Abstract

We use the t’Hooft-Wilson method for the generation of static fermions potential in order to derive a class of confining potentials which can describe the quark confinement. A general pattern for the construction of propagators through the localization of non-local actions is uncovered.

1 Introduction

One of the most interesting open problems of contemporary physics is the quarks and gluons confinement [1]. Confinement is a property believed to hold in QCD associated to the non-abelian gauge symmetry, and is searched to explain the absence of colored particles from the QCD spectrum [2, 3]. The role of gauge copies in the dynamical setup underlining the confinement has been uncovered years ago [4–7].

Different criteria to identify the confinement have been proposed, according to the distinct characteristics of the elementary particles, such as spin and statistics. In the gluons case, the criterion mostly used is the breakdown of the propagator’s positivity [5, 8, 9], which is a quantum mechanical
requirement. In the quarks case, confinement is associated to the t’Hooft-Wilson criterion \[10,11\]. This is ultimately quantum mechanical, although its fundamentals can be understood classically. It allows for the identification of the interaction potential and the very simple idea emerges that an infinite energy would be necessary to separate the interacting elements. To get a clear idea of the attractiveness of this criterion, imagine two particles at rest, separated by the distance \(r_0\), and submitted to the action of an attractive Coulomb potential depending on the distance \(r\) between the charges, 

\[
U = -\frac{1}{4\pi} \frac{q^2}{r}
\]

(1)

To separate these particles up to an infinite distance, we demand the exponent of an energy minimum given by \(\Delta U = \frac{1}{4\pi} \frac{q^2}{r_0}\). Contrast the situation, for example, with the one when the attractive potential is

\[
U = \alpha^2 r.
\]

(2)

In this case, an infinite amount of energy would be required to completely separate the particles. This behavior is believed to occur in QCD through the formation of a flux tube of color fields linking the particles. Then, if the energy necessary to pull them apart increases indefinitely, it becomes more energetically efficient for the system to create new particles than to deconfine the first ones. These are clearly confining potentials.

The static potentials are due to vectorial bosons mediated quark interactions. This is seen through a procedure which first-quantizes the fermions while the boson are fully second-quantized. As a result the interaction potential between two fermions of charges \(q_x\) and \(q_y\) located in the spatial positions \(\vec{x}\) and \(\vec{y}\) is given by \[11\] \[12\]

\[
V(\vec{r}) = \lim_{T \to \infty} \frac{2\pi^2}{T} \text{tr} \int d^4p \tilde{J}^0\mu(p) \tilde{M}_{\mu\nu}^{-1ab}(p) \tilde{J}^b\nu(-p),
\]

(3)

where \(\tilde{J}^0\mu\) is the Fourier transform of the four current associated to the static fermions \[\]

\[
J^a_\mu(\vec{z}) = q_x \delta_{\mu0} \delta(\vec{z} - \vec{x}) T^a + q_y \delta_{\mu0} \delta(\vec{z} - \vec{y}) T^a
\]

(4)

and \(\vec{r} = \vec{x} - \vec{y}\). Here \(\tilde{M}_{\mu\nu}^{-1ab}(p)\) is the inverse vector field propagator Fourier transform. As an example, for the photon case there are no color indexes \((T^a = 1)\),

\[
\tilde{M}_{\mu\nu}^{-1}(p) = \frac{1}{4\pi p^2} \left( \delta_{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{p^2} \right),
\]

(5)

---

\(^1T^a\) are the SU(N) group generators.
and the coulomb potential \(^2\)

\[ V(r) = \frac{1}{4\pi} \frac{q_x q_y}{r}, \quad (6) \]

is obtained.

The aim of this paper is to study the potentials that are associated to models that have been recently proposed. This paper is organized as follows. In section 2, we present the propagator that gives rise to a polynomial generalization of the linear Cornell potential \[^{12,13}\]. In section 3, we obtain the potential for the refined Gribov-Zwanziger propagator. In section 4, we review the actions associated to the Gribov-Zwanziger propagator and then obtain those that generate the polynomial potentials.

## 2 Polynomial Potentials

In order to assure the confinement through the t’Hoft-Wilson criterion, the potential must grow with the distance between the constituents. The most simple confining potential is the linear one, but higher polynomials can even do a better job. Note that the dynamical input in equation \(^3\) lies essentially on the propagator, which is inherent to the model. Different propagators, however, have emerged as the result of quantum computations of Yang-Mills theory. In this setting, interesting simple generalizations of the electromagnetic free propagator have been considered. This prompts us to study a class of propagators that leads to polynomial potentials.

The electromagnetic-like propagator

\[ M_{\mu\nu}^{(1)}(p) = \frac{\delta^{ab}}{4\pi} \left( \frac{1}{p^2 + \xi m^2} \right) \left( \delta_{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{p^2} \right) \quad (7) \]

will be our starting point\[^4\]. Propagators of type \(^7\) appear naturally from the localization processes of actions with non local terms. The importance of this procedure in the study of gluon confinement has already been stressed \[^{14}\]. Let us include another constant \( \zeta = \pm 1 \) to control the relative signs of the charges, \( q_x = q \) and \( q_y = \zeta q \). The current turns out to be,

\[ \tilde{J}_a^{\mu}(p) = \frac{1}{2\pi} q \delta(p_0) \left( e^{-i\vec{p}.\vec{x}} + \zeta e^{-i\vec{p}.\vec{y}} \right) \delta_{\mu0} T^a \quad (8) \]

\[^2\]except for an infinite factor of self-energy

\[^3\]We introduced a constant \( \xi = \pm 1 \). Its role will soon be clear
and the equation (3) takes the form

\[ V(\vec{x} - \vec{y}) = \frac{C_2(R)q^2}{(2\pi)^3} \int \frac{d^3p}{p^2} \left( 1 + \frac{\xi m^n}{p^n} \right) \left( 1 + \zeta \cos(\vec{p} \cdot (\vec{x} - \vec{y})) \right), \tag{9} \]

where \( C_2(R)I = T^a T^a \) (\( I \) is the identity matrix for the \( SU(N) \) group representation). Keeping the notation \( \vec{r} = \vec{x} - \vec{y} \) and extracting the \( r \) independent self energy piece

\[ \frac{q^2 C_2(R)}{2\pi^2} \int dp \left( 1 + \frac{\xi m^n}{p^n} \right), \tag{10} \]

we obtain

\[ V(\vec{r}) = \frac{C_2(R)\zeta q^2}{2\pi^2} \int_{-\infty}^{\infty} dp \left( \frac{1}{2ipr} e^{ipr} + \frac{\xi m^n}{2ip^{(n+1)r}} e^{ipr} \right). \tag{11} \]

The complex integration process leads to the potential

\[ V(\vec{r}) = \frac{\zeta C_2(R) q^2}{4\pi} \frac{1}{r} + \frac{\xi \zeta C_2(R) i^n m^n}{4\pi n!} q^2 r ^{n-1}. \tag{12} \]

The simplest case is the Coulomb potential and it corresponds to \( n = 0 \) or \( m = 0 \), so that (7) leads to (5).

The case \( n = 2 \) is particularly important

\[ V(\vec{r}) = \frac{\zeta C_2(R) q^2}{4\pi} \frac{1}{r} - \frac{\xi \zeta m^2 C_2(R)}{8\pi} q^2 r. \tag{13} \]

This potential (Coulomb + linear) presents the standard confining properties. The case, when \( \xi = 1 \) and \( \zeta = -1 \) is known as Cornell’s potential \[15\] and was obtained in [12, 13]. The presence of the linear rising term in the potential is sometimes termed as the emergence of a ”magnetic phase” of QCD, and implies an area-law exponential decay of the Wilson-Loop \[16\]. This situation is the case where the two terms, Coulomb and linear, are attractive.

Another interesting case happens, still in the \( n = 2 \) scenario, when \( \xi = -1 \) e \( \zeta = 1 \). Here the Coulomb term is repulsive and the linear is attractive. The later term is dominant for large distances while for short distances the Coulomb one dominates. There is an intermediate point of equilibrium which describes the confined system distance of stability. This gives the measure of the resultant composed particle size. There is always the possibility of the existence of other interactions, but we see that it is just

\[ C_2(R) \] is the value of the Casimir in the representation G.
enough to consider the propagator (7) and, of course, just one interaction field, to obtain confinement and stability of the composed resultant particle.

Polynomial potentials of order \( n > 1 \), which behave at large distance as \( r^n \), lead to stronger confinement character than the linear potential. In order to obtain the reality of the energy, \( n \) must be even. The parity of these terms is related to the even number of derivatives in the Lagrangians kinetic terms.

Let us stress that the existence of a balance point between the interactions, analogous to that obtained in the \( \xi = -1 \) and \( \zeta = 1 \) linear potential case, can be obtained in a larger set of situations. Note that in order to keep the Coulombian interaction repulsive (\( \zeta = 1 \)), and the polynomial part attractive, we must impose \( \xi r^n > 0 \). Since \( \xi = \pm 1 \), there is no restriction for the values of \( n \), apart from being even, which gives rise to a wide spectrum of possible forms of these confining potentials.

3 The Refined Gribov Zwanziger Potential

We now discuss to what extent do the vector propagators that confine gluons by the positivity breakdown generate confinement of fermions. We briefly review the potential calculation for the Gribov-Zwanziger (GZ) propagator. For the case of pure GZ [5],

\[
\widetilde{M}^{-1ab}_{\mu\nu}(p) = \frac{\delta^{ab} p^2}{4\pi(p^4 + \gamma^4)} \left( \delta_{\mu\nu} - \left(1 - \alpha\right) \frac{p_\mu p_\nu}{p^2} \right),
\]

(14)

the quantum corrections to the potential were analyzed in [17], where \( \gamma \) is determined by Gribov’s process and takes significance only in the infrared regime.

The GZ model needs, however, to be refined when renormalization is taken into account. New massive constants appear when the counter-term is built in the quantization procedure. The propagator takes the form [18]

\[
\widetilde{M}^{-1ab}_{\mu\nu}(p) = \frac{\delta^{ab}(p^2 + M^2)}{4\pi(p^4 + (M^2 + m^2)p^2 + 2g^2N\gamma^4 + M^2m^2)} \times \left( \delta_{\mu\nu} - \left(1 - \alpha\right) \frac{p_\mu p_\nu}{p^2} \right).
\]

(15)

In this case, the potential (3) is given by

\[
V(\vec{r}) = \frac{C_2(R)g^2}{2\pi^2} I,
\]

(16)
where

\[ I = \int_{-\infty}^{\infty} \frac{p^2 (p^2 + M^2)}{p^4 + (M^2 + m^2)p^2 + 2g^2 N\gamma^4 + M^2 m^2} \left\{ 1 + \frac{i}{2pr} e^{ipr} \right\} dp. \]  (17)

The self-energy results from the part of the above integral given by

\[ \frac{C_2(R)g^2}{2\pi^2} \int_{-\infty}^{\infty} \frac{p^2 (p^2 + M^2)}{p^4 + (M^2 + m^2)p^2 + 2g^2 N\gamma^4 + M^2 m^2} dp, \]  (18)

and will be discarded.

If we define

\[ \beta^2 = M^2 + m^2, \quad \lambda^4 = 2g^2 N\gamma^4 + M^2 m^2, \]  (19)

the potential (16), apart from the self-energy, takes the form

\[
V(\vec{r}) = \frac{C_2(R)g^2}{2\pi r} e^{-\frac{r}{2}\sqrt{2\lambda^2 + \beta^2}} \left\{ \frac{2M^2 - \beta^2}{2\sqrt{4\lambda^2 - \beta^2}} \sin \left( \frac{r}{2} \sqrt{2\lambda^2 - \beta^2} \right) + \frac{1}{2} \cos \left( \frac{r}{2} \sqrt{2\lambda^2 - \beta^2} \right) \right\},
\]  (20)

Taking the limit \( \beta \to 0 \) and \( M \to 0 \), it results in

\[
V(\vec{r}) = \frac{C_2(R)g^2}{2\pi r} e^{-\lambda r/2} \cos \frac{\lambda \sqrt{2r}}{2}.
\]  (21)

The decreasing exponential terms present in (20) and (21) points to the non confining character of the fermions in the refined Gribov-Zwanziger theory. The potential in (21) is exactly the same as presented in [17] showing that the new scales introduced by renormalization in the refined theory do not impact the non confining character of the fermions in Gribov-Zwanziger theory.

4 The actions

We will provide, in this section, the actions generating the GZ propagator and confining polynomial potentials. The complete treatment of such actions, including the full discussion of the renormalization, will not be dealt with here. We will restrict ourselves to show the starting point used in the GZ renormalization, adapted to each polynomial potential generating action.
4.1 The Gribov-Zwanziger case

The GZ action was born from the search for the elimination of copies of gauge still present in Yang-Mills model, even after the naive fixation of the gauge, which we assumed here to be Landau. The starting action is

\[ S_{YM} = \int d^4 x \left( \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i b^a \partial^\mu A^a_\mu + \bar{\omega}^a \partial^\mu D^{ab}_\mu c^b \right), \tag{22} \]

where

\[ F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{23} \]

and

\[ D^{ab}_\mu = \delta^{ab} \partial_\mu - g f^{abc} A^c_\mu. \tag{24} \]

The action is invariant under the BRST transformations

\[ sA^a_\mu = -D^{ab}_\mu c^b, \quad sc^a = g \frac{2}{g} f^{abc} c^b c^c, \quad s\bar{\omega}^a = i b^a, \quad sb^a = 0. \tag{25} \]

Following the Gribov procedure [4], the elimination of the persistent copies is performed studying the configurations defined by the operator

\[ M^{ab} = -\partial^\mu D^{ab}_\mu \tag{26} \]

It was initially assumed that the first Gribov region (Ω),

\[ \Omega := \{ A^a_\mu | \partial^\mu A^a_\mu = 0, M^{ab}(A) > 0 \}. \tag{27} \]

is free of copies. As pointed out by Zwanziger [5], the implementation of the action that eliminates the copies includes a non-local term of the form

\[ S_{GZ} = S_{YM} + \gamma^4 g^2 \int d^4 x d^4 y f^{abc} A^{ab}(x)[M^{-1}]^{ad}(x,y) f^{dec} A^e_\mu. \tag{28} \]

The localization process is done by means of the quartet

\[ s\bar{\omega}^a_\mu = \bar{\phi}^a_\mu, \quad s\bar{\phi}^a_\mu = 0, \quad s\varphi^a_\mu = 0, \quad s\varphi^a_\mu = \omega^a_\mu. \tag{29} \]

The gauge group is SU(N).

Now it is known that there are copies inside the first region [8,19]. The more restrictive modular region, within the first Gribov region, has to be considered in order to try to get rid of the copies.

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where \((\bar{\varphi}, \varphi)\) are a pair of complex commutating fields, while \((\bar{\omega}, \omega)\) are anti-commutating ones. The key point is the perception that

\[
s(\bar{\omega}^{\nu a} \partial^\mu D_\mu \varphi^a_\nu - \bar{\varphi}^{\nu a} \partial^\mu D_\mu \omega^a_\nu + gf^{abc} \bar{\omega}^a_\mu \partial^\nu (D_\nu c)^b \varphi^{c}_\mu) = \bar{\varphi}^{\nu a} \partial^\mu D_\mu \varphi^a_\nu - \bar{\omega}^{\nu a} \partial^\mu D_\mu \omega^a_\nu + gf^{abc} \bar{\omega}^a_\mu \partial^\nu (D_\nu c)^b \varphi^{c}_\mu . \tag{30}
\]

Now, the local version of the GZ action is then given by:

\[
S_{GZ}^{local} = S_{YM} + \int d^4x \left[ -\bar{\varphi}^{\nu a} \partial^\mu D_\mu \varphi^a_\nu + \bar{\omega}^{\nu a} \partial^\mu D_\mu \omega^a_\nu + gf^{abc} \bar{\omega}^a_\mu \partial^\nu (D_\nu c)^b \varphi^{c}_\mu + \frac{\gamma^2}{\sqrt{2}} (\bar{\varphi}^{a}_\mu + \bar{\varphi}^{a}_\mu) A^{a\mu} \right] . \tag{31}
\]

The last term of the rhs in (31) breaks the BRST symmetry. To deal with this problem, Zwanziger introduced two (non dynamical) BRST sources, also in a doublet. These sources are taken to their physical values after the study of renormalization \[20\].

A simpler way of handling the cases under consideration emerges by taking the linear transformation of the fields

\[
\begin{align*}
v^{a}_\mu & = \bar{\varphi}^{a}_\mu - \varphi^a_\mu \\
u^{a}_\mu & = \bar{\varphi}^{a}_\mu + \varphi^a_\mu ,
\end{align*}
\tag{32}
\]

so that the \(v^a_\mu\) field decouples up to the quadratic terms. The action takes the form

\[
S_{GZ}^{local} = S_{YM} + \int d^4x \left[ -\frac{1}{4} u^{a\mu} \partial^\nu u^a_\nu + \frac{1}{4} v^{a\mu} \partial^\nu v^a_\nu + \bar{\omega}^{a\mu} \partial^\nu D_\nu \omega^a_\mu + \frac{g}{4} f^{abc} \omega^{a\mu} (D_\nu c)^b (u^c_\mu - v^c_\mu) + \frac{1}{\sqrt{2}} \gamma^2 u^{a\mu} A^{a\mu} \right] . \tag{33}
\]

and the propagators are

\[
\begin{align*}
\langle A^a_\mu(-p) A^b_\nu(p) \rangle &= \frac{\delta^{ab}}{4\pi} \frac{p^2}{p^4 + \gamma^4} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) , \tag{34} \\
\langle A^a_\mu(-p) u^b_\nu(p) \rangle &= \frac{\delta^{ab}}{4\pi} \frac{m^2}{p^4 + \gamma^4} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) , \tag{35} \\
\langle u^a_\mu(-p) u^b_\nu(p) \rangle &= -\frac{\delta^{ab}}{4\pi} \frac{p^2}{p^4 + \gamma^4} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) . \tag{36}
\end{align*}
\]
The action (33) presents in a more simplified way the diagonal and mixed terms, which makes the identification of propagators more direct. Indeed, the terms that matter for the propagator of the vector field $A_\mu$ are the bilinear contributions $u^2 u$ and $u A$. The resource to $u^a_\mu$ and $v^a_\mu$, instead of the original fields $\bar{\varphi}^a_\mu$ and $\varphi^a_\mu$ of the GZ model, simplifies the search for the actions leading to the desired propagators.

The standard procedure for the study of the renormalization of this model takes into account the fact that (33) can be put into the form

$$ S_{\text{local}}^{\text{GZ}} = S_{\text{YM}} + \int d^4 x \left[ s \Omega + \Psi \right], $$

where

$$ \Omega = -\frac{1}{2} \bar{\omega}^{\nu a} \partial^\mu D^\mu (u^a_\nu - v^a_\nu) $$

and

$$ \Psi = \frac{1}{\sqrt{2}} \gamma^2 u^a_\mu A^a_\mu. $$

The term (39) is the one which breaks the BRST symmetry. According to the standard treatment just mentioned, BRST sources should be introduced such that

$$ sM = V \quad \text{and} \quad sV = 0 $$

and the term (39) should be substituted by

$$ s \left( \frac{1}{\sqrt{2}} M u^a_\mu A^a_\mu \right). $$

This means that the resultant action is YM plus an exact BRST variation. The action is renormalized and, after that, the sources are taken to their physical values. This means,

$$ V = \gamma^2 \quad \text{and} \quad M = 0, $$

so that the starting action is re-obtained. In this sense, the action (31) should be considered as part of a larger one.

This procedure has been severely questioned recently [21]. Although new improvements have been proposed defending the procedure [22], no definitive answer exists in the literature until now. If it is valid, the GZ action can be considered renormalized and its use is justified. As we shall see, analogous conclusions can be reached for the case of the actions whose potentials were discussed in section 2.
4.2 The linear potential action

We take here the derivation of the GZ action presented in the previous section as a paradigm keeping the term highlighted in (30) but replacing the simple derivatives by covariant ones. Further, we will add the matter contribution to the transformations (29), resulting in

\[ s \bar{\omega}_\mu^a = \bar{\varphi}_\mu^a + gf^{abc} c^b \bar{\omega}_\mu^c, \quad s \bar{\varphi}_\mu^a = gf^{abc} c^b \bar{\varphi}_\mu^c, \]
\[ s \varphi_\mu^a = \omega_\mu^a + gf^{abc} c^b \varphi_\mu^c, \quad s \omega_\mu^a = gf^{abc} c^b \omega_\mu^c. \] (43)

It turns out that

\[ s(\bar{\omega}_\mu^\alpha D^\nu D^\nu \varphi_\mu^a) = \bar{\varphi}_\mu^\alpha D^\nu D^\nu \varphi_\mu^a - \bar{\omega}_\mu^\alpha D^\nu D^\nu \varphi^a_\mu. \] (44)

Our first task here is to construct an action that generates a propagator of the type

\[ \langle A^a_\mu (p) A^b_\nu (p) \rangle = \frac{\delta^{ab}}{4\pi} \left( \frac{1}{p^2 - m^2} \right) \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \] (45)

where we have set \( \xi = -1 \), in (7), which selects potentials generating equilibrium points between the Coulomb and the linear term (13). In terms of the fields \( u_\mu^a \) and \( v_\mu^a \) in (32), we are led to consider the action:

\[ S_{\text{local}} = S_{\text{YM}} + \frac{1}{2} \int d^4x \left[ u_\mu^a D^\nu D_\nu u^a_\mu - v_\mu^a D^\nu D_\nu v^a_\mu \right. \]
\[ \left. - 4 \bar{\omega}_\mu^\alpha D^\nu D^\nu \omega^a_\mu + m^2 (v^a_\mu + A^a_\mu)^2 \right]. \] (46)

This action compares with the GZ case (37) through the replacements

\[ \Omega = \omega_\mu^\alpha D^\mu D_\mu (u^a_\mu - v^a_\nu) \] (47)

and

\[ \Psi = \frac{1}{2} m^2 (v^a_\mu + A^a_\mu)^2. \] (48)

The fundamental aspect of the renormalizability of this model is now being investigated, as well as the GZ case, and will be left for further discussion.

Let us stress a point that bears some similarity to the GZ case. Considering the use of the equations of motion for the auxiliary fields, we see that (46) originates from the localization process of a non-local action of the form

\[ S_{\tilde{n}\text{local}} = S_{\text{YM}} + \frac{1}{2} m^2 \int d^4x A_\mu^a N \delta^{ab} \delta_{\mu\nu} A^b_\nu, \] (49)
where
\[ N = 1 + \frac{m^2}{D^2} - \frac{m^4}{4(D^2 + m^2)^2} + \frac{m^6}{4D^4(D^2 + m^2)}. \] (50)

Observe the similarity of this linear potential case with the GZ case in which two doublets were also needed, \( \{ \bar{\varphi}^a_{\mu}, \bar{\omega}^a_{\mu} \} \) and \( \{ \varphi^a_{\mu}, \omega^a_{\mu} \} \), as shown in (31). The structure of treatment is the same. We will see that for non-linear potentials cases the situation is quite different.

### 4.3 Cubic potential action

We will discuss how to obtain the action for the more confining cubic potential
\[ V(\vec{r}) = C_2(R)\zeta \left( \frac{q^2}{r} + \frac{\xi m^4}{4} q^2 r^3 \right), \] (51)

that is generated by the gauge field propagator of the form
\[ \langle A^b_a(-p)A^b_a(p) \rangle = \frac{\delta^{ab}}{4\pi} \left( \frac{1}{p^2} + \frac{\xi m^4}{p^2} \right) \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \] (52)

which can be obtained from (12) and (7) respectively by choosing \( n = 4 \).

First, we consider the case where \( \xi = 1 \). Within the same spirit of the last section we consider now the action
\[ S_{local} = S_{YM} + \frac{1}{2} \int d^4x \left[ \epsilon^{\mu\nu\rho\sigma} D_\nu v^a_\rho - u^a_\rho D_\nu u^a_\mu + 4\bar{\omega}^a_\mu D_\nu \omega^a_\mu \right. \\
\left. + m^2 \left( A^a_\mu + u^a_\mu \right) v^a_\mu \right]. \] (53)

The terms with double covariant derivatives of the auxiliary fields in (53) are very similar to those in the linear potential case. They are seen to be BRST trivial by the use of (44). In this sense the possibility of the BRST quantization should take into account the replacement of (38) by
\[ \Omega = -\bar{\omega}^{\nu a} D_\nu D_\mu (u^a_\mu - v^a_\mu) \] (54)

and (39) by
\[ \Psi = \frac{1}{2} m^2 v^{\nu a} (A^a_\mu + u^a_\mu). \] (55)

Again, the last term of (53) breaks the BRST symmetry.
The genesis of the action [53] from the localization of a non-local action, analogously to the linear case [49], implies \( \mathcal{N} \) to be

\[
\mathcal{N} = \frac{m^6 D^2}{(D^4 + m^4)^2}.
\]

Let’s look now at the case \( \xi = -1 \). This case is more interesting because it provides a point of equilibrium between the contributions of the two terms of the potential, i.e., the force coming from the Coulombian term cancels the originated from the cubic term, what can lead to a prediction of the nucleon radius.

In this case the kinetic terms of the localizing fields have no relative sign. This implies the impossibility in constructing the kinetic term of the auxiliary fields sector with a quartet. Indeed we need two quartets of localizing fields. They are introduced with the transformations below,

\[
\begin{align*}
    s\tilde{\omega}_\mu^a &= \tilde{\varphi}_\mu^a + gf^{abc} c^b \omega_\mu^c, \\
    s\tilde{\varphi}_\mu^a &= \varphi_\mu^a + gf^{abc} c^b \tilde{\varphi}_\mu^c, \\
    s\tilde{\xi}_\mu^a &= \tilde{\psi}_\mu^a + gf^{abc} c^b \xi_\mu^c, \\
    s\tilde{\psi}_\mu^a &= \psi_\mu^a + gf^{abc} c^b \tilde{\psi}_\mu^c.
\end{align*}
\]

Note that

\[
\begin{align*}
    s(\tilde{\omega}^{\mu a} D^\mu D_\nu \tilde{\varphi}_\nu^a) &= \tilde{\varphi}^{\mu a} D^\mu D_\nu \varphi_\nu^a - \tilde{\omega}^{\mu a} D^\mu D_\nu \omega_\nu^a, \\
    s(\tilde{\xi}^{\mu a} D^\mu D_\nu \tilde{\psi}_\nu^a) &= \tilde{\psi}^{\mu a} D^\mu D_\nu \psi_\nu^a - \tilde{\xi}^{\mu a} D^\mu D_\nu \xi_\nu^a.
\end{align*}
\]

Finally, introducing the field transformations

\[
\begin{align*}
    v_\mu^a &= \tilde{\varphi}_\mu^a - \varphi_\mu^a, \\
    u_\mu^a &= \tilde{\psi}_\mu^a + \psi_\mu^a, \\
    k_\mu^a &= \tilde{\psi}_\mu^a - \psi_\mu^a, \\
    t_\mu^a &= \tilde{\psi}_\mu^a + \psi_\mu^a,
\end{align*}
\]

the action we are looking for takes the form

\[
\begin{align*}
    S_{local} &= S_{YM} + \frac{1}{2} \int d^4x \left[ v^{\mu a} A^\nu D_\nu v_\mu^a - u^{\mu a} A^\nu D_\nu u_\mu^a + k^{\mu a} A^\nu D_\nu k_\mu^a - t^{\mu a} A^\nu D_\nu t_\mu^a + \\
    &\quad - 2\tilde{\omega}^{\mu a} D^\nu D_\nu \tilde{\omega}_\mu^a + 2\tilde{\xi}^{\mu a} D^\nu D_\nu \tilde{\xi}_\mu^a + 2m^2 A^{\mu a} v_\mu^a + 2m^2 k^{\mu a} \tilde{\psi}_\mu^a \right].
\end{align*}
\]

This action localizes a nonlocal term which, in the form [49], has \( \mathcal{N} \) given by

\[
\mathcal{N} = \frac{1}{4} (D^4 + m^4) \left( \frac{mD}{2D^4 - m^4} \right)^2.
\]
The similarity with the GZ case can be seen if we look at (37) taking
\[ \Omega = -2\bar{\epsilon}_a^\nu D_\mu \varphi^a_\nu - 2\bar{\xi}_a^\nu D_\mu \psi^a_\nu \]  \hspace{1cm} (62)
and
\[ \Psi = 2A^\mu a_\mu + 2k^a_\mu v^a_\mu. \]  \hspace{1cm} (63)

The generalization to actions generating arbitrary even powers in $1/p^{n+2}$ written in (7) (with $\xi = -1$), demands $n$ doublets. In the case presently analyzed we needed four doublets and just two of these fields couple to the gauge field.

5 Conclusion

We investigate in this work the 't Hoft - Wilson method [10] and its application to the construction of potentials arising from interactions between fermions, mediated by gauge fields. With the use of the first and second quantization processes, one is able to identify the potential of static charges in general cases. We apply these ideas to calculate the potential for the refined GZ model and to generalize the linear Cornell potential [15] to polynomial ones $r^{n-1}$, being $n$ an even number.

Within this scenario we establish the interaction described by a vector field $A_\mu$ that can lead to situations of quarks confinement. In general these contributions are non local terms added to the Yang-Mills action.

In particular, along our analysis, we showed the non-confining character of the refined GZ model in the sense of the 't Hooft-Wilson criteria. Anyway, inspired by the GZ action, we used the same localization process to build local actions leading to the polynomial confining potentials that we studied previously. The quartet structure that is typical of the GZ construction then needs to be adapted and generalized in order to generate confining potentials taking into account the necessity of a point of stable equilibrium. This will be associated to the average radius of the fermionic condensed state.

Obviously, when we look at the legitimacy of such Lagrangians, other physical demands need to be considered. The renormalizability of these models becomes a fundamental question. An important point is that the Zwanziger-Sorella treatment of the renormalization that we briefly showed in section (4.1) is now being questioned by recent results [21]. Although an alternative treatment has been constructed [22] (with the introduction of a Stuckelberg auxiliary field) we think that the renormalization of these models is still an open question.
It is worth mentioning that the calculation of the potential for the original GZ model was done in [17] and showed its nonconfining character. As we said a moment ago the improved GZ theory displays the same feature. We understand that this happens because the propagator that breaks the positivity, a basic ingredient for gluon confinement, does not generate a potential which increases with the distance between the fermions. But our ultimate goal was achieved, as we succeeded in describing local actions with a confining character in ’t Hooft sense by the use of the same machinery characteristic of the GZ construction. This opens a possible path in the direction of building an action with both effects in the same context, aiming a simultaneous confinement of quarks and gluons.

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