_1 + \frac{\mu_0}{x_1} + \frac{\mu_1}{x_1 + 1} + \mu_\infty x_1 \right] + \left[ Y_2^2 + Y_2 + \frac{\mu_0}{x_2} + \frac{\mu_1}{x_2 + 1} + \mu_\infty x_2 \right] = 0. \]

This equation holds in \( k(X \times X) \) and since the variables \((x_1, Y_1)\) and \((x_2, Y_2)\) are separated, each of the two terms equals zero. So we can drop the indices and we obtain an equation
\[
Y^2 + y = \mu_\infty x + \frac{\mu_0}{x} + \frac{\mu_1}{x + 1} = \frac{\mu_\infty x^3 + \mu_\infty x^2 + (\mu_0 + \mu_1)x + \mu_0}{x(x + 1)}, \quad \text{(3.3)}
\]
which has to be equivalent (after applying an automorphism of \( \mathbb{A}^2_K \)) to the normal form \((2.6)\) of the equation of \( X \). The automorphism is given by \( i_s \) (see section \( \text{2.4} \)) with \( S(x) = s \in \overline{K} \) satisfying \( s^2 + s = \mu_1 \). Hence \((2.3)\) is equivalent to \( Y^2 + Y = R(x) \) with \( R(x) = \frac{\mu_\infty x^3 + (\mu_0 + \mu_1)x^2 + \mu_0 x + \mu_0}{x(x + 1)} \).

Hence by uniqueness of the normal form, we obtain \( a = \mu_\infty, b = \mu_1, c = \mu_0 \) and therefore also the relations claimed in the proposition.

\[\Box\]

### 4 Deformation of genus 2 curves

Let \( X/k \) be a smooth curve with Hasse-Witt invariant equal to 1. We introduce a family \( \mathcal{X} \) over \( R = k[[t]] \) such that the special fibre \( \mathcal{X}_0 \) is isomorphic to \( \mathcal{X} \) and the generic fibre \( \mathcal{X}_\eta \) is an ordinary genus 2 curve. Here \( \eta \) (resp. 0) is the generic (resp. closed) point of \( \text{Spec}(R) \). In section 5.1 we give an example of a family \( \mathcal{X} \) with given special fibre \( X \). Let \( \mathcal{JX} \) be its associated Jacobian scheme over \( \text{Spec}(R) \).

#### 4.1 2-divisible groups

Let \( \mathcal{JX}(2) \) be the 2-divisible group of \( \mathcal{JX} \), which is finite and flat over \( \text{Spec}(R) \). We consider the canonical exact sequence
\[
0 \longrightarrow \mathcal{JX}(2)^0 \longrightarrow \mathcal{JX}(2) \longrightarrow \mathcal{JX}(2)^{et} \longrightarrow 0, \quad \text{(4.1)}
\]
where \( \mathcal{JX}(2)^0 \) (resp. \( \mathcal{JX}(2)^{et} \)) is a connected (resp. étale) 2-divisible group. Taking again the connected component of the Cartier dual of \( \mathcal{JX}(2)^0 \) we obtain a filtration
\[
\mathcal{JX}(2)^{00} \subset \mathcal{JX}(2)^0 \subset \mathcal{JX}(2),
\]
with quotients given by the 2-divisible groups
\[
\mathcal{JX}(2)/\mathcal{JX}(2)^0 = \mathcal{JX}(2)^{et} \cong \mathbb{Q}_2/\mathbb{Z}_2, \quad \mathcal{JX}(2)^0/\mathcal{JX}(2)^{00} \cong \mathbb{G}_m(2).
\]
The 2-divisible group \( J(X)(2)_{\mathfrak{o}} \) is self-dual, of dimension 1 and of height 2. Because of the uniqueness of 2-divisible groups over \( k \) with these properties (see e.g. [4] Examples page 93), the special fibre \( J(X)(2)_{\mathfrak{o}} \) (\( = J(X)(2)_{\mathfrak{o}} \otimes_R k \)) is isomorphic to the 2-divisible group associated to the supersingular elliptic curve \( E^{ss}/k \). We recall that there exists a unique (up to isomorphism) supersingular curve \( E^{ss} \), which is defined by \( j = 0 \). Therefore by a theorem of Serre-Tate [4], there exists an elliptic curve \( E_X \) over \( \text{Spec}(R) \) such that \( (E_X)_0 \cong E^{ss} \) and the associated 2-divisible group \( E_X(2) \) is isomorphic to \( J(X)(2)_{\mathfrak{o}} \) over \( \text{Spec}(R) \).

### 4.2 Deformation of elliptic curves

In this section we compute the linear action of the 2-torsion point of \( E \) on the space of second order theta functions \( H^0(E, 2\Theta) \) for a family of elliptic curves \( E/\text{Spec}(R) \) with supersingular special fibre \( E_0 \cong E^{ss} \) and ordinary generic fibre \( E_\eta \).

#### 4.2.1 Addition on an ordinary elliptic curve

Let \( E \) be an ordinary elliptic curve defined over a field \( K \) by the homogeneous equation

\[
Y^2Z + a_1XYZ = X^3 + a_2X^2Z + a_4XZ^2
\]

(4.2)

with \( a_1, a_2, a_4 \in K \) and \( a_1 \neq 0 \). We take as origin the inflection point \( \infty \) with projective coordinates \((0 : 1 : 0)\). The projection with center \( \infty \) gives a 2:1 morphism \( E \to P^1_K \), with \( X, Z \) projective coordinates on \( P^1_K \). The Abel-Jacobi map \( E \to JE, e \mapsto O_E(e - \infty) \) identifies \( E \) with \( JE \). Under this identification the theta divisor \( \Theta_B \), associated to the canonical theta-characteristic \( B = O_E(P - \infty) \in JE[2] \), becomes the 2-torsion point \( P \) with projective coordinates \((0 : 0 : 1)\). Moreover, using this identification, we have \( O_{JE}(2\Theta) = O_E(2P) = \pi^*O_{P^1}(1) \).

The point \( B \in JE[2] \) induces a linear involution, denoted by \( g \), on the space

\[
W = H^0(JE, O(2\Theta)) = H^0(E, \pi^*O_{P^1}(1)),
\]

(4.3)

such that for all nonzero \( s \in W \) we have \( T_B \text{div}(s) = \text{div}(gs) \). Here \( T_B \) denotes translation in \( JE \) by the point \( B \). The space \( W \) has two distinguished bases: first the coordinate functions \( \{X, Z\} \) and secondly the Theta basis \( \{X_0, X_1\} \) (see [5] section 2). Since the canonical section \( X_0 \in W \) (associated to the divisor \( \Theta_B \)) is proportional to \( X \), there exists a nonzero \( a \in K \) such that

\[
g.X = aZ, \quad g.Z = a^{-1}X.
\]

In order to determine \( a \) in terms of the coefficients \( a_i \in K \), we choose one of the two points on \( E \) with projective coordinates of the form \((1 : Y : 1)\) and call it \( A \). By construction we have \( A \in \text{div}(X + Z) \) and, after applying \( T_B \), we obtain \( A + P \in T_B \text{div}(X + Z) \). Since

\[
T_B \text{div}(X + Z) = \text{div}(g.X + g.Z) = \text{div}(aZ + a^{-1}X),
\]

we deduce that

\[
\left(\begin{array}{c} X \\ Z \end{array}\right) (A + P) = a^2.
\]

Now the addition formula for elliptic curves (see e.g. [4] page 59) implies that \( \left(\begin{array}{c} X \\ Z \end{array}\right) (A + P) = a_4 \).

Hence \( a = (a_4)^{1/2} \) and the Theta basis of \( W \) is given by

\[
X_0 = X, \quad X_1 = g.X_0 = (a_4)^{1/2}Z.
\]

(4.4)
4.2.2 An example

We consider the family of elliptic curves $\mathcal{E}$ over $\text{Spec}(R)$ defined by the homogeneous equation

$$V^2 Z + t^4 U V Z + V Z^2 = U^3 + U Z^2 \quad (4.5)$$

with origin $\infty = (0 : 1 : 0)$. The generic fiber $\mathcal{E}_\eta$ is an ordinary elliptic curve over $\text{Spec}(K)$, with $K = k((t))$, and the special fiber is supersingular, i.e., $\mathcal{E}_0 \cong E^{ss}$. The 2-torsion point $P_\eta$ of $\mathcal{E}_\eta$ has projective coordinates

$$P_\eta = t^6(u_0 : v_0 : 1),$$

with $u_0 = t^{-4}$, $v_0 = t^{-2} + t^{-6}$, which specializes to $\infty_0 \in \mathcal{E}_0$.

The $R$-module $H^0(\mathcal{E}, 2\Theta)$ is free, of rank 2, and an $R$-basis is given by $\{U, Z\}$. In order to compute the linear action $g$ of the 2-torsion point $P \in \mathcal{E}$ on $H^0(\mathcal{E}, 2\Theta)$, we consider the generic fibre $\mathcal{E}_\eta$. The change of variables $X = U + u_0 Z$ and $Y = V + v_0 Z$ transforms equation (4.5) into (4.3), with $a_1 = t^4$, $a_2 = u_0$, and $a_4 = 1 + u_0^2 + t^4 v_0 = 1 + t^{-8} + t^2 + t^{-2}$. With the notation of section 4.2.1 we find $a = t^4(1 + t^3 + t^4 + t^5) = (a_4)^{1/2}$. Therefore the action of $g$ on $H^0(\mathcal{E}, 2\Theta)$ is given by the formulae

$$g.U = \frac{1}{1 + t^3 + t^4 + t^5} U + \frac{t^2 + t^4 + t^6}{1 + t^3 + t^4 + t^5} Z,$$

$$g.Z = \frac{t^4}{1 + t^3 + t^4 + t^5} U + \frac{1}{1 + t^3 + t^4 + t^5} Z,$$

and, using (4.4), the theta functions $X_0$ and $X_1$ can be expressed in the $R$-basis $\{U, Z\}$ as follows

$$X_0 = t^4 U + Z,$$
$$X_1 = g.X_0 = (1 + t^3 + t^4 + t^5) Z.$$

Note that the two sections $X_0 \otimes R k$ and $X_1 \otimes R k$ coincide at the special fibre $H^0(\mathcal{E}_0, 2\Theta|_{\mathcal{E}_0}) \cong H^0(\mathcal{E}, 2\Theta) \otimes_R k$.

5 Equations of $\tilde{V}$ for non-ordinary $X$

5.1 Specializing an ordinary curve

Let $X/k$ be a smooth genus 2 curve with Hasse-Witt invariant equal to 1. Following e.g. [L] $X$ is birational to an affine curve given by an equation of the form

$$y^2 + x y = \lambda x^5 + \mu x^3 + x,$$

with $\lambda, \mu \in k$ and $\lambda \neq 0$. The projection $(x, y) \mapsto x$ defines a separable double cover $X \to \mathbb{P}^1_k$ ramified at 0 and $\infty$. Let $\mathbb{P}^1_R$ be the projective line over $R = k[[s]]$ with affine coordinate $x$. We introduce the family $\mathcal{X} \to \mathbb{P}^1_R$ defined by the projective closure of the affine curve with equation

$$y^2 + (sx^2 + x)y = \lambda x^5 + \mu x^3 + x.$$

The special fibre $\mathcal{X}_0/k$ equals $X$ and the generic fibre $\mathcal{X}_\eta/K$ of the family $\mathcal{X}$ is a smooth ordinary curve of genus 2, which is birational to the curve (defined over a finite extension of $K$) given by the standard equation (2.3) with coefficients

$$a = \lambda/s^3, \quad b = \alpha^2 + \alpha, \quad c = s, \quad \text{and} \quad \alpha^2 = \lambda/s^3 + \mu/s + s. \quad (5.1)$$
Let \( \mathcal{J} \mathcal{X} \) be the associated Jacobian scheme and \( \mathcal{J} \mathcal{X}[2]/R \) be the group scheme of 2-torsion points. Then we have the following isomorphisms

\[
\mathcal{J} \mathcal{X}[2]_\eta \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2^2/K, \quad \mathcal{J} \mathcal{X}[2]_0 \cong JX[2] \cong (\mathbb{Z}/2\mathbb{Z}) \times \mu_2 \times G_{1,1}/k,
\]

where \( G_{1,1} \) is the unique self-dual local-local group scheme of dimension 1 and length 4. Note that \( G_{1,1} \) is isomorphic to the group scheme of 2-torsion points \( E^{ss}[2] \) (see section 4.2.2). The étale parts of both fibres can be described in terms of Weierstrass points as follows.

The 3 Weierstrass points of \( \mathcal{X}_\eta \to \mathbb{P}^1_K \) are \( 0_\eta, \infty_\eta \), and \( 1_\eta \) with affine coordinate \( 1/s \), which specialize to \( 0, \infty \) and \( \infty \) respectively. We obtain by (5.2) the three nonzero elements of \( \mathcal{J} \mathcal{X}[2]^{et}_\eta \), which we denote by \([0]_\eta, [1]_\eta \) and \([\infty]_\eta \). At the special fibre the nonzero 2-torsion point in \( JX[2]^{et}_\eta \cong \mathbb{Z}/2\mathbb{Z} \) equals \( AJ(0 + \infty) \), which we denote by \([1]_0 \). We see that \([1]_\eta \) and \([\infty]_\eta \) specialize to \([1]_0 \in JX[2]^{et}, \) and \([0]_\eta \) specializing to 0.

### 5.2 Decomposing Heisenberg groups

We are interested in the linear action of the Heisenberg group scheme \( \mathcal{H}/R \), which is a central extension (see [M] page 221)

\[
0 \longrightarrow \mu_2 \longrightarrow \mathcal{H} \longrightarrow \mathcal{J} \mathcal{X}[2] \longrightarrow 0, \hspace{1cm} (5.2)
\]
on the free \( R \)-module \( \mathcal{W} = H^0(\mathcal{J} \mathcal{X}[2], 2\Theta) \) of rank 4. We choose the splitting over \( R \) of the connected-étale exact sequence

\[
0 \longrightarrow \mathcal{J} \mathcal{X}[2]^0 \longrightarrow \mathcal{J} \mathcal{X}[2] \longrightarrow \mathcal{J} \mathcal{X}[2]^{et} \longrightarrow 0
\]
determined by the nonzero 2-torsion point \([1] = AJ(0 + \infty) \in \mathcal{J} \mathcal{X}[2] \). Note that \( \mathcal{J} \mathcal{X}[2]^{et} \cong \mathbb{Z}/2\mathbb{Z} \) and that \([\infty] \in \mathcal{J} \mathcal{X}[2] \) determines a different splitting. Passing to the Cartier dual we obtain a decomposition over \( R \),

\[
\mathcal{J} \mathcal{X}[2] = \mathbb{Z}/2\mathbb{Z} \times \mu_2 \times \mathcal{J} \mathcal{X}[2]^{00}.
\]

Pulling-back the central extension (5.2) by the canonical inclusions of \( \mathbb{Z}/2\mathbb{Z} \times \mu_2 \) and \( \mathcal{J} \mathcal{X}[2]^{00} \) into \( \mathcal{J} \mathcal{X}[2] \) we obtain the two Heisenberg groups \( \mathcal{H}^{et} \) and \( \mathcal{H}^{0} \)

\[
0 \longrightarrow \mu_2 \longrightarrow \mathcal{H}^{et} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mu_2 \longrightarrow 0, \quad 0 \longrightarrow \mu_2 \longrightarrow \mathcal{H}^{0} \longrightarrow \mathcal{J} \mathcal{X}[2]^{00} \longrightarrow 0.
\]

It is clear that the Heisenberg group scheme \( \mathcal{H} \) is isomorphic to the quotient \( \mathcal{H}^0 \times \mathcal{H}^{et}/\mu_2 \), where \( \mu_2 \) acts diagonally on \( \mathcal{H}^0 \times \mathcal{H}^{et} \). Let \( \mathcal{W}^{et} \) and \( \mathcal{W}^0 \) be the sub-\( R \)-modules of \( \mathcal{W} \) fixed by the subgroups \( \mathcal{H}^0 \) and \( \mathcal{H}^{et} \) of \( \mathcal{H} \). By general theory of Heisenberg groups, \( \mathcal{W} \) is the unique irreducible \( \mathcal{H} \)-module of weight 1, which implies an \( \mathcal{H} \)-isomorphism

\[
\mathcal{W} \cong \mathcal{W}^0 \otimes \mathcal{W}^{et}.
\]

Moreover \( \mathcal{W}^0 \) (resp. \( \mathcal{W}^{et} \)) is the unique irreducible \( \mathcal{H}^0 \) (resp. \( \mathcal{H}^{et} \))-module of weight 1.

Let \( H/k \) be the Heisenberg group scheme associated to \( \mathbb{Z}/2\mathbb{Z} \times \mu_2 \) (\( \cong E[2] \) for any ordinary elliptic curve \( E/k \)) and let \( W \) be the unique irreducible \( H \)-module of weight 1. Note that \( W \) is isomorphic (as \( H \)-module) to the space \( \{1, 3\} \). It is clear that we have the following isomorphisms

\[
\mathcal{H}^{et} \cong H \otimes_k R, \quad \mathcal{W}^{et} \cong W \otimes_k R.
\]

We will denote by \( \{Z_0, Z_1\} \) the “constant” \( R \)-basis of \( \mathcal{W}^{et} \) induced by the Theta basis \( \{X_0, X_1\} \) of \( W \), i.e., \( Z_i := X_i \otimes_k 1 \).
The structure of the $\mathcal{H}^0$-module $\mathcal{W}^0$ is determined by analyzing the group scheme $\mathcal{J}\mathcal{X}[2]^{00}$. In section 4.1 we considered the 2-divisible group $\mathcal{J}\mathcal{X}(2)^{00}$ and we showed the existence of an elliptic curve $\mathcal{E}_X/R$ such that $\mathcal{E}_X(2) \cong \mathcal{J}\mathcal{X}(2)^{00}$. In particular $\mathcal{E}_X[2] \cong \mathcal{J}\mathcal{X}[2]^{00}$. We observe that the $j$-invariants of the elliptic curves $\mathcal{E}_X/R$ and $\mathcal{E}/R$ (section 4.2.2) lie in the maximal ideal of $R$, because $(\mathcal{E}_X)_0 \cong \mathcal{E}_0 \cong E^{ss}$. Therefore there exists a relation of the form

$$s^n = ut^m$$

(5.3)

between the two uniformizing parameters $s$ in (5.1) and $t$ in (1.3), with $u$ invertible in $k[[t]]$ and $n, m \in \mathbb{N}^*$, and we can assume, after passing to the ramified cover given by (5.3), that the $\mathcal{H}^0$-module $\mathcal{W}^0$ equals $H^0(\mathcal{E}, 2\Theta)$.

In order to have a consistent notation we denote the $R$-basis $\{U, Z\}$ of $\mathcal{W}^0 = H^0(\mathcal{E}, 2\Theta)$ by $\{Z_0, Z_1\}$ and recall from section 4.2.2 the transition formulae

$$X_0 = t^4 Z_0 + Z_1, \quad X_1 = (1 + t^3 + t^4 + t^5) Z_1.$$  

(5.4)

Let $\{x_i\}$ and $\{z_i\}$ denote the dual $K$-bases of $\{X_i\}$ and $\{Z_i\}$ in both spaces $\mathcal{W}^0$ and $\mathcal{W}^*$. Then the 4 tensors $z_{ij} := z_i \otimes z_j \in \mathcal{W}^*$ form an $R$-basis and the dual Theta functions $x_{ij} := x_i \otimes x_j$ can be expressed as follows (after chasing denominators)

$$x_{00} = (1 + t^3 + t^4 + t^5) z_{00}, \quad x_{10} = z_{00} + t^4 z_{10},$$

$$x_{01} = (1 + t^3 + t^4 + t^5) z_{01}, \quad x_{11} = z_{01} + t^4 z_{11}.$$  

(5.5)

Note that the coordinate $x_{10}$ specializes to $x_{00}$ and $x_{11}$ specializes to $x_{01}$. Via the level 2 structure (2.3) this parallels the specialization of the 2-torsion points $[0], [1]$, and $[\infty]$ (section 5.1).

### 5.3 Specializing the quadrics

It can be shown as in [LP] section 5 that the identification $\mathcal{M}_X \to |2\Theta|$ extends to the relative case $\mathcal{X} \to \text{Spec}(R)$, i.e., we have an isomorphism $\mathcal{M}_X \to \mathbb{P}(\mathcal{W})$ over $\text{Spec}(R)$. Therefore the relative Frobenius morphism $\mathcal{X} \to \mathcal{X}_1$ (over $\text{Spec}(R)$) induces by pull-back a rational map

$$\mathbb{P}(\mathcal{W}_1) \to \mathbb{P}(\mathcal{W})$$

$$\downarrow \quad \mathcal{V} \quad \downarrow$$

$$\text{Spec}(R)$$

with $\mathcal{W}_1 = H^0(\mathcal{J}\mathcal{X}_1, 2\Theta_1)$. We recall that the map $\mathcal{V}$ is given by a linear system of 4 quadrics. Over the generic point $\eta \in \text{Spec}(R)$ the equations of the map $\mathcal{V}_\eta : \mathbb{P}(\mathcal{W}_1)_\eta \to \mathbb{P}(\mathcal{W})_\eta$ are of the form ([LP] Proposition 3.1 (3))

$$\mathcal{V} : x = (x_{ij}) \mapsto (\lambda_{00} P_{00}(x) : \lambda_{01} P_{01}(x) : \lambda_{10} P_{10}(x) : \lambda_{11} P_{11}(x))$$

(5.6)

with

$$P_{00}(x) = x_{00}^2 + x_{01}^2 + x_{20}^2 + x_{21}^2, \quad P_{10}(x) = x_{00} x_{10} + x_{01} x_{11},$$

$$P_{01}(x) = x_{00} x_{01} + x_{10} x_{11}, \quad P_{11}(x) = x_{00} x_{11} + x_{10} x_{01}.$$  

Here we use the $K$-basis of Theta coordinates $x_{ij}$ on $\mathbb{P}(\mathcal{W}_1)_\eta$ and $\mathbb{P}(\mathcal{W})_\eta$. Moreover Proposition 3.1 relates the coefficients $\lambda_{ij}$ (defined up to a scalar) to the coefficients $a, b, c \in K$ ([LP]),

$$(\lambda_{00} : \lambda_{01} : \lambda_{10} : \lambda_{11}) = (\sqrt{abc} : \sqrt{b} : \sqrt{c} : \sqrt{a}).$$
Since the Theta coordinates $x_{ij}$ are no longer independent after specialization, we express the equations (5.6) of $\tilde{V}$ in the $R$-basis $\{z_{ij}\}$ using the transition formulae (5.7). A straight-forward computations shows that the map

$$\tilde{V} : z = (z_{ij}) \mapsto (R_{00}(z) : R_{01}(z) : R_{10}(z) : R_{11}(z))$$

is given by the quadrics

$$
\begin{align*}
R_{00}(z) &= \frac{\sqrt{abc}}{1 + t^c + t^d + t^e} [(t^{12} + t^{16} + t^{20})(z_{00}^2 + z_{01}^2) + t^{16}(z_{10}^2 + z_{11}^2)], \\
R_{01}(z) &= \frac{\sqrt{abc}}{1 + t^c + t^d + t^e} [(t^{12} + t^{16} + t^{20})z_{00}z_{01} + t^8(z_{00}z_{11} + z_{10}z_{01}) + t^{16}z_{10}z_{11}], \\
R_{10}(z) &= \frac{1}{t^4} [R_{00} + \sqrt[2]{c}(1 + t^d + t^8 + t^{10}) (z_{00}^2 + z_{01}^2 + t^8(z_{00}z_{10} + z_{10}z_{01}))], \\
R_{11}(z) &= \frac{1}{t^4} [R_{01} + \sqrt[2]{c}(1 + t^d + t^8 + t^{10})t^8(z_{00}z_{11} + z_{01}z_{10})].
\end{align*}
$$

Note that we square the coefficients in (5.5) when considering coordinates on $\mathbb{P}(V_1)$. At the special fibre the map (5.7) specializes to $\tilde{V}$ coincides with the Verschiebung $\tilde{V}$ of the curve $X$, because both maps extend to rational maps over $R$ and coincide over $K$. Since the image of $\tilde{V}$ is non-degenerate (it contains the Kummer surface $K\text{um}_X \subset |2\Theta|$), the lowest valuations for each of the 4 quadrics $R_{ij}$ coincide (otherwise the image of $\tilde{V}$ is contained in a hyperplane).

We work out the specialization of the quadrics as follows. We write $\nu = \frac{\nu}{(s)}$ and replace $s$ by $vt^s$, with $v \in k[[t]]$ invertible, in the expression of the coefficients $a, b, c$ of (5.1). Note that the (rational) valuations of $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are $-\frac{3}{2}\nu, -\frac{5}{2}\nu, \frac{3}{2}\nu$ respectively. First we observe that the valuations of $R_{01}$ and $R_{00}$ equal $8 - \frac{3}{2}\nu$ and $12 - \frac{5}{2}\nu$ respectively. Since they coincide, we obtain $\nu = 4$, i.e.,

$$R_{00} = t^2(z_{00}^2 + z_{01}^2) + \text{h.o.t.}, \quad R_{01} = t^2(z_{00}z_{11} + z_{10}z_{01}) + \text{h.o.t.},$$

up to some multiplicative nonzero constants. Next we see that the expansions of $R_{10}$ and $R_{11}$ are given by

$$R_{10} = t^2(z_{00}^2 + z_{01}^2 + z_{10}^2 + z_{11}^2) + \text{h.o.t.}, \quad R_{11} = t^2(z_{00}z_{01}) + \text{h.o.t.},$$

up to some multiplicative nonzero constants and some multiple of $R_{00}$ and $R_{01}$ respectively. Thus we have shown

5.1. Theorem. Let $X$ be a smooth curve with Hasse-Witt invariant equal to 1. There exist coordinates $\{z_{ij}\}$ on $|2\Theta_1|$ and $\{y_{ij}\}$ on $|2\Theta|$ such that the equations of $\tilde{V}$ are given by

$$|2\Theta_1| \xrightarrow{\tilde{V}} |2\Theta|, \quad z = (z_{ij}) \mapsto y = (y_{ij}) = (\lambda_{00}Q_{00}(z) : \lambda_{01}Q_{01}(z) : \lambda_{10}Q_{10}(z) : \lambda_{11}Q_{11}(z))$$

with

$$Q_{00}(z) = z_{00}^2 + z_{01}^2, \quad Q_{01}(z) = z_{00}z_{11} + z_{01}z_{10}, \quad Q_{10}(z) = z_{00}^2 + z_{01}^2 + z_{10}^2 + z_{11}^2, \quad Q_{11}(z) = z_{00}z_{01}$$

and the $\lambda_{ij}$’s are nonzero constants depending on the curve $X$.

5.2. Remark. We note that the equations of $\tilde{V}$ given in Theorem 5.1 are written in two sets of coordinates on $|2\Theta|$ and $|2\Theta_1|$ which do not necessarily correspond under the $k$-semi-linear isomorphism $JX_1 \rightarrow JX$.

5.3. Remark. In case $X$ is a non-ordinary curve with Hasse-Witt invariant equal to 0, i.e., $X$ is supersingular, we observe that the 2-divisible group $JX(2) = JX(2)^0$ (see section 4.1) is self-dual, of dimension 2 and height 4. There exists a finite number of isomorphism classes of such 2-divisible groups over $k$ (see [4] page 93). Moreover one can show that $JX(2)$ cannot be isomorphic to the product $E^{ss}(2) \times E^{ss}(2)$. 

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5.4 Applications

As in [LP] section 6 we can easily deduce from Theorem [5.1] a full description of the Verschiebung $\tilde{V}$. Since the computations are analogous to those of [LP] Proposition 6.1, we leave them to the reader.

5.4. Proposition. Let $X$ be a smooth genus 2 curve with Hasse-Witt invariant equal to 1.

1. There exists a unique stable bundle $E_{BAD} \in M_{X_1}$, which is destabilized by the Frobenius map, i.e., $F^*E_{BAD}$ is not semi-stable. We have $E_{BAD} = F_*B^{-1}$ and its projective coordinates are $(0 : 0 : 1 : 1)$.

2. Let $H_1$ be the hyperplane in $|2\Theta_1|$ defined by $z_{00} + z_{01} = 0$. The map $\tilde{V}$ contracts $H_1$ to the conic $Kum_X \cap H$, where $H$ is the hyperplane in $|2\Theta|$ defined by $y_{00} = 0$.

3. The fiber of $\tilde{V}$ over a point $[E] \in M_X$ is
   \begin{itemize}
   \item a non-degenerate $\mathbb{Z}/2\mathbb{Z}$-orbit of a point $[E_1] \in M_{X_1}$, if $[E] \notin H$
   \item empty, if $[E] \in H \setminus (H \cap Kum_X)$
   \item a projective line passing through $E_{BAD}$, if $[E] \in H \cap Kum_X$
   \end{itemize}

   In particular, $\tilde{V}$ is dominant and non-surjective. The separable degree of $\tilde{V}$ is 2.

6 Equations of $\tilde{V}$ in characteristic 3

Let $X$ be a smooth curve of genus 2 defined over a field of characteristic $p = 3$. The main result of this section is

6.1. Theorem. 1. There exists an embedding $\alpha : Kum_X \hookrightarrow |2\Theta_1|$ such that the equality of hypersurfaces in $|2\Theta_1|$ $$\tilde{V}^{-1}(Kum_X) = Kum_{X_1} \cup \alpha(Kum_X)$$ holds set-theoretically.

2. The cubic equations of $\tilde{V}$ are given by the 4 partials of the quartic equation of the Kummer surface $\alpha(Kum_X) \subset |2\Theta_1|$. In other words, $\tilde{V}$ is the polar map of the surface $\alpha(Kum_X)$.

Proof. Given $L \in JX$ and a section $\varphi \in H^0(X, \omega \otimes L^2)$, we can consider, using adjunction and relative duality for the map $F$, the vector bundle map

$$F_*L \xrightarrow{\varphi} L \otimes \omega,$$

where we also write $L$ for the pull-back $\iota^*L$ under the $k$-semi-linear isomorphism $\iota : X_1 \to X$. We define $E_L := \ker(\varphi)$. Hence there is an exact sequence of vector bundles over $X_1$

$$0 \to E_L \to F_*L \xrightarrow{\varphi} L \otimes \omega \to 0. \quad (6.1)$$

It is straight-forward to show that $E_L$ has determinant $O_{X_1}$, is semi-stable, that $E_L = E_{L^{-1}}$, (if $L^{-1} \neq L$) and that $F^*E_L$ is $S$-equivalent to $L \oplus L^{-1}$. Indeed, by adjunction, the inclusion $E_L \to F_*L$ induces a nonzero map $F^*E_L \to L$.

We observe that for $L$ such that $L^2 \neq O$, $\dim H^0(\omega L^2) = 1$, hence $\varphi$ is uniquely defined (up to a scalar). For $L$ such that $L^2 = O$, we obtain a projective line $\mathbb{P}_L^1$ of rank 2 vector bundles $E_{L,\varphi}$.
with $\varphi \in |\omega|$. The variety of pairs $(L, \varphi)$ is isomorphic to the blowing-up $\text{Bl}_2(JX)$ of $JX$ at the 16 2-torsion points $L \in JX[2]$. Hence we obtain a morphism

$$e : \text{Bl}_2(JX) \longrightarrow M_{X_1} \cong |2\Theta_1|, \quad (L, \varphi) \mapsto E_L.$$  

We will determine the image of the morphism $e$. First we consider the image of the 16 exceptional divisors $\mathbb{P}^1_L$ for $L \in JX[2]$. We identify $H^0(X_1, \omega_1) = H^0(X, \omega)$.

**6.2. Lemma.** Given $L \in JX[2]$ and $w$ a Weierstrass point, i.e., $2w \in |\omega|$. Then we have $E_{L,\varphi} \cong B \otimes L^{-1}(-w)$ with $\text{Div}(\varphi) = 2w$.

**Proof.** We tensorize the exact sequence (6.1) with $L$, using $F_sL \otimes L = F_s(L \otimes F^*L) = F_s\mathcal{O}_X$ (projection formula)

$$0 \longrightarrow E_{L,\varphi} \otimes L \longrightarrow F_s\mathcal{O}_X \longrightarrow \omega_{X_1} \longrightarrow 0$$

The vertical arrows form an exact sequence (see [R] section 4.1). The upper diagonal map defined as composite map $\mathcal{O}_{X_1} \rightarrow F_s\mathcal{O}_X \rightarrow \omega_{X_1}$ equals $\varphi$, which implies that the lower diagonal map $\hat{\varphi} : E_{L,\varphi} \otimes L \rightarrow B$ vanishes at $w$. Hence, using stability of $B$ and $\det B = \omega$, we obtain $E_{L,\varphi} \otimes L \cong B(-w)$. \hfill $\square$

We observe that $L(w)$ is a theta-characteristic $\kappa$. Moreover the image of $\mathbb{P}^1_L$ in $|2\Theta_1| = \mathbb{P}^3$ is a conic and it is straight-forward to establish that the 16 points $B \otimes \kappa^{-1}$ (with $\kappa$ varying over the set of theta-characteristics) and the 16 hyperplanes spanned by the 16 conics $e(\mathbb{P}^1_L)$ form a classical $16_6$-configuration of nodes and tropes, whose associated Kummer surface equals $\text{Kum}_X$ (see [GL]). We recover the curve $X$ e.g. by taking the $2 : 1$ cover over a $\mathbb{P}^1_L$ branched at the 6 nodes on $\mathbb{P}^1_L$. Since deg $e(\text{Bl}_2(JX)) = 4$ and $e(\text{Bl}_2(JX))$ is singular at the 16 points $B \otimes \kappa^{-1}$, the image $e(\text{Bl}_2(JX))$ coincides with $\alpha(\text{Kum}_X)$ for an embedding $\alpha : \text{Kum}_X \hookrightarrow M_{X_1}$.

By Proposition 1.2 the map $\tilde{V}$ is given by a linear system $|\mathcal{L}|$ of 4 cubics on $|2\Theta_1|$. The key fact underlying Theorem 6.1 is a striking relationship between cubics and quartics on $|2\Theta_1|$ ([vG] Proposition 2): the 4 cubics in $|\mathcal{L}|$ are the 4 partial derivatives of a Heisenberg invariant quartic, whose zero divisor we denote by $Q \subset |2\Theta_1|$.\hfill $\square$

**6.3. Remark.** 1. The rational map $e : \text{Kum}_X \longrightarrow M_{X_1}$ (defined away from the 16 nodes of $\text{Kum}_X$) is the birational inverse of $\tilde{V}$.

2. One has the following scheme-theoretical equality (among divisors in $|2\Theta_1|$)

$$\tilde{V}^{-1}(\text{Kum}_X) = \text{Kum}_{X_1} + 2\alpha(\text{Kum}_X).$$

**6.4. Corollary.** 1. The map $\tilde{V}$ has exactly 16 base points, which correspond bijectively to the
• 16 nodes of the surface $\alpha(Kum_X) \subset |2\Theta_1|

• 16 stable rank 2 vector bundles $B \otimes \kappa^{-1} \in M_{X_1} \cong |2\Theta_1|$, where $B$ is the bundle of locally exact differentials and $\kappa$ a theta-characteristic of $X_1$.

2. The map $\tilde{V}$ is surjective, separable and of degree 11.

Proof. We only have to show part 2, since part 1 is clear from the proof of Theorem 6.1. We recall that the rational map $\tilde{V}$, which is defined away from the 16 points $B \otimes \kappa^{-1}$, coincides with the polar map of the Kummer surface $Kum_X$. It is well-known that $\tilde{V}$ can be resolved into a morphism $V : Bl(|2\Theta_1|) \rightarrow |2\Theta|$ by blowing-up these 16 points in $|2\Theta_1|$. We denote by $E_\kappa$ the exceptional divisor over $B \otimes \kappa^{-1}$ and by $H_\kappa \subset |2\Theta|$ the hyperplane $V(E_\kappa)$. Note that the $H_\kappa$ are the 16 tropes of $Kum_X \subset |2\Theta|$ and that $V|_{E_\kappa}$ is a linear isomorphism. It is clear that the image of $\tilde{V}$ contains the complement of the 16 hyperplanes $H_\kappa$.

Let us check that the $H_\kappa$ are also contained in the image of $\tilde{V}$: a simple computation shows that the cubic $C_\kappa := \tilde{V}^{-1}(H_\kappa) \subset |2\Theta_1|$ is singular at the point $B \otimes \kappa^{-1}$ and that the restriction of $\tilde{V}$ to the cubic $C_\kappa$ coincides with the (birational) projection with center $B \otimes \kappa^{-1}$. Moreover the projectivized tangent cone at $B \otimes \kappa^{-1}$ to $C_\kappa$ is the conic $Q_\kappa \subset H_\kappa$ through the 6 nodes (recall that $2Q_\kappa = H_\kappa \cap Kum_X$). Therefore any point in $H_\kappa \setminus Q_\kappa$ lies in the image of $\tilde{V}$. To finish the argument we observe that $Q_\kappa \subset Kum_X$ and that $\tilde{V} : Kum_{X_1} \rightarrow Kum_X$ is surjective. \qed

6.5. Remark. 1. We recall ([LP] Remark 6.2) that surjectivity only holds for $S$-equivalence classes (not isomorphism classes!). In fact, there always exist semi-stable bundles $E$ which do not descend by Frobenius.

2. The number of base points and the degree of $\tilde{V}$ was also obtained in [O] by computing the number of connections (on certain unstable bundles) with zero $p$-curvature.

3. It would be interesting to have an explicit description of the 11 vector bundles in a general fiber $\tilde{V}^{-1}(E)$ of the polar map $\tilde{V}$.

7 Appendix: base points of $\tilde{V}$

In this section we consider a smooth curve $X$ of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. We denote by $M_X(r)$ (resp. $M_{X_1}(r)$) the moduli space of semi-stable rank $r$ vector bundles over $X$ (resp. $X_1$) and by $L$ (resp. $L_1$) the determinant line bundle over $M_X(r)$ (resp. $M_{X_1}(r)$). The relative Frobenius map $F : X \rightarrow X_1$ induces by pull-back a rational map

$$V : M_{X_1}(r) \rightarrow M_X(r),$$

called the Verschiebung. Let $I$ be the indeterminacy locus of $V$, i.e., the closed subscheme of $M_{X_1}(r)$ consisting of semi-stable bundles $E$ such that $F^*E$ is not semi-stable. Let $U = M_{X_1}(r) \setminus I$ be the open subset where $V$ is a morphism.

7.1 General facts

7.1. Proposition. We have an isomorphism $V^*(L) \cong (L_1^\otimes p)|_U$. 

**Proof.** Let $\mathcal{M}_X(r)$ and $\mathcal{M}_{X_1}(r)$ be the moduli stacks of rank $r$ vector bundles over $X$ and $X_1$ and let $\mathcal{E}$ and $\mathcal{E}_1$ be the universal bundles with trivialized determinant on $X \times \mathcal{M}_X(r)$ and $X_1 \times \mathcal{M}_{X_1}(r)$. It is well-known that the inverses of the determinant of cohomology, which we denote by $\det Rp_* \mathcal{E}$ and $\det Rp_1_* \mathcal{E}_1$ descend (after restriction to the semi-stable loci) to the line bundles $\mathcal{L}$ and $\mathcal{L}_1$ on the moduli spaces $M_X(r)$ and $M_{X_1}(r)$. Now, since $\det Rp_*$ commutes with base change, we have an isomorphism over the moduli stack $\mathcal{M}_{X_1}(r)$

$$V^*(\det Rp_* \mathcal{E}) \cong \det Rp_* ((F \times \text{Id})^* \mathcal{E}_1).$$

Moreover we have a commutative diagram

$$\begin{array}{ccc}
X \times \mathcal{M}_{X_1}(r) & \xrightarrow{F \times \text{Id}} & X_1 \times \mathcal{M}_{X_1}(r) \\
p & & \downarrow^{p_1} \\
\mathcal{M}_{X_1}(r) & & \mathcal{M}_{X_1}(r)
\end{array}$$

where $p$ and $p_1$ denote the projections on the second factor. Since $F \times \text{Id}$ is an affine morphism, we have $R^1(F \times \text{Id})_* = 0$. Hence

$$\det Rp_* ((F \times \text{Id})^* \mathcal{E}_1) \cong \det Rp_1_* ((F \times \text{Id})_* (F \times \text{Id})^* \mathcal{E}_1) \cong \det Rp_1_* (\mathcal{E}_1 \boxtimes F_* \mathcal{O}_X).$$

The last equality follows from the projection formula. Using a filtration by line bundles of the rank $p$ bundle $F_* \mathcal{O}_X$ and by showing that $\det Rp_1_* (\mathcal{E}_1 \boxtimes \mathcal{O}_{X_1}(D)) = \det Rp_1_* (\mathcal{E}_1)$ for an effective divisor $D$, we show that $\det Rp_1_* (\mathcal{E}_1 \boxtimes F_* \mathcal{O}_X) \cong (\det Rp_1_* \mathcal{E}_1)^{\otimes p}$. We obtain the isomorphism of the lemma by descent on $U$. $\square$

**7.2. Proposition.** If $g = 2$ and $r = 2$, then $\dim \mathcal{I} = 0$ and the rational map $\tilde{V}$ is given by polynomials of degree $p$.

**Proof.** The fact that $\dim \mathcal{I} = 0$ is proved in Theorem 3.2 [JX]. This implies that $V^*(\mathcal{L})$ extends uniquely to $\mathcal{L}_1^{\otimes p}$ over $M_{X_1}$ and the lemma follows from the isomorphism $\mathcal{L}_1 \cong \mathcal{O}_{\mathbb{P}^3}(1)$. $\square$

**7.3. Remark.** For general $g, r, p$ we do not know an estimate of the dimension of $\mathcal{I}$.

**7.2 Existence of base points**

**7.4. Theorem.** The indeterminacy locus $\mathcal{I}$ is non-empty.

**Proof.** First it will be enough to show non-emptiness of $\mathcal{I}$ in the case $r = 2$, since taking direct sums with the trivial bundle implies non-emptiness for arbitrary $r$. Secondly it suffices to show non-emptiness of $\mathcal{I}$ after a field extension $k'/k$, with $k'$ algebraically closed.

Let $\overline{\mathcal{M}}_g$ be the coarse moduli space of stable genus $g$ curves defined over $k$, which is an irreducible projective variety [DM]. Let $\eta$ be the generic point of $\overline{\mathcal{M}}_g$. The choice of a geometric point $\overline{\eta}$ over $\eta$ defines a smooth curve $\mathcal{X}_{\overline{\eta}}$ over $\overline{k}(\eta)$, the algebraic closure of the function field $k(\eta)$ of $\overline{\mathcal{M}}_g$. The curve $\mathcal{X}_{\overline{\eta}}$ is defined over a finite extension $K$ of $k(\eta)$ and we denote by $\mathcal{X}_K$ some model of $\mathcal{X}_{\overline{\eta}}$, i.e., $\mathcal{X}_K \times_K \overline{k}(\eta) \cong \mathcal{X}_{\overline{\eta}}$.

The curve $X/k$ defines a $k$-rational point $x$ of $\overline{\mathcal{M}}_g$, which lies in the closure of $\eta$. The local ring $A_x$ at the generic point of the exceptional divisor of the blowing-up of $\overline{\mathcal{M}}_g$ at the point $x$ is a discrete valuation ring with fraction field $k(\eta)$ and residue field containing $k$. By the stable reduction theorem (Corollary 2.7 [DM]) there exists a finite extension $L$ of $K$, and therefore also
of $k(\eta)$, such that $\mathcal{X}_L$ is the generic fibre of a stable curve $\mathcal{X}$ over the integral closure $A$ of $A_k$ in $L$. Note that $A$ is a discrete valuation ring with fraction field $L$ and with residue field, denoted by $k(s)$, containing $k$. Moreover the diagram

$$
\begin{array}{ccc}
\text{Spec}(A) & \xrightarrow{\mathcal{X}} & \overline{\mathcal{M}}_g \\
\downarrow & & \\
\text{Spec}(A_x) & \xrightarrow{x} & \overline{\mathcal{M}}_g
\end{array}
$$

commutes when restricted to $\text{Spec}(L) \hookrightarrow \text{Spec}(A)$ and therefore commutes because $\overline{\mathcal{M}}_g$ is separated. It follows that the special point $s \in \text{Spec}(A)$ maps to $x$, i.e., there exists an isomorphism $X \times_k \overline{k}(s) \cong \mathcal{X}_s \times_{k(s)} \overline{k}(s)$.

In summary, we have constructed a stable curve $\mathcal{X}$ over a discrete valuation ring $A$ with generic fibre $\mathcal{X}_L$ and geometric special fibre isomorphic to $X \times_k \overline{k}(s)$. The fraction field of $A$ is $L$ and its residue field $k(s)$.

We now choose a tree of $\mathbb{P}^1$’s, denoted by $X'$, defining a closed point $x'$ in the boundary of $\overline{\mathcal{M}}_g$. Repeating the above construction with $x'$ instead of $x$, we obtain a stable curve $\mathcal{X}'$ over a discrete valuation ring $A'$ with generic fibre $\mathcal{X}'_L$ and geometric special fibre isomorphic to $X' \times_k \overline{k}(s')$. The fraction field of $A'$ is $L'$, a finite extension of $k(\eta)$, and its residue field is $k(s')$. Moreover the isomorphism $X' \times_k \overline{k}(s') \cong \mathcal{X}'_s \times_{k(s')} \overline{k}(s')$ is defined over a finite extension of $k(s')$.

We choose a finite extension of $k(\eta)$ containing both $L$ and $L'$, which we call again $L$, and take the integral closures in $L$ of $A$ and $A'$, which we call again $A$ and $A'$. Thus we have constructed two stable curves $\mathcal{X}$ and $\mathcal{X}'$ over $A$ and $A'$ such that $\mathcal{X}_L \cong \mathcal{X}'_L$ and which specialize to $X$ and $X'$ respectively.

Let $\hat{L}$ be the fraction field of the completion $\hat{A}'$ of $A'$. By construction the curve $\mathcal{X}_L \cong \mathcal{X}'_L$ is a Mumford-Tate curve and, by the main result of [3], there exists a stable rank 2 vector bundle $\hat{\mathcal{E}}$ over $\mathcal{X}_L$ such that $F^*\hat{\mathcal{E}}$ is not semi-stable.

7.5. Lemma. There exists a finite extension $L_1$ of $L$ contained in the field $\hat{L}$ and a stable bundle $\mathcal{E}_1$ over $\mathcal{X}_{L_1}$ such that $\mathcal{E}_1 \otimes_{L_1} \hat{L} \cong \hat{\mathcal{E}}$ and $F^*\mathcal{E}_1$ is not semi-stable.

Proof. Let $\hat{\pi} : F^*\hat{\mathcal{E}} \to \hat{L}$ be a maximal destabilizing quotient of $\hat{\mathcal{E}}$. There exist a models $\mathcal{E}_{k(S)}$, $\mathcal{L}_{k(S)}$ and $\pi_{k(S)}$ of $\hat{\pi}$ over $\mathcal{X}_{k(S)}$, where $k(S)$ is an extension of finite type of $L$. The field $k(S)$ is the function field for some algebraic variety $S$ over $L$. Shrinking $S$ if necessary, one can assume that $\pi_{k(S)}$ comes from

$$
\pi_S : F^*\mathcal{E}_S \to \mathcal{L}_S,
$$

where $\mathcal{E}_S$ is a family of stable bundles over $\mathcal{X}_L$ parametrized by $S$ (stability is an open condition). We now choose a closed point $s \in S$ and pull-back the family $\mathcal{E}_S$ under the inclusion $s \hookrightarrow S$. We thus obtain a stable bundle $\mathcal{E}_{L_1}$ over $\mathcal{X}_{L_1}$, where $L_1$ is the residue field at the point $s$, which is a finite extension of $L$.

Again we take the integral closures $A_1$ and $A'_1$ of the discrete valuation rings $A$ and $A'$ in $L_1$. By the previous lemma we have a stable bundle $\mathcal{E}_1$ and a destabilizing quotient $\mathcal{L}_1$ over $\mathcal{X}_{L_1} = \mathcal{X}'_{L_1}$

$$
\pi_{L_1} : F^*\mathcal{E}_1 \to \mathcal{L}_1.
$$

After possibly taking a finite extension of $L_1$, we can assume [4] that $\mathcal{E}_1$ and $\mathcal{L}_1$ have models over $\mathcal{X} \to \text{Spec}(A)$ with $(\mathcal{E}_1)_\pi$ semi-stable over $X \times k \overline{k}(s)$. By semi-continuity, we have

$$
\text{Hom}(F^*\mathcal{E}_1\pi, \mathcal{L}_\pi) \neq 0,
$$

15
which shows that $F^*E_{17}$ is not semi-stable.

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