FREE SUBGROUPS OF ONE-RELATOR RELATIVE PRESENTATIONS
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Suppose that $G$ is a nontrivial torsion-free group and $w$ is a word over the alphabet $G \cup \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$. It is proved that for $n \geq 2$ the group $\widetilde{G} = \langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle$ always contains a nonabelian free subgroup. For $n = 1$ the question about the existence of nonabelian free subgroups in $\widetilde{G}$ is answered completely in the unimodular case (i.e., when the exponent sum of $x_1$ in $w$ is one). Some generalisations of these results are discussed.

Key words: relative presentations, one-relator groups, free subgroups.

MSC: 20F05, 20E06, 20E07.

0. Introduction
The following theorem was stated in [Ma32]; as far as we know, the proof appeared for the first time in [Mo69].

Free subgroup theorem for one-relator groups. A one-relator group

$$\langle x_1, x_2, \ldots, x_n \mid w = 1 \rangle$$

contains no nonabelian free subgroups if and only if it is either cyclic or isomorphic to the Baumslag–Solitar group

$$G_{1,k} = \langle x, y \mid y^{-1}xy = x^k \rangle$$

for some $k \in \mathbb{Z} \setminus \{0\}$.

We try to solve the same problem for relative one-relator presentations, i.e., for groups of the form

$$\widetilde{G} = \langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle \overset{\text{def}}{=} (G * F(x_1, x_2, \ldots, x_n))/\langle \langle w \rangle \rangle.$$

Here $G$ is a group and $w$ is an element of the free product of $G$ and the free group $F(x_1, \ldots, x_n)$. In this paper, we assume that the group $G$ is torsion-free.

In the case $n \geq 2$, the answer is as expected.

Theorem 1. If $G$ is a nontrivial torsion-free group and $n \geq 2$, then the group $\widetilde{G} = \langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle$ contains a nonabelian free subgroup.

Note that the existence of free subgroups in $\widetilde{G}$ for $n \geq 3$ follows immediately from the free subgroup theorem for one-relator groups. Thus, Theorem 1 is nontrivial only for $n = 2$.

The most difficult case is $n = 1$. An important role in this situation is played by the exponent sum of the generator in the relator. A word $w = \prod g_i t_i^s \in G * (t)_{\infty}$ is called unimodular if $\sum s_i = 1$.

If the exponent sum of the generator in the word $w$ equals to any number $p \neq \pm 1$, then it is unknown even whether the group $G$ embeds into $\widetilde{G} = \langle G, t \mid w = 1 \rangle$; in other words, it is unknown when the group $\widetilde{G}$ is different from $\mathbb{Z}/p\mathbb{Z}$. There are a lot of papers on this subject, but the answer is known only under additional strong restrictions on the group $G$ or/and on the word $w$ (see, e.g., [BS4], [KP95], [C02], [C03], [CG95], [CG00], [EH91], [FeR98], [GR62], [IK00], [Le62], [Ly80], [S87]). For this reason, in this paper we study only unimodular presentations.

The injectivity of the natural mapping $G \to \widetilde{G}$ in the unimodular case was proved in [Kl93] (see also [FeR96]). More delicate properties of the group $\widetilde{G}$ can be found in [CR01], [FoR03], [Kl06], and [Kl05].

Theorem 2. If $G$ is a torsion-free group and a word $w \in G * (t)_{\infty}$ is unimodular, then the group $\widetilde{G} = \langle G, t \mid w = 1 \rangle$ contains a nonabelian free subgroup, except in the following two cases:

1) $w \equiv g_1 t_2 g_2$, where $g_1, g_2 \in G$ (so $\widetilde{G} \cong G$), and the group $G$ contains no nonabelian free subgroups;
2) the group $G$ is cyclic and $\widetilde{G}$ is isomorphic to the Baumslag–Solitar group $G_{1,2} = \langle g, t \mid g^{-1}tg = t^2 \rangle$.

In [Kl06], we suggested a generalisation of the notion of unimodularity to the case when the word $w$ is an element of the free product of a group $G$ and any (not necessarily cyclic) group $T$. A word $w \equiv g_1 t_1 \ldots g_n t_n \in G * T$ is called unimodular if

1) $\prod t_i$ is an element of infinite order in the group $T$;
2) the cyclic subgroup $\langle \prod t_i \rangle$ is normal in $T$;
3) the quotient group $T/\langle \prod t_i \rangle$ is a group with the strong unique-product property.

Recall that a group $H$ is called a UP-group, or a group with the unique product property, if the product $XY$ of any two finite nonempty subsets $X, Y \subseteq H$ contains at least one element, which decomposes uniquely into the product

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of an element from $X$ and an element from $Y$. Some time ago, there was the conjecture that any torsion-free group is UP (the converse is, obviously, true). However, it turned out that there exist counterexamples ([P88], [RS87]).

We say that a group $H$ has the strong unique product property if the product $XY$ of any two finite nonempty subsets $X, Y \subseteq H$ such that $|Y| \geq 2$ contains at least two uniquely decomposable elements $x_1y_1$ and $x_2y_2$ such that $x_1, x_2 \in X$, $y_1, y_2 \in Y$, and $y_1 \neq y_2$.

As far as we know, all known examples of UP-groups have the strong UP-property. In particular, right orderable groups, locally indicable groups, diffuse groups in the sense of Bowditch have the strong UP property.

The following theorem, on the one hand, generalises (to be more precise, complements) Theorem 2, and on the other hand, is a key step in the proof of Theorem 1.

**Theorem 3.** If a group $G$ is torsion-free, a group $T$ is noncyclic, and a word $w \in G \ast T$ is unimodular, then the group $\tilde{G} = \langle G, T \mid w = 1 \rangle$ contains no nonabelian free subgroups if and only if $G$ is cyclic, $T$ contains no nonabelian free subgroups, and $w$ is conjugate in $G \ast T$ to a word of the form $gt$, where $t \in T$ and $g$ is a generator of $G$.

The notation which we use is mainly standard. If $x$ and $y$ are elements of a group, and $X$ is a subset of this group, then $x^y$ means $y^{-1}xy$, commutator $[x, y]$ is understood as $x^{-1}y^{-1}xy$, and the symbols $(X)$ and $\langle X \rangle$ denote, respectively, the subgroup generated by the set $X$ and the normal subgroup generated by the set $X$. The symbol $|X|$ denotes the cardinality of $X$.

1. Proof of Theorem 1

Suppose that the word $w$ has the form $w = g_1x_1^{\varepsilon_1}g_2x_2^{\varepsilon_2}\ldots g_p x_p^{\varepsilon_p}$ and the word $w' \in F(x_1, \ldots, x_n)$ is obtained from $w$ by erasing coefficients: $w' = x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_p^{\varepsilon_p}$.

**Case 1:** $w'$ is a proper power in the free group $F(x_1, \ldots, x_n)$. In this case, $\tilde{G}$ contains a nonabelian free subgroup, because, according to the free subgroup theorem for one-relator groups, a nonabelian free subgroup exists in the one-relator group $T_1 = \langle x_1, \ldots, x_n \mid w' = 1 \rangle$, which is a homomorphic image of $G$.

**Case 2:** $w'$ is not a proper power. Consider the groups

$$T = \langle x_1, \ldots, x_n \mid [x_1, w'] = \ldots = [x_n, w'] = 1 \rangle \quad \text{and} \quad T_1 = \langle x_1, \ldots, x_n \mid w' = 1 \rangle = T / \langle w' \rangle.$$ 

The group $T$ is the free central extension of the one-relator group $T_1$. It is well known that, if $w'$ is not a proper power in the free group $F(x_1, \ldots, x_n)$, then the group $T_1$ is locally indicable ([B84]) and, therefore, has the strong unique product property. The element $w'$ has infinite order in the group $T$ (see [LS77]). Thus, the word $w$ considered as an element of the free product $G \ast T$ is unimodular. The group $T$ is noncyclic, because its commutator quotient is a free abelian group of rank $n$ and $n \geq 2$. It remains to note that the group $\langle G, T \mid w = 1 \rangle$ is a homomorphic image of $G$ and, thus, the assertion of Theorem 1 follows immediately from Theorem 3.

2. Proof of Theorem 2

Following [FoR03], we say that an element $v \in G \ast \langle t \rangle_\infty$ with cyclically reduced form $v = g_1t^{\varepsilon_1}\ldots g_nt^{\varepsilon_n}$, where $\varepsilon_i \in \{\pm 1\}$ and $g_i \in G$, has complexity 0 if all exponents $\varepsilon_i$ are equal (i.e., the word is either positive or negative); we say that the complexity of the word $v$ is 1 if the cyclic sequence $(\varepsilon_1, \ldots, \varepsilon_n)$ contains both positive and negative exponents, but either two successive exponents are never both positive, or two successive exponents are never both negative. In all other cases, we say that the complexity of $v$ is higher than 1. The complete definition can be found in [FoR03].

**Minimal complexity theorem** [FoR03]. If a group $G$ is torsion-free, a cyclically reduced word $w \in G \ast \langle t \rangle_\infty$ is unimodular and the complexity of a word $v \in G \ast \langle t \rangle_\infty$ is lower than that of $w$, then $v \neq 1$ in the group $\tilde{G} = \langle G, t \mid w = 1 \rangle$.

This theorem implies immediately that, in the case when the complexity of $w$ is higher than 1, $\tilde{G}$ contains the free square of the group $G$ (because $\langle G, G' \rangle = G \ast G'$) and, hence, a nonabelian free subgroup.

It remains to consider a presentation with the relator of complexity 1:

$$\tilde{G} = \langle G, t \mid ct \prod_{i=0}^{m} (b_{i}a_{i}^{c}) = 1 \rangle,$$  

where $m \geq 0$, $a_{i}, b_{i} \in G \setminus \{1\}$, $c \in G$.  

(1)

In this case, we use the following lemma.
Lemma [Kl05, Lemma 22]. If a group $G$ is torsion-free and $\bar{G}$ has presentation (1), then there exists a $d \in \{2,3\}$ such that

$$u \equiv \prod_{i=1}^{s} y_{i}x_{i}^{t_{i}} \neq 1 \text{ in } \bar{G}$$

for any positive integer $s$ and any $x_{i}, y_{i} \in G$ for which $\left|\{i \mid x_{i} \in \langle a_{m} \rangle\}\right| + \left|\{i \mid y_{i} \in \langle b_{0} \rangle\}\right| \leq 2$ and $u \neq 1$ in $G \ast \langle t \rangle_{\infty}$.

This lemma implies that the elements $g_{1}h_{1}^{d_{1}}$ and $h_{2}^{d_{2}}g_{2}$ of the group $\bar{G}$ generate a free subgroup of rank 2 for any $g_{1}, g_{2}, h_{1}, h_{2} \in G$ such that $h_{1}, h_{2}, h_{1}h_{2} \notin \langle a_{m} \rangle$ and $g_{2}, g_{1}, g_{2}g_{1} \notin \langle b_{0} \rangle$. Therefore, the absence of nonabelian free subgroups in $\bar{G}$ implies that the group $G$ is virtually cyclic ($|G : \langle a_{m} \rangle| \leq 2$ or $|G : \langle b_{0} \rangle| \leq 2$) and, hence, $G$ is cyclic (because $G$ is torsion-free).

If the group $G$ is cyclic, then $\bar{G}$ is a one-relator group. The free subgroup theorem for one-relator groups implies that either $\bar{G}$ contains a nonabelian free subgroup, $\bar{G}$ is cyclic (which is impossible for $m \geq 0$), or $\bar{G}$ is isomorphic to $G_{1,k}$. In the last case, the unimodularity of the relation $w$ implies that the number $k$ must be 2. It is easy to verify this by considering the commutator quotient. Theorem 2 is proven.

3. Proof of Theorem 3

Suppose that the word $w$ has the form $w = g_{1}t_{1} \ldots g_{n}t_{n}$.

**Case 1:** $n = 1$. Clearly, in this case $|T/\langle t \rangle_{1}| = \infty$,

$$\bar{G} \simeq \left\{ \begin{array}{ll}
G \ast T & \text{if } g_{1} \neq 1 \text{ in } G, \\
G \ast (T/\langle t \rangle_{1}) & \text{if } g_{1} = 1 \text{ in } G,
\end{array} \right.$$

and the assertion of the theorem is obtained from the following well-known simple facts:

1. An amalgamated product contains a nonabelian free subgroup if the amalgamated subgroup is proper in each factor and its index is larger than 2 in one of the factors.
2. Let $\langle a \rangle$ be a cyclic normal subgroup of a group $A$. Then $A$ contains a nonabelian free subgroup if and only if such a subgroup exists in the quotient group $A/\langle a \rangle$.

**Case 2:** $n > 1$ and the group $\langle t_{1}, \ldots, t_{n} \rangle$ is cyclic (and, therefore, is generated by the element $t = \prod t_{i}$ by virtue of unimodularity). In this case, $\bar{G}$ is the amalgamated product:

$$\bar{G} \simeq \langle G, t \mid w = 1 \rangle \ast T_{\langle t \rangle}.$$

This free product always contains a nonabelian free subgroup, because the amalgamated subgroup has infinite index in both factors. The infinity of the index of $\langle t \rangle$ in the first factor follows from the minimal complexity theorem, which, in particular, claims that $G \cap \langle t \rangle = \{1\}$ for $n > 1$. The equality $|T : \langle t \rangle| = \infty$ holds because $w$ is unimodular and $T$ is noncyclic.

**Case 3:** The group $\langle t_{1}, \ldots, t_{n} \rangle$ is noncyclic. In this case, without loss of generality we can assume that $T = \langle t_{1}, \ldots, t_{n} \rangle$.

Put $t = \prod t_{i}$. Let us decompose $T$ into the union of cosets:

$$T = \bigcup_{x \in T/\langle t \rangle} c_{x} \langle t \rangle_{1}, \quad \text{where } c_{1} = 1,$$

and rewrite the word $w$ in the form

$$w \equiv t \prod_{t} g_{i}^{c_{i}t_{k}^{i}} = 1. \quad (2)$$

Let $X_{1} = \{x_{i} \}$ be the set of all $x \in T/\langle t \rangle$ occurring in the reduced expression (2). Note that $|X_{1}| > 1$, because the group $T = \langle t_{1}, \ldots, t_{n} \rangle$ is noncyclic. In [Kl06], it was shown that in the group $\bar{G}$ we have the decomposition

$$H_{1} = \langle \{G^{y} \mid y \in X_{1} \} \rangle = \ast_{y \in X_{1}} G^{y}.$$

This implies immediately that $\bar{G}$ contains the free square of the group $G$ and, hence, a nonabelian free subgroup.
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