\section{Introduction}

Let $M$ be a compact 3-manifold. $M$ is \emph{irreducible} if each 2-sphere in $M$ bounds a 3-ball. $M$ is \emph{$\partial$-irreducible} if $\partial M$ is incompressible. $M$ is \emph{anannular} if $M$ does not contain essential annuli. $M$ is \emph{atoroidal} if $M$ does not contain essential tori. A compact, orientable 3-manifold $M$ is said to be \emph{simple} if it is irreducible, $\partial$-irreducible, anannular and atoroidal. By Thurston’s Geometrization Theorem, a Haken manifold $M$ is hyperbolic if and only if $M$ is simple.

Let $M$ be a compact 3-manifold, and $F$ be a component of $\partial M$. A \emph{slope} $\alpha$ on $F$ is an isotopy class of essential simple closed curves on $F$. Denote by $\Delta(\alpha, \beta)$ the minimal geometric intersection number among all the curves representing the slopes $\alpha$ and $\beta$. For a slope $\alpha$ on $F$, denote by $M(\alpha)$ the manifold obtained by attaching a 2-handle to $M$ along a regular neighborhood of $\alpha$ on $F$, then capping off a possible 2-sphere component of the resulting manifold by a 3-ball. Note that if $F$ is a torus, then $M(\alpha)$ is the Dehn filling along $\alpha$. Given a simple 3-manifold $M$, a slope $\alpha$ is \emph{degenerating} if $M(\alpha)$ is non-simple.

It is known that there are finitely many degenerating Dehn fillings. This result is precisely stated on the upper bound of $\Delta(\alpha, \beta)$ for two degenerating slopes $\alpha$ and $\beta$. There are 10 cases as follows. C. Gordon and J. Luecke showed that $\Delta(\alpha, \beta) \leq 1$ if $M(\alpha)$ and $M(\beta)$ are reducible \cite{GordonLuecke}. M. Scharlemann showed that $\Delta(\alpha, \beta) = 0$ if $M(\alpha)$ is reducible and $M(\beta)$ is $\partial$-reducible \cite{Scharlemann}. R. Qiu and Y. Wu independently showed that $\Delta(\alpha, \beta) \leq 2$ if $M(\alpha)$ is reducible and $M(\beta)$ is anannular \cite{QiuWu1, QiuWu2}. Y. Wu and S. Oh independently showed that $\Delta(\alpha, \beta) \leq 3$ if $M(\alpha)$ is reducible and $M(\beta)$ is toroidal \cite{WuOh1, WuOh2}. Y. Wu showed that $\Delta(\alpha, \beta) \leq 1$ if $M(\alpha)$ and $M(\beta)$ are $\partial$-reducible \cite{Wu}. C. Gordon and Y. Wu showed that $\Delta(\alpha, \beta) \leq 2$ if $M(\alpha)$ is $\partial$-reducible and $M(\beta)$ is anannular \cite{GordonWu}. C. Gordon and J. Luecke showed that $\Delta(\alpha, \beta) \leq 2$ if $M(\alpha)$ is $\partial$-reducible and $M(\beta)$ is toroidal \cite{GordonLuecke2}. C. Gordon showed that $\Delta(\alpha, \beta) \leq 5$ if $M(\alpha)$ and $M(\beta)$ are anannular, $\Delta(\alpha, \beta) \leq 5$ if $M(\alpha)$ is anannular and $M(\beta)$ is toroidal and $\Delta(\alpha, \beta) \leq 8$ if $M(\alpha)$ and $M(\beta)$ are toroidal \cite{Gordon}.

Suppose that the genus of $F$ is at least two. M. Scharlemann and Y. Wu gave an example that has infinitely many degenerating slopes \cite{ScharlemannWu}. Let $\alpha$ be a degenerating slope on $F$. If either $\alpha$ is separating or there are no separating degenerating curves
coplanar with $\alpha$, $\alpha$ is a basic degenerating slope. M. Scharlemann and Y. Wu proved that there are finitely many basic degenerating slopes. Furthermore, $\Delta(\alpha, \beta)$ for two separating degenerating slopes $\alpha$ and $\beta$ has an upper bound. Specially, $\Delta(\alpha, \beta) = 0$ if $M(\alpha)$ is reducible and $M(\beta)$ is $\partial$-reducible [11]. R. Qiu and M. Zhang showed that $\Delta(\alpha, \beta) \leq 2$ if $M(\alpha)$ and $M(\beta)$ are both reducible [12].

The case that $M(\alpha)$ and $M(\beta)$ are both $\partial$-reducible was also considered by Y. Li, R. Qiu and M. Zhang. They showed that $\Delta(\alpha, \beta) \leq 2$ when the $\partial$-reducing disks have boundaries on $\partial M - F$ [13]. In this paper, we consider the case with no limits on the $\partial$-reducing disks.

**Theorem 1.1.** Suppose that $M$ is a simple 3-manifold and $F$ is a component of $\partial M$ of genus at least 2. Let $\alpha$ and $\beta$ be separating slopes on $F$ such that $M(\alpha)$ and $M(\beta)$ are $\partial$-reducible. Then $\Delta(\alpha, \beta) \leq 8$.

2. **Preliminaries**

Suppose that $M$ is a simple 3-manifold, and $F$ is a component of $\partial M$ of genus at least two. Let $\alpha$ and $\beta$ be two separating slopes on $F$ such that $M(\alpha)$ and $M(\beta)$ are both $\partial$-reducible. Note that if one of $M(\alpha)$ and $M(\beta)$ is reducible, then $\Delta(\alpha, \beta) = 0$. Thus we can assume that $M(\alpha)$ and $M(\beta)$ are irreducible. We denote by $H_{\alpha}$ (resp. $H_{\beta}$) the 2-handle attached to $M$ along $\alpha$ (resp. $\beta$) to obtain $M(\alpha)$ (resp. $M(\beta)$). Let $P$ (resp. $Q$) be an essential disk in $M(\alpha)$ (resp. $M(\beta)$) such that $|\tilde{P} \cap H_{\alpha}|$ (resp. $|\tilde{Q} \cap H_{\beta}|$) is minimal among all essential disks in $M(\alpha)$ (resp. $M(\beta)$). Let $P = \tilde{P} \cap M$ and $Q = \tilde{Q} \cap M$. Obviously, $P \cap H_{\alpha}$ (resp. $Q \cap H_{\beta}$) consists of disks with boundary components having slope $\alpha$ (resp. $\beta$).

**Lemma 2.1.** $P$ (resp. $Q$) is incompressible and $\partial$-incompressible in $M$.

**Proof.** Suppose that $P$ is compressible in $M$. Let $D$ be a compressing disk for $P$ in $M$. Then $\partial D$ bounds a disk $D'$ in $\tilde{P}$. $D \cup D'$ is a 2-sphere which bounds a 3-ball in $M(\alpha)$ since $M(\alpha)$ is irreducible. By isotoping $D'$ to $D$, we reduce $|\tilde{P} \cap H_{\alpha}|$, contradicting that $|\tilde{P} \cap H_{\alpha}|$ is minimal. Thus $P$ is incompressible in $M$.

Suppose that $P$ is $\partial$-compressible. Let $D$ be a $\partial$-compressing disk for $P$ in $M$. $\partial D = u \cup v$, $u \subset F$ and $v \subset P$.

Case 1: $v$ connects two components of $\partial P - \partial \tilde{P}$.

After $\partial$-compressing $P$ along $D$, we can get an essential disk $\tilde{P}_1$ in $M(\alpha)$.

Case 2: $\partial v$ is in a component of $\partial P - \partial \tilde{P}$.

By $\partial$-compressing $P$ along $D$, we get two planar surface $P_1$ and $P_2$. Suppose that $\partial \tilde{P} \subset \partial P_2$. Then we get an essential disk $\tilde{P}_1$ by capping off the components of $\partial P_1$ having slope $\alpha$.

Case 3: $\partial v$ is in $\partial \tilde{P}$.

Let $P_1$ and $P_2$ be planar surfaces obtained by $\partial$-compressing $P$ along $D$. After capping off the components of $\partial P_1$ having slope $\alpha$, we get an essential disk $\tilde{P}_1$.

Case 4: $v$ connects $\partial \tilde{P}$ and one component of $\partial P - \partial \tilde{P}$.

Let $P_1$ be the planar surface obtained by $\partial$-compressing $P$ along $D$. After capping off the components of $\partial P_1$ having slope $\alpha$, we get a disk $\tilde{P}_1$ and $\partial \tilde{P}_1 = \partial \tilde{P}$ since $\alpha$ is a trivial slope in $\partial M(\alpha)$. Then $P_1$ is an essential disk in $M(\alpha)$.

In each case we have $|\tilde{P}_1 \cap H_{\alpha}| < |\tilde{P} \cap H_{\alpha}|$, contradicting that $|\tilde{P} \cap H_{\alpha}|$ is minimal. Thus $P$ is $\partial$-incompressible in $M$. □
We may assume that $|P \cap Q|$ is minimal. Then each component of $P \cap Q$ is either
an essential arc or an essential simple closed curve on both $P$ and $Q$.

R. Litherland and C. Gordon first used the results in graph theory to study Dehn
fillings. See [13], [15]. Then M. Scharlemann and Y. Wu introduced these methods
to study handle additions. See [11], [16].

Let $\Gamma_P$ be a graph on $\hat{P}$. The vertices of $\Gamma_P$ are components of
$\partial P$ and the edges of $\Gamma_P$ are arc components of $P \cap Q$. Each component of $\partial P - \partial \hat{P}$ is called
an interior vertex of $\Gamma_P$. An edge incident to $\partial \hat{P}$ is called a boundary edge
of $\Gamma_P$. Similarly, we can define $\Gamma_Q$ in the disk $\hat{Q}$. Note that by taking
$\partial \hat{P}$ (resp. $\partial \hat{Q}$) also
as a vertex $\Gamma_P$ (resp. $\Gamma_Q$) is a spherical graph.

By Lemma 2.1, we have the following Lemma 2.2.

**Lemma 2.2.** There are no 1-sided disk faces in both $\Gamma_P$ and $\Gamma_Q$.

**Lemma 2.3.** There are no edges that are parallel in both $\Gamma_P$ and $\Gamma_Q$.

**Proof.** See Lemma 2.1 in [11]. $\square$

**Lemma 2.4.** There are no $3q$ parallel edges in $\Gamma_P$.

**Proof.** Suppose that there are $3q$ parallel edges in $\Gamma_P$. Let $\Gamma'$ be the subgraph of $\Gamma_Q$ formed by these edges and all the vertices of $\Gamma_Q$. Let $e$ be the number of edges of $\Gamma'$ and $f$ be the number of disk faces of $\Gamma'$. By Euler characteristic formula,

$$q - e + f \geq 1,$$

then $f \geq e - q + 1$.

Suppose that there are no 2-sided disk faces of $\Gamma'$. Then $2e \geq 3f$. Thus $\frac{2}{3}e \geq f \geq e - q + 1$. Then $e \leq 3q - 3$, contradicting with $e = 3q$. Now there is a 2-sided disk face of $\Gamma'$. The two edges bounding this disk face are parallel in both $\Gamma_P$ and $\Gamma_Q$. This contradicts Lemma 2.3. $\square$

Let $F'$ be the component of $F - (\partial P - \partial \hat{P})$ such that $\partial \hat{P} \subset F'$. Number the
components of $\partial P - \partial \hat{P}$ as $\partial_1 P, \partial_2 P, \ldots, \partial_u P, \ldots, \partial_p P$ consecutively on $F$ such that

$\partial_1 P = \partial F'$. This means that $\partial_u P$ and $\partial_{u+1} P$ bound an annulus in $F$ with interior disjoint from $P$. See Figure 1. Similarly, number the components of $\partial Q - \partial \hat{Q}$ as $\partial_1 Q, \partial_2 Q, \ldots, \partial_i Q, \ldots, \partial_q Q$. These give the corresponding labels of the vertices of $\Gamma_P$ and $\Gamma_Q$. Since $M$ is simple, then $p, q > 1$.

Let $x$ be an endpoint of an arc in $P \cap Q$. If $x$ belongs to $\partial_u P \cap \partial_i Q$, then we label it as $(u, i)$. If $x$ belongs to $\partial P \cap \partial_i Q$, then we label it as $(*, i)$. If $x$ belongs to $\partial_u P \cap \partial \hat{Q}$, then we label it as $(u, *)$. If $x$ belongs to $\partial \hat{P} \cap \partial \hat{Q}$, then we label it as $(*, *)$. On $\Gamma_P$, the labels are written only by the second index $i$ (or $*$) for
short. And on $\Gamma_Q$, the labels are written only by the first index $u$ (or $*$) for short. Here, $*$ is called the boundary label. Around each vertex of $\Gamma_P$, the labels appear as $q, q - 1, \cdots, 1, *, \cdots, *, 1, 2, \cdots, q$ in clockwise direction or anticlockwise direction. By giving a sign to the numbered labels as in [17], we can assume that

Assumption 2.5. $-q, -(q - 1), \cdots, -1, *, \cdots, *, +1, +2, \cdots, +q$ (resp. $-p, -(p - 1), \cdots, -1, *, \cdots, *, +1, +2, \cdots, +p$) appear in clockwise direction around each vertex of $\Gamma_P$ (resp. $\Gamma_Q$).

Note. By taking $\Gamma_P$ as a graph on disk $\hat{P}$, the labels $-q, -(q - 1), \cdots, -1, *, \cdots, *, +1, +2, \cdots, +q$ appear on $\partial \hat{P}$ in anticlockwise direction. See Figure 2.

On $\Gamma_P$ (resp. $\Gamma_Q$), each edge has a label pair $(i, j)$ of its two endpoints, $i, j \in \{+1, +2, \cdots, +q, -q, \cdots, -1\}$. As Lemma 3.3 in [17], we have a parity rule.

Lemma 2.6. Let $e$ be an edge in $\Gamma_P$ (resp. $\Gamma_Q$). $e$ cannot have label pair $(i, i)$, $i \in \{+1, +2, \cdots, +q, -q, \cdots, -1\}$.

Let $x$ be a signed label in $\{+1, +2, \cdots, +q, -q, \cdots, -1\}$. An $x$-edge is an edge in $\Gamma_P$ with label $x$ at one endpoint. Let $B_x^P$ denote the subgraph of $\Gamma_P$ consisting of all the vertices of $\Gamma_P$ and all the $x$-edges. Let $x$ be a signed label in $\{+1, +2, \cdots, +p, -p, \cdots, -1\}$, the definition for $B_x^Q$ is the same.

A cycle of $B_x^P$ which bounds a disk face of $\Gamma_P$ and contains no edges with boundary labels (at the endpoints) is called a virtual Scharlemann cycle. In a virtual Scharlemann cycle, the label pair of each edge is the same, which is called the label pair of the virtual Scharlemann cycle. A virtual Scharlemann cycle with label pair $(i, j)$ is called a Scharlemann cycle if $i \neq -j$.

As in [12], we have the following Lemma 2.7 and Lemma 2.8.

Lemma 2.7. Suppose $S = \{e_i | i = 1, 2, \ldots, n\}$ is a set of parallel edges in $\Gamma_P$. Each edge in $S$ has no boundary labels. If one of the edges, say $e_k$, has opposite labels at its two endpoints, then each edge in $S$ has opposite labels at its two endpoints.

Lemma 2.8. Let $x \in \{+1, +2, \cdots, +q, -q, \cdots, -1\}$. Let $D$ be a disk face of $B_x^P$ and no edges in $D$ have boundary labels. Then there is a virtual Scharlemann cycle lying in $D$. 
Lemma 2.9. There are no Scharlemann cycles in $\Gamma_P$ or $\Gamma_Q$.

Proof. See Lemma 2.5.2 (a) in [18]. □

The following Lemma 2.10 is Lemma 2.6.5 in [18].

Lemma 2.10. Let $\Gamma$ be a graph in a disk $D$ with no trivial loops or parallel edges, such that every vertex of $\Gamma$ belongs to a boundary edge. Then $\Gamma$ has a vertex of valency at most 3 which belongs to a single boundary edge.

3. Proof of Theorem 1.1

Let $R_P = \Delta(\alpha, \partial Q)$, $\Delta = \Delta(\alpha, \beta)$.

Lemma 3.1. Suppose that $\Delta \geq 6$. For any label $v \in \{+1, \ldots, +p, -p, \ldots, -1\}$, there are at least $(\frac{\Delta}{2} q + \frac{R_P}{2} - 3q + 3)$ 2-sided disk faces of $B^v_Q$.

Proof. Denote the number of edges in $B^v_Q$ by $e$. Then $e = \frac{\Delta}{2} q + \frac{R_P}{2}$. Denote the number of 2-sided disk faces of $B^v_Q$ by $f_1$ and the number of disk faces with at least 3 sides by $f_2$. By Euler characteristic formula, $q - e + f_1 + f_2 \geq 1$. Since $2f_1 + 3f_2 \leq 2e$, $f_2 \leq \frac{2e - 2f_1}{3}$. Then $q - e + f_1 + \frac{2e - 2f_1}{3} \geq 1$. Thus $f_1 \geq e - 3q + 3 = \frac{\Delta}{2} q + \frac{R_P}{2} - 3q + 3$. □

Let $D$ be a 2-sided disk face of $B^v_Q$. See Figure 3. The edges of $\Gamma_Q$ in $D$ are parallel. Suppose that these edges are incident to two subarcs $z$ and $z'$ of $\partial Q$. When we go from top to bottom along $z$ (resp. from bottom to top along $z'$), the labels appear in the direction of $+1, +2, \ldots, +p, -p, \ldots, -1,*$. Let $X(D)$ (resp. $Y(D)$) be the collection of labels in $z$ (resp. $z'$). There are three cases of 2-sided disk face $D$ of $B^v_Q$. In case 1, two labels $v$ are both in $X(D)$. Note that it is the same after rotating the figure by $\pi$ if two labels $v$ are both in $Y(D)$. See Figure 3 (1). In case 2, the top label (resp. lowest label) in $z$ (resp. $z'$) is $v$. See Figure 3 (2). In case 3, the lowest label (resp. top label) in $z$ (resp. $z'$) is $v$. See Figure 3 (3).

We have the following Fact 3.2.

Fact 3.2. There are no edges in int$D$ with label $v$.

If a interior vertex of $\Gamma_P$ is incident to boundary edges or edges with both endpoints in it, then the vertex is good. Otherwise the vertex is bad.

Lemma 3.3. Suppose that $\Delta \geq 6$. Vertex 1 in $\Gamma_P$ is good.
Proof. Let $D$ be a 2-sided disk face of $B_Q^{+1}$. Then $D$ is as in Figure 4. Denote the two edges in $\partial D$ by $e_1$ and $e_n$ respectively. If one of $w$ and $w'$ is a boundary label, then this lemma holds. Thus we assume that both $w$ and $w'$ are not boundary labels.

Case 1: $D$ is as in Figure 4 (1).
If there are boundary labels in $Y(D)$, then there is label +1 in $Y(D)$ by Assumption 2.5. If there are no boundary labels in $Y(D)$, then there is label +1 in $Y(D)$ since there are at least $2p+1$ edges in $D$. By Lemma 2.6, $w, w' \neq +1$. Then there is an edge with label +1 in int$D$, contradicting with Fact 3.2.

Case 2: $D$ is as in Figure 4 (2).
If there are boundary labels in $X(D)$ or $Y(D)$, then there is an edge with label +1 in int$D$ by Assumption 2.5 and Lemma 2.6, contradicting with Fact 3.2. Thus there are no boundary labels in both $X(D)$ and $Y(D)$. By Lemma 2.8 and Lemma 2.9, there is a virtual Scharlemann cycle $\Sigma$ with label pair $(+1, -1)$ or $(+p, -p)$ in $D$. By Lemma 2.7, $e_1$ has label pair $(+1, -1)$, then $e_1$ is an edge with both endpoints in vertex 1 in $\Gamma_P$.

Case 3: $D$ is as in Figure 4 (3).
There must be label −1 in both $X(D)$ and $Y(D)$ since $w$ and $w'$ are not boundary labels. Assume that edge $e_i$ (resp. $e_j$) has label −1 in $Y(D)$ (resp. $X(D)$). If $e_j$ is higher than $e_i$, $e_i$ and $e_j$ must have boundary labels by Assumption 2.5. Then $e_i$ and $e_j$ are boundary edges incident to vertex 1 in $\Gamma_P$, which means that vertex 1 is good. Now assume that $e_i$ is higher than $e_j$. See Figure 4 (3). $e_i$ and $e_j$ bound a 2-sided disk face $D'$ of $B_Q^{+1}$. By Fact 3.2, the edges in $D'$ have no boundary labels. Then by Lemma 2.8 and Lemma 2.9, there is a virtual Scharlemann cycle $\Sigma$ with label pair $(+1, -1)$ or $(+p, -p)$ in $D'$. By Lemma 2.7, $e_i$ has label pair $(+1, -1)$, contradicting with Fact 3.2.
Thus vertex 1 in $\Gamma_P$ is good.

Lemma 3.4. For a bad vertex $v$ ($v > 1$) of $\Gamma_P$, let $D$ be a 2-sided disk face of $B_Q^{+v}$. Then there are labels +1 and −1 in both $X(D)$ and $Y(D)$.

Proof. Since $v$ is bad, then the 2-sided disk face of $B_Q^{+v}$ is as in Figure 3 where $w$ and $w'$ are not boundary labels.

Case 1: $D$ is as in Figure 3 (1).
By Assumption 2.5, there are labels +1 and −1 in $X(D)$.
Suppose that there are no boundary labels in $Y(D)$. Since there are at least $2p+1$ edges in $D$, then there is label $+v$ in $Y(D)$, contradicting with Fact 3.2 or
Lemma 2.6. Thus there are boundary labels in \(Y(D)\). Then there are labels +1 and −1 in \(Y(D)\) by Assumption 2.5.

Case 2: \(D\) is as in Figure 3 (2).

Suppose that there are no boundary labels in both \(X(D)\) and \(Y(D)\). By Lemma 2.8 and Lemma 2.9, there is a virtual Scharlemann cycle with label pair \((+1, −1)\) or \((+p, −p)\) in \(D\). By Lemma 2.7, \(e_1\) has label pair \((v, −v)\). Then \(e_1\) is an edge with both endpoints in vertex \(v\) in \(\Gamma_P\), contradicting that vertex \(v\) is bad.

Now there are boundary labels in \(X(D)\) or \(Y(D)\). Without loss of generality, we assume that there are boundary labels in \(X(D)\). Thus there are labels +1 and −1 in \(X(D)\) by Assumption 2.5. Then there are at least \(2p − v + 3\) edges in \(D\). See Figure 5 (1). By counting the labels in \(Y(D)\) from bottom to top, label −1 must appear in \(Y(D)\). Since \(w\) is not *, label +1 must appear in \(Y(D)\).

Case 3: \(D\) is as in Figure 3 (3).

Suppose that there are no boundary labels in both \(X(D)\) and \(Y(D)\). Similar to the proof in Case 2, we get a contradiction. Now suppose that there are boundary labels in \(X(D)\). Thus there are labels +1 and −1 in \(X(D)\). Then there are at least \((v + 2)\) edges in \(D\). See Figure 5 (2). By a similar argument as in Case (2), labels +1 and −1 must appear in \(Y(D)\).

\(\Box\)

Lemma 3.5. Suppose \(v (v > 1)\) is a bad vertex of \(\Gamma_P\). Then any two 2-sided disk faces of \(B_{Q,v}^+\) are not parallel.

Proof. We first prove that any two 2-sided disk faces \(D_1\) and \(D_2\) of \(B_{Q,v}^+\) cannot share a common edge. Suppose not. See Figure 6. Without loss of generality, assume that \(X(D_1) \cap X(D_2) = \{+v\}\). By Lemma 3.4, there is an edge \(e_1\) in \(D_1\) with label −1 in \(Y(D_1)\) and an edge \(e_2\) in \(D_2\) with label +1 in \(Y(D_2)\). By Assumption 2.5, there is an edge \(e_3\) between \(e_1\) and \(e_2\) with label +v in \(Y(D_1) \cup Y(D_2)\). By Lemma 2.6, \(w \neq +v\). Thus \(e_3\) is in \(\text{int}D_1\) or \(\text{int}D_2\), contradicting with Fact 3.2.

Now suppose that there are two parallel but not adjacent 2-sided disk faces \(D_1\) and \(D_2\) of \(B_{Q,v}^+\). See Figure 7. There must be another 2-sided disk face \(D_3\) which shares a common edge with \(D_2\), a contradiction. Hence this lemma holds.

\(\Box\)

Lemma 3.6. Suppose that \(\Delta \geq 10\). Then each interior vertex in \(\Gamma_P\) is good.
Figure 6.

Figure 7.

Proof. Suppose otherwise that there is a bad vertex $v$ of $\Gamma_P$. By Lemma 3.3 $v > 1$. By taking each 2-sided disk face of $B_{Q}^{+v}$ as an edge, we have a graph $D_{Q}^{+v}$. Denote the number of edges in $D_{Q}^{+v}$ by $e$ and the number of disk faces by $f$. By Lemma 3.1 $e \geq \frac{3}{2}q + \frac{R}{2} - 3q + 3 \geq 5q + \frac{R}{2} - 3q + 3 = 2q + \frac{R}{2} + 3$. By Lemma 3.5, there are no 2-sided disk faces of $D_{Q}^{+v}$.

Suppose that there are no 3-sided disk faces of $D_{Q}^{+v}$. Then $4f \leq 2e$, that is $f \leq \frac{e}{2}$. By Euler characteristic formula, $q - e + f \geq 1$, $e - q + 1 \leq f \leq \frac{e}{2}$. Hence $e \leq 2q - 2$, a contradiction.

Now suppose that $D$ is a 3-sided disk face of $D_{Q}^{+v}$ bounded by 2-sided disk faces $D_1$, $D_2$ and $D_3$. Suppose that $z$, $z'$ and $z''$ are three subarcs of $\partial Q$ which contain the endpoints of all the edges in $D$, $D_1$, $D_2$ and $D_3$. See Figure 8. Suppose that there is an edge with label $+v$ in $int D$. Then there are two 2-sided disk faces of $B_{Q}^{+v}$ sharing a common edge, contradicting with Lemma 3.5. Thus there are no edges in $int D$ with label $+v$. Then $D$ is a 3-sided disk face of $B_{Q}^{+v}$ bounded by edges $e_1$, $e_2$ and $e_3$.

By Lemma 3.4 there are labels $+1$ and $-1$ in both $X(D_i)$ and $Y(D_i)$, $i = 1, 2, 3$. Then at least one of $a_2$ (resp. $b_2$, $c_2$) and $a_3$ (resp. $b_3$, $c_3$) is $+v$ since there are no edges with label $+v$ in $int D$ and $int D_i$, $i = 1, 2, 3$. We claim that exactly one of $a_2$ and $a_3$ is $+v$. Otherwise, both $a_2$ and $a_3$ are $+v$. By Lemma 2.6 $b_3, c_2 \neq +v$. Thus both $b_2$ and $c_3$ are $+v$, contradicting with Lemma 2.6. Similarly, exactly one of $b_2$ (resp. $c_2$) and $b_3$ (resp. $c_3$) is $+v$.
Suppose that there is an edge in \( \text{int} D \) with label +1 between \( a_2 \) and \( a_3 \). Then there are three labels +1 in \( z \). By Assumption 2.5, there are two labels +\( v \) between \( a_1 \) and \( a_4 \). Then both \( a_2 \) and \( a_3 \) are labels +\( v \), which contradicts with the last paragraph. Thus no edges in \( \text{int} D \) have label +1 between \( a_2 \) and \( a_3 \). Similarly, no edges in \( \text{int} D \) have label −1 between \( b_2 \) and \( b_3 \) and between \( c_2 \) and \( c_3 \).

Suppose that there is a boundary label \( \ast \) between \( a_2 \) and \( a_3 \). Since vertex \( v \) is bad, \( a_i \), \( b_i \) and \( c_i \) are not boundary label \( \ast \), \( i = 1, 2, 3 \). Then \( a_2 \) is −1 and \( a_3 \) is +1, contradicting that exactly one of \( a_2 \) and \( a_3 \) is +\( v \). Thus there are no edges in \( \text{int} D \) with boundary label \( \ast \) between \( a_2 \) and \( a_3 \). The same is true for \( b_2 \), \( b_3 \) and \( c_2 \), \( c_3 \).

Now \( D \) is a disk face of \( B^+_Q \) which contains no boundary labels. By Lemma 2.8 and lemma 2.9 there is a virtual Scharlemann cycle \( \Sigma \) with label pair (+1, −1) or (+\( p \), −\( p \)) in \( D \). By Lemma 2.7 \( e_1 \) has label pair (+\( v \), −\( v \)). Then \( e_1 \) is an edge with two endpoints in vertex \( v \) in \( \Gamma P \), a contradiction. Thus each vertex of \( \Gamma P \) is good.

The proof of Theorem 1.1.

Suppose that \( \Delta \geq 10 \). By Lemma 3.6 each vertex of \( \Gamma P \) is good, i.e, it incident to boundary edges or edges with both endpoints in it. Suppose that there is a vertex \( v \) incident to an edge \( e \) with both endpoints in it. Then \( e \) bounds a disk \( D \) in \( \hat{P} \). If there is a vertex \( v' \) in \( D \), then \( v' \) also has an edge \( e' \) with both endpoints in it. And \( e' \) also bounds a disk \( D' \) in \( \hat{P} \). By repeating the above process, we finally get an innermost 1-sided disk face, which contradicts Lemma 2.2. Thus each vertex of \( \Gamma P \) is incident to boundary edges.

Let \( \tilde{\Gamma}_P \) be the reduced graph of \( \Gamma P \). By Lemma 2.10 there is a vertex \( v \) of valency at most 3 in \( \Gamma P \). Then there are at most three families of parallel edges incident to \( v \) in \( \Gamma P \). Since \( \Delta \geq 10 \), one of them has more than 3\( q \) edges, contradicting with Lemma 2.4. Thus \( \Delta \leq 8 \). We complete the proof of Theorem 1.1.

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School of Mathematical Sciences, Dalian University of Technology, Dalian, People’s Republic of China, 116024
E-mail address: lh0720@mail.dlut.edu.cn

School of Mathematical Sciences, Dalian University of Technology, Dalian, People’s Republic of China, 116024
E-mail address: zhangmx@dlut.edu.cn