RESTRICTED WEAK TYPE INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL OPERATOR IN HIGHER DIMENSIONS

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Abstract. We give a quantitative characterization of the pairs of weights \((w, v)\) for which the dyadic version of the one-sided Hardy-Littlewood maximal operator satisfies a restricted weak \((p, p)\) type inequality for \(1 \leq p < \infty\). More precisely, given any measurable set \(E_0\), the estimate

\[
w(\{x \in \mathbb{R}^n : M^{+d}(\lambda E_0)(x) > t\}) \leq \frac{C[\{(w, v)\}]_{A^+_{p,d}(\mathcal{R})}^p}{t^p} v(E_0)
\]

holds if and only if the pair \((w, v)\) belongs to \(A^+_{p,d}(\mathcal{R})\), that is,

\[
\frac{|E|}{|Q|} \leq [(w, v)]_{A^+_{p,d}(\mathcal{R})} \left( \frac{v(E)}{w(Q)} \right)^{1/p}
\]

for every dyadic cube \(Q\) and every measurable set \(E \subset Q^+\). The proof follows some ideas appearing in S. Ombrosi (2005). We also obtain a similar quantitative characterization for the non-dyadic case in \(\mathbb{R}^2\) by following the main ideas in L. Forzani, F.J. Martín-Reyes, S. Ombrosi (2011).

Keywords: restricted weak type; one-sided maximal operator

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1. Introduction

In 1986 Sawyer in [10] started the theory of one-sided weights. Namely, he introduced the class of weights \(A^+_p\) and showed that this class is necessary and sufficient for the weighted boundedness of the one-sided Hardy-Littlewood maximal function.

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Some extensions and generalizations were given consequently in the articles [5], [6] and [7], among others. In [9], the author characterizes the functions $w$ for which the one-sided Hardy-Littlewood maximal operator

$$M^+_v f(x) = \sup_{h > 0} \frac{\int_{x}^{x+h} |f| v}{\int_{x}^{x+h} v}$$

verifies a restricted weak $(p, p)$ type on the real line, that is, a weak type inequality applied to the function $f = \chi_E$, where $E$ is an arbitrary measurable set. More precisely, the inequality

$$w(\{x \in \mathbb{R}: M^+_v(\chi_E)(x) > t\}) \leq C \frac{1}{t^p} v(E)$$

holds if and only if $w \in A^+_p(\mathbb{R})(v \, dx)$. This set corresponds to the class of weights that satisfy a restricted $A^+_p$ condition with respect to the measure $d\mu = v(x) \, dx$, see the section below for details.

Although the theory in this setting was deeply developed and the main results were improved and generalized, most of the results were set on $\mathbb{R}$.

In [8], Ombrosi characterized the pair of weights $(w, v)$ for which the inequality

$$(1.1) \quad w(\{x \in \mathbb{R}^n: M^{+,d} f(x) > t\}) \leq C \frac{1}{t^p} \int_{\mathbb{R}^n} |f|^p v$$

holds for every positive $t$, and where $1 \leq p < \infty$. The operator $M^{+,d}$ is a dyadic version of $M^+$ defined on $\mathbb{R}^n$. A similar result was also obtained for $M^{-,d}$.

It is well-known that the operators $M^+ f$ and $M^{+,d} f$ are pointwise equivalent on $\mathbb{R}$, see [6]. However, this result is false in general in higher dimensions. This means that a non-dyadic version of (1.1) cannot be obtained directly from the dyadic case, and the problem of finding such an estimate remained open.

In [1], Forzani, Martín-Reyes and Ombrosi proposed a way to generalize the operators $M^+$ and $M^-$ to higher dimensions and solved the problem discussed above on $\mathbb{R}^2$. The technique used, although newfangled and quite delicate, relied heavily upon the dimension. This means that the corresponding problem for $n \geq 3$ still remains open.

Related to strong estimates in dimension greater than one, some partial results were obtained in [4]. At this point we would also like to mention interesting applications of this theory to parabolic differential equations obtained by Kinnunen and Saari in [2] and [3].

In this article we use some ideas of [1] and [8] to give a characterization of the pairs of weights for which the one-sided Hardy-Littlewood maximal operator satisfies a restricted weak type inequality in higher dimensions.
Concretely, for the dyadic case we have the following result.

**Theorem 1.1.** Let \((w, v)\) be a pair of weights and \(1 \leq p < \infty\). Then the following statements are equivalent:

(a) The operator \(M^{+,d}\) is of restricted weak \((p, p)\) type with respect to \((w, v)\), that is, there exists a positive constant \(C\) such that the inequality

\[
    w(\{x \in \mathbb{R}^n : M^{+,d}(X_E)(x) > t\}) \leq \frac{C[(w, v)]^p_{A^{+,d}_p(\mathbb{R})}}{t^p} v(E)
\]

holds for every positive \(t\) and every measurable set \(E\).

(b) \((w, v)\) belongs to \(A^{+,d}_p(\mathbb{R})\).

For the non-dyadic case we prove the next theorem.

**Theorem 1.2.** Let \((w, v)\) be a pair of nonnegative measurable functions defined in \(\mathbb{R}^2\) and \(1 \leq p < \infty\). The following conditions are equivalent:

(a) The operator \(M^+\) is of restricted weak \((p, p)\) type with respect to \((w, v)\), that is, there exists a positive constant \(C\) such that the inequality

\[
    w(\{x \in \mathbb{R}^2 : M^+(X_E)(x) > t\}) \leq \frac{C[(w, v)]^p_{A^+_p(\mathbb{R})}}{t^p} v(E)
\]

holds for every positive \(t\) and every measurable set \(E\).

(b) \((w, v)\) belongs to \(A^+_p(\mathbb{R})\).

The article is organized as follows. In Section 2 we give the preliminaries and definitions required for these main results. In Sections 3 and 4 we prove Theorems 1.1 and 1.2, respectively.

2. Preliminaries and basic definitions

We shall deal with dyadic cubes with sides parallel to the coordinate axes. Given a dyadic cube \(Q = \prod_{i=1}^{n} [a_i, b_i)\), we denote with \(Q^+ = \prod_{i=1}^{n} [b_i, 2b_i - a_i)\) and \(Q^- = \prod_{i=1}^{n} [2a_i - b_i, a_i)\).

Given a positive number \(s\), we denote \((Q)^{s,+} = \prod_{i=1}^{n} [a_i, a_i + sh)\), where \(h = b_i - a_i\).

Similarly, we denote \((Q)^{s,-} = \prod_{i=1}^{n} [b_i - sh, b_i)\).

For \(x = (x_1, \ldots, x_n)\) in \(\mathbb{R}^n\) and \(h > 0\), we denote \(Q_{x,h} = \prod_{i=1}^{n} [x_i, x_i + h)\) and \(Q_{x,h^-} = \prod_{i=1}^{n} [x_i - h, x_i)\). The one-sided Hardy-Littlewood maximal operators are
given by
\[
M^+ f(x) = \sup_{h > 0} \frac{1}{|Q_{x,h}|} \int_{Q_{x,h}} |f(y)| \, dy, \quad \text{and} \quad M^- f(x) = \sup_{h > 0} \frac{1}{|Q_{x,h^-}|} \int_{Q_{x,h^-}} |f(y)| \, dy.
\]

We shall consider the dyadic version of these operators, that is,
\[
M^{+,d} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q^+} |f(y)| \, dy \quad \text{and} \quad M^{-,d} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q^-} |f(y)| \, dy,
\]
where the supremum is taken over dyadic cubes.

Given \(1 < p < \infty\), we say that a pair of weights \((w, v)\) belongs to \(A^+_p\) if there exists a positive constant \(C\) such that the inequality
\[
\left( \int_Q w \right) \left( \int_{Q^+} v^{1-p'} \right)^{p-1} \leq C |Q|^p
\]
holds for every cube \(Q\) in \(\mathbb{R}^n\).

When \(p = 1\), we say that \((w, v)\) belongs to \(A^+_1\) if there exist a positive constant \(C\) that verifies
\[
M^- w(x) \leq C v(x)
\]
for almost every \(x\). The smallest constant for which these inequalities hold is denoted by \([[(w, v)]_{A^+_p}]\).

Similarly, we say that \((w, v)\) belongs to \(A^{+,d}_p\) if the inequalities above hold for every dyadic cube \(Q\) and when \(p = 1\), the involved operator is \(M^{-,d}\). In this case, the corresponding smallest constant is denoted by \([[(w, v)]_{A^{+,d}_p}]\).

For \(1 \leq p < \infty\) we say that \((w, v) \in A^{+,d}_p(\mathbb{R})\) if there exists a positive constant \(C\) such that the inequality
\[
(2.1) \quad \frac{|E|}{|Q|} \leq C \left( \frac{v(E)}{w(Q)} \right)^{1/p}
\]
holds for every dyadic cube \(Q\) and every measurable set \(E \subset Q^+\). The smallest constant \(C\) for which the inequality above holds will be denoted by \([[(w, v)]_{A^{+,d}_p(\mathbb{R})}]\).

We say that a pair of weights \((w, v)\) belongs to \(A^+_p(\mathbb{R})\), \(1 \leq p < \infty\), if inequality (2.1) holds for every cube \(Q\) and every measurable subset \(E\) of \(Q^+\).

**Remark 2.1.** By replacing \(Q^+\) by \(Q^-\) and \(M^-\) by \(M^+\) we can define the \(A^-_p\) classes for \(1 \leq p < \infty\). The dyadic version of these classes, \(A^{-,d}_p\), are defined by considering dyadic cubes on their definitions. The same occurs for \(A^+_R(\mathbb{R})\) and \(A^{-,d}_p(\mathbb{R})\).

Throughout the paper we shall present the results for \(M^+\), but the same arguments can be adapted to get the corresponding versions for \(M^-\).
The novelty of considering restricted weak type inequalities relies on that although we take a particular function \( f \), we consider a wider class of weights. This property is contained in the following proposition.

**Proposition 2.1.** \( A_p^+ \subset A_p^+(\mathbb{R}) \) for every \( 1 < p < \infty \), and \( A_1^+ = A_1^+(\mathbb{R}) \).

**Proof.** Let \( 1 < p < \infty \) and assume that \( (w, v) \in A_p^+ \). Fix a cube \( Q \) and a measurable subset \( E \) of \( Q^+ \) with \( |E| > 0 \). Then

\[
|E| \leq \left( \int_E v \right)^{1/p} \left( \int_{Q^+} v^{1-p'} \right)^{1/p'} \leq [(w, v)]_{A_p^+}^{1/p} \left( \frac{v(E)}{w(Q)} \right)^{1/p'} |Q|,
\]

which implies that \( (w, v) \in A_p^+ \) and \( [(w, v)]_{A_p^+(\mathbb{R})} \leq [(w, v)]_{A_p^+}^{1/p} \).

On the other hand, set \( p = 1 \) and assume that \( (w, v) \in A_1^+ \). Fix a cube \( Q \) and a measurable set \( E \subset Q^+ \) with positive measure. Then for every \( x \in E \) we have that

\[
\frac{1}{|Q|} \int_{(Q^+)^-} w \leq [(w, v)]_{A_1^+} v(x),
\]

which implies that

\[
\frac{w(Q)}{|Q|} \leq [(w, v)]_{A_1^+} \frac{v(E)}{|E|},
\]

and then \( (w, v) \in A_1^+ \). Conversely, fix \( x \) and \( h > 0 \). Let \( Q = Q_{x,h}^- \), \( \lambda > \text{ess inf } v \) and \( E = \{ y \in Q_{x,h} : v(y) < \lambda \} \). Then we have that

\[
\frac{w(Q_{x,h}^-)}{|Q_{x,h}^-|} \leq [(w, v)]_{A_1^+(\mathbb{R})} \lambda.
\]

By letting \( \lambda \to \text{ess inf } v \) and then taking supremum over \( h \) we get that

\[
M^- w(x) \leq [(w, v)]_{A_1^+(\mathbb{R})} v(x).
\]

\[\square\]

The following lemma states a useful property for weights on the \( A_p^+(\mathbb{R}) \) class.

**Lemma 2.1.** Let \( 1 \leq p < \infty \), \( (w, v) \) be a pair of weights in \( A_p^+(\mathbb{R}) \) and \( a, b \) two positive constants. Then

(a) if \( a \leq b \), then \( (w_0, v_0) = (\max\{w, a\}, \max\{v, b\}) \) belongs to \( A_p^+(\mathbb{R}) \) and

\[
[(w_0, v_0)]_{A_p^+(\mathbb{R})} \leq 2 \max\{1, [(w, v)]_{A_p^+(\mathbb{R})}\};
\]

(b) \( (w_1, v_1) = (\min\{w, a\}, \max\{v, b\}) \) belongs to \( A_p^+(\mathbb{R}) \) and

\[
[(w_1, v_1)]_{A_p^+(\mathbb{R})} \leq [(w, v)]_{A_p^+(\mathbb{R})}.
\]

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Proof. Let us first prove statement (a). Fix a cube $Q$ and a measurable subset $E$ of $Q^+$. We have to show that there exists a positive constant $C$, independent of $Q$ and $E$, such that
\[
\frac{w_0(Q)}{v_0(E)} \leq C \left( \frac{|Q|}{|E|} \right)^p.
\]
We write
\[
w_0(Q) = \int_{Q \cap \{w \geq a\}} w_0 + \int_{Q \cap \{w < a\}} w_0 = w(Q \cap \{w \geq a\}) + a|Q \cap \{w < a\}|,
\]
and therefore,
\[
\frac{w_0(Q)}{v_0(E)} = \frac{w(Q \cap \{w \geq a\})}{v_0(E)} + \frac{a|Q \cap \{w < a\}|}{v_0(E)} = I + II.
\]
Now observe that
\[
I \leq \frac{w(Q)}{v(E)} \leq [(w, v)]_{A^+_p(R)} \left( \frac{|Q|}{|E|} \right)^p.
\]
On the other hand,
\[
II \leq \frac{a|Q|}{b|E|} \leq \left( \frac{|Q|}{|E|} \right)^p,
\]
since $a \leq b$ and $|Q| \geq |E|$. Therefore, $(w_0, v_0) \in A^+_p(R)$ and $[(w_0, v_0)]_{A^+_p(R)} \leq 2 \max\{1, [(w, v)]_{A^+_p(R)}\}$.

For the proof of statement (b), observe that $w_1 \leq w$ and $v_1 \geq v$, so
\[
\frac{w_1(Q)}{v_1(E)} \leq \frac{w(Q)}{v(E)} \leq [(w, v)]_{A^+_p(R)} \left( \frac{|Q|}{|E|} \right)^p,
\]
which shows that $(w_1, v_1) \in A^+_p(R)$ with $[(w_1, v_1)]_{A^+_p(R)} \leq [(w, v)]_{A^+_p(R)}$. \qed

3. Restricted weak $(p, p)$ type of $M^{+, d}$ in $\mathbb{R}^n$

We devote this section to proving Theorem 1.1. We start with the following lemma which will be useful for this purpose. This result is an adaptation of Lemma 2.1 in [8].

Lemma 3.1. Let $1 \leq p < \infty$, $(w, v) \in A^{+, d}_p(R)$ and $\mu > 0$. Let $E$ be a measurable set such that $0 < |E| < \infty$ and $\{Q_j\}_{j \in \Gamma_\mu}$ a disjoint family of dyadic cubes such that for every $j \in \Gamma_\mu$ we have
\[
\mu < \frac{|E \cap Q_j^+|}{|Q_j|} \leq 2\mu.
\]
Then we have that
\[
\sum_{j \in \Gamma_\mu} w(Q_j) \leq C [(w, v)]^p_{A^{+, d}_p(R)} v \left( E \cap \left( \bigcup_{j \in \Gamma_\mu} Q_j^+ \right) \right).
\]
Proof. For \( m \geq 0 \) we define the sets

\[ i_m = \{ j \in \Gamma_{\mu} : \text{there exist exactly} \ m \ \text{cubes} \ Q_s^+ : Q_j^+ \subseteq Q_s^+ \ \text{with} \ s \in \Gamma_{\mu} \} \]

and also

\[ \sigma_m = \bigcup_{j \in i_m} Q_j^+. \]

Also, we define \( E_j^+ = E \cap Q_j^+ \) and \( F_m = \bigcup_{j \in i_m} E_j^+ \).

Notice that \( \Gamma_{\mu} = \bigcup_{m \geq 0} i_m \) and if \( j_1 \) and \( j_2 \) belong to \( i_m \) for some \( m \), then \( Q_{j_1}^+ \cap Q_{j_2}^+ = \emptyset \). This yields

\[ |F_m| = \sum_{j \in i_m} |E_j^+|. \]

On the other hand, \( \sigma_{m+1} \subset \sigma_m \) for every \( m \geq 0 \), so

\[ F_{m+1} \subset F_m \quad \text{and} \quad |F_{m+1}| \leq |F_m|. \]

For fixed \( m_0 \) and \( j_0 \in i_{m_0} \), if \( Q_j^+ \subseteq Q_{j_0}^+ \), then \( j \in i_m \) with \( m > m_0 \) and \( Q_j \subset Q_{j_0}^2 \). Therefore,

\[ \bigcup_{m > m_0} \bigcup_{j \in i_m : Q_j^+ \subseteq Q_{j_0}^+} Q_j \subset (Q_{j_0}^+)_{2,+} \]

and this implies that

\[ \sum_{m > m_0} \sum_{j \in i_m : Q_j^+ \subseteq Q_{j_0}^+} |Q_j| \leq |(Q_{j_0}^+)_{2,+}| = 2^n |Q_{j_0}|, \]

since the cubes \( Q_j \) are disjoint. Thus, by (3.1) we get

\[ \sum_{m > m_0} |F_m \cap Q_{j_0}^+| = \sum_{m > m_0} \sum_{j \in i_m : Q_j^+ \subseteq Q_{j_0}^+} |E_j^+| \leq 2\mu \sum_{m > m_0} \sum_{j \in i_m : Q_j^+ \subseteq Q_{j_0}^+} |Q_j| \]

\[ \leq 2^{n+1} \mu |Q_{j_0}| \leq 2^{n+1} |E_{j_0}^+|. \]

This last estimate implies that

\[ \sum_{m = m_0 + 1}^{m_0 + 2^{n+2}} |F_m \cap Q_{j_0}^+| < 2^{n+1} |E_{j_0}^+| \]

and then there must be an index \( m, m_0 + 1 \leq m \leq m_0 + 2^{n+2} \) such that

\[ |F_m \cap Q_{j_0}^+| < \frac{|E_{j_0}^+|}{2}. \]
By (3.2) we get
\[ |F_{m_0+2^n+2} \cap Q_{j_0}^+| \leq |F_m \cap Q_{j_0}^+| < \frac{|E_{j_0}^+|}{2}, \]
and consequently,
\[ \frac{|Q_{j_0}^+ \cap (E \setminus F_{m_0+2^n+2})|}{|Q_{j_0}|} > \frac{1}{2} \frac{|E_{j_0}^+|}{|Q_{j_0}|} > \frac{\mu}{2}. \]
Now, we can estimate
\[
\sum_{j \in \Gamma_m} w(Q_j) = \sum_{m=0}^{\infty} \sum_{j \in \Gamma_m} w(Q_j) \\
< \left( \frac{2}{\mu} \right)^p \sum_{m=0}^{\infty} \sum_{j \in \Gamma_m} w(Q_j) \left( \frac{|Q_j^+ \cap (E \setminus F_{m_0+2^n+2})|}{|Q_j|} \right)^p \\
\leq \left( \frac{2}{\mu} \right)^p \sum_{m=0}^{\infty} \sum_{j \in \Gamma_m} [(w, v)]_{A_p^{+,d}(\mathcal{R})}^p \int_{\sigma_m-\sigma_{m+2^n+2}}^\infty \mathcal{X}_E v \\
= \left( \frac{2}{\mu} \right)^p [(w, v)]_{A_p^{+,d}(\mathcal{R})}^p \sum_{k=0}^{2^{n+1}-1} \sum_{m=0}^{\infty} \int_{\sigma_{2^{n+2}m+k}-\sigma_{2^{n+2}(m+1)+k}}^\infty \mathcal{X}_E v \\
\leq \left( \frac{2}{\mu} \right)^p [(w, v)]_{A_p^{+,d}(\mathcal{R})}^p \sum_{k=0}^{2^{n+1}-1} \int_{\sigma_k}^\infty \mathcal{X}_E v \\
\leq \frac{2^{n+p+2}[(w, v)]_{A_p^{+,d}(\mathcal{R})}^p}{\mu^p} \mathcal{X}_E v(E \cap \sigma_0). \]

\[ \square \]

**Proof of Theorem 1.1.** We shall first prove that (a) implies (b). Fix a dyadic cube \( Q \) and a measurable subset \( E \) of \( Q^+ \). Assume that \( |E| > 0 \), since otherwise the condition follows immediately. For every \( x \) in \( Q \) we have that
\[
M^{+,-}(x) \geq \frac{1}{|Q|} \int_{Q^+} \mathcal{X}_E = \frac{|E|}{|Q|},
\]
which implies that \( Q \subset \{ x : M^{+,-}(x) > |E|/(2|Q|) \} \). By using statement (a) we get
\[
w(Q) \leq C \left( \frac{|Q|}{|E|} \right)^p v(E),
\]
which shows that \( (w, v) \in A_p^{+,d}(\mathcal{R}) \).
Now we prove that (b) implies (a). Fix a measurable set $E$ and assume, without loss of generality, that $0 < |E| < \infty$. For fixed $t > 0$, let $\mathcal{F}$ be the family of dyadic cubes $Q$ such that $|E \cap Q^+|/|Q| > t$ and let $\{Q_j\}_j$ be the family of the maximal cubes of $\mathcal{F}$. It follows that the cubes $Q_j$ are disjoint and

$$\{x: M^{+, d} X_E(x) > t\} = \bigcup_j Q_j.$$ 

We shall consider a partition of this family of cubes. Given $k \geq 0$, we set

$$C_k = \{j: 2^k t < \frac{|E \cap Q_j^+|}{|Q_j|} \leq 2^{k+1} t\}$$

and apply Lemma 3.1 to the family $C_k$ with $\mu = 2^k t$ for every $k$. Therefore,

$$\sum_{j \in C_k} w(Q_j) \leq \frac{C[(w, v)]_{A_p^{+, d}(\mathbb{R})}}{(2^k t)^p} v\left(\bigcup_{j \in C_k} E_j^+\right).$$

This yields

$$w(\{x: M^{+, d} X_E(x) > t\}) = \sum_j w(Q_j) = \sum_{k=0}^{\infty} \sum_{j \in C_k} w(Q_j) \leq \sum_{k=0}^{\infty} \frac{C[(w, v)]_{A_p^{+, d}(\mathbb{R})}}{(2^k t)^p} v(E) = \frac{C[(w, v)]_{A_p^{+, d}(\mathbb{R})}}{t^p} v(E),$$

which completes the proof. \hfill \square

4. Restricted weak $(p, p)$ type of $M^+$ in $\mathbb{R}^2$

We devote this section to the proof of Theorem 1.2. Along this section we shall assume that the space, where we work, is $\mathbb{R}^2$. We begin by introducing some specifics in this setting.

We say that a square $Q$ has dyadic size if $l(Q) = 2^k$ for an integer $k$. Let $l(Q)$ denote the length of the sides of $Q$. Given a square $Q$, $\alpha Q$ will denote the square with the same center as $Q$ and sides of length $\alpha l(Q)$.

For $h > 0$ and $Q = [a, a + h] \times [b, b + h]$, we set $\tilde{Q}$ the dilation of $Q$ to the right and to the bottom in $\frac{1}{2} l(Q)$. That is, $\tilde{Q} = [a, a + \frac{3}{2} h] \times [b - \frac{1}{2} h, b + h]$. 

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Given \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( h > 0 \) recall that \( Q_{x,h} = [x_1, x_1 + h] \times [x_2, x_2 + h] \).

We shall consider the following partition of a cube \( Q_{x,h} \):

\[
Q_{x,h} = Q_{x,h/2} \cup Q_{1}^{1,h} \cup Q_{2}^{2,h} \cup Q_{3}^{3,h},
\]

where

\[
Q_{1}^{1,h} = [x_1 + \frac{1}{2}h, x_1 + h] \times [x_2 + \frac{1}{2}h, x_2 + h],
\]
\[
Q_{2}^{2,h} = [x_1 + \frac{1}{2}h, x_1 + h] \times [x_2, x_2 + \frac{1}{2}h],
\]
\[
Q_{3}^{3,h} = [x_1, x_1 + \frac{1}{2}h] \times [x_2 + \frac{1}{2}h, x_2 + h].
\]

The proof of Theorem 1.2 relies on the following covering lemma, that is, a consequence of Lemma 3.1 stated and proved in [1], when we take \( f = x_E \). This result contains a covering argument that is related to the subcube \( Q_{2}^{2,h} \). For the main proof we will require the corresponding versions for \( Q_{1}^{1,h} \) and \( Q_{3}^{3,h} \), which can be achieved by following similar ideas.

**Lemma 4.1.** Let \( t > 0 \) and \( E \) be a measurable set such that \( 0 < |E| < \infty \). Let \( A = \{x_j\}_{j=1}^N \) be a finite set of points in \( \mathbb{R}^2 \). Suppose that for every \( 1 \leq j \leq N \), we have a square of dyadic size \( Q_j \) with \( x_j \) as its upper right corner and that satisfies

\[
\frac{t}{4} < \frac{|E \cap Q_j^{+2}|}{|Q_j|}.
\]
Then there exists a set $\Gamma \subset \{1, \ldots, N\}$ such that

\[(4.1) \quad A \subset \bigcup_{i \in \Gamma} \tilde{Q}_i\]

and

\[(4.2) \quad \frac{t}{4} < \frac{|E \cap \tilde{Q}_j^+|}{|Q_j|}.

Moreover, for every $i, j \in \Gamma$ with $i \neq j$ we have $\tilde{Q}_i \not\subseteq \tilde{Q}_j$ and the squares $\tilde{Q}_i$, $i \in \Gamma$, of the same size are almost disjoint, that is, there exists a positive constant $C$ such that for every $l$

$$\sum_{i \in \Gamma, l(Q_i) = l} X_{\tilde{Q}_i}(x) \leq C.$$ 

This implies that the squares $(\tilde{Q}_i)^+$ are almost disjoint, too. Further, if

$$\frac{|E \cap (\tilde{Q}_j)^+|}{|Q_j|} \leq 8t,$$

then there exists a family of sets $\{F_j\}_{j \in \Gamma}$ with $F_j \subset (\tilde{Q}_j)^+$ such that

$$\frac{t}{8} < \frac{|E \cap F_j|}{|Q_j|}$$

and they are almost disjoint, that is, there exists a positive constant $C$ (independent of everything) such that

$$\sum_{j \in \Gamma} X_{F_j}(x) \leq C.$$

Proof of Theorem 1.2. The fact that (a) implies (b) can be achieved in a similar way to Theorem 1.1. Let us prove that (b) implies (a). The operator $M^+$ is pointwise equivalent to the operator

$$M^+ f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|Q_{x, 2^k}|} \int_{Q_{x, 2^k}} |f|,$$

that is, the one-sided maximal operator defined over squares of dyadic size. We shall consider for $i = 1, 2, 3$ the operators

$$M^{+i} f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|Q_{x, 2^k}^i|} \int_{Q_{x, 2^k}^i} |f|,$$

where the cubes $Q_{x, 2^k}^i$ are depicted in Figure 2.
Let us fix a measurable set $E$ with $0 < |E| < \infty$. Let $(w, v)$ be a pair of weights in $A_p^+(\mathcal{R})$. We shall prove that

$$w(\{x \in \mathbb{R}^2: \mathcal{M}^+(\lambda_E)(x) > t\}) \leq \frac{C}{tp}v(E)$$

for every $t > 0$. It will be enough to show that

$$w(\{x \in \mathbb{R}^2: t < \mathcal{M}^+(\lambda_E)(x) \leq 2t\}) \leq \frac{C}{tp}v(E),$$

and this also reduces to proving that

$$(4.3) \quad w(\{x \in \mathbb{R}^2: t < \mathcal{M}^+(\lambda_E)(x), \mathcal{M}^+(\lambda_E)(x) \leq 2t\}) \leq \frac{C}{tp}v(E)$$

for $i = 1, 2, 3$. We show the proof for $i = 2$, being similar for the other indices.

Given a positive number $\xi$ we consider the truncated maximal operator defined by

$$M^{+2}_\xi(\lambda_E)(x) = \sup_{\substack{h = 2^k > \xi \\ k \in \mathbb{Z}}} \frac{4|E \cap Q^2_x, h|}{h^2}.$$ 

Observe that $M^{+2}_\xi(\lambda_E) \uparrow M^{+2}(\lambda_E)$ when $\xi \to 0^+$. Therefore, it will be enough to prove that

$$(4.4) \quad w(\{x \in \mathbb{R}^2: t < M^{+2}_\xi(\lambda_E)(x), M^+(\lambda_E)(x) \leq 2t\}) \leq \frac{C}{tp}v(E)$$

for every $t > 0$ and with $C$ independent of $\xi$, $E$ and $t$.

By virtue of Lemma 2.1 we can assume that $w \in L^1_{\text{loc}}$ and also that there exists a positive constant $\gamma$ such that $0 < \gamma \leq w(x)$ for every $x \in \mathbb{R}^2$.

Let $\Omega_t = \{x \in \mathbb{R}^2: t < M^{+2}_\xi(\lambda_E)(x), M^+(\lambda_E)(x) \leq 2t\}$. The measure $d\mu(x) = w(x) \, dx$ is finite over compact sets since we are assuming $w \in L^1_{\text{loc}}$. Therefore, inequality (4.4) follows if we prove that

$$w(K) \leq \frac{C}{tp}v(E)$$

for every compact set $K \subset \Omega_t$ and with $C$ independent of $K$.

Fix a compact set $K \subset \Omega_t$. For every $x = (x_1, x_2) \in K$ there exists a square $Q_x = [x_1 - l, x_1] \times [x_2 - l, x_2]$ with $\xi \leq l, l = 2^k$ for some $k \in \mathbb{Z}$ and

$$\frac{t}{4} < \frac{|E \cap Q_x^{+2}|}{|Q_x|}. $$

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Let $Q_{x,2l} = [x_1, x_1 + 2l] \times [x_2, x_2 + 2l]$. We have that $(\tilde{Q}_x)^+ \subset Q_{x,2l}$ (see Figure 3) and thus,

$$\frac{|E \cap (\tilde{Q}_x)^+|}{|Q_x|} \leq \frac{|E \cap Q_{x,2l}|}{|Q_x|} = \frac{4|E \cap Q_{x,2l}|}{|Q_{x,2l}|} \leq 4\mathcal{M}^+(\mathcal{X}_E)(x) \leq 8t.$$

![Figure 3. $(\tilde{Q}_x)^+ \subset Q_{x,2l}$.

Therefore, we have that for every $x \in K$ there exists a square $Q_x = [x_1 - l, x_1] \times [x_2 - l, x_2]$ such that $\xi \leq l$,

$$\frac{t}{4} < \frac{|E \cap Q_x^+|}{|Q_x|} \quad \text{and} \quad \frac{|E \cap (\tilde{Q}_x)^+|}{|Q_x|} \leq 8t.$$

We have also that there exists a positive constant $M$, depending on $t$ and $E$, such that $l \leq M$ since

$$|Q_x| \leq \frac{4|E \cap Q_x^+|}{t} \leq \frac{4|E|}{t} < \infty.$$

This implies that there exists a square $R$ such that $\bigcup_{x \in K} \tilde{Q}_x \subset R$. We shall consider the square $2R$. Since $w$ is integrable in $2R$, there exists $0 < \varepsilon < 1$ such that if $Q \subset R$ is a square, then

$$w((1 + \varepsilon)Q \setminus Q) \leq \gamma \xi^2.$$

If $Q \subset R$ verifies $l(Q) \geq \xi$, then

$$w((1 + \varepsilon)Q \setminus Q) \leq \gamma \xi^2 \leq \gamma |Q| \leq w(Q).$$

This yields

$$w((1 + \varepsilon)Q) \leq 2w(Q)$$

for every $Q \subset R$ with $l(Q) \geq \xi$. Particularly,

$$w((1 + \varepsilon)\tilde{Q}_x) \leq 2w(\tilde{Q}_x) \quad \text{for every } x \in K.$$
Let $B_x(r)$ be the ball of radius $r$ centered at $x$. We have that $K \subset \bigcup_{x \in K} B_x(\frac{1}{2}\xi \varepsilon)$, and then there exist $x_1, x_2, \ldots, x_s \in K$ such that $K \subset \bigcup_{j=1}^{s} B_{x_j}(\frac{1}{2}\xi \varepsilon)$, since $K$ is compact.

We apply now Lemma 4.1 to the set $A = \{x_j\}_{j=1}^{s}$ and the squares $\{Q_j\}_{j=1}^{s}$ associated to the points $x_j$. Then there exists a set $\Gamma \subset \{1, \ldots, s\}$ that verifies $A \subset \bigcup_{i \in \Gamma} \widetilde{Q}_{x_i}$, and there also exist $\{F_{x_i} : i \in \Gamma\}$, $F_{x_i} \subset (\widetilde{Q}_{x_i})^+$, (4.5)

$$\frac{t}{8} < \frac{|E \cap F_{x_i}|}{|Q_{x_i}|}$$

and

$$\sum_{i \in \Gamma} \mathcal{X}_{F_{x_i}}(x) \leq C.$$

Observe that if $x_j \in A$, there exists $i \in \Gamma$ such that $x_j \in \widetilde{Q}_{x_i}$. Then $B_{x_j}(\frac{1}{2}\xi \varepsilon) \subset (1 + \varepsilon)\widetilde{Q}_{x_i}$. In fact, this is straightforward if we assume $0 < \xi < 1$. Consequently, we have that

$$K \subset \bigcup_{j=1}^{s} B_{x_j}(\frac{\xi \varepsilon}{2}) \subset \bigcup_{i \in \Gamma}(1 + \varepsilon)\widetilde{Q}_{x_i},$$

which implies that

$$w(K) \leq \sum_{i \in \Gamma} w((1 + \varepsilon)\widetilde{Q}_{x_i}) \leq 2 \sum_{i \in \Gamma} w(\widetilde{Q}_{x_i}).$$

Thus, by using (4.5) and the $A^+_p(\mathcal{R})$ condition of $(w, v)$, we obtain

$$w(K) \leq 2 \sum_{i \in \Gamma} w(\widetilde{Q}_{x_i}) \leq \frac{C}{t^p} \sum_{i \in \Gamma} w(\widetilde{Q}_{x_i}) \left(\frac{|E \cap F_{x_i}|}{|Q_{x_i}|}\right)^p$$

$$= \frac{C}{t^p} \sum_{i \in \Gamma} w(\widetilde{Q}_{x_i}) \left(\frac{|(Q_{x_i})^+|}{|Q_{x_i}|}\right)^p \left(\frac{|E \cap F_{x_i}|}{|(Q_{x_i})^+|}\right)^p$$

$$\leq \frac{C}{t^p} [(w, v)]^p_{A^+_p(\mathcal{R})} \sum_{i \in \Gamma} v(E \cap F_{x_i})$$

$$\leq \frac{C}{t^p} [(w, v)]^p_{A^+_p(\mathcal{R})} v\left(E \cap \left(\bigcup_{i \in \Gamma} F_{x_i}\right)\right)$$

$$\leq \frac{C}{t^p} [(w, v)]^p_{A^+_p(\mathcal{R})} v(E).$$

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