Remark on the Lifespan of Solutions to 3-D Anisotropic Navier Stokes Equations

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\textbf{Abstract.} The goal of this article is to provide a lower bound for the lifespan of smooth solutions to 3-D anisotropic incompressible Navier-Stokes system, which in particular extends a similar type of result for the classical 3-D incompressible Navier-Stokes system.

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\section{Introduction}

In this article, we shall investigate the lifespan for smooth enough solutions to the following 3-D anisotropic incompressible Navier-Stokes system:

\begin{equation}
\begin{aligned}
\partial_t u + u \cdot \nabla u - \Delta_h u &= -\nabla p, \\
\text{div} u &= 0, \\
|u|_{t=0} &= u_0,
\end{aligned}
\end{equation}

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where $\Delta_h \overset{\text{def}}{=} \partial_1^2 + \partial_2^2$, $u$ designates the velocity of the fluid and $p$ the scalar pressure function which guarantees the divergence free condition of the velocity field.

Systems of this type appear in geophysical fluid dynamics (see for instance [5, 16]). In fact, meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $-\mu_h \Delta_h - \mu_3 \partial_3^2$, where $\mu_h$ and $\mu_3$ are empirical constants, and $\mu_3$ is usually much smaller than $\mu_h$. We refer to the book of Pedlovsky [16], Chap. 4 for a complete discussion about this model.

We remark that for the classical Navier-Stokes system (NS), which corresponds to (1.1) with $\Delta_h$ there being replaced by $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, Leray [12] proved the global existence of weak solutions to (NS) in 1934. Yet the uniqueness and regularity to such weak solutions are still open. In [6], Chemin and Gallagher showed that: let $u_0$ be a regular solenoidal vector field, then the classical Navier-Stokes system (NS) has a unique regular solution on $[0,T]$. Let $T^* (u_0)$ be the maximal time of existence of this regular solution. Then for any $\gamma \in (0,1/4)$, a positive constant $C_\gamma$ exists so that

$$T^* (u_0) \geq C_\gamma \| u_0 \|^{-\frac{1}{2} + 2\gamma}.$$  

(1.2)

In the special case when $\gamma = \frac{1}{4}$, this type of result goes back to the seminal work of Leray [12]. Lately the same type of result has been proved for 3-D inhomogeneous incompressible Navier-Stokes system in [17] by the second author.

Considering that the system (1.1) has only horizontal dissipation, it is reasonable to use anisotropic Sobolev space defined as follows:

**Definition 1.1.** For any $(s,s')$ in $\mathbb{R}^2$, the anisotropic Sobolev space $\dot{H}^{s,s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution $a$ such that

$$\| a \|^2_{\dot{H}^{s,s'}} = \int_{\mathbb{R}^3} |\xi_h|^s |\xi_3|^{2s'} |\hat{a}(\xi)|^2 \, d\xi < \infty \quad \text{with} \quad \xi_h = (\xi_1, \xi_2).$$

Mathematically, Chemin et al. [4] first studied the system (1.1). In particular, Chemin et al. [4] and Iftimie [11] proved that (1.1) is locally well-posed with initial data in $L^2 \cap \dot{H}^{0,1+\varepsilon}$ for some $\varepsilon > 0$, and is globally well-posed if in addition

$$\| u_0 \|_{L^2} \| u_0 \|^{-\varepsilon}_{\dot{H}^{0,1+\varepsilon}} \leq c$$

(1.3)

for some sufficiently small constant $c$.

Paicu [14] improved the well-posedness result in [4, 11] to be the critical case, namely, with initial data in the critical anisotropic Besov space, which basically
corresponds to $\varepsilon = 0$ in [4, 11]. Chemin and the second author [8] introduced an anisotropic Besov-Sobolev type space with negative index and proved the global well-posedness of (1.1) with initial data being sufficiently small in this space. Paicu and the second author [15] improved further the global well-posedness result in [8] by requiring only two components of the initial data to be small in such negative anisotropic Besov spaces. Lately Liu and the second author proved the global well-posedness of (1.1) by requiring only one directional derivative of the initial data to be sufficiently small in some critical spaces. One may check [13] and the references therein concerning the recent progresses on the well-posedness of this system (1.1).

The goal of this article is to extend similar result as (1.2) for the lifespan of solutions to the classical Navier-Stokes system to the case of (1.1). The main result states as follows:

**Theorem 1.1.** Let $s = 2\gamma + \frac{1}{2}$ with $\gamma \in (0, 1/4)$. Let $u_0 \in H^s$ be a solenoidal vector field with $\partial_3 u_0 \in H^{s-1,0} \cap H^{-1,s}$. Then (1.1) has a unique solution $u \in C([0,T];H^{0,s})$ with $\nabla_h u \in L^2((0,T);H^{0,s})$ for some $T > 0$. Moreover, if $T^*(u_0)$ designates the lifespan of this solution, there exists a positive constant $C_\gamma$ so that

$$T^*(u_0) \geq C_\gamma [u_0]_s^{-\frac{1}{s}} \quad \text{with} \quad [u_0]_s \overset{\text{def}}{=} \|u_0\|_{L^s}^2 + \|\partial_3 u_0\|_{L^{s-1,0}}^2 + \|\partial_3 u_0\|_{L^{-1,s}}^2. \quad (1.4)$$

Let us end this section with the notations that we shall use in this context.

**Notations:** Let $A, B$ be two operators, we denote $[A; B] = AB - BA$, the commutator between $A$ and $B$, for $a \lesssim b$, we means that there is a uniform constant $C$, which may be different in each occurrence, such that $a \leq Cb$. We shall denote by $(a|b)_{L^2}$ the $L^2(\mathbb{R}^3)$ inner product of $a$ and $b$. We always denote $(c_\ell)_{\ell \in \mathbb{Z}}$ to be a nonnegative generic element of $\ell^2(\mathbb{Z})$ so that $\sum c_\ell^2 = 1$. Finally, we denote $L^r_T(L^q_{h}(L^p_{x}))$ the space $L^r([0,T];L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2};L^q(\mathbb{R}_{x_3})))$, and $\nabla_h \overset{\text{def}}{=} (\partial_{x_1}, \partial_{x_2})$, $\text{div}_h = \partial_{x_1} + \partial_{x_2}$.

## 2 Littlewood-Paley analysis and product laws

In the rest of this paper, we shall frequently use Littlewood-Paley decomposition in the vertical variable. For the convenience of the readers, we collect some basic facts on anisotropic Littlewood-Paley theory in this section. Let us first recall from [1] that

$$\begin{align*}
\Delta^h_k a &= \mathcal{F}^{-1}(\phi(2^{-k}|\xi_h|)\hat{a}), \\
\Delta^h_\ell a &= \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_\ell|)\hat{a}), \\
S^h_k a &= \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\hat{a}), \\
S^h_\ell a &= \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_\ell|)\hat{a}),
\end{align*} \quad (2.1)$$

S. Liang, P. Zhang and R. Zhu / Commun. Math. Res., 36 (2020), pp. 31-50 33
where \( \hat{\xi} = (\xi_0, \xi_3) \), \( \mathcal{F}a \) and \( \hat{a} \) denote the Fourier transform of the distribution \( a \), and \( \chi(\tau) \), \( \varphi(\tau) \) are smooth functions such that

\[
\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \tau) = 1,
\]

\[
\text{Supp } \chi \subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{k \geq 0} \varphi(2^{-k} \tau) = 1.
\]

Next we recall the following anisotropic Bernstein inequalities from [8, 14]:

**Lemma 2.1.** Let \( B_h \) (resp. \( B_v \)) a ball of \( \mathbb{R}^2_h \) (resp. \( \mathbb{R}^2_v \)), and \( C_h \) (resp. \( C_v \)) a ring of \( \mathbb{R}^2_h \) (resp. \( \mathbb{R}^2_v \)); let \( 1 \leq p_2 \leq p_1 \leq \infty \) and \( 1 \leq q_2 \leq q_1 \leq \infty \). Then there holds

\[
\begin{align*}
\text{if Supp } \hat{a} & \subset 2^k B_h \Rightarrow \|\partial_{\xi}^a a\|_{L^p_h(L_q^1)} \lesssim 2^{k(|a| + \frac{2}{p_2} - \frac{2}{p_1})} \|a\|_{L^p_h(L_q^1)}; \\
\text{if Supp } \hat{a} & \subset 2^k B_v \Rightarrow \|\partial_{\xi}^b a\|_{L^p_v(L_q^1)} \lesssim 2^{k(b + \frac{1}{p_2} - \frac{1}{p_1})} \|a\|_{L^p_v(L_q^1)}; \\
\text{if Supp } \hat{a} & \subset 2^k C_h \Rightarrow \|a\|_{L^p_h(L_q^1)} \lesssim 2^{-kN} \sup_{|a| = N} \|\partial_{\xi}^a a\|_{L^p_h(L_q^1)}; \\
\text{if Supp } \hat{a} & \subset 2^k C_v \Rightarrow \|a\|_{L^p_v(L_q^1)} \lesssim 2^{-kN} \|\partial_{\xi}^a a\|_{L^p_v(L_q^1)}.
\end{align*}
\]

Before preceding, we recall Bony’s decomposition for the vertical variable from [2]:

\[ ab = T^v_a b + R^v(a, b) \quad \text{with} \quad T^v_a b = \sum_{\ell \in \mathbb{Z}} S^v_{\ell-1} a \Delta^v_{\ell} b, \quad R^v(a, b) = \sum_{\ell \in \mathbb{Z}} \Delta^v_{\ell} a S^v_{\ell+2} b. \quad (2.2) \]

Let us now apply the above basic facts on Littlewood-Paley theory to prove the following laws of product:

**Lemma 2.2.** Let \( s \in \left( \frac{1}{4}, 1 \right) \), one has

\[
\|\Delta^v_s(ab)\|_{L^2(L^2_h)} \lesssim c_s 2^{-\ell s} \left( \|a\|_{H^0_s} \|\nabla_h a\|_{H^0_s} \|b\|_{L^2}^{1 - \frac{s}{2}} \right)_{L^2}^{\frac{s}{2}} + \left( \|a\|_{L^2} \|\nabla_h a\|_{L^2}^{\frac{1}{2}} + \|a\|_{H^0_s} \|\nabla_h a\|_{H^0_s} \right) \|b\|_{H^0_s}^{1 - \frac{s}{2}}. \quad (2.3)
\]

**Proof.** By applying Bony’s decomposition (2.2) in the vertical variable to \( ab \), we find

\[ ab = T^v_a b + R^v(a, b). \]
Considering the support properties to the Fourier transform of the terms in $T_a^\nu b$, we write
\[
\| \Delta_\ell^\nu (T_a^\nu b) \|_{L^2_\ell(L^4_h)} \lesssim \sum_{|j-\ell| \leq 5} \| S_{j-1}^\nu \|_{L^\nu(L^4_h)} \| \Delta_j^\nu b \|_{L^2_h}
\]
\[
\lesssim |a| \| L_\nu(L^4_h) \| \left( \sum_{|j-\ell| \leq 5} \| \Delta_j^\nu b \|_{L^2_h} \right)
\]
\[
\lesssim |a| \| L_\nu(L^4_h) \| \| \nabla_h a \| \| \Delta_j^\nu b \|_{L^2_h} \left( \sum_{|j-\ell| \leq 5} \| \Delta_j^\nu b \|_{L^2_h} \right),
\]
from which, and
\[
\| f \|_{L^\nu(L^4_h)} \lesssim \| f \|_{L^2_\ell(L^4_h)} \lesssim \| f \|_{L^2_\ell(B^s_{2,1})_v} \lesssim \| f \|_{L^2_\ell(L^4_h)}^{1-\frac{k}{2}} \| f \|_{H^s_{0,v}}, \tag{2.4}
\]
We deduce that
\[
\| \Delta_j^\nu (T_a^\nu b) \|_{L^2_\ell(L^4_h)} \lesssim c_j 2^{-js} \left( |a| \| L^2_h \| \| \nabla_h a \| \| \Delta_j^\nu b \|_{L^2_h} \right)^{\frac{1}{2}} \left( |a| \| L^2_{H^{0,s}} \| \| \nabla_h a \| \| \Delta_j^\nu b \|_{L^2_h} \right)^{\frac{1}{2}} \| b \|_{L^2_{H^{0,s}}}, \tag{2.5}
\]
Along the same line to the proof of (2.5), we infer
\[
\| \Delta_\ell^\nu (R^\nu (a,b)) \|_{L^2_\ell(L^4_h)} \lesssim \sum_{j \geq \ell - N_0} \| S_{j+2}^\nu \|_{L^\nu(L^4_h)} \| \Delta_j^\nu a \|_{L^2_\ell(L^4_h)}
\]
\[
\lesssim \| b \|_{L^\nu(L^4_h)} \left( \sum_{j \geq \ell - N_0} \| \Delta_j^\nu a \|_{L^2_\ell(L^4_h)} \right)
\]
\[
\lesssim \| b \|_{L^2_\ell(L^4_h)}^{1-\frac{k}{2}} \| b \|_{H^s_{0,v}} \left( \sum_{j \geq \ell - N_0} \| \Delta_j^\nu a \|_{L^2_\ell(L^4_h)} \right)^{\frac{1}{2}} \| \nabla_h a \|_{H^s_{0,v}} \sum_{j \geq \ell - N_0} c_j 2^{-js}.
\]
Notice that
\[
\left\| \left( \sum_{j \geq \ell - 5} c_j 2^{-js} \right)_{\ell \in \mathbb{Z}} \right\|_{\ell^2} \lesssim \| 1_{(-\infty,5]}^{2s} \|_{\ell^1} \| c_j \|_{\ell^2} \lesssim 1, \tag{2.6}
\]
we conclude that
\[
\| \Delta_\ell^\nu (R^\nu (a,b)) \|_{L^2_\ell(L^4_h)} \lesssim c_j 2^{-\ell s} \| b \|_{L^2_\ell(L^4_h)}^{1-\frac{k}{2}} \| b \|_{H^s_{0,v}} \| a \|_{H^s_{0,v}} \| \nabla_h a \|_{H^s_{0,v}}, \tag{2.7}
\]
Combining (2.5) with (2.7) leads to (2.3). \qed
**Remark 2.1.** The proof of Lemma 2.2 also implies that

\[
\left\| \Delta_L^y(ab) \right\|_{L^2_x(L^4_h)} \lesssim c_L 2^{-\ell s} \left( \|a\|_{L^\infty_y(L^4_h)} \, \|b\|_{\dot{H}^{0,s}} + \|a\|_{\dot{H}^{0,s}} \|\nabla_h a\|_{L^2} \|b\|_{L^2} \right),
\]

and

\[
\left\| \Delta_L^y(ab) \right\|_{L^2_x(L^4_h)} \lesssim c_L 2^{-\ell s} \left( \|a\|_{L^\infty_y(L^4_h)} \, \|b\|_{\dot{H}^{0,s}} + \|a\|_{\dot{H}^{0,s}} \|\nabla_h a\|_{L^2} \|b\|_{L^2} \right).
\]

**Lemma 2.3.** Let \( a = (a^h, a^3) \) be a solenoidal vector field. Then for any \( s \in (\frac{1}{2}, 1) \), one has

\[
\left\| (\Delta_L^y(a^3 \partial_3 b), \Delta_L^y b) \right\|_{L^2} \lesssim c_L 2^{-2\ell s} \left( \|\nabla_h a^h\|_{L^2} \|\nabla_h a^h\|_{\dot{H}^{0,s}} \|b\|_{\dot{H}^{0,s}} \|\nabla_h b\|_{\dot{H}^{0,s}} + \|\nabla_h a^h\|_{\dot{H}^{0,s}} \|b\|_{L^2} \frac{1}{2} \right)\left( \|b\|_{\dot{H}^{0,s}} \|\nabla_h b\|_{\dot{H}^{0,s}} \right)^{\frac{1}{2} + \frac{1}{s}} \right).
\]

**Proof.** By applying Bony’s decomposition (2.2) in the vertical variable to \( a^3 \partial_3 b \), we write

\[a^3 \partial_3 b = T_{a^3}^\partial \partial_3 b + R^\partial (a^3, \partial_3 b)\]

We first deduce from the support properties to the Fourier transform of the terms in \( R^\partial (a^3, \partial_3 b) \) and Lemma 2.1 that

\[
\left\| (\Delta_L^y(R^\partial (a^3, \partial_3 b)) \right\|_{L^2} \lesssim \sum_{j \geq -N_0} 2^j \|\Delta_L^y a^3\|_{L^2} \|S_{j+2} b\|_{L^\infty_y(L^4_h)} \|\Delta_L^y b\|_{L^2_x(L^4_h)}
\]

\[
\lesssim \sum_{j \geq -N_0} \|\Delta_L^y \partial_3 a^3\|_{L^2} \|b\|_{L^2} \|\nabla_h b\|_{\dot{H}^{0,s}} \|\Delta_L^y b\|_{L^2} \|\Delta_L^y \nabla_h b\|_{L^2},
\]

which together with (2.4) and \( \partial_3 a^3 = -\text{div}_h a^h \) ensures that

\[
\left\| (\Delta_L^y(R^\partial (a^3, \partial_3 b)) \right\|_{L^2} \lesssim c_L 2^{-\ell s} \left( \|b\|_{L^2} \|\nabla_h b\|_{L^2} \right)^{\frac{1}{2} + \frac{1}{s}} \left( \|b\|_{\dot{H}^{0,s}} \|\nabla_h b\|_{\dot{H}^{0,s}} \right)^{\frac{1}{2} + \frac{1}{s}} \left( \|b\|_{\dot{H}^{0,s}} \|\nabla_h b\|_{\dot{H}^{0,s}} \right)^{\frac{1}{2} + \frac{1}{s}} \right).
\]

where in the last step we used (2.6) once again.
To handle the term \((\Delta^\gamma T^\gamma_{a^3} \partial_3 b) | \Delta^\gamma b\rangle_{L^2}\), we get, by using a standard commutator’s process (see for instance [4]), that
\[
\Delta^\gamma_{\ell}(T^\gamma_{a^3} \partial_3 b) = S_{\ell-1}^\gamma a^3 \partial_3 \Delta^\gamma b + \sum_{|j-\ell| \leq 5} |\Delta^\gamma_{\ell}; S_{j-1}^\gamma a^3| \partial_3 \Delta^\gamma b \\
+ \sum_{|j-\ell| \leq 5} (S_{j-1}^\gamma a^3 - S_{\ell-1}^\gamma a^3) \partial_3 \Delta^\gamma_{\ell} \Delta^\gamma b.
\] (2.12)

Corresponding to the first term in (2.12), we get, by applying (2.4) and \(\text{div} a = 0\), that
\[
\left| (S_{\ell-1}^\gamma a^3 \partial_3 \Delta^\gamma b | \Delta^\gamma b)_{L^2} \right| = \frac{1}{2} \left| (S_{\ell-1}^\gamma a^3 \partial_3 \Delta^\gamma b | \Delta^\gamma b)_{L^2} \right|
\leq \|\partial_3 a^3\|_{L^\infty(L_h^2)} \|\Delta^\gamma b\|_{L^2(L_h^4)}^2
\leq \|\nabla h a^h\|_{L^\infty(L_h^2)} \|\Delta^\gamma b\|_{L^2} \|\Delta^\gamma \nabla h b\|_{L^2}
\leq c_2^2 2^{-2\ell s} \|\nabla h a^h\|_{L^2} \|\nabla h a^h\|_{L^2} \|\nabla h b\|_{L^2} \|\nabla h b\|_{L^2}.
\]
Similarly, corresponding to the last term in (2.12), we get, by applying Lemma 2.1, that
\[
\sum_{|j-\ell| \leq 5} \left| (S_{j-1}^\gamma - S_{\ell-1}^\gamma a^3) \partial_3 \Delta^\gamma_{\ell} \Delta^\gamma b | \Delta^\gamma_{\ell} b)_{L^2} \right|
\leq \sum_{|j-\ell| \leq 5} \|\Delta^\gamma a^3\|_{L^\infty(L_h^2)} 2^\ell \|\Delta^\gamma b\|_{L^2(L_h^4)}^2
\leq \sum_{|j-\ell| \leq 5} \|\Delta^\gamma \partial_3 a^3\|_{L^\infty(L_h^2)} \|\Delta^\gamma b\|_{L^2} \|\Delta^\gamma \nabla h b\|_{L^2}
\leq c_2^2 2^{-2\ell s} \|\nabla h a^h\|_{L^2} \|\nabla h a^h\|_{L^2} \|\nabla h b\|_{L^2} \|\nabla h b\|_{L^2}.
\]
Finally let us deal with the commutator term. We observe that
\[
\sum_{|j-\ell| \leq 5} \left| (|\Delta^\gamma_{\ell}; S_{j-1}^\gamma a^3| \partial_3 \Delta^\gamma b | \Delta^\gamma b)_{L^2} \right|
\leq \sum_{|j-\ell| \leq 5} \|\Delta^\gamma_{\ell}; S_{j-1}^\gamma a^3| \partial_3 \Delta^\gamma b\|_{L^2(L_h^4)} \|\Delta^\gamma b\|_{L^2(L_h^4)}.
\]
Let $h \overset{\text{def}}{=} F^{-1} \varphi$ and $h_1(z) \overset{\text{def}}{=} zh(z)$. Then again due to $\partial_3 a^3 = -\text{div}_h a^h$, we find

$$\| [\Delta^y_j; S^y_{j-1} a^3] \partial_3 \Delta^y_j b(x_h, x_3) \|_{L^2_h(L^4_h)}$$

$$= 2^\ell \int_{\mathbb{R}} h(2^\ell y_3) (S^y_{j-1} a^3(x_h, x_3 - y_3) - S^y_{j-1} a^3(x_h, x_3)) \partial_3 \Delta^y_j b(x_h, x_3 - y_3) dy_3$$

$$= 2^\ell \int_{\mathbb{R} \times [0, 1]} y_3 h(2^\ell y_3) S^y_{j-1} a^3(x_h, x_3 - \tau y_3) \partial_3 \Delta^y_j b(x_h, x_3 - y_3) d\tau dy_3$$

$$= \left| \int_{\mathbb{R} \times [0, 1]} h_1(2^\ell y_3) S^y_{j-1} \text{div}_h a^h(x_h, x_3 - \tau y_3) \partial_3 \Delta^y_j b(x_h, x_3 - y_3) d\tau dy_3 \right|.$$

When $|j - \ell| \leq 5$, taking the $L^2_h(L^4_h)$ norm to the above quantity gives rise to

$$\| [\Delta^y_j; S^y_{j-1} a^3] \partial_3 \Delta^y_j b \|_{L^2_h(L^4_h)} \lesssim \int_{\mathbb{R} \times [0, 1]} |h_1(2^\ell y_3)| \| S^y_{j-1} \text{div}_h a^h(\cdot - \tau y_3) \|_{L^4_h(L^4_h)} \| \partial_3 \Delta^y_j b(\cdot - \tau y_3) \|_{L^4_h(L^4_h)} d\tau dy_3$$

$$= 2^\ell \| S^y_{j-1} \text{div}_h a^h(\cdot, \cdot - \tau y_3) \|_{L^4_h(L^4_h)} \| \Delta^y_j b \|_{L^2_h(L^4_h)} \int_{\mathbb{R}} |h_1(2^\ell y_3)| dy_3$$

$$\lesssim \| \text{div}_h a^h \|_{L^4_h(L^4_h)} \| \Delta^y_j b \|_{L^2_h} \| \Delta^y_j \nabla_h b \|_{L^2_h}.$$

Then by virtue of (2.4), we deduce that

$$\sum_{|j - \ell| \leq 5} \| [\Delta^y_j; S^y_{j-1} a^3] \partial_3 \Delta^y_j b \|_{L^2} \leq c_\ell^2 2^{-2s} \| \nabla_h a^h \|_{L^2}^{1 - \frac{1}{3}} \| \nabla_h a^h \|_{L^4_h(L^4_h)}^{\frac{1}{3}} \| \nabla_h b \|_{H^0,s} \| \nabla_h b \|_{H^0,s}.$$

As a result, it comes out

$$\| (\Delta^y_j(T^y_{a^3} \partial_3 b) | \Delta^y_j b) \|_{L^2} \leq c_\ell^2 2^{-2s} \| \nabla_h a^h \|_{L^2}^{1 - \frac{1}{3}} \| \nabla_h a^h \|_{L^4_h(L^4_h)}^{\frac{1}{3}} \| \nabla_h b \|_{H^0,s} \| \nabla_h b \|_{H^0,s}.$$

Along with (2.11), we complete the proof of (2.10). \hfill \square

**Remark 2.2.** The proof of Lemma 2.3 implies that

$$\| (\Delta^y_j(a^3 \partial_3 b) | \Delta^y_j b) \|_{H^0,s} \leq c_\ell^2 2^{-2s} \left( \| \nabla_h a^h \|_{L^4_h(L^4_h)} \| b \|_{H^0,s} \| \nabla_h b \|_{H^0,s} \right. \left. + \| \nabla_h a^h \|_{H^0,s} (\| b \|_{L^2} \| \nabla_h b \|_{L^2})^{\frac{1}{2}} \left( \| b \|_{H^0,s} \| \nabla_h b \|_{H^0,s} \right)^{\frac{1}{2} + \frac{1}{4 s}} \right). \quad (2.13)$$
3 Scaled $L^2$ energy estimate

Inspired by the lecture notes of Chemin [3], we are going to present a scaled energy estimate for smooth enough solutions of (1.1). We remark that estimate of this type was first proposed by Chemin and Plamchon in [7] for the classical 3-D Navier-Stokes system (see also [6]). The main result of this section states as follows:

**Proposition 3.1.** Let $s \in \left(\frac{1}{4}, 1\right)$ and $u_L \overset{\text{def}}{=} e^{tB_h}u_0$. Let $u \overset{\text{def}}{=} u_L + w$ be a smooth enough solution of (1.1) on $[0, T^*)$. Then for any $t < T^*$, one has

$$
\left\| w(t) \right\|_{L^2}^2 \leq \int_0^t \left( \frac{s}{2} \left\| w(t') \right\|_{L^2}^2 + \frac{\left\| \nabla_h w(t') \right\|_{L^2}^2}{t^{1+s}} \right) dt' \leq C_s t^{s-\frac{1}{2}} \left( \left\| u_0 \right\|_{L^2}^2 + \left\| \partial_3 u_0 \right\|_{L^2}^2 \right) \exp\left( C_s t^{\frac{2s-1}{2}} \left( \left\| u_0 \right\|_{L^2}^4 + \left\| \partial_3 u_0 \right\|_{L^2}^4 \right) \right).
$$

**Proof.** Due to $u \overset{\text{def}}{=} w + u_L$, by virtue of (1.1), $w$ verifies

$$
\begin{align*}
\partial_t w + (u_L + w) \cdot \nabla w + w \cdot \nabla u_L - \Delta h w &= -u_L \cdot \nabla u_L - \nabla p, \\
\text{div} w &= 0, \\
\left. w \right|_{t=0} &= 0.
\end{align*}
$$

By taking $L^2$ inner product of (3.2) with $w$ and dividing the resulting equality by $t^s$, we find

$$
\frac{1}{2} \frac{d}{dt} \left( \frac{\left\| w(t) \right\|_{L^2}^2}{t^{1+s}} \right) + \frac{s}{2} \frac{\left\| w(t) \right\|_{L^2}^2}{t^{1+s}} + \frac{\left\| \nabla_h w(t) \right\|_{L^2}^2}{t^s} = -\frac{\left\| w \cdot \nabla u_L \right\|_{L^2}}{t^s} - \frac{\left( u_L \cdot \nabla u_L \right) w}{t^s}.
$$

(3.3)

In what follows, we shall separate the estimate of the terms on the right-hand side of (3.3) with vertical derivative and without vertical derivative. Notice that for any $p > 2$, we have

$$
\left\| f \right\|_{L^2}^{2p} \leq C \left\| f \right\|_{L^2}^{1 - \frac{1}{p}} \left\| \nabla_h f \right\|_{L^2}^{\frac{1}{p}},
$$

(3.4)

from which, we infer

$$
\left| (w^h, \nabla_h u_L | w)_{L^2} \right| \leq \left\| \nabla_h u_L \right\|_{L^2(\Omega')} \left\| w \right\|_{L^2}^{2p} \left\| \nabla_h w \right\|_{L^2}^{2p} \leq C \left\| \nabla_h u_L \right\|_{L^2(\Omega')} \left\| w \right\|_{L^2}^{2p} + \frac{1}{8} \left\| \nabla_h w \right\|_{L^2}^{2p}.
$$
Finally applying Young’s inequality gives rise to
\[
|\langle w^h \cdot \nabla_h u_L, w \rangle_{L^2}| \leq C \| \nabla_h u_L \|_{L^\infty(L^2_h)} \| w \|_{L^2}^2 + \frac{1}{8} \| \nabla_h w \|_{L^2}^2. \tag{3.5}
\]

While observing that \( \text{div} \, w = 0 \) and
\[
\| f \|_{L_h^2(L^\infty)} \leq \| f \|_{L^2} \| \partial_3 f \|_{L^2},
\]
we deduce that
\[
|\langle w^3 \partial_3 u_L, w \rangle_{L^2}| \leq \| w^3 \|_{H^1(L^\infty)} \| \partial_3 u_L \|_{L^\infty(L^2)} \| w \|_{L^2}
\leq \| w^3 \|_{L^2}^2 \| \partial_3 w^3 \|_{L^2} \| \partial_3 u_L \|_{L^\infty(L^2)} \| w \|_{L^2}
\leq C \| \partial_3 u_L \|_{L^\infty(L^2)} \| w \|_{L^2}^2 + \frac{1}{8} \| \nabla_h w \|_{L^2}^2. \tag{3.6}
\]

Finally applying Young’s inequality gives rise to
\[
\left| \frac{\langle u_L \cdot \nabla u_L, w \rangle}{t^s} \right| \leq \frac{1}{t^2} \| u_L \cdot \nabla u_L \|_{L^2} \| w \|_{L^2}^2
\leq \frac{t^{1-s}}{s} \| u_L \cdot \nabla u_L \|_{L^2}^2 + \frac{s \| w \|_{L^2}^2}{4t^{1+s}}. \tag{3.7}
\]

Inserting the estimates (3.5), (3.6) and (3.7) into (3.3), we achieve
\[
\frac{d}{dt} \left( \frac{\| w(t) \|_{L^2}^2}{t^s} \right) + \frac{s \| w(t) \|_{L^2}^2}{t^{1+s}} + \frac{\| \nabla_h w(t) \|_{L^2}^2}{t^s}
\leq C \left( \| \nabla_h u_L \|_{L^\infty(L^2)}^4 + \| \partial_3 u_L \|_{L^\infty(L^2)}^4 \right) \frac{\| w \|_{L^2}^2}{t^s} + \frac{2}{s} t^{1-s} \| u_L \cdot \nabla u_L \|_{L^2}^2.
\]

Applying Gronwall’s inequality gives rise to
\[
\frac{\| w(t) \|_{L^2}^2}{t^s} + \int_0^t \left( \frac{s}{4} \| w(t') \|_{L^2}^2 + \frac{\| \nabla_h w(t') \|_{L^2}^2}{(t')^s} \right) dt'
\leq \frac{2}{s} \int_0^t (t')^{1-s} \| u_L \cdot \nabla u_L(t') \|_{L^2}^2 dt'
\times \exp \left( C \int_0^t \left( \| \nabla_h u_L(t') \|_{L^\infty(L^2)}^4 + \| \partial_3 u_L(t') \|_{L^\infty(L^2)}^4 \right) dt' \right). \tag{3.8}
\]

Let us now handle term by term in (3.8).
Estimate of $\int_0^t \| \nabla_h u_L (t') \|^4_{L_h^4 (L_\infty^\infty)} dt'$

We first observe from Lemma 2.4 of [1] and Lemma 2.1 that

$$\int_0^t \| \nabla_h u_L (t') \|^4_{L_h^4 (L_\infty^\infty)} dt' \leq C \int_0^t \left( \sum_{k \in \mathbb{Z}} e^{c_k t' 2^{2k}} \frac{2}{2^k} \sum_{\ell \in \mathbb{Z}} \| \Delta_k^h \Delta_\ell^h u_0 \|^4_{L_2^2} \right)^{\frac{1}{2}} dt'.$$

Yet it follows from Lemma 4.3 of [9] that $\dot{H}^s (\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}_h (B_{2,1}^1)$, so that

$$\sum_{\ell \in \mathbb{Z}} 2^{\frac{s}{2}} \| \Delta_k^h \Delta_\ell^h u_0 \|_{L_2} \lesssim c_k 2^{-k (s - \frac{1}{2})} \| u_0 \|_{\dot{H}^s}, \quad \text{(3.9)}$$

where $(c_k)_{k \in \mathbb{Z}}$ is a generic element of $\ell^2 (\mathbb{Z})$ so that $\sum_{k \in \mathbb{Z}} c_k^2 = 1$. As a result, we obtain

$$\int_0^t \| \nabla_h u_L (t') \|^4_{L_h^4 (L_\infty^\infty)} dt' \leq C \int_0^t \left( \sum_{k \in \mathbb{Z}} e^{-c_k t' 2^{2k}} 2^{k (2 - s)} \right)^{\frac{1}{2}} dt' \| u_0 \|^\frac{1}{2}_{\dot{H}^s}.$$

On the other hand, it follows from Lemma 2.35 of [1] that

$$\sup_{t > 0} \left( \sum_{k \in \mathbb{Z}} t^{1 - \frac{s}{2}} 2^{k (2 - s)} e^{-c_k t 2^{2k}} \right) \overset{\text{def}}{=} M_s < \infty.$$

Hence we obtain

$$\int_0^t \| \nabla_h u_L (t') \|^4_{L_h^4 (L_\infty^\infty)} dt' \leq C M_s^\frac{4}{3} \int_0^t (t')^{-\frac{3}{4} + \frac{2s}{3}} dt' \| u_0 \|^\frac{1}{2}_{\dot{H}^s} \leq C_s t^{\frac{2s-1}{3}} \| u_0 \|^\frac{4}{3}_{\dot{H}^s}. \quad \text{(3.10)}$$

Estimate of $\int_0^t \| \partial_3 u_L (t') \|^4_{L_h^4 (L_\infty^\infty)} dt'$

Notice that $H^{s-1,0} \hookrightarrow L_2^4 (B_{2,2}^{-\frac{s}{2}})$, we deduce from Theorem 2.34 of [1] that

$$\int_0^t \| \partial_3 u_L (t') \|^4_{L_h^4 (L_\infty^\infty)} dt' \leq \left( \int_0^t (t')^{2 (s - 1)} dt' \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \int_0^t (t')^{1 - s} \| \partial_3 u_L (t', \cdot, z) \|^2_{L_\infty^\infty} dt' dz \right)^{\frac{3}{2}} \leq C_s t^{\frac{2s-1}{3}} \left( \int_{\mathbb{R}^3} \| \partial_3 u_0 \|^2_{L_2^2} dz \right)^{\frac{3}{2}} \leq C_s t^{\frac{2s-1}{3}} \| \partial_3 u_0 \|^\frac{4}{3}_{\dot{H}^{s-1,0}}. \quad \text{(3.11)}$$
Estimate of $\int_0^t (t')^{1-s} \| u_L \cdot \nabla u_L (t') \|_{L^2}^2 dt'$

We first get, by applying Hölder’s inequality, that

$$\int_0^t (t')^{1-s} \| u_L^h \cdot \nabla h u_L (t') \|_{L^2}^2 dt' \leq \left( \int_0^t \| u_L^h (t) \|^4_{L^4} (L^4_h) dt' \right)^{\frac{1}{4}} \left( \int_0^t (t')^{2(1-s)} \| \nabla h u_L (t') \|^4_{L^4 (L^4_h)} dt' \right)^{\frac{1}{4}}.$$

Observing that $\dot{H}^s_{-1,0} \hookrightarrow L^2_v (B^{s-\frac{3}{2}}_4) \hookrightarrow L^2_v (B^{s-\frac{3}{2}}_4)$, and thanks to Theorem 2.34 of [1], we infer

$$\left( \int_0^t (t')^{2(1-s)} \| \nabla h u_L^h (t') \|^4_{L^4 (L^4_h)} dt' \right)^{\frac{1}{4}} \leq \left\| \left( \int_0^t (t')^{2(1-s)} \| \nabla h u_L^h (t',\cdot, z) \|^4_{L^4_h} dt' \right)^{\frac{1}{4}} \right\|_{L^2_v} \lesssim \| \nabla h u_0 \|_{\dot{H}^{-1,0}}.$$

Whereas we deduce from (3.9) that

$$\int_0^t \| u_L^h (t) \|^4_{L^4_h (L^4_v)} dt' \lesssim \int_0^t \left( \sum_{k \in \mathbb{Z}} e^{-ct 2^k} \sum_{\ell \in \mathbb{Z}} 2^\frac{k}{2} \| \Delta_k^h \Delta \gamma u_0 \|_{L^2_v} \right)^4 dt' \lesssim \int_0^t (t')^{2(s-1)} \sup_{r > 0} \left( \sum_{k \in \mathbb{Z}} (t')^{\frac{s-1}{2}} 2^{k(1-s)} \right)^4 dt' \| u_0 \|^4_{\dot{H}^s} \leq C_s t^{2s-1} \| u_0 \|^4_{\dot{H}^s}. \quad (3.12)$$

As a result, it comes out

$$\int_0^t (t')^{1-s} \| u_L^h \cdot \nabla h u_L (t') \|_{L^2}^2 dt' \lesssim t^{s-\frac{1}{2}} \| u_0 \|^4_{\dot{H}^s}. \quad (3.13)$$

Exactly along the same line, we have

$$\int_0^t (t')^{1-s} \| u_L^3 \partial_3 u_L (t') \|_{L^2}^2 dt' \lesssim t^{s-\frac{1}{2}} \| \partial_3 u_0 \|^2_{\dot{H}^{-1,0}} \| u_0 \|^2_{\dot{H}^s}. \quad (3.14)$$

By inserting the estimates (3.10), (3.11), (3.13) and (3.14) into (3.8), we achieve (3.1). This finishes the proof of Proposition 3.1. \qed
4 The proof of Theorem 1.1

The goal of this section is to present the proof of Theorem 1.1. The key ingredient will be the following proposition:

**Proposition 4.1.** Let $s \in (1/2, 1)$ and $w$ be a smooth enough solution of (3.2) on $[0, T^*)$. Then for any $t < T^*$, we have

\[
\frac{d}{dt} \|w(t)\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2 \\
\leq \|\nabla u_L\|_{L^2}^2 + C \left( \|\partial_3 u_L\|_{L^2}^2 \|u_L\|_{L^2}^2 \|\nabla w\|_{L^2}^2 \right) \\
+ \left( \|u_L\|^2_{L^4} + \|\nabla u_L\|_{L^2}^2 + \|\partial_3 u_L\|_{L^2}^4 \right) \left( \|w\|^2_{L^2} + \|u_L\|^2_{L^2} \right) \\
+ \|\nabla_h w\|_{L^2}^2 \left( \|w\|_{L^2} \frac{1}{t^{1/2}} + \|w\|_{L^2}^2 \right) + \|w\|^2_{L^2} \|w\|_{L^2}^2 \left( \frac{2(2s+1)}{t^{1/4}} \right). \tag{4.1}
\]

**Proof.** By applying the operator $\Delta^s_h$ to (3.2) and then taking $L^2$ inner product of the resulting equations with $\Delta^s_h w$, we find

\[
\frac{1}{2} \frac{d}{dt} \|\Delta^s_h w(t)\|_{L^2}^2 + \|\Delta^s_h \nabla_h w\|_{L^2}^2 \\
= -\langle \Delta^s_h ((u_L + w) \cdot \nabla) \Delta^s_h w \rangle_{L^2} \\
- \langle \Delta^s_h (w \cdot \nabla u_L) \Delta^s_h w \rangle_{L^2} - \langle \Delta^s_h (u_L \cdot \nabla u_L) \Delta^s_h w \rangle_{L^2}. \tag{4.2}
\]

Let us now handle term by term in (4.2). We first observe that

\[
\langle \Delta^s_h (w \cdot \nabla) \Delta^s_h w \rangle_{L^2} = \left( \Delta^s_h (w^h \cdot \nabla_h w) \Delta^s_h w \right)_{L^2} + \left( \Delta^s_h (w^3 \partial_3 w) \Delta^s_h w \right)_{L^2}.
\]

Applying Lemma 2.2 yields

\[
\left| \left( \Delta^s_h (w^h \cdot \nabla_h w) \Delta^s_h w \right)_{L^2} \right| \leq \|\Delta^s_h (w^h \cdot \nabla_h w)\|_{L^2(L^2)} \|\Delta^s_h w\|_{L^2(L^2)} \\
\leq \|\Delta^s_h (w^h \cdot \nabla_h w)\|_{L^2(L^2)} \|\Delta^s_h w\|_{L^2(L^2)} + \|w\|_{L^2}^{1+\frac{1}{2s}} \|\nabla_h w\|_{L^2} \|\nabla_h w\|_{L^2}^{1-\frac{1}{2s}} \\
+ \|w\|_{L^2} \|\nabla_h w\|_{L^2}^{\frac{1}{2}} \left( \|w\|_{L^2} \|\nabla_h w\|_{L^2} \right)^{1-\frac{1}{2s}}.
\]
Applying Lemma 2.3 to \((\Delta^\gamma_w v^3 \partial_3 v) | \Delta^\gamma_w w)_{L^2}\) gives rise to the same estimate. Therefore, we achieve

\[
\left| (\Delta^\gamma_w (w \cdot \nabla v) | \Delta^\gamma_w w)_{L^2} \right| \lesssim c^2(t) 2^{-2\ell s} \left( \|w\|_{\dot{H}^{0,s}} \|\nabla_h w\|_{L^2}^{1 + \frac{1}{2s}} \|\nabla_h w\|_{L^2}^{1 - \frac{1}{2s}} + \|w\|_{\dot{H}^{0,s}} \|\nabla_h w\|_{L^2}^{\frac{1}{2} + \frac{1}{4s}} \right),
\]

which implies that

\[
\sum_{\ell \in \mathbb{Z}} 2^{2\ell s} \left| (\Delta^\gamma_w (w \cdot \nabla v) | \Delta^\gamma_w w)_{L^2} \right| \lesssim \|w\|_{\dot{H}^{0,s}} \|\nabla_h w\|_{L^2}^{1 - \frac{1}{2s}} \|\nabla_h w\|_{L^2}^{1 + \frac{1}{2s}} + \|w\|_{\dot{H}^{0,s}} \frac{1}{2} \|\nabla_h w\|_{L^2}^{\frac{3}{2} + \frac{1}{4s}}.
\]

Applying Young’s inequality gives rise to

\[
\sum_{\ell \in \mathbb{Z}} 2^{2\ell s} \left| (\Delta^\gamma_w (w \cdot \nabla v) | \Delta^\gamma_w w)_{L^2} \right| \leq C \left( \|w\|_{\dot{H}^{0,s}} \frac{1}{2s} + \|w\|_{L^2}^2 \|w\|_{\dot{H}^{0,s}}^{\frac{2}{1 + 2s}} \right) \|\nabla_h w\|_{L^2}^2 + \frac{1}{8} \|\nabla_h w\|_{L^2}^2. \tag{4.3}
\]

Similarly, we write

\[
(\Delta^\gamma(w_L \cdot \nabla v) | \Delta^\gamma w)_{L^2} = \left( \Delta^\gamma(u_L^3 \cdot \nabla_h w) | \Delta^\gamma w \right)_{L^2} + \left( \Delta^\gamma(u_L^3 \partial_3 w) | \Delta^\gamma w \right)_{L^2}.
\]

By applying the law of product (2.8), we get

\[
\left| (\Delta^\gamma(u_L^3 \cdot \nabla_h w) | \Delta^\gamma w)_{L^2} \right| \leq \|\Delta^\gamma(u_L^3 \cdot \nabla_h w)\|_{L^2} \|\Delta^\gamma w\|_{L^2} \leq \|\Delta^\gamma(u_L^3 \cdot \nabla_h w)\|_{L^2} \|\Delta^\gamma w\|_{L^2} \leq \left( \|u_L\|_{\dot{H}^{0,s}(L^2)} \|w\|_{\dot{H}^{0,s}} \|\nabla_h w\|_{L^2}^{\frac{1}{2}} + \|u_L\|_{\dot{H}^{0,s}} \|\nabla_h u_L\|_{L^2} \|\nabla_h w\|_{L^2} \|\nabla_h w\|_{L^2} \|\nabla_h w\|_{L^2} \right).
\]

Whereas applying the law of product (2.13) gives

\[
\left| (\Delta^\gamma(u_L^3 \partial_3 w) | \Delta^\gamma w)_{L^2} \right| \leq c^2(t) 2^{-2\ell s} \left( \|\nabla_h u_L\|_{L^2} \|w\|_{\dot{H}^{0,s}} \|\nabla_h w\|_{L^2}^{\frac{1}{2}} + \|\nabla_h w\|_{L^2} \|\nabla_h w\|_{L^2} \|\nabla_h w\|_{L^2} \right).
\]
As a result, it comes out
\[
\left| \left( \Delta_{t}^{Y} (u_{L} \cdot \nabla w) \right) \right| \Delta_{t}^{Y} w \right|_{L^{2}} \leq C \left( \| \nabla_{h} w \|_{L_{t}^{2}(L_{h}^{4})} \right)^{2} \left( \| u_{L} \|_{L_{t}^{6}(L_{h}^{1})} \right) \left( \| \Delta_{t}^{Y} w \|_{L_{t}^{2}(L_{h}^{4})} \right)^{2} + \| u_{L} \|_{L_{t}^{6}(L_{h}^{1})} \left( \| \nabla_{h} u_{L} \|_{L_{t}^{2}(L_{h}^{4})} \right)
\]

Then we get, by a similar derivation of (4.3), that
\[
\sum_{\ell \in \mathbb{Z}} 2^{2\ell s} \left| \left( \Delta_{t}^{Y} (u_{L} \cdot \nabla w) \right) \Delta_{t}^{Y} w \right|_{L^{2}} \leq C \left( \| \nabla_{h} w \|_{L_{t}^{2}(L_{h}^{4})} \right)^{2} \left( \| u_{L} \|_{L_{t}^{6}(L_{h}^{1})} \right) \left( \| \Delta_{t}^{Y} w \|_{L_{t}^{2}(L_{h}^{4})} \right)^{2} + \| u_{L} \|_{L_{t}^{6}(L_{h}^{1})} \left( \| \nabla_{h} u_{L} \|_{L_{t}^{2}(L_{h}^{4})} \right)
\]
and
\[
\sum_{\ell \in \mathbb{Z}} 2^{2\ell s} \left| \left( \Delta^\alpha (u_L \cdot \nabla_h u_L) | \Delta^\alpha w \right|_{L^2} \right|
\leq C \left( \left( \|u_L\|_{L^2(L^4)}^4 + \|\nabla_h u_L\|_{L^2(L^2)}^2 \right) \|w\|_{H^{0,1}}^2 + \|\nabla u_L\|_{L^2(L^2)}^2 \|u_L\|_{H^{0,1}}^2 \right)
+ \frac{1}{8} \left( \|\nabla_h w\|_{H^{0,1}}^2 + \|\nabla_h u_L\|_{H^{0,1}}^2 \right).
\]  

(4.6)

To handle the last two terms in (4.2) involving vertical derivative, we need the following lemma:

Lemma 4.1. Let \( a = (a^h, a^3) \) be a solenoidal vector field. Then we have
\[
\left| \left( \Delta^\alpha (a^3 \partial_3 u_L) | \Delta^\alpha b \right|_{L^2} \right| \lesssim c_2^2 \xi \left( \|a^3\|_{L^t(L^6)} \|\partial_3 u_L\|_{H^{0,1}} \|b\|_{H^{0,1}} \right) \frac{3}{8} \|\nabla_h b\|_{H^{0,1}} \frac{1}{8}
+ \|\partial_3 u_L\|_{L^2(L^6)} \|a^3\|_{H^{0,1}} \left( \|w\|_{H^{0,1}} \|\nabla_h w\|_{H^{0,1}} \right) \frac{1}{8} + \frac{1}{8},
\]

(4.7)

Let us postpone the proof of this lemma till we finish the proof of this proposition.

Applying (2.4) and Lemma 4.1 yields
\[
\left| \left( \Delta^\alpha (w^3 \cdot \partial_3 u_L) | \Delta^\alpha w \right|_{L^2} \right| 
\lesssim c_2^2(t) 2^{-2\ell s} \left( \|\partial_3 u_L\|_{H^{0,1}} \|w\|_{H^{0,1}} \|\nabla_h w\|_{H^{0,1}} \right) \frac{3}{8} \|\nabla_h w\|_{H^{0,1}} \frac{1}{8}
+ \left( \|w^3\|_{L^2} \|\nabla_h w^3\|_{L^2} \right) \frac{1}{8} \|\partial_3 u_L\|_{H^{0,1}} \left( \|w\|_{H^{0,1}} \|\nabla_h w\|_{H^{0,1}} \right) \frac{1}{8} + \frac{1}{8},
\]

and
\[
\left| \left( \Delta^\alpha (u_L^3 \partial_3 u_L) | \Delta^\alpha w \right|_{L^2} \right| \lesssim c_2^2(t) 2^{-2\ell s} \left( \|u_L^3\|_{L^t(L^6)} \|\partial_3 u_L\|_{H^{0,1}} \|w\|_{H^{0,1}} \|\nabla_h w\|_{H^{0,1}} \right) \frac{3}{8}
+ \left( \|u_L^3\|_{L^2(L^6)} \|\nabla_h u_L^3\|_{H^{0,1}} \|w\|_{H^{0,1}} \right) \frac{1}{8} + \frac{1}{8}.
\]

Then a similar derivation of (4.3) gives rise to
\[
\sum_{\ell \in \mathbb{Z}} 2^{2\ell s} \left| \left( \Delta^\alpha (w^3 \partial_3 u_L) | \Delta^\alpha w \right|_{L^2} \right| \lesssim C \left( \|\partial_3 u_L\|_{H^{0,1}} \|w\|_{H^{0,1}} \|\nabla_h w\|_{L^2} \right) \frac{2^{(2\ell + 1)}}{2^{(2\ell + 1)}}
+ \frac{1}{8} \left( \|\partial_3 u_L\|_{H^{0,1}} \|\nabla_h w\|_{H^{0,1}} \right),
\]

(4.8)
and

$$
\sum_{\ell \in \mathbb{Z}} 2^{2\ell s} \left| \left( \Delta_\ell Y (u_3^3 \partial_3 u_L) \right) \Delta_\ell Y w \right|_{L^2} \\
\leq C \left( \| u_L \|_{L^4_y(L^1_x)} \| w \|_{H_{0,s}^1}^2 + \| \partial_3 u_L \|_{L^4_y(L^1_x)} \| u_L \|_{H_{0,s}^1} \| w \|_{H_{0,s}^1}^\frac{3}{2} \right) \\
+ \frac{1}{8} \left( \| \nabla_h u_L \|_{H_{0,s}^1}^2 + \| \nabla_h w \|_{H_{0,s}^1}^2 + \| \partial_3 u_L \|_{H_{0,s}^1}^2 \right). 
$$

(4.9)

Multiplying (4.2) by $2^{2\ell s}$ and summing up the resulting inequalities over $\ell \in \mathbb{Z}$, and then inserting the estimates (4.3), (4.4), (4.5), (4.6), (4.8) and (4.9) into the resulting inequality, we obtain (4.1). This completes the proof.

Proposition 4.1 is proved provided that we present the proof of Lemma 4.1.

Proof of Lemma 4.1. We first get, by applying Bony’s decomposition (2.2) in the vertical variable $a^3 \partial_3 u_L$, that

$$
a^3 \partial_3 u_L = T_{a_3}^Y \partial_3 u_L + R^Y (a^3, \partial_3 u_L).
$$

Considering the support properties to the Fourier transform of the terms in $T_{a_3}^Y \partial_3 u_L$, we infer

$$
\left| (\Delta_\ell Y (T_{a_3}^Y \partial_3 u_L) \mid \Delta_\ell Y b) \right|_{L^2} \lesssim \sum_{|j - \ell| \leq 5} \| S_{j-1}^Y a^3 \|_{L^\infty_y(L^4_x)} \| \Delta_\ell Y \partial_3 u_L \|_{L^2} \| \Delta_\ell Y b \|_{L^2_y(L^4_x)} \\
\lesssim \sum_{|j - \ell| \leq 5} \| a^3 \|_{L^\infty_y(L^4_x)} \| \Delta_\ell Y \partial_3 u_L \|_{L^2} \| \Delta_\ell Y b \|_{L^2_y(L^4_x)} \| \Delta_\ell Y \nabla_h b \|_{L^2_y(L^4_x)} \\
\lesssim c_2 2^{-2\ell s} \| a^3 \|_{L^\infty_y(L^4_x)} \| \partial_3 u_L \|_{H_{0,s}^1} \| b \|_{H_{0,s}^1} \| \nabla_h b \|_{H_{0,s}^1}.
$$

Along the same line, due to $\partial_3 a^3 = -\text{div}_h a^h$, we deduce that

$$
\left| \left( \Delta_\ell Y (R^Y (a^3, \partial_3 u_L)) \mid \Delta_\ell Y b \right) \right|_{L^2} \\
\lesssim \sum_{j \geq \ell - N_0} \| \Delta_j Y a^3 \|_{L^\infty_y(L^4_x)} \| S_{j+2}^Y \partial_3 u_L \|_{L^\infty_y(L^4_x)} \| \Delta_\ell Y b \|_{L^2_y(L^4_x)} \\
\lesssim \sum_{j \geq \ell - N_0} \| \Delta_j Y a^3 \|_{L^2_y(L^4_x)} \| \Delta_j Y \partial_3 a^3 \|_{L^2_y(L^4_x)} \| \partial_3 u_L \|_{L^\infty_y(L^4_x)} \| \Delta_\ell Y b \|_{L^2_y(L^4_x)} \| \Delta_\ell Y \nabla_h b \|_{L^2_y(L^4_x)} \\
\lesssim c_2 2^{-2\ell s} \| \partial_3 u_L \|_{H_{0,s}^1(L^4_x)} \| a^3 \|_{H_{0,s}^1} \| \nabla_h a^h \|_{H_{0,s}^1} \| b \|_{H_{0,s}^1}.
$$

This leads to (4.7).
Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The local well-posedness part of Theorem 1.1 has been proved in [4, 11]. Let us denote $T^*$ be the maximal existence of such a solution. It remains to prove (1.4). Indeed, we first get, by a similar derivation of (3.10) that

$$
\int_0^t \| \nabla_h u_L(t') \|_{L_h^2(L^\infty)}^2 dt' \leq C \int_0^t \left( \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} e^{-ct'2^{2k}} \| \Delta_k^h \Delta_\ell u_0 \|_{L^2} \right)^2 dt'
$$

$$
\leq C \int_0^t \| \nabla_h u_L(t') \|_{L_h^2(L^\infty)}^2 dt' \sup_{t' \in [0,t]} \left( \sum_{k \in \mathbb{Z}} e^{-ct'2^{2k}} \| \Delta_k^h \Delta_\ell u_0 \|_{L^2} \right)^2
$$

$$
\leq C_s t^{s-\frac{1}{2}} \| u_0 \|_{H^s}^2.
$$

(4.10)

Now let us define

$$
T_1^* \overset{\text{def}}{=} \sup_{0 \leq t \leq T} \left\{ T \leq T^*(u_0); \sup_{0 \leq t \leq T} \left( \| w(t) \|_{H^{0,s}}^2 + \| \nabla_h w \|_{L^2_h(H^{0,s})}^2 \right) \leq 4 \| u_0 \|_s^2 \right\}
$$

with

$$
\| u_0 \|_s^2 \overset{\text{def}}{=} \| u_0 \|_{H^{0,s}}^2 + \| \partial_3 u_0 \|_{H^{s-1,0}}^2 + \| \partial_3 u_0 \|_{H^{s-1,0}}^2.
$$

(4.11)

Then for $t \leq T_1^*$, by virtue of (3.11) and (3.12), we have

$$
\int_0^t \| \partial_3 u_L \|_{L_h^2(L^4)}^4 \| u_L \|_{H^{0,s}}^2 \| w \|_{H^{s-1,0}}^2 dt' \leq C_s t^{2s-1} \| \partial_3 u_0 \|_{H^{s-1,0}}^1 \| u_0 \|_s^2,
$$

(4.12)

and

$$
\int_0^t \left( \| u_L \|_{L^4(L^4)}^4 + \| \nabla_h u_L \|_{L^2(L^6)}^2 + \| \partial_3 u_L \|_{L^2(L^4)}^4 \right) \left( \| w \|_{H^{0,s}}^2 + \| u_L \|_{H^{0,s}}^2 \right) dt' \leq C_s \left( t^{2s-1} \| u_0 \|_{H^s}^4 + t^{2s-\frac{1}{2}} \| \partial_3 u_0 \|_{H^{s-1,0}}^4 + t^{s-\frac{1}{2}} \| u_0 \|_{H^s}^4 \right) \| u_0 \|_s^2.
$$

(4.13)

In order to deal with the integral of the last term in (4.1), we define

$$
T_2^* \overset{\text{def}}{=} \min \left( T_1^*, \varepsilon \| u_0 \|_s^{-\frac{1}{2}} \right).
$$

(4.14)

Then for $t \leq T_2^*$, we deduce from Proposition 3.1 that

$$
\frac{\| w(t) \|_{L^2}}{t^s} + \int_0^t \frac{\| \nabla_h w(t') \|_{L^2}}{(t')^s} dt' \leq C t^{s-\frac{1}{2}} \| u_0 \|_{H^s}^4.
$$

(4.15)
As a result, we find

\[
\int_0^t \| \nabla_h w \|^2_{L^2} \, dt' \leq C \left( t^8 \| u_0 \|^4_{H^{0, s}} + C t^{2s} \| u_0 \|_{\dot{H}^{2s}} \sup_{t' \in [0, t]} \frac{\| w(t') \|^2_{L^2}}{(t')^5} \right) \int_0^t \| \nabla_h w(t') \|^2_{L^2} \, dt' \leq C \left( t^{2s-\frac{1}{2}} \| u_0 \|_{\dot{H}^{0, s}}^{\frac{4s}{s^2}} + t^{4s-1} \| u_0 \|_{\dot{H}^{2s}}^{\frac{12s-2}{s^2}} \right) \| u_0 \|_{\dot{H}^{0, s}}^4.
\]

(4.16)

Let \( s = \frac{1}{2} + 2\gamma \) for \( \gamma \in (0, 1/4) \), for \( t \leq T_2^* \), by integrating (4.1) over \([0, t]\) and then inserting the estimates, (4.12), (4.13) and (4.16), into the resulting inequality, we obtain

\[
\| w(t) \|^2_{L^2} + \| \nabla_h w \|^2_{L^2} \leq 2 \| u_0 \|^2_{H^{0, s}} + \| \partial_3 u_0 \|^2_{\dot{H}^{-1, s}} + C_s \left( t^{4\gamma} \| u_0 \|_{\dot{H}^{0, s}}^{\frac{4}{s^2}} + t^{2\gamma} \| u_0 \|_{\dot{H}^{2s}}^{\frac{2}{s^2}} + t^{1+8\gamma} \| u_0 \|_{\dot{H}^{2s}}^{\frac{1+8\gamma}{s^2}} + t^{1+8\gamma} \| u_0 \|_{\dot{H}^{2s}}^{\frac{1+8\gamma}{s^2}} \right) \| u_0 \|_{\dot{H}^{0, s}}^2.
\]

(4.17)

Therefore as long as \( \varepsilon \) is sufficiently small in (4.14), we deduce from (4.17) that

\[
\| w(t) \|^2_{L^2} + \| \nabla_h w \|^2_{L^2} \leq 3 \| u_0 \|_{\dot{H}^{0, s}}^2 \quad \text{for} \quad t \leq T_2^*.
\]

(4.18)

Then a standard continuous argument shows that \( T_1^* \geq \varepsilon \| u_0 \|_{\dot{H}^{0, s}}^{-\frac{1}{7}} \). Hence we conclude that

\[
T^*(u_0) \geq \varepsilon \| u_0 \|_{\dot{H}^{0, s}}^{-\frac{1}{7}}.
\]

This completes the proof of Theorem 1.1.

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