1-PARAMETER SUBGROUPS, FILTRATIONS, AND RATIONAL INJECTIVITY

ERIC M. FRIEDLANDER*

Abstract. We begin with an evident filtration on rational \( G_a \)-modules over an algebraically closed field of characteristic \( p > 0 \). We then extend this “filtration by degree” to apply to rational modules for a unipotent algebraic group \( U \) over \( k \) and use this filtration to characterize rationally injective \( U \)-modules. A more elaborate construction using 1-parameter subgroups leads to the “filtration by exponential degree” which applies to rational \( G \)-modules for any linear algebraic group \( G \) over \( k \) with a structure of exponential type. This filtration has many good properties, leading once again to a characterization of rationally injective \( G \)-modules. The terms of these filtrations are determined by sub-coalgebras of the natural coalgebra structure of the coordinate algebra of \( G \).

0. Introduction

The purpose of this paper is to introduce a filtration on rational \( G \)-modules for \( G \) a linear algebraic group of exponential type over a field \( k \) of characteristic \( p > 0 \). There are few such filtrations (by rational \( G \)-submodules), so this filtration by exponential degree has potential to inform our understanding of rational \( G \)-modules.

In some sense, this paper is a sequel to the author’s recent paper [4] in which a theory of support varieties \( M \mapsto V(G)_M \) was constructed for rational \( G \)-modules. The construction of the filtration by exponential degree \( \{ M_0 \subset M_1 \subset \cdots \subset M \} \) uses restrictions of \( M \) to 1-parameter subgroups \( G_a \to G \), and thus is based upon actions of \( G \) on \( M \) at \( p \)-unipotent elements of \( G \). The role of 1-parameter subgroups to study rational \( G \)-modules was introduced in [3]: the property of \( p \)-unipotent degree introduced in [3, 2.5] is the precursor to our filtration by exponential degree. The origins of this approach to filtrations lie in considerations of support varieties for modules for infinitesimal group schemes, varieties which are defined in terms of \( p \)-nilpotent actions on such modules.

The basic theme of this paper is to investigate rational \( G \)-modules through their restrictions via 1-parameter subgroups \( G_a \to G \). Whereas the support variety construction \( M \mapsto V(G)_M \) is defined in terms of restrictions of \( M \) to a \( p \)-nilpotent operator associated to each 1-parameter subgroup of \( G \), our present approach involves the full information of the restriction of \( M \) to each 1-parameter subgroup. We give one specific application: a necessary and sufficient condition for rational

* partially supported by the NSF grant DMS-0909314 and DMS-0966589.
injectivity of a rational $G$-module, something we have not succeeded in doing using support varieties. We expect further applications will be given in the near future.

Perhaps the groups of most interest are reductive groups, especially simple groups of classical type. For such groups $G$, it is instructive to compare the approach in this paper and in [1] with traditional considerations of weights for the action of a maximal torus $T \subset G$ on a rational $G$-module. Whereas consideration of weights for $T$ are highly suitable in classifying irreducible rational $G$-modules, the action at $p$-unipotent elements has the potential of recognizing extensions of such modules. Although our filtration by exponential degree involves actions at $p$-unipotent elements of $G$, Example 5.4 shows that a bound on the $T$-weights for a rational $G$-module $M$ determines a bound on the exponential degree of $M$ for $G$ reductive (where $T$ is a maximal torus for $G$). We emphasize that the filtration by exponential degree applies to rational modules for unipotent groups (see Theorem 2.13) for which one has no useful torus action.

We sketch the contents of this paper. We begin with the most elementary example $G = G_a$. Indeed, this effort was in part motivated by establishing a “local criterion” for rationally injective $G_a$-modules using support varieties. As seen in Section 3, there are counter-examples to our first proposed criterion (thereby emphasizing some of the complexity of rational $G_a$-modules), but Proposition 3.3 does provide a necessary and sufficient condition for rational injectivity in terms of the “degree filtration” of Definition 1.8. This filtration is determined by a filtration of the coordinate algebra $k[G_a]$ by sub-coalgebras $k[G_a]_{<d}$. As observed in Proposition 1.11, the category of comodules for the sub-coalgebra $k[G_a]_{<p^r}$ is naturally isomorphic to the category of rational modules for the infinitesimal kernels $G_{a(r)}$ of the linear algebraic group $G_a$.

In Section 2, we extend the degree filtration for rational $G_a$-modules introduced in the previous section to a degree filtration for rational $U$-modules, where $U \subset U_N$ is a unipotent algebraic group. For non-abelian $U$, the sub-coalgebras $k[U]_{<p^r} \subset k[U]$ which we use to define this degree filtration are not well related to the coordinate algebras of infinitesimal kernels $U_{(r)}$; nevertheless, Proposition 2.10 provides some comparison. Section 3 investigates rationally injective $U$-modules: these are easy to describe, but not by “local” data.

The key construction of this paper is that of the sub-coalgebras $(k[G])_{[d]} \subset k[G]$ in Definition 4.5. As shown in Theorem 2.13 and made explicit in Example 4.7 for many unipotent algebraic groups $G$, we have $(k[U])_{[d-1]} \subset (k[G])_{<d} \subset (k[U])_{[(p-1)d]}$ so that the degree filtration based on the sub-coalgebras $k[U]_{<d}$ is very closely related to the filtration by exponential degree based on the sub-coalgebras $(k[G])_{[d]}$ as in Definition 5.1. As mentioned in Example 4.8, the dual of the Schur algebra $S(n, d)$ for $GL_n$ has a natural embedding as a sub-coalgebra of $k[GL_n]_{[(p-1)d]}$.

Theorem 5.3 provides a list of properties for the filtration $\{M_{[d]} \mid d \geq 0\}$ of a rational $G$ module $M$ with a structure of exponential type. In particular, this is a filtration by rational $G$-submodules of $M$ and is independent of the structure of exponential type on $G$. The filtration is finite for finite dimensional rational modules and has expected functoriality properties. Proposition 5.6 gives a relation of this filtration to the theory of support varieties for $G$ as formulated in [1]. We apply the filtration in Proposition 5.7 to give a necessary and sufficient condition for $L$ to be a rationally injective $G$-module in terms of its filtration $\{L_{[d]} \mid d \geq 0\}$.
Proposition 5.6 initiates the study of full subcategories of the category of rational $G$-modules associated to the filtration of exponential degree, categories of comodules for the coalgebras $(k[G])_{[q]}$. We conclude with Grothendieck spectral sequences in Proposition 5.8 relating rational cohomology of rational $G$-modules to these subcategories.

We thank Julia Pevtsova, Paul Sobaje, and Andrei Suslin for conversations related to the contents of this paper.

1. Rational modules for the additive group $\mathbb{G}_a$

We recall that $\mathbb{G}_a$ (the additive group) has coordinate algebra $k[T]$ equipped with the coproduct

$$\Delta : k[T] \to k[T] \otimes k[T], \quad T \mapsto (T \otimes 1 + 1 \otimes T).$$

In particular, this coproduct on $k[T]$ gives $k[T]$ the structure of a rational $\mathbb{G}_a$-module (which is rationally injective). One can view that action as follows: for every commutative $k$-algebra $R$ and for every $a \in \mathbb{G}_a(R) = R$, the action of $a$ on $f(T) \in R[T]$ is given by $a \circ f(T) = f(a + T)$.

The $r$-th Frobenius kernel $\mathbb{G}_{a(r)}$ of $\mathbb{G}_a$,

$$i_r : \mathbb{G}_{a(r)} \subset \mathbb{G}_a,$$

is the closed subgroup scheme with coordinate algebra given by $i_r^* : k[\mathbb{G}_a] = k[T] \to k[T]/T^{p^r} = k[\mathbb{G}_{a(r)}]$ and group algebra (i.e., $k$-linear dual of $k[\mathbb{G}_a]$) denoted by $k\mathbb{G}_{a(r)}$. Using the notation of [10], we let $v_0, \ldots, v_{p^r-1}$ be the $k$-basis of $k\mathbb{G}_{a(r)}$ dual to the standard basis $\{T^j, 0 \leq j < p^r\}$ of $k[T]/T^{p^r}$. Denote $v_p$ by $u_s$. If $j = \sum_{\ell=0}^{s-1} j_\ell p^\ell$, $0 \leq j_\ell < p$, then

$$v_j = \frac{u_0^{j_0} \cdots u_{r-1}^{j_{r-1}}}{j_0! \cdots j_{r-1}!}.$$

Notation 1.1. (see [10]) With notation as above,

$$k\mathbb{G}_{a(r)} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

For any $r, s > 0$, the quotient maps

$$q_{r,s} : k[\mathbb{G}_{a(r)}] \cong k[T]/T^{p^r} \to k[\mathbb{G}_{a(r+s)}] \cong k[T]/T^{p^r},$$

sending $T$ to $T$ are Hopf algebra maps, whose duals we denote by

$$i_{r,s} : k\mathbb{G}_{a(r)} \to k\mathbb{G}_{a(r+s)}, \quad u_i \mapsto u_i, \quad i < r.$$

The colimit

$$k\mathbb{G}_a \equiv \lim_{\longrightarrow} k\mathbb{G}_{a(r)} \simeq k[u_0, \ldots, u_n, \ldots]/(u_0^p, \ldots, u_n^p, \ldots),$$

is the group algebra (or hyperalgebra or algebra of distributions at the identity) of $\mathbb{G}_a$.

The following evident lemma makes explicit the action of $k\mathbb{G}_a$ on a rational $G$-module $M$.

Lemma 1.2. Let $M$ be a rational $\mathbb{G}_a$-module given by the comodule structure $\Delta_M : M \to M \otimes k[\mathbb{G}_a]$. For $\phi \in k\mathbb{G}_a$,

$$\phi(m) = ((1 \otimes \phi) \circ \Delta_M)(m).$$
Consequently, the action of \( v_j \in kG_\alpha \) on the rational \( G_\alpha \)-module \( M \) is determined by the formula
\[
\Delta_M(m) = \sum_j v_j(m) \otimes T^j, \quad v_0(m) = m.
\]

In particular, the action of \( v_j \) on \( f(T) = \sum_{n \geq 0} a_n T^n \in k[G_\alpha] \cong k[T] \) is given by
\[
v_j(f(T)) = \sum_{n \geq j} a_n \binom{n}{j} T^{n-j}.
\]

This specializes to \( u_s(T^n) = \delta_{p,s,n} \).

Using (1.2.2), we immediately identify those \( kG_\alpha \)-modules which arise as rational \( G_\alpha \)-modules.

**Proposition 1.3.** Let \( M \) be a \( kG_\alpha \)-module satisfying the following condition:
\[
\forall \ m \in M, \ \exists \text{ only finitely many } v_j \text{ acting non-trivially on } m.
\]

Then the \( kG_\alpha \)-module structure on \( M \) arises from the rational \( G_\alpha \)-module structure
\[
M \to M \otimes k[G_\alpha], \quad m \in M \mapsto \sum_j v_j(m) \otimes T^j.
\]

Conversely, any rational \( G_\alpha \)-module satisfies condition (1.3.1).

We make explicit the following useful consequence of Proposition 1.3.

**Corollary 1.4.** Let \( M \) be a rational \( G_\alpha \)-module and \( S \subseteq M \) be a subset. Then the rational \( G_\alpha \)-submodule generated by \( S \), \( (G_\alpha \cdot S) \), is spanned by \( \{v_j(s); s \in S, j \geq 0\} \).

In particular, the rational \( G_\alpha \)-submodule \( (G_\alpha \cdot f(T)) \subseteq k[G_\alpha] \cong k[T] \) generated by \( f(T) \) is the subspace of \( k[T] \) spanned by \( \{v_j(f(T))\} \) as given in (1.2.3).

**Proof.** The span of \( \{v_j(s); s \in S, j \geq 0\} \) is clearly a \( kG_\alpha \)-submodule of \( M \). Thus, the corollary follows from Proposition 1.3. \( \square \)

**Example 1.5.** We can easily construct many non-isomorphic rational \( G_\alpha \)-module structures on the underlying vector space of \( k[T] \). Namely, for each \( i \geq 0 \), choose \( g_i(\underline{u}) \in kG_\alpha \cong k[u_0, u_1, \ldots]/(u_0^n, u_1^n, \ldots) \) such that \( g_i(\underline{u}) \) is a polynomial in the \( u_j \)'s with \( j \geq n_i \). If \( \lim_{n_i} n_i = \infty \), then the \( kG_\alpha \)-module structure on \( k[T] \) given by setting the action of \( u_i \) on \( k[T] \) to be that of \( g_i(\underline{u}) \) on \( k[G_\alpha] \cong k[T] \) with its canonical (injective) structure is a rational \( G_\alpha \)-module by Proposition 1.3; namely, only finite many \( g_i(\underline{u})'s \) act non-trivially on a given \( T^n \).

**Remark 1.6.** Different choices of \( g_i(\underline{u}) \) in the preceding example can lead to isomorphic rational \( G_\alpha \)-modules. For example, let \( \theta : N \to N \) be a bijection and set \( g_i(\underline{u}) = u_{\theta(i)} \). The resulting module structure on \( k[T] \) is isomorphic to that of \( k[G_\alpha] \) via the isomorphism \( \Theta_S : k[T] \to k[T] \) sending \( T^p \) to \( T^{\theta(i)} \).

The following elementary proposition justifies the functor \((-)_{\leq d} \) of Definition 1.3.

**Proposition 1.7.** For any rational \( G_\alpha \)-module \( M \) and any \( \phi \in kG_\alpha \),
\[
M_\phi \equiv \{m \in M : \phi(m) = 0\} \subset M
\]
is a rational \( G_\alpha \)-submodule of \( M \). Moreover, if \( f : M \to N \) is a map of rational \( G_\alpha \)-modules, then \( f \) restricts to \( M_\phi \to N_\phi \).
Proof. To show that $M_\phi \subset M$ is a rational $G_a$-submodule it suffices by Proposition 1.9 to show that $\psi \cdot M_\phi \subset M_\phi$ for any $\psi \in kG_a$. This follows immediately from the commutativity of $G_a$ (implying the commutativity of $kG_a$).

The second assertion concerning a map $f : M \to N$ of rational $G_a$-modules follows from the fact that $f$ necessarily commutes with the action of $\phi$. □

Proposition 1.10 enables the formulation of many natural filtrations on $(G_a - Mod)$. The motivation for considering the following is given by Proposition 1.10.

Definition 1.8. For any $d \geq 1$, we define the idempotent endo-functor

$(-)_{<d} : (G-Mod) \to (G-Mod), \quad M \mapsto M_{<d} \equiv \{m \in M : v_j(m) = 0, j \geq d\}.$

In other words, $m \in M_{<d}$ if and only if $\Delta(m) \in M \otimes k[T]_{<d}$.

For any rational $G_a$-module $M$, we consider the degree filtration

$$M_{<1} \subset M_{<2} \subset M_{<3} \subset \cdots \subset M$$

of $M$ by rational $G_a$-submodules.

The following proposition, established in [4, 2.6], follows easily from the observation that the coproduct $\Delta_M : M \to M \otimes k[G_a]$ defining the rational $G_a$-module structure on $M$ sends a finite dimensional subspace of $M$ to a finite dimensional subspace of $M \to M \otimes k[G_a]$.

Proposition 1.9. Each finite dimensional rational $G_a$-module lies in the image of $(-)_{<d}$ for $d$ sufficiently large. Consequently,

1. For any rational $G_a$-module $M$, $M = \bigcup_d M_{<d}$.
2. If $M$ is finite dimensional, then $M = M_{<d}$ for $d >> 0$.

Unlike for other linear groups considered in later sections, the filtration on the coordinate algebra $k[T]$ of $G_a$ can be viewed as a coalgebra splitting of restriction maps $k[G_a] \to k[G_a(r)]$ as observed in the next proposition.

Proposition 1.10. For each $d > 0$, the rational submodule $j_d : k[T]_{<d} \subset k[T]$ is an embedding of coalgebras. Moreover,

$$pr_d \circ j_d : k[T]_{<d} \subset k[T] \to k[T]/T^d$$

is an isomorphism of coalgebras.

Proof. The fact that $j_d$ is an embedding of coalgebras follows from the form of the coproduct $\Delta : k[T] \to K[T] \otimes k[T]$ which sends $T^n$ to $(T \otimes 1 + 1 \otimes T)^n$; thus $\Delta$ applied to a polynomial $f(T)$ of degree $< d$ is mapped to $\sum_i g_i \otimes h_i$ with the degree of each $g_i$ and each $h_i < d$.

The fact that $pr_d \circ j_d$ is injective (and thus an isomorphism by dimension considerations) is evident by inspection. □

We summarize some of the relationships between various functors on rational $G_a$-modules. We denote the abelian category of such rational modules either by $(G_a - Mod)$ or by $(kG_a)-coMod$; we denote the category of rational modules for the infinitesimal group scheme $G_a(r)$ either by $(G_a(r)-Mod)$ or by $Mod(kG_a(r))$ or by $(kG_a(r))-coMod$.

We denote by

$$\rho_r : (G_a-Mod) \to Mod(kG_a(r))$$
the restriction functor (sending a rational $G$-module $M$ with coproduct $M \to M \otimes k[G_a]$ to the comodule for $k[\mathfrak{g}_a(r)]$ with coproduct defined by composition with the projection $pr_{\rho'} : k[G_a] \to k[\mathfrak{g}_a(r)]$).

**Proposition 1.11.** Consider the full subcategory $\iota_{d} : (k[\mathfrak{g}_a]_{<d}\Mod) \subset (k[\mathfrak{g}_a]\Mod)$ of rational $G$-modules $M$ whose coproduct is of the form $M \to M \otimes k[\mathfrak{g}_a]_{<d}$.

1. The image of $\iota_{d}$ consists of rational $G$-modules $M$ such that $M = M_{<d}$.
2. $\iota_{d}$ is left adjoint to $(-)_{<d}$ for any $d > 0$.
3. The composition $\rho_{\iota} \circ \iota_{<d} : (G_a\Mod)_{<d} \overset{\sim}{\to} \Mod(k[\mathfrak{g}_a(r)])$
   
   is an equivalence of categories.

**Proof.** The first statement is essentially a tautology. The fact that $\iota_{d}$ is left adjoint to $(-)_{<d}$ follows from the observation that if $f : M \to N$ is a map of $k[\mathfrak{g}_a]$-modules and if $\phi \in k[\mathfrak{g}_a]$ vanishes on $M$, then $f$ factors (uniquely) through $N_{<d} \subset N$.

The last statement is a consequence of the isomorphism $pr_{\rho'} \circ j_{\rho'} : k[T]_{<d} \overset{\sim}{\to} k[T]/T'$ of Proposition 1.10. Namely, viewing $\rho_{\iota}$ and $\iota_{<d}$ as functors on categories of comodules, $\iota_{<d}$ is determined on comodules by composing with $j_{d'}$ and $\rho_{\iota}$ is determined by composing with $pr_{\rho'}$. \qed

2. Rational modules for unipotent groups

Let $U_{N}$ denote the unipotent radical of the standard (upper triangular) Borel subgroup of the general linear group $GL_{N}$. Then $k[U_{N}]$ is a polynomial algebra on the strictly upper triangular coordinate functions $\{x_{i,j} : 1 \leq i < j \leq N\}$. We equip $k[U_{N}]$ with the grading determined by setting the degree of each $x_{i,j}$ equal to 1. For any $d > 0$, we set $k[U_{N}]_{<d} \subset k[U_{N}]$ equal to the subspace of polynomials (functions on $U_{N}$) of degree $< d$.

**Proposition 2.1.** Let $i : U \to U_{N}$ be a closed embedding of linear algebraic groups. Set $k[U]_{<d} \subset k[U]$ equal to the image under $i^{*}$ of $k[U_{N}]_{<d} \subset k[U_{N}]$. The map of Hopf algebras $i^{*} : k[U_{N}] \to k[U]$ induces for each $d > 0$ a map of coalgebras $k[U_{N}]_{<d} \to k[U]_{<d}$.

**Proof.** The coproduct $\Delta_{U_{N}} : k[U_{N}] \to k[U_{N}] \otimes k[U_{N}]$ of the Hopf algebra $k[U_{N}]$ is a map of algebras, determined by $\Delta_{k[U_{N}]}(x_{i,j}) = (x_{i,j} \otimes 1) + \left( \sum_{i < j} x_{i,t} \otimes x_{t,j} \right) + (1 \otimes x_{i,j})$.

In particular, if $f \in k[U_{N}]$ has degree $< d$ and if $\Delta_{U_{N}}(f) = \sum f_{i}^{'} \otimes f_{i}^{''}$, then each $f_{i}^{'}$ and each $f_{i}^{''}$ also has degree $< d$.

Because $i^{*}$ is a map of Hopf algebras, $i^{*}$ determines a commutative square of algebras

(2.1.1) \[
\begin{array}{ccc}
    k[U_{N}] & \Delta_{U_{N}} & k[U_{N}] \otimes k[U_{N}] \\
    i^{*} & \downarrow & i^{*} \otimes i^{*} \\
    k[U] & \Delta_{U} & k[U] \otimes k[U]
\end{array}
\]
A simple diagram chase implies that (2.1.2) restricts to a commutative square

\[
\begin{array}{ccc}
k[U_N]_{<d} & \xrightarrow{\Delta_{U_N}} & k[U_N]_{<d} \otimes k[U]_{<d} \\
\downarrow i^* & & \downarrow i^* \otimes i^* \\
k[U]_{<d} & \xrightarrow{\Delta_U} & k[U]_{<d} \otimes k[U]_{<d}
\end{array}
\]

In particular, each subspace \(k[U]_{<d} \subset k[U]\) is a rational \(U\)-module with coaction \(k[U]_{<d} \rightarrow k[U]_{<d} \otimes k[U]\) given by coalgebra structure on \(k[U]_{<d}\).

**Definition 2.2.** Let \(U\) be a linear algebraic group provided with a closed embedding \(i : U \rightarrow U_N\) for some \(N\). For any rational \(U\)-module \(M\) and any \(d > 0\), we define

\[
M_{<d} \equiv \{m \in M : \Delta_M(m) \in M \otimes k[U]_{<d}\}.
\]

The **degree filtration** on \(M\) is the filtration

\[
M_{<1} \subset M_{<2} \subset M_{<3} \subset \cdots \subset M.
\]

If \(M = M_{<d}\), then we say that \(M\) has filtration degree \(< d\).

Although this filtration on \(M\) depends upon the embedding \(i : U \rightarrow U_N\), it does not change if we compose this embedding with a **linear** embedding \(U_N \rightarrow U_N\).

The following proposition will prove useful at many points; in particular, it implies that the degree filtration of (2.2.1) is a filtration by rational \(\mathbb{G}_m\)-modules.

**Proposition 2.3.** Let \(C\) be a coalgebra over \(k\) and \(i : B \subset C\) a right coideal (i.e., \(\Delta_C : C \rightarrow C \otimes C\) restricts to \(\Delta_B : B \rightarrow B \otimes C\)). For any right \(C\)-comodule \(M\) (i.e., \(\Delta_M : M \rightarrow M \otimes C\)), the subspace

\[
M' \equiv \Delta_M^{-1}(M \otimes B) \subset M
\]

is a right \(C\)-subcomodule of \(M\). Moreover, if \(i : B \subset C\) is a sub-coalgebra, then \(M'\) is a right \(B\)-comodule.

In particular, let \(G\) be a linear algebraic group and let \(B \subset k[G]\) be a right coideal (i.e., a rational \(G\)-submodule of \(k[G]\)). Then for any rational \(G\)-module \(M\), the subspace \(M' \equiv \Delta_M^{-1}(M \otimes B) \subset M\) is a rational \(G\)-submodule.

**Proof.** The comodule structure map \(\Delta_M : M \rightarrow M \otimes C\) for \(M\) is a map of right \(C\)-comodules provided that the right \(C\)-comodule structure on \(M \otimes C\) is given by sending \(m \otimes c\) to \(m \otimes \Delta_C(c)\). Since \(i : B \subset C\) is a right coideal, \(1 \otimes i : M \otimes B \subset M \otimes C\) is a right \(C\)-comodule. We claim that the pre-image \(M' \equiv \Delta_M^{-1}(M \otimes B)\) (in the abelian category of right \(C\)-modules) of the right \(C\)-subcomodule \(M \otimes B \subset M \otimes C\) with respect to the map \(\Delta_M : M \rightarrow M \otimes C\) of right \(C\)-comodules is a right \(C\)-comodule as asserted. Namely, the kernel \(K \subset M \otimes (M \otimes B)\) of the map of right \(C\)-modules \(\Delta_M - (1 \otimes i) : M \otimes (M \otimes B) \rightarrow M \otimes C\) maps isomorphically via projection onto the first factor to \(M' \subset M\) since \(1 \otimes i : M \otimes B \rightarrow M \otimes C\) is injective. Furthermore, the right \(C\)-coproduct \(\Delta_{M'} : M' \rightarrow M' \otimes C\) (the restriction of \(\Delta_M\)) has image in \(M \otimes B\) by definition of \(M'\).

If \(B \subset C\) is a sub-coalgebra, then the right \(C\)-comodule structure on \(M \otimes B\), \(\Delta_{M \otimes B} : M \otimes B \rightarrow M \otimes B \otimes C\), is a right \(B\)-comodule structure and thus restricts to a right \(B\)-comodule structure on \(M'\).
Specializing the previous argument to $C = k[G]$, we get the second assertion concerning rational $G$-modules.

**Remark 2.4.** To understand the statement of Proposition 2.3 it may be useful to consider the special case in which $G$ is a discrete group acting on a $k$-vector space $M$, and $B \subset C$ is taken to be the inclusion of group algebras $k[G/H] \subset k[G]$ for some normal subgroup $H \subset G$. In this case, $M' \subset M$ is the subspace of elements $m' \in M$ with the following property: if $gm = m'$ for some $g \in G, m \in M$, then $ghm = m'$ for every $h \in H$.

Specializing $C$ to $k[U]$ and $B$ to $k[U]_{<d}$ in Proposition 2.3, we conclude the following.

**Corollary 2.5.** For any $d > 0$, the subspace $M_{<d} \subset M$ of a rational $U$-module $M$ as in (2.2.1) is a rational $U$-submodule of $M$ equipped with a natural structure of a $k[U]_{<d}$-comodule.

In particular, the degree filtration $\{M_{<d}, d > 0\}$ is a filtration of $M$ by rational $U$-submodules.

**Example 2.6.** Let $M$ be a rational $G$-module, with $G = U_2$ (isomorphic to $\mathbb{G}_a$). Then the degree filtration on $M$ as formulated in Definition 2.2 equals that in Definition 1.8.

The following example should be compared to Example 4.8.

**Example 2.7.** Let $M$ be a polynomial $GL_N$-module, homogeneous of degree $d-1$ (i.e., a comodule for $k[M_{d-1}] \subset k[GL_N]$), and consider $M$ via restriction as a rational $U_N$-module. Then $M$ has filtration degree $< d$. This follows immediately by observing that restriction of $M$ to $U_N$ has coproduct $M \rightarrow M \otimes k[U_N]$ equal to the composition $(1_M \otimes \pi) \circ \Delta_M : M \rightarrow M \otimes k[M_N] \rightarrow M \otimes k[U_N]$, where $\pi : k[M_N] \rightarrow k[U_N]$ sends $x_{i,j}$ with $i < j$ to $x_{i,j}$; $x_{i,j}$ to 1; and $x_{i,j}$ with $i > j$ to 0.

**Proposition 2.8.** Let $U$ be a linear algebraic group provided with a closed embedding $U \hookrightarrow U_N$ for some $N$. Then for any $d > 0$ and any rational $U$-module $M$

1. $M_{<d} = (M_{<d})_{<d}$ (where $M_{<d}$ is given in (2.2.1));
2. the natural embedding given by the inclusion of coalgebras $k[U]_{<d} \hookrightarrow k[U]$, $t_d : (k[U]_{<d}) \rightarrow (k[U])_{<d}$,

is left adjoint to the functor

\[-d] : (k[U]) \rightarrow (k[U]_{<d})_{<d}, \quad M \mapsto M_{<d}.

**Proof.** By Proposition 2.3, $\Delta_{M_{<d}} : M_{<d} \rightarrow M_{<d} \otimes k[U]$ has image in $M_{<d} \otimes k[U]_{<d}$. Thus, $(M_{<d})_{<d} = (\Delta_{M_{<d}})^{-1}(M_{<d} \otimes k[U]_{<d}) = M_{<d}$.

If $M = M_{<d}$ and $N$ are rational $U$-modules, then any map $f : M \rightarrow N$ of rational $U$-modules fits in a commutative square

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta_M} & M \otimes k[U]_{<d} \\
\downarrow{f} & & \downarrow{f \otimes i_d} \\
N & \xrightarrow{\Delta_N} & N \otimes k[U]_{<d}.
\end{array}
\]
Consequently, $f$ factors uniquely through $N_{<d}$; this means that $(-)_{<d}$ is right adjoint to $\iota_d$. \hfill \Box

The fact that $(-)_{<d}$ admits an exact left adjoint immediately implies that it sends injectives to injectives as we make explicit in the following corollary of Proposition 2.8.

**Corollary 2.9.** For any rationally injective $U$-module $L$ and any $d > 0$, $L_{<d}$ is an injective object of $(k[U]_{<d}-\text{coMod})$.

For any linear algebraic group $G$, the coordinate algebra of the $r$-th Frobenius kernel $G_{(r)}$ of $G$ is given by $k[G_{(r)}] = k[G]/(f^{p^r}, f \in m_e)$, the quotient of $k[G]$ by the ideal generated by $p^r$-th powers of elements in the maximal ideal at the identity (i.e., generated by $f^{p^r}$ for all $f \in k[G]$ with $f(1) = 0$). The quotient map $k[G] \rightarrow k[G_{(r)}]$ is a map of Hopf algebras.

The following proposition is the natural extension of Proposition 1.10 as indicated, the second part of this proposition requires that the closed embedding $i : U \rightarrow U_N$ be linear.

**Proposition 2.10.** Let $U$ be a connected linear algebraic group provided with a closed embedding $i : U \rightarrow U_N$ for some $N$. Let $m$ denote the dimension of $U$. Then for any $r > 0$

1. The composition $k[U]_{<p^r} \subset k[U] \rightarrow k[U_{(r)}]$ is injective.
2. The dimension of $k[U_{(r)}]$ equals $p^rm$.
3. If the closed embedding $i : U \subset U_N$ is linear, then the composition $k[U]_{<p^{r+s}} \subset k[U] \rightarrow k[U_{(r)}]$ is surjective whenever $p^s \geq \dim(U)$.
4. If the closed embedding $i : U \subset U_N$ is linear, then the dimension of $k[U]_{<p^r}$ equals $(m+p^r)$; this is never a $p$-th power, and is not divisible by $p$ if $m < p^r$.

**Proof.** Write $k[U_{(r)}] = k[U]/(y^{p^r}, y \in m_e)$, where $m_e$ is the maximal ideal of functions vanishing at the identity. Clearly, $(y^{p^r}, y \in m_e) \subset m_e^{p^r}$, so that $k[U_{(r)}]$ maps onto $k[U]/m_e^{p^r}$. All homogeneous elements of $k[U_N]$ of degree $1$ must map via $i^*$ to elements of $m_e \backslash m_e^{p^r}$. Thus, the composition $k[U]_{<p^r} \rightarrow k[U] \rightarrow k[U_{(r)}] \rightarrow k[U]/m_e^{p^r}$ is injective, thereby justifying (1).

The computation in (2) of the dimension of $k[U_{(r)}]$ can be verified as follows. For $r = 1$, $k[U_{(1)}]$ is dual to the restricted enveloping algebra of $u = \text{Lie}(U)$ and therefore has dimension equal to $p^m$. Furthermore, the quotient $U_{(r)}/U_{(r-1)}$ is isomorphic to $U_{(1)}$ for $r > 1$, so the computation is concluded using induction on $r$.

Linearity of $i$ implies that $m_e$ is generated by elements $y_1, \ldots, y_m$, $m = \dim(U)$ which are images of homogeneous elements of $k[U_N]$ of degree $1$. Thus, $k[U_{(r)}]$ is spanned by $y_1^{e_1} \cdots y_m^{e_m}$, $0 \leq e_i < p^r$. This implies (3).

To prove (4), observe that the dimension of those polynomial functions of degree $< p^r$ in $m$ variables equals the dimension of those homogeneous polynomial functions of degree $p^r$ in $m + 1$ variables. One checks recursively that the latter dimension equals $(m + p^r)$ which equals $((m + p^r) + \cdots + (1 + p^r))/m!$. Clearly, this is not divisible by $p$ if $m < p^r$ and is never a $p$-th power. \hfill \Box

**Corollary 2.11.** If the closed embedding $i : U \subset U_N$ is linear as in Proposition 2.10(4), then the dimension of $k[U_{(r)}]$ is not divisible by the dimension of $k[U]_{<p^r}$.

In order to motivate our filtration by exponential degree (see Definition 2.11), we investigate how the degree filtration $\{M_{<d}, d > 0\}$ of Definition 2.2 behaves with
Thus, the ring homomorphism $E$ is defined by:

\[ \mathcal{E} : \mathcal{N}_p(u_N) \times \mathbb{G}_a \to U_N, \quad (B, t) \mapsto \exp_B(t) \equiv 1 + tX + \frac{(tX)^2}{2} + \cdots + \frac{(tX)^{p-1}}{(p-1)!} \]

defines a structure of exponential type on $U_N$ (as defined in Definition 2.1).

Furthermore, if $i : U \subset U_N$ is closed embedding of exponential type (see Definition 2.11) and if $u = \mathrm{Lie}(U)$, then the restriction of (2.12.1) determines a structure of exponential type on $U$

\[ \mathcal{E} : \mathcal{N}_p(u) \times \mathbb{G}_a \to U, \quad (B, t) \mapsto \exp_B(t). \]

The following theorem is reformulated in Example 4.7 as asserting that

\[ (k[U])_{[d-1]} \subset k[U]_{<d} \subset (k[U])_{(p-1)(d-1)}, \]

thereby relating the degree filtration of Definition 2.2 for unipotent algebraic groups to the exponential degree filtration of Definition 4.1 for algebraic groups equipped with a structure of exponential type.

**Theorem 2.13.** Let $i : U \subset U_N$ be a closed embedding of exponential type.

1. $\mathcal{E}_B(x_{i,j}) \in kU$ acts trivially on $k[U]_{<d}$ for all $B \in \mathcal{N}_p(u)$, $j > (p-1)(d-1)$.
2. $\mathcal{E}^* : k[U]_{<d} \to k[\mathcal{N}_p(u)] \otimes k[\mathbb{G}_a]$ has image contained in $k[\mathcal{N}_p(u)] \otimes k[T]_{<(p-1)(d-1)+1}$.
3. If $U$ is the unipotent radical of a “restricted parabolic” of $G_{NW}$ (so that $U$ has nilpotent class $< p$) and if $\mathcal{E}_B(x_{ij}) \in kU$ acts trivially on $f$ for all $B \in \mathcal{N}_p(u)$ and all $j \geq d$, then $f \in k[U]_{<d}$.

**Proof.** In the special case $U = U_N$, $\mathcal{E}_B(x_{i,j})$, $1 \leq i < j \leq N$, is the polynomial in $T$ whose coefficient of $T^n$ is $1/1!$ times the $(i, j)$-th entry of $B^n$ for any $n, 1 \leq n < p$. Thus, the ring homomorphism $\mathcal{E}_B$ sends a polynomial in the $x_{i,j}$ of degree $\leq d$ to a polynomial in $T$ of degree $\leq (p-1)(d-1)$. Hence $\mathcal{E}_B(x_{ij})$ applied to $f \in k[U_N]_{<d}$ is 0 for $j > (p-1)(d-1)$. For $U \subset U_N$ of exponential type, this argument restricts to $U$, thereby implying (1).

The fact that (1) is equivalent to (2) is a consequence of the facts that $k[\mathcal{N}_p(u)]$ is reduced (i.e., has no non-trivial nilpotent elements) and that $k$ is algebraically closed. Namely, the Hilbert Nullstellensatz implies that the coefficient of $T^n$ of $\mathcal{E}^*(f) = \sum g_m \cdot T^m \in (k[\mathcal{N}_p(u)][T]$ is 0 if and only if $g_n(B) = 0$ for all $B \in \mathcal{N}_p(u)$.

To prove (3), we first recall that if $U$ is the unipotent radical of a “restricted parabolic” of a simple algebraic group such as $SL_N$, then

\[ \epsilon : u = \mathcal{N}_p(u) \to U, \quad B \mapsto \mathcal{E}_B(1) \]
is an isomorphism, a so-called “Springer isomorphism” (see [4.3]). Write $B_u \equiv \epsilon^{-1}(u) \in u$ for any $u \in U$. If $f \in k[U]$ is not constant, then there exists some $u \in U$ such that $f(u) \neq f(0)$; this implies that $\mathcal{E}_B^*(f) \in k[G_a]$ is not a constant polynomial in $T$. On the other hand, since $\mathcal{E}_B^*(f)$ is a ring homomorphism, the degree of $\mathcal{E}_B^*(f)$ must be greater or equal to the degree of some $f \in k[U]$ mapping to $f$. Consequently, if $f \notin k[U]_{<d}$, then there exists some $u \in U$, $j \geq d$ such that $\mathcal{E}_B^*(x_{ij})$ acts non-trivially on $f$. □
As introduced in [1, 4.5] for $G$ a linear algebraic group equipped with the structure of exponential type and $M$ a rational $G$-module, $M$ is said to have exponential degree $< p^r$ if $\mathcal{E}_{B_+}(u_s)$ acts trivially on $M$ for all $s \geq r$, all $B \in \mathcal{N}_p(g)$. The following corollary follows immediately from Theorem 2.13.

**Corollary 2.14.** If $M$ is a rational $U$-module such that $M = M_{<d}$ and if $p^r > (p-1)(d-1)$, then $M$ has exponential degree $< p^r$ in the sense of [1].

As shown in [1, 4.6.1], the condition on $M$ of having exponential degree $< p^r$ implies that the support variety $V(U)_M$ as defined in [1] has the special from mentioned in the following corollary.

**Corollary 2.15.** ([1, 4.6.1]) Let $i : U \subset U_N$ be a closed embedding of exponential type, let $M$ be a rational $U$-module such that $M = M_{<d}$, and let $r$ satisfy $(p^r > (p-1)(d-1))$. Then the support variety $V(U)_M$ of $M$ satisfies

$$V(U)_M = \Lambda^{-1}_r(V_r(U)_M(k))$$

where $V_r(U)_M$ is the support variety of $M$ as a $kU_{p^r}$-module and $V_r(U)_M(k)$ is its associated Zariski topological space of $k$-rational points.

The following simple example makes clear that the condition that $M = M_{<d}$ is not equivalent to some condition on the support variety of $M$. Conceptually, the support variety of $M$ is the locus of 1-parameter subgroups $\psi$ at which the $p$-nilpotent action at $\psi$ is not free (see [1]), whereas the condition $M = M_{<d}$ is the condition on the triviality of the action of $\mathcal{E}_{B_+}(u_s)$ on $M$ for all $B \in \mathcal{N}_p(g)$ and all $s$, $p^s > (p-1)(d-1)$.

**Example 2.16.** Consider the 2-dimensional rational $\mathbb{G}_a$-module $Y_r$ with basis \{\(v, w\)\} whose $k\mathbb{G}_a$-module structure is given by $u_s(w) = 0, \forall s \geq 0$; $u_s(v) = w, s \leq r$, $u_s(v) = 0, \forall s > r$. If $p > 2$, then $V(\mathbb{G}_a)_{Y_R} = V(\mathbb{G}_a)$. On the other hand, $Y_R \neq (Y_R)_{<p^r}$.

### 3. Rationally injective modules for unipotent groups

Since $k[U]$ is injective as a rational $U$-module (i.e., rationally injective), $V \otimes k[U]$ is also rationally injective for any $k$-vector space $V$.

**Proposition 3.1.** Let $L$ be a rationally injective $U$-module and set $L_0 = H^0(U, L)$. Then a map $f : L \to L_0 \otimes k[U]$ of rational $U$-modules is an isomorphism if and only if the induced map $H^0(f) : H^0(U, L) \to H^0(U, L_0 \otimes k[U])$ is an isomorphism of vector spaces.

**Proof.** Clearly, the condition that $H^0(f)$ be an isomorphism is necessary for $f$ to be an isomorphism.

Since $U$ is unipotent, $H^0(U, L)$ is the socle of both $L$ and $L_0 \otimes k[U]$. Thus, the condition that $H^0(f)$ be injective implies that $f$ is itself injective because a non-trivial kernel of $f$ would have to meet the socle of $L$ non-trivially. If $f$ is injective, the rational injectivity of $L$ implies the existence of some $g : L_0 \otimes k[U] \to L$ with $g \circ f = id$. On the other hand, if $L_0 \otimes k[U] \simeq L \oplus L'$ with $L' \neq 0$, then $H^0(f)$ is not surjective. \[\square\]

For some time, we had tried to prove that a rational $\mathbb{G}_a$-module $M$ must be rationally injective if $V(\mathbb{G}_a)_M = 0$. The latter condition is equivalent to the condition that the restriction to $\mathbb{G}_a(r)$ is free for all $r > 0$ (see [1, 6.1]). In the following,
we provide natural examples of rational \( U \)-modules \( M \) which are not rationally injective whose restrictions to each \( kU_{(r)} \) are free.

**Example 3.2.** By [4 2.1.4.5], if \( H \subset G \) is a closed embedding of linear algebraic groups, then \( k[G] \) is injective as a rational \( H \)-module if and only if \( G/H \) is affine. In particular, if \( U \) is unipotent and is given a closed embedding into some \( SL_n \), then the restriction of \( k[SL_n] \) to \( U \) is not rationally injective. On the other hand, \( k[SL_n] \) is rationally injective as a \( SL_n \)-module and thus injective as an \( kSL_{N(r)} \)-module for any \( r > 0 \). Because injective \( kSL_{N(r)} \)-modules are projective, we conclude that \( k[SL_n] \) is projective as a \( kSL_{N(r)} \)-module. Since \( kSL_{N(r)} \) is free as a \( kU_{(r)} \)-module (as well as if and only if they are free, we conclude that \( k[SL_n] \) is not rationally injective as a \( U \)-module, but each of its restrictions to \( kU_{(r)} \) is free.

To analyze whether or not a rational \( U \)-module \( L \) is rationally injective, it is natural to consider submodules (for example, in trying to proceed by some sort of induction on the dimension of the socle). With this in mind, the following criterion for rational injectivity is somewhat natural.

**Proposition 3.3.** Let \( U \) be a linear algebraic group provided with a closed embedding \( i : U \to U_N \) for some \( N \). Then a rational \( U \)-module \( L \) is rationally injective if and only if \( L \subset L_{<p^r} \) is injective as a \( k[U]_{<p^r} \)-comodule for each \( r > 0 \).

In particular, if \( U \cong \mathbb{G}_a \), then a rational \( \mathbb{G}_a \)-module \( L \) is rationally injective if and only if for all \( r > 0 \) the restriction of \( L \subset L_{<p^r} \) to \( k\mathbb{G}_a(r) \) is free.

**Proof.** By Corollary [2.9] the restriction of a rationally injective \( U \)-module is injective as a \( k[U]_{<p^r} \)-comodule for each \( r > 0 \). For \( U \cong \mathbb{G}_a \), Proposition [2.11.3] tells us that injective \( k[\mathbb{G}_a]_{<p^r} \)-comodules are injective (equivalently, free) \( k\mathbb{G}_a(r) \)-modules.

To prove the converse, consider some rational \( U \)-module \( L \) such that \( L \subset L_{<p^r} \) is injective as a \( k[U]_{<p^r} \)-comodule for each \( r > 0 \). Let \( L_0 \) denote the socle of \( L \). Since \( L_0 \) is the socle of \( L \) and \( L_0 \otimes k[U] \) is rationally injective, there exists some embedding of rational \( U \)-modules \( f : L \to L_0 \otimes k[U] \) extending \( L_0 \subset L_0 \otimes k[U] \). Restricting \( f \), we obtain injections \( f_r : L_{<p^r} \to L_0 \otimes k[U]_{<p^r} \). We proceed inductively to exhibit

\[
h_r : L_0 \otimes k[U]_{<p^r} \to L_{<p^r} \text{ extending } h_{r-1}, \quad h_r \circ f_r = id.
\]

We set \( h_0 : L_0 \to L \) to be natural inclusion. We construct \( h_r \) extending \( h_{r-1} \) by applying the assumed injectivity property of \( L_{<p^r} \) to the inclusion of \( k[U]_{<p^r} \)-comodules

\[
i_{p^r} + f_r : L_{<p^r} + L_0 \otimes k[U]_{<p^r} \cong L_0 \otimes k[U]_{<p^r},
\]

using the fact that \( i_{p^r} \) is left adjoint to \((-)_{p^r} \). Taking the colimit of the maps \( h_r \), we obtain \( h : L_0 \otimes k[U] \to L \) whose composition with \( f \) is the identity. Because \( h \) restricts to an isomorphism on socles, it must be injective (as well as surjective), and is therefore an isomorphism.

**Remark 3.4.** Corollary [2.11] implies that if the dimension of \( U \) is greater than 1 then \( k[U]_{<p^r} \) is not free as a \( kU_{(r)} \)-module.
4. Sub-coalgebras of \( k[G] \) defined in terms of 1-parameter subgroups

Throughout this section, \( G \) denotes a connected linear algebraic group with Lie algebra \( \mathfrak{g} \). We denote by \( \mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \) the variety of \( r \)-tuples \( (B_0, \ldots, B_{r-1}) \) of pairwise commuting, \( p \)-nilpotent elements of \( \mathfrak{g} \); in other words, each \( B_i \) satisfies \( B_i^{[p]} = 0 \) and each pair \( B_i, B_j \) satisfies \( [B_i, B_j] = 0 \). We denote by \( \mathcal{N}_p(\mathfrak{g})(k) \) the Zariski space of \( k \)-rational points of \( \mathcal{N}_p(\mathfrak{g}) \). Furthermore, we denote by \( V_r(G) \) the variety of height \( r \) infinitesimal 1-parameter subgroups \( \mathbb{G}_{a(r)} \to G \) of \( G \).

We begin by recalling from \([3, 1.6]\) the definition of a structure of exponential type on a linear algebraic group, a definition which extends the formulation in \([10]\) of an embedding \( G \subset GL_n \) of exponential type. Up to isomorphism (as made explicit in \([4, 1.7]\)), if such a structure exists then it is unique.

**Definition 4.1.** \([3, 1.6]\) Let \( G \) be a linear algebraic group with Lie algebra \( \mathfrak{g} \). A (\( G \)-equivariant) structure of exponential type on \( G \) is a (\( G \)-equivariant) morphism of \( k \)-schemes

\[
\mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \to G, \quad (B, s) \mapsto \mathcal{E}_B(s)
\]

such that

1. For each \( B \in \mathcal{N}_p(\mathfrak{g})(k) \), \( \mathcal{E}_B : \mathbb{G}_a \to G \) is a 1-parameter subgroup.
2. For any pair of commuting \( p \)-nilpotent elements \( B, B' \in \mathfrak{g} \), the maps \( \mathcal{E}_B, \mathcal{E}_{B'} : \mathbb{G}_a \to G \) commute.
3. For any commutative \( k \)-algebra \( A \), any \( \alpha \in A \), and any \( s \in \mathbb{G}_a(A) \), \( \mathcal{E}_{\alpha \cdot s}(B) = \mathcal{E}_B(\alpha \cdot s) \).
4. For any \( r > 0 \), \( V_r(G) \) admits the identification

\[
C_r(\mathcal{N}_p(\mathfrak{g})) \xrightarrow{\sim} V_r(G), \quad B \mapsto \mathcal{E}_B \circ i_r = \prod_{s=0}^{r-1} (\mathcal{E}_{B_{i_s}} \circ F^s) \circ i_r.
\]

Moreover, \( H \subset G \) is said to be an embedding of exponential type if \( H \) is equipped with the structure of exponential type given by restricting that provided to \( G \); in particular, we require \( \mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \to G \) to restrict to \( \mathcal{E} : \mathcal{N}_p(\mathfrak{h}) \times \mathbb{G}_a \to H \).

In what follows, we shall not require that our structures of exponential type are \( G \)-equivariant.

Because \( \mathcal{E}_B : \mathbb{G}_a \to G \) is a homomorphism of algebraic groups, \( \mathcal{E}_B(r) \in G(R) \) is \( p \)-unipotent for any commutative \( k \)-algebra \( R \) and any \( r \in R \).

**Example 4.2.** If \( U \) is the unipotent radical of a parabolic subgroup of \( SL_N \) or a product of commuting root groups in \( SL_n \), then \( U \subset U_N \) is a closed linear embedding of exponential type.

**Example 4.3.** As another class of examples, we may consider a unipotent algebraic group \( U \) of nilpotent class \( < p \). Let \( \mathfrak{u} \) denote the Lie algebra of \( U \). Then the Campbell-Hausdorff-Baker formula gives an isomorphism of algebraic groups

\[
\exp : \mathfrak{u} \xrightarrow{\sim} U.
\]

A structure of exponential type on \( U \) is given by sending \( (X, t) \in \mathfrak{u} \times \mathbb{G}_a \) to \( \exp(tX) \in U \).
Remark 4.4. Subject to a (relatively weak) condition on $p$, simple algebraic groups, their parabolic subgroups, and unipotent radicals of these parabolic subgroups admit a structure of exponential type. For simple groups of classical type, no condition of $p$ is required. See [10], [8], [9] for details.

The following construction provides a reasonable analogue of $k[U]<d$ for any $G$ equipped with a structure of exponential type.

Definition 4.5. Let $G$ be a linear algebraic group equipped with a structure of exponential type, $\mathcal{E} : \mathcal{N}_p(g) \times G_a \to G$. For any $d \geq 0$, we define $(k[G])_d \subset k[G]$ as follows:

$$(k[G])_d \equiv \{ f \in k[G] : (\mathcal{E}_B(v_j))(f) = 0, \forall B \in \mathcal{N}_p(g), j > d \}.$$ 

In other words, $(k[G])_d$ is the pre-image under $\mathcal{E}^* : k[G] \to k[\mathcal{G}_a] \otimes k[\mathcal{N}_p]$ of $(k[\mathcal{N}_p])[T]_{<d+1}$.

The proof of Proposition 2.13 establishing the equivalence of the two formulations of the condition for $f \in k[U]$ to lie in $k[U]_{<d}$ applies equally to justify the description of $(k[G])_d$ as the pre-image under $\mathcal{E}^* : k[G] \to k[\mathcal{G}_a] \otimes k[\mathcal{N}_p]$ of $(k[\mathcal{N}_p])[T]_{<d+1}$.

Proposition 4.6. Let $G$ be a linear algebraic group equipped with a structure of exponential type. For any $d \geq 0$, $(k[G])_d \subset k[G]$ is a sub-coalgebra. In particular, $(k[G])_d$ is a rational $G$-submodule of $k[G]$.

Moreover, $(k[G])_0$ is a sub-Hopf algebra of $k[G]$.

Proof. Let $f \in k[T]$, $B \in \mathcal{N}_p(g)$. Because $\mathcal{E}_B : \mathcal{G}_a \to G$ is a morphism of algebraic groups,

$$\mathcal{E}_B^*(-\Delta_G(f)) = \Delta_{\mathcal{G}_a}(\mathcal{E}_B^*(f)) \in k[\mathcal{G}_a] \otimes k[\mathcal{G}_a].$$

On the other hand, if $p(T) = \mathcal{E}_B(f) \in k[\mathcal{G}_a]$ has degree $< d+1$, then $\Delta_{\mathcal{G}_a}(p(T)) = \sum_i p_i(T) \otimes p'_i(T)$ with each $p_i(T)$, $p'_i(T)$ having degree $< d+1$. Thus, the coproduct of $k[G]$ restricts to a coproduct on $(k[G])_d$.

The multiplicative structure of the commutative $k$-algebra $k[G]$ restricts to a multiplicative structure $(k[G])_0 \otimes (k[G])_0 \to (k[G])_0$ (since the product of two polynomials in $k[T]$ of degree $< 1$ is again of degree $< 1$), thereby verifying that $(k[G])_0$ is a sub-Hopf algebra of $k[G]$. \hfill \qedsymbol

Example 4.7. Let $i : U \subset U_N$ be a closed embedding of exponential type. Then Theorem 2.13(1) tells us that

$$k[U]_{<d} \subset (k[U])_{(p-1)(d-1)}.$$ 

If $U$ is the unipotent radical of a “restricted parabolic” of $GL_N$, then Theorem 2.13(3) tells us that

$$(k[U])_{d-1} \subset k[U]_{<d};$$

in particular, $(k[U])_d$ is finite dimensional.

Example 4.8. The dual of the Schur algebra $S(N, d)$ is the coalgebra $k[M_N]_d$, the vector space of polynomials homogeneous of degree $d$ in the variables $\{x_{i,j}, 1 \leq i, j \leq N \}$. We observe that

$$(4.8.1)$$

$$k[M_N]_d \leftrightarrow k[GL_N]_{(p-1)d}.$$ 

Namely, if $B$ is a $p$-nilpotent, $N \times N$ matrix, then $exp_B(x_{i,j}) \in k[T] = k[\mathcal{G}_a]$ is the $(i, j)$-th entry of the matrix $1 + BT + B^2T^2/2 + \cdots + B^{p-1}T^{p-1}/(p-1)!$ which has
degree \leq p - 1 \) (as a polynomial in \( T \)). Thus, if \( f \in k[GL_N] \) is homogenous of degree \( d \) in the \( x_{i,j} \) (i.e., in the image of \( k[M_N]_d \)), then \( \exp_B(f) \) has degree \( \leq (p - 1)d \).

One consequence of the following proposition is that \((k[G])_d\) is never finite dimensional if \( G \) is a non-trivial reductive algebraic group, since for reductive \( G \) the kernel ideal of \( k[G] \to k[U_p] \) is an infinite dimensional vector space.

**Proposition 4.9.** Let \( G \) be a linear algebraic group of exponential type and \( U_p(G) \subset G \) denote the closed variety of \( p \)-unipotent elements of \( G \). Then \( \mathcal{E}^* : k[G] \to k[N_p(g)] \otimes k[G_a] \) factors through an embedding
\[
\mathcal{E}^* : k[U_p(G)] \hookrightarrow k[N_p(g)] \otimes k[G_a].
\]

Consequently, the augmentation ideal of \((k[G])_0 \subset k[G]\) equals the ideal in \( k[G] \) of those functions on \( G \) which vanish on \( U_p(G) \subset G \).

**Proof.** If \( f \in k[G] \) vanishes on \( U_p(G) \) (i.e. lies in the ideal defining the closed subvariety \( U_p(G) \subset G \)), then \( \mathcal{E}^*_B(f) = 0 \) for all \( B \in N_p(g) \) since \( \mathcal{E}_B : G_a \to G \) factors through \( U_p \). Thus, \( \mathcal{E}^* \) factors through \( k[U_p(G)] \). On the other hand, if \( 0 \neq \bar{f} \in k[U_p(G)] \), then there exists some \( u \in U_p(G) \) with \( \bar{f}(u) \neq 0 \). Let \( \mathcal{E}_{B_n} : G_a \to G \) be chosen so that \( \mathcal{E}_{B_n}(1) = u \) as in the proof of Theorem 2.13(3). Then \( \mathcal{E}^*_{B_n}(\bar{f}) \neq 0 \), since \( (\mathcal{E}^*_{B_n}(\bar{f}))(1) = \bar{f}(u) \). Thus we conclude the embedding (4.9.1).

In particular, we have shown that the kernel of \( \mathcal{E}^* : k[G] \to k[G_a] \otimes k[N_p(g)] \) equals the kernel of the restriction map \( k[G] \to k[U_p] \) which equals the ideal of those functions on \( G \) which vanish on \( U_p(G) \subset G \). On the other hand, \((k[G])_0\) consists of those \( f \in k[G] \) such that \( \mathcal{E}^*_B(f) \) is constant (i.e., lies in \( k \) for all \( B \)). Therefore, the augmentation ideal of \((k[G])_0\) equals the ideal of \( U_p(G) \).

**Remark 4.10.** Since \((k[G])_0\) is a sub-Hopf algebra of \( k[G] \), we might view \((k[G])_0\) as the coordinate algebra of some “quotient group scheme” \( G/G_0 \) of \( G \). We may “visualize” this quotient group scheme using the following commutative diagram:
\[
\begin{array}{ccc}
N_p(g) \times G_a & \xrightarrow{\mathcal{E}} & G \\
& \searrow & \downarrow \mathcal{E} \\
& & U_p(G)
\end{array}
\]

Since \((k[G])_0\) is typically not finitely generated as an algebra, \( G/G_0 \) is not a familiar group scheme.

5. **Filtrations on rational \( G \)-modules in terms of 1-parameter subgroups**

We introduce the filtration by exponential degree on a rational \( G \)-module, an “extension” of the degree filtration on a rational \( U \)-module given in Definition 2.22.

**Definition 5.1.** Let \( G \) be a linear algebraic group equipped with a structure of exponential type and let \( M \) be a rational \( G \)-module. For any \( d \geq 0 \), we define
\[
M_{[d]} \equiv \{ m \in M : \Delta_M(m) \in M \otimes (k[G])_d \}.
\]
Equivalently, \( M_{[d]} \subset M \) consists of those elements \( m \in M \) such that \((\mathcal{E}_B)_*(v_j) \in kG \) vanishes on \( m \) for all \( B \in N_p(g) \) and all \( j > d \).
The filtration by exponential degree on \( M \) is the filtration
\[
M_{[0]} \subset M_{[1]} \subset \cdots \subset M.
\]

We say that \( M \) has exponential degree \( \leq d \) if \( M = M_{[d]} \). This is consistent with the terminology “\( M \) has exponential degree \( \leq p^n \)” of \([4, 4.5]\).

**Example 5.2.** Let \( M \) be a polynomial representation of \( GL_N \), which is homogeneous of degree \( d \) (i.e., a comodule for \( k[M_N]_d \)). By Example 4.8.1, \( M = M_{[(p-1)d]} \).

We list various properties of our filtration by exponential degree of rational \( G \)-modules.

**Theorem 5.3.** Let \( G \) be a linear algebraic group equipped with a structure of exponential type and let \( M \) be a rational \( G \)-module.

1. The abelian category of comodules for the coalgebra \( (k[G])_{[d]} \) equals the full subcategory of \((G\text{-Mod})\) consisting of those rational \( G \)-modules of exponential degree \( \leq d \).
2. The filtration \( \{M_{[d]}, \; d \geq 0\} \) of \( M \) is independent of the choice of structure of exponential type for \( G \).
3. If \( M \) is a finite dimensional rational \( G \)-module, then \( M = M_{[d]} \) for \( d >> 0 \).
4. \( M = \cup_d M_{[d]} \) for any rational \( G \)-module \( M \).
5. The exponential filtration of \( k[G] \), \( \{k[G]_{[d]}, \; d \geq 0\} \), is finite only if \( N_p(g) = \{0\} \).
6. If \( M \) has exponential degree \( \leq d \), then its Frobenius twist \( M^{(1)} \) has exponential degree \( \leq pd \).
7. If \( M' \) has exponential degree \( \leq d \) and if \( f : M' \to M \) is a map of rational \( G \)-modules, then \( f(M') \subset M_{[d]} \).
8. If \( f : M' \to M \) is an inclusion of rational \( G \)-modules and if \( m' \in M'[\setminus M'_{[d]}] \), then \( f(m') \in M_{[d]} \).
9. If \( j : H \subset G \) is an embedding of exponential type and if \( M \) has exponential degree \( \leq d \) as a rational \( G \)-module, then the restriction to \( H \) of \( M \) has exponential degree \( \leq d \) as a rational \( H \)-module.

**Proof.** Property (1) is merely a rephrasing of the condition that a rational \( G \)-module \( M \) satisfies the condition \( M = M_{[d]} \).

Property (2) follows from \([1, 1.7]\); property (3) is established in \([4, 2.6]\). Since any rational \( G \)-module is a union of its finite dimensional submodules, property (4) follows from property (3).

If \( \psi : G_a \to G \) is a non-trivial 1-parameter subgroup, then \( \psi^* : k[G] \to k[G_a] \) has infinite dimensional image. Thus, for every \( d > 0 \) there exists some \( e(d) > d \) and some \( f \in k[G] \) such that \( \psi_{e(d)} \) is non-zero on \( \psi^*(f) \); in other words, \( f \notin (k[G])_{[e(d)]} \). This verifies property (5).

As discussed in \([5]\), the Frobenius twist \( V^{(1)} \) of a \( k \)-vector space \( V \) is the \( k \)-vector space defined as the base change of \( V \) along the \( p \)-th power map \( \phi : k \to k \), denoted \( k \otimes_{\phi} V \). The Frobenius twist of a rational \( G \)-module \( M \), \( M^{(1)} \), has as its coproduct structure \( \Delta_{M^{(1)}} : M^{(1)} \to M^{(1)} \otimes k[G] \) the composition
\[
(1_M \otimes F) \circ (\Delta_M)^{(1)} : M^{(1)} \to M^{(1)} \otimes k[G]^{(1)} \to M^{(1)} \otimes k[G]
\]
where \( F : k[G]^{(1)} \to k[G] \) is the \( k \)-linear map sending \( \alpha \otimes f \in k \otimes_{\phi} k[G] \) to \( \alpha f^p \in k[G] \). For any 1-parameter subgroup \( \psi : G_a \to G \), \( \psi^*(f^p) = (\psi^*(f))^p \in k[G_a] \), so
that the image under $F : k[G]^{(1)} \to k[G]$ of $(k[G])_{(d)}^{(1)}$ lies in $(k[G])_{[pd]}$, thereby establishing property (6).

Properties (7) and (8) are easy consequences Definition 5.3 and the fact that a map $f : M' \to M$ is a map of $k[G]$-comodules.

Finally, the condition that $j : H \subset G$ be an embedding of exponential type (see Definition 4.1) implies the commutativity of the square

$$(5.3.1) \begin{array}{c}
k[G] \xrightarrow{\varepsilon^*} k[N_p(g)] \otimes k[G_a] \\
\downarrow j^* \quad \quad \quad \quad \quad \quad \downarrow j^* \otimes 1 \\
k[H] \xrightarrow{\varepsilon^*} k[N_p(h)] \otimes k[G_a].
\end{array}$$

The surjectivity of $j^*$ together with the commutativity of (5.3.1) implies that $j^*$ restricts to $j^*_d : (k[G])_{[d]} \to (k[H])_{[d]}$. Thus, if the coproduct $\Delta_M : M \to M \otimes k[G]$ for the rational $G$-module $M$ factors through $M \otimes (k[G])_{[d]}$, then the coproduct for $M$ restricted to $H$ factors through $M \otimes (k[H])_{[d]}$. $\square$

**Example 5.4.** Let $G$ be a reductive group with a structure of exponential type and let $M$ be a rational $G$-module all of whose high weights $\mu$ satisfy the condition that $2 \sum_{j=1}^r (\mu, \omega_j) < p^r$. Here, $\{\omega_1, \ldots, \omega_r\}$ is the set of fundamental dominant weights of $G$ (with respect to some $T \subset B \subset G$) and $\omega_j = 2\omega_j/(\alpha_j, \alpha_j)$. As argued in [3, 5.7] following [1, 4.6.2], $M$ has exponential degree $< p^r$.

As in Corollary 2.15 the condition on $M$ of having exponential degree $< p^r$ is shown in [3, 4.6.1] to imply that the support variety $V(G)_M$ as defined in [4] is of very special from (determined by the restriction of $M$ to $G_{(r)}$).

**Proposition 5.5.** ([4, 4.6.1]) Let $G$ be a linear algebraic group equipped with a structure of exponential type and let $M$ be a rational $G$-module such that $M = M_{[d]}$. If $p^r > d$ (so that $M$ has exponential degree $< p^r$), then the support variety $V(G)_M$ of $M$ satisfies

$$V(G)_M = \Lambda_{p^{-1}}(V_r(G)_M(k)).$$

The abelian categories $((k[G])_{[d]}\text{coMod})$ are worthy of further study. In the following proposition, we give a few elementary properties.

**Proposition 5.6.** Let $G$ be a linear algebraic group equipped with a structure of exponential type.

1. The functor $(-)_{[d]} : (G\text{-Mod}) \to (G\text{-Mod})$ sending $M$ to $M_{[d]}$ is idempotent with image $((k[G])_{[d]}\text{-coMod})$.

2. The functor $(-)_{[d]} : (G\text{-Mod}) \to (G\text{-Mod})$ is left exact.

3. The natural embedding $\iota_d : ((k[G])_{[d]}\text{-coMod}) \subset (G\text{-Mod})$ is left adjoint to $(-)_{[d]}$.

4. The category $((k[G])_{[d]}\text{-coMod})$ has enough injectives; in other words, for every rational $G$-module $M$ of exponential degree $< d$, there exists an inclusion of rational $G$-modules $M \hookrightarrow L$ of exponential degree $< d$ with $L$ an injective object of $((k[G])_{[d]}\text{-coMod})$.

Proof. The proof of (1) is an easy consequence of Theorem 5.3.1. The proof of Proposition [2.3] applies with merely notational changes to prove (2) and (3).

To prove (4), recall that $(G\text{-Mod})$ has enough injectives. If $M$ is a rational $G$-module of exponential degree $\leq d$ and if $j : M \to I$ is an embedding of $M$ into
a rationally injective $G$-module $I$, then $j$ factors through $L \equiv I_{[d]}$ (by Theorem 5.3). Since $\iota_d$ is an exact left adjoint to $(-)_{[d]}$, $L$ is an an injective object of $((k[G])_{[d]}{\text{-coMod}})$. \hfill \Box

The following necessary and sufficient criterion for rational injectivity is an extension of Proposition 5.3.

**Proposition 5.7.** Let $G$ be a linear algebraic group equipped with a structure of exponential type. Then a rational $G$-module $L$ is rationally injective if and only if for all $d > 0$ the rational $G$-submodule $L_{[d]} \subset L$ is an injective object of $((k[G])_{[d]}{\text{-coMod}})$.

**Proof.** If $L$ is injective, then Proposition 5.6(2) implies that $L_{[d]} \subset L$ is an injective object of $((k[G])_{[d]}{\text{-coMod}})$. Namely, this is a formal consequence of the fact that $(-)_{[d]}$ has an exact left adjoint.

Assume now that the rational $G$-module has the property that each $L_{[d]} \subset L$ is an injective object of $((k[G])_{[d]}{\text{-coMod}})$. Let $M' \to M$ be an inclusion of rational $G$-modules and $f' : M' \to L$ be a map of rational $G$-modules. We proceed to extend $f'$ to $f : M \to L$.

Set $N_d = (M' \cap M_{[d]} + M_{[d-1]} \hookrightarrow M_{[d]}$ for any $d > 0$ (and set $M_{[0]} = 0$). We inductively define $f_d : N_d \rightarrow L_{[d]} \subset L$ as some extension of $f'_d |_{M' \cap M_{[d]}} + f_{d-1}$ using the injectivity property of $L_{[d]}$. Then we obtain $f$ as the colimit of the $F_d$’s, defined on all of $M$ by Theorem 5.3. \hfill \Box

We conclude with Grothendieck spectral sequences relating rational cohomology to the structures we have considered.

**Proposition 5.8.** Let $G$ be a linear algebraic group equipped with a structure of exponential type. Then we have a natural identification of functors

$$H^0(G, -) \simeq H^0(G/G_{[0]}, -) \circ (-)_{[0]} : (G{-\text{-Mod}}) \to (vsp/k)$$

leading to a spectral sequence

$$H^s(G/G_{[0]}, R^t(-)_{[0]}(M)) \Rightarrow H^{s+t}(G, M).$$

More generally, for any $d \geq 0$, there is a natural identification of functors

$$H^0(G, -) \simeq H^0_{\text{hom}}((k[G])_{[d]}(k, -) \circ (-)_{[d]} : (G{-\text{-Mod}}) \to (vsp/k)$$

leading to a spectral sequence

$$R^s H^0_{\text{hom}}((k[G])_{[d]}(k, -) \circ R^t(-)_{[d]}(M) \Rightarrow H^{s+t}(G, M).$$

**Proof.** The asserted identification of the composition of functors $H^0_{\text{hom}}((k[G])_{[d]}(k, -) \circ (-)_{[d]}$ with $H^0(G, -)$ is made by observing that both send a rational $G$-module $M$ to the subspace $M^G$ of invariant elements (which consists of those $m \in M$ such that $\Delta_M(m) = m \otimes 1 \in M \otimes k[G]$).

Since the functor $(-)_{[d]}$ has an exact left adjoint by Proposition 5.6(3) and therefore sends injectives to injectives, the Grothendieck spectral sequence for a composition of left exact functors applies and has the asserted form (see [11]). \hfill \Box
References

[1] J. Carlson, Z. Lin, and D. Nakano, Support varieties for modules over Chevalley groups and classical Lie algebras, Trans. A.M.S. 360 (2008), 1870 - 1906.

[2] E. Cline, B. Parshall, L. Scott, Induced Modules and Affine Quotients, Math. Ann. 30 (1977), no. 1, 1–14.

[3] E. Friedlander, Restrictions to $G(F_p)$ and $G(r)$ of rational $G$-modules, Compos. Math. 147 (2011), no. 6, 1955–1978.

[4] E. Friedlander, Support varieties for rational representations. To appear in Compositio Math. [http://arxiv.org/abs/1406.7499]

[5] E. Friedlander, A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (1997), 209-270.

[6] J.C. Jantzen, Representations of Algebraic groups, Academic Press, (1987).

[7] G. Seitz, Unipotent elements, tilting modules, and saturation, Invent. Math. 141 (2000), 467-502.

[8] P. Sobaje, On exponentiation and infinitesimal one-parameter subgroups of reductive groups, J. Algebra 385 (2013), 14-26.

[9] P. Sobaje, Springer isomorphisms in characteristic p, [http://arxiv.org/abs/1210.4629]

[10] A. Suslin, E. Friedlander, C. Bendel, Infinitesimal 1-parameter subgroups and cohomology, J. Amer. Math. Soc. 10 (1997), 693-728.

[11] C. Weibel, An Introduction to Homological Algebra Cambridge University Press, (1995).

Department of Mathematics, University of Southern California, Los Angeles, CA
E-mail address: ericmf@usc.edu
E-mail address: eric@math.northwestern.edu