A note on devising HDG+ projections on polyhedral elements

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Abstract

In this note, we propose a simple way of constructing HDG+ projections on polyhedral elements. The projections enable us to analyze the Lehrenfeld-Schöberl HDG (HDG+) methods in a very concise manner, and make many existing analysis techniques of standard HDG methods reusable for HDG+. The novelty here is an alternative way of constructing the projections without using $M$-decomposition as a middle step. This extends our previous results [S. Du and F.-J. Sayas, SpringerBriefs in Mathematics (2019)] (elliptic problems) and [S. Du and F.-J. Sayas, arXiv:1903.11766] (elasticity) to polyhedral meshes.

1 Introduction

The Lehrenfeld-Schöberl HDG (HDG+) methods [16] have recently gained considerable interest since they superconverge on polyhedral meshes in addition to the easiness of implementation. In [12] (elliptic problems) and [13] (elasticity), we proposed mathematical tools to incorporate the analysis of the HDG+ methods into the projection-based error analysis setting [8]. In this way, we can reuse existing analysis techniques and avoid repeated or unnecessary arguments. In [12] and [13], the projections were devised for simplicial elements. In this paper, we extend the results to polyhedral elements.

To motivate the discussion, let us review some existing works. For mixed finite element methods (or simply mixed methods), the core in their design and analysis is the local projection operators; see, for instance, [22] for the Raviart-Thomas (RT) projection, [1] for the Brezzi-Douglas-Marini (BDM) projection, and [17, 18] for the Nédélec projection. These projections satisfy certain commutativity properties that can be used to analyze the numerical methods in a very concise way. Inspired by the mixed method projections, the first HDG projection was devised in [6]. It enables us to analyze a wide class of HDG methods in an unified, and also simple and concise manner. Since for both the mixed methods and the HDG methods, the core

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in their error analysis is the specially devised projections that are tailored to the numerical schemes, this way of analysis is often referred to as the “projection-based error analysis” (PBEA).

PBEA has been widely used to analyze HDG methods. See, for instance, the error analysis of the HDG methods for heat/fractional diffusion [2, 9], acoustic waves [10, 5], Stokes equations [7], Helmholtz equations [13]. On the other hand, new HDG projections have been devised, incorporating more variants of HDG methods into the PBEA setting; see the work of $M$-decomposition [4], an mathematical tool to systematically devise superconvergent HDG methods on polyhedral meshes. Since all $M$-decomposition HDG methods have associated HDG projections, all of their analysis can be incorporated into the PBEA setting.

Despite the wide and successful applications of HDG projections, the error analysis of some important HDG methods cannot be incorporated into the PBEA setting until very recently. An important example is the HDG+ method, proposed first by Lehrenfeld and Schöberl [16] and then analyzed by Oikawa [19] in the setting of elliptic diffusion. The method uses $P^d_k - P_{k+1} - P_k$ to approximate the flux-primal-trace triplet, and it achieves optimal convergence for all variables on general polyhedral meshes. Compared to the standard $P^d_k - P_k - P_k$ HDG method, the HDG+ method is as efficient as the standard method, since the two methods share the same size of the global systems. Moreover, the HDG+ method does not suffer from the problem of losing convergence order, which is observed for the standard HDG method on non-simplicial polyhedral meshes, or for elastic problems with strong symmetric stress formulation. Finally, the HDG+ method is extremely easy to implement, since it is a simple tweak of the standard HDG method.

As is mentioned before, most of the existing error analysis of the HDG+ methods (see, for instance, [15, 19, 20, 21]) cannot be incorporated into the PBEA setting. This makes their error analysis less concise compared to those HDG methods that can be analyzed by HDG projections. More importantly, this leads to a scattered style of error analysis and makes it hard for us to reuse the existing projection-based analysis techniques that were established in a decade. All the above indicates the necessity to develop mathematical tools to incorporate the error analysis of HDG+ methods into the PBEA setting. In this way, many existing works using HDG projections, such as the analysis of the HDG methods for various types of evolutionary equations and Helmholtz equations (see, for instance, [2, 9, 10, 14, 5]), can be automatically reused for the design and analysis of the HDG+ methods.

Following this idea, we have devised the HDG+ projections in [13] for elliptic problems and in [13] for elasticity with strong symmetric stress formulation. We have successfully used the projections, combined with some existing analysis techniques of the standard HDG methods, to derive the error estimates of the HDG+ methods for heat diffusion and acoustic waves in [12], and for time-harmonic and transient elastic waves in [13]. For simplicity, we have limited the discussions on simplicial meshes in [12, 13]. In this paper, we extend the results to polyhedral meshes by using an alternative way of constructing the projections without using $M$-decomposition [4] as a middle step.

We finally give an outline for the rest of the paper. In Section 2, we devise the HDG+ projection for elliptic problems. We also demonstrate how to use the
projection to analyze the HDG+ method for a model problem. In Section 3, we devise the HDG+ projection for elasticity. We will not demonstrate its usage, since this has been done in [13]. The projection we devise here satisfies [13, Theorem 2.1] and it will render all the analysis and estimates in [13, Sections 5,6&7] valid for general polyhedral meshes.

2 The projection for elliptic problems

In this section, we devise the HDG+ projection and demonstrate how to use it to derive the error estimates for the HDG+ method. Note that the first analysis of the HDG+ method was obtained in [19]. However, our proof here is quite different from the proof in [19]. Instead, as we will demonstrate in Section 2.3, the proof we obtained is very similar to those used in [8], thanks to the introduction of the HDG+ projection. In this way, we are able to reuse the existing projection-based error analysis to analyze the HDG+ method in a very concise way. Consequently, we can unify the analysis of the standard HDG and HDG+ methods.

Notation. Let us first introduce some notation that will be used throughout the paper. Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) be a polyhedral domain with Lipschitz continuous boundary. We consider a triangulation of \( \Omega \) denoted by \( \mathcal{T}_h \). For each element \( K \in \mathcal{T}_h \), we use the standard notation \( h_K \) as the diameter of \( K \), and introduce the parameter \( \gamma_K \) that describes the shape-regularity of \( K \), as is done in [11, Appendix]. Let \( E_K \) and \( E_h \) denote the collections of all the faces of \( K \) and \( \mathcal{T}_h \), respectively. We assume that there is a fixed positive constant \( \gamma_0 \) such that \( \gamma_0 \geq \gamma_K \) for all \( K \in \mathcal{T}_h \) (consequently the shape-regularity of \( \mathcal{T}_h \) is controlled). We write \( h := \max_{K \in \mathcal{T}_h} h_K \) as the mesh-size and \( h_{\text{min}} := \min_{K \in \mathcal{T}_h} h_K \) as the smallest diameter among all elements.

Let \( P_k(X) \) denote the polynomial space of degree \( k \) on \( X \) and let \( \Pi_k : L^2(X) \rightarrow P_k(X) \) be the corresponding \( L^2 \) projection. Here \( X \) can be an element \( K \) or a face of \( K \). Let \( \mathcal{R}_k(\partial K) := \prod_{F \in \partial K} P_k(F) \) and let \( \mathcal{P}_M : \prod_{K \in \mathcal{T}_h} L^2(\partial K) \rightarrow \prod_{K \in \mathcal{T}_h} \mathcal{R}_k(\partial K) \) be the corresponding \( L^2 \) projection. We finally introduce the following notation for the discrete inner products on \( \mathcal{T}_h \) and \( \partial \mathcal{T}_h \):

\[
(*_1, *_2)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (*_1, *_2)_K, \quad (*_1, *_2)_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (*_1, *_2)_{\partial K},
\]

where \( (\cdot, \cdot)_K \) and \( (\cdot, \cdot)_{\partial K} \) denote the \( L_2 \) inner products on \( K \) and \( \partial K \), respectively.

Model problem. In this section, we consider the following steady-state diffusion equations:

\[
\begin{align*}
\kappa^{-1} \mathbf{q} + \nabla u &= 0 \quad \text{in } \Omega, \quad \text{(1a)} \\
\nabla \cdot \mathbf{q} &= f \quad \text{in } \Omega, \quad \text{(1b)} \\
\mathbf{u} &= g \quad \text{on } \Gamma := \partial \Omega, \quad \text{(1c)}
\end{align*}
\]

where the parameter \( \kappa \in L^\infty(\Omega) \) is uniformly positive, the forcing term \( f \in L^2(\Omega) \)
and the Dirichlet data \( g \in H^{1/2}(\Gamma) \). We also introduce the notion of elliptic regularity:

\[
\|\kappa \nabla u\|_{1,\Omega} + \|u\|_{2,\Omega} \leq C_{\text{reg}} \|\nabla \cdot (\kappa \nabla u)\|_{\Omega}
\]  \tag{2} \{eq:ell_reg\}

holds for any \( u \in H^1(\Omega) \) such that the right term of the above inequality is finite, where \( C_{\text{reg}} \) is a positive constant depending only on \( \kappa \) and \( \Omega \).

**HDG+ method.** Let us first define the approximation spaces:

\[
V_h := \prod_{K \in \mathcal{T}_h} \mathcal{P}_k(K)^d, \quad W_h := \prod_{K \in \mathcal{T}_h} \mathcal{P}_{k+1}(K), \quad M_h := \prod_{F \in \mathcal{E}_h} \mathcal{P}_k(F).
\]

The HDG+ scheme is defined as follows: find \((q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h\) such that

\[
\begin{align*}
(k^{-1} q_h, r)_{\mathcal{T}_h} - (u_h, \nabla \cdot r)_{\mathcal{T}_h} + \langle \tilde{u}_h, r \cdot n \rangle_{\partial \mathcal{T}_h} &= 0, \quad \tag{3a} \{eq:HDG_s_1\} \\
(\nabla \cdot q_h, w)_{\mathcal{T}_h} + \langle \tau \mathcal{P}_m(u_h - \tilde{u}_h), w \rangle_{\partial \mathcal{T}_h} &= (f, w)_{\mathcal{T}_h}, \quad \tag{3b} \{eq:HDG_s_2\} \\
-\langle q_h \cdot n + \tau (u_h - \tilde{u}_h), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma} &= 0, \quad \tag{3c} \{eq:HDG_s_3\} \\
\langle \tilde{u}_h, \mu \rangle_{\Gamma} &= \langle g, \mu \rangle_{\Gamma}, \quad \tag{3d} \{eq:HDG_s_4\}
\end{align*}
\]

for all \((r, w, \mu) \in V_h \times W_h \times M_h\). The stabilization function \( \tau \in \prod_{K \in \mathcal{T}_h} \mathcal{R}_0(\partial K) \) and it satisfies \( c_1 h_K^{-1} \leq \tau |_{\partial K} \leq c_2 h_K^{-1} \) for all \( K \in \mathcal{T}_h \), where \( c_1 \) and \( c_2 \) are two fixed positive constants.

### 2.1 Main results

We now present the main results in this section – the HDG+ projection (Theorem 2.1) and its application (Theorem 2.2). Their proofs can be found in Section 2.2 and Section 2.3.

**HDG+ projection.** The HDG+ projection is defined as follows:

\[
\Pi_K(q, u) := (\Pi_k q, \Pi_{k+1} u) \in \mathcal{P}_k(K)^d \times \mathcal{P}_{k+1}(K), \quad \tag{4a} \{eq:def_hdg+_ell\}
\]

where \( \Pi_k : H^1(K)^d \to \mathcal{P}_k(K)^d \) is defined by solving

\[
\begin{align*}
(\Pi_k q - q, r)_K &= 0 \quad \forall r \in \nabla \mathcal{P}_k(K) \oplus (\nabla \mathcal{P}_{k+1}(K))^\perp_k, \quad \tag{4b} \{eq:def_hdg+_ell_1\} \\
(\Pi_k^d q - q, \nabla w)_K &= \langle \mathcal{P}_m(q \cdot n) - q \cdot n, w \rangle_{\partial K} \quad \forall w \in \mathcal{P}_k(K)^{k+1}. \quad \tag{4c} \{eq:def_hdg+_ell_2\}
\end{align*}
\]

In the above equations, \((\cdot)^\perp_m\) represents the orthogonal complement in the background space \( \mathcal{P}_k(K)^d \) (for \( 4b \)) or \( \mathcal{P}_m(K) \) (for \( 4c \)).

We also define an operator:

\[
\delta_{\pm \tau}^{\Pi K}(q, u) := \Pi_K q \cdot n - \mathcal{P}_m(q \cdot n) \pm \tau (\mathcal{P}_m \Pi_K u - \mathcal{P}_m u) \in \mathcal{R}_k(\partial K). \quad \tag{4d} \{eq:def_hdg+_ell_3\}
\]

We call \( \delta_{\pm \tau}^{\Pi K} \) the boundary remainder of \( \Pi_K \).
Theorem 2.1 (HDG+ projection). The projection $\Pi_K$ and the remainder $\delta^\Pi_{\pm\tau}(q, u)$ are well defined by \eqref{eq:hdgp_pj_ell_eqn_1} and they satisfy

$$ (\Pi_K u - u, v)_K = 0 \quad \forall v \in \mathcal{P}_{k-1}(K), \quad \forall \mu \in \mathcal{R}_k(\partial K), $$

\begin{align}
\langle \Pi_K q \cdot n - q \cdot n \pm \tau (\Pi_K u - u), \mu \rangle_{\partial K} &= \langle \delta^\Pi_{\pm\tau}(q, u), \mu \rangle_{\partial K}, \\
(\nabla \cdot (\Pi_K q - q), w)_K + \langle \tau P_M(\Pi_K u - u), w \rangle_{\partial K} &= \langle \delta^\Pi_{\pm\tau}(q, u), w \rangle_{\partial K} \quad \forall w \in \mathcal{P}_{k+1}(K).
\end{align}

Furthermore,

$$ \|\Pi_K q - q\|_K + h_K^{-1} \|\Pi_K u - u\|_K + h_K^{1/2} \|\delta^\Pi_{\pm\tau}(q, u)\|_{\partial K} \leq Ch_K^m(|q|_{m,K} + |u|_{m+1,K}), \quad \forall q \in \Pi^0, u \in \Pi^BDM $$

where $m \in [1, k + 1]$. Here, the constant $C$ depends only on $k$, $\gamma_K$, and $c_2$.

Note that in Theorem 2.1, equations \eqref{eq:hdgp_pj_ell_eqn_3} do not define the HDG+ projection. However, they are exactly what we need for the error analysis. The boundary remainder operator $\delta^\Pi_{\pm\tau}$ can be regarded as an indicator for how much the projection $\Pi_K$ resembles a HDG projection or a mixed method projection. For instance, if $\Pi_K$ is the classical HDG projection \cite{huang2020}, then we have $\delta^\Pi_{r=0} = 0$. This can be easily obtained by using \cite{huang2020} Eqn. (2.1c). Similarly, we have $\delta^\Pi_{r=0} = 0$ and $\delta^\Pi_{r=0} = 0$, where $\Pi^H$ and $\Pi^BDM$ represent the Raviart-Thomas and the BDM projection, respectively.

The key idea behind the HDG+ projection is to find weaker but still sufficient conditions to carry out a projection-based error analysis. For the classical HDG projection, the boundary remainder is zero, and the equations that define the projection are also the equations that we use for the error analysis. However, these two properties are not necessary, especially if we want to extend the projection-based error analysis to more variants of HDG methods. Taking the HDG+ method as an example, the guideline for devising the projection now becomes the following: among all the projections that satisfy the equations \eqref{eq:hdgp_pj_ell_eqn_3}, find one such that its approximation property is optimal, and its boundary remainder is as small as possible. As we will see soon, there is no need to enforce the boundary remainder to be zero, which is the case the standard HDG projection. In fact, a small enough boundary remainder is sufficient for optimal convergence of the method. In this way, we can devise HDG projections more flexibly, and generalize the classical projection-based error analysis of HDG methods \cite{huang2020}.

Error estimates. By using \eqref{eq:hdgp_pj_ell_eqn_conv}, we define the element-wise projections and the boundary remainder of the exact solutions:

$$ \Pi q := \prod_{K \in \mathcal{T}_h} \Pi_K q, \quad \Pi u := \prod_{K \in \mathcal{T}_h} \Pi_K u, \quad \delta_r(q, u) := \prod_{K \in \mathcal{T}_h} \delta^\Pi_{\pm\tau}(q, u). $$

We also define the norm $\| \cdot \|_h$ by $\| \mu \|_h := \sum_{K \in \mathcal{T}_h} h_K \| \mu \|_{\partial K}$ for any $\mu \in L^2(\partial \mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} L^2(\partial K)$.
Theorem 2.2. There holds
\[ \| \Pi q - q_h \|_{\mathcal{T}_h} \leq C_1 \left( \| \Pi q - q \|_{\mathcal{T}_h} + \| \tau^{-1/2} \delta_r(q, u) \|_{\mathcal{G}_h} \right). \]  \( \text{(7)} \) \{eq:est_qh\}
If \( k \geq 1 \), and the regularity assumption (2) holds, then we have
\[ \| \Pi u - u_h \|_{\mathcal{T}_h} \leq C_2 h \left( \| \Pi q - q_h \|_{\mathcal{T}_h} + \| \tau^{-1/2} \delta_r(q, u) \|_{\mathcal{G}_h} \right), \]  \( \text{(8)} \) \{eq:est_uh\}
\[ \| P_M u - u_h \|_h \leq C_2 h \left( 1 + \frac{h}{h_{\min}} \right) \left( \| \Pi q - q_h \|_{\mathcal{T}_h} + \| \tau^{-1/2} \delta_r(q, u) \|_{\mathcal{G}_h} \right). \]  \( \text{(9)} \) \{eq:est_uhat\}
Here, \( C_1 \) depends only on \( \kappa \), and \( C_2 \) depends additionally on \( k, \gamma_0 \), and \( C_{\text{reg}} \).

Note that by (6) and the fact that \( \tau \|_{\partial K} \approx h_K^{-1} \), we have \( \| \tau^{-1/2} \delta_r(q, u) \|_{\mathcal{G}_h} \lesssim h^{k+1} \) for smooth enough exact solutions. Therefore, Theorem 2.2 shows that the HDG+ method is optimal for both \( q_h \) and \( u_h \), and \( \| Pu - \hat{u}_h \|_h \) achieves the superconvergence rate of \( \mathcal{O}(h^{k+2}) \) since \( \hat{u}_h \vert_{K} \in \mathcal{P}_k(F) \).

Theorem 2.2 can be easily proved by adopting a very similar analysis used in [8], combined with the HDG+ projection. We show how this is done in Section 2.3.

### 2.2 Proof of Theorem 2.1

In this subsection, we prove Theorem 2.1. We begin by presenting a lemma that gives a collection of lifting/inverse inequalities and convergence properties about \( L^2 \) projections. These inequalities will be used extensively in the paper.

**Lemma 2.1.** If \( u \in \mathcal{P}_k(K) \), then
\[ \| u \|_{\partial K} \leq C h_K^{-1/2} \| u \|_K, \quad \| \nabla u \|_K \leq C h_K^{-1} \| u \|_K. \]  \( \text{(10a)} \) \{eq:bs_inq_1\}
If \( u \in H^1(K) \), then
\[ \| \Pi_k u - u \|_K \leq C h^m_K | u |_{m,K}, \quad \| \Pi_k u - u \|_{\partial K} \leq C h^{m-1/2}_K | u |_{m,K}. \]  \( \text{(10b)} \) \{eq:bs_inq_2\}
Here, \( m \in [1, k + 1] \), and the constant \( C \) depends only on \( k \) and \( \gamma_K \).

**Proof.** See [11, Appendix]. \[ \square \]

We next prove that the projection \( \Pi_k^\# \) is well defined by (11b) and (11c), and it converges optimally.

**Proposition 2.1.** The projection \( \Pi_k^\# \) is well defined by (11b) and (11c), and we have
\[ h^{1/2}_K \| \Pi_k^\# q - q \|_{\partial K} + \| \Pi_k^\# q - q \|_K \leq C h^m_K | q |_{m,K}, \]  \( \text{(11)} \) \{eq:g+_conv\}
where \( m \in [1, k + 1] \). Here, the constant \( C \) depends only on \( k \) and \( \gamma_K \).

**Proof.** In this proof, we use the sign ‘\( \lesssim \)’ to hide a constant that depends only on \( k \) and \( \gamma_K \). First note that (11b) and (11c) define a square system. We next prove
We now decompose $\varepsilon_q := \Pi_q^0 q - \Pi_q q \in \mathcal{P}_k(K)^d$. By (11) and (4c), we have

\[
(\varepsilon_q, r)_K = 0 \quad \forall r \in \nabla \mathcal{P}_k(K) \oplus (\nabla \mathcal{P}_{k+1}(K))^\perp,
\]
\[
(\varepsilon_q, \nabla w)_K = \langle P_M(q \cdot n) - q \cdot n, w \rangle_{\partial K} \quad \forall w \in \mathcal{P}_k(K)^{k+1}.
\]

The above equations imply that

\[
(\varepsilon_q, \nabla w)_K = \langle P_M(q \cdot n) - q \cdot n, w \rangle_{\partial K} \quad \forall w \in \mathcal{P}_{k+1}(K).
\]

We now decompose $\varepsilon_q$ into the summation $\varepsilon_q = \varepsilon_q^1 + \varepsilon_q^2$, where $\varepsilon_q^1 \in \nabla \mathcal{P}_{k+1}(K)$ and $\varepsilon_q^2 \in (\nabla \mathcal{P}_{k+1}(K))^{k+1}$. By (12a) we have $\|\varepsilon_q\|_K^2 = (\varepsilon_q, \varepsilon_q)_K$. Since $\varepsilon_q^1 \in \nabla \mathcal{P}_{k+1}(K)$, we can write $\varepsilon_q^1 = \nabla (p + c)$ for some $p \in \mathcal{P}_{k+1}(K)$ and arbitrary constant $c$. This with (13) gives

\[
\|\varepsilon_q\|_K^2 = (\varepsilon_q, \nabla (p + c))_K = \langle P_M(q \cdot n) - q \cdot n, p + c \rangle_{\partial K}
\]
\[
\leq h_K^{-1/2} \left\| P_M(q \cdot n) - q \cdot n \right\|_{\partial K} \| p + c \|_K.
\]

We now choose the constant $c = -\Pi_p p$ and obtain

\[
\|\varepsilon_q\|_K^2 \leq h_K^{-1/2} \| P_M(q \cdot n) - q \cdot n \|_{\partial K} \| \nabla p \|_K \leq h_K^{-1/2} \| P_M(q \cdot n) - q \cdot n \|_{\partial K} \|\varepsilon_q\|_K.
\]

This completes the proof. \(\square\)

We are now ready to prove Theorem 2.1. By Proposition 2.1, we know $\Pi_K$ and $\delta_{\Pi_K}$ are well defined. We next prove that $\Pi_K$ satisfies equations (15). Equation (5a) holds obviously since $\Pi_K u = \Pi_{k+1} u$. Equation (5b) holds by the definition (4d).

To prove (5c), first note that

\[
(\nabla \cdot (\Pi^0 q - q), w)_K \pm \langle \tau P_M(\Pi_{k+1} u - u), w \rangle_{\partial K}
\]
\[
= \langle (\Pi^0 q - q) \cdot n + \tau P_M(\Pi_{k+1} u - u), w \rangle_{\partial K} - (\Pi^0 q - q, \nabla w)_K,
\]

for all $w \in \mathcal{P}_{k+1}(K)$. By (4b) and (4c), we have

\[
(\Pi^0 q - q, \nabla w)_K = \langle P_M(q \cdot n) - q \cdot n, w \rangle_{\partial K} \quad \forall w \in \mathcal{P}_{k+1}(K).
\]

Now (5a) follows by using (14) and (15).

We next prove (6). By (10d) and (11), we know that $\Pi_K q = \Pi^0 q$ and $\Pi_K u = \Pi_{k+1} u$ converge optimally. It only remains to estimate the boundary remainder. By the definition (4d) and the fact that $\|\tau\|_{L^\infty(\partial K)} \leq c_2 h_K^{-1}$, we have

\[
\|\delta_{\Pi_K}(q, u)\|_{\partial K} \leq \| P_M(\Pi^0 q \cdot n - q \cdot n) \|_{\partial K} + c_2 h_K^{-1} \| P_M(\Pi_{k+1} u - u) \|_{\partial K}.
\]

By (10b) and (11) again, we complete the proof.
2.3 Proof of Theorem 2.2

In this subsection, we give a step-by-step proof for Theorem 2.2. The proof will be very similar to those used in [8], thanks to the introduction of the HDG+ projection. In this way, we are able to reuse the existing projection-based error analysis for the analysis of the HDG+ method.

Step 1: Error equations. We first define the error terms:

\[ \varepsilon_h^q := \Pi q - q_h \in V_h, \quad \varepsilon_h^u := \Pi u - u_h \in W_h, \quad \tilde{\varepsilon}_h^u := P_M u - \hat{u}_h \in M_h. \]

Now, by testing (1) with \((r, w, \mu) \in V_h \times W_h \times M_h\) and then using (3), we obtain the projection equations:

\[ (\kappa^{-1} \Pi q, r)_{\mathcal{T}_h} - (\Pi u, \nabla \cdot r)_{\mathcal{T}_h} + \langle P_M u, r \cdot n \rangle_{\partial \mathcal{T}_h} = (\kappa^{-1} (\Pi q - q), r)_{\mathcal{T}_h}, \]  
\[ (\nabla \cdot \Pi q, w)_{\mathcal{T}_h} + \langle \tau P_M (\Pi u - P_M u), w \rangle_{\partial \mathcal{T}_h} = (f, w)_{\mathcal{T}_h} + \langle \delta_r(q, u), w \rangle_{\partial \mathcal{T}_h}, \]  
\[ -\langle \Pi q \cdot n + \tau (\Pi u - u), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = -\langle \delta_r(q, u), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma}, \]  
\[ \langle P_M u, \mu \rangle_{\Gamma} = \langle g, \mu \rangle_{\Gamma}, \]

for all \((r, w, \mu) \in V_h \times W_h \times M_h\). In the above equations, (16a), (16b), and (16c) are obtained by using (5a), (5b), and (5c), respectively. The equation (16d) holds obviously since \(P_M|_{\partial K}\) is the \(L^2\) projection to \(R_K(\partial K)\) for all \(K \in \mathcal{T}_h\).

By taking the difference between (10) and (3), we obtain the error equations:

\[ (\kappa^{-1} \varepsilon_h^q, r)_{\mathcal{T}_h} - (\varepsilon_h^u, \nabla \cdot r)_{\mathcal{T}_h} + \langle \tilde{\varepsilon}_h^u, r \cdot n \rangle_{\partial \mathcal{T}_h} = (\kappa^{-1} (\Pi q - q), r)_{\mathcal{T}_h}, \]  
\[ (\nabla \cdot \varepsilon_h^q, w)_{\mathcal{T}_h} + \langle \tau P_M (\varepsilon_h^u - \tilde{\varepsilon}_h^u), w \rangle_{\partial \mathcal{T}_h} = \langle \delta_r(q, u), w \rangle_{\partial \mathcal{T}_h}, \]  
\[ -\langle \varepsilon_h^q \cdot n + \tau (\varepsilon_h^u - \tilde{\varepsilon}_h^u), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = -\langle \delta_r(q, u), \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma}, \]  
\[ \langle \tilde{\varepsilon}_h^u, \mu \rangle_{\Gamma} = 0, \]

for all \((r, w, \mu) \in V_h \times W_h \times M_h\).

Step 2: Energy identity. By testing the error equations with \(r = \varepsilon_h^q, w = \varepsilon_h^u, \mu = \tilde{\varepsilon}_h^u\) in (17a)-(17c) and adding the equations, then using (17d), which suggests that \(\tilde{\varepsilon}_h^u|_{\Gamma} = 0\), we obtain the following energy identity:

\[ (\kappa^{-1} \varepsilon_h^q, \varepsilon_h^q)_{\mathcal{T}_h} + \langle \tau P_M (\varepsilon_h^u - \tilde{\varepsilon}_h^u), \varepsilon_h^u - \tilde{\varepsilon}_h^u \rangle_{\partial \mathcal{T}_h} = (\kappa^{-1} (\Pi q - q), \varepsilon_h^q)_{\mathcal{T}_h} + \langle \delta_r(q, u), \varepsilon_h^u - \tilde{\varepsilon}_h^u \rangle_{\partial \mathcal{T}_h}. \]  

By using the energy identity (18), we easily obtain

\[ \|\kappa^{-1/2} \varepsilon_h^q\|_{\mathcal{T}_h}^2 + \|\tau^{1/2} P_M (\varepsilon_h^u - \tilde{\varepsilon}_h^u)\|_{\partial \mathcal{T}_h}^2 \leq \|\kappa^{-1/2} (\Pi q - q)\|_{\mathcal{T}_h}^2 + \|\tau^{-1/2} \delta_r(q, u)\|_{\partial \mathcal{T}_h}^2. \]

This proves (17). We are next going to prove (8) and (9).

Step 3: Duality identity. We first introduce the duality equations of (1):

\[ \kappa^{-1} \psi - \nabla \phi = 0 \quad \text{in } \Omega, \]  
\[ -\nabla \cdot \psi = \theta \quad \text{in } \Omega, \]  
\[ \phi = 0 \quad \text{on } \Gamma, \]  

where \(\psi, \phi, \theta \in H^1(\Omega)\), \(\psi, \phi \in H^2(\Omega)\), and \(\phi \in C(\partial \Omega)\).
We next define the projections and the boundary remainder of the solutions of the duality equations (20):

\[
\Pi \psi := \prod_{K \in T_h} \Pi_K \psi, \quad \Pi \phi := \prod_{K \in T_h} \Pi_K \phi, \quad \delta_{-\tau}(\psi, \phi) := \prod_{K \in T_h} \delta_{-\tau}^\Pi (\psi, \phi).
\]

Note that we used \(-\tau\) to define the boundary remainder. By testing (20) with \((r, w, \mu) \in V_h \times W_h \times M_h\) and then using (15), we obtain the following equations in a similar way as (16):

\[
(\kappa^{-1} \Pi \phi, r)_{T_h} + (\Pi \phi, \nabla \cdot r)_{T_h} - \langle P_M \phi, r \cdot n \rangle_{\partial T_h} = (\kappa^{-1} (\Pi \psi - \psi), r)_{T_h}, \quad \text{(21a)} \tag{eq:dual pj e}
\]

\[
-\langle \nabla \cdot \Pi \phi, w \rangle_{T_h} + \langle \tau P_M (\Pi \phi - P_M \phi), w \rangle_{\partial T_h} = (\theta, w)_{T_h} - \langle \delta_{-\tau}(\psi, \phi), w \rangle_{\partial T_h}, \quad \text{(21b)} \tag{eq:dual pj e}
\]

\[
(\Pi \psi \cdot n - \tau (\Pi \phi - \phi), \mu)_{\partial T_h \setminus \Gamma} = (\delta_{-\tau}(\psi, \phi), \mu)_{\partial T_h \setminus \Gamma}, \quad \text{(21c)} \tag{eq:dual pj e}
\]

\[
\langle P_M \phi, \mu \rangle_\Gamma = 0, \quad \text{(21d)} \tag{eq:dual pj e}
\]

for all \((r, w, \mu) \in V_h \times W_h \times M_h\). Now we test (17a)-(17d) with \(r = \Pi \psi, w = \Pi \phi, \mu = P_M \phi\), test (21a)-(21d) with \(r = \psi_h, w = \psi_h, \mu = \hat{\psi}_h\), and use (17d) and (21d), which imply \(\hat{\psi}_h^u p | = P_M \phi p | = 0\). Comparing the two sets of equations, we obtain

\[
(\kappa^{-1} (\Pi \psi - q), \Pi \psi)_{T_h} + (\delta_{-\tau}(q, u), \Pi \phi - P_M \phi)_{\partial T_h}
= (\kappa^{-1} (\Pi \psi - \psi), \psi_h)_{T_h} + (\theta, \dot{\psi}_h)_{T_h} - \langle \delta_{-\tau}(\psi, \phi), \dot{\psi}_h - \hat{\psi}_h^u \rangle_{\partial T_h}.
\]

Assuming \(k \geq 1\) and rearranging the terms of the above identity, we have the following duality identity:

\[
(\theta, \dot{\psi}_h)_{T_h} = ((\Pi q - q), \nabla \phi - \Pi_0 \nabla \phi)_{T_h} + (\kappa^{-1} (\Pi \psi - \psi), \dot{q}_h - q)_{T_h}
+ (\delta_{-\tau}(q, u), P_M \Pi \phi - P_M \phi)_{\partial T_h} + (\delta_{-\tau}(\psi, \phi), P_M \dot{\psi}_h - \hat{\psi}_h^u \rangle_{\partial T_h}.
\]

Assuming \(u_h \in H^1(\Omega) \cap H_0^\text{div}(\Omega)\) and \(h \leq h\), we obtain

\[
(\theta, \dot{\psi}_h)_{T_h} \leq \langle (\Pi q - q), \nabla \phi - \Pi_0 \nabla \phi \rangle_{T_h} + (\kappa^{-1} (\Pi \psi - \psi), \dot{q}_h - q)_{T_h}
+ (\delta_{-\tau}(q, u), P_M \Pi \phi - P_M \phi)_{\partial T_h} + (\delta_{-\tau}(\psi, \phi), P_M \dot{\psi}_h - \hat{\psi}_h^u \rangle_{\partial T_h}.
\]

Taking \(\theta = \dot{\psi}_h\) in (22), we have

\[
\| \dot{\psi}_h \|_{T_h} \leq h \left( (\Pi q - q)_{T_h} + \| q_h - q \|_{T_h} + \| \bar{\delta}_{-\tau}(q, u) \|_{\partial T_h} + \| P_M \dot{\psi}_h - \hat{\psi}_h^u \|_{\partial T_h} \right).
\]

The above inequality with (19) implies (5).

It now only remains to estimate the term \(\| P u - \hat{u}_h \|_{h}\). First note that

\[
\| \tilde{\psi}_h^u \|^2 = \sum_{K \in T_h} h_K \| \tilde{\psi}_h^u \|^2_{K} \approx \sum_{K \in T_h} h_K \tau^{-1} \| \tilde{\psi}_h^u \|^2_{K} \leq h^2 \tau^{-1} \| \tilde{\psi}_h^u \|^2_{T_h}.
\]

By (19), we have

\[
\| \tau^{-1} \tilde{\psi}_h^u \|_{\partial T_h} \leq \| \tau^{-1} P_M \tilde{\psi}_h^u \|_{\partial T_h} + \| \Pi q - q \|_{T_h} + \| \bar{\delta}_{-\tau}(q, u) \|_{T_h}.
\]
By using (8), we can estimate the term \( \| \tau^{1/2} P_M e_h \|_{\partial T_h} \) as follows:

\[
\| \tau^{1/2} P_M e_h \|_{\partial T_h}^2 = \sum_{K \in T_h} \| \tau^{1/2} P_M (\Pi u - u_h) \|_{\partial K}^2 \lesssim h_K^{-2} \Pi u - u_h \|_{K}^2 \lesssim h^{-2} \min h^2 (\| \Pi q - q \|_{T_h} + \| \tau^{-1/2} \delta_\tau (q, u) \|_{\partial T_h})^2.
\]

Combining (23), (24), and (25), we obtain (9). This completes the proof.

3 The projection for elasticity

3.1 Main results

In [13], we devised the HDG+ projection for elasticity on simplicial elements. In this section, we extend the projection (see [13] Theorem 2.1)) to polyhedral elements. This new projection will render all the analysis and estimates in [13] Sections 5, 6, & 7 valid for general polyhedral meshes. (The three sections in [13] cover the error analysis of the HDG+ methods for steady-state elasticity, time-harmonic elastodynamics, and transient elastic waves, respectively.)

We define the HDG+ projection for elasticity as follows:

\[
\Pi_K (\sigma, u) := (\Pi_k^{sg} \sigma, \Pi_{k+1} u) \in \mathcal{P}_k(K; \mathbb{R}^{d \times d}_{sym}) \times \mathcal{P}_{k+1}(K; \mathbb{R}^d),
\]

where \( \Pi_k^{sg} : H^1(K; \mathbb{R}^{d \times d}_{sym}) \to \mathcal{P}_k(K; \mathbb{R}^{d \times d}_{sym}) \) is defined by solving

\[
(\Pi_k^{sg} \sigma - \sigma, \theta)_K = 0 \quad \forall \theta \in \varepsilon(\mathcal{P}_k(K; \mathbb{R}^d)) \oplus \varepsilon(\mathcal{P}_{k+1}(K; \mathbb{R}^d))^{-1},
\]

\[
(\Pi_k^{sg} \sigma - \sigma, \varepsilon(v))_K = \langle \mathcal{P}_M (\sigma u) - \sigma n, v \rangle_{\partial K} \quad \forall v \in \mathcal{P}_k(K; \mathbb{R}^d)^{-1}.
\]

In the above equations, \((\cdot)^\perp_m\) represents the orthogonal complement in the background space \( \mathcal{P}_m(K; \mathbb{R}^{d \times d}_{sym}) \) (for (26a)) or \( \mathcal{P}_m(K; \mathbb{R}^d) \) (for (26b)), the notation \( \varepsilon(v) := \nabla v + (\nabla v)^\perp \) represents the symmetric gradient, and \( \mathcal{P}_M : L^2(\partial K; \mathbb{R}^d) \to \mathcal{R}_k(\partial K; \mathbb{R}^d) \) is the \( L^2 \) projection to the range space.

We define the associated boundary remainder as follows:

\[
\delta^{\Pi_k}_{\pm \tau} (\sigma, u) := - (\Pi_K \sigma n - \mathcal{P}_M (\sigma n)) \pm \tau (\mathcal{P}_M \Pi_K u - \mathcal{P}_M u) \in \mathcal{R}_k(\partial K; \mathbb{R}^d),
\]

where \( \tau \in \mathcal{R}_0(\partial K; \mathbb{R}^{d \times d}_{sym}) \) satisfying [13] Eqn. (2.1), namely, \( c_1 h^{-1}_K \| \mu \|_{\partial K}^2 \leq \langle \tau \mu, \mu \rangle_{\partial K} \leq c_2 h^{-1}_K \| \mu \|_{\partial K}^2 \) for all \( \mu \in L^2(\partial K; \mathbb{R}^d) \) and two fixed positive constants \( c_1 \) and \( c_2 \).

The main result in this section is the following theorem. For notational convenience, we hide the dependence of \( \delta^{\Pi_k}_{\pm \tau} \) on \((\sigma, u)\).

**Theorem 3.1** (HDG+ projection for elasticity). The projection \( \Pi_K \) and the remainder \( \delta^{\Pi_k}_{\pm \tau} \) are well defined by (26) and they satisfy

\[
\langle (\Pi_K u - u, v) \rangle_K = 0 \quad \forall v \in \mathcal{P}_{k-1}(K; \mathbb{R}^d), \tag{27a}
\]

\[
\langle -(\Pi_K \sigma n - \sigma n) \pm \tau (\Pi_K u - u), \mu \rangle_{\partial K} = \langle \delta^{\Pi_k}_{\pm \tau}, \mu \rangle_{\partial K} \quad \forall \mu \in \mathcal{R}_k(\partial K; \mathbb{R}^d), \tag{27b}
\]

\[
- \langle \nabla \cdot (\Pi_K \sigma - \sigma), w \rangle_{K} \pm \langle \mathcal{P}_M (\Pi_K u - u), w \rangle_{\partial K} = \langle \delta^{\Pi_k}_{\pm \tau}, w \rangle_{\partial K} \quad \forall w \in \mathcal{P}_{k+1}(K; \mathbb{R}^d). \tag{27c}
\]
Furthermore,
\[ \| \Pi_K \sigma - \sigma \|_K + h_K^{-1} \| \Pi_K u - u \|_K + h_K^{1/2} \| \delta_{\Pi_K} \|_{\partial K} \leq C h_K^m (|\sigma|_{m,K} + |u|_{m+1,K}), \]  
where \( m \in [1, k+1] \). Here, the constant \( C \) depends only on \( k, \gamma_K \), and \( c_2 \).

Note that the two boundary remainders \( \delta_{\Pi_K} \) and \( \delta_{\Pi_K} \) correspond to the HDG+ projection and the adjoint projection in [13, Theorem 2.1], respectively. We also remark that we have used the HDG+ projection to define the initial velocity for the semi-discrete HDG+ scheme in [13]. Therefore, equations (26) provide a way of calculating the initial conditions for the semi-discrete scheme for elastic waves.

### 3.2 Proof of Theorem 3.1

In this subsection, we prove Theorem 3.1. The proof here will be similar to the proof of Theorem 2.1 in Section 2.2.

**Proposition 3.1.** The projection \( \Pi^s \) is well defined by (26b) and (26c), and we have
\[ h^{1/2} \| \Pi_k \sigma - \sigma \|_{\partial K} + \| \Pi^s_k \sigma - \sigma \|_K \leq C h_k^m (|\sigma|_{m,K}), \]  
where \( m \in [1, k+1] \). Here, the constant \( C \) depends only on \( k \) and the shape-regularity constant \( \gamma_K \).

**Proof.** We can easily verify that (26b) and (26c) define a square system. We next prove the convergence equation (29), from which the unique solvability of (26b) and (26c) follows automatically. Let \( \varepsilon \) := \( \Pi^s_k \sigma - \Pi_k \sigma \). By (26b) and (26c), we have
\[ (\varepsilon, \theta)_K = 0 \quad \forall \theta \in \varepsilon(P_k(K; \mathbb{R}^d)) \oplus \varepsilon(P_{k+1}(K; \mathbb{R}^d)^{1-k}, \]
\[(\varepsilon, \varepsilon(v))_K = \langle P_M(\sigma n) - \sigma n, v \rangle_{\partial K} \quad \forall v \in P_k(K; \mathbb{R}^d)^{1-k}. \]

The above equations imply that
\[ (\varepsilon, \varepsilon(v))_K = \langle P_M(\sigma n) - \sigma n, v \rangle_{\partial K} \quad \forall v \in P_{k+1}(K; \mathbb{R}^d). \]

We now decompose \( \varepsilon \) into the summation \( \varepsilon = \varepsilon^1 + \varepsilon^2 \), where \( \varepsilon^1 \in \varepsilon(P_{k+1}(K; \mathbb{R}^d)) \) and \( \varepsilon^2 \in \varepsilon(P_{k+1}(K; \mathbb{R}^d)^{1-k} \). Since \( \varepsilon^1 \in \varepsilon(P_{k+1}(K; \mathbb{R}^d)) \), we can write \( \varepsilon^1 = \varepsilon(p + m) \) for some \( p \in P_{k+1}(K; \mathbb{R}^d) \) and arbitrary rigid motion \( m \in M \). By (30a) and (31) we have
\[ \| \varepsilon \|_K^2 = (\varepsilon, \varepsilon^1)_K = (\varepsilon, \varepsilon(p + m))_K = \langle P_M(\sigma n) - \sigma n, p + m \rangle_{\partial K}. \]

We next apply [20, Lemma 4.1] to the term \( p + m \) and then obtain
\[ \| \varepsilon \|_K^2 \leq h^{1/2} \| P_M(\sigma n) - \sigma n \|_{\partial K} \| \varepsilon(p) \|_K \leq h^{1/2} \| P_M(\sigma n) - \sigma n \|_{\partial K} \| \varepsilon \|_K. \]

This completes the proof.
Let us now prove Theorem 3.1. By Proposition 3.1, we know $\Pi_K$ and $\delta^{\Pi_K}$ are well defined by (26). We next prove equations (27). Equations (27a) and (27b) hold obviously by the definitions (26a) and (26d).

To prove (27c), first note that

$$- (\nabla \cdot (\Pi^g_k \sigma - \sigma), w)_K \pm \langle \tau P_M(\Pi_{k+1} u - u), w \rangle_{\partial K}$$

$$= \langle - (\Pi^g_k \sigma n - \sigma n) \pm \tau P_M(\Pi_{k+1} u - u), w \rangle_{\partial K} + \langle \Pi^g_k \sigma - \sigma, \varepsilon(w) \rangle_K,$$

(32) \{eq:pf_17\}

for all $w \in P_{k+1}(K; \mathbb{R}^d)$. By (26b) and (26c), we obtain

$$\langle \Pi^g_k \sigma - \sigma, \varepsilon(v) \rangle_K = \langle P_M(\sigma n) - \sigma n, v \rangle_{\partial K} \quad \forall v \in P_{k+1}(K; \mathbb{R}^d).$$

(33) \{eq:pf_19\}

Equations (32) and (33) imply (27c).

The convergence property (28) holds because of equations (29) and (10b), and the fact that $\langle \tau \mu, \mu \rangle_{\partial K} \leq c_2 h_K^{-1} \| \mu \|_{\mathcal{E}^{k+1}}^2$ for all $\mu \in L^2(\partial K; \mathbb{R}^d)$. This completes the proof.

**Conclusions**

We have devised two new HDG+ projections on polyhedral elements, extending our previous results in [12] for elliptic problems and the results in [13] for elasticity to polyhedral meshes. The projections here are constructed in a different way without using the $M$-decomposition as a middle step. Consequently, the construction is more straightforward. Future work of interest involves incorporating the HDG methods with Hybrid-High Order stabilization functions [3] into the projection-based error analysis setting.

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