Abstract

We describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. We introduce an equation which we suggest to be the Schrödinger equation of the Schwarzschild black hole corresponding to this model. We solve the Schrödinger equation explicitly and obtain the mass spectrum of the black hole as such as it can be observed by an observer very far away and at rest relative to the black hole. Our equation implies that there is no singularity inside the Schwarzschild black hole, and that the black hole has a certain ground state in which its mass is non-zero.

1. Introduction

One of the basic requirements every physical theory must satisfy is that the theory must be able to predict the possible outcomes of measurements. In order to be a reliable physical theory this requirement must be satisfied even by as esoteric a theory as quantum gravity.

So far the quantum theory of gravity has given rather few direct physical predictions. Perhaps the most important of them are the existence of the so called Hawking radiation emitted by black holes,[1] and the result given by Ashtekar, Rovelli and Smolin, which says that area is quantized.[2] Quite a lot of effort has been spent on the study of quantum cosmology. An application of quantum gravity to quantum cosmology, however, meets with grave conceptual difficulties, such as the interpretation of the wave function if the

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observers are assumed to be a part of the physical system under consideration, and the problem of time.[3]

In this paper an attempt is made to use quantum gravity in order to predict the physical properties of the Schwarzschild black hole as such as they are observed by an observer who is very far away and at rest relative to the black hole. We assume that the Schwarzschild black hole is a quantum mechanical system, and that its physical states can therefore be described by a certain wave function \( \psi \). Since we assume that the observer is very far away from the black hole, he can consider himself, at least to a very good approximation, as an observer external to the physical system, the black hole, under consideration. Moreover, spacetime around him is, at least to a very good approximation, flat, and so he can use the standard interpretation of quantum mechanics, and he has a well-defined time coordinate. In other words, the conceptual problems related to quantum gravity are absent.

At least classically, the only thing an external observer can observe on the properties of the Schwarzschild black hole is its mass \( M \). Because of that, our object is to predict the possible masses of the Schwarzschild black hole as such as they are observed by an external observer very far away and at rest relative to the black hole. The mass \( M \) measured by this kind of an observer is the same mass as the one written in the usual expression of the Schwarzschild metric. In terms of the mass \( M \), the observer can define the concept of energy \( E \) of the black hole as:

\[
E = Mc^2, \tag{1.1}
\]

where \( c \) is the velocity of light. It is obvious that if the observer can predict the energies of the black hole, he can also predict its possible masses.

In order to predict the energies of the black hole, the observer writes down the time-independent Schrödinger equation

\[
\hat{H}\psi = E\psi \tag{1.2}
\]

of the black hole. In this equation, \( \hat{H} \) is the Hamiltonian operator of the black hole, and \( \psi \) is its wave function. The possible outcomes \( E_n \) of the measurements of the energy are the eigenvalues of \( \hat{H} \), and the eigenfunctions \( \psi_n \) give the corresponding probability amplitudes. In other words, if we can construct the Schrödinger equation of the Schwarzschild black hole, we can predict its possible masses.

In this paper we describe the gravitational degrees of freedom of the black hole by one free variable. We introduce an equation which we suggest to be the time-independent Schrödinger equation of the Schwarzschild black hole corresponding to our model as such as it can be used by an observer very far away and at rest relative to the black hole. It turns out that our Schrödinger equation can be solved explicitly, and so we can predict the mass spectrum of the Schwarzschild black hole. The mass spectrum, in turn, can be calculated if we can calculate the spectrum of the area

\[
A_S := \frac{16\pi G^2}{c^4}M^2, \tag{1.3}
\]

of the event horizon of the Schwarzschild black hole. It turns out that the eigenvalues of \( A_S \) are an integer plus \((1/2)\) times a certain area, whose order of magnitude is \( 10^{-68}m^2 \).
This result is entirely in harmony with the results of Ashtekar, Rovelli and Smolin on the quantization of area in quantum gravity.\cite{2} It is also compatible with the studies made by Bekenstein and others on the properties of black holes.\cite{4-8}

We shall also see that our Schrödinger equation, if true, solves the fundamental problem on whether there is a singularity inside the Schwarzschild black hole. The answer turns out to be negative, since if there were a singularity inside the black hole, then our Schrödinger equation would have no physically acceptable solutions. Moreover, it turns out that there is a certain ground state in which the mass of the black hole is non-zero.

\section{Schrödinger Equation}

When searching for the Schrödinger equation of the Schwarzschild black hole, one must first find its classical Hamiltonian from the point of view of an observer at rest very far away. This problem brings us to the Hamiltonian dynamics of asymptotically flat spacetimes.

An extensive study of the Hamiltonian dynamics of asymptotically flat spacetimes was made long ago by Regge and Teitelboim\cite{9}. They found that in asymptotically flat spacetimes certain \textit{surface integrals} at spatial infinity play a decisive role. For example, the true Hamiltonian of an asymptotically flat spacetime is not the Hamiltonian $H_0$ of spatially compact spacetimes written in terms of the lapse $N$, the shift $N^i$, and the Hamiltonian and the diffeomorphism constraints $H$ and $H_i$ as:

$$H_0 := \int d^3x (N \dot{H} + N^i \dot{H}_i),$$

but the correct Hamiltonian is

$$H := H_0 + E_{\text{ADM}},$$

where $E_{\text{ADM}}$ is the so called \textit{ADM energy}. If the spacetime coordinates have been chosen in such a way that the spacetime metric $g_{\mu\nu}$ becomes to the flat Minkowski metric $\eta_{\mu\nu}$ at spatial infinity, then $E_{\text{ADM}}$ can be written as a surface integral at spatial infinity:

$$E_{\text{ADM}} = \frac{c^4}{16\pi G} \int d^2s_k (g_{ik,i} - g_{ii,k}),$$

where $i, k = 1, 2, 3$. It can be easily shown that for the Schwarzschild black hole at rest we have:

$$E_{\text{ADM}} = M c^2.$$  

Indeed, this can be considered as the energy of the black hole. In this paper, when we talk about the energy of the Schwarzschild black hole, we always mean its ADM energy.

It was one of the main results of Regge and Teitelboim that if one fixes the coordinate system by fixing the lapse $N$ and the shift $N^i$ then, in this fixed coordinate system,
the correct Hamiltonian of an asymptotically flat spacetime is obtained by inserting the solution of the Hamiltonian and the diffeomorphism constraints

\[ \mathcal{H} = 0, \quad (2.5.a) \]
\[ \mathcal{H}_i = 0, \quad (2.5.b) \]

into the surface integral (2.3). More precisely, between the phase space coordinates of spacetime and the numerical value of the ADM energy (2.3) there is a certain relationship which can be solved from the constraints (2.5) in the fixed coordinate system. If the numerical value of the ADM energy is written in terms of the phase space coordinates, by using the relationship found from the constraints (2.5), we get the spacetime Hamiltonian in our coordinate system. In our case, this means that we must first find an appropriate variable \( a \) describing the gravitational degrees of freedom of the Schwarzschild black hole, and then write the constraints (2.5) in terms of \( a \) and its canonical momentum \( p \). From the constraints (2.5) we find how the quantity \( M c^2 \), which is the ADM energy of our spacetime, depends on \( a \) and \( p \). In this way we get the classical Hamiltonian of the Schwarzschild black hole.

To begin with, we write down the spacetime metric of the Schwarzschild black hole in terms of the Schwarzschild coordinates \( r \) and \( t \), and the spherical angles \( \theta \) and \( \phi \). Outside the black hole horizon the metric is:

\[ ds^2 = - \left( 1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \frac{1}{1 - \frac{2GM}{c^2 r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.6.a) \]

whereas inside the black hole horizon the metric is:

\[ ds^2 = \left( \frac{2GM}{c^2 r} - 1 \right) c^2 dt^2 - \frac{1}{\frac{2GM}{c^2 r} - 1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.6.b) \]

We are outside the black hole horizon when \( r \) is greater than the Schwarzschild radius

\[ R_S := \frac{2GM}{c^2}, \quad (2.7) \]

and we are inside the black hole horizon when \( r \) is smaller than the Schwarzschild radius.

We must now find an appropriate coordinate system. Since the coordinates \( r \) and \( t \) behave remarkably badly when \( r = R_S \), and inside the black hole horizon \( t \) is not a timelike coordinate, the coordinates \( r \) and \( t \) are out of question. We have plenty of choice because the ADM energy is independent of the chosen coordinate system provided that the coordinate system is at rest at spatial infinity relative to the black hole, and the time coordinate coincides with the time coordinate \( t \) at infinity. In this regard, an appropriate choice is to use the so called Novikov coordinates[10], and we begin with a brief review on the properties of these coordinates.

The basic idea of the Novikov coordinates is to relate the spacetime coordinates to observers in a radial free fall towards the center of the black hole such that all these observers are at rest relative to the black hole when \( t = 0 \), and at that moment of the
time $t$ they are released into a free fall. The proper times of these observers give the time coordinate, and a certain variable related to the positions of these observers when $t = 0$, gives one of the spatial coordinates for each point in space and time. Since an observer in a radial free fall is always at rest relative to the black hole at spatial infinity, and his proper time coincides with the time $t$ at spatial infinity, the Novikov coordinates behave in the desired way.

It can be easily shown that for an observer in a radial free fall we have, in general:

$$
\left(1 - \frac{2GM}{c^2r}\right) \dot{t} = \text{constant} := \frac{\chi}{\sqrt{1 + \chi^2}},
$$

where the dot means proper time derivative, and the constant $\chi \geq 0$. Because of that, we find that when $r$ goes from $r$ to $r - dr$, then the proper time $\tau$ of the observer in a free fall goes from $\tau$ to $\tau + d\tau$ such that

$$
c^2 d\tau^2 = -\frac{\frac{\chi^2}{2GMc^2}}{1 - \frac{c^2}{1 + \chi^2}} c^2 d\tau^2 + \frac{dr^2}{2GMc^2 - 1},
$$

and we find that the equation of motion of the observer in a free fall is

$$
\dot{r}^2 = \frac{2GM}{r} - \frac{c^2}{1 + \chi^2}.
$$

As one can see, the observer is at rest relative to the black hole at the point where

$$
r = r_{\text{max}} := (1 + \chi^2) \frac{2GM}{c^2},
$$

and so the constant $\chi$ can be written in terms of $r_{\text{max}}$ as:

$$
\chi = \left(\frac{r_{\text{max}}}{R_S} - 1\right)^{1/2}.
$$

The Novikov coordinates are now the coordinates $\tau$ and $\chi$ such that $r = r_{\text{max}}$ and $t = 0$ when $\tau = 0$. Using Eqs.(2.8) and (2.10) we can, at least in principle, express the coordinates $r$ and $t$ in terms of $\tau$ and $\chi$. In other words, we have $r = r(\tau, \chi)$, and $t = t(\tau, \chi)$. Because we have:

$$
dt = \frac{\partial t}{\partial \tau} d\tau + \frac{\partial t}{\partial \chi} d\chi, \quad \text{(2.13.a)}
$$

$$
dr = \frac{\partial r}{\partial \tau} d\tau + \frac{\partial r}{\partial \chi} d\chi, \quad \text{(2.13.b)},
$$

we find from Eq.(2.6) that if we can express $r$ and $t$ in terms of $\tau$ and $\chi$, the spacetime metric becomes to:

$$
ds^2 = -\left(1 - \frac{2GM}{c^2r}\right) \left(c \frac{\partial t}{\partial \tau} d\tau + c \frac{\partial t}{\partial \chi} d\chi\right)^2 + \frac{1}{1 - \frac{2GM}{c^2r}} \left(\frac{\partial r}{\partial \tau} d\tau + \frac{\partial r}{\partial \chi} d\chi\right)^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad \text{(2.14)}$$

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The precise relationship between $r$, $\tau$ and $\chi$, although in an implicit form, can be calculated from Eq.(2.10). We get:

$$\tau = \frac{1}{c}(1 + \chi^2)\left(R_S r - \frac{r^2}{1 + \chi^2}\right)^{1/2} + \frac{R_S}{c}(1 + \chi^2)^{3/2}\cos^{-1}\left[\left(\frac{r/R_S}{1 + \chi^2}\right)^{1/2}\right]. \quad (2.15)$$

Differentiating the both sides of this equation with respect to $\chi$ we get an expression for $(\partial r/\partial \chi)$ in terms of $r$ and $\chi$:

$$\frac{\partial r}{\partial \chi} = 3R_S\chi - \frac{\chi r}{1 + \chi^2} + 3R_S(1 + \chi^2)^{1/2}\left(\frac{R_S}{r} - \frac{1}{1 + \chi^2}\right)^{1/2}\cos^{-1}\left[\left(\frac{r/R_S}{1 + \chi^2}\right)^{1/2}\right], \quad (2.16)$$

and from Eqs.(2.8) and (2.10) we get:

$$\frac{\partial t}{\partial \tau} = \frac{1}{1 - \frac{R_S}{r}\sqrt{1 + \chi^2}}, \quad (2.17.a)$$

$$\frac{\partial r}{\partial \tau} = -c\left(\frac{R_S}{r} - \frac{1}{1 + \chi^2}\right)^{1/2}. \quad (2.17.b)$$

The quantity $(\partial t/\partial \chi)$ can also be expressed in terms of $r$ and $\chi$. As it is well known, the relationship between $t$ and $r$ can be expressed in a parametrized form[10]:

$$r = \frac{1}{2}R_S(1 + \chi^2)(1 + \cos \eta), \quad (2.18.a)$$

$$t = \frac{R_S}{c}\ln\left|\frac{\chi + \tan\left(\frac{\eta}{2}\right)}{\chi - \tan\left(\frac{\eta}{2}\right)}\right| + \frac{R_S}{c}\chi[\eta + \frac{1}{2}(1 + \chi^2)(\eta + \sin \eta)]. \quad (2.18.b)$$

If one solves the parameter $\eta$ from Eq.(2.18.a) in terms of $r$ and $\chi$, and inserts the result into Eq.(2.18.b), one gets an expression for $t$ in terms of $r$ and $\chi$. From that expression one can calculate $(\partial t/\partial \chi)$ in terms of $\chi$, $r$ and $(\partial r/\partial \chi)$, and with the help of Eq.(2.16) one gets $(\partial t/\partial \chi)$ in terms of $r$ and $\chi$. If one inserts Eqs.(2.16) and (2.17), and the expression of $(\partial t/\partial \chi)$ into the metric (2.14), one gets an expression of the metric written in terms of $r$ and $\chi$. It turns out to be[10]:

$$ds^2 = -c^2\,d\tau^2 + \frac{1 + \chi^2}{\chi^2}\left(\frac{\partial r}{\partial \chi}\right)^2\,d\chi^2 + r^2(d\theta^2 + \sin^2 \theta\,d\phi^2), \quad (2.19)$$

and from Eq.(2.16) it follows that the metric takes the form:

$$ds^2 = -c^2\,d\tau^2 + (1 + \chi^2)$$

$$\times \left\{3R_S - \frac{r}{1 + \chi^2} + 3R_S(1 + \chi^2)^{1/2}\left(\frac{R_S}{r} - \frac{1}{1 + \chi^2}\right)^{1/2}\cos^{-1}\left[\left(\frac{r/R_S}{1 + \chi^2}\right)^{1/2}\right]\right\}^2\,d\chi^2$$

$$+ r^2(d\theta^2 + \sin^2 \theta\,d\phi^2). \quad (2.20)$$
As one can see, this metric behaves perfectly well in the black hole horizon \( r = R_S \). When written in this form, it does not involve any explicit \( \tau \)-dependence, but all \( \tau \)-dependence is included into the function \( r(\tau, \chi) \).

We have now found an appropriate coordinate system for the whole spacetime. Our next task is to find an appropriate variable describing the geometrical degrees of freedom of the Schwarzschild black hole, and to write the spacetime metric in terms of this variable. This variable should contain all \( \tau \)-dependence of the metric.

We define the variable \( a(\tau) \) describing the geometrical degrees of freedom of the Schwarzschild black hole as:

\[
a(\tau) := r(\tau, 0),
\]

for all \( \tau \geq 0 \). This definition can be understood as a boundary condition to the differential equation (2.16). If one solves Eq.(2.16) with the boundary condition (2.21), one gets \( r(\tau, \chi) \) in terms of \( a(\tau) \) and \( \chi \). It is easy to see that the general solution of Eq.(2.16) is:

\[
(1 + \chi^2) \left( R_S r - \frac{r^2}{1 + \chi^2} \right)^{1/2} + R_S (1 + \chi^2)^{3/2} \cos^{-1} \left( \frac{r/R_S}{1 + \chi^2} \right)^{1/2} = \text{constant} := C,
\]

and therefore the solution satisfying the boundary condition (2.21) is:

\[
(1 + \chi^2) \left( R_S r - \frac{r^2}{1 + \chi^2} \right)^{1/2} + R_S (1 + \chi^2)^{3/2} \cos^{-1} \left( \frac{r/R_S}{1 + \chi^2} \right)^{1/2} = (R_S a - a^2)^{1/2} + R_S \cos^{-1} \left( \frac{a}{R_S} \right)^{1/2}.
\]  

(2.23)

To gain some insight into the geometric meaning of the variable \( a(\tau) \), let us write the metric when \( \chi = 0 \). Since it follows from Eqs.(2.16) and (2.17) that

\[
\frac{\partial r}{\partial \chi} = \frac{\partial t}{\partial \tau} = 0,
\]

\[
\frac{\partial r}{\partial \tau} = -c \left( \frac{R_S}{r} - 1 \right)^{1/2},
\]

when \( \chi = 0 \), it follows from Eq.(2.14) and from the boundary condition (2.21) that the metric can be written as:

\[
ds^2 = -c^2 d\tau^2 + \left( \frac{2GM}{c^2 a(\tau)} - 1 \right) \left( \frac{\partial t}{\partial \chi} \right)^2 c^2 d\chi^2 + a^2(\tau)(d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(2.25)

If we assume that \( 0 < a(\tau) < R_S \), we can use \( t \) as one of the spatial coordinates, and we get:

\[
ds^2 = -c^2 d\tau^2 + \left( \frac{2GM}{c^2 a(\tau)} - 1 \right) c^2 dt^2 + a^2(\tau)(d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(2.26)
Comparing Eqs. (2.6.b) and (2.26) we find that the quantity \( a(\tau) \) is, actually, the radius of the throat of the black hole as such as it is observed by an observer inside the black hole horizon. We use \( a(\tau) \) as a minisuperspace-type variable of our model, and the idea is that \( a(\tau) \) carries, through Eq. (2.23), all information about the time evolution of the spacelike hypersurface \( \tau = \text{constant} \) of spacetime. In other words, if we know how \( a \) depends on the chosen time coordinate \( \tau \), we can calculate from Eq. (2.23) the value of the function \( r(\tau, \chi) \) for every \( \tau \) and \( \chi \). Inserting \( r(\tau, \chi) \) into the metric (2.20) we obtain the time evolution of the spacelike hypersurface. We assume that \( a \) can be any function of the time coordinate \( \tau \). However, if Einstein’s equations written in terms of \( a(\tau) \) are satisfied, then the precise relationship between \( a(\tau) \) and \( \tau \) is the same as the one between \( r(\tau, 0) \) and \( \tau \) in Eq. (2.15).

Since it follows from Eq. (2.11) that, classically, \( a(\tau) \) is always smaller than, or equal to, the Schwarzschild radius \( R_S \) of the black hole, we find that \( a(\tau) \) is an ideal choice for a variable describing the gravitational degrees of freedom inside the black hole horizon.

Our next task is to write the constraints (2.5) and to find an expression of the ADM energy \( M c^2 \) in terms of \( a \) and its canonical momentum \( p \). The Hamiltonian constraint of spacetime without matter fields is, in general:

\[
\mathcal{H} = \frac{c^4}{16\pi G} \sqrt{q} (K_{ij} K^{ij} - K^2 - \mathcal{R}) = 0, \quad (2.27)
\]

where \( q \) is the determinant of the metric on the hypersurface \( \tau = \text{constant} \), \( K_{ij} \) is the exterior curvature tensor, \( K \) its trace, and \( \mathcal{R} \) is the Riemannian scalar on the hypersurface. We find, by using the metric (2.19), that \( \mathcal{R} \) and \( K_{ij} K^{ij} - K^2 \) can be written as:

\[
\mathcal{R} = -\frac{4}{r} \frac{\chi}{(1 + \chi^2)^2} \left( \frac{\partial r}{\partial \chi} \right)^{-1} + \frac{2}{r^2} \frac{1}{1 + \chi^2}, \quad (2.28.a)
\]
\[
K_{ij} K^{ij} - K^2 = -\frac{4}{c^2r} \left( \frac{\partial r}{\partial \chi} \right)^{-1} \frac{\partial r}{\partial \tau} \frac{\partial r}{\partial \tau} \left( \frac{\partial r}{\partial \chi} \right) - \frac{2}{c^2r^2} \left( \frac{\partial r}{\partial \tau} \right)^2. \quad (2.28.b)
\]

Because all time dependence is included in \( a \), we have:

\[
\frac{\partial r}{\partial \tau} = \frac{\partial r}{\partial a} \dot{a}, \quad (2.29)
\]

and because it follows from Eq. (2.23) that:

\[
\frac{\partial r}{\partial a} = \left( \frac{R_S}{r} - \frac{1}{1 + \chi^2} \right)^{1/2}, \quad (2.30)
\]

we find that the Hamiltonian constraint can be written in an arbitrary point on the spacelike hypersurface \( \tau = \text{constant} \) as:

\[
\frac{3c^4}{8\pi G} \sin \theta \left\{ \frac{R_S}{(1 + \chi^2)^{1/2}} - \frac{r}{(1 + \chi^2)^{3/2}} + R_S \left( \frac{R_S}{r} - \frac{1}{1 + \chi^2} \right)^{1/2} \cos^{-1} \left[ \left( \frac{r/R_S}{1 + \chi^2} \right)^{1/2} \right] \right\} \times \left[ \frac{1}{c^2} \left( \frac{R_S}{a} - 1 \right)^{-1} a^2 - 1 \right] = 0. \quad (2.31)
\]
The solution of this constraint is, in every hypersurface point:

$$\dot{a}^2 = c^2 \left( \frac{R_S}{a} - 1 \right),$$

(2.32)

and we find that the mass $M$ of the black hole can be written in terms of $a$ and its time derivative $\dot{a}$ as:

$$M = \frac{1}{2G} (a\dot{a}^2 + c^2 a).$$

(2.33)

The diffeomorphism constraints do not give us anything new, and we find that the ADM energy, and hence the classical Hamiltonian $H$ of the Schwarzschild black hole, can be written in terms of $a$ and $\dot{a}$ as:

$$H = \frac{c^2}{2G} a\dot{a}^2 + \frac{c^4}{2G} a.$$

(2.34)

The first term on the right hand side can now be considered as the “kinetic energy”, and the second term as the “potential energy” of the black hole, and therefore the black hole Lagrangian is:

$$L = \frac{c^2}{2G} a\dot{a}^2 - \frac{c^4}{2G} a.$$

(2.35)

The canonical momentum conjugate to $a$ is therefore

$$p := \frac{\partial L}{\partial \dot{a}} = \frac{c^2}{G} a\dot{a},$$

(2.36)

and the classical Hamiltonian of the Schwarzschild black hole takes in terms of $a$ and $p$ a form:

$$H = \frac{G}{2c^2 a} \frac{1}{p^2} + \frac{c^4}{2G} a.$$

(2.37)

We can now obtain the time-independent Schrödinger equation (1.2) of the Schwarzschild black hole by replacing the classical Hamiltonian $H$ by the corresponding operator $\hat{H}$. To find a correct expression to the operator $\hat{H}$ we must first specify the inner product between the states $|\psi\rangle$ represented by the wave functions $\psi = \psi(a)$ of the black hole. Since $a$ can be thought to describe the distance from the center of the black hole, a natural inner product between arbitrary states $\psi_1$ and $\psi_2$ is:

$$\langle \psi_1 | \psi_2 \rangle := \int_0^\infty \psi_1^*(a) \psi_2(a) a^2 da.$$  

(2.38)

As it is well known from elementary quantum mechanics, the correct expression to the operator $\hat{p}^2$ corresponding to this kind of inner product is:[11]

$$\hat{p}^2 = -\hbar^2 \left( \frac{d^2}{da^2} + \frac{2}{a} \frac{d}{da} \right),$$

(2.39)
and therefore we get:

$$\left[ -\frac{\hbar^2 G}{2\epsilon^2} \frac{1}{a} \left( \frac{d^2}{da^2} + 2 \frac{d}{a} \frac{da}{d} \right) + \frac{c^4}{2G} a \right] \psi(a) = E\psi(a). \quad (2.40)$$

We suggest that, within our minisuperspace-type model, this is the time-independent Schrödinger equation of the Schwarzschild black hole.

### 3. Eigenvalues and Eigenstates

Our next task is to solve the time-independent Schrödinger equation (2.40) of the Schwarzschild black hole. To begin with, we denote:

$$\psi(a) := \frac{1}{a} u(a), \quad (3.1)$$

and we get:

$$\left[ -\frac{1}{2k^4} \frac{d^2}{da^2} + \frac{1}{2} \left( a^2 - R_S a \right) \right] u(a) = 0, \quad (3.2)$$

where $k$ is, essentially, the inverse of the Planck length:

$$k := \sqrt{\frac{c^3}{\hbar G}}, \quad (3.3)$$

and $R_S$ is the Schwarzschild radius. If we denote:

$$\xi := a - \frac{1}{2} R_S, \quad (3.4)$$

we get a very interesting equation:

$$\left( -\frac{1}{2k^4} \frac{d^2}{d\xi^2} + \frac{1}{2} \xi^2 \right) u(\xi) = \frac{1}{8} R_S^2 u(\xi). \quad (3.5)$$

As one can see, we have obtained the eigenvalue equation to the quantity $(1/8)R_S^2$, and hence, in essence, to the area $A_S$ of the event horizon of the Schwarzschild black hole. Moreover, this equation is something every physicist knows very well. It is the Schrödinger equation of linear harmonic oscillator. Its solutions are, in general, of the form:

$$u_n(\xi) = N_n H_n(k\xi)e^{-\frac{1}{4}k^2\xi^2}, \quad (3.6)$$

where $n = 0, 1, 2...$, $H_n$’s are Hermite polynomials, and $N_n$ is an appropriate normalization factor. The corresponding eigenvalues of $(1/8)R_S^2$ are:

$$\frac{1}{8} R_S^2(n) := (n + \frac{1}{2}) \frac{1}{k^2}. \quad (3.7)$$
In other words, the area of the event horizon of the Schwarzschild black hole has a discrete spectrum.

At this point, however, we meet with a very delicate problem, which is related to the boundary conditions of the function $u(a)$. It follows from Eq.(3.1) defining the function $u(a)$ that when $a$ goes to zero, then $u(a)$ must go to zero at least as fast as $a$, since otherwise the wave function $\psi(a)$ would have a singularity when $a = 0$. When $u$ is written in terms of $\xi$, it follows from Eqs.(3.4) and (3.7) that in the state $n$ the function $u_n(\xi)$ must go to zero in the point

$$\xi_n := -\frac{1}{2} R_S(n) = -\sqrt{2n + \frac{1}{k}}$$

(3.8)

at least as fast as the function $\xi + (1/2) R_S(n)$. However, none of the solutions $u_n(\xi)$ expressed in Eq.(3.6) to the eigenvalue equation (3.4) has this property. This can be seen if one recalls that Eq.(3.4) is similar to the Schrödinger equation of linear harmonic oscillator such that $(1/8) R_S^2$ takes the place of energy. The “classically allowed region” is the one in which

$$\frac{1}{8} R_S^2 \geq \frac{1}{2} \xi^2;$$

(3.9)

or,

$$-\frac{1}{2} R_S \leq \xi \leq \frac{1}{2} R_S.$$  

(3.10)

Now, if the function $u_n(\xi)$ went to zero in the point $\xi = -(1/2) R_S$, then that would mean that the wave function of the harmonic oscillator would be precisely zero in the boundary of its classically allowed region, which we know is not true. Because the only physically acceptable solutions to the eigenvalue equation (3.5) are those in Eq.(3.6), we come into the remarkable conclusion that Eq.(3.5) has no solutions satisfying the given boundary conditions, and we must ask ourselves: Where is the mistake?

The mistake is in the boundary condition. If we dismiss the requirement that $u(a)$ must go to zero when $a$ goes to zero, and instead state that $u(a)$ goes to zero at some point $a_{\min} > 0$, which is not quite zero, and that $u(a)$ is identically zero for all $0 \leq a < a_{\min}$, then -if $a_{\min}$ is chosen appropriately- the functions $u_n(\xi)$ of Eq.(3.6) are solutions to the eigenvalue equation (3.5), and everything is well.

As an example, consider the solution, where $n = 1$. Because $H_1(x) = 2x$, we have

$$u_1(\xi) = N_1 \xi e^{-\frac{1}{2} k^2 \xi^2}.$$  

(3.11)

If we now state the boundary condition in such a way that if $a$ goes to zero then $u_1(a)$ goes to zero, then that would mean that $u_1(\xi)$ goes to zero when $\xi$ goes to $\xi_1 = -\sqrt{3} (1/k)$, and we readily find that $u_1(\xi)$ does not satisfy this boundary condition. However, if we state the boundary condition in such a way that $u_1(a)$ goes to zero when $a$ goes to

$$a_{\min}(1) := \sqrt{\frac{1}{k}},$$  

(3.12)

and vanishes identically for all $a < a_{\min}(1)$, then that would mean that $u_1(\xi)$ goes to zero when $\xi$ goes to zero and vanishes identically when $\xi < 0$. We readily observe that if we
define \( u_1(\xi) \) in such a way that it is the function \( u_1(\xi) \) of Eq.(3.11), when \( \xi > 0 \), and that it vanishes identically otherwise, then this kind of \( u_1(\xi) \) satisfies the given boundary condition, and it is also a solution to the eigenvalue equation (3.5). In other words, it is the \textit{physical state} of the quantum black hole, which determines the boundary condition of the wave function.

The example we had above gives us a clue to the boundary conditions of the wave function of a general stationary state of the Schwarzschild black hole. Since the functions \( u_n(\xi) \) of Eq.(3.6) are the only solutions to the eigenvalue equation (3.5), we must state our boundary condition in such a way that the functions \( u_n(\xi) \) of Eq.(3.6) satisfy it. In other words, if we pick up some point \( \tilde{\xi} \), and say that \( u_n(\xi) \equiv 0 \) for all \( \xi \leq \tilde{\xi} \), then the point \( \tilde{\xi} \) must be a zero of \( u_n(\xi) \).

Now, because every function \( u_n(\xi) \) has \( n \) zeros, we have plenty of choice. The most natural choice is to choose the smallest of these zeros. Since the zeros of the function \( u_n(\xi) \) are \( 1/k \) times the zeros of the Hermite polynomial \( H_n(x) \), we denote the smallest zero of \( H_n(x) \) by \( h_n \), and state the boundary condition of the function \( u_n(\xi) \) in the following way:

\[
\forall \xi \leq \frac{1}{k} h_n.
\] (3.13)

Using Eqs.(3.1), (3.4), (3.6) and (3.7) we can at last write the wave functions \( \psi_n(a) \) corresponding to the stationary states of the Schwarzschild black hole: If

\[
a > \frac{1}{k} (h_n + \sqrt{2n+1}),
\] (3.14.a)

then

\[
\psi_n(a) = N_n H_n(ka - \sqrt{2n+1}) \frac{1}{a} \exp\left[-\frac{1}{2}k^2(a - \frac{1}{k} \sqrt{2n+1})^2\right],
\] (3.14.b)

and if

\[
a \leq \frac{1}{k} (h_n + \sqrt{2n+1}),
\] (3.15.a)

then

\[
\psi_n(a) \equiv 0.
\] (3.15.b)

Because \( H_0 \) has no zeros, we must have \( n = 1, 2, 3... \)

It should be noted that, if our Schrödinger equation is true, we have now solved the singularity problem of the Schwarzschild black hole. \textit{There is no singularity inside the black hole.} This can be seen from Eq.(3.15), which states that below some positive value of \( a \), which depends on the quantum state of the black hole, the wave function vanishes identically. This means that in a given quantum state the radius \( a \) of the throat of the Schwarzschild black hole can never take values below a certain fixed positive value of \( a \). In other words, an observer inside the Schwarzschild black hole can never fall into the singularity, where \( a = 0 \), because no such singularity does exist.

After this lengthy talk about the boundary conditions of the wave function we can now go into the physical predictions given by our model. It follows from Eqs.(3.3) and (3.7) that the eigenvalues of the area

\[
A_S = 4\pi R_S^2
\] (3.16)
of the event horizon of the Schwarzschild black hole are:

\[ A_S(n) = (n + \frac{1}{2}) \frac{32\pi}{c^3} hG, \quad (3.17) \]

where \( n = 1, 2, 3... \) When Eq.(3.17) is put in numbers, we get:

\[ A_S(n) = (n + \frac{1}{2}) \times 2.63 \times 10^{-68} m^2. \quad (3.18) \]

In other words, the area of the event horizon of the black hole can take only discrete values such that the quanta of the area are of the same order of magnitude as the Planck area. This result is similar to the one given by Bekenstein and others.[4−8] It is also in harmony with the results on the quantization of area in quantum gravity in general obtained by Ashtekar, Rovelli and Smolin.[2] In this paper we obtained the similar result for the Schwarzschild black hole by means of an explicit calculation.

From Eqs.(1.3) and (3.17) it follows that the mass eigenvalues of the Schwarzschild black hole are:

\[ M_n = \sqrt{2n + 1} \sqrt{\frac{hc}{G}} = \sqrt{2n + 1} \times 2.18 \times 10^{-8} kg, \quad (3.19) \]

and so the corresponding energy eigenvalues are:

\[ E_n = \sqrt{2n + 1} \sqrt{\frac{hc^5}{G}} = \sqrt{2n + 1} \times 1.96 \times 10^9 J. \quad (3.20) \]

As one can see, the Schwarzschild black hole has a certain ground state, where \( n = 1 \). The energy of this ground state is

\[ E_1 = 3.39 \times 10^9 J. \quad (3.21) \]

According to our model, this is the lowest energy state the Schwarzschild black hole can have.

It was shown by Hawking long ago that black holes are not completely black but they can emit radiation.[1] When they emit radiation they lose their mass. It is now very interesting to see, what kind of predictions could be made of the spectrum on this so called Hawking radiation, using our simple model. It follows from Eq.(1.3) that when the area is changed from \( A_S \) to \( A_S + dA_S \), the mass \( M \) is changed from \( M \) to \( M + dM \) such that:

\[ dM = \frac{c^4}{32\pi G^2} \frac{1}{M} dA_S. \quad (3.22) \]

Because it follows from Eq.(3.16) that \( dA_S \) is some integer times a certain area, we see that the energy emitted by a macroscopic black hole when it performs a transition from state \( n_1 \) to the state \( n_2 = n_1 - n \), is:

\[ \Delta E_n = n \frac{c^3 h}{G M} = n \times 42.6 J kg^{-1} M. \quad (3.23) \]
As an example, let us assume that the mass $M$ of the black hole is ten solar masses, or $2.0 \times 10^{31} \text{kg}$. In that case we find that the energies of the quanta of the Hawking radiation are of the form

$$\Delta E_n = n \times 1.3 \times 10^{-11} \text{eV},$$

and the corresponding frequencies are:

$$\nu_n = n \times 3.2k\text{Hz}.$$  \hspace{1cm} (3.25)

The curious fact that the differences between the frequencies of the quanta are of the same order of magnitude as the resolving power of an ordinary radio receiver, raises some hopes about a possibility to observe, in some very distant future, genuine quantum gravitational effects by measuring the spectrum of Hawking radiation, although this problem, of course, deserves a much more detailed analysis.[12]

\section*{4. Conclusion}

In our analysis on the quantum mechanical properties of the Schwarzschild black hole there were three main points. They were an adoption of the point of view of an observer very far away and at rest relative to the black hole, the use of the Hamiltonian dynamics of asymptotically flat spacetimes in a form developed long ago by Regge and Teitelboim[9], and our decision to describe the gravitational degrees of freedom of the Schwarzschild black hole by one free variable. This variable was the radius $a$ of the throat of the black hole as such as it is observed by an observer in a radial free fall inside the black hole horizon. By using the formalism of Regge and Teitelboim, we wrote the classical Hamiltonian $H$ of the Schwarzschild black hole in terms of $a$ and its canonical momentum $p$. We then constructed the quantum mechanical Hamiltonian operator $\hat{H}$ corresponding to our minisuperspace-type model from the classical Hamiltonian $H$. The eigenvalues of $\hat{H}$ are the ADM energies of the black hole. By writing the eigenvalue equation we obtained the time-independent Schrödinger equation of the black hole.

Our Schrödinger equation, when written in terms of $a$, turned out to be surprisingly simple. Indeed, we were able to solve that equation explicitly. The solutions, which gave the mass and energy eigenstates of the black hole, implied the quantization of the area of the event horizon in a manner which is in harmony with the results obtained by Bekenstein and others [4-8]. The result is also entirely in harmony with the general results of Ashtekar, Rovelli and Smolin on the area quantization in quantum gravity[2]. Moreover, it was found that the black hole has a certain ground state in which its mass is non-zero.

Perhaps the most striking result of our analysis, however, was the conclusion that there is no singularity inside the black hole horizon. Indeed, we found that if there were a singularity, then our Schrödinger equation would not have any physically acceptable solutions. It was also very interesting to observe that, in a certain sense, the physical states of the quantum black hole determine their own boundary conditions.
The physical reliability of the results mentioned above depends, of course, on whether one accepts that the geometrical degrees of freedom of the Schwarzschild black hole can be described, at least qualitatively, by means of the variable we used in our analysis, and on whether one accepts the analysis which lead to the Schrödinger equation (2.40). However, if one accepts the Schrödinger equation, one is also compelled to accept the results, and the fact that at least the results related to area quantization have also been obtained by some others, suggests that perhaps our model is not completely erroneous.

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