Littlewood–Paley Equivalence and Homogeneous Fourier Multipliers

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Abstract. We consider certain Littlewood–Paley operators and prove characterization of some function spaces in terms of those operators. When treating weighted Lebesgue spaces, a generalization to weighted spaces will be made for Hörmander’s theorem on the invertibility of homogeneous Fourier multipliers. Also, applications to the theory of Sobolev spaces will be given.

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1. Introduction

Let \( \psi \) be a function in \( L^1(\mathbb{R}^n) \) such that

\[
\int_{\mathbb{R}^n} \psi(x) \, dx = 0.
\]

(1.1)

We consider the Littlewood–Paley function on \( \mathbb{R}^n \) defined by

\[
g_\psi(f)(x) = \left( \int_0^\infty |f \ast \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2},
\]

(1.2)

where \( \psi_t(x) = t^{-n} \psi(t^{-1}x) \). The following result of Benedek, Calderón and Panzone [2] on the \( L^p \) boundedness, \( 1 < p < \infty \), of \( g_\psi \) is well-known.

Theorem A. We assume (1.1) for \( \psi \) and

\[
|\psi(x)| \leq C (1 + |x|)^{-n-\epsilon}, \quad (1.3)
\]

\[
\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)| \, dx \leq C |y|^{\epsilon} \quad (1.4)
\]

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for some positive constant $\epsilon$. Then $g_\psi$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$:

$$\|g_\psi(f)\|_p \leq C_p \|f\|_p,$$

(1.5)

where

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}.$$

By the Plancherel theorem, it follows that $g_\psi$ is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$, where $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t$, which is a homogeneous function of degree 0. Here the Fourier transform is defined as

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x)e^{-2\pi i \langle x, \xi \rangle} \, dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$

be the Poisson kernel on the upper half space $\mathbb{R}^n \times (0, \infty)$ and $Q(x) = [(\partial/\partial t)P_t(x)]_{t=1}$. Then, we can see that the function $Q$ satisfies the conditions (1.1), (1.3) and (1.4). Thus by Theorem A $g_Q$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

Let $H(x) = \text{sgn}(x)\chi_{[-1,1]}(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$ on $\mathbb{R}$ (the Haar function), where $\chi_E$ denotes the characteristic function of a set $E$ and $\text{sgn}(x)$ the signum function. Then $g_H(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^2}\right)^{1/2},$$

where $F(x) = \int_0^x f(y) \, dy$. Also, we can easily see that Theorem A implies that $g_H$ is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

Further, we can consider the generalized Marcinkiewicz integral $\mu_\alpha(f)$ ($\alpha > 0$) on $\mathbb{R}$ defined by

$$\mu_\alpha(f)(x) = \left(\int_0^\infty |S_\alpha^f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$S_\alpha^f(x) = \frac{\alpha}{t} \int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1} (f(x-u) - f(x+u)) \, du.$$

We observe that $\mu_\alpha(f) = g_{\varphi^{(\alpha)}}(f)$ with

$$\varphi^{(\alpha)}(x) = \alpha|1 - |x||^{\alpha-1} \text{sgn}(x)\chi_{(-1,1)}(x).$$

(1.6)

The square function $\mu_1$ coincides with the ordinary Marcinkiewicz integral $\mu$. When $\psi$ is compactly supported, relevant sharp results for the $L^p$ boundedness of $g_\psi$ can be found in [6, 8, 20].

We can also consider Littlewood–Paley operators on the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$. We consider a dense subspace $S_0(\mathbb{R}^n)$ of $H^p(\mathbb{R}^n)$ consisting of those functions $f$ in $S(\mathbb{R}^n)$ which satisfy $\hat{f} = 0$ near the origin,
where $S(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing smooth functions. Let $f \in S_0(\mathbb{R})$. Then, if $2/(2\alpha + 1) < p < \infty$ and $\alpha > 0$, we have $\|\mu_\alpha(f)\|_p \simeq \|f\|_{H^p}$, which means

$$c_p \|f\|_{H^p} \leq \|\mu_\alpha(f)\|_p \leq C_p \|f\|_{H^p}$$

(1.7)

with some positive constants $c_p, C_p$ independent of $f$ (see [19,27]).

To state results about the reverse inequality of (1.5), we first recall a theorem of Hörmander [12]. Let $m \in L^\infty(\mathbb{R}^n)$ and define

$$T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi.$$  

(1.8)

We say that $m$ is a Fourier multiplier for $L^p$ and write $m \in M_p$ if there exists a constant $C > 0$ such that $\|T_m(f)\|_p \leq C \|f\|_p$ for all $f \in L^2 \cap L^p$. Then the result of Hörmander [12] can be stated as follows.

**Theorem B.** Let $m$ be a bounded function on $\mathbb{R}^n$ which is homogeneous of degree 0. Suppose that $m \in M_p$ for all $p \in (1, \infty)$. Suppose further that $m$ is continuous and does not vanish on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Then, $m^{-1} \in M_p$ for every $p \in (1, \infty)$.

See [2,5] for related results. Applying Theorem B, we can deduce the following (see [12, Theorem 3.8]).

**Theorem C.** Suppose that $g_\psi$ is bounded on $L^p$ for every $p \in (1, \infty)$. Let $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 dt/t$. If $m$ is continuous and strictly positive on $S^{n-1}$, then we have

$$\|f\|_p \leq c_p \|g_\psi(f)\|_p,$$

and hence $\|f\|_p \simeq \|g_\psi(f)\|_p$, $f \in L^p$, for all $p \in (1, \infty)$.

In this note we shall generalize Theorems B and C to weighted $L^p$ spaces with $A_p$ weights of Muckenhoupt (see Theorems 2.5, 2.9 and Corollaries 2.6, 2.11). Our proof of Theorem 2.5 has some features in common with the proof of Wiener–Lévy theorem in [30, vol. I, Chap. VI]. We also consider a discrete parameter version of $g_\psi$:

$$\Delta_\psi(f)(x) = \left( \sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2 \right)^{1/2}.$$  

(1.9)

We shall have $\Delta_\psi$ analogues of results for $g_\psi$ (see Theorem 3.5 and Corollary 3.7). We formulate Theorems 2.9 and 3.5 in general forms so that they include unweighted cases as special cases, while Corollaries 2.11 and 3.7 may be more convenient for some applications.

In the unweighted case, we shall prove some results on $H^p$ analogous to Corollaries 2.11 and 3.7 for $p$ close to 1, $p \leq 1$, in Sect. 4 under a certain regularity condition for $\psi$ (Theorems 4.7 and 4.8). We shall consider functions $\psi$ including those which cannot be treated directly by the theory of [28]. As a
result, in particular, we shall be able to give a proof of the second inequality of (1.7) for \(1/2 < \alpha < 3/2\) and \(2/(2\alpha+1) < p \leq 1\) by methods of real analysis which does not depend on the Poisson kernel.

Here we recall some more background materials on \(\mu_\alpha\). When \(p < 1\) and \(1/2 < \alpha < 1\), we know proofs for the first and the second inequality of (1.7) which use pointwise relations \(\mu_\alpha(f) \geq cg_0(f)\) and \(\mu_\alpha(f) \sim g_\lambda^*(f)\) with \(\lambda = 1 + 2\alpha\), respectively, and apply appropriate properties of \(g_0\) and \(g_\lambda^*\). Also, we note that a proof of the inequality \(\|\mu(f)\|_1 \leq C\|f\|_{H^{1}}\) using a theory of vector valued singular integrals can be found in [10, Chap. V] (see also [17]). We have assumed that \(\text{supp}(\hat{f}) \subset [0, \infty)\) in stating \(\mu_\alpha(f) \sim g_\lambda^*(f)\) and \(g_0(f)\), \(g_\lambda^*(f)\) are the Littlewood–Paley functions defined by

\[
\begin{align*}
g_0(f)(x) &= \left( \int_0^\infty |(\partial/\partial x)u(x, t)|^2 t\,dt \right)^{1/2}, \\
g_\lambda^*(f)(x) &= \left( \iint_{\mathbb{R} \times (0, \infty)} \left( \frac{t}{t + |x-y|} \right)^\lambda |\nabla u(y,t)|^2\,dy\,dt \right)^{1/2}
\end{align*}
\]

with \(u(y,t)\) denoting the Poisson integral of \(f\): \(u(y,t) = P_t * f(y)\) (see [19,27] and references therein, and also [13,15] for related results).

In [28], a proof of \(\|f\|_{H^p} \leq C\|g_Q(f)\|_p\) on \(\mathbb{R}^n\) is given without the use of harmonicity (see [9] for the original proof using properties of harmonic functions). Also, when \(n = 1\), a similar result is shown for \(g_0\). It is to be noted that, combining this with the pointwise relation between \(g_0\) and \(\mu_\alpha\) mentioned above, we can give a proof of the first inequality of (1.7) for the whole range of \(p\), \(\alpha\) in such a manner that a special property of the Poisson kernel is used only to prove the pointwise relation.

In Sect. 5, we shall apply Corollaries 2.11 and 3.7 to the theory of Sobolev spaces. In [1], the operator

\[
U_\alpha(f)(x) = \left( \int_0^\infty \left| f(x) - \int_{B(x,t)} f(y)\,dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0, \quad (1.10)
\]

was studied, where \(\int_{B(x,t)} f(y)\,dy\) is defined as \(|B(x,t)|^{-1} \int_{B(x,t)} f(y)\,dy\) with \(|B(x,t)|\) denoting the Lebesgue measure of a ball \(B(x,t)\) in \(\mathbb{R}^n\) of radius \(t\) centered at \(x\). The operator \(U_1\) was used to characterize the Sobolev space \(W^{1,p}(\mathbb{R}^n)\).

**Theorem D.** Let \(1 < p < \infty\). Then, the following two statements are equivalent:

1. \(f\) belongs to \(W^{1,p}(\mathbb{R}^n)\),
2. \(f \in L^p(\mathbb{R}^n)\) and \(U_1(f) \in L^p(\mathbb{R}^n)\).

Furthermore, from either of the two conditions (1), (2) it follows that

\[
\|U_1(f)\|_p \simeq \|\nabla f\|_p.
\]

This may be used to define a Sobolev space analogous to \(W^{1,p}(\mathbb{R}^n)\) in metric measure spaces. We shall also consider a discrete parameter version of \(U_\alpha\):
\[ E_\alpha(f)(x) = \left( \sum_{k=-\infty}^{\infty} \left| f(x) - \int_{B(x,2^k)} f(y) \, dy \right|^2 2^{-2k\alpha} \right)^{1/2}, \quad \alpha > 0, \quad (1.11) \]

and prove an analogue of Theorem D for \( E_\alpha \). Further, we shall consider operators generalizing \( U_\alpha, E_\alpha \) and show that they can be used to characterize the weighted Sobolev spaces, focusing on the case \( 0 < \alpha < n \).

2. Invertibility of Homogeneous Fourier Multipliers and Littlewood–Paley Operators

We say that a weight function \( w \) belongs to the weight class \( A_p, 1 < p < \infty \), of Muckenhoupt on \( \mathbb{R}^n \) if

\[ [w]_{A_p} = \sup_B \left( |B|^{-1} \int_B w(x) \, dx \right) \left( |B|^{-1} \int_B w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty, \]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^n \). Also, we say that \( w \in A_1 \) if \( M(w) \leq Cw \) almost everywhere, with \( M \) denoting the Hardy-Littlewood maximal operator

\[ M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| \, dy, \]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^n \) containing \( x \); we denote by \([w]_{A_1}\) the infimum of all such \( C \).

Let \( m \in L_\infty(\mathbb{R}^n) \) and \( w \in A_p, 1 < p < \infty \). Let \( T_m \) be as in (1.8). We say that \( m \) is a Fourier multiplier for \( L_p^w \) and write \( m \in M^p_w \) if there exists a constant \( C > 0 \) such that

\[ \|T_m(f)\|_{p,w} \leq C\|f\|_{p,w} \quad (2.1) \]

for all \( f \in L^2 \cap L_p^w \), where

\[ \|f\|_{p,w} = \|f\|_{L_p^w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p}. \]

We also write \( L^p(w) \) for \( L_p^w \). Define

\[ \|m\|_{M^p_w} = \inf C, \]

where the infimum is taken over all the constants \( C \) satisfying (2.1). Since \( L^2 \cap L_p^w \) is dense in \( L_p^w \), \( T_m \) uniquely extends to a bounded linear operator on \( L_p^w \) if \( m \in M^p_w \). In this note we shall confine our attention to the case of \( L_p^w \) boundedness of \( T_m \) with \( w \in A_p \). We note that \( M^p(w) = M^{p'}(\tilde{w}^{-p'/p}) \) by duality, where \( 1/p + 1/p' = 1 \) and \( \tilde{w}(x) = w(-x) \).

If \( w \in A_p, 1 < p < \infty \), then \( w^s \in A_r \) for some \( s > 1 \) and \( r < p \) (see [10]). In applying interpolation arguments it is useful if sets of those \( (r,s) \) are specially denoted.
Definition 2.1. Let \( w \in A_p, 1 < p < \infty \). If \( 0 < \sigma < p - 1, \tau > 0 \) and if \( w^s \in A_r \) for all \( r \in (p - \sigma, p + \sigma) \) and all \( s \in [1, 1 + \tau) \), we define a set \( U(w, p) \) by

\[
U(w, p) = U(w, p, \sigma, \tau) = (p - \sigma, p + \sigma) \times [1, 1 + \tau).
\]

Let \( \mathcal{F}(w, p) \) be the family of all such \( U(w, p) \). We write \( m \in M(U(w, p)) \) if \( m \in M^r(w^s) \) for all \( (r, s) \in U(w, p) \).

We need a relation of \( \|m\|_{M^p(w)} \) and \( \|m\|_\infty \) in the following.

Proposition 2.2. Let \( w \in A_p, 1 < p < \infty \). Suppose that \( m \in M(U(w, p)) \) for some \( U(w, p) \in \mathcal{F}(w, p) \). Then

\[
\|m\|_{M^p(w)} \leq \|m\|_{M^{p+\delta}(w^{1+\epsilon})}^{1-\theta} \|m\|_{M^{p-\delta}(w^{1+\epsilon})}^{\theta},
\]

for some \( \delta, \epsilon > 0 \). Thus we can choose \( \epsilon, \delta > 0 \) so that \( \epsilon + \delta > 0 \). This completes the proof for \( p \in (1, 2) \).

Suppose that \( 2 < p < \infty \) and \( w \in A_p \). Then there exist \( \epsilon_0 \) and \( \delta_0 > 0 \) such that \( (p + \delta, 1 + \epsilon) \in U(w, p) \) for all \( \epsilon \in (0, \epsilon_0) \) and \( \delta \in (0, \delta_0) \). Let \( 1/p = (1 - \theta)/2 + \theta/(p - \delta) \), \( \delta \in (0, \delta_0) \). Then, since \( m \in M^{p-\delta}(w^{1+\epsilon}) \), by interpolation with change of measures of Stein–Weiss (see [3]) between \( L^2 \) and \( L^{p-\delta}(w^{1+\epsilon}) \) boundedness, we have

\[
\|m\|_{M^p(w)} \leq \|m\|_{M^p(w^{\theta(1+\epsilon)/(p-\delta)})} \|m\|_{M^p(w^{\theta(1+\epsilon)/p-\delta})} \|m\|_{M^{p-\delta}(w^{1+\epsilon})}.
\]

To prove a weighted version of Theorem B, we need an approximation result for Fourier multipliers in \( M^p(w) \).
Proposition 2.4. Let $1 < p < \infty$, $w \in A_p$ and $m \in L^\infty(\mathbb{R}^n)$. We assume that $m$ is dyadically homogeneous of degree 0 and continuous on the closed annulus $B_0 = \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 2\}$. We further assume that there exists $U(w, p) \in \mathcal{F}(w, p)$ such that $m \in M(U(w, p))$. Then, for any $\epsilon > 0$, there exists $\ell \in M^p(w)$ which is dyadically homogeneous of degree 0 and in $C^\infty(\mathbb{R}^n\setminus\{0\})$ such that $\|m - \ell\|_\infty < \epsilon$ and $\rho_{p,w}(m - \ell) < \epsilon$.

Proof. Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of functions on the orthogonal group $O(n)$ such that

- each $\varphi_j$ is infinitely differentiable and non-negative,
- for any neighborhood $U$ of the identity in $O(n)$, there exists a positive integer $N$ such that $\text{supp}(\varphi_j) \subset U$ if $j \geq N$,
- $\int_{O(n)} \varphi_j(A) \, dA = 1$, where $dA$ is the Haar measure on $O(n)$.

Also, let $\{\psi_j\}_{j=1}^\infty$ be a sequence of non-negative functions in $C^\infty(\mathbb{R})$ such that $\text{supp}(\psi_j) \subset [1 - 2^{-j}, 1 + 2^{-j}]$ and $\int_0^\infty \psi_j(t) \, dt / t = 1$. Define

$$m_j(\xi) = \int_0^\infty \int_{O(n)} m(tA\xi) \varphi_j(A) \psi_j(t) \, dA \, dt / t.$$ 

Then $m_j$ is dyadically homogeneous of degree 0, infinitely differentiable and $m_j \to m$ uniformly in $\mathbb{R}^n\setminus\{0\}$ by the continuity of $m$ on $B_0$. This can be shown similarly to [12, pp. 123–124], where we can find the case when $m$ is homogeneous of degree 0. Also, for a positive integer $k$, the derivatives of $m_j^k$ satisfy

$$|\partial^\gamma m_j(\xi)| \leq C_{j,k,M} \|m\|^k_b |\xi|^{-|\gamma|}, \quad \partial^\gamma_x = (\partial / \partial \xi_1)^{\gamma_1} \cdots (\partial / \partial \xi_n)^{\gamma_n}$$

for all multi-indices $\gamma$ with $|\gamma| \leq M$, where $M$ is any positive integer, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, $\gamma_j \in \mathbb{Z}$, $\gamma_j \geq 0$ and we have $C_{j,k,M} \leq C_{j,M} k^M$ with a constant $C_{j,M}$ independent of $k$. By (2.2), if $M$ is sufficiently large, it follows that

$$\|m_j^k\|_{M^p(w)} \leq C_{j,k,M} \|m\|^{k_b}_\infty, \quad w \in A_p, 1 < p < \infty$$

with a constant $C$ independent of $k$, which is well-known (see [4,14] for related results). Thus, by the evaluation of $C_{j,k,M}$ we have

$$\rho_{p,w}(m_j) \leq \|m\|_\infty.$$ (2.3)

Since $m, m_j \in M(U(w, p))$, by Proposition 2.2, we can find $r$ close to $p$, $s > 1$ with $(r, s) \in U(w, p)$ and $\theta \in (0, 1)$ such that

$$\|(m - m_j)^k\|_{M^r(w)} \leq \|(m - m_j)^k\|_\infty^{1-\theta} \|(m - m_j)^k\|_{M^r(w)}^\theta.$$ 

It follows that

$$\rho_{p,w}(m - m_j) \leq \|m - m_j\|_\infty^{1-\theta} \rho_{r,w^s}(m - m_j)^\theta.$$

Thus, by (2.3) we have

$$\rho_{p,w}(m - m_j) \leq \|m - m_j\|_\infty^{1-\theta} (\rho_{r,w^s}(m) + \rho_{r,w^s}(m_j))^\theta.$$

This completes the proof since $\|m - m_j\|_\infty \to 0$ as $j \to \infty$. \qed
Applying Proposition 2.4, we can generalize Theorem B as follows.

**Theorem 2.5.** Suppose that $1 < p < \infty, w \in A_p$ and that $m \in L^\infty(\mathbb{R}^n)$ fulfills the hypotheses of Proposition 2.4. Also, suppose that $m(\xi) \neq 0$ for every $\xi \neq 0$. Let $\varphi(z)$ be holomorphic in $\mathbb{C}\setminus\{0\}$. Then we have $\varphi(m(\xi)) \in M^p(w)$.

**Proof.** Define $\epsilon_0 > 0$ by

$$4\epsilon_0 = \min_{\xi \in \mathbb{R}^n \setminus \{0\}} |m(\xi)| = \min_{1 \leq |\ell| \leq 2} |m(\xi)|.$$

By Proposition 2.4, there is $\ell \in M^p(w)$ which is dyadically homogeneous of degree 0 and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$ such that $\|m - \ell\|_{\infty} < \epsilon_0$ and $\rho_{p,w}(m - \ell) < \epsilon_0$. If we consider a curve $C : \ell(\xi) + 2\epsilon_0 e^{i\theta}, 0 \leq \theta \leq 2\pi$, Cauchy’s formula can be applied to represent $\varphi(m(\xi))$ by a contour integral as follows:

$$\varphi(m(\xi)) = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)}{\zeta - m(\xi)} \, d\zeta = \frac{\epsilon_0}{\pi} \int_0^{2\pi} \frac{\varphi(\ell(\xi) + 2\epsilon_0 e^{i\theta})}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} e^{i\theta} \, d\theta, \quad \xi \neq 0.$$

Note that

$$\frac{e^{i\theta}}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} = \frac{1}{2\epsilon_0} \sum_{k=0}^{\infty} \left( \frac{m(\xi) - \ell(\xi)}{2\epsilon_0 e^{i\theta}} \right)^k;$$

the series converges uniformly in $\theta \in [0, 2\pi]$ since $|m(\xi) - \ell(\xi)| < \epsilon_0$. Thus

$$\varphi(m(\xi)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left( \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right)^k M_k(\xi)$$

uniformly in $\mathbb{R}^n \setminus \{0\}$, where

$$M_k(\xi) = \int_0^{2\pi} \varphi(\ell(\xi) + 2\epsilon_0 e^{i\theta}) e^{-ik\theta} \, d\theta.$$

Since $|\ell(\xi) + 2\epsilon_0 e^{i\theta}| \geq \epsilon_0$, we can see that $M_k(\xi)$ is dyadically homogeneous of degree 0 and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$; also the derivative satisfies

$$|\partial^\gamma M_k(\xi)| \leq C_\gamma |\xi|^{-|\gamma|}$$

for every multi-index $\gamma$ with a constant $C_\gamma$ independent of $k$, which follows from homogeneity. This implies that $\|M_k\|_{M^p(w)} \leq C$ with a constant $C$ independent of $k$, similarly to the estimate for $\|m_j^k\|_{M^p(w)}$ in the proof of Proposition 2.4. Thus we have $\varphi(m) \in M^p(w)$ and

$$\|\varphi(m)\|_{M^p(w)} \leq \frac{1}{2\pi} \sum_{k=0}^{\infty} (2\epsilon_0)^{-k} \|m - \ell\|^k_{M^p(w)} \|M_k\|_{M^p(w)},$$

since the series converges, for $\|m - \ell\|^k_{M^p(w)} \leq \epsilon_0^k$ if $k$ is sufficiently large. \(\square\)

**Corollary 2.6.** Let $1 < p < \infty$ and $w \in A_p$. Let $m$ be a dyadically homogeneous function of degree 0 such that $m \in M^r(v)$ for all $r \in (1, \infty)$ and all $v \in A_p$. We assume that $m$ is continuous on $B_0$ and does not vanish there. Then $m^{-1} \in M^p(w)$.

Theorem 2.5 in particular implies the following.
We have applications of Theorem 2.5 and Corollary 2.6 to the theory of Littlewood–Paley operators. Let $w \in A_p$, $1 < p < \infty$. We say that $g_\psi$ of (1.2) is bounded on $L^p_w$ if there exists a constant $C$ such that $\|g_\psi(f)\|_{p,w} \leq C\|f\|_{p,w}$ for $f \in L^2_w \cap L^2$. The unique sublinear extension on $L^p_w$ is also denoted by $g_\psi$. The $L^p_w$ boundedness for $\Delta_\psi$ of (1.9) is considered similarly.

Let $\mathcal{H}$ be the Hilbert space of functions $u(t)$ on $(0, \infty)$ such that $\|u\|_{\mathcal{H}} = (\int_0^\infty |u(t)|^2 \, dt/t)^{1/2} < \infty$. We consider weighted spaces $L^p_{w,\mathcal{H}}$ of functions $h(y,t)$ with the norm

$$\|h\|_{p,w,\mathcal{H}} = \left( \int_{\mathbb{R}^n} \|h_{\psi}^{\mathcal{H}}(y)\|^p_{p,\mathcal{H}} \, dy \right)^{1/p},$$

where $h_{\psi}(t) = h(y,t)$. If $w = 1$ identically, the spaces $L^p_{w,\mathcal{H}}$ will be written simply as $L^p_{\mathcal{H}}$.

Define

$$E^\psi_{\mathcal{H}}(h)(x) = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-y)h(y,t) \, dy \, \frac{dt}{t}, \quad (2.4)$$

where $h \in L^2_{\mathcal{H}}$ and $h_{\psi}(y,t) = h(y,t)\chi_{(\epsilon,1)}(t)$, $0 < \epsilon < 1$, and we assume that $\psi \in L^1(\mathbb{R}^n)$ with (1.1).

Then we have the following.

Lemma 2.7. Let $1 < r < \infty$ and $v \in A_r$. We assume that

$$\|g_\psi(f)\|_{r',w-\epsilon/r} \leq C_0(r,v)\|f\|_{r',w-\epsilon/r}. \quad (1.9)$$

Then, if $h \in L^r_{v,\mathcal{H}} \cap L^2_{\mathcal{H}}$, we have

$$\sup_{\epsilon \in (0,1)} \|E^\psi_{\mathcal{H}}(h)\|_{r,v} \leq C_0(r,v)\|h\|_{r,v,\mathcal{H}},$$

where $\bar{\psi}$ denotes the complex conjugate.

Proof. For $f \in S(\mathbb{R}^n)$, we see that

$$\left| \int_{\mathbb{R}^n} E^\psi_{\mathcal{H}}(h)(x)f(x) \, dx \right| = \left| \int_{\mathbb{R}^n} \left( \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \bar{\psi_t}(x-y)h(y,t) \, dy \, \frac{dt}{t} \right) f(x) \, dx \right|$$

$$= \left| \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \bar{\psi_t} f(y)h(y,t) \, dy \, \frac{dt}{t} \right|$$

$$\leq \int_{\mathbb{R}^n} g_\psi(\bar{f})(y)\|h_{\psi}^{\mathcal{H}}\|_{\mathcal{H}} \, dy.$$
By applying Lemma 2.7, we have the following.

**Proposition 2.8.** Suppose that \( g_\psi \) satisfies the hypothesis of Lemma 2.7 with \( r \in (1, \infty) \) and \( v \in A_r \). Also, we assume that
\[
\|g_\psi(f)\|_{r,v} \leq C_1(r,v)\|f\|_{r,v}.
\]
Put
\[
m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t}.
\]
Then \( \|m\|_{M^r(v)} \leq C_0(r,v)C_1(r,v) \).

**Proof.** We first note that an interpolation with change of measures between
the \( L^r(v) \) and \( L^{r'}(v^{-r'/r}) \) boundedness of \( g_\psi \) implies the \( L^2 \) boundedness of \( g_\psi \). Thus we have \( m \in L^\infty(\mathbb{R}^n) \).

Let \( F(y,t) = f \ast \psi_t(y), f \in L^p_w \cap L^2 \). Then
\[
E^\xi_\psi(F)(x) = \int_\epsilon^{\infty} \int_{\mathbb{R}^n} \psi_t * f(y)\psi_t(y-x) \frac{dydt}{t} = \int_{\mathbb{R}^n} \Psi(\epsilon)(x-z)f(z)dz,
\]
where
\[
\Psi(\epsilon)(x) = \int_\epsilon^{\infty} \int_{\mathbb{R}^n} \psi_t(x+y)\psi_t(y) \frac{dydt}{t}.
\]
We see that
\[
\widehat{\Psi}(\epsilon)(\xi) = \int_\epsilon^{\infty} \hat{\psi}(t\xi)\hat{\psi}(-t\xi) \frac{dt}{t} = \int_\epsilon^{\infty} |\hat{\psi}(t\xi)|^2 \frac{dt}{t}.
\]
Thus
\[
\int_{\mathbb{R}^n} \Psi(\epsilon)(x-z)f(z)dz = T_{m(\epsilon)}f(x), \quad m(\epsilon)(\xi) = \int_\epsilon^{\infty} |\hat{\psi}(t\xi)|^2 \frac{dt}{t}.
\]
From Lemma 2.7 and the \( L^r_v \) boundedness of \( g_\psi \) it follows that
\[
\|T_{m(\epsilon)}f\|_{r,v} = \|E^\xi_\psi(F)\|_{r,v} \leq C_0(r,v)\|g_\psi(f)\|_{r,v} \leq C_0(r,v)C_1(r,v)\|f\|_{r,v}.
\]
(2.5)

Letting \( \epsilon \to 0 \) and noting \( m(\epsilon) \to m \), we see that \( m \in M^r(v) \) and \( \|m\|_{M^r(v)} \)
can be evaluated by (2.5).

Now we can state a weighted version of Theorem C.

**Theorem 2.9.** Let \( g_\psi \) be as in (1.2). Let \( w \in A_p, 1 < p < \infty \). Suppose that there exists \( U(w,p) \in F(w,p) \) such that \( g_\psi \) fulfills the hypotheses of Proposition 2.8 on the weighted boundedness for all \( r, v = w^s, (r,s) \in U(w,p) \). Further, suppose that \( m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \) is continuous and does not vanish on \( S^{n-1} \). Then we have
\[
\|f\|_{p,w} \leq C_{p,w}\|g_\psi(f)\|_{p,w}
\]
for \( f \in L^p_w \).

Obviously, this implies Theorem C when \( w = 1 \).
Proof of Theorem 2.9. We first note that by Proposition 2.8 \( m \in M(U(w,p)) \).
Thus from Theorem 2.5 with \( \varphi(z) = 1/z \) and our assumptions, we see that \( m^{-1} \in M_{p}(w) \). Since \( f = T_{m^{-1}}T_{m}f, f \in L_{w}^{p} \cap L^{2} \), we have
\[
\|f\|_{p,w} \leq C\|T_{m}f\|_{p,w}.
\]
Also, by (2.5) it follows that
\[
\|T_{m}f\|_{p,w} \leq C\|g_{\psi}(f)\|_{p,w}.
\]
Combining results we have the desired inequality. \( \square \)

From Theorem 2.9 the next result follows.

Theorem 2.10. Suppose the following.

1. \( \|g_{\psi}(f)\|_{r,v} \leq C_{r,v}\|f\|_{r,v} \) for all \( r \in (1,\infty) \) and all \( v \in A_{r} \);
2. \( m(\xi) = \int_{0}^{\infty} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \) is continuous and strictly positive on \( S^{n-1} \).

Then, if \( f \in L_{w}^{p} \), we have
\[
\|f\|_{p,w} \leq C_{p,w}\|g_{\psi}(f)\|_{p,w}
\]
for all \( p \in (1,\infty) \) and \( w \in A_{p} \).

The following result is known (see [18]).

Theorem E. Suppose that

1. \( B_{\epsilon}(\psi) < \infty \) for some \( \epsilon > 0 \), where \( B_{\epsilon}(\psi) = \int_{|x| > 1} |\psi(x)||x|^\epsilon \, dx \);
2. \( C_{u}(\psi) < \infty \) for some \( u > 1 \) with \( C_{u}(\psi) = \int_{|x| < 1} |\psi(x)|^u \, dx \);
3. \( H_{\psi} \in L^{1}(\mathbb{R}^{n}) \), where \( H_{\psi}(x) = \sup_{|y| \geq |x|} |\psi(y)| \).

Then
\[
\|g_{\psi}(f)\|_{p,w} \leq C_{p,w}\|f\|_{p,w}
\]
for all \( p \in (1,\infty) \) and \( w \in A_{p} \).

By Theorems 2.10 and E we have the following result, which is useful in some applications.

Corollary 2.11. Suppose that \( \psi \) satisfies the conditions (1), (2), (3) of Theorem E and the non-degeneracy condition: \( \sup_{t > 0} |\hat{\psi}(t\xi)| > 0 \) for all \( \xi \neq 0 \).
Then \( \|f\|_{p,w} \simeq \|g_{\psi}(f)\|_{p,w}, f \in L_{w}^{p}, \) for all \( p \in (1,\infty) \) and \( w \in A_{p} \).

Proof. Let \( m(\xi) = \int_{0}^{\infty} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \). Then by our assumption \( m(\xi) \neq 0 \) for \( \xi \neq 0 \). Thus by Theorems E and 2.10, it suffices to show that \( m \) is continuous on \( S^{n-1} \). From [18] we have
\[
\int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \leq C \min \left( 2^{ke}, 2^{-k\epsilon} \right)
\]
with some \( \epsilon > 0 \) for \( \xi \in S^{n-1} \) and \( k \in \mathbb{Z} \). Thus it can be seen that \( \int_{\epsilon}^{\epsilon^{-1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \to m(\xi) \) uniformly on \( S^{n-1} \) as \( \epsilon \to 0 \). Since \( \int_{\epsilon}^{\epsilon^{-1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \) is continuous on \( S^{n-1} \) for each fixed \( \epsilon > 0 \), the continuity of \( m \) follows. \( \square \)
Remark 2.12. Let $1 < p < \infty$, $w \in A_p$. Suppose that $g_{\psi}$ is bounded on $L^p_w$ and $\psi$ is a radial function with $\int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t = 1$ for every $\xi \neq 0$. Then we have $\|f\|_{p,w} \leq C\|g_{\psi}(f)\|_{p,w}$ if $g_{\psi}$ is also bounded on $L^{p'_{w-p'/p}}$. This is well-known and follows from the proofs of Lemma 2.7 and Proposition 2.8. Also, this can be proved by applying arguments of [10, Chap. V, 5.6 (b)].

3. Discrete Parameter Littlewood–Paley Functions

Let $\psi \in L^1(\mathbb{R}^n)$ with (1.1) and let $\Delta_{\psi}$ be as in (1.9). We first give a criterion for the boundedness of $\Delta_{\psi}$ on $L^p_w$ analogous to Theorem E.

**Theorem 3.1.** Let $B_{\epsilon}(\psi)$, $H_{\psi}$ be as in Theorem E. Suppose that

1. $B_{\epsilon}(\psi) < \infty$ for some $\epsilon > 0$;
2. $|\hat{\psi}(\xi)| \leq C|\xi|^{-\delta}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ with some $\delta > 0$;
3. $H_{\psi} \in L^1(\mathbb{R}^n)$.

Let $1 < p < \infty$. Then

$$\|\Delta_{\psi}(f)\|_{p,w} \leq C_{p,w}\|f\|_{p,w}$$

for every $w \in A_p$.

We assume the pointwise estimate of $\hat{\psi}$ in (2), which is not required in Theorem E.

**Proof of Theorem 3.1.** We apply methods of [7]. Define

$$\widehat{D_j(f)}(\xi) = \Psi(2^j \xi) \hat{f}(\xi) \quad \text{for} \quad j \in \mathbb{Z},$$

where $\Psi \in C^\infty$ satisfies that $\text{supp}(\Psi) \subset \{1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j=\pm\infty} \Psi(2^j \xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

We write

$$f * \psi_{2k}(x) = \sum_{j=\pm\infty} D_{j+k}(f * \psi_{2k})(x),$$

where we initially assume that $f \in \mathcal{S}(\mathbb{R}^n)$. Let

$$L_j(f)(x) = \left( \sum_{k=\pm\infty} |D_{j+k}(f * \psi_{2k})(x)|^2 \right)^{1/2}.$$

Then

$$\Delta_{\psi}(f)(x) \leq \sum_{j \in \mathbb{Z}} L_j(f)(x).$$

We note that the condition (1) and (1.1) imply that $|\hat{\psi}(\xi)| \leq C|\xi|^{\epsilon'}$, $\epsilon' = \min(1, \epsilon)$ (see [18, Lemma 1]). So, since the Fourier transform of $D_j(f * \psi_{2k})$
is supported in \( E_j = \{2^{-1-j} \leq |\xi| \leq 2^{1-j}\} \), the Plancherel theorem and the conditions (1), (2) imply that
\[
\|L_j(f)\|_2^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |D_{j+k} (f * \psi_{2^k}) (x)|^2 \, dx
\leq \sum_{k \in \mathbb{Z}} C \int_{E_{j+k}} \min (|2^k \xi|^{\epsilon}, |2^k \xi|^{-\epsilon}) \left| \hat{f}(\xi) \right|^2 \, d\xi
\leq C 2^{-\epsilon |j|} \sum_{k \in \mathbb{Z}} \int_{E_{j+k}} \left| \hat{f}(\xi) \right|^2 \, d\xi
\leq C 2^{-\epsilon |j|} \left\| f \right\|_2^2,
\] (3.1)
for some \( \epsilon > 0 \), where to get the last inequality we also use the fact that the sets \( E_j \) are finitely overlapping.

By the condition (3), we see that \( \sup_{t > 0} |f * \psi_t| \leq CM(f) \) (see [24, pp. 63–64]). Thus, if \( w \in A_2 \), by the Hardy–Littlewood maximal theorem and the Littlewood–Paley inequality for \( L_w^2 \), we see that
\[
\|L_j(f)\|_{L^2_w}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |D_{j+k} (f * \psi_{2^k}) (x)|^2 w(x) \, dx
\leq \sum_{k \in \mathbb{Z}} C \int_{\mathbb{R}^n} |M(D_{j+k}(f))(x)|^2 w(x) \, dx
\leq \sum_{k \in \mathbb{Z}} C \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 w(x) \, dx
\leq C \left\| f \right\|_{L^2_w}^2.
\] (3.2)

Interpolation with change of measures between (3.1) and (3.2) implies that
\[
\|L_j(f)\|_{L^2_{w^u}} \leq C 2^{-\epsilon (1-u)|j|/2} \| f \|_{L^2_{w^u}}
\]
for \( u \in (0, 1) \). Choosing \( u \), close to 1, so that \( w^{1/u} \in A_2 \), we have
\[
\|L_j(f)\|_{L^2_w} \leq C 2^{-\epsilon (1-u)|j|/2} \| f \|_{L^2_w},
\]
and hence
\[
\|\Delta \psi(f)\|_{L^2_w} \leq \sum_{j \in \mathbb{Z}} \|L_j(f)\|_{L^2_w} \leq C \| f \|_{L^2_w}.
\]
Thus the conclusion follows from the extrapolation theorem of Rubio de Francia [16].

\( \square \)

Remark 3.2. Under the hypotheses of Theorem 3.1, \( g_\psi \) is also bounded on \( L_p^w \) for all \( p \in (1, \infty) \) and \( w \in A_p \). This can be seen from the proof of Theorem E in [18].

Let \( \mathcal{K} \) be the Hilbert space of functions \( v(k) \) on \( \mathbb{Z} \) such that
\[
\|v\|_\mathcal{K} = \left( \sum_{k=-\infty}^{\infty} |v(k)|^2 \right)^{1/2} < \infty.
\]
We define spaces \( L^p_w \) similarly to \( L^p_w \). Also, we use notation similar to the one used when \( E^p_w(h) \) is considered. We define
\[
L^N_\psi(l)(x) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \psi_{2^k}(x - y) l_{(N)}(y, k) \, dy,
\]
(3.3)
where \( l \in L^2_\psi \), \( l_{(N)}(x, k) = l(x, k) \chi_{[-N, N]}(k) \) for a positive integer \( N \).

Then, we have the following result.

**Lemma 3.3.** Suppose that \( 1 < r < \infty \), \( v \in A_r \) and that
\[
\|\Delta_\psi(f)\|_{r', v^{-r'/r}} \leq C_0(r, v) \|f\|_{r', v^{-r'/r}}.
\]
Then, we have \( \sup_{N \geq 1} \|L^N_\psi(l)\|_{r, v} \leq C_0(r, v) \|l\|_{r, v, \mathcal{K}} \), that is,
\[
\sup_{N \geq 1} \left( \int_{\mathbb{R}^n} |L^N_\psi(l)(x)|^r v(x) \, dx \right)^{1/r} 
\leq C_0(r, v) \left( \int_{\mathbb{R}^n} \left( \sum_{k = -\infty}^{\infty} |l(x, k)|^2 \right)^{r/2} v(x) \, dx \right)^{1/r}
\]
for \( l \in L^r_w \cap L^2_\psi \).

This is used to prove the following.

**Proposition 3.4.** We assume that \( \Delta_\psi \) satisfies the hypothesis of Lemma 3.3 with \( r \in (1, \infty) \) and \( v \in A_r \). Further, we assume that
\[
\|\Delta_\psi(f)\|_{r, v} \leq C_1(r, v) \|f\|_{r, v}.
\]
Set
\[
m(\xi) = \sum_{k = -\infty}^{\infty} |\hat{\psi}(2^k \xi)|^2.
\]
Then, we have \( \|m\|_{M^{r}(v)} \leq C_0(r, v) C_1(r, v) \).

Proposition 3.4 and Theorem 2.5 are applied to prove the following.

**Theorem 3.5.** We assume the following.
(1) \( \|\Delta_\psi(f)\|_{r, v} \leq C_{r, v} \|f\|_{r, v} \) for all \( r \in (1, \infty) \) and all \( v \in A_r \);

We note that \( m \) is dyadically homogeneous of degree 0 and that, under the assumptions of Theorem 3.5, \( m \in M(U(w, p)) \).

Theorem 3.5 implies the next result.

**Theorem 3.6.** We assume the following.
(1) \( \|\Delta_\psi(f)\|_{r, v} \leq C_{r, v} \|f\|_{r, v} \) for all \( r \in (1, \infty) \) and all \( v \in A_r \);
(2) $m$ is continuous and strictly positive on $B_0$, where $m$ is defined as in Theorem 3.5.

Let $w \in A_p$, $1 < p < \infty$. Then we have

$$\|f\|_{p,w} \leq C_{p,w}\|\Delta_\psi(f)\|_{p,w}, \quad f \in L^p_w.$$  

Lemma 3.3, Proposition 3.4, Theorems 3.5 and 3.6 are analogous to and can be proved similarly to Lemma 2.7, Proposition 2.8, Theorems 2.9 and 2.10, respectively. We omit their proofs.

We also have an analogue of Corollary 2.11.

**Corollary 3.7.** Suppose that $\psi$ satisfies the conditions (1), (2), (3) of Theorem 3.1 and the non-degeneracy condition: $\sup_{k \in \mathbb{Z}}|\hat{\psi}(2^k \xi)| > 0$ for all $\xi \neq 0$. Then $\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}$, for all $p \in (1, \infty)$ and $w \in A_p$.

**Proof.** By the assumption $m(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\psi}(2^k \xi)|^2 > 0$ for $\xi \neq 0$. Therefore, by Theorem 3.6, to prove a reverse inequality of the conclusion of Theorem 3.1 it suffices to show that $m$ is continuous on $B_0$. From the estimate $|\hat{\psi}(\xi)| \leq C \min(|\xi|^\epsilon, |\xi|^{-\epsilon})$ for some $\epsilon > 0$, which follows from (1) and (2) of Theorem 3.1 (see [18, Lemma 1]), it can be seen that $\sum_{k=-N}^{N} |\hat{\psi}(2^k \xi)|^2 \rightarrow m(\xi)$ uniformly on $B_0$ as $N \rightarrow \infty$. Since $\sum_{k=-N}^{N} |\hat{\psi}(2^k \xi)|^2$ is continuous on $B_0$ for each fixed $N$, we can conclude that $m$ is also continuous on $B_0$. \hfill \square

4. Littlewood–Paley Operators on $H^p$, $0 < p \leq 1$, with $p$ Close to 1

Let $0 < p \leq 1$. We consider the Hardy space of functions on $\mathbb{R}^n$ with values in $\mathcal{K}$, which is denoted by $H^p_\mathcal{K}(\mathbb{R}^n)$. Choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi(x) \, dx = 1$. Let $h \in L^2_\mathcal{K}(\mathbb{R}^n)$. We say $h \in H^p_\mathcal{K}(\mathbb{R}^n)$ if $\|h\|_{H^p_\mathcal{K}} = \|h^*\|_{L^p} < \infty$ with

$$h^*(x) = \sup_{s > 0} \left( \int_0^{\infty} |\varphi_s \ast h^t(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where we write $h^t(x) = h(x, t)$. Similarly, we consider the Hardy space $H^p_\mathcal{K}(\mathbb{R}^n)$ of functions $l$ in $L^2_\mathcal{K}(\mathbb{R}^n)$ such that $\|l\|_{H^p_\mathcal{K}} = \|l^*\|_{L^p} < \infty$, where

$$l^*(x) = \sup_{s > 0} \left( \sum_{j=-\infty}^{\infty} |\varphi_s \ast l^j(x)|^2 \right)^{1/2}, \quad l^j(x) = l(x, j).$$

Let $\psi \in L^1(\mathbb{R}^n)$ with (1.1) and let $E_\psi^\ast(h)$ be defined as in (2.4).

**Theorem 4.1.** Suppose that

1. $\int_0^{\infty} |\hat{\psi}(\xi)|^2 \, d\xi = C$ with a constant $C$;
2. there exists $\tau \in (0, 1]$ such that if $|x| > 2|y|$, \n$$\left( \int_0^{\infty} |\hat{\psi}_t(x - y) - \hat{\psi}_t(x)|^2 \frac{dt}{t} \right)^{1/2} \leq C \frac{|y|^\tau}{|x|^{n+\tau}}.$$
Then
\[ \sup_{\epsilon \in (0,1)} \| E_{\psi}^\epsilon (h) \|_{H^p} \leq C \| h \|_{H^p_{\infty}} \]
if \( n/(n + \tau) < p \leq 1 \), where \( H^p = H^p(\mathbb{R}^n) \) is the ordinary Hardy space on \( \mathbb{R}^n \).

Recall that we say \( f \in S'(\mathbb{R}^n) \) (the space of tempered distributions) belongs to \( H^p(\mathbb{R}^n) \) if \( \| f \|_{H^p} = \| f^* \|_p < \infty \), where \( f^*(x) = \sup_{t>0} |\varphi_t * f(x)| \), with \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) satisfying \( \int \varphi(x) \, dx = 1 \) (see [9]).

We also have a similar result for \( L^N_{\psi}(l) \).

**Theorem 4.2.** Let \( L^N_{\psi}(l) \) be defined as in (3.3). We assume the following conditions:

1. \( \sum_{k=-\infty}^{\infty} |\tilde{\psi}(2^k \xi)|^2 \leq C \) with a constant \( C \);
2. if \( |x| > 2 |y| \), we have
   \[ \left( \sum_{k=-\infty}^{\infty} |\psi_{2^k}(x - y) - \psi_{2^k}(x)|^2 \right)^{1/2} \leq C \frac{|y|^\tau}{|x|^{n+\tau}} \]
   with some \( \tau \in (0,1) \).

Then
\[ \sup_{N \geq 1} \| L^N_{\psi}(l) \|_{H^p} \leq C \| l \|_{H^p_{\infty}} \] for \( n/(n + \tau) < p \leq 1 \).

To prove these theorems we apply atomic decompositions.

Let \( a \) be a \( (p, \infty) \) atom in \( H^p_{\infty}(\mathbb{R}^n) \). Thus

1. \( \left( \int_0^\infty |a(x,t)|^2 \, dt/t \right)^{1/2} \leq |Q|^{-1/p} \), where \( Q \) is a cube in \( \mathbb{R}^n \) with sides parallel to the coordinate axes;
2. \( \text{supp}(a(\cdot, t)) \subset Q \) uniformly in \( t > 0 \), where \( Q \) is the same as in (i);
3. \( \int_{\mathbb{R}^n} a(x, t) x^\gamma \, dx = 0 \) for all \( t > 0 \) and \( |\gamma| \leq [n(1/p - 1)] \), where \( x^\gamma = x_1^{\gamma_1} \ldots x_n^{\gamma_n} \) and \([a]\) denotes the largest integer not exceeding \( a\).

To prove Theorem 4.1 we use the following.

**Lemma 4.3.** Let \( h \in L^2_{\infty}(\mathbb{R}^n) \). Suppose that \( h \in H^p_{\infty}(\mathbb{R}^n) \). Then there exist a sequence \( \{a_k\} \) of \( (p, \infty) \) atoms in \( H^p_{\infty}(\mathbb{R}^n) \) and a sequence \( \{\lambda_k\} \) of positive numbers such that \( h = \sum_{k=1}^{\infty} \lambda_k a_k \) in \( H^p_{\infty}(\mathbb{R}^n) \) and in \( L^2_{\infty}(\mathbb{R}^n) \), and \( \sum_{k=1}^{\infty} \lambda_k^p \leq C \| h \|_{H^p_{\infty}}^p \), where \( C \) is a constant independent of \( h \).

See [10,26] for the case of \( H^p(\mathbb{R}^n) \); the vector valued case can be proved similarly. We apply Lemma 4.3 for \( p \in (n/(n+1), 1] \). We also need the following.

**Lemma 4.4.** Let \( \varphi \) be a non-negative \( C^\infty \) function on \( \mathbb{R}^n \) supported in \( \{|x| < 1\} \) which satisfies \( \int \varphi(x) \, dx = 1 \). Suppose that \( \psi \in L^1(\mathbb{R}^n) \) satisfies the conditions (1),(2) of Theorem 4.1. Let \( \Psi_{s,t} = \varphi_s * \psi_t, s, t > 0 \). Then, if \( |x| > 3 |y| \), we have
\[ \left( \int_0^\infty |\Psi_{s,t}(x - y) - \Psi_{s,t}(x)|^2 \, dt/t \right)^{1/2} \leq C \frac{|y|^\tau}{|x|^{n+\tau}} \]
with a constant \( C \) independent of \( s > 0 \).
Proof. We note that
\[ \Psi_{s,t}(x-y) - \Psi_{s,t}(x) = \int_{|z|<|x|} (\psi_t(x-y-z) - \psi_t(x-z)) \varphi_s(z) \, dz. \]
Hence
\[ \left| \Psi_{s,t}(x-y) - \Psi_{s,t}(x) \right|^2 \leq \int_{|z|<|x|} \left| \psi_t(x-y-z) - \psi_t(x-z) \right|^2 \varphi_s(z) \, dz \]
and
\[ \int_{|z|<|x|} \left| \psi_t(x-y-z) - \psi_t(x-z) \right|^2 \varphi_s(z) \, dz \leq C \int_{|z|<|x|} \frac{|y|^\tau}{|x-z|^{n+\tau}} \varphi_s(z) \, dz \]
for some constant $C$. Let $0 < s < |x|/4$. Then, if $|x| > 3|y|$ and $|z| < s$, we have $|x-z| \geq (3/4)|x| \geq 2|y|$. Thus by the Minkowski inequality and (2) of Theorem 4.1 we see that
\[ \left( \int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x)|^2 \frac{dt}{t} \right)^{1/2} \leq C \int_{|z|<|x|} \frac{|y|^\tau}{|x-z|^{n+\tau}} \varphi_s(z) \, dz \]
(4.1)
To deal with the case $s \geq |x|/4$, we write
\[ \Psi_{s,t}(x-y) - \Psi_{s,t}(x) = \int \hat{\varphi}(s\xi) \hat{\psi}(t\xi) e^{2\pi i \langle x, \xi \rangle} e^{2\pi i \langle y, \xi \rangle} (e^{2\pi i \langle y, \xi \rangle} - 1) \, d\xi. \]
Applying Minkowski’s inequality again and using Theorem 4.1 (1) we see
\[ \left( \int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x)|^2 \frac{dt}{t} \right)^{1/2} \leq C |y| \int |\varphi(s\xi)||\xi| \, d\xi \]
\[ \leq C |y| s^{-n-1} \int |\varphi(\xi)||\xi| \, d\xi \]
\[ \leq C \frac{|y|^\tau}{|x|^{n+\tau}}, \quad (4.2) \]
By 4.1 and 4.2 we get the desired estimates.

Proof of Theorem 4.1. Let $a$ be a $(p, \infty)$ atom in $H^p_{\mathcal{M}}(\mathbb{R}^n)$ supported on the cube $Q$ of the definition of the atom. Let $y_0$ be the center of $Q$. Let $	ilde{Q}$ be a concentric enlargement of $Q$ such that $3|y - y_0| < |x - y_0|$ if $y \in Q$ and $x \notin \mathbb{R}^n \setminus \tilde{Q}$. Let $\varphi$ be as in Lemma 4.4. Then, using Lemma 4.4, the properties of an atom and the Schwarz inequality, for $x \in \mathbb{R}^n \setminus \tilde{Q}$ we have
\[ |\varphi_s * E^c_\psi(a)(x)| \]
\[ = \left| \int \int (\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)) a_{(t)}(y, t) \, dy \frac{dt}{t} \right| \]
\[ \leq \int_Q \left( \int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)|^2 \frac{dt}{t} \right)^{1/2} \left( \int_0^\infty |a(y, t)|^2 \frac{dt}{t} \right)^{1/2} \, dy \]
\[ \int_{\mathbb{R}^n} \sup_{s>0} |\varphi_s \ast E_\psi^\varepsilon(a)(x)|^p \, dx \leq C |Q|^{-1+p+p\tau/n} \int_{\mathbb{R}^n \setminus \tilde{Q}} |x-y_0|^{-p(n+\tau)} \leq C. \tag{4.3} \]

The condition (1) implies the $L^2$ boundedness of $g_\psi$ and hence by Lemma 2.7 we can see that

\[ \sup_{\varepsilon \in (0,1)} \|E_\psi^\varepsilon(h)\|_2 \leq C \|h\|_{L^2_{3\ell}} , \quad h \in L^2_{3\ell}(\mathbb{R}^n). \]

So, by Hölder’s inequality and the properties (i), (ii) of $a$, we get

\[ \int_{\tilde{Q}} \sup_{s>0} |\varphi_s \ast E_\psi^\varepsilon(a)(x)|^p \, dx \leq C |Q|^{1-p/2} \left( \int_{\tilde{Q}} |M(E_\psi^\varepsilon(a)(x)|^2 \, dx \right)^{p/2} \leq C |Q|^{1-p/2} \left( \int_{Q} \int_0^\infty |a(y,t)|^2 \frac{dt}{t} \, dy \right)^{p/2} \leq C. \tag{4.4} \]

Combining (4.3) and (4.4), we have

\[ \int_{\mathbb{R}^n} \sup_{s>0} |\varphi_s \ast E_\psi^\varepsilon(a)(x)|^p \, dx \leq C. \tag{4.5} \]

Let $h \in H^p_{3\ell}(\mathbb{R}^n)$ and the decomposition $h = \sum_{k=1}^\infty \lambda_k a_k$ be as in Lemma 4.3. We easily see that $|\varphi_s \ast E_\psi^\varepsilon(h)(x)| \leq C_\varepsilon \|\psi\|_1 \|\varphi_s\|_2 \|h\|_{L^2_{3\ell}}$. Thus, since $h = \sum_{k=1}^\infty \lambda_k a_k$ is in $L^2_{3\ell}(\mathbb{R}^n)$, we have

\[ \varphi_s \ast E_\psi^\varepsilon(h)(x) = \sum_{k=1}^\infty \lambda_k \varphi_s \ast E_\psi^\varepsilon(a_k)(x). \]
By this and (4.5) we can prove
\[
\int_{\mathbb{R}^n} \sup_{s>0} |\varphi_s * E^e_{\psi}(h)(x)|^p \, dx \leq C\|h\|_{H^p}^p.
\]
\(\square\)

Theorem 4.2 can be shown similarly.

Also, we can prove the following mapping properties of \(g_{\psi}\) and \(\Delta_{\psi}\) on \(H^p(\mathbb{R}^n)\) in the same way.

**Theorem 4.5.** Suppose that \(\psi\) fulfills the hypotheses of Theorem 4.1. We define \(F(\psi, f)(x, t) = f * \psi_t(x)\). Then if \(n/(n + \tau) < p \leq 1\),
\[
\|F(\psi, f)\|_{H^p} \leq C\|f\|_{H^p}.
\]
for \(f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\).

**Theorem 4.6.** We assume that \(\psi\) fulfills the hypotheses of Theorem 4.2. Let \(G(\psi, f)(x, k) = f * \psi_{2k}(x)\). Then
\[
\|G(\psi, f)\|_{H^p_{\infty}} \leq C\|f\|_{H^p}, \quad f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),
\]
if \(n/(n + \tau) < p \leq 1\).

**Proof of Theorem 4.5.** The proof is similar to that of Theorem 4.1. By the atomic decomposition, it suffices to show that \(\|F(\psi, a)\|_{H^p_{\infty}} \leq C\), where \(a\) is a \((p, \infty)\) atom in \(H^p(\mathbb{R}^n)\) such that \(\|a\|_{\infty} \leq |Q|^{-1/p}\), \(\text{supp}(a) \subset Q\) with a cube \(Q\) and \(\int a = 0\). Let \(y_0\) be the center of \(Q\) and let \(\tilde{Q}, \varphi_s, \Psi_{s,t}\) be as in the proof of Theorem 4.1. Then, using Minkowski’s inequality and Lemma 4.4, for \(x \in \mathbb{R}^n \setminus \tilde{Q}\) we have
\[
\left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t}\right)^{1/2} = \left(\int_0^\infty \left|\int_0^\infty (\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)) a(y) \, dy\right|^2 \, \frac{dt}{t}\right)^{1/2}
\leq C|Q|^{-1/p} \int_{\tilde{Q}} \left(\int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)|^2 \, \frac{dt}{t}\right)^{1/2} \, dy
\leq C|Q|^{-1/p+1+\tau/n}|x-y_0|^{-n-\tau}.
\]
Therefore, as in (4.3), for \(p > n/(n + \tau)\), we have
\[
\int_{\mathbb{R}^n} \sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t}\right)^{p/2} \leq C.
\]
Since by the Minkowski inequality we easily see that
\[
\sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t}\right)^{1/2} \leq \sup_{s>0} \varphi_s * g_{\psi}(a)(x) \leq CM(g_{\psi}(a))(x),
\]
as in (4.4) we have
\[
\int_{\tilde{Q}} \sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t}\right)^{p/2} \leq C.
\]
Collecting results, we have the desired estimates.

The proof of Theorem 4.6 is similar. Using Theorems 4.1, 4.5 and 4.2, 4.6, we can show analogues of Corollaries 2.11 and 3.7 for $p \leq 1$.

**Theorem 4.7.** Suppose that $\psi$ fulfills the hypotheses of Theorem 4.1. Put $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t}$. We assume that $m$ does not vanish in $\mathbb{R}^n \setminus \{0\}$ and $m \in C^k(\mathbb{R}^n \setminus \{0\})$, where $k$ is a positive integer satisfying $k/n > 1/p - 1/2$, with $n/(n + \tau) < p \leq 1$. Then we have

$$\|F(\psi, f)\|_{H^p_{\psi}} \simeq \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, where $F(\psi, f)$ is as in Theorem 4.5.

**Theorem 4.8.** We assume that $\psi$ fulfills the hypotheses of Theorem 4.2. Set $m(\xi) = \sum_{n=-\infty}^\infty |\hat{\psi}(2^n\xi)|^2$. Let $n/(n + \tau) < p \leq 1$. We assume that $m(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $m \in C^k(\mathbb{R}^n \setminus \{0\})$ with a positive integer $k$ as in Theorem 4.7. Let $G(\psi, f)$ be as in Theorem 4.6. Then we have

$$\|G(\psi, f)\|_{H^p_{\psi}} \simeq \|f\|_{H^p}, \quad f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

**Proof of Theorem 4.7.** By Theorem 4.5 we have $\|F(\psi, f)\|_{H^p_{\psi}} \leq C\|f\|_{H^p}$. To prove the reverse inequality we note that $f = T_{m^{-1}}T_m f$. Since $m^{-1} \in C^k(\mathbb{R}^n \setminus \{0\})$ and it is homogeneous of degree 0, $m^{-1}$ is a Fourier multiplier for $H^p$ by [10, pp. 347–348]. Thus

$$\|f\|_{H^p} \leq C\|T_m f\|_{H^p} \leq C\liminf_{\epsilon \to 0} \|T_{m^{(\epsilon)}} f\|_{H^p}, \quad (4.6)$$

and by the proof of Proposition 2.8 we see that

$$T_{m^{(\epsilon)}} f = E^{\epsilon}_{\psi}(F),$$

where $m^{(\epsilon)}$, $F$ are defined as in the proof of Proposition 2.8. Thus by Theorem 4.1 we have

$$\|T_{m^{(\epsilon)}} f\|_{H^p} = \|E^{\epsilon}_{\psi}(F)\|_{H^p} \leq C\|F(\psi, f)\|_{H^p_{\psi}}.$$

which combined with (4.6) implies the reverse inequality.

Theorem 4.8 can be proved similarly.

We note that Theorems 4.5 and 4.6 imply that $\|g(\psi, f)\|_p \leq C\|f\|_{H^p}$, $\|\Delta(\psi, f)\|_p \leq C\|f\|_{H^p}$. Under the assumptions of Theorems 4.7 and 4.8, the reverse inequalities, which would improve results, are not available at present stage of the research. For related results which can handle Littlewood–Paley operators like $g_{Q}$, we refer to [28].

Let $\varphi^{(\alpha)}$ on $\mathbb{R}^1$ be as in (1.6). Then we can show that

$$\left(\int_0^\infty |\varphi^{(\alpha)}(x - y) - \varphi^{(\alpha)}(x)|^2 \frac{dt}{t}\right)^{1/2} \leq C \frac{|y|^\sigma}{|x|^{1+\sigma}}, \quad \sigma = (2\alpha - 1)/2, \quad (4.7)$$

if $2|y| < |x|$, where $1/2 < \alpha < 3/2$. Also, it is not difficult to see that the condition (1) of Theorem 4.1 is valid for $\varphi^{(\alpha)}$. Thus, from Theorem 4.5 we in particular have the second inequality of (1.7) for $1/2 < \alpha < 3/2$, $2/(2\alpha + 1) < p \leq 1$. We shall give a proof of the estimate (4.7) in Sect. 6 for completeness.
5. Applications to the Theory of Sobolev Spaces

Let $0 < \alpha < n$ and

$$T_{\alpha}(f)(x) = \left( \int_{0}^{\infty} \left| I_{\alpha}(f)(x) - \int_{B(x,t)} I_{\alpha}(f)(y) \, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad (5.1)$$

where $I_{\alpha}$ is the Riesz potential operator defined by

$$\hat{I}_{\alpha}(f)(\xi) = \left( \frac{2}{\pi} |\xi| \right)^{-\alpha} \hat{f}(\xi). \quad (5.2)$$

Then, from [1] we can see the following result.

**Theorem F.** Suppose that $1 < p < \infty$ and $n \geq 2$. Let $T_{\alpha}$ be as in (5.1). Then

$$\|T_{1}(f)\|_{p} \simeq \|f\|_{p}, \quad f \in S(\mathbb{R}^{n}).$$

In [1] this was used to prove Theorem D in Sect. 1 when $n \geq 2$. Theorem F is generalized to the weighted $L^{p}$ spaces (see [11,21]).

We consider square functions generalizing $U_{\alpha}$ and $T_{\alpha}$ in (1.10) and (5.1).

Let

$$U_{\alpha}(f)(x) = \left( \int_{0}^{\infty} |f(x) - \Phi_{t} * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0, \quad (5.3)$$

with $\Phi \in M^{\alpha}$, where we say $\Phi \in M^{\alpha}$, $\alpha > 0$, if $\Phi$ is a bounded function on $\mathbb{R}^{n}$ with compact support satisfying $\int_{\mathbb{R}^{n}} \Phi(x) \, dx = 1$; if $\alpha \geq 1$, we further assume that

$$\int_{\mathbb{R}^{n}} \Phi(x) x^{\gamma} \, dx = 0 \quad \text{for all } \gamma \text{ with } 1 \leq |\gamma| \leq [\alpha]. \quad (5.4)$$

When $1 \leq \alpha < 2$, (5.4) is satisfied if $\Phi$ is even; in particular, we note that $\chi_{0} = \chi_{B(0,1)}/|B(0,1)| \in M^{\alpha}$ for $0 < \alpha < 2$ and if $\Phi = \chi_{0}$ in (5.3), we have $U_{\alpha}$ of (1.10).

We also consider

$$T_{\alpha}(f)(x) = \left( \int_{0}^{\infty} |I_{\alpha}(f)(x) - \Phi_{t} * I_{\alpha}(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad (5.5)$$

where $0 < \alpha < n$ and $\Phi \in M^{\alpha}$. If we set $\Phi = \chi_{0}$ in (5.5), we get $T_{\alpha}$ of (5.1).

We prove the following.

**Theorem 5.1.** Suppose that $T_{\alpha}$ is as in (5.5) and $0 < \alpha < n$, $1 < p < \infty$. Let $w \in A_{p}$. Then

$$\|T_{\alpha}(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in S(\mathbb{R}^{n}).$$

By Theorem 5.1 we see that $U_{\alpha}$ can be used to characterize the weighted Sobolev spaces.

Let $J_{\alpha}$ be the Bessel potential operator defined as $J_{\alpha}(g) = K_{\alpha} * g$ with

$$\hat{K}_{\alpha}(\xi) = (1 + 4\pi^{2}|\xi|^{2})^{-\alpha/2} \quad (\text{see } [24]).$$

Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_{p}$. The weighted Sobolev space $W^{\alpha,p}_{w}(\mathbb{R}^{n})$ is defined to be the collection of all the functions $f$ which can be expressed as $f = J_{\alpha}(g)$ with $g \in L^{p}_{w}(\mathbb{R}^{n})$ and its norm is defined by $\|f\|_{p,\alpha,w} = \|J_{\alpha}(g)\|_{p,w}$. 

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∥g∥_{p,w}. The weighted $L^p$ norm inequality for the Hardy-Littlewood maximal operator with $A_p$ weights (see [10]) implies that $J_\alpha(g) \in L^p_w$ if $g \in L^p_w$, since it is known that $|J_\alpha(g)| \leq CM(g)$ (see [24,25]). We also note that $J_\alpha$ is injective on $L^p_w$. So, the norm $∥f∥_{p,\alpha,w}$ is well-defined.

Applying Theorem 5.1, we have the following.

**Corollary 5.2.** Let $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < n$. Let $U_\alpha$ be as in (5.3). Then $f \in W^{\alpha,p}_w(\mathbb{R}^n)$ if and only if $f \in L^p_w$ and $U_\alpha(\mathbf{f}) \in L^p_w$; furthermore,

$$\|f\|_{p,\alpha,w} \approx \|f\|_{p,w} + \|U_\alpha(f)\|_{p,w}.$$  

For the case $n = 1$ and $\alpha = 1$, see Remark 5.7 below. We refer to [22,23,25,29] for relevant results. See [11] for characterization of the weighted Sobolev space $W^{1,p}_w$ using square functions.

Also, we consider discrete parameter versions of $T_\alpha$ and $U_\alpha$:

$$D_\alpha(f)(x) = \left( \sum_{k=-\infty}^{\infty} |I_\alpha(f)(x) - \Phi_2 \ast I_\alpha(f)(x)|^2 2^{-2k\alpha} \right)^{1/2}, \quad (5.6)$$

with $0 < \alpha < n$;

$$E_\alpha(f)(x) = \left( \sum_{k=-\infty}^{\infty} |f(x) - \Phi_2 \ast f(x)|^2 2^{-2k\alpha} \right)^{1/2}, \quad \alpha > 0, \quad (5.7)$$

where $\Phi \in M^\alpha$. If we put $\Phi = \chi_0$ in (5.7), we have $E_\alpha$ of (1.11). We have discrete parameter analogues of Theorem 5.1 and Corollary 5.2.

**Theorem 5.3.** Let $0 < \alpha < n$ and $1 < p < \infty$. Let $D_\alpha$ be as in (5.6). Then

$$\|D_\alpha(f)\|_{p,w} \approx \|f\|_{p,w}, \quad f \in S(\mathbb{R}^n),$$

where $w$ is any weight in $A_p$.

**Corollary 5.4.** Let $E_\alpha$ be as in (5.7). Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < n$. Then $f \in W^{\alpha,p}_w(\mathbb{R}^n)$ if and only if $f \in L^p_w$ and $E_\alpha(f) \in L^p_w$; also,

$$\|f\|_{p,\alpha,w} \approx \|f\|_{p,w} + \|E_\alpha(f)\|_{p,w}.$$  

A version of Theorem 5.1 for $0 < \alpha < 2$ and $n \geq 2$ is shown in [21], where $\Phi$ is assumed to be radial. Combining the arguments of [21] with Corollary 2.11, we can relax the assumption that $\Phi$ is radial.

Here we give proofs of Theorem 5.3 and Corollary 5.4; Theorem 5.1 and Corollary 5.2 can be shown similarly.

**Proof of Theorem 5.3.** Recall that $\tilde{L}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$, $0 < \alpha < n$, if $L_\alpha(x) = \tau(\alpha)|x|^{\alpha-n}$ with

$$\tau(\alpha) = \frac{\Gamma(n/2 - \alpha/2)}{\pi^{n/2}2^\alpha \Gamma(\alpha/2)}.$$  

Let

$$\psi(x) = L_\alpha(x) - \Phi \ast L_\alpha(x).$$
Then, we have \( D_\alpha(f) = \Delta_\psi(f), f \in S(\mathbb{R}^n) \), by homogeneity of \( L_\alpha \), where \( D_\alpha \) is as in (5.6). We observe that \( \psi \) can be written as
\[
\psi(x) = \int (L_\alpha(x) - L_\alpha(x - y)) \Phi(y) \, dy. \tag{5.8}
\]
Because \( \Phi \) is bounded and compactly supported and \( L_\alpha \) is locally integrable, we see that
\[
\sup_{|x| \leq 1} \left| \int L_\alpha(x - y) \Phi(y) \, dy \right| \leq C
\]
for some constant \( C \). Using this inequality in the definition of \( \psi \), we have
\[
|\psi(x)| \leq C|x|^{\alpha-n} \text{ for } |x| \leq 1. \tag{5.9}
\]
By applying Taylor’s formula and (5.4), we can easily deduce from (5.8) that
\[
|\psi(x)| \leq C|x|^{\alpha-n-\lfloor \alpha \rfloor - 1} \text{ for } |x| \geq 1. \tag{5.10}
\]
Taking the Fourier transform, we see that
\[
\hat{\psi}(\xi) = (2\pi|\xi|)^{-\alpha} \left( 1 - \hat{\Phi}(\xi) \right). \tag{5.11}
\]
By (5.4) this implies \( |\hat{\psi}(\xi)| \leq C|\xi|^{\alpha+1-\alpha} \), from which the condition (1.1) follows, since \( |\alpha| + 1 - \alpha > 0 \). It is easy to see that the conditions (1), (2) and (3) of Theorem 3.1 follow from the estimates (5.9), (5.10) and (5.11). Also, obviously we have \( \sup_{k \in \mathbb{Z}} |\hat{\psi}(2^k \xi)| > 0 \) for all \( \xi \neq 0 \). (This can be easily seen by noting that \( \hat{\Phi}(\xi) \to 0 \) as \( |\xi| \to \infty \).) Thus we can apply Corollary 3.7 to get the equivalence of the \( L^p_w \) norms claimed. \( \Box \)

**Proof of Corollary 5.4.** Riesz potentials and Bessel potentials are related as follows.

**Lemma 5.5.** Let \( \alpha > 0, 1 < p < \infty \) and \( w \in A_p \).

1. We have
\[
(2\pi|\xi|)^\alpha = \ell(\xi)(1 + 4\pi^2|\xi|^2)^{\alpha/2}
\]
with a Fourier multiplier \( \ell \) for \( L^p_w \).

2. There exists a Fourier multiplier \( m \) for \( L^p_w \) such that
\[
(1 + 4\pi^2|\xi|^2)^{\alpha/2} = m(\xi) + m(\xi)(2\pi|\xi|)^\alpha.
\]

To prove this we note that
\[
|\partial_\xi^\gamma \ell(\xi)| \leq C_\gamma |\xi|^{-|\gamma|}, \quad \xi \in \mathbb{R}^n \setminus \{0\},
\]
for all multi-indices \( \gamma \) and similar estimates for \( m(\xi) \). So, by a theorem on Fourier multipliers for \( L^p_w \) we can get the conclusion, as in the case of the estimate for \( ||M_k||_{L^p(w)} \) in the proof of Theorem 2.5. See also [23, Lemma 4].

When \( g \in L^p_w, w \in A_p, 1 < p < \infty \) and \( 0 < \alpha < n \), we show that
\[
||E_\alpha(J_\alpha(g))||_{p,w} + ||J_\alpha(g)||_{p,w} \simeq ||g||_{p,w}. \tag{5.12}
\]
We first prove (5.12) for \( g \in S_0(\mathbb{R}^n) \). Since \( E_\alpha(J_\alpha(g)) = D_\alpha(I^{-\alpha}J_\alpha(g)) \) and \( I^{-\alpha}J_\alpha(g) \in S(\mathbb{R}^n) \), when \( g \in S_0(\mathbb{R}^n) \), by Theorem 5.3 we have
\[
||E_\alpha(J_\alpha(g))||_{p,w} \simeq ||I^{-\alpha}J_\alpha(g)||_{p,w}, \tag{5.13}
\]
where $I_{-\alpha}$ is defined by (5.2) with $-\alpha$ in place of $\alpha$. Part (1) of Lemma 5.5 implies that
\[ \|I_{-\alpha}J_\alpha(g)\|_{p,w} \leq C\|g\|_{p,w} \]
and hence
\[ \|E_\alpha(J_\alpha(g))\|_{p,w} \leq C\|g\|_{p,w}. \] (5.14)

On the other hand, by part (2) of Lemma 5.5 and (5.13) we have
\[ \|g\|_{p,w} = \|J_{-\alpha}J_\alpha(g)\|_{p,w} \leq C\|J_\alpha(g)\|_{p,w} + C\|I_{-\alpha}J_\alpha(g)\|_{p,w} \]
\[ \leq C\|J_\alpha(g)\|_{p,w} + C\|E_\alpha(J_\alpha(g))\|_{p,w}, \] (5.15)
where we recall that the Bessel potential operator $J_\beta$ is defined on $S(\mathbb{R}^n)$ for any $\beta \in \mathbb{R}$ by $\hat{J_\beta}(f)(\xi) = (1 + 4\pi^2|\xi|^2)^{-\beta/2}\hat{f}(\xi)$. Also we have
\[ \|J_\alpha(g)\|_{p,w} \leq C\|M(g)\|_{p,w} \leq C\|g\|_{p,w}. \] (5.16)

Combining (5.14), (5.15) and (5.16), we have (5.12) for $g \in S_0(\mathbb{R}^n)$.

Now we show that (5.12) holds for any $g \in L_w^p$. For a positive integer $N$, let
\[ E_\alpha^{(N)}(f)(x) = \left( \sum_{k=-N}^{N} |f(x) - \Phi_{2k} \ast f(x)|^2 2^{-2k\alpha} \right)^{1/2}. \]
Then $E_\alpha^{(N)}(f) \leq C_N M(f)$, which implies that $E_\alpha^{(N)}$ is bounded on $L_w^p$. We can take a sequence $\{g_k\}$ in $S_0(\mathbb{R}^n)$ such that $g_k \to g$ in $L_w^p$ and $J_\alpha(g_k) \to J_\alpha(g)$ in $L_w^p$ as $k \to \infty$. By (5.12) for $S_0(\mathbb{R}^n)$ we see that
\[ \|E_\alpha^{(N)}(J_\alpha(g_k))\|_{p,w} \leq C\|g_k\|_{p,w}. \]

Letting $k \to \infty$, by $L_w^p$ boundedness and sublinearity of $E_\alpha^{(N)}$ we have
\[ \|E_\alpha^{(N)}(J_\alpha(g))\|_{p,w} \leq C\|g\|_{p,w}. \]
Thus, letting $N \to \infty$, we get
\[ \|E_\alpha(J_\alpha(g))\|_{p,w} \leq C\|g\|_{p,w}. \]

Therefore, we have
\[ \lim_{k \to \infty} \|E_\alpha(J_\alpha(g)) - E_\alpha(J_\alpha(g_k))\|_{p,w} \leq \lim_{k \to \infty} \|E_\alpha(J_\alpha(g - g_k))\|_{p,w} \]
\[ \leq C \lim_{k \to \infty} \|g - g_k\|_{p,w} = 0. \]
Consequently, letting $k \to \infty$ in the relation
\[ \|E_\alpha(J_\alpha(g_k))\|_{p,w} + \|J_\alpha(g_k)\|_{p,w} \simeq \|g_k\|_{p,w}, \]
which we have already proved, we can obtain (5.12) for any $g \in L_w^p$.

To complete the proof of Corollary 5.4, it thus only remains to show that $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if $f \in L_w^p$ and $E_\alpha(f) \in L_w^p$. To prove this it is convenient to note the following.

**Lemma 5.6.** Suppose that $f \in L_w^p$, $w \in A_p$, $1 < p < \infty$, $g \in S(\mathbb{R}^n)$ and $\alpha > 0$. Then we have the following.

1. $K_\alpha \ast (f \ast g)(x) = (K_\alpha \ast f) \ast g(x) = (K_\alpha \ast g) \ast f(x)$ for every $x \in \mathbb{R}^n$;
(2) \( \int_{\mathbb{R}^n} (K_\alpha * f)(y) g(y) \, dy = \int_{\mathbb{R}^n} (K_\alpha * g)(y) f(y) \, dy. \)

Proof. To prove part (1), by Fubini’s theorem it suffices to show that

\[
I = \iint K_\alpha(x - z - y) |f(y)||g(z)| \, dy \, dz < \infty.
\]

This is obvious, for

\[
I \leq C \int M(f)(x - z) |g(z)| \, dz = C \int M(f)(z) |g(x - z)| \, dz
\]

\[
\leq C \|M(f)\|_{p,w} \left( \int |g(x - z)|^{p'} w(z)^{-p'/p} \, dz \right)^{1/p'}
\]

\[
\leq C \|f\|_{p,w} \left( \int |g(x - z)|^{p'} w(z)^{-p'/p} \, dz \right)^{1/p'},
\]

where the last integral is finite since \( g \in \mathcal{S}(\mathbb{R}^n) \), \( w^{-p'/p} \in A_{p'} \) and \( \mathcal{S}(\mathbb{R}^n) \subset L^r_v \) for \( v \in A_r \), \( 1 < r < \infty \) (see [10, p. 412] for a related result).

Part (2) follows from part (1) by putting \( x = 0 \) since \( K_\alpha \) is radial. \( \square \)

Let \( f \in L^p_w \) and \( E_\alpha(f) \in L^p_w \). We take \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) with \( \int \varphi(x) \, dx = 1 \). Let \( f^{(\epsilon)}(x) = \varphi_\epsilon \ast f(x) \) and \( g^{(\epsilon)}(x) = J_{-\alpha}(\varphi_\epsilon) \ast f(x) \). Then, note that \( g^{(\epsilon)} \in L^p_w \) and \( f^{(\epsilon)} = J_{\alpha}(g^{(\epsilon)}) \) by part (1) of Lemma 5.6.

By (5.12) we have

\[
\|E_\alpha(f^{(\epsilon)})\|_{p,w} + \|f^{(\epsilon)}\|_{p,w} \lesssim \|g^{(\epsilon)}\|_{p,w}.
\]  

(5.17)

We note that

\[
\sup_{\epsilon > 0} \|f^{(\epsilon)}\|_{p,w} \leq C \|M(f)\|_{p,w} \leq C \|f\|_{p,w}.
\]  

(5.18)

Also, Minkowski’s inequality implies that

\[
E_\alpha(f^{(\epsilon)})(x) = \left( \sum_{k=-\infty}^{\infty} |\varphi_\epsilon \ast f(x) - \Phi_{2k} \ast \varphi_\epsilon \ast f(x)|^2 2^{-2k\alpha} \right)^{1/2}
\]

\[
\leq \int_{\mathbb{R}^n} |\varphi_\epsilon(y)| \left( \sum_{k=-\infty}^{\infty} |f(x - y) - \Phi_{2k} \ast f(x - y)|^2 2^{-2k\alpha} \right)^{1/2} \, dy
\]

\[
\leq CM(E_\alpha(f))(x).
\]

Thus

\[
\sup_{\epsilon > 0} \|E_\alpha(f^{(\epsilon)})\|_{p,w} \leq C \|M(E_\alpha(f))\|_{p,w} \leq C \|E_\alpha(f)\|_{p,w},
\]

which combined with (5.17) and (5.18) implies that \( \sup_{\epsilon > 0} \|g^{(\epsilon)}\|_{p,w} < \infty \).

Therefore we can choose a sequence \( \{g^{(\epsilon_k)}\} \), \( \epsilon_k \to 0 \), which converges weakly in \( L^p_w \). Let \( g^{(\epsilon_k)} \to g \) weakly in \( L^p_w \). Then, since \( \{f^{(\epsilon_k)}\} \) converges to \( f \) in \( L^p_w \), we can conclude that \( f = J_\alpha(g) \). To show this, let \( \Lambda_h(f) = \int f(x) h(x) \, dx \) for \( h \in \mathcal{S}(\mathbb{R}^n) \). Then it is easy to see that \( \Lambda_h \) is a bounded linear functional on \( L^p_w \) for every \( h \in \mathcal{S}(\mathbb{R}^n) \), since \( |\Lambda_h(f)| \leq \|f\|_{p,w} \|h\|_{p',w^{-p'/p}} \) by
Hölder’s inequality. Thus, for any \( h \in S(\mathbb{R}^n) \), applying part (2) of Lemma 5.6 and noting \( J_\alpha(h) \in S(\mathbb{R}^n) \), we have
\[
\int f(x) h(x) \, dx = \lim_k \int f^{(\epsilon_k)}(x) h(x) \, dx = \lim_k \int J_\alpha(g^{(\epsilon_k)})(x) h(x) \, dx
\]
\[
= \lim_k \int g^{(\epsilon_k)}(x) J_\alpha(h)(x) \, dx = \int g(x) J_\alpha(h)(x) \, dx
\]
\[
= \int J_\alpha(g)(x) h(x) \, dx.
\]
This implies that \( f = J_\alpha(g) \) and hence \( f \in W^{\alpha,p}_w(\mathbb{R}^n) \). This completes the proof of Corollary 5.4.

\[\square\]

Remark 5.7. Let \( \psi = \text{sgn} - \text{sgn} * \Phi \) on \( \mathbb{R} \), where \( \Phi \in \mathcal{M}^1 \). We note that \( \hat{\psi}(\xi) = -i\pi^{-1} \xi^{-1} (1 - \hat{\Phi}(\xi)) \). We have results analogous to Theorems 5.1 and 5.3 for \( g_\psi \) and \( \Delta_\psi \), respectively, with similar proofs. They can be applied to prove results generalizing Corollaries 5.2 and 5.4 to the case \( n = 1 \) and \( \alpha = 1 \) by arguments similar to those used for the corollaries.

6. Proof of (4.7)

In this section we give a proof of the estimate (4.7) for completeness. Put \( \psi = \varphi^{(\alpha)} \). To prove (4.7), assuming \(|y| < |x|/2\), we write
\[
L = \int_0^\infty |t^{-1} \psi((x - y)/t) - t^{-1} \psi(x/t)|^2 \frac{dt}{t}.
\]
We first assume \( x > 0 \) and \( y > 0 \). By the change of variables \( x/t = u \) we have
\[
L = x^{-2} \int_0^\infty |\psi(u - uy/x) - \psi(u)|^2 u \, du = I + II,
\]
where
\[
I = x^{-2} \int_0^1 |\psi(u - uy/x) - \psi(u)|^2 u \, du,
\]
\[
II = x^{-2} \int_1^\infty |\psi(u - uy/x) - \psi(u)|^2 u \, du.
\]
We estimate \( I \) and \( II \) separately. We see that
\[
II = x^{-2} \int_1^\infty |\psi(u - uy/x)|^2 u \, du
\]
\[
= \alpha^2 x^{-2} \int_1^{x/(x-y)} (1 - |u(1 - y/x)|)^{2(\alpha-1)} u \, du.
\]
Thus, by the change of variables \( w = u(x - y)/x \), we have
\[
II = \alpha^2 (x - y)^{-2} \int_0^1 (1 - w)^{2(\alpha-1)} w \, dw
\]
\[
\leq \alpha^2 (x - y)^{-2} \int_0^1 (1 - w)^{2(\alpha-1)} \, dw,
\]
which implies that
\[ II \leq \alpha^2(x - y)^{-2}(2\alpha - 1)^{-1}(y/x)^{2\alpha-1} \leq C\alpha y^{2\alpha-1}x^{-1-2\alpha}. \] (6.1)

To deal with $I$, we write
\[ I = \alpha^2x^{-2} \int_0^1 |(1 - u(1 - y/x))^{\alpha-1} - (1 - u)^{\alpha-1}|^2 u \, du = I_1 + I_2, \]
where
\[ I_1 = \alpha^2x^{-2} \int_0^{1-2y/x} |(1 - u(1 - y/x))^{\alpha-1} - (1 - u)^{\alpha-1}|^2 u \, du, \]
\[ I_2 = \alpha^2x^{-2} \int_{1-2y/x}^1 |(1 - u(1 - y/x))^{\alpha-1} - (1 - u)^{\alpha-1}|^2 u \, du. \]

We observe that
\[ \int_{1-2y/x}^1 (1 - u(1 - y/x))^{2(\alpha-1)} \, du = x(x - y)^{-1} \int_{(1-2y/x)(x-y)/x}^{(x-y)/x} (1 - w)^{2(\alpha-1)} \, dw \]
\[ \leq C\alpha \left( (1 - (1 - 2y/x)(x-y)/x)^{2\alpha-1} - (1 - (x-y)/x)^{2\alpha-1} \right) \]
\[ \leq C\alpha(y/x)^{2\alpha-1}. \] (6.2)

Also, we have
\[ \int_{1-2y/x}^1 (1 - u)^{2(\alpha-1)} \, du \leq C\alpha(y/x)^{2\alpha-1}. \] (6.3)

By (6.2) and (6.3) we see that
\[ I_2 \leq C\alpha x^{-2}(y/x)^{2\alpha-1} = C\alpha y^{2\alpha-1}x^{-1-2\alpha}. \] (6.4)

To estimate $I_1$ we recall that $1/2 < \alpha < 3/2$. By the mean value theorem,
\[ I_1 \leq Cx^{-2}(y/x)^2 \int_0^{1-2y/x} (1 - u)^{2(\alpha-2)} \, du \]
\[ \leq Cx^{-2}(y/x)^2(2y/x)^{2\alpha-3} = Cy^{2\alpha-1}x^{-2\alpha-1}. \] (6.5)

The estimate $I \leq C\alpha y^{2\alpha-1}x^{-1-2\alpha}$ follows from (6.4) and (6.5), which combined with (6.1) implies
\[ L \leq C\alpha y^{2\alpha-1}x^{-1-2\alpha}, \]
when $x > 0, y > 0$.

Next we deal with the case $x > 0, y < 0$. In this case we also consider the analogous decomposition $L = I + II$. Since $\psi$ is supported in $[-1,1]$ and $x > 0, y < 0$, we see that $II = 0$. Also, $I = I_1 + I_2$, where
\[ I_1 = \alpha^2x^{-2} \int_0^{1-2|y|/x} |(1 - u(1 - y/x))^{\alpha-1} - (1 - u)^{\alpha-1}|^2 u \, du, \]
\[ I_2 = x^{-2} \int_{1-2|y|/x}^1 |\psi(u(1 - y/x)) - \psi(u)|^2 u \, du. \]
To estimate $I_2$, we see that
\[
\int_{1-2|y|/x}^{1} |\psi(u(1 - y/x))|^2 u \, du
\leq \alpha^2 \int_{1-2|y|/x}^{x/(x-y)} |1 - u(1 - y/x)|^{2(\alpha-1)} \, du
\]
\[
= \alpha^2 x(x-y)^{-1} \int_{x-y(x+2y)/x^2}^{1} (1 - w)^{2(\alpha-1)} \, dw
\]
\[
= \alpha^2 x(x-y)^{-1}(2\alpha - 1)^{-1}(|y|/x + 2(y/x)^2)^{2\alpha - 1}
\leq C_\alpha |y/x|^{2\alpha - 1}.
\]
Similarly,
\[
\int_{1-2|y|/x}^{1} (1 - u)^{2(\alpha-1)} u \, du \leq C_\alpha |y/x|^{2\alpha - 1}.
\]
Thus
\[
I_2 \leq C|y|^{2\alpha - 1}x^{-2\alpha - 1}.
\] (6.6)

On the other hand, by the mean value theorem,
\[
I_1 \leq \alpha^2 x^{-2} \int_{0}^{1-2|y|/x} (|y|^{-1}|\alpha - 1|(1 - u(1 - y/x)))^{\alpha-2} \, du
\leq C y^2 x^{-4} x(x-y)^{-1} \int_{0}^{(x+2y)(x-y)/x^2} (1 - u)^{2(\alpha-2)} \, du
\]
\[
= C y^2 x^{-3} (x-y)^{-1}(3 - 2\alpha)^{-1} \left((|y|/x + 2(y/x)^2)^{2\alpha - 3} - 1\right)
\leq C_\alpha |y|^{2\alpha - 1}x^{-2\alpha - 1}.
\] (6.7)

The estimates (6.6), (6.7) imply that $I \leq C|y|^{2\alpha - 1}x^{-2\alpha - 1}$ for $x > 0, y < 0$.

Since $\psi$ is odd, we observe that
\[
L = \int_{0}^{\infty} \left|t^{-1}\psi((-x+y)/t) - t^{-1}\psi(-x/t)\right|^2 \frac{dt}{t}.
\]
Thus, the results for the cases $x < 0, y > 0$ and $x < 0, y < 0$ will follow from the results for the cases $x > 0, y < 0$ and $x > 0, y > 0$, respectively.

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