A note on Closure Operators in Category of Groups

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Abstract

We give some applications of closure operators in category of groups and link them with the join problem of subnormal subgroups.

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Notion of closure operators (operations, systems, functions, relations) is known to us from algebra, logic, lattice theory and topology. Categorical view of closure operators play an important role in various branches of mathematics. In an arbitrary category \( \mathcal{X} \) with suitable axiomatically defined notion of subobjects, a (categorical) closure operator \( c \) is defined to be a family \( (c_X)_{X \in \mathcal{X}} \) satisfying the properties of extension, monotonicity and continuity. Closure operators are proved to be useful in study of Galois equivalence between certain factorization systems. In category of \( \text{R-mod} \) of R-modules closure operators correspond to preradicals. For more details, see [2]. In this article we establish a few results and examples in the category of groups by means of closure operators. In section 2, Theorem 2.1.3 provides an interesting link between the join problem of two subnormal subgroups and additive closure operators defined on \( \text{Grp} \) the category of groups. In subsection 2.2 we use the notion of closure operator induced by a subcategory \( \mathcal{A} \) of \( \text{Ab} \) the category of abelian groups to characterize the homomorphisms between the quotient group \( G/H \) and a group \( A \in \mathcal{A} \). This characterization provide useful methods for determining the relation between epimorphisms and surjective homomorphisms in many algebraic categories.

Subcategories are always assumed to be full and isomorphism closed.

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1 Preliminaries

Throughout this paper we consider a category \( \mathcal{X} \) and a fixed class \( \mathcal{M} \) of monomorphisms in \( \mathcal{X} \) which contains all isomorphisms of \( \mathcal{X} \). It is assumed that

- \( \mathcal{M} \) is closed under composition;
- \( \mathcal{X} \) is finite \( \mathcal{M} \)-complete.

A closure operator \( c \) on the category \( \mathcal{X} \) with respect to class \( \mathcal{M} \) of subobjects is given by a family \( c = (c_X)_{X \in \mathcal{X}} \) of maps \( c_X : \mathcal{M}/X \rightarrow \mathcal{M}/X \) such that for every \( X \in \mathcal{X} \)

1. \( m \leq c(m) \);
2. \( m \leq m' \Rightarrow c(m) \leq c(m') \); and
3. for every \( f : X \rightarrow Y \) and \( m \in \mathcal{M}/X \), \( f(c_X(m)) \leq c_Y(f(m)) \).

For each \( m \in \mathcal{M} \) we denote by \( c(m) \) the closure of \( m \).

An \( \mathcal{M} \)-morphism \( m \in \mathcal{M}/X \) is called \( c \)-closed if \( m \equiv c_X(m) \). A closure operator \( c \) is said to be idempotent if \( c(c(m)) \equiv c(m) \). In case \( c(m \lor n) \equiv c(m) \lor c(n) \) we say \( c \) to be additive. An \( \mathcal{M} \)-subobject \( m \) of \( X \) is called \( c \)-dense in \( X \) if \( c_X(m) \equiv 1_X \).

For a subcategory \( A \) of \( \mathcal{X} \), a morphism \( f : X \rightarrow Y \) is an \( A \)-regular monomorphism if it is the equalizer of two morphisms \( h, k : Y \rightarrow A \) with \( A \in A \).

Let \( \mathcal{M} \) contain the class of regular monomorphisms of \( \mathcal{X} \). For \( m : M \rightarrow X \) in \( \mathcal{M} \) define

\[
c_A(m) = \bigwedge \{ r \in \mathcal{M} \mid r \geq m \text{ and } r \text{ is } A \text{-regular} \}
\]

which is a closure operator of \( \mathcal{X} \). These closure operators are called regular and \( c_A(m) \) is called the \( A \)-closure of \( m \). In case \( A = \mathcal{X} \) we denote \( c_A(m) \) by \( c(m) \).

2 Closure operators in category of groups

In this section we will see application of closure operators in category of groups.

2.1

Let \( \mathcal{X} = \text{Grp} \) the category of groups and let \( \mathcal{M} \) be the class of all monomorphisms of \( \mathcal{X} \). In this case clearly \( \mathcal{X} \) is finite \( \mathcal{M} \)-complete. For an object \( G \) of \( \text{Grp} \), \( \mathcal{M}/G \) can be identified with the set of all subgroups of \( G \).

Let \( H \) be a subgroup of \( G \), we define

\[
c_G(H) = \{ g^{-1}hg \mid h \in H, g \in G \}
\]
the least normal subgroup of $G$ containing $H$.
It is easy to prove that $c_G$ is a closure operator on $X$ which is also an idempotent operator.
Let $[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$ be the commutator subgroup of $G$.
Define
$$c'_G(H) = [G, G] \cdot H$$
This gives a closure operator on Grp which is normal in $G$. For trivial subgroup $(e)$ of $G$, $c'_G(e) = [G, G]$ while $c_G(e) = (e)$.
A preradical $r$ in Grp is the subfunctor of the identity functor in Grp. For $G \in$ Grp, $r(G)$ is a normal subgroup of $G$. We define two more closure operators on Grp as follows:
$$c''_G(H) = H \cdot r(G) = H \cdot r(G)$$
and
$$c'''_G(H) = \pi^{-1}(r(G)/c_G(H))$$
where $\pi : G \longrightarrow G/c_G(H)$ is the canonical projection.
Closure operator $c''_G$ (in general) is not normal in $G$, but $c'''_G$ is always normal in $G$. (see [2])
We can observe that closure of a subgroup $H$ of $G$ can be converted to a normal closure of $H$ in $G$ and vise-versa. For example,
$$c_G(c''_G(H)), \ c'_G(c''_G(H)), \ c'''_G(c''_G(H))$$
are normal in $G$, but
$$c''_G(c_G(H)), \ c''_G(c'_G(H)), \ c''_G(c'''_G(H))$$
are (in general) not normal in $G$.

Next result is obvious.

**Proposition 2.1.1** Let $G, H$ be objects in $X = \text{Grp}$. Let $f : G \longrightarrow H$ be a non-zero homomorphism from $G$ to a simple group $H$. $f$ is onto if and only if $c_{f(G)}(K) = f(G)$ for all subgroups $(e) \neq K$ of $f(G)$.

Recall that a subgroup $H$ of a group $G$ is said to be subnormal in $G$ if there are a non-negative integer $m$ and a series
$$H = H_m \triangleleft H_{m-1} \triangleleft H_{m-2} \triangleleft \ldots \triangleleft H_0 = G$$
of subgroups of $G$. In this situation we write $H \triangleleft^m G$ and $H \triangleleft G$. The smallest such $m$ is called the defect of subnormal subgroup $H$ of $G$. 

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In finite group theory subnormal subgroups are precisely those subgroups which occur as terms of composition series, the factors of which are of paramount importance in describing a group’s structure. In 1939 H. Wielandt proved the celebrated join theorem for finite groups. Twenty years later H. Zassenhaus showed that Wielandt’s join theorem can fail to hold in infinite groups. The determination of interesting necessary and sufficient conditions for a join of subnormal subgroups (i.e. the subgroup generated by two subnormal subgroups) to be subnormal is probably the most important unsolved problem in this area of group theory [4]. In following we establish an interesting link between subnormal subgroups and closure operators which provides a solution of subnormal subgroups join problem.

Let \( c_G(H) \) denote the closure of \( H \) in \( G \). Set \( H_0 = G, H_1 = c_{H_0}(H), H_2 = c_{H_1}(H), \ldots, H_{m+1} = c_{H_m}(H) \).

**Proposition 2.1.2** ([4]) Let \( H \) be a subgroup of \( G \). Then \( H \trianglelefteq^n G \) if and only if \( H \) coincides with its \( n \)th normal closure in \( G \).

Denote by \( T = \langle H, K \rangle \) group generated by two subnormal subgroups and \( T_{m,n} = \langle H_m, K_n \rangle \) for \( m, n = 0, 1, 2, \ldots \) where \( H_{m+1} = c_{H_m}(H) \) and \( K_{n+1} = c_{K_n}(K) \).

**Theorem 2.1.3** Let \( H \) and \( K \) be subnormal subgroups of a group \( G \) in \( \mathcal{X} \). Then following implications hold.

1. The class \( \mathcal{M}^c_G \) of \( c \)-closed elements in \( \mathcal{M}/G \) is closed under binary suprema for every object \( G \) in \( \mathcal{X} \).
2. \( c \) is additive.
3. \( c_{T_{m,n}}(T) \) is subnormal in \( G \).

1. \( \iff \) 2. \( \implies \) 3.

**Proof.** (sketch) 1 \( \iff \) 2 straightforward.

2 \( \implies \) 3. Let

\[
H \leq \ldots c_{H_m}(H) \triangleleft c_{H_{m-1}}(H) \triangleleft c_{H_{m-2}}(H) \triangleleft \ldots \triangleleft c_{H_1}(H) \triangleleft c_G(H) \triangleleft G
\]

and

\[
K \leq \ldots c_{K_n}(K) \triangleleft c_{K_{n-1}}(K) \triangleleft c_{K_{n-2}}(K) \triangleleft \ldots \triangleleft c_{K_1}(K) \triangleleft c_G(K) \triangleleft G
\]

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be the series of $H$ and $K$ respectively. Since $c$ is additive, therefore we have 

\[ c_G(H \vee K) = c_G(H) \vee c_G(K) \] i.e. $c_G(<H,K>) = <c_G(H), c_G(K)>$. 

Also we have 

\[ c_G(<c_H^{-1}(H), c_K^{-1}(K)>) = <c_G(c_H^{-1}(H)), c_G(c_K^{-1}(K))> \] for all $i, 1 \leq i \leq m$ and $j, 1 \leq i \leq n$, and for all group $G$ in $\mathcal{X}$.

Clearly we have 

\[ c_{T_{m,n}}(<H,K>) \subseteq c_{T_{m,n}}(<c_{H_{m}}^{-1}(H), c_{K_{n}}^{-1}(K)> ) \] 
\[ \subseteq c_{T_{m-1,n-1}}(<c_{H_{m-1}}^{-1}(H), c_{K_{n-1}}^{-1}(K)> ) \] 
\[ \subseteq c_{T_{m-2,n-2}}(<c_{H_{m-2}}^{-1}(H), c_{K_{n-2}}^{-1}(K)> ) \] 
\[ \subseteq \ldots \subseteq c_{T_{1,1}}(<c_{H_{1}}^{-1}(H), c_{K_{1}}^{-1}(K)> ) \subseteq G. \]

By the additivity of $c$ and the above expression we get 

\[ c_{T_{m,n}}(T) \triangleleft <c_{H_{m-1}}^{-1}(H), c_{K_{n-1}}^{-1}(K)>) \triangleleft <c_{H_{m-2}}^{-1}(H), c_{K_{n-2}}^{-1}(K)>) \] 
\[ \triangleleft \ldots \triangleleft <c_{H_{1}}^{-1}(H), c_{K_{1}}^{-1}(K)> \triangleleft G. \]

This proves that $c_{T_{m,n}}(T)$ is subnormal in $G$. $\square$

**Remark 2.1.4** If $c$ is a normal closure operator in $G$, then it satisfy the conditions 1 and 2 of the Theorem 2.1.3. This means that join problem of two subnormal subgroups is reduced to find the suitable conditions when $H = c_{H_{m}}^{-1}(H)$ and $K = c_{K_{n}}^{-1}(K)$ (cf. Prop. 2.1.2).

**Remark 2.1.5** Normal subgroups are not only stable under intersection, but also under arbitrary join in the subgroup lattice. Therefore all the closure operators which are normal as a subgroup of $G$ are additive. We observe that the normal subgroups produced by $c'_G, c''_G$ and $c'''_G$ will be larger than of the normal subgroups produced by $c_G$. So there will be possibility of a fast termination of subnormal series induced by these operators.

As a corollary to above theorem we have following result of Wielandt (cf. [4])

**Corollary 2.1.6** If $H$ and $K$ are subnormal subgroups of of a finite group $G$ in $\mathcal{X}$. Then $T = <H,K>$ is subnormal in $G$. 

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2.2

Let \( \mathcal{A} \) be a subcategory of \( \mathcal{X} = \text{Ab} \) the category of abelian groups and let \( \mathcal{M} \) be the class of all monomorphisms of \( \mathcal{X} \). Notice that in \( \text{Ab} \) and \( \text{Grp} \) the strong monomorphisms coincide with monomorphisms. In this case \( \mathcal{X} \) is \( \mathcal{M} \)-complete. For an object \( G \in \text{Ab} \), \( \mathcal{M}/G \) can be identified with the set of all subgroups of \( G \).

**Theorem 2.2.1** Let \( H \) be a subgroup of \( G \).

(a) (cf. [1]) \( H \) is \( \mathcal{A} \)-dense in \( G \), i.e., \( c_{\mathcal{A}}(H) = G \) if and only if \( \text{Hom}(G/H, A) = (0) \) for every \( A \in \mathcal{A} \).

(b) \( H \) is \( \mathcal{A} \)-closed, i.e., \( c_{\mathcal{A}}(H) = H \) if and only if \( \text{Hom}(G'/H, A) \neq (0) \) for some \( A \in \mathcal{A} \) and for every non-zero subgroup \( G'/H \) of \( G/H \).

(c) Let \( f : G \to T \) be an \( \mathcal{A} \)-morphism. The morphism \( f \) is not epic if and only if \( c_{\mathcal{A}}(f(G)) = f(G) \).

In particular \( c_{\mathcal{A}}(f(G)) = f(G) \) implies \( f \) is not surjective.

**Proof.** (a) (cf. [1])

(b) Without loss of generality we assume that \( H \) is not \( \mathcal{A} \)-dense in \( G \). Let \( G'/H \) be a non-zero subgroup of \( G/H \). Since \( H \) is not \( \mathcal{A} \)-dense in \( G \), we have \( \text{Hom}(G/H, A) \neq (0) \) for some \( A \in \mathcal{A} \). Let \( 0 \neq f \in \text{Hom}(G/H, A) \). One can get a non-zero morphism \( f' : G'/H \to A \) in obvious sense i.e., \( f' = f \cdot j \) where \( j : G'/H \to G/H \) is just the inclusion map, which implies that \( \text{Hom}(G'/H, A) \neq (0) \).

Conversely, suppose \( H \) a proper subgroup of \( c_{\mathcal{A}}(H) \). Since closure of \( H \) is \( c_{\mathcal{A}}(H) \), \( H \) is dense in \( c_{\mathcal{A}}(H) \) (treating \( H \) as a subobject fo \( c_{\mathcal{A}}(H) \)). This implies \( \text{Hom}(c_{\mathcal{A}}(H)/H, A) = (0) \), but this contradicts our hypothesis, therefore we must have \( c_{\mathcal{A}}(H) = H \).

(c) Since \( f(G) \) is closed in \( T \) implies \( \text{Hom}(T/f(G), A) \neq (0) \) for some \( A \in \mathcal{A} \). This implies \( f \) is not an epimorphism.

Conversely, if \( f \) is not an epimorphism implies \( c_{\mathcal{A}}(f(G)) \neq T \) (cf. [1]) which gives \( \text{Hom}(T'/f(G), A) \neq (0) \) for some \( A \in \mathcal{A} \) and for every non-zero subgroup \( T'/f(G) \) of \( T/f(G) \). But then from (b) we get \( c_{\mathcal{A}}(f(G)) = f(G) \). \( \square \)

**Remark 2.2.2** Above results can be used in case of following subcategories of \( \text{Ab} \). (cf. [3])

1. Category of torsion free abelian groups; 2. category of reduces abelian groups;
3. category of free abelian groups; 4. category of topological abelian groups, etc.
References
[1] G. Castellini, Closure operators, monomorphisms and epimorphisms in categories of groups, Cahiers de Topologie et Geometric Differentielle Categoriques Vol. XXVII -2(1986)151-167.
[2] D. Dikranjan and W. Tholen, Categorical structure of closure operators with applications to Topology, Algebra and Discrete Mathematics (Kluwer, Dordrecht, 1994).
[3] L. Fuchs, Infinite abelian groups Vol. 1 (Academic Press, New York, 1970)
[4] J.C. Lennox and S.E. Stonehewer, Subnormal subgroups of groups (Clarendon Press, Oxford, 1987).