A NEW THETA CYCLE FOR $GSp_4$ AND AN EDIXHOVEN TYPE THEOREM

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Abstract. In this paper, we investigate a new theta cycle for $GSp_4/\mathbb{Q}$ by using author’s theta operators defined in the previous work. In the course of the construction, we also modify the theta operators to work on any characteristic including $p = 2$ and any weight. As an application, we discuss an Edixhoven type theorem for $GSp_4/\mathbb{Q}$.

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1. Introduction

Mod $p$ modular forms in a broad sense play an important role of the arithmetic study of Galois representations (cf. [16], [17], and [4] among others). After Serre-Katz-Jochnowitz’s works (cf. [10] and [11]), several people have studied a variant of the theta operator which is a mod $p$ analogue of Maass-Shimura differential operator (or the Ramanujan-Serre differential in the case of elliptic modular forms). The author refers [5] as a genuine reference around this topic including a comprehensive introduction.

In this paper we recast the theta operators for $GSp_4/Q$ constructed in [21] and construct a new “big” theta operator acting on any geometric Siegel modular forms over $\mathbb{F}_p$ of degree 2. A novelty is we allow any weight and any characteristic of the base field. Let us fix some notation to explain main results and refer the appropriate sections for details. Let $p$ be any prime including 2 and $N \geq 3$ be an integer with $p \nmid N$. For each pair $k = (k_1, k_2)$ of integers with $k_1 \geq k_2$ where we allow $k_2$ negative, we denote by $M_k(N, \mathbb{F}_p)$ be the space of the geometric (Siegel) modular forms over $\mathbb{F}_p$ of weight $k$ with respect to the principal congruence subgroup $K(N) \subset GSp_4(\mathbb{Z})$. Put

$$m_k := \begin{cases} (0, 0) & \text{if } p = 2, \\ (p - 1, p - 1) & \text{if } p > 2 \text{ and } k_1 - k_2 \leq 1, \\ (2p - 2, 2p - 2) & \text{if } p > 2 \text{ and } k_1 - k_2 > 1. \end{cases}$$

Our first main result is the following (see Theorem 2.11 and Theorem 2.13):

**Theorem 1.1.** There is an $\mathbb{F}_p$-linear operator $\Theta_k : M_k(N, \mathbb{F}_p) \rightarrow M_{k+(2,2)+m_k}(N, \mathbb{F}_p)$ satisfying the properties below:

1. if $f \in M_k(N, \mathbb{F}_p)$ is a Hecke eigenform outside $pN$ (outside $N$ if $k_2 \geq 2$), then so is $\Theta_k(f)$;
2. if $f \in M_k(N, \mathbb{F}_p)$ is a Hecke eigen cusp form outside $pN$ and $\Theta_k(f)$ is not identically zero, then $\overline{\Theta_k(f),p} \sim \overline{\chi_p} \otimes \overline{\theta_{f,p}}$ for the corresponding mod $p$ Galois representations (cf. [18], [19], [20]) of $GQ := \text{Gal}(\overline{Q}/Q)$. Here $\chi_p$ stands for the mod $p$ cyclotomic character of $GQ$.

A motivation to construct $\Theta_k$ is to study the filtration of $f \in M_k(N, \mathbb{F}_p)$ which is defined by

$$w(f) := \min\{k - i(p - 1, p - 1) \in \mathbb{Z}^2 \mid f \in (H_{p-1})^i \cdot M_{k-i(p-1,p-1)}(N, \mathbb{F}_p)\}$$

with respect to the lexicographic order on $\mathbb{Z}^2$ so that the second entries are firstly compared. Here $H_{p-1}$ is the Hasse invariant of degree 2 and it can be regarded as a geometric Siegel modular form over $\mathbb{F}_p$ of parallel weight $(p - 1, p - 1)$ with level one. The filtration of $f$ is well-defined by Theorem 2.11 (2). Since the Hasse invariant has a scalar weight, it seems natural to ask the construction of some differential operator increasing the weight by the scalar weights and also interacting with Hecke eigenvalues. In author’s previous work [21], he carried out it only when $k_1 = k_2$ under the assumption $p \geq 5$. In contrast our big theta $\Theta_k$ works for any weight $k = (k_1, k_2)$ with $k_1 \geq k_2$ and any prime $p$.

For each $f \in M_k(N, \mathbb{F}_p)$, the theta cycle of $f$ with respect to $\Theta = \Theta_k$ is defined by

$$\text{Cyc}(f) = \begin{cases} (w(\Theta(f)), w(\Theta^2(f)), \ldots, w(\Theta^{p-1}(f))) & \text{if } p > 2, \\ (w(\Theta(f))) & \text{if } p = 2. \end{cases}$$
We will give details of the cycle for any $f$ satisfying some condition to guarantee the non-vanishing of $\Theta^i(f)$ for any positive integer $i$. Combining with the automorphy lifting theorems due to Gee-Geraghty [9] which are extended by the author [22] we prove the following:

**Theorem 1.2.** Let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{GSp}_4(\overline{\mathbb{F}}_p)$ be a mod $p$ Galois representation satisfying

- $p \geq 3$;
- $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is irreducible and Im$(\bar{\rho})$ is adequate;
- if $\bar{\rho}|_{G_{\mathbb{Q}p}}$ is irreducible, up to the twists by finite characters of $G_{\mathbb{Q}p}$, it is of form $\text{Ind}_{\text{GSp}_4}^{G_{\mathbb{Q}p}} \omega_w^a$, $a = a_0 + a_1 p + a_2 p^2 + a_3 p^3$ with distinct integers $0 \leq a_0, a_1, a_2, a_3 \leq p - 1$;
- $\bar{\rho} \sim \bar{\rho}_{f,p}$ for some cuspidal Hecke eigen form $f$ in $M_2(N, \overline{\mathbb{F}}_p)$ with some weight $k' = (k'_1, k'_2)$, $k'_1 \geq 3$ and some $N \geq 3$ with $p \nmid N$.

Then there exist a cuspidal Hecke eigen form $g$ over $\overline{\mathbb{F}}_p$, unramified outside $N$ of the classical Serre weight $(k_1(\bar{\rho}), k_2(\bar{\rho}))$ and an integer $w(\bar{\rho})$ defined in Section 4 such that $\bar{\rho} \sim \chi^w(\bar{\rho}) \otimes \bar{\rho}_{g,p}$. Further, if $g$ is not weakly $p$-singular, then

$$\bar{\rho} \sim \begin{cases} \bar{\rho}_{\Theta^j(g),p} & \text{if } w(\bar{\rho}) \equiv 2j \mod p - 1 \text{ with } 0 \leq j < \frac{p-1}{2}, \\ \bar{\rho}_{3 \Theta^j(g),p} & \text{if } w(\bar{\rho}) \equiv 2j + 1 \mod p - 1 \text{ with } 0 \leq j < \frac{p-1}{2}, \end{cases}$$

where the filtration of $\Theta^j(g)$ appears in the theta cycle $\text{Cyc}(g)$ of $g$. Here $\Theta_3$ stands for the small theta operator defined in Section 2.4.1 (see Definition 3.1 for weakly $p$-singular forms).

The geometric modular form $g$ in the above claim, in fact, comes from the reduction of a geometric modular form of such a weight over a field of characteristic zero.

If $\bar{\rho}|_{G_{\mathbb{Q}p}}$ is irreducible, it is well-known that it has a potentially diagonalizable lift of Hodge-Tate weight $\{0, 1, 2, 3\}$. Therefore, if we do not require the lift to be crystalline, then we have a similar result as recorded in Theorem 6.1. A theorem of this kind is well-known after [1] with [8, 6].

The classical Serre weight for $\bar{\rho}$ is defined in Section 4 for $p > 2$ and in Section 4 for $p = 2$ according to the shape of $\bar{\rho}|_{G_{\mathbb{Q}p}}$. More precisely, assume $\bar{\rho}|_{G_{\mathbb{Q}p}}$ is $*$-ordinary where $* \in \{\text{Borel}, \text{Siegel}, \text{Klingen}\}$. Then the classical Serre weight will be closely related to the Hodge-Tate weights of a potentially diagonalizable crystalline lift of $\bar{\rho}|_{G_{\mathbb{Q}p}}$ which preserves $*$-ordinary. In [22], the classical Serre weights are defined but not explicit in some cases. Here we give a precise definition as in [17, 4].

This paper is organized as follows. In Section 2 after preliminaries for geometric objects, we construct our theta operators (big theta and small theta) which work for any characteristic and any weight. A detailed study of the local behavior of the theta operator is a hallmark of this paper. A new phenomena is observed when we discuss the entire extension of the theta operator. The theta cycle is defined by using this operator and basic properties are discussed in Section 3. In Section 4 and 5 we give the definition of the classical Serre weights for a given $\rho$. Finally, we give a proof of Theorem 1.2 in Section 6.

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2. Theta operators

In this section we give a modification of theta operators defined in Section 3 of [21] to work on any characteristic and any weight. We refereed Section 2 and 3 of [21] for the notation.
We denote by $GSp_4$ the symplectic group with respect to $J = \begin{pmatrix} 0_2 & s \\ -s & 0_2 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with the similitude. This is a smooth group scheme over $\mathbb{Z}$.

2.1. Geometric modular forms. Let $N \geq 3$ be an integer and $S_{K(N)}$ be the Siegel modular threefold over $\mathbb{Z}[1/N]$ with respect to the principal congruence subgroup $K(N) \subset GSp_4(\hat{\mathbb{Z}})$. Let $f : \mathcal{A} \to S_N$ be the universal abelian surface and we define the Hodge bundle $\mathcal{E} := f_*\Omega^1_{\mathcal{A}/S_N}$ which is a locally free sheaf on $S_N$ of rank 2. Put $\omega := \det(\mathcal{E})$. For each pair $\underline{k} = (k_1, k_2)$ of integers with $k_1 \geq k_2$ where we allow $k_2$ non-positive. Put $\mathcal{E}_{\underline{k}} := \text{Sym}^{k_1-k_2}\mathcal{E} \otimes_{\mathcal{O}_{S,N}} \omega^{k_2}$ (this is denoted by $\omega_{\underline{k}}$ in [21]) and for any $\mathbb{Z}[1/N]$-algebra $R$, we define $M_{\underline{k}}(N, R) := H^0(S_N \otimes_{\mathbb{Z}[1/N]} R, \mathcal{E}_{\underline{k}} \otimes_{\mathbb{Z}[1/N]} R)$. Each element of $M_{\underline{k}}(N, R)$ is said to be a geometric (Siegel) modular form over $R$ of weight $\underline{k}$ with respect to $K(N)$.

**Theorem 2.1.** Keep the notation being as above. It holds

(1) if $k_2 < 0$, then $M_{\underline{k}}(N, \mathbb{C}) = 0$, and

(2) for any $p$ and any negative integer $k_2$ with $|k_2|$ sufficiently large, $M_{\underline{k}}(N, \overline{\mathbb{F}_p}) = 0$.

**Proof.** The first claim follows from [17]. The second claim follows from the first claim, Serre duality, and Serre’s vanishing theorem with the argument in the proof of Corollary 4.3 in [12].

2.2. Gauss-Manin connection. Let $\mathbb{H}^1_{dR}(\mathcal{A}/S_N)$ be the algebraic de Rham cohomology sheaf on $S_N$. Let $\nabla : \mathbb{H}^1_{dR}(\mathcal{A}/S_N) \to \mathbb{H}^1_{dR}(\mathcal{A}/S_N) \otimes_{\mathcal{O}_{S_N}} \Omega^1_{S_N}$ be the Gauss-Manin connection. It yields the Kodaira-Spencer isomorphism

$$KS : \text{Sym}^2\mathcal{E} \cong \Omega^1_{S_N}, \quad \omega_1 \otimes \omega_2 \mapsto \langle \omega_1, \nabla \omega_2 \rangle_{dR}$$

where $\langle *, * \rangle_{dR}$ stands for the alternating pairing on $\mathbb{H}^1_{dR}(\mathcal{A}/S_N)$. We remark that the formation of $KS$ is compatible with the base change to any $\mathbb{Z}[1/N]$-algebra.

2.3. A non-canonical projection $p_1$. Fix a local basis $e_1, e_2$ of $\mathcal{E}$. Put $u_i = e_1^i e_2^{-i}$, $i = 0, 1, 2$ with the convention $e_0 = e_2^0 := 1$ which make up a local basis of $\text{Sym}^2\mathcal{E}$. To avoid confusion, we prepare additional symbols $v_i = e_1^i e_2^{2-i}$, $i = 0, 1, 2$ which play the same role. We introduce a non-canonical projection

$$(2.1) \quad p_1 : \text{Sym}^2\mathcal{E} \otimes_{\mathcal{O}_{S_N}} \text{Sym}^2\mathcal{E} \to \omega^2$$

as $\text{Aut}_{\mathcal{O}_{S_N}}(\mathcal{E})$-modules. This can be given explicitly as follows. For any local section $x = \sum_{0 \leq i, j \leq 2} a_{ij} u_i \otimes v_j$ of $\text{Sym}^2\mathcal{E} \otimes_{\mathcal{O}_{S_N}} \text{Sym}^2\mathcal{E}$ we define

$$(2.2) \quad p_1(x) = (2a_{20} - a_{11} + 2a_{02})(e_1 \wedge e_2)^2.$$ 

By direct computation, for any local section $\gamma$ of $\text{Aut}_{\mathcal{O}_{S_N}}(\mathcal{E})$ we see $p_1(\gamma x) = \det(\gamma)^2 p_1(x)$. We also remark that the formation of $p_1$ is compatible with the base change to any $\mathbb{Z}[1/N]$-algebra.
2.4. The theta operator for the projection $p_1$. Let $p$ be any prime including 2 and $N \geq 3$ be a positive integer with $p \nmid N$. Let $S_{N,p}$ be a connected component of the special fiber $S_{K(N) \otimes \mathbb{Z}[1/N]} \mathbb{F}_p$ at $p$. We work locally on the ordinary locus of $S_{N,p}$. By abusing notation we use the same symbols $A, E, \omega, E_k$ to denote their base change to $\mathbb{F}_p$ and two maps $\Psi_k \otimes$ with respect to this decomposition. We are now ready to define $F = E_{k \otimes \mathbb{F}_p}$, $i(2.3)$

The theta operator for the projection $k \nabla$ our “pre”-theta operator $\tilde{\Theta}_k$ and put $k \nabla$ Frobenius map $C$ and $k \nabla$ our “pre”-theta operator $\tilde{\Theta}_k$ and put $k \nabla$ Frobenius map $C$ and $k \nabla$ Frobenius map $C$. We first compute the local behavior of $\tilde{\Theta}_k$. Fix a local basis $e_1, e_2$ of $E$. Put $u_i = e_i^2, v_i = e_i^{2-i}, i = 0, 1, 2$ (resp. $v_i = e_i^{2-i}, i = 0, 1, 2$) with the convention $e_0^0 = e_2^0 := 1$ which make up a local basis of the first (resp. second) $\text{Sym}^2 E$ in the target of $\Psi_k^{(2)}$. We also consider $\delta_n := e_1^{k_1-k_2-n} e_2^n (e_1 \wedge e_2)^{k_2}$ for $0 \leq n \leq r := k_1 - k_2$. Then $\{\delta_n\}_{n=0}^r$ makes up a local basis of $E_k$. For $1 \leq i \leq j \leq 2$, let $\nabla_{ij} = \nabla(D_{ij})$ with $D_{ij} := \langle e_i, \nabla(e_j) \rangle_{\text{dr}}$. As in (3.21) of Section 3.3 of [21], we consider

$$ F^*((\nabla_{11}(e_1))^{(p)}, (\nabla_{22}(e_2))^{(p)}) = (e_1, e_2, \nabla_{11}(e_1), \nabla_{22}(e_2)) \begin{pmatrix} B \\ A \end{pmatrix}, $$

(2.4)

$$ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(O_{S,N,p}) $$

and put

$$ C = BA^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. $$

(2.5)

Notice that $A$ is nothing but the Hasse matrix of the universal abelian surface and it is nowhere vanishing on the ordinary locus of $S_{N,p}$. We first compute the local behavior of $\Psi_k^{(1)}$.

**Proposition 2.2.** Recall $r = k_1 - k_2$. Let $F = \sum_{n=0}^r F_n \delta_n$ be a local section of $E_k$. It holds that

$$ \psi_k^{(1)}(F) = \left( \sum_{n=0}^r b_{2n}^{(n)} \delta_n \otimes u_2 \right) + \left( \sum_{n=0}^r b_{1n}^{(n)} \delta_n \otimes u_1 \right) + \left( \sum_{n=0}^r b_{0n}^{(n)} \delta_n \otimes u_0 \right) $$

$$ \quad F = \sum_{n=0}^r F_n \delta_n $$

where $b_{2n}^{(n)}, b_{1n}^{(n)}, b_{0n}^{(n)}$ are the entries of the Hasse matrix $C$.
modulo $\nabla L \otimes \mathcal{O}_{S_N^p} R_{(2,0)}(U)$ where

\begin{align*}
b^{(n)}_2 &= \nabla_{11}(F_n) - (k_1 - n)F_n c_{11} - (r + 1 - n)F_{n-1}c_{12} \\
b^{(n)}_1 &= \nabla_{12}(F_n) - (n + 1)F_{n+1}c_{11} - F_n \{(k_1 - n)c_{21} + (k_2 + n)c_{12}\} - (r + 1 - n)F_{n-1}c_{22} \\
b^{(n)}_0 &= \nabla_{22}(F_n) - (k_2 + n)F_n c_{22} - (n + 1)F_{n+1}c_{21}.
\end{align*}

Proof. By definition, we have

\begin{align*}
\Psi_k^{(1)}(F) &= \left(\sum_{n=0}^{r} (\nabla_{11}(F_n)\delta_n + F_n \nabla_{11}(\delta_n)) \otimes u_2\right) + \left(\sum_{n=0}^{r} (\nabla_{12}(F_n)\delta_n + F_n \nabla_{12}(\delta_n)) \otimes u_1\right) \\
&\quad + \left(\sum_{n=0}^{r} (\nabla_{22}(F_n)\delta_n + F_n \nabla_{22}(\delta_n)) \otimes u_0\right).
\end{align*}

The claim follows from Proposition 3.4-(1) of [21].

Before going further, we need the following lemmas for the local sections appearing in [2.4] and [2.5].

Lemma 2.3. For the local sections appearing in [2.4] and [2.5], it holds that

\begin{align*}
\nabla_{11}(\det(A)) &= -c_{11} \cdot \det(A), \quad \nabla_{12}(\det(A)) = -(c_{12} + c_{21}) \cdot \det(A), \quad \nabla_{22}(\det(A)) = -c_{22} \cdot \det(A)
\end{align*}

Proof. The claim follows from Proposition 3.4-(1) of [21].

Lemma 2.4. Suppose that all $\nabla_{ij}$'s $(1 \leq i \leq j \leq 2)$ commute with each other. Then it holds that

\begin{align*}
\nabla_{11}(c_{11}) &= c_{11}^2 \\
\nabla_{12}(c_{11}) &= (c_{12} + c_{21})c_{11} \\
\nabla_{22}(c_{11}) &= c_{12}c_{21} \\
\nabla_{11}(c_{12}) &= c_{11}c_{12} \\
\nabla_{12}(c_{12}) &= c_{11}c_{22} + c_{12}^2 \\
\nabla_{22}(c_{12}) &= c_{22}c_{12} \\
\nabla_{11}(c_{21}) &= c_{11}c_{21} \\
\nabla_{12}(c_{21}) &= c_{11}c_{22} + c_{21}^2 \\
\nabla_{22}(c_{21}) &= c_{22}c_{21} \\
\nabla_{11}(c_{22}) &= c_{12}c_{22} \\
\nabla_{12}(c_{22}) &= (c_{12} + c_{21})c_{22} \\
\nabla_{22}(c_{22}) &= c_{22}^2.
\end{align*}

Proof. For each $1 \leq i \leq 2$, let $\eta_i = c_{ij} \cdot \det(A)$. For each $1 \leq k \leq l \leq 2$, we have

\[\nabla_{kl}(c_{ij}) = \frac{\nabla_{kl}(\eta_i)}{\det(A)} - \frac{\nabla_{kl}(\det(A))}{\det(A)} c_{ij}.\]

The first term will be computed by Proposition 3.4-(1),(2) of [21] with the assumption on $\nabla_{kl}$'s and the second term is done by Lemma 2.3.

The commutativity of the differential operators yields an important property as below.

Lemma 2.5. Keep the notation in [2.4] and [2.5]. Suppose that all $\nabla_{ij}$'s $(1 \leq i \leq j \leq 2)$ commute with each other. Then it holds $c_{12} = c_{21}$.

Proof. For $i = 0$, let $\eta_i$ be the local section of $R^1 f_* \mathcal{O}_A$ corresponding to $e_i$ under the Serre duality. By using Lemma 3.2 of [21] and the Leibniz rule, one can check

\[\nabla_{ij}(\nabla_{11}(e_1), \nabla_{22}(e_2))_{\text{dR}} = 0, \quad 1 \leq i \leq j \leq 2.\]

It follows from this that $\langle \nabla_{11}e_1, \nabla_{22}e_2 \rangle_{\text{dR}} = 0$. By (3.28) of [21], we have

\[\nabla_{11}e_1 = \eta_1 - (c_{11}e_1 + c_{12}e_2), \quad \nabla_{22}e_2 = \eta_2 - (c_{21}e_1 + c_{22}e_2)\]
where \( \eta_i \) (\( i = 0, 1 \)) is a lift of \( \tilde{\eta}_i \) to \( V \). It yields
\[
0 = (\nabla_{11}(e_1), \nabla_{22}(e_2))_{dR} = c_{12} - c_{21}
\]
which gives us the claim. \( \square \)

Let \( F = \sum_{n=0}^{r} F_n \delta_n \) be a local section of \( E_k \). The computation of \( \tilde{\Theta}_k(F) = (id \otimes p_1) \circ \Psi_k^{(2)}(F) \) goes as follows:

1. first we compute \( \nabla_{kl}(b_i^{(n)}) \) and \( \nabla_{kl}(\delta_n) \) for \( 1 \leq k \leq l \leq 2, \ i = 0, 1, 2, \) and \( 0 \leq n \leq r = k_1 - k_2; \)
2. next we collect the coefficients of \( u_2 \otimes v_0, \ u_1 \otimes v_1, u_0 \otimes v_2 \) to compute \( id \otimes p_1. \)

The resulting terms involve some of \( \nabla_{kl}(c_{ij}), c_{ij} \) and we use Lemma 2.4 and Lemma 2.5 to simplify the equation. Summing up, we have the following explicit form of the local behavior of \( \tilde{\Theta}_k. \)

**Proposition 2.6.** Recall \( r = k_1 - k_2. \) Suppose that all \( \nabla_{ij}'s \) (\( 1 \leq i \leq j \leq 2 \)) commute with each other. Let \( F = \sum_{n=0}^{r} F_n \delta_n \) be a local section of \( E_k \) on each open subscheme of the ordinary locus of \( S_{N,p}. \) Let \( \tilde{\Theta}_k(F) = \sum_{n=0}^{r} A_n \delta_n(\varepsilon_1 \wedge \varepsilon_2)^2. \) Then it holds
\[
A_n = \det \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{12} & 2\nabla_{22} \end{pmatrix} (F_n) + 2 \{ (k_1(-1 + 2k_2) + (k_1 - k_2 - n)n) \det(C)F_n \\
-2(-1 + 2k_2 + 2n)c_{22}\nabla_{11}(F_n) + 2(-1 + k_1 + 2k_2)c_{12}\nabla_{12}(F_n) + 2(1 - 2k_1 + 2n)c_{11}\nabla_{22}(F_n) \\
-\{ (k_1 - k_2 - 1)(k_1 - k_2) - 6(k_1 - k_2 - n)n \} c_{12}^2 F_n \\
+2(1 + k_1 - k_2 - n)c_{22}\nabla_{12}(F_{n-1}) - 4(1 + k_1 - k_2 - n)c_{12}\nabla_{22}(F_{n-1}) \\
- 4(n + 1)c_{12}\nabla_{11}(F_{n+1}) + 2(n + 1)c_{11}\nabla_{12}(F_{n+1}) \\
+2(1 + k_1 - k_2 - n)(-1 + k_2 - k_1 + 2n)c_{12}\nabla_{22}(F_{n-1}) \\
-2(1 + n)(-k_1 + k_2 + 1 + 2n)c_{12}F_{n+1} \\
-(1 + k_1 - k_2 - n)(2 + k_1 - k_2 - n)c_{22}^2 F_{n-2} - (1 + n)(2 + n)c_{11}^2 F_{n+2} \}
\]
where the terms involving \( \nabla_{kl}(F_{n+i}) \) or \( F_{n+i} \) with \( i \in \{0, \pm 1, \pm 2\} \) and \( 1 \leq k \leq l \leq 2 \) are ignored if \( n + i \) is out of the range for the index.

**Remark 2.7.** The symbolic computation in Proposition 2.6 is done by using Wolfram Mathematica 12.1.

We write down the formula in Proposition 2.6 in the case when \( r = 0 \) or \( r = 1 \) respectively. It turns out later that these cases have a special feature among others.

**Corollary 2.8.** Keep the notation and the assumption being as above. Suppose \( k := k_1 = k_2 \) so that \( r = 0. \) Let \( F = F_0 \delta_0 \) be a local section of \( E_{(k,k)} = \omega^{\otimes k} \) and put \( \tilde{\Theta}_k(F) = A_0(\delta_0 \otimes (\varepsilon_1 \wedge \varepsilon_2)^2). \) Then it holds that
\[
A_0 = \det \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{12} & 2\nabla_{22} \end{pmatrix} (F_0) + 2k(2k - 1) \det(C)F_0 \\
-2(2k - 1)\{ c_{22}\nabla_{11}(F_0) - c_{12}\nabla_{12}(F_0) + c_{11}\nabla_{22}(F_0) \}. 
\]
Corollary 2.9. Keep the notation and the assumption being as above. Suppose \((k_1, k_2) = (k + 1, k)\) so that \(r = 1\). Let \(F = \delta_0 + F_1\delta_1\) be a local section of \(E_{k+1, k}\) and put \(\tilde{\Theta}_F(G) = A_0(\delta_0 \otimes (e_1 \wedge e_2)^2) + A_1(\delta_1 \otimes (e_1 \wedge e_2)^2)\). Then it holds that

\[
A_0 = \det \begin{pmatrix} 2\nabla_{11} & \nabla_{12} \\ \nabla_{12} & 2\nabla_{22} \end{pmatrix}(F_0) + 2(k + 1)(2k - 1)\det(C)F_0
-2(2k - 1)c_{22}\nabla_{11}(F_0) + 4kc_{12}\nabla_{12}(F_0) - 2(2k + 1)c_{11}\nabla_{22}(F_0) - 4c_{12}\nabla_{11}(F_1) + 2c_{11}\nabla_{12}(F_1)
\]

and

\[
A_1 = \det \begin{pmatrix} 2\nabla_{11} & \nabla_{12} \\ \nabla_{12} & 2\nabla_{22} \end{pmatrix}(F_1) + 2(k + 1)(2k - 1)\det(C)F_1
-2(1 + 2k)c_{22}\nabla_{11}(F_1) + 2(1 + 4k)c_{12}\nabla_{12}(F_1) + 2(1 - 2k)c_{11}\nabla_{22}(F_1)
\]

\[+ 2c_{22}\nabla_{12}(F_0) - 4c_{12}\nabla_{22}(F_0).\]

Let \(H_{p-1} = \det(A)\) be the Hasse invariant which can be regarded as a non-zero element of \(\text{H}^0(S_{N,p}, \omega^{\otimes(p-1)})\).

Definition 2.10. For each pair \(k = (k_1, k_2)\) with \(k_1 \geq k_2\), define the theta operator for the weight \(k\) by

\[
\Theta_k := \begin{cases} \\
\tilde{\Theta}_k & \text{if } p = 2 \\
H_{p-1} \cdot \tilde{\Theta}_k & \text{if } p > 2 \text{ and } k_1 - k_2 \leq 1 \\
H_{p-1}^2 \cdot \tilde{\Theta}_k & \text{if } p > 2 \text{ and } k_1 - k_2 > 1.
\end{cases}
\]

Put

\[
m_k := \begin{cases} \\
(0, 0) & \text{if } p = 2 \\
(p - 1, p - 1) & \text{if } p > 2 \text{ and } k_1 - k_2 \leq 1 \\
(2p - 2, 2p - 2) & \text{if } p > 2 \text{ and } k_1 - k_2 > 1
\end{cases}
\]

and \(M_k(N) = M_k(N, \overline{\mathbb{F}_p})\) for simplicity.

Theorem 2.11. Keep the notation being as above. The theta operator \(\Theta_k\) is holomorphically extended to the whole space of \(S_{N,p}\). Further, it holds that for each \(F \in M_k(N)\), \(\Theta_k(F)\) is an element of \(M_{k+(2,2)+m_k}(N)\). Further it satisfies the following properties:

(1) if \(F\) is a cusp form, then so is \(\Theta_k(F)\);

(2) if \(F\) is a Hecke eigenform outside \(pN\) (outside \(N\) if \(k_2 \geq 2\)), then so is \(\Theta_k(F)\). In this case, if \(F\) is a cusp form and \(\Theta_k(F)\) is non-zero, then

\[
\overline{p}_\Theta(F,p) \sim \overline{\chi}^0_p \otimes \overline{\rho}_{F,p}
\]

for the corresponding mod \(p\) Galois representations of \(G_{\overline{\mathbb{Q}}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). Here \(\overline{\chi}_p\) stands for the mod \(p\) cyclotomic character.

Proof. The coefficient \(A_n\) in Proposition 2.6 may have poles along zeroes of \(H_{p-1} = \det(A)\) except for \(p = 2\). The possible poles come from \(c_{ij}\), \(1 \leq i \leq j \leq 2\) and \(\det(A)c_{ij}\) is holomorphic by Lemma 2.3. Since \(A_n\) contains at most quadratic monomials in \(c_{ij}\)’s. The holomorphically extension follows from this when \(r = k_1 - k_2 > 1\). In the case when \(r = 0\) or \(r = 1\), the similar claim follows from Corollary 2.8 and Corollary 2.9 since \(A_n\) contains only linear terms in \(c_{ij}\)’s.

When \(p = 2\), we see easily by Proposition 2.6 that \(\Theta_k = \tilde{\Theta}_k\) is already holomorphically extended to the whole space.
The claim (1) follows from \( q \)-expansion principle and Proposition 3.10 of [21] yields the claim (2).

\[\Box\]

Remark 2.12. When \( p > 3 \), it is known, by Theorem 3.4.1 of [5], that \( H_{p-1}^2 \Theta_k \) is extended to the whole space \( S_{N,p} \) without any explicit computation. Their method is conceptual and works for many interesting cases. However, it seems difficult to check the extension of our theta operator \( \Theta_k \) when \( k_1 - k_2 \in \{0,1\} \) or \( p \) is any small prime.

2.4.1. The (small) theta operator \( \theta_3 = \Theta_3^k \): revisited. Recall the contents in Section 3.3.1 and Section 3.3.2 of [21]. The author constructed three (small) theta operators. Among all, \( \theta_3 = \Theta_3^k \) can be constructed for any \((k_1,k_2)\) with \( k_1 \geq k_2 \) and any prime \( p \). When \( k_1 - k_2 \geq 2 \), we may apply the explicit projection (7.1) in Appendix of [21] while we use \( \tilde{\theta} \) in Section 3.3.1 of loc.cit. when \( k_1 = k_2 \) and (3.32), (3.33) in Section 3.3.2 of loc.cit. when \( k_1 - k_2 = 1 \). Notice that the construction works for any \( p \). Therefore, we have obtained the following:

Theorem 2.13. There is an \( \mathbb{F}_p \)-linear map \( \theta_3^k : M_{k}(N) \rightarrow M_{k+(p+1,p-1)}(N) \) satisfying the properties below:

1. if \( f \in M_k(N) \) is a Hecke eigenform outside \( pN \) (outside \( N \) if \( k_2 \geq 2 \)), then so is \( \theta_3^k(f) \);
2. if \( f \in M_k(N) \) is a Hecke eigen cusp form outside \( pN \) and \( \theta_3^k(f) \) is not identically zero, then \( \theta_3^k(f) \sim \overline{p}_{f,(f),p} \) for the corresponding \( \mathbb{Z}_p \)-Galois representations of \( G_{\mathbb{Q}} \).

Proof. The argument in the proof of Proposition 3.13 of [21] works also for this setting.

Recall the definition of [11].

Theorem 2.14. Put \( r = k_1 - k_2 \). Let \( f \in M_k(N) \) be a non-zero element satisfying \( k = w(f) \). Then \( \theta_3^k(f) \) is not identically zero.

Proof. We assume \( r = k_1 - k_2 \geq 2 \). By definition, the coefficient of \( \theta_3^k(f) \) in the basis \( f_{r+2}^{(0)} \) in the notation of (7.1) in Appendix of [21] is nothing but

\[ b_2^{(r)} = \nabla_{11}(F_r) - k_2 F_r c_{11} - F_{r-1} c_{12} \]

in the notation of Proposition 2.2. Applying the action of \( \text{GL}_2(\mathbb{F}_p) \) on \( E_k \) if necessary, we may assume \( F_{r-1} \) is not divisible by \( H_{p-1} \) by the assumption \( k = w(f) \). As in the proof of Theorem 4.7 of [21], by using the local deformation at some point, we see that \( b_2^{(r)} \) is non-zero.

When \( k_1 - k_2 \in \{0,1\} \), we apply a similar argument to \( \theta \) (see Proposition 3.5 of [21]) and (3.33) of loc.cit..

\[\Box\]

3. The theta cycle

In this section we study the theta cycle defined by \( \Theta_k \).

3.1. Non-vanishing results. Let \( r = k_1 - k_2 \). For each \( F = \sum_{n=0}^r F_n \delta_n \in M_k(N) \) we denote by

\[ F(q_N) := \sum_{n=0}^r \sum_{T \in \text{Sym}^2(\mathbb{Z})} A_{F_n}(T)q_N^T \delta_n = \sum_{n=0}^r \sum_{T \in \text{Sym}^2(\mathbb{Z})_{\geq 0}} A_{F_n}(T)q_N^T \delta_n = \sum_{T \in \text{Sym}^2(\mathbb{Z})_{\geq 0}} \left( \sum_{n=0}^r A_{F_n}(T) \delta_n \right) q_N^T \]


the \( q \)-expansion of \( F \) at the Mumford’s semi-abelian scheme over \( \mathbb{F}_p[[q^{1/N}, q^{1/N}, q^{1/N}]] \) (cf. Section 2.5 of [21]). Here \( \text{Sym}^2(\mathbb{Z}) = \left\{ \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \) and \( \text{Sym}^2(\mathbb{Z})_{\geq 0} = \{ T \in \text{Sym}^2(\mathbb{Z}) \mid T \geq 0 \} \).

For each \( T = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \), we write \( q^T_N = q_{11}^{a/N} q_{12}^{b/N} q_{22}^{c/N} \). Since \( H^M_{p-1} F \) is liftable to a (holomorphic) Siegel modular form of characteristic zero for a sufficiently large \( M \) and \( H_{p-1}(q_N) = 1 \), the coefficient \( A_F(T) := \sum_{n=0}^{r} A_{F_n}(T) \delta_n \) vanishes unless \( T \in \text{Sym}^2(\mathbb{Z})_{\geq 0} \).

**Definition 3.1.** An element \( F \in M_k(N) \) is said to be weakly \( p \)-singular if \( A_F(T) = 0 \) for any \( T \in \text{Sym}^2(\mathbb{Z})_{\geq 0} \) with \( p | \det(T) \).

This notation is slightly different from \( p \)-singular forms in [21]. The author expects that Hecke eigen weakly \( p \)-singular forms can be characterized by means of the image of the corresponding mod \( p \) Galois representations.

Let \( S_{(0,0)} \) be the superspecial locus of \( S_{N,p} \).

**Theorem 3.2.** Let \( F = \sum_{i=0}^{r} F_n \delta_n \in M_k(N) \). It holds:

1. if \( F \) is not weakly \( p \)-singular, then \( \Theta_k(F) \) is neither zero nor weakly \( p \)-singular;
2. if \( p > 2 \), \( (k_1, k_2) = (k, k) \), and \( F|_{S_{(0,0)}} \neq 0 \), then \( \Theta_k(F)|_{S_{(0,0)}} \) is non-zero unless \( p \mid k(2k-1) \) and ;
3. if \( p > 2 \), \( (k_1, k_2) = (k + 1, k) \), and \( F|_{S_{(0,0)}} \neq 0 \), then \( \Theta_k(F)|_{S_{(0,0)}} \) is non-zero unless \( p \mid (k + 1)(2k-1) \);
4. if \( p > 2 \) and \( k_1 - k_2 > 1 \) and \( F \) is not a multiple of \( H_{p-1} \), then \( \Theta_k(F) \) is neither zero nor a multiple of \( H_{p-1} \).

**Proof.** Let \( F(q_N) = \sum_{T \in \text{Sym}^2(\mathbb{Z})_{\geq 0}} A_F(T) q_N^T \) be the \( q \)-expansion of \( F \). Since \( H_{p-1}(q_N) = 1 \), \( c_{ij}(q_N) = 0 \) by Lemma 2.3. By Proposition 2.6 we see that

\[
\Theta_k(F)(q_N) = \sum_{n=0}^{r} \det\left( \begin{array}{cc} 2q_{11}^{d/dq_{11}} & q_{12}^{d/dq_{12}} \\ q_{12}^{d/dq_{12}} & 2q_{22}^{d/dq_{22}} \end{array} \right) (F_n) \delta_n = \frac{1}{N^2} \sum_{T \in \text{Sym}^2(\mathbb{Z})_{\geq 0}} \det(T) A_F(T) q_N^T.
\]

The first claim follows from this formula.

The second and the third claims follow from the argument in the proof of Theorem 4.7-(2) in [21] and Corollary 2.3.2.9.

For the last claim, by assumption, we may assume \( \alpha := F_2|_X \in \mathbb{F}_p \) is non-zero for some \( X \in S_{(0,0)} \) by using the action of \( \text{GL}_2(\mathbb{F}_p) \) on \( \mathcal{E}_k \) if necessary. Let \( t_{11}, t_{12}, t_{22} \) be local parameters of \( S_{N,p} \) at \( X \). Let \( R = \mathbb{F}_p[[t_{11}, t_{12}, t_{22}]] \) with the maximal ideal \( m_R \) and \( I_X := (t_{11}t_{22} - t_{12}^2)R \). Let \( A_n(X) \) be the local expansion at \( X \) where \( A_n \) is the coefficient in Proposition 2.4. By the argument in the proof of Theorem 4.7-(2) in [21] again,

\[
H^2_{p-1}(X) A_0(X) \equiv \beta_1 t_{12}^2 + \beta_2 t_{12} t_{22} + \beta_3 t_{12} t_{11} + \beta_4 t_{12} - 2at_{11}^2 \mod m_R^3
\]

for some \( \beta_i \in \mathbb{F}_p \) (\( 1 \leq i \leq 4 \)). The right hand side is clearly non-zero and it yields the claim. \( \Box \)
3.2. Filtration. We follow the contents in Section 6 of [21]. For each \( f \in M_k(N) \) one can write \( f = H_p^m \cdot g \) for some \( g \in M_k(m(p-1,p-1)(N) \) and \( m \in \mathbb{Z}_{\geq 0} \). We define the filtration \( w(f) \) of \( f \) to be \( k - m(p-1,p-1) \) for the biggest \( m \) satisfying the above condition. If \( m_0 := \max \{ m \in \mathbb{Z}_{\geq 0} \mid g \in M_k(m(p-1,p-1)(N) \) s.t. \( g(qN) = f(qN) \} \), then we have

\[
w(f) = k - m_0(p - 1, p - 1).
\]

For a multiple \( M \) of \( N \), let \( T_M \) be the (abstract) Hecke ring over \( \mathbb{Z} \) outside \( M \) acting on \( M_k(N) \). Put

\[
T = \begin{cases} 
T_N & \text{if } k_2 \geq 2 \\
T_{Np} & \text{if } k_2 < 2.
\end{cases}
\]

Then one can define the usual Hecke action of \( T \) on \( M_k(N) \). The assumption \( k_2 \geq 2 \) is necessary to guarantee that the formal Hecke action is defined over \( \mathbb{Z} \) (in general the factor \( p^{k_2-2} \) appears in the formula, cf. (2.10) of [21]).

We set the following convention for Hecke eigen forms \( f_1 \in M_k(N) \) and \( f_2 \in M_{k'}(N) \) where two weights \( k \) and \( k' \) are allowed to be different:

\[
w(f_1) = w(f_2) \text{ if } p > 2 \text{ and the Hecke eigensystems of } f_1 \text{ and } f_2 \text{ for } T \text{ are the same as each other.}
\]

Recall

\[
m_k := \begin{cases} 
(0, 0) & \text{if } p = 2 \\
(p - 1, p - 1) & \text{if } p > 2 \text{ and } k_1 - k_2 \leq 1 \\
(2p - 2, 2p - 2) & \text{if } p > 2 \text{ and } k_1 - k_2 > 1.
\end{cases}
\]

We now study the filtration under \( \Theta_k \). Henceforth we sometimes write \( \Theta = \Theta_k \) for simplicity and accordingly, \( \Theta^2 \) means \( \Theta_{k+(2,2)+m_k} \circ \Theta_k \). Similarly, for each integer \( j \geq 1 \), \( \Theta^j \) is also inductively defined in that sense. For pairs \( (a, b), (c, d) \in \mathbb{Z}^2 \) with \( a - b = c - d \). We write \( (a, b) \leq (c, d) \) if \( b \leq d \). The equality holds exactly when \( (a, b) = (c, d) \).

Theorem 3.3. Suppose that \( f \in M_k(N) \) is not weakly \( p \)-singular and further it is not identically zero on \( S_{(0,0)} \) if \( p > 2 \). Then

1. \( w(\Theta_k(f)) \leq w(f) + (2, 2) + m_k \) for any \( p \);

2. suppose \( p > 2 \) and then it holds
   - if \( (k_1, k_2) = (k, k) \), the equality of (1) and \( \Theta_k(f)|_{S_{(0,0)}} \neq 0 \) hold unless \( p | (k(2k - 1)) \);
   - if \( (k_1, k_2) = (k+1, k) \), the equality of (1) and \( \Theta_k(f)|_{S_{(0,0)}} \neq 0 \) hold unless \( p | (k+1)(2k - 1)) \);
   - if \( k_1 - k_2 > 1 \), the equality of (1) always holds;

3. \( w(\Theta^{\frac{m_k}{2}}(f)) = w(\Theta(f)) \) if \( p > 2 \) and \( w(\Theta(f)) = w(f) + (2, 2) \) if \( p = 2 \).

Proof. The inequality follows by definition. As explained in the proof of Theorem 5.2 it follows from the proof of Theorem 4.7-(2) in [21] that \( \Theta_k(f) \) is non-zero at some point in \( S_{(0,0)} \) under the assumption on \( k \). Further, under the assumption the equality holds in the case when \( k_1 - k_2 \in \{0, 1\} \) since the Hasse invariant is identically zero on \( S_{(0,0)} \). When \( k_1 - k_2 > 1 \), the local expansion (3.1) shows that \( \Theta(f) \) is not a multiple of the Hasse invariant (otherwise, the right hand side of (3.1) has to be a multiple of \( t_{11}t_{22} - t_{12}^2 \) in the notation there).

The second claim is a consequence of Theorem 2.11-(2) with the convention (3.2) when \( p > 2 \). When \( p = 2 \), by Proposition 2.6 and Lemma 2.8 it is easy to see \( \Theta(H_{p-1}F) = H_{p-1}(\Theta(F)) \). The claim follows from this.

\( \square \)
3.3. A new theta cycle. Keep the notation in the previous subsection. For each \( f \in M_k(N) \) satisfying the assumption in Theorem \ref{thm_3.3} the theta cycle of \( f \) with respect to \( \Theta_k \) is defined by

\[
\text{Cyc}(f) := \begin{cases} 
(w(\Theta(f)), w(\Theta^2(f)), \ldots, w(\Theta^{k-1}(f)) & \text{if } p > 2 \\
(w(\Theta(f))) & \text{if } p = 2
\end{cases}
\]

For the second projection \( p_2 : \mathbb{Z}^2 \rightarrow \mathbb{Z}, \ (x, y) \mapsto y \), we also define

\[
p_2(\text{Cyc}(f)) := \begin{cases} 
(p_2(w(\Theta(f))), p_2(w(\Theta^2(f))), \ldots, p_2(w(\Theta^{k-1}(f)))) & \text{if } p > 2 \\
(p_2(w(\Theta(f)))) & \text{if } p = 2
\end{cases}
\]

Notice that any \( \Theta^j(f) \) in the above cycle is not identically zero by Theorem \ref{thm_3.2}. Since \( w(\Theta(f)) = w(\Theta^{\frac{k-1}{2}}(f)) \) if \( f \) is a Hecke eigenform and \( p > 2 \) under our convention \ref{convention_5.2}, and \( w(\Theta^2(f)) = w(\Theta(f)) \) if \( p = 2 \), actually it makes up a “cycle” in some sense.

When \( p = 2 \) or \( k_1 - k_2 > 1 \), the theta cycle is easily computed by Theorem \ref{thm_3.3}. Therefore, we focus on the case when \( k_1 - k_2 \in \{0, 1\} \) and \( p > 2 \).

**Definition 3.4.** Assume \( p > 2 \). Let \( f \) be an element in \( M_k(N) \) satisfying the assumption in Theorem \ref{thm_3.3}. Suppose \((k_1, k_2) = (k, k) \) or \((k + 1, k) \). Put \( r = k_1 - k_2 \) so that \( r \in \{0, 1\} \).

1. We say \( \Theta^i(f) \) is a low point of the first type (resp. the second type) if \( p_2(w(\Theta^{i-1}(f))) + r \equiv 0 \mod p \) (resp. \( 2p_2(w(\Theta^{i-1}(f))) - 1 \equiv 0 \mod p \)). If \( f_i := \Theta^{c_i}(f) \) is a low point for some integer \( c_i > 0 \), then the number \( c_i \) means one of times we add \( p + 1, p + 1 \) to \( w(f) \). We say \( c_i \) the low number of the low point \( \Theta^{c_i}(f) \). We say \( c_i \) the low number for \( f_i \). We write \( c_i = c_i^{(1)} \) (resp. \( c_i = c_i^{(2)} \)) if the low point is of the first type (resp. the second type).

2. We define the number \( b_i \) such that

\[
b_i(p - 1) = p_2(w(\Theta^{c_i-1}(f))) + (p + 1) - p_2(w(\Theta^{c_i}(f)))
\]

which means the amount falling the filtration at the low point \( f_i \) with the next application of \( \Theta \). We say \( b_i \) the jumping number of the low point \( \Theta^{c_i}(f) \). As in \( c_i \), we also write \( b_i = b_i^{(1)} \) or \( b_i = b_i^{(2)} \) according to the first type or the second type respectively.

We illustrate the notion of low points as below.

\[
f \quad \Theta(f) \quad \cdots \quad \Theta^{c_1^{(1)}-1}(f) \quad \Theta^{c_1^{(1)}}(f) \quad \cdots \quad \Theta^{c_2^{(2)}-1}(f)
\]

where \( j_1, j_2, \ldots \in \{1, 2\} \). The variant of the filtration goes as follows:

\[
k := p_2(w(f)) \quad p_2(w(\Theta(f))) = k + (p + 1) \quad \cdots \quad p_2(w(\Theta^{c_1^{(1)}-1}(f))) = k + (c_1^{(1)} - 1)(p + 1)
\]

The weight falls \( b_1^{(1)}(p - 1) \)

\[
p_2(w(f_1^{(j_1)})) \quad p_2(w(\Theta(f_1^{(j_1)}))) \quad \cdots \quad p_2(w(\Theta^{c_2^{(2)}-1}(f_1^{(j_1)})))
\]

The weight falls \( b_2^{(2)}(p - 1) \)
Then the number \( c_1^{(j_1)} \) means that the first number such that

\[
(3.6) \quad p_2(w(\Theta^j(f))) = p_2(w(f)) + j(p + 1) \text{ for } 0 \leq j \leq c_1^{(j_1)} - 1
\]

and

\[
(3.7) \quad \begin{cases} 
  p_2(w(\Theta_1^{(j_1)}(f))) + r \equiv 0 \mod p & \text{if } j_1 = 1 \text{ (a low point of the first type)} \\
  2p_2(w(\Theta_1^{(j_1)}(f))) - 1 \equiv 0 \mod p & \text{if } j_1 = 2 \text{ (a low point of the second type)}.
\end{cases}
\]

Let \( \{c_1^{(j_i)}\}_{i=1}^s \) (resp. \( \{b_1^{(j_i)}\}_{i=1}^s \)) be the collection of all low numbers (jumping numbers) for \( f \). We define \( f_i^{(j_i)}, 1 \leq i \leq s \) inductively such that \( f_{i+1}^{(j_{i+1})} = \Theta_{i+1}^{(j_{i+1})} (f_i^{(j_i)}) \) and \( f_1^{(j_1)} := \Theta_1^{(j_1)} (f) \). Since the length of the theta cycle of \( f \) is \( \frac{p-1}{2} \), one has

\[
(3.8) \quad \sum_{i=1}^s c_i^{(j_i)} = \frac{p-1}{2}.
\]

The total amount of the varying weights in the theta cycle is \( (p + 1) \frac{(p-1)}{2} \). It follows from this that \( \sum_{i=1}^s b_i^{(j_i)}(p - 1) = (p + 1) \frac{(p-1)}{2} \). Hence we have

\[
(3.9) \quad \sum_{i=1}^s b_i^{(j_i)} = \frac{p+1}{2}.
\]

Further, by definition we have

\[
(3.10) \quad p_2(w(\Theta_1^{(j_{i+1})-1} f_i^{(j_i)})) = p_2(w(f_i^{(j_i)})) + (c_i^{(j_{i+1})} - 1)(p + 1), \quad j_i, j_{i+1} \in \{1, 2\}.
\]

Further, the equations \( (3.6) \) and \( (3.7) \) turn out to be

\[
(3.11) \quad b_i^{(j_i)}(p - 1) = p_2(w(\Theta_1^{(j_{i+1})-1} f_{i+1}^{(j_i)-1})) + (p + 1) - p_2(w(f_i^{(j_i)}))
\]

and for each \( 1 \leq i \leq s \),

\[
(3.12) \quad \begin{cases} 
  p_2(w(\Theta_1^{(j_{i-1})-1} f_{i-1}^{(j_i-1)}))) + r \equiv 0 \mod p & \text{if } j_i = 1 \text{ (a low point of the first type)} \\
  2p_2(w(\Theta_1^{(j_{i-1})-1} f_{i-1}^{(j_i-1)}))) - 1 \equiv 0 \mod p & \text{if } j_i = 2 \text{ (a low point of the first type)}.
\end{cases}
\]

respectively.

3.3.1. The case when \( r = 0 \). In this case, the theta cycle \( \text{Cyc}(f) \) was computed in Section 6.0.1-6.0.2 in [21].

3.3.2. The case when \( r = 1 \). The computation is quite similar to the case when \( r = 0 \) but we give the details for reader’s convenience.

It follows from \( (3.12) \) and \( (3.11) \) that

\[
(3.13) \quad \begin{cases} 
  p_2(w(f_i^{(j_i)})) \equiv b_i^{(j_i)} \mod p & \text{if } j_i = 1 \\
  p_2(w(f_i^{(j_i)})) \equiv b_i^{(j_i)} + \frac{p+3}{2} \mod p & \text{if } j_i = 2.
\end{cases}
\]

Further, the condition \( (3.12) \) and \( (3.10) \) yield

\[
(3.14) \quad p_2(w(f_i^{(j_i)})) + c_{i+1}^{(j_{i+1})} \equiv \begin{cases} 
  -1 \mod p & \text{if } j_{i+1} = 1 \\
  \frac{p+1}{2} \mod p & \text{if } j_{i+1} = 2.
\end{cases}
\]
Further, since $p_i$ and $s$ to be $s$ (3.15)

It follows from (3.13) and (3.14) that the theta cycle is computed as

$$c_{i+1}^{(j+1)} + b_i^{(j)} \equiv 0 \mod p$$

(3.15)

This also shows $c_{i+1}^{j+1} + b_i^{j} \geq \min\{\frac{p-3}{2}, 3\}$ for any $1 \leq i \leq s$ and $j_i, j_{i+1} \in \{1, 2\}$. If $s \geq 2$, it has to be $s = 2$ since

$$p = \sum_{i=1}^{s} (b_i^{(j_i)} + c_i^{(j_i)}) = b_s^{(j_s)} + c_1^{(j_1)} + \sum_{i=1}^{s-1} (b_i^{(j_i)} + c_{i+1}^{(j_{i+1})})$$

and $p \geq 3$. Therefore, the number of low points $s$ is less than or equal to 2. Assume $f$ is non semi-ordinary. Then $p_2(w(\Theta^{p-1}f)) = p_2(w(f))$. Hence, a jump happens at least one time and thus $s \geq 1$. Further, the last jump necessarily happens at $w(\Theta^{p-2}f)$ to conclude $w(\Theta^{p-1}f) = w(f)$. Further, since $w(f)$ appears in $\text{Cyc}(f)$, the next step $w(\Theta(f))$ is automatically outputted in the computation below. This is a phenomena in the case of non semi-ordinary.

When $s = 1$, by (3.18) and (3.19) we have

$$c_1^{(j_1)} = \frac{p-1}{2}, \ b_1^{(j_1)} = \frac{p+1}{2}, \ j_1 \in \{1, 2\}.$$ 

Put $k := p_2(w(f)) = ap + k_0$ with $a \in \mathbb{Z}_{\geq 0}$ and $1 \leq k_0 \leq p$. When $j_1 = 1$, since $p_2(w(\Theta^{(1)\ldots(1)}f)) = k + (c_1^{(1)} - 1)(p+1) \equiv -1 \mod p, k_0 \equiv \frac{k+1}{2} \mod p$. Hence $k_0 = \frac{k+1}{2}$. As a check, by formula (3.11) we have $p_2(w(\Theta^{(1)}f)) = k + (c_1^{(1)} - 1)(p+1) + (p+1) - b_1^{(1)}(p-1) = k$. Similarly, when $j_1 = 2$, we have $k_0 = 2$. In either of cases, we have

$$c_1^{(j_1)-1} = \frac{p-3}{2},$$

$$p_2(\text{Cyc}(f)) = (k + (p+1), \ldots, k + \frac{(p-3)}{2}(p+1), k), \ k = ap + k_0$$

with $\frac{p+1}{2}$ if $j_1 = 1$ and $k_0 = 2$ if $j_1 = 2$.

Henceforth we assume $s = 2$ (this implies $p \geq 5$ since the length of the theta cycle is $\frac{p-1}{2}$). Since

$$p = \sum_{i=1}^{2} (b_i^{(j_i)} + c_i^{(j_i)}) = (b_2^{(j_2)} + c_1^{(j_1)}) + (b_1^{(j_1)} + c_2^{(j_2)}) \geq 2 + (b_1^{(j_1)} + c_2^{(j_2)}).$$

Combining it with the congruence relation (3.11), we have

$$(j_1, j_2) = (1, 2) \text{ or } (2, 1).$$

Put $k := p_2(w(f)) = ap + k_0$ with $a \in \mathbb{Z}_{\geq 0}$ and $1 \leq k_0 \leq p$. When $(j_1, j_2) = (1, 2)$, we have $b_1^{(1)} + c_2^{(2)} = \frac{p+3}{2}$. Since $p_2(w(\Theta^{(1)^{\ldots(1)}}f)) = k + (c_1^{(1)} - 1)(p+1) \equiv -1 \mod p, k_0 + c_1^{(1)} \equiv 0 \mod p$. It follows from $2 \leq k_0 + c_1^{(1)} \leq p + \frac{p-1}{2} < 2p$ that $k_0 + c_1^{(1)} = p$. Then we have

$$c_1^{(1)} = p - k_0, \ c_2^{(2)} = k_0 - \frac{p+1}{2}, \ b_1^{(1)} = p + 2 - k_0, \ b_2^{(2)} = k_0 - \frac{p+3}{2}.$$  

Since these integers are positive integers, we should have $\frac{p+5}{2} \leq k_0 \leq p - 1$ and also $p \geq 7$. Then the theta cycle is computed as

$$p_2(\text{Cyc}(f)) = (k + (p+1), \ldots, k + (p-1 - k_0)(p+1)).$$
\[
k_1^{(1)} := k + 2 - 2k_0, k_1^{(1)} + (p + 1), \ldots, k_1^{(1)} + (k_0 - \frac{p + 3}{2})(p + 1), k).
\]

When \((j_1, j_2) = (2, 1)\), we have \(b_1^{(2)} + c_2^{(1)} = \frac{p - 3}{2}\). A similar argument shows \(k_0 + c_1^{(2)} = \frac{p + 3}{2}\). Hence we have
\[
c_1^{(2)} = \frac{p + 3}{2} - k_0, \quad c_2^{(1)} = k_0 - 2, \quad b_1^{(2)} = \frac{p + 1}{2} - k_0, \quad b_2^{(1)} = k_0
\]
with \(3 \leq k_0 \leq \frac{p - 1}{2}\) and also \(p \geq 7\). Then the theta cycle is computed as
\[
p_2(\text{Cyc}(f)) = (k + (p + 1), \ldots, k + \left(\frac{p + 1}{2} - k_0\right)(p + 1), k)
\]
\[
k_1^{(2)} := k + 2 + 2p - 2k_0, k_1^{(2)} + (p + 1), \ldots, k_1^{(2)} + (k_0 - 3)(p + 1), k).
\]

Summing up, we have obtained the following result for the theta cycle:

**Theorem 3.5.** Assume \((k_1, k_2) = (k + 1, k)\). Let \(f \in M_k(N)\) satisfying the assumption in Theorem 3.3 Assume \(f\) is not semi-ordinary. Put \(k := p_2(w(f)) = ap + k_0\) with \(a \in \mathbb{Z}_{\geq 0}\) and \(1 \leq k_0 \leq p\). Let \(k_1^{(1)} := k + 2 - 2k_0\) and \(k_2^{(1)} := k + 2 + 2p - 2k_0\). If \(p > 2\), then \(p_2(\text{Cyc}(f))\) satisfies either of the followings

\[
\begin{align*}
(k + (p + 1), \ldots, k + \left(\frac{p - 3}{2}\right)(p + 1), k) & \quad \text{if } k_0 = \frac{p + 1}{2} \text{ with } j_1 = 1 \text{ or } k_0 = 2 \text{ with } j_1 = 2 \\
(k + (p + 1), \ldots, k + (p - 1 - k_0)(p + 1)) & \quad k_1^{(1)}, k_1^{(1)} + (p + 1), \ldots, k_1^{(1)} + (k_0 - \frac{p + 3}{2})(p + 1), k) \\
& \quad \text{if } p \geq 7 \text{ and } k_0 \in [\frac{p + 5}{2}, p - 1] \\
(k + (p + 1), \ldots, k + (\frac{p + 1}{2} - k_0)(p + 1)) & \quad k_2^{(1)}, k_2^{(1)} + (p + 1), \ldots, k_2^{(1)} + (k_0 - 3)(p + 1), k) \\
& \quad \text{if } p \geq 7 \text{ and } k_0 \in [3, \frac{p - 1}{2}].
\end{align*}
\]

If \(p = 2\), \(p_2(\text{Cyc}(f)) = (p_2(w(f)) + 2)\).

When \(s = 2\), the low points happens so that

1. the low point of the second kind comes after the low point of the first kind in which case \((j_1, j_2) = (1, 2)\) and
2. the low point of the first kind comes after the low point of the second kind in which case \((j_1, j_2) = (2, 1)\).

Therefore, once \(w(f)\) is given, one can check which case of two happens.

**Corollary 3.6.** Keep the notation being in Theorem 3.3 If \(f\) is not semi-ordinary and \(p\) is odd, then \(p_2(w(f)) \neq 1, \frac{p + 3}{2}, p\).

Finally we discuss when \(f\) is semi-ordinary. In this case we do not know if \(w(f)\) appears in \(\text{Cyc}(f)\). Therefore, we need to observe the first step \(w(\Theta(f))\) to start the theta cycle. This is a bit cumbersome part in the case of semi-ordinary.

Let \(k := p_2(w(f)) = ap + k_0\). Suppose \(p \nmid (k_0 + 1)(2k_0 - 1)\). Then \(p_2(w(\Theta(f))) = k + p + 1 = (a + 1)p + (k_0 + 1)\). Let \(g := \Theta(f)\). Then \(w(\Theta(\Theta(g)) = w(g)\). Applying Theorem 3.6 to \(g := \Theta(f)\) and \(p_2(w(\Theta(f)))\), we have the following:
Theorem 3.7. Assume \((k_1, k_2) = (k + 1, k)\). Let \(f \in M_k(N)\) satisfying the assumption in Theorem 3.5. Assume \(f\) is semi-ordinary. Put \(k := p_2(w(f)) = ap + k_0\) with \(a \in \mathbb{Z}_{\geq 0}\) and \(1 \leq k_0 \leq p\). Let \(k_1^{(1)} := k + (p + 1) - 2k_0\) and \(k_1^{(2)} := k + (p + 1) + 2p - 2k_0\). Suppose \(p \mid (k_0 + 1)(2k_0 - 1)\). If \(p > 2\), then \(p_2(\text{Cyc}(f))\) satisfies either of the followings

\[
\begin{align*}
(k + (p + 1), \ldots, k + (p + 1),) & \quad \text{if } k_0 = 1 \text{ or } p = 2, \\
(k + (p + 1), \ldots, k + (p + 1),) & \quad \text{if } p \geq 7 \text{ and } k_0 \in \left[\frac{p + 3}{2}, p - 2\right], \\
(k + (p + 1), \ldots, k + (p + 1),) & \quad \text{if } p \geq 7 \text{ and } k_0 \in \left[2, \frac{p - 3}{2}\right], \\
(k + (p + 1), k + 2(p + 1), \ldots, k + (p + 1)) & \quad \text{otherwise}.
\end{align*}
\]

If \(p = 2\), \(p_2(\text{Cyc}(f)) = (p_2(w(f)) + 2)\).

The remaining cases are \(p \mid (k_0 + 1)\) or \(p \mid (2k_0 - 1)\) when \(p > 2\). In this case, once we obtain \(b \in \mathbb{Z}_{\geq 0}\) such that \(p_2(w(\Theta(f))) = p_2(w(f)) + (p + 1) - b(p - 1)\), then we may apply Theorem 3.7 to \(p_2(w(\Theta(f)))\). Notice \(p_2(w(\Theta(f)))\) is congruent to \(b\) or \(\frac{p - 3}{2} + b\). We would try to determine \(b\) somewhere else.

4. Classical Serre weights for \(p > 2\)

Assume \(p > 2\). For each positive integer \(n\), we denote by \(\omega_n : G_{Q_p^n} \rightarrow \mathbb{F}_p^\times\) the fundamental character of level \(n\) where \(Q_p^n\) is a unique unramified extension of \(Q_p\) of degree \(n\). Note that \(\omega_1 = \varepsilon\) where, by abusing notation, \(\varepsilon\) stands for the mod \(p\) cyclotomic character of \(G_{Q_p}\).

According to Section 10 of [22], we give an explicit description of the classical Serre weight \((k_1(\overline{\pi}), k_2(\overline{\pi}), w(\overline{\pi}))\) for a given mod \(p\) Galois representation \(\pi : G_{Q} \rightarrow \text{GSp}_4(\mathbb{F}_p)\). Put \(\overline{\pi}_p := \pi|_{G_{Q_p}}\). For a character \(\chi : G_{Q_p} \rightarrow \mathbb{F}_p^\times\), a class of the Galois cohomology \(H^1(G_{Q_p}, \chi)\) is said to be trè s ramifiée (ramified) if \(\chi = \varepsilon\) and it is a trè s ramifiée (ramified) class in the sense of Definition 9.10 of [22]. If a class is neither trè s ramifiée nor ramified, then we say it peu ramifiée.

Henceforth the characters \(\overline{\psi}_i : G_{Q_p} \rightarrow \mathbb{F}_p^\times\), \(0 \leq i \leq 2\) always mean finite unramified characters. By Proposition 7.2 of [21] we have five types of \(\overline{\pi}_p\) whose images take the values in \(\text{GSp}_4(\mathbb{F}_p)\):

(1) (Borel ordinary case)

\[
\overline{\pi}_p \simeq \overline{\psi}_0 \varepsilon \otimes \begin{pmatrix} \overline{\pi}_1 & B \\ 0 & \overline{\psi}_2 \varepsilon^a \end{pmatrix}, \quad \overline{\pi}_1 = \begin{pmatrix} \overline{\psi}_1 \varepsilon^{a+b} & \overline{\pi}_0 \\ 0 & \overline{\psi}_2 \varepsilon^a \end{pmatrix}, \quad B \in H^1(Q_p, (\overline{\psi}_1 \varepsilon^{a+b})^{-1} \text{Sym}^2(\overline{\pi}_1));
\]

(2) (Siegel ordinary case)

\[
\overline{\pi}_p \simeq \overline{\psi}_0 \varepsilon \otimes \begin{pmatrix} \overline{\psi}_2 \varepsilon^{a+b} & \overline{\pi}_3 \\ 0 & \overline{\pi}_1 \overline{\pi}_2 \overline{\pi}_3 \\ 0 & 0 & 1 \end{pmatrix}
\]

where \(\overline{\pi}_1 \simeq \overline{\psi}_1 \otimes \text{Ind}_{G_{Q_p}}^{G_{Q_p}} \omega_2^b a^c\) with \(0 \leq b < a \leq p, 0 \leq c \leq p - 2\), and \(\overline{\psi}_1, \overline{\psi}_2\) satisfy \(\overline{\psi}_1 = \det(\overline{\pi}_1) \varepsilon^{-(a+b)} = \overline{\psi}_2\);

(3) (Klingen ordinary case)

\[
\overline{\pi}_p \simeq \begin{pmatrix} \overline{\pi}_1 & * \\ 0_2 & \overline{\psi}_0 \overline{\pi}_2 \overline{\pi}_3 \end{pmatrix}
\]
where $\overline{\rho}_1 \simeq \overline{\psi}_1 \otimes \text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p'}} \omega_2^{b+ap}$ with $0 \leq b < a \leq p$, $c \in \mathbb{Z}$ and $\overline{\rho}_2 \simeq \overline{\rho}_1'$;

(4) (Endoscopic case)
$$\overline{\rho}_p \simeq \overline{\psi}_1 \otimes \overline{\rho}_2$$

where $\overline{\psi}_i \simeq \overline{\psi}_1 \otimes \text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p'}} \omega_2^{b_i+pa_i}$ with $0 \leq b_i < a_i < p$ for $i = 1, 2$ and $\det(\overline{\rho}_1) = \det(\overline{\rho}_2)$.

(5) (Irreducible case)
$$\overline{\rho}_p \simeq \overline{\psi}_0^{-r} \otimes \text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p'}} \omega_4^a, \ a \in \mathbb{Z}$$

where $a \not\equiv 0 \mod (p^i - 1)/(p^i - 1)$ for $i = 1, 2$ but $a \equiv 0 \mod p + 1$.

In what follows we will define the classical Serre weights and it is defined according to how $\overline{\rho}_p$ is lifted to a crystalline (potentially diagonalizable) representation of $G_{\mathbb{Q}_p}$. The classical weights consist of a triple $(k_1(\overline{\rho}), k_2(\overline{\rho}), w(\overline{\rho}))$ of integers with $k_1(\overline{\rho}) \geq k_2(\overline{\rho}) \geq 3$, $w(\overline{\rho}) \in \mathbb{Z}$.

4.1. Local Galois cohomologies. This is a preliminary to define the classical Serre weights. For each character $\chi: G_{\mathbb{Q}_p} \rightarrow \overline{\mathbb{F}}_p^\times$ and each class $\alpha \in H^1(G_{\mathbb{Q}_p}, \chi)$, put

$$r_{\chi, \alpha} := \begin{cases} p-1 & \text{if } \alpha \text{ is très ramifiée} \\ 0 & \text{otherwise} \end{cases}$$

(see Definition 9.14 of [22]). For integers $i, j$ and a prime $p$, we define an element of $\mathbb{Z}^2$ by

$$\delta^p_{ij} := \begin{cases} (p-1, p-1) & \text{if } i = j \\ (0, 0) & \text{otherwise.} \end{cases}$$

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$. Let $\overline{M}$ be a finite dimensional representation over $\mathbb{F}$ of $G_{\mathbb{Q}_p}$ and $M$ be a lift of $\overline{M}$ to some $p$-adically integral ring $\mathcal{O}$ whose residue field is $\mathbb{F}$. Here $M$ is considered as an $\mathcal{O}[G_{\mathbb{Q}_p}]$-module. The local Galois cohomology $H^1(G_{\mathbb{Q}_p}, M)$ is a finite $\mathcal{O}$-module and may have torsion elements. We denote by $H^1(G_{\mathbb{Q}_p}, M)$ the quotient of $H^1(G_{\mathbb{Q}_p}, M)$ by all torsion elements. Suppose $\mathbb{F}(\overline{\sigma}) \subset \overline{M}$ as an $\mathbb{F}[G_{\mathbb{Q}_p}]$-module. Then it yields $H^1(G_{\mathbb{Q}_p}, \mathbb{F}(\overline{\sigma})) \rightarrow H^1(G_{\mathbb{Q}_p}, \overline{M})$. Let $M^* := M^!(1) = M^!(\overline{\sigma})$ and $\langle \ast, \ast \rangle$ be the perfect pairing on $H^1(G_{\mathbb{Q}_p}, \overline{M}) \times H^1(G_{\mathbb{Q}_p}, \overline{M})$ defined by the local Tate duality. The natural surjection $\overline{M}^* \rightarrow (\mathbb{F}(\overline{\sigma}))^* = \mathbb{F}$ induces $H^1(G_{\mathbb{Q}_p}, \overline{M}^*) \rightarrow H^1(G_{\mathbb{Q}_p}, \mathbb{F}) = \text{Hom}(\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p, \mathbb{F})$. Let $e_{\text{ur}}$ (resp. $e_{\text{ram}}$) be a non-zero element in the RHS such that $e_{\text{ur}}(p) = 1$ and zero on the units (resp. $e_{\text{ram}}(1 + p) = 1$). Since $p > 2$, they make up a basis of $H^1(G_{\mathbb{Q}_p}, \mathbb{F})$. For $\bullet \in \{\text{ur}, \text{ram}\}$, pick a lift $\widetilde{e}_\bullet$ of $e_\bullet$ to $H^1(G_{\mathbb{Q}_p}, \overline{M})$ under $e_{\text{ur}}$. If there does not exist any lift, we put $\widetilde{e}_\bullet := 0$. For each class $\alpha$ of $H^1(G_{\mathbb{Q}_p}, \overline{M})$ we define

$$a_{\text{ur}}(\alpha) := \langle \alpha, \widetilde{e}_{\text{ur}} \rangle, \ a_{\text{ram}}(\alpha) := \langle \alpha, \widetilde{e}_{\text{ram}} \rangle$$

Similarly, if $\mathbb{F} \subset \overline{M}$ as an $\mathbb{F}[G_{\mathbb{Q}_p}]$-module, then we have

$$H^1(G_{\mathbb{Q}_p}, \overline{M}^*) \rightarrow H^1(G_{\mathbb{Q}_p}, \mathbb{F}(\overline{\sigma})) \simeq \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p$$

where the above last isomorphism is the Kummer map. Then we have the dual basis $e_{\text{ur}}^*, e_{\text{ram}}^*$ of $H^1(G_{\mathbb{Q}_p}, \mathbb{F}(\overline{\sigma}))$. For each class $\alpha$ of $H^1(G_{\mathbb{Q}_p}, \overline{M})$ we also define

$$a_{\text{ur}, \ast}(\alpha) := \langle \alpha, \widetilde{e}_{\text{ur}}^* \rangle, \ a_{\text{ram}, \ast}(\alpha) := \langle \alpha, \widetilde{e}_{\text{ram}}^* \rangle$$

where lifts $\widetilde{e}_{\text{ur}}^*, \widetilde{e}_{\text{ram}}^*$ of $e_{\text{ur}}^*, e_{\text{ram}}^*$ to $H^1(G_{\mathbb{Q}_p}, \overline{M})$ are similarly defined as above.
Definition 4.1. Keep the notation being as above. Assume $p$ is odd. For each class $\alpha \in H^1(G_{Q_p}, \overline{M})$, the pair $(a_{ur}(\alpha), a_{ram}(\alpha))$ defined in (4.2) is said to be the marking of $\alpha$ with respect to $\{\overline{e}_{ur}, \overline{e}_{ram}\}$.

Similarly, the pair $(a_{ur,*}(\alpha), a_{ram,*}(\alpha))$ defined in (4.3) is said to be the marking of $\alpha$ with respect to $\{\overline{e}_{ur}^*, \overline{e}_{ram}^*\}$.

4.2. Borel ordinary case. Recall $B = \begin{pmatrix} b_2 & b_1 \\ b_3 & b_2 \end{pmatrix}$ belongs to $H^1(Q_p, (\psi_1^{1+a+b})^{-1} \text{Sym}^2(\overline{p}_1))$ where

\[
(\psi_1^{1+a+b})^{-1} \text{Sym}^2(\overline{p}_1) = \begin{pmatrix} \psi_1^{1+a+b} & 2\overline{p}_0 \\ \psi_1^{1+a+b} & \psi_1^{-1}\psi_2^{1+b}\overline{p}_0 \\ \psi_1^{-1}\psi_2^{1+b}\overline{p}_0 & 0 \end{pmatrix}.
\]

Put

\[
\overline{x}_1 := \psi_1^{1+a+b}, \quad \overline{x}_2 := \psi_1^{1+a+b}, \quad \overline{x}_3 := \psi_1^{-1}\psi_2^{1+b},
\]

\[
\overline{p}_3 := (\psi_1^{1+a+b})^{-1} \text{Sym}^2(\overline{p}_1)), \quad \overline{p}_2 := \begin{pmatrix} \overline{x}_2 & \psi_1^{-1}\psi_2^{1+b}\overline{p}_0 \\ 0 & \overline{x}_3 \end{pmatrix}
\]

for simplicity. The exact sequence $0 \rightarrow \overline{x}_1 \rightarrow \overline{p}_3 \rightarrow \overline{p}_2 \rightarrow 0$ yields an exact sequence

\[
H^1(G_{Q_p}, \overline{x}_1) \rightarrow H^1(G_{Q_p}, \overline{p}_3) \rightarrow H^1(G_{Q_p}, \overline{p}_2).
\]

By choosing a conjugation of $\overline{p}_3$ if necessary, we have a homomorphism $H^1(G_{Q_p}, \overline{p}_3) \rightarrow H^1(G_{Q_p}, \overline{x}_1)$ which sends $B$ to $b_i$ for each $1 \leq i \leq 3$. Regarding the usage of (4.1), $b_i$ is considered in this manner whenever we involve $r_{\overline{x}_i,b_i}$ in the discussion below.

4.2.1. The case of the trivial extension. First we consider the case when $\overline{p}_0 = 0$. We have $H^1(G_{Q_p}, \overline{p}_3) = \bigoplus_{1 \leq i \leq 3} H^1(G_{Q_p}, \overline{x}_i)$. By Proposition 9.12 of [22] one can lift $b_i$ to a crystalline extension in $H^1(G_{Q_p}, \overline{x}_i)$ for some crystalline lift $\chi_i$ of $\overline{x}_i$. Then we define $w(\overline{p}) = c$ and

\[
(k_1(\overline{p}), k_2(\overline{p})) = \begin{cases} (1, 2) + (a + \max_{1 \leq i \leq 3} \{r_{\chi_i,b_i}\}, b) + \delta^p_{\overline{p}_0} & \text{if } a > b \\ (1, 2) + (a + p - 1, a) + \delta^p_{\overline{p}_0} & \text{if } a = b \\ (1, 2) + (a + p - 1 + r_{\overline{x}_i,b_i}, b) & \text{if } a < b. \end{cases}
\]

We remark that when $a < b$, the ramified case does not happen by definition.

4.2.2. The case of the non-trivial peu ramifiée extensions. Let us keep the notation in the previous subsection. To define the classical Serre weights, we apply the argument in Section 9.3.3 in [22].

In this case, since $\overline{p}_0$ is peu ramifiée one can lift $\overline{p}_2$ to a (potentially diagonalizable) crystalline lift $\rho_{2,m_2,m_3}$ of Hodge-Tate weights $\{m_1,m_2\}$ with $m_2 > m_3 > 0$ satisfying $(m_2,m_3) \equiv (a,a-b)$ modulo $p-1$. Let $B_2$ be the image of $B$ under (4.3). Applying Proposition 9.16 of [21], we first lift $B_2$ to an element $B_3$ in $H^1(Q_p, \rho_{2,s_1,s_2})$ by suitably choosing $\rho_{2,s_1,s_2}$ and also choosing an extension $\rho$ of it by a crystalline lift $\chi_1$ of $\overline{x}_1$ such that

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(G_{Q_p}, \chi_1)_{tt} & \longrightarrow & H^1(G_{Q_p}, \rho)_{tt} & \longrightarrow & H^1(G_{Q_p}, \rho_{2,m_2,m_3})_{tt} & \longrightarrow & 0 \\
& & \downarrow \epsilon_1 & & \downarrow \epsilon_3 & & \downarrow \epsilon_2 & & \\
& & H^1(G_{Q_p}, \overline{x}_1) & & H^1(G_{Q_p}, \overline{p}_3) & & H^1(G_{Q_p}, \overline{p}_2) & & \end{array}
\]
where the vertical arrows mean the reduction maps. Note that none of them is necessarily surjective. When either of $\chi_2, \chi_3$ is trivial or $\varepsilon$, by using the notion in Definition 4.1 and Proposition 9.12 in [22], one can lift $B_2$ by suitably choosing a crystalline lift $\tilde{\mathcal{P}}_{2,m_2,m_3}$ of $\mathcal{P}_2$ as above. Let $\tilde{B}_2$ be such a lift of $\tilde{B}_2$ to $H^1(G_{\mathbb{Q}_p}, \rho)_1$. When $\chi_1$ is the trivial character or $\varepsilon$, by using the marking in Definition 4.1 again and Proposition 9.12 of [21], one can lift $B$ to $H^1(G_{\mathbb{Q}_p}, \rho)$ as a crystalline extension. In fact, lifts of $\iota_3(\tilde{B}_3)$ and $B$ differ by an element in the line of $H^1(G_{\mathbb{Q}_p}, \chi_1)$ defined by the marking. Since $H^1(G_{\mathbb{Q}_p}, \chi_1)_1$ is of rank one when $\chi_1$ is chosen suitably and its reduction is surjective onto that line. This makes sure the existence of a lift. According to the above procedure, we can choose $(m_2, m_3)$ for $\rho_{2,m_2,m_3}$ as follows:

$$(m_2, m_2 - m_3) = \begin{cases} 
(a + \max_{2 \leq i \leq 3} \{r_{\chi_i,b_i}\}, b) + \delta_{b_0}^p & \text{if } a > b \\
(a + p - 1, a) & \text{if } a = b \\
(a + p - 1 + r_{\chi_3,b_3}, b) & \text{if } a < b.
\end{cases}$$

Note that when $a < b$, it holds that $r_{\chi_3,b_3} = p - 1$ if and only if $(a, b) = (0, p - 2)$ and $b_3$ is très ramifié.

According to the above Hodge-Tate numbers, we define $w(\mathcal{P}) = c$ and

$$(k_1(\mathcal{P}), k_2(\mathcal{P})) = \begin{cases} 
(1, 2) + (a + \max_{1 \leq i \leq 3} \{r_{\chi_i,b_i}\}, b) + \delta_{b_0}^p & \text{if } a > b \\
(1, 2) + (a + p - 1, a) & \text{if } a = b \\
(1, 2) + (a + p - 1 + r_{\chi_3,b_3}, b) & \text{if } a < b.
\end{cases}$$

4.2.3. The case of the trés ramifié or ramified extensions. In this case, since $\chi_0$ is trés ramifié or ramified, $b = 1$ or $0$ respectively. Further, $r_{\chi_3,1}^{-1} \chi_2, \tau_0 = p - 1$ by definition and we keep to use $r_{\chi_3,1}^{-1} \chi_2, \tau_0$ to clarify how $\chi_0$ affects to the classical Serre weights. As in the previous case one can lift $\mathcal{P}_2$ to a (potentially diagonalizable) crystalline lift $\rho_{2,m_2,m_3}$ of Hodge-Tate weights $\{m_1, m_2\}$ as follows:

$$(m_2, m_2 - m_3) = \begin{cases} 
(a + \max_{2 \leq i \leq 3} \{r_{\chi_i,b_i}\} + r_{\chi_3,1}^{-1} \chi_2, \tau_0, b + r_{\chi_3,1}^{-1} \chi_2, \tau_0) + \delta_{b_0}^p & \text{if } a > b \\
(a + p - 1 + r_{\chi_3,1}^{-1} \chi_2, \tau_0, a + r_{\chi_3,1}^{-1} \chi_2, \tau_0) + \delta_{b_0}^p & \text{if } a = b \\
(a + p - 1 + r_{\chi_3,b_3} + r_{\chi_3,1}^{-1} \chi_2, \tau_0, b + r_{\chi_3,1}^{-1} \chi_2, \tau_0) & \text{if } a < b.
\end{cases}$$

According to the above Hodge-Tate numbers, we define $w(\mathcal{P}) = c$ and

$$(k_1(\mathcal{P}), k_2(\mathcal{P})) = \begin{cases} 
(1, 2) + (a + \max_{1 \leq i \leq 3} \{r_{\chi_i,b_i}\} + r_{\chi_3,1}^{-1} \chi_2, \tau_0, b + r_{\chi_3,1}^{-1} \chi_2, \tau_0) + \delta_{b_0}^p & \text{if } a > b \\
(1, 2) + (a + p - 1 + r_{\chi_3,1}^{-1} \chi_2, \tau_0, a + r_{\chi_3,1}^{-1} \chi_2, \tau_0) + \delta_{b_0}^p & \text{if } a = b \\
(1, 2) + (a + p - 1 + r_{\chi_3,b_3} + r_{\chi_3,1}^{-1} \chi_2, \tau_0, b + r_{\chi_3,1}^{-1} \chi_2, \tau_0) & \text{if } a < b.
\end{cases}$$

4.2.4. A uniform formula. In the case of Borel ordinary, in general, we may define $w(\mathcal{P}) = c$ and

$$(k_1(\mathcal{P}), k_2(\mathcal{P})) = \begin{cases} 
(1, 2) + (a + \max_{1 \leq i \leq 3} \{r_{\chi_i,b_i}\} + r_{\chi_3,1}^{-1} \chi_2, \tau_0, b + r_{\chi_3,1}^{-1} \chi_2, \tau_0) + \delta_{b_0}^p & \text{if } a > b \\
(1, 2) + (a + p - 1 + r_{\chi_3,1}^{-1} \chi_2, \tau_0, a + r_{\chi_3,1}^{-1} \chi_2, \tau_0) + \delta_{b_0}^p & \text{if } a = b \\
(1, 2) + (a + p - 1 + r_{\chi_3,b_3} + r_{\chi_3,1}^{-1} \chi_2, \tau_0, b + r_{\chi_3,1}^{-1} \chi_2, \tau_0) & \text{if } a < b.
\end{cases}$$

4.3. Siegel ordinary case. As explained in Section 9.4.2 of [22], the extension $\tau_2$ is related to $\tau_1$ under an isomorphism between local Galois cohomologies. Therefore, we may consider

$$\mathcal{P}_3 := \begin{pmatrix} \chi & \tau_1 \\ 0 & \mathcal{P}_1 \end{pmatrix}, \ \chi := \psi_{\tau_2} \cdot \omega^b.$$
Since $\tau_1$ is liftable to a crystalline extension since $\tau_1$ is irreducible by Proposition 9.3.2 of [22]. Therefore, we have only to consider $\tau_3$ which is controlled by a line in $H^1(G_{Q_p}, \chi)$ as in the case of Borel ordinary. Notice that $\omega_2^{b+p_\rho}$ does not change when we replace $(a, b)$ with $(a, b) + m(p-1, p-1)$ for any $m \in \mathbb{Z}$. Then, in this case, we define $w(\overline{\rho}) = c$ and

$$ (k_1(\overline{\rho}), k_2(\overline{\rho})) = (1, 2) + (a + r_{\chi,\tau}, a - b, c - r_{\chi,\tau} - a) + \delta_{b0}^p. $$

4.4. Klingen ordinary case. Put $\overline{\rho}_3 := (\overline{\psi_0}^{e})^{-1}ad^0(\overline{\rho}_1)$. The extension class corresponding to $\overline{\rho}$, say $B$, belongs to $H^1(G_{Q_p}, \overline{\rho}_3)$. Since $p > 2$,

$$ ad^0(\text{Ind}_{G_{Q_p}}^{G_{K_3}}(\overline{\rho}_3^{b+ap})) = \overline{\delta} \oplus \text{Ind}_{G_{Q_p}}^{G_{K_3}}(\overline{\rho}_3^{(p-1)(a-b)}) $$

where $\overline{\delta} : G_{Q_p} \rightarrow \mathbb{F}_p^\times$ is the unramified quadratic character. The second component is irreducible if and only if $(p+1) \nmid 2(a-b)$. If $(p+1) | 2(a-b)$, put $m = \frac{2(a-b)}{p+1}$. Since $0 < 2(a-b) < 2(p-1)$, $m = 1$. Then $\text{Ind}_{G_{Q_p}}^{G_{K_3}}(\overline{\rho}_3^{(p-1)(a-b)}) = \overline{\delta} \oplus \overline{\delta_1}$. We can apply the argument in Section 4.3 for the former case and in Section 4.2.1 for the latter case respectively.

Put $\overline{\chi} = (\psi_0^{e})^{-1}\overline{\delta}$ if $(p+1) | 2(a-b)$ and we denote by $\overline{\chi}$ the extension class corresponding to $B$ under the projection $H^1(G_{Q_p}, \overline{\rho}_3) \rightarrow H^1(G_{Q_p}, \overline{\chi})$ defined by the decomposition of $ad^0(\overline{\rho}_1)$. In this case we define $(k_1(\overline{\rho}), k_2(\overline{\rho}), w(\overline{\rho}))$ by

$$ \begin{cases} 
(1, 2, 0) + (a + b - c + r_{\chi,\tau}, a - b, c - r_{\chi,\tau} - a) & \text{if } 2b > c - r_{\chi,\tau} \\
(1, 2, 0) + (a + 2(p-1) - c + r_{\chi,\tau}, a - b, c - r_{\chi,\tau} - a) & \text{if } 2b \leq c - r_{\chi,\tau}.
\end{cases} $$

When $(p+1) | 2(a-b)$, hence $a-b = \frac{p+1}{2}$, put

$$ \overline{\chi}_1 = (\psi_0^{e})^{-1}\overline{\delta}, \overline{\chi}_2 = (\psi_0^{e})^{-1}\overline{\delta_1}, \overline{\chi}_3 = (\psi_0^{e})^{-1}\overline{\delta_2} $$

For each $1 \leq i \leq 3$, we denote by $\overline{\tau}_i$ the extension class corresponding to $B$ under the projection $H^1(G_{Q_p}, \overline{\rho}_3) \rightarrow H^1(G_{Q_p}, \overline{\chi}_i)$ defined by the decomposition of $ad^0(\overline{\rho}_1)$ as before. Put

$$ r_{\overline{\tau}_i, B} := \max_{1 \leq i \leq 3} \{ \overline{\tau}_i, a \} $$

for simplicity. Then we define $(k_1(\overline{\rho}), k_2(\overline{\rho}), w(\overline{\rho}))$ by

$$ \begin{cases} 
(1, 2, 0) + (a + b - c + r_{\overline{\tau}_1, B}, a - b, c - r_{\overline{\tau}_1, B} - a) & \text{if } 2b > c - r_{\overline{\tau}_1, B} \\
(1, 2, 0) + (a + 2(p-1) - c + r_{\overline{\tau}_1, B}, a - b, c - r_{\overline{\tau}_1, B} - a) & \text{if } 2b \leq c - r_{\overline{\tau}_1, B}.
\end{cases} $$

4.5. Endoscopic case. In this case, We define by $SW(\overline{\rho})$ the subset of $\mathbb{Z}_{\geq 0}^3$ consisting of all triples $(k_1, k_2, w)$ satisfying

- there exists a potentially diagonalizable crystalline lift $\rho$ of $\overline{\rho}$ which takes the values in $GSp_4$

such that $\rho$ has regular Hodge-Tate weights and

$$ HT(\rho) = \{ k_1 + k_2 - 3 + w, k_1 - 1 + w, k_2 - 2 + w, w \} $$

where $HT(\rho)$ stands for the Hodge-Tate weights of $\rho$ which is a multi-subset of $\mathbb{Z}$. By Proposition 9.15 and the proof of Lemma 1.4.3 of [11] one can construct a potentially diagonalizable crystalline lift $\rho_i$ of $\overline{\tau}_1$ with $det(\rho_1) = det(\rho_2)$ such that $HT(\rho_i) = (a_i, b_i) + c_i(p-1)$ for some $c_i \in \mathbb{Z}_{\geq 0}$. Therefore, $SW(\overline{\tau})$ is non-empty. We put the lexicographic order on $SW(\overline{\rho})$ so that we first compare $k_2$, next $k_1$, and finally $w$. Then we define

$$ (k_1(\overline{\rho}), k_2(\overline{\rho}), w(\overline{\rho})) = \min(SW(\overline{\rho})). $$
4.6. Irreducible case. In this case, as in endoscopic case, we similarly define $SW(\overline{\varphi})$ for $\varphi$. However, it may be hard to check $SW(\overline{\varphi})$ is non-empty. Therefore, one can define $(k_1(\overline{\varphi}), k_2(\overline{\varphi}), w(\overline{\varphi}))$ to be a $(k_1(\overline{\varphi}), k_2(\overline{\varphi}), w(\overline{\varphi})) = \min(SW(\overline{\varphi}))$

provided if $SW(\overline{\varphi})$ is non-empty. Any further development of the integral $p$-adic Hodge theory would be necessary.

Recall $\overline{\varphi}_p \simeq \overline{\psi}_p \otimes \text{Ind}_{G_{q,p}}^{G_{q,4}} \omega_1^a$. We write $a = a_0 + a_1 p + a_2 p^2 + a_3 p^3$ with $0 \leq a_0, a_1, a_2, a_3 \leq p - 1$. Suppose the integers $0 \leq a_0, a_1, a_2, a_3 \leq p - 1$ are distinct each other. Then we can easily lift $\overline{\varphi}_p$ to be a potentially diagonalizable crystalline lift of regular Hodge-Tate weights (cf. the proof of Lemma 2.1.12 of [8] but we may put a crystalline $\chi = 1$ because of the assumption on $a_0, a_1, a_2, a_3$).

5. Classical Serre weights for $p = 2$

In this case $\varphi = 1$ and $H^1(G_{q,2}, \mathbb{F}) \simeq \text{Hom}((\mathbb{Q}_2^\times)^2, \mathbb{F})$ is generated by three classes. By using these classes, we make markings as in Definition 4.1 to lift extension classes. Then the same argument mostly works and the classical Serre weights similarly are defined except for the case of Klingen. In this case, in the notation of Section 4.3

$$0 \longrightarrow (\overline{\varphi}_1)^{(2)} \longrightarrow \overline{\varphi}_3 \longrightarrow \mathbb{F}(\overline{\psi}_1 \overline{\psi}_0^{-1}) \longrightarrow 0$$

where $(\overline{\varphi}_1)^{(2)}$ stands for the Frobenius twist defined by 2-th power map. Even though this sequence is non-split, applying Proposition 9.15 of [22], $\overline{\varphi}_3$ is liftable to a (potentially diagonalizable) crystalline lift. Put $\overline{\psi} = \overline{\psi}_1 \overline{\psi}_0^{-1}$. We denote by $\varphi$ the extension class corresponding to $B$ under the projection $H^1(G_{q,p}, \overline{\varphi}_3) \longrightarrow H^1(G_{q,p}, \overline{\psi})$ defined by the above sequence. Notice that $\text{Ind}_{G_{q,2}}^{G_{q,2}} \omega_2 \simeq \text{Ind}_{G_{q,2}}^{G_{q,2}} \omega_2^2$. Therefore, we may assume $(a, b) = (1, 0)$. In this case we define $(k_1(\overline{\varphi}), k_2(\overline{\varphi}), w(\overline{\varphi}))$ as in the former case of Section 4.4 by substituting $(a, b) = (1, 0)$.

6. A proof and a supplemental result

6.1. A proof of Theorem 1.2 By the construction in Section 4 one can take a potentially diagonalizable crystalline lift of $\overline{\varphi}|_{G_{q,p}}$ with regular Hodge-Tate weights

$$\{k_1(\overline{\varphi}) + k_2(\overline{\varphi}) - 3 + w(\overline{\varphi}), k_1(\overline{\varphi}) - 1 + w(\overline{\varphi}), k_2(\overline{\varphi}) - 2 + w(\overline{\varphi}), w(\overline{\varphi})\}$$

satisfying $k_1(\overline{\varphi}) \geq k_2(\overline{\varphi}) \geq 3$. Applying Theorem 1.5 of [22], we have the claim. The remaining part for the theta cycle follows from the contents in Section 3.

6.2. A potentially diagonalizable lift of the minimal Hodge-Tate weights. If $\overline{\varphi}|_{G_{q,p}}$ is irreducible, then by Lemma 2.1.12 of [8], it has, up to the twist by a power of $p$-adic cyclotomic character, a potentially diagonalizable lift of Hodge-Tate weight $\{0, 1, 2, 3\}$. Combining it with Theorem 1.5 of [22] we have the following:

**Theorem 6.1.** Let $\overline{\varphi} : G_{q} \longrightarrow \text{GSp}_4(\mathbb{F}_p)$ be a mod $p$ Galois representation satisfying

- $p \geq 3$;
- $\overline{\varphi}|_{G_{q,(p)}}$ is irreducible and $\text{Im}(\overline{\varphi})$ is adequate;
- $\overline{\varphi}|_{G_{q,p}}$ is irreducible;
- $\overline{\varphi} \sim \overline{\psi}_{f,p}$ for some cuspidal Hecke eigenform $f$ in $M_{k'}(N)$ with some weight $k' = (k'_1, k'_2)$, $k'_1 \geq 3$.

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Then there exist a cuspidal Hecke eigen form $g$ unramified outside $pN$ of weight $(3, 3)$ and an integer $w(\bar{p})$ such that $\bar{p} \sim \chi^{w(\bar{p})} \otimes \bar{p}_{g,p}$ and $\rho_{g,p}|_{G_{\mathbb{Q}_p}}$ is potentially diagonalizable.

This would suggest to study the mod $p$ reductions of potentially crystalline representations of the minimal regular Hodge-Tate weights $\{0, 1, 2, 3\}$ as it is done for $GL_3$ in [13],[13].

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