A Quantum Broadcasting Problem in Classical Low Power Signal Processing

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Abstract
We pose a problem called “broadcasting Holevo-information”: given an unknown state taken from an ensemble, the task is to generate a bipartite state transferring as much Holevo-information to each copy as possible.

We argue that upper bounds on the average information over both copies imply lower bounds on the quantum capacity required to send the ensemble without information loss. This is because a channel with zero quantum capacity has a unitary extension transferring at least as much information to its environment as it transfers to the output.

For an ensemble being the time orbit of a pure state under a Hamiltonian evolution, we derive such a bound on the required quantum capacity in terms of properties of the input and output energy distribution. Moreover, we discuss relations between the broadcasting problem and entropy power inequalities.

The broadcasting problem arises when a signal should be transmitted by a time-invariant device such that the outgoing signal has the same timing information as the incoming signal had. Based on previous results we argue that this establishes a link between quantum information theory and the theory of low power computing because the loss of timing information implies loss of free energy.
1 Introduction

Quantum information theory and the theory of low-power processing are currently quite different scientific disciplines. Even though future low power computers will operate more and more on the nanoscale and therefore in the quantum regime (e.g. single electron transistors, spintronic networks [1]), superpositions of logically different states being crucial for quantum computing [2], are not intended to occur in low-power computing devices.

On the other hand, quantum computing research is little interested in issues of low power processing. The control of quantum systems involves large laboratory equipment and even power consumption rates for logical operations that are in the magnitude of usual classical chips seem currently to be out of reach.

To understand limitations of low-power information processing it is useful to construct theoretical models of computers which process information without consuming energy, i.e., the process is implemented in an energetically closed physical system. In our opinion, discussions on fundamental issues like bounds on power consumption require a quantum theoretical description. Interesting quantum models of computers being closed physical systems can be found in Refs. [3, 4, 5, 6, 7]. Remarkably, it is common to all these models that the synchronization is based upon some propagating wave or particle and that the quantum uncertainty of its position leads to an ill-defined logical state of the computer. In other words, the clock is entangled with the data register. It seems as if the clocking issue brings some aspects of quantum information theory into the field of low power computing. This is not surprising for the following reason: the states of a quantum mechanical system have a consistent classical description only if the attention is restricted to a set of mutually commuting density matrices. But the Hamiltonian dynamics automatically generated non-commuting density matrices from a given one. Hence the dynamical aspect makes it necessary to include quantum superpositions into the description. Note that this is also in the spirit of Hardy’s paper “Quantum theory from five reasonable axioms” [8], saying that every statistical theory that satisfies some very natural axioms is quantum, as soon as it makes continuous reversible dynamical evolution possible.

If signal propagation in future low-power devices takes place in a system being (approximately) thermodynamically closed it must be described by a quantum Hamiltonian evolution. The idea of this article is that processing
such signals leads to quantum broadcasting problems for two reasons:

First, it is a natural problem to distribute signals (like clock signals) to several devices. The timing information carried by a signal whose quantum state is a density operator within a family of non-commuting states cannot be considered as classical information, its distribution is therefore some kind of broadcasting problem. The results in [9, 10] indicate that no-broadcasting theorems are expected to get relevant for the distribution when the signal energy is reduced to a scale where quantum energy-time uncertainty becomes the limiting factor for the accuracy of clocking.

The second reason why broadcasting problems come into play is more subtle. If such a clocking signal enters a device and triggers the transmission of an output signal we may desire that the output should have as much timing information as the input (in a sense that will be further specified later). Whether channels with zero quantum capacity are able to satisfy this requirement is a question that is linked to quantum broadcasting.

The intention of this article is to describe a special kind of broadcasting problem. In contrast to the usual setting [11], the task is not to obtain output states that are close to the inputs. The problem is to broadcast the Holevo-information of an ensemble of non-commuting quantum density matrices such that each party gets almost the same amount of Holevo-information as the original ensemble possessed. The use of entropy-like information measures makes it possible to draw connections to thermodynamics.

In this paper, the ensemble of non-commuting states will always be given by the Hamiltonian time evolution of a given state. Even though the problem of broadcasting Holevo-information makes also sense for general ensembles, time evolution is the most obvious way how non-commuting ensembles occur in devices that are not designed to do quantum information processing.

It seems to be hard to derive general bounds on the information loss of each copy when the Holevo-information is broadcast. Thus, we will only conjecture that it is not possible for non-commuting ensembles to get full Holevo-information for both copies. The intention of this article is therefore rather to pose the broadcasting problem and show its relevance than to solve it. However, for pure input states we will give one lower bound on the loss that depends on the energy distribution of input and output signals.

The paper is organized as follows. In Section 2 we introduce time-invariant signal processing devices and argue that in this setting timing information is a resource that can never be increased. In Section 3 we formally state the problem of broadcasting Holevo-information in the general case and
in the case of timing information. In Section 4 we argue that the broadcasting problem leads to the question how the Holevo-information of an ensemble of bipartite states is related to the information of the ensembles of the corresponding reduced states. We discuss this information deficit for the special case of pure product states where the problem is related to the entropy power inequalities of classical information theory. In Section 5 we derive a bound on the information deficit that depends on the energy spectral measure of input and output signal. In Section 6 we show that the results imply lower bounds on the quantum capacity required for lossless transmission of signals having small energy uncertainty in a time-covariant way. Section 7 derives lower bounds on the loss of free energy implied by the loss of timing information caused by a channel with too little quantum capacity. This describes an even tighter link between quantum information theory and the theory of low-power signal processing.

2 Quantum model of time-invariant signal processing devices

As already stated, the problem of transmitting non-commuting ensembles of quantum states arises most naturally for ensembles that are given by the Hamiltonian time evolution of a given state. Such an ensemble may, for instance, describe the density matrix of a propagating signal before or after it is processed by the device. If all clocking signals that enter a given device are included into the formal description, the quantum operation mapping the input onto the output is time-invariant. As we will describe below, such a device cannot increase the timing information. The latter is therefore considered as a resource. The idea that devices with non-zero quantum capacity seem to deal with this resource more carefully than classical channels is essential for this article.

Now we introduce the abstract description of time-invariant devices. Here a device may be a transistor, an optical element or some other system with input and output signals. The signal may, for instance, be an electric pulse, a light pulse, or an acoustical signal. We consider it as a physical system with some Hilbert space $\mathcal{H}$ and the state is a density operator $\rho$ acting on $\mathcal{H}$. For the examples mentioned above, the space $\mathcal{H}$ will typically be infinite dimensional since one may e.g. think of position degrees of freedom that are
encoded into $\rho$. Before and after the signal is processed in the device its free time evolution is generated by a Hamiltonian $H$ (i.e. a densely defined self-adjoint operator on $\mathcal{H}$) and reads

$$\alpha_t(\rho) := e^{-iHt} \rho e^{iHt}. \quad (1)$$

We assume that input state $\rho$ and its output $G(\rho)$ are related by some completely positive trace-preserving map $G$ satisfying the covariance condition

$$G(\alpha_t(\rho)) = \alpha_t(G(\rho)) \quad \forall \rho. \quad (2)$$

There are rather different situations where the covariance condition is satisfied. One example would be if the interactions between signal and device are weak. A more interesting justification is the following. Consider a signal propagating towards the device by its own autonomous Hamiltonian time evolution until it begins to interact with the latter. Then it leaves the device (as a possibly modified signal) and as soon as the interaction with the device is negligible it is again subjected to its Hamiltonian only. Such a process should be considered as a quantum stochastic analogue of a scattering process (see [12] for details) and the time covariance condition (2) is then a generalization of the statement that the $S$-"matrix" of a scattering process commutes with the free Hamiltonian evolution of the incoming and outgoing particle [13]. Note that the existence of an unitary $S$ operator would require devices which preserve the purity of the input.

In [12] we have given a quite explicit description of the set of CP maps satisfying this covariance condition. Here it is more interesting to discuss the implications of covariance. We first rephrase the definition of timing information used in [14] (see also [15] for a more general context).

Recall that the Holevo-information of an ensemble of quantum states $\rho_x$ with probability measure $p$ (denoted by $\{p(x), \rho_x\}_x$) is defined by [2]

$$I := S\left( \int \rho_x dp(x) \right) - \int S(\rho_x) dp(x),$$

where the measure-theoretic integral reduces to sums when $p$ is supported by a countable set of points. Here

$$S(\rho) = -tr(\rho \log \rho) \quad (3)$$

is the von-Neumann entropy and the base of the logarithm remains unspecified. In the sequel we will measure entropy in bits or nats since sometimes one unit is more natural and sometimes the other.
Timing information refers to a specific ensemble, namely the orbit with respect to a unitary one-parameter group:

**Definition 1 (Timing Information)**

Let $\rho$ be the state of a quantum system whose Hamiltonian $H$ has discrete spectrum. Then its timing information is defined as

$$I := S(\overline{\rho}) - S(\rho),$$

where $\overline{\rho}$ denotes the time average

$$\overline{\rho} := \lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha_t(\rho) dt = \sum_x R_x \rho R_x,$$

$\alpha_t$ is defined as in eq. (1) and $(R_x)$ is the family of spectral projections (with eigenvalues $x$ corresponding to the system Hamiltonian. For pure states $\rho = |\psi\rangle\langle\psi|$ we have $S(\rho) = 0$. Thus, $I$ is the entropy of $\overline{\rho}$ which is then exactly the entropy of a classical random variable $X$ describing the distribution of energy values with $P(X = x) := \langle \psi | R_x | \psi \rangle$.

Note that it is a well-known question to what extent information on reference frames in time or space requires quantum communication or profits from it and to which degree shared reference frames are resources that are comparable to shared quantum states [16, 17, 18, 19, 20]. In this article we want to understand to what extent timing information should be considered as quantum information by exploring the information loss occurring when it is copied. In [9] we have derived lower bounds on the loss of timing information in terms of Fisher-information for the broadcasting problem. To our knowledge, no results in terms of Holevo-information can be found in the literature.

## 3 Broadcasting timing information

Before we pose the problem of broadcasting timing information (which we have motivated from the time-covariant transmission of signals) we first state the more general problem of “broadcasting Holevo-information”. It is defined as follows:
Definition 2 (Broadcasting Holevo-Information)

Given an ensemble \( \{ p(x), \rho(x) \}_x \) of quantum states acting on some Hilbert space \( \mathcal{H} \). Let \( I \) be its Holevo-information. Find an optimal broadcasting map in the following sense:

Let \( \mathcal{H}_A \) and \( \mathcal{H}_B \) be some arbitrary additional Hilbert spaces and \( G \) be a completely positive trace-preserving operation from the density operators on \( \mathcal{H} \) on the density operators acting on \( \mathcal{H}_A \otimes \mathcal{H}_B \). Let \( I_A, I_B \) denote the Holevo-information of the ensembles given by the reduced states \( \text{tr}_B(G(\rho_x)) \) and \( \text{tr}_A(G(\rho_x)) \), respectively.

Maximize the average information

\[
\frac{1}{2}(I_A + I_B)
\]

over all \( \mathcal{H}_A \otimes \mathcal{H}_B \) and \( G \) such that it gets as close to \( I \) as possible.

We call

\[
\Delta := I - \frac{1}{2}(I_A + I_B)
\]

the broadcasting loss of a given broadcasting operation. Let \( \Delta_{\min} \) be the minimal loss over all broadcasting operations for a given ensemble. In the context of timing information we will also use the terminology "\( \Delta_{\min} \) of a state \( \rho \)" when actually referring to the information loss of the ensemble \( \{ \alpha_t(\rho) \}_{t \in [0,\tau]} \) with uniform probability distribution over the whole time period.

Due to the monotonicity of the Holevo-information of an arbitrary ensemble with respect to CP maps \[14\] we certainly have \( I_1 \leq I \) and \( I_2 \leq I \). It is natural to conjecture that \( I = I_1 = I_2 \) can only be achieved if all density matrices commute, which is exactly the case where usual broadcasting is possible \[11\]. It is furthermore obvious that there are maps that provide both parties with the accessible information \[2\] by applying a measurement to the input state and sending mutually orthogonal quantum states representing the results to both parties.

Now we apply the definition of broadcasting to an ensemble given by the time orbit \( \{ \rho_t \}_{t \in [0,\tau]} \) of a dynamical evolution with period \( \tau \) with uniform distribution over the whole interval. Then the task is to optimally broadcast the timing information in the sense of Definition \[1\]. Note that information differences like that one in eq. \[4\] maybe well-defined for systems with continuous spectrum where the timing information itself is infinite. By appropriate limits, one could therefore define the question on the information
loss in broadcasting operations also for systems possessing no time average state.

To give an impression on the problem of broadcasting timing information we consider the phase-covariant cloning of an equatorial qubit state, i.e., a state

$$|\psi_t\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{it}|1\rangle)$$

with unknown \( t \in [0, 2\pi) \). In the usual quantum cloning problem one tries, for instance, to obtain two copies whose states get as close to the original as possible with respect to the Fidelity. As shown in [21] one can generate two copies as mixed states whose Bloch vectors point in the same direction as that of the original, but are shorter than the original by the factor \( 1/\sqrt{2} \). Thus, the density matrices of the copies have the eigenvalues \( 1/2 \pm 1/(2\sqrt{2}) \). The Holevo-information of each copy is then given by the entropy of the time average (which is still one bit) minus the above binary entropy when inserting the above eigenvalues:

$$I = 1 + \frac{1}{2}(1 + \frac{1}{\sqrt{2}})\log_2 \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) + \frac{1}{2}(1 - \frac{1}{\sqrt{2}})\log_2 \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) \approx 0.399 \text{ bit.}$$

The information of the original was 1. Here, even the sum of the amount of information over both copies is less than the original amount. In other words, the average information over both copies is even smaller than it was if we had given one party the original and the other an arbitrary state that is independent from the input.

4 Information deficit in pure product states and entropy power inequalities

In the following we will not explicitly consider the broadcasting operation that generates a bipartite state from the original. Since this operation can never increase the information we focus to the following problem: Given an ensemble of bipartite states, compare the Holevo-information of the two ensembles \( I_A \) and \( I_B \) defined by the restrictions to the subsystems to the information \( I \) of the joint system. Call \( I - (I_A + I_B)/2 \) the information deficit. In other words, the information deficit is the broadcasting loss if the broadcasting map is the identity and the original is already a bipartite state.
Remarkably, the determination of the deficit is non-trivial even when the
bipartite state is a product state. Given the state
\[ |\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \]
where each subsystem is subjected to its own Hamiltonian \( H_A \) and \( H_B \), respec-
respectively. We may assume without loss of generality that both Hamiltonians
are diagonal and non-degenerate (since we restrict the attention to the time
orbits of each state). The distribution of energy values in the state \( |\psi_A\rangle \otimes |\psi_B\rangle \)
defines a joint distribution of two stochastically independent classical random
variables \( X, Y \) by
\[
P(X = x, Y = y) := \langle \psi_A | R_x | \psi_A \rangle \langle \psi_B | Q_y | \psi_B \rangle,
\]
where \( R_x \) is defined as in Definition 1 and \( Q_y \) similarly. Since \( H_A \otimes 1 + 1 \otimes H_B \)
is the Hamiltonian of the joint system, its timing information is given by
\[
\mathcal{I} = S(X + Y),
\]
where we have decided to use the same symbol for the entropy of classical
random variables as for the von-Neumann entropy of quantum states. The
subsystem timing information is given by
\[
\mathcal{I}_A = S(X) \quad \text{and} \quad \mathcal{I}_B = S(Y).
\]
Note that it is a well-known problem in classical information theory to relate
the entropy of the distributions of two independent random variables to the
entropy of their sum since it addresses the question how the entropy of a
real-valued signal changes when subjected to an additive noise. We rephrase
the following result that applies to continuous distributions. For probability
densities \( P(X) \) the continuous entropy is defined by
\[
S(X) = - \int P(x) \ln P(x) dx + c,
\]
with an unspecified constant \( c \). For two independent random variables, i.e.,
when their density satisfies \( P(x, y) = P(x)P(y) \), we have the entropy power
inequality \([22]\]
\[
e^{2S(X+Y)} \geq e^{2S(X)} + e^{2S(Y)},
\]
and hence
\[
2S(X + Y) \geq \ln \left( \frac{1}{2} (2e^{2S(X)} + 2e^{2S(Y)}) \right) \\
\geq \frac{1}{2} \left( \ln 2e^{2S(X)} + \ln 2e^{2S(Y)} \right) \\
= \ln 2 + S(X) + S(Y),
\]
where the second inequality follows from the concavity of the logarithm. Assuming that the spectral measures of \( H_A \) and \( H_B \) are sufficiently distributed over many energy eigenvalues we can approximate the discrete entropy with the continuous expression for appropriate densities. After using eqs. (\ref{eq:1}) and (\ref{eq:2}) we obtain
\[
I \geq \frac{1}{2} (\ln 2 + I_A + I_B).
\]
Note that \( \ln 2 \) corresponds exactly to one bit of information since the entropy power inequality refers to entropy measured in natural units. We conclude that for continuous spectrum and product states the timing information of the joint system is at least half a bit more than the average timing information over both systems.

5 Information deficit for pure entangled states

To estimate the information deficit for entangled states we will also use the joint distribution of \( X \) and \( Y \) on \( \mathbb{R}^2 \) given by
\[
P(X = x, Y = y) := tr(\rho(R_x \otimes Q_y)),
\]
with the spectral projections \( R_x \) and \( Q_y \). If the bipartite system is in an entangled state, eq. (\ref{eq:3}) is no longer true. Moreover, we cannot assume that both Hamiltonians are “without loss of generality” non-degenerate since the reduced states may be mixed even within a specific degenerate energy eigenspace. However, eq. (\ref{eq:4}) still holds for pure states. We replace eq. (\ref{eq:4}) by
\[
\mathcal{I}_A = S(\rho_A) - S(\rho_A),
\]
where \( \rho_A \) denotes the reduced state on system \( A \) and obtain \( \mathcal{I}_B \) in a similar way. To derive upper bounds on the timing information of the subsystems we need the following Lemma.
Lemma 1 (Average Entropy of Post-Measurement States)
Let \((R_j)_j\) be a complete family of orthogonal projections defining a measurement and \(\sigma\) be an arbitrary quantum state. Let \(S(p)\) be the Shannon entropy of the outcome probabilities \(p_j := \text{tr}(R_j\sigma)\). Then we have
\[
S\left(\sum_j R_j\sigma R_j\right) \leq S(\sigma) + S(p)
\]

Proof: The statement is equivalent to
\[
\sum_j p_j S\left(\frac{1}{p_j} R_j\sigma R_j\right) \leq S(\sigma)
\]
(10)

Let \(\sigma = \sum_i q_i\sigma_i\) be a decomposition of \(\sigma\) into pure states. We can consider \(S(\sigma)\) as the Holevo-information of the ensemble \(\{q_i, \sigma_i\}_i\). Then the left hand side of eq. (10) is equal to the Holevo-information of the ensemble after the measurement has been applied. It can certainly be not greater than the Holevo-information of the original ensemble \(\square\).

For our derivation of an upper bound on the information of the subsystems the following Lemma will be crucial.

Lemma 2 (Timing Information is less than Conditional Entropy)
Let \(\rho\) be a (possibly mixed) state on a bipartite system. Then the timing information of \(A\) and \(B\) satisfies
\[
\mathcal{I}_A \leq S(X|Y), \quad \mathcal{I}_B \leq S(Y|X),
\]
respectively, where the joint distribution of \(X\) and \(Y\) is defined by eq. (9).

Proof: By Definition \(\square\) the timing information of system \(A\) is given by
\[
\mathcal{I}_A = S(\overline{\rho_A}) - S(\rho_A).
\]
We decompose \(\rho_A\) into
\[
\rho_A = \sum_y p(y)\rho_{A,y},
\]
where $\rho_{A,y}$ denotes the conditional state given that we had measured the energy value $y$ on system $B$. Since $I_A$ is the Kullback-Leibler distance between $\rho_A$ and $\rho_A$ (see [23]) it is convex and we get

$$I_A \leq \sum_y p(y)(S(\rho_{A,y}) - S(\rho_{A,y})).$$

For each specific value $y$ of $Y$

$$S(\rho_{A,y}) - S(\rho_{A,y}) \leq S(X|y)$$

holds due to Lemma 1. Taking the convex sum of this inequality over all $y$ with weights $p(y)$ completes the proof. □

Note that there are conditions known [24], where the joint probability density of two dependent random variables satisfies the entropy power inequality

$$e^{2S(X+Y)} \geq e^{2S(X|Y)} + e^{2S(Y|X)}.$$

Under such conditions we obtain the same lower bound on the information deficit as in Section 3.

In the general case we have to use other methods to derive more explicit bounds from the bounds of Lemma 2. For doing so, we will need the following lemma.

**Lemma 3 (Information Deficit and Classical Mutual Information)**

The information deficit of a bipartite system being in a pure state satisfies

$$\Delta \geq \frac{1}{2} \left( I(X : X+Y) + I(Y : X+Y) \right) = S(X+Y) - \frac{1}{2} \left( S(X|Y) + S(Y|X) \right),$$

where $I(.,.)$ denotes the mutual information between classical random variables [25].

Proof: Note that the equation $I = S(X+Y)$ holds also for pure entangled states. Using Lemma 2 we obtain

$$2I - I_A - I_B \geq 2S(X+Y) - S(X|Y) - S(Y|X)$$

$$= 2S(X+Y) - S(X + Y|Y) - S(X + Y|X)$$

$$= I(X + Y : Y) + I(X + Y : X).$$

□
It is possible to derive bounds on the information loss based on Lemma 3, since the term on the right hand vanishes only in the trivial case $S(X+Y) = 0$ in which the joint system contains no timing information at all. To show this we observe that there is no joint measure where $X$ and $Y$ are both uncorrelated to $X + Y$. This is seen from

$$C(X, X + Y) + C(Y, X + Y) = V(X + Y),$$

where $C(.,.)$ denotes the covariance and $V(.)$ the variance. However, to derive lower bounds on the mutual information based on these covariance terms requires additional assumptions on the distribution. We will deal with this point later.

In order to apply the bounds of Lemma 3 it can be convenient to relate them to other information-theoretic quantities:

**Lemma 4 (Mutual Information and Relative Entropy)**

Let $X$ and $Y$ be two real-valued random variables and $P$ the corresponding joint distribution on $\mathbb{R}^2$ with discrete support. Let $P_X$ and $P_{X+Y}$ denote the marginal distribution for $-X$ and $X+Y$, respectively. Denote the convolution of both by $P_X \ast P_{X+Y}$. Then we have

$$I(X : X + Y) \geq K(P_Y \| P_X \ast P_{X+Y})$$

and

$$I(Y : X + Y) \geq K(P_X \| P_Y \ast P_{X+Y}).$$

Moreover, we have the symmetrized statement

$$I(X : X + Y) + I(Y : X + Y) \geq K\left(\frac{1}{2}(P_X + P_Y) \big\| \frac{1}{2}(P_{-X} + P_{-Y}) \ast P_{X+Y}\right).$$

**Proof:** We define measures on $\mathbb{R}^2$ by

$$Q(X = a, Y = b) := P(X + Y = a + b)P(Y = b)$$

and

$$R(X = a, Y = b) := P(X + Y = a + b)P(X = a).$$

Then we can rewrite the mutual information on the left hand side as Kullback-Leibler distances:

$$I(X : X + Y) = K(P \| R)$$

13
and
\[ I(Y : X + Y) = K(P \| Q). \]

Due to the monotonicity of relative entropy distance under marginalization \[26\] we have
\[ K(P \| Q) \geq K(P_X \| Q_X) \]
where \( P_X \) and \( Q_X \) denote the marginal distribution of \( X \) according to \( P \) and \( Q \), respectively, i.e., \( Q_X(X = a) := Q(X = a) \). Similarly
\[ K(P \| R) \geq K(P_Y \| R_Y). \]

We have
\[
Q(X = a) \quad = \quad \sum_b Q(X = a, Y = b) \\
= \quad \sum_b P(X + Y = a + b)P(Y = b) \\
= \quad \sum_c P(X + Y = c)P(Y = c - a) \\
= \quad \sum_c P(X + Y = c)P(X = c - a). 
\]

Hence the marginal distribution \( Q_X \) of \( Q \) is the convolution product \( P_{X+Y} * P_{-X} \) and the marginal distribution \( R_Y \) of \( R \) is the product \( P_{X+Y} * P_{-Y} \). This proves inequalities \[12\] and \[13\].

We obtain the symmetrized statement from the convexity of relative entropy distance \[25\]. □

After applying Lemma 4 and and Lemma 3 we conclude:

**Theorem 1 (Information Deficit for Pure States)**

*Given a pure state of a bipartite system \( A \times B \). Let \( P_X, P_Y \) and \( P_{X+Y} \) denote the probability distributions for the energy of \( A, B \) and \( A \times B \), respectively. Then the difference between the joint timing information and the average information of the subsystems satisfies*

\[
\Delta \quad \geq \quad K(P_X \| P_{-Y} * P_{X+Y}) + K(P_Y \| P_{-X} * P_{X+Y}) \\
\geq \quad K\left(\frac{1}{2}(P_X + P_Y) \bigg\| \frac{1}{2}(P_{-X} + P_{-Y}) * P_{X+Y}\right). 
\]
The intuitive content of Theorem 1 is the following. If the energy uncertainty of $A$ and $B$ are both on the same scale as the uncertainty of $X + Y$, the convolution with $P_{X+Y}$ adds a non-negligible amount of uncertainty to $(P_X + P_Y)/2$, which implies that the new distribution obtained by adding additional noise cannot be close to the original distribution of $X$.

It is often helpful to consider measures that are symmetric with respect to exchanging $X$ and $Y$, i.e., $P(X = x, Y = y) = P(X = y, Y = x)$. The following Lemma shows that lower bounds on $I(X : X + Y) + I(Y : X + Y)$ for symmetric joint measures automatically provide bounds for asymmetric measures:

**Lemma 5 (Symmetrization)**

Let $P$ be a joint distribution of $X$ and $Y$ and $ar{P}$ its symmetrization $\bar{P} := (P + P')/2$, where

$$P'(X = x, Y = y) := P(Y = x, Y = y).$$

Then we have

$$I_P(X : X + Y) + I_P(Y : X + Y) \geq I_{\bar{P}}(X : X + Y) + I_{\bar{P}}(Y : X + Y),$$

where $I_P(.,.)$ refers to the mutual information induced by the measure $P$.

Proof: We write

$$P(X = x, Y = y) = P(X = x|X + Y = x + y)P(X + Y = x + y).$$

We obtain such a representation also for $P'$ by replacing only the conditional $P(X|X + Y)$ since the marginal distribution on $X + Y$ coincides for $P$ and $P'$. Then the Lemma follows already from the convexity of mutual information with respect to convex sums of conditionals with fixed marginals (Theorem 2.7.3 in [25]).

A simple bound on the information deficit can be provided in terms of the fourth moments of the signal energies:

**Theorem 2 (Information Deficit in Terms of Energy)**

Given a pure bipartite state on $A \times B$. Let $(\Delta E)^2$ denote the variance of the total energy and $\langle E_4^4 \rangle$ denote the 4th moment of the energy of system
\( j = A,B \) and \( \langle E^4 \rangle \) be the fourth moment of the total energy. Then the information deficit (measured in natural units) satisfies

\[
\Delta \geq \frac{(\Delta E)^8}{64 \langle \langle E_A^4 \rangle + \langle E_B^4 \rangle \rangle \langle E^4 \rangle}.
\]

Proof: Let \( P \), as above, be the discrete probability measure on \( \mathbb{R}^2 \) describing the energy distribution of the bipartite system. We begin by assuming that \( P \) is symmetric (see Lemma 5). Then we have \( C(X, X + Y) = V(X + Y)/2 \) (see eq. (11)). We define a measure \( R \) as in the proof of Theorem 1 and we can rewrite the covariance as

\[
C(X, X + Y) = \sum_{xy} x(x + y)(P(x, y) - R(x, y))
\]

With \( Z := X + Y \) we have

\[
\frac{1}{4} V^2(X + Y) = C(X, Z)^2 = \left| \sum_{xz} xz\sqrt{(P(x, z - x) - R(x, z - x))} \right|^2
\]

\[
\leq \sum_{xz} x^2 z^2 |P(x, z - x) - R(x, z - x)|
\]

\[
\times \sum_{xz} |P(x, z - x) - R(x, z - x)|
\]

\[
\leq \sum_{xz} x^2 z^2 (P(x, z - x) + R(x, z - x))\|P - R\|_1
\]

\[
= (\langle X^2 Z^2 \rangle + \langle X^2 \rangle (\langle Z^2 \rangle)) \|P - R\|_1
\]

\[
\leq 2\sqrt{\langle X^4 \rangle \langle Z^4 \rangle} \|P - R\|_1.
\]

From the first line to the second we have used the Cauchy Schwarz inequality which shows also \( \langle X^2 Z^2 \rangle \leq \sqrt{\langle X^4 \rangle \langle Z^4 \rangle} \) as well as \( \langle X^2 \rangle \leq \sqrt{\langle X^4 \rangle} \).

We recall the bound

\[
K(P||R) \geq \frac{1}{2}\|P - R\|^2
\]

(see Lemma 12.6.1 in [25]) for the relative entropy measured in natural units. Then we obtain

\[
\frac{1}{2}\|P - R\|^2 \geq \frac{V(X + Y)^4}{128 \langle (X + Y)^4 \rangle}.
\]
This implies
\[ I(X : X + Y) \geq \frac{V(X + Y)^4}{128 \langle X^4 \rangle \langle (X + Y)^4 \rangle}. \]

If we consider an asymmetric measure \( P \) we have to symmetrize it first. Then we replace \( \langle X^4 \rangle \) with \( (\langle X^4 \rangle + \langle Y^4 \rangle)/2 \) since the fourth moment of \( Y \) with respect to the original measure \( P \) coincides with the fourth moment of \( X \) when calculated with respect to the reflected measure \( P'(X = x, Y = y) := P(X = y, Y = x) \). Using Lemma 3 this proves the statement when replacing the statistical moments of \( X, Y \), and \( X + Y \) with the more physical terms \( \langle E_A^4 \rangle, \langle E_B^4 \rangle \) and \( \langle E^4 \rangle \). □

6 Quantum capacity required for lossless transmission

In this section we will derive lower bounds on the quantum capacity required to transmit an ensemble with some fixed maximal information loss. The idea of the argument is the following. Assume that the timing information of \( G(\rho) \) is exactly the same as that of \( \rho \). Assume furthermore that \( G \) has zero quantum capacity. This implies, roughly speaking, that \( G \) can be modeled by a unitary that copies as much information to the environment as the amount of information that passes the channel. But if this would be the case we had perfect broadcast of Holevo-information, an operation that we consider unlikely to be possible for non-commuting ensembles like time orbits. To put this argument on a solid basis, we rephrase the following result of Devetak [27]. Recall that the private information capacity (see [27] for a formal definition) is the maximal number of encoded qubits per transmitted qubits, that two parties, the sender Alice and the receiver Bob, can asymptotically achieve in a protocol where a potential eavesdropper Eve, having access to the full environment of the channel, gets a vanishing amount of information. The following theorem relates the private information capacity to the information the environment obtains when the channel is represented by a unitary acting on the system and an abstract environment being in a pure state. \(^1\)

\(^1\)One should emphasize that the unitary extension gives only upper bounds on the information transferred to the environment. Real environments are usually in mixed states and can therefore destroy quantum superpositions without receiving information from the system (see [28] for details).
Theorem 3 (Private Channel Capacity)
Let $G$ be a quantum channel mapping density operators acting on $\mathcal{H}$ to density operators acting on the same space. Let $\mathcal{H}_E$ be an additional Hilbert space thought of as the space of the environment. Moreover, let $U$ be a unitary acting on $\mathcal{H} \otimes \mathcal{H}_E$ and $|\phi\rangle \in \mathcal{H}_E$ be a state such that
$$G(\rho) = \tr_2(U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger).$$

Let $\rho_x$ with $x \in \mathcal{X}$ be some finite family of input states (sent by Alice with probability $p(x)$) and $\sigma_x := U(\rho_x \otimes |\phi\rangle\langle\phi|)U^\dagger$ be the corresponding joint states of the environment and the receiver’s (i.e Bob’s) system. Denote the restrictions to these subsystems by $\sigma_x^B$ and $\sigma_x^E$, respectively. Set
$$I(X : B) := S\left(\sum_x p(x)\sigma_x^B\right) - p(x)\sum_x S(\sigma_x^B),$$
and $I(X : E)$ similarly. Define the single copy private channel capacity by
$$C_1(G) := \sup\{I(X : B) - I(X : E)\},$$
where the supremum is taken over all ensembles $(p(X), \rho_x)$. Let $G^{\otimes l}$ be the $l$-fold copy of $G$. Then the private channel capacity is given by
$$C_p(G) = \lim_{l \to \infty} \frac{1}{l} C_1(G^{\otimes l}).$$

Certainly, we have $C_p(G) \geq C_1(G)$. This is seen by transmitting independently distributed product states through the copies of channels. We observe:

Theorem 4 (Information Loss in Classical Channels)
Let $\{p(x), \rho_x\}_x$ be an ensemble of quantum states with Holevo-information
$$I(X : A) = S\left(\sum_x p(x)\rho_x\right) - \sum_x p(x)S(\rho_x),$$
with minimal broadcasting loss $\Delta_{\text{min}}$. Let $G$ be some channel with
$$I(X : B) = S\left(\sum_x p(x)G(\rho_x)\right) - \sum_x p(x)S(G(\rho_x)).$$

Then the private channel capacity of $G$ can be bounded from below by
$$C_p(G) \geq 2\left(\Delta_{\text{min}} - (I(X : A) - I(X : B))\right).$$
Note that \( I(X : A) - I(X : B) \) is the information loss caused by the channel because it is the difference between input and output Holevo-information. Given a bound for broadcasting the Holevo-information of the considered ensemble, we have a lower bound on the quantum capacity to transmit them without loss.

Proof (of Theorem 4): Given some unitary operation \( U \) extending the channel \( G \). We have
\[
\Delta := I(X : A) - \frac{1}{2}(I(X : B) + I(X : E)) \geq \Delta_{\text{min}}
\]
by definition of \( \Delta_{\text{min}} \) and
\[
C_p(G) \geq I(X : B) - I(X : E),
\]
by Theorem 3. Then simple calculations yield the stated inequality. \( \square \)

The theorem shows that for states with non-zero \( \Delta_{\text{min}} \) (which is probably every non-stationary state \( \rho \)) the covariant lossless transmission requires a channel with non-zero quantum capacity. Instead of deriving lower bounds on \( \Delta_{\text{min}} \), i.e., the minimum over all \( \Delta \) we will use the bound from Theorem 2 and only obtain bounds in terms of the fourth moments of the energy distribution. However, using this theorem is not straightforward for the following reason: Given some assumptions on the energy distribution of the input and output signals of a device we want to derive lower bounds on the quantum capacity required to transmit the signal without information loss. To this end, we use the unitary extension of the CP map formalizing the device because we have only derived bounds for \textit{pure} bipartite states. However, the usual construction of the unitary extension uses an abstract environment Hilbert space where no “environment Hamiltonian” is specified. And, even worse, given that we had specified an arbitrary “environment Hamiltonian”, the unitary that models the channel could have lead to arbitrary energy distributions for system plus environment and we obtained no useful statements on the fourth moments.

The following Lemma shows that we can construct the unitary extension such that it is energy conserving in the constructed joint system. We have here considered a finite dimensional system for technical reasons.

\textbf{Lemma 6 (Unitary Extension of Covariant Operations)}
\begin{quote}
Let \( G \) be a completely positive trace-preserving map on the set of \( d \times d \) density
matrices that satisfies the covariance condition (2) with respect to the time evolution generated by a Hamiltonian \( H \) acting on \( \mathbb{C}^d \).

Then there is a (not necessarily finite dimensional) Hilbert space \( \mathcal{H}_E \), a densely defined Hamiltonian \( H_E \) on \( \mathcal{H}_E \) with purely discrete spectrum and an eigenstate \( |\phi\rangle \) of \( H_E \) with eigenvalue 0 such that the following condition holds:

There exists a unitary \( U \) on \( \mathbb{C}^n \otimes \mathcal{H}_E \) commuting with the extended Hamiltonian \( H \otimes 1 + 1 \otimes H_E \) which satisfies

\[
G(\rho) = \text{tr}_2(U(\rho \otimes |\phi\rangle \langle \phi|)U^\dagger)
\]

for all density matrices \( \rho \).

Proof: We assume without loss of generality that \( H \) is diagonal with respect to the canonical basis. Let

\[
G(\rho) = \sum_{j=1}^{k} A_j \rho A_j^\dagger
\]

be the Kraus representation of \( G \) (see [29]). Define \( \Sigma := \{ x - y \mid x, y \in \text{spec}(H) \} \) where \( \text{spec}(H) \) denotes the spectrum of \( H \). As shown in (eqs. (14) in [12]) we can choose the Kraus operators such that for every \( A_j \) there is some real number \( \sigma_j \in \Sigma \) with

\[
[H, A_j] = \sigma_j A_j.
\]

In other words, the operator \( A_j \) implements a shift of energy values by \( \sigma_j \) in the sense that it maps eigenstates of \( H \) with eigenvalue \( \lambda \) onto states with energy \( \lambda + \sigma_j \). The idea is to choose a unitary extension such that the energy shift caused by \( A_j \) is compensated by the opposite shift in the environment.

Thus, we define the Hamiltonian \( H_E \) of the environment such that all values in \( \Sigma \) occur as spectral gaps in \( H_E \). Set \( \mathcal{H}_E := l^2(\mathbb{Z})^\otimes k \) and

\[
H_E = \sum_{j=1}^{k} \sigma_j M_j,
\]

where \( M_j \) is the multiplication operator acting on the \( j \)th component

\[
M_j := 1^\otimes j - 1 \otimes \text{diag}(\ldots, -1, 0, 1, \ldots) \otimes 1^\otimes k - j.
\]
Let \( S_j := 1^\otimes j - 1 \otimes S \otimes 1^\otimes k-j \) be the unitary left shift on \( l^2(\mathbb{Z}) \) acting on the \( j \)th tensor component via \( S|n\rangle := |n - 1\rangle \) for each \( n \in \mathbb{Z} \). Define
\[
U := \sum_{j=1}^{k} A_j \otimes S_j.
\]
To see that \( U \) is indeed unitary we consider basis states
\[
|l\rangle \otimes |z\rangle,
\]
where \( l = 0, \ldots, d-1 \) and \( z \) is in the \( k \)th fold cartesian product \( \mathbb{Z}^\times k \). They are all mapped onto unit vectors because \( \sum_j \langle l|A_jA_j^\dagger|l\rangle = 1 \). The images of different basis states are clearly mutually orthogonal whenever they correspond to different \( k \)-tuples \( z \). If they have \( z \) in common, they are also orthogonal since we obtain then the inner product
\[
\begin{align*}
\sum_j \langle l|A_jA_j^\dagger|\tilde{l}\rangle \langle z_1, \ldots, z_j + 1, \ldots, z_k |z_1, \ldots, z_j + 1, \ldots, z_k\rangle &= \\
\sum_j \langle l|A_jA_j^\dagger|\tilde{l}\rangle = \langle l|\tilde{l}\rangle &= 0.
\end{align*}
\]
To see that \( U \) commutes with the total Hamiltonian \( H_T := H \otimes 1 + 1 \otimes H_E \) we observe that for every eigenstate \( |l\rangle \) of \( H \) with eigenvalue \( \lambda_l \) we have
\[
A_j|l\rangle = |\phi_{l,j}\rangle,
\]
where \( |\phi_{l,j}\rangle \) is some state with
\[
H|\phi_{l,j}\rangle = (\lambda_l + \sigma_j)|\phi_{l,j}\rangle.
\]
We have
\[
(A_j \otimes S_j)(|l\rangle \otimes |z\rangle) = |\phi_{l,j}\rangle \otimes |z_1, \ldots, z_j - 1, \ldots, z_k\rangle,
\]
which is also an eigenstate of \( H_T \) for the eigenvalue \( \lambda + \sum_j \sigma_j z_j \) as \( |l\rangle \otimes |z\rangle \) is. That is, \( U \) maps energy basis states onto energy basis states with the same eigenvalues, i.e., it commutes with \( H_T \). We can now choose \( |\phi\rangle := |0\rangle \) as the state of the environment. \( \square \)
Note that the state $U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger$ appearing in the extension of Theorem 6 has the same energy distribution with respect to the extended Hamiltonian as $\rho$ has with respect to the original system Hamiltonian. This implies that the distribution of energy values in the joint state of system plus environment is given in terms of the distribution of input and output energies. Hence we may now apply our bounds on the information deficit to the problem of transmitting the states when only limited quantum capacity is available:

**Theorem 5 (Information Loss and Quantum Capacity)**

Given a covariant completely positive map $G$ with private channel capacity $C_p(G)$. Then the difference between the timing information of input and output satisfies

$$\mathcal{I}_{in} - \mathcal{I}_{out} \geq \frac{(\Delta E_{in})^8}{64(9\langle E_{out}^4 \rangle + 8\langle E_{in}^4 \rangle)} - \frac{1}{2} C_p(G),$$

where $(\Delta E_{in})^2$ and $\langle E_{in}^4 \rangle$ refer to the variance and the fourth moment of the incoming signal and similarly, $\langle E_{out}^4 \rangle$ denotes the fourth moment of the outgoing signal.

**Proof:** Construct a unitary energy conserving extension of $G$ according to Lemma 6. Let $E_{out} := X$ denote the energy of the output signal and $Y$ the energy of the environment. This implies that $E_{in} := X + Y$ is the initial energy. To get a bound for $\langle Y^4 \rangle = \langle (E_{in} - E_{out})^4 \rangle$ we use $|E_{in} - E_{out}| \leq |E_{in}| + |E_{out}|$ and hence $(E_{in} - E_{out})^4 \leq 8(E_{in}^4 + E_{out}^4)$. Then we obtain the statement using Theorem 2. □

### 7 Implications for the Energy Loss

In this section we want to explain why we expect the broadcasting problem to be specific to low-power devices. One reason is, certainly, that in current technology, information processing devices are not Hamiltonian systems. Since the system is not closed, a unitary description of the signal propagation is not justified. Furthermore, quantum broadcasting gets only relevant when the time inaccuracy of a clock signal is not dominated by classical noise of highly mixed density operators. In the latter case, the energy-time uncertainty is irrelevant. This is in agreement with the results
in Ref. [9] showing (in terms of Fisher-information) that quantum bounds on broadcasting timing information get relevant when the signal energy times the considered timing accuracy is on the scale of \( \hbar \). However, there is also another link between energy consumption of information processing devices and broadcasting problems that we have not mentioned before. The idea is that loss of timing information inevitably leads to loss of free energy in covariant devices. This is shown in [14]. We describe the relevant results.

First, we need the notion of passive devices, i.e., devices having no additional energy source apart from the considered incoming signal. In other words, all energy resources are explicitly included into the description.

**Definition 3 (Passive Device)**

A device with quantum input state \( \rho \) and output \( G(\rho) \) is called passive if \( G \) is implemented without energy supply, i.e.,

\[
F(G(\rho)) \leq F(\rho) \quad \forall \rho
\]

where \( F(\rho) := \text{tr}(\rho H) - kT S(\rho) \) is the free energy of the system in the state \( \rho \) with reference temperature \( T \) and Boltzmann constant \( k \).

We have shown in [14] that covariant passive channels that decrease the timing information decrease also the free energy. We rephrase this result formally.

**Theorem 6 (Loss of Timing Information Implies Free Energy Loss)**

Let \( G \) be a completely positive trace-preserving map describing a covariant passive device. The free energy loss caused by \( G \) can be bounded from below by the loss of timing information:

\[
F(\rho) - F(G(\rho)) \geq kT \left( I(\rho) - I(G(\rho)) \right).
\]

This shows that the channel can only be thermodynamically reversible if it does not subject the signal to a stochastically fluctuating time delay, i.e., it has to conserve the timing information. The result is less trivial than it may seem at first sight. The increase of signal entropy caused by the additional time delay could in principle be compensated by an increase of its inner energy such that the free energy of the system is conserved. The covariance condition is indeed required to show [14] that the free energy splits up into the following two components

\[
F(\rho) = kT I(\rho) + F(\bar{\rho}),
\]
which cannot be converted into each other.

Together with Theorem 6 we even obtain statements of the thermodynamical irreversibility of the signal transmission:

**Theorem 7 (Free Energy Loss in Classical Channels)**

Let \( \rho \) be a quantum state whose timing information has the broadcasting loss \( \Delta_{\text{min}} \). Then every channel \( G \) satisfies

\[
C_p(G) \geq 2\left( \Delta_{\text{min}} - \frac{1}{kT}(F(\rho) - F(G(\rho))) \right).
\]

In particular, for every channel with capacity \( C_p(G) = 0 \) we have

\[
F(\rho) - F(G(\rho)) \geq \frac{2}{kT}\Delta_{\text{min}}.
\]

We may combine Theorem 7 and Theorem 5 and obtain the following result:

**Theorem 8 (Free Energy Conservation and Quantum Capacity)**

Given a passive covariant device \( G \) with private channel capacity \( C_p(G) \). Let \( G \) be applied to a pure input state \( \rho \). Then the free energy loss caused by applying \( G \) to \( \rho \) satisfies

\[
F(\rho) - F(G(\rho)) \geq kT \left( \frac{(\Delta E_{\text{in}})^8}{64(9\langle E_{\text{out}}^4 \rangle + 8\langle E_{\text{in}}^4 \rangle \langle E_{\text{in}}^4 \rangle)} - \frac{1}{2} C_p(G) \right),
\]

with \( E_{\text{in}} \) and \( E_{\text{out}} \) as in Theorem 5.

It would be desirable to find similar results for mixed states. However, it seems to be hard to provide general bounds. Nevertheless, Theorem 8 shows why time covariance brings aspects of quantum information theory into the theory of low-power signal processing. In the context of synchronization protocols we have already described in [30] why covariance gives rise to additional limitations of thermodynamically reversible information transfer with classical channels.
8 Conclusions

We have described a quantum broadcasting problem that arises naturally in classical low power signal processing. If a time-invariant device transmits a signal such that the output signal contains the same amount of Holevo-information about an absolute time frame as the input the following two alternatives are possible: Either the channel has non-zero quantum capacity or it has internally solved a quantum broadcasting problem and copied the same amount of information to its environment. But this is not possible provided that (as we conjecture) the Holevo-information of non-commuting ensembles cannot be broadcast without loss. It is therefore likely that the time-covariant transmission of signals in a way that causes no stochastic time delay of the signal requires devices with non-zero quantum capacity. But avoiding stochastic time delays is, as we have argued a necessary requirement in order to avoid loss of free energy. Thus, we have described a link between quantum information theory and the theory of classical low-power processing.

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