ON THE HESSIAN-CSCK EQUATIONS

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Abstract. In this paper, we propose a coupled system of complex Hessian equations which generalizes the equation for constant scalar curvature Kähler (cscK) metrics. We show this system can be realized variationally as the Euler-Lagrange equation of a Hessian version of the Mabuchi K-energy in an infinite dimensional space of $k$-Hessian potentials, which can be seen as an infinite dimensional Riemannian manifold with negative sectional curvature. Finally, we prove an a priori $C^0$-estimate for this system which depends on the Entropy, which generalizes a fundamental result of Chen and Cheng [1] for cscK metrics.

1. Introduction

There has been increasing interest in recent years in two directions in the theory of geometric partial differential equations: on one hand, on systems consisting of a non-linear equation coupled with its linearization, of which the constant scalar curvature Kähler metric (cscK) is an example in complex geometry [1], and the affine Plateau problem an example from real geometry [16]; on the other hand, consideration of other elliptic equations besides the most familiar examples of the Laplacian and the Monge-Ampère equations (e.g. Harvey and Lawson on the Lagrangian equation [9], Collins-Yau on the deformed HYM equation [3], Phong-Picard-Zhang on the Fu-Yau equation [11]). In this paper, we consider a coupled system which stands at the crossroads of the above two broad lines of development, namely the coupled system of a complex Hessian equation with its linearization. We show that this system admits a natural interpretation in terms of a generalized notion of curvature, and that this notion of curvature admits, just as the standard notion, an interpretation in terms of Deligne pairings [13]. We establish the $C^0$ estimate for this Hessian coupled system, and give an interpretation of this coupled system as a variational problem for an energy functional in an infinite dimensional Riemannian manifold of negative sectional curvature, generalizing the constructions of Donaldson [4], Mabuchi [10], and Semmes [15].

We now describe the coupled system we are interested in. Let $(X, \omega)$ be a compact Kähler manifold. We consider the following coupled system of equations for a pair of smooth functions $(\varphi, F)$:

\[
\begin{align*}
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^k \wedge \omega^{n-k} &= e^F \omega^n, \quad \sup_X \varphi = 0 \\
\Delta_G F &= -\overline{\alpha} + \text{tr}_G \alpha,
\end{align*}
\]

where $G^{ij} = k \frac{\sqrt{-1}}{\omega^n} \frac{d_{\omega}^{i} \wedge d_{\bar{\omega}}^{\bar{j}} \wedge \omega_{\varphi}^{k-1} \wedge \omega^{n-k}}{\omega^k \wedge \omega^{n-k}}$ and $\Delta_G = G^{ij} \partial_i \partial_j$ is the linearized operator associated to the nonlinear operator $\varphi \mapsto \log \frac{\omega^k \wedge \omega_{\varphi}^{n-k}}{\omega^n}$. $\alpha$ is a smooth $(1,1)$-form and $\overline{\alpha} = \frac{1}{V} \int_X \alpha \wedge \omega_{\varphi}^{k-1} \wedge \omega^{n-k}$ is a constant making the second equation of (1.1) compatible. For this system to be...
elliptic (i.e. $G^{ij} > 0$), we require $\varphi$ to be admissible, which means it satisfies the condition
$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^j \wedge \omega^{n-j} > 0 \text{ for } j = 1, \ldots, k.$$Alternatively, $\varphi$ is admissible if and only if the eigenvalues of $\omega + \sqrt{-1} \partial \bar{\partial} \varphi$ with respect to the Kähler metric $\omega$ is in the $\Gamma_k$-cone, where $\Gamma_k \subset \mathbb{R}^n$ is given by
$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}$$and $\sigma_k$ is the elementary symmetric polynomial of degree $k$ on $\mathbb{R}^n$. It is important to note that in general, the condition of being admissible for $k < n$ will depend on the background Kähler metric $\omega$, in particular, it may not be invariant under biholomorphic maps.

We state some basic well-known properties of the cone $\Gamma_k$ which will be used later. For a more thorough description of the properties of $\Gamma_k$, we refer the readers to [17].

**Lemma 1.** For $\lambda \in \Gamma_k$, we have
\begin{itemize}
  \item[(1)] $\sigma_{k-1,i}(\lambda) > 0$ for any $1 \leq i \leq n$, where $\sigma_{k-1,i}(\lambda) = \sigma_{k-1}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$.
  \item[(2)] (Garding’s inequality) For any $\mu, \lambda \in \Gamma_k$, we have the inequality
$$\sum_{i=1}^n \mu_i \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} \geq C(n, k) \sigma_k(\mu)^{\frac{1}{k}} \sigma_k(\lambda)^{\frac{k-1}{k}}.$$for some explicit constant $C(n, k)$.
\end{itemize}

2. Energy functionals

A key aspect about the cscK equation is that the equation for a cscK metric can be realized as the Euler-Lagrange equation of a Mabuchi $K$-energy in the space of Kähler potentials. This is true for the system (1.1) as well. In this section, we will introduce the analogue of the Mabuchi $K$-energy and show that it is critical points correspond precisely the solutions of (1.1).

**Definition 1.** We define the (generalized) Ricci curvature of $\omega_\varphi$ (relative to $\omega$) as
$$\widehat{\text{Ric}}(\omega_\varphi) = -\sqrt{-1} \partial \bar{\partial} \log \omega_\varphi^k \wedge \omega^{n-k} = -\sqrt{-1} \partial \bar{\partial} f + \text{Ric}(\omega)$$where $f = \log \frac{\omega_\varphi^k \wedge \omega^{n-k}}{\omega^n}$ and $\text{Ric}(\omega)$ is the usual Ricci curvature of $\omega$.

The generalized Ricci curvature of $\omega_\varphi$ lies in the first Chern class $c_1(X)$. In fact, the equation (1.1) can be viewed as an equation for the generalized Ricci curvature $\widehat{\text{Ric}}(\omega_\varphi)$. Indeed (1.1) is equivalent to the equation
$$\text{tr}_G(\widehat{\text{Ric}}(\omega_\varphi) - \text{Ric}(\omega) + \alpha) = \bar{\alpha},$$(2.1)

Viewing $\text{tr}_G(\widehat{\text{Ric}}(\omega_\varphi))$ as the generalized Scalar curvature, we can define the analogue of the Mabuchi energy, whose critical points are precisely potentials of constant generalized Scalar curvature. We will write $SH_k(X, \omega)$ the set of functions $\varphi$ such that $\omega^{-1} \cdot \omega_\varphi \in \Gamma_k$ on $X$. 

Definition 2. Given a constant \( \lambda \in \mathbb{R} \), we define the Hessian Mabuchi energy \( \mu_k \) by its variation: for a family of \( \phi_t \in SH_k(X, \omega) \), we have

\[
\frac{d}{dt} \mu_k(\phi_t) = -k \int_X \phi_t (\hat{Ric}(\omega_{\phi_t}) - \lambda \omega_{\phi_t}) \wedge \omega^{k-1}_{\phi_t} \wedge \omega^{n-k}_{\phi_t}. \tag{2.2}
\]

Since we can always add a constant to \( \mu_k \) without changing the variation, we will often also choose a normalization so that \( \mu_k(0) = 0 \).

It’s clear from this definition that the critical points of \( \mu_k \) is precisely the solution of equation 2.1 with \( \alpha = Ric(\omega) \), which is exactly those \( \omega_{\phi} \) with constant generalized Scalar curvature. However, it is not clear from this definition that \( \mu_k \) is well-defined. The following Theorem shows that \( \mu_k(\cdot) \) is well-defined and gives and explicit formula for \( \mu_k \).

**Theorem 1.** \( \mu_k(\varphi) \) can be expressed as follows:

\[
\mu_k(\varphi) = \frac{1}{V} \int_X \left( \log \frac{\omega^k_{\varphi} \wedge \omega^{n-k}_{\varphi}}{\omega^n_{\varphi}} + \lambda \varphi \right) \omega^k_{\varphi} \wedge \omega^{n-k}_{\varphi} - \frac{\lambda}{V(k+1)} \left( \int_X \varphi \sum_{j=0}^k \omega_{\varphi}^{k-j} \wedge \omega^{n-k+j}_{\varphi} \right)
- \frac{1}{V} \int_X \varphi \sum_{j=1}^k (Ric(\omega) - \lambda \omega) \wedge \omega_{\varphi}^{k-j} \wedge \omega^{n-k+j-1} \tag{2.3}
\]

We remark that the first integral in (2.3) corresponds to the “entropy term”, the second one to the usual \( J \)-functional, and the last one to the \( J_{Ric(\omega)-\lambda \omega} \)-functional in Kahler geometry.

**Proof.** It suffices for us to show the variation of \( \mu_k \) is given by the formula 2.2. For simplicity we omit the subscript \( t \) in \( \phi_t \) and write \( \phi = \phi_t \), then we compute the variation of \( \mu_k \) as defined above

\[
\frac{d}{dt} \mu_k(\phi_t) = \frac{1}{V} \int_X (\Delta_G \phi + \lambda \phi) \omega^k_{\phi} \wedge \omega^{n-k}_{\phi} + \frac{1}{V} \int_X \Delta_G \phi \left( \log \frac{\omega^k_{\phi} \wedge \omega^{n-k}_{\phi}}{\omega^n_{\phi}} + \lambda \phi \right) \omega^k_{\phi} \wedge \omega^{n-k}_{\phi}
- \frac{1}{V} \left( \int_X \lambda \phi \omega^k_{\phi} \wedge \omega^{n-k}_{\phi} \right) - \frac{1}{V} \int_X \phi \sum_{j=1}^k (Ric(\omega) - \lambda \omega) \wedge \omega^{k-j}_{\phi} \wedge \omega^{n-k+j-1}
- \frac{1}{V} \int_X \phi (Ric(\omega) - \lambda \omega) \wedge (k \omega^{k-1}_{\phi} \wedge \omega^{n-k}_{\phi} - \sum_{j=1}^k \omega^{k-j}_{\phi} \wedge \omega^{n-k+j-1})
= \frac{k}{V} \int_X \phi (Ric(\omega) - \hat{Ric}(\omega_{\phi}) + \lambda \sqrt{-1} \partial \bar{\partial} \phi) \wedge \omega^{k-1}_{\phi} \wedge \omega^{n-k}_{\phi}
- \frac{k}{V} \int_X \phi (Ric(\omega) - \lambda \omega) \wedge \omega^{k-1}_{\phi} \wedge \omega^{n-k}_{\phi}
= -\frac{k}{V} \int_X \phi (\hat{Ric}(\omega_{\phi}) - \lambda \omega_{\phi}) \wedge \omega^{k-1}_{\phi} \wedge \omega^{n-k}_{\phi} \]

\[\square\]
Given the definition of $\frac{d}{dt} \mu_k$ in (2.2), we can also compute the second variation of $\mu_k$, which is given by

$$\frac{d^2}{dt^2} \mu_k(\varphi_t) = -\frac{k}{V} \int_X \phi \left( \widehat{\operatorname{Ric}}(\omega_\varphi) - \lambda \omega_\varphi \right) \wedge \omega_\varphi^{-k} \wedge \omega^{n-k} - \frac{k}{V} \int_X \phi \left( -\sqrt{-1} \partial \bar{\partial} \Delta_G \phi - \lambda \sqrt{-1} \partial \bar{\partial} \phi \right) \wedge$$

$$\wedge \omega_\varphi^{-k} \wedge \omega^{n-k} - \frac{k(k-1)}{V} \int_X \phi \left( \widehat{\operatorname{Ric}}(\omega_\varphi) - \lambda \omega_\varphi \right) \wedge \sqrt{-1} \partial \bar{\partial} \phi \wedge \omega_\varphi^{-2} \wedge \omega^{n-k}$$

$$= -\frac{k}{V} \int_X \phi \left( \widehat{\operatorname{Ric}}(\omega_\varphi) - \lambda \omega_\varphi \right) \wedge \omega_\varphi^{-k} \wedge \omega^{n-k} + \frac{1}{V} \int_X \Delta_G \phi \left( \Delta_G \phi + \lambda \phi \right) \omega_\varphi^k \wedge \omega^{n-k}$$

$$- \frac{k(k-1)}{V} \int_X \phi \left( \widehat{\operatorname{Ric}}(\omega_\varphi) - \lambda \omega_\varphi \right) \wedge \sqrt{-1} \partial \bar{\partial} \phi \wedge \omega_\varphi^{-2} \wedge \omega^{n-k}$$

$$= -\frac{k}{V} \int_X \left( \phi - |\partial \phi|^2_G \right) \left( \widehat{\operatorname{Ric}}(\omega_\varphi) - \lambda \omega_\varphi \right) \wedge \omega_\varphi^{-k} \wedge \omega^{n-k} + \frac{1}{V} \int_X |\Delta_G \phi|^2 \omega_\varphi^k \wedge \omega^{n-k}$$

$$+ \frac{k(k-1)}{V} \int_X \sqrt{-1} \partial \bar{\partial} \phi \wedge \bar{\operatorname{Ric}}(\omega_\varphi) \wedge \omega_\varphi^{-2} \wedge \omega^{n-k} - \frac{k}{V} \int_X |\partial \phi|^2_G \bar{\operatorname{Ric}}_G \wedge \omega_\varphi^{-1} \wedge \omega^{n-k}$$

(2.4)

**Remark 1.** More generally, we can also consider the $\alpha$-twisted Hessian Mabuchi energy (with $\lambda = \bar{\alpha}/k$)

$$\mu_{\alpha,k}(\varphi) = \mu_k(\varphi) + \frac{k}{V} \int_0^1 \int_X \phi \left( \operatorname{Ric}(\omega) - \alpha \right) \wedge \omega_\varphi^{-k} \wedge \omega^{n-k}$$

$$= \frac{1}{V} \int_X \left( \log \frac{\omega_\varphi^k \wedge \omega^{n-k}}{\omega^n} + \alpha \varphi \right) \omega_\varphi^k \wedge \omega^{n-k} - \frac{\lambda}{V(k+1)} \left( \int_X \varphi \sum_{j=0}^k \omega_\varphi^{-j} \wedge \omega^{n-k+j} \right)$$

$$- \frac{1}{V} \int_X \varphi \sum_{j=1}^k (\alpha - \lambda \varphi) \wedge \omega_\varphi^{-j} \wedge \omega^{n-k+j-1}.$$  

(2.5)

whose variation is given by

$$\frac{d}{dt} \mu_{k,\alpha}(\phi_t) = -\frac{k}{V} \int_X \phi_t \left( \widehat{\operatorname{Ric}}(\omega_{\phi_t}) - \lambda \omega_{\phi_t} - \operatorname{Ric}(\omega) + \alpha \right) \wedge \omega_{\phi_t}^{-k} \wedge \omega^{n-k}.$$  

(2.6)

It follows that an $\omega_\varphi$ satisfying (2.1) is a critical point of $\mu_{k,\alpha}$.

### 2.1. The Deligne pairing and the energy $\mu_k$.

Similar to the interpretation of the Mabuchi $K$-energy as the metric of some line bundle from the Deligne pairing, as shown in Phong and Sturm [13, 14] (see also [12]), we explain in this section that when $X$ is a projective manifold, the energy functional $\mu_k(\cdot)$ in (2.2) can be regarded as the metric on some $\mathbb{R}$-line bundle from the Deligne pairing. Suppose $L_0, \ldots, L_n$ are holomorphic Hermitian line bundles on $X$, then the Deligne pairing

$$\langle L_0, \ldots, L_n \rangle$$
is a Hermitian line bundle over a point (for the precise definition, we refer to [13]). The change of metric formula (c.f. (2.10) in [14]) states that

$$\langle L_0 \otimes O(\phi_0), \ldots, L_n \otimes O(\phi_n) \rangle = \langle L_0, \ldots, L_n \rangle \otimes O(E)$$

where $O(f)$ denotes the trivial line bundle equipped with the Hermitian metric $he^{-f}$, and $E$ is given by

$$E = \int_X \sum_{j=0}^n \phi_j \wedge_{k<j} c_1(L_k \otimes O(\phi_k)) \wedge \wedge_{j<k \leq n} c_1(L_k)$$

and $c_1(L \otimes O(\phi)) = c_1(L) + \sqrt{-1} \partial \bar{\partial} \log h \in c_1(L)$. Let $K_X$ be the canonical line bundle on $X$. The Kähler metric $\omega$ induces a metric $\frac{1}{V} \langle L, \ldots, L \rangle$ on $K_X$. We define a metrized $\mathbb{R}$-line bundle

$$\mathcal{M}_{k,h} = \langle K_X, L, \ldots, L \rangle^{1/V} \langle L, \ldots, L \rangle^{\lambda k/V(1+k)}$$

where $L$ is given the metric $h$ and $K_X$ is equipped with the metric $\frac{1}{\omega^n}$. With the help of the formula (2.6), we can easily verify that

$$\langle K_X \otimes O(\Theta), L, \ldots, L \rangle^{k \text{ terms}} \langle L, \ldots, L \rangle^{(k+1 \text{ terms})}$$

$$= \langle K_X, L, \ldots, L \rangle^{1/V} \langle L, \ldots, L \rangle^{\lambda k/V(1+k)} \otimes O(\mu_k(\varphi)),$$

where $\Theta = \log \frac{\omega^k \wedge \omega^{n-k}}{\omega^n} \in C^\infty(X, \mathbb{R})$, and $L_\varphi = L \otimes O(\varphi)$ is the line bundle $L$ equipped with the Hermitian metric $he^{-\varphi}$. From (2.7) and (2.3) we see that the functional $\mu_k(\varphi)$ can be interpreted as the change of a Hermitian metric on the Deligne pairing line bundle $\mathcal{M}_{k,h}$.

2.2. Space of $k$-Hessian potentials. Following [10], we can define a Riemannian structure on the space of $k$-potentials and investigate its geometry. For simplicity we will assume in this section that $\int_X \omega^n = 1$. Set

$$\mathcal{H}_k(X, \omega) = \{ u \in C^\infty(X) \mid \omega^j_u \wedge \omega^{n-j} > 0 \text{ for } j = 1, \ldots, k \}. \quad (2.8)$$

This is an open set of $C^\infty(X)$, hence we can identify the tangent space $T_u \mathcal{H}_k$ with the space of smooth functions on $X$. Let us define an inner product of two tangent vectors $\varphi, \psi \in T_u \mathcal{H}_k$ by

$$\langle \varphi, \psi \rangle = \int_X \varphi \psi \omega^k_u \wedge \omega^{n-k}. \quad (2.9)$$

With this inner product, we can formally view $\mathcal{H}_k$ as an infinite dimensional Riemannian manifold.
2.3. Geodesic equation. Now we want to define a connection $D$ that is compatible with (2.9). This means that if $\varphi, \psi$ are tangent vector fields along a curve $u_t \in \mathcal{H}_k$, we should require

$$\frac{d}{dt} \langle \varphi, \psi \rangle = \langle D_{\dot{u}} \varphi, \psi \rangle + \langle \varphi, D_{\dot{u}} \psi \rangle. \quad (2.10)$$

Integrating by parts shows that

$$\frac{d}{dt} \langle \varphi, \psi \rangle = \int_X \left\{ \left( \dot{\varphi} - \frac{1}{2} G_{p\bar{q}}(\dot{u}_p \varphi_p + \varphi_p \dot{u}_p) \right) \psi + \varphi \left( \dot{\psi} - \frac{1}{2} G_{p\bar{q}}(\dot{u}_p \psi_p + \psi_p \dot{u}_p) \right) \right\} \omega^k_u \wedge \omega^{n-k}. \quad (2.11)$$

Combining (2.10) and (2.11) motivates the following definition of a connection on $\mathcal{H}_k(X, \omega)$:

$$D_{\dot{u}} \varphi = \dot{\varphi} - \frac{1}{2} G_{p\bar{q}}(\dot{u}_p \varphi_p + \varphi_p \dot{u}_p). \quad (2.12)$$

Thus we see that the equation $D_{\dot{u}} \ddot{u} = 0$ for a geodesic in this setting is

$$\ddot{u} - G_{p\bar{q}} \dot{u}_p \dot{u}_p = 0. \quad (2.13)$$

Similar to [10, 15, 4], we can recast this as a degenerate Hessian equation on a product manifold of dimension $n + 1$. First we complexify the time variable $t$ by adding an imaginary part, and assume everything is independent of the imaginary part of $t$. Then multiplying equation (2.13) by the term $\sqrt{-1} dt \wedge d\bar{t} \wedge \omega^k_u \wedge \omega^{n-k}$, we get

$$(u_{\bar{t}} - G_{p\bar{q}} u_{\bar{q}} u_p) \sqrt{-1} dt \wedge d\bar{t} \wedge \omega^k_u \wedge \omega^{n-k} = 0,$$

which is equivalent to the following degenerate, homogenous Hessian equation on the product manifold $M \times \mathbb{C}_t$:

$$(\omega + \sqrt{-1} \partial \bar{\partial} u)^{k+1} \wedge \omega^{n-k} = 0. \quad (2.14)$$

Thus equation for geodesics segments joining two potentials $\varphi_0, \varphi_1$ can be formulated as the following boundary value problem for a homogenous degenerate Hessian equation.

$$\begin{cases} 
(\omega + \sqrt{-1} \partial \bar{\partial} u)^{k+1} \wedge \omega^{n-k} = 0 \\
u(\cdot, t) = \varphi_0 \text{ for } \text{Re } t = 0 \\
u(\cdot, t) = \varphi_1 \text{ for } \text{Re } t = 1
\end{cases} \quad (2.15)$$

There are some major analytic difficulties for solving the geodesic equation arising from the fact that it is degenerate in two different ways: not only is data on the right hand side of the equation is zero, but also the $(1,1)$-form $\omega$ is also degenerate in the $t$ direction. This causes a lot of analytic difficulties, and we plan to investigate the existence and regularity of geodesics for this system in a subsequent work.

2.4. Curvature of $\mathcal{H}_k$. In the rest of this section, we compute the curvature of $D$ and prove the following theorem.

**Theorem 2.** The sectional curvature of $D$ on $\mathcal{H}_k$ is non-positive.
Proof. First we write the connection on $H_k$ as

$$D_q \varphi = \dot{\varphi} - Q(\nabla \dot{u}, \nabla \varphi).$$

where

$$Q(\nabla \dot{u}, \nabla \psi) = \frac{1}{2} G^{pq}(u_q \varphi_p + \varphi_q \dot{u}_p) = \frac{k \sqrt{-1(\partial \varphi \wedge \bar{\partial} \varphi + \partial \varphi \wedge \bar{\partial} \bar{\partial} \varphi)} \wedge \omega^{-k} \wedge \omega^{n-k}}{\omega^{k} \wedge \omega^{n-k}}.$$

Suppose that $u(x, t, s)$ is a family of $k$-potentials, and let $R$ be the curvature of $D$, then we compute $R(u_t, u_s)\eta$,

$$R(u_t, u_s)\eta = D_t(\eta_t - Q(\nabla \eta, \nabla u_t)) - D_s(\eta_t - Q(\nabla \eta, \nabla u_t))$$

$$= -\frac{\partial}{\partial t} Q(\nabla \eta, \nabla u_t) + Q(\nabla u_t, \nabla (\eta_t - Q(\nabla \eta, \nabla u_t)))$$

$$+ Q(\nabla u_t, \nabla (\eta_t - Q(\nabla \eta, \nabla u_t)))$$

$$= \frac{k(k-1)}{2} \frac{\sqrt{-1(\partial \eta \wedge \bar{\partial} \eta_t + \partial \eta_t \wedge \bar{\partial} \eta) \wedge \sqrt{-1\partial \bar{\partial} u_s \wedge \omega^{-k} \wedge \omega^{n-k}}}{\omega^{k} \wedge \omega^{n-k}}$$

$$- \frac{k(k-1)}{2} \frac{\sqrt{-1(\partial \eta \wedge \bar{\partial} u_t + \partial u_t \wedge \bar{\partial} \eta) \wedge \sqrt{-1\partial \bar{\partial} u_s \wedge \omega^{-k} \wedge \omega^{n-k}}}{\omega^{k} \wedge \omega^{n-k}}$$

$$+ Q(\nabla \eta, \nabla u_t) \Delta_G u_t - Q(\nabla \eta, \nabla u_t) \Delta_G u_s$$

$$+ Q(\nabla u_t, \nabla Q(\nabla \eta, \nabla u_s)) - Q(\nabla u_t, \nabla Q(\nabla \eta, \nabla u_t))$$

$$= \int_X u_t \sqrt{-1(\partial u_s \wedge \bar{\partial} u_t + \partial u_t \wedge \bar{\partial} u_s) \wedge \sqrt{-1\partial \bar{\partial} u_s \wedge \omega^{-k} \wedge \omega^{n-k}}}$$

$$- k(k-1) \int_X u_t \sqrt{-1} \partial u_s \wedge \bar{\partial} u_s \wedge \sqrt{-1}\partial \bar{\partial} u_s \wedge \omega^{n-k}$$

$$+ \int_X (Q(\nabla u_s, \nabla u_t)^2 - Q(\nabla u_s, \nabla u_s) Q(\nabla u_t, \nabla u_t)) \omega^{k} \wedge \omega^{n-k}$$

$$= k(k-1) \int_X (\sqrt{-1} \partial u_s \wedge \bar{\partial} u_s) \wedge (\sqrt{-1} \partial u_t \wedge \bar{\partial} u_t) \wedge \omega^{k-2} \wedge \omega^{n-k}$$

$$+ \int_X (Q(\nabla u_s, \nabla u_t)^2 - Q(\nabla u_s, \nabla u_s) Q(\nabla u_t, \nabla u_t)) \omega^{k} \wedge \omega^{n-k}$$

If we fix a normal coordinate where $g_{ij} = \delta_{ij}$, $u_{ij} = (\lambda_{i} - 1) \delta_{ij}$ and $\partial u_t = X_t$, $\partial u_s = Y_t$, then we have

$$\langle R(u_t, u_s)u_s, u_t \rangle = k(n-k)! \int_X \sum_{ij} \sigma_{k-2,ij} \frac{1}{2}(|X_i|^2|Y_j|^2 + |X_j|^2|Y_i|^2 - X_i Y_i Y_j X_j - Y_i X_i X_j Y_j)$$

$$+ k(n-k)! \int_X \sum_{ij} \sigma_{k-1,ij} \frac{1}{4} \sigma_k (X_i Y_i X_j Y_j + X_i Y_j X_j Y_i + Y_i X_i X_j Y_j + Y_i X_j X_j Y_i).$$
\[-k!(n-k)! \int_X \sum_{ij} \frac{\sigma_{k-1,i} \sigma_{k-1,j}}{\sigma_k} |X_i|^2 |Y_j|^2 \]
\[= k!(n-k)! \int_X \sum_{ij} \sigma_{k-2,ij} (|X_i|^2 |Y_j|^2 - X_i \bar{Y}_j X_j) \]
\[+ k!(n-k)! \int_X \sum_{ij} \frac{\sigma_{k-1,i} \sigma_{k-1,j}}{2\sigma_k} (\text{Re}(X_i \bar{Y}_i X_j Y_j) + X_i \bar{Y}_i Y_j X_j) \]
\[= k!(n-k)! \int_X \sum_{ij} \left( \sigma_{k-2,ij} - \frac{\sigma_{k-1,i} \sigma_{k-1,j}}{\sigma_k} \right) \left( \frac{1}{2} (|X_i|^2 |Y_j|^2 + |Y_i|^2 |X_j|^2) - X_i \bar{Y}_i Y_j X_j) \right) \]

**Lemma 2.** For any \(i, j\), the terms
\[\sigma_{k-2,ij} - \frac{\sigma_{k-1,i} \sigma_{k-1,j}}{\sigma_k}\]
are non-positive.

**Proof.** The inequality we have to show is equivalent to
\[\sigma_{k-2,ij} \sigma_k \leq \sigma_{k-1,i} \sigma_{k-1,j}.\]
If we expand both sides, the terms on each side that contains either a multiple \(\lambda_i\) or a multiple of \(\lambda_j\) will cancel, and it suffices to look at only terms that doesn’t contain \(\lambda_i\) or \(\lambda_j\). Therefore the inequality is equivalent to
\[\sigma_{k-2,ij} \sigma_{k,ij} \leq \sigma_{k-1,ij}^2\]
which follows from Newton’s inequality. \(\square\)

We remark that Lemma 2 is stronger than the well-known fact that \(\log \sigma_k(\lambda)\) is concave in \(\Gamma_k\). By Young’s inequality, for each \(i, j\) we have
\[\frac{1}{2} (|X_i|^2 |Y_j|^2 + |X_j|^2 |Y_i|^2) \geq |X_i||Y_j||Y_i||X_j|\]
Therefore each term in the sum
\[\sum_{ij} \left( \sigma_{k-2,ij} - \frac{\sigma_{k-1,i} \sigma_{k-1,j}}{\sigma_k} \right) \left( \frac{1}{2} (|X_i|^2 |Y_j|^2 + |Y_i|^2 |X_j|^2) - \text{Re}(X_i \bar{Y}_i Y_j X_j) \right)\]
is non-positive, and hence the sectional curvature \(R\) of the \(L^2\) metric on \(\mathcal{H}_k\) is non-positive, which proves Theorem 2. \(\square\)

3. \(C^0\)-estimate

In this section, we prove a \(C^0\) estimate for the system (1.1) depending on a generalized entropy. The main result of this section is a direct counterpart of Theorem 5.1 in [1].

**Theorem 3.** Let \((\varphi, F)\) be a smooth solution to (1.1), then a bound on \(\int_X e^{\text{tr} F} |F| \omega^n\) implies a bound for \(\|F\|_\infty\) and \(\|\varphi\|_\infty\).
Before we present the proof, let us remark that by a result from [5, 6], using just the first equation in (1.1), we can bound the $L^\infty$ norm of $\varphi$ if the $e^F$ is bounded in $L^p$ for $p > \frac{n}{k}$. However, this estimate fails when $p = \frac{n}{k}$. Our result can be seen as a refinement of their result for the coupled system.

3.1. Proof. We begin with the following lemma which gives a lower bound on $\det G^{ij}$.

**Lemma 3.** There is a constant $C = C(n, k)$ such that

$$\det(G^{ij}) \geq C(n, k)\sigma_k^{-\frac{n}{k}}.$$  

**Proof.** We denote the eigenvalues of $\omega_\varphi$ (w.r.t. $\omega$) by $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$. Then $G^{ij} = \sigma_k(\lambda)^{-1} \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} \delta_{ij}$. By Garding’s inequality (Lemma 1) we know that if $\mu = (\mu_1, \ldots, \mu_n) \in \Gamma_k$, then

$$\sum_{i=1}^n \mu_i \frac{\partial \sigma_k}{\partial \lambda_i} \geq C(n, k)\sigma_k(\mu)^{\frac{1}{n}}\sigma_k(\lambda)^{-\frac{k}{n}}.$$  

Taking infimum over all $\mu \in \Gamma_n$ with $\prod_{i=1}^n \mu_i = 1$, we see the LHS of the above becomes $n(\prod_{i=1}^n \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i})^{1/n}$, and the RHS is bounded below by

$$C(n, k)\sigma_n(\mu)^{\frac{1}{n}}\sigma_k(\lambda)^{-\frac{k-1}{n}} = C(n, k)\sigma_k(\lambda)^{-\frac{k-1}{n}}.$$  

Combining these inequalities the lemma follows straightforwardly.

As in [1], we introduce an auxiliary complex Monge-Ampère equation to prove the $C^0$ estimate. For notation convenience we denote $\Phi(F) = \sqrt{F^2 + 1}$ and

$$A_F = \int_X e^{\frac{n}{2} F} \Phi(F) \omega^n.$$  

$A_F$ is bounded by our assumption. We consider the complex MA equation

$$\begin{cases} 
\omega^n = \frac{e^{\frac{n}{2} F} \Phi(F)}{A_F} \omega^n, & \sup_X \psi = 0 \\
\omega_\psi = \omega + \sqrt{-1} \partial \bar{\partial} \psi > 0, 
\end{cases}$$  

which admits a unique solution by Yau’s theorem [18]. The lemma below is the key step to obtain the $L^\infty$ estimate of $\varphi$, and the proof follows closely the ABP-type argument in [1] (see also [6]).

**Lemma 4.** For any $\varepsilon > 0$, there exist constants $\lambda > 0$ and $C = C(n, k, \varepsilon) > 0$ such that

$$F + \varepsilon \varphi - \lambda \varphi \leq C.$$  

**Proof.** For notation convenience we denote $\phi(t) = \phi_\delta(t) = t + \sqrt{t^2 + \delta} > 0$ which converges to $\max(t, 0)$ as $\delta \to 0$. Here $\delta > 0$ is a small number which will go to zero. We also denote $f := F + \varepsilon \varphi - \lambda \varphi$ for notation simplicity and we will look at $\phi(f)$ which converges to $f_+$ as $\delta \to 0$ and is a regularization of $2f_+$. 
We define a \textit{smooth} function

\[ H = \phi(f)^q, \]

where \( q = 1 + \frac{1}{2n} > 1 \) is constant. Since \( X \) is compact, we may assume \( H \) achieves its maximum at \( x_0 \) and \( \max_X H = M > 1 \). Let \( r = r(X, \omega) > 0 \) be the injectivity radius of \((X, \omega)\) as a Riemannian manifold. So we can identify the geodesic ball \( B_r(x_0) \) as the Euclidean ball \( B_{\mathbb{C}^n}(0, r) \) for simplicity. Let \( \theta \in (0, 1) \) be

\[ \theta := \min\left\{ \frac{r^2}{1000M^1/n}, \frac{1}{10n} \right\} < \frac{1}{10}. \] (3.2)

Choose an auxiliary smooth function \( \eta \) defined on \( B_r(x_0) \) so that \( \eta = 1 \) on \( B_{r/4}(x_0) \) and \( \eta = 1 - \theta \) on \( B_{3r/4}(x_0) \), and \( \eta \) also satisfies

\[ |\nabla \eta|_g^2 \leq \frac{100\theta^2}{r^2}, \quad |\nabla^2 \eta|_g \leq \frac{10\theta}{r^2}, \]

where we identify \( \omega \) with its associated Riemannian metric \( g \).

We calculate

\[ \Delta_G(H \eta) = \eta \Delta_G H + H \Delta_G \eta + 2\text{Re}(G^{j\bar{i}} \nabla_j \nabla_i \eta). \] (3.3)

Observe that (below for a function \( f \), \( |\nabla f|_G^2 = G^{j\bar{i}} \nabla_j f \nabla_i f \))

\[ H \Delta_G \eta = H \text{tr}_G \sqrt{-1} \partial \bar{\partial} \eta \geq -H \frac{10\theta}{r^2} \text{tr}_G \omega. \]

And

\[ \text{Re}(G^{j\bar{i}} \nabla_j \nabla_i \eta) = q \phi(f)^q - 1 \text{Re}(G^{j\bar{i}} \nabla_j \phi(f) \nabla_i \eta) \]

\[ \geq - \frac{q(q - 1)}{4} \phi(f)^{q-2} |\nabla \phi(f)|_G^2 - \frac{q}{q - 1} \phi(f)^q |\nabla \eta|_G^2; \]

\[ \geq - \frac{q(q - 1)}{4} \phi(f)^{q-2} |\nabla \phi(f)|_G^2 - \frac{q}{q - 1} \phi(f)^q \frac{10\theta^2}{r^2} \text{tr}_G \omega. \]

And

\[ \eta \Delta_G H = q \eta \phi(f)^q - 1 \Delta_G \phi(f) + q(q - 1) \eta \phi(f)^{q-2} |\nabla \phi(f)|_G^2; \]

\[ = q \eta \phi(f)^q - 1 \phi'(f) \Delta_G f + q \eta \phi(f)^{q-1} \phi''(f) |\nabla f|_G^2 + q(q - 1) \eta \phi(f)^{q-2} |\nabla \phi(f)|_G^2. \]

We note that the middle term above is nonnegative due to the fact that

\[ \phi''(t) = \frac{1}{\sqrt{t^2 + \delta}} - \frac{t^2}{(\sqrt{t^2 + \delta})^3} > 0 \]

For the first term we calculate

\[ \Delta_G f = \Delta_G (F + \epsilon \psi - \lambda \varphi) = \text{tr}_G \alpha - \bar{\alpha} + \epsilon \text{tr}_G \omega \psi - \epsilon \text{tr}_G \omega - \lambda k + \lambda \text{tr}_G \omega \]

\[ \geq (\lambda - \epsilon) \text{tr}_G \omega + \text{tr}_G \alpha - C(n, k, \alpha) + \epsilon n (\text{det} G \cdot \det \omega_\psi)^{1/n} \]

\[ \geq (\lambda - \epsilon) \text{tr}_G \omega + \text{tr}_G \alpha - C(n, k, \alpha) + \epsilon n (\text{det} G \cdot \text{det} \Phi(F) A_F^{-1})^{1/n} \]

\[ \geq (\lambda - \epsilon) \text{tr}_G \omega + \text{tr}_G \alpha - C(n, k, \alpha) + c(n, k) \epsilon \Phi(F)^{1/n} A_F^{-1/n}, \]
where in the second inequality we use Lemma 3. Plugging these inequalities into (3.3), if we choose \( \lambda = 10 + \sup_X |\alpha|_\omega \)

\[
\Delta_G(H\eta) \geq -\frac{10\theta}{r^2} \phi(f)^q \text{tr}_G \omega - \frac{q}{q-1} \phi(f)^q \frac{10\theta^2}{r^2} \text{tr}_G \omega \\
+ q\phi(f)^{q-1}\phi'(f) (\text{tr}_G((\lambda - \varepsilon)\omega + \alpha) + c(n, k)\varepsilon\Phi(F)^{1/n}A_F^{-1/n} - C)
\]

\[
\geq q\phi(f)^{q-1}\left(2\phi'(f) \text{tr}_G \omega - \frac{10\theta}{r^2} \phi(f) \text{tr}_G \omega \right)
\]

\[
- \frac{10\theta^2}{(q-1)r^2} \phi(f) \text{tr}_G \omega + c(n, k)\phi'(f)\varepsilon\Phi(F)^{1/n}A_F^{-1/n} - C\phi'(f).
\]

To deal with RHS in the equation (3.4), note that on the set \( \{ f \leq 0 \} \)

\[
\phi(f) = f + \sqrt{f^2 + \delta} = \frac{\delta}{\sqrt{f^2 + \delta} - f} \leq \sqrt{\delta}.
\]

and

\[
1 \geq \phi'(f) = 1 + \frac{f}{\sqrt{f^2 + \delta}} = \frac{\phi(f)}{\sqrt{f^2 + \delta}} \geq 0.
\]

So the in the set \( \{ f \leq 0 \} \) RHS of (3.4) is

\[
\geq q\phi(f)^{q-1}\left(\frac{10\theta}{r^2} \sqrt{\delta} \text{tr}_G \omega - \frac{10\theta^2}{(q-1)r^2} \sqrt{\delta} \text{tr}_G \omega - C\right)
\]

On the other, on the set \( \{ f > 0 \} \), we know \( \phi'(f) > 1 \), so the RHS of (3.4) is

\[
\geq q\phi(f)^{q-1}\left(c(n, k)\varepsilon\Phi(F)^{1/n}A_F^{-1/n} - C\right),
\]

where in the last inequality we use the choice of \( \theta \) in (3.2) and the fact that \( \phi(f) \leq M^{1/q} \).

Combining the above two cases, we obtain that

\[
\Delta_G(H\eta) \geq q\phi(f)^{q-1}(c(n, k)\varepsilon\Phi(F)^{1/n}A_F^{-1/n} - C)\chi_{\{f > 0\}}
\]

\[
- q\phi(f)^{q-1} \left(\frac{10\theta}{r^2} \sqrt{\delta} \text{tr}_G \omega + \frac{10\theta^2}{(q-1)r^2} \sqrt{\delta} \text{tr}_G \omega + C\right)\chi_{\{f \leq 0\}},
\]

where \( \chi_E \) denotes the characteristic function of a given set \( E \).

We now apply the ABP maximum principle ([8]) on the (Euclidean) ball \( B_r(x_0) \), and we get

\[
\sup_{B_r(x_0)} (H\eta) \leq \sup_{\partial B_r(x_0)} (H\eta) + C(n)r \left\{ \int_{B_r(x_0) \cap \{ f > 0 \}} \left(\frac{\phi(f)^{q-1}}{(\det G)^2}\right)^{2n} + \int_{B_r(x_0) \cap \{ f \leq 0 \}} \left(\frac{\phi(f)^{q-1}}{(\det G)^2}\right)^{2n} \right\}^{1/2n}
\]

\[
+ \left\{ \int_{B_r(x_0) \cap \{ f > 0 \}} \left(\frac{\phi(f)^{q-1}}{(\det G)^2}\right)^{2n} + \int_{B_r(x_0) \cap \{ f \leq 0 \}} \left(\frac{\phi(f)^{q-1}}{(\det G)^2}\right)^{2n} \right\}^{1/2n}
\]

\[
+ C(n, F, G, \omega)\delta^{n(q-1)} \right\}^{1/2n}
\]

\[e^{-2nF/k}\]
where the constant $C(n, F, G, \omega)$ is not uniformly bounded, but this is not a concern, since later on we will let $\delta \to 0$. We observe that the integral above is in fact integrated over the set where $c(n, k) \varepsilon \Phi(F)^{1/n} A_F^{-1/n} - C < 0$ and $F + \varepsilon \psi - \lambda \varphi > 0$, and over this set $F \leq C(n, k, A_F, \varepsilon)$ by the definition of $\Phi(F) = \sqrt{F^2 + 1}$. On the other hand, on this set $F + \varepsilon \psi - \lambda \varphi \leq C - \lambda \varphi$.

Therefore, we have

$$M = \sup_{B_r(x_0)} (H \eta) \leq (1 - \theta) \sup_{\partial B_r(x_0)} H + C \left( \int_{B_r(x_0)} (C - \lambda \varphi) \omega^n + C(n, F, G, \omega) \delta^{n(q-1)} \right)^{1/2n}$$

$$\leq (1 - \theta) M + C + C(n, F, G, \omega) \delta^{(q-1)/2}$$

where we have use the inequality that $\int_{X} (-\varphi) \omega^n \leq C$ which follows by the Green formula, $n + \Delta \omega \varphi > 0$ and the normalization condition $\sup_X \varphi = 0$. Hence we conclude that

$$M^{1 - \frac{1}{q}} \leq C + C(n, F, G, \omega) \delta^{(q-1)/2}, \Rightarrow M \leq C + C(n, F, G, \omega) \delta^{2q},$$

which says that

$$\sup_X 2 f^+ \leq \sup_X \phi(f) \leq C + C(n, F, G, \omega) \delta^{2q}$$

letting $\delta \to 0$ gives the desired estimate.

\[\square\]

**Proof of Theorem 3.** Lemma 4 shows that for any $p > n/k$, we have

$$e^{p F} = e^{p(F + \varepsilon \psi - \lambda \varphi) - p \varepsilon \psi + p \lambda \varphi} \leq C e^{-p \varepsilon \psi},$$

if $\varepsilon$ is chosen small enough so that $p \varepsilon < \alpha(X, \omega)$, the $\alpha$-invariant of the Kahler manifold $(X, \omega)$. We can get the $L^\infty$ bound of $\varphi$ by a result of Dinew-Kolodziej ([5], see also [6, 7]). Kolodziej’s $L^\infty$-estimate for complex MA equations implies the $L^\infty$-bound of $\psi$. Plugging these estimates to Lemma 4 again, we get the upper bound of $F$, i.e. $F \leq C$.

To see the lower bound of $F$, we calculate (if we take $A = 1 + \sup |\alpha|_\omega$)

$$\Delta_G(F + A \varphi) = \text{tr}_G \alpha - \alpha + A \text{tr}_G \bar{\omega} - A \text{tr}_G \omega$$

$$\leq - \text{tr}_G \omega + Ak - \alpha$$

$$\leq - n (\det G \cdot \det \omega)^{1/n} + Ak - \alpha$$

$$\leq - c(n, k) e^{- \frac{K}{n}} + Ak - \alpha.$$

At the minimum of $F + A \varphi$, we have $F > -C(n, k, \omega, \alpha)$. Then the lower bound of $F$ follows easily from the $L^\infty$ estimate of $\varphi$.

\[\square\]

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