Privacy Games: Optimal Protection Mechanism Design for Bayesian and Differential Privacy

Reza Shokri
ETH Zurich, Switzerland
firstname.lastname@inf.ethz.ch

Abstract—Perturbing information, before being shared with untrusted entities, is an effective and widely proposed approach to protect users’ privacy. However, the privacy of users and the utility of the obfuscated information are at odds with each other, and increasing one results in decreasing the other. In this paper, we propose a methodology for designing protection mechanisms that optimally trade utility for privacy, by maximizing one and guaranteeing a lower-bound on the other, while anticipating the optimal inference attack. We formulate the optimization problem of maximizing user’s utility and guaranteeing her privacy as a non zero-sum Stackelberg game. The defender (user) leads the game by designing and committing to a protection mechanism, and the adversary follows by making inference on the shared information. The solution of this game is optimal against any possible inference attack. We show that these games can be solved using linear programming. Our second contribution is to design optimal protection mechanisms using the $\epsilon$-differential privacy metric. We find the values of $\epsilon$ that maximize privacy under utility constraints. Inversely, we design mechanisms that optimize utility for a given value of $\epsilon$, as the bound on privacy. For a generic distance function between secrets, we design these optimal mechanisms for differential privacy using linear and quadratic programming. The Bayesian and differential privacy metrics complement each other, as the former measures the absolute privacy level of user due to a protection mechanism, and the latter measures the relative information leakage due to observation from the protection mechanism. A bound on one does not guarantee a bound on the other. Our third contribution is to combine the two notions. We design optimal obfuscation mechanisms that guarantee both Bayesian and differential privacy and maximize utility, or guarantee one of the privacy metrics and maximize the other under utility constraints. Our work fills the gap between Bayesian and differential privacy, and is the first work, to the best of our knowledge, that unifies different privacy metrics and provides a methodology to design optimal protection mechanisms in a generic case. Using simulation, we show that optimal differential protection mechanisms impose more utility cost, yet they are more robust to inference attacks and adversaries with accurate background knowledge. We show that the optimal joint Bayesian-differential mechanism is indeed superior to the two mechanisms individually.

Keywords—Privacy Protection Mechanism Design; Obfuscation; Perturbation; Bayesian Privacy; Differential Privacy; Utility; Stackelberg Game; Optimization; Linear Programming

I. INTRODUCTION

There are variety of ways a user’s information can leak to some untrusted entities. Information leakage usually happens in two ways: 1) when each user is independently sharing her data with an entity (e.g., an online service provider), perhaps to receive personalized service (i.e., the result of some processing on the user’s data), and 2) when aggregated information about data entries of multiple users is shared with an entity. In case the entity is untrusted and users are sensitive towards their personal data, their privacy is endangered due to the shared information. The untrusted entity not only can obtain the shared sensitive data, but is also able to infer more information about the user(s) through linkage of the observed information with his background knowledge (or side-channel information).

To protect privacy of users, we can introduce an obfuscation mechanism that intercepts the information flow and perturbs the sensitive data that is going to be shared with any untrusted entities. Depending on the data sharing architecture, this obfuscation is done either on the input or on the output of data processing mechanisms, known as input/output perturbation [1], [2]. Figure 1 illustrates both architectures for which input and output perturbation is recommended. We use an abstract model of an information channel between the user(s) and the observer entity (adversary), interrupted and modified by an obfuscation function, to model both such systems. The more noise the obfuscation mechanism adds to this channel, the higher the privacy gain of users will be. However, the privacy of users and the usefulness (utility) of the shared (obfuscated) information are at odds with each other, and increasing one results in decreasing the other.

Hence, in designing an obfuscation function, we should consider the tradeoff between privacy and utility, if we seek useful protection mechanisms. But, more importantly, we must anticipate the inference attack that follows the observation: an adversary observes the output of the obfuscation mechanism and tries to infer the private sensitive information (secret). A protection mechanism can be designed such that it is optimal against a fixed inference attack (e.g., Bayesian inference [3], [4] using as privacy attacks [5], [6]). However, it is not guaranteed that its promised privacy level can be achieved. This is due to the fact that an adversary can also design the inference attack that is optimal against the protection mechanism.

In this paper, we advocate the design of privacy protection mechanisms that are optimal against any inference attack (notably, the adaptive and strategic inference attack that is optimal against the protection mechanism used by the user). We need to design a robust and secure protection mechanism that anticipates the best attack against itself for inferring the user’s secret. This is, in nature, similar to the problem of adversarial machine learning [7], [8] and the design of security mechanism, such as intelligent spam detection algorithms [9].
Moreover, the expected distance between the observables and
based on the prior distribution we assume on the secrets.
for the user. We call it the Bayesian metric as it is computed
expected error of the inference attack as the privacy metric
the prior distribution on the secrets.
Bayesian Stackelberg game, where the adversary’s belief is
which is unknown to the adversary. However, the adversary
take obfuscation function). These actions depend on the secret,
who can later choose an optimal course of attack against it.
The defender’s action set consists of the observables (output of
the obfuscation function). These actions depend on the secret,
which is unknown to the adversary. However, the adversary
has a belief on what the secret can be. Such a game is a
Bayesian Stackelberg game, where the adversary’s belief is
the prior distribution on the secrets.

As the user wants to hide her secret, we consider the
expected error of the inference attack as the privacy metric
for the user. We call it the Bayesian metric as it is computed
based on the prior distribution we assume on the secrets.
Moreover, the expected distance between the observables and
the secret is the utility metric [18], [19], [20]. We consider
two scenarios for finding the right balance between utility and privacy: 1) optimizing privacy under a maximum utility
loss constraint, 2) optimizing utility under a minimum privacy
level constraint. The solution to these games is not achievable
through a straightforward optimization, due to the conflicting
nature of the objective. The optimization problem for finding
the protection strategy (for the first player) depends on the
optimal inference strategy (of the second player), and vice versa.
More precisely, if we know the strategy of one of the
players, we can find the optimal strategy for the other using
simple optimization techniques.

The Stackelberg game for the first scenario is a zero-
sup game as, by definition, the objective of adversary is to
reveal what the user protects (and to minimize what the user
maximizes – i.e., her privacy). The solution to this zero-sum
game is equivalent to the player’s minimax strategy [16], [21].
Hence, we can solve the game and find the optimal strategies
using linear programming, as it has already been shown in the
literature [13], [22], [23], [24]. In fact, linear programming is
the most efficient solution for this problem [25].

However, the Stackelberg game for the second scenario,
where we aim at minimizing the utility cost of protection
mechanism and guaranteeing a minimum privacy level for the
user, is not a zero-sum game. The objective function of the
defender is the utility cost metric, whereas the objective of
the attacker is his secret estimation error (which is the user’s
privacy metric). We formalize this game, and we prove that
there is also a linear programming solution for this non-zero-
sup Bayesian game. We turn the objective of adversary into a
constraint in the user’s optimization problem, and we find the
obfuscation strategy that is best response to any (including the
best) following inference strategy of the adversary. Besides our
unified formal framework, this is our first main contribution.

The Bayesian privacy metric measures the absolute privacy
level of the user, assuming a prior on the secrets, a protection
mechanism, and an inference attack. We maximize/guarantee
privacy against any inference attack, but not over any prior. So,
there is a danger that the adversary in the real-world has some
additional side information that we did not incorporate into
the secret’s prior. As a related piece of work, in [26], authors
study the effect of side-channel information on anonymity in
databases. To prevent this of harming the user’s privacy, we
make use of the differential privacy metric.

Differential privacy has been originally proposed for mea-
suring privacy of output perturbation methods in statistical
databases [27], [28], [29], [30]. The metric is sensitive only to
the difference between the probabilities of obfuscating multi-
ple secrets to the same observation. A protection mechanism
designed based on differential privacy guarantees a minimum
uncertainty for the adversary in distinguishing two secrets
being linkable to any observable. Assuming two statistical
databases to be neighbor if they differ only in one entry,
and for the case of counting queries, [18], [31] designs utility maximizing perturbation mechanisms. However, [19] presents some impossibility results of extending this approach to other types of database queries. In [32], [33], authors propose different approaches to design perturbation mechanisms for counting queries under differential privacy.

In this paper, we consider a generic formulation of differential privacy that can be applied on any distance metric on the secrets [20], [34], [35]. Under this definition, the required indistinguishability between two secrets for an observation decreases as their distance increases. In this paper, we formalize the two mentioned privacy-utility tradeoff scenarios. By using a differential privacy metric, we disentangle the optimization problems of the user and adversary. So, we can optimize the user’s objective independently from that of the adversary. We design a linear program whose solution is a utility maximizing differential–privacy protection mechanism. We also solve the privacy maximizing utility preserving mechanism. Depending on whether we use a multiplicative or additive difference metric to compare probability distributions, we show that the optimal privacy maximizing obfuscation can be constructed by solving either a quadratic or linear program. We also propose optimal inference attacks. Finding optimal differentially private obfuscation mechanisms and attacks, for generic secret distance functions and utility metrics, is our second main contribution in this paper.

Differential privacy metric, as opposed to the Bayesian privacy metric, is robust against adversary with an arbitrary background knowledge. However, it does not reflect the absolute privacy level of the user. Bayesian metric, as opposed to the differential metric, measures how close the estimate of an adversary with certain prior knowledge to the user’s secret is. Hence, the two metrics complement each other. As our third contribution in this paper, we propose a methodology to design optimal obfuscation mechanisms that take both the absolute and differential aspects of the user’s privacy into account. Thanks to our reduction of the separate optimization problems to linear programs, we can construct new linear programs that concern the new two dimensional privacy metric. For example, we can design an obfuscation mechanism that maximizes the Bayesian privacy of user, guarantees some differential privacy for her, and bounds her utility cost due to obfuscation.

We analyze different aspects of optimal protection mechanisms, through simulating on a real location data set. Our evaluation results give an insight on the differences between Bayesian and differential mechanisms. Our simulation analysis shows that an optimal differential (and joint) privacy mechanism imposes a higher utility cost compared with an optimal Bayesian privacy mechanism, for achieving the same level of Bayesian privacy (adversary’s expected error). Our results also provide a justification for this cost: optimal differential (and joint) privacy is more robust to inference attacks. So, the expected error of an adversary in estimating the user’s secret, in the case of using optimal differential (or joint) mechanism, is higher than that of optimal Bayesian mechanism. We show that the utility cost of guaranteeing optimal joint Bayesian-differential privacy is at most as high as that of the optimal differential protection mechanism. We also see from the results that the privacy guaranteed by an optimal joint mechanism is at least as high as the privacy offered by each of the optimal Bayesian or differential protection mechanisms. We also analyze the effect of adversary’s prior knowledge on privacy of users using various optimal protection mechanisms. Our results indicate that adversary’s error decreases as he assumes a more accurate prior on the secrets. However, the optimal differential and joint protection mechanisms are more robust to such knowledgable adversaries.

The rest of the paper is organized as follows. In Section II, we present our definitions, notations, and assumptions. In Section III we state the problem of optimally trading privacy and utility. In Section IV we suggest our generic game theoretic solution. In Section V and VI we present our solutions for Bayesian and differential privacy metrics, respectively. In Section VII we solve the joint Bayesian-differential privacy optimization problems. In Section VIII we study our optimal protection mechanisms in various scenarios.

II. Definitions and Assumptions

In this section, we define different parts of our model, before we formalize the problem of designing optimal privacy protection mechanisms against optimal inference attacks. We abstract away the details of the information sharing system model, and we present the problem for both input and output perturbation mechanisms. The protection mechanism can act as an input perturbation, for example, in the case of sharing the location of a mobile user with location-based services. It can also act as an output perturbation, for example, in the case of sharing/publishing some (statistical) properties of a sensitive dataset. In both cases, we refer to the sensitive data/information of the user that needs to be protected as her secret. Figure 2 illustrates the framework of information flow that we assume in this paper.

The input to the protection mechanism is a secret $s \in S$, where $S$ is the set of all possible values that $s$ can take (e.g., diseases that the user is susceptible to, the locations that the user is visiting, or the individuals that she is acquainted with). We assume that $s$ is distributed according to the following probability distribution.

\[ \pi(s) = \Pr \{ S = s \} \]  

A. Protection Mechanism

We assume the user wants to preserve her privacy with respect to $s$. Therefore, if $s$ is revealed to the untrusted observer (adversary), the user has no privacy. However, sharing $s$ brings the maximum utility (usefulness) of the information (to the user). So, privacy and utility, with respect to a secret, are at odds with each other, where the definition of the utility determines the magnitude of this conflict.

To protect her privacy, we assume that the user employs an obfuscation mechanism that transforms the secret $s$ to some noisy, yet observable, information $o \in O$ that can be shared (with the untrusted entity). We assume the broad class of
probabilistic protection mechanisms, in which the observable \(o\) is sampled according to the following conditional probability distribution.

\[
p(o|s) = \Pr\{O = o|S = s\} \tag{2}
\]

Thus, we model the privacy preserving mechanism as a noisy channel between the user and the untrusted observer. This is similar to the model used in quantitative information flow and quantitative side-channel analysis [36], [37]. The output, the set of observables \(O\), can in general be a subset of the power set of \(S\). As an example, in the most basic case, \(O = S\), i.e., the protection mechanism can only perturb the secret by replacing it with another possible value for the secret. This can happen through adding noise to \(s\). In a more generic case, the members of \(O\) can also contain a subset of secrets. For example, the protection mechanism can generalize the secret or attach some fake information to it.

### B. Utility Cost

Let a distance function \(c(o, s)\) determine the utility cost (information usefulness degradation) due to replacing a secret \(s\) with an observable \(o\). The cost function is very dependent on the application of the revealed information, and on the specific service that is provided to the user.

We compute the expected utility cost of a protection mechanism \(p\) as

\[
\sum_s \pi(s) \sum_o p(o|s) \cdot c(o, s), \tag{3}
\]

We can also compute the worst (maximum) utility cost over all possible secrets as

\[
\max_s \sum_o p(o|s) \cdot c(o, s). \tag{4}
\]

Depending on whether the user wants to minimize/limit the average utility cost of the protection mechanism or its worst-case utility cost, we will use metric (3) or (4), respectively.

### C. Inference Attack

We stated that the user is sensitive with respect to her secret and she wants to protect it against untrusted observers. To be consistent with this, we define the adversary as the entity who aims at finding the user secret. We assume that the adversary observes the outcome of the protection mechanism. For any observation \(o\), the adversary determines the probability distribution over the possible secrets \(\hat{s} \in S\) as to be the true secret of the user.

\[
q(\hat{s}|o) = \Pr\{S = \hat{s}|O = o\} \tag{5}
\]

As we don’t protect privacy by obscurity, we assume that the adversary is aware of the protection mechanism. Hence, the strategy of an adversary is to design an inference algorithm \(q\) that can invert a given protection mechanism \(p\). The error of adversary, in this estimation process, determines the effectiveness of his inference attack.

### D. Bayesian Privacy Metric

As stated above, the user’s privacy and the adversary’s inference error are two sides of the same coin. We define the privacy gain of the user with secret \(s\), when the adversary’s estimation is \(\hat{s}\), as a (sensitivity) distance between the two values: \(d(\hat{s}, s)\). The distance function \(d\) determines the sensitivity of the user towards each secret \(s\). The higher the distance between \(s\) and an estimation \(\hat{s}\) is, the less harmful the estimation is to the privacy of the user. In other words, the user is less worried about revealing \(o \sim p(o|s)\), if by observing \(o\), the adversary portrays the user’s secret \(s\) as \(\hat{s}\) such that \(d(\hat{s}, s)\) is large.

We compute the user privacy obtained through a protection mechanism \(p\), with respect to a given inference attack \(q\), for a specific secret \(s\) as

\[
\sum_o p(o|s) \sum_{\hat{s}} q(\hat{s}|o) \cdot d(\hat{s}, s). \tag{6}
\]

By averaging this value over all possible secrets, we can compute the expected privacy of the user as follows. As this metric assumes a prior distribution \(\pi(s)\) on the secret, we call it the Bayesian privacy metric. This metric also shows the average estimation error of the assumed adversary.

\[
\sum_s \pi(s) \sum_o p(o|s) \sum_{\hat{s}} q(\hat{s}|o) \cdot d(\hat{s}, s) \tag{7}
\]

In cases where the user applies the protection mechanism not on her secret, but on an attribute or a function \(x\) of the secret, one further layer of Bayesian inference must be added. This is to convert the estimate of the adversary on \(x\) to an estimate of the secret.

\[
\sum_{s,x} \pi(s) \cdot p(x|s) \sum_o p(o|x) \sum_{\hat{x},\hat{s}} q(\hat{x}|o) \cdot p(\hat{s}|\hat{x}) \cdot d(\hat{s}, s) \tag{8}
\]

where \(p(x|s)\), and also conversely \(p(s|x)\), model the relation between secret \(s\) and its attribute \(x\). More complex relations between \(s\) and \(x\) (represented using a graphical model, such as Bayesian networks) can be incorporated similarly, as a product of factors summed over the random variables.
E. Differential Privacy Metric

Privacy of the user can also be quantified independently from the attack strategy $q$. For example, the differential privacy metric, originally proposed for protecting privacy in statistical databases \cite{27, 28, 29, 30}, is sensitive only to the difference between the probabilities of obfuscating multiple secrets to the same observation (which is the input to the attack).

According to the original definition of differential privacy, a randomized function $\mathcal{K}$ (that acts as the privacy protection mechanism) provides $\epsilon$-differential privacy if for all data sets $D$ and $D'$, that differ on at most one element, and all $Y \subseteq \text{Range}(\mathcal{K})$, the following inequality holds.

\[
\Pr\{\mathcal{K}(D) \in Y\} \leq \exp(\epsilon) \Pr\{\mathcal{K}(D') \in Y\} \leq \exp(\epsilon)
\]  

(9)

This notion can be simply used for measuring information leakage \cite{38}. Using our notation, a protection mechanism is defined to be differentially private if for all neighbor secrets $s, s' \in S$ (two secrets are neighbor if $d(s, s') = 1$), and all observables $o \in O$, the following inequality holds.

\[
\frac{p(o|s)}{p(o|s')} \leq \exp(\epsilon \cdot d(s, s'))
\]  

(10)

In this paper, we use a generic definition of differential privacy, assuming arbitrary distance function $d(\cdot)$ on the secrets \cite{20, 34, 35}. In this generic form, a protection mechanism is differentially private if for any secrets $s, s' \in S$, whose distance is determined by $d(s, s')$, and for all observables $o \in O$, the (multiplicative) difference between obfuscation probabilities $p(o|s)$ and $p(o|s')$ is

\[
\frac{p(o|s)}{p(o|s')} \leq \exp(\epsilon \cdot d(s, s'))
\]  

(11)

Alternatively, considering the additive form of the difference between obfuscation probabilities, the inequality can be formulated as

\[
p(o|s) - p(o|s') \leq \delta \cdot d(s, s')
\]  

(12)

Both (11) and (12), in different ways, guarantee that the smaller the distance between two secrets is, the more indistinguishable their obfuscation probability distribution functions (on all observables) will be. Note that the additive difference, as opposed to the multiplicative difference, is not sensitive to the absolute values of the probabilities. Parameters $\epsilon$ and $\delta$ determine the extent to which the distance function $d(\cdot)$ can affect the bound.

Note that, if the user shares $x$, an attribute of the secret or its related information, both the protection mechanism (similar to the case of Bayesian metric) and the distance function will be defined on $x$. In fact, the differential privacy metric guarantees that, given the observation, there is not enough convincing evidence to prefer one value of the obfuscation mechanism input among a set (of close values).

III. Problem Statement

Maximum privacy is achieved through an obfuscation mechanism that minimizes the correlation between the observation and the secret. The strongest protection mechanism, however, imposes a considerable cost on the utility of shared information, to the extent that it is of no benefit for the user. Conversely, maximum utility is achieved through sharing the secret without any noise, which results in minimum privacy for the user. The problem that we address in this paper is to identify the optimal balance between privacy and utility. We also want to build the protection mechanism that achieves the optimal point. We assume that the adversary is able to design the optimal inference attack such that it minimizes his error in estimating the true value of the user’s secret.

The user might want to guarantee a minimum threshold for her privacy and optimize her utility. Conversely, a user might prefer to optimize her privacy under the constraint of a guarantee on the utility. We formalize both problems, however, in this paper, our main focus will be on the former.

A. Maximize Utility and Guarantee Privacy

Assume we have a utility cost function $c(\cdot)$ and a (secret sensitivity) distance function $d(\cdot)$ as described in Section II. The problem is to find a probability distribution function $p^*(\cdot)$ such that it minimizes utility cost of the user, on average,

\[
p^*(\cdot) = \arg\min_{p(\cdot)} \sum_s \pi(s) \sum_o p(o|s) \cdot c(o, s)
\]  

(13)

or, alternatively, over all the secrets

\[
p^*(\cdot) = \arg\min_{p(\cdot)} \max_s \sum_o p(o|s) \cdot c(o, s)
\]  

(14)

under the user’s privacy constraint.

1) Bayesian Privacy: Let $d_m$ be the minimum desired Bayesian privacy level of the user. The user’s average Bayesian privacy is guaranteed if the optimal obfuscation mechanism $p^*(\cdot)$ satisfies the following inequality.

\[
\sum_s \pi(s) \sum_o p^*(o|s) \sum_s q^*(s|o) \cdot d(s, s) \geq d_m
\]  

(15)

where $q^*$ is the optimal inference attack against $p^*$.

2) Differential Privacy: Let $\epsilon_m$ (for multiplicative form) and $\delta_m$ (for additive form) represent the differential privacy bound associated with the minimum desired privacy of the user. The user’s privacy is guaranteed if $p^*$ satisfies the following inequality.

\[
p^*(o|s) \leq \exp(\epsilon_m \cdot d(s, s')) \cdot p^*(o|s'), \forall s, s', o
\]  

(16)

or, alternatively, in the case of an additive differential privacy bound, it satisfies the following.

\[
p^*(o|s) \leq \delta_m \cdot d(s, s') + p^*(o|s'), \forall s, s', o
\]  

(17)
B. Maximize Privacy and Guarantee Utility

Let \( c_m \) be the maximum utility cost that the user can tolerate. The optimal protection mechanism \( p^*(\cdot) \) must satisfy the following inequality

\[
\sum_s \pi(s) \sum_o p^*(o|s) \cdot c(o, s) \leq c_m \tag{18}
\]

to guarantee a utility averaged over all the secrets. Or, if alternatively the constraint is defined over each secret, the optimal protection mechanism \( p^*(\cdot) \) should satisfy

\[
\sum_o p^*(o|s) \cdot c(o, s) \leq c_m, \forall s \tag{19}
\]

1) Bayesian Privacy: The problem is to maximize the expected error of the adversary. In this case, \( p^*(\cdot) \) is determined by solving the following optimization problem, under the utility constraint (18) or (19).

\[
\max_{p(\cdot)} \sum_s \pi(s) \sum_o p(o|s) \sum_{\hat{s}} q^*(\hat{s}|o) \cdot d(\hat{s}, s) \tag{20}
\]

where \( q^* \) is the optimal inference attack against \( p^* \).

2) Differential Privacy: The privacy level in differential privacy is represented by the bounds \( \epsilon \) (or, \( \delta \)). Thus, the optimal \( p^*(\cdot) \) is the mechanism that minimizes \( \epsilon \) in (11), or alternatively, minimizes \( \delta \) in (12).

Hence, we formulate the optimization problem, for multiplicative differential privacy bound, as

\[
\min_{p(\cdot)} \epsilon \tag{21a}
\]

s.t. \( p(o|s) \leq \exp(\epsilon \cdot d(s, s')) \cdot p(o|s'), \forall s, s', o \tag{21b}
\]

under the constraint of (18), or (19).

Similarly, for the case of an additive bound, the optimization problem is

\[
\min_{p(\cdot)} \delta \tag{22a}
\]

s.t. \( p(o|s) \leq \delta \cdot d(s, s') + p(o|s'), \forall s, s', o \tag{22b}
\]

under the constraint of (18), or (19).

IV. SOLUTION: GAME OF PRIVACY

The flow of information starts from the user where the secret is generated. The user then selects a protection mechanism, commits to it, and obfuscates her secret according to its probabilistic function. After the adversary observes the output, he has the power to design the optimal inference attack against the committed protection mechanism. This helps the adversary to invert the obfuscation mechanism and estimates the secret.

The best protection mechanism for the user, hence, is the one that anticipates the adversary’s de-obfuscation action. And, it is not primarily designed against only one fixed inference attack, but an adaptive attack which is tailored for each protection mechanism. So, assuming that the adversary designs the best inference attack against each protection mechanism, the user’s goal (as the defender) is to design the obfuscation mechanism that maximizes the user’s objective despite an adversary that optimizes his own (conflicting) objective of guessing the secret.

For each obfuscation mechanism there is an inference attack that optimizes the adversary’s objective and leads to a certain (privacy/utility) payoff for the user. The optimal obfuscation mechanism for the user is the one that brings the maximum payoff for her (under its corresponding optimal inference attack). Enumerating all pairs of user-attacker mechanisms to find the optimal obfuscation function is simply infeasible.

We model this optimization problem as a leader-follower (Stackelberg) game between the user and the adversary. More precisely, the user leads the game by choosing the protection mechanism \( p(\cdot) \), and the adversary follows by designing the inference attack \( q(\cdot) \). The solution to this game is the pair of users’ best response strategies \( p^*(\cdot) \) and \( q^*(\cdot) \) which are mutually optimal against each other. If the user implements \( p^*(\cdot) \), there would be no inference attack that can degrade her privacy/utility below the optimal.

The players of the games are the user (player 1) and the adversary (player 2). For any secret \( s \in S \), the strategy space of player 1 (the user) is the set of observables \( O \). For any observable \( o \in O \), the strategy space of player 2 (the adversary) is the set of secrets \( S \), as an adversary’s estimate \( \hat{s} \) is a member of \( S \). For a given secret \( s \in S \), we represent a mixed strategy for player 1 by a vector \( p(\cdot|s) = (p(o_1|s), p(o_2|s), \cdots, p(o_n|s)) \), where \( \{o_1, o_2, \cdots, o_m\} = O \). Similarly, a mixed strategy for player 2, for a given observable \( o \in O \) is a vector \( q(\cdot|o) = (q(s_1|o), q(s_2|o), \cdots, q(s_n|o)) \), where \( \{s_1, s_2, \cdots, s_n\} = S \).

Let the vectors \( p(\cdot|s) \) and \( q(\cdot|o) \) are respectively the conditional probability distribution functions associated with an obfuscated function for a secret \( s \) and an inference algorithm for an observable \( o \). Let \( P \) and \( Q \) be the sets of mixed strategies of players 1 and 2, respectively.

\[
P = \{p(\cdot|s) = (p(o_1|s), p(o_2|s), \cdots, p(o_n|s)), \forall s \in S : p(o_i|s) \geq 0, \forall o_i, \forall s \in S, \sum_i p(o_i|s) = 1\} \tag{23}
\]

\[
Q = \{q(\cdot|o) = (q(s_1|o), q(s_2|o), \cdots, q(s_n|o)), \forall o \in O : q(s_j|o) \geq 0, \forall s_j, \sum_j q(s_j|o) = 1\} \tag{24}
\]

A member vector of sets \( P \) or \( Q \) with a 1 for the \( k \)th component and zeros elsewhere is the pure strategy of choosing action \( k \). For example, an obfuscation function \( p(\cdot|s) \) for which \( p(o_i|s) = 0, \forall i \neq k \) and \( p(o_k|s) = 1 \) is the pure strategy of exclusively and deterministically outputting observable \( o_k \) for secret \( s \). Thus, the set of pure strategies of a player is a subset of mixed strategies of the player.

In the case of the Bayesian privacy metric, the game can be formulated as a Bayesian Stackelberg game. In this game, we assume a prior \( \pi(\cdot) \) on the secrets and we find \( p^*(\cdot) \in P \) and \( q^*(\cdot) \in Q \). In the case of a Differential privacy metric, as the metric is disconnected to the adversary’s inference attack, the optimization loop between finding \( p^*(\cdot) \) and \( q^*(\cdot) \) is broken. Nevertheless, it is still the user who plays first by choosing
that relies on the Bayes ruls to combines the prior and the value (25). This is missing in a Bayesian inference attack, distance function Bayesian inference attack. The optimal attack (26) takes the optimal protection mechanism that is, by definition, the best mechanism against the optimal inference attack. In the following sections, we solve these games and provide solutions on how to design the optimal user/adversary strategies using linear programming.

V. BAYESIAN STACKELBERG PRIVACY GAMES

We assume that the nature draws secret $s$ according to a prior distribution $\pi(s)$. Given $s$, then, player 1 (the user) draws $o$ according to her protection mechanism $p(o|s)$, and sends it to player 2 (the adversary). Given observation $o$, player 2 draws $\hat{s}$ according to his inference attack $q(\hat{s}|o)$. We assume that $\pi(s)$ is known to both players. We want to find the mutually optimal $(p^*(\cdot), q^*(\cdot))$: The solution of the Bayesian Stackelberg privacy game.

To this end, we first design the optimal inference attack against any given protection mechanism $p(\cdot)$. This will be the best response of the adversary to the user’s strategy. Then, we design the optimal protection mechanisms for the user according to her objective and constraints, as stated in Section III. These will be the user’s best strategies that anticipate the adversary’s best response, and maximize privacy/utility of the user.

A. Optimal Inference Attack

We assume that the adversary’s objective is to minimize his secret estimation error. Given a secret $s$, the distance function $d(\hat{s}, s)$ determines the error of adversary in estimating the secret as $\hat{s}$. In fact, this distance is exactly what the user wants to maximize (or put a lower bound on).

We compute the expected error of the adversary as

$$
\sum_s \pi(s) \sum_{\hat{s}} \Pr\{\hat{s}|s\} \cdot d(\hat{s}, s) =
\sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s)
$$

(25)

Therefore, we design the following linear program, through which we can compute the adversary’s inference strategy that, given a prior $\pi(\cdot)$ and obfuscation $p(\cdot)$, minimizes adversary’s expected error with respect to a distance function $d(\cdot)$.

$$
\begin{align*}
\min_{q} & \quad \sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s) \\
\text{s.t.} & \quad \sum_{\hat{s}} q(\hat{s}|o) = 1, \forall o \\
& \quad q(\hat{s}|o) \geq 0, \forall \hat{s}, o
\end{align*}
$$

(26a)

where the constraints make sure that the solution is a proper conditional probability distribution function.

The optimal attack on a protection mechanism, that considers a Bayesian privacy metric, should not be confused with a Bayesian inference attack. The optimal attack (26) takes the distance function $d(\cdot)$ into account and minimizes its expected value (25). This is missing in a Bayesian inference attack, that relies on the Bayes rules to combines the prior and the obfuscation function to build a posterior belief on the secrets. Given a prior $\pi(\cdot)$ and obfuscation mechanism $p(\cdot)$, a Bayesian attack computes inference strategy $q(\cdot)$ as follows.

$$
q(\hat{s}|o) = \frac{\pi(\hat{s}) \cdot p(o|\hat{s})}{\Pr(o)} = \sum_s \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s)
$$

(27)

B. Optimal Protection Mechanism: Maximize Utility and Guarantee Privacy

In this case, we assume the user would like to minimize her utility cost (13) under a (lower bound) constraint on her privacy (15). Therefore, we can formulate the problem as solving the following linear program.

$$
\begin{align*}
\min_{p(\cdot)} & \quad \sum_{s, o} \pi(s) p(o|s) \cdot c(o, s) \\
\text{s.t.} & \quad \sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s) \geq d_m \\
& \quad \sum_{o} p(o|s) = 1, \forall s \\
& \quad p(o|s) \geq 0, \forall o, s
\end{align*}
$$

(28a)

However, solving this optimization problem requires us to know the optimal $q^*(\cdot)$ against $p^*(\cdot)$, for which we need to know $p^*(\cdot)$ as formulated in (26). So, we have two linear programs (one for the user and one for the adversary) to solve. But, the solution of each one is a requirement in solving the other. This optimization loop reflects the game-theoretic concept of mutual best response of the two players.

The game is a nonzero-sum Stackelberg game as the user (leader player) and adversary (follower player) have different objectives. We prove that the user’s best strategy can be constructed by linear programming.

**Theorem 1**: Given a prior distribution $\pi(\cdot)$, the distance functions $d(\cdot)$ and $c(\cdot)$, and the threshold $d_m$, the solution to the following linear program is the optimal protection strategy $p^*(\cdot)$ for the user, that is the solution to (28) with respect to adversary’s best response (26).

$$
\begin{align*}
\min_{p(\cdot)} & \quad \sum_{s, o} \pi(s) p(o|s) \cdot c(o, s) \\
\text{s.t.} & \quad \sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) \geq x(o), \forall o, \hat{s} \\
& \quad \sum_{o} x(o) \leq d_m \\
& \quad \sum_{o} p(o|s) = 1, \forall s \\
& \quad p(o|s) \geq 0, \forall o, s
\end{align*}
$$

(29a)

**Proof**: We construct (29) from (28). In (28), we condition the optimal protection mechanism $p^*(\cdot)$ on its corresponding optimal inference (best response) attack $q^*(\cdot)$. So, for any observable $o$, the inference strategy $q^*(\cdot|o)$ is the one that, by definition of the best response, minimizes the expected error

$$
\sum_{\hat{s}} q(\hat{s}|o) \cdot \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s)
$$

(30)
Thus,
\[
\sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot q^*(\hat{s}|o) \cdot d(\hat{s}, s) = \\
= \sum_{o} \min_{q(.|o)} \sum_{\hat{s}} q(\hat{s}|o) \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\]
(31)

Note that (30) is an average of \(\sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s)\) over \(\hat{s}\), and thus it must be larger or equal to the smallest value of it for a particular \(\hat{s}\).

\[
\min_{q(.|o)} \sum_{\hat{s}} q(\hat{s}|o) \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\geq \min_{\hat{s}} \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\]
(32)

Let \(q'(., o)\) be a conditional probability distribution function such that for any given observable \(o\),
\[
q'(s'|o) = \begin{cases} 
1 & \text{if } s' = \arg\min_{\hat{s}} \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) \\
0 & \text{otherwise}
\end{cases}
\]

Note that \(q'(.) \in Q\) is a pure strategy that represents one particular inference attack. Moreover, (31) constructs \(q^*(.)\) such that it optimizes (30) over the set of all mixed strategies \(Q\) which includes all the pure strategies. The minimum value for the optimization over the set of mixed strategies is less than or equal to the minimum value for the optimization over its subset (the pure strategies). Thus the following inequality holds.

\[
\sum_{\hat{s}} q^*(\hat{s}|o) \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) = \\
= \min_{q(.|o)} \sum_{\hat{s}} q(\hat{s}|o) \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\leq \sum_{\hat{s}} q'(\hat{s}|o) \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
= \min_{\hat{s}} \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\]
(33)

Thus, from (32) and (33) we have
\[
\sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot q^*(\hat{s}|o) \cdot d(\hat{s}, s) = \\
= \sum_{o} \min_{\hat{s}} \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
= \sum_{o} x(o) 
\]
(34)

where \(x(o) = \min_{\hat{s}} \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s)\), or equivalently \(x(o) \leq \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s), \forall \hat{s}\).

Thus, constraint (28b) in the linear program (28) is equivalent to constraints (29b) and (29c) in the linear program (29).

The linear program is infeasible for \(d_m\) greater than some positive \(d_m^{\text{max}}\). We will compute \(d_m^{\text{max}}\) in (38) after we explain how to compute privacy maximizing protection mechanism.

C. Optimal Protection Mechanism: Maximize Privacy and Guarantee Utility

In this case, we assume the user wants to maximize her privacy (20) under a maximum utility cost constraint (13). So, we need to solve the following optimization problem in order to construct \(p^*(.)\).

\[
\max_{p()} \sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot q^*(\hat{s}|o) \cdot d(\hat{s}, s) 
\]
(35a)
\[
\text{s. t.} \sum_{s, o} \pi(s) \cdot p(o|s) \cdot c(o, s) \leq c_m \quad \forall s \in S 
\]
(35b)
\[
\sum_{o} p(o|s) = 1, \forall s, o \in S 
\]
(35c)
\[
p(o|s) \geq 0, \forall o, s 
\]
(35d)

For solving this problem, we also require \(q^*(.)\) which is unknown a priori. However, this game, as opposed to the game for optimization problem (28), is a zero-sum game as \(26b\) maximizes the negative of (35a). So, according to the Minimax theorem there is a polynomial algorithm for this game (16), (21), and it has been shown that we can solve it using linear programming (22), (25), (23), (24), (13). The user’s optimal strategy is equivalent to her minimax strategy that minimizes the maximum expected payoff the adversary can obtain. For the sake of completeness, we briefly repeat two of these solutions here.

For every estimate \(\hat{s}\) (as a pure strategy for adversary), we compute a mixed strategy for user that maximizes her privacy under the constraint that playing \(\hat{s}\) is the best response for adversary. We can compute such a mixed strategy, for our problem, using the following linear program.

\[
\max_{p()} \sum_{s, o, \hat{s}} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\]
(36a)
\[
\text{s. t.} \sum_{s, o} \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s) 
\leq \sum_{s, o} \pi(s) \cdot p(o|s) \cdot d(\hat{s}', s), \forall \hat{s}', s \in S 
\]
(36b)
\[
\sum_{s, o} \pi(s) \cdot p(o|s) \cdot c(o, s) \leq c_m 
\]
(36c)
\[
\sum_{o} p(o|s) = 1, \forall s 
\]
(36d)
\[
p(o|s) \geq 0, \forall o, s 
\]
(36e)

The optimal strategy \(p^*(.)\) is the solution for this linear program that maximizes the solution value over all \(\hat{s}\). In more details, among the feasible programs for (pure) strategy \(\hat{s}\) of the follower (adversary), we chooses a strategy \(\hat{s}^*\) that maximizes the solution value of the linear program, which is the leader’s payoff (user’s privacy). The user’s optimal strategy is equivalent to her minimax strategy that minimizes the maximum expected payoff the adversary can obtain. For the sake of completeness, we briefly repeat two of these solutions here.

As proposed in (13), we can find user’s optimal mixed strategy \(p^*(.)\) using a single linear program that combines all linear programs of (36). It finds the best response against any
advocacy’s strategy $\hat{s}$ that maximizes the user’s privacy.

$$\max_{p(\cdot)} \sum_o z(o)$$  \hspace{1cm} (37a)

s. t. $z(o) \leq \sum_s \pi(s) \cdot p(o|s) \cdot d(\hat{s}, s), \forall o, \hat{s}$ \hspace{1cm} (37b)

$$\sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o, s) \leq c_m$$ \hspace{1cm} (37c)

$$\sum_o p(o|s) = 1, \forall s$$ \hspace{1cm} (37d)

$$p(o|s) \geq 0, \forall o, s$$ \hspace{1cm} (37e)

Note that the solution to both (36) and (37) is the same optimal user strategy $p^*()$.

It has been proved that the problem cannot be solved more efficiently, as any linear program with a probability constraint on its variables can be reduced to a problem of computing the optimal mixed strategy in a two player normal-form game [25]. We can use (37) to address what we mentioned in the previous subsection about the infeasibility of the linear program (29) for $d_m$ greater than some $d_m^{\max}$. By relaxing the utility constraint of (37), we can compute $d_m^{\max}$ as follows.

$$d_m^{\max} = \max_{p(\cdot)} \sum_o z(o)$$  \hspace{1cm} (38a)

s. t. $z(o) \leq \sum_s \pi(o) \cdot p(o|s) \cdot d(\hat{s}, s), \forall o, \hat{s}$ \hspace{1cm} (38b)

$$\sum_o p(o|s) = 1, \forall s$$ \hspace{1cm} (38c)

$$p(o|s) \geq 0, \forall o, s$$ \hspace{1cm} (38d)

The value of $d_m^{\max}$ is the highest achievable Bayesian privacy for the user. It can be achieved when the observation does not give any information the user, e.g., in the case of a uniform distribution obfuscation.

VI. DIFFERENTIAL PRIVACY OPTIMIZATIONS

In this section, we design optimal differentially private protection mechanisms, for the standard multiplicative difference metric as well as the additive difference metric. We solve the optimization problems for maximizing utility under privacy constraint. We also solve optimization problems to find the optimal parameter $\epsilon$ (and $\delta$ for the additive metric) that leads to the maximum privacy under utility constraint.

A. Optimal Protection Mechanism: 
Maximize Utility and Guarantee Privacy

We design the following linear program to find the user strategy that guarantees user differential privacy (16), for a given parameter $\epsilon_m$, and minimizes the utility cost (13) of the obfuscation mechanism. The solution is $q^*(\cdot)$ and its value is the least utility cost imposed by the protection mechanism.

$$\min_{p(\cdot)} \sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o, s)$$  \hspace{1cm} (39a)

s. t. $p(o|s) \leq p(o|s') \cdot \exp(\epsilon_m \cdot d(s, s')), \forall s, s', o$ \hspace{1cm} (39b)

$$\sum_o p(o|s) = 1, \forall s$$ \hspace{1cm} (39c)

$$p(o|s) \geq 0, \forall s, o$$ \hspace{1cm} (39d)

We can guarantee differential privacy in the case of an additive difference metric (17) by appropriately updating its privacy constraint.

$$\min_{p(\cdot)} \sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o, s)$$  \hspace{1cm} (40a)

s. t. $p(o|s) \leq \delta_m \cdot d(s, s'), \forall s, s', o$ \hspace{1cm} (40b)

$$\sum_o p(o|s) = 1, \forall s$$ \hspace{1cm} (40c)

$$p(o|s) \geq 0, \forall s, o$$ \hspace{1cm} (40d)

In both cases of (39) and (40), we minimize the average utility cost considering the prior $\pi(\cdot)$. Note that this does not affect the differential privacy metric, and the guaranteed privacy is still independent of the assumed prior distribution on the secrets.

If the user wants to minimize the utility cost on each secret separately, as opposed to minimizing the average cost, we can incorporate (14) by updating the optimization objective (39a).

$$\min_{p(\cdot)} \sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o, s)$$  \hspace{1cm} (41a)

s. t. $p(o|s) \leq x, \forall s$ \hspace{1cm} (41b)

$p(o|s) \leq p(o|s') \cdot \exp(\epsilon_m \cdot d(s, s')), \forall s, s', o$ \hspace{1cm} (41c)

$$\sum_o p(o|s) = 1, \forall s$$ \hspace{1cm} (41d)

$p(o|s) \geq 0, \forall s, o$ \hspace{1cm} (41e)

This minimizes the maximum utility cost over all secrets. We can repeat the same update for (40a) as well.

B. Optimal Protection Mechanism: 
Maximize Privacy and Guarantee Utility

The parameters $\epsilon$, in (11), and $\delta$, in (12), measure the differential privacy level of a protection mechanism. The smaller these values are the higher the user’s privacy is. So, to maximize privacy of user, we design optimization problems that find optimal protection mechanisms that minimizes these parameters.

For the standard multiplicative form of differential privacy, we design the protection mechanism using the following
quadratic program.

\[
\min \exp(\epsilon) \\
\text{s.t.} \quad \sum_{s,o} p(s) \cdot p(o|s) \cdot c(o,s) \leq c_m \\
p(o|s) \leq p(o|s') \cdot (\exp(\epsilon) + \exp(d(s', s))) , \forall s, s', o \\
\sum_{o} p(o|s) = 1, \forall s \\
p(o|s) \geq 0, \forall s, o
\]  

(42a)

(42b)

(42c)

(42d)

(42e)

The variables are \( p(o|s), \forall o, s \) and \( \exp(\epsilon) \). The solution is optimal user strategy \( p^*(o|s) \) and the natural logarithm of the solution value is the optimal \( \epsilon \).

Similarly, the following linear program solves the same problem in the case of additive difference metric.

\[
\min \delta \\
\text{s.t.} \quad \sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o,s) \leq c_m \\
p(o|s) - p(o|s') \leq \delta \cdot d(s', s) \leq 0, \forall s, s', o \\
\sum_{o} p(o|s) = 1, \forall s \\
p(o|s) \geq 0, \forall s, o
\]

where the solution value is the optimal \( \delta \).

C. Optimal Inference Attack

Given the user’s protection mechanism \( p^*(o|s) \), the inference attack (26) is still a valid strategy for the adversary, as there is no dependency between the defender and attacker strategies in the case of differential privacy metric (they are disentangled by definition).

However, as the differential privacy metric (used in the protection mechanism) does not include the prior distribution on secrets, we design an inference attack whose objective is to minimize the conditional expected error

\[
E_s = \sum_{\hat{s}} \Pr(\hat{s}|s) \cdot d(\hat{s}, s)
\]

\[
= \sum_{o,\hat{s}} p^*(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s)
\]

(44)

for all secrets \( s \). This is a multi-objective optimization problem [49] that does not prefer any of the \( E_s \) (for any secret) to another. Under no such preferences, the objective is to minimize \( \sum_s E_s \), using weighted sum method with equal weight for each secret.

Thus, the following linear program constitutes the optimal inference attack, under the mentioned assumptions.

\[
\min_{q()} \sum_{s,o,\hat{s}} p^*(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s) \\
\text{s.t.} \quad \sum_{\hat{s}} q(\hat{s}|o) = 1, \forall o \\
q(\hat{s}|o) \geq 0, \forall \hat{s}, o
\]

(45a)

(45b)

(45c)

As all the weights of \( E_s \) are positive (= 1), the minimum of (45) is Pareto optimal [40]. This means that minimizing (45) is sufficient for Pareto optimality. The optimal point in a multi-objective optimization (as in our case) is Pareto optimal “if there is no other point that improves at least one objective function without detriment to another function” [59], [41].

An alternative approach is to use the min-max formulation, and minimize the maximum conditional expected error \( E_s \) over all secrets \( s \). For this, we introduce a new unknown parameter \( x \) (that will be the maximum \( E_s \)). The following linear program solves the optimal inference attack using the min-max formulation. This also provides a necessary condition for Pareto optimality [42].

\[
\min_{q()} x \\
\text{s.t.} \quad \sum_{o,\hat{s}} p^*(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s) \leq x, \forall s \\
\sum_{\hat{s}} q(\hat{s}|o) = 1, \forall o \\
q(\hat{s}|o) \geq 0, \forall \hat{s}, o
\]

(46a)

(46b)

(46c)

(46d)

We can also consider the expected error conditioned on both secret \( s \) and estimate \( \hat{s} \) as the adversary’s objective to minimize. So, we can use \( E_{s,\hat{s}} = \Pr(\hat{s}|s) \cdot d(\hat{s}, s) \) instead of \( \sum_{\hat{s}} \Pr(\hat{s}|s) \cdot d(\hat{s}, s) \) in (44), and use the same approach as in [40]. The following linear program finds the optimal inference attack that minimizes the conditional expected estimation error over all \( s \) and \( \hat{s} \), using the min-max formulation.

\[
\min_{q()} x \\
\text{s.t.} \quad \sum_{o,\hat{s}} p^*(o|s) \cdot q(\hat{s}|o) \cdot d(\hat{s}, s) \leq x, \forall s, \hat{s} \\
\sum_{\hat{s}} q(\hat{s}|o) = 1, \forall o \\
q(\hat{s}|o) \geq 0, \forall \hat{s}, o
\]

(47a)

(47b)

(47c)

(47d)

Overall, we prefer the linear program (45) as it has the least number of constraints among the above three. We can also use (26) for comparison of optimal protection mechanisms based on Bayesian and differential metrics.

VII. JOINT BAYESIAN-DIFFERENTIAL PRIVACY: OPTIMAL MECHANISM DESIGN

Obfuscation mechanisms designed based on Bayesian and differential privacy protect the user’s privacy in two different ways. There is no guarantee that a mechanism with a bound on Bayesian privacy holds a bound on differential privacy, and vice versa.

Bayesian privacy metric reflects the absolute privacy of the user, assuming a prior distribution on the secrets. However, the optimal mechanism designed upon this metric is not robust against a real-world adversary who has a more accurate prior distribution on the secrets. Thus, the privacy level that is guaranteed by such a mechanism cannot be fully trusted. Note that the protection mechanism is still optimal against any inference algorithm, but under the assumed prior.
Differential privacy metric reflects the relative information leakage of each observation about the secret. However, it is not a measure on the extent to which an adversary, who already has some knowledge about the secret, can guess the secret correctly. So, the inference might be very successful despite the fact that the obfuscation in place is a differentially private mechanism.

As both Bayesian and differential metrics guarantee certain privacy requirements of the user, it is of a great interest to achieve them both in a protection mechanism. This assures that not only the information leakage is limited, but also the absolute privacy level is at the minimum required level. Thanks to our unified formulation of privacy optimization problems as linear programs, the problem of jointly optimizing and guaranteeing privacy with different metrics can also be formulated as a linear program.

The solution to the following linear program is a protection mechanism \( p^*() \) that maximizes the user’s utility and guarantees a minimum Bayesian privacy \( d_m \) and a minimum differential privacy \( \epsilon_m \), assuming prior \( \pi() \) and distance functions \( c() \) and \( d() \). The value of the optimal solution is the utility cost imposed by the obfuscation mechanism.

\[
\begin{align*}
\min_{p()} & \sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o,s) \\
\text{s.t.} & \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s},s) \geq x(o), \forall o, \hat{s} \quad (48a) \\
& \sum_{o} x(o) \geq d_m \quad (48b) \\
& p(o|s) \leq p(o|s') \cdot \exp(\epsilon_m \cdot d(s,s')) \quad \forall s, s', o \quad (48c) \\
& \sum_{o} p(o|s) = 1, \forall s \quad (48d) \\
& p(o|s) \geq 0, \forall s, o \quad (48e)
\end{align*}
\]

Note that, as in \( (29) \), there is a possibility of no feasible solution to this optimization problem, for example, when there is little randomness in the secret’s prior distribution.

The following linear program designs an obfuscation mechanism that maximizes the user’s Bayesian privacy and guarantees a maximum utility cost \( c_m \) and a minimum differential privacy \( \epsilon_m \). The value of the optimal solution is the Bayesian privacy level achieved through obfuscation.

\[
\begin{align*}
\max_{p()} & \sum_{o} z(o) \\
\text{s.t.} & z(o) \leq \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s},s), \forall o, \hat{s} \quad (49a) \\
& \sum_{s,o} \pi(s) \cdot p(o|s) \cdot c(o,s) \leq c_m \quad (49b) \\
& p(o|s) \leq p(o|s') \cdot \exp(\epsilon_m \cdot d(s,s')) \quad \forall s, s', o \quad (49c) \\
& \sum_{o} p(o|s) = 1, \forall s \quad (49d) \\
& p(o|s) \geq 0, \forall o, s \quad (49e)
\end{align*}
\]

Similarly, we can design an obfuscation function that maximizes user’s differential privacy and guarantees a minimum Bayesian privacy \( d_m \) under a utility constraint \( c_m \). In the case of a multiplicative difference metric, the solution to the following quadratic programming is the optimal obfuscation function.

\[
\begin{align*}
\max_{p()} & \exp(\epsilon_m) \\
\text{s.t.} & \sum_{s} \pi(s) \cdot p(o|s) \cdot d(\hat{s},s) \geq x(o), \forall o, \hat{s} \quad (50a) \\
& \sum_{o} x(o) \geq d_m \quad (50b) \\
& p(o|s) \leq p(o|s') \cdot (\exp(\epsilon) + \exp(d(s',s))), \forall s, s', o \quad (50c) \\
& \sum_{o} p(o|s) = 1, \forall s \quad (50d) \\
& p(o|s) \geq 0, \forall s, o \quad (50e)
\end{align*}
\]

Note that if we consider an additive difference metric for differential privacy, the solution can be found using a linear programming. This can be achieved by appropriately updating the objective and the also constraint.

VIII. Analysis

We have implemented all our linear program solutions in a software tool. The tool is available online, and can be used to process data for different applications, in different settings.\(^1\) In this section, we use our library to design privacy protection mechanism, and also to make a comparison between different optimal mechanisms, i.e., Bayesian, Differential, and joint Bayesian-Differential privacy preserving mechanisms. We study the properties of these mechanism against optimal inference attacks, and we show how robust they are to inference attack algorithms as well as the adversary’s prior knowledge. We also investigate their utility cost for protecting privacy. Furthermore, we show that what we introduced as the optimal joint Bayesian-differential mechanism are more robust than the two mechanisms separately. Last but not least, we briefly explain different ways of approximating the linear program solutions in the case of a high number of constraints.

We use a real data-set location access profiles of mobile users. The location information belong to a 15 × 8km area. We split the area into 20 × 15 cells. We consider location of a mobile user in a cell as a secret. Hence, the set of secrets is equivalent to the set of location cells. We assume the set of observables also to be the set of cells, so the users obfuscate their location by perturbation. We consider 10 users, for which we compute the prior \( \pi() \) based on their location access profiles.

In our analysis, we focus on the following optimization scenario: a user wants to minimize the utility cost of her protection mechanism under a minimum privacy level constraint. We consider three types of optimal protection mechanism:

- **Optimal Bayesian Protection**, that is computed using linear program \( (29) \).

\(^1\) We have omitted the reference to the tool due to anonymous submission.
A. Comparing Protection Mechanisms’ Cost and Benefit

Our first goal is to have a fair comparison between optimal Bayesian privacy protection mechanism and optimal differential protection mechanism. For this, we set the differential privacy parameter $\epsilon_m$ to $\{0.1, 0.2, \ldots, 0.9\}$. For each value of $\epsilon_m$, 

1) We compute the optimal differential privacy mechanism using (39). Let $p_{e_m}^*$ be the optimal mechanism.
2) We run optimal Bayesian attack (26) on $p_{e_m}^*$, and we compute the user Bayesian privacy as $OBP(p_{e_m}^*)$.
3) We compute the optimal Bayesian privacy mechanism using (29). For this, we set the privacy lower-bound to be guaranteed $d_m$ to $OBP(p_{e_m}^*).$ Let $p_{e_m}^*$ be the optimal mechanism. So, we want to design an optimal Bayesian protection mechanism that guarantees what our optimal differential privacy mechanism provides.
4) We compute the optimal joint Bayesian-differential privacy mechanism using (48). We set the privacy lower-bounds to $\epsilon_m$ and $d_m$ for the differential and Bayesian constraints, respectively. Let $p_{e_m,d_m}^*$ be the optimal mechanism.
5) We run optimal Bayesian attack (26) on $p_{e_m}^*_{d_m}$ and $p_{e_m,d_m}^*$, and we compute the user Bayesian privacy as $OBP(p_{e_m}^*_{d_m})$ and $OBP(p_{e_m,d_m}^*)$, respectively.
6) We run basic Bayesian inference attack (27) on the three optimal mechanisms $p_{e_m}^*_{d_m}, p_{e_m}^*, and p_{e_m,d_m}^*$. Let $BBP(p_{e_m}^*_{d_m}), BBP(p_{e_m}^*), and BBP(p_{e_m,d_m}^*)$ be the user privacy against the basic attack, respectively.

Figure 3 shows the results of our analysis, explained above. Figure 3(a) shows how expected privacy of users converges down to below one (km).
**Privacy: Basic Bayesian Inference Attack**

**Privacy: Optimal Bayesian Attack**

**Joint Differential Bayesian**

*Fig. 5.* Bayesian privacy of users using any of the three optimal mechanisms against the basic Bayesian inference attack (27) versus their privacy against the optimal Bayesian attack (26). Each dot represents privacy of one user for one value of $\epsilon_m$.

**Privacy: Optimal Joint Protection**

**Privacy: Optimal (Differential/Bayesian) Protection**

*Fig. 6.* Users’ privacy using optimal differential or Bayesian protection mechanism versus using optimal joint protection mechanism. Bayesian privacy is computed using optimal Bayesian attack (26). The lower-bound privacy parameters of the optimal joint mechanism is set to $\epsilon_m$ and $d_m$ associated respectively with the analyzed differential and Bayesian mechanisms.

**Privacy: Optimal Differential Protection**

**Privacy: Optimal Bayesian Protection**

*Fig. 7.* Users’ privacy using optimal differential protection versus using optimal Bayesian protection. Each circle represents privacy of a user for a different $\epsilon_m$ and for a different prior assumed in the attack.

As we set $d_m$ to $\text{OBP}(p^*_{\epsilon_m})$, the user’s Bayesian privacy for using optimal Bayesian and optimal differential mechanism is the same, when we confront them with the optimal Bayesian attack (26). In Figure 4, however, we compare the effectiveness of these two mechanisms against basic Bayesian inference attack (27). It is interesting to observe that the optimal differential mechanism is more robust to such attacks compared with the optimal Bayesian mechanisms. This justifies the extra utility cost that we need to spend on optimal differential mechanisms.

In Figure 5, we compare the effectiveness of basic Bayesian inference attack (27) and optimal Bayesian attack (26). We show the results for all three optimal protection mechanisms. It is clear that optimal attack outperforms the basic attack, as users have a relatively higher privacy level under the basic
Bayesian inference. However, the difference is much more clear for the case of differential protection and joint protection mechanisms. The basic attack overestimates users’ privacy, as it does not take the distance function $d()$ into account, whereas the optimal Bayesian attack minimizes the expected value of $d()$ over all secrets and estimates.

In this paper, we introduced the optimal joint Bayesian-differential protection mechanisms to provide us with the benefits of both mechanisms. Figure 3(b) shows that the optimal joint mechanism is not more costly than the two optimal Bayesian and differential mechanisms. It also shows that it guarantees the highest privacy for a utility cost. To further study the effectiveness of optimal joint mechanisms, we run the following evaluation scenario. We compute optimal differential privacy for some values of $\epsilon_m$. And, we compute optimal Bayesian mechanism for some values of $d_m$ that provides higher privacy level than what provided by those differential privacy mechanisms. Figure 6 shows how the optimal joint mechanism adapts itself to guarantee the maximum of the privacy levels guaranteed by optimal Bayesian and optimal differential mechanisms individually. This is clear from the fact that users’ privacy for the optimal joint mechanism is equal to their privacy for Bayesian mechanism (that in our scenario are higher than that of differential mechanism). Thus, by adding the Bayesian privacy constraints in the design of optimal mechanism, we can increase the privacy that can be achieved only using differential mechanism, with the same utility cost.

**B. Evaluating the Effects of Prior**

When using Bayesian metric to quantify and also to privacy, we make an assumption about the prior distribution $\pi$ over the secrets. Differential privacy metric, however, does not make any such assumption. In the optimal Bayesian attack against various protection mechanism, we also make use of a prior distribution. The implicit assumption of optimal Bayesian protection mechanism is that the prior assumed by the adversary is the same as the one assumed by the defender. In this subsection, we evaluate to what extent a more informed adversary can harm privacy of users further than what is promised by the optimal protection mechanism.

To perform this analysis, we consider a scenario in which the adversary’s assumption on $\pi$, for each user, has a lower level of uncertainty compare to $\pi$. This can happen in the real world when an adversary has collected more evidence about a user’s secret than it is incorporated by the user herself for computing $\pi$. Let $\hat{\pi}$ be the estimation of adversary for $\pi$, for a given user. For the sake of our analysis, we generate some $\hat{\pi}$, such that the entropy of $\hat{\pi}$ is less than that of $\pi$. Entropy is a metric for uncertainty [43]. We compute entropy of $\pi$ as $-\sum_s \pi(s) \cdot \log_2(\pi(s))$.

We construct our protection mechanisms assuming $\pi$, and we attack them by both optimal and basic Bayesian inference, but assuming the lower entropy $\hat{\pi}$ priors. Figure 7 illustrates privacy of users against these attacks. Figure 7(a) and Figure 7(b) show privacy of users for different assumptions of $\hat{\pi}$, against optimal Bayesian attack and basic Bayesian inference attack, respectively. Both figures show user’s privacy using optimal differential protection versus optimal Bayesian protection (assuming $\pi$). We observe that a more informed adversary has a lower expected error no matter if the user employs an optimal differential mechanism or an optimal Bayesian mechanism. However, the figures further show that an optimal differential protection mechanism is more robust to attacks of a knowledgable adversary than an optimal Bayesian mechanism. Note that we set $d_m$ to $OBP(p^*_{e_m})$, as explained in Section VII-A. So, in the case $\hat{\pi} = \pi$, both optimal protection mechanisms guarantee the same level of privacy. However, as there is more information in $\hat{\pi}$ than in $\pi$, more information can be inferred from the optimal Bayesian mechanism compared with the differential mechanism.

**C. Approximating the Optimal Protection Mechanisms**

In the end, we briefly discuss the computational aspects of the design of optimal protection mechanisms. Although the solution to linear programs provide us with the optimal protection mechanism, their computation cost is quadratic (for Bayesian mechanisms) and cubic (for differential mechanisms) in the cardinality of the set of secrets and observables. Providing privacy for a large set of secrets needs a high computation budget. To establish a balance between our computation budget and our privacy requirements, we can make use of approximation techniques to design optimal protection mechanisms. In this subsection, we explore some possible approaches.

Linear programming [44] is one of the fundamental areas of mathematics and computer science, and there are variety of algorithms to solve a linear program. Surveying those algorithms and evaluating their efficiencies is out of the scope of this paper. These algorithms search the set of feasible solutions of a problem for finding the optimal solution that meets the constraints. Many of these algorithms are iterative and they converge to the optimal solution as the number of iterations increases [45], [46]. Thus, a simple approximation method is to stop the iterative algorithm when our computation budget is over. Other approximation methods exist. For example, [47] suggests a sampling algorithm to select a subset of constraints in an optimization problem to speed up the computation.

We can implement those approximation techniques to solve approximately optimal protection mechanisms in an affordable time. Furthermore, we can rely on the definition of privacy to find the constraints that have a minor contribution to the design of the protection mechanism. In this subsection, we study one approximation method: we remove the constraints for which the distance $d(s, s')$ is larger than a threshold. We can justify this by observing that, in the definition of differential privacy metric (11), the privacy is more protected when for secrets $s, s'$, the distance $d(s, s')$ is small. To put this in perspective, note that if we use the original definition of differential privacy, there would not be any constraint for $d(s, s') > 1$ (i.e., if $s$ and $s'$ are not neighbors). We can extend this to the distance between an observable and a secret, and use it in the design on optimal Bayesian mechanism.
In Figure 8 we show the privacy loss of users due to approximation. As we increase the approximation threshold, the approximation error goes to zero. This suggests that, for a large set of secrets, if we choose a relatively small threshold the designed protection mechanism provides almost the same privacy level as the optimal solution would.

IX. CONCLUSION

In this paper, we study the problem of designing optimal privacy protection mechanisms, and we address the tradeoff between privacy and utility. We propose a generic framework for quantitative privacy and utility, using which we formalize the problems of maximizing users’ utility (privacy) under a lower-bound constraint on their privacy (utility). There are very few solutions, yet for some specific privacy optimization problems. We solve these optimization problems for both Bayesian and differential privacy metrics, for the generic case of any distance function between the secrets, using linear programming. We also suggest and solve optimal joint Bayesian-differential privacy protection mechanisms that has the benefits of both mechanisms. We provide linear and quadratic program solutions for optimization problems. We implemented our solutions as a tool, using which we study different aspects of optimal mechanisms, and compare them under various scenarios. Our tool can be used as a benchmark for evaluating and comparing protection mechanisms.

REFERENCES

[1] N. R. Adam and J. C. Worthmann, “Security-control methods for statistical databases: a comparative study,” ACM Computing Surveys (CSUR), 1989.
[2] S. Chawla, C. Dwork, F. McSherry, A. Smith, and H. Wee, “Toward privacy in public databases,” in Theory of Cryptography, 2005.
[3] J. O. Berger, Statistical decision theory and Bayesian analysis. Springer, 1985.
[4] D. J. MacKay, Information theory, inference and learning algorithms. Cambridge university press, 2003.
[5] R. Shokri, G. Theodorakopoulos, J.-Y. Le Boudec, and J.-P. Hubaux, “Quantifying location privacy,” in Proceedings of the IEEE Symposium on Security and Privacy, 2011.
[6] C. Troncoso and G. Danezis, “The bayesian traffic analysis of mix networks,” in Proceedings of the 16th ACM conference on Computer and communications security, 2009.
[7] M. Barreno, B. Nelson, R. Sears, A. D. Joseph, and J. Tygar, “Can machine learning be secure?,” in Proceedings of the ACM Symposium on Information, computer and communications security, 2006.
[8] L. Huang, A. D. Joseph, B. Nelson, B. I. Rubinstein, and J. Tygar, “Adversarial machine learning,” in Proceedings of the 4th ACM workshop on Security and artificial intelligence, 2011.
[9] W. Liu and S. Chawla, “A game theoretical model for adversarial learning,” in IEEE International Conference on Data Mining Workshops (ICDM 2009), 2009.
[10] M. Brückner and T. Scheffer, “Stackelberg games for adversarial prediction problems,” in 17th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD 2011), 2011.
[11] D. Korzhyyk, Z. Yin, C. Kiekintveld, V. Conitzer, and M. Tambe, “Stackelberg vs. Nash in security games: An extended investigation of interchangeability, equivalence, and uniqueness,” Journal of Artificial Intelligence Research, vol. 41, pp. 297–327, May–August 2011.
[12] P. Paruchuri, J. P. Pearce, J. Marecki, M. Tambe, F. Ordóñez, and S. Kraus, “Efficient algorithms to solve Bayesian Stackelberg games for security applications,” in Conference on Artificial Intelligence, 2008.
[13] R. Shokri, G. Theodorakopoulos, C. Troncoso, J.-P. Hubaux, and J.-Y. Le Boudec, “Protecting location privacy: optimal strategy against localization attacks,” in Proceedings of the ACM conference on Computer and communications security, 2012.
[14] M. Manshaei, Q. Zhu, T. Alpcan, T. Basar, and J.-P. Hubaux, “Game theory meets network security and privacy,” ACM Computing Surveys, vol. 45, no. 3, 2012.
[15] R. B. Myerson, Game theory: analysis of conflict. Harvard university press, 1990.
[16] M. J. Osborne and A. Rubinstein, A course in game theory. Cambridge, Mass.: MIT Press, 1994.
[17] H. Von Stackelberg, Markform und Gleichgewicht. J. Springer, 1934.
[18] A. Ghosh, T. Roughgarden, and M. Sundararajan, “Universally utility-maximizing privacy mechanisms,” in Proceedings of the 41st annual ACM symposium on Theory of computing, pp. 351–360, ACM, 2009.
[19] H. Brenner and K. Nissim, “Impossibility of differentially private universally optimal mechanisms,” in Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on, pp. 71–80, IEEE, 2010.
[20] K. Chatzikokolakis, M. E. André, N. E. Bordenabe, and C. Palamidessi, “Broadening the scope of differential privacy using metrics,” in Privacy Enhancing Technologies, pp. 82–102., Springer, 2013.
[21] S. Dasgupta, C. Papadimitriou, and U. Vazirani, Algorithms. New York, NY: McGraw-Hill, 2008.
[22] I. Adler, “The equivalence of linear programs and zero-sum games,” International Journal of Game Theory, 2013.
[23] R. D. Luce and H. Raiffa, Games and Decisions. New York, NY: John Wiley and Sons, 1957, Dover republication 1989.
[24] J.-P. Ponssard and S. Sorin, “The lp formulation of finite zero-sum games with incomplete information,” *International Journal of Game Theory*, 1980.

[25] V. Conitzer and T. Sandholm, “Computing the optimal strategy to commit to,” in *Proceedings of the 7th ACM conference on Electronic commerce*, 2006.

[26] S. R. Ganta, S. P. Kasiviswanathan, and A. Smith, “Composition attacks and auxiliary information in data privacy,” in *Proceedings of the ACM SIGKDD international conference on Knowledge discovery and data mining*, 2008.

[27] C. Dwork, “Differential privacy,” in *Automata, languages and programming*, pp. 1–12, Springer, 2006.

[28] C. Dwork, F. McSherry, K. Nissim, and A. Smith, “Calibrating noise to sensitivity in private data analysis,” in *Theory of Cryptography*, pp. 265–284, Springer, 2006.

[29] C. Dwork, “Differential privacy: A survey of results,” in *Theory and Applications of Models of Computation*, 2008.

[30] C. Dwork and J. Lei, “Differential privacy and robust statistics,” in *Proceedings of the 41st annual ACM symposium on Theory of computing*, 2009.

[31] A. Ghosh, T. Roughgarden, and M. Sundararajan, “Universally utility-maximizing privacy mechanisms,” *SIAM Journal on Computing*, vol. 41, no. 6, pp. 1673–1693, 2012.

[32] M. Gupta and M. Sundararajan, “Universally optimal privacy mechanisms for minimax agents,” in *Proceedings of the twenty-ninth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, 2010.

[33] C. Li, M. Hay, V. Rastogi, G. Miklau, and A. McGregor, “Optimizing linear counting queries under differential privacy,” in *Proceedings of the twenty-ninth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pp. 123–134, ACM, 2010.

[34] J. Reed and B. C. Pierce, “Distance makes the types grow stronger: a calculus for differential privacy,” *ACM Sigplan Notices*, 2010.

[35] G. Barthe, B. Köpf, F. Olmedo, and S. Zanella Béguelin, “Probabilistic relational reasoning for differential privacy,” *ACM SIGPLAN Notices*, 2012.

[36] S. A. Mario, K. Chatzikokolakis, C. Palamidessi, and G. Smith, “Measuring information leakage using generalized gain functions,” 2012 *IEEE 25th Computer Security Foundations Symposium*, 2012.

[37] B. Köpf and D. Basin, “An information-theoretic model for adaptive side-channel attacks,” in *Proceedings of the 14th ACM conference on Computer and communications security*, 2007.

[38] M. S. Alvim, M. E. André, K. Chatzikokolakis, and C. Palamidessi, “Quantitative information flow and applications to differential privacy,” in *Foundations of security analysis and design VI*, 2011.

[39] R. T. Marler and J. S. Arora, “Survey of multi-objective optimization methods for engineering,” *Structural and multidisciplinary optimization*, vol. 26, no. 6, pp. 369–395, 2004.

[40] L. Zadeh, “Optimality and non-scalar-valued performance criteria,” *Automatic Control, IEEE Transactions on*, vol. 8, 1963.

[41] V. Pareto, *Manuale di economia politica*, vol. 13. Societa Editrice, 1906.

[42] K. Miettinen, *Nonlinear multiobjective optimization*, vol. 12. Springer, 1999.

[43] T. M. Cover and J. A. Thomas, *Elements of information theory*. John Wiley & Sons, 1994.

[44] S. P. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[45] Y. E. Nesterov and A. Nemirovskii, *Interior point polynomial methods in convex programming: Theory and algorithms*. SIAM Publications. SIAM, Philadelphia, USA, 1993.

[46] M. Grötschel, L. Lovász, and A. Schrijver, “The ellipsoid method and its consequences in combinatorial optimization,” *Combinatorica*, 1981.

[47] V. F. Farias and B. Van Roy, “Tetris: A study of randomized constraint sampling,” in *Probabilistic and Randomized Methods for Design Under Uncertainty*, 2006.