Abstract
We introduce a family of local deformations for meromorphic connections on $\mathbb{P}^1$ in the neighborhood of a higher rank (simple) singularity. Following the scheme in Malgrange [16-18] we use these local models to prove that the zeros of the tau function, introduced by Jimbo, Miwa, and Ueno in their pioneering work on “Birkhoff” deformations at irregular singular points [12], occur at precisely those points in the deformation space at which a certain Birkhoff-Riemann-Hilbert problem fails to have a solution.

§1 Introduction

The Riemann-Hilbert problem and monodromy preserving deformations. Suppose that $A(x)$ is an $p \times p$ matrix with entries that are rational functions of $x$ on $\mathbb{P}^1$. Linear differential equations

$$
\frac{d\psi}{dx} = A(x)\psi,
$$

arise in many important applications and have been studied intensively for more than 100 years. The nature of the singularities in the coefficient matrix $A(x)$ at its poles has a lot to do with the character of the solutions to (1.0). For example, in the neighborhood of a point $a$ where $A(x)$ has but a simple pole it is well known that the equation (1.0) has a fundamental solution, $\Psi(x)$, with polynomially limited growth as $x \to a$ (by which we mean that solutions are dominated near $x = a$ by $c|x - a|^{-N}$ where $c$ and $N$ are constants). Higher order poles in $A(x)$ typically produce local fundamental solutions with more complicated exponential-polynomial growth near the pole. Fundamental solutions to the equation (1.0) are usually not single valued. If one has a fundamental solution $\Psi(x)$ defined for $x$ in some small ball not containing any of the poles, $\{a_1, a_2, \ldots, a_n\}$, of $A(x)$ then it is possible to analytically continue $\Psi(x)$ along paths in $\mathbb{P}^1$ which avoid these poles. The analytic continuation depends only on the homotopy class of the path, so that the resulting function, $\Psi(x)$, is defined not on $\mathbb{P}^1$ but for $x$ in the simply connected covering space, $\mathcal{R}(X)$, of $X := \mathbb{P}^1 \backslash \{a_1, a_2, \ldots, a_n\}$ with projection $\pi : \mathcal{R}(X) \to X$. If
\( x_0 \in \mathcal{R}(X) \) and \( \gamma \) is a closed path in \( X \) with base point \( \pi(x_0) \) then

\[
\Psi(\gamma \cdot x_0) = \Psi(x_0)M(\gamma)^{-1},
\]

where \( x_0 \rightarrow \gamma \cdot x_0 \) is the natural action of the homotopy class of the path \( \gamma \) on \( x_0 \in \mathcal{R}(X) \), and \( \gamma \rightarrow M(\gamma) \) is an \( n \) dimensional representation of the fundamental group \( \pi_1(X, \pi(x_0)) \). Riemann’s analysis of the solutions of the hypergeometric equation in terms of its monodromy representation \( M(\cdot) \) led to the general problem of determining if any representation of the fundamental group of the punctured sphere could arise as the monodromy representation for a linear differential equation (1.0). It is clear from simple examples, that there will not be a unique association between a representation of \( \pi_1 \) and a differential equation (1.0) without some further restriction on the differential equation. It is possibly most natural to look for a differential equation with simple poles to realize a given monodromy representation. It is not always possible to solve this problem \([1]\) but if one relaxes the condition on the differential equation to admit regular singular points then then it is classical that the inverse monodromy problem always has a solution (a good discussion of the confusion about the status of the solution to this problem can also be found in \([1]\)). A pole \( a \) of \( A(x) \) in (1.0) is said to be a regular singular point if fundamental solutions to (1.0) have at worst polynomial growth at \( a \). Because the fundamental solutions are defined on the simply connected covering and paths that wind around the point \( a \) pick up powers of the local monodromy, the precise notion of polynomial growth requires some restriction to sectors in the covering space \([1]\). More important for us, however, is a deformation variant of the simple pole condition. Suppose that for some collection of points \( \{ a_0^1, a_0^2, \ldots, a_0^n \} \) a given representation of

\[
\pi_1 \left( \mathbb{P}^1 \setminus \{ a_0^1, a_0^2, \ldots, a_0^n \}, a_0 \right)
\]

can be realized by a differential equation (1.0) with simple poles at \( a_0^j \ j = 1, 2, \ldots, n \) and a fundamental solution \( \Psi(x) \) normalized to \( \Psi(a_0) = I \) at the base point \( a_0 \). In this case, we write \( A^0(x) \) for the matrix coefficient in the differential equation (1.0) which realizes the appropriate monodromy representation. Since \( A^0(x) \) has simple poles,

\[
A^0(x) = \sum_{\nu=1}^{n} \frac{A^0_\nu}{x - a_\nu^0}
\]

The problem, first formulated by Schlesinger \([19]\), is to ask whether it is possible to deform the coefficients \( A^0_\nu \) in \( A^0(x) \) as functions of the pole locations \( a_\nu \) so that the differential equation (1.0) with coefficient matrix

\[
A(x) = \sum_{\nu=1}^{n} \frac{A_\nu(a)}{x - a_\nu}
\]
realizes the same monodromy representation as as the differential equation with coefficient matrix (1.1) (we will be more precise about what this means later on). Note that we’ve written \( a = (a_1, a_2, \ldots, a_n) \) and we want

\[ A_\nu(a^0) = A_\nu(a_1^0, a_2^0, \ldots, a_n^0) = A_\nu^0. \]

When the point at infinity is a regular point and \( a_0 = \infty \), Schlesinger showed that if such a deformation exits the coefficients \( A_\nu \) must satisfy a non-linear system of differential equations

\[ dA_\mu = -\sum_{\nu \neq \mu} \frac{A_\mu A_\nu - A_\nu A_\mu}{a_\mu - a_\nu} d(a_\mu - a_\nu) \]

now called the Schlesinger equations. A modern treatment of the existence question can be found in [17]. If we write \( a \in C^n \) then as might be guessed from looking at the Schlesinger equations it is important to remove the points at which \( a_\mu = a_\nu \) from consideration. Let

\[ \Delta_{\mu \nu} = \{ a \in C^n | a_\nu = a_\mu \}, \]

and define

\[ Z^n = C^n \setminus \cup_{\nu \neq \mu} \Delta_{\mu \nu}. \]

As observed by Malgrange an appropriate place to seek monodromy preserving deformations is the simply connected covering space \( \mathcal{R}(Z^n) \to Z^n \). Because this same space will enter our considerations with a different significance later on, it will be useful at this point to introduce distinctive notation for elements in \( \mathcal{R}(Z^n) \) and their projection onto \( Z^n \). We will write \( t \in \mathcal{R}(Z^n) \) and \( a(t) \in \mathcal{R}(Z^n) \) for the projection of \( t \) on \( \mathcal{R}(Z^n) \subset C^n \). We write \( a_j(t) \) for the \( j^{th} \) component of \( a(t) \).

As pointed out in [17] and [11] the space \( \mathcal{R}(Z^n) \) has a number of advantages as a deformation space. Not only is it simply connected but since \( Z^n \) is contractible [7] the long exact sequence associated with the fiber bundle projection

\[ a : \mathcal{R}(Z^n) \to Z^n \]

shows that all the higher homotopy groups of \( \mathcal{R}(Z^n) \) are also trivial and hence that \( \mathcal{R}(Z^n) \) is contractible. Since \( Z^n \) is the complement of the zero set of an analytic function on \( C^n \) it is a Stein space and since \( \mathcal{R}(Z^n) \) is an unramified covering of \( Z^n \) it too is a Stein space [9]. The consequent triviality of the sheaf cohomology \( H^1(\mathcal{R}(Z^n), O^*) \) plays a role in giving a global definition of tau functions following [11]. Finally, although it will not be the principal focus of the deformations we consider in this paper we indicate the crucial property used to construct the
Schlesinger deformations considered in [17] and [11]. Let $Y_k$ denote the subset of $\mathbb{P}^1 \times \mathcal{R}(\mathbb{Z}^n)$ given by

$$Y_k = \{(x, t) | x = a_k(t)\}.$$ 

Let

$$Y_\infty = \{ (\infty, t) | t \in \mathcal{R}(\mathbb{Z}^n) \},$$

and

$$Y = Y_\infty \cup Y_1 \cup Y_2 \cup \cdots \cup Y_n.$$ (1.1)

Then the property, which is exploited in [17] to construct the Schlesinger deformations, is that for any choice of $t_0 \in \mathcal{R}(\mathbb{Z}^n)$, the injection

$$\mathbb{P}^1 \setminus \{a_1^0, a_2^0, \ldots, a_n^0, \infty\} \ni x \to (x, t^0) \in \mathbb{P}^1 \times \mathcal{R}(\mathbb{Z}^n) \setminus Y$$

induces an isomorphism of fundamental groups (where $a_j^0 = a_j(t^0)$ is the $j^{th}$ coordinate of the projection). Using the correspondence between flat connections and representations of the fundamental group from [6], this allows one to prolong the original connection from $\mathbb{P}^1 \setminus \{a_1^0, a_2^0, \ldots, a_n^0, \infty\}$ to $\mathbb{P}^1 \times \mathcal{R}(\mathbb{Z}^n) \setminus Y$ in a holonomy preserving fashion. The crux of the existence proof for the Schlesinger deformations is then to extend this connection to a neighborhood of $Y$ so that it has logarithmic poles along $Y$. In [17] (and by a similar construction in [11]) this is accomplished by exhibiting local deformations in a neighborhood of each $Y_k$ and then proving that these can be fit together to provide a solution to the deformation problem if and only if a certain Fredholm integral equation has a solution. We will consider a related construction later on in this paper.

Some further analysis then leads to the existence of a holomorphic function $\tau$ defined on $\mathcal{R}(\mathbb{Z}^n)$ whose 0 set is an exceptional set for the solution of the original deformation problem. In this paper we wish to use similar constructions to examine a somewhat different class of deformations. Our goal will be to show that the $\tau$ function introduced by Jimbo, Miwa, and Ueno [12] in a study of such deformations has a similar property. Namely, that the zeros of these tau functions are exceptional sets for the solution of a deformation problem.

**Birkhoff deformations.** The deformations we wish to consider are associated with Birkhoff’s generalization of the Riemann-Hilbert problem [5]. To understand what is involved it is useful to review the analysis of solutions to linear differential equations

$$\frac{d\psi}{dx} = A(x)\psi,$$

with a rational matrix valued coefficient $A(x)$ in a neighborhood of an irregular singular point. A helpful modern review of this subject can be found in [23]. Suppose that $x = a$ is a pole of $A(x)$ of order $r + 1$ then we have

$$A(x) = A_r(x-a)^{-r-1} + A_{r-1}(x-a)^{-r} + \cdots$$
The integer \( r \) is called the rank of the singularity and for the moment we will confine our attention to the case \( r \geq 1 \). The theory of formal solutions to (1.0) is much simplified by one further assumption which we will make from now on. We require that

**Standing Assumption (1.2).** The coefficient, \( A_r \), of the leading singularity in the Laurent expansion of \( A(x) \) at \( x = a \) has distinct eigenvalues.

This assumption, of course, guarantees that \( A_r \) can be diagonalized by a non singular matrix \( G \)

\[
G^{-1} A_r G = \Lambda_r
\]

where \( \Lambda_r \) is a diagonal matrix with distinct complex entries. In such circumstances (1.0) has a unique formal fundamental solution

\[
\hat{\Psi}(x, t) = G\hat{\alpha}(x)e^{H(x)}
\]

where \( \hat{\alpha}(x) \) is an \( n \times n \) matrix valued formal power series

\[
\hat{\alpha}(x) = I + \beta_1(x-a)^1 + \beta_2(x-a)^2 + \cdots
\]

and

\[
H(x) = \Lambda_r \frac{(x-a)^{-r}}{-r} + \Lambda_{r-1} \frac{(x-a)^{-r+1}}{-r+1} \cdots + \Lambda_1 \frac{(x-a)^{-1}}{-1} + \Lambda_0 \log(x-a),
\]

where all the matrices \( \Lambda_k \) are diagonal with diagonal entries

\[
\Lambda_{k,j} := (\Lambda_k)^{jj}.
\]

The construction of this formal solution hinges on the inversion of \( \text{ad}(A_r) \) acting on the off diagonal matrices. Since the eigenvalues of \( \text{ad}(A_r) \) acting on the off diagonal matrices are differences of distinct eigenvalues for \( A_r \), our standing assumption guarantees that this can be done. It is quite typical that the series for \( \hat{\alpha} \) does not converge and this leads to some complication in making a connection between the formal solution to (1.0) and genuine solutions to this equation. Before we turn to this matter we mention a slightly different way of looking at this result which will make it simpler for the reader to connect this way of thinking with the developments in [17], [11], and [23].

Now write \( \partial = \frac{\partial}{\partial x} \) and \( \overline{\partial} = \frac{\partial}{\partial \overline{x}} \) and instead of the differential equation (1.0) one regards the connection

\[
dx \otimes (\partial - A(x)) + dx \otimes \overline{\partial}, \quad (1.3)
\]
on the trivial bundle
\[ \mathbb{P}^1 \times \mathbb{C}^p \to \mathbb{P}^1, \]
as the fundamental object. If \( \{a_1, a_2, \ldots, a_n\} \) is the set of poles for \( A(x) \) then flat sections, \( \psi \), for (1.3) defined locally in \( \mathbb{P}^1 \setminus \{a_1, a_2, \ldots, a_n\} \) are solutions to the differential equation (1.0).

Gauge transformations (e.g., multiplication by smooth invertible matrix valued functions of \( x \)) which are holomorphic (or meromorphic) in \( x \) do not change the \( d\bar{x} \otimes \bar{\partial} \) part of the connection (at least away from the singularities) and the solution of the differential equation (1.0) is effectively accomplished by gauging \( \partial - A(x) \) into diagonal form by such a gauge transformation. The formal gauge substitution
\[ \psi \leftarrow \hat{G} \hat{\alpha} \psi \]

can then be seen to reduce the connection (1.3) to the diagonal form
\[ dx \otimes (\partial - h(x)) + d\bar{x} \otimes \bar{\partial}, \]
where \( h(x) := \frac{dH}{dx} \). A fundamental solution to the differential equation
\[ (\partial - h(x)) \psi = 0, \quad (1.4) \]
is given by
\[ \Psi = e^{H(x)}, \]
which is well defined modulo the possible appearance of a multivalued log term in \( H(x) \). This accounts for the structure of the formal fundamental solution above but the relation between this formal solution and a genuine fundamental solution is complicated by Stokes’ phenomena which we will now describe.

We first adopt some notation and definitions from [23]. \( \Sigma \) will denote a sector with vertex at the origin, consisting of points \( r e^{i\theta} \) with \( r > 0 \) and argument \( \theta \in (a, b) \) with \( 0 \leq a < b < 2\pi \). For \( \delta > 0 \), we write \( \Sigma_\delta \) for the subset of \( \Sigma \) with \( r < \delta \). If \( \Sigma \) and \( \Sigma' \) are two sectors we write \( \Sigma' \subset \subset \Sigma \) if the bounding rays for \( \Sigma' \) are contained in \( \Sigma \). An open set \( \Omega \subset \Sigma \) is said to be asymptotic to the sector \( \Sigma \), if for each \( \Sigma' \subset \subset \Sigma \) we have \( \Sigma' \subset \subset \Omega \) for all sufficiently small \( \delta \). We introduce \( \mathcal{A}(\Sigma) \), a complex algebra of germs of analytic functions defined on open sets asymptotic to \( \Sigma \) consisting of functions which are asymptotic to formal meromorphic series. The asymptotic condition on \( f \in \mathcal{A}(\Sigma) \) is understood to mean that there exists a formal series
\[ \hat{f}(x) \sim \sum_{k \geq m} f_k x^k, \]
with \( m > -\infty \) so that for any \( \Sigma' \subset \subset \Sigma \) and any integer \( M \) one has

\[
f(x) = \sum_{k \geq m}^{M} f_k x^k + O(|x|^{M+1}) \text{ as } x \to 0 \text{ in } \Sigma'.
\]

The basic local existence result for solutions of (1.0) near \( x = a \) (and we take \( a = 0 \) for convenience) is that if a sector \( \Sigma \) is chosen appropriately, there is a function (or germ) \( \alpha_{\Sigma} \in \mathcal{A}(\Sigma) \) which is asymptotic to \( \tilde{\alpha} \) with the property that the gauge transformation by \( \alpha_{\Sigma}^{-1} \) reduces the differential equation (1.0) to the diagonal form (1.4) in an open set \( \Omega \) asymptotic to \( \Sigma \). To describe the non-uniqueness for \( \alpha_{\Sigma} \), upon which the Stokes’ phenomena hinges, we introduce

\[
\alpha_{\Sigma,k} = k^{th} \text{ column of } \alpha_{\Sigma},
\]

and

\[
H_k(x) = k^{th} \text{ diagonal entry of } H(x).
\]

Another way to state the existence result mentioned above is that the \( n \) vector valued functions

\[
\psi_k(x) = \alpha_{\Sigma,k}(x)e^{H_k(x)},
\]

are independent solutions to (1.0) in some open set \( \Omega \) asymptotic to the sector \( \Sigma \). Because the functions \( \alpha_{\Sigma,k}(x) \) are asymptotic to power series the “growth” of the functions \( \psi_k(x) \) as \( x \to 0 \) is controlled by the exponential factors \( e^{H_k(x)} \) which have absolute value \( e^{\Re H_k(x)} \) (where \( \Re x \) = real part of \( x \)). Now let

\[
\Delta H_{jk}(x) = \Re(H_j(x) - H_k(x)).
\]

The curves along which

\[
\Delta H_{jk}(x) = 0,
\]

play an important role in understanding the relationship between formal solutions and analytic solutions near the singularity at \( x = 0 \). To get an idea of what such curves look like near 0 it is enough to consider the leading order equivalent of (1.5). This is

\[
\Re(\Lambda_{r,j} - \Lambda_{r,k})x^{-r} = 0.
\]

Since each difference \( \Lambda_{r,j} - \Lambda_{r,k} \neq 0 \) it follows that there are \( 2r \) rays emanating from 0 which satisfy (1.6) given by

\[
\arg x = \frac{1}{r} \left( \arg(\Lambda_{r,j} - \Lambda_{r,k}) + (n + \frac{1}{2})\pi \right), \text{ for } n = 0, \ldots, 2r - 1.
\]
These rays are called Stokes’ lines. Near \( x = 0 \) the family of solutions to (1.5) consists of \( 2r \) curves each asymptotic to one of the Stokes’ lines (1.6). The property of the Stokes’ lines that will be important for us is that any open sector which contains one of the Stokes’ lines (1.6) will contain points with \( \Delta H_{jk} < 0 \) and also points with \( \Delta H_{jk} > 0 \).

Now suppose that one crosses such a Stokes’ line going from \( \Delta H_{jk} < 0 \) to \( \Delta H_{jk} > 0 \). One moves from a region in which \( \psi_k(x) \) dominates \( \psi_j(x) \) as \( x \to 0 \) to a region in which this dominance relation is reversed. For the purpose of illustration assume that \( \Delta H_{jk}(x) < 0 \) for \( x \) in some truncation \( \Sigma_\delta \) of \( \Sigma \). Then for any constant \( c \) one finds

\[
\psi_k + c\psi_j = (\alpha_{\Sigma,j,k} + c\alpha_{\Sigma,j}e^{H_j - H_k})e^{H_k},
\]

and because \( e^{H_j - H_k} \) is exponentially small in \( \Sigma_\delta \) it follows that

\[
\alpha_{\Sigma,j,k} + c\alpha_{\Sigma,j}e^{H_j - H_k} \sim \hat{\alpha}_k
\]

where \( \hat{\alpha}_k \) is the \( k^{th} \) column of the formal series \( \hat{\alpha} \). Thus the less dominant solution \( \psi_j \) may be freely mixed in with \( \psi_k \) without effecting the asymptotics of \( \alpha_{\Sigma,k} \). In this way one sees that genuine solutions with a given exponential behavior are not uniquely determined by the asymptotics of their “power series component”. One obvious way to “cure” this particular non-uniqueness would be to include a Stokes’ line from the family \( \Delta H_{jk} = 0 \) in the sector \( \Sigma \). The exchange of dominance across the line makes it impossible to alter \( \psi_k \) by adding in multiples of \( \psi_j \) without changing the asymptotics of \( \alpha_{\Sigma,k} \) and vice versa. In fact, if one includes exactly one Stokes’ line from each of the families (1.6) for \( j < k \) then a simple argument \([22]\) shows that a genuine fundamental solution

\[
\Psi = G\alpha_{\Sigma}e^H,
\]

in \( \Sigma_\delta \) is uniquely determined by the condition that the asymptotic expansion of \( \alpha_{\Sigma} \) is given by \( \hat{\alpha} \). What’s more there is also an existence result for such sectors which can be proved using a variant of the usual integral equation technique but employing different contours for the each of the matrix elements in the solution (see the references in \([23]\)). Following \([16]\) we call such sectors good sectors. It is not hard to see that a punctured neighborhood of 0 can be covered by \( 2r \) (truncated) good sectors \( \Sigma_{i,\delta} \) which we will take to be arranged in counterclockwise order starting with \( \Sigma_{1,\delta} \). Because each good sector contains exactly one Stokes’ line from each family (1.5), it follows that the intersections \( \Sigma_{i,\delta} \cap \Sigma_{i+1,\delta} \) do not contain any Stokes’ lines. On this overlap the two local fundamental solutions \( \alpha_{\Sigma_{i}e^H} \) and \( \alpha_{\Sigma_{i+1}e^H} \) of (1.0) necessarily differ by a constant invertible \( p \times p \) matrix \( S_{i,i+1}, \)

\[
\alpha_{\Sigma_{i+1}e^H} = \alpha_{\Sigma_{i}e^H}S_{i,i+1}.
\]
The matrices $S_{i,i+1}$ are called Stokes’ multipliers and must satisfy a triangularity property which we will now explain. Since $\Sigma_{i,\delta} \cap \Sigma_{i+1,\delta}$ does not contain any Stokes’ lines, it follows that for each fixed choice of $(j, k)$ the quantity $\Delta H_{jk}$ is either always positive or always negative in $\Sigma_{i,\delta} \cap \Sigma_{i+1,\delta}$ (at least if $\delta$ is small enough so that the Stokes’ lines are good “stand ins” for the curves $\Delta H_{jk} = 0$). Thus there is a fixed dominance ordering

\[
\Re \Lambda_{r,i_1}x^{-r} > \Re \Lambda_{r,i_2}x^{-r} > \cdots > \Re \Lambda_{r,i_n}x^{-r},
\]

for $x \in \Sigma_{i,\delta} \cap \Sigma_{i+1,\delta}$, and some permutation $(i_1, i_2, \ldots, i_n)$ of $1, 2, \ldots, n$. If we write (1.7) in matrix form relative to the ordered basis $\{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ then the matrix of $S_{i,i+1}$ relative to this ordered basis must be lower triangular with 1’s on the diagonal in order that $S_{i,i+1}$ should only alter $\alpha_{\Sigma_{i}}$ by exponentially small terms. Another (basis independent) version of the same observation is that

\[
e^H S_{i,i+1}e^{-H} = I + O(\|x\|^N), \text{ for } x \to 0 \text{ in } \Sigma_{i,\delta} \cap \Sigma_{i+1,\delta}, \tag{1.8}\]

for some $\epsilon > 0$ and all positive integers $N$. The relation (1.7) allows us to construct a fundamental solution $\Psi$ to (1.0) in a neighborhood of $x = a$ by analytically continuing the fundamental solution $\alpha_{\Sigma_{1}}e^{H}$ from $\Sigma_{1,\delta}$ to $\Sigma_{2,\delta}$ to $\Sigma_{3,\delta}$ and etc. The result is

\[
\Psi(x) = \begin{cases} 
\alpha_{\Sigma_{1}}(x)e^{H(x)} & \text{for } x \in \Sigma_{1,\delta} \\
\alpha_{\Sigma_{2}}(x)e^{H(x)}S^{-1}_{1,2} & \text{for } x \in \Sigma_{2,\delta} \\
\cdots \\
\alpha_{\Sigma_{2r}}(x)e^{H(x)}S^{-1}_{1,2r} & \text{for } x \in \Sigma_{2r,\delta}
\end{cases} \tag{1.8}
\]

where we’ve written

\[S_{1,k} = S_{1,2}S_{2,3} \cdots S_{k-1,k},\]

and it is understood that the logarithmic term in $H(x)$ is analytically continued from $\Sigma_{1}$ to $\Sigma_{2}$ to ... to $\Sigma_{2r}$. If we write $S_{2r,1}$ for the Stokes’ multiplier connecting $\Sigma_{2r}$ with $\Sigma_{1}$ then it is not hard to see that the analytic continuation of $\alpha_{\Sigma_{1}}(x)e^{H(x)}$ around $x = a$ comes back to its original value multiplied on the right by,

\[
e^{H(e^{2\pi i}x)-H(x)}(S_{1,2r}S_{2r,1})^{-1} = e^{2\pi i\Lambda_0}(S_{1,2r}S_{2r,1})^{-1}. \tag{1.9}\]

The matrix (1.9) is thus the local monodromy for the fundamental solution (1.8). We will refer to the exponent of formal monodromy $\Lambda_0$, together with the Stoke’s multipliers $S_{i,i+1}$ as the generalized monodromy data at each of the singularities for (1.0) at $x = a$. Roughly speaking the deformations of (1.0) we are interested in are those that fix the local generalized monodromy data at each of the singularities for (1.0) and fix the global
monodromy for (1.0) but permit the formal expansion coefficients \( \{ \Lambda_r, \Lambda_{r-1}, \ldots, \Lambda_1 \} \) at each singularity to vary. The global monodromy is precisely the representation of the fundamental group described earlier and we will say a more about this later on following the presentation in Malgrange [17] and Helmink [11]. Generalizations of the Schlesinger deformations in which the location of the poles are varied are also quite interesting; however, the issues we wish to pursue have already been treated for these deformations in [17] and [11], and so for the moment we confine our attention to deformations of the local exponents \( \{ \Lambda_r, \Lambda_{r-1}, \ldots, \Lambda_1 \} \). Our strategy in studying these deformations will follow closely the developments in [18]. In fact, in [18] Malgrange already proves existence results for such deformations in the irregular singular case. However, we did not understand how to make use of his results to establish the connection with the JMU tau function that is the principal object in this paper. Instead, we will adapt an integral equation technique from Flaschka and Newell [8] to produce local models for the desired deformation at each of the poles. These local deformations are then fit together by solving the same Toeplitz integral equations that one finds in [17]. The tau function is introduced by identifying its log-derivative as the connection one form for a flat connection on an appropriate determinant bundle. A computation shows that this connection one form differs from the JMU connection one form by a regular term and so the tau function we’ve introduced and the JMU tau functions have the same 0 set. We show that this 0 set is precisely the exceptional set for the existence of the deformations we are considering.

**Local analysis and Stokes’ multipliers.** It will be useful at this point to be a little more precise about the nature of the generalized local monodromy data that is to be “fixed” under the deformations that are of interest to us. These deformations concern the local model for our connections. Suppose that one starts with a holomorphic connection \( \nabla^0 \) defined on a trivial bundle over a punctured neighborhood of the point \( a \in \mathbb{P}^1 \) with a singularity of type \( r \) at \( a \) (note: we will put a bar over connections defined on subsets of \( \mathbb{P}^1 \) to distinguish them from the connections in many variables that will soon appear as deformations). Suppose that the connection satisfies our standing assumption and that the trivialization is chosen so that the leading singularity in the one form for \( \nabla^0 \) has a diagonal matrix coefficient. Then \( \nabla^0 \) is formally gauge equivalent to the diagonal form

\[
d_x - d_x H_0 - \Lambda_0 \frac{dx}{x - a}
\]

where

\[
H_0 = \sum_{j=1}^{r} \frac{\Lambda_j^0}{-j} (x - a)^{-j}
\]
where $\Lambda^0_j$ and $\Lambda_0$ are diagonal matrices. More precisely there is a formal gauge transformation
\[
\tilde{\alpha}^0(x) = I + \beta_1^0(x-a) + \beta_2^0(x-a)^2 + \ldots
\]
so that
\[
\nabla^0 = \tilde{\alpha}^0 \cdot \left[ d_x - d_x H_0 - \Lambda_0 \frac{dx}{x-a} \right],
\]
where $\tilde{\alpha} \cdot [X] = \tilde{\alpha}[X]\tilde{\alpha}^{-1}$. Note that it is actually the inverse of $\tilde{\alpha}^0$ which reduces $\nabla^0$ to diagonal form. This is just how things work out if the relationship between $\tilde{\alpha}^0$ and a fundamental solution to $\nabla^0$ is given along the lines explained above.

The local analytic equivalence class of $\nabla^0$ is determined by further data which can be specified by choosing a covering of the punctured neighborhood of $a$ by good sectors, $\Sigma_1, \Sigma_2, \ldots, \Sigma_{2r}$ (to somewhat unburden the notation we will write $\Sigma_k$ for the truncated sector $\Sigma_k, \delta$ when the precise value of $\delta$ is not an issue). For simplicity in the following discussion we will always suppose that such a covering is obtained by first choosing a good sector $\Sigma_1$. The other sectors $\Sigma_j$ are obtained from $\Sigma_1$ by rotating counterclockwise by $\frac{j}{r}$ radians. The Stokes’ multipliers $S_{j,j+1}$ which connect local fundamental solutions with fixed asymptotics in the sectors $\Sigma_j$ and $\Sigma_{j+1}$ (with $\Sigma_{2r+1} = \Sigma_1$) then determine the local holomorphic equivalence class of $\nabla^0$. The deformations we wish to focus on allow the local model
\[
\nabla_{\Lambda} := d_x - d_x H - \Lambda_0 \frac{dx}{x-a} \quad (1.12)
\]
with
\[
H = \sum_{j=1}^{r} \frac{\Lambda_j}{-j} (x-a)^{-j} \quad (1.13)
\]
to vary in the sense that the diagonal coefficient matrices $\Lambda_r, \Lambda_{r-1}, \ldots, \Lambda_1$ vary in a neighborhood of $\Lambda_r^0, \Lambda_{r-1}^0, \ldots, \Lambda_1^0$ with $\Lambda_r$ maintaining distinct eigenvalues and $\Lambda_0$ held fixed. In what follows $\Lambda_0$ will always denote a fixed diagonal $p \times p$ matrix.

The Stokes’ multipliers are to remain fixed under our deformations. However, since the Stokes’ multipliers are significant relative to some choice of a covering by good sectors $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{2r}\}$ we must either fix this covering for all values of the deformation parameters (this is what is done in [22] and [12]) or say how the choice of covering affects the notion of “fixed” Stokes’ multipliers. We follow Malgrange in adopting the second alternative. The choice of a good sector $\Sigma_1$ is nearly equivalent to the selection of a suitable collection of “consecutive” Stokes’ lines (one from each of the families (1.6)). Once one has selected such a collection of Stokes’ lines any open sector which contains these Stokes’ lines could serve as a good sector. However, we wish to put a further limitation on our good sectors. No Stokes’ line
should lie on the boundary of a good sector. The reason for this is that for a fixed connection, the Stokes’ multipliers associated with such a good sector are not stable under small rotations of the sector; Stokes’ lines can rotate in or out the sector and this changes the associated Stokes’ multipliers. Another way of saying this is that the local analytic equivalence class associated with a choice of a covering by good sectors and associated Stokes’ multipliers is not stable under small rotations of the good sectors unless the sectors do not have Stokes’ lines on their boundary. We will say that a covering \( \{ \Sigma_1, \Sigma_2, \ldots, \Sigma_{2r} \} \) of a punctured neighborhood of \( a \) by good sectors is \textit{stable} if the good sector \( \Sigma_1 \) has no Stokes’ lines on its boundary.

We now define a configuration space, \( \mathcal{C} \), for our local models (1.12),

\[
\mathcal{C} := \mathbb{Z}^p \times \mathbb{C}^p \times \cdots \times \mathbb{C}^p,
\]

where there are \( r - 1 \) factors \( \mathbb{C}^p \). The first factor \( \mathbb{Z}^p \) is, of course, the configuration space for the leading coefficient \( \Lambda_r \), and the remaining factors \( \mathbb{C}^p \) are associated with the \( \Lambda_j, j = r - 1, \ldots, 1 \). Following Malgrange, we now introduce a fiber bundle, \( \mathcal{M} \to \mathcal{C} \). The fiber \( \mathcal{M}_\Lambda \) over each point \( \Lambda := (\Lambda_r, \Lambda_{r-1}, \ldots, \Lambda_1) \) in \( \mathcal{C} \) is the moduli space of all holomorphic gauge equivalence classes of locally defined type \( r \) connections on the trivial bundle \( \{ x : |x - a| < \epsilon \} \times \mathbb{C}^p \) (for some \( \epsilon > 0 \)) which are formally equivalent to the connection (1.12-1.13). Or put another way, if \( \nabla_1 \) and \( \nabla_2 \) are two type \( r \) connections at \( a \) and they are formally equivalent to \( \nabla_\Lambda \) via \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \), then \( \nabla_1 \simeq \nabla_2 \) if \( \hat{\alpha}_2^{-1}\hat{\alpha}_1 \) is convergent in a neighborhood of \( a \) (see [18]).

Next we want to show that

\[
\pi : \mathcal{M} \to \mathcal{C},
\]

is a fiber bundle with a natural flat connection. This connection will play a role in providing a global sense to monodromy preserving deformations. We first discuss local trivializations for (1.14). Suppose that \( \Lambda^0 \in \mathcal{C} \) and write \( \nabla^0 \) for the connection (1.13) associated with \( \Lambda^0 \). Suppose that \( \{ \Sigma_1, \Sigma_2, \ldots, \Sigma_{2r} \} \) is a covering of a punctured neighborhood of \( a \) by stable good sectors for \( \nabla^0 \). Then we can find a neighborhood \( U \) of \( \Lambda^0 \) in \( \mathcal{C} \) so that \( \{ \Sigma_1, \Sigma_2, \ldots, \Sigma_{2r} \} \) is a stable covering at \( a \) for all \( \nabla_\Lambda \) with \( \Lambda \in U \) (in fact, it is clear that \( U \) can be taken to be of the form \( U_r \times \mathbb{C}^p \times \cdots \times \mathbb{C}^p \) where \( U_r \) is a sufficiently small neighborhood of \( \Lambda^0_r \) in \( \mathbb{Z}^p \)). It is a result of Malgrange and Sibuya, [18,20], that for any \( \Lambda \in U \) and suitable choice of Stokes’ multipliers \( S_{j,j+1} \) there exists a type \( r \) connection at \( a \) with local model \( \nabla_\Lambda \) and Stokes’ multipliers \( S_{j,j+1} \) relative to the covering \( \{ \Sigma_1, \Sigma_2, \ldots, \Sigma_{2r} \} \). This connection is not unique but its local holomorphic gauge equivalence class is unique. To be “suitable” the Stokes’ multipliers must be 1 on the diagonal and lower triangular relative to the dominance ordering of \( \Re(\Lambda_{r,j}x) \) for \( x \in \Sigma_j \cap \Sigma_{j+1} \).
This identifies the fiber of $\mathcal{M}$ as $\mathbb{C}^{rp(p-1)}$, since there are $2r$ intersections $\Sigma_j \cap \Sigma_{j+1}$ and $\frac{p(p-1)}{2}$ arbitrary coefficients for each Stokes’ multiplier. The choice of a neighborhood $U$ and a covering $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{2r}\}$ which is stable for all $\nabla_\Lambda$ with $\Lambda \in U$, thus produces a trivialization
\[ \pi^{-1}(U) \simeq U \times \mathbb{C}^{rp(p-1)}. \]

Note: The fiber $\mathcal{M}_\Lambda$ is more invariantly defined in [18], as $H^1(S^1, \text{St}(\nabla_\Lambda))$, the first cohomology of the Stokes’ sheaf associated with the connection $\nabla_\Lambda$ (see also [23]).

We will now show that $\mathcal{M}$ is a fiber bundle by determining that the transition maps between trivializations are given by polynomial diffeomorphisms in the fiber. Suppose that $U'$ is a neighborhood in $\mathbb{C}$ with $\{\Sigma'_1, \Sigma'_2, \ldots, \Sigma'_{2r}\}$ a covering by sectors at $a$ which is stable for all $\nabla_\Lambda$ with $\Lambda \in U'$. Now suppose that $\Lambda \in U \cap U'$. Then since $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{2r}\}$ and $\{\Sigma'_1, \Sigma'_2, \ldots, \Sigma'_{2r}\}$ are both good stable covering for $\nabla_\Lambda$ it follows that, up to small changes in the opening angle of $\Sigma'_1$, which do not effect the Stokes’ multipliers, $\Sigma'_1$ can be obtained from $\Sigma_1$ by rotating $\Sigma_1$ counterclockwise through an angle $\theta$. To emphasize this we will write $\Sigma'_j = \Sigma_\theta j$. Now let $\nabla$ denote a connection of type $r$ at $a$ which is formally equivalent to $\nabla_\Lambda$ (i.e., $\pi(\nabla) = \Lambda$). Suppose $\hat{\alpha}$ is a formal series with
\[ \nabla = \hat{\alpha} \cdot [\nabla_\Lambda] \]
in the sense of formal series at $a$. Suppose that $\alpha^\theta_j \in \mathcal{A}(\Sigma^\theta_j)$ is asymptotic to $\hat{\alpha}$ and $\nabla = \alpha^\theta_j \cdot [\nabla_\Lambda]$ analytically in the sector $\Sigma^\theta_j$. Then the Stokes’ multipliers $S^\theta_{j,j+1}$ are defined by,
\[ \alpha^\theta_{j+1}(x)e^{H_\Lambda(x)} = \alpha^\theta_j(x)e^{H_\Lambda(x)}S^\theta_{j,j+1}, \quad (1.15) \]
where,
\[ H_\Lambda(x) = \sum_{j=1}^r \frac{\Lambda_j}{-j}(x-a)^{-j} + \Lambda_0 \log(x-a). \quad (1.16) \]

There is a slight ambiguity in the definition of the Stokes’ multipliers in (1.15) associated with the choice of $x \to \log(x-a)$ in (1.16). To deal with this ambiguity we require that the data which specifies a local trivialization for $\mathcal{M} \to \mathcal{C}$ includes not only the choice of a stable good covering $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{2r}\}$ but also a branch of the function $x \to \log(x-a)$ in the sector $\Sigma_1$. One may analytically continue this choice from $\Sigma_1$ to $\Sigma_2$ to $\Sigma_3$ and so on to fix a choice of log in (1.16) and render (1.15) an unambiguous defining relation for $S^\theta_{j,j+1}$. In the rank $r = 1$ case there are only two sectors $\Sigma_1$ and $\Sigma_2$ and the intersection $\Sigma_1 \cap \Sigma_2$ is disconnected. In this case we suppose that the analytic continuation from $\Sigma_1$ to $\Sigma_2$ is accomplished
so that the function is smooth on counterclockwise oriented circles passing from $\Sigma_1$ into $\Sigma_2$.

In the special case $\theta = 0$ we will simply write $\alpha_j^0 = \alpha_j$ and $S_{j,j+1}^0 = S_{j,j+1}$.

Now we turn to the proof that $M \to C$ is a fiber bundle. As noted above our trivializations depend on a choice of $\log(x - a)$ in $\Sigma_1^{\theta}$. However, different choices will simply alter $S_{j,j+1}^{\theta}$ by conjugation with powers of $\exp(2\pi i \Lambda_0)$. Since this is a linear transformation in the fiber, constant in the base we may as well suppose that this choice of $\log(x - a)$ has been fixed in $\Sigma_1^{\theta}$ (and $\Sigma_1$). The Stokes’ multipliers (1.15) are thus well defined and the transition map we wish to compute takes $\{S_{j,j+1}\}$ to $\{S_{j,j+1}^{\theta}\}$. It is enough to compute the transition in the fiber for $0 < \theta < \pi$ since a larger rotation may be realized as a composition of rotations satisfying this condition. Suppose now that $0 < \theta < \pi$. Because the rotation $\theta$ is smaller than $\pi$ it follows that $\Sigma_j \cap \Sigma_{j+1}$ is not empty. We may thus compare the local fundamental solutions $\alpha_j e^{H\Lambda}$ and $\alpha_j^{\theta} e^{H\Lambda}$ on this intersection,

$$\alpha_j e^{H\Lambda} = \alpha_j^{\theta} e^{H\Lambda} S_j(\theta), \quad (1.17)$$

where $S_j(\theta)$ is a constant $p \times p$ matrix. Combining (1.16) and (1.17) one finds that

$$S_{j,j+1}^{\theta} = S_j(\theta) S_{j,j+1} S_{j+1}(\theta)^{-1} \quad (1.18)$$

Thus to find $S_{j,j+1}^{\theta}$ in terms of the Stokes’ multipliers $\{S_{k,k+1}\}$ it will suffice to determine $S_k(\theta)$ in terms of $\{S_{k,k+1}\}$.

By relabeling the eigenvalues, $\Lambda_{r,j} = 1, \ldots, p$ , of $\Lambda_r$ we may suppose that the dominance ordering in $\Sigma_j \cap \Sigma_{j+1}$ is the “natural” one

$$\Re(\Lambda_{r,1} x) < \Re(\Lambda_{r,2} x) < \cdots < \Re(\Lambda_{r,p} x)$$

for $x \in \Sigma_j \cap \Sigma_{j+1}$. It is not hard to see what happens to this dominance ordering as one rotates $\Sigma_j \cap \Sigma_{j+1}$ counterclockwise. As $\Sigma_j \cap \Sigma_{j+1}$ crosses a “simple” Stokes’ line, say

$$\Re(\Lambda_{r,1} x) < \Re(\Lambda_{r,2} x) = \Re(\Lambda_{r,3} x) < \Re(\Lambda_{r,3} x),$$

then the dominance ordering permutes 2 and 3 leaving the rest of the indices in sequence. If one crosses a Stokes’ line with “higher multiplicity”, say

$$\Re(\Lambda_{r,1} x) < \Re(\Lambda_{r,2} x) = \Re(\Lambda_{r,3} x) = \Re(\Lambda_{r,4} x) < \Re(\Lambda_{r,5} x),$$

then the $(2,3,4)$ part of the ordering is inverted to $(4,3,2)$.

For the purpose of illustration suppose that $p = 5$ and that in going from $\Sigma_j \cap \Sigma_{j+1}$ to $\Sigma_j^{\theta} \cap \Sigma_{j+1}^{\theta}$ one passes through the simple Stokes’ line $\Re(\Lambda_{r,1} x) =$
\( \Re(\Lambda_{r,2x}) \) and the higher multiplicity Stokes’ line \( \Re(\Lambda_{r,3x}) = \Re(\Lambda_{r,4x}) = \Re(\Lambda_{r,5x}) \). Then it is not hard to see that the Stokes’ multiplier for \( \Sigma_j^\theta \cap \Sigma_{j+1}^\theta \) must have the following “triangularity”,

\[
S_{j,j+1}^\theta = \begin{bmatrix}
1 & * & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
* & * & 1 & * & * \\
* & * & 0 & 1 & * \\
* & * & 0 & 0 & 1
\end{bmatrix}
\]

relative to the basis for \( \mathbb{C}^5 \) in which \( S_{j,j+1} \) is lower triangular. The \( * \)'s represent possibly non-zero entries. Next consider (1.17). Since \( \alpha_j \) and \( \alpha_j^\theta \) have the same asymptotics in the sector \( \Sigma_j \cap \Sigma_j^\theta \) it follows that \( S_j(\theta) \) must have 1's on the diagonal and cannot have non zero off diagonal elements for any of the pairs associated with Stokes’ lines in the intersection \( \Sigma_j \cap \Sigma_j^\theta \). In the example (1.19) this means that the \((\ell, m)\) matrix elements for \( S_j(\theta) \) are zero for \((\ell, m) \in \{(k, 1), (k, 2), (1, k), (2, k) : k = 3, 4, 5 \} \).

Since \( \theta < \frac{\pi}{2} \) it follows that \( \Sigma_j \cap \Sigma_j + 1 \) is contained in \( \Sigma_j \cap \Sigma_j^\theta \) (provided at least one Stokes’ line is crossed, which is the only interesting case). Thus one can also say of \( S_j(\theta) \) that it is lower triangular with respect to the dominance ordering in \( \Sigma_j \cap \Sigma_{j+1} \). Thus for our example the matrix of \( S_j(\theta) \) must have the form,

\[
S_j(\theta) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
a & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & b & 1 & 0 \\
0 & 0 & c & d & 1
\end{bmatrix}
\]

In a similar fashion \( \Sigma_j^\theta \cap \Sigma_{j+1}^\theta \) is contained in \( \Sigma_{j+1} \cap \Sigma_{j+1}^\theta \) and it follows that \( S_{j+1}(\theta) \) must be lower triangular with respect to the dominance ordering for \( \Sigma_j^\theta \cap \Sigma_{j+1}^\theta \). In our example, if we rewrite (1.17) in the form \( S_{j,j+1}^\theta S_{j+1}(\theta) = S_j(\theta) S_{j,j+1} \) and make use of the lower triangularity of the product \( S_{j,j+1}^\theta S_{j+1}(\theta) \) with respect to the dominance ordering in \( \Sigma_j^\theta \cap \Sigma_{j+1}^\theta \). We find

\[
\begin{bmatrix}
1 & * & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
* & * & 1 & * & * \\
* & * & 0 & 1 & * \\
* & * & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
a & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & b & 1 & 0 \\
0 & 0 & c & d & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
* & * & 1 & * & 0 \\
* & * & 0 & * & 1 \\
* & * & 0 & 0 & 1
\end{bmatrix}
\]

(1.21)
The \((2,1)\) matrix element of \(S_j(\theta)\) can be determined simply by equating the \((2,1)\) matrix elements on both sides of (1.21). One finds
\[
a = S_j(\theta)_{2,1} = -(S_{j,j+1})_{2,1}.
\]
The same thing can be done for the matrix elements \(b\) and \(d\) for \(S_j(\theta)\) on the subdiagonal. For example
\[
d = S_j(\theta)_{5,2} = -(S_{j,j+1})_{5,2}.
\]
Once one knows the subdiagonal elements one can move out to the diagonal below the subdiagonal. Equating \((5,3)\) matrix elements for (1.21) one finds
\[
0 = c + d(S_{j,j+1})_{4,3} + (S_{j,j+1})_{5,3}.
\]
From the earlier relation for \(d\) one finds that \(c\) is a polynomial function of the matrix elements of \(S_{j,j+1}\). Thus the entries of \(S_j(\theta)\) are polynomials in the entries for \(S_{j,j+1}\) and it is clear that this is true quite generally and does not depend on anything special in our example. From (1.18) it then follows that \(S_{j,j+1}^\theta\) is a polynomial in the entries of \(S_{j,j+1}\) and \(S_{j+1,j+2}\). Since this relation is invertible by construction it follows that the map from \(\{S_{k,k+1}\}\) to \(\{S_{k,k+1}^\theta\}\) is a polynomial diffeomorphism on \(\mathbb{C}^{rp(p-1)}\).

We’ve seen that \(\pi : \mathcal{M} \to \mathcal{C}\) is a fiber bundle and that there is a natural family of trivializations for this bundle which are related by polynomial diffeomorphisms in the fiber that are constant in the base. Recall that the vertical vectors in the tangent space to \(\mathcal{M}\) at \(p, T_p(\mathcal{M})\), are those killed by \(d\pi_p\). A connection on \(\mathcal{M}\), is determined by a one form \(\omega\) on \(\mathcal{M}\) whose value \(\omega_p(v)\) at a vector \(v \in T_p(\mathcal{M})\) is a vertical vector in \(T_p(\mathcal{M})\) at \(p\); \(\omega\) must have the further property that \(\omega_p(v) = v\) if \(v\) is a vertical vector in \(T_p(\mathcal{M})\). Our trivializations single out a flat connection on \(\mathcal{M}\). If \(\pi^{-1}(U)\) has trivialization \(U \times \mathbb{C}^N\) and \(f_1, f_2, \ldots f_N\) are the natural coordinate functions on \(\mathbb{C}^N\) then it is easy to see that
\[
\omega = \sum_{k=1}^{N} \frac{\partial}{\partial f_k} df_k
\]
defines a connection one form independent of the choice of trivialization (among the special class of trivializations that we have been considering for \(\mathcal{M}\) in which the fiber coordinates transform among themselves). The curve \(\tilde{\gamma}(t)\) in \(\mathcal{M}\) is the horizontal lift of \(\gamma(t) = \pi \tilde{\gamma}(t)\) provided that \(\omega(\tilde{\gamma}'(t)) = 0\). Relative to one of our distinguished trivializations this translates into
\[
\tilde{\gamma}(t) = (\gamma(t), f)
\]
where \( f \in \mathbb{C}^N \) is constant in \( t \). The curvature of this connection is clearly 0 and locally parallel sections look like

\[
\sigma(x) = (x, f)
\]

where \( f \) is independent of \( x \). Now let \( \mathcal{R}(\mathcal{C}) \) denote the simply connected covering space of \( \mathcal{C} \). As was the case for \( \mathcal{R}(\mathbb{Z}^n) \) above it will be convenient to introduce special notation for projection from \( \mathcal{R}(\mathcal{C}) \) to \( \mathcal{C} \). We will write \( \lambda \in \mathcal{R}(\mathcal{C}) \) and \( \Lambda(\lambda) \in \mathcal{C} \) for the projection of \( \lambda \) onto \( \mathcal{C} \).

The bundle \( \mathcal{M} \) over \( \mathcal{C} \) with flat connection \( \omega \) pulls back to a bundle over \( \mathcal{R}(\mathcal{C}) \) (which we will continue to denote by \( \mathcal{M} \)) with a flat connection (which we will continue to denote by \( \omega \)). Because \( \mathcal{R}(\mathcal{C}) \) is simply connected the bundle

\[
\mathcal{M} \to \mathcal{R}(\mathcal{C})
\]

has globally defined parallel sections. The existence of these global sections, which are the analogues of the local notion of “constant Stokes’ multipliers” in \( \mathcal{C} \) is the reason for working here in \( \mathcal{R}(\mathcal{C}) \) instead of just locally in \( \mathcal{C} \) (note: the theory of local systems explained in [23] also applies to \( \mathcal{M} \) and could be used to bypass the introduction of a connection in the fiber bundle \( \mathcal{M} \)).

**Local models for integrable deformations.** Before we state our global result for \( \mathcal{R}(\mathcal{C}) \) deformations we introduce some notation from [18] concerning connections in several complex variables. Suppose that \( X \) is a complex analytic variety of dimension \( n \), \( Y \) is a smooth hypersurface in \( X \), \( E \) is a rank \( p \) complex vector bundle over \( X \), \( \nabla \) is a holomorphic connection on \( X \setminus Y \) and \( \Omega \) is the one form for \( \nabla \) in a local frame for \( E \).

**Definition 1.22.** For \( r \geq 0 \) an integer one says that \( \nabla \) is of type \( r \) along \( Y \), if in a system of local coordinates \( x_1, x_2, \ldots, x_n \) for \( X \) with \( Y \) locally defined by \( x_1 = 0 \) one has

\[
\Omega = \sum_{j=1}^{n} M_j dx_j,
\]

where \( M_1 \) has a pole of order \( r+1 \) in \( x_1 \), \( M_j \) for \( j \geq 2 \) has a pole of order \( r \) in \( x_1 \) and \( M_j \) is holomorphic in \( x_2, x_3, \ldots, x_n \) for all \( j \).

Suppose that

\[
M_1 = \sum_{-r-1}^{\infty} M_{1,k} x_1^k.
\]

Our standing assumption (1.2) for type \( r \geq 1 \) connections, translates into this situation as the assumption that \( M_{1,-r-1}(x_2, x_3, \ldots, x_n) \) has distinct eigenvalues.
We will refer to such connections as *simple* type \( r \) connections along \( Y \). When \( r = 0 \) we will say that a type 0 connection is *simple* if \( M_{1,-1}(x_2, x_3, \ldots, x_n) \) has distinct eigenvalues which in addition *do not differ by integers*. Malgrange has shown how to develop the theory of Schlesinger deformations without this assumption, but it will be convenient for us to insist on it so that we may work with regular and irregular singular points in parallel. Propositions (1.23a) and (1.23b) below are the principal reason that this is possible and will permit us to present the “unified” formula for \( d \log \tau \) which can be found in [12].

We recall some results of Malgrange for simple *integrable* type \( r \) connections. Suppose that one has a simple integrable type \( r \) connection \( \nabla \) along \( Y \) and that, as above, \( Y \) is locally defined by \( x_1 = 0 \). Let \( y = (x_2, x_3, \ldots, x_n) \) denote the local coordinates on \( Y \). Then one has the following local version of Proposition 1.3 in [18].

**Proposition 1.23a** Suppose that \( (E, \nabla) \) is a vector bundle with a simple integrable connection \( \nabla \) of type \( r \geq 1 \) along \( Y \). There exists a local trivialization in a neighborhood of \( x = 0 \) so that \( M_{1,-r-1} \) is diagonal in this trivialization. Let \( \Omega \) denote the connection form for \( \nabla \) in this trivialization. Then in a sufficiently small neighborhood \(|y| < \epsilon\) there exists a formal power series \( \hat{\alpha} \) in \( x_1 \)

\[
\hat{\alpha}(x) = I + \beta_1(y)x_1 + \beta_2(y)x_1^2 + \cdots
\]

with matrix coefficients \( \beta_j(y) \) which are holomorphic in \( y \) for \(|y| < \epsilon\), with the property that

\[
\hat{\alpha} \cdot [\nabla] = d + \Omega
\]

where \( \hat{\alpha} \cdot [\nabla] = \hat{\alpha}\nabla\hat{\alpha}^{-1} \) is the formal gauge transformation by \( \hat{\alpha} \),

\[
\nabla \Lambda = d - d(H) - \Lambda_0 \frac{dx_1}{x_1},
\]

\( d \) is the exterior derivative in the \( x \) variables, \( \Lambda_0 \) is a constant diagonal matrix,

\[
H = \sum_{j=1}^{r} \Lambda_j(y) \frac{x_1^{-j}}{-j},
\]

with \( \Lambda_j(y) \) a diagonal matrix with entries that are holomorphic functions of \( y \), and \( \Lambda_r(y) \) a diagonal matrix with distinct entries (note: in the statement of this Proposition \( \Lambda \) does not denote the projection \( \mathcal{R}(C) \to C \) described above).

There is an analogue of this result for simple type 0 connections, which is the principal reason we work with such connections.
Proposition 1.23b. Suppose that \((E, \nabla)\) is a vector bundle with a simple integrable type 0 connection \(\nabla\) along \(Y\). There exists a local trivialization near \(x = 0\) so that \(M_{1,-1}\) is diagonal. Let \(\Omega\) denote the one form for \(\nabla\) in this trivialization. Then in a sufficiently small neighborhood of \(x = 0\) there exists a \(\text{GL}(p, \mathbb{C})\) valued holomorphic function

\[
\alpha(x) = I + \beta_1(y)x_1 + \beta_2(y)x_1^2 + \cdots
\]

so that

\[
\alpha \left( d - \frac{\Lambda_0}{x_1} \right) \alpha^{-1} = d + \Omega.
\]

where \(\Lambda_0\) is a diagonal matrix which is independent of \(x\).

Remark: The conclusion of this theorem is simply that the connection \(\nabla\) is given by \(d - \frac{\Lambda_0}{x_1}\) in a suitable local trivialization. The analogy with Proposition 1.23a is not so apparent in this formulation, however.

Proof (of Proposition 1.23b). Let \(\Omega = \sum_{j=1}^{n} M_j dx_j\) denote the connection one form for \(\nabla\) relative to some trivialization in a neighborhood of \(x = 0\). Suppose \(M_{1,-1}(y)\) is the residue of \(M_1\) at \(x_1 = 0\). Then the matrix \(M_{1,-1}(0)\) has distinct eigenvalues, so this remains true for \(M_{1,-1}(y)\) for \(x\) in a sufficiently small neighborhood of 0. Because its eigenvalues are distinct one may diagonalize \(M_{1,-1}(y)\) by a holomorphic similarity transformation

\[
Q(y)M_{1,-1}(y)Q(y)^{-1} = \Lambda_0(y),
\]

where \(Q(y)\) is holomorphic and \(\Lambda_0(y)\) is diagonal. Making a gauge transformation by \(Q\) one finds that the connection \(\nabla\) becomes

\[
d - \frac{\Lambda_0(y)}{x_1}dx_1 - \sum_{j=1}^{n} B^j(x)dx_j,
\]

where the \(B^j(x)\) is holomorphic in \(x\). By assumption the eigenvalues of \(\Lambda_0(0)\) do not differ by integers and so the same is true for \(\Lambda_0(y)\) if \(y\) remains in a sufficiently small neighborhood of 0. Now let \(z = x_1\) and consider the connection

\[
d_z - \frac{\Lambda_0(y)}{z}dz - B^1(z, y)dz,
\]

depending on the parameter \(y\). Since the eigenvalues of \(\Lambda_0(y)\) are distinct and do not differ by integers, the standard construction of a fundamental solution (which depends on the inversion of \(\text{ad}(\Lambda_0) - nI\) where \(n = 1, 2, \ldots\) is an integer) shows that
one can find a gauge transformation \( \beta(z, y) = I + O(z) \) which depends holomorphically on the parameters \( y \) and which transforms this connection to

\[
d_z - \frac{\Lambda_0(y)}{z} dz.
\]

The gauge transformation of the complete connection by this transform gives one

\[
d - \frac{\Lambda_0(y)}{z} dz - \sum_{j=1}^{n-1} B^{j+1}_0(z, y) dy_j,
\]

(1.23)

where \( y_j = x_{j+1} \). Now we express the integrability of this connection \( (d\Omega + \Omega \wedge \Omega = 0) \) expanding \( B^k(z, y) \) in powers of \( z \),

\[
B^k(z, y) = \sum_{j=0}^{\infty} B^k_j(y) z^j.
\]

Equating the coefficients of \( z^{-1} \) in the integrability condition we find that

\[
\frac{\partial \Lambda_0(y)}{\partial y_j} = [B^j_0, \Lambda_0].
\]

Since \( \Lambda_0 \) is diagonal the right hand side vanishes on the diagonal and this shows that \( \Lambda_0(y) = \Lambda_0 = \text{constant} \). The left hand side vanishes off the diagonal and since the entries of \( \Lambda_0 \) are distinct this implies that \( B^j_0 \) must be diagonal. Equating the coefficients of \( z^j dz \wedge dy_j \) in the integrability condition one finds

\[
n B^j_n = [\Lambda_0, B^j_n] \quad \text{for} \quad n = 1, 2, \ldots
\]

which implies that \( B^j_n = 0 \) for \( n = 1, 2, \ldots \). Thus the connection \( \nabla \) has the form

\[
d_z - \frac{\Lambda_0}{z} dz + d_y - \sum_{j=1}^{n-1} B^{j+1}_0(y) dy_j.
\]

The connection

\[
d_y - \sum_{j=1}^{n-1} B^{j+1}_0(y) dy_j
\]

is integrable and so a gauge transformation, \( \gamma(y) \), in the \( y \) variables reduces this connection to \( d_y \). Since \( B^{j+1}_0(y) \) is diagonal this gauge transformation may be
chosen to be diagonal and also may be chosen so that \( \gamma(y) = I + O(y) \). Since \( \gamma \) is diagonal it does not alter the \( dz \) part of the connection. Composing \( \beta \) and \( \gamma \) we get a gauge transformation \( \alpha = I + O(z) \) which reduces (1.23) to

\[
d - \frac{\Lambda_0}{z} dz,
\]

and this finishes the proof. QED

Although we don’t require the result until the next section it will be convenient here to recall Theorem 2.1 from [18]. Suppose that \( (E, \nabla) \) is a vector bundle with a connection \( \nabla \) that has a simple type \( r \) pole along the hypersurface \( Y \) defined by \( x_1 = 0 \). Let \( y = (x_2, \ldots, x_n) \) denote the coordinates along \( Y \). Suppose that in a neighborhood of a point \( x^0 \in Y \), with coordinates \( x_1 = 0 \) and \( y = 0 \), and relative to some local trivialization of the bundle one has a formal isomorphism,

\[
\nabla = \tilde{\alpha} \cdot [\Lambda],
\]

where \( \nabla \) is the diagonal connection described in Proposition 1.26a and \( \tilde{\alpha} \) is the formal power series described in that same proposition. Suppose that \( \Sigma \) is a good stable sector (in the \( x_1 \) variable) for the connection \( \nabla \) restricted to \( y = 0 \). Suppose that \( \epsilon > 0 \) is chosen small enough so that for all \( y_1 \) with \( |y_1| < \epsilon \), the sector \( \Sigma \) remains a good stable sector for the restriction of \( \nabla \) to \( y = y_1 \).

Then one has (Theorem 2.1 in [18])

**Proposition 1.23c.** There exists a uniquely determined invertible holomorphic map \( \alpha_\Sigma \in A(\Sigma_\epsilon \times |y| < \epsilon) \) such that on \( \Sigma_\epsilon \times |y| < \epsilon \) and in an appropriate trivialization for \( E \) one has

\[
\nabla = \alpha_\Sigma \cdot [\Lambda],
\]

and such that the map \( \alpha_\Sigma \) extends \( \tilde{\alpha} \) in the sense that \( \alpha_\Sigma \) has an asymptotic development along \( Y \) which is equal to \( \tilde{\alpha} \).

Remark. The consequence of this result that is of interest for us is that the formal isomorphism class \( \nabla \) together with the Stokes’ multipliers determine the local holomorphic equivalence class of a simple, integrable type \( r \) connection. Two collections \( \alpha_{\Sigma_k} \) and \( \alpha'_{\Sigma_k} \) associated with the same good stable cover \( \{\Sigma_1, \ldots, \Sigma_{2r}\} \) with the same Stokes’ multipliers and the same asymptotics \( \tilde{\alpha} \) clearly differ by an invertible holomorphic map.

We are now prepared to state an existence result for a global version of a “Stokes’ multiplier preserving deformation” which is however, local in the \( x_1 \) variable. The space we will work on is \( D \times R(C) \), where \( D \) is the unit disk about
$x = a$ in $\mathbb{C}$ and the connection whose existence we wish to demonstrate is a simple integrable type $r$ connection along $\{a\} \times \mathcal{R}(\mathcal{C})$ which has formal reduction to

$$\nabla_\lambda = d - d(H) - \Lambda_0 \frac{dx}{x-a}$$

(1.24)

where $d = d_x + d_\lambda$ is the exterior derivative in the $(x, \lambda) \in D \times \mathcal{R}(\mathcal{C})$ variables and

$$H = \sum_{j=1}^{r} \Lambda_k(\lambda) \frac{(x-a)^{-j}}{-j}.$$  

(1.25)

Note that $\nabla_\lambda = e^H de^{-H}$, is the gauge transform of the connection $d$ by $e^H$, with $H$ given by $H + \Lambda_0 \log(x-a)$. However, since $H$ is singular at $x = a$ and multivalued this is not properly a global statement but does make sense locally.

**Theorem 1.26** Suppose that $\sigma$ is a parallel section for $\mathcal{M} \to \mathcal{R}(\mathcal{C})$. Let $D$ denote the unit disk in $\mathbb{C}$ centered at $a$. Then there exists a holomorphic integrable connection $\nabla$ defined on the trivial vector bundle

$$D \times \mathcal{R}(\mathcal{C}) \times \mathbb{C}^p \to D \times \mathcal{R}(\mathcal{C}),$$

which has a singularity of type $r \geq 1$ along the hypersurface $Y = \{a\} \times \mathcal{R}(\mathcal{C})$ such that the restriction of the connection $\nabla$ to $D \setminus \{a\} \times \mathcal{R}(\mathcal{C})$ is formally equivalent to the diagonal model $\nabla_\lambda$ defined in (1.24) and such that the the holomorphic equivalence class of the restriction of $\nabla$ to $(D \setminus \{a\}) \times \{\lambda\}$ is given by $\sigma(\lambda)$.

Proof. We first recall a result of Malgrange and Sibuya, for which one can also find a detailed proof in [2]. Suppose that $\lambda \in \mathcal{R}(\mathcal{C})$ and $\sigma \in \mathcal{M}_\lambda$, where $\mathcal{M}_\lambda$ is the fiber in $\mathcal{M}$ over $\lambda$. Then on the trivial bundle $D \times \mathbb{C}^p \to D$ there exists a simple type $r$ connection $\nabla$ singular at $x = a$ in $D$, which has formal reduction to the diagonal model

$$\nabla_\lambda := d_x - d_x H(\lambda) - \Lambda_0 \frac{dx}{x-a},$$

(1.28)

and which has “Stokes’ multipliers” given by $\sigma$. In the version of this result that is proved in [2] the connection is shown to exist on a disk, $D_\delta$, of small radius $\delta$. It is not difficult to use the Birkhoff factorization theorem to produce a connection defined on $D \setminus \{a\}$ with the same properties. Suppose then one has a connection $\nabla$ defined on the trivial bundle over $D_\delta$ and satisfying the conditions above. There exists a connection $\nabla_{ext}$ defined on the trivial bundle $\mathbb{C} \setminus \{a\} \times \mathbb{C}^p \to \mathbb{C} \setminus \{a\}$ with the same holonomy about $x = a$ as the connection $\nabla$ (it is easy to produce such a connection with a logarithmic pole at $x = a$). Because the holonomy of $\nabla$ and
of $\nabla_{ext}$ about $x = a$ are equal it follows that there is an annulus $A$ containing the circle, $S_{\delta/2}$, of radius $\delta/2$ on which the two connections are gauge equivalent. Thus there exists a holomorphic map $g : A \to \text{GL}(p, \mathbb{C})$ such that
\[ g \cdot [\nabla] = \nabla_{ext}. \]
The Birkhoff theorem gives us a factorization,
\[ g = g_{\infty}^{-1} (x - a)^N g_0 \]
where $g_0$ is holomorphic and invertible in a neighborhood of $x = a$ containing the annulus $A$, $g_{\infty}$ is holomorphic and invertible in a neighborhood of $\infty$ which contains the annulus, $A$, and $N$ is a diagonal matrix with integer entries. The equality
\[ g_0 \cdot [\nabla] = (x - a)^{-N} g_{\infty} \cdot [\nabla_{ext}] \]  
(1.29)
on the annulus $A$, shows that the connection $g_0 \cdot [\nabla]$ extends to the punctured unit disk and since it is in the same local holomorphic equivalence class as $\nabla$, we have finished the demonstration that we may work on the unit disk, $D$, rather than $D_\delta$.

To complete the proof of Theorem 1.26 we proceed in two steps. First we show that if we confine our attention to a sufficiently small neighborhood $U$ of $\lambda \in \mathcal{R}(\mathbb{C})$ then we can find a connection $\nabla_U$ defined over $D \setminus \{a\} \times U$ which satisfies the conclusions of Theorem 1.26, with $\sigma_U$ the unique local flat section of $\mathcal{M}$ with $\sigma_U(\lambda) = \sigma$. We will prove this using a variant of the Flaschka-Newell integral equation to produce a “perturbation” of the connection $\nabla^0$. We defer the proof of this result to Proposition 1.35 below. The second step is to put together the “local” solutions $\nabla_U$ to get something defined on all of $\mathcal{R}(\mathbb{C})$. We will now show how to do this.

Let $\lambda \to \sigma(\lambda)$ denote a flat section of $\mathcal{M} \to \mathcal{R}(\mathbb{C})$. For each point $\lambda \in \mathcal{R}(\mathbb{C})$ there exists a neighborhood $U(\lambda, \sigma)$ of $\lambda$ in which the construction of Proposition 1.35 applies. Let $\mathcal{U}$ denote a subcollection of such open neighborhoods which is a covering for $\mathcal{R}(\mathbb{C})$ and for which $U \cap V$ is contractible for each pair $U, V \in \mathcal{U}$. We also suppose that each neighborhood $U \in \mathcal{U}$ is chosen sufficiently small so that for $\delta > 0$ small enough there exists a sectorial covering $\{\Sigma_{1,\delta}, \Sigma_{2,\delta}, \ldots, \Sigma_{2r,\delta}\}$ of the punctured neighborhood $D_\delta \setminus \{a\}$ which is stable and good for the all connections
\[ d_x - d_x \mathbf{H}(\lambda) - \Lambda_0 \frac{dx}{x - a} \]
with $\lambda \in U$. The construction of Proposition 1.35 shows that there exist holomorphic maps
\[ a_k^U : \Sigma_{k,\delta} \times U \to \text{GL}(p, \mathbb{C}), \]
so that
\[ \alpha_k^U \cdot [\nabla_\lambda] = \nabla_U \text{ on } \Sigma_{k,\delta} \times U. \]

Furthermore the maps \( \alpha_k^U \) are related to one another
\[ \alpha_{k+1}^U = \alpha_k^U S_{k,k+1}(x,\lambda) \]
where \( S_{k,k+1} : \Sigma_{k,\delta} \cap \Sigma_{k+1,\delta} \times U \to \text{GL}(p,\mathbb{C}) \) is a gauge automorphism of \( \nabla_\lambda \) that is asymptotic to the identity to all orders at \( x = a \). Recall that
\[ S_{k,k+1}(x,\lambda) = e^H S_{k,k+1} e^{-H}, \]
where \( S_{k,k+1} \) is a constant matrix and \( e^H \) is well defined once a choice of \( x \to \log(x-a) \) is made for \( x \in \Sigma_{1,\delta} \).

Suppose that \( U,V \in \mathcal{U} \), then for \( \lambda \in U \cap V \), the fact that \( S_{k,k+1} \) does not depend on \( U \) implies that the collection of holomorphic maps
\[ \alpha_k^U (\alpha_k^V)^{-1} \text{ for } k = 1,2,\ldots,2r \]
defines a holomorphic map, \( g_{UV} \), in a punctured neighborhood, \( D_\delta \setminus \{a\} \times U \cap V \), and the fact that \( S_{k,k+1}(x,\lambda) \) is asymptotic to the identity to all orders in \( (x-a) \) implies that \( g_{UV} \) asymptotic to a power series near \( x = a \) that does not depend on the sector. This implies that \( g_{UV} \) is actually holomorphic on \( D_\delta \times U \cap V \). By construction
\[ g_{UV} \cdot [\nabla_V] = \nabla_U \text{ on } D_\delta \setminus \{a\} \times U \cap V. \quad (1.30) \]

This shows that the holonomy of the connection \( \nabla_U \) and the holonomy of the connection \( \nabla_V \) agree on \( D \setminus \{a\} \times U \cap V \) (since \( U \cap V \) is contractible the fundamental group of the product \( D \setminus \{a\} \times U \cap V \) is determined by the first factor) and hence that they are holomorphically equivalent on all of \( D \setminus \{a\} \times U \cap V \). The map \( g_{UV} \) has an invertible holomorphic extension to all of \( D \times U \cap V \) with the property that \( g(0,\lambda) = I \) for all \( \lambda \in U \cap V \). It is not difficult to see that the gauge transformation \( g_{UV} \) is uniquely determined by (1.30) and this normalization. Because of this \( g_{UV} g_{VW} = g_{UW} \). Thus the collection \( \{g_{UV} | U,V \in \mathcal{U}\} \) is a collection of transition functions for a holomorphic vector bundle over \( D \times \mathcal{R}(\mathcal{C}) \). But \( D \times \mathcal{R}(\mathcal{C}) \) is contractible since both factors are, and every bundle on \( D \times \mathcal{R}(\mathcal{C}) \) is thus topologically trivial. But \( D \times \mathcal{R}(\mathcal{C}) \) is a Stein space. It is a theorem of Grauert that on a Stein space every topologically trivial holomorphic bundle is holomorphically trivial [10]. Thus the bundle defined by these transition functions is holomorphically trivial. Thus there exists invertible holomorphic maps \( g_U : D \times U \to \text{GL}(p,\mathbb{C}), \) so that
\[ g_{UV} = g_U^{-1} g_V. \]

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Equation (1.30) becomes

\[ g_V \cdot [\nabla V] = g_U \cdot [\nabla U], \]

and we see that \( g_U \cdot [\nabla U] \) defines a global connection on \( D \setminus \{a\} \times \mathcal{R}(\mathbb{C}) \) which satisfies the conditions of Theorem 1.26. QED

Now we turn to the proof of the perturbation result used in the proof of the preceding theorem. Suppose that \( \nabla^0 \) is a simple type \( r \) connection on the trivial bundle

\[ D \times \mathbb{C}^p \to D, \]

with singularity at \( x = a \) in \( D \). For the purpose of a technical result later on it will be convenient to choose a special trivialization in which to consider the connection \( \nabla^0 \). Choose \( r < 1 \) and let \( A \) denote the annulus

\[ A = \{ x : r < |x - a| < 1 \} \subset D. \]

Choose a point \( p \in A \) and let \( M \) denote the \( p \times p \) invertible matrix which gives the holonomy of the connection \( \nabla^0 \) (with respect to the initial trivialization) along a counterclockwise oriented circle of radius \( |p - a| \) about \( a \). Let

\[ m := \frac{1}{2\pi i} \log M, \]

for some choice of a logarithm for \( M \). The connection

\[ \nabla^\infty := dx - \frac{m}{x - a}, \]

defined on the trivial bundle \( \mathbb{P}^1 \times \mathbb{C}^p \) has regular singular points at 0 and \( \infty \) and its restriction to \( A \) has the same holonomy representation as \( \nabla^0 \). Hence there exists an invertible holomorphic map \( g : A \to \text{GL}(p, \mathbb{C}) \) so that

\[ g \cdot \nabla^0 = \nabla^\infty \text{ on } A. \]

Let \( g = g_{\infty}^{-1}(x - a)^{-N}g_0 \) denote the Birkhoff factorization of \( g \), with \( g_0 \) holomorphic and invertible in \( D \), \( g_{\infty} \) holomorphic and invertible in \( \{ x : |x - a| > r \} \cup \{ \infty \} \), and \( N \) a diagonal matrix with integer entries. Thus

\[ g_0 \cdot \nabla^0 = (x - a)^N g_{\infty} \cdot \nabla^\infty, \]

and we see that by adjusting the trivialization on \( D \times \mathbb{C}^p \) by \( g_0 \) we may suppose that our connection \( \nabla^0 \) extends to a connection on the trivial bundle \( \mathbb{P}^1 \times \mathbb{C}^p \to \mathbb{P}^1 \).
with a regular singular point at $\infty$ (the resulting connection may not be of simple type at $\infty$, however). In what follows we suppose that we are looking at $\nabla^0$ in just such a trivialization.

Now suppose that the leading singularity in the connection one form for $\nabla^0$ is diagonal and that for the formal power series
\[
\hat{\alpha}^0 = I + \beta_1^0(x-a) + \beta_2^0(x-a)^2 + \cdots
\]
we have
\[
\nabla^0 = \hat{\alpha}^0 \cdot \left[ dx - d_x H(\lambda^0) - \Lambda_0 \frac{dx}{x-a} \right],
\]
where $H$ is given by (1.25) and $\lambda^0$ is some fixed element in $\mathcal{R}(\mathcal{C})$ covering $\Lambda(\lambda^0)$. Note that the projection $\Lambda^0 = \Lambda(\lambda^0)$ is determined by $\nabla^0$ through (1.31). Let $\{\Sigma_1, \delta, \Sigma_2, \delta, \ldots, \Sigma_{2r}, \delta\}$ denote a stable good covering of a punctured neighborhood of $x = a$ for the diagonal connection $\nabla_{\lambda^0}$ (see 1.28). Let $\alpha_k^0 = \alpha^0_{\Sigma_k} \in \mathcal{A}(\Sigma_k)$ be a holomorphic function whose asymptotics are given by $\hat{\alpha}$ and for which one has the analytical relation,
\[
\nabla^0 = \alpha_k^0 \cdot \left[ dx - d_x H(\lambda^0) - \Lambda_0 \frac{dx}{x-a} \right],
\]
in the sector $\Sigma_{k, \delta}$. Finally suppose that on $\Sigma_{k, \delta} \cap \Sigma_{k+1, \delta}$ we have
\[
\alpha_{k+1}^0 = \alpha_k^0 S_{k,k+1}(x, \lambda^0)
\]
where
\[
S_{k,k+1}(x, \lambda) = e^{H(\lambda)} S_{k,k+1}^0 e^{-H(\lambda)},
\]
Here $H$ is given by (1.16) and a choice of $\log(x-a)$ is fixed for $x \in \Sigma_{1, \delta}$ to make $H$ well defined. The Stokes’ multiplier $S_{k,k+1}^0$ is independent of $x$ and $\lambda^0$. Let $\Omega^0$ denote the connection form for $\nabla^0$ and let $pr$ denote the natural projection
\[
D \times \mathcal{R}(\mathcal{C}) \to D.
\]
Define
\[
\nabla^0 = d + pr^* \Omega^0,
\]
with $d = dx + d\lambda$, so that $\nabla^0$ defines a connection on the trivial bundle,
\[
D \times \mathcal{R}(\mathcal{C}) \times \mathbb{C}^p \to D \times \mathcal{R}(\mathcal{C}).
\]
The following proposition demonstrates the existence of local “Birkhoff deformations” of the connection $\nabla^0$ in the space $D \times \mathcal{R}(C)$.

**Proposition 1.35.** For a sufficiently small neighborhood $U$ of $\lambda^0$ there exists a simple integrable type $r$ connection $\nabla_U$ defined on the trivial bundle

$$D \times U \times \mathbb{C}^p \to D \times U$$

so that

(i) $\nabla_U$ is formally reducible to the diagonal form (1.28),

$$\nabla_U = \hat{\alpha} \cdot [\nabla_{\lambda}]$$

where

$$\hat{\alpha} = I + \beta_1(\lambda)(x-a) + \beta_2(\lambda)(x-a)^2 + \cdots$$

is a formal power series with holomorphic matrix valued coefficients $\beta_k(\lambda)$. The sectors $\Sigma_k \times U$ are good stable sectors for $\nabla_U$; there exist holomorphic maps

$$\alpha_k \in \mathcal{A}(\Sigma_k, U),$$

with asymptotics given by $\hat{\alpha}$ so that on open sets asymptotic to $\Sigma_k \times U$,

$$\nabla_U = \alpha_k \cdot [\nabla_{\lambda}], \quad (1.39)$$

(ii) $\nabla_U$ is a “Birkhoff deformation” of $\nabla^0$ in that the restriction of $\nabla_U$ to $D \setminus \{a\} \times \{\lambda^0\}$ is equivalent to $\nabla^0$ and

$$\alpha_{k+1}(x, \lambda) = \alpha_k(x, \lambda)S_{k,k+1}(x, \lambda) \quad (1.40)$$

on $\Sigma_{k,\delta} \cap \Sigma_{k+1,\delta} \times U$ (see 1.34)

(iii) On the punctured neighborhood $D \setminus \{a\} \times U$, the connections $\nabla_U$ and $\nabla^0$ are gauge equivalent

$$\nabla_U = \Phi \cdot [\nabla^0] \quad (1.41)$$

by a gauge transformation $x \to \Phi(x, \lambda)$ which is holomorphic in the exterior of the disk $D$ and asymptotic to $I$ as $x \to \infty$.

(iv) If $\nabla^0$ extends to a connection on the trivial bundle $\mathbb{P}^1 \times \mathbb{C}^p \to \mathbb{P}^1$ with a regular singular point at infinity, then

$$d_{\lambda} \text{Res}_{x=a} \text{Tr} \{\hat{\alpha}^{-1} d_{\lambda} \hat{\alpha} d_{\lambda} H(\lambda)\} = 0. \quad (1.42)$$

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Proof. To construct $\nabla_U$ through equation (1.39) it will suffice to construct functions $\alpha_k$ satisfying (1.40) with $\alpha_k(x,\lambda^0) = \alpha_k^0(x)$. Our strategy will be to look for solutions

$$
\alpha_k(x, \lambda) = \varphi_k(x, \lambda)\alpha_k^0(x)
$$

with $\varphi_k : \Sigma_{k,\delta} \times U \to \text{GL}(p, \mathbb{C})$ a holomorphic map with appropriate asymptotics. Condition (1.40) for \{\alpha_k\} translates into

$$
\varphi_{k+1}(x, \lambda) = \varphi_k(x, \lambda)(I + \Delta s_{k,k+1}(x, \lambda)),
$$

(1.43)

for $(x, \lambda)$ in $\Sigma_{k,\delta} \cap \Sigma_{k+1,\delta} \times U$, where

$$
I + \Delta s_{k,k+1}(x, \lambda) = \alpha_k^0(x)S_{k,k+1}(x, \lambda)\alpha_k^0(x)^{-1}.
$$

The important property of $\Delta s_{k,k+1}$ for us is that for any fixed $\delta > 0$ one can make $\Delta s_{k,k+1}$ as close to zero as one likes by choosing $\lambda \in U$, with $U$ a sufficiently small neighborhood of $\lambda^0$.

The condition (1.40) (or its translation (1.43)) does not determine $\alpha_k$ uniquely. We will impose a further condition on $\alpha_k$ which will uniquely determine it. The extra condition is that the “fundamental solutions” $\Psi(x, \lambda)$ and $\Psi^0(x)$ associated to \{\alpha_k\} and \{\alpha_k^0\} by (1.8) differ on the circle of radius $\delta$ by a map which is holomorphic in the exterior of the circle of radius $\delta$ about $x = a$ and asymptotic to the identity at $\infty$. More precisely $\Psi(x, \lambda)\Psi^0(x)^{-1}$ should be holomorphic in $x$ outside the circle of radius $\delta$ and

$$
\Psi(x, \lambda)\Psi^0(x)^{-1} = I + O(x^{-1}).
$$

Note that $\Phi(x, \lambda) = \Psi(x, \lambda)\Psi^0(x)^{-1}$ will be the gauge transformation in (iii) of Proposition 1.35. We will now translate the conditions we’ve outlined into an integral equation for the functions $\varphi_k$. It will be useful to begin by describing the pieces that make up the integration contours we will use. Let $\sigma_{k,k+1}$ denote an oriented ray segment that lies in the intersection $\Sigma_{k,\delta} \cap \Sigma_{k+1,\delta}$, joins the point $a$ to the circle of radius $\delta$ about $a$, and separates the Stokes’ lines in the sector $\Sigma_{k,\delta}$ from the Stokes’ lines in the sector $\Sigma_{k+1,\delta}$ for all $\lambda \in U$ (Stokes’ lines are associated with the function $\Lambda_r(\lambda)$). It is always possible to do this if $\delta$ is small enough and $U$ is chosen to be a sufficiently small neighborhood of $\lambda^0$. Let $\gamma_k$ denote the counterclockwise oriented segment of the circle of radius $\delta$ about $a$ which joins the end point of $\sigma_{k-1,k}$ to the end point of $\sigma_{k,k+1}$. Finally let $\sigma_k$ denote the open wedge that is bounded by $\sigma_{k-1,k}$, $\gamma_k$, and $\sigma_{k,k+1}$. Then clearly $\sigma_k \subset \Sigma_{k,\delta}$ and the oriented boundary of $\sigma_k$ is given by

$$
\partial \sigma_k = \sigma_{k-1,k} + \gamma_k - \sigma_{k,k+1}.
$$
If we now compare $\Psi$ and $\Psi^0$ defined by (1.8) on the circle of radius $\delta$ about $a$ we find

$$\Psi(x, \lambda)\Psi^0(x)^{-1} = \varphi_k(x, \lambda)(I + \Delta m_k(x, \lambda)) \text{ for } x \in \gamma_k$$

(1.44)

where $k = 1, 2, \ldots, 2r$ with

$$I + \Delta m_k(x, \lambda) := \alpha_k^0(x)e^{H(x, \lambda) - H(x, \lambda^0)}/\alpha_k^0(x)^{-1}. \quad (1.45)$$

We’ve written (1.45) in the special form $I + \Delta m_k$ to emphasize the fact that for fixed $\delta$ the right hand side of (1.45) can be made as close to the identity as one pleases by choosing $\lambda \in U$, with $U$ a sufficiently small neighborhood of $\lambda^0$. Next we will obtain a system of integral equations for $\varphi_k$ following Flaschka and Newell [8]. Suppose that $y \in \sigma_1$, then by Cauchy’s theorem

$$\varphi_1(y, \lambda) = \int_{\partial \sigma_1} \frac{\varphi_1(x, \lambda)}{x - y} \frac{dx}{2\pi i}$$

$$= \int_{\sigma_{0,1}} \varphi_1(x, \lambda) \frac{dx}{x - y} 2\pi i + \int_{\gamma_1} \varphi_1(x, \lambda) \frac{dx}{x - y} 2\pi i - \int_{\sigma_{1,2}} \varphi_1(x, \lambda) \frac{dx}{x - y} 2\pi i \quad (1.46)$$

On $\sigma_{1,2}$ we can use (1.43) to write

$$\varphi_1(x, \lambda) = \varphi_1(x, \lambda) - \varphi_2(x, \lambda) + \varphi_2(x, \lambda)$$

$$= -\varphi_1 \Delta s_{1,2}(x, \lambda) + \varphi_2(x, \lambda), \quad (1.47)$$

where for brevity we’ve written $\varphi_1 \Delta s_{1,2}(x, \lambda)$ for $\varphi_1(x, \lambda)\Delta s_{1,2}(x, \lambda)$ (we will use this notation without further comment in what follows). Substituting this expression in the $\sigma_{1,2}$ integral in (1.46) one finds

$$\varphi_1(y, \lambda) = \int_{\sigma_{0,1}} \varphi_1(x, \lambda) \frac{dx}{x - y} 2\pi i + \int_{\gamma_1} \varphi_1(x, \lambda) \frac{dx}{x - y} 2\pi i + \int_{\sigma_{1,2}} \varphi_1 \Delta s_{1,2}(x, \lambda) \frac{dx}{x - y} 2\pi i$$

$$- \int_{\sigma_{1,2}} \varphi_2(x, \lambda) \frac{dx}{x - y} 2\pi i \quad (1.48)$$

Since $\varphi_2(x, \lambda)$ is holomorphic in $\sigma_2$ and $y \in \sigma_1$ which is outside of $\sigma_2$ it follows that

$$- \int_{\sigma_{1,2}} \varphi_2(x, \lambda) \frac{dx}{x - y} 2\pi i = \int_{\gamma_2} \varphi_2(x, \lambda) \frac{dx}{x - y} 2\pi i - \int_{\sigma_{2,3}} \varphi_2(x, \lambda) \frac{dx}{x - y} 2\pi i. \quad (1.49)$$

We substitute (1.49) for the last integral to appear in (1.48). In the expression that results we observe that the integral,

$$\int_{\sigma_{2,3}} \varphi_2(x, \lambda) \frac{dx}{x - y} 2\pi i = - \int_{\sigma_{2,3}} \varphi_2 \Delta s_{2,3}(x, \lambda) \frac{dx}{x - y} 2\pi i + \int_{\sigma_{2,3}} \varphi_3(x, \lambda) \frac{dx}{x - y} 2\pi i,$$
and one may continue this last integral to $\gamma_3$ and $\sigma_{3,4}$ by Cauchy’s theorem as above. Proceeding all the way around the circle in this fashion one finds

$$\varphi_1(y, \lambda) = -\sum_{k=1}^{2r} \left\{ \int_{\gamma_k} \frac{\varphi_k(x, \lambda)}{x-y} \frac{dx}{2\pi i} + \int_{\sigma_{k,k+1}} \frac{\varphi_k \Delta s_{k,k+1}(x, \lambda)}{x-y} \frac{dx}{2\pi i} \right\} = 0 \quad (1.50)$$

Now we formulate the condition that the right hand side of (1.44) should be holomorphic in the exterior of the circle of radius $\delta$ about $a$ and asymptotic to the identity at $\infty$. This can be expressed as

$$\sum_{k=1}^{2r} \int_{\gamma_k} \frac{\varphi_k(x, \lambda)(I + \Delta m_k(x, \lambda))}{x-y} \frac{dx}{2\pi i} = I. \quad (1.51)$$

Adding this result to the preceding equation one finds

$$\varphi_1(y, \lambda) + K_1 \varphi(y, \lambda) = I \quad (1.52)$$

where

$$K_1 \varphi(y, \lambda) = \sum_{k=1}^{2r} \left\{ \int_{\gamma_k} \frac{\varphi_k \Delta m_k(x, \lambda)}{x-y} \frac{dx}{2\pi i} - \int_{\sigma_{k,k+1}} \frac{\varphi_k \Delta s_{k,k+1}(x, \lambda)}{x-y} \frac{dx}{2\pi i} \right\} \quad (1.53)$$

and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{2r})$. We “derived” (1.52) with $y$ chosen to be in the interior of $\sigma_1$. However, we now choose to think of (1.52) as an integral equation for the restriction of $\varphi_1$ to $\partial \sigma_1$. In this case the integral operator $K_1$ defined in (1.52) is understood to involve non tangential limits for $y$ on $\partial \sigma_1$ from the interior of $\sigma_1$. There is was nothing special about $\varphi_1$ in the arguments above and so we find that the vector $\varphi$ satisfies the system of integral equations

$$\varphi_k(y, \lambda) + K_k \varphi(y, \lambda) = I \quad (1.54)$$

where the integral operators $K_k$ are defined by the same formula as $K_1$ but the $y$ variable which occurs in (1.53) takes values in $\partial \sigma_k$ (with the integral operator defined by non-tangential limits from the interior). It is well known that non-tangential limits for the Cauchy kernel $(x-y)^{-1}$ determine a bounded operator on $L^2(\partial \sigma_k)$ in case both $x$ and $y$ are in $\partial \sigma_k$. It is not hard to see from this that the integral operator

$$K \varphi = (K_1 \varphi, K_2 \varphi, \ldots, K_{2r} \varphi)$$

is a bounded operator on

$$\mathcal{H} = L^2(\partial \sigma_1) \oplus L^2(\partial \sigma_2) \oplus \cdots \oplus L^2(\partial \sigma_{2r}).$$
Furthermore it is clear that because $\Delta m_k(x, \lambda)$ and $\Delta s_{k,k+1}(x, \lambda)$ can be made uniformly small by choosing $\lambda$ close enough to $\lambda_0$, the system of integral equations (1.54) has a unique solution in $\mathcal{H}$ provided the neighborhood $U$ is small enough.

Next we wish to show that the solution of (1.54) satisfies (1.43). Suppose then that $\varphi$ is a solution to (1.54) in $\mathcal{H}$. One calculates that for $y \in \sigma_{j,j+1}$,

$$
\varphi_{j+1}(y, \lambda) - \varphi_j(y, \lambda) = K_j \varphi(y, \lambda) - K_{j+1} \varphi(y, \lambda)
= \int_{\sigma_{j,j+1}} \left\{ \frac{\varphi_j \Delta s_{j,j+1}(x, \lambda)}{x - y(\sigma_j)} - \frac{\varphi_j \Delta s_{j,j+1}(x, \lambda)}{x - y(\sigma_{j+1})} \right\} \frac{dx}{2\pi i}
$$

(1.55)

where we’ve written $y(\sigma_k)$ for the boundary value on $\sigma_{j,j+1}$ taken from the interior of $\sigma_k$ for $k = j$ and $k = j + 1$. Since

$$
\frac{1}{2\pi i} \left\{ \frac{1}{x - y(\sigma_j)} - \frac{1}{x - y(\sigma_{j+1})} \right\} = \delta(x - y) \text{ for } y \in \sigma_{j,j+1}
$$

it follows from (1.55) that

$$
\varphi_{j+1}(y, \lambda) - \varphi_j(y, \lambda) = \varphi_j(y, \lambda) \Delta s_{j,j+1}(y, \lambda) \text{ for } y \in \sigma_{j,j+1}.
$$

(1.56)

This is a “boundary value” version of (1.43), which we will now extend to a sectorial neighborhood of $\sigma_{j,j+1}$. As a simple consequence of satisfying the integral equation (1.54) we know that $\varphi_j(y, \lambda)$ for $y \in \sigma_{j,j+1}$ is the boundary value of a holomorphic function $\varphi_j(y, \lambda)$ for $y \in \sigma_j$. Also $\varphi_{j+1}(y, \lambda)$ for $y \in \sigma_{j,j+1}$ is the boundary value of a holomorphic function $\varphi_{j+1}(y, \lambda)$ for $y \in \sigma_{j+1}$. Solving (1.56) for $\varphi_{j+1}(y, \lambda)$ we see this function has an analytic continuation into a sector containing $\sigma_{j,j+1}$, since equation (1.43) shows that the function $I + \Delta s_{j,j+1}(y, \lambda)$ (and its inverse) has an analytic continuation into such a sector. One can use Morera’s theorem to show that the function obtained by gluing together $L^2$ boundary values along $\sigma_{j,j+1}$ is actually holomorphic in a neighborhood of $\sigma_{j,j+1}$. The same argument works for $\varphi_j(y, \lambda)$ and equation (1.56) extends to a sectorial neighborhood of $\sigma_{j,j+1}$. This is (1.43).

The iterative solution to (1.54) produces an analytic function of $\lambda$ with values in $\mathcal{H}$. The asymptotics for $\varphi_j$ are obtained by substituting

$$
\frac{1}{x - y} = \frac{1}{x} \sum_{n=0}^{N} \left( \frac{y}{x} \right)^n + y^{N+1} \frac{x^{-N-1}}{x - y}
$$

into the integral equation (1.54) and noting that the functions

$$
\varphi_j(x, \lambda)x^{-k} \Delta s_{j,j+1}(x, \lambda)
$$
are integrable in $x$ on $\sigma_{j,j+1}$ for all integers $k \geq 0$ with integrals that are analytic in $\lambda$.

This finishes the proof of (i), (ii). To establish (iii) note that we’ve showed that each solution $\varphi_k$ to (1.54) extends to a holomorphic function in a sector containing $\sigma_k$, with controlled asymptotic behavior as $x \to a$. This is all that is needed to establish the analogue of (1.50) for $\varphi_k$. Subtracting this from (1.54) one obtains the analogue of (1.51) for $\varphi_k$. As noted above this is an expression of the holomorphic character of the gauge transformation $\Phi(x, \lambda) := \Psi(x, \lambda)\Psi^0(x)^{-1}$ in the exterior of $D$ and this finishes the proof of (iii).

To establish (iv) (which will play an important role in a tau function calculation in section 3) write

$$\Omega = \Omega_x + \Omega_\lambda,$$

for the one form associated with $\nabla_U$. Here $\Omega_x$ is the $dx$ term in the one form and $\Omega_\lambda$ is a sum

$$\Omega_\lambda = \sum_k \Omega_{\lambda,k} d\lambda_k.$$

The $d\lambda$ component of the formal equivalence $\nabla_U = \hat{\alpha} \cdot [\nabla_\lambda]$ is

$$-d\hat{\alpha}\hat{\alpha}^{-1} - \hat{\alpha}dH\hat{\alpha}^{-1} = \Omega_\lambda,$$

or

$$d\hat{\alpha} = -\hat{\alpha}dH - \Omega_\lambda\hat{\alpha}. \quad (1.57)$$

For simplicity we write $d = d\lambda$ and calculate,

$$d\text{Res}_{x=a} \text{Tr} \left( \hat{\alpha}^{-1} d_x \hat{\alpha} dH \right)$$

$$= \text{Res}_{x=a} \text{Tr} \left( -\hat{\alpha}^{-1} d\hat{\alpha}\hat{\alpha}^{-1} d_x \hat{\alpha} dH - \hat{\alpha}^{-1} d_x d\hat{\alpha} dH \right)$$

$$= \text{Res}_{x=a} \text{Tr} \left( d_x (dH) dH + \hat{\alpha}^{-1} d_x \Omega_\lambda\hat{\alpha} dH \right),$$

where to get from the second to the the third line we substituted (1.57) for $d\hat{\alpha}$ did an obvious cancellation in the result and made use of the fact that $dHdH = 0$ since $dH$ is diagonal. But

$$\text{Res}_{x=a} \text{Tr} \left( d_x (dH) dH \right) = 0,$$

since the Laurent series for $d_x (dH) dH$ begins with terms $C(x-a)^{-3}$ and so we find

$$d\text{Res}_{x=a} \text{Tr} \left( \hat{\alpha}^{-1} d_x \hat{\alpha} dH \right) = \text{Res}_{x=a} \text{Tr} \left( d_x \Omega_\lambda\hat{\alpha} dH\hat{\alpha}^{-1} \right). \quad (1.58)$$

A straightforward calculation now shows that
\[
\text{Res}_{x=a} \text{Tr} \left( d_x (\hat{\alpha} dH \hat{\alpha}^{-1}) \hat{\alpha} dH \hat{\alpha}^{-1} \right) = \text{Res}_{x=a} \text{Tr} \left( d_x (dH) dH \right) = 0
\]  
(1.59)

In (1.59) we replace \( \hat{\alpha} dH \hat{\alpha}^{-1} \) by \(-d\hat{\alpha}^{-1} - \Omega_\lambda \) from (1.57), and find

\[
\text{Res}_{x=a} \text{Tr} \left( d_x (\Omega_\lambda) \Omega_\lambda \right) + 2\text{Res}_{x=a} \text{Tr} \left( d_x (\Omega_\lambda) d\hat{\alpha} \hat{\alpha}^{-1} \right) = 0,
\]  
(1.60)

where we made use of the fact that \( d_x (d\hat{\alpha} \hat{\alpha}^{-1}) d\hat{\alpha} \hat{\alpha}^{-1} \) is “regular” at \( x = a \) and so has 0 residue, and that

\[
\text{Res}_{x=a} d_x \text{Tr} \left( \Omega_\lambda d\hat{\alpha} \hat{\alpha}^{-1} \right) = 0.
\]

Now substitute \(-d\hat{\alpha}^{-1} - \Omega_\lambda \) for \( \hat{\alpha} dH \hat{\alpha}^{-1} \) in (1.58) and make use of (1.60) to get,

\[
 d\text{Res}_{x=a} \text{Tr} \left( \hat{\alpha}^{-1} d_x dH \right) = -\frac{1}{2} \text{Res}_{x=a} \text{Tr} \left( d_x (\Omega_\lambda) \Omega_\lambda \right).
\]  
(1.61)

Now for the first time we use the fact that \( \nabla^0 \) is looked at in a trivialization in which it extends to a connection on \( \mathbb{P}^1 \) with an additional regular singularity at \( \infty \). We see from this and (iii) that,

\[
\Omega_\lambda = -d\Phi \Phi^{-1},
\]  
(1.60)

is holomorphic in a neighborhood of \( \infty \). Since \( \text{Tr}(d_x (\Omega_\lambda) \Omega_\lambda) \) is meromorphic on \( \mathbb{P}^1 \) with a single pole at \( x = a \), it follows that the residue at this pole must be 0. With (1.61) this finishes the proof of (iv) (incidentally, the argument here follows the argument in [12] used to show that the Jimbo, Miwa, Ueno expression for \( d \log \tau \) is closed). QED.

§2 The Vector Bundle Deformation of Malgrange

**Representations of the fundamental group and flat connections.** In this section we are interested in constructing an integrable deformation of a connection on a bundle over \( \mathbb{P}^1 \) which is monodromy preserving and which respects the local character of the connection near its singular points.

The principal tool in the construction of this deformation away from the singular set is a correspondence between representations of the fundamental group and vector bundles with flat connections. More precisely, suppose that \( X \) is a connected complex manifold with base point \( x^0 \). Suppose that \( E \to X \) is a complex vector bundle with a flat holomorphic connection \( \nabla \). Suppose that \( \gamma : [0,1] \to X \) is a
piecewise smooth curve in $X$ and let $\mathcal{P}_\nabla(\gamma)$ denote parallel translation with respect to $\nabla$ along $\gamma$. Then

$$\mathcal{P}_\nabla(\gamma) : E_{\gamma(0)} \to E_{\gamma(1)},$$

is a linear isomorphism between the fibers of $E$ at the endpoints $\gamma(0)$ and $\gamma(1)$. Now suppose that $\gamma$ is a piecewise smooth closed loop based at $x^0$ and let $g = [\gamma]$ denote the homotopy class of $\gamma$. Then

$$\rho(g) := \mathcal{P}_\nabla(\gamma)^{-1}, \quad (2.1)$$

defines a representation of $\pi_1(X, x^0)$ on $E_{x^0}$. The right hand side depends only on the homotopy class of $\gamma$ because the curvature of $\nabla$ is zero. The equivalence class of the representation $\rho$ actually determines the pair $(E, \nabla)$ up to isomorphism. Before we turn to the main theorem of this section we digress to sketch a construction that takes one from $\rho$ to $(E, \nabla)$. Let $\pi : \mathcal{R}(X) \to X$ denote the simply connected covering space of $X$ and suppose that $(E, \nabla)$ is a vector bundle with flat connection over $X$ as above. The pull back bundle $\pi^*(E) \to \mathcal{R}(X)$ is necessarily trivial since the base $\mathcal{R}(X)$ is simply connected. The natural projection $\tilde{\pi} : \pi^*(E) \to E$ is a local diffeomorphism and since $d\tilde{\pi}^*$ is an isomorphism of tangent spaces which maps the vertical vectors in $T_{\tilde{\pi}}(\mathcal{R}(X))$ bijectively onto the vertical vectors in $T_p(E)$ we may use $d\tilde{\pi}^*$ to lift the horizontal subspace in $T_p(E)$ that comes from $\nabla$ to a horizontal subspace in $T_{\tilde{\pi}}(\mathcal{R}(X))$. We write $\pi^*(\nabla)$ for the resulting connection on $\pi^*(E)$, and note that since the pull back connection $\pi^*(\nabla)$ is related to $\nabla$ by a local diffeomorphism it is also a flat connection. We may thus produce a trivialization for $\pi^*(E)$ consisting of flat sections for $\pi^*(\nabla)$. Let $\tilde{x}^0$ denote a base point in $\mathcal{R}(X)$ such that $\pi(\tilde{x}^0) = x^0$. For $u \in \pi^*(E)_{\tilde{x}^0}$ let $\tilde{x} \to \mathcal{P}(\tilde{x}, u) \in \pi^*(E)_{\tilde{x}}$ denote the parallel section of $\pi^*(E)$ which agrees with $u$ at $\tilde{x}^0$. Writing $E_0 := \pi^*(E)_{\tilde{x}^0}$ (which is naturally isomorphic to $E_{x^0}$) we have

$$\mathcal{R}(X) \times E_0 \ni (x, u) \to \mathcal{P}(x, u) \in \pi^*(E),$$

is an isomorphism between the bundle $\pi^*(E)$ and the trivial bundle $\mathcal{R}(X) \times E_0$. If we compose this map $\mathcal{P}$ with the vector bundle projection

$$\tilde{\pi} : \pi^*(E) \to E,$$

one obtains a vector bundle map

$$\tilde{\pi}\mathcal{P} : R(X) \times E_0 \to E, \quad (2.2)$$

which covers the projection $\pi : \mathcal{R}(X) \to X$. 34
The representation $\rho$ from (2.1) determines a left action of $\pi_1(X, x^0)$ on $R(X) \times E_0$ given by
\[ g \cdot (\tilde{x}, u) = (g \cdot \tilde{x}, \rho(g)u), \tag{2.3} \]
where $g \cdot \tilde{x}$ is just the usual action of $\pi_1$ on the simply connected covering space $\mathcal{R}(X)$. We will show that the quotient bundle $\pi_1(X, x^0) \backslash \mathcal{R}(X) \times E_0 \to \pi_1(X, x^0) \backslash \mathcal{R}(X)$ is isomorphic to $E \to X$ as a vector bundle with connection.

We begin by showing that the map $\tilde{\pi}P$ is equivariant for this action of $\pi_1(X, x^0)$. To see this suppose that $\tilde{x} = [\chi]$ where $\chi$ is a smooth curve joining $x^0$ to $x$ in $X$. Let $\tilde{x}^0$ denote the class of the constant path starting and ending at $x^0$. Write $\tilde{\chi}$ for the lift of $\chi$ into $\mathcal{R}(X)$ with initial point $\tilde{x}^0$. Let $g = [\gamma]$ where $\gamma$ is a smooth closed path in $X$ based at $x^0$. Then one finds
\[
\tilde{\pi}P(g \cdot \tilde{x}, \rho(g)u) = \tilde{\pi}P_{\pi^*\nabla}(\tilde{\gamma}\tilde{\chi}) \rho(g)u = \mathcal{P}_\nabla(\tilde{\gamma}\tilde{\chi}) \rho(g)u = \mathcal{P}_\nabla(\tilde{\gamma}\tilde{\chi})u = \tilde{\pi}P(\tilde{x}, u),
\]
which shows the equivariance of the map $\tilde{\pi}P$. In the third and last equality we used the fact that $\tilde{\pi}P_{\pi^*\nabla}(\tilde{\gamma})u = \mathcal{P}_\nabla(\gamma)u$, where $\tilde{\gamma}$ is a lift of $\gamma$ and $u \in E_{\gamma(0)}$. This is obvious if the curve $\gamma$ stays in a neighborhood $U$ in $X$ which is evenly covered by the projection on $X$ from $\mathcal{R}(X)$; the general result follows from the fact that parallel translation is an anti-homomorphism under homotopy composition.

Since the vector bundle action (2.3) covers the standard left action of $\pi_1(X, x^0)$ on $\mathcal{R}(X)$ and since $\pi_1(X, x^0) \backslash \mathcal{R}(X) \simeq X$ one finds that
\[
\pi_1(X, x^0) \backslash \mathcal{R}(X) \times E_0 := \mathcal{R}(X) \times _\rho E_0,
\]
is a vector bundle over $X$ isomorphic to $E$ through the map induced by (2.2). Since the construction of the bundle
\[
\mathcal{R}(X) \times _\rho E_0 \to X \tag{2.4}
\]
depends only on the representation $\rho$ it follows that this representation determines the bundle $E \to X$ up to isomorphism. In fact the bundle (2.4) has a naturally defined flat connection $\nabla_\rho$ so that the map induced by (2.2) determines an isomorphism,
\[
(\mathcal{R}(X) \times _\rho E_0, \nabla_\rho) \simeq (E, \nabla). \tag{2.5}
\]
In order to define $\nabla_\rho$ we introduce a family of trivializations for the vector bundle,
\[
\pi_\rho : \mathcal{R}(X) \times _\rho E_0 \to X. \tag{2.6}
\]
Let $\mathcal{F}$ denote a covering of $X$ by open sets, with the following properties,

1. If $U \in \mathcal{F}$ then $U$ is evenly covered by $\pi : \mathcal{R}(X) \to X$. That is there exist disjoint sets $U_\alpha \subset X$ such that $\pi^{-1}(U) = \cup \alpha U_\alpha$ and $\pi : U_\alpha \to U$ is a diffeomorphism for each $\alpha$.

2. If $U, V \in \mathcal{F}$ and $U \cap V \neq \emptyset$ then $U \cup V$ is evenly covered by $\pi$.

The existence of such a covering is an easy consequence of the fact that $X$ has a metric topology. Now suppose that $U \in \mathcal{F}$ and $U_\alpha \subset \mathcal{R}(X)$ is such that $\pi : U_\alpha \to U$ is a diffeomorphism. We define a trivialization, $\phi(U_\alpha)$ of $\pi^{-1}_\alpha(U)$ by sending each equivalence class,

$$[(\tilde{x}, u)] := \{(y, v) : (\tilde{y}, v) = g \cdot (\tilde{x}, u) \text{ for } g \in \pi_1(X, x^0)\}$$

in $\pi^{-1}_\alpha(U)$ into the unique representative of the form $(\tilde{x}, u)$ with $\tilde{x} \in U_\alpha$ and $u \in E_0$. Then

$$\phi(U_\alpha)([\tilde{x}, u]) := (\pi(\tilde{x}), u) = (x, u) \in U \times E_0.$$

Now suppose that $U, V \in \mathcal{F}$ and $\pi : U_\alpha \to U$ and $\pi : V_\beta \to V$ are diffeomorphisms (note that $U = V$ is a possibility). Then if $U \cap V \neq \emptyset$ it follows that there exists a unique $g_{\alpha\beta} \in \pi_1(X, x^0)$ so that $U_\alpha \cap g_{\alpha\beta}V_\beta \neq \emptyset$. The existence of such a $g_{\alpha\beta}$ is trivial and uniqueness is equivalent to the assertion that if $U_\alpha \cap V_\beta \neq \emptyset$ and $U_\alpha \cap gV_\beta \neq \emptyset$ for some $g \in \pi_1(X, x^0)$ then $g = 1$. Suppose then that $U_\alpha \cap V_\beta \neq \emptyset$ and $U_\alpha \cap gV_\beta \neq \emptyset$. Then since $U \cup V$ is evenly covered we have $(U_\alpha \cup V_\beta) \cap g(U_\alpha \cup V_\beta) = \emptyset$ if $g \neq 1$. But evidently,

$$U_\alpha \cap gV_\beta \subset (U_\alpha \cup V_\beta) \cap g(U_\alpha \cup V_\beta) = \emptyset,$$

so $U_\alpha \cap gV_\beta = \emptyset$ if $g \neq 1$. Uniqueness follows. One may now easily compute

$$\phi(U_\alpha)\phi(V_\beta)^{-1}(x, u) = (x, \rho(g_{\alpha\beta}u)). \quad (2.7)$$

Since these transition functions are constant in the base variables there is a globally defined flat connection $\nabla_\rho$ on $\mathcal{R}(X) \times_\rho E_0$ which is obtained by gluing together the exterior derivative in the base variables defined in each of the trivializations $\phi(U_\alpha)$. In these trivializations it is not hard to check the isomorphism of connections (2.5).
Then parallel translation of a vector $u$ in $E_0$ along $\gamma$ is constant in the trivializations $\phi(U_{\alpha_j})$ for $j = 0, \ldots, n - 1$. To compute the holonomy one must only compute what the vector $u$ in the trivialization $\phi(U_{\alpha_{n-1}})$ looks like in the trivialization $\phi(U_{\alpha_0})$. However, it is clear from the construction that $g U_{\alpha_0} \cap U_{\alpha_{n-1}} \neq \phi$. Thus the parallel transport of $u \in E_0$ along $\gamma$ gives $\rho(g)^{-1} u \in E_0$. This finishes our sketch of the reconstruction of a vector bundle with flat connection (up to equivalence) from its holonomy representation. We now begin to explain the setting for Theorem 2.9 below.

**The vector bundle deformation.** Suppose that \{a^0_1, a^0_2, \ldots, a^0_n\} is a collection of $n$ distinct points in $\mathbb{C}$. In this section we will construct a global deformation for a connection $\nabla^0$ defined on the trivial bundle $E^0 := \mathbb{P}^1 \times \mathbb{C}^p$, with a simple type $r_j$ singularity at $a^0_j$ and a regular point or simple type $r_{\infty}$ singularity at $\infty$. For simplicity in stating results, when $\nabla^0$ has a regular point at $\infty$ we put $r_{\infty} = -1$ and say that $\nabla^0$ has a simple type -1 singularity.

Roughly speaking the deformation we consider will preserve the local type of the singularities for $\nabla^0$ and also the local and global monodromy data for the connection $\nabla^0$. We now make this more precise.

By relabeling the points if necessary we may suppose that $r_j \geq 1$ for $j = 1, 2, \ldots, m$ and $r_j = 0$ for $j = m + 1, m + 2, \ldots, n$. It could happen that $m = 0$ or $m = n$. The point $\infty$ is somewhat special in this context since it does not contribute to the space of pole deformations, $\mathcal{R}(\mathbb{Z}^n)$, which we defined in section 1. We will mention special considerations concerning $\infty$ when we encounter them.

For $j = 1, \ldots, m$ let

$$C_j := \mathbb{Z}^p \times \mathbb{C}^p \times \cdots \times \mathbb{C}^p$$

with $r_j - 1$ factors $\mathbb{C}^p$, denote the local configuration space at $a^0_j$, as described in section 1. Recall that each point in $C_j$ corresponds to a formal equivalence class for a simple type $r_j$ connection at $a^0_j$. Write $C_\infty$ for the corresponding configuration space at $\infty$, defined if $r_{\infty} \geq 1$.

Write $\Lambda_j^0 \in C_j$ for the data associated to $\nabla^0$ at $a^0_j$, for $j = 1, \ldots, m$. Recall that $\mathcal{R}(X)$ is just the simply connected cover of $X$, and define

$$\mathcal{D} := \mathcal{R}(\mathbb{Z}^n) \times \prod_{j=1}^m \mathcal{R}(C_j) \times \mathcal{R}(C_\infty),$$

(2.8)

where the product is just the Cartesian product, and the final factor $\mathcal{R}(C_\infty)$ only appears if $r_{\infty} \geq 1$.

The space $\mathcal{D}$ will serve as our “deformation space”. In the first factor, $\mathbb{Z}^n$, is the space of pole locations and in each of the subsequent factors, $C_j$, is the space
of local formal equivalence classes at \(a_j^0\). We must pass to the simply connected cover in \(\mathcal{D}\) to guarantee global existence for the sort of deformation we are about to describe.

Let \(\mathcal{M}_j \to \mathcal{C}_j\) denote the fiber bundle over \(\mathcal{C}_j\) whose fiber over \(\Lambda \in \mathcal{C}_j\) is the holomorphic equivalence class of connections formally equivalent to the diagonal model (1.12) associated to the base point \(\Lambda\) (defined in section 1). As in section 1 we also write \(\mathcal{M}_j \to \mathcal{R}(\mathcal{C}_j)\) for the pull back of \(\mathcal{M}_j\) under the projection \(\mathcal{R}(\mathcal{C}_j) \to \mathcal{C}_j\).

Let \(\sigma_j^0 \in \mathcal{M}_j\) denote the point in the fiber associated to the class of \(\nabla^0\) in a neighborhood of \(x = a_j^0\) (\(\sigma_\infty^0\) is also defined if \(r_\infty \geq 1\)). Let \(\lambda \to \sigma_j(\lambda)\) denote the unique flat section of \(\mathcal{M}_j \to \mathcal{R}(\mathcal{C}_j)\) with \(\sigma_j(\Lambda_0^0) = \sigma_j^0\) (\(\sigma_\infty(\lambda)\) is defined in a similar fashion if \(r_\infty \geq 1\)).

Recall that \(Y_k\) is the subset of points \((x,t) \in \mathbb{P}^1 \times \mathcal{R}(\mathbb{Z}^n)\) with \(x = a_k(t)\), and for \(j = 1, 2, \ldots, n\) define

\[
\mathcal{Y}_k = Y_k \times \prod_{j=1}^m \mathcal{R}(\mathcal{C}_j) \times \mathcal{R}(\mathcal{C}_\infty) \subset \mathbb{P}^1 \times \mathcal{D},
\]

where the factor \(\mathcal{R}(\mathcal{C}_\infty)\) is present only if \(r_\infty \geq 1\). Let \(t^0 \in \mathcal{R}(\mathbb{Z}^n)\) denote a point in the covering space of \(\mathbb{Z}^n\) such that \(a_j(t^0) = a_j^0\), let \(\lambda_j^0 \in \mathcal{R}(\mathcal{C}_j)\) denote a point in the covering space of \(\mathcal{C}_j\) such that \(\Lambda_j^0 = \Lambda_j(\lambda_j^0)\) (where \(\Lambda_j\) is the projection from \(\mathcal{R}(\mathcal{C}_j)\) to \(\mathcal{C}_j\)) and write

\[
\mathbb{P}^1 \ni x \to i(x) := (x, t^0, \lambda^0) \in \mathbb{P}^1 \times \mathcal{D},
\]

where

\[
\lambda^0 = (\lambda_1^0, \ldots, \lambda_m^0, \lambda_\infty^0),
\]

and as above \(\lambda_\infty^0\) only occurs when \(r_\infty \geq 1\).

The following theorem is due to Malgrange ([18] theorem 3.1).

**Theorem 2.9** There exists a rank \(p\) holomorphic vector bundle \(E \to \mathbb{P}^1 \times \mathcal{D}\) and an integrable connection \(\nabla\) on \(E\) with a simple type \(r_j\) singularity along \(\mathcal{Y}_j\) for \(j = 1, \ldots, n\) and a simple type \(r_\infty\) singularity along \(\mathcal{Y}_\infty\) such that the restriction of \((E, \nabla)\) to \(\mathbb{P}^1 \times \{(t^0, \lambda^0)\}\) is equivalent to \((E^0, \nabla^0)\) (that is, \(i^*(E, \nabla) \simeq (E^0, \nabla^0)\)). Furthermore for \(j = 1, \ldots m\) the restriction of \((E, \nabla)\) to \(\mathbb{P}^1 \times \{(t, \lambda)\}\) is formally equivalent to the model connection \(\nabla_{\Lambda_j(\lambda_j)}\) (1.12) near \(x = a_j(t)\) and is in the holomorphic equivalence class \(\sigma_j(\lambda_j) \in \mathcal{M}_j\).

Proof. We will prove this result as Malgrange does by first constructing the deformation in the complement of

\[
\mathcal{Y} = \bigcup_{j=1}^n \mathcal{Y}_j \cup \mathcal{Y}_\infty \subset \mathbb{P}^1 \times \mathcal{D},
\]

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and then extending the connection $\nabla^0$ to a tubular neighborhood $\mathcal{T}(\mathcal{Y}_j)$ ($\mathcal{T}(\mathcal{Y}_\infty)$) of each singular set $\mathcal{Y}_j$ ($\mathcal{Y}_\infty$) so that it has the right local characteristics. In particular one finds that the two constructions must be holomorphically equivalent on $\mathcal{T}(\mathcal{Y}_k) \setminus \mathcal{Y}_k$ and this equivalence allows one to define a bundle over all $\mathbb{P}^1 \times \mathcal{D}$ together with a connection that has the right global and local properties.

An important result for the construction of the deformation on $\mathbb{P}^1 \times \mathcal{D} \setminus \mathcal{Y}$ is the following observation of Malgrange. Choose some point $x^0 \in \mathbb{C}$ so that $x^0 \neq a_j(t^0)$ for all $j = 1, \ldots, n$. Define $p^0 = (x^0, t^0, \lambda^0)$. Then the map

$$
\mathbb{P}^1 \setminus \{a_1^0, a_2^0, \ldots, a_n^0, \infty\} \ni x \to (x, t^0, \lambda^0) \in \mathbb{P}^1 \times \mathcal{D} \setminus \mathcal{Y},
$$

induces an isomorphism of fundamental groups

$$
\pi_1 \left( \mathbb{P}^1 \setminus \{a_1^0, a_2^0, \ldots, a_n^0, \infty\}, x^0 \right) \simeq \pi_1 \left( \mathbb{P}^1 \times \mathcal{D} \setminus \mathcal{Y}, p^0 \right). \tag{2.11}
$$

This is explained in both [17] and [11] where the deformation space does not include the factors $\mathcal{R}(\mathcal{C}_j)$. However, the product of these factors is simply connected so it does not influence the result. The holonomy of the connection $\nabla^0$ at the base point $x^0$ determines a representation, $\rho$, of $\pi_1 \left( \mathbb{P}^1 \setminus \{a_1^0, a_2^0, \ldots, a_n^0, \infty\}, x^0 \right)$ on $GL(p, \mathbb{C})$. The isomorphism (2.11) and the representation $\rho$ determines a $GL(p, \mathbb{C})$ representation of $\pi_1(\mathbb{P}^1 \times \mathcal{D} \setminus \mathcal{Y}, p_0)$ which we continue to denote by $\rho$. Associated with this representation is a vector bundle $E_\rho := (\mathbb{P}^1 \times \mathcal{D} \setminus \mathcal{Y}) \times_\rho \mathbb{C}^p$ with connection $\nabla_\rho$ whose holonomy representation is given by $\rho$.

Next we turn to the construction of the local deformations. Suppose that $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and let $\delta(a)$ denote the minimum of the distances, $\{|a_i - a_j|, |a_i^0 - a_j^0|\}_{i \neq j}$ (the reason for insisting that $\delta(a) \leq \min\{|a_i - a_j^0|\}_{i \neq j}$ will appear below). Let $D_j(a)$ denote the disk of radius $\delta(a)/3$ about the point $a_j$. Let $D_\infty(a)$ denote the open complement of the closed disk of radius $\delta(a) + \max_i \{|a_i|, |a_i^0|\}$. It is clear by construction that the disks $\{D_\infty(a), D_j(a), j = 1, \ldots, n\}$ are pairwise disjoint for each $a \in \mathbb{Z}^n$. Define a tubular neighborhood, $\mathcal{T}(\mathcal{Y}_j)$, of $\mathcal{Y}_j$ by

$$
\mathcal{T}(\mathcal{Y}_j) = \{(x, t, \lambda) \in \mathbb{P}^1 \times \mathcal{D} | x \in D_j(a(t))\},
$$

and a tubular neighborhood of $\mathcal{Y}_\infty$ by

$$
\mathcal{T}(\mathcal{Y}_\infty) := \{(x, t, \lambda) \in \mathbb{P}^1 \times \mathcal{D} | x \in D_\infty(a(t))\}.
$$

Then the neighborhoods $\{\mathcal{T}(\mathcal{Y}_\infty), \mathcal{T}(\mathcal{Y}_j), j = 1, \ldots, n\}$ are pairwise disjoint. Following the scheme that can be found in Malgrange [17] we define a connection on the trivial bundle $\mathcal{T}(\mathcal{Y}_j) \times \mathbb{C}^p \to \mathcal{T}(\mathcal{Y}_j)$ by lifting the connection on the trivial bundle over $D_j(a^0) \times \mathcal{R}(\mathcal{C}_j)$ that one obtains from theorem 1.26 above. Recall
that $\lambda \rightarrow \sigma_j(\lambda)$ is the unique flat section of $M_j \rightarrow \mathcal{R}(C_j)$ with $\sigma_j(\Lambda^0_j) = \sigma^0_j$. For $j = 1, \ldots, m$ let $\nabla_j$ denote the integrable connection on the trivial bundle

$$D_j(a^0) \times \mathcal{R}(C_j) \times \mathbb{C}^p \rightarrow D_j(a^0) \times \mathcal{R}(C_j)$$

with a simple type $r_j$ singularity along $\{a^0_j\} \times \mathcal{R}(C_j)$, whose existence is guaranteed by theorem 1.26. This connection naturally extends to a connection on the trivial bundle

$$D_j(a^0) \times \mathcal{R}(C) \times \mathbb{C}^p \rightarrow D_j(a^0) \times \mathcal{R}(C),$$

by pulling back the connection one form under the natural projection

$$D_j(a_0) \times \mathcal{R}(C) \rightarrow D_j(a_0) \times \mathcal{R}(C_j),$$

where we’ve written

$$\mathcal{R}(C) := \prod_{j=1}^m \mathcal{R}(C_j) \times \mathcal{R}(C_\infty).$$

We continue to denote this connection by $\nabla_j$. Now define a map

$$\text{pr}_j : \mathcal{T}(Y_j) \rightarrow D_j(a^0) \times \mathcal{R}(C), \quad (2.12)$$

by $\text{pr}_j(x, t, \lambda) = (x - a_j(t) + a^0_j, \lambda)$ (the extra condition that $\delta(a) \leq \min \{|a_i^0 - a_j^0|\}_{i \neq j}$ now guarantees that $x - a_j(t) + a^0_j \in D_j(a^0)$ for $x \in D_j(a^0)$. Let $\Omega_j(x, \lambda_j)$ denote the one form for $\nabla_j$ (and we’ve written $\Omega_j(x, \lambda_j)$ to emphasize the fact that $\Omega_j$ only depends on the variables $(x, \lambda_j)$),

$$\nabla_j = d + \Omega_j(x, \lambda_j), \quad (2.13)$$

and define a connection (which we again call $\nabla_j$!) on the trivial bundle $E_j := \mathcal{T}(Y_j) \times \mathbb{C}^p \rightarrow \mathcal{T}(Y_j)$ by

$$\nabla_j := d + \text{pr}_j^*(\Omega_j), \quad (2.14)$$

where $d$ denotes the exterior derivative on $\mathcal{T}(Y_j)$ (acting on $\mathbb{C}^p$ valued functions). It is easy to check that connection (2.14) is integrable and has a simple type $r_j$ singularity along $Y_j$ as a consequence of (2.13) being integrable with a simple type $r_j$ singularity along $\{a^0_j\} \times \mathcal{R}(C)$. Now we wish to determine the holonomy for $\nabla_j$ on $\mathcal{T}(Y_j) \setminus Y_j$. The map

$$\mathcal{T}(Y_j) \setminus Y_j \ni (x, t, \lambda) \rightarrow t \in \mathcal{R}(\mathbb{Z}^n), \quad (2.15)$$

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is surjective with fiber over $t$ given by $D_j(a(t))\setminus\{a(t)\} \times \mathcal{R}(C)$ (which is homeomorphic to $D_j(a^0)\setminus\{a_j^0\} \times \mathcal{R}(C)$). The first part of the homotopy exact sequence for this fiber bundle reads

$$\rightarrow \pi_2(\mathcal{R}(Z^n)) \rightarrow \pi_1(\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j) \rightarrow \pi_1(D_j(a^0)\setminus\{a_j^0\}) \rightarrow \pi_1(\mathcal{R}(Z^n)) \rightarrow 0.$$ 

where we substituted $\pi_1(D_j(a^0)\setminus\{a_j^0\})$ for $\pi_1(D_j(a^0)\setminus\{a_j^0\} \times \mathcal{R}(C))$. Thus we have

$$\pi_1(\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j) \cong \pi_1(D_j(a^0)\setminus\{a_j^0\}). \tag{2.16}$$

Let

$$(\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j)_{t=t_0} = \{(x, t^0, \lambda) \in \mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j \} \cong D_j(a^0)\setminus\{a_j^0\} \times \mathcal{R}(C)$$

Then the restriction of $pr_j$,

$$pr_j : (\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j)_{t=t_0} \rightarrow D_j(a^0)\setminus\{a_j^0\} \times \mathcal{R}(C), \tag{2.17}$$

is essentially the identity. Since (2.16) shows that representatives of all the homotopy classes of curves in $\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j$ can be found among the loops that stay in the section $(\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j)_{t=t_0}$ it follows that the holonomy of the connection $\nabla_j$ can be computed from its restriction to $(\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j)_{t=t_0}$. But the identification (2.16) shows that this holonomy is the same as the holonomy of the connection coming from theorem (1.26) on the the space $D_j(a^0)\setminus\{a_j^0\} \times \mathcal{R}(C)$. Since $\pi_1(\mathcal{R}(C)) = 0$ we may compute this holonomy by restricting to $\lambda = \lambda_0$ where the connection agrees with $\nabla_0$ by construction.

We also need to construct a model connection in the tubular neighborhoods $\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j$ for $j = m + 1, \ldots, n$ and $\mathcal{T}(\mathcal{Y}_{\infty})\setminus\mathcal{Y}_{\infty}$. In the first instance, we are looking for a connection on $\mathcal{T}(\mathcal{Y}_j)$ with a simple type 0 singularity along $\mathcal{Y}_j$, which is equivalent to $\nabla_0$ in a neighborhood of $a_j^0$. Since $\nabla_0$ is a simple type 0 connection, Proposition 1.25b shows that in a neighborhood of $x = a_j^0$ there exists a diagonal matrix $\Lambda_j$ so that in the appropriate local trivialization about $x = a_j^0$, $\nabla_0$ is represented by

$$d_x + \Omega_j$$

on the trivial bundle

$$D_j(a^0) \times \mathbb{C}^p \rightarrow D_j(a^0), \tag{2.18}$$

where

$$\Omega_j = -\Lambda_j d_x(x - a_j^0) \frac{x - a_j^0}{x - a_j^0}.$$
Define a connection $\nabla_j$ on the trivial bundle over $\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j$ by

$$\nabla_j = d - \Lambda_j d(x - a_j(t)) \over x - a_j(t),$$

where $d = d_x + d_t + d_{\lambda}$ is the exterior derivative on $\mathcal{T}(\mathcal{Y}_j)$. It is easy to check that $\nabla_j$ is a simple integrable connection with type 0 singularity along $\mathcal{Y}_j$ which has a restriction to $D_j(a^0) \times \{(t^0, \lambda^0)\}$ that is equivalent to $\overline{\nabla}^0$ by construction. The same argument given above for $j = 1, \ldots, m$ applies here and one sees that the connection $\nabla_j$ defined on the trivial bundle,

$$\mathcal{T}(\mathcal{Y}_j) \times \mathbb{C}^p \to \mathcal{T}(\mathcal{Y}_j),$$

has holonomy determined by its restriction to $t = t^0$ and $\lambda = \lambda^0$, where it is essentially $\overline{\nabla}^0$. To extend the connection to $\mathcal{T}(\mathcal{Y}_\infty)\setminus\mathcal{Y}_\infty$ so that it is regular along $\mathcal{Y}_\infty$, or has a logarithmic pole along $\mathcal{Y}_\infty$, or a higher rank simple singularity one proceeds as above with some simplification arising from the fact that $\infty$ adds no component to the space of pole deformations. One should use the local parameter $w = \frac{1}{x}$ with $pr_\infty(w, t, \lambda) = (w, \lambda)$.

We leave the details to the reader.

What we have now is a bundle $E_\rho$ over $\mathbb{P}^1 \times \mathcal{D}\setminus\mathcal{Y}$, bundles $E_j$ over $\mathcal{T}(\mathcal{Y}_j)$, and $E_\infty$ over $\mathcal{T}(\mathcal{Y}_\infty)$ with bundle isomorphisms,

$$b_j : E_\rho|_{\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j} \to E_j|_{\mathcal{T}(\mathcal{Y}_j)\setminus\mathcal{Y}_j}, \quad (2.19)$$

and

$$b_\infty : E_\rho|_{\mathcal{T}(\mathcal{Y}_\infty)\setminus\mathcal{Y}_\infty} \to E_\infty|_{\mathcal{T}(\mathcal{Y}_\infty)\setminus\mathcal{Y}_\infty}, \quad (2.20)$$

which take the connection $\nabla_\rho$ into $\nabla_j$ and $\nabla_\infty$. We define a vector bundle $E$ over $\mathbb{P}^1 \times \mathcal{D}$ by forming the union $E_\rho \cup E_1 \cup \cdots \cup E_n \cup E_\infty$ modulo the equivalence relations determined by (2.19) and (2.20). By construction the bundle $E$ and the connection $\nabla$ obtained by gluing together the connections $\nabla_\rho$, $\nabla_j$, and $\nabla_\infty$ have the properties asserted in the formulation of Theorem 2.9. This finishes the proof.

QED

§3. Tau functions

In this section we will first follow Helmink [11] to define a tau function associated with the deformation construction of Theorem 2.0 above. Let $(E, \nabla)$ be
the integrable deformation of \((E^0, \nabla^0)\) constructed in Theorem 2.9. Recall that 
\(D = \mathcal{R}(Z^n) \times \mathcal{R}(C)\) and define 
\[ \Theta := \{(t, \lambda) \in D, \ E|_{P^1 \times \{t, \lambda\}} \text{is non-trivial}\} \]

**Theorem 3.0.** There exists a non vanishing holomorphic map \(\tau : D \to \mathbb{C}\) so that \(\Theta\) is equal to the zero set of \(\tau\).

This is basically a result of Malgrange [18] and Helmink [11] which we will sketch a proof of following the arguments for Proposition 3.2 in [11].

Sketch of Proof (more details can be found in [11]). Choose \(0 < \rho_1 < 1 < \rho_2\) and set 
\[ D_1 := \{x|x \in P^1, |x| > \rho_1\}, \quad D_2 := \{x|x \in P^1, |x| < \rho_2\}. \]

Then \(D_k \times D\) for \(k = 1, 2\) is a contractible Stein space, and hence \(E|_{D_k \times D}\) is holomorphically trivial for \(k = 1, 2\). Let \(f := (f_1, f_2, \ldots, f_p)\) be a row vector of sections for the restriction of \(E\) to \(D_1 \times D\) which trivializes this restriction. Let \(g := (g_1, g_2, \ldots, g_p)\) be a row vector of sections for the restriction of \(E\) to \(D_2 \times D\) which trivializes this restriction. Then there exists a holomorphic map \(S : D_1 \cap D_2 \times D \to GL(p, \mathbb{C})\) so that 
\[ g = fS \]
on \(D_1 \cap D_2 \times D\). Write \(S(x, t, \lambda) = S_{t, \lambda}(x)\). Then the restriction of \(E\) to \(P^1 \times \{(t, \lambda)\}\) will be trivial if and only if there are holomorphic maps \(S_{t, \lambda}^- : D_1 \to GL(p, \mathbb{C})\) and \(S_{t, \lambda}^+ : D_2 \to GL(p, \mathbb{C})\) so that for all \(x \in D_1 \cap D_2\),
\[ S_{t, \lambda}(x) = S_{t, \lambda}^-(x)S_{t, \lambda}^+(x)^{-1}. \quad (3.1) \]

A \(p\) vector of sections that trivializes \(E\) is then given by the appropriate extension of 
\[ gS_{t, \lambda}^+ = fS_{t, \lambda}^- \]
from \(D_1 \cap D_2\). Let \(S^1\) denote the unit disk, write \(\mathcal{H} := L^2(S^1, \mathbb{C}^p)\) and let \(\mathcal{H}_+\) be the closed subspace of \(\mathcal{H}\) consisting of those functions with are boundary values functions holomorphic inside the unit disk. Let \(\mathcal{H}_- = \mathcal{H}_+^\perp\). Suppose that \(S^1 \ni x \to S(x)\) is a smooth \(p \times p\) matrix valued function on the circle and let \(\mathcal{H} \ni f \to Sf\) denote the associated multiplication operator on \(\mathcal{H}\). Let 
\[ S = \begin{bmatrix} a(S) & b(S) \\ c(S) & d(S) \end{bmatrix}, \]
denote the decomposition of $S$ relative to the direct sum decomposition $H = H_+ \oplus H_-$. It is well known [1] that the factorization (3.1) exists if and only if $a(S_{t,\lambda}) : H_+ \to H_+$ is invertible. Since $(t, \lambda) \to a(S_{t,\lambda})$ is continuous in the uniform norm topology for $a(S_{t,\lambda})$ and $a(S)$ is known to be Fredholm if $S$ is smooth, it follows that the index of $a(S_{t,\lambda})$ is independent of $(t, \lambda)$. Since $a(S_{t,\lambda})$ is invertible by construction it follows that the index of $a(S_{t,\lambda})$ is 0 for all $(t, \lambda) \in \mathcal{D}$. Fix $(t, \lambda)$ for the moment. Then since $a(S_{t,\lambda})$ has index 0, there exists a finite rank operator $k : H_+ \to H_+$ so that $k + a(S_{t,\lambda})$ is invertible. In fact there exists a neighborhood $V_{t,\lambda}$ of $(t, \lambda)$ so that for all $(s, \mu) \in V_{t,\lambda}$ the operator $q_{s,\mu} := k + a(S_{s,\mu})$ is invertible. Note that $q_{s,\mu}$ is a parametrix for $a(S_{s,\mu})$ for $(s, \mu) \in V_{t,\lambda}$ in that

$$a(S_{s,\mu})q_{s,\mu}^{-1} = I + F_{s,\mu}$$

where $F_{s,\mu}$ is a finite rank operator that depends holomorphically on $(s, \mu)$. Now we show that it is possible to make a coherent choice of parametrices $q_{s,\mu}$ following the argument in Helmink [11]. Let $\{V_i\}$ be a locally finite covering of $\mathcal{D}$ with $q_i(s, \mu)$ a holomorphic parametrix for $a(S_{s,\mu})$ for all $(s, \mu) \in V_i$. For each $i, j$ such that $V_i \cap V_j \neq \emptyset$ define $q_i q_j^{-1} = \phi_{ij}$. Note that each $\phi_{ij}$ is a finite rank perturbation of the identity. The maps $V_i \cap V_j \ni (s, \mu) \to \det \phi_{ij}(s, \mu)$ are the transition functions for a holomorphic line bundle on $\mathcal{D}$. Since $\mathcal{D}$ is a Stein space we have $H^1(\mathcal{D}, \mathcal{O}^*) = 0$, and it follows that this line bundle must be trivial. Hence there exist holomorphic maps $\tau_i : V_i \to \mathbb{C}^*$ so that $\tau_i^{-1}\tau_j = \det \phi_{ij}$. Now define $t_i : H_+ \to H_+$ by $t_i1 = \tau_i$ and $t_i(e^{in\theta}) = e^{in\theta}$ for $n = 1, 2, \ldots$. Now define $q_i = t_i q_i$. Then $q_i(s, \mu)$ remains a holomorphic parametrix for $a(S_{s,\mu})$ for $(s, \mu) \in V_i$ and since

$$\det q_i q_j^{-1} = \det t_i t_j^{-1} \det q_i q_j^{-1} = \tau_i \tau_j^{-1} \det \phi_{ij} = 1,$$

it follows that

$$\tau(s, \mu) := \det(a(S_{s,\mu})q_i(s, \mu)^{-1})$$

is a well defined holomorphic function on all of $\mathcal{D}$ whose 0 set is equal to $\Theta$. QED

Next will make a connection with the tau function defined by Jimbo, Miwa and Ueno [12]. We follow Malgrange by computing the regularized logarithmic derivative of the determinant of a Toeplitz operator, the invertibility of which determines if $E|_{\mathbb{P}^1 \times \{(t, \lambda)\}}$ is trivial or not. The bundle $E$ is first realized in terms of a system of transition functions which relate local models for the connection $\nabla$ in a neighborhood of the singularities to a model for the connection $\nabla$ in a complement of a neighborhood of the singularities. We will spend some effort to choose the local models carefully (so that (3.7) below is satisfied) even though this is not important for the calculation of the regularized logarithmic derivative of the Toeplitz operator. We do it because it will simplify a curvature calculation later on.
First fix a point \((t^0, \lambda^0) \in D\). We will examine the restriction of \(E\) to \(\mathbb{P}^1 \times W\) where \(W\) is a suitably small neighborhood of \((t^0, \lambda^0)\). We will cover \(\mathbb{P}^1 \times W\) by neighborhoods \(B_j \times W\) which contain the singular sets \(x = a_j(t)\) (including \(x = \infty\) for \(j = \infty\)) and a complementary set \(B_{\infty} \times W\). We will choose trivializations for \(E\) over each of these sets and understand the bundle \(E\) in terms of the transition maps between these trivializations.

Now we turn to the identification of a suitable local model for the connection \(\nabla\) in a neighborhood of each singularity.

For each \(j = 1, \ldots, m\) choose a connected, simply connected product neighborhood \(W_j = U_j \times V_j\) of \((t^0_j, \lambda^0_j)\), with compact closure. For \(j = m + 1, \ldots, n (j = \infty)\) choose a connected, simply connected neighborhood of \(t^0_j (\lambda^0_\infty)\) with compact closure. Let

\[
W = \prod_{j=1}^n W_j \times W_\infty.
\]

Let \(D_j(a^0)\) denote the disk of radius \(\delta(a^0)/3\) defined in section 2. Choose a trivialization, \(g^j\), for the restriction \(E|_{D_j(a^0) \times W}\) and suppose that relative to this trivialization the connection \(\nabla\) is given by

\[
d + \Omega_j.
\]

Let \(\Omega_j^0\) denote the restriction of \(\Omega_j\) to \((t_j, \lambda_j) = (t^0_j, \lambda^0_j)\). As in the developments preceding Theorem 1.35 it is possible to adjust the trivialization \(g^j\) by a gauge transformation in the \(x\) variables alone so that \(d_x + \Omega_j^0\) extends to a connection on the trivial bundle \(\mathbb{P}^1 \times \mathbb{C}^p \to \mathbb{P}^1\) with a simple type \(r\) singularity at \(a_j^0\) and a regular singularity at \(\infty\). In what follows we suppose that this has been done. For \(j = 1, \ldots, m\) choose \(V_j\) small enough so that the Birkhoff deformation of \(d_x + \Omega_j^0\) constructed in Proposition 1.35 exists on \(D_j(a^0) \times V_j\). Thus there exists an integrable connection

\[
d_x + d_{\lambda_j} + \Omega_j^{loc},
\]

which is a Birkhoff deformation of \(d_x + \Omega_j^0\) in the sense of Proposition 1.35 (i) and (ii) and such that on \(D_j(a^0) \setminus \{a_j^0\} \times V_j\) one has the gauge equivalence

\[
d_x + d_{\lambda_j} + \Omega_j^{loc} = \varphi_j \cdot [d_x + d_{\lambda_j} + \Omega_j^0]
\]

where \(x \to \varphi_j(x, \lambda_j)\) is holomorphic in the punctured disk \(D_j(a^0) \setminus \{a_j^0\}\) and asymptotic to the identity as \(x \to \infty\). Now choose the neighborhood \(U_j\) of \(\{t^0_j\}\) small enough so that the map, \(p_j\) defined by

\[
D_j(a^0) \times U_j \times V_j \ni (x, t, \lambda) \to p_j(x, t, \lambda) := (x - a_j(t) + a_j^0, \lambda)
\]

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maps into $D_j(a^0) \times V_j$. Let $d_j = d_x + d_{t_j} + d_\lambda$, where $d_\lambda$ is the exterior derivative on $\prod_{k=1}^n R(\mathcal{C}_k) \times R(\mathcal{C}_\infty)$ and define the connection
\[ d_j + p_j^* \Omega_j^{\text{loc}}, \] (3.4)
on the trivial $\mathbb{C}^p$ bundle over
\[ X_j := D_j(a^0) \times W. \]

It is a consequence of Proposition 1.26c above that the connection (3.4) on the trivial bundle over $X_j$ is equivalent to $\nabla$ on $E|_{X_j}$ (they have the same formal reduction and Stokes’ multipliers). Possibly shrinking the open neighborhood $U_j$ we can insure that there is an annular region
\[ A_j = \{ x | \rho_j < |x - a^0_j| < \rho'_j \} \subset D_j(a^0) \]
with the property that $x - a_j(t) \neq 0$ for $(x, t) \in A_j \times U_j$. The integrable connection (3.4) and the integrable connection
\[ d_j + \Omega_j^{\text{loc}} \]
have the same holonomy on $A_j \times W_j$. They are thus related by a holomorphic gauge transformation, $\psi_j$, defined on $A_j \times W_j$ so that
\[ d_j + p_j^* \Omega_j^{\text{loc}} = \psi_j \cdot [d_j + \Omega_j^{\text{loc}}], \] (3.5)
which can be normalized so that $\psi_j(x, t^0_j, \lambda^0_j) = I$ for $x \in A_j$. By choosing $W_j$ small enough we may insure that $\psi_j$ is a sufficiently small perturbation of the identity so that it has a “canonical factorization” $\psi_j = \psi^+_j \psi^-_j$, where $\psi^+_j(x, t_j, \lambda_j)$ is holomorphic for $|x - a^0_j| < \rho'_j$ and $\psi^-_j(x, t_j, \lambda_j)$ is holomorphic for $|x - a^0_j| > \rho_j$, with $\psi^-_j(\infty, t_j, \lambda_j) = I$. Now define the gauge transform of $d_j + p_j^* \Omega_j^{\text{loc}}$ by $(\psi^+_j)^{-1}$,
\[ \nabla_j := (\psi^+)^{-1} \cdot [d_j + p_j^* \Omega_j^{\text{loc}}] = \psi^-_j \cdot [d_j + \Omega_j^{\text{loc}}]. \] (3.6)
Combining (3.6) with the extension of (3.3) to
\[ d_j + \Omega_j^{\text{loc}} = \phi_j \cdot [d_j + \Omega_j^0], \]
which follows from (3.3) since $\phi_j$ does not depend on $t_j$, or $\lambda_k$ for $k \neq j$ we also have,
\[ \nabla_j = \phi_j \cdot [d_j + \Omega_j^0], \] (3.7)
where $\phi_j(x, t_j, \lambda_j)$ is holomorphic for $|x - a^0_j| > \rho_j$. The connection $\nabla_j$ is equivalent to the restriction of $\nabla$ to $E|_{X_j}$ and so is a good local model for $\nabla$. Thus we can choose a trivialization $F^j$ for $E|_{X_j}$ so that in this trivialization $\nabla$ is represented by $\nabla_j$.

Now we wish to do something similar for $j = m+1, \ldots, n$. $W_j$ should be small enough so the the map

$$D_j(a^0) \times W_j \ni (x, t) \to p_j(x, t) := x - a_j(t) + a^0_j,$$

maps into $D_j(a^0)$. Choose a trivialization for $E|_{D_j(a^0) \times W}$ and write $\Omega_j$ for the connection one form for $\nabla$ in this trivialization. Thus $\nabla$ is represented by $d + \Omega_j$.

in this trivialization.

Let $\Omega^0_j$ denote the restriction of $\Omega_j$ to $(t, \lambda) = (t^0, \lambda^0)$. Again using the same argument to be found in the preliminaries to Proposition 1.35 we may suppose that the trivialization of $E|_{D_j(a^0) \times W}$ has been chosen so that $d_x + \Omega^0_j$ extends to a meromorphic connection on the trivial bundle $P^1 \times \mathbb{C}^p \to P^1$ with a simple type 0 singularity at $x = a$ and a regular singularity at $x = \infty$.

As above consider the connection

$$d_j + p_j^* \Omega^0_j.$$

One can choose $W_j$ small enough so that there exists an annular region $A_j = \{x| \rho_j < |x - a^0_j| < \rho'_j\}$ contained in $D_j(a^0)$ with the property that $x - a_j(t) \neq 0$ for $(x, t_j) \in A_j \times W_j$. The connection $d_j + p_j^* \Omega^0_j$ and $d_j + \Omega^0_j$ then have the same holonomy over $A_j \times W$ and so there exists a holomorphic gauge transformation $\psi_j$ defined on $A_j \times W$ so that

$$d_j + p_j^* \Omega^0_j = \psi_j \cdot [d_j + \Omega^0_j].$$

One can normalize $\psi_j$ so that $\psi_j(x, t^0, \lambda^0) = I$ and by choosing $W$ sufficiently small we can guarantee that $\psi_j$ has a canonical factorization $\psi_j^+ \psi_j^-$ as above. We define

$$\nabla_j := (\psi_j^+)^{-1} \cdot [d_j + p_j^* \Omega^0_j] = \psi_j^- \cdot [d_j + \Omega^0_j],$$

where $\psi_j^-(x, t, \lambda)$ is holomorphic for $|x - a^0_j| > \rho_j$ and asymptotic to the identity $I$ as $x \to \infty$. As above we can find a trivialization $f^j$ for the restriction of $E$ to $D_j(a^0) \times W$ so that $\nabla$ is represented by $\nabla_j$. 

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The local connection at infinity, $\nabla_\infty$, might be regular, or have a simple type $r_\infty$ singularity. It does not matter much for the calculation we are going to do, but for definiteness we suppose $r_\infty \geq 1$. Then $W_\infty$ is a neighborhood of $\lambda^0_\infty$ (no pole deformation parameters). Proceeding in close analogy with the first case discussed above we choose a trivialization $g^\infty$ for $E$ restricted to $D_\infty(a^0) \times W$ so that the connection $\nabla$ is represented by

$$d_x + d_\lambda + \Omega_\infty.$$

We further suppose that the trivialization $g^\infty$ has been chosen so that the restriction of this connection to $\lambda = \lambda^0$ is given by $d_x + \Omega^0_\infty$, extends to a connection on the trivial bundle $\mathbb{P}^1 \times \mathbb{C}^p \to \mathbb{P}^1$ with a simple type $r_\infty$ singularity at $\infty$ and a regular singular point at 0.

Now let

$$d_x + d_\lambda + \Omega^0_\infty$$

denote the Birkhoff deformation of $d_x + \Omega^0_\infty$ constructed in Proposition 1.35 (with the slight alterations needed to locate the singularity at $\infty$). Let $p_\infty$ denote the projection $(x, \lambda) \to (x, \lambda_\infty)$ and define

$$\nabla_\infty := d_x + d_\lambda + p_\infty^* \Omega^\text{loc}_\infty.$$  \hfill (3.9)

Choose $\rho_{\infty}$ and $\rho'_{\infty}$ so that $\nabla_\infty$ and $d_x + d_\lambda + \Omega^0_\infty$ are holomorphically equivalent on the annulus

$$A_\infty = \{x | \rho_{\infty} < x < \rho'_{\infty}\} \subset D_\infty(a^0),$$

by a gauge transformation $\phi_{\infty}$,

$$\nabla_\infty = \phi_{\infty} \cdot \left[ d_x + d_\lambda + \Omega^0_\infty \right],$$  \hfill (3.10)

which can be chosen so that $\phi_{\infty}(x, \lambda)$ is holomorphic for $|x| < \rho'_{\infty}$.

In contrast Theorem 2.9 we do not assume that the restriction

$$E|_{\mathbb{P}^1 \times \{t^0, \lambda^0\}}$$

is holomorphically trivial. Let $B_j(\rho) = \{x : |x - a^0_j| < \rho\}$ denote the open ball of radius $\rho$ about $a^0_j$. Let $B_{\infty}(\rho) = \{x : |x| > \rho\}$, and define

$$B_j := B_j(\rho'_{j}),$$

$$B_{\infty} := B_{\infty}(\rho_{\infty}),$$

$$B := B_1(\rho_1) \cup B_2(\rho_2) \cup \cdots \cup B_n(\rho_n) \cup B_{\infty}(\rho'_{\infty}),$$

$$B_{\text{ex}} := \mathbb{P}^1 \setminus \overline{B},$$\hfill (3.11)
where $\overline{X}$ denotes the closure of $X$. Then

$$\{B_1, B_2, \ldots, B_n, B_\infty, B_{ex}\}$$

is an open covering of $\mathbb{P}^1$.

We now wish to show that there is a holomorphic trivialization of the bundle $E$ over $B_{ex} \times W$. Since we’ve seen that $\Theta$ is the zero set of a nonvanishing holomorphic function, $\tau$, it follows that there is some $(t^1, \lambda^1) \in W$ so that $E|_{\mathbb{P}^1 \times \{(t^1, \lambda^1)\}}$ is holomorphically trivial. Since $B_{ex} \times W$ does not intersect any of the singular sets (3.2) or $\{\infty\} \times W$ it follows that the integrable connection $\nabla$ on $E$ is smooth over $B_{ex} \times W$, and since $W$ is connected and simply connected one may integrate $\nabla$ over $W$ to extend the trivialization over $B_{ex} \times \{(t^1, \lambda^1)\}$ to a holomorphic trivialization of $E$ over $B_{ex} \times W$. We wish to pick a trivialization so that the connection form for $\nabla$ is particularly simple. Suppose that relative to some choice of a trivialization for the restriction of $E$ to $\mathbb{P}^1 \times \{(t^1, \lambda^1)\}$ the connection $\nabla$ is,

$$d_x + \Omega_{ex}(x),$$

where $\Omega_{ex}(x)$ depends only on $x$. Write $d = dx + dt + d\lambda$; then since $W$ is connected and simply connected it is easy to see that the connection,

$$d + \Omega_{ex}(x),$$

defined on the trivial bundle over $B_{ex} \times W$ with fiber $\mathbb{C}^p$ has the same holonomy representation as $\nabla$ on the restriction of $E$ to $B_{ex} \times W$. The holomorphic equivalence of these two connections implies that we can choose a trivialization $f_{ex}$ for $E|_{B_{ex} \times W}$ so that relative to this trivialization the connection $\nabla$ is given by (3.13). Observe that this connection form has no $dt$ or $d\lambda$ components.

For each $j = 1, \ldots, n$ there exists a holomorphic map $S^j : A_j \times W \to GL(p, \mathbb{C})$ so that

$$f_{ex}^j = f^j S^j,$$

and a holomorphic map $S^\infty : A_\infty \times W \to GL(p, \mathbb{C})$ so that

$$f_{ex}^\infty = f^\infty S^\infty,$$

where we think of each trivialization $f = (f_1, \ldots, f_p)$ as a row vector of sections.

Define

$$B_{in} = \cup_j B_j \cup B_\infty,$$
Let $S$ denote the holomorphic map from $B_{\text{in}} \cap B_{\text{ex}} \times W$ into $GL(p, \mathbb{C})$ which restricts to $S^j$ on $A_j \times W$ and $S^\infty$ on $A_\infty \times W$. Then for $(t, \lambda) \in W$, $E_{(t, \lambda)}$ will be holomorphically trivial if and only if there exists a factorization,

$$S(x, t, \lambda) = \Phi_{\text{in}}(x, t, \lambda)^{-1}\Phi_{\text{ex}}(x, t, \lambda),$$ \hspace{1cm} (3.16)

where $x \to \Phi_{\text{in}}(x, t, \lambda)$ is holomorphic and invertible in $B_{\text{in}}$ and $x \to \Phi_{\text{ex}}(x, t, \lambda)$ is holomorphic and invertible in $B_{\text{ex}}$. Let $C_j$ ($C_\infty$) denote a counterclockwise oriented circle contained in $A_j$ ($A_\infty$), and define the oriented curve

$$C = C_\infty - C_1 - C_2 - \cdots - C_n.$$

Choose a point $z_0 \in B_{\text{ex}}$ with $z_0 \notin \overline{B_{\text{in}}}$ and define a projection $P_{\text{ex}}$ on $L^2(C)$ by,

$$P_{\text{ex}}f(z) = \int_C f(x) \left\{ \frac{1}{x - z} - \frac{1}{x - z_0} \right\} \frac{dx}{2\pi i}.$$

This projects onto the subspace of $L^2(C)$ which has a holomorphic extension into the connected part of $B_{\text{ex}}$ bounded by the curve $C$, with the further property that this holomorphic extension vanishes at $z = z_0$. We can make the factorization (3.6) unique (when it exists) by normalizing

$$\Phi_{\text{ex}}(z_0, t, \lambda) = \text{identity} = I.$$

Rewriting (3.16) we have

$$\Phi_{\text{in}}S = \Phi_{\text{ex}},$$

and writing $P_{\text{in}} = I - P_{\text{ex}}$ we find

$$P_{\text{in}}(\Phi_{\text{in}}S) = I.$$ \hspace{1cm} (3.17)

Regarding this as an equation for the rows of $\Phi_{\text{in}}$, the solution of this equation is equivalent to the existence of the factorization (3.16). Suppose that (3.16) has a solution $\Phi_{\text{in}}$, $\Phi_{\text{ex}}$. Then the Toeplitz operator $T_S$ defined by

$$T_Sf = P_{\text{in}}(fS),$$

where $f$ is a row vector, has inverse,

$$T_S^{-1}g = P_{\text{in}}(g\Phi_{\text{ex}}^{-1})\Phi_{\text{in}}.$$

Following Malgrange we now calculate the regularized trace

$$\omega := \text{Tr} \left( T_S^{-1}T_{dS} - T_{dSS^{-1}} \right),$$ \hspace{1cm} (3.18)
where for brevity we write $d = d_{t, \lambda}$.

Note: because the multiplication operator in our Toeplitz operator is acting on the right it is $T_{RS} - T_S T_R$ which is compact when $R$ and $S$ are smooth.

We will eventually show that (3.18) differs from $d \log \tau$, defined above, by a regular term and this will allow us to make a connection with the formula for the tau function given by Jimbo, Miwa and Ueno in [12].

We compute

$$T_S^{-1}T_{dS}f = P_{in} ((fdS) \Phi_{ex}^{-1}) \Phi_{in}$$
$$= P_{in} (fdS \Phi_{ex}^{-1}) \Phi_{in}$$
$$= P_{in} (f (\Phi_{in}^{-1}d\Phi_{ex} \Phi_{ex}^{-1} - \Phi_{in}^{-1}d\Phi_{in} \Phi_{in}^{-1})) \Phi_{in},$$

and

$$T_{dSS^{-1}}f = P_{in} (fdSS^{-1})$$
$$= P_{in} (f (\Phi_{in}^{-1}d\Phi_{ex} \Phi_{ex}^{-1} \Phi_{in} - \Phi_{in}^{-1}d\Phi_{in})).$$

Now let

$$R_q f =fq,$$

denote right multiplication by $q$, and define

$$Q = \Phi_{in}^{-1}d\Phi_{ex} \Phi_{ex}^{-1} - \Phi_{in}^{-1}d\Phi_{in} \Phi_{in}^{-1}.$$

Then from (3.9) and (3.10) one sees that

$$T_S^{-1}T_{dS} - T_{dSS^{-1}} = R_{\Phi_{in}} P_{in} R_Q - P_{in} R_{\Phi_{in}} R_Q$$
$$= [R_{\Phi_{in}}, P_{in}] R_Q$$
$$= [P_{ex}, R_{\Phi_{in}}] R_Q.$$

Thus the trace of interest is

$$\text{Tr} [P_{ex}, R_{\Phi_{in}}] R_Q$$

(3.21)

and one computes

$$[P_{ex}, R_{\Phi_{in}}] R_Q f(z) =$$
$$\int_C f(x)Q(x)(\Phi_{in}(x) - \Phi_{in}(z)) \left\{ \frac{1}{x-z} - \frac{1}{x-z_0} \right\} \frac{dx}{2\pi i}.$$
Writing \( d_x \Phi = \Phi' dx \), the integral of the (finite dimensional) trace over the diagonal is then

\[
\int_C \text{Tr} (Q(x)\Phi'_{in}(x)) \frac{dx}{2\pi i} = \int_C \text{Tr} (d\Phi_{ex}\Phi^{-1}_{ex} \Phi'_{in} - d\Phi_{in}\Phi^{-1}_{in} \Phi'_{in}) \frac{dx}{2\pi i} \tag{3.22}
\]

where the second equality follows from Cauchy’s theorem and the fact that \( \Phi_{in} \) is holomorphic in \( B_{in} \). To make a connection between (3.22) and the JMU [12] formula for the log derivative of the tau function we replace \( \Phi_{in} \) and \( \Phi_{ex} \) in (3.22) with quantities more intimately related to the connection \( \nabla \). First choose \( j \in \{1, \ldots, n, \infty\} \), and let \( \nabla_j \) denote the representation for \( \nabla \) in the trivialization \( f_j \).

Let \( \nabla_{ex} \) denote the representation of \( \nabla \) in the trivialization \( f_{ex} \). Then in the trivialization \( f_j \Phi^{-1}_{in} = f_{ex} \Phi^{-1}_{ex} \) for the restriction of \( E \) to \( P^1 \times (W\setminus\Theta) \) we find that the connection \( \nabla \) is given by,

\[
\nabla = \Phi_{ex} [\nabla_{ex}] = \Phi_j \cdot [\nabla_j] \tag{3.23}
\]

on \( A_j \times W\setminus\Theta \) (or \( A_\infty \times W\setminus\Theta \) for \( j = \infty \)).

Now fix \( j \in \{1, \ldots, m, \infty\} \) and let \( \hat{\alpha}_j \) be the formal power series near \( x = a_j(t) \) (or \( \infty \) if \( j = \infty \)) for which

\[
\hat{\alpha}_j \cdot [\nabla_{\lambda_j}] = \nabla \tag{3.24}
\]

Let \( \hat{\alpha}_j(\text{loc}) \) denote the formal power series near \( x = a_j(t) \) such that

\[
\hat{\alpha}_j(\text{loc}) \cdot [\nabla_{\lambda_j}] = \nabla_j \tag{3.25}
\]

Equating (3.24) with the first term of (3.23) and recalling that \( \Omega_{ex} \) does not have any \( dt \) or \( d\lambda \) terms, one finds

\[
d\Phi_{ex} \Phi^{-1}_{ex} = d\hat{\alpha}_j \hat{\alpha}_j^{-1} - \hat{\alpha}_j dH_j \hat{\alpha}_j^{-1}, \tag{3.26}
\]

which should be understood in the following sense. Replacing \( \hat{\alpha}_j \) by \( \alpha_{\Sigma,j} \) defined in a suitable sector \( \Sigma \) with vertex at \( x = a_j(t) \) one finds a sectorial version of (3.26). This shows that the function \( d\Phi_{ex} \Phi^{-1}_{ex} \) which is analytic in an annular region about \( x = a_j(t) \) extends holomorphically into the sector \( \Sigma \). Since this is true for a collection of sectors which cover a punctured neighborhood of \( x = a_j(t) \) it follows that \( d\Phi_{ex} \Phi^{-1}_{ex} \) is holomorphic in a punctured neighborhood of \( x = a_j(t) \).
Equation (3.26) may then be understood as an equality of formal Laurent series (which in fact converge since the left hand side has a convergent Laurent series).

Applying \( \hat{\Phi}_\text{in} \) (the formal power series associated to \( \Phi_\text{in} \)) to both sides of (3.25) and comparing the result with (3.23) and (3.24) one finds

\[
\hat{\Phi}_\text{in} \hat{\alpha}_j(\text{loc}) = \hat{\alpha}_j c_j
\]

where \( c_j \) is a diagonal constant matrix (the only gauge automorphisms of \( \nabla_{\lambda_j} \) are diagonal constants). Thus

\[
\hat{\Phi}_\text{in} = \hat{\alpha}_j c_j \hat{\alpha}_j(\text{loc})^{-1},
\]

(3.27) for \( j = 1, 2, \ldots, m, \infty \). This is to be understood in the sense of formal power series at \( x = a_j(t) \).

For \( j = m + 1, \ldots, n \) let \( \alpha_j \) denote the local holomorphic gauge transformation constructed in Proposition 1.25b such that

\[
\alpha_j \cdot \left[ d_x + d - \frac{\Lambda_j}{x - a_j(t)} d(x - a_j(t)) \right] = \nabla,
\]

(3.28) where \( \Lambda_j \) is a constant diagonal matrix. Let \( \alpha_j(\text{loc}) \) denote the local holomorphic gauge transformation so that

\[
\alpha_j(\text{loc}) \cdot \left[ d_x + d - \frac{\Lambda_j}{x - a_j(t)} d(x - a_j(t)) \right] = \nabla_j.
\]

(3.29)

Comparing this with the first term in (3.23) and making use of the fact that \( \Omega_{\text{ex}} \) has no \( dt \) or \( d\lambda \) components one finds

\[
d\Phi_{\text{ex}} \Phi_{\text{ex}}^{-1} = d\alpha_j \alpha_j^{-1} + \alpha_j \frac{\Lambda_j d a_j(t)}{x - a_j(t)} \alpha_j^{-1}.
\]

(3.30)

If we write \( dH_j = \frac{\Lambda_j}{x - a_j(t)} d(x - a_j(t)) \) for \( j = m + 1, \ldots, n \), (remember \( d = d_t + d_\lambda \)) then (3.30) can be written

\[
d\Phi_{\text{ex}} \Phi_{\text{ex}}^{-1} = d\alpha_j \alpha_j^{-1} - \alpha_j dH_j \alpha_j^{-1},
\]

(3.31)

which is analogous to (3.26).

Applying \( \Phi_\text{in} \) to both sides of (3.29) and comparing with (3.28) we find

\[
\Phi_\text{in} = \alpha_j c_j \alpha_j(\text{loc})^{-1} \text{ for } j = m + 1, \ldots, n
\]

(3.32)
which is analogous to (3.27).

The formal power series expansion for (3.26) and the powers series expansion for (3.30) shows that the integral,

$$
\int_{C_j} \text{Tr} \left( d\Phi \Phi^{-1}_x \Phi'^{-1} \right) \frac{dx}{2\pi i},
$$
can be “done” by residues to get,

$$
\pm \text{Res}_{x=a_j(t)} \text{Tr} \left( d\Phi \Phi^{-1}_x \Phi'^{-1} \right),
$$
with the + choice for \( j = 1, \ldots, n \) and the - choice for \( j = \infty \). We now substitute (3.26), (3.27), (3.31) and (3.32) into (3.33). The first term in (3.26) and (3.30) does not make a contribution to the formal residue in (3.28) since \( d\hat{\alpha}_j^{-1}\hat{\alpha}'_j \) and \( \Phi^{-1}_x \Phi'^{-1} \) both have formal power series expansions at \( x = a_j(t) \). After some simplification (using \( c^{-1}_j dH_j = dH_j \)) one finds

$$
\int_{C_j} \text{Tr} \left( d\Phi \Phi^{-1}_x \Phi'^{-1} \right) \frac{dx}{2\pi i}
$$

$$
= \pm \text{Res}_j \text{Tr} \left( \hat{\alpha}^{-1}_j \hat{\alpha}'_j dH_j - \hat{\alpha}_j (loc)^{-1} \hat{\alpha}'_j (loc)dH_j \right),
$$
where \( \text{Res}_j \) is the residue at \( x = a_j(t) \) or \( \infty \) if \( j = \infty \), and the sign \( \pm \) is + for \( j = 1, \ldots, n \) and - for \( j = \infty \). Combining (3.22) with (3.34) one finds that

$$
\text{Tr} \left( T^{-1}_S T_{ds} - T_{dSS^{-1}} \right)
$$

$$
= -\sum_j \text{Res}_j \text{Tr} \left( \hat{\alpha}^{-1}_j \hat{\alpha}'_j dH_j - \hat{\alpha}_j (loc)^{-1} \hat{\alpha}'_j (loc)dH_j \right)
$$

(3.35)

where the sum is over \( j \in \{1, \ldots, n, \infty\} \).

Next we make use of the special choice we made for the local models \( \nabla_j \) that is reflected in (3.7), (3.8) and (3.10). Following the arguments for the proof of part (iv) of Proposition 1.35 we find that

$$
d_{t, \lambda} \text{Res}_j \text{Tr} \left( \hat{\alpha}_j (loc)^{-1} \hat{\alpha}'_j (loc)dH_j \right) = 0,
$$

(3.36)

for \( j = 1, \ldots, n, \infty \) follows from (3.7), (3.8), and (3.10) in the same way that (1.42) follows from (1.41).

We will now make use of (3.36) to show that the one form in (3.35) is closed off the singular set \( \Theta \). One easily computes

$$
d \text{Tr} \left( T^{-1}_S T_{ds} - T_{dSS^{-1}} \right)
$$

$$
= -\frac{1}{2} \sum_{j > k} \text{Tr} \left( [T^{-1}_S \partial_k S, T^{-1}_S \partial_j S] + T_{[\partial_j SS^{-1}, \partial_k SS^{-1}]} \right) \, ds_j \wedge ds_k,
$$

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where $s := (t, \lambda)$ and $\partial_k = \frac{\partial}{\partial s_k}$. This last expression can be computed as in Malgrange [17] and one finds

$$\frac{1}{2} \sum_{j > k} \int_C \frac{dx}{2\pi i} \Tr \left( \partial_j S S^{-1} \left( \partial_k S S^{-1} \right)' \right) ds_j \wedge ds_k. \tag{3.37}$$

If $(t^0, \lambda^0) \notin \Theta$ then we may take $(t^1, \lambda^1) = (t^0, \lambda^0)$ in the construction above. Note that equation (3.35) shows that $\Tr \left( T_S^{-1} T_{dS} - T_{dSS^{-1}} \right)$ is actually independent of the choice of $(t^1, \lambda^1)$, though it does depend on $(t^0, \lambda^0)$ through $\hat{\alpha}_j(\text{loc})$. With this choice for $(t^1, \lambda^1)$ equations (3.7), (3.8) and (3.10) take on added significance. In this case one may choose a global trivialization for $E|_{\mathbb{P}^1 \times \{(t^0, \lambda^0)\}}$. If $\Omega_0^j$ denotes the one form for $\nabla$ relative to this trivialization then (3.7), (3.8) and (3.9) show that the transition function $S_j$ can be chosen so that it has a holomorphic continuation into the exterior of $C_j$ in $\mathbb{P}^1$ (including $j = \infty$). Each of the integrals

$$\int_{C_j} \frac{dx}{2\pi i} \Tr \left( \partial_k S_j S^{-1} \left( \partial_k S_j S^{-1} \right)' \right),$$

then vanishes by Cauchy’s theorem. This shows that if $(t^0, \lambda^0) \notin \Theta$ then,

$$d\Tr \left( T_S^{-1} T_{dS} - T_{dSS^{-1}} \right) = 0, \tag{3.38}$$

at least for the special constructions associated with $(t^0, \lambda^0) = (t^1, \lambda^1)$. But we can now use (3.35) and (3.36) to show that $\Tr \left( T_S^{-1} T_{dS} - T_{dSS^{-1}} \right)$ is closed even without this restriction. Define

$$\omega_{JMU} := - \sum_j \text{Res}_j \Tr \left( \hat{\alpha}_j^{-1} \hat{\alpha}_j' dH_j \right),$$

which is the one form introduced by Jimbo, Miwa and Ueno in [12]. Then (3.38), (3.35) and (3.36) together show that

$$d\omega_{JMU} = 0, \tag{3.39}$$

for $(t, \lambda) \notin \Theta$. This result and (3.36) (which is not restricted to the special choice $(t^0, \lambda^0) = (t^1, \lambda^1)$) then show that the right hand side of (3.35) is closed in general and we conclude that $\Tr \left( T_S^{-1} T_{dS} - T_{dSS^{-1}} \right)$ is closed even when $(t^1, \lambda^1)$ is different from $(t^0, \lambda^0)$. We are now prepared to state the principal result of this paper.

**Theorem 3.40.** Suppose that $\nabla^0$ is a connection on the trivial bundle, $\mathbb{P}^1 \times C^p \to \mathbb{P}^1$, with simple type $r_j$ singularities at the distinct points $a_j \in \mathbb{P}^1$ for
\( j \in \{1, 2, \ldots, n, \infty\} \) with \( a_\infty := \infty \). Let \((E, \nabla)\) denote the vector bundle with connection constructed as a deformation of \( \nabla^0\) in Theorem 2.9. Let \( \Theta \) denote the set of \((t, \lambda) \in \mathcal{D}\) such that \( E|_{\mathbb{P}^1 \times \{(t, \lambda)\}}\) is not trivial. For \((t, \lambda) \notin \Theta\) let \( \hat{\alpha}_j \) and \( \alpha_j \) be defined as in (3.24) and (3.28). Then

(i) The form defined by

\[
\omega_{\text{JMU}} = - \sum_j \text{Res}_j \text{Tr} \left( \hat{\alpha}_j^{-1} \hat{\alpha}'_j dH_j \right),
\]

is a closed one form on \( \mathcal{D} \setminus \Theta \) and there exists a holomorphic function \( \tau_{\text{JMU}} \) on \( \mathcal{D} \) such that

\[
\omega_{\text{JMU}} = d \log \tau_{\text{JMU}}.
\]

(ii) The point \((t, \lambda) \in \mathcal{D}\) is a zero of \( \tau_{\text{JMU}} \) if and only if \((t, \lambda) \in \Theta\).

Proof. Let \( S \) denote the transition function defined by (3.14) and (3.15). Then as in the proof of Theorem 3.0 one can find an invertible holomorphic parametrix,

\[
W \ni (t, \lambda) \mapsto q(t, \lambda),
\]

so that

\[
T_{S(t, \lambda)} q(t, \lambda)^{-1} = I + \text{trace class},
\]

for \((t, \lambda) \in W\). Define

\[
\tau_q(t, \lambda) := \det \left( T_{S(t, \lambda)} q(t, \lambda)^{-1} \right).
\]

Then it is clear that \( \tau_q(t, \lambda) = 0 \) if and only if \((t, \lambda) \in \Theta\). As above write \( d = dt + d\lambda\). Then the usual formula for the derivative of a determinant gives

\[
d \log \tau_q = \text{Tr} \left( T^{-1}_S T_{dS} - dq^{-1} \right),
\]

off the singular set \( \Theta \). Comparing this with \( \omega \) defined by (3.18) above we find that:

\[
\omega - d \log \tau_q = \text{Tr} \left( dq^{-1} - T_{dSS^{-1}} \right).
\]

The left hand side is a closed form on \( W \setminus \Theta \) and the right hand side is holomorphic on \( W \). Thus the right hand side of (3.43) is a closed form on \( W \). Since \( W \) is simply connected it follows that there exists a holomorphic function \( \phi \) on \( W \) so that

\[
\omega - d \log \tau_q = d\phi.
\]
Consulting the definition (3.41) for $\omega_{JMU}$ and the result (3.35) for $\omega$ one finds

$$\omega_{JMU} - \omega = - \sum_j \text{Res}_j \text{Tr} \left( \hat{\alpha}_j (\text{loc})^{-1} \hat{\alpha}'_j (\text{loc})^{-1} dH_j \right) \quad (3.45)$$

The right hand side of (3.45) is a holomorphic one form on $W$ and by (3.36) it is closed. Thus there exists a holomorphic function $\varphi$ on $W$ so that

$$\omega_{JMU} - \omega = d\varphi. \quad (3.46)$$

Adding (3.44) and (3.46) and writing $\Phi = \phi + \varphi$ we find

$$\omega_{JMU} = d \log \tau_q + d\Phi = d \log (e^{\Phi} \tau_q). \quad (3.47)$$

Thus for some constant, $c$, we have

$$\omega_{JMU} = ce^{\Phi} \tau_q.$$

From this it follows immediately that $\omega_{JMU}(t, \lambda) = 0$ if and only if $(t, \lambda) \in \Theta$ and this finishes the proof of the theorem. QED

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