STRONGLY SUMMABLE FIBONACCI DIFFERENCE GEOMETRIC SEQUENCES DEFINED BY ORLICZ FUNCTIONS

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ABSTRACT. The purpose of this paper is to introduce the space of geometric sequences that are strongly summable with respect to an Orlicz function and the Fibonacci difference sequences. Also some topological properties and inclusion relations between the resulting geometric sequence spaces are discussed here.

1. Introduction and preliminaries

Let \( \Lambda = (\lambda_n) \) be a non-decreasing sequence of positive reals tending to infinity and \( \lambda_1 = 1 \) and \( \lambda_{n+1} \leq \lambda_n + 1 \). The generalized de la Vallee-Pousin means is defined by

\[
t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k \quad \text{where} \quad I_n = [n - \lambda_n + 1, n].
\]

A sequence \( x = (x_k) \) is said to be \((V, \lambda)\)-summable to a number \( \ell \) if \( t_n(x) \to \ell \) as \( n \to \infty \). \((V, \lambda)\)-summability reduces to \((C, 1)\)-summability where \( \lambda_n = n \) for all \( n \). We write

\[
[C, 1] = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell| = 0 \text{ for some } \ell \right\}
\]

and

\[
[V, \lambda] = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - \ell| = 0 \text{ for some } \ell \right\}
\]

for the sets of the sequences \( x = (x_k) \) which are strongly Cesaro summable and strongly \((V, \lambda)\)-summable to \( \ell \) i.e. \( x_k \to \ell[C, 1] \) and \( x_k \to \ell[V, \lambda] \) respectively.

Let \( X \) and \( Y \) be two sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real numbers \( a_{nk} \), where \( n, k \in \mathbb{N} \).

A defines a matrix mapping from \( X \) into \( Y \) and is written as \( A : X \to Y \) if for every sequence \( x = (x_k)_{k=0}^{\infty} \in X \), the sequence \( A(x) = \{A_n(x)\}_{n=0}^{\infty} \), the \( A \)-transform of \( x \) is in \( Y \), where

\[ A_n(x) = \sum_{k \in I_n} a_{nk}x_k \quad (n \in \mathbb{N}). \]

For simplicity in notation, throughout this paper summation without limits runs from 0 to \( \infty \).

The matrix domain \( X_A \) of an infinite matrix \( A \), is a sequence space \( X \) is defined
by
\[ X_A = \{ x = (x_k) \in \omega : Ax \in X \} \]
which is a sequence space. The approach of constructing a new sequence space by means of matrix domain of particular limitation method has been employed by several authors \[1, 3, 4, 5, 7, 17, 18, 22, 23, 24, 25, 26].
The matrix domain \( \mu \Delta \) is called the difference sequence space whenever \( \mu \) is a normed or paranormed sequence space. The idea of difference sequence spaces was first introduced by Kızmaz \[10\]. In fact he has defined the sequence spaces \( X(\Delta) = \{ x \in \omega : \Delta x \in X \} \) where \( \Delta x = (\Delta x_k) = (x_k - x_{k+1}) \), for \( X = \ell_\infty, c \) and \( c_0 \). Atlay and Basar \[2\] introduced the difference sequence spaces \( bv_p \), consisting of all sequences \( (x_k) \) such that \( (x_k - x_{k-1}) \) is in \( \ell_p \).

The operators \( \Delta^m : \omega \to \omega \), for \( m \in \mathbb{N} \), is defined by \( \Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1} \). Et and Çolak \[9\] generalized the sequence spaces \( X(\Delta^m) = \{ x \in \omega : \Delta^m x \in X \} \) for \( X = \ell_\infty, c \) and \( c_0 \).

Then Et and Bektas \[8\] introduced the sequence spaces \( (C, 1)(\Delta^m) = \{ x \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_k - \ell| = 0 \ for \ some \ \ell \} \)
\( [C, 1](\Delta^m) = \{ x \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (\Delta^m x_k - \ell) = 0 \ for \ some \ \ell \} \)
\( (V, \lambda)(\Delta^m) = \{ x \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} (\Delta^m x_k - \ell) = 0 \ for \ some \ \ell \} \)
\( [V, \lambda](\Delta^m) = \{ x \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta^m x_k - \ell| = 0 \ for \ some \ \ell \} \)
and studied their various topological properties.

Savas \[27\] introduced the classes
\( [V, \lambda]_0 = \{ x = x_k : \lim_{n} \frac{1}{n} \sum_{k \in I_n} |x_k| = 0 \} \)
\( [V, \lambda] = \{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - \ell| = 0, \ for \ some \ \ell \in C \} \) and
\( [V, \lambda]\infty = \{ x = x_k : \sup_{n} \frac{1}{n} \sum_{k \in I_n} |x_k| < \infty \} \)
which are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method. In the special case where \( \lambda = n \) for \( n = 1, 2, 3, \ldots \), the sets \([V, \lambda]_0, [V, \lambda],[V, \lambda]\infty\) reduce to the sets, \( \omega_0, \omega \), and \( \omega_\infty \) respectively introduced by Maddox. \[16\].

An Orlicz function \( M \) is a continuous, convex, non-decreasing function defined for \( x \geq 0 \) such that \( M(0) = 0 \) and \( M(x) \geq 0 \) for \( x > 0 \). If convexity of Orlicz function \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \) then the function is called
a modulus function, defined and discussed by Nakano [19], Ruckle [21], Maddox [15] and others. Lindenstrauss and Tzafriri [14] used the idea of Orlicz function to construct the sequence space

\[ \ell_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\} \]

with the norm \( \|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \)

which becomes a Banach space and is called an Orlicz sequence space. For \( M(x) = x^p, 1 \leq p \leq \infty \), the space \( \ell_M \), coincides with the classical sequence space \( \ell_p \).

Parashar and Choudhary [20] have introduced the sequence spaces

\[ W(M, p)_0 = \left\{ x \in \omega : \frac{1}{n} \sum_{k=1}^{n} \left( M\left(\frac{|x_k|}{\rho}\right) \right)^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for some } \rho \text{ and } \ell > 0 \right\} \]

\[ W(M, p) = \left\{ x \in \omega : \frac{1}{n} \sum_{k=1}^{n} \left( M\left(\frac{|x_k - \ell|}{\rho}\right) \right)^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for some } \rho \text{ and } \ell > 0 \right\} \]

\[ W(M, p)_\infty = \left\{ x \in \omega : \sup_{n} \frac{1}{n} \sum_{k=1}^{n} \left( M\left(\frac{|x_k|}{\rho}\right) \right)^{p_k} < \infty \text{ for some } \rho > 0 \right\} \]

which are complete paranormed spaces and these classes generalize the strongly summable sequence spaces \([C, 1, p]_0, [C, 1, p] \] and \([C, 1, p]_\infty \).

Let \( p = (p_k) \) be any sequence of strictly positive real numbers. Then Savas and Savas [27] introduced the spaces

\[ [V, M, p]_0 = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} = 0 \text{ for some } \rho > 0 \right\} \]

\[ [V, M, p] = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k - \ell|}{\rho}\right)\right]^{p_k} = 0 \text{ for some } \ell \text{ and } \rho > 0 \right\} \]

\[ [V, M, p]_\infty = \left\{ x = x_k : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} < \infty \text{ for some } \rho > 0 \right\} \]

where \( M \) is an Orlicz function.

**Definition 1.1.** A sequence \( x = (x_k) \) is said to be \( \lambda - \) statistical convergent or \( S_\lambda - \) convergent to \( \ell \) if for every \( \varepsilon > 0 \)

\[ \lim_{n \to \infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : |y_k - \ell| \geq \varepsilon \right\} \right| = 0. \]
where the vertical bars indicate the number of elements in the enclosed set.
In this case we write $S_{\lambda} - \lim x$ or $x_k \to \ell(S_{\lambda})$, and

$$S_{\lambda} = \{x \in \omega : S_{\lambda} - \lim x = \ell \text{ for some } \ell\}$$

Et and Bektas showed that $S(\Delta^m) \subset S_{\lambda}(\Delta^m)$ if and only if $\lim_{n \to \infty} \inf \frac{\lambda_n}{n} > 0$.

2. Fibonacci difference sequence spaces

The sequence $\{f_n\}_{n=0}^\infty$ of Fibonacci numbers satisfies $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geq 2$ and has applications in arts, sciences and architecture. Some basic properties of Fibonacci numbers \([11]\) are given as follows:

$$\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \alpha \text{ (golden ratio)}$$

$$\sum_{k=0}^n f_k = f_{n+2} - 1 \quad (n \in \mathbb{N})$$

$$\sum_k \frac{1}{f_k} \text{ converges,}$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \text{ (Cassini formula).}$$

Substituting for $f_{n+1}$ in Cassini formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

Let $f_n$ be the $n$th Fibonacci number for every $n \in \mathbb{N}$. Then we define the infinite matrix $\hat{F} = (\hat{f}_{nk})$ by

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & (k = n - 1), \\ \frac{f_n}{f_{n+1}} & (k = n), \\ 0 & (0 \leq k < n - 1 \text{ or } k > n), \end{cases}$$

Let $X$ be a sequence space. Then $X$ is called:

(i) Solid (or normal) if $(a_kx_k) \in X$ whenever $(x_k) \in X$, for all sequences $a_k$, scalars with $|a_k| \leq 1$.

(ii) Monotone provided $X$ contains the canonical preimage of all its step spaces.

(iii) Perfect if $X = X^\alpha$.

It is well known that $X$ is perfect $\Rightarrow X$ is normal $\Rightarrow X$ is monotone.

Non-Newtonian or Geometric calculus also known as multiplicative calculus was introduced by Grossman and Katz \([?]\). It come up with differentiation and integration tool based on multiplication rather than addition.

Türkmen and Başar \([?]\) introduced geometric sequence spaces for $X = c, c_0, l_\infty, l_p$ as

$$\omega(G) = \{x = (x_k) : x_k \in C(G), \text{ for all } k \in \mathbb{N}\}$$

$$l_\infty(G) = \left\{ x = (x_k) \in \omega(G) : \sup_{k \in \mathbb{N}} |x_k|^G < \infty \right\}$$

$$c(G) = \left\{ x = (x_k) \in \omega(G) : G \lim_{k \to \infty} |x_k|^G = 1 \right\}$$

$$c_0(G) = \left\{ x = (x_k) \in \omega(G) : G \lim_{k \to \infty} x_k = 1 \right\}$$

$$l_p(G) = \left\{ x = (x_k) \in \omega(G) : G \sum_{k=0}^\infty |x_k|^G < \infty \right\}$$
and the geometric complex number

\[ C(G) := \{ e^z : z \in C \} \]
\[ = C/\{0\} \]

where \((C(G), \oplus, \odot)\) is a field with geometric zero 1 and geometric identity \(e\), and we define the geometric addition, subtraction see \([6, 28, 29, 30]\).

Now we define the following Fibonacci difference geometric sequence spaces,

\[
[V, \lambda, M, \hat{F}, p^G]_0^G = \left\{ x = (x_k) \in \omega(G) : G \lim_{n \to \infty} \frac{1}{\lambda_n} G \sum_{k \in I_n} M \left( \frac{|f_{k+1} x_k \oplus f_k x_{k-1}|}{\rho} \right) \right\}^p = \{ 1, \rho > 1 \}
\]

\[
[V, \lambda, M, \hat{F}, p^G]_1^G = \left\{ x = (x_k) \in \omega(G) : G \lim_{n \to \infty} \frac{1}{\lambda_n} G \sum_{k \in I_n} M \left( \frac{|(f_{k+1} x_k \oplus f_k x_{k-1}) \odot \ell|}{\rho} \right) \right\}^p = \{ 1, \rho > 1 \}
\]

\[
[V, \lambda, M, \hat{F}, p^G]_\infty^G = \left\{ x = (x_k) \in \omega(G) : G \lim_{n \to \infty} \frac{1}{\lambda_n} G \sum_{k \in I_n} M \left( \frac{|f_{k+1} x_k \oplus f_k x_{k-1}|}{\rho} \right) \right\}^p < \infty, \rho > 1
\]

which are the set of all sequences whose \(\hat{F}\)-transforms are in the space

\[ [V, M, p]^G, [V, M, p]^G \text{ and } [V, M, p]^G \text{ respectively i.e.} [V, \lambda, M, \hat{F}, p]^G = ([V, \lambda, M, p]^G)_{\hat{F}} \text{ and so on.} \]

3. Main Results

**Theorem 3.1.** For any Orlicz function \(M\) and any sequence \(p^G = (p_k)^G\) of strictly positive real numbers

\[ [V, \lambda, M, \hat{F}, p]^G, [V, \lambda, M, \hat{F}, p]^G \text{ and } [V, \lambda, M, \hat{F}, p]^G \text{ are linear spaces over the set of complex numbers } C(G). \]

**Proof.** We shall prove the theorem only for \([V, \lambda, M, \hat{F}, p]^G\) and others can be proved with similar techniques.

Let \(x, y \in [V, \lambda, M, \hat{F}, p]^G\) and \(\alpha, \beta \in C(G)\). Then there exist some positive numbers \(\rho_1\) and \(\rho_2\) such that

\[
G \lim_{n \to \infty} \frac{1}{\lambda_n} G \sum_{k \in I_n} M \left( \frac{|f_{k+1} x_k \oplus f_k x_{k-1}|}{\rho_1} \right) = 1
\]

and

\[
G \lim_{n \to \infty} \frac{1}{\lambda_n} G \sum_{k \in I_n} M \left( \frac{|f_{k+1} y_k \oplus f_k y_{k-1}|}{\rho_2} \right) = 1.
\]

Let \(\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)\). Since \(M\) is non-decreasing and convex, so we have

\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} M \left( \frac{|f_{k+1}(\alpha x_k) \oplus f_k(\alpha x_{k-1}) \oplus f_{k+1}(\beta y_k) \oplus f_k(\beta y_{k-1})|}{\rho_3} \right) p_k^G
\]
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1}(\alpha x_k) \ominus f_k x_k|}{\rho} \right) \right]^{p_k^G} \leq \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1} x_k \ominus f_k x_k|}{\rho} \right) \right]^{p_k^G} \right)^{\frac{1}{p_k^G}} \leq e, \text{ for } e = 2e, 3e, \ldots
\]

as \( n \to \infty \), where \( B = \max(e, 2H^{-1}), H = \sup p_k \). Hence \( \alpha x \oplus \beta y \in [V, \lambda, M, \hat{F}, p]^G_0 \).

**Theorem 3.2.** The space \([V, \lambda, M, \hat{F}, p]^G_0\) is a paranormed space, (not totally paranormed), paranormed.

by, \( g(x) = \inf \left\{ \rho^\frac{p^G}{\hat{p}} G : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1} x_k \ominus f_k x_k|}{\rho} \right) \right]^{p_k^G} \right) \right\} \leq e, n = e, 2e, 3e, \ldots \)

where \( H = \max(e, \sup p_k^G) \).

**Proof.** Now \( g(x) = g(\oplus x) \). From linearity we have if \( \alpha = \beta = e \) then \( g(x \oplus y) \leq g(x) \oplus g(y) \).

Since \( \frac{1}{\lambda_n} M(1) = 1, \) we get \( \inf \left\{ \rho^\frac{p^G}{\hat{p}} \right\} = 1 \) for \( x = 1 \)

Conversely, suppose \( g(x) = 1, \) then

\[
g(x) = \inf \left\{ \rho^\frac{p^G}{\hat{p}} G : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1} x_k \ominus f_k x_k|}{\rho} \right) \right]^{p_k^G} \right) \right\} \leq e \}
\]

This implies that for a given \( \epsilon > 1 \), there exists some \( \rho_\epsilon (1 < \rho_\epsilon < \epsilon) \) such that

\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1} x_k \ominus f_k x_k|}{\rho} \right) \right]^{p_k^G} \right) \leq 1
\]

\[
\Rightarrow \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1} x_k \ominus f_k x_k|}{\epsilon} \right) \right]^{\frac{p_k^G}{\hat{p}}} \right) \leq \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|f_{k+1} x_k \ominus f_k x_k|}{\rho} \right) \right]^{p_k^G} \right)^{\frac{1}{\hat{p}}}
\]

\[
\leq e \quad \text{for each } n
\]
Suppose that \( y_k \neq 1 \) for some \( k \in I_n \) where \( y_k = \frac{f_k}{f_{k+1}}x_k \oplus \frac{f_{k+1}}{f_k}x_{k-1} \).

Let \( \epsilon \to 1 \). Then

\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \left| \frac{y_k}{\epsilon} \right| \right) \right] p_k^G \right)^{\frac{1}{n}} \neq 1
\]

which is a contradiction. Therefore \( y_k = 1 \) for each \( k \). Now, we prove that scalar multiplication is continuous.

Let \( \mu \) be any complex number, consider

\[
g(\mu \odot x) = \inf \left\{ \rho^{\frac{m}{n}} : \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \left| \frac{\mu(f_k x_k \oplus f_{k+1} x_{k-1})}{\rho} \right| \right) \right] p_k^G \right)^{\frac{1}{n}} \leq e, \ n = e, 2e, \ldots \right\}
\]

\[
= \inf \left\{ (|\mu| \odot s)^{\frac{m}{n}} : \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \left| \frac{f_k x_k \oplus f_{k+1} x_{k-1}}{s} \right| \right) \right] p_k^G \right)^{\frac{1}{n}} \leq e, \ n = e, 2e, \ldots \right\}
\]

where \( s = \frac{\rho}{|\mu|} \). Since \( |\mu|^{p_k^G} \leq \max(e, |\mu|^{\sup p_k^G}) \),

we have \( g(\mu \odot x) \leq (\max(e, |\mu|^{\sup p_k^G}))^{\frac{1}{n}} \)

\[
\times \inf \left\{ s^{\frac{m}{n}} : \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \left| \frac{f_k x_k \oplus f_{k+1} x_{k-1}}{s} \right| \right) \right] p_k^G \right)^{\frac{1}{n}} \leq e, \ n = e, 2e, \ldots \right\}
\]

which converges to one as \( g(x) \) converges to one in \([V, \lambda, M, \hat{F}, p]_0^G\).

Now, suppose \( \mu_m \to 1 \) as \( m \to \infty \) and \( y_k \) be a sequence fixed in \([V, \lambda, M, \hat{F}, p]_0^G\). For arbitrary \( \epsilon \).

Let \( N \) be a positive integer such that

\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \left| \frac{f_k x_k \oplus f_{k+1} x_{k-1}}{\rho} \right| \right) \right] p_k^G \leq \left( \frac{\epsilon}{2} \right)^H \quad \text{for some } \rho > 1 \text{ and all } n > N.
\]

This implies that

\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \left| \frac{f_k x_k \oplus f_{k+1} x_{k-1}}{\rho} \right| \right) \right] p_k^G \leq \frac{\epsilon}{2} \quad \text{for some } \rho > 1 \text{ and all } n > N.
\]

Let \( 1 < |\mu| < e \), using convexity of \( M \), for \( n > N \), we get

\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \left| \frac{f_k x_k \oplus f_{k+1} x_{k-1}}{\rho} \right| \right) \right] p_k^G < \frac{\epsilon}{2} \quad \text{for some } \rho > 1 \text{ and all } n > N.
\]

Let \( 1 < |\mu| < e \), using convexity of \( M \), for \( n > N \), we get

\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ |\mu|M \left( \left| \frac{f_k x_k \oplus f_{k+1} x_{k-1}}{\rho} \right| \right) \right] p_k^G < \left( \frac{\epsilon}{2} \right)^H
\]
Since \( M \) is continuous everywhere in \([1, \infty)\), then for \( n \leq N \) consider any scalar \( t \)
\[
 f(t \odot x) = \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \frac{\left| \frac{f_k}{f_{k+1}} x_k \ominus \frac{f_{k+1}}{f_k} x_{k-1} \right|}{\rho} \right) \right]^{p_k^G}
\]
is continuous at 1. So there is \( e > \delta > 1 \) such that \( |f(t)| < \left( \frac{t}{2} \right)^H \) for \( 1 < t < \delta \).
Let \( \tilde{K} \) be such that \( |\mu_m| < \delta \) for \( m > \tilde{K} \) then for \( m > \tilde{K} \) and \( n \leq N \)
\[
 \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \frac{|\mu_m| \left( \frac{f_k}{f_{k+1}} x_k \ominus \frac{f_{k+1}}{f_k} x_{k-1} \right)|}{\rho} \right) \right]^{p_k^G} \right)^{\frac{1}{p}} < \frac{\epsilon}{2}
\]
\[
 \Rightarrow \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \frac{|\mu_m| \left( \frac{f_k}{f_{k+1}} x_k \ominus \frac{f_{k+1}}{f_k} x_{k-1} \right)|}{\rho} \right) \right]^{p_k^G} \right)^{\frac{1}{p}} < \epsilon.
\]
for \( m > \tilde{K} \) and all \( n \), so that \( g(\mu \odot x) \to 1 \)
We know \( g \{ (\mu_m \odot x^m) \odot (\mu \odot x) \} \leq \{ (\mu_m \odot \mu) \odot g(x^m) \} \odot \{ |\mu| \odot g(x^m \odot x) \} \)

**Theorem 3.3.** The sequence spaces \([V, \lambda, M, \widehat{F}, p]^G_0\) and \([V, \lambda, M, \widehat{F}, p]^G_\infty\) are solid.

**Proof.** We give the proof for \([V, \lambda, M, \widehat{F}, p]^G_0\) and for \([V, \lambda, M, \widehat{F}, p]^G_\infty\) will be done in a similar way.
Let \((y_k) \in [V, \lambda, M, \widehat{F}, p]^G_0\) and \(\alpha_k\) be any sequence of scalars such that \( |\alpha_k| \leq e \) for all \( k \in \mathbb{N} \).
where \( y_k = \frac{f_k}{f_{k+1}} x_k \ominus \frac{f_{k+1}}{f_k} x_{k-1} \). Then we have
\[
 \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \frac{|\alpha_k y_k|}{\rho} \right) \right]^{p_k^G} < \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[ M \left( \frac{|y_k|}{\rho} \right) \right]^{p_k^G} \to 1 \quad (n \to \infty)
\]
Hence \((\alpha_k \odot y_k) \in [V, \lambda, M, \widehat{F}, p]^G_0\) for all sequences of scalars \((\alpha_k)\) with \( |\alpha_k| \leq e \) for all \( k \in \mathbb{N} \),
whenever \((y_k) \in [V, \lambda, M, \widehat{F}, p]^G_0\).

**Corollary 3.4.** (i) The sequence spaces \([V, \lambda, M, \widehat{F}, p]^G_0\) and \([V, \lambda, M, \widehat{F}, p]^G_\infty\) are monotone.

4. \(\lambda\)–Statistical Convergence

A sequence \( x = (x_k) \) is said to be \( S^G_\lambda(\widehat{F}) \)–convergent to \( \ell \) if for every \( \varepsilon > 1 \)
\[
 G \lim_{n \to \infty} \frac{1}{\lambda_n} \left| \left\{ k \leq n : \left( \left| \frac{f_k}{f_{k+1}} x_k \ominus \frac{f_{k+1}}{f_k} x_{k-1} \ominus \ell \right| \geq \varepsilon \right) \right\} \right|^G = 1.
\]
where the bars indicate the cardinality of the set. Write for simplicity \( y_k = \frac{f_k}{f_{k+1}} x_k \ominus \frac{f_{k+1}}{f_k} x_{k-1} \).
Theorem 4.1. For any Orlicz function M, \([V, \lambda, M, \hat{F}]^G \subset S^G(\hat{F})\)

Proof. Let \(x \in [V, \lambda, M, \hat{F}]^G\) and \(\varepsilon > 1\) be given. Then for \(y_k = \frac{f_k}{f_{k+1}} x_k \oplus \frac{f_{k+1}}{f_k} x_{k-1}\) consider
\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} M\left(\frac{|y_k \oplus \ell|}{\rho}\right) \geq \frac{1}{\lambda_n} G \sum_{k \in I_n, |y_k \oplus \ell| \geq \varepsilon} M\left(\frac{|y_k \oplus \ell|}{\rho}\right) > \frac{1}{\lambda_n} \bigoplus M\left(\frac{\varepsilon}{\rho}\right) \|k \in I_n : |y_k \oplus \ell| \geq \varepsilon\|.
\]
Hence \(x \in S^G(\hat{F})\) \(\square\)

Definition 4.2. ([12]) An Orlicz function M is said to satisfy \(\Delta_2\)-condition for all values of \(u\), if there exists a constant \(K > 0\) such that \(M(2u) \leq KM(u), u \geq 0\), where always \(K > 2\). The \(\Delta_2\)-condition is equivalent to the satisfaction of inequality \(M(lu) \leq K(l)M(u)\), for all values of \(u\) and for \(l > 1\).

Theorem 4.3. For any Orlicz function M which satisfies \(\Delta_2\)-condition, we have \([V, \lambda, \hat{F}, p]^G \subset [V, \lambda, M, \hat{F}, p]^G\).

Proof. Let \(x \in [V, \lambda, \hat{F}, p]^G\) so that
\[
A_n \equiv \frac{1}{\lambda_n} G \sum_{k \in I_n} \left|\left(\frac{f_k}{f_{k+1}} x_k \oplus \frac{f_{k+1}}{f_k} x_{k-1}\right) \oplus \ell\right| \to 1
\]
as \(n \to \infty\) for some \(\ell\).

Let \(\varepsilon > 1\) and choose \(\delta\) with \(1 < \delta < \varepsilon\) such that \(M(t) < \varepsilon\) for \(1 \leq t \leq \delta\).

Let \(y_k = \left|\left(\frac{f_k}{f_{k+1}} x_k \oplus \frac{f_{k+1}}{f_k} x_{k-1}\right) \oplus \ell\right|\) and consider
\[
\frac{1}{\lambda_n} G \sum_{k \in I_n} M(|y_k|) = \frac{1}{\lambda_n} G \sum_{k \in I_n, |y_k| \leq \delta} M(|y_k|) + \frac{1}{\lambda_n} \sum_{k \in I_n, |y_k| > \delta} M(|y_k|)
\]

Since M is continuous, \(\frac{1}{\lambda_n} G \sum_{k \in I_n, |y_k| \leq \delta} M(|y_k|) < \lambda_n \bigoplus \varepsilon\). and for \(y_k \geq \delta\) we use \(y_k < \frac{\lambda_n}{\delta} < 1 \oplus \frac{\lambda_n}{\delta}\).

Since M is non-decreasing and convex, then it becomes
\[
M(y_k) < M(1 \oplus \delta^{-1} y_k) < \frac{1}{2} M(2) \oplus \frac{1}{2} M(2 \delta^{-1} ty_k)_G.
\]

Since M satisfies \(\Delta_2\)-condition there is a constant \(K > 2\) such that \(M(2\delta^{-1} y_k)_G \leq \left(\frac{1}{2} K\delta^{-1} y_k M(2)\right)_G\), therefore
\[
M(y_k) < \left(\frac{1}{2} K\delta^{-1} y_k M(2)\right)_G \oplus \left(\frac{1}{2} K\delta^{-1} y_k M(2)\right)_G = \left(\frac{1}{4} K\delta^{-1} y_k M(2)\right)_G.
\]

Hence \(\frac{1}{\lambda_n} G \sum_{k \in I_n, |y_k| \geq \delta} M(|y_k|) \leq \frac{1}{4} K\delta^{-1} y_k M(2) \lambda_n A_n\).

which together with \(\frac{1}{\lambda_n} G \sum_{k \in I_n, |y_k| \leq \delta} M(|y_k|) \leq \varepsilon \lambda_n\) yields \([V, \lambda, \hat{F}, p]^G \subset [V, M, \hat{F}, p]^G\).
Similarly the result holds for 
\[ [V, \lambda, \hat{F}, p]^G_0 \subset [V, \lambda, M, \hat{F}, p]^G, \]  
and 
\[ [V, \lambda, \hat{F}, p]^G_\infty \subset [V, \lambda, M, \hat{F}, p]^G_\infty. \] □

**Theorem 4.4.** Let \( 1 \leq p_k \leq q_k \) and \((\frac{p_k}{q_k})\) be bounded. Then 
\[ [V, \lambda, M, \hat{F}, q]^G \subset [V, \lambda, M, \hat{F}, p]^G. \]

**Proof.** Let \( x = (x_k) \in [V, \lambda, M, \hat{F}, q]^G \).and let 
\[ t_k = \left[ M\left( \frac{f_k x_k \ominus \rho}{f_k} \right) \right]^{q_k} \]
and 
\[ \mu_k = \frac{p_k}{q_k}, \text{ for all } k \in \mathbb{N}. \]

Then \( 1 < \mu_k \leq e \) for all \( k \in \mathbb{N} \). Write \( 0 < \mu_k < \mu \) for all \( k \in \mathbb{N} \).

Define the \( \mu_k \) as follows:
For \( t_k \geq e \), let \( u_k = t_k \) and \( v_k = 1 \). Then for all \( k \in \mathbb{N} \), we have 
\[ t_k = u_k \oplus v_k, \quad t_k^{\mu_k} = u_k^{\mu_k} \oplus v_k^{\mu_k}. \]

Now, it follows that 
\[ u_k^{\mu_k} \leq u_k \leq t_k \]  
and 
\[ v_k^{\mu_k} \leq v_k. \]

Therefore, 
\[ \frac{1}{\lambda_n} G \sum_{k \in I_n} t_k^{\mu_k} = \frac{1}{\lambda_n} G \sum_{k \in I_n} (u_k^{\mu_k} \oplus v_k^{\mu_k}) \]
\[ \leq \frac{1}{\lambda_n} G \sum_{k \in I_n} t_k \oplus \frac{1}{\lambda_n} \sum_{k \in I_n} v_k^{\mu_k} \]

Now for each \( k \),
\[ \frac{1}{\lambda_n} G \sum_{k \in I_n} v_k^{\mu_k} = \sum_{k \in I_n} \left( \frac{1}{\lambda_n} v_k^{\mu} \right) \cdot \left( \frac{1}{\lambda_n} \right)^{1-\mu} \]
\[ \leq \left( \sum_{k \in I_n} \left[ \left( \frac{1}{\lambda_n} v_k \right)^{\mu} \right] \right) \cdot \left( \sum_{k \in I_n} \left[ \frac{1}{\lambda_n} \right]^{1-\mu} \right)^{1-\mu} \]
\[ = \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} v_k \right)^{\mu}, \]

and 
\[ \frac{1}{\lambda_n} G \sum_{k \in I_n} t_k^{\mu_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} t_k \oplus \left( \frac{1}{\lambda_n} G \sum_{k \in I_n} v_k \right)^{\mu}. \]

Hence, \( x = (x_k) \in [V, \lambda, M, \hat{F}, p]^G. \) □

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