Perturbative renormalization of the $\Delta B = 2$ four-quark operators in lattice NRQCD

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We present a perturbative calculation of the renormalization constants for the $\Delta B = 2$ four-quark operators, which is needed to evaluate the bag parameters $B_B$ and $B_S$ in lattice QCD. The one-loop coefficients are calculated for the $O(1/M)$ NRQCD heavy quark action and the $O(a)$-improved Wilson action for light quark. Using these coefficients, which remove $O(\alpha_s/aM)$ error, we reanalyze our previous calculation of the bag parameters.

1. Introduction

In the lattice calculation of the $B$ meson $B$-parameters, the use of the effective theory for heavy quark is essential to obtain results with controlled systematic errors. The static approximation in which the heavy quark mass is sent to infinity has been used by many authors, and the next step is to incorporate the correction from finite $b$ quark mass. In this direction, we performed an exploratory lattice calculation using the NRQCD action for heavy quark, in which we found that the $1/m_B$ correction could be sizable for $B_B$ and for $B_S$. A drawback in these calculations was, however, that the perturbative renormalization of the four-quark operators were not available for the NRQCD action and we used the one-loop coefficients for the static action. As a result our results contained large systematic error of $O(\alpha_s/aM)$, which is about 10% if evaluated with an order counting at $\beta = 5.9$.

In this paper, we present a perturbative calculations of renormalization constants of the heavy-light four quark operators with the $O(1/M)$ lattice NRQCD action. The results connect the missing link in our previous calculations, and we present a reanalysis of $B_B$ and $B_S$. A full description of our results is published in [5].

2. Operator Definitions

The $B$ parameters in the $B_q - \bar{B}_q$ are defined through

\begin{align*}
\langle \bar{B}_q | O_{\text{MS}}^L(M_B) | B_q \rangle &= \frac{8}{3} M_{B_q}^2 f_{B_q}^2 B_{L_q}^\text{MS}(\mu), \\
\langle \bar{B}_q | O_{\text{MS}}^S(M_B) | B_q \rangle &= -\frac{5}{3} M_{B_q}^2 f_{B_q}^2 B_{S_q}^\text{MS}(\mu) \left( \frac{M_{B_q}}{m_b(\mu)} + \frac{m_q(\mu)}{m_b(\mu)} \right)^2,
\end{align*}

where a subscript $q$ denotes light quark flavor, i.e. $d$ or $s$. $B_L$ is usually called $B_B$, but we use a notation $B_L$ in this paper to remind that it parameterizes a matrix element of the operator $O_L$. In the above equations, the operators $O_{\text{MS}}^L(\mu)$ and $O_{\text{MS}}^S(\mu)$ are defined in the continuum theory with the MS scheme. Their Dirac structure is

\begin{align*}
O_L &= \bar{b} \gamma_\mu P_L q \bar{b} \gamma_\mu P_L q, \\
O_S &= \bar{b} P_L q \bar{b} P_L q,
\end{align*}

where $P_L = 1 - \gamma_5$. The totally anti-commuting convention for $\gamma_5$ is assumed in the dimensional regularization.

The matching of these continuum operators is done against the lattice NRQCD. Namely we consider the $O(1/M)$ NRQCD action for heavy quark and the $O(a)$-improved Wilson fermion for light quark. For gluons the standard plaquette action is used.

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For $O_L$ and $O_S$ the following heavy-light four-quark operators appear in the matching relation:

$$
O_L = [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_L q],
O_R = [\bar{b}\gamma_{\mu} P_R q] [\bar{b}\gamma_{\mu} P_R q],
O_S = [\bar{b} P_L q] [\bar{b} P_L q],
O_N = 2 [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_R q] + 4 [\bar{b} P_L q] [\bar{b} P_R q],
O_M = 2 [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_R q] - 4 [\bar{b} P_L q] [\bar{b} P_R q],
O_P = 2 [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_R q] + 12 [\bar{b} P_L q] [\bar{b} P_R q],
O_T = (2 + N_c) [\bar{b}\gamma_{\mu} P_R q] - 2(3N_c^2 - 2N_c - 4) [\bar{b} P_L q] [\bar{b} P_R q],
$$

where $P_R = 1 + \gamma_5$. The heavy quark field $b$ has four Dirac components, and it is obtained from the two-component spinor non-relativistic quark (anti-quark), $Q$ ($\chi$), through the Foldy-Wouthuysen-Tani transformation,

$$b = \left[ 1 - \frac{\vec{\gamma} \cdot \vec{D}}{2M} \right] \left( Q \chi \right).$$

In the static limit $M \to \infty$, where algebra is greatly reduced, we consider the $O(\alpha)$-improvement of the lattice four-quark operators through $O(\alpha_s \alpha)$. In doing that we also define the following dimension seven operators:

$$O_{LD} = [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_L (a\vec{D} \cdot \vec{\gamma}) q],
O_{SD} = [\bar{b} P_L q] [\bar{b} P_L (a\vec{D} \cdot \vec{\gamma}) q],
O_{ND} = 2 [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_R (a\vec{D} \cdot \vec{\gamma}) q]
+ 4 [\bar{b} P_L q] [\bar{b} P_R (a\vec{D} \cdot \vec{\gamma}) q],
O_{PD} = 2 [\bar{b}\gamma_{\mu} P_L q] [\bar{b}\gamma_{\mu} P_R (a\vec{D} \cdot \vec{\gamma}) q]
+ 12 [\bar{b} P_L q] [\bar{b} P_R (a\vec{D} \cdot \vec{\gamma}) q].$$

3. Matching

At one-loop level, the matching of the continuum operator $O_{\text{MS}}(\mu)$ is given by

$$O_{\text{MS}}(\mu) = \left[ 1 + \frac{\alpha_s}{4\pi} \zeta_{X,X} (\mu; 1/a) \right] O^\text{lat}_Z (1/a)
+ \sum_{Z=\text{dim.7}} \frac{\alpha_s}{4\pi} \zeta_{X,Y} (\mu; 1/a) O^\text{lat}_Y (1/a),$$

where the summations with $Y$ and $Z$ run over dimension six and seven operators respectively. $\mu$ is the renormalization scale in the $\overline{\text{MS}}$ scheme and $a$ is lattice spacing.

The coefficients $\zeta$ are determined so that the on-shell scattering amplitudes in the continuum and lattice theories agree with each other. For the external state, zero spatial momentum heavy and light quark suffice to determine the coefficients for dimension six operators. A derivative in terms of spatial momenta have to be considered to obtain the dimension seven coefficients.
3.1. $O_L$

The matching relation for $O_L$ is

$$O_L^{\text{MS}}(\mu) = \left[ 1 + \frac{\alpha_s}{4\pi} \zeta_{L,L}(\mu; 1/a) \right] O_L^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{L,S}(1/a) O_S^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{L,N}(1/a) O_N^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{L,M}(1/a) O_M^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{L,R}(1/a) O_R^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{L,LD} O_{LD}^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{L,ND} O_{ND}^{\text{lat}}(1/a).$$

(9)

The coefficient $\zeta_{L,L}(\mu; 1/a)$ of the leading operator $O_L^{\text{lat}}$ contains logarithmic terms $6 \ln(a^2 M^2) - 2 \ln(a^2 \mu^2)$, while others do not have $\mu$ dependence. The operator $O_M$ does not exist in the matching of the static operator, but it becomes necessary for NRQCD.

Figure 1 shows the heavy quark mass dependence of $\zeta_{L,X}$ for the dimension six operators. Filled symbols at $1/a M_0 = 0$ are calculated for the static action. We find a large slope in $1/a M_0$ for $\zeta_{L,L}$ as shown in the top panel of Figure 1. Most of the mass dependence comes from factorized diagrams, i.e. the four-quark operator can be split into axial-vector currents and the gluon propagator does not connect these two. As a result, the coefficient $\zeta_{L,L}$ is much reduced if we subtract the contributions of two axial vector current, as plotted in the lower panel ($\zeta_{L,L} - 2 \zeta_A$). This combination will appear in the analysis of the $B$ parameters, when we consider the ratio $\langle O_L \rangle / (A_0)^2$.

The coefficients for the dimension seven operators are obtained in the static limit as $\zeta_{L,LD} = -17.20$ and $\zeta_{L,ND} = -9.20$.

3.2. $O_S$

The matching of $O_S$ is given by

$$O_S^{\text{MS}}(\mu) = \left[ 1 + \frac{\alpha_s}{4\pi} \zeta_{S,S}(\mu; 1/a) \right] O_S^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{S,L}(\mu; 1/a) O_L^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{S,P}(1/a) O_P^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{S,R}(1/a) O_R^{\text{lat}}(1/a)$$

$$+ \frac{\alpha_s}{4\pi} \zeta_{S,T}(1/a) O_T^{\text{lat}}(1/a)$$

$$+ \alpha_s O_{P D}^{\text{lat}}(1/a)$$

$$+ \alpha_s O_{S,SD}^{\text{lat}}(1/a)$$

$$+ \alpha_s O_{S,LD}^{\text{lat}}(1/a)$$

$$+ \alpha_s O_{S,ND}^{\text{lat}}(1/a).$$

(10)

where $O_T$ is new for the NRQCD action. The coefficients $\zeta_{S,X}$ are plotted in Figure 2. Like $\zeta_{L,L}$ in the matching of $O_L$, the leading operator $O_S$ has large coefficient, which is reduced by subtracting the factorized contribution, as shown in the upper panel.

The coefficients for the dimension seven operators are $\zeta_{S,SD} = -6.88$, $\zeta_{S,LD} = 2.58$ and $\zeta_{S,PD} = 1.15$ in the static limit.
4. Reanalysis of $B$ Parameters

With these perturbative coefficients, we reanalyze our previous lattice calculation for the $B$ parameters, $B_L$ and $B_S$. $\text{[3]}$.

4.1. $B_L$

The $B$ parameter $B_L$ is calculated through

$$B_{\text{MS}}^L(\mu) = \sum_X Z_{L,X}/A^2(\mu; 1/a)B^{\text{lat}}_X(1/a),$$

(11)

where $X$ runs over $L, S, N, M$ and $R$. The renormalization constants are

$$Z_{L,L}/A^2(\mu; 1/a) = 1 + \frac{\alpha_s}{4\pi}(\zeta_{L,L}(\mu; 1/a) - 2\zeta_A(1/a)),$$

$$Z_{L,S}/A^2(1/a) = \frac{\alpha_s}{4\pi}\zeta_{L,S}(1/a),$$

$$Z_{L,N}/A^2(1/a) = \frac{\alpha_s}{4\pi}\zeta_{L,N}(1/a),$$

$$Z_{L,M}/A^2(1/a) = \frac{\alpha_s}{4\pi}\zeta_{L,M}(1/a),$$

$$Z_{L,R}/A^2(1/a) = \frac{\alpha_s}{4\pi}\zeta_{L,R}(1/a),$$

and the lattice $B$ parameters $B^{\text{lat}}_X(1/a)$ were measured in the numerical simulations through

$$B^{\text{lat}}_X(1/a) = \frac{(B_q|O^{\text{lat}}_X(1/a)|B_q)}{\bar{\sum}(B_q|A^{\text{lat}}_X(1/a)|0\rangle\langle 0|A^{\text{lat}}_X(1/a)|B_q\rangle},$$

(12)

in our previous paper $\text{[3]}. \text{[4]}$. Since the dimension seven operators are not included in the calculation, we could not remove the error of $O(\alpha_s a\Lambda_{QCD})$.

Figure $\text{[3]}$ shows the individual contributions $Z_{L,X}/A^2(\mu; 1/a)B^{\text{lat}}_X(1/a)$ for the $B$ parameter $B_L$. The $V$-scheme coupling constant $\alpha_V(q^*)$ is used with $q^* = 2/a$. Although there is a sizable mass dependence in $Z_{L,X}/A^2(\mu; 1/a)$ as we can see from the plot of coefficients (Figure 1), it is canceled by the $1/m_B$ corrections in the matrix elements on the lattice $B^{\text{lat}}_X(1/a)$, and there is little mass dependence in the products shown in Figure $\text{[3]}$.

Our result for $B_{L,\text{MS}}^\text{MS}(\mu)$ is plotted in Figure $\text{[3]}$ as a function of $1/M_F$ with filled circles. The statistical error shown by solid error bars is very small except for the heaviest heavy quark, while the systematic error given by dotted error bars is more significant. We estimate the systematic errors with an order counting, in which the systematic errors of $O(\alpha_s^2), O(\alpha_s a\Lambda_{QCD}), O(\alpha_s^2\Lambda_{QCD}/M), O(\alpha_s^2\Lambda_{QCD}/M)^2$, and $O((\Lambda_{QCD}/M)^2)$ are considered and added in quadrature. For comparison, we also show a result with one-loop coefficients calculated for the static heavy quark by open circles, for which a large systematic error of $O(\alpha_s/(aM))$ is added. The effect of the NRQCD renormalization constants is sizable, but still it is within the estimated systematic error.

The numerical results for $B_L$ at physical $B$ meson mass are

$$B_{L,\text{MS}}^\text{MS}(m_b) = 0.85(3)(11),$$

(13)

$$B_{L,\text{MS}}^\text{MS}(m_b) = 0.87(2)(11),$$

(14)

where the first error is statistical and the second is systematic one ($\sim 13\%$). The SU(3) breaking effect is

$$\frac{B_{L,\text{MS}}^\text{MS}(m_b)}{B_{L,\text{MS}}^\text{MS}(m_b)} = 1.01(1),$$

(15)

where only the statistical error is quoted.
4.2. $B_S$

Using Eq. (10), $B_S$ is obtained as

$$
\frac{B_{S/M}^{MS}(\mu)}{\mathcal{R}_d(\mu)^2} = \sum X Z_{S,X/\Lambda^2}(\mu; 1/a) B_{X/A}^{lat}(1/a), \tag{16}
$$

where $X$ runs over $S, L, R, P$ and $T$. In the LHS, we define a ratio of heavy-light matrix elements

$$
\mathcal{R}_q(\mu) = \left| \frac{\langle 0 | A_{S,R}^{MS} | B_q \rangle}{\langle 0 | A_{S,R}^{lat}(\mu) | B_q \rangle} \right|. \tag{17}
$$

The renormalization constants are

$$
Z_{S,S/\Lambda^2}(\mu; 1/a) = 1 + \frac{\alpha_s}{4\pi} (\zeta_{S,S}(\mu; 1/a) - 2\zeta_A(1/a)),
$$

$$
Z_{S,L/\Lambda^2}(\mu; 1/a) = \frac{\alpha_s}{4\pi} \zeta_{S,L}(1/a),
$$

$$
Z_{S,R/\Lambda^2}(1/a) = \frac{\alpha_s}{4\pi} \zeta_{S,R}(1/a),
$$

$$
Z_{S,P/\Lambda^2}(1/a) = \frac{\alpha_s}{4\pi} \zeta_{S,P}(1/a),
$$

$$
Z_{S,T/\Lambda^2}(1/a) = \frac{\alpha_s}{4\pi} \zeta_{S,T}(1/a).
$$

For $B_S$ we define the lattice $B$ parameters as

$$
B_{X/A}^{lat}(1/a) = \frac{\langle B_q | O_{X/A}^{lat}(1/a) | B_q \rangle}{\langle B_q | A_4^{lat}(1/a) | B_q \rangle} \langle B_q | A_4^{lat}(1/a) | B_q \rangle, \tag{18}
$$

The mass dependence of $B_{S/M}^{MS}/\mathcal{R}_s$ is plotted in Figure 4 (filled squares). Open squares are our previous results calculated with the static renormalization constants. The mass dependence is less significant in the results with the correct (NRQCD) renormalization constants.

The numerical results are

$$
\frac{B_{S,M}^{MS}(m_b)}{\mathcal{R}_d(m_b)^2} = 1.24(3)(16), \tag{19}
$$

$$
\frac{B_{S,R}^{MS}(m_b)}{\mathcal{R}_s(m_b)^2} = 1.24(3)(16), \tag{20}
$$

and

$$
\frac{B_{S,M}^{MS}(m_b)\mathcal{R}_d(m_b)^2}{B_{S,R}^{MS}(m_b)\mathcal{R}_s(m_b)^2} = 1.003(4). \tag{21}
$$

for the SU(3) breaking ratio.

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