Generalized Weierstrass representation for surfaces and Lax-Phillips scattering theory for automorphic functions

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Abstract

Relation between generalized Weierstrass representation for conformal immersion of generic surfaces into three-dimensional space and Lax-Phillips scattering theory for automorphic functions is considered.

It is well-known that Poincare plane Π, i.e., the upperhalf plane

\[ y > 0, \quad -\infty < x < \infty, \quad z = x + iy \]

be the model of Lobachevsky geometry, where the role of motion group played the group \( G = SL(2, \mathbb{R}) \) of fractional linear transformations

\[ z \rightarrow zg = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (1) \]

where \( a, b, c, d \in \mathbb{R}, \quad ad - bc = 1 \).

The group \( SL(2, \mathbb{R}) \) has a great number of so-called discrete subgroups. The subgroup \( \Gamma \) is called discrete if the identical transformation is isolate from the other transformations \( \gamma \in \Gamma \). For example, a modular group consisting of transformations with integer \( a, b, c, d \) is discrete subgroup. Further, a fundamental domain \( F \) of discrete subgroup \( \Gamma \) be an any domain on Poincare plane such that the every point of \( \Pi \) may be transfered into a closing \( \bar{F} \) of domain \( F \) by means of some transformation \( \gamma \in \Gamma \), at the same time no
there exists the point from $F$ which transferred to the other point of $F$ by such transformation. The function $f$ defined on $\Pi$ is called *automorphic* with reference to discrete subgroup $\Gamma$ if

$$f(\gamma z) = f(z), \quad \gamma \in \Gamma.$$ 

Further, generalized Weierstrass representation for surfaces was proposed by Konopelchenko in 1993 \[1, 2\] is defined by the following formulae

$$X^1 + iX^2 = i \int_\epsilon (\bar{\psi}'dz' - \bar{\varphi}'d\bar{z'}),$$

$$X^1 - iX^2 = i \int_\epsilon (\varphi'dz' - \psi'd\bar{z'}),$$

$$X^3 = -\int_\epsilon (\psi\bar{\varphi}dz' + \varphi\bar{\psi}d\bar{z'}),$$

(2)

where $\epsilon$ is arbitrary curve in $\mathbb{C}$, $\psi$ and $\varphi$ are complex-valued functions on variables $z, \bar{z} \in \mathbb{C}$ satisfying to the linear system (two-dimensional Dirac equation):

$$\psi_z = U\varphi,$$

$$\varphi_{\bar{z}} = -U\psi,$$

(3)

where $U(z, \bar{z})$ is a real-valued function. If to interpret the functions $X^i(z, \bar{z})$ as coordinates in a space $\mathbb{R}^{3,0}$, then the formulae (2), (3) define a conformal immersion of surface into $\mathbb{R}^{3,0}$ with induced metric

$$ds^2 = D(z, \bar{z})^2dzd\bar{z}, \quad D(z, \bar{z}) = |\psi(z, \bar{z})|^2 + |\varphi(z, \bar{z})|^2,$$

at this the Gaussian and mean curvature are

$$K = -\frac{4}{D^2}[\log D]_{z\bar{z}}, \quad H = \frac{2U}{D}.$$ 

(4)

Let us consider a closed surface with genus $> 1$, and let $F : \Sigma \rightarrow \mathbb{R}^{3,0}$ be an immersion of surface with genus $> 1$ given by (2)-(3). It is well-known that every closed oriented surface $\Sigma$ with positive genus is uniformizable:

$$\rho : M \rightarrow \Sigma,$$

where a surface $M$ is conformal covering. Hence it immediately follows that a factor-space $M/\Gamma$ is conformally equivalent to the surface $\Sigma$, where $\Gamma$ is a
discrete subgroup of a group of isometries of $M$. In our case a space $M$ is isometric to the Poincare plane $\Pi$. The group of isometries of $\Pi$ is the group $G = SL(2, \mathbb{R})$, the transformations of which are defined by (1).

According to [3] (Proposition 4) we have that a surface $\Sigma$ with genus $> 1$ immersing into $\mathbb{R}^3$ by formulas (2)-(3) is conformally equivalent to a surface $\Pi/\Gamma$, where $\Gamma$ is a discrete subgroup of $SL(2, \mathbb{R})$. The functions $\psi$ and $\varphi$, the metric tensor $D(z)^2$ and potential $U(z)$, are transformed by elements of $\Gamma$ as follows

$$
\psi(\gamma(z)) = (c\bar{z} + d)\psi(z),
\varphi(\gamma(z)) = (cz + d)\varphi(z),
D(\gamma(z)) = |cz + d|^2 D(z),
U(\gamma(z)) = |cz + d|^2 U(z).
$$

Hence it immediately follows from (4) that

$$
H(\gamma(z)) = H(z), \quad \gamma \in \Gamma.
$$

Therefore, the mean curvature is automorphic function.

Further, follows to [4] let us consider the discrete subgroup $\Gamma \subset SL(2, \mathbb{R})$ satisfying to the following requirements:

1. A space $SL(2, \mathbb{R})/\Gamma$ is noncompact.
2. $\Gamma$ contains the only one parabolic subgroup.

The fundamental domain $F_\Gamma = F$ for the group $\Gamma$ choosing as follows

a. $F$ lies on a strip $-X < x < X$, $y > Y > 0$.

b. Intersection $F \cap \{ y > d \}$ at the some $d$, $d > 1$, is coincide with a strip $-X_1 < x < X_1$, $y > d$.

c. A boundary of $F$ is smooth and consists of geodesic segments with a finite number of corner points.

In Hilbert space $L_2(F, d\mu)$, where $d\mu = y^{-2} dx dy$ is a measure, consider a symmetric operator $L$ defined by the differential expression

$$
L = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{4}
$$

(6)
on all sufficiently smooth and uniformly restricted in $F$ the functions $H$, which satisfying to automorphy condition (5). A spectrum of the operator $L$ consists of finite set of own values: $\lambda_0 = -\frac{1}{4}$ (this own number corresponds to the unit representation of group $SL(2, \mathbb{R})$), the numbers $\lambda_l = -\mu_l^2$, $l = 1, 2, \ldots, N$ are belong to $(-\frac{1}{4}, 0)$ (additional series), the set of positive own values $\lambda_l$, $l = N + 1, N + 2, \ldots$, $\lambda_l \in (0, \infty)$ (basic series), and the branch of absolutely continuous spectrum $\lambda = k^2$ on $[0, \infty]$.

If we assume that functions $H$ compose the basis of Hilbert space $L_2(F, d\mu)$, then the automorphic wave equation may be written as

$$H_{tt} + LH = 0,$$

where operator $L$ has the form (6). This equation naturally defines a group $V_t$ of transformations (smooth and finite) of Cauchy data $U(z, t) = (u(z, t), \partial_t u(z, t))$, the action of which expressed by the formula

$$U(z, t) = V_t U(z, 0).$$

The group $V_t$ has orthogonal in- and out-spaces $D_-$, $D_+$ are satisfying to conditions

1) $\cap_{t<0} V_t D_- \subset D_-$, $t < 0$,
2) $\cap_{t>0} V_t D_+ \subset D_+$, $t > 0$,
3) $\cup_{t<0} V_t D_- = \cup_{t>0} V_t D_+$,
4) $D_- \perp D_+.$

These conditions allow to apply the Lax-Phillips framework [3] and to find generalized own functions $e(z)$ of automorphic wave equation (7) which are expressed via the Eisenstein series, and also to define the spectrum representation of operator $L$ and scattering matrix.

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