A local proof of the dimensional Prékopa’s theorem

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1. Introduction

Prékopa’s theorem [11] says that marginals of log-concave functions are log-concave, i.e., if \( \varphi : \mathbb{R}^{n+1} \to \mathbb{R} \) is convex, then the function \( \phi \) defined by

\[
\phi(t) = -\log \left( \int_\mathbb{R^n} e^{-\varphi(t,x)} \, dx \right)
\]

is convex on \( \mathbb{R} \). By modifying \( \varphi \) if necessary, we can replace \( \mathbb{R}^{n+1} \) by any of its open convex subsets \( \Omega \), and the integration in (1.1) is taken in the section \( \Omega(t) = \{ x \in \mathbb{R}^n : (t,x) \in \Omega \} \). Prékopa’s theorem is a direct consequence of the Prékopa–Leindler inequality which can be seen as the functional form of the Brunn–Minkowski inequality (see [6]). The Brunn–Minkowski inequality is known to be one of the most important tools in analysis and geometry. It states that if \( A, B \) are non-empty measurable subsets of \( \mathbb{R}^n \) then

\[
|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.
\]
where $A + B = \{ a + b : a \in A, b \in B \}$ and $|\cdot|$ denotes the Lebesgue measure of the measurable set (see [4,6,8,9] for the proofs and applications of the Brunn–Minkowski inequality).

A new proof of the Prékopa theorem is recently given in [1,2]. In these papers, the authors proved a local formulation for the second derivative of the function $\phi$ above. By using the convexity of $\varphi$, they show that $\phi''$ is nonnegative. This local approach was also used by D. Cordero-Erausquin (see [5]) to generalize a result of Berndtsson concerning Prékopa’s theorem for plurisubharmonic functions (see [3]).

In this paper, we adapt the local approach given in [1,2] to find an expression for the second derivative of the function $\phi$ defined by

$$
\phi(t) = \left( \int_V \varphi(t, x)^{-\beta} \, dx \right)^{-\frac{1}{\beta n}}, \quad \beta \neq n,
$$

where $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^n$ are open bounded subsets, boundary of $V$ is $C^\infty$-smooth, and $\varphi : U \times V \to \mathbb{R}_+$ is a $C^2$-smooth function on $U \times V$. For this purpose, we denote for each $t \in U$

$$
d\mu_t = \frac{\varphi(t, x)^{-\beta} \, dx}{\int_V \varphi(t, x)^{-\beta} \, dx}
$$

the probability measure on $V$. We also denote the corresponding symmetric diffusion operation with the invariant measure $\mu_t$ by

$$
L_t u(x) = \Delta u(x) - \beta \frac{\langle \nabla_x \varphi(t, x), \nabla u(x) \rangle}{\varphi(t, x)},
$$

where $u$ is any function in $C^2(V)$. By using integration by parts, we have

$$
\int_V L_t u(x)v(x) \, d\mu_t(x) = - \int_V \langle \nabla u(x), \nabla v(x) \rangle \, d\mu_t(x) + \int_{\partial V} v(x) \frac{\partial u}{\partial \nu}(x) \, d\mu_t(x),
$$

where $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ is the outer normal vector to $x \in \partial V$.

Since $\partial V$ is $C^\infty$-smooth, then $\nu$ is $C^\infty$-smooth on $\partial V$ and it can be extended to a $C^\infty$-smooth map on a neighborhood of $\partial V$. Hence the second fundamental form $II$ of $\partial V$ at $x \in \partial V$ is defined by

$$
II_x(X, Y) = \sum_{i,j=1}^n X_iY_j \partial_i(\nu_j)(x),
$$

for any two vector fields $X = (X_1, \cdots, X_n)$ and $Y = (Y_1, \cdots, Y_n)$ in $\partial V$.

In the sequel, we denote by $\nabla f$ and $\nabla^2 f$ the gradient and Hessian matrix of a function $f$, respectively. We also denote by $\| \cdot \|_{HS}$ the Hilbert–Schmidt norm on the space of square matrices. When $f$ is function of the variables $t$ and $x$, we write $\nabla_x f$ and $\nabla^2_x f$ for the gradient and Hessian matrix of $f$ which are taken only on $x$, respectively.

Our first main theorem of this paper is the following:

**Theorem 1.1.** Suppose that $V$ has $C^\infty$-smooth boundary, and $\varphi$ is $C^\infty$-smooth up to boundary of $U \times V$. Let $\phi$ be defined by (1.2) then
\[
\frac{\phi''(t)}{\phi(t)} = \frac{\beta}{\beta - n} \int_V \frac{((\nabla_t \phi)X, X)}{\varphi} \, d\mu_t + \frac{\beta^2}{\beta - n} \int_V \left( \|\nabla^2 u\|_{HS}^2 - \frac{1}{n}(\Delta u)^2 \right) \, d\mu_t \\
+ \frac{\beta}{|\beta - n|} \int_V \left( \sqrt{|\beta - n|} \Delta u - \text{sign}(\beta - n) \sqrt{\frac{n}{|\beta - n|}} \int_V \frac{\partial_t \varphi}{\varphi} \, d\mu_t \right)^2 \, d\mu_t \\
+ \frac{\beta^2}{\beta - n} \int_{\partial V} H(\nabla u, \nabla u) \, d\mu_t,
\]

(1.3)

where \( u \) is the solution of the equation

\[
L_t u = \frac{\partial_t \varphi(t, \cdot)}{\varphi(t, \cdot)} - \int_V \frac{\partial_t \varphi(t, x)}{\varphi(t, x)} \, d\mu_t(x) \quad \text{and} \quad \frac{\partial u(x)}{\partial \nu(x)} = 0, \quad x \in \partial V,
\]

(1.4)

and \( X \) denotes the vector field \((1, \beta \nabla u(x))\) in \( \mathbb{R}^{n+1} \).

Since \( \frac{\partial u(x)}{\partial \nu(x)} = 0 \) for every \( x \in \partial V \), hence \( \nabla u(x) \in T_x(\partial V) \) (the tangent space to \( \partial V \) at \( x \in \partial V \)). This implies that \( H(\nabla u, \nabla u) \) is well-defined on \( \partial V \). Theorem 1.1 is proved in the next section. We will need the following classical fact about the existence of the solution of the elliptic partial differential equation (see [7] and references therein):

**Lemma 1.2.** If \( V \) has \( C^\infty \)-smooth boundary \( \partial V \), and \( \varphi \) is \( C^\infty \)-smooth up to boundary of \( V \), then for any function \( f \in C^\infty(\nabla V) \), \( \int_V f(x) \, d\mu_t(x) = 0 \) there exists a function \( u \in C^\infty(\nabla V) \) such that \( L_t u = f \) and \( \frac{\partial u(x)}{\partial \nu(x)} = 0 \) on \( \partial V \).

Our second main theorem of this paper is the dimensional Prékopa’s theorem which is considered as a direct consequence of Theorem 1.1 and stated in the following theorem. The first part of this theorem concerns the convex case, and the second part concerns the concave case.

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^{n+1} \) be a convex open subset, and let \( \varphi : \Omega \to \mathbb{R}_+ \) be a \( C^2 \)-smooth function up to boundary of \( \Omega \). For \( t \in \mathbb{R} \), we define the section \( \Omega(t) = \{ x \in \mathbb{R}^n : (t, x) \in \Omega \} \). Then the following assertions hold:

(i) If \( \varphi \) is convex on \( \Omega \), and \( \beta > n \), then the function \( \phi \) defined by

\[
\phi(t) = \left( \int_{\Omega(t)} \varphi(t, x)^{-\beta} \, dx \right)^{-\frac{1}{n-\beta}},
\]

is convex on \( \mathbb{R} \).

(ii) If \( \varphi \) is concave on \( \Omega \), and \( \beta > 0 \), then the function \( \phi \) defined by

\[
\phi(t) = \left( \int_{\Omega(t)} \varphi(t, x)^\beta \, dx \right)^{\frac{1}{n+\beta}},
\]

is concave on \( \mathbb{R} \).

Finally, we remark that Prékopa’s theorem can be deduced from Theorem 1.3 by letting \( \beta \) tend to infinity since
\[
\begin{align*}
\lim_{\beta \to \infty} (\beta - n)[\left( \int_{\Omega(t)} \left( 1 + \frac{\varphi(t,x)}{\beta} \right) - \beta \right)^{\frac{1}{\beta - n}} - 1] &= -\log \left( \int_{\Omega(t)} e^{-\varphi(t,x)} \, dx \right), \\
\lim_{\beta \to \infty} (\beta + n)[\left( \int_{\Omega(t)} \left( 1 - \frac{\varphi(t,x)}{\beta} \right) + \beta \right)^{\frac{1}{\beta + n}} - 1] &= \log \left( \int_{\Omega(t)} e^{-\varphi(t,x)} \, dx \right),
\end{align*}
\]

where \( a_+ = \max\{a, 0\} \) denotes the positive part of \( a \).

2. Proof of main theorems

We begin this section by giving the proof of Theorem 1.1. Our proof is direct and similar to the method used in [10].

**Proof of Theorem 1.1.** If \( \beta = 0 \), then (1.3) is evident since \( \phi \) is a constant function.

If \( \beta \neq 0 \), then (1.3) is equivalent to

\[
\frac{\beta - n \phi''(t)}{\beta \phi(t)} = \int_V \frac{\langle \nabla_x^2 \varphi, X \rangle}{\varphi} \, d\mu_t + \beta \int_V \left( \frac{\|\nabla^2 u\|_{HS}^2}{\varphi} - \frac{1}{n} \frac{(\Delta u)^2}{\varphi} \right) \, d\mu_t \\
+ \text{sign}(\beta - n) \int_V \left( \sqrt{\frac{\beta - n}{n}} \Delta u - \text{sign}(\beta - n) \sqrt{\frac{n}{\beta - n}} \int_V \frac{\partial \varphi}{\varphi} \, d\mu_t \right)^2 \, d\mu_t \\
+ \beta \int_{\partial V} \langle \nabla_x (\partial_t \varphi) \rangle \varphi \, d\mu_t. \tag{2.1}
\]

By a direct computation, we easily get

\[
\frac{\beta - n \phi''(t)}{\beta \phi(t)} = \int_V \frac{\partial^2 \varphi(t,x)}{\varphi(t,x)} \, d\mu_t(x) - (\beta + 1) \var_{\mu_t} \left( \frac{\partial \varphi(t, \cdot)}{\varphi(t, \cdot)} \right) \\
+ \frac{n}{\beta - n} \left( \int_V \frac{\partial \varphi(t, x)}{\varphi(t, x)} \, d\mu_t(x) \right)^2, \tag{2.2}
\]

where \( \var_{\mu_t}(f) := \int_V f^2 \, d\mu_t - (\int_V f \, d\mu_t)^2 \) denotes the variance of any function \( f \) on \( V \) with respect to \( \mu_t \).

Let \( u \in C^\infty(V) \) be the solution of Eq. (1.4). Since \( \mu_t \) is a probability measure on \( V \), then we have

\[
\var_{\mu_t} \left( \frac{\partial \varphi(t, \cdot)}{\varphi(t, \cdot)} \right) = -\int_V (L_t u)^2 \, d\mu_t + 2 \int_V \left( \frac{\partial \varphi}{\varphi} - \int_V \frac{\partial \varphi}{\varphi} \, d\mu_t \right) L_t u \, d\mu_t.
\]

Using integration by parts and the fact \( \int_V L_t u \, d\mu_t = 0 \), we get

\[
\int_V \left( \frac{\partial \varphi}{\varphi} - \int_V \frac{\partial \varphi}{\varphi} \, d\mu_t \right) L_t u \, d\mu_t = -\int_V \frac{\nabla_x (\partial \varphi), \nabla u}{\varphi} \, d\mu_t + \int_V \frac{\partial \varphi}{\varphi} \frac{\nabla_x \varphi, \nabla u}{\varphi} \, d\mu_t. \tag{2.3}
\]
It follows from integration by parts (see also the proof of Theorem 1 in [10]) that
\[
\int_V (L_t u)^2 \, d\mu_t = \int_V \|\nabla^2 u\|^2_{HS} \, d\mu_t + \beta \int_V \left\langle \frac{(\nabla_x^2 \varphi) \nabla u}{\varphi}, \nabla u \right\rangle \, d\mu_t
\]
\[- \beta \int_V \frac{\langle \nabla_x \varphi, \nabla u \rangle^2}{\varphi^2} \, d\mu_t - \int_{\partial V} \left\langle \frac{(\nabla_x^2 \varphi) \nabla u}{\varphi}, \nu \right\rangle \, d\mu_t.
\]
(2.4)

From (2.3) and (2.4), we get an expression of \( \text{Var}_{\mu_t} (\partial_t \varphi(t, \cdot))/\varphi(t, \cdot) \) as follows
\[
\text{Var}_{\mu_t} \left( \frac{\partial_t \varphi(t, \cdot)}{\varphi(t, \cdot)} \right) = -2 \int_V \frac{\langle \nabla_x (\partial_t \varphi), \nabla u \rangle}{\varphi} \, d\mu_t + 2 \int_V \frac{\partial_t \varphi \langle \nabla_x \varphi, \nabla u \rangle}{\varphi} \, d\mu_t
\]
\[- \int_V \|\nabla^2 u\|^2_{HS} \, d\mu_t - \beta \int_V \left\langle \frac{(\nabla_x^2 \varphi) \nabla u, \nabla u}{\varphi} \right\rangle \, d\mu_t
\]
\[+ \beta \int_V \frac{\langle \nabla_x \varphi, \nabla u \rangle^2}{\varphi^2} \, d\mu_t + \int_{\partial V} \left\langle \frac{(\nabla_x^2 \varphi) \nabla u}{\varphi}, \nu \right\rangle \, d\mu_t.
\]
(2.5)

It follows from the definition of \( L_t \) that
\[
\beta^2 \int_V \frac{\langle \nabla_x \varphi, \nabla u \rangle^2}{\varphi^2} \, d\mu_t = \int_V [(L_t u)^2 + (\Delta u)^2] \, d\mu_t - 2 \int_V \Delta u L_t u \, d\mu_t.
\]
(2.6)

Plugging (2.4) and (1.4) into (2.6), we obtain
\[
\beta (\beta + 1) \int_V \frac{\langle \nabla_x \varphi, \nabla u \rangle^2}{\varphi^2} \, d\mu_t = \int_V \|\nabla^2 u\|^2_{HS} \, d\mu_t + \int_V (\Delta u)^2 \, d\mu_t
\]
\[+ \beta \int_V \frac{\langle (\nabla_x^2 \varphi) \nabla u, \nabla u \rangle}{\varphi} \, d\mu_t - 2 \int_V \frac{\partial_t \varphi}{\varphi} \Delta u \, d\mu_t
\]
\[+ 2 \int_V \Delta u \left( \int_V \frac{\partial_t \varphi}{\varphi} \, d\mu_t \right) \, d\mu_t - \int_{\partial V} \left\langle \frac{(\nabla_x^2 \varphi) \nabla u, \nu}{\varphi} \right\rangle \, d\mu_t.
\]
(2.7)

Moreover, using again integration by parts, we have
\[
\int_V \frac{\partial_t \varphi}{\varphi} \left\langle \nabla_x \varphi, \nabla u \right\rangle \, d\mu_t = -\frac{1}{\beta} \int_V \frac{\partial_t \varphi(t, x)}{\varphi(t, x)} \langle \nabla_x (\varphi(t, x)^{-\beta}), \nabla u(x) \rangle \, dx
\]
\[= \frac{1}{\beta} \int_V \frac{\langle \nabla_x (\partial_t \varphi), \nabla u \rangle}{\varphi} \, d\mu_t - \frac{1}{\beta} \int_V \frac{\partial_t \varphi \langle \nabla_x \varphi, \nabla u \rangle}{\varphi} \, d\mu_t
\]
\[+ \frac{1}{\beta} \int \frac{\partial_t \varphi}{\varphi} \Delta u \, d\mu_t.
\]
(2.8)
Plugging (2.5), (2.7), and (2.8) into (2.2), we obtain
\[
\frac{\beta - n \phi''(t)}{\beta \phi(t)} = \int_V \frac{\nabla_{\beta}^2 \phi}{\phi} \, d\mu_t + 2\beta \int_V \frac{\nabla_x(\partial_t \phi), \nabla u}{\phi} \, d\mu_t + \beta^2 \int_V \frac{\langle \nabla_{\beta}^2 u, \nabla u \rangle}{\phi} \, d\mu_t
\]
\[+\beta \int_V \left\| \nabla^2 u \right\|_{HS}^2 \, d\mu_t - 2 \int_V \Delta u \left( \int_V \frac{\partial_t \phi}{\phi} \, d\mu_t \right) \, d\mu_t - \int_V (\Delta u)^2 \, d\mu_t
\]
\[+\frac{n}{\beta - n} \left( \int_V \frac{\partial_t \phi(t, x)}{\phi(t, x)} \, d\mu_t(x) \right)^2 - \beta \int_V \langle \nabla^2 u, \nabla u, \nu \rangle \, d\mu_t
\]
\[= \int_V \frac{\partial_t \phi}{\phi} \, d\mu_t + 2\beta \int_V \frac{\nabla_x(\partial_t \phi), \nabla u}{\phi} \, d\mu_t + \beta^2 \int_V \frac{\langle \nabla_{\beta}^2 u, \nabla u \rangle}{\phi} \, d\mu_t
\]
\[+\beta \int_V \left( \left\| \nabla^2 u \right\|_{HS}^2 - \frac{1}{n} (\Delta u)^2 \right) \, d\mu_t + \frac{\beta - n}{n} \int_V (\Delta u)^2 \, d\mu_t
\]
\[+2 \int_V \Delta u \left( \int_V \frac{\partial_t \phi}{\phi} \, d\mu_t \right) \, d\mu_t + \frac{n}{\beta - n} \left( \int_V \frac{\partial_t \phi(t, x)}{\phi(t, x)} \, d\mu_t(x) \right)^2
\]
\[= \beta \int_V \langle \nabla^2 u, \nabla u, \nu \rangle \, d\mu_t.
\]

To finish our proof, we need to treat the term on boundary in (2.9). Since \( \frac{\partial u}{\partial \nu} = 0 \) on \( \partial V \), then \( \nabla u(x) \in T_x(\partial V) \) for every \( x \in \partial V \), and
\[
\langle \nabla^2 u(x), \nabla u(x), \nu(x) \rangle = -II_x(\nabla u(x), \nabla u(x)), \quad x \in \partial V.
\]
Combining (2.9) and (2.10), and denoting \( X(t, x) = (1, \beta \nabla u(x)) \) with \( (t, x) \in U \times V \), we get (2.1). Then Theorem 1.1 is completely proved. \( \square \)

In the following, we use Theorem 1.1 to prove the dimensional Prékopa’s theorem (Theorem 1.3).

**Proof of Theorem 1.3.** By using an approximation argument, we can assume that \( \Omega \) is bounded and \( \varphi \) is \( C^\infty \)-smooth up to boundary of \( \Omega \).

**Part (i):** We first prove when \( \Omega = U \times V \) with \( U \subset \mathbb{R} \), and \( V \subset \mathbb{R}^n \) has \( C^\infty \)-smooth boundary \( \partial V \). Since \( \beta > n \), then applying Theorem 1.1, we have
\[
\frac{\beta - n \phi''(t)}{\beta \phi(t)} = \int_V \langle \nabla_{\beta}^2 u, X, X \rangle \, d\mu_t + \beta \int_V \left( \left\| \nabla^2 u \right\|_{HS}^2 - \frac{1}{n} (\Delta u)^2 \right) \, d\mu_t
\]
\[+\beta \int_V \left( \sqrt{\frac{\beta - n}{n}} \Delta u - \sqrt{\frac{n}{\beta - n}} \int_V \frac{\partial_t \phi}{\phi} \, d\mu_t \right)^2 \, d\mu_t
\]
\[+\beta \int_{\partial V} II(\nabla u, \nabla u) \, d\mu_t,
\]
where \( II \) denotes the second fundamental form of \( \partial V \), and \( u \) is the \( C^\infty \)-smooth solution of Eq. (1.4) with \( L_t = \Delta - \beta \langle \nabla x \varphi, \cdot \rangle \varphi \), and \( X \) denotes the vector field \( (1, \beta \nabla u) \) in \( \mathbb{R}^{n+1} \).
We have \( II_x(\nabla u(x), \nabla u(x)) \geq 0 \), \( x \in \partial V \) because of the convexity of \( V \). By the Cauchy–Schwartz inequality, we have

\[
\|\nabla^2 u\|_{HS}^2 \geq \frac{1}{n} (\Delta u)^2.
\]

As a consequence of the convexity of \( \varphi \), we obtain \( \nabla^2 \varphi \geq 0 \) in the sense of symmetric matrix. All the integrations on the right hand side of (2.11) hence are nonnegative. This implies that \( \phi'' \geq 0 \), or \( \phi \) is convex.

In the general case, there exists an increasing sequence of \( C^\infty \)-smooth open convex \( \Omega_k \) such that

\[
\Omega_k = \{(t, x) : \rho_k(t, x) < 0\},
\]

where \( \rho_k \in C^\infty(\mathbb{R}^{n+1}) \), \( k = 1, 2, \ldots \), are convex functions, and \( \Omega = \bigcup_k \Omega_k \). Hence, by using an approximation argument, we can assume that

\[
\Omega = \{(t, x) : \rho(t, x) < 0\}
\]

with a \( C^\infty \)-smooth convex function \( \rho \), and \( \varphi \) is defined in a neighborhood of \( \Omega \). Since the convexity is local, it is enough to prove that \( \phi \) is convex in a neighborhood of each \( t \). Fix \( t_0 \), choose a small enough neighborhood \( U \) of \( t_0 \) such that

\[
(U \times \mathbb{R}^n) \cap \overline{\Omega} \subset U \times V
\]

and \( \rho, \varphi \) are defined in \( U \times V \), where \( V \) is convex subset of \( \mathbb{R}^n \) and has \( C^\infty \)-smooth boundary \( \partial V \). Define \( \rho_0 = \max\{\rho, 0\} \), then \( \rho_0 \) is a convex function in \( U \times V \). With \( N > 0 \), we know that the function

\[
\phi_N(t) = \left( \int_V (\varphi(t, x) + N \rho_0(t, x))^{-\beta} \, dx \right)^{-\frac{1}{\beta}}
\]

is convex in \( U \). Moreover, \( \phi_N(t) \to \phi(t) \) in \( U \) as \( N \) tends to infinity, then \( \phi \) is convex in \( U \). This finishes the proof of the convexity of \( \phi \).

Part (ii): As explained in the proof of the part (i) above, it suffices to prove the part (ii) in the case \( \Omega = U \times V \) where \( U \subset \mathbb{R} \) and \( V \subset \mathbb{R}^n \) are bounded open convex subsets, and \( \partial V \) is \( C^\infty \)-smooth. Since \( \beta > 0 \), by applying Theorem 1.1 to \( -\beta \) instead of \( \beta \), we have

\[
\frac{\beta + n}{\beta} \phi''(t) = \int_V \frac{((\nabla^2 (t,x)\varphi) X, X)}{\varphi} \, d\mu_t - \beta \int_V \left( \|\nabla^2 u\|_{HS}^2 - \frac{1}{n} (\Delta u)^2 \right) \, d\mu_t
\]

\[
- \int_V \left( \sqrt{\frac{\beta + n}{n}} \Delta u + \sqrt{\frac{n}{\beta + n}} \int_V \frac{\partial_t \varphi}{\varphi} \, d\mu_t \right)^2 \, d\mu_t
\]

\[
- \beta \int_{\partial V} II(\nabla u, \nabla u) \, d\mu_t.
\]

Using the arguments in the proof of part (i) and the concavity of \( \varphi \), we get \( \phi''(t) \leq 0 \) from (2.12), or \( \phi \) is concave. \( \square \)
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