Existence of Time Operator for a Singular Harmonic Oscillator

V. Mikuta-Martinis and M. Martinis
Theoretical Physics Division, Rudjer Bošković Institute
10002 Zagreb, Croatia
e-mail:vmikuta@irb.hr

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Abstract

The time operator for a quantum singular oscillator of the Calogero-Sutherland type is constructed in terms of the generators of the SU(1,1) group. In the space spanned by the eigenstates of the Hamiltonian, the time operator is not self-adjoint. We show, that the time-energy uncertainty relation can be given the meaning within the Barut-Girardello coherent states defined for the singular oscillator. We have also shown the relationship with the time-of-arrival operator of Aharonov and Bohm.

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1 Introduction

The existence of a self-adjoint time operator conjugate to a given Hamiltonian is a longstanding problem of quantum mechanics. The unequal role played by time as an observable in classical and quantum mechanics is the main source of controversy. The problem arises because we expect observables to be represented in quantum mechanics by self-adjoint operators.\cite{1,2}. In an attempt to promote time to be an observable, we have to face a well-known argument of Pauli\cite{3} that such an operator cannot be self-adjoint if the spectrum of a self-adjoint Hamiltonian is bounded from below. As a consequence, the time-energy uncertainty relation cannot be deduced from the same kinematical point of view as the position-momentum uncertainty relation. Nevertheless, the search for various time operators and the analysis of their self-adjointness and associated time-energy uncertainty relations have been the subject of a number of papers\cite{4}. The general consensus seems to be that no such self-adjoint operator exists.

Recently, the validity of Pauli’s objections has been critically evaluated\cite{5}, with the conclusion that there is no a priori reason to exclude the existence of self-adjoint time operators for semibounded Hamiltonians.

In this work we consider the problem of constructing a self-adjoint time operator for a singular harmonic oscillator.

2 The problem

The singular harmonic oscillator of the Calogero-Sutherland type\cite{6} is described by the Hamiltonian

\[ H_{CS} \equiv 2\omega K_3 = \omega^2 K + H, \quad (1) \]

where \( K = x^2/2 \) and

\[ H = \frac{1}{2} (p^2 + \frac{g}{x^2}), \quad g > 0. \quad (2) \]

is the Calogero-Moser\cite{7} scale invariant Hamiltonian. We have identified \( H_{CS} \) Hamiltonian with the compact generator, \( K_3 \), of the SU(1,1) group, which is the dynamical group of this problem. Two other generators of SU(1,1) are

\[ K_1 = \frac{1}{2} (\omega K - \frac{1}{\omega} H), \]

\[ K_2 = D, \quad (3) \]

where \( D = -(xp + px)/4 \) is the scale operator.
The group generators $K_3$ and $K_\pm = K_1 \pm iK_2$ satisfy the standard commutation relations of the su(1,1) algebra:

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3.$$  

Our objective is to construct an operator $\hat{T}$ in terms of the generators $K_3, K_\pm$ that is conjugate to the Hamiltonian $H_{CS}$ and satisfies $[H_{CS}, \hat{T}] = i$.

Let us denote by $|n, k>$, $n = 0, 1, 2, ...$ the complete orthonormal basis states, which diagonalize the compact generator $K_3 = H_{CS}/2\omega$ [8]. The Bargman index $k = \frac{1}{2}(1 + \sqrt{1 + 4g})$ is related to the eigenvalue $k(k - 1) = g$ of the quadratic Casimir operator $\hat{C}_2 = K_3^2 - K_1^2 - K_2^2$ of the SU(1,1) group. These states are obtained from $|0, k>$ by n-fold application of $K_+$:

$$|n, k> = \sqrt{\frac{\Gamma(2k + n)}{\Gamma(2k + n)!}} (K_+)^n |0, k>, \quad K_- |0, k> = 0, \quad K_3 |n, k> = (n + k) |n, k>, \quad n = 0, 1, 2, ...$$  

In the space spanned by the eigenstates of the generator $K_3$, we immediately encounter the problem. The matrix elements of $[H_{CS}, \hat{T}]$ in the basis $|n, k>$,

$$<n, k|[H_{CS}, \hat{T}]|m, k> = 2\omega(n - m) <n, k|\hat{T}|m, k>$$  

vanish for $n = m$. This implies $[H_{CS}, \hat{T}] \neq i$ if $<n, k|\hat{T}|m, k> \neq 0$. This relation is correct only if the operation by $\hat{T}$ on a state $|n, k>$ is of the form

$$\hat{T}|n, k> = \sum_m t_{mn} |m, k>.$$  

However, $\hat{T}$ does not have that property if it is conjugate to $H_{CS}$. In the next Section, we shall study $[H_{CS}, \hat{T}]$ commutator in the time-variable representation, i.e., in the representation in which $\hat{T}$ is diagonal.

### 3 Construction of $\hat{T}$

Let us observe that from $[K_3, K^n] = -nK^n$, we can make a simple ansatz that $\hat{T}$ is some power series function of $K_-$ and $K_+$ such that

$$[K_3, F(K_\pm)] = \pm K_\pm F'(K_\pm) = \frac{i}{4\omega}.$$  

A possible solution is

$$\hat{T} = \frac{1}{4i\omega} (lnK_+ - lnK_-),$$  

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which is easily represented in the coherent state representation of the operator $K_-$.

States which diagonalize the operator $K_-$

$$K_- |z, k> = z |z, k>,$$

where $z$ is an arbitrary complex number, are known as the Barut-Girardello (BG) coherent states [9], [10]. The expansion of these states over the orthonormal basis $|n, k> is

$$|z, k> = \frac{z^{k-1/2}}{\sqrt{I_{2k-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(2k+n)}} |n, k>,$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. The above BG states are normalized to unity, they resolve the identity operator, but are not mutually orthogonal

$$<z_1, k|z_2, k> = I_{2k-1}(2\sqrt{z_1^*z_2})[I_{2k-1}(2|z_1|)I_{2k-1}(2|z_2|)]^{-1/2}.\quad (12)$$

Due to this property any quantum state $|\psi> can be represented by the analytic function

$$f_\psi(z) = \sqrt{I_{2k-1}(2|z|)}(z^{1/2-k}) <k, z^*|\psi>.$$

The operators $K_\pm$ and $K_3$ act in the Hilbert space of analytic functions $f_\psi(z)$ as linear differential operators

$$K_+ = z, \quad K_- = 2k \frac{d}{dz} + z \frac{d^2}{dz^2}, \quad K_3 = k + z \frac{d}{dz}.\quad (14)$$

In terms of BG coherent states, the time operator $\hat{T}$ is

$$\hat{T} = (4\pi i)^{-1} \int d\mu(z, k) ln(\frac{z}{z^*}) |z, k><z, k|,$$

where $d\mu(z, k) = 2K_{2k-1}(2|z|)I_{2k-1}(2|z|)d^2z/\pi$.

### 4 Discussion

BG coherent states can also be written as an exponential operator acting on the vacuum state of $K_-,$

$$|z, k> = e^{zK_+(K_3+k)^{-1}}|0, k>.$$

In deriving this expression, we have used an operator identity

$$[K_+(K_3+k)^{-1}]^n = K_+^n \frac{\Gamma(K_3+k)}{\Gamma(K_3+k+n)}.$$

$$3$$
Note also that the operator $K_+(K_3 + k)^{-1}$ is canonical to $K_-$:

$$[K_-, K_+(K_3 + k)^{-1}] = 1. \quad (18)$$

It is easy to see that $H = \omega(K_3 - K_1)$ is related to $K_-$:

$$e^{-\omega K} H e^{\omega K} = -2\omega K_- \quad (19)$$

Therefore, the energy eigenstates of $H |E> = E |E>$ are proportional to the BG coherent states [9,11] if $z = -E/2\omega$:

$$|E> = e^{\omega K} |E>, k >. \quad (20)$$

Note also, that the state $<x|E>$, in the limit $E \to 0$, is not normalizable, since $\lim_{E \to 0} <x|E> = <x|\omega^K[0,k] > \propto \omega^k x^{2k-1/2}$. The difficulty arises from the oscillating behavior of $<x|E>$ at large distances [12].

Finally, we consider an explicit construction of time operator for $H_CS$ using the method developed in [13,14,15]. We first observe that there exists a singular similarity transformation between $H_CS$ and the Hamiltonian of the ordinary harmonic oscillator, $H_h = H_CS(g = 0)$:

$$H_CS S = S H_h, \quad S = e^{-K_-} e^{K^0} \quad (21)$$

where $K^0 = K_-(g = 0)$. The time operator for $H_h$ was constructed and discussed in [14, 15, 16]. Its construction is simple if we observe that the Casimir operator with $k = 3/4$ can be used to express the operator $K$ in the form

$$K = T_0 H_0 T_0 + \frac{1}{16 H_0} \quad (22)$$

$$Q = -T_0 + \frac{i}{4 H_0} \quad (23)$$

where

$$T_0 = -\frac{1}{2} \left( x \frac{1}{p} + \frac{1}{p} x \right)$$

is the time-of-arrival operator of Aharonov and Bohm [16], and $H_0 = p^2/2$. Then the Hermitian operator

$$T_h = \frac{1}{2} (T_h(Q) + T_h^\dagger(Q)) \quad (24)$$

satisfies $[H_h, T_h] = i$, where

$$T_h(Q) = \frac{1}{\omega} \arctg(\omega Q). \quad (25)$$
It is now easy to see that the time operator for the Hamiltonian $H_{CS}$ is

$$T_{CS} = ST^{-1}_hS^{-1}, \quad S^{-1} \neq S^1.$$  \hspace{1cm} (26)

Note that in this construction $T_{CS} \neq T_{CS}^\dagger$. Formally, in the limit $\omega \to 0$ we obtain the time operator for the scale invariant Hamiltonian $H$ [17].

5 Conclusions

In conclusion, we have presented an algebraic method of constructing Hermitian operators conjugate to a Hamiltonian with $SU(1, 1)$ dynamical symmetry. In terms of generators of $SU(1, 1)$, the time operator for a singular harmonic oscillator is constructed explicitly, and shown that it can be related to the time-of-arrival operator, $T_0$ of Aharanov and Bohm. The question whether time operators thus constructed are self-adjoint operators in Hilbert space requires a careful examination of their spectra and eigenfunctions. The time-energy commutation relation is studied in the energy and the time domains. The eigenvalue problem of the operator $T_{CS}$ can be solved in the time domain using BG coherent states. It is not self-adjoint and its eigenfunctions are not orthogonal. Therefore, the problem of finding self-adjoint $T_h$ and $T_{CS}$ is still open [5,18]. Let us also mention that the problem of time-operator for a repulsive singular potential of the Calogero-Moser type [7] is interesting for several reasons. It is scale invariant and has the full conformal group $SO(2, 1)$ as a dynamical symmetry group [12] with the generators $H, D$ and $K$. The spectrum of $H$, for $g > 0$ is positive, continuous, and bounded from below, but with a non-normalizable ground state. $H$ can be easily extended to the well-known one-dimensional N-body problem of Calogero-Moser [7]. Recently, it has been observed that the dynamics of scalar particles near the horizon of a black hole is also associated with this Hamiltonian [17,19,20,21].

It is important to point out here that the solution of the formal equation $[H, \hat{T}] = i$ is not unique. Any $T' = \hat{T} + \phi(H)$, with arbitrary $\phi$, satisfies the same canonical commutation relation.

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