Abstract. \( n\mathbb{Z} \)-cluster tilting subcategories are an ideal setting for higher dimensional Auslander–Reiten theory. We give a complete classification of \( n\mathbb{Z} \)-cluster tilting subcategories of module categories of Nakayama algebras. In particular, we show that there are three kinds of Nakayama algebras that admit \( n\mathbb{Z} \)-cluster tilting subcategories: finite global dimension, selfinjective and non-Iwanaga–Gorenstein. Only the selfinjective ones can admit more than one \( n\mathbb{Z} \)-cluster tilting subcategory. It has been shown by the second author, that each such \( n\mathbb{Z} \)-cluster tilting subcategory induces an \( n\mathbb{Z} \)-cluster tilting subcategory of the corresponding singularity category. For each Nakayama algebra in our classification, we describe its singularity category, the canonical functor from its module category to its singularity category, and provide a complete comparison of \( n\mathbb{Z} \)-cluster tilting subcategories in the module category and the singularity category. This relies heavily of results by Shen, who described the singularity categories of all Nakayama algebras.

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1. Introduction

Auslander–Reiten theory is a fundamental tool to study representation theory from a homological point of view. A higher version of Auslander–Reiten theory was introduced by Iyama [Iya07] for each \( n \geq 1 \), with \( n = 1 \) corresponding to the classical version. This theory has connections to algebraic geometry [IW11, IW13, IW14, HIMO20], combinatorics [OT12, HJr21, Wil22], higher category theory and algebraic K-theory [DJW19], representation theory, [HI11, IO11, Miz13], and to wrapped Floer theory in symplectic geometry [DJL21]. It is also a crucial ingredient in the recent proof of the Donovan–Wemyss conjecture as announced by Keller [JMK] (see Conjecture 1.4 in [DW16] for the

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statement of the conjecture). In higher Auslander–Reiten theory one changes perspective from studying some category $\mathcal{A}$, say finitely generated modules over a finite dimensional algebra or its bounded derived category, to studying some suitable subcategory $\mathcal{C} \subseteq \mathcal{A}$. Commonly $\mathcal{C}$ is an $n$-cluster tilting subcategory, possibly with additional properties.

In Definition 2.3, we recall the definition of $n$-cluster tilting subcategories of module categories, but they can be defined in many different settings (See [Iya07, GKO13, Jas16, HLN22].) Depending on the setting, $n$-cluster tilting subcategories give rise to higher notions of various types of categories of interest in homological algebra. For instance if $\mathcal{A}$ is abelian and $\mathcal{C} \subseteq \mathcal{A}$ is $n$-cluster tilting, then $\mathcal{C}$ is $n$-abelian in the sense of [Jas16]. If $\mathcal{A}$ is triangulated and $\mathcal{C} \subseteq \mathcal{A}$ is $n$-cluster tilting with the additional property that $\mathcal{C}$ is closed under $n$-fold suspension then $\mathcal{C}$ is $(n+2)$-angulated in the sense of [GKO13].

Let $\Lambda$ be a finite dimensional algebra and $\Lambda \mod$ the category of finitely generated left $\Lambda$-modules. In this paper we are concerned with $n$-cluster tilting subcategories $\mathcal{C} \subseteq \Lambda \mod$. The existence of such a $\mathcal{C}$ imposes a strong restriction on $\Lambda$ and in the vast majority of known cases $\Lambda$ exhibits very regular homological behaviour. For instance the case when $\text{gl. dim } \Lambda = n$ has been extensively studied (See for instance [Iya11, IO11, IO13, HI11b, HI11a].) In this case $\mathcal{C}$ is unique if it exists. Moreover, $\mathcal{C}$ gives rise to an $n$-cluster tilting subcategory $\text{add}\{X[in] \mid X \in \mathcal{C}, i \in \mathbb{Z}\} \subseteq \text{D}^b(\Lambda \mod)$ of the bounded derived category $\text{D}^b(\Lambda \mod)$, which is $(n+2)$-angulated by [GKO13].

For $\Lambda$ with $\text{gl. dim } \Lambda > n$ much fewer results are known. One of the cases that has received some attention is when $\Lambda$ is selfinjective (see [EH08, IO13, DI20]). In this case the stable category $\Lambda \mod$ is triangulated and the image of $\mathcal{C}$ in $\Lambda \mod$ is $n$-cluster tilting but not necessarily $(n+2)$-angulated. Some algebras $\Lambda$ with $\text{gl. dim } \Lambda > n$, which are not selfinjective but admit $n$-cluster tilting subcategories have been found by the third author [Vas19, Vas20, Vas21, Vas23]. Another notable family of algebras admitting $n$-cluster tilting subcategories and exhibiting a variety of homological properties are the higher Nakayama algebras introduced in [JKPK19].

Iyama and Jasso introduced the notion of $n\mathbb{Z}$-cluster tilting in [IJ17], which resolves certain issues that appear in particular when $\text{gl. dim } \Lambda > n$. This property is significantly stronger than $n$-cluster tilting (see Definition 2.1 for the definition). In particular, if $\text{gl. dim } \Lambda < \infty$ and there is $\mathcal{C} \subseteq \Lambda \mod$, $n\mathbb{Z}$-cluster tilting, then $\text{gl. dim } \Lambda \in n\mathbb{Z}$. On the other hand if $\text{gl. dim } \Lambda = n$, then any $n$-cluster tilting subcategory is $n\mathbb{Z}$-cluster tilting. One of the benefits of this notion was found by the second author, who showed in [Kva21] that every $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda \mod$ gives rise to an $n\mathbb{Z}$-cluster tilting subcategory of the singularity category $D_{\text{sing}}(\Lambda)$ (see Section 5 for relevant definitions). In the triangulated setting $n\mathbb{Z}$-cluster tilting means precisely closure under $n$-fold suspension and so $(n+2)$-angulated categories are obtained in this way. Note that if $\text{gl. dim } \Lambda < \infty$, then $D_{\text{sing}}(\Lambda) = 0$ and if $\Lambda$ is selfinjective, then $D_{\text{sing}}(\Lambda) = \Lambda \mod$.

For these reasons given above we are motivated to search for algebras $\Lambda$ admitting $n\mathbb{Z}$-cluster tilting subcategories $\mathcal{C} \subseteq \Lambda \mod$. In particular, we are interested in cases when $\text{gl. dim } \Lambda = \infty$ and $\Lambda$ is not selfinjective.

We limit our search to Nakayama algebras (see Section 2.3 for relevant definitions) as it is a class of algebras that is not too large and for which computations can be done easily. Still it allows for algebras of the complex homological behaviour we are looking for. Our main result is a classification of all Nakayama algebras that admit an $n\mathbb{Z}$-cluster tilting subcategory. Moreover, for each algebra in our classifying list we describe all its $n\mathbb{Z}$-cluster tilting subcategories explicitly. We note that although there are many results giving examples of algebras with $n$-cluster tilting subcategories as mentioned above, there are few that give a complete classification for a given family of algebras. Notable exceptions for which classification results have been obtained are selfinjective Nakayama algebras [DI20], radical square zero algebras [Vas23] and gentle algebras [HJS22].

Our classification is subdivided into four cases depending on the quiver with relation describing the Nakayama algebra $\Lambda$. With respect to the quiver we subdivide depending on if it acyclic or not. With respect to the relations we subdivide depending on if the relations are homogeneous (meaning that the relations are given by all paths of some fixed length) or not. The classification for homogeneous relations
The classification for non-homogeneous relations is given in Theorem 4.20 (acyclic) and Theorem 4.29 (cyclic).

In Section 5 we let Λ be a Nakayama algebra with an $n\mathbb{Z}$-cluster tilting subcategory. In summary it goes as follows. To obtain a Nakayama algebra Λ with an $n\mathbb{Z}$-cluster tilting subcategory, decide if you prefer $\text{gl.dim}\, \Lambda = rn$ for some $r \geq 1$, for Λ to be non-Iwanaga–Gorenstein or for Λ to be selfinjective. In the first case pick any $r$ Nakayama algebras of global dimension $n$, each admitting an $n\mathbb{Z}$-cluster tilting module, and glue them together in any order. For the second option do the same and then self-glue the result. In these cases the $n\mathbb{Z}$-cluster tilting subcategory is unique. For Λ selfinjective, make sure that $n$ divides the number of simples and Λ has Loewy length 2 or $n+2$. In that case Λ admits precisely $n$ distinct $n\mathbb{Z}$-cluster tilting subcategories. These are the only options as every other Nakayama algebra does not admit an $n\mathbb{Z}$-cluster tilting subcategory.

In Section 5 we let Λ be a Nakayama algebra with an $n\mathbb{Z}$-cluster tilting subcategory and study its singularity category $D^\text{sing}(\Lambda)$. We describe the canonical functor from the module category to the singularity category and the image of its $n\mathbb{Z}$-cluster tilting subcategory. If Λ is acyclic, then $\text{gl.dim}\, \Lambda < \infty$ and its singularity category is zero, so there is nothing to discuss. If Λ is selfinjective then the singularity category coincides with the stable module category and there is a bijection between $n\mathbb{Z}$-cluster tilting subcategories in the module category and in the singularity category. The final case to consider is when Λ is cyclic and non-homogeneous. We use a description of singularity categories of Nakayama algebras due to Shen [She15], to show that $D^\text{sing}(\Lambda)$ is equivalent to the stable category of a radical square zero cyclic Nakayama algebra Γ, see Corollary 5.11. By our previous results we know that Γ admits precisely $n$ distinct $n\mathbb{Z}$-cluster tilting subcategories and we determine which of them corresponds to the unique $n\mathbb{Z}$-cluster tilting subcategory coming from Λ.

By our results we know exactly which $n\mathbb{Z}$-cluster tilting subcategories appear in singularity categories of Nakayama algebras. Since they all have finitely many indecomposable objects they can equally be described as $n\mathbb{Z}$-cluster tilting objects by taking the direct sum of all indecomposables. It has recently been shown by Jasso and Muro [JMK] that any algebraic triangulated category with an $n\mathbb{Z}$-cluster tilting objects is determined by the endomorphism algebra of that object (under the assumption that this endomorphism algebra is finite dimensional over a perfect ground field). As a consequence any algebraic triangulated category admitting an $n\mathbb{Z}$-cluster tilting object with the same endomorphism algebra as one appearing in this paper must be equivalent to a singularity category of a Nakayama algebra.

2. Preliminaries

2.1. Conventions and notation. Let us start by setting conventions and introducing notation. Throughout this paper we fix a positive integer $n \geq 2$ and a ground field $k$. All subcategories are assumed to be full. By algebra we mean associative $k$-algebra and by module we mean left module. We denote by $D$ the duality $\text{Hom}_k(-, k)$.

Let Λ be a finite dimensional algebra. We consider the category $\Lambda\text{-mod}$ of finitely generated left Λ-modules. We denote by $\Lambda\text{-mod}$ the projectively stable module category of Λ, that is the category with objects the same as $\Lambda\text{-mod}$ and morphisms given by $\text{Hom}(M, N) = \text{Hom}(M, N)/\mathcal{P}(M, N)$ where
\(P(M, N)\) denotes the subspace of morphisms factoring through projective modules. We denote by \(\Omega: \Lambda-\text{mod} \to \Lambda-\text{mod}\) the syzygy functor defined by \(\Omega(M)\) being the kernel of a projective cover \(P \to M\). The injectively stable module category \(\Lambda-\text{mod}\) and the cosyzygy functor \(\Omega^-: \Lambda-\text{mod} \to \Lambda-\text{mod}\) are defined dually. We consider the \(n\)-Auslander–Reiten translations \(\tau_n: \Lambda-\text{mod} \to \Lambda-\text{mod}\) and \(\tau_n^-: \Lambda-\text{mod} \to \Lambda-\text{mod}\) defined by \(\tau_n = \tau\Omega^{n-1}\) and \(\tau_n^- = \tau^-\Omega^(-(n-1))\), where \(\tau\) and \(\tau^-\) denote the usual Auslander–Reiten translations.

For a module \(M \in \Lambda-\text{mod}\) we denote by \(\text{add}(M)\) the subcategory of \(\Lambda-\text{mod}\) containing all direct summands of finite direct sums of \(M\). For \(i = 1, \ldots, s\) let \(M_i \in \Lambda-\text{mod}\) be a module and let \(C_i = \text{add}(M_i)\). We define

\[
\text{add}(C_1, \ldots, C_s) := \text{add}(M_1, \ldots, M_s) := \text{add}\left(\bigoplus_{i=1}^s M_i\right).
\]

For a finite quiver \(Q\) we denote its path algebra over \(k\) by \(kQ\). If \(\alpha\) is an arrow in \(Q\) from \(i\) to \(j\) we write \(i \to j\). For each vertex \(i\) in \(Q\) we denote the path of length 0 at \(i\) by \(\epsilon_i\). Multiplication in \(kQ\) is defined by concatenation of paths in such a way that if \(i \to j\) is an arrow, then \(\epsilon_j\alpha\epsilon_i = \alpha\). We denote the two sided \(kQ\)-ideal generated by all arrows by \(R_Q = R\). A two sided \(kQ\)-ideal \(I\) is called admissible if \(R^2 \supseteq I \supseteq R^N\) for some \(N \geq 2\).

2.2. \(n\mathbb{Z}\)-cluster tilting subcategories. Let \(D\) be a subcategory of a category \(C\) and let \(x \in C\). A right \(D\)-approximation of \(x\) is a morphism \(f: a \to x\) with \(a \in D\) such that all morphisms \(g: b \to x\) with \(b \in D\) factor through \(f\). If every \(x \in C\) admits a right \(D\)-approximation, then we say that \(D\) is contravariantly finite (in \(C\)). The notions left \(D\)-approximation and covariantly finite are defined dually. We say that \(D\) is functorially finite (in \(C\)) if \(D\) is both contravariantly and covariantly finite. Notice that if \(M \in \Lambda-\text{mod}\), then \(\text{add}(M)\) is functorially finite.

We recall the following definition from [IJ17].

**Definition 2.1.**
\((a)\) We call a subcategory \(C\) of \(\Lambda-\text{mod}\) an \(n\)-cluster tilting subcategory if it is functorially finite and

\[
\mathcal{C} = \{X \in \Lambda-\text{mod} \mid \text{Ext}_\Lambda^i(X, X) = 0 \text{ for all } 0 < i < n\}
\]

\[
= \{X \in \Lambda-\text{mod} \mid \text{Ext}_\Lambda^i(X, \mathcal{C}) = 0 \text{ for all } 0 < i < n\}.
\]

If moreover \(\text{Ext}_\Lambda^i(X, \mathcal{C}) \neq 0\) implies that \(i \in n\mathbb{Z}\), then we call \(C\) an \(n\mathbb{Z}\)-cluster tilting subcategory.

\((b)\) Let \(M \in \Lambda-\text{mod}\). If \(\text{add}(M)\) is an \(n\)-cluster tilting subcategory (respectively \(n\mathbb{Z}\)-cluster tilting subcategory) of \(\Lambda-\text{mod}\), then we call \(M\) an \(n\)-cluster tilting module (respectively \(n\mathbb{Z}\)-cluster tilting module).

We collect some basic properties of \(n\mathbb{Z}\)-cluster tilting subcategories in the following proposition.

**Proposition 2.2.** Let \(\Lambda\) be a finite dimensional algebra and let \(\mathcal{C} \subseteq \Lambda-\text{mod}\) be an \(n\)-cluster tilting subcategory. Then the following statements hold.

\((a)\) \(\Lambda \in \mathcal{C}\) and \(D(\Lambda) \in \mathcal{C}\).

\((b)\) Denote by \(\mathcal{C}_P\) and \(\mathcal{C}_I\) the sets of isomorphism classes of indecomposable non-projective respectively non-injective modules in \(\mathcal{C}\). Then \(\tau_n\) and \(\tau_n^-\) induce mutually inverse bijections

\[
\mathcal{C}_P \xrightarrow{\tau_n} \mathcal{C}_I,
\]

\[
\mathcal{C}_I \xrightarrow{\tau_n^-} \mathcal{C}_P.
\]

\((c)\) \(\Omega^iM\) is indecomposable for all \(M \in \mathcal{C}_P\) and \(0 < i < n\).

\((d)\) \(\Omega^{-i}N\) is indecomposable for all \(N \in \mathcal{C}_I\) and \(0 < i < n\).

Moreover, if \(\Lambda\) is representation-directed, then any subcategory of \(\Lambda-\text{mod}\) which is closed under finite direct sums and summands and satisfies \((a)-(d)\) is an \(n\)-cluster tilting subcategory.
Proof. Part (a) follows immediately by the definition of an $n$-cluster tilting subcategory and the facts that $\text{Ext}^1_k(\Lambda, M) = 0$ and $\text{Ext}^1_k(M, D(\Lambda)) = 0$ for all $M \in \Lambda \text{-mod}$. For part (b) we refer to [Iya07, Section 1.4.1]. For parts (c) and (d) we refer to [Vas19, Corollary 3.3]. For the reverse implication if $\Lambda$ is representation-directed, we refer to [Vas19, Theorem 1].

Corollary 2.3. If $\Lambda$ is representation-directed, then any $n$-cluster tilting subcategory of $\Lambda \text{-mod}$ is unique and given by $C = \text{add} \{ \tau^{-r}(\Lambda) \mid r \geq 0 \}$.

Proof. If $D \subseteq \Lambda \text{-mod}$ is an $n$-cluster tilting subcategory, then $C \subseteq D$ by Proposition 2.2(a) and (b). Since $D$ is an $n$-cluster tilting subcategory, it follows that $C$ satisfies Proposition 2.2(b)–(d). Since we also have $\Lambda \in C$, it follows that $C$ is $n$-cluster tilting by [Vas19, Theorem 1]. Since $C \subseteq D$ are both $n$-cluster tilting subcategories, we conclude that $C = D$.

Proposition 2.4. Let $\Lambda$ be a finite dimensional algebra and let $C \subseteq \Lambda \text{-mod}$ be an $n$-cluster tilting subcategory. Then the following statements are equivalent.

(a) $C$ is $n\mathbb{Z}$-cluster tilting.
(b) $\Omega^n(C) \subseteq C$.
(c) $\Omega^{-n}(C) \subseteq C$.

In this case, the following statements hold.

(d) If $M \in C_P$, then $\Omega^{-\tau}(M) \in C$.
(e) If $N \in C_I$, then $\Omega^{-\tau}(N) \in C$.

Proof. For the equivalence between (a), (b) and (c) we refer to [IJ17, Section 2.2]. For part (d) notice that if $M \in C_P$, then $M \cong \tau^{-n}(N)$ for some $N \in C_I$ by Proposition 2.2(b). By Proposition 2.2(d) it follows that $\Omega^{-(n-1)}(N)$ is indecomposable and not injective so

$$\tau(M) \cong \tau \tau^{-n}(N) \cong \tau \Omega^{-n}(N) \cong \Omega^{-(n-1)}(N).$$

Applying $\Omega^{-}$ in the above we have

$$\Omega^{-\tau}(M) \cong \Omega^{-n}(N),$$

where $\Omega^{-n}(N) \in C$ by (c). Hence $\Omega^{-\tau}(M) \in C$. Part (e) follows similarly.

2.3. Nakayama algebras. In this section we discuss (connected) Nakayama algebras in terms of quivers with relations as well as the shape of their Auslander–Reiten quivers. Note that everything stated is essentially known, but we still provide some proofs for the reader’s convenience.

Let $m \geq 1$ be a positive integer and $Q_m \in \{ A_m, \tilde{A}_m \}$, where $A_m$ and $\tilde{A}_m$ are the following quivers:

$$A_m: \quad m \xrightarrow{\alpha_m} m - 1 \xrightarrow{\alpha_{m-1}} m - 2 \xrightarrow{\alpha_{m-2}} \cdots \xrightarrow{\alpha_3} 2 \xrightarrow{\alpha_2} 1$$

$$\tilde{A}_m: \quad \alpha_m \quad \alpha_m \quad \alpha_m \quad \alpha_{m-1}$$

We say that an algebra $\Lambda$ is a Nakayama algebra if $\Lambda = KQ_m/I$ for some admissible ideal $I$. We call $\Lambda$ cyclic if $Q_m = \tilde{A}_m$ and acyclic if $Q_m = A_m$. Moreover, we say that $\Lambda$ is a Nakayama algebra with homogeneous relations or simply a homogeneous Nakayama algebra if $I = R^l_{Q_m}$ for some $l \geq 2$. Note that in both cases $Q_m$ has $m$ vertices, which is perhaps non-standard but means that $A_m$ is obtained from $\tilde{A}_m$ by removing the arrow $\alpha_1$, which is useful for our purposes.
Our aim is to study \( n\mathbb{Z}\)-cluster tilting subcategories for Nakayama algebras. The case \( \Lambda = \mathbb{k} \) is not very interesting as \( \Lambda \) is semisimple and \( C = \Lambda \text{-mod} \) is the unique \( n\mathbb{Z}\)-cluster tilting subcategory for any \( n \). To simplify the discussion we therefore exclude this case and assume from now on that the quiver of \( \Lambda \) has at least one arrow. Note that the restriction to connected Nakayama algebras is mainly for convenience as it does not affect the existence of \( n\mathbb{Z}\)-cluster tilting subcategories in any essential way.

The representation theory of Nakayama algebras is well-understood, see for example [ASS06, Chapter V]. In particular, indecomposable modules over Nakayama algebras are uniserial, see [ASS06, Corollary V.3.6]. To describe all indecomposable modules over a Nakayama algebra \( \Lambda \) we first introduce a \( \mathbb{k}\mathcal{A}_m \)-module \( M(i, j) \) for each pair of integers \( i \leq j \). We define \( M(i, j) \) to have a basis \( \{ b_t \mid i \leq t \leq j \} \) and \( \mathbb{k}\mathcal{A}_m \)-action defined by

\[
\epsilon_s b_t = \begin{cases} b_t & \text{if } s - t \in m\mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \alpha_s b_t = \begin{cases} b_{t-1} & \text{if } s - t \in m\mathbb{Z} \text{ and } i \leq t - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Note that \( M(i + m, j + m) \cong M(i, j) \) by renaming basis vectors in the obvious way. For simplicity we will consider this isomorphism as an identity. Similarly, if \( i \leq k \leq j \), then there is a monomorphism \( M(i, k) \to M(i, j) \) and an epimorphism \( M(i, j) \to M(k, j) \) obtained by forgetting some of the basis vectors. Hence we consider \( M(i, k) \) and \( M(k, j) \) as a submodule respectively quotient module of \( M(i, j) \) in this case.

To deal with acyclic Nakayama algebras we will consider \( M(i, j) \) as a \( \mathbb{k}\mathcal{A}_m \)-module whenever \( \alpha_1 M(i, j) = 0 \). In this situation we may as well assume \( 1 \leq i \leq j \leq m \).

Now let \( \Lambda \) be a Nakayama algebra. If \( \Lambda = \mathbb{k}\mathcal{A}/I \text{ and } IM(i, j) = 0 \), then \( M(i, j) \) defines a \( \Lambda \)-module. Similarly, if \( \Lambda = \mathbb{k}\mathcal{A}/I \text{ and } \alpha_1 M(i, j) = 0 \) as well as \( IM(i, j) = 0 \), then \( M(i, j) \) again defines a \( \Lambda \)-module. We will indicate either of these situations simply by writing \( M(i, j) \in \Lambda \text{-mod} \) as a condition on the pair \( (i, j) \).

The modules \( M(i, j) \in \Lambda \text{-mod} \) classify all indecomposable \( \Lambda \)-modules up to isomorphism. Moreover, almost split sequences in \( \Lambda \text{-mod} \) are straightforward to compute for instance as in [ASS06, Theorem V.4.1]. Thus we obtain the following description of the Auslander–Reiten quiver of \( \Lambda \).

**Proposition 2.5.** Let \( \Lambda = \mathbb{k}\mathcal{Q}/I \) be a Nakayama algebra where \( \mathcal{Q} \in \{ A_m, \mathcal{A}_m \} \). Then the Auslander–Reiten quiver \( \Gamma = \Gamma(\Lambda) \) of \( \Lambda \) can be described as follows.

- The set of vertices \( \Gamma_0 = \{(i, j) \mid (i, j) \in \mathbb{Z}^2/\mathbb{Z}(m, m) \mid i \leq j \text{ and } M(i, j) \in \Lambda \text{-mod} \} \).
- For each \( (i, j) \in \Gamma_0 \) there is an arrow \( (i, j) \to (i, j + 1) \) if \( (i, j + 1) \in \Gamma_0 \) and an arrow \( (i, j) \to (i + 1, j) \) if \( (i + 1, j) \in \Gamma_0 \).

Moreover, all arrows \( (i, j) \to (i, j + 1) \) correspond to monomorphisms, all arrows \( (i, j) \to (i + 1, j) \) correspond to epimorphisms and the Auslander–Reiten translation is given by \( \tau(i, j) = (i - 1, j - 1) \), whenever it is defined.

To navigate the Auslander–Reiten quiver it is convenient to encode its shape as follows.

**Definition 2.6.** Let \( \Gamma \) be the Auslander–Reiten quiver of a Nakayama algebra \( \Lambda \).

(a) For \( i \in \mathbb{Z} \) define \( r_i \in \mathbb{Z} \) such that \( (i, r_i) \in \Gamma_0 \) with \( r_i - i \) maximal.

(b) For \( j \in \mathbb{Z} \) define \( \ell_j \in \mathbb{Z} \) such that \( (\ell_j, j) \in \Gamma_0 \) with \( j - \ell_j \) maximal.

As an illustration we consider the following examples

**Example 2.7.**
(a) Let \( \Lambda = \mathbb{k}\mathcal{A}_7/\langle b_2 a_3, \alpha_4 \rangle \). The Auslander–Reiten quiver \( \Gamma(\Lambda) \) of \( \Lambda \) is

\[
\begin{array}{cccc}
(2, 5) & (2, 4) & \cdots & (3, 5) \\
(3, 4) & (3, 3) & \cdots & (4, 5) \\
(4, 2) & (4, 1) & \cdots & (5, 6) \\
(4, 5) & (4, 4) & \cdots & (5, 7) \\
(5, 6) & (5, 5) & \cdots & (6, 7) \\
(6, 7) & (6, 6) & \cdots & (7, 7) \\
\end{array}
\]

(b) Let \( \Lambda = \mathbb{k}\mathcal{A}_7/\langle a_2 a_3, \alpha_4 \rangle \). The Auslander–Reiten quiver \( \Gamma(\Lambda) \) of \( \Lambda \) is

\[
\begin{array}{cccc}
(2, 5) & (2, 4) & \cdots & (3, 5) \\
(3, 4) & (3, 3) & \cdots & (4, 5) \\
(4, 2) & (4, 1) & \cdots & (5, 6) \\
(4, 5) & (4, 4) & \cdots & (5, 7) \\
(5, 6) & (5, 5) & \cdots & (6, 7) \\
(6, 7) & (6, 6) & \cdots & (7, 7) \\
\end{array}
\]
Moreover,
\[ r_1 = 3, \quad r_2 = 5, \quad r_3 = 5, \quad r_4 = 5, \quad r_5 = 7, \quad r_6 = 7, \quad r_7 = 7 \]
and
\[ \ell_1 = 1, \quad \ell_2 = 1, \quad \ell_3 = 1, \quad \ell_4 = 2, \quad \ell_5 = 2, \quad \ell_6 = 5, \quad \ell_7 = 5. \]

(b) Let \( \tilde{\Lambda} = k\tilde{\Lambda}/(\alpha_2\alpha_3\alpha_4; \alpha_5\alpha_6; \alpha_7\alpha_1; \alpha_1\alpha_2\alpha_3) \) be a cyclic Nakayama algebra. The Auslander–Reiten quiver \( \Gamma(\tilde{\Lambda}) \) of \( \tilde{\Lambda} \) is

\[ \begin{array}{c}
(1,3) \quad (2,5) \\
(1,2) \quad (2,3) \quad (3,4) \quad (4,5) \\
(5,6) \quad (6,7) \quad (7,8) \quad (1,2) \\
(1,1) \quad (2,2) \quad (3,3) \quad (4,4) \quad (5,5) \quad (6,6) \quad (7,7) \quad (1,1) \\
\end{array} \]

where \((1,1), (1,2)\) and \((1,3)\) have been drawn twice. Note that by our labelling convention \((7,7) = (0,0), (7,8) = (0,1)\) and \((7,9) = (0,2)\). Moreover,
\[ r_1 = 3, \quad r_2 = 5, \quad r_3 = 5, \quad r_4 = 5, \quad r_5 = 7, \quad r_6 = 7, \quad r_7 = 9 \]
and
\[ \ell_1 = 0, \quad \ell_2 = 0, \quad \ell_3 = 1, \quad \ell_4 = 1, \quad \ell_5 = 2, \quad \ell_6 = 5, \quad \ell_7 = 5. \]

The following results gives a general description of the shape of the Auslander–Reiten quiver.

**Proposition 2.8.** Let \( \Lambda \) be a Nakayama algebra with Auslander–Reiten quiver \( \Gamma \).

(a) If \( (i, j) \in \Gamma_0 \), then \( (i', j') \in \Gamma_0 \) for all \( i \leq i' \leq j \).

(b) If \( i \leq i' \), then \( r_i \leq r_{i'} \). Similarly, if \( j' \leq j \), then \( \ell_{j'} \leq \ell_j \).

(c) We have \( r_i + m = r_i + m \) and \( \ell_i + m = \ell_i + m \) for all \( i, j \in \mathbb{Z} \).

(d) If \( \Lambda \) is cyclic, then \( r_i \geq i + 1 \) and \( \ell_j \leq j - 1 \) for all \( i, j \in \mathbb{Z} \).

(e) If \( \Lambda \) is acyclic, then \( r_i = m \) and \( i + 1 \leq r_i \leq m \) for all \( 1 \leq i \leq m - 1 \). Similarly \( \ell_1 = 1 \) and \( \ell_i = 1 \) for all \( 1 \leq i \leq j - 1 \) for all \( 2 \leq j \leq m \).

**Proof.** Part (a) follows from the fact that \( M(i', j') \) is a subquotient of \( M(i, j) \) if \( i \leq i' \leq j' \leq j \). Part (b) follows from (a). Part (c) follows from \( M(i, j) = M(i + m, j + m) \). Part (d) follows from the admissibility of the ideal \( I \) in \( \Lambda = k\Lambda_m/I \). Part (e) similarly follows from admissibility and the fact that \( \alpha_1 M(i, j) = 0 \) for all \( M(i, j) \in \Lambda - \text{mod} \) if \( \Lambda \) is acyclic. \( \square \)

Note that as a consequence of Proposition 2.8(a), it is enough to know either the numbers \( r_i \) or the numbers \( \ell_j \) to recover the shape of the Auslander–Reiten quiver.

Next we observe that \( \Gamma \) has a particularly nice shape in case the Nakayama algebra is homogeneous.

**Proposition 2.9.** (a) Let \( \Lambda = k\Lambda_m/I \) be an acyclic Nakayama algebra. Then \( I = R_{\Lambda_m}^i \) if and only if \( r_i = \min\{i + l - 1, m\} \) for all \( 1 \leq i \leq m \), and if and only if \( \ell_j = \max\{j - l + 1, 1\} \) for all \( 1 \leq j \leq m \).

(b) Let \( \Lambda = k\tilde{\Lambda}_m/I \) be a cyclic Nakayama algebra. Then \( I = R_{\tilde{\Lambda}_m}^i \) if and only if \( r_i = i + l - 1 \) for all \( i \in \mathbb{Z} \), and if and only if \( \ell_j = j - l + 1 \) for all \( j \in \mathbb{Z} \).

**Proof.** The claims follow from the fact that \( R_{\Lambda_m}^i M(i, j) = 0 \) if and only if \( j - i + 1 \leq l \). \( \square \)

The following proposition shows how a number of computations can be done simply by considering the shape of the Auslander–Reiten quiver.

**Proposition 2.10.** Let \( \Lambda \) be a Nakayama algebra and \( M(i, j) \in \Lambda - \text{mod} \).

(a) We have \( \text{top}(M(i, j)) = M(j, j) \) and \( \text{soc}(M(i, j)) = M(i, i) \). In particular, \( M(i, j) \) is simple if and only if \( i = j \).

(b) The projective cover of \( M(i, j) \) is \( M(\ell_j, j) \). In particular, \( M(i, j) \) is projective if and only if \( i = \ell_j \). Otherwise we have \( \Omega(M(i, j)) \cong M(\ell_j, i - 1) \).
(c) The injective hull of $M(i, j)$ is $M(i, r_i)$. In particular, $M(i, j)$ is injective if and only if $j = r_i$.

Otherwise we have $\Omega^- (M(i, j)) \cong M(j + 1, r_i)$.

**Proof.** For part (a) note that $M(i, j)$ is uniserial and has $M(j, j)$ as a quotient module of dimension 1. Hence $\text{top}(M(i, j)) = M(j, j)$. Similarly $\text{soc}(M(i, j)) = M(i, i)$.

For part (b) one may compute that $M(\ell_j, j) \cong \Lambda e_j$, which shows that $M(\ell_j, j)$ is projective and indecomposable. Moreover there is an epimorphism $M(\ell_j, j) \to M(i, j)$ whose kernel is $M(\ell_j, i - 1)$ if $i > \ell_j$. The claim follows.

Part (c) follows similarly to (b). \hfill \□

Note that by Proposition 2.10, the sequences $(j - \ell_j + 1)j$ and $(r_i - i + 1)i$ are just the Kupisch series of $\Lambda$ and $\Lambda^\text{op}$ originally introduced in [Kup58].

Finally we observe which Nakayama algebras are selfinjective.

**Corollary 2.11.** Let $\Lambda$ be a Nakayama algebra. Then $\Lambda$ is selfinjective if and only if it is cyclic and homogeneous.

**Proof.** If $\Lambda$ is cyclic and homogeneous then $\Lambda$ is selfinjective by Proposition 2.10(b) and Proposition 2.10(b) and (c). If $\Lambda$ is selfinjective, then $\Lambda$ is clearly cyclic since otherwise $\Lambda$ has finite global dimension. Since $\Lambda$ is selfinjective, we have for all $i \in \mathbb{Z}$ that $r_{i+1} = r_i + 1$ (otherwise either $M(i+1, r_i)$ is injective but not projective or $M(i+1, r_i + 1)$ is projective but not injective). Hence for all $i \in \mathbb{Z}$ we have $r_i = i + r_1 - 1$ and $\Lambda$ is homogeneous by Proposition 2.10(b). \hfill \□

**Example 2.12.** (a) Let $\Lambda = \mathbb{k}A_2/R^3$ be a homogeneous acyclic Nakayama algebra. Then the Auslander–Reiten quiver $\Gamma(\Lambda)$ of $\Lambda$ is

\[
\begin{array}{cccccc}
(1,3) & (2,4) & (3,5) & (4,6) & (5,7) \\
(1,2) \cdots (2,3) \cdots (3,4) \cdots (4,5) \cdots (5,6) \cdots (6,7) \\
(1,1) \cdots (2,2) \cdots (3,3) \cdots (4,4) \cdots (5,5) \cdots (6,6) \cdots (7,7).
\end{array}
\]

(b) Let $\bar{\Lambda} = \mathbb{k}\bar{A}_6/R^3$ be a homogeneous cyclic Nakayama algebra. Then the Auslander–Reiten quiver $\Gamma(\bar{\Lambda})$ of $\bar{\Lambda}$ is

\[
\begin{array}{cccccc}
(1,3) & (2,4) & (3,5) & (4,6) & (5,7) & (6,8) & (1,3) \\
(1,2) \cdots (2,3) \cdots (3,4) \cdots (4,5) \cdots (5,6) \cdots (6,7) \cdots (1,2) \\
(1,1) \cdots (2,2) \cdots (3,3) \cdots (4,4) \cdots (5,5) \cdots (6,6) \cdots (1,1)
\end{array}
\]

where $(1,1)$, $(1,2)$ and $(1,3)$ have been drawn twice. Notice that $\bar{\Lambda}$ is selfinjective as claimed in Corollary 2.11.

3. $n\mathbb{Z}$-cluster tilting subcategories for Nakayama algebras

3.1. Computations. The aim of this paper is to classify all Nakayama algebras that admit an $n\mathbb{Z}$-cluster tilting subcategory for some $n$. In this section we perform some computations that will be useful to achieve this aim.

Since $n\mathbb{Z}$-cluster tilting subcategories are closed under $\Omega^n$, $\Omega^{-n}$, $\tau_n$ and $\tau_n^-$ it is crucial to describe the action of these functors on the Auslander–Reiten quiver. We start by computing the action of iterated syzygies and cosyzygies.

**Lemma 3.1.** Let $\Lambda$ be Nakayama algebra and $k \in \mathbb{Z}$. Let $i_1 \leq j_1$ and $i_2 \leq j_2$ be such that $M(i_1, j_1), M(i_2, j_2) \in \Lambda\text{-mod}$ and $\Omega^k(M(i_1, j_1)), \Omega^k(M(i_2, j_2))$ are non-zero. Then there are $i_1' \leq j_1'$ and $i_2' \leq j_2'$ such that $\Omega^k(M(i_1, j_1)) \cong M(i_1', j_1')$, $\Omega^k(M(i_2, j_2)) \cong M(i_2', j_2')$ and

(a) if $k$ is even, the following implications hold

- $i_1 = i_2 \Rightarrow i_1' = i_2'$,
- $i_1 \leq i_2 \Rightarrow i_1' \leq i_2'$,
- $j_1 = j_2 \Rightarrow j_1' = j_2'$,
- $j_1 \leq j_2 \Rightarrow j_1' \leq j_2'$.

(b) if $k$ is odd, then $i_1' = i_2'$ and $j_1' = j_2'$. \hfill \□
(b) if $k$ is odd, the following implications hold
\[ i_1 = i_2 \Rightarrow j'_1 = j'_2, \quad i_1 \leq i_2 \Rightarrow j'_1 \leq j'_2, \]
\[ j_1 = j_2 \Rightarrow i'_1 = i'_2, \quad j_1 \leq j_2 \Rightarrow i'_1 \leq i'_2. \]

**Proof.** We assume $k \geq 0$ as the case $k \leq 0$ is similar.

(a) The statement is trivial for $k = 0$ as we can choose $(i'_1, j'_1) = (i_1, j_1)$ and $(i'_2, j'_2) = (i_2, j_2)$.

By induction it is enough to consider the case $k = 2$. By Proposition 2.8(b) we have $\Omega(M(i, j)) \cong M(\ell_j, i - 1)$ and $\Omega^2(M(i, j)) \cong \Omega M(\ell_j, i - 1) \cong M(\ell_{j - 1}, \ell_j - 1)$. Now by Proposition 2.8(b) the maps $i \mapsto \ell_{i - 1}$ and $j \mapsto \ell_j - 1$ are weakly increasing so the claim follows.

(b) By (a) it is enough to consider the case $k = 1$. Again by Proposition 2.8(b) we have $\Omega(M(i, j)) \cong M(\ell_j, i - 1)$ and since the maps $i \mapsto i - 1$ and $j \mapsto \ell_j$ are weakly increasing the claim follows. \qed

As a corollary we get a similar result for the $n$-Auslander–Reiten translations.

**Corollary 3.2.** Let $\Lambda$ be a Nakayama algebra. Let $i_1 \leq i_2$ and $i_2 \leq j_2$ be such that $M(i_1, j_1), M(i_2, j_2) \in \Lambda \mod$ and $\tau_n(M(i_1, j_1)), \tau_n(M(i_2, j_2))$ are non-zero. Then there are $i'_1 \leq j'_1$ and $i'_2 \leq j'_2$ such that $
abla_n(M(i_1, j_1)) \cong M(i'_1, j'_1), \nabla_n(M(i_2, j_2)) \cong M(i'_2, j'_2)$ and

(a) if $n$ is even, the following implications hold
\[ i_1 = i_2 \Rightarrow j'_1 = j'_2, \quad i_1 \leq i_2 \Rightarrow j'_1 \leq j'_2, \]
\[ j_1 = j_2 \Rightarrow i'_1 = i'_2, \quad j_1 \leq j_2 \Rightarrow i'_1 \leq i'_2. \]

(b) if $n$ is odd, the following implications hold
\[ i_1 = i_2 \Rightarrow i'_1 = i'_2, \quad i_1 \leq i_2 \Rightarrow i'_1 \leq i'_2, \]
\[ j_1 = j_2 \Rightarrow j'_1 = j'_2, \quad j_1 \leq j_2 \Rightarrow j'_1 \leq j'_2. \]

The same is true replacing $\tau_n$ by $\tau_n^-$ everywhere.

**Proof.** By Lemma 3.1 it is enough to note that $\tau M(i, j) = M(i - 1, j - 1)$ and $\tau^- M(i, j) = M(i + 1, j + 1)$ if $M(i, j)$ is not projective respectively not injective. \qed

Finally we give a sufficient condition for non-vanishing of $\text{Ext}^1_{\Lambda}(i, j)$ that is useful to exclude possible $n\mathbb{Z}$-cluster tilting subcategories.

**Proposition 3.3.** Let $\Lambda$ be a Nakayama algebra and $M(i, j) \in \Lambda \mod$. Further let $i', j' \in \mathbb{Z}$ be such that $i + 1 \leq i' \leq j + 1 \leq j' \leq r$. Then $M(i', j') \in \Lambda \mod$ and $\text{Ext}^1_{\Lambda}(M(i', j'), M(i, j)) \neq 0$.

**Proof.** First note that $j' \leq r_i$ implies $M(i, j') \in \Lambda \mod$. Since $i < i'$ we have that $M(i', j')$ is a proper quotient module of $M(i, j')$ which gives that $M(i', j')$ is a non-projective $\Lambda$-module. Next we apply the Auslander–Reiten formula to obtain
\[ \text{Ext}^1_{\Lambda}(M(i', j'), M(i, j)) \cong \text{DHom}_{\Lambda}(M(i, j), \tau M(i', j')) = \text{DHom}_{\Lambda}(M(i, j), M(i' - 1, j' - 1)). \]

Now $i \leq i' - 1$ and $j \leq j' - 1$ tell us that there is a non-zero morphism $M(i, j) \rightarrow M(i' - 1, j' - 1)$ obtained as the composition of the quotient $M(i, j) \rightarrow M(i' - 1, j)$ and the inclusion $M(i' - 1, j) \rightarrow M(i' - 1, j' - 1)$. Assume towards a contradiction that this morphism factors through an injective $\Lambda$-module. Then it must factor through the inclusion of $M(i, j)$ in its injective hull, which is $M(i, r_i)$ by Proposition 2.10(c). This contradicts $j' - 1 < r_i$ since $M(i, j) = R^{r_i-j}M(i, r_i)$, but the image of $M(i, j)$ in $M(i' - 1, j' - 1)$ is $M(i' - 1, j) = R^{j' - 1 - i}M(i' - 1, j' - 1)$, which is strictly larger than $R^{r_i-j}M(i' - 1, j' - 1)$. \qed

3.2. **Necessary conditions.** In this section we introduce some necessary conditions for the existence of an $n\mathbb{Z}$-cluster tilting subcategory for a Nakayama algebra.

First we observe the following rather strong condition that holds assuming just the existence of an $n$-cluster tilting subcategory.

**Lemma 3.4.** Let $\Lambda$ be a Nakayama algebra that admits an $n$-cluster tilting subcategory. If $M(i, j) \in \Lambda \mod$, then $M(i - 1, j - 1) \in \Lambda \mod$ or $M(i + 1, j + 1) \in \Lambda \mod$. 

Lemma 3.6(b) can be used for any indecomposable injective non-projective module. Later we will need to consider some special cases.

Lemma 3.5. Let \( \Lambda \) be a Nakayama algebra and let \( C \subseteq \Lambda\text{-mod} \) be an \( n \)-cluster tilting subcategory.

(a) \( \Omega M \) is indecomposable for all \( M \in C_p \) and \( 0 \leq i \leq n \).

(b) \( \Omega^{-1}N \) is indecomposable for all \( N \in C_t \) and \( 0 \leq i \leq n \).

Proof. We only prove (a) as (b) is similar. For \( i = 0 \) the result is clear by definition and for \( 0 < i < n \) the result follows by Proposition 2.5. Since \( \Lambda \) is a Nakayama algebra and \( \Omega^{n-1}M \) is indecomposable, it follows that \( \Omega^{n-1}M \) is indecomposable or zero. Hence it is enough to show that \( \Omega^{n-1}M \) is not projective. Assume towards a contradiction that \( \Omega^{n-1}M \) is projective. Then \( \text{Ext}^{\Lambda}_{\Lambda}^{n-1}(M, \Omega^{n-1}M) \neq 0 \), contradicting that \( C \) is an \( n \)-cluster tilting.

Lemma 3.6. Let \( \Lambda \) be a Nakayama algebra and let \( C \subseteq \Lambda\text{-mod} \) be an \( n \)-cluster tilting subcategory.

(a) Assume for some \( i < j \) that \( M(i, j) \neq M(i, j + 1) \). Then \( \{M(i, j), M(i, j + 1), M(i, i), M(i, j - 1)\} \subseteq C \).

(b) Assume for some \( i < j \) that \( M(i, j) \neq M(i, j + 1) \). Then \( \{M(i, j), M(i, j + 1), M(i, j), M(i, j + 1)\} \subseteq C \).

Proof. We only prove (a) as (b) is similar. We will use Proposition 2.10(b)(c) repeatedly to identify projective and injective modules as well as compute syzygies and cosyzygies. Recall also that the Auslander–Reiten translation can be computed using Proposition 2.5.

By Proposition 2.8(a) \( M(i, j + 1) \in C \) implies \( M(i, j) \in C \). Also, by the same proposition \( M(i, j) \notin C \) implies \( M(i, j + 1) \notin C \). Hence \( M(i, j) \) is projective and \( M(i, j) \) is projective non-injective. In particular,

\[
\{M(i, j), M(i, j + 1)\} \subseteq C.
\]

Since \( M(i, j) \) is not injective, we have \( \Omega^{\ell_j} M(i, j) \in C \) by Proposition 2.11(c). We compute that

\[
C \ni \Omega^{\ell_j} M(i, j) \cong \Omega(M(i + 1, j + 1)) \cong M(\ell_{j+1}, i + 1) - 1 = M(i, i)
\]
since \( M(i, j + 1) \) is projective.

It remains to show \( M(i, j - 1) \in C \). If \( M(i, j - 1) \notin C \), then \( M(i, j - 1) \) is projective and so \( M(i, j - 1) \notin C \). Otherwise, assume \( M(i, j - 1) \in \Lambda\text{-mod} \). This implies \( M(i - 1, i) \in \Lambda\text{-mod} \) by Proposition 2.8(a) and so \( \ell_i \leq i - 1 \). In particular \( M(i, i) \) is not projective. Thus Proposition 2.4(d) gives \( \Omega^{-1}M(i, i) \in C \). Again we compute that

\[
C \ni \Omega^{-1} M(i, i) \cong \Omega^{-1} M(i - 1, j - 1) \cong M(i - 1, i) - 1 = M(i, j - 1)
\]
since \( M(i - 1, j - 1) \notin \Lambda\text{-mod} \) implies that \( M(i - 1, j - 1) \) is injective. Hence in this case it also follows that \( M(i - 1, j - 1) \in C \).

The strength of Lemma 3.6(a) is that for each indecomposable projective non-injective module \( M(i, j) \), we get two more indecomposable modules in \( C \), namely \( M(i, i) \) and \( M(i, j - 1) \). Similarly Lemma 3.6(b) can be used for any indecomposable injective non-projective module. Later we will show that this drastically reduces the possible \( n \)-cluster tilting subcategories of \( \Lambda\text{-mod} \) and we only need to consider some special cases.
Lemma 3.7. Let $\Lambda$ be a Nakayama algebra, $C \subseteq \Lambda\text{–mod}$ an $n\mathbb{Z}$-cluster tilting subcategory and $M(i, j) \in C$.

(a) If $M(i, j + 1) \in C$, then $M(i, j + 1)$ is projective.
(b) If $M(i - 1, j) \in C$, then $M(i - 1, j)$ is injective.

Proof. We only prove (a) as (b) is similar. Assume towards a contradiction that $M(i, j + 1)$ is not projective. By Proposition 2.10(d) we have that $\Omega^{-\tau}(M(i, j + 1)) \in C$. Using Proposition 2.10(c) we compute

$$\Omega^{-\tau}(M(i, j + 1)) \cong \Omega^{-\tau}(M(i - 1, j)) \cong M(j + 1, r_{i-1}).$$

We claim that $r_{i-1} = j + 1$. Indeed, if $r_{i-1} > j + 1$, then Proposition 3.3 gives

$$\text{Ext}^1_\Lambda(M(j + 1, r_{i-1}), M(i, j + 1)) \neq 0$$
as $r_{i-1} \leq r_1$ by Proposition 2.8(b), which contradicts that $C$ is $n$-cluster tilting.

Hence $M(j + 1, j + 1) \in C$. But $r_{i-1} \geq j + 1$ so Proposition 3.3 gives $\text{Ext}^1_\Lambda(M(j + 1, j + 1), M(i, j)) \neq 0$ contradicting that $C$ is $n$-cluster tilting.

Lemma 3.8. Let $\Lambda$ be a Nakayama algebra, $C \subseteq \Lambda\text{–mod}$ an $n\mathbb{Z}$-cluster tilting subcategory and $M(i, j) \in C$.

(a) Assume $M(i, j)$ is not projective and $\tau_n(M(i, j)) \cong M(i', j')$. If $M(i + 1, j) \in C$, then

$$\tau_n(M(i + 1, j)) \cong \begin{cases} M(i', j' + 1) & \text{if } n \text{ is even,} \\ M(i' + 1, j') & \text{if } n \text{ is odd.} \end{cases}$$

(b) Assume $M(i, j)$ is not injective and $\tau_n^-(M(i, j)) \cong M(i', j')$. If $M(i, j - 1) \in C$, then

$$\tau_n^-(M(i, j - 1)) \cong \begin{cases} M(i' - 1, j') & \text{if } n \text{ is even,} \\ M(i', j' - 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We only prove (a) as (b) is similar. Since $M(i, j) \in \Lambda\text{–mod}$ we get that $M(i + 1, j)$ is not projective by Proposition 2.10(b). Now Proposition 2.8(c) implies that $\Omega^kM(i, j) = M(i_k, j_k)$ and $\Omega^kM(i + 1, j) = M(i'_k, j'_k)$ for all $0 \leq k \leq n - 1$ and some $i_k \leq j_k$, $i'_k \leq j'_k$. We show by induction on $k$, that

$$M(i'_k, j'_k) = \begin{cases} M(i_k + 1, j_k) & \text{if } k \text{ is even,} \\ M(i_k, j_k + 1) & \text{if } k \text{ is odd.} \end{cases}$$

Then (a) follows from the case $k = n - 1$.

Note that the case $k = 0$ is trivial. Now assume the statement holds for some $0 \leq k < n - 1$.

Proposition 2.10(b) gives $M(i_k + 1, j_k + 1) = M(\ell_{j_k}, i_k - 1)$ and $M(i'_k + 1, j'_k + 1) = M(\ell_{j'_k}, i'_k - 1)$. Hence by induction hypothesis

$$M(i'_k + 1, j'_k + 1) = \begin{cases} M(\ell_{j_k}, i_k) & \text{if } k \text{ is even,} \\ M(\ell_{j_k + 1}, i_k - 1) & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand we need to show

$$M(i'_k + 1, j'_k + 1) = \begin{cases} M(i_k + 1, j_k + 1) & \text{if } k + 1 \text{ is even,} \\ M(i_k + 1, j_k + 1) & \text{if } k + 1 \text{ is odd,} \end{cases}$$

$$= \begin{cases} M(\ell_{j_k + 1}, i_k - 1) & \text{if } k + 1 \text{ is even,} \\ M(\ell_{j_k}, i_k) & \text{if } k + 1 \text{ is odd.} \end{cases}$$

Hence it suffices to show that $\ell_{j_k + 1} = \ell_{j_k} + 1$ if $k$ is odd.

First assume towards a contradiction that $\ell_{j_k + 1} < \ell_{j_k} + 1$. Then $\ell_{j_k + 1} = \ell_{j_k}$ by Proposition 2.8(b). Hence $M(i'_k + 1, j'_k + 1) = M(\ell_{j_k}, i_k - 1) = M(\ell_{j_k + 1}, j_k + 1)$. But then

$$\tau_n^-(M(i, j)) = \tau\Omega^{n-k-2}M(i_k + 1, j_k + 1) = \tau\Omega^{n-k-2}M(i'_k + 1, j'_k + 1) = \tau_n^-(M(i + 1, j))$$

contradicting Proposition 2.2(b).
Next assume towards a contradiction that $\ell_{j_k+1} > \ell_{j_k} + 1$. Then
\[
\{M(\ell_{j_k+1}, j_k + 1), M(\ell_{j_k+1} - 2, j_k)\} \subseteq \Lambda\text{-mod} \quad \text{and} \quad M(\ell_{j_k+1} - 1, j_k + 1) \not\subseteq \Lambda\text{-mod},
\]
so by Lemma 3.8(b) we get $M(\ell_{j_k+1}, j_k) \in C$. Now recall $M(i'_k, j'_k) = M(i_k, j_k + 1)$, which is not projective implying that $\ell_{j_k+1} < i_k$. This allows us to apply Proposition 3.3 to obtain
\[
\text{Ext}_1^M(M(i_k, j_k + 1), M(\ell_{j_k+1}, j_k)) \neq 0
\]
(note that $j_k + 1 \leq r_{\ell_{j_k+1}}$ holds since $t \leq r_t$ is true for all $t \in \mathbb{Z}$). But then
\[
\text{Ext}_1^M(M(i + 1, j), M(\ell_{j_k+1}, j_k)) \cong \text{Ext}_1^M(\Omega k M(i + 1, j), M(\ell_{j_k+1}, j_k)) \cong
\]
\[
\text{Ext}_1^M(M(i'_k, j'_k), M(\ell_{j_k+1}, j_k)) \cong \text{Ext}_1^M(M(i_k, j_k + 1), M(\ell_{j_k+1}, j_k)) \neq 0,
\]
which contradicts that $C$ is an $\mathbb{Z}$-cluster tilting.

\textbf{Lemma 3.9.} Let $\Lambda$ be a Nakayama algebra and $C \subseteq \Lambda\text{-mod}$ an $n\mathbb{Z}$-cluster tilting subcategory. Assume for some $i < j$ that $\{M(i, j), M(i - 1, j - 1)\} \subseteq \Lambda\text{-mod}$ and $M(i - 1, j) \notin \Lambda\text{-mod}$. Then both $M(i, j)$ and $M(i - 1, j - 1)$ are both projective and injective unless $n$ is even and $j = i + 1$.

\textbf{Proof.} Since $M(i - 1, j) \notin \Lambda\text{-mod}$ we have that $M(i, j)$ is projective and $M(i - 1, j - 1) = \Lambda\text{-mod}$. We only show that $M(i, j)$ projective. The proof that $M(i - 1, j - 1)$ is projective is similar. Note that it suffices to show that $M(i, j + 1) \notin \Lambda\text{-mod}$. Thus we assume towards a contradiction that $M(i, j + 1) \in \Lambda\text{-mod}$.

Then by Lemma 3.6(a)
\[
\{M(i, j), M(i - 1, j - 1), M(i, j + 1), M(i, i), M(i, j - 1)\} \subseteq C.
\]
Moreover, since $M(i, j + 1) \in \Lambda\text{-mod}$, we get $\{M(i, j), M(i, i), M(i, j - 1)\} \subseteq C$. Recall from Proposition 2.2(b), that there are mutually inverse bijections
\[
\begin{array}{c}
\text{C}_P \xrightarrow{\tau_n^{-1}} \text{C}_I.
\end{array}
\]
We use these repeatedly in the rest of the proof. In particular note that
\[
\{\tau_n^{-1}(M(i, j)), \tau_n^{-1}(M(i, i)), \tau_n^{-1}(M(i, j - 1))\} \subseteq \text{C}_P.
\]
Now assume that $n$ is odd. By Lemma 3.8(b) we get $\tau_n^{-1}(M(i, j)) \cong M(i', j')$ and $\tau_n^{-1}(M(i, j - 1)) \cong M(i', j' - 1)$ for some $i' < j'$. But then Lemma 3.7(a) implies $M(i', j')$ is projective contradicting $M(i', j') \notin \text{C}_P$.

Next assume that $n$ is even and $j > i + 1$ so that $M(i, i) \neq M(i, j - 1)$. By Lemma 3.8(b) and Corollary 3.2(a) we get
\[
\tau_n^{-1}(M(i, j)) \cong M(i', j')
\]
\[
\tau_n^{-1}(M(i, j - 1)) \cong M(i' - 1, j')
\]
\[
\tau_n^{-1}(M(i, i)) \cong M(k, j')
\]
for some $i', j', k$ satisfying $i' < j'$ and $k \leq i' - 1$. Moreover, the above modules are non-projective and distinct so in fact $k < i' - 1$. In particular, $M(i' - 2, j') \in \Lambda\text{-mod}$ is not projective. Since $\{M(i', j'), M(i' - 1, j')\} \subseteq \text{C}$ it follows that $M(i' - 1, j')$ is injective by Lemma 3.7(b), and therefore so is also $M(i', j')$. In particular $M(i' - 2, j') \in \text{C}_P$. Now $\tau_n(M(i' - 1, j')) \cong \tau_n(\tau_n^{-1}(M(i, j - 1))) \cong M(i, j - 1)$, so by Lemma 3.8(a) we have that $\tau_n(M(i' - 2, j')) \cong M(i, j - 2) \in \text{C}$.

Hence we have $M(i, j - 2), M(i, j - 1) \in \text{C}$ and $M(i - 1, j - 1) \in \Lambda\text{-mod}$. By Lemma 3.7(a) we have that $M(i, j - 1)$ is projective. But this contradicts $M(i - 1, j - 1) \in \Lambda\text{-mod}$. □

Lemma 3.8 provides strong restrictions on which $\Lambda$ admit an $n\mathbb{Z}$-cluster tilting subcategory. For instance we can now show the following.
Corollary 3.10. Let Λ be a Nakayama algebra and let \( C \subseteq \Lambda \text{-mod} \) be an \( n\mathbb{Z} \)-cluster tilting subcategory. If \( n \) is odd, then the algebra \( \Lambda \) is a homogeneous Nakayama algebra.

**Proof.** Assume towards a contradiction that \( \Lambda \) is not homogeneous. Then by Proposition 2.8 and Proposition 2.9 there exists \( i < j \) such that \( \{M(i-1,j-1), M(i,j)\} \subseteq \Lambda \text{-mod}, M(i-1,j) \notin \Lambda \text{-mod} \) and such that \( M(i-2,j-1) \in \Lambda \text{-mod} \) or \( M(i,j+1) \in \Lambda \text{-mod} \). Hence \( M(i-1,j-1) \) is not projective or \( M(i,j) \) is not injective. In either case Lemma 3.9 is contradicted. \( \square \)

Corollary 3.11. Let \( \Lambda \) be a Nakayama algebra and assume \( \Lambda \text{-mod} \) has at least one \( n\mathbb{Z} \)-cluster tilting subcategory. If \( \Lambda \) is not homogeneous, then there exists a simple module \( M(i,i) \in \Lambda \text{-mod} \) which is neither projective nor injective, such that \( M(i-1,i+1) \notin \Lambda \text{-mod} \) and such that any \( n\mathbb{Z} \)-cluster tilting subcategory of \( \Lambda \) contains \( M(i,i) \).

**Proof.** Since \( \Lambda \) is not homogeneous, there exist \( i, j \in \mathbb{Z} \) with \( i < j \) and such that
\[
\{M(i-1,j-1), M(i,j)\} \subseteq \Lambda \text{-mod}, M(i-1,j) \notin \Lambda \text{-mod}
\]
and at least one of the conditions
\[
M(i,j+1) \in \Lambda \text{-mod}, M(i-2,j-1) \in \Lambda \text{-mod}
\]
holds. Assume that \( M(i,j+1) \in \Lambda \text{-mod}; \) the other case is similar. Then it follows by Lemma 3.9 that \( j = i+1 \) and so \( M(i-1,i+1) = M(i-1,j) \notin \Lambda \text{-mod} \). It follows by Lemma 3.6(a) that \( M(i,i) \) belongs to any \( n\mathbb{Z} \)-cluster tilting subcategory of \( \Lambda \text{-mod} \). Finally, since \( \{M(i-1,i), M(i,i+1)\} = \{M(i-1,j-1), M(i,j)\} \subseteq \Lambda \text{-mod} \), it follows that \( M(i,i) \) is neither projective nor injective. \( \square \)

4. Classification

In this section we classify all Nakayama algebras admitting an \( n\mathbb{Z} \)-cluster tilting subcategory.

4.1. Homogeneous relations. We begin by restricting our attention to homogeneous Nakayama algebras. Homogeneous acyclic Nakayama algebras admitting an \( n \)-cluster tilting subcategory were classified in [Vas19]. Homogeneous cyclic (i.e. self-injective) Nakayama algebras admitting an \( n \)-cluster tilting subcategory were classified in [DI20]. We recall these classifications.

**Theorem 4.1.** Let \( \Lambda = kQ_m/R^l \) be a homogeneous Nakayama algebra where \( Q_m \in \{A_m, \tilde{A}_m\} \) and \( l \geq 2 \). Then \( \Lambda \) admits an \( n \)-cluster tilting subcategory if and only if at least one of the following conditions is satisfied.

\[
\begin{align*}
(a) & \quad Q_m = A_m, \ l = 2 \text{ and } n \mid m - 1. \\
(b) & \quad Q_m = A_m, \ n \text{ is even and } (l(n-1)+2) \mid m-1-\frac{n}{2}. \\
(c) & \quad Q_m = \tilde{A}_m \text{ and } ((l(n-1)+2) \mid 2m. \\
(d) & \quad Q_m = \tilde{A}_m \text{ and } (l(n-1)+2) \mid tn, \text{ where } t = \gcd(n+1,2(l-1)).
\end{align*}
\]

**Proof.** Parts (a) and (b) are [Vas19 Theorem 2]. Parts (c) and (d) are [DI20 Theorem 5.1]. \( \square \)

Our aim is now to determine which of the above algebras \( \Lambda \) admit not only an \( n \)-cluster tilting subcategory but an \( n\mathbb{Z} \)-cluster tilting subcategory. Since \( \Lambda \) is homogeneous Proposition 2.9 gives explicit formulas for \( r_i \) and \( \ell_j \), which we can use to determine projectives and injectives, as well as compute syzygies and co-syzygies using Proposition 2.10. We will use these results freely in this section without further reference.

We start with the acyclic case. Note that then \( \Lambda \) is representation-directed and so there is at most one \( n \)-cluster tilting subcategory in \( \Lambda \text{-mod} \) by Corollary 2.8

**Proposition 4.2.** Let \( \Lambda = kA_m/R^l \) be a homogeneous acyclic Nakayama algebra. There exists an \( n\mathbb{Z} \)-cluster tilting subcategory \( C \subseteq \Lambda \text{-mod} \) if and only if one of the following conditions is satisfied.

\[
\begin{align*}
(a) & \quad l = 2 \text{ and } n \mid m - 1. \\
(b) & \quad l \geq 3, \ l \mid m - 1 \text{ and } n = 2^{m-1}.
\end{align*}
\]
Lemma 4.3.

More explicitly we have in case (a) that

\[ C = \text{add}(\{\Lambda \}) \cup \{\tau_n(M(1,1)) | 1 \leq k \leq \frac{2n-1}{n}\} = \text{add}(\{\Lambda \}) \cup \{M(1 + kn, 1 + kn) | 1 \leq k \leq \frac{2n-1}{n}\} \]

and in case (b) that \( C = \text{add}(\Lambda \oplus D(\Lambda)) \).

Proof. Assume first that \( C \subseteq \Lambda - \text{mod} \) is \( n \)-cluster tilting. In particular, \( C \) is \( n \)-cluster tilting.

If \( l = 2 \), then by Theorem 4.1(a)(b) we have that at least one of the conditions

(i) \( n | m - 1 \), or
(ii) \( 2n | m - 1 - n \)

is satisfied. If (ii) is satisfied, then \( n | m - 1 - n \) and so \( n | m - 1 \). Either way condition (a) is satisfied.

Now assume that \( l \geq 3 \). From Theorem 4.1 it follows that \( n \) is even. Since \( \Lambda \) is homogeneous we have that \( r_1 = l \) and \( \ell_j = 1 \) for \( 1 \leq j \leq l \). Thus \( M(1, l) \) is projective and injective, while \( M(1, j) \) is projective and non-injective for \( 1 \leq j \leq l - 1 \). In particular we have \( M(1, j) \in C_I \) and \( \tau_n(M(1, j)) \in C_P \) for \( 1 \leq j \leq l - 1 \) by Proposition 2.2(a)(b).

Write \( \tau_n(M(1, l - 1)) \cong M(i', j') \) for some \( i' \leq j' \). By Lemma 3.8(b) we get that

\[ \tau_n(M(1, l - t)) \cong M(i' - t + 1, j') \quad \text{for} \quad 1 \leq t \leq l - 1. \]

Next we use this equation to show \( i' = j' = m \). For \( t = l - 1 \) we obtain \( \tau_n(M(1, 1)) \cong M(i' - l + 2, j') \) which is not projective. Thus \( i' - l + 2 > \ell_j = \max\{j' - l + 1, 1\} \). Since \( i' \leq j' \) the only possibility is \( i' = j' \). For \( t = 2 \) we obtain \( \tau_n(M(1, l - 2)) \cong M(i' - 1, j') \in C \) which is injective by Lemma 3.7(b).

Hence \( i' = \ell_j - 1 = \min\{i' + l - 2, m\} \). Since \( l \geq 3 \) we get \( i' = m \). We now obtain

\[ \tau_n(M(l, 1 - t)) \cong M(m - t + 1, m) \quad \text{for} \quad 1 \leq t \leq l - 1. \]

On the other hand, for homogeneous acyclic Nakayama algebras there is a formula for computing \( \tau_n \) given in [Vas19] Lemma 4.8(b)] which translated to our notation reads

\[ \tau_n(M(i, j)) \cong M(j + \frac{2l - 2}{2}, i + \frac{l - 2}{2}). \]

Comparing this formula to the one above gives \( n = \frac{2l - 2}{2} \), as required.

For the converse implication assume that one of the conditions (a) or (b) is satisfied.

If (a) is satisfied, then there exists a unique \( n \)-cluster tilting subcategory \( C \subseteq \Lambda - \text{mod} \) by Theorem 4.1(a). We show that \( C \) is actually \( n \)-\( \mathbb{Z} \)-cluster tilting. Let \( 2 \leq j \leq m \). Since \( l = 2 \) we get \( \ell_j = j - 1 \) and

\[ \Omega(M(j, j)) = M(\ell_j, j - 1) = M(j - 1, j - 1) = \tau M(j, j). \]

Note that the above formula applies to every non-projective indecomposable since \( l = 2 \). Hence \( \Omega^n(M) \cong \tau_n(M) \) for every \( M \in C_P \). By Proposition 2.2(b) and Proposition 2.3(b) it follows that \( C \) is \( n \)-\( \mathbb{Z} \)-cluster tilting. The explicit form follows from Corollary 2.3 by noticing that \( M(1, 1) \) is the unique indecomposable projective non-injective \( \Lambda \)-module.

Finally, assume that (b) is satisfied. By Theorem 3 in Section 5.2 in [Vas19] there exists a unique \( n \)-cluster tilting subcategory \( C = \text{add}(\Lambda \oplus D(\Lambda)) \subseteq \Lambda - \text{mod} \) and \( \text{gl.dim}(\Lambda) = n \). In particular, \( C \) is \( n \)-\( \mathbb{Z} \)-cluster tilting. \( \square \)

We now turn our attention to the cyclic homogeneous case. Thus for the rest of this section let \( \Lambda = kA_n/R^l \) for some \( l \geq 2 \). Then \( \Lambda \) is selfinjective and \( M(i, j) \) is projective (or injective) if and only if \( j - i = l - 1 \). In other words, \( r_i = i + l - 1 \) and \( \ell_j = j - l + 1 \). Using this we get the following convenient formulas for \( \Omega, \Omega^-, \tau \) and \( \tau^- \).

Lemma 4.3. Let \( i \leq j \leq i + l - 2 \). Then

(a) \( \Omega(M(i, j)) \cong M(j - l + 1, i - 1) \).
(b) \( \Omega^-(M(i, j)) \cong M(j + 1, i + l - 1) \).
(c) \( \Omega \Omega^-(M(i, j)) \cong \Omega^-\Omega M(i, j) \cong M(i, j) \).
(d) \( \tau \tau^-(M(i, j)) \cong \tau^-\tau M(i, j) \cong M(i, j) \).
(e) \( \Omega \tau^-(M(i, j)) \cong \tau^-\Omega M(i, j) \cong M(j - l + 2, i) \).
(f) \( \Omega^-(M(i, j)) \cong \tau \Omega^- M(i, j) \cong M(j, i + l - 2) \).
Proof. Let $C \subseteq \Lambda \mod$. Since $M(i, i + l - 1)$ is projective (and injective) we have $M(i, i + l - 1) \in C$ for all $i$. Hence to describe $C$ we need to determine for which $i \leq j \leq i + l - 2$ we have $M(i, j) \in C$. As we shall see next, it turns out that only the extreme cases $i = j$ and $j = i + l - 2$ (which coincide for $l = 2$) can occur. In other words $C_P$ will only contain simples and (co-)syzygies of simples. We also remark that $C_P \neq \emptyset$ as $\Lambda$ is not semi-simple.

**Lemma 4.4.** Let $C \subseteq \Lambda \mod$ be an $nZ$-cluster tilting.

(a) If $M(i, j) \in C_P$, then $j = i$ or $j = i + l - 2$.

(b) $M(i, i) \in C_P$ if and only if $M(i, i + l - 2) \in C_P$.

*Proof. (a) Assume towards a contradiction that $i < j < i + l - 2$. By Proposition 2.3(d) and Proposition 4.3(i) we have $C_P \ni \Omega^-\tau(M(i, j)) \cong M(j, i + l - 2)$. Then $\text{Ext}^1_\Lambda(M(j, i + l - 2), M(i, j)) \neq 0$ by Proposition 4.3 contradicting that $C$ is $n$-cluster tilting.

(b) Apply Proposition 4.3 to compute

$$
\Omega^-\tau(M(i, j)) \cong M(i, i + l - 2), \quad \Omega^-\tau(M(i, i + l - 2)) \cong M(i + l - 2, i + l - 2)
$$

and

$$
\Omega^-\Omega^-\tau(M(i + l - 2, i + l - 2)) \cong M(i, i).
$$

The claim follows by Proposition 2.3(d).\hfill \Box

Note that Lemma 4.3 is independent of the number $n$. We now involve $n$ to get another constraint.

**Lemma 4.5.** Let $C \subseteq \Lambda \mod$ be an $nZ$-cluster tilting, let $k \in \mathbb{Z}$ be an integer, and assume $M(i, j) \in C_P$. Then $M(i + k, j + k) \in C_P$ if and only if $n \mid k$.

*Proof. By Proposition 4.3

$$
M(i + k, j + k) \cong \tau^{-k}(M(i, j)) \cong \Omega^{-k}(\Omega^{-r})^k(M(i, j))
$$

where $(\Omega^{-r})^k(M(i, j)) \in C_P$ by Proposition 2.3(c).

For $k = n$ we get $M(i + n, j + n) \in C_P$ by Proposition 2.3(c).

For $1 \leq k \leq n - 1$ we get

$$
\text{Ext}^k_\Lambda(M(i + k, j + k), (\Omega^{-r})^k(M(i, j))) \neq 0
$$

which implies $M(i + k, j + k) \notin C_P$.

For $k \geq 0$ the claim now follows by induction. The case $k \leq 0$ is similar.\hfill \Box

Together Lemma 4.3 and Lemma 4.5 pose very strong restrictions on $l$, $m$, and $n$.

**Corollary 4.6.** If $\Lambda \mod$ admits an $nZ$-cluster tilting subcategory then $n \mid m$ and $n \mid l - 2$.

*Proof. Let $C \subseteq \Lambda \mod$ be an $nZ$-cluster tilting. Since $C_P \neq \emptyset$ there are $i \leq j \leq i + l - 2$ such that $M(i, j) \in C_P$. Then $M(i + m, j + m) = M(i, j) \in C_P$, together with Lemma 4.3 implies $n \mid m$.

By Lemma 4.3 we have that $j = i$ or $j = i + l - 2$ and regardless we get $\{M(i, i), M(i + l - 2, i + l - 2)\} \subseteq C_P$. Again Lemma 4.3 implies $n \mid l - 2$.\hfill \Box

Note that the condition $n \mid l - 2$ is trivially satisfied for $l = 2$ and implies $l \neq 3$. It turns out that for $l \geq 4$ we must have $n = l - 2$. To show this we need to get better control over Ext-groups involving $M(i, i)$ and $M(i, i + l - 2)$.

**Lemma 4.7.** Let $i \in \mathbb{Z}$ and $X \in \Lambda \mod$ be indecomposable.

(a) $\text{Ext}^1_\Lambda(X, M(i, i)) \neq 0$ if and only if $X \cong M(i + 1, i + t)$ for some $1 \leq t \leq l - 1$.

(b) $\text{Ext}^1_\Lambda(X, M(i, i + l - 2)) \neq 0$ if and only if $X \cong M(i + t, i + l - 1)$ for some $1 \leq t \leq l - 1$.

(c) $\text{Ext}^1_\Lambda(M(i, i), X) \neq 0$ if and only if $X \cong M(i - t, i - 1)$ for some $1 \leq t \leq l - 1$.

(d) $\text{Ext}^1_\Lambda(M(i, i + l - 2), X) \neq 0$ if and only if $X \cong M(i - 1, i + l - 2 - t)$ for some $1 \leq t \leq l - 1$. 

Proof. We only prove (a) and (b) as (c) and (d) are similar.

(a) By the Auslander–Reiten formula
\[ \text{Ext}^1_{\Lambda}(X, M(i, i)) \cong D\text{Hom}_{\Lambda}(\tau^{-1}(M(i, i)), X) \cong D\text{Hom}_{\Lambda}(M(i + 1, i + 1), X). \]
By Proposition 2.10(a), \(M(i + 1, i + 1)\) is simple. Moreover, \(X\) is indecomposable and hence uniserial. It follows that \(D\text{Hom}_{\Lambda}(M(i + 1, i + 1), X) \neq 0\) if and only if \(\text{soc} X \cong M(i + 1, i + 1)\) and \(X\) is not projective-injective. By Proposition 2.10 this is equivalent to \(X \cong M(i + 1, j)\) for some \(i + 1 \leq j \leq i + l - 1\). The claim follows.

(b) By Proposition 4.3(b) we have that \(\Omega^{-}(M(i - 1, i - 1)) \cong M(i, i + l - 2)\). Hence
\[ \text{Ext}^1_{\Lambda}(X, M(i, i + l - 2)) \cong \text{Ext}^1_{\Lambda}(X, \Omega^{-}(M(i - 1, i - 1))) \cong \text{Ext}^1_{\Lambda}(\Omega X, M(i - 1, i - 1)) \]
and the claim follows from (a) and Proposition 4.3. \(\square\)

Lemma 4.8. Assume that \(l \geq 4\) and \(C \subseteq \Lambda \text{--mod}\) is an \(n\mathbb{Z}\)-cluster tilting subcategory. Then \(n = l - 2\).

Proof. Since \(C_P \neq \emptyset\), Lemma 4.4 gives \(M(i, i) \in C_P\) for some \(1 \leq i \leq m\) and by Lemma 4.5 we have that \(M(i + n, i + n) \in C_P\) as well. By Corollary 4.6 we have that \(n \mid l - 2\) and so \(n \leq l - 2\). Assume towards a contradiction that \(n < l - 2\). Then \(M(i, i + n) \not\in C_P\) by Lemma 4.7 and so there is \(C \in C_P\) such that \(0 \neq \text{Ext}^{k}(M(i, i + n), C) \cong \text{Ext}^{1}_{\Lambda}(M(i, i + n), \Omega^{-}(k^{-1}) C)\) for some \(1 \leq k \leq n - 1\). By Lemma 4.7 it follows that \(C\) is simple or a syzygy of a simple and by Lemma 4.3 is \(\Omega^{-}(k^{-1}) C\).

Assume first that \(\Omega^{-}(k^{-1}) C\) is simple. Then there is \(0 \leq i' < n - 1\) such that \(\Omega^{-}(k^{-1}) C \cong M(i', i')\).

By Lemma 4.7(a) we get that \(i' + l = i\) and so \(\Omega^{-}(k^{-1}) C \cong M(i - 1, i - 1)\). Thus
\[ \text{Ext}^{1}_{\Lambda}(M(i, i), C) \cong \text{Ext}^{1}_{\Lambda}(M(i, i), \Omega^{-}(k^{-1}) C) \cong \text{Ext}^{1}_{\Lambda}(M(i, i), M(i - 1, i - 1)) \neq 0 \]
(again by Lemma 4.7(a), which contradicts that \(C\) is \(n\mathbb{Z}\)-cluster tilting.

Secondly assume that \(\Omega^{-}(k^{-1}) C\) is a syzygy of a simple. Then we can use Lemma 4.7(b) to show in a similar way that \(\Omega^{-}(k^{-1}) C \cong M(i + n - l + 1, i + n - 1)\) and \(\text{Ext}^{k}_{\Lambda}(M(i + n, i + n), C) \cong \text{Ext}^{1}_{\Lambda}(M(i + n, i + n), M(i + n - l + 1, i + n - 1)) \neq 0\). Again this contradicts that \(C\) is \(n\mathbb{Z}\)-cluster tilting.

Now we are ready to deal with homogeneous cyclic Nakayama algebras.

Proposition 4.9. Let \(\Lambda = k\tilde{A}_{m}/R^{l}\) be a homogeneous cyclic Nakayama algebra. There exists an \(n\mathbb{Z}\)-cluster tilting subcategory \(C \subseteq \Lambda \text{--mod}\) if and only if one of the following conditions is satisfied.

(a) \(l = 2\) and \(n \mid m\).
(b) \(l \geq 4\), \(n = l - 2\) and \(n \mid m\).

Then there is a unique \(1 \leq i \leq n\) such that \(M(i, i) \subseteq C\). More explicitly we have in case (a) that
\[ C = \text{add} \{ \Lambda \} \cup \{ \tau_{n}^{k}(M(i, i)) \mid k \in \mathbb{Z} \} = \text{add} \{ \Lambda \} \cup \{ M(i + kn, i + kn) \mid k \in \mathbb{Z} \} \]
and in case (b) that
\[ C = \text{add} \{ \Lambda \} \cup \{ \Omega^{-}\tau_{n}^{k}(M(i, i)) \mid k \in \mathbb{Z} \} \]
\[ = \text{add} \{ \Lambda \} \cup \{ M(i + kn, i + kn) \oplus M(i + kn, i + kn) \mid k \in \mathbb{Z} \}. \]

Proof. First note that if there exists an \(n\mathbb{Z}\)-cluster tilting subcategory \(C \subseteq \Lambda \text{--mod}\), then (a) or (b) hold by Corollary 1.6 and Lemma 4.5. Moreover, Lemma 4.4 and Lemma 4.5 imply that there is unique \(1 \leq i \leq n\) such that \(M(i, i) \subseteq C\). Thus it remains to show that if (a) or (b) holds, then there exists an \(n\mathbb{Z}\)-cluster tilting subcategory \(C \subseteq \Lambda \text{--mod}\), and it has the desired form.

Assume that condition (a) is satisfied. Then
\[ l(n - 1) + 2 = 2n \mid 2m \]
and hence there exists an \(n\)-cluster tilting subcategory \(C \subseteq \Lambda \text{--mod}\) by Theorem 1.4. Note that since \(l = 2\) every indecomposable \(\Lambda\)-module is either simple or projective-injective. Let \(M(i, i) \in C_P\). Using Proposition 4.3 we compute
\[ \tau_{n} M(i, i) = \tau^{\Omega^{n-1}} M(i, i) \cong \tau M(i - n + 1, i - n + 1) \cong M(i - n, i - n) \cong \Omega^{n} M(i, i) \]
Hence \( C \) is \( n\mathbb{Z} \)-cluster tilting by Proposition 2.2(b) and Proposition 2.4 (b). The second part of the claim also follows.

Next assume that condition (b) is satisfied. To simplify notation we define

\[
F(M(i, j)) := M(j, i + n) = M(j, i + l - 2)
\]

for all \( i \leq j \leq i + n \) so that \( F(M(i, j)) \cong \Omega^{-1} \tau(M(i, j)) \) by Lemma 4.3.

Let \( 1 \leq i \leq n \) and set

\[
C = \text{add}(\{\Lambda \cup (\Omega^{-1})^k(M(i, i)) \mid k \in \mathbb{Z}\})
\]

Using the notation just introduced we get

\[
C = \text{add}(\{\Lambda \cup \{\Lambda_i^k(M(i, i)) \mid k \in \mathbb{Z}\})
\]

\[
= \text{add}(\{\Lambda \cup \{\Lambda_i + kn, i + kn \} \mid k \in \mathbb{Z}\}.
\]

In particular, we note that \( \text{Ext}^2_n(M(i, i)) = M(i + m, i + m) = M(i, i) \). Hence \( C \) has exactly \( \frac{2m}{n} \) elements, half of which are simple.

To complete the proof we show that \( C \) is \( n \)-cluster tilting. Then it follows that \( C \) is \( n\mathbb{Z} \)-cluster tilting as for \( M(i, j) \in C \) we have by Lemma 4.3 that

\[
\Omega^k(M(i, j)) = \tau \Omega^{-1} \Omega^{-1}(M(i, j)) = \tau_n F^{-1}(M(i, j)) \in C
\]

Moreover, any \( n\mathbb{Z} \)-cluster tilting subcategory containing \( M(i, i) \) must contain \( C \) (and thus be equal to \( C \)) by Proposition 2.3(d) and (e).

We show

\[
C = \{X \in \Lambda \mod \text{Ext}^1(X, C) = 0 \text{ for } 1 \leq s \leq n - 1\}
\]

is similar. Set

\[
X = \{M(i + s, i + s + t) \mid 1 \leq s \leq n - 1, 0 \leq t \leq n\},
\]

and let \( X = M(i', j') \) for some \( 1 \leq i' \leq j' \leq i' + n \). It is straightforward to show that \( X \in F^k X \) if and only if \( X \notin C \). On the other hand, we claim that \( X \notin X \) if and only if \( \text{Ext}^1(X, F^{s-1}(M(i, i))) \neq 0 \) for some \( 1 \leq s \leq n - 1 \). Indeed, this follows from the calculation

\[
\text{Ext}^1(X, F^{s-1}(M(i, i))) \cong \text{Ext}^1(\Omega^{s-1}(X), \Omega^{s-1} \Omega^{-1} M(i, i)) \cong \text{Ext}^1(X, M(i + s - 1, i + s - 1))
\]

and Lemma 4.7(a). Hence \( X \in F^k X \) if and only if \( \text{Ext}^1(X, F^{s+1}(M(i, i))) \neq 0 \) for some \( k \in \mathbb{Z} \) and \( 1 \leq s \leq n - 1 \), which is equivalent to

\[
X \notin \{Y \in \Lambda \mod \text{Ext}^1(Y, C) = 0 \text{ for } 1 \leq s \leq n - 1\}
\]

The claim follows. \( \square \)

**Remark 4.10.** Note that in Proposition 4.9 any \( n\mathbb{Z} \)-cluster tilting subcategory \( C \subseteq \Lambda \mod \) is determined by \( 1 \leq i \leq n \) such that \( M(i, i) \in C \). By symmetry any such \( i \) can appear. Thus for a homogeneous cyclic Nakayama algebras there are precisely \( n \) distinct \( n\mathbb{Z} \)-cluster tilting subcategories if any at all.

We now combine our results to achieve the classification of all homogeneous Nakayama algebras admitting \( n\mathbb{Z} \)-cluster tilting subcategories.

**Theorem 4.11.** Let \( \Lambda = kQ_m/R^l \) be a homogeneous Nakayama algebra where \( Q_m \in \{A_m, \tilde{A}_m\} \) and \( l \geq 2 \). Then \( \Lambda \) admits an \( n\mathbb{Z} \)-cluster tilting subcategory if and only if one of the following conditions is satisfied.

(a) \( Q_m = A_m, l = 2 \) and \( n \mid m - 1 \).

(b) \( Q_m = A_m, l \geq 3 \) and \( l \mid m - 1 \) and \( n = 2^{m-1} \).

(c) \( Q_m = A_m, l = 2 \) and \( n \mid m \).

(d) \( Q_m = A_m, l \geq 4 \), \( n = l - 2 \) and \( n \mid m \).

**Proof.** This follows immediately by Proposition 4.2 and Proposition 4.9. \( \square \)
Example 4.12. We give examples of Theorem 4.11 below. In each case the additive closure of the modules corresponding to vertices in $\Gamma(\Lambda)$ marked by rectangles forms an $n\mathbb{Z}$-cluster tilting subcategory.

(a) Let $\Lambda = \mathbb{k}A_2/R^2$ and $n = 4$. Then $\Gamma(\Lambda)$ is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

(b) Let $\Lambda = \mathbb{k}A_2/R^3$ and $n = 4$. Then $\Gamma(\Lambda)$ is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

(c) Let $\tilde{\Lambda} = \tilde{\mathbb{k}}A_2/R^2$ and $n = 4$. Then $\Gamma(\Lambda)$ is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

where (1, 1) has been drawn twice.

(d) Let $\Lambda = \mathbb{k}A_5/R^3$ and $n = 3$. Then $\Gamma(\Lambda)$ is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

where (1, j), has been drawn twice for $1 \leq j \leq 5$.

4.2. Non-homogeneous relations: acyclic case. To give the classification in this case, we first need to recall the notion of gluing from [Vas21]. Since we shall deal with several Nakayama algebras at the same time, it is convenient to use a different notation for their quivers. To this end, for $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$ we denote by $A_{[m_1, m_2]}$ the quiver

\[
A_{[m_1, m_2]} : m_2 \xrightarrow{\alpha_{m_2}} m_2 - 1 \xrightarrow{\alpha_{m_2 - 1}} \cdots \xrightarrow{\alpha_{m_1 + 1}} m_1.
\]

If $\Lambda = \mathbb{k}A_{[m_1, m_2]}/I$ is an acyclic Nakayama algebra and $M(i, j) \in \Lambda\text{-mod}$, then we have that $M(i, j) \cong M(i + m_2 - m_1 + 1, j + m_2 - m_1 + 1)$ where we consider this isomorphism as the identity. To avoid certain technicalities arising from this identification, from now on we drop our assumption that $i \in \mathbb{Z}$ and we only allow $m_1 \leq i \leq m_2$. We also denote by $\text{ind}(\Lambda)$ the set

\[
\text{ind}(\Lambda) \coloneqq \{M(i, j) \in \Lambda\text{-mod} \mid m_1 \leq i \leq m_2\},
\]

which is then a complete and irredundant set of representatives of isomorphism classes of indecomposable $\Lambda$-modules. If $\mathcal{C} \subseteq \Lambda\text{-mod}$ is a subcategory, we set $\text{ind}(\mathcal{C}) = \{M(i, j) \in \mathcal{C} \mid m_1 \leq i \leq m_2\}$.

Definition 4.13. Let $m_1, m_2, m_3 \in \mathbb{Z}$ with $m_1 < m_2 < m_3$ and let $\Lambda_1 = \mathbb{k}A_{[m_1, m_2]}/I_1$ and $\Lambda_2 = \mathbb{k}A_{[m_2, m_3]}/I_2$ be two acyclic Nakayama algebras. We define the gluing of $\Lambda_2$ and $\Lambda_1$ to be the acyclic Nakayama algebra $\Lambda = \Lambda_1 \triangle \Lambda_2$ given by $\Lambda = \mathbb{k}A_{[m_1, m_3]}/I_\Lambda$ where $I_\Lambda$ is the ideal generated by $I_1 \cup I_2 \cup \{\alpha_{m_2}\alpha_{m_2+1}\}$.

We immediately have the following lemma.

Lemma 4.14. Let $m_1, m_2, m_3 \in \mathbb{Z}$ with $m_1 < m_2 < m_3$ and let $\Lambda = \mathbb{k}A_{[m_1, m_3]}/I_\Lambda$ be an acyclic Nakayama algebra. Assume that $M(m_2 - 1, m_2 + 1) \notin \Lambda\text{-mod}$. Let $I_1 = \mathbb{k}A_{[m_1, m_2]} \cap I_\Lambda$ and $I_2 = \mathbb{k}A_{[m_2, m_3]} \cap I_\Lambda$. Set $\Lambda_1 = \mathbb{k}A_{[m_1, m_2]}/I_1$ and $\Lambda_2 = \mathbb{k}A_{[m_2, m_3]}/I_2$. Then $\Lambda = \Lambda_1 \triangle \Lambda_2$. 

Proposition 4.15(b). By Lemma 3.5(a) and since $M_2^\Lambda \subseteq \{ \text{Vas21, Corollary 3.39} \}$. □

and (c). Parts (d) and (e) follow from part (a), Proposition 2.5 and Proposition 2.10(b) and (c). For more details we refer to [Vas21] Section 3. It also follows directly from the definition that

$$(\Lambda_1 \triangleleft \Lambda_2) \triangle \Lambda_3 = \Lambda_1 \triangle (\Lambda_2 \triangle \Lambda_3),$$

and hence gluing is associative.

In the following proposition we collect some basic properties of gluing.

**Proposition 4.15.** Let $m_1, m_2, m_3 \in \mathbb{Z}$ with $m_1 < m_2 < m_3$ and let $\Lambda_1 = kA_{[m_1, m_2]} / I_1$ and $\Lambda_2 = kA_{[m_2, m_3]} / I_2$ be two acyclic Nakayama algebras. Let $\Lambda = \Lambda_1 \triangle \Lambda_2$.

(a) $\text{ind}(\Lambda) = \text{ind}(\Lambda_1) \cup \text{ind}(\Lambda_2) \text{ and } \text{ind}(\Lambda_1) \cap \text{ind}(\Lambda_2) = \{ M(m_2, m_2) \}$.

(b) $M(i, j)$ is a projective $\Lambda$-module if and only if exactly one of the following conditions hold

(i) either $M(i, j)$ is a projective $\Lambda_1$-module, or

(ii) $M(i, j)$ is a projective $\Lambda_2$-module different from $M(m_2, m_2)$.

(c) $M(i, j)$ is an injective $\Lambda$-module if and only if exactly one of the following conditions hold

(i) either $M(i, j)$ an injective $\Lambda_2$-module, or

(ii) $M(i, j)$ is an injective $\Lambda_1$-module different from $M(m_2, m_2)$.

(d) If $M(i, j) \in \text{ind}(\Lambda_1)$, then $\tau_\Lambda(M(i, j)) = \tau_{\Lambda_1}(M(i, j))$ and $\Omega_\Lambda(M(i, j)) = \Omega_{\Lambda_1}(M(i, j))$. If moreover $M(i, j) \neq M(m_2, m_2)$, then $\tau_\Lambda(M(i, j)) = \tau_{\Lambda_1}(M(i, j))$ and $\Omega_\Lambda(M(i, j)) = \Omega_{\Lambda_1}(M(i, j))$.

(e) If $M(i, j) \in \text{ind}(\Lambda_2)$, then $\tau_\Lambda(M(i, j)) = \tau_{\Lambda_2}(M(i, j))$ and $\Omega_\Lambda(M(i, j)) = \Omega_{\Lambda_2}(M(i, j))$. If moreover $M(i, j) \neq M(m_2, m_2)$, then $\tau_\Lambda(M(i, j)) = \tau_{\Lambda_2}(M(i, j))$ and $\Omega_\Lambda(M(i, j)) = \Omega_{\Lambda_2}(M(i, j))$.

**Proof.** Part (a) follows from definition. Parts (b) and (c) follow from part (a) and Proposition 2.10(b) and (c). Parts (d) and (e) follow from part (a), Proposition 2.5 and Proposition 2.10(b) and (c). For a more detailed proof of a more general version of this proposition see [Vas21] Corollary 3.37 and [Vas21] Corollary 3.39. □

With this we can show the following facts about $n\mathbb{Z}$-cluster tilting subcategories of glued acyclic Nakayama algebras.

**Proposition 4.16.** Let $m_1, m_2, m_3 \in \mathbb{Z}$ with $m_1 < m_2 < m_3$ and let $\Lambda_1 = kA_{[m_1, m_2]} / I_1$ and $\Lambda_2 = kA_{[m_2, m_3]} / I_2$ be two acyclic Nakayama algebras. Let $\Lambda = \Lambda_1 \triangle \Lambda_2$. If $\Lambda_1$ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C}_i$, for $i = 1, 2$, then $\mathcal{C}_i = \text{add}\{ \mathcal{C}_{i_1}, \mathcal{C}_{i_2} \}$ is an $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda$-mod.

**Proof.** That $\mathcal{C}_i$ is an $n$-cluster tilting subcategory of $\Lambda$-mod follows by [Vas21] Corollary 4.17. To show that $\mathcal{C}_i$ is an $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda$-mod, let $M(i, j) \in \text{ind}(\mathcal{C}_i)$ and we show that $\Omega^k_\Lambda(M(i, j)) \in \mathcal{C}_i$. Clearly we may assume that $M(i, j)$ is not projective and so $\Omega^k_\Lambda(M(i, j))$ is indecomposable by Lemma 3.5(a). We consider the cases $M(i, j) \in \mathcal{C}_1$ and $M(i, j) \in \mathcal{C}_2 \setminus \{ M(m_2, m_2) \}$ separately (notice that $M(m_2, m_2)$ is not injective as a $\Lambda_1$-module).

If $M(i, j) \in \mathcal{C}_1$, then $M(i, j)$ is not projective as a $\Lambda_1$-module by Proposition 4.15(b). Hence $\Omega^k_\Lambda(M(i, j)) \in \text{ind}(\mathcal{C}_1)$ for all $0 \leq k \leq n$ by Lemma 3.5(a). By Proposition 4.15(d) and since $\mathcal{C}_1$ is an $n\mathbb{Z}$-cluster tilting, it follows that

$$\Omega^k_\Lambda(M(i, j)) = \Omega^k_\Lambda(M(i, j)) \in \mathcal{C}_i \subseteq \mathcal{C}_1,$$

as required.

If $M(i, j) \in \mathcal{C}_2$ and $M(i, j) \neq M(m_2, m_2)$, then $M(i, j)$ is not projective as a $\Lambda_2$-module by Proposition 4.15(b). By Lemma 3.5(a) and since $M(m_2, m_2)$ is a projective $\Lambda_2$-module it follows that
and Proposition 4.15(d). Then Proposition 2.2(b) holds for non-injective \( \Lambda \). Theorem 4.20.

Let \( m_1, m_2, m_3 \in \mathbb{Z} \) with \( m_1 < m_2 < m_3 \) and let \( \Lambda_1 = kA_{[m_1, m_2]}/I_1 \) and \( \Lambda_2 = kA_{[m_2, m_3]}/I_2 \) be two acyclic Nakayama algebras. Let \( \Lambda = \Lambda_1 \oplus \Lambda_2 \). If \( \Lambda \) admits an \( n\mathbb{Z} \)-cluster tilting subcategory \( \mathcal{C}_\Lambda \) such that \( M(m_2, m_2) \in \mathcal{C}_\Lambda \), then

\[
\mathcal{C}_{\Lambda_1} := \mathcal{C}_\Lambda - \text{mod} \cap \mathcal{C}_\Lambda \text{ and } \mathcal{C}_{\Lambda_2} := \mathcal{C}_\Lambda - \text{mod} \cap \mathcal{C}_\Lambda
\]

are \( n\mathbb{Z} \)-cluster tilting subcategories of \( \Lambda_1 - \text{mod} \) and \( \Lambda_2 - \text{mod} \), respectively.

**Proposition 4.17.** Let \( m_1, m_2, m_3 \in \mathbb{Z} \) with \( m_1 < m_2 < m_3 \) and let \( \Lambda_1 = kA_{[m_1, m_2]}/I_1 \) and \( \Lambda_2 = kA_{[m_2, m_3]}/I_2 \) be two acyclic Nakayama algebras. Let \( \Lambda = \Lambda_1 \oplus \Lambda_2 \). If \( \Lambda \) admits an \( n\mathbb{Z} \)-cluster tilting subcategory \( \mathcal{C}_\Lambda \) such that \( M(m_2, m_2) \in \mathcal{C}_\Lambda \), then

\[
\mathcal{C}_{\Lambda_1} := \mathcal{C}_\Lambda - \text{mod} \cap \mathcal{C}_\Lambda \text{ and } \mathcal{C}_{\Lambda_2} := \mathcal{C}_\Lambda - \text{mod} \cap \mathcal{C}_\Lambda
\]

are \( n\mathbb{Z} \)-cluster tilting subcategories of \( \Lambda_1 - \text{mod} \) and \( \Lambda_2 - \text{mod} \), respectively.

**Proof.** We only show that \( \mathcal{C}_{\Lambda_1} = \Lambda_1 - \text{mod} \cap \mathcal{C}_\Lambda \) is an \( n\mathbb{Z} \)-cluster tilting subcategory of \( \Lambda_1 - \text{mod} \) as the claim about \( \mathcal{C}_{\Lambda_2} \) can be proved dually. We first claim that for all \( M(i, j) \in \text{ind}(\mathcal{C}_{\Lambda_1}) \) such that \( M(i, j) \) is not injective as a \( \Lambda_1 \)-module and for all \( 1 \leq k \leq n-1 \) we have

\[
\Omega_{\Lambda_1}^{-k}(M(i, j)) = \Omega_{\Lambda_1}^{-k}(M(i, j)) \neq M(m_2, m_2).
\]

By Proposition 4.15(d) it is enough to show that \( \Omega_{\Lambda_1}^{-k}(M(i, j)) \neq M(m_2, m_2) \) for all \( 1 \leq k \leq n-1 \). But this follows since \( \{M(i, j), M(m_2, m_2)\} \subseteq \mathcal{C}_\Lambda \) and so \( \text{Ext}_k^\Lambda(M(m_2, m_2), M(i, j)) = 0 \) for \( 1 \leq k \leq n-1 \).

To show that \( \mathcal{C}_{\Lambda_1} \) is \( n\mathbb{Z} \)-cluster tilting, it is enough to show that the statements (a)-(d) in Proposition 2.2 and statement (b) in Proposition 2.4 hold. That Proposition 2.2(a) holds follows from Proposition 4.15(b) and (c) and since \( M(m_2, m_2) \in \mathcal{C}_\Lambda \). That Proposition 2.2(c) and Proposition 2.4(b) hold follows from Proposition 4.15(d). That Proposition 2.2(d) holds follows from (4.1).

Finally it remains to show that Proposition 2.2(b) holds for \( \mathcal{C}_{\Lambda_1} \). For any non-projective \( \Lambda_1 \)-module \( M(i, j) \in \text{ind}(\Lambda_1) \) we have that \( (\tau_n)_i(M(i, j)) = (\tau_n)_i(M(i, j)) \) by Proposition 4.15(d). For any non-injective \( \Lambda_1 \)-module \( M(i, j) \in \text{ind}(\Lambda_1) \) we have that \( (\tau_n)_i(M(i, j)) = (\tau_n)_i(M(i, j)) \) by 4.11 and Proposition 4.15(d). Then Proposition 2.2(b) holds for \( \mathcal{C}_{\Lambda_1} \) since it holds for \( \mathcal{C}_\Lambda \).

Using Proposition 4.15 we can also get some control of how global dimension behaves under gluing.

**Proposition 4.18.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be two acyclic Nakayama algebras. Then

\[
\text{gl. dim}(\Lambda_1 \oplus \Lambda_2) \leq \text{gl. dim}(\Lambda_1) + \text{gl. dim}(\Lambda_2).
\]

**Proof.** This result follows from [Vas21] Corollary 2.38. We include a proof for the reader’s convenience.

Set \( \text{gl. dim}(\Lambda_1) = d_1 \), \( \text{gl. dim}(\Lambda_2) = d_2 \) and \( \Lambda = \Lambda_1 \oplus \Lambda_2 \). As before let \( M(m_2, m_2) \) be the simple that satisfies \( M(m_2, m_2) \in \text{ind}(\Lambda_1) \cap \text{ind}(\Lambda_2) \).

Let \( M(i, j) \in \text{ind}(\Lambda) \). If \( M(i, j) \in \text{ind}(\Lambda_1) \), then \( \text{pr. dim}_{\Lambda_1} M(i, j) = \text{pr. dim}_{\Lambda_1} M(i, j) \leq d_1 \) by Proposition 4.15(d). If \( M(i, j) \in \text{ind}(\Lambda_2) \), then by Proposition 4.15(e) there is \( 0 \leq k \leq d_2 \) such that \( \Omega_{\Lambda_2}^{-k}(M(i, j)) = \Omega_{\Lambda_2}^{-k}(M(i, j)) \) is either projective or equal to \( M(m_2, m_2) \). In the first case \( \text{pr. dim}_{\Lambda_1} M(i, j) = k \leq d_2 \). In the second case \( \text{pr. dim}_{\Lambda_1} M(i, j) = k + \text{pr. dim}_{\Lambda_1} M(m_2, m_2) = k + \text{pr. dim}_{\Lambda_1} M(m_2, m_2) \leq d_2 + d_1 \).

To describe acyclic Nakayama algebras with non-homogeneous relations which admit \( n\mathbb{Z} \)-cluster tilting subcategories we introduce the following notion.

**Definition 4.19.** We call an acyclic Nakayama algebra \( \Lambda \) piecewise homogeneous if \( \Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r \), where each \( \Lambda_k \) is a homogeneous acyclic Nakayama algebra.

**Theorem 4.20.** Let \( \Lambda \) be an acyclic Nakayama algebra. Then \( \Lambda \) admits an \( n\mathbb{Z} \)-cluster tilting subcategory \( \mathcal{C}_\Lambda \) if and only if \( \Lambda \) is piecewise homogeneous

\[
\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r,
\]

where \( \Lambda_k \) for \( 1 \leq k \leq r \) is a homogeneous acyclic Nakayama algebra which admits an \( n\mathbb{Z} \)-cluster tilting subcategory \( \mathcal{C}_{\Lambda_k} \). In this case we have \( \mathcal{C}_\Lambda = \text{add}\{\mathcal{C}_{\Lambda_k} \mid 1 \leq k \leq r\} \).
Proof. Assume first that $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$ and that each $\Lambda_k$ for $1 \leq k \leq r$ is a homogeneous acyclic Nakayama algebra which admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda_k}$. Set $C_\Lambda = \{C_{\Lambda_k} \mid 1 \leq k \leq r\}$. Write $\Lambda = (\Lambda_1 \oplus \cdots \oplus \Lambda_{r-1}) \oplus \Lambda_r$. By an induction on $r$ and by using Proposition 4.15 it follows that $C_\Lambda$ is an $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda$-mod.

Next we assume that $\Lambda$ admits an $n\mathbb{Z}$-cluster tilting subcategory $C_\Lambda$ and show that there exist homogeneous acyclic Nakayama algebras $\Lambda_1, \ldots, \Lambda_r$ such that $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$ and each $\Lambda_k$ admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda_k}$. We prove the claim by induction on the number of simple modules of $\Lambda$. If $\Lambda$ is homogeneous there is nothing to show. Assume that $\Lambda$ is not homogeneous and write $\Lambda = kA_{[m_1, m_2]} / I_\Lambda$ where $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$. By Corollary 4.11 there exists $i \in \mathbb{Z}$ with $m_1 < i < m_2$ and such that $M(i - 1, i + 1) \not\subseteq \Lambda$-mod and $M(i, i) \subseteq C_\Lambda$. By Lemma 4.14 we have that $\Lambda = \Lambda \oplus \Lambda'$ where $\Lambda = kA_{[m_1, i]} / I_{\Lambda}$ and $\Lambda' = kA{i, m_2} / IB$ for suitable ideals $I_\Lambda$ and $I_B$. By Proposition 4.17 we conclude that $\Lambda$-mod and $\Lambda'$-mod admit $n\mathbb{Z}$-cluster tilting subcategories $C_\Lambda$ and $C_{\Lambda'}$. The result follows by the induction hypothesis applied to $A$ and $B$. □

Remark 4.21. Let $\Lambda$ be an acyclic Nakayama algebra which admits an $n\mathbb{Z}$-cluster tilting subcategory. Then $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$ where $\Lambda_k = kA_{m_k, m_{k+1}} / R_k$ for $1 \leq k \leq r$ is a homogeneous acyclic Nakayama algebra which admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda_k}$.

(a) By Proposition 4.2 we have that either $l_k = 2$ and $n \mid m_{k+1} - m_k$ or $l_k \geq 3$, $l_k \mid m_{k+1} - m_k$ and $n = 2\frac{m_{k+1} - m_k}{l_k}$. By [Vas19, Theorem 3] we have that $\text{gl.dim}(\Lambda_k) = n$ if and only if $n = 2\frac{m_{k+1} - m_k}{l_k}$ and only if $C_{\Lambda_k} = \text{add}(\Lambda_k \oplus D(\Lambda_k))$.

(b) The algebras $\Lambda_1, \ldots, \Lambda_r$ are not necessarily unique. Indeed, let $\Lambda = kA_{[1, m]} / R^2$ and $m = 1 - \text{rn}$ for some $r \in \mathbb{Z}_{\geq 2}$. Then $\Lambda$ is a homogeneous acyclic Nakayama algebra which admits an $n\mathbb{Z}$-cluster tilting subcategory by Proposition 4.2. Moreover, for $1 \leq k \leq r$, set $\Lambda_k = kA_{1+(k-1)n, 1+kn} / R^2$. Then $\Lambda_k$ admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda_k} = \text{add}(\Lambda_k \oplus M(1 + kn, 1 + k\text{rn}))$ and $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$. Notice also that in this case $\text{gl.dim}(\Lambda_k) = n$.

(c) By (a) the only case where a simple module which is neither projective nor injective belongs to $C_{\Lambda_k}$ is the case $l_k = 2$ and $m_{k+1} - m_k = r'n$ for some $r' \in \mathbb{Z}_{\geq 2}$ which is equivalent to $\text{gl.dim}(\Lambda_k) > n$. By (b), in this case, we can write $\Lambda_k$ as the gluing of $r'$ homogeneous acyclic Nakayama algebras of global dimension equal to $n$ in a unique way. Hence if we require that the algebras $\Lambda_1, \ldots, \Lambda_r$ have global dimension equal to $n$, then the decomposition $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$ is unique. Moreover, in this case $\text{gl.dim}(\Lambda) = \text{rn}$. Indeed, by Proposition 4.15, $\text{gl.dim}(\Lambda) \leq \text{rn}$. Furthermore, one may compute that $\Omega^{-\text{rn}}M(m_1, m_1) = M(m_{k+1}, m_{k+1})$ for $0 \leq k \leq r$ by Proposition 4.15 and induction. In particular $\Omega^{-\text{rn}}M(m_1, m_1) \neq 0$ and $\text{gl.dim}(\Lambda) = \text{rn}$.

Example 4.22. Let $\Lambda_1 = kA_{[1, 7]} / R^3$ and $\Lambda_2 = kA_{[7, 15]} / R^2$. Then $\Lambda_1$ and $\Lambda_2$ admit 4Z-cluster subcategories $C_{\Lambda_1}$ and $C_{\Lambda_2}$ respectively by Theorem 4.11 (see also Example 4.14). Let $\Lambda = \Lambda_1 \oplus \Lambda_2$. Then $\Lambda$ admits a 4Z-cluster tilting subcategory by Theorem 4.24. Indeed, the Auslander–Reiten quiver $\Gamma(\Lambda)$ of $\Lambda$ is

where the rectangles indicate the 4Z-cluster tilting subcategory $C_{\Lambda}$. Notice that $\text{gl.dim}(\Lambda_1) = 4$ while $\text{gl.dim}(\Lambda_2) = 8$. Hence, following Remark 4.21, we can write $\Lambda_2$ as a gluing of homogeneous acyclic Nakayama algebras of global dimension equal to 4 which admit 4Z-cluster tilting subcategories. Indeed, if $\Lambda'_2 = kA_{[7, 11]} / R^2$ and $\Lambda''_2 = kA_{[11, 15]} / R^2$, then $\Lambda_2 = \Lambda'_2 \oplus \Lambda''_2$ and $\Lambda'$ and $\Lambda''$ both admit 4Z-cluster tilting subcategories. In this way we have $\Lambda = \Lambda_1 \oplus \Lambda'_2 \oplus \Lambda''_2$ and this is the unique decomposition of $\Lambda$ in acyclic homogeneous Nakayama algebras of global dimension equal to 4 such that each of them admits a 4Z-cluster tilting subcategory. Also note that $\text{gl.dim}(\Lambda) = 12$.

4.3. Non-homogeneous relations: cyclic case. To give the classification in this case, we first need to recall the notion of self-gluing from [Vas20].
Definition 4.23. Let $m \geq 2$. Note that the arrows in $A_{0,m}$ and $\tilde{A}_m$ have the same labels. The correspondence given by this labelling extends to a multiplicative linear map $R_{A_{0,m}} \to R_{\tilde{A}_m}$, which induces a bijection $R_{A_{0,m}} \to R_{\tilde{A}_m}/(\alpha_m \alpha_1)$. Thus every admissible ideal $I \subseteq kA_{0,m}$ corresponds bijectively to an admissible ideal $\tilde{I} \subseteq k\tilde{A}_m$ such that $\alpha_m \alpha_1 \in \tilde{I}$. We define the self-gluing of $\Lambda = kA_{0,m}/I$ to be the cyclic Nakayama algebra $\tilde{\Lambda} = k\tilde{A}_m/\tilde{I}$, where $\tilde{I}$ is the ideal generated by $I \cup \{\alpha_m \alpha_1\}$.

We immediately have the following lemma.

Lemma 4.24. Let $m \geq 2$ and $\tilde{\Lambda}$ be a cyclic Nakayama algebra with $m$ vertices. Assume that $M(-1,1) \notin \tilde{\Lambda}$-mod. Then $\tilde{\Lambda}$ is the self-gluing of an acyclic Nakayama algebra $\Lambda = kA_{0,m}/I$, i.e., $\tilde{\Lambda} = k\tilde{A}_m/\tilde{I}$.

Proof. The condition $M(-1,1) \notin \tilde{\Lambda}$-mod means precisely $\alpha_m \alpha_1 = 0$ in $\tilde{\Lambda}$, and so the claim follows directly from Definition 4.23. □

Definition 4.23 is a special case of the methods described in [Vas20, Section 5.2]. In this case, no full and faithful embedding exists between the module category of an acyclic Nakayama algebra and its self-gluing. However, we can still compute syzygies, cosyzygies and Auslander–Reiten translations in $\tilde{\Lambda}$-mod using the corresponding concepts in $\Lambda$-mod.

In the following, for an indecomposable $\Lambda$-module $M(i,j)$ we write $\tilde{M}(i,j)$ or $M(i,j)^\sim$ for the corresponding $\tilde{\Lambda}$-module. This is useful for distinguishing the modules in $\Lambda$-mod and $\tilde{\Lambda}$-mod.

Proposition 4.25. Let $m \geq 2$ and $\Lambda = kA_{0,m}/I$ be an acyclic Nakayama algebra. Let $\tilde{\Lambda}$ be the self-gluing of $\Lambda$. Then
(a) The set
$$\text{ind}(\tilde{\Lambda}) := \{\tilde{M}(i,j) \mid M(i,j) \in \text{ind}(\Lambda) \text{ and } 0 \leq i \leq m\}$$
is a complete and irredundant set of representatives of isomorphism classes of indecomposable $\tilde{\Lambda}$-modules.

If moreover $i \in \mathbb{Z}$ with $0 \leq i \leq m$, then
(b) $\tilde{M}(i,j)$ is a projective $\tilde{\Lambda}$-module if and only if $M(i,j)$ is a projective $\Lambda$-module different from $M(0,0)$.
(c) $\tilde{M}(i,j)$ is an injective $\tilde{\Lambda}$-module if and only if $M(i,j)$ is an injective $\Lambda$-module different from $M(m,m)$.
(d) If $M(i,j) \in \Lambda$-mod with $M(i,j) \neq M(0,0)$, then $\tau(\tilde{M}(i,j)) = (\tau(M(i,j)))^\sim$ and $\Omega(\tilde{M}(i,j)) = (\Omega(M(i,j)))^\sim$.
(e) If $M(i,j) \in \Lambda$-mod with $M(i,j) \neq M(m,m)$, then $\tau^-(\tilde{M}(i,j)) = (\tau^-(M(i,j)))^\sim$ and $\Omega^-(\tilde{M}(i,j)) = (\Omega^-(M(i,j)))^\sim$.

Proof. Similar to the proof of Proposition 4.15 we refer to [Vas20, Section 5.2] for more details. □

Remark 4.26. (a) Proposition 4.25(a) implies that the set $\text{ind}(\tilde{\Lambda})$ has one less element than the set $\text{ind}(\Lambda)$. This is due to the fact that $M(0,0) = M(m,m)$ but $M(0,0) \neq M(m,m)$.
(b) Using Proposition 4.25(d) we can also compute $\tau(\tilde{M}(0,0)) = (\tau(M(m,m)))^\sim = (\Omega(M(m,m)))^\sim$.
(c) Using Proposition 4.25(e) we can also compute $\tau^-(\tilde{M}(m,m)) = (\tau^-(M(0,0)))^\sim = (\Omega^-(M(1,1)))^\sim$.

With this we can show the following results about $n\mathbb{Z}$-cluster tilting subcategories of self-gluing of acyclic Nakayama algebras.

Proposition 4.27. Let $m \geq 2$ and $\Lambda = kA_{0,m}/I$ be an acyclic Nakayama algebra. Let $\tilde{\Lambda}$ be the self-gluing of $\Lambda$. If $\Lambda$ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C}_\Lambda$, then $\tilde{\Lambda}$ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C}_{\tilde{\Lambda}}$ where
$$\mathcal{C}_{\tilde{\Lambda}} = \text{add}\{\tilde{M}(i,j) \mid M(i,j) \in \mathcal{C}_\Lambda \text{ and } 0 \leq i \leq m\}$$
Proof. That $C_{A}$ is an $n$-cluster tilting subcategory of $\Lambda\mod$ follows by [Vas20, Corollary 5.13]; see also [Vas20, Corollary 6.9]. To show that $C_{A}$ is an $n\mathbb{Z}$-cluster tilting, we can use Proposition 4.27 and follow the proof of Proposition 4.28. We leave the details to the reader. \hfill \Box

Proposition 4.28. Let $m \geq 2$ and $\Lambda = kA_{0, m}/I$ be an acyclic Nakayama algebra. Let $\tilde{\Lambda}$ be the self-gluing of $\Lambda$. If $\tilde{\Lambda}$ admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\tilde{\Lambda}}$ such that $M(m, m) \in C_{\tilde{\Lambda}}$, then

$$C_{\tilde{\Lambda}} = \text{add}\{M(i, j) \mid M(i, j) \in C_{\tilde{\Lambda}} \text{ and } 0 \leq i \leq m\}$$

is an $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda\mod$.

Proof. To show that $C_{\tilde{\Lambda}}$ is an $n\mathbb{Z}$-cluster tilting it is enough to show that the statements (a)–(d) in Proposition 4.22 and statement (b) in Proposition 4.24 hold. These can be easily verified using Proposition 4.25 and Remark 4.26, similarly to the proof of Proposition 4.17. \hfill \Box

We can now give the main result for this section.

Theorem 4.29. Let $\Lambda$ be a cyclic Nakayama algebra and assume that $\tilde{\Lambda}$ is not homogeneous. Then $\tilde{\Lambda}$ admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\tilde{\Lambda}}$ if and only if $\tilde{\Lambda}$ is the self-gluing of an acyclic Nakayama algebra $\Lambda$ which admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda}$. In this case, $\Lambda$ is piecewise homogeneous, the $n\mathbb{Z}$-cluster tilting subcategory $C_{\tilde{\Lambda}}$ is unique and $C_{\tilde{\Lambda}} = \text{add}\{M(i, j) \mid M(i, j) \in C_{\tilde{\Lambda}} \text{ and } 0 \leq i \leq m\}$.

Proof. Let $\Lambda = k\tilde{\Lambda}_{m}/I$ for some $m \geq 1$. If $m = 1$, then $\tilde{\Lambda}$ is necessarily homogeneous. Hence we have that $m \geq 2$.

Assume first that $\tilde{\Lambda}$ is the self-gluing of an acyclic Nakayama algebra $\Lambda$ which admits an $n\mathbb{Z}$-cluster tilting subcategory. Then $\tilde{\Lambda}$ admits an $n\mathbb{Z}$-cluster tilting subcategory $C_{\tilde{\Lambda}}$ by Proposition 4.27.

Next assume that $\tilde{\Lambda}$ is a piecewise $n\mathbb{Z}$-cluster tilting subcategory. By Corollary 3.14 there exists a simple module $M(i, i) \in \tilde{\Lambda}\mod$, such that $M(i - 1, i + 1) \notin \Lambda\mod$ and such that $M(i, i)$ belongs to any $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda\mod$. By possibly relabeling the vertices of $\tilde{\Lambda}_{m}$, we may assume that $i = m$. By Lemma 4.24 it then follows that $\tilde{\Lambda}$ is the self-gluing of an acyclic Nakayama algebra $\Lambda$. Hence, there is a bijection between the $n\mathbb{Z}$-cluster tilting subcategories of $\Lambda\mod$ and $\Lambda\mod$ by Proposition 4.27 and Proposition 4.28. Since there is a unique $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda}$ in $\Lambda\mod$ by Corollary 3.14, it follows that $\Lambda\mod$ has a unique $n\mathbb{Z}$-cluster tilting subcategory $C_{\Lambda}$. Finally, by Proposition 4.27 we get that $C_{\Lambda} = \text{add}\{M(i, j) \mid M(i, j) \in C_{\tilde{\Lambda}} \text{ and } 0 \leq i \leq m\}$. \hfill \Box

Remark 4.30. (a) Consider $\tilde{\Lambda}$ as in in Theorem 4.29. Similarly to Remark 4.21(c) one may compute that the simples in $C_{\tilde{\Lambda}}$ form an orbit under $\Omega^{-n}$. In particular, gl. dim $\tilde{\Lambda} = \infty$. In fact, we shall see in Corollary 5.18 that $\tilde{\Lambda}$ is not Iwanaga–Gorenstein.

(b) The algebra $\Lambda$ in Theorem 4.29 is not necessarily unique, as can be seen in the following example.

Example 4.31. Let $\Lambda_{1}$, $\Lambda_{2}$ and $\Lambda = \Lambda_{1} \triangleleft \Lambda_{2}$ be as in Example 4.22. Let $\tilde{\Lambda}$ be the self-gluing of $\Lambda$. Then $\tilde{\Lambda}$ admits a $4\mathbb{Z}$-cluster tilting subcategory and hence $\tilde{\Lambda}$ also admits a $4\mathbb{Z}$-cluster tilting subcategory by Theorem 4.29. Indeed, the Auslander–Reiten quiver $\Gamma(\Lambda)$ of $\Lambda$ is

$$\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \text{16}
\end{array}$$

where (1, 1) is drawn twice and the rectangles indicate the $4\mathbb{Z}$-cluster tilting subcategory $C_{\tilde{\Lambda}}$.

Now let $\Lambda_{2}'$ and $\Lambda_{2}''$ be as in Example 4.22. Then it is easy to see that $\tilde{\Lambda}$ is also the self-gluing of $\Lambda' = \Lambda_{2}'' \triangleleft \Lambda_{1} \triangleleft \Lambda_{2}'$, up to relabelling the vertices, but $\Lambda \not\cong \Lambda'$. 

5. Singularity categories

It was shown in [Kva21] that if a finite-dimensional algebra $\Lambda$ admits an $n\mathbb{Z}$-cluster tilting subcategory, then the singularity category of $\Lambda$ admits an $n\mathbb{Z}$-cluster tilting subcategory. Our goal in this section is to describe the singularity category and its $n\mathbb{Z}$-cluster tilting subcategories for Nakayama algebras admitting an $n\mathbb{Z}$-cluster tilting subcategory. Furthermore, we describe the canonical functor from the module category to the singularity category, and its restriction to the $n\mathbb{Z}$-cluster tilting subcategories. To do this we rely heavily on results from [She15], where the author studies the singularity category for general Nakayama algebras.

5.1. Cluster tilting in singularity categories. We start by recalling the definition of cluster tilting in triangulated categories.

**Definition 5.1.** Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{C}$ of $\mathcal{T}$ is called $n$-cluster tilting if it is functorially finite and

$$
\mathcal{C} = \{ T \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(\mathcal{C}, T[i]) = 0 \text{ for all } 0 < i < n \}
$$

$$
= \{ T \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(T, \mathcal{C}[i]) = 0 \text{ for all } 0 < i < n \}.
$$

If moreover $\text{Hom}_\mathcal{T}(\mathcal{C}, \mathcal{C}[i]) \neq 0$ implies that $i \in n\mathbb{Z}$, then we call $\mathcal{C}$ an $n\mathbb{Z}$-cluster tilting subcategory.

Analogously to module categories, we have that an $n$-cluster tilting subcategory $\mathcal{C}$ of $\mathcal{T}$ is $n\mathbb{Z}$-cluster tilting if and only if $\mathcal{C}[n] \subseteq \mathcal{C}$.

Now fix a finite-dimensional algebra $\Lambda$, and let $D^b(\Lambda\text{-mod})$ denote the bounded derived category of finitely generated $\Lambda$-modules. Furthermore, let $\text{perf} \Lambda \subseteq D^b(\Lambda\text{-mod})$ denote the subcategory of perfect complexes, i.e. complexes which are quasi-isomorphic to bounded complexes of finitely generated projective $\Lambda$-modules. The singularity category of $\Lambda$, denoted $\text{D}_\text{sing}(\Lambda)$, is the triangulated category defined by the Verdier quotient

$$
\text{D}_\text{sing}(\Lambda) := D^b(\Lambda\text{-mod})/ \text{perf} \Lambda.
$$

Note that we have a canonical functor

$$
\Lambda\text{-mod} \to \text{D}_\text{sing}(\Lambda)
$$

given by the composite $\Lambda\text{-mod} \to D^b(\Lambda\text{-mod}) \to \text{D}_\text{sing}(\Lambda)$ where $\Lambda\text{-mod} \to D^b(\Lambda\text{-mod})$ sends a module $M$ to the stalk complex which is equal to $M$ in degree 0, and where $D^b(\Lambda\text{-mod}) \to \text{D}_\text{sing}(\Lambda)$ is the canonical localization functor. By abuse of notation we write $M$ both for a module in $\Lambda\text{-mod}$, and for its image in $\text{D}_\text{sing}(\Lambda)$.

The following result relates $n\mathbb{Z}$-cluster tilting subcategories of $\Lambda\text{-mod}$ and $\text{D}_\text{sing}(\Lambda)$, and motivates our study of the singularity category.

**Theorem 5.2.** Let $\Lambda$ be a finite-dimensional algebra and let $\mathcal{C}$ be an $n\mathbb{Z}$-cluster tilting subcategory of $\Lambda\text{-mod}$. Then the subcategory

$$
\{ X \in \text{D}_\text{sing}(\Lambda) \mid X \cong \mathcal{C}[ni] \text{ for some } \mathcal{C} \in \mathcal{C} \text{ and } i \in \mathbb{Z} \}
$$

is an $n\mathbb{Z}$-cluster tilting subcategory of $\text{D}_\text{sing}(\Lambda)$.

**Proof.** This follows from [Kva21] Theorem 1.2. $\square$

5.2. Singularity categories of Nakayama algebras. We recall the description of the singularity category of a Nakayama algebra obtained in [She15]. To do this we recall the definition of the resolution quiver, which was first introduced in [Rin13].

**Definition 5.3.** Let $\Lambda$ be a Nakayama algebra. The resolution quiver $R(\Lambda)$ of $\Lambda$ is given as follows:

- The vertices of $R(\Lambda)$ correspond to isomorphism classes of simple $\Lambda$-modules.
- Let $i$ and $j$ be two vertices of $R(\Lambda)$, corresponding to the simple $\Lambda$-modules $S_i$ and $S_j$, respectively. Then there is an arrow $i \to j$ if and only if $S_j \cong \tau \text{soc } P_i$, where $P_i$ is the projective cover of $S_i$.

A simple $\Lambda$-module is called cyclic if it is contained in a cycle of $R(\Lambda)$.
Following [She15], we let $\mathcal{S}_c$ denote the class of simple cyclic $\Lambda$-modules, and we let $\mathcal{X}_c$ denote the class of indecomposable $\Lambda$-modules $M$ for which $\text{top} M \in \mathcal{S}_c$ and $\tau \text{soc} M \in \mathcal{S}_c$. Finally, we let $F = \text{add} \mathcal{X}_c$ denote the additive closure of $\mathcal{X}_c$.

The following theorem is one of the main results in [She15]. Here we consider the canonical functor $F \to D\text{sing}(\Lambda)$ given by the composite $F \to \Lambda-\text{mod} \to D\text{sing}(\Lambda)$ where $F \to \Lambda-\text{mod}$ is the inclusion functor.

**Theorem 5.4.** Let $\Lambda$ be a Nakayama algebra of infinite global dimension. The following hold:

1. $F$ is a wide subcategory of $\Lambda-\text{mod}$, i.e. it is closed under extensions, kernels and cokernels.
2. $F$ is a Frobenius abelian category. Hence, the stable category $\mathcal{F}$ of the abelian category $F$ is triangulated.
3. The functor $F \to D\text{sing}(\Lambda)$ kills the projective objects in $F$ and induces an equivalence $\mathcal{F} \cong D\text{sing}(\Lambda)$ of triangulated categories.

**Proof.** This follows from [She15, Proposition 3.5], [She15, Proposition 3.8] and [She15, Theorem 3.11].

Hence, to describe $D\text{sing}(\Lambda)$, it suffices to determine the objects in $F$.

5.3. **Singularity categories of Nakayama algebras admitting $n\mathbb{Z}$-cluster tilting categories.**

Let $\tilde{\Lambda}$ be a Nakayama algebra admitting an $n\mathbb{Z}$-cluster tilting subcategory. We want to describe $D\text{sing}(\tilde{\Lambda})$ and its $n\mathbb{Z}$-cluster tilting subcategories. If $\tilde{\Lambda}$ is acyclic, then $\tilde{\Lambda}$ has finite global dimension, and hence $D\text{sing}(\tilde{\Lambda}) \cong 0$ is trivial. If $\tilde{\Lambda}$ is a cyclic homogeneous Nakayama algebra, then $\tilde{\Lambda}$ is self-injective by Corollary 2.11, and hence we have an equivalence $\tilde{\Lambda}-\text{mod} \cong D\text{sing}(\tilde{\Lambda})$ by Buchweitz theorem [Buc21]. It follows that the singularity category of $\tilde{\Lambda}$ is easy to describe in this case. For example, it is well known that there is a bijection between $n\mathbb{Z}$-cluster tilting subcategories of $\tilde{\Lambda}-\text{mod}$ and $\tilde{\Lambda}-\text{mod}$.

The remaining case we need to consider is when $\tilde{\Lambda}$ is a cyclic non-homogeneous Nakayama algebra with an $n\mathbb{Z}$-cluster tilting subcategory. By Theorem 4.20, Remark 4.21(c), and Theorem 4.29 it follows that $\tilde{\Lambda}$ is the self-gluing of an acyclic Nakayama algebra of the form

$$\Lambda = \Lambda_1 \triangledown \cdots \triangledown \Lambda_r,$$

where $\Lambda_k = \mathbb{k}A_{m_k-1}/R^k$ for $1 \leq k \leq r$ is a homogeneous acyclic Nakayama algebra which admits an $n\mathbb{Z}$-cluster tilting subcategory and satisfies $\text{gl. dim}(\Lambda_k) = n$. Fix such algebras $\Lambda$ and $\Lambda_k$ for the remainder of this section. For convenience we also set $l_{r+1} := l_1$. We make the following observations:

- By Remark 4.20 $\text{gl. dim } \tilde{\Lambda} = \infty$ so Theorem 5.3 applies.
- If $l_k \geq 3$, then $l_k \mid m_{k+1} - m_k$ and $n = \frac{m_{k+1} - m_k}{l_k}$ by Proposition 4.2(b). Since $\tilde{\Lambda}$ is not homogeneous, there must exist a $k$ with $l_k \geq 3$, and therefore $n$ must be even.
- We claim that $l_k \mid m_{k+1} - m_k$ and $n = \frac{m_{k+1} - m_k}{l_k}$ also hold when $l_k = 2$. Indeed, we have

$$n = m_{k+1} - m_k = 2 \frac{m_{k+1} - m_k}{l_k}$$

since $\text{gl. dim}(\Lambda_k) = n$. This together with the fact that $n$ is even implies that $l_k \mid m_{k+1} - m_k$.

We start by computing the resolution quiver of the algebras $\Lambda_k$.

**Lemma 5.5.** Let $Q^k$ be the quiver given by the disjoint union

$$Q^k = \bigcup_{j=0}^{l_k-1} \hat{Q}^{k,j}$$
where $\tilde{Q}^{k,j}$ is a linearly oriented $A_\infty$ quiver when $j \neq 0$ and a linearly oriented $A_\infty$ quiver when $j = 0$. Then $Q^k$ is the resolution quiver of $\Lambda_k$, where the $i$th vertex of $Q^{k,j}$ corresponds to the simple $\Lambda_k$-module $M(m_k + (i-1)l_k + j, m_k + (i-1)l_k + j)$.

**Proof.** We write a vertex of $Q^k$ as a pair $(i, j)$ where $0 \leq j \leq l_k - 1$ and $i$ is a vertex of $\tilde{Q}^{k,j}$. With this notation, it is clear that the association

$$(i, j) \mapsto S_{m_k + (i-1)l_k + j} = M(m_k + (i-1)l_k + j, m_k + (i-1)l_k + j)$$

gives a bijection between the vertices of $Q^k$ and the isomorphism classes of simple $\Lambda_k$-modules. Hence, we only need to check that the arrows of $Q^k$ coincide with the arrows of the resolution quiver under this bijection. If $i > 1$ then the projective cover of $S_{m_k + (i-1)l_k + j}$ is

$$P_{m_k + (i-1)l_k + j} = M(m_k + (i-2)l_k + j + 1, m_k + (i-1)l_k + j)$$

by Proposition 2.10 (a) and Proposition 2.10 (b). It follows that

$$\tau \text{ soc } P_{m_k + (i-1)l_k + j} = \tau M(m_k + (i-2)l_k + j + 1, m_k + (i-1)l_k + j + 1)
= M(m_k + (i-2)l_k + j, m_k + (i-2)l_k + j)
= S_{m_k + (i-1)l_k + j}$$

where the first equality follows from Proposition 2.10 (a) and the second equality from Proposition 2.9. Since there is a unique arrow $(i, j) \to (i-1, j)$ in $Q^k$, we see that the arrows of $Q^k$ coincide with the arrows of the resolution quiver when $i > 1$. Finally, if $i = 1$ then the projective cover of $S_{m_k + j}$ is

$$P_{m_k + j} = M(m_k, m_k + j)$$

by Proposition 2.10 (a) and Proposition 2.10 (b), and so we get

$$\tau \text{ soc } P_{m_k + j} = \tau M(m_k, m_k) \cong 0$$

where the last isomorphism follows from the fact that $M(m_k, m_k)$ is projective. Since $(1, j)$ is a sink in $Q^k$, we conclude that $Q^k$ is the resolution quiver of $\Lambda_k$.

We now compute the resolution quiver of $\tilde{\Lambda}$.

**Lemma 5.6.** Let $Q$ denote the quiver obtained as follows:

1. First take the disjoint union

$$\bigcup_{k=1}^{r} \bigcup_{j=0}^{l_k-1} Q^{k,j}$$

where $Q^{k,j}$ is a linearly oriented $A_\infty$ quiver for all $k$ and $j$.

2. Then add an arrow from the sink of $Q^{k+1,0}$ to the source of $Q^{k,0}$ for all $1 \leq k \leq r$ (where $Q^{r+1,0} := Q^{1,0}$).

3. Finally, add an arrow from the sink of $Q^{k+1,j}$ to the source of $Q^{k,l_k-1}$ for all $1 \leq k \leq r$ and $0 < j < l_k+1$ (where $Q^{r+1,j} := Q^{1,j}$).

Then $Q$ is the resolution quiver of $\tilde{\Lambda}$, where the $i$th vertex of $Q^{k,j}$ corresponds to the simple $\tilde{\Lambda}$-module $M(m_k + (i-1)l_k + j, m_k + (i-1)l_k + j)$.

**Proof.** By Lemma 5.6, we know the resolution quiver of $\Lambda_k$. Hence, we only need to determine the effect on resolution quivers for gluing and self-gluing of acyclic Nakayama algebras. To this end, let $\Lambda'$ and $\Lambda''$ be two acyclic Nakayama algebras with resolution quivers $Q'$ and $Q''$, respectively. Then, by Proposition 4.13 the resolution quiver of $\Lambda' \land \Lambda''$ is given as follows:

1. First take the disjoint union $Q' \cup Q''$.

2. Then identify the unique vertex of $Q'$ corresponding to a simple injective module with the unique vertex of $Q''$ corresponding to a simple projective module.

3. Finally add an arrow from each non-projective sink vertex of $Q''$ to the vertex of $Q'$ corresponding to the $\tau$-translate of the simple injective $\Lambda'$-module.

Similarly, by Proposition 1.25 the resolution quiver of the self-gluing of $\Lambda'$ is obtained as follows:
(1) First identify the unique vertex of \( Q' \) corresponding to a simple injective module with the unique vertex of \( Q' \) corresponding to a simple projective module.

(2) Then add an arrow from each non-projective sink vertex of \( Q' \) to the vertex of \( Q' \) corresponding to the \( \tau \)-translate of the simple injective module.

We want to apply these constructions to obtain the resolution quiver of the self-gluing \( \tilde{\Lambda} \) of \( \Lambda_1 \triangleleft \cdots \triangleleft \Lambda_r \) from the resolution quivers \( Q^k \) of the algebras \( \Lambda_k \). Similar to the proof of Lemma 5.5 we let a vertex in \( Q^k \) be denoted by a pair \((i, j)\) where \( 0 \leq j \leq l_k - 1 \) and \( i \) is a vertex of \( Q^k \). With this notation we see that the unique vertex of \( Q^k \) corresponding to the simple projective module \( M(m_k, m_k) \) is \((1, 0)\), and the unique vertex corresponding to the simple injective module \( M(m_{k+1}, m_{k+1}) \) is \((\frac{1}{2} + 1, 0)\). Using these observations and the description of the resolution quiver of gluings and self-glueings given above, we get that the resolution quiver of \( \tilde{\Lambda} \) is obtained as follows:

(1) First take the disjoint union \( \bigcup_{k=1}^{r} Q^k \).

(2) Then identify the vertex \((1, 0)\) of \( Q^{k+1} \) with the vertex \((\frac{1}{2} + 1, 0)\) of \( Q^k \) for each \( 1 \leq k \leq r \) (where \( Q^{r+1} := Q^1 \)).

(3) Finally, add an arrow from the vertex \((1, j)\) of \( Q^{k+1} \) to the vertex \((\frac{1}{2}, l_k - 1)\) of \( Q^k \) for each \( 1 \leq k \leq r \) and \( 0 < j < l_k - 1 \).

This is precisely the quiver \( Q \) described in the lemma, which proves the claim. \( \square \)

**Example 5.7.** Let \( \Lambda = \Lambda_1 \triangleleft \Lambda_2 \triangleleft \Lambda_3 \) where \( \Lambda_1 = kA_{1, 7}/R^1, \Lambda_2 = kA_{7, 11}/R^2 \) and \( \Lambda_3 = kA_{11, 15}/R^2 \) (see also Example 4.22). Let \( \tilde{\Lambda} \) be the self-gluing of \( \Lambda \). Then the resolution quiver \( \tilde{Q} \) of \( \tilde{\Lambda} \) is the quiver

\[
S_{13} \xrightarrow{} S_{11} \xrightarrow{} S_9 \xrightarrow{} S_7 \xrightarrow{} S_4 \xrightarrow{} S_1 \\
S_5 \xrightarrow{} S_2 \xrightarrow{} S_{14} \xrightarrow{} S_{12} \xrightarrow{} S_{10} \xrightarrow{} S_8 \xrightarrow{} S_6 \xrightarrow{} S_3.
\]

We obtain the following result which describes the cyclic simple modules for a cyclic non-homogeneous Nakayama algebra with an \( n\mathbb{Z} \)-cluster tilting subcategory.

**Lemma 5.8.** Let \( \tilde{M}(i, \ell) \) be a simple \( \tilde{\Lambda} \)-module where \( m_k \leq i < m_{k+1} \) for some \( 1 \leq k \leq r \). Then \( \tilde{M}(i, \ell) \) is cyclic if and only if

\[
i \equiv m_k \text{ (mod } l_k) \quad \text{or} \quad i \equiv m_k - 1 \text{ (mod } l_k).\]

**Proof.** Assume \( i \equiv m_k \) or \( i \equiv m_k - 1 \) (mod \( l_k \)). Then \( \tilde{M}(i, \ell) \) corresponds to a vertex in \( \bigcup_{k} Q^{k,0} \cup \bigcup_{k} Q^{k,l_k-1} \), using the notation of Lemma 5.6. From the description of the resolution quiver \( Q \) in Lemma 5.6 we see that any vertex in \( \bigcup_{k} Q^{k,0} \cup \bigcup_{k} Q^{k,l_k-1} \) has a unique predecessor and a unique successor in \( \bigcup_{k} Q^{k,0} \cup \bigcup_{k} Q^{k,l_k-1} \). Since \( \bigcup_{k} Q^{k,0} \cup \bigcup_{k} Q^{k,l_k-1} \) has finitely many vertices, it follows that \( \bigcup_{k} Q^{k,0} \cup \bigcup_{k} Q^{k,l_k-1} \) is a union of cycles. This shows that \( \tilde{M}(i, \ell) \) is cyclic.

Conversely, assume \( i \not\equiv m_k \) and \( i \not\equiv m_k - 1 \) (mod \( l_k \)). Then \( \tilde{M}(i, \ell) \) corresponds to a vertex in \( Q^{k,j} \) where \( j \neq 0 \) and \( j \neq l_k - 1 \). From the description of the resolution quiver \( Q \) in Lemma 5.6 we see that any vertex in \( Q^{k,j} \) is either a source vertex or has a unique predecessor which is in \( Q^{k,j} \). Since \( Q^{k,j} \) is a linearly oriented \( A_{2r} \)-quiver, it contains no cycles. This shows that \( \tilde{M}(i, \ell) \) is not cyclic. \( \square \)

**Example 5.9.** Let \( \tilde{\Lambda} \) be as in Example 5.7. Notice that \( l_2 = l_3 = 2 \). By Lemma 5.8 it follows that for every \( m_2 \leq i \leq m_4 \) we have that \( \tilde{M}(i, \ell) \) is cyclic. Since \( l_1 = 3 \), again by Lemma 5.8 we conclude that the only non-cyclic simple modules are \( M(2, 2) \) and \( M(5, 5) \). The resolution quiver \( \tilde{Q} \) of \( \tilde{\Lambda} \) computed in Example 5.7 confirms our computation.

We can now determine the subcategories \( \mathcal{X}_c \) and \( \mathcal{F} = \text{add } \mathcal{X}_c \) of \( \tilde{\Lambda} \)-mod described in subsection 5.2.

**Theorem 5.10.** The subcategory \( \mathcal{X}_c \) consists of the following types of indecomposables \( \tilde{\Lambda} \)-modules, where we run through all pairs of integers \((k, i)\) where \( 1 \leq k \leq r \) and where \( m_k \leq i < m_{k+1} \) and \( i \equiv m_k \) (mod \( l_k \)):
Proof. Let \( \tilde{M}(i,j) \) be an indecomposable \( \bar{\Lambda} \)-module with \( m_k \leq i \leq j \leq m_{k+1} \). By definition, we have that \( \tilde{M}(i,j) \in \mathcal{X}_c \) if and only if top \( \tilde{M}(i,j) \) and \( \tau \text{soc } \tilde{M}(i,j) \) are cyclic simple \( \bar{\Lambda} \)-modules. Since

\[
\text{top } \tilde{M}(i,j) \equiv \tilde{M}(j,j) \quad \text{and} \quad \tau \text{soc } \tilde{M}(i,j) \equiv \tilde{M}(i-1,i-1)
\]

by Proposition \[2.6\] and Proposition \[2.10\] (a), this is equivalent to requiring \( \tilde{M}(j,j) \) and \( \tilde{M}(i-1,i-1) \) to be cyclic simple \( \bar{\Lambda} \)-modules. Now by Lemma \[5.8\] this is equivalent to

\[
i - 1 \equiv m_k \text{ or } i - 1 \equiv m_k - 1 \pmod{l_k}.
\]

Analyzing the different cases for \( i \) and \( j \), and using that \( 0 \leq j - i \leq l_k - 1 \), we get the list of indecomposable \( \bar{\Lambda} \)-modules given in the theorem.

Finally, we prove that \( \tilde{M}(i,j) \) is projective in \( \mathcal{F} \) if and only if is of type (3) or (4). Indeed, since \( r_i = i + l_k - 1 \) when \( m_k \leq i \) and \( i + l_k - 1 \leq m_{k+1} \), it follows that any module in the set described by (3) or (4) is projective in \( \Lambda \text{-mod} \) by Proposition \[2.10\] (b). Hence any such module must also be projective in \( \mathcal{F} \). On the other hand, for any module \( \tilde{M}(i+1, i+l_k-1) \) of type (2) we have a non-split epimorphism

\[
\tilde{M}(i,i+l_k-1) \rightarrow \tilde{M}(i+1, i+l_k-1)
\]

where \( \tilde{M}(i,i+l_k-1) \) is in the set described by (3). This shows that modules of type (2) are not projective in \( \mathcal{F} \). Similarly, for any module \( \tilde{M}(i,i) \) of type (1) where \( m_k < i \leq m_{k+1} \) we have a non-split epimorphism

\[
\tilde{M}(i-l_k+1,i) \rightarrow \tilde{M}(i,i)
\]

where \( \tilde{M}(i-l_k+1,i) \) is in the set described by (4). This shows that modules of type (1) are not projective in \( \mathcal{F} \).

We can now describe the singularity category of \( \bar{\Lambda} \), using the subcategory \( \mathcal{F} \) of \( \Lambda \text{-mod} \) which we computed in Theorem \[5.10\]

**Corollary 5.11.** There exists an equivalence \( \mathcal{F} \cong \Gamma \text{-mod} \) where \( \Gamma = k\bar{\Lambda}_m/R^2 \) is a homogeneous Nakayama algebra with \( m = rn \). In particular, we have an equivalence

\[D_{\text{sing}}(\bar{\Lambda}) \cong \Gamma \text{-mod}\]

of triangulated categories.

**Proof.** Since \( \mathcal{F} \) is an abelian category with enough projectives, we have that \( \mathcal{F} \cong \mathcal{P}^{\text{op}} \text{-mod} \) where \( \mathcal{P} \) is the subcategory of projective objects in \( \mathcal{F} \) and \( \mathcal{P}^{\text{op}} \text{-mod} \) is the category of finitely presented \( k \)-linear functors from \( \mathcal{P}^{\text{op}} \) to \( \text{-mod} \), see [Kra99, Proposition 2.3]. By Theorem \[5.10\] we know that \( \mathcal{P} \) is the additive closure of modules of type (3) and (4). We enumerate the modules in (3) and (4) by letting

\[
t_s = 2 \sum_{k=1}^{s} \frac{m_{k+1} - m_k}{l_k},
\]

for each \( 1 \leq s \leq r \) and setting

\[
P_t = \tilde{M}(m_s + il_s, m_s + (i + 1)l_s - 1) \quad \text{and} \quad P_{t+1} = \tilde{M}(m_s + il_s + 1, m_s + (i + 1)l_s)
\]

where \( 0 \leq i < \frac{m_{s+1} - m_s}{l_s} \). With this notation we have the indecomposable projective modules

\[P_0, P_1, \ldots, P_{m-1}\]

in \( \mathcal{F} \), since

\[
2 \sum_{k=1}^{r} \frac{m_{k+1} - m_k}{l_k} = 2 \sum_{k=1}^{r} \frac{n}{2} = rn = m
\]
Now for $0 \leq i \leq m - 1$ we have that
\[
\text{Hom}_{\tilde{\Lambda}}(P_i, P_j) = \begin{cases} 
  k & \text{if } j = i \text{ or } j = i + 1, \\
  0 & \text{otherwise}
\end{cases}
\]
where $P_m := P_0$. Hence $\mathcal{P}$ is equivalent to the category $\text{add}\Gamma$ of projective $\Gamma$-modules. Therefore we have that
\[
\mathcal{P}^{\text{op}}-\text{mod} \cong (\text{add} \Gamma)^{\text{op}}-\text{mod} \cong \Gamma^{\text{op}}-\text{mod} \cong \Gamma-\text{mod}
\]
where the last equivalence follows from the fact that $\Gamma \cong \Gamma^{\text{op}}$. This proves the claim. \qed

**Remark 5.12.** We note that there can be more $n\mathbb{Z}$-cluster tilting subcategories in the singularity category than in the module category. Indeed, by Theorem 4.29 we know that $\tilde{\Lambda}$ has a unique $n\mathbb{Z}$-cluster tilting subcategory, and hence the singularity category of $\tilde{\Lambda}$ has $n$ different $n\mathbb{Z}$-cluster tilting subcategories by Remark 4.10. These are precisely the subcategories of $\Gamma-\text{mod}$ consisting of a simple module $S$ and its $n$-syzygies $\Omega^k(S)$ for $0 \leq k \leq r - 1$. Under the isomorphism $\mathcal{F} \cong D_{\text{sing}}(\tilde{\Lambda})$, the $n\mathbb{Z}$-cluster tilting subcategory of $D_{\text{sing}}(\tilde{\Lambda})$ given in Theorem 5.2 corresponds to the subcategory of $\mathcal{F}$ consisting of the simple modules $\tilde{M}(m_k, m_k)$ for all $1 \leq k \leq r$.

On the other hand, the different $n\mathbb{Z}$-cluster tilting subcategories in $D_{\text{sing}}(\tilde{\Lambda})$ are all related by applying some automorphism of $D_{\text{sing}}(\tilde{\Lambda})$. Based on this observation we pose the following question, to which we have no counter example.

**Question 5.13.** Given a finite dimensional algebra $A$ with $\mathcal{C} \subseteq D_{\text{sing}}(A)$ an $n\mathbb{Z}$-cluster tilting subcategory. Does there exist a finite dimensional algebra $B$ together with $\mathcal{D} \subseteq B-\text{mod}$ an $n\mathbb{Z}$-cluster tilting subcategory and an equivalence $D_{\text{sing}}(B) \to D_{\text{sing}}(A)$ such that the $n\mathbb{Z}$-cluster tilting subcategory of $D_{\text{sing}}(B)$ corresponding to $\mathcal{D}$ is sent to $\mathcal{C}$?

**Example 5.14.** Let $\tilde{\Lambda}$ be as in Example 5.7. In Example 4.11 we have seen that $\tilde{\Lambda}$ admits a 4-cluster tilting subcategory $\mathcal{C}_{\Lambda}$. Using Theorem 5.10 we can compute $\mathcal{X}_{\tilde{\Lambda}}$. Indeed, the Auslander–Reiten quiver of $\tilde{\Lambda}$ was computed in Example 4.11 to be

$$
\begin{array}{cccccccccccc}
\text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \text{16} & \text{17} & \text{18} & \text{19} & \text{20} & \text{21} & \text{22} \\
\text{11} & (21) & (22) & (31) & (32) & (41) & (42) & (51) & (52) & (61) & (62) & (71) \\
\text{11} & (51) & (52) & (61) & (62) & (71) & (72) & (81) & (82) & (91) & (92) & (101) \\
\text{11} & (51) & (52) & (61) & (62) & (71) & (72) & (81) & (82) & (91) & (92) & (101) \\
\end{array}
$$

where $\mathcal{X}_{\tilde{\Lambda}}$ consists of the encircled modules and $\mathcal{C}_{\tilde{\Lambda}}$ is the additive closure of all modules inside a rectangle. By Corollary 5.11 we have that $\mathcal{F} \cong \Gamma-\text{mod}$ where $\Gamma = k\tilde{\Lambda}_{12}/R^{\mathbb{Z}}$. The 4-cluster tilting subcategory $\mathcal{C}_{\tilde{\Lambda}}$ of $\tilde{\Lambda}-\text{mod}$ gives rise to the 4-cluster tilting subcategory $\mathcal{C}_{\mathcal{F}} = \mathcal{C}_{\tilde{\Lambda}} \cap \mathcal{F}$ of $\mathcal{F} \cong \Gamma-\text{mod}$. However, there are 3 different 4-cluster tilting subcategories inside $\Gamma-\text{mod}$ which give rise to 3 different 4-cluster tilting subcategories inside $D_{\text{sing}}(\tilde{\Lambda}) \cong \Gamma-\text{mod}$.

Finally we describe the functor $\tilde{\Lambda}-\text{mod} \to D_{\text{sing}}(\tilde{\Lambda})$, using the subcategory $\mathcal{F}$ of $\tilde{\Lambda}-\text{mod}$.

**Proposition 5.15.** Let $\tilde{M}(i, j)$ be an indecomposable $\tilde{\Lambda}$-module where $m_k \leq i \leq j \leq m_{k+1}$. The following hold:

1. $\tilde{M}(i, j)$ is nonzero in $D_{\text{sing}}(\tilde{\Lambda})$ if and only if $j - i < l_k - 1$ and either 
   \[i \equiv m_k + 1 \pmod{l_k} \quad \text{or} \quad j \equiv m_k \pmod{l_k}\]
   hold.

2. If $j - i < l_k - 1$ and $i \equiv m_k + 1 \pmod{l_k}$, then the inclusion $\tilde{M}(i, j) \to \tilde{M}(i, i + l_k - 2)$ becomes an isomorphism in $D_{\text{sing}}(\tilde{\Lambda})$.

3. If $j - i < l_k - 1$ and $j \equiv m_k \pmod{l_k}$, then the projection $\tilde{M}(i, j) \to \tilde{M}(j, j)$ becomes an isomorphism in $D_{\text{sing}}(\tilde{\Lambda})$. 
Remark 5.16. Note that the codomains of the maps in part (2) and (3) of Proposition 5.15 are non-projective objects in \( D \) by Theorem 5.10. Also, any morphism \( \tilde{M}(i_1, j_1) \to \tilde{M}(i_2, j_2) \) which is not of the form given in Proposition 5.15 (2) or Proposition 5.15 (3) must be zero in \( D_{sing}(\tilde{\Lambda}) \), since it factors through a module \( \tilde{M}(i, j) \) which is not of the form given in Proposition 5.15 (1). This implies that Proposition 5.15 contains all the information about the behavior of the functor \( \tilde{\Lambda} - \text{mod} \to D_{sing}(\tilde{\Lambda}) \cong D \) on the objects and morphisms of \( \tilde{\Lambda} - \text{mod} \).

Proof of Proposition 5.15. Let \( D_1 \) be the set of indecomposable \( \tilde{\Lambda} \)-modules \( \tilde{M}(i, j) \) where

\[
m_k \leq i \leq j \leq m_{k+1} \quad \text{and} \quad j - i < l_k
\]

and where

\[
i \equiv m_k + 1 \pmod{l_k} \quad \text{or} \quad j \equiv m_k \pmod{l_k}
\]

for some \( 1 \leq k \leq r \). Let \( D_2 \) be the set of indecomposable \( \tilde{\Lambda} \)-modules \( \tilde{M}(i, j) \) which are not contained in \( D_1 \). We want to show that the modules in \( D_1 \) (resp \( D_2 \)) become non-zero (resp zero) in the singularity category. We first show that up to isomorphism \( D_1 \) and \( D_2 \) are closed under syzygies. To this end, let \( \tilde{M}(i, j) \) be a \( \tilde{\Lambda} \)-module where \( m_k \leq i \leq j \leq m_{k+1} \) and where \( (i, j) \neq (m_k, m_k) \). If \( j \leq m_k + l_k - 1 \), then by Proposition 2.10 (b) we have that

\[
\Omega \tilde{M}(i, j) \cong \tilde{M}(m_k, i - 1)
\]  \hspace{1cm} (5.1)

Note that \( j \equiv m_k \pmod{l_k} \) cannot hold in this case, and \( i \equiv m_k + 1 \pmod{l_k} \) implies that \( i = m_k + 1 \). Hence \( \tilde{M}(i, j) \) is contained in \( D_1 \) if and only if \( i = m_k + 1 \), which is equivalent to \( \tilde{M}((m_k, i - 1) \) being contained in \( D_1 \). This shows that \( D_1 \) and \( D_2 \) are closed under syzygies when \( j \leq m_k + l_k - 1 \). Now assume \( j > m_k + l_k - 1 \). Then by Proposition 2.10 (b) we have an isomorphism

\[
\Omega \tilde{M}(i, j) \cong \tilde{M}(j - l_k + 1, i - 1)
\]  \hspace{1cm} (5.2)

Since \( i \equiv m_k + 1 \pmod{l_k} \) if and only if \( i - 1 \equiv m_k \pmod{l_k} \), and \( j \equiv m_k \pmod{l_k} \) if and only if \( j - l_k + 1 \equiv m_k + 1 \pmod{l_k} \), we see that \( \tilde{M}(i, j) \) is in \( D_1 \) if and only if \( \tilde{M}(j - l_k + 1, i - 1) \) is in \( D_1 \). This shows that \( D_1 \) and \( D_2 \) are closed under syzygies.

Now let \( \tilde{M}(i, j) \) be a \( \tilde{\Lambda} \)-module where \( m_k \leq i \leq j \leq m_{k+1} \). We claim that there exists a syzygy of \( \tilde{M}(i, j) \) which is isomorphic to \( \tilde{M}(m_k, t) \) for some \( t \geq m_k \). Indeed, if \( i = m_k \), then we are done, so assume \( i > m_k \). Repeatedly using formula (5.2), we see that there exists an integer \( s \geq 0 \) for which \( \Omega^s \tilde{M}(i, j) \cong \tilde{M}(i', j') \) where \( m_k \leq i' \leq j' \leq m_k + l_k - 1 \). Hence, by (5.1) it follows that

\[
\Omega^{s+1} \tilde{M}(i, j) \cong \tilde{M}(m_k, i' - 1).
\]

Setting \( t = i' - 1 \), this proves the claim.

Now assume \( \tilde{M}(i, j) \) is in \( D_2 \), and let \( t \) be an integer such that \( \tilde{M}(m_k, t) \) is isomorphic to a syzygy of \( \tilde{M}(i, j) \). Since \( D_2 \) is closed under syzygies, \( \tilde{M}(m_k, t) \) must be in \( D_2 \). Therefore \( t > m_k \), which implies that \( \tilde{M}(m_k, t) \) is a projective \( \tilde{\Lambda} \)-module by Proposition 4.15 (b) and Proposition 4.25 (b). Hence, \( \tilde{M}(m_k, t) \) is zero in \( D_{sing}(\tilde{\Lambda}) \), so \( \tilde{M}(i, j) \) must also be zero in \( D_{sing}(\tilde{\Lambda}) \). This shows the "only if" direction of part (1) of the proposition.

To prove the remaining part of the proposition, we need to consider the set \( D_3 \) consisting of the identity morphisms between objects in \( D_1 \), the morphisms in part (2) of the proposition, and the morphisms and in part (3) of the proposition. We claim that up to isomorphism \( D_3 \) is closed under syzygies. Clearly this is true for the identity morphisms, since \( D_1 \) is closed under syzygies. Now let \( \tilde{M}(i, j) \to \tilde{M}(i, i + l_k - 2) \) be a morphism as in (2). We consider two cases separately:

- If \( j \leq m_k + l_k - 1 \), then taking the syzygy and using (5.1), we get the identity map

\[
\tilde{M}(m_k, i - 1) \to \tilde{M}(m_k, i - 1)
\]

which is in \( D_3 \);

- If \( j > m_k + l_k - 1 \), then taking the syzygy and using (5.2), we get the projection map

\[
\tilde{M}(j - l_k + 1, i - 1) \to \tilde{M}(i, i - 1).
\]

This is of type (3) since \( i - 1 \equiv m_k \pmod{l_k} \), and hence it must be in \( D_3 \).
Finally, let $\tilde{M}(i,j) \to \tilde{M}(i,j)$ be a morphism as in (3). We can assume that $(i,j) \neq (m_k,m_k)$ since otherwise we just have the identity map, in which case we already know the claim is true. Therefore $j > m_k$, and since $j \equiv m_k \pmod{l_k}$, we must have that $j > m_k + l_k - 1$. Hence, taking the syzygy and using the formula (5.2), we get the inclusion map

$$\tilde{M}(j - l_k + 1, i - 1) \to \tilde{M}(j - l_k + 1, j - 1).$$

This is of type (2) since $j - l_k + 1 \equiv m_k + 1 \pmod{l_k}$, and hence it must be in $\mathcal{D}_3$. This shows that $\mathcal{D}_3$ is closed under syzygies.

Now let $\tilde{M}(i_1, j_1) \to \tilde{M}(i_2, j_2)$ be a map in $\mathcal{D}_3$ with $m_k \leq i_1 \leq j_1 \leq m_{k+1}$ and $m_k \leq i_2 \leq j_2 \leq m_{k+1}$. As shown above, we can find an integer $s \geq 0$ such that $\Omega^s \tilde{M}(i_1, j_1) \cong \tilde{M}(m_k, t)$. Now since $\mathcal{D}_1$ is closed under syzygies, $\tilde{M}(m_k, t)$ must be in $\mathcal{D}_1$, so $t = m_k$. Furthermore, since $\mathcal{D}_3$ is closed under syzygies, the map

$$\Omega^s \tilde{M}(i_1, j_1) \to \Omega^s \tilde{M}(i_2, j_2)$$

must be isomorphic to the identity map

$$\tilde{M}(m_k, m_k) \cong \tilde{M}(m_k, m_k)$$

since this is the only map in $\mathcal{D}_3$ with domain isomorphic to $\tilde{M}(m_k, m_k)$. Using that the syzygy functor extends to an autoequivalence on the singularity category, we get that the map $\tilde{M}(i_1, j_1) \to \tilde{M}(i_2, j_2)$ must also be an isomorphism in $\mathcal{D}_{\text{sing}}(\tilde{A})$. This proves part (2) and (3) of the proposition. Finally, the "if" direction of part (1) follows from the fact that the codomain of the maps in (2) and (3) are non-projective objects in $\mathcal{F}$ by Theorem 5.10. □

**Remark 5.17.** Let $C_1$ denote the $n\mathbb{Z}$-cluster tilting subcategory of $\tilde{A}$. By Theorem 4.20 (a) and Theorem 4.29 we know that the category $C_1$ consists of the projective $\tilde{A}$-modules, the injective $\tilde{A}$-modules, and the $\tilde{A}$-modules of the form $\tilde{M}(m_k, m_k)$ where $1 \leq k \leq r$. Note that the modules $\tilde{M}(m_k, m_k)$ are in $\mathcal{F}$ by Theorem 5.10. Also, any injective and non-projective indecomposable module is of the form $\tilde{M}(i, m_{k+1})$ for $m_{k+1} - l_k + 1 < i < m_{k+1}$. By Proposition 5.15 (c) such a module becomes isomorphic in $\mathcal{D}_{\text{sing}}(\tilde{A})$ to its top via the map

$$\tilde{M}(i, m_{k+1}) \to \tilde{M}(m_{k+1}, m_{k+1}).$$

Since the projective modules vanish in $\mathcal{D}_{\text{sing}}(\tilde{A})$, we get a complete description of the functor

$$C_1 \to \mathcal{D}_{\text{sing}}(\tilde{A}) \cong \mathcal{E}.$$

The following corollary shows that $\tilde{A}$ is not Iwanaga–Gorenstein. As a consequence, $\mathcal{D}_{\text{sing}}(\tilde{A})$ cannot be computed by considering the stable category of Gorenstein projective modules as in Buchweitz theorem, see [BOJ15] Theorem 3.6.

**Corollary 5.18.** A non-homogeneous cyclic Nakayama algebra with an $n\mathbb{Z}$-cluster tilting subcategory is not Iwanaga–Gorenstein.

**Proof.** We show this for $\tilde{A}$. It suffices to prove that there exists an injective $\tilde{A}$-module which is non-zero in $\mathcal{D}_{\text{sing}}(\tilde{A})$. Since $\tilde{A}$ is non-homogeneous, there exists a $k$ where $l_k \geq 3$. Hence, $\tilde{M}(m_{k+1} - l_k, m_{k+1})$ is a $\tilde{A}$-module which is injective and not projective. By Proposition 5.15 (3) this module is isomorphic to $\tilde{M}(m_{k+1}, m_{k+1})$ in $\mathcal{D}_{\text{sing}}(\tilde{A})$. Since $\tilde{M}(m_{k+1}, m_{k+1})$ is non-zero in $\mathcal{D}_{\text{sing}}(\tilde{A})$ by Theorem 5.10 this proves the claim. □

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