Optimal error estimate of a conservative Fourier pseudo-spectral method for the space fractional nonlinear Schrödinger equation

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Abstract

In this paper, we consider the error analysis of a conservative Fourier pseudo-spectral method that conserves mass and energy for the space fractional nonlinear Schrödinger equation. We give a new fractional Sobolev norm that can construct the discrete fractional Sobolev space, and we also can prove some important lemmas for the new fractional Sobolev norm. Based on these lemmas and energy method, a priori error estimate for the method can be established. Then, we are able to prove that the Fourier pseudo-spectral method is unconditionally convergent with order $O(\tau^2 + N^{\alpha/2-r})$ in the discrete $L^\infty$ norm, where $\tau$ is the time step and $N$ is the number of collocation points used in the spectral method. Numerical examples are presented to verify the theoretical analysis.

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1 Introduction

The nonlinear fractional Schrödinger equation is a generalization of the classical Schrödinger equation. It has found several applications in physics, such as nonlinear optics [21], propagation

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dynamics and water wave dynamics. In this paper, we consider the following space fractional nonlinear Schrödinger (FNLS) equation

\[
iu_t - (-\Delta)^\frac{\alpha}{2} u + \beta |u|^2 u = 0, \quad x \in \Omega, \quad 0 < t \leq T,
\]  

(1.1)

with the periodic boundary condition

\[
u(x,t) = u(x + L,t), \quad x \in \Omega, \quad 0 < t \leq T,
\]  

(1.2)

and the initial condition

\[
u(x,0) = \varphi(x), \quad x \in \Omega,
\]  

(1.3)

where \( i = \sqrt{-1}, 1 < \alpha \leq 2, \Omega = [a,b] \) and \( L = b - a \). \( u(x,t) \) is a complex-valued wave function, parameter \( \beta \) is a real constant, and \( \varphi(x) \) is a complex-value initial data. The fractional Laplacian \( (-\Delta)^\frac{\alpha}{2} \) acting on periodic function defined by

\[
(-\Delta)^\frac{\alpha}{2} u = \sum_{k \in \mathbb{Z}} |\mu_k|^\alpha \hat{u}_k e^{i\mu_k x}, \quad \mu = \frac{2\pi}{L},
\]  

(1.4)

where

\[
u = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{i\mu_k x}, \quad \hat{u}_k = \frac{1}{L} \int_{\Omega} u(x)e^{-i\mu_k x} dx.
\]  

(1.5)

When \( \alpha = 2 \), the equation (1.1) reduces to the classical nonlinear Schrödinger (NLS) equation. Due to self-adjoint property of the fractional Laplacian, the solution of (1.1)-(1.3) satisfies the following mass and energy conservation laws:

\[
\text{Mass : } M(t) = \int_{\Omega} |u(x,t)|^2 dx = M(0),
\]  

(1.6)

\[
\text{Energy : } E(t) = \int_{\Omega} |(-\Delta)^\frac{\alpha}{2} u(x,t)|^2 - \frac{\beta}{2} |u(x,t)|^4 dx = E(0).
\]  

(1.7)

Various numerical methods have been developed in the literatures for the space FNLS equation, including finite difference methods, finite element methods, spectral methods. In the past few decades, structure-preserving methods which can inherit the intrinsic geometric properties of the given dynamical system have attracted a lot of interest due to the superior properties in long time numerical simulation over traditional methods. For more details, readers can refer to. Recently, structure-preserving numerical methods have been extended to solve the space FNLS equation. For example, in, Wang et al. first constructed a mass conservative Crank-Nicolson difference scheme, and they further proposed a linearly implicit difference scheme that conserves mass and energy in. In, the modified mass and energy conservative Crank-Nicolson difference schemes were presented. Other works related to the conservative method can be found in.
Spectral and pseudo-spectral methods have been proved to be an efficient and high order numerical method in solving smooth problems [25]. Over the past few years, though structure-preserving spectral methods have been widely used to solve Hamiltonian PDEs [3, 6, 9, 17]. Only in very recently years, structure-preserving Fourier pseudo-spectral methods are extended to solve the space FNLS equation. For instance, in [32], Wang and Huang proposed the symplectic and multi-symplectic Fourier pseudo-spectral methods for the space FNLS equation. In [22], a mass and energy conservative Fourier pseudo-spectral method was constructed. The numerical results show that the conservative Fourier pseudo-spectral method is efficient and stable for long-term numerical simulation. However the unconditionally convergent results on the conservative Fourier pseudo-spectral method for the space fractional PDEs have not been obtained. Actually, with the help of the defined fractional Sobolev norm and the discrete uniform Gagliardo-Nirenberg inequality in [16], we can easily prove that the conservative Fourier pseudo-spectral method for the space FNLS equation is unconditionally convergent in the discrete $L^2$ norm, but the challenge problem is error estimate in $L^\infty$ norm. For the classical NLS equations, in [10], Gong first established the semi-norm equivalence between the finite difference method and the Fourier pseudo-spectral method and thus obtained the unconditionally convergent results on the Fourier pseudo-spectral in the discrete $L^2$ norm. Then based on this equivalence, the error estimates of the Fourier pseudo-spectral method in the discrete $L^\infty$ norm were obtained in [15]. However, this error analysis technique for establishing semi-norm equivalence can not extend to the FNLS equations. By reading the finite difference methods for the fractional PDEs with fractional Laplacian [12, 31, 36], we know that under the homogeneous Dirichlet boundary condition, the Riesz derivative is discreted instead of fractional Laplacian due to the equivalence between Riesz derivative and fractional Laplacian. But this equivalence does not hold in the case of periodic boundary condition. That’s why even now there is no corresponding finite difference method has been used to solve the space fractional PDEs under the periodic boundary condition. Therefore, for the FNLS equation, it is impossible to establish semi-norm equivalence between the finite difference method and the Fourier pseudo-spectral method in the error analysis.

To obtain the $L^\infty$ norm error estimates of the Fourier pseudo-spectral method for the space FNLS equation, in this paper, we introduce the discrete fractional Sobolev space $H^{\alpha/2}_h$ with a new discrete fractional Sobolev norm. We establish several lemmas for the new discrete fractional norm, based on these important lemmas and the energy method, a prior estimate for the method is estimated. Then we can prove that the conservative Fourier pseudo-spectral method is unconditionally convergent with order of $O(\tau^2 + N^{\alpha/2-r})$ in the discrete $L^\infty$ norm.

The rest of the paper is organized as following. In section 2, we construct the discrete fractional Sobolev space by introducing a new fractional Sobolev norm and we also prove some
important lemmas for the new fractional Sobolev norm. In section 3, a conservative Fourier pseudo-spectral scheme for the FNLS equation is given, we show the numerical scheme satisfies discrete conservation laws and obtain a priori estimate. In section 4, the convergence property of the scheme is analyzed. Subsequently in section 5, we carry out some numerical experiments to confirm our theoretical results and show the efficiency of the scheme. Finally, we give a conclusion in section 6.

2 Fourier pseudo-spectral method

Let $N$ be an even integer, we define step size in space: $h = L/N$. Then, the spatial grid points are defined as follows: $\Omega_h = \{x_j = a + jh, j = 0, 1, \ldots, N\}$. For any positive integer $N_t$, we define the time-step: $\tau = T/N_t$. Then grid points in space and time are given by $\Omega_{ht} = \Omega_h \times \Omega_t$, where $\Omega_t = \{t_n = n\tau, n = 0, 1, \ldots, N_t\}$. For a grid function $u = \{u^n_j\}_{(x_j, t_n) \in \Omega_{ht}}$, we introduce the following notations:

$$
\delta^n_x u^n_j = \frac{u^n_{j+1} - u^n_j}{h}, \quad u^n_{j+\frac{1}{2}} = \frac{u^n_{j+1} + u^n_j}{2}, \quad \delta^n_t u^n_j = \frac{u^n_{j+1} - u^n_j}{\tau}.
$$

Let $\mathcal{V}_h = \{u|u = (u_j), x_j \in \Omega_h\}$ be the space of grid functions defined on $\Omega_h$. For any grid function $u, v \in \mathcal{V}_h$, we define the discrete inner product and associated $L^2$ norm

$$
(u, v)_h = \frac{1}{N} \sum_{j=0}^{N-1} u_j v_j, \quad \|u\|_h^2 = (u, u)_h. \tag{2.1}
$$

We also define the discrete $L^p$ norm as

$$
\|u\|_h^p = \frac{1}{N} \sum_{j=0}^{N-1} |u_j|^p, \quad 1 \leq p < +\infty, \tag{2.2}
$$

and the discrete $L^\infty$ norm as

$$
\|u\|_h^\infty = \max_{0 \leq j \leq N-1} |u_j|. \tag{2.3}
$$

2.1 Disctete Fractional Sobolev norm

We define a function space $S_N$ by

$$
S_N = \text{span}\{g_j(x), \ j = 0, 1, \ldots, N - 1\},
$$

where $g_j(x)$ is a trigonometric polynomial defined by

$$
g_j(x) = \frac{1}{N} \sum_{k=-N/2}^{N/2} \frac{1}{c_k} e^{i\mu(x-x_j)}, \tag{2.4}
$$
where
\[
c_k = \begin{cases} 
1, & |k| < N/2, \\
2, & |k| = N/2,
\end{cases}
\]
\[\mu = \frac{2\pi}{L}.
\]

Then, we define the interpolation operator \( I_N : L^2(\Omega) \rightarrow S_N \) by
\[
I_N u(x) = \sum_{j=0}^{N-1} u_j g_j(x) = \sum_{k=-N/2}^{N/2} \hat{u}_k e^{ik\mu x},
\]
where
\[
\hat{u}_k = \frac{1}{Nc_k} \sum_{j=0}^{N-1} u_j e^{-ik\mu x_j}, \quad -N/2 \leq k \leq N/2,
\]
and \( \hat{u}_{N/2} = \hat{u}_{-N/2} \) for \( k = \frac{N}{2} \). Therefore we have the inverse transformation
\[
u_j = (I_N u)(x_j) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k e^{ik\mu x_j}.
\]

For any \( u \in l^2_h := \{ u|u \in V_h, \|u\|_h^2 < \infty \} \), we have \( \hat{u} \in l^2 := \{ x = \{ x_k \} | \sum_{k=-\infty}^{\infty} x_k^2 < \infty \} \), and the Parseval’s theorem gives
\[
(u, v)_h = \sum_{k=-N/2}^{N/2-1} \hat{u}_k \bar{v}_k.
\]

Given a constant \( \sigma \in [0, 1] \), we define the discrete fractional Sobolev norm \( \| \cdot \|_{H^\sigma_h} \) and semi-norm \( | \cdot |_{H^\sigma_h} \) as
\[
|u|_{H^\sigma_h}^2 = \sum_{k=-N/2}^{N/2-1} |\mu k|^{2\sigma} |\hat{u}_k|^2, \quad \|u\|_{H^\sigma_h}^2 = \sum_{k=-N/2}^{N/2-1} (1 + |\mu k|^{2\sigma}) |\hat{u}_k|^2.
\]

Clearly, \( \|u\|_{H^\sigma_h}^2 = \|u\|_h^2 + |u|_{H^\sigma_h}^2, \|u\|_{H^0_h}^2 = \|u\|_h^2 \). We can easily prove that the discrete Sobolev spaces is the normed linear spaces according to the norm \( \|u\|_{H^\sigma_h} \) defined in (2.9). Next, we introduce the following lemmas, which are important for unconditional convergence analysis of the conservative Fourier pseudo-spectral method.

**Lemma 2.1** (Discrete uniform Sobolev inequality). *For any \( \frac{1}{2} < \sigma \leq 1 \), there exists a constant \( C = C(\sigma) > 0 \) independent of \( h > 0 \) such that*
\[
\|u\|_{l^\infty_h} \leq C\|u\|_{H^\sigma_h}.
\]

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Lemma 2.2 For \( \|u\|_{H^0_h} \) the proof is completed.

\[ \|u\|_{H^0_h} \leq \sum_{k=-N/2}^{N/2-1} |\tilde{u}_k| = \sum_{k=-N/2}^{N/2-1} \frac{1}{1 + |\mu k|^{2\sigma}} \left(1 + |\mu k|^{2\sigma}\right)^{1/2} |\tilde{u}_k| \]
\[ \leq \left( \sum_{k=-N/2}^{N/2-1} \frac{1}{1 + |\mu k|^{2\sigma}} \right)^{1/2} \left( \sum_{k=-N/2}^{N/2-1} \left(1 + |\mu k|^{2\sigma}\right)|\tilde{u}_k|^2 \right)^{1/2} \]
\[ \leq \left( \sum_{k=-N/2}^{N/2-1} \frac{1}{1 + |\mu k|^{2\sigma}} \right)^{1/2} \|u\|_{H^0_h} \tag{2.11} \]

For \( \frac{1}{2} < \sigma \leq 1 \), this implies (2.10) and thus the proof is completed.

Lemma 2.2 For \( 0 \leq \sigma_0 \leq \sigma \leq 1 \), there exist a constant \( C \in [1, 2] \) such that
\[ \|u\|_{H^0_h} \leq C \|u\|_{H^0_h}^{\sigma_0} \|u\|_{H^0_h}^{1-\sigma_0}. \tag{2.12} \]

Proof. From the definition of \( \|u\|_{H^0_h} \) and the Hölder’s inequality, we have
\[ \|u\|_{H^0_h}^2 = \sum_{k=-N/2}^{N/2-1} (1 + |\mu k|^{2\sigma_0})|\tilde{u}_k|^2 \]
\[ = \sum_{k=-N/2}^{N/2-1} \left(1 + |\mu k|^{2\sigma_0}\right)|\tilde{u}_k|^2 \sigma_0 \left(|\tilde{u}_k|^2\right)^{1-\sigma_0} \left(1 + |\mu k|^{2\sigma_0}\right)^{\sigma_0 \sigma_0} (1 + |\mu k|^{2\sigma_0})^{\sigma_0} \frac{(1 + |\mu k|^{2\sigma_0})^{\sigma_0}}{(1 + |\mu k|^{2\sigma_0})^{\sigma_0}} \]
\[ \leq C \left( \sum_{k=-N/2}^{N/2-1} (1 + |\mu k|^{2\sigma_0})|\tilde{u}_k|^2 \right)^{\sigma_0 \sigma_0} \left( \sum_{k=-N/2}^{N/2-1} |\tilde{u}_k|^2 \right)^{1-\sigma_0} \tag{2.13} \]
\[ = C \left( \sum_{k=-N/2}^{N/2-1} (1 + |\mu k|^{2\sigma_0})|\tilde{u}_k|^2 \right)^{\sigma_0 \sigma_0} \left( \sum_{k=-N/2}^{N/2-1} |\tilde{u}_k|^2 \right)^{1-\sigma_0} \]
\[ = C \left( \|u\|_{H^0_h}^{\sigma_0} \|u\|_{H^0_h}^{1-\sigma_0} \right)^2, \]
where the inequality holds due to the fact \( \frac{1}{2} (1 + a^\mu) \leq (1 + a)^\mu \leq (1 + a^\mu) \) for \( a > 0, \ 0 \leq \mu \leq 1 \). Thus the proof is completed.

Lemma 2.3 (Hausdorff-Young inequality). If \( 1 \leq q \leq 2, \ \frac{1}{q} + \frac{1}{p} = 1 \), then
\[ \left( \frac{1}{h} \sum_{j=0}^{N-1} |u_j|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=-N/2}^{N/2-1} |\tilde{u}_k|^q \right)^{\frac{1}{q}}. \tag{2.14} \]
Proof. From the inverse transformation \((2.7)\), we have

\[
\sup_{0 \leq j \leq N-1} |u_j| \leq \sum_{k=-N/2}^{N/2-1} |\hat{u}_k|,
\]

the Parseval’s identity gives

\[
h \sum_{j=0}^{N-1} |u_j|^2 = \sum_{k=-N/2}^{N/2-1} |\hat{u}_k|^2.
\]

Then using the Riesz-Thorin Interpolation theorem (see Theorem 8.6 in [3, page 316]), we can obtain the conclusion.

**Lemma 2.4** For any \(\frac{v-2}{2p} < \sigma_0 \leq 1\), there exists a constant \(C_{\sigma_0} = C(\sigma_0) > 0\) independent of \(h > 0\), such that

\[
\|u\|_{L^p} \leq C_{\sigma_0} \|u\|_{H^{\sigma_0}}^{\frac{\sigma_0}{p}} \|u\|_{H^\sigma}^{1-\frac{\sigma_0}{p}}, \quad 2 \leq p \leq +\infty, \quad \sigma_0 \leq \sigma \leq 1.
\]

**Proof.** By Lemma 2.3 and Hölder’s inequality, for \(1 \leq q \leq 2\) such that \(\frac{1}{p} + \frac{1}{q} = 1\), we have

\[
\left( \sum_{j=0}^{N-1} |u_j|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=-N/2}^{N/2-1} |\hat{u}_k|^q \right)^{\frac{1}{q}} = \left( \sum_{k=-N/2}^{N/2-1} \frac{1}{1 + |\mu k|^{2\sigma_0}} \left(1 + |\mu k|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \leq \left( \sum_{k=-N/2}^{N/2-1} \frac{1}{1 + |\mu k|^{2\sigma_0}} \frac{2-q}{2q} \right)^{\frac{2-q}{2q}} \leq \|u\|_{H^{\sigma_0}} \left( \sum_{k=-N/2}^{N/2-1} \frac{1}{1 + |\mu k|^{2\sigma_0}} \right)^{\frac{2-q}{2q}}.
\]

Then for \(\frac{v-2}{2p} < \sigma_0 \leq 1\), we have

\[
\|u\|_{L^p} \leq \tilde{C}_{\sigma_0} \|u\|_{H^{\sigma_0}},
\]

where \(\tilde{C}_{\sigma_0} = \tilde{C}(\sigma_0) > 0\) is independent of \(h\). Combining the above inequality with \((2.12)\) gives \((2.17)\) and thus completes the proof.

### 2.2 Discrete fractional Laplacian

Applying the fractional Laplacian \((-\Delta)^{\frac{\sigma}{2}}\) to the interpolated function \((2.5)\) yields

\[
(-\Delta)^{\frac{\sigma}{2}} I\!_{N} u(x) = \frac{1}{N} \sum_{j=0}^{N-1} u_j \sum_{p=-N/2}^{N/2-1} \frac{1}{c_p} \mu_p^{\sigma} e^{i\mu_p(x-x_j)},
\]

(2.20)
and thus
\begin{equation}
(-\Delta)^{\frac{\alpha}{2}}I_N(u(x_k)) = \sum_{p=-N/2}^{N/2-1} d_p \left( \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-\frac{2\pi ikj}{N}} \right) e^{\frac{2\pi ikp}{N}},
\end{equation}
(2.21)
where
\begin{equation}
d_p = |\mu_p|^\alpha, \quad -N/2 \leq p \leq N/2 - 1.
\end{equation}
(2.22)
For \( U \in \mathcal{V}_h \), we define a discrete fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}_d\) by
\begin{equation}
((-\Delta)^{\frac{\alpha}{2}}_dU)_k = \sum_{p=-N/2}^{N/2-1} d_p \left( \frac{1}{N} \sum_{j=0}^{N-1} U_j e^{-\frac{2\pi ikj}{N}} \right) e^{\frac{2\pi ikp}{N}},
\end{equation}
(2.23)
By using the notation of the discrete Fourier transform and its inverse:
\begin{equation}
(F_dU)_k = \frac{1}{N} \sum_{j=0}^{N-1} U_j e^{-\frac{2\pi ikj}{N}}, \quad (F_d^{-1}\hat{U})_j = \sum_{k=-N/2}^{N/2-1} \hat{U}_k e^{\frac{2\pi ikj}{N}},
\end{equation}
(2.24)
the discrete fractional Laplacian can be expressed as
\begin{equation}
(-\Delta)^{\frac{\alpha}{2}}_dU = F_d^{-1}\Lambda_\alpha F_dU,
\end{equation}
(2.25)
where \( \Lambda_\alpha = \text{diag}(d_{-\frac{N}{2}}, d_{-\frac{N}{2}-1}, \ldots, 0, \ldots, d_{\frac{N}{2}-1}) \). Next, we give several lemmas that show the relationship between discrete fractional Sobolev semi-norm and fractional Laplacian.

**Lemma 2.5** For any grid function \( u \in \mathcal{V}_h \), we have
\begin{equation}
(D_\alpha u, u)_h = |u|^2_{H^{\alpha/2}_h}, \quad 1 < \alpha \leq 2.
\end{equation}
(2.26)

**Proof.** Using the Parseval’s identity (2.8), we have
\begin{align}
(D_\alpha u, u)_h &= (F_d^{-1}\Lambda_\alpha F_d u, u)_h \\
&= \sum_{k=-N/2}^{N/2-1} (\Lambda_\alpha F_d u)_k (\Lambda_\alpha \overline{F_d u})_k \\
&= \sum_{k=-N/2}^{N/2-1} d_k \tilde{u}_k \overline{\tilde{u}}_k = |u|^2_{H^{\alpha/2}_h}.
\end{align}
(2.27)

**Lemma 2.6** For any two grid functions \( u, v \in \mathcal{V}_h \), we have
\begin{equation}
(D_\alpha u, v)_h = (D_{\alpha/2} u, D_{\alpha/2} v)_h, \quad 1 < \alpha \leq 2.
\end{equation}
(2.28)

**Proof.** Using the Parseval’s identity (2.8), we have
\begin{align}
(D_\alpha u, v)_h &= (F_d^{-1}\Lambda_\alpha F_d u, v)_h \\
&= \sum_{k=-N/2}^{N/2-1} (\Lambda_\alpha F_d u)_k (\Lambda_\alpha \overline{F_d v})_k \\
&= \sum_{k=-N/2}^{N/2-1} (\Lambda_{\alpha/2} F_d u)_k (\Lambda_{\alpha/2} F_d v)_k = (D_{\alpha/2} u, D_{\alpha/2} v)_h.
\end{align}
(2.29)
Lemma 2.7 For any two grid functions \( u, v \in \mathcal{V}_h \), we have

\[
(D_\alpha u, v)_h \leq |u|_{H^{\alpha/2}_h}/|v|_{H^{\alpha/2}_h}, \quad 1 < \alpha \leq 2.
\] (2.30)

Proof. Using the Parseval’s identity (2.8), we have

\[
(D_\alpha u, v)_h = (F_\alpha^{-1} \Lambda_\alpha F_d u, v)_h = \sum_{k=-N/2}^{N/2-1} (\Lambda_\alpha F_d u)_k (F_d v)_k = \sum_{k=-N/2}^{N/2-1} d_k \hat{u}_k \hat{v}_k.
\] (2.31)

Therefore

\[
(D_\alpha u, v)_h \leq \left( \sum_{k=-N/2}^{N/2-1} d_k |\hat{u}_k|^2 \right)^{1/2} \left( \sum_{k=-N/2}^{N/2-1} d_k |\hat{v}_k|^2 \right)^{1/2} = |u|_{H^{\alpha/2}_h}/|v|_{H^{\alpha/2}_h}.
\] (2.32)

For simplicity, we denote \( u^n_j = u(x_j, t_n) \) and \( U^n_j \) as the exact value of \( u(x, t) \) and its numerical approximation at \((x_j, t_n)\), respectively.

3 Solution existence and conservation of the scheme

We discretize the FNLS equation (1.1)-(1.3) using the Fourier pseudo-spectral method in space and the Crank-Nicolson method in time to arrive at a fully discrete system:

\[
i \delta_t^+ U^n_j - (D_\alpha U^{n+1/2})_j + \frac{\beta}{2} (|U^n_j|^2 + |U^{n+1}_j|^2) U^{n+1/2}_j = 0, \quad U^n \in \mathcal{V}_h,
\] (3.1)

where \( D_\alpha = (-\Delta)^{\alpha/2}_d = F_\alpha^{-1} \Lambda_\alpha F_d \), \( j = 0, 1, \ldots, N - 1 \). For convenience, scheme (3.1) can be written in an equivalent form

\[
i \delta_t^+ U^n - D_\alpha U^{n+1/2} + F(U^n, U^{n+1}) = 0, \quad U^n \in \mathcal{V}_h,
\] (3.2)

where \( U^n = (U^n_j) \), \( F(U^n, U^{n+1}) = F(U^n_j, U^{n+1}_j) = (\frac{\beta}{4} (|U^n_j|^2 + |U^{n+1}_j|^2) (U^n_j + U^{n+1}_j)).

Lemma 3.1 For the approximation \( U^n \in \mathcal{V}_h \), there exist identities:

\[
\text{Im}(D_\alpha U^{n+1/2}, U^{n+1/2})_h = 0,
\] (3.3)

\[
\text{Re}(D_\alpha U^{n+1/2}, \delta_t^+ U^n)_h = \frac{1}{2\tau} (|U^{n+1/2}|_{H^{\alpha/2}_h}^2 - |U^n|_{H^{\alpha/2}_h}^2),
\] (3.4)

According to Lemma 2.5 and Lemma 2.6, we can get the results immediately. Here “Im(s)” and “Re(s)” mean taking the imaginary part and real part of a complex number \( s \), respectively.
3.1 Conservation

**Theorem 3.1** The scheme (3.2) is conservative in the sense that

\[
M^n = M^0, \quad 0 \leq n \leq N, \tag{3.5}
\]
\[
E^n = E^0, \quad 0 \leq n \leq N, \tag{3.6}
\]

where

\[
M^n := \|U^n\|_h^2, \quad E^n = |U|^2_{H_h^{\alpha/2}} - \frac{\beta}{2}\|U^n\|_h^4 \tag{3.7}
\]

**Proof.** Computing the discrete inner product of (3.2) with \(U^{n+1/2}\), then taking the imaginary part, we obtain

\[
\frac{1}{2\tau}((U^{n+1/2})^2_h - \|U^n\|_h^2) = 0, \quad t_n \in \Omega, \tag{3.8}
\]

where Lemma 3.1 is used. This gives (3.5).

Computing the discrete inner product of (3.2) with \(\delta_t U^n\), then taking the real part, we obtain

\[
-\frac{1}{2\tau}((U^{n+1/2})^2_h - \|U^n\|_h^2) - (\|U^n\|_h^4 - \frac{\beta}{2}\|U^n\|_h^4) = 0, \quad t_n \in \Omega, \tag{3.9}
\]

where Lemma 3.1 is used. This yields (3.6).

3.2 A priori estimate

**Theorem 3.2** Then numerical solution of scheme (3.2) is bounded in the following sense

\[
\|U^n\|_h \leq C_1, \quad |U^n|_{H_h^{\alpha/2}} \leq C_2, \quad \|U^n\|_{l_h^\infty} \leq C_3, \quad 0 \leq n \leq N, \tag{3.10}
\]

where \(C_1, C_2, C_3\) are some positive constants.

**Proof.** The proof is similar to that in [33, Theorem 3.2]. The mass conservation (3.5) implies the first inequality in (3.10) immediately if we choose \(\|U^0\|_h \leq C_1\).

Next, we prove the second inequality by the energy conservation (3.6). If \(\beta \leq 0\), according to the second term of (3.7) and energy conservation (3.6), we can get the result straightforwardly. If \(\beta > 0\), In the view of Lemma 2.4 with \(\frac{1}{4} < \sigma_0 < \frac{\alpha}{4}\), Young’s inequality and the first inequality in (3.10), we obtain

\[
\|U^n\|_h^{\frac{4}{\alpha}} \leq C_{\sigma_0}\|U^n\|_{H_h^{\alpha/2}}^{\frac{\sigma_0}{\alpha}}\|U^n\|_h^{1 - \frac{\sigma_0}{\alpha}} \leq C_{\sigma_0}(\varepsilon\|U^n\|_{H_h^{\alpha/2}}^2 + \varepsilon\|U^n\|_h^2 + C(\varepsilon)), \tag{3.11}
\]

where \(\varepsilon\) is any arbitrary positive constant. Combing the second term of (3.7) with (3.11), the energy conservation (3.6) imply

\[
\|U^n\|_{H_h^{\alpha/2}}^2 = \frac{\beta}{2}\|U^n\|_h^4 + E^0 \leq \frac{\beta}{2}C_{\sigma_0}(\varepsilon\|U^n\|_{H_h^{\alpha/2}}^2 + \varepsilon\|U^n\|_h^2 + C(\varepsilon)) + E^0. \tag{3.12}
\]
Taking $\varepsilon = \frac{1}{\beta C_{\sigma_0}}$, we have

$$|U^n|_{H^0_h}^2 \leq \|U^0\|_h^2 + \beta C_{\sigma_0} C(\varepsilon) + 2E^0 := C_2^2, \quad 0 \leq n \leq N_t.$$  

(3.13)

This implies the second inequality of (3.7).

Finally, combining the first two inequality in (3.10) with Lemma 2.1, we get the third inequality in (3.10), that is,

$$\|U^n\|_{l^\infty_h}^2 \leq C_2^2 (\|U^n\|_h^2 + |U^n|_{H^{\alpha/2}_h}^2) \leq C_2^2 (C_1^2 + C_2^2) := C_3^2, \quad 0 \leq n \leq N_t.$$  

(3.14)

Thus the proof is completed.

3.3 Existence

Theorem 3.3 The nonlinear equation system in scheme (3.2) is solvable.

Proof. The argument of the existence for the solution relies on the Browder fixed point theorem (see [1, 11]). Here we omit the proof for brevity.

4 Convergence of the scheme

In this section, we will establish error estimate of (3.2) in the discrete $L^\infty$ norm. For simplicity, we let $\Omega = [0, 2\pi]$ and assume that $C_p^\infty(\Omega)$ is a set of infinitely differentiable functions with $2\pi$-period defined on $\Omega$. $H^r_p(\Omega)$ is the closure of $C_p^\infty(\Omega)$ in $H^r(\Omega)$. The semi-norm and the norm of $H^r_p(\Omega)$ are denoted by $|\cdot|$ and $\|\cdot\|_r$, respectively.

For the given even $N$, we introduce the projection space

$$S_N = \{u|u(x) = \sum_{|l| \leq N/2} \hat{u}_l e^{ilx}\},$$

and the interpolation space

$$S'_{N} = \{u|u(x) = \sum_{|l| \leq N/2} \frac{\hat{u}_l}{c_l} \hat{u}_l e^{ilx}, \quad \hat{u}_{-N/2} = \hat{u}_{N/2}, \}$$

where $c_l = 1, |l| < \frac{N}{2}, c_{-N} = c_{N} = 2$.

It is clear that $S'_{N} \subseteq S_N$. We denote by $P_N : L^2(\Omega) \rightarrow S_N$ as the orthogonal projection operator and recall the interpolation operator $I_N : L^2(\Omega) \rightarrow S'_{N}$. Further, $P_N$ and $I_N$ satisfy:

1. $P_N \partial_x u = \partial_x P_N u, \quad I_N \partial_x u \not= \partial_x I_N u.$

2. $P_N u = u, \quad \forall u = S_N, I_N u = u, \forall u \in S'_{N}.$
Lemma 4.1 ([(10)]) For \( u \in S_N^r \), \( \| u \| \leq \| u \|_h \leq 2 \| u \|.

Lemma 4.2 ([(3)]) If \( 0 \leq t \leq r \) and \( u \in H^r_h(\Omega) \), then
\[
\| P_N u - u \|_t \leq C N^{1-r} |u|_r, \]
\[
\| P_N u \|_t \leq C \| u \|_t. \tag{4.1}
\]

In addition, if \( r > \frac{1}{2} \), then
\[
\| I_N u - u \|_t \leq C N^{1-r} |u|_r, \]
\[
\| I_N u \|_t \leq C \| u \|_t. \tag{4.2}
\]

Lemma 4.3 ([(10)]) For \( u \in H^r_h(\Omega), r > \frac{1}{2} \), let \( u^* = P_{N-2} u \), then \( \| u^* - u \|_h \leq C N^{-r} |u|_r \).

Lemma 4.4 For \( u \in H^r_p(\Omega), r > \frac{1}{2} \), let \( u^* = P_{N-2} u \), then \( |u^* - u|_{H_{h/2}^r} \leq C N^{\alpha/2-r} |u|_r \).

Proof. According to Lemma 2.5 we have
\[
|u^* - u|_{H_{h/2}^r} = \| (\Delta)^{\frac{\alpha}{2}} (u^* - u), u^* - u) \|_{h/2}^{\frac{1}{2}}
\leq \| (\Delta)^{\frac{\alpha}{2}} (u^* - u) \|_{h}^{\frac{1}{2}} \| u^* - u \|_{h}^{\frac{1}{2}} \tag{4.5}
\leq \| (\Delta)^{\frac{\alpha}{2}} (I_N(u^* - u)) \|_{h}^{\frac{1}{2}} \| u^* - u \|_{h}^{\frac{1}{2}}.
\]

Together with Lemma 4.1 and Lemma 4.2 we can deduce
\[
\| (\Delta)^{\frac{\alpha}{2}} (I_N(u^* - u)) \|_{h} = \| I_N[(\Delta)^{\frac{\alpha}{2}} (I_N(u^* - u))] \|_{h}
\leq \sqrt{2} \| I_N[(\Delta)^{\frac{\alpha}{2}} (I_N(u^* - u))] \|
\leq C \| (\Delta)^{\frac{\alpha}{2}} (I_N(u^* - u)) \|
\leq C \| (I_N(u^* - u))\|_{\alpha}
\leq C \| u^* - u \|_{\alpha} \leq C N^{\alpha-r} |u|_r. \tag{4.6}
\]

Then, we can deduce from (4.6) and Lemma 4.3 that
\[
|u^* - u|_{H_{h/2}^r} \leq C N^{\alpha/2-r} |u|_r.
\]

Lemma 4.5 ([(27)]) For time sequence \( w = \{w^0, w^1, \cdots, w^n\}, g = \{g^1, g^2, \cdots, g^{n-\frac{1}{2}}\} \)
\[
|2 \tau \sum_{t=1}^{n} g^{l-\frac{1}{2}} \cdot \delta^+_t w^t| \leq (\tau \sum_{t=1}^{n-1} |\delta^+_t g^{l-\frac{1}{2}}|^2 + \tau \sum_{t=1}^{n-1} |w^t|^2)
+ |g^{n-\frac{1}{2}}|^2 + |w^n|^2 + |g^{\frac{1}{2}}|^2 + |w^0|^2. \tag{4.7}
\]
Lemma 4.6 (Gronwall Inequality([37])). Suppose that the discrete function \( \{ \omega^n \mid n = 0, 1, 2 \cdots N \} \) is nonnegative and satisfies the recurrence formula

\[
\omega^n - \omega^{n-1} \leq A \omega^n \tau + B \omega^{n-1} \tau + C_n \tau,
\]

where \( A, B \) and \( C_n(n = 1, 2, \cdots) \) are nonnegative constants. Then

\[
\max_{0 \leq n \leq N} |\omega^n| \leq (\omega^0 + \sum_{k=1}^{N} C_k \tau)e^{2(A+B)T},
\]

where \( \tau \) is satisfied \( (A + B) \tau \leq \frac{N-1}{2N}(N > 1) \).

Lemma 4.7 (Gronwall Inequality([37])). Suppose that the discrete function \( \{ \omega^n \mid n = 0, 1, 2 \cdots N \} \) is nonnegative and satisfies the inequality

\[
\omega^n \leq A + \tau \sum_{l=1}^{n} B_l \omega^l,
\]

where \( A \) and \( B_l(l = 1, 2, \cdots) \) are nonnegative constants. Then

\[
\max_{0 \leq n \leq N} |\omega^n| \leq A e^{2 \sum_{i=1}^{N} B_i \tau},
\]

where \( \tau \) is satisfied \( \tau\left( \max_{l=0,1,\cdots,N} B_l \right) \leq \frac{1}{2} \).

Lemma 4.8 ([27]) For any complex numbers \( U, V, u, v \), the following inequality holds

\[
||U||^2 - |u|^2 |v| \leq \left( \max\{|U|, |V|, |u|, |v|) \right)^2 \cdot (2|U - u| + |V - v|).
\]

Theorem 4.1 We assume that the continuous solution \( u \) of (1.1) satisfies

\[
u(x, t) \in C^4(0, t; H^r_p(\Omega)), \quad r > 1,
\]

then the solution \( U^n \) of (3.2) is unconditionally convergent with order of \( O(\tau^2 + N^{\alpha/2-r}) \) in the discrete \( L^\infty \) norm.

Proof. We denote

\[
uu^* = P_{N-2} u, \quad f = f(u) = \beta |u|^2 u, \quad f^* = P_{N-2} f.
\]

The projection equation of (1.1) is

\[
i \partial_t \nu^* - (\Delta)^{\frac{1}{2}} \nu^* + f^* = 0.
\]

We define

\[
\xi_j^{n+\frac{1}{2}} = i \delta_t^+ u_j^{n+\frac{1}{2}} - (D_\alpha u^{*(n+1/2)})_j + f_j^*.
\]
Since \( u^* \in S_N \), \((-\Delta)^{\frac{3}{2}} u^*(x_j, t_n) = (D_\alpha u^{*(n+1/2)})_j\), we obtain
\[
\xi_j^{n+\frac{1}{2}} = i(\delta^+_t u^*_j - \partial_t u^{*(n+1/2)}),
\] (4.17)
and
\[
\delta^+_t \xi_j^n = \frac{\xi_j^{n+\frac{1}{2}} - \xi_j^{n-\frac{1}{2}}}{\tau}
= \frac{i}{\tau^2}(u_j^{n+2} - 2u_j^{n+1} + u_j^n) - \frac{i}{2\tau}(\partial_t u_j^{n+2} - \partial_t u_j^n).
\] (4.18)
Using the Taylor expansion, we obtain
\[
|\xi_j^{n+\frac{1}{2}}| \leq C\tau^2,
\] (4.19)
and
\[
|\delta^+_t \xi_j^n| \leq C\tau^2.
\] (4.20)
for some constant \( C \).

Denote \( e^n_j = u^*_j - U^n_j \). Subtracting (3.2) from (4.15) yields the following error equation
\[
\xi^{n+\frac{1}{2}} = i\delta^+_t e^n - D_\alpha e^{n+1/2} + G^n,
\] (4.21)
\[
e^0 = u^* - u^0,
\] (4.22)
where
\[
G^n_j = f_j^{*n+1/2} - F(U^n_j, U^{n+1}_j).
\]
Denoting
\[
(G_1)^n_j = f_j^{*n+1/2} - f_j^{n+1/2}, \quad (G_2)^n_j = f_j^{n+1/2} - F(u^n_j, u^{n+1}_j),
\]
\[
(G_3)^n_j = F(u^n_j, u^{n+1}_j) - F(u_j^n, u_j^{n+1}), \quad (G_4)^n_j = F(u^n_j, u^{n+1}_j) - F(U^n_j, U^{n+1}_j),
\] (4.23)
we have \( G^n_j = (G_1)^n_j + (G_2)^n_j + (G_3)^n_j + (G_4)^n_j \).
According to Lemma 4.2, we have
\[
\|G_1^n\|_h \leq CN^{-r}.
\] (4.24)
By the Taylor expansion, we can see that
\[
\|G_2^n\|_h \leq C\tau^2.
\] (4.25)
From Lemma 4.8, we deduced that
\[
|(G_3)^n_j| \leq C(|u^n_j - u^*_j| + |u_j^{n+1} - u_j^{n+1}|).
\] (4.26)
This, together with Lemma 4.3, gives
\[
\|G_3^n\|_h \leq CN^{-r}.
\] (4.27)
From the definition of $G^n_4$, we have
\[
(G_4^n)_j = \frac{\beta}{2} (|u_j e_n|^2 + |u_j e_n + 1|^2) u_j^{e_n + \frac{1}{2}} - \frac{\beta}{2} (|U^n_j|^2 + |U^n_j + 1|^2) U^n_j^{e_n + \frac{1}{2}} \\
= \frac{\beta}{2} (|u_j e_n|^2 + |u_j e_n - 1|^2) - |u_j e_n + 1|^2 - |u_j e_n + 1|^2) u_j^{e_n + \frac{1}{2}} \\
= \frac{\beta}{2} (|u_j e_n|^2 - |u_j e_n|^2 - |u_j e_n + 1|^2) u_j^{e_n + \frac{1}{2}} \\
= \frac{\beta}{2} (|u_j e_n|^2 - |u_j e_n|^2 - |u_j e_n + 1|^2) u_j^{e_n + \frac{1}{2}} + \frac{\beta}{2} (|U^n_j|^2 + |U^n_j + 1|^2) U^n_j^{e_n + \frac{1}{2}} \\
:= (G_{41})^n_j + (G_{42})^n_j,
\]
where
\[
(G_{41})^n_j = \frac{\beta}{2} (|u_j e_n|^2 + |u_j e_n + 1|^2 - |u_j e_n|^2 - |u_j e_n + 1|^2) u_j^{e_n + \frac{1}{2}} \\
= \frac{\beta}{2} (|u_j e_n|^2 + |u_j e_n| + 1 + |u_j e_n + 1|^2 + |u_j e_n + 1|^2) - |e_n|^2 - |e_n + 1|^2) u_j^{e_n + 1/2}. \\
(G_{42})^n_j = \frac{\beta}{2} (|U^n_j|^2 + |U^n_j + 1|^2) e_j^{\frac{n+1}{2}}.
\]

Computing the discrete inner product of (4.21) with $e^{n+1/2}$, then taking the imaginary part, we obtain
\[
\frac{1}{2\tau} (\|e^{n+1}\|^2_h - \|e^n\|^2_h) + \text{Im}(G^n_1 + G^n_2 + G^n_3 + G^n_4, e^{n+1/2})_h = \text{Im}(\xi^{e_n + 1/2}, e^{n+1/2})_h,
\]
Using Cauchy-Schwartz inequality, we obtain
\[
|\langle (G^n_s, e^{n+1/2})_h \rangle| \leq \frac{1}{2} \|G^n_s\|_h^2 + \frac{1}{4} (\|e^n\|^2_h + \|e^{n+1}\|^2_h), \\
|\text{Im}(G^n_4, e^{n+1/2})_h| = |\text{Im}(G^n_{41}, e^{n+1/2})_h| \leq C(\|e^n\|^2_h + \|e^n\|_h^4 + \|e^{n+1}\|^2_h + \|e^{n+1}\|_h^4), \\
|\langle (\xi^{e_n + 1/2}, e^{n+1/2})_h \rangle| \leq \frac{1}{2} \|\xi^{e_n + 1/2}\|^2_h + \frac{1}{4} (\|e^n\|^2_h + \|e^{n+1}\|^2_h).
\]

According to theorem 3.2, Lemma 4.1 and Lemma 4.2 we can get
\[
\|e^n\|_h \leq \|u^n - U^n\|_h \leq \|u^n\|_h + \|U^n\|_h \leq \sqrt{2} \|u^n\| + \|U^n\|_h \leq C. \\
|e^n|_{H^{1/2}_h} \leq \|u^n - U^n\|_{H^{1/2}_h} \leq \|u^n\|_{H^{1/2}_h} + \|U^n\|_{H^{1/2}_h} \leq C.
\]

From (4.34)-(4.35) and Lemma 2.4, we have
\[
\|e^n\|^2_h \leq C\|e^n\|^2_h.
\]
This, together with (4.32), gives
\[
|\text{Im}(G^n_4, e^{n+1/2})_h| \leq C(\|e^n\|^2_h + \|e^{n+1}\|_h), \\
\|G^n_4\|^2_h \leq C(\|e^n\|^2_h + \|e^{n+1}\|_h).
\]

Substituting (4.31), (4.33), (4.37) into (4.30) yields
\[
\frac{1}{2\tau} (\|e^{n+1}\|^2_h - \|e^n\|^2_h) \leq C(\|e^n\|^2_h + \|e^{n+1}\|^2_h) + \frac{1}{2} (\|G^n_1\|^2_h + \|G^n_2\|^2_h + \|G^n_3\|^2_h + \|\xi^{e_n + 1/2}\|^2_h).
\]
This together with Lemma 4.6 and (4.24), (4.25), (4.27), gives that, for a sufficiently small $\tau$,

$$
\|e^n\|_h^2 \leq (\|e^0\|_h^2 + CT(N^{-2r} + \tau^4)e^{4CT}).
$$

(4.40)

This together with

$$
\|e^0\|_h = \|u^{*0} - u^0\|_h \leq CN^{-r},
$$

(4.41)

gives

$$
\|e^n\|_h \leq C(N^{-r} + \tau^2).
$$

(4.42)

With Lemma 4.3 and (4.42), we can get

$$
\|u^n - U^n\|_h \leq \|u^n - u^{*n}\|_h + \|u^{*n} - U^n\|_h \leq C(N^{-r} + \tau^2).
$$

(4.43)

Computing the discrete inner product of (4.21) with $\delta_t^+ e^n$, and taking the real part, we obtain

$$
\frac{1}{\tau} (|e^{n+1/2}|_{H^{\alpha/2}}^2 - |e^n|_{H^{\alpha/2}}^2) = -2\text{Re}(G^n, \delta_t^+ e^n)_h + 2\text{Re}(\xi^{n+\frac{1}{2}}, \delta_t^+ e^n)_h,
$$

(4.44)

where (2.27) is used. From (4.21), we have

$$
\delta_t^+ e^n = -iD_\alpha e^{n+1/2} + iG^n - i\xi^{n+\frac{1}{2}}.
$$

(4.45)

Substituting (4.45) into the first term on the right side of (4.44), we have

$$
2\text{Re}(G^n, \delta_t^+ e^n) = 2\text{Re}(G^n, -iD_\alpha e^{n+1/2} + iG^n - i\xi^{n+\frac{1}{2}})_h
$$

$$
= 2\text{Im}(G^n, D_\alpha e^{n+1/2})_h + 2\text{Im}(G^n, \xi^{n+\frac{1}{2}})_h.
$$

(4.46)

By virtue of (2.30), we have

$$
2\text{Im}(G^n, D_\alpha e^{n+1/2})_h \leq 2|G^n|_{H^{\alpha/2}}|e^{n+1/2}|_{H^{\alpha/2}}^2
$$

$$
\leq |G^n|_{H^{\alpha/2}}^2 + |e^{n+1/2}|_{H^{\alpha/2}}^2,
$$

(4.47)

and

$$
2\text{Im}(G^n, \xi^{n+\frac{1}{2}})_h \leq 2\|G^n\|_h \|\xi^{n+\frac{1}{2}}\|_h \leq \|G^n\|_h^2 + \|\xi^{n+\frac{1}{2}}\|_h^2.
$$

(4.48)

The estimate of $|G^n|_{H^{\alpha/2}}$ is established as follows. According to Lemma 4.4, we have

$$
|G^n|_{H^{\alpha/2}} \leq CN^{\alpha/2-r}.\tag{4.49}
$$

By Taylor formula, we can see that

$$
|G^n|_{H^{\alpha/2}} \leq C\tau^2.\tag{4.50}
$$

With the similar argument for $G_3^n$ and $G_4^n$ as above, we can deduce that

$$
|G_3^n|_{H^{\alpha/2}} \leq CN^{\alpha/2-r},
$$

(4.51)
and

$$|G_4|^2_{H_h^{n/2}} \leq C(\|e^n\|^2_h + \|e^{n+1}\|^2_h + |e^n|_{H_h^{n/2}}^2 + |e^{n+1}|_{H_h^{n/2}}^2),$$

(4.52)

where Lemma 4.4 is used. Therefore, it follows from (4.49) and (4.52) that

$$|G_4|^2_{H_h^{n/2}} \leq C(\|e^n\|^2_h + \|e^{n+1}\|^2_h + |e^n|_{H_h^{n/2}}^2 + |e^{n+1}|_{H_h^{n/2}}^2)$$

$$+ C(N^{\alpha-2r} + \tau^4).$$

(4.53)

Hence we get

$$2\text{Re}(G^n, \delta^+_i e^n) \leq C(\|e^n\|^2_h + \|e^{n+1}\|^2_h + |e^n|_{H_h^{n/2}}^2 + |e^{n+1}|_{H_h^{n/2}}^2)$$

$$+ C(N^{\alpha-2r} + \tau^4).$$

(4.54)

Substituting (4.54) into (4.44), we have

$$|e^{n+1}|_{H_h^{n/2}}^2 - |e^n|_{H_h^{n/2}}^2 \leq C\tau(\|e^n\|^2_h + \|e^{n+1}\|^2_h + |e^n|_{H_h^{n/2}}^2 + |e^{n+1}|_{H_h^{n/2}}^2)$$

$$+ 2\tau\text{Re}(\xi^{n+\frac{1}{2}}, \delta^+_i e^n)_h + C\tau(N^{\alpha-2r} + \tau^4).$$

(4.55)

Summing up the superscript $n$ from 0 to $M$ and then replacing $M$ by $n$, we have

$$|e^{n+1}|_{H_h^{n/2}}^2 - |e^0|_{H_h^{n/2}}^2 \leq \tau \sum_{l=0}^n C(\|e^l\|^2_h + \|e^{l+1}\|^2_h + |e^l|_{H_h^{n/2}}^2 + |e^{l+1}|_{H_h^{n/2}}^2)$$

$$+ 2\tau \sum_{l=0}^n \text{Re}(\xi^{l+\frac{1}{2}}, \delta^+_i e^l)_h + CT(N^{\alpha-2r} + \tau^4).$$

(4.56)

With the Lemma 4.5 (4.19) and (4.20), we have

$$|2\tau \sum_{l=0}^n \text{Re}(\xi^{l+\frac{1}{2}}, \delta^+_i e^l)_h| = |\text{Re}(2\tau \sum_{l=0}^n \sum_{j=0}^{N-1} \xi^{l+\frac{1}{2}} j (\delta^+_i e^l_j))|

\leq \left( h \sum_{j=0}^{N-1} 2\tau \sum_{l=1}^n \xi^{l+\frac{1}{2}} (\delta^+_i e^l_j) \right)

\leq \left( \tau \sum_{l=1}^n \|\delta^+_i e^{l-\frac{1}{2}}\|_h + \tau \sum_{l=0}^n \|e^l\|^2_h \right)

+ \|\xi^{n+\frac{1}{2}}\|^2_h + \|e^{n+1}\|^2_h + \|\xi^{\frac{1}{2}}\|^2_h + \|e^0\|^2_h

\leq C(N^{-r} + \tau^4).$$

(4.57)

By virtue of (4.43), we deduce from (4.56) and (4.57)

$$|e^{n+1}|_{H_h^{n/2}}^2 \leq |e^0|_{H_h^{n/2}}^2 + \tau \sum_{l=0}^n C(|e^l|^2_{H_h^{n/2}} + |e^{l+1}|^2_{H_h^{n/2}}) + C(N^{\alpha-2r} + \tau^4)$$

$$\leq \tau \sum_{l=1}^{n+1} C|e^l|^2_{H_h^{n/2}} + C(N^{\alpha-2r} + \tau^4),$$

(4.58)
where $|e^0|_{H^{\alpha/2}_h} \leq CN^{\alpha-2r}$ is used.

Applying Lemma 4.7 to (4.58), we get

$$|e^n|_{H^{\alpha/2}_h} \leq (N^{\alpha-2r} + \tau^4)e^2\sum_{k=1}^{N} C\tau,$$

(4.59)

where $\tau$ is sufficiently small, such that $C\tau \leq \frac{1}{2}$. With Lemma 4.4 and (4.59), we can prove

$$|u^n - U^n|_{H^{\alpha/2}_h} \leq |u - u^*|_{H^{\alpha/2}_h} + |u^* - U|_{H^{\alpha/2}_h} \leq C(N^{\alpha-2r} + \tau^4).$$

(4.60)

Finally, thanks to Lemma 2.1, we obtain

$$\|u^n - U^n\|_{l^\infty_h} \leq C(N^{\alpha/2-r} + \tau^2).$$

(4.61)

This completes the proof.

5 Numerical examples

In this section, we test the numerical accuracy and discrete conservation laws of the fully discrete pseudo-spectral scheme (3.2). Similar to that in [10], we can use the fixed point iteration method and the fast Fourier transform (FFT) to solve the nonlinear system defined in scheme (3.2). For the convergence rate, we use the formula

$$Order = \frac{\ln(error_1/error_2)}{\ln(\tau_1/\tau_2)},$$

(5.1)

where $\tau_l, error_l, (l = 1, 2)$ are step sizes and corresponding errors, respectively. The relative errors of energy and mass are defined as

$$RH^n = |(H^n - H^0)/(H^0)|, \quad RM^n = |(M^n - M^0)/(M^0)|.$$

(5.2)

5.1 Example 1

Consider the FNLS equation (1.1) with plane wave solution [7]

$$u(x, t) = A\exp(i(\lambda x - \omega t)), \quad with \ \omega = |\lambda|^\alpha - \beta|A|^2,$$

(5.3)

where the problem is solved on domain $[-\pi, \pi]$ with $\lambda = 4, A = 1, \beta = -2$.

Firstly, we test the accuracy and efficiency of numerical scheme (3.2) for different power $\alpha$. Table 1 indicates that the method is of second-order in time. Table 2 shows that the spatial error is very small and almost negligible, and the error is dominated by the time discretization error. It confirms that, for sufficiently smooth problem, the Fourier pseudo-spectral method is
of arbitrary order of accuracy. These numerical results confirm the accuracy of the numerical scheme \((3.2)\) in Theorem 4.1.

Secondly, we testify the discrete conservation laws. Considering the iteration error and the slow-varying process of the solution, it is easy to select large-step \(\tau = 0.05\) in relative errors test of energy and mass. In Figure 1, we depict the relative errors of energy and mass in a longer time interval. It is observed that the scheme \((3.2)\) conserves both energy and mass very well.

| \(\alpha\) | \(\tau\) | \(L^\infty\) | Rate | \(L^2\) | Rate |
|---|---|---|---|---|---|
| 1.4 | 0.05 | 0.1529 | - | 0.3831 | - |
| 0.025 | 0.00382 | 2.0017 | 0.0957 | 2.0017 |
| 0.0125 | 0.0095 | 2.0095 | 0.0238 | 2.0095 |
| 0.00625 | 0.0024 | 2.0069 | 0.0059 | 2.0069 |
| 1.7 | 0.05 | 0.4062 | - | 1.0183 | - |
| 0.025 | 0.1041 | 1.9644 | 0.2609 | 1.9644 |
| 0.0125 | 0.0260 | 2.0013 | 0.0652 | 2.0013 |
| 0.00625 | 0.0065 | 2.0049 | 0.0162 | 2.0049 |
| 1.9 | 0.05 | 0.7866 | - | 1.9718 | - |
| 0.025 | 0.2104 | 1.9027 | 0.5273 | 1.9027 |
| 0.0125 | 0.0530 | 1.9900 | 0.1328 | 1.9900 |
| 0.00625 | 0.0132 | 2.0024 | 0.0331 | 2.0024 |
| 2.0 | 0.05 | 1.0794 | - | 2.7056 | - |
| 0.025 | 0.3011 | 1.8420 | 0.7547 | 1.8420 |
| 0.0125 | 0.0763 | 1.9805 | 0.1912 | 1.9806 |
| 0.00625 | 0.0191 | 2.0004 | 0.0478 | 2.0004 |

### 5.2 Example 2

In this example, we study the dynamics of solitions in the one-dimensional FNLS equation

\[
iu_t - (-\Delta)^{\alpha/2}u + |u|^2u = 0,\tag{5.4}
\]

with initial condition

\[
u(x, 0) = \text{sech}(\sqrt{2}x/2)\exp(ix/2), \quad x \in [-20, 20].\tag{5.5}
\]

For \(\alpha = 2\), the exact solution is the solition on the whole real axis. In our simulations, we choose the spatial grid size \(N = 320\).
Table 2: Convergence test in space for different $\alpha$ with $\tau=1.0e-6$ at $T = 1$.

| $\alpha$ | N  | $L^\infty$   | $L^2$       |
|---------|----|--------------|-------------|
| 1.4     | 16 | 1.5612e-10   | 3.5615e-10  |
|         | 32 | 3.0115e-10   | 7.3557e-10  |
|         | 64 | 1.0579e-10   | 2.0756e-10  |
|         | 128| 1.0232e-10   | 2.0650e-10  |
| 1.7     | 16 | 1.2158e-10   | 2.9206e-10  |
|         | 32 | 9.9284e-11   | 2.4356e-10  |
|         | 64 | 2.0598e-10   | 4.4501e-10  |
|         | 128| 1.7674e-10   | 4.0082e-10  |
| 1.9     | 16 | 3.1580e-10   | 7.8269e-10  |
|         | 32 | 1.9745e-10   | 4.8347e-10  |
|         | 64 | 4.0320e-10   | 9.4242e-10  |
|         | 128| 3.7175e-10   | 8.9973e-10  |
| 2.0     | 16 | 6.0690e-10   | 1.5007e-09  |
|         | 32 | 1.9745e-10   | 4.8347e-10  |
|         | 64 | 4.0320e-10   | 9.4242e-10  |
|         | 128| 3.7175e-10   | 8.9973e-10  |

Figure 1: Relative errors of energy and mass with $N = 32$, $\tau=0.05$ for different $\alpha$. 
In Figure 2, the relative errors of energy and mass are plotted, which show that energy and mass are conserved very well. Figure 3 presents the time evolution of the density $|u(x, t)|$ of the FNLS with different power $\alpha$. In the classical NLS with $\alpha = 2$, the shape and velocity of soliton solutions are unchanged. In the FNLS, the shape of soliton solutions presents a slow changing process. When $\alpha$ decreases, the propagation of waves in the time-axis direction slows down with the elapse of time. We can also see that the solution of the FNLS behaves more like a wave with effects that might be described as “interference” arising from the long-range interactions of the fractional Laplacian. All these properties of FNLS may better simulate the shape of waves in physics.

6 Conclusions

In this paper, unconditional convergence analysis of a conservative Fourier pseudo-spectral method for solving the FNLS equation is established. We introduce the discrete fractional Sobolev space $H^{\alpha/2}_{\text{d}}$ with a new discrete fractional Sobolev norm for the first time, and we also prove some lemmas for the new Sobolev norm. Based on these lemmas and energy method, a priori error estimate for the method can be estimated. Then, we can prove that the conservative Fourier pseudo-spectral method is unconditionally convergent with order of $O(\tau^2 + N^{\alpha/2-r})$ in the discrete $L^\infty$ norm. In fact, here we adopt a more direct analysis method so that unconditionally convergent results can be obtained without establishing semi-norm equivalence in error analysis. Furthermore, The method of error analysis in the discrete $L^\infty$ for the presented conservative Fourier pseudo-spectral scheme can be extended to other PDEs involving fractional Laplacian, for example, the space fractional Allen-Cahn equation [26] and the Klein-Gordon-Schrödinger equation [13].
Figure 3: Evolution of the solitons with $N = 320$, $\tau = 0.01$ for different $\alpha$. 

(a) $\alpha = 2, \tau = 0.01$

(b) $\alpha = 1.95, \tau = 0.01$

(c) $\alpha = 1.9, \tau = 0.01$

(d) $\alpha = 1.7, \tau = 0.01$

(e) $\alpha = 1.5, \tau = 0.01$

(f) $\alpha = 1.3, \tau = 0.01$
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