Functional Erdős-Rényi laws for the increments of Lévy processes

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Received: date / Accepted: date

Abstract

We establish functional forms of the Erdős-Rényi law of large numbers for the increments of Lévy processes. The proofs treat in parallel the cases of two different natural assumptions of finiteness of exponential moments and are based on functional large deviations principles.

Keywords Large deviations - Lévy processes - Erdős-Rényi laws.

1 Introduction

Let \((X_i)_{i \geq 1}\) be an i.i.d. sequence of centered \(\mathbb{R}\)-valued random variables. For any \(t \geq 1\), set \(S(t) := \sum_{i=1}^{\lfloor t \rfloor} X_i\) where \(\lfloor t \rfloor\) is the integer part of \(t\). By the Erdős-Rényi law of large numbers (see [12]), if \(\mathbb{E}[\exp(\theta X_1)] < \infty\) for all \(\theta \in \mathbb{R}\), then for \(A_n := \lfloor c \log n \rfloor\) with \(c > 0\) and \(n \in \mathbb{N}^*\), there exists a constant \(\alpha_c\) such that

\[
\max \left\{ \frac{S(j + A_n) - S(j)}{A_n} : j \in [0, n - A_n] \cap \mathbb{N} \right\} \xrightarrow{n \to \infty} \alpha_c \quad \text{a.s.} \quad (1.1)
\]

It follows from (1.1) that for \(a_T := c \log T\) with \(c > 0\) and \(T > 1\),

\[
\sup_{0 \leq x \leq T - a_T} \frac{S(x + a_T) - S(x)}{a_T} \xrightarrow{T \to \infty} \alpha_c \quad \text{a.s.} \quad (1.2)
\]

Let \(\{S(t) : t \geq 0\}\) be the partial sum process. For \(x \geq 0\), consider the whole path of \(S(\cdot)\) between \(x\) and \(x + a_T\), that is introduce the standardized increment function \(\gamma_{x, a_T}\) of \(S(\cdot)\) defined for \(s \in [0, 1]\) by

\[
\gamma_{x, a_T}(s) = \frac{S(x + sa_T) - S(x)}{a_T}.
\]

A functional Erdős-Rényi law [FERL] is a strong limit theorem, as \(T \to \infty\), for sets of functions of that form. If \(\mathbb{E}[\exp(\theta X_1)] < \infty\) for \(\theta\) in a neighborhood of 0, a FERL is obtained in [6], for the sets \((G_T)_{T>1}\) defined by

\[
G_T := \{\gamma_{x, a_T} : 0 \leq x \leq T - a_T\}.
\]

This result extends FERL’s proved in [3] and [20], under a more general assumption, and implies (1.1). Such a FERL is derived in [14], when \(X_1\) has a semiexponential distribution. More recently, in [10] and [16], FERL’s have been obtained for some processes which are variants of \(\{S(t) : t \geq 0\}\).

In the present paper, we establish FERL’s for sets of standardized increment functions of Lévy processes. Given a Lévy process \(\{Z(t) : t \geq 0\}\), setting \(A_n := \lfloor c \log n \rfloor\) with \(c > 0\) and \(n \in \mathbb{N}^*\), these functions are defined by

\[
\eta_{x, A_n}(s) = \frac{Z(x + sA_n) - Z(x)}{A_n}, \quad s \in [0, 1], \quad x \geq 0. \quad (1.3)
\]
We obtain FERL’s for the sets \((L_n)_{n>1}\) defined by
\[
L_n := \{\eta_{x,A_n} : 0 \leq x \leq n - A_n\},
\]
under each of assumptions \((C)\) and \((A)\) below, on the moment-generating function \([\text{mgf}]\ \Phi\) of \(Z(1)\), where \(\Phi(\theta) = E[\exp(\theta Z(1))] \in (0, \infty]\), for \(\theta \in \mathbb{R}\).

\((C)\) \(\Phi(\theta) < \infty\) for all \(\theta \in \mathbb{R}\).

\((A)\) \(\Phi(\theta) < \infty\) for \(\theta\) in a neighborhood of 0.

Our proofs are based on large deviations principles \([\text{LDP}]\) in functional spaces. Under \((C)\), in \([24]\), such LDP’s are proved in the Skorohod space \(D(0,1)\). If only \((A)\) holds, under additional conditions on the Lévy process, its sample paths have a.s. finite variations. In this case, such LDP’s have been obtained in \([17]\), in the space \(BV_0(0,1)\) of functions \(f \in D(0,1)\) of bounded variations on \([0,1]\) with \(f(0) = 0\), endowed with the weak topology. We emphasize on the similarity of structure of the proofs under each assumption \((C)\) and \((A)\), which extends that of the functional spaces \(D(0,1)\) and \(BV_0(0,1)\), itself pointed out at the end of Sect. 3. Thus, the main steps of these proofs are treated in parallel.

Instead of viewing our FERL’s as generalizations of \((1.1)\), one could consider them directly in the framework of functional limit theorems \([\text{FLT}]\)’s. In that setting, for any increments \((\alpha_n)_{n>1}\), we introduce the increment functions \(\nu_{x,\alpha_n}\) defined by
\[
\nu_{x,\alpha_n}(s) = Z(x + s\alpha_n) - Z(x), \quad s \in [0,1], \ x \geq 0.
\]
Assume that \(\frac{\alpha_n}{\log n} \to \delta \in (0, \infty)\) as \(n \to \infty\). We say that the increments \(\alpha_n\) are large (resp. intermediate) if \(\delta = \infty\) (resp. \(\delta < \infty\)). Given a real sequence \((\beta_n)_{n>1}\), we seek FLT’s for sets of rescaled increment functions \((M_n)_{n>1}\) of the form
\[
M_n := \{\beta_n^{-1} \nu_{x,\alpha_n} : 0 \leq x \leq n - \alpha_n\}, \quad \text{for } n > 1.
\]
For large increments, we obtain FLT’s for \((M_n)_{n>1}\), with a specific choice of \((\beta_n)_{n>1}\), by applying an invariance principle \([\text{IP}]\) which is derived from results of \([9]\). This approach is detailed in the Appendix (See Proposition 6). However, it is not possible to follow this method for intermediate increments (See Remark 2 in the Appendix). For such increments, our approach by LDP’s appears as an alternative to the IP one and our FERL’s are FLT’s for \((M_n)_{n>1}\), with \(\beta_n = \alpha_n = \lceil c \log n \rceil\), for any \(c > 0\).

The remainder of the present paper is organized as follows. In Sect. 2 and Sect. 3, we present general results on Lévy processes and functional spaces which will be needed in our proofs. Our main results are stated in Sect. 4, with proofs detailed in Sect. 5. Some technical results are deferred to the Appendix.

## 2 Lévy Processes

In this paper, a stochastic process is a family \(\{Z(t) : t \geq 0\}\) of \(\mathbb{R}\)-valued random variables defined on a common probability space \((\Omega, \mathcal{F}, P)\). For all \(t \geq 0\), the image of \(\omega \in \Omega\) by the random variable \(Z(t)\) is denoted by \(Z(t, \omega)\). For all \(\omega \in \Omega\), the sample path \(t \in [0, \infty) \mapsto Z(t, \omega)\) is denoted by \(Z(\cdot, \omega)\).

### 2.1 Construction of Lévy Processes

#### Definition 1.

For any interval \(I\), \(D(I)\) is the set of functions on \(I\) which are right-continuous with left-hand limits. On \(D([0,1])\), denoted by \(D(0,1)\), the Skohorod topology \(S\) is induced by the distance
\[
d_S(f,g) = \inf_{\nu \in \Lambda} \left\{ \max \{\|\nu - I\| ; \|f - g \circ \nu\|\} \right\},
\]
(2.6)
where \( \Lambda \) is the class of strictly increasing, continuous mappings of \([0,1]\) onto itself.

**Definition 2.** A Lévy process \( \{Z(t) : t \geq 0\} \) is a stochastic process such that:
(i) \( Z(0) = 0 \) a.s. (ii) The process has stationary and independent increments and is stochastically continuous. (iii) For all \( \omega \) in some set of probability 1, the sample path \( Z(\cdot, \omega) \) lies in \( D([0, \infty)) \).

**Definition 3.** For any interval \( I \), let \( \mathbb{R}^I \) be the set of all functions from \( I \) to \( \mathbb{R} \). Let \( C_I \) be the class of cylinder sets, that is sets of the form
\[
\bigcap_{i=1}^n \left\{ f \in \mathbb{R}^I : f(t_i) \in B_i \right\},
\]
for \( n \geq 1, (t_i)_{1 \leq i \leq n} \in I^n \) and \( (B_i)_{1 \leq i \leq n} \in (\mathcal{B}_\mathbb{R})^n \), where \( \mathcal{B}_\mathbb{R} \) is the Borel \( \sigma \)-algebra of \( \mathbb{R} \).

Let \( \Omega = \mathbb{R}^{[0, \infty)} \). Let \( \mathcal{F} \) be the \( \sigma \)-algebra on \( \Omega \) generated by \( C_{[0, \infty)} \).

**Theorem 1.** For any Lévy process \( \{Z(t) : t \geq 0\} \), the distribution \( P_{Z(1)} \) of \( Z(1) \) is an infinitely divisible distribution [IDD]. Conversely, given an IDD \( \mu \) on \( \mathbb{R} \), there exists a unique probability measure \( P \) on \((\Omega, \mathcal{F})\) and a Lévy process \( \{Z(t) : t \geq 0\} \) on \((\Omega, \mathcal{F}, P)\) such that \( P_{Z(1)} = \mu \) and for all \( \omega \in \Omega \), the path \( Z(\cdot, \omega) \) lies in \( D([0, \infty)) \).

**Proof.** For all \( t \geq 0 \), let \( X(t) \) be the measurable map from \((\Omega, \mathcal{F})\) to \((\mathbb{R}, \mathcal{B}_\mathbb{R})\) defined by \( X(t, \omega) = \omega(t) \). Then, by Section 11 of [21], there exists a unique probability measure \( P \) on \((\Omega, \mathcal{F})\) such that \( \{X(t) : t \geq 0\} \) is a Lévy process with \( P_{X(1)} = \mu \). Now, it follows from Chapter 14 of [4] that one can define on \((\Omega, \mathcal{F}, P)\) a process \( \{Z(t) : t \geq 0\} \) such that (2.7) holds and for all \( t \geq 0 \), \( P(Z(t) = X(t)) = 1 \). Therefore, \( \{Z(t) : t \geq 0\} \) is also a Lévy process with \( P_{Z(1)} = \mu \).

Throughout the sequel, we will assume that all stochastic processes are defined on \((\Omega, \mathcal{F}, P)\) and that for all Lévy process \( \{Z(t) : t \geq 0\} \),
\[
\forall \omega \in \Omega, \ Z(\cdot, \omega) \in D([0, \infty)).
\] (2.7)

### 2.2 Generating triplet

By the Lévy-Khintchine formula, there is a one-to-one correspondence between IDD’s and triplets \((A, \nu, \gamma)\) where \( A \geq 0, \nu \) is a measure on \( \mathbb{R} \) such that \( \nu\{\{0\}\} = 0 \) and \( \int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty \), and \( \gamma \in \mathbb{R} \). Then, by Theorem 1, a Lévy process \( \{Z(t) : t \geq 0\} \) is associated to a triplet \((A, \nu, \gamma)\), called its generating triplet. \( A \) is the variance of the Gaussian component of the process and \( \nu \) is its Lévy measure.

**Proposition 1.** Let \( \{Z(t) : t \geq 0\} \) be a Lévy process with generating triplet \((A, \nu, \gamma)\), associated to an IDD \( \mu \). Assume that \((A)\) holds. Then, for all \( t \geq 0 \),
\[
\mathbb{E}[Z(t)] = t\mathbb{E}[Z(1)].
\] (2.8)

**Proof.** Jensen’s inequality implies that for all \( \theta > 0 \),
\[
\exp\left(\theta \mathbb{E}[\|Z(1)\|]\right) \leq \mathbb{E}\left[\exp(\theta|Z(1)|)\right] \leq \mathbb{E}\left[\exp(\theta Z(1))\right] + \mathbb{E}\left[\exp(-\theta Z(1))\right].
\]
Therefore, under \((A)\), \( \mathbb{E}[\|Z(1)\|] < \infty \). We deduce from Theorem 25.3 in [21] that \( \int_{|x| \geq 1} |x| \nu(dx) < \infty \), which implies in turn that for all \( t \geq 0 \), \( \mathbb{E}[\|Z(t)\|] < \infty \). Now, for any \( t \geq 0 \), the characteristic function of \( P_{Z(t)} \) is \( \hat{\mu}^t \), where \( \hat{\mu} \) is the characteristic function of \( \mu = P_{Z(1)} \). Since \( \mathbb{E}[\|Z(1)\|] < \infty \) and \( \mathbb{E}[|Z(t)|] < \infty \),
\[
\mathbb{E}[Z(t)] = \left. \frac{1}{i} \frac{\partial (\hat{\mu}(z))^t}{\partial z} \right|_{z=0} = t \left. \frac{1}{i} \frac{\partial \hat{\mu}(z)}{\partial z} \right|_{z=0} = t\mathbb{E}[Z(1)].
\]
Theorem 2. Let \( \{Z(t) : t \geq 0\} \) be a Lévy process with generating triplet \((A, \nu, \gamma)\). Assume that \( A = 0 \) and \( \int_{|x| \leq 1} |x| \nu(dx) < \infty \). Then, a.s., the sample path \( Z(\cdot, \omega) \) has finite variation on \((0,t]\), for any \( t \in (0, \infty) \).

**Proof.** See Theorem 21.9 in [21]. \[ \square \]

### 2.3 Increment functions

Let \( x \geq 0 \) and \( \lambda > 0 \). For \( \omega \in \Omega \), the increment function \( \eta_{x,\lambda}(\cdot, \omega) \) is defined on \([0,1]\) by

\[
\eta_{x,\lambda}(s, \omega) = \frac{Z(x+s\lambda, \omega) - Z(x, \omega)}{\lambda}, \quad s \in [0,1].
\]

#### 2.3.1 Measurability issues

**Lemma 1.** The Borel \( \sigma \)-algebra \( \mathcal{B}_S \) of \((D(0,1), \mathcal{S})\) is generated by the class of sets \( \mathcal{C}^D_{[0,1]} := D(0,1) \cap \mathcal{C}_{[0,1]} \).

**Proof.** See Theorem 12.5 in [2]. \[ \square \]

**Lemma 2.** Let \( x \geq 0 \) and \( \lambda > 0 \). Then, for any \( B \in \mathcal{B}_S \),

\[
\{ \omega \in \Omega : \eta_{x,\lambda}(\cdot, \omega) \in B \} \in \mathcal{F}. \tag{2.9}
\]

**Proof.** By Lemma 1, it is enough to prove that for any \( \Gamma \in \mathcal{C}^D_{[0,1]} \),

\[
\{ \omega \in \Omega : Z(x + \cdot, \lambda, \omega) \in \Gamma \} \in \mathcal{F} \quad \text{and} \quad \{ \omega \in \Omega : Z(x, \omega) \in \Gamma \} \in \mathcal{F}.
\]

Set \( F := \{ \omega \in \Omega : Z(x + \cdot, \lambda, \omega) \in \Gamma \} \). It follows from the definition of \( \mathcal{C}^D_{[0,1]} \) that we can write \( F = F_1 \cap F_2 \), with

\[
F_1 = \{ \omega \in \Omega : Z(x + \cdot, \lambda, \omega) \in D(0,1) \} \quad \text{and} \quad F_2 = \bigcap_{i=1}^n \{ \omega \in \Omega : Z(x + t_i \lambda, \omega) \in B_i \},
\]

where \( n \geq 1 \), \((t_i)_{1 \leq i \leq n} \in [0,1]^n \) and \((B_i)_{1 \leq i \leq n} \in (\mathcal{B}_R)^n \). Then, (2.7) implies that \( F_1 = \Omega \), while clearly, \( F_2 \in \mathcal{F} \). So, \( F \in \mathcal{F} \). We conclude the proof since, on the other hand,

\[
\{ \omega \in \Omega : Z(x, \omega) \in \Gamma \} = \bigcap_{i=1}^n \{ \omega \in \Omega : Z(x, \omega) \in B_i \} \in \mathcal{F}.
\]

\[ \square \]

**Lemma 3.** Let \( x \geq 0 \). For any \( u > 0 \),

\[
F_{x,u} := \left\{ \sup_{0 \leq t \leq 1} |Z(x + t) - Z(x)| > u \right\} \in \mathcal{F} \tag{2.10}
\]

**Proof.** By (2.7), for all \( \omega \in \Omega \), \( Z(x + \cdot, \omega) - Z(x) \) is right-continuous. So,

\[
F_{x,u} = \left\{ (Z(x + \cdot) - Z(x)) \in \bigcup_{k \in \mathbb{N}} \{ f \in D(0,1) : |f(\tau_k)| > u \} \right\},
\]

where \( \{ \tau_k : k \in \mathbb{N} \} := [0,1] \cap \mathbb{Q} \). We deduce as in the proof of Lemma 2 that (2.10) holds, since

\[
\bigcup_{k \in \mathbb{N}} \{ f \in D(0,1) : |f(\tau_k)| > u \} \in \sigma \left( \mathcal{C}^D_{[0,1]} \right) = \mathcal{B}_S.
\]

\[ \square \]
2.3.2 Invariance

**Lemma 4.** Let $x \geq 0$ and $\lambda > 0$. Then, for any $B \in \mathcal{B}_S$,

$$P(\eta_{x,\lambda} \in B) = P(\eta_{0,\lambda} \in B) = P\left(\frac{Z(\cdot \lambda)}{\lambda} \in B\right).$$  \hspace{1cm} (2.11)

**Proof.** Proposition 10.7. in [21] implies that for all $B \in \mathcal{C}_{[0, \infty)}$,

$$P(Z(x + \lambda) - Z(x) \in B) = P(Z(\cdot \lambda) \in B).$$  \hspace{1cm} (2.12)

So, (2.12) holds for all $B \in \mathcal{C}_{[0, 1]} \subset \mathcal{C}_{[0, \infty)}$ and so for all $B \in \sigma\left(\mathcal{C}_{[0,1]}\right) = \mathcal{B}_S$. \hfill \square

**Lemma 5.** Let $x \geq 0$. For any $u > 0$,

$$P\left(\sup_{0 \leq \tau \leq 1} |Z(x + \tau) - Z(x)| > u\right) = P\left(\sup_{0 \leq \tau \leq 1} |Z(\tau)| > u\right).$$  \hspace{1cm} (2.13)

**Proof.** We deduce (2.13) from the proofs of Lemmas 3 and 4. \hfill \square

3 Functional spaces

**Proposition 2.** For all $x \geq 0$ and $\lambda > 0$,

$$\forall \omega \in \Omega, \, \eta_{x,\lambda}(\cdot, \omega) \in D(0, 1).$$

**Proof.** This follows from Theorem 1. \hfill \square

**Definition 4.** Let $BV_0(0, 1)$ be the set of functions $f$ on $[0, 1]$ such that

(i) $f(0) = 0$.

(ii) $f$ is of bounded variations on $[0, 1]$.

(iii) $f$ is right-continuous on $[0, 1]$.

**Proposition 3.** Under the assumptions of Theorem 2, for all $x \geq 0$ and $\lambda > 0$,

$$\eta_{x,\lambda}(\cdot, \omega) \in BV_0(0, 1) \; \text{a.s.}$$

**Proof.** Since $BV_0(0, 1) \subset D(0, 1)$, this follows from Theorem 2 and Proposition 2. \hfill \square

In this section, we study the properties of the functional spaces in which the increment functions lie. Let $\mathcal{E}$ be a functional space and $d$ a distance defined on $\mathcal{E}$. For $f \in \mathcal{E}$ and $\epsilon > 0$, let $B^\mathcal{E}_d(f, \epsilon)$ be the following open ball:

$$B^\mathcal{E}_d(f, \epsilon) := \{g \in \mathcal{E} : d(f, g) < \epsilon\}.$$  

Let $C(0, 1)$ be the set of continuous functions on $[0, 1]$, on which we define the uniform distance defined by

$$d_U(f, g) = \sup_{s \in [0, 1]} |f(s) - g(s)|.$$  

We will often replace the distance $d$ by a notation for the topology induced by $d$. Sometimes, we will omit the reference to the interval $[0, 1]$ for the functional space. For example, $B^{D(0,1)}_{d_U}$ will be denoted by $B^{D(0,1)}_{d_U}$. 

5
3.1 Uniform topology on $D(0,1)$

On $D(0,1)$, we present the relationships between the Skorohod and the uniform topology $\mathcal{U}$, induced by $d_\mathcal{U}$. We denote by $\mathcal{U}_0$ be the $\sigma$-algebra generated by the open balls of $(D(0,1),\mathcal{U})$.

**Lemma 6.** (i) On $D(0,1)$, $\mathcal{U}$ is stronger than $\mathcal{S}$.
(ii) The restriction of $\mathcal{S}$ to $C(0,1)$ coincides there with $\mathcal{U}$.
(iii) $\mathcal{U}_0 = \mathcal{B}_\mathcal{S}$.

**Proof.** In [2], see Section 12 for (i), (ii) and Section 15 for (iii). \hfill \square

**Lemma 7.** Let $K$ be a compact subset of $(C(0,1),\mathcal{U})$ and $\epsilon > 0$. Then, there exists $\zeta > 0$ such that for all $g \in K$,

$$B^\mathcal{U}_\mathcal{U}(g, \epsilon) \supseteq B^\mathcal{S}_\mathcal{S}(g, \zeta).$$

(3.14)

Therefore,

$$K^{\epsilon,\mathcal{U}} = \bigcup_{g \in K} B^\mathcal{U}_\mathcal{U}(g, \epsilon) \supseteq \bigcup_{g \in K} B^\mathcal{S}_\mathcal{S}(g, \zeta) = K^{\zeta,\mathcal{S}}.$$  

(3.15)

**Proof.** For any $f \in C(0,1)$, let $\omega_f$ be the modulus of continuity of $f$, defined for $\delta > 0$ by

$$\omega_f(\delta) = \sup \{|f(x) - f(y)| : |x - y| \leq \delta\}.$$  

By the Arzelà-Ascoli theorem, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\sup \{\omega_g(\delta_\epsilon) : g \in K\} < \frac{\epsilon}{2}. \quad (3.16)$$

Let $g \in K$ and $\epsilon > 0$. Set $\zeta := \min \{\delta_\epsilon; \frac{\epsilon}{2}\}$. By (2.6), for all $h \in B^\mathcal{S}_\mathcal{S}(g, \zeta)$, there exists $\nu_h \in \Lambda$ satisfying

$$\|\nu_h - I\| < \zeta \leq \delta_\epsilon \quad \text{and} \quad \|h - g \circ \nu_h\| < \zeta \leq \frac{\epsilon}{2}. \quad (3.17)$$

The first part of (3.17) combined to (3.16) imply that $\|g \circ \nu_h - g\| \leq w_g(\|\nu_h - I\|) < \frac{\epsilon}{2}$, which combined to the second part of (3.17) implies that

$$\|h - g\| \leq \|h - g \circ \nu_h\| + \|g \circ \nu_h - g\| < \epsilon.$$  

Therefore, $h \in B^\mathcal{U}_\mathcal{U}(g, \epsilon)$. \hfill \square

3.2 The space $BV_0(0,1)$

**Definition 5.** Let $\mathfrak{M}_f(0,1)$ be the space of totally bounded signed measures on $[0,1]$.

**Theorem 3.** For any $\mu \in \mathfrak{M}_f(0,1)$, let $F_\mu$ be its distribution function, defined by

$$F_\mu(x) = \mu([0,x]), \quad x \in [0,1].$$

Then, $F_\mu \in BV_0(0,1)$ and the map $\mathcal{D} : \mu \mapsto F_\mu$ is a bijection between $\mathfrak{M}_f(0,1)$ and $BV_0(0,1)$. Moreover, the inverse function $\mathcal{D}^{-1}$ is defined by $\mathcal{D}^{-1}(F) = dF$.

**Proof.** See (4.3) of chapter 0 in [19]. \hfill \square
3.2.1 The weak topology on $BV_0(0,1)$

**Definition 6.** Let $C(0,1)^*$ be the topological dual space of $(C(0,1), \mathcal{U})$, that is the space of continuous linear functionals from $(C(0,1), \mathcal{U})$ to $\mathbb{R}$. The weak-* topology is the coarsest topology on $C(0,1)^*$ such that for all $\phi \in C(0,1)$, the map $T_\phi$ from $C(0,1)^*$ to $\mathbb{R}$ defined by $T_\phi(\Lambda) = \langle \Lambda, \phi \rangle$ remains continuous.

For any $F \in BV_0(0,1)$, let $\Lambda_F$ be the element of $C(0,1)^*$ such that for all $\phi \in C(0,1)$, $\langle \Lambda_F, \phi \rangle$ is the following Riemann-Stieltjes integral:

$$\langle \Lambda_F, \phi \rangle = \int_0^1 \phi(x) dF(x).$$

**Theorem 4.** The map $\Xi : F \mapsto \Lambda_F$ is a bijection from $BV_0(0,1)$ to $C(0,1)^*$. 

*Proof.* By the Riesz representation theorem, for any $\mu \in \mathfrak{M}_f(0,1)$, there exists a unique $\Lambda(\mu) \in C(0,1)^*$ such that for all $\phi \in C(0,1)$,

$$\langle \Lambda(\mu), \phi \rangle = \int_0^1 \phi(x)(dx).$$

In other words, the map $\mathfrak{M} : \mu \mapsto \Lambda(\mu)$ is a bijection between $\mathfrak{M}_f(0,1)$ and $C(0,1)^*$. Then, $\Xi$ is a bijection, since $\Xi = \mathfrak{M} \circ \mathcal{D}^{-1}$. 

**Definition 7.** Consider the bijection $\Xi : BV_0(0,1) \rightarrow C(0,1)^*$ and endow $C(0,1)^*$ with the weak-* topology. Then, the weak topology $\mathcal{W}$ on $BV_0(0,1)$ is the inverse image topology, so that $\Xi$ defines a homeomorphism.

**Corollary 1.** A net $(f_\alpha)$ in $BV_0(0,1)$ convergent to $\hat{f}$ in $(BV_0(0,1), \mathcal{W})$ if and only if

$$\forall \phi \in C(0,1), \int \phi(x) df_\alpha(x) \rightarrow \int \phi(x) d\hat{f}(x).$$

3.2.2 The space $(BV_{0,M}(0,1), \mathcal{W})$

**Definition 8.** For $f \in BV_0(0,1)$, let $|f|_v$ be its total variation. For $M > 0$, let

$$BV_{0,M}(0,1) := \{ f \in BV_0(0,1) : |f|_v \leq M \}.$$

Recall that $(BV_0(0,1), \mathcal{W})$ is not metrizable (see Remark 1.2. in [7]). However, it follows from the Banach-Alaoglu theorem that for any $M > 0$, the restriction of $\mathcal{W}$ to $BV_{0,M}(0,1)$ is metrizable. By the Corollary below, the following distance $d_\mathcal{H}$ on $BV_0(0,1)$ provides such a metric.

$$d_\mathcal{H}(f,g) = \int_0^1 |f(u) - g(u)| du + |f(1) - g(1)|. \tag{3.18}$$

We will denote by $\mathcal{H}$ the topology induced by $d_\mathcal{H}$ on any set on which $d_\mathcal{H}$ is defined.

**Lemma 8.** A net $(f_\alpha)$ in $BV_0(0,1)$ is weakly convergent to $\hat{f} \in BV_0(0,1)$ if and only if both following conditions hold.
1) There exists a positive constant $M < \infty$ such that $f_\alpha$ is ultimately in $BV_{0,M}(0,1)$.
2) $d_\mathcal{H} \left(f_\alpha, \hat{f}\right) \rightarrow 0$.

*Proof.* See [15]. 

**Corollary 2.** For any $M > 0$, $d_\mathcal{H}$ metrizes $(BV_{0,M}(0,1), \mathcal{W})$.

**Lemma 9.** For any $M > 0$, $BV_{0,M}(0,1)$ is a compact subset of $(BV_0(0,1), \mathcal{W})$.

*Proof.* See Proposition 1.4. in [7].
3.2.3 Comparison of $\mathcal{W}$ and $\mathcal{H}$

**Lemma 10.** Let $g \in BV_0(0,1)$. Then, for all $\xi > 0$, the ball

$$
B_{\mathcal{H}}^{BV}(g, \xi) := \{f \in BV_0(0,1) : d_{\mathcal{H}}(f, g) < \xi\}
$$

is an open subset of $(BV_0(0,1), \mathcal{W})$.

**Proof.** It is enough to prove that for all $g \in BV_0(0,1)$, the map $\theta_g$ from $(BV_0(0,1), \mathcal{W})$ to $[0, \infty)$ defined by $\theta_g(f) = d_{\mathcal{H}}(f, g)$ is continuous. Let $(f_\alpha)$ be a net in $BV_0(0,1)$ weakly converging to $\hat{f}$ in $BV_0(0,1)$. Then, by Lemma 8,

$$
\left| d_{\mathcal{H}}(f_\alpha, g) - d_{\mathcal{H}}(\hat{f}, g) \right| \leq d_{\mathcal{H}}(f_\alpha, \hat{f}) \to 0.
$$

Therefore, the net $(\theta_g(f_\alpha))$ converges to $\theta_g(\hat{f}) \in \mathbb{R}$, and so $\theta_g$ is continuous. \hfill \Box

3.2.4 Comparison of $\mathcal{W}$ and $\mathcal{S}$

**Lemma 11.** (i) Let $M > 0$. Then, on $BV_{0,M}(0,1)$, $\mathcal{S}$ is stronger than $\mathcal{H}$.

(ii) For any $M > 0$, $BV_{0,M}(0,1)$ is a closed subset of $(D(0,1), \mathcal{S})$.

**Proof.** Let $(f_n)_{n \geq 1}$ be a sequence of functions in $D(0,1)$ and $f \in D(0,1)$. Then,

$$
d_{\mathcal{S}}(f_n, f)_{n \to \infty} \to 0 \implies d_{\mathcal{H}}(f_n, f)_{n \to \infty} \to 0,
$$

which can be found in [6]. This proves (i). For (ii), let $(f_n)_{n \geq 1}$ be any sequence of elements of $BV_{0,M}(0,1)$ which is $\mathcal{S}$-convergent to $f \in D(0,1)$. First, Lemma 9 implies that there exists a subsequence $(f_{\phi(n)})_{n \geq 1}$ which converges weakly to some $\tilde{f} \in BV_{0,M}(0,1)$. By Lemma 8, $d_{\mathcal{H}}(f_{\phi(n)}, \tilde{f})_{n \to \infty} \to 0$.

On the other hand, it follows from (3.19) that $d_{\mathcal{H}}(f_{\phi(n)}, f)_{n \to \infty} \to 0$. So, $\tilde{f} = f \in BV_{0,M}(0,1)$, which proves (ii). \hfill \Box

**Lemma 12.** Let $B_{\mathcal{W}}$ be the Borel $\sigma$-algebra of $(BV_0(0,1), \mathcal{W})$. Then,

$$
B_{\mathcal{W}} \subset B_{\mathcal{S}}
$$

**Proof.** It is enough to prove that for any closed subset $\Gamma$ of $(BV_0(0,1), \mathcal{W})$ and for all $M \in \mathbb{N}$, $\Gamma_M := \Gamma \cap BV_{0,M}(0,1) \in B_{\mathcal{S}}$. Now, for all $M \in \mathbb{N}$, $\Gamma_M$ is a closed subset of $(BV_{0,M}(0,1), \mathcal{W})$. Combining Corollary 2 and Lemma 11 (i), we deduce that $\Gamma_M$ is a closed subset of $(BV_{0,M}(0,1), \mathcal{S})$ and so of $(D(0,1), \mathcal{S})$, by Lemma 11 (ii). \hfill \Box

**Corollary 3.** Let $\mathcal{H}_0$ be the $\sigma$-algebra generated by the open balls of $(BV_0(0,1), \mathcal{H})$. Then,

$$
\mathcal{H}_0 \subset B_{\mathcal{W}} \subset B_{\mathcal{S}}.
$$

**Proof.** This follows from Lemmas 10 and 12. \hfill \Box

3.3 Functional large deviations

**Definition 9.** Let $\mathcal{E}$ be a topological space, endowed with a topology $\mathcal{T}$ and its Borel $\sigma$-algebra, denoted by $B_{\mathcal{T}}$. A function $\mathcal{I} : \mathcal{E} \to [0, \infty)$ is a rate function if $\mathcal{I}$ is lower semicontinuous. Moreover, we say that $\mathcal{I}$ is a good rate function if for all $\alpha < \infty$, the sublevel set $\mathcal{K}_\alpha := \{f \in \mathcal{E} : \mathcal{I}(f) \leq \alpha\}$ is compact.
Definition 10. A family of probability measures $(P_\lambda)_{\lambda \geq 0}$ on $(\mathcal{E}, \mathcal{B}_T)$ satisfies a large deviations principle [LDP] in $(\mathcal{E}, T)$, with rate function $I$ when, for any closed (resp. open) subset $F$ (resp. $G$) of $T$,

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log P_\lambda(F) \leq -I(F)$$

and

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log P_\lambda(G) \geq -I(G),$$

where for any non-empty subset $A$ of $\mathcal{E}$, $I(A) := \inf_{f \in A} I(f)$. Then, $(3.21)$ and $(3.22)$ are called respectively the upper bound and the lower bound of the LDP.

For $\lambda > 0$, let $P_\lambda$ be the distribution of $Z_\lambda(\cdot)$, where for $s \in [0, 1]$,

$$Z_\lambda(s) := \frac{1}{\lambda} Z(\lambda s).$$

Then, we state hereunder the LDP results for the distributions $(P_\lambda)_{\lambda > 0}$, on which our proofs rely. The rate functions depend on the Legendre transform $\Psi$ of $\Phi$ defined by

$$\Psi(\alpha) := \sup_{\{\theta: \Phi(\theta) < \infty\}} \{\alpha \theta - \log \Phi(\theta)\}, \text{ for } \alpha \in \mathbb{R}.$$ 

Theorem 5. Assume that $(C)$ holds. Then, the distributions $(P_\lambda)_{\lambda > 0}$ satisfy a LDP in $(D(0,1), S)$, with good rate function $I$ defined on $D(0,1)$ by

$$I(f) = \begin{cases} \int_0^1 \Psi \left( \frac{df}{d\theta} (s) \right) \, ds & \text{if } f \in AC(0,1) \text{ and } f(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. See [24].

When only $(A)$ holds, we need to introduce the following notations. For $f \in BV_0(0,1)$, write $f = f_+ - f_-$, where $df = df_+ - df_-$ is the Hahn-Jordan decomposition of $df$. For $g \in BV_0(0,1)$, write $g = g^A + g^S$, where $dg = dg^A + dg^S$ is the Lebesgue decomposition of $dg$ into an absolutely continuous and a singular component. Then, define the function $J$ on $BV_0(0,1)$ by

$$J(f) = \int_0^1 \Psi \left( \frac{df^A}{ds} (s) \right) \, ds + \theta_0 f^S_+(1) - \theta_1 f^S_-(1),$$

where $\theta_0 := \sup \{\theta : \Phi(\theta) < \infty\} > 0$ and $\theta_1 := \inf \{\theta : \Phi(\theta) < \infty\} < 0$.

Theorem 6. Let $\{Z(t) : t \geq 0\}$ be a Lévy process with generating triplet $(A, \nu, \gamma)$. Assume that $A = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$, so that for all $\lambda > 0$, the sample paths $Z_\lambda(\cdot, \omega)$ lie a.s. in $BV_0(0,1)$. Then, under $(A)$, the distributions $(P_\lambda)_{\lambda > 0}$ satisfy a LDP in $(BV_0(0,1), \mathcal{W})$, with good rate function $J$. 

Proof. See [17].
3.4 Summary

We summarize in the following array the features which are common to the functional spaces \((D(0, 1), S)\) and \((BV_0(0, 1), W)\). They are described in points (1)-(6) below, which follow from the preceding results. Throughout the sequel, we will denote by \(E\) one or the other of these spaces of functions on \([0, 1]\).

| \((E, \mathcal{T})\) | \(d\) | \(E\) | \(\mathcal{I}\) |
|---------------------|-------|-------|-------|
| \((D(0, 1), S)\)    | \(d_U\) | \(C(0, 1)\) | \(I\) |
| \((BV_0(0, 1), W)\) | \(d_H\) | \(BV_0, W(0, 1)\) | \(J\) |

(1) For all \(x \geq 0\) and \(\lambda > 0\), \(\eta_{x, \lambda} \in E\) a.s.
(2) \(\mathcal{T}\) is a topology on \(E\) such that \((E, \mathcal{T})\) is a topological vector space.
(3) \(d\) is a distance on \(E\), derived from a norm.
(4) \(E\) is a convex subset of \(E\) such that the restriction of \(\mathcal{T}\) to \(E\) is induced by \(d\).
(5) \(\mathcal{I}\) is a convex good rate function on \(E\).
(6) The distributions \((P_\lambda)_{\lambda > 0}\) defined by (3.23) satisfy a LDP in \((E, \mathcal{T})\) with rate function \(\mathcal{I}\).

For any subset \(A \subset E\) and \(\epsilon > 0\), set

\[
A^{\epsilon,d} := \bigcup_{f \in A} B^E_d(f, \epsilon). \tag{3.25}
\]

In the sequel, for all \(\alpha > 0\), \(K_\alpha\) denotes either a sublevel set associated to \(I\) or \(J\). Thus, we set generically

\[
K_\alpha := \{f \in E : \mathcal{I}(f) \leq \alpha\}.
\]

**Lemma 13.** For all \(\alpha > 0\), \(K_\alpha \subset E\).

**Proof.** If \(E = D(0, 1)\), this follows from the definition of \(I\). If \(E = BV_0(0, 1)\), this is (3.9) in [6]. \(\Box\)

**Lemma 14.** For all \(\alpha > 0\), \(K_\alpha\) is a compact subset of \((E, d)\).

**Proof.** Let \(V\) be an open subset of \((E, d)\). Then, \(V \cap E\) is an open subset of \((E, d) = (E, \mathcal{T}_E)\), by (4). So, \(V \cap E = U \cap E\), where \(U\) is an open subset of \((E, \mathcal{T})\). We deduce readily that a compact subset of \((E, \mathcal{T})\) contained in \(E\) is a compact subset of \((E, d)\). Now, (5) and Lemma 13 imply that for all \(\alpha > 0\), \(K_\alpha\) is a compact subset of \((E, \mathcal{T})\) contained in \(E\), which concludes the proof. \(\Box\)

**Lemma 15.** For all \(\alpha > 0\) and \(\epsilon > 0\),

\[
(K_\alpha)^{\epsilon,d} \in \mathcal{B}_S. \tag{3.26}
\]

**Proof.** By Lemma 14, \(K_\alpha\) is separable in \((E, d)\). Then, Lemma 1 in Section 6 of [2] implies that \((K_\alpha)^{\epsilon,d} \in \mathcal{D}_0\), where \(\mathcal{D}_0\) is the \(\sigma\)-algebra generated by the open balls of \((E, d)\). If \((E, d) = (D(0, 1), d_U)\), then \(\mathcal{D}_0 = \mathcal{B}_S\), by Lemma 6 (iii). If \((E, d) = (BV_0(0, 1), d_H)\), then \(\mathcal{D}_0 = \mathcal{B}_S\), by Corollary 3. This proves (3.26) in both cases. \(\Box\)

4 Functional Erdős-Rényi laws

Let \(c > 0\). For \(n > 1\), set

\[
A_n := \lfloor c \log n \rfloor,
\]

and for \(G \subset [0, \infty)\), set

\[
\mathcal{L}_n^G := \{\eta_{x,A_n} : x \in [0, n - A_n] \cap G\}.
\]
If $G = [0, \infty)$, we will write indifferently $L_n^G$ or $L_n$. Throughout the sequel, $G$ will be a subset of $[0, \infty)$. For any sequence $(E_n)_{n>1}$ of subsets of $\Omega$, set

$$
\{E_n \text{ ult.}\} := \bigcup_{n>1} \bigcap_{k \geq n} E_n \quad \text{and} \quad \{E_n \text{ i.o.}\} := \bigcap_{n>1} \bigcup_{k \geq n} E_n.
$$

where ultimately (resp. infinitely often) has been abbreviated as ult. (resp. i.o.).

4.1 Measurability issues

Lemma 16. Assume that $G$ is at most countable. Then, for all $\alpha > 0$ and $\epsilon > 0$,

$$
\mathcal{M}_\epsilon^G := \left\{ L_n^G \subset (K_\alpha)^{\epsilon \text{dul.}} \right\} \in \mathcal{F}.
$$

Proof. By Lemmas 2 and 15, for all $\epsilon > 0$, $\alpha > 0$ and $x \geq 0$,

$$
\left\{ \omega \in \Omega : \eta_{x,A_n}((\cdot),\omega) \in (K_\alpha)^{\epsilon \text{d}} \right\} \in \mathcal{F}.
$$

Now, $\mathcal{M}_\epsilon^G$ is an at most countable intersection of such sets. Therefore, $\mathcal{M}_\epsilon^G \in \mathcal{F}$. □

Lemma 17. Assume that $G$ is at most countable. Then, for all $g \in K_\alpha$ and $\epsilon > 0$,

$$
\left\{ g \in \left( L_n^G \right)^{\epsilon \text{d}} \cup \text{ult.} \right\} \in \mathcal{F}.
$$

Proof. Set $I_n^G := [0, n - A_n] \cap G$. For all $g \in K_\alpha$, $\epsilon > 0$ and $n > 1$,

$$
\left\{ g \in \left( L_n^G \right)^{\epsilon \text{d}} \right\} = \bigcup_{x \in I_n^G} \{ \eta_{x,A_n} \in \mathcal{B}_d^\epsilon(g, \epsilon) \}.
$$

We deduce from Lemma 2 that it is enough to prove that $\mathcal{B}_d^\epsilon(g, \epsilon) \in \mathcal{B}_S$. Then, Lemma 6 (iii) implies that $\mathcal{B}_d^\epsilon(g, \epsilon) \in \mathcal{U}_0 = \mathcal{B}_S$ and it follows from Corollary 3 that $\mathcal{B}_d^\epsilon(g, \epsilon) \in \mathcal{B}_S$. □

4.2 Preliminary results

We make the choice not to consider the completion of the probability space $(\Omega, \mathcal{F}, P)$, in order to avoid confusions. We will say that an event $M \subset \Omega$ (not necessarily in $\mathcal{F}$) happens almost surely (a.s.) if $M \supset M'$ with $M' \in \mathcal{F}$ and $P(M') = 1$.

Definition 11. We say that the FERL holds for $(L_n^G)_{n>1}$ in $(\mathcal{E}, d)$ with limit set $K_\alpha$ if, for all $\epsilon > 0$, a.s.

$$
L_n^G \subset (K_\alpha)^{\epsilon \text{dul.}} \quad \text{(4.28)}
$$

and

$$
K_\alpha \subset \left( L_n^G \right)^{\epsilon \text{dul.}} \quad \text{(4.29)}
$$

Hereabove, (4.28) and (4.29) will be called respectively the upper and lower bound for $L_n^G$. The reason is that their proofs rely respectively on the upper and the lower bound in functional LDP’s.
Proposition 4. For $\mu \in \mathbb{R}$, let $\{Z^\mu(t) : t \geq 0\}$ be the Lévy process defined by
$$Z^\mu(t) = Z(t) + \mu t, \quad \text{for } t \geq 0.$$  
For $G \subset [0, \infty)$ and $n > 1$, set $(L^G_n)^\mu := \left\{ \eta^\mu_{x,A_n} : x \in [0, n - A_n] \cap G \right\}$, where
$$\eta^\mu_{x,A_n}(s) = \frac{Z^\mu(x + A_n s) - Z^\mu(x)}{A_n}, \quad s \in [0, 1].$$
If the FERL holds for $(L^G_n)^\mu_{n>1}$ with limit set $K_\alpha := \{ f \in \mathcal{E} : I(f) \leq \alpha \}$, then it holds for $(L^G_n)^\mu_{n>1}$ with limit set $(K_\alpha)^\mu := \{ f \in \mathcal{E} : I(f) \leq \alpha \}$. 

Proof. See Appendix. 

Corollary 4. Assume that (A) holds. Then, in order to establish the FERL, one can assume that $\{Z(t) : t \geq 0\}$ is centered, which means that for all $t \geq 0$, $E[Z(t)] = 0$. 

Proof. Under (A), by Proposition 1, for all $t \geq 0$, $E[Z(t)] = tE[Z(1)]$. For $t \geq 0$, set $\overline{Z}(t) := Z(t) - tE[Z(1)]$. If the FERL holds for the centered Lévy process $\{\overline{Z}(t) : t \geq 0\}$, then Proposition 4, applied with $\mu = E[Z(1)]$, implies that it holds for $\{Z(t) : t \geq 0\}$. 

Proposition 5. Assume that the FERL holds for $(L^G_n)^\mu_{n>1}$ in $(\mathcal{E}, d)$, with limit set $K_\alpha$. Then, for any continuous map $\Theta : (\mathcal{E}, d) \rightarrow \mathbb{R}$,
$$\sup \left\{ \Theta(f) : f \in L^G_n \right\} \xrightarrow{n \rightarrow \infty} \sup \left\{ \Theta(f) : f \in K_\alpha \right\} \quad \text{a.s.} \quad (4.30)$$

Proof. See Appendix. 

4.3 Main results

Theorem 7. Assume that (C) holds. Then, for any $c > 0$, the FERL holds for $(L_n)^\mu_{n>1}$ in $(D(0,1), d_U)$, with limit set
$$K_{1/c} := \{ f \in D(0,1) : I(f) \leq 1/c \}, \quad \text{where } I \text{ is defined in Theorem 5.}$$

Proof. Since (C) implies (A), by Corollary 4, one can assume that $\{Z(t) : t \geq 0\}$ is centered. Then, Lemma 24 (resp. 26) provides the upper (resp. lower) bound. 

Theorem 8. Let $\{Z(t) : t \geq 0\}$ be a Lévy process with generating triplet $(A, \nu, \gamma)$ such that $A = 0$ and $\int_{|x| \leq 1} |x|^2 \nu(dx) \leq \infty$. Assume that (A) holds. Then, for any $c > 0$, the FERL holds for $(L^G_n)^\mu_{n>1}$ in $(BV_0(0,1), d_H)$, with limit set
$$L_{1/c} := \{ f \in BV_0(0,1) : J(f) \leq 1/c \}, \quad \text{where } J \text{ is defined in Theorem 6.}$$

Proof. Lemma 21 provides the upper bound. By Corollary 4, one can assume that $\{Z(t) : t \geq 0\}$ is centered. So, Lemma 27 gives the lower bound. 

Corollary 5. Let $c > 0$. Under (C), for any continuous map $\Theta : (D(0,1), U) \rightarrow \mathbb{R}$,
$$\sup \left\{ \Theta(\eta_{x,A_n}) : x \in [0, n - A_n] \right\} \xrightarrow{n \rightarrow \infty} \sup \left\{ \Theta(f) : f \in K_{1/c} \right\} \quad \text{a.s.}$$

In particular, since $\Theta : f \mapsto f(1)$ is such a map,
$$\sup_{0 \leq x \leq n - A_n} \frac{Z(x + A_n) - Z(x)}{A_n} \xrightarrow{n \rightarrow \infty} \sup \left\{ f(1) : f \in K_{1/c} \right\} \quad \text{a.s.}$$

Under the assumptions of Theorem 8, for a continuous map $\Theta : (BV_0(0,1), d_H) \rightarrow \mathbb{R}$,
$$\sup \left\{ \Theta(\eta_{x,A_n}) : x \in [0, n - A_n] \cap \mathbb{N} \right\} \xrightarrow{n \rightarrow \infty} \sup \left\{ \Theta(f) : f \in K_{1/c} \right\} \quad \text{a.s.}$$

Proof. This follows from Proposition 5, and Theorems 7 and 8.
4.4 Examples

4.4.1 Continuous paths

Let \( \{Z(t) : t \geq 0\} \) be a Lévy process with continuous paths, that is a brownian motion with drift. So, Theorem 7 yields the FERL for \( \{Z(t) : t \geq 0\} \), since it satisfies (C).

4.4.2 Compound Poisson process

Let \( \{Y_i : i \geq 1\} \) be a sequence of i.i.d. random variables. Let \( \{N(t) : t \geq 0\} \) be a homogeneous, right-continuous Poisson process of parameter \( \lambda \), which is assumed to be independent of \( \{Y_i : i \geq 1\} \). For any \( t \geq 0 \), set

\[
S_N(t) = \sum_{1 \leq i \leq N(t)} Y_i.
\]  

(4.31)

The compound Poisson process \( \{S_N(t) : t \geq 0\} \) is a Lévy process with generating triplet such that

\[
\int_{|x| \leq 1} |x| \nu(dx) \leq \nu(\mathbb{R}) = \lambda < \infty \text{ and } A = 0.
\]

The mgf \( \Phi \) is such that for \( \theta \in \mathbb{R} \), \( \Phi(\theta) = \exp[\lambda(\Phi_{Y_1}(\theta) - 1)] \), where \( \Phi_{Y_1} \) is the mgf of \( Y_1 \). Therefore, if \( \Phi_{Y_1}(\theta) < \infty \) for all \( \theta \) in \( \mathbb{R} \) (resp. in a neighborhood of 0) then (C) (resp. (A)) holds.

4.5 Discussion and open problems

4.5.1 Upper bound for \( \mathcal{L}_n \) under (A)

We obtain the FERL for \( \mathcal{L}_n^a \) under (A) in Theorem 8, which contains the lower bound for \( \mathcal{L}_n \). Since the tail of \( Z(1) \) is heavier than under (C), the fluctuations of the increments are more often wide, so that it is more difficult to derive the upper bound for the whole \( \mathcal{L}_n \). By Corollary 6 in Sect. 5, it would be enough to prove that for all \( \epsilon > 0 \),

\[
\left\{ \Delta^H_n(\epsilon) \text{ i.o. in } j \right\} \in \mathcal{N},
\]  

(4.32)

where for all \( n > 1 \), \( \Delta^H_n(\epsilon) := \bigcup_{x \in [0,n-A_n]} \{d_H(\eta_{x,A_n}, \eta_{[x],A_n}) \geq \epsilon \} \). As illustrated by Remark 1 in Sect. 5, the proof of (4.32) is still an open question. However, Theorem 9 below, proved in [13], is a manifestation of the FERL for \( \mathcal{L}_n \) under (A).

Theorem 9. Assume that (A) holds. For any \( c \) large enough, set \( a_T := c\log(T) \). Then, there exists a constant \( \beta_c \) such that

\[
\sup_{0 \leq x \leq T - a_T} \frac{Z(x + a_T) - Z(x)}{a_T} \rightarrow \beta_c \quad \text{a.s.}
\]  

(4.33)

4.5.2 Rate of convergence

For the result of [12], an almost sure central limit theorem has been obtained in [8], yielding a rate of convergence of order \( \frac{\log(k_n)}{k_n} \) in (1.1). For functional versions, it is natural to study the rate of convergence, which is called rate of clustering in the framework of FLT’s (see [11]). For our FERL’s, such a rate would be provided by a sequence \( (\epsilon_n)_{n \geq 1} \) converging to 0 such that, with the notations of Definition 11,

\[
\left\{ \omega \in \Omega : \mathcal{L}_n^G \subset \mathcal{K} \text{ ult.} \right\} \supset \mathcal{M}(\epsilon_n), \text{ with } P(\mathcal{M}(\epsilon_n)) = 1.
\]

This is an open question, included for the FERL’s for partial sums recalled in the Introduction.
5 Proof of main results

Throughout the sequel, we will denote by $\mathcal{N}$ the class of negligible sets, that is

$$\mathcal{N} = \{ N \subset \Omega : \exists N' \in \mathcal{F} \text{ such that } N \subset N' \text{ and } P(N') = 0 \}.$$

Let $n > 1$. Let $\mathcal{D}$ (resp. $\mathcal{D}_n$) be the set of all maps from $[0, \infty)$ (resp. $[0, n-A_n]$) to $D(0,1)$. Given a subset $E$ of $\mathcal{D}$, $\mathcal{D}_n \cap E$ is viewed as the set of restrictions to $[0, n-A_n]$ of the elements of $E$. For $\omega \in \Omega$, $\mathcal{L}_n(\omega) := \{ \eta_x,A_n(\cdot,\omega) : 0 \leq x \leq n-A_n \}$ is identified with the map of $\mathcal{D}$ defined by $x \mapsto \eta_x,A_n(\cdot,\omega)$.

For any integer $j > 1$, set $n_j := \max \{ n : A_n = j \}$, so that

$$\exp\left( \frac{j}{c} \right) \leq n_j < \exp\left( \frac{j+1}{c} \right).$$

**Lemma 18.** Let $E \subset \mathcal{D}$. Assume that one of the following conditions holds.

(i) $\left\{ \mathcal{L}^G_{n_j} \in \mathcal{D}_{n_j} \cap E \text{ ult. in } j \right\} \in \mathcal{N}^c$.

(ii) For all $j > 1$, $\left\{ \mathcal{L}^G_{n_j} \in \mathcal{D}_{n_j} \cap E \right\} \in \mathcal{F}$ and $\sum_{j>1} P\left( \mathcal{L}^G_{n_j} \notin \mathcal{D}_{n_j} \cap E \right) < \infty$.

Then, $\left\{ \mathcal{L}^G_n \in \mathcal{D}_n \cap E \text{ ult.} \right\} \subset \mathcal{N}^c$.

**Proof.** By the Borel-Cantelli lemma, (ii) implies (i). Now, assume that (i) holds. For $j > 1$, by definition of $n_j$, if $n_j < n \leq n_{j+1}$, then $A_n = A_{n_{j+1}} = j + 1$. Therefore, for all $x \in [0, n-A_n] \subset [0, n_{j+1} - A_{n_{j+1}} ]$, we have that $\eta_x,A_n = \eta_x,A_{n_{j+1}}$. So, for such $n$, we have that $\mathcal{L}_n \subset \mathcal{L}_{n_{j+1}}$. So, $\left\{ \mathcal{L}^G_n \in \mathcal{D}_n \cap E \text{ ult. } \right\} \subset \left\{ \mathcal{L}^G_{n_j} \in \mathcal{D}_{n_j} \cap E \text{ ult. in } j \right\}$. \hfill $\square$

5.1 Upper bounds for $\mathcal{L}^N_n$

5.1.1 A general result on rate functions

**Lemma 19.** Let $(\mathcal{E},d)$ be a metric space. Let $\mathcal{I}$ be a good rate function on $(\mathcal{E},d)$, so that for all $\alpha > 0$, $\mathcal{K}_\alpha := \{ f \in \mathcal{E} : \mathcal{I}(f) \leq \alpha \}$ is compact. Then, for all $\epsilon > 0$,

$$\mathcal{I}_{\alpha,\epsilon} := \inf \left\{ \mathcal{I}(x) : x \notin (\mathcal{K}_{\alpha})^\epsilon d \right\} > \alpha.$$

**Proof.** By definition of $\mathcal{K}_\alpha$, $\mathcal{I}_{\alpha,\epsilon} \geq \alpha$. Suppose that $\mathcal{I}_{\alpha,\epsilon} = \alpha$. Then, there exists a sequence $(x_n)$ such that, for all $n \geq 1$, $x_n \notin (\mathcal{K}_{\alpha})^\epsilon d$ and $\mathcal{I}(x_n) \leq \alpha$. So there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathcal{I}(x_n) \leq \alpha + 1$, so that $x_n \in \mathcal{K}_{\alpha+1}$ which is a compact set. Hence there exists a subsequence $(x_{nk})_{k \geq 1}$ which converges to $\ell \in \mathcal{K}_{\alpha+1}$, as $k \to \infty$. Since $\mathcal{I}$ is lower semicontinuous, $\mathcal{I}(\ell) \leq \lim_{k \to \infty} \mathcal{I}(x_{nk}) = \alpha$, so that $\ell \in \mathcal{K}_\alpha$. However, recall that for all $n \geq 1$, $x_n \notin (\mathcal{K}_{\alpha})^\epsilon d$ which implies that for all $k \geq 1$, $d(x_{nk}, \ell) \geq \epsilon$. This leads to a contradiction. \hfill $\square$

5.1.2 Upper bound for $\mathcal{L}^N_n$ under $(\mathcal{C})$

**Lemma 20.** Assume that $(\mathcal{C})$ holds. Then, for any $\epsilon > 0$,

$$P\left( \mathcal{L}^N_n \subset (K_{1/c})^{\epsilon d} \text{ ult. } \right) = 1.$$

**Proof.** Fix $\epsilon > 0$. First, by Lemma 16, $\left\{ \mathcal{L}^N_n \subset (K_{1/c})^{\epsilon d} \text{ ult. } \right\} \in \mathcal{F}$. Since for all $c > 0$, $K_{1/c}$ is a compact subset of $(C(0,1),\mathcal{U})$, Lemma 7 implies that there exists $\zeta > 0$ such that $$(K_{1/c})^{\epsilon d} \supset (K_{1/c})^{\zeta S}.$$
Let $F_{\zeta}^S$ be the complement in $D(0,1)$ of $(K_{1/c})^{c\mathcal{S}}$. Then, for all $n$ large enough,
\[
P \left( \mathcal{L}_n^N \not\subseteq (K_{1/c})^{c\mathcal{H}} \right) \leq P \left( \mathcal{L}_n^N \not\subseteq (K_{1/c})^{c\mathcal{S}} \right)
\leq \sum_{m=0}^{n-A_n} P \left( \eta_{m,A_n} \in F_{\zeta}^S \right)
= (n-A_n+1) P \left( \frac{Z(A_n)}{A_n} \in F_{\zeta}^S \right).
\]
The equality hereabove is justified by Lemma 4. Since (C) holds, we can apply Theorem 5. $F_{\zeta}^S$ being a closed subset of $(D(0,1),\mathcal{S})$, for any $\theta > 0$, we have for all $n$ large enough,
\[
P \left( \frac{Z(A_n)}{A_n} \in F_{\zeta}^S \right) \leq \exp \left[ A_n \left( -I(F_{\zeta}^S) + \theta \right) \right]. \quad (5.34)
\]
Since $I$ is a good rate function, we can apply Lemma 19 with $(\mathcal{E},d) = (D(0,1),\mathcal{S})$. Therefore, we have that $I(F_{\zeta}^S) = \frac{1}{c} + \delta$ with $\delta > 0$. So applying (5.34) with $\theta = \frac{\delta}{4}$, we have for all $n$ large enough,
\[
P \left( \mathcal{L}_n^N \not\subseteq (K_{1/c})^{c\mathcal{H}} \right) \leq n \exp \left[ A_n \left( -\frac{1}{c} - \frac{3\delta}{4} \right) \right].
\]
Applying this inequality with $n = n_j$, so that $A_n = j$, we obtain that
\[
P \left( \mathcal{L}_{n_j}^N \not\subseteq (K_{1/c})^{c\mathcal{H}} \right) \leq n_j \exp \left[ -j \left( \frac{1}{c} + \frac{3\delta}{4} \right) \right]
< \exp \left( \frac{j+1}{c} \right) \exp \left[ -j \left( \frac{1}{c} + \frac{3\delta}{4} \right) \right]
= \exp \left( \frac{1}{c} - j \frac{3\delta}{4} \right).
\]
Set $E := \{ f \in \mathcal{D} : \forall x \in \mathbb{N}, f(x) \in (K_{1/c})^{c\mathcal{H}} \}$. We conclude by Lemma 18, since
\[
\sum_{j>1} P \left( \mathcal{L}_{n_j}^N \not\subseteq \mathcal{D}_{n_j} \cap E \right) = \sum_{j>1} P \left( \mathcal{L}_{n_j}^N \not\subseteq (K_{1/c})^{c\mathcal{H}} \right) < \infty.
\]
\[\square\]

5.1.3 Upper bound for $\mathcal{L}_n^N$ under (A)

Lemma 21. Let $\{Z(t) : t \geq 0\}$ be a Lévy process with generating triplet $(A,\nu,\gamma)$ such that $A = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$. Assume that (A) holds. Then, for any $\epsilon > 0$,
\[
P \left( \mathcal{L}_n^N \subseteq (L_{1/c})^{c\mathcal{H}} \text{ ult.} \right) = 1.
\]

Proof. Fix $\epsilon > 0$. First, Lemma 16 implies that $\left\{ \mathcal{L}_n^N \subseteq (L_{1/c})^{c\mathcal{H}} \text{ ult.} \right\} \in \mathcal{F}$. Then, set $\Gamma_{\epsilon} := \left[ (L_{1/c})^{c\mathcal{H}} \right]^{\epsilon}$. By Lemma 15, $\Gamma_{\epsilon} \in \mathcal{B}_S$. So, by Lemma 4, for any $n > 1$,
\[
P \left( \mathcal{L}_n^N \not\subseteq (L_{1/c})^{c\mathcal{H}} \right) \leq \sum_{m=0}^{n-A_n} P \left( \eta_{m,A_n} \in \Gamma_{\epsilon} \right)
= (n-A_n+1) P \left( \frac{Z(A_n)}{A_n} \in \Gamma_{\epsilon} \right).
\]
By Lemma 10, \( \Gamma_\epsilon \) is closed in of \((BV_0(0,1), W)\). By Theorem 6, for any \( \theta > 0 \), for all \( n \) large enough,

\[
P \left( \frac{Z(\cdot, A_n)}{A_n} \in \Gamma_\epsilon \right) \leq \exp \left[ A_n \left( -J(\Gamma_\epsilon) + \theta \right) \right].
\]

By Lemma 19 with \((\mathcal{E}, d) = (BV_0(0,1), d_H), J(\Gamma_\epsilon) = \frac{1}{\epsilon^2} + \delta \) for some \( \delta > 0 \). We conclude as in the proof of Lemma 20.

\[ \square \]

5.2 Upper bounds for \( \mathcal{L}_n \)

5.2.1 A general result

Lemma 22. For \( \epsilon > 0 \), and \( n > 1 \), set

\[
\Lambda_n^d(\epsilon) := \left\{ \mathcal{L}_n \not\subset (\mathcal{K}_\alpha)^{\epsilon,d} \right\} \cap \left\{ \mathcal{L}_n^N \subset (\mathcal{K}_\alpha)^{\epsilon/2,d} \right\}.
\]

Then, for all \( n > 1 \),

\[
\Lambda_n^d(\epsilon) \subset \bigcup_{x \in [0,n-A_n]} \left\{ d(\eta_{x,A_n}, \eta_{[x],A_n}) \geq \epsilon/2 \right\}. \tag{5.35}
\]

Proof. Let \( \epsilon > 0 \). Since \((\mathcal{K}_\alpha)^{\epsilon/2,d} \subset (\mathcal{K}_\alpha)^{\epsilon,d}\), we have that for all \( n > 1 \),

\[
\Lambda_n^d(\epsilon) \subset \bigcup_{x \in [0,n-n-A_n] \cap N^c} \left\{ \eta_{x,A_n} \not\in (\mathcal{K}_\alpha)^{\epsilon,d} \right\}. \tag{5.36}
\]

So, by taking the intersection with \( \left\{ \mathcal{L}_n^N \subset (\mathcal{K}_\alpha)^{\epsilon/2,d} \right\} \) in both sides of (5.36),

\[
\Lambda_n^d(\epsilon) \subset \bigcup_{x \in [0,n-n-A_n] \cap N^c} \left\{ \eta_{x,A_n} \not\in (\mathcal{K}_\alpha)^{\epsilon,d} \right\} \cap \left\{ \mathcal{L}_n^N \subset (\mathcal{K}_\alpha)^{\epsilon/2,d} \right\}
\subset \bigcup_{x \in [0,n-n-A_n] \cap N^c} \left\{ \eta_{x,A_n} \not\in (\mathcal{K}_\alpha)^{\epsilon,d} \right\} \cap \left\{ \eta_{[x],A_n} \in (\mathcal{K}_\alpha)^{\epsilon/2,d} \right\}
\subset \bigcup_{x \in [0,n-n-A_n]} \left\{ d(\eta_{x,A_n}, \eta_{[x],A_n}) \geq \epsilon/2 \right\}.
\]

Indeed, if \( \eta_{[x],A_n} \in (\mathcal{K}_\alpha)^{\epsilon/2,d} \), then there exists \( g \in \mathcal{K}_\alpha \) such that \( d(\eta_{[x],A_n}, g) < \frac{\epsilon}{2} \). If simultaneously \( \eta_{x,A_n} \not\in (\mathcal{K}_\alpha)^{\epsilon,d} \), then \( d(\eta_{x,A_n}, g) \geq \epsilon \). By the triangle inequality, \( d(\eta_{x,A_n}, \eta_{[x],A_n}) \geq \frac{\epsilon}{2} \). Therefore, (5.35) holds.

\[ \square \]

Corollary 6. We keep the assumptions of Lemma 22. For all \( u > 0 \) and \( n > 1 \), set

\[
\Delta_n^d(u) := \bigcup_{x \in [0,n-n-A_n]} \left\{ d(\eta_{x,A_n}, \eta_{[x],A_n}) \geq u \right\} \tag{5.37}
\]

Suppose that for all \( \epsilon > 0 \),

\[
\left\{ \Delta_n^d(\epsilon) \ i.o. \ in \ j \right\} \in \mathcal{N}. \tag{5.38}
\]

Then, the upper bound for \( \left( \mathcal{L}_n^N \right)_{n>1} \) in \((\mathcal{E}, d)\) implies it for \( \left( \mathcal{L}_n \right)_{n>1} \).
Proof. Let $\epsilon > 0$. Set $E := \{f \in D : \exists x \in [0, \infty), \ d(f(x), f([x])) \geq \epsilon/2\}$. Then, for all $j > 1$, $\Delta_{nj}(\epsilon/2) = \{L_{nj} \in D_{nj} \cap E\}$. So, by Lemma 18 and (5.38),

$$\left\{ \Delta_{nj}(\epsilon/2) \text{ i.o.} \right\} \in N. \quad (5.39)$$

By Lemma 22, for all $n > 1$, $\Lambda_{nj}(\epsilon) \subset \Delta_{nj}(\epsilon/2)$. Now, we deduce from the upper bound for $(L_{nj})_{n>1}$ that there exists $N \in N$ such that

$$\left\{ L_n \not\subseteq (K_n)^{\text{cd}} \text{ i.o.} \right\} = \left\{ \left\{ L_n \not\subseteq (K_n)^{\text{cd}} \text{ i.o.} \right\} \cap \left\{ \sum_{n=1}^{N} L_{nj} \subset (K_n)^{\text{cd}} \text{ ult.} \right\} \right\} \cup N$$

$$\subseteq \left\{ \Lambda_{nj}(\epsilon) \text{ i.o.} \right\} \cup N$$

$$\subseteq \left\{ \Delta_{nj}(\epsilon/2) \text{ i.o.} \right\} \cup N.$$

This, combined with (5.39), implies that $\left\{ L_n \subset (K_n)^{\text{cd}} \text{ ult.} \right\} \in N^c$.  \hfill \Box

5.2.2 Upper bound for $L_n$ under $\mathcal{C}$

Lemma 23. Let $n > 1$. Then, for all $u > 0$,

$$\Delta_n(u) \subset \bigcup_{i=0}^{n+1} \left\{ \sup_{0 \leq \tau \leq 1} |Z(i+\tau) - Z(i)| > \frac{uA_n}{9} \right\}. \quad (5.40)$$

Proof. By the triangle inequality,

$$\Delta_n(u) \subset \bigcup_{x \in [0,n-A_n]} \left\{ \|Z(x \cdot A_n) - Z([x] \cdot A_n)\|_{\mathcal{U}} + \|Z(x) - Z([x])\|_{\mathcal{U}} \geq uA_n \right\}$$

$$\subset \bigcup_{x \in [0,n-A_n]} \left\{ \|Z(x \cdot A_n) - Z([x] \cdot A_n)\|_{\mathcal{U}} > \frac{uA_n}{3} \right\} \bigcup \left\{ |Z(x) - Z([x])| > \frac{uA_n}{3} \right\}$$

$$= \bigcup_{x \in [0,n-A_n]} \left[ \bigcup_{s \in [0,1]} \left\{ |Z(x+sA_n) - Z([x] + sA_n)| > \frac{uA_n}{3} \right\} \right]$$

$$\subset \bigcup_{y \in [0,n]} \left[ \bigcup_{a \in [0,1]} \left\{ |Z(y+a) - Z(y)| > \frac{uA_n}{3} \right\} \right].$$

Now, for any $y \in [0,n]$ and $a \in [0,1]$, two cases occur.

First case: If $y + a \leq y + 1$, then

$$|Z(y+a) - Z(y)| \leq |Z(y+a) - Z([y])| + |Z(y) - Z([y])| \leq 2 \sup_{0 \leq \tau \leq 1} |Z(y + \tau) - Z([y])|.$$

Second case: If $y + a > y + 1$, then $0 < (y + a) - (y + 1) < (y + 1) - y = 1$, so that

$$|Z(y+a) - Z(y)| \leq |Z(y+a) - Z([y] + 1)| + |Z([y] + 1) - Z([y])| + |Z(y) - Z([y])| \leq \sup_{0 \leq \tau \leq 1} |Z([y] + 1 + \tau) - Z([y] + 1)| + 2 \sup_{0 \leq \tau \leq 1} |Z([y] + \tau) - Z([y])|. $$
So, (5.40) holds, since we obtain from both cases that
\[
\sup_{y \in [0,n]} \left\{ \sup_{0 \leq a \leq 1} |Z(y + a) - Z(y)| \right\} \leq 3 \max_{0 \leq i \leq n+1} \left\{ \sup_{0 \leq \tau \leq 1} |Z(i + \tau) - Z(i)| \right\}.
\]
\(\square\)

**Lemma 24.** Assume that \((C)\) holds and that \(Z\) is centered. Then, for all \(\epsilon > 0\),
\[
\{ L_n \subset (K_{1/c}) \in \mathcal{N} \} \in \mathcal{N}^c.
\]
Proof. For all \(n > 1\) and \(u > 0\), \(\bigcup_{i=0}^{n+1} \left\{ \sup_{0 \leq \tau \leq 1} |Z(i + \tau) - Z(i)| > \frac{uA_n}{9} \right\} \in \mathcal{F}\), which follows from Lemma 3. Set
\[
\pi_n(u) := P\left( \bigcup_{i=0}^{n+1} \left\{ \sup_{0 \leq \tau \leq 1} |Z(i + \tau) - Z(i)| > \frac{uA_n}{9} \right\} \right).
\]
Fix \(\epsilon > 0\). By Lemma 5, for all \(j\) large enough,
\[
\pi_{n_j}(\epsilon) \leq \sum_{i=0}^{n_j+1} P\left( \sup_{0 \leq \tau \leq 1} |Z(i + \tau) - Z(i)| > \frac{j\epsilon}{9} \right) \leq 2n_j P\left( \sup_{0 \leq \tau \leq 1} |Z(\tau)| > \frac{j\epsilon}{9} \right).
\]
Now, for all \(\theta > 0\) and \(v > 0\),
\[
P\left( \sup_{0 \leq \tau \leq 1} |Z(\tau)| > v \right) \leq P\left( \sup_{0 \leq \tau \leq 1} \exp(\theta Z(\tau)) > \exp(\theta v) \right) + P\left( \sup_{0 \leq \tau \leq 1} \exp(-\theta Z(\tau)) > \exp(\theta v) \right).
\]
Since \(\{Z(t) : t \geq 0\}\) is centered, it is a martingale. We deduce that the processes \(\{\exp[\theta Z(t)] : t \geq 0\}\) and \(\{\exp[-\theta Z(t)] : t \geq 0\}\) are nonnegative submartingales. Now, \((C)\) implies that for all \(\theta > 0\), \(\Phi(\theta) + \Phi(-\theta)\) is finite. Then, by Doob’s inequality, for all \(\theta > 0\) and \(j > 1\),
\[
\pi_{n_j}(\epsilon) \leq 2n_j [\Phi(\theta) + \Phi(-\theta)] \exp\left( -\frac{\theta j \epsilon}{9} \right)
\]
\[
< 2 \left[ \Phi(\theta) + \Phi(-\theta) \right] \exp\left( \frac{1}{c} \right) \exp\left( -j \left( \frac{\theta \epsilon}{9} - \frac{1}{c} \right) \right).
\]
Indeed, by definition, \(n_j < \exp\left( \frac{j+1}{c} \right)\). By \((C)\), we can choose \(\theta = \theta(\epsilon,c)\) such that
\[
\theta \frac{\epsilon}{9} > \frac{1}{c} \quad \text{and} \quad \Phi(\theta) + \Phi(-\theta) < \infty.
\]
We deduce from (5.42) and (5.43) that \(\sum_{j>1} \pi_{n_j}(\epsilon) < \infty\). Applying Lemma 23 and then the Borel-Cantelli lemma, we obtain that
\[
\{ \Delta_{n_j}(\epsilon) \text{ i.o. in } j \} \subset \left\{ \bigcup_{i=0}^{n_j+1} \left\{ \sup_{0 \leq \tau \leq 1} |Z(i + \tau) - Z(i)| > \frac{j\epsilon}{9} \right\} \text{ i.o. in } j \right\} \in \mathcal{N}.
\]
We conclude the proof by Corollary 6 and Lemma 20. \(\square\)
Remark 1. Notice that for all \( f, g \in D(0,1) \),
\[
d_H(f, g) \leq 2d_I(f, g).
\]
(5.44)
Therefore, for all \( n > 1 \), \( \{\Delta^H_n(\epsilon)\} \subset \{\Delta^U_n(\epsilon)\} \), so that \( \{\Delta^H_{n_1}(\epsilon) \text{ i.o. in } j\} \) is included in \( \{\Delta^U_{n_2}(\epsilon) \text{ i.o. in } j\} \).

Then, by Corollary 6, one could try to use the arguments of the proof of Lemma 24 to derive the upper bound for \( L_n \) under \( (A) \). However, if only \( (A) \) holds, for \( \epsilon > 0 \) small enough, it is not possible to choose \( \theta \) such that \( \theta^2_n > \frac{1}{\epsilon} \) and \( \Phi(\theta) < \infty \). So, (5.44) does not provide this upper bound.

5.3 Lower bound
5.3.1 A general result on rate functions

Lemma 25. Let \((\mathcal{E}, \|\cdot\|)\) be a normed vector space. Denote by \( d \) the distance on \( \mathcal{E} \) induced by \( \|\cdot\| \). Let \( I : \mathcal{E} \to [0, \infty) \) be a convex function with \( I(0_{\mathcal{E}}) = 0 \). For all \( \alpha > 0 \), set \( \mathcal{K}_\alpha := \{ f \in \mathcal{E} : I(f) \leq \alpha \} \).

Then, for all \( g \in \mathcal{K}_\alpha \) and \( \rho > 0 \),
\[
\inf \{ I(f) : f \in B^\rho_d(g, \rho) \} < \alpha.
\]
(5.45)

Proof. Consider the map \( \phi : [0,1] \to \mathbb{R} \) defined by \( \phi(\lambda) = \|\lambda g - g\| \). Then, \( \phi \) is continuous and \( \phi(1) = 0 \). So, for all \( \rho > 0 \), there exists \( \lambda_0 < 1 \) such that \( \phi(\lambda_0) < \rho \), that is \( \lambda_0 g \in B^\rho_d(g, \rho) \). Since \( I \) is convex and \( I(0_{\mathcal{E}}) = 0 \), we have that \( I(\lambda_0 g) \leq \lambda_0 I(g) \). Then, since \( \lambda_0 < 1 \) and \( g \in \mathcal{K}_\alpha \), we have that \( \lambda_0 I(g) < I(g) \leq \alpha \). Therefore, \( \lambda_0 I(g) < \alpha \), which proves (5.45).

For \( n > 1 \), set \( R_n := [(n - A_n)/A_n] \) and for \( G = \{ rA_n : r \in \mathbb{N} \} \),
\[
Q_n := \mathcal{L}^G_n = \{ \eta_{rA_n} : 0 \leq r \leq R_n - 1 \}.
\]

5.3.2 Lower bound under \( (C) \)

Lemma 26. Assume that \( (C) \) holds and that \( \{Z(t) : t \geq 0\} \) is centered. Then, for any \( c > 0 \) and \( \epsilon > 0 \),
\[
\left\{ K_{1/c} \subset (\mathcal{L}_n)^{\epsilon/2d} \text{ ult.} \right\} \in \mathcal{N}^c.
\]

Proof. Let \( g \in K_{1/c} \) and \( \epsilon > 0 \). By Lemma 17, for all \( n > 1 \), \( \{g \notin (Q_n)^{\epsilon/2d} \} \in \mathcal{F} \). Now, by Lemma 7, there exists \( \zeta > 0 \) such that \( B^\rho_I(g, \epsilon/2) \supset B^\rho_{S}(g, \zeta) \). Therefore,
\[
P \left( g \notin (Q_n)^{\epsilon/2d} \right) = \left( \bigcap_{r=0}^{R_n-1} \left\{ \eta_{rA_n} \notin B^\rho_I(g, \epsilon/2) \right\} \right)
\leq \left( \bigcap_{r=0}^{R_n-1} \left\{ \eta_{rA_n} \notin B^\rho_{S}(g, \zeta) \right\} \right)
= \left[ 1 - P \left( \eta_{rA_n} \in B^\rho_{S}(g, \zeta) \right) \right]^{R_n},
\]
where the last two lines follow from the mutual independence of \( (\eta_{rA_n}, A_n) \) for \( 1 \leq r \leq R_n \) and Lemma 4. By Theorem 5, for any \( \theta > 0 \), there exists \( N_\theta \in \mathbb{N} \) such that for all \( n \geq N_\theta \),
\[
P \left( \frac{Z \cdot A_n}{A_n} \in B^\rho_{S}(g, \zeta) \right) \geq \exp \left( A_n \left( -I \left( B^\rho_{S}(g, \zeta) \right) - \theta \right) \right)
\geq \exp \left( A_n \left( -I \left( B^\rho_{I}(g, \zeta) \right) - \theta \right) \right).
\]
(5.46)
Indeed, by definition of $d_S$, for all $f, g \in D(0, 1)$, $d_S(f, g) \leq d_H(f, g)$. Therefore, $B^D_{H}(g, \zeta) \subset B^D_S(g, \zeta)$ and $I\left( B^D_S(g, \zeta) \right) \leq I\left( B^D_{H}(g, \zeta) \right)$. Now, we apply Lemma 25 with $\mathcal{E} = D(0, 1)$, $d = d_H$ and $I = I$. Notice that $I(0_{D(0,1)}) = 0$, since $Z$ is centered. Therefore,

$$I\left( B^D_{U}(g, \zeta) \right) < \frac{1}{c}. $$

So, we can write $I(B^D_{U}(g, \zeta)) = \frac{1}{c} - \delta$ with $\delta > 0$. Taking $\theta = \frac{\delta}{4}$ in (5.47), we obtain that for all $n \geq N_\theta$,

$$P \left( g \notin (Q_n)^{\epsilon/2;H} \right) \leq \left[ 1 - \exp \left( A_n \left( -\frac{1}{c} + \frac{3\delta}{4} \right) \right) \right]^{R_n}. $$

Consequently, $\sum P \left( g \notin (Q_n)^{\epsilon/2;H} \right) < \infty$ and by the Borel-Cantelli lemma,

$$P \left( g \in (Q_n)^{\epsilon/2;H} \text{ ult.} \right) = 1. \quad (5.48)$$

Now, by Lemma 14, $K_{1/c}$ is a compact subset of $(D(0, 1), d_H)$. So, we can find $d < \infty$ and functions $(g_q)_{q=1, \ldots, d}$ in $K_{1/c}$ such that $K_{1/c} \subset \bigcup_{q=1}^d B^D_{U}(g_q, \epsilon/2)$. By the triangle inequality, for all $n > 1$,

$$\left\{ \{g_q : q = 1, \ldots, d\} \subset (Q_n)^{\epsilon/2;H} \right\} \subset \left\{ K_{1/c} \subset (Q_n)^{\epsilon/2H} \right\}. $$

By (5.48) applied to each $g_q$, $P\left( \{g_q : q = 1, \ldots, d\} \subset (Q_n)^{\epsilon/2;H} \text{ ult.} \right) = 1$. Therefore,

$$\left\{ K_{1/c} \subset (\mathcal{L}_n)^{\epsilon;H} \text{ ult.} \right\} \supset \left\{ K_{1/c} \subset (Q_n)^{\epsilon;H} \text{ ult.} \right\} \in \mathcal{N}^c. $$

$\square$

3.3 Lower bound under $(\mathcal{A})$

Lemma 27. Assume that $(\mathcal{A})$ holds and that $\{Z(t) : t \geq 0\}$ is centered. Then, for any $c > 0$ and $\epsilon > 0$,

$$\{ L_{1/c} \in (\mathcal{L}_n)^{\epsilon;H} \text{ ult.} \} \in \mathcal{N}^c. \quad (5.49)$$

Proof. Let $g \in L_{1/c}$ and $\epsilon > 0$. By Lemma 17, for all $n > 1$, $\{ g \notin (Q_n)^{\epsilon/2;H} \} \in \mathcal{F}$ and as in the beginning of the proof of Lemma 26,

$$P \left( g \notin (Q_n)^{\epsilon/2;H} \right) = P \left( \bigcap_{r=0}^{R_n-1} \left\{ d_H (g, \eta_{r,A_n,A_n}) \geq \epsilon/2 \right\} \right)$$

$$= \prod_{r=0}^{R_n-1} \left[ 1 - P \left( \eta_{r,A_n,A_n} \in B^{BV}_{H} (g, \epsilon/2) \right) \right]. $$

By Corollary 3, $B^{BV}_{H} (g, \epsilon/2) \in \mathcal{B}_S$. Then, it follows from Lemma 4 that

$$\forall r \in \{0, \ldots, R_n - 1\}, \quad P \left( \eta_{r,A_n,A_n} \in B^{BV}_{H} (g, \epsilon/2) \right) = P \left( \frac{Z(\cdot; A_n)}{A_n} \in B^{BV}_{H} (g, \epsilon/2) \right). $$

By Lemma 10, $B^{BV}_{H} (g, \epsilon/2)$ is an open subset of $(BV_0(0,1), \mathcal{W})$. So, by Theorem 6,

$$P \left( \frac{Z(\cdot; A_n)}{A_n} \in B^{BV}_{H} (g, \epsilon/2) \right) \geq \exp \left[ A_n \left( -J \left( B^{BV}_{H} (g, \epsilon/2) \right) - \theta \right) \right]. $$

Now, we apply Lemma 25 with $\mathcal{E} = BV_0(0,1)$, $d = d_H$ and $I = I$. Notice that $J(0_{BV_0(0,1)}) = 0$, since $Z$ is centered. Therefore,

$$J \left( B^{BV}_{H} (g, \epsilon/2) \right) < \frac{1}{c}. $$

By Lemma 14, $L_{1/c}$ is compact in $(BV_0(0,1), d_H)$ so we conclude as in the end of the proof of Lemma 26. $\square$
6 Appendix

6.1 Strong invariance principle

Let \( AC(0, 1) \) be the set of absolutely continuous functions on \([0, 1]\), endowed with the uniform distance \( d_U \). Let \( S \) be the Strassen-type set defined by

\[
S = \left\{ f \in AC(0, 1) : \int_0^1 f(s)^2 ds \leq 1 \text{ and } f(0) = 0 \right\}.
\]

Let \( a_T \) be a function of \( T > 1 \), with \( 0 < a_T \leq T \). Set

\[
b_T := \left[ 2a_T \left( \log(Ta_T^{-1}) + \log(\log T) \right) \right]^{1/2}.
\]

For a Lévy process \( \{ Z(t) : t \geq 0 \} \), let \( \mathcal{H}_T^Z \) be the following set of increment functions.

\[
\mathcal{H}_T^Z := \left\{ b_T^{-1} \nu_{x,a_T}^Z : 0 \leq x \leq T - a_T \right\}
\]

where \( \nu_{x,a_T}^Z : s \in [0, 1] \mapsto Z(x + sa_T) - Z(x) \).

Then, we deduce the following Proposition by combining the strong invariance principle of Theorem 10 below, from [9], and Theorem 11 hereunder, from [18].

**Theorem 10.** Let \( \{ Z(t) : t \geq 0 \} \) be a Lévy process such that (A) holds. Assume that for all \( t \geq 0 \), \( \mathbb{E}[Z(t)] = 0 \) and \( \text{Var}(Z(t)) = t \). Then, there exists a probability space on which one can define a standard Wiener process \( \{ W(t) : t \geq 0 \} \) jointly with the process \( \{ Z(t) : t \geq 0 \} \), in such a way that, as \( T \to \infty \),

\[
\sup_{0 \leq t \leq T} |Z(t) - W(t)| = \mathcal{O}(\log T) \quad \text{a.s.}
\]

**Theorem 11.** Let \( \{ W(t) : t \geq 0 \} \) be a standard Wiener process. Assume that \( a_T \) and \( Ta_T^{-1} \) are non-decreasing and that \( \frac{\log(Ta_T^{-1})}{\log(\log T)} \to \infty \). Then, for all \( \epsilon > 0 \), a.s.

\[
\mathcal{H}_T^W \subset S^{\epsilon \text{ ult.}} \quad \text{and} \quad S \subset (\mathcal{H}_T^W)^{\epsilon \text{ ult.}}.
\]

**Lemma 28.** Assume that \( \frac{\log(Ta_T^{-1})}{\log(\log T)} \to \infty \). Then,

\[
\frac{b_T}{\log T} \sim \sqrt{2} \left( \frac{a_T}{\log T} \right)^{1/2} \quad \text{as } T \to \infty.
\]

**Proof.** The assumption implies that \( \frac{b_T}{a_T} \sim \left[ 2a_T \log(Ta_T^{-1}) \right]^{1/2} =: u_T \) as \( T \to \infty \). Clearly, \( u_T = \sqrt{2} \left[ \frac{\log T}{a_T} - \frac{\log a_T}{a_T} \right]^{1/2} \sim \sqrt{2} \left[ \frac{\log T}{a_T} \right]^{1/2} \) as \( T \to \infty \). Therefore,

\[
\frac{b_T}{\log T} = \left( \frac{b_T}{a_T} \right) \left( \frac{a_T}{\log T} \right) \sim \sqrt{2} \left[ \frac{\log T}{a_T} \right]^{1/2} \left( \frac{a_T}{\log T} \right) = \sqrt{2} \left( \frac{a_T}{\log T} \right)^{1/2}.
\]

\[\Box\]

**Proposition 6.** Let \( \{ Z(t) : t \geq 0 \} \) be a Lévy process satisfying the assumptions of Theorem 10. Assume that \( a_T \) fulfills the conditions of Theorem 11 and that

\[
a_T(\log T)^{-1} \to \infty, \quad \text{as } T \to \infty.
\]

Then, for all \( \epsilon > 0 \), a.s.

\[
\mathcal{H}_T^Z \subset S^{\epsilon \text{ ult.}} \quad \text{and} \quad S \subset \left( \mathcal{H}_T^Z \right)^{\epsilon \text{ ult.}}.
\]
According to (6.51), if (6.52) holds, then $b_T^{-1} \log T = o(1)$ as $T \to \infty$. We conclude the proof by applying (6.50), which implies that, as $T \to \infty$,
$$2b_T^{-1} \sup_{0 \leq t \leq T} |Z(t) - W(t)| = \mathcal{O}(b_T^{-1} \log T) = o(1). \quad (6.53)$$

**Remark 2.** Assumption (6.52) means that we consider "large increments" $a_T$. We cannot invoke the argument of the preceding proof to derive a FLT for increment functions of the form $\nu^Z_{x,a_T}$ with $a_T$ of order $\log T$. Indeed, in that case, (6.51) implies that $b_T$ is also of order $\log T$. On the other hand, it is well known that rates as in (6.50) cannot be reduced to $o(\log T)$.

### 6.2 Proof of Proposition 4

**Proof.** Let $I_d$ be the identity function on $\mathcal{E}$. For $x \geq 0$ and $s \in [0,1]$, we have that $\eta^\mu_{x,A_n}(s) = \eta_{x,A_n}(s) + \mu s$. Therefore,
$$\left(\mathcal{L}_n^G\right)^\mu = \mathcal{L}_n^G + \mu I_d. \quad (6.54)$$

We write with a superscript $\mu$ all elements concerning $\{Z^\mu(t) : t \geq 0\}$. Therefore, for all $\theta \in \mathbb{R}$, $\Phi^\mu(\theta) = \exp(\mu \theta) \Phi(\theta)$. Consequently, for all $a \in \mathbb{R}$, $\Psi^\mu(a) = \Psi(a - \mu)$. So, for all $f \in AC(0,1)$,
$$\int_0^1 \Psi^\mu \left( \frac{df}{ds} \right) ds = \int_0^1 \Psi \left( \frac{df}{ds} - \mu \right) ds = \int_0^1 \Psi \left( \frac{df - \mu I_d}{ds} \right) ds \quad \text{and for all } f \in \mathcal{E}, \mathcal{I}^\mu(f) = \mathcal{I}(f - \mu I_d),$$
so that for all $\alpha > 0$,
$$(K_\alpha)^\mu = K_\alpha + \mu I_d. \quad (6.55)$$

Let $T_{\mu I_d} : f \in \mathcal{E} \mapsto f + \mu I_d \in \mathcal{E}$. Since $d$ is derived from a norm, $T_{\mu I_d}$ is an isometry of $(\mathcal{E}, d)$. So, for all $g \in \mathcal{E}$ and $\epsilon > 0$, $T_{\mu I_d} \left( B^\epsilon_d(g, \epsilon) \right) = B^\epsilon_d(g + \mu I_d, \epsilon)$. Consequently, for $\mathcal{G} = \mathcal{L}_n^G$ or $\mathcal{G} = K_\alpha$,
$$\left(\mathcal{G}^\mu\right)^{c,d} = (\mathcal{G} + \mu I_d)^{c,d} = \bigcup_{g \in \mathcal{G}} B^\epsilon_d(g + \mu I_d, \epsilon) = \left[ \bigcup_{g \in \mathcal{G}} B^\epsilon_d(g, \epsilon) \right] + \mu I_d = \mathcal{G}^{c,d} + \mu I_d. \quad (6.56)$$

We conclude since (6.54), (6.55), (6.56) imply that (4.28) and (4.29) hold for $\left(\mathcal{L}_n^G\right)^\mu$. \qed

### 6.3 Proof of Proposition 5

**Proof.** In this proof, $K_\alpha$ is abbreviated as $K$. Set $S_K := \sup \{ \Theta(f) : f \in K \}$ and for all $n \geq 1$, set
$$S(n) := \sup \{ \Theta(f) : f \in \mathcal{L}_n^G \}.$$

**First step:** For all $\omega \in \Omega$, there exists a subsequence $(S(\phi(n)))_{n \geq 1}$ of $(S(n))_{n \geq 1}$ such that $S(\phi(n)) \xrightarrow{n \to \infty} \varprojlim S(n)$.

So, for all $n \geq 1$, there exists $f_\phi(n) \in \mathcal{L}_n^G$ such that
$$\Theta(f_\phi(n)) \xrightarrow{n \to \infty} \varprojlim S(n).$$

Set $R_1 := \bigcap_{k \geq 1} \{ \mathcal{L}_n^G \subset K^{1/k; d} \text{ ult.} \}$. For all $\omega \in R_1$ and $k \geq 1$, there exists $N_k \in \mathbb{N}$ and $\lambda_k \in K$ such that $d \left( f_\phi(N_k), \lambda_k \right) < \frac{1}{k}$. Since $K$ is compact, there exists a subsequence $(\lambda_\Psi(k))_{k \geq 1}$ of $(\lambda_k)_{k \geq 1}$ which converges to some $\lambda \in K$. Then, for all $k \geq 1$,
$$d \left( f_\phi(N_\Psi(k)), \lambda \right) \leq d \left( f_\phi(N_\Psi(k)), \lambda_\Psi(k) \right) + d \left( \lambda_\Psi(k), \lambda \right) < \frac{1}{\Psi(k)} + d \left( \lambda_\Psi(k), \lambda \right).$$

22
So, for all \( \omega \in \mathcal{R}_1 \), \( f_{\phi(N(\omega,k))} \to \lambda \) and \( \lim_{k \to \infty} S(n) = \lim_{k \to \infty} \Theta\left(f_{\phi(N(\omega,k))}\right) = \Theta(\lambda) \leq S_K \). Now, the upper bound implies that for all \( k \geq 1 \), there exists \( M(1/k) \in \mathcal{N}^c \) such that

\[
\bigcap_{k \geq 1} M(1/k) \subset \mathcal{R}_1 \subset \{\lim S(n) \leq S_K\}.
\]

Since \( \bigcap_{k \geq 1} M(1/k) \in \mathcal{N}^c \), we deduce that \( \{\lim S(n) \leq S_K\} \in \mathcal{N}^c \).

**Second step:** \( \Theta \) is continuous, and \( \mathcal{K} \) is compact. So, there exists \( \lambda \in \mathcal{K} \) such that

\[
\Theta(\lambda) = S_K. \tag{6.57}
\]

Since \( \Theta \) is continuous, for all \( k \geq 1 \), there exists \( \epsilon_k > 0 \) such that for \( g \in \mathcal{E} \),

\[
d(g, \lambda) < \epsilon_k \implies \Theta(g) > \Theta(\lambda) - \frac{1}{k}.
\]

Set \( \mathcal{R}_2 := \bigcap_{k \geq 1} \{ \mathcal{K} \subset (\mathcal{L}_n^\mathcal{G})^\psi \} \). For all \( \omega \in \mathcal{R}_2 \) and \( k \geq 1 \), there exists an integer \( N(\omega,k) \) such that for all \( n \geq N(\omega,k) \), there exists \( f_n(\omega,k) \in \mathcal{L}_n^\mathcal{G} \) satisfying

\[
d(f_n(\omega,k), \lambda) < \epsilon_k.
\]

This implies that \( \Theta(f_n(\omega,k)) > \Theta(\lambda) - \frac{1}{k} \). Therefore, for all \( \omega \in \mathcal{R}_2 \), \( k \geq 1 \), and \( n \geq N(\omega,k) \), \( S(n) > S_K - \frac{1}{k} \). So, for all \( \omega \in \mathcal{R}_2 \), \( \lim S(n) \geq S_K \). Now, the lower bound implies that for all \( k \geq 1 \), there exists \( M(\epsilon_k) \in \mathcal{N}^c \) such that

\[
\bigcap_{k \geq 1} M(\epsilon_k) \subset \mathcal{R}_2 \subset \{\lim S(n) \geq S_K\}.
\]

Since \( \bigcap_{k \geq 1} M(\epsilon_k) \in \mathcal{N}^c \), we deduce that \( \{\lim S(n) \geq S_K\} \in \mathcal{N}^c \). \( \square \)

**Acknowledgements**

The author wishes to thank Prof. Paul Deheuvels, who suggested this problem to him, and Prof. Zhan Shi for helpful discussions.

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