Entanglement at a Scale and Renormalization Monotones

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We study the amount of information in quantum field theory states reduced to a region, that cannot be recovered from its subregion density matrices. We reconstruct the density matrix from its subregions using two approaches: scaling maps and recovery maps. The vacuum of a conformal field theory is the fixed point of both transformations. We define the entanglement of scaling and the entanglement of recovery as measures of entanglement that are intrinsic to the continuum limit. Both measures increase monotonically under the renormalization group flow. This provides a unifying information-theoretic structure underlying the different approaches to the renormalization monotones in various dimensions.

In recent years, the techniques and intuitions from quantum information-theory have proven to be immensely helpful in the study of many-body quantum systems. The entanglement structure of the low energy states of local Hamiltonians is a key concept in simulating lattice systems in condensed matter, the study of order parameters in phase-transitions, and constructing renormalization monotones in relativistic quantum field theories.

The renormalization group (RG) flow is the process in which one integrates out the ultraviolet (UV) high energy degrees of freedom, and compensates for them by adjusting the coupling constants such that the low energy physics is unchanged. Since the information about the UV modes are washed out, one might expect that the RG flow is irreversible. RG monotones are functions that reflect this irreversability as they change monotonically under the flow.

The study of RG monotones in relativistic quantum field theory (QFT) was started by the seminal work of Zamolodchikov [1], where he showed that the two point function of stress tensor in 2d QFT is a monotonic function of scale. In four dimensions, it was conjectured by Cardy in [2], and later proved in [3], that the a-anomaly term is an RG monotone. In two and three dimensions, the strong subadditivity (SSA) of entropy was used to show that there are universal terms in the entanglement entropy of vacuum in QFT reduced to a ball-shaped region that are RG monotones [4]. At the moment, the approaches to construct RG monotones seem to depend on the dimensionality of the spacetime, and a framework that works for all dimensions is missing.

In field theory, scaling is a unitary operation that allows us to compare the reduced density matrices on subsystems of different size. In this paper, we use scaling and the recovery maps of quantum information theory to quantify the amount of long-range quantum correlations at a scale. As a crucial step, we show that the vacuum of a conformal field theory reduced to a null cone can be recovered perfectly from its subregions using both maps. That is to say, it is a scale-invariant quantum Markov state. We define the entanglement of scaling and the entanglement of recovery as two measures whose first derivative quantifies the long-range entanglement. Both of these functions increase monotonically under the RG flow. Our monotonic functions are generalizations of the 2d and 3d entanglement monotones to higher dimensions. They provide a unifying information-theoretic approach to RG monotones in various dimensions. Furthermore, it points to a connection between recovery maps in quantum information theory and the RG transformation of states that goes beyond the construction of monotones.

1 We start by reviewing some notions and tools in quantum information theory.

Measuring asymmetry

Consider a many-body finite quantum system split into $n$ non-overlapping regions $A_1$ to $A_n$, with isomorphic Hilbert spaces on $A_i$. The relabeling of the subsystem index $i$ is a unitary operation in the global Hilbert space: $\otimes_{i=1}^{n} \mathcal{H}_i$. A simple example of such a unitary is the translation defined by $i \rightarrow i + 1 \mod n$:

$$U = \sum_{a_1 \cdots a_n} |a_2 \cdots a_n a_1\rangle \langle a_1 \cdots a_n|,$$

where $\{a_i\}$ is the basis that spans $\mathcal{H}_i$. The density matrix $\rho_i$ on $A_i$ is mapped to $A_{i+1}$ with the local unitary

$$\rho_{i+1} = \mathcal{E}(\rho_i) = U_i^\dagger \rho_i U_i$$

$$U_i = \sum_{a_i,a_{i+1}} |a_{i+1}\rangle \langle a_i|.$$  \hspace{1cm} (1)

If the transformation sends a subsystem $A$ to $\tilde{A}$, and the state is asymmetric under this transformation, some information about $\rho_A$ will be lost. The relative entropy $S(\rho_A || \mathcal{E}(\rho_A))$ is a measure of the amount of information in $\rho_A$ that is lost. It is non-negative, and vanishes if and only if $\rho_A$ is symmetric under the transformation.

1 While this manuscript was in preparation, the papers [5, 6] appeared, which have overlaps with some results presented here.
Measuring non-Markovianity

Imagine that we are probing the global state with detectors that are localized in $A_1A_2$. The von Neumann entropy $S(\rho_{12})$ is a measure of the amount of quantum information $\rho_{12}$ is missing about a pure global state. If we made a larger detector that allows us access to the region $A_1A_2A_3$, then the new detector teaches us $S(A_3|A_1A_2)$ more qubits of information. The quantity $S(A|A') \equiv S(AA') - S(A)$ is the conditional entropy. Another way to gain more information is by moving our detectors to adjacent sites $A_2A_3$. This gives us access to both $\rho_{12}$ and $\rho_{23}$; however, we are still missing the long-range correlations between $A_1$ and $A_3$. We would like to quantify the amount of quantum information ("entanglement") about in $\rho_{123}$ that is neither in $\rho_{12}$ nor in $\rho_{23}$. Naively, one can say that by moving the detector we have learned $S(A_3|A_2)$ but there are still more qubits in $\rho_{123}$ that we are missing. This quantity is the conditional mutual information (CMI), and is non-negative by the SSA inequality [2].

A careful study of the operational question of how well one can guess $\rho_{123}$ from the knowledge of $\rho_{12}$ and $\rho_{23}$ (the marginals) suggests that this naive estimate (CMI) is, indeed, a good measure of the amount of quantum information. All the correlations between $A_1A_2\ldots A_n$ are quantum Markov states, and satisfy the following property:

$$S(\phi_{123}) = I(A_1 : A_3 | A_2).$$

In our terminology, these Markov states have no entanglement at any scale larger than $r$.

Intuitively, a quantum Markov chain is scale-invariant, in the sense that all the information in a density matrix of size $R$ can be recovered perfectly from subsystems of size $r < R$. This suggests that quantum Markov states should appear naturally as the fixed points of the renormalization group flow.

**ENTANGLEMENT OF SCALING**

The states of a quantum field theory are wavefunctionals of fields: $\Psi(\phi(x))$. The transformations $f : x^\mu \mapsto x'^\mu + \xi^\mu$ (diffeomorphisms) are the generalization of the relabeling operation in finite systems to the continuum limit. Analogously, diffeomorphisms act on the global state as unitary operators: $|\tilde{\psi}\rangle = e^{i f} d^{d+1}x d\xi e^{-i T_{\mu\nu}} |\psi\rangle$, where $\Sigma$ is the spacelike surface where the state lives, and $T_{\mu\nu}$ is the stress tensor. If we split the degrees of freedom into a subregion $A$ and the complement, then the unitary that maps the reduced state on $A$ to the reduced state on $\bar{A}$ is:

$$U = \int [D\phi]_\phi |(f^{-1})^* \phi\rangle \langle \phi|$$

where $(f^{-1})^*$ is the pull-back of functions from $A$ to $\bar{A}$.
A familiar example of such diffeomorphisms is the generalization of translations in finite systems to the continuum limit. In quantum field theory, the translations are described by the unitaries $U = e^{i\alpha P}$, which map $\rho_A$ to $\hat{\rho}_A$:
\[
\langle \phi_a(x_{\in A})|\rho_{A,g}|\phi_b(x_{\in A}) \rangle = \langle (f^{-1})^*\phi_a|\rho'_{A,g}|(f^{-1})^*\phi_b \rangle,
\]
where $\hat{g} = (f^{-1})^*g$ is the transformed metric. If the translation is a symmetry of the background metric, and the state then the density matrix changes only by a unitary rotation.

In the remainder of this work, we will be interested in how local dilatations act on null cones. In polar coordinates, this maps $f: (t,r) \rightarrow (e^{\lambda(r)}t,e^{\lambda(r)}r)$, and leaves the perpendicular directions $\Omega$ untouched; see figure 1. Take a ball on the time slice $t = R$ centered at $r = 0$. The boundary of this ball is on the null cone defined by $r - t = 0$. The dilatation $f$ with constant $\lambda$ rescales the size of the ball from $R$ to $e^\lambda R$, and moves it from $t = R$ to $t = e^\lambda R$. The metric transforms by an overall conformal factor: $\hat{g} = e^{2\lambda}g$. If the state is conformally invariant, for instance in a conformal field theory (CFT) vacuum in flat space, one can ignore the change of the metric, and the state remains unchanged up to a unitary. To simplify the notation, we denote the unitarily scaled density matrix from $R$ to $R'$ by
\[
\hat{\rho}_{R'} \equiv \mathcal{E}(\rho_R) = U^\dagger \rho_R U,
\]
where $R'$ has been suppressed in the notation, and will be clear from the context.

We are interested in a quantum field theory that is a deformation of a CFT by a relevant operator of scaling dimension $\Delta < d$
\[
S_{QFT} = S_{CFT} + \lambda_0 \int d^d x \mathcal{O}(x),
\]
where $\lambda_0 = \mu^{d-\Delta} g_0$ is the dimensionful coupling at the UV length scale $\mu$. Diffeomorphism invariance allows us to compare $\rho_R$, the reduced states on a ball of size $R$, to a smaller ball $\rho_r$ rescaled back to $R$. In the UV $(r/\mu \ll 1)$, the state $\rho_r$ can be approximated well by the CFT vacuum state with corrections proportional to the coupling $\lambda_0$. The modular operator of $\rho_r$ can be computed in the conformal perturbation theory. It remains local in spacetime, to the first order in $\lambda_0$. The relative entropy $S(\rho_R||\mathcal{E}(\rho_r))$ is a measure of the amount of distinguishability lost under the dilatation. We define the entanglement of scaling to be
\[
S_{sc}(\rho_R) = \lim_{r \to 0} S(\rho_R||\mathcal{E}(\rho_r)).
\]
The entanglement of scaling is, by definition, non-negative. It can be defined for any region of space whose boundary lies on a null cone. Similar to the entanglement entropy, the entanglement of scaling is invariant under any unitary operations: $S_{sc}(\rho) = S_{sc}(U^\dagger \rho U)$.

**MARKOV STATES IN QFT**

Take a quantum field theory density matrix $\rho_R$. If it is a quantum Markov state\(^2\), it can be perfectly recovered from its smaller marginals $\rho_r$, for any $r < R$. This suggests that there is no new physics at any length scale in between the $r$ and $R$. In other words, it is scale-invariant in that range. One might expect that the CFT vacuum reduced to ball-shaped regions are quantum Markov states. In this section, we show that this intuition is indeed correct.

Start with a ball-shaped region $A$ in a CFT vacuum state, and make two geometric deformations $f_a$ and $f_b$. The state will be Markovian if the CMI $I(\delta A_a;\delta A_b|A)$ vanishes for any finite size deformation. This quantity was computed in a perturbation theory in small deformations by [10]. They find the CMI to be
\[
I(\delta A_a;\delta A_b|A) = \delta A_a^i \delta A_b^j 2\pi^2 C_T (d + 1) R^2 |\Omega_a - \Omega_b|^{2(d - 1)}
\]
where $\eta_{ij}$ and $\delta A_a^i$ and $\delta A_b^j$ are, respectively, the metric and the area elements in the $t,r$, directions, and $C_T$ is the coefficient in the two-point function of the stress tensor. For a generic deformation, this CMI is non-zero. However, if we take the deformed ball to be on a null cone, that is $\xi = \xi^a(\Omega)\partial_a$, the CMI is proportional to $\eta_{uu}$ which is zero in flat space. This leaves the possibility that for null deformations the vacuum state is Markovian. To demonstrate this, we use the method of the Euclidean path-integrals to construct the density matrix to all orders in perturbation theory in the deformation size, and show that it changes only

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\(^2\) In the remainder of this paper, we use the words Markov chain and Markov states synonymously.
by a unitary transformation. In fact, it is pedagogical to start with a simpler example:

**Ex. 1: QFT vacuum on half-space:**

As the first example, we show that the QFT vacuum in flat space reduced to a half-space is a quantum Markov state with respect to null deformations; see figure 1. Consider the vacuum of a $d > 2$ dimensional QFT in flat space $ds^2 = du^2 + dv^2 + dz_dz^d$, with $u = y + t$ and $v = y - t$ the null directions. We reduce the state to the region A, the $y > 0$ half-space. The modular operator of this region, $K_A \equiv -\log \rho_A$, is local \cite{17}. On the null surface $v = 0$, it has the form

$$K_A \equiv -\log \rho_A = \int d^{d-3}z \int_0^\infty du u T_{uu}. \quad (13)$$

In Euclidean QFT, the density matrix $\rho_A$ is represented by a path-integral on $\mathbb{R}^d$, with boundary conditions above and below A in the Euclidean time; i.e. $(\tau_E = 0^\pm, y > 0)$ \cite{13}. One can split the $x$ direction into $n$ slabs $A_i = (x_i, x_{i+1})$, and insert the resolutions of identity in between slabs; see figure 2

$$\rho = \prod_{i=1}^N \langle D[\phi_i] \rho_i(\phi_i, \phi_{i+1}), \rho_{i,i+1}(\phi_i, \phi_{i+1}) = \langle \phi_i | \rho_i | \phi_{i+1} \rangle. \quad (14)$$

Here, $\rho_i(\phi_i, \phi_{i+1})$ is an operator (transfer) matrix that acts only on the subsystem $A_i$.

We apply a diffeomorphism that is non-zero only at $A_a$ and $A_b$, and deforms $A$ to $\tilde{A} = A + \delta A + \delta \tilde{A}$. The density matrix of $\tilde{A}$ is given by $\rho_{\tilde{A},\tilde{\eta}} = U^\dagger \rho_{A,\eta} U$, where $g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \partial_\alpha \xi_\mu \partial_\alpha \xi_\nu$, and $\eta$ is the flat metric \cite{13}. We take $f$ to be a translation in a null direction localized on two slabs $I_a$ and $I_b$:

$$f_a : u \mapsto u + \lambda f(x_a), \quad (15)$$

with $f(x_a)$ a function that has a peak at the center of $A_a$, and goes to zero on the boundaries of $I_a$ at $x_a$ and $x_a + 1$.\footnote{One might worry about the fact that the function $f$ is not infinitely differentiable. We will be ignorant of such subtleties here.} The flat metric changes by $g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \partial_\alpha \xi_\mu \partial_\alpha \xi_\nu$, which is nonzero only inside the slab $I_a$ and vanishes on the boundaries $\partial I_a$. Partitioning the path-integral of $\tilde{\rho}_{\tilde{A}}$ according to (14) and comparing with $\rho_A$, only the transfer matrices $\rho_a$ and $\rho_b$ have changed. Let us focus on the matrix elements of one of these operators, $\tilde{\rho}_a$:

$$\langle \phi^1 | \partial I^+_a \rangle | \tilde{\rho}_a(\phi_a, \phi_{a+1}) | \phi^2 | \partial I^+_a \rangle = \int \phi(x_a) = \phi + \phi(x_{a+1}) = \phi(a^+_1, \phi(a^+_2) = \phi^2 | D[\phi] e^{-S[\phi, g]} \rangle, \quad (16)$$

where $\partial I^+_a$ are the boundaries at $x \in A_a$ and $\tau_E = 0^\pm$; see figure 2. The path-integral above is on $I_a$ that has five boundaries in the Euclidean $\mathbb{R}^{d+1}$. Two boundaries at $x = x_a$, $x = x_{a+1}$, two boundaries at $\partial I_a^+$ and $\partial I_a^-$, and a fifth boundary at $y^2 + \tau_E^2 = \epsilon$ which is a small cylinder cut around $y = \tau_E = 0$.

The only difference between the path-integrals for $\tilde{\rho}_a$ and $\rho_a$ is in the metric that goes into the action. We Taylor expand the action around the flat space

$$S[\phi, g] = \exp \left( \int_{I_a} \partial^\mu \xi^\nu \frac{\delta}{\delta g^{\mu\nu}} \right) S[\phi, \eta] = \exp \left( -\int_{I_a} \xi^\nu \partial_\mu \frac{\delta}{\delta g^{\mu\nu}} + \int_{\partial I_a} dS_\mu \xi^\nu \frac{\delta}{\delta g^{\mu\nu}} \right) S[\phi, \eta],$$

where we have used the integration by parts, and $dS_\mu$ is the normal to the boundary $\partial I_a$. The term with the integral over $I_a$ vanishes, due to the fact that $\partial_\mu \frac{\delta}{\delta g^{\mu\nu}} S[\phi, g] = \partial_\nu T^{\mu\nu}$, which is identically zero.

The change in the metric under the diffeomorphism by $f_a$ is in the $g^{ux}$ component, and since $\xi^u$ has only $u$ components, only the two boundaries at constant $x$ contribute to (17). However, we chose $\xi$ to vanish on these boundaries; therefore $S[\phi, g]$ on $I_a$ can be replaced with its flat space value $S[\phi, \eta]$. Hence, the transfer matrices in the partitioned path-integral in (14) do not change:

$$\tilde{\rho}_a(\phi_a, \phi_{a+1}) = \rho_a(\phi_a, \phi_{a+1}). \quad (17)$$

Hence, there is a unitary that rotates the overall density matrix $\rho_A$ to $\tilde{\rho}_{\tilde{A}}$:

$$\tilde{\rho}_{\tilde{A}} = (\mathbb{I} \otimes U^\dagger_a \otimes U^\dagger_b) \rho_A (\mathbb{I} \otimes U_a \otimes U_b).$$

On the null surface ($x = t$) the algebra of operators factorizes \cite{19}, and the operators $\rho_i$ and $U^\dagger \rho_i U$ commute. Hence, one can take the logarithm of the above identity to find the modular operator of the deformed region:

$$K_{A + \delta_a A} = K + (U^\dagger_a K_a U_a - K_a) + (U^\dagger_b K_b U_b - K_b).$$

**FIG. 2:** (a) Partitioning the Euclidean path-integral into slabs in the $x$ directions. (b) The path-integral over each slab has five boundaries. Two boundaries at $x_1$ and $x_{i+1}$, two at $\partial I^+$ and $\partial I^−$ where the state lives, and one infinitesimal cylinder cut around the origin at $y = \tau_E = 0$. 
As a result the modular operator for two null deformation of half-space \( A \) by \( \delta_a A \) and \( \delta_b A \) satisfies

\[
K_{A+\delta_a A+\delta_b A} = K_A + \delta_a A + K_A + \delta_b A - K_A
\] (18)

This establishes that the QFT vacuum on half-space deformed in null-directions is a quantum Markov state.

**Ex. 2: CFT vacuum on a null cone:**

There is a conformal transformation that maps the causal development of a half-space \( A \) to the causal development of a ball \( B \) [17]. If \( K_A \) and \( K_B \) are, respectively, the modular operators of subsystems \( A \) and \( B \), there exists a unitary such that \( K_B = U^\dagger K_A U \). Under this conformal transformation, the deformed half-space \( A + \delta_a A \) is mapped to a deformed ball \( B + \delta_a B \); see figure [1]. Deformations on the null surface in \( A \) are sent to deformations of \( B \) on the null-cone. The equation (18) with \( A \) replaced with \( B \) can be written for null deformations of a ball in CFTs of arbitrary dimensions. As a result, the vacuum of a \( d \)-dimensional CFT is a quantum Markov state with respect to deformations on a null cone.

In 2d CFTs, any state that is a descendant of vacuum with arbitrary time-dependence is related to vacuum by a conformal transformation, and remains a quantum Markov state. It is straightforward to check that SSA is saturated in these states from the expressions in [20].

**Near Markov States**

Before applying the SSA inequality to the states of a quantum field theory, we would like to have an analogue of CMI that is insensitive to the ultraviolet details. We replace the entanglement entropies in CMI with the entanglement of scaling:

\[
I_{sc}(A_1 : A_2) \equiv S_{sc}(\rho_{12}) + S_{sc}(\rho_{23}) - S_{sc}(\rho_2) - S_{sc}(\rho_{123})
\]

\[
= I_{\rho R}(A_1 : A_2) - \lim_{r \to 0} I_{\rho R}(A_1 : A_2, A_3)
\]

\[
= I(A_1 : A_2) \geq 0,
\] (19)

where we have used the fact that the UV CFT state is Markovian.

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*ENTANGLEMENT AT A SCALE*

In this section, for simplicity we restrict to vacuum state of QFTs in flat space.\(^5\) The goal is to find a “good” information-theoretic measure that quantifies the entanglement at a scale, that is to say it is insensitive to the UV, and it has an operational interpretation. A measure of entanglement at scale \( R \) is a function that \( \rho_R \) and its derivatives \( \partial_R^n \rho_R \). Here, we compare three candidate measures that appear natural from an information-theory point of view:

1. The obvious candidate is the relative entropy \( S(\rho_{R+\delta R}|E(\rho_R)) \). This quantity vanishes at the first order in \( \delta R \), due to the smoothness of relative entropy. At the second order, it becomes the quantum Fisher information which is a metric in the space of density matrices:

\[
S(\rho_{R+\delta R}|\rho_R) = (\delta R)^2(\delta R, \delta R)R + O((\delta R)^3).
\]

It is finite, non-negative at any \( R \), and vanishes in CFTs. It is a metric, and hence satisfies the triangle inequality. Quantum Fisher information has an interpretation in terms of distinguishability, as it is the variation of a relative entropy.

2. The second candidate is the derivative \( \partial_R S_{sc}(\rho_R) \). It is finite, and non-negative at any \( R \) (see the supplementary material for a proof):

\[
\partial_R S_{sc}(R) \geq 0.
\] (20)

As we argued before this quantity is expected to be insensitive to the UV details, and has the benefit that its integral, \( S_{sc} \), resembles a smoothed-out version of \( S_{UV} - S_{IR} \).

3. The third candidate, the information-theorist’s favorite, is based on recovery maps and SSA. The task is to quantify how well one can recover the state \( \rho_{R+\delta R} \) from the knowledge of all balls of size \( R \) within the causal development of \( \rho_{R+\delta R} \). That is to say, we want to build a ball of size \( R + \delta R \) from the iteration of a recovery map which acts on balls of size \( R \). One way to do this was introduced in [4]. Take two balls with boundaries on a null cone. As we bring the balls close in the angular directions on the cone, the distance between \( \delta_a A \) and \( \delta_b A \) tends to \( R \). The CMI measures the entanglement at scale \( R \). To obtain the larger \( \rho_{R+\delta R} \) we have to apply the recovery map many times following [4], and add up the CMI contributions at each step. The total sum

\(^5\) The generalization of the measures introduced here to arbitrary states requires minor, but straightforward modifications.
of the CMI we obtain as we repeat this recipe is the quantity that we define to be the derivative of the entanglement of recovery

$$\partial_R S_{rec}(\rho_R) \equiv ((d - 3)\partial_R + R\partial^2_R)S_{sc}(R) \geq 0. \quad (21)$$

It is a measure of the entanglement in the vacuum of QFTs at the scale $R$, that has an operational interpretation in terms of recovery. It vanishes in a CFT vacuum. Integrating this quantity from the UV to the scale $R$ we obtain

$$S_{rec}(R) = (d - 2 - R\partial_R)S_{sc}(\rho_R). \quad (22)$$

**RENORMALIZATION MONOTONES**

We are encouraged by [21] to look for an RG monotone in arbitrary dimensions that has the following properties

1. It is a finite dimensionless quantity, and regularization independent.

2. It decreases monotonically along the flow.

3. If the flow ends in an IR fixed point, the value of the function can only depend on quantities that are intrinsic to the UV and IR fixed points.

Both the entanglement of scaling and the entanglement of recovery satisfy the first two criteria:

$$\partial_R S_{sc}(R) \geq 0$$
$$\partial_R S_{rec}(R) \geq 0. \quad (23)$$

In all the known examples in $2d$ and $3d$ they also satisfy the third criterion. It is unclear to us, whether this continues to be the case in all dimensions.

In $2d$ and $3d$ they do indeed reduce to all the known monotones. The entanglement of scaling, $S_{sc}(R)$, is a smoothed version of the RG monotone defined in [22], which is the relative entropy of vacua in two different CFTs. While intuitive, the smoothness of $S_{sc}(R)$ deserves further investigation. We believe that studying the entanglement of scaling in more detail can shed light on the UV divergences in the quantity in [22] for the particular range of the deformation scaling dimensions $\Delta > (d + 2)/2$.

The entanglement of recovery, $S_{rec}(R)$, is a smoothed version of the entanglement monotones in $2d$ and $3d$ introduced in [4] generalized to arbitrary dimension. As this work was in its final stages, we learned about the work in [6] that generalizes the previous entanglement proof to the a-theorem in four dimensions. It is of great interest to relate the entanglement of recovery to other known quantities of CFTs in $d > 4$.

**CONCLUSIONS**

In this work, we studied a connection between recovery maps in quantum information theory, and the renormalization group flow in quantum field theories. Applying information-theoretic tools, and taking advantage of the diffeomorphism invariance of QFT, we constructed candidate functions for the entanglement at a scale. Two new entanglement measures intrinsic to the continuum limit, the entanglement of scaling and the entanglement of recovery were defined. They are built to be insensitive to the UV details, and the first derivative of both measures quantify the amount of entanglement at scale. However, the more natural quantity from the point of view of the recovery maps is the entanglement of recovery. Both quantities are monotonic under a change of scale.

A better understanding of the RG monotones in higher dimensions can be achieved by studying these quantities and relating them to the properties of the IR conformal fixed point.

It is tempting to rewrite the entanglement of scaling in the language of the algebraic QFT as

$$\lim_{\lambda \to 0} (|\Omega\rangle \Delta_{\Omega, U_\lambda'|\Omega⟩}, \quad (24)$$

and avoid referring to the density matrix. Here, $|\Omega⟩$ is the state of a QFT, and $\Delta_{\Omega, U_\lambda'}$ is the relative modular operator of the two states with respect to a region, and $U_\lambda$ generates dilatation by factor $\lambda$. We postpone a further investigation of this, and potential connections between the entanglement of scaling and the renormalized entanglement entropy [23] to future work. Furthermore, since our approach views RG as an operation on a QFT state, the RG monotones we find characterize a particular flow from the UV to the IR. An interesting question to explore is whether this quantity can be read off, directly from a CFT Hilbert space.

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The entanglement of scaling is monotonic

We are interested in the derivative:

$$\lim_{\mu \to 0} \partial_R S(\rho_R \| E(\rho_\mu)) \geq 0. \quad (25)$$

We start by proving that the operations, $E$ and $N$ commute: $N(\epsilon(\rho)) = \epsilon(N(\rho))$. Split the system in two
parts: the part that is traced out $A$, and the remaining part $B$. The matrix elements of $\mathcal{E}(tr_A \rho)$ are

\[ \int [D\psi_A]_A \langle \psi_A(f^{-1})^* \phi_B^+ | \rho | \psi_A(f^{-1})^* \phi_B^- \rangle. \] (26)

After a change of variables this is equal to

\[ \int [D(f^{-1})^* \psi_A] \langle (f^{-1})^* \psi_A(f^{-1})^* \phi_B^+ | \rho | (f^{-1})^* \psi_A(f^{-1})^* \phi_B^- \rangle. \]

which is nothing but $tr_A \mathcal{E}(\rho)$.

Relative entropy is monotonic under a partial trace: $\mathcal{N}_{R\to R-\delta R}$. We have

\[ S(\rho_R || \mathcal{E}(\rho_\mu)) \geq S(\mathcal{N}(\rho_R) || \mathcal{N}\mathcal{E}(\rho_\mu)) \]
\[ = S(\mathcal{N}(\rho_R) || \mathcal{N}\mathcal{E}(\rho_\mu)) \]
\[ = S(\rho_{R-\delta R} || \mathcal{E}(\rho_\mu) + \mu \mathcal{E}(\delta \rho_\mu)) \] (27)

Taking the limit $\mu \to 0$ we establish that

\[ \partial_R S_{sc}(R) \geq 0. \] (28)

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