Quantum noise effects with Kerr nonlinearity enhancement in coupled gain-loss waveguides

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It is generally difficult to study the dynamical properties of a quantum system with both inherent quantum noises and non-perturbative nonlinearity. Due to the possibly drastic intensity increase of an input coherent light in the gain-loss waveguide couplers with parity-time (PT) symmetry, the Kerr effect from a nonlinearity added into the systems can be greatly enhanced, and is expected to create the macroscopic entangled states of the output light fields with huge photon numbers. Meanwhile, the quantum noises also coexist with the amplification and dissipation of the light fields. Under the interplay between the quantum noises and nonlinearity, the quantum dynamical behaviors of the systems become rather complicated. However, the important quantum noise effects have been mostly neglected in the previous studies about nonlinear PT-symmetric systems. Here we present a solution to this non-perturbative quantum nonlinear problem, showing the real-time evolution of the system observables. The enhanced Kerr nonlinearity is found to give rise to a previously unknown decoherence effect that is irrelevant to the quantum noises, showing the real-time evolution of the system observables. The enhanced Kerr nonlinearity is found to give rise to a previously unknown decoherence effect that is irrelevant to the quantum noises, imposing a limit on the emergence of macroscopic nonclassicality. In contrast to what happen in the linear systems, the quantum noises exert significant impact on the system dynamics, and can create the nonclassical light field states in conjunction with the enhanced Kerr nonlinearity. This first study on the noise involved quantum nonlinear dynamics of the coupled gain-loss waveguides can help to better understand the quantum noise effects in the broad nonlinear systems.

I. INTRODUCTION

Originating from the uncertainty relation in quantum mechanics, quantum noises are ubiquitous in open quantum systems coupled to their environment. Those accompanying light dissipation are unavoidable in any realistic quantum optical system [1], while the noise going together with light amplification determines the quantum limit of an amplifier [2]. These most commonly encountered quantum noises can be regarded as the random drives from the associated external reservoirs interacting with a quantum system, and they also follow the law of quantum mechanics. The solvability of linear Heisenberg-Langevin equations makes it possible to describe the quantum noise effects in systems with quadratic Hamiltonians, but the situations of nonlinear quantum systems are much more complicated. In classical nonlinear systems, noises as the fluctuations of system parameters or random external forces are known to give rise to various interesting physical effects such as stochastic resonance [3], noise-induced phase transition [4] and phase synchronization [5], etc. However, the effects of quantum noises, especially those in the systems with non-perturbative nonlinearity, remained to be uncovered yet.

Open quantum systems with Kerr nonlinearity are meaningful examples for studying quantum noise effects. Kerr nonlinearity is considered as a prerequisite for generating macroscopic photonic states [6] and operating deterministic quantum logic devices [7]. The Kerr coefficient in a natural material is typically small, though it can be enhanced in coherently prepared atomic ensembles [8] or Josephson junctions [8,10]. The straightforward way to get a larger Kerr nonlinear effect on an input light is to strengthen its intensity. Such seemingly trivial practice of enhancing Kerr nonlinearity can lead to interesting phenomena in a simple system as illustrated in Fig. 1. Here the two coupled waveguides are with the balanced gain and loss rates, respectively, realizing an optical analogue of parity-time (PT) symmetric quantum mechanics [11,12]. This model has attracted intensive researches in recent years, and some recent experiments with similar systems [13,16] have demonstrated its interesting light transmission properties. In such systems the light intensity undergoes a drastic transition from periodic oscillation to exponential increase when they are tuned into the PT symmetry broken regime. If one of the waveguides is also added with a weak Kerr nonlinearity, one will see its significant influence on the light field dynamics because its effects are greatly enhanced to those in the non-perturbative nonlinear regime after breaking the PT symmetry. In a Kerr medium without gain or loss, an input coherent light will evolve into a so-called Kerr state that might manifest macroscopic nonclassicality [17,18], and its evolution to a photonic Schrödinger cat state of a few photons has been demonstrated with a circuit QED setup that realizes an effective strong Kerr nonlinearity [19]. By making use of the above mentioned Kerr effect enhancement from light amplification, two-mode macroscopic nonclassical states (such as entangled states of light fields) are expected to be generated. On the other hand, the existing quantum noises with the gain and loss of the light fields could destroy such nonclassicality as in other quantum systems [20]. To clarify these
possibilities, it is necessary to find a complete solution to the system dynamics including the noise effects.

The similar setups with the assumed strong nonlinearity \[21\, 28\] were theoretically studied in the context of all-optical signal control such as non-reciprocal light propagation, and other classical Kerr type nonlinear PT-symmetric systems have also been explored (see, e.g. \[29\, 34\]). In addition to the above researches on the classical aspect of the nonlinear PT-symmetric systems, the quantum nonlinear properties of another type PT-symmetric system have found the application of implementing a phonon laser recently \[33\]. However, except for the noise-induced spontaneous photon generation in linear couplers \[36\], little was known about the effects of the quantum noises in these PT-symmetric nonlinear systems. An obvious difficulty in approaching the dynamics of these systems is the non-integrability of their dynamical equations in the presence of the noise terms. Since the significant enhancement of Kerr effect is from the exponentially growing light intensity after breaking the PT symmetry, there is no steady-state solution in the course of evolution to the non-perturbative nonlinear regime. So far the linearization of the dynamical equations around the steady-states is the most commonly adopted approach to a quantum nonlinear system \[37\], but it is not workable for studying the dynamical process in the systems. Generally, few approaches except for numerical simulation \[38\] have been reported for dealing with the dynamics of a nonlinear system in the non-perturbative regime and under the quantum noise effects at the same time.

Here, we present a first solution to such challenging dynamical problem for a coherent light sent into the nonlinear coupler in Fig. 1. A novel phenomenon found by our approach is the decoherence of an input coherent light going together with the Kerr nonlinearity enhancement in the symmetry breaking regime, but it does not originate from the quantum noises as in other quantum systems. On the other hand, the quantum noises, which can be hardly dealt with in other approaches, significantly influence the dynamics of this nonlinear system, leading to a much more complicated system evolution than what is predicted by the simplified model neglecting the noises.

Moreover, analogous to the synergic effects in the classical nonlinear systems \[3\, 38\], the nonclassical states of the output light fields can only be realized by the enhanced nonlinearity under the joint quantum noise action.

## II. QUANTUM DYNAMICS OF COUPLED GAIN-LOSS WAVEGUIDES

Under the full dynamics including the light field coupling under the balanced gain and loss, as well as the nonlinearity of the Kerr coefficient \(\chi\), the Heisenberg-Langevin equations for the two waveguide modes in Fig. 1 read

\[
\frac{d}{dt} \hat{a}(t) = \kappa \hat{a}(t) - iJ \hat{b}(t) + \sqrt{2\kappa} \hat{\xi}_a(t),
\]

\[
\frac{d}{dt} \hat{b}(t) = -\kappa \hat{b}(t) - iJ \hat{a}(t) - i\chi \hat{b}^\dagger(t) \hat{b}(t) + \sqrt{2\kappa} \hat{\xi}_b(t).
\]

Here the quantum noise \(\hat{\xi}_a(t), \hat{\xi}_b(t)\) coexisting with the light field amplification (dissipation) at the rate \(\kappa\) also acts on the waveguide mode coupling at the rate \(J\) with the other one, and the noise operator satisfies the commutation relation \([\hat{\xi}_c(t), \hat{\xi}_c(t')^\dagger] = \delta(t - t')\) and the correlations \([\hat{\xi}_c(t), \hat{\xi}_c(t')] = 0\) for \(c = a, b\).

Note that the notation for the amplification noise here is different from that in \[31\, 33\], so that the correlations for both amplification and dissipation noise can be written in the unified forms. The input light we work with is in the coherent state \(|\alpha\rangle\), and initially enters the gain channel without loss of generality. We assume \(|\alpha| \gg 1\) with an aim to create macroscopic nonclassical states, but the product \(|\alpha|^2\) is small so that the Kerr effect is still weak from the input.

The properties of the dynamical process in \[1\] can be seen better from the attempts to solve the problem by the standard methods. The similar equations were discussed in a mean field approach which replaces the mode operator \(\hat{a} (\hat{b})\) with its expectation value \(\langle \hat{a} (\hat{b}) \rangle \) (\(\beta = \langle \hat{b} \rangle\)). Though the resulting nonlinear Langevin equations for the similar processes can be reduced to the solvable ones (see, e.g. \[21\, 32\]), the quantum noise terms are averaged out so that it is not appropriate to see many meaningful quantum effects in this way. Another straightforward solution following the practice of numerical simulation \[38\] gives the iterative forms of the evolved modes

\[
\hat{a}[n + 1] = (\sum_{c = a, b} M_{a,c} \hat{c}[n]) + \sum_{\beta = \zeta, \zeta'} N_{a,\beta} \hat{\beta}[n] \delta t,
\]

\[
\hat{b}[n + 1] = (\sum_{c = a, b} M_{b,c} \hat{c}[n]) + \sum_{\beta = \zeta, \zeta'} N_{b,\beta} \hat{\beta}[n] \delta t + e^{-i\chi \hat{b}^\dagger[n] \hat{b}[n]} \hat{\xi}_b[n].
\]
modes plus the amplification and dissipation of the light fields, and the matrix $\mathcal{N}$ is due to the noise drives. For this quantum system, however, the linear coupling between the modes, as well as to the noises, and the action of the Kerr nonlinearity are not commutative. The above procedure is therefore consistent with the real system evolution only in the limit of infinite number of iterative steps ($\delta t \to 0$). The errors from the non-commutativity of the linear and nonlinear actions will accumulate for any finite $\delta t$.

One could also take an alternative route in the Schrödinger picture. The commonly used tool is the quantum master equation

$$\dot{\rho} = -i [J(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) + \frac{1}{2} \sqrt{\kappa^2 + J^2} \rho] + \mathcal{L}_a \rho + \mathcal{L}_d \rho \tag{3}$$

for the density matrix $\rho$ of the system, where the superoperator operation in the Lindblad form

$$\mathcal{L}_a \rho = -\frac{1}{2} (\hat{a}^\dagger \rho \hat{a} - 2 \hat{a}^\dagger \rho \hat{a}) \tag{4}$$

describes the amplification process, and the corresponding one

$$\mathcal{L}_d \rho = -\frac{1}{2} (\hat{b}^\dagger \rho \hat{b} - 2 \hat{b}^\dagger \rho \hat{b}) \tag{5}$$

is about the dissipation process. The analytical solutions to the master equations with Kerr nonlinearity were found only for the single mode situation thus far (see, e.g. [39, 41]). The more severe disadvantage of the master equation approach is with the amplification process. According to Eq. 3, a simple input as the vacuum state $|0\rangle_s |0\rangle_b$ will not change with time; it can be seen from the exact solution of the master equation in the limit $J \to 0$.

Then the zero output photon number calculated as the average $\text{Tr}[\hat{a}^\dagger \hat{a} \rho(t)]$ contrasts with the phenomenon of spontaneous photon generation [38], i.e. the generation of photons from a vacuum input state. The reason for such discrepancy will be explained below.

### III. TRANSITION FROM PERTURBATIVE TO NON-PERTURBATIVE NONLINEAR DYNAMICS

Our approach to this dynamical problem is based on the observation of the following Kerr nonlinearity transition across the threshold of $PT$ symmetry. One sees the jump of the light intensity from the solution

$$\left( \begin{array}{c} \hat{A}(t) \\ \hat{B}(t) \end{array} \right) = \left( \begin{array}{c} -ie^{\lambda_1(t-\tau)} \hat{n}_1 \hat{A}_1(t) + ie^{\lambda_2(t-\tau)} \hat{n}_2 \hat{A}_2(t) \\ e^{\lambda_1(t-\tau)} \hat{B}_1(t) + e^{\lambda_2(t-\tau)} \hat{n}_2 \hat{B}_2(t) \end{array} \right)$$

$$+ \sqrt{2} \kappa \int_0^t d\tau \left( \begin{array}{c} -ie^{\lambda_1(t-\tau)} \hat{n}_1 \hat{A}_1(\tau) + ie^{\lambda_2(t-\tau)} \hat{n}_2 \hat{A}_2(\tau) \\ e^{\lambda_1(t-\tau)} \hat{B}_1(\tau) + e^{\lambda_2(t-\tau)} \hat{n}_2 \hat{B}_2(\tau) \end{array} \right)$$

$$\tag{6}$$

to \(1\) in the linear coupler limit with $\chi = 0$, where $\hat{o}(\hat{n})_1 = \frac{i}{\eta_1 + \eta_2} \hat{a}(\hat{\xi}_1(t)) + \frac{\eta_2}{\eta_1 + \eta_2} \hat{b}(\hat{\xi}_2(t))$ and $\hat{o}(\hat{n})_2 = -i \frac{J_0}{\eta_1 + \eta_2} \hat{a}(\hat{\xi}_1(t)) + \frac{\eta_2}{\eta_1 + \eta_2} \hat{b}(\hat{\xi}_2(t))$ with $\eta_1(2) = \mp \kappa + \sqrt{\kappa^2 - J^2}$ and $\lambda_1(2) = \mp \sqrt{\kappa^2 - J^2}$. The linear coupler mode $\hat{A}(\hat{B})$ oscillates in the $PT$-symmetric regime of $\kappa < J$, but will exponentially grow if the symmetry is broken when $\kappa > J$. Given a weak Kerr nonlinearity as in Fig. 1, such drastic increase of the light intensity across the threshold $\kappa = J$ can greatly enhance its effects. Meanwhile, one noise component in $[40]$ also gets much larger in the symmetry broken regime, contributing to a more significant spontaneous photon generation.

Then we reformulate the process in Eq. 1 in terms of the stochastic Hamiltonian

$$H_L(t) = J(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) + i \sqrt{2 \kappa} \{ \hat{a}^\dagger \hat{\xi}_1(t) - \hat{a} \hat{\xi}_2(t) \}$$

$$+ i \sqrt{2 \kappa} \{ \hat{b}^\dagger \hat{\xi}_3(t) - \hat{b} \hat{\xi}_4(t) \},$$

$$\tag{7}$$

which does not commute with the additional Kerr nonlinear term $H_{NL} = 1/2 \{ \hat{b} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \}$. The system evolves under the joint operation $U_L(t)$ as a time-ordered exponential $\mathcal{T} e^{-i \int_0^t d\tau H_L(\tau)}$ ($\hbar = 1$) on both system and reservoirs, together with a perturbation of $H_{NL}$ when its effect is weak. Different from the last term involving the dissipation noise in $[41]$, a vacuum state will not be kept invariant if acting the second term about the gain process on it. This property explains the phenomenon of spontaneous photon generation, but cannot be reflected by the master equation $[43]$ about the reduced density matrix of the system alone, hence the limitation of the master equation in the presence of the amplification noise. The quantum master equation $[43]$ can be derived from averaging out the noise-reservoir part in the momentary action $e^{-i \{ H_L(t) + H_{NL} \} \delta t}$ of the stochastic Hamiltonian on a joint state of the system and reservoir (see, e.g. $[41]$), so it is actually equivalent to the Langevin equations about the mean values $\langle \hat{a} \rangle$, $\langle \hat{b} \rangle$ of the system modes.

To solve the dynamics of the quantum nonlinear system, we expand its evolution operator in terms of the perturbative Hamiltonian $H_{NL}$ as follows:

$$U(t) = \mathcal{T} e^{-i \int_0^t d\tau (H_L + H_{NL})(\tau)}$$

$$= U_L(t) \{ I - i \int_0^t ds_1 U_L^\dagger(s_1) H_{NL} U_L(s_1) - \int_0^t ds_1 U_L^\dagger(s_1)$$

$$\times H_{NL} U_L(s_1) \int_0^{s_1} ds_2 U_L^\dagger(s_2) H_{NL} U_L(s_2) + \cdots \}.$$  \(\tag{8}\)
Using the infinite product form \( \prod_{\ell=1}^{\infty} e^{-i U_{NL}(t)H_{NL}U_{NL}(t)\delta t} \) of the nonlinear action \( U_{NL}(t) \), one will find the exact forms of the evolved modes with Eq. (8) as follows:

\[
\begin{align*}
U^\dagger(t) \hat{a}U(t) &= U^\dagger_{\text{NL}}(t) \hat{A} U_{\text{NL}}(t) = \hat{A}(t) \\
- i \chi \int_0^t \text{d}r \text{c}_{\text{ab}}(\tau) U^\dagger_{\text{NL}}(t, \tau) \hat{B}(\tau) \hat{B}(\tau) U_{\text{NL}}(t, \tau), \\
U^\dagger(t) \hat{b}U(t) &= U^\dagger_{\text{NL}}(t) \hat{B}(t) U_{\text{NL}}(t) = \hat{B}(t) \\
- i \chi \int_0^t \text{d}r \text{c}_{\text{ab}}(\tau) U^\dagger_{\text{NL}}(t, \tau) \hat{B}(\tau) \hat{B}(\tau) U_{\text{NL}}(t, \tau),
\end{align*}
\]

(9)

where \( \text{c}_{\text{ab}}(t, t') = [\hat{A}(t), \hat{B}(t')] \), \( \text{c}_{\text{ab}}(t, t') = [\hat{B}(t), \hat{B}(t')] \) are the commutators of the linear coupler modes in (8), and \( U_{\text{NL}}(t, \tau) = T e^{-i \int_0^t \text{d}r U^\dagger_{\text{NL}}(\tau) H_{\text{NL}} U_{\text{NL}}(\tau)} \). Iteratively applying the transformation

\[
\begin{align*}
U^\dagger_{\text{NL}}(t, \tau) \hat{B}(\tau) U_{\text{NL}}(t, \tau) &= \hat{B}(\tau) \\
- i \chi \int_\tau^t \text{d}t' \text{c}_{\text{ab}}(\tau, t') U^\dagger_{\text{NL}}(t, t') \hat{B}(t') \hat{B}(t') U_{\text{NL}}(t, t')
\end{align*}
\]

(10)

in Eq. (9) leads to two series expansions of the folded integrals to all orders of the Kerr coefficient \( \chi \). These general forms of the evolved modes are valid in any regime of the system parameters.

A nontrivial result with the above approach is that, in the symmetry broken regime, the evolved modes can be reduced to the closed forms

\[
\begin{align*}
U^\dagger(t) \hat{a}U(t) &= \frac{\eta_2}{4} e^{-\iota(\zeta_1 + \zeta_2)} \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \hat{B}(\tau) \\
&+ \frac{\eta_2}{4} e^{-\iota(\zeta_1 + \zeta_2)} \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \hat{B}(\tau) - 1) \\
&\times \left( \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \hat{B}(\tau) \right) \int_0^t \text{d}t' e^{2 \lambda \tau} \hat{B}(t') \hat{B}(t'), \\
U^\dagger(t) \hat{b}U(t) &= e^{-\iota(\zeta_1 + \zeta_2)} \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \hat{B}(\tau) \\
&+ \left( e^{-\iota(\zeta_1 + \zeta_2)} \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \hat{B}(\tau) - 1) \\
&\times \left( \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \hat{B}(\tau) \right) \int_0^t \text{d}t' e^{2 \lambda \tau} \hat{B}(t') \hat{B}(t') \hat{B}(t'),
\end{align*}
\]

(11)

where \( \hat{\nu}(t, t') = \sqrt{2 \kappa} \int_0^t \text{d}r e^{\lambda(\tau - \tau)} \hat{n}_2(\tau) \) is a noise operator. The derivation of the result and the estimation of the corrections to the solution are given in Appendix B and C, respectively. The coefficient \( \zeta_1 = \frac{\eta_1^2 + J^2}{(\eta_1 + \eta_2)^2} \) in the phase operator on the right sides of the above equations comes from the system operator \( \hat{a}_2 \) in (10), but the other one \( \zeta_2 = \frac{\kappa}{\lambda^2} \frac{\eta_1^2 - J^2}{(\eta_1 + \eta_2)^2} \) is due to the accompanying noise operator \( \hat{n}_2 \) including both amplification and dissipation noise. In the current problem, therefore, both of the system and reservoir degrees of the freedom contribute to the nonlinearity induced phase in Eq. (11). Another interesting feature of this non-perturbative solution is that the mode \( U^\dagger(t) \hat{a}U(t) \) out of the gain channel happens to be \( i \eta_2/J \) times of the mode \( U^\dagger(t) \hat{b}U(t) \) out of the loss channel, due to the forms of the evolved linear coupler modes in Eq. (10).

To illustrate the quantum dynamics of the system, one should find the expectation values of the quantum operators evolving under the full dynamics including the quantum noise effects. For our input in the continuous-variable (CV) state \( \rho_{in} = |\alpha_0\rangle_\alpha \langle \alpha_0| \), a suitable operator that conveys much information is the quadrature \( \hat{x}_\phi(\phi) = 1/2(\hat{a}e^{-\iota \phi} + \hat{a}^\dagger e^{\iota \phi}) \) of the evolving field modes \( \hat{c}(t) = \hat{a}(t) + \hat{b}(t) \). Going back to the Schrödinger picture, the non-zero expectation of a quadrature means the existence of the quantum coherence from the superposition of the Fock basis, i.e., the presence of the off-diagonal elements \(|n\rangle \langle n + 1| \) and \(|n + 1\rangle \langle n| \) \((n \geq 0)\) of the evolved density matrix. We apply the mean quadrature values in the different direction \( \phi \) to test if the initial coherent state could evolve to a less coherent one. In finding its expectation value \( \langle \hat{x}_\phi(\phi) \rangle \) we need to average over both system and reservoir degrees of freedom. It can be done by first tracing out the reservoir part as in the following procedure:

\[
\langle \hat{b}(t) \rangle = \text{Tr}_{S,R}[U^\dagger(t) \hat{b}U(t) \times |\alpha_0\rangle_\alpha \langle \alpha_0| \otimes \rho_R]
\]

\[
= a \langle \alpha_0| e^{-i(\zeta_1 + \zeta_2)} x \int_0^t \text{d}r e^{2 \lambda \tau} \hat{a}_2 \hat{a}_2 + e^{2 \lambda \tau} \sigma(\tau) \rangle \langle \alpha_0|_a,
\]

(12)

where

\[
\sigma(\tau) = \frac{\kappa}{\lambda_2} \frac{J}{\eta_1 + \eta_2} e^{2 \lambda \tau} - 1
\]

(13)

and

\[
\sigma(\tau, t) = e^{2 \lambda \tau} \frac{\kappa}{\lambda_2} \frac{J}{\eta_1 + \eta_2} \right)^2 e^{2 \lambda \tau} \left( 1 - e^{-2 \lambda \tau} \right)
\]

(14)

In (12), the expectation value of an evolving waveguide mode, the first term is obtained by taking the average of the noise operators inside the phase operator \( e^{-i(\zeta_1 + \zeta_2)} x \int_0^t \text{d}r e^{2 \lambda \tau} \hat{B}(\tau) \) in (11) over the reservoir state \( \rho_R \), while the second is found by averaging the noise part in the operator \( \hat{B}(t) \) on the right side of (11) with its counterpart in this phase operator. The term containing the noise operator \( \hat{\nu}(t, t') \) in (11) does not contribute to the expectation value. We use Wick’s theorem to obtain the average. The corresponding expectation value after neglecting the quantum noise effects [assuming the noise operators to be zero in the derivations of Eqs. (11) and (12)] for the dynamical process is

\[
\langle \hat{b}(t) \rangle = a \langle \alpha_0| e^{-i \chi x} x \int_0^t \text{d}r e^{2 \lambda \tau} \hat{a}_2 \hat{a}_2 + e^{2 \lambda \tau} \hat{a}_2 \hat{a}_2 |\alpha_0\rangle_\alpha.
\]

(15)

Its difference from the complete quantum expectation value in Eq. (12) indicates the nontrivial role of the noises to the system dynamics.
FIG. 2: Kerr nonlinearity induced change of a mean quadrature in the \(\mathcal{PT}\)-symmetric regime. The definition for the ratio shown on the vertical axis of the plots is given in the text, and its distribution is over the time and in the parameter space. Here we use the quadratic mean of \(\langle \hat{X}_b(\pi) \rangle\) over an oscillation period to avoid the singularities from its vanishing values, and the corrections are calculated to the first order of the Kerr coefficient \(\chi\) in Eq. (3). The system parameters are \(\chi = 10^{-9} \kappa\) and \(\alpha_0 = 10^3\).

FIG. 3: Change of a mean quadrature under the enhanced Kerr nonlinearity in the symmetry broken regime. The upper panels, one as the front view along the time axis and the other as the corresponding back view, are about the realistic situation under the quantum noise effects. The lower ones describe the simplified model without quantum noise. The platforms of the unit ratio indicate where the mean quadrature disappears due to decoherence, and the valleys of zero ratio are where there is no Kerr effect. Here we take the Kerr coefficient and input coherent state amplitude in Fig. 2.

IV. KERR NONLINEARITY ENHANCEMENT AND QUANTUM NOISE EFFECTS

Though it is conceivable that the \(\mathcal{PT}\) symmetry breaking will enhance the Kerr effect with the much amplified light intensity, how it works in the system is a main issue we should clarify. For this purpose we illustrate the change ratio \(\Delta|\langle \hat{X}_b(\pi) \rangle/|\langle \hat{X}_b(\pi) \rangle|\), where \(\Delta|\langle \hat{X}_b(\pi) \rangle|\) is the absolute difference between the mean quadrature \(\langle \hat{X}_b(\pi) \rangle\) under the Kerr effect and \(\langle \hat{X}_b(\pi) \rangle\) without the Kerr nonlinearity (\(\chi = 0\)). In the \(\mathcal{PT}\)-symmetric regime of \(\kappa < J\), the correction to the average quadrature due to the Kerr nonlinearity can be found by the perturbative expansion according to \(\chi\); see Fig. 2. From Eq. (11) we can also obtain the corresponding ratios in the symmetry broken regime. The light field is magnified with the factor \(e^{\lambda_2 t} = e^{\sqrt{\kappa^2 - J^2} t}\) in the symmetry broken regime, so it would take a longer time to enhance the Kerr nonlinearity at a larger \(J\). However, the higher coupling rate \(J\) enables more light to enter the channel filled with the Kerr medium, making its effect larger. These tendencies combined give rise to the illustrated change ratio distribution in the upper panels of Fig. 3. Surprisingly, there will appear the unit ratio platforms (see the upper right panel of Fig. 3), on which the mean quadrature \(\langle \hat{X}_b(\pi) \rangle\) is totally eliminated after a seemingly irregular evolution period. As a contrast, the mean quadrature \(\langle \hat{X}(\phi) \rangle\) of the input coherent state can never vanish for arbitrary \(\phi\). A decoherence process thus occurs during the time evolution of an input coherent light. One question is whether it is connected with the quantum noise effects boosted in the symmetry broken regime?

To analyze the \(\mathcal{PT}\) symmetry broken regime more thoroughly, we cut across one point on the axis \(J/\kappa\) of Fig. 3 to see the evolution of four mean quadratures in

FIG. 4: Time evolution of the mean quadratures. Here we use the notations \(\hat{X}_c(0) = \hat{X}_C\) and \(\hat{X}_c(\phi) = \hat{P}_C\), for \(C = A\) (the gain channel) or \(B\) (the loss channel). The solid curves represent the average quadrature values given the Kerr nonlinearity of \(\chi = 10^{-9} \kappa\), and the dashed curves stand for those of the linear coupler with \(\chi = 0\). The evolutions take place at \(J = 0.1 \kappa\) for the coherent light of \(\alpha_0 = 10^3\). In (d) a refined view of the exponentially accelerating oscillation is shown in the inserted plot. Due to our choice of initially sending the light into the gain channel, the dashed curves coincide with the horizontal axis in (b) and (c).
Fig. 4. The dashed curves in the figure describe the linear coupler situation [36], in which the quantum noises do not affect the mean quadrature evolutions at all. There is a pretty symmetry between the mean quadrature evolution in the gain and loss channel, due to the proportionality of the evolved modes in (11). During the beginning period their time evolutions show no difference from those without the Kerr nonlinearity. In addition to deviating these mean quadratures from those of a linear coupler, the gradually enhanced nonlinear action brings an oscillation pattern to their evolutions and, interestingly, the oscillation becomes exponentially fast with time. The seemingly irregular areas of the change ratio distribution in Fig. 3 are where such exponentially accelerating oscillation exists. We can track down its cause in Eq. (12).

The terms in this equation are proportional to the factor

\[ a \langle 0 | e^{-i(\zeta_1 + \zeta_2) t} \sigma_1 \sigma_2 | 0 \rangle_a, \]

which is actually the overlap between the input state \(| 0 \rangle_a \) and the product \( e^{-i(\zeta_1 + \zeta_2) t} \sigma_1 \sigma_2 | 0 \rangle_a \) of two transformed coherent states. Under the broken PT symmetry giving a real number \( \lambda_2 \), the operator-valued phase factor before the input state oscillates at exponentially increasing frequency with time, hence the same behavior of the overlap.

The overlap \( \langle \beta_2(t) | 0 \rangle \) defined in the above tends to zero with its exponentially accelerating oscillation, leading to the decoherence indicated by the vanishing mean quadratures. This happens because such oscillation from the continually enhanced nonlinear action will instantaneously repeat any close to zero value in the limit of its vanishing oscillation period, and the mean values \( \langle \hat{a}^n \rangle \) and \( \langle \hat{b}^n \rangle \) \((n \geq 1)\) will be killed in this way. The decoherence phenomenon we have illustrated is irrelevant to the quantum noises, as it also exists in the assumed situation without quantum noise; see the lower panels of Fig. 3. The change ratio distribution based on the “noiseless” expectation value in (15) contains a connected unit ratio platform indicating the decoherence, and it comes into being much earlier than those under the noise effects in the upper panels of Fig. 3. Due to such decoherence, the light field state of an input coherent state evolving in the symmetry broken regime is very different from a Kerr state [7] generated under a constant Kerr nonlinearity. The coherence of a Kerr state exhibited by its mean quadrature \( \langle X(\phi) \rangle \neq 0 \) with arbitrary \( \phi \) can keep for long time and revive periodically, but an input coherent light will eventually evolve to a decohered one in our concerned nonlinear coupler.

In such open quantum system the noise components will inevitably enter the evolved light field modes as in [9] and [11], so some of the system observables can gain the extra contributions from taking the averages of the involved amplification noise operators over their associated reservoir state. One effect manifested by the evolved photon number operators like this is spontaneous photon generation, which destroys the nonclassicality of quantum light sent into a linear coupler [36] but is negligible to our system of strong light fields. The more nontrivial way that the quantum noises affect the system dynamics is through their interplay with the Kerr nonlinearity. In each small step of evolution like that with \( \delta t \to 0 \) in Eq. (2), the noise components enter the phase induced by the nonlinearity, while the nonlinear term containing the noise contribution significantly modifies the evolution from that of a linear coupler. The result on a mean quadrature evolution can be seen from comparing the “noisy” and “noiseless” change ratio distributions in Fig. 3. Totally different from a linear coupler in which the quantum noises do not affect a mean quadrature evolution, the existing quantum noises significantly alter its evolution in the nonlinear coupler. The effects of the Kerr nonlinearity can be even canceled by the quantum noises along the vanishing change ratio “valleys” in Fig. 3. The change ratio distribution is pretty symmetric between the mean quadratures under the noise effect, and the thinner (blue) ones are those in the simplified “noiseless” situation. The plots in (a) are found with \( J = 0.3 \kappa \), when the quantum noises cancel the Kerr effect as in the vanishing change ratio “valleys” shown in Fig. 3. Those in (b) are obtained with \( J = 0.9 \kappa \), for which it should take much a longer time to have Kerr nonlinearity enhancement under the noise effects. The inserted plot in both (a) and (b) shows a longer time realistic evolution under the noise effects. The one in (b) shows an obvious slowdown of the nonlinearity enhancement by the quantum noises. The Kerr coefficient and input coherent state amplitude are the same as in the previous figures.

FIG. 5: Quantum noise induced deviation in a mean quadrature evolution. The thicker (purple) curves stand for the mean quadratures under the noise effect, and the thinner (blue) ones are those in the simplified “noiseless” situation. The plots in (a) are found with \( J = 0.3 \kappa \), when the quantum noises cancel the Kerr effect as in the vanishing change ratio “valleys” shown in Fig. 3. Those in (b) are obtained with \( J = 0.9 \kappa \), for which it should take much a longer time to have Kerr nonlinearity enhancement under the noise effects. The inserted plot in both (a) and (b) shows a longer time realistic evolution under the noise effects. The one in (b) shows an obvious slowdown of the nonlinearity enhancement by the quantum noises. The Kerr coefficient and input coherent state amplitude are the same as in the previous figures.
V. LIGHT FIELD ENTANGLEMENT AND NONCLASSICALITY

A possible application of the nonlinear coupler illustrated in Fig. 1 is to generate the macroscopic nonclassical states of strong light fields, since the Kerr effect is greatly enhanced after breaking the \( \mathcal{PT} \) symmetry. Usually strong Kerr nonlinearity is required to entangle coherent states without light amplification (see, e.g., [48]). One should know the proper regimes for the existence of the nonclassicality such as light field entanglement. Here we apply the entanglement criterion for CV quantum state disentanglement due to the noises in other systems (see, e.g., [20]), the loss of quantum entanglement and nonclassicality, when their coefficients happen to be the necessary and sufficient condition for the general nonclassicality of the evolved light field states, when their \( P \) functions fail to be classical probability distributions [47]. The macroscopic nonclassical states can be used to generate other types of entanglement by coupling them with different photonic states through the beam-splitter [18, 49].

In Fig. 6(b) we also plot the possible evolution of the quantity \( D_3 \), where there was no quantum noise. As we see in the evolutions of a mean quadrature depicted in the lower panels of Fig. 3, the decoherence from the exponentially accelerating oscillation is the only governing factor alongside the nonlinearity enhancement in this simplified situation. Then this quantity evolves in a much simpler way, from that in a linear coupler \( (D_3 = 0) \) to the one of the completely decohered light fields \( (D_3 = 1) \). The comparison shows that the input light will become nonclassical only under a proper joint action of the existing quantum noises and enhanced Kerr nonlinearity. This phenomenon can be regarded as an analog of the synergic effects of noise and nonlinearity in classical systems, a typical example of which is stochastic resonance [3].

VI. DISCUSSION

We have found the solution to the dynamical problem about a coherent light evolving in the nonlinear \( \mathcal{PT} \)-symmetric coupler in Fig. 1, focusing on the effects from the enhanced Kerr nonlinearity, as well as the inherent quantum noises. By appearance the enhancement of Kerr nonlinearity due to much strengthened light intensity in the symmetry broken regime is a rather straightforward result. When it comes to its quantum properties, however, the system exhibits rich and unexpected phenomena related to this Kerr nonlinearity enhancement. The coexistence of field coupling plus amplification and dissipation, nonlinearity, as well as quantum noises, makes

\[
\langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{b}^\dagger\hat{b}\rangle D_3 = \langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{b}^\dagger\hat{b}\rangle + \langle \hat{a}\rangle\langle \hat{b}\rangle\langle \hat{a}^\dagger\hat{b}\rangle + \langle \hat{a}^\dagger\hat{b}\rangle\langle \hat{a}\rangle - \langle \hat{a}\rangle\langle \hat{b}\rangle\langle \hat{a}^\dagger\hat{b}\rangle - \langle \hat{a}^\dagger\hat{b}\rangle\langle \hat{a}\rangle - \langle \hat{a}\rangle\langle \hat{b}\rangle\langle \hat{a}^\dagger\hat{b}\rangle, \tag{16}
\]

which is normalized with respect to the product of output photon numbers. The different terms in the above equation are found by averaging the evolved operators over both system and reservoir degrees of freedom. Under the full dynamics including the noise effects, the quantity \( D_3 \) begins to take the negative values with the gradually enhanced Kerr nonlinearity. The negativity mainly comes from an extra term in the correlator \( \langle \hat{a}\hat{b}\rangle \) and its conjugate, which is similar to the second term in \( |12⟩ \), due to the interplay between the noises and nonlinearity. Meanwhile, the coefficient \( \zeta_2 \) from the noise action slows down the decoherence due to the exponentially accelerating oscillation developing with the enhanced Kerr nonlinearity. These factors work together to realize the regimes for the nonclassical states. Eventually the quantity \( D_3 \) will become positive, as the decoherence leaves the photon generation of increasing \( \langle \hat{a}^\dagger\hat{a}\rangle \) and \( \langle \hat{b}^\dagger\hat{b}\rangle \) to be the dominant process in the end. The above evolution of the quantity \( D_3 \) is illustrated in Fig. 6(a). Different from the well known quantum state disentanglement due to the noises in other systems (see, e.g., [20]), the loss of entanglement in this system is caused by the enhanced Kerr nonlinearity itself. Inside the proper regimes the realized macroscopic entangled states can have more than \( 10^{10} \) photons in our example. Thanks to the proportionality between the evolved modes in [11], the negativity of
this simple system a unique platform for studying complicated quantum dynamics. These factors build up the highly symmetric and unusual evolution of the mean quadratures of the waveguide modes. Instead of originating from the quantum noises, the decoherence of an input coherent light is caused by the enhanced Kerr nonlinearity itself. The regimes for the macroscopic entanglement of light fields are found to exist in where the Kerr nonlinearity is enhanced but has not reached the degree of giving rise to the decoherence. Such decoherence determines the final state of an input coherent light.

The important findings of the study are the effects of the quantum noises in the $\mathcal{PT}$ symmetry broken regime, where the coupler is a typical quantum system with non-perturbative nonlinearity. Quantum systems with both strong nonlinearity and noises were generally not tractable with the standard methods. The approach presented here makes it feasible to find the physical observables of the related intricate systems with both nonlinearity and quantum noises, when there is also no steady-state solution for the dynamical processes. Compared with the simplified model without quantum noise, the noises considerably slow down the Kerr nonlinearity enhancement and can even eliminate the Kerr effect. Another interesting phenomenon involving the quantum noises is the creation of the nonclassicality of the output fields, when they act together with the enhanced nonlinearity. This is analogous to various synergic phenomena involving classical noises in classical nonlinear systems. In stark contrast to the physics of classical noises that has been well explored in classical nonlinear systems, the research on quantum noises in the systems with non-perturbative nonlinearity is only at the beginning stage. The current work may stimulate the development in this direction.

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Appendix A: Expansion of the nonlinear action

The joint evolution operator $U(t)$ as the time-ordered exponential $\mathcal{T} e^{-i \int_0^t d\tau (H_L + H_{NL}) (\tau)}$ for the nonlinear coupler satisfies the following differential equation

$$\frac{dU(t)}{dt} = -i (H_L(t) + H_{NL}) U(t), \quad (A-1)$$

with the initial condition $U(0) = I$, the unit operator. On the other hand, the process $U_L(t)$ solely under the first linear Hamiltonian $H_L$ is the solution of the differential equation

$$\frac{dU_L(t)}{dt} = -i H_L(t) U_L(t) \quad (A-2)$$

with the same initial condition. Two variations of these equations are the quantum stochastic differential equations (QSDE) in Stratonovich and Itô form [1]. We consider the following differential

$$\frac{d}{dt} \{ U_L^\dagger(t) U(t) \} = -i U_L^\dagger(t) H_{NL} U(t). \quad (A-3)$$

The integral of the above equation from 0 to $t$ gives

$$U(t) = U_L(t) \{ I - i \int_0^t ds_1 U_L^\dagger(s_1) H_{NL} U(s_1) \}. \quad (A-4)$$

Substituting the same expression for the operator $U(s_1)$ into the above leads to the second order formula

$$U(t) = U_L(t) \{ I - i \int_0^t ds_1 U_L^\dagger(s_1) H_{NL} U(s_1)$$

$$- \int_0^s ds_1 U_L^\dagger(s_1) H_{NL} U_L(s_1) \int_s^t ds_2 U_L^\dagger(s_2) H_{NL} U_L(s_2) \}. \quad (A-5)$$

Iteratively applying the above procedure in the above equation, one will obtain the expansion in Eq. (8) of the main text. In the symmetry broken regime where the transformed Hamiltonian $H_{NL}^\prime(t) = 1/2 \chi (\hat{B}^\dagger(t))^2 (\hat{B}(t))^2$ becomes non-perturbative, each term in the expansion should be well taken account for the dynamical process.

Appendix B: Non-perturbative solution in the symmetry broken regime

Eq. (8) in the main text enables one to find the evolved mode operators. The first step is with the linear action $U_L(t)$ to get

$$U_L^\dagger(t) \hat{A} U_L(t) = \hat{A}(t) = -i e^{-\lambda} \eta_1 \frac{\eta_2}{\eta_1 + \eta_2} \hat{a} + \frac{\eta_2}{\eta_1 + \eta_2} \hat{b} + i e^{\lambda} \frac{\eta_2}{\eta_1 + \eta_2} \hat{b}$$

$$+ i e^{\lambda} \frac{\eta_1}{\eta_1 + \eta_2} \hat{a} + \frac{\eta_1}{\eta_1 + \eta_2} \hat{b}$$
\[ T \] 

reduced to the commutator, if one also includes the contribution from the noise term pairing with the decay mode in (B-1). Then the second-order term of the action \( U_{NL}(t) \) will be reduced to \( (-i \int_0^t dt H'_{NL}(\tau))^{2}/2! \). Generalizing to all orders of the nonlinear action, one will replace the time-ordered exponential \( e^{-i \int_0^t dt H'_{NL}(\tau)} \) by an ordinary exponential \( e^{-i \int_0^t dt H^o_{NL}(\tau)} \), as the effective Hamiltonian \( H^o_{NL}(t) \) becomes commutative at the different time \( t \).

Then one of the evolved modes can be found as follows:

\[
\hat{a}(t) = U_{NL}^\dagger(t) \hat{A}(t) U_{NL}(t) \\
= \hat{A}(t) + i \int_0^t dt_1 H_{NL}^o(t_1), \hat{A}(t) + \frac{1}{2!} [i \int_0^t dt_2 H_{NL}^o(t_2), [i \int_0^t dt_1 H_{NL}^o(t_1), \hat{A}(t)]] + \cdots \\
= \hat{A}(t) + \sum_{n=1}^\infty \frac{(-i)^n}{n!} \int_0^\infty dt_1 \chi c_{ab}(t, t_1) \hat{B}^o(t_1) \hat{B}(t_1) \cdots \int_0^\infty dt_n c_{bb}(t_n-1, t_n) \chi \hat{B}^o(t_n) \hat{B}(t_n) \hat{B}(t_n),
\]

where

\[
c_{ab}(t, t') = [\hat{A}(t), \hat{B}^o(t')] = \frac{\eta_2}{J}(\zeta_1 + \zeta_2) e^{\lambda t - \lambda t'},
\]

and \( c_{bb}(t, t') \) the corresponding commutator for the linear coupler mode of the loss channel. In the commutator \( c_{ab}(t, t') \), the factor \( \zeta_1 = \frac{k^2 - J^2}{\eta_1 + \eta_2} \) arises from the exponentially increasing term \( e^{\lambda t} \), while \( \zeta_2 = \frac{2k^2 - J^2}{2(\eta_1 + \eta_2)} \) is from the noise term \( \hat{n}(t) \). There will be a correction term \(-i \eta_2 \chi \zeta_3 e^{-\lambda t - \lambda t'}\) with \( \zeta_3 = \frac{2k^2 - J^2}{\lambda(\eta_1 + \eta_2)} \) to this commutator, if one also includes the contribution from the other noise term pairing with the decay mode in (B-1).

But its contribution to the evolved mode is small as we will see from that of the similar correction in the next section.

The first term in the commutator (B-5) is factorizable with respect to \( t \) and \( t' \). We only consider this term in \( c_{ab}(t, t') \) and \( c_{bb}(t, t') \) for the time being. From Eq. (B-1) this part of contribution can be found as

\[
\hat{a}_0(t) = i \frac{\eta_2}{J} \hat{B}(t) + i \frac{\eta_2}{J} \sum_{n=1}^\infty \frac{(-i)^n}{n!} (\zeta_1 + \zeta_2)^n \chi \int_0^t dt e^{2\lambda t} \hat{B}^o(t) \hat{B}(t) \hat{B}(t) \\
= i \frac{\eta_2}{J} \exp(-i(\zeta_1 + \zeta_2) \chi \int_0^t dt e^{2\lambda t} \hat{B}^o(t)) \hat{B}(t) \\
+ i \frac{\eta_2}{J} (\exp(-i(\zeta_1 + \zeta_2) \chi \int_0^t dt e^{2\lambda t} \hat{B}^o(t)) \hat{B}(t) - 1) \left( \int_0^t dt e^{2\lambda t} \hat{B}^o(t) \hat{B}(t) \hat{B}(t) \right)^{-1} \int_0^t dt' e^{2\lambda t'} \hat{B}^o(t') \hat{B}(t') \hat{B}(t')
\]
Kerr nonlinearity is considerably enhanced in the course of system evolution in the symmetry broken regime. The second term in the above can be regarded as a noise correction to the first main term. The other evolved mode operator \( \hat{b}_0(t) \) can be obtained in the same way.

Appendix C: Correction to the non-perturbative solution

The corrections to the evolved mode operators from the other term in Eq. (B-5) can be evaluated as follows. We first let only one of commutators \( c_{ab} \) or \( c_{bb} \) in (B-4) be replaced by other terms in (B-5), e.g. the one proportional to \( e^{\lambda |t-t'|} \), while still keeping the dominant factor \( e^{\lambda (t+t')} \) in all others as in deriving (B-6). The exact form of the first order correction is therefore found as

\[
\hat{a}_1(t) = -i \frac{\eta_2}{\zeta_2} \int_0^t dt' \chi(\hat{a}^{\dagger}(t-t') e^{\lambda t'} \hat{B}^{\dagger}(t') \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(-i \chi(\zeta_1 + \zeta_2) \int_0^t dt' \chi e^{2\lambda t'} \hat{B}^{\dagger}(t) \right)^n \hat{B}(t) \\
- \frac{\eta_2}{\zeta_2} \sum_{n=1}^{\infty} \frac{\lambda^{n+1}(\zeta_1 + \zeta_2)^n}{(n+1)!} \left\{ \int_0^t dt_1 \int_0^t dt' \chi e^{\lambda(t+t_1)} e^{\lambda t'} \hat{B}^{\dagger}(t_1) \hat{B}(t') \right. \\
\times \left. \int_0^t dt_2 e^{2\lambda t_2} \hat{B}^{\dagger}(t_2) \cdots \int_0^t dt_n e^{2\lambda t_n} \hat{B}^{\dagger}(t_n) + \cdots \right\} \\
+ \int_0^t dt' e^{\lambda(t+t')} \hat{B}^{\dagger}(t_1) \cdots \int_0^t dt_n e^{\lambda t_n} \hat{B}^{\dagger}(t_n) \hat{B}(t_1) \hat{B}(t') \},
\]

FIG. C-1: Coefficients in the first order correction to the evolved mode operators. Here we take \( J = 0.1\kappa \) as an example.

This first order correction to the main term of (B-6) can be approximated as

\[
\hat{a}_1(t) \approx -i \frac{\eta_2}{\zeta_2} \int_0^t dt' e^{\lambda(t-t')} e^{\lambda t'} \hat{B}^{\dagger}(t') \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(-i \chi(\zeta_1 + \zeta_2) \int_0^t dt' \chi e^{2\lambda t'} \hat{B}^{\dagger}(t) \right)^n \hat{B}(t) \\
- \frac{\eta_2}{\zeta_2} \sum_{n=1}^{\infty} \frac{\lambda^{n+1}(\zeta_1 + \zeta_2)^n}{(n+1)!} \left\{ \int_0^t dt_1 \int_0^t dt' e^{\lambda(t+t_1)} e^{\lambda t'} \hat{B}^{\dagger}(t_1) \hat{B}(t') \right. \\
\times \left. \int_0^t dt_2 e^{2\lambda t_2} \hat{B}^{\dagger}(t_2) \cdots \int_0^t dt_n e^{2\lambda t_n} \hat{B}^{\dagger}(t_n) + \cdots \right\} \\
+ \int_0^t dt' e^{\lambda(t+t')} \hat{B}^{\dagger}(t_1) \cdots \int_0^t dt_n e^{\lambda t_n} \hat{B}^{\dagger}(t_n) \hat{B}(t_1) \hat{B}(t') \}.
\]

The coefficients \( f_1(t) \) and \( f_2(t) \) in the correction gradually tend to zero as shown in Fig. (C-1), implying the negligible contribution of the first order correction after a period of time. The higher order corrections proportional to \( f_1^n(t) \) and \( f_2^n(t) (n > 1) \) are even smaller. Therefore, one can take the dominant term like that in Eq. (B-6) only, when the Kerr nonlinearity is considerably enhanced in the course of system evolution in the symmetry broken regime.
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