Gravitation and Electromagnetism as Geometrical Objects of a Riemann-Cartan Spacetime Structure

J. Fernando T. Giglio\(^{(1)}\) and Waldyr A. Rodrigues Jr.\(^{(2)}\).

\(^{(1)}\) FEA-CEUNSP. 13320-902 - Salto, SP, Brazil.
jfernandotg@gmail.com

\(^{(2)}\) Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP
13083-859 Campinas, SP, Brazil
walrod@ime.unicamp.br or walrod@mpc.com.br

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Abstract

In this paper we first show that any coupled system consisting of a gravitational plus a free electromagnetic field can be described geometrically in the sense that both Maxwell equations and Einstein equation having as source term the energy-momentum of the electromagnetic field can be derived from a geometrical Lagrangian proportional to the scalar curvature \(R\) of a particular kind of Riemann-Cartan spacetime structure. In our model the gravitational and electromagnetic fields are identified as geometrical objects of the structure. We show moreover that the contorsion tensor of the particular Riemann-Cartan spacetime structure of our theory encodes the same information as the one contained in Chern-Simons term \(A \wedge dA\) that is proportional to the spin density of the electromagnetic field. Next we show that by adding to the geometrical Lagrangian a term describing the interaction of an electromagnetic current with a general electromagnetic field plus the gravitational field, together with a term describing the matter carrier of the current we get Maxwell equations with source term and Einstein equation having as source term the sum of the energy-momentum tensors of the electromagnetic and matter terms. Finally modeling by dust charged matter the carrier of the electromagnetic current we get the Lorentz force equation. Moreover, we prove that our theory is gauge invariant. We also briefly discuss our reasons for the present enterprise.
1 Introduction

Since the geometrization of gravitation by General Relativity (GR) where the gravitational field, generated by an energy-momentum tensor $T \in \text{sec} T^2_0 M$, is represented by a particular Lorentzian spacetime structure $\langle M, g, \bar{\nabla}, \tau g, \uparrow \rangle$ together with Einstein equation, several classical models have been proposed which try to geometrize the description of the electromagnetic field, with the obvious interest in describing both fields, i.e., gravitational and electromagnetic, through a unique principle: the geometric one.

This has been tried by generalizing the Lorentzian spacetime structure, i.e., utilizing more general geometries incorporating additional degrees of freedom, which hopefully permits in principle a description of the electromagnetic field as some aspect of the underlying chosen geometry. In this sense, there has been many lines of investigation of this problem, some of the most well known are:

(i) Weyl theory, where non metric compatible symmetric connections $[51, 52]$ are used. The resulting geometry has non zero Riemann tensor and a null torsion tensor and is now known as Weyl geometries $[58]$. We mention also in this class Eddington unified theory which is described by a non metric compatible connection with non null Riemann and torsion tensors $[15]$.

(ii) Theories based on spacetimes with more than four dimensions $[33, 36]$, known as Kaluza-Klein theories. These theories have been studied in the last decades in connection with fiber bundle formulations of the four fundamental interactions $[3, 12]$.

(iii) Introduction of metric compatible connections in four dimensions, other than that the Levi-Civita connection, like, e.g., in the so-called Riemann-Cartan geometries $[56, 9]$, which in general have non zero Riemann and torsion tensors.

(iv) The non symmetric metric theory of Einstein $[16]$.

(v) Theories on Finslerian spaces $[18, 6, 60, 61]$.

(vi) Einstein teleparallel theory.

The fact is that all these theories are problematic. According to the majority view, Weyl theory received what is thought to be a knockdown by Einstein $[11]$ but according to Eddington, non metricity can not be completely ruled out by experiment if due care is taken $[15]$. Concerning (ii) it remains always a problem to explain why the extra dimensions are not observable, or why they compactify $[12]$. Concerning (iii), several theories that includes Cartan’s torsion in GR have been proposed. One motivation was to obtain a unified geometrical description

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1 In this paper the notation $\text{sec} T^r_0 M$ means section the the $T^r_0 M$ bundle. Also, $\text{sec} \Lambda^r T^* M$ means section of the bundle of $r$-form fields.

2 In the structure $\langle M, g, \bar{\nabla}, \tau g, \uparrow \rangle$, the pair $\langle M, g \rangle$ is called a Lorentzian manifold, $M$ being a 4-dimensional Hausdorff paracompact locally compact manifold and $g \in \text{sec} T^2_0 M$ a Lorentzian metric of signature $(1, -1, -1, -1)$. $\bar{\nabla}$ is the Levi-Civita connection of $g$, $\tau g \in \text{sec} \Lambda^4 T^* M$ and $(\uparrow)$ define respectively a spacetime orientation and a time orientation for $M$. More details about time orientation may be found, e.g., in $[54]$. See complete list of Einstein papers on teleparallelism in $[53]$.

3 The problem exists also in modern string theory, for no reason is given for the compactification of the extra dimensions.
of electromagnetism and gravitation [18, 25]. These theories, and that of (i) and (ii), at least to what refers to a classical unified description of the gravitational and electromagnetic fields, are not in general totally accepted because of their failure in obtaining simultaneously the electromagnetic field equations (Maxwell equations), the energy-momentum tensor of the electromagnetic field and the Lorentz force equation for the motion of charged matter as they are known in the physical situations involving gravitation and electromagnetism in a four dimensional universe. Also, some authors [4] are of the opinion that Riemann-Cartan geometry is necessary to describe besides gravitation, also the spinning matter, which is supposed to be the source of the torsion field [9, 28, 25, 29, 30] and it seems also that a non vanishing torsion tensor appears as a necessary ingredient in the gauge formulation of GR when the Poincaré invariance is taken locally [26, 59, 34, 55, 39]. Besides that in string theory, in the low energy limit, the effective Lagrangian has an antisymmetric field that is interpreted as torsion [19]. Torsion derived from scalar field, vector field or antisymmetric tensor field can be found in Hammond [20] and references cited therein. The non-symmetric theory of Einstein has recently been developed by Moffat [40] with a very different interpretation aiming to describe dark matter, but will not be commented here, nor will we discuss the status of theories that use Finslerian spacetimes. However we comment that Einstein’s teleparallel theory, that he originally interpreted as a unified theory of the gravitational plus the electromagnetic field is indeed a non sequitur. The case is that Einstein’s preferred version of his teleparallel theory has a Lagrangian that is equivalent to the Einstein-Hilbert Lagrangian of GR [6]. This according to [53] has been informed by Lanczos to Einstein and put, so to say, an end to teleparallelism as an unified field theory

Moreover, it is now known [50] that a theory of the gravitational field can be formulated for the gravitational potentials \( g^a \in \text{sec} \wedge^1 T^* M \) (with at least one of the \( g^a \) for \( a = 0, 1, 2, 3 \) non closed, i.e., \( F^a = \delta g^a \neq 0 \)) living on Minkowski spacetime \( \langle M, \eta, D, \tau_\eta, \uparrow \rangle \) and satisfying field equations derived from a postulated Lagrangian density, thus dispensing the geometrical interpretation of gravitation as a Lorentzian or a teleparallel spacetime. The field equations of the theory are easily seem to be equivalent to Einstein’s equations once we introduce a field \( g = \eta_{ab} g^a \otimes g^b \in \text{sec} T^2_0 M \) together with its Levi-Civita connection \( \bar{\nabla} \) and interpret the structure \( \langle M, g, \bar{\nabla}, \tau_g, \uparrow \rangle \) as an effective Lorentzian spacetime [7].

It is also the case that for the formulation of the electromagnetic field theory we do not even need a metric field defined on a manifold that serves as support

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5Well, this is indeed a polemical view, not endorsed, e.g., by Weinberg. See his exchange of letters with Hehl in Physics Today at: http://ptonline.aip.org/journals/doc/PHTOAD-9/vol_60/iss_3/16.2.shtml?bypassSSO=1

 Anyway there are some proposals in the literature to observe experimentally the existence of torsion, see, e.g., [20, 57].

6This is clear, e.g., in [2] where the torsion tensor is used to describe the gravitational field of Einstein’s GR in a particular Riemann-Cartan spacetime structure known nowadays as Weitzenböck (or teleparallel) spacetime \( \langle M, g, \nabla, \tau_g, \uparrow \rangle \) where the curvature tensor of \( \nabla \) is null, and the torsion tensor of \( \nabla \) is non null. See also [50].

7There are other possibilities also involving more complicated geometrical structures, see, e.g., [44].
for that field \([31]\). Indeed, it is well known that Maxwell equations can be written in a star shape manifold \([11]\) as the compatibility equations for a closed 2-form field \(F \in \text{sec} \bigwedge^2 T^* M\) \((dF = 0)\) and a closed current \(J \in \text{sec} \bigwedge^3 T^* M\) \((dJ = 0)\). These equations imply in a star shape manifold the existence of a 1-form field \(A \in \text{sec} \bigwedge^1 T^* M\) and a 2-form field \(G \in \text{sec} \bigwedge^2 T^* M\) such that \(F = da\) and \(J = -dG\) in such a way that Maxwell equations read

\[
dF = 0, \quad dG = -J
\]  

(1)

It is also well known that in general \(G\) is related to \(F\) through the so-called constitutive equations of the medium. In that sense the gravitational field in GR modeled by a Lorentzian spacetime serves as an effective medium for the propagation of the electromagnetic field and the constitutive equations are given simple by

\[
G = *_g F,
\]  

(2)

and thus defining \(J = *_g J \in \text{sec} \bigwedge^1 T^* M\) we can write Maxwell equations as

\[
dF = 0, \quad g \delta F = -J,
\]  

(3)

where \(*_g\) is the Hodge dual operator and \(\delta\) is the Hodge coderivative operator\([8]\). The intrinsic Maxwell equations even when expressed in a Lorentzian or Riemann-Cartan spacetime structures do not need the use of the covariant derivative operator of those structures for their writing, although they can be formulated with such operators (see below\([9, 10]\)).

Having said all that we can ask: is it possible, in the same sense that the gravitational field can be describe by the potentials \(g^a\) living in Minkowski spacetime or by a Lorentzian structure \(\langle M, g, \nabla, \tau_g, \uparrow \rangle\) or a teleparallel structure \(\langle M, g, \nabla, \tau_g, \uparrow \rangle\) to describe gravitation and electromagnetism geometrically, i.e., through a particular Riemann-Cartan spacetime structure where those fields are represented by some of the geometrical objects associated with that structure?

As we shall see, the answer is positive once we use a Riemann-Cartan structure equipped with a particular connection whose contortion tensor is given by Eq.\((23)\) below\([11]\).

In our very simple model we are able to obtain a unified description of the gravitational and free electromagnetic fields as geometrical aspects of a particular Riemann-Cartan spacetime, with the Einstein and the free Maxwell
equations being derived from a geometrical Lagrangian, i.e., a Lagrangian proportional to the scalar curvature $R$ of a particular Riemann-Cartan connection. The main mathematical tools for doing that is presented in Section 2 where we also show that the information contained in the contortion tensor of the particular Riemann-Cartan spacetime structure of our theory is the same as the one contained in the Chern-Simons term $A \wedge dA$ that as well known now [48, 21] is proportional to the spin density of the electromagnetic field.

Moreover, we show in Section 3 that by adding to the geometrical Lagrangian an interaction term proportional to $J \cdot A$ describing the source of the electromagnetic field and its interaction with that field and a term describing the matter carrier of the current we get Maxwell equations with source term and Einstein equations having as source term the sum of the energy-momentum tensors of the electromagnetic and matter terms. In Section 4, modeling by dust charged matter the carrier of the electromagnetic current we get from the nullity of Riemann-Cartan covariant derivative of the sum of energy-momentum tensor of matter plus the electromagnetic field the Lorentz force equation. In Section 5 we show that our theory is well defined by proving its gauge invariance. Finally, in Section 6 we present our conclusions.

2 Maxwell Equations

In what follows $\langle M, g \rangle$ as defined above is a Lorentzian manifold. Let $\nabla$ and $\nabla$ be respectively the Levi-Civita connection and a particular metric compatible Riemann-Cartan [43] connection of $g$ on $M$. Let $U \subset M$ and $\langle x^\mu \rangle$ be a coordinates for $U \subset M$, and $\langle e_\mu = \partial/\partial x^\mu \rangle$ a basis of $TU$ ($\mu = 0, 1, 2, 3$) and $\langle \vartheta_\mu = dx^\mu \rangle$ the corresponding dual basis i.e., a basis for $T^*U$. We also introduce the reciprocal basis $\langle e^\mu \rangle$ of $\langle e_\mu \rangle$ for $TM$ and the reciprocal basis $\langle \vartheta^\mu \rangle$ of $\langle \vartheta_\mu \rangle$ for $T^*M$, such that

$$g = g_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu = g^{\mu\nu} \partial_\mu \otimes \partial_\nu, \quad g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu,$$

$$e^\mu = g^{\mu\nu} e_\nu, \quad \vartheta^\mu = g^{\mu\nu} \vartheta_\nu.$$

Moreover we introduce as metric for the cotangent bundle the object $g \in \text{sec} T^*_2M$,

$$g = g^{\mu\nu} e_\mu \otimes e_\nu = g_{\mu\nu} e^\mu \otimes e^\nu,$$

and define the scalar product of arbitrary 1-form fields $X$ and $Y$ by

$$X \cdot Y = g(X, Y)$$

Let moreover $\Gamma^{-\lambda}_{\mu\nu}$ and $\bar{\Gamma}^{-\lambda}_{\mu\nu}$ be the connection coefficients of $\nabla$ and $\nabla$ in the coordinate basis just introduced, i.e., $\nabla_\partial_\mu \partial_\nu = \Gamma^{-\lambda}_{\mu\nu} \partial_\lambda$ and $\nabla_\partial_\mu \partial_\nu = \bar{\Gamma}^{-\lambda}_{\mu\nu} \partial_\lambda$. As it is well known (see, e.g., [37] [48]), $\Gamma^{-\lambda}_{\mu\nu}$ and $\bar{\Gamma}^{-\lambda}_{\mu\nu}$ are related by

$$\Gamma^{-\lambda}_{\mu\nu} = \bar{\Gamma}^{-\lambda}_{\mu\nu} + K^{-\lambda}_{\mu\nu},$$

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where the connection coefficients $\Gamma_{\mu\nu}^\lambda$ of the Levi-Civita connection are given by:

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

(7)

and the $K_{\mu\nu}^\lambda$ are the components of the contorsion tensor $K = K_{\mu\nu}^\lambda e_\beta \otimes dx^\mu \otimes dx^\nu \in TM \otimes \text{sec} T^2 M$ defined by [22].

$$K_{\mu\nu}^\lambda := \frac{1}{2} (g^{\lambda\beta} g_{\mu\rho} T_{\rho\nu}^\beta - g^{\lambda\beta} g_{\nu\rho} T_{\rho\mu}^\beta - g^{\lambda\beta} g_{\mu\nu} T_{\rho\rho}^\beta)$$

(8)

where the connection coefficients $\Gamma_{\mu\nu}^\lambda$ of the Levi-Civita connection are given by:

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

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(8)

However, taking into account that we have the (bastard [23]) symmetry $K_{\mu\nu}^\lambda := g_{\beta\lambda} K_{\mu\nu}^\beta = -K_{\mu\nu}^\lambda$ we prefer in what follows to take the contorsion as the object $K \in \Lambda^1 T^* M \otimes \Lambda^2 T^* M$ (which carries the same information as $K$) defined by

$$K = \frac{1}{2} K_{\mu\nu}^\lambda \vartheta^\mu \otimes \vartheta^\nu \wedge \vartheta^\lambda$$

(9)

where $\theta = \frac{1}{2} \Gamma_{\mu\nu}^\lambda e_\lambda \otimes \vartheta^\mu \wedge \vartheta^\nu \in \text{sec} TM \otimes \Lambda^2 T^* M$ are the components of the torsion tensor $\Theta = \frac{1}{2} T_{\mu\nu}^\lambda \vartheta^\mu \otimes \vartheta^\nu \in \Lambda^1 T^* M \otimes \Lambda^2 T^* M$ which encodes the same information than $\Theta$.  

We now proceed by introducing a particular Riemann-Cartan spacetime structure $\langle M, g, \nabla, \tau g, \uparrow \rangle$ where the contorsion tensor is defined by

$$K := -C B \otimes F \in \text{sec} \Lambda^1 T^* M \otimes \Lambda^2 T^* M$$

(11)

with components

$$K_{\mu\nu}^\lambda = -C B_\mu F_\nu^\lambda, \quad K_{\mu\nu}^\lambda = -C B_\mu F_\nu^\lambda = -K_{\mu\nu}^\lambda$$

(12)

and where the constant $C$ and the $B_\mu$, which are the components of $B \in \text{sec} \Lambda^1 T^* M$, are to be determined and where the $F_\nu^\lambda := g^{\lambda\alpha} F_{\nu\alpha} = -g^{\alpha\lambda} F_{\nu\alpha} = -F_{\nu}^\lambda$ are the components of $F \in \text{sec} \Lambda^1 T^* M$, i.e.,

$$F := \frac{1}{2} F_{\mu\nu} \vartheta^\mu \wedge \vartheta^\nu = \frac{1}{2} F_{\mu\nu} \vartheta^\mu \wedge \vartheta^\nu$$

(13)

In what follows we propose to give a physical interpretation for those objects associated to the structure $\langle M, g, \nabla, \tau g, \uparrow \rangle$.

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12Note that this differs from the definition in [28] by a signal and a factor 1/2 due to conventions used here with are the ones in [48].

13This form is analogous to such that is taken in [8] for torsion.
Before proceeding to build our theory we recall some formulas that will be used latter. We start calculating the covariant derivative $\nabla_{\mu} F^{\mu\nu}$ 14. Using the connection given by Eq. (6) we obtain

$$\nabla_{\mu} F^{\mu\nu} = \bar{\nabla}_{\mu} F^{\mu\nu} + K^{\cdot \nu}_{\mu \delta} F^{\delta \nu} + K^{\cdot \nu}_{\mu \delta} F^{\mu \delta}. \quad (14)$$

But the last two terms in Eq. (14) cancel out due to Eq. (12), and we have

$$\nabla_{\mu} F^{\mu\nu} = \bar{\nabla}_{\mu} F^{\mu\nu} = \frac{1}{(-\det g)^{\frac{1}{2}}} \partial_{\mu} [(-\det g)^{\frac{1}{2}} F^{\mu\nu}]. \quad (15)$$

Next, we observe that from Eq. (14) it follows immediately that

$$\nabla_{[\mu} F_{\nu \lambda]} = \nabla_{\mu} F_{\nu \lambda} + \nabla_{\nu} F_{\lambda \mu} + \nabla_{\lambda} F_{\mu \nu} = \bar{\nabla}_{[\mu} F_{\nu \lambda]} = \partial_{\mu} F_{\nu \lambda} + \partial_{\nu} F_{\lambda \mu} + \partial_{\lambda} F_{\mu \nu}. \quad (16)$$

From Eqs. (15) and (16), a natural assumption is to define $F = dA$, where $A \in \text{sec} \Lambda^{1} T^{*} M$, and of course,

$$F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}. \quad (17)$$

This suggests to interpret $F$ as the electromagnetic field and $A$ as its potential, and we are going to show that this is indeed the case.

Also, Eq. (15) defines in general a conserved current 1-form field $J = J_{\mu} = J^{\nu} \partial_{\nu} \in \text{sec} \Lambda^{1} T^{*} M$ such that ($c$ being the velocity of light in vacuum)

$$\frac{4\pi}{c} J^{\nu} := \nabla_{\mu} F^{\mu\nu} = \bar{\nabla}_{\mu} F^{\mu\nu} = (-\det g)^{\frac{1}{2}} \partial_{\mu} [(-\det g)^{\frac{1}{2}} F^{\mu\nu}] \quad (18)$$

and of course,

$$\partial_{\nu} [(-\det g)^{\frac{1}{2}} J^{\nu}] = 0. \quad (19)$$

With our choise of $F$ the second member of Eq. (16) is null. We then recognize Eq. (15) and Eq. (18) as Maxwell equations written on a Lorentzian spacetime [38, 49].

3 Action Principle, Maxwell and Einstein Equations and their Source Terms

In a Riemann-Cartan spacetime the curvature tensor can be written as (see, e.g., [37, 38])

$$R_{\mu \nu \lambda}^{\cdot \cdot \chi} = \bar{R}_{\mu \nu \lambda}^{\cdot \cdot \chi} + \nabla_{\mu} K_{\nu \chi}^{\cdot \cdot \lambda} - \nabla_{\nu} K_{\chi \lambda}^{\cdot \cdot \chi} + K_{\nu \lambda}^{\cdot \cdot \rho} K_{\rho \chi}^{\cdot \cdot \chi} - K_{\chi \rho}^{\cdot \cdot \chi} K_{\rho \mu}^{\cdot \cdot \lambda}, \quad (20)$$

14More precisely we write, e.g., for a tensor $t = t_{\alpha}^{\mu} \partial_{\mu} \otimes dx^{\alpha} \in \text{sec} TM \otimes T^{*} M$, $\nabla \partial_{\mu} t = (\nabla \partial_{\mu} t_{\alpha}^{\nu}) \partial_{\nu} \otimes dx^{\alpha}$ where $\nabla \partial_{\mu} t_{\alpha}^{\nu} = \partial_{\mu} t_{\alpha}^{\nu} + \Gamma_{\mu \delta}^{\cdot \cdot \chi} t_{\chi}^{\nu} - \Gamma_{\nu \delta}^{\cdot \cdot \chi} t_{\chi}^{\nu}$. 

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where the bars as already said above refers to quantities defined with the Levi-Civita connection. The last two terms in Eq. (20) cancel out because of Eq. (12) and then from Eq. (20) we have for the scalar curvature:

\[ R = g^{\nu\lambda} R_{\nu\lambda} = \bar{R} + 2 \bar{\nabla}_\mu K^{\nu\mu}. \]  

Doing the evaluation of \( \bar{\nabla}_\mu K^{\nu\mu} \) and taking into account that \( \bar{\nabla}_\mu B_\nu - \bar{\nabla}_\nu B_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu \) we get

\[ R = \bar{R} + C (\partial_\mu B_\nu - \partial_\nu B_\mu) F^{\mu\nu} + \frac{8\pi C}{e} B_\mu J^\mu. \]  

Eq. (22) suggests to us to identify the \( B_\mu \) in \( K^{\nu\lambda} \) with the components of the electromagnetic potential \( A_\mu = A_\mu dx^\mu \), i.e., we take from now on

\[ K^{\nu\lambda} := -CA_\mu F^{\nu\lambda}, \]  

since this identification will permit us to interpret (a factor apart) the first and second terms on the r.h.s. of Eq. (22) as the gravitational field and the free electromagnetic field \((J_\mu = 0)\) Lagrangian in GR. Indeed, for that case we are in position of interpreting those fields as parts of a Riemann-Cartan spacetime structure \( \langle M, g, \nabla, \tau_g, \uparrow \rangle \) by taking as Lagrangian of the system

\[ L := -\frac{c^3}{16\pi G} \bar{R} \]  

and the action is

\[ S = \frac{-c^3}{16\pi G} \int (\bar{R} + C F^{\mu\nu} F_{\mu\nu})(-\det g)^{\frac{3}{2}} d^4 x. \]  

We can now give a geometrical model for the interaction of the electromagnetic field \( F \), its current \( J \) and the gravitational field \( g \) by interpreting those fields as parts of a Riemann-Cartan spacetime structure \( \langle M, g, \nabla, \tau_g, \uparrow \rangle \). Variation \( \delta S \) of \( S \) in Eq. (25) with respect to the contravariant components of \( g \) gives

\[ \delta S = \frac{-c^3}{16\pi G} \delta \int (\bar{R}(-g))^{\frac{3}{2}} d^4 x + C \int F^{\mu\nu} F_{\mu\nu}(-\det g)^{\frac{3}{2}} d^4 x \]
\[ = \frac{-c^3}{16\pi G} \int (\bar{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \bar{R} - 8\pi C T_{\mu\nu}) \delta g^{\mu\nu}(-\det g)^{\frac{3}{2}} d^4 x, \]  

where

\[ T_{\mu\nu} = \frac{1}{4\pi} (-F_\beta F^{\mu\beta} + \frac{1}{4} g_{\mu\nu} F^\alpha_\beta F_{\alpha\beta}), \]  

are the components of the energy-momentum tensor of electromagnetic electromagnetic field, showing moreover that we must take \( C \) as

\[ C = \frac{G}{c^4}. \]
Remark 1. This completes the proof of our claim in the introduction that any coupled system consisting of a gravitational field and electromagnetic field can be fully geometrized by a special Riemann-Cartan spacetime structure.

Remark 2. We observe that if the contorsion tensor just introduced and whose information is contained in $K$ is the same as the one contained in the Chern-Simons object $A \wedge dA \in \text{sec}\wedge^3 T^*M$. Indeed,

\[
C = A \wedge dA = A \wedge F = \frac{1}{2} A_\mu F_{\nu \lambda} \partial^\mu \wedge \partial^\nu \wedge \partial^\lambda = \frac{1}{3!} (A_\mu F_{\nu \lambda} + A_\lambda F_{\mu \nu} + A_\nu F_{\lambda \mu}) \partial^\mu \wedge \partial^\nu \wedge \partial^\lambda \tag{30}
\]

It is eventually opportune to observe that $A \wedge dA$ has been called in \cite{35} the topological torsion, although in \cite{48} it has been argued that this was not a good nomenclature since this object is proportional to the spin density of the electromagnetic field. This result seems to be endorsed by the nice analysis in \cite{21}.

Remark 3. Before proceeding we note that \cite{45} $\delta g \int A_\mu J^\mu ( - \det g ) \frac{1}{2} d^4 x = 0$. Indeed, since $J$ is a time-like 1-form field there must be (at least) one coordinate system where $J = J^0 := \rho \delta_0$ and thus $A_\mu J^\mu = A_0 \delta_0$. Consequently we have that

\[
\int A_\mu J^\mu ( - \det g ) \frac{1}{2} d^4 x = Q \int A_0 dx^0, \tag{31}
\]

where $Q$ is the total charge in space and then $\delta g \int A_\mu J^\mu ( ( - \det g ) \frac{1}{2} ) d^4 x = 0$.

Based on the last remark, we proceed with the building of our theory by postulating a Lagrangian for the gravitational and electromagnetic fields and their sources which must include an electromagnetic current $J$ and a material medium carrying that current (which as in GR cannot be geometrized) as

\[
L := -\frac{c^3}{16\pi G} ( R + \frac{8\pi C}{c} A_\mu J^\mu ) + L_m. \tag{32}
\]

Thus the total action for our theory is

\[
S_t = S + S_m = -\frac{c^3}{16\pi G} \int ( R + C F_{\mu \nu} F^{\mu \nu} + \frac{16\pi C}{c} A_\mu J^\mu ) ( - \det g ) \frac{1}{2} d^4 x + \frac{1}{c} \int L_m ( - \det g ) \frac{1}{2} d^4 x, \tag{33}
\]

where $G$ is the gravitational constant and we require as usual in field theories that the equations of motion of the theory are giving by
\[ \delta(S + S_m) = 0. \quad (34) \]

Now, the energy-momentum of the material charge distribution is defined by \[38\]

\[ \delta g_{S_m} := \frac{1}{c} \int \tilde{T}_{\mu\nu} \delta g^{\mu\nu} \left( - \det g \right)^{1/2} d^4x, \]

\[ \frac{1}{2} \tilde{T}_{\mu\nu} \left( - \det g \right)^{1/2} = \partial \left[ \left( - \det g \right)^{1/2} L_m \right] - \frac{\partial}{\partial x^\lambda} \left[ \left( - \det g \right)^{1/2} L_m \right], \quad (35) \]

and performing the \( g \) variation of \((S + S_m)\) we get:

\[ \tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} + \tilde{T}_{\mu\nu} \right), \quad (36) \]

where \( \tilde{G}_{\mu\nu} \) are the components of the Einstein tensor associated with the Levi-Civita connection.

Eq.\( (36) \) (Einstein equation in components form) gives the well known relation between the Einstein tensor and the energy-momentum tensor of the electromagnetic plus the energy-momentum tensor of matter on a Lorentzian spacetime \[38\].

Varying Eq.\( (32) \) with respect to and \( A_\mu \) gives, as well known \[38\], Eq. \( (18) \), the non homogeneous Maxwell equation\[15\], and this completes the proof of our claim that with a special Riemann-Cartan connection it is possible to present the electromagnetic and gravitational fields as parts of the Riemann-Cartan structure \( \langle M, g, \nabla, \tau g, \uparrow \rangle \) with a Lagrangian giving by Eq.\( (32) \).

### 4 Lorentz Force Equation

For simplicity, we will consider in what follows, a continuous distribution of non-interacting incoherent charged matter, or “dust” as the material support of the electromagnetic current \( J \). Let \( \tilde{T} = \rho_0 c^2 V \otimes V \in \sec T^2_0 M \) be the energy momentum of the charged “dust” where \( V = V^\mu \partial_\mu \) is the 1-form velocity field of the dust \( (g(V, V) = 1) \) and \( \rho_0 \) is its proper charged mass density.

We calculate now the components of the covariant Riemann-Cartan covariant derivative of \( (T + \tilde{T}) \), i.e.,

\[ \nabla_\mu (T^{\mu\nu} + \tilde{T}^{\mu\nu}) = \nabla_\mu (T^{\mu\nu} + \tilde{T}^{\mu\nu}) + K^\mu_{\mu\delta} (T^{\delta\nu} + \tilde{T}^{\delta\nu}) + K^\mu_{\mu\delta} (T^{\mu\delta} + \tilde{T}^{\mu\delta}). \quad (37) \]

Since from Eq.\( (36) \) it is:

\[ \nabla_\mu \tilde{G}^{\mu\nu} = \frac{8\pi G}{c^4} \nabla_\mu (T^{\mu\nu} + \tilde{T}^{\mu\nu}) = 0, \quad (38) \]

\[ \text{15The homogeneous Maxwell equations follows trivially form } F = dA. \]
we get recalling Eq. (23) that
\[ K_{\mu \delta} \cdot T^{\delta \nu} + K_{\nu \delta} \cdot T^{\mu \delta} = 0. \quad (39) \]

Then, we can write
\[ \nabla_\mu (T^{\mu \nu} + \tilde{T}^{\mu \nu}) = K_{\mu \delta} \cdot \tilde{T}^{\delta \nu} + K_{\nu \delta} \cdot \tilde{T}^{\mu \delta}. \quad (40) \]

Now, recall that from Maxwell equations it follows trivially that
\[ \nabla_\mu T^{\mu \nu} = \frac{1}{c} F^{\mu \nu} J_\mu. \quad (41) \]

Using this result in Eq. (40) we have
\[ \rho_0 c^2 V_\mu \nabla_\mu V^\nu + V^\nu \nabla_\mu (\rho_0 c^2 V_\mu) = -\frac{1}{c} F^{\mu \nu} J_\mu + \frac{G}{c^4} \rho_0 c^2 A_\mu V^\delta (F_{\delta \nu} V^\mu + F_{\delta \mu} V^\nu). \quad (42) \]

Due to \( V_\mu V^\mu = 1 \) and the skew symmetry of \( F_{\mu \nu} \), we get contracting Eq. (42) with \( V_\nu \) that
\[ \nabla_\mu (\rho_0 c^2 V_\mu) = \frac{G}{c^4} \rho_0 c^2 A_\mu F_{\nu}^{\nu} V^\nu. \quad (43) \]

With Eq. (43), we have from Eq. (42),
\[ \rho_0 c^2 V_\mu \nabla_\mu V^\nu = -\frac{1}{c} F^{\mu \nu} J_\mu + \frac{G}{c^4} \rho_0 c^2 A_\mu V^\delta F_{\delta}^{\nu} V^\nu. \quad (44) \]

But for each integral line \( \sigma \) (parametrized by proper time \( s \)) with tangent vector field \( \sigma^s \) at \( \sigma(s) \) of the flow defined by the velocity field \( V \) we can write (with \( g(\sigma^s, \cdot) = V \))
\[ V^\mu \nabla_\mu V^\nu = \frac{dV^\nu}{ds} + \tilde{\Gamma}_{\mu \delta}^{\nu} V^\mu V^\delta + K_{\mu \delta}^{\nu} V^\mu V^\delta, \quad (45) \]
and then, substituting Eq. (45) in Eq. (41) and using Eq. (23), we obtain
\[ \frac{dV^\nu}{ds} + \tilde{\Gamma}_{\mu \delta}^{\nu} V^\mu V^\delta - \rho_q \rho_0 c^2 F_{\mu}^{\nu} V^\mu = 0 , \quad (46) \]
where have used that \( J_\mu = c\rho_q V^\mu \) with the rest charge density given by \( \rho_q \).

Eq. (46) is then identified as the Lorentz force law on a Lorentzian spacetime \[38\]. Note also that evaluating explicitly the covariant derivative Eq. (43), with the aid of Eq. (15), we get
\[ \nabla_\mu (\rho_0 c^2 V_\mu) = \partial_\mu [(-g)^{\frac{1}{2}} \rho_0 c^2 V^\mu] = 0 , \quad (47) \]
i.e., in this model we have matter conservation.

**Remark 4** We observe here that if we model the matter as a Dirac-field living in the Riemann-Cartan background we may obtain like in [46] that the torsion tensor is also a source of the spin density. This will be discussed elsewhere.
5 The Gauge Invariance

Recall that in our model the Lagrangian (excluding the electromagnetic coupling of the electromagnetic potential with the electromagnetic current) for the gravitational plus the electromagnetic field is geometrized, i.e., it is given by the scalar curvature of the Riemann-Cartan connection according to Eq. (22)

\[ \bar{R} + \frac{G}{c^4} F_{\mu\nu} F^{\mu\nu} = R - \frac{8\pi G}{c^4} A_{\mu} J^{\mu}. \] (48)

Now, we investigate what happens if we make a gauge transformation \( A \mapsto A + d\phi \) in the definition of the contorsion. That transformation changes \( K \mapsto K \), where the components of \( K \) are

\[ K^{\cdot \cdot \cdot \cdot \cdot \cdot \lambda}_{\mu \nu} = K^{\cdot \cdot \cdot \cdot \cdot \cdot \lambda}_{\mu \nu} + \frac{G}{c^4} (\partial_{\mu} \phi) F^{\cdot \cdot \cdot \cdot \cdot \cdot \lambda}_{\nu}. \] (49)

Then, we get a new Riemann-Cartan curvature tensor whose components are

\[ \bar{R}^{\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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whose contortion tensor is given by Eq. (23), although this is not apparent in the usual physical equations. Also the contortion tensor of our theory encodes the same information as the one that is contained in the Chern-Simons term $A \wedge dA$, which is proportional to the spin density of the electromagnetic field. Finally we observe that despite the fact that the contortion tensor and the Riemann-Cartan curvature tensor of our theory is not gauge invariant, the resulting field equations obtained through the complete Lagrangian involving the coupled interacting system consisted of gravitational field plus the electromagnetic fields and the charge current produce equations for those fields and equations of motion of the charged matter that are gauge invariant, as they should be. So, for each gauge we have a different, but equivalent Riemann-Cartan spacetime structure defined by another gauge.

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