Path-dependent Kyle equilibrium model.

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Abstract

We consider an auction type equilibrium model with an insider in line with the one originally introduced by Kyle in 1985 and then extended to the continuous time setting by Back in 1992. The novelty introduced with this paper is that we deal with a general price functional depending on the whole past of the aggregate demand, i.e. we work with path-dependency. By using the functional Itô calculus, we provide necessary and sufficient conditions for the existence of an equilibrium. Furthermore, we consider both the cases of a risk-neutral and a risk-averse insider.

Key words: Kyle model, market microstructure, equilibrium, insider trading, stochastic control, semimartingales, functional Itô calculus.

JEL-Classification C61 · D43 · D44 · D53 · G11 · G12 · G14

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1 Introduction

It is well known that insider information and informational asymmetries are everywhere in the real economy. In his pioneering work, Kyle (1985) constructed a model in a discrete time setting with market makers, uninformed traders and one insider, who knows the fundamental value of an asset at a certain fixed released time. Also, the model included a price functional relating market prices and the total demand. Back (1992), extends Kyle’s model to the continuous time case. Since these worked appeared, several generalisations and extensions have been produced. To mention some, Back and Pedersen (1998), who consider a dynamic fundamental price and Gaussian noises with time varying volatility; Cho (2003) who considers pricing functions depending on the path of the demand process and also studies the case when the informed trader is risk-averse; Lasserre (2004), who considers a multivariate setting; Back and Baruch (2004), where the market depth (i.e. the marginal effect on price of the volume traded) depends on the market price of the stock; Aase, Bjuland, and Øksendal (2012a, 2012b), who put emphasis on filtering techniques to solve the equilibrium problem; Campi and Çetin (2007), who consider a defaultable bond instead of a stock as in the Kyle-Back model and also consider the knowledge of the default time as the insider’s privileged information; Danilova (2010), who deals with non-regular pricing rules; Caldentey and Stacchetti (2010) who take a random release time into account; Campi, Çetin, and Danilova (2013), who consider again a defaultable bond, but this time they consider the privileged information to be represented by some dynamic signal related with the default time; and Collin-Dufresne and Fos (2016) where the market depth depends on the (random) volatility of the noise in the market. In Corcuera and Di Nunno (2018), the authors propose a general framework to include all the particular extensions mentioned above and study the general characteristics of the equilibria. Recently Corcuera, Di Nunno, and Fajardo (2019) have also considered the same general situation, but with a random price pressure and a random release time of information.

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In this paper we propose a step even further and we consider a price functional that depends on the whole path of the aggregate demand. We study the properties of the equilibrium and sufficient and necessary conditions for the existence of equilibria. Also we study both the case of a risk-neutral and risk-averse insider. With this work we extend substantially the present frontiers of the literature on this theme. We also note that the analysis of price functional of these type will involve the recently introduced functional calculus, see e.g. Cont and Fournié (2013), and this work represents a good venture to see these new mathematical techniques applied in economics and finance.

The paper is structured as follows. In the next section we describe the model and we define the equilibrium for the admissible strategies. Section 3 presents some needed background material from functional Itô calculus. In the Section 4, we also suggest some general results that allow to reduce the set of admissible strategies on which the insider can find its optimal performance and hence describe the equilibrium. Also, we give necessary and sufficient conditions to obtain an equilibrium under very general classes of pricing rules. In this section, we consider the two cases of a risk-neutral and a risk adverse insider. Section 5 is devoted to the study of necessary conditions for an equilibrium, without fixing a priori, up to smoothness conditions, the set of pricing rules. We observe that these latter results motivate and justify those restrictions imposed on the classes of pricing rules considered in the study of Section 4. The last two sections are dedicated to examples of classes of pricing rules and examples of equilibrium models, correspondingly.

2 The model and equilibrium

We consider a market with two assets, a stock and a bank account with interest rate \( r \) equal to zero for the sake of simplicity. The trading is continuous in time over the period \([0, \infty)\) and it is order driven. There is a (possibly random) release time of information \( \tau < \infty \) a.s., when the fundamental value of the stock is revealed. The fundamental value process \( V \) represents the actual value of the asset, which would be the same as the market price of the asset only if all the information was public. We could say, with Malkiel (2011), that the fundamental value is the intrinsic value of a stock, via an analysis of the balance sheet, the expected future dividends, and the growth prospects of a company. The fundamental value process is denoted by \( V \).

We shall denote the market price of the stock at time \( t \) by \( P_t \). This represents the market evaluation of the asset. Just after the revelation time \( \tau \), the price of the stock coincides with the fundamental value. Then we consider \( P_t \) defined only on \( t \leq \tau \). Obviously, it is possible that \( P_t \neq V_t \) for \( t \leq \tau \).

We assume that all the random variables and processes mentioned are defined in the same complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{H}, \mathbb{P})\) where the filtration \( \mathbb{H} \) and any other filtration considered in this present work are complete and right-continuous by taking, when necessary, the usual augmentation.

There are three kinds of traders. A number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who has privileged information about the firm and can deduce its fundamental value, and the market makers, who set the market price and clear the market.

2.1 The agents and the equilibrium

At time \( t \), the insider’s information is the full information \( \mathcal{H}_t \) and her flow of information is represented by the filtration \( \mathbb{H} = (\mathcal{H}_t)_{t \geq 0} \). Since this is also the filtration with respect to which all the processes considered in the present work are adapted, we shall omit to write it in the notation. A random release time of information \( \tau \) is considered from insider’s perspective to be of one of these types:

- \( \tau \) it is bounded and predictable,
- \( \tau \) it is not a predictable stopping time, but it is independent of the observable variables.

We assume that the fundamental value \( V \) is a continuous martingale such that \( \sigma^2_V(t) := \frac{d[V,V]}{dt} \) is well defined.
Hereafter we describe in detail the three types of agents involved in this market model, namely their role, their demand process, and their information. Let $Z$ be the aggregate demand process of the liquidity traders. We recall that there are a large number of traders motivated by liquidity or hedging reasons. They are perceived by the insider as constituting noise in the market, thus also called noise traders. It is assumed that $Z$ is a continuous martingale, starting at zero, independent of $V$, and such that $\sigma_Z^2(t) := \frac{d[Z,Z]_t}{dt}$ is well defined. As it is shown in Corcuera \textit{et al.} (2010), if $Z$ had jumps, an equilibrium would not be possible.

Remark 1 In this equilibrium model, the time $\tau$ and the processes $V$ and $Z$ are exogenously given.

Market makers clear the market giving the market prices. They rely on the information given by the total aggregate demand $Y$, which they observe, and the release time $\tau$, that is a stopping time for them. Hence, their information flow is: $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t = \sigma(Y_s, \tau \wedge s, 0 \leq s \leq t)$. Here $\sigma$ denotes the $\sigma$-field corresponding to the usual augmentation of the natural filtration.

The total aggregate demand is defined as $Y := X + Z$, where $X$ denotes the insider demand process, which is naturally assumed to be a predictable process and also a càdlàg semimartingale:

\[(A1) \quad X_t = M_t + A_t + \int_0^t \theta_s \, ds, \quad t \geq 0,\]

where $M$ is a continuous martingale with $M_0 = 0$,

$A$ a bounded variation predictable process with $A_t = \sum_{0 < s \leq t} (X_s - X_{s-})$ and $A_0 = 0$,

$\theta$ is a càdlàg adapted process.

Strategies $X$ satisfying $(A.1)$ are called admissible. Market makers provide liquidity and fix the market prices $P_t$, for all $t$, based on the total demand $Y$, resulting in the functionals:

$$P_t = P_t(Y_s, 0 \leq s \leq t), \quad 0 \leq t \leq \tau.$$ 

It is natural to assume that prices are strictly increasing with the total demand $Y$. We shall precise this condition in the next section. From the economic point of view, due to the competition among market makers, the market prices are competitive, in the sense that

$$P_t = \mathbb{E}(V_t | \mathcal{F}_t), \quad 0 \leq t \leq \tau. \tag{1}$$

Therefore $(P_t)_{0 \leq t \leq \tau}$ is an $\mathcal{F}$-martingale.

Definition 1 The couple $(P, X)$ is an equilibrium if market prices admit a pricing rule (i.e. a functional of $Y$), that we shall name equilibrium pricing rule,

$$P_t = P_t(Y_s, 0 \leq s \leq t), \quad 0 \leq t \leq \tau,$$

such that, at the same time, the market prices $P$ are competitive given $X$, i.e.

$$P_t = \mathbb{E}(V_t | \mathcal{F}_t), \quad 0 \leq t \leq \tau,$$

and the strategy $X$ is optimal for the insider given the prices $P$.

Now we have to make precise what an optimal strategy for the insider is. The informed trader aims at maximising the expected final utility of her wealth. Let $W$ be the wealth process corresponding to the insider’s portfolio $X$. To obtain the formula for the insider’s wealth assume that trades occur at times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N = \tau$. If at time $t_{i-1}$ there is an order to buy $X_{t_i} - X_{t_{i-1}}$ shares, its cost will be $P_{t_i} \times (X_{t_i} - X_{t_{i-1}})$, so there is a change in the insider’s bank account given by

$$-P_{t_i} \times (X_{t_i} - X_{t_{i-1}}) = -P_{t_{i-1}} \times (X_{t_i} - X_{t_{i-1}}) - (P_{t_i} - P_{t_{i-1}}) \times (X_{t_i} - X_{t_{i-1}}),$$

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where the second term in the right-hand side accounts for the impact of the demand on the current price. Due to the fact that the price of the asset equals its fundamental value at the release time $\tau$, there is, in addition, the extra income $X_r V_r$. Then the total wealth at $\tau$ is given by

$$W_\tau = - \sum_{i=1}^{N} P_{t_{i-1}} \times (X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^{N} (P_{t_i} - P_{t_{i-1}}) \times (X_{t_i} - X_{t_{i-1}}) + X_r V_\tau,$$

so taking the limit with the time between trades going to zero, we have

$$W_\tau = - \int_0^\tau P_t \cdot dX_t - [P, X]_\tau + X_r V_\tau$$

where (here and throughout the whole article) $P_t := \lim_{s \uparrow t} P_s$ a.s.

Then the informed trader aims at maximising

$$\mathbb{E}(U(W_\tau) | H_0) = \mathbb{E} \left( U \left( - \int_0^\tau P_t \cdot dX_t - [P, X]_\tau + X_r V_\tau \right) \right | H_0)$$

for a given utility function $U$, that is, a strictly increasing and concave function satisfying the Inada conditions. The case when $U$ is the identity function corresponds to the so called risk-neutral case. The insider’s strategy $X$ of type (A.1) providing the maximum is called optimal.

### 3 Regularity of the functionals. The functional Itô formula

Trading is developed in the context of imperfect competition, in the sense that prices are affected by the demand, that is $P_t = P_t(Y_s, 0 \leq s \leq t)$. Here and in the sequel, we shall write $Y_t$ to indicate the path of the process $Y$ from zero to $t$:

$$Y_t(s) := Y_s, \ 0 \leq s \leq t.$$  

Notice that we also can look at $Y_t$ as the process $Y$ stopped at $t$, in such a way that

$$Y_t(s) := \begin{cases}  Y_s & \text{for } 0 \leq s \leq t \\  Y_t & \text{for } t \leq s \leq \tau \end{cases}$$

Therefore we can write, alternatively, $P_t = P_t(Y_t)$ and to consider $P_t$ as a functional of the process $Y$ stopped at $t$. We recall that this functional is assumed to be strictly increasing in the total aggregate demand $Y$, see Definition 7. We shall also add some regularity on the functionals we are going to consider when needed. We shall also consider the following perturbation of a process $Y$. For $h \in \mathbb{R}$, we define

$$Y_t^h(s) := \begin{cases}  Y_s & \text{for } 0 \leq s < t \\  Y_t + h & \text{for } t \leq s \leq \tau \end{cases}.$$ 

In the sequel, we are going to use a functional version of the Itô formula obtained by functional Itô calculus. We give here some definitions and the results necessary to our scopes as can be found in Cont and Fournié (2013).

To include more functionals in the class of regular ones we shall consider nonanticipative functionals as maps

$$P : (t, Y_t, A_t) \rightarrow P(t, Y_t, A_t) = P_t(Y_t, A_t)$$

where $A_t := \frac{d[Y,Y_t]}{dt}$, and given two stopped processes $Y_t, Z_{t'}$ we consider the distance defined by

$$d_\infty((Y_t, A_t), (Z_{t'}, B_{t'})) = \|Y_t - Z_{t'}, A_t - B_{t'}\|_\infty + |t - t'|.$$ 

where $B_t := \frac{d[Z,Z_t]}{dt}$ and $\|\cdot\|_\infty$ is the sup-norm.
Definition 2 A nonanticipative functional $P$ is said to be left-continuous at $t$ if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $0 \leq t' \leq t \leq \tau$

\[ d_\infty((Y_{t'}, A_{t'}), (Z_{t'}, B_{t'})) < \eta \implies |P_t(Y_{t'}, A_{t'}) - P_t(Z_{t'}, B_{t'})| < \varepsilon \]

$P$ is said to be left-continuous if it is left-continuous at any $(t, Y_t, A_t)$. Right-continuity is defined analogously. Continuity means that left- and right-continuity occur at the same time. If, in the previous definition, we consider only times $t' = t$ then we say that the functional is said to be continuous at fixed times. It can be seen that continuity at fixed times implies that the process $(P_t(Y_t, A_t))_{0 \leq t \leq T}$ is adapted if $Y$ is adapted.

Since the space of càdlàg functions is not separable under the sup-norm, we need the following additional regularity, even for the continuous functionals defined above.

Definition 3 A functional $P$ is said to be boundedness preserving if for every constants $K$, $R$ and $t_0 \leq T$ there exists a constant $C_{K,R,t_0}$ such that for all $t \leq t_0 \leq T$, with $\|Y_t\|_\infty < K$.

\[ \|A_t\|_\infty < R \implies |P_t(Y_t, A_t)| < C_{K,R,t_0}. \]

Definition 4 We call horizontal derivative of the functional $P$ at $(t, Y_t, A_t)$, the limit given by

\[ \mathcal{D}_t P_t := \lim_{\Delta t \to 0} \frac{P_{t+\Delta t}(Y_t, A_t) - P_t(Y_t, A_t)}{\Delta t}, \]

provided it exists.

Example 1 From the definition, it is easy to see that

\[ \mathcal{D}_t G(Y_t) = 0, \quad \mathcal{D}_t F(t, Y_t) = \partial_t F(t, Y_t), \]

for any smooth functions $G, F$. Also, for any integrable $h$, we have

\[ \mathcal{D}_t \int_0^t h(Y_s)ds = h(Y_t). \]

Furthermore, for a smooth integrable function $f$ and a Brownian motion $W$, we obtain

\[ \mathcal{D}_t \left( \int_0^t f(s, W_s)dW_s \right) = -\frac{1}{2}\partial_y f(t, W_t). \]

A justification of the last statement comes from the usual Itô formula assuming that $f(\cdot, \cdot) = \partial_y F(\cdot, \cdot)$. In fact, it is enough to take horizontal derivatives in both sides of

\[ \int_0^t f(s, W_s)dW_s = F(t, W_t) - F(0, W_0) - \int_0^t \partial_x F(s, W_s)ds - \frac{1}{2} \int_0^t \partial_y f(s, W_s)ds, \]

to obtain the result.

Definition 5 We call vertical derivative of the functional $P$ at $(t, Y_t)$ the limit, provided it exists, given by

\[ \nabla_Y P_t := \lim_{h \to 0} \frac{P_t(Y^h, A_t) - P_t(Y_t, A_t)}{h}. \]

Example 2 If, as above, we consider some smooth functions $G, F$, we have

\[ \nabla_Y G(Y_t) = \frac{d}{dy} G(Y_t), \quad \nabla_Y F(t, Y_t) = \partial_y F(t, Y_t) \]
and for an integrable \( h \) we have
\[
\nabla_Y \int_0^t h(Y_s)ds = 0.
\]
Moreover, with integrability assumptions on \( f \) and for the Brownian motion \( W \) we have
\[
\nabla_W \left( \int_0^t f(s, W_s)dW_s \right) = f(t, W_t).
\]
In fact, assuming as in the previous example that \( f(\cdot, \cdot) = \partial_y F(\cdot, \cdot) \),
the classical Itô formula gives
\[
\int_0^t f(s, W_s)dW_s = F(t, W_t) - F(0, W_0) - \int_0^t \partial_y F(s, W_s)ds - \frac{1}{2} \int_0^t \partial_y f(s, W_s)ds,
\]
then, taking the vertical derivative on both sides, we have
\[
\nabla_W \left( \int_0^t f(s, W_s)dW_s \right) = \partial_y F(t, W_t) = f(t, W_t).
\]

**Remark 2** In general \( D \) and \( \nabla \) do not commute. If we set
\[
L_t := [D_t, \nabla] = D_t \nabla - \nabla D_t,
\]
then we have that
\[
L_t \left( \int_0^t f(s, W_s)dW_s \right) = [D_t, \nabla_W] \left( \int_0^t f(s, W_s)dW_s \right)
= D_t \nabla_W \left( \int_0^t f(s, W_s)dW_s \right) - \nabla_W D_t \left( \int_0^t f(s, W_s)dW_s \right)
= D_t f(t, W_t) + \frac{1}{2} \nabla^2_W f(t, W_t) \neq 0,
\]
if \( (f(t, W_t))_{t \geq 0} \) is not a local martingale.

**Definition 6** We say that a left-continuous functional belongs to \( C^{i,k}_b \) if it is \( j \)-times horizontally differentiable with derivatives continuous at fixed points and boundedness preserving, and it is \( k \)-times vertically differentiable with left-continuous and boundedness preserving derivatives.

**Theorem 1** (Functional Itô’s formula). If \( Y \) is a continuous semimartingale and \( P \in C^{1,2}_b \) with \( P_t(Y_t, A_t) = P_t(Y_{t-}, A_{t-}) \), then
\[
P_t(Y_t, A_t) = P_0(Y_0, A_0) + \int_0^t D_s P_s(Y_s, A_s)ds + \int_0^t \nabla_Y P_s(Y_s, A_s)dY_s + \frac{1}{2} \int_0^t \nabla^2_Y P_s(Y_s, A_s)d[Y,Y]_s, \text{ } P\text{-a.s. } 0 \leq t \leq \tau.
\]

**Proof.** See Theorem 4.1 in Cont and Fournié (2013) ■

**Definition 7** We say that \( P \) is strictly increasing at "point" \( \{Y_s, 0 \leq s \leq t\} \) if, for any \( t \leq \tau, \nabla_Y P_t(Y_t) > 0 \).

In the following we shall assume that \( P \) is strictly increasing for all \( \{Y_s, 0 \leq s \leq t\} \), that is \( \nabla_Y P_t(Y_s) > 0 \) for all \( \{Y_s, 0 \leq s \leq t\} \) and \( t \leq \tau \).
4 Necessary and sufficient conditions for an equilibrium

In this section we present necessary and sufficient conditions for the existence of an equilibrium when the release time $\tau$ and the pricing functional satisfy some conditions. The nature of these conditions will be further studied in the next section. In this analysis we shall consider both a risk-neutral insider and a risk-averse insider with exponential utility function.

First, we have the following result that reduces the set of strategies in which we find the optimum in the optimisation problem here above considered. An analogous result is given in Corcuera et al. (2019).

**Proposition 1** Assume that the functional $P$ is left-continuous, bounded preserving and strictly increasing, then admissible strategies $X$ with a continuous martingale part or jumps are suboptimal in the class of all predictable semimartingale strategies.

**Proof.** Let $F(t, Y_t)$ be a smooth functional, $F \in C^{1,2}$, with $\nabla Y F_t = P_t$. Then, assume that $X$ has a continuous martingale part and jumps. Take, by simplicity and admissible strategies $X$.

\[ F(T, Y_T) - F(0, Y_0) = \int_0^T P_t \, dY_t + \int_0^T \partial_t F \, dt + \frac{1}{2} \int_0^T \nabla Y P_t \, d[Y^c, Y^c]_t \]

\[ + \sum_{t: \Delta X \neq 0} \Delta F_t - \nabla Y F_t - \Delta Y_t, \]

now, since

\[ d[Y^c, Y^c]_t = d[X^c, X^c]_t + 2d[X^c, Z]_t + d[Z, Z]_t \]

and

\[ [P, X] = \int_0^T \nabla Y P_t \, d[X^c, Z]_t + \int_0^T \nabla Y P_t \, d[X^c, X^c]_t + \sum_{t: \Delta X \neq 0} \Delta P_t \Delta X_t \]

we have

\[ W_T = -\int_0^T P_t \, dX_t - [P, X] + X_T V \]

\[ = -F(T, Y_T) + F(0, Y_0) + \int_0^T \partial_t F \, dt \]

\[ + \frac{1}{2} \int_0^T \nabla Y P_t \, d[X^c, X^c]_t + \int_0^T \nabla Y P_t \, d[X^c, Z]_t + \frac{1}{2} \int_0^T \nabla Y P_t \, d[Z, Z]_t \]

\[ - \int_0^T \nabla Y P_t \, d[X^c, Z]_t - \int_0^T \nabla Y P_t \, d[X^c, X^c]_t - \sum_{t: \Delta X \neq 0} \Delta P_t \Delta X_t + X_T V + \int_0^T P_t \, dZ_t \]

\[ + \sum_{t: \Delta X \neq 0} \Delta F_t - P_t - \Delta Y_t \]

\[ = -F(T, Y_T) + F(0, Y_0) + \int_0^T \partial_t F \, dt + \frac{1}{2} \int_0^T \nabla Y P_t \, d[Z, Z]_t + X_T V \]

\[ + \int_0^T P_t \, dZ_t - \frac{1}{2} \int_0^T \nabla Y P_t \, d[X^c, X^c]_t + \sum_{t: \Delta X \neq 0} \Delta F_t - P_t - \Delta Y_t, \]

The contribution of the last two terms is always negative because $\nabla^2 Y F_t = \nabla Y P_t > 0$ and the other terms can be approximate as far as we want by replacing $Y = Z + X$ by $\tilde{Y} = Z + \tilde{X}$ where $\tilde{X}$ is an absolutely continuous approximation to $X$.

Since we are going to consider left-continuous boundedness preserving functionals in the sequel, the set of admissible insider’s strategies (A1) can be reduced to those strategies $X$ satisfying

\[(A1') \quad X_t = \int_0^t \theta_s \, ds, \text{ for all } t \geq 0, \text{ where } \theta \text{ is a càdlàg adapted process.}\]
Furthermore, the goal of the insider becomes to maximise the performance

\[ J(X) := \mathbb{E}(U(W_\tau) | \mathcal{H}_0) = \mathbb{E} \left( U \left( \int_0^\tau (V_\tau - P_t) dX_t \right) | \mathcal{H}_0 \right) \]  

(3)

over the set of admissible strategies \( X \) satisfying (A1').

**Remark 3** Observe that, in view of (A1'), we have \( d[Y,Y]_t = \sigma^2_Z(t) dt \).

We also have a general result in the case when \( \tau \) is a predictable stopping time for the insider. The same result is given in Corcuera et al. (2019).

**Proposition 2** If \( \tau \) is a predictable stopping time for the insider and \( X \) is an optimal strategy then

\[ V_\tau = P_\tau \text{ a.s.} \]

**Proof.** If the insider’s strategy is such that \( V_\tau - P_\tau \neq 0 \) then it is suboptimal since the insider could approximate a jump at \( \tau \) with the same sign of \( V_\tau - P_\tau \) by an absolutely continuous strategy and improving her wealth.

\[ \blacksquare \]

**Remark 4** From the economic point of view, due to Bertrand’s type competition among market makers, in the equilibrium market prices are rational, or competitive, in the sense that the competitive price is a price such that the expectation of the market maker’s profit equals zero. In fact, the total final wealth \( W^M_\tau \) of the market makers is given by

\[ W^M_\tau := -Y_\tau (V_\tau - P_\tau) - \int_0^\tau Y_t dP_t, \]

then, if \( P_t = \mathbb{E}(V_t | \mathcal{F}_t) \), \( 0 \leq t \leq \tau \) see (1), under the assumption that \( \mathbb{E}(\int_0^T Y^2_t d[P,P]_t) < \infty \), we have that \( \mathbb{E}(W^M_\tau) = 0 \).

4.1 Main results

First, we consider the risk-neutral case.

**Theorem 2** Suppose that \( \tau = T \) and that for all \( t < T \), the price functional \( P \) is \( C^{1,3} \) and such that

\[ D_t P_t + \frac{1}{2} \nabla^2 Y P_t \sigma^2_Z(t) = 0, \]

(4)

with

\[ \mathcal{L} P_t := [D_t, \nabla Y] = 0 \]

(5)

and

\[ \nabla Y P_t = G(t, P_t), \]

(6)

where \( G(t,y_t) \geq C > 0 \) is \( C^{1,2} \). Assume that \( \mathbb{E}\left(\int_0^T (P_t - V_t)^2 (\sigma^2_Z(t) + \sigma^2_Y(t)) dt\right) < \infty \) and \( \mathbb{E}\left(\int_0^T G^2(t, P_t) \sigma^2_Z(t) dt\right) < \infty \).

Then there is an equilibrium in the risk-neutral case if and only if

(i) \( P_T = V_T \),

(ii) \( Y \) is an \( \mathbb{F} \)-martingale,

(iii) \( G(t,y_t) = g(t,y_t) \)

where \( g(t,y) \geq C > 0 \) is \( C^{1,2} \).
Proof. Firstly, we prove that (i) and (ii) and (iii) are sufficient conditions. Set the functional

$$I(t, y_t, v) := \int_{y_t}^{y} \frac{z - v}{\nabla P_t (y_z)} \, dz$$

Where

$$\nabla P_t (y_z) := G(t, y_z)$$

and $$y_z = P_t (\omega)$$, for $$0 \leq t \leq T$$. Then we have

$$\nabla P I(t, P_t, V_t) = \frac{P_t - V_t}{\nabla P_t (P_t)}$$

and by the chain rule

$$\nabla^2 P I(t, P_t, V_t) = \frac{\nabla P_t (P_t) - (P_t - V_t) \nabla P (\nabla P_t (P_t))}{\nabla P_t (P_t)^2}.$$  \hfill (7)

Consequently

$$\left( \nabla P_t \left( P_t^{z-P_t} \right) - (z - V_t) \partial_z \nabla P_t \left( P_t^{z-P_t} \right) \right) \bigg|_{z=P_t} = \nabla^2 P I(t, P_t, V_t) (\nabla P_t (P_t))^2,$$

where

$$\nabla P_t \left( P_t^{z-P_t} \right) := \nabla P_t (y_z^{y_z-P_t}) \bigg|_{y_z=P_t},$$

Then, we can write

$$\nabla^2 P I(t, P_t, V_t) (\nabla P_t (P_t))^2 = - \int_{V_t}^{P_t} (z - V_t) \partial_z^2 \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) \, dz + \nabla P_t \left( P_t^{y_z-P_t} \right).$$

On the other hand

$$\mathcal{D}_t I(t, P_t, V_t) = - \int_{V_t}^{P_t} \frac{z - V_t}{\left( \nabla P_t \left( P_t^{z-P_t} \right) \right)^2} \mathcal{D}_t \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) \, dz$$ \hfill (9)

in such a way that,

$$\mathcal{D}_t I(t, P_t, V_t) + \frac{1}{2} \nabla^2 P I(t, P_t, V_t) (\nabla P_t (P_t))^2 \sigma^2(t)$$

$$= - \int_{V_t}^{P_t} (z - V_t) \left( \mathcal{D}_t \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) \right)^2 \frac{1}{2} \partial_z^2 \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) \sigma^2(t) \, dz$$

$$+ \frac{1}{2} \nabla P_t \left( P_t^{y_z-P_t} \right) \sigma^2(t).$$

$$= \int_{V_t}^{P_t} (z - V_t) \left( \mathcal{D}_t \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) \right)^2 \frac{1}{2} \nabla^2 P \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) \sigma^2(t) \, dz$$

$$+ \frac{1}{2} \nabla P_t \left( P_t^{y_z-P_t} \right) \sigma^2(t).$$

Notice that

$$\partial_z \left( \nabla P_t \left( P_t^{z-P_t} \right) \right) = \nabla P \left( \nabla P_t \left( y_z \right) \right) \bigg|_{y_z=P_t}.$$}

Now, conditions (4) and (5) imply that

$$\mathcal{D}_t \left( \nabla P \right) + \frac{1}{2} \nabla^2 P \left( \nabla P_t \right) \sigma^2(t) = 0.$$
In this expression we are considering \( \nabla_Y P \) as a functional of \( Y \), so to indicate that \( D_t \) is taken in this situation we shall write \( D_t^Y \) and we use \( D_t \) to indicate that \( D_t \) is taken freezing \( P \). So we have

\[
D_t^Y (\nabla_Y P) + \frac{1}{2} \nabla_Y^2 (\nabla_Y P_t) \sigma_Z^2(t) = 0
\]

Now if we consider \( \nabla_Y P \) as a functional of \( P \), that is at the same time a functional of \( Y \), by applying the chain rule we have

\[
\nabla_Y^2 (\nabla_Y P_t) = \nabla_Y^2 (\nabla_Y P_t) (\nabla_Y P_t)^2 + (\nabla_P (\nabla_Y P_t))^2 \nabla_Y P_t
\]

\[
= \nabla_Y^2 (\nabla_Y P_t) (\nabla_Y P_t)^2 + \frac{(\nabla_Y^2 P_t)^2}{\nabla_Y P_t}
\]

\[
D_t^Y (\nabla_Y P) = D_t (\nabla_Y P) + \nabla_P (\nabla_Y P) D_t^Y P
\]

\[
= D_t (\nabla_Y P) - \frac{1}{2} \nabla_Y^2 P_t \nabla_Y P_t \sigma_Z^2(t),
\]

where in this last equality we use (4). So we obtain that

\[
D_t (\nabla_Y P) + \frac{1}{2} \nabla_P (\nabla_Y P_t) (\nabla_Y P_t)^2 \sigma_Z^2(t) = 0.
\]

Note that we are considering functionals satisfying this condition for any continuous trajectory with quadratic variation \( \sigma_Z^2 \), so by continuity of the functionals we obtain that

\[
D_t (\nabla_Y P_t \left( P_t^{z-P_t} \right)) + \frac{1}{2} \sigma_Z^2(t) \nabla_Y^2 \left( \nabla_Y P_t \left( P_t^{z-P_t} \right) \right)
\]

\[
= 0.
\]

and

\[
D_t I(t, P_t, V_t) + \frac{1}{2} \nabla_P^2 I(t, P_t, V_t) (\nabla_Y P_t (P_t))^2 \sigma_Z^2(t)
\]

\[
= \frac{1}{2} \nabla_Y P_t \left( P_t^{\gamma_t-P_t} \right) \sigma_Z^2(t).
\] (10)

Finally if we apply the functional Ito formula we have, since \( V \) and \( Z \) are independent and we consider only absolutely continuous strategies, see (A1'),

\[
I(T, P_T, V_T) = I(0, P_0, V_0) + \int_0^T D_t I(t, P_t, V_t) dt + \int_0^T \nabla_P I(t, P_t, V_t) dP_t + \int_0^T \nabla_V I(t, P_t, V_t) dV_t
\]

\[
+ \frac{1}{2} \int_0^T \nabla_P^2 I(t, P_t, V_t) (\nabla_Y P_t (P_t))^2 \sigma_Z^2(t) dt + \frac{1}{2} \int_0^T \nabla_V^2 I(t, P_t, V_t) \sigma_V^2(t) dt,
\]

Also, we have the fact that

\[
d[P_t]_t = (\nabla_Y P_t (P_t))^2 d[Y, Y]_t = (\nabla_Y P_t (P_t))^2 \sigma_Z^2(t) dt.
\]

Then by (4) (7) and (10)

\[
I(T, P_T, V_T) = I(0, P_0, V_0) + \int_0^T (P_t - V_t) dY_t + \int_0^T \nabla_V I(t, P_t, V_t) dV_t
\]

\[
+ \frac{1}{2} \int_0^T \nabla_Y P_t \left( P_t^{\gamma_t-P_t} \right) \sigma^2(t) dt + \frac{1}{2} \int_0^T \nabla_V^2 I(t, P_t, V_t) \sigma_V^2(t) dt,
\]

10
and
\[
\int_0^T (V_t - P_t) \, dX_t - \left( I(0, P_0, V_0) + \frac{1}{2} \int_0^T \nabla^2 \tilde{I}(t, P_t, V_t) \sigma^2(t) \, dt + \frac{1}{2} \int_0^T \nabla P_t \left( P^V_{i=1} - P_i \right) \sigma^2(t) \, dt \right)
= -I(T, P_T, V_T) + \int_0^T (P_t - V_t) \, dZ_t + \int_0^T \nabla V \, I(t, P_t, V_t) \, dV_t.
\] (11)

We have that
\[
|\nabla V I(t, P_t, V_t)| = \left| \int_{V_t}^{P_t} \frac{-1}{\nabla V P_t \left( P^V_{i=1} - P_i \right)} \, dz \right| < \frac{|P_t - V_t|}{C},
\]
therefore
\[
\mathbb{E} \left( \int_0^T (\nabla V I(t, P_t, V_t))^2 \sigma^2(t) \, dt \right) < \frac{1}{C^2} \mathbb{E} \left( \int_0^T (P_t - V_t)^2 \sigma^2(t) \, dt \right) < \infty,
\]
and consequently \( \mathbb{E} \left( \int_0^T \nabla V I(t, P_t, V_t) \, dV_t \right) = 0 \). Also \( \mathbb{E} \left( \int_0^T (P_t - V_t) \, dZ_t \right) = 0 \) since \( \mathbb{E} \left( \int_0^T (P_t - V_t)^2 \sigma^2(t) \, dt \right) < \infty \).

If (iii)
\[
G(t, y_t) = g(t, y_t)
\]
\[
\nabla V P_t \left( P^V_{i=1} - P_i \right) = g(t, V_t)
\]
Finally,
\[
\nabla^2 V I(t, P_t, V_t) = \frac{1}{\nabla V P_t \left( P^V_{i=1} - P_i \right)} = \frac{1}{g(t, V_t)} < \frac{1}{C}
\]
in a way that
\[
\frac{1}{2} \int_0^T \nabla^2 V I(t, P_t, V_t) \sigma^2(t) \, dt, \quad \text{and} \quad \frac{1}{2} \int_0^T \nabla V P_t \left( P^V_{i=1} - P_i \right) \sigma^2(t) \, dt
\]
only depend on \( V \). From this, it is easy to see that
\[
I(t, P_t, V_t) = I(t, P_t, V_t)
\]

By (7) and (i), we have
\[
\partial_2 I(T, P_T, V_T) = \frac{P_T - V_T}{g(T, P_T)} = 0
\]
and, by (8) and (i), we obtain
\[
\partial_{22} I(T, P_T, V_T) = \frac{1}{g(T, P_T)} - \frac{(P_T - V_T) \partial_2 g(T, P_T)}{g^2(T, P_T)} = \frac{1}{g(T, P_T)} > 0.
\]
So we have a maximum of \(-\mathbb{E}(I(T, P_T, V_T))\). We also have that \( P \) is an \( \mathbb{F} \)-martingale from (4), the integrability condition and (i). Therefore by (i) and since \( V \) is an \( \mathbb{H} \)-martingale, we obtain that
\[
P_t = \mathbb{E}(P_T|\mathcal{F}_t) = \mathbb{E}(V_T|\mathcal{F}_t) = \mathbb{E}(\mathbb{E}(V_T|\mathcal{H}_t)|\mathcal{F}_t) = \mathbb{E}(V_t|\mathcal{F}_t).
\]

Now we show that (i) and (ii) are necessary conditions. In fact (i) is necessary by Proposition 2. By (4) and the functional Itô’s formula, we have
\[
dP_t = \nabla V P_t \, dY_t,
\]
then the result follows from the fact that \( P \) is an \( \mathbb{F} \)-martingale and \( \nabla V P_t \geq C > 0 \).

Condition (iii) is also necessary if we want to maximize \( \mathbb{E} \left( \int_0^T (V_t - P_t) \, dX_t \bigg| \mathcal{H}_0 \right) \). In fact, by (11)
\[
\int_0^T (V_t - P_t) \, dX_t - I(0, P_0, V_0)
= -I(T, P_T, V_T) + \frac{1}{2} \int_0^T J_i(t, P_t, V_t) \, dt + \int_0^T (P_t - V_t) \, dZ_t + \int_0^T \nabla V I(t, P_t, V_t) \, dV_t.
\] (12)
with
\[ J_t(t, P_t, V_t) := \nabla^2 V I(t, P_t, V_t) \sigma^2_Y(t) + \nabla Y P_t \left( P_t V_t - P_t \right) \sigma^2_Z(t) \]
\[ = \frac{1}{\nabla Y P_t \left( P_t V_t - P_t - P_t \right)} \sigma^2_Y(t) + \nabla Y P_t \left( P_t V_t - P_t - P_t \right) \sigma^2_Z(t). \]

then, for fixed \( \omega \), \( J_t \) is an unbounded convex function of \( \nabla Y P_t \left( P_t V_t - P_t - P_t \right) \). Therefore we can modify the strategy in order to get \( \nabla Y P_t \left( P_t V_t - P_t - P_t \right) \) as large (or small if \( \sigma^2_Y(t) \neq 0 \)) as we want and at the same time keeping \( P_T = V_T \). That shows that the optimal wealth is not bounded and there is not equilibrium in such a situation except if \( \nabla Y P_t(y_t) = g(t, y_t) \), that it is a function of the spot value. In other words if
\[ G(t, y_t) = g(t, y_t) \]
where \( g(t, y) \geq C > 0 \) is \( C^{1,2} \). ■

We can obtain an analogous result to Theorem 2 for the non-risk-neutral case when the utility function is \( U(x) = \gamma e^{\gamma x}, \gamma < 0 \) and when \( V_t \equiv V \).

**Theorem 3** Let \( V_t \equiv V \) and \( \tau = T \). For any \( t < T \), let \( P \in C^{1,3}_b \) be a price functional such that
\[ \mathcal{D}_t P_t + \frac{1}{2} \nabla^2_Y P_t \sigma^2_Z(t) = 0, \quad (13) \]
with
\[ \mathcal{L} P_t := \mathcal{D}_t \nabla Y P_t - \nabla Y \mathcal{D}_t P_t = \gamma \sigma^2_Z(t) \left( \nabla Y P_t \right)^2 \quad (14) \]
and
\[ \nabla Y P_t = G(t, P_t), \]
where \( G(t, y_t) \geq C > 0 \) is \( C^{1,2} \). Assume that \( \mathbb{E} \left( \exp \left\{ \frac{1}{2} \gamma^2 \int_0^T (P_t - V)^2 \sigma^2_Z(t) dt \right\} \right) < \infty \) and \( \mathbb{E} \left( \int_0^T G^2(t, P_t) \sigma^2_Z(t) dt \right) < \infty \).

Then there is an equilibrium in the non-risk-neutral case, with utility function \( U(x) = \gamma e^{\gamma x} \), if and only if
(i) \( P_T = V \),  
(ii) \( Y \) is an \( \mathbb{F} \)-martingale  
(iii) \( G(t, y_t) = g(t, y_t) \)
where \( g(t, y) \geq C > 0 \) is \( C^{1,2} \).

**Proof.** As in the previous proof, let
\[ I(t, y, t, v) := \int_v^{y_t} \frac{z - v}{\nabla Y P_t \left( y_t - y_t \right)} dz, \]
then, by the functional Itô formula,
\[ I(T, P_T, V) = I(0, P_0, V) + \int_0^T \mathcal{D}_t I(t, P_t, V) dt + \int_0^T \nabla Y I(t, P_t, V) P_t dt \]
\[ + \frac{1}{2} \int_0^T \nabla^2 Y I(t, P_t, V) \left( \nabla Y P_t \right)^2 \sigma^2_Z(t) dt, \]
and by (13) and (14)
\[ \mathcal{D}_t \left( \nabla Y P \right) + \frac{1}{2} \nabla^2_Y \left( \nabla Y P_t \right) \left( \nabla Y P_t \right)^2 \sigma^2_Z(t) = \gamma \sigma^2_Z(t). \]
Analougously to the previous theorem we have

\[ D_t I(t, P_t, V_t) + \frac{1}{2} \nabla_t^2 I(t, P_t, V_t) \left( \nabla Y P_t (P_t) \right)^2 \sigma_Z^2(t) \]

\[ = -\gamma \int_V^{P_t} (z - v) dz + \nabla Y P_t \left( P_t^V - P_t \right) \sigma_Z^2(t) \]

\[ = -\frac{\gamma}{2} (P_t - V)^2 \sigma_Z^2(t) + \nabla Y P_t \left( P_t^V - P_t \right) \sigma_Z^2(t) \]  

(15)

And we get the maximum value of

\[ I \left( t, P_t, V_t \right) = I(t, P_T, V) \]

and consequently

\[ \partial_2 I(T, P_T, V_T) = \frac{P_T - V_T}{g(T, P_T)} = 0 \]

\[ \partial_{22} I(T, P_T, V) = \frac{1}{g(T, P_T)} - (P_T - V) \partial_2 g(T, P_T) = \frac{1}{g(T, P_T)} > 0. \]

So the minimum value of \( I(T, P_T, V) \) is when \( P_T = V \) and its value is \( I(T, P_T, V) := \int_V^{P_T} \frac{V - v}{g(t, v)} dv = 0. \) Then, since \( \gamma < 0, \)

\[ \mathbb{E} \left( \gamma \exp \left\{ \gamma \int_0^T (V - P_t) dX_t \right\} \exp \left\{ -\gamma I(0, P_0, V) - \gamma \int_0^T g(t, V) dt \right\} \right) \]

\[ \leq \gamma \mathbb{E} \left( \exp \left\{ \gamma \int_0^T (P_t - V) dZ_t - \frac{1}{2} \gamma^2 \int_0^T (P_t - V)^2 \sigma_Z^2(t) dt \right\} \right) = \gamma. \]

And we get the maximum value of \( \mathbb{E} \left( \gamma \int_0^T (V - P_t) dX_t \right) \) when \( P_T = V. \) The rest of the proof is analogous to the one of the previous theorem. \( \blacksquare \)

5 Necessary conditions for the equilibrium pricing rules

In this section we study general necessary conditions to obtain an equilibrium and we see that the classes of price functionals of the previous section, characterised by the relationships (4) and (5) for the risk-neutral insider and (13) and (14) for the risk-averse one, are actually justified by the arguments that follow. Note that in this section, the release time of information \( \tau \) is assumed predictable and bounded. A remark at the end of the session deals with the case of \( \tau \) independent of the observable variables.
Here below we study the effect of an $\varepsilon$-perturbation of the insider strategies:

$$dX_t^{(\varepsilon)} := dX_t + \varepsilon \beta_t dt,$$

where $\beta$ is a bounded adapted process, in the prices $P_t = P_t(Z_t + X_t)$.

From now on, we are going to assume that there exist a strictly positive $B(\mathbb{R}_+) \otimes \mathcal{F}$-measurable function $K(s, t)(\omega)$, $0 \leq s \leq t \leq \tau$, $\omega \in \Omega$, continuous for all $0 \leq s \leq t \leq \tau$, such that, for a.a. $t$,

$$P_t^{(\varepsilon)} - P_t = \varepsilon \int_0^t K(s, t) \beta_s ds + o(\varepsilon) R_t,$$

when we make an $\varepsilon$-perturbation of the strategies. Here above $P_t^{(\varepsilon)} := P_t(Z_t + X_t^{(\varepsilon)})$, and $R$ is a bounded progressively measurable process, independent of $\beta$. Observe that the random variables $K(s, t)$ are strictly positive because $P_t = P_t(Y_t)$ is a strictly increasing functional, see Definition (7). Note that, as a consequence of (R), we have that

$$\lim_{\varepsilon \to 0} \frac{P_t^{(\varepsilon)} - P_t}{\varepsilon} = \int_0^t K(s, t) \beta_s ds.$$

**Proposition 3** Assume that $P$ is continuous for fixed times and that, for any bounded adapted process $\beta$, (R) holds by means of the kernels $K$ described above. Then

$$\nabla_Y P_t = K(t, t).$$

**Proof.** Set, for fixed $t$ and with $r < t$,

$$\beta^{(r)}_s := \frac{1}{t-r} 1_{[r,t]}(s).$$

Taking limits in (R) when $r \to t$ we have that, a.s. $\mathbb{P} \otimes \text{Leb},$

$$P_t(Y_t^{(\varepsilon)}) - P_t(Y_t) = \varepsilon K(t, t) + o(\varepsilon) R_t$$

By this we can conclude. ■

The next result presents a factorisation property of the kernel and a sufficient condition to obtain it.

**Proposition 4** Let $G$ and $F$ be $C^{1,2}$. Assume that

$$\nabla_Y P_t = G(t, P_t),$$

and

$$D_t P_t + \frac{1}{2} \nabla_Y^2 P_t \sigma_Z^2(t) = F(t, P_t)$$

hold. Then the kernel $K$ admits factorisation

$$K(s, t) = K_1(s) K_2(t),$$

with

$$K_2(t) = \mathcal{E} \left( \int_0^t \partial_2 G(s, P_s) dY_s \right) \exp \left( \int_0^t \partial_2 F(s, P_s) ds \right),$$

where $\mathcal{E}$ is the stochastic exponential, and

$$K_1(t) = \frac{G(t, P_t)}{K_2(t)}.$$

Moreover $[K_1, K_1] \equiv 0$. 

$^1\mathcal{F}$ denotes the $\mathcal{F}$-predictable $\sigma$-field.
Proof. Since
\[ P_t = P_0 + \int_0^t \nabla_Y P_s dY_s + \int_0^t \left( \mathcal{D}_s P_s + \frac{1}{2} \nabla_Y^2 P_s \sigma_Z^2(s) \right) ds \]
we have that,
\[
\frac{dP^{(e)}_t}{d\varepsilon} \bigg|_{\varepsilon=0} = \int_0^t \partial_2 G \frac{dP^{(e)}_s}{d\varepsilon} \bigg|_{\varepsilon=0} dY_s + \int_0^t G \beta_s ds + \int_0^t \partial_2 F \frac{dP^{(e)}_s}{d\varepsilon} \bigg|_{\varepsilon=0} ds.
\]
Therefore
\[
\frac{dP^{(e)}_t}{d\varepsilon} \bigg|_{\varepsilon=0} = \mathcal{E} \left( \int_0^t \partial_2 G(s, P_s) dY_s \right) \exp \left( \int_0^t \partial_2 F(s, P_s) ds \right) \times \int_0^t \mathcal{E} \left( \int_0^s \partial_2 G(u, P_u) dY_u \right) \exp \left( \int_0^s \partial_2 F(u, P_u) du \right) ds.
\]
This is easy to be verified by showing that the differentials and the values at \( t = 0 \) of \( \frac{dP^{(e)}_t}{d\varepsilon} \bigg|_{\varepsilon=0} \) in (20) and (21) are the same. Finally, by a uniqueness argument, we have that
\[
\frac{dP^{(e)}_t}{d\varepsilon} \bigg|_{\varepsilon=0} = \int_0^t K_1(s) K_2(t) \beta_s ds,
\]
with
\[ K_2(t) = \mathcal{E} \left( \int_0^t \partial_2 G(s, P_s) dY_s \right) \exp \left( \int_0^t \partial_2 F(s, P_s) ds \right), \]
and
\[ K_1(t) = \frac{G(t, P_t)}{\mathcal{E} \left( \int_0^t \partial_2 G(u, P_u) dY_u \right) \exp \left( \int_0^t \partial_2 F(u, P_u) du \right)}. \]
Finally it is easy to see that
\[
dK_1(t) = \frac{\partial_1 G + \frac{1}{2} G^2 \partial_{22} G \sigma_Z^2(t)}{K_2(t)} dt + \partial_2 G \frac{\mathcal{D}_t P_t + \frac{1}{2} \nabla_Y^2 P_t}{K_2(t)} dt - \frac{G \partial_2 F}{K_2(t)} dt.
\]
\[ \blacksquare \]
In particular we obtain the following

**Proposition 5** Let \( P \) be a price functional such that (17) holds and (18) holds for \( F \equiv 0 \), i.e.
\[ \mathcal{D}_t P_t + \frac{1}{2} \nabla_Y^2 P_t \sigma_Z^2(t) = 0. \]
Then
\[ \mathcal{L} P_t = K_2(t) \frac{d}{dt} K_1(t). \]

Proof. By (23) we have
\[ \mathcal{L} P_t = \mathcal{D}_t \nabla_Y P_t + \frac{1}{2} \nabla_Y \left( \nabla_Y^2 P_t \right) P_t \sigma_Z^2(t) \]
\[ = \partial_1 G + \frac{1}{2} G^2 \partial_{22} G \sigma_Z^2(t) + \frac{1}{2} \partial_2 G \left( \mathcal{D}_t P_t + \frac{1}{2} \nabla_Y^2 P_t \sigma_Z^2(t) \right) \]
\[ = \partial_1 G + \frac{1}{2} G^2 \partial_{22} G \sigma_Z^2(t). \]
Now by (22) and since $F \equiv 0$, we have that
\[
\mathcal{L} P_t = K_2(t) \frac{dK_1(t)}{dt}.
\]

We have obtain a general result with a necessary condition for the an optimal strategy.

**Theorem 4** Assume that for all $\beta$ bounded (R) holds in terms of the kernel $K$ as above. If $X$ is optimal, then we have
\[
1_{[0, \tau]}(t) E \left( U'(W_t) (V_t - P_t) \mid \mathcal{H}_t \right) - E \left( \int_{t \wedge \tau} \mathbb{E} \left( U'(W_{t'}) \mid \mathcal{H}_{t'} \right) K(t, s) dX_s \bigg| \mathcal{H}_t \right) = 0, \text{ a.s.-}\mathbb{P} \otimes \text{Leb} \tag{24}
\]

**Proof.** Take $dX_t^{(\varepsilon)} := dX_t + \varepsilon \beta_t dt$, where $\beta$ is a bounded adapted processes, then,
\[
E \left( U(W_t^{(\varepsilon)}) - U(W_t) \right)
= E \left( U \left( \int_0^T (V_t - P_t^{(\varepsilon)}) dX_t^{(\varepsilon)} \right) - U (W_t) \right)
= \varepsilon E \left( U'(W_t) \left( \int_0^T (V_t - P_t) \beta_t dt - \int_0^T \left( \int_0^t K(s, t) \beta_s ds \right) dX_t \right) \right) + o(\varepsilon)
= \varepsilon E \left( U'(W_t) \left( \int_0^T (V_t - P_t) \beta_t dt - \int_t^T K(t, s) dX_s \beta_t dt \right) \right) + o(\varepsilon).
\]

Note that, by Fubini’s theorem,
\[
\int_0^T \left( \int_0^t K(s, t) \beta_s ds \right) dX_t = \int_0^T \left( \int_t^T K(t, s) dX_s \right) \beta_t dt.
\]

Then
\[
\left. \frac{dE \left( U \left( W_t^{(\varepsilon)} \right) \right)}{d\varepsilon} \right|_{\varepsilon = 0} = 0
\]

implies that
\[
E \left( \int_0^T U'(W_t) \left( V_t - P_t - \int_t^T K(t, s) dX_s \right) \beta_t dt \right) = 0.
\]

Since we can take $\beta_t = \alpha_u 1_{[u, u+h]}(t)$, with $\alpha_u$ measurable and bounded and $\tau$ is a stopping time, we have that
\[
1_{[0, \tau]}(t) E \left( U'(W_t) \left( V_t - P_t - \int_t^T K(t, s) dX_s \right) \mid \mathcal{H}_t \right) = 0,
\text{ a.s.-}\mathbb{P} \otimes \text{Leb}.
\]

And, from the Law of Iterated Expectations
\[
1_{[0, \tau]}(t) E \left( U'(W_t) (V_t - P_t) \mid \mathcal{H}_t \right) - E \left( \int_{t \wedge \tau} \mathbb{E} \left( U'(W_{t'}) \mid \mathcal{H}_{t'} \right) K(t, s) dX_s \bigg| \mathcal{H}_t \right) = 0
\]

The result above allows us to give some necessary conditions for an equilibrium.

**Proposition 6** In the conditions of the Theorem 4 and assuming that (17) and (18) hold, we have that if $(P, X)$ is an equilibrium, then
\[
0 = E \left( U'(W_t) (V_t - P_t) \mid \mathcal{H}_t \right) \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) - \frac{E \left( U'(W_t) \mid \mathcal{H}_t \right)}{K_1(t)} \left( D_t P_t + \frac{1}{2} \nabla^2 P_t \sigma_Z^2(t) \right)
- \frac{1}{K_1(t)} \frac{d}{dt} \left[ P_t E \left( U'(W_t) \mid \mathcal{H}_t \right) \right]_0^t. \tag{25}
\]
Proof. Thanks to (A1’), (R), the factorisation property (19), and by means of Theorem 1 and Proposition 3, we have that

\[
\mathbb{E} \left( \int_t^T \mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s) K(t, s) dX_s \mid \mathcal{H}_t \right)
\]

\[= K_1(t) \mathbb{E} \left( \int_t^T \frac{1}{K_1(s)} \mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s) K(t, s) dY_s \bigg| \mathcal{H}_t \right)
\]

\[= K_1(t) \mathbb{E} \left( \int_t^T \frac{1}{K_1(s)} \mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s) \left( dP_s - \left( \mathcal{D}_s P_s + \frac{1}{2} \nabla^2 Y P_s \sigma^2_2(s) \right) ds \right) \bigg| \mathcal{H}_t \right)
\]

\[= K_1(t) \mathbb{E} \left( \int_t^T \frac{1}{K_1(s)} \mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s) dP_s \bigg| \mathcal{H}_t \right) - K_1(t) \mathbb{E} \left( \int_t^T \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s)}{K_1(s)} \left( \mathcal{D}_s P_s + \frac{1}{2} \nabla^2 Y P_s \sigma^2_2(s) \right) ds \bigg| \mathcal{H}_t \right).
\]

Moreover, observe that

\[\int_0^t \frac{1}{K_1(s)} \mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s) dP_s = \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_t) P_t}{K_1(t)} - \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_0) P_0}{K_1(0)} - \int_0^t P_s d \left( \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s)}{K_1(s)} \right) - \left[ P_s \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s)}{K_1(s)} \right]_0^t.
\]

Hence, taking (24) into account, we obtain

\[
1_{[0,\tau]}(t) \left( \frac{\mathbb{E}(U'(W_\tau) (V_\tau - P_t) \mid \mathcal{H}_t)}{K_1(t)} \right) + \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_{t \wedge \tau}) P_{t \wedge \tau}}{K_1(t \wedge \tau)} - \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_0) P_0}{K_1(0)}
\]

\[= \int_0^{t \wedge \tau} P_s d \left( \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s)}{K_1(s)} \right) - \left[ P_s \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s)}{K_1(s)} \right]_0^{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_s)}{K_1(s)} \left( \mathcal{D}_s P_s + \frac{1}{2} \nabla^2 Y P_s \sigma^2_2(s) \right) ds
\]

\[= 0.
\]

Then by the uniqueness of the canonical decomposition in the previous equation (notice that the jump of \(1_{[0,\tau]}(t)\) is killed in the case that \(\tau\) is predictable), we have

\[0 = \mathbb{E}(U'(W_\tau) (V_\tau - P_t) \mid \mathcal{H}_t) \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) - \frac{\mathbb{E}(U'(W_\tau) \mid \mathcal{H}_t)}{K_1(t)} \left( \mathcal{D}_t P_t + \frac{1}{2} \nabla^2 Y P_t \sigma^2_2(t) \right)
\]

\[- \frac{1}{K_1(t)} \frac{d}{dt} \left[ P_t \mathbb{E}(U'(W_\tau) \mid \mathcal{H}_t) \right]_0^t + \frac{d}{dt} \left[ \mathbb{E}(U'(W_\tau) (V_\tau - P_t) \mid \mathcal{H}_t) \frac{1}{K_1(t)} \right]_0^t.
\]

Finally the last term vanishes by Proposition 4. □

Moreover, we have the following specific conditions in the risk-neutral and risk-averse (exponential) cases.

**Proposition 7** In the risk-neutral case, under the assumptions of Proposition 6, if \((P, X)\) is an equilibrium, then

\[
\mathcal{D}_t P_t + \frac{1}{2} \nabla^2 Y P_t \sigma^2_2(t) = 0
\]

holds. Also, if \(V_t \neq P_t\), a.s. \(\mathbb{P} \otimes \text{Leb}\), we have that

\[
\mathcal{L} P_t = 0.
\]

**Proof.** As a consequence of Proposition 6 we have that, in the risk neutral case, for the functionals above,

\[0 = (V_t - P_t) \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) - \frac{1}{K_1(t)} \left( \mathcal{D}_t P_t + \frac{1}{2} \nabla^2 Y P_t \frac{d[Z]}{dt} \right)
\]

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By the competitiveness of prices $E(V_t|F_t) = P_t$, so by taking conditional expectations w.r.t $F_t$ we obtain that
\[ D_t P_t + \frac{1}{2} \nabla^2 V P_t \frac{d|Z|}{dt} = 0 \]
and if $V_t \neq P_t$, a.s. $\mathbb{P} \otimes \text{Leb}$,
\[ \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) = 0. \tag{27} \]

Now by Proposition 5 we obtain (26). \[ \square \]

Consider the risk-averse case when $U(x) = \gamma e^{\gamma x}$ with $\gamma < 0$. If the noise traders total demand $Z$ is Gaussian we can apply the following relationship between vertical and Fréchet or Malliavin derivatives (see Theorem 6.1 in Cont and Fournié (2013)):
\[ E \left( D^2 V(W_r) | \mathcal{H}_t \right) = \nabla_Z E (U(W_r) | \mathcal{H}_t). \]

Then by (24)
\[ E \left( U'(W_r) (V_r - P_t) + D^2 V(W_r) \right) | \mathcal{H}_t) = 0, \]
we have that
\[ \nabla_Z E (U(W_r) | \mathcal{H}_t) = -E \left( U'(W_r) (V_r - P_t) | \mathcal{H}_t \right). \]

Since $U'(x) = \gamma U(x)$, we obtain
\[ \nabla_Z E (U'(W_r) | \mathcal{H}_t) = -\gamma E \left( U'(W_r) (V_r - P_t) | \mathcal{H}_t \right) \]
\[ \frac{d}{dt} \left[ P_t E (U'(W_r) | \mathcal{H}_t) \right] = \nabla_Y P_t \nabla_Z E \left( U'(W_r) | \mathcal{H}_t \right) \sigma_Z^2(t) = K(t,t) E \left( U'(W_r) (V_r - P_t) | \mathcal{H}_t \right) \sigma_Z^2(t). \]

Then (25) becomes
\[ E \left( U'(W_r) (V_r - P_t) | \mathcal{H}_t \right) \left( \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) + \gamma K_2(t) \sigma_Z^2(t) \right) - \frac{E (U'(W_r) | \mathcal{H}_t)}{K_1(t)} \left( D_t P_t + \frac{1}{2} \nabla^2 V P_t \sigma_Z^2(t) \right) = 0. \]

Furthermore, if $V_t \equiv V$ we have that
\[ (V - P_t) \left( \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) + \gamma K_2(t) \sigma_Z^2(t) \right) - \frac{1}{K_1(t)} \left( D_t P_t + \frac{1}{2} \nabla^2 V P_t \sigma_Z^2(t) \right) = 0. \]

Taking the conditional expectations w.r.t $F_t$, by the competitiveness of prices $E (V | F_t) = P_t$, we obtain that
\[ D_t P_t + \frac{1}{2} \nabla^2 Y P_t \sigma_Z^2(t) = 0. \]
Provided that $V \neq P_t$, a.s. $\mathbb{P} \otimes \text{Leb}$, we have that
\[ \frac{d}{dt} \left( \frac{1}{K_1(t)} \right) + \gamma K_2(t) \sigma_Z^2(t) = 0. \tag{28} \]

Then we have the following proposition.

**Proposition 8** Consider the risk-averse case with utility function is $U(x) = \gamma e^{\gamma x}$, $\gamma < 0$. Let $V_t \equiv V$ and assume that (17) and (18) hold. Also assume that $Z$ is Gaussian. If $(P, X)$ is an equilibrium, we have (23):
\[ D_t P_t + \frac{1}{2} \nabla^2 Y P_t \sigma_Z^2(t) = 0 \]
and, if $V \neq P_t$, a.s. $\mathbb{P} \otimes \text{Leb}$, we have that
\[ \mathcal{L} P_t = \gamma \sigma_Z^2 (\nabla_Y P_t)^2. \]
Proof. By (28)
\[ \frac{1}{K_2^2(t)} \frac{d}{dt} K_1(t) = \gamma K_2(t) \sigma_Z^2(t), \]
now by Proposition 5 and Proposition 3
\[ \mathcal{L} P_t = \gamma K_1(t) K_2(t) \sigma_Z^2(t) = \gamma (\nabla_Y P_t)^2 \sigma_Z^2(t). \]

\[ \therefore \]

Remark 5 Finally, according to Corcuera et al. (2019) and for the above functionals if the horizon \( \tau \) is random and independent of the rest of processes involved, in an equilibrium situation we have
\[ \frac{d}{dt} \mathbb{P}(\tau > t) = 0, \]
then
\[ \frac{d}{dt} K_1(t) = K_1(t) \frac{d}{dt} \mathbb{P}(\tau > t), \]
and by Proposition 5
\[ \mathcal{L} P_t = K_2(t) \frac{dK_1(t)}{dt} = K_1(t) K_2(t) \frac{d}{dt} \mathbb{P}(\tau > t) \]
\[ = \nabla_Y P_t \frac{d}{dt} \mathbb{P}(\tau > t). \]

6 Examples of equilibrium pricing rules

Consider the following class of functionals
\[ P_t = H(t, \xi_t), \quad t \geq 0, \quad \xi_t := \int_0^t \lambda(s, P_s) dY_s, \]
where \( \lambda \in C^{1,2} \) is a strictly positive function and \( H \in C^{1,3} \) with \( H(t, \cdot) \) strictly increasing for every \( t \geq 0 \).

Then, by using the Itô’s formula and omitting the arguments in the functions, we have
\[ dP_t = \partial_2 H \lambda dY_t + \left( \partial_1 H + \frac{1}{2} \partial_{22} H \lambda^2 \sigma_Z^2 \right) dt. \]
Furthermore, we have that
\[ D_t P_t = \partial_1 H + \partial_2 H D_t \xi_t = \partial_1 H - \frac{1}{2} \partial_2 H \nabla_Y \xi_t \sigma_Z^2 \]
\[ = \partial_1 H - \frac{1}{2} \partial_2 H \nabla_Y \lambda \sigma_Z^2 \]
\[ = \partial_1 H - \frac{1}{2} \partial_2 H \partial_2 \lambda \nabla_Y P_t \sigma_Z^2 \]
and
\[ \nabla_Y^2 P_t = \partial_2 \lambda \nabla_Y P_t \partial_2 H + \lambda \partial_{22} H \nabla_Y \xi_t \]
\[ = \partial_2 \lambda \nabla_Y P_t \partial_2 H + \lambda^2 \partial_{22} H. \]
Consequently,
\[ D_t P_t + \frac{1}{2} \nabla_Y^2 P_t \sigma_Z^2 = \partial_1 H + \frac{1}{2} \partial_{22} H \lambda^2 \sigma_Z^2. \]
Then, under the condition
\[ \mathcal{D}_t P_t + \frac{1}{2} \nabla^2_Y P_t \sigma^2_Z = 0, \]
we have that
\[ \partial_1 H + \frac{1}{2} \partial_{22} H \lambda^2 \sigma^2_Z = 0, \]
and by Proposition 4,
\[ K(s, t) = \frac{\lambda(s, P_s)}{\eta_s} \partial_2 H(t, \xi_t) \eta_t, \]
where
\[ \eta_t := E \left( \int_0^t \partial_2 H \partial_2 \lambda dY_s \right). \]
Therefore \( K(s, t) = K_1(s) K_2(t), \) with
\[ K_1(s) = \frac{\lambda(s, P_s)}{\eta_s}, K_2(t) = \partial_2 H(t, \xi_t) \eta_t. \]
By using the Itô formula we obtain that
\[
\begin{align*}
\frac{1}{K_1(s)} d \left( \frac{1}{K_1(s)} \right) &= \eta_s \left( \frac{1}{\lambda} \right) + \frac{1}{\lambda} d \eta_s + d \left[ \eta, \frac{1}{\lambda} \right]_s \\
&= -\eta_s \frac{\partial_1 \lambda}{\lambda^2} ds - \eta_s \frac{\partial_1 \lambda}{\lambda^2} \partial_2 H \lambda dY_s - \frac{1}{2} \eta_s \lambda^2 \partial_{22} \lambda - 2 \left( \frac{\partial_1 \lambda}{\lambda} \right) \left( \partial_2 H^2 \right) \lambda \sigma^2_Z ds \\
& \quad + \frac{1}{\lambda} \eta_s \partial_1 \lambda \partial_2 H dY_s - \left( \frac{\partial_1 \lambda \partial_2 H}{\lambda} \right)^2 \sigma^2_Z ds \\
&= -\eta_s \frac{\partial_1 \lambda}{\lambda^2} ds - \frac{1}{2} \partial_{22} \lambda \left( \partial_2 H^2 \right) \sigma^2_Z ds \\
& \quad - \frac{1}{2} \partial_{22} \lambda \left( \partial_2 H^2 \right) \sigma^2_Z ds \\
&= -\eta_s \frac{\partial_1 \lambda}{\lambda^2} ds - \frac{1}{2} \partial_{22} \lambda \left( \partial_2 H^2 \right) \sigma^2_Z ds.
\end{align*}
\]
Then, we have that
\[
\mathcal{L} P_t = K_2(t) \frac{d}{dt} K_1(t) = K_2(t) K_1^2(t) \eta_t \left( \frac{1}{2} \partial_{22} \lambda \left( \partial_2 H^2 \right) \sigma^2_Z + \frac{\partial_1 \lambda}{\lambda^2} \right)
\]
\[ = \partial_2 H \lambda^2 \left( \frac{1}{2} \partial_{22} \lambda \left( \partial_2 H^2 \right) \sigma^2_Z + \frac{\partial_1 \lambda}{\lambda^2} \right)
\]
\[ = \partial_2 H \left( \partial_1 \lambda + \frac{1}{2} \sigma^2_Z \left( \lambda \partial_2 H \right)^2 \right). \]
Hence, we will have an equilibrium price rule, in the risk-neutral case, if
\[ \partial_1 H + \frac{1}{2} \partial_{22} H \lambda^2 \sigma^2_Z = 0, \]
\[ \partial_1 \lambda + \frac{1}{2} \sigma^2_Z \left( \lambda \partial_2 H \right)^2 \partial_{22} \lambda = 0. \]
and in the non risk-neutral case, for the exponential risk aversion, if
\[ \partial_1 H + \frac{1}{2} \partial_{22} H \lambda^2 \sigma^2_Z = 0, \]
and
\[
\mathcal{L} P_t = \partial_2 H \left( \partial_1 \lambda + \frac{1}{2} \sigma^2_Z \left( \lambda \partial_2 H \right)^2 \partial_{22} \lambda \right) = \gamma (\nabla_Y P_t)^2 \sigma^2_Z
\]
\[ = \gamma K_1^2(t) K_2^2(t) \sigma^2_Z = \gamma \left( \partial_2 H \lambda \right)^2 \sigma^2_Z. \]
that is
\[
\frac{\partial_1 \lambda}{\lambda^2} + \frac{1}{2} \sigma_Z^2 (\partial_2 H)^2 \partial_{22} \lambda = \gamma \partial_2 H \sigma_Z^2.
\]

(29)

We can identify some particular cases.

**For the risk-neutral case**

- \(\lambda(t, x) = \lambda > 0\), and \(H(t, x)\) harmonic with \(H(, x)\) strictly increasing. Notice that in this case it is sufficient to require that \(H(t, x)\) is \(C^{1,2}\).
- If we take \(H(t, x) = x\) and \(\lambda(t, x) = \lambda > 0\), we have
  \[P_t = P_0 + \lambda Y_t,\]
  that corresponds to the Bachelier model for \(Z\) Gaussian.
- If \(H(t, x) = x\) and \(\lambda(t, x) = \lambda x\), we have
  \[P_t = P_0 e^{\lambda Y_t - \frac{1}{2} \lambda^2 t}\]
  that is the Black-Scholes model.

**For the non risk-neutral model**

- Note that \(H(t, x)\) harmonic and \(\lambda\) constant cannot be an equilibrium. Therefore equilibrium prices cannot be a function of the spot aggregate demand.
- If we take \(H(t, x) = x\) and \(\lambda(t, x) = Cx(1 - x)\), with \(C > 0\), we have that (29) becomes
  \[\frac{1}{2} \partial_{xx} \lambda = \gamma\]
  that is \(\gamma = -C\). This model will give prices in \((0, 1)\) and if \(Y\) is a Brownian motion \(B\) we have that
  \[dP_t = CP_t (1 - P_t) dB_t\]
  and this is the well-known Kimura model in population genetics, see Kimura (1964).

7 Examples of equilibrium models

It is apparent that depending on the behaviour of the fundamental value and the aggregate demand of the noise traders we can have an equilibrium with one or another equilibrium pricing rule. If the aggregate demand \(Z\) of the noise traders is a Brownian motion with variance \(\sigma_Z^2\), \(Y = X + Z\) will be also an \(\mathbb{F}\)-Brownian motion with variance \(\sigma_Z^2\), because of Theorem 2 and the Lévy Theorem. Consequently, we will have an equilibrium if the pricing rule \(P_t(Y_t)\) is such that \(P_T(Y_T) = V_T\). Note also that the strategy \(X\) will be just obtained as the canonical decomposition of the \(\mathbb{F}\)-Brownian motion \(Y\) under the filtration \(\mathbb{F}\).

Consider the case where \(Z\) is a Brownian motion with variance \(\sigma^2\) and \(V_t \equiv V\). In such a situation we have a necessary and sufficient condition for and equilibrium for both, the risk-neutral case and the risk-adverse case under the exponential utility. Also in both cases the equilibrium pricing rules give prices that are continuous diffusions:
\[dP_t = \lambda(t, P_t) dY_t\]
where \(dY_t = \sigma dW_t\) and \(W\) is a standard Brownian motion. In the risk-neutral case \(\lambda(t, x)\) satisfies
\[\partial_t \lambda + \frac{1}{2} \lambda^2 \sigma^2 \partial_{xx} \lambda = 0,\]
and in the risk-adverse case
\[ \partial_t \lambda + \frac{1}{2} \lambda^2 \sigma^2 \partial_{xx} \lambda = \gamma \lambda^2 \sigma^2. \]
In any case the additional necessary and sufficient condition to have an equilibrium model is to find a strategy such that \( P_T = V \) and at the same time \( Y \) is certainly a Brownian motion with variance \( \sigma^2 \). We have to find \( \alpha_t(V) \) with \( \alpha_t(x) \mathcal{F}_t \)-measurable, such that the equation
\[ dY_t = \alpha_t(V)dt + dZ_t, \quad 0 \leq t \leq T, \]
with \( V = P_T \) and independent of \( Z \), has a strong solution. In order to do so, we can look for certain \( \alpha(t, x, Y_t) \), where \( x \) is a fixed value of \( P_T \) and try to find a strong solution of
\[ dY_t = \alpha_t(x)dt + dZ_t, \quad 0 \leq t \leq T, \]
later we can insert \( V \) instead of \( y \), but we need \( Y \) to be a Brownian motion with variance \( \sigma^2 \). Sufficient conditions to have a strong solution are given, e.g., in Theorem 4.6, Liptser and Shiryaev (2001). Then \( \alpha_t(x) \) has to be the drift in the canonical decomposition of \( Y \) when \( Y_t \) and \( Z_t \) are known at time \( t \). The following propositions are useful to find \( \alpha \), here we assume that \( \mathcal{F}_t = \bar{\sigma}(W_t, t \geq 0) \), \( \bar{\sigma} \) denotes the \( \sigma \)-field corresponding to the usual augmentation of the natural filtration.

**Proposition 9** Assume that for any bounded and measurable function \( f \) there exists a \( \mathcal{B}[0, T] \otimes \mathcal{F}_t \)-measurable process \( \xi \), independent of \( f \), such that
\[ f(P_T) = \mathbb{E}(f(P_T)) + \int_0^T \mathbb{E}(f(P_T)\xi_t|\mathcal{F}_t) \, dW_t, \]
with \( \int_0^T |\xi_t| \, dt < \infty \). Then \( W - \int_0^T \alpha_t(P_T) \, dt \) is an \( (\mathcal{F}_t \cup \sigma(P_T)) \)-Brownian motion with \( \alpha_t(P_T) = \mathbb{E}(\xi_t|\mathcal{F}_t \cup \sigma(P_T)) \).

**Proof.** Let \( f \) be a measurable and bounded function and \( A \in \mathcal{F}_s \), with \( s \leq t \). Then
\[ \mathbb{E}((W_t - W_s)1_A f(P_T)) = \mathbb{E} \left( 1_A \int_s^t \mathbb{E}(f(P_T)\xi_u|\mathcal{F}_u) \, du \right) \]
\[ = \mathbb{E} \left( 1_A f(P_T) \int_s^t \mathbb{E}(\xi_u|\mathcal{F}_u \cup \sigma(P_T)) \, du \right). \]

**Proposition 10** Suppose that
\[ \text{d}P_{P_T|\mathcal{F}_t}(x|\mathcal{F}_t) = L_T(x; W_t)\mu(\text{d}x) \]
is a regular version of the conditional probability of \( P_T \) given \( \mathcal{F}_t \), \( \mu \) being a reference measure and such that
\begin{enumerate}
  \item \( L_T(x; W_t) > 0 \) for all \( (x, \omega) \mu \otimes \mathbb{P} \)-a.s.,
  \item \( \nabla_W \int_{\mathbb{R}} f(x) L_T(x; W_t) \mu(\text{d}x) = \int_{\mathbb{R}} f(x) \nabla_W L_T(x; W_t) \mu(\text{d}x). \)
\end{enumerate}
Then \( W - \int_0^T \alpha_t(P_T) \, dt \) is an \( (\mathcal{F}_t \cup \sigma(P_T)) \)-Brownian motion with \( \alpha_t(x) = \nabla_W \log L_T(x; W_t) \), provided that \( \log L_T(x; W_t) \in \mathbb{C}^1 \).

**Proof.** Let \( f \) be a measurable and bounded function
\[ \nabla_W \mathbb{E}(f(P_T)|\mathcal{F}_t) = \nabla_W \int_{\mathbb{R}} f(x) L_T(x; W_t) \mu(\text{d}x) \]
\[ = \int_{\mathbb{R}} f(x) \nabla_W L_T(x; W_t) \mu(\text{d}x) \]
\[ = \int_{\mathbb{R}} f(x) \nabla_W \log L_T(x; W_t) L_T(x; W_t) \mu(\text{d}x) \]
\[ = \mathbb{E}(f(P_T) \nabla_W \log L_T(P_T; W_t)|\mathcal{F}_t). \]
Example 3 Assume that \( P_t = P_0 + \sigma W_t \). Then \( P_T | F_t \sim N(P_0 + \sigma W_t, \sigma^2(T-t)) \), that is
\[
dP_{P_t | F_t}(x | F_t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp \left\{ -\frac{1}{2\sigma^2(T-t)} (x - P_0 - \sigma W_t)^2 \right\} dx,
\]
then
\[
\alpha_t(x) = \frac{\sigma (x - P_0 - \sigma W_t)}{\sigma^2(T-t)},
\]
that is
\[
\alpha_t(P_T) = \frac{W_T - W_t}{T-t}.
\]

Example 4 Assume that \( P_t = P_0 + \int_0^t G(u, P_u) dW_u \), \( G \in C^{1,2}, \mathbb{E} \left( \int_0^T G^2(t, P_t) dt \right) < \infty \). Let \( p_{s,t}(x, y) \) the transition density corresponding to the Markov process \( P \). Then according to the previous proposition
\[
\begin{align*}
\alpha_t(y) &= \nabla_w \log p_{s,t}(P_t, y) \\
&= \partial_1 \log p_{s,t}(P_t, y) \nabla_w P_t \\
&= \partial_1 \log p_{s,t}(P_t, y) G(t, P_t).
\end{align*}
\]
For instance, we can consider the simple case where \( P_t = P_0 + \int_0^t \sigma P_u dW_u \), then, for \( s \leq t \), \( P_t = P_s \exp \{ \sigma (W_t - W_s) - \frac{1}{2} \sigma^2 (t-s) \} \), and
\[
p_{s,t}(x, y) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp \left\{ -\frac{1}{2\sigma^2(T-t)} \left( \log y - \log x - \frac{1}{2} \sigma^2 (t-s) \right)^2 \right\} \frac{1}{y},
\]
consequently
\[
\alpha_t(y) = \partial_1 \log p_{s,t}(P_t, y) G(t, P_t) = \sigma \frac{\log y - \log P_t - \frac{1}{2} \sigma^2 (t-s)}{\sigma^2(T-t)},
\]
that is
\[
\alpha_t(P_T) = \frac{W_T - W_t}{T-t}.
\]
Notice that in this case we obtain the same result as in the previous example. This is not surprising since in both cases to know \( P_T \) is the same as to know \( W_T \) since \( P_T \) is an increasing function of \( W_T \). Obviously this will not be the case for a general diffusion. The simplest case where this does not happen is the equilibrium price model
\[
P_t = P_0 + \int_0^t \lambda(s) dW_s,
\]
with
\[
\partial_t \lambda = \gamma \lambda^2 \sigma^2.
\]
arising in the non-risk-neutral model. Now \( P_T | P_t \sim N \left( P_0, \int_0^T \lambda^2(s) ds \right) \), then
\[
\alpha_t(x) = \nabla_w \log L_T(x; W_t) \\
= \frac{\lambda(t) (x - P_t)}{\int_0^t \lambda^2(s) ds}.
\]
If we consider the Kimura model, with risk-aversion parameter \( \gamma < 0 \),
\[
P_t = P_0 - \gamma \int_0^t P_t (1 - P_t) dW_t,
\]
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then the transition density is given by (Kimura, 1964)

\[
p_{t,T}(P_t, x) = \frac{1}{\sqrt{2\pi\gamma^2(T-t)}} \left(\frac{P_t(1-P_t)}{\sqrt{x(1-x)}}\right)^3 \exp \left\{-\frac{\gamma^2}{8}(T-t) - \frac{\log \frac{1-P_t}{1-x}P_t^2}{2\gamma^2(T-t)} \right\},
\]

and we have that

\[
\alpha_t(y) = \nabla_W \log L_T(x; W_t) = \frac{1}{2}(1-2P_t) + \frac{\log \frac{1-P_t}{1-x}P_t}{\gamma^2(T-t)}.
\]

**Example 5** We can consider the case where the privilege information is the time, say \(\tau\), where a Brownian motion reaches for the first time a level \(a\). Then, assume that

\[
P_T = h(T \wedge \tau)
\]

for a measurable and bounded function \(h\). Now we have that

\[
P_T = P_0 + \int_0^T \nabla_W \mathbb{E}(h(T \wedge \tau)|\mathcal{F}_t) \, dW_t,
\]

then since

\[
f_{\tau}(u|\mathcal{F}_t) = \frac{W_t - a}{\sqrt{2\pi(u-t)^3}} \exp \left\{-\frac{(W_t - a)^2}{2(u-t)} \right\} 1_{\{\tau > t\}},
\]

we obtain that

\[
\alpha_t(u) = \nabla_W \log f_{\tau}(u|\mathcal{F}_t) = \left(\frac{1}{W_t - a} - \frac{W_t - a}{u-t}\right) 1_{\{\tau > t\}}.
\]

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