Smooth initial conditions for the Navier-Stokes equation without solution

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An analytic example of smooth initial condition for the incompressible Navier-Stokes equation without a smooth solution at $t = 0$

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Abstract

The proof of existence and uniqueness of the Navier-Stokes equation is a famous challenging problem. It is known that if the initial velocity field fulfills the compatibility condition, a smooth solution exists for a certain time. Usually, this condition is associated with smooth, divergence-free fields which are compatible with the boundary conditions. However, such a field does not necessarily fulfill the compatibility condition in the Navier-Stokes equation. In this paper, a smooth initial condition is presented in a periodic channel that violates the condition. It is calculated analytically that the given problem has no smooth solution at the initial time.

1. Introduction

Some mathematical proofs about the existence of a smooth Navier-Stokes solution were derived in the last century. The first theorems were summarized in the book of Ladyzhenskaia [1]. The problem in strong form has a unique solution in two dimensions. Unfortunately, this statement has not been proved yet in three dimensions. At the same time, partial results were achieved. The solution must exist for a certain time $T^*$ on unbounded or periodic domains. However, if the domain is bounded (e.g. a wall presents or the velocity is prescribed), the initial condition of the problem cannot be arbitrary. It is briefly discussed in the aforementioned book, and it was investigated thoroughly by Temam [2]. He proved that a smooth solution exists in three-dimensional space at $t = 0$, if and only if the initial condition fulfills the compatibility condition. He was the first who gave all the necessary and sufficient conditions for smooth solutions. The condition will be presented here in Section 2. Later, he gave a physical explanation for these mathematical results [3]. The problem is that if the velocity is prescribed at the boundaries (Dirichlet boundary condition), the calculation of pressure leads to an overdetermined problem. The usual method is using the wall-normal momentum equation to obtain a Neuman boundary condition for the pressure. Still, the calculated pressure can violate the tangential component of the momentum equation in an arbitrary case. Using the tangential component of the momentum equation to obtain a boundary condition would lead to different pressure field [4] unless the compatibility condition is fulfilled. The same problem is discussed from a numerical point of view by Gresho and Sani [5]. They investigated the Poisson equation of the pressure. They suggest multiple methods to handle the problem where the condition is not fulfilled. Furthermore, they showed numerical examples. Gallavotti [6] discussed the same problem in Chapter 2.1. He suggests a theoretically possible numerical procedure to handle the issue by extending the domain where the viscosity can slow down the flow. However, he admits that the real numerical solution of the suggested method is challenging. It must be mentioned that these techniques were developed to obtain numerical results, but according to the theorem of Temam [2], the solution does not exist in these cases at $t = 0$.

Although the Navier-Stokes equation rarely has an analytic solution, there are many counterexamples. They are derived usually with some approximations or assumptions. In addition, some mathematical papers have discussed the extraordinary analytic solutions of the Navier-Stokes equation in the weak form. For example, Heywood [7] investigated the uniqueness of the problem. He showed that multiple solutions exist in the case of flow through a hole. He pointed out the need a further auxiliary condition for a unique solution. At the same time, the solution must be unique in the case of the strong form. It can be smooth if it satisfies the above-mentioned compatibility condition. According to the author’s best knowledge, no one gives an analytic example for a smooth initial velocity field that violates the condition. Therefore, a smooth solution does not exist at the initial time. Here, the example is given in a channel configuration. In that case, the steady-state solution is the well-known Poiseuille flow. Recently, Józsa [8] obtained

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\[ x_2 = l \]
\[ x_2 = -l \]

**Figure 1:** The channel geometry.

an analytical solution for controlled channel flow. Here, the analytic calculation method and the example are shown in Section 2.

2. The calculation method of the analytic solution

An incompressible Newtonian fluid can be described with the continuity equation

\[ \frac{\partial u_i}{\partial x_i} = 0 \]  \hspace{1cm} (1)

and the Navier-Stokes equations

\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j^2}, \]  \hspace{1cm} (2)

where \( u_i \) is the non-dimensional velocity, \( p \) is the non-dimensional pressure, \( Re \) is the Reynolds number.

The solution domain is a cuboid: \( x_1 \in [0, L_x], x_2 \in [-1, 1], x_3 \in [0, L_z]. \) It represents a channel. The boundary conditions are the followings. The velocity field is periodic in the \( x_1 \) and \( x_3 \) directions:

\[ u_i(x_1, x_2, x_3, t) = u_i(x_1 + L_x, x_2, x_3, t), \]  \hspace{1cm} (3)

\[ u_i(x_1, x_2, x_3, t) = u_i(x_1, x_2, x_3 + L_z, t). \]  \hspace{1cm} (4)

Usually, these directions are called streamwise and spanwise directions, respectively. At \( x_2 = -1 \) and \( x_2 = 1 \), the no-slip wall condition is prescribed, meaning that

\[ u_i(x_1, x_2 = -1, x_3, t) = u_i(x_1, x_2 = 1, x_3, t) = 0. \]  \hspace{1cm} (5)

\( x_2 \) is the wall-normal coordinate. The initial velocity field is given as

\[ u_i(x_1, x_2, x_3, t = 0) = u_{i,0}(x_1, x_2, x_3), \]  \hspace{1cm} (6)

which fulfills (1) and the boundary conditions. The initial velocity field example will be shown later.

During the solution, the curl of the Navier-Stokes equation is used, which known as vortex transport or vorticity equation,

\[ \frac{\partial \omega_i}{\partial t} = -u_j \frac{\partial \omega_i}{\partial x_j} + \omega_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \omega_i}{\partial x_j^2}, \]  \hspace{1cm} (7)

where \( \omega_i \) is the vorticity, the curl of \( u_i \). The benefit of this procedure is the elimination of the pressure. \( \omega_{i,0} \) is the curl of the initial condition \( (u_{i,0}) \) that can be easily calculated for a given initial velocity. The temporal partial derivative of the vorticity at the initial time \( (t = 0) \) can be determined using the vorticity equation (7)

\[ \frac{\partial \omega_i}{\partial t} \bigg|_{t=0} = -u_{j,0} \frac{\partial \omega_{i,0}}{\partial x_j} + \omega_{j,0} \frac{\partial u_{i,0}}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \omega_{i,0}}{\partial x_j^2}. \]  \hspace{1cm} (8)
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Furthermore, the well-known vector algebra identity,
\[ \nabla \times (\nabla \times \mathbf{u}_i) = \nabla (\nabla \cdot \mathbf{u}_i) - \Delta \mathbf{u}_i \]
and the symmetry of second derivatives lead to
\[ \nabla \times \left( \frac{\partial \mathbf{u}_i}{\partial t} \right) = -\Delta \left( \frac{\partial \mathbf{u}_i}{\partial t} \right), \]
since the temporal change of the velocity must be a solenoid field. \( \nabla \) is the Nabla, \( \Delta \) is the Laplace operator. At the initial time, the temporal derivative of vorticity can be obtained from (8) and (10) leads to three independent Poisson equations for the temporal derivative of each velocity component:
\[ \frac{\partial^2}{\partial x_j^2} \left( \frac{\partial u_i}{\partial t} \right) = f_i(x_1, x_2, x_3), \]
where
\[ f_k(x_1, x_2, x_3) = -\nabla \times \left( -u_{j,0} \frac{\partial \omega_{i,0}}{\partial x_j} + \omega_{j,0} \frac{\partial u_{i,0}}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \omega_{i,0}}{\partial x_j^2} \right) \]

The velocity boundary conditions of this problem are the same as the original problem, Eqs. (3)-(5). Since the Poisson equation is a linear differential equation, it has only one solution in the case of the Dirichlet boundary condition. However, it will be shown that for certain initial values, the solution of Eq. (11) is not divergence-free! (It is mentioned that \( f_k \) (12) is undoubtedly divergence-free.) The sole solution is not acceptable, meaning that the problem has no smooth solution at the initial time!

Temam [2] proved (and later physically discussed [3]) that the solution exists at the initial time, if and only if it fulfils the compatibility condition. In the case of the linear heat equation [3], the compatibility condition expresses that the initial value of the problem does not contradict the boundary condition. At the same time, the compatibility condition in the case of the Navier-Stokes equation is more complicated. The condition is that the expression [2]
\[ \frac{1}{Re} P(\Delta u_{i,0}) - P \left( u_{j,0} \frac{\partial u_{i,0}}{\partial x_j} \right) + g_i(0) = 0 \]
must fulfil at the boundaries; otherwise, the solution does not exist at \( t = 0 \). (The sign of the second term is corrected since it was positive in the cited paper.) \( P \) is the Leray projector that gives the divergence-free part of a vector field. The projector can be expressed with the Helmholtz decomposition. \( \Delta \) is the Laplace operator, \( g_i \) is an arbitrary volume source in the momentum equation, which is 0 in this study. Temam [3] proposed a calculation method for calculating the compatibility condition. First, the following equation is solved.
\[ \Delta p_0 = -\frac{\partial u_{j,0}}{\partial x_j} \frac{\partial u_{i,0}}{\partial x_i} \]
which is basically the divergence of the momentum equation. It is also known as the Poisson equation for the pressure. (In the cited paper, the negative sign is omitted, which leads to the same condition, but in this form, \( p \) is the widely used physical pressure.) The boundary condition is
\[ \frac{\partial p_0}{\partial x_i} n_i = \left( \frac{1}{Re} \Delta u_{i,0} \right) n_i \]
in the case of a no-slip wall, which can be obtained from the momentum equation (2) at the stationary walls (5) in the wall-normal direction. \( n_i \) is the wall-normal vector. In the channel, it is non-zero only for \( i = 2 \). After solving
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Figure 2: The slice \( z = 0 \) of the example velocity field. The parameters \( \alpha = 1, \beta = 1, \text{Re} = 80 \) (a) \( u_{2,0} \) (18); (b) \( u_{3,0} \) Eq. (19).

Figure 3: The divergence of \( \frac{\partial u_i}{\partial t} \) (Eq. 43) as the function of \( x, y \) for the parameters \( \alpha = 1, \beta = 1, \text{Re} = 80 \) at \( z = 0 \).

The problem, Temam [2] states that the smooth temporal derivative of the velocity field exists at \( t = 0 \), if the tagential components of \( \frac{\partial p_0}{\partial n_i} \) are equal to the tagential components of \( \frac{1}{\text{Re}} \Delta u_{i,0} \):

\[
\frac{\partial p_0}{\partial n_i} t_i = \left( \frac{1}{\text{Re}} \Delta u_{i,0} \right) t_i,
\]

where \( t_i \) is arbitrary vector perpendicular to \( n_i \). The condition is actually the evaluation of the momentum equation at the wall in a tangential direction. This condition means physically that the pressure field should prevent the tangential acceleration (the tangential movement) of the wall. The problem is that this additional condition cannot be prescribed for the Poisson equation; since then, it would become overdetermined. This condition is a constraint for the initial velocity field! If it does not hold, there is no solution which fulfills the stationary wall condition and the solenoid property at the same time. Such a velocity field can exist only in mathematical abstraction. It cannot evolve physically since neither a smooth previous state nor a next state of the fluid does not exist. However, this statement is valid only for a fully incompressible flow, which is an assumption. A real fluid is always compressible.

In the paper, the calculation for the following velocity field is presented

\[
\begin{align*}
    u_{1,0} &= 0 \\
    u_{2,0} &= \cos (\alpha x + \beta z) \left( y^2 - 1 \right)^2
\end{align*}
\]
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\[ u_{3,0} = -\frac{4y \sin(\alpha x + \beta z)}{\beta} \left( y^2 - 1 \right), \]  
(19)

where \( x = x_1, y = x_2, z = x_3, \alpha = 2\pi/L_x, \beta = 2\pi/L_z \). This is one of the simplest velocity fields, which is divergence-free and fulfils the boundary conditions. (Of course, \( u_{1,0} \) is not necessarily 0. This case is analyzed, for the sake of simplicity, since the analytical terms can be extremely long for publication.) The velocity field is plotted in Figure 2 at the section \( z = 0 \). The analytic calculation of the temporal change of the velocity, its divergence, and the compatibility condition is presented in the appendix. The calculation is straightforward, but the long terms become more lengthy in the case of a more complex initial field. Matlab 2019b Symbolic toolbox is used to reduce the solution time and the risk of miscalculation. (The calculation is repeated with different numerical functions for certain parameters. The analytic and numerical computation results are practically identical, meaning that no error is made in the calculation.) This initial velocity field does not fulfil the compatibility condition, and the corresponding problem has no smooth solution at \( t = 0 \). The divergence of the velocity change is shown in Figure 3 for certain parameters, and its value is clearly non-zero. The calculated terms and their derivation is described in the appendix. The presented solution fulfils the boundary condition, but it violates the divergence-free condition. It is possible to obtain a divergence-free solution but it violates some component of the velocity boundary conditions at the wall.

Some further analysis was done to investigate which initial fields fulfil the condition and have solution. The velocity field components have the form \( a(y) \cos(\alpha x + \beta z) + b(y) \sin(\alpha x + \beta z) \) in all cases, where \( a(y) \) and \( b(y) \) are polynomial functions. In this form, the periodic boundary conditions (3)-(4) are automatically fulfilled and the coefficients of the polynomials should be properly chosen to fulfil the wall boundary condition (5) and the continuity equation (1). A possible choice is \( c(y) = \tilde{c}(y)(y^2 - 1) \) for \( u_{1,0} \) and \( c(y) = \tilde{c}(y)(y^2 - 1)^2 \) for \( u_{2,0} \). The last velocity component, \( u_{3,0} \) can be calculated with the continuity equation.

After analyzing multiple initial velocity fields, the following conclusions can be drawn. In the case of \( u_{2,0} = 0 \), the solution of Equation (11) is divergence-free in every case. If the initial velocity field has not the wall-normal velocity component, the smooth solution of the corresponding problem exists. In the case of \( u_{2,0} \neq 0 \), many attempts are made to obtain an analytical velocity field that fulfils the compatibility condition. The procedure is failed since increasing the order of the polynomials leads to a very complex analytical problem that cannot be solved on a personal computer. At the same time, the equations can be easily solved numerically. First, the asymptotic modes of the corresponding Orr-Sommerfeld problem are used as initial conditions. These modes are the solution of the linearised Navier-Stokes equation where the base flow is the Poiseuille flow. The eigenfunctions are obtained with the method of Juniper et al. [9], where the domain is discretized with \( N = 40 \) Chebyshev polynomials. In this case, a spatially non-oscillating base flow, the Poiseuille flow, is part of the velocity field. The eigenvectors fulfil the compatibility condition, and the solution exists at \( t = 0 \). This is not surprising since the temporal evaluation of the velocity is known from the eigenvalue in the case of sufficiently low amplitudes. However, after increasing the amplitude, the condition fulfils numerically.

Furthermore, using the same Chebyshev collocation method, the velocity field in the previously discussed form is calculated, which fulfils the compatibility condition. This is achieved with the built non-linear optimization \( \text{fsolve} \) function, where the coefficients of the polynomials are varied to obtain a divergence-free temporal change of the velocity. An example is:

\[ a_{u_1} = (0.8147) y^4 + 0.9058 y^3 + 0.127 y^2 + 0.9134 y + 0.6324 \left( y^2 - 1 \right) \]  
(20)

\[ b_{u_1} = (0.09754) y^4 + 0.2785 y^3 + 0.5469 y^2 + 0.9575 y + 0.9649 \left( y^2 - 1 \right) \]  
(21)

\[ a_{u_2} = (-0.6011) y^4 - 0.1986 y^3 + 0.7068 y^2 + 0.4689 y + 1.599 \left( y^2 - 1 \right)^2 \]  
(22)

\[ b_{u_2} = (0.1063) y^4 + 0.2864 y^3 + 0.8618 y^2 + 0.3238 y + 1.537 \left( y^2 - 1 \right)^2 \]  
(23)

at \( \alpha = 1, \beta = 1, Re = 80 \) and the subscript of the Fourier coefficients notes the velocity field. The third velocity component can be easily obtained with the continuity equation. Although these calculations are carried out with numerical techniques, there are probably analytical initial velocity fields, where \( u_{2,0} \neq 0 \), and the corresponding Navier-Stokes equation has a solution at \( t = 0 \).
3. Conclusion

The solution of the Navier-Stokes equation is a challenging problem. In the last century, mathematicians proved that the smooth solution exists for a finite time. However, in the case of physically relevant bounded domains, the solution exists if and only if the compatibility condition holds. In this paper, an analytic example is shown in a channel flow, when the initial velocity field is smooth and divergence-free. At the same time, it does not fulfil the compatibility condition, and the corresponding Navier-Stokes equation has no solution at $t = 0$. According to the author’s best knowledge, this is the first analytic example. The problem is related to the presence of walls where the pressure Poisson equation can be overdetermined. In the absence of a wall-normal velocity component, the condition always fulfils in the presented configuration. Some further numerical calculation suggests that velocity fields with the non-zero wall-normal component can also fulfil the condition. However, the analytical construction of such a field proved to be difficult.

The results of the paper are a direct consequence of mathematical proof. At the same time, the physical consequence is surprising. There exist mathematically many smooth velocity fields that fulfil the boundary conditions and the continuity equation, but they violate the governing equation in a non-trivial way. In these cases, the Navier-Stokes equation has no smooth solution.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

A. The calculation of temporal derivative in an analytic example

First, the curl of the initial velocity (17)-(19) is calculated:

$$\omega_{1,0} = \frac{\sin(\alpha x + \beta z)}{\beta} \left( \beta^2 y^4 - 2 \beta y^2 + \beta^2 - 12 y^2 + 4 \right)$$

(24)

$$\omega_{2,0} = \frac{4 \alpha y \cos(\alpha x + \beta z)}{\beta} \left( y^2 - 1 \right)$$

(25)

$$\omega_{3,0} = -\alpha \sin(\alpha x + \beta z) \left( y^2 - 1 \right)^2.$$  

(26)

The temporal derivative of the vorticity ($\frac{\partial \omega_i}{\partial t} \bigg|_{t=0}$) is obtained from Eq. (8). These terms are not shown here. Then the opposite and the curl of $\frac{\partial \omega_i}{\partial t} \bigg|_{t=0}$ is calculated according to Eq. (12). It is convenient to express the term in a Fourier series in the following form:

$$f_k = A_{k,0}(y) + \sum_{j=1}^{2} \left\{ A_{k,j}(y) \cos(j(\alpha x + \beta z)) + B_{k,j}(y) \sin(j(\alpha x + \beta z)) \right\},$$

(27)

where $A_{k,j}(y), B_{k,j}(y)$ are polynomials.

The coefficients are in the presented case:

$$A_{1,0} = 0$$

(28)

$$A_{1,1} = 0$$

(29)

$$B_{1,1} = 0$$

(30)

$$A_{1,2} = 0$$

(31)
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\begin{align}
B_{1,2} &= -8 a \left( y^2 - 1 \right)^2 \left( y^2 + 1 \right) \\
A_{2,0} &= 0 \\
A_{2,1} &= -\frac{1}{\text{Re}} \left( -a^4 - 2 a^2 \beta^2 - \beta^4 \right) y^4 + \left( 2 a^4 + 4 a^2 \beta^2 + 24 a^2 + 2 \beta^4 + 24 \beta^2 \right) y^2 \\
&\quad - a^4 - 2 a^2 \beta^2 - 8 a^2 - \beta^4 - 8 \beta^2 - 24 \\
B_{2,1} &= 0 \\
A_{2,2} &= 8 \, y \left( 3 y^2 + 1 \right) \left( y - 1 \right) \left( y + 1 \right) \\
B_{2,2} &= 0 \\
A_{3,0} &= 0 \\
A_{3,1} &= 0 \\
B_{3,1} &= \frac{4 \, y \left( a^2 + \beta^2 \right) \left( -a^2 y^2 + a^2 - \beta^2 y^2 + \beta^2 + 12 \right)}{\text{Re} \, \beta} \\
A_{3,2} &= 0 \\
B_{3,2} &= \frac{4 \left( 2 a^2 y^6 - 2 a^2 y^4 - 2 a^2 y^2 + 2 a^2 - 15 y^4 + 6 y^2 + 1 \right)}{\beta}
\end{align}

The temporal derivative of the velocity must have similar form:

\begin{equation}
\frac{\partial u_k}{\partial t} \Bigg|_{t=0} = a_{k,0}(y) + \sum_{j=1}^{2} \left\{ a_{k,j}(y) \cos \left( j(ax + \beta z) \right) + b_{k,j}(y) \sin \left( j(ax + \beta z) \right) \right\}. \tag{43}
\end{equation}

Its Laplacian is

\begin{align}
\frac{\partial^2}{\partial x^2} \left( \frac{\partial u_k}{\partial t} \Bigg|_{t=0} \right) &= \frac{d^2 a_{k,0}(y)}{dy^2} + \sum_{j=1}^{2} \left( -j^2(a^2 + \beta^2)a_{k,j}(y) + \frac{d^2 a_{k,j}(y)}{dy^2} \right) \cos \left( j(ax + \beta z) \right) \\
&\quad + \sum_{j=1}^{2} \left( -j^2(a^2 + \beta^2)b_{k,j}(y) + \frac{d^2 b_{k,j}(y)}{dy^2} \right) \sin \left( j(ax + \beta z) \right). \tag{44}
\end{align}

The solution of the Poisson Eq. (11) for the temporal change of velocity is equivalent to solving (27)=(44). The three equations can be split for each periodic component. The original Poisson equations become five independent 1D boundary value problems for each component of the vector field in the series expansion. For example:

\begin{equation}
\frac{d^2 a_{1,0}(y)}{dy^2} = A_{1,0}(y) \tag{45}
\end{equation}

or

\begin{equation}
\left( -a^2 - \beta^2 \right) a_{1,1}(y) + \frac{d^2 a_{1,1}(y)}{dy^2} = A_{1,1}(y). \tag{46}
\end{equation}

The periodic boundary conditions are automatically fulfilled. The no slip boundary condition (5) can be written as \( a_{k,j}(y = 1) = a_{k,j}(y = -1) = 0 \) for \( j = \{0, 1, 2\} \) and similarly \( b_{k,j}(y = 1) = b_{k,j}(y = -1) = 0 \) for \( j = \{1, 2\} \). The solutions of the 15 differential equations are carried out with Matlab symbolic and \( a_{k,j}, b_{k,j} \) polynomials are obtained: The solution of the Poisson equation is:

\begin{align}
a_{1,0} &= 0 \tag{47} \\
a_{1,1} &= 0 \tag{48} \\
b_{1,1} &= 0 \tag{49}
\end{align}
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\[ a_{1,2} = 0 \]
\[ b_{1,2} = \frac{2 \alpha y^6}{\alpha^2 + \beta^2} - \frac{\alpha (-4 \alpha^6 - 12 \alpha^4 \beta^2 + 2 \alpha^4 - 12 \alpha^2 \beta^4 + 4 \alpha^2 \beta^2 + 6 \alpha^2 - 4 \beta^6 + 2 \beta^4 + 6 \beta^2 - 45)}{\alpha^2 + \beta^2} = 2 (\alpha^2 + \beta^2)^4 \]
\[ - \frac{\alpha y^4 (2 \alpha^2 + 2 \beta^2 - 15)}{(\alpha^2 + \beta^2)^2} - \frac{\alpha y^2 (2 \alpha^4 + 4 \alpha^2 \beta^2 + 6 \alpha^2 + 2 \beta^4 + 6 \beta^2 - 45)}{(\alpha^2 + \beta^2)^3} \]
\[ - \frac{\alpha e^{-2 y \sqrt{\alpha^2 + \beta^2}} (16 \alpha^4 + 32 \alpha^2 \beta^2 + 84 \alpha^2 + 16 \beta^4 + 84 \beta^2 + 45)}{2 \left(e^{-2 \sqrt{\alpha^2 + \beta^2}} + e^{2 \sqrt{\alpha^2 + \beta^2}}\right) (\alpha^2 + \beta^2)^4} \]
\[ - \frac{\alpha e^{2 y \sqrt{\alpha^2 + \beta^2}} (16 \alpha^4 + 32 \alpha^2 \beta^2 + 84 \alpha^2 + 16 \beta^4 + 84 \beta^2 + 45)}{2 \left(e^{-2 \sqrt{\alpha^2 + \beta^2}} + e^{2 \sqrt{\alpha^2 + \beta^2}}\right) (\alpha^2 + \beta^2)^4} \]

\[ a_{2,0} = 0 \]
\[ a_{2,1} = \frac{2 \alpha^2 (\alpha^2 + \beta^2 + 6)}{\text{Re}} - \frac{8 \epsilon^{(y+1)} \sqrt{\sqrt{\alpha^2 + \beta^2}}}{\text{Re} \left(e^{2 \sqrt{\alpha^2 + \beta^2}} + 1\right)} - \frac{y^4 (\alpha^2 + \beta^2)}{\text{Re}} = \frac{\alpha^2 + \beta^2 + 4}{\text{Re}} \]
\[ - \frac{8 e^{-y \sqrt{\alpha^2 + \beta^2}}}{\text{Re} \left(e^{-\sqrt{\alpha^2 + \beta^2}} + e^{\sqrt{\alpha^2 + \beta^2}}\right)} \]

\[ b_{2,1} = 0 \]
\[ a_{2,2} = \frac{2 \alpha y^2 (2 \alpha^2 + 2 \beta^2 - 15)}{(\alpha^2 + \beta^2)^2} - \frac{6 y^5}{\alpha^2 + \beta^2} + \frac{y (2 \alpha^4 + 4 \alpha^2 \beta^2 + 6 \alpha^2 + 2 \beta^4 + 6 \beta^2 - 45)}{(\alpha^2 + \beta^2)^3} \]
\[ + \frac{3 e^{-2 y \sqrt{\alpha^2 + \beta^2}} (8 \alpha^2 + 8 \beta^2 + 15)}{\left(e^{-2 \sqrt{\alpha^2 + \beta^2}} - e^{2 \sqrt{\alpha^2 + \beta^2}}\right) (\alpha^2 + \beta^2)^3} - \frac{3 e^{2 y \sqrt{\alpha^2 + \beta^2}} (8 \alpha^2 + 8 \beta^2 + 15)}{\left(e^{-2 \sqrt{\alpha^2 + \beta^2}} - e^{2 \sqrt{\alpha^2 + \beta^2}}\right) (\alpha^2 + \beta^2)^3} \]

\[ b_{2,2} = 0 \]

\[ a_{3,0} = 0 \]
\[ a_{3,1} = \frac{2 \alpha^2 (\alpha^2 + \beta^2 + 6)}{\text{Re}} - \frac{8 \epsilon^{(y+1)} \sqrt{\sqrt{\alpha^2 + \beta^2}}}{\text{Re} \left(e^{2 \sqrt{\alpha^2 + \beta^2}} + 1\right)} - \frac{y^4 (\alpha^2 + \beta^2)}{\text{Re}} = \frac{\alpha^2 + \beta^2 + 4}{\text{Re}} \]
\[ - \frac{8 e^{-y \sqrt{\alpha^2 + \beta^2}}}{\text{Re} \left(e^{-\sqrt{\alpha^2 + \beta^2}} + e^{\sqrt{\alpha^2 + \beta^2}}\right)} \]

\[ b_{3,1} = 0 \]
\[ a_{3,2} = \frac{2 \alpha y^3 (2 \alpha^2 + 2 \beta^2 - 15)}{(\alpha^2 + \beta^2)^2} - \frac{6 y^5}{\alpha^2 + \beta^2} + \frac{y (2 \alpha^4 + 4 \alpha^2 \beta^2 + 6 \alpha^2 + 2 \beta^4 + 6 \beta^2 - 45)}{(\alpha^2 + \beta^2)^3} \]
\[ + \frac{3 e^{-2 y \sqrt{\alpha^2 + \beta^2}} (8 \alpha^2 + 8 \beta^2 + 15)}{\left(e^{-2 \sqrt{\alpha^2 + \beta^2}} - e^{2 \sqrt{\alpha^2 + \beta^2}}\right) (\alpha^2 + \beta^2)^3} - \frac{3 e^{2 y \sqrt{\alpha^2 + \beta^2}} (8 \alpha^2 + 8 \beta^2 + 15)}{\left(e^{-2 \sqrt{\alpha^2 + \beta^2}} - e^{2 \sqrt{\alpha^2 + \beta^2}}\right) (\alpha^2 + \beta^2)^3} \]
Smooth initial conditions for the Navier-Stokes equation without solution

\( b_{3,2} = 0 \). (61)

The divergence of \( \frac{du}{dt} \bigg|_{t=0} \) (Eq. 43) is taken, its Fourier coefficients are:

\[
da_0 = 0
\]

\[
da_1 = \frac{8e^{-y\sqrt{a^2+\beta^2}}(e^{(2y+1)\sqrt{a^2+\beta^2}} - e^{\sqrt{a^2+\beta^2}})}{\text{Re}(e^{\sqrt{a^2+\beta^2}} - 1)} C_1
\]

\[
C_1 = 3e^{2\sqrt{a^2+\beta^2} + \sqrt{a^2 + \beta^2} - e^{2\sqrt{a^2+\beta^2}}} \sqrt{a^2 + \beta^2} + 3
\]

\( db_1 = 0 \)

\[
da_2 = -\frac{e^{-2(y-1)\sqrt{a^2+\beta^2}}(e^{4y\sqrt{a^2+\beta^2}} + 1)}{(e^{8\sqrt{a^2+\beta^2}} - 1)(a^2 + \beta^2)^4} C_2
\]

\[
C_2 = \left( 16e^{4\sqrt{a^2+\beta^2}} - 16 \right) (a^2 + \beta^2)^3 - \left( 90e^{4\sqrt{a^2+\beta^2}} + 90 \right) (a^2 + \beta^2)^{3/2}
\]

\[
+ a^2 \left( 45e^{4\sqrt{a^2+\beta^2}} - 45 \right) + a^4 \left( 84e^{4\sqrt{a^2+\beta^2}} - 84 \right) + \beta^2 \left( 45e^{4\sqrt{a^2+\beta^2}} - 45 \right)
\]

\[
+ \beta^4 \left( 84e^{4\sqrt{a^2+\beta^2}} - 84 \right) - a^2 \left( 48e^{4\sqrt{a^2+\beta^2}} + 48 \right)(a^2 + \beta^2)^{3/2}
\]

\[
- \beta^2 \left( 48e^{4\sqrt{a^2+\beta^2}} + 48 \right)(a^2 + \beta^2)^{3/2} + a^2 \beta^2 \left( 168e^{4\sqrt{a^2+\beta^2}} - 168 \right)
\]

\( db_2 = 0 \)

For the parameters \( \alpha = 1, \beta = 1, \text{Re} = 80 \) the divergence was plotted as the function of \( x, y \) at \( z = 0 \) in Fig. 3.

The compatibility condition is obtained similarly. The pressure Poisson eq. (14) is solved with the boundary condition (15) for \( i = 2 \) for each periodic component. The compatibility condition (16) is evaluated at the walls \( (y = \pm 1) \) in only the two main tangential directions: the streamwise and spanwise directions. Here, the difference between the two sides of Eq. (16) is presented with the Fourier coefficients. Due to the symmetry property of the example, the Fourier coefficients at the top and bottom walls are almost the same; only the signs are different at some components.

The Fourier coefficients of the difference at the top \( (y = 1) \) evaluating the compatibility condition in the streamwise \( (x) \) direction:

\[
CCxa_0 = 0
\]

\[
CCxa_1 = 0
\]

\[
CCxb_1 = \frac{8\alpha}{\text{Re}(e^{2\sqrt{a^2+\beta^2}} + 1)\sqrt{a^2 + \beta^2}}
\]

\[
CCxa_2 = 0
\]

\[
CCxb_2 = -2\alpha C_3
\]

\[
C_3 = \frac{2a^2 + 2\beta^2 - 15}{2(a^2 + \beta^2)^2} - \frac{1}{a^2 + \beta^2} + \frac{2a^4 + 4a^2\beta^2 + 6a^2 + 2\beta^4 + 6\beta^2 - 45}{2(a^2 + \beta^2)^3}
\]

\[
+ \frac{-4a^6 - 12a^4\beta^2 + 2a^4 - 12a^2\beta^4 + 4a^2\beta^2 + 6a^2 - 4\beta^6 + 2\beta^4 + 6\beta^2 - 45}{4(a^2 + \beta^2)^4}
\]

\[
+ \frac{24a^2 + 24\beta^2 + 45}{2(e^{4\sqrt{a^2+\beta^2}} - 1)(a^2 + \beta^2)^{7/2}} + \frac{3e^{4\sqrt{a^2+\beta^2}}(8a^2 + 8\beta^2 + 15)}{2(e^{4\sqrt{a^2+\beta^2}} - 1)(a^2 + \beta^2)^{7/2}}
\]
Smooth initial conditions for the Navier-Stokes equation without solution

The difference coefficients at the top \((y = 1)\) evaluating the compatibility condition in the spanwise \((z)\) direction:

\[
CC z a_0 = 0 \quad (77)
\]
\[
CC z a_1 = 0 \quad (78)
\]
\[
CC z b_1 = \frac{8 \beta \left( e^{2 \sqrt{a^2 + \beta^2}} - 1 \right)}{\text{Re} \left( e^{2 \sqrt{a^2 + \beta^2}} + 1 \right) \sqrt{a^2 + \beta^2}} - \frac{24}{\text{Re} \beta} \quad (79)
\]
\[
CC z a_2 = 0 \quad (80)
\]
\[
CC z b_2 = -2 \beta C_3 \quad (81)
\]

The coefficients are zero if the compatibility condition fulfils. However, it does not hold in this case.

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