The explicit formulae for the distributions of nonoverlapping words and its applications to statistical tests for pseudo random numbers

Hayato Takahashi Random Data Lab. Inc.
1210062 Tokyo Japan
Email: hayato.takahashi@ieee.org

Abstract

The distributions of the number of occurrences of words (the distributions of words for short) play key roles in information theory, statistics, probability theory, ergodic theory, computer science, and DNA analysis. Bassino et al. 2010 and Regnier et al. 1998 showed generating functions of the distributions of words for all sample sizes. Robin et al. 1999 presented generating functions of the distributions for the return time of words and demonstrated a recurrence formula for these distributions. These generating functions are rational functions; except for simple cases, it is difficult to expand them into power series. In this paper, we study finite-dimensional generating functions of the distributions of nonoverlapping words for each fixed sample size and demonstrate the explicit formulae for the distributions of words for the Bernoulli models. Our results are generalized to nonoverlapping partial words. We study statistical tests that depend on the number of occurrences of words and the number of block-wise occurrences of words, respectively. We demonstrate that the power of the test that depends on the number of occurrences of words is significantly large compared to the other one. Finally, we apply our results to statistical tests for pseudo random numbers.

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Index Terms

exact distribution, words, combinatorics, statistical tests, pseudo random numbers

I. INTRODUCTION

We study the distributions of the number of occurrences of words (the distribution of words for short) of the finite alphabet, which play key roles in statistical science, probability theory, information theory, ergodic theory, computer science, and DNA analysis, see \[3, 9\]–[11], \[13\], \[16\], \[17\], \[19\]–[22].

Classical studies on the distributions of words \[1, 5, 7\]–[9], \[14\], \[16\], \[20\] concern the generating functions of the distributions of words for all sample sizes and present them as rational functions. The asymptotic formulae for the distributions of words are obtained from rational generating functions \[6, 9\]. Robin et al. 1999 \[16\] studied rational generating functions of the distributions for the return time of words and demonstrated a recurrence formula for these distributions.

For example, let \( F(z) := \sum_{n} f(n)z^n \), where \( f(n) \) is the number of binary strings of length \( n \) that do not contain the word \( 01 \). The generating function \( F \) is given by (1) for \( |z| < 1 \) (a special case of the theorem in Guibas and Odlyzko \[8\], see also pp.61 \[6\]). We have

\[
F(z) = \frac{1}{1 - 2z + z^2}
\]

\[
= \frac{1}{(1 - z)^2} = (\sum_{n} z^n)^2 = \sum_{n} (n + 1)z^n
\]

and \( f(n) = n + 1 \). For example, \( f(3) = 4 \) and the four words, 000, 100, 110, and 111 do not contain the word 01 among the strings of length 3.

However, it is difficult to expand rational functions into power series except for simple cases. For example, to obtain the coefficient of the expansion of rational functions by the partial fraction expansion, we need to obtain the all roots of the denominator and it is impractical to do so except for simple cases, see pp. 275 Feller \[5\].

We study the finite-dimensional generating functions for the probabilities of the number of nonoverlapping words for each fixed sample size and present the explicit formulae for the exact distributions of nonoverlapping words (Theorem \[1\]). We give all orders of moments of the distributions nonoverlapping words (Theorem \[2\]. In Section \[3\] our results are generalized to nonoverlapping partial words. In Section \[4\] we study statistical tests that depend on
the number of occurrences of words and the number of block-wise occurrences of words, respectively. We show that the power of the test that depends on the number of occurrences of words is significantly large compared to the other one. In Section V, we apply our results to statistical tests for pseudo random numbers.

II. DISTRIBUTIONS OF NONOVERLAPPING WORDS

Let \( X_n := X_1 \ldots X_n \) be random variables that take value in finite alphabet \( \mathcal{A} \) and \( N(w_1, \ldots, w_l; X^n) \) the number of the appearances of the words \( w_1, \ldots, w_l \) in an arbitrary position of \( X^n \). For example \( N(10, 11; 1011101) = (2, 2) \).

Let \( |x| \) be the length of \( x \). A word \( x \) is called overlapping if there is a word \( z \) such that \( x \) appears at least 2 times in \( z \) and \( |z| < 2|x| \) otherwise \( x \) is called nonoverlapping. A pair of words \( x, y \) is called overlapping if there is a word \( z \) such that \( x \) and \( y \) appear in \( z \) and \( |z| < |x| + |y| \) otherwise, the pair is called nonoverlapping. A word \( x \) is overlapping if and only if \( (x, x) \) is overlapping. A finite set of words \( S \) is called nonoverlapping if every pair \( (x, y) \) for \( x, y \in S \) are nonoverlapping, otherwise, \( S \) is called overlapping. For example, sets of words, \{11\}, \{10, 01\}, and \{00, 11\} are overlapping, and \{10\} and \{0011, 0010\} are nonoverlapping.

**Theorem 1** Let \( X_1X_2 \cdots X_n \) be an i.i.d. process with finite alphabet. Let \( w_1, \ldots, w_l \) be the set of nonoverlapping words. Let \( m_i := |w_i| \) and \( P(w_i) \) be the probability of \( w_i \) for \( i = 1, \ldots, l \). Let

\[
A(k_1, \ldots, k_l) = \left( n - \sum_{i=1}^{l} m_i k_i + \sum_{i=1}^{l} k_i \right) \prod_{i=1}^{l} P(k_i(w_i)),
\]

\[
B(k_1, \ldots, k_l) = P(\sum_{i=1}^{n} I_{X_i = w_j} = k_j, j = 1, \ldots, l),
\]

\[
F_A(z_1, \ldots, z_l) = \sum_{k_1, \ldots, k_l} A(k_1, \ldots, k_l) z^{k_1} \cdots z^{k_l}, \text{ and}
\]

\[
F_B(z_1, \ldots, z_l) = \sum_{k_1, \ldots, k_l} B(k_1, \ldots, k_l) z^{k_1} \cdots z^{k_l}.
\]

Then

\[
F_A(z_1, z_2, \ldots, z_l) = F_B(z_1 + 1, z_2 + 1, \ldots, z_l + 1),
\]
and

\[ P(N(w_1, \ldots, w_l; X^n) = (s_1, \ldots, s_l)) = \sum (-1)^{\sum_i k_i - s_i} \left( \begin{array}{c} n - \sum_i m_i k_i + \sum_i k_i \\ s_1, \ldots, s_l, k_1 - s_1, \ldots, k_l - s_l \end{array} \right) \prod_{i=1}^l P^k_i(w_i). \] (3)

Proof) For simplicity, we prove the theorem for \( l = 1 \). The proof of the general case is similar. Let \( m = |w| \). Since \( w \) is nonoverlapping, the number of possible allocations such that \( k \) times the appearance of \( w \) in the string of length \( n \) is

\[ \left( \begin{array}{c} n - mk + k \\ k \end{array} \right). \]

This is because if we replace each \( w \) with additional extra symbol \( \alpha \) in the string of length \( n \) then the problem reduces to choosing \( k \alpha \)'s among the string of length \( n - mk + k \). Let

\[ A(k) := \left( \begin{array}{c} n - mk + k \\ k \end{array} \right) P^k_i(w). \] (4)

The function \( A \) is not the probability of \( k \) \( w \)'s occurrences in the string, since we allow any letters in the remaining place except for the appearance of \( w \). The function \( A \) may count the event that \( w \) appears more than \( k \) times. Let \( B(t) \) be the probability that \( w \) appears \( k \) times. We have the following identity,

\[ A(k) = \sum_{k \leq t} B(t) \binom{t}{k}. \]

Let \( F_A(z) := \sum_k A(k) z^k \) and \( F_B(z) := \sum_k B(k) z^k \). Then

\[ F_A(z) = \sum_k z^k \sum_{k \leq t} B(t) \binom{t}{k} = \sum_t B(t) \sum_{k \leq t} \binom{t}{k} z^k = \sum_t B(t)(z + 1)^t = F_B(z + 1). \]
We have

\[ F_\mathcal{B}(z) = F_\mathcal{A}(z - 1) \]

\[ = \sum_{k, m_k \leq n} \binom{n - m_k + k}{k} (z - 1)^k P^k(w) \]

\[ = \sum_{k, t: m_k \leq n} \binom{n - m_k + k}{k} \binom{k}{t} z^t (-1)^{k-t} P^k(w) \]

\[ = \sum_t z^t \sum_{k, m_k \leq n} (-1)^{k-t} \binom{n - m_k + k}{t, k-t} P^k(w), \]

and (3) \quad \blacksquare.

Regnier et al. [14] showed expectation, variance, and central limit theorems (CLTs) for the occurrences of words.

We give all orders of moments for nonoverlapping words. Let

\[ A_{t,s} = \sum_r \binom{s}{r} r^t (-1)^{s-r} \]

for all \( t, s = 1, 2, \ldots \).

Then \( A_{t,s} \) is the number of surjective functions from \( \{1, 2, \ldots, t\} \rightarrow \{1, 2, \ldots, s\} \) for all \( t, s \), see pp.100 Problem 1 [15].

**Theorem 2** Let \( w \) be a nonoverlapping word.

\[ E(N^t(w; X^n)) = \min\{T, t\} \sum_{s=1}^{\min\{T, t\}} A_{t,s} \binom{n - s|w| + s}{s} P^s(w) \]

for all \( t = 1, 2, \ldots \), where \( T = \max\{t \mid t|w| \leq n\} \).

Proof) Let \( Y_i = I_{X^{i+|w|}-1\leq w} \). We say that \( \{i, i+1, \ldots, i+|w| - 1\} \) is the support of \( Y_i \). Random variables \( Y_{n(1)}, \ldots, Y_{n(s)} \) are called disjoint if their support are disjoint, where \( 1 \leq n(1), \ldots, n(s) \leq i + |w| - 1 \). Since \( w \) is nonoverlapping, we have

\[ E(Y_i Y_j) = \begin{cases} 
  P(w) & \text{if } i = j, \\
  P^2(w) & \text{if } Y_i \text{ and } Y_j \text{ are disjoint,} \\
  0 & \text{else.} 
\end{cases} \quad (5) \]

Let \( Y_{j,i} = Y_i \) for all \( 1 \leq j \leq t \). Then

\[ E(N^t(w; X^n)) = E((\sum_i Y_i)^t) = E(\prod_{j=1}^t \sum_i Y_{j,i}) \]

\[ = E(\sum_{n(1), \ldots, n(t)} \prod_{j=1}^t Y_{j,n(j)}). \quad (6) \]
By (5), $E(\prod_{j=1}^{t} Y_{j,n(j)}) = P^*(w)$ if and only if there is a disjoint set $Y_{i(1)}, \ldots, Y_{i(s)}$ such that \( \{Y_{1,n(1)}, \ldots, Y_{t,n(t)}\} = \{Y_{1(1)}, \ldots, Y_{t(s)}\} \).

The number of possible combination of $s$ disjoint $\{Y_{i(1)}, \ldots, Y_{i(s)}\}$ is $\left(\begin{array}{c} n - s|w| + s \\ s \end{array}\right)$. If $n < s|w|$ then there is no $s$ disjoint $Y_i$. For each disjoint $\{Y_{i(1)}, \ldots, Y_{i(s)}\}$, the number of possible combination of $n_1, \ldots, n_t$ such that $\{Y_{1,n(1)}, \ldots, Y_{t,n(t)}\} = \{Y_{1(1)}, \ldots, Y_{t(s)}\}$ is $A_{t,s}$. By (6), we have the theorem.

**Example 3** Consider the three bits strings. Then 000, 100, 110, and 111 have no 01 and the others have one 01.

On the other hand, let $P$ be a fair coin-flipping, $n = 3$, and $w = 01$ in Theorem 1. Then

$$P(N(01; X^3) = s) = \sum_{s \leq k, 2k \leq 3} (-1)^{k-s}\left(\frac{3 - k}{s, k - s}\right)2^{-2k}.$$  

We have $P(N = 0) = 1/2$ and $P(N = 1) = 1/2$.

### III. DISTRIBUTIONS OF NONOVERLAPPING PARTIAL WORDS

We introduce the symbol $?$ to represent arbitrary symbols. Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ the set of words consisting of alphabet $\mathcal{A}$. A word consists of extended alphabet $\mathcal{A} \cup \{?\}$ is called partial word [2]. We say the word $w' \in \mathcal{A}^*$ realization of the partial word $w$ if $w'$ coincides with $w$ except for the symbol $?$. A partial word is called nonoverlapping if the set of the realization is nonoverlapping. For example, 001?1 is a nonoverlapping partial word with $\mathcal{A} = \{0, 1\}$ and its realizations are 00101 and 00111.

The probability of partial word $w$ is defined by

$$P(w) := \sum_{w': \text{realization of } w} P(w').$$

For example, $P(1?1) := P(101) + P(111)$ if $\mathcal{A} = \{0, 1\}$.

**Corollary 4** Theorem [2] and Theorem [2] holds for nonoverlapping partial words.

Proof) The proof is the same as Theorem [1] and Theorem [2].

We can find many nonoverlapping partial words. For example, $0^{m+1}(1?)^n1$ are nonoverlapping for all $n, m$. Here $w^m$ is the $m$ times concatenation of the word $w$. For example, $0^3(1?)^21 = 0001??1??1$. We can construct large-size partial words that have large probabilities.
Proposition 5 Let $A := \{0,1\}$, $P$ the fair-coin flipping, and

$$w(m) := 0^m (1^m - 1)^{m-1}.$$  \hspace{1cm} (7)

Then $w(m)$ is nonoverlapping for all $m$, $|w(m)| = m^2 + 1$, and $P(w(m)) = 2^{-2m}$.

Proof) Since $0^m$ is not overlapping for all realization of $1^m - 1$, $w(m)$ is nonoverlapping for all $m$. The latter part is obvious. \hfill \blacksquare

IV. Power Function

In this section, we study tests of statistical hypothesis: Null hypothesis is $P(w) = \theta^*$ vs alternative hypothesis is $P(w) < \theta^*$, where $w$ is a nonoverlapping word and $0 < \theta^* \leq 1$ is a fixed constant.

Let $A := \{0,1\}$, $N_w := N(w; X^n)$, and

$$N'_w := \sum_{i=1}^{[n/|w|]} I_{X_{i+1}^{|w|}} I_{X_{i+1}^{|w|} I_{X_{i+1}^{|w|} - 1} = w}. $$

We compare the power of tests that depend on $N_w$ and $N'_w$.

Let

$$E(N_w) = (n - |w| + 1)P(w)$$

and

$$V(N_w) = E(N_w) + (n - 2|w| + 2)(n - 2|w| + 1)P^2(w).$$

Then we have CLTs for the occurrences of words [4], [14]. For all $x$

$$\lim_{n \to \infty} P\left( \frac{N_w - E(N_w)}{\sqrt{V(N_w)}} < x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} \, ds.$$

By CLT, we construct tests that depend on $N_w$ and $N'_w$, respectively. First, let $E_\theta := E(N_w)$ and $V_\theta := V(N_w)$ if $\theta = P(w)$. Reject null hypothesis if and only if $N_w < E_\theta - 5\sqrt{V_\theta}$.

Next, let $E'_\theta = [n/|w|] \theta$ and $V'_\theta = [n/|w|] \theta(1 - \theta)$ if $\theta = P(w)$. Reject null hypothesis if and only if $N'_w < E'_\theta - 5\sqrt{V'_\theta}$.

Let

$$\text{Pow}(\theta) := P_\theta(N_w < E_\theta - 5\sqrt{V_\theta})$$

and

$$\text{Pow}'(\theta) := P_\theta(N'_w < E'_\theta - 5\sqrt{V'_\theta})$$

for $\theta \leq \theta^*$.\hfill \blacksquare
The functions $\text{Pow}$ and $\text{Pow}'$ are called power function.

Let $w = 10$, $\theta^* = 0.25$, and $n = 500$. The following table shows $\text{Pow}(P(10))$ and $\text{Pow}'(P(10))$ at $P(10) = 0.2$ and $0.175$, respectively. Figure 1 shows the graph of $\text{Pow}(P(10))$ and $\text{Pow}'(P(10))$. We see that the power of the test that depends on $N_w$ is significantly large compared to that of $N'_w$.

| $P(10)$ | 0.2  | 0.175 |
|---------|------|-------|
| $\text{Pow}(P(10))$ | 0.316007 | 0.928624 |
| $\text{Pow}'(P(10))$ | 0.000295 | 0.004982 |

Figure 1: The graph of power functions for null hypothesis $P(10) = 0.25$ with $n = 500$. The black line is the graph of $\text{Pow}(P(10))$ and the red line is that of $\text{Pow}'(P(10))$.

V. EXPERIMENTS ON STATISTICAL TESTS FOR PSEUDO RANDOM NUMBERS

In [18], a battery of statistical tests for pseudo random number generators is proposed, and the chi-square test is recommended to test the pseudo random numbers with $N_w$ for a nonoverlapping word $w$.

In this section, we apply the Kolmogorov-Smirnov test to the empirical distributions of pseudo random numbers with the true distributions of $N_w$ for a nonoverlapping word $w$. Let sample size $n = 1600$ and $l = 1$ in [3] and null hypothesis $P$ be fair-coin flipping. For each nonoverlapping words $w = 10$ and $w = 11110$, we consider the true
distributions (3) of $N_w$ and empirical distributions of $N_w$ generated by the Monte Carlo method with BSD RNG.

Figure 2 shows the graph of these distributions for $w = 11110$.

![Graph of distributions](image)

Figure 2: Comparison of distributions: The black line is the distribution of the number of occurrences of the word 11110 in sample size 1600. The Red line is the corresponding empirical distribution generated by the Monte Carlo simulation with 200000 BSD RNG random samplings.

From the experiment for $w = 11110$, we have obtained

$$\sup_{0 \leq x < \infty} |F_t(x) - F(x)| = 0.302073,$$

where $F_t(x)$ is the empirical cumulative distribution generated by the Monte Carlo method with $t = 200000$ BSD RNG random samplings. $F(x)$ is the true cumulative distributions of (3). The p-value of the Kolmogorov-Smirnov test is almost zero,

$$P\left( \sup_{0 \leq x < \infty} |F_t(x) - F(x)| \geq 0.302513 \right) \approx 0,$$

where $P$ is the null hypothesis. The p-values of the Kolmogorov-Smirnov test for BSD RNG with $w = 10$, $w = 11110$, $t = 200000$, and $n = 1600$ are summarized in the following table.
The p-values of the Kolmogorov-Smirnov test for MT RNG [12] with $w = 10$, $w = 11110$, $t = 200000$, and $n = 1600$ are summarized in the following table.

| BSD RNG | $w=10$ | $w=11110$ |
|---------|--------|-----------|
| $\sup_{0 \leq x < \infty} |F_t(x) - F(x)|$ | 0.012216 | 0.302073 |
| p-value | 0.000000 | 0.000000 |

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