STABILITY OF NON-MONOTONE AND BACKWARD WAVES FOR DELAY NON-LOCAL REACTION-DIFFUSION EQUATIONS

Abraham Solar
Instituto de Física, Facultad de Física
P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
and
Departamento de Matemática y Física Aplicadas
Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile
(Communicated by Hirokazu Ninomiya)

Abstract. This paper deals with the stability of semi-wavefronts to the following delay non-local monostable equation:
\[ \dot{v}(t, x) = \Delta v(t, x) - v(t, x) + \int_{\mathbb{R}^d} K(y)g(v(t-h, x-y))dy, x \in \mathbb{R}^d, t > 0; \]
where \( h > 0 \) and \( d \in \mathbb{Z}^+ \). We give two general results for \( d \geq 1 \): on the global stability of semi-wavefronts in \( L^p \)-spaces with unbounded weights and the local stability of planar wavefronts in \( L^p \)-spaces with bounded weights. We also give a global stability result for \( d = 1 \) which yields to the global stability in Sobolev spaces with bounded weights. Here \( g \) is not assumed to be monotone and the kernel \( K \) is not assumed to be symmetric, therefore non-monotone semi-wavefronts and backward semi-wavefronts appear for which we show their stability. In particular, the global stability of critical wavefronts is stated.

1. Introduction. We study the following non-local equation with delay
\[ \dot{v}(t, x) = \Delta v(t, x) - v(t, x) + \int_{\mathbb{R}^d} K(y)g(v(t-h, x-y))dy \quad x \in \mathbb{R}^d, t > 0, \] (1)
for \( h > 0, 0 \leq K \in L^1(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} K(y)dy = 1 \).

Equation (1) is an important model in population dynamics [29, 33, 38, 34, 23, 44, 11, 42, 3, 35], where the parameter \( h \) is the sexual mature period of some species with birth rate \( g \) which has only two fixed points: 0 and \( \kappa > 0 \). The non-local interaction between individuals is determined by the kernel \( K \) while the quantity \( v(t, x) \) stands for the mature population at some time \( t \) and location \( x \). In this context, a kind of colonization waves with constant propagation speed appears which are called planar semi-wavefronts, i.e., solutions \( v(t, x) = \phi_\nu(\nu \cdot x + ct) \) with speed \( c \in \mathbb{R} \), \( \nu \in S^{d-1} \) and the profile \( \phi_\nu : \mathbb{R} \to \mathbb{R}_+ \) satisfying \( \phi_\nu(-\infty) = 0 \) (or \( \phi_\nu(+\infty) = 0 \)) and \( \lim_{z \to +\infty} \phi_\nu(z) > 0 \) (or \( \lim_{z \to -\infty} \phi_\nu(z) > 0 \)); if \( \phi_\nu(+\infty) = \kappa \) (or \( \phi_\nu(-\infty) = \kappa \)) then the semi-wavefronts are called planar wavefronts. Due to a possible asymmetry of \( K \), the class of profiles satisfying \( \psi_\nu(+\infty) = 0 \) are not necessarily obtained by the change \( \psi_\nu(z) := \phi_{-\nu}(-z) = 0 \), for some semi-wavefront with speed \( -c \) satisfying \( \phi_{-\nu}(-\infty) = 0 \), therefore we must expect two critical speeds for the existence of semi-wavefronts which could be non-opposite [39, 19, 11, 42]. Consequently, backwards.
semi-wavefronts can appear, i.e., semi-wavefronts $v(t, x) = \phi_c(\nu \cdot x + ct)$ such that $\lim_{t \to \infty} v(t, x) = 0$ for all $x \in \mathbb{R}^d$.

In the non-delayed local case, i.e., when we take $K(y) = \delta(y)$ and $h = 0$ in (1), semi-wavefronts are monotone wavefronts and the study of existence, uniqueness, asymptotic spreading speeds and stability is widely documented [7, 12, 17, 3, 26, 32, 36]. Broadly speaking, it has been shown that the asymptotic propagation speed of solutions only depends on the asymptotic behavior of initial datum at the trivial equilibrium, i.e., two initial data could coincide on some domain $(N, +\infty]$, for arbitrary $N \in \mathbb{R}$, but if their asymptotic behavior at $-\infty$ are different then they will be propagated with different speeds. Particularly, Kolmogorov, Petrovskii and Piskunov [18] showed that if the initial datum is the Heaviside step function the solution is propagated with the critical speed $c^*$. Due to this result, the critical wavefronts have been one of the main edges in the research on this subject. The model in [18] satisfies the subtangential property $g(u) \leq g'(0)u$, for all $u \geq 0$, which implies that the critical wavefronts are propagated with the linear speed $c^* = 2\sqrt{g'(0)} - 1$, those critical wavefronts we will consider in this paper.

Many delay models were presented after [18] and the research was addressed to similar problems [9, 20, 22, 23]. One of the most cited models is the Nicholson’s blowflies model (see [20, 6, 31] and references therein) that, in its non-local diffusive version, is

$$\dot{v}(t, x) = \Delta v(t, x) - \delta v(t, x) + p \int_{\mathbb{R}^d} K(x - y)v(t - h, y)e^{-v(t, y)}dy, \quad (2)$$

for some positive parameters $\delta$ and $p$, which is reduced to (3) by an appropriate rescaling of variables.

Since the nonlinearity in (2) satisfies $|g|_{Lip} = g'(0)$ (here we use $|g|_{Lip}$ to denote the Lipschitz constant of $g$), the uniqueness (up to translation) of semi-wavefronts to (2) is a consequence from [1, Theorem 7]. Alternatively, we give a result on the uniqueness of semi-wavefronts to (1)(see Corollary 2.4). Otherwise, the existence of wavefronts (monotone and non-monotone) to (1) has been studied, e.g., in [39, 19, 33, 34, 42] and results for the existence of semi-wavefronts has been given, e.g., in [34, Theorem 4] and [11, Theorem 18] (by providing the existence of a critical speed when $|g|_{Lip} = g'(0)$) where the kernel $K$ is not assumed to be symmetric. A more complete discussion on the existence of wavefronts and semi-wavefronts is given in Subsection 2.2.

In general, the study of delayed case mainly presents two difficulties. The first one is concerned with the asymptotic behavior of semi-wavefronts at the positive equilibrium $\kappa$ since the associated characteristic equations have infinity solutions and semi-wavefronts could oscillate around $\kappa$. Indeed, non-monotone wavefronts to (1) have been observed [34, 40, 11, 42] as well as backwards wavefronts [11, Theorem 18] and [42, Theorem 4.4] when $K$ is non-symmetric. Otherwise, the second inconvenient is that the associated semi-flow to (1) is not monotone in general. This lack complicates the construction of sub and super-solutions, an approach widely used when $h = 0$ or $g$ is monotone to prove the existence and stability of wavefronts [19, 38, 23, 15]. Also, the spectral technique has been used in order to obtain the local stability of wavefronts [9, 17, 26, 27], however, the maximum principle arguments to reaction-diffusion equations frequently imply the global stability of wavefronts [36, 23, 31, 15]. Our approach is a combination of maximum principle arguments and Fourier analysis for linear delay PDE’s.
For local equations, with \( d = 1 \) and unimodal \( g \) (i.e. when \( g \) has a unique point of local extreme) satisfying \( |g'(u)| \leq g'(0), u \in \mathbb{R} \), the local exponential stability of wavefronts, in suitable Sobolev spaces, was given by Lin et al [20] (for non-critical wavefronts) and Chern et al [6] (for critical wavefronts) under the condition \( |g'(\kappa)| < 1 \) for any delay or \( g'(\kappa) < -1 \) for small delay. These results also were recently extended for global perturbations of non-monotone wavefronts in certain Sobolev sub-spaces for a local equation by Mei, Zhang K. and Zhang Q. [24] and for a non-local equation [41] by Xu et al, these results give an exponential decay for non-critical wavefronts and an algebraic decay for critical-wavefronts. The authors in [41] consider equation (1) with subtagentional \( g \), an even kernel \( K \) but non-local diffusion. At the same time, in [30] the algebraic stability of semi-wavefronts with speed \( c \geq c(|g|_{Lip}) \), some speed \( c(|g|_{Lip}) \), on any domain of the form \((-\infty, N], N \in \mathbb{R} \), was proved in [30, Theorem 3] without assumptions on neither subtagentionality of \( g \) nor the size of derivative on equilibrium \( \kappa \). In particular, when \( |g|_{Lip} = g'(0) \) semi-wavefronts (including the critical and asymptotically periodic semi-wavefronts) are stable on any domain \((-\infty, N] \). This limitation on the stability domain is by the use of an unbounded weight so that the control of the stability of semi-wavefronts on its whole domain yields to the stability with a bounded weight. With respect to the local stability, in [30, Corollary 17] was showed that the size of local perturbations depends on the size of neighborhoods of \( \kappa \) where \( g \) is a contractive application and one of these neighborhoods of \( \kappa \) is attractor and therefore the global exponential stability of non-critical semi-wavefronts was also established in [30, Corollary 11], which includes non-monotone wavefronts for typical models such as local Nicholson’s blowflies model (when \( p/\delta \in [1, e^2] \)) and Mackey-Glass’ model [23, 20, 6, 31]. However, the stability of critical wavefronts was not addressed in [30] so that we also study the global stability of critical wavefronts in this paper.

With respect to the non-local equations (1), when \( d = 1 \) the stability of wavefronts has also been studied for bistable nonlinearity without delay (see, e.g., [5]) and with delay (see,e.g., [37] and [22]). In the monostable case with delay, the global stability of the monotone wavefronts with monotone \( g \) was satisfactorily answered by Mei, Ou and Zhao in [23] when \( K \) is a heat kernel. Similar global results, for more general equation, were established by Lv and Wang [21] by using unbounded and bounded weights. Also, the global stability with unbounded weights has recently been establish for the delayed Fisher-KPP (which includes non-monotone semi-wavefronts) by Benguria and Solar [4]. Next, a close model to (1) is a paper of Wang, Li and Ruan [38] where the authors proved the global stability of non-critical (under minimal conditions on the initial data) when \( g \) is monotone and \( K \) is a symmetric kernel. Also, in a recent paper of Huang et al [16] the authors consider (1) with \( n = 1 \) and \( K \) symmetric but with non-local dispersion and \( g \) possibly non-monotone, for such model they showed the stability of small perturbations of non-critical wavefronts in weighted Sobolev spaces.

Otherwise, for \( d \geq 1 \), we should mention a very interesting work for dispersal equations presented by Huang, Mei and Wang [15] where the global stability of monotone planar wavefronts (critical and non-critical) was stated and the study of the convergence rate was also dealt; here \( K \) is a multidimensional heat kernel. So that, as much as we know the study of stability of wavefronts for the non-local equation (1) either assumes the monotonicity of \( g \) or the symmetry of \( K \). Thus, our aim is to prove the global stability of critical and non-critical wavefronts for (1) which could be backward wavefronts or oscillatory wavefronts. In particular, our
global stability result for non-symmetric kernel implies a change of behavior in the problem of speeds selection for the equation (1), i.e., to determine the asymptotic speed propagation of solutions generated by an initial data by only knowing the asymptotic behavior of the initial data at the trivial equilibrium (see Remark 2.9).

The convergence rate of solutions to critical semi-wavefronts in our results is comparable to Gallay work [9] for local equations without delay. More precisely, in the study of the local stability in [9] the disturbances space turns out the subspace of our disturbance space by adding a quadratic factor. Although, in our space the convergence is $O(t^{-1/2})$ while in the subspace considered in [9] is $O(t^{-3/2})$. Also, the convergence rate in our global stability result extends the pioneering result from [23] on Sobolev spaces for (1) (see Corollary 2.10 below). This paper is matched with a recent work of Benguria and Solar [3] where it is showed that the algebraic convergence rate for critical wavefronts obtained in this paper is optimal in the underline weighted space and it also has a close relation with the convergence rate obtained in [15].

We organize this paper in the following way. In the Section 2 we present and discuss the main results, in the Section 3 we state an existence and regularity result for the Cauchy problem, in the Section 4 we prove the stability results for $d \geq 1$ (stability on semi-intervals and local stability). Finally, in Section 5 we prove the global stability result for $d = 1$.

2. Main results and discussion.

2.1. **Global stability with exponential weight on** $\mathbb{R}^d$. Now, in order to study the stability of semi-wavefronts with speed $c$ in the direction $\nu \in S^{d-1}$ we make the change of variables $z := x + ct\nu$ and $u(t, z) := v(t, z - ct\nu)$, so that we have the following equation for $u$

\[ \dot{u}(t, z) = \Delta u(t, z) - cv \cdot \nabla u(t, z) - u(t, z) + K \ast [g \circ u](t - h, z - ch\nu), \]  

(3)

for which the planar semi-wavefronts $v(t, x) = \phi_c(\nu \cdot x + ct)$ with speed $c$, $\phi_c : \mathbb{R} \rightarrow \mathbb{R}_+$, are stationary solutions $u(t, z) = \phi_c(\nu \cdot z)$ to the following equation

\[ \phi''_c(\nu \cdot z) - c\phi'_c(\nu \cdot z) - \phi_c(\nu \cdot z) + \int_{\mathbb{R}^d} K(z - ch\nu - y) g(\phi_c(\nu \cdot y)) dy = 0. \]  

(4)

In our first stability result we do not assume neither smoothness nor subtangentiality on $g$. Our general assumption on $g$ is the following

(L) The function $g : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuos with constant $|g|_{Lip}$.

By denoting $\xi_\lambda(z) := e^{-\lambda \cdot z}$, for some $\lambda \in \mathbb{R}^d$, we have the following linear equation associated with (3)

\[ \dot{r}(t, z) = \Delta r(t, z) + (2\lambda - c\nu) \cdot \nabla r(t, z) + p_\lambda r(t, z) + |g|_{Lip} e^{-\lambda \cdot ch\nu} \xi_\lambda K \ast r(t - h, z - ch\nu), \]

(5)

where $p_\lambda = p_\lambda(c) = |\lambda|^2 - c\nu \cdot \lambda - 1$. Also, we denote by

\[ q_\lambda = q_\lambda(c) = |g|_{Lip} e^{-\lambda \cdot ch\nu} \int_{\mathbb{R}^d} K(y) e^{-\lambda \cdot y} dy \]

The behavior of solutions to (5), in the state space $L^1(\mathbb{R})$ with certain exponential weight has been studied, see [3]. Otherwise, for $r \in \mathbb{Z}_+ \cup \{0\}$ and $1 \leq p \leq \infty$ we denote the weighted Lebesgue spaces $L^{r,p}_\lambda := \{ u : \mathbb{R}^d \rightarrow \mathbb{R}, \text{ such that } |u|_{L^{r,p}_\lambda} := || \xi_\lambda u ||_{L^{r,p}(\mathbb{R}^d)} < \infty \}$
and
\[ L^p_{h,\lambda} := \{ u \in C([-h, 0], L^p_\lambda) \}, \text{ with norm } |u|_{L^p_{h,\lambda}} = \max_{s \in [-h, 0]} |u(s)|_{L^p_\lambda}. \]

Analogously, we define weighted Hölder spaces
\[ C^{\alpha}_{h,\lambda} := \{ u : \mathbb{R}^d \to \mathbb{R}, \text{ such that } |u|_{C^{\alpha}_{h,\lambda}} := |||\xi u|||_{C^{\alpha}_{h,\lambda}(\mathbb{R}^d)} < \infty \}, \]
and
\[ C^{r,\alpha}_{h,\lambda} := \{ u \in C([-h, 0], C^{r,\alpha}_{\lambda}), \text{ with norm } |u|_{C^{r,\alpha}_{h,\lambda}} = \max_{s \in [-h, 0]} |u(s)|_{C^{r,\alpha}_{\lambda}} \}. \]

Also, to simplify we put
\[ L^p_\lambda = L^0_{0,\lambda}, \quad C^\alpha_\lambda = C^0_{0,\lambda}, \quad L^p_{h,\lambda} = L^0_{h,\lambda} \quad \text{and} \quad C^\alpha_{h,\lambda} = C^0_{h,\lambda}. \]

Finally, for some function \( u : [a-h, b] \to X \), some Banach space \( X \) and \( a, b \in \mathbb{R} \), we define \( u_t : [-h, 0] \to X \) for each \( t \in [a, b] \) as \( u_t(s) = u(t+s) \).

Now, we state our first result on the stability of solutions to (3).

**Theorem 2.1.** Assume (L) and fix \( c \in \mathbb{R} \) and \( (\lambda, X, \lambda_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \) such that \( K \in L_1^\lambda \cap L_\infty^\lambda \cap L_h^\lambda \). Consider \( u_0 \in L^\infty_{h,\lambda} \cap L^\infty_{h,\lambda} \cap C^{\alpha}_{h,\lambda} \), some \( 1 \leq p \leq +\infty \) and \( \alpha \in (0, 1) \), then there exist a unique solution \( u(t, z) \) of (3) such that \( u(s, z) = u_0(s, z) \) for all \( (s, z) \in [-h, 0] \times \mathbb{R}^d \) and this solution also satisfies \( u_t \in L^\infty_p \cap L^\infty_{h,\lambda} \cap C^\alpha_{h,\lambda} \) for all \( t \geq 0 \). Moreover, for any solutions \( u(t, z) \) and \( \psi(t, z) \) of (3) such that \( u_0, \psi_0 \in L^\infty_{h,\lambda_0} \) and
\[ r_0(s, z) := |u_0(s, z) - \psi_0(s, z)| \in L^1_{h,\lambda}, \quad (6) \]
then
\[ |u(t, z) - \psi(t, z)| \leq r(t, z) \leq A_\lambda r_0|e^{-\gamma_\lambda t} \frac{e^{\gamma_\lambda z}}{t^{d/2}} | \text{ for all } t > h/2, \quad z \in \mathbb{R}^d, \quad (7) \]
where \( r(t, z) \) is the solution to (5) with the initial datum \( \xi_\lambda r_0 \),
\[ A_\lambda := \left( \frac{1 + hq_\lambda e^{\gamma_\lambda \lambda_0}}{4\pi} \right)^{d/2} \quad (8) \]
and \( \gamma_\lambda \) is the unique real solution of the following equation
\[ \gamma + p_\lambda + q_\lambda e^{b\gamma} = 0, \quad (8) \]
In particular, if \( \phi_\epsilon \) is a stationary solution of (3) and \( q_\lambda \leq -p_\lambda \), i.e. \( \gamma_\lambda \leq 0 \), then \( \phi_\epsilon \) is globally stable in \( L^1_{h,\lambda} \cap L^\infty_{h,\lambda} \) with rate convergence \( O(t^{-d/2} e^{-\gamma_\lambda t}) \).

**Remark 2.2.** When the initial datum is taken in \( BUC(\mathbb{R}^d) \) it is possible to prove the existence of a mild solution of (1) on \( (0, +\infty) \times \mathbb{R} \), see Remark 3.3. Moreover, it also is possible to prove that for \( t > h \) that solution is a classic solution of (1), see Proposition 3.6 and compare with [38, Theorem 3.3].

**Remark 2.3.** Suppose that \( |g|_{L^p} > 1 \). Next, for each \( y \in \mathbb{R}^d \) we write \( y = (\tilde{y}, y_d) \) with \( \tilde{y} \in \mathbb{R}^{d-1} \) and \( y_d \in \mathbb{R} \). We fix \( \nu = (\tilde{\nu}, \nu_d) \) and \( \tilde{\lambda} \in \mathbb{R}^{d-1} \), then we define
\[ E_c(\lambda_\nu) := \lambda_\nu - c\nu_d \lambda_d - 1 + |\tilde{\lambda}| - c\tilde{\nu} \cdot \tilde{\lambda} + q^* e^{-c\lambda_d \nu_d} \int_{\mathbb{R}} e^{-\lambda_d y_d \mathcal{K}(y_d)}dy_d, \]
where \( \mathcal{K}(s) := \int_{\mathbb{R}^{d-1}} e^{-\tilde{\lambda} \cdot \tilde{y}} K(y_1, ..., y_d-1, s)dy \in L^1(\mathbb{R}) \) and \( q^* := |g|_{L^p} \ e^{-c\tilde{\nu} \cdot \tilde{\lambda}}. \)
Next, if
\[ 0 < c\tilde{\nu} \cdot \tilde{\lambda} + 1 + |\tilde{\lambda}| < q^* ||\mathcal{K}||_{L^1(\mathbb{R})} \quad (9) \]
we can invoke [11, Lemma 22] in order to obtain two numbers \( c^-_\nu = c^-_\nu(\bar{\nu}, \bar{\lambda}) \) and \( c^+_{\nu} = c^+_{\nu}(\bar{\nu}, \bar{\lambda}) \) such that if \( c \geq c^+_{\nu}/\nu_d \) or \( c \leq c^-_{\nu}/\nu_d \) then there exist at least a number \( \lambda^*_d \) such that

\[
q(\tilde{\lambda}, \lambda^*_d) \leq - p(\tilde{\lambda}, \lambda^*_d);
\]

In particular, if \( \bar{\lambda} = 0 \) and \( |g|_{Lip} = g'(0) \) then (9) is satisfied and therefore each stationary solution of (3) with \( c \geq c^+_{\nu}/\nu_d \) or \( c \leq c^-_{\nu}/\nu_d \) is globally stable in \( L^p_{h,\nu}(0,\lambda^*_d) \).

We note that due to [1, Theorem 3], Theorem 2.1 shows that two semi-wavefronts are equal by a translation whenever their first-order asymptotic terms coincide, i.e. the condition (6).

**Corollary 2.4** (Uniqueness of semi-wavefronts). Assume the condition (L). If \( \phi_c \) and \( \tilde{\phi}_c \) are stationary solutions of (3) such that \( p_{\lambda'}(c) + q_{\lambda'}(c) \leq 0 \) and \( \phi_c - \tilde{\phi}_c \in L^1_{\lambda'} \) for some \( \lambda' \in \mathbb{R}^d \), then \( \phi_c(\cdot + z_0) = \tilde{\phi}_c(\cdot) \) for some \( z_0 \in \mathbb{R}^d \).

Naturally, because of (7) the perturbation is maintained in the space \( L^1_{h,\nu} \cap L^\infty_{h,\nu} \) for all \( t > hd/2 \). This fact is true for all \( t \geq -h \) as it is showed in Proposition 3.2. For \( g \in W^{1,\infty}(\mathbb{R}) \) the second-order derivates of \( u(t, \cdot) \) belong to \( L^p \cap L^\infty \) for \( t > h \), this is showed in Proposition 3.6.

This result generalizes [30, Theorem 3] which is referred to local equations. In the case \( d = 1 \) and \( |g|_{Lip} = g'(0) \), equation (3) admits a semi-wavefront \( \phi_c \) with speed \( c \in \mathcal{C} \) (see, e.g. Proposition 2.6 below) and \( \phi_c(z) = A_{\phi_c}(-z) + e^{\lambda_1(c)z} + O(e^{\lambda_1(c)+\epsilon}z) \), for some positive numbers \( A_{\phi_c} \) and \( \epsilon \), and \( j_c = 0, 1 \) where \( j_c = 0 \) if only if \( c \in \mathcal{C} \setminus \{ c^-_\nu, c^+_{\nu} \} \) (see e.g., [1, Theorem 3]). Therefore, the semi-wavefront \( \phi_c \) can be found, on any domain \( (-\infty, N], N \in \mathbb{R} \), by means of the evolution of any initial datum to (3) in the form \( u_0(z) = A_{\phi_c}e^{\lambda_1(c)z} + O(e^{\lambda_1(c)+\epsilon}z) \) with bounds explicitly given in (7) where the convergence rate is \( O(t^{-1/2}e^{-\gamma_\lambda t}) \) for \( \gamma_\lambda \geq 0 \) determined by some \( \lambda \in (\lambda_1(c), \lambda_1(c) + \epsilon) \) in (8). For local equations, the numerical simulations for the approximation to critical wavefronts done in [6, Section 7] can also be used with the distance controlled by (7). Also, in [2] there are numerical simulations for monotone wavefronts to equation (1).

We note that the Cauchy problem to (3) is well posed for non negative initial data \( u_0 \) since an application of maximum principle on unbounded domains (see, [25, Theorem 10, Chapter 3]) implies that \( u(t, \cdot) \) is positive for \( t \in [0, h] \) and by repeating this argument to intervals \([h, 2h], [2h, 3h], \ldots\) we can conclude that \( u(t, \cdot) \) is positive for all \( t > 0 \).

### 2.2. Existence of d-dimensional planar semi-wavefronts.

The results on the existence of wavefronts to (4) for monotone \( g \) in an abstract setting are well known [39, 19, 33]. For non-monotone \( g \), the existence of non monotone wavefronts to (4) has been studied when \( d = 1 \) [14, 34, 40, 11, 42]. For \( d > 1 \), consider the following condition on \( K \)

**(K)** There exist non-negative functions \( K_i \in L^1(\mathbb{R}) \), for \( i = 1, \ldots, d \), such that

\[
K(s_1, \ldots, s_d) = \prod_{i=1}^d K_i(s_i) \quad \text{and} \quad \int_{\mathbb{R}^d} K(y) dy = 1.
\]

Also, the function

\[
\lambda \in \mathbb{R} \mapsto \int_{\mathbb{R}} K_i(y) e^{-\lambda y} dy,
\]
is defined on some maximal open interval \((a_i, b_i) \ni 0\).

If in equation (4) we take \(\nu = e_j\), where \(e_j\) is some canonic vector of \(\mathbb{R}^d\), then under the hypothesis \((K)\) the existence result for planar semi-wavefronts given in [11, Theorem 18] can be applied to (4). More precisely, associated to equation (4), for each \(c \in \mathbb{R}\), we have the characteristic function \(E_c^i : (a_i, b_i) \to \mathbb{R}\) defined by
\[
E_c^i(\lambda) := \lambda^2 - c\lambda - 1 + g'(0)e^{-\lambda y} \int_{\mathbb{R}} K_i(y)e^{-\lambda y} dy.
\]
Without restriction of \((K)\) we can take \(\int_{\mathbb{R}} K_i(s)ds = 1\) for \(i = 1, \ldots, d\) and according to [11, Lemma 22] we can make the following definition

**Definition 2.5.** Denote by \(c_i^- := c_i^-(e_j) < c_i^+(e_j) =: c_i^+(e_j)\) the two real numbers such that if \(c \in (c_i^-, c_i^+)\) then \(E_c^i(\lambda) > 0\) for all \(\lambda \in (a_i, b_i)\) and for each \(c \in \mathcal{C} := (-\infty, c_i^-) \cup [c_i^+, \infty)\) the function \(E_c^i(\lambda)\) either (i) has exactly two real solutions \(\lambda_1(c) \leq \lambda_2(c)\) or (ii) has exactly one real solution \(\lambda_1(c)\).

Also, from [11, Lemma 22], if \(g'(0) > 1\) then for \(c \leq c_i^-\) the zeros of \(E_c^i\) are negative while for \(c \geq c_i^+\) the zeros of \(E_c^i\) are positive. Therefore, since \(E_c^i(0) > 0\), \(E_c^i\) is not positive on \([\lambda_1(c), \lambda_2(c)]\).

Now, in order to establish the next results we make the following mono-stability condition.

\((M)\) The function \(g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) is bounded and the equation \(g(u) = u\) has exactly two solutions: 0 and \(\kappa > 0\). Moreover, \(g \in C^{1,\alpha}\) in some \(\delta_0\)-neighborhood of zero and \(g\) is a Lipschitz continuous function with \(|g|_{Lip} > 1\).

Under the conditions \((M)\) and \((K)\) the existence of semi-wavefronts was established, e.g., in [11, Theorem 18] and we present it as follow.

**Proposition 2.6.** Suppose that \(g\) satisfies \((M)\) with \(|g|_{Lip} = g'(0)\) and \(K\) satisfies \((K)\). Then for each \(c \in \mathcal{C}\) the equation (1) has a planar semi-wavefront \(v(t, x) = \phi_c(x; e_j + ct)\). Moreover, if \(c \leq c_i^-\) then \(\phi_c(+\infty) = 0\) and if \(c \geq c_i^+\) then \(\phi_c(-\infty) = 0\). Also, if for some \(\zeta_2 = \sup_{s \geq 0} g(s)\) the equilibrium \(\kappa\) is a global attractor of the map \(g : (0, \zeta_2) \to (0, \zeta_2)\), then each semi-wavefront is in fact a wavefront.

In the particular case when \(g\) is monotone, Proposition 2.6 says that semi-wavefronts for non-local equation (1) are wavefronts, indeed these are monotone wavefronts (see Remark 5.5). The problem in determining the condition for which \(\kappa\) is a global attractor for \(g : (0, \zeta_2) \to (0, \zeta_2)\) was dealt in [43] where the following condition characterizes this globally property.

\((G)\) The application \(g^2\) has a unique fix point \(\kappa\) on \((0, \zeta_2)\).

In this sense, under condition \((G)\) and an additional hypothesis on \(K\) (which can be dropped by Proposition 2.6) the authors in [42, Theorem 4.4] have stated the existence of a minimal speed for the existence of wavefronts which coincides with \(c_i^+\). Thus, if \(g\) satisfies \((G)\) then all planar semi-wavefronts are actually planar wavefronts under conditions \(|g|_{Lip} = g'(0) > 1\) and \((K)\).

In another cases it is also possible to determine whether a semi-wavefronts is actually a wavefront. For instance, since by [11, Remark 12] we have \([m, M] \subset g([m, M])\) where \(m = \lim \inf_{s \to +\infty} \phi_c(s)\) and \(M = \lim \sup_{s \to +\infty} \phi_c(s)\) for each semi-wavefront \(\phi_c\) such that \(c \geq c_i^+\) (a similar conclusion is obtained for \(c \leq c_i^-\)) it is a straightforward exercise to check that if \(|g|_{Lip} < 1\) and \(\kappa \in [m, M]\) then \(\phi_c\) is a
wavefront under the condition (M). This situation occurs in our next two stability results.

Otherwise, if we do not assume $|g|_{Lip} = g'(0)$ in (M) but we assume $(a_j, b_j) = \mathbb{R}$ in (K) the authors in [34, Theorem 4] show the existence of a speed $\hat{c}$ for the existence of semi-wavefronts $\phi_c$ such that $\phi_c(-\infty) = 0$ whenever $c \geq \hat{c}$. Moreover, if $|g|_{Lip} = g'(0)$ then $\hat{c} = c^+_0$.

Finally, note that the case $c^+_0 < 0$ is possible. For instance, by taking $K(s) = e^{-(s+\rho)^2}/\sqrt{4\pi}$, with $h = 2$, $g'(0) = 2$ and $\rho = 5$, the authors in [11, page 16] show that $c^-_0 = 2.7$ and $c^+_0 = 0, 7, ...$. Thus, in this case the equation (1) has stationary semi-wavefronts (for $c = 0$) and backwards semi-wavefronts (for $c \in (0, c^-_0)$).

2.3. Local stability of $d$-dimensional planar waves. Now, by following the same notation from Subsection 2.1 and Subsection 2.2 we state our stability result for local perturbations of planar wavefronts.

**Theorem 2.7.** Suppose (L) with $|g|_{Lip} > 1$ and $\rho_\epsilon := |g|_{|x_0-\epsilon e+\epsilon|} < 1$ for some $\epsilon > 0$. Consider a planar wavefront $\phi_c(e_j \cdot z)$, with $c \geq c^+_0$, for which there exists $z_t \in \mathbb{R}$ such that

$$
\phi_c(z_t) \in [\kappa - \epsilon/2, \kappa + \epsilon/2] \quad \text{for all } z_t \geq z_c,
$$

and take $\lambda' = \lambda e_j$ with $\lambda \in \mathbb{R}$ such that $K \in L^1_\lambda$ and $-p_\lambda \geq q_\lambda$ (according to Theorem 2.1). If the non-negative initial datum $u_0 \in C^0_{\alpha} \cap L^1_\lambda$ satisfies

$$
|u_0 - \phi_c|_{L^1_{\lambda}, \epsilon}, \quad |u_0 - \phi_c|_{L^\infty} \leq \epsilon C_\epsilon,
$$

for certain $C_\epsilon \in (0, 1/2)$, then the following assertions are true

(i) If $-p_\lambda > q_\lambda$ then for each $0 < \gamma_* \leq \gamma_\lambda$ satisfying $\rho_\epsilon e^{\gamma_*} < 1 - \gamma_\epsilon$ we have

$$
|u(t, z) - \phi_c(z_t)| \leq \frac{e^{\gamma_*} e^{(d+2)h/2}}{2} e^{-\gamma_* t} \quad \text{for all } (t, z) \in [-h, \infty) \times \mathbb{R}^d.
$$

(ii) If $-p_\lambda = q_\lambda$, then there exists $\delta_* = \delta_*(\rho_\epsilon) > 1 + (d+2)h/2$ such that

$$
|u(t, z) - \phi_c(z_t)| \leq \frac{e}{2[t + \delta_* - (d+2)h/2]^{d/2}} \quad \text{for all } (t, z) \in [0, \infty) \times \mathbb{R}^d.
$$

It is instructive to compare Theorem 2.7 with a work of Gallay [9] about the local perturbations of critical wavefronts for a local equation with $h = 0$. Note that in [9, Theorem 1.1] the perturbation is additionally weighted with quadratic function in the trivial equilibrium and the exponential convergence to the positive equilibrium is assumed. Although, in this subspace considered by Gallay the rate the convergence is $O(t^{-3/2})$. Otherwise, note that for non-critical wavefronts the convergence rate depends on the weighted space where the perturbation is taken attaining an algebraic convergence rate when the perturbation is in $L^1_{h_\lambda} \cap L^\infty_h$, for $\lambda = \lambda_j$, $j = 1, 2$, and an exponential convergence rate if $\lambda \in (\lambda_1(e), \lambda_2(e))$.

2.4. Global stability of wavefronts on the line. In this section we take $d = 1$ and give a global stability result in the sense that the wavefronts are attractors for the following class of initial data

**Theorem 2.1.** The continuous initial datum $u_0 : [-h, 0] \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ to (3) is a bounded function such that

$$
\pm c \geq \pm c^+_0 \quad \text{implies } u_0(s, \pm z) \geq \sigma \quad \text{for all } s \in [-h, 0] \text{ and } z \geq z_0.
$$
We note that in Theorem 2.7 it is necessary that ε < κ and φε(z) ≥ κ - ε/2 for z ≥ zϵ, therefore an initial datum satisfying the condition (11) meets the condition (IC) with s = κ - ε and z0 = zϵ.

Next, denote M_g := max_{u∈[0,κ]} g(u), m_g := min_{u∈[κ,M_g]} g(u) and I_g := [m_g, M_g]. Also we define the following weighted Sobolev space

\[ W_{t,λ}^{r,p} := \{u : \mathbb{R} → \mathbb{R}, \text{ such that } |u|_{L_{t,λ}^{r,p}} := ||ξλu||_{W^{r,p}(\mathbb{R})} < ∞\} \]

\[ W_{h,λ}^{r,p} := \{u ∈ C([-h,0],W_{t,λ}^{r,p}) \text{ with norm } |u|_{W_{h,λ}^{r,p}} = \max_{s∈[-h,0]} |u(s)|_{W_{t,λ}^{r,p}}\} \]

Theorem 2.8. Suppose (K), (M) with |g|_{Lip} = g'(0) and ρ := |g|_{Lip} < 1.

Consider c ∈ C and λ ∈ [λ1(c), λ2(c)]. If the initial datum u0 ∈ C_{h,1}^{0,α} satisfies the condition (IC) and u0 - φc ∈ L_{h,λ}^1 ∩ L_{h,λ}^∞ then the following assertions are true

(i) If λ ∈ (λ1(c), λ2(c)) then for any 0 < γ∗ ≤ γλ satisfying ρeγ∗ < 1 - γ there exists C = C(g,c,u0) > 0 such that

\[ |u(t,z) - φc(z)| ≤ Ce^{-γ∗t} \quad ∀(t,z) ∈ [-h,∞) × \mathbb{R}. \]  

(ii) If λ = λj(c), j = 1, 2, then there exists C > 0 such that

\[ |u(t,z) - φc(z)| ≤ C \sqrt{t} \quad ∀(t,z) ∈ (0,∞) × \mathbb{R}. \]

Remark 2.9 (Speeds selection problem). Consider ±c ≥ ±c∗± and an initial datum v0 ∈ C_{h,0}^{α} to (1) satisfying (IC) and in the form v0(s,x) = A(-z)βe^{λj(c)z} + O(e^{(λj(c)±c)x}) for some A, β ∈ R+ and where j_c = 0, 1 and j_c = 1 if and only if c = c*_+ or c = c*_-, then note that for each β ∈ (0,κ) the associated level set for v(t,.) is asymptotically propagated with speed c. More precisely, by (7) and (14)-(15) for large t the set \{x ∈ \mathbb{R} : v(t,x) = β\} is not empty and if c ≥ c*_± this set also is lower bounded therefore it has a infimum m(t), then by evaluating in (14)-(15) at z = m(t) + ct we necessarily have \{m(t) + ct, t ≥ 0\} is bounded so that \[|c + m(t)/t| = O(1/t), \text{ i.e., } m(t) \text{ is propagated with speed } -c. \] Also, note that if c*± < 0 and c ∈ (c*+, 0) the level set will move to +∞ contrary to symmetric case. A similar situation occurs when c ≤ c*±.

Since the condition φc(·) - u0(s,·) ∈ W^{1,p}(\mathbb{R}_±), uniformly for s ∈ [-h,0] and ±c ≥ ±c*_±, implies (IC) we get

Corollary 2.10 (Global stability in Sobolev spaces ). Assume that u0 ∈ C_{h,1}^{0,α}, u0 - φc ∈ L_{h,λ}^1 ∩ L_{h,λ}^∞ and

\[ |u0(s,z) - φc(z)| ≤ \frac{1}{\min\{1, c_λ z\}} \]  

for some λ ∈ (a,b) and 1 ≤ p < ∞. Then, λ ∈ (λ1(c), λ2(c)) implies (14) and λ = λ_j, j = 1, 2, implies (15).

Note that Corollary 2.10 generalizes [23, Theorem 2.2] when p = 2. We also have the following result for non-local Nicholson’s model

Corollary 2.11 (Nicholson’s model). Suppose that p/δ ∈ [1, c^2] in (2). If φc is a wavefront with speed c ∈ C to (2) then the perturbations of φc either are globally algebraically stable in L_{h,1}^1 ∩ L_{h,1}^∞ when λ = λ_j, j = 1, 2, or are globally exponentially stable in L_{h,1}^1 ∩ L_{h,1}^∞ when λ ∈ (λ1(c), λ2(c)), in the sense of Theorem 2.8.
For a local version of (2) in [10, Theorem 2.3] it was demonstrated that for \( p/\delta \in [e, e^2] \) this equation has non-monotone wavefronts with speed arbitrarily large. Under this restriction on the parameters \( p \) and \( \delta \), Solar and Trofimchuk have demonstrated the global stability of non-critical wavefronts for the local Nicholson equation [31, Corollary 3]. Thus, the global stability of critical wavefronts for local equations in Corollary 2.11 is a complement to the result obtained in [31].

3. A regularity result. We start giving a result on the persistence of disturbances in the underlying space for the following equation

\[
\dot{u}(t, z) = \Delta u(t, z) + d_1 \cdot \nabla u(t, z) + d_2 u(t, z) + \int_{\mathbb{R}^d} K(z-y)d_3(t, y)u(t-h, y)dy.
\]

**Proposition 3.1.** Suppose \( d_1 \in \mathbb{R}^d \), \( d_2 \in \mathbb{R} \) and \( d_3 \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \). Consider a solution \( u(t, z) \) of (16) and \( \lambda, \lambda' \in \mathbb{R}^d \) such that \( K \in L^1_{\lambda} \cap L^\infty_{\lambda'} \). If the continuous initial datum \( u_0 \) holds \( u_0 \in L^p_{h, \lambda} \cap L^\infty_{h, \lambda} \), some \( 1 \leq p \leq \infty \), then \( u_t \in L^p_{h, \lambda'} \cap L^\infty_{h, \lambda} \) for all \( t \geq 0 \) and the following estimate holds

\[
|u_{kh}|_{L^p_{h, \lambda'}} \leq \theta^k |u_0|_{L^p_{h, \lambda'}} \quad \text{for} \quad k = 0, 1, 2, ...
\]

and for some \( \theta = \theta(\lambda') > 1 \).

**Proof.** By making the change of variables \( \tilde{u}(t, z) := u(t, z)e^{-\lambda'z} \) the equation (16) is transformed to

\[
\dot{\tilde{u}}(t, z) = \Delta \tilde{u}(t, z) + d'_1 \cdot \nabla \tilde{u}(t, z) + d'_2 \tilde{u}(t, z) + \int_{\mathbb{R}} K'(z-y)d_3(t, y)\tilde{u}(t-h, y)dy,
\]

where \( d'_1 = 2\lambda + d_1 \), \( d'_2 = |\lambda'|^2 + d_1 \cdot \lambda' + d_2 \) and \( K'(y) = K(y)e^{-\lambda'y} \). Next, by the change of variable \( \tilde{u}(t, z) := \tilde{u}(t, z - d'_1t)e^{-d'_2t} \) the equation (18) is reduced to inhomogeneous heat equation,

\[
\dot{\tilde{u}}(t, z) = \Delta \tilde{u}(t, z) + f(t, z) \quad \text{for all} \quad (t, z) \in [0, h] \times \mathbb{R}^d
\]

where

\[
f(t, z) = e^{-d'_2h} \int_{\mathbb{R}^d} K'(y)d_3(t, z - d'_1t - y)\tilde{u}(t-h, z-y - d'_1h)dy,
\]

for \( (t, z) \in [0, h] \times \mathbb{R} \). Now, note that

\[
|f(t, \cdot)|_{L^\infty} \leq e^{-(d'_2+\Lambda d'_1)t}|d_3|_{L^\infty}|K|_{L^1}|u_0|_{L^\infty_{h, \lambda}},
\]

where \( \Lambda := \lambda - \lambda' \) and \( K(y) := K(y)e^{-\lambda'y} \). So that, we have

\[
\tilde{u}(t) := \Gamma_t * \tilde{u}(0) + \int_0^t \Gamma_{t-s} * f(s)ds,
\]

where \( \Gamma_t \) is the \( d \)-dimensional heat kernel. Similarly,

\[
|f(t, \cdot)|_{L^p} \leq e^{-d'_2h}|K'|_{L^1}|d_3|_{L^\infty}|\tilde{u}_0|_{L^p_{h, \lambda}}, \quad \text{for all} \quad t \in [0, h]
\]

Then, for \( t \in [0, h] \) and \( t_n \to t \) we have

\[
|\tilde{u}(t) - \tilde{u}(t_n)|_{L^p} \leq |\Gamma_t - \Gamma_{t_n}|_{L^1} |\tilde{u}(0)|_{L^p} + \int_0^t |\Gamma_{t-s} - \Gamma_{t_n-s}|_{L^1} |f(s)|_{L^p}ds + \int_{t_n}^t |\Gamma_{t-s}|_{L^1} |f(s)|_{L^p}ds,
\]

for all \( t \in [0, h] \).
and by using (23),

$$|\bar{u}(t) - \bar{u}(t_n)|_{L^p} \leq (|\Gamma_1 - \Gamma_{t_n}|_{L^1} + \int_0^t |\Gamma_{t-s} - \Gamma_{t_n-s}|_{L^1} ds + |t - t_n|)R|\tilde{u}_0|_{L^p}^\gamma,$$

(25)

where $R = \max\{1, e^{-d_2^\gamma t}|K'|_{L^1}|d_3|_{L^\infty}\}$. Since $|\Gamma_{t_n}|_{L^1} = |\Gamma_{t}|_{L^1} = 1$ the last inequality implies $|\bar{u}(t) - \bar{u}(t_n)| \to 0$ as $t_n \to t$, therefore $u(t, \cdot) \in L^p_h$ for all $t \in [0, h]$, so that $u_t \in L^p_{h, \lambda} \cap L^\infty_{h, \lambda}$ for all $t \in [0, h]$. Similarly, by using (22),

$$|\bar{u}(t)|_{L^p} \leq |\bar{u}(0)|_{L^p} + h \max_{s \in [-h, 0]} |f(s)|_{L^p} \leq (1 + he^{-d_2^\gamma t}|d_3|_{L^\infty}|K'|_{L^1})|\bar{u}_0|_{L^p}^\gamma,$$

therefore if we multiply by $e^{-d_2^\gamma t}$ the last inequality then (17) follows for $k = 1$ by taking

$$\theta := e^{2h|d_2^\gamma|}(1 + he^{-d_2^\gamma t}|d_3|_{L^\infty}|K'|_{L^1}).$$

Analogously, by using $u(t + h, \cdot), u(t + 2h, \cdot), ..., t \in (0, h]$, for the intervals $[h, 2h], [2h, 3h]...$ we obtain $u_{t+k}h \in L^p_{h, \lambda} \cap L^\infty_{h, \lambda}$ and (17) for $k = 2, 3, ...$

**Proposition 3.2.** Suppose that $g$ is globally Lipschitz continuous and $K \in L^1_{\lambda'} \cap L^1_{\lambda}$ for some $\lambda', \lambda \in \mathbb{R}^d$. If the initial datum $u_0 \in L^\infty_{h, \lambda} \cap C^0_{h, \lambda}$ then there exists a unique solution $u(t, z)$ to the nonlinear equation (3) and this solution also satisfies $u_t \in L^\infty_{h, \lambda} \cap C^0_{h, \lambda'}$ for all $t \geq 0$. Moreover, if $u_0 \in L^p_{h, \lambda_0}$ for some $\lambda_0$ such that $K \in L^1_{\lambda_0}$, then the solution $u(t, z)$ satisfies the estimation (17).

**Proof.** In the same notation of Proof of Proposition 3.1, consider equation (19) with $f$ given by (20) and with $d_3(t, z) = g(u(t-h, z-vch))/u(t-h, z-vch)$ for $t \in [0, h]$. Note that

$$|f(t, \cdot)|_{C^0, \alpha} \leq |g|_{L^p, \alpha} e^{-d_2^\gamma t}|K'|_{L^1} |\bar{u}_0|_{C^0_{h, \alpha}}, \quad \text{for all } t \in [0, h].$$

(26)

Then, by (21) and (26) we can apply [8, Chapter 1, Theorem 12 and Theorem 16] to conclude that the equation (19) has a unique solution $\bar{u}(t, z)$, for all $t \in [0, h]$, which implies the existence of a solution $u(t, z)$ to (3), for $t \in [0, h]$, with initial datum $u_0$. Moreover, by proceeding as in (24) and (25) we conclude that $u_t \in L^\infty_{h, \lambda} \cap C^0_{h, \lambda'}$ for all $t \in [0, h]$. Thus, by repeating this process on the intervals $[2h, 3h], [3h, 4h], ...$ we obtain a unique solution $u(t, z)$ of (19) such that $u_t \in C^0_{h, \lambda'}$ for all $t \geq 0$. Finally, by Proposition 3.1 with $d_3(t, z) = g(u(t-h, z-vch))/u(t-h, z-vch)$ we get the estimation (17). \qed

**Remark 3.3.** Note that the same procedure used in Proof of Proposition 3.2 can be applied to the equation (22) only requiring that $u_0 \in L^\infty_{h, \lambda}$, which gives a mild solution $u(t, z)$ of (3) for all $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^d$.

**Lemma 3.4.** Assume $d_1, d_2$ and $d_3$ satisfy the hypothesis of Proposition 3.1. Consider $u : [-h, h] \times \mathbb{R} \to \mathbb{R}$ satisfying equation (16) for $(t, z) \in [0, h] \times \mathbb{R}$. If $u_0 \in L^p_{h, \lambda}$ for some $\lambda \in \mathbb{R}^d$ such that $K \in L^1_{\lambda}$, then

$$|u_{z_i}(t, \cdot)|_{L^p_{\lambda'}} \leq \left( \frac{\theta_1}{\sqrt{t}} + \sqrt{t} \theta_2 \right)|u_0|_{L^p_{h, \lambda}}, \quad t \in [0, h] \text{ and } i = 1, ..., d$$

(27)

for some positive numbers $\theta_1 = \theta_1(\lambda')$ and $\theta_2 = \theta_2(\lambda')$. Furthermore, $|u_{z_i}(t, \cdot)|_{L^p_{\lambda'}}$ continuously depends on $t$. 


Remark 3.5. By using Proposition 3.1, \( u_0 \in L^p_{h,\lambda'} \) implies \( u(\cdot + h, \cdot), u(\cdot + 2h, \cdot) \ldots \in L^p_{h,\lambda'} \), therefore Lemma 3.4 implies that for \( k = 1, 2, 3, \ldots \) and \( t \in [0, h] \) we have \( u_{z_i}(t + kh, \cdot) \in L^p_{\lambda'} \) for \( i = 1, \ldots, d \). More precisely, for \( k = 1, 2, 3, \ldots \) Proposition 3.1 implies

\[
|u_{z_i}(t + kh, \cdot)|_{L^p_{\lambda'}} \leq \left( \frac{\theta_1}{t} + \sqrt{t} \theta_2 \right) \theta^{k - 1} |u_0|_{L^p_{h,\lambda'}} \quad t \in (0, h) \text{ and } i = 1, \ldots, d. \tag{28}
\]

with \( |u_{z_i}(t + kh, \cdot)|_{L^p_{\lambda'}} \) continuously depending on \( t \).

Proof. If \( t > 0 \) from (22) it follows

\[
\tilde{u}_{z_i}(t, z) = \int_{\mathbb{R}^d} \frac{(y_i - z_i)e^{-|z - y|^2/4t}}{2^{d+1}t(\pi t)^{d/2}} \tilde{u}(0, y) dy - \int_0^t \int_{\mathbb{R}^d} \frac{(z_i - y_i)e^{-(z - y)^2/4(t-s)}}{2^{d+1}((t-s))[\pi(t-s)]^{d/2}} f(s, y) dy ds
\]

therefore, by using (23), for each \( t \in (0, h] \) we get

\[
|\tilde{u}_{z_i}(t, z)|_{L^p} \leq \frac{|u_0|_{L^p}}{\sqrt{\pi t}} \int_{\mathbb{R}^d} |y_i| e^{-|y|^2} dy + 2 \sqrt{t} \int_{\mathbb{R}^d} |y_i| e^{-|y|^2} dy |f(\cdot, \cdot)|_{L^p} \leq \frac{|u_0|_{L^p}}{\sqrt{\pi t}} + 2 \sqrt{t} |d_3|_{L^\infty} |K'|_{L^1} e^{-d^2 h} \tilde{u}(\cdot, \cdot)|_{L^p_{\lambda'}} \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} |y_i| e^{-|y|^2} dy \leq \frac{1}{\sqrt{t}} + 2 \sqrt{t} |d_3|_{L^\infty} |K'|_{L^1} e^{-d^2 h} |u_0|_{L^p_{h,\lambda'}} \sqrt{\pi},
\]

which implies (27) by taking \( \theta_1 = 1/\sqrt{\pi} \) and \( \theta_2 = 2 |d_3|_{L^\infty} |K'|_{L^1} e^{-d^2 h}/\sqrt{\pi} \).

Finally, by using (29) and proceeding as in (24) and (25) we obtain that \( |u_{z_i}(t, \cdot)|_{L^p_{\lambda'}} \) continuously depends on \( t \).

Proposition 3.6 (Lp-Regularity). Consider a solution \( u \) of \((3)\) such that the initial datum \( u_0 \) satisfies \( u_0 \in L^p_{h} \cap L^\infty_{h} \) for some \( p \) such that \( 1 \leq p \leq \infty \). If \( g(t, \cdot) \in L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}^d)) \) then

\[
D^2 u(t, \cdot) \in L^p(\mathbb{R}) \quad \text{for all } t \in (h, +\infty), \tag{30}
\]

uniformly, in norm, on compact sets of \((h, +\infty)\).

Proof. By Remark 3.5, \( u_{z_i}(t, \cdot) \in L^p(\mathbb{R}) \) for each \( t \in (0, +\infty) \), uniformly (in norm) on compacts. In particular, if \( T > h \) then \( u_{z_i}(t + T, \cdot) \in L^p_h \) then differentiating (3) we obtain that \( u_{z_i} \) satisfies (16) with \( d_3(t, z) := g'(u(t - h, z - ch)) \) and therefore we can conclude \( u_{z_i, z_i}(t, \cdot) \in L^p \), for each \( t \in (h, +\infty) \), uniformly (in norm) on compacts.

4. Proof of Theorem 2.1 and Theorem 2.7. We denote the Fourier transform of \( u : \mathbb{R}^d \to \mathbb{R} \) by

\[
\hat{u}(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot y} u(y) dy.
\]

Next, we define the function \( l : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \) by the equation

\[
l_\lambda(\zeta) = -|\zeta|^2 + p_\lambda + L g e^{-\lambda u_{z_i}} |\xi_{\lambda} K(\zeta)| e^{-h_\lambda(z)} \zeta, \tag{31}
\]

Now, we will estimate the function \( l_\lambda(\zeta) \). For \( \epsilon_h = 1/[1 + h q_{\lambda} e^{h\gamma}] > 0 \) we define the function

\[
\alpha_h(\zeta) := -\frac{1}{h} \log(1 + h \epsilon_h |\zeta|^2),
\]

if \( |\zeta|^2 < h \epsilon_h^{-1} \), otherwise \( \alpha_h(\zeta) := 0 \).
and we denote by $\tilde{q}_\lambda(\xi) := L_\delta e^{-\lambda v\delta}|\xi\hat{R}(\xi)|$.

**Lemma 4.1.** The function $l_\lambda$ meets the following inequalities

$$-\epsilon_\delta|\xi|^2 - \gamma_\lambda \leq l_\lambda(\xi) \leq \alpha_\delta(\xi) - \gamma_\lambda \quad \text{for all } \xi \in \mathbb{R}.\tag{32}$$

**Remark 4.2.** In the local case (when $\kappa$ is formally a constant $q$) we have $e^{l(\xi)} \sim -q/\xi^2$ (see [30, Lemma 13]) but in the non local case, because of Riemann-Lebesgue Lemma, the estimations for $l(\xi)$ can be improved.

**Proof.** Let us denote $\beta(\xi) = l_\lambda(\xi) - \alpha_\delta(\xi) + \gamma_\lambda$. Then $\beta(\xi)$ satisfies the following equation

$$\beta(\xi) = -|\xi|^2 + \frac{1}{h} \log(1 + h\epsilon_\delta|\xi|^2) + \gamma_\lambda + p_\lambda + \tilde{q}_\lambda(\xi)e^{h\gamma_\lambda}(1 + h\epsilon_\delta|\xi|^2)e^{-h\beta(\xi)}.$$

From Lemma [30, Lemma 12] we have that $\beta(\xi) \leq 0$ if and only if:

$$|\xi|^2 - \frac{1}{h} \log(1 + h\epsilon_\delta|\xi|^2) - \gamma_\lambda - p_\lambda \geq \tilde{q}_\lambda(0)e^{h\gamma_\lambda}(1 + h\epsilon_\delta|\xi|^2).\tag{33}$$

Now, by using $\log(1 + x) \leq x$, for all $x \geq 0$, then in order to obtain (33) it is enough to have

$$|\xi|^2 - \epsilon_\delta|\xi|^2 - \gamma_\lambda - p_\lambda \geq \tilde{q}_\lambda(0)e^{h\gamma_\lambda}(1 + h\epsilon_\delta|\xi|^2) \quad \text{for all } \xi \in \mathbb{R}$$

$$\iff (1 - \epsilon_\delta - q_\lambda h\epsilon_\delta e^{h\gamma_\lambda})|\xi|^2 - \gamma_\lambda - p_\lambda - q_\lambda e^{h\gamma_\lambda} = 0 \quad \text{for all } \xi \in \mathbb{R}$$

This proves (32). \qed

**Proof of Theorem 2.1.** Note that by Proposition 3.2, if $u_0 \in L^p_{h,\lambda} \cap L^\infty_{h,\lambda} \cap C^{0,\alpha}_{h,\lambda}$ then there exist a unique solution $u(t, \cdot)$ of (3) which satisfies $u_t \in L^p_{h,\lambda} \cap L^\infty_{h,\lambda} \cap C^{0,\alpha}_{h,\lambda}$ for all $t \geq 0$.

Otherwise, by making the following change of variable $\tilde{u}(t, z) = u(t, z)e^{-\lambda z}$ we have

$$\tilde{u}(t, z) = \Delta \tilde{u}(t, z) + (2\lambda - c \cdot \nu)\nabla \tilde{u}(t, z) + p_\lambda \tilde{u}(t, z) + e^{-\lambda z}[K + g(e^{\lambda \cdot (-c)\nu})\tilde{u}(t - h, \cdot - c\nu)](z).$$

Now, if we denote the linear operator

$$L_0^\delta(t, z) := \Delta \delta(t, z) + (2\lambda - c \cdot \nu)\nabla \delta(t, z) + p_\lambda \delta(t, z) - \delta(t, z),$$

and $\delta_\lambda(t, z) := \pm[u(t, z) - \tilde{u}(t, z)] - r(t, z)$ then for $(t, z) \in [0, h] \times \mathbb{R}^d$ we have

$$(L_0^\delta \pm)(t, z) = e^{-\lambda z}K^*\{g[e^{\lambda \cdot (-c)\nu}]\psi(t-h, \cdot - c\nu) - g[e^{\lambda \cdot (-c)\nu}]\tilde{u}(t-h, \cdot - c\nu)](z) - L_0^r(t, z)$$

$$\geq -L_0^\mu e^{-\lambda ch}K_\xi^\lambda [\psi(t-h, \cdot - c\nu) - \tilde{u}(t-h, \cdot - c\nu)](z) - L_0^r(t, z).$$

Because of $|\psi(t-h, z) - \tilde{u}(t-h, z)| \leq r(t-h, z)$ for $(t, z) \in [0, h] \times \mathbb{R}^d$ then

$$(L_0^\delta)(t, z) \geq -L_0^\mu e^{-\lambda ch}K_\xi^\lambda [r(t-h, z) - \tilde{u}(t-h, \cdot - c\nu)](z) - L_0^r(t, z) = 0.\tag{34}$$

Now, by Proposition 3.2 the function $\delta(t, z)$ is exponentially bounded in the variable $z$, uniformly for $t$ on each interval $[0, h], [h, 2h], ..., $ so that by Phragmén-Lindelöf principle [25, Chapter 3, Theorem 10] we obtain

$$\pm[u(t, z) - \tilde{u}(t, z)] \leq r(t, z) \quad \text{for all } (t, z) \in [0, h] \times \mathbb{R}.$$ 

By repeating the same process for the intervals $[h, 2h], [2h, 3h], ...$ we conclude

$$\pm[u(t, z) - \tilde{u}(t, x)] \leq r(t, z) \quad \text{for all } (t, z) \in [-h, +\infty) \times \mathbb{R}.\tag{35}$$
Now, we globally estimate the function \( r \). Next, by Proposition 3.6 we have \( r, r_{x_1}, r_{x_2} \in L^1(\mathbb{R}) \) for all \( t > h \). Then, by applying Fourier’s transform to (5) we obtain

\[
\hat{r}(t, z) = \left( |z|^2 + i(2\lambda - \omega) \cdot z + p_\lambda \right) \hat{r}(t, z) + L_{\gamma} e^{-\lambda \nu \chi \xi(x \cdot) z} e^{-i \chi(x \cdot) \nu \chi(x \cdot) t} e(t - h, z),
\]

for all \( (t, z) \in (2h, +\infty) \times \mathbb{R} \). So, due to [30, Lemma 11], by using \( l_{\lambda} + \gamma_\lambda \leq 0 \), we get

\[
e^{-\lambda t} |\hat{r}(t, z)| \leq |r_0|_{L^1_h} e^{(l_{\lambda}(z) + \gamma_\lambda)t} \quad \text{for all} \quad (t, z) \in (2h, +\infty) \times \mathbb{R},
\]

and by Lemma 4.1 we have

\[
l_{\lambda}(z) + \gamma_\lambda \leq -\frac{1}{h} \log(1 + \h \varepsilon \abs{z}^2) \quad \forall z \in \mathbb{R}^d.
\]

Then, by (36) and (37)

\[
\int_{\mathbb{R}^d} |\hat{r}(t, y)| dy \leq |r_0|_{L^1_h} \int_{\mathbb{R}^d} e^{l_{\lambda}(y) t} dy \leq |r_0|_{L^1_h} e^{-\gamma_\lambda t} \int_{\mathbb{R}^d} \frac{dy}{(1 + \h \varepsilon \abs{y}^2)^{\pi}}
\]

\[
= |r_0|_{L^1_h} e^{-\gamma_\lambda t} \int_0^{+\infty} \int_{\partial B(0, r)} [1 + \h \varepsilon \abs{y}^2]^{-\pi} \, dS \, dr
\]

\[
= \frac{2\pi^{d/2}}{\Gamma(d/2)} |r_0|_{L^1_h} e^{-\gamma_\lambda t} \int_0^{+\infty} \frac{r^{d-1} dr}{[1 + \h \varepsilon \abs{y}^2]^{\pi/\h}}
\]

therefore we obtain that \( \hat{r}(t, \cdot) \in L^1(\mathbb{R}^d) \) for \( t > h/2 \) and by using Fourier’s inversion formula we have

\[
|r(t, z)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{r}(t, y)| dy \leq |r_0|_{L^1_h} e^{-\gamma_\lambda t} \int_0^{+\infty} \frac{r^{d-1} dr}{[1 + \h \varepsilon \abs{y}^2]^{\pi/\h}}
\]

\[
= \frac{|r_0|_{L^1_h} e^{-\gamma_\lambda t}}{2^{d-1} \Gamma(d/2) (\pi \varepsilon \abs{y}^2)^{d/2}} \int_0^{+\infty} \frac{r^{d-1} dr}{[1 + \h \varepsilon \abs{y}^2]^{d/\h}}
\]

\[
\leq \frac{|r_0|_{L^1_h} e^{-\gamma_\lambda t}}{2^{d-1} \Gamma(d/2) (\pi \varepsilon \abs{y}^2)^{d/2}} \int_0^{+\infty} r^{d-1} e^{-r^2} dr
\]

\[
= \frac{|r_0|_{L^1_h} e^{-\gamma_\lambda t}}{2^{d-1} \Gamma(d/2) (\pi \varepsilon \abs{y}^2)^{d/2}} \frac{1}{2} \Gamma(d/2)
\]

\[
= \frac{|r_0|_{L^1_h} e^{-\gamma_\lambda t}}{2^{d-1} (\pi \varepsilon \abs{y}^2)^{d/2}}
\]

\[
\Box
\]

**Proof of Theorem 2.7.** We will give the proof to the case \( c \geq c^*_+ \) since the proof for the case \( c \leq c^*_- \) is completely analogous.

(i) Note that by (7) we get

\[
e^{-\lambda x_j} \abs{u(t, z)} \leq \frac{|r_0|_{L^1_h}}{A \lambda t^{d/2}} e^{-\gamma_\lambda t} \quad \forall \ t > h/2, z \in \mathbb{R}^d
\]

Now, by an application of Proposition 3.2, with \( p = \infty \), we can take \( C_\varepsilon \in (0, 1/2] \) small enough, which will be fixed below, such that \( \abs{u_0 - \phi_c} \leq \varepsilon C_\varepsilon \) implies

\[
\abs{u(t, \cdot) - \phi_c}_{L^\infty} \leq \frac{\varepsilon}{2} \quad \text{for all} \quad t \in [-h, (d + 2)h/2],
\]
so that, in particular we get
\[ u(t, z) \in [\kappa - \epsilon, \kappa + \epsilon] \text{ for all } (t, z) \in [-h, (d+2)h/2] \times \{ z \in \mathbb{R}^d : z_j \geq z \epsilon \} \quad (40) \]

Next, we consider the function \( r : [-h, +\infty) \to \mathbb{R}_+ \) given by \( r(t) := \frac{\epsilon}{2} e^{-\gamma_* t} \) and define \( \delta_\pm(t, z) := \pm[u(t + (d+2)h/2, z) - \phi_c(z)] - r(t) \). Then, by (39) we obtain
\[ \delta_\pm(s, z) \leq 0 \text{ for } (s, z) \in [-h, 0] \times \mathbb{R}^d. \]

And if \((t, z) \in [0, h] \times \mathbb{R}^d \) then by (38) and (40) we get
\[
(L \delta_\pm)(t, z) = \pm \int_{\mathbb{R}^d} K(z - c u - y)\{g(\phi_c(y)) - g(u(t + dh/2, y))\}dy - L r(t) \\
\geq - \frac{|g|_{L^p} |\rho_0|_{L^1/w} e^{-\gamma_* t} }{A_{\lambda} [t + dh/2]^{d/2}} \int_{y \in \mathbb{R}^d : y_j \leq \zeta} e^{\lambda y_j} K(z - c u - y)dy \\
+ \rho_* \int_{y \in \mathbb{R}^d : y_j \geq \zeta} K(z - c u - y)r(t - h)dy - L r(t)
\]

But, by [11, Lemma 22] we have \( \lambda > 0 \), so that
\[
(L \delta_\pm)(t, z) \geq - \frac{\epsilon}{2} e^{-\gamma_* t} \frac{|g|_{L^p} |\rho_0|_{L^1/w} e^{-\gamma_* t (d+1)h/2}}{A_{\lambda} [dh/2]^{d/2}} e^{\lambda z} + \rho_* e^{\gamma_* h} - 1 + \gamma_*
\]

Now, in the last inequality since \( \rho_* e^{\gamma_* h} < 1 - \gamma_* \) we can fix \( C_\epsilon \) small enough such that \( |\rho_0|_{L^1/w} \leq C_\epsilon \) implies \( L \delta_\pm(t, z) \geq 0 \) for all \((t, z) \in [0, h] \times \mathbb{R}^d\), so that Phragmén-Lindelöf principle implies \( \delta_\pm(t, z) \leq 0 \) for \((t, z) \in [0, h] \times \mathbb{R}^d\).

Since \(|u(t + 2h, z) - \phi_c(v \cdot z)| \leq r(t) \) for all \((t, z) \in [0, h] \times \mathbb{R}^d\) implies \( u(t + 2h, z) \in [\kappa - \epsilon, \kappa + \epsilon] \) for all \((t, z) \in [0, h] \times \mathbb{R}^d : z_1 \geq z \epsilon \) it is possible to repeat the process, by using (38), for the intervals \([h, 2h], [2h, 3h], ...\) in order to obtain \( \delta(t, z) \leq 0 \) for all \((t, z) \in [-h, \infty) \times \mathbb{R}^d\).

(ii) We take \( \delta_* > 1 + (d+2)h/2 \) large enough satisfying
\[
\rho_* \frac{(t + \delta_*)^{d/2}}{(t - \delta_*)^{d/2}} + \frac{d}{2(t + \delta_*)} < 1 \quad t \geq -h.
\]

Now, we consider \( r : [-h, +\infty) \to \mathbb{R}_+ \) given by \( r(t) := \epsilon/(2(t + \delta_*))^{d/2} \). Next, similarly part (i), we can take \( C_\epsilon \), which we will fix below, such that \(|u_0 - \phi_c|_{L^\infty} \leq C_\epsilon \) implies
\[
|u(t, \cdot) - \phi_c|_{L^\infty} \leq \epsilon/2\delta_*^{d/2} \quad \text{for all } t \in [-h, (d+2)h/2].
\]

So that, if we define \( \delta_\pm(t, z) := \pm[u(t + (d+2)h/2, z) - \phi_c(z)] - r(t) \) then by (43) we have
\[ \delta_\pm(s, z) \leq 0 \text{ for all } (s, z) \in [-h, 0] \times \mathbb{R}. \]

And if \((t, z) \in [0, h] \times \mathbb{R}^d\), by (38) and (40) we get
\[
L \delta_\pm(t, z) = \pm \int_{\mathbb{R}^d} K(z - c u - y)\{g(\phi_c(y)) - g(u(t + dh/2, y))\}dy - L r(t) \\
\geq - \frac{|g|_{L^p} |\rho_0|_{L^1/w}}{A_{\lambda} [t + dh/2]^{d/2}} \int_{y \leq \zeta} e^{\lambda y_j} K(z - c u - y)dy \\
- \rho_* \int_{y \geq \zeta} K(z - c u - y)r(t - h)dy - L r(t)
\]
Proof of Theorem 2.8.

5. Proof of Theorem 2.8.

5.1. Monotone case. We begin this section with some results which generalize those found in [31] and [30]. In this section, \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is a monotone function and \( C^1 \) on \( (0,0] \).

**Definition 5.1.** Continuous function \( u_+ : [-h, +\infty) \times \mathbb{R} \to \mathbb{R} \) is called a super-solution for (3), if, for some \( \delta(t,z) \geq 0 \) for all \( (t,z) \in [0, h] \times \mathbb{R}^d \), so that

\[
\delta(t,z) \leq 0 \text{ for } (t,z) \in [h, 0] \times \mathbb{R}^d.
\]

By repeating the process in the intervals \([h, 2h], [2h, 3h], \ldots\) we obtain \( \delta(t,z) \leq 0 \) for all \( (t,z) \in [-h, \infty) \times \mathbb{R}^d \).

\[\square\]

5. Proof of Theorem 2.8.

5.1. Monotone case. We begin this section with some results which generalize those found in [31] and [30]. In this section, \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is a monotone function and \( C^1 \) on \( (0,0] \).

**Definition 5.1.** Continuous function \( u_+ : [-h, +\infty) \times \mathbb{R} \to \mathbb{R} \) is called a super-solution for (3), if, for some \( \delta(t,z) \geq 0 \) for all \( (t,z) \in [0, h] \times \mathbb{R}^d \), so that

\[
\delta(t,z) \leq 0 \text{ for } (t,z) \in [h, 0] \times \mathbb{R}^d.
\]

By repeating the process in the intervals \([h, 2h], [2h, 3h], \ldots\) we obtain \( \delta(t,z) \leq 0 \) for all \( (t,z) \in [-h, \infty) \times \mathbb{R}^d \).

\[\square\]

5. Proof of Theorem 2.8.

5.1. Monotone case. We begin this section with some results which generalize those found in [31] and [30]. In this section, \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is a monotone function and \( C^1 \) on \( (0,0] \).

**Definition 5.1.** Continuous function \( u_+ : [-h, +\infty) \times \mathbb{R} \to \mathbb{R} \) is called a super-solution for (3), if, for some \( \delta(t,z) \geq 0 \) for all \( (t,z) \in [0, h] \times \mathbb{R}^d \), so that

\[
\delta(t,z) \leq 0 \text{ for } (t,z) \in [h, 0] \times \mathbb{R}^d.
\]

By repeating the process in the intervals \([h, 2h], [2h, 3h], \ldots\) we obtain \( \delta(t,z) \leq 0 \) for all \( (t,z) \in [-h, \infty) \times \mathbb{R}^d \).

\[\square\]
and
\[
\frac{\partial \delta_\pm(t, z_\pm)}{\partial z} - \frac{\partial \delta_\pm(t, z_\mp)}{\partial z} = \pm \left( \frac{\partial u_\pm(t, z_\pm)}{\partial z} - \frac{\partial u_\pm(t, z_\mp)}{\partial z} \right) > 0. \tag{45}
\]

We claim that \(\delta_\pm(t, z) \leq 0\) for all \(t \in [0, h]\), \(z \in \mathbb{R}\). Indeed, otherwise there exists \(r_0 > 0\) such that \(\delta(t, z)\) restricted to any rectangle \(\Pi_r = [-r, r] \times [0, h]\) with \(r > r_0\), reaches its maximal positive value \(M_r > 0\) at at some point \((t', z') \in \Pi_r\).

We claim that \((t', z')\) belongs to the parabolic boundary \(\partial \Pi_r\) of \(\Pi_r\). Indeed, suppose on the contrary, that \(\delta(t, z)\) reaches its maximal positive value at some point \((t', z')\) of \(\Pi_r \setminus \partial \Pi_r\). Then clearly \(z' > z_\ast\), because of \((45)\). Suppose, for instance that \(z' > z_\ast\). Then \(\delta(t, z)\) considered on the subrectangle \(\Pi = [z_\ast, r] \times [0, h]\) reaches its maximal positive value \(M_r\) at the point \((t', z') \in \Pi \setminus \partial \Pi\). Then the classical results \([25, \text{Chapter 3, Theorems 5,7}]\) show that \(\delta_\pm(t, z) \equiv M_r > 0\) in \(\Pi\), a contradiction.

Hence, the usual maximum principle holds for each \(\Pi_r\), \(r \geq r_0\), so that we can appeal to the proof of the Phragmèn-Lindelöf principle from \([25]\) (see Theorem 10 in Chapter 3 of this book), in order to conclude that \(\delta_\pm(t, z) \leq 0\) for all \(t \in [0, h]\), \(z \in \mathbb{R}\).

But then we can again repeat the above argument on the intervals \([h, 2h]\), [2\(h, 3h]\), ... establishing that the inequality \(u_-(t, z) \leq u(t, z) \leq u_+(t, z), z \in \mathbb{R}\), holds for all \(t \geq -h\).

\[\Box\]

Now, if \(g\) meets (M) then, as in \([31, \text{formula (16)}\) and \((17)]\), for given \(q^* > 0\), \(q_\ast \in (0, \kappa)\), there are \(\delta^* < \delta_0, \gamma^* > 0\) such that
\[
\begin{align*}
g(u) - g(u - qe^{\gamma h}) & \leq g(1 - 2\gamma), & (u, q, \gamma) \in \Pi_- = [\kappa - \delta^*, \kappa + \delta^*] \times [0, q_\ast] \times [0, \gamma^*]; \quad (46) \\
g(u) - g(u + qe^{\gamma h}) & \geq -g(1 - 2\gamma), & (u, q, \gamma) \in \Pi_+ = [\kappa - \delta^*, \kappa + \delta^*] \times [0, q_\ast] \times [0, \gamma^*]. \quad (47)
\end{align*}
\]

For \(c \geq c_\ast^+\) and a wavefront \(\phi_c\) we fix \(z^\pm = z^\pm(\phi_c)\) such that \(\phi_c(z) \in [\kappa - \delta^*, \kappa + \delta^*]\) for all \(z \geq z^\pm\) and if \(c \leq c_\ast^-\) we fix \(z^- = z^-(\phi_c)\) such that \(\phi_c(z) \in [\kappa - \delta^*, \kappa + \delta^*]\) for all \(z \leq z^-\). Also, for \(\gamma \in (0, g'(0))\) we define \(b^+_\gamma = b^+_\gamma(\phi_c)\) and \(b^-_\gamma = b^-_\gamma(\phi_c)\) by
\[
g'(0) \int_{b^+_\gamma - z^+ - ch}^{+\infty} K(y) dy = g'(0) \int_{-\infty}^{b^-_\gamma - z^- - ch} K(y) dy = g e^{\gamma h}. \tag{48}
\]

**Theorem 5.3.** Suppose that \(g\) is non-decreasing function satisfying (M), \(K\) satisfies (K) and \(\gamma \in (0, \gamma^*)\) satisfies \((46)-(47)\). If for \(\pm c > \pm c_\ast^\pm\) and \(\lambda \in (\lambda_1(c), \lambda_2(c))\) we have
\[
\gamma + p_\lambda \leq e^{\gamma h} q_\lambda \tag{49}
\]
then
\[
u_0(s, z) \leq \phi_c(z) + q_\lambda(z - b), \quad z \in \mathbb{R}, \quad s \in [-h, 0],
\]
with \(q \in (0, q^*)\) and \(\pm b \geq \pm b_\ast^\pm\) implies
\[
u(t, z) \leq \phi_c(z) + q e^{-\gamma t} \eta \lambda(z - b), \quad z \in \mathbb{R}, \quad t \geq -h, \tag{50}
\]
Similarly, the inequality
\[
\phi_c(z) - q_\lambda(z - b) \leq \nu_0(s, z), \quad z \in \mathbb{R}, \quad s \in [-h, 0],
\]
with some \(0 < q \leq q_\ast\) and \(\pm b \geq \pm b_\ast^\pm\) implies
\[
\phi_c(z) - q e^{-\gamma t} \eta \lambda(z - b) \leq \nu(t, z), \quad z \in \mathbb{R}, \quad t \geq -h, \tag{52}
\]
Finally, if $K$ has compact support the conclusions above are true for $\pm c \geq c^*_+ \text{ and } \lambda = \lambda_j(c), j = 1, 2$, by taking $\gamma = 0$ and the weight $\eta\lambda_j(c)(-b)$ with $\pm b \geq b_0^+$ for some $b_0^+ \in \mathbb{R}$.

**Remark 5.4.** It is instructive to compare Theorem 5.3 with [23] for asymptotic stability of non-critical wavefronts. Due to the continuous embedding $H^1_\eta(q)(\mathbb{R}) \subset C^0(\mathbb{R}) \cap C^0_{0.1/2}(\mathbb{R})$ if we take an initial datum $u_0$ like in [23] then $u_0$ is convergent at $+\infty$, so that $u_0(s, +\infty) = \kappa$ uniformly for $s \in [-h, 0]$ and therefore for suitable $q \in (0, q^*)$ the initial datum $u_0$ holds (51). Also, the weight function $\eta(-x_0)$ in [23] is defined by $x_0 = x_0(\phi_\omega)$ ($x_0 \geq b_0^+$ in our case) such that the wavefront $\phi_\omega$ belongs to a suitable neighborhood of $\kappa$. In our case the number $b_0^+$ also depends upon $\gamma > 0$ due to the kernel $K$ could have no compact support which constrain us to do the integral small enough in (48) (see formula (55) below). So that, when $K$ has compact support, in particular, we get the local stability of the critical wavefronts which is a generalization of the local case (compare with [31, Lemma 2]).

**Proof.** Let $c > c^*_+$ be. Set $u_{\pm}(t, z) = \phi_\omega(z) \pm q e^{-\gamma t} \eta\lambda(z - b)$, Then, for $t > 0$ and $z \in \mathbb{R} \setminus \{b\}$, after a direct calculation we find that

$$Nu_{\pm}(t, z) = \pm q e^{-\gamma t} [-\gamma \eta\lambda(z - b) + c\eta^o(z - b) - \eta^o(z - b) + \eta\lambda(z - b)]$$

$$+ \int_{\mathbb{R}} K(y)[g(\phi(z - ch - y)) - g(u_{\pm}(t - h, z - ch - y))] dy.$$  

By (49), it is clear that if $z < b$ it holds that

$$\pm Nu_{\pm}(t, z) \geq q e^{-\gamma t} [e^{\lambda(z - b)}(-\gamma + c\lambda - \lambda^2 + 1) - g'(0) e^{\gamma h} \int_{\mathbb{R}} K(y) e^{\kappa(z - ch - b - y)} dy]$$

$$\geq q e^{-\gamma t + \lambda(z - b)} [-\gamma + c\lambda - \lambda^2 + 1 - g'(0) e^{-\lambda ch + \gamma h} \int_{\mathbb{R}} K(y) e^{-\gamma y} dy] \geq 0.$$  

Similarly, if $c < c^*_-$ we have $\pm Nu_{\pm}(t, z) \geq 0$ for $z > b$ and $t > 0$.

Now, for $c > c^*_+$, $z > b$ and $q \in (0, q^*)$, then

$$\pm Nu_{\pm}(t, z) = q e^{-\gamma t} [-\gamma + 1] \pm [I^+_{\pm}(t, z) + I^\pm_{\pm}(t, z)],$$

where

$$I^+_{\pm}(t, z) = \int_{-\infty}^{b - ch + b} K(y)[g(\phi(z - ch - y)) - g(u_{\pm}(t - h, z - ch - y))] dy,$$  

and

$$I^\pm_{\pm}(t, z) = \int_{-z - ch + b}^{b - ch + b} K(y)[g(\phi(z - ch - y)) - g(u_{\pm}(t - h, z - ch - y))] dy.$$  

If we use formula (47) to estimate $|I^+_{\pm}|$ and (48) to estimate $|I^\pm_{\pm}|$ then for $q > 0$ we have

$$Nu_{\pm}(t, z) \geq q e^{-\gamma t}[1 - \gamma - (1 - 2\gamma)] - g'(0) e^{\gamma h} \int_{-z - ch + b}^{+\infty} K(y) dy] \geq 0,$$

for all $(t, z) \in [-h, \infty) \times [b, +\infty)$. Similarly, if $q \in (0, q^*)$ from (46) and (48) we obtain that

$$-Nu_{\pm}(t, z) \geq 0 \forall (t, z) \in [-h, \infty) \times [b, +\infty).$$

The same arguments are used for $c < c^*_-$ replacing $z^+$ by $z^-$. Next, since

$$\pm \left( \frac{\partial u_{\pm}(t, b^+)}{\partial z} - \frac{\partial u_{\pm}(t, b^-)}{\partial z} \right) = -q \lambda \cdot e^{-\gamma t} < 0,$$
we conclude that $u_{±}(t, z)$ is a pair of super- and sub-solutions for equation (3). So, an application of Lemma 5.2 completes the proof for case $±c > ±c^∗_w$.

Finally, for the case $λ = λ_j$ and $K$ compactly supported we can take $b_0^∗$ large enough in (55) in order to get $I^±_κ = 0$ and therefore the proof of (50) and (52) with $γ = 0$ and $η_{λ_j}(z) = (−b), b ≥ b_0^∗$, is obtained by following the same arguments above. □

**Remark 5.5. [Monotonicity of wavefronts]** Following the abstract setting developed in [19], for $t ≥ 0$ we define $Q_t : [0, κ] → [0, κ]$ (this map is well defined since in Lemma 5.2 we can take $u_− = 0$ and $u_+ = κ$ as $Q_t(u_0)(x) := u(t, x)$ where $u(t, x)$ is the solution to (3) with initial datum $u_0 ∈ [0, κ]$. Next, we note that the hypothesis (K1)-(K5) in [19] are trivially satisfied with $K := [0, κ] ≤ L^∞(R)$ and $O_{n}$ defined by mean $≪_{B}$ (see formula (2.1) of [19, page 861]). Also, for $Q = Q_1$ we see that hypothesis (A1), (A2), (A4) and (A6) in [19] are trivially satisfied. Then, due to Remark 3.5 (with $λ' = 0$) for any family of functions $U$ of $K = [0, κ] ≤ L^∞(R)$, by Arzelà-Ascoli Theorem, we have that $Q_1(U)_I ⊂ K$ is relatively compact and therefore (A3) is satisfied. Finally, note that if in (53) we take $ϕ_c = κ$ and $u_−(t, z) = κ − qe^{−γt}, q ∈ [0, q_∗]$ and $γ ∈ [0, γ_∗]$ satisfying (46)-(47) we have $N u_−(t, z) ≤ 0$ for all $(t, z) ∈ [0, +∞) × R$, so that if we take an initial datum $u_0$ such that $u_0 > 0$ we can find $q_0 ∈ [0, q_∗]$ such that $κ − q_0e^{−γt} ≤ u_0(s, z)$ for all $(s, z) ∈ [−h, 0] × R$ and Lemma 5.2 implies $κ − q_0e^{−γt} ≤ u(t, z)$ for all $(t, z) ∈ [0, +∞) × R$ and therefore the condition (A5) in [19] is satisfied. Thus, by [19, Theorem 4.1 and Theorem 4.2] all wavefronts of (1) are monotone if $g$ is monotone. Note that the hypothesis $L_g = g′(0)$ is no mandatory by using these arguments.

5.2. Attractivity of an optimal neighborhood of $κ$.

**Lemma 5.6 (Comparison Lemma).** Let consider $g_1$ and $g_2$ satisfying (L) and $K_1, ξ_λ K_2 ∈ L^1(\mathbb{R})$, some $λ ∈ \mathbb{R}$. Suppose that for some $R ∈ \mathbb{R}_+ \cup \{+∞\}$:

$$K_1(y)g_1(u) ≤ K_2(y)g_2(u) \text{ for all } (y, u) ∈ \mathbb{R} × (-∞, R).$$  \hspace{1cm} (56)

Denote by $v_1$ and $v_2$ the solutions to (3), generated by the initial data $v_1^0$ and $v_2^0$ with $Kg = K_1g_1$ and $Kg = K_2g_2$, respectively.

Moreover, if $R < +∞$ we suppose

$$v_2(t, z) ≤ R \text{ for all } (t, z) ∈ [-h, +∞) × \mathbb{R},$$  \hspace{1cm} (57)

while if $R = +∞$ we suppose

$$|v_2^0(s, z)| ≤ Ne^{λz} \text{ for all } (s, z) ∈ [-h, 0] × \mathbb{R},$$  \hspace{1cm} (58)

for some $N > 0$.

If $g_1$ or $g_2$ is a non-decreasing function, then

$$0 ≤ v_1^0(s, z) ≤ v_2^0(s, z) \text{ for all } (s, z) ∈ [-h, 0] × \mathbb{R},$$ \hspace{1cm} (59)

implies

$$v_1(t, z) ≤ v_2(t, z) \text{ for all } (t, z) ∈ \mathbb{R}_+ × \mathbb{R}.$$  

**Proof.** We take $δ(t, z) = v_1(t, z) − v_2(t, z)$. Let us note that if $(t, z) ∈ [0, h] × \mathbb{R}$ then

$$\mathcal{L}δ(t, z) = \int_{\mathbb{R}} K_2(y)g_2(v_2(t, z) − ch) dy − \int_{\mathbb{R}} K_1(y)g_1(v_1(t, z) − ch) dy.$$  

If $g_2$ is a non-decreasing function by (59) we have

$$\mathcal{L}δ(t, z) ≥ \int_{\mathbb{R}} K_2(y)g_2(v_1(t, z) − ch) dy − \int_{\mathbb{R}} K_1(y)g_1(v_1(t, z) − ch) dy.$$
But, (59) and (57) imply \( v_1(t-h, -) \leq R \) so by (56)
\[
L \delta(t, z) \geq 0.
\]

Analogously, if \( g_1 \) is a non-decreasing function, we have
\[
L \delta(t, z) \geq \int_{\mathbb{R}} K_2(y) g_2(v_2(t-h, z - ch - y)) dy - \int_{\mathbb{R}} K_1(y) g_1(v_2(t-h, z - ch - y)) dy \geq 0.
\]

Finally, by using Proposition 3.1 with \( d_3(t, x) := g_j(v_j(t-h, x-ch))/v_j(t-h, x-ch) \), \( j = 1, 2 \), we conclude that the function \( \delta(t, z) \) is exponentially bounded on \([0, h] \times \mathbb{R} \). Then, since \( \delta(0, z) \leq 0 \) for all \( z \in \mathbb{R} \), the Phragmèn-Lindelöf principle [25] (Chapter 3, Theorem 10) implies that \( \delta(t, z) \leq 0 \) for \( (t, z) \in [0, h] \times \mathbb{R} \). The argument is repeated for intervals \([h, 2h], [2h, 3h] \)...

\[\square\]

**Lemma 5.7.** If \( v(t, z) \) is a solution of (3) with initial datum \( u_0(s, z) \) satisfying (IC) and such that
\[
u_\infty := \sup_{(t, z) \in [-h, +\infty) \times \mathbb{R}} u(t, z) < \infty.
\]

then there exit \( \sigma' > 0 \) and \( z'_0 \in \mathbb{R} \) such that
\[
u(t, z) \geq \sigma' \quad \forall (t, z) \in [-h, \infty) \times [z'_0, \infty).
\]

**Proof.** Fix \( N > 0 \) and define \( K^-(z) := \tau_{\chi_{[-N, N]}} K(z) \) for some \( \tau \geq 1 \) such that \( |K^-|_{L^1} = 1 \). Next, define the monotone function \( g^- : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) by \( g^-(u) = \tau^{-1} \min_{x \in [u, u_\infty]} g(x) \) and \( g^-(u) = g(u_\infty) \) for \( u \geq u_\infty \). Clearly, \( g^- \) holds (M) with positive equilibrium \( \kappa^- = \min_{x \in [0, u_\infty]} g(x) \) and \( Lg^- = \tau^{-1} g'(0) \). Also, \( g^-(u) \leq g(u) \) for all \( u \in [0, u_\infty] \).

Next, by denoting \( q_\ast := \kappa^- - \sigma > 0 \) without loss of generality, due to the asymptotic behavior of wavefronts in \( -\infty \) (see [1, Theorem 3 and Theorem 7]) there are a wavefront (which is monotone by Remark 5.5) \( \phi^- \) to (3) (where \( K \) and \( g \) are replaced by \( K^- \) and \( g^- \), respectively) such that
\[
\phi^-_c(z) - q_s \eta_{\lambda_j(c)}(z - b) \leq 0 \leq u_0(s, \pm z) \quad \text{for all } (s, z) \in [-h, 0] \times (-\infty, z_0].
\]

where \( \pm b \geq \pm b_0^\pm \) and \( j = 1 \) if and only if \( c \geq c^+_\ast \). By (IC) we also have
\[
\phi^-_c(z) - q_s \eta_{\lambda_j(c)}(z - b) \leq u_0(s, \pm z) \quad \text{for all } (s, z) \in [-h, 0] \times [z_0, +\infty).
\]

Thus, for all \( \pm c \geq \pm c^\pm_\ast \) we get
\[
\phi^-_c(z) - q_s \eta_{\lambda_j(c)}(z - b) \leq u_0(s, \pm z) \quad \text{for all } (s, z) \in [-h, 0] \times \mathbb{R}
\]

Now, denote by \( u^-(t, z) \) the solution to (3), with \( g = g^-, K = K^- \) and \( u^-(s, z) = u_0(s, z) \) for \( (s, z) \in [-h, 0] \times \mathbb{R} \). Then, because of Theorem 5.3 for \( \pm b \geq \pm b_0^\pm \) we have
\[
\phi^-_c(z) - q_s \eta_{\lambda_j(c)}(z - b) \leq u^-(t, z) \quad \forall (t, z) \in [-h, \infty) \times \mathbb{R}
\]

Thus, there are \( \sigma' > 0 \) and \( z'_0 \in \mathbb{R} \) such that
\[
\nu^-(t, \pm z) \geq \sigma' \quad \forall (t, z) \in [-h, \infty) \times [z'_0, \infty), \quad \pm c \geq \pm c^\pm_\ast.
\]

However, Lemma 5.6 (with \( R = u_\infty \)) implies
\[
\nu(t, \pm z) \geq u^-(t, \pm z) \quad \forall (t, z) \in [-h, \infty) \times \mathbb{R}, \quad \pm c \geq \pm c^\pm_\ast.
\]

Thus, the Lemma follows from (63) and (65).

\[\square\]
Now, we prove a key result in order to obtain our global stability result

**Lemma 5.8.** If $u_0$ satisfies (IC) then for each $\epsilon \in (0, m_g)$ there exist $T_\epsilon = T_\epsilon(u_0) > 0$ such that $\pm c \geq \pm c^\pm_\epsilon$ implies

$$m_g - \epsilon \leq u(t, z) \leq M_g + \epsilon \quad \text{for all } (t, z) \in [T_\epsilon, \infty)^2.$$  

(66)

**Proof.** We define $m_\epsilon := \min_{x \in [\kappa, M_\epsilon + \epsilon/2g'(0)]} g(x)$, 

$$\bar{g}(u) := \max_{x \in [0, u]} g(x) \quad \text{and} \quad \underline{g}(u) := \min_{x \in [u, M_\epsilon + \epsilon/2g'(0)]} g(x).$$

It is clear that these monotone functions (we define $g(u) = g(M_g + \epsilon/2g'(0))$ for all $u \geq M_g + \epsilon/2g'(0)$) satisfy (M) with positive equilibrium $M_g$ and $m_\epsilon$, respectively. Also, $L_\bar{g} = L_\underline{g} = \bar{g}(0)$ and

$$\bar{g}(u) \leq \bar{g}(u) \text{ for all } u \geq 0 \quad \text{and} \quad \underline{g}(u) \leq g(u) \text{ for all } u \in [0, M_g + \epsilon/2g'(0)].$$  

(67)

By denoting $\bar{u}(t)$ as the homogeneous solution to (3) with $g = \bar{g}$ and initial datum $\bar{u}(s) = |u_0|_{L^\infty}$ for $s \in [-h, 0]$, because of Lemma 5.6 (with $R = +\infty$) we have

$$u(t, z) \leq \bar{u}(t) \quad \forall (t, z) \in [-h, \infty) \times \mathbb{R}.$$  

(68)

Thus, because of $M_g$ is the global attractor to $\bar{g}$ there is $t_\epsilon > 0$ such that

$$u(t, z) \leq M_g + \epsilon/2g'(0) \quad \forall (t, z) \in [t_\epsilon, \infty) \times \mathbb{R},$$  

(69)

Now, we proceed to obtain the lower estimation. Denoting by $u(t, z)$ the solution to (3) with $g = \bar{g}$ and initial datum $\bar{u}(s, z) = u(s + h + t_\epsilon, z)$ for $(s, z) \in [-h, 0] \times \mathbb{R}$, then by (69) and Lemma 5.6 (with $R = M_g + \epsilon/2g'(0)$)

$$u(t + t_\epsilon + h, z) \geq u(t, z) \quad \forall (t, z) \in [0, \infty) \times \mathbb{R}.$$  

(70)

Next, by (69) we have

$$u_\infty := \sup_{(t, z) \in [-h, \infty) \times \mathbb{R}} u(t, z) < \infty,$$

so that Lemma 5.7 implies there exist $\sigma' > 0$ and $z'_0 \in \mathbb{R}$ such that

$$u(t, z) \geq \sigma' \quad \forall (t, z) \in [-h, +\infty) \times [z'_0, \infty)$$

(71)

Then, we define

$$0 < \alpha := \frac{m_\epsilon - \epsilon/4}{\bar{g}(m_\epsilon - \epsilon/4)} < 1,$$

for $\epsilon$ enough small such that $g_\alpha := \alpha g$ satisfies (M) with positive equilibrium $m_\epsilon - \epsilon/4$. Next, we set $\beta(t)$ as the solution to the problem

$$\begin{align*}
\beta'(t) &= -\beta(t) + g_\alpha(\beta(t - h)) \quad t > 0 \\
\beta(s) &= \sigma' \quad s \in [-h, 0],
\end{align*}$$

Without restriction: $\sigma' < m_\epsilon - \epsilon/4$. So by [28, Corollary 2.2, p. 82] $\beta(t)$ converges monotonetly to $m_\epsilon - \epsilon/4$.

Then, by Theorem 2.1 and Proposition 2.6, for each $N > 0$ there are $t_N > t_\epsilon$ and $z' \geq z'_0$ such that

$$u(t + t_N, \pm z) \geq m_\epsilon - \frac{\epsilon}{4} > \beta(t) \quad \forall (t, z) \in [0, \infty) \times [z', z' + N], \quad \pm c \geq \pm c^\pm_\epsilon.$$  

(72)

Now we consider $c \geq c^\pm_\epsilon$ and we fix $N$ large enough, such that

$$\alpha \leq \int_{-\infty}^{N-\epsilon ch} K(y)dy.$$  

(73)
We define $\delta(t, z) := \beta(t) - w(t + t_N + h, z)$. So, by (71) we obtain

$$\delta(s, z) \leq 0 \quad \forall (s, z) \in [-h, 0) \times [z', \infty),$$ (74)

and for $(t, z) \in [0, h] \times [z' + N, \infty)$ because of (74) and (73) we have

$$\mathcal{L} \delta(t, z) = \int_\mathbb{R} K(y)g(w(t + t_N, z - ch - y))dy - \alpha g(\beta(t - h)) \geq \int_{-\infty}^{N-ch} K(y)[g(w(t + t_N, z - ch - y)) - g(\beta(t - h))]dy \geq 0.$$

So by (72) and Phragmèn-Lindelöf principle we conclude that

$$\delta(t, z) \leq 0 \quad \forall (t, z) \in [0, h) \times [z' + N, \infty),$$ (75)

Therefore using again (72) and (75) instead of (74) we can repeat the process for

$$\beta(t) \leq w(t + t_N + h, z) \quad \forall (t, z) \in [0, \infty) \times [z', \infty).$$ (76)

Finally, by (70) and (76) there exist $T_\epsilon(u_0) \geq t_N + t_\epsilon$

$$m_g - g'(0)\epsilon/2g'(0)] - \epsilon/2 \leq m_\epsilon - \epsilon/2 \leq u(t, z) \quad (t, z) \in [T_\epsilon(u_0), \infty)^2.$$

Otherwise, for $c \leq c^-_\epsilon$ if we use the inequality (65) and the same function $\beta(t)$ then the situation is completely analogous and therefore (66) can be obtained. \(\square\)

5.3. Proof of Theorem 2.8. We will give the proof to the case $c \geq c^+_\epsilon$ since the proof for the case $c \leq c^-_\epsilon$ is completely analogous.

Proof. (i) We take small $\epsilon_0 > 0$ such that $\rho_{\epsilon_0} := L_g([m_g - \epsilon_0, M_g + \epsilon_0]) < 1$ and

$$\rho_{\epsilon_0} e^{\gamma_{\epsilon_0} t} < 1 - \gamma_{\epsilon_0}.$$

Note that by (7) we get

$$e^{-\lambda z} |u(t, z) - \phi_c(z)| \leq |r_0| L_{h, \lambda} e^{-\gamma_{\epsilon_0} t} \quad \forall t > h, z \in \mathbb{R}$$ (77)

Now, we consider a function $r : [-h, +\infty) \rightarrow \mathbb{R}_+$ given by $r(t) := q_0 e^{-\gamma_{\epsilon_0} t}$ where $q_0 \geq m_g + M_g$ will be fixed below. Then, for

$$T_0 := \max\{T_{\epsilon_0}(u_0), T_{\epsilon_0}(\phi_c), h\},$$

(according to Lemma 5.8) we define $\delta_{\pm}(t, z) := \pm [u(t + T_0, h, z) - \phi_c(z)] - r(t)$. So, by (69) we obtain

$$\delta_{\pm}(s, z) \leq 0 \quad \text{for } (s, z) \in [-h, 0] \times \mathbb{R}.$$

And, if $(t, z) \in [0, h] \times \mathbb{R}$ then by (77) and Lemma 5.8 we get

$$\mathcal{L} \delta_{\pm}(t, z) = \pm \int_\mathbb{R} K(z - ch - y)[g(\phi_c(z)) - g(u(t + T_0, y))] - \mathcal{L}r(t)$$ (78)

$$\geq -\frac{g'(0)|r_0| L_{h, \lambda} e^{-\gamma_{\epsilon_0} t}}{A_h \sqrt{t + T_0}} \int_{-\infty}^{T_0} e^{\lambda y} K(z - ch - y)dy$$

$$+ \rho_{\epsilon_0} \int_{T_0}^{+\infty} K(z - ch - y)r(t - h)dy - \mathcal{L}r(t).$$
Thus,
\[ \mathcal{L}\delta_\pm(t, z) \geq -q_0 e^{-\gamma_t} \left[ \frac{g'(0)|r_0|L_{1, c\lambda_0}^{-1} e^{-\gamma_{T_0}}}{q_0 A_h \sqrt{t + T_0}} + e^{\gamma_h \rho_{c_0}} \int_{T_0}^{+\infty} K(z - ch - y) dy - 1 + \gamma_s \right] \]

Now, in the last inequality since \( \rho_{c_0} e^{\gamma_h} < 1 - \gamma_s \) we can choose \( q_0 \) large enough such that \( \mathcal{L}\delta_\pm(t, z) \geq 0 \) for all \((t, z) \in [0, h] \times \mathbb{R}\), so that Phragmèn-Lindelöf principle implies \( \delta_\pm(t, z) \leq 0 \) for \((t, z) \in [0, h] \times \mathbb{R}\).

Analogously, by using (77) and Lemma 5.8 it is possible to repeat the process for the intervals \([h, 2h], [2h, 3h], \ldots\) in order to obtain \( \delta(t, z) \leq 0 \) for all \((t, z) \in [-h, \infty) \times \mathbb{R}\).

Finally, by Proposition 3.2, with \( p = \infty \) and \( \lambda_0 = 0 \), we obtain
\[ \sup_{(t, z) \in [-h, T_0] \times \mathbb{R}} |u(t, z) - \phi_c(z)| \leq q_0^{[T_0/h] + 1} \]

By taking \( C = \max\{q_0, qD[T_0/h] + 1 e^{\gamma_{T_0}}\} \) the result is followed.

(ii) We take \( \epsilon_0 > 0 \) such that \( \rho_{c_0} := L_g([m_g - \epsilon_0, M_g + \epsilon_0]) < 1 \) and choose \( d_\ast > h \) satisfying
\[ \frac{\rho_{c_0} \sqrt{t + d_\ast}}{\sqrt{t + d_\ast - h}} + \frac{1}{2(t + d_\ast)} < 1 \quad t \geq -h. \tag{79} \]

Now, we consider \( r : [-h, +\infty) \to \mathbb{R}_+ \) given by \( r(t) := q_0 / \sqrt{t + d_\ast} \), where \( q_0 \geq (m_g + M_g) / \sqrt{t_0} \) will be fixed below, and for \( T_0' = \max\{T_0, d_\ast\} \) (\( T_0' \) is taken like in part (i)) we define \( \delta_\pm(t, z) := \pm[u(t + T_0' + h, z) - \phi_c(z)] - r(t) \).

So, by (66) we have
\[ \delta_\pm(s, z) \leq 0 \quad \text{for all } (s, z) \in [-h, 0] \times \mathbb{R}. \]

And if \((t, z) \in [0, h] \times \mathbb{R}\), by (7) and Lemma 5.8
\[
\mathcal{L}\delta_\pm(t, z) = \pm \int_{\mathbb{R}} K(z - ch - y)[g(\phi_c(y)) - g(u(t + T_0', y))] - \mathcal{L}r(t) \\
\geq -[g'(0)|r_0|L_{1, c\lambda_0}^{-1} A_h \sqrt{t + T_0'} e^{\gamma_h y} K(z - ch - y) dy \\
+ \rho_{c_0} \int_{T_0'}^{+\infty} K(z - ch - y)r(t - h) dy + \mathcal{L}r(t)] \\
\geq - \frac{q_0}{\sqrt{t + d_\ast}} \left[ g'(0)|r_0|L_{1, c\lambda_0}^{-1} A_h \sqrt{t + T_0'} + \rho_{c_0} \frac{\sqrt{t + d_\ast}}{\sqrt{t + d_\ast - h}} - 1 + \frac{1}{2(t + b)} \right].
\]

However, by (79), in the last inequality we can choose \( q_0 \) large enough such that \( \mathcal{L}\delta_\pm(t, z) \geq 0 \) for all \((t, z) \in [0, h] \times \mathbb{R}\), so that Phragmèn-Lindelöf principle implies \( \delta_\pm(t, z) \leq 0 \) for \((t, z) \in [0, h] \times \mathbb{R}\). Repeating the process in the intervals \([h, 2h], [2h, 3h], \ldots\) we obtain \( \delta(t, z) \leq 0 \) for all \((t, z) \in [-h, \infty) \times \mathbb{R}\). The rest of the proof is similar to part (i).

\[ \square \]

**Acknowledgments.** It is a pleasure to thank Rafael Benguria for many valuable discussions. This work has been supported by Fondecyt (Chile) Project 3160473.
REFERENCES

[1] M. Aguerrea, C. Gomez and S. Trofimchuk, On uniqueness of semi-wavefronts, Math. Ann., 354 (2012), 73–109.
[2] M. Bani-Yaghoub, The traveling wavefront for a nonlocal delayed reaction-diffusion equation, J. Appl. Math. Comput., 53 (2017), 77–94.
[3] R. Benguria and A. Solar, An estimation of level set for a non-local KPP equation with delay, Nonlinearity, 32 (2019), 777–799.
[4] R. Benguria and A. Solar, An iterative estimation for disturbances of semi-wavefronts to the delayed Fisher-KPP equation, Proc. Amer. Math. Soc., 147 (2019), 2495-2501.
[5] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolutions equations, Advances in Differential Equations, 2 (1997), 125–160.
[6] I.-L. Chern, M. Mei, X. Yang and Q. Zhang, Stability of non-monotone critical traveling waves for reaction-diffusion equation with time-delay, J. Diff. Eqns, 259 (2015), 1503–1541.
[7] U. Ebert and W. van Saarloos, Front propagation into unstable states: Universal algebraic convergence towards uniformly translating pulled fronts, Phys. D, 146 (2000), 1–99.
[8] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs (N.J., USA), 1964.
[9] T. Gallay, Local stability of critical fronts in nonlinear parabolic partial differential equations, Nonlinearity, 7 (1994), 741–764.
[10] A. Gomez and S. Trofimchuk, Global continuation of monotone wavefronts, J. Lond. Math. Soc., 89 (2014), 47–68.
[11] C. Gomez, H. Prado and S. Trofimchuk, Separation dichotomy and wavefronts for a nonlinear convolution equation, J. Math. Anal. Appl., 420 (2014), 1–19.
[12] S. A. Gourley, J. So and J. Wu, Non-locality of reaction-diffusion equations induced by delay: Biological modeling and nonlinear dynamics, J. Math. Sci., 124 (2004), 5119–5153.
[13] A. Halanay, Differential Equations: Stability, Oscillations, Time Lags, Academic Press, New York (N.Y., USA), 1966.
[14] S.-B. Hsu and X.-Q. Zhao, Spreading speeds and traveling waves for nonmonotone integrodifference equations, SIAM J. Math. Anal., 40 (2008), 776–789.
[15] R. Huang, M. Mei and Y. Wang, Planar traveling waves for nonlocal dispersion equation with monostable nonlinearity, Discret Contin. Dyn. Syst., 32 (2012), 3621–3649.
[16] R. Huang, M. Mei, K. Zhang and Q. Zhang, Asymptotic stability of non-monotone traveling waves for time-delayed nonlocal dispersion equations, Discrete Contin. Dyn. Syst., 36 (2016), 1331–1353.
[17] K. Kirchgassner, On the nonlinear dynamics of travelling fronts, J. Diff. Eqns., 96 (1992), 256–278.
[18] A. Kolmogorov, I. Petrovskii and N. Piskunov, Étude de l’équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, (French) [Study of a Diffusion Equation That Is Related to the Growth of a Quality of Matter and Its Application to a Biological Problem], Moscow Univ. Bull. Math., 1 (1937), 1–25.
[19] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Funct. Anal., 259 (2010), 857–903.
[20] C.-K. Lin, C.-T. Lin, Y. Lin and M. Mei, Exponential stability of nonmonotone traveling waves for Nicholson’s blowflies equation, SIAM J. Math. Anal., 46 (2014), 1053–1084.
[21] G. Lv and M. Wang, Nonlinear stability of travelling wave fronts for delayed reaction diffusion equations, Nonlinearity, 23 (2010), 845–873.
[22] S. Ma and J. Wu, Existence, uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation, J. Dyn. Diff. Eqns., 19 (2007), 391–436.
[23] M. Mei, Ch. Ou and X.-Q. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations, SIAM J. Math. Anal., 42 (2010), 2762–2790.
[24] M. Mei, K. Zhang and Q. Zhang, Global stability of critical traveling waves with oscillations for time-delayed reaction-diffusion equation, Int. J. Numer. Anal. Model., 16 (2019), 375–397.
[25] M. Protter and H. Weinberger, Maximum Principles in Differential Equations, Prentice Hall, Englewood Cliffs (N.J., USA), 1967.
[26] D. Sattinger, On the stability of waves of nonlinear parabolic systems, Adv. Math., 22 (1976), 312–355.
[27] K. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, Trans. Am. Math. Soc., 302 (1987), 587–615.
[28] H. Smith, Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems, AMS, Providence (R.I. USA), 1995.
[29] J. W.-H. So, J. Wu and X. Zou, A reaction-diffusion model for a single species with age structure I. Travelling wavefronts on unbounded domains, Proc. R. Soc. A, 457 (2001), 1841–1853.
[30] A. Solar, Stability of semi-wavefronts for delayed reaction-diffusion equations, preprint, arXiv:1704.03011.
[31] A. Solar and S. Trofimchuk, Speed selection and stability of wavefronts for delayed monostable reaction-diffusion equations, J. Dyn. Diff. Eqns., 28 (2016), 1265–1292.
[32] A. Stokes, On two types of moving front in quasilinear diffusion, Math. Biosciences, 31 (1976), 307–315.
[33] H. Thieme and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models, J. Diff. Eqns., 195 (2003), 430–470.
[34] E. Trofimchuk, P. Alvarado and S. Trofimchuk, On the geometry of wave solutions of a delayed reaction-diffusion equation, J. Diff. Eqns., 246 (2009), 1422–1444.
[35] E. Trofimchuk, M. Pinto and S. Trofimchuk, Monotone waves for non-monotone and non-local monostable reaction-diffusion equations, J. Diff. Eqns, 261 (2016), 1203–1236.
[36] K. Uchiyama, The behavior of solutions of some nonlinear diffusion equations for large time, J. Math. Kyoto Univ., 18 (1978), 453–508.
[37] Z. Wang, W. Li and S. Ruan, Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, J. Diff. Eqns., 238 (2007), 153–200.
[38] Z. Wang, W. Li and S. Ruan, Travelling fronts in monostable equations with nonlocal delayed effects, J. Dyn. Differ. Eqns., 20 (2008), 573–607.
[39] H. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal., 13 (1982), 353–396.
[40] S.-L. Wu, W.-T. Li and S.-Y. Liu, Oscillatory waves in reaction-diffusion equations with nonlocal delay and crossing-monostability, Non. Lin. Anal.: Real World App., 10 (2009), 3141–3151.
[41] T. Xu, S. Ji, M. Mei and J. Yin, Theoretical and numerical studies on global stability of traveling waves with oscillations for time-delayed nonlocal dispersion equations, preprint, arXiv:1810.07484.
[42] T. Yi and X. Zou, Asymptotic behavior, spreading speeds, and traveling waves of nonmonotone dynamical systems, SIAM J. Math. Anal., 47 (2015), 3005–3034.
[43] T. Yi and X. Zou, Map dynamics versus dynamics of associated delay reaction-diffusion equations with a Neumann condition, Proc. R. Soc. A, 466 (2010), 2955–2973.
[44] T. Yi, Y. Chen and J. Wu, Unimodal dynamical systems: Comparison principles, spreading speeds and travelling waves, J. Diff. Eqns, 254 (2013), 3538–3572.