A PROPOSAL FOR A DIFFERENTIAL
CALCULUS IN QUANTUM MECHANICS

E. Gozzi♭ and M. Reuter♯

♭ Dipartimento di Fisica Teorica, Università di Trieste,
Strada Costiera 11, P.O.Box 586, Trieste, Italy
and INFN, Sezione di Trieste.

♯ Deutsches Elektronen-Synchrotron DESY,
Notkestrasse 85, W-2000 Hamburg 52, Germany

ABSTRACT

In this paper, using the Weyl-Wigner-Moyal formalism for quantum mechanics, we develop a \textit{quantum-deformed} exterior calculus on the phase-space of an arbitrary hamiltonian system. Introducing additional bosonic and fermionic coordinates we construct a supermanifold which is closely related to the tangent and cotangent bundle over phase-space. Scalar functions on the super-manifold become equivalent to differential forms on the standard phase-space. The algebra of these functions is equipped with a Moyal super-star product which deforms the pointwise product of the classical tensor calculus. We use the Moyal bracket algebra in order to derive a set of quantum-deformed rules for the exterior derivative, Lie derivative, contraction, and similar operations of the Cartan calculus.
1. INTRODUCTION

The physics community has recently witnessed a growing interest in quantum groups\textsuperscript{[1]}. These are deformations of classical Lie algebras which first appeared in the context of the quantum inverse scattering method\textsuperscript{[2]}. A lot of efforts\textsuperscript{[3]} has already gone into the investigation of the differential structures associated to quantum groups, but the program of developing a quantum deformed differential calculus and investigating its impact on physics is certainly still in its infancy. It became clear by now that there is a close relationship between quantum groups and the general framework of non-commutative geometry\textsuperscript{[4] [5]} which, loosely speaking, deals with spaces whose coordinates are non-commuting objects. It is one of the basic credos of non-commutative geometry that these spaces should not be investigated by visualizing them as a set of points, but rather by studying the algebra of functions defined on them. An important example\textsuperscript{[6]} of a non-commutative manifold which can be investigated in this framework is the quantum mechanical phase-space. Canonical quantization turns the c-number coordinates of the classical phase-space into non-commuting operators so that it is not clear a priori in which sense quantum phase-space can be considered a ”manifold”. A first step towards a non-commutative geometry of phase-space was taken long ago by Moyal\textsuperscript{[7]} and by Bayen et al.\textsuperscript{[8]} who, building upon the work of Weyl and Wigner\textsuperscript{[9]}, reformulated quantum mechanics in terms of functions on phase-space. The concept of ”quantum” phase-space employed here is classical in the sense that the coordinates commute, but a non-classical feature is introduced via a new non-commutative product, referred to as the star-product, which replaces the classical pointwise multiplication of functions on phase-space. It was shown that the full machinery of quantum mechanics can be reformulated by working with the algebra of functions on phase-space, whereby the algebra-multiplication is provided by the star-product. Since the star-product, in the classical limit, reduces to the pointwise product, the former may be considered a ”quantum deformation” of the latter. Clearly this deformation-theory approach to quantization\textsuperscript{[8]} is very much in the spirit of modern non-commutative geometry: the transition from classical to quantum phase-space is achieved by deforming the algebra of functions on the space under consideration.

Moyal’s phase space formulation of quantum mechanics makes use of the so-called symbol calculus\textsuperscript{[10]} which associates, in a one-to-one manner, ordinary functions to the operators on some Hilbert space. In this way the observables and the density operators of the standard Hilbert space formulation of quantum mechanics are turned into functions on phase-space; they are called the ”symbols” of the respective operators. Operator products correspond to
star-products of symbols then, and commutators of operators go over into the Moyal bracket, which is the commutator with respect to star-multiplication. The Moyal bracket is a deformation\(^7\) of the classical Poisson bracket, to which it reduces in the limit \(\hbar \to 0\). Moyal has studied the quantum dynamics of scalar pseudo-densities (Wigner functions\(^9\)) in this language. In doing so he introduced a quantum deformed version of the classical hamiltonian vector field. It is the purpose of the present paper to investigate more general geometric objects and operations such as vectors, forms, Lie derivatives, etc., in this framework. In a previous paper\(^{11}\) we have reformulated the classical exterior calculus on symplectic manifolds (Cartan calculus) in a hamiltonian language. This means that operations such as Lie derivatives, exterior derivatives, contractions, etc. were expressed in terms of a novel type of Poisson bracket defined for functions on an extended phase-space. The extended phase-space is a supermanifold\(^{12}\) which is closely related to the (co)-tangent bundle over the standard phase-space. In this way, scalar functions on the extended phase-space are equivalent to tensors on the standard phase-space. However, as discussed above, we know how to deform the algebra of (scalar) functions on any phase-space, therefore we should arrive at a kind of "quantum exterior calculus" if we apply the Moyal deformation not to the standard phase-space, but rather to the extended one. In order to implement this program we first review in section 2 the relevant material on the Moyal deformation. Then, in section 3, we introduce the extended phase-space and describe the classical Cartan calculus in terms of the associated extended Poisson bracket structure. Finally, in sections 4 through 7, we study the deformed calculus resulting from the Moyal deformation of the extended Poisson bracket structure.

2. SYMBOL CALCULUS AND WEYL-WIGNER-MOYAL FORMALISM

The basic idea behind the "symbol calculus"\(^{7-10}\) is to set up a linear one-to-one map between the operators\(^*\) \(\hat{A}, \hat{B}, \cdots\) on some Hilbert space \(\mathcal{V}\) and the complex-valued functions \(A, B, \cdots \in \text{Fun}(\mathcal{M})\) defined on an appropriate finite-dimensional manifold \(\mathcal{M}\). The operator \(\hat{A}\) is uniquely represented by the function \(A\) which is called the symbol of \(\hat{A}\). For the "symbol map" relating the function \(A\) to the operator \(\hat{A}\) we write \(A = \text{symb}(\hat{A})\). It has a well-defined inverse \(\hat{A} = \text{symb}^{-1}(A)\). The space of symbols, \(\text{Fun}(\mathcal{M})\), is equipped with the so called "star-product" \(*\) which implements the operator multiplication at the level of

\* In this section the caret (\(\hat{\cdot}\)) is used to denote operators.
symbols. It is defined by the requirement that the symbol map is an algebra homomorphism, \(i.e.,\) that

\[ \text{symb}(\hat{A}\hat{B}) = \text{symb}(\hat{A}) \ast \text{symb}(\hat{B}) \]  

for any pair of operators \(\hat{A}\) and \(\hat{B}\). Since operator multiplication is non-commutative in general, but associative, the same is also true for the star-multiplication:

\[ A \ast B \neq B \ast A \]  

\[ A \ast (B \ast C) = (A \ast B) \ast C \]

As we shall see, the star-product may be considered a deformation \(^8\) of the ordinary pointwise product of functions. Here "deformation" is meant in the sense of ref.\(^{13}\) to which we refer the reader for further details.

Let us now be more specific and let us assume that the Hilbert space \(\mathcal{V}\) is the state space of an arbitrary quantum mechanical system with \(N\) degrees of freedom, and that the manifold \(\mathcal{M} = \mathcal{M}_{2N}\) is the \(2N\)-dimensional classical phase-space pertaining to this system. Then the quantum mechanical operator \(\hat{A}\) is represented by a function \(A = A(\phi)\), where \(\phi^a = (p^1, \ldots, p^N, q^1, \ldots, q^N),\ a = 1, \ldots, 2N\) are canonical coordinates on the phase-space \(\mathcal{M}_{2N}\). For the sake of simplicity we assume that canonical coordinates can be introduced globally. This implies that the symplectic two form \(^{14}\) on \(\mathcal{M}_{2N}\), \(\omega = \frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b\), has constant components:

\[ \omega_{ab} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \]  

The inverse matrix, denoted by \(\omega^{ab}\), reads

\[ \omega^{ab} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \]  

Using \(\omega^{ab}\) we define the Poisson bracket for any pair of functions \(A, B \in \text{Fun}(\mathcal{M}_{2N})\):

\[ \{A, B\}_{pb}(\phi) \equiv \partial_a A(\phi) \omega^{ab} \partial_b B(\phi) \]  

Here \(\partial_a \equiv \frac{\partial}{\partial \phi^a}\).
There exists a variety of possibilities of associating an operator 
\( \hat{A}(\hat{p}, \hat{q}) = \text{symb}^{-1}(A(p, q)) \) to the function \( A(p, q) \). Typically, different definitions of the
symbol-map correspond to different operator ordering prescriptions\[^{[10]}\]. In the following we shall mainly work with the Weyl symbol\[^{[9]}\] which has the property that if \( A(p, q) \) is a polynomial in \( p \) and \( q \), the operator \( \hat{A}(\hat{p}, \hat{q}) \) is the symmetrically ordered polynomial in \( \hat{p} \) and \( \hat{q} \), e.g., \( \text{symb}^{-1}(pq) = \frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) \). The Weyl symbol \( A(\phi^a) \) of the operator \( \hat{A} \) is given by\[^{[10]}^{[15]}\]

\[
A(\phi^a) = \int \frac{d^2N \phi_0}{(2\pi\hbar)^N} \exp\left[\frac{i}{\hbar} \phi_0^{ab} \omega_{ab} \phi^b\right] Tr\left[ \hat{T}(\phi_0) \hat{A} \right]
\]  

(2.7)

where

\[
\hat{T}(\phi_0) = \exp\left[\frac{i}{\hbar} \phi^a \omega_{ab} \phi^b_0\right] \equiv \exp\left[\frac{i}{\hbar} (p_0\hat{q} - q_0\hat{p})\right]
\]

The inverse map reads

\[
\hat{A} = \int \frac{d^2N \phi d^2N \phi_0}{(2\pi\hbar)^2N} A(\phi) \exp\left[\frac{i}{\hbar} \phi^{ab} \omega_{ab} \phi^b_0\right] \hat{T}(\phi_0)
\]  

(2.8)

Eq.(2.8) is due to Weyl\[^{[9]}\]. It expresses the fact that the operators \( \hat{T}(\phi_0) \) form a complete and orthogonal (with respect to the Hilbert-Schmidt inner product) set of operators in terms of which any operator \( \hat{A} \) can be expanded\[^{[15]}\].

A particularly important class of operators are the density operators \( \hat{\rho} \). Their symbols \( \rho = \text{symb}(\hat{\rho}) \) are given by eq. (2.7) for \( \hat{A} = \hat{\rho} \). In particular, for pure states \( \hat{\rho} = |\psi\rangle \langle \psi| \) one obtains the Wigner function\[^{[9]}\]

\[
\rho(p, q) = \int d^N x \exp\left[\frac{i}{\hbar} px\right] \psi(q + \frac{1}{2} x) \psi^*(q - \frac{1}{2} x)
\]

(2.9)

The symbol \( \rho(\phi^a) \) is the quantum mechanical analogue of the classical probability density \( \rho_c(\phi^a) \) used in classical statistical mechanics. However, differently than in classical mechanics, the quantum symbol \( \rho(\phi) \) is not positive definite and is therefore referred to as a "pseudodensity". The usual positive definite quantum mechanical distributions over position or momentum space, respectively, are recovered as

\[
|\psi(q)|^2 = \int \frac{d^N p}{(2\pi\hbar)^N} \rho(p, q)
\]

\[
|\tilde{\psi}(p)|^2 = \int \frac{d^N q}{(2\pi\hbar)^N} \rho(q, p)
\]
Similarly, the expectation value of any observable $\hat{O}$ is given by

$$\langle \psi | \hat{O} | \psi \rangle = \int \frac{d^{2N} \phi}{(2\pi \hbar)^{2N}} \varrho(\phi) O(\phi)$$

In this way quantum mechanics can be formulated in a "classically-looking" manner involving only c-number functions on $\mathcal{M}_{2N}$.

The star product which makes the algebra of Weyl symbols isomorphic to the operator algebra is given by

$$(A \ast B)(\phi) = A(\phi) \exp\left[\frac{i}{\hbar} \omega^a \partial_a \partial_b \right] B(\phi)$$

$$\equiv \exp\left[\frac{i}{\hbar} \omega^a \partial_a \partial_b \right] A(\phi_1) B(\phi_2) |_{\phi_1=\phi_2=\phi}$$

(2.10)

with $\frac{1}{\hbar} \partial_a = \frac{\partial}{\partial \phi^a_1,2}$, or more explicitly

$$(A \ast B)(\phi) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{i}{\hbar} m \omega^{a_1 b_1} \cdots \omega^{a_m b_m} (\partial_{a_1} \cdots \partial_{a_m} A)(\partial_{b_1} \cdots \partial_{b_m} B)$$

(2.11)

We see that to lowest order in $\hbar$ the star-product of two functions reduces to the ordinary pointwise product. For non-zero values of $\hbar$ this multiplication is "deformed" in such a way that the resulting $\ast$-product remains associative but non-commutative in general.

The Moyal bracket [7] of two symbols $A, B \in Fun(\mathcal{M}_{2N})$ is defined as their commutator (up to a factor of $i\hbar$) with respect to star-multiplication:

$$\{A, B\}_{mb} = \frac{1}{i\hbar} (A \ast B - B \ast A)$$

$$= symb\left(\frac{1}{i\hbar} [\hat{A}, \hat{B}]\right)$$

(2.12)

Using (2.10) this can be written as

$$\{A, B\}_{mb} = A(\phi) \frac{2}{\hbar} \sin\left[\frac{\hbar}{2} \omega^a \partial_a \partial_b \right] B(\phi)$$

$$= \{A, B\}_{pb} + O(\hbar^2)$$

(2.13)

In the classical limit ($\hbar \to 0$) the Moyal bracket reduces to the classical Poisson bracket. The Moyal bracket $\{\cdot, \cdot\}_{mb}$ is a "deformation" of $\{\cdot, \cdot\}_{pb}$ which preserves two important
properties of the Poisson bracket:

(i) The Moyal bracket obeys the Jacobi identity.

(ii) For every \( A \in \text{Fun}(\mathcal{M}_{2N}) \) the operation \( \{A, \cdot \}_{mb} \) is a derivation on the algebra \((\text{Fun}(\mathcal{M}_{2N}), \ast)\), i.e., it obeys a Leibniz rule\(^*\) of the form

\[
\{A, B_1 \ast B_2\}_{mb} = \{A, B_1\} \ast B_2 + B_1 \ast \{A, B_2\}_{mb}
\]  

(2.14)

It is also a well known fact\(^[8]\) that all derivations \( D \) of the Moyal bracket are "inner derivation", i.e., for any \( D \) one can find an element \( X \in \text{Fun}(\mathcal{M}_{2N}) \) such that \( DA = \{X, A\}_{mb} \).

An analogous statement holds true for the commutator algebra, but not for the classical Poisson-bracket algebra. This difference of the algebraic properties of the Moyal and Poisson brackets is also at the heart of the Groenwald-Van Hove obstruction to quantization\(^[16]\).

The standard correspondence rules of quantum mechanics as postulated by Dirac try to associate operators \( \hat{A} \) to phase functions \( A(\phi) \) in such a way that the Poisson bracket algebra is matched by the operator algebra. In view of the above discussion, which shows that actually it is the Moyal bracket algebra which is equivalent to the operatorial one, it is clear that the Dirac correspondence can be implemented only for the very narrow class of observables for which the higher derivatives of the RHS of eq.(2.13) are ineffective, i.e., for functions at most quadratic in \( \phi^a \).

The symbol calculus suggests that the process of "quantization" can be understood as a smooth deformation of the algebra of classical observables \((\{\cdot, \cdot\}_{pb} \to \{\cdot, \cdot\}_{mb})\) rather than as a radical change in the nature of the observables \((\text{c-numbers} \to \text{operators})\). This point of view has been advocated in refs.[8] and in Moyal’s original paper\(^[7]\) where also the time evolution of the pseudodensities \( \varrho(\phi, t) \) has been studied. At the operatorial level we have von Neumann’s equation

\[
 i\hbar \partial_t \hat{\varrho} = -[\hat{\varrho}, \hat{H}]
\]

which goes, via the symbol map, in

\[
\partial_t \varrho(\phi^a, t) = -\{\varrho, H\}_{mb}
\]  

(2.15)

where \( H(\phi) \) is the symbol of the hamiltonian operator \( \hat{H} \). For pure states with \( \hat{\varrho} = \langle \psi \rangle \langle \psi | \) this equation is equivalent to the Schrödinger equation for \( |\psi\rangle \). In the classical limit

---

\(^*\) Recall that, for any algebra with elements \( A, B, \cdots \) and a product \( \circ \), a derivation \( D \) has the property \( D(A \circ B) = (DA) \circ B + A \circ (DB) \). In the present case, \( A \circ B \equiv \{A, B\}_{mb} \).
eq. (2.15) becomes the well-known Liouville equation for a (positive-definite) probability density $\varrho$:

$$\partial_t \varrho(\phi^a, t) = - \{\varrho, H\}_{pb} \equiv - h^a(\phi) \partial_a \varrho \equiv - l_\hbar \varrho$$  \hspace{1cm} (2.16)

In the second line of eq. (2.16) we used the components

$$h^a(\phi) \equiv \omega^{ab} \partial_b H(\phi)$$  \hspace{1cm} (2.17)

of the hamiltonian vector field $h \equiv h^a \partial_a$, which coincides with the Lie derivative $l_\hbar$ when acting on scalars (zero-forms) $\varrho(\phi)$. Comparing eqs. (2.15) and (2.16) we may say that the Moyal bracket gives rise to the notion of a quantum deformed hamiltonian vector field or, equivalently, of a quantum deformed Lie derivative for zero-forms.

In classical mechanics it is well known how to generalize eq. (2.16) to higher p-form valued ”densities” of the type

$$\varrho = \frac{1}{p!} \varrho_{a_1 \cdots a_p}(\phi) \, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$$  \hspace{1cm} (2.18)

They are pulled along the hamiltonian flow according to the equation of motion

$$\partial_t \varrho = - l_\hbar \varrho$$  \hspace{1cm} (2.19)

where now $l_\hbar$ is the Lie derivative appropriate for p-forms. So far no quantum mechanical analogue of eq. (2.19) has been constructed along the lines of Moyal. It is exactly this problem which we shall address in the following sections. As mentioned in the introduction, our strategy is to introduce an extended phase-space, denoted by $\mathcal{M}_{8N}$, such that scalars on $\mathcal{M}_{8N}$ represent antisymmetric tensors on the standard phase-space $\mathcal{M}_{2N}$, and to apply the Moyal deformation to the extended phase-space.
3. FROM ORDINARY PHASE-SPACE TO EXTENDED PHASE-SPACE

In this section we describe how the conventional exterior calculus on phase-space can be reformulated in a hamiltonian language which lends itself to a deformation à la Moyal. In the present section we introduce the relevant classical structures. Their Moyal deformation will be discussed later on.

Let us suppose we are given a $2N$-dimensional symplectic manifold $\mathcal{M}_{2N}$ endowed with a closed non-degenerate two form $\omega$. For simplicity we assume again that we can introduce canonical coordinates globally so that the components $\omega_{ab}$ are given by (2.4). Furthermore we pick some Hamiltonian $H \in \text{Fun}(\mathcal{M}_{2N})$. It gives rise to the vector field $h^a$ of eq. (2.17) in terms of which Hamilton’s equations read

$$\dot{\phi}^a(t) = h^a(\phi(t))$$

In the following we consider $h^a$ in its role as the generator of symplectic diffeomorphisms (canonical transformations). Under the transformation

$$\delta \phi^a = -h^a(\phi)$$

the components of any tensors change according to

$$\delta T^{a_1a_2\ldots} = l_h T^{a_1a_2\ldots}$$

where

$$l_h T^{a\ldots} = h^c \partial_c T^{a\ldots} + \partial_b h^c T^{a\ldots} - \partial_c h^a T^{c\ldots} + \cdots$$

is the classical Lie-derivative\textsuperscript{[14]}. As time evolution in classical mechanics is a special symplectic diffeomorphism, the time-evolution of the p-form density (2.18) is given by

$$\partial_t \varrho_{a_1\ldots a_p}(\phi, t) = -l_h \varrho_{a_1\ldots a_p}(\phi, t)$$

As we mentioned already, for $p = 0$ eq. (3.5) coincides with Liouville’s equation (2.16). In this case we were able to give a hamiltonian interpretation to the RHS of the evolution equation: it was the Poisson bracket of $H$ with $\varrho$. We shall now introduce the extended phase-space $\mathcal{M}_{8N}$ in such a manner that, even for $p > 0$, the RHS of eqn. (3.5) can be expressed as a generalized Poisson bracket.
The extended phase-space $\mathcal{M}_{8N}$ is a $4N + 4N$ dimensional supermanifold\cite{12} with $4N$ bosonic and $4N$ fermionic dimensions. It is coordinatized by the $8N$-tuples $(\phi^a, \lambda_a, c^a, \bar{c}_a), \ a = 1 \cdots 2N$. Here $\phi^a$ are coordinates on the standard phase-space $\mathcal{M}_{2N}$ which we now identify with the hypersurface in $\mathcal{M}_{8N}$ on which $\lambda_a = 0$ and $c^a = 0 = \bar{c}_a$. The $\lambda_a$’s are additional bosonic variables, and the $c^a$’s and $\bar{c}_a$’s are anticommuting Grassmann numbers. As indicated by the positioning of the indices, $\lambda_a$ and $\bar{c}_a$ are assumed to transform, under a diffeomorphism on $\mathcal{M}_{2N}$, like the derivatives $\partial_a$, while $c^a$ transform like the coordinate differentials $d\phi^a$. Let us define the extended Poisson bracket (epb) structure on $\mathcal{M}_{8N}$ as follows

\[
\{ \phi^a, \lambda_b \}_{\text{epb}} = \delta^a_b, \ \
\{ \phi^a, \phi^b \}_{\text{epb}} = 0 = \{ \lambda_a, \lambda_b \} \\
\{ c^a, \bar{c}_b \}_{\text{epb}} = -i\delta^a_b, \ \text{all others} = 0
\]

(3.6)

With respect to the epb-structure, the auxiliary variables $\lambda_a$ can be thought of as “momenta” conjugate to the $\phi^a$’s. Note also that the $\phi^a$’s have vanishing extended Poisson brackets among themselves, whereas their conventional Poisson bracket on $\mathcal{M}_{2N}$ is different from zero:

\[
\{ \phi^a, \phi^b \}_{pb} = \omega^{ab}
\]

(3.7)

Eq. (3.6) implies the following bracket for $A, B \in \text{Fun}(\mathcal{M}_{8N})$, i.e., for functions $A = A(\lambda_a, \phi^a, \bar{c}_a, c^a), \ldots$,

\[
\{ A, B \}_{\text{epb}} = A\left[ \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \lambda_a} - \frac{\partial}{\partial \lambda_a} \frac{\partial}{\partial \phi^a} - i\left( \frac{\partial}{\partial \bar{c}_a} \frac{\partial}{\partial c^a} + \frac{\partial}{\partial c^a} \frac{\partial}{\partial \bar{c}_a} \right) \right] B
\]

(3.8)

(Note that this is a $\mathbb{Z}_2$-graded bracket whose symmetry character depends on whether $A$ and $B$ are even or odd elements of the Grassmann algebra.) The epb-structure (3.6) was first introduced in refs.[11] where we gave a path-integral representation of classical hamiltonian mechanics. In particular, it was shown that the Grassmann variables $\bar{c}_a$ form a basis in the tangent space $T_\phi \mathcal{M}_{2N}$ and that, similarly, the $c^a$’s form a basis in the cotangent space $T^*_\phi \mathcal{M}_{2N}$ thus playing the role of the differentials $d\phi^a$. This fact can be exploited as follows. Let us assume we are given an arbitrary, completely antisymmetric tensor field on $\mathcal{M}_{2N}$:

\[
T = T_{a_1 \cdots a_p}(\phi) \partial_{b_1} \wedge \cdots \wedge \partial_{b_q} \ d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}
\]

(3.9)

From $T$ we can construct the following function $\hat{T} \in \text{Fun}(\mathcal{M}_{8N})$:

\[
\hat{T} = T_{\bar{c}_1 \cdots \bar{c}_r}(\phi) \ \bar{c}_{b_1} \cdots \bar{c}_{b_q} \ c^{a_1} \cdots c^{a_p}
\]

(3.10)

Under diffeomorphisms on $\mathcal{M}_{2N}$ the function $\hat{T}$ transforms as a scalar. Here and in the following the caret $\hat{\cdot}$ does not indicate operators, but rather that $\partial_a$ and $d\phi^a$ have been
replaced by $\tilde{c}_a$ and $c_a$, respectively. Sometimes we refer to this substitution as the ”hat map”. For example, the p-form of eq. (2.18) becomes
\[
\hat{\omega} = \frac{1}{p!} \varrho_{a_1 \ldots a_p}(\phi) \, c^{a_1} \cdots c^{a_p}
\] (3.11)

We mentioned already that we would like to express the RHS of eq. (3.5) like a Poisson bracket as it was done in eq. (2.16) for zero forms. To this end we try to find a ”super-Hamiltonian” $\tilde{\mathcal{H}} \in \text{Fun}(\mathcal{M}_{SN})$ with the following two properties
\[
\{ \tilde{\mathcal{H}}, \varrho \} \pmb{pb} = \{ H, \varrho \} \pmb{pb} = -l_h \varrho \tag{3.12}
\]
\[
\{ \tilde{\mathcal{H}}, \varrho_{a_1 \ldots a_p}(\phi)c^{a_1} \cdots c^{a_p} \} \pmb{pb} = -(l_h \varrho_{a_1 \ldots a_p})c^{a_1} \cdots c^{a_p} \tag{3.13}
\]

Eq. (3.12) guarantees that for zero-form $\varrho = \varrho(\phi)$ the dynamics given by the new Hamiltonian $\tilde{\mathcal{H}}$ together with the extended Poisson bracket coincides with the one obtained from the standard Hamiltonian together with the standard Poisson bracket. Eq. (3.13) generalizes eq. (3.12) for higher forms. It allows us to rewrite eq. (3.5) in Hamiltonian form:
\[
\partial_t \hat{\omega} = -\{ \hat{\omega}, \tilde{\mathcal{H}} \} \pmb{pb} \tag{3.14}
\]

In ref.[11] we showed that the solution to eqs. (3.12) and (3.13) is provided by the following super-Hamiltonian
\[
\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_B + \tilde{\mathcal{H}}_F \tag{3.15}
\]
where
\[
\tilde{\mathcal{H}}_B = \lambda_a h^a(\phi) \tag{3.16}
\]
and
\[
\tilde{\mathcal{H}}_F = i\tilde{c}_a \partial_b h^a(\phi)c^b \tag{3.17}
\]

The super-Hamiltonian $\tilde{\mathcal{H}}$ has vanishing $ep$-bracket with the following conserved charges:
\[
Q = ic^a \lambda_a
\]
\[
\bar{Q} = i\bar{c}_a \omega^{ab} \lambda_b
\]
\[
Q_g = c^a \bar{c}_a
\]
\[
K = \frac{1}{2} \omega_{ab} c^a c^b
\]
\[
\bar{K} = \frac{1}{2} \omega^{ab} \bar{c}_a \bar{c}_b
\]

Under the extended Poisson bracket they form a closed algebra isomorphic to $\text{ISp}(2)$, whose inhomogeneous part is generated$^{[11]}$ by the BRS operator $Q$ and the anti-BRS operator $\bar{Q}$. 

11
It is interesting that $\widetilde{H}$ is a pure BRS-variation,

$$\widetilde{H} = i\{Q, \hat{Q}, H\}_{epb}$$

which is typical of topological field theories\[17\]. The use of it in this context has been explored in ref.[18]. Furthermore, $\widetilde{H}$ possesses a N=2 supersymmetry\[19\] justifying the term "super-Hamiltonian" for $\widetilde{H}$. However the SUSY will play no important role in the following.

The five conserved charges (3.18) are the essential tool in reformulating the classical Cartan calculus in hamiltonian form. To illustrate this point, consider the following tensors and their counterparts in $Fun(\mathcal{M}_{8N})$:

$$v = v^a \partial_a \mapsto \hat{v} = v^\alpha \bar{c}_\alpha$$

$$\alpha = \alpha_a d\phi^a \mapsto \hat{\alpha} = \alpha_a c^a$$

$$F^{(p)} = \frac{1}{p!} F_{a_1 \cdots a_p} d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p} \mapsto \hat{F}^{(p)} = \frac{1}{p!} F_{a_1 \cdots a_p} c^{a_1} \cdots c^{a_p}$$

$$V^{(p)} = \frac{1}{p!} V^{a_1 \cdots a_p} \partial_{a_1} \wedge \cdots \wedge \partial_{a_p} \mapsto \hat{V}^{(p)} = \frac{1}{p!} V^{a_1 \cdots a_p} \bar{c}_{a_1} \cdots \bar{c}_{a_p}$$

All coefficients $v^a, \alpha_a, F^{a_1 \cdots a_p}, etc.$ appearing in the above formulas are functions of $\phi$. By this "hat map" $\hat{\cdot}$, the exterior derivative of p-forms $F^{(p)}$ goes over into the $ep$-bracket with the BRS charge $Q$:

$$(dF^{(p)})^\wedge = i\{Q, \hat{F}^{(p)}\}_{epb}$$

The exterior co-derivative of p-vectors $V^{(p)}$ is given by the $ep$-bracket with the anti-BRS operator $\bar{Q}$

$$(dV^{(p)})^\wedge = i\{\bar{Q}, \hat{V}^{(p)}\}_{epb}$$

$$\bar{d}V^{(p)} = \omega^{ab} \partial_b V^{a_1 \cdots a_p} \partial_a \wedge \partial_{a_1} \cdots \wedge \partial_{a_p}$$

In symplectic geometry vectors and forms can be related by contraction with $\omega_{ab}$ or $\omega^{ab}$. This operation is realized\[11,14\] as the $ep$-bracket with $K$ and $\bar{K}$:

$$(v^\flat)^\wedge = i\{K, \hat{v}\}_{epb} \quad (v^\flat)^a = \omega_{ac} v^c$$

$$(\alpha^\flat)^\wedge = i\{\bar{K}, \hat{\alpha}\}_{epb} \quad (\alpha^\flat)^a = \omega^{ac} \alpha_c$$

The contractions with vectors and 1-forms translates into the following brackets

$$(i(v)F^{(p)})^\wedge = i\{\hat{v}, \hat{F}^{(p)}\}_{epb}$$

$$(i(\alpha)V^{(p)})^\wedge = i\{\hat{\alpha}, \hat{V}^{(p)}\}_{epb}$$
The Lie derivative along the hamiltonian vector field is

\[(l_H T)^\wedge = -\{\tilde{\mathcal{H}}, \tilde{T}\}_{epb}\]

(3.26)

where \(T\) can be any antisymmetric tensor. The \(epb\)-representation of the various classical tensor manipulations are summarized in table 1.

### 4. MOYAL DEFORMATION ON EXTENDED PHASE-SPACE

In the previous section we realized the classical Cartan calculus in terms of \(ep\)-brackets on the extended phase-space \(\mathcal{M}_{8N}\). Let us now try to deform the extended Poisson bracket to an extended Moyal bracket. Following Berezin\(^{[10]}\), we define the extended star product\(^{*}\) on \(Fun(\mathcal{M}_{8N})\) as

\[
A \ast_e B \equiv A \exp\left[\frac{i}{2} \left( \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \lambda^a} - \frac{\partial}{\partial \lambda^a} \frac{\partial}{\partial \phi^a} \right) + \frac{\partial}{\partial c^a} \frac{\partial}{\partial \bar{c}^a} \right] B
\]

(4.1)

where \(A(\lambda, \phi, \bar{c}, c), \text{etc.}\). The extended Moyal bracket is introduced as the graded commutators with respect to \(\ast_e\)-multiplication:

\[
\{A, B\}_{emb} = \frac{1}{i} \left[ A \ast_e B - (-)^{[A][B]} B \ast_e A \right]
\]

(4.2)

Here \([A] = 0, 1\) denotes the grading of \(A\). The sign factor on the RHS of (4.2) guarantees that the \(em\)-bracket has the same symmetry properties as the graded commutator:

\[
\{A, B\}_{emb} = -(-1)^{[A][B]} \{B, A\}_{emb}
\]

It can be checked that the bracket (4.2) obeys the graded Jacobi identity and that \(\{A, \cdot\}_{emb}\) is a graded derivation of the algebra \((Fun(\mathcal{M}_{8N}), \ast_e)\) for any \(A \in Fun(\mathcal{M}_{8N})\):

\[
\{A, B_1 \ast_e B_2\}_{emb} = \{A, B_1\}_{emb} B_2 + (-1)^{[A][B_1]} B_1 \ast_e \{A, B_2\}_{emb}
\]

(4.3)

This is a consequence of the associativity of the extended star product. Eq. (4.3) is very important for our purposes because we would like to rewrite the derivations \(d, i_v, l_h, \text{etc.}\), as extended Moyal brackets. Comparing (4.1) to (2.10) we see that the transition from standard phase-space to extended phase-space entails the replacements \(\frac{\partial}{\partial q} \rightarrow \frac{\partial}{\partial \phi}, \frac{\partial}{\partial p} \rightarrow \frac{\partial}{\partial \lambda}\) and the

\* To keep with the supersymmetry jargon we should call this product the "super-star product".
addition of the Grassmannian piece. For a reason which will become clear shortly, we have set $\hbar = 1$ for the deformation parameter in eq. (4.1). Reinstating $\hbar$ and letting $\hbar \to 0$, the extended Moyal bracket (4.2) reduces to the extended Poisson bracket (3.8). The fundamental $em$-brackets among $\phi^a, \lambda_a, c^a$ and $\bar{c}_a$ coincide with the respective $ep$-brackets given in eq. (3.6) because in this case the higher derivative terms vanish. In practical calculations, involving the extended star product, the following alternative representation has proven helpful:

$$ (A \ast_e B)(\phi, \lambda, \bar{c}, c) = A(\lambda_a - \frac{i}{2} \frac{\partial}{\partial \phi^a}, \phi^a + \frac{i}{2} \frac{\partial}{\partial \lambda_a}, \bar{c}_a, c^a + \frac{\partial}{\partial \bar{c}_a}) |_{\tilde{\lambda} = \lambda, \tilde{\phi} = \phi, \tilde{\bar{c}} = \bar{c}} $$.  

The above equation is easily proven by Fourier-transforming the functions $A$ and $B$. The Grassmannian variables $c^a$ and $\bar{c}_a$ are the Wick symbols of fermionic creation and annihilation operators. This means that, for example, the symbol $\bar{c}_a c^b = -c^b \bar{c}_a$ represents the operator $\hat{c}_a \hat{c}^b$. The symbol for $\hat{c}^b \bar{c}_a$ has an additional commutator term therefore. In fact, eq. (4.1) yields

$$ c^a \ast_e c^b = c^a c^b $$.  
$$ \bar{c}_a \ast_e \bar{c}_b = \bar{c}_a \bar{c}_b $$.  
$$ \bar{c}_a \ast_e c^b = \bar{c}_a c^b $$
$$ c^b \ast_e \bar{c}_a = c^b \bar{c}_a + \delta^b_a $$

where it appears a normal-ordering term on the RHS of the last equation.

5. DEFORMED SUPER-HAMILTONIAN: BOSONIC SECTOR

Following the same strategy as in the classical case, we now try to find a deformed super-Hamiltonian $\tilde{H}^h \in Fun(M_{8N})$ which, with respect to the extended Moyal bracket, gives rise to the same time-evolution of zero-forms as the standard Hamiltonian $H(\phi)$ with respect to the standard Moyal bracket. Let us first try to solve the deformed version of the zero-form equation

$$ \{ \tilde{H}^h, \varphi(\phi) \}_{emb} = \{ H(\phi), \varphi(\phi) \}_{mb} $$.  

For the time being we ignore the Grassmann variables. The p-form generalization of eq. (5.1) will be investigated in section 6. Here we look for a deformation of the bosonic part (3.16)
only, \( \tilde{\mathcal{H}}_B = \lambda_a \hbar^a(\phi) \). However, as we prove in appendix A, the equation

\[
\{ \tilde{\mathcal{H}}_B^\hbar(\lambda, \phi), \varrho(\phi) \}_\text{emb} = \{ H(\phi), \varrho(\phi) \}_\text{mb} \tag{5.2}
\]

does not possess any solution for \( \tilde{\mathcal{H}}_B^\hbar \). At this point classical mechanics does not offer any hint of how to proceed and some new input is required. We shall weaken the condition on \( \tilde{\mathcal{H}}_B^\hbar \) by requiring that eq. (5.2) holds true only on the hypersurface where \( \lambda = 0 \). This is certainly a sensible choice, since it is exactly the \( \lambda = 0 \)-hypersurface which is to be identified with the standard phase-space \( \mathcal{M}_{2N} \), and only there the ordinary Moyal formalism fixes the dynamics. So let us replace eq. (5.2) by

\[
\mathcal{P}\{ \tilde{\mathcal{H}}_B^\hbar(\lambda, \phi), \varrho(\phi) \}_\text{emb} = \{ H(\phi), \varrho(\phi) \}_\text{mb} \tag{5.3}
\]

where the projection operator \( \mathcal{P} \) acts on any function \( F(\lambda, \phi) \) according to

\[
\mathcal{P}F(\lambda, \phi) = F(\lambda = 0, \phi) \tag{5.4}
\]

Of course \( \lambda \) is set to zero in eq. (5.3) only after the derivatives with respect to \( \lambda \) have been taken. (In the language of Dirac’s theory of constraints, \( \lambda \) is set to zero “weakly”.) In appendix A we show that the most general solution to eq. (5.3) is given by

\[
\tilde{\mathcal{H}}_B^\hbar(\lambda, \phi) = \frac{1}{\hbar} \sinh [\hbar \lambda_a \omega^{ab} \partial_b] H(\phi) + \frac{1}{\hbar} Y(\lambda, \phi) \tag{5.5}
\]

Here \( Y(\lambda, \phi) \) is an arbitrary function which is even in \( \lambda \) and of order \( \hbar^2 \). Therefore, in the classical limit, the second term on the RHS of (5.5) vanishes and the first one reproduces the classical result:

\[
\lim_{\hbar \to 0} \tilde{\mathcal{H}}_B^\hbar(\lambda, \phi) = \lambda_a \omega^{ab} \partial_b H = \lambda_a h^a = \tilde{\mathcal{H}}_B(\lambda, \phi) \tag{5.6}
\]

In our conventions \( \tilde{\mathcal{H}}_B^\hbar \) is explicitly dependent on the deformation parameter \( \hbar \), but the extended Moyal bracket (4.1) is not, so that formally, even in the classical limit, all higher derivative terms are retained. Effectively these terms are all irrelevant, however, since the classical \( \tilde{\mathcal{H}}_B \) is only linear in \( \lambda \). (By rescaling \( \lambda \) we could transfer the \( \hbar \)-dependence from \( \mathcal{H} \) to the bracket.)
The arbitrary function \( Y(\lambda, \phi) \) parametrizes the manner in which we extrapolate the dynamics away from the \( \lambda = 0 \)-surface. It is instructive to consider the following examples:

\[
Y_\pm(\lambda, \phi) = \pm \cosh[\hbar \lambda_a \omega^{ab} \partial_b] H(\phi) \\
Y_0(\lambda, \phi) = 0
\]

(5.7)

The power series of \( Y_{0,\pm} \) start with a term \( \propto \hbar^2 \) and contain only even powers of \( \lambda \). It is remarkable that the resulting super-Hamiltonians are given by the application of certain finite-difference operators to the standard Hamiltonian \( H(\phi) \). In fact, let us define, for any function \( F(\phi) \), the following operations:

\[
(D_+ F)(\phi) = \frac{1}{\hbar} [F(\phi_a + \hbar \omega^{ab} \lambda_b) - F(\phi_a)] \\
(D_- F)(\phi) = \frac{1}{\hbar} [F(\phi_a) - F(\phi_a - \hbar \omega^{ab} \lambda_b)] \\
(D_0 F)(\phi) = \frac{1}{2\hbar} [F(\phi_a + \hbar \omega^{ab} \lambda_b) - F(\phi_a - \hbar \omega^{ab} \lambda_b)]
\]

(5.8)

Then eq. (5.5) yields the following super-Hamiltonians for the examples (5.7):

\[
\tilde{H}_B^h_+ (\lambda, \phi) = - D_+ H(\phi) \\
\tilde{H}_B^h_0 (\lambda, \phi) = - D_0 H(\phi) \\
\tilde{H}_B^h_- (\lambda, \phi) = - D_- H(\phi)
\]

(5.9)

The operators \( D_{\pm,0} \) are strongly reminiscent of a forward, backward and symmetric lattice derivative, respectively. They evaluate the difference of some \( F \in \text{Fun}(\mathcal{M}_{2N}) \) at two points with a finite separation \( \hbar \omega^{ab} \lambda_b \). Here \( \lambda_a \) acts as a parameter, external to \( \mathcal{M}_{2N} \), which defines how the ”links” of the lattice are imbedded into \( \mathcal{M}_{2N} \). In the classical limit the operators in (5.8) become identical and coincide with the directional derivative along the vector \( \omega^{ab} \lambda_a \):

\[
\lim_{\hbar \to 0} D_{\pm,0} = \omega^{ab} \lambda_b \partial_a
\]

(5.10)

In order to further illuminate this lattice structure, let us look at the equations of motion of \( \phi^a(t) \) and \( \lambda_a(t) \). From

\[
\frac{d}{dt} \phi^a(t) = \{ \phi^a, \tilde{H}_B^h \}_{\text{emb}} = \frac{\partial}{\partial \lambda_a} \tilde{H}_B^h(\lambda, \phi) \\
\frac{d}{dt} \lambda_a(t) = \{ \lambda_a, \tilde{H}_B^h \}_{\text{emb}} = - \frac{\partial}{\partial \phi^a} \tilde{H}_B^h(\lambda, \phi)
\]

(5.11)
One easily obtains

\[
\frac{d}{dt} \phi^a(t) = \begin{cases} 
  h^a(\phi^b + \hbar \omega^{bc} \lambda_c) 
  & , \\
  \frac{1}{2}[h^a(\phi^b + \hbar \omega^{bc} \lambda_c) + h^a(\phi^b - \hbar \omega^{bc} \lambda_c)] 
  & , \\
  h^a(\phi^b - \hbar \omega^{bc} \lambda_c) 
  & ,
\end{cases}
\]  

(5.12)

where \( \hat{H}^\hbar_B +, \hat{H}^\hbar_B 0 \) and \( \hat{H}^\hbar_B - \) has been used, respectively. Similarly

\[
\frac{d}{dt} \lambda^a(t) = \begin{cases} 
  \frac{1}{\hbar} \left[ \partial_a H(\phi^b + \hbar \omega^{bc} \lambda_c) - \partial_a H(\phi^b) \right] 
  & , \\
  \frac{1}{2\hbar} \left[ \partial_a H(\phi^b + \hbar \omega^{bc} \lambda_c) - \partial_a H(\phi^b - \hbar \omega^{bc} \lambda_c) \right] 
  & , \\
  \frac{1}{\hbar} \left[ \partial_a H(\phi^b) - \partial_a H(\phi^b - \hbar \omega^{bc} \lambda_c) \right] 
  & ,
\end{cases}
\]  

(5.13)

The trajectory \((\phi^a(t), \lambda^a(t))\) is the solution of the coupled system of equations (5.12), (5.13). Contrary to classical mechanics\(^{[11]}\) where \(\lambda^a(t)\) does not influence the dynamics of \(\phi^a(t)\), we see that here the \(\phi\)-dynamics depends on \(\lambda\). In a sense, \(\phi^a(t)\) evolves on a ”dynamical lattice” on \(\mathcal{M}_{2N}\): by eq. (5.13) a fixed trajectory \(\phi^a(t)\) leads to a certain solution \(\lambda^a(t)\) which defines a kind of lattice which, in turn, governs the evolution of \(\phi^a\) according to (5.12). In fact, \(\phi^a(t)\) does not feel the ”drift force” \(h^a\) at the point \(\phi^a\) as in the classical case, but rather at the ”sites” \(\phi^b \pm \hbar \omega^{bc} \lambda_c\) sitting at the ends of the ”link” given by the vector \(\hbar \omega^{bc} \lambda_c\). Both in the classical limit \(\hbar \to 0\) and upon projection on the \(\lambda = 0\)-surface (for \(\hbar \neq 0\) ) all choices for \(\tilde{H}^\hbar_B\) are equivalent, and eq.(5.12) becomes

\[
\frac{d}{dt} \phi^a(t) = h^a(\phi(t))
\]  

(5.14)

Depending on whether we set \(\hbar = 0\) or \(\lambda = 0\) at \(\hbar \neq 0\) the interpretation of eq. (5.14) is different. For \(\hbar = 0\), eq. (5.14) is Hamilton’s classical equation of motion. For \(\lambda = 0\), \(\hbar \neq 0\) eq.(5.14) is the equation for the symbol \(\phi^a(t)\) of the Heisenberg operator \(\hat{\phi}^a(t)\). This equation looks like Hamilton’s equation because Heisenberg’s equation has the same form as the classical one.

Also the equations (5.13) become identical in the limit \(\hbar \to 0\) where we recover the classical result

\[
\frac{d}{dt} \lambda^a(t) = -\partial_a h^b(\phi(t))\lambda_b
\]  

(5.15)

It is important to note that for \(\hbar \neq 0\) and for any choice of \(\hat{H}^\hbar_B\) eq. (5.13) always admits the solution \(\lambda^a(t) = 0\), for all \(t\). This is essential for the consistency of our approach because it shows that the ”constraint” \(\lambda_a = 0\) is compatible with the time evolution of the extended Moyal formalism: If the point \((\phi^a, \lambda^a)(t)\) is on the hypersurface \((\lambda = 0)\) representing \(\mathcal{M}_{2N}\) at \(t = 0\), it will stay there at any time \(t > 0\).
It would be interesting to keep $\lambda \neq 0$ and from eq. (5.12) and (5.13), which are modified Heisenberg equations of motion for the symbols $\phi$ and $\lambda$, derive the associated modified Schrödinger equation. The need for a modified Schrödinger equation may arise at extremely high energies like in the realm of quantum gravity where pure states might evolve into mixed ones. For sure this modified Schrödinger equation will bring new light on the field $\lambda$. This field is similar to the response field of statistical mechanics and plays a central role in the fluctuation dissipation theorem (FDT). The fact that, to recover standard quantum mechanics, we have to restrict ourselves to $\lambda = 0$ may imply that the modified form of quantum mechanics violates the FDT. Also worth noticing is the fact that the $\lambda$-field appeared already in stochastic processes where it effectively takes the place of the noise. So its presence now in modified quantum Heisenberg evolution could indicate that this equation can be turned into the standard Heisenberg equation coupled to noise on the line of the recent interesting investigation reviewed in. We hope to come back to these topics in the future.

Let us now turn to less speculative topics and analyze in detail the various choices of the $Y$-functions of (5.7). We shall show that the choice $Y = 0$, leading to the symmetric "lattice" derivative $D_0$, is singled out uniquely by a variety of special features. First of all, looking at the second expression contained in eqs. (5.12) and (5.13), we see that for that choice the equations of motion are invariant under the transformation

$$ t \rightarrow -t, \; \phi^a \rightarrow \phi^a, \; \lambda_a \rightarrow -\lambda_a, \; h^a \rightarrow -h^a \quad (5.16) $$

For any other choice of $Y$ this would not be the case. Second, let us introduce the variables

$$ X^a_+ = \phi^a + \hbar \omega^{ab} \lambda_b $$
$$ X^a_- = \phi^a - \hbar \omega^{ab} \lambda_b \quad (5.17) $$

and let us derive their equations of motion. By adding and subtracting the second equation of (5.13) and (5.12) we find that

$$ \frac{d}{dt} X^a_+(t) = h^a(X_+(t)) $$
$$ \frac{d}{dt} X^a_-(t) = h^a(X_-(t)) \quad (5.18) $$

Remarkably, $X^a_+$ and $X^a_-$ do not mix under time evolution and, even for $\hbar \neq 0$ and $\lambda \neq 0$, they separately obey the same equation as $\phi^a$ in the classical limit, see eq. (5.14). This can
be traced back to the fact that the symmetric Hamiltonian \( \tilde{H}_{B0}^{h} = -D_0 H \) can be written as

\[
\tilde{H}_{B0}^{h}(\lambda, \phi) = \frac{1}{2\hbar}[H(\phi^a - \hbar \omega^{ab} \lambda_b) - H(\phi^a + \hbar \omega^{ab} \lambda_b)]
\]  

whence

\[
\tilde{H}_{B0}^{h} = \frac{1}{2\hbar}[H(X^a) - H(X^a_+)]
\]  

and that the extended Moyal bracket decomposes similarly. For functions \( A(X_+, X_-) \) and \( B(X_+, X_-) \) the star-product (4.1) can be written as

\[
A \ast_e B = A \exp\left[+i\hbar \left( \frac{\partial}{\partial X_-^a} \omega^{ab} \left. \frac{\partial}{\partial X_-^b} \right) - \frac{\partial}{\partial X_+^a} \omega^{ab} \left. \frac{\partial}{\partial X_+^b} \right) \right] B
\]  

Each one of the two terms in the bracket on the RHS of (5.21) has the same structure as the operator \( \frac{\partial}{\partial a} \omega^{ab} \frac{\partial}{\partial b} \) appearing in the ordinary star product (2.10) with \( \phi^a \) replaced by \( X_-^a \) and \( X_+^a \), respectively. The overall factor is different, however. This suggests to introduce the following modified star product and Moyal bracket on \( Fun(\mathcal{M}_{2N}) \):

\[
(A \ast_2 B)(\phi) = A(\phi) \exp\left[i\hbar \left( \frac{\partial}{\partial X_-^a} \omega^{ab} \frac{\partial}{\partial X_-^b} - \frac{\partial}{\partial X_+^a} \omega^{ab} \frac{\partial}{\partial X_+^b} \right) \right] B(\phi)
\]  

They differ from the \( * \) of (2.10) and the \( \{ \cdot, \cdot \} \) of (2.12) only by the rescaling \( h \rightarrow 2h \).

Let us now consider functions which depend only on either \( X_-^a \) or \( X_+^a \). In a slight abuse of language we shall call them holomorphic and antiholomorphic, respectively. These functions have the property that their extended star-product can be expressed in terms of the modified star-product \( \ast_2 \) on \( Fun(\mathcal{M}_{2N}) \). Eq. (5.21) implies that

\[
A(X_-) \ast_e B(X_-) = A(X_-) \exp\left[+i\hbar \left( \frac{\partial}{\partial X_-^a} \omega^{ab} \frac{\partial}{\partial X_-^b} \right) \right] B(X_-) = A(\phi) \ast_2 B(\phi)|_{\phi = X_-}
\]  

and that the \( \ast_e \)-product of a holomorphic with an antiholomorphic function reduces to
ordinary pointwise multiplication:

\[ A(X_\mp) \ast_e B(X_\pm) = A(X_\mp)B(X_\pm) \]  \hspace{1cm} (5.25)

This entails the following relations for the *em*-brackets of (anti)-holomorphic functions

\[
\begin{align*}
\{A(X_\pm), B(X_\pm)\}_{em} &= \mp 2\hbar \{A(\phi), B(\phi)\}_{mb}^{(2)} |_{\phi = X_\pm} \\
\{A(X_\pm), B(X_\mp)\}_{em} &= 0
\end{align*}
\]  \hspace{1cm} (5.26)

In particular it follows that

\[
\begin{align*}
\{X_-^a, X_-^b\}_{em} &= + 2\hbar \omega^{ab} \\
\{X_+^a, X_+^b\}_{em} &= - 2\hbar \omega^{ab} \\
\{X_-^a, X_+^b\}_{em} &= 0
\end{align*}
\]  \hspace{1cm} (5.27)

i.e., up to a factor of \(\pm 2\hbar\), the *em*-algebra of \(X_-^a\) and \(X_+^a\) coincides with the standard *mb*-algebra \(\{\phi^a, \phi^b\}_{mb} = \omega^{ab}\). Thus the holomorphic and the antiholomorphic functions form two closed and mutually commuting algebras with respect to the *em*-bracket which are essentially equivalent to \(Fun(\mathcal{M}_{2N})\) equipped with the standard Moyal bracket. This fact is important for various reasons. First of all, it explains the simple form of the equations (5.18): neither the Hamiltonian (5.20) nor the *em*-brackets couple the \(X_+\)-dynamics to the \(X_-\)-dynamics. It has to be stressed, however, that this is true only for \(Y = 0\); any other choice would spoil the decoupling of \(X_+\) and \(X_-\).

Another special property of the choice \(Y = 0\) is related to the composition rule for two consecutive symplectic diffeomorphisms on \(\mathcal{M}_{2N}\). Denoting the generating functions for these transformations \(G_1(\phi)\) and \(G_2(\phi)\), the associated Hamiltonian vector fields are \(h_{1,2} = (dG_{1,2})^\sharp \equiv \omega^{ab}\partial_b G_{1,2}\partial_a\) and their Lie brackets form the following algebra

\[
[h_1, h_2] = -h_3 \hspace{1cm} h_3 = (d\{G_1, G_2\}_{pb})^\sharp
\]  \hspace{1cm} (5.28)

In the extended Poisson bracket formalism, this equation is equivalent to

\[
\{\tilde{\mathcal{H}}[G_1], \tilde{\mathcal{H}}[G_2]\}_{epb} = \tilde{\mathcal{H}}[\{G_1, G_2\}_{pb}]
\]  \hspace{1cm} (5.29)

where \(\tilde{\mathcal{H}}[G_{1,2}]\) is the classical super-Hamiltonian (3.15) with \(H(\phi)\) replaced by \(G_{1,2}(\phi)\). It is important to assess whether also the deformed (bosonic) super-Hamiltonians form a
closed algebra under the extended Moyal bracket. For \( Y = 0 \) we take

\[
\tilde{\mathcal{H}}^h_{B_0}[G_{1,2}] = \frac{1}{2\hbar}[G_{1,2}(X_-) - G_{1,2}(X_+)]
\]  

so that eq. (5.26) implies

\[
\{ \tilde{\mathcal{H}}^h_{B_0}[G_1], \tilde{\mathcal{H}}^h_{B_0}[G_2] \}_{\text{emb}} = \left( \frac{1}{2\hbar} \right)^2 \left( \{ G_1(X_-), G_2(X_-) \}_{\text{emb}} + \{ G_1(X_+), G_2(X_+) \}_{\text{emb}} \right)
\]

\[
= \left( \{ G_1(\phi), G_2(\phi) \}_{mb} \big| _{\phi = X_-} \right) - \left( \{ G_1(\phi), G_2(\phi) \}_{mb} \big| _{\phi = X_+} \right)
\]

\[
= \frac{1}{2\hbar}[G_3(X_-) - G_3(X_+)]
\]  

(5.31)

with \( G_3 = \{ G_1, G_2 \}_{mb}^{(2)} \). We conclude that the Hamiltonians \( \tilde{\mathcal{H}}^h_{B_0} \) indeed form a closed algebra under the \( \text{emb} \)-bracket:

\[
\left\{ \tilde{\mathcal{H}}^h_{B_0}[G_1], \tilde{\mathcal{H}}^h_{B_0}[G_2] \right\}_{\text{emb}} = \tilde{\mathcal{H}}^h_{B_0}[\{ G_1, G_2 \}_{mb}^{(2)}]
\]  

(5.32)

For \( Y \neq 0 \) no comparable result can be proven in general. In the classical limit the algebra (5.32) reduces to (5.28), so we could call the (5.32) the algebra of the quantum deformed hamiltonian vector fields.

At first sight it might seem puzzling that the modified bracket \( \{ G_1, G_2 \}_{mb}^{(2)} \) appears on the RHS of (5.32). In fact, in the ordinary Moyal formalism, one would expect the standard bracket \( \{ G_1, G_2 \}_{mb} \) to appear. However, we have to recall that \( \{ \tilde{\mathcal{H}}^h, \cdot \}_{\text{emb}} \) coincides with \( \{ H, \cdot \}_{mb} \) only after the projection on the \( \lambda = 0 \)-surface. Therefore iterated Moyal brackets \( \{ G_1, \{ G_2, \cdot \}_{mb} \}_{mb} \) become \( \mathcal{P}\{ \tilde{\mathcal{H}}^h_{B_0}[G_1], \mathcal{P}\{ \tilde{\mathcal{H}}^h_{B_0}[G_2], \cdot \}_{\text{emb}} \}_{\text{emb}} \) where the \( \mathcal{P} \) forbids a naive application of Jacobi’s identity which would allow to combine the \( \tilde{\mathcal{H}}^h \)’s into one bracket. Nevertheless, eq. (5.32) shows that the \( \text{emb} \)-algebra of the super-Hamiltonians closes even without intermediate projections. The only change is the rescaling \( \hbar \to 2\hbar \). Therefore, once we have passed over from the ordinary Moyal formalism to the extended one, if we interpret \( 2\hbar \) as the "physical" value of the deformation parameter, we may study the quantum analog of the classical canonical transformations as \( \text{emb} \)-operations without any subsequent projection.

To close this section, we remark that the symmetric super-Hamiltonian was also obtained by Marinov\cite{23} in a completely different manner, namely by constructing a path-integral solution to eq. (2.15). An alternative derivation of Marinov’s path-integral is as
follows. One starts from a density matrix operator \( \hat{\rho} = |\psi\rangle \langle \psi| \) and expresses its time-evolution via the product of two Feynman path-integrals. Then, after an appropriate change of variables, \( symb(\hat{\rho}) \) can be seen to evolve according to Marinov’s path-integral involving \( \tilde{\mathcal{H}}_B^h(X_+, X_-) \). In this derivation it becomes clear that the transformation (5.16) which interchanges \( X^a_+ \) with \( X^a_- \) corresponds to interchanging vectors \( |\psi\rangle \) with dual vectors \( \langle \psi| \). Therefore eq. (5.16) should be thought of as a kind of ”modular conjugation” as it is explained in detail elsewhere\[24\].

6. DEFORMED SUPER-HAMILTONIAN: FERMIONIC SECTOR

In the previous section we derived the bosonic Hamiltonian \( \tilde{\mathcal{H}}_B^h \), which is the quantum deformed version of the classical super-Hamiltonian \( \tilde{\mathcal{H}}_B = \lambda_a h^a \), or, equivalently of the hamiltonian vector field \( h^a \partial_a \). Thus, in a sense, \( \tilde{\mathcal{H}}_B^h \) is the appropriate notion of a ”quantum hamiltonian vector field”. In order to be able to deform more general geometric objects, we have to generalize the fermionic part\(^*\) of the super-Hamiltonian, \( \tilde{\mathcal{H}}_F \). Then the complete deformed hamiltonian

\[
\tilde{\mathcal{H}}^h = \tilde{\mathcal{H}}_B^h + \tilde{\mathcal{H}}_F^h \in Fun(\mathcal{M}_{8N})
\] (6.1)

will define a ”quantum Lie derivative \( L_h \)” by an equation analogous to (3.12):

\[
\{ \tilde{\mathcal{H}}^h, \hat{\rho} \}_{emb} = -(L_h \hat{\rho})^\wedge
\] (6.2)

Here the ”hat-map \((\cdot)^\wedge\)” is defined as in the classical case, see eqs. (3.9) and (3.10). Even though the natural multiplication of the \( c^a \)’s is via the extended star product \( \ast_e \) now, the classical wedge product remains undeformed basically, because, by virtue of (4.5),

\[
\bar{c}_a \ast_e \bar{c}_b \cdots \ast_e c^e \ast_e c^f \ast_e \cdots = \bar{c}_a \bar{c}_b \cdots c^e c^f \cdots
\] (6.3)

(A similar result was found in the third of ref.[6] in a different framework.)

Because the new variables \( c^a \) and \( \bar{c}_a \) are not present in conventional quantum mechanics, it is difficult to have any intuition on how \( \tilde{\mathcal{H}}_F^h \) should be chosen. Of course we require that

\[
\lim_{\hbar \to 0} \tilde{\mathcal{H}}_F^h = \tilde{\mathcal{H}}_F = i\bar{c}_a \partial_b h^a c^a
\] (6.4)

but we are still left with a variety of possible choices leading to different deformed calculi for \( \hbar \neq 0 \). In this paper we study a very simple choice of \( \tilde{\mathcal{H}}_F^h \) which is inspired by our experience with the classical case. We require \( \tilde{\mathcal{H}}_F^h \) to have the following two properties:

\(^*\) Also other authors\[25\] have inserted Grassmannian variables in the standard path-integral, but their techniques and goals were different from the ones presented here.
The complete Hamiltonian $\tilde{H} = \tilde{H}_B + \tilde{H}_F$ is assumed to have the same BRS symmetry as its classical ancestor. This symmetry is generated by the nilpotent charge $Q = i\epsilon^{a} \lambda_{a}$. Both by $ep$- and by $em$- brackets it induces the following transformations on the fields ($\epsilon$ is an anticommuting constant):
\begin{align*}
\delta \phi^{a} &= \epsilon \epsilon^{a} \\
\delta \bar{c}_{a} &= i \epsilon \lambda_{a} \\
\delta c^{a} &= 0 = \delta \lambda_{a}
\end{align*}
(6.5)

We assume that $\tilde{H}_F$ is bilinear in $c^a$ and $\bar{c}_a$, i.e., that, for some function $W_{b}^{a}$,
\begin{equation}
\tilde{H}_F = i \bar{c}_{a} W^{a}_{b}(\lambda, \phi) c^{b}
\end{equation}
(6.6)
We show in appendix B that for any bosonic Hamiltonian of the form
\begin{equation}
\tilde{H}_B^{h}(\lambda, \phi) = F(\lambda \omega^{ab} \partial_{b})H(\phi)
\end{equation}
(6.7)
(with $F(x) = x$ in the classical limit) the two conditions (F1) and (F2) fix the fermionic piece $\tilde{H}_F$ uniquely. The answer we find is
\begin{equation}
\tilde{H}^{h} = F(\lambda \omega \partial)H(\phi) + i \bar{c}_{a} \omega^{ac} \partial_{b} \partial_{c} \frac{F(\lambda \omega \partial)}{(\lambda \omega \partial)}H(\phi) c^{b}
\end{equation}
(6.8)
where $\lambda \omega \partial \equiv \lambda_{a} \omega^{ab} \partial_{b}$. In section 5 we argued that a preferred choice for $\tilde{H}_B$ is the symmetric one, i.e., $Y = 0$ in eq. (5.5). Therefore we shall use from now on:
\begin{equation}
F(x) = \frac{1}{\hbar} \sinh(\hbar x)
\end{equation}
(6.9)
Note that $F(x) = x + O(\hbar^2)$.

The Hamiltonian (6.8) is BRS invariant by construction. It is remarkable that it is also invariant under the transformations generated by all the other charges listed in (3.18):
\begin{equation}
\{\tilde{H}^{h}, \Omega\}_{emb} = 0
\end{equation}
(6.10)
\begin{equation}
\Omega \equiv Q, \bar{Q}, Q_{g}, K, \bar{K}
\end{equation}
(6.11)
It is particularly interesting that the charge $K$ is conserved. In fact, in the classical case, it was shown\cite{11} that this conservation is equivalent to Liouville’s theorem. In quantum mechanics we have an analogous ”formal” conservation because the Heisenberg equations of
motion (or the corresponding equations for the symbols) have the same form as the classical Hamiltonian equations. Of course this does not mean that in quantum mechanics we have a Liouville theorem \(\text{i.e., that the volume of phase-space is conserved under time evolution}\). The formal analogy between Heisenberg and Hamilton equations holds quantum mechanically only at the operatorial level and not at the level of averaged quantities.

The generators \(\Omega\) in equation (6.11) act on \(A \in Fun(\mathcal{M}_{8N})\) via the em-brackets:

\[
\delta A = i\{e\Omega, A\}_{\text{emb}} \tag{6.12}
\]

As the \(\Omega\)'s are only quadratic in \(\phi^a, \lambda_a, \cdots\), the expressions for \(\delta\phi^a, \delta\lambda_a, \cdots\) are the same as in the classical case\(^{[11]}\). The \(\Omega\)'s obey the following algebra:

\[
\{Q, Q\}_{\text{emb}} = \{Q, \bar{Q}\}_{\text{emb}} = \{\bar{Q}, \bar{Q}\}_{\text{emb}} = 0 \tag{6.13}
\]

\[
i \{Q_g, Q\}_{\text{emb}} = Q, \quad i \{Q_g, \bar{Q}\}_{\text{emb}} = -\bar{Q}
\]

\[
i \{K, Q\}_{\text{emb}} = 0, \quad i \{K, \bar{Q}\}_{\text{emb}} = \bar{Q}
\]

\[
i \{\bar{K}, Q\}_{\text{emb}} = \bar{Q}, \quad i \{\bar{K}, \bar{Q}\}_{\text{emb}} = 0 \tag{6.14}
\]

\[
i \{Q_g, K\}_{\text{emb}} = 2K, \quad i \{Q_g, \bar{K}\}_{\text{emb}} = -2\bar{K}
\]

\[
i \{K, \bar{K}\}_{\text{emb}} = Q_g + N
\]

This is the same \(ISp(2)\) algebra as in the classical case\(^{[11]}\) except for the last equation which was \(i \{K, \bar{K}\}_{\text{epb}} = Q_g\) there. The new term \(+N\) is easily removed reordering the \(c\)'s in the ”ghost-charge” \(Q_g\). We replace \(Q_g = c^a \bar{c}_a \equiv c^a * e \bar{c}_a - 2N\) by the symmetric combination

\[
\tilde{Q}_g = \frac{1}{2} \left( c^a * e \bar{c}_a - \bar{c}_a * e c^a \right) = Q_g + N \tag{6.15}
\]

so that the \(+N\) disappears from the last equation without changing the other ones.

Comparing the Hamiltonian (6.8) to its classical limit given in eqs.(3.16), (3.17), we observe that the only effect of the quantum deformation consists of replacing the classical Hamiltonian \(H(\phi)\) by its ”quantum lift”

\[
H^h(\lambda, \phi) = \mathcal{F}(\lambda \omega \partial) (\lambda \omega \partial) H(\phi) \tag{6.16}
\]

where \(\mathcal{F}\) is given by (6.9). In fact, we can re-write (6.8) as

\[
\tilde{H}^h = \lambda_a h^a(\lambda, \phi) + i\bar{c}_a \partial_b h^a(\lambda, \phi) c^b
\]

with the following deformed components of the hamiltonian vector field

\[
h^a(\lambda, \phi) = \omega^{ab} \partial_b H(\lambda, \phi) \tag{6.18}
\]

We refer to \(H^h(\lambda, \phi)\) as the quantum lift of \(H(\phi)\) because, contrary to \(H\), it depends also
on $\lambda$ so it is lifted from the $\text{Fun}(\mathcal{M}_{2N})$ to $\text{Fun}(\mathcal{M}_{4N})$. In our interpretation the bosonic subspace $\mathcal{M}_{4N} \equiv \{(\lambda, \phi)\}$ of $\mathcal{M}_{8N}$ is identified with the tangent bundle $T\mathcal{M}_{2N}$ over standard phase-space. Thus the deformation lifts the Hamiltonian from a function on $\mathcal{M}_{2N}$ to a function on $T\mathcal{M}_{2N}$. Similarly, $h^a \partial_a$ is lifted to a vector field $h^a \partial_a$ on $\mathcal{M}_{4N}$. This vector field is horizontal in the sense that a generic vector field on $\mathcal{M}_{4N}$ has the structure

$$v = v_1^a \frac{\partial}{\partial \phi^a} + v_2^a \frac{\partial}{\partial \lambda_a}$$

but the $\frac{\partial}{\partial \lambda_a}$-piece is missing in $h^a \partial_a$.

**Eq. (6.16)** can be rewritten in a rather intriguing manner. Exploiting the identity

$$\frac{\sinh(hx)}{(hx)} = \frac{1}{2} \int_{-1}^{1} ds \exp[-hx]$$

and the fact that $\exp[-h(\lambda \omega \partial)]s$ is a shift operator on $\mathcal{M}_{2N}$, we arrive at

$$H_h(\lambda, \phi) = \frac{1}{2} \int_{-1}^{1} ds \left( H(\phi^a + h^a \omega \lambda_b s) \right)$$

We see that the deformed Hamiltonian $H_h$ is a kind of ”average” of $H$, which is performed along a straight line centered at $\phi$ and connecting $(\phi^a - h^a \omega \lambda_b)$ to $(\phi^a + h^a \omega \lambda_b)$, i.e., along a ”link” of the lattice mentioned earlier. As the length of the link is proportional to $h$, the semiclassical limit has an obvious geometrical interpretation in this language: for $h \to 0$ the averaging is over very short line segments so that $H_h(\lambda, \phi) \approx H(\phi)$, and the dependence on $\lambda$ disappears. Conversely, in this framework it is easy to understand where the non-local features of quantum mechanics come from: as we turn on $h$, the relevant Hamiltonian is not $H(\phi)$ anymore, but $H_h(\lambda, \phi)$ which is a ”smeread” version of $H(\phi)$. It is remarkable that this smearing is performed along a one-dimensional line only, and that the orientation of this line is determined by $\lambda_a$ which, as we mentioned before, plays a role similar to the response field in statistical mechanics.

We remark that also the deformed super-Hamiltonian is a pure BRS variation. It is in fact easy to show that

$$\tilde{H}^h = \{Q, h^a \tilde{c}_a\}_{\text{emb}} \equiv \{Q, \tilde{h}_h\}_{\text{emb}}$$

with $h^a_h$ given by (6.18). This suggests that one might obtain a one-dimensional topological field theory once $\tilde{H}^h$ is inserted into a path-integral with BRS invariant boundary conditions on the line of what we did in the classical case.
Having added a Grassmannian piece to $\widetilde{\mathcal{H}}_B^h$, we have to make sure that the flow induced by $\widetilde{\mathcal{H}}_B^h$ leaves invariant the $\lambda = 0$-hypersurface which we identified with the standard phase-space $\mathcal{M}_{2n}$. The equations of motion, $\frac{d}{dt} A = \{ A, \widetilde{\mathcal{H}}_B^h \}_{emb}$, for $A = \phi^a, \lambda_a, c^a, \bar{c}_a$, respectively, read

\[ \dot{\phi}^a = \frac{1}{2} \left[ h^a(\phi + \hbar \omega \lambda) + h^a(\phi - \hbar \omega \lambda) \right] \]

\[ + \frac{i}{2} h_{ab}^{cd} \omega^{cd} \int_{-1}^{+1} ds \, \partial_b \partial_c \partial_e H(\phi + \hbar \omega \lambda) \, \bar{c}_d c^e \]  

(6.23)

\[ \dot{\lambda}_a = \frac{1}{2\hbar} \left[ \partial_a H(\phi + \hbar \omega \lambda) - \partial_a H(\phi - \hbar \omega \lambda) \right] \]

\[ - \frac{i}{2} \int_{-1}^{+1} ds \, \partial_a \partial_b h^c(\phi + \hbar \omega \lambda) \, \bar{c}_c \, c^b \]  

(6.24)

\[ \dot{c}^a = \frac{1}{2} \int_{-1}^{+1} ds \, \partial_b h^a(\phi + \hbar \omega \lambda) c^b \]  

(6.25)

\[ \dot{\bar{c}}_a = -\frac{1}{2} \int_{-1}^{+1} ds \, \partial_a h^b(\phi + \hbar \omega \lambda) \, \bar{c}_b \]  

(6.26)

$\lambda_a \equiv 0$ is not in general a solution of the coupled set of equations because of the Grassmannian piece on the RHS of (6.24); the bilinear $\bar{c}c$ acts as a source for $\lambda$. In order to eliminate this term, we supplement the "constraint" $\lambda = 0$ by its Grassmannian counterpart $\bar{c}_a = 0$. It is easy to see that $\lambda_a(t) = 0$ and $\bar{c}_a(t) = 0$ is always a solution of the above equations of motion, and that in this case the equation for $\phi^a$ and $c^a$ reduce to Hamilton’s and Jacobi’s equation, respectively:

\[ \dot{\phi}^a = h^a(\phi) \]

\[ \dot{c}^a = \partial_b h^a(\phi) \, c^b \]  

(6.27)

We conclude that the hypersurface consisting of the points $(\phi, \lambda = 0, c, \bar{c} = 0) \in \mathcal{M}_{8N}$ is preserved under the hamiltonian flow, and that the symbol $\phi^a(t)$ correctly follows the Heisenberg dynamics of $\hat{\phi}^a(t)$. The new feature is the symbol $c^a(t)$ parametrizing nearby $\phi^a$-trajectories. Stated differently, the function $c^a(t)$ is an element of the tangent space to the space of bosonic paths $\phi^a(t)$.
7. THE DEFORMED EXTERIOR CALCULUS

In this section we apply the extended Moyal deformation outlined in sections 4, 5 and 6 to the hamiltonian formulation of the exterior calculus which we introduced in section 3. Even in the deformed case, the charges \( \Omega \equiv Q, \bar{Q}, Q_g, K, \bar{K} \) have vanishing \( \text{em} \)-brackets with \( \widehat{H}^h \) and form a closed \( ISp(2) \) algebra. This enables us to set up a deformed exterior calculus by paralleling the classical construction.

The ”hat map” is defined as in the classical case. It replaces \( \partial_a \) and \( d\phi^a \) by \( \bar{c}_a \) and \( c^a \), and it maps tensors of the type (3.9) to functions \( \hat{T} \in \text{Fun}(\mathcal{M}_{8N}) \) as given in (3.10). It will be helpful to rewrite eq. (3.10) as

\[
\hat{T} = T_{a_1 \cdots a_q}^b(\phi) \bar{c}_{b_1} \star \cdots \star \bar{c}_{b_q} \star c^{a_1} \star \cdots \star c^{a_p}
\]

(7.1)

where (6.3) has been used. The algebra of functions \( \text{Fun}(\mathcal{M}_{8N}) \) is equipped with the star product \( \star \) or, equivalently, the extended Moyal bracket. The operations of the deformed tensor calculus will be implemented by \( \text{em} \)-brackets between elements of \( \text{Fun}(\mathcal{M}_{8N}) \) representing tensors on \( \mathcal{M}_{8N} \) and the \( ISp(2) \) generators. In order to describe the deformed calculus, we consider the special tensors \( v, \alpha, F^{(p)} \) and \( V^{(p)} \) defined in eq. (3.20). We define a quantum exterior derivative of \( p \)-forms \( F^{(p)} \) as the extended Moyal bracket with the BRS charge \( Q \)

\[
(df)^\wedge \equiv i\{Q, \hat{F}^{(p)}\}_{\text{emb}}
\]

(7.2)

In particular, for zero forms

\[
(df)^\wedge \equiv i\{Q, f(\phi)\}_{\text{emb}} = \partial_a f(\phi)c^a
\]

(7.3)

Because \( Q = ic^a\lambda_a \equiv ic^a \star_e \lambda_a \) is quadratic there is no difference between \( \{Q, \cdot\}_{\text{emb}} \) and the classical \( \{Q, \cdot\}_{\text{epb}} \). However, we can extend the definition of ”\( d \)” to any \( \hat{T} \in \text{Fun}(\mathcal{M}_{8N}) \) of the form (7.1) with the coefficients possibly depending also on \( \lambda \):

\[
d\hat{T} = i\{Q, \hat{T}\}_{\text{emb}}
\]

(7.4)

Even through this formula looks classical, there is an important difference with respect to the Leibniz rule obeyed by the deformed ”\( d \)”: it is a graded derivation on the algebra \( (\text{Fun}(\mathcal{M}_{8N}), \star_e) \) instead of the classical \( (\text{Fun}(\mathcal{M}_{8N}), \cdot) \). Eq. (4.3) implies that

\[
\{Q, \hat{T}_1 \star_e \hat{T}_2\}_{\text{emb}} = \{Q, \hat{T}_1\}_{\text{emb}} \star_e \hat{T}_2 + (-1)^{[\hat{T}_1]}\hat{T}_1 \star_e \{Q, \hat{T}_2\}_{\text{emb}}
\]

(7.5)

There is no simple way of expressing the exterior derivative of the pointwise product \( \hat{T}_1 \cdot \hat{T}_2 \).
\(\hat{T}_2\), which, in this context, is a quite unnatural object. It is easy to see that the relation \(\{Q, Q\} = 0\) implies the nilpotency of \(d\), i.e., \(d^2 = 0\).

Similar remarks apply to the exterior coderivative
\[
(dV(p))^\wedge = i\{Q, \hat{V}(p)\}_{emb} \quad (7.6)
\]
On zero-forms it acts as \(\bar{d}f = (df)^\sharp = \omega^{ab}\partial_b f\partial_a\), like in the classical case. Note that, as a consequence of \(\{Q, \bar{Q}\}_{emb} = 0\), we have \(dd + \bar{d}d = 0\). (This is different from Riemannian geometry, where the corresponding anticommutator yields the Laplace-Beltrami operator.)

Further graded derivations obeying a Leibniz rule similar to (7.5) include the interior products of p-forms and p-vectors with vectors and 1-forms, respectively:
\[
\begin{align*}
(i(v)F(p))^\wedge &= i\{\hat{\nu}, \bar{F}(p)\}_{emb} \\
(i(\alpha)\nu(p))^\wedge &= i\{\hat{\alpha}, \bar{\nu}(p)\}_{emb}
\end{align*}
\]
They, too, can be extended to any \(\hat{T} \in Fun(M_{8N})\). The maps relating vectors \(v\) to 1-forms \(\nu^\flat\) and 1-forms \(\alpha\) to vectors \(\alpha^\sharp\) are realized as the brackets with \(K\) and \(\bar{K}\) again:
\[
\begin{align*}
(v^\flat)^\wedge &= i\{K, \hat{\nu}\}_{emb} \\
(\alpha^\sharp)^\wedge &= i\{\bar{K}, \hat{\alpha}\}_{emb}
\end{align*}
\]
Finally we turn to the quantum Lie derivative \(L_h\) along the hamiltonian vector field. Choosing a fermionic Hamiltonian \(\tilde{H}_F^h\) amounts to deciding for a specific form of the Lie derivative. For any \(\hat{T} \in Fun(M_{8N})\) we define
\[
L_h\hat{T} = -\{\tilde{H}_F^h, \hat{T}\}_{emb} \quad (7.9)
\]
If \(\hat{T}\) represents a tensor of the form (7.1) its Lie derivative, considered as an operation acting on tensors on \(M_{2N}\), is obtained by projecting (7.9) on the \(\lambda = 0\)-surface
\[
(L_h\hat{T})^\wedge = -P\{\tilde{H}_F^h, \hat{T}\}_{emb} \quad (7.10)
\]
Clearly \(L_h\) of (7.9) obeys
\[
L_h(\hat{T}_1 *_e \hat{T}_2) = (L_h\hat{T}_1) *_e \hat{T}_2 + \hat{T}_1 *_e (L_h\hat{T}_2)
\]
but the projected quantity (7.10) has no simple composition properties anymore. We can
show that (7.9), without projection to \( \lambda = 0 \), obeys

\[
L_h = di(h_h) + i(h_h)d
\]

which implies that \( dL_h = L_hd \). The proof of (7.11) makes use of (7.2), (7.7), (6.22) and the Jacobi identity for the \( \text{em} \)-bracket:

\[
di(h_h)\hat{T} + i(h_h)d\hat{T} = i\{Q, i\{\hat{h}_h, \hat{T}\}_\text{emb}\}_\text{emb} \\
+ i\{\hat{h}_h, i\{Q, \hat{T}\}_\text{emb}\}_\text{emb} \\
= - \{Q, \hat{h}_h\}_\text{emb}, \hat{T}\}_\text{emb} \\
= - \{\hat{H}_h, \hat{T}\}_\text{emb} = L_h\hat{T}
\]

Eq. (7.11) suggests the introduction of the operation

\[
I(v) \equiv i(v_h)
\]

consisting of the lift \( v \mapsto v_h \) followed by the contraction "i". Here \( v = v^a(\phi)\partial_a \) is any (not necessarily hamiltonian) vector field and

\[
v_h(\lambda, \phi) = \frac{F(\lambda \omega \partial)}{\lambda \omega \partial} v^a(\phi)
\]

is its quantum lift. Then, for arbitrary vector fields,

\[
L_v = dI(v) + I(v)d
\]

with \( L_v \) defined by (7.9) where \( \hat{H}_h \) is given by (6.17) with \( h^q_h \) replaced by \( v^q_h \). In terms of \( \text{em} \)-brackets we have

\[
I(v)\hat{T} = i\left\{\frac{F(\lambda \omega \partial)}{\lambda \omega \partial} v^a(\phi)\right\}_\text{emb}
\]

As an example we present the contraction of \( \hat{v} \) with \( \hat{F}(p) \) as defined in eq. (3.20). Using eq. (4.4) it is easy to show that for any \( \lambda \)

\[
I_v\hat{F}(p) = \frac{1}{(p-1)!}G([\lambda a - \frac{1}{2} \partial_a]\omega^{ab}) v^a(\phi_1) F_{a_1 \ldots a_p}(\phi_2)_{\phi_1,2=\phi} c^{a_2} \ldots c^{a_p}
\]

where

\[
G(x) \equiv \frac{F(x)}{x} \equiv \frac{\sinh(hx)}{(hx)}
\]

approaches unity in the classical limit so that \( I_v \rightarrow i_v \). For the special case of a hamiltonian
vector field $v^a = \omega^{ab} \partial_b G \equiv (dG)^a$ contracted with a gradient $F_a^{(1)} = \partial_a f = (df)_a$, the $\lambda = 0$-projection of (7.17) can be expressed as a standard Moyal bracket:

$$\mathcal{P} I ((dG)^a) \hat{d} f = \{ f, G \}_{mb}$$ (7.19)

In this case the $(hx)^{-1}$-piece from $\mathcal{G}$ cancels the derivatives in $v^a$ and $F_a^{(1)}$ and we are left with $\sin \left( \frac{\hbar}{2} \partial_a \omega^{ab} \partial_b \right)$ leading to a normal Moyal bracket. Hence Moyal’s equation of motion for the time-evolution of pseudodensities may be re-written as

$$-\partial_t \varrho = \{ \varrho, H \}_{mb} = \mathcal{P} I (h) \hat{d} \varrho$$ (7.20)

which generalizes the classical result$^{[11,14]}$

$$-\partial_t \varrho = \{ \varrho, H \}_{pb} = i(h) \hat{d} \varrho$$ (7.21)

We stress once more that the RHS of eq. (7.21) is automatically independent of $\lambda$ but not so the RHS of eq. (7.20). Using eq. (4.4) we can convince ourselves that eq. (7.20) is equivalent to

$$\{ \varrho, H \}_{mb} = \mathcal{P} h^a h^b (\lambda, \phi) \ast_e \partial_a \varrho (\phi)$$ (7.22)

Comparing this to the classical formula $\{ \varrho, H \}_{pb} = h^a \partial_a \varrho$ one might be tempted to consider the pseudodifferential operator

$$D(h) \equiv h^a h^b (\lambda, \phi) \partial_a \ast_e$$ (7.23)

acting on $Fun(\mathcal{M}_{4N})$, as the deformed version of the first order operator $h^a \partial_a$ to which $D(h)$ reduces in the classical limit. However, this interpretation is unnatural for the following reason. The classical vector fields $h^a \partial_a$ form a closed algebra (5.28), but the algebra of the $D$’s does not close. For two hamiltonian vector fields $h_{1,2}^a = \omega^{ab} \partial_b G_{1,2}$ one obtains

$$[D(h_1), D(h_2)] = D(h_3) + i \{ h_{1,2}^a, h_{2,3}^b \}_{emb} \ast_e \partial_a \partial_b$$ (7.24)

with

$$h_{3,4}^a \equiv h_{1,4}^a \ast_e \partial_b h_{2,4}^b - h_{2,4}^b \ast_e \partial_b h_{1,4}^a$$ (7.25)

The first term on the RHS of (7.24) resembles the familiar Lie bracket, but the second one is new and spoils the closure of the algebra. It vanishes in the limit $\hbar \to 0$. When we try to
re-express the \( em \)-bracket of the associated Hamiltonian \( \tilde{\mathcal{H}}^h_{\mathcal{B}0}[G_{1,2}] = h^a_{1,2;h} \) in terms of \( h^a_{1,2;h} \), we obtain a formula similar to (7.24) suffering from the same problem:

\[
\{ \tilde{\mathcal{H}}^h_{\mathcal{B}0}[G_1], \tilde{\mathcal{H}}^h_{\mathcal{B}0}[G_2] \}_{\text{emb}} = -h^a_{3,h} \lambda_a + \{ h^a_{1,h}, h^b_{2,h} \} \ast_e \lambda_a \lambda_b
\]  

(7.26)

In Section 5 we have seen that the algebra of the \( \tilde{\mathcal{H}}^h_{\mathcal{B}} \)'s does indeed close. Eq. (5.32) can be re-written as

\[
\{ h^a_{1,h}, h^b_{2,h} \}_{\text{emb}} = v^a_{12} \lambda_a
\]

(7.27)

where the new vector field \( v^a_{12} \) is the quantum lift of \( \omega^{ab} \partial_b \{ G_1, G_2 \}_{mb} \),

\[
v^a_{12} = \frac{\mathcal{F}(\lambda \omega \partial)}{(\lambda \omega \partial)} \left[ \{ h^a_{1}, G_2 \}_{mb}^{(2)} + \{ G_1, h^a_{2} \}_{mb}^{(2)} \right]
\]

(7.28)

Composing two consecutive canonical transformations by the rule (7.28) leads to a closed algebra, but using eq. (7.25) it does not. The lesson to be learned from this is that, in the deformed case, it is unnatural to decompose the hamiltonian vector field as the product of the "components" \( h^a_{h} \) with "basis elements" \( \partial_a \) or \( \lambda_a \). As \( h^a_{h}(\lambda, \phi) \) contains arbitrary powers of \( \lambda \), it is meaningless to separate off a single factor of \( \lambda \). (Clearly the situation is different in the classical case where \( \tilde{\mathcal{H}}^h_B = h^a \lambda_a \) is always linear in \( \lambda \).) In quantum mechanics it seems appropriate to call \( \tilde{\mathcal{H}}^h_B \) as a whole the "hamiltonian vector field". Generally speaking it is a complicated function on the tangent bundle over \( \mathcal{M}_{2N} \); only in the classical limit it becomes a vector field on \( \mathcal{M}_{2N} \). In more physical terms, the higher powers of \( \lambda_a \) or \( \partial_a \) are an expression of the nonlocal nature of quantum mechanics.

As an example we give the explicit form of the quantum Lie derivative for tensors of the form (7.1). The \( em \)-bracket of \( \tilde{T} \) with the super-Hamiltonian is given by

\[
\{ \tilde{\mathcal{H}}^h, \tilde{T} \}_{\text{emb}} = \left( i \left[ \mathcal{F}(\lambda + \frac{i}{2} \partial) \omega \frac{\partial}{\partial \omega} - \mathcal{F}(\lambda - \frac{i}{2} \partial) \omega \frac{\partial}{\partial \omega} \right] \tilde{T}_{a_1 \cdots a_p}(\phi_1) \tilde{H}(\phi_2) \right.

+ \sum_{j=1}^{q} \left[ G(\lambda \omega \frac{\partial}{\partial \omega} + \frac{i}{2} \partial \omega \frac{\partial}{\partial \omega}) \tilde{T}_{b_1 \cdots b_{j-1} c_{j+1} \cdots b_q}(\phi_1) \tilde{\lambda}^2 \tilde{h}^{b_j}(\phi_2) \right.

- \sum_{j=1}^{p} \left[ G(\lambda \omega \frac{\partial}{\partial \omega} - \frac{i}{2} \partial \omega \frac{\partial}{\partial \omega}) \tilde{T}_{b_1 \cdots b_q}(\phi_1) \tilde{\lambda}^2 \tilde{h}^{b_j}(\phi_2) \right.

+ \left. \left[ G(\lambda \omega \frac{\partial}{\partial \omega} - \frac{i}{2} \partial \omega \frac{\partial}{\partial \omega}) - G(\lambda \omega \frac{\partial}{\partial \omega} + \frac{i}{2} \partial \omega \frac{\partial}{\partial \omega}) \right] \cdot \tilde{T}_{a_1 \cdots a_p}(\phi_1) \tilde{\partial}_a \tilde{h}^{c_0}(\phi_2) \right]

\cdot \tilde{c}_{b_1} \cdots \tilde{c}_{b_q} \tilde{c}^{a_1} \cdots \tilde{c}^{a_p}

\]

(7.29)

with \( \lambda \omega \partial \equiv \lambda_a \omega^{ab} \partial_b \), etc., and \( \mathcal{F} \) and \( \mathcal{G} \) defined in (6.9) and (7.18), respectively. Some
details of the calculations are given in appendix C. The point to be noticed is that, because of the term involving $\bar{c}_e c f$, the bracket with $\hat{\mathcal{H}}^b$ maps a $(p,q)$-tensor to the sum of a $(p,q)$ and a $(p+1,q+1)$-tensor. Upon projection on the $\lambda = 0$ surface the $(p+1,q+1)$-piece vanishes. In fact, putting $\lambda = 0$, the two terms inside the square brackets of the last term on the RHS of (7.29) cancel because $\mathcal{G}$ is an even function. (This would not be the case for the Hamiltonians $\bar{\mathcal{H}}^b B$.) Since the tensor structure of $\mathcal{P}\{\hat{\mathcal{H}}^b, \hat{T}\}_{\text{emb}}$ matches that of $\hat{T}$, we may strip off the $\bar{c}$’s and the $c$’s. This leaves us with

$$L_h T_{a_1 \ldots a_p}^{b_1 \ldots b_q} (\phi) = \{T_{a_1 \ldots a_p}^{b_1 \ldots b_q} (\phi), H\}_{mb} - \sum_{j=1}^{q} \sin \left[ \frac{\hbar}{2} \frac{1}{\partial} \frac{\partial}{\partial \phi^2} \right] T_{a_1 \ldots a_p}^{b_j \ldots b_q} (\phi) \frac{2}{\partial_\phi} h^b_j(\phi_2) |_{\phi_1,2=\phi} \right] + \sum_{j=1}^{p} \frac{\sin \left[ \frac{\hbar}{2} \frac{1}{\partial} \frac{\partial}{\partial \phi^2} \right]}{\partial_\phi} T_{a_1 \ldots a_p}^{b_1 \ldots b_q} (\phi) \frac{2}{\partial_\phi} h^c_j(\phi_2) |_{\phi_1,2=\phi} \right] (7.30)$$

The index structure in (7.30) is the same as in the classical case but the tensor components are multiplied by $\partial_\phi h^b$ by means of a new nonlocal product. In particular for a 1-form $\varrho_a$ one finds

$$L_h \varrho = \{\varrho_a, H\}_{mb} + \frac{\sin \left[ \frac{\hbar}{2} \frac{1}{\partial} \frac{\partial}{\partial \phi^2} \right]}{\partial_\phi} \varrho_a h^b(\phi_1) \frac{2}{\partial_\phi} h^b(\phi_2) |_{\phi_1,2=\phi} \right] (7.31)$$

Taking the partial derivative on both sides of eq. (2.15) and comparing to (7.31) shows that the derivative of zero-forms, $\varrho_a \equiv \partial_\phi \varrho$, evolves as a 1-form, exactly as it happened in classical mechanics. That would not be the case for any other bosonic super-Hamiltonian different from the symmetric one $\bar{\mathcal{H}}^b B_0$. This is a further reason to choose this form for the quantum hamiltonian vector field.

Finally we have to ask in which sense the quantum Lie derivatives form a closed algebra. In our approach the action of $L_h$ on tensor-fields is represented as the Moyal bracket of a certain super-Hamiltonian with those tensors; therefore the algebra of the $L_h$’s closes by construction on the space of all generating functions on $\mathcal{M}_{8N}$. However, restricting the generating functions to the $\hat{\mathcal{H}}^b$-type, there is no guarantee that the algebra still closes. In eq. (5.32) we have seen that the $\text{emb}$-algebra of the bosonic parts $\bar{\mathcal{H}}^b_B$ closes nevertheless, and it is also known \cite{11} that for $\hbar = 0$ the $\text{ebp}$-bracket algebra of the full $\mathcal{H} = \bar{\mathcal{H}}_B + \bar{\mathcal{H}}_F$ is closed as well. It turns out that for $\hbar \neq 0$ the algebra does not close on the space of the super-Hamiltonians $\hat{\mathcal{H}}^h$, but only on a slightly larger one. To see this, let us try to generalize
by adding the fermionic piece $\tilde{\mathcal{H}}^b_F$ and let us look at the term $\{\tilde{\mathcal{H}}^b_F[G_1], \tilde{\mathcal{H}}^b_F[G_2]\}_{\text{emb}}$. It contains a term with four ghosts

$$-\tilde{c}_a \tilde{c}_d c^e \{\partial_h h^a_1, \partial_c h^d_2\}_{\text{emb}}$$

(7.32)

which prevents the algebra from closing on another $\tilde{\mathcal{H}}^b$. The term (7.32) vanishes in the classical limit and also for $\lambda_a = 0$. But, of course, in order to make sure that the transformations $L_h \tilde{T} = \{\tilde{T}, \tilde{\mathcal{H}}^h\}_{\text{emb}}$ close, we are not allowed to set $\lambda = 0$, $\tilde{c} = 0$ before having acted on $\tilde{T}$. Let us denote by $\mathcal{C}$ the subspace of $\text{Fun}(\mathcal{M}_{8N})$ consisting of the functions $\tilde{\Gamma}$ which can be obtained by taking repeated $em$-brackets of the super-Hamiltonians $\tilde{\mathcal{H}}^h$. Then the algebra of the (generalized) Lie derivatives $L(\tilde{\Gamma}) = \{\cdot, \tilde{\Gamma}\}_{\text{emb}}$ closes trivially on the enlarged space $\mathcal{C}$, which contains the old super-Hamiltonians as well as new types of functions with more than one $\bar{c}c$-pair. It seems quite natural to call the transformation $L(\tilde{\Gamma}) = \{\cdot, \tilde{\Gamma}\}_{\text{emb}}$ for any $\tilde{\Gamma} \in \mathcal{C}$, a quantum canonical transformation, even though there are many more $\tilde{\Gamma}$’s than classical generating functions $G \in \text{Fun}(\mathcal{M}_{2N})$. In fact, every $\tilde{\Gamma} \in \mathcal{C}$ is of the form

$$\tilde{\Gamma} = \tilde{\mathcal{H}}^h[G] + \Delta \tilde{\Gamma} \ , \ \Delta \tilde{\Gamma} = O(h)$$

(7.33)

i.e., for $h \to 0$ each $\tilde{\Gamma}$ equals a conventional classical super-Hamiltonian for some $G \in \text{Fun}(\mathcal{M}_{2N})$, but there can be many $\tilde{\Gamma}$’s which have the same classical limit. In the classical case a (symplectic) diffeomorphism is equivalent to a vector field, and two consecutive transformations are again equivalent to a vector field, the Lie bracket of the original ones. In the deformed case the product of two canonical transformations (in the ”narrow” sense of $\tilde{\mathcal{H}}^h$) can be something more general than a vector field. In fact, applying a term like (7.32) to a tensor $\tilde{T}$ leads (after projection) to tensors of the type

$$T^b_{cd} \cdots a_p(\phi) \star e \{\partial_{a_1} h^c_{1,h}, \partial_{a_2} h^d_{2,h}\}_{\text{emb}}|_{\lambda=0}$$

(7.34)

This is a contribution to $L(\tilde{\Gamma})\tilde{T}$ which cannot be parametrized by a single vector field. It should be compared with the RHS of (7.30). The novel feature is that we can act with $\partial_{a} h^b_{h}$ on more than one tensor index. In general the $\Delta \tilde{\Gamma}$-terms induce rather complicated terms at the level of tensor components and we shall give no explicit formulas here. It has to be remarked, however, that the functions $\tilde{\Gamma}$, though more complicated than $\tilde{\mathcal{H}}^h$, still have a very particular form, since they are invariant under the full $ISp(2)$ group: $\{\tilde{\Gamma}, \Omega\}_{\text{emb}} = 0$. This follows from the fact that the $\tilde{\Gamma}$’s are obtained as $em$-brackets of $\tilde{\mathcal{H}}^h$’s.
Classically tensors are representations of the group of diffeomorphisms on $\mathcal{M}_{2N}$ which is essentially the same as $\text{Fun}(\mathcal{M}_{2N})$ in the symplectic case. In our approach "quantum $p$-forms" are representations of the algebra generated by $\{\tilde{\Gamma}, \cdot \}$, which is larger than $\text{Fun}(\mathcal{M}_{2N})$. Therefore specifying a classical canonical transformation does not uniquely fix a quantum canonical transformation for $p \geq 1$. We encounter a kind of "holonomy effect" on $\mathcal{C}$. What we mean is that we can associate to $G \in \text{Fun}(\mathcal{M}_{2N})$ directly a $\tilde{\mathcal{H}}_F^\hbar[G] \equiv \tilde{\Gamma}$ or we can reach a quantum transformation associated to the same classical $G$ via some intermediate transformations $G_1$ and $G_2$, and the $\tilde{\Gamma}$ we shall obtain will be different from the previous one. This "holonomy effect" might be at the heart of QM irrespective of the quantum-tensor calculus we choose. It may be that this overall attempt to understand the geometry of QM, by defining quantum-forms and similar structures, sheds some light on the old problem of the non-local nature of quantum mechanics. Work is in progress on this issue especially in the direction of getting a $\tilde{\mathcal{H}}_F^\hbar$ from more physical requirements.

Table 2 summarizes the $emb$-representations of the various quantum-tensor manipulations and it should be compared with table 1 where the classical-tensor manipulations were summarized. In view of the above it should be kept in mind that there are more general quantum Lie derivatives than the $L_h$ displayed in the table.

8. CONCLUSIONS

In this paper we have made a proposal for a quantum exterior calculus which stays as close as possible to the classical one. We have been able to find a calculus in which the pointwise product of functions was deformed, but the wedge-product was left unaltered. The quantum deformed hamiltonian vector field and Lie derivative are, in some respect, surprisingly similar to their classical counterparts, the only difference being the non-locality creeping in through non-local star products and "lattice" quantities. Moreover some extra variables, $\lambda, c, \bar{c}$, appeared which were needed in order to unfold the geometrical properties of quantum mechanics. A crucial role is played by the auxiliary variable $\lambda_a$. Though formally equivalent to the response field of statistical mechanics, in quantum mechanics its physics seems to be much more involved. In particular it seems to be at the heart of the "foamy" structure of quantum phase-space: it partitions phase-space into Planck cells of size $\Delta p \Delta q \sim \hbar$.

Leaving aside the interesting mathematical properties of the new variables $\lambda_a, c^a$, and $\bar{c}_a$, it is tempting to speculate about their possible role in physics. It is intriguing that
the theory might have a consistent interpretation even away from the $\lambda = 0$-surface. One could envisage a situation in which, allowing for $\lambda \neq 0$, smoothens out certain (space-time) singularities present for $\lambda = 0$. It seems very likely that this would allow for an evolution of pure states into mixed states as it might possibly occur in the late stage of black hole evaporation\textsuperscript{[27]}. It is also conceivable that the inclusion of the fermionic sector improves the renormalizability properties of some models.

An important step towards a physical understanding of the extended theory presented here would be a reformulation of the formalism in terms of wave-functions and Hilbert spaces. To achieve that it is probably mandatory to choose the fermionic sector (i.e., $\tilde{\mathcal{H}}^F$) in such a way that the holomorphic decomposition is manifest also at the level of $c$ and $\bar{c}$. In our present formulation this is not yet the case. A related problem is that the universal supersymmetry (SUSY) present in the classical case\textsuperscript{[11,19]}, contrary to the $ISp(2)$, does not seem to survive the quantum deformation. We have shown in ref.[19] how that classical SUSY was strictly related to the concept of classical ergodicity\textsuperscript{[28]} and how it could nicely reproduce the classical KMS conditions\textsuperscript{[29]}. It would be interesting to find (possibly for a different $\tilde{\mathcal{H}}^F$) a deformed SUSY which could be used to study quantum ergodicity\textsuperscript{[30]} and the quantum KMS conditions.

In conclusion we can say that in our approach a tensor calculus is selected by choosing a specific quantum-Lie-derivative (or super-Hamiltonian). Its bosonic part, $\mathcal{H}_B$, is essentially unique and represents Moyal’s deformed hamiltonian vector field. For the fermionic part $\tilde{\mathcal{H}}^F$ many choices are possible a priori and future work will have to show their physical relevance, and their relation to other approaches.\textsuperscript{[31]}

\underline{Acknowledgements:} This research has been supported in part by grants from INFN, MURST and NATO. M.R. acknowledges the hospitality of the Dipartimento di Fisica Teorica, Università di Trieste while this work was in progress.
APPENDIX A

In this appendix we solve eq. (5.3) and show that the original equation (5.2) has no solution. The RHS of eq. (5.2) reads, using (2.13),

\[\{H(\phi), \varrho(\phi)\}_{mb} = \frac{2}{\hbar} \sin \left[ \frac{\hbar}{2} \omega^{ab} \frac{1}{2} \partial_a \partial_b \right] H(\phi_1) \varrho(\phi_2)|_{\phi_{1,2}=\phi} \]

\[= \frac{2}{\hbar} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left( \frac{\hbar}{2} \right)^{2m+1} \omega^{a_1 b_1} \cdots \omega^{a_{2m+1} b_{2m+1}} \partial_{a_1} \cdots \partial_{a_{2m+1}} H(\phi) \quad (A.1)\]

In a similarly way the LHS of eq. (5.2) follows from (4.2) and (4.1):

\[\{ \tilde{H}^h_B(\lambda, \phi), \varrho(\phi) \}_{emb} = 2 \sin \left[ \frac{1}{2} \partial_{\phi_1} \partial_{\phi_2} \right] \tilde{H}^h_B(\lambda_1, \phi_1) \varrho(\phi_2)|_{\phi_{1,2}=\phi} \]

\[= -2 \sin \left[ \frac{1}{2} \partial_{\lambda_1} \partial_{\phi_2} \right] \tilde{H}^h_B(\lambda, \phi) \varrho(\tilde{\phi})|_{\tilde{\phi}=\phi} \]

\[= -2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left( \frac{1}{2} \right)^{2m+1} \partial_{\lambda_{b_1}} \cdots \partial_{\lambda_{b_{2m+1}}} \tilde{H}^h_B(\lambda, \phi) \partial_{b_1} \cdots \partial_{b_{2m+1}} \varrho(\phi) \quad (A.2)\]

As \( \varrho(\phi) \) is an arbitrary function, we can compare the coefficients of \( \partial_{b_1} \cdots \partial_{b_{2m+1}} \varrho \) in (A.2) and (A.1), so that eq. (5.2) is equivalent to the set of equations

\[( h \omega^{a_1 b_1} \partial_{b_1} ) ( h \omega^{a_2 b_2} \partial_{b_2} ) \cdots ( h \omega^{a_{2m+1} b_{2m+1}} \partial_{b_{2m+1}} ) H(\phi) = \]

\[= h \frac{\partial}{\partial \lambda_{a_1}} \frac{\partial}{\partial \lambda_{a_2}} \cdots \frac{\partial}{\partial \lambda_{a_{2m+1}}} \tilde{H}^h_B(\lambda, \phi) \quad (A.3)\]

where \( m = 0, 1, 2, \cdots \). Before studying (A.3) in general, let us check its classical limit. For \( \hbar \to 0 \) one finds

\[\frac{\partial}{\partial \lambda_a} \tilde{H}_B = \omega^{ab} \partial_b H = h^a \quad (A.4)\]

\[\frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_{2m+1}} \tilde{H}_B = 0 \quad m = 1, 2, 3, \cdots \quad (A.5)\]

The unique solution to (A.4), (A.5) is

\[\tilde{H}_B = \lambda_a h^a \quad (A.6)\]

which, as expected, coincides with (3.16).
Returning to $\hbar \neq 0$, it is quite obvious that the infinite system of equations in (A.3) has no solution. Setting $m = 0$ we find (A.4) again. Its solution (A.6) has vanishing, second, third, etc. derivatives with respect to $\lambda$. Therefore it does not solve (A.3) for $m = 1, 2, \cdots$ and for a generic Hamiltonian $H$. Hence, as discussed in section 5, we only require the weaker condition (5.3) which is equivalent to (A.3) with $\lambda$ put to zero on the RHS of (A.3) after the derivatives have been taken.

We assume that $\tilde{\mathcal{H}}_B^\hbar(\lambda, \phi)$ is analytic in $\lambda$ and make the ansatz

$$
\tilde{\mathcal{H}}_B^\hbar(\lambda, \phi) = \sum_{l=0}^{\infty} \frac{1}{l!} \Theta^{a_1 \cdots a_l}(\phi) \lambda_{a_1} \cdots \lambda_{a_l} \tag{A.7}
$$

with the symmetric coefficients

$$
\Theta^{a_1 \cdots a_l}(\phi) = \frac{\partial}{\partial \lambda_{a_1}} \cdots \frac{\partial}{\partial \lambda_{a_l}} \tilde{\mathcal{H}}_B^\hbar(\lambda, \phi)|_{\lambda=0} \tag{A.8}
$$

Inserting (A.7) into (A.3) the coefficients $\Theta^{l}(\phi)$, for even values of $l$, are left unconstrained and those with odd values $l \equiv 2m + 1$, $m = 0, 1, 2, \cdots$ are fixed to be

$$
\Theta^{a_1 \cdots a_{2m+1}}(\phi) = \frac{1}{\hbar} (\hbar \omega a_1 b_1 \partial_{b_1}) \cdots (\hbar \omega a_{2m+1} b_{2m+1} \partial_{b_{2m+1}}) H(\phi) \tag{A.9}
$$

For the choice $\Theta^{l}(\phi) = 0$ for $l$ even, (A.7) with (A.9) yields

$$
\tilde{\mathcal{H}}_B^\hbar(\lambda, \phi) = \frac{1}{\hbar} \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} (\hbar \lambda a b \partial_{b})^{2m+1} H(\phi) \tag{A.10}
$$

As the $\Theta^{l}(\phi)$’s with $l$ even are not fixed by (A.3) we are free to add to (A.10) any function which is even in $\lambda$. This leads to eq. (5.5) given in section 5.
APPENDIX B

In this appendix we show that the conditions $F_1$ and $F_2$ of section 6 imply the super-Hamiltonian of eq. (6.8). We start from the ansatz

$$\tilde{H}^h = \mathcal{F}(\lambda \omega \partial) H(\phi) + i \bar{c}_a W^a_b(\lambda, \phi) c^b \quad (B.1)$$

and we require $\tilde{H}^h$ to be BRS invariant. Applying the transformations (6.5) to (B.1) one finds

$$\delta \tilde{H}^h = \epsilon c^b \mathcal{F}(\lambda \omega \partial) \partial_b H(\phi) - \epsilon \lambda_a W^a_b(\lambda, \phi) c^b - i \epsilon \bar{c}_a \partial_c W^a_b(\lambda, \phi) c^c c^b \quad (B.2)$$

From $\delta \tilde{H}^h = 0$ it follows that

$$\partial_c W^a_b - \partial_b W^a_c = 0 \quad (B.3)$$

and

$$\mathcal{F}(\lambda \omega \partial) \partial_b H(\phi) = \lambda_a W^a_b(\lambda, \phi) \quad (B.4)$$

Assuming that $M_{2N}$ is topologically trivial, eq. (B.3) implies that

$$W^a_b(\lambda, \phi) = \partial_b W^a(\lambda, \phi) = \omega^{ac} \partial_c U^c(\lambda, \phi) \quad (B.5)$$

for some function $W^a \equiv \omega^{ac} U^c$. Inserting (B.5) into (B.4) yields

$$\left(\lambda_a \omega^{ac} \partial_c\right) \partial_b \frac{\mathcal{F}(\lambda \omega \partial)}{(\lambda \omega \partial)} H(\phi) = \lambda_a \omega^{ac} \partial_b U^c(\lambda, \phi) \quad (B.6)$$

On the LHS of this equation $\lambda$ appears only in the combination $\lambda_a \omega^{ac} \partial_c$, so the same must be true for the RHS also. This implies that $U^c(\lambda, \phi) = \partial_c U(\phi, \lambda)$. Inserting this into (B.6) we find that (up to an irrelevant constant)

$$U(\lambda, \phi) = \frac{\mathcal{F}(\lambda \omega \partial)}{(\lambda \omega \partial)} H(\phi) \quad (B.7)$$

and therefore

$$W^a_b(\lambda, \phi) = \omega^{ac} \partial_b \partial_c \frac{\mathcal{F}(\lambda \omega \partial)}{(\lambda \omega \partial)} H(\phi) \quad (B.8)$$

Eq. (B.1) with (B.8) is the result (6.8) given in section 6. Note that $W^a_a(\lambda, \phi) = 0$ so that, as in the classical case, $\hat{e}_a W^a_b \hat{c}^b$ is free from ordering ambiguities.
APPENDIX C

In this appendix we give some details of the derivation of eq. (7.29). The evaluation of \( \{\hat{\mathcal{H}}^h, \hat{T}\}_\text{emb} \), with \( \hat{T} \) defined in (7.1), proceeds by repeated application of eq. (4.3). In a first step we write

\[
\{\hat{\mathcal{H}}^h, \hat{T}\}_\text{emb} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3
\]  

(C.1)

with

\[
\mathcal{R}_1 = \{\hat{\mathcal{H}}^h_B, T^{b_1 \cdots b_q}_{a_1 \cdots a_p}\}_\text{emb} *_c \bar{c}_{b_1} \cdots \bar{c}_{b_q} c^{a_1} \cdots c^{a_p}
\]

\[
\mathcal{R}_2 = T^{b_1 \cdots b_q}_{a_1 \cdots a_p} *_c \left[ \bar{c}_{b_1} \cdots \bar{c}_{b_q} *_c \{\hat{\mathcal{H}}^h_F, c^{a_1} \cdots c^{a_p}\}_\text{emb} + \{\hat{\mathcal{H}}^h_F, \bar{c}_{b_1} \cdots \bar{c}_{b_q}\}_\text{emb} *_c c^{a_1} \cdots c^{a_p} \right]
\]

\[
\mathcal{R}_3 = \{\hat{\mathcal{H}}^h_F, T^{b_1 \cdots b_q}_{a_1 \cdots a_p}\}_\text{emb} *_c \bar{c}_{b_1} \cdots \bar{c}_{b_q} c^{a_1} \cdots c^{a_p}
\]  

(C.2)

Using (6.7) and (4.4) we obtain for the first piece

\[
\mathcal{R}_1 = i \left[ \mathcal{F}([\lambda + \frac{i}{2} \frac{1}{\partial}]\omega \frac{2}{\partial}) - \mathcal{F}([\lambda - \frac{i}{2} \frac{1}{\partial}]\omega \frac{2}{\partial}) \right] T^{b_1 \cdots b_q}_{a_1 \cdots a_p}(\phi_1) H(\phi_2)|_{\phi_{1,2}=\phi} \bar{c}_{b_1} \cdots \bar{c}_{b_q} c^{a_1} \cdots c^{a_p}
\]  

(C.3)

which is the first term on the RHS of (7.29). For \( \mathcal{R}_2 \) we need:

\[
\{\hat{\mathcal{H}}^h_F, c^{a_1} \cdots c^{a_p}\}_\text{emb} = \sum_{j=1}^p c^{a_1} \cdots c^{a_{j-1}} *_c \{\hat{\mathcal{H}}^h_F, c^{a_j}\}_\text{emb} *_c c^{a_{j+1}} \cdots c^{a_p}
\]

\[
= - \sum_{j=1}^p \partial_b h^a_{\hat{h}} c^{a_1} \cdots c^{a_{j-1}} c^b c^{a_{j+1}} \cdots c^{a_p}
\]  

(C.4)

and a similar formula for \( \bar{c}_{b_1} \cdots \bar{c}_{b_q} \). This leads to

\[
\mathcal{R}_2 = \left[ \sum_{j=1}^p T^{b_1 \cdots b_{j-1} b_{j+1} \cdots b_q}_{a_1 \cdots a_p} *_c \partial_c h^b_{\hat{h}} \right] \bar{c}_{b_1} \cdots \bar{c}_{b_q} c^{a_1} \cdots c^{a_p}
\]

\[
- \sum_{j=1}^q T^{b_1 \cdots b_{j-1} e a_{j+1} \cdots a_p}_{a_1 \cdots a_p} *_c \partial_a h^e_{\hat{h}} \bar{c}_{b_1} \cdots \bar{c}_{b_q} c^{a_1} \cdots c^{a_p}
\]  

(C.5)

Because the quantum lift \( h^a_{\hat{h}} = \mathcal{G}(\lambda \omega \partial) h^a \) depends on \( \lambda \), the star products between the tensor components and \( \partial_a h^a_{\hat{h}} \) are non-trivial. We use (4.4) again:

\[
T^{:::}(\phi) *_c \partial_a h^b_{\hat{h}}(\lambda, \phi) = \mathcal{G}(\lambda \omega \partial + \frac{i}{2} \frac{1}{\partial} \omega \frac{2}{\partial}) T^{:::}(\phi_1) \partial_a h^b_{\hat{h}}(\phi_2)|_{\phi_{1,2}=\phi}
\]  

(C.6)

Eqs. (C.5) with (C.6) yields the second and the third term on the RHS of eq. (7.29). The term \( \mathcal{R}_3 \) is a typical non-classical feature. The bracket \( \{\hat{\mathcal{H}}^h_F, T^{:::}\}_\text{emb} \) is non-zero only
because $\hbar^2$ depends on $\lambda$. In the classical limit this bracket vanishes. We have

$$ R_3 = i\tilde{c}_e c^f \{ G(\lambda \omega \partial) \partial_f h^e(\phi), T_{a_1 \cdots a_p}^{b_1 \cdots b_q}(\phi) \}_{\text{emb} \ast \tilde{c}_{b_1} \cdots \tilde{c}_{b_q} c^{a_1} \cdots c^{a_p}} $$

(C.7)

Applying eq. (4.4) this yields the last term on the RHS of eq. (7.29).

REFERENCES

1. V.Drinfeld, in: ”Proceedings of the International Congress of Mathematicians” (Berkeley), Acad.Press., Vol. 1 (1986) 798;
   M.Jimbo, Lett.Math.Phys. 11 (1986) 247

2. L.D.Faddeev, N.Y.Reshetikhin, L.A.Takhtajian, Preprint LOMI E-14-87

3. S.L.Woronowicz, Comm.Math.Phys. 122 (1989) 125
   D.Bernard, ”Quantum Lie Algebras and differential calculus on quantum groups”
   Saclay-sphl-90-124, Sept 1990.
   To appear in the proceedings of the 1990 Yukawa international seminar ”Common
   trends in mathematics and quantum field theory”, Kyoto 1990

4. A.Connes, ”Non-commutative differential geometry”, Publ.Math.IHES, 62 (1985) 41

5. Yu.I.Manin, ”Quantum groups and non-commutative geometry”,
   Montreal Univ. CRM-1561 (1988)

6. J.Wess and B.Zumino, Nucl.Phys. B (Proc.Suppl.) 18 (1990) 302;
   M.Dubois-Violette, in ”Differential Geometric Methods in Theoretical Physics”,
   C.Bertocci et al. (Eds.), Lecture-Notes in Physics, no.275, Springer, New York (1991);
   A.Dimakis and F.Müller-Hoissen, J.Phys.A 25 (1992) 5625

7. J.E.Moyal, Proc.Cambridge Phil.Soc. 45 (1949) 99

8. F.Bayen et al., Ann.of Phys. 111 (1978) 61; ibid. 111 (1978) 111

9. H.Weyl, Z.Phys.46 (1927) 1;
   E.Wigner, Phys.Rev.40 (1932) 740

10. F.A.Berezin, Sov.Phys.Usp. 23 (1980) 1981;
    L.Hörmander, Comm. Pure and Applied Math. 32 (1979) 359

11. E.Gozzi, M.Reuter, W.D.Thacker, Phys.Rev.D 40 (1989) 3363; ibid. D46 (1992) 757
12. B.DeWitt, ”Supermanifolds”, Cambridge University Press, 1984
13. M.Gerstenhaber, Ann.of Math. 79 (1964) 59
14. V.I. Arnold, ”Mathematical Methods of Classical Mechanics”, Springer, New York, 1978;
    R. Abraham and J. Marsden, ”Foundations of Mechanics”, Benjamin, New York, 1978
15. R.G. Littlejohn, Phys.Rep. 138 (1986) 193
16. H.J. Groenwald, Physica 12 (1946) 405;
    L. Van Hove, Mem. Acad. R. Belg. 26 (1951) 26
17. E. Witten, Com.Math. Phys. 117 (1988) 353; ibid. 118 (1988) 411;
    D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Phys.Rep. 209 (1991) 130
18. E. Gozzi, M. Reuter, Phys.Lett. B 240 (1990) 137
19. E. Gozzi, M. Reuter, Phys.Lett. B 233 (1989) 383; ibid. B 238 (1990) 451
20. P.C. Martin, E.D. Siggia and H.A. Rose, Phys.Rev. A 8 (1973) 423
21. E. Gozzi, Suppl. Prog. Theor. Phys. 111 (1993) 115
22. G.C. Ghirardi, A. Rimini, in ”Sixty-Two Years of Uncertainty”,
    A.I. Miller (Ed.), Plenum, New York, 1990
23. M.S. Marinov, Phys.Lett. A 153 (1991) 5
24. E. Gozzi, M. Reuter, ”Quantum deformed canonical transformations,
    \( W_\infty \)-algebras and unitary transformations”, UTS-DFT-93-11 preprint.
25. M. Blau, E. Keski-Vakkuri, A. Niemi, Phys.Lett. 246B (1990) 92
26. E. Gozzi, M. Reuter, work in progress
27. S. Hawking, Com.Math.Phys. 43 (1975) 199;
    J. Ellis et al. Nucl.Phys. B241 (1984) 381
28. V.I. Arnold and A. Avez, ”Ergodic problems of classical mechanics”,
    Benjamin, New York, 1968
29. G. Gallavotti and E. Verboven, Nuovo Cimento 28 (1975) 274
30. J. von Neumann, Z.Phys. 57 (1929) 30
31. J. Wess, B. Zumino, O. Ogievetskii, W. B. Schmidke, Munich preprint 93-192;
    J. Wess et al., Munich preprint 93-0194
|                | Classical Cartan’s Rules | $\{\cdot, \cdot\}_{\text{epb}}$-Rules |
|----------------|--------------------------|----------------------------------|
| **vector-fields** | $v = v^a \partial_a$       | $v = v^a \bar{c}_a$              |
| **1-forms**     | $\alpha = \alpha_a d\phi^a$ | $\alpha = \alpha_a c^a$          |
| **p-forms**     | $F^{(p)} = \frac{1}{p!} F_{a_1 \ldots a_p} d\phi^{a_1} \wedge \ldots \wedge d\phi^{a_p}$ | $F^{(p)} = \frac{1}{p!} F_{a_1 \ldots a_p} c^{a_1} \ldots c^{a_p}$ |
| **p-vectors**   | $V^{(p)} = \frac{1}{p!} V^{a_1 \ldots a_p} \partial_{a_1} \wedge \ldots \wedge \partial_{a_p}$ | $V^{(p)} = \frac{1}{p!} V^{a_1 \ldots a_p} \bar{c}_{a_1} \ldots \bar{c}_{a_p}$ |
| **ext. deriv.** | $dF^{(p)}$               | $i\{Q, F^{(p)}\}_{\text{epb}}$ |
| **int. product**| $i(v)F^{(p)}$             | $i\{v, F^{(p)}\}_{\text{epb}}$ |
| **Ham. vec. field** | $h = \omega^{ab} \partial_b H \partial_a$ | $h = i\omega^{ab} \partial_b H \lambda_a$ |
| **Lie-deriv.**  | $l_h = di_h + i_h d$      | $i\mathcal{H} = i\lambda_a h^a - \bar{c}_a \partial_b h^b c^b$ |
| **Lie-deriv. on $T=F^{(p)}, V^{(p)}$** | $l_h T$                   | $-\{\mathcal{H}, T\}_{\text{epb}}$ |
| Cartan’s Rules | quantum-Cartan’s Rules |
|----------------|------------------------|
| **vector-fields** | $v = v^a \partial_a$ | $\mathring{v} = v^a \mathring{c}_a$ |
| **1-forms** | $\alpha = \alpha_a d\phi^a$ | $\mathring{\alpha} = \alpha_a c^a$ |
| **forms** | $F^{(p)} = \frac{1}{p!} F_{a_1 \cdots a_p} d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$ | $\mathring{F}^{(p)} = \frac{1}{p!} F_{a_1 \cdots a_p} c^{a_1} \ldots c^{a_p}$ |
| **tensors** | $V^{(p)} = \frac{1}{p!} V^{a_1 \cdots a_p} \partial_{a_1} \wedge \cdots \wedge \partial_{a_p}$ | $\mathring{V}^{(p)} = \frac{1}{p!} V^{a_1 \cdots a_p} \mathring{c}_{a_1} \ldots \mathring{c}_{a_p}$ |
| **ext-deriv.** | $dF^{(p)}$ | $i\{Q, \mathring{F}^{(p)}\}_{emb}$ |
| **int.product** | $i(v)F^{(p)}$ | $i\{v, \mathring{F}^{(p)}\}_{emb}$ |
| **Ham-Vec-field** | $h = \omega^{ab} \partial_b H \partial_a$ | $h^a_h \equiv \frac{\sinh(h\lambda \omega \partial)}{(h\lambda \omega \partial)} h^a(\phi)$ |
| **Lie-deriv.** | $L_h = di_h + i_h d$ | $i\mathring{\mathcal{H}}^h = i\lambda_a h^a_h - \mathring{c}_a \partial_b h^a_h e^b$ |
| **Lie-deriv. on $T = F^{(p)}, V^{(p)}$** | $L_h T$ | $-\{\mathring{\mathcal{H}}^h, T\}_{emb}$ |