The Relationship between Extremum Statistics and Universal Fluctuations

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The normalized probability density function (PDF) of global measures of a large class of highly correlated systems has previously been demonstrated to fall on a single non Gaussian “universal” curve. We derive the functional form of the “global” PDF in terms of the “source” PDF of the individual events in the system. A single parameter distinguishes the global PDF and is related to the exponent of the source PDF. When normalized, the global PDF is shown to be insensitive to this parameter and importantly we obtain the previously demonstrated “universality” from an uncorrelated Gaussian source PDF. The second and third moments of the global PDF are more sensitive, providing a powerful tool to probe the degree of complexity of physical systems.

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The study of systems exhibiting non Gaussian statistics is of considerable current interest. These statistics are observed to arise in finite sized many body systems exhibiting correlation over a broad range of scales. The apparent ubiquitous nature of this behavior has led to interest in self organized criticality [1,2] as a paradigm; other highly correlated systems include fluid turbulence. Two recent results have highlighted the connection between extremum statistics and highly correlated systems. The probability density function (PDF) of fluctuations in power needed to drive an enclosed rotating turbulent fluid at constant angular frequency has been measured over 2 decades in Reynolds number. Intriguingly, when the PDF \( P(E) \) of these series of experiments were normalized to the first two moments they were found to fall on a single non Gaussian “universal” curve [3,4]. This same universal curve was later identified in a study of the two dimensional X-Y model, a numerical model for magnetization near the critical point [5]. To obtain the universal curve, the PDF of a global measure, namely the magnetization summed over the entire system, is again normalized to the first two moments. It was suggested that these two disparate systems share the same statistics as they are both critical. The functional form of the “universal” curve was found for the X-Y model and was shown to be of the form [5,6,7]

\[
P(E) = K(e^{y-e^y})^a \quad \text{with} \quad y = b(E - s) \quad (1)
\]

with \( a = \pi/2 \) and \( K, b, s \) obtained by normalizing the curve to the first two moments. Crucially, it was then demonstrated that this curve was also in reasonable agreement with appropriately chosen normalized global measures for a range of numerical models of highly correlated systems. It was suggested that this behavior is related to the extremum statistics that arises from a process that is highly correlated.

In this Letter we give a comprehensive analysis of extremum statistics in the context of finite sized systems. Our aim is to determine the relationship between the underlying “source” PDF of a given process and the PDF of some global measure. Given that events occur over a range of sizes, and that each event represents some quantity, magnetization, or energy dissipation say, we obtain a relationship between the “source” PDF of the event size, and the PDF of a global measure, the total magnetization, or energy dissipation over the system. We find, as suggested in [1,2], that the global PDF, when normalized to the first and second moments is essentially of the form of equation (1). Crucially however we find that the “universal” curve for the global PDF, that is, equation (1) with \( a = \pi/2 \) is not uniquely a property of a source PDF of a correlated process. Instead, in a finite sized system, distributions of this form with \( a \) in the range \([1,2]\) arise from uncorrelated samples from a source PDF ranging from exponential through Gaussian to power law, the value of \( a \) being determined by the source PDF. When normalized to the first and second moments these curves are only distinguishable asymptotically. Hence in reality the “universal” curve describes, to within typical experimental or numerical statistical uncertainties, distribution (1) with \( a \) in the range \([1,2]\).

In many physical situations it is relatively straightforward to measure the PDF of some global quantity such as power dissipation in the driven turbulent fluid. In order to understand the underlying process we require details of the distribution of the source PDF. In particular, if this process is highly correlated, the source PDF of individual events is anticipated to be power law and we wish to i) distinguish this unambiguously from an uncorrelated Gaussian process and ii) measure the exponent. A direct measurement of the source PDF requires the challenging measurement of event sizes over many decades, but if we can relate the power law exponent to the form of the global PDF there is the possibility to remote sense this exponent. Normalizing the global PDF to the first and second moments is an insensitive method to find \( a \); we show that for finite sized systems the higher order moments provide a more feasible method.

The first step is to obtain the PDF of some global quantity from that of the source PDF that describes individual events. Consider a finite sized system of dimension \( D \) which at any instant in time has patches of activity on various length scales up to the system size \( L_B \). The patches are drawn from the (time independent) source probability \( N(L) \) of a patch of length \( L \). These patches can represent sites involved in an avalanche in a sandpile, vortices in a turbulent fluid, ignited trees in a forest fire, or sites with nonzero magnetization in the X-Y model. Associated with the active sites is some quantity of interest, \( Q \) say, for example energy or magnetization, which we take to be given by \( Q = L^D \). There will be some maximum \( Q_B \) corresponding to the (extremely rare) configuration with the highest possible value of \( Q \), that is, highest energy or magnetization, that can be realized by the system. The total value of \( Q \) over the system at any instant arises from the distribution of the patches at that instant \( N_j(L) \):

\[
\bar{Q}_j = \int_0^{Q_B} Q N_j(Q) dQ = \int_0^{L_B} L^D N_j(L) dL \quad (2)
\]

where \( N_j \) is the distribution of an (unknown) realizable ensemble of patches (continuous limit \( N(L) \)) that fits within the finite sized system, and the integral is over the system. Since \( N \) is normalized, \( N(L)dL = N(Q)dQ \). We now wish to evaluate the PDF of the \( \bar{Q}_j \). This arises from the many ensembles of the system, for the \( j \)th ensemble the total value of \( Q \) can alternatively be written as a sum over the \( M_i \) (unknown) individual patches \( \{L_i\} \), \( 1 \leq i \leq M_j \). If \( N_j(L) \) is monotonically decreasing (from maximum \( N_{0j} \) to zero) we can generate each of the
\{L_i\}_i$ by choosing $M_j$ random numbers $N_i$ in the range $[0, N_{0j}]$, with uniform probability distribution $P(N_i)$. If we then insist that $P(N_i) = P(L_i)$, for each realization the random $N_i$ will each lie in one of the $M_j$ uniform intervals $\delta N_i$, giving $L_i$ patches which lie in corresponding (nonuniform) intervals $\delta L_i$ obtainable in principle by inverting $N_j(L_i)$. We can then write the sum of the patches in the $j^{th}$ ensemble:

$$\bar{Q}_j = \sum_{i=1}^{M_j} \{L_i^D\}_i = \sum_{i=1}^{M_j} P(L_i)\delta L_i L_i^D \tag{3}$$

If the gradient of $N(L)$ is near monotonic, $\delta N_i/\delta L_i \simeq -(dN/dL)$ so that

$$\bar{Q}_j \simeq \sum_{i=1}^{M_j} P(N_i) \delta N_i L_i^D \equiv \int_{N_{0j}}^{1} L_i^D dN/dL \tag{4}$$

For a source PDF $N(L)$ that is exponential, Gaussian or inverse power law for large $L \ dN/dL << L^D$ for small $N$, that is, large $L$ (large $Q$). Hence the dominant contribution to $\bar{Q}_j$ is that of the largest patch of activity. Thus the statistics of the PDF of $Q$, $P(Q)$ will be extremum statistics, $P(Q) = P_m(Q)$, the normalized PDF of the maximum drawn from the ensembles. Given that the maximum for the $j^{th}$ ensemble is given by $Q_j^* = max\{Q_1,..,Q_{M_j}\}$, where $Q_{M_j} \leq Q_B$, that is, $M_j$ finite, the PDF for $Q^*$ is given by

$$P_m(Q^*) = MN(Q^*)(1 - N_>(Q^*))^{M-1} \tag{5}$$

where $M$ is the average of $M_j$ over the ensembles and

$$N_>(Q^*) = \int_{Q^*}^{\infty} N(Q)dQ \simeq \int_{Q^*}^{\infty} N(Q)dQ \tag{6}$$

We now obtain $P_m$ for large finite $M, Q$. For a general PDF $N(Q)$, $1 - N_>(Q^*)^{M} = \exp(-Mg(Q^*))$ where

$$g(Q^*) = -\ln(1 - N_>(Q^*)) \sim N_> + \frac{N_>^2}{2} \tag{7}$$

We now choose a characteristic value of $Q^*$, namely $\bar{Q}^*$, such that for any of the $j$ ensembles

$$q = Mg(\bar{Q}^*) = MN_>(\bar{Q}^*) + M\frac{N_>^2(\bar{Q}^*)}{2} + \cdots \tag{8}$$

Using this definition and the form for $g(Q^*)$ we obtain $g'(Q^*) = -N(\bar{Q}^*)$ to lowest order in an expansion in $q/M$.

We now consider specific source PDF $N(Q)$. If $N(Q)$ is Gaussian or exponential we can consider lowest order only giving $g(\bar{Q}^*) \sim N_> \tag{0}$ and $q = MN_>(\bar{Q}^*)$. After some algebra, expanding in $Q^*$ near $\bar{Q}^*$ gives

$$P(\bar{Q}) = P_m(Q) \equiv P_m(Q^*) \sim (e^{u} - e^a)^a \tag{9}$$

with

$$a = \frac{N'(\bar{Q}^*)N_>(\bar{Q}^*)}{N^2(\bar{Q}^*)} \tag{10}$$

$$u = \ln(MN_>(\bar{Q}^*)) + \frac{N(\bar{Q}^*)}{N_>(\bar{Q}^*)}\Delta Q^* \tag{11}$$

where $\Delta Q^* = Q - \bar{Q}^*$. For $N(Q)$ exponential (11) gives $a \equiv 1$ (see 0). For $N(Q)$ Gaussian we cannot obtain $a$ exactly but as we shall see it is instructive to make an estimate. Given $N(Q) = N_0 \exp(-\lambda Q^2)$ in the above we obtain $P_m = P_m(\exp(R(u))$ with

$$R = -\frac{\ln^2(q)}{4\lambda Q^2} + \bar{u} \left( 1 + \frac{2\ln(q)}{4\lambda Q^2} \right) - \frac{\bar{u}^2}{4\lambda Q^2} - ee^u \tag{12}$$

where we have used $u = -2\bar{Q}^* \Delta Q^*$ and $\bar{u} = u + \ln(q)$. To lowest order in $\Delta Q^*/Q^*$ (i.e. $Q^* \rightarrow \infty$) we have PDF $\bar{Q}$ with $a = 1$, but to next order, that is, neglecting the term in $\bar{u}^2$ only in (12) we have this PDF with

$$a \equiv 1 + \frac{2\ln(q)}{4\lambda Q^2} = 1 \tag{13}$$

Power law source PDF $N(Q)$ fall off sufficiently slowly with $Q$ that we need to go to next order in $\Delta Q^*/\bar{Q}^*$. If we consider normalizable source PDF

$$N(Q) = \frac{N_0}{(1 + Q^2)^t} \tag{14}$$

then for large $Q$ the above method yields that $P(Q)$ is given by the form (0) but with

$$u = -\ln(a) - \ln(q) - (2k - 1)\frac{\Delta Q^*}{Q^*}(1 - \frac{\Delta Q^*}{2Q^*}) \tag{15}$$

and $a = 2k/2k - 1$. To lowest order, neglecting the $(\Delta Q^*/Q^*)^2$ term (15) reduces to (11). Hence a power law source PDF has maximal statistics $P_m(Q)$ which, when evaluated to next order, have distribution (0) with a correction that is non negligible at the asymptotes, consistent with the well known result due to Frechet (11).

The above results should be contrasted with that of Fischer and Tippett 9. Central to 0 and later derivations is that a single ensemble of $NM$ patches has the same statistics as the $N$ ensembles (of $M$ patches), of which it is comprised. The fixed point of this expression for arbitrarily large $N$ and $M$ is $a = 1$ for the exponential and Gaussian PDF, and the Frechet result for power law PDF. Here, we consider a finite sized system so that although the number of realizable ensembles of the system can be taken arbitrarily large, the number of patches $M$ per ensemble is always large but finite. Importantly, the rate of convergence with $M$ depends on the PDF $N(L)$. For an exponential or power law PDF we are able to resum the above expansion exactly to obtain $a$; and convergence will then just depend on terms $O(1/M)$ and above.
This procedure is not possible for $N(Q)$ Gaussian, instead we consider the characteristic $Q^*$, that is $Q^*$ which for $M$ arbitrarily large should be large also. Rearranging $N(Q)$ to lowest order for $N(Q) = N_0 \exp(-M Q^2)$ yields $\sqrt{M Q^*} \sim \sqrt{\ln(M)}$ implying significantly slower convergence.

We now have the intriguing result that for a wide range of source PDF the PDF of a global measure $P(Q)$ is essentially a family of curves that are approximately Gumbel in form and are asymmetric with a handedness that just depends on the sign of $Q$; we have assumed $Q$ positive whereas one could choose $Q$ negative (with $L$ positive) in which case $N(Q) \to N(|Q|)$. The single parameter $a$ that distinguishes the global PDF then just depends on the source PDF of the individual events. For $N(Q)$ exponential we recover the well known result $\beta / G$.

We can now plot the “universal” curves, that is, normalized to the first two moments. Experimental measurements of a global PDF $P(E)$ normalized to $M_0$ would be plotted $M_0 P$ versus $(E-M_1)/M_2$. For the Frechet it is straightforward to show that the moments of order $a$ exist for $2k > n + 1$ and therefore these curves exist for power law of index $\infty > 2k > 3$ i.e. $1 < a < 3/2$. This is significant since processes exhibiting long range correlations typically have $k$ lower than this $\frac{1}{3}$. Inset in Figure 1 we plot the normalized Frechet PDF for $k = 3, 5, 100$ and the PDF $\frac{1}{\beta} \ln(M)$ with $a = 1$. In the limit $k \to \infty$, $a \to 1$ and the normalized Frechet PDF tends to the $a = 1$ limit of $\frac{1}{\beta} \ln(M)$, hence for $k = 100$ these are indistinguishable and differences between the PDF appear on such a plot around the mean for $k < 3$ approximately. In the main plot we show normalized distributions of the form $\frac{1}{\beta} \ln(M)$ for $a = 1, \pi/2$ and $2$. It is immediately apparent that the curves are difficult to distinguish for several decades in $P(y)$ and either numerical or real experiments would require good statistics over a dynamic range of about 4 decades which is not readily achievable.

Since the second moment $M_2$ does not exist for $k \leq 3/2$ we cannot consider curves of $a \geq 3/2$ generated by power law source PDF; however such values (in particular $a = \pi/2$) were identified for the “universal” curves in turbulence experiments and a variety of models of correlated systems. We now demonstrate that these are straightforward to produce. On Figure 1 we have over plotted (*) the global PDF generated by a source PDF that is uncorrelated Gaussian, calculated numerically. We randomly select $M$ uncorrelated variables $Q_j, j = 1, M$ and to specify the handedness of the extremum distribution, the $Q_j$ are defined negative and $N(Q)$ is normally distributed. This would physically correspond to a system where the global quantity $Q$ is negative, i.e. power consumption in a turbulent fluid, as opposed to power generation. To construct the global PDF we generate $T$ ensembles, that is select $T$ samples of the largest negative number $Q_i = \min(Q_1, \cdots, Q_M), i = 1, T$. For the data shown in the figure $M = 10^5$ and $T = 10^6$; this gives $\sqrt{Q^*} \sim \sqrt{\ln(M)} \approx 3$ so that for the Gaussian we are far from the $a = 1$ limit $\frac{1}{\beta} \ln(M)$.

The numerically calculated PDF lies close to $a = \pi/2$. Such a value of $a$ on these “universal” curves is therefore not strong evidence of a correlated process as suggested by $\frac{1}{\beta} \ln(M)$. Generally, plotting data in this way is an insensitive method for determining $a$ and thus distinguishing the statistics of the underlying physical process.

The question of interest is whether we can determine the form of the source PDF from the global PDF from data with a reasonable dynamic range. We consider two possibilities here. First, a uniformly sampled process will have the most statistically significant values on the universal curve near the peak. For both the PDF the peak
is at $u = 0$ and is at $\tilde{P}(u = 0) = Ke^{-a}$ with $K$ given by [13] and (21) respectively. The latter applies to $k > 3/2$; for smaller $k > 1$ we may use $M_0 = 1, M_1 = 0$ plus a condition on $\tilde{P}(u = 0)$ to obtain $a$. A more sensitive indicator may be the third moment of $\tilde{P}$ which after some algebra can be written as

$$M_3 = -\frac{\Psi''(a)}{(\Psi'(a))^3}$$

for a Gaussian or exponential source PDF i.e. with (14) and

$$M_3 = \left[ \frac{\Gamma(1 + \frac{a}{2}) - 3\Gamma(1 + \frac{2}{3})\Gamma(1 + \frac{3}{a}) + 2\Gamma^3(1 + \frac{3}{a})}{\left[ \Gamma(1 + \frac{a}{2}) - \Gamma^2(1 + \frac{1}{a}) \right]^3} \right]$$

for a power law source PDF i.e. with (15); the latter converging for $2k > 4$. Again these refer to one of the two possible solutions for $P(Q)$; the other solution corresponding to $y \rightarrow -y, M_1 \rightarrow -M_1$. We can compare these two methods by noting that for PDF of the form (1) with $a = 1, 2$ the corresponding values of $P_m$ differ by $\sim 7.9\%$ whereas $M_3$ differs by $\sim 32\%$. For Frechet PDF, the variation in $\tilde{P}(u = 0)$ is most significant for smaller $k$, for example with $k = 3, 4$ $\tilde{P}(u = 0)$ differs by $\sim 15\%$ whereas $M_3$ differs by $\sim 30\%$.

In conclusion, we have shown that the statistics of fluctuations in a global measure of a finite sized system, such as total energy dissipation in a turbulent fluid, or total magnetization in a ferromagnet are generally given by extremum statistics. The PDF of the global measure is then one of a family of curves whose moments have been determined in terms of a single parameter $a$ which in turn quantifies the PDF of the underlying “source” process, such as the PDF of individual energy release events or patches of magnetization. When normalized to the first and second moments these curves are insensitive to $a$ and fall close to the single “universal” curve previously identified as a property of a large class of highly correlated systems [12], over the range achieved by previous real or numerical experiments. In particular, we find that the global PDF of an uncorrelated Gaussian process is ‘Gumbel’ [13] distributed with $a \simeq \pi/2$, providing a straightforward explanation for the previously demonstrated “universality”. Finally we suggest that the peak, or the third moment of the global PDF is a more sensitive indicator of the source PDF. This is a powerful tool to probe the exponents of physical systems where the source PDF is difficult to measure but provides a signature of the degree of complexity of the system.

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FIG. 1. The normalized PDF (1) with $a = 2, \pi/2, 1$ (the right hand asymptotes of the curves intersect the ordinate in that order from left to right). Overplotted is the numerically evaluated global PDF of an uncorrelated Gaussian process. Inset are Frechet PDF normalized to the first two moments for source PDF exponents $2k, k = 3, 5, 100$, on the same scale.
