Peculiarity of the $^{12}$C($0^+$) and $^{12}$C($2^+$) energy spectrum in a $3\alpha$ model

E. M. Tursunov

Institute of Nuclear Physics, Academy of Sciences, 100214, Ulugbek, Tashkent, Uzbekistan

I. Mazumdar

2 Dept. of Nuclear & Atomic Physics, Tata Institute of Fundamental Research, Mumbai 400 005, India

Abstract

Lowest energy spectrum of the $^{12}$C nucleus is analyzed in the $3\alpha$ cluster model with a deep $\alpha\alpha$-potential of Buck, Friedrich and Wheatley with Pauli forbidden states in the $S$ and $D$ waves. The direct orthogonalization method is applied for the elimination of the $3\alpha$-Pauli forbidden states. The effects of possible first order quantum phase transition are shown in the lowest $^{12}$C($0^+_1$) and $^{12}$C($2^+_1$) states from weakly bound phase to a deep phase. The ground and lowest $2^+$ states of the $^{12}$C nucleus in the deep phase are created by the critical eigen states of the Pauli projector for the $0^+$ and $2^+$ three-alpha functional spaces, respectively.

PACS numbers: 11.10.Ef,12.39.Fe,12.39.Ki
I. INTRODUCTION

The studies in structure and structural evolution of atomic nuclei continue to be one of the major aspects of low and medium energy nuclear physics. The richness of nuclear structure and the spectrum of various modes of excitations have led to the development of different models to understand the underlying mechanisms guiding the response of the nucleus. The plethora of nuclear models developed from as early as the thirties have been guided by experimental data that show both single particle and collective behavior of the nuclei. In addition, the richness is further enhanced by the observation of interplay of single particle and collective behavior. The evolution of nuclear structure with physical observables like, angular momentum, isospin, temperature are associated with phase transitions, both continuous and discontinuous. The studies in different types of phase transitions happen to be one of the central themes of nuclear structure studies. These include both Quantum Phase Transitions (QPT) at zero temperature \[1\text{–}4\] and phase transitions associated with nuclei at finite temperature and angular momentum \[5\text{–}8\]. Abrupt changes in the ground state shapes from spherical or very small deformations to large deformed shapes with change in neutron (N) numbers for a given element have been observed and theoretically explained to be due to quantum phase transition \[9\]. Such changes are associated with massive reorganization of the proton and neutron orbitals and is understood in terms of a QPT. Variation in some non-thermal control parameter (say, number of nucleons) is responsible for inducing such abrupt phase transitions. QPT driven by variation in some defined or order parameter has also been studied in cases of spontaneous fission of heavy nuclei \[10\].

The behavior of atomic nuclei have resulted in variety of nuclear models. The alpha cluster model has been associated with the light even-even nuclei which can justifiably be modeled as an ensemble of alpha particles. The $^{12}$C nucleus occupies a pre-eminent position in the list of nuclei which support alpha-cluster states. Arguably, the 7.65 MeV, $0^+$ excited state, well known as the Hoyle state, is the most famous cluster state of the atomic nucleus. There has been a large body of work trying to describe the cluster states in nuclei like $^{12}$C and other self conjugate nuclei like $^{16}$O, $^{20}$Ne etc. One of the interesting properties of these nuclei is their special structure, associated with the Bose-Einstein condensation \[11\]. Another special structure is connected with the QPT found in the \textit{ab-initio} calculations \[1\]. Broadly, the cluster models can be categorized into microscopic and macroscopic approaches.
with their fair share of success, simplicity, failure and difficulties. On the one side, within the microscopic models the ground and the first $2^+$ excited states of $^{12}\text{C}$ are strongly over-bound by about 4-6 MeV in comparison with the experimental energy values \[12\]. Only a complicated nonlocal $\alpha\alpha$ potential derived from the resonating group model calculations is able to reproduce the energies of the ground state and the Hoyle ($0^+_2$) resonance \[13\]. The macroscopic models treat the system like $^{12}\text{C}$ as an ensemble of structure-less alpha particles and use both shallow and deep local potentials and also non-local potentials for the binary $\alpha - \alpha$ systems. The success of these methods in producing the $\alpha - \alpha$ phase shifts and $^8\text{Be}$ resonances and difficulties in terms of producing experimental binding energies and removing spurious, redundant states in the $^{12}\text{C}$ have been studied and discussed in detail by several authors \[11, 13–16\].

Although the $3\alpha$ cluster model for the structure of the lowest $^{12}\text{C}$ states seems very natural due to strong binding of nucleons inside the $\alpha$-clusters, there are serious problems, associated with a realistic modeling of Pauli forces. Repulsive local $\alpha\alpha$-potentials, both $l$-dependent and $l$-independent, strongly underestimate the bound states of the $^{12}\text{C}$ nucleus \[15\]. The application of the alternative local deep $\alpha\alpha$-interaction potential of Buck-Friedrich-Wheatley (BFW) \[17\] requires a careful treatment of the Pauli forbidden states (FS). The method of orthogonalizing pseudopotentials (OPP) \[18\] is a powerful technic for the elimination of forbidden states in a three-body system. The wave functions of the $^6\text{He}$ and $^6\text{Li}$ nuclei calculated in the $\alpha + N + N$ three-body model based on the OPP method, have been successfully applied to the study of the beta decay of $^6\text{He}$ halo nucleus into the $\alpha + d$ continuum \[19, 20\], and the astrophysical capture reaction $\alpha + d \rightarrow ^6\text{Li} + \gamma \[21, 23\]$. However, the $3\alpha$ quantum system is strongly different from the $\alpha + N + N$, which contains a single $\alpha$-particle. Here $\alpha\alpha$-Pauli forbidden states play a decisive role in the description of dynamics of the $3\alpha$ system. Indeed, within the OPP method it was found \[16, 24\] that the energy spectrum of the ground $0^+_1$ and first excited $2^+_1$ states is highly sensitive to the description of the $\alpha\alpha$-Pauli forbidden states. From the results of the calculations it was not possible to understand how to fix the energies of the ground and excited levels, since a convergence in respect to the projecting constant $\lambda$ was not clear. When passing values of $\lambda = 10^4 - 10^6$ MeV the energies of the $0^+$ states show a non-analytical behaviour: for the $\lambda = 10^4$ MeV the energies of lowest states -16.106 MeV, -0.422 MeV, 1.353 MeV change to the values -0.435 MeV, 1.407 MeV and 3.316 MeV for $\lambda = 10^6$ MeV. In other words, the lowest state with the energy
-16.106 MeV was lost, which is not usual. The energies of the lowest $2^+$ states show similar behaviour. Again the lowest state with the energy $E(2_1^+) = -15.649$ was lost.

For understanding the enigmatic behaviour of the $^{12}\text{C}(0^+)$ spectrum the authors of Ref. [16] have applied more transparent direct orthogonalization method of eliminating Pauli forbidden states from the $3\alpha$ functional space. They have found that there are two so-called almost forbidden states (AFS) as eigen states of the three-body Pauli projector, which play a decisive role for the $^{12}\text{C}(0^+)$ spectrum. So, if these AFS are included into the $3\alpha$ allowed functional subspace, then the energy of the ground state is about -20 MeV, while it is -0.20 MeV if these AFS are accepted as forbidden $3\alpha$ basis states. In order to avoid the AFS problem, the authors of above work suggested to use the $\alpha\alpha$- forbidden states derived from underlying microscopic theory and not to use the FS of the BFW potential. Such a way gives normal three-body FS (as in other three-body systems like $\alpha + 2N$) contrary to the three-body FS derived from the initial $\alpha\alpha$-potential. However, they still yield a strong overbinding of the $3\alpha$ ground state. Moreover, from physical viewpoint, this way is not realistic, since the forbidden states should be associated with two-body potentials which describe the experimental data, energy spectrum of bound and resonance states, and phase shifts. Since the BFW potential yields a very nice description of the experimental data for the $\alpha\alpha$- scattering and the $^{8}\text{Be}$ resonances, the specific properties of the $3\alpha$ spectrum associated with the Pauli projecting could be connected with a strong physics which is still not well understood.

The aim of present work is to study peculiar properties of the $^{12}\text{C}(0^+)$ and $^{12}\text{C}(2^+)$ energy spectrum associated with removing Pauli forbidden states from the $3\alpha$ functional space. A deep $\alpha\alpha$-potential of BFW will be employed. Differently from the Faddeev equation method in Ref. [16] we use a variational method on symmetrized Gaussian basis. For the elimination of the $3\alpha$ Pauli forbidden states we use the same direct orthogonalization method from Ref. [16] where only $^{12}\text{C}(0^+)$ lowest states have been studied. We will examine a similarity of the $0^+$ and $2^+$ spectrum including the Hoyle band. As a possible origin of above mentioned non-analytical behaviour of the $^{12}\text{C}$ spectrum, consequences of the QPT in the $^{12}\text{C}$ nucleus will be discussed.

The theoretical model is described in Section 2. Sections 3 and 4 contain the numerical results for the $^{12}\text{C}(0^+)$ and $^{12}\text{C}(2^+)$ spectrum, respectively. A discussion of the results is given in Section 5 and conclusions are drawn in the last section.
II. THEORETICAL MODEL

The direct orthogonalization method \[16\] is based on the separation of the complete Hilbert functional space into two parts. The first subspace \(L_Q\), which we call allowed subspace, is defined by the kernel of the complete three-body projector. The rest subspace \(L_P\) contains 3\(\alpha\) states forbidden by the Pauli principle. After the separation of the complete Hilbert functional space of 3\(\alpha\) states into the \(L_Q\) (allowed) and \(L_P\) (forbidden) subspaces, at next step we solve the three-body Schrödinger equation in \(L_Q\).

The \(\alpha\alpha\)-interaction potential of Buck-Friedrich-Wheatley (BFW) \[17\] has a simple Gaussian form

\[
V(r) = V_0 \exp(-\eta r^2) + 4e^2 erf(br)/r, \tag{1}
\]

with parameters \(V_0=-122.6225\) MeV, \(\eta = 0.22\) fm\(^{-2}\) for the nuclear part and \(b=0.75\) fm\(^{-1}\) for the Coulomb part. This choice of the potential parameters yields a very good description of the experimental phase shifts \(\delta_L(E)\) for the \(\alpha\alpha\)-elastic scattering in the partial waves \(L = 0, 2, 4\) within the energy range up to 40 MeV and the energy positions and widths of the \(^8\)Be resonances. Hereafter we use a value \(\hbar^2/m_\alpha = 10.4465\) MeV fm\(^2\) for comparison with the results of Ref.\[16\]. This potential contains two Pauli forbidden states in the \(S\) wave with energies \(E_1 = -72.6257\) MeV and \(E_2 = -25.6186\) MeV, and a single forbidden state in the \(D\) wave with \(E_3 = -22.0005\) MeV. For the realistic description of the system one has to eliminate all FS from the solution of the three-body Schrödinger equation.

The three-body Hamiltonian in the 3\(\alpha\)-cluster model reads:

\[
\hat{H} = \hat{H}_0 + V(r_{23}) + V(r_{31}) + V(r_{12}), \tag{2}
\]

where \(\hat{H}_0\) is the kinetic energy operator and \(V(r_{ij})\) is the interaction potential between the \(i\)-th and \(j\)-th particles. A solution of the Schrödinger equation

\[
\hat{H}\Psi_{sJ}^{JM} = E\Psi_{sJ}^{JM}, \quad \Psi_{sJ}^{JM} \in L_Q. \tag{3}
\]

should belong to the allowed subspace \(L_Q\) of the complete 3\(\alpha\) functional space.

The wave function of the 3\(\alpha\)-system is expanded in the series of symmetrized Gaussian functions \[24\]:

\[
\Psi_{sJ}^{JM} = \sum_{\gamma_j} C_j^{(\lambda J)} \varphi_{\gamma_j}, \tag{4}
\]
where \( \varphi_{\gamma j} = \varphi_{\gamma j}(1; 2, 3) + \varphi_{\gamma j}(2; 3, 1) + \varphi_{\gamma j}(3; 1, 2) \),

\[
\varphi_{\gamma j}(k; l, m) = N^{(\lambda l)}_{j} x_{k}^{\lambda} y_{k}^{l} \exp(-\alpha_{\lambda j} x_{k}^{2} - \beta_{lj} y_{k}^{2}) F_{JM}^{\lambda l}(\tilde{x}_{k}, \tilde{y}_{k})
\]  

(5)

Here \((k; l, m) = (1; 2, 3), (2; 3, 1), (3; 1, 2)\), \(\gamma = (\lambda, l, J, M) = (\gamma_{0}, J, M)\); \(\tilde{x}_{k}, \tilde{y}_{k}\) are the normalized Jacobi coordinates in the \(k\)-set:

\[
\tilde{x}_{k} = \sqrt{\mu/\hbar}(\vec{r}_{l} - \vec{r}_{m}) \equiv \tau^{-1} \vec{r}_{l,m};
\]

\[
\tilde{y}_{k} = \frac{2\sqrt{\mu/\hbar}}{\sqrt{3\hbar}}(\vec{r}_{l} + \vec{r}_{m}/2) - \vec{r}_{k} \equiv \tau^{-1} \vec{\rho}_{k},
\]

(6)

\(N^{(\lambda l)}_{j}\) is a normalizing multiplier. The nonlinear variational parameters \(\alpha_{\lambda j}, \beta_{lj}\) are chosen as the nodes of the Chebyshev grid:

\[
\alpha_{\lambda j} = \alpha_{0} \text{tg}(\frac{2j - 1}{2N_{\lambda}} \pi), j = 1, 2, ... N_{\lambda},
\]

\[
\beta_{lj} = \beta_{0} \text{tg}(\frac{2j - 1}{2N_{l}} \pi), j = 1, 2, ... N_{l},
\]

(7)

where \(\alpha_{0}\) and \(\beta_{0}\) are scale parameters for each \((\lambda l)\) partial component of the complete wave function. The angular part of the Gaussian basis is factorized as:

\[
F_{JM}^{\lambda l}(\tilde{x}_{k}, \tilde{y}_{k}) = \{Y_{\lambda}(\tilde{x}_{k}) \bigotimes Y_{l}(\tilde{y}_{k})\}_{JM} \phi(1) \phi(2) \phi(3),
\]

(8)

where \(\phi(i)\) is the internal wave functions of the \(\alpha\)-particles. Here the orbital momenta \(\lambda\) and \(l\) are conjugate to the Jacobi coordinates \(\tilde{x}_{k}\) and \(\tilde{y}_{k}\), respectively.

The kinetic energy operator of the Hamiltonian can be expressed in the normalized Jacobi coordinates in a simple form as

\[
\hat{H}_{0} = -\frac{\partial^{2}}{\partial x_{k}^{2}} - \frac{\partial^{2}}{\partial y_{k}^{2}}
\]

(9)

within any choice of \((\tilde{x}_{k}, \tilde{y}_{k})\), \(k = 1, 2, 3\). The matrix elements of the kinetic energy operator and the interaction potentials have been given in Ref. [25].

In order to separate the complete \(3\alpha\) functional space into the \(L_{Q}\) and \(L_{P}\) we calculate eigen states and corresponding eigen values of the projecting operator [16]

\[
\hat{P} = \sum_{i=1}^{3} \hat{P}_{i},
\]

(10)

where each \(\hat{P}_{i}, (i = 1, 2, 3)\) is the sum of Pauli projectors \(\hat{\Gamma}^{(f)}_{i}\) on the partial \(f\) wave forbidden states (1S, 2S, and 1D) in the \(i\)-th \(\alpha\alpha\)-subsystem:

\[
\hat{P}_{i} = \sum_{f} \hat{\Gamma}^{(f)}_{i},
\]

(11)
with the forbidden state function expanded into the Gaussian basis:

\[
\varphi_{fm_f}(\vec{x}_i) = x_i^f \sum_m N_m^{(f)} b_m^{(f)} \exp\left(-\frac{r_i^2}{2r_0^{(f)2}}\right) Y_{fm_f}(\hat{\vec{x}}_i).
\]  

Here \(r_0\) is the "projector radius" and \(N_m^{(f)}\) is the normalizing multiplier:

\[
N_m^{(f)} = 2^{f+7/4} \frac{\alpha_m^{(2f+3)/4}}{\pi^{1/4}[(2\lambda + 1)!]^{1/2}}, \quad \alpha_m = \tau^2/(2r_0^{(f)})^2.
\]

III. \(^{12}\text{C}(0^+)\) SPECTRUM

First we calculate the \(^{12}\text{C}(0^+)\) spectrum. The three body channels \((\lambda, \ell) = (0, 0), (2, 2), (4, 4)\) contain up to 280 Gaussian functions. Convergence is fast due to the use of symmetrized basis functions. We reproduced the spectrum of the operator \(\hat{P}\) in the \(3\alpha\) functional space with \(J^\pi = 0^+\), firstly calculated in Ref.\[16\]. It belongs to the interval from 0 to 3. As was noted above, the allowed subspace \(L_Q\) is defined by the eigen states of the operator \(\hat{P}\), corresponding to its zero eigen value: \(L_Q = \text{kern}(\hat{P})\). However, in the \(3\alpha\) system this procedure is not easy due to a high sensitivity of the energy on the description of \(\alpha\alpha\)-Pauli forbidden states \[24\]. As in mentioned work, there are two eigen states of the operator \(\hat{P}\) among other eigen states, which play a decisive role for the structure of the \(^{12}\text{C}(0^+)\) lowest states. The first special eigen state \(\Phi_1\) corresponds to a small eigen value \(\epsilon_1=1.35333 \times 10^{-5}\): \(\hat{P}\Phi_1 = \epsilon_1 \Phi_1\). The second eigen state \(\Phi_2\) of the operator \(\hat{P}\) with corresponding eigen value \(\epsilon_2=1.07152 \times 10^{-3}\) is especially important for the structure of the \(^{12}\text{C}\) nucleus. Both eigen values are close to, but not exactly zero, nevertheless they strongly influence on the structure of the carbon nucleus. This makes the corresponding eigen states of the operator \(\hat{P}\) very special and in above work \[16\] they are called "almost forbidden states" (AFS). In Fig.1 we display the calculated \(0^+\) spectrum of the lowest \(^{12}\text{C}\) states as a function of \(\epsilon\), the maximal allowed eigen value of the operator \(\hat{P}\). As can be seen from the figure, the first AFS \(\Phi_1\) influences only the lowest \(^{12}\text{C}(0^+)\) spectrum. Other levels are not affected by the three-body projector, indicating that they belong to the continuum spectra or correspond to a resonance \[26\]. The first AFS \(\Phi_1\) decreases the lowest \(0^+\) energy from \(-0.278\) MeV to \(-0.627\) MeV. The next AFS \(\Phi_2\) creates a new \(3\alpha\) state with the energy of \(-19.897\) MeV, which becomes
FIG. 1. Energy spectrum of the lowest $^{12}$C($0^+$) states in dependence on the maximally allowed eigen values $\epsilon$ of the operator $\hat{P}$.

the ground state of the $^{12}$C nucleus. It is strongly bound with binding energy of about 20 MeV or underbound with the energy $E = -0.627$ MeV in respect to the $3\alpha$ threshold in dependence of that $\Phi_2$ belongs to $L_Q$ (allowed) or $L_P$ (forbidden). In Ref. [16] the situation was not accepted as physically possible. On the other hand, a non-analytical behavior of the lowest $^{12}$C($0^+$) states around the critical point $\Phi_2$ can be interpreted as a first-order quantum phase transition (QPT) from the weakly bound phase $\Psi_0$ with the energy $-0.627$ MeV to the deep phase $\Psi_1$ with the energy of $-19.897$ MeV. The situation is very close to the finding of Ref. [1] stating about the "nature near a quantum phase transition". Indeed, the QPT occurs due-to the quantum fluctuation between the two phases, which are rooted in the Heisenberg uncertainty principle [27].

Beyond the critical point $\Phi_2$ the energy of the Hoyle state $\Psi_2$ increases to $E = -0.458$ MeV, which is, however, lower than the experimental energy value $E_{exp}(0^+_2) = 0.380$ MeV. Also it can be found that on the left side of the critical point $\Phi_2$ in Fig. 1 the ground and Hoyle states coexist in the preliminary weakly bound phase $\Psi_0$. The overlap of $\Psi_0$ with the Hoyle state $\Psi_2$ is 0.992, while its overlap with the ground state in the deep phase $\Psi_1$ is 0.109. This means that the ground state deep phase is mostly created by the critical $\Phi_2$ eigen state of the operator $\hat{P}$ from the continuum. The theoretical energy of $-19.897$ MeV is significantly lower than the experimental value $E_{exp} = -7.274$ MeV [28], which indicates that the ground state can be in the deep phase $\Psi_2$ with a probability, smaller than 1. The
IV. $^{12}$C($2^+$) SPECTRUM

Now we go to the $^{12}$C($2^+$) spectrum. In this case the $2^+$ functional space of the $3\alpha$ system is built in the three-body channels $(\lambda, \ell) = (0, 2), (2, 0), (2, 2)$. Numerical calculations have been done with up to 340 symmetrized Gaussian basis functions which yield a good convergence of the results. Exactly as in the case of the $0^+$ spectrum, the projector $\hat{\mathcal{P}}$ contains two AFS $\Phi_3$ and $\Phi_4$ with corresponding eigen values $\epsilon_3 = 6.74419 \times 10^{-6}$ and $\epsilon_4 = 3.83029 \times 10^{-4}$, which play a decisive role for the $^{12}$C($2^+$) spectrum. As can be seen in Fig. 2, a behavior of the $2^+$ spectrum is very close to the behavior of the lowest $0^+$ states. The eigen state $\Phi_3$ of the operator $\hat{\mathcal{P}}$ corresponding to $\epsilon_3$, changes the second $2^+_2$ state energy from 2.578 MeV to 1.873 MeV. Again as in the previous case, the critical point $\epsilon_4$ (eigen state $\Phi_4$ of the operator $\hat{\mathcal{P}}$) creates a new deep phase with the energy $E = -16.572$ MeV, much lower than the experimental energy value $E_{exp}(2^+_1) = -2.834$ MeV. In other words, we again have a possible first order quantum phase transition from weakly bound phase $\Psi_0(2^+)$ to the deep phase $\Psi_1(2^+)$. On the right-hand side of the critical point $\Phi_4$ (or $\epsilon_4 = 3.83029 \times 10^{-4}$) the Hoyle analog state $\Psi_2(2^+)$ energy increases to $E = 2.279$ MeV, slightly lower than the experimental value $E_{exp}(2^+_2) = 2.596$ MeV. Again as in the case of the $0^+$ spectrum, on the left side of the critical point $\Phi_4$ in Fig. 2 the lowest $2^+_1$ and Hoyle analog $2^+_2$ states coexist in the preliminary weakly bound phase $\Psi_0(2^+)$. The overlap of $\Psi_0(2^+)$ with the Hoyle analog state $\Psi_2(2^+)$ is 0.876, while its overlap with the lowest $2^+_1$ phase $\Psi_1(2^+)$ is -0.334. The two $2^+$ levels in Fig. 2 with energies $\epsilon_3$ and $\epsilon_4$ are not affected by the projecting procedures, hence belong to the $3\alpha$ continuum spectrum.

V. DISCUSSION

The energy of the Hoyle state $E(0^+_2) = 0.38$ MeV [28] can be reproduced with the help of a weak three-body potential $V(\rho) = 23 \exp(-0.1 \rho^2)$ MeV, where \(\rho\) is the hyperradius. The
FIG. 2. Energy spectrum of the lowest $^{12}$C($2^+$) states in dependence on the maximally allowed eigen values $\epsilon$ of the operator $\hat{P}$.

The same potential yields for the Hoyle analog state energy a value $E_{th}(2^+_2) = 2.567$ MeV which is very close to its experimental value of 2.596 MeV.

Now we can discuss the situation that happened in Ref. [24] when applying the OPP method for the elimination of Pauli forbidden states in the 3$\alpha$ system. In this method the Hamiltonian reads

$$\tilde{H} = H_0 + \tilde{V}(r_{12}) + \tilde{V}(r_{23}) + \tilde{V}(r_{31}),$$

(15)

where the pseudopotentials

$$\tilde{V}(r_{ij}) = V(r_{ij}) + \sum_f \lambda_f \hat{\Gamma}^{(f)}_{ij}.$$  

(16)

Here $\lambda_f$ is the projecting constant, $\hat{\Gamma}^{(f)}_{ij}$ is the projecting operator to the $f$-wave forbidden state in the two-body subsystem $(i + j)$, $(i, j, k) = (1, 2, 3)$, and their cyclic permutations.

When increasing $\lambda_f$ one should have more and more repulsive interaction and all the forbidden states should go out of the allowed three-body functional space. The sensitivity of the energy on alpha-alpha forbidden states can be seen from Tables 2 and 3 of Ref. [24] for the values of projecting constant $\lambda_f = 10^4 - 10^6$ MeV. The method of OPP allows to eliminate all the Pauli forbidden states in the 3-body system with the increasing projecting constant $\lambda_f$ up to infinity. This works well for the $\alpha + N + N$ system, but does not work for the 3$\alpha$ system. Indeed, when increasing projecting constant $\lambda_f$ from $10^4$ MeV to $10^6$ MeV,
the energy of the ground state changes from -16.106 MeV to -0.435 MeV, and the energy of
the $2^+_1$ state changes from -15.65 MeV to a positive energy value (the binding was lost).

In previous sections of present paper, as well as in Ref. [16] for understanding the strong
sensitivity of the $3\alpha$ bound state energies on the description of the Pauli forbidden states a
different, direct orthogonalization method was applied which is based on the separation of
the complete Hilbert functional space into two parts: allowed and forbidden subspaces. This
method is more transparent than the OPP method. Here we can understand well what is
happening in the OPP method when increasing $\lambda_f$ to infinity. The situation was understood
as coming from the so called ”almost forbidden states” (AFS) which play a decisive role for
the spectrum of the $^{12}$C nucleus. When $\lambda_f$ goes to infinity, one completely removes these
”almost forbidden states” from the functional model space of relative motion. The situation
is in strong contrast to other systems like $^6$Li = $\alpha + p + n$, $^6$He = alpha + n + n, where one has
a convergent energy value with increasing $\lambda_f$ to infinity. In other words, in the $3\alpha$ system,
if we remove these AFS from the model space, then we have a strong underbinding. And
contrary, if we include these AFS into the model space, then we have a strong overbinding.

Thus we can state that the above effect, which we treat as possible quantum phase
transition, is not an artifact of the specific procedure used. Since the effect was seen also
with the OPP method earlier, but was fully understood only with a more transparent direct
orthogonalization method from Ref. [16]. And the Hamiltonian of OPP method contains a
parameter $\lambda_f$ (projecting constant): when increasing it one has more repulsive interaction,
when decreasing it one has more attractive interaction. In the direct orthogonalization
method we have a parameter $\epsilon$, the maximal allowed eigen value of the operator $\hat{P}$: when
increasing it one has more attractive interaction, when decreasing it one has more repulsive
interaction in the Hamiltonian. In this sense we state that the situation is very close to the
finding of Ref. [1]

An important and difficult question is, how the above possible quantum phase transition
effects in the $^{12}C(0^+_1)$ and $^{12}C(2^+_1)$ lowest states can be directly detected in the experiment.
For the nuclear interaction time of order $10^{-23} - 10^{-22}$ s the quantum fluctuation can occur
with the energy shift $\Delta E \approx \hbar/\Delta t \approx 6.6 - 66$ MeV. Our model calculations gives an energy
shift values of about 19.27 MeV and 18.45 MeV for the possible QPT in the $^{12}C(0^+)$ and
$^{12}C(2^+)$ spectrum, respectively. These estimations are very consistent with above (6.6 - 66)
MeV interval for the energy shift of quantum fluctuations.
VI. CONCLUSION

In summary, the energy spectrum of the $^{12}\text{C}$ nucleus has been analyzed within the $3\alpha$ model. The Pauli forbidden states were treated by the exact orthogonalization method. An evidence of possible first order quantum phase transition has been examined. It was shown that there are effects of possible QPT in the lowest $^{12}\text{C}(0^+_1)$ and $^{12}\text{C}(2^+_1)$ states from the weakly bound phase to a deep phase. For the $0^+$ spectrum there is a critical eigen function (critical point) and corresponding critical eigen value of the three-body projector, which is responsible for the quantum phase transition. On the left hand side of the critical point the lowest $0^+$ state mostly presents the Hoyle state, while on the right hand side of the critical point the lowest state becomes the ground state of the $^{12}\text{C}$ nucleus in the deep phase. An overlap of the critical eigen function of the three-body projector with the ground state is close to unity, while its overlap with the Hoyle state is almost zero. This means that the ground state of the $^{12}\text{C}$ nucleus in the deep phase is created by the critical eigen function of the Pauli projector. A behavior of the $2^+$ levels is analogous. The lowest $2^+$ state in a deep phase is created by the critical eigen function of the Pauli projector for the $2^+$ levels.

Main physical result is that the origin of a possible quantum phase transition is strong Pauli forces in the $3\alpha$ system. If one calculates the spectrum of the $3\alpha$, $4\alpha$, $5\alpha$, etc quantum systems in the alpha-cluster model, the same QPT effects can be seen. These QPT effects are not specific due-to the orthogonalization method, but rather due to the Pauli forces. They can be seen in any alpha-cluster model with exact treatment of the Pauli principle.

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