Quantization-based approximation of reflected BSDEs with extended upper bounds for recursive quantization

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Abstract

We establish upper bounds for the $L^p$-quantization error, $p \in (1, 2 + d)$, induced by the recursive Markovian quantization of a $d$-dimensional diffusion discretized via the Euler scheme. We introduce a hybrid recursive quantization scheme, easier to implement in the high-dimensional framework, and establish upper bounds to the corresponding $L^p$-quantization error. To take advantage of these extensions, we propose a time discretization scheme and a recursive quantization-based discretization scheme associated to a Reflected Backward Stochastic Differential Equation and estimate $L^p$-error bounds induced by the space approximation. We will explain how to numerically compute the solution of the reflected BSDE relying on the recursive quantization and compare it to other types of quantization.

Keywords: reflected backward stochastic differential equation, recursive quantization, optimal quantization, Euler scheme, hybrid schemes, $L^p$-error bounds, Markov chain.

1 Introduction

We are interested in the discretization and the computation of the solution of the following reflected backward stochastic differential equation RBSDE with maturity $T$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s.dW_s, \quad t \in [0, T], \quad (1)$$

$$Y_t \geq h(t, X_t) \quad \text{and} \quad \int_0^T (Y_s - h(s, X_s))dK_s = 0. \quad (2)$$

$(X_t)_{t \geq 0}$ is a Brownian diffusion process taking values in $\mathbb{R}^d$ and solution to the SDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad X_0 = x_0 \in \mathbb{R}^d, \quad (3)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the matrix diffusion coefficient $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{M}(d, q)$ are Lipschitz continuous in $(t, x)$ so that $b(., 0)$ and $\sigma(., 0)$ are bounded on $[0, T]$ and satisfy the linear growth condition

$$\|\sigma(., x)\| + \|b(., x)\| \leq L_{b, \sigma}(1 + \|x\|)$$

with $L_{b, \sigma} = \max ([b]_{\text{Lip}}, [\sigma]_{\text{Lip}}, \|b(., 0)\|_{\sup}, \|\sigma(., 0)\|_{\sup})$ and $\| \cdot \|$ denoting any norm on $\mathbb{R}^d$. $(W_t)_{t \geq 0}$ is a $q$-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with its augmented

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natural filtration \((\mathcal{F}_t)_{t \geq 0}\) where \(\mathcal{F}_t = \sigma(W_s, s \leq t, \mathcal{N}_t)\), \(\mathcal{N}_t\) denotes the class of all \(\mathbb{P}\)-negligible sets of \(\mathcal{A}\). The solution of this equation is defined as a \(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+\)-valued triplet \((Y_t, Z_t, K_t)\) of \(\mathcal{F}_t\)-progressively measurable square integrable processes. \(K_t\) is continuous, non-decreasing, such that \(K_0 = 0\) and grows exclusively on \(\{t : Y_t = h(t, X_t)\}\). The driver \(f(t, x, y, z) : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) is \([f]_{\text{Lip}}\)-Lipschitz continuous with respect to \((t, x, y, z)\), \(g(X_T)\) is the terminal condition where \(g : \mathbb{R}^d \to \mathbb{R}\) is \([g]_{\text{Lip}}\)-Lipschitz continuous and \(h : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is \([h]_{\text{Lip}}\)-Lipschitz continuous such that \(g \geq h\) for every \(t\) and \(x\). Under these assumptions on \(b, \sigma, h, g\) and \(f\), the RBSDE (1) and the SDE (3) admit both a unique solution. The existence of a process \((Y_t, Z_t, K_t)\), solution of (1), was established in [16] where the authors also showed that this solution satisfies the following property

\[
\left\| \sup_{t \in [0,T]} |Y_t| \right\|_{2_p} \vee \|K_T\|_{2p} \vee \left\| \int_0^T |Z_t|^2 dt \right\|_p < \gamma_0
\]

(4)

for a finite constant \(\gamma_0\) (see also [2]). In general, these solutions admit no closed form. Approximation schemes are needed to approximate them. In the literature, many authors studied different types of RBSDEs, for example, in [2, 12, 16, 29, 30] and many approximation schemes were investigated: Feynman-Kac type representation formula were given in [29] for the solutions of RBSDEs, a four step algorithm was developed in [31] to solve FBSDEs, a random time scheme in [1]. We can also cite the papers devoted to time(-space) discretization of RBSDE, the driver does not depend on the process \(Z\) in the literature, the expectation is usually applied to the driver \(f\) for BSDE (without reflection) in [6, 19], the multi-step schemes methods (see [5]), a hybrid approach combining Picard iterates with a decomposition in Wiener chaos (see [8]), a connection with the semi-linear PDE associated to the BSDE (see [23]) and Monte Carlo simulations with Malliavin calculus (see [6, 11, 24, 20]). Another approach is optimal quantization introduced for RBSDEs in [4] and then
developed in a series of papers ([2, 3, 25, 39] for example), quantization-based discretization schemes have also been used in [13] for fully coupled Forward-Backward SDEs. In this paper, we will rely on the recursive quantization of the time-discretized Euler scheme \((\tilde{X}_{t_k}^n)_{0 \leq k \leq n}\). This method, originally introduced in [35] and then studied deeply in [32] and [37] for one-dimensional diffusions, consists in building a Markov chain having values into a grid (or quantizer) \(\Gamma_k\) of the discrete Euler scheme \(\tilde{X}_{t_k}\) at time \(t_k\). The grids \(\Gamma_k\) can be optimized in a recursive way as a kind of embedded procedure.

In order to explain the principle of this recursive Markovian quantization, let us first recall briefly what optimal quantization is. Assume that \(\mathbb{R}^d\) is equipped with a norm \(\| \cdot \|\) (usually the canonical Euclidean norm for our purpose). Let \(X \in L^p_{\mathbb{R}^d}(\Omega, \mathcal{A}, \mathbb{P})\) and let \(N \geq 1\) be a quantization level. The aim of \(L^p\)-optimal quantization is to find the best approximation of \(X\) in \(L^p(\mathbb{P})\) by a random vector \(Y\) defined on \((\Omega, \mathcal{A}, \mathbb{P})\) taking at most \(N\) values. As a first step, we may consider the grid (or quantization grid) \(\Gamma^N = Y(\Omega) = \{x_1, \ldots, x_N\}\) (with possibly repeated elements). One easily checks that, \(\Gamma^N\) being fixed, the best possible choice is given by a (Borel) nearest neighbor projection of \(X\) on \(\Gamma^N\). It is called a Voronoï quantization of \(X\) defined by

\[
\tilde{X}^{\Gamma^N} = \text{Proj}_{\Gamma^N}(X) := \sum_{i=1}^N x_i \mathbb{I}_{\gamma_i(\Gamma^N)}(X)
\]

where \((\gamma_i(\Gamma^N))_{1 \leq i \leq N}\) is a Borel partition of \(\mathbb{R}^d\) satisfying

\[
\gamma_i(\Gamma^N) \subset \{\xi \in \mathbb{R}^d : \|\xi - x_i\| \leq \min_{j \neq i} \|\xi - x_j\|\}, \quad i = 1, \ldots, N.
\]

The \(N\)-tuple \((\gamma_i(\Gamma^N))_{1 \leq i \leq N}\) is called the Voronoï partition induced by \(\Gamma^N\). The induced \(L^p\)-quantization error associated to the grid \(\Gamma^N\) is defined by

\[
e^p(\Gamma^N, X) = \|X - \tilde{X}^{\Gamma^N}\|_p
\]

where \(\| \cdot \|_p\) denotes the \(L^p(\mathbb{P})\)-norm. The optimal quantization problem boils down to finding the grid \(\Gamma^N\) that minimizes this error i.e. solving the problem

\[
e_{p,N}(X) := \inf_{|\Gamma| \leq N} e_p(\Gamma, X).
\]

where \(|\Gamma|\) denotes the cardinality of the grid \(\Gamma\). A solution to this problem exists, as established in [22, 33, 34] for example, and is called an \(L^p\)-optimal quantization grid of (the distribution of) \(X\). The corresponding quantization error converges to 0 as \(N\) goes to \(+\infty\) and its rate of convergence is given by two well known results exposed in the following theorem.

**Theorem 1.1.** (a) Zador’s Theorem (see [41]): Let \(X \in L^{p+\eta}_{\mathbb{R}^d}(\mathbb{P})\), \(\eta > 0\), with distribution \(P\) having the following decomposition \(P = h.\lambda_d + \nu\) where \(\lambda_d\) denotes the Lebesgue measure on \((\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))\) and \(\nu \perp \lambda_d\) (singular). Then,

\[
\lim_{N \to +\infty} N^{\frac{1}{2}} e_{p,N}(X) = \tilde{J}_{p,d}\mathbb{E}\|\varphi\|_{L^p(\lambda_d)}^{\frac{1}{p}}
\]

where \(\tilde{J}_{p,d} = \inf_{N \geq 1} N^{\frac{1}{2}} e_{p,N}(\mathcal{U}([0,1]^d)) \in (0, +\infty)\).

(b) Extended Pierce’s Lemma (see [26, 34]): Let \(p, \eta > 0\). There exists a constant \(\kappa_{d,p,\eta} \in (0, +\infty)\) such that, for any random vector \(X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d\),

\[
\forall N \geq 1, \quad e_{p,N}(X) \leq \kappa_{d,p,\eta}\sigma_p(X)N^{-\frac{\eta}{2}}
\]

where, for every \(p \in (0, +\infty)\), \(\sigma_p(X) = \inf_{a \in \mathbb{R}^d} \|X - a\|_p\) is the \(L^p\)-(pseudo-)standard deviation of \(X\).
An important property, shared by quadratic optimal quantizers, is the stationarity property: an $L^2$-optimal quantizer $\Gamma^N$ is said to be stationary if

$$\mathbb{E}(X|\hat{X}^{\Gamma^N}) = \hat{X}^{\Gamma^N}. \quad (15)$$

Let us now explain what recursive quantization is. If we define the Euler operator with step $\Delta$ by

$$\mathcal{E}_k(x, \varepsilon_{k+1}) = x + \Delta h(t_k, x) + \sqrt{\Delta} \sigma(t_k, x) \varepsilon_{k+1}$$

where $(\varepsilon_k)_{0 \leq k \leq n}$ is an i.i.d. sequence of random variables with distribution $\mathcal{N}(0, I_d)$, then the recursive quantization $(\hat{X}_{t_k})_{0 \leq k \leq n}$ of $(X_{t_k})_{0 \leq k \leq n}$ is defined by $\hat{X}_{t_0} = \hat{X}_{t_0}^n = x_0$ and

$$\begin{cases}
\hat{X}_{t_k} = \mathcal{E}_{k-1}(\hat{X}_{t_{k-1}}^{\Gamma_{k-1}}, \varepsilon_k), \\
\hat{X}_{t_k}^{\Gamma_k} = \text{Proj}_{\Gamma_k}(\hat{X}_{t_k}),
\end{cases} \quad \forall k = 1, \ldots, n \quad (16)$$

where $(\Gamma_k)_{0 \leq k \leq n}$ is a sequence of optimal quantizers of $(\hat{X}_{t_k})_{0 \leq k \leq n}$ of size $N_k$, $k = 0, \ldots, n$. The optimal quantizers $(\Gamma_k)_{1 \leq k \leq n}$ can be either quadratic or $L^p$-optimal quantizers, we will detail the difference between these two frameworks later in the paper. The main advantage of this method is that it preserves the Markov property of the Euler scheme with respect to the filtration $(\mathcal{F}_{t_k})_{0 \leq k \leq n}$, the process $\hat{X}_{t_k}$ is $\mathcal{F}_{t_k}$-measurable for every $k \in \{0, \ldots, n\}$. In fact, the transition matrices $(p^k_{ij})_{1 \leq i, j \leq N_k}$ where $p^k_{ij} = \mathbb{P}(\hat{X}_{t_{k+1}} \in C_j(\Gamma_{k+1}) | \hat{X}_{t_k} \in C_i(\Gamma_k))$ and the initial distribution characterize the distribution of the Markov chain $(\hat{X}_{t_k})_{k \geq 0}$, which was not the case with the optimal quantization in [37] for example. This Markov property will bring much help to carry on computations of the weights $p^k_{ij}$ of the Voronoï cells and the transition weights $p^k_{ij}$, as well as with the quantized scheme of the RBSDE itself.

Going back to our problem, we consider, in this paper, the recursive quantization scheme associated to (6)-(7)-(8)-(9) based on the recursive quantization $(\hat{X}_{t_k})_{0 \leq k \leq n}$ of the Euler scheme $(\hat{X}_{t_k}^n)_{0 \leq k \leq n}$. It is defined recursively in a backward way as follows:

$$\begin{align*}
\hat{Y}^n_T &= g(\hat{X}_T) \\
\hat{c}^n_{t_k} &= \frac{1}{\Delta} \mathbb{E}(\hat{Y}^n_{t_{k+1}}(W_{t_{k+1}} - W_{t_k}) | \mathcal{F}_{t_k}), \quad k = 0, \ldots, n - 1, \\
\hat{Y}^n_{t_k} &= \max\left( h_k(\hat{X}_{t_k}), \mathbb{E}(\hat{Y}^n_{t_{k+1}} | \mathcal{F}_{t_k}) + \Delta f(t_k, \hat{X}_{t_k}, \mathbb{E}(\hat{Y}^n_{t_{k+1}} | \mathcal{F}_{t_k}, \hat{c}^n_{t_k})) \right), \quad k = 0, \ldots, n - 1, \quad (19)
\end{align*}$$

where $(\hat{X}_{t_k})_{0 \leq k \leq n}$ is the recursively quantized Euler scheme associated to $(\hat{X}_{t_k}^n)_{0 \leq k \leq n}$ given by (16). As a preliminary step, we are interested in estimating the $L^p$-quantization error $\|\hat{X}_{t_k} - \hat{X}_{t_k}^n\|_p$, not only for $p = 2$ like in [37] but for any $p \in (1, 2 + d)$. The fact that we are limited to $p < 2 + d$ will become clear later in the paper, as well as the type of optimal quantizers $\Gamma_k$ of $\hat{X}_{t_k}$ needed to obtain satisfactory upper bounds for the $L^p$-quantization error. Note that in the quadratic case $p = 2$, the proof was based on a Pythagoras property which cannot be applied in a general framework. Furthermore, we introduce a kind of hybrid recursive quantization where the white noise $(\varepsilon_k)_{0 \leq k \leq n}$ is replaced by its (already computed) quantized version $(\tilde{\varepsilon}_k)_{0 \leq k \leq n}$.

In a second part, we will proceed with the time and space discretization of the RBSDE (1), as explained briefly before, and give more details about these schemes. We establish a priori estimates for the time discretization error $\|\hat{Y}_{t_k} - \hat{Y}_{t_k}^n\|_2$ in a quadratic case. Although time discretization have already been studied in the literature (see [2, 6, 29, 39, 42]), our approach is still different mostly because of the combination of the reflection in the backward SDE and the conditional expectation applied directly to $\hat{Y}_{t_k}^n$ and $\tilde{Y}_{t_k}^n$ inside the driver $f$ depending itself on the process $\hat{Z}_t$ (its approximations). Likewise, estimates for the space discretization error $\|\hat{Y}_{t_k} - \tilde{Y}_{t_k}^n\|_p$ in $L^p$ for $p \in (1, 2 + d)$ will
be established. To illustrate these theoretical results, we detail the numerical techniques available to compute the recursive quantization $\hat{X}^n_{t_k}$ of $X^n_{t_k}$, for every $k \in \{1, \ldots, n\}$, their distributions and the corresponding transition weight matrices. Moreover, we will explain how to compute numerically the solution of the discretized scheme (17)-(18)-(19) associated to the RBSDE (1). These computations will be useful to carry on numerical tests and experiments illustrating the above error bounds. One of the most important applications of these quantization-based discretizations is the pricing of American options for which the driver $f$ is equal to 0, among other examples (with a non-zero driver) that will be presented at the end of this paper. This link between BSDEs and the pricing of financial options have been first introduced in [17].

Throughout this paper, we will replace, for convenience, the indices $t_k$ by $k$ for $k \in \{0, \ldots, n\}$, i.e. we will use, for example, $\tilde{X}_k$ instead of $\tilde{X}_{t_k}$. Also, we will replace $f(t_k, x, y, z)$ by $f_k(x, y, z)$, $b(t_k, \cdot)$ by $b_k(\cdot)$ and $\sigma(t_k, \cdot) = \sigma_k(\cdot)$. And, we will omit the $n$ in $\tilde{Y}^n_{k+1}, \tilde{X}^n_{k+1}$, etc.

This paper is organized as follows: In section 2, we provide some short background on recursive quantization and establish the new $L^p$-error bounds for $p \in (1, 2 + d)$, of the recursive quantization error as well as those of the hybrid recursive quantization error. Section 3 is devoted to the time discretization of the RBSDE and to the estimation of the corresponding error. The space discretization of the RBSDE will be treated in Section 4. In Section 5, we will present the numerical techniques to compute the recursive quantizers and the solution of the RBSDE. Finally, Section 6 is devoted to several numerical examples.

2 Recursive Quantization: background, $L^p$-error bounds and hybrid schemes.

In this section, we study the discretization of the forward process $(X_t)_{t \geq 0}$. It is a Brownian diffusion process taking values in $\mathbb{R}^d$, solution to the SDE (3) given in the introduction and recalled below

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad X_0 = x_0 \in \mathbb{R}^d.$$ 

First, we start by the time discretization and we present the Euler scheme $(\tilde{X}_{t_k})_{0 \leq k \leq n}$, with uniform mesh $t_k = k\Delta$ for $k \in \{0, \ldots, n\}$ and $\Delta = \frac{T}{n}$, associated to the process $(X_t)_{t \in [0, T]}$ which is recursively given by

$$\tilde{X}_{t_{k+1}} = \tilde{X}_{t_k} + \Delta b_k(\tilde{X}_{t_k}) + \sigma_k(\tilde{X}_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad \tilde{X}_0 = X_0 = x_0,$$

where $W_{t_{k+1}} - W_{t_k} = \sqrt{\Delta} \epsilon_{k+1}$, for every $k \in \{0, \ldots, n-1\}$ and $(\epsilon_k)_{0 \leq k \leq n}$ is a sequence of i.i.d. random variables with distribution $\mathcal{N}(0, I_d)$. Its continuous counterpart, the genuine Euler scheme, is given by

$$d\tilde{X}_t = b(t, \tilde{X}_t)dt + \sigma(t, \tilde{X}_t)dW_t$$

where $t = t_k$ when $t \in [t_k, t_{k+1})$. This process satisfies for every $p \in (0, +\infty)$ and every $n \geq 1$, (see [7])

$$\left\| \sup_{t \in [0, T]} X_t \right\|_p + \sup_{n \geq 1} \left\| \sup_{t \in [0, T]} \tilde{X}_t \right\|_p \leq C_{b, T, \sigma} (1 + |x_0|) \quad \text{and} \quad \left\| \sup_{t \in [0, T]} |X_t - \tilde{X}_t| \right\|_p \leq C_{b, T, \sigma} \sqrt{\Delta} (1 + |x_0|)$$

where $C_{b, T, \sigma}$ is a positive constant depending on $p, T, b$ and $\sigma$.

After the time discretization, one must proceed with space discretization schemes. As introduced, we consider in this paper the approximation of the Euler scheme $(\tilde{X}_{t_k})_{0 \leq k \leq n}$ by recursive quantization.
2.1 Background

Our aim is to design, for \( k \in \{0, \ldots, n\} \), optimal quantizers \( \Gamma_k \) of size \( N_k \) of a function of the discrete Euler scheme \( (\bar{X}_k)_{0 \leq k \leq n} \). In other words, we want to find the grid \( \Gamma_k \) that minimizes the \( L^p \)-distortion function \( G_k^p(\Gamma) = \mathbb{E}[\text{dist}(\epsilon_k(\bar{X}_{k-1}), \Gamma)^p] \) corresponding to \( \epsilon_k(\bar{X}_{k-1}, \epsilon_k) \) where

\[
\mathcal{E}_{k-1}(x, \epsilon_k) = x + \Delta b_k(x) + \sqrt{\Delta \sigma_k(x)} \epsilon_k
\]

and \((\epsilon_k)\) is an i.i.d. sequence of \( \mathcal{N}(0, I_q) \)-distributed random vectors independent from \( X_0 \).

Since \( \bar{X}_0 = X_0 = x_0 \) is fixed, its quantizer is given by \( \Gamma_0 = \{x_0\} \). Then, we compute \( \bar{X}_1 = \mathcal{E}_0(\bar{X}_0^{\Gamma_0}, \epsilon_1) \) and we build an optimal quantization grid \( \Gamma_1 \) of size \( N_1 \) that minimizes \( G_1^p(\bar{X}_1, \Gamma) \) on the set of grids \( \Gamma \) of size \( N_1 \) (see Section 5). Doing so, we are able to define the quantization of \( \bar{X}_1 \) by \( \bar{X}_1 = \text{Proj}_{\Gamma_1}(\bar{X}_1) \). Repeating this procedure, we define a(n optimized) recursive quantization of \((\bar{X}_k)_{0 \leq k \leq n}\) by the following recursion: \( \bar{X}_0 = \bar{X}_0 = x_0 \) and

\[
\begin{align*}
\bar{X}_{k+1} &= \mathcal{E}_{k}(\bar{X}_k^{\Gamma_k}, \epsilon_k), \\
\bar{X}_k^{\Gamma_k} &= \text{Proj}_{\Gamma_k}(\bar{X}_k), \quad \forall k = 1, \ldots, n.
\end{align*}
\]

In practice, we ask the grids \( \Gamma_k \) to share some optimality properties, typically to be \( L^p \)-optimal or in higher dimension to be a product grid with optimal marginals, etc. For that purpose, the following identities play a crucial role: the \( L^p \)-distortion function associated to \( \Gamma_k = (x_1^k, \ldots, x_N^k) \) is approximated by

\[
G_k^p(x_1^k, \ldots, x_N^k) = \mathbb{E}[\text{dist}(\bar{X}_k; \{x_1^k, \ldots, x_N^k\})^p] = \sum_{i=1}^{N_k} \mathbb{E}[\text{dist}(\mathcal{E}_{k-1}(x_i^{k-1}, \epsilon_k), x_i^k)^p] \mathbb{P}(\bar{X}_k^{\Gamma_k} \in C_i(\Gamma_k))
\]

where \( \mathbb{P}(\bar{X}_k^{\Gamma_k} \in C_i(\Gamma_k)) \) is the weight of the Voronoi cell of centroid \( x_i^k \in \Gamma_k \). Note that one can write the distortion function as a function of the grid \( \Gamma_k \) but writing it as a function of an \( N_k \)-tuple is needed in order to talk about its differentiability. In fact, if the \( N_k \)-tuple \( (x_1^k, \ldots, x_N^k) \) has pairwise distinct components and the boundaries of the Voronoi diagram \( (\partial C_i(\Gamma_k))_{1 \leq i \leq N_k} \) are negligible w.r.t. the distribution of \( \bar{X}_k \), then the gradient of the differentiable \( L^p \)-distortion function is given by

\[
\nabla G_k^p(x_1^k, \ldots, x_N^k) = p \mathbb{E} \left[ \mathbb{I}_{\bar{X}_k \in C_i(\Gamma_k)} (x_i^k - \bar{X}_k)^{p-1} \right]_{1 \leq i \leq N_k}.
\]

Note that since the grid \( \Gamma_k \) has pairwise distinct components for every \( k \in \{0, \ldots, n\} \), the distribution of \( \bar{X}_k \) exists as soon as \( \sigma \sigma^t \) is invertible. From now on, we denote \( \bar{X}_k \) instead of \( \bar{X}_k^{\Gamma_k} \) for simplicity.

2.2 \( L^p \)-error bounds for recursive quantization

Our aim is to establish \( L^p \)-upper bounds for the recursive quantization error \( \|\bar{X}_{tk} - \bar{X}_{tk}\|_p \) for \( p \in (1, 2+d) \) and \( k \in \{0, \ldots, n\} \). As explained, the recursive quantization schemes of \( \bar{X}_{tk} \) are based on optimal quantization sequences of \( \bar{X}_{tk} \) which can be either quadratic or \( L^p \)-quantization sequences, \( p \neq 2 \). The more interesting case is when we rely on \( L^2 \)-optimal quantization because, from an algorithmic point of view, one has direct access to optimal quadratic quantizers since they are stationary and the algorithms used to produce optimal quantizers are either directly based on the stationarity property or easier to manage in a quadratic framework. Nevertheless, establishing an upper bound for the error \( \|\bar{X}_{tk} - \bar{X}_{tk}\|_p \) where \( \bar{X}_{tk} \) is itself an \( L^p \)-optimal quantizer of \( \bar{X}_{tk} \) still seems a natural track to consider.

\( L^2 \)-optimal quantization
We consider the case where, for every $k \in \{1, \ldots, n\}$, $\hat{X}_{t_k}$ is a quadratic optimal quantization of $X_{t_k}$, hence it is stationary in the sense of (15) (see [34]). In the following, we assume that $\Delta \in [0, \Delta_{\text{max}})$, $\Delta_{\text{max}} > 0$. Note that for the Euler scheme, one can have $\Delta_{\text{max}} = \frac{T}{n_0}$ if we consider schemes with step $\Delta = \frac{T}{n}$ and a number of steps $n > n_0$ for some $n_0 > 0$.

**Theorem 2.1.** Let $p \in (1, 2 + d)$, $(\hat{X}_k)_{0 \leq k \leq n}$ defined by (20) and $(\hat{\tilde{X}}_k)_{0 \leq k \leq n}$ the corresponding recursive quantization sequence defined by (22). Assume that, for every $k \in \{0, \ldots, n\}$, $\hat{X}_k$ is a stationary quadratic optimal quantization of $\tilde{X}_k$ of size $N_k$ in the sense of (15), with $\tilde{X}_0 = \tilde{X}_0 = x_0 \in \mathbb{R}^d$. For every $k \in \{1, \ldots, n\}$ and every $\delta \in (0, 1]$, 

$$
\|\hat{X}_k - \tilde{X}_k\|_p \leq (K_{d,2,2+\delta,p}) \sum_{l=1}^{k} (\mathcal{E}_k)_{l_{\text{Lip}}} C_{2+\delta,b,\sigma,T}(I) \left(\frac{1}{\pi^{d/2}} N_{l}^{-\frac{d}{2}} \right)
$$

where $K_{d,2,\delta}$ is the constant from Pierce’s Lemma 1.1(b),

$$
\tilde{K}_{d,2,2+\delta,p} \leq 2^{\frac{p(2+\delta)}{2+d}}\frac{1}{d} \kappa_{X,r} \min_{\varepsilon \in (0, 1]} \left[ (1 + \varepsilon) \varphi(\varepsilon) \right] \left( \int_{\mathbb{R}^d} (1 + \|x\|)^{\left(\frac{d+2-p(2+\delta)}{p}\right)} dx \right) \frac{1}{\pi^{d/2}}
$$

with $\kappa_{X,r}$ a finite positive constant independent from $N_l$, $V_d$ the volume of the hyper-unit ball and $\varphi_2(u) = \left(\frac{1}{3u} - u^2\right) u^d$,

$$
[\mathcal{E}_k]_{\text{Lip}} = \begin{cases} 
\Delta \left((s[b]_{\text{Lip}} + c_{p})_{e,s,D_{\max},e,k+1}^{(3)} \sigma_{\text{Lip}}^{(3)} / p \right) & \text{if } p \in (1, 2) \\
\Delta \left((s[b]_{\text{Lip}} + c_{p})_{e,s,D_{\max},e,k+1}^{(1)} \sigma_{\text{Lip}}^{(1)} / p \right) & \text{if } p \in [2, 2 + d) 
\end{cases}
$$

with $s = p + 1 > 2$, $c_{p}^{(3)} = 2^{(p-3)+\frac{(p-1)(p-2)}{2}}$ and $c_{p}^{(3)}_{e,s,D_{\max},e,k+1}^{(3)} = 2^{(p-3)+\frac{(p-1)\Delta_{\text{max}}}{p}} (1 + \frac{p}{2} \Delta_{\text{max}})$ and 

$$
C_{2+\delta,b,\sigma,T}(I) = e^{\frac{1}{2} b_{1+c_{2}}} |x_0|^{2+\delta} + \frac{C_3}{C_1} \left( e^{b_{1+c_{2}}} - 1 \right)
$$

where $C_1$, $C_2$ and $C_3$ are defined in Lemma 2.4.

Before sharing the proof, we need to present some a priori useful results, mainly the distortion mismatch problem and two lemmas. We reconsider the notations where we replace the indices $t_k$ by $k$ to alleviate notations.

**$(L^r, L^s)$-Problem or Distortion Mismatch Problem**

Let $r, s \in (0, +\infty)$, the $(L^r, L^s)$-problem, also called distortion mismatch problem, consists in determining whether the optimal rate of $L^r$-optimal quantizers holds for $L^s$-quantizers for $s \neq r$, i.e. whether an $L^r$-optimal quantizer $\Gamma_N$ of size $N$ of a random vector $X$ has an $L^s$-optimal convergence rate for $s \neq r$. For $s < r$, it is clear that an $L^r$-optimal quantizer is $L^s$-rate optimal due to the monotony of $r \to \|\cdot\|_r$. When $s$ becomes greater than $r$, we do not have such direct results. This problem was first introduced and treated in [21, 22] for radial density distributions on $\mathbb{R}^d$ and then generalized in [39] for all random vectors satisfying a certain moment condition. In the following theorem, we sum up this result and give a universal non-asymptotic Pierce type optimality result (in the sense of (14)).

**Theorem 2.2** (Extended Pierce’s Lemma). (a) Let $r > 0$ and $X$ be an $\mathbb{R}^d$-valued random vector such that $\mathbb{E}[|X|^r] < +\infty$ for some $r' > r$. Assume that its distribution $\mathbb{P}_X$ has a non-zero absolutely continuous component and let $(\Gamma_N)_{N \geq 1}$ be a sequence of $L^r$-optimal quantizers of $X$. Then, for every $s \in \left(0, \frac{(d+r)r'}{d+r} \right)$,

$$
e_s(\tilde{X}^N, X) \leq \tilde{K}_{d,r,r',s} \sigma_s(X) N^{-\frac{1}{d}}
$$

(24)
where \( \sigma_{r'}(X) = \inf_{a \in \mathbb{R}^d} \|X - a\|_{r'} \) is the \( L^{r'} \)-standard deviation of \( X \) and

\[
\widetilde{K}_{d,r',s} \leq 2^{\frac{s}{r+d-2d}} V_d \frac{1}{\kappa_{X,r}} \min_{\varepsilon \in (0,\frac{1}{2})} \Psi_r(\varepsilon) \left( \int (1 \wedge \|x\|)^{-\frac{(d+r-s)r'}{s}} dx \right)^{1/r+d}
\]

with \( \kappa_{X,r} \) a finite positive constant independent from \( N, V_d \) the volume of the hyper-unit ball and \( \Psi_r(u) = (1 + u) \left( \frac{1}{\sqrt{r}} - u \right)^{-\frac{1}{d+r}} u^{-\frac{d}{d+r}} \).

(b) In particular if \( X \) has finite polynomial moments at any order, then (24) is satisfied for every \( s \in (r, d + r) \) and \( r' > \frac{sd}{d+r-s} \).

The following lemma is a technical one used repeatedly in the proofs in this paper. Its proof will be postponed to the appendix.

**Lemma 2.3.** Let \( r \in [2, +\infty) \) and \( h_0 > 0 \). Let \( Z \in L^r_{\mathbb{P}}(\mathbb{R}) \) with \( \mathbb{E}Z = 0 \) and let \( a \in \mathbb{R}^d, A \in \mathcal{M}(d, q, \mathbb{R}) \).

Then for every \( h \in (0, h_0) \),

\[
\mathbb{E} |a + \sqrt{h}AZ|^r \leq |a|^r (1 + c_{r,1}^{(1)} h + c_{r,0}^{(2)} h) \|A\|^r \mathbb{E} |Z|^r
\]

where \( c_{r,1}^{(1)} = 2^{(r-3)+} \frac{(r-1)(r-2)}{2} \), \( c_{r,0}^{(2)} = 2^{(r-3)+} (r - 1) (1 + \frac{r}{2} h^{\frac{r}{2}-1}) \) and \( \|A\| \) is the operator norm.

The following lemma is important for the proof of Theorem 2.1.

**Lemma 2.4.** Consider \( (X_k)_{0 \leq k \leq n} \) defined by (20) and \( (\bar{X}_k)_{0 \leq k \leq n} \) its recursive quantization sequence defined by (22). Assume that, for every \( k \in \{0, \ldots, n\} \), \( \bar{X}_k \) is a stationary quadratic optimal quantization of \( X_k \) of size \( N_k \) in the sense of (15), with \( \bar{X}_0 = X_0 = x_0 \in \mathbb{R}^d \). For every \( r \geq 2 \) and every \( k \in \{1, \ldots, n\} \),

\[
\mathbb{E} |\bar{X}_k|^r \leq e^{tk(C_1+C_2)} |x_0|^r + \frac{C_3}{C_1 + C_2} \left( e^{tk_{-1}(C_1+C_2)} - 1 \right).
\]

where

\[
C_1 = rL_{b,\sigma} + (r - 1)2^{r-2} + c_{r,1}^{(1)}, \quad C_2 = 2^{r-1} r L_{b,\sigma} \mathbb{E} |Z|^r \Delta_{\max}^r \Delta_{\max}^{r-1} \Delta_{\max} := L_{b,\sigma} 2^{r-1} \Delta_{\max}^r \Delta_{\max} \quad \text{and} \quad C_3 = C_2 + 2^{r-2} L_{b,\sigma} (1 + r \Delta_{\max}^{-1}) (1 + c_{r,1}^{(1)} \Delta_{\max}) \quad \text{with} \quad c_{r,1}^{(1)} \quad \text{and} \quad c_{r,0}^{(2)} \quad \text{defined in Lemma 2.3}.
\]

**Proof.** The starting point is to use inequality (25) with \( a = x + \Delta b(t, x) \) and \( A = \sigma(t, x) \). On the one hand, we notice that

\[
|a| \leq |x| + \Delta L_{b,\sigma}(1 + |x|) \leq |x|(1 + \Delta L_{b,\sigma}) + \Delta L_{b,\sigma}.
\]

Then, using the fact that, for every \( \varepsilon > 0 \),

\[
(\alpha + \beta)^r \leq \alpha^r + r \beta (\alpha + \beta)^{r-1} \\
\leq \alpha^r + r 2^{r-2} \left( (\varepsilon \alpha)^{r-1} \frac{\beta}{\varepsilon r-1} + \beta^r \right) \\
\leq \alpha^r + r 2^{r-2} \left( \beta^r + \varepsilon^r \alpha^r (r - 1) + \frac{\beta^r}{\varepsilon^r(r-1)} \right) \quad \text{(Young’s inequality with \( \frac{r}{r-1} \) and \( r \))} \\
\leq \alpha^r \left( 1 + (r - 1) 2^{r-2} \varepsilon^r \right) + 2^{r-2} \beta^r \left( r + \frac{1}{\varepsilon^r(r-1)} \right),
\]

one has, by considering \( \alpha = |x|(1 + \Delta L_{b,\sigma}) \) and \( \beta = \Delta L_{b,\sigma} \), that

\[
|a|^r \leq |x|^r (1 + \Delta L_{b,\sigma})^r \left( 1 + (r - 1) 2^{r-2} \varepsilon^r \right) + 2^{r-2} \Delta L_{b,\sigma}^r \left( r + \frac{1}{\varepsilon^r(r-1)} \right).
\]
On the other hand,
\[
\|A\| \leq \Delta L_{b,\sigma}(1 + |x|) \quad \text{so that} \quad \|A\|^r \leq 2^{r-1} \Delta^r L_{b,\sigma}^r(1 + |x|^r).
\]

Consequently, Lemma 2.3 yields
\[
E|a + A\sqrt{\Delta Z}|^r \leq |x|^r(1 + \Delta L_{b,\sigma})^r \left(1 + (r - 1)2^{-2} \varepsilon^r\right) \left(1 + c_r^{(1)} \Delta\right) + L_{b,\sigma}^r 2^{r-1} E|Z|^r \Delta^{r+1} c_{r,\Delta_{\max}} |x|^r
+ \left(1 + c_r^{(1)} \Delta\right) 2^{r-2} L_{b,\sigma}^r \Delta^r \left(r + \frac{1}{\varepsilon_r(r-1)}\right) + L_{b,\sigma}^r 2^{r-1} E|Z|^r \Delta^{r+1} c_{r,\Delta_{\max}}^2.
\]

At this stage, we are interested in considering a particular value of \(\varepsilon\) to avoid any explosion at infinity in the rest of the proof. The best choice (up to a multiplicative constant) is
\[
\varepsilon = \Delta^{\frac{1}{2}}.
\]

Now, we recall that \(\Delta \in [0, \Delta_{\max})\), \(\Delta_{\max} > 0\) and denote
\[
C_1 := C_1(r) = r L_{b,\sigma} + (r - 1)2^{-2} + c_r^{(1)} \Delta
\]
\[
C_2 := C_2(r, L_{b,\sigma}, \Delta, \Delta_{\max}) = 2^{-1} L_{b,\sigma}^r E|Z|^r \Delta_{\max}^{r+1} c_{r,\Delta_{\max}} := L_{b,\sigma}^r 2^{r-1} \Delta_{\max}^{r+1} c_{r,\Delta_{\max}, Z}
\]
\[
C_3 := C_3(r, L_{b,\sigma}, \Delta, \Delta_{\max}) = C_2 + 2^{r-2} L_{b,\sigma}^r (1 + r \Delta_{\max}^{-1})(1 + c_r^{(1)} \Delta_{\max})
\]

Having \(1 + x \leq e^x\) yields
\[
E|a + A\sqrt{\Delta Z}|^r \leq |x|^r e^{C_1 \Delta} + \Delta (C_2 |x|^r + C_3) \leq |x|^r e^{C_1 \Delta} (1 + \Delta C_2 e^{-C_1}) + \Delta C_3 \leq e^{(C_1 + C_2)} |x|^r + \Delta C_3.
\]

Thus, since \(E|\tilde{X}_k|^r = E|\tilde{X}_{k-1}(\tilde{X}_{k-1}, \varepsilon_k)|^r\), one can write
\[
E|\tilde{X}_k|^r \leq e^{(C_1 + C_2)} E|\tilde{X}_{k-1}|^r + \Delta C_3.
\]

Using the fact that \(\tilde{X}_{k-1}\) is a stationary quadratic optimal quantization of \(\tilde{X}_{k-1}\) and Jensen inequality yield
\[
E|\tilde{X}_{k-1}|^r = E[E(\tilde{X}_{k-1}|\tilde{X}_{k-1})]^r \leq E[E(|\tilde{X}_{k-1}|^r|\tilde{X}_{k-1})] \leq E|\tilde{X}_{k-1}|^r.
\]

Therefore,
\[
E|\tilde{X}_k|^r \leq e^{(C_1 + C_2)} E|\tilde{X}_{k-1}|^r + \Delta C_3.
\]

Finally, it follows by induction that
\[
E|\tilde{X}_k|^r \leq e^{k \Delta (C_1 + C_2)} E|\tilde{X}_0|^r + \Delta C_3 \sum_{j=0}^{k-1} e^{j \Delta (C_1 + C_2)}
\]
\[
\leq e^{k \Delta (C_1 + C_2)} |x_0|^r + \Delta C_3 \frac{e^{(k-1) \Delta (C_1 + C_2)} - 1}{e^{\Delta (C_1 + C_2)} - 1}
\]
\[
\leq e^{k \Delta (C_1 + C_2)} |x_0|^r + \frac{C_3}{C_1 + C_2} \left(e^{(k-1) \Delta (C_1 + C_2)} - 1\right).
\]

The result is obtained by noting that \(k\Delta = k \frac{T}{n} = t_k\). \(\square\)

**Proof of Theorem 2.1.** The first step of the proof is to show that the function \(\xi_k(\cdot, \varepsilon_{k+1})\) is \(L^p\)-lipschitz continuous with Lipschitz coefficient \(\|\xi_k\|_{\text{Lip}}\) for every \(k \in \{0, \ldots, n - 1\}\). We consider two cases depending on the values of \(p\).

- If \(p \in [2, 2 + d)\): For every \(x, x' \in \mathbb{R}^d\),
\[
E|\xi_k(x, \varepsilon_{k+1}) - \xi_k(x', \varepsilon_{k+1})|^p = E|x - x' + \Delta(b_k(x) - b_k(x')) + \sqrt{\Delta}\varepsilon_{k+1}(\sigma_k(x) - \sigma_k(x'))|^p.
\]

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Since $p \geq 2$, one applies Lemma 2.3 with $a = x - x' + \Delta (b_k(x) - b_k(x'))$ and $A = \sigma_k(x) - \sigma_k(x')$. We have

$$|a|^p \leq (|x - x'| + \Delta [b]_{Lip} |x - x'|)^p \leq |x - x'|^p (1 + \Delta [b]_{Lip})^p \leq |x - x'|^p e^{p\Delta [b]_{Lip}}$$

and

$$\|A\|^p \leq |\sigma|_{Lip}^p |x - x'|^p.$$ 

At this stage, reusing the constants $c_p^{(1)} = 2(p-3)/(p-2)$ and $c_p^{(3)} = 2(p-3)/(p-1)$, one deduces

$$\|A\|^p \|x - x'|^p.$$ 

Consequently, $E_k$ is $L^p$-lipschitz continuous with $|E_k|_{Lip} = e^{\Delta (p[|b]_{Lip} + c_p^{(1)}|) + \Delta (|\sigma|_{Lip}^p c_p^{(3)} e^{p\Delta [b]_{Lip}} + |\sigma|_{Lip}^p c_p^{(1)} e^{p\Delta [b]_{Lip}})}$, for every $k \in \{1, \ldots, n\}$ and $p \in [2, 2 + d)$.

- If $1 < p < 2$: Consider $s = p + 1 > 2$ so that $p - s < 0$. One has

$$\|E_k(x, \varepsilon_{k+1}) - E_k(x', \varepsilon_{k+1})\| = \|E_k(x, \varepsilon_{k+1}) - E_k(x', \varepsilon_{k+1})\|^s \|E_k(x, \varepsilon_{k+1}) - E_k(x', \varepsilon_{k+1})\|^{p - s}.$$

On the one hand,

$$|E_k(x, \varepsilon_{k+1}) - E_k(x', \varepsilon_{k+1})|^{p - s} \leq |x - x'|^{p - s} (1 + \Delta [b]_{Lip} + \sqrt{\Delta} |\sigma(x) - \sigma(x')| |\varepsilon_{k+1}|)^{p - s}$$

$$\leq |x - x'|^{p - s} e^{(p - s)(1 + \Delta [b]_{Lip} + \sqrt{\Delta} |\sigma(x) - \sigma(x')| |\varepsilon_{k+1}|)}$$

(since $1 + x \leq e^x$)

$$\leq |x - x'|^{p - s}.$$ 

On the other hand, one uses inequality (71) from the proof of Lemma 2.3 (see Appendix) and denotes $a = x - x' + \Delta [b]_{Lip} (x - x')$ and $AZ = (\sigma(x) - \sigma(x')) \varepsilon_{k+1}$, to obtain

$$|E_k(x, \varepsilon_{k+1}) - E_k(x', \varepsilon_{k+1})|^s \leq |a|^s (1 + \Delta c^{(1)}_s) + s \left( |a|^{s-1} a / |a| \right) |A\sqrt{\Delta Z}| + \Delta c^{(2)}_{s, \varepsilon|\max|} |AZ|^s.$$ 

At this stage, one notices that $|a|^{s} \leq |x - x'|^{s} (1 + \Delta [b]_{Lip})^{s}$ and that $|AZ|^s = |\sigma|_{Lip}^s |x - x'|^s |\varepsilon_{k+1}|^s$. Then, using $1 + x \leq e^x$, one deduces

$$|E_k(x, \varepsilon_{k+1}) - E_k(x', \varepsilon_{k+1})|^s \leq |x - x'|^s (1 + \Delta c^{(1)}_s) (1 + \Delta [b]_{Lip})^{s} + s \left( |a|^{s-1} a / |a| \right) |A\sqrt{\Delta Z}| + \Delta c^{(2)}_{s, \varepsilon|\max|} |\sigma|_{Lip}^s |x - x'|^s |\varepsilon_{k+1}|^s.$$
Consequently, applying the expectation and keeping in mind that \( \mathbb{E}|AZ| = 0 \), we obtain

\[
\mathbb{E}|\mathcal{E}_k(x, \varepsilon_{k+1}) - \mathcal{E}_k(x', \varepsilon_{k+1})|^p \leq e^{\Delta e(s^{(1)} + \delta[b_{L,\text{lip}}]|x - x'|^p + \Delta c^{(2)}_{s, \Delta_{\text{max}}}[\sigma]^s_{\text{Lip}}|x - x'|^p\mathbb{E}(|\varepsilon_{k+1}|^s)}
\leq |x - x'|^p e^{\Delta e(s^{(1)} + \delta[b_{L,\text{lip}}])(1 + \Delta c^{(2)}_{s, \Delta_{\text{max}}}[\sigma]^s_{\text{Lip}}\mathbb{E}(|\varepsilon_{k+1}|^s) - \Delta e^{(1)} + \delta[b_{L,\text{lip}}])}
\leq |x - x'|^p e^{\Delta e(s^{(1)} + \delta[b_{L,\text{lip}}] + c^{(2)}_{s, \Delta_{\text{max}}}[\sigma]^s_{\text{Lip}}\mathbb{E}(|\varepsilon_{k+1}|^s)}.
\]

Consequently, \( \mathcal{E}_k \) is \( L^p \)-Lipschitz continuous, for every \( k \in \{1, \ldots, n\} \) and \( p \in (1, 2) \), with \( [\mathcal{E}_k]_{\text{Lip}} = e^{\Delta e(s^{(1)} + \delta[b_{L,\text{lip}}] + c^{(2)}_{s, \Delta_{\text{max}}}[\sigma]^s_{\text{Lip}}\mathbb{E}(|\varepsilon_{k+1}|^s)} / p \).

For the second step, we first note that

\[
\|\bar{X}_{k+1} - \bar{X}_{k+1}\|_p = [\mathcal{E}_k]_{\text{Lip}}\|\bar{X}_k - \bar{X}_k\|_p
\leq [\mathcal{E}_k]_{\text{Lip}}\|\bar{X}_k - \bar{X}_k\|_p + [\mathcal{E}_k]_{\text{Lip}}\|\bar{X}_k - \bar{X}_k\|_p.
\]

Then, we show by induction, since \( \bar{X}_0 = \bar{X}_0 \), that

\[
\|\bar{X}_k - \bar{X}_k\|_p \leq \sum_{l=1}^{k-1}[\mathcal{E}_k]_{\text{Lip}}\|\bar{X}_l - \bar{X}_l\|_p.
\]

Consequently,

\[
\|\bar{X}_k - \bar{X}_k\|_p \leq \|\bar{X}_k - \bar{X}_k\|_p + \|\bar{X}_k - \bar{X}_k\|_p \leq \sum_{l=1}^{k}[\mathcal{E}_k]_{\text{Lip}}\|\bar{X}_l - \bar{X}_l\|_p.
\]

Now relying on the fact that \( \bar{X}_l \) is an \( L^2 \)-optimal quantizer of \( \bar{X}_l \) for every \( l \in \{1, \ldots, k\} \), we distinguish two cases: one the one hand, if \( p \in (1, 2) \), we use the monotony of \( p \mapsto \|\cdot\|_p \) and Pierce’s Lemma (14) to conclude that, for every \( l \in \{1, \ldots, k\} \),

\[
\|\bar{X}_l - \bar{X}_l\|_p \leq \|\bar{X}_l - \bar{X}_l\|_2 \leq \kappa_{d, 2, \delta}\|\bar{X}_l\|_{2 + \delta}N_t^{-\frac{1}{2}},
\]

for some \( \delta > 0 \), and, on the other hand, if \( p \in [2, 2 + d) \), we note that \( \bar{X}_l = F_l(\bar{X}_{l-1}, \varepsilon_l) \) has finite polynomial moments at any order since the innovations \( (\varepsilon_k)_{0 \leq k \leq n} \) in the Euler operators are with Gaussian distribution and hence have finite polynomial moments at any order, so one uses section (b) of the distortion mismatch Theorem 2.2 to conclude that the quantization \( \bar{X}_l \) of \( \bar{X}_l \) is \( L^p \)-rate optimal for every \( p \in [2, 2 + d) \), in other words, we consider \( \delta > 0 \) such that \( r' = 2 + \delta > \frac{pd}{\delta + 2 - p} > 2 \) so that

\[
\|\bar{X}_l - \bar{X}_l\|_p \leq \kappa_{d, 2, 2 + \delta}p\|\bar{X}_l\|_{2 + \delta}N_t^{-\frac{1}{2}}.
\]

Hence, for every \( p \in (1, 2 + d) \),

\[
\|\bar{X}_k - \bar{X}_k\|_p \leq (\kappa_{d, 2, 2 + \delta}p \vee \kappa_{d, 2, \delta})\sum_{l=1}^{k}[\mathcal{E}_k]_{\text{Lip}}\|\bar{X}_l\|_{2 + \delta}N_t^{-\frac{1}{2}}.
\]

The result is obtained by plugging (26) for \( r = 2 + \delta > 2 \) in this last inequality. \( \square \)
Remark 2.5. In higher dimensions, an approach to obtain the quantization grid of a multidimensional random variable is by taking the tensor product of one-dimensional quantization grids, that is the independent marginals of the distribution. The product quantization grid hence obtained by independent optimal one-dimensional quantizers is stationary and so this problem is solved in the multidimensional case. However, in most cases, the components of the diffusion $X_i$ are not independent so this is not a very useful technique in practice.

Remark 2.6. We assume that $\tilde{X}_k$ is an $L^p$-optimal quantizer of $X_k$ for every $k \in \{1, \ldots, n\}$. What differs from $L^2$-optimal quantizers is that $L^p$-optimal quantizers are not usually stationary, a property that was very useful in the quadratic case. The beginning of the study is exactly similar to the quadratic framework until we obtain

\[
\mathbb{E}|\tilde{X}_k|^r \leq c^{(C_1+C_2)} \mathbb{E}|\tilde{X}_{k-1}|^r + \Delta C_3.
\]

At this stage, one cannot use the stationarity property. Instead, applying inequality (27) yields

\[
\mathbb{E}|\tilde{X}_{k-1}|^r \leq \mathbb{E}(|\tilde{X}_{k-1} - \tilde{X}_{k-1}| + |\tilde{X}_{k-1}|)^r \leq \mathbb{E}|\tilde{X}_{k-1} - \tilde{X}_{k-1}|^r c^{C_1 \Delta} + \mathbb{E}|\tilde{X}_{k-1}|^r 2^{r-2} \left(r + \frac{1}{\Delta^{r-1}}\right)
\]

where we took $\varepsilon = \Delta^{\frac{1}{r}}$ and denoted $C_4 = (r-1)2^{r-2}$. Then,

\[
\mathbb{E}|\tilde{X}_k|^r \leq \mathbb{E}|\tilde{X}_{k-1} - \tilde{X}_{k-1}|^r c^{(C_1+C_2)\Delta} + e^{(C_1+C_2)\Delta} \mathbb{E}|\tilde{X}_{k-1}|^r 2^{r-2} \left(r + \frac{1}{\Delta^{r-1}}\right) + \Delta C_3
\]

and an induction yields

\[
\mathbb{E}|\tilde{X}_k|^r \leq e^{k(C_1+C_2)\Delta} \left[2^{r-2} \left(r + \frac{1}{\Delta^{r-1}}\right)\right]^k \mathbb{E}|X_0|^r
\]

\[
+ \sum_{i=0}^k e^{(k-i)(C_1+C_2)\Delta} \left(\mathbb{E}|\tilde{X}_{k-1} - \tilde{X}_{k-1}|^r e^{(C_1+C_2)\Delta} + \Delta C_3\right) \left[2^{r-2} \left(r + \frac{1}{\Delta^{r-1}}\right)\right]^k
\]

which clearly diverges as $n$ goes to infinity. The fact that it seems impossible to get rid of the factor $\frac{1}{\Delta}$, without the stationarity property, leads to conclude that we do not obtain satisfactory $L^p$-error bounds with a non-stationary $L^p$-optimal quantizer $\tilde{X}_k$ of $X_k$. However, this is not really problematic since this is a very rare situation in practice because, as mentioned previously, one usually uses quadratic optimal quantizers for numerical purposes.

2.3 Hybrid recursive quantization

When the dimension becomes greater than 1, computing the distribution (grids and transition matrices) of $(\tilde{X}_k)_{0 \leq k \leq n}$ via the recursive formulas (22) cannot be achieved via closed formulas and deterministic optimization procedures. Multi-dimensional extensions can be found in [18] based on product quantization but this approach becomes computationally demanding when the dimension grows, an alternative being to implement a massive "embedded" Monte Carlo simulation. We propose here a third approach based on the quantization of the white noise (here a Gaussian one). This quantization can be part of a pre-processing and kept off line. In the case of a Gaussian noise, highly accurate quantization grids of $\mathcal{N}(0, I_d)$ distribution for dimensions $d = 1$ up to 10 and regularly sampled sizes from $N = 1$ to 1000 can be downloaded from the quantization website www.quantize.maths-fi.com (for non-commercial purposes). In other words, we consider, instead of (22), the following recursive scheme

\[
\begin{cases}
\tilde{X}_k & = \varepsilon_{k-1}(\tilde{X}_{k-1}, \hat{\varepsilon}_k), \\
\hat{X}_k & = \text{Proj}_{\Gamma_k}(\tilde{X}_k),
\end{cases}
\quad \forall k = 1, \ldots, n.
\]

(28)

where $(\hat{\varepsilon}_k)_{k}$ is now a sequence of optimal quantizers of the Normal distribution $\mathcal{N}(0, I_d)$, which are already computed and kept off line. The main advantage of this approach is that using quantization
grids of small size $N_k^\varepsilon$ approaching the Gaussian random vectors $\varepsilon_k$ gives the same precision as a Monte Carlo simulation of much larger size, always having in mind that the optimal quantizers can be computed offline and called when needed. This is a great gain in cost.

In the following, we establish $L^p$-error bounds of this hybrid recursive quantization scheme, for $p \in (1, 2 + \delta)$, in terms of the error between $\hat{X}_k$ and $\hat{X}_k$ and the quantization error between $\varepsilon_k$ and $\hat{\varepsilon}_k$ simultaneously. We recall that $\Delta \in [0, \Delta_{\text{max}})$, $\Delta_{\text{max}} > 0$.

**Theorem 2.7.** Let $p \in (1, 2 + \delta)$ and $\delta > 0$. Consider $(\hat{X}_k)_{0 \leq k \leq n}$ defined by (20) and $(\hat{X}_k)_{0 \leq k \leq n}$ its hybrid recursive quantization sequence defined by (28). Assume that, for every $k \in \{0, \ldots, n\}$, $\hat{X}_k$ is a stationary $L^2$-optimal quantization of $X_k$ of size $N_k^X$ in the sense of (15) with $\hat{X}_0 = X_0 = x_0 \in \mathbb{R}^d$ and $(\hat{\varepsilon}_k)_{0 \leq k \leq n}$ an $L^p$-optimal quantization sequence of the Gaussian distributed sequence $(\varepsilon_k)_{0 \leq k \leq n}$ of size $N_k^\varepsilon$. For every $k \in \{1, \ldots, n\}$,

$$
\|\hat{X}_k - \hat{X}_k\|_p \leq (\tilde{K}_{d,2,2+\delta,p} \vee \kappa_{d,2,\delta}) \sum_{l=1}^{k} [F^p_{\varepsilon,l+k-l} C^{2p-1}_{2+\delta,b,\sigma,T}(N_k^\varepsilon)^{-\frac{1}{2}} + \sum_{l=1}^{k-1} \kappa_{d,p,\delta}[F^p_{\varepsilon,l+k-l} \|\varepsilon_l\|_p (N_k^\varepsilon)^{-\frac{1}{2}}
$$

where $\kappa_{d,2,\delta}$ is the constant given by Pierce’s Lemma, $\tilde{K}_{d,2,2+\delta,p}$ is given in Theorem 2.2,

$$
C_{2+\delta,b,\sigma,T} = e^{t_k(C_1+C_2)} \|x_0\|^{2+\delta} + \frac{C_3}{C_1+C_2} \left(e^{t_k(C_1+C_2)} - 1\right)
$$

with $C_1, C_2$ and $C_3$ are defined in Lemma 2.4.

$$
[F^p_{\varepsilon}]_{\text{Lip}} = \begin{cases} 
\frac{1}{p} \left(\frac{1}{p} \left(c^{(1)}_{p} + L_{b,\sigma} (p + 2^{p-1} \Delta_{\text{max}})\right)^{\frac{1}{p}} \right) & \text{if } p \in [2, 2 + \delta) \\
\frac{1}{p} \left(\frac{1}{p} \left(c^{(1)}_{p} + L_{b,\sigma} (p + 2^{p-1} \Delta_{\text{max}})\right)^{\frac{1}{p}} \right) & \text{if } p \in (1, 2)
\end{cases}
$$

and

$$
[F^p_{\varepsilon}]_{\text{Lip}} = \begin{cases} 
\Delta_p \left(2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} L_{b,\sigma}\right)^{\frac{1}{p}} & \text{if } p \in [2, 2 + \delta) \\
\Delta_p \left(\frac{1}{p} 2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} L_{b,\sigma}\right)^{\frac{1}{p}} & \text{if } p \in (1, 2)
\end{cases}
$$

where $s = p + 1$, $c^{(1)}_p$ and $c^{(2)}_p,\Delta_{\text{max}}$ are defined in Lemma 2.3.

**Proof.** We start by showing that $\mathcal{E}_k$ is Lipschitz continuous with respect to its two variables. For every $x, x' \in \mathbb{R}^d$ and $\mathbb{R}^d$-valued r.v. $\varepsilon$ and $\varepsilon'$ with standard Normal distribution, we consider two cases depending on the values of $p$.

- If $p \in [2, 2 + \delta)$: Always keeping in mind that $\Delta < \Delta_{\text{max}}$, Lemma 2.3 yields

$$
\mathbb{E}[\mathcal{E}_k(x, \varepsilon) - \mathcal{E}_k(x', \varepsilon')]^p \leq \mathbb{E} \left| x - x' + \Delta (b(x) - b(x')) + \sqrt{\Delta} (\sigma(x) \varepsilon - \sigma(x') \varepsilon') \right|^p \\
\leq \mathbb{E} \left| x - x' + (b(x) - b(x')) \right|^p (1 + c^{(1)}_p \Delta) + \Delta c^{(2)}_{p,\Delta_{\text{max}}} \mathbb{E} \left| \sigma(x) \varepsilon - \sigma(x') \varepsilon' \right|^p \\
\leq \mathbb{E} \left| x - x' \right|^p (1 + \Delta \|b\|_{\text{Lip}}^p (1 + c^{(1)}_p \Delta) + \Delta c^{(2)}_{p,\Delta_{\text{max}}} \mathbb{E} \left| \sigma(x) \varepsilon - \sigma(x') \varepsilon' \right|^p
$$

where $c^{(1)}_p$ and $c^{(2)}_{p,\Delta_{\text{max}}}$ are defined in Lemma 2.3. Now, noticing that $\|\sigma(x) \varepsilon - \sigma(x') \varepsilon'\| = \|\sigma(x) \varepsilon - \sigma(x') \varepsilon'\| + \|\sigma(x) \varepsilon - \sigma(x') \varepsilon'\|$ and using $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ yield

$$
\mathbb{E}[\mathcal{E}_k(x, \varepsilon) - \mathcal{E}_k(x', \varepsilon')]^p \leq \mathbb{E} \left| x - x' \right|^p (1 + \Delta \|b\|_{\text{Lip}}^p (1 + c^{(1)}_p \Delta) \\
+ 2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} \Delta \mathbb{E} \left| \sigma(x) \varepsilon - \sigma(x') \varepsilon' \right|^p + \mathbb{E} \left| \sigma(x') \varepsilon' - \sigma(x') \varepsilon' \right|^p
\leq \mathbb{E} \left| x - x' \right|^p (1 + \Delta \|b\|_{\text{Lip}}^p (1 + c^{(1)}_p \Delta) + 2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} \mathbb{E} \left| \sigma \right|_{\text{Lip}} \mathbb{E} \left| \varepsilon' \right|^p
\leq \mathbb{E} \left| x - x' \right|^p (1 + \Delta \|b\|_{\text{Lip}}^p (1 + c^{(1)}_p \Delta) + 2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} \|\sigma\|_{\infty} \mathbb{E} \left| \varepsilon' \right|^p
\leq \mathbb{E} \left| x - x' \right|^p (1 + \Delta \|b\|_{\text{Lip}}^p (1 + c^{(1)}_p \Delta) + 2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} \|\sigma\|_{\infty} \mathbb{E} \left| \varepsilon' \right|^p
\leq 2^{p-1} c^{(2)}_{p,\Delta_{\text{max}}} \mathbb{E} \left| \sigma \right|_{\infty} \mathbb{E} \left| \varepsilon' \right|^p.
$$

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Now, using the fact that $1 + x \leq e^x$ yields

$$\mathbb{E}[\xi_k(x, \varepsilon) - \xi_k(x', \varepsilon')]^p \leq e^{\overline{C}A} |x - x'|^p + \Delta \overline{C}\mathbb{E}|\varepsilon - \varepsilon'|^p$$

where $\overline{C} = p[b]_{\text{Lip}} + c_p^{(1)} + 2(p-3)_{+} + p-1(1 + \frac{p}{2}\Delta_{\text{max}}^s)[\sigma]_{\text{Lip}}$ and $\overline{C} = 2(p-3)_{+} + p-1(1 + \frac{p}{2}\Delta_{\text{max}}^s)||\sigma||_{\infty}$. Then, applying $(a + b)^\frac{1}{p} \leq a^\frac{1}{p} + b^\frac{1}{p}$ for $a, b > 0$ and $p > 1$ yields

$$\|\xi_k(x, \varepsilon) - \xi_k(x', \varepsilon')\|_p \leq e^{\frac{\overline{C}}{p}} \|x - x'\|_p + (\Delta \overline{C})^\frac{1}{p} \|\varepsilon - \varepsilon'\|_p.$$
Using the fact that $\varepsilon$ is independent of $\{x, x'\}$ and applying Young inequality with the conjugate exponents $\frac{p}{s}$ and $\frac{p}{s-1}$ to $E[|\varepsilon - \varepsilon'|^s | x - x'|^{p-s}]$ yields

$$
E|c_k(x, \varepsilon) - c_k(x', \varepsilon')|^p \leq E|x - x'|^p \left( e^{\Delta(c_1^{(1)} + s|b|_{\text{Lip}})} + \Delta c_2^{(2)} \right) 2^{s-1} |\sigma|_{\text{Lip}}^s E|\varepsilon|^s \\
+ \Delta c_2^{(2)} 2^{s-1} |\sigma|_{\text{Lip}}^s \left( \frac{s}{p} E|\varepsilon - \varepsilon'|^p + \frac{p - s}{p} E|x - x'|^p \right) \\
\leq E|x - x'|^p \left( e^{\Delta(c_1^{(1)} + s|b|_{\text{Lip}})} + \Delta \tilde{\kappa}_1 \right) + \Delta \tilde{\kappa}_2 E|\varepsilon - \varepsilon'|^p \\
\leq E|x - x'|^p e^{\Delta(c_1^{(1)} + s|b|_{\text{Lip}})} (1 + \Delta \tilde{\kappa}_1 e^{-\Delta(c_1^{(1)} + s|b|_{\text{Lip}})}) + \Delta \tilde{\kappa}_2 E|x - x'|^p \\
\leq E|x - x'|^p e^{\Delta(c_1^{(1)} + s|b|_{\text{Lip}})} + \Delta \tilde{\kappa}_2 E|\varepsilon - \varepsilon'|^p \\
where \tilde{\kappa}_1 = c_2^{(2)} 2^{s-1} \left( |\sigma|_{\text{Lip}}^s E|\varepsilon|^s + |\sigma|_{\text{Lip}}^s \frac{p-s}{p} \right) and \tilde{\kappa}_2 = c_2^{(2)} 2^{s-1} |\sigma|_{\text{Lip}}^s \frac{p}{p}. Then,

$$
\left\| c_k(x, \varepsilon) - c_k(x', \varepsilon') \right\|_p \leq \left\| x - x' \right\|_p e^{\Delta \kappa_1} + \left\| \varepsilon - \varepsilon' \right\|_p \Delta \tilde{\kappa}_2 \frac{1}{p} \\
where \kappa_1 = (c_1^{(1)} + s|b|_{\text{Lip}} + \tilde{\kappa}_1)/p and \kappa_2 = \frac{1}{\tilde{\kappa}_2}. Consequently, $c_k$ is lipschitz continuous for $k \in \{1, \ldots, n\}$ with Lipschitz coefficients $[F^x]_{\text{Lip}} \leq e^{\Delta \kappa_1}$ and $[F^\varepsilon]_{\text{Lip}} \leq \Delta \frac{1}{\tilde{\kappa}_2}$, for $p \in (1, 2)$.

For the section step, the Lipschitz continuity of $c_k$ yields

$$
\left\| \tilde{X}_{k+1} - \tilde{X}_{k+1} \right\|_p \leq \left\| c_k(\tilde{X}_k, \varepsilon_k) - c_k(\tilde{X}_k, \tilde{\varepsilon}_k) \right\|_p \\
\leq [F^x]_{\text{Lip}} \left\| \tilde{X}_k - \tilde{X}_k \right\|_p + [F^\varepsilon]_{\text{Lip}} \left\| \varepsilon_k - \tilde{\varepsilon}_k \right\|_p \\
\leq [F^x]_{\text{Lip}} \left\| \tilde{X}_k - \tilde{X}_k \right\|_p + [F^\varepsilon]_{\text{Lip}} \left\| \varepsilon_k - \tilde{\varepsilon}_k \right\|_p.
$$

Then, by induction, one has

$$
\left\| \tilde{X}_k - \tilde{X}_k \right\|_p \leq \sum_{l=1}^{k-1} [F^x]_{\text{Lip}}^{k-l} \left\| \tilde{X}_l - \tilde{X}_l \right\|_p + [F^\varepsilon]_{\text{Lip}}^{k-l} \left\| \varepsilon_l - \tilde{\varepsilon}_l \right\|_p
$$

so that

$$
\left\| \tilde{X}_k - \tilde{X}_k \right\|_p \leq \left\| \tilde{X}_k - \tilde{X}_k \right\|_p + \left\| \tilde{X}_k - \tilde{X}_k \right\|_p \leq \sum_{l=1}^{k} [F^x]_{\text{Lip}}^{k-l} \left\| \tilde{X}_l - \tilde{X}_l \right\|_p + \sum_{l=1}^{k-1} [F^\varepsilon]_{\text{Lip}}^{k-l} \left\| \varepsilon_l - \tilde{\varepsilon}_l \right\|_p.
$$

Now, since $\tilde{\varepsilon}_l$ is an optimal quantization of $\varepsilon_l$ of size $N_l^\varepsilon$, then Pierce’s Lemma 1.1(b) yields

$$
\left\| \tilde{X}_k - \tilde{X}_k \right\|_p \leq \sum_{l=1}^{k} [F^x]_{\text{Lip}}^{k-l} \left\| \tilde{X}_l - \tilde{X}_l \right\|_p + \sum_{l=1}^{k-1} [F^\varepsilon]_{\text{Lip}}^{k-l} \left\| \varepsilon_l + \tilde{\varepsilon}_l \right\|_p + \frac{1}{\tilde{\kappa}_2} N_l^\varepsilon - \tilde{\varepsilon}.
$$

As for the error terms $\|\tilde{X}_l - \tilde{X}_l\|_p$, one uses the same techniques as in the end of the proof of Theorem 2.1, namely the distortion mismatch Theorem 2.2 and Lemma 2.4, to deduce the result. \hfill \Box

### 3 Time discretization of the RBSDE

We consider the reflected backward stochastic differential equation RBSDE (1) with maturity $T$ given in the introduction and recalled below

$$
Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s.dW_s, \quad t \in [0, T],
$$
\[ Y_t \geq h(t, X_t) \quad \text{and} \quad \int_0^T (Y_s - h(s, X_s))dK_s = 0 \]

where \((W_t)_{t \geq 0}\) is a \(q\)-dimensional Brownian motion independent of \(X_0\) and \((X_t)_{t \geq 0}\) is an \(\mathbb{R}^d\)-valued Brownian diffusion process solution to the SDE (3) given in the introduction and recalled below

\[ X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad X_0 = x_0 \in \mathbb{R}^d, \]

As explained, we need to approximate the solutions of these equations by discretization schemes. The time and space discretization of the forward process \((X_t)_{t \in [0,T]}\) have already been investigated and detailed in Section 2. We proceed now with the time discretization of the solution of the RBSDE. Plugging the time-discretized process \((\tilde{X}_{tk})_{0 \leq k \leq n}\) in (1) will not make it possible to find an exact solution for the RBSDE. Another approximation is needed, in which we discretize the term \(Z_t\) itself: considering a sequence \((\varepsilon_k)_{0 \leq k \leq n}\) of i.i.d. normally distributed random variables, the time discretization scheme associated to \((Y_t, Z_t)\) is given by the following backward recursion

\[
\begin{align*}
\hat{Y}_T &= g(\bar{X}_T) \\
\hat{Y}_{tk} &= \mathbb{E}(\hat{Y}_{tk+1} | \mathcal{F}_{tk}) + \Delta \mathcal{E}_k(\bar{X}_{tk}, \mathbb{E}(\hat{Y}_{tk+1} | \mathcal{F}_{tk}), \bar{Z}_{tk}) , \quad k = 0, \ldots, n - 1 \\
\bar{Z}_{tk} &= \frac{1}{\sqrt{\Delta}} \mathbb{E}(\hat{Y}_{tk+1} \varepsilon_{k+1} | \mathcal{F}_{tk}), \quad k = 0, \ldots, n - 1, \\
\hat{Y}_{tk} &= \hat{Y}_{tk} \lor h_k(\bar{X}_{tk}), \quad k = 0, \ldots, n - 1.
\end{align*}
\]

As stated previously, this scheme differs from what was previously studied in the literature (see the references in the Introduction) since the conditional expectation is applied directly to \(\hat{Y}_{tk+1}\) inside the driver function which depends itself on the discretization \(\bar{Z}_{tk}\) of \(Z_t\). That is why it is interesting to establish a priori estimates for the error induced by the approximation with such a time discretization scheme. We note that, among others, time discretization errors for RBSDEs with a driver independent of \(Z_t\) were established in [2], errors for BSDEs (without reflection) with a driver depending on \(Z_t\) and on the conditional expectation of \(\hat{Y}_t\) in [39] and those for BSDEs (without reflection) with a driver depending on \(Z_t\) but where the conditional expectation is applied to the whole function \(f\) were studied in [42].

Since \(\bar{X}_{tk}\) is a Markov chain, one shows that there exists, for every \(k \in \{0, \ldots, n\}\), Borel functions \(\bar{y}_{tk}, \bar{y}_{tk}^*\) and \(\bar{z}_{tk}\) such that \(\bar{Y}_{tk} = \bar{y}_{tk}(\bar{X}_{tk})\), \(\bar{Y}_{tk}^* = \bar{y}_{tk}^*(\bar{X}_{tk})\) and \(\bar{Z}_{tk} = \bar{z}_{tk}(\bar{X}_{tk})\) and defined by

\[
\begin{align*}
\bar{y}_{tk}(x) &= g(x) \\
\bar{y}^*_{tk}(x) &= E(\bar{y}_{tk+1}(\mathcal{E}_k(x, \varepsilon_{k+1}))) \quad \Delta \mathcal{E}_k(x, \mathbb{E}(\bar{y}_{tk+1}(\mathcal{E}_k(x, \varepsilon_{k+1})))\bar{z}_{tk}(x)) \\
\bar{z}_{tk}(x) &= \frac{1}{\sqrt{\Delta}} \mathbb{E}(\bar{y}_{tk+1}(\mathcal{E}_k(x, \varepsilon_{k+1}))\varepsilon_{k+1}) \\
\bar{y}_{tk}(x) &= \bar{y}_{tk}(x) \lor h_k(x) \\
\bar{y}^*_{tk}(x) &= \bar{y}^*_{tk}(x) \lor h_k(x),
\end{align*}
\]

where \(\mathcal{E}_k(x, \varepsilon_{k+1}) = x + \Delta b_k(x) + \sqrt{\Delta} \sigma_k(x) \varepsilon_{k+1}\) and \((\varepsilon_k)_{k \geq 0}\) are i.i.d random variables with distribution \(\mathcal{N}(0, I_q)\).

In order to establish error bounds between \((Y_t, Z_t)\) and \((\bar{Y}_{tk}, \bar{Z}_{tk})\), it is useful to introduce a time continuous process which extends \(\bar{Y}_{tk}\). In fact, one notes that since the variable \(\sum_{k=1}^{n-1} \bar{Y}_{tk+1} - E(\bar{Y}_{tk+1} | \mathcal{F}_{tk})\) is square integrable and measurable with respect to the augmented Brownian filtration \(\mathcal{F}_{tk}\), then, by the martingale representation Theorem, it can be considered as the terminal value of a Brownian martingale \(\int_0^T \bar{Z}_s dW_s\) where the process \(\bar{Z}_t\) is such that \(\mathbb{E}\sup_{t \in [0,T]} |\bar{Z}_t|^2 \leq \gamma_1 < +\infty\) for a finite constant \(\gamma_1\). So,

\[
\bar{Y}_{tk+1} - E(\bar{Y}_{tk+1} | \mathcal{F}_{tk}) = \int_{tk}^{tk+1} \bar{Z}_s dW_s \quad \text{for} \quad k = 0, \ldots, n - 1.
\]
Theorem 3.1. Let $Y_t$ be the solution of (1) and $(\tilde{Y}_k)_{0 \leq k \leq n}$ the corresponding time discretized process defined by (33). Assume that the functions $f$ and $h$ are Lipschitz continuous. Then, for every $k \in \{1, \ldots, n\}$, 

$$\mathbb{E}|Y_k - \tilde{Y}_k|^2 \leq C_{b,\sigma,f,h,T} \left( \Delta + \int_0^T \mathbb{E}|Z_s - \tilde{Z}_s|^2 ds \right)$$

where $s = t_k$ if $s \in [t_k, t_{k+1})$ and $C_{b,\sigma,f,h,T}$ is a real positive constant.

Furthermore, there exists a finite constant $C > 0$ such that 

$$\int_0^T \mathbb{E}|Z_s - \tilde{Z}_s|^2 ds \leq C\sqrt{\Delta}.$$

The second part of the theorem is established in [29], see Theorem 6.3. The proof of the first part is postponed to the appendix (see Appendix B).

4 Space discretization of the RBSDE

After the time discretization, we move to the space discretization schemes to approximate the solution of the RBSDE. We rely on the recursive quantization $(\tilde{X}_{t_k})_{0 \leq k \leq n}$ of the time-discretized scheme.
To obtain the recursive quantization scheme associated to (30)-(31)-(32)-(33). If we consider a sequence $(\varepsilon_k)_{0 \leq k \leq n}$ of i.i.d. random variables with distribution $\mathcal{N}(0, I_d)$, this scheme is defined recursively by

$$
\hat{Y}_T = g(\hat{X}_T) \\
\tilde{c}_k = \frac{1}{\sqrt{\Delta}} \mathbb{E}_k (\hat{Y}_{t_{k+1}} - \varepsilon_{k+1}) , \quad k = 0, \ldots, n - 1 ,
$$

$$
\tilde{Y}_{t_k} = \max \left( h_k(\hat{X}_{t_k}), \mathbb{E}_k \hat{Y}_{t_{k+1}} + \Delta \varepsilon_k (\hat{X}_{t_k}, \mathbb{E}_k \hat{Y}_{t_{k+1}}, \tilde{c}_k) \right) , \quad k = 0, \ldots, n - 1 .
$$

where $(\hat{X}_k)_{0 \leq k \leq n}$ is the recursively quantized process associated to $(\hat{X}_k)_{0 \leq k \leq n}$ given by (22) or (28). This quantization scheme is different than the optimal (or marginal) quantization schemes that were usually applied before in these situations, in [2, 25, 39] for example. The main difference is that since recursive quantization preserve the Markov property, the process $\hat{Y}_{t_k}$ is $\mathcal{F}_k$-measurable for every $k \in \{0, \ldots, n\}$ where $\mathcal{F}_k = \sigma(W_{t_1}, \ldots, W_{t_k}, \mathcal{N}_{T_k})$ which is not the case for optimal quantization. More details on the utility of this scheme of recursive quantization will be presented in Section 5.

In the following, we will reconsider the notations with the indices $k$ instead of $t_k$ for every $k \in \{0, \ldots, n\}$, and we establish an upper bound for the quantization error induced by approximating $\hat{Y}_k$ by $\hat{Y}_k$ in $L^p$ for $p \in (1, 2 + d)$ and $k \in \{1, \ldots, n\}$. We recall that $\Delta \in [0, \Delta_{\text{max}})$, $\Delta_{\text{max}} > 0$.

**Theorem 4.1.** Let $(\hat{Y}_k)_{0 \leq k \leq n}$ be the time-discretized process defined by (33) and $(\hat{Y}_k)_{0 \leq k \leq n}$ the corresponding recursive quantized process defined by (45). For every $p \in (1, 2 + d)$ and every $k \in \{1, \ldots, n\}$,

$$
\| \hat{Y}_k - \hat{Y}_k \|_p \leq \left( \kappa_1 \left( e^{(p-1)\Delta} - 1 \right) + \kappa_2 \left( [g]_{\text{Lip}}^p + [h]_{\text{Lip}}^p \right) \right) \max_{k \leq t \leq n} \| \hat{X}_t - \hat{X}_t \|_p
$$

where $\kappa_1 \geq p \Delta + (p - 1)2^{p-2}$, $\kappa_2 \geq 2^{p-2}[f]_{\text{Lip}}^p (1 + p\Delta^{p-1})$ and $\kappa = \kappa_1 + [g]_{\text{Lip}}^p + [h]_{\text{Lip}}^p c_{\Delta_{\text{max}}}^{(3)}$. The positive finite constants $c_{\Delta_{\text{max}}}^{(3)}$ and $c_{\Delta_{\text{max}}, \Delta_{\text{max}}^{2}}^{(3)}$ are defined in Lemmas 2.3 and 2.4.

**Remark 4.2.** The norms $\| \hat{X}_t - \hat{X}_t \|_p$ are recursive quantization errors established in Theorems 2.1 and 2.7 for $p \in (1, 2 + d)$. We recall that, for every $l \in \{1, \ldots, n\}$, one has $\| \hat{X}_l - \hat{X}_l \|_p = O(N_l^{-\frac{1}{2}})$ where $N_l$ is the size of the quantization grid corresponding to $\hat{X}_l$.

**Proof.** For every $k \in \{1, \ldots, n\}$, we use the inequality $|\max(a, b) - \max(a', b')| \leq \max(|a - a'|, |b - b'|)$ and have

$$
|\hat{Y}_k - \hat{Y}_k| \leq \max \left( |h_k(\hat{X}_k) - h_k(\hat{X}_k)|, \left| \mathbb{E}_k \hat{Y}_{t_{k+1}} - \mathbb{E}_k \hat{Y}_{t_{k+1}} + \Delta \varepsilon_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{t_{k+1}}, \tilde{c}_k) - \varepsilon_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{t_{k+1}}, \tilde{c}_k) \right| \right)
$$

We denote $\beta_k = \mathbb{E}_k (\hat{Y}_{k+1} - \hat{Y}_{k+1}) + \Delta \left( \mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k) - \varepsilon_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k) \right)$ and we have

$$
\beta_k = \mathbb{E}_k (\hat{Y}_{k+1} - \hat{Y}_{k+1}) + \Delta \left( \hat{A}_k (\hat{X}_k - \hat{X}_k) + \hat{B}_k \mathbb{E}_k (\hat{Y}_{k+1} - \hat{Y}_{k+1}) + \frac{\hat{C}_k}{\Delta} \mathbb{E}_k (\hat{Y}_{k+1} - \hat{Y}_{k+1}) \varepsilon_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k) \right)
$$

where

$$
\hat{A}_k = \frac{\mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k) - \mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k)}{\hat{X}_k - \hat{Y}_k} \mathbb{1}_{\hat{X}_k \neq \hat{X}_k},
$$

$$
\hat{B}_k = \frac{\mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k) - \mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k)}{\mathbb{E}_k (\hat{Y}_{k+1} - \hat{Y}_{k+1})} \mathbb{1}_{\hat{Y}_{k+1} \neq \hat{Y}_{k+1}},
$$

$$
\hat{C}_k = \frac{\mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k) - \mathcal{E}_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k)}{\mathbb{E}_k (\hat{Y}_{k+1} - \hat{Y}_{k+1}) \varepsilon_k (\hat{X}_k, \mathbb{E}_k \hat{Y}_{k+1}, \tilde{c}_k)} \mathbb{1}_{\tilde{c}_k \neq \tilde{c}_k}.
$$
It is clear that \( \max(|\hat{A}_k|, |\hat{B}_k|, |\hat{C}_k|) \leq [f]_{\text{Lip}} \), so one has
\[
|\beta_k| \leq \Delta[f]_{\text{Lip}}|\hat{X}_k - \hat{X}_k| + E_k |(1 + \Delta \hat{B}_k + \sqrt{\Delta \hat{C}_k \varepsilon_{k+1}})(\hat{Y}_{k+1} - \hat{Y}_{k+1})|.
\]
At this stage, we consider two conjugate exponents \( r \in (1, 2 \wedge p) \) and \( s = \frac{r}{r - 1} > 2 \) and we apply conditional Hölder’s inequality
\[
E_k |(1 + \Delta \hat{B}_k + \sqrt{\Delta \hat{C}_k \varepsilon_{k+1}})(\hat{Y}_{k+1} - \hat{Y}_{k+1})| \leq (E_k |1 + \Delta \hat{B}_k + \sqrt{\Delta \hat{C}_k \varepsilon_{k+1}}|^s)^{\frac{1}{s}} (E_k |\hat{Y}_{k+1} - \hat{Y}_{k+1}|^r)^{\frac{1}{r}}.
\]
Since \( s > 2 \), one can apply Lemma 2.3 with \( a = 1 + \Delta \hat{B}_k \) and \( A = \hat{C}_k \) and obtains
\[
E_k |1 + \Delta \hat{B}_k + \sqrt{\Delta \hat{C}_k \varepsilon_{k+1}}|^s \leq (1 + \Delta[f]_{\text{Lip}})(1 + c_s^{(1)} \Delta) + \Delta[f]_{\text{Lip}}^{\epsilon_{s, \Delta_{\text{max}}, \varepsilon_{k+1}}} \\
\leq e^{\Delta[c_s^{(1)} + \Delta[f]_{\text{Lip}}^{\epsilon_{s, \Delta_{\text{max}}, \varepsilon_{k+1}}}} \\
\leq e^{\Delta[c_s^{(1)} + \Delta[f]_{\text{Lip}}]} (1 + \Delta[f]_{\text{Lip}}^{\epsilon_{s, \Delta_{\text{max}}, \varepsilon_{k+1}}}) e^{-\Delta(c_s^{(1)} + [f]_{\text{Lip}})} \\
\leq e^{\Delta[c_s^{(1)} + \Delta[f]_{\text{Lip}} + [f]_{\text{Lip}}^{\epsilon_{s, \Delta_{\text{max}}, \varepsilon_{k+1}}}}
\]
where \( c_s^{(1)} \) and \( c_s^{(3), \Delta_{\text{max}}, \varepsilon_{k+1}} \) are real constants defined in Lemmas 2.3 and 2.4. Therefore,
\[
|\beta_k| \leq \Delta[f]_{\text{Lip}}|\hat{X}_k - \hat{X}_k| + e^{\Delta} \left(E_k |\hat{Y}_{k+1} - \hat{Y}_{k+1}|^r\right)^{\frac{1}{r}},
\]
where \( \kappa = \frac{c_s^{(1)} + \Delta[f]_{\text{Lip}} + [f]_{\text{Lip}}^{\epsilon_{s, \Delta_{\text{max}}, \varepsilon_{k+1}}}}{s} \), and
\[
|\hat{Y}_k - \hat{Y}_k|^p \leq \max \left([h]_{\text{Lip}}^p |\hat{X}_k - \hat{X}_k|^p, |\beta_k|^p\right).
\]
Now, using inequality (27) yields
\[
|\beta_k|^p \leq \kappa^{\frac{1}{p}} \left(E_k |\hat{Y}_{k+1} - \hat{Y}_{k+1}|^r\right)^{\frac{1}{p}} \left(1 + (p - 1)2^{p-2}e^{p}\right) + 2^{p-2}f^p_{\text{Lip}} |\hat{X}_k - \hat{X}_k|^p \Delta^p \left(p + \frac{1}{e^{p(p-1)}}\right).
\]
We choose \( \varepsilon = \Delta^{\frac{1}{p}} \) so that \( \Delta^p \left(p + \frac{1}{e^{p(p-1)}}\right) = \Delta(1 + p\Delta^{p-1}) \) and hence
\[
|\beta_k|^p \leq \kappa^{\frac{1}{p}} \left(E_k |\hat{Y}_{k+1} - \hat{Y}_{k+1}|^r\right)^{\frac{1}{p}} + \Delta \kappa_2 |\hat{X}_k - \hat{X}_k|^p
\]
where \( \kappa_1 = p\kappa + (p - 1)2^{p-2} \) and \( \kappa_2 = 2^{p-2}f^p_{\text{Lip}} \). Moreover, by our choice of \( r \), we have that \( \frac{p}{r} > 1 \) so we apply Jensen’s inequality and obtain
\[
|\beta_k|^p \leq e^{\kappa_1 \Delta} E_k |\hat{Y}_{k+1} - \hat{Y}_{k+1}|^p + \Delta \kappa_2 |\hat{X}_k - \hat{X}_k|^p.
\]
Hence, having in mind that \( \hat{X}_k, \hat{X}_k, \hat{Y}_k \) and \( \hat{Y}_k \) are all \( \mathcal{F}_k \)-measurable processes, one has
\[
E_k |\hat{Y}_k - \hat{Y}_k|^p \leq \max \left([h]_{\text{Lip}}^p E_k |\hat{X}_k - \hat{X}_k|^p, e^{\kappa_1 \Delta} E_k |\hat{Y}_{k+1} - \hat{Y}_{k+1}|^p + \Delta \kappa_2 E_k |\hat{X}_k - \hat{X}_k|^p\right). \quad (47)
\]
At this stage, we aim to prove that \( E_k |\hat{Y}_k - \hat{Y}_k|^p \) satisfies the following backward induction
\[
E_k |\hat{Y}_k - \hat{Y}_k|^p \leq e^{(n-k)\kappa_1 \Delta}( [g]^p_{\text{Lip}} \vee [h]^p_{\text{Lip}}) E_k \max_{k \leq i \leq n} |\hat{X}_i - \hat{X}_i|^p + \Delta \kappa_2 \sum_{i=k}^{n-1} e^{(i-k)\kappa_1 \Delta} E_k |\hat{X}_i - \hat{X}_i|^p. \quad (48)
\]
First, it is clear that \( \mathbb{E}_n|\bar{Y}_n - \bar{Y}_n|^p \leq |g|^p_{\text{Lip}} \mathbb{E}_n |\bar{X}_n - \bar{X}_n|^p \) so the induction is satisfied for \( k = n \). We assume that (48) is true for \( k + 1 \), i.e.

\[
\mathbb{E}_{k+1} |\bar{Y}_{k+1} - \bar{Y}_{k+1}|^p \leq e^{(n-k-1)\Delta} \left( |g|^p_{\text{Lip}} \lor |h|^p_{\text{Lip}} \mathbb{E}_{k+1} \max_{k+1 \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \right) + \Delta \kappa_2 \sum_{i=k+1}^{n-1} e^{(i-k-1)\kappa_1 \Delta} |\bar{X}_i - \bar{X}_i|^p \tag{49}
\]

and show it for \( k \). In fact, since \( \mathbb{E}_k \mathbb{E}_k(\cdot) = \mathbb{E}_k(\cdot) \), one has, by merging (47) with (49), that

\[
\mathbb{E}_k |\bar{Y}_k - \bar{Y}_k|^p \leq \max \left( |h|^p_{\text{Lip}} \mathbb{E}_k |\bar{X}_k - \bar{X}_k|^p, e^{\kappa_1 \Delta} \mathbb{E}_k \mathbb{E}_{k+1} |\bar{Y}_{k+1} - \bar{Y}_{k+1}|^p + \Delta \kappa_2 \mathbb{E}_k |\bar{X}_k - \bar{X}_k|^p \right) \\
\leq \max \left( |h|^p_{\text{Lip}} \mathbb{E}_k |\bar{X}_k - \bar{X}_k|^p, e^{\kappa_1 \Delta} \mathbb{E}_k \mathbb{E}_{k+1} \max_{k+1 \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \right) + e^{(n-k)\kappa_1 \Delta} \left( |g|^p_{\text{Lip}} \lor |h|^p_{\text{Lip}} \mathbb{E}_k \mathbb{E}_{k+1} \max_{k \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \right) \\
\leq \max \left( |h|^p_{\text{Lip}} \mathbb{E}_k |\bar{X}_k - \bar{X}_k|^p, e^{(n-k)\kappa_1 \Delta} \left( |g|^p_{\text{Lip}} \lor |h|^p_{\text{Lip}} \mathbb{E}_k \mathbb{E}_{k} \max_{k \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \right) \right) \\
+ \Delta \kappa_2 \sum_{i=k}^{n-1} e^{(i-k)\kappa_1 \Delta} |\bar{X}_i - \bar{X}_i|^p 
\]

since \( \max_{k \leq i \leq n} \alpha_i = \max_{k \leq i \leq n} \alpha_i \) for \( \alpha_i > 0 \). Furthermore, noticing that

\[
|h|^p_{\text{Lip}} \mathbb{E}_k |\bar{X}_k - \bar{X}_k|^p \leq \left( |g|^p_{\text{Lip}} \lor |h|^p_{\text{Lip}} \mathbb{E}_k \max_{k \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \right) \leq e^{(n-k)\kappa_1 \Delta} \left( |g|^p_{\text{Lip}} \lor |h|^p_{\text{Lip}} \mathbb{E}_k \mathbb{E}_{k} \max_{k \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \right) 
\]

because \( e^{(n-k)\kappa_1 \Delta} > 1 \), one concludes the induction (48). This yields

\[
\mathbb{E}_k |\bar{Y}_k - \bar{Y}_k|^p \leq e^{|\Delta|} \mathbb{E}_k \mathbb{E}_{k} \max_{k \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p + \Delta \kappa_2 \mathbb{E}_k \mathbb{E}_{k} \max_{k \leq i \leq n} |\bar{X}_i - \bar{X}_i|^p \sum_{i=k}^{n-1} e^{(i-k)\kappa_1 \Delta}. \tag{50}
\]

Finally, since \( e^x - 1 \geq x \) for \( x \geq 0 \), one has

\[
\sum_{i=k}^{n-1} e^{(i-k)\kappa_1 \Delta} = \frac{e^{(n-k)\kappa_1 \Delta} - 1}{e^{\kappa_1 \Delta} - 1} \leq \frac{e^{|\Delta|} - 1}{\Delta \kappa_1}
\]

and then deduces the result by taking the expectation in (50).

\[\square\]

## 5 Algorithmics

Our aim is to write \( \tilde{Y}_k, \tilde{\tilde{Y}}_k \), which approximates the solution of the RBSDE (1), in a form that allows us to compute their values. For this, we first note that \( (\bar{X}_k)_{0 \leq k \leq n} \) and \( (\bar{X}_k)_{0 \leq k \leq n} \) are both Markov chains where \( F_{t_k} = \sigma(W_s, s \leq t_k, \mathbb{N}_P) \), for every \( k \in \{0, \ldots, n\} \), with respective transitions \( P_k(x, dy) = \mathbb{P}(\bar{X}_{k+1} \in dy | \bar{X}_k = x) \) and \( P_k(x, dy) = \mathbb{P}(\bar{X}_{k+1} \in dy | \bar{X}_k = x) \). The main advantage of recursive quantization is that it preserves the Markovian property of \( (\bar{X}_k)_{0 \leq k \leq n} \) with respect to the filtration \( \mathcal{F}_{t_k} \). Note that, for optimal quantization, the trick was to force the Markov property by conditioning with respect to the filtration \( \tilde{F}_{t_k} = \sigma(\bar{X}_0, \ldots, \bar{X}_k) \) instead of \( \mathcal{F}_{t_k} \) in (44)-(45). The price to pay is that the approximations \( \| \bar{X}_k - \bar{X}_k \|_p \), for every \( k \in \{1, \ldots, n\} \), are less accurate (but not in a drastic way). This point is discussed in details in [39].
For every bounded or non-negative Borel function \( f \), one has \( P_k f(x) = \int_{\mathbb{R}^d} f(y) P_k(x, dy) \), so that

\[
\mathbb{E}(f(\bar{X}_{k+1}) \mid \mathcal{F}_k) = P_k f(\bar{X}_k) \quad \text{and} \quad \mathbb{E}(f(\bar{X}_{k+1}) \mid \mathcal{F}_k) = \tilde{P}_k f(\bar{X}_k).
\]

Moreover, we introduce

\[
Q_k f(\bar{X}_k) = \frac{1}{\sqrt{\Delta}} \mathbb{E}(f(\bar{X}_{k+1}) \varepsilon_{k+1} \mid \mathcal{F}_k) \quad \text{and} \quad \tilde{Q}_k f(\bar{X}_k) = \frac{1}{\sqrt{\Delta}} \mathbb{E}(f(\bar{X}_{k+1}) \varepsilon_{k+1} \mid \mathcal{F}_k)
\]

where \( \varepsilon_{k} \) are i.i.d. with Normal distribution \( \mathcal{N}(0, I_d) \).

Similarly to the functions \( (\bar{y}_k)_{0 \leq k \leq n} \) defined by (37), one shows that there exists Borel functions \( (\bar{y}_k)_{0 \leq k \leq n} \) such that \( \bar{Y}_k = \bar{y}_k(\bar{X}_k) \) for every \( k \in \{0, \ldots, n \} \). They are defined recursively by the following Backward Dynamic Programming Principle (BDPP)

\[
\begin{aligned}
&\bar{y}_n = h_n \\
&\bar{y}_k = \max \left\{ h_k, \tilde{P}_k \bar{y}_{k+1} + \Delta \varepsilon_k, \tilde{Q}_k \bar{y}_{k+1} \right\}, \quad k = 0, \ldots, n-1,
\end{aligned}
\]

This BDPP can also be written in distribution, one can write \( (\bar{y}_k)_{0 \leq k \leq n} \) as

\[
\begin{aligned}
&\bar{y}_n = h_n \\
&\bar{y}_k = \max \left\{ h_k, P_k \bar{y}_{k+1} + \Delta \varepsilon_k, P_k \bar{y}_{k+1}, Q_k \bar{y}_{k+1} \right\}, \quad k = 0, \ldots, n-1,
\end{aligned}
\]

The fact that \( \bar{Y}_k = \bar{y}_k(\bar{X}_k) \) and \( \hat{Y}_k = \hat{y}_k(\hat{X}_k) \) can easily be checked by a backward induction relying on (30)-(31)-(33) and (43)-(45) respectively. Furthermore, there exists functions \( \bar{z}_k \) and \( \hat{z}_k \) such that \( \bar{z}_k = \bar{z}_k(\bar{X}_k) \) and \( \hat{z}_k = \hat{z}_k(\hat{X}_k) \), defined by

\[
\bar{z}_k = Q_k \bar{y}_{k+1} \quad \text{and} \quad \hat{z}_k = \hat{Q}_k \hat{y}_{k+1}.
\]

In order to compute \( \hat{Y}_k \) and \( \hat{z}_k \), we first need to compute the optimal (or at least optimized) recursive quantization \( \hat{X}_k \) of \( \bar{X}_k \) for every \( k \in \{0, \ldots, n\} \) and the corresponding transition weights. We will consider the quadratic case \( p = 2 \) for all numerical aspects.

### 5.1 Computation of the recursive quantizers

As defined previously, the recursive quantization of \( (\bar{X}_k)_{0 \leq k \leq n} \) is realized via (22) (or (28)). In a quadratic framework, the computation of the optimal quantization grids \( \Gamma_k \) of \( \bar{X}_k \) of size \( N_k \), at each time step \( t_k \), is achieved by algorithms such as CLVQ (Competitive Learning Vector Quantization), Lloyd’s algorithm or Newton-Raphson. These algorithms are presented in details in [36] for example. Here, we expose a variant of Lloyd’s algorithm for recursive quantization.

For \( k \in \{1, \ldots, n\} \), computing an optimal quantizer \( \hat{X}_k^{\Gamma_k} \) of \( \hat{X}_k \) consists in computing the grid \( \Gamma_k \) solution to the minimization problem

\[
\Gamma_k \in \arg \min \left\{ \| \hat{X}_k^{\Gamma} - \bar{X}_k \|_2^2, \Gamma \subset \mathbb{R}^d, \text{card}(\Gamma) \leq N_k \right\}.
\]

The construction of these grids is performed recursively at each step \( t_k \) in a forward way. It is somehow an embedded optimization. We suppose that, at time \( t_k \), the grid \( \Gamma_k = \{ x_1^k, \ldots, x_{N_k}^k \} \) is already computed (optimized) and that \( \bar{X}_k \) has been quantized by \( \bar{X}_k = \sum_{i=1}^{N_k} x_i^k 1_{C_i(\Gamma_k)} \) where \( (C_i(\Gamma_k))_{1 \leq i \leq N_k} \) is the Voronoi diagram associated to \( \bar{X}_k \) and defined by (11). Then, at time step \( t_{k+1} \), we build the grid \( \Gamma_{k+1} \) that minimizes the quadratic distortion \( G_{k+1}^2(\Gamma) \) defined by (23) and written as a function
of the grid $\Gamma_k = \{x^k_1, \ldots, x^k_{N_k}\}$ computed at the previous step. So, if $\Gamma_{k+1} = \{x^{k+1}_1, \ldots, x^{k+1}_{N_{k+1}}\}$, then one has, for every $j \in \{1, \ldots, N_{k+1}\}$,

$$x^{k+1}_j = \mathbb{E}\left(\bar{X}_{k+1} \mid \bar{X}_{k+1} \in C_j(\Gamma_{k+1})\right) = \frac{\sum_{i=1}^{N_k} p^k_{ij} \mathbb{E}\left(\mathcal{E}_k(x^k_i, \varepsilon_{k+1}) \mathbbm{1}_{\{\mathcal{E}_k(x^k_i, \varepsilon_{k+1}) \in C_j(\Gamma_{k+1})\}}\right)}{p^k_{ij}}. \quad (52)$$

Recalling that $\mathcal{E}_k(x, \varepsilon_{k+1}) = x + \Delta b_k(x) + \sqrt{\Delta \sigma_k(x)} \varepsilon_{k+1}$, it is important to notice that, for every $k \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, N_k\}$, $\mathcal{E}_k(x^k_i, \varepsilon_{k+1}) \sim \mathcal{N}(m^k_i, \Sigma^k_i)$ where $m^k_i = x^k_i + \Delta b_k(x^k_i)$ and $\Sigma^k_i = \sqrt{\Delta \sigma_k(x^k_i)}$.

We are interested in more than just computing the distribution of $(\bar{X}_k)_{0 \leq k \leq n}$; the computation of the transition matrices $P_k = (p^k_{ij})_{ij}$ is even more fundamental among the companion parameters in view of our applications. For every $k \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots, N_k\}$, the transition probability $p^k_{ij}$ from $x^k_i$ to $x^k_j$ is given by

$$p^k_{ij} = \mathbb{P}\left(\bar{X}_{k+1} \in C_j(\Gamma_{k+1}) \mid \bar{X}_k \in C_i(\Gamma_k)\right) = \mathbb{P}\left(\mathcal{E}_k(x^k_i, \varepsilon_{k+1}) \in C_j(\Gamma_{k+1})\right). \quad (53)$$

This identity allows the computation of the weights $p^k_{ij}$ of the Voronoï cells $C_j(\Gamma_{k+1})$, for every $j \in \{1, \ldots, N_{k+1}\}$, via the classical (discrete time) forward Kolmogorov equation. They are given by

$$p^k_{ij} = \mathbb{P}(\bar{X}^{k+1} \in C_j(\Gamma_{k+1})) = \sum_{i=1}^{N_k} p^k_{ij} \mathbb{P}(\mathcal{E}_k(x^k_i, \varepsilon_{k+1}) \in C_j(\Gamma_{k+1})). \quad (54)$$

**One-dimensional setting** $q = d = 1$: The transition weights $p^k_{ij}$ can be computed in a direct way as follows: for every $i \in \{1, \ldots, N_k\}$ and $j \in \{1, \ldots, N_{k+1}\}$

$$p^k_{ij} = \mathbb{P}\left(\bar{X}_{k+1} \leq x^{k+1}_j \mid \bar{X}_k = x^k_i\right) - \mathbb{P}\left(\bar{X}_{k+1} \leq x^{k+1}_{j-\frac{1}{2}} \mid \bar{X}_k = x^k_i\right) = \Phi_0(x^{k+1}_{i,j+}) - \Phi_0(x^{k+1}_{i,j-})$$

where $\Phi_0$ is the cumulative distribution function of the standard Normal distribution $\mathcal{N}(0, 1)$ and

$$x^{k+1}_{i,j+} = \frac{x^{k+1}_j - x^k_i - \Delta b_k(x^k_i)}{\sqrt{\Delta \sigma_k(x^k_i)}} \quad \text{and} \quad x^{k+1}_{i,j-} = \frac{x^{k+1}_{j-\frac{1}{2}} - x^k_i - \Delta b_k(x^k_i)}{\sqrt{\Delta \sigma_k(x^k_i)}}$$

with $x^{k+1}_{j+\frac{1}{2}} = \frac{x^{k+1}_j + x^{k+1}_{j+1}}{2}$, $x^{k+1}_{j-\frac{1}{2}} = -\infty$ and $x^{k+1}_{j+\frac{1}{2}} = +\infty$.

**General setting:** In order to approximate the transition probabilities and the weights of the Voronoï cells when $d > 1$, one may proceed with Monte Carlo simulations or rely on Markovian and componentwise product quantization (see [18]). A very interesting alternative is the hybrid recursive quantization, studied in Section 2.3, where we replaced the white Gaussian noise by its optimal quantization sequences. The principle on which we rely to design the hybrid recursive quantizers is the same as the one for the standard recursive quantization. The only difference is with the computation of the expectations and probabilities in (52),(53) and (54). Instead of resorting to large and slow Monte Carlo simulations, we consider sequences of optimal quantizers $(\hat{\mathcal{E}}_l^{k})_{1 \leq l \leq N_e}$ of size $N_e$ of the Gaussian distribution $\mathcal{N}(0, I_d)$, available on the quantization website www.quantize.maths-fi.com, and compute the sequence and its companion parameters based on the following formulas

$$\mathbb{E}\left(\mathcal{E}_k(x^k_i, \varepsilon_k) \mathbbm{1}_{\mathcal{E}_k(x^k_i, \varepsilon_k) \in C_j(\Gamma_{k+1})}\right) = \sum_{l=1}^{N_e} p^k_{il} \mathbb{E}\left(\mathcal{E}_k(x^k_l, \varepsilon_l) \mathbbm{1}_{\mathcal{E}_k(x^k_l, \varepsilon_l) \in C_j(\Gamma_{k+1})}\right). \quad (55)$$
and

\[ P(\mathcal{E}_k(x^k_i, \varepsilon_k) \in C_j(\Gamma_{k+1})) = \sum_{l=1}^{N_k} p_{e_l}^k \mathbb{I}_{\mathcal{E}_k(x^k_i, \varepsilon_k) \in C_j(\Gamma_{k+1})} \]  

(56)

where \( p_{e_l}^k \) is the weight of the Voronoï cell of centroid \( \varepsilon^k_i \), also available on the quantization website.

### 5.2 Computation of the quantized solution of the RBSDE

Having already computed the recursive quantization \( (\tilde{X}_k)_{0 \leq k \leq n} \) of \( (X_k)_{0 \leq k \leq n} \) as described in the previous section 5.1, as well as the corresponding companion parameters (the weights \( (p^k)_{1 \leq i \leq N_k} \) of Voronoï cells and the transition weights \( (p^k)_{1 \leq i \leq N_k, 1 \leq j \leq N_k+1} \)), we proceed with the computation of \( (\tilde{Y}_k)_{0 \leq k \leq n} \) and rely on the BDPP (51) allowing us to compute \( \tilde{Y}_k = \tilde{y}_k(\tilde{X}_k) \) as a function of the quantizer \( \Gamma_k = \{x^k_1, \ldots, x^k_{N_k}\} \). For every \( k \in \{0, \ldots, n-1\} \) and \( i \in \{1, \ldots, N_k\} \), we denote

\[ \hat{\alpha}_k(x^k_i) = \sum_{j=1}^{N_{k+1}} \tilde{y}_{k+1}(x^{k+1}_j)p_{ij}^k \quad \text{and} \quad \hat{\beta}_k(x^k_i) = \frac{1}{\Delta} \sum_{j=1}^{N_{k+1}} \tilde{y}_{k+1}(x^{k+1}_j)\pi_{ij}^k \]

where

\[ \pi_{ij}^k = \sqrt{\frac{\Delta}{p_i^k}} \mathbb{E}
\left( \varepsilon_{k+1} \mathbb{I}_{\{\tilde{X}_{k+1} = x^{k+1}_j, \tilde{X}_k = x^k_i\}} \right) = \sqrt{\Delta} \mathbb{E}
\left( \varepsilon_{k+1} \mathbb{I}_{\mathcal{E}_k(x^k_i, \varepsilon_k) \in C_j(\Gamma_{k+1})} \right) \]

(57)

and \( \mathcal{E}_k(x, \varepsilon_k) = x + \Delta h_k(x) + \sqrt{\Delta} \sigma_k(x)\varepsilon_k + 1. \) Note that the quantities \( (\pi_{ij}^k)_{1 \leq i, j \leq N_k} \) are computed online at the same time as the transition weight matrices \( (p_{ij}^k)_{1 \leq i, j \leq N_k} \) for every \( k \in \{0, \ldots, n-1\} \), so that they can be stored and used instantly in the computations of the solution of the RBSDE.

Therefore, the solution \( Y_0 \) of the RBSDE is approximated by the value \( \hat{y}_0 \) at time \( t_0 \) of the following recursive quantized scheme

\[
\begin{cases}
\hat{y}_n(x^n_i) = h_n(x^n_i), & i = 1, \ldots, N_n, \\
\hat{y}_k(x^k_i) = \max\left(h_k(x^k_i), \hat{\alpha}_k(x^k_i), \hat{\beta}_k(x^k_i) + \Delta \mathcal{E}_k(x^k_i)\right), & i = 1, \ldots, N_k,
\end{cases}
\]

(58)

And, the function \( \hat{z}_k \) used to approximate \( \hat{c}_k \) is computed via the following sum

\[ \hat{z}_k(x^k_i) = \frac{1}{\Delta} \sum_{j=1}^{N_{k+1}} \tilde{y}_{k+1}(x^{k+1}_j)\pi_{ij}^k. \]

**Remark 5.1.** One should mention that, once the recursive quantization grids and the corresponding companion parameters are computed, the computation of the solution of the RBSDE is almost instantaneous, we can even say that its computational cost is negligible.

### 6 Numerical examples

We carry out some numerical experiments to illustrate the rate of convergence of the recursive quantization-based discretized scheme and to compare its performances with other schemes based on optimal quantization, greedy quantization, and greedy recursive quantization. We start by explaining how to obtain the quantizers and their companions parameters (Voronoï and transition weights) by optimal, greedy and recursive greedy quantization. Concerning the time discretization, we consider the Euler scheme of the forward diffusion \( (X_t)_{0 \leq t \leq T} \) defined by (20).
6.1 Various quantization methods

6.1.1 Quanization tree with optimal marginal quantization

In this section, we aim to build optimal quantizers $\hat{X}_k^\Gamma$ of $X_k$ for every $k \in \{0, \ldots, n\}$. At time $t_0$, we start with $\hat{X}_0 = X_0 = x_0 \in \mathbb{R}^d$. Then, at each time step $t_k$, we rely on a sequence of optimal quantizers $(z^k_i)_{1 \leq i \leq N_k}$ of size $N_k$ of the Normal distribution $\mathcal{N}(0, I_d)$ and we compute the quantizer $\Gamma_k = (x^k_1, \ldots, x^k_{N_k})$ via

$$x^k_i = x_0 + t_k b(x_0) + \sqrt{t_k} \sigma(x_0) z^k_i, \quad i \in \{1, \ldots, N_k\}.$$  

In particular, if $(\hat{X}_k)_{0 \leq k \leq n}$ evolves following a Black-Scholes model with interest rate $r$ and volatility $\sigma$, then the quantizers are computed as follows

$$x^k_i = x_0 \exp \left((r - \frac{\sigma^2}{2}) t_k + \sigma \sqrt{t_k} z^k_i \right).$$

The weights of the Voronoï cells are obtained by the forward Kolmogorov equation (54). In the one-dimensional case, they are easily computed relying on the c.d.f. of the Gaussian distribution.

The challenge in this method is the computation of the transition weights $p^k_{ij}$, which are mandatory for our cause. By optimal quantization, $(\hat{X}_k)_{0 \leq k \leq n}$ is not a Markov chain so one cannot use its distribution to compute $p^k_{ij}$ like for recursive quantization. One usually compute them by Monte Carlo simulations, but, in the one-dimensional case, there exist some closed formulas. In the following, we present such closed formulas in the case of a Black-Scholes model (the case that interests us the most for our numerical examples), i.e. a case where, for an the interest rate $r$ and a volatility $\sigma$, the process is given by

$$\hat{X}_k = \hat{X}_0 \exp \left((r - \frac{\sigma^2}{2}) t_k + \sigma \sqrt{t_k} \varepsilon_k \right)$$

where $(\varepsilon_k)_{1 \leq k \leq n}$ is an i.i.d. sequence of random variables with distribution $\mathcal{N}(0, 1)$.

**Exact computation of the transition weights**  Assume that the quantizers $\Gamma_k = (x^k_i)_{1 \leq i \leq N_k}$ of size $N_k$ of $\hat{X}_k$ are already computed for every $k \in \{1, \ldots, n\}$ and that the sizes of the grids $N_k$, $k = 1, \ldots, n$, are all equal to $N \in \mathbb{N}$. Note that this hypothesis is not optimal but turns out to be optimal in terms of complexity for a given budget $N_1 + \cdots + N_n$. It is not sharp in terms of error estimates (up to a multiplicative constant) but remains a good compromise which is convenient in practice for the implementation. The goal is to compute the transition weights $p^k_{ij}$.

$$p^k_{ij} = \mathbb{P} \left( \hat{X}_{k+1} = x^k_{j+1} \mid \hat{X}_k = x^k_i \right) = \frac{\tilde{p}^k_{ij}}{p^k_{i}}$$

where

$$\tilde{p}^k_{ij} = \mathbb{P} \left( \hat{X}_{k+1} = x^k_{j+1}, \hat{X}_k = x^k_i \right) \quad \text{and} \quad p^k_{i} = \mathbb{P} (\hat{X}_k = x^k_i).$$

The weights $p^k_{i}$ are computed via the forward Kolmogorov equation, using the transition weights $p^k_{ij}$, as follows

$$p^k_{i} = \sum_{i=1}^{N_k} p^k_{ij} \frac{N_k}{i},$$

keeping in mind that the Voronoï weight at time $t_0$ (i.e. $k = 0$) is equal to 1 since $\hat{X}_0 = X_0 = x_0$ is deterministic. So, our main concern is the computation of $\tilde{p}^k_{ij}$ for every $k \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots, N\}$. We start by noticing that

$$\hat{X}_{k+1} = \hat{X}_k (1 + rh + \sigma \sqrt{h} \varepsilon_k)$$
where \( h = \frac{T}{n} \) is the time step of the discretization scheme. Note that highly accurate quantization grids of \( \mathcal{N}(0,1) \) for regularly sampled sizes from \( N = 1 \) to 1000 are available and can be downloaded from the quantization website www.quantize.maths-fi.com (for non-commercial purposes). Then, considering two independent random variables \( z_1 \) and \( z_2 \) with distribution \( \mathcal{N}(0,1) \), one has

\[
\tilde{p}_{ij}^k = \mathbb{P} \left( \bar{X}_{k+1} \in [x_{k+1}^{j-\frac{1}{2}}, x_{k+1}^{j+\frac{1}{2}}] \right), \quad \bar{X}_k \in [x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]
\]

\[
= \mathbb{P} \left( \bar{X}_k(1 + rh + \sigma \sqrt{h} z) \in C_j(\Gamma_{k+1}), z_1 \in \left[ x_{k}^{j}, x_{k}^{j+1} \right] \right)
\]

where

\[
x_{k}^{j} = \frac{\ln \left( x_{k}^{j} \right) + \left( \frac{a^2}{2} - r \right) t_k - \ln(x_0)}{\sigma \sqrt{t_k}} \quad \text{and} \quad x_{k}^{j+1} = \frac{\ln \left( x_{k}^{j+1} \right) + \left( \frac{a^2}{2} - r \right) t_k - \ln(x_0)}{\sigma \sqrt{t_k}},
\]

(59)

Then, the independence of \( z_1 \) and \( z_2 \) yields

\[
\tilde{p}_{ij}^k = \int_{x_{k}^{i}}^{x_{k}^{i+1}} \mathbb{P} \left( x_0 (1 + rh + \sigma \sqrt{h} z) \exp \left( (r - \frac{a^2}{2}) t_k + \sigma \sqrt{t_k} z \right) \in \left[ x_{k+1}^{j-\frac{1}{2}}, x_{k+1}^{j+\frac{1}{2}} \right] \right) e^{-\frac{x^2}{2}} \frac{dz}{\sqrt{2\pi}}
\]

\[
= \int_{x_{k}^{i}}^{x_{k}^{i+1}} \mathbb{P} \left( z \in \left[ \frac{x_{k+1}^{j} e^{\left( \frac{a^2}{2} - r \right) t_k - \sigma \sqrt{t_k} z - x_0 - rhx_0}}{\sigma x_0 \sqrt{h}}, \frac{x_{k+1}^{j+1} e^{\left( \frac{a^2}{2} - r \right) t_k - \sigma \sqrt{t_k} z - x_0 - rhx_0}}{\sigma x_0 \sqrt{h}} \right] \right) e^{-\frac{x^2}{2}} \frac{dz}{\sqrt{2\pi}}
\]

\[
= \int_{x_{k}^{i+1}}^{x_{k}^{i+1}} \left( \Phi_0 \left( x_{k+1}^{j+1} \right) - \Phi_0 \left( x_{k}^{i+1} \right) \right) e^{-\frac{x^2}{2}} \frac{dz}{\sqrt{2\pi}},
\]

(60)

These integrals can be computed via Gaussian quadrature formulas, mainly Gauss-Legendre quadrature formulas for integrals on closed intervals and Gauss-Laguerre quadrature formulas for integrals on semi-closed intervals. So, if \( i = 1 \) or \( i = N \), one uses Gauss-Laguerre formulas since the Voronoï cells (over which we are integrating) are of the form \((\infty, a)\) or \((a, +\infty)\) for some \( a \in \mathbb{R} \). Otherwise, the Voronoï cells are closed intervals so one relies on Gauss-Legendre quadrature formula. Let us detail these computations.

▷ Integration on a closed interval \([a, b]\): Gauss Legendre formula

Considering \( f(z) = \left( \Phi_0 \left( x_{k+1}^{j+1} \right) - \Phi_0 \left( x_{k}^{i+1} \right) \right) e^{-\frac{x^2}{2}} \), \( a = x_{k}^{i+1} \) and \( b = x_{k+1}^{j+1} \), the goal is to compute

\[
I = \int_{a}^{b} f(z) dz.
\]

Applying the change of variables \( z = \frac{b-a}{2} x + \frac{a+b}{2} \), \( I \) can be written and computed as follows

\[
I = \frac{b-a}{2} \int_{-1}^{1} f \left( \frac{b-a}{2} x + \frac{a+b}{2} \right) dx = \frac{b-a}{2} \sum_{i=1}^{n} w_i f \left( \frac{b-a}{2} x_i + \frac{a+b}{2} \right)
\]

where \( (x_i)_{1 \leq i \leq n} \) are the roots of the \( n \)th Legendre polynomial \( P_n(x) = \frac{1}{2^n} \sum_{j=0}^{n} \binom{2n}{n} (-1)^j \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} \)

and the weights \( (w_i)_{1 \leq i \leq n} \) are given by

\[
w_i = \frac{2}{(1-x_i^2)P_n'(x_i)^2} = \frac{2(1-x_i^2)}{(n+1)^2 P_{n+1}(x_i)^2}.
\]

▷ Integration on intervals of the form \([a, +\infty)\) or \((\infty, a]\): Gauss Laguerre quadrature
We consider \( f(z) = \Phi_0(x_j^{k+1}) - \Phi_0(x_j^{k+1}) \) and distinguish two cases.

- **Integration on \([a, +\infty)\)**
  
  The goal is to compute \( I = \int_a^\infty f(z)e^{-\frac{z^2}{2}}dz \) where \( a = x_j^k \). Applying the change of variables \( x = \frac{z^2}{2} \) and denoting \( g(x) = \frac{f(x)}{x} \) yield
  
  \[
  I = \int_{\frac{a^2}{2}}^{+\infty} \frac{f(\sqrt{2x})}{\sqrt{2x}} e^{-x}dx = \int_{\frac{a^2}{2}}^{+\infty} g(\sqrt{2x})e^{-x}dx = e^{-\frac{a^2}{2}} \int_{0}^{+\infty} g \left( \sqrt{2x + a^2} \right) e^{-x}dx
  \]
  
  where we applied in the last equality the change of variables \( y = x - \frac{a^2}{2} \). Hence, we use Gauss-Legendre quadrature formula to obtain
  
  \[
  I = e^{-\frac{a^2}{2}} \sum_{i=1}^{N} w_i g \left( \sqrt{2x_i + a^2} \right)
  \]
  
  where \((x_i)_{1 \leq i \leq n}\) are the roots of the \(n\)th Laguerre polynomial \(L_n(x) = \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!(n-k)!} x^k\) and the weights \((w_i)_{1 \leq i \leq n}\) are given by
  
  \[
  w_i = \frac{1}{(n+1)L_n'(x_i)L_{n+1}(x_i)} = \frac{x_i}{(n+1)^2 L_{n+1}(x_i)^2}.
  \]  
  (62)

- **Integration on \((-\infty, a)\)**
  
  The goal is to compute \( I = \int_{-\infty}^{a} f(x)e^{-\frac{x^2}{2}}dx \) where \( a = x_i^k \). Similarly to the previous case, \( I \) can be written as follows
  
  \[
  I = \int_{-\infty}^{+\infty} f(-x)e^{-\frac{x^2}{2}}dx = \int_{\frac{a^2}{2}}^{+\infty} \frac{f(-\sqrt{2z})}{\sqrt{2z}} e^{-z}dz = \int_{\frac{a^2}{2}}^{+\infty} g(\sqrt{2z})e^{-z}dz = e^{-\frac{a^2}{2}} \int_{0}^{+\infty} g \left( \sqrt{2z + a^2} \right) e^{-z}dz
  \]
  
  where \( g(x) = \frac{f(-x)}{x} \). Hence, Gauss-Legendre quadrature formula yields
  
  \[
  I = e^{-\frac{a^2}{2}} \sum_{i=1}^{N} w_i g \left( \sqrt{2x_i + a^2} \right)
  \]
  
  where \((x_i)_{1 \leq i \leq n}\) are the roots of \(L_n(x)\) and \((w_i)_{1 \leq i \leq n}\) are given by (62).

**Approximation of the transition weights**  

If the goal is not necessarily the highest level of precision, then one approximates the transition weights \( p_{ij}^k \) by \( g_j(z^k) \) where the function \( g_j(z) \) is defined by

\[
 g_j(z) = \Phi_0(x_j^{k+1}) - \Phi_0(x_j^{k+1}).
\]  
(63)

and \( x_j^{k+1} \) and \( x_j^{k+1} \) are given by (61). In fact, based on (60) and then applying Taylor-Lagrange formula, one has

\[
 p_{ij}^k = \int_{z_i^{k+1}}^{z_j^{k+1}} g_j(z)e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}
\]

\[
 = g_j(z_i^{k}) p_{ij}^k + g_j'(z_i^{k}) \int_{z_i^{k}}^{z_j^{k}} (z - z_i^{k}) e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} + \int_{z_i^{k}}^{z_j^{k}} g_j''(\xi(z)) \frac{(z - z_i^{k})^2}{2} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}.
\]

Since \((z_i^{k})_{1 \leq i \leq N}\) is a quadratic optimal quantization sequence of the standard Normal distribution, then it is stationary and the second term of the above inequality is equal to 0. Moreover,

\[
 g_j'(z) = \frac{k}{x_0 \sqrt{2\pi}} \left( x_{j+\frac{1}{2}} e^{-\sigma \sqrt{t} z - \frac{1}{2} \sigma t} - x_{j-\frac{1}{2}} e^{-\sigma \sqrt{t} z - \frac{1}{2} \sigma t} \right)
\]
and

\[ g_j''(z) = \frac{k}{x_0 \sqrt{2\pi}} \left[ \sigma \sqrt{t_k} e^{-\sigma \sqrt{t_k} z} \left( x_{j+1} - x_j e^{-\frac{1}{2} z^2} e^{\frac{1}{4} t_k^2} \right) + \frac{k}{x_0} e^{-2\sigma \sqrt{t_k} z} \left( x_{j+1}^2 \bar{p}^2 e^{-\frac{1}{2} z^2} - x_j^2 \frac{1}{2} z^2 e^{-\frac{1}{2} z^2} \right) \right]. \]

At this stage, one notices that \( \gamma(z) := \exp(-2z - \frac{1}{2} e^{-2z}) \leq \kappa \) for every \( z \in \mathbb{R} \) for some finite positive constant \( \kappa \) and that \(|g_j''(z)| \leq \bar{\kappa} \) for a finite positive constant \( \bar{\kappa} \). Consequently, \(|p_{ij}^k - g_j(z_k^j)| \) is bounded.

It is important to note that when we estimate the transition weight by \( g_j(z_k^j) \), we formally get the transition weight from \( x_i^k \) to \( x_j^{k+1} \) obtained by recursive quantization, even though they are not the same grids.

**Remark 6.1.** For local volatility models (CEV models for example), it becomes more complicated to establish such closed formulas for the computations of the transition matrix. One tends to approximate them by Monte Carlo simulations, for example.

6.1.2 Greedy quantization

Another technique is greedy vector quantization introduced in [27] and developed in [15]. It consists in building a sequence of points \((a_n)_{n \geq 1}\) in \( \mathbb{R}^d \) recursively optimal step by step, in the following greedy sense: having computed the first \( n \) points \( a_1, \ldots, a_n \) of the sequence and defining the resulting grid \( a^{(n)} = \{a_1, \ldots, a_n\} \) for \( n \geq 1 \), we compute the \((n+1)\)-th point as a solution to the minimization problem

\[ a_{n+1} = \arg\min_{\xi \in \mathbb{R}^d} e_p(a^{(n)} \cup \{\xi\}, X), \]

with the convention \( a^{(0)} = \emptyset \). Quadratic greedy quantization sequences are obtained by implementing "freezing" avatars of usual stochastic optimization algorithms used for optimal quantization, these variants are exposed in details in [28]. In this paragraph, we give a quick idea on the computation of the greedy quantization sequence of \((X_k)_{0 \leq k \leq n}\). Starting at \( \bar{X}_0 = \bar{X}_0 = x_0 \), the process \( \bar{X}_k \) can be written, for every \( k \in \{1, \ldots, n\} \), as follows

\[ \bar{X}_k = x_0 + t_k b(x_0) + \sqrt{t_k} \sigma(x_0) \epsilon_k \]

where \( \epsilon_k \) is a random variable with distribution \( \mathcal{N}(0, I_d) \). So \( \bar{X}_k \) is with Normal distribution \( \mathcal{N}(m_k, \Sigma_k) \) where \( m_k = x_0 + t_k b(x_0) \) and \( \Sigma_k = \sqrt{t_k} \sigma(x_0) \) and hence this is the distribution that needs to be discretized by greedy quantization. The transition weights in the one-dimensional case are computed via Gaussian quadrature formula like explained for the optimal quantization, and the weights of the Voronoi cells by the forward Kolmogorov equation.

In the high-dimensional framework \((d \geq 2)\), the computations become too demanding. So, instead of designing pure greedy quantization sequences, one tends to build greedy product quantization sequences which are obtained as a result of the tensor product of one-dimensional sequences, when the target law is a tensor product of its independent marginal laws. We refer to [15] for further details.

6.1.3 Greedy recursive quantization

In the algorithm described in Section 5, the recursive quantization scheme (22) is based on an optimal quantization of the sequences \((\bar{X}_k)_{0 \leq k \leq n}\) at each time step \( t_k \). Here, we consider, as an alternative, greedy optimal quantization grids \( \bar{X}_k \) of \( \bar{X}_k \). They are designed as follows: At time \( t_{k+1} \), assuming that the \( N_k \)-tuple \((x_1^k, \ldots, x_{N_k}^k)\) and its companion parameters are already computed, one needs to build, step by step by greedy quantization, the \( N_{k+1} \)-tuple \((x_1^{k+1}, \ldots, x_{N_{k+1}}^{k+1})\) which approaches best \( \bar{X}_{k+1} = \bar{E}_k(\bar{X}_k, \epsilon_{k+1}) \). Since \( \bar{E}_k(x_i^k, \epsilon_{k+1}) \sim \mathcal{N}(m_i^k, \Sigma_i^k) \) with \( m_i^k = x_i^k + \Delta b_k(x_i^k) \) and \( \Sigma_i^k = \sqrt{\Delta} \sigma_k(x_i^k) \),
the first point of the sequence is \( x_{i1}^{k+1} = \mathbb{E} [\tilde{X}_k + \Delta b_k (\tilde{X}_k)] = \sum_{i=1}^{N_k} p_i^k (x_i^k + \Delta b_k (x_i^k)) \) and then, at each iteration \( N, N \in \{2, \ldots, N_{k+1}\} \), one adds one point \( x_{ij}^{k+1} \) following the steps of the greedy variant of Lloyd’s algorithm detailed in [28]. One should take in consideration that the local interpoint inertia are computed, at each time step \( t_{k+1} \), by

\[
\sigma_j^2 = \sum_{i=1}^{N_k} p_i^k \left( \int_{x_{ij}^{k+1,N}}^{x_{ij+1/2}^{k+1,N}} (\xi - x_{ij+1/2}^{k+1,N})^2 P(d\xi) + \int_{x_{ij+1/2}^{k+1,N}}^{x_{ij+1,N}} (\xi - x_{ij+1/2}^{k+1,N})^2 P(d\xi) \right) := \sum_{i=1}^{N_k} p_i^k s_{ij}
\]

where \( x_{ij+1/2}^{k+1,N} = \frac{x_{ij}^{k+1,N} + x_{ij+1}^{k+1,N}}{2} \) with \( x_0^{k+1,N} = x_{k+1,N} = -\infty \) and \( x_{N}^{k+1,N} = x_{N-1/2}^{k+1,N} = +\infty \). Likewise, the recurrence of the algorithm is given by

\[
x_{ij+1} = \frac{\sum_{i=1}^{N_k} p_i^k \mathbb{E} \left( E_k (x_i^k, \varepsilon_{k+1}) \mathbb{1} \{ E_k (x_i^k, \varepsilon_{k+1}) \in C_j (\Gamma_{k+1}) \} \right)}{\sum_{i=1}^{N_k} p_i^k \mathbb{P} \left( E_k (x_i^k, \varepsilon_{k+1}) \in C_j (\Gamma_{k+1}) \right)},
\]

The companion parameters are computed following the same principle as for the standard recursive quantization.

### 6.2 Examples

#### 6.2.1 American call option in a market with bid-ask spread on interest rates

We are interested in the valuation of an American call option with maturity \( T \) in a market with a bid-ask spread on interest rates with a borrowing rate \( r \) and a lending rate \( R \leq r \). The stock price is represented by the process \((X_t)_{t \in [0,T]}\) given by the SDE (3) and the dynamics of the portfolio are given by

\[
-dY_t = \left( -r Y_t - \frac{b_t(X_t) - r}{\sigma_t(X_t)} Z_t - (R - r) \min \left( Y_t - \frac{Z_t}{\sigma_t(X_t)}, 0 \right) \right) dt - Z_t dW_t
\]

\[
Y_T = h(X_T) \quad \text{and} \quad Y_t \geq g(X_t)
\]

where \( h(x) = g(x) = \max (x - K, 0) \), \( K \) being the strike price.

**Black-Scholes model** We consider that \((X_t)_{t \in [0,T]}\) evolves following the Black-Scholes dynamics and is time discretized following the Euler scheme, i.e. for every \( k \in \{0, \ldots, n - 1\} \),

\[
\bar{X}_{k+1} = \bar{X}_k + \mu \Delta \bar{X}_k + \sigma \sqrt{\Delta \bar{X}_k} \varepsilon_{k+1}
\]

where \( \mu \) is the drift and \( \sigma \) is the volatility. The space discretization is established via recursive quantization (RQ), optimal quantization (OQ), greedy quantization (GQ) and greedy recursive quantization (GRQ). We consider \( n = 20 \) time steps and build corresponding quantization grids of size \( N = 100 \) and their companion parameters as explained in the different sections previously in the paper. Then, we rely on the backward recursion (58) to compute the value \( Y_0 \) of the underlying option. Note that the quantities \( \pi^{k}_{ij} \) are computed, for every \( k \in \{1, \ldots, n\} \), as a companion parameter with the diffusion \( \tilde{X}_k \) via a Monte Carlo simulation of size \( 10^6 \). We consider the following parameters

\[
X_0 = 100, \quad T = 0.25, \quad \sigma = 0.2, \quad \mu = 0.05, \quad r = 0.01, \quad R = 0.06
\]

and we compare the values obtained by the different methods for different values of \( K \) varying between 100 and 120. As a benchmark, we will assume that the optimal quantization converges to the exact value and, under this hypothesis, we consider the fastest and most accurate version of optimal quantization, which is the quantization-based Richardson-Romberg extrapolation. The idea is the following:
If the goal is to approximate $\mathbb{E}f(X)$ for a function $f$ and a random variable $X$, one considers two optimal quantization sequences $\hat{X}^{N_1}$ of size $N_1$ and $\hat{X}^{N_2}$ of size $N_2$ of the random variable $X$ and hence $\mathbb{E}f(X)$ is given by

$$\mathbb{E}f(X) = \frac{N_2^2\mathbb{E}f(\hat{X}^{N_2}) - N_1^2\mathbb{E}f(\hat{X}^{N_1})}{N_2^2 - N_1^2}. \quad (68)$$

From a practical point of view, one usually considers $N_1 = N$ and $N_2 = \frac{N}{2}$. Furthermore, when the dimension $d = 1$, the standard quantization error is of the form

$$e_2(X, \mu) \approx c_1 \sqrt{n} + c_2 \sqrt{n}N^{-1}$$

and the Romberg-quantization error is of the form

$$e_2(X, \mu) \approx c_2 \sqrt{n} \left( \frac{1}{N_1} - \frac{1}{N_2} \right) \approx \frac{c_1 \sqrt{n}}{2N_1}.$$

So, by studying the values of this error for different values of $n$ and $N_1$, we realize that the best technique is to consider a small number of time steps $n$ and a large size $N$ of the quantizer.

In our example, we consider an optimal quantization-based Richardson Romberg extrapolation with $n = 5$ and $N = 1000$. We observe in Table 1 the results and the errors obtained by the various methods. Here, we emphasize on the computational time of these simulations which are performed on a CPU 2.7 GHz and 8 GB memory computer. The optimal quantizer and its companion parameters are obtained in about 40 seconds while the greedy quantization sequence and its companions in about 30 seconds. This is approximately a 25% gain in time in favor of greedy quantization whose results are comparable (a little less precise) than optimal quantization. As for the recursive quantization, the standard simulations (RQ) are obtained in about 2.3 minutes and the greedy simulations (GRQ) in about 2 minutes. Hence, the greedy character introduced in the recursive algorithm brings a 13% gain in time. The additional cost in time is compensated by the preservation of the Markovian property and the precision of the results.

Figure 1 depicts the convergence of the error induced by the approximation of $Y_0$ based on a recursive quantization of the forward process $\hat{X}_k$. For this illustration, we consider a strike $K = 100$ and we make the size $N$ of the grids vary between 10 and 100. The graph is represented in a log-log-scale and an $O(N^{-1})$ rate of convergence is clearly observed.

### CEV model

We consider a local volatility model, the CEV model, in which $(X_t)_{0 \leq t \leq T}$ evolves following

$$dX_t = \mu X_t dt + \sigma X_t^\delta dW_t, \quad X_0 = x_0,$$  \quad (69)
Figure 1: Convergence rate of the error induced by the approximation of the Bid-ask spread Call option in a Black-Scholes model discretized by recursive quantization for different sizes \( N = 10, \ldots, 100 \). (logarithmic scale)

for some \( \delta \in (0, 1) \) and \( \theta \in (0, \overline{\theta}] \) with \( \overline{\theta} > 0 \). \( \sigma(x) = \theta x^\delta \) is the local volatility function. The discretized Euler scheme associated to \( (X_t)_{t \in [0,T]} \) is given, for every \( k \in \{0, \ldots, n-1\} \), by

\[
\bar{X}_{k+1} = \bar{X}_k + \mu \Delta \bar{X}_k + \theta \bar{X}_k^\delta \sqrt{\Delta} \epsilon_k
\]

where \((\epsilon_k)_{1 \leq k \leq n}\) is an i.i.d sequence of random variables with distribution \( N(0, 1) \).

The construction of the quantizers and the computation of the companion parameters by recursive and greedy recursive quantization is similar to what was done for the Black-Scholes model. As for optimal and greedy quantization, closed forms for the companion parameters are no longer available in this model, we estimate them by Monte Carlo simulations of size \( 10^5 \) coupled with a nearest neighbor search. We build corresponding quantization grids of size \( N = 150 \) and consider \( n = 15 \) time steps. The parameters are the following

\[
X_0 = 100, \quad T = 0.25, \quad \theta = 4, \quad \delta = 0.5, \quad \epsilon = 1, \quad \mu = 0.05, \quad r = 0.01, \quad R = 0.06
\]

and we compare the values obtained by the different methods for different values of \( K \) between 100 and 120. The benchmark is given by an optimal quantization-based Richardson-Romberg extrapolation (68). We observe in Table 2 the results and errors obtained by such comparisons. As for the computation time, we note that the optimal quantizer and its companion parameters are obtained in about 100 seconds while the greedy quantization sequence and its companions in about 70 seconds. The fact that these computations take more time for the CEV model than for the Black-Scholes model is due to the non-existence of closed formulas for the computation of the companion parameters in the CEV model, the computation of the quantizers themselves is almost instantaneous. Moreover, the recursive quantizer and its companions are computed in about 3.5 minutes while the greedy recursive quantizers in about 3 minutes.

6.2.2 Two-dimensional American exchange options

We are interested in pricing an American exchange option with exchange rate \( \mu \) and maturity \( T \). This price is given by the value \( Y_0 \) at time \( t_0 \) of the solution of the RBSDE (1) with driver \( f = 0 \) and \( h_t(x) = g_t(x) = \max(e^{-\lambda t} X_t^1 - \mu X_t^2, 0) \). \( X_t^1 \) and \( X_t^2 \) are two assets, such that \( X_t^1 \) is with a geometric dividend rate \( \lambda \) and \( X_t^2 \) is without dividend, both following a Black-Scholes model. The discretized
Table 2: Pricing of an American call option in a market with bid-ask spread for interest rates in a CEV model by recursive (RQ), greedy recursive (GRQ), optimal (OQ) and greedy (GQ) quantization.

Euler scheme \((\tilde{X}^1_k, \tilde{X}^2_k)\) is given, for every \(k \in \{0, \ldots, n-1\}\), by

\[
\begin{align*}
\tilde{X}^1_{k+1} &= X^1_k e^{(r - \frac{\sigma^2}{2}) \Delta + \sigma \sqrt{\Delta} \varepsilon^1_k} \\
\tilde{X}^2_{k+1} &= X^2_k e^{(r - \frac{\sigma^2}{2}) \Delta + \sigma \sqrt{\Delta} (\rho \varepsilon^1_k + \sqrt{1 - \rho^2} \varepsilon^2_k)}
\end{align*}
\]

where \(r\) is the interest rate, \(\sigma\) the volatility, \(\rho\) is a correlation coefficient and \((\varepsilon^1_k, \varepsilon^2_k)_{1 \leq k \leq n}\) is a sequence of i.i.d. random variables with distribution \(\mathcal{N}(0, I_2)\).

From a numerical point of view, we discretize in \(n = 10\) time steps, build quantizers of size \(N_X = 100\) and consider the following parameters

\[X^1_0 = 40, \quad T = 1, \quad r = 0, \quad \sigma = 0.2, \quad \lambda = 0.05, \quad \mu = 1.\]

In high dimensions \((d > 1)\), the implementation of the recursive quantization algorithm is too expensive and its cost in time is very high. We consider, instead, the hybrid recursive quantization, introduced in Section 2.3 and use sequences of optimal quantizers \((\varepsilon^1_k)_{1 \leq k \leq N}\) of size \(N^c = 1000\) to compute the sequence and the companion parameters as detailed in Section 5. We also build optimal quantizers and greedy product quantization sequences (see Section 6.1.2). We compute the price of the option by these methods for \(X^2_0 \in \{36; 44\}\) and \(\rho \in \{-0.8; 0; 0.8\}\) and compare the results obtained to those computed by a finite difference algorithm in [40] and expose the errors hence induced in Table 3.

Similarly to the one-dimensional Example 6.2.1, a gain in the computation time appears in favor of the greedy quantization. In fact, greedy product quantization sequences are obtained in about 55 seconds whereas optimal and hybrid recursive quantizers in about 70 seconds and 3.75 minutes respectively, and hence the gain is about 20% compared to optimal quantization and 75% compared to hybrid recursive quantization. Moreover, we remark that hybrid recursive quantization gives the most precise results while an expected gain in precision for optimal quantization compared to greedy quantization is observed.
References

[1] Bally V. (1997). Approximation scheme for solutions of BSDE. Backward Stochastic Differential Equations (N. El Karoui and L. Mazliak, eds.) 177–191. Pitman, London.

[2] Bally V. & Pagès G. (2003). Error analysis of the quantization algorithm for obstacle problems, Stochastic Processes & Their Applications., 1: 1-40.

[3] Bally V. & Pagès G. (2003). A quantization algorithm for solving discrete time multidimensional optimal stopping problems, Bernoulli., 6: 1003-1049.

[4] Bally V., Pagès G. & Printemps J. (2001). A stochastic quantization method for non-linear problems, Monte Carlo Methods and Appl., 1: 21-34.

[5] Bender C. & Denk R. (2007). A forward scheme for backward SDEs, Stochastic Processes & Their Applications., 117(12): 1793-1812.

[6] Bouchard B. & Touzi N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations, Stochastic Processes & Their Applications., 111(2): 175-206.

[7] Bouleau N., Lépingle D. (1993). Numerical Methods for Stochastic Processes, Wiley-Interscience.

[8] Briand P. & Labart C. (2014). Simulation of BSDEs by Wiener chaos expansion, Ann. Appl. Probab., 24(3): 1129-1171.

[9] Callegaro G., Fiorin L. & Grasselli M. (2017). Pricing via recursive quantization in stochastic volatility models, Quantitative Finance, 17 (6):855-872.

[10] Chassagneux J.F., & Richou A. (2016). Numerical simulation of quadratic BSDEs, Ann. Appl. Probab., 26 (1):262-304.

[11] Crisan D., Manolarakis K. & Touzi N. (2010). On the Monte Carlo simulation of BSDE’s: an improvement on the maliavin weights., Stochastic Processes and their Applications., 120: 1133-1158.

[12] Cvitanić J, & Ma J.(2001) Reflected forward–backward SDEs and obstacle problems with boundary conditions, J. Appl. Math. Stochastic Anal., 14 (2):113-138.

[13] Delarue F., & Menozzi S.(2006) A Forward-Backward stochastic algorithm for quasi-linear PDEs, The Annals of Applied Probability, 16 (1):140-184.

[14] Delarue F., & Menozzi S.(2008) An interpolated stochastic algorithm for quasi-linear PDEs, Math. Comp., 77 (26):125-158.

[15] El Nmeir R., Lushgy H. & Pagès G. (2020). New approach to greedy vector quantization, ArXiv (available at http://arxiv.org/abs/2003.14145).

[16] El Karoui N., Kapoudjian, C., Pardoux, E., Peng, S., Quenez, M.C. (1997). Reflected solutions of Backward Stochastic Differential Equations and related obstacle problems for PDEs., Ann. Probab., 25(2): 702-737.

[17] El Karoui N., Peng S. & Quenez M.C. (1997). Backward stochastic differential equations in Finance, Math. Finance, 7(1): 1-71.

[18] Fiorin L., Pagès G. & Sagna A. (2019). Product Markovian quantization of a diffusion process with applications to finance, Methodol. Comput. Appl. Probab., 21(4): 1087-1118.

[19] Gobet E., Lopez-Salas J., Turkedjiev P. & Vasquez C. (2016). Stratified regression Monte-Carlo scheme for semilinear PDEs and BSDEs with large scale parallelization on GPUs., SIAM Journal on Scientific Computing, 38(6): C652-C677.

[20] Gobet E. & Turkedjiev P. (2016). Approximation of backward stochastic differential equations using Malliavin weights and least-squares regression, Bernoulli, 22(1): 530-562.

[21] Graf S., Lushgy H. and Pagès G. (2008). Distortion mismatch in the quantization of probability measures, ESAIM P&ES, 12: 127-154.

[22] Graf S. & Lushgy H. (2000). Foundations of Quantization for Probability Distributions, Lectures Notes in Math. 1730. Springer, Berlin.
[23] Henry-Labordère P., Tan X. & Touzi N. (2014). A numerical algorithm for a class of BSDEs via the branching process, *Stochastic Process. Appl.* 124(2):1112-1140.

[24] Hu Y., Nualart T. & Song X. (2011). Malliavin calculus for backward stochastic differential equations and applications to numerical solutions, *The Annals of Applied Probability*, 21(6):2379–2423.

[25] Illand C. (2012). Contrôle stochastique par quantification et applications à la finance, PhD thesis, UPMC.

[26] Luschgy H. & Pagès G. (2008). Functional quantization rate and mean regularity of processes with an application to Lévy processes, *Annals of Applied Probability*, 18(2):427-469.

[27] Lushgy H. & Pagès G. (2015). Greedy vector quantization, *Journal of Approximation Theory*, 198: 111-131.

[28] Lushgy H. & Pagès G. (2015). Greedy vector quantization (extended version), *ArXiv*. (Available at https://arxiv.org/abs/1409.0732)

[29] Ma J. & Zhang J. (2005). Representation and regularities for solutions to BSDEs with reflections, *Stochastic Processes and their applications*, 115:539-569.

[30] Ma J. & Wang Y. (2009). On Variant Reflected Backward SDEs, with Applications, *J. Appl. Math. Stochastic Anal.*, Vol. Art. ID 854768, pp. 26.

[31] Ma J. & Wang Y. (1994). Solving forward–backward stochastic differential equations explicitly—a four step scheme, *Probab. Theory Relat/Fields*, 98:339-359.

[32] McWalter A., Rudd R., Kienitz J. & Platen E. (2018). Recursive marginal quantization of higher-order schemes, *Quantitative Finance*, 18(4):693-706.

[33] Pagès G. (2015). Introduction to optimal vector quantization and its applications for numerics. CEMRACS 2013-modelling and simulation of complex systems : Stochastic and deterministic approaches. *ESAIM*.

[34] Pagès G. (2018). Numerical probability: An introduction with applications to finance, Springer-Verlag, xvi+579p.

[35] Pagès G., Pham H. & Printemps J. (2004). Optimal quantization methods and applications to numerical problems in finance, Rachev S.T. (eds) Handbook of Computational and Numerical Methods in Finance. Birkhäuser, Boston, MA.

[36] Pagès G., Printemps J. (2003). Optimal quadratic quantization for numerics: the Gaussian case, *Monte Carlo Methods Appl.*, 9(2): 135-165.

[37] Pagès G., Sagna A. (2014). Recursive marginal quantization of the Euler scheme of a diffusion. *Appl. Math. Finance*, 22 (5), 463–498.

[38] Pagès G., Sagna A. (2016). Improved error bounds for quantization based numerical schemes for BSDE and nonlinear filtering, (extended version). Available at: http://Arxiv.Org/Abs/1510.01048.

[39] Pagès G., Sagna A. (2018). Improved error bounds for quantization based numerical schemes for BSDE and nonlinear filtering, *Stochastic Processes and their Applications*, 128 847-883.

[40] Villeneuve S., Zanette A. (2002). Parabolic A.D.I. methods for pricing American option on two stocks, *Math. Oper. Res.*, 27121-149.

[41] Zador P.L. (1982). Asymptotic quantization error of continuous signals and the quantization dimension, *IEEE Trans. Inform. Theory*, IT-28(2):139-14.

[42] Zhang J. (2004). A numerical scheme for BSDEs, *Ann. Appl. Probab.*, 14(1):459-488.
7 Appendix

7.1 Appendix A: The proof of Lemma 2.3

First note that the function \( f : u \mapsto |u|^r \) satisfies (since \( r \geq 2 \))

\[
\nabla|u|^r = r|u|^{r-1} \frac{u}{|u|} \quad \text{and} \quad \nabla^2|u|^r = r|u|^{r-2} \left( (r-2) \frac{u}{|u|} \frac{u}{|u|} + I_d \right)
\]

(convention \( \frac{0}{0} = 0 \)). Consequently, Taylor’s Theorem with Lagrange remainder applied to \( f \) reads

\[
f(u + v) = f(u) + \langle \nabla f(u), v \rangle + \frac{1}{2} v^* \nabla^2 f(u) v
\]

for some \( \xi_{u,v} = \lambda_{u,v} u + (1 - \lambda_{u,v}) (u + v) \), \( \lambda_{u,v} \in (0,1) \). Note that

\[
v^* \nabla^2 f(u) v = r|\xi_{u,v}|^2 (r-2) \frac{\langle u, \xi_{u,v} \rangle^2}{|\xi_{u,v}|^2} + |v|^2 \leq r|\xi_{u,v}|^2 (r-1)|v|^2
\]

owing to Cauchy-Schwartz inequality. Then, noting that \( |\xi_{u,v}| \leq |u| \vee |u + v| \leq |u| + |v| \), we obtain

\[
|u + v|^r \leq |u|^r + \left( r|u|^{r-1} \frac{u}{|u|} , v \right) + \frac{r(r-1)}{2} (|u| + |v|)^{r-2} |v|^2
\]

\[
\leq |u|^r + \left( r|u|^{r-1} \frac{u}{|u|} , v \right) + \frac{r(r-1)}{2} 2^{(r-3)} + (|u|^{r-2} + |v|^{r-2}) |v|^2
\]

\[
= |u|^r + \left( r|u|^{r-1} \frac{u}{|u|} , v \right) + \frac{r(r-1)}{2} 2^{(r-3)} + (|u|^{r-2} + |v|^{r})
\]

Applying the above inequality to \( u = a \) and \( v = \sqrt{h} AZ \) yields

\[
|a + A \sqrt{h} Z|^r \leq |a|^r + \left( |a|^{r-1} \frac{a}{|a|}, A \sqrt{h} Z \right) + 2^{(r-3)} + \frac{r(r-1)}{2} (h|a|^{r-2} |AZ|^2 + h^{r/2} |AZ|^r)
\]

Applying Young’s inequality (when \( r \geq 2 \)) to the product \( |a|^{r-2} |AZ|^2 \) with conjugate exponents \( r' = \frac{r}{r-2} \) and \( s' = \frac{r}{2} \) yields

\[
|a + A \sqrt{h} Z|^r \leq |a|^r + \left( |a|^{r-1} \frac{a}{|a|}, A \sqrt{h} Z \right) + 2^{(r-3)} + \frac{r(r-1)}{2} \left( \frac{h}{r} (r-2)|a|^r + |AZ|^r \right)
\]

\[
\leq |a|^r \left( 1 + 2^{(r-3)} + \frac{r(r-1)}{2} \frac{h}{r} \right) + \left( |a|^{r-1} \frac{a}{|a|}, A \sqrt{h} Z \right) + 2^{(r-3)} + (r-1) h \|A\| |Z|^r \left( 1 + \frac{r}{2} \frac{1}{h^{r/2}} \right)
\]

Finally, taking expectation and using that \( \mathbb{E} Z = 0 \) and \( h < h_0 \) yields the announced result.

7.2 Appendix B: Proof of Theorem 3.1

To get into the core of the proof of the first part of Theorem 3.1, we need to show some properties of the functions \( \bar{y}_k \) and \( \bar{z}_k \).

Lemma 7.1. The functions \( \bar{y}_k \) and \( \bar{z}_k \) defined by (36)-(37) are Lipschitz continuous with \( |\bar{y}_k|_{\text{Lip}} \) and \( |\bar{z}_k|_{\text{Lip}} \) their respective Lipschitz coefficients given by

\[
|\bar{y}_k|_{\text{Lip}} \leq |h|_{\text{Lip}} + \Delta_{\max} (1 + \Delta_{\max}) |f|_{\text{Lip}} + e^{(1+C_{f}+C_{b,s})} \Delta_{\max} |\bar{y}_{k+1}|_{\text{Lip}}
\]

and

\[
|\bar{z}_k|_{\text{Lip}} \leq \frac{1}{\sqrt{\Delta}} |\bar{y}_{k+1}|_{\text{Lip}} e^{C_{b,s} \Delta}
\]

where \( C_{b,s} = 1 + \Delta_{\max} (2|h|_{\text{Lip}} + |\sigma_{k}|_{\text{Lip}}) + \Delta_{\max}^{2} |h|_{\text{Lip}}^{2} \) and \( C_{f} = 2 |f|_{\text{Lip}} + |f|_{\text{Lip}}^{2} \).

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Proof. STEP 1: We show that \( \tilde{y}_k \) and \( \tilde{y}_k \) are Lipschitz continuous. We rely on a backward induction. In this part, we denote \( \mathcal{E}_k^x = \mathcal{F}_k(x, \varepsilon_{k+1}) \) for every \( x \) to alleviate notations. It is clear that \( [\tilde{y}_n]_{\text{Lip}} = [g]_{\text{Lip}} \). We assume that \( \tilde{y}_{k+1} \) is \([\tilde{y}_{k+1}]_{\text{Lip}}\)-Lipschitz continuous and show the Lipschitz continuity of \( \tilde{y}_k \). For every \( x, x' \), we start by noticing that

\[
|\tilde{y}(x) - \tilde{y}(x')| = \left| \mathbb{E}_k \tilde{y}_{k+1}(\mathcal{E}_k^x) - \mathbb{E}_k \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) + \Delta \left( A_k(x - x') + B_k \mathbb{E}_k \left( \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right) \right) \right|
\]

where

\[
A_k = \frac{\mathbb{E}_k \tilde{y}_{k+1}(\mathcal{E}_k^x)}{x - x'} \mathbb{1}_{x \neq x'},
\]

\[
B_k = \frac{\mathbb{E}_k \tilde{y}_{k+1}(\mathcal{E}_k^x)}{x - x'} \mathbb{1}_{x \neq x'},
\]

\[
C_k = \frac{\mathbb{E}_k \tilde{y}_{k+1}(\mathcal{E}_k^x)}{x - x'} \mathbb{1}_{x \neq x'},
\]

It is clear that these quantities are \( \mathcal{F}_k \)-measurable and that max \( \{|A_k|, |B_k|, |C_k|\} \leq |f|_{\text{Lip}} \) so

\[
|\tilde{y}(x) - \tilde{y}(x')| \leq \Delta |f|_{\text{Lip}}|x - x'| + \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \left( 1 + B_k + C_k \sqrt{\Delta} \right) \right).
\]

Now, using the inequality \( (a + b)^2 \leq a^2(1 + \Delta) + b^2(1 + \frac{1}{\Delta}) \), one obtains

\[
|\tilde{y}(x) - \tilde{y}(x')|^2 \leq \Delta^2 |f|_{\text{Lip}}^2 |x - x'|^2 (1 + \frac{1}{\Delta}) + (1 + \Delta) \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \left( 1 + B_k + C_k \sqrt{\Delta} \right) \right)^2
\]

\[
\leq \Delta^2 (1 + \Delta) |f|_{\text{Lip}}^2 |x - x'|^2 + (1 + \Delta) \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \right)^2 \mathbb{E}_k \left( 1 + B_k + C_k \sqrt{\Delta} \right)^2.
\]

Since, \( (\varepsilon_k)_{k \geq 0} \) is a sequence of i.i.d. random variables, then

\[
\mathbb{E}_k \left( 1 + B_k + C_k \sqrt{\Delta} \right)^2 = (1 + |f|_{\text{Lip}} \Delta)^2 + \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \right)^2 \leq 1 + 2\Delta |f|_{\text{Lip}} + \Delta |f|_{\text{Lip}}^2 \leq |f|_{\text{Lip}}^2,
\]

so that

\[
|\tilde{y}(x) - \tilde{y}(x')|^2 \leq (1 + \Delta) |f|_{\text{Lip}}^2 |x - x'|^2 + e^{(C_k + 1)\Delta} \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \right)^2.
\]

At this stage, one notes that if \( a, b \geq 0 \), then max \( (a, b)^2 \leq \max(a^2, b^2) \) so

\[
|\tilde{y}(x) - \tilde{y}(x')|^2 \leq \max \left( |h_k(x) - h_k(x')|^2, |\tilde{y}(x) - \tilde{y}(x')|^2 \right)
\]

\[
\leq \max \left( |h_k|^2_{\text{Lip}} |x - x'|^2, \Delta (1 + \Delta) |f|_{\text{Lip}}^2 |x - x'|^2 + e^{(1 + C_k + \Delta)\Delta} \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \right)^2 \right).
\]

We use the fact that \( \tilde{y}_k \) is Lipschitz continuous and write

\[
\mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \right)^2 \leq \mathbb{E}_k \left[ |h_k(x) - \tilde{y}_{k+1}(\mathcal{E}_k^x)| x - x' + \Delta |b_k(x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'})| \right]^2 \right.
\]

\[
\leq \left[ \mathbb{E}_k \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) \right| x - x'^2 (1 + \Delta) |h_k|_{\text{Lip}}^2 \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) \right| x - x'^2 \right) + \Delta^2 |h_k|_{\text{Lip}}^2 \right]
\]

\[
\leq \left[ \mathbb{E}_k \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) \right| x - x'^2 \right] \Delta \Delta \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) \right| x - x'^2 \right)
\]

where \( C_{b, \sigma} = 2|h_k|_{\text{Lip}} + |\sigma_k|_{\text{Lip}} + \Delta \max |b_k|_{\text{Lip}}^2 \). Therefore, one has

\[
|\tilde{y}(x) - \tilde{y}(x')|^2 \leq \max \left( |h_k|^2_{\text{Lip}} |x - x'|^2, \Delta (1 + \Delta) |f|_{\text{Lip}} |x - x'|^2 + e^{(1 + C_k + C_{b, \sigma})\Delta} \mathbb{E}_k \left( \left| \tilde{y}_{k+1}(\mathcal{E}_k^x) - \tilde{y}_{k+1}(\mathcal{E}_k^{x'}) \right| \right)^2 \right).
\]

Now, since \( \Delta \leq \Delta_{\text{max}} \), one deduces that \( \tilde{y}_k \) is \([\tilde{y}_k]_{\text{Lip}}\)-Lipschitz continuous with

\[
|\tilde{y}_k|_{\text{Lip}} \leq |h|_{\text{Lip}} + \Delta_{\text{max}} (1 + \Delta_{\text{max}}) |f|_{\text{Lip}} + e^{(1 + C_k + C_{b, \sigma})\Delta_{\text{max}}} |\tilde{y}_{k+1}|_{\text{Lip}}.
\]
STEP 2: For the Lipschitz continuity of $\bar{z}_k$, we will use the same property of $\bar{y}_{k+1}$, more precisely inequality (72). For every $x, x' \in \mathbb{R}^d$,

$$|\bar{z}_k(x) - \bar{z}_k(x')|^2 \leq \frac{1}{\sqrt{\Delta}} \mathbb{E} \left[ (\bar{y}_{k+1}(E_k^x) - \bar{y}_{k+1}(E_k^{x'})) |\bar{z}_k(x) - \bar{z}_k(x')| \right] \leq \frac{1}{\sqrt{\Delta}} |\bar{y}_{k+1}|_{\text{Lip}} e^{C_b \Delta} |x - x'|^2.$$ 

\qed

**Proof of Theorem 3.1.** We denote $\delta V_t = \bar{V}_t - \bar{V}_t$ for any process $V$. We consider the following stopping times

$$\tau^c = \inf \left\{ u \geq t; \int_t^u 1_{\delta Y_s > 0} dK_s > 0 \right\} \wedge T, \quad (73)$$

$$\tau^d = \min \left\{ t_j \geq t; \int_{t_j}^t (h_i(X_i) - \bar{Y}_i)_+ > 0 \right\} \wedge T \quad (74)$$

and

$$\tau = \tau^c \wedge \tau^d.$$ 

Keeping in mind that $(\bar{Y}_i)_t$ is a càglàd process (see (41)), we use Itô’s formula between $t$ and $\tau$ to write

$$|\delta Y_\tau|^2 = |\delta Y_t|^2 + 2 \int_{[t, \tau)} \delta Y_s d\delta Y_s + \int_{[t, \tau)} |\delta Z_s|^2 ds + \sum_{t \leq s < \tau} (\delta Y_s - \delta Y_{s-})^2$$

$$= |\delta Y_t|^2 - 2 \int_{[t, \tau)} \delta Y_s (f(\Theta_s) - f(\bar{\Theta}_s)) ds - 2 \int_{[t, \tau)} \delta Y_s dK_s + 2 \int_{[t, \tau)} \delta Y_s d\bar{K}_s$$

$$+ \int_{[t, \tau)} (Z_s - \bar{Z}_s) dW_s + \int_{[t, \tau)} |\delta Z_s|^2 ds + \sum_{t \leq s < \tau} (\delta Y_s - \delta Y_{s-})^2$$

where $\Theta_s = (X_s, Y_s, Z_s)$, $\bar{\Theta}_s = (\bar{X}_s, \mathbb{E} \bar{Y}_s, \bar{Z}_s)$, $s = t_i$ and $s = \bar{t}_{i+1}$ if $s \in (t_i, t_{i+1})$. One notes that $(\delta Y_s - \delta Y_{s-})^2 = (\bar{Y}_s - \bar{Y}_s)^2$ so that, by the definition of the process $K_s$, one has

$$|\delta Y_\tau|^2 = |\delta Y_t|^2 + 2 \int_{[t, \tau)} \delta Y_s (f(\Theta_s) - f(\bar{\Theta}_s)) ds + 2 \int_{[t, \tau)} \delta Y_s dK_s - \int_{[t, \tau)} (Z_s - \bar{Z}_s) dW_s$$

$$- \int_{[t, \tau)} |\delta Z_s|^2 ds - \sum_{t \leq s < \tau} (2\delta Y_s h_i(X_i) - \bar{Y}_i)_+ + (\bar{Y}_i - \bar{Y}_i)^2). \quad (75)$$

For every $t_i < \tau$, we set $\alpha_i = 2\delta Y_i (h_i(X_i) - \bar{Y}_i)_+ + (\bar{Y}_i - \bar{Y}_i)^2$ for convenience. It can be written as follows:

$$\alpha_i = 2(Y_i - \bar{Y}_i) (h_i(X_i) \vee \bar{Y}_i - \bar{Y}_i) + (\bar{Y}_i - \bar{Y}_i)^2$$

$$= 2(Y_i - \bar{Y}_i) (\bar{Y}_i - \bar{Y}_i) + (Y_i - \bar{Y}_i)^2$$

$$= (Y_i - \bar{Y}_i)^2 - (Y_i - \bar{Y}_i)^2$$

$$= (Y_i - \bar{Y}_i)^2 - (\delta Y_i)^2.$$ 

where we used, in the third line, the equality $2(a - b)(b - c) + (b - c)^2 = (a - c)^2 - (a - b)^2$.

Let us evaluate this term $\alpha_i$. For every $t_i < \tau < \tau^d$, we have, by (74), two choices: Either $h_i(X_i) < \bar{Y}_i$ so that $\bar{Y}_i = \bar{Y}_i$ and hence, $\delta Y_i = Y_i - \bar{Y}_i$, and $(\delta Y_i)^2 = (Y_i - \bar{Y}_i)^2$, or, $\delta Y_i > 0$ so, since $\bar{Y}_i < \bar{Y}_i$ for every $t$, we have $\bar{Y}_i - Y_i < \bar{Y}_i - Y_i < 0$ and then, $(\delta Y_i)^2 < (Y_i - \bar{Y}_i)^2$.

Consequently, for every $t_i \in [t, \tau[$,

$$\alpha_i = (Y_i - \bar{Y}_i)^2 - (\delta Y_i)^2 \geq 0.$$ 

Moreover, for $s \in [t, \tau]$, $s < \tau^c$ so that, by (73), we have $\delta Y_s < 0$ $dK_s$-a.e. Hence,

$$\int_{[t, \tau)} \delta Y_s dK_s < 0.$$ 

This yields

$$|\delta Y_\tau|^2 = |\delta Y_t|^2 + 2 \int_{[t, \tau)} \delta Y_s (f(\Theta_s) - f(\bar{\Theta}_s)) ds - \int_{[t, \tau)} (Z_s - \bar{Z}_s) dW_s - \int_{[t, \tau)} |\delta Z_s|^2 ds.$$ 

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Now, we evaluate $|\delta Y_\tau|^2$ depending on the value of $\tau$.

- If $\tau = \tau^d$, then, by (74), $\delta Y_\tau < 0$ and $h_\tau(X_\tau) > \bar{Y}_\tau$. This means $\bar{Y}_\tau = h_\tau(X_\tau)$ and, since $Y_t \geq h_t(X_t)$ for every $t \in [0, T]$,  
  \[ 0 \leq |\delta Y_\tau| = \bar{Y}_\tau - Y_\tau = h_\tau(X_\tau) - Y_\tau \leq h_\tau(X_\tau) - h_\tau(X_\tau). \]
  Hence, $|\delta Y_\tau|^2 \leq |h|_{\text{Lip}}^2|X_\tau - \bar{X}_\tau|^2$.

- If $\tau = \tau^c$, then, by (73), $\delta Y_\tau > 0$ and $K_\dot{s}$ changes its value so $Y_\tau = h_\tau(X_\tau)$. Consequently,  
  \[ 0 \leq |\delta Y_\tau| = Y_\tau - \bar{Y}_\tau = h_\tau(X_\tau) - \bar{Y}_\tau \leq h_\tau(X_\tau) - h_\tau(X_\tau) \]
  since $\bar{Y}_\tau \geq h_t(X_t)$ for every $t \in [0, T]$. So, $|\delta Y_\tau|^2 \leq |h|_{\text{Lip}}^2|X_\tau - \bar{X}_\tau|^2$.

- If $\tau = T$, $\delta Y_T = g(X_T) - g(\bar{X}_T)$ so $|\delta Y_\tau|^2 \leq |g|_{\text{Lip}}^2|X_\tau - \bar{X}_\tau|^2$. Consequently, for all the possible values of $\tau$, we have  
  \[ |\delta Y_\tau|^2 \leq C_{h,g,b,T,\sigma} \Delta \]

where $C_{h,g,b,T,\sigma}$ is a constant related to the Euler discretization error and depending on $h$ and $g$. Thus, taking the conditional expectation with respect to $t$ leads to  
\[
E_t \left( |\delta Y_\tau|^2 + \int_{t}^{\tau} |\delta Z_s|^2 ds \right) \leq C_{h,g,b,T,\sigma} \Delta + 2E_t \int_{t}^{\tau} \delta Y_s (f_s(\Theta_s) - f_\bar{s}(\bar{\Theta}_s)) - E_t \int_{t}^{\tau} (Z_s - \bar{Z}_s)dW_s. \tag{76}
\]

It remains to study the term $2E_t \int_{t}^{\tau} \delta Y_s (f_s(\Theta_s) - f_\bar{s}(\bar{\Theta}_s))$. As $f$ is Lipschitz continuous, we use Young’s inequality $ab \leq \frac{a^2}{2a} + \frac{b^2}{2}$ and the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ to write  
\[
2E_t \int_{t}^{\tau} \delta Y_s (f_s(\Theta_s) - f_\bar{s}(\bar{\Theta}_s)) \leq \frac{3|f|_{\text{Lip}}}{\alpha} \left( \int_{t}^{\tau} E_t |X_s - \bar{X}_s|^2 ds + \int_{t}^{\tau} E_t |Y_s - \bar{Y}_s|^2 ds \right) + \alpha|f|_{\text{Lip}} E_t \int_{t}^{\tau} |\delta Y_s|^2 ds. \tag{77}
\]

On the one hand,  
\[
E_t |X_s - \bar{X}_s|^2 \leq 2E_t |X_s - X_\bar{sl}|^2 + 2E_t |X_s - \bar{X}_s| \]
where $E_t |X_s - X_\bar{sl}|^2$ is bounded as follows: from (3) taken between $\bar{s}$ and $s$, we have  
\[
E_t |X_s - X_\bar{sl}|^2 \leq 2E_t \int_{\bar{s}}^{s} b_u(X_u)^2 du + 2E_t \int_{\bar{s}}^{s} \sigma_u(X_u)^2 du \leq 4L_{b,\sigma}^2 E_t \int_{\bar{s}}^{s} (1 + |X_u|)^2 du \leq 4L_{b,\sigma}^2 \Delta E_t \sup_{\bar{s} \leq u \leq s} (1 + |X_u|)^2.
\]
Hence, denoting $C_X = 4L_{b,\sigma}^2 (\tau - t)$,  
\[
\int_{t}^{\tau} E_t |X_s - \bar{X}_s|^2 ds \leq C_X \Delta E_t \sup_{\bar{s} \leq u \leq s} (1 + |X_u|)^2 + 2 \int_{t}^{\tau} E_t |X_s - \bar{X}_s|^2 ds. \tag{78}
\]

On the other hand,  
\[
E_t |Y_s - \bar{Y}_s|^2 \leq 2E_t |Y_s - \bar{Y}_s|^2 + 4E_t |Y_s - \bar{Y}_s|^2 + 4E_t \bar{Y}_s - \bar{Y}_\bar{s}|^2. \tag{79}
\]
For every $v, v'$ such that $v < v'$ and $|v - v'| \leq \Delta$, (41) at $v$ and $v'$ yields  
\[
\bar{Y}_v - \bar{Y}_{v'} = (v' - v) f(v, \bar{X}_v, E_v \bar{Y}_v, \bar{\zeta}_\bar{v}) - \int_{v}^{v'} \bar{Z}_dW_s + \bar{K}_{v'} - \bar{K}_v
\]
so that taking the conditional expectations w.r.t. $t$ yields  
\[
E_t |\bar{Y}_v - \bar{Y}_{v'}|^2 \leq 2(v' - v)^2 E_t f(\bar{\Theta}_\bar{v})^2 + 2E_t \left( \int_{v}^{v'} \bar{Z}_dW_s \right)^2 + 2E_t (\bar{K}_{v'} - \bar{K}_v)^2 \leq 2\Delta^2 E_t f(\bar{\Theta}_\bar{v})^2 + 2E_t \left( \int_{\bar{v}}^{v'} \bar{Z}_dW_s \right)^2 + 2E_t (\bar{K}_{v'} - \bar{K}_v)^2.
\]
Since $\bar{K}_v \geq 0$ for every $v \in [0, T]$, we have $\bar{K}_v < \bar{K}_v$ so that $\bar{K}_v - \bar{K}_v < \bar{K}_v + \bar{K}_v$. Then, owing the fact that $\bar{K}_v$ is non-decreasing, $\bar{K}_v - \bar{K}_v \geq 0$ so

$$(\bar{K}_v - \bar{K}_v)^2 \leq (\bar{K}_v - \bar{K}_v)(\bar{K}_v + \bar{K}_v) = \bar{K}_v^2 - \bar{K}_v^2.$$

Hence, noting that $f(\bar{\Theta}_s) = f(\bar{X}_s, E_s \bar{Y}_s, \bar{z}_s)$ is a composition of the functions $f$, $\bar{y}_s$ and $\bar{z}_s$ which are all Lipschitz continuous according to Lemma 7.1 and recalling that if a function $g$ is Lipschitz continuous then it has linear growth i.e. there exists a finite constant $C_0$ such that $g(x) \leq C(1 + |x|)$, one has

$$E_t[|\bar{Y}_v - \bar{Y}_v|^2 \leq 2\Delta^2 C_0 E_t[1 + \sup_{|x| \leq v} |\bar{X}_s|^2] + 2E_t \left( \int_{v}^{t} \bar{Z}_s dW_s \right)^2 + 2E_t(\bar{K}_v^2 - \bar{K}_v^2).$$

Combining this with (79) twice yields

$$\int_t^\tau E_t[|Y_s - E_s \bar{Y}_s|^2] ds \leq 2 \int_t^\tau E_t[|\delta Y_s|^2] ds + 4 \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} \int_{t_i}^{t_{i+1}} E_t[|Y_s - \bar{Y}_s|^2] ds + 4 \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} \int_{t_i}^{t_{i+1}} E_t[|Y_s - \bar{Y}_s|^2] ds$$

$$\leq 2 \int_t^\tau E_t[|\delta Y_s|^2] ds + 8\Delta^2 C_0(\tau - t)E_t[1 + \sup_{s \leq \tau} |\bar{X}_u|)^2$$

$$+ 8 \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} \int_{t_i}^{t_{i+1}} E_t(\bar{K}_s^2 - \bar{K}_s^2) + E_t(\bar{K}_s^2 - \bar{K}_s^2)$$

$$+ 8 \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} \int_{t_i}^{t_{i+1}} E_t(\int_{\tau}^s \bar{Z}_u dW_u)^2 ds + 8 \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} \int_{t_i}^{t_{i+1}} E_t(\int_{\tau}^s \bar{Z}_u dW_u)^2 ds$$

$$\leq 2 \int_t^\tau E_t[|\delta Y_s|^2] ds + 8\Delta(\tau - t)E_t[|\bar{K}_T|^2 + 8\Delta^2 C_0(\tau - t)E_t[1 + \sup_{s \leq \tau} |\bar{X}_u|)^2$$

$$+ 8 \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} \int_{t_i}^{t_{i+1}} \left( E_t(\int_{\tau}^s \bar{Z}_u dW_u)^2 + E_t(\int_{\tau}^s \bar{Z}_u dW_u)^2 \right) ds \tag{80}$$

where we used the fact that $\bar{K}_t$ is a non-decreasing positive process so for every $t \in [0, T]$, $\bar{K}_t < \bar{K}_T$ and the fact that $\sup_{s \leq \tau} \alpha_u \leq \sup_{s \leq \tau} \alpha_u$.

Thirdly,

$$E_t \int_t^\tau |Z_s - \bar{z}_s|^2 ds \leq \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} E_t \int_{t_i}^{t_{i+1}} |Z_s - \bar{z}_s|^2 ds$$

$$\leq \sum_{i=\tau/\Delta}^{\tau/\Delta - 1} E_t \left( 4 \int_{t_i}^{t_{i+1}} |Z_s - Z_s|^2 ds + 4 \int_{t_i}^{t_{i+1}} |Z_s - \bar{z}_s|^2 ds + 2 \int_{t_i}^{t_{i+1}} |\bar{z}_s - \bar{z}_s|^2 ds \right).$$

By the definitions (40) and (39) of $\zeta_s$ and $\bar{\zeta}_s$, we have

$$|Z_s - \zeta_s|^2 = |Z_s - \frac{1}{\Delta} E_s \int_{\tau}^s Z_s ds|^2 = \frac{1}{\Delta^2} \left| E_s \int_{\tau}^s (Z_s - Z_s) \right|^2 \leq \frac{1}{\Delta} E_s \int_{\tau}^s |Z_s - Z_s|^2 ds$$

where the last inequality was obtained by using Cauchy-Schwarz inequality. Hence, we use Fubini’s Theorem to deduce

$$\int_{t_i}^{t_{i+1}} |Z_s - \bar{z}_s|^2 ds \leq E_s \int_{\tau}^s |Z_s - Z_s|^2 ds.$$

Likewise,

$$|\zeta_s - \bar{\zeta}_s|^2 = \left| \frac{1}{\Delta} E_s \int_{\tau}^s (Z_s - Z_s) \right|^2 \leq \frac{1}{\Delta} E_s \int_{\tau}^s |Z_s - Z_s|^2 ds.$$
Consequently,
\[ \mathbb{E}_t \int_0^\tau |Z_k - \bar{Z}_k|^2 ds \leq 8 \mathbb{E}_t \int_0^\tau |Z_k - Z_k|^2 ds + 2 \mathbb{E}_t \int_0^\tau |Z_s - \bar{Z}_s|^2 ds. \] (81)

At this stage, we merge the 3 equations (78), (80) and (81) with (77) and take the expectation to obtain
\[ \mathbb{E}\left(|\delta Y_i|^2 + \int_t^\tau |\delta Z_k|^2\right) \leq \Delta C_{h,g,b,T,\sigma} + \frac{6 \int |f|_{\text{Lip}}}{\alpha} (C_X + C_0(\bar{\tau} - \frac{\tau}{2}))\mathbb{E}(1 + \sup_{\frac{\tau}{2} \leq u \leq \frac{3\tau}{2}} |X_{u}|^2) + \frac{12 \int |f|_{\text{Lip}}}{\alpha} (\bar{\tau} - \frac{\tau}{2})\mathbb{E}|K|^2 \]

Moreover,
\[ \sum_{i=t/\Delta}^{(\bar{\tau}/\Delta)-1} \int_t^{t+1} \left( \mathbb{E}\int_{\frac{\tau}{2}}^{s} |Z_k|^2 ds + \mathbb{E}\int_{\frac{\tau}{2}}^{s} |\bar{Z}_k|^2 ds \right) ds \leq \sum_{i=t/\Delta}^{(\bar{\tau}/\Delta)-1} \int_t^{t+1} 2\Delta \mathbb{E}\sup_{\frac{\tau}{2} \leq u \leq \frac{3\tau}{2}} |Z_u|^2 \leq 2\Delta^2 (\bar{\tau} - \frac{\tau}{2})\gamma_1. \]

Hence, if we consider \( \alpha = 6 \int |f|_{\text{Lip}} \) and denote \( \tilde{C} = C_{h,g,b,T,\sigma} + C_X + (\bar{\tau} - \frac{\tau}{2})(2\gamma_0 + 4\gamma_1 \Delta_{\text{max}} + (1 + C_0)C_{h,T,\sigma}(1 + |x_0|^2)) \) and \( \tilde{C} = 1 + 12 \int |f|_{\text{Lip}}^2 \), then we obtain
\[ \mathbb{E}\left(\int_t^\tau |\delta Y_i|^2 + \int_t^\tau |\delta Z_k|^2\right) \leq \Delta \tilde{C} + \tilde{C} \int_t^\tau \mathbb{E}|\delta Y_k|^2 ds + 4\mathbb{E}\int_t^\tau |Z_k - Z_k|^2 ds \]
\[ + \mathbb{E}\int_t^\tau |Z_k - \bar{Z}_k|^2 ds + \mathbb{E}\int_t^\tau |Z_k - Z_k|^2 ds + \mathbb{E}\int_t^\tau |Z_k - \bar{Z}_k|^2 ds. \]

Consequently,
\[ \mathbb{E}|\delta Y_i|^2 \leq \tilde{C} \int_t^\tau \mathbb{E}|\delta Y_k|^2 ds + K \]
where \( K = \Delta \tilde{C} + 4\mathbb{E}\int_t^\tau |Z_k - Z_k|^2 ds + 4\mathbb{E}\int_t^\tau |Z_k - \bar{Z}_k|^2 ds + \mathbb{E}\int_t^\tau |Z_k - Z_k|^2 ds. \) Let us denote \( f(t) = \mathbb{E}|\delta Y_i|^2 \). This function satisfies
\[ f(t) \leq \tilde{C} \int_t^\tau f(s) ds + K. \]

We consider \( g(t) = f(T - t) \) which satisfies also
\[ g(t) \leq \tilde{C} \int_0^t g(s) ds + K. \]

Hence, Gronwall’s Lemma yields \( g(t) \leq e^{\tilde{C}t} K \) so that
\[ f(t) \leq e^{\tilde{C}(T-t)} K. \]
Consequently,

\[ E|Y_t - \bar{Y}_t|^2 \leq e^{\tilde{C}(T-t)} \left( \Delta \tilde{C} + 4E \int_0^T |Z_s - Z_s'|^2 ds + E \int_t^T |Z_s - \bar{Z}_s|^2 ds + E \int_{\tau}^{\bar{\tau}} |Z_s - \bar{Z}_s|^2 ds \right). \]

In particular, if \( t = t_k \) and \( \tau = t_{k'} \), \( k, k' \in \{1, \ldots, n\} \), then \( t = t \) and \( \bar{\tau} = \tau \) so

\[ E|Y_k - \bar{Y}_k|^2 \leq e^{\tilde{C}(T-t_k)} \left( \Delta \tilde{C} + 4 \int_0^T E|Z_s - Z_s'|^2 ds + 0 + 0 \right). \]

This completes the proof. □