SEMIPERFECT AND COREFLEXIVE COALGEBRAS

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Abstract. We study non-counital coalgebras and their dual non-unital algebras, and introduce the finite dual of a non-unital algebra. We show that a theory that parallels in good part the duality in the unital case can be constructed. Using this, we introduce a new notion of left coreflexivity for counital coalgebras, namely, a coalgebra is left coreflexive if $C$ is isomorphic canonically to the finite dual of its left rational dual $\text{Rat}(C^\ast C^\ast)$. We show that right semiperfectness for coalgebras is in fact essentially equivalent to this left reflexivity condition, and we give the connection to usual coreflexivity. As application, we give a generalization of some recent results connecting dual objects such as quiver or incidence algebras and coalgebras, and show that Hopf algebras with non-zero integrals (compact quantum groups) are coreflexive.

1. Introduction and Preliminaries

Let $\mathbb{K}$ be a field and let $\Gamma$ be a quiver. Two combinatorial objects that are important in representation theory are associated with $\Gamma$: the quiver algebra $\mathbb{K}[\Gamma]$, and the path coalgebra $\mathbb{K}\Gamma$. The quiver algebra is an algebra with enough idempotents, but does not have a unit unless $\Gamma$ is finite, while the path coalgebra $\mathbb{K}\Gamma$ is a coalgebra with counit. In [5] it was investigated how one of the two objects can be recovered from the other one; a finite dual coalgebra $A^0$ was constructed for an algebra $A$ with enough idempotents, and it was proved that $\mathbb{K}\Gamma$ is isomorphic to $\mathbb{K}[\Gamma]^0$ provided that $\Gamma$ has no oriented cycles and between any two vertices of $\Gamma$ there are finitely many arrows. On the other hand, the algebra $\mathbb{K}[\Gamma]$ can be recovered as the left (or right) rational part of $(\mathbb{K}\Gamma)^\ast$, provided that for any vertex $v$ of $\Gamma$ there are finitely many paths starting at $v$ and finitely many paths ending at $v$ (see [5]). A consequence of these results is that $\mathbb{K}\Gamma$ is coalgebra isomorphic to $((\mathbb{K}\Gamma)^{rat})^0$ for certain general enough $\Gamma$. Parallel results are obtained for incidence (co)algebras of a locally finite partially ordered set $X$, in which case the role of $\mathbb{K}\Gamma$, $(\mathbb{K}\Gamma)^\ast$ and $\mathbb{K}[\Gamma]$ is played by the incidence coalgebra $\mathbb{K}X$, the incidence algebra $IA(X)$, and the finite incidence algebra $FIA(X)$, consisting of functions in $IA(X)$ of finite support. It is natural to ask whether a general context that unifies these structures and results exists, for arbitrary algebras and coalgebras.

The first step in obtaining such a context is having a well behaved duality for non-unital algebras and coalgebras, that would naturally include the (co)unital case. We extend the construction of the finite dual coalgebra $A^0$ to the case of an arbitrary algebra $A$, not necessarily with enough idempotents, and extend the coalgebra theory to coalgebras without counit. We show that several results on coalgebras, including the fundamental theorem of coalgebras, carry over to non-counital coalgebras. These are needed and used for the above mentioned unification, but may also present some interest in their own. Some of the results from coalgebra theory can be extended to the non-counital case with parallel proofs, but other need new arguments avoiding the use of the counit property. We explain how some results in the non-counital case can be obtained by using a counitalization construction for coalgebras, and the transfer of properties.

2010 Mathematics Subject Classification. 16T15, 16T05, 05C38, 06A11, 16T30.

Key words and phrases. non-unital algebra, semiperfect coalgebra, coreflexive coalgebra, coalgebra, Hopf algebra with non-zero integral.
through it. In Section 3 we construct \( A^0 \) for an arbitrary algebra \( A \), and we give equivalent characterizations for it. As in the unital case, we show that locally finite representations of \( A \) are just corepresentations over its finite dual \( A^0 \).

(Co)reflexivity for algebras and coalgebras was introduced and studied by R. Heyneman, D.E. Radford and E. Taft ([6, 14, 18, 19]) in tight connection to the duality between algebras and coalgebras. A coalgebra \( C \) with counit is called coreflexive if the canonical map \( C \to (C^*)^0 \) is bijective. In other words, the coalgebra is recovered from its dual algebra via the finite dual construction, i.e., it is its own double dual. Coreflexivity is a finiteness condition, since it is shown to be equivalent to every finite dimensional left (equivalently, right) \( C^* \)-module being rational. Motivated by finding a unifying context for the above mentioned combinatorial results connecting the algebra and coalgebra structures associated with quivers and partially ordered sets, in Section 4 we consider another coreflexivity-type condition for a coalgebra \( C \) with counit.

Instead of looking at the dual algebra of \( C \), we look at the rational left (or right) dual of \( C \), regarded as a non-unital algebra \( \text{Rat}(C{-}C^*) \) (or \( \text{Rat}(C^*_r) \)), and consider the natural map \( \phi_0 : C \to (\text{Rat}(C{-}C^*))^0 \). We call a coalgebra \( C \) left coreflexive if \( \phi_0 \) is an isomorphism. We first note a connection with another well studied property for coalgebras, namely, the morphism \( \phi_0 \) is injective if and only if \( C \) is right semiperfect. Recall that a coalgebra is right semiperfect if the category of right comodules has enough projective objects, or equivalently, indecomposable injective left comodules are finite dimensional. Moreover, if \( C \) is right semiperfect and the coradical \( C_0 \) is coreflexive, we prove that \( \phi_0 \) is an isomorphism. In this case, we show that \( C \) is also coreflexive in the sense of Radford and Taft. This situation appears in [5], for the case of coalgebras associated to certain general quivers or PO-sets. We note that the condition that \( C_0 \) is coreflexive is not very restrictive, and in fact \( C_0 \) is essentially always coreflexive if \( \mathbb{K} \) is infinite. Thus, this effectively gives an unexpected interpretation of semiperfect coalgebras, as equivalent to a coreflexivity (or self double dual) type of condition. Finally, we give an example showing that a coreflexive coalgebra is not necessarily left (nor right) coreflexive.

If \( C \) is left and right semiperfect, we show that \( \phi_0 \) is always bijective (independent of the supplementary assumption on \( C_0 \)), thus unifying and extending the results about path coalgebras and incidence coalgebras mentioned above.

We also give an application and interpretation of our results as a a double dual property of Hopf algebras with non-zero integrals; many interesting quantum groups are such Hopf algebras.

There are many well known duality pairings for classes of Hopf algebras and quantum groups. Let \( G \) be an algebraic group over \( \mathbb{C} \), and let \( L \) be its Lie algebra. Let \( H \) be the Hopf algebra of functions on \( G \). If \( G \) is simply connected, then \( H \) is the finite dual of \( U(L) \), i.e., \( H \cong U(L)^0 \). This duality is also preserved at the quantum level, for example, there is an isomorphism \( SL_n(q) \cong U(sl_n(q))^0 \) ([10]). The group \( G \) can be recovered as the group \( G(H^0) \) of grouplike elements in \( H^0 \), and, in fact, we again have an isomorphism \( H^0 \cong \mathbb{C}[G] \succsim U(L) \), a smash product, by the theorems of Cartier, Gabriel, Kostant, Milnor and Moore (see also [3] Remark 3.7.3). For example, when \( G \) is the additive algebraic group \( \mathbb{C} \), then \( H = \mathbb{C}[X] \), and \( H^0 = \mathbb{C}[G] \succsim U(L) \), where \( L \) is the abelian Lie algebra of dimension 1. This example is considered also in [18] page 1123 (for countable algebraically closed fields), but one sees that the coalgebra (Hopf algebra) \( H^0 \) for \( H = \mathbb{C}[X] \) here is coreflexive (see [18]).

It is natural to ask if there is some reflexivity (duality) property of any of these classes of algebras. One motivation is the duality between commutative Hopf algebras and algebraic groups, as noted above: \( G \leftrightarrow H = O(G) = \text{algebra of functions on } G \), \( H \leftrightarrow G = G(H^0) \). In fact the above mentioned example of coreflexive Hopf algebra in [18] holds more generally for every finite dimensional Lie algebra \( L \). Namely, using results in [6, 9], it is not difficult to see
that if $L$ is a finite dimensional (say over $\mathbb{C}$), $U(L)$ is coreflexive. Moreover, with notations for $G$ and $H$ as above, we see that $H^0 = \bigoplus_G U(L)$ is a coreflexive coalgebra too (see Proposition [4.7]).

We show that there is such a reflexivity for Hopf algebras with nonzero integral, which are a generalization of algebras of functions on compact groups (i.e. compact quantum groups). Namely, we observe that such Hopf algebras are coreflexive in all the possible meanings, i.e. both in the sense of Lin, Radford and Taft, provided that the set of simple subcoalgebras satisfy a certain non-restrictive condition (which is always true over infinite fields), as well as with respect to our new reflexivity notion.

We work over a field $\mathbb{K}$. By algebra we mean an associative algebra, not necessarily with unit. For such an algebra $A$ we denote by $A - Mod$ the category of left $A$-modules (with no unital condition, not even in the case where $A$ has a unit). If $A$ is an algebra with unit, we denote by $\mathcal{A}Mod$ the category of unital left $A$-modules. By coalgebra we mean a coassociative coalgebra which does not necessarily have a counit. If $C$ is such a coalgebra, with comultiplication $\Delta$, then a right $C$-comodule is a space $M$ together a linear map $\rho : M \to M \otimes C$ such that $(\rho \otimes I)\rho = (I \otimes \delta)\rho$. The category of right $C$-comodules is denoted by $Comod - C$. If $C$ has a counit, then the category of counital $C$-comodules (i.e. satisfying the counit property) is denoted by $\mathcal{M}C$. For basic terminology and notation about coalgebras and comodules we refer to [4], [12], and [16].

2. Non-counital coalgebras

If $A$ is an algebra, then the unitalization of $A$ is the algebra $A^1$ with identity, obtained by adjoining to $A$ a unit element $u$, i.e. $A^1 = A \oplus \mathbb{K}u$, with multiplication defined by $(a + \alpha u)(b + \beta u) = ab + \alpha b + \beta a + \alpha \beta u$ for any $a, b \in A$ and $\alpha, \beta \in \mathbb{K}$. Then the inclusion map $i : A \to A^1$ is an algebra map satisfying the following universal property. If $B$ is an algebra with unit, then any algebra map $f : A \to B$ extends uniquely to a map $F : A^1 \to B$ of algebras with unit. The categories $A - Mod$ and $A^1 Mod$ are isomorphic. Indeed, a left $A$-module $M$ can be viewed as a unital left $A^1$-module by extending the action of $A$ with the condition that $u$ acts as identity on $M$, while a unital left $A^1$-module is a left $A$-module by restricting scalars via $i$. Moreover, the lattices $\mathcal{L}_{A-Mod}(M)$ and $\mathcal{L}_{A^1 Mod}(M)$ of subobjects of $M$ in these two categories coincide. We also note that any finite dimensional algebra $A$ embeds in a matrix algebra. Indeed, if $\dim(A) = n$, then $\pi : A \to \text{End}(A^1) \cong M_{n+1}(\mathbb{K})$, $\pi(a)(z) = az$ for any $a \in A$, $z \in A^1$, is an injective morphism of algebras.

We show that there are dual constructions and results for coalgebras.

**Proposition 2.1.** (i) Every coalgebra is a quotient of a coalgebra with counit. (ii) Every finite dimensional coalgebra $(C, \delta)$ is a quotient of a comatrix coalgebra. Consequently, there are elements $(c_{ij})_{i,j=1,...,n}$ which span $C$ such that $\delta(c_{ij}) = \sum_{k=1}^{n} c_{ik} \otimes c_{kj}$.

**Proof.** (i) Let $(C, \delta)$ be a coalgebra, and write $\delta(c) = \sum c_1 \otimes c_2$. Let $C^1 = C \oplus \mathbb{K}e$, the direct sum of $C$ and a 1-dimensional space. Then it is easy to check that $C^1$ has a structure of a coalgebra with counit with comultiplication $\Delta$ defined by

$$\Delta(c + \alpha e) = c \otimes e + e \otimes c + \alpha e \otimes e + \sum c_1 \otimes c_2$$

and counit $\epsilon$, $\epsilon(c + \alpha e) = \alpha$ for any $c \in C, \alpha \in \mathbb{K}$. Moreover, the map $\pi : C^1 \to C$ defined by $\pi(c + \alpha e) = c$, is a surjective morphism of coalgebras.

(ii) The dual algebra $C^*$ embeds in a matrix algebra $M_n(\mathbb{K})$, and then $C$ is a quotient of the
matrix coalgebra $M_n^c(\mathbb{K})$. For the last part, if $\theta : M_n^c(\mathbb{K}) \to C$ is a surjective morphism of coalgebras and $e_{ij}$ is a comatrix basis of $M_n^c(\mathbb{K})$, then it suffices to take $c_{ij} = \theta(e_{ij})$. □

The following describes basic properties of the construction $C \mapsto C^1$, the "counitalization" of a coalgebra.

**Proposition 2.2.** Let $(C, \delta)$ be a coalgebra, and let $(C^1, \Delta, \epsilon)$ be the coalgebra with counit constructed in Proposition 2.1 with the coalgebra map $\pi : C^1 \to C$. The following assertions are true.

(i) For any coalgebra with counit $D$ and any coalgebra morphism $f : D \to C$, there exists a unique morphism of coalgebras with counit $\overline{f} : D \to C^1$ such that $\pi \overline{f} = f$. Thus the counitalization functor $(-)^1$ is a right adjoint to the forgetful functor from the category of counital coalgebras to the category of coalgebras.

(ii) The category $\text{Comod} - C$ of right $C$-comodules is isomorphic to the category $\mathcal{M}C^1$-comodules via the corestriction of scalars functor associated to $\pi$.

Proof. (i) It is easy to check that the map $\overline{f} : D \to C^1$ defined by $\overline{f}(d) = f(d) + \epsilon_D(d)e$ for any $d \in D$ satisfies the required conditions.

(ii) If $M$ is a right $C$-comodule with comodule structure map $\rho : M \to M \otimes C$, then $M^1$ can be viewed as a counital right $C^1$-comodule with comodule structure map $\rho^1 : M \to M \otimes C^1$, $\rho^1(m) = \rho(m) + m \otimes e$. By corestricting scalars for this $C^1$-comodule via $\pi$, we obtain the initial $C$-comodule $M^1$.

Now if $M$ is a right $C^1$-comodule with comodule structure map $m \mapsto \sum m_{[0]} \otimes m_{[1]}$, then it becomes a right $C$-comodule via $\pi$, and further a right $C^1$-comodule with comodule structure map $m \mapsto m \otimes e + \sum m_{[0]} \otimes \pi(m_{[1]})$. If we write $m_{[1]} = m'_{[1]} + m''_{[1]} e$, with $m'_{[1]} \in C, m''_{[1]} \in \mathbb{K}$, then $m'_{[1]} = \pi(m_{[1]})$ and $m = \sum m'_{[1]} m_{[0]}$ by the counit property. Therefore, $m \otimes e + \sum m_{[0]} \otimes \pi(m_{[1]}) = \sum m_{[0]} m_{[1]} \otimes e + \sum m_{[0]} \otimes m'_{[1]} = \sum m_{[0]} \otimes m_{[1]}$, so we obtain the initial $C^1$-comodule structure on $M^1$.

It is clear that the subobjects of $M$ in the two categories are the same. □

**Corollary 2.3.** Let $C$ be a coalgebra, and $(M, \rho)$ a right $C$-comodule. Then the subcomodule generated by any element $m \in M$ is finite dimensional.

Proof. The result is known for counital comodules over coalgebras with counit (see [4, Theorem 2.1.7]). Now everything is clear since $\mathcal{L}_{\text{Comod}-C}(M) = \mathcal{L}_{\mathcal{M}C^1}(M)$ shows that the $C$-subcomodule of $M$ generated by $m$ is the same to the $C^1$-subcomodule of $M$ generated by $m$. □

**Proposition 2.4.** Let $C$ be a coalgebra. Then $(C^*)^1 \simeq (C^1)^*$ as algebras with unit.

Proof. Let $\pi : C^1 \to C$ be the natural projection and $\pi^* : C^* \to (C^1)^*$ its dual map. We show that the pair $((C^1)^*, \pi^*)$ satisfies the universal property of the unitalization of the algebra $C^*$, and the desired isomorphism follows. If $B$ is an algebra with unit $1_B$, and $f : C^* \to B$ is an algebra map, let $\overline{f} : (C^1)^* \to B$ be defined by $\overline{f}(\phi) = f(\phi|_C) + \phi(e)1_B$ for any $\phi \in (C^1)^*$, where $\phi|_C$ is the restriction of $\phi$ to $C$, where we consider $C^1 = C \oplus \mathbb{K}e$. It is then easy to show $\overline{f}$ is a map of algebras with identity satisfying $\overline{f}\pi^* = f$, and that $\overline{f}$ is unique with these properties.

Alternatively, one can show the isomorphism directly using the construction of the unitalization and counitalization. □

**Corollary 2.5.** Let $C$ be a (not necessarily counital) coalgebra, and let $M$ be a right $C$-comodule. Then the lattices $\mathcal{L}_{\text{Comod}-C}(M)$ and $\mathcal{L}_{C^{-\text{Mod}}}(M)$ coincide.
Proof. We consider the functors

\[ \text{Comod} - C \rightarrow \mathcal{M}^{C^1} \rightarrow (\mathcal{C}^{1})^* \rightarrow (C^*)^1 \mathcal{M} \rightarrow C^*_{-\text{Mod}} \mathcal{M} \]

where the first functor is the isomorphism of categories from Proposition 2.2 (ii), the second functor is the usual one that regards comodules as modules over the dual algebra, the third functor is the isomorphism of categories associated to the isomorphism of algebras from Proposition 2.3, and the fourth functor is the isomorphism of categories explained at the beginning of this section for an arbitrary algebra. It is easy to check that the composition of these functors is just the usual functor \( \text{Comod} - C \rightarrow C^*_{-\text{Mod}} \mathcal{M} \). It follows that

\[ \mathcal{L}_{\text{Comod} - C}(M) = \mathcal{L}_{\mathcal{M}^{C^1}}(M) = \mathcal{L}_{(\mathcal{C}^{1})^* \mathcal{M}}(M) = \mathcal{L}_{(C^*)^1 \mathcal{M}}(M) = \mathcal{L}_{C^*_{-\text{Mod}}}(M) \]

Next we note that the result stating that the category of right \( C \)-comodules is isomorphic to the category of rational left \( C^* \)-modules holds in the non-counital case, too. The proof is as in the case where \( C \) has counit, see for example [4, Theorem 2.2.5].

Now we can extend the fundamental theorem of coalgebras to the non-counital case.

**Proposition 2.6.** Let \( C \) be a coalgebra (not necessarily with counit). Then the subcoalgebra generated by an element \( c \in C \) is finite dimensional.

Proof. A subcoalgebra of \( C \) is the same as a sub-bicomodule of \( C \). Regarding \( C \)-bicomodules as \( C \otimes C^0 \)-right comodules, the statement follows from Corollary 2.3 \( \square \)

### 3. The finite dual of a non-unital algebra

If \( A \) is an algebra with identity, then \( A^0 \) is the subspace of \( A^* \) consisting of all elements \( f \) with the property that \( \ker(f) \) contains a two-sided ideal of \( A \) of finite codimension. This property of \( f \) is equivalent to the existence of two finite families \((g_j)_j, (h_j)_j \in A^* \) such that \( f(ab) = \sum g_j(a)h_j(b) \) for any \( a, b \in A \). For such an \( f \), the families \((g_j)_j\) and \((h_j)_j\) can be chosen in \( A^0 \), and the mapping \( f \mapsto \sum g_j \otimes h_j \) defines a coassociative comultiplication on \( A^0 \). Thus \( A^0 \) is a coalgebra with counit \( f \mapsto f(1_A) \), called the finite dual of \( A \).

In this section, we extend the construction of the finite dual of an algebra to the non-unital case. Let \( A \) be an arbitrary algebra, \( A^1 \) its unitalization, and \( i : A \rightarrow A^1 \) the inclusion map. Let \( i^* : (A^1)^* \rightarrow A^* \) be the dual map. We consider the finite dual \( (A^1)^0 \) of \( A^1 \), which has a structure of a coalgebra with counit as above, with comultiplication denoted by \( \Delta \).

**Proposition 3.1.** \( i^*((A^1)^0) \) has a unique coalgebra structure making the restriction of \( i^* \) to \( (A^1)^0 \) a coalgebra map.

Proof. We show that there exists a unique linear map \( \delta : i^*((A^1)^0) \rightarrow i^*((A^1)^0) \otimes i^*((A^1)^0) \) making the following diagram commutative.

\[
\begin{array}{ccc}
(A^1)^0 & \xrightarrow{\Delta} & (A^1)^0 \otimes (A^1)^0 \\
\downarrow i^* & & \downarrow i^* \otimes i^* \\
i^*((A^1)^0) & \xrightarrow{\delta} & i^*((A^1)^0) \otimes i^*((A^1)^0)
\end{array}
\]

Note that for simplicity we kept the notation \( i^* \) for the restriction of this map to \( (A^1)^0 \). Indeed, let \( f \in (A^1)^0 \) such that \( i^*(f) = 0 \), i.e. the restriction \( f|_A \) of \( f \) to \( A \) is zero. Let \( \Delta(f) = \sum j g_j \otimes h_j \).
thus \( f(st) = \sum_j g_j(z)h_j(t) \) for any \( z, t \in A^1 \). Then \((i^* \otimes i^*)\Delta(f) = \sum_j g_j|A \otimes h_j|A\). If \( \theta : A^* \otimes A^* \to (A \otimes A)^* \) is the natural embedding, then \( \theta(\sum_j g_j|A \otimes h_j|A)(a \otimes b) = \sum_j g_j(a)h_j(b) = f(ab) = 0 \). We obtain that \((i^* \otimes i^*)\Delta(f) = 0\), and so the existence and uniqueness of \( \delta \) follows.

Now we have that

\[
(\delta \otimes I)\delta i^* = (i^* \otimes i^* \otimes i^*)(\Delta \otimes I)\Delta
\]

and

\[
(I \otimes \delta)\delta i^* = (i^* \otimes i^* \otimes i^*)(I \otimes \Delta)\Delta
\]

Since \( \Delta \) is coassociative and \( i^* \) is an epimorphism, we must have \((\delta \otimes I)\delta = (I \otimes \delta)\delta \). \( \square \)

We can derive the following description of \( i^*((A^1)^0) \).

**Proposition 3.2.** \( i^*((A^1)^0) \) is the set of all elements \( f \in A^* \) such that \( \text{Ker}(f) \) contains a two-sided ideal of \( A \) of finite codimension.

**Proof.** If \( f \in i^*((A^1)^0) \), then \( f = g_i|A \) for some \( g \in (A^1)^0 \). Let \( J \) be a two-sided ideal of \( A^1 \) of finite codimension such that \( J \subseteq \text{Ker}(g) \). Then \( \text{Ker}(f) \) contains \( A \cap J \), a two-sided ideal of \( A \) of finite codimension.

Conversely, if \( \text{Ker}(f) \) contains a two-sided ideal \( I \) of \( A \) of finite codimension, let \( g \in (A^1)^* \) be a linear map such that \( g_i|A = f \). Then \( I \) is also a two-sided ideal of \( A^1 \), it has finite codimension in \( A^1 \), and \( I \subseteq \text{Ker}(g) \). Thus \( g \in (A^1)^0 \) and \( f = i^*(g) \in i^*((A^1)^0) \). \( \square \)

We denote \( i^*((A^1)^0) \) by \( A^0 \), and we call the coalgebra \((A^0, \delta)\) the finite dual of \( A \). **Proposition 3.2** and the description of \( \delta \) in **Proposition 3.1** shows that there is no inconsistency in the notation, since in the case where \( A \) is an algebra with identity, \( i^*((A^1)^0) \) is just the usual finite dual of the algebra with identity \( A \).

If \( \phi : A \to B \) is an algebra map, then \( \phi^*(B^0) \subseteq A^0 \). Indeed, this follows from the fact that the inverse image of a finite codimensional two-sided ideal of \( B \) through \( \phi \) is a finite codimension two-sided ideal of \( A \). We denote by \( \phi^0 : B^0 \to A^0 \) the map induced by \( \phi^* \). The mappings \( A \to A^0 \) and \( \phi \mapsto \phi^0 \) define a contravariant functor \((-)^0 : \text{alg} \to \text{coalg} \), where \( \text{alg} \) is the category of algebras, and \( \text{coalg} \) is the category of coalgebras. As in the (co)unital case (see for example [1, Theorem 1.5.22]) one sees that \((-)^0 \) is a left adjoint for the ”dual algebra” functor \((-)^* : \text{coalg} \to \text{alg} \).

Next we give several characterizations of the finite dual \( A^0 \). Some of their proofs carry over from the unital case in a straightforward way, but some other need more attention.

We first introduce a notation. If \( \eta : A \to \text{End}(V) \) is a finite dimensional representation of \( A \), let \( v_1, \ldots, v_n \) be a basis of \( V \) and let \( \phi : \text{End}(V) \cong M_n(\mathbb{K}) \) be the algebra isomorphism associated to this basis. Let \( \rho = \phi \eta : A \to M_n(\mathbb{K}) \) be the resulting matrix representation, and \( \rho_{ij} \) be the coefficient functions of \( \rho \). Thus \( \rho(a) = (\rho_{ij}(a))_{i,j} \) and \( \eta(a)(v_j) = \sum_i \rho_{ij}(a)v_i \) for any \( a \in A \) and \( 1 \leq i, j \leq n \). Let \( R(A) \subset A^* \) be the span of all possible such coefficient functions \( \rho_{ij} \), for arbitrary representations \( \eta \) and arbitrary choice of basis \( (v_i) \). This is called the set of representative functions on \( A \). In the unital case, this is a classical notion, with roots in representations of compact groups, and of algebraic groups. For such groups, it is well known now that the space of representative functions is the same as the representative functions on an apropriate (group) algebra, and, for compact and algebraic groups (or group schemes), it forms a Hopf algebra. Now we can give the equivalent characterizations of \( A^0 \); the equivalent statements of the following proposition are well known for unital algebras.
Proposition 3.3. Let $A$ be an algebra (not necessarily with unit), and let $f : A \to \mathbb{K}$ be a linear map. The following are equivalent.

(i) $f \in A^0$.

(ii) $\ker(f)$ contains a left ideal of finite codimension.

(iii) $\ker(f)$ contains a right ideal of finite codimension.

(iv) $Af$ is finite dimensional.

(v) $fA$ is finite dimensional.

(vi) The bimodule generated by $f$, $\mathbb{K}f + Af + fA + AfA$, is finite dimensional.

(vii) There exist $g_i, h_i \in A^*, i = 1, \ldots, n$ such that $f(ab) = \sum_{i=1}^n g_i(a)h_i(b)$ for any $a, b \in A$.

(viii) $f \in R(A)$.

Proof. The equivalence of the first seven conditions can be proved by adapting the arguments from the unital case, see for example the proof of [12, Lemma 9.1.1]. One change is that the $A$-submodule generated by $f$ in the left $A$-module $A^*$ is $Af + \mathbb{K}f$, which change does not affect the finiteness of the dimension. Similarly $fA$ must be replaced by $fA + \mathbb{K}f$, and $AfA$ by $AfA + Af + fA + \mathbb{K}f$ in the proof. Another difference is at (ii)⇒(i), where we have to show that a left ideal $I$ of finite codimension contains a two-sided ideal of finite codimension. For this, we take the finite dimensional left $A$-module $A/I$, and the associated representation $\phi : A \to \text{End}(A/I)$, $\phi(a)(\hat{b}) = \hat{ab}$, where the hat indicates the class modulo $I$. Then it is easy to check that $I \cap \ker(\phi)$ is a two-sided ideal of $A$ of finite codimension.

(viii)⇒(i) It is enough to show that any coefficient functions $\rho_{ij}$ associated to a representation of $A$ lies in $A^0$. Since $\rho(ab) = \rho(a)\rho(b)$, we obtain that $\rho_{ij}(ab) = \sum_r \rho_{ir}(a)\rho_{rj}(b)$ for any $a, b \in A$, so $\rho_{ij} \in A^0$.

(i)⇒(viii) Let $I$ be a two-sided ideal contained in $\ker(f)$. For a linear map $h : A \to \mathbb{K}$ with $I \subseteq \ker(h)$ write $\overline{h}$ for the induced map to $A/I$. Let $B = (A/I)^1$ be the unitalization of $A/I$, and let $u : A/I \hookrightarrow B$ be the inclusion map. Let $v_1, \ldots, v_n$ be a basis of $B$. Consider $B$ as a representation of $A$, via $\eta : A \to \text{End}(B)$, and let $\rho : A \to M_n(\mathbb{K})$ be the corresponding matrix representation associated to the basis $v_1, \ldots, v_n$. Since $1_B = \sum i \alpha_j v_j$ for some scalars $\alpha_j$, we see that $\hat{a} = \sum j \alpha_j \hat{a} v_j = \sum j \alpha_j \overline{\pi}_ij(\hat{a}) v_i = \sum i \alpha_i \overline{\pi}_{ij}(\hat{a}) v_i$ for any $a \in A$, where $\hat{a}$ denotes the class of $a$ in $A/I$. Since $\hat{a} = \sum i \overline{v}_i^*(\hat{a}) v_i$, where $(\overline{v}_i^*)_i$ is the dual basis of $(v_i)_i$, we obtain that $v_i^*(\hat{a}) = \sum_j \overline{\pi}_{ij}(\hat{a})$, so $u^*(v_i^*) = \sum j \overline{\pi}_{ij}$. As $u^*$ is surjective, $(u^*(v_i^*))_i$, spans $(A/I)^*$, and then $(\overline{\pi}_{ij})_{i,j}$ also spans $(A/I)^*$. Therefore $\overline{f} = \sum_{i,j} \beta_{ij} \overline{\pi}_{ij}$ for some scalars $\beta_{ij}$, and this shows that $f = \sum_{i,j} \beta_{ij} \rho_{ij}$, since $I \subseteq \ker(f)$ and $I \subseteq \ker(\rho)$. Hence, $f \in R(A)$. \hfill \square

Thus, we note that $A^0 = R(A)$ for every not necessarily unital algebra. By the previous proposition, the coalgebra structure of $R(A)$ can also be obtained as $R(A) = \lim_{\rightarrow} (A/I)^*$, with the limit being taken over all cofinite ideals $I$.

The following shows that the unitalization, the counitalization and the finite dual functors are compatible in some sense.

Proposition 3.4. Let $A$ be an algebra. Then $(A^1)^0 \simeq (A^0)^1$ as coalgebras with counit.
Proof. Let \( i^0 : (A^1)^0 \to A^0 \) be the image of the inclusion \( i : A \to A^1 \) through the functor \((-)^0\). We show that \(( (A^1)^0 , i^0 ) \) satisfies the universal property of the counitalization \(( (A^0)^1 , \pi ) \) of \( A^0 \); the desired isomorphism (which moreover is compatible with \( i^0 \) and \( \pi \)) follows from this.

Let \( D \) be a coalgebra with counit, and let \( f : D \to A^0 \) be a map of coalgebras. Let \( \overline{f} : D \to (A^1)^0 \), 
\[ \overline{f}(d)(a + \alpha u) = f(d)(a) + \alpha \varepsilon_D(d) \]
for any \( d \in D, a \in A, \alpha \in \mathbb{K} \). Since \( f(d) \in A^0 \), there exists an ideal \( I \) of \( A \) of finite codimension with \( I \subseteq \text{Ker}(f(d)) \). Then \( I \) is also an ideal of finite codimension in \( A^1 \) and \( I \subseteq \text{Ker}(\overline{f}(d)) \), and this shows that indeed \( \overline{f}(d) \in (A^1)^0 \).

We now show that \( \overline{f} \) is a coalgebra map. Let \( \gamma : (A^1)^0 \otimes (A^1)^0 \to (A^1 \otimes A^1)^0 \) be the natural embedding. We have that
\[
(\gamma \Delta_{(A^1)^0}) \overline{f}(d)((a + \alpha u) \otimes (b + \beta u)) = \sum \overline{f}(d_1)(a + \alpha u)\overline{f}(d_2)(b + \beta u)
= \overline{f}(d)((a + \alpha u)(b + \beta u))
= \overline{f}(d)(ab + \alpha b + \beta a + \alpha \beta u)
= f(d)(ab + \alpha b + \beta a + \alpha \beta \varepsilon_D(d)
\]
and
\[
(\gamma(f \otimes \overline{f})) \Delta_D(d)((a + \alpha u) \otimes (b + \beta u)) = \sum \overline{f}(d_1)(a + \alpha u)\overline{f}(d_2)(b + \beta u)
= \sum (f(d_1)(a) + \alpha \varepsilon_D(d_1))(f(d_2)(b) + \beta \varepsilon_D(d_2))
= \beta f(d)(a) + \alpha f(d)(b) + \alpha \beta \varepsilon_D(d) + \sum f(d_1)(a)f(d_2)(b)
\]
Since \( \sum f(d_1)(a)f(d_2)(b) = \sum f(d_1)(a)f(d_2)(b) = f(d)(ab) \), we obtain that \( \gamma \Delta_{(A^1)^0} \overline{f} = \gamma(f \otimes \overline{f}) \Delta_D \). Since \( \gamma \) is injective, this shows that \( \Delta_{(A^1)^0} \overline{f} = (f \otimes \overline{f}) \Delta_D \), i.e. \( \overline{f} \) is a coalgebra map.

It is clear that \( i^0 \overline{f} = f \). Moreover, if \( \tilde{f} : D \to (A^1)^0 \) is another map of coalgebras with counit such that \( i^0 \tilde{f} = f \), then for any \( d \in D \) we have that \( \tilde{f}(d)(u) = \varepsilon_{(A^1)^0}(\tilde{f}(d)) = \varepsilon_D(d) \), and \( \tilde{f}(d)(a) = \tilde{f}(d)(i(a)) = (i^0 \tilde{f}(d))(a) = f(d)(a) \) for any \( a \in A \). We obtain that \( \tilde{f}(d)(a + \alpha u) = f(d)(a) + \alpha \varepsilon_D(d) = \overline{f}(d)(a + \alpha u) \), so \( \tilde{f} = \overline{f} \).

Now we give an extension of the result saying that locally finite unital representations of an algebra \( A \) with identity are just corepresentations of the finite dual \( A^0 \), see [1] Chapter 3, page 126. If \( A \) is an arbitrary algebra, a module \( M \) is called locally finite if the submodule generated by any element is finite dimensional. We denote by \( \text{Locfin } A - \text{Mod} \) the full subcategory of \( A - \text{Mod} \) whose objects are the locally finite modules.

**Proposition 3.5.** Let \( A \) be an algebra. Then the categories \( \text{Locfin } A - \text{Mod} \) and \( \text{Comod} - A^0 \) are isomorphic. In the case where \( A \) is unital, this isomorphism of categories restricts to an isomorphism between the category of locally finite unital left \( A \)-modules and the category of counital right \( A^0 \)-comodules.

**Proof.** Let \( M \) be a right \( A^0 \)-comodule, with comodule structure given by \( m \mapsto \sum m_0 \otimes m_1 \). Then \( M \) is a left \( A \)-module with action given by \( am = \sum m_1(a)m_0 \), and this is locally finite, since the \( A \)-submodule generated by \( m \) is \( Am + \mathbb{K}m \), which is finite dimensional since it is contained in the span of \( m \) and all \( m_0 \)'s.
Conversely, if $M$ is a locally finite $A$-module, let $m \in M$, and let $m_1, \ldots, m_n$ be a basis of $Am + \mathbb{K}m$, the $A$-submodule generated by $m$. Define $a_1^*, \ldots, a_n^* \in A^*$ by $am = \sum_{i=1}^{n} a_i^*(a)m_i$ for any $a \in A$. Then it is easy to check that $a_i^* \in A^0$ for any $i$, that $\sum_i m_i \otimes a_i^*$ does not depend on the choice of the basis $m_1, \ldots, m_n$, and the mapping $m \mapsto \sum_i m_i \otimes a_i^*$ defines a right $A^0$-comodule structure on $M$.

These correspondences are compatible with morphisms and they define an isomorphism of categories. The last part is straightforward. \hfill \Box

Let us consider now a bialgebra $H$, in the non-unital non-counital sense. This means that $H$ is an algebra (not necessarily with unit) and a coalgebra (not necessary with counit), such that the comultiplication is an algebra morphism. If moreover $H$ is a counital coalgebra and the comultiplication is an algebra morphism, then $H$ is called a counital coalgebra. Similarly one defines unital bialgebras.

**Proposition 3.6.** Let $H$ be a bialgebra. Then $H^0$, the finite dual of the underlying algebra structure, is a subalgebra of $H^*$. Moreover, $H^0$ is a bialgebra. If $H$ is counital (respectively unital), then $H^0$ is unital (respectively counital).

*Proof.* Let $f, g \in H^0$, and let $\delta(f) = \sum f_1 \otimes f_2$, $\delta(g) = \sum g_1 \otimes g_2$, where $\delta$ is the co-
multiplication of $H^0$. Then for any $a, b \in H$ we have that $(f(\delta)(ab)) = \sum (f(a_1b_1)g(a_2b_2) = \sum f_1(a_1)f_2(b_1)g_1(a_2)g_2(b_2) = \sum (f_1g_1)(a_1)(f_2g_2)(b_2)$, and this shows that $fg \in H^0$ and $\delta(fg) = \sum f_1g_1 \otimes f_2g_2$. Thus $H^0$ is a bialgebra. The last part is obvious. \hfill \Box

**Examples 3.7.**

1. Let $S$ be a semigroup. Then the semigroup algebra $\mathbb{K}S$ is a counital bialgebra. In fact, since $(\mathbb{K}S)^* \simeq \text{Fun}(S, \mathbb{K})$, the algebra of functions on $S$ (with values in $\mathbb{K}$), then $(\mathbb{K}S)^0$ is just the set $R_{\mathbb{K}}(S)$ of representative functions, i.e. the set of all functions $f : S \to \mathbb{K}$ for which there exist finite families of functions $(u_i)_i$ and $(v_i)_i$ on $S$ such that $f(xy) = \sum u_i(x)v_i(y)$ for any $x, y \in S$. By Proposition 3.6, $R_{\mathbb{K}}(S)$ is a unital bialgebra.

2. Let $\Gamma$ be a quiver and let $\mathbb{K}[\Gamma]$ be the associated quiver algebra, which is unital if and only if $\Gamma$ has finitely many vertices. It seems to be a difficult problem to describe $\mathbb{K}[\Gamma]^0$ explicitly for an arbitrary $\Gamma$. Indeed, consider the case of $\Gamma$ consisting of one vertex $v$ and $n$ arrows ($n$ loops at $v$). In this situation, $\mathbb{K}[\Gamma] \simeq \mathbb{K} < X_1, \ldots, X_n >$, the noncommutative algebra of polynomials in $n$ variables. Determining the finite dual of this algebra would involve the classifying in some way the cofinite ideals of $\mathbb{K} < X_1, \ldots, X_n >$, or equivalently, the annihilators of finite dimensional representations. Classification of finite dimensional $\mathbb{K} < X_1, \ldots, X_n >$-modules is however a “wild” problem, and it is to be expected that describing the finite dual of this non-commutative polynomial algebra should be of similar difficulty. In fact, it is known that $\mathbb{K} < X_1, \ldots, X_n >^0$ is the cofree coalgebra over a finite dimensional vector space of dimension $n$ (see [1] Theorem 2.4.2). More generally, if the quiver has oriented cycles, the representation theory will bear similarities to that of $\mathbb{K} < X_1, \ldots, X_n >$. In particular, if at least to different such cycles exist, the category of finite dimensional representations is wild.

Under certain conditions or in particular cases the description is known. For instance, it is proved in [3, Theorem 3.3] that if $\Gamma$ has no oriented cycles and there are only finitely many arrows between any two vertices, then $\mathbb{K}[\Gamma]^0$ is isomorphic to the path coalgebra $\mathbb{K}[\Delta]$ associated to $\Gamma$. Another interesting situation is when $\Gamma$ has just one vertex and just one arrow (a loop), in which case $\mathbb{K}[\Gamma]$ is just the polynomial algebra $\mathbb{K}[X]$. In this case, $\mathbb{K}[\Gamma]^0$ is identified with the linearly recursive functions on $\mathbb{K}[X]$, see [12, Example 9.1.7]. It can precisely be described as
\[ K[X]^0 = \bigoplus_{f \text{ irreducible}} \left[ \lim_{n \to \infty} (K[X]/(f^n))^* \right] \]

(3) Let \((X, \preceq)\) be a partially ordered set which is locally finite, i.e., the set \(\{ z \mid x \preceq z \preceq y \}\) is finite for any \(x \preceq y\) in \(X\). Let \(\mathbb{K}X\) be the incidence coalgebra of \(X\). This is the vector space with basis \(\{ e_{x,y} \mid x, y \in X, x \leq y \}\), comultiplication defined by \(\Delta(e_{x,y}) = \sum_{x \leq z \leq y} e_{x,z} \otimes e_{z,y}\), and counit defined by \(\epsilon(e_{x,y}) = \delta_{x,y}\) for any \(x, y \in X\) with \(x \leq y\). The dual algebra of \(\mathbb{K}X\) is isomorphic to the incidence algebra \(IA(X)\), which is the space of all functions \(f : \{(x, y) \mid x, y \in X, x \leq y\} \to K\), with multiplication given by \((fg)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)\) for any \(f, g \in IA(X)\) and any \(x, y \in X, x \leq y\). Let \(FIA(X)\) be the set of all elements of \(IA(X)\) of finite support. Then \(FIA(X)\) is a subalgebra of \(IA(X)\), non-unital when \(X\) is infinite. It is proved in [5] Theorem 4.2 that \(FIA(X)^0 \simeq \mathbb{K}X\) as coalgebras.

4. Applications to coalgebras

We give an application of the above constructions to coalgebras. We note a connection between semiperfect coalgebras and notions of coreflexivity for coalgebras. In this section \(C\) will be a coalgebra with counit. Then there is a decomposition into indecomposable left comodules \(C = \bigoplus_i E(S_i)\), where \(S_i\) are simple left \(C\)-comodules and \(E(S_i)\) is an injective hull of \(S_i\) contained in \(C\). A similar decomposition holds on the right: \(C = \bigoplus_j E(T_j)\). Recall from [11] that a coalgebra is left (right) semiperfect if the right (left) injective indecomposable comodules \(E(T_j)\) (respectively, \(E(S_i)\)) are finite dimensional. We recall from [4] Theorem 3.2.3 that the following are equivalent for a coalgebra \(C\): (i) \(C\) is right semiperfect; (ii) The category \(\mathcal{M}^C\) has enough projectives; (iii) \(\text{Rat}(\mathcal{C}^*, C^*)\) is dense in \(C^*\) in the finite topology of \(C^*\); (iv) Any finite dimensional right \(C\)-comodule has a projective cover.

We also recall from [6, 13, 14] that a coalgebra is called coreflexive if the canonical map \(C \to (C^*)^0\) is bijective (it is always injective). Given a coalgebra \(C\), besides the dual algebra \(C^*\), there are also two versions of dual (co)modules, namely \(\text{Rat}(\mathcal{C}^*, C^*)\), the left rational dual module (which is a right comodule), and \(\text{Rat}(C^*, \mathcal{C}^*)\), the right rational dual module. These have been considered before in relation to self-duality properties of coalgebras and Hopf algebras (see for example, [7, 8, 15, 16, 17]), and play an important role in the theory of Hopf algebras with non-zero integral. These duals are ideals of \(C^*\), so they can also be regarded as non-unital algebras. We can define a map \(\phi_l : C \to (\text{Rat}(\mathcal{C}^*, C^*))^0\) in the natural way by \(\phi_l(c)(c^*) = e^*(c)\). We note that \(\ker \phi_l(c) = c^* \cap \text{Rat}(\mathcal{C}^*, C^*)\) has finite codimension in \(\text{Rat}(\mathcal{C}^*, C^*)\), since it contains \(D^* \cap \text{Rat}(\mathcal{C}^*, C^*)\), where \(D\) is the subcoalgebra of \(C\) generated by \(c\). Since \(D\) is finite dimensional, then \(D^*\) has finite codimension in \(C^*\), and then \(D^* \cap \text{Rat}(\mathcal{C}^*, C^*)\) has finite codimension in \(\text{Rat}(\mathcal{C}^*, C^*)\). Thus \(\phi_l(c)\) lies indeed in \((\text{Rat}(\mathcal{C}^*, C^*))^0\). It is easy to see that \(\phi_l\) is a morphism of coalgebras. We introduce the following

Definition 4.1. Let \(C\) be a coalgebra. We call \(C\) left coreflexive if the map \(\phi_l\) is bijective.

Proposition 4.2. If \(C\) is left coreflexive, then any finite dimensional left \(\text{Rat}(\mathcal{C}^*, C^*)\)-module \(M\) has a structure of a right \(C\)-comodule such that the associated left \(C^*\)-module structure gives by restriction of scalars the initial \(\text{Rat}(\mathcal{C}^*, C^*)\)-module structure of \(M\).

Proof. Since \(M\) has finite dimension, it is locally finite, and then by Proposition 3.5 \(M\) is a right \(\text{Rat}(\mathcal{C}^*, C^*)^0\)-comodule with coaction \(m \mapsto \sum m_{[0]} \otimes m_{[1]}\) such that the \(\text{Rat}(\mathcal{C}^*, C^*)\)-action on \(M\) is given by \(c^* m = \sum m_{[1]} (c^*) m_{[0]}\) for any \(c^* \in \text{Rat}(\mathcal{C}^*, C^*)\) and \(m \in M\). Then \(M\) becomes a
right $C$-comodule (not necessarily counital) via the coalgebra isomorphism $\phi_l$, i.e. the comodule structure is given by $m \mapsto \sum m_0 \otimes m_1 = \sum m_{[0]} \otimes \phi_l^{-1}(m_{[1]})$. This right $C$-comodule structure induces a new left $C^*$-module structure on $M$, with action denoted by $c^* \cdot m$ for $c^* \in C^*$ and $m \in M$. We have that $c^* \cdot m = \sum c^*(m_1)m_0 = \sum\phi_l^{-1}(m_{[1]}))m_{[0]}$ for any $c^* \in C^*$ and $m \in M$.

In the case where $c^* \in \text{Rat}(C, C^*)$ we have that $c^*(\phi_l^{-1}(m_{[1]})) = \phi_l(\phi_l^{-1}(m_{[1]}))(c^*) = m_{[1]}(c^*)$, so then $c^* \cdot m = \sum c^*(\phi_l^{-1}(m_{[1]}))m_{[0]} = \sum m_{[1]}(c^*)m_{[0]} = c^*m$. Thus restricting this $C^*$-action on $M$ to $\text{Rat}(C, C^*)$ gives the initial action.

The following explains and extends results of [5] about path coalgebras and incidence coalgebras.

**Theorem 4.3.** Let $C$ be a coalgebra. Then the following assertions are true.

(i) $C$ is right semiperfect if and only if $\phi_l$ is injective. In particular if $C$ is left coreflexive, then $C$ is right semiperfect.

(ii) If $C$ is right semiperfect and the coradical $C_0$ of $C$ is coreflexive, then $C$ is left coreflexive.

Proof. (i) The map $\phi_l$ is injective if and only if $\text{Ker} \phi_l = \text{Rat}(C, C^*)^\bot = \{c \in C| c^*(c) = 0, \forall c^* \in \text{Rat}(C, C^*)\} = 0$, which means that $\text{Rat}(C, C^*)$ is dense in $C^*$ (see, for example, [4] Corollary 1.2.9), i.e. $C$ is right semiperfect.

(ii) It remains to prove the surjectivity of $\phi_l$. Since $C_0$ is coreflexive and $C$ is right semiperfect, by [9] Corollary 4.10] we see that $C$ is coreflexive. Let $C = \bigoplus_{j \in J} E(S_j)$ be the decomposition into indecomposable left comodules as before. Since $C$ is right semiperfect we have that each $E(S_j)$ is finite dimensional. Then $C^* \simeq \prod_j E(S_j)^*$ as left $C^*$-modules. In fact we identify $C^*$ with $\prod_j E(S_j)^*$, by regarding $E(S_j)^*$ as the set of elements $c^* \in C^*$ such that $c^*(E(S_p)) = 0$ for any $p \neq j$. For any $j \in J$ we denote by $e_j$ the element of $C^*$ which agrees with $\epsilon$ on $E(S_j)$ and vanishes on any $E(S_p)$ with $p \neq j$. Then $E(S_j)^* = C^*e_j$ for any $j$, and $(e_j)_{j \in J}$ is a set of orthogonal idempotents in $C^*$.

Denote $R = \text{Rat}(C, C^*)$; we have that $\bigoplus_j E(S_j)^* \subseteq R$. Let $f \in R^0$, and let $I$ be a cofinite two-sided ideal of $R$ such that $I \subseteq \ker(f)$. The right $C^*$-module $R/I$ is finite dimensional, so it is rational, since $C$ is coreflexive. Let $\rho : R/I \to C \otimes_R I$ be a left coaction compatible with the right $C^*$-module structure. The coalgebra of coefficients $D = ef(R/I)$ of $R/I$ is finite dimensional, so there is a finite set $F \subset J$ such that $D \subseteq \bigoplus_{j \in F} E(S_j)$. Let $e = \epsilon - \sum_{j \in F} e_j$. Note that $R/I \cdot e = 0$, so $Re \subseteq I$.

Now $f_{\bigoplus_{j \in F} E(S_j)^*} \in (\bigoplus_{j \in F} E(S_j)^*)^* \simeq (\bigoplus_{j \in F} E(S_j))^**$. Since

$$\gamma : \bigoplus_{j \in F} E(S_j) \to (\bigoplus_{j \in F} E(S_j))^**, \gamma(c^*) = c^*(c)$$

is an isomorphism, we obtain that there is $x \in \bigoplus_{j \in F} E(S_j)$ such that $f_{\bigoplus_{j \in F} E(S_j)^*} = \gamma(x)$, i.e. $f(c^*) = c^*(x)$ for any $c^* \in \bigoplus_{j \in F} E(S_j)^*$. We show that $f(u) = u(x)$ for all $u \in R$. Let $g = \sum_{j \in F} e_j$. Then $u = ug + we$, $ug \in \bigoplus_{j \in F} E(S_j)^*$ and $we \in Re \subseteq I$. Thus $f(ug) = (ug)(x)$, $f(we) = 0$ and $f(u) = (ug)(x)$. On the other hand $\delta(x) = \sum x_1 \otimes x_2 \in C \otimes \bigoplus_{j \in F} E(S_j)$, and we have $(we)(x) = \sum u(x_1)e(x_2) = 0$. Therefore, we get $f(u) = (ug)(x) = (ug)(x) + (ue)(x) = (ug + we)(x) = u(x)$, and this ends the proof.

**Corollary 4.4.** Let $C$ be a coalgebra such that the coradical $C_0$ is coreflexive. Then $C$ is left coreflexive if and only if $C$ is right semiperfect. In this case, $C$ is also coreflexive.

Proof. The equivalence follows from the previous Theorem 4.3. If $C_0$ is coreflexive and $C$ is right semiperfect, then $C$ is coreflexive by [9] Corollary 4.10].
Let to motivate it we recall here a class of coreflexive Hopf algebras considered in the introduction. With the above notations, if \( C \) is coreflexive. Indeed, let \( \Gamma \) be the infinite line quiver, i.e. the vertices of \( \Gamma \) are the integers, and there is an arrow from \( n \) to \( n+1 \) for any integer \( n \). Then \( C_0 \) is the grouplike coalgebra of the set \( \mathbb{Z} \) of integers, so it is coreflexive by \( [5, \text{Proposition 3.1.4}] \), but \( C \) is not left coreflexive. By \([5, \text{Proposition 5.4}]\), \( C \) is also coreflexive. On the other hand, for any integer \( n \) there exist infinitely many paths starting from \( n \) and infinitely many paths ending at \( n \), so \( C \) is neither left semiperfect nor right semiperfect, see \([2, \text{Corollary 6.3}]\). Therefore \( C \) is not left coreflexive.

We note that the condition that \( C_0 \) is coreflexive in Theorem \( 4.3 \) (ii) is not a restrictive condition. Using the terminology of \([6, \text{Section 3.7}]\), a set \( X \) is called reasonable if every ultrafilter on \( X \) closed under countable intersections is principal. Such sets are called non-measurable in set theoretic terminology. Countable sets are reasonable, power sets and subsets of reasonable sets are reasonable, and unions and direct products of families of reasonable sets, indexed by reasonable sets, are also reasonable. Thus the class of reasonable sets is very large, essentially every set we work with is reasonable. It is proved in \([6, \text{Theorem 3.7.5}]\) that if the basefield is infinite and the set of simple subcoalgebras of \( C \) is reasonable, then \( C_0 \) is coreflexive.

We see by Corollary \( 4.4 \) that if the set of simple subcoalgebras of \( C \) is reasonable and \( k \) is infinite, then \( C \) is left coreflexive if and only if \( C \) is right semiperfect, thus the two concepts coincide. Under the same conditions, we also have that if \( C \) is left coreflexive, then \( C \) is coreflexive.

In the case where \( C \) is left and right semiperfect, the assumption on \( C_0 \) in Corollary \( 4.4 \) or on the set of simple subcoalgebras of \( C \) being reasonable in the discussion above are no longer necessary.

**Theorem 4.6.** Let \( C \) be a left and right semiperfect coalgebra. Then \( C \) is left coreflexive. Moreover, \( \phi_l \) is an isomorphism of counital coalgebras.

**Proof.** Since \( C \) is left and right semiperfect, we have that \( R = \text{Rat}(C, C^*) = \bigoplus_{j \in J} E(S_j)^* \), and this is an algebra with enough idempotents \((e_j)_{j \in J}\) (see \([4, \text{Section 3.3}]\)), where \( e_j \) is defined as in the proof of Theorem \( 4.3 \).

Let \( f \in R^0 \) and \( I \) an ideal of finite codimension in \( R \) such that \( I \subseteq \text{Ker}(f) \). By \([5, \text{Lemma 2.1}]\), only finitely many of the idempotents \((e_j)_{j \in J} \) can lie outside \( I \). Then there exists a finite subset \( F \) of \( J \) such that \( \bigoplus_{j \in J \setminus F} E(S_j)^* \subseteq I \). As in the proof of Theorem \( 4.3 \) (ii), there exists \( x \in \bigoplus_{j \in F} E(S_j) \) such that \( f(c^*) = c^*(x) \) for any \( c^* \in \bigoplus_{j \in F} E(S_j)^* \). Now if \( c^* \in \bigoplus_{j \in J \setminus F} E(S_j)^* \), then \( c^* \in I \), so \( f(c^*) = 0 \), and \( c^*(x) = 0 \) since \( x \in \bigoplus_{j \in F} E(S_j) \). Thus \( f(c^*) = c^*(x) \), so \( f = \phi_l(x) \). This shows that \( \phi_l \) is bijective.

By \([5, \text{Proposition 2.4}]\) the coalgebra \( R^0 \) has counit \( E \) defined by \( E(f) = \sum_j f(e_j) \). Then \( (E\phi_l)(c) = \sum_j \phi_l(c)(e_j) = \sum_j e_j(c) = \varepsilon(c) \), so \( \phi_l \) is a morphism of counital coalgebras. \( \square \)

**4.1. Examples in Hopf algebras.** Before applying the above result to Hopf algebras, in order to motivate it we recall here a class of coreflexive Hopf algebras considered in the introduction. Let \( G \) be an algebraic group over \( \mathbb{C} \), and let \( L \) be its Lie algebra. Let \( H \) be the algebra of functions on \( G \).

**Proposition 4.7.** With the above notations, if \( L \) is a finite dimensional Lie algebra (in particular in the case where \( G \) is an affine algebraic group), then the Hopf algebras \( U(L) \) and \( H^0 \) are coreflexive.
Proof. If \( L \) is a finite dimensional Lie algebra over \( \mathbb{C} \), by [6], \( U(L) \) is coreflexive, since the coradical of \( U(L) \) is 1-dimensional and the space of primitives is finite dimensional (see also the results of [9, Sections 2 & 4]). With the notations for \( G \) and \( H \) as above, since \( H^0 = \mathbb{C}[G] \otimes U(L) \) as coalgebras. To show that \( H^0 \) is coreflexive, one can notice that \( H^0 \) is cocommutative, and the space of primitives is finite dimensional (and the set of grouplikes is \( G \)); thus, [6, 5.1.3 Corollary] applies and \( H^0 \) is coreflexive. Alternatively, one can use [6, Theorem 3.4.3] to get that \( U(L) \) is strongly coreflexive, and since \( \mathbb{C}[G] \) is coreflexive, by [14, Theorem 4.4] \( \mathbb{C}[G] \otimes U(L) \) is coreflexive.

It is also possible to check the coreflexivity of \( H^0 \) directly, using the fact that \( H^0 = \mathbb{C}[G] \otimes U(L) = \bigoplus_G U(L) \) as coalgebras. \( \square \)

Since a Hopf algebra with non-zero integrals is left and right semiperfect as a coalgebra, we obtain as a consequence of Theorem 4.6 the following.

**Corollary 4.8.** Let \( H \) be a Hopf algebra with non-zero integrals. Then \( H \simeq (H^{srat})^0 \) as coalgebras.

Let \( H \) be a Hopf algebra with non-zero left integral \( t \) on \( H \) (in particular \( H \) can be any compact quantum group). Then \( H^{srat} = H \rightharpoonup t \), where \( \rightharpoonup \) denotes the standard left \( H \)-action on \( H^* \), induced by the right \( H \)-module structure of \( H \) (given by the multiplication of \( H \)). Then the above result shows that the coalgebra structure of \( H \) can be reconstructed from the algebra structure of \( H \) and a non-zero integral on \( H \). Thus knowing a non-zero integral is quite a strong information about the structure of \( H \).

We also note that, by the discussion above, any Hopf algebra with non-zero integral is coreflexive as a coalgebra, provided that \( k \) is infinite and the set of simple subcoalgebras of \( H \) is reasonable. This essentially covers all usual examples.

**Acknowledgment**

The research was supported by the UEFISCDI Grant PN-II-ID-PCE-2011-3-0635, contract no. 253/5.10.2011 of CNCSIS. The research of the second author was also supported by the strategic grant POSDRU/89/1.5/S/58852, Project “Postdoctoral program for training scientific researchers” cofinanced by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007-2013.

**References**

[1] E. Abe, Hopf Algebras, Cambridge University Press, Cambridge, 1977.
[2] W. Chin, M. Kleiner, D. Quinn, *Almost split sequences for comodules*, J. Algebra 249 (2002), 1-19.
[3] P. Cartier, *A Primer of Hopf Algebras*, Lecture Notes IHES, [http://preprints.ihes.fr/2006/M/M-06-40.pdf](http://preprints.ihes.fr/2006/M/M-06-40.pdf)
[4] S. Dăscălescu, C. Năstăescu, Ş. Raianu, Hopf algebras. An introduction, Marcel Dekker, New York, 2001.
[5] S. Dăscălescu, M.C. Iovanov, C. Năstăescu, *Quiver algebras, path coalgebras, and coreflexivity*, Pac. J. Math. 262 (2013), 49-79.
[6] R. Heyneman, D.E. Radford, *Reflexivity and Coalgebras of Finite Type*, J. Algebra 28 (1974), 215-246.
[7] M.C. Iovanov, *Co-Frobenius Coalgebras*, J. Algebra 303 (2006), no. 1, 146-153.
[8] M.C.Iovanov, *Generalized Frobenius Algebras and Hopf Algebras*, preprint arXiv, to appear, Can. J. Math.
[9] M.C.Iovanov, *On Extensions of Rational Modules*, preprint arxiv, to appear, Israel J. Math.
[10] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Springer 1995.
[11] B.J-Peng Lin, *Semiperfect coalgebras*, J. Algebra 30 (1974), 559-601.
[12] S. Montgomery, Hopf algebras and their actions on rings, *CBMS Reg. Conf. Series* 82, American Mathematical Society, Providence, 1993.
[13] D.E. Radford, *On the Structure of Ideals of the Dual Algebra of a Coalgebra*, Trans. Amer. Math. Soc. Vol. 198 (Oct 1974), 123-137.
[14] D.E. Radford, *Coreflexive coalgebras*, J. Algebra 26 (1973), 512-535.
[15] J.B. Sullivan, *The uniqueness of integrals for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras*, J. Algebra 19 (1971), 426-440.
[16] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[17] M.E. Sweedler, *Integrals for Hopf algebras*, Ann. Math 89 (1969), 323-335.
[18] E.J. Taft, *Reflexivity of Algebras and Coalgebras*, American Journal of Mathematics, Vol. 94, No. 4 (Oct 1972), 1111-1130.
[19] E.J. Taft, *Reflexivity of algebras and coalgebras. II*. Comm. Algebra 5 (1977), no. 14, 1549–1560.

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