Critical exponents from parallel plate geometries subject to periodic and antiperiodic boundary conditions

José B. da Silva Jr.* and Marcelo M. Leite†

Laboratório de Física Teórica e Computacional,
Departamento de Física,
Universidade Federal de Pernambuco,
50670-901, Recife, PE, Brazil

Abstract

We introduce a renormalized 1PI vertex part scalar field theory setting in momentum space to computing the critical exponents $\nu$ and $\eta$, at least at two-loop order, for a layered parallel plate geometry separated by a distance $L$, with periodic as well as antiperiodic boundary conditions on the plates. We utilize massive and massless fields in order to extract the exponents in independent ultraviolet and infrared scaling analysis, respectively, which are required in a complete description of the scaling regions for finite size systems. We prove that fixed points and other critical amounts either in the ultraviolet or in the infrared regime dependent on the plates boundary condition are a general feature of normalization conditions. We introduce a new description of typical crossover regimes occurring in finite size systems. Avoiding these crossovers, the three regions of finite size scaling present for each of these boundary conditions are shown to be indistinguishable in the results of the exponents in periodic and antiperiodic conditions, which coincide with those from the (bulk) infinite system.

PACS numbers: 64.60.an; 64.60.F-; 75.40.Cx

* e-mail:jborba@df.ufpe.br
† e-mail:mleite@df.ufpe.br
I. INTRODUCTION

Finite-size effects manifest themselves generically whenever particles or fields are confined within a given volume whose limiting surfaces are separated by a certain distance $L$. Their size and shape can affect key properties of the system in comparison with those obtained from the $L \to \infty$ limit ("bulk system"). Perhaps the most investigated aspects are related to critical properties of finite systems [1, 2], where field-theoretic methods can be employed in the vicinity of the phase transitions taking place in the system under consideration. Experimentally, the simplest realization of such critical behavior and the role played by the finite size corrections show up in parallel plate geometries, for instance, in coexistence curves of critical films of certain fluids [3] as well as superfluid transition features (e.g., specific heat amplitudes) in confined $^4$He [4, 5]. From the theoretical viewpoint, field theory studies have been put forth to explain these effects not only for $^4$He [6], but also in thin slabs [7, 8] formed by wetting phenomena [9]. The Casimir effect has also been investigated in superfluid wetting films [10]. Plus, the recent study of some microscopic properties of finite-length cobalt nanowires [11] reveals that the influence of the finiteness is a ubiquitous theme in several properties of physical systems.

Momentum space $\epsilon$-expansion description [12] of critical properties for finite size systems was presented some time ago by Nemirovsky and Freed (NF) [13, 14]. The simplest approach uses a parallel plate layered geometry, namely, a (slab) volume of material whose limiting surfaces (plates) are of infinite extent along $(d-1)$ spatial directions and are separated by a distance $L$. This parallelepiped-shaped (e.g., magnetic) material possess a field-theoretical description of its critical behavior in momentum space which requires continuous momenta components parallel to the $(d-1)$ spatial directions and discrete “quasimomenta” along the finite size direction of the material. It is basically a combination of effects coming from volume (bulk), finite size and surface phenomena. The first two are dominant whenever the absolute value of the order parameter (field) is chosen to have the same (not specified $a$ priori) value at the limiting plates. (If an external field is allowed in addition to the bulk order parameter, and kept at a fixed value at the limiting surfaces, surface effects will become proeminent in the discussion of the subsequent criticality, beyond the simpler volume (bulk) plus finite size corrections pattern.)

These geometric restrictions can be realized as many different boundary conditions im-
plemented in the bare free propagator. The above mentioned simpler finite size correction shall interest us throughout and can be modelled when periodic and antiperiodic boundary conditions are employed. (Dirichlet and Neumann boundary conditions mimic free surfaces, are appropriate to explain finite size plus surface effects and shall not concern us in what follows.) The limitation caused by the boundary conditions provides a scaling variable \( \frac{L}{\xi_{\infty}} \), where \( \xi_{\infty} \) is the (bulk) correlation length of the infinite system. Many computations up to first order in \( \epsilon \) of amplitudes connected to Green functions have been carried out within this massive framework, as well as some universal amplitude ratios of certain thermodynamical potentials [15]. Within this context, three scaling regions induced by the limitation have been proposed. The first one is characterized by \( \frac{L}{\xi_{\infty}} > 1 \) where perturbative methods can be applied and the physics is quasi \( d \)-dimensional, characterized by bulk critical exponents but limitation dependent amplitudes. The second region corresponds to \( \frac{L}{\xi_{\infty}} \sim 1 \) and it was conjectured that the critical behavior is neither \( d \)-dimensional nor \( (d-1) \)-dimensional. The third region is associated to values of the variable \( \frac{L}{\xi_{\infty}} < 1 \). It was also argued that in this regime the physics is almost \( (d-1) \)-dimensional and usual perturbation expansions break down [14]. Another prediction stated that the normalization functions and the exponents would be the same as those found in the infinite system for the boundary conditions above mentioned.

In this work we introduce a one-particle irreducible (1PI) renormalized field-theoretic version of the \( NF \) formalism in order to investigate finite size corrections to normalization functions, fixed points, etc., at higher order in a perturbative loop expansion which are dependent upon the boundary condition on the plates. Concrete applications for periodic (PBC) and antiperiodic (ABC) boundary conditions are explored through the computations of the critical exponents \( \eta \) and \( \nu \) in finite size scaling using the diagrammatic method in momentum space, at least up to two-loop order. We improve the understanding of the three scaling regions and show that the finite size effects related to the limitation caused by the boundary conditions do not show up in the exponents themselves, although they modify the ingredients required to compute them.

We utilize massive fields obeying these boundary conditions on the plates for nonvanishing values of \( L \) corresponding to fixed finite values of the bulk correlation length. Both first region \( (L > \xi_{\infty}) \) and the second one associated to finite values of \( L (\to \xi_{\infty}) \) can be described satisfactorily within this massive framework. The remaining region is treated with massless
fields having infinite bulk correlation length. In that case, second region is realized through the limit $L \sim \xi_{\infty} \to \infty$. The third region naturally describes arbitrary finite values of $L$ and can only be approached using massless fields. The universal results obtained are shown to be valid for the three regions determined by the boundedness variable $L/\xi_{\infty}$ which interpolates from infinite to finite (not so small) values of $L$. The failure of the finite size phenomenological scaling arguments regarding the second and especially the third scaling region is demonstrated for the first time.

From our analytical expressions described essentially in terms of elementary primitives, we demonstrate that the dominant contribution of the finite size correction goes with the inverse power of $L$ only for periodic boundary condition, where dimensional crossover starts to set in the critical behavior. This pattern occurs for both massive ($t > 0$) and massless ($t = 0$) regimes, although with a larger coefficient in the last situation. Antiperiodic boundary conditions have the usual crossover at $t < 0$ in the massive theory as previously discussed by Nemirovsky and Freed [14]. Furthermore, our analytical method shows clearly the existence of a new type of crossover which takes place for $ABC$ at $t > 0$ when a term proportional to $\ln L$ becomes important for small values of $L$. On the other hand, $ABC$ in the massless regime $t = 0$ presents a power law of the type $L^{-2}$ whenever $L$ is small. Therefore massless and massive crossover regimes are completely different for $ABC$, which is demonstrated here for the first time. We show that, as long as we avoid these crossover regions for very small values of $L$, there is no breakdown of the $\epsilon$-expansion into third region and demonstrate the validity of the computation of the exponents. As far as critical exponents are concerned, the physics of the systems in the three regions is actually quasi $d$-dimensional, for the bulk critical exponents are recovered from the finite size evaluation irrespective of the boundary condition and the value of $L$, i.e., independent of the limitation variable $L/\xi_{\infty}$.

The paper is organized as follows. In Section II we describe the formalism of massive fields with fixed finite correlation length (mass). The case $L \to \infty$ corresponding to region a) is shown to smoothly reproduce the bulk exponents. An introduction to the $L \to 0$ limit and how it is related to dimensional crossover is presented as well as the result of the solution to higher loop diagrams away from the dimensional crossover region. Section III presents the computation of the critical exponents $\eta$ and $\nu$ using normalization conditions utilizing the Feynman diagrams outlined in the previous section, at least up to two-loop level. We show that they are $L$-independent.
We set the massless framework in Section IV, using normalization conditions as well as minimal subtraction. We discuss the behavior of the integral for certain values of $L$ and compute them in the form suitable for each renormalization scheme. The exponents obtained from the setting of massless fields are computed in Section V. For infinite values of $L$, we show that our results correspond to region b). Finite values of $L$ are shown to be equivalent to region c). The complete equivalence with the exponents computed using massive fields is established.

In Section VI we discuss our results and point out future potential applications of the method to approaching other types of critical behaviors with simple boundary conditions. Higher loop Feynman integrals are presented in the appendixes. The massive integrals in normalization conditions are described in Appendix A. In Appendix B we display the massless integrals in normalization conditions and in minimal subtraction.

II. MASSIVE FIELDS FOR PBC AND ABC IN THE NF APPROACH

In this section we begin with a quick review of the NF setup [14] in order to describe our computation of the critical exponents explicitly for periodic and antiperiodic boundary conditions. These boundary conditions realize the simplest situation in the discussion of finite size effects inasmuch they do not include the effect of free surfaces.

The layered system can be described by the bare Lagrangian density

$$L = \frac{1}{2} |\nabla \phi_0|^2 + \frac{1}{2} \mu_0^2 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4,$$  

(1)

where $\phi_0$, $\mu_0$ and $g_0$ are the bare order parameter, mass (where $\mu_0^2 = t_0$ is the bare reduced temperature) and coupling constant, respectively [16–18]. The coordinates are decomposed in the form $x = (\bar{\rho}, z)$ where $\bar{\rho}$ is a $(d - 1)$-dimensional vector characterizing the surface of each plate and the $z$ direction corresponds to the region perpendicular to them. The plates are parallel and layered in the region between $z = 0$ and $z = L$. The field satisfies $\phi_0(z = 0) = \phi_0(z = L)$ for periodic boundary conditions, whereas $\phi_0(z = 0) = -\phi_0(z = L)$ for antiperiodic boundary conditions. The order parameter can be expanded in Fourier modes as $\phi_0(\bar{\rho}, z) = \sum_{j=-\infty}^{\infty} \int d^{d-1}k \exp(i\bar{k}.\bar{\rho}) u_j(z) \phi_0j(\bar{k})$, where $\bar{k}$ is the momentum vector associated to the $(d - 1)$-dimensional space, $u_j(z)$ are the normalized eigenfunctions of the operator $\frac{d^2}{dz^2}$ whose eigenvalues $\kappa_j$ defined by $-\frac{d^2u_j(z)}{dz^2} = \kappa_j^2 u_j(z)$ are called the quasi-momentum along the
z-direction. In addition, the eigenfunctions obey the relations 
\[ \sum_{j=-\infty}^{\infty} u_j(z)u_j(z') = \delta(z - z') \]
and \[ \int_0^L dz u_j(z)u_j^*(z) = \delta_{jj'} \]. Note that \( \kappa_j = \sigma(j + \tau) \), where \( \sigma = \frac{2\pi}{L} \), \( j = 0, \pm 1, \pm 2, \ldots \), the label \( \tau = 0 \) corresponds to PBC and \( \tau = \frac{1}{2} \) to ABC. The free bare massive propagator \( (\mu_0^2 \neq 0) \) in momentum space for either boundary condition is given by the expression
\[ G_{0j}^{(\tau)}(k, j) = \frac{1}{k^2 + \sigma^2(j + \tau)^2 + \mu_0^2}. \]

Since a typical Feynman integral involves the product of many bare propagators \( G_{0j}^{(\tau)} \), the Feynman rules are modified as follows: beyond the standard ten sorial couplings of the infinite theory corresponding to a \( N \) component order parameter, each momentum line (propagator) must be multiplied by the tensor \( S_{j_1j_2j_3j_4}^{(\tau)} = \int_0^L dz u_{j_1}(z)u_{j_2}(z)u_{j_3}(z)u_{j_4}(z) \). Furthermore, each momentum loop integral in the finite system can be obtained from the infinite system through the substitution \( \int d^dk \to \sum_{j=-\infty}^{\infty} \sigma \int d^{d-1}k \). The eigenfunctions actually depend on \( \tau \) and can be written as \( u_{j_1}^{(\tau)}(z) = L^{-\frac{1}{2}} \exp(ik_jz) \) which implies \( S_{j_1j_2}^{(\tau)} = \delta_{j_1+j_2,0} \) and \( S_{j_1j_2j_3j_4}^{(\tau)} = L^{-1}\delta_{j_1+j_2+j_3+j_4,0} \).

This means that the quasi-momentum is "conserved" along the \( z \) direction for periodic and antiperiodic boundary conditions.

Let us define the renormalized 1PI vertex parts from the NF construction. Although they do depend on the boundary conditions, we shall not introduce this additional label on them. Consequently, considering an arbitrary 1PI divergent (but regularized, say, by a cutoff \( \Lambda \)) bare vertex part including composite operators \( \Gamma^{(N,M)} ((N, M) \neq (0, 2)) \), the statement of multiplicative renormalizability amounts to finding renormalization functions \( Z_\phi^{(\tau)}, Z_{\phi^2}^{(\tau)} \) such that the vertex parts defined by
\[ \Gamma_R^{(N,M)}(p_l, i_l, Q_l, i'_l, g, \mu) = (Z_\phi^{(\tau)})^N (Z_{\phi^2}^{(\tau)})^M \Gamma^{(N,M)}(p_l, i_l, Q_l, i'_l, \lambda_0, \mu_0, \Lambda), \]
are automatically finite (when the regulator \( \Lambda \) is taken to infinity).

In the massive framework, the primitive divergent vertex parts of this \( \lambda \phi^4 \) field theory are chosen to be renormalized in the standard way \[ \{16\] , but now they are explicitly dependent on the boundary condition, even though we omit the label \( \tau \) in all vertex parts. Then, we choose the following normalization conditions at zero external momenta (and quasi-
momenta), namely

\[ \Gamma_R^{(2)}(k = 0, j = 0, g, \mu) = \mu^2 + \sigma^2 \tau^2, \quad (3a) \]

\[ \frac{\partial \Gamma_R^{(2)}(k, j = 0, g, \mu)}{\partial k^2} \bigg|_{k^2 = 0} = 1, \quad (3b) \]

\[ \Gamma_R^{(4)}(k_i = 0, i_t = 0, g, \mu) = g, \quad (3c) \]

\[ \Gamma_R^{(2,1)}(k = 0, j = 0, Q = 0, j' = 0, g, \mu) = 1. \quad (3d) \]

Note that the normalization condition on the two-point function above amounts to choosing the renormalized mass \( \mu \) independent of the boundary condition \([6]\). These conditions are sufficient to formulate all vertex parts which can be renormalized multiplicatively.

First let us discuss the situation at the critical dimension \( d = 4 \). In that case, utilize implicitly a cutoff \( \Lambda \) to regularization of the integrals and suppose that after the renormalization procedure is defined the limit of infinite cutoff can be taken. We can obtain a Callan-Symanzik equation which describes the scaling regime through the following steps: i) apply the derivative \( \frac{\partial}{\partial \mu^2} \) over the bare vertex part \( \Gamma^{(N,M)}(p_l, i_l; Q_l, i'_l; 0; \mu_0, \Lambda) \) at fixed \( \mu_0, \Lambda \) which produces the vertex function \( \Gamma^{(N,M+1)}(p_l, i_l; Q_l, i'_l, g, \mu) \); ii) rewrite the remaining bare vertex parts in terms of the renormalized ones. This results in the following expression

\[ (2\rho \frac{\partial}{\partial \mu^2} + \frac{\alpha}{\mu^2} \frac{\partial}{\partial g} - \frac{1}{2} N \frac{\kappa}{\mu^2} - M \frac{\pi}{\mu^2})\Gamma^{(N,M)}(p_l, i_l; Q_l, i'_l, g, \mu) = (4) \]

\[ \Gamma^{(N,M+1)}(p_l, i_l; Q_l, i'_l, 0, g, \mu), \]

where \( 2\rho = \frac{\partial^2}{\partial \mu^2} Z^{(\tau)}(\mu, g), \quad \frac{\alpha}{\mu^2} = Z^{(\tau)}(\frac{\partial g}{\partial \mu^2}), \quad \frac{\kappa}{\mu^2} = Z^{(\tau)}(\frac{\partial \ln Z^{(\tau)}}{\partial \mu^2}), \quad \frac{\pi}{\mu^2} = Z^{(\tau)}(\frac{\partial \ln Z^{(\tau)}}{\partial \mu^2}). \) Let the flow functions be defined by the expressions \( \beta^{(\tau)}(\mu, g)(\equiv \frac{\alpha}{\rho}) = \mu \frac{\partial g}{\partial \mu}, \quad \gamma^{(\tau)}(\phi^2)(\equiv \frac{\kappa}{\rho}) = \mu \frac{\partial \ln Z^{(\tau)}}{\partial \mu} \) and \( \gamma_{\phi^2}^{(\tau)}(\equiv -\frac{\pi}{\rho}) = -\mu \frac{\partial \ln Z^{(\tau)}}{\partial \phi^2}. \) Multiplying last equation by \( 2\rho \frac{\partial}{\partial \mu^2}, \) we obtain its equivalent form in terms of these redefinitions as

\[ (\mu \frac{\partial}{\partial \mu} + \beta^{(\tau)}(\mu, g) - \frac{N}{2} \gamma^{(\tau)}(\phi^2) + M \gamma_{\phi^2}^{(\tau)})\Gamma^{(N,M)}(p_l, i_l; Q_l, i'_l, g, \mu) = (5) \]

\[ 2\mu^2 \frac{\partial^2}{\partial \mu^2} [Z^{(\tau)}(\phi^2)]^{-1} \Gamma^{(N,M+1)}(p_l, i_l; Q_l, i'_l; 0, g, \mu). \]

Now, taking \( N = 2 \) at zero external momenta and quasi-momenta and using the normalization conditions Eq.\((3a)\) and Eq.\((3d)\) we obtain the Callan-Symanzik equation for finite size
functions of the dimensionful quantities. For instance, let $\Gamma_{CS}$ be the function which governs the flow of the coupling constant in parameter space. In order to get rid of undesirable dimensionful parameters when $d = 4 - \epsilon$, define the Gell-Mann-Low function $[\beta(g, \mu)]_{GL} = -\epsilon g + \beta(g, \mu)$. Using the Gell-Mann-Low function into the $CS$ equation, it is easy to find that all dimensionful parameters turn into dimensionless quantities. For instance, let $\lambda = \mu^\epsilon u_0$ be the dimensionful bare coupling constant written in terms of the bare dimensionless coupling $u_0$ and $g$ the renormalized dimensionful counterpart written in terms of the dimensionless renormalized coupling $u$ as $g = \mu^\epsilon u$. Those definitions imply that $[\beta(g, \mu)]_{GL} \frac{\partial}{\partial g} = \beta(u) \frac{\partial}{\partial u}$, i.e., we get a description entirely in terms of the dimensionless renormalized coupling constant, which has a well defined scaling limit $19, 20$. The Callan-Symanzik equation can be expressed in the form

$$
(\mu \frac{\partial}{\partial \mu} + \beta^{(\tau)} g, \mu \frac{\partial}{\partial u} - \frac{N}{2} \gamma^{(\tau)}(g) + M \gamma^{(\tau)}(g, \mu) = \Gamma^{(N,M)}(p, i, q, i', g, \mu) = \Gamma^{(N,M+1)}(p, i, q, i'; 0, g, \mu),
$$

where $\gamma^{(\tau)}(g, \mu) = \gamma^{(\tau)}(g, \mu) + \gamma^{(\tau)}(g, \mu)$. The definition $Z^{(\tau)}(g, \mu)$ can be used to write down another function, namely $Z^{(\tau)}(g, \mu) = \beta^{(\tau)}(\frac{\partial \ln Z^{(\tau)}}{\partial u})$, which shall be useful to our purposes. The solution of the Callan-Symanzik equation is analogous to the infinite systems version and we shall not discuss it here; instead, we shall use the results of previous analysis in order to discuss the salient features which naturally leads to the ultraviolet fixed points along with the critical exponents for finite systems satisfying various boundary conditions.

Recalling that the infrared divergences are absent in the massive theory, we analyze the theory at the ultraviolet region where the momentum of the internal propagators in
arbitrary loop graphs are very large, i.e., at the scaling region $\frac{p}{\mu} \to \infty$ [17, 18]. This means that the right hand side can be neglected order by order in perturbation theory just like in the field-theoretic description of infinite systems.

Let us turn now our attention to the computation of the Feynman integrals corresponding to one-, two- and three-loop diagrams required to getting the critical exponents $\eta$ and $\nu$ perturbatively.

The one-loop integral contributing to the four-point function is then given by:

$$I_2^{(\tau)}(k, i; \sigma, \mu) = \sigma \sum_{j = -\infty}^{\infty} \int d^{d-1}q \frac{1}{[(q)^2 + (\sigma)^2(j + \tau)^2 + \mu^2]^2} \times \frac{1}{[(q + k)^2 + (\sigma)^2(j + i + \tau)^2 + \mu^2]^2}. $$

(8)

Remember that $\mu = \frac{t^\frac{1}{2}}{\xi^\frac{1}{2}} = \xi^{-1}$ at tree level, where $t$ is the renormalized reduced temperature. Performing the transformation $p = \frac{q}{\mu}$ in all momenta present in the diagram ($k' = \frac{k}{\mu}$, restoring $k' \to k$) and defining $r \equiv \frac{\sigma}{\mu} = \left(\frac{2\pi\xi}{L}\right)$, we use a Feynman parameter $x$ before resolving the integral over $p$, or in other words,

$$I_2^{(\tau)}(k, i; \sigma, \mu) = r \mu^{-\epsilon} \sum_{j = -\infty}^{\infty} \int_0^1 dx \int d^{d-1}p \frac{1}{[p^2 + 2xkp + xk^2 + r^2((j + \tau + ix)^2 + x(1 - x)i^2) + 1]^2}. $$

(9)

A typical result within our conventions (see Ref. [16]) appropriate to dimensionally regularized integrals is expressed by the formula

$$\int \frac{d^d q}{(q^2 + 2k.q + m^2)^\alpha} = \frac{1}{2} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2} - \frac{m^2 - k^2}{2}\right)\Gamma(-\alpha/2 - \frac{m^2 - k^2}{2})}{\Gamma(\alpha)} S_d, $$

(10)

where $S_d$ is the area of the $d$-dimensional unit sphere. Using this relation we get to

$$I_2^{(\tau)}(k, i; \sigma, \mu) = r \mu^{-\epsilon} \frac{1}{2} S_{d-1} \Gamma\left(\frac{d - 1}{2}\right) \Gamma\left(2 - \frac{(d - 1)}{2}\right) \times \int_0^1 dx \sum_{j = -\infty}^{\infty} [x(1 - x)(k^2 + x^2 r^2) + r^2(j + \tau + ix)^2 + 1]^\frac{-\alpha}{2}.$$ 

(11)

Notice that $r^{-1} \propto \frac{L}{\xi}$ here is the boundedness variable in the massive theory, where $\xi$ is the fixed bulk correlation length. After factoring out the $r^2$ term in the last integral, we can proceed in the computation by noticing that the remaining summation can be identified
with the generalized thermal function \[21\]

\[
D_\alpha(a, b) = \sum_{n=-\infty}^{\infty} [(n + a)^2 + b^2]^{-\alpha} = \frac{\sqrt{\pi}}{\Gamma(\alpha)} \left[ \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha - 1)} + f_\alpha(a, b) \right], \tag{12}
\]

where

\[
f_\alpha(a, b) = 4 \sum_{m=1}^\infty \cos(2\pi ma) \frac{(\pi m)^{\alpha - \frac{1}{2}}}{b} K_{\alpha - \frac{1}{2}}(2\pi mb), \tag{13}
\]

and \(K_\nu(x)\) is the modified Bessel function of the second kind. The identifications \(a(x) = \tau + ix, b(x) = r^{-1} \sqrt{(k^2 + r^2)x(1-x) + 1}\) and \(\epsilon = 4 - d\) permit us to write

\[
I_2^{(\tau)}(k, i; \sigma, \mu) = \mu^{-\epsilon} \frac{1}{2} S_{d-1} \Gamma\left(\frac{d-1}{2}\right) \sqrt{\pi} \left[ \int_0^1 dx \times \left[ \Gamma\left(\frac{\epsilon}{2}\right)[x(1-x)(k^2 + i^2r^2) + 1]^{-\frac{\epsilon}{2}} + f_{\frac{1}{2}+\frac{\epsilon}{2}}(a, b) \right] \right], \tag{14}
\]

Now, using the identity \(\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right) S_{d-1} = \Gamma\left(\frac{d}{2}\right) S_d\) and expanding in \(\epsilon\) the argument of the gamma function, we can rewrite last integral as

\[
I_2^{(\tau)}(k, i; \sigma, \mu) = S_d \mu^{-\epsilon} \left[ \frac{1}{\epsilon} (1 - \frac{\epsilon}{2}) \right] \int_0^1 dx [x(1-x)(k^2 + i^2r^2) + 1]^{-\frac{\epsilon}{2}}
+ \frac{1}{2} r^{-\epsilon} \Gamma(2 - \frac{\epsilon}{2}) \int_0^1 dx f_{\frac{1}{2}+\frac{\epsilon}{2}}(\tau + ix, r^{-1} \sqrt{x(1-x)(k^2 + i^2r^2) + 1}). \tag{15}
\]

Whenever we perform a loop integral, the area of the unit sphere \(S_d\) naturally takes place and this angular factor can be neutralized in a redefinition of the coupling constant. We adopt this procedure henceforward in all loop integrals and suppress this overall factor. We then find

\[
I_2^{(\tau)}(k, i; r) \equiv \frac{I_2^{(\tau)}(k, i; \sigma, \mu)}{S_d} = \frac{\mu^{-\epsilon}}{\epsilon} \left[ (1 - \frac{\epsilon}{2}) \right] \int_0^1 dx [x(1-x)(k^2 + r^2i^2) + 1]^{-\frac{\epsilon}{2}}
+ \frac{1}{2} r^{-\epsilon} \Gamma(2 - \frac{\epsilon}{2}) F_\frac{1}{2}^{(\tau)}(k, i; r)), \tag{16}
\]

where

\[
F_\alpha^{(\tau)}(k, i; r) = r^{-2\alpha} \int_0^1 dx f_{\frac{1}{2}+\alpha}(\tau + xi, h(k, i, r)), \tag{17}
\]

and

\[
h(k, i, r) = r^{-1} \sqrt{x(1-x)(k^2 + r^2i^2) + 1}. \tag{18}
\]

Since we are going to use normalization conditions in this massive setting, we are interested in the simplest situation which occurs for vanishing external momenta and quasi-momenta...
(k = 0, i = 0). In that case, $F^{(r)}_{\tau}(r) = r^{-\epsilon} f_{\frac{1}{\tau} + \frac{1}{r}}(\tau, r^{-1})$. Furthermore, recalling that the finite size contribution is $O(\epsilon^0)$, the one-loop integral can be written as

$$I^{(r)}_{2}(r) = I^{(r)}_{2}(k = 0, i = 0, r) = \mu^{-\epsilon} \left[ \frac{1}{\epsilon} (1 - \frac{\epsilon}{2}) + \frac{1}{2} f_{\frac{1}{\tau} + \frac{1}{r}}(\tau, r^{-1}) \right]. \quad (19)$$

In this massive approach, so long as $r^{-1}$ takes finite nonzero values, last equation is suitable to compute amplitudes and other observables and demonstrates explicitly that the one-loop bubble at the symmetry point is decomposable in the form bulk contribution plus finite size correction.

On the other hand, in the computation of the critical exponents, in practice we use this form without any specification of the correction function, since it does not have singular behavior for these values of $r^{-1}$ and the divergence structure of this diagram is governed by the first term (bulk) in that expression. In that case, the $\epsilon$-expansion is well defined and we can proceed to the computation of loop diagrams of primitively divergent vertex parts in order to renormalize the theory and obtain those universal quantities. Note that even though the last expression is written in terms of $f_{\frac{1}{\tau} + \frac{1}{r}}(\tau, r^{-1})$, we prefer to write the correction in terms of $F^{(r)}_{\tau}(k = 0, i = 0; r)$ anticipating our future discussion of the massless case. In addition, applications of the present method might be important to compute amplitudes.

We would like to understand the importance of the finite size correction in limit values of the boundedness variable in calculating an arbitrary amplitude. This amounts to figuring out the approach to the regions $r^{-1} \to 0$ (or $r \to \infty$) as well as $r^{-1} \to \infty$ and what are the limits of validity of the $\epsilon$-expansion.

In order to describe these asymptotic values and its effects on the finite size correction, we write the latter in the form

$$F^{(r)}_{\tau}(k = 0, i = 0; r) = 4r^{-\epsilon} \sum_{n=1}^{\infty} \cos(2\pi n \tau) (\pi n)^{\frac{1}{2}} K_{\tau}^{0}(2\pi nr^{-1}). \quad (20)$$

This expression shows clearly that the correction has no poles in $\epsilon$. Since it is well behaved, take $\epsilon = 0$ in this whole expression in order to rewrite Eq.\(19\) at the symmetry point as

$$I^{(r)}_{2SP}(r) = \mu^{-\epsilon} \left[ \frac{1}{\epsilon} (1 - \frac{\epsilon}{2}) + 2 \sum_{n=1}^{\infty} \cos(2\pi n \tau) K_{0}(2\pi nr^{-1}) \right]. \quad (21)$$

The limit $r^{-1} \to \infty$ corresponds to $\frac{\lambda}{\xi} \to \infty$, whereas $0 < r^{-1} < \infty$ represent finite values of $\frac{\lambda}{\xi}$. Let us focus our attention in the limit $r^{-1} \to \infty$. Using the asymptotic form of the
Bessel function for $x \to \infty$, namely $K_{\alpha}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}[1 + O(1/x)]$, for $r^{-1} \to \infty$ one learns that the correction term has the behavior

$$\lim_{r^{-1} \to \infty} \left( \frac{\epsilon}{2} F_{\tau}(k = 0, i = 0; r) \right) = \frac{2\epsilon}{(r-1)^{1/2}} \sum_{n=1}^{\infty} \cos(2\pi n \tau)n^{-1/2}e^{(-2\pi nr^{-1})}. \quad (22)$$

Using a trivial inequality to simplify our task, we can show that last term vanishes in the wanted limit as follows

$$\lim_{r^{-1} \to \infty} \left( \frac{1}{r^{1/2}} \sum_{n=1}^{\infty} \cos(2\pi n \tau)n^{-1/2}e^{(-2\pi nr^{-1})} \right) < \lim_{r^{-1} \to \infty} \left( \frac{1}{r^{-1/2}} \sum_{n=1}^{\infty} \cos(2\pi n \tau)e^{(-2\pi nr^{-1})} \right) = \lim_{r^{-1} \to \infty} \left[ \frac{1}{r^{1/2}} \left( \frac{1}{1 - e^{-2\pi(r^{-1} - i\tau)}} + \frac{1}{1 - e^{-2\pi(r^{-1} + i\tau)}} - 2 \right) \right] \to 0. \quad (23)$$

Therefore, the integral turns out to reproduce the (bulk) value from the massive theory of the infinite system \[^{17}\]. We can identify region $\frac{L}{\xi} \to \infty$ with usual bulk critical behavior $L \to \infty$. The region $\frac{L}{\xi} > 1$ interpolates from finite size corrections to the bulk critical behavior.

As $\frac{L}{\xi}$ decreases, the finite size correction will increase until it will eventually become as big as the pole in $\epsilon$, modifying the leading singularity of the four-point function. To see this let us consider the potential trouble which is hidden in the different values of $r^{-1}$ and, in particular, in the limit $r^{-1} \to 0 (L \to 0)$.

Let us perform the sum which appears explicitly in the correction term. From Ref.\[^{22}\], the identity

$$\sum_{n=1}^{\infty} K_{0}(nx)\cos(nx) = \frac{1}{2} \left[ \gamma + \ln \left( \frac{x}{4\pi} \right) \right] + \frac{\pi}{2x\sqrt{1+t^{2}}} + \frac{\pi}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{x^{2} + (2n\pi + tx)^{2}}} - \frac{1}{2n\pi} \right], \quad (24)$$

which is valid for positive finite values of the variable $x$, along with the identifications $x = 2\pi r^{-1}, t = r\tau$ (and $\gamma = 0.57721566...$ is the Euler-Mascheroni constant) implies the following result to the one-loop graph

$$I_{2}(k = 0, i = 0, r^{-1}) = \mu^{-\epsilon} \left[ \frac{\epsilon}{2} (1 - \frac{\epsilon}{2}) + \gamma + \ln \left( \frac{r^{-1}}{2} \right) + \frac{1}{2\sqrt{r^{-2} + \tau^{2}}} \right. \left. + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{r^{-2} + (n + \tau)^{2}}} - \frac{1}{n} \right] + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{r^{-2} + (n - \tau)^{2}}} - \frac{1}{n} \right] \right]. \quad (25)$$
The simplest way to prove that the two infinite series are convergent in the limit $r^{-1} \to 0$ is to set directly $r^{-1} = 0$ and compute this correction [23]. It becomes

$$
\lim_{r^{-1} \to 0} \left( \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{n+\tau} - \frac{1}{n} \right] \right] + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{n-\tau} - \frac{1}{n} \right] \right). \tag{26}
$$

Now, from the definition of the dilogarithm function $\psi(1 + z) = -\gamma + \psi(1 + z) = \psi(z) + \frac{1}{2}z$ and the value $\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2$, we easily obtain

$$
\lim_{r^{-1} \to 0} \left( \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{n+\tau} - \frac{1}{n} \right] \right] \right) = (2\ln 2 - 1)^2 \delta_{\tau, \frac{1}{2}}, \tag{27}
$$

which is finite as advertised. Therefore, for small values of $r^{-1}$ we can write the one-loop bubble as

$$
I_2(k = 0, i = 0, r^{-1}) = \mu^{-\epsilon} \left[ \frac{1}{\epsilon} (1 - \frac{\epsilon}{2}) + \gamma + \ln(\frac{r^{-1}}{2}) + \frac{1}{2\sqrt{r^{-2} + \tau^2}} \right] + (2\ln 2 - 1)^2 \delta_{\tau, \frac{1}{2}} + O(r^{-1}). \tag{28}
$$

This expression for the one-loop four point function depends on the boundary condition.

It gives the support to identify two types of crossover in finite systems presenting these simple boundary conditions away from the critical point ($t \neq 0$) as follows.

Firstly, our analytical expression above is a transliteration of the analysis performed in Refs.[13, 14] concerning the breakdown of the expansion in $\epsilon = 4 - d$, namely, when the argument of the square root term in the above expression vanishes. Indeed, for periodic boundary conditions $\tau = 0$ and perturbation theory is invalid in the limit $r^{-1} = \frac{L}{2\pi \xi} \to 0$.

For antiperiodic boundary conditions, however, if the temperature is below the bulk critical temperature ($t < 0$), whenever $r^{-2} = -\frac{L^2}{(2\pi \xi)^2} = \frac{1}{2}$ the inverse square root blows up. This effect was denominated “dimensional crossover” as discussed previously by those authors.

Secondly, if the value of $r^{-1}$ is decreased further for fixed $t > 0$, i.e., diminishing $L$, the logarithmic term starts to become important for antiperiodic boundary condition when its argument becomes around the same order of magnitude that the dimensional pole $\frac{1}{\epsilon}$. If we switch to cutoff regularization for a moment, the ultraviolet regime is characterized by $\frac{1}{\epsilon} \to \ln(\frac{4}{\mu^2}) = \ln(\Lambda \xi)$ with $\Lambda \xi \gg 1$. The logarithm contribution will eventually become comparable with the ultraviolet dimensional pole, whenever $(\frac{4}{\xi}) \sim \frac{1}{(\Lambda \xi)^2}$, i.e., when $L \Lambda \sim 1$.

In terms of a lattice parameter $a$, $\Lambda \sim \frac{1}{a}$ which implies $L \sim a$. It is the reduction of $L$ for fixed $t > 0$ in this massive framework which is responsible for this new effect. This is a novel
type of crossover which only happens for antiperiodic boundary condition at \( t > 0 \) and is straightforward from our analytical expression given purely in terms of elementary primitive functions. This new type of crossover starts when instead of a large number of parallel plates, there are only two parallel plates (the limiting surfaces) and the bulk description is no longer reliable. Note that this behavior is also there for periodic boundary condition, but the square root term proportional to \( \left( \frac{\xi}{L} \right)^{-1} \) is overwhelming in that limit.

A word of caution here. It is dangerous to take the limit \( t = 0 \) (or \( \xi \to \infty \)) in the above expression. The reason this limit is inconsistent in this massive framework is that the scale invariance of the renormalized theory only takes place in the ultraviolet regime. The most appropriate strategy would be to start from scratch with massless fields which are scale invariant at this infrared regime, renormalize the theory at nonvanishing external momenta scale and push forward all the consequences which follow from this approach. We are going to study this case later on and shall prove from a full two-loop calculation that the phenomenological scaling theory, which states that \( \epsilon \)-expansion results have meaningless results at \( t = 0 \) is incorrect. We postpone this discussion to Secs. IV and V.

Without loss of generality we can choose \( \mu^2 = 1 \) which is equivalent to a fixed (arbitrary but finite) correlation length, such that \( r = \sigma = \frac{2\pi}{L} \). (In our subsequent discussion we can reconstruct the \( \xi \) dependence through its multiplication by \( L^{-1} \).) In fact we could have started directly from this choice for the mass scale, and it will define all other massive loop integrals yet to be discussed. We have only to keep in mind that this choice makes \( L \) dimensionless.

From now on, we stay away from the region of crossover in order to compute the higher loop integrals. These objects can be computed analogously to our previous one-loop discussion and the reader is advised to consult Appendix A for details. One typical example is the integral contributing to the four-point function at two-loops, namely

\[
I_4(k_1, k_2, k_3, k_4, i_1, i_2, i_3, i_4, \sigma, \mu) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1} q_1 d^{d-1} q_2}{(q_1^2 + \sigma^2 (j_1 + \tau)^2 + \mu^2)} \frac{1}{(q_2^2 + \sigma^2 (j_2 + \tau)^2 + \mu^2) [((q_1 - q_2 + k_3)^2 + \sigma^2 (j_1 - j_2 + i_3 + \tau)^2 + \mu^2]} \frac{1}{[(P - q_1)^2 + \sigma^2 (p - j_1 + \tau)^2 + \mu^2]},
\]

where \( P = k_1 + k_2 \) is the external momenta along the plates and \( p = i_1 + i_2 \) is a discrete “external” quasi-momentum label. Taking \( \mu = 1 \) at zero external momenta and quasi-
momenta, let the integral be denoted by \( I_{4SP}^{(τ)}(0, 0; σ) \). From the result computed in Appendix A, the outcome to this diagram is

\[
I_{4SP}^{(τ)}(σ) = \frac{1}{2e^2} \left[ (1 - \frac{ε}{2}) + εf_4^{(τ)}(σ, σ^{-1}) \right].
\]  

(30)

The integrals contributing to the two-point function at two- and three-loops, respectively, turn out to be

\[
I_3(k, i, σ, μ) = σ^2 \sum_{j_1, j_2 = -∞}^{∞} \int \frac{d^{d-1}q_1 d^{d-1}q_2}{(q_1^2 + σ^2(j_1 + τ)^2 + μ^2)(q_2^2 + σ^2(j_2 + τ)^2 + μ^2)} \times \frac{1}{[(q_1 + q_2 + k)^2 + σ^2(j_1 + j_2 + i + τ)^2 + μ^2]},
\]  

(31)

and

\[
I_5(k, i, σ, μ) = σ^2 \sum_{j_1, j_2 = -∞}^{∞} \int \frac{d^{d-1}q_1 d^{d-1}q_2 d^{d-1}q_3}{(q_1^2 + σ^2(j_1 + τ)^2 + μ^2)(q_2^2 + σ^2(j_2 + τ)^2 + μ^2)(q_3^2 + σ^2(j_3 + τ)^2 + μ^2)} \times \frac{1}{[(q_1 + q_2 + k)^2 + σ^2(j_1 + j_2 + i + τ)^2 + μ^2]} \times \frac{1}{[(q_1 + q_3 + k)^2 + σ^2(j_1 + j_3 + i + τ)^2 + μ^2]},
\]  

(32)

In passing, we note that the massless diagrams to be studied later, can be obtained from the above expressions by setting \( μ = 0 \), although we work in the massive case at \( μ = 1 \).

The objects required for our purposes are the derivative of those integrals with respect to the external momenta, setting the external momenta at the symmetry point in the end of the process. Let the derivatives in relation to \( k^2 \) be \( I_3^{(τ)} \) and \( I_5^{(τ)} \) (at null \( k \)). It is demonstrated in Appendix A that they are given by:

\[
I_{3SP}^{(τ)}(σ) = -\frac{1}{8ε} \left[ 1 - \frac{ε}{4} + εW^{(τ)}(σ) \right],
\]  

(33)

and

\[
I_{5SP}^{(τ)}(σ) = -\frac{1}{6ε^2} \left[ 1 - \frac{ε}{4} + \frac{3ε}{2}W^{(τ)}(σ) \right],
\]  

(34)

where \( W^{(τ)}(σ) = C^{(τ)}(σ) + H^{(τ)}(σ) - 4rF_0^{(τ)}(σ) - F_0^{(τ)}(σ), C^{(τ)}(σ) \) and \( H^{(τ)}(σ) \) are defined in Appendix A, Eqs. (A7), (A14a) and (A14b), respectively, computed at \( ε = 0 \). We are not going to write them down explicitly since we shall show soon that they will be eliminated during the renormalization process.

With the information at hand, we can proceed to compute the normalization functions, Wilson functions, the repulsive fixed point, the anomalous dimension of the field and of the composite operator. This task shall be tackled in the next section.
III. CRITICAL EXPONENTS FROM FINITE SIZE WITH PBC AND ABC

Now we describe the normalization functions and Wilson functions in terms of the loop integrals. These are the fundamental quantities needed to uncover the diagrammatic computation of universal quantities.

The occurrence of a nontrivial ultraviolet fixed point, the scaling limit in the ultraviolet regime and the simplification achieved for the renormalized vertex parts computed at the fixed point are important aspects with consequences in this sort of computation.

As previously discussed, the ultraviolet flow in momentum space can be described in terms of dimensionless coupling constants. In this way, we can write the dimensionless bare coupling constant and normalization function $Z_\phi, \tilde{Z}_\phi^2$ as power series in the dimensionless renormalized coupling constant as $u_0^{(\tau)} = u(1 + a_1^{(\tau)}u + a_2^{(\tau)}u^2)$, $Z_\phi^{(\tau)} = 1 + b_2^{(\tau)}u^2 + b_3^{(\tau)}u^3$ and $\tilde{Z}_\phi^{(\tau)} = 1 + c_1^{(\tau)}u + c_2^{(\tau)}u^2$. The divergence structure of these objects are dimensional poles appearing as inverse powers of $\epsilon(= 4 - d)$. In order to figure out explicitly each coefficient as parameters depending on the loop Feynman integrals computed (at symmetry point defined at zero external momenta and quasimomenta) so far, express them in the form

$$u_0 = u[1 + \frac{(N + 8)}{6} I_{2SP}^{(\tau)} u + \frac{[(N + 8) I_{2SP}^{(\tau)}]^2}{18}]
- \frac{(N^2 + 6N + 20)(I_{2SP}^{(\tau)})^2}{36} + \frac{(5N + 22)I_{4SP}^{(\tau)}}{9} - \frac{(N + 2)I_{3SP}^{(\tau)}}{9} u^2], \quad (35a)$$

$$Z_\phi^{(\tau)} = 1 + \frac{(N + 2)I_{2SP}^{(\tau)} u}{18} + \frac{(N + 2)(N + 8)(I_{2SP}^{(\tau)})^2}{54} - \frac{(N + 2)I_{3SP}^{(\tau)}}{9} u^2, \quad (35b)$$

$$\tilde{Z}_\phi^{(\tau)} = 1 + \frac{(N + 2)I_{2SP}^{(\tau)} u}{6}
+ \frac{(N^2 + 7N + 10)I_{2SP}^{(\tau)}}{18} - \frac{(N + 2)(N + 8)(I_{2SP}^{(\tau)})^2}{6} + \frac{(N + 2)I_{4SP}^{(\tau)} + I_{3SP}^{(\tau)}}{6} u^2. \quad (35c)$$

The flow functions describing the parameter space as the momentum scale varies are $\beta^{(\tau)}(u)$, $\gamma_\phi^{(\tau)}(u)$ and $\tilde{\gamma}_\phi^{(\tau)}$. When they are written as series expansions in terms of $u$, we obtain explicitly

$$\beta^{(\tau)}(u) = -\epsilon u[1 - a_1^{(\tau)}u + 2((a_1^{(\tau)})^2 - a_2^{(\tau)})u^2], \quad (36a)$$

$$\gamma_\phi^{(\tau)} = -\epsilon u[2b_2^{(\tau)}u + (3b_3^{(\tau)} - 2b_2^{(\tau)}a_1^{(\tau)})u^2], \quad (36b)$$

$$\tilde{\gamma}_\phi^{(\tau)} = \epsilon u[c_1^{(\tau)} + (2c_2^{(\tau)} - (c_1^{(\tau)})^2 - a_1^{(\tau)}c_2^{(\tau)})u]. \quad (36c)$$
We then employ the results for the integrals presented in the last section for periodic and antiperiodic boundary conditions in order to find the values of each coefficient.

Looking at Eq. (36) and comparing it with the expansions of the dimensionless bare coupling constant, normalization constant of the field and that from the composite operator, we can read off their results. The fact of the matter is that the utilization of normalization conditions provokes the appearance of correction terms in those functions which depend explicitly on the boundary conditions. As is well known, this is a prevalent artifact taking place in this renormalization scheme. Thus, the renormalization functions at two loops will not be equal to those provenient from the bulk system. Nevertheless, these nonuniversal corrections are going to cancel out in the expression for universal quantities as we shall see in the remainder of this section.

Working out the details using this prescription, it is not difficult to prove that they are given by the following expressions

\[ a_1^{(\tau)} = \frac{(N + 8)}{6\epsilon}[1 - \frac{1}{2}\epsilon + \frac{1}{2}f_2^{(\tau, \sigma^{-1})}\epsilon], \]  

\[ a_2^{(\tau)} = \frac{(N + 8)^2}{6\epsilon} - \frac{N^2 + 21N + 86}{36\epsilon} + \left(\frac{N^2 + 16N + 64}{36\epsilon}\right)f_2^{(\tau, \sigma^{-1})} \]
\[ + \left(\frac{N + 2}{72\epsilon}\right)(1 - \frac{\epsilon}{4} + \epsilon W^{(\tau)}(\sigma)), \]  

\[ b_2^{(\tau)} = -\frac{(N + 2)}{144\epsilon}[1 - \frac{\epsilon}{4} + \epsilon W^{(\tau)}(\sigma)], \]  

\[ b_3^{(\tau)} = -\frac{(N + 2)(N + 8)}{1296\epsilon^2}[1 - \frac{7\epsilon}{4} + \frac{3\epsilon}{2}f_2^{(\tau, \sigma^{-1})}\epsilon], \]  

\[ c_1^{(\tau)} = \frac{(N + 2)}{6\epsilon}[1 - \frac{1}{2}\epsilon + \frac{1}{2}f_2^{(\tau, \sigma^{-1})}\epsilon], \]  

\[ c_2^{(\tau)} = \frac{(N + 2)(N + 5)}{36\epsilon^2} - \frac{2N^2 + 17N + 26}{72\epsilon} + \left(\frac{N^2 + 7N + 10}{36\epsilon}\right)f_2^{(\tau, \sigma^{-1})}. \]  

Using the coefficients \( a_1^{(\tau)} \) as well as \( a_2^{(\tau)} \) into Eq. (36) we get to

\[ \beta^{(\tau)}(u) = -\epsilon u + \frac{(N + 8)}{6}(1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}f_2^{(\tau, \sigma^{-1})}\epsilon)u^2 - \frac{3N + 14}{12}u^3. \]  

Furthermore, we can also find the solutions for the functions related to the anomalous dimension of the field and composite operator. They read

\[ \gamma_\phi^{(\tau)} = u^2\frac{(N + 2)}{72}\left[1 - \frac{\epsilon}{4} + \epsilon W^{(\tau)}(\sigma) - \frac{(N + 8)}{6}(1 + W^{(\tau)}(\sigma) - f_2^{(\tau, \sigma^{-1})})u\right], \]  

\[ \gamma_{\phi^2}^{(\tau)} = \frac{(N + 2)}{6}(1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}f_2^{(\tau, \sigma^{-1})})u - \frac{(N + 2)}{12}u^2. \]
The eigenvalue condition \( \beta^{(\tau)}(u_\infty) = 0 \) yields the repulsive ultraviolet fixed point which is given by
\[
    u_\infty = \frac{6}{(N + 8)} \epsilon [1 + \epsilon \left( \frac{9N + 42}{(N + 8)^2} + \frac{1}{2} (1 - f_{1/2}(\tau, \sigma^{-1})) \right)].
\]  
As usual we identify the fixed point value of the anomalous dimension of the field with the critical exponent \( \eta \), i.e.
\[
    \eta = \gamma^{(\tau)}(u_\infty) = \frac{(N + 2) \epsilon^2}{(N + 8)^2} [1 + \left( \frac{6(3N + 14)}{(N + 8)^2} - \frac{1}{4} \right) \epsilon].
\]  
Similarly, it is straightforward to show that
\[
    \zeta^{(\tau)}_{\phi^2}(u_\infty) = \frac{(N + 2) \epsilon}{(N + 8)} [1 + \left( \frac{6N + 18}{(N + 8)^2} \right) \epsilon],
\]
which in conjunction with the equation \( \zeta^{(\tau)}_{\phi^2}(u_\infty) = 2 - \eta - \nu^{-1} \) yields the value of the correlation length exponent
\[
    \nu = \frac{1}{2} + \frac{(N + 2)}{4(N + 8)} \epsilon + \frac{(N + 2)(N^2 + 23N + 60)}{8(N + 8)^3} \epsilon^2.
\]  
Note that these exponents are indeed independent of the boundary conditions and are exactly the same as those obtained from the bulk system through an analogous utilization of diagrammatic methods. If we use the remaining bulk scaling relations, we find that all the critical exponents for these simple boundary conditions reproduce those from the bulk confirming the one loop analysis by Nemirovsky and Freed and extending it to the present higher loop correction.

We have thus succeeded in formulating 1PI renormalized vertex parts for the massive theory in order to compute the exponents within the perturbation expansion in \( \epsilon \). We shall introduce the framework of massless fields in the calculation of the critical exponents in the next section.

### IV. NF METHOD FOR PBC AND ABC USING MASSLESS FIELDS

We start by describing the method of renormalization for massless fields, given by the previous Lagrangian density. The \((d-1)\)-dimensional momentum space lies along the plates and characterizes directions perpendicular to the finite size direction \( L \), which is represented by the quasimomenta. The free bare critical (massless) propagator is \( G_{0j}^{(\tau)}(k, j) = \frac{1}{k^2 + \sigma^2(j + \tau)^2} \).

The former definitions for the tensors \( S_{i_1i_2} \) and \( S_{i_1i_2j_3j_4} \) hold for the massless case as well.
Let us focus now in the renormalization scheme to be chosen in the massless situation. The massless theory has infrared divergences at zero external momenta, so it would be worthy to define the renormalized theory of the few bare primitively divergent vertex parts to be finite after using the divergent normalization constants. It is very simple to employ two independent renormalization schemes in order to compute critical exponents: normalization conditions and minimal subtraction. We emphasize that the integrals involved can be resolved for arbitrary external momenta so that the most convenient form of the results can be used to pursue the computation of the critical exponents in either renormalization scheme. We postpone this discussion to the next section. For the time being we shall analyze basic facts regarding the structure of the normalization conditions. In the present section, a thorough analysis of the one-loop four-point diagram in different limits shall be worked out along with the quotation of higher loop diagrams (extracted from Appendix B).

Normalization conditions is appealing due to its simplicity after the choice of the symmetry(subtraction) point for the external momenta taken at nonzero value. Let $k_i$ be the external momenta of a $(d-1)$-dimensional transversal space and let $\kappa$ be the external momentum scale along the plates where the renormalized theory is defined. At the symmetry point we choose $k_i, k_j = \frac{\kappa^2}{4}(4\delta_{ij} - 1)$ leading to $(k_i + k_j)^2 = \kappa^2$. We fix the external momentum scale of the two-point function at $k^2 = \kappa^2 = 1$. The multiplicative renormalization can be achieved through conditions on the primitively divergent bare vertices at zero mass, such that their renormalized versions take the following values at the symmetry point:

\begin{align}
\Gamma^{(2)}_R (k = 0, j = 0, g, 0) &= \sigma^2 \tau^2, \\
\frac{\partial \Gamma^{(2)}_R (k = \kappa, j = 0, g, 0)}{\partial k^2} &\Big|_{k^2 = \kappa^2} = 1,
\Gamma^{(4)}_R (k_i, i_l = 0, g, 0) &\Big|_{SP} = g, \\
\Gamma^{(2,1)}_R (k_1, i_1 = 0, k_2, i_2 = 0, Q, j_l = 0, g, 0) &\Big|_{SP} = 1,
\end{align}

It is important to mention that the symmetry point implies that the insertion momentum in last equation satisfies $Q^2 = (k_1 + k_2)^2$.

Multiplicative renormalization arguments can be most easily implemented when the bare theory is regularized through the ultraviolet cutoff. Indeed, when the normalization conditions given above are replaced into the renormalized vertex parts defined by

\begin{equation}
\Gamma^{(N,M)}_R (p_n, i_n, Q_m, i'_m, g, 0) = (Z_{\phi}^{(\tau)})^N (Z_{\phi^2}^{(\tau)})^M \Gamma^{(N,M)} (p_n, i_n, Q_m, i'_m, \lambda_0, \Lambda),
\end{equation}
they turn out to render them automatically finite when the regulator $\Lambda$ is taken to infinity. Although we mentioned the cutoff, we could also use another regularization scheme. In fact, we shall shift the argument to consider dimensionally regularized diagrams. We shall utilize the cutoff whenever we can successfully simplify the point under consideration.

Imposing that the bare theory does not depend on the momentum scale where the renormalized theory is defined we find a renormalization group equation in terms of dimensionful quantities. At the critical dimension the coupling constant is dimensionful just as discussed in the massive setting. Away from the critical dimension, similar arguments can be devised to go from dimensionful quantities to dimensionless amounts.

Let the flow functions be defined by the expressions

$$
\beta^{(\tau)}(\kappa, g) = \kappa \frac{\partial g}{\partial \kappa}, \quad \gamma_\phi^{(\tau)} = \kappa \frac{\partial \ln Z_\phi^{(\tau)}}{\partial \kappa}
$$

and

$$
\gamma_{\phi^2}^{(\tau)} = -\kappa \frac{\partial \ln Z_{\phi^2}^{(\tau)}}{\partial \kappa}.
$$

The renormalized (bare) dimensionful coupling constant is defined in terms of $\kappa$ as

$$
g = \kappa \epsilon u(\lambda = \kappa \epsilon u_0),
$$

where $u(u_0)$ is the dimensionless renormalized coupling constant. In order to get rid of undesirable dimensionful parameters when $d = 4 - \epsilon$, define the Gell-Mann-Low function $[\beta(g, \kappa)]_{GL} = -\epsilon g + \beta(g, \kappa)$. Consequently, we find that $[\beta(g, \kappa)]_{GL} \frac{\partial}{\partial g} = \beta(u) \frac{\partial}{\partial u}$, and the renormalization group equation for the multiplicatively renormalized vertex parts read

$$
(k \frac{\partial}{\partial \kappa} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \gamma_\phi^{(\tau)} + M \gamma_{\phi^2}^{(\tau)}) \Gamma_{R}^{(N,M)}(p_n, i_n, Q_m, i'_m, u, 0) = 0,
$$

where $\beta^{(\tau)}(u) = -\epsilon \left( \frac{\partial \ln u_0^{(\tau)}}{\partial u} \right)$, $\gamma_\phi^{(\tau)}(u) = \beta^{(\tau)} \left( \frac{\partial \ln Z_\phi^{(\tau)}}{\partial \kappa} \right)$ and $\gamma_{\phi^2}^{(\tau)} = \beta^{(\tau)} \left( \frac{\partial \ln Z_{\phi^2}^{(\tau)}}{\partial \kappa} \right)$. The combinations $\bar{Z}_\phi^{(\tau)} = Z_\phi^{(\tau)} Z_\phi^{(\tau)}$ and $\bar{\gamma}_{\phi^2}^{(\tau)} = \beta^{(\tau)} \left( \frac{\partial \ln \bar{Z}_{\phi^2}^{(\tau)}}{\partial \kappa} \right)$ shall also be employed. We emphasize that the dynamic variable now is the external momentum scale where the renormalized theory is defined. The solution is identical to that from the ordinary $\phi^4$ theory and it is not going to be discussed here. As discussed in the massive theory, we employ solely the definitions above for the sake of determination of fixed points and other universal quantities via diagrammatic tools.

To begin with, we write down the one-loop contribution for the four-point function, namely

$$
I_2^{(\tau)}(k; i; \sigma) = \sigma \sum_{j=-\infty}^{\infty} \int d^{d-1} q \frac{1}{[(q)^2 + (\sigma)^2 (j + \tau)^2]} \times \frac{1}{[(q + k)^2 + (\sigma)^2 (j + \tau + i)^2]}.
$$

We utilize Feynman parameters to solve the integral over the continuous momentum using
standard formulae for dimensional regularization. Using Eqs. (10) and (12) along with the identity \( \sqrt{\pi} \Gamma(\frac{d-1}{2d})S_{d-1}=\Gamma(\frac{d}{2})S_d \), where \( S_d \) is the area of the \( d \)-dimensional unit sphere, we obtain a result proportional to \( S_d \). As before, we neutralize this angular factor appearing in each loop integral by absorbing it in a redefinition of the coupling constant. Collecting this steps together we determine the massless expression for the four-point one-loop contribution in the form

\[
I_2^{(\tau)}(k, i; \sigma) = \frac{I_2^{(\tau)}(k; i; \sigma)}{S_d} = \frac{1}{\epsilon} \left( (1 - \frac{\epsilon}{2}) \int_0^1 dx [x(1 - x)(k^2 + \sigma^2i^2)^{-\frac{\epsilon}{2}} + \frac{\epsilon}{2} \Gamma(2 - \frac{\epsilon}{2}) F_{\frac{\tau}{2}}^{(\tau)}(k, i; \sigma) \right),
\]

where

\[
F_{\alpha}^{(\tau)}(k, i; \sigma) = \sigma^{-2\alpha} \int_0^1 dx f_{\frac{\tau}{2} + \alpha}\left( \tau + x i, h'(k, i, \sigma) \right),
\]

and

\[
h'(k, i, \sigma) = \sigma^{-1} \sqrt{x(1 - x)(k^2 + \sigma^2i^2)}.
\]

Note that Eqs. (49) and (50) are the massless counterparts of the massive definitions Eqs. (17) and (18), respectively. Using the representation (13), the above definitions lead to

\[
F_{\frac{\tau}{2}}^{(\tau)}(k, i; \sigma) = 4\sigma^{-\epsilon} \sum_{n=1}^{\infty} \int_0^1 dx \cos(2\pi n(\tau + ix)\sqrt{\frac{\sigma^2}{[x(1 - x)(k^2 + i^2\sigma^2)]^{\frac{\epsilon}{2}}}}) \times K_{\frac{\tau}{2}}(2\pi n\sigma^{-1}[x(1 - x)(k^2 + i^2\sigma^2)]^{\frac{1}{2}}).
\]

For both normalization conditions and minimal subtraction, the external quasi-momentum label can be taken as the zero mode value \( (i = 0) \) without loss of generality, which simplifies our task. Recalling that the finite size correction is \( O(e^0) \) and neglecting \( O(\epsilon) \) terms, we can rewrite Eq.(48) as

\[
I_2^{(\tau)}(k, i = 0; L) = \frac{1}{\epsilon} \left( 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} \int_0^1 dx \ln[x(1 - x)k^2] \right)
+ 2 \sum_{n=1}^{\infty} \int_0^1 dx \cos(2\pi n\tau) K_0(nL[x(1 - x)k^2]^{\frac{1}{2}}).
\]

We can readily take the limit \( L \rightarrow \infty (\sigma = \frac{2\pi}{L}) \) by considering the Bessel function for large values of its argument. Its asymptotic value is given by \( K_0(z') = \sqrt{\frac{2\pi}{z'}}e^{-z'}[1 + O(1/z')] \) with \( z' = nLk\sqrt{x(1 - x)} \equiv nB \) and \( n = 1, 2, \ldots \). Therefore, the correction becomes

\[
2 \sum_{n=1}^{\infty} \int_0^1 dx \cos(2\pi n\tau) K_0(nL[x(1 - x)k^2]^{\frac{1}{2}}) = \sqrt{\frac{2\pi}{Lk}} \int_0^1 dx [x(1 - x)]^{-\frac{1}{4}}
\times \sum_{n=1}^{\infty} \cos(2\pi n\tau) \exp(-nB).
\]
The remaining series is a geometric one that can be computed when we express the cosine in terms of the complex exponents. We can write it in terms of an upper bound through the inequality
\[
\sum_{n=1}^{\infty} \cos(2\pi n\tau) e^{-nB} = \left[ \frac{1}{1 - e^{-B}\delta_{\tau,0} + e^{-B}\delta_{\tau,\frac{1}{2}}} - 1 \right] < 1. \tag{54}
\]
If we take directly the limit \( L \to \infty \) before performing the integral we see that this term vanishes and multiplies the prefactor which is also zero. Instead, if we use the upper bound we can estimate the integral which in the limit \( L \to \infty \) implies the result
\[
2 \left[ \sum_{n=1}^{\infty} \int_0^1 dx \cos(2\pi n\tau) K_0(nL[x(1-x)k^2]) \right] < \sqrt{\frac{2\pi}{Lk} \left( \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{2})} \right)^2} \to 0. \tag{55}
\]
Therefore, the finite size correction interpolates from the contribution for large but finite values of \( L \) and vanishes to infinite values of \( L \) even in the massless case. Sometimes, it is useful to define the bulk parametric integral \( i(k) = \int_0^1 dx \ln[x(1-x)k^2] \), take \( k = \sqrt{k^2} \) and consider the correction term at \( \epsilon = 0 \).

Let us try to comprehend the limit \( L \to 0 \). Using Eq.\((24)\) with the replacements \( x \to z = L\sqrt{x(1-x)k^2}, \ t = \frac{2\pi \tau}{z} \) and taking into account the above observations, we find at \( i = 0 \) a simple expression useful for minimal subtraction
\[
I_{2\tau}(k, i = 0; L) = \frac{1}{\epsilon} \left( 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} i(k) \right) + \left[ \gamma + \int_0^1 dx \ln \left( \frac{Lk(x(1-x))^{\frac{1}{2}}}{2\pi} \right) \right]
\]
\[
+ \frac{1}{2} \int_0^1 dx \frac{1}{\sqrt{(\tau^2 + x(1-x)[Lk/2\pi]^2)}} + \frac{1}{2} \sum_{n=1}^{\infty} \int_0^1 dx \left[ \frac{1}{\sqrt{(x-\tau)^2 + x(1-x)[Lk/2\pi]^2}}} \right] - \frac{1}{n}, \tag{56}
\]
whereas computing it at the symmetry point convenient for normalization conditions (\( k^2 = 1 \)), we obtain
\[
I_{2\sigma}(\sigma) = \frac{1}{\epsilon} \left( 1 + \frac{\epsilon}{2} \right) + \left[ \gamma + \int_0^1 dx \ln \left( \frac{L(x(1-x))^{\frac{1}{2}}}{2\pi} \right) \right]
\]
\[
+ \frac{1}{2} \int_0^1 dx \frac{1}{\sqrt{(\tau^2 + x(1-x)[Lk/2\pi]^2)}} + \frac{1}{2} \sum_{n=1}^{\infty} \int_0^1 dx \left[ \frac{1}{\sqrt{(x+\tau)^2 + x(1-x)[Lk/2\pi]^2}}} \right] - \frac{1}{n}, \tag{57}
\]
Except for the integrals, the $L$-dependence in last equation has pretty much the same form as Eq.(25) has in $r^{-1} = \frac{L}{2\pi \xi}$ described in the massive case. In that case the theory renormalized at “mass” $\xi = 1$ (fixed and finite bulk correlation length) and zero external momenta is completely analogous to our renormalized massless theory at the symmetry $\kappa^2 = 1$ (infinite bulk correlation length). After those choices we just have to recall the variable $L$ is dimensionless. Except for minor modifications like the extra integral on the Feynman parameter $x$ the discussion of the various terms parallels that for the massive case.

In order to study the limit $L \to 0$, let us start with the last two integrals. It is licit to take $L = 0$ inside both of them, such that their summation produces precisely Eq.(27). The first integral is straightforward and its result is $\ln \left[ \frac{L}{4\pi} \right] - 1$. We are then left with the task of evaluating the second integral. The identity

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{-1}{\sqrt{-c}} \arcsin \left[ \frac{2cx + b}{\sqrt{-\Delta}} \right],$$

is valid for $c < 0$ and $\Delta < 0$, where $\Delta = 4ac - b^2$. Performing the identifications $a = \tau^2$ and $b = -c = \frac{L^2}{4\pi^2}$, we get to

$$\int_0^1 \frac{dx}{\sqrt{\tau^2 + \frac{L^2}{4\pi^2} x(1 - x)}} = \frac{4\pi}{L} \arcsin \left[ \frac{1}{\sqrt{1 + \frac{16\pi^2 \tau^2}{L^2}}} \right].$$

Collecting these steps together, we can rewrite the one-loop contribution to the four-point function in the form

$$I_{2SP}^{(\tau)}(\sigma) = \frac{1}{\epsilon} \left( 1 + \frac{\epsilon}{2} \right) + \left[ \gamma + (2\ln 2 - 1) \delta_{\tau,1/2} + \ln \left[ \frac{L}{4\pi} \right] - 1 \right]$$

$$+ \frac{2\pi}{L} \arcsin \left[ \frac{1}{\sqrt{1 + \frac{16\pi^2 \tau^2}{L^2}}} \right].$$

Last equation shows that last term inside the finite size contribution for periodic boundary condition $\tau = 0$ becomes exactly $\frac{\pi^2}{L}$ which differs from the expression in the massive case (with fixed $\xi = 1$) by a factor of $\pi$ due to the effect of performing the integral over the Feynman parameter $x$. The new situation occurs for antiperiodic boundary condition ($\tau = \frac{1}{2}$) in the last term, whose limit $L \to 0$ becomes $\frac{\pi}{L}$. The logarithm is still there just as before in the massive case, but now these two different power-law behavior present in both boundary conditions are dominant in the $L \to 0$ limit. Therefore, the finite size correction is generically enhanced for both boundary conditions in the critical massless theory consistent with the enhancement of fluctuations at this regime. At the critical point crossover starts earlier.
in antiperiodic than in periodic boundary condition as evidenced by our analytical result in above equation. Thus, the massless case has nontrivial aspects in comparison with the massive case, as shown here for the first time, since the second type of crossover discussed in Sec. II for \( t > 0 \) in antiperiodic boundary condition is now absent for \( t = 0 \).

Provided we stay away from the crossover regions characterized by very small values of \( L \), the two descriptions are almost equivalent, even though the equivalence is not complete, as far as the finite size contribution is concerned. The \( \epsilon \)-expansion is well defined in both situations, if the crossover regions are precluded from our analysis. What is really remarkable from the massless and massive analysis is the bulk correlation length independence of the finite size correction. From last equation, finite values of \( L \) persist in the correction even when the starting point of the massless theory corresponds to \( \frac{L}{\xi} \to 0 \). Thus, region c) is available to our scrutiny. Consequently, the previous phenomenological conjecture that the massless limit cannot be understood in terms of \( \epsilon \)-expansion is unfounded. From now on, we are going to consider finite (but not too small values) of \( L \), since the finite size correction has a good behavior and \( \epsilon \)-expansion methods can be utilized without further problems.

We leave the task of computing the higher loop integrals to Appendix B. Here we simply quote the results. The massless counterpart of the integral which contributes to the four-point function at two-loops can be extracted from Eq. (29) by setting \( \mu = 0 \), namely

\[
I_4(k_1, k_2, k_3, k_4, i_1, i_2, i_3, i_4, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2}{(q_1 + \sigma^2(j_1 + \tau)^2)} \times \frac{1}{(q_2 + \sigma^2(j_2 + \tau)^2)} \times \frac{1}{[(P - q_1)^2 + \sigma^2(p - j_1 + \tau)^2]}.
\]

At zero external quasimomenta, there is no loss of generality to approaching the minimal subtraction scheme with arbitrary external momenta or normalization conditions with fixed nonvanishing external momenta. From our discussion in Appendix B we obtain

\[
I_4^{(\tau)}(k_i, 0; \sigma) = \frac{1}{2\epsilon^2} \left( 1 - \frac{\epsilon}{2} - \epsilon i(P) + \epsilon F^{(\tau)}(\frac{PL}{2\pi}, 0) \right).
\]

At the symmetry point \( P^2 = \kappa^2 = 1 \) it takes the simpler form by setting \( \epsilon = 0 \) into the finite size correction

\[
I_{4SP}^{(\tau)}(\sigma) = \frac{1}{2\epsilon^2} \left( 1 + \frac{3\epsilon}{2} + \epsilon F_0^{(\tau)}(\sigma) \right).
\]
Analogously, the integrals contributing to the two-point function at two- and three-loops can be read off from Eqs. (31) and (32) at $\mu = 0$, respectively, see Appendix B. $I_3(k, i = 0; \sigma)$ in a form appropriate to minimal subtraction reads

$$I_3^{(\tau)}(k, \sigma) = -\frac{1}{8\epsilon}((k^2 + \sigma^2 \tau^2)[1 + \frac{\epsilon}{4} - 2\epsilon i_3(k^2 + \sigma^2 \tau^2)] - 2\epsilon \tilde{F}_\epsilon^{(\tau)}(k, i = 0; \sigma) - 4\epsilon F_{\tau,1}(k, i = 0; \sigma)).$$  

(64)

In normalization conditions the derivative of this integral with respect to $k^2$ computed at the symmetry point $k^2 = 1$ can be written as

$$I_{3SP}^{(\tau)}(\sigma) = -\frac{1}{8\epsilon}(1 + \frac{5\epsilon}{4} - 2\epsilon W_0(\sigma)).$$  

(65)

Using a similar reasoning, the solution for $I_5$ appropriate within minimal subtraction of dimensional poles was found to be

$$I_5^{(\tau)}(k, \sigma) = -\frac{1}{6\epsilon^2}((k^2 + \sigma^2 \tau^2)[1 + \frac{\epsilon}{2} - 3\epsilon i_3(k^2 + \sigma^2 \tau^2)] - 3\epsilon \tilde{F}_\tau^{(\tau)}(k, i = 0; \sigma) - 6\epsilon F_{\tau,1}(k, i = 0; \sigma)).$$  

(66)

On the other hand, its derivative in relation to $k^2$ at the symmetry point $k^2 = 1$ is required when applying normalization conditions as our renormalization scheme. Hence,

$$I_{5SP}^{(\tau)}(\sigma) = -\frac{1}{6\epsilon^2}\left(1 + 2\epsilon - 3\epsilon W_0(\sigma)\right).$$  

(67)

Note that we can safely replace the values of the subscript of the additional functions appearing as finite size corrections at $\epsilon = 0$ into Eqs. (62), (64) and (66) when we employ the minimal subtraction scheme. This will facilitate the computations of the normalization functions, since many different functions become identical at $\epsilon = 0$. We have now all the required integrals to compute critical exponents in normalization conditions or minimal subtraction, to which we turn our attention next.

V. CRITICAL EXPONENTS FROM THE MASSLESS APPROACH

The results displayed in last section can be substantiated in the calculation of the critical exponents $\nu$ and $\eta$ at two- and three-loop order, respectively, through diagrammatic (perturbative) methods. We are going to compute the critical indices and show that they are independent of the renormalization scheme, using either normalization conditions or minimal subtraction.
A. **Normalization conditions**

As before, we write the dimensionless bare coupling constant and normalization function $Z_\phi$, $\bar{Z}_{\phi^2}$ as power series in the dimensionless renormalized coupling constant as $u^{(\tau)}_0 = u(1 + a^{(\tau)}_1 u + a^{(\tau)}_2 u^2)$, $Z^{(\tau)}_\phi = 1 + b^{(\tau)}_2 u^2 + b^{(\tau)}_3 u^3$ and $\bar{Z}^{(\tau)}_{\phi^2} = 1 + c^{(\tau)}_1 u + c^{(\tau)}_2 u^2$. A considerable labor can be saved by noting that the structure of these equations are the same as those previously discussed in Sec.III for the massive case. Indeed, using Eqs. (35) from Sec.III and replacing the values of the massless integrals computed at the symmetry point, we can determine the above coefficients in the massless theory as well. Following this trend, we encounter the following values for the coefficients

\[
\begin{align*}
    a^{(\tau)}_1 &= \frac{(N + 8)}{6\epsilon} [1 + \frac{\epsilon}{2} + \frac{\epsilon}{2} F^{(\tau)}_0(\sigma)], \\
    a^{(\tau)}_2 &= \frac{(N + 8)^2}{6\epsilon} + \frac{2N^2 + 23N + 86}{72\epsilon} + \frac{(N + 8)^2 F^{(\tau)}_0(\sigma)}{\epsilon}, \\
    b^{(\tau)}_2 &= -\frac{(N + 2)}{144\epsilon} [1 + \frac{5\epsilon}{4} - 2\epsilon W^{(\tau)}_0(\sigma)], \\
    b^{(\tau)}_3 &= -\frac{(N + 2)(N + 8)}{1296\epsilon^2} [1 + \frac{5\epsilon}{4} + \frac{3\epsilon}{2} F^{(\tau)}_0(\sigma)], \\
    c^{(\tau)}_1 &= \frac{(N + 2)}{6\epsilon} [1 + \frac{\epsilon}{2} + \frac{\epsilon}{2} F^{(\tau)}_0(\sigma)], \\
    c^{(\tau)}_2 &= \frac{(N + 2)(N + 5)}{36\epsilon^2} + \frac{2N^2 + 11N + 14}{72\epsilon} + \frac{(N^2 + 7N + 10)}{36\epsilon} F^{(\tau)}_0(\sigma).
\end{align*}
\]

Now, introducing the coefficients $a^{(\tau)}_1$ as well as $a^{(\tau)}_2$ appropriate to the massless formulation into Eqs. (36) from Sec. III, we first determine the flow function in the zero mass limit which yields

\[
\beta^{(\tau)}(u) = -\epsilon u + \left(\frac{N + 8}{6}\right)(1 + \frac{\epsilon}{2} + \frac{\epsilon}{2} F^{(\tau)}_0(\sigma))u^2 - \frac{3N + 14}{12} u^3.
\]

The attractive nontrivial infrared fixed point at two-loop order is found out from the condition $\beta^{(\tau)}(u^*) = 0$, namely

\[
u^* = \left(\frac{6}{N + 8}\right)\epsilon [1 + \left(\frac{9N + 42}{(N + 8)^2} - \frac{1}{2} - \frac{1}{2} F^{(\tau)}_0(\sigma)\right)].
\]

Let us now utilize Eqs. (39a) and (39b) in order to determine the Wilson functions for the massless formalism. Inserting the coefficients determined above into that equation and proceeding along the same lines as before, we get the expressions

\[
\begin{align*}
    \gamma^{(\tau)}_\phi &= u^2 \frac{(N + 2)}{72} [1 + \frac{5\epsilon}{4} - 2\epsilon W^{(\tau)}_0(\sigma)] - \frac{(N + 8)}{12} (1 - 4W^{(\tau)}_0(\sigma) - 2F^{(\tau)}_0(\sigma))u, \\
    \tilde{\gamma}^{(\tau)}_{\phi^2} &= \frac{(N + 2)}{6} [1 + \frac{\epsilon}{2} + \frac{\epsilon}{2} F^{(\tau)}_0(\sigma)]u - \frac{(N + 2)}{12} u^2.
\end{align*}
\]
Computing these functions at the nontrivial infrared fixed point $u^*$, we find that, i) $\eta = \gamma_\phi^{(r)}(u^*)$ is identical to Eq.(41) and ii) $\bar{\gamma}_{\phi^2}^{(r)}(u^*)$ is equal to the expression in Eq.(42), which leads to the exponent $\nu$ from Eq.(43).

The massive treatment in the ultraviolet regime is therefore completely equivalent to the massless framework at the infrared region, for they originate the same critical indices, even though the intermediate steps are completely distinct in the two formalisms, in compliance with universality. This thorough treatment of massless fields shall be concluded in a moment with the computation using minimal subtraction.

### B. Minimal subtraction

Here we are not going to calculate explicitly the critical exponents. Instead, we are going to calculate the fixed point as well as the functions $\gamma_\phi^{(r)}$ and $\bar{\gamma}_{\phi^2}^{(r)}$ at the fixed point. As these functions at the fixed point are universal, they should be equal to the ones obtained using normalization conditions, leading to the same exponents in either renormalization scheme.

The dimensionless bare couplings and the renormalization functions are defined in minimal subtraction by

\begin{equation}
    u_0^{(r)}(\tau) = u[1 + \sum_{i=1}^{\infty} a_i^{(r)}(\epsilon)u^i],
\end{equation}

\begin{equation}
    Z_\phi^{(r)} = 1 + \sum_{i=1}^{\infty} b_i^{(r)}(\epsilon)u^i,
\end{equation}

\begin{equation}
    \bar{Z}_{\phi^2}^{(r)} = 1 + \sum_{i=1}^{\infty} c_i^{(r)}(\epsilon)u^i.
\end{equation}

The renormalized vertices

\begin{equation}
    \Gamma_R^{(2)}(k, u, \kappa) = Z_\phi^{(r)}\Gamma_R^{(2)}(k, u_0^{(r)}, \kappa),
\end{equation}

\begin{equation}
    \Gamma_R^{(4)}(k_1, u, \kappa) = (Z_\phi^{(r)})^2 \Gamma_R^{(4)}(k_1, u_0^{(r)}, \kappa),
\end{equation}

\begin{equation}
    \Gamma_R^{(2,1)}(k_1, k_2, p; u, \kappa) = \bar{Z}_{\phi^2}^{(r)}\Gamma_R^{(2,1)}(k_1, k_2, p, u_0^{(r)}, \kappa),
\end{equation}

should be finite when $\epsilon \to 0$ to any desired order in $u$. Observe that the external momenta into the bare vertices are multiplied by $\kappa^{-1}$ and all the external quasimomenta of the diagrams are set to zero in order to simplify matters. The coefficients $a_i^{(r)}(\epsilon)$, $b_i^{(r)}(\epsilon)$ and $c_i^{(r)}(\epsilon)$ are obtained by requiring that the poles in $\epsilon$ be minimally subtracted. The bare vertices can
now be expressed as

\[
\Gamma^{(2)}(k, u_0^{(\tau)}, \kappa) = k^2(1 - B_2^{(\tau)}(u_0^{(\tau)})^2 + B_3^{(\tau)}(u_0^{(\tau)})^3), \tag{74a}
\]

\[
\Gamma^{(4)}(k, u_0^{(\tau)}, \kappa) = \kappa^{\epsilon} u_0^{(\tau)} [1 - A_2^{(\tau)} u_0^{(\tau)} + (A_2^{(\tau)}(1) + A_2^{(\tau)}(2))(u_0^{(\tau)})^2], \tag{74b}
\]

\[
\Gamma^{(2,1)}(k_1, k_2, p; u_0^{(\tau)}, \kappa) = 1 - C_1^{(\tau)} u_0^{(\tau)} + (C_2^{(\tau)}(1) + C_2^{(\tau)}(2))(u_0^{(\tau)})^2. \tag{74c}
\]

We recognize that \(B_2^{(\tau)}\) is proportional to the integral \(I_3^{(\tau)}\) and \(B_3^{(\tau)}\) is proportional to \(I_5^{(\tau)}\). In the remainder we shall suppress the upper indices in the integrals referring to the boundary condition, but keeping them implicitly. Explicitly, the coefficients can be written in terms of integrals like

\[
A_1^{(\tau)} = \frac{(N + 8)}{18} [I_2(k_1 + k_2) + I_2(k_1 + k_3) + I_2(k_2 + k_3)], \tag{75a}
\]

\[
A_2^{(\tau)}(1) = \frac{(N^2 + 6N + 20)}{108} [I_2^2(k_1 + k_2) + I_2^2(k_1 + k_3) + I_2^2(k_2 + k_3)], \tag{75b}
\]

\[
A_2^{(\tau)}(2) = \frac{(5N + 22)}{54} [I_4(k_1) + 5 \text{ permutations}], \tag{75c}
\]

\[
B_2^{(\tau)} = \frac{(N + 2)}{18} I_3(k_1), \tag{75d}
\]

\[
B_3^{(\tau)} = \frac{(N + 2)(N + 8)}{108} I_5(k_1), \tag{75e}
\]

\[
C_1^{(\tau)} = \frac{N + 2}{18} [I_2(k_1 + k_2) + I_2(k_1 + k_3) + I_2(k_2 + k_3)], \tag{75f}
\]

\[
C_2^{(\tau)}(1) = \frac{(N + 2)^2}{108} [I_2^2(k_1 + k_2) + I_2^2(k_1 + k_3) + I_2^2(k_2 + k_3)], \tag{75g}
\]

\[
C_2^{(\tau)}(2) = \frac{N + 2}{36} [I_4(k_1) + 5 \text{ permutations}]. \tag{75h}
\]

Firstly substitute Eqs. (75) inside Eqs. (74). Next, utilize Eq. (72a) into Eqs. (74). Finally, impose that the renormalized vertex parts expressed as Eqs. (73) are finite via minimal subtraction of dimensional poles. Interestingly, all the logarithmic integrals in the external momenta as well as the finite size corrections appearing in \(I_2, I_3, I_4, \) and \(I_5\) cancel each other in the algorithm of renormalization. This results in the determination of the coefficients in minimal subtraction, or in other words

\[
u_0^{(\tau)} = u(1 + \frac{(N + 8)}{6\epsilon}u + [\frac{(N + 8)^2}{36\epsilon^2} - \frac{(3N + 14)}{24\epsilon}]u^2), \tag{76a}
\]

\[
\hat{Z}_\phi^{(\tau)} = 1 - \frac{N + 2}{144\epsilon}u^2 + [-\frac{(N + 2)(N + 8)}{1296\epsilon^2} + \frac{(N + 2)(N + 8)}{5184\epsilon}]u^3, \tag{76b}
\]

\[
\hat{Z}_\phi^{(\tau)} = 1 + \frac{N + 2}{6\epsilon}u + [\frac{(N + 2)(N + 5)}{36\epsilon^2} - \frac{(N + 2)}{24\epsilon}]u^2). \tag{76c}
\]
What is amazing is that in minimal subtraction the above mentioned renormalization functions do not depend on the boundary condition explicitly, since all that dependence cancelled out naturally, i.e., they do not appear in the right hand side of the above equations.

Furthermore, from the renormalization functions one can obtain:

\[ \gamma_{\phi}^{(\tau)} = \frac{(N + 2)}{72} u^2 - \frac{(N + 2)(N + 8)}{1728} u^3, \]  
\[ \bar{\gamma}_{\phi^2}^{(\tau)} = \frac{(N + 2)}{6} u[1 - \frac{1}{2} u]. \]  

(77)  
(78)

The fixed point is defined by \( \beta^{(\tau)}(u^*) = 0 \). Then, we find:

\[ u^* = \frac{6}{8 + N} \epsilon \left\{ 1 + \epsilon \frac{(9N + 42)}{(8 + N)^2} \right\}. \]

(79)

Substitution of this result into the renormalization constants will give at the fixed point \( \gamma_{\phi}^{* (\tau)} = \eta \), whereas, in addition, we have

\[ \bar{\gamma}_{\phi^2}^{* (\tau)} = \frac{(N + 2)}{(N + 8)} \epsilon \left[ 1 + \frac{6(N + 3)}{(N + 8)^2} \epsilon \right]. \]

(80)

This expression is equal to Eq.(42) obtained using the massive method and consequently lead to the same exponent \( \nu \) from Eq.(43). Notice that Eqs.(76)-(80) are the same as their counterpart obtained in minimal subtraction for the usual bulk theory. All the construction of renormalization schemes developed in the present work are thus consistent with universality, which states that critical exponents (among other universal quantities) are scheme independent as we have shown herein.

VI. DISCUSSION OF THE RESULTS AND CONCLUSION

We have computed critical exponents at higher loop order from finite size layered systems subject to periodic and antiperiodic boundary conditions on the limiting surfaces of the slab (parallel plate) geometry by defining a one-particle irreducible (1PI) vertex parts formalism to the previous field-theoretic framework of Green functions for those systems introduced by Nemirovsky and Freed earlier. In order to do that, we determine normalization functions as well as fixed points and show that they depend on the boundary conditions whenever we use normalization conditions either in the massive or massless methods. In minimal subtraction, however, we find that these quantities are independent of the boundary condition.
We confirm that for large values of $L$ and in the $L \to \infty$ all the finite size corrections are under control and the critical exponents obtained in this way are identical to those from $d$-dimensional (bulk) universality class. In the case of periodic boundary conditions we proved that dimensional crossover only occurs at very small values of $L$, the behavior of the finite size correction is proportional to $L^{-1}$ in the massive and massless cases, being independent of the value of the bulk correlation length, although the coefficient of this term is larger in the massless case. This extends the previous analysis by $NF$ performed solely in the massive case. As far as the crossover regimes are concerned, antiperiodic boundary conditions do not present the simple behavior from $PBC$. In addition to the “dimensional crossover” previously discussed by $NF$ below the critical temperature $t < 0$, we have found a new regime of crossover for $ABC$ which exists only for $t > 0$ characterized by much shorter values of $L$ than its dimensional crossover counterpart occurring in $PBC$, i.e., when the lattice constant is of the same order of magnitude of the distance between the limiting plates. Moreover, the crossover in the massless case $t = 0$ for $ABC$ starts earlier than in the massive case. Actually, the finite size correction is proportional to $L^{-2}$ in that case, which starts to modify the bulk critical behavior for larger values of $L$ than its counterpart in the $PBC$ case. Thus, fluctuations at the critical point enhance the effect of crossover in finite systems in these simple boundary conditions.

Let us discuss the connection between one previous two-loop calculation using $NF$ formulation with our work of higher loop integrals computed in whole detail in Appendix B. Actually, Krech and Dietrich Ref. [7] used $NF$ formulation for massless fields in their computation of Casimir amplitudes (see Appendix A therein). However, only one two loop diagram was computed for the free energy, which actually corresponds to a squared tadpole diagram. But this is equivalent to a one loop computation, since normalization constants, fixed points, etc., were computed only at one-loop order. Unfortunately, perhaps the lack of a better representation for the function $f_\alpha(a,b)$ at the time of the writing of that paper prevented them to obtain simple answers in terms of elementary primitive functions. Consequently, they abandoned $NF$ method in momentum space in their subsequent work with their collaborators. The work presented here, on the other hand, permits us to go beyond the simple conclusions of previous analysis: even though the exponents are identical to those from the bulk system when we avoid the crossover region which is certainly not too exciting, the crossover regimes assessed by the analytical results described in the present paper sheds
new light on the fundamental difference between massive and massless regimes to finite size systems criticality, and how fluctuations enhance the effect of finiteness in the latter. Moreover, a consistent description in terms of massless fields with inequivalent crossover regions in comparison with the massive case for $ABC$ is certainly a step forward which cannot be underestimated. Since the consistency of the critical regime for both boundary conditions implies that phenomenological scaling relying on the failure of $\epsilon$-expansion results in the region $\frac{L}{\xi} \to 0$ is incorrect, the present study should be considered the starting point to widen the subject and put it on new grounds, such as computing higher order universal quantities like amplitudes in order to improve our present knowledge of critical finite systems.

Away from the crossover regimes, we have shown a complete equivalence between the formulations using either massive or massless fields, where the renormalized mass scale plays the analogue role of the external momenta scale used to fix the (new) massless theory in normalization conditions. A step further is the minimal subtraction treatment for massless fields.

Thus, our critical exponents results confirm the previous expectations pointed out by Nemirovsky and Freed concerning a behavior identical to those describing the bulk system in higher order loop computations. Contrary to previous conjectures, it is not the boundedness variable $\frac{L}{\xi} < 1$ which makes the $\epsilon$-expansion invalid, but small values of $L$ decreasing below a given threshold which are responsible for crossover. This crossover description is far from being completely figured out, but the resources developed in our trend here should be encouraging to tackle this problem. We hope our discussion in the present work can serve as an introduction to this subject and might be valuable when new perturbative methods to treat the limit $L \to 0$ become available.

A rather interesting topic is to extend the $1PI$ method at two-loop order to treat systems with Neumann and Dirichlet boundary conditions, since they are more appealing from the phenomenological viewpoint. They characterize free surfaces \cite{24}, which disturb further the system due to the breaking of translational invariance along the finite directions. First, the quasi-momentum are not conserved in the treatment of these boundary conditions. Consequently, in order to renormalize the theory we have to introduce distinct external fields, one in the bulk and other in the limiting surfaces. This implies that the surface parameter becomes relevant and requires a new normalization function to renormalize it. In spite of these additional aspects, we expect that this topic can be investigated along a
similar line of reasoning to that employed in the present work.

Another intriguing perspective is to consider the finite size approach to competing systems of the Lifshitz type \[25–27\]. It remains to be seen if the competing axes with arbitrary momentum powers permit exact results when the finite size direction points along any of them. The last few years have witnessed promising new applications of this kind of field theory from quantum field theory to quantum gravity and cosmology. The most direct application is to study aspects of space(time)s with one compact spatial dimension, called “Lifshitz space(time)”, e. g., in the Horava-Lifshitz theory of gravity \[28\] and other simpler quantum field theories \[29\].

Finally, it is possible that the results obtained in the present paper must be used to update certain computations of amplitude ratios of certain thermodynamical potentials. Other aspects like extension of the present method in the analysis of semi-infinite systems are also worthwhile.

VII. ACKNOWLEDGMENTS

JBSJ acknowledges financial support by CNPq from Brazil.

Appendix A: Higher order massive integrals in dimensional regularization

Since the relevant one-loop integral integral contributing to the four-point was discussed in detail in the main text, we shall discuss only two- and three-loop diagrams, using the one-loop result extracted from the text whenever possible.

The required integrals are computed at unity mass, which makes \( r \equiv \frac{\sigma}{\mu} = \frac{2\pi}{L} \). The simpler contributions come from the two- and three-loop diagrams of the two-point function, respectively, given by the following expressions

\[
I_3(k, i, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2}{(q_1^2 + \sigma^2(j_1 + \tau)^2 + 1)(q_2^2 + \sigma^2(j_2 + \tau)^2 + 1)} \times \frac{1}{[(q_1 + q_2 + k)^2 + \sigma^2(j_1 + j_2 + i + \tau)^2 + 1]},
\]

(A1)
\[ I_5(k, i, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2 d^{d-1}q_3}{(q_1^2 + \sigma^2(j_1 + \tau)^2 + 1)(q_2^2 + \sigma^2(j_2 + \tau)^2 + 1)} \]
\[ \times \frac{1}{(q_3^2 + \sigma^2(j_3 + \tau)^2 + 1)((q_1 + q_2 + k)^2 + \sigma^2(j_1 + j_2 + i + \tau)^2 + 1)} \]
\[ \times \frac{1}{[(q_1 + q_3 + k)^2 + \sigma^2(j_1 + j_3 + i + \tau)^2 + 1]} \] \hfill (A2)

The remaining integral is the nontrivial contribution to the four-point function at two loops, namely

\[ I_4(k_1, k_2, k_3, k_4, i_1, i_2, i_3, i_4, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2}{(q_1^2 + \sigma^2(j_1 + \tau)^2 + 1)} \]
\[ \times \frac{1}{(q_2^2 + \sigma^2(j_2 + \tau)^2 + 1)((q_1 - q_2 + k)^2 + \sigma^2(j_1 - j_2 + i_3 + \tau)^2 + 1)} \]
\[ \times \frac{1}{[(P - q_1)^2 + \sigma^2(p - j_1 + \tau)^2 + 1]} \] \hfill (A3)

where \( P = k_1 + k_2 \) is the external momenta along the plates and \( p = i_1 + i_2 \) is a discrete “external” quasi-momentum label. For convenience, we shall compute all the integrals with all external quasi-momenta set to zero.

The systematics to solve \( I_3 \) and \( I_5 \) is very similar: in the former there appears a four-point one-loop subdiagram, whilst in the latter a squared of that object takes place. We first solve the internal bubble(s) belonging to \( I_3 \) (\( I_5 \)), and use Feynman parameters to solve the external resulting bubble. It is important to divide each loop integral by the unit area of the \( d \)-dimensional sphere \( S_d \), a standard procedure within this technique.

The objects required to our purposes are the derivative of those integrals with respect to the external momenta, setting the external momenta at the symmetry point in the end of the process.

Let us apply this general strategy to perform a detailed computation of \( I_3' \) at zero external momentum and quasi-momentum. First, set the external quasi-momentum \( i = 0 \) inside Eq.\((A1)\) in order to simplify our task. Let us rewrite this integral in terms of the one-loop subdiagram \( I_2^{(r)}(k, i = 0, r) \) in the form

\[ I_3^{(r)}(k, i = 0; \sigma) = \sigma \sum_{j = -\infty}^{\infty} \int d^{d-1}q \frac{I_2^{(r)}(q + k, j; \sigma)}{[(q)^2 + \sigma^2(j + \tau)^2 + 1]} \] \hfill (A4)

Sometimes, it is appropriate to perform the change of variables \( q' = q + k \), such that the external momenta is exchanged to the external subdiagram; see below. Now using Eq.\((16)\)
along with the definitions Eqs. (17), (18) given in the text in order to solve the internal bubble, we find
\[
I_{3}^{(\tau)}(k, i = 0; \sigma) = \sigma \frac{1}{\epsilon} \left( 1 - \frac{\epsilon}{2} \right) \int_{0}^{1} dx \sum_{j = -\infty}^{\infty} \int \frac{d^{d-1}q}{[(q)^{2} + \sigma^{2}(j + \tau)^{2} + 1][x(1-x)][(q+k)^{2} + \sigma^{2}j^{2} + 1]^{\frac{1}{2}}} + \frac{\epsilon}{2} \Gamma(2 - \frac{\epsilon}{2}) \int \frac{d^{d-1}q}{[(q)^{2} + \sigma^{2}(j + \tau)^{2} + 1]} F_{\frac{\tau}{2}}^{(\tau)}(q, k, i = 0; \sigma) .
\] (A5)

Before going ahead define the objects
\[
F_{\alpha, \beta}^{(\tau)}(k, i; \sigma) \equiv \frac{1}{S_{d}} \sigma \sum_{j = -\infty}^{\infty} \int d^{d-1}q \frac{F_{\alpha}^{(\tau)}(q + k, j + i; \sigma)}{[(q)^{2} + \sigma^{2}(j + \tau)^{2} + 1]^{\frac{1}{2}}},
\] (A6)
\[
F_{\alpha}^{(\tau)}(\sigma) \equiv \frac{\partial F_{\alpha, 1}^{(\tau)}(k, i; \tau)}{\partial k^{2}} |_{(k, i) = 0},
\] (A7)
\[
i_{3}(k, \sigma, x) = \sum_{j = -\infty}^{\infty} \int \frac{d^{d-1}q}{[q^{2} + \sigma^{2}(j + \tau)^{2} + 1][q + k^{2} + \sigma^{2}j^{2} + 1]} F_{\frac{\tau}{2}}^{(\tau)}(q, k, i = 0; \sigma) .
\] (A8)

which shall be important in what follows. Inserting last equations into the expression for \( I_{3}^{(\tau)} \), the latter can be rewritten as
\[
I_{3}^{(\tau)}(k, i = 0; \sigma) = \frac{1}{\epsilon} \left( \sigma \left( 1 - \frac{\epsilon}{2} \right) \int_{0}^{1} dx [x(1-x)]^{-\frac{\epsilon}{2}} i_{3}(k, \sigma, x) \right.
\]
\[
+ S_{d} \frac{\epsilon}{2} \Gamma(2 - \frac{\epsilon}{2}) F_{\frac{\tau}{2}}^{(\tau)}(k, i = 0; \sigma) \Bigg). \] (A9)

Next, let us compute \( i_{3}(k, \sigma, x) \). First, perform the change of variables \( q' = q + k \). We utilize another Feynman parameter which leads to
\[
i_{3}(k, \sigma, x) = \frac{\Gamma(1 + \frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})} \int_{0}^{1} dy (1 - y)^{\frac{\epsilon}{2} - 1} \sum_{j = -\infty}^{\infty} \int d^{d-1}q' \]
\[
\times \frac{1}{\left( q'^{2} - 2ykq' + yk^{2} + y + \sigma^{2}(j + y\tau)^{2} + y(1-y)\sigma^{2}\tau^{2} + \frac{1-y}{x(1-x)} \right)^{\frac{1}{2}}}. \] (A10)

Resolving the momentum integral using Eq. (A10), we obtain after expanding in \( \epsilon = 4 - d \) the result
\[
i_{3}(k, \sigma, x) = \frac{S_{d-1} \Gamma(d-1)(-\frac{1}{2} + \epsilon)}{2 \Gamma(\frac{\epsilon}{2})} \int_{0}^{1} dy (1 - y)^{\frac{\epsilon}{2} - 1} \]
\[
\sum_{j = -\infty}^{\infty} \left[ y(1-y)[k^{2} + \sigma^{2}\tau^{2}] + \sigma^{2}(j + y\tau)^{2} + y + \frac{1-y}{x(1-x)} \right]^{\frac{1}{2} - \epsilon}. \] (A11)
Utilize the identity $\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)S_{d-1} = \Gamma\left(\frac{d}{2}\right)S_d$, absorb $S_d$ in the redefinition of the coupling constant just as explained for the one-loop four-point graph in the main text and expand in $\epsilon$. Plugging these steps in $I_3^{(\tau)}(k, i = 0; \sigma)$, it is easy to show that

$$I_3^{(\tau)}(k, i = 0, \sigma) = \frac{1}{\epsilon} \left[ \sigma(1 - \frac{\epsilon}{2}) \frac{\Gamma(2 - \frac{\epsilon}{2}) \Gamma\left(-\frac{\epsilon}{2}\right)}{2\sqrt{\pi}} \right] \int_0^1 dx [x(1 - x)]^{-\frac{\epsilon}{2}}$$

$$\times \int_0^1 dy (1 - y)^{-\frac{\epsilon}{2} - 1} \sum_{j = -\infty}^{\infty} \left[ y(1 - y)[k^2 + \sigma^2 r^2] + \sigma^2(j + y\tau)^2 + y + \frac{1 - y}{x(1 - x)} \right]^{\frac{1}{2} - \epsilon}$$

$$+ \frac{\epsilon}{2} \Gamma(2 - \frac{\epsilon}{2}) F_{\tau, 1}^{(\tau)}(k, i = 0; \sigma).$$

Now, perform the derivative with respect to $k^2$ at $k^2 = 0$. Define $I_{3SP}^{(\tau)}(\sigma) = \frac{\partial I_3^{(\tau)}(k, i = 0, \sigma)}{\partial k^2}|_{k^2 = 0}$, expanding the argument of the $\Gamma$-functions $\epsilon = 4 - d$, employing the representations Eqs. (12) and (13) to resolving the remaining summation and neglecting higher order terms in $\epsilon$, we get to

$$I_{3SP}^{(\tau)}(\sigma) = -\frac{1}{4\epsilon} \left[ (1 - \epsilon) \int_0^1 dx [x(1 - x)]^{-\frac{\epsilon}{2}} \int_0^1 dy y(1 - y)^{\frac{\epsilon}{2}} \left[ y(1 - y)\tau^2 + y\sigma^{-2} \right. \right.$$

$$\left. + \frac{1 - y}{x(1 - x)\sigma^2} \right]^{\epsilon} + \epsilon \int_0^1 dx [x(1 - x)]^{-\frac{\epsilon}{2}} \int_0^1 dy y(1 - y)^{\frac{\epsilon}{2}}$$

$$\times f_{\frac{\epsilon}{2} + \epsilon}(y\tau, \sqrt{y(1 - y)\tau^2 + \sigma^{-2}y + \frac{1 - y}{x(1 - x)\sigma^{-2}}}) - 2\epsilon F_{\tau, 2}^{(\tau)}(\sigma) \right].$$

This is the explicit form that can be reduced further by noticing that $O(\epsilon)$ terms in this expression can be computed at $\epsilon = 0$. We can simplify the final steps when we write last equation in terms of the following parametric integrals

$$G^{(\tau)}(r) = -2 \int dx dy y \times$$

$$\ln \left[ y(1 - y)\tau^2 + y + \frac{1 - y}{x(1 - x)} \right] - \frac{1}{2},$$

$$H^{(\tau)}(r) = 2 \int dx dy y \times$$

$$f_{\frac{\epsilon}{2} + \epsilon}(y\tau, \sqrt{y(1 - y)\tau^2 + r^{-2}y + \frac{r^{-2}(1 - y)}{x(1 - x)}}),$$

with $r \equiv \sigma$. Inserting these definitions in the remainder of last integral we finally obtain

$$I_{3SP}^{(\tau)}(\sigma) = -\frac{1}{8\epsilon} \left( 1 - \frac{\epsilon}{4} + \epsilon W^{(\tau)}(\sigma) \right),$$

where $W^{(\tau)}(\sigma) = G^{(\tau)}(\sigma) + H^{(\tau)}(\sigma) - 4F_{0}^{(\tau)}(\sigma)$. Let us compute the three-loop contribution for the two-point function following the same reasoning. We can write that integral in the
form
\[ I_5^{(r)}(k, i = 0; \sigma) = \sigma \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{[I_2^{(r)}(q, j; \sigma)]^2}{[(q - k)^2 + \sigma^2(j + \tau)^2 + 1]} \] (A16)

Making use of the result for the one-loop bubble and neglecting higher order corrections in \( \epsilon \) we obtain the following intermediary result
\[ I_5^{(r)}(k, i = 0; \sigma) = \sigma \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{d^{d-1}q}{[(q - k)^2 + \sigma^2(j + \tau)^2 + 1]} \left[ 1 - \epsilon \times \left( \int_0^1 \frac{dx}{[x(1 - x)q^2 + j^2\tau^2 + 1]^2} \right)^2 + \epsilon F_{\frac{j}{2}}^{(r)}(q, \sigma) \left( \int_0^1 dx [x(1 - x)q^2 + j^2\tau^2 + 1]^2 \right) \right]. \] (A17)

It is not difficult to prove that we can rewrite last equation as
\[ I_5^{(r)}(k, i = 0; \sigma) = \frac{1}{\epsilon^2} \left[ \sigma(1 + \epsilon) \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{d^{d-1}q}{[(q - k)^2 + \sigma^2(j + \tau)^2 + 1]} \right] \left( \int_0^1 \frac{dx}{[q^2 + \sigma^2j^2 + \frac{1}{x(1 - x)}]} + \epsilon S_d F_{\frac{j}{2}, 1}^{(r)}(k, \sigma) + O(\epsilon^2) \right). \] (A18)

Define the subintegral
\[ i_5(k, \sigma, x) = \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{d^{d-1}q}{[(q - k)^2 + \sigma^2(j + \tau)^2 + 1][q^2 + \sigma^2j^2 + \frac{1}{x(1 - x)}]}. \] (A19)

We utilize another Feynman parameter in last integral in order to compute the momentum integral employing Eq.(10). The result in terms of parametric integrals reads
\[ i_5(k, \sigma, x) = \frac{S_{d-1} \Gamma(\frac{d-1}{2}) \Gamma(-\frac{1}{2} + \frac{3\sigma}{2})}{2 \Gamma(\epsilon)} \int_0^1 dy (1 - y)^{\epsilon - 1} \sum_{j=-\infty}^{\infty} \left[ y(1 - y)[k^2 + \sigma^2\tau^2] + \sigma^2(j + y\tau)^2 + y + \frac{1 - y}{x(1 - x)} \right]^{\frac{1}{2} - \frac{3\sigma}{2}}. \] (A20)

Once again, use the identity \( \sqrt{\pi} \Gamma(\frac{d-1}{2})S_{d-1} = \Gamma(\frac{d}{2})S_d \) and expand in \( \epsilon \). Replacing it back into \( I_5^{(r)}(k, \sigma) \) and absorbing the factor \( S_d \), we find
\[ I_5^{(r)}(k, i = 0, \sigma) = \frac{1}{\epsilon^2} \left[ \sigma \frac{\Gamma(2 - \frac{1}{2}) \Gamma(-\frac{1}{2} + \frac{3\sigma}{2})}{2 \sqrt{\pi} \Gamma(\epsilon)} \int_0^1 dy (1 - y)^{\epsilon - 1} \sum_{j=-\infty}^{\infty} \left[ y(1 - y)[k^2 + \sigma^2\tau^2] + \sigma^2(j + y\tau)^2 + y + \frac{1 - y}{x(1 - x)} \right]^{\frac{1}{2} - \frac{3\sigma}{2}} \right. \]
\quad \left. + \epsilon F_{\frac{j}{2}, 1}^{(r)}(k, i = 0; \sigma) \right]. \] (A21)
Now performing the derivative and defining $I_{5SSP}^{(r)} = \frac{\partial I_{5SSP}^{(r)}(k, i = 0, \sigma)}{\partial k^2}|_{k^2 = 0}$ results in the expression

$$I_{5SSP}^{(r)}(\sigma) = \frac{1}{\epsilon^2} \left[ \frac{\Gamma(2 - \frac{\epsilon}{2})\Gamma(-\frac{1}{2} + \frac{3\epsilon}{2})}{2\sqrt{\pi}\Gamma(\epsilon)} \right] (1 + \epsilon) \left( \frac{1}{2} - \frac{3\epsilon}{2} \right) \int_0^1 dx \times \int_0^1 dy \left( 1 - y \right)^\epsilon \sum_{j = -\infty}^{\infty} \left[ y(1 - y)\sigma^2 r^2 + \sigma^2 (j + y r)^2 + y + \frac{1 - y}{x(1 - x)} \right]^{\frac{1}{2} - \frac{3\epsilon}{2}} + \epsilon F_{\frac{7}{2}}^{(r)}(\sigma).$$

(A22)

Performing the summation we can rewrite this integral as

$$I_{5SSP}^{(r)}(\sigma) = \frac{1}{\epsilon^2} \left[ \frac{\Gamma(2 - \frac{\epsilon}{2})\Gamma(-\frac{1}{2} + \frac{3\epsilon}{2})}{2\sqrt{\pi}\Gamma(\epsilon)} \right] (1 + \epsilon) \left( \frac{1}{2} - \frac{3\epsilon}{2} \right) \int_0^1 dx \int_0^1 dy \left( 1 - y \right)^\epsilon \times \left[ \Gamma\left( \frac{3\epsilon}{2} \right) \right] y(1 - y) \tau^2 + y\sigma^2 + \frac{1 - y}{x(1 - x)\sigma^2} \right]^{\frac{3\epsilon}{2}} + \epsilon F_{\frac{7}{2}}^{(r)}(\sigma) \right].$$

(A23)

Expanding the $\Gamma$-functions and setting $\epsilon = 0$ in the subscripts of the functions which remain just as we proceeded in the calculation of the two-loop contribution we obtain the result

$$I_{5}^{(r)}(\sigma) = -\frac{1}{6\epsilon^2} (1 - \frac{\epsilon}{4} + \frac{3\epsilon}{2} W^{(r)}(\sigma)).$$

(A24)

The integral $I_{4SP}(\sigma)$ can be solved along similar steps. Realizing that, it can be written at the symmetry point in terms of the four-point one-loop subdiagram as

$$I_{4SP}^{(r)}(\sigma) = \sigma \sum_{j = -\infty}^{\infty} \int_{0}^{1} dx \sum_{j = -\infty}^{\infty} \int_{0}^{1} dy \left[ \left( \frac{I_{2}^{(r)}(q, j; \sigma)}{[(q)^2 + (j + \tau)^2 + 1]^2} \right] \right. $$

(A25)

we can solve the internal bubble, and obtain

$$I_{4SP}^{(r)}(\sigma) = \sigma \frac{1}{\epsilon^2} \left( 1 - \frac{\epsilon}{2} \right) \int_0^1 dx \sum_{j = -\infty}^{\infty} \int_{0}^{1} dy \left[ \left( \frac{1}{[(q)^2 + (j + \tau)^2 + 1]^2} \right] \right. $$

$$\times \left. \left[ \left( \frac{1}{(q^2 + \sigma^2 \tau^2) x(1 - x) + 1]^{\frac{3}{2}}} - \frac{1}{2} F_{\frac{7}{2}}^{(r)}(\sigma) S_d. \right. \right. $$

(A26)

Before proceeding, let us prove that last term is convergent, does not contribute to the ultraviolet divergences of $I_{4SP}^{(r)}(\sigma)$ and therefore can be neglected in the consideration of its singularities (dimensional poles in $\epsilon$). Explicitly, it is given by Eq. (A6)

$$F_{\frac{7}{2}}^{(r)}(\sigma) \equiv \frac{1}{S_d} \sigma \sum_{j = -\infty}^{\infty} \int_{0}^{1} dy \left[ \frac{F_{\frac{7}{2}}^{(r)}(q, j; \sigma)}{[q^2 + \sigma^2 (j + \tau)^2 + 1]^2} \right].$$

(A27)
Since this is a massive integral, the potential singularities come from the region of high momentum. From simple power counting together with our previous discussions, this integral will be divergent if, in the limit \( q \to \infty \), the \( F_\pm^{(r)}(q, j; \sigma) \) behavior is \( O(q^0) \) or proportional to a positive power of \( q \). Thus, it suffices to prove that this object is proportional to a negative power of \( q \), which we shall show next. From the definitions given in the main text Eq.(17) and the representation in terms of sum involving the product of cosine and Bessel function, we can write it as

\[
F_\pm^{(r)}(\tau, \epsilon; q, j; \sigma) = 4\sigma^{-\epsilon} \int_0^1 dx \sum_{m=1}^{\infty} \cos[2\pi m(\tau + ix)] \left( \frac{\pi m}{\sigma^{-1} \sqrt{x(1-x)(q^2 + \sigma^2 i^2) + 1}} \right)^{\frac{\epsilon}{2}} \times K_{\frac{\epsilon}{2}} \left( 2\pi m \sigma^{-1} \sqrt{x(1-x)(q^2 + \sigma^2 i^2) + 1} \right).
\]

(A28)

In order to attain maximal simplicity, take \( i = 0 \) in the above expression, since it is obvious that \( F_\pm^{(r)}(q, i; \sigma) < F_\pm^{(r)}(q, i = 0; \sigma) \). Note that the integrand is symmetric around \( x = \frac{1}{2} \) and we can write

\[
F_\pm^{(r)}(q, i = 0; \sigma) = 8\sigma^{-\epsilon} \int_0^{\frac{1}{2}} dx \sum_{m=1}^{\infty} \cos(2\pi m \tau) \left( \frac{\pi m}{\sigma^{-1} \sqrt{x(1-x)q^2 + 1}} \right)^{\frac{\epsilon}{2}} \times K_{\frac{\epsilon}{2}} \left( 2\pi m \sigma^{-1} \sqrt{x(1-x)q^2 + 1} \right).
\]

(A29)

In the limit \( q \to \infty \), choose a small real parameter \( \lambda \ll 1 \) with the property \( \lambda q^2 \to \infty \). The idea is to split the integration limits into two pieces: in the first one we use the Bessel function and in the second piece we replace its asymptotic form for large values of the argument, namely

\[
\lim_{q \to \infty} F_\pm^{(r)}(q, i = 0; \sigma) = 8\sigma^{-\epsilon} \sum_{m=1}^{\infty} \left[ \int_0^{\lambda} dx \cos(2\pi m \tau) \left( \frac{\pi m}{\sigma^{-1} \sqrt{xq^2 + 1}} \right)^{\frac{\epsilon}{2}} \times K_{\frac{\epsilon}{2}} \left( 2\pi m \sigma^{-1} \sqrt{xq^2 + 1} \right) \right. + \int_{\frac{1}{2}}^{\lambda} dx \cos(2\pi m \tau) \left( \frac{\pi m}{\sigma^{-1} \sqrt{x(1-x)q^2 + 1}} \right)^{\frac{\epsilon}{2}} \times \\
\left. \sqrt{4\pi m \sigma^{-1} \sqrt{x(1-x)q^2 + 1}} \exp \left( -2\pi m \sigma^{-1} \sqrt{x(1-x)q^2 + 1} \right) \right] \times K_{\frac{\epsilon}{2}} (2\pi m \sigma^{-1} \sqrt{xq^2 + 1}).
\]

(A30)

The second term can be neglected in this limit. Performing the change of variables \( y = 1 + xq^2 \), we can rewrite last expression taking into account these observations in the form

\[
\lim_{q \to \infty} F_\pm^{(r)}(q, i = 0; \sigma) = \frac{8\sigma^{-\epsilon}}{q^2} \sum_{m=1}^{\infty} \cos(2\pi m \tau)(\pi m)^{\frac{\epsilon}{2}} \int_1^{\infty} dy y^{-\frac{\epsilon}{4}} \times K_{\frac{\epsilon}{2}} (2\pi m \sigma^{-1} y).
\]

(A31)
Using the identity \[22\]
\[
\int_{1}^{\infty} dx x^{-\frac{1}{2}}(x-1)^{\mu-1}K_{\nu}(a\sqrt{x}) = \Gamma(\mu)2^{\mu-\mu}K_{\nu-\mu}(a),
\tag{A32}
\]
we are able to prove that
\[
\lim_{q \to \infty} F_{r,m}^{(r)}(q, i = 0; \sigma) = \frac{8\sigma^{1-\frac{1}{2}}}{q^{2}} \sum_{m=1}^{\infty} \cos(2\pi m \tau)(\pi m)^{\frac{1}{2} - 1}K_{\frac{1}{2} - 1}(2\pi m \sigma^{-1}),
\tag{A33}
\]
which is clearly regular in \(\epsilon\) completing our task in proving that in the ultraviolet region this object and therefore the desired integral involving it are both finite. Then it is safe to neglect that term in the computation of \(I_{4Sp}(\sigma)\).

When we neglect the correction from Eq. (A26) as just discussed, we can proceed from the following expression
\[
I_{4Sp}(\sigma) = \frac{1}{\epsilon} (1 - \frac{\epsilon}{2}) \int_{0}^{1} dx [x(1-x)]^{\frac{1}{2}} \sum_{j=-\infty}^{\infty} \int_{0}^{1} dy y^{(1-y)}^{\frac{1}{2} - 1}
\times \int \frac{d^{d-1} q}{[(q^{2} + \sigma^{2} \tau^{2}) + y^{2 - \frac{1}{2}}]^{2 + \frac{1}{2}}}. \tag{A34}
\]

We introduce an additional Feynman parameter \(y\) to rewrite last equation as
\[
I_{4Sp}(\sigma) = \frac{1}{\epsilon} \Gamma\left(\frac{2 + \frac{1}{2}}{2}\right) (1 - \frac{\epsilon}{2}) \int_{0}^{1} dx [x(1-x)]^{\frac{1}{2}} \sum_{j=-\infty}^{\infty} \int_{0}^{1} dy y^{(1-y)}^{\frac{1}{2} - 1}
\times \int \frac{d^{d-1} q}{[(q^{2} + \sigma^{2} \tau^{2}) + y^{2 - \frac{1}{2}}]^{2 + \frac{1}{2}}}. \tag{A35}
\]

Utilizing Eq. (10), we resolve the momentum integral and express the result in the form
\[
I_{4Sp}(\sigma) = \frac{\Gamma\left(\frac{1 + \frac{1}{2}}{2}\right)S_{d-1} \Gamma\left(\frac{d-1}{2}\right)}{2 \Gamma\left(\frac{d}{2}\right)} (1 - \frac{\epsilon}{2}) \int_{0}^{1} dx [x(1-x)]^{\frac{1}{2}} \int_{0}^{1} dy y^{(1-y)}^{\frac{1}{2} - 1}
\times \sum_{j=-\infty}^{\infty} \left[\sigma^{2} (j + y \tau)^{2} + y (1-y) \sigma^{2} \tau^{2} + y + \frac{1-y}{x(1-x)}\right]^{\frac{1}{2} - \epsilon}. \tag{A36}
\]

Performing the summation using Eq. (12), employing the identity \(\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)S_{d-1} = \Gamma\left(\frac{d}{2}\right)S_{d}\), expanding the argument of the \(\Gamma\)-function in \(\epsilon\) and absorbing the factor \(S_{d}\), we obtain the following result
\[
I_{4Sp}(\sigma) = \frac{1}{\epsilon} \left[\frac{(1 - \frac{\epsilon}{2}) \sigma^{2} \tau^{2}}{2 \Gamma\left(\frac{d}{2}\right)} \int_{0}^{1} dx [x(1-x)]^{\frac{1}{2}} \int_{0}^{1} dy y^{(1-y)}^{\frac{1}{2} - 1}
\times \left[\Gamma(\epsilon) \left(y(1-y) \tau^{2} + y \sigma^{2} - \frac{(1-y) \sigma^{2}}{x(1-x)}\right)^{-\epsilon}
\right.
\left. + f_{2 + \epsilon}\left(\frac{y \tau}{\sqrt{y(1-y) \tau^{2} + y \sigma^{2} - \frac{(1-y) \sigma^{2}}{x(1-x)}}}\right)\right] + \frac{\epsilon}{2} F_{r,m}^{(r)}(\sigma). \tag{A37}
\]

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Note that the integral over \( y \) possesses a pole in \( y = 1 \). A standard procedure is to compute the bracket which multiplies the integral at \( y = 1 \) \([16]\), which facilitates the computation and retains the pole contribution which we are interested. Expanding in \( \epsilon \) and neglecting \( O(\epsilon^0) \) terms, we finally obtain

\[
I_{4SP}^{(\tau)}(\sigma) = \frac{1}{2\epsilon^2} \left( (1 - \frac{\epsilon}{2}) + \epsilon f_2(\tau, \sigma^{-1}) \right). \tag{A38}
\]

These integrals and all the nomenclature defined here are utilized in Secs. II and III.

**Appendix B: Massless integrals in dimensional regularization**

The higher-loop massless integrals are even simpler to evaluate than those occurring in the massive setting discussed above, with the difference that they need to be calculated at nonvanishing external momenta. The massless counterparts of the integrals discussed in the previous Appendix are given by

\[
I_3(k, i, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2}{(q_1^2 + \sigma^2(j_1 + \tau)^2)(q_2^2 + \sigma^2(j_2 + \tau)^2)} \\
\times \frac{1}{[(q_1 + q_2 + k)^2 + \sigma^2(j_1 + j_2 + i + \tau)^2]}, \tag{B1}
\]

\[
I_4(k_1, k_2, k_3, k_4, i_1, i_2, i_3, i_4, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2}{(q_1^2 + \sigma^2(j_1 + \tau)^2)} \\
\times \frac{(q_2^2 + \sigma^2(j_2 + \tau)^2)[(q_1 - q_2 + k_3)^2 + \sigma^2(j_1 - j_2 + i_3 + \tau)^2]}{1} \\
\times \frac{1}{[(P - q_1)^2 + \sigma^2(p - j_1 + \tau)^2]}, \tag{B2}
\]

\[
I_5(k, i, \sigma) = \sigma^2 \sum_{j_1, j_2 = -\infty}^{\infty} \int \frac{d^{d-1}q_1 d^{d-1}q_2 d^{d-1}q_3}{(q_1^2 + \sigma^2(j_1 + \tau)^2)(q_2^2 + \sigma^2(j_2 + \tau)^2)} \\
\times \frac{(q_3^2 + \sigma^2(j_3 + \tau)^2 + 1)[(q_1 + q_2 + k)^2 + \sigma^2(j_1 + j_2 + i + \tau)^2]}{1} \\
\times \frac{1}{[(q_1 + q_3 + k)^2 + \sigma^2(j_1 + j_3 + i + \tau)^2]].} \tag{B3}
\]

Firstly, let us discuss \( I_3 \) at arbitrary external momentum, which shall be important in the evaluation of critical exponents using minimal subtraction. As we are going to compute the exponents in normalization conditions as well, we shall discuss its derivative with respect
to squared nonvanishing external momenta for that sake. All integrals will be computed at zero external quasimomenta from now on. Since many developments discussed in the previous Appendix will have their exact analogy here, we will be more economical in the steps to compute the integrals. First, write $I_3$ in terms of the one-loop subdiagram as

$$I_3^{(\tau)}(k, i = 0; \sigma) = \sigma \sum_{j = -\infty}^{\infty} \int d^{d-1}q I_2^{(\tau)}(q, j; \sigma) \frac{I_2^{(\tau)}(q, j; \sigma)}{[(q - k)^2 + \sigma^2(j + \tau)^2]}.$$  \hfill (B4)

Replacing the value of the subdiagram computed in Sec. IV through the use of Eqs. (48)-(50) defined for the massless case, we get

$$I_3^{(\tau)}(k, i = 0; \sigma) = \frac{1}{\epsilon} \left( \sigma(1 - \frac{\tau}{2}) \int_0^1 dx [x(1 - x)]^{-\frac{d}{2}} \sum_{j = -\infty}^{\infty} \int \frac{d^{d-1}q}{[(q-k)^2 + \sigma^2(j+\tau)^2][q^2 + \sigma^2 j^2]^\frac{d}{2}} \right) \left( \tau \Gamma(2 - \frac{\tau}{2}) \sigma \int \frac{d^{d-1}q}{[(q-k)^2 + \sigma^2(j+\tau)^2]} F^{(\tau)}(q, i = 0; \sigma) \right).$$  \hfill (B5)

For the sake of minimal subtraction, define the parametric integral

$$L_3(k^2 + \sigma^2 \tau^2) = \int_0^1 dy \ln[y(1 - y)(k^2 + \sigma^2 \tau^2)].$$  \hfill (B6)

Other useful definitions are the massless functions

$$F_{\alpha, \beta}^{(\tau)}(k, i; \sigma) \equiv \frac{1}{S_d} \sigma \sum_{j = -\infty}^{\infty} \int d^{d-1}q \frac{F_{\alpha, \beta}^{(\tau)}(q + k, j + i; \sigma)}{[q^2 + r^2(j + \tau)^2]^\beta},$$  \hfill (B7a)

$$F_{\alpha, \beta}^{(\tau)}(k, i; \sigma) \equiv \frac{\partial F_{\alpha, \beta}^{(\tau)}(k, i; \sigma)}{\partial k^2} \bigg|_{(k^2 = 1, i = 0)},$$  \hfill (B7b)

$$i_3(k, \sigma) = \sum_{j = -\infty}^{\infty} \int d^{d-1}q \frac{d^{d-1}q}{[q^2 + \sigma^2(j + \tau)^2][(q + k)^2 + \sigma^2 j^2]^\frac{d}{2}}.$$  \hfill (B8)

Let us compute $i_3(k, \sigma)$, which is analogous to the expression for the massive case and much simpler. We introduce a Feynman parameter $y$ in order to express the two denominators as a single one, solve the momentum integral using Eq. (10) and expand in $\epsilon$ afterward. This set of steps are identical to those which led to Eq. (A11) and the reader can check that it results in the following expression

$$i_3(k, \sigma) = \frac{S_{d-1} \Gamma\left(\frac{d-1}{2}\right)}{2 \Gamma\left(\frac{d}{2}\right)} \int_0^1 dy (1 - y)^{\frac{d}{2} - 1} \sum_{j = -\infty}^{\infty} \left[ y(1 - y)(k^2 + \sigma^2 \tau^2) + \sigma^2(j + y \tau)^2 \right]^{\frac{d}{2} - \epsilon}.$$  \hfill (B9)
The summation can be performed as before and we find

$$i_3(k, \sigma) = \frac{S_{d-1} \Gamma(d-1)}{2 \Gamma(\frac{d}{2})} \int_0^1 dy (1-y)^{\frac{d}{2}-1} \sqrt{\pi} \left[ \sigma^{-1} \Gamma(\epsilon - 1) (y(1-y)[k^2 + \sigma^2 \tau^2])^{1-\epsilon} + \sigma^{1-2\epsilon} f_{\sigma^2 \tau^2}^2 (y, \sigma^{-1} \sqrt{y(1-y)[k^2 + \sigma^2 \tau^2]}) \right].$$

(B10)

Utilize the identity $\sqrt{\pi} \Gamma(d-1) S_{d-1} = \Gamma(d-1) S_d$, absorb $S_d$ in the redefinition of the coupling constant just as explained for the one-loop four-point graph in the main text and expand in $\epsilon$. Integrating over the Feynman parameter $x$ and substituting into the expression of $I_3^{(r)}(k, i = 0, \sigma)$, we obtain

$$I_3^{(r)}(k, i = 0; \sigma) = \frac{1}{\epsilon} \left( (1 - \frac{\epsilon}{2}) \frac{\Gamma^2(1 - \frac{\epsilon}{2})}{2 \Gamma(\frac{d}{2}) \Gamma(2 - \epsilon)} \int_0^1 dy (1-y)^{\frac{d}{2}-1} \left[ \Gamma(\epsilon - 1) \times \left( y(1-y)[k^2 + \sigma^2 \tau^2])^{1-\epsilon} + \sigma^{2-2\epsilon} f_{\sigma^2 \tau^2}^2 (y, \sigma^{-1} \sqrt{y(1-y)[k^2 + \sigma^2 \tau^2]}) \right) \right. \right.$$

$$+ \frac{\epsilon}{2} F_{\sigma^2 \tau^2}^{(r)}(k, i = 0, \sigma) \left). \right)$$

(B11)

Next, use the property $\Gamma(x+1) = x\Gamma(x)$ in order to get a result purely in terms of $\epsilon$ when higher order terms are neglected. We then have

$$I_3^{(r)}(k, i = 0; \sigma) = \frac{1}{\epsilon} \left( \frac{1}{4} (1 + \epsilon) \int_0^1 dy (1-y)^{\frac{d}{2}-1} \left[ \left( y(1-y)[k^2 + \sigma^2 \tau^2] \right)^{1-\epsilon} - \sigma^{2-2\epsilon} f_{\sigma^2 \tau^2}^2 (y, \sigma^{-1} \sqrt{y(1-y)[k^2 + \sigma^2 \tau^2]}) \right. \right.$$

$$+ \frac{\epsilon}{2} F_{\sigma^2 \tau^2}^{(r)}(k, i = 0, \sigma) \left). \right)$$

(B12)

We introduce two new parametric functions which will be useful to our future manipulations, whose expressions are given by

$$F_{\alpha}^{(r)}(k, i = 0; \sigma) = \sigma^{2-2\alpha} \int_0^1 dy (1-y)^{\frac{d}{2}-1} f_{\sigma^2 \tau^2}^\alpha (y, \sigma^{-1} \sqrt{y(1-y)[k^2 + \sigma^2 \tau^2]),} \quad \text{(B13a)}$$

$$F_{\alpha}^{(r)}(k, i = 0; \sigma) = \sigma^{-2\alpha} \int_0^1 dy (1-y)^{\frac{d}{2}} f_{\sigma^2 \tau^2}^{-\alpha} (y, \sigma^{-1} \sqrt{y(1-y)[k^2 + \sigma^2 \tau^2]).} \quad \text{(B13b)}$$

In terms of these functions, we can express the solution for the integral $I_3(k, i = 0; \sigma)$ in a form appropriate to minimal subtraction, namely

$$I_3^{(r)}(k, \sigma) = -\frac{1}{8\epsilon} \left( (k^2 + \sigma^2 \tau^2)[1 + \frac{\epsilon}{4} - 2\epsilon L_3(k^2 + \sigma^2 \tau^2)] - 2\epsilon F_{\epsilon}^{(r)}(k, i = 0; \sigma) \right.$$

$$- 4\epsilon F_{\sigma^2 \tau^2}^{(r)}(k, i = 0; \sigma)) \quad \text{(B14)}$$

If we prefer to employ the normalization condition scheme, we need the derivative of this integral with respect to $k^2$ computed at the symmetry point $k^2 = 1$. Henceforth we denote
the argument of a function \((k, i = 0, \sigma)\) at \(k^2 = 1\) by \((\sigma)\). Using the recursion relation for Bessel functions

\[
\frac{d}{dz} \left[ z^{-\alpha} K_{\alpha}(\beta z) \right] = -\beta z^{-\alpha-1} K_{\alpha+1}(\beta z),
\]

and employing the explicit representation of \(f_a(a, b)\) in terms of the product involving the Bessel function and cosine, we learn that

\[
I_{5SP}^{(r)}(\sigma) = -\frac{1}{8\epsilon} \left(1 + \frac{5\epsilon}{4} - 2\epsilon \ln(1 + \sigma^2 \tau^2) + 2\epsilon \tilde{F}_e^{(r)}(\sigma) - 4\epsilon F_2^{(r)}(\sigma)\right).
\] (B16)

Let us conclude our analysis of the two-point function by calculating the three-loop integral \(I_5^{(r)}(k, i = 0; \sigma)\). First, write it as

\[
I_5^{(r)}(k, i = 0; \sigma) = \sigma \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{[I_2^{(r)}(q, j; \sigma)]^2}{[(q - k)^2 + \sigma^2(j + \tau)^2]}. \tag{B17}
\]

When we use the explicit expression for the four-point one-loop subdiagram along with the value of the integral over the parameter originally appearing there, expanding the \(\Gamma\) functions in \(\epsilon\) and neglecting higher order terms, we obtain

\[
I_5^{(r)}(k, i = 0; \sigma) = \frac{1}{2\epsilon} \left(\sigma(1 + \epsilon) \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{[I_2^{(r)}(q, j; \sigma)]^2}{[(q - k)^2 + \sigma^2(j + \tau)^2]} + \epsilon \sigma \int \frac{d^{d-1}q}{[(q - k)^2 + \sigma^2(j + \tau)^2 + 1]} \tilde{F}_e^{(r)}(q, i = 0; \sigma)\right). \tag{B18}
\]

Using completely similar steps in the calculation of \(I_3^{(r)}(k, i = 0; \sigma)\) with minor modifications, we can proceed henceforth quite analogously. Define the functions

\[
\tilde{F}_e^{(r)}(k, i = 0; \sigma) = \sigma^{2-3\epsilon} \int_0^1 dy (1 - y)^{\epsilon-1} f_{\frac{d-1}{2}, \frac{d+1}{2}}(y, \sigma^{-1} \sqrt{y(1-y)(k^2 + \sigma^2 \tau^2)}), \tag{B19a}
\]

\[
\tilde{F}_3^{(r)}(k, i = 0; \sigma) = \sigma^{-3\epsilon} \int_0^1 dy y (1 - y)^{\epsilon-1} f_{\frac{d-1}{2}, \frac{d+1}{2}}(y, \sigma^{-1} \sqrt{y(1-y)(k^2 + \sigma^2 \tau^2)}). \tag{B19b}
\]

The solution for \(I_5\) useful in utilizing minimal subtraction scheme can be written as

\[
I_5^{(r)}(k, \sigma) = -\frac{1}{\epsilon^2} \left(\epsilon \ln(1 + \sigma^2 \tau^2) [1 + \frac{\epsilon}{2} - 3\epsilon L_3(k^2 + \sigma^2 \tau^2)] - 3\epsilon \tilde{F}_3^{(r)}(k, i = 0; \sigma) - 6\epsilon \tilde{F}_3^{(r)}(k, \sigma)\right). \tag{B20}
\]

Instead, if we wish to employ normalization conditions, we have to compute the derivative in relation to \(k^2\) at the symmetry point \(k^2 = 1\) and we get to

\[
I_{5SP}^{(r)}(\sigma) = -\frac{1}{\epsilon^2} \left(1 + 2\epsilon - 3\epsilon \left[\frac{1}{2} \ln(1 + \sigma^2 \tau^2) - \tilde{F}_3^{(r)}(\sigma) + 2\epsilon F_2^{(r)}(\sigma)\right]\right). \tag{B21}
\]
It is worthy to mention at this point that the finite size corrections for \( I^{(\tau)}_{3SP}(\sigma) \) and \( I^{(\tau)}_{5SP}(\sigma) \) become simpler when computed at \( \epsilon = 0 \). In fact, defining the massless quantity
\[
W^{(\tau)}_0(\sigma) = \frac{1}{2} \ln(1 + \sigma^2 \tau^2) - F^{(\tau)}_0(\sigma) + 2F^{(\tau)}_0(\sigma),
\]
those integrals can be written simply as
\[
I^{(\tau)}_{3SP}(\sigma) = -\frac{1}{8\epsilon}(1 + 5\epsilon^2 - 2\epsilon W_0(\sigma)), \quad (B22a)
\]
\[
I^{(\tau)}_{5SP}(\sigma) = -\frac{1}{6\epsilon^2}(1 + 2\epsilon - 3\epsilon W_0(\sigma)). \quad (B22b)
\]

The massless counterpart of the integral which contributes to the four-point function at two-loops can be written in terms of \( I_2 \) and reads:
\[
I^{(\tau)}_4(k, i; \sigma) = \sigma \sum_{j=-\infty}^{\infty} \int d^{d-1}q \frac{I^{(\tau)}_2(q + k_3, j, i; \sigma)}{[q^2 + \sigma^2(j + \tau)^2][(q - P)^2 + \sigma^2(j - p + \tau)^2]}.
\] (B23)

Once again we shall restrict ourselves to the simplest expressions for this integrals, namely those with zero mode \( (p = 0) \) characterizing the external quasimomentum associated to the finite size direction, perpendicular to the plate surfaces. At the symmetry point, all primitively divergent vertex parts depend on only one external momenta scale. For this reason we are going to list the last integral at a certain external momentum \( P \) and display the results in the most convenient form either using minimal subtraction or normalization conditions renormalization schemes.

We solve the internal bubble, i.e., we compute the integral over the momenta. Then, use a Feynman parameter \( x \) to melt the integer powers of the propagators in a single denominator. We are left with an integral over the Feynman parameter \( x \) as follows
\[
I^{(\tau)}_4(k, i; \sigma) = \sigma \sum_{j=-\infty}^{\infty} \int_0^1 dx \sum_{j=-\infty}^{\infty} \int \frac{d^{d-1}q}{[q^2 - 2xPq + xP^2 + \sigma^2(j + \tau)^2]^2} \times \frac{1}{[q + k_3]^2 + \sigma^2 j^2]^{\frac{d}{2}}} + \frac{1}{2} G^{(\tau)}_2(P, k_3, 0, \sigma) S_d, \quad (B24)
\]

where
\[
G^{(\tau)}_2(P, k_3, 0, \sigma) = \frac{\sigma}{S_d} \sum_{j=-\infty}^{\infty} \int \frac{d^{d-1}q F^{(\tau)}_0(q + k_3, j, \sigma)}{[q - P]^2 + \sigma^2(j + \tau)^2][q^2 + \sigma^2(j + \tau)^2]^2}. \quad (B25)
\]

We shall be concerned with the behavior of \( F^{(\tau)}_0(q, j, \sigma) \) in the limit \( q \to \infty \). In that limit, if this object behaves as an inverse power of \( q \) all is well, since last integral will have no dimensional poles in \( \epsilon \) using simple power counting arguments. Therefore, it can be neglected in our consideration of poles from \( I_4(k, i = 0, \sigma) \). Let us turn our attention to this
issue. From its explicit form $F_\alpha(q, j, \sigma) < F_\alpha(q, j = 0, \sigma)$ and we just have to discuss the latter in the appropriate limit.

From our discussion in the main text Sec. IV, we already know that

$$F^{(\tau)}(k, i = 0; \sigma) = 4\kappa \sum_{n=1}^{\infty} \cos(2\pi n \tau)(\pi n)^{-\frac{3}{2}} \int_0^1 dx [x(1 - x)]^{-\frac{3}{2}} \times K_\frac{3}{2}(2\pi n \sigma^{-1}[x(1 - x)]^{\frac{1}{2}}).$$  (B26)

The integrand is symmetric around $x = \frac{1}{2}$. Then we can multiply the integral by two and use the integration limits at $(0, \frac{1}{2})$. Next, perform the change of variables $x(1 - x) = \left(\frac{y^2}{2}\right)$. Thus, we can write the integral in the form

$$\int_0^1 dx [x(1 - x)]^{-\frac{3}{2}} K_\frac{3}{2}(2\pi n \sigma^{-1}[x(1 - x)]^{\frac{1}{2}}) = 2^{\frac{3}{2}} \int_0^1 dy y^{-\frac{3}{2}}(1 - y^2)^{-\frac{3}{2}} K_\frac{3}{2}(\pi n \sigma^{-1} y).$$  (B27)

After some manipulation with special functions using Ref. [22], last integral can be put in the form

$$\int_0^{\pi n \sigma^{-1}} \sinh \frac{t}{t} dt - \sinh(\pi n \sigma^{-1}) \left(\gamma + \ln(\pi n \sigma^{-1}) + \int_0^{\pi n \sigma^{-1}} \cosh t - 1 \frac{1}{t} dt\right).$$  (B28)

The first integral in the right-hand side is defined as the integral hyperbolic sine and denoted by $\text{sh}(\pi n \sigma^{-1})$. The complete term multiplying $\sinh(\pi n \sigma^{-1})$ is defined as the integral hyperbolic cosine whose symbol is $\text{ch}(\pi n \sigma^{-1})$. Using the identity $\text{cosec}(\frac{\pi \epsilon}{2}) = 2^{\frac{\pi \epsilon}{2}} \frac{\pi \epsilon}{2} + O(\epsilon^2)$, the above expression becomes regular in the limit $\epsilon \to 0$. Now, the finite size contribution is $O(\epsilon^0)$ (regular) and we can set $\epsilon = 0$ inside its expression in order to obtain

$$F^{(\tau)}(k, i = 0; \sigma) = \frac{4\sigma}{\pi k} \sum_{n=1}^{\infty} \cos(2\pi n \tau)(\pi n)^{-\frac{3}{2}} \left[\cosh(\pi n \sigma^{-1}) \text{sh}(\pi n \sigma^{-1}) - \sinh(\pi n \sigma^{-1}) \text{ch}(\pi n \sigma^{-1})\right] + O(\epsilon).$$  (B29)

Consider the terms inside the brackets and define $z = \pi n \sigma^{-1}$. For example, from our definitions, we can write

$$\cosh z \sinh z - \sinh z \cosh z = \frac{e^z}{2}(\text{sh} z - \text{ch} z) + \frac{e^{-z}}{2}(\text{sh} z + \text{ch} z).$$  (B30)

We can work out these equations further in order to reduce them in terms of the incomplete Gamma function defined by $\Gamma(0, z) = \int_z^{\infty} dt \frac{e^{-t}}{t}$, such that

$$\cosh z \sinh z - \sinh z \cosh z = \frac{e^z}{2} \Gamma(0, z) + \frac{e^{-z}}{2} \Gamma(0, -z).$$  (B31)
Now the limit $k \to \infty$ is the same as $z \to \infty$. From Ref. [22], the asymptotic value for real $z$ is given by $lim_{z \to} \Gamma(0, z) = \frac{e^{-z}}{|z|}$. Therefore, we find

$$lim_{z \to} (\cosh z h z - \sinh z h z) = \frac{1}{z}.$$  \hspace{1cm} (B32)

Thus, we conclude that

$$lim_{k \to} F^{(\tau)}(k, i = 0; \sigma) = \frac{4\sigma^2}{\pi^2 k^2} \sum_{n=1}^{\infty} \frac{\cos(2\pi n \tau)}{n^2},$$ \hspace{1cm} (B33)

showing that this behavior guarantees that the integral $G^{(\tau)}(k, i = 0, \sigma)$ is regular and can be neglected in the determination of the coefficients of the pole terms contained in this diagram.

In the following we outline the method of computation of $I^{(\tau)}_4(k, i = 0, \sigma)$. It does not bring essential new features in comparison with the massive case, being actually simpler its evaluation. Consider the first term in the last expression $I^{(\tau)}_4(k, i = 0, \sigma)$. Using an additional Feynman parameter $y$ to fold the remaining terms into a single denominator in order to compute the integral over the momenta $q$, the resulting parametric integral over $y$ has a pole in $y = 1$. In analogy to the massive case in Appendix A, we keep the prefactor (depending only on $y$) in that integral, and set $y = 1$ in the overall term which depends on $(x, y, P, \sigma)$. We then find

$$I^{(\tau)}_4(k, i = 0; \sigma) = \frac{1}{2\epsilon^2} \left(1 - \frac{\epsilon}{2} - \epsilon i(P) + \epsilon F^{(\tau)}_e(\frac{PL}{2\pi}, 0)\right),$$ \hspace{1cm} (B34)

which at the symmetry point $P^2 = \kappa^2 = 1$ takes a simpler form and simplifies further if the finite size correction is computed at $\epsilon = 0$, namely

$$I^{(\tau)}_{4SP}(\sigma) = \frac{1}{2\epsilon^2} \left(1 + \frac{3\epsilon}{2} + \epsilon F^{(\tau)}_0(\sigma)\right).$$ \hspace{1cm} (B35)

We are going to use these results in Sections IV and V in the massless computation of the critical indices. Notice that the results are such that $F^{(\tau)}_0(\sigma)$ used in the present appendix is the appropriate object for the massless theory, given in the text by Eq. (10).
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