NANHUA XI
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THE BASED RING OF THE LOWEST TWO-SIDED CELL
OF AN AFFINE WELYL GROUP, II

BY NANHUA XI

ABSTRACT. - We show that the lowest based ring of an affine Weyl group \( W \) is very interesting to understand some simple representations of the corresponding Hecke algebra \( H_{q_0}(q_0 \in \mathbb{C}^*) \) even when \( q_0 \) is a root of 1.

Let \( H_{q_0} \) be the Hecke algebra (over \( \mathbb{C} \)) attached by Iwahori and Matsumoto [IM] to an affine Weyl group \( W \) and to a parameter \( q_0^2 \in \mathbb{C}^* \).

When \( q_0 \) is not a root of 1 or \( q_0^2 = 1 \), the simple \( H_{q_0} \)-modules have been classified (see [KL2]). However we know little about the simple \( H_{q_0} \)-modules when \( q_0 \) is a root of 1. In this paper we give some discussion to the representations of \( H_{q_0} \) with \( q_0 \) a root of 1.

Namely, let \( J \) be the asymptotic Hecke algebra defined in [L3, III]. There exists a natural injection \( \phi_{q_0}: H_{q_0} \rightarrow J \). Let \( K(J) \) [resp. \( K(H_{q_0}) \)] be the Grothendieck group of \( J \)-modules (resp. \( H_{q_0} \)-modules) of finite dimension over \( \mathbb{C} \), then \( \phi_{q_0} \) induces a surjective homomorphism \( (\phi_{q_0})_*: K(J) \rightarrow K(H_{q_0}) \), when \( q_0 \) is not a root of 1 or \( q_0^2 = 1 \), \( (\phi_{q_0})_* \) is an isomorphism (loc. cit.). For each two-sided cell \( c \) of \( W \), we can define the direct summand \( K(J_c) \) [resp. \( K(H_{q_0})_c \)] of \( K(J) \) [resp. \( K(H_{q_0}) \)]. Thus \( (\phi_{q_0})_* \) induces a homomorphism \( (\phi_{q_0})_{*,c}: K(J_c) \rightarrow K(H_{q_0})_c \). The map \( (\phi_{q_0})_{*,c} \) remains surjective and is an isomorphism if \( q_0 \) is not a root of 1 or \( q_0^2 = 1 \). In this paper we mainly discuss the map \( (\phi_{q_0})_{*,c_0} \) where \( c_0 \) is the lowest two-sided cell of \( W \).

1. Introduction

1.1. Let \( G \) be a simply connected, almost simple complex algebraic group and \( T \) a maximal torus. Let \( P \leq X = \text{Hom}(T, \mathbb{C}^*) \) be the root lattice. The Weyl group \( W_0 = N_G(T)/T \) of \( G \) acts on \( X \) in a natural way and this action is stable on \( P \). Thus we can form the affine Weyl group \( W_a = W_0 \times P \), which is a normal subgroup of the extended affine Weyl group \( W = W_0 \times X \). There exists a finite abelian subgroup \( \Omega \) of \( W \) such that \( W = \Omega \times W_a \). Let \( S = \{ r_0, r_1, \ldots, r_n \} \) be the set of simple reflections of \( W_a \) with \( r_0 \notin W_0 \). Then we have a standard length function \( l \) on \( W_a \) which can be extended

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to $W$ by defining $l(\alpha w) = l(w)$ for any $\alpha \in \Omega$, $w \in W_a$. We keep the same notation for the extension of $l$.

1.2. For any $u = \alpha_1 u_1$, $w = \alpha_2 w_1$, $\alpha_1, \alpha_2 \in \Omega$, $u_1, w_1 \in W_a$, we define $P_{u, w}$ to be $P_{u_1, w_1}$, as in [KL I] if $\alpha_1 = \alpha_2$ and $P_{u, w}$ to be zero if $\alpha_1 \neq \alpha_2$. We say that $u \leq L_R w$ or $u \leq L_R w$ if $u_1 \leq L_R w_1$, or $u_1 \leq L_R w_1$ in the sense of [KL I], we say that $u \leq w$ if $u^{-1} \leq w^{-1}$. These equivalence classes are called two-sided cells, left cells, right cells of $W$, respectively. The relation $\leq$ (resp. $\leq$, $\leq$) in $W$ then induces a partial order $\leq$ (resp. $\leq$, $\leq$) in the set of two-sided (resp. left, right) cells of $W$. We extend the Bruhat order $\leq$ in $W_a$ to $W$ by defining $u \leq w$ if and only if $\alpha_1 = \alpha_2$ and $u_1 \leq w_1$.

Let $q$ be an indeterminate and let $A = \mathbb{C}[q, q^{-1}]$. Let $H$ be the Hecke algebra of $W$ over $A$, that is a free $A$-module with basis $T_w (w \in W)$ and multiplication defined by

$$(T_r - q^a)(T_s + 1) = 0 \quad \text{if} \quad r \in S \quad \text{and} \quad T_w T_{w'} = T_{ww'} \quad \text{if} \quad l(ww') = l(w) + l(w').$$

For each $w \in W$, let

$$C_w = q^{-l(w)} \sum_{u \leq w} P_{u, w} (q^a) T_u \in H.$$  

And we write

$$C_w C_u = \sum_z h_{w, u, z} C_z \in H, \quad h_{w, u, z} \in A.$$  

For each $z \in W$, there is a well defined integer $a(z) \geq 0$ such that

$$q^{a(z)} h_{w, u, z} \in \mathbb{C}[q] \quad \text{for all} \quad w, u \in W,$$

$$q^{a(z)-1} h_{w, u, z} \notin \mathbb{C}[q] \quad \text{for some} \quad w, u \in W$$

(see [L 3, I, 7.3]). We have $a(z) \leq l(w_0)$, where $w_0$ is the longest element of $W_0$. It is known that

$$C_0 = \{ w \in W \mid a(w) = l(w_0) \}$$

is a two-sided cell of $W$ (see [S, I]) which is the lowest one for the partial order $\leq$.

1.3. Let $\gamma_{w, u, z}$ be the constant term of $q^{a(z)} h_{w, u, z} \in \mathbb{C}[q]$. We have $\gamma_{w, u, z} \in \mathbb{N}$. Moreover (see [L 3, II])

(a) \hspace{1cm} $\gamma_{w, u, z} \neq 0 \quad \Rightarrow \quad w \sim L u^{-1}, \quad u \sim L z, \quad w \sim R z.$

Let $J$ be the $C$-vector space with basis $(t_u)_{u \in W}$. This is an associative $C$-algebra with multiplication

$$t_u t_v = \sum_z \gamma_{w, u, z} t_z.$$
It has a unit element $1 = \sum_{d \in \mathcal{D}} t_d$, where $\mathcal{D} = \{ d \in W_a | a(d) = l(d) - 2 \deg P_e, d \}$ (e is the unit of $W$) (see [L 3, II]).

For each two-sided cell $c$ of $W$, let $J_c$ be the subspace of $J$ spanned by $t_{we}, w \in c$, then $J = \bigoplus J_c$, where the sum is over the set of all two-sided cells of $W$. By (a) we see that $J_c$ is a two-sided ideal of $J$ and in fact is an associative $C$-algebra with unit $\sum_{d \in \mathcal{D} \cap c} t_d$.

1.4. For each $q_0 \in \mathbb{C}^*$, we denote $H_{q_0} = H \otimes_A C$, where $C$ is an $A$-algebra with $q$ acting as scalar multiplication by $q_0$. We shall denote $T_w \otimes 1, C_w \otimes 1$ in $H_{q_0}$ again by $T_w, C_w$. We also use the notation $h_{w, u, z}$ for the specialization at $q_0 \in \mathbb{C}^*$ of $h_{w, u, z}$.

The $A$-linear map $\phi : H \to J \otimes_C A$ defined by

$$\phi(C_w) = \sum_{d \in \mathcal{D}} \sum_{z \in W} h_{w, d, z} t_z$$

is a homomorphism of $A$-algebra with $1$ (see [L 3, II]). Let $\phi_{q_0} : H_{q_0} \to J$ be the induced homomorphism for any $q_0 \in \mathbb{C}^*$.

Any (left) $J$-module $E$ gives rise, via $\phi_{q_0} : H_{q_0} \to J$, to a (left) $H_{q_0}$-module $E_{q_0}$. We denote by $K(J)$ (resp. $K(H_{q_0})$) the Grothendieck group of (left) $J$-modules (resp. $H_{q_0}$-modules) of finite dimension over $C$. The correspondence $E \mapsto E_{q_0}$ defines a homomorphism $(\phi_{q_0})_* : K(J) \to K(H_{q_0})$.

We similarly define $K(J_c)$ for any two-sided cell $c$ of $W$. Then we have $K(J) = \bigoplus_{c} K(J_c)$, where the sum is over the set of all two-sided cells of $W$. Now we define $K(H_{q_0})_c$. For any simple $H_{q_0}$-module $M$, we attach to $M$ a two-sided cell $c_M$ of $W$ by the following two conditions:

$$C_w M \neq 0 \text{ for some } w \in c_M$$
$$C_w M = 0 \text{ for any } w \text{ in a two-sided cell } c \text{ with } c \subseteq c_M, c \neq c_M.$$

Then $c_M$ is well defined since there are only a finite number of two-sided cells in $W$. Let $K(H_{q_0})_c$ be the subgroup of $K(H_{q_0})$ spanned by simple $H_{q_0}$-modules $M$ with $c_M = c$. Obviously we have $K(H_{q_0}) = \bigoplus_c K(H_{q_0})_c$. Thus for a two-sided cell $c$ of $W$, $(\phi_{q_0})_* c : K(J_c) \to K(H_{q_0})_c$ induces a homomorphism

$$(\phi_{q_0})_{* c} : K(J_c) \to K(H_{q_0})_c.$$ 

The following result is due to Lusztig (see [L 3, III, 1.9 and 3.4]).

**Proposition 1.5.** - The map $(\phi_{q_0})_{* c}$ is surjective for any $q_0 \in \mathbb{C}^*$, moreover, $(\phi_{q_0})_{* c}$ is an isomorphism when $q_0$ is not a root of $1$ or $q_0^* = 1$. 

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE
Now we state a conjecture.

**CONJECTURE 1.6.** The map \((\phi_{q_0})_{*,c}\) is injective if \((\phi_{q_0})_{*,c'}\) is injective for some two-sided cell \(c'\) of \(W\) with \(c' \leq c\).

By proposition 1.6 one knows that \((\phi_{q_0})_{*,c}\) is injective is equivalent to that \((\phi_{q_0})_{*,c'}\) is bijective.

We mainly discuss \((\phi_{q_0})_{*,c_0}\), where \(c_0\) is the lowest two-sided cell of \(W\). We prove that if \(\sum_{w \in W_0} q_0^{2l(w)} \neq 0\), then \((\phi_{q_0})_{*,c_0}\) is injective (see Theorem 3.4) and show that \((\phi_{q_0})_{*,c_0}\) is likely not injective if \(\sum_{w \in W_0} q_0^{2l(w)} = 0\) (see Theorem 3.6).

**1.7.** Let \(H'_q\) be the subalgebra of \(H_{q_0}\) spanned by \(T_w, w \in W_0\). And let \(J'\) be the subspace of \(J\) spanned by \(t_w, w \in W_0\). \(J'\) is a \(C\)-algebra with unit \(\sum_{d \in S \cap W_0} t_d\). Let \(\phi'_{q_0} : H'_q \rightarrow J'\) be defined by

\[
\phi'_{q_0}(C_w) = \sum_{d \in S \cap W_0} h_{w, d, z}(q_0) t_z, \quad w \in W_0,
\]

then \(\phi'_{q_0}\) is a \(C\)-algebra homomorphism preserving 1.

As in 1.4 we define \(K(H'_q), K(J'), K(H'_{q_0})_{c'}, K(J'_{c'}), (\phi'_{q_0})_{*}, (\phi'_{q_0})_{*,c'},\) etc., where \(c'\) is a two-sided cell of \(W_0\). We also have

**PROPOSITION 1.8.** \((\phi'_{q_0})_{*,c'}\) is surjective for any \(q_0 \in C^*\). Moreover \((\phi'_{q_0})_{*,c'}\) is an isomorphism when \(q_0\) is not a root of 1 or \(q_0^2 = 1\).

**CONJECTURE 1.9.** \((\phi'_{q_0})_{*,c'}\) is injective if \((\phi'_{q_0})_{*,c'}\) is injective for some two-sided cell \(c''\) of \(W_0\) with \(c'' \leq c'\).

When \(c'\) is the lowest two-sided cell of \(W_0\), it is easy to see that \((\phi'_{q_0})_{*,c'}\) is injective if and only if \(\sum_{w \in W_0} q_0^{2l(w)} \neq 0\).

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### 2. The two-side cell \(c_0\) and the ring \(J_{c_0}\)

In this section we recall and prove some results on \(c_0\) and \(J_{c_0}\).

**2.1.** We denote by \(w_0\) the longest element in \(W_0\). Let

\[
\mathbb{S} = \{ w \in S \mid |l(w)| = |l(w_0)| + |l(w_0)| \text{ and } \text{ for any } r \in S \cap W_0 \}.
\]

Then \(\mathbb{S} = \mathbb{D} \cap c_0\) and \(\mathbb{S} = \{ w \in W \mid w = w_0^{-1} \} \) and \(|\mathbb{S}| = |W_0|\) (see [S, II]).

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4e série - tome 27 - 1994 - n° 1.
Let $X^+ = \{ w \in W \mid l(rx) > l(x) \text{ for any } r \in S' \}$, where $S' = S \cap W_0$. Let $x_i \in X^+$ ($i \in \{1, 2, \ldots, n\} = I_0$) be the $i$-th basic dominant weight, then $x_i$ has the properties: $l(x_i r_i) < l(x_i)$, $x_i r_j = r_j x_i$, $l(x_i r_i) = l(x_i) + 1$ if $i \neq j \in I_0$. We have
\[ c_0 = \{ w' w_0 xw^{-1} \mid w, w' \in \Sigma, x \in X^+ \} \] (see [S, II]).

Moreover $l(w' w_0 xw^{-1}) = l(w') + l(w_0) + l(x) + l(w^{-1})$.

**Lemma 2.2.** Let $u \in c_0$, then $C_u = h C_{w_0} h'$ for some $h, h' \in H_\mathfrak{g}_0$, i.e., the two-sided ideal $\bigoplus_{u \in c_0} C_u$ of $H_\mathfrak{g}_0$ is generated by the element $C_{w_0}$.

**Proof.** Write $u = w' w_0 w$ for some $w', w \in W$ such that $l(u) = l(w') + l(w_0) + l(w)$. We use induction on $l(u)$ to prove that $C_u$ is in the two-sided ideal $N$ of $H_\mathfrak{g}_0$ generated by $C_{w_0}$.

When $l(u) = l(w_0)$, then $C_u = C_{w_0} C_{w_0} C_{w_0}$ for some $\omega, \omega' \in \Omega$. Now assume that $l(w') > 0$. Let $s \in S$ be such that $sw \leq w$. Then
\[ C_\omega C_{w_0} = C_\omega + \sum_{i < l(u)} a_i C_{\omega}, \quad a_i \in \mathbb{N} \] (see [KL 1]).

By induction hypothesis we know that $C_\omega \in N$. Similarly we can prove that $C_\omega \in N$ if $l(w) > 0$. The lemma is proved.

**Corollary 2.3.** For a simple $H_\mathfrak{g}_0$-module $M$, we have $c_M = c_0$ if and only if $C_{w_0} M \neq 0$.

For $w \in W$, set $L(w) = \{ r \in S \mid rw \leq w \}$ and $R(w) = \{ r \in S \mid wr \leq w \}$.

**Lemma 2.4.** (i) Let $w'$ be the longest element in the Weyl group generated by $L(w)$ (or $R(w)$), then $w = w' w''$ (or $w = w'' w'$) for some $w'' \in W$ and $l(w) = l(w') + l(w'')$.

(ii) Let $w'$ be the longest element in the Weyl group $W'$ generated by $S - L(w)$ [resp. $S - R(w)$], then $l(ww') = l(w' + l(w')$ [resp. $l(ww') = l(w')$].

**Proof.** (i) follows from $T_{w'} C_w = q^l(w') C_w$ or $C_w T_{w'} = q^l(w') C_w$.

(ii) follows from the fact that $w$ is the shortest element in $W' w$ or $w W'$.

Let $\Gamma_0$ be the left cell in $c_0$ containing $w_0$, then
\[ \Gamma_0 = \{ w w_0 x \mid x \in X^+, w \in \Sigma \} \]
\[ = \{ w \in W \mid R(w) = S' \} \]

**Lemma 2.5.** Any element $u \in \Gamma_0$ has the form $w x w_j$, where $w \in W_0$, $x = \prod_{i=1}^{n} x_i^j \in X^+$. $w_j$ is the longest element in $W_j = \langle r_j \mid a_j = 0, j \in I_0 \rangle$, moreover $l(u) = l(w) + l(x) + l(w_j)$.

**Proof.** Choose $x = \prod_{i=1}^{n} x_i^j \in X^+$ such that $u \in \Gamma_0 \cap W_0 x W_0$. 

*ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPERIEURE*
Then the shortest element in $W_0 \times W_0$ is $xw_jw_0$ and the shortest element in $\Gamma_0 \cap W_0 \times W_0$ is $xw_j$ by lemma 2.4 (i), where $w_j$ is the longest element in $W_j = \langle r_j | a_j = 0, j \in I_0 \rangle$. The lemma is proved.

**Lemma 2.6.** (i) Let $J \subseteq K \subseteq I_0$, then in $H_{q_0}$ we have $C_{w_j}C_{w_k} = C_{w_j}C_{w_k} = \eta_j C_{w_k}$, where $w_j$, $w_k$ are the longest element in $W_j = \langle r_j | j \in J \rangle$, $W_k = \langle r_k | k \in K \rangle$, respectively, $\eta_j = q_{0}^{-l(w_j)} \sum_{w \in W_j} q_{0}^{l(w)}$.

(ii) $C_{w_j} = h C_{w_j}$, $C_{w_j} w' = C_{w_j} h'$ for some $h$, $h' \in H_{q_0}$ if $l(ww_j) = l(w) + l(w_j)$, $l(w_j w') = l(w_j) + l(w')$.

**Proof.** First we prove (ii). We use induction on $l(w)$. Assume that $l(w) > 0$. Choose $r \in S$ such that $r w \leq w$, then

$$C_r C_{r w w_j} = C_{w w_j} + \sum_{z \in W} a_z C_z, a_z \in \mathbb{N} \text{ (see [KL 1]).}$$

Moreover $a_z \neq 0$ implies that $z \leq r w w_j$. So $R(z) \supseteq \{ r_j | j \in J \}$ (see [KL 1]).

By Lemma 2.4 we see that $z = z' w_j$ for some $z' \in W$ and $l(z) = l(z') + l(w_j)$. By induction hypothesis we know that $C_{w_j} = h C_{w_j}$ for some $h \in H_{q_0}$. Similarly we have $C_{w_j} w' = C_{w_j} h'$ for some $h' \in H_{q_0}$.

(i) follows from $C_j C_j = \eta_j C_j$ and (ii).

**Corollary 2.7.** Let $x, w_j$ be as in 2.5, then in $H_{q_0}$ we have

$$C_{w_0} C_{w_j} = \eta_j \sum_{y \in X^*} a_{x, y} C_{w_0 x} \in \mathbb{C}, a_{x, y} \in \mathbb{C} \text{ and } a_{x, x} = 1.$$ 

**Proof.** By 2.1 and 2.6(ii) we see that $C_{w_0} = C_{w_j} = C_{w_j} h$, where

$$h = \sum_{w \in W} a_w T_w, a_w \in \mathbb{C}, a_x = q_0^{-l(x)}.$$ 

By (2.6(i) we know that

(a) $C_{w_0} C_{w_j} = C_{w_0} C_{w_j} = \eta_j C_{w_0} h.$

Note that $h_{w_0, w_j} x \neq 0$ implies that $z \sim x w_j$, $z \sim w_0$ (see [L 3, I]), we have $z \in \Gamma_0 \cap \Gamma_0^{-1} = \{ w_0 y | y \in X^* \}$. So by (a) we get

$$C_{w_0} C_{w_j} = \eta_j \sum_{y \in X^*} a_{x, y} C_{w_0 y}, a_{x, y} \in \mathbb{C}.$$
Since \( a_w = q_0^{-1}(w) \) and \( l(w) < l(x) \) if \( a_w \neq 0 \), \( w \neq x \). We have \( a_{x, y} = 1 \) and \( a_{x, y} = 0 \) if \( l(y) > l(x) \) or \( l(y) = l(x) \) but \( x \neq y \). Let \( w \in W \) be such that \( a_w \neq 0 \). Consider the expression
\[
C_{w_0} \cdot T_w = \sum_{z^{-1} \in \Gamma_0} b_z C_z, \quad b_z \in \mathbb{C}.
\]
Since \( w \leq w_j \), we have \( b_z \neq 0 \) implies that \( z \leq w_0 \). Thus by (a) we know that \( a_{x, y} \neq 0 \) implies that \( w_0 \leq w_0 \). The Corollary is proved.

2.8. For any \( x \in X \), we choose \( x' \), \( x'' \in X^+ \) such that \( x = x' x''^{-1} \) and then define \( \mathcal{T}_x = q_0^{-1}(x) T_x (q_0^{-1(x')} T_{x''})^{-1} \). \( \mathcal{T}_x \) is independent of the choices \( x' \) and \( x'' \). We denote the conjugacy class of \( x \in X \) in \( W \) by \( O_x \) and let \( z_x = \sum_{x'' \in O_x} \mathcal{T}_{x''} \). \( z_x \) is in the center of \( H_{q_0} \). For \( x \in X^+ \), denote \( d(x', x) \) the dimension of the \( x' \)-weight space \( V(x)_x \) of \( V(x) \), where \( V(x) \) is the irreducible representation of \( G \) with highest weight \( x \). We set \( S_x = \sum_{x' \in X^+} d(x', x) z_{x'}, \ x \in X^+ \).

**Lemma 2.9.** (see [X]). In \( H_{q_0} \) we have \( C_{w' w_0 w^{-1}} S_x = S_x C_{w' w_0 w^{-1}} = C_{w' w_0 x w^{-1}} \) for any \( w' \), \( w \in \mathbb{S} \), \( x \in X^+ \).

**Lemma 2.10.** Let \( u \in \Gamma_0 \), then
\[
C_u = \sum_{y \in X^+} h_{1, y} C_{y w_1} S_y,
\]
where \( h_{1, y} \in H_{q_0} = \bigoplus_{w \in W_0} \mathbb{C} T_w = \bigoplus_{w \in W_0, x_1 = \prod_{i \in I} x_i} \mathbb{C}, \ I' = I_0 - I. \)

**Proof.** By 2.5 we see that \( u = w x w_j \), where \( w \in W_0 \), \( x = \prod_{i \in I} x_i, J = \{ j \in I_0 \mid a_j = 0 \} \).

We use induction on \( l(u) \), when \( w = e \), by 2.9 we see that \( C_u = C_{s y w_j} S_y \), where \( J' = I_0 - J, y = \prod_{j \in J'} x_j^{l_j^{-1}}, i.e. \) the lemma is true. Now assume that \( l(w) > 0 \) and choose \( r \in S' \) such that \( rw \leq w \), then
\[
C_r C_{r x w x w} = C_{x x w x w} + \sum_{z \in \Gamma_0 \setminus \{ z \} \leq l(w x w_j)} a_z C_z, \qquad a_z \in \mathbb{N}.
\]
By induction hypothesis we know that there exists \( h_{1, y} \in H_{q_0} \) such that \( C_u = \sum_{y \in X^+} h_{1, y} C_{y w_1} S_y \). The lemma is proved.

2.11. Let \( R_G \) be the ring of the rational representations ring of \( G \) tensor with \( \mathbb{C} \). Then \( R_G \) is a \( C \)-algebra with a \( C \)-basis \( V(x), x \in X^+ \). Let \( M_{\mathbb{S} \times \mathbb{S}}(R_G) \) be the \( \mathbb{S} \times \mathbb{S} \) matrix.
ring over $R_G$. Then we have

**Theorem 2.12 (see [X]).** — There is a natural isomorphism $J_{c_0} \simeq M_{\mathfrak{g} \times \mathfrak{g}}(R_G)$ such that

$$m_{w_1,w_2} = \begin{cases} V(x) & \text{if } w_1 = w', \ w_2 = w \\ 0 & \text{otherwise} \end{cases}$$

Hereafter we identify $J_{c_0}$ with $M_{\mathfrak{g} \times \mathfrak{g}}(R_G)$.

### 3. The homomorphism $(\phi_{q_{0}})_{*,c_0}$

**3.1.** For any semisimple conjugacy class $s$ in $G$, we have a simple representation $\psi_s$ of $J_{c_0} \simeq M_{\mathfrak{g} \times \mathfrak{g}}(R_G)$:

$$\psi_s : M_{\mathfrak{g} \times \mathfrak{g}}(R_G) \to M_{\mathfrak{g} \times \mathfrak{g}}(C)$$

$$(m_{w,w'}) \to (\text{tr}(s, m_{w,w'})), \ w, w' \in \mathfrak{g}.$$  

Any simple representation of $J_{c_0}$ is isomorphic to some $\psi_s$ (see [X]). Let $E_s$ be the simple $J_{c_0}$-module providing the representation $\psi_s$. $E_s$ gives rise, via

$$\phi_{q_{0},c_{0}} : H_{q_0} \to J \to J_{c_0},$$

to an $H_{q_0}$-module $E_{s,q_{0}}$. Note that $\phi_{q_{0},c_{0}}(S_x) = \sum_{w \in \mathfrak{g}} t_{w_wq_{0}w_1}^{-1}$ for any $x \in X^+$, we see that $S_x$ acts on $E_{s,q_{0}}$ by scalar $\text{tr}(s, V(x))$.

**Proposition 3.2.** — The set $\Lambda = \{(\phi_{q_{0}})_{*,c_{0}}(E_s) | s \text{ semisimple conjugacy class of } G \} - \{0\}$ is a base of $K(H_{q_{0}})_{c_{0}}$.

**Proof.** — It is easy to see that $(\phi_{q_{0}})_{*,c_{0}}(E_s) = \sum_{M} a_{M} M$, where the sum is over the set of composition factors $M$ of $E_{s,q_{0}}$ with $c_{M} = c_{0}$ and $a_{M}$ is the multiplicity of $M$ in $E_{s,q_{0}}$.

Now let $F_i = (\phi_{q_{0}})_{*,c_{0}}(E_s) \in \Lambda$, $1 \leq i \leq k$, and suppose that $\sum_{i=1}^{k} m_i F_i = 0$, $m_i \in \mathbb{Z}$. Let $F_i = \sum_{M_{ij}} a_{M_{ij}} M_{ij}$, $M_{ij}$ simple $H_{q_{0}}$-module with $c_{M_{ij}} = c_{0}$. Since $S_x$ acts on $E_{s_{ij},q_{0}}$ by scalar $\text{tr}(s_i, V(x))$. $S_x$ acts on $M_{ij}$ by scalar $\text{tr}(s_i, V(x))$ if $a_{M_{ij}} \neq 0$. $F_i \in \Lambda$ implies that $a_{M_{ij}} \neq 0$ for some $M_{ij}$. Therefore $m_i = 0$. By 1.6 we know that $(\phi_{q_{0}})_{*,c_{0}}$ is surjective, hence $\Lambda$ is a base of $K(H_{q_{0}})_{c_{0}}$. The proposition is proved.

**Corollary 3.3.** — $E_{s,q_{0}}$ has at most one composition factor to which the attached two-sided cell is $c_{0}$. Moreover the multiplicity $a_{M}$ is 1 if $E_{s,q_{0}}$ has such a composition factor $M.$
THEOREM 3.4. — If \( \sum_{w \in W_0} q_0^{2l(w)} = q_0^{l(w)} \eta_{l_0} \neq 0 \), then \( (\phi_{q_0})_{*, c_0} \) is injective, so \( (\phi_{q_0})_{*, c_0} \) is an isomorphism by 1.6.

Proof. — We have

\[
\phi_{q_0, c_0}(C_{w_0}) = \sum_{w \in \mathcal{S}} h_{w_0, wwo, w^{-1}, w0xw^{-1}} I_{w0xw^{-1}} \in J_{c_0}.
\]

We identify \( J_{c_0} \) with \( M_{\mathcal{S} \times \mathcal{S}}(R_G) \), then \( \phi_{q_0, c_0}(C_{w_0}) = (m_{w', w}) \in M_{\mathcal{S} \times \mathcal{S}}(R_G) \) and

\[
m_{w', w} = \begin{cases} 
\sum_{x \in X^+} h_{w_0, wwo, w^{-1}, w0xw^{-1}} V(x) & \text{if } w' = e \\
0 & \text{if } w' \neq e
\end{cases}.
\]

Note that \( C_{w_0} \) is the unit in \( W \). So \( C_{w_0} \) is not zero for any semisimple conjugacy class \( s \) of \( G \) since \( \psi_s \phi_{q_0, c_0}(C_{w_0}) \neq 0 \). Now let \( 0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_i = E_s, q_0 \) be a composition series of \( E_s, q_0 \) and let \( i \) be the integer such that \( C_{w_0} F_i \neq 0 \) and \( C_{w_0} F_{i-1} = 0 \). Then \( C_{w_0} M \neq 0 \) where \( M = F_i/F_{i-1} \), otherwise, \( C_{w_0} M \neq 0 \) and we have \( C_{w_0} M \neq 0 \). A contradiction, so \( C_{w_0} M \neq 0 \), i.e., \( c_M = c_0 \). That is to say \( (\phi_{q_0})_{*, c_0}(E_s) \neq 0 \). The theorem follows from proposition 3.2.

3.5. In the subsequent part of this section we assume that \( \eta_{l_0} = 0 \), i.e., \( \sum_{w \in W_0} q_0^{2l(w)} = 0 \).

Let \( \Delta_{q_0} = \{ I \subseteq I_0 \mid \eta_{l} \neq 0 \text{ but } \eta_{l'} \cup \{i\} = 0 \text{ for any } i \in I \} \). Here we use the convention that \( I' \) always denotes the complement of \( I \) in \( I_0 \), i.e., \( I' = I_0 - I \).

THEOREM 3.6. — Let \( s \) be a semisimple conjugacy class of \( G \), then \( (\phi_{q_0})_{*, c_0}(E_s) = 0 \) if and only if \( \alpha_I = 0 \) for all \( I \subseteq \Delta_{q_0} \), where

\[
\alpha_I = \sum_{x \in X^+} h_{w_0, x_1 w_1, x_2 w_2} tr(s, V(x)) \text{ for any } I \subseteq I_0.
\]

We need two lemmas.

LEMMA 3.7. — The following conditions are equivalent.

(i) \( C_{w_0} E_{s, q_0} = 0 \).

(ii) \( \psi_s \phi_{q_0, c_0}(C_{w_0}) = 0 \).

(iii) \( \sum_{x \in X^+} h_{w_0, wwo, w^{-1}, w0xw^{-1}} tr(s, V(x)) = 0 \) for all \( w \in \mathcal{S} \).

(iv) \( \sum_{x \in X^+} h_{w_0, wwo, w0x} tr(s, V(x)) = 0 \) for all \( w \in \mathcal{S} \).

(v) \( \alpha_I = 0 \) for all \( I \subseteq I_0 \).

(vi) \( \alpha_I = 0 \) for all \( I \subseteq I_0 \).

ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE
Proof. — (i) and (ii) are obviously equivalent.
Note that \( h_{w_0, w w_0 w^{-1}, z} \neq 0 \) implies that \( z = w_0 x w^{-1} \) for some \( x \in X^+ \) and that
\[
\phi_{q_0, c_0}(C_{w_0}) = (m_{w'}, w),
\]
\[
m_{w', w} = \begin{cases} \sum_{x \in X^+} h_{w_0, w w_0 w^{-1}, w_0 x w^{-1} V(x), w'} & \text{if } w' = e \\ 0, & \text{otherwise} \end{cases}
\]
we see that (ii) \( \Leftrightarrow \) (iii).

By theorem 2.9 in [X] we have \( h_{w_0, w w_0, w_0 x} = h_{w_0, w w_0 w^{-1}, w_0 x w^{-1}} \). So we have (iii) \( \Leftrightarrow \) (iv).

By Lemma 2.4 (i) we see that \( x_i w_t = w w_0 \) for some \( w \in W \). Using the method in [S] one knows that \( w \in \mathcal{W} \). Thus we have (iv) \( \Rightarrow \) (v). Now we show that (v) \( \Rightarrow \) (iv). Let \( w \in \mathcal{W} \), then \( w w_0 \in \Gamma_0 \), hence by 2.10
\[
C_{w w_0} = \sum_{\begin{array}{c} \gamma \in X^+ \\ i \in I_0 \end{array}} h_{i, y} C_{x_i w_i} S_{\gamma}, \quad h_{i, y} \in H_{q_0}.
\]
Since \( C_{w_0} h_{i, y} = a_{l, y} C_{w_0} \) for some \( a_{l, y} \in \mathbb{C} \), we have
\[
\sum_{x \in X^+} h_{w_0, w w_0, w_0 x} tr(s, V(x)) = \sum_{\begin{array}{c} \gamma \in X^+ \\ i \in I_0 \end{array}} a_{l, y} \alpha_i tr(s, V(\gamma)) = 0.
\]
Finally we prove that (v) and (vi) are equivalent.
One direction is obvious. Now assume that (vi) holds. Let \( J \subseteq I_0 \). We use induction on \( l(x_j) \) to prove that \( \alpha_j = 0 \). When \( \eta_J = 0 \) or \( J \not\subseteq \Delta_{q_0} \) we have \( \alpha_j = 0 \) by 2.7 or by (vi). Suppose \( \eta_J \neq 0 \) and \( J \not\subseteq \Delta_{q_0} \). Choose \( j \in J \) such that \( \eta_{J \cup \{j\}} \neq 0 \). Let \( K = J - \{j\} \), then \( K' = J' \cup \{j\} \). We have
\[
C_{w_0} C_{x_i w_j} = \frac{1}{\eta_K} C_{w_0} C_{w_K} C_{x_i w_j} \quad \text{(by 2.6)}
\]
\[
= \frac{\eta_J}{\eta_K} C_{w_0} (C_{w_K} C_{x_i w_j} + \sum_{\begin{array}{c} \gamma \in X^+ \\ i \in I_0 \end{array}} h_{i, y} C_{x_i w_i} S_{\gamma}), \quad h_{i, y} \in H'_{q_0} \quad \text{(by 2.6, 2.10)}.
\]
Let \( C_{w_0} h_{i, y} = a_{l, y} C_{w_0}, \quad a_{l, y} \in \mathbb{C} \). By 2.7 we see that \( a_{l, y} \eta_J \neq 0 \) implies that \( l(x_i y) < l(x_j) \). Obviously \( l(x_K) < l(x_j) \). Using induction hypothesis we get
\[
\alpha_j = \frac{\eta_J}{\eta_K} (\alpha_i tr(s, V(x_i)) + \sum_{\begin{array}{c} \gamma \in X^+ \\ i \in I_0 \end{array}} a_{l, y} \alpha_i tr(s, V(\gamma))) = 0.
\]
The lemma is proved.

Lemma 3.8. — \((\phi_{q_0, c_0}(F_y) = 0 \text{ if and only if } C_{w_0} F_{y, q_0} = 0)\).

Proof. — The "if" part is obvious. The "only if" part need to do a little more.
Assume that $C_w \neq 0$. By 3.7 we see that $\alpha_i \neq 0$ for some $I \subseteq I_0$. As in [LX] we define an automorphism $\alpha : W \to W$ by

$$\alpha(wx) = w_0wx^{-1}w_0, \quad w \in W_0, \quad x \in X.$$ 

One verifies that $\alpha$ leaves stable $W_0$, $X$, $S$, $S'$. In particular, $\alpha$ induces a bijection $\alpha : I_0 \to I_0$ and an automorphism $\sigma : H_{q_0} \to H_{q_0}$ by defining $C_u \to C_{\alpha(u)}$, $u \in W$. Let $J = \alpha(I)$, we have $\alpha(x_i) = x_j$, $\alpha(w_i) = w_j$. Consider

$$\psi_s \phi_{q_0, c_0}(C_{x^{-1}j}w) = (n_{w', w}) \in M_{\mathbb{S} \times \mathbb{S}}(\mathbb{C}).$$ 

By 2.4 and 2.12, we know that $n_{w', w} = 0$ if $w' \neq e$ and

$$n_{e, w} = \sum_{x \in X^+} h_{x^{-1}jw', w_0xw^{-1}, w_0x} tr(s, V(x)).$$

In particular,

$$n_{e, e} = \sum_{x \in X^+} h_{x^{-1}jw', w_0x} tr(s, V(x)).$$

We claim that $n_{e, e} = \alpha_i$. In fact, let $\iota$ be the antiautomorphism of $H_{q_0}$ defined by $C_u \to C_{u^{-1}}$, $u \in W$. Apply $\iota$ to the equality

$$C_{w_0} C_{xjw} = \sum_{x \in X^+} h_{w_0, xjw_0x} C_{w_0x}.$$ 

We get

$$C_{x^{-1}jw} C_{w_0} = \sum_{x \in X^+} h_{w_0, xjw_0x} C_{x^{-1}w}.$$ 

Apply $\sigma$ to the above identity we obtain

$$C_{x^{-1}jw} C_{w_0} = \sum_{x \in X^+} h_{w_0, xjw_0x} C_{w_0x}.$$ 

Therefore $h_{x^{-1}jw_0, w_0x} = h_{w_0, w_0x}$ and $n_{e, e} = \alpha_i \neq 0$. By this and $n_{w', w} = 0$ if $w' \neq e$ we see that $\alpha_i$ is an eigenvalue of $\psi_s \phi_{q_0, c_0}(C_{x^{-1}jw})$. Let $0 \neq v \in E_{x, q_0}$ be such that $C_{x^{-1}jw} v = \alpha_i v$. Let $F$ be the $H_{q_0}$-submodule of $E_{x, q_0}$ generated by $v$. Then $F$ has a maximal $H_{q_0}$-submodule $F_0$ which doesn't contain $v$. $F/F_0$ is an irreducible $H_{q_0}$-module. Moreover $C_{x^{-1}jw}(F/F_0) \neq 0$ since $v \notin F_0$. We have proved that $(\phi_{q_0})_{*, c_0}(E_x) \neq 0$.

Theorem 3.6 follows from 3.7 and 3.8.

3.9. There are two special cases. One is that $\eta_{i_0} = 0$ but $\eta_i \neq 0$ for any proper subset $I$ of $I_0$. In this case we have $A_{q_0} = \{i \mid i \in I_0\}$. Let $I' = I - \{i\}$. By 2.7 we have $h_{w_0, xw_i, w_0x} = \eta_i a_i x$ for some $a_i x \in \mathbb{C}$. Moreover, $a_i x \neq 0$ implies that $w_0 x \leq w_0 x_i$ and $a_i x_i = 1$. By this we see that the equation system

$$\alpha_i = \eta_i \sum_{x \in X^+} a_i x tr(s, V(x)) = 0, \quad i \in I_0$$

ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPERIEURE
uniquely determines \( tr(s, V(x)) \), \( i \in I_0 \). In other words, there exists a unique semisimple conjugacy class \( s \) of \( G \) such that \( \alpha_{(i)} = 0 \) for all \( i \in I_0 \). By 3.6 we have got the following.

**Proposition.** — There exists a unique semisimple conjugacy class \( s \) of \( G \) such that \( (\phi_{\alpha_0})_{\ast, 0}(E_\delta) = 0 \) when \( \eta_0 = 0 \) but \( \eta_1 \neq 0 \) for any proper subset \( I \) of \( I_0 \).

When \( W \) is of type \( \widetilde{A}_n \). We can determine the semisimple conjugacy class \( s \) in the proposition explicitly. We have \( a_i = 0 \) if \( x \neq x_i \) since \( x_i \) is a minimal dominant weight for any \( i \in I_0 \). So \( \alpha_{(i)} = \eta_i \cdot tr(s, V(x_i)) \). Let \( T \) be the diagonal subgroup of \( G = SL_{n+1}(\mathbb{C}) \). We may require that \( x_i \in Hom(T, \mathbb{C}^*) \) is defined by \( x_i(t) = t_1 t_2 \ldots t_i \) where \( t = \text{diag}(t_1, t_2, \ldots, t_{n+1}) \in T \). Thus, we have

\[
tr(s, V(x_i)) = \sum_{j_i \in I_0 \cup (n+1)} t_{j_1} t_{j_2} \ldots t_{j_i},
\]

where \( t = \text{diag}(t_1, t_2, \ldots, t_{n+1}) \in s \cap T, s \) a semisimple conjugacy class of \( G \). Hence, \( tr(s, V(x_i)) = 0 \), \( 1 \leq i \leq n \) is equivalent to that \( t_i (1 \leq i \leq n+1) \) is the solution of the equation \( \lambda^{n+1} + (-1)^{n+1} = 0 \). So if \( \eta_0 = 0 \) but \( \eta_1 \neq 0 \) for any proper subset \( I \) of \( I_0 \),

\[
(\phi_{\alpha_0})_{\ast, 0}(E_\delta) = 0 \text{ if and only if the eigenpolynomial of } s \text{ is } \lambda^{n+1} + (-1)^{n+1}.
\]

Another special case is that \( q_0 + q_0^{-1} = 0 \). In this case \( \Delta_{q_0} = \{ I_0 \} \). So \( (\phi_{q_0})_{\ast, 0}(E_\delta) = 0 \) if and only if \( \alpha_{q_0} = 0 \). If we identify the set \{semisimple conjugacy classes of \( G \} \) with \( \mathbb{C}^n \) through the bijection

\[
s \rightarrow (tr(s, V(x_1)), tr(s, V(x_2)), \ldots, tr(s, V(x_n))),
\]

then \( \alpha_{q_0} = 0 \) defines a hypersurface in \( \mathbb{C}^n \). That is to say, the set \{semisimple conjugacy class \( s \) of \( G \mid (\phi_{\alpha_0})_{\ast, 0}(E_\delta) = 0 \} \) is a variety of dimension \( n-1 \).

When \( W_0 \) is of rank 2, if \( \eta_0 = 0 \), then either \( \eta_1 \neq 0 \) for any proper subset \( I \subseteq I_0 \) or \( q_0 + q_0^{-1} = 0 \). The above discussion shows that \( (\phi_{q_0})_{\ast, 0} \) is an isomorphism if and only if \( \eta_0 \neq 0 \).

3.10. In general it is difficult to compute \( C_{w_0} C_{zw_1} \) in \( H \). Now we compute it for the simplest case: \( x_1 \) is the highest short root.

When \( x_1 \in X^+ \) is the highest short root, \( x_1 w_1 = r_0 w_0 \), and \( w_0 x = w_0 x_1 \), \( x \in X^+ \) implies that \( x = e \) or \( x_1 \). So by 2.7, in \( H \) we have

\[
C_{w_0} C_{r_0 w_0} = C_{w_0} C_{z w_1} = \sigma_T (C_{w_0 z_1} + a C_{w_0}),
\]

where \( \sigma_T \in A = \mathbb{C}[q, q^{-1}] \) is determined by \( C_{w_1} C_{w_1} = \sigma_T, C_{w_1}, a \in A \). We need to determine the coefficient \( a \). Comparing the coefficient of \( T_e \) in both sides we get

\[
q^{-l(w_0)} - 1 \sigma_{l_0} = q^{-l(w_0 w_1)} \sigma_T P_{w_0, w_0 w_1}(q) + a q^{-l(w_0)} \sigma_T.
\]

i.e.

\[
\sigma_{l_0} = q^{l-1(x)} \sigma_T P_{w_0, w_0 w_1}(q^2) + a \sigma_T.
\]
Using the formula 8.10 in [L2] we get the following

**Proposition 3.11.** If \( x_i \) is the highest short weight, then

\[
P_{w_0, w_0 x_1} = \begin{cases} 
\sum_{i=1}^{n} q^{i-1} & \text{for type } \tilde{A}_n, \tilde{B}_n, \tilde{E}_n, \\
1 & \text{for type } \tilde{C}_n, \tilde{G}_2, \\
q^2(n-1) - 1 & \text{for type } \tilde{F}_4, \\
q^2 - 1 & \text{for type } \tilde{F}_4.
\end{cases}
\]

where \( e_1, \ldots, e_n \) are the exponents of \( W_0 \).

By the proposition and 3.10(a) we obtain the following

**Proposition 3.12.** In \( H \) we have

\[
C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x_1 w_1} = \sigma_1 C_{w_0 x_1} + \frac{\sigma_{10}}{[e_n + 1]} [e_n] C_{w_0},
\]

where \( e_n \) is the largest exponent of \( W_0 \) and \([i] = (q^i - q^{-i})/(q - q^{-1})\) for any \( i \in \mathbb{N} \).

**3.13.** When \( W \) is of type \( \tilde{A}_n \), the highest short weight is \( x_n \).

\[
\eta_{10} = \eta_{+1} = \cdots = \eta_{n+1} = 1.
\]

Now suppose \([n]_{q_0} = 0 \) but \([i]_{q_0} \neq 0 \) for \( i, 1 \leq i \leq n - 1 \), then \( \Lambda_{q_0} = \{1, n\}, \{2\}, \{3\}, \ldots, \{n - 1\} \).

**4. Examples**

**4.1.** Type \( \tilde{A}_1 \). In this case \( G = SL_2(\mathbb{C}), S = \{r_0, r_1\}, x_1 = r_0 \omega, \Omega = \{e, \omega\}, \eta_{10} = q_0 + q_0^{-1}, c_0 = \{w \in W \vert l(w) > 0\}. \) Another two-sided cell \( e \) of \( W \) is \( \Omega \).

\( J_e \) has two irreducible modules \( F_0, F_1 \). Both have dimension 1 and \( t_w \) acts on \( F_i \) by scalar \( (-1)^i, i = 0, 1 \). Via, \( \phi_{q_0} \colon H_{q_0} \to J \to J_e, F_i \to H_{q_0} \)-module \( F_{i, q_0} \). \( T_w \) acts on \( F_{i, q_0} \) by scalar \( (-1)^i \) and \( T_{r_1} \) acts on \( F_{i, q_0} \) by scalar \( -1 \). \( \phi_{q_0} \) is an isomorphism for any \( q_0 \in \mathbb{C}^* \).
For \( c_0 \), we have \( J_{c_0} = M_{2 \times 2} (R_G) \) and
\[
\phi_{q_0, c_0} (C_{r_1}) = \begin{pmatrix} \eta_{t_0} & V(x_1) \\ 0 & 0 \end{pmatrix},
\phi_{q_0, c_0} (C_{r_0}) = \begin{pmatrix} 0 & 0 \\ V(x_1) & \eta_{t_0} \end{pmatrix},
\phi_{q_0, c_0} (C_{a_0}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Suppose that \( \eta_{t_0} \neq 0 \). Let \( s \) be the semisimple conjugacy class of \( G \) containing \( (t, 0, 0) \in G \). Then \( E_{s, q_0} \) is irreducible if and only if \( \eta_{t_0} \neq \pm (t + t^{-1}) \). When \( \eta_{t_0} = \pm (t + t^{-1}) \), \( E_{s, q_0} = F_{s, q_0} \simeq M_{2 \times 2} (R_G) \) a\(^a\), \( \phi_{q_0} \) acts on \( M_{2 \times 2} (R_G) \) by scalar \((-1)^{i+1} \) and \( T_{r_1} \) acts on \( M_{2 \times 2} (R_G) \) by scalar \( q_0 \). \( \phi_{q_0}^* (E_{s, q_0}) = E_{s, q_0} \) if \( \eta_{t_0} \neq \pm (t + t^{-1}) \), \( \phi_{q_0}^* (E_{s, q_0}) = M_{2 \times 2} (R_G) \) if \( \eta_{t_0} = \pm (t + t^{-1}) \). In particular, when \( \eta_{t_0} \neq 0 \), \( \phi_{q_0}^* \) is an automorphism.

When \( \eta_{t_0} = 0 \), one verifies that \( E_{s, q_0} \) is irreducible if \( t + t^{-1} \neq 0 \) and \( E_{s, q_0} = F_{0, q_0} \oplus F_{1, q_0} \) if \( t + t^{-1} = 0 \). In particular \( \text{rank ker} (\phi_{q_0}) = 1 \).

4.2. Type \( \tilde{A}_2 \). In this case we have \( G = SL_3 (C) \), \( S = \{ r_0, r_1, r_2 \} \), \( \Omega = \{ 1, \omega, \omega^2 \} \) and \( \omega r_0 = r_1, \omega r_1 = r_2, \omega r_2 = r_0, \omega x_1 = r_0 r_1 r_2 x_1, \omega x_2 = r_0 r_1 r_2 x_2 \). \( W \) has three two-sided cells: \( c = \Omega, c_0, c' = W \cup c_0 \cup c_0 \). \( c' \) is the two-sided cell of \( W \) containing \( r_0, r_1, r_2 \).

It is obviously \( \phi_{q_0}^* \) is an automorphism.

Now consider \( J_c \). Any element in \( c' \) has one of the following forms: \( \omega^i r_1 x_1^a \omega^j, \omega^i r_1 x_1^a \omega^j, \omega^i r_2 x_2^a \omega^j, \omega^i r_2 x_2^a \omega^j, \omega^i r_2 x_2^a \omega^j, i, j = 0, 1, 2 \). We define a \( C \)-linear map \( \theta : J_c \to M_{3 \times 3} (A) \), \( A = C [q, q^{-1}] \), by \( \theta (t_0) = (M_{ab}) \in M_{3 \times 3} (A) \), \( w \in c' \). Assume that \( w \) is one of the above forms, then \( m_{ab} = 0 \) except \( (a, b) = (i + 1, j + 1) \) and
\[
\begin{align*}
m_{i+1, j+1} &= \begin{cases} q^{2a} & \text{if } w = \omega^i r_1 x_1^a \omega^j \\ q^{2a-1} & \text{if } w = \omega^i r_1 x_1^a \omega^j \\ q^{-2a} & \text{if } w = \omega^i r_1 x_1^a \omega^j \\ q^{-2a+1} & \text{if } w = \omega^i r_2 x_2^a \omega^j \\ q^{2a+1} & \text{if } w = \omega^i r_2 x_2^a \omega^j \\ q^{-2a+1} & \text{if } w = \omega^i r_2 x_2^a \omega^j \\ q^{2a+1} & \text{if } w = \omega^i r_2 x_2^a \omega^j. \end{cases}
\end{align*}
\]

By [L 1, 3. 8] we know that \( \theta \) is a \( C \)-algebra isomorphism. We have
\[
\begin{align*}
\theta (\phi_{q_0, c} (C_{r_1})) &= \begin{pmatrix} [2]_{q_0} & q^{-1} \\ 0 & 0 & 0 \end{pmatrix}, \\
\theta (\phi_{q_0, c} (C_{a_0})) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
\theta (\phi_{q_0, c} (C_{a_0})) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

Specialize \( q \) to \( a \in C^* \), we get a simple representation \( \psi_a \) of \( J_{c_0} = M_{3 \times 3} (A) \) and any simple representation of \( J_{c_0} \) is isomorphic to some \( \psi_a, a \in C^* \). Let \( E_a \) be a simple \( J_{c_0} \)-module providing \( \psi_a \).
A little surprisingly, the homomorphism $(\phi_{q_0})_{*,\cdot}: K(J_c) \rightarrow K(H_{q_0})_{*,\cdot}$ is an isomorphism for any $q_0 \in \mathbb{C}^*$. In fact, via $\phi_{q_0}: H_{q_0} \rightarrow J \rightarrow J_c$, $E_a$ gives rise to an $H_{q_0}$-module $E_{a,q_0}$. One verifies that $E_{a,q_0}$ has a unique quotient $M_{a,q_0}$ such that the attached two-sided cell is $c'$ and $(\phi_{q_0})_{*,\cdot}(E_a) = M_{a,q_0}$; moreover, $M_{a,q_0}$ is not isomorphic to $M_{b,q_0}$ whenever $a \neq b$.

When $\eta_1 = [2]_{q_0} [3]_{q_0} \neq 0$, $(\phi_{q_0})_{*,c_0}$ is an isomorphism by 3.4. So $(\phi_{q_0})_*$ is an isomorphism. When $[3]_{q_0} = 0$, by 3.9 we see that $(\phi_{q_0})_{*,c_0}(E_a) = 0$ if and only if

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \in s,$$

here we regard $\omega$ as a 3-th primitive root of 1 in $\mathbb{C}$. When $[2]_{q_0} = q_0 + q_0^{-1} = 0$, by 3.13 we see that $(\phi_{q_0})_{*,c_0}(E_a) = 0$ if and only if the eigenpolynomial of $s$ has the form $\lambda^3 - a \lambda^2 + a^{-1} \lambda - 1$, $a \in \mathbb{C}^*$.

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N. Xi,
Institute for Advanced Study,
School of Mathematics,
Princeton, NJ 08540.
Permanent address:
Institute of Mathematics,
Academia Sinica,
Beijing 100080, China.

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