Affine Gindikin–Karpelevich Formula via Uhlenbeck Spaces

Alexander Braverman, Michael Finkelberg, and David Kazhdan

Abstract We prove a version of the Gindikin–Karpelevich formula for untwisted affine Kac–Moody groups over a local field of positive characteristic. The proof is geometric and it is based on the results of [Braverman, Finkelberg, and Gaitsgory, Uhlenbeck spaces via affine Lie algebras, Progr. Math., 244, 17–135, 2006] about intersection cohomology of certain Uhlenbeck-type moduli spaces (in fact, our proof is conditioned upon the assumption that the results of [Braverman, Finkelberg, and Gaitsgory, Uhlenbeck spaces via affine Lie algebras, Progr. Math., 244, 17–135, 2006] are valid in positive characteristic; we believe that generalizing [Braverman, Finkelberg, and Gaitsgory, Uhlenbeck spaces via affine Lie algebras, Progr. Math., 244, 17–135, 2006] to the case of positive characteristic should be essentially straightforward but we have not checked the details). In particular, we give a geometric explanation of certain combinatorial differences between finite-dimensional and affine case (observed earlier by Macdonald and Cherednik), which here manifest themselves by the fact that the affine Gindikin–Karpelevich formula has an additional term compared to the finite-dimensional case. Very roughly speaking, that additional term is related to the fact that the loop group of an affine Kac-Moody group (which should be thought of as some kind of “double loop group”) does not behave well from algebro-geometric point of view; however, it
has a better behaved version, which has something to do with algebraic surfaces. A uniform (i.e. valid for all local fields) and unconditional (but not geometric) proof of the affine Gindikin–Karpelevich formula is going to appear in [Braverman, Kazhdan, and Patnaik, The Iwahori-Hecke algebra for an affine Kac-Moody group (in preparation)].

Dedicated to S. Patterson on the occasion of his 60th birthday.

1 The Problem

1.1 Classical Gindikin–Karpelevich Formula

Let \( \mathcal{H} \) be a non-archimedian local field with ring of integers \( \mathcal{O} \) and let \( G \) be a split semi-simple group over \( \mathcal{O} \). The classical Gindikin–Karpelevich formula describes explicitly how a certain intertwining operator acts on the spherical vector in a principal series representation of \( G(\mathcal{O}) \).\(^1\) In more explicit terms, it can be formulated as follows.

Let us choose a Borel subgroup \( B \) of \( G \) and an opposite Borel subgroup \( B_\perp \); let \( U, U_\perp \) be their unipotent radicals. In addition, let \( \Lambda \) denote the coroot lattice of \( G \), \( R_+ \subset \Lambda \) – the set of positive coroots, \( \Lambda_+ \) – the subsemigroup of \( \Lambda \) generated by \( R_+ \). Thus any \( \gamma \in \Lambda_+ \) can be written as \( \sum a_i \alpha_i \), where \( \alpha_i \) are the simple roots. We shall denote by \(|\gamma|\) the sum of all the \( a_i \).

Set now \( \text{Gr}_G = G(\mathcal{H})/G(\mathcal{O}) \). Then it is known that \( U(\mathcal{H}) \)-orbits on \( \text{Gr} \) are in one-to-one correspondence with elements of \( \Lambda \) (this correspondence will be reviewed in Sect. 2); for any \( \mu \in \Lambda \), we shall denote by \( S^\mu \) the corresponding orbit. The same thing is true for \( U_\perp(\mathcal{H}) \)-orbits. For each \( \gamma \in \Lambda \), we shall denote by \( T^\gamma \) the corresponding orbit. It is well known that \( T^\gamma \cap S^\mu \) is non-empty if \( \mu - \gamma \in \Lambda_+ \) and in that case the above intersection is finite. The Gindikin–Karpelevich formula allows one to compute the number of points in \( T^{-\gamma} \cap S^0 \) for \( \gamma \in \Lambda_+ \) (it is easy to see that the above intersection is naturally isomorphic to \( T^{-\gamma+\mu} \cap S^\mu \) for any \( \mu \in \Lambda \)). The answer is most easily stated in terms of the corresponding generating function:

\[ \sum_{\gamma \in \Lambda_+} #(T^{-\gamma} \cap S^0) q^{-|\gamma|} e^{-\gamma} = \prod_{\alpha \in R_+} \frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}}. \]

\(^1\)More precisely, the Gindikin-Karpelevich formula answers the analogous question for real groups; its analog for \( p \)-adic groups (usually also referred to as Gindikin-Karpelevich formula) is proved e.g., in Chap. 4 of [6].
1.2 Formulation of the Problem in the General Case

Let now $G$ be a split symmetrizable Kac–Moody group functor in the sense of [8] and let $\mathfrak{g}$ be the corresponding Lie algebra. We also let $\hat{G}$ denote the corresponding “formal” version of $G$ (cf. page 198 in [8]). The notations $\Lambda, \Lambda_+, R_+, \text{Gr}_G, S^\mu, T^\gamma$ make sense for $\hat{G}$ without any changes (cf. Sect. 2 for more detail).

**Conjecture 1.** For any $\gamma \in \Lambda_+$, the intersection $T^{-\gamma} \cap S^0$ is finite.

This conjecture will be proved in [2] when $G$ is of affine type. In this paper, we are going to prove the following result:

**Theorem 2.** Assume that $\mathcal{K} = k((t))$ where $k$ is finite. Then Conjecture 1 holds.

So now (at least when $\mathcal{K}$ is as above) we can ask the following

**Question:** Compute the generating function$^2$

$$I_{\mathfrak{g}}(q) = \sum_{\gamma \in \Lambda_+} \#(T^{-\gamma} \cap S^0) \ q^{-|\gamma|} e^{-\gamma}. $$

One possible motivation for the above question is as follows: when $G$ is finite-dimensional, Langlands [6] has observed that the usual Gindikin–Karpelevich formula (more precisely, some generalization of it) is responsible for the fact that the constant term of Eisenstein series induced from a parabolic subgroup of $G$ is related to some automorphic $L$-function. Thus, we expect that generalizing the Gindikin–Karpelevich formula to general Kac-Moody group will eventually become useful for studying Eisenstein series for those groups. This will be pursued in further publications.

We do not know the answer for general $G$. In the case when $G$ is finite-dimensional, the answer is given by Theorem 1. In this paper we are going to reprove that formula by geometric means and give a generalization to the case when $G$ is untwisted affine.

1.3 The Affine Case

Let us now assume that $\mathfrak{g} = \mathfrak{g}'_{\text{aff}}$, where $\mathfrak{g}'$ is a simple finite-dimensional Lie algebra. The Dynkin diagram of $\mathfrak{g}$ has a canonical (“affine”) vertex and we let $\mathfrak{p}$ be the corresponding maximal parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^\vee$ denote the Langlands dual algebra and let $\mathfrak{p}^\vee$ be the corresponding dual parabolic. We denote by $n(\mathfrak{p}^\vee)$ its (pro)nilpotent radical.

Let $(e, h, f)$ be a principal $sl(2)$-triple in $(\mathfrak{g}')^\vee$. Since the Levi subalgebra of $\mathfrak{p}^\vee$ is $\mathbb{C} \oplus \mathfrak{g}' \oplus \mathbb{C}$ (where the first multiple is central in $\mathfrak{g}^\vee$ and the second is responsible

$^2$The reason that we use the notation $I_{\mathfrak{g}}$ rather than $I_G$ is that it is clear that this generating function depends only on $\mathfrak{g}$ and not on $G$. 

for the “loop rotation”), this triple acts on \( n(p^\vee) \) and we let \( \mathcal{W} = (n(p^\vee))^f \) (the centralizer of \( f \) in \( n(p^\vee) \)). We are going to regard \( \mathcal{W} \) as a complex (with zero differential) and with grading coming from the action of \( h \) (thus, \( \mathcal{W} \) is negatively graded). In addition, \( \mathcal{W} \) is endowed with an action of \( \mathbb{G}_m \), coming from the loop rotation in \( g^\vee \). In the case when \( g' \) is simply laced we have \( (g')^\vee \simeq g' \) and \( n(p^\vee) = t \cdot g'[t] \) (i.e., \( g' \)-valued polynomials, which vanish at 0). Hence, \( \mathcal{W} = t \cdot (g')^f[t] \) and the above \( \mathbb{G}_m \)-action just acts by rotating \( t \).

Let \( d_1, \ldots, d_r \) be the exponents of \( g' \) (here \( r = \text{rank}(g') \)). Then \( (g')^f \) has a basis \( (x_1, \ldots, x_r) \), where each \( x_i \) is placed in the degree \( -2d_i \). We let \( Fr \) act on \( \mathcal{W} \) by requiring that it acts by \( q^{i/2} \) on elements of degree \( i \). Additionally, for any \( n \in \mathbb{Z} \), let \( \mathcal{W}(n) \) be the same graded vector space but with Frobenius action multiplied by \( q^{-n} \).

Consider now \( \text{Sym}^*(\mathcal{W}) \). We can again consider it as a complex concentrated in degrees \( \leq 0 \) endowed with an action of \( Fr \) and \( \mathbb{G}_m \). For each \( n \in \mathbb{Z} \), let \( \text{Sym}^*(\mathcal{W})_n \) be the part of \( \text{Sym}^*(\mathcal{W}) \) on which \( \mathbb{G}_m \) acts by the character \( z \mapsto z^n \).

This is a finite-dimensional complex with zero differential, concentrated in degrees \( \leq 0 \) and endowed with an action of \( Fr \).

We are now ready to formulate the main result. Let \( \delta \) denote the minimal positive imaginary coroot of \( g \). Set

\[
\Delta_{\mathcal{W}}(z) = \sum_{n=0}^\infty \text{Tr}(Fr, \text{Sym}^*(\mathcal{W})_n)z^n.
\]

In particular, when \( g' \) is simply laced we have

\[
\Delta(z) = \prod_{i=1}^r \prod_{j=0}^\infty (1 - q^{-d_i}z^j)^{-1}.
\]

**Theorem 3.** (Affine Gindikin–Karpelevich formula)

Assume that the results of [1] are valid over \( k \) and let \( \mathcal{X} = k((t)) \). Then

\[
I_\mathfrak{g}(q) = \frac{\Delta_{\mathcal{W}}(e^{-\delta})}{\Delta_{\mathcal{W}(1)}(e^{-\delta})} \prod_{\alpha \in R^+} \left( \frac{1 - q^{-1}e^{-\alpha}}{1 - e^{-\alpha}} \right)^{m_\alpha}.
\]

Here, \( m_\alpha \) denotes the multiplicity of the coroot \( \alpha \).

**Remark.** Although formally the paper [1] is written under the assumption that \( \text{char} k = 0 \), we believe that adapting all the constructions of [1] to the case \( \text{char} k = p \) should be more or less straightforward. We plan to discuss it in a separate publication.

Let us make two remarks about the above formula: first, we see that it is very similar to the finite-dimensional case (of course in that case \( m_\alpha = 1 \) for any \( \alpha \)) with the exception of a “correction term” (which is equal to \( \frac{\Delta_{\mathcal{W}}(e^{-\delta})}{\Delta_{\mathcal{W}(1)}(e^{-\delta})} \)). Roughly speaking, this correction term has to do with imaginary coroots of \( g \). The second
remark is that the same correction term appeared in the work of Macdonald [7] from purely combinatorial point of view (cf. also [3] for a more detailed study). The main purpose of this note is to explain how the term $\frac{\Delta_W(e^{-\delta})}{\Delta_W(1)e^{-\delta}}$ appears naturally from geometric point of view (very roughly speaking it is related to the fact that affine Kac–Moody groups over a local field of positive characteristic can be studied using various moduli spaces of bundles on an algebraic surface). The relation between the present work and the constructions of [3] and [7] will be discussed in [2].

2 Interpretation via Maps from $\mathbb{P}^1$ to $\mathcal{B}$

2.1 Generalities on Kac–Moody Groups

In what follows all schemes will be considered over a field $k$ which at some point will be assumed to be finite. Our main reference for Kac–Moody groups is [8]. Assume that we are given a symmetrizable Kac–Moody root data and we denote by $G$ (resp. $\widehat{G}$) the corresponding minimal (resp. formal) Kac–Moody group functor (cf. [8], page 198); we have the natural embedding $G \hookrightarrow \widehat{G}$. We also let $W$ denote the corresponding Weyl group and we let $\ell: W \to \mathbb{Z}_{\geq 0}$ be the corresponding length function.

The group $G$ is endowed with closed subgroup functors $U \subset B, U_- \subset B_-$ such that the quotients $B/U$ and $B_-/U_-$ are naturally isomorphic to the Cartan group $H$ of $G$; also $H$ is isomorphic to the intersection $B \cap B_-$. Moreover, both $U_-$ and $B_-$ are still closed as subgroup functors of $\widehat{G}$. On the other hand, $B$ and $U$ are not closed in $\widehat{G}$ and we denote by $\widehat{B}$ and $\widehat{U}$ their closures.

The quotient $G/B$ has a natural structure of an ind-scheme which is ind-proper; the same is true for the quotient $\widehat{G}/\widehat{B}$ and the natural map $G/B \to \widehat{G}/\widehat{B}$ is an isomorphism. This quotient is often called the thin flag variety of $G$. Similarly, one can consider the quotient $\mathcal{B} = \widehat{G}/B_-$; it is called the thick flag variety of $G$ or Kashiwara flag scheme. As is suggested by the latter name, $\mathcal{B}$ has a natural scheme structure. The orbits of $B$ on $\mathcal{B}$ are in one-to-one correspondence with the elements of the Weyl group $W$; for each $w \in W$, we denote by $\mathcal{B}_w$ the corresponding orbit. The codimension of $\mathcal{B}_w$ is $\ell(w)$; in particular, $\mathcal{B}_e$ is open. There is a unique $H$-invariant point $y_0 \in \mathcal{B}_e$. The complement to $\mathcal{B}_e$ is a divisor in $\mathcal{B}$ whose components are in one-to-one correspondence with the simple roots of $G$.

In what follows $\Lambda$ will denote the coroot lattice of $G, R_+ \subset \Lambda$ – the set of positive coroots, $\Lambda_+ –$ the subsemigroup of $\Lambda$ generated by $R_+$. Thus, $\gamma \in \Lambda_+$ can be written as $\sum a_i \alpha_i$ where $\alpha_i$ are the simple coroots. We shall denote by $|\gamma|$ the sum of all the $a_i$.

In what follows we shall assume that $G$ is “simply connected,” which means that $\Lambda$ is equal to the full cocharacter lattice of $H$. 

2.2 Some Further Notations

For any variety $X$ and any $\gamma \in \Lambda_+$ we shall denote by $\text{Sym}^\gamma X$ the variety parametrizing all unordered collections $(x_1, \gamma_1), \ldots, (x_n, \gamma_n)$, where $x_j \in X, \gamma_j \in \Lambda_+$ such that $\sum \gamma_j = \gamma$.

Assume that $k$ is finite and let $\mathcal{S}$ be a complex of $\ell$-adic sheaves on a variety $X$ over $k$. We set

$$\chi_k(\mathcal{S}) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Fr}, H^i(\overline{X}, \mathcal{S})),$$

where $\overline{X} = X \times \text{Spec} \overline{k}$.

We shall denote by $(\mathbb{Q}_\ell)_X$ the constant sheaf with fiber $\mathbb{Q}_\ell$. According to the Grothendieck–Lefschetz fixed point formula, we have

$$\chi_k((\mathbb{Q}_\ell)_X) = \# X(k).$$

2.3 Semi-Infinite Orbits

As in the introduction, we set $\mathcal{X} = k((t)), \mathcal{O} = k[[t]]$. We let $\text{Gr} = \hat{G}(\mathcal{X})/\hat{G}(\mathcal{O})$, which we are just going to consider as a set with no structure. Each $\lambda \in \Lambda$ is a homomorphism $\mathbb{G}_m \to H$; in particular, it defines a homomorphism $\mathcal{X}^* \to H(\mathcal{X})$.

We shall denote the image of $t$ under the latter homomorphism by $t^\lambda$. Abusing the notation, we shall denote its image in $\text{Gr}$ by the same symbol. Set

$$S^\lambda = \hat{U}(\mathcal{X}) \cdot t^\lambda \subset \text{Gr}; \quad T^\lambda = U_-(\mathcal{X}) \cdot t^\lambda \subset \text{Gr}.$$

**Lemma 1.** $\text{Gr}$ is equal to the disjoint union of all the $S^\lambda$.

**Proof.** This follows from the Iwasawa decomposition for $G$ of [5]; we include a different proof for completeness. Since $\Lambda \simeq \hat{U}(\mathcal{X}) \setminus \hat{B}(\mathcal{X})/\hat{B}(\mathcal{O})$, the statement of the lemma is equivalent to the assertion that the natural map $\hat{B}(\mathcal{X})/\hat{B}(\mathcal{O}) \to \hat{G}(\mathcal{X})/\hat{G}(\mathcal{O})$ is an isomorphism; in other words, we need to show that $\hat{B}(\mathcal{X})$ acts transitively on $\text{Gr}$. But this is equivalent to saying that $\hat{G}(\mathcal{O})$ acts transitively on $\hat{G}(\mathcal{X})/\hat{B}(\mathcal{X})$, which means that the natural map $\hat{G}(\mathcal{O})/\hat{B}(\mathcal{O}) \to \hat{G}(\mathcal{X})/\hat{B}(\mathcal{X})$ is an isomorphism. However, the left-hand side is $(\hat{G}/\hat{B})(\mathcal{O})$ and the right-hand side is $(\hat{G}/\hat{B})(\mathcal{X})$ and the assertion follows from the fact that the ind-scheme $\hat{G}/\hat{B}$ satisfies the valuative criterion of properness.

The statement of the lemma is definitely false if we use $T^\mu$’s instead of $S^\lambda$’s since the scheme $\hat{G}/B_-$ does not satisfy the valuative criterion of properness. Let us say that an element $g(t) \in \hat{G}(\mathcal{X})$ is good if its projection to $\mathcal{B}(\mathcal{X}) = B_-(\mathcal{X}) \setminus \hat{G}(\mathcal{X})$
comes from a point of $B(O)$. Since $B(O) = B_-(O) \setminus \hat{G}(O)$, it follows that the set of good elements of $\hat{G}(\mathcal{H})$ is just equal to $B_-(\mathcal{H}) \cdot G(O)$, which immediately proves the following result:

**Lemma 2.** The preimage of $\bigcup_{\gamma \in \Lambda} T^\gamma$ in $\hat{G}(\mathcal{H})$ is equal to the set of good elements of $\hat{G}(\mathcal{H})$.

### 2.4 Spaces of Maps

Recall that the Picard group of $B$ can be naturally identified with $\Lambda^\vee$ (the dual lattice to $\Lambda$). Thus for any map $f : \mathbb{P}^1 \to B$, we can talk about the degree of $f$ as an element $\gamma \in \Lambda$. The space of such maps is non-empty iff $\gamma \in \Lambda_+$. We say that a map $f : \mathbb{P}^1 \to B$ is based if $f(\infty) = y_0$. Let $\mathcal{M}^\gamma$ be the space of based maps $f : \mathbb{P}^1 \to B$ of degree $\gamma$. It is shown in the Appendix to [1] that this is a smooth scheme of finite type over $k$ of dimension $2|\gamma|$. We have a natural (“factorization”) map $\pi^\gamma : \mathcal{M}^\gamma \to \text{Sym}^\gamma \mathbb{A}^1$, which is related to how the image of a map $\mathbb{P}^1 \to B$ intersects the complement to $B_e$. In particular, if we set $F^\gamma = (\pi^\gamma)^{-1}(\gamma \cdot 0)$,

then $F^\gamma$ consists of all the based maps $f : \mathbb{P}^1 \to B$ of degree $\gamma$ such that $f(x) \in B_e$ for any $x \neq 0$.

**Theorem 4.** There is a natural identification $F^\gamma(k) \simeq T^{-\gamma} \cap S^0$.

Since $F^\gamma$ is a scheme of finite type over $k$, it follows that $F^\gamma(k)$ is finite and thus Theorem 4 implies Theorem 2.

The proof of Theorem 4 is essentially a repetition of a similar proof in the finite-dimensional case, which we include here for completeness.

**Proof.** First of all, let us construct an embedding of the union of all the $F^\gamma(k)$ into $S^0 = \hat{U}(\mathcal{H})/\hat{U}(O)$. Indeed, an element of $\bigcup_{\gamma \in \Lambda_+} F^\gamma$ is uniquely determined by its restriction to $G_m \subset \mathbb{P}^1$; this restriction is a map $f : G_m \to B_e$ such that $\lim_{x \to \infty} f(x) = y_0$. We may identify $B_e$ with $\hat{U}$ (by acting on $y_0$). Thus, we get

$$\bigcup_{\gamma \in \Lambda_+} F^\gamma \subset \{u : \mathbb{P}^1 \setminus \{0\} \to \hat{U} | u(\infty) = e\}. \tag{1}$$

We have a natural map from the set of $k$-points of the right-hand side of (1) to $\hat{U}(\mathcal{H})$; this map sends every $u$ as above to its restriction to the formal punctured neighbourhood of 0. We claim that after projecting $\hat{U}(\mathcal{H})$ to $S^0 = \hat{U}(\mathcal{H})/\hat{U}(O)$, this map becomes an isomorphism. Recall that $\hat{U}$ is a group-scheme, which can be written as a projective limit of finite-dimensional unipotent group-schemes $U_i$;
moreover, each $U_i$ has a filtration by normal subgroups with successive quotients isomorphic to $\mathbb{G}_m$. Hence, it is enough to prove that the above map is an isomorphism when $U = \mathbb{G}_a$. In this case, we just need to check that any element of the quotient $k((t))/k[[t]]$ has unique lift to a polynomial $u(t) \in k[t,t^{-1}]$ such that $u(\infty) = 0$, which is obvious.

Now Lemma 2 implies that a map $u(t)$ as above extends to a map $\mathbb{P}^1 \to \mathcal{B}$ if and only if the corresponding element of $S^0$ lies in the intersection with some $T^{-\gamma}$.

It remains to show that $\mathcal{F}(\gamma)(k)$ is exactly equal to $S^0 \cap T^{-\gamma}$ as a subset of $S^0$. Let $\Lambda^\vee$ be the weight lattice of $G$ and let $\Lambda^\vee_+$ denote the set of dominant weights of $G$. For each $\lambda^\vee \in \Lambda^\vee_+$, we can consider the Weyl module $L(\lambda^\vee)$, defined over $\mathbb{Z}$; in particular, $L(\lambda^\vee)(\mathcal{H})$ and $L(\lambda^\vee)(\mathcal{O})$ make sense. By the definition $L(\lambda^\vee)$ is the module of global sections of a line bundle $\mathcal{L}(\lambda^\vee)$ on $\mathcal{B}$. Moreover, we have a weight decomposition

$$L(\lambda^\vee) = \bigoplus_{\mu^\vee \in \Lambda^\vee} L(\lambda^\vee)_{\mu^\vee},$$

where each $L(\lambda^\vee)_{\mu^\vee}$ is a finitely generated free $\mathbb{Z}$-module and $L(\lambda^\vee)_{\lambda^\vee} := l_{\lambda^\vee}$ has rank one. Geometrically, $l_{\lambda^\vee}$ is the fiber of $\mathcal{L}(\lambda^\vee)$ at $y_0$ and the corresponding projection map from $L(\lambda^\vee) = \Gamma(\mathcal{B}, \mathcal{L}(\lambda^\vee))$ to $l_{\lambda^\vee}$ is the restriction to $y_0$.

Let $\eta_{\lambda^\vee}$ denote the projection of $L(\lambda^\vee)$ to $l_{\lambda^\vee}$. This map is $U_-$-equivariant (where $U_-$ acts trivially on $l_{\lambda^\vee}$).

**Lemma 3.** The projection of a good element $g \in G(\mathcal{H})$ lies in $T^v$ (for some $v \in \Lambda$) if and only if for any $\lambda^\vee \in \Lambda^\vee$ we have:

$$\eta_{\lambda^\vee}(g(L(\lambda^\vee)(\mathcal{O}))) \subset t^{(v,\lambda^\vee)}l_{\lambda^\vee}(\mathcal{O}); \quad \eta_{\lambda^\vee}(g(L(\lambda^\vee)(\mathcal{O}))) \not\subset t^{(v,\lambda^\vee)-1}l_{\lambda^\vee}(\mathcal{O}).$$

(2)

**Proof.** First of all, we claim that if the projection of $g$ lies in $T^v$ then the above condition is satisfied. Indeed, it is clearly satisfied by $t^v$; moreover, (2) is clearly invariant under left multiplication by $U_-(\mathcal{H})$ and under right multiplication by $G(\mathcal{O})$. Hence any $g \in U_-(\mathcal{H}) \cdot t^v \cdot G(\mathcal{O})$ satisfies (2).

On the other hand, assume that a good element $g \in G(\mathcal{H})$ satisfies (2). Since $g$ lies in $U_-(\mathcal{H}) \cdot t^v \cdot G(\mathcal{O})$ for some $v'$, it follows that $g$ satisfies (2) when $v$ is replaced by $v'$. However, it is clear that this is possible only if $v = v'$.

It is clear that in (2) one can replace $g(L(\lambda^\vee)(\mathcal{O}))$ with $g(L(\lambda^\vee)(k))$ (since the latter generates the former as an $\mathcal{O}$-module).

Let now $f$ be an element of $\mathcal{F}(\gamma)$. Then $f^*\mathcal{L}(\lambda^\vee)$ is isomorphic to the line bundle $\mathcal{L}((\gamma, \lambda^\vee))$ on $\mathbb{P}^1$. On the other hand, the bundle $\mathcal{L}(\lambda^\vee)$ is trivialized on $\mathcal{B}_\gamma$ by means of the action of $U$; more precisely, the restriction of $\mathcal{L}(\lambda^\vee)$ is canonically identified with the trivial bundle with fiber $l_{\lambda^\vee}$. Let now $s \in L(\lambda^\vee)(k)$; we are going to think of it as a section of $L(\lambda^\vee)$ on $\mathcal{B}$. In particular, it gives rise to a function $\tilde{s}: \mathcal{B}_\gamma \to l_{\lambda^\vee}$. Let also $u(t)$ be the element of $U(\mathcal{H})$, corresponding to $f$. Then $\eta_{\lambda^\vee}(u(t)(s))$ can be described as follows: we consider the composition $\tilde{s} \circ f$ and restrict it to the formal neighbourhood of $0 \in \mathbb{P}^1$ (we get an element of $l_{\lambda^\vee}(\mathcal{H})$).
On the other hand, since \( f \in \mathcal{F}_\gamma \), it follows that \( f^* \mathcal{L}(\lambda^\vee) \) is trivialized away from 0 and any section of it can be thought of as a function \( \mathbb{P}^1 \setminus \{0\} \) with pole of order \( \leq \langle \gamma, \lambda^\vee \rangle \) at 0. Hence, \( \widetilde{s} \circ f \) has pole of order \( \leq \langle \gamma, \lambda^\vee \rangle \) at 0.

To finish the proof it is enough to show that for some \( s \) the function \( \widetilde{s} \circ f \) has pole of order exactly \( \langle \gamma, \lambda^\vee \rangle \) at 0 (indeed if \( f \in T_{-\gamma}' \) for some \( \gamma' \in \Lambda \), then by (2) \( \widetilde{s} \circ f \) has pole of order \( \leq \langle \gamma', \lambda^\vee \rangle \) at 0 and for some \( s \), it has pole of order exactly \( \langle \gamma', \lambda^\vee \rangle \), which implies that \( \gamma = \gamma' \)). To prove this, let us note that since \( \mathcal{L}(\lambda^\vee) \) is generated by global sections, the line bundle \( f^* \mathcal{L}(\lambda^\vee) \) is generated by sections of the form \( f^* s \), where \( s \) is a global section of \( \mathcal{L}(\lambda^\vee) \). This implies that for any \( s \in \Gamma(\mathbb{P}^1, f^* \mathcal{L}(\lambda^\vee)) \) there exists a section \( s \in \Gamma(\mathcal{B}, \mathcal{L}(\lambda^\vee)) \) such that the ratio \( s/s \) is a rational function on \( \mathbb{P}^1 \), which is invertible at 0. Taking \( s \) such that its pole with respect to the above trivialization of \( f^* \mathcal{L}(\lambda^\vee) \) is exactly equal to \( \langle \gamma', \lambda^\vee \rangle \) and taking \( s \) as above, we see that the pole of \( f^* s \) with respect to the above trivialization of \( f^* \mathcal{L}(\lambda^\vee) \) is exactly equal to \( \langle \gamma', \lambda^\vee \rangle \).

3 Proof of Theorem 1 via Quasi-Maps

3.1 Quasi-Maps

We shall denote by \( \mathcal{D} \mathcal{M}^\gamma \) the space of based quasi-maps \( \mathbb{P}^1 \to \mathcal{B} \). According to [4], we have the stratification

\[
\mathcal{D} \mathcal{M}^\gamma = \bigcup_{\gamma' \leq \gamma} \mathcal{M}^\gamma \times \text{Sym}^{\gamma - \gamma'} \mathbb{A}^1.
\]

The factorization morphism \( \pi^\gamma \) extends to the similar morphism \( \overline{\pi}^\gamma : \mathcal{D} \mathcal{M}^\gamma \to \text{Sym}^\gamma \) and we set \( \mathcal{F}^\gamma = (\overline{\pi}^\gamma)^{-1}(0) \). Thus, we have

\[
\mathcal{F}^\gamma = \bigcup_{\gamma' \leq \gamma} \mathcal{F}^\gamma.
\]

There is a natural section \( i^\gamma : \text{Sym}^\gamma \mathbb{A}^1 \to \mathcal{D} \mathcal{M}^\gamma \). According to [4], we have

Theorem 5. 1. The restriction of \( \text{IC} \mathcal{D} \mathcal{M}^\gamma \) to \( \mathcal{F}^\gamma \) is isomorphic to \( (\overline{\pi}^\gamma)^{-1}(0) \otimes \text{Sym}^* (n^\gamma_+[2](1))_{\gamma - \gamma'} \).

2. There exists a \( \mathbb{G}_m \)-action on \( \mathcal{D} \mathcal{M}^\gamma \), which contracts it to the image of \( i^\gamma \). In particular, it contracts \( \mathcal{F}^\gamma \) to one point (corresponding to \( \gamma' = 0 \) in (3)).

3. Let \( s_\gamma \) denote the embedding of \( \gamma : 0 \) into \( \text{Sym}^\gamma \mathbb{A}^1 \). Then

\[
s^\gamma_\gamma \text{IC} \mathcal{D} \mathcal{M}^\gamma = \text{Sym}^* (n^\gamma_+)_{\gamma}
\]

(here the right hand is a vector space concentrated in cohomological degree 0 and with trivial action of \( \text{Fr} \)).
The assertion (2) implies that $\pi^\gamma_! IC_{\mathcal{M}^\gamma} = i^\gamma_! IC_{\mathcal{M}^\gamma}$ and hence

$$H^s_c(\mathcal{F}, IC_{\mathcal{M}^\gamma} |_{\mathcal{F}^\gamma}) = s^\gamma_! i^\gamma_! IC_{\mathcal{M}^\gamma} = s^\gamma_! i^\gamma_! IC_{\mathcal{M}^\gamma} = Sym^*(n_+)_{\gamma}.$$ 

Thus, setting, $\mathcal{S}_\gamma = IC_{\mathcal{M}^\gamma} \mid_{\mathcal{F}^\gamma}$ we get

$$\sum_{\gamma \in \Lambda^+} \chi_k(\mathcal{S}_\gamma)e^{-\gamma} = \prod_{\alpha \in R^+} \frac{1}{1-e^{-\alpha}}. \quad (4)$$

On the other hand, according to (1) we have

$$\chi_k(\mathcal{S}_\gamma) = \sum_{\gamma' \leq \gamma} (\#\mathcal{F}^\gamma) q^{-|\gamma'|} \text{Tr}(\text{Fr}, \text{Sym}^*(n_{\gamma'}^\gamma(2)(1))_{\gamma'-\gamma}),$$

which implies that

$$\sum_{\gamma \in \Lambda^+} \chi_k(\mathcal{S}_\gamma)e^{-\gamma} = \frac{\sum_{\gamma \in \Lambda^+} \#\mathcal{F}^\gamma(k) q^{-|\gamma|} e^{-\gamma}}{\prod_{\alpha \in R^+} 1 - q^{-1} e^{-\alpha}} = \frac{I_g(q)}{\prod_{\alpha \in R^+} 1 - q^{-1} e^{-\alpha}}. \quad (5)$$

Hence,

$$I_g(q) = \prod_{\alpha \in R^+} \frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}}.$$

### 4 Proof of Theorem 3

#### 4.1 Flag Uhlenbeck Spaces

We now assume that $G = (G')_{\text{aff}}$ where $G'$ is some semi-simple simply connected group. We want to follow the pattern of Sect. 3. Let $\gamma \in \Lambda_+$. As is discussed in [1], the corresponding space of quasi-maps behaves badly when $G$ is replaced by $G'_{\text{aff}}$. However, in this case one can use the corresponding flag Uhlenbeck space $\mathcal{U}^\gamma$. In fact, as was mentioned in the Introduction, in [1] only the case of $k$ of characteristic 0 is considered. In what follows we are going to assume that the results of loc. cit. are valid also in positive characteristic.

The flag Uhlenbeck space $\mathcal{U}^\gamma$ has properties similar to the space of quasi-maps $\mathcal{M}^\gamma$ considered in the previous section. Namely, we have:

a. $\mathcal{U}^\gamma$ is an affine variety of dimension $2|\gamma|$, which contains $\mathcal{M}^\gamma$ as a dense open subset.

b. There is a factorization map $\pi^\gamma : \mathcal{U}^\gamma \to \text{Sym}^\gamma \mathbb{A}^1$; it has a section $i_\gamma : \text{Sym}^\gamma \mathbb{A}^1 \to \mathcal{U}^\gamma$.

c. $\mathcal{U}^\gamma$ is endowed with a $\mathbb{G}_m$-action, which contracts $\mathcal{U}^\gamma$ to the image of $i_\gamma$. 

These properties are identical to the corresponding properties of $\mathcal{M}_\gamma$ from the previous section. The next (stratification) property, however, is different (and it is in fact responsible for the additional term in Theorem 3). Namely, let $\delta$ denote the minimal positive imaginary coroot of $G'_\text{aff}$. Then we have

d. There exists a stratification

$$\mathcal{U}^\gamma = \bigcup_{\gamma' \in \Lambda^+, \gamma \in \Lambda^+, n \in \mathbb{Z}} (\mathcal{M}_\gamma - \gamma' - n\delta \times \text{Sym}^\gamma \mathbb{A}_1) \times \text{Sym}^n (\mathbb{G}_m \times \mathbb{A}_1).$$

(6)

In particular, if we now set $\mathcal{F}^\gamma = (\pi^\gamma)^{-1}(\gamma \cdot 0)$, we get

$$\mathcal{F}^\gamma = \left( \bigcup_{\gamma' \in \Lambda^+, \gamma \in \Lambda^+, n \in \mathbb{Z}} \mathcal{F}^\gamma \right) \times \text{Sym}^n (\mathbb{G}_m).$$

(7)

### 4.2 Description of the IC-Sheaf

In [1], we describe the IC-sheaf of $\mathcal{U}^\gamma$. To formulate the answer, we need to introduce some notation. Let $P(n)$ denote the set of partitions of $n$. In other words, any $P \in P(n)$ is an unordered sequence $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$ such that $\sum n_i = n$. We set $|P| = k$. For a variety $X$ and any $P \in P(n)$, we denote by $\text{Sym}^P(X)$ the locally closed subset of $\text{Sym}^n(X)$ consisting of all formal sums $\sum n_i x_i$ where $x_i \in X$ and $x_i \neq x_j$ for $i \neq j$. The dimension of $\text{Sym}^P(X)$ is $|P| \cdot \dim X$. Let also $\text{Sym}^* (\mathbb{W}[2])(P) = \bigotimes_{i=1}^k \text{Sym}^* (\mathbb{W}[2](1))_{n_i}$.

**Theorem 6.** The restriction of $\text{IC}_{\mathcal{U}^\gamma}$ to $\mathcal{M}_\gamma - \gamma' - n\delta \times \text{Sym}^\gamma (\mathbb{A}_1) \times \text{Sym}^P (\mathbb{G}_m \times \mathbb{A}_1)$ is isomorphic to constant sheaf on that scheme tensored with

$$\text{Sym}^* (\mathbb{W}[2](1))_P = \bigotimes_{i=1}^k \text{Sym}^* (\mathbb{W}[2](1))_{n_i}.$$

**Corollary 1.** The restriction of $\text{IC}_{\mathcal{U}^\gamma}$ to $\mathcal{F}^\gamma - \gamma' - n\delta \times \text{Sym}^P (\mathbb{G}_m)$ is isomorphic to the constant sheaf tensored with

$$\text{Sym}^* (\mathbb{W}[2](1))_P [2|\gamma - \gamma' - n\delta|][|\gamma - \gamma' - n\delta|].$$

Let now $\mathcal{I}^\gamma$ denote the restriction of $\text{IC}_{\mathcal{U}^\gamma}$ to $\mathcal{F}^\gamma$. Then as in (4) we get

$$\sum_{\gamma' \in \Lambda^+} \chi_k(\mathcal{I}^\gamma) e^{-\gamma} = \prod_{\alpha \in R_+} \frac{1}{(1 - e^{-\alpha})^{m_\alpha}}.$$  

(8)
On the other hand, arguing as in (5) we get that

$$\sum_{\gamma \in \Lambda^+} \chi_k(\mathcal{Y}^\gamma) e^{-\gamma} = A(q) \prod_{\alpha \in R^+} \frac{I_B(q)}{(1 - q^{-1} e^{-\alpha})^{m_\alpha}},$$

(9)

where

$$A(q) = \sum_{n=0}^{\infty} \sum_{p \in \mathcal{P}(n)} \text{Tr}(\text{Fr}, H^*_c(\text{Sym}^p(G_m), \overline{Q}_l) \otimes \text{Sym}^*(\mathcal{W}[2](1))_p) e^{-n\delta}.$$ 

This implies that

$$I_B(q) = A(q) \prod_{\alpha \in R^+} \left( \frac{1-q^{-1} e^{-\alpha}}{1-e^{-\alpha}} \right)^{m_\alpha}.$$ 

It remains to compute $A(q)$. However, it is clear that

$$A(q) = \sum_{n=0}^{\infty} \text{Tr}(\text{Fr}, \text{Sym}^n(H_c^*(G_m)) \otimes \mathcal{W}[2](1)) e^{-n\delta} = \frac{\Delta_{\mathcal{W}}(e^{-\delta})}{\Delta_{\mathcal{W}}(1)(e^{-\delta})}.$$ 

(10)

This is true since $H^i_c(G_m) = 0$ unless $i = 1, 2$, and we have

$$H^1_c(G_m) = \overline{Q}_l, \quad H^2_c(G_m) = \overline{Q}_l(-1),$$

and thus if we ignore the cohomological $\mathbb{Z}$-grading, but only remember the corresponding $\mathbb{Z}_2$-grading, then we just have

$$\text{Sym}^*(H^*_c(G_m) \otimes \mathcal{W}[2](1)) = \text{Sym}^*(\mathcal{W}) \otimes \Lambda^*(\mathcal{W}(1)),$$

whose character is exactly the right-hand side of (10).

Acknowledgements We thank I. Cherednik, P. Etingof and M. Patnaik for very helpful discussions. A. B. was partially supported by the NSF grant DMS-0901274. M. F. was partially supported by the RFBR grant 09-01-00242 and the Science Foundation of the SU-HSE awards No.T3-62.0 and 10-09-0015. D. K. was partially supported by the BSF grant 037.8389.

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Contributions in Analytic and Algebraic Number Theory
Festschrift for S. J. Patterson
Blomer, V.; Mihăilescu, P. (Eds.)
2012, XVIII, 290 p., Hardcover
ISBN: 978-1-4614-1218-2