Abstract

We consider the Klein-Gordon equation with minimal coupling in FRW-like spacetimes, with compact space sections (not necessarily isotropic nor homogeneous). In such spaces, the bi-scalar kernel allowing to select the positive-frequency part of any solution is developed on modes, using the eigenfunctions of the three-dimensional Laplacian. Of course this kernel is not unique, but any choice of it turns out to be manifestly invariant under the isometries of space sections. In the generic case (excluding a special law of evolution for the scale factor), spacetime has no more symmetries (connected with the identity) than those inherited from space sections. Therefore, in this case, any choice of the kernel is also invariant under the spacetime isometries connected with identity. It is associated with a definition of hermitian scalar product that enjoys the same invariance properties, entailing that spacetime symmetries connected with the identity are unitarily represented. Since all possible kernels are invariant and related one to another by a unitary transformation, we contemplate the possibility that all these definitions of the vacuum correspond to different representations with the same physical content.

1 Quantization in curved spacetime
1.1 Isometric Invariance Principle

Our goal is a theory of quantum free fields, in a given spacetime; before undertaking the construction of Fock space we need to define the Hilbert space suitable for describing the motion of a single particle. Therefore we consider here the Klein-Gordon (KG) equation

\[(\nabla^2 + m^2)\Psi = 0\] (1)

for a complex valued wave function \(\Psi(x)\), describing the minimal coupling of a scalar particle with gravity. The sesquilinear form

\[(\Phi; \Psi) = \int j^\mu(\Phi, \Psi) d\Sigma_\mu\] (2)

constructed from the Gordon current \(j^\nu(\Phi, \Psi)\) is conservative with respect to changes of hypersurface \(\Sigma\) provided \(\Phi\) and \(\Psi\) are solutions to the KG equation. But it is not positive definite. In order to exhibit a candidate for one-particle Hilbert space, the linear space of solutions must be split in two subspaces. In one of them (further identified as positive-frequency space) the restriction of \((\Phi; \Psi)\) must be definite positive.

The trouble is that, in nonstationary spacetimes, such splitting is not unique. Nevertheless, criteria for its determination have been given soon, either in terms of finding a real linear operator \(J\) with \(J^2 = -\text{Id}\), determining a complex structure in the space of real solutions \([1]\), or alternatively in terms of a projector \(\Pi^+ = \frac{1}{2}(1 + iJ)\) which projects any complex solution into the positive-frequency subspace. Application of \(J\) or \(\Pi^+\) to a solution \(\Psi\) is carried out with the help of a kernel \([2]\), existence of which was proved by C. Moreno \([3]\) under very general global assumptions.

But in spite of the enormous amount of literature up to now devoted to quantization in curved background, it seems that the role of spacetime isometries has not received all the attention it deserves \([4]\), with an exception for de Sitter spacetime \([5]\).

In view of the fundamental role played by Poincaré group with respect to quantum mechanics in Minkowski spacetime, we propose that quantization in any curved spacetime should satisfy this Principle:

Quantum mechanics of free particles must be invariant under all spacetime symmetries continuously connected with the Identity.

For simplicity we consider here only continuous isometries, and postpone a discussion about discrete ones.

1.2 The positive-frequency kernel

The theory of retarded and advanced Green functions has been thoroughly investigated for many years. These objects are unambiguously defined under very

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general assumptions (global hyperbolicity).
More problematic is the kernel $D^\pm$ which allows for defining positive-frequency (resp. negative-frequency) solutions of the KG equation through the formula

$$(\Psi^\pm(y) = (D^\pm_y; \Psi) = \int J^\alpha(D^\pm_y, \Psi) d\Sigma^\alpha \quad (3)$$

where $\Psi^\pm = \Pi^\pm \Psi$ is the positive-frequency part of $\Psi$, and $y$ is an arbitrary point of $V_4$. Of course, $D^\pm$ must satisfy the KG equation in both arguments; the notation with subscript $y$ indicates that integration is performed on the variable $x$. Kernel $D^+$ has fundamental importance, for the quantum field of the particle must be defined as annihilation operator associated with the one-particle state $D^+$. By isometric invariance we mean

$$D^+(x,y) = D^+(T x, T y)$$

for any metric-preserving transformation $T$ of the connected isometry group.

When spacetime has the topology of $\mathbf{R} \times V_3$, with $V_3$ compact and connected, Moreno’s theorem ensures that many $D^+$ actually exist (whatever may be the local form of the metric). Two such kernels are related by a unitary transformation. But if spacetime admits Killing vectors, we cannot yet be satisfied with any choice of $D^+$, before we answer the question whether this kernel is invariant by action of the isometries of $(V_4, g)$. This issue remains an open problem for arbitrary metric, but in the sequel we focus on a class of spacetimes where isometries are under control and the KG equation is separable.

2 Quantization in Generalized FRW spacetimes

We assume that the spacetime manifold is $V_4 = \mathbf{R} \times V_3$ with connected and compact $V_3$, and for some time scale $t$

$$ds^2 = B^0 dt^2 - B^2 d\sigma^2 \quad (4)$$

where $B$ is a strictly positive function of $x^0 = t$, and $d\sigma^2 = \gamma_{ij}(x^k)dx^i dx^j$ defines an elliptic metric. Notice that $(V_3, \gamma)$ may have arbitrary curvature. The form (4) provides a generalization of Friedmann-Robertson-Walker line element. Lorentzian manifolds with such a metric have been systematically studied. Beyond the conventional FRW case, they can represent a universe filled with a non-perfect fluid characterized by anisotropic pressures. They have two nice properties:

— In the generic case $(V_4, g)$ has no Killing vector beside trivial ones corresponding to the isometries of $(V_3, \gamma)$. Cases where additional Killing vectors arise require exceptional laws of evolution for the scale factor; they naturally include the de Sitter manifold.
—The KG equation is separable for special solutions associated with "modes"; the time dependence of mode solutions is determined by an ordinary differential equation of second order. This situation stems from existence of a certain first integral, proportional to kinetic energy. Indeed the three-dimensional Laplacian $\Delta_3$ associated with metric $\gamma$ commutes with $\nabla^2$, thus also with the operator in l.h.s. of (1). This fact permits to reduce the wave equation into a one dimensional problem for time dependence, supplemented with an elliptic spectral problem for space dependence.

2.1 Kernel at a given mode

By extension of the usual terminology we define Mode $n$ as the linear space $H_n$ of solutions to (1) that are also eigenfunctions of $-\Delta_3$ with eigenvalue $\lambda_n$. We also refer to $H_n$ as "kinetic-energy shell". Separation of frequencies is supposed to respect the mode decomposition, therefore in each shell $H_n$ we look for projectors $\Pi^\pm_n$ associated with kernels $D^\pm_n$. Notice that $H_n$ is orthogonal to $H_l$ for $l \neq n$ in the sense of the sesquilinear form (2).

In this Section kinetic energy is kept fixed and the label $n$ referring to a determined eigenvalue $\lambda_n$ is provisionally dropped.

Let us consider a mode solution, say $\Phi$. For some nonnegative $\lambda \in \text{Spec}(V)$ we have $\Delta_3 \Phi = -\lambda \Phi$ so (1) reduces to

$$(\partial^0 \partial_0 + \lambda B^{-2} + \mu) \Phi = 0$$

with $\partial^0 = B^{-6} \partial_0$ and $\mu = m^2$. The space variables $x^j$ can be ignored in solving (5) for $\Phi$. This equation is always of second order. It is well-known [10] that the eigenspace $E$ of $-\Delta_3$ in $C^\infty(V)$, associated with the eigenvalue $\lambda$ has finite dimension, say $r$.

Let $S$ be the two-dimensional space of $C^\infty$ complex-valued functions of (single variable) $t$ satisfying the equation $(\partial^0 \partial_0 + \lambda B^{-2} + \mu) f = 0$ and let the functions $f_1(t)$ and $f_2(t) = f_1^*$ form a basis of $S$. They respectively span the one-dimensional subspaces $S^{(1)}$ and $S^{(2)}$.

We can always chose our notation and normalize the basis of $S$ according to the Wronskian condition $W(f_1, f_2) = -i$, which amounts to associate $f_1, f_2$ respectively with positive and negative frequencies. Call admissible such a basis.

With this convention we proved, Prop.3 in [9], that the restriction of $(\Phi; \Phi)$ to $S^{(1)} \otimes E$ is positive definite (resp. negative, in $S^{(2)} \otimes E$).

The Hilbertian scalar product is defined as $\langle \Phi, \Psi \rangle = \pm (\Phi; \Psi)$ respectively in $S^{(1)} \otimes E$ and $S^{(2)} \otimes E$. It crucially depends on the splitting we have performed (the choice of an admissible basis) in $S$. Notice that $S^{(1)} \otimes E$ and $S^{(2)} \otimes E$ are mutually orthogonal in the sense of this scalar product as well as in the sense of the sesquilinear form (2).
If $\Psi$ is a positive-frequency solution in mode $n$ we must have, restoring now the mode label

$$((D_n^+)_{y};\Psi) = \Psi(y)$$

where the notation $(D_n^+)_{y}$ indicates that $D_n^+$ is considered as a function of $x$ which additionally depend on $y$. Let $E_{1,n},...,E_{r,n}$ be a real orthonormal basis of $E_n$. Use the notation $x = (t,\xi)$, $y = (u,\eta)$ with $\xi,\eta \in V_3$.

The only possibility for $D_n^+$ takes on the standard form

$$D_n^+(y, x) = f_1^*(u)f_{1,n}(t)\Gamma_n(\eta, \xi)$$

where the expression

$$\Gamma_n(\eta, \xi) = \sum E_{a,n}(\eta)E_{a,n}(\xi)$$

is real and does not depend on the choice of a real orthonormal basis in $E_n$. It is intrinsically determined by the spectral properties of $V_3$.

It is straightforward to check that expression (6) of $D_n^+$ actually satisfies equation (3) as it should. The only arbitrariness in formula (6) is in the factor $f_1^*(u)f_{1,n}(t) = f_{2,n}(u)f_1(t)$ which depends on the choice of an admissible basis in the two-dimensional space $S_n$. For usual FRW spacetimes ($V_3$ is of constant curvature), a basis of $E_n$ can be found in the literature [10].

Let us now turn to isometric invariance. If $T$ is an isometry of the spatial metric $\gamma_{ij}$, it acts on functions according to $T F = F(T\xi)$. Invariance of $\Delta_3$ entails that each eigenspace $E_n$ is globally invariant. Moreover $T$ leaves invariant the three-dimensional scalar product $((F, G))$. Thus $E_{1,n}(T\xi),...,E_{r,n}(T\xi)$ is another real orthogonal basis of $E_n$. Finally $\Gamma_n(T\eta, T\xi) = \Gamma_n(\eta, \xi)$ and we can write $D_n^+(T y, T x) = D_n^+(y, x)$.

In the generic case all isometries of $V_4$ are lifts of spatial isometries, so we summarize:

The only kernel $D_n^+$ solution of the wave equation (1), eigenfunction of $-\Delta_3$ for eigenvalue $\lambda_n$ and satisfying (3) where $\Psi$ is a solution in mode $n$, is given by (6) and defined up to a change of admissible basis in the two-dimensional complex space $S_n$ (Bogoliubov transformation). In the generic case, it is isometrically invariant.

### 2.2 Sum over modes

Define $\mathcal{H}^+ = \bigoplus S_n^{(1)} \otimes E_n$. The infinite sum $\Phi = \sum_{n=1}^{\infty} \Phi_n$ where $\Phi_n \in \mathcal{H}_n$, always exists in the distributional sense, if we define as test functions the sums $\Psi = \sum \Phi_n$ having an arbitrary but finite number of terms (terminating sums). This definition is invariant under spatial isometries. In this sense we can assert:
The only kernel solution of the wave equation, mode-wise defined and satisfying (3) is given by $D^+ = \sum_{n=0}^{\infty} D^+_n$. It is defined up to a unitary transformation $U = \bigoplus U_n$ where $U_n$ is an arbitrary Bogolubov transformation in mode $n$. In the generic case, $D^+$ is invariant under the continuous isometries of $V_4$.

3 Conclusion

As expected from the work of several authors, the positive-frequency kernel is defined up to a unitary transformation. For generalized FRW spacetimes we have additionally proved that this kernel is invariant under all spacetime isometries connected with the identity, except perhaps for very special forms of the scale-factor evolution. de Sitter manifold is one such exception. This does not contradict existence of a de Sitter invariant vacuum, but rather indicates that a mode-wise definition of $D^+$ is no longer satisfactory when the separation of space from time fails to be unique.

Returning to the generic case, it can be easily read off from (6) that the connected group of spacetime isometries acts unitarily in $H^+$ whatever is the choice of admissible basis made in each $S_n$. This fact strongly suggests to consider that all isometrically invariant definitions of the one-particle sector are equivalent representations of the same physics.

When expansion is statically bounded in past and future, this point of view amounts to by-pass the customary "in and out" vacua in favor of a unique class of observers, not submitted to asymptotic conditions. This scheme at least provides a reasonable approximation insofar as the curvature gradient in $V_3$ remains small and the scale factor does not vary too rapidly.

Notice that in some particular cases, among all equivalent definitions, a distinguished vacuum may arise after all, in agreement with recent works.

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