Malliavin Matrix of Degenerate SDE and Gradient Estimate

Dong Zhao†, Xuhui Peng‡

Academy of Mathematics and Systems Science, Chinese Academy of Sciences,
Beijing, 100190, P.R.China.

Abstract

In this article, we prove that the inverse of Malliavin matrix belong to \( L^p(\Omega, \mathbb{P}) \) for a kind of degenerate stochastic differential equation (SDE) under some conditions, which like to Hörmander condition, but don’t need all the coefficients of the SDE are smooth. Furthermore, we obtain a locally uniform estimation for Malliavin matrix, a gradient estimate, and prove that the semigroup generated by the SDE is strong Feller. Also some examples are given.

Keywords: Degenerate stochastic differential equation; Gradient estimate; Strong Feller; Malliavin calculus; Hörmander condition.

AMS 2000: 60H10, 60H07.

1 Introduction and Notations

In this article, we consider the following degenerate stochastic differential equations (SDE)

\[
\begin{align*}
x_t &= x + \int_0^t a_1(x_s, y_s)ds, \\
y_t &= y + \int_0^t a_2(x_s, y_s)ds + \int_0^t b(x_s, y_s)dW_s.
\end{align*}
\]

where \( x \in \mathbb{R}^m, y \in \mathbb{R}^n, b \in \mathbb{R}^{n \times d}, W_s \) is a \( d \)-dimensional standard Brownian motion. Eq. (1.1) is a model for many physical phenomenons. For example, \( x_t \) represents the position of an object and \( y_t \) represents the momentum of the object. When a random force affects the object, the momentum of the object changes firstly, then that would lead to the variety of the object position. Thus the equation which describes the movement of the object is

†Email: dzhao@amt.ac.cn
‡Corresponding author. Email: pengxuhui@amss.ac.cn
*The authors were Supported by 973 Program, No. 2011CB808000 and Key Laboratory of Random Complex Structures and Data Science, No.2008DP173182, NSFC, No.:10721101, 11271356, 11371041.
naturally degenerate as Eq. (1.1). To understand the long time behavior of the movement of the object, we need to study the ergodicity of Eq. (1.1). For this reason, the gradient estimate of the semigroup and the strongly Feller property associated to the solution should be considered, and the solution is ergodic if one also knows that the solution is topological irreducible and has an invariant probability measure.

Let $\mathbb{P}_{x,y}$ be the law of the solution to equation Eq. (1.1) with initial value $(x,y)$, and $P_t$ be the transition semigroup of Eq. (1.1)

$$P_tf(x,y) := \mathbb{E}_{x,y}f(x_t,y_t), \ f \in \mathcal{B}_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}),$$

where $\mathcal{B}_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ denotes the collection of bounded Borel measurable functions and $\mathcal{B}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ denotes the collection of Borel measurable functions.

For the general SDE

$$X_t = x + \int_0^t V_0(X_s)ds + \sum_{j=1}^d \int_0^t V_j(X_s) \circ dW_j(s), \ x \in \mathbb{R}^{m+n}. \quad (1.2)$$

The Hörmander condition (H) is that the vector space spanned by the vector fields

$$\text{(H)} \quad V_1, \cdots, V_d, \ [V_i, V_j], 0 \leq i, j \leq d, \ [[V_i, V_j], V_k], 0 \leq i, j, k \leq d, \cdots,$$

at point $x$ is $\mathbb{R}^{m+n}$. The coefficients are infinitely differentiable functions with bounded partial derivatives of all order. If the Hörmander condition (H) holds for any $x \in \mathbb{R}^{m+n}$, the process $X_t$ has a smooth density and the transition semigroup of Eq. (1.2) is strong Feller (see [9], [11], [14], [17] etc).

Let $V = (V_1, \cdots, V_d)$ and $P_t(x, \cdot)$ be the transition probabilities of the $X_t$ in (1.2). When $VV^*$, where $* \text{ means the transpose of the matrix, being uniformly elliptic, the two-sided bounds of the density for } P_t(x, \cdot) \text{ were given in [18] by using stochastic control tools. There also many other excellent works when } VV^* \text{ is non-degenerate.}$

There is also many works in the hypoelliptic setting. For the special case $V_0 \equiv 0$, in [11], Kusuoka and Stroock gave the two-sided bounds of the density for $P_t(x, \cdot)$ under some conditions which need some uniformity on $V_1, \cdots, V_d$. Recently, in [4], Delarue and Menozzi considered the following SDE,

$$\begin{cases}
X_t^1 = x_1 + \int_0^t F_1(s, X_s^1, \cdots, X_s^n)ds + \int_0^t b(s, X_s^1, \cdots, X_s^n)dW_s, \\
X_t^2 = x_2 + \int_0^t F_2(s, X_s^1, \cdots, X_s^n)ds, \\
X_t^3 = x_3 + \int_0^t F_3(s, X_s^2, \cdots, X_s^n)ds, \\
\vdots \\
X_t^n = x_n + \int_0^t F_n(s, X_s^{n-1}, X_s^n)dt.
\end{cases} \quad (1.3)$$
If the spectral of the $A(t, x) = [bb^*](t, x)$, is included in $[\Lambda^{-1}, \Lambda]$ for some $\Lambda \geq 1$ and $D_{x_i=1}F_i(t, x_{i-1}, x_i, \cdots, x_n)$ is non-degenerate, uniformly in space and time, they gave the two-sided bounds of the density for the solution to Eq. (1.3). Another work is that in [2], the authors considered the SDE as

$$X^i_t = x_i + W^i_t, \quad i \in [1,n], \quad X^{n+1}_t = x_{n+1} + \int_0^t |X^i_{s,n}|^k ds, \quad (1.4)$$

here $X^{1,n}_t = (X^1_s, \cdots, X^n_s)$ and they gave the two-sided bounds estimation for the transition function $p(t, x, \cdot)$ in [2].

There are also many other researches on the special case of Eq. (1.1), such as [13], [10], [20] and so on. In [13] and [20], the authors studied the ergodicity and in [10], the author studied the recurrence and invariant measure.

In most of the above works, the coefficients are smooth or some uniform conditions are needed. Since our aim in this article is to prove the strong Feller property and give a gradient estimate of the semigroup, we don’t need the smooth conditions for all the coefficients or some uniform conditions. Instead of the Hörmander conditions, we give some new conditions, which are equivalent to the Hörmander condition if the coefficients are smooth, and proved that the inverse of the Malliavin matrix is $L^p$ integrable for any $p \geq 0$. Furthermore, our new conditions also ensure that we can obtain a gradient estimate and the strong Feller property.

We haven’t obtained the smoothness of the density or the two-sided bounds of the density for the lack of smoothness or some uniform conditions on the coefficients.

Before we give the organization of this article, we introduce some notations. For $j \in \mathbb{N}$, let $C^j(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ be the collection of functions which have continuous derivatives up to order $j$ and $C^j_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ be the collection of functions in $C^j(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ with bounded derivatives. Sometimes, we will use $C^j_b$ and $C^j$ instead of them for the convenience of writing. For $l \in \mathbb{N}$, $k = (k_1(x,y), \cdots, k_l(x,y))^* \in C^1(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$, $x = (x_1, \cdots, x_m)^*$, $y = (y_1, \cdots, y_n)^*$,

$$\nabla_x k = \left( \frac{\partial k_1}{\partial x_1}, \cdots, \frac{\partial k_l}{\partial x_l} \right)^*, \quad i = 1, \cdots, m, \quad \nabla_x k = (\nabla_x k_1, \cdots, \nabla_x k_l),$$

$$\nabla_y k = \left( \frac{\partial k_1}{\partial y_1}, \cdots, \frac{\partial k_l}{\partial y_l} \right)^*, \quad j = 1, \cdots, n, \quad \nabla_y k = (\nabla_y k_1, \cdots, \nabla_y k_l),$$

and $\nabla k = (\nabla_x k, \nabla_y k)$. If $a_1 \in C^{j_0}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ for some $j_0 \in \mathbb{N}$, we define vector fields:

$$A_1 = \{ \nabla_y a_1, \quad j = 1, \cdots, n, \},$$

$$A_l = \{ \nabla_y a_k, \quad j = 1, \cdots, n, \quad \nabla_x a_1 \cdot k + \nabla_x k \cdot a_1 : k \in A_{l-1} \}, \quad l = 2, \cdots, j_0.$$ 

Assume $a_1 = (a_1^1, \cdots, a_1^m)^*$, $a_2 = (a_2^1, \cdots, a_2^j)^*$, $a = (a_1^*, a_2^*)^*$. $\mathbb{N} = \{1, \cdots, \}$. Let det$(A)$ be the determinant of the matrix $A = (a_{ij})$, $||A||^2 = \sum_{i,j} a_{ij}^2$. Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product and $| \cdot |$ be the Euclidean norm. For any $x_0 \in \mathbb{R}^{m+n}$ and $R > 0$, $B(x_0, R) = \{ x \in \mathbb{R}^{m+n}, |x - x_0| \leq R \}$, $B^0(x_0, R) = \{ x \in \mathbb{R}^{m+n}, |x - x_0| < R \}$. $\|k\|_\infty$ denotes the essential
The supreme norm for the function $k$ defined on Euclidean space. We use $C(d)$ or $c_0(d)$ to denote a positive and finite constant depending on $d$, $\|\nabla a\|_\infty$ and $\|\nabla b\|_\infty$. This constant may change from line to line. Sometimes, we will use $C$ instead of $C(d)$ for the convenience of writing.

Without otherwise specified, in this article, $(x_t, y_t)$ is the solution for Eq.(1.1) and $(x, y)$ is its initial value. Let $M_t$ be the Malliavin matrix for $(x_t, y_t)$. Then (c.f. [14])

$$M_t = J_t \int_0^t J_s^{-1} \begin{pmatrix} 0 & 0 \\ b(x_s, y_s) & b(x_s, y_s) \end{pmatrix}^* (J_s^{-1})^* ds J_t^*, \quad (1.5)$$

here $J_t^{-1}$ satisfies

$$J_t^{-1} = I_{m+n} - \int_0^t J_s^{-1} \begin{pmatrix} 0 & 0 \\ \nabla_x b_j & \nabla_y b_j \end{pmatrix} (x_s, y_s) dW_j(s)$$

$$- \int_0^t J_s^{-1} \left[ \begin{pmatrix} \nabla_x a_1 & \nabla_y a_1 \\ \nabla_x a_2 & \nabla_y a_2 \end{pmatrix} (x_s, y_s) \\ - \sum_{j=1}^d \begin{pmatrix} 0 & 0 \\ \nabla y b_j \nabla x b_j & \nabla y b_j \nabla y b_j \end{pmatrix} (x_s, y_s) \right] ds, \quad (1.6)$$

and $J_t$ satisfies

$$J_t = I_{m+n} + \int_0^t \begin{pmatrix} \nabla_x a_1 & \nabla_y a_1 \\ \nabla_x a_2 & \nabla_y a_2 \end{pmatrix} (x_s, y_s) J_s ds$$

$$+ \sum_{j=1}^d \int_0^t \begin{pmatrix} 0 & 0 \\ \nabla x b_j & \nabla y b_j \end{pmatrix} (x_s, y_s) J_s dW_j(s). \quad (1.7)$$

Our article is organized as follows. In section 2, we prove the key theorem of this article Theorem 2.1 under the Hypothesis 2.1. In Hypothesis 2.1 we only need $a_2 \in C^1$, $b \in C^2$ and $a_1 \in C^{j_0+2}$ for some $j_0 \in \mathbb{N}$. Compare with Hörmander condition, the functions $a_2$ and $b$ are only required to be $C^1$ and $C^2$ respectively. Our method to prove Theorem 2.1 is similar to that in [14], but it also has some differences. These differences depend heavily on the special form of the Eq.(1.1). In [14], $J_t^{-1}$ is regarded as a whole. Here, we divide $J_t^{-1}$ into $\begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ and do more elaborate estimates.

In section 3, we firstly give a local uniform estimate for Malliavin matrix under the Hypothesis 3.1 and then give a gradient estimate in Theorem 3.1. The local uniform estimate for Malliavin matrix is a key point to prove Theorem 3.1.

In section 3, we have proved $P_t$ is strong Feller under some conditions which need all the coefficients of Eq. (1.1) are in $C^2_t$. Since there are bounded conditions on the coefficients and their derivatives, it seems too strong to apply, for example, the Hamiltonian systems, so we weaken this bounded conditions in section 4. In the section 4, we mainly use the localization method to prove $P_t$ is strong Feller, our hypothesis is the Hypothesis 4.1.

In section 5, we apply the above results to some examples, such as the Langevin SDEs, the stochastic Hamiltonian systems and high order stochastic differential equations.
2 The $L^p$ Integrability of the Inverse of Malliavin Matrix

In subsection 2.1, we give the key Theorem 2.1 and put its proof in subsection 2.2.

2.1 The Main Theorem and Its Relations with Hörmander Theorem

**Hypothesis 2.1.** $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and there exists a $j_0 := j_0(x, y) \in \mathbb{N}$ such that:

(i) $a_1 \in C^1_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^m) \cap C^{j_0+2}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^m), a_2 \in C^1_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n);$

(ii) $\det(b(x, y) \cdot b^*(x, y)) \neq 0, b \in C^1_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^d) \cap C^2(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^d);$

(iii) The vector space spanned by $\bigcup_{k=1}^{j_0} A_k$ at point $(x, y)$ has dimension $m.$

**Theorem 2.1.** Let the Hypothesis 2.1 hold, $T > 0$, then $\det(M_T^{-1}) \in L^p(\Omega, \mathbb{P}_{x,y})$ for any $p > 0.$

**Remark 2.1.** If the coefficients $a_1, a_2, b$ in Eq.(2.1) also depend on $t$ and for any $T > 0$, $t \to (a_1(t, 0), a_2(t, 0))$ and $t \to b(t, 0)$ are bounded on $[0, T]$, then the Theorem 2.1 Theorem 3.1 and Theorem 4.1 hold also.

There is a natural relation between Hörmander conditions (H) and Hypothesis 2.1 from the well-known geometric interpretation of Hörmander conditions. This relation can be proved directly by tedious calculations also.

**Remark 2.2.** Assume $a_1 \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^m), a_2 \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n), b \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d), n \leq d, \det(b(x, y) \cdot b^*(x, y)) \neq 0.$ Then the Hörmander conditions (H) is equivalent to Hypothesis 2.1.

But Hypothesis 2.1 is weaker than Hörmander conditions in some sense, the followings are three examples.

**Example 2.1.** A concrete example is the following stochastic differential equation

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_2(t)dt + y_t dt \\
\frac{dx_2(t)}{dt} &= x_1(t)dt \\
\frac{dx_3(t)}{dt} &= x_2(t)dt + x_3(t)dt \\
\frac{dy_t}{dt} &= a_2(x_1, y_t)dt + bdW_t
\end{align*}
\]

where $x_t = (x_1(t), x_2(t), x_3(t))^* \in \mathbb{R}^3, y_t \in \mathbb{R}, a_2(x_1, x_2, x_3, y)$ only has one order derivatives and $b \in \mathbb{R}^1 \setminus \{0\}$ is a constant, then the Hypothesis 2.1 holds, but the Hörmander conditions (H) can’t be applied directly.
Proof. Set \( a_1(x_1, x_2, x_3, y) = (x_2 + y, x_1, x_2 + x_3)^* \), then

\[
\nabla_x a_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \nabla_y a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

In this example, by calculating,

\[
A_1 = \nabla_y a_1, \quad A_2 = -\nabla_x a_1 \nabla_y a_1, \quad A_3 = (+\nabla_x a_1)^2 \nabla_y a_1
\]

and

\[
\nabla_y a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad -\nabla_x a_1 \nabla_y a_1 = - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (+\nabla_x a_1)^2 \nabla_y a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

So the vector space spanned by \( \{A_j, \ j = 1, 2, 3\} \) at any point \((x, y)\) is \( \mathbb{R}^3 \). \( \square \)

The following example is a special case for the SDE considered in [4] with \( n = 3 \).

Example 2.2. Consider the following SDE

\[
\begin{align*}
X_t^1 &= x_1 + \int_0^t F_1(s, X_s^1, X_s^2, X_s^3)ds + \int_0^t \sigma(s, X_s^1, X_s^2, X_s^3)dW_s, \\
X_t^2 &= x_2 + \int_0^t F_2(s, X_s^1, X_s^2, X_s^3)ds, \\
X_t^3 &= x_3 + \int_0^t F_3(s, X_s^2, X_s^3)ds.
\end{align*}
\]

If \( \det(\sigma(0, x_1, x_2, x_3)\sigma^*(0, x_1, x_2, x_3)) \neq 0 \), by calculating,

\[
A_1 = \begin{pmatrix} \nabla_x F_2 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \nabla_x F_2 \\ 0 \end{pmatrix}, \quad \left( \frac{G(x_1, x_2, x_3)}{\nabla_x F_2 \cdot \nabla_x F_2} \right),
\]

for some function \( G \). The condition in [4] is \( \nabla_x F_2 \cdot \nabla_x F_3 \neq 0 \). So the (iii) in Hypothesis 2.1 is the same as that in [4]. And the Hörmander conditions (H) can’t be applied directly.

The following example shows the condition (ii) in Hypothesis 2.1 is necessary in some sense.

Example 2.3. \( W_t \) is a one dimension standard Brownian motion and \( X_t = (X_t^1, X_t^2), Y_t = (Y_t^1, Y_t^2) \) satisfy the following equations

\[
\begin{align*}
X_t &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y_t dt, \\
Y_t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Y_t dt + \begin{pmatrix} 1 \\ 1 \end{pmatrix} dW_t.
\end{align*}
\]
If set
\[ V = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 2 & 1 & -1 \\ 2 & -3 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_t^1 \\ X_t^2 \\ \bar{Y}_t^1 \\ \bar{Y}_t^2 \end{pmatrix} = V \begin{pmatrix} X_t^1 \\ X_t^2 \\ \bar{Y}_t^1 \\ \bar{Y}_t^2 \end{pmatrix}, \]
then \( \bar{Y}_t^2 = 0 \), so the Malliavin matrix for \((X_t^1, X_t^2, Y_t^1, Y_t^2)\) is singular a.s. For \( V \) is invertible, so the Malliavin matrix for \((X_t, Y_t)\) is singular also. But the Malliavin matrix for \(Y_t\) is invertible by Hörmander Theorem, and the (i), (iii) in the Hypothesis 2.1 hold.

### 2.2 Proof of Theorem 2.1

In [14], the inverse of Jacobian matrix \( J_t^{-1} \) is regarded as a whole. In this subsection, we divide \( J_t^{-1} \) into four parts \( \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \) and do more elaborate estimates, we obtain \( \det(M_T^{-1}) \in L^p(\Omega, \mathbb{P}, \mathcal{F}_T) \), \( \forall p, T > 0 \) under some conditions weaker than Hörmander conditions. Our main method is similar to that in [14], but it also has some differences and its proof is more complicated. The differences depend heavily on the special form of the Eq. (2.1). Before we prove the Theorem 2.1 we introduce some notations and list the Lemmas which will be used in the proof of Theorem 2.1.

Assume \( J_t^{-1} = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \), \( A_t \) is a matrix with dimension \( m \times m \), then

\[
\begin{align*}
DA_t &= -\sum_{j=1}^{d} B_t \nabla_x b_j dW_j(t) - (A_t \nabla_x a_1 + B_t \nabla_x a_2) dt + \sum_{j=1}^{d} B_t \nabla_y b_j \nabla_x b_j dt, \\
DB_t &= -\sum_{j=1}^{d} B_t \nabla_y b_j dW_j(t) - (A_t \nabla_y a_1 + B_t \nabla_y a_2) dt + \sum_{j=1}^{d} B_t \nabla_y b_j \nabla_y b_j dt, \\
DC_t &= -\sum_{j=1}^{d} D_t \nabla_x b_j dW_j(t) - (C_t \nabla_x a_1 + D_t \nabla_x a_2) dt + \sum_{j=1}^{d} D_t \nabla_y b_j \nabla_x b_j dt, \\
DD_t &= -\sum_{j=1}^{d} D_t \nabla_y b_j dW_j(t) - (C_t \nabla_y a_1 + D_t \nabla_y a_2) dt + \sum_{j=1}^{d} D_t \nabla_y b_j \nabla_y b_j dt.
\end{align*}
\]

(2.1)

For the vector space spanned by \( \cup_{k=1}^{d} A_k \) at point \((x, y)\) has dimension \( m \), then there exist two positive constants \( R_1 \) and \( \epsilon \) such that

\[ \sum_{j=1}^{d_0} \sum_{V \in A_j} (v^* V(x', y'))^2 \geq \epsilon \]

(2.2)

holds for all \( v \in \mathbb{R}^m, |v| = 1 \) and \(|(x', y') - (x, y)| \leq R_1\).
Fix $R_2 = \frac{1}{100}$, define the stopping time as

$$S = S(x, y) := \inf \left\{ s \geq 0 : \sup_{0 \leq u \leq s} |(x_u, y_u) - (x, y)| \geq R_1 \text{ or } \sup_{0 \leq u \leq s} |J_u^{-1} - I_{m+n}| \geq R_2 \right\}.$$  (2.3)

Define the adapted process

$$\lambda(s) = \inf \left\{ v^*b(x_s, y_s)b^*(x_s, y_s)v \right\}.$$  (2.4)

For $|\inf_v a_v - \inf_v b_v| \leq \sup_v |a_v - b_v|$, so

$$|\lambda(s) - \lambda(t)| \leq \|b(x_s, y_s)b^*(x_s, y_s) - b(x_t, y_t)b^*(x_t, y_t)\|.$$  (2.5)

Then $\lambda(s)$ is continuous w.r.t $s$. Since $\det(b(x,y)b^*(x,y)) \neq 0$, so $\lambda(0) > 0$. For $R_3 = \lambda(0)/2$, we define the stopping times

$$\tau' = \inf \{ s > 0 : |\lambda(s) - \lambda(0)| \geq R_3 \},$$

$$\tau = \tau' \wedge S \wedge T.$$  (2.6)

Let $j_0$ be as in Hypothesis 2.1, $v = (v_1^*, v_2^*) \in \mathbb{R}^m \times \mathbb{R}^n$ with $|v| = 1$. Fix $q > 8$ and set

$$F = \left\{ \sum_{j=1}^{d} \int_0^T |(v_1^*B_s + v_2^*D_s)b_j|^2 ds \leq \epsilon^{qj_0+6} \right\},$$

$$E_j = \left\{ \sum_{K \in A_j} \int_0^\tau |(v_1^*A_s + v_2^*C_s)K(x_s, y_s)|^2 ds \leq \epsilon^{qj_0+3-3j} \right\}, \quad j = 1, \ldots, j_0,$n

$$E = F \cap E_1 \cap E_2 \cdots \cap E_{j_0}.$$

Denote by

$$\|v_1^*B_s + v_2^*D_s\|^2 := \sup_{s, r \in [0, \tau]} \left| \frac{|(v_1^*B_s + v_2^*D_s)^2 - |v_1^*B_r + v_2^*D_r|^2|}{|s-r|^{\frac{1}{4}}} \right|.$$  (2.8)

**Remark 2.3.** In the definition of $S$, $R_2 = \frac{1}{100}$ is a technique skill. In the Lemma 2.12, we essentially need $R_2$ small enough, and we need $R_2$ finite in other places. Here, $R_1, R_3$ and $c$ depend on $(x, y)$.

From the definition of $S$, when $s \leq S$, $|(x_s, y_s) - (x, y)| \leq R_1$. So from (2.2),

$$\sum_{j=1}^{j_0} \sum_{V \in A_j} (v^*V(x_s, y_s))^2 \geq c$$  (2.9)

holds for all $s \leq S$ and $v \in \mathbb{R}^m$ with $|v| = 1$. 

8
Lemma 2.1. (c.f. Lemma 6.14, [3]). Let $f : [0, T_0] \to \mathbb{R}$ be continuous differentiable and $\alpha \in (0, 1]$. Then

$$\|\partial_t f\|_\infty = \|f\|_1 \leq 4 \| f \|_\infty \max \left\{ \frac{1}{T_0}, \| f \|_\infty \frac{1}{1+\alpha}, \| \partial_t f \|_1 \frac{1}{1+\alpha} \right\},$$

where $\|f\|_\alpha = \sup_{s, t \in [0, T_0], s \neq t} \frac{|f(s) - f(t)|}{|t-s|^\alpha}$.

Lemma 2.2. (c.f. Corollary 2.2.1, [14]). Let the Hypothesis 2.1 hold, then for any $p, T > 0$, there exists a finite constant $C(T, p, x, y)$ such that

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} |(x_t, y_t)|^p \right\} \leq C(T, p, x, y).$$

Lemma 2.3. Let the Hypothesis 2.1 hold, then for any $p, T > 0$, there exists a finite constant $C(T, p, x, y)$ such that

$$\mathbb{E} \left\{ \sup_{0 \leq s \leq T} \| J_s^{-1} \|^p \right\} \leq C(T, p, x, y),$$

$$\mathbb{E} \left\{ \sup_{0 \leq s \leq T} \| J_s \|^p \right\} \leq C(T, p, x, y).$$

Proof. It directly follows from (1.6) (1.7) and Lemma 2.2.1 in [14].

Lemma 2.4. Let the Hypothesis 2.1 hold, then for any $p > 0$, there exists a finite constant $C(p, x, y)$ such that

$$\mathbb{P}\{ S < \epsilon \} \leq C(p, x, y)\epsilon^p, \forall \epsilon > 0.$$

Proof. For any $p > 0$, it holds that

$$\mathbb{P}\{ S < \epsilon \} \leq \mathbb{P}\left\{ \sup_{0 \leq s \leq \epsilon} |(x_s, y_s) - (x, y)| \geq R_1 \right\} + \mathbb{P}\left\{ \sup_{0 \leq s \leq \epsilon} |J_s^{-1} - I_{m+n}| \geq R_2 \right\} \leq C(p, x, y)\mathbb{E} \left\{ \sup_{0 \leq s \leq \epsilon} |(x_s, y_s) - (x, y)|^{2p} \right\} + C(p, x, y)\mathbb{E} \left\{ \sup_{0 \leq s \leq \epsilon} |J_s^{-1} - I_{m+n}|^{2p} \right\}.$$

Then this Lemma comes from Burkholder’s and Hölder’s inequalities.

Lemma 2.5. Let the Hypothesis 2.1 hold, then for any $p > 0$, there exists a finite constant $C(p, T, x, y)$ such that

$$\mathbb{P}\{ \tau < \epsilon \} \leq C(p, T, x, y)\epsilon^p, \forall \epsilon > 0.$$

Proof. From Lemma 2.4 and the fact

$$\mathbb{P}\{ \tau < \epsilon \} \leq \mathbb{P}\{ S < \epsilon \} + \mathbb{P}\{ \tau' < \epsilon \} + \mathbb{P}\{ T < \epsilon \},$$

9
we only need to estimate $\mathbb{P}\{\tau' < \epsilon\}$. For any $p > 0$

$$
\mathbb{P}\{\tau' < \epsilon\} \leq \mathbb{P}\left\{ \sup_{0 \leq s \leq \epsilon} |\lambda(s) - \lambda(0)| \geq R_3 \right\} 
\leq C(p, x, y) \mathbb{E}\left\{ \sup_{0 \leq s \leq \epsilon} |\lambda(s) - \lambda(0)|^{2p} \right\}. 
$$

(2.10)

Due to $|\inf_v a_v - \inf_v b_v| \leq \sup_v |a_v - b_v|$, 

$$
\mathbb{E}\left\{ \sup_{0 \leq s \leq \epsilon} |\lambda(s) - \lambda(0)|^{2p} \right\} 
\leq C(p) \sum_{i,k=1,...,n} \mathbb{E}\left\{ \sup_{0 \leq s \leq \epsilon} \left| b_{kj}(x,s,y)b_{ij}(x,s) - b_{kj}(x,y)b_{ij}(x,y) \right|^{2p} \right\}. 
$$

(2.11)

Note that 

$$
b_{kj}(x,s,y)b_{ij}(x,s) - b_{kj}(x,y)b_{ij}(x,y) 
= (b_{kj}(x,s,y) - b_{kj}(x,y))(b_{ij}(x,s) - b_{ij}(x,y)) + b_{kj}(x,y)(b_{ij}(x,s) - b_{ij}(x,y)) + b_{ij}(x,y)(b_{kj}(x,s) - b_{kj}(x,y)).
$$

(2.12)

From (2.10) (2.11) (2.12), 

$$
\mathbb{P}\{\tau' < \epsilon\} \leq C(p, x, y) \left[ \mathbb{E}\left\{ \sup_{0 \leq s \leq \epsilon} |b_{ij}(x,s,y) - b_{ij}(x,y)|^{2p} \right\} + \mathbb{E}\left\{ \sup_{0 \leq s \leq \epsilon} |b_{ij}(x,s,y) - b_{ij}(x,y)|^{4p} \right\} \right].
$$

So this Lemma comes from Burkholder’s and Hölder’s inequalities and the fact 

$$b_{ij}(x,s,y) - b_{ij}(x,y) = \langle \nabla b_{ij}(\xi, \eta), (x,s,y) - (x,y) \rangle,$$

here $(\xi, \eta)$ is some point depending on $(x,s,y)$ and $(x,y)$. ~\(\square\)

**Lemma 2.6.** Let $\sigma$ be a finite stopping time with bound $c_\sigma < \infty$, and there exists $\bar{p} > 0$ such that 

$$
\mathbb{P}\{\sigma < \epsilon\} \leq C(c_\sigma, \bar{p})\epsilon^{\bar{p}}, \ \forall \epsilon > 0,
$$

holds for some constant $C(c_\sigma, \bar{p})$. Assume $\gamma(t) = (\gamma_1(t), ..., \gamma_d(t))$, $u(t) = (u_1(t), ... u_d(t))$ are continuous adapted processes, $W(t) = (W_1(t), \cdots, W_d(t))^\top$ is a standard Wiener process, $a(t), \tilde{y}(t) \in \mathbb{R}$ and for $t \in [0, c_\sigma]$

$$
a(t) = \alpha + \int_0^t \beta(s)ds + \int_0^t \gamma(s)dW(s), \\
\tilde{y}(t) = \tilde{y} + \int_0^t a(s)ds + \int_0^t u(s)dW(s).
$$
Suppose for some $p, \tilde{c} > 0$,
\[
\mathbb{E}\left\{ \sup_{0 \leq t \leq \sigma} (|\tilde{\beta}(t)| + |\gamma(t)| + |a(t)| + |u(t)|)^p \right\} \leq \tilde{c} < \infty.
\]

Then for any three positive numbers $(q, r, v)$ satisfy $2q - 36r - 9v > 16$, there exists $\epsilon_0 = \epsilon_0(c_\sigma, q, r, v)$ such that for any $\epsilon < \epsilon_0$,
\[
\mathbb{P}\left( \int_0^\sigma \tilde{y}(t)^2 dt < \epsilon^q, \int_0^\sigma (|a(t)|^2 + |u(t)|^2)dt \geq \epsilon \right) \leq \tilde{c} \epsilon^r + \exp(-\epsilon^{-\frac{r}{2}}) + C(c_\sigma, \tilde{p}) \epsilon^{\tilde{p}}.
\]

The proof of Lemma 2.6 is postponed to Appendix A.

**Lemma 2.7.** Let $\sigma$ be a finite stopping time with bound $c_\sigma < \infty$, and there exists $\tilde{p} > 2$, such that
\[
\mathbb{P}\{\sigma < \epsilon\} \leq C(c_\sigma, \tilde{p}) \epsilon^{\tilde{p}}, \ \forall \epsilon > 0 \tag{2.13}
\]
holds for some constant $C(c_\sigma, \tilde{p})$. Consider the following one dimensional stochastic differential equation
\[
\tilde{y}(t) = \tilde{y} + \int_0^t a(s)ds + \int_0^t u(s)dW(s), \ t \in [0, c_\sigma], \tag{2.14}
\]
where $u(s) = (u_1(s), \cdots, u_d(s))$ is a continuous adapted process, $W(t) = (W_1(t), \cdots, W_d(t))^*$ is a $d$-dimensional standard Wiener process. $a(t), u(t)$ satisfy
\[
\mathbb{E}\left\{ \sup_{0 \leq t \leq \sigma} (|a(t)| + |u(t)|)^p \right\} \leq \tilde{c} < \infty, \tag{2.15}
\]
for some $p, \tilde{c} > 0$.

Then for any three positive numbers $(q, r, v)$ satisfy $2q > 8 + 20r + v$, there exists $\epsilon_0 = \epsilon_0(c_\sigma, q, r, v)$ such that for any $\epsilon \leq \epsilon_0$,
\[
\mathbb{P}\left( \int_0^\sigma \tilde{y}(t)^2 dt < \epsilon^q, \int_0^\sigma |u(t)|^2 dt \geq \epsilon \right) \leq \tilde{c} \epsilon^r + \exp\{-\epsilon^{-\frac{r}{2}}\} + C(c_\sigma, \tilde{p}) \epsilon^{\tilde{p}}.
\]

The proof of Lemma 2.7 is postponed to Appendix A.

**Lemma 2.8.** Let the Hypothesis 2.1 hold and $C_0 = 2/\lambda(0)$, then for any $p > 0$, there exists a constant $C = C(p, T, x, y, q)$ such that
\[
\mathbb{P}\left\{ ||v_1^* B_s + v_2^* D_s||_3^2 > \frac{1}{4^p C_0^2} \epsilon^{-\frac{3\lambda_0 + 6}{8}} \right\} \leq C(p, T, x, y, q) \epsilon^p, \ \forall \epsilon > 0.
\]

**Proof.** From (2.11) and Itô’s formula
\[
d|v_1^* B_s + v_2^* D_s|^2 = -2((v_1^* B_s + v_2^* D_s), (v_1^* B_s + v_2^* D_s) \nabla x a_0)ds
\]
\[
-2((v_1^* B_s + v_2^* D_s), (v_1^* A_s + v_2^* C_s) \nabla x a_1)ds
\]
Due to $\gamma$, then this Lemma is obtained by setting $C$. Thus, $\forall$ such that $\rho > \epsilon > 0$, $\forall p' > 0$

$$\mathbb{P}\left\{ \|v_1^s B_r + v_2^s D_r\|_4^2 < \frac{1}{4\epsilon C_0^4} e^{-\frac{1}{4}} \right\} \leq C(p') e^{\frac{1}{8}} \mathbb{E}\left[ \|v_1^s B_r + v_2^s D_r\|_4^2 \right]^{p'}.$$

From BDG inequality, Lemma 2.3 and and the above equation, for any $p > 0$, there exists a constant $C = C(p, T, x, y)$ such that for any $s, r \in [0, T]$,

$$\mathbb{E} \left[ (v_1^s B_s + v_2^s D_s)^2 - |v_1^r B_r + v_2^r D_r|^2 \right]^{2p} \leq C|s - r|^p.$$

Set $\gamma = 2p$, $\epsilon = p - 1$, $T_0 = T$ in Theorem 2.1, [16], for any $p > 2$,

$$C_{p, T, x, y} := \mathbb{E} \left[ \|v_1^s B_r + v_2^s D_r\|_4^2 \right]^{2p} < \infty.$$

Thus, $\forall \epsilon > 0$, $\forall p' > 0$

$$\mathbb{P}\left\{ \|v_1^s B_r + v_2^s D_r\|_4^2 > \frac{1}{4\epsilon C_0^4} e^{-\frac{1}{4}} \right\} \leq C(p') e^{\frac{1}{8}} \mathbb{E}\left[ \|v_1^s B_r + v_2^s D_r\|_4^2 \right]^{p'}.$$

Then this Lemma is obtained by setting $p' = \frac{8p}{q^2 + \alpha}$ in (2.16), \hfill $\square$

**Lemma 2.9.** Let the Hypothesis 2.1 hold, then for any $p > 0$ there exists a constant $C(p, T, x, y, q)$ such that

$$\mathbb{P}\left\{ F \cap \left\{ \sup_{s \in [0, \tau]} |v_1^s B_s + v_2^s D_s|^2 > \epsilon e^{\frac{1}{10}} \right\} \right\} \leq C(p, T, x, y, q)p^p, \quad \forall \epsilon > 0.$$

**Proof.** Due to $\tau \leq \tau'$ and $\omega \in F$, there exists a constant $C_0 = 2/\lambda(0) = C(x, y) > 0$ such that

$$\int_0^\tau |v_1^s B_s + v_2^s D_s|^2(\omega)ds \leq C_0 e^{\frac{1}{10}}.$$

Set $f(s) = \int_0^s |v_1^s B_u + v_2^s D_u|^2du$, $T_0 = \tau(\omega)$ and $\alpha = \frac{1}{4}$ in Lemma 2.1 then

$$\sup_{s \in [0, T]} |v_1^s B_s + v_2^s D_s|^2 \leq \max\left\{ \frac{4}{\tau} \int_0^\tau |v_1^s B_u + v_2^s D_u|^2du, \frac{1}{4} \left( \int_0^\tau |v_1^s B_u + v_2^s D_u|^2du \right)^\frac{5}{4} \right\}.$$

Thus

$$\mathbb{P}\left\{ F \cap \left\{ \sup_{s \in [0, \tau]} |v_1^s B_s + v_2^s D_s|^2 > \epsilon e^{\frac{1}{10}} \right\} \right\}$$

12
Define $C$ then there exists a constant $C$ such that

$$p \leq p \left\{ \|v_1^* B_s + v_2^* D_s\|_p^2 > \frac{1}{4} \right\} \epsilon^{-\frac{3\eta_0 + 6}{8}} + p \left( \tau < 4C_0 \epsilon^{\frac{3\eta_0 + 6}{8}} \right). \quad (2.17)$$

Due to (2.17), Lemma 2.5 and Lemma 2.8 for any $p > 0$, there exists a constant $C(p, T, x, y, q)$ such that

$$p \{ F \cap \{ \sup_{s \in [0, \tau]} |v_1^* B_s + v_2^* D_s|^2 > \epsilon^{-\frac{3\eta_0 + 6}{10}} \} \} \leq C(p, T, x, y, q) \epsilon^p, \quad \forall \epsilon > 0.$$

\[ \square \]

**Lemma 2.10.** Let the Hypothesis 2.11 hold, then for any $p > 0$, there exists positive constant $C(p, T, x, y, q)$ such that

$$p \left\{ \sum_{j=1}^d \int_0^\tau |(v_1^* B_s + v_2^* D_s) b_j|^2 ds \leq \epsilon^{-\frac{3\eta_0 + 6}{10}} \right\} \leq C(p, T, x, y, q) \epsilon^p, \quad \forall \epsilon > 0.$$

**Proof.** From 2.11,

$$d(v_1^* B_s + v_2^* D_s) = -(v_1^* B_s + v_2^* D_s) \nabla_y a_2 ds - (v_1^* A_s + v_2^* C_s) \nabla_y a_1 ds$$

$$- \sum_{j=1}^d (v_1^* B_s + v_2^* D_s) \nabla_y b_j dW_j(t) + \sum_{j=1}^d (v_1^* B_s + v_2^* D_s) \nabla_y b_j \nabla_y b_j ds.$$

From $\det(\text{det}(b(x, y)b^*(x, y)) \neq 0$ and the definition of $\tau$, if

$$\sum_{j=1}^d \int_0^\tau |(v_1^* B_s + v_2^* D_s) b_j|^2(\omega) ds \leq \epsilon^{-\frac{3\eta_0 + 6}{10}},$$

then there exists a constant $C = C(x, y)$ such that

$$\int_0^\tau |v_1^* B_s + v_2^* D_s|^2(\omega) ds \leq C \epsilon^{-\frac{3\eta_0 + 6}{10}} . \quad (2.18)$$

Define

$$\tilde{y}(s) := (v_1^* B_s + v_2^* D_s) + \int_0^s (v_1^* B_u + v_2^* D_u) \nabla_y a_2 du - \sum_{j=1}^d \int_0^s (v_1^* B_u + v_2^* D_u) \nabla_y b_j \nabla_y b_j du,$$

then

$$d\tilde{y}(s) = -(v_1^* A_s + v_2^* C_s) \nabla_y a_1 ds - \sum_{j=1}^d (v_1^* B_s + v_2^* D_s) \nabla_y b_j dW_j(s). \quad (2.19)$$
Due to Hölder inequality, (2.18) and (2.19), there exists a constant $C(T, x, y)$ such that
\[
\int_0^T |\tilde{y}(s)|^2 ds \leq C(T, x, y) \int_0^T |v_1^* B_s + v_2^* D_s|^2 ds
\]
This implies that
\[
\left\{ \sum_{j=1}^d \int_0^T |(v_1^* B_s + v_2^* D_s)b_j|^2 ds \leq \epsilon_0^{q^2/2}, \int_0^T |v_1^* A_s + v_2^* C_s)\nabla_y a_1|^2 ds > \epsilon_0^{q^2/2} \right\}
\subseteq \left\{ \int_0^T |\tilde{y}(s)|^2 ds \leq C(T, x, y)\epsilon_0^{q^2/2}, \int_0^T |v_1^* A_s + v_2^* C_s)\nabla_y a_1|^2 ds \geq \epsilon_0^{q^2/2} \right\}.
\]
The probability of the above event can be estimated by Lemma 2.6 and Lemma 2.5.

**Lemma 2.11.** Let the Hypothesis [2.1] hold, then for any $p > 0$, there exists constants $C = C(p, T, x, y, q)$, $\epsilon_0 = \epsilon_0(q, x, y)$ such that for $j = 1, \cdots, j_0 - 1$,
\[
P\{F \cap E_j \cap E_{j+1}^c\} \leq C(p, T, x, y, q)\epsilon_0^p, \forall \epsilon \leq \epsilon_0.
\]
**Proof.** For any $K \in A_j$, by calculating,
\[
d(v_1^* A_s + v_2^* C_s)K(x_s, y_s)
\]
\[
= - \sum_{i=1}^d ((v_1^* B_s + v_2^* D_s)\nabla_x b_i, \nabla_y K b_i)ds + \sum_{i=1}^d (v_1^* B_s + v_2^* D_s)\nabla_y b_i \nabla_x b_i K(x_s, y_s)ds
\]
\[
+ \sum_{i=1}^d \left( (v_1^* A_s + v_2^* C_s)\nabla_y K(x_s, y_s)b_i - (v_1^* B_s + v_2^* D_s)\nabla_x b_i K(x_s, y_s) \right) \cdot dW_i(s)
\]
\[
+ (v_1^* A_s + v_2^* C_s)\nabla_y K(x_s, y_s)a_2(x_s, y_s)ds - (v_1^* B_s + v_2^* D_s)\nabla_x a_2 K(x_s, y_s)ds
\]
\[
+ (v_1^* A_s + v_2^* C_s)(( - \nabla_x a_1(x_s, y_s)K(x_s, y_s) + \nabla_x K(x_s, y_s)a_1(x_s, y_s))ds
\]
\[
+ \frac{1}{2}(v_1^* A_s + v_2^* C_s)\sum_{i=1}^d (\nabla_y (\nabla_y K \cdot b_i)b_i)ds,
\]
and
\[
P(F \cap E_j \cap E_{j+1}^c) = P \left( F \cap \sum_{K \in A_j} \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right)
\leq \sum_{K \in A_{j+1}} P \left( F \cap \sum_{K \in A_j} \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right)
\leq \sum_{K \in A_j} P \left( F \cap \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right) - \sum_{K \in A_{j+1}} \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds
\geq \frac{\epsilon^{3j_0 - 3j}}{n(j)} \right) \subseteq B_1 \cup B_2,
\]
where \(n(j)\) denotes the cardinality of the set \(A_j\).

It is not difficult to prove there exists a constant \(\epsilon_0 = \epsilon_0(g, x, y)\), such that when \(\epsilon < \epsilon_0\)
\[
F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right\}
\cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds + \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds \geq \frac{\epsilon^{3j_0 - 3j}}{n(j)} \right\} \subseteq B_1 \cup B_2,
\]
and \(B_1 \subseteq B_1'\), \(B_2 \subseteq B_3 \cup B_4, B_4 \subseteq \bigcup_{i=1}^d B_{4i}^i, B_{4i}^i \subseteq B_{4}^{ii}\),
here,
\[
B_1 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right\},
\]
\[
B_1' = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds \geq \frac{\epsilon^{3j_0 + 3 - 3j}}{n(j)} \right\},
\]
\[
B_2 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right\}
\]
\[
\int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds \geq \frac{\epsilon^{3j_0 + 3 - 3j}}{n(j)} \right) \subseteq B_1 \cup B_2,
\]

and
\[
B_1 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right\},
\]
\[
B_1' = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds \geq \frac{\epsilon^{3j_0 + 3 - 3j}}{n(j)} \right\},
\]
\[
B_2 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 + 3 - 3j} \right\}
\]
\[
\int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \right|^2 ds \geq \frac{\epsilon^{3j_0 + 3 - 3j}}{n(j)} \right) \subseteq B_1 \cup B_2,
\]
\[ B_3 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) (\nabla_x a_1 \cdot K + \nabla_x K \cdot a_1) \right|^2 ds \leq \epsilon^{3j_0 - 3j} \right\}, \]

\[ B_4 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) K(x_s, y_s) \right|^2 ds \leq \epsilon^{3j_0 - 3j} \right\}, \]

\[ B'_4 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \nabla_y K(x_s, y_s) b_i \right|^2 ds \leq \epsilon^{3j_0 + 2 - 3j} \right\}, \]

\[ B''_4 = F \cap \left\{ \int_0^\tau \left| (v_1^* A_s + v_2^* C_s) \nabla_y (\nabla_y K(x_s, y_s) b_i) \right|^2 ds \leq \epsilon^{3j_0 + 2 - 3j} \right\}, \]

From Lemma 2.7

\[ \mathbb{P}(B_1) \leq \mathbb{P}(B'_1) \leq C(p, T, x, y) \epsilon^p, \quad \forall \epsilon > 0. \]

The estimation of \( \mathbb{P}(B''_4) \) is similar to the estimation of \( \mathbb{P}(B_1) \). For \( \mathbb{P}(B_3) \), define

\[ \tilde{y}(s) = (v_1^* A_s + v_2^* C_s) K(x_s, y_s) - \frac{1}{2} \sum_{i=1}^d \int_0^s (v_1^* A_r + v_2^* C_r) \nabla_y (\nabla_y K(x_r, y_r) b_i) b_i dr \]

\[ - \int_0^s (v_1^* A_r + v_2^* C_r) \nabla_y K(x_r, y_r) a_2(x_r, y_r) dr + \int_0^s (v_1^* B_r + v_2^* D_r) \nabla_x a_2 K(x_r, y_r) dr \]
\[- \sum_{i=1}^{d} \int_0^s (v_1^i B_r + v_2^i D_r) \nabla y b_i \nabla x b_i K(x_r, y_r) dr + \sum_{i=1}^{d} \int_0^s \langle (v_1^i B_r + v_2^i D_r) \nabla x b_i, \nabla y K b_i \rangle dr,\]

then from the equation \((v_1^i A_s + v_2^i C_s) K(x_s, y_s)\) satisfied, we know

\[d \tilde{y}(s) = \sum_{i=1}^{d} \left[ (v_1^i A_s + v_2^i C_s) \nabla y K(x_s, y_s) b_i + (v_1^i B_s + v_2^i D_s) \nabla x b_i K(x_s, y_s) \right] dW_i(s) + (v_1^i A_s + v_2^i C_s) (- \nabla x a_1 K + \nabla y a_1) (x_s, y_s) ds.\]

If \(\omega \in B_3\), from the definitions of \(\tilde{y}(s)\) and \(\tau\), there exists a constant \(C(p, T, x, y)\) such that

\[\int_0^\tau \tilde{y}(s)^2 ds \leq C(p, T, x, y) \epsilon^{q^{3j_0+1-3j}}.\]

Thus \(B_3\) is a subset of

\[\left\{ \int_0^\tau \tilde{y}(s)^2 ds \leq C(p, T, x, y) \epsilon^{q^{3j_0+1-3j}}, \right.\]

\[\left. \int_0^\tau \left| (v_1^i A_s + v_2^i C_s) (- \nabla x a_1 K + \nabla y a_1) \right|^2 ds \geq \frac{\epsilon^{q^{3j_0-3j}}}{2n(j)} \right\}. \tag{2.20}\]

The estimate of the above set is from Lemma 2.6 and Lemma 2.5. This finishes the proof of the Lemma 2.13.

**Lemma 2.12.** Let the Hypothesis 2.1 hold, then there exists a constant \(\epsilon_0 = \epsilon_0(q, x, y)\) such that

\[E \cap \{ \tau \geq \epsilon^q \} \cap \left\{ \sup_{s \in [0, \tau]} |v_1^i B_s + v_2^i D_s| \leq \epsilon^{q^{3j_0+6}} \right\} = \emptyset, \forall \epsilon < \epsilon_0.\]

**Proof.** If \(\omega \in E \cap \{ \tau \geq \epsilon^q \} \cap \left\{ \sup_{s \in [0, \tau]} |v_1^i B_s + v_2^i D_s| \leq \epsilon^{q^{3j_0+6}} \right\},\) from (2.9), for some \(c > 0,\)

\[\sum_{j=1}^{j_0} \sum_{V \in A_j} \int_0^\tau \left| (v_1^i A_s + v_2^i C_s) V(x_s, y_s)(\omega) \right|^2 ds \]

\[= \int_0^\tau \sum_{j=1}^{j_0} \sum_{V \in A_j} \left( (v_1^i A_s + v_2^i C_s) V(x_s, y_s)(\omega) \right)^2 \cdot |v_1^i A_s + v_2^i C_s|^2 ds \]

\[\geq c \int_0^\tau |v_1^i A_s + v_2^i C_s|^2 ds. \tag{2.21}\]

For \(\omega \in \left\{ \sup_{s \in [0, \tau]} |v_1^i B_s + v_2^i D_s| \leq \epsilon^{q^{3j_0+6}} \right\},\) let \(s = 0,\)

\[|v_2| \leq \epsilon^{q^{3j_0+6}} \leq \frac{1}{100}, \quad |v_1| = \sqrt{1 - |v_2|^2} > \frac{9}{10}. \tag{2.22}\]
Due to (2.22) and the fact when $s \leq \tau$, $\|A_s - I_m\| \leq \frac{1}{100}$, $\|C_s\| \leq \frac{1}{100}$,
\[
\int_0^\tau |v_1^s A_s + v_2^s C_s|^2 ds \geq \frac{1}{8} \tau \geq \frac{1}{8} \epsilon^q. \tag{2.23}
\]
From (2.21) and (2.23),
\[
\sum_{j=1}^{j_0} \sum_{V \in A_j} \int_0^\tau \left| (v_1^s A_s + v_2^s C_s) V(x_s, y_s)(\omega) \right|^2 ds \geq \frac{c}{8} \epsilon^q. \tag{2.24}
\]
In the following part, we will prove this is impossible when $\epsilon$ is small enough. Set $\epsilon_0(q, x, y)$ such that when $\epsilon < \epsilon_0(q, x, y)$,
\[
\epsilon \frac{\epsilon^{3j_0 + 6}}{10} \leq \frac{1}{100} \sum_{j=1}^{j_0} \epsilon^{q^{3j_0 + 3-3j}} < \frac{c}{8} \epsilon^q.
\]
For $\omega \in E \subseteq E_j$, then
\[
\sum_{K(x,y) \in A_j} \int_0^\tau \left| (v_1^s A_s + v_2^s C_s) K(x_s, y_s)(\omega) \right|^2 ds \leq \epsilon^{q^{3j_0 + 3-3j}},
\]
so when $\epsilon < \epsilon_0$,
\[
\sum_{j=1}^{j_0} \sum_{V \in A_j} \int_0^\tau \left| (v_1^s A_s + v_2^s C_s) V(x_s, y_s)(\omega) \right|^2 ds \leq \sum_{j=1}^{j_0} \epsilon^{q^{3j_0 + 3-3j}} < \frac{c}{8} \epsilon^q,
\]
this contradict with (2.24).

Thus $E \cap \{\tau \geq \epsilon\} \cap \{\sup_{s \in [0, \tau]} |v_1^s B_s + v_2^s D_s|^2 \leq \epsilon^{\frac{3j_0 + 6}{10}}\} = \emptyset$ when $\epsilon < \epsilon_0$. \hfill \Box

We are now in a position to give

Proof. The proof of Theorem 2.1 Since
\[
M_T = J_T \tilde{M}_T J_T^*, \tag{2.25}
\]
where
\[
\tilde{M}_T = \int_0^T J_s^{-1} \left( \begin{array}{cc} 0 & 0 \\ b(x_s, y_s) & b(x_s, y_s) \end{array} \right) (J_s^{-1})^* ds, \tag{2.26}
\]
we only need to prove the $L^p$ integrability of $\det(\tilde{M}_T^{-1})$. For this purpose, we need to prove for any $p > 0$, there exists constant $C(p)$, such that
\[
\sup_{|v| = 1} \mathbb{P}\{v^* \tilde{M}_T v \leq \epsilon\} \leq C(p) \epsilon^p, \quad \forall \epsilon > 0. \tag{2.27}
\]
It is easy to check that (2.27) is equivalent to for any $p > 0$, $v \in \mathbb{R}^{m+n}, |v| = 1$, there exists positive constants $\epsilon_0(p), \, C(p)$ such that
\[
\mathbb{P}\{v^* \tilde{M}_T v \leq \epsilon\} \leq C(p) \epsilon^p, \quad \forall \epsilon \leq \epsilon_0(p). \tag{2.28}
\]
For $v = (v_1^*, v_2^*)^* \in \mathbb{R}^m \times \mathbb{R}^n$, $J^{-1}(s) = \begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix}$ and due to (2.20),
\[ v^* \tilde{M}_T v = \sum_{j=1}^d \int_0^T |(v_1^* B_s + v_2^* D_s)b_j|^2 ds. \]

Here, we recall the definitions of $E, F, E_j, \tau$ which are given in the beginnings of this subsection. Then (2.25) is equivalent to for any $p > 0$ and $v \in \mathbb{R}^{m+n}, |v| = 1$, there exists constants $C(p)$ and $\epsilon_0(p)$ such that
\[ \mathbb{P}(F) \leq C(p)\epsilon^p, \forall \epsilon \leq \epsilon_0(p). \]

For
\[ F \subseteq (F \cap E_1^c) \cup (F \cap E_1 \cap E_2^c) \cup (F \cap E_2 \cap E_3^c) \cup \cdots \cup (F \cap E_j \cap E_{j+1}^c) \cup E, \]
so
\[ \mathbb{P}(F) \leq \mathbb{P}(E) + \sum_{j=1}^{j=j_0} \mathbb{P}(F \cap E_j \cap E_{j+1}^c) + \mathbb{P}(F \cap E_1^c). \quad (2.29) \]

From Lemma 2.10 and Lemma 2.11 for any $p > 0$ and $v \in \mathbb{R}^{m+n}, |v| = 1$, there exists positive constants $C(p, T, x, y, q), \epsilon_0(q, x, y)$ such that for any $\epsilon \leq \epsilon_0(q, x, y)$,
\[ \sum_{j=1}^{j=j_0} \mathbb{P}(F \cap E_j \cap E_{j+1}^c) + \mathbb{P}(F \cap E_1^c) \leq C(p, T, x, y, q)\epsilon^p. \quad (2.30) \]

For estimating $\mathbb{P}(E)$, we note that
\[ \mathbb{P}(E) \leq \mathbb{P}(E \cap \{ \tau \geq \epsilon^q \}) + \mathbb{P}(\tau < \epsilon^q) \]
\[ \leq \mathbb{P}\left( F \cap \{ \tau \geq \epsilon^q \} \cap \left\{ \sup_{s \in [0, \tau]} |v_1^* B_s + v_2^* D_s| > \epsilon^{3j_0 + 6} \right\} \right) \]
\[ + \mathbb{P}\left( E \cap \{ \tau \geq \epsilon^q \} \cap \left\{ \sup_{s \in [0, \tau]} |v_1^* B_s + v_2^* D_s| \leq \epsilon^{3j_0 + 6} \right\} \right) \]
\[ + \mathbb{P}(\tau < \epsilon^q). \]

So, due to Lemma 2.9, Lemma 2.12 and Lemma 2.5 there exists constants $C(p, T, x, y, q)$ and $\epsilon_0 = \epsilon_0(q, x, y)$ such that
\[ \mathbb{P}(E) \leq C(p, T, x, y, q)\epsilon^p, \forall \epsilon \leq \epsilon_0. \quad (2.32) \]

So from (2.29) (2.30) (2.32), we know for any $p > 0$ and $v \in \mathbb{R}^{m+n}, |v| = 1$, there exists constants $C(p, T, x, y, q)$ and $\epsilon_0(q, x, y)$ such that
\[ \mathbb{P}(F) \leq C(p, T, x, y, q)\epsilon^p, \forall \epsilon \leq \epsilon_0(q, x, y). \]

Since $T, x, y, q$ are all fixed, this theorem has been proved. \qed
3 Gradient Estimate

In this section, we give a gradient estimate. The Hypothesis and main Theorem in this section is

**Hypothesis 3.1.** There exists $j_0 \in \mathbb{N}$ and $R > 0$ such that

(i) $a_1 \in C^2_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}) \cap C^{j_0+2}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$, $a_2 \in C^2_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$;

(ii) $b \in C^2_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R} \times \mathbb{R}^d)$, $\det(b(x, y) \cdot b^*(x, y)) \neq 0$, $\forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ with $|(x, y)| \leq R$;

(iii) $\forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^n, |(x, y)| \leq R$, the vector space spanned by $\cup_{k=1}^{j_0} \mathcal{A}_k$ at point $(x, y)$ has dimension $m$.

**Theorem 3.1.** Let the Hypothesis 3.1 hold, then for any $t > 0$, there exists a constant $C = C(R, t)$ such that for any $f \in C^1_b(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$, $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ with $|(x, y)| \leq R$,

$$|
abla P_t f(x, y)| \leq C(R, t) \|f\|_\infty.$$

In order to prove this Theorem, we need the following Lemmas. These Lemmas give some estimations of $J_1, J_1^{-1}, (x_t, y_t)$ and $M_t$. Especially, we give a uniform estimation of $M_t$ in Lemma 3.3. In the end of this section, we give the proof of Theorem 3.1. The method to prove Theorem 3.1 is standard.

Before all of this, we introduce some notations. Let $D(x_t, y_t)$ denote the Malliavin derivative of $(x_t, y_t)$ and $H = L^2([0, \infty), ds)$. $\delta$ denotes the divergence operator.

**Lemma 3.1.** Let the Hypothesis 3.1 hold, then for any $T, p > 0$,

$$\sup_{|(x, y)| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{t \in [0, T]} |(x_t, y_t)|^p \right\} < \infty, \quad (3.1)$$

$$\sup_{|(x, y)| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{t \in [0, T]} \|J_t\|^p \right\} < \infty, \quad (3.2)$$

$$\sup_{|(x, y)| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{t \in [0, T]} \|J_t^{-1}\|^p \right\} < \infty, \quad (3.3)$$

$$\sup_{|(x, y)| \leq R} \mathbb{E}_{x,y} \|M_T\|^p < \infty, \quad (3.4)$$

**Proof.** For any $(x, y)$ fixed, $\mathbb{E}_{x,y} \left\{ \sup_{s \in [0, T]} |(x_s, y_s)|^p \right\} < \infty$, and $h(x, y) := \mathbb{E}_{x,y} \left\{ \sup_{t \in [0, T]} |(x_t, y_t)|^p \right\}$ is continuous w.r.t $(x, y)$, so (3.1) holds.

For any $p > 2$, set $f(t) = \mathbb{E}_{x,y} \left\{ \sup_{s \in [0, t]} \|J_s\|^p \right\}$, due to (1.7), there exists constants $C(p), C(p, T)$ such that

$$f(t) \leq C(p) + C(p, T) \int_0^t f(s)ds, \forall t \in [0, T].$$

Then, (3.2) comes from Gronwall inequality and the proof of (3.3) is similar. (3.4) follows by (3.1), (3.2), (3.3) and (1.5).
Lemma 3.2. Let the Hypothesis \(3.1\) hold, then for any \(T, p > 0\),

\[
\sup_{|x,y| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{t \in [0,T]} \| D_r(x_t, y_t) \|^p \right\} < \infty, \tag{3.5}
\]

\[
\sup_{|x,y| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{t \in [0,T]} \| D_r J_t^{-1} \|^p \right\} < \infty, \tag{3.6}
\]

\[
\sup_{|x,y| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{t \in [0,T]} \| D_r J_t \|^p \right\} < \infty, \tag{3.7}
\]

\[
\sup_{|x,y| \leq R} \mathbb{E}_{x,y} \left\{ \sup_{r_1, r_2 \leq t \leq T} \| D_{r_1, r_2} X(t) \|^p \right\} < \infty. \tag{3.8}
\]

Proof. \((3.5), (3.8)\) are given in Theorem 2.2.1, Theorem 2.2.2, [13]. The other two estimations are similar. \(\square\)

Lemma 3.3. Let the Hypothesis \(3.1\) hold, then for any \(p, T > 0\), there exists a constant \(C(T, p, R)\) such that

\[
\sup_{|x,y| \leq R} \mathbb{E}_{x,y} \left| \det(M_T^{-1}) \right|^p \leq C(p, R, T) < \infty.
\]

Proof. From \((2.25)\) and \((2.26)\), it only need to prove for any \(p > 0\), there exists a constant \(C(p, R, T)\) such that

\[
\sup_{|x,y| \leq R} \mathbb{P}_{x,y} \{ v^* M_T v \leq \epsilon \} \leq C(p, R, T) \epsilon^p, \quad \forall \epsilon > 0. \tag{3.9}
\]

All the constants appeared in the proof of Theorem 2.1, we can choose them depending on \(R\) but independent of the \((x, y) \in \overline{B}(0, R)\) if the Hypothesis \(3.1\) holds. So, \((3.9)\) holds. In the following paragraphs, we will list the changes in the proof of Theorem 2.1 when we prove this Lemma.

(1) \(R_1\) in \((2.2), (2.3), c\) in \((2.2)\) and Lemma 2.12 Define

\[
\Lambda(x, y) := \inf_{|v| = 1} \left( \sum_{j=1}^J \sum_{V \in A_j} (v^* V(x, y) V^*(x, y) v) \right).
\]

For any \((x, y) \in \overline{B}(0, R)\), \(\Lambda(x, y) > 0\). And also for \(\Lambda(x, y)\) is continuous w.r.t \((x, y)\) (the reason is the same as that in \((2.5)\)), there exists a constant \(R_1\) such that

\[
\inf_{|x,y| \leq R + R_1} \Lambda(x, y) > c := \frac{1}{2} \inf_{|x,y| \leq R} \Lambda(x, y) > 0.
\]

If we choose \(c\) and \(R_1\) as above, we can prove that the following inequality holds,

\[
\sum_{j=1}^J \sum_{V \in A_j} (v^* V(x', y'))^2 \geq c, \quad \forall (x', y') \in \overline{B}(x, y), R_1), \quad (x, y) \in \overline{B}(0, R), \quad |v| = 1.
\]

(2) \(R_3\) in \((2.6), C_0\) in Lemma 2.9 and Lemma 2.8 set

\[
R_3 = \frac{1}{C_0} := \frac{1}{2} \inf_{(x, y) \in \overline{B}(0, R)} \inf_{|v| = 1} (v^* b(x, y) b^*(x, y) v) > 0.
\]
From the choosing of $R_{3}$ and the definition of $\tau'$, if the Hypothesis 3.1 holds, then for any $s \leq \tau'$ and process $(x_{s}, y_{s})$ with initial value $(x, y) \in \overline{B}(0, R)$, the following inequality holds,

$$|v|^2 \leq C_0 \sum_{j=1}^{d}|v^*b_j(x_{s}, y_{s})|^2, \forall v \in \mathbb{R}^n.$$ 

From the above fact and $\tau \leq \tau'$, we can also choose the following constants depending on $R$, but independent of the $(x, y) \in \overline{B}(0, R)$.

(3) The estimate of $\tau$ in Lemma 2.5. From Lemma 3.1 for these constants appeared in the proof of Lemma 2.4 and Lemma 2.5, we can choose them depending on $R$ but independent of the $(x, y) \in \overline{B}(0, R)$.

(3) The using of Lemma 2.6 and Lemma 2.7 in Lemma 2.10 and Lemma 2.11. From Lemma 2.6, we need to estimate the following probability for some constant $C(R)$,

$$\mathbb{P} \left\{ \int_0^\tau |\tilde{y}(s)|^2 ds \leq (1 + T^2C(R))e^{q^3/2} + \int_0^\tau |(v^*_1 A_s + v^*_2 C_s)\nabla_y a_1|^2 ds \geq e^{q^10} \right\},$$

here

$$d\tilde{y}(s) = -(v^*_1 A_s + v^*_2 C_s)\nabla_y a_1 ds - \sum_{j=1}^{d}(v^*_1 B_s + v^*_2 D_s)\nabla_y b_j \cdot dW_j(s).$$

We need to check the condition (2.13) when using Lemma 2.6. Assume $d(v^*_1 A_s + v^*_2 C_s)\nabla_y a_1(x_s, y_s) = K_{1}(s)ds + \sum_{j=1}^{d}K_{2j}(s)dW_j(s)$. From Lemma 3.1 and the fact when $s \leq \tau, |(x, y)| \leq R$ and $|(x_s, y_s)| \leq R + R_1$, for any $p > 0$, there exists a constant $C = C(T, p, R)$ such that,

$$\mathbb{E}_{x,y} \sup_{0 \leq t \leq \tau} \left( |K_{1}(s)|^p + |K_{2}(s)|^p + |(v^*_1 A_s + v^*_2 C_s)\nabla_y a_1(x_s, y_s)| + \sum_{j=1}^{d}|(v^*_1 B_s + v^*_2 D_s)\nabla_y b_j| \right)^p \leq C(p, R, T) < \infty.$$ 

(4) For the other constants appeared in the proof of Theorem 2.1, we can also choose them depending on $R$ but independent of the $(x, y) \in \overline{B}(0, R)$.

We are now in a position to give

**Proof. The proof of Theorem 3.1.** For any $\xi \in \mathbb{R}^{m+n}$, 

$$\langle \nabla P_t f(x, y), \xi \rangle = \mathbb{E}_{x,y} \nabla f(x_t, y_t) J_t \xi.$$ 

Assume $x_t = (x^1_t, \cdots, x^n_t)$ and $y_t = (y^1_t, \cdots, y^n_t)$, then from (2.29), (2.30) in [14],

$$\mathbb{E}_{x,y} \left\{ \nabla_i f(x_t, y_t) J_t \xi \right\}$$

22
\[
\sum_{k=1}^{m} \mathbb{E}\left\{ f(x_t, y_t) \delta(J_t \xi(M^{-1})^{i,k}x_t^k) \right\} + \sum_{k=m+1}^{m+n} \mathbb{E}\left\{ f(x_t, y_t) \delta(J_t \xi(M^{-1})^{i,k}y_t^{k-m}) \right\}.
\]

So, this Theorem comes from Lemma 3.1, Lemma 3.2, Lemma 3.3 and Proposition 1.5.8 in [14].

In the end of this section, we give a Proposition which is supplementary to this article.

**Proposition 3.1.** Let \( a_1, a_2, b \in C^2_b \) and the Hypothesis 2.1 hold, then the law of \((x_t, y_t)\) with initial value \((x, y)\) is absolutely continuous with respect to Lebesgue measure and its density function \( p(t, (u, v)) \) is continuous w.r.t \((u, v) \in \mathbb{R}^m \times \mathbb{R}^n\) for fixed \( t \). Furthermore, the following estimation holds

\[
\sup_{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n} |p(t, (u, v))| < \infty.
\]

**Proof.** It directly comes from the Theorem 5.9 in [17] and the Theorem 2.1 in this article.

4 Strong Feller Property

In this section, we prove that the semigroup \( P_t \) associated with Eq. (1.1) is strong Feller under some conditions. From Theorem 3.1, \( P_t \) is strong Feller under some conditions which need all the coefficients for Eq. (1.1) are in \( C^2_b \). But in the Hamiltonian systems, the diffusion and drift part are polynomial growth, so the Theorem 3.1 can’t apply directly. But if the SDE has global solution, we can also prove \( P_t \) is strong Feller without the bounded conditions.

The followings are our Hypothesis and Theorem in this section.

**Hypothesis 4.1.** There exists \( j_0 \in \mathbb{N} \) such that:

(i) \( a_1 \in C^{j_0+2}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^m) \), \( a_2 \in C^2(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n) \);

(ii) \( b \in C^2(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^d) \), \( \det(b(x, y) \cdot b^*(x, y)) \neq 0 \), \( \forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \);

(iii) \( \forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \), the vector space spanned by \( \cup_{k=1}^{j_0} A_k \) at point \((x, y)\) has dimension \( m \);

(iv) The solution to equation (1.1) globally exists for any initial value \((x, y) \in \mathbb{R}^m \times \mathbb{R}^n\).

**Remark 4.1.** If there exists a Liapunov function \( W \) such that \( LW \leq cW \) for some \( c > 0 \), then the (iv) in Hypothesis 4.1 holds by Theorem 5.9, [12]. Here

\[
L = \sum_{i=1}^{m} a_1^i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} a_2^i \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{n} (b \cdot b^*)_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}.
\]

**Theorem 4.1.** Let the Hypothesis 4.1 hold, then \( P_t \) is strong Feller.
For the convenience of writing, we will use $x$ instead of $(x, y)$ in the rest of this section. Let $X^l_t = (x_l, y_l)$ be the solution of (1.1) with initial value $x \in \mathbb{R}^m \times \mathbb{R}^n$. In the following part, we would like to use the localization to prove Theorem 1.1.

For any fixed $l \in \mathbb{N}$, set $a(x) = (a_1(x), a_2(x))^\top$, $g_l(x) = h_l(x)a(x)$, $q_l(x) = h_l(x)b(x)$, $h_l(x) \in \mathbb{R}$ is a smooth function with compact support and $h_l(x) = 1$ on $B^c(0, l)$. Let $X^l_s(x)$ be the solution to the following equation,

$$X^l_s(x) = x + \int_0^s g_l(X^l_r(x))dr + \int_0^s \left(0 \begin{bmatrix} 0 \\ q_l(X^l_r(x)) \end{bmatrix}\right) dW_r. \quad (4.1)$$

Define a sequence of stopping time

$$S_l(x) = \inf\{s > 0, \ X^l_s(x) \notin B^c(0, l)\}, \ l \geq 1.$$ 

If the Hypothesis 4.1 holds, then for any $x \in \mathbb{R}^{m+n}$, the following properties holds a.s.

$$S_l(x) < S_{l+1}(x), \quad (4.2)$$

$$X^l_s(x) = X^l_{s+1}(x), \forall s \in [0, S_l(x)), \quad (4.3)$$

$$X^l_s = X^l_0(x), \quad \forall s \in [0, S_l(x)), \quad (4.4)$$

$$\sup_l S_l(x) = \infty. \quad (4.5)$$

In order to prove Theorem 1.1, we also need the following Lemmas.

**Lemma 4.1.** Let the Hypothesis 4.1 hold, then for any $x_0 \in \mathbb{R}^{m+n}$, $l \geq 2$, $t > 0$

$$\lim_{x \to x_0} \sup_{t > S_l(x)} I_{\{t > S_l(x)\}} \leq I_{\{t \geq S_{l-1}(x_0)\}}, \text{ a.s.}$$

**Proof.** Let $\Gamma$ be a measurable set with $\mathbb{P}(\Gamma^c) = 0$ and such that $X^l_s(x, \omega)$ is continuous w.r.t. $s$ and $x$ for $\omega \in \Gamma$. For $\omega \in \Gamma$, the conclusion is apparent if

$$\lim_{x \to x_0} \sup_{t > S_l(x)} I_{\{t > S_l(x)\}}(\omega) = 0 \text{ or } I_{\{t \geq S_{l-1}(x_0)\}}(\omega) = 1.$$

Assume that $\lim_{x \to x_0} \sup_{t > S_l(x)} I_{\{t > S_l(x)\}}(\omega) = 1$ and $I_{\{t \geq S_{l-1}(x_0)\}}(\omega) = 0$, then

$$\sup_{s \in [0, l]} |X^{l-1}_s(x_0, \omega)| \leq l - 1. \quad (4.6)$$

Furthermore, by (4.3),

$$\sup_{s \in [0, l]} |X^l_s(x_0, \omega)| \leq l - 1.$$ 

Since $\lim_{x \to x_0} \sup_{t > S_l(x)}(\omega) = 1$, then there exist $\{x_n\} \subset \mathbb{R}^{m+n}$ with $x_n \to x_0$ as $n \to \infty$, such that for $n$ large enough

$$\sup_{s \in [0, l]} |X^l_s(x_n, \omega)| \geq l. \quad (4.7)$$

24
From $t < S_{l-1}(x_0)$, \((4.3)\) and $\sup_{s \in [0,t]} |X_s^{l-1}(x_0, \omega)| \leq l - 1$,
\[
\sup_{s \in [0,t]} |X_s^l(x_0, \omega)| \leq l - 1. \tag{4.8}
\]

For $X_s^l(x, \omega)$ is continuous w.r.t. $s$ and $x$ and $[0, t] \times \overline{B}(0, 1) \subseteq [0, \infty) \times \mathbb{R}^{m+n}$ is a compact set, so for $\epsilon_0 = \frac{1}{2}$ there exists $\delta_0 > 0$ such that for any $|x - x_0| \leq \delta_0$ and $s \in [0, t]$
\[
|X_s^l(x, \omega) - X_s^l(x, \omega)| \leq \frac{1}{2}.
\]
That means when $|x - x_0| \leq \delta_0$,
\[
\sup_{s \in [0,t]} |X_s^l(x, \omega)| \leq \sup_{s \in [0,t]} |X_s^l(x_0, \omega)| + \frac{1}{2},
\]
this contradict \((4.7)\) and \((4.8)\). \qed

Let \(\{P_t^l\}_{t \geq 0}\) be the transition semigroup of \((4.3)\).

**Lemma 4.2.** Let the Hypothesis \((4.1)\) hold, then for any $f \in \mathcal{B}_b(\mathbb{R}^{m+n}; \mathbb{R})$, $P_t^lf$ is continuous on $B^\circ(0, l)$.

**Proof.** For any $0 < l_0 < l$, since $b_t = b$ on $\overline{B}(0, l)$, then
\[
\inf_{y \in B(0, l_0)} \inf_{|a| = 1} \left( |aA|^2 + |a \nabla b_t(y)A|^2 \right) > 0.
\]
In the Hypothesis \((3.1)\) and the proof of Theorem \((3.1)\) let $R = l_0$ and substitute $b_t$ for $b$, then
\[
|\nabla P_t^lf(y)| \leq C(l_0, t)\|f\|_\infty, \quad \forall f \in C_b, \forall y \in \overline{B}(0, l_0).
\]
\qed

We are now in a position to give

**Proof. The proof of Theorem 4.1.** For $f \in \mathcal{B}_b(\mathbb{R}^{m+n})$ with $f \geq 0$ and $x_0 \in \mathbb{R}^{m+n}$ with $|x_0| < l$
\[
\lim \sup_{x \to x_0} \mathbb{E}f(X_t^x) = \lim \sup_{x \to x_0} \mathbb{E}\{f(X_t^x)I\{t \leq S_l(x)\}\} + \lim \sup_{x \to x_0} \mathbb{E}\{f(X_t^x)I\{t > S_l(x)\}\}
\]
\[
\leq \lim \sup_{x \to x_0} \mathbb{E}\{f(X_t^x(x))I\{t \leq S_l(x)\}\} + \|f\|_\infty \lim \sup_{x \to x_0} \mathbb{P}(t > S_l(x))
\]
\[
\leq \lim \sup_{x \to x_0} \mathbb{E}f(X_t^l(x)) + \|f\|_\infty \lim \sup_{x \to x_0} \mathbb{P}(t > S_l(x))
\]
\[
= \mathbb{E}f(X_t^l(x_0)) + \|f\|_\infty \lim \sup_{x \to x_0} \mathbb{P}(t > S_l(x)), \tag{4.9}
\]
where we use (4.4) in the second inequality and Lemma 4.2 in the last equality. It follows (4.9) and (4.13) that

\[
\limsup_{x \to x_0} \mathbb{E} f(X_t^x) \\
\leq \mathbb{E} \left\{ f(X_t^x(x_0)) \right\} + \mathbb{E} \{ f(X_t^x(x_0)) I_{\{t > S_t(x_0)\}} \} + \left\| f \right\|_{\infty} \limsup_{x \to x_0} \mathbb{P}(t > S_t(x)) \\
\leq \mathbb{E} f(X_t^{x_0}) + \left\| f \right\|_{\infty} \mathbb{P}(t > S_t(x_0)) + \left\| f \right\|_{\infty} \limsup_{x \to x_0} \mathbb{P}(t > S_t(x)).
\]

Let \( l \to \infty \) in the above inequality and by Lemma 4.1 we obtain

\[
\limsup_{x \to x_0} \mathbb{E} f(X_t^x) \leq \mathbb{E} f(X_t^{x_0}) + \left\| f \right\|_{\infty} \lim_{l \to \infty} \limsup_{x \to x_0} \mathbb{P}(t > S_t(x)) \\
\leq \mathbb{E} f(X_t^{x_0}) + \left\| f \right\|_{\infty} \lim_{l \to \infty} \mathbb{E} \limsup_{x \to x_0} I_{\{t > S_t(x)\}} \\
\leq \mathbb{E} f(X_t^{x_0}) + \left\| f \right\|_{\infty} \lim_{l \to \infty} \mathbb{E} I_{\{t > S_t(x_0)\}} \\
\leq \mathbb{E} f(X_t^{x_0}).
\]

For \( g \in \mathcal{B}_b(\mathbb{R}^{m+n}) \), repeating the above procedure with \( \| g \|_{\infty} - g \) and \( \| g \|_{\infty} + g \), one arrives at

\[
\limsup_{x \to x_0} \mathbb{E} \left\{ \| g \|_{\infty} - g(X_t^x) \right\} \leq \| g \|_{\infty} - \mathbb{E} g(X_t^{x_0}), \quad (4.10) \\
\limsup_{x \to x_0} \mathbb{E} \left\{ \| g \|_{\infty} + g(X_t^x) \right\} \leq \| g \|_{\infty} + \mathbb{E} g(X_t^{x_0}). \quad (4.11)
\]

Therefore, for any \( g \in \mathcal{B}_b(\mathbb{R}^{m+n}) \)

\[
\lim_{x \to x_0} \mathbb{E} g(X_t^x) = \mathbb{E} g(X_t^{x_0}).
\]

\[
\square
\]

**Remark 4.2.** In [4], the authors considered the following SDE

\[
\begin{align*}
X_t^1 &= x_1 + \int_0^t F_1(s, X_s^1, \ldots, X_s^n)ds + \int_0^t \sigma(s, X_s^1, \ldots, X_s^n)dW_s, \\
X_t^2 &= x_2 + \int_0^t F_2(s, X_s^1, \ldots, X_s^n)ds, \\
X_t^3 &= x_3 + \int_0^t F_3(s, X_s^2, \ldots, X_s^n)ds, \\
&\vdots \\
X_t^n &= x_n + \int_0^t F_n(s, X_s^{n-1}, X_s^n)dt.
\end{align*}
\]

the authors proved that \( X_t \) has a density \( p(t, x, y) \) and gave the upper and lower bounds of \( p(t, x, y) \) if the spectrum of the matrix-valued function \( A = \sigma \cdot \sigma^* \) is included in \([\Lambda^{-1}, \Lambda] \) for some \( \Lambda \geq 1 \). In our article, we can’t obtain such strong results since in our condition is \( \text{det} (\sigma(x) \cdot \sigma^*(x)) \neq 0 \), which is weaker than that in [4].
5 Some Applications

The strong Feller property is very useful when we prove the uniqueness of invariant measure. If $X_t \in \mathbb{R}^n$, $t \in [0, +\infty)$, $n \in \mathbb{N}$ is a continuous Markov process. The following theorem is classical.

**Hypothesis 5.1.** Let $P_t$ be the semigroup associated with $X_t$, and

- the Markov process $X_t$ is irreducible, i.e,
  $$P_t(x, A) > 0, \text{ for all } t > 0, x \in \mathbb{R}^n, \text{ open set } A.$$

- $P_t$ is strong Feller.

**Theorem 5.1.** (c.f. [19] [3] [6]) Let the Hypothesis 5.1 hold, then $P_t$ exists at most one invariant measure.

5.1 The Langevin Equation

This example is extended from the one in [13]. Let $W_t, t \geq 0$ be a standard d-dimensional Brownian Motion and $F : \mathbb{R}^d \to R$, $\sigma \in \mathbb{R}^{d \times d}$ invertible. Consider the Langevin SDE for $q, p \in \mathbb{R}^d$ the position and momenta of particle of unit mass, namely

$$\begin{aligned}
\left\{ 
\begin{array}{l}
dq = pdt, \\
dp = \gamma pdt - \nabla F(q) dt + \sigma dW_t.
\end{array}
\right.
\end{aligned}
$$

(5.1)

**Hypothesis 5.2.** The function $F \in C^3(\mathbb{R}^d, \mathbb{R})$ and satisfy

- $F(q) \geq 0$ for all $q \in \mathbb{R}^d$;

- There exists an $\alpha > 0$ and $\beta \in (0, 1)$ such that
  $$\frac{1}{2} \langle \nabla F(q), q \rangle \geq \beta F(q) + \gamma^2 \frac{\beta(2 - \beta)}{8(1 - \beta)} ||q||^2 - \alpha.$$

**Proposition 5.1.** Let the Hypothesis 5.2 hold, then the semigroup $P_t$ associated with the Langevin SDE is strong Feller and has a unique invariant measure.

**Proof.** First, the Hypothesis 4.1 holds for $j_0 = 1$, so $P_t$ is strong Feller by Theorem 4.1. Second, by the Lemma 3.4 in [13], we know that $P_t$ is irreducible. So $P_t$ has at most one invariant measure. Third, by the Corollary A.5 in [13], the invariant measure for $P_t$ exists. \[\square\]
5.2 Stochastic Hamiltonian Systems

This example is extended from the one in [20]. Consider a stochastic differential system of the type

\[
\begin{align*}
    X_t &= X_0 + \int_0^t \partial_y H(X_s, Y_s) ds, \\
    Y_t &= Y_0 - \int_0^t \left[ \partial_x H(X_s, Y_s) + F(X_s, Y_s) \partial_y H(X_s, Y_s) \right] ds + W_t,
\end{align*}
\]

(5.2)

where \(X_t, Y_t, W_t\) belong to \(\mathbb{R}^d\).

In the following Hypothesis, we don’t need \(F\) and \(H \in C^\infty\) as in [20].

**Hypothesis 5.3.** There exist strictly positive numbers \(\nu, M, \delta\), there exists a function \(R(x, y)\) on \(\mathbb{R}^{2d}\) with second derivatives having polynomial growth at infinity, such that

- \(F \in C^2, H \in C^4\);
- \(0 < \nu|\xi|^2 \leq \sum_{i,j=1}^d \partial_{y_i y_j} H(x, y) \xi_i \xi_j, \forall x, y, \xi;\)
- \(H(x, y) + R(x, y) + M \geq \delta(|x|\nu + |y|\nu);\)
- \(LH(x, y) + LR(x, y) \leq -\delta(H(x, y) + R(x, y)) + M;\)
- \(|\partial_y H(x, y) + \partial_y R(x, y)|^2 \leq M(H(x, y) + R(x, y) + 1).\)

**Proposition 5.2.** Let the Hypothesis 5.3 holds, then the semigroup \(P_t\) associated with the equation (5.2) is strong Feller and has a unique invariant measure.

**Proof.** First, for \(0 < \nu|\xi|^2 \leq \sum_{i,j=1}^d \partial_{y_i y_j} H(x, y) \xi_i \xi_j, \forall x, y, \xi,\) we have the Hypothesis 4.1 is satisfied for \(j_0 = 1\). Thus \(P_t\) is strong feller by Theorem 4.1. Second, by the Lemma 2.2 in [20], We know the \(P_t\) is irreducible. So the invariant for \(P_t\) is at most one. Third, by the Lemma 2.1 and Corollary 2.1 in [20], the invariant measure for \(P_t\) exists. \(\square\)

5.3 High Order Stochastic Differential Equations

Consider the following Stochastic Differential Equations with order \(n\),

\[
x_t^{(n)} = f(x_t^{(n-1)}, \ldots, x_t) + b(x_t^{(n-1)}, \ldots, x_t) B_t,
\]

(5.3)

where \(x_t^{(k)} = \frac{d^k x_t}{dt^k}, k = 1, \ldots, n, x_t \in \mathbb{R}^m, b \in \mathbb{R}^{m \times d}, B_t \in \mathbb{R}^d.\)

Set \(y_t(i) = x_t^{(i-1)}, 1 \leq i \leq n,\) then \(y_t = (y_1(t), \ldots, y_n(t))\) satisfy the following stochastic differential equation:

\[
\begin{align*}
    dy_1(t) &= y_2(t) dt, \\
    & \vdots \\
    dy_{n-1}(t) &= y_n(t) dt, \\
    dy_n(t) &= f(y_n, y_{n-1}, \ldots, y_1) dt + b(y_n, y_{n-1} \ldots, y_1) dB_t.
\end{align*}
\]

(5.4)
Proposition 5.3. Let $x_t^j$ be the solution of equation (5.3) with initial value $x = (x_0, \cdots, x_0^{(n-1)}) \in \mathbb{R}^{m \times n}$, $P_t$ be the semigroup associated with (5.3),

1. If $f \in C^2_b(\mathbb{R}^{m \times n}; \mathbb{R}), b \in C^2_b(\mathbb{R}^{m \times n}; \mathbb{R})$ and $\det(b(x)b^*(x)) \neq 0$, then the law of $x_t^j$ is absolutely continuous with respect to Lebesgue measure, and its density $p(t, x, y)$ is continuous with respect to $y$ and $\sup_y |p(t, x, y)| < \infty$.

2. If $f \in C^2(\mathbb{R}^{m \times n}; \mathbb{R}), b \in C^2(\mathbb{R}^{m \times n}; \mathbb{R})$ and for any $x \in \mathbb{R}^{m \times n}$, $\det(b(x)b^*(x)) \neq 0$ and the solution to equation (5.3) with initial value $x$ is globally exists, then the semigroup $P_t$ is strong Feller.

Proof. The Hypothesis holds for $j = 1$, so (1) follows from Proposition 3.1 And (2) follows from Theorem 4.1.

Specially, if we consider the following stochastic differential equation

$$x_t^{(n)} + a_{n-1}(x_t)x_t^{(n-1)} + \cdots + a_0(x_t)x_t + c(x_t) + \frac{b(x_t)dB_t}{dt} = 0, \quad (5.5)$$

where $x_t^{(k)} = \frac{d^k x_t}{dt^k}$, $x_t \in \mathbb{R}^m$, $B_t \in \mathbb{R}^d$, $b(x_t) \in \mathbb{R}^{m \times d}$, $c \in \mathbb{R}^m$, $a_0, \cdots, a_{n-1} \in \mathbb{R}^{m \times m}$.

Corollary 5.1. Let $x_t^j$ be the solution of equation (5.3) with initial value $x = (x_0, \cdots, x_0^{(n-1)}) \in \mathbb{R}^{m \times n}$.

1. If $a_0, \cdots, a_{n-1} \in C^2_b(\mathbb{R}^m; \mathbb{R}^{m \times m}), b \in C^2_b(\mathbb{R}^m; \mathbb{R}^{m \times d}), c \in C^2_b(\mathbb{R}^m; \mathbb{R}^m)$, and $\det(b(x_0)b^*(x_0)) \neq 0$, then the law of $x_t^j$ is absolutely continuous with respect to Lebesgue measure, and its density $p(t, x, y)$ is continuous with respect to $y$ and $\sup_y |p(t, x, y)| < \infty$.

2. If $a_0, \cdots, a_{n-1} \in C^2(\mathbb{R}^m; \mathbb{R}^{m \times m}), b \in C^2(\mathbb{R}^m; \mathbb{R}^{m \times d}), c \in C^2(\mathbb{R}^m; \mathbb{R}^m)$, and for any $x = (x_0, \cdots, x_0^{(n-1)}) \in \mathbb{R}^{m \times n}$, $\det(b(x_0)b^*(x_0)) \neq 0$ and $x_t^j$ is globally exists, then the semigroup $P_t$ is strong Feller.

Proof. It can be obtained by Proposition 5.3.

Appendix A  Proof of Lemma 2.6 and Lemma 2.7

The proof of Lemma 2.6 and Lemma 2.7 are very similar to the proof of Norris Lemma (c.f. Lemma 2.3.1, [14]), so we only give the proof of Lemma 2.7 here.

Proof. Proof of Lemma 2.7: Define stopping time as

$$\zeta = \inf \left\{ t \geq 0 : \sup_{0 \leq s \leq t} (|a(s)| + |u(s)|) > \epsilon^{-r} \right\} \land \sigma,$$

then

$$B = \left\{ \int_0^\sigma \tilde{y}(t)^2 dt < \epsilon^q, \int_0^\sigma |u(t)|^2 dt \geq \epsilon \right\} \subseteq A_1 \cup A_2 \cup A_3,$$
where

\[ A_1 = \left\{ \int_0^\sigma \tilde{y}(t)^2 dt < \epsilon^q, \int_0^\sigma |u(t)|^2 dt \geq \epsilon, \zeta = \sigma, \sigma \geq \epsilon \right\} , \]
\[ A_2 = \{ \zeta < \sigma \} , \]
\[ A_3 = \{ \sigma < \epsilon \} . \]

Obviously, 

\[ \mathbb{P}\{A_2\} \leq \tilde{c}\epsilon^{rp}, \quad \mathbb{P}\{A_3\} \leq C(c_\sigma, \tilde{p})\epsilon^{\tilde{p}}, \]

so we only need to estimate \( \mathbb{P}(A_1) \).

Introduce the following notation

\[ N_t = \sum_{i=1}^d \int_0^t \tilde{y}(s)u_i(s)dW_i(s), \]
\[ M_t = \sum_{i=1}^d \int_0^t u_i(s)dW_i(s), \]
\[ B = \left\{ \langle N \rangle_{\sigma} < \rho_1, \sup_{0 \leq s \leq \sigma} |N_s| \geq \delta_1 \right\} , \]

where \( \rho_1 = \epsilon^{q-2r}, \delta_1 = \epsilon^{\frac{q}{2}+r-\frac{\delta}{q}}. \)

We will prove that there exists \( \epsilon_0 = \epsilon_0(c_\sigma, q, r, v) \), such that

\[ A_1 \subseteq B, \text{ for all } \epsilon \leq \epsilon_0. \]

If this has been proved, then

\[ \mathbb{P}\{A_1\} \leq \mathbb{P}\{B\} \leq 2 \exp\left\{-\frac{\delta_1^2}{2\rho_1}\right\} \leq \exp\{-\epsilon^{\frac{q}{2}}\}, \]

thus this Lemma holds.

In the below, we will to prove: there exists \( \epsilon_0 = \epsilon_0(c_\sigma, q, r, v) \), such that

\[ A_1 \subseteq B, \text{ for all } \epsilon \leq \epsilon_0. \]

Set \( \epsilon_0 = \epsilon_0(c_\sigma, q, r, v) \), such that for \( \epsilon \leq \epsilon_0(c_\sigma, q, r, v) \), the following inequalities hold.

\[ \epsilon^q + 2c_\sigma(\sqrt{c_\sigma\epsilon^{\frac{q}{2}+r}} + \delta_1) \leq \epsilon^{\frac{q}{2}+r-\frac{\delta}{q}}(1+2c_\sigma), \]
\[ \epsilon^{\frac{q}{2}+r-\frac{\delta}{q}}(1+2c_\sigma) + \epsilon^{\frac{q}{2}} - \frac{\delta}{q} < \epsilon. \]

We only need to prove for any \( \epsilon \leq \epsilon_0(c_\sigma, q, r, v) \), \( \omega \in B^c \) implies \( \omega \in A_1^c \).

Let \( \omega \in B^c \), \( \int_0^\sigma \tilde{y}(t)^2 dt < \epsilon^q \), \( \sigma(\omega) = \zeta(\omega) \geq \epsilon \), then similar to the estimations of \( \sup_{0 \leq s \leq T} \left| \int_0^s Y_s dY_s \right|, \int_0^T (M)_t dt \) and \( (M)_T \) in the proof of Lemma 2.3.2 in [14], we can obtain

\[ \sup_{t \leq \sigma} \left| \int_0^t \tilde{y}_s d\tilde{y}_s \right| \leq \sqrt{c_\sigma} \epsilon^{\frac{q}{2}-r} + \delta_1. \]
\[
\int_0^\sigma \langle M \rangle_t dt \leq \epsilon^{\frac{q}{2} - r - \frac{v}{4}} (1 + 2c_\sigma),
\]
\[
\langle M \rangle_\sigma \leq \gamma^{-1} \epsilon^{\frac{q}{2} - r - \frac{v}{4}} (1 + 2c_\sigma) + \gamma \epsilon^{-2r}, \quad \forall \gamma \in (0, \sigma).
\] (A.1)

Choosing \( \gamma = \epsilon^{\frac{1}{2} \left( \frac{q}{2} - r - \frac{v}{4} \right)} < \epsilon \leq \sigma \) in (A.1). Since \( 2q > 8 + 20r + v \), we have \( \langle M \rangle_\sigma < \epsilon \), i.e. \( \omega \in A^\epsilon_1 \).

\[\square\]

Acknowledgements

The authors thanks for Professor Fuzhou Gong, Zhiming Ma, Pei Xu, Tusheng Zhang and Xicheng Zhang and Dr. Pei Yan Li for their valuable discussions. Especially, we also thanks for Dr. Jian Zhou for his much valuable discussions on his doctor period.

References

[1] Cattiaux, P., Leon, J. R. and Prieur, C. (2012). Estimation for Stochastic Damping Hamiltonian Systems under Partial Observation. I. Invariant density[J].

[2] Cinti, C., Menozzi, S & Polidoro, S. (2012). Two-sided bounds for degenerate processes with densities supported in subsets of \( \mathbb{R}^N \). arXiv preprint arXiv:1203.4918.

[3] Daprato, G. and Zabczyk, J. (1996). Ergodicity for infinite dimensional systems, Cambridge University Press.

[4] Delarue, F. and Menozzi, S. (2010). Density estimates for a random noise propagating through a chain of differential equations. Journal of Functional Analysis. 259(6) 1577-1630.

[5] Flandoli, F. (1994). Dissipativity and invariant measures for stochastic Navier-Stokes equations. Non-linear Differential Equations Appl. 1 403-423.

[6] Flandoli, F. and Maslowski, B. (1995). Ergodicity of the 2-D Navier-Stokes Equation Under Random Perturbations. Commun.Math.Phys. 172 119-141.

[7] Hairer, M. and Mattingly, J.C. (2006). Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. Annals of Mathematics. 164 993-1032.

[8] Hairer, M. and Mattingly, J. C. (2011). A theory of Hypoellipticity and Unique Ergodicity for Semilinear Stochastic PDEs. Electronic Journal of Probability. 16 658-738.

[9] Ichihara, Kanji. and Kunita, Hiroshi. (1974). A classification of the second order degenerate elliptic operators and its probabilistic characterization Probability Theory and Related Fields. 30(3) 235-254.
[10] Kliemann, K. (1987). Recurrence and invariant measure for degenerate diffusions. *The annals of probability* 15(2) 690-707.

[11] Kusuoka, S. and Stroock, D. (1987). Applications of the Malliavin calculus, III, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 34 391C442.

[12] L, Rey-Bellet. Ergodic properties of Markov processes, Open Quantum Systems II 1881 (2006): 1-39.

[13] Mattingly, J. C., Stuartb, A. M. and Higham, D. J. (2002). Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise. *Stochastic Processes and Their Applications.* 101 185-232.

[14] Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, Springer.

[15] Protter, P.E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed. Springer.

[16] Revuz, D. and Yor, M. (2005). *Continuous Martingales and Brownian Motion*, 3nd ed. Springer.

[17] Shigekawa, I. (2004). *Stochastic Analysis, Translations of Mathematical Monographs*. Vol, 224.

[18] Sheu, S.J. (1991). Some estimates of the transition density of a nondegenerate diffusion Markov process[J]. *The Annals of Probability.* 19(2) 538-561.

[19] Stettner, L. (1994). Remarks on Ergodic Conditions for Markov Processes on Polish Spaces. *Bulletin Polish Acad. Sci. Math.* 42 103-114.

[20] Talay, D. (2002). Stochastic Hamiltonian Systems: Exponential Convergence to the Invariant Measure, and Discretization by the Implicit Euler Scheme. *Markov Processes and Related Fields.* 8 163-198.