Geometries for Possible Kinematics

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The algebras for all possible Lorentzian and Euclidean kinematics with so(3) isotropy except static ones are re-classified. The geometries for algebras are presented by contraction approach. The relations among the geometries are revealed. Almost all geometries fall into pairs. There exists $t \leftrightarrow 1/(\nu^2 t)$ correspondence in each pair. In the viewpoint of differential geometry, there are only 9 geometries, which have right signature and geometrical spatial isotropy. They are 3 relativistic geometries, 3 absolute-time geometries, and 3 absolute-space geometries.

Keywords: possible kinematics, geometries, contraction, time duality

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The contraction is a powerful method in mathematical physics, which reveals the relations among groups and geometries. By the Inönü-Wigner contraction method \[1\], Bacry and Lévy-Leblond show that there are 11 kinematical groups of 8 types according to their algebras under the assumption that a kinematical group should possess (i) an \(SO(3)\) isotropy generated by \(J\), (ii) parity (\(\Pi : H \rightarrow H, P \rightarrow -P, K \rightarrow -K, J \rightarrow J\)) and time-reversal (\(\Theta : H \rightarrow -H, P \rightarrow P, K \rightarrow -K, J \rightarrow J\)) automorphism, and (iii) non-compact one-dimensional subgroup generated by each boost \(K_i\) \[2\]. These groups are the Poincaré (\(P\)), de Sitter (\(dS\)), anti-de Sitter (\(AdS\)), Inhomogenous \(SO(4)\) (\(E'^1\)), para-Poincaré (\(P'\)), Galilei (\(G\)), Newton-Hooke (\(NH_+\)), anti-Newton-Hooke (\(NH_-\)), para-Galilei (\(G'\)), Carroll (\(C\)), and static (\(S\)) groups. Releasing the third condition, three geometrically kinematical groups — Euclid (\(Euc\)), Riemann (\(Riem\)), and Lobachevsky (\(Lob\)) groups — should be added. Among these 14 kinematical groups, the geometries corresponding to \(Riem, Lob, dS, AdS, Euc, P, NH_\pm, G\) and \(C\) groups are clear. But, the others still need to be clarified though the correspondence between 2d possible kinematics and 9 Cayley-Klein geometries \[3\] have been set up \[4\]. Different from the higher dimension cases, each direction in 2d case can be identified as the timelike direction and the other as the spacelike one when the signature is \((+,-)\), which results in that different kinematics’ correspond to the same Cayley-Klein geometry.

The possible kinematical algebras can also be obtained from very different approach — combinatorial approach \[5\]. The foundation of the new approach is the principle of relativity with an invariant speed \(c\) and an invariant length \(l\), denoted as the \(PoR_{c,l}\) \[5–7\]. It is well known that the principle of relativity is the foundation of physics, which is closely related to the symmetry of space and time. Based on \(PoR_{c,l}\), the triality of special relativity with \(dS\)-, \(AdS\)-, and \(P\)-invariance can be established \[6\]. It has been shown that the general form of transformations preserving \(PoR_{c,l}\) is the linear fractional transformation \[8, 9\]. All linear fractional transformations form a group, known as \(PGL(5, \mathbb{R})\) (or “inertial-motion” group \(IM(4)\)) \[5\]. The group contains 24 possible kinematical or geometrical (sub-)groups with the same \(SO(3)\) isotropy. In addition to the 11 possible kinematics in \[2\] and 3 geometrical kinematics, there are additional 10 groups. A natural question appears: what are the meanings of these additional possible kinematical groups or what do these additional possible

\[1\] It is sometimes denoted as \(P'_+\) in the literature.
kinematical groups represent?

The purposes of the present paper are two-fold. The first is to present the geometrical structures for all possible kinematics revealed in [2, 5] but static ones. Seeing that the different kinematical algebras are linked together by the contraction, their corresponding geometries should also be linked by the same contraction. Therefore, the unknown geometries can be obtained from known geometries by the contraction approach. The obtained geometrical structures are required to be invariant under the given 10-parameter transformations and are not identically vanishing or divergent everywhere. The second is further to explore the relations among the geometries with different kinematical groups and to pick out the geometries for the genuine possible kinematics.

The paper will be organized in the following way. In the next section, we shall review all the possible kinematical algebras and clarify their relation to the \( R_{\text{iem}} \), \( Lob \), \( dS \) and \( AdS \) algebras. In sections III, we shall “derive” the geometries from the geometries for \( R_{\text{iem}} \), \( Lob \), \( dS \), and \( AdS \) algebras and/or their contractions. In section IV, we shall further explore the relations among the geometries. The concluding remarks are given in the last section. In the section, we shall make some comments on the requirements to select the possible kinematics proposed by Bacry and Lévy-Leblond and modify the requirements. Under the modified requirements, there are only 9 possible kinematical algebras.

II. POSSIBLE KINEMATICAL ALGEBRAS

In [5], we show that there are 24 possible kinematical or geometrical (sub-)algebras with the same \( \mathfrak{so}(3) \) isotropy in \( \mathfrak{pgl}(5, \mathbb{R}) \). They are listed in the TABLE I, in which the generators are defined by

\[
\begin{align*}
H = \partial_t, \quad H' = -\nu^2 t x^\mu \partial_\mu, \quad H^\pm = \partial_t \mp \nu t x^\mu \partial_\mu, \quad x^0 = ct, \quad \nu = c/l; \\
P_i = \partial_i, \quad P_i' = -l^{-2} x^i x^\mu \partial_\mu, \quad P_i^\pm = \partial_i \mp l^{-2} x^i x^\mu \partial_\mu, \\
K_i = t \partial_i - c^2 x_i \partial_t, \quad K_i^g = t \partial_i, \quad K_i^c = -c^2 x_i \partial_t, \quad N_i = t \partial_i + c^2 x_i \partial_t \\
J_i = \frac{1}{2} \epsilon_{i}^{jk} (x_j \partial_k - x_k \partial_j),
\end{align*}
\]

where \( c \), \( l \) and thus \( \nu \) are invariant parameters with dimensions of speed, length and the inverse of time, respectively. Hereafter, the lowercase Greek letters \( \mu, \nu, \kappa, \rho, \sigma, \cdots \) in indices run from 0 to 3 while the lowercase Latin letters \( i, j, k, l, m, n, \cdots \) in indices run from 1 to 3. \( H, H' \), and \( H^\pm \) are known as the (algebraic) translation, pseudo-translation, and Beltrami translation of time, respectively. \( P, P' \), and \( P^\pm \) are the (algebraic) translation, pseudo-translation, and Beltrami translation of space, respectively. \( K, K^g, K^c \), and \( N \) are the Lorentz, Galilei, Carroll, and geometrical boosts, respectively. The other generators in \( \mathfrak{pgl}(5, \mathbb{R}) \) may transform the generators in one kinematical algebra to the ones for another.

According to the classification of Bacry and Lévy-Leblond, the first four algebras in TABLE I are purely geometrical ones and the remaining 20 are kinematical algebras. All possible kinematical and geometrical algebras are related together in the two extremely different approaches. One is the combinatorial method [5] and the other is the contraction method [2].
TABLE I: All possible kinematical and geometrical algebras

| Algebra | Symbol | Generator set | Algebra | Symbol | Generator set |
|---------|--------|---------------|---------|--------|---------------|
| Riemann | \( \tau \) | \((H^-, P_i^+, N_i, J_i)\) | Euclid  | \( c \) | \((H, P_i, N_i, J_i)\) |
| Lobachevsky | \( \iota \) | \((H^+, P_i^+, N_i, J_i)\) | Poincaré | \( p \) | \((H, P_i, K_i, J_i)\) |
| \( dS \) | \( \sigma_+ \) | \((H^+, P_i^+, K_i, J_i)\) | \( \mathcal{p}_2 \) | \( (H', P_i, K_i, J_i) \) |
| \( AdS \) | \( \sigma_- \) | \((H^-, P_i^-, K_i, J_i)\) | Galilei  | \( g \) | \((H, P_i, K_i^0, J_i)\) |
| \( NH_+ \) | \( n_+ \) | \((H^+, P_i^+, K_i^0, J_i)\) | Carroll  | \( c \) | \((H, P_i, K_i^0, J_i)\) |
| \( n_{+2} \) | \((H^+, P_i^+, K_i, J_i)\) | \( \mathcal{g}_2 \) | \((H', P_i, K_i^0, J_i)\) |
| \( NH_- \) | \( n_- \) | \((H^-, P_i, K_i^0, J_i)\) | \( \mathcal{g}'_2 \) | \((H', P_i, K_i^0, J_i)\) |
| \( n_{-2} \) | \((-H^-, P_i^+, K_i, J_i)\) | \( \mathcal{g}' \) | \((H', P_i, K_i^0, J_i)\) |
| \( HN_+^b \) | \( h_+ \) | \((H, P_i^+, K_i^c, J_i)\) | \( \mathcal{g}' \) | \((H', P_i, K_i^0, J_i)\) |
| \( c' \) | \((H^+, P_i^+, K_i^0, J_i)\) | \( \mathcal{g}'_2 \) | \((H', P_i, K_i^0, J_i)\) |
| \( HN_-^b \) | \( h_- \) | \((H, P_i^-, K_i^0, J_i)\) | \( \mathcal{g}' \) | \((H', P_i, K_i^0, J_i)\) |
| \( p' \) | \((-H', P_i^-, K_i, J_i)\) | \( \mathcal{g}'_2 \) | \((H', P_i, K_i^0, J_i)\) |

\(^a\)The second versions of algebras here have different meaning from the one in Ref. [10]. Say, the second Poincaré group here is the semi-product of Lorentz group with pseudo-translations, while in [10] the second Poincaré group is the semi-product of Lorentz group with special conformal transformations.

\(^b\)\(HN_\pm^b\) are historically called the para-Poincaré algebras [2]. They are called Hooke-Newton algebras because they are different from the Newton-Hooke algebras by the replacement \(H^\pm \leftrightarrow H, P_i \leftrightarrow P_i^\pm, K_i^0 \leftrightarrow K_i^c\). From the geometrical point of view (see: Sec. \textit{III}D), the second version of \(HN_\pm\) algebra are called para-Euclid and para-Poincaré algebra, respectively.

\(^c\)The generator \(H^s\) is meaningful only when the central extension is considered.

From \(dS\) and \(AdS\) algebras \(\sigma_\pm\), one may obtain the generators of the Poincaré algebra \(p\) and the second Poincaré algebra \(p\) by the simple summation or subtraction,

\[
p : \quad H = \frac{1}{2}(H^+ + H^-), \quad P = \frac{1}{2}(P^+ + P^-), \quad K, \quad J, \quad (2.2)
\]

and

\[
p_2 : \quad H' = \frac{1}{2}(H^+ - H^-), \quad P' = \frac{1}{2}(P^+ - P^-), \quad K, \quad J, \quad (2.3)
\]

respectively. One may also obtain them by the contraction of the generators of \(\sigma_\pm\),

\[
p : \quad H = \lim_{l_r \to \infty} H^\pm_{l_r}, \quad P = \lim_{l_r \to \infty} P^\pm_{l_r}, \quad K, \quad J, \quad (2.4)
\]

and

\[
p_2 : \quad H' = \pm \lim_{l_r \to 0} \frac{l_r^2}{2} H^\pm_{l_r}, \quad P' = \pm \lim_{l_r \to 0} \frac{l_r^2}{2} P^\pm_{l_r}, \quad K, \quad J, \quad (2.5)
\]

respectively. Here, \(l_r\) is a running parameter of dimension \(L\). It replaces \(l\) in the definition of generators and thus the generators is denoted with a subscript \(r\). The contraction prescription used here is slightly different from that used by Bacry and Lévy-Leblond, in which the contraction for \(p\), for example, is defined by

\[
H \to \varepsilon H, \quad P \to \varepsilon P, \quad K, \quad J, \quad \varepsilon \to 0.
\quad (2.6)
\]
In comparison of Eq. (2.5) with Eq. (2.6), the Bacry-Lévy-Leblond contraction gives $p_2$ if $\epsilon = \pm l_r/l^2$ is taken.

Similarly, from Ricm algebra $r$ and Lob algebra $l$, one may attain the generators of the Euc algebra $e$ and the second Euc algebra $e_2$ by the summation or substraction,

\[ e : \quad H = \frac{1}{2}(H^+ + H^-), \quad P = \frac{1}{2}(P^+ + P^-), \quad N, \quad J, \quad (2.7) \]

and \[ e_2 : \quad -H' = \frac{1}{2}(H^- - H^+), \quad P' = \frac{1}{2}(P^+ - P^-), \quad N, \quad J, \quad (2.8) \]

respectively, or by the contraction of the generators of $r$ and $l$,

\[ e : \quad H = \lim_{l_r \to \infty} H_r^\pm, \quad P = \lim_{l_r \to \infty} P_r^\pm, \quad N, \quad J, \quad (2.9) \]

and \[ e_2 : \quad -H' = \pm \lim_{l_r \to 0} \frac{l_r^2}{l^2} H_r^\pm, \quad P' = \pm \lim_{l_r \to 0} \frac{l_r^2}{l^2} P_r^\pm, \quad N, \quad J, \quad (2.10) \]

respectively.

From $\mathfrak{d}_+$, $l$ and $\mathfrak{d}_-$, $r$, one may get the generators of $NH_\pm$ algebras $n_\pm$ and the second $NH_\pm$ algebras $n_{\pm 2}$ by the combinatory method, respectively,

\[ n_\pm : \quad H^\pm, \quad P = \frac{1}{2}(P^+ + P^-), \quad K^g = \frac{1}{2}(K + N), \quad J, \quad (2.11) \]

and \[ n_{\pm 2} : \quad \pm H^\pm, \quad P' = \frac{1}{2}(P^+ - P^-), \quad K^c = \frac{1}{2}(K - N), \quad J. \quad (2.12) \]

The generators of $n_+$ and $n_{+ 2}$ can also be acquired by the contraction from $\mathfrak{d}_+$ or ($\Pi$ of) $l$,

\[ n_+ : \quad H^+ = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} H_r^+, \quad P = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} P_r^+, \quad K^g = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} K_r(\text{or } N_r), \quad J, \quad (2.13) \]

and \[ n_{+ 2} : \quad H^+ = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} H_r^+, \quad P' = \pm \lim_{c_r, l_r \to 0, \nu \text{ fixed}} \frac{l_r^2}{l^2} P_r^+, \quad K^c = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} \frac{c_r^2}{c^2} K_r(\text{or } -\frac{c_r^2}{c^2} N_r), \quad J. \quad (2.14) \]

where $c_r$ is a running parameter of dimension $LT^{-1}$. In comparison of Eq. (2.14) with Eq. (9a) in [2], the Bacry-Lévy-Leblond contraction gives $n_{+ 2}$ if $\epsilon = l_r^2/l^2 = c_r^2/c^2$ is taken. Similarly, the generators of $n_-$ and $n_{- 2}$ can also be acquired by the contraction from ($\Theta\Pi$ of) $\mathfrak{d}_-$ or ($\Theta$ of) $r$,

\[ n_- : \quad H^- = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} H_r^-, \quad P = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} P_r^-, \quad K^g = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} K_r(\text{or } N_r), \quad J, \quad (2.15) \]

and \[ n_{- 2} : \quad -H^- = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} H_r^-, \quad P' = \mp \lim_{c_r, l_r \to 0, \nu \text{ fixed}} \frac{l_r^2}{l^2} P_r^-, \quad K^c = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} \frac{c_r^2}{c^2} K_r(\text{or } -\frac{c_r^2}{c^2} N_r), \quad J. \quad (2.16) \]

The generators of $HN_+$ algebra $\mathfrak{h}_+$ and para-Euclid [isomorphic to iso(4)] algebra $e'$ (or $HN_-$ algebra $\mathfrak{h}_-$ and para-Poincare algebra $p'$) are obtained from those of $\mathfrak{d}_+$ and $r$ (or $\mathfrak{d}_-$ and $l$) by the combinatory method,

\[ \mathfrak{h}_\pm : \quad H = \frac{1}{2}(H^+ + H^-), \quad P^\pm, \quad K^c = \frac{1}{2}(K - N), \quad J, \quad (2.17) \]

and \[ e', p' : \quad \pm H' = \pm \frac{1}{2}(H^+ - H^-), \quad P^\pm, \quad K^g = \frac{1}{2}(K + N), \quad J. \quad (2.18) \]
respectively. They are related to \( \mathfrak{d}_\pm \) by

\[
\mathfrak{h}_\pm: \quad H = \lim_{c_r \to 0} H_r^\pm, \quad P^\pm, K^c = \lim_{c_r \to 0} \frac{C_r^2}{C^2} K_r, \quad J, \tag{2.19}
\]

and \( \mathfrak{c}', \mathfrak{p}' \):

\[
\pm H' = \lim_{c_r \to \infty} \frac{c_r^2}{C^2} H_r^\pm, \quad P^\pm, K^g = \lim_{c_r \to \infty} K_r, \quad J, \tag{2.20}
\]

respectively. \( \mathfrak{h}_\pm, \mathfrak{c}' \), and \( \mathfrak{p}' \) with \( H' \) replaced by \(-H'\) are also the contractions of \((\Theta \text{ of}) \mathfrak{r} \) and \((\Theta \text{ of}) \mathfrak{l} \) in the limit \( c_r \to 0 \) and \( c_r \to \infty \), respectively.

The generators of Galilei/Carroll algebras \( \mathfrak{g}/\mathfrak{c} \) are the linear combination of generators of \( \mathfrak{d}_\pm, \mathfrak{r}, \) and \( \mathfrak{l} \), respectively,

\[
\mathfrak{g}/\mathfrak{c}: \quad H = \frac{1}{2}(H^+ + H^-), \quad P = \frac{1}{2}(P^+ + P^-), \quad K^{g/c} = \frac{1}{2}(K \mp N), \quad J, \tag{2.21}
\]

and \( \mathfrak{g}_2/\mathfrak{c}_2 \):

\[
H' = \frac{1}{2}(H^+ - H^-), \quad P' = \frac{1}{2}(P^+ - P^-), \quad K^{c/g} = \frac{1}{2}(K \mp N), \quad J, \tag{2.22}
\]

or as the contraction from \( \mathfrak{d}_\pm, \)

\[
\mathfrak{g}: \quad H = \lim_{c_r \to \infty, l_r \to \infty, \nu_r \to 0} H_r^\pm, \quad P = \lim_{c_r \to \infty, l_r \to \infty, \nu_r \to 0} P^\pm, \quad K^g = \lim_{c_r \to \infty, l_r \to \infty, \nu_r \to 0} \frac{C_r^2}{C^2} K_r, \quad J, \tag{2.23}
\]

\[
\mathfrak{c}: \quad H = \lim_{l_r \to \infty, \nu_r \to 0} H_r^\pm, \quad P = \lim_{l_r \to \infty, \nu_r \to 0} P^\pm, \quad K^c = \lim_{l_r \to \infty, \nu_r \to 0} \frac{c_r^2}{C^2} K_r, \quad J, \tag{2.24}
\]

\[
\mathfrak{g}_2: \quad H' = \pm \lim_{c_r \to \infty, l_r \to \infty, \nu_r \to 0} \frac{\nu_r^2}{C^2} H_r^\pm, \quad P' = \pm \lim_{c_r \to \infty, l_r \to \infty, \nu_r \to 0} \frac{l_r^2}{C^2} P^\pm, \quad K^c = \lim_{c_r \to \infty, l_r \to \infty, \nu_r \to 0} \frac{c_r^2}{C^2} K_r, \quad J, \tag{2.25}
\]

and \( \mathfrak{c}_2 \):

\[
H' = \pm \lim_{l_r \to 0, \nu_r \to \infty} \frac{\nu_r^2}{C^2} H_r^\pm, \quad P' = \pm \lim_{l_r \to 0, \nu_r \to \infty} \frac{l_r^2}{C^2} P^\pm, \quad K^g = \lim_{l_r \to 0, \nu_r \to \infty} K_r, \quad J. \tag{2.26}
\]

Obviously, \( \mathfrak{g} \) can also be derived by the contraction from \( \mathfrak{r} \) or \( \mathfrak{l} \) in the limit of \( c_r, l_r \to \infty, \nu_r \to 0, \)

\[
\mathfrak{g}: \quad H = \lim_{c_r, l_r \to \infty, \nu_r \to 0} H_r^\pm, \quad P = \lim_{c_r, l_r \to \infty, \nu_r \to 0} P^\pm, \quad K^g = \lim_{c_r, l_r \to \infty, \nu_r \to 0} N_r, \quad J; \tag{2.27}
\]

from \( \mathfrak{c} \) and \( \mathfrak{p} \) in the limit of \( c_r \to \infty \); and from \( \mathfrak{n}_\pm \) in the limit of \( \nu_r \to 0. \ \mathfrak{g}_2 \) can also be derived by the contraction from \((\Theta \text{ of}) \mathfrak{r} \) or \((\Pi \text{ of}) \mathfrak{l} \) in the limit of \( c_r, l_r \to 0, \nu_r \to \infty, \)

\[
\mathfrak{g}_2: \quad H' = \mp \lim_{c_r, l_r \to \infty, \nu_r \to \infty} \frac{\nu_r^2}{C^2} H_r^\pm, \quad P' = \pm \lim_{c_r, l_r \to \infty, \nu_r \to \infty} \frac{l_r^2}{C^2} P^\pm, \quad K^c = \lim_{c_r, l_r \to \infty, \nu_r \to \infty} \frac{c_r^2}{C^2} N, \quad J; \tag{2.28}
\]

from \((\Theta \text{ of}) \mathfrak{c}_2 \) or \( \mathfrak{p}_2 \) in the limit of \( c_r \to 0 \); and from \( \mathfrak{n}_{\pm 2} \) in the limit of \( \nu_r \to \infty. \) Similarly, \( \mathfrak{c} \) and \( \mathfrak{c}_2 \) are the direct contractions of \( \mathfrak{h}_\pm \) in the limit \( l_r \to \infty \) and \( \mathfrak{e}', \mathfrak{p}' \) in the limit \( l_r \to 0, \) respectively, or the direct contractions of \( \mathfrak{p} \) in the limit \( c_r \to 0 \) and \( \mathfrak{p}_2 \) in the limit of \( c_r \to \infty, \) respectively. \( \mathfrak{c} \) and \( \mathfrak{c}_2 \) with \( H' \) replaced by \(-H'\) are also the contraction of \((\Theta \text{ of}) \mathfrak{c} \) in the limit \( c_r \to 0 \) and \( \mathfrak{e}_2 \) in the limit \( c_r \to \infty, \) respectively.
FIG. 1: Contraction scheme for the possible kinematics. (The degeneracy in each algebra and its partner has been released.)

The combination of the generators of \( d_\pm, r \) and \( l \) can define the generators of para-Galilei algebra \( g' \) in the following way.

\[
g': \ H' = \frac{1}{2}(H^+ - H^-), \quad P = \frac{1}{2}(P^+ + P^-), \quad K^g = \frac{1}{2}(K + N), \quad J, \quad (2.29)
\]

and

\[
g'_2: \ H = \frac{1}{2}(H^+ + H^-), \quad P' = \frac{1}{2}(P^+ - P^-), \quad K^c = \frac{1}{4}(K - N), \quad J. \quad (2.30)
\]

g' and \( g'_2 \) can be deduced from the contraction from \( d_+ \),

\[
g': \ H = \lim_{l_r, c_r \to \infty, v \to \infty} \frac{v^2}{l_r^2} H_r^+, \quad P = \lim_{l_r, c_r \to \infty, v \to \infty} \frac{P^+}{l_r^2}, \quad K^g = \lim_{l_r, c_r \to \infty, v \to \infty} K_r, \quad J, \quad (2.31)
\]

and

\[
g'_2: \ H = \lim_{l_r, c_r \to 0, v \to \infty} H_r^+, \quad P' = \lim_{l_r, c_r \to 0, v \to \infty} \frac{P^+}{l_r^2}, \quad K^c = \lim_{l_r, c_r \to 0, v \to \infty} \frac{c^2}{l_r} K_r, \quad J. \quad (2.32)
\]

The contraction from \( I \) in the limit of \( c_r, l_r, v \to \infty \) can also give rise to \( g' \),

\[
g': \ H' = \lim_{l_r, c_r \to \infty, v \to \infty} \frac{v^2}{l_r^2} H_r^+, \quad P = \lim_{l_r, c_r \to \infty, v \to \infty} P^-, \quad K^g = \lim_{l_r, c_r \to \infty, v \to \infty} N_r, \quad J. \quad (2.33)
\]
Besides, $g'$ is the results of the contractions from $n_\pm$ in the limit $\nu_r \to \infty$ or from $e'$ in the limit $l_r \to \infty$, $g'$ with $H'$ replaced by $-H'$ is the result of the contraction from $d_-$ and $r$ in the limit $l_r, c_r, \nu_r \to \infty$, from $n_-$ in the limit $\nu_r \to \infty$, or from (II of) $p'$ in the limit $l_r \to \infty$. Similarly, the contraction from (II of) I in the limit of $c_r, l_r, \nu_r \to 0$ results in $g_2'$,

$$g_2': \quad H = \lim_{l_r, c_r \to 0, \nu_r \to 0} H^+_{r}, \quad P' = -\lim_{l_r, c_r \to 0, \nu_r \to 0} \frac{\rho}{2} P_{r}^{-}, \quad K' = -\lim_{l_r, c_r \to 0, \nu_r \to 0} \frac{c_r^2}{2} N_r, \quad J. \quad (2.34)$$

Finally, $g_2'$ is the results of the contractions from $n_{+2}$ in the limit $\nu_r \to 0$ or from $h_\pm$ in the limit $l_r \to 0$, $g_2'$ with $H'$ replaced by $-H'$ is the result of the contraction from (II$\Theta$ of) $d_-$ and ($\Theta$ of) $r$ in the limit $l_r, c_r, \nu_r \to 0$, from $n_{-2}$ in the limit $\nu_r \to 0$ or from II of $h_-$ in the limit $l_r \to 0$.

The contraction scheme for the possible kinematics are shown in FIG. 1 in the similar way as in Ref.[2]. In the figure the degeneracy in the each algebra and its partner has been released and the diagram is not two perfect cubes with a common vertex. FIG. 2 presents the contraction scheme for possible kinematics in a more symmetric way, where the two static algebras as exceptions are ignored, whose time-translation generator is meaningful only when the central extension is taken into consideration.
III. GEOMETRIES FOR THE POSSIBLE KINEMATICAL ALGEBRAS

A. Geometries for \( r, l, \epsilon, \) and \( \epsilon_2 \)

The metrics of 4d Riemann and Lobachevsky spaces in a Beltrami coordinate system are well known,

\[
\begin{align*}
\text{ds}_E^2 &= \frac{1}{\sigma_E^\pm} \left( \delta_{\mu\nu} \mp \frac{\delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda}{l^2 \sigma_E^\pm} \right) dx^\mu dx^\nu, \\
\text{ds}_{L^2} &= \lim_{l \to 0} \frac{1}{\sigma_{E,r}^\pm} \left( \delta_{\mu\nu} \mp \frac{\delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda}{l^2 \sigma_{E,r}^\pm} \right) dx^\mu dx^\nu = \delta_{\mu\nu} dx^\mu dx^\nu,
\end{align*}
\]

where

\[
\sigma_E^\pm = 1 \pm l^{-2} \delta_{\kappa\lambda} x^\kappa x^\lambda > 0.
\]

(If the antipodal identification is not taken, more Beltrami coordinate charts are needed to cover all 4d Riemann sphere, for example \cite{7}. Here, we do not plan to discuss the problem in details in this paper.) They are invariant under the transformations generated by \( r \) and \( l \), respectively. \( \sigma_E^+ > 0 \) is automatically satisfied. \( \sigma_E^- > 0 \) puts the constraint on the domain. Obviously, if \( \sigma_E^- < 0 \), the metric

\[
\text{ds}_E^2 = \frac{1}{\sigma_E^-} \left( \delta_{\mu\nu} + \frac{\delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda}{l^2 \sigma_E^-} \right) dx^\mu dx^\nu, \tag{3.3}
\]

is also invariant under the transformations generated by \( l \). But, it has the signature \((-+, +, +, +)\) and thus is the alternative representation of \( dS \) space-time though the 1d sub-groups generated by each boost is compact. For brevity, it is referred to as Lobachevsky-Beltrami-de Sitter (\( \text{LBdS} \)) space-time later and the line-element is denoted by \( \text{ds}_{\text{LBdS}}^2 \).

Both \( \text{Riem} \) and \( \text{Lob} \) geometries contract to the Euclid metric in the limit of \( l_r \to \infty \),

\[
\text{ds}_E^2 = \lim_{l_r \to \infty} \frac{1}{\sigma_{E,r}^\pm} \left( \delta_{\mu\nu} \mp \frac{\delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda}{l_r^2 \sigma_{E,r}^\pm} \right) dx^\mu dx^\nu = \delta_{\mu\nu} dx^\mu dx^\nu, \tag{3.4}
\]

where \( \sigma_{E,r}^\pm \) is the \( \sigma_E^\pm \) with a running parameter \( l_r \), namely,

\[
\sigma_{E,r}^\pm = 1 \pm l_r^{-2} \delta_{\kappa\lambda} x^\kappa x^\lambda > 0. \tag{3.5}
\]

The inequality \( \sigma_{E,r}^- < 0 \) will be violated in the limiting process of \( l_r \to \infty \). Thus, \( \text{LBdS} \) space-time is not contractible in this way\(^2\). Obviously, these metrics contain all local geometrical information of the given spaces because of non-degeneracy.

On the other hand, in the limiting process of \( l_r \to 0 \), \( l_r^2 \sigma_{E,r}^+ > 0 \) and \( l_r^2 \sigma_{E,r}^- < 0 \) are always hold, while \( l_r^2 \sigma_{E,r}^- > 0 \) is not. Thus, the \( \text{Lob} \) geometry is not contractible in this limit, while both \( \text{Riem} \) and \( \text{LBdS} \) geometries give rise to

\[
\text{ds}_{E_2}^2 = \pm \lim_{l_r \to 0} \frac{l_r^2}{\sigma_{E,r}^\pm} \left( \delta_{\mu\nu} \mp \frac{\delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda}{l_r^2 \sigma_{E,r}^\pm} \right) dx^\mu dx^\nu = l_r^2 \delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda = l_r^2 \frac{\delta_{\mu\nu} \delta_{\kappa\lambda} x^\kappa x^\lambda}{(\delta_{\sigma\tau} x^\kappa x^\tau)^2} dx^\mu dx^\nu =: g^{E_2}. \tag{3.6}
\]

\(^2\) Please note that “contractible” here is different from the usual concept in topology.
Unfortunately, Eq. (3.6) defines a degenerate metric. It does not contain enough geometrical information to determine a 4d geometry uniquely. Also, it may possess larger symmetry with more than 10 parameters. To determine the geometry, one should consider the limit of the inverse metric and the connection with the limit of connection coefficients. For the contraction of Riem geometry,

\[
\left( \frac{\partial}{\partial s} \right)^2_{E_2} = \lim_{l_r \to 0} \frac{l_r^4}{l_r^4} \sigma^+_{E_r}(\delta^\mu\nu + l_r^{-2} x^\mu x^\nu) \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = l_r^{-4} \delta_{\sigma\tau} x^\sigma x^\tau x^\mu x^\nu \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} =: h_{E_2}, \quad (3.7)
\]

\[
\Gamma^\lambda_{E_2 \mu\nu} = \lim_{l_r \to 0} \frac{\delta^\lambda_{\mu\nu} + \delta^\lambda_{\nu\mu} - \delta^\lambda_{\mu\nu}}{l_r^2 \sigma^+_{E_r}} = \frac{-(\delta^\lambda_{\mu\nu} + \delta^\lambda_{\nu\mu}) x^\kappa}{\delta_{\sigma\tau} x^\sigma x^\tau}.
\]

(3.8)

It can be shown that the connection is compatible to \( g^E_{\mu\nu} \) and \( h^E_{\mu\nu} \), i.e.

\[
\nabla^E_{\lambda} g^E_{\mu\nu} = \partial_{\lambda} g^E_{\mu\nu} - \Gamma^E_{\lambda\kappa} g^E_{\mu\nu} - \Gamma^E_{\mu\lambda} g^E_{\nu\kappa} - 2 \Gamma^E_{\mu\nu} g^E_{\lambda\kappa} = 0,
\]

\[
\nabla^E_{\lambda} h^E_{\mu\nu} = \partial_{\lambda} h^E_{\mu\nu} + \Gamma^E_{\mu\lambda} h^E_{\nu\kappa} + \Gamma^E_{\nu\lambda} h^E_{\mu\kappa} = 0,
\]

and that the curvature of the connection has the form for constant curvature space,

\[
R^E_{\mu\nu\lambda\kappa} = \partial_\mu \Gamma^E_{\nu\lambda\kappa} - \partial_\nu \Gamma^E_{\mu\lambda\kappa} + \Gamma^E_{\nu\mu} \Gamma^E_{\lambda\kappa} - \Gamma^E_{\mu\nu} \Gamma^E_{\lambda\kappa} = -l_r^{-2} (\delta^\sigma_{\nu} g^E_{\mu\rho} - \delta^\sigma_{\rho} g^E_{\mu\nu}) \quad (3.11)
\]

\[
R^E_{\mu\nu} = -3l_r^{-2} g^E_{\mu\nu}.
\]

(3.12)

It can be checked that \( \{ M^E_2, g^E_2, h^E_2, \nabla^E_2 \} \) is invariant under \( E_2 \) transformations, namely, \( \forall \xi \in \mathfrak{e}_2 \subset TM^E_2 \)

\[
\mathcal{L}_{\xi} g^E_2 = g^E_{\mu\nu} \partial_{\xi}^\lambda + g^E_{\mu\lambda} \partial_{\nu}^\lambda + g^E_{\lambda\nu} \partial_{\mu}^\lambda = 0,
\]

\[
\mathcal{L}_{\xi} h^E_2 = h^E_{\mu\nu} \partial_{\xi}^\lambda - h^E_{\mu\lambda} \partial_{\nu}^\lambda - h^E_{\lambda\nu} \partial_{\mu}^\lambda = 0,
\]

\[
[\mathcal{L}_{\xi}, \nabla^E_2] = 0,
\]

\[
\text{or Eqs.}(3.6), (3.7), (3.8) \text{ are invariant under the coordinate transformation}
\]

\[
x' = \frac{Sx}{1 + l_r^{-1} b^T x},
\]

\[
(3.16)
\]

and its inverse transformation,

\[
x = \frac{S^{-1} x'}{1 + l_r^{-1} b^T S^{-1} x'}, \quad \frac{S^{-1} x'}{1 + l_r^{-1} b^T S^{-1} x'} = \frac{S^{-1} x'}{1 + l_r^{-1} (b')^T x'},
\]

\[
(3.17)
\]

where \( x, x' \), and \( b \) are the \( 4 \times 1 \) matrixes, \( b^T \) is the transpose of \( b \), \( x \) and \( x' \) have the dimension of length while \( b \) is dimensionless, \( S \in SO(4) \), \( b' = S b \). The points satisfying

\[
1 + l_r^{-1} b^T x = 0
\]

\[
(3.18)
\]
will be transformed to infinity in the new coordinate system $x'$ under the coordinate transformation (3.16). Therefore, the infinity point should be in the manifold $M^{E_2}$. In contrast, the origin $x = 0$ is an invariant point under the transformation, and so has to be detached from the manifold.

In order to see the manifold more transparently, consider the coordinate transformations,

$$
\begin{align*}
\begin{cases}
x^0 = l^2\rho^{-1}\cos \chi \\
x^1 = l^2\rho^{-1}\sin \chi \sin \theta \cos \phi \\
x^2 = l^2\rho^{-1}\sin \chi \sin \theta \sin \phi \\
x^3 = l^2\rho^{-1}\sin \chi \cos \theta,
\end{cases}
\end{align*}
$$

(3.19)

where $\bar{x}^0 = \rho \in (-\infty, +\infty)$, $\bar{x}^1 = \chi \in [0, \pi]$, $\bar{x}^2 = \theta \in [0, \pi]$, $\bar{x}^3 = \phi \in [0, 2\pi)$. Under the coordinate transformation, Eqs.(3.6), (3.7), and (3.8) become

$$
g^{E_2} = l^2(d\chi^2 + \sin^2 \chi d\Omega_2^2) = \bar{g}^{E_2}_{ij} d\bar{x}^i d\bar{x}^j,
$$

(3.20)

$$
h_{E_2} = \left(\frac{\partial}{\partial \rho}\right)^2,
$$

(3.21)

$$
\begin{align*}
\bar{\Gamma}^{0}_{E_2 ij} &= l^{-2} \rho g_{ij}, \\
\bar{\Gamma}^{i}_{E_2 jk} &= -\sin \chi \cos \chi \delta_{1}^{i}((\delta_{j}^{2}\delta_{k}^{2} + \sin^2 \theta \delta_{j}^{3}\delta_{k}^{3}) + 2 \cot \theta (\delta_{j}^{1} - \delta_{j}^{i}\delta_{l}^{1})\delta_{k}^{l}) \\
&+ 2 \cot \theta \delta_{j}^{2}\delta_{l}^{2}\delta_{k}^{3} - \sin \theta \cos \theta \delta_{j}^{2}\delta_{l}^{3}\delta_{k}^{3},
\end{align*}
$$

(3.22)

respectively. Then, Eqs.(3.11) and (3.12) become

$$
\bar{R}_{E_2}^{\sigma}_{\mu \rho \nu} = -l^{-2}(\delta_{\rho}^{\sigma} g^{E_2}_{\mu \nu} - \delta_{\mu}^{\sigma} g^{E_2}_{\rho \nu}),
$$

(3.23)

$$
\bar{R}_{E_2}^{\sigma}_{\mu \nu} = -3l^{-2} \rho \bar{g}^{E_2}_{\mu \nu}.
$$

(3.24)

All these equations show that the manifold is $S_3 \times \mathbb{R}$, where 3d sphere has the radius $l$. The signature of $g^{E_2}$ and $h^{E_2}$ are $(+, +, +)$ and $(+)$, respectively, and denoted by $(+, +, +; +)$ for brevity.

For the contraction of $LBDs$ space-time,

$$
\lim_{l \to 0} \frac{l^4}{l^2} \frac{\sigma^{E_2} R^{E_2}}{l^2} = \frac{\delta^{\mu} x^{\nu}}{\delta^{\nu} x^{\mu}} \frac{\partial}{\partial x^{\mu}} \otimes \frac{\partial}{\partial x^{\nu}} = -h^{E_2}.
$$

(3.25)

$$
\lim_{l \to 0} \frac{l^2}{l^2} \frac{\sigma^{E_2} R^{E_2}}{l^2} = -\frac{(\delta_{\mu}^{\lambda} \delta_{\nu}^{\kappa} + \delta_{\nu}^{\lambda} \delta_{\mu}^{\kappa}) x^{\kappa}}{\delta_{\sigma}^{\tau} x^{\sigma} x^{\tau}} = \Gamma^{E_2}_{\mu \nu}.
$$

(3.26)

The resulting space-time possesses the same topology, the same symmetry, and the same connection and curvature as the $E_2$ geometry. And the 4d geometry is also split into 3d and 1d same geometries. The only difference is the opposite sign in $h$. Therefore, it may also be denoted by $E_2$. In order to see the contraction path and for convenience in the following discussion, however, we denote it by $E_{2-}$ and the signature by $(+, +, +; -)$. (We shall treat other degenerate geometries in the similar way in the following discussion.)
B. Geometries for $\partial_{\pm}$, $p$, and $p_2$

The metrics of 4d $dS$ and $AdS$ space-times in a Beltrami coordinate system are \cite{7, 11}

\[
    ds^2_{\pm} = \frac{1}{\sigma^{\pm}} \left( \eta_{\mu\nu} \pm \eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda \right) dx^\mu dx^\nu,
\]

\[\text{(3.27)}\]

respectively, where

\[
    \sigma^{\pm} = 1 \mp l^{-2} \eta_{\kappa\lambda} x^\kappa x^\lambda > 0.
\]

\[\text{(3.28)}\]

If the domain condition (3.28) is changed to

\[
    \sigma^{\pm} < 0,
\]

\[\text{(3.29)}\]

the geometries become

\[
    ds^2_{\pm, <} = \pm \frac{1}{\sigma^{\pm}} \left( \eta_{\mu\nu} \pm \eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda \right) dx^\mu dx^\nu.
\]

\[\text{(3.30)}\]

They are also invariant under the transformations generated by $\partial_{\pm}$. The metric with $\sigma_+ < 0$ has the signature $(+,-,+,-)$. It is a $BdS$ model of Lob space. Hence, it may be named as $BdSL$ space. The metric with $\sigma_- < 0$ has the signature $(+,-,+,-)$. It may be called the double time $dS$ space-time, denoted as $DTdS$. All these metrics are non-degenerate and thus contain all local geometrical information.

Form either $dS$ or $AdS$ space-time, take $l_r \to \infty$ limit, we get the familiar Minkowski ($Min$) metric

\[
    ds^2_{Min} = \lim_{l_r \to \infty} \frac{1}{\sigma^+_r} \left( \eta_{\mu\nu} \pm \eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda \right) dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu,
\]

\[\text{(3.31)}\]

where

\[
    \sigma^+_r = 1 \mp l^{-2} \eta_{\kappa\lambda} x^\kappa x^\lambda > 0.
\]

\[\text{(3.32)}\]

In the limit of $l_r \to \infty$, the conditions $\sigma^+_r < 0$ will be violated. Thus, $BdSL$ space and $DTdS$ space-time are uncontractible.

In the limit of $l_r \to 0$,

\[
    l^2 \sigma^+_r = -\eta_{\kappa\lambda} x^\kappa x^\lambda > 0,
\]

\[\text{(3.33)}\]

the metrics of $dS$ and $AdS$ space-times reduce to the degenerate ones

\[
    ds^2_{p^2_{\pm}} = \lim_{l_r \to 0} \frac{l^2}{l^2 \sigma^+_r} \left( \eta_{\mu\nu} \pm \eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda \right) dx^\mu dx^\nu = \pm l^4 \eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda dx^\mu dx^\nu = g^{p^2_{\pm}}.
\]

\[\text{(3.34)}\]

The upper sign corresponds to the limit of $dS$ space-time while the lower sign corresponds to the limit of $AdS$ space-time. Similar to the second Euclid case, to determine the geometry completely, we should consider the limit of the inverse metrics

\[
    \left( \frac{\partial}{\partial s} \right)^2_{p^2_{\pm}} = \lim_{l_r \to 0} \frac{l^4}{l^4 \sigma^+_r} \left( \eta^{\mu\nu} \mp l^{-2} x^\mu x^\nu \right) \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = l^{-4} \eta_{\sigma\tau} x^\sigma x^\tau x^\mu x^\nu \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} =: h_{p^2_{\pm}}
\]

\[\text{(3.35)}\]
and the connection with the limit of connection coefficients

\[
\Gamma_{P^2 \pm \mu \nu} = \pm \lim_{l \to 0} \left( \frac{(\delta^\lambda_{\mu} \eta_{\nu \kappa} + \delta^\lambda_{\nu} \eta_{\mu \kappa}) x^\kappa}{l^2 \sigma^\pm_r} \right) = \mp \frac{(\delta^\lambda_{\mu} \eta_{\nu \kappa} + \delta^\lambda_{\nu} \eta_{\mu \kappa}) x^\kappa}{\eta_{\sigma \tau} x^\sigma x^\tau}. \tag{3.36}
\]

It is easy to check that the connection \( \nabla^{P^2 \pm} \) with coefficients (3.36) is compatible with Eq. (3.34) and (3.35), i.e.

\[
\nabla^{P^2 \pm}_{\lambda} g_{\mu \nu}^{P^2 \pm} = \partial_{\lambda} g_{\mu \nu}^{P^2 \pm} - \Gamma^{P^2 \pm}_{\lambda \mu \nu} g_{\rho \kappa}^{P^2 \pm} - \Gamma^{P^2 \pm}_{\lambda \nu \rho} g_{\mu \kappa}^{P^2 \pm} = 0 \tag{3.37}
\]

and

\[
\nabla^{P^2 \pm}_{\lambda} h_{\mu \nu}^{P^2 \pm} = \partial_{\lambda} h_{\mu \nu}^{P^2 \pm} + \Gamma^{P^2 \pm}_{\lambda \mu \kappa} h_{\nu \rho}^{P^2 \pm} + \Gamma^{P^2 \pm}_{\lambda \nu \rho} h_{\mu \kappa}^{P^2 \pm} = 0, \tag{3.38}
\]

respectively. It can also be shown that \( \{ M^{P^2 \pm}, g^{P^2 \pm}, h^{P^2 \pm}, \nabla^{P^2 \pm} \} \) is invariant under \( P^2 \) transformations, namely, \( \forall \xi \in p_2 \subset TM^{P^2 \pm} \)

\[
\mathcal{L}_\xi g_{\mu \nu}^{P^2 \pm} = g_{\mu \nu, \lambda}^{P^2 \pm} + g_{\mu \lambda}^{P^2 \pm} \partial_{\nu} \xi^\lambda + g_{\nu \lambda}^{P^2 \pm} \partial_{\mu} \xi^\lambda = 0, \tag{3.39}
\]

\[
\mathcal{L}_\xi h_{\mu \nu}^{P^2 \pm} = \xi^\lambda \partial_{\lambda} h_{\mu \nu}^{P^2 \pm} - h_{\mu \lambda}^{P^2 \pm} \partial_{\nu} \xi^\lambda - h_{\nu \lambda}^{P^2 \pm} \partial_{\mu} \xi^\lambda = 0, \tag{3.40}
\]

\[
[\mathcal{L}_\xi, \nabla^{P^2 \pm}] = 0, \tag{3.41}
\]

or Eqs.(3.34), (3.35), and (3.36) are invariant under the coordinate transformations

\[
x' = \frac{Lx}{1 + l^{-1}b^T x} \tag{3.42}
\]

and its inverse transformation,

\[
x = \frac{L^{-1}x'}{1 + l^{-1}b^T L^{-1} x'} = \frac{L^{-1} x'}{1 + l^{-1}b'^T x'}, \tag{3.43}
\]

where \( L \) is the Lorentz transformation, \( b \) is dimensionless, \( b' = Lb \), a superscript \( T \) stands for the transpose under the metric \( \eta_{\mu \nu} = \text{diag}(1, -1, -1, -1) \).

By definition, the curvature tensor is

\[
R^{P^2 \pm \sigma \mu \rho \nu} = \pm l^{-2} (\delta^\sigma_{\nu} g_{\mu \rho}^{P^2 \pm} - \delta^\sigma_{\rho} g_{\mu \nu}^{P^2 \pm}). \tag{3.44}
\]

The Ricci curvature tensor is then

\[
R_{\mu \nu}^{P^2 \pm} = R^{P^2 \pm \sigma \mu \rho \nu} = \pm 3l^{-2} g_{\mu \nu}^{P^2 \pm}. \tag{3.45}
\]

They are obviously invariant under \( P^2 \) transformation.

The structure of the space-times has been analyzed in [12]. The manifold is \( dS_3 \times \mathbb{R} \) for \( x \cdot x < 0 \) and \( \mathbb{H}_3 \times \mathbb{R} \) for \( x \cdot x > 0 \) and has the signature \( (+, -, -; -) \) and \( (-, -, -; +) \), respectively.
In the limit of $l_r \to 0$, the $BdSL$ space and $DTdS$ space-time contract to

$$\frac{d^2s_{EP_{2-}DTP_{2+}}}{\lambda} = \pm \lim_{l_r \to 0} \frac{l^2_r}{r^2 \sigma_r^\pm} \left( \eta_{\mu\nu} \pm \frac{\eta_{\mu\nu} \eta_{\lambda x^\kappa}}{l^2_r \sigma_r^\pm} \right) dx^\mu dx^\nu = \frac{l^2 r^2 \sigma_r^\pm}{(\eta_{\sigma^\tau x^\kappa x^\lambda}^x)^2} dx^\mu dx^\nu$$

$$= \left\{ \begin{array}{ll}
\frac{l^2}{r^2} \left( \frac{(d(r/c)^2)}{1 - r^2/c^2 l^2} \right)^2 + \frac{r^2/c^2 l^2}{r^2/c^2 l^2} d\Omega_2^2 & \eta_{\mu\nu} x^\mu x^\nu > 0 \\
\frac{l^2}{r^2} \left( \frac{(d(r/c)^2)}{r^2/c^2 l^2 - 1} \right)^2 - \frac{r^2/c^2 l^2}{r^2/c^2 l^2 - 1} d\Omega_2^2 & \eta_{\mu\nu} x^\mu x^\nu < 0
\end{array} \right. , \quad (3.46)$$

$$(\frac{\partial}{\partial s})_{EP_{2-}DTP_{2+}}^2 = \pm \lim_{l_r \to 0} \frac{l^4_r}{l^2 \sigma_r^\pm} \left( \eta_{\mu\nu} \pm l^{-2}_r x^\mu x^\nu \right) \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = \pm l^{-4}_r \eta_{\sigma^\tau x^\kappa x^\lambda} x^\nu \eta_{\mu\nu} x^\mu x^\nu \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}$$

$$= \left\{ \begin{array}{ll}
\frac{\partial}{\partial (c^2 t^2 - r^2)^{-1/2}} \otimes \left( \frac{\partial}{\partial (c^2 t^2 - r^2)^{-1/2}} \right) & \eta_{\mu\nu} x^\mu x^\nu > 0 \\
\frac{\partial}{\partial (r^2 - c^2 t^2)^{-1/2}} \otimes \left( \frac{\partial}{\partial (r^2 - c^2 t^2)^{-1/2}} \right) & \eta_{\mu\nu} x^\mu x^\nu < 0
\end{array} \right. , \quad (3.47)$$

$$\Gamma_{EP_{2-}DTP_{2+}}^{\lambda \mu \nu} = \pm \lim_{l_r \to 0} \frac{(\delta^\lambda_\mu \eta_{\nu} + \delta^\lambda_\nu \eta_{\mu}) x^\kappa}{l^2_r \sigma_r^\pm} = \frac{(\delta^\lambda_\mu \eta_{\nu} + \delta^\lambda_\nu \eta_{\mu}) x^\kappa}{\eta_{\sigma^\tau x^\kappa x^\lambda}} \quad (3.48)$$

where $r^2 := \delta_{ij} x^i x^j$ and $d\Omega_2^2$ is the line-element of 2d unit sphere. They have the forms of $g_{P_{2+}}^0$, $\pm h_{P_{2+}}$, and $\Gamma_{P_{2+}}$ but with

$$l^2_r \sigma_r^\pm \rightarrow \mp \eta_{\kappa \lambda} x^\kappa x^\lambda < 0. \quad (3.49)$$

The contraction of $BdSL$ space is $\mathbb{H}_3 \times \mathbb{R}$ with the signature $(+, +, +; +)$. It is the Euclidean version of $P_{2-}$ space-time, denoted by $EP_{2-}$. The contraction of $DTdS$ space is $dS_3 \times \mathbb{R}$ with the signature $(+, -, -; +)$. It is the double time version of $P_{2+}$ space-time, denoted by $DT P_{2+}$.

C. Geometries for $n_{\pm}$ and $n_{\pm 2}$

The $NH_{\pm}$ space-times can be obtained from $dS$ and $AdS$ space-times by the contraction in the limit of $c_r, l_r \to \infty$ but $\nu = c_r/l_r$ fixed [13]. When $c_r, l_r \to \infty$ and $\nu = c_r/l_r = c/l$,

$$\sigma_r^\pm := \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \sigma_r = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \left[ 1 \mp l_r^{-2} \left( c_r^2 l^2 - \delta_{ij} x^i x^j \right) \right] = 1 \mp l^{-2} (x^0)^2 > 0. \quad (3.50)$$

the metrics, inverse metrics, and connection coefficients become

$$d^2s_{NH_{\pm}}^2 = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \frac{c^2}{c_r^2} \frac{1}{\sigma_r^\pm} \left( \eta_{\mu\nu} \pm \frac{\eta_{\mu\nu} \eta_{\lambda x^\kappa}}{l^2_r \sigma_r^\pm} \right) dx^\mu dx^\nu = \frac{c^2}{(\sigma_r^\pm)^2} d^2x =: g_{NH_{\pm}}, \quad (3.51)$$

$$\left( \frac{\partial}{\partial s} \right)_{NH_{\pm}}^2 = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \sigma_r^\pm \left( \eta_{\mu\nu} \pm l_r^{-2} x^\mu x^\nu \right) \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = -\sigma_r^\pm \delta_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} =: h_{NH_{\pm}}. \quad (3.52)$$
\[ \Gamma_{NH \pm \mu \nu}^\lambda = \pm \lim_{c_r, l_r \to \infty} \frac{c_r (\delta^\lambda_\mu \eta_{\nu \kappa} + \delta^\lambda_\nu \eta_{\mu \kappa}) x_r^\kappa}{l_r^2 \sigma_n^\pm} = \pm \frac{(\delta^\lambda_\mu \eta_{\nu 0} + \delta^\lambda_\nu \eta_{\mu 0}) x^0}{l^2 \sigma_n^\pm}, \] (3.53)

where \( x_r^\mu \) means \((c_r t, x^i)\). The nonzero connection coefficients are\(^3\)

\[ \Gamma_{NH \pm 00}^0 = \pm \frac{2x^0}{l^2 \sigma^\pm_n}, \quad \Gamma_{NH \pm 0j}^i = \Gamma_{NH \pm j0}^i = \pm \frac{x^0}{l^2 \sigma^\pm_n} \delta^i_j. \] (3.54)

The non-zero components of curvature tensor and Ricci tensor are given by

\[ R_{\lambda \mu \nu}^\rho = \pm l^{-2}(\delta^\rho_\lambda g_{\mu \nu}^\pm - \delta^\lambda_\mu g_{\rho \nu}^\pm), \] (3.55)

\[ R_{00} = \pm 3l^{-2} g_{NH \pm 00}, \] (3.56)

respectively. The \( NH \pm \) space-times are the non-relativistic space-times. They are the generalization of Galilei space-time and have absolute-time. In terms of Beltrami coordinates, the time is “curved” and the space is conformally flat. The conformal factor is independent of point in the space. It has been shown that \( \{ M^{NH \pm}, g^{NH \pm}, h_{NH \pm}, \nabla^{NH \pm} \} \) is invariant under \( NH \pm \) transformations [13]. The mechanics on the \( NH \pm \) space-times has been studied in details in [13, 14].

The contractions of 4d \( Lob \) and 4d \( Riem \) spaces in the same limit give the same domain conditions

\[ \sigma_n^\pm = \lim_{c_r, l_r \to \infty} \sigma_{E, r}^\pm = 1 \mp l^{-2}(x^0)^2 > 0, \] (3.57)

the same covariant degenerate metrics

\[ ds^2_{ENH \pm} = \lim_{c_r, l_r \to \infty} \frac{1}{\sigma_{E, r}^\pm} \left( \delta_{\mu \nu} \pm \delta_{\mu \kappa} \delta_{\nu \lambda} x_r^\kappa x_r^\lambda \right) dx_r^\mu dx_r^\nu = \frac{c^2}{(\sigma_n^\pm)^2} dt^2 =: g^{ENH \pm} = g^{NH \pm}, \] (3.58)

and the same connection coefficients

\[ \Gamma_{ENH \pm \mu \nu}^\lambda = \pm \lim_{c_r, l_r \to \infty} \frac{c_r (\delta^\lambda_\mu \delta_{\nu \kappa} + \delta^\lambda_\nu \delta_{\mu \kappa}) x_r^\kappa}{l_r^2 \sigma_{E, r}^\pm} = \pm \frac{(\delta^\lambda_\mu \delta_{\nu 0} + \delta^\lambda_\nu \delta_{\mu 0}) x^0}{l^2 \sigma_n^\pm} = \Gamma_{NH \pm \mu \nu}. \] (3.59)

The contravariant degenerate metrics are

\[ \lim_{c_r, l_r \to \infty} \sigma_{E, r}^\pm (\delta_{\mu \nu} \mp l^{-2} x_r^\mu x_r^\nu) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = \sigma_n^\pm \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} =: h_{ENH \pm} = -h_{NH \pm}. \] (3.60)

Since the signatures of \( g^{NH \pm} \) and \( -h_{NH \pm} \) are \((+)\) and \((+, +, +)\), respectively, the geometries obtained from the contraction of \( Lob \) and \( Riem \) geometries are called Euclidean \( NH \pm \) \((ENH \pm)\) space-times.

The domain conditions

\[ \lim_{c_r, l_r \to \infty} \sigma_{E, r}^\pm \left. , \lim_{c_r, l_r \to \infty} \sigma_{r}^+ \right\} = 1 - l^{-2}(x^0)^2 = \sigma_n^+ < 0 \] (3.61)

\(^3\) \( \Gamma_{NH \pm \mu \nu}^\lambda = c \Gamma_{NH \pm 00}^0 \), \( \Gamma_{NH \pm \mu \nu}^\rho = c \Gamma_{NH \pm \rho 0}^0 \) are used in [13].
means $t^2 > \nu^{-2}$. In this case, the $LBdS$ space-time contracts to

$$ds^2_{NH^*} = -\lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \frac{1}{\sigma_r} \left( \delta_{\mu\nu} + \frac{\delta_{\mu\nu}\delta_{\nu\lambda} x^\lambda_r x^\nu_r}{l_r^2 \sigma_r} \right) dx^\mu_r dx^\nu_r = -\frac{e^2 dt^2}{(\sigma^+_n)^2}, \quad (3.62)$$

$$\left( \frac{\partial}{\partial s} \right)_{NH^*}^2 = -\lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \sigma_r (\delta_{\mu\nu} - l_r^{-2} x^\mu_r x^\nu_r) \frac{\partial}{\partial x^\mu_r} \otimes \frac{\partial}{\partial x^\nu_r} = -\sigma^n_+ \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad (3.63)$$

$$\Gamma^{NH^*}_{\nu\mu\nu} = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \frac{c_r}{c} \frac{\delta^\lambda_{\mu\nu} + \delta^\lambda_{\nu\mu}}{l_r^2 \sigma_r} \left( \frac{\delta^\lambda_{\nu\delta\eta} x^\delta_r x^\eta_r}{l^2 \sigma^+_n} \right) = \left( \frac{\delta^\lambda_{\mu\nu} + \delta^\lambda_{\nu\mu}}{l^2 \sigma^+_n} \right). \quad (3.64)$$

Except an overall minus, the degenerate covariant and contravariant metrics have the same form as $ENH^*_r$, but the same signature as $NH^+$ because $\sigma^+_n < 0$. In order to distinguish it from $NH^+$ space-time, it may be called para-$NH^+$ ($NH^*_r$) space-time. The $BdSL$ space-time contracts to

$$ds^2_{ENH^*_r} = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \frac{1}{\sigma_r} \left( \eta_{\mu\nu} + \frac{\eta_{\mu\kappa}\eta_{\nu\lambda} x^\kappa_r x^\lambda_r}{l_r^2 \sigma_r} \right) dx^\mu_r dx^\nu_r = \frac{e^2 dt^2}{(\sigma^+_n)^2}, \quad (3.65)$$

$$\left( \frac{\partial}{\partial s} \right)_{ENH^*_r}^2 = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \sigma^+_n (\eta_{\mu\nu} - l_r^{-2} x^\mu_r x^\nu_r) \frac{\partial}{\partial x^\mu_r} \otimes \frac{\partial}{\partial x^\nu_r} = -\sigma^+_n \delta_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad (3.66)$$

$$\Gamma^{NH^*_r}_{\nu\mu\nu} = \lim_{c_r, l_r \to \infty, \nu \text{ fixed}} \frac{c_r}{c} \frac{\delta^\lambda_{\mu\nu} + \delta^\lambda_{\nu\mu}}{l_r^2 \sigma_r} \left( \frac{\delta^\lambda_{\nu\delta\eta} x^\delta_r x^\eta_r}{l^2 \sigma^+_n} \right) = \left( \frac{\delta^\lambda_{\mu\nu} + \delta^\lambda_{\nu\mu}}{l^2 \sigma^+_n} \right). \quad (3.67)$$

The signature is $(+;+;+,+)$. It is the Euclidean version of para-$NH^+$ space-time, denoted by $ENH^*_r$. Finally, $DTdS$ is uncontrollable in this case because $\sigma^-_r < 0$ is not preserved in the limit.

On the other hand, in the limit $c_r, l_r \to 0$ but $\nu = c_r/l_r$ fixed,

$$\lim_{c_r, l_r \to 0, \nu \text{ fixed}} l^2 \sigma^+_r = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} l^2 \left( 1 \mp l^2 (c_r^2 t^2 - \delta_{ij} x^i x^j) \right) = \pm l^2 \delta_{ij} x^i x^j = \sigma^+_n. \quad (3.68)$$

The inequalities $\sigma^+_r > 0, \sigma^-_r < 0, \sigma^+_E, r > 0$, and $\sigma^-_r < 0$ are always valid in the limit. Thus, the second $NH$ geometry ($NH_2$), the second para-$NH$ geometry ($NH'_2$), the second Euclidean $NH$ geometry ($ENH_2$), and the second double time $NH$ space-time ($DTNH_2$) can be obtained from the contraction of $dS, LBdS, Riem$, and $DTdS$ geometries, respectively.

For the $NH_2$ geometry, the covariant degenerate metric, contravariant degenerate metric, and connection coefficients are

$$ds^2_{NH_2} = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} \frac{l^2}{l^2 \sigma^+_r} \left( \eta_{\mu\nu} + \frac{\eta_{\mu\kappa}\eta_{\nu\lambda} x^\kappa_r x^\lambda_r}{l_r^2 \sigma_r} \right) dx^\mu_r dx^\nu_r = l^2 \left( \delta_{ij} x^i x^j + \delta_{ij} \delta_{ij} x^i x^j \right) = g^NH_2, \quad (3.69)$$

$$\left( \frac{\partial}{\partial s} \right)_{NH_2}^2 = \lim_{c_r, l_r \to 0, \nu \text{ fixed}} \frac{l^4}{l^4 \sigma_r} (\eta_{\mu\nu} - l^{-2} x^\mu_r x^\nu_r) \frac{\partial}{\partial x^\mu_r} \otimes \frac{\partial}{\partial x^\nu_r} = \frac{\delta_{mn} x^n x^m}{l^4} \left( \nu^2 (1 - \nu^2 t^2) \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} - x^i x^j \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} - 2tx^i \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial x^j} \right) =: h_{NH_2}, \quad (3.70)$$
\[ \Gamma_{NH2}^{\lambda}_{\mu\nu} = \lim_{\nu \to 0} \frac{(\delta^\lambda_{\mu} \eta_{\nu\kappa} + \delta^\lambda_{\nu} \eta_{\mu\kappa}) x^\kappa}{l_r^2 \sigma_r^+} = \frac{(\delta^\lambda_{\mu} \eta_{\nu i} + \delta^\lambda_{\nu} \eta_{\mu i}) x^i}{\delta_{mn} x^m x^n}, \]  

(3.71)

respectively. The rank of the degenerate metric (3.69) is 2, which may serve as the non-degenerate metric of the sphere \( S_2 \) with the radius \( l \). The rank of the degenerate “inverse” metric (3.70) is also 2. Their signatures are \((-,-)\) and \((+, -)\), respectively, denoted by \((-,-;+,-)\). The coordinate \( c^2 t^2 \) serves as a timelike coordinate. The non-zero connection coefficients are

\[ 
\Gamma_{NH20i}^0 = \Gamma_{NH2i0}^0 = -\frac{\delta_{ij} x^j}{\delta_{mn} x^m x^n}, \quad \Gamma_{NH2jk}^i = -\frac{(\delta^i_j \delta_{kl} + \delta^i_k \delta_{jl}) x^l}{\delta_{mn} x^m x^n}. 
\]  

(3.72)

The nonzero components of curvature tensor and Ricci tensor are

\[ R_{NH2}^{0\,i0j} = -R_{NH2}^{0\,j0i} = -l^{-2} g_{ij}^{NH2}, \quad R_{NH2}^{k\,iij} = l^{-2} (\delta^k_j \delta_{il}^{NH2} - \delta^k_i g_{ij}^{NH2}), \]  

(3.73)

\[ R_{ij}^{NH2} = 3l^{-2} g_{ij}^{NH2}, \]  

(3.74)

respectively.

Under the coordinate transformation

\[ 
\begin{align*}
  x^0 &= \psi r/l \\
  x^1 &= r \sin \theta \cos \phi \\
  x^2 &= r \sin \theta \sin \phi \\
  x^3 &= r \cos \theta,
\end{align*} 
\]  

(3.75)

Eqs.(3.69) and (3.70) become

\[ ds^2_{NH2} = -l^2 (d\theta^2 + \sin^2 \phi d\phi^2), \]  

(3.76)

\[ \left( \frac{\partial}{\partial s} \right)^2_{NH2} = \frac{\partial}{\partial \psi} \otimes \frac{\partial}{\partial \psi} - \frac{r^4}{l^4} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r}, \]  

(3.77)

respectively. Obviously, in \( NH_2 \) space-time, there is no geometrical SO(3) isotropy with respect to any fixed point in \( M^{NH2} \). The \( NH_2 \) space-time describes the Beltrami coordinate region \( \delta_{ij} x^i x^j > 0 \).

It implies that the spatial point \( x^1 = x^2 = x^3 = 0 \) is not in the manifold.

The contraction from the \( LBdS \) gives rise to the \( NH_2' \) geometry:

\[ ds^2_{NH2'} = -\lim_{\nu \to 0} \frac{l^2}{l_r^2 \sigma_r^-} \left( \delta_{\mu\nu} + \delta_{mn} \delta_{ij} x^m x^n \frac{x^\mu x^\nu}{l_r^2 \sigma_{E,r}} \right) dx^\mu dx^\nu = l^2 \frac{(\delta^i_j \delta_{kl} - \delta^i_k \delta_{jl}) x^k x^l}{(\delta_{mn} x^m x^n)^2} dx^i dx^j = -g^{NH2}, \]  

(3.78)

\[ \left( \frac{\partial}{\partial s} \right)^2_{NH2'} = -\lim_{\nu \to 0} \frac{l^4}{l_r^4 \sigma_{E,r}} \left( \delta_{\mu\nu} - l^{-2} x^\mu x^\nu \frac{\partial}{\partial x_r^-} \otimes \frac{\partial}{\partial x_r^-} \right) \]  

\[ = \delta_{mn} x^m x^n \frac{\nu^{-2} (1 - \nu^2 l^2) \partial}{\partial t} \otimes \frac{\partial}{\partial t} - x^i x^j \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} - 2t x^i \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial x^j} = h_{NH2'}, \]  

(3.79)
\[ \Gamma_{NH2}^{\lambda}_{\mu \nu} = \lim_{c_r, l_r \to 0} \frac{(\delta^{\lambda}_{\mu} \delta_{\nu k} + \delta^{\lambda}_{\nu} \delta_{\mu k}) x^k}{l_r^2 \sigma_{E,r}^2} = -\frac{(\delta^{\lambda}_{\mu} \delta_{\nu i} + \delta^{\lambda}_{\nu} \delta_{\mu i}) x^i}{\delta_{mn} x^m x^n}. \] (3.80)

The resulting \(-g^{NH2}\) and \(h_{NH2}\) have rank 2 each and signature \((+, +)\) and \((+, -)\), respectively, denoted by \((+, +; +, -)\). In this case, the coordinate \(\rho = (\delta_{\mu \nu} x^\mu x^\nu)^{1/2}\) rather than \(x^0 = ct\) serves as a timelike coordinate.

The contraction of Riem geometry defines the \(ENH\) geometry:

\[ ds^2_{ENH2} = \lim_{c_r, l_r \to 0} \frac{l_r^4}{l_r^2 \sigma_{E,r}^2} (\delta_{\mu \nu} - \frac{\delta_{\mu \nu} \delta_{\nu k} x^k}{l_r^2 \sigma_{E,r}^2}) dx^\mu dx^\nu = l^2 (\frac{\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) x^k x^l}{(\delta_{mn} x^m x^n)^2} dx^i dx^j = -g^{NH2}, \] (3.81)

\[ \left( \frac{\partial}{\partial s} \right)^2_{ENH2} = \lim_{c_r, l_r \to 0} \frac{l_r^4}{l_r^2 \sigma_{E,r}^2} (\delta^{\mu \nu} + l_r^{-2} x^\mu x^\nu) \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = \frac{\delta_{mn} x^m x^n}{l^4} \left( \nu^{-2}(1 + \nu^2 l^2) \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + x^i x^j \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} + 2 t x^i \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial x^j} \right) = h_{ENH2}, \] (3.82)

\[ \Gamma_{ENH2}^{\lambda}_{\mu \nu} = -\lim_{c_r, l_r \to 0} \frac{\delta^{\lambda}_{\mu} \delta_{\nu k} x^k}{l_r^2 \sigma_{E,r}^2} = -\frac{(\delta^{\lambda}_{\mu} \delta_{\nu i} + \delta^{\lambda}_{\nu} \delta_{\mu i}) x^i}{\delta_{mn} x^m x^n}. \] (3.83)

The resulting \(-g^{NH2}\) and \(h_{ENH2}\) have rank 2 and signature \((+, +)\) and \((+, +)\), respectively. It describes a pure geometry and may serve as the Euclidean version of both \(NH2\) and \(NH2'\) geometries.

The similar contraction of \(DTdS\) space-time gives

\[ ds^2_{DTNH2} = -\lim_{c_r, l_r \to 0} \frac{l_r^4}{l_r^2 \sigma_r^2} (\eta_{\mu \nu} - \frac{\eta_{\mu \nu} \eta_{\nu k} x^k}{l_r^2 \sigma_r^2}) dx^\mu dx^\nu = l^2 (\frac{\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) x^k x^l}{(\delta_{mn} x^m x^n)^2} dx^i dx^j = g^{NH2}, \] (3.84)

\[ \left( \frac{\partial}{\partial s} \right)^2_{DTNH2} = -\lim_{c_r, l_r \to 0} \frac{l_r^4}{l_r^2 \sigma_r^2} (\eta^{\mu \nu} + l_r^{-2} x^\mu x^\nu) \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = \frac{\delta_{mn} x^m x^n}{l^4} \left( \nu^{-2}(1 + \nu^2 l^2) \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + x^i x^j \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} + 2 t x^i \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial x^j} \right) = h_{DTNH2}, \] (3.85)

\[ \Gamma_{DTNH2}^{\lambda}_{\mu \nu} = -\lim_{c_r, l_r \to 0} \frac{(\delta^{\lambda}_{\mu} \eta_{\nu k} + \delta^{\lambda}_{\nu} \eta_{\mu k}) x^k}{l_r^2 \sigma_r^2} = -\frac{(\delta^{\lambda}_{\mu} \eta_{\nu i} + \delta^{\lambda}_{\nu} \eta_{\mu i}) x^i}{\delta_{mn} x^m x^n}. \] (3.86)

The resulting \(g^{NH2}\) and \(h_{DTNH2}\) have rank 2 each and the signature \((-,-)\) and \((+,-)\), respectively. It describes a double time space-time, named \(DTNH2\) space-time.
D. Geometries for \( h_\pm, e', \) and \( p' \)

The \( HN_\pm \) space-times can be obtained from the \( dS \) and \( AdS \) space-times by the contraction in the limit of \( c_r \to 0 \). When \( c_r \to 0 \),

\[
\lim_{c_r \to 0} \sigma^\pm_r = \lim_{c_r \to 0} [1 \mp \frac{l^2 (c_i^2 l^2 - \delta_{ij} x^i x^j)}{\sigma^\pm_r}] = 1 \pm \frac{t^2 \delta_{ij} x^i x^j = \sigma^\pm_{E,3} (x^i),}{}
\]

which is 3d \( \sigma^\pm_{E} \). The metrics and “inverse” metrics become

\[
d^2 s^2_{HN_\pm} = \lim_{c_r \to 0} \frac{1}{\sigma^2_r} \left( \eta_{\mu
u} \pm \frac{\eta_{\mu\nu} \lambda x^\nu x^\lambda}{l^2 \sigma^\pm_r} \right) dx^\mu dx^\nu = \frac{1}{\sigma^2_{E,3}} \left( \delta_{ij} \pm \frac{\delta_{ik} \delta_{jl} x^k x^l}{l^2 \sigma^\pm_{E,3}} \right) dx^i dx^j =: g^{HN_\pm},
\]

or

\[
\left( \frac{\partial}{\partial s} \right)^2_{HN_\pm} = \lim_{c_r \to 0} \frac{c_r^2}{c^2} \left( \sigma^\pm_r (\eta^\mu\nu \pm \frac{l^2 x^\mu x^\nu}{\sigma^\pm_r}) \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \right) = \sigma^\pm_{E,3} \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} =: h_{HN_\pm},
\]

Namely, the space-time is split into 3d space and 1d time. The 3d space is a Riemann sphere or a Lobachevsky space. The scaling factor of the 1d time depends on the position in space.

In comparison, \( NH_\pm \) space-times are split into 1d time and 3d space. The 3d space is conformally flat. The conformal factor depends on time. From this point of view, the geometries for \( HN_\pm \) is in contrast to the geometries for \( h_\pm \). It is one of the reasons that the geometries are referred to as (anti-)Hooke-Newton geometries.

The connection are

\[
\Gamma_{HN_\pm, \mu\nu}^\lambda = \pm \lim_{c_r \to 0} \frac{(\delta^\lambda_{\mu} \eta_{\nu\kappa} + \delta^\lambda_{\nu} \eta_{\mu\kappa}) x^\kappa}{l^2 \sigma^\pm_r} = \pm \frac{(\delta^\lambda_{\mu} \eta_{\nu \kappa} + \delta^\lambda_{\nu} \eta_{\mu \kappa}) x^\kappa}{l^2 \sigma^\pm_{E,3}}.
\]

The nonzero coefficients are

\[
\Gamma_{HN_\pm, 0 \mu}^0 = \Gamma_{HN_\pm, 0 i}^0 = \mp \frac{\delta_{il} x^l}{l^2 \sigma^\pm_{E,3}}, \quad \Gamma_{HN_\pm, jk} = \mp \frac{(\delta^i_j \delta_{kl} + \delta^i_k \delta_{jl}) x^l}{l^2 \sigma^\pm_{E,3}}.
\]

The nonzero components of curvature tensor and Ricci tensor are

\[
R_{HN_\pm, 0 \mu 0 \nu} = -R_{HN_\pm, 0 \mu i 0} = \pm l^{-2} g^{HN_\pm}_{ij}, \quad R_{HN_\pm, k i j 0} = \mp l^{-2} (\delta_{ik} g^{HN_\pm}_{lj} - \delta_{il} g^{HN_\pm}_{kj}),
\]

\[
R_{ij}^{HN_\pm} = \mp l^{-2} g^{HN_\pm}_{ij},
\]

respectively.

When \( c_r \to 0, \sigma^\pm_{E, r} \) also tend to \( \sigma^\pm_{E,3} (x^i) \). The \( \text{Riem} \) and \( \text{Lob} \) spaces tend to

\[
d^2 s^2_{EHN_\pm} = \lim_{c_r \to 0} \frac{1}{\sigma^\pm_{E,r}} \left( \delta_{\mu\nu} \pm \frac{\delta_{\mu\nu} \lambda x^\nu x^\lambda}{l^2 \sigma^\pm_{E,r}} \right) dx^\mu dx^\nu = \frac{1}{\sigma^\pm_{E,3}} \left( \delta_{ij} \pm \frac{\delta_{ik} \delta_{jl} x^k x^l}{l^2 \sigma^\pm_{E,3}} \right) dx^i dx^j = -g^{HN_\pm},
\]
\[
\left( \frac{\partial}{\partial s} \right)^2_{EHN_\pm} = \lim_{c_r \to 0} \frac{c_r^2}{c^2} \left( \frac{\sigma^\pm_{E,r}}{\sigma^2_{E,r}} \right) \left( \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) = \sigma^\pm_{E,3} \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} = h_{HN_\pm}
\]

(3.96)

with \( \sigma^\pm_{E,3} > 0 \). The signatures of \(-g_{HN_\pm}\) and \(h_{HN_\pm}\) are \((+,+,+),(+,+,-)\), respectively. Therefore, they describe the Euclidean version of \(HN_\pm\) space-times.

For \(\sigma^-_{E,r} < 0\) and \(\sigma^-_r < 0\), the \(LBdS\) and \(DTdS\) space-times contract to

\[
d s_{HN_+}^2 = - \lim_{c_r \to 0} \frac{1}{\sigma^+_E} \left( \delta^{ij} + \frac{\delta_m \delta_\nu x^m x^\nu}{l^2 \sigma^-_{E,r}} \right) dx^i dx^j \\
d s^2_{DTdS} = - \lim_{c_r \to 0} \frac{1}{\sigma^-_r} \left( \delta^{ij} + \frac{\delta_m \delta_\nu x^m x^\nu}{l^2 \sigma^-_{E,3}} \right) dx^i dx^j
\]

(3.97)

\[
\left( \frac{\partial}{\partial s} \right)^2_{HN_+} = - \lim_{c_r \to 0} \frac{c_r^2}{c^2} \left( \frac{\sigma^-_{E,r}}{\sigma^2_{E,r}} \right) \left( \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) = - \sigma^-_{E,3} \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} = h_{HN_+},
\]

(3.98)

with \(\sigma^-_{E,3} < 0\), where \(g_{3}^{LBdS}\) is the metric of the 3d \(LBdS\) space-time. \((g_{3}^{LBdS}, h_{HN_+})\) have the signatures \((\mp,\pm,\pm;+)(\mp,\pm,\pm;+)\) and thus are called para-\(HN_+\) space-time and double time \(HN_+\) space-time, respectively. \(\sigma^+_r < 0\) is not valid in the limit and thus \(BdSL\) space-time is uncontractible in this way.

On the other hand, the para-Euclid space (or \(HN_{+2}\)) can be obtained from the 4d Riemann sphere by the contraction in the limit of \(c_r \to \infty\). As \(c_r\) is running,

\[
\sigma^+_E = \nu^2 t^2 + \sigma^+_E,3 > 0.
\]

(3.100)

The metric becomes

\[
d s_E^2 = \lim_{c_r \to \infty} \frac{c_r^2}{c^2} \frac{1}{t^2} \left[ \left( c_r^2 dt^2 + \delta_{ij} dx^i dx^j - \frac{1}{c_r^2 t^2} \right) \right] = \frac{1}{\nu^2 t^2} \left( \frac{t^2 \sigma^+_E,3}{l^2} dt^2 + \delta_{ij} dx^i dx^j - \frac{2}{t} \delta_{ij} x^i dt dx^j \right) = g^E_r
\]

(3.101)

which is non-degenerate. It provides all local information of the space-time. For example, the inverse metric

\[
h^{-1} = l^{-2} \nu^2 t^2 \left( \frac{t^2}{\partial t} \frac{\partial}{\partial t} + \left( l^2 \delta_{ij} + x^i x^j \right) \frac{\partial}{\partial x^i} \right. \frac{\partial}{\partial x^j} + 2 t x^i \frac{\partial}{\partial t} \frac{\partial}{\partial x^j},
\]

(3.102)
and the connection coefficients
\[ \Gamma_{E'}^{00} = -\frac{2}{ct}, \quad \Gamma_{E'}^{i0} = \Gamma_{E'}^{0i} = -\frac{1}{ct}\delta^i_j, \quad \text{others vanish}, \] (3.103)

which can also be obtained from the limit
\[ \left( \frac{\partial}{\partial s} \right)^2_{E'} = \lim_{c_r \to \infty} \frac{c^2_r}{c^2_r} \nu^2_r t^2 \left( \partial_{\mu} \right) \partial_{\nu} \partial_{\mu} \partial_{\nu} \] (3.104)

and
\[ \Gamma_{E'}^{\lambda \mu \nu} = -\lim_{c_r \to \infty} \frac{c_r (\delta^\lambda_\mu \delta_\nu^\kappa + \delta^\lambda_\nu \delta_\mu^\kappa) x^\kappa_r}{t^2 \sigma^+_{E, r}}, \] (3.105)

respectively. The space (3.101) has the signature (+, +, +, +) and vanishing curvature. Therefore, it is better to be referred to as para-Euclid geometry rather than the HN+2 or P′+ geometry.

Similarly, the para-Poincaré space-time can be obtained from the AdS space-times by the contraction in the limit of \( c_r \to \infty \). As \( c_r \) is running,
\[ \sigma_r^- = \nu^2_r t^2 + \sigma_{E, 3} > 0. \] (3.106)

The metric reads
\[ d\sigma^2_{P'} = \lim_{c_r \to \infty} \frac{c^2_r}{c^2_r} \nu^2_r t^2 \left[ (c^2_r dt^2 - \delta_{ij} dx^i dx^j - \frac{1}{c^2_r t^2} (1 - \frac{\sigma_{E, 3}}{\nu^2_r t^2} (c^2_r dt^2 - \delta_{ij} dx^i dx^j))^2 \right] \]
\[ = \frac{1}{\nu^2 t^2} \left( l^2 \sigma_{E, 3} dt^2 - \delta_{ij} dx^i dx^j + \frac{2}{t} \delta_{ij} x^i dt dx^j \right) =: g_{P'} \] (3.107)

which is also non-degenerate. The inverse metric is
\[ \left( \frac{\partial}{\partial s} \right)^2_{P'} = \lim_{c_r \to \infty} \frac{c^2_r}{c^2_r} \nu^2_r t^2 \left( \partial_{\mu} \right) \partial_{\nu} \partial_{\mu} \partial_{\nu} \]
\[ = l^{-2} \nu^2 t^2 \left( l^2 \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} - (l^2 \delta_{ij} - x^i x^j) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} + 2t x^i \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial x^i} \right) \]
\[ = g^{-1}_{P'} =: h_{P'}. \] (3.108)

The nonzero connection coefficients are
\[ \Gamma_{P'}^{00} = -\frac{2}{ct}, \quad \Gamma_{P'}^{i0} = \Gamma_{P'}^{0i} = -\frac{1}{ct}\delta^i_j. \] (3.109)

The geometry (3.107) has the signature (+, −, −, −) and vanishing curvature. It is better to be referred to as para-Poincaré geometry as the nomenclature in Ref. [2].

When \( c_r \to \infty \),
\[ \sigma_{E, r}^- \to -\nu^2_r t^2 + \sigma_{E, 3} < 0, \] (3.110)
\[ \sigma_{r}^+ \to -\nu^2_r t^2 + \sigma_{E, 3} < 0. \] (3.111)
The $L BdS$ space-time and $BdSL$ space also contract to, respectively, the $P'$ space-time,

\[ -\lim_{c_r\to\infty} \frac{c_r^2}{\sigma_{E,r}} \frac{1}{c_r^2} \left( \delta_{\mu\nu} + \frac{\delta_{\mu\lambda} \delta_{\nu\lambda} x_r^\lambda}{l^2 \sigma_{E,r}^-} \right) dx^\mu_r dx^\nu_r \]

\[ = \lim_{c_r\to\infty} \frac{c_r^2}{\nu_r^2 l^2} \left[ c_r^2 dt^2 + \delta_{ij} dx^i dx^j - \frac{1}{c_r^2 l^2} \left( 1 + \frac{\sigma_{E,3}}{\nu_r^2 l^2} \right) (c_r^2 dt + \delta_{ij} x^i dx^j)^2 \right] \]

\[ = \frac{1}{\nu_r^2 l^2} \left( -\frac{l^2 \sigma_{E,3}}{t^2} dt^2 + \delta_{ij} dx^i dx^j - \frac{2}{l} \delta_{ij} x^i dt dx^j \right) = -g^{P'} \]  

(3.112)

and the $E'$ space,

\[ \lim_{c_r\to\infty} \frac{c_r^2}{\sigma_r} \frac{1}{c_r^2} \left( \eta_{\mu\nu} + \frac{\eta_{\mu\lambda} \eta_{\nu\lambda} x_r^\lambda}{l^2 \sigma_r^+} \right) dx_r^\mu dx_r^\nu \]

\[ = \lim_{c_r\to\infty} \frac{c_r^2}{\nu_r^2 l^2} \left[ c_r^2 dt^2 - \delta_{ij} dx^i dx^j - \frac{1}{c_r^2 l^2} \left( 1 + \frac{\sigma_{E,3}}{\nu_r^2 l^2} \right) (c_r^2 dt - \delta_{ij} x^i dx^j)^2 \right] \]

\[ = \frac{1}{\nu_r^2 l^2} \left( -\frac{l^2 \sigma_{E,3}}{t^2} dt^2 - \delta_{ij} dx^i dx^j - \frac{2}{l} \delta_{ij} x^i dt dx^j \right) = -g^{E'} . \]

(3.113)

E. Geometries for $g$, $c$, $g_2$, and $c_2$

It is well known that the $G$ geometry can be obtained from the $Min$ space-time:

\[ ds^2_G = \lim_{c_r\to\infty} \frac{c_r^2}{c_r^2} \left( c_r^2 dt^2 - \delta_{ij} dx^i dx^j \right) = c_r^2 dt^2 =: g^G . \]

(3.114)

\[ \left( \frac{\partial}{\partial s} \right)_G^2 = \lim_{c_r\to\infty} \left( \frac{1}{c_r^2} \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} - \delta_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \right) = -\delta_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} =: h_G . \]

(3.115)

Similarly, the $C$ geometry can also be obtained from the $Min$ space-time:

\[ ds^2_C = \lim_{c_r\to0} \left( c_r^2 dt^2 - \delta_{ij} dx^i dx^j \right) = -\delta_{ij} dx^i dx^j =: g^C , \]

(3.116)

\[ \left( \frac{\partial}{\partial s} \right)_C^2 = \lim_{c_r\to0} \frac{c_r^2}{c_r^2} \left( \frac{1}{c_r^2} \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} - \delta_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \right) = \frac{1}{c_r^2} \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} =: h_C . \]

(3.117)

For both $G$ and $C$ geometries, the spaces and times are flat:

\[ \Gamma_{G/C}^{\lambda}_{\mu\nu} = 0, \quad R_{G/C}^{\lambda}_{\mu\nu} = 0 . \]

(3.118)

The $G$ and $C$ geometries can also be obtained from $dS$ and $AdS$ space-times directly. In the contraction procedure to obtain $G$ geometry from $dS$ and $AdS$ space-times, both $c_r$ and $l_r$ tend to infinity but $\nu_r = c_r/l_r \to 0$. Without loss of generality, we may suppose $c_r^2/l_r$ keeps finite in the limit. Then

\[ ds^2_G = \lim_{c_r, l_r \to \infty, c_r^2/l_r \text{ fixed}} \frac{c_r^2}{c_r^2 \sigma_r^+} \left( \eta_{\mu\nu} + \frac{\eta_{\mu\lambda} \eta_{\nu\lambda} x_r^\lambda}{l_r^2 \sigma_r^+} \right) dx_r^\mu dx_r^\nu , \]

(3.119)
The contraction from $dS$ and $AdS$ space-times to $C$ space-time is easier:

$$ds_C^2 = \lim_{l_r \to \infty, c_r \to 0} \frac{1}{c_r^2} \left( \eta_{\mu\nu} \frac{\eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda}{l_r^2 \sigma_r^+} \right) dx^\mu_r dx^\nu_r,$$

(3.121)

$$\left( \frac{\partial}{\partial s} \right)_C^2 = \lim_{l_r \to \infty, c_r \to 0} \frac{c_r^2}{\sigma_r^+} \left( \eta_{\mu\nu} - l_r^2 x^\mu_r x^\nu_r \right) \frac{\partial}{\partial x^\mu_r} \otimes \frac{\partial}{\partial x^\nu_r}.$$

(3.122)

In addition, the direct contractions from Riem and Lob geometries can give the Euclidean $G$ space-time (denoted by $EG$) and Euclidean $C$ space-time (denoted by $EC$). Since the inequalities

$$\sigma_{E,r}^- < 0, \quad \sigma_{C,r}^+ < 0$$

(3.123)

are not valid either in the limit $c_r, l_r \to \infty, \nu_r \to 0$ or in the limit of $l_r \to \infty, c_r \to 0$, the $LBdS$, $BdSL$ and $DTdS$ geometries are not contractible in the two ways.

In the similar way, the $G_2$ and $C_2$ geometries are related to the $P_{2\pm}$ space-time.

If the inequality (3.33) is always valid (i.e. $\mp \eta_{\kappa\lambda} x^\kappa x^\lambda > 0$) when $c_r \to 0$, only the upper sign (i.e. $P_{2+}$) is meaningful. Therefore,

$$ds^2_{EG_2} = \lim_{c_r \to 0} \frac{1}{c_r^2} \left( \eta_{\mu\nu} \frac{\eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda}{l_r^2 \sigma_r^+} \right) dx^\mu_r dx^\nu_r = l_r^2 \delta_{ij} \delta_{kl} x^i x^j dx^i dx^j =: g^{EG_2} = g^{NH_2}.$$

(3.124)

$$\left( \frac{\partial}{\partial s} \right)_{EG_2}^2 = \lim_{c_r \to 0} l_r^{-4} \delta_{\sigma\tau} x^\sigma_r x^\tau_r \frac{\partial}{\partial x^\mu_r} \otimes \frac{\partial}{\partial x^\nu_r} = -l_r^{-4} \delta_{mn} x^m x^n x^\mu_r x^\nu_r \frac{\partial}{\partial x^\mu_r} \otimes \frac{\partial}{\partial x^\nu_r}.$$

(3.125)

$$\Gamma_{EG_2\mu\nu}^\lambda = -\lim_{c_r \to 0} \left( \frac{\delta^\lambda_{\mu} \eta_{\kappa\lambda} x^\kappa_r + \delta^\lambda_{\nu} \eta_{\kappa\lambda} x^\kappa_r}{\eta_{\sigma\tau} x^\sigma_r x^\tau_r} \right) = \left( \frac{\delta^\lambda_{\mu} \eta_{\kappa\lambda} x^\kappa_r + \delta^\lambda_{\nu} \eta_{\kappa\lambda} x^\kappa_r}{\delta_{mn} x^m x^n} \right) = \Gamma_{NH_2\mu\nu}^\lambda.$$

(3.126)

The curvature tensor and Ricci tensor are the same as those for the $NH_2$ geometry. It should be noted that the sum of ranks of $g^{G_2}$ and $h_{G_2}$ is only 3. They have the signature $(-, -)$ and $(-)$, respectively. Therefore, they describe a 3d Euclidean geometry with the free parameter $\psi = l x^0 / \sqrt{\delta_{ij} x^i x^j}$, denoted by $EG_2$ geometry. The $EG_2$ geometry can also be obtained from the direct contraction of $dS$ space-time. Since both $c_r$ and $l_r$ tend to 0 but $\nu_r = c_r / l_r \to \infty$, we may suppose $c_r^2 / l_r$ keeps finite in the limit without loss of generality. Then,

$$ds^2_{EG_2} = \lim_{c_r^2 / l_r \to 0} \frac{l_r^2}{c_r^2 \sigma_r^+} \left( \eta_{\mu\nu} \frac{\eta_{\mu\nu} \eta_{\kappa\lambda} x^\kappa x^\lambda}{l_r^2 \sigma_r^+} \right) dx^\mu_r dx^\nu_r.$$

(3.127)
The direct contraction from the Riem geometry in the limit of \( c_r, l_r \to 0 \) and \( c^2_r/l_r = c^2/l \) gives rise to \(-g^{EG_2}\) and \(-h_{EG_2}\) and the free parameter \( \psi = lx^0/\sqrt{\delta_{ij} x^i x^j} \). The contractions of \( LBdS \) and \( DTdS \) space-times are, respectively,

\[
\left( \frac{\partial}{\partial s} \right)^2_{EG_2} = - \lim_{c_r, l_r \to 0, c^2_r/l_r \text{ fixed}} \frac{l^2}{l_r^2} \frac{1}{\sigma_{r}} \left( \delta_{\mu\nu} + \frac{\delta_{\mu\nu}}{l_r^2} \right) dx^\mu_r dx^\nu_r = -g^{NH_2}, \quad (3.128)
\]

\[
\left( \frac{\partial}{\partial s} \right)^2_{G_2} = - \lim_{c_r, l_r \to 0, c^2_r/l_r \text{ fixed}} \frac{l^2}{l_r^2} \frac{1}{\sigma_{r}} \left( \eta_{\mu\nu} - \frac{\eta_{\mu\nu} \eta_{\rho\sigma} x^\rho_r x^\sigma_r}{l_r^2} \right) dx^\mu_r dx^\nu_r = g^{EG_2}, \quad (3.129)
\]

\[
\left( \frac{\partial}{\partial s} \right)^2_{G_2} = - \lim_{c_r, l_r \to 0, c^2_r/l_r \text{ fixed}} \frac{l^2}{l_r^2} \frac{1}{\sigma_{r}} \left( \eta_{\mu\nu} + l^{-2} x^\mu_r x^\nu_r \right) dx^\mu_r dx^\nu_r = h_{EG_2}, \quad (3.130)
\]

and

\[
\left( \frac{\partial}{\partial s} \right)^2_{G_2} = - \lim_{c_r, l_r \to 0, c^2_r/l_r \text{ fixed}} \frac{l^2}{l_r^2} \frac{1}{\sigma_{r}} \left( \eta_{\mu\nu} - \frac{\eta_{\mu\nu} \eta_{\rho\sigma} x^\rho_r x^\sigma_r}{l_r^2} \right) dx^\mu_r dx^\nu_r = g^{EG_2}, \quad (3.131)
\]

\[
\left( \frac{\partial}{\partial s} \right)^2_{G_2} = - \lim_{c_r, l_r \to 0, c^2_r/l_r \text{ fixed}} \frac{l^2}{l_r^2} \frac{1}{\sigma_{r}} \left( \eta_{\mu\nu} + l^{-2} x^\mu_r x^\nu_r \right) dx^\mu_r dx^\nu_r = h_{EG_2}, \quad (3.132)
\]

with the free parameter \( \psi = lx^0/\sqrt{\delta_{ij} x^i x^j} \). The domain condition is again \( \delta_{ij} x^i x^j > 0 \). The Lob, \( AdS \) and \( BdSL \) geometries are not contractible in this limit.

If the inequality (3.33) is always valid as \( c_r \to \infty \), only the \( P_{2-} \) geometry is taken.

\[
ds^2_{C_2} = - \lim_{c_r \to \infty} \frac{l^2}{c_r^2} \frac{1}{\sigma_r} \left( \eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\rho} \eta_{\nu\sigma} \right) x^\mu_r x^\nu_r dx^\mu_r dx^\nu_r = -l^2 \delta_{ij} d(x^i_{ct})d(x^j_{ct}) = : g^{C_2}, \quad (3.133)
\]

\[
\left( \frac{\partial}{\partial s} \right)^2_{C_2} = \lim_{c_r \to \infty} \frac{l^2}{c_r^2} \frac{1}{\sigma_r} \left( \eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\rho} \eta_{\nu\sigma} \right) x^\mu_r x^\nu_r dx^\mu_r dx^\nu_r = l^{-4} c^2 l^2 x^\mu_r x^\nu_r \frac{\partial}{\partial x^\mu_{ct}} \frac{\partial}{\partial x^\nu_{ct}} = l^{-4} c^2 l^2 x^\mu_r x^\nu_r \frac{\partial}{\partial x^\mu_{ct}} \frac{\partial}{\partial x^\nu_{ct}} = : h_{C_2}, \quad (3.134)
\]

\[
\Gamma^0_{C_2ij} = - \frac{2}{ct}, \quad \Gamma^i_{C20j} = \Gamma^i_{C2j0} = - \frac{1}{ct} \delta^i_j, \quad \text{others vanish}. \quad (3.135)
\]

\[
R^\lambda_{\mu\nu}\rho^\mu = 0. \quad (3.136)
\]

The \( C_2 \) geometry can also be directly obtained from the contraction of \( AdS \) and \( LBdS \) space-times:

\[
ds^2_{C_2} = \begin{cases} & \lim_{l_r \to 0, c_r \to \infty} \frac{l^2}{c_r^2} \frac{1}{\sigma_r} \left( \eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\rho} \eta_{\nu\sigma} \right) x^\mu_r x^\nu_r dx^\mu_r dx^\nu_r = -l^2 \delta_{ij} d(x^i_{ct})d(x^j_{ct}) \\
& - \lim_{l_r \to 0, c_r \to \infty} \frac{l^2}{c_r^2} \frac{1}{\sigma_E,r} \left( \delta_{\mu\nu} + \frac{\delta_{\mu\nu} \eta_{\rho\sigma} x^\rho_r x^\sigma_r}{l_r^2} \right) dx^\mu_r dx^\nu_r = l^2 \delta_{ij} d(x^i_{ct})d(x^j_{ct}) \end{cases}, \quad (3.137)
\]
\[
\left( \frac{\partial}{\partial s} \right)^2_{c_2} = \begin{cases}
\lim_{l_r, c_r \to \infty} \frac{l_r^2 \nu_r^2}{l_r^2 \nu_r^2} \left( \eta^{\mu \nu} + l_r^{-2} x_r^\mu x_r^\nu \right) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = l_r^{-4} c^2 l_r^2 x_r^\mu \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu}, \\
\lim_{l_r, c_r \to \infty} \frac{l_r^2 \nu_r^2}{l_r^2 \nu_r^2} \left( \delta^{\mu \nu} - l_r^{-2} x_r^\mu x_r^\nu \right) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = -l_r^{-4} c^2 l_r^2 x_r^\mu \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu},
\end{cases}
\tag{3.138}
\]

for
\[
0 < \sigma_r^- \\
0 > \sigma_{E,r}
\rightarrow \pm \nu_r^2 l_t^2 (1 \mp \frac{\delta_{ij} x^i x^j}{c_r^2 l_t^2}).
\tag{3.139}
\]

The Riem space and BdSL space contract to \(-g^{C_2}, h_{C_2}\), which has the signature \((+,+,+;+\)) and thus is named EC\(_2\) space. Lob and DTdS as well as dS are not contractible in this limit.

### F. Geometries for \(g'\) and \(g'_2\)

The inequalities \(\sigma^+_r > 0, \sigma^-_r < 0, \) and \(\sigma^-_{E,r} > 0\) do not keep valid in the limiting process of \(l_r, c_r, \nu_r \to \infty\), and the inequalities \(\sigma^+_r < 0, \sigma^-_r > 0, \) and \(\sigma^-_{E,r} > 0\) do not keep valid in the limiting process of \(l_r, c_r, \nu_r \to 0\). Therefore, dS, DTdS, and Lob geometries and BdSL, AdS, and Lob geometries cannot define new geometries by the contraction approach in the two ways, respectively.

The inequalities \(\sigma^+_r > 0, \sigma^-_{E,r} < 0, \sigma^-_r < 0, \) and \(\sigma^-_r > 0\) in the limit of \(l_r, c_r, \nu_r \to \infty\) require \(\nu_r^2 l_t^2 (1 \pm \frac{1}{c_r^2 l_t^2}) > 0\). It means that the hypersurface at \(t = 0\) should be removed from the manifold. Without loss of generality, \(c_r/l_r^2\) is supposed to be fixed when \(c_r, l_r \to \infty\) in these cases. Then, Riem, LBdS, BdSL, and AdS geometries contract to, respectively,

\[
d s^2_{EG'} = \lim_{l_r, c_r \to \infty} \frac{1}{\sigma_{E,r}} \left( \eta^{\mu \nu} \mp \eta_{\mu k} \eta_{\lambda \nu \rho} x_r^\rho x_r^\lambda \right) dx_r^\mu dx_r^\nu = \pm l_t^2 [d(\frac{1}{\nu t})]^2 =: g^{EG'}
\tag{3.140}
\]

\[
d s^2_{EG'_2} = \lim_{l_r, c_r \to \infty} \frac{1}{\sigma_{E,r}} \left( \eta^{\mu \nu} \pm \eta_{\mu k} \eta_{\lambda \nu \rho} x_r^\rho x_r^\lambda \right) dx_r^\mu dx_r^\nu = l_t^2 [d(\frac{1}{\nu t})]^2 = g'^{2}
\tag{3.141}
\]

\[
\left( \frac{\partial}{\partial s} \right)^2_{EG'} = \lim_{l_r, c_r \to \infty} \frac{\nu_r^2}{c_r^2 l_t^2} (\pm \sigma_{E,r}^\pm ) (\delta^{\mu \nu} \pm l_r^{-2} x_r^\mu x_r^\nu) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = \delta^{ij} \frac{\partial}{\partial (x^i / \nu t)} \otimes \frac{\partial}{\partial (x^j / \nu t)} = \pm h_{EG'}
\tag{3.142}
\]

\[
\left( \frac{\partial}{\partial s} \right)^2_{EG'_2} = \lim_{l_r, c_r \to \infty} \frac{\nu_r^2}{c_r^2 l_t^2} (\pm \sigma_{E,r}^\pm ) (\delta^{\mu \nu} \pm l_r^{-2} x_r^\mu x_r^\nu) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = \pm h_{EG'}
\tag{3.143}
\]

\[
\Gamma^\lambda_{\mu \nu}^{EG'} = \pm \lim_{l_r, c_r \to \infty} \frac{c_r}{c_r^2 l_t^2} \left( \delta^{\lambda}_{\mu k} \delta^{\nu}_{\lambda \rho} x_{r}^{\rho} \right) = -\frac{\delta^{\lambda}_{\mu \rho} + \delta^{\lambda}_{\nu \rho}}{c r}
\tag{3.144}
\]

\[
\Gamma^\lambda_{\mu \nu}^{EG'_2} = \pm \lim_{l_r, c_r \to \infty} \frac{c_r}{c_r^2 l_t^2} \left( \delta^{\lambda}_{\mu k} \delta^{\nu}_{\lambda \rho} x_{r}^{\rho} \right) = -\frac{\delta^{\lambda}_{\mu \rho} + \delta^{\lambda}_{\nu \rho}}{c r}.
\tag{3.145}
\]
### TABLE II: Algebras and their corresponding geometries.

| Alg | Geom. name | Geometrical variables$^a$ | $(g, h)$ Ranks | Signature | Contraction | Domain |
|-----|------------|---------------------------|----------------|------------|-------------|--------|
| r   | Riem       | $g^{Riem} = \frac{1}{\sigma_E} \left( (dx \cdot dx)_E - \frac{(x \cdot dx)^2}{l^2\sigma_E^2} \right)$ | (4, 4) | (+, +, +, +) | No | $\sigma_E^+ > 0$ |
| t   | Lob        | $g^{Lob} = \frac{1}{\sigma_E} \left( (dx \cdot dx)_E + \frac{(x \cdot dx)^2}{l^2\sigma_E^2} \right)$ | (4, 4) | (+, +, +, +) | No | $\sigma_E^- > 0$ |
| t   | LBdS       | $g^{LBdS} = -\frac{1}{\sigma_E} \left( (dx \cdot dx)_E + \frac{(x \cdot dx)^2}{l^2\sigma_E^2} \right)$ | (4, 4) | (−, +, +, +) | No | $\sigma_E^- < 0$ |
| e   | Euc        | $g^{Euc} = (dx \cdot dx)_E$ | (4, 4) | (+, +, +, +) | $l_r \to \infty$ | arbitrary |
| e   | $E_2$      | $h_{E_2} = l_2^{-2}(x \cdot x)_E \left( x^\mu \partial_\mu \right)^2$ | (3, 1) | (+, +, +; +) | $l_r \to 0$ | $(x \cdot x)_E > 0$ |
| e   | $E_{2-}$   | $h_{E_{2-}} = l_2^{-2}(x \cdot x)_E \left( x^\mu \partial_\mu \right)^2$ | (3, 1) | (+, +, +; −) | $l_r \to 0$ | $(x \cdot x)_E > 0$ |
| d+  | dS         | $g^{dS} = \frac{1}{\sigma^2} \left( (dx \cdot dx) + \frac{(x \cdot dx)^2}{l_2^2\sigma^2} \right)$ | (4, 4) | (+, −, −, −) | No | $\sigma^+ > 0$ |
| d+  | BdSL       | $g^{BdSL} = \frac{1}{\sigma^2} \left( (dx \cdot dx) + \frac{(x \cdot dx)^2}{l_2^2\sigma^2} \right)$ | (4, 4) | (+, +, +, +) | No | $\sigma^+ < 0$ |
| d−  | AdS        | $g^{AdS} = \frac{1}{\sigma} \left( (dx \cdot dx) - \frac{(x \cdot dx)^2}{l_2^2\sigma^2} \right)$ | (4, 4) | (+, −, −, −) | No | $\sigma^- > 0$ |
| d−  | DTdS       | $g^{DTdS} = \frac{-1}{\sigma} \left( (dx \cdot dx) - \frac{(x \cdot dx)^2}{l_2^2\sigma^2} \right)$ | (4, 4) | (+, +, −, −) | No | $\sigma^- < 0$ |
| p   | Min        | $g^{Min} = dx \cdot dx$ | (4, 4) | (+, −, −, −) | $l_r \to \infty$ | arbitrary |
| p   | $P_{2\pm}$ | $h_{P_{2\pm}} = l_2^{-4}(x \cdot x)(x^\mu \partial_\mu)^2$ | (3, 1) | (+, −, −; −) | $l_r \to 0$ | $x \cdot x < 0$ |
| p   | $E_{P_{2-}}$ | $h_{E_{P_{2-}}} = l_2^{-4}(x \cdot x)(x^\mu \partial_\mu)^2$ | (3, 1) | (+, +, +; +) | $l_r \to 0$ | $x \cdot x > 0$ |
| p   | $E_{P_{2+}}$ | $h_{E_{P_{2+}}} = l_2^{-4}(x \cdot x)(x^\mu \partial_\mu)^2$ | (3, 1) | (+, −, −; +) | $l_r \to 0$ | $x \cdot x < 0$ |
| p   | $DTP_{2\pm}$ | $h_{DTP_{2\pm}} = l_2^{-4}(x \cdot x)(x^\mu \partial_\mu)^2$ | (3, 1) | (+, −, −; +) | $l_r \to 0$ | $x \cdot x < 0$ |

$^a$For brevity, the following shorthands are used. $(x \cdot x)_E := \delta_{\mu\nu} x^\mu x^\nu$; $x \cdot x := \eta_{\mu\nu} x^\mu x^\nu$; $x \cdot x := \delta_{ij} x^i x^j$; $\partial_\mu \partial_\nu := \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}$, etc.
TABLE II: Algebras and their corresponding geometries (Cont.).

| Alg name | Geometrical variables | \((g, h)\) Ranks | Signature | Contraction | Domain |
|----------|-----------------------|-------------------|-----------|-------------|--------|
| \(n_+ NH_\pm\) | | \(|\Gamma_{NH_\pm} = (\sigma_n^\pm)^{-2} 2\hat{c} dt^2\) | | \(l_r, c_r \to \infty\) | \(|\sigma_n^\pm > 0|\) |
| \(n_+ ENH_\pm\) | | \(|\Gamma_{ENH_\pm} = (\sigma_n^\pm)^{-2} 2\hat{c} dt^2\) | | \(l_r, c_r \to \infty\) | \(|\sigma_n^\pm > 0|\) |
| \(n_+ NH'_\pm\) | | \(|\Gamma_{NH'_\pm} = 2\hat{c} dt^2\) | | \(l_r, c_r \to \infty\) | \(|\sigma_n^\pm > 0|\) |
| \(n_+ ENH'_\pm\) | | \(|\Gamma_{ENH'_\pm} = 2\hat{c} dt^2\) | | \(l_r, c_r \to \infty\) | \(|\sigma_n^\pm > 0|\) |

\(n_{+2} NH_2\)

\(|\Gamma_{NH_2} = \frac{\nu^2}{\hat{c}} (x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)\) | \(l_r, c_r \to \infty\) |

\(n_{-2} ENH_2\)

\(|\Gamma_{ENH_2} = \frac{\nu^2}{\hat{c}} (x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)\) | \(l_r, c_r \to \infty\) |
TABLE II: Algebras and their corresponding geometries (Cont.).

| Alg name | Geometrical variables | (g, h) Ranks | Signature | Contraction | Domain |
|----------|-----------------------|--------------|-----------|-------------|--------|
| $b_{\pm} HN_{\pm}$ | $g^{HN_{\pm}} = -\frac{1}{\sigma_{E,3}} \left[ dx \cdot dx \mp \frac{(x \cdot dx)^2}{l^2 \sigma_{E,3}} \right]$ | | (3, 1) | $c_r \to 0$ | $\sigma_{E,3} > 0$ |
| | | $h_{HN_{\pm}} = \sigma_{E,3} \partial^2_t$ | | | |
| | $\Gamma_{HN_{\pm} t0i}^0 = \Gamma_{HN_{\pm} i0}^0 = \mp \frac{x^i}{l^2 \sigma_{E,3}}$ | | | | |
| | $\Gamma_{HN_{\pm} ijk}^i = \mp \frac{\delta^i_j x^k + \delta^i_k x^j}{l^2 \sigma_{E,3}}$ | | | | |
| | $g^{EHN_{\pm}} = \frac{1}{\sigma_{E,3}} \left[ dx \cdot dx \mp \frac{(x \cdot dx)^2}{l^2 \sigma_{E,3}} \right]$ | | (3, 1) | $c_r \to 0$ | $\sigma_{E,3} > 0$ |
| | | $h_{EHN_{\pm}} = \sigma_{E,3} \partial^2_t$ | | | |
| | $\Gamma_{EHN_{\pm} t0i}^0 = \Gamma_{EHN_{\pm} i0}^0 = \mp \frac{x^i}{l^2 \sigma_{E,3}}$ | | | | |
| | $\Gamma_{EHN_{\pm} ijk}^i = \mp \frac{\delta^i_j x^k + \delta^i_k x^j}{l^2 \sigma_{E,3}}$ | | | | |
| $b_{-} HN_{-}$ | $g^{HN_{-}} = -\frac{1}{\sigma_{E,3}} \left[ dx \cdot dx + \frac{(x \cdot dx)^2}{l^2 \sigma_{E,3}} \right]$ | | (3, 1) | $c_r \to 0$ | $\sigma_{-E,3} < 0$ |
| | $h_{HN_{-}} = -\sigma_{E,3} \partial^2_t$ | | | | |
| | $\Gamma_{HN_{-} t0i}^0 = \Gamma_{HN_{-} i0}^0 = \frac{x^i}{l^2 \sigma_{-E,3}}$ | | | | |
| | $\Gamma_{HN_{-} ijk}^i = \mp \frac{\delta^i_j x^k + \delta^i_k x^j}{l^2 \sigma_{E,3}}$ | | | | |
| $b_{-} DTHN$ | $g^{DTHN} = \frac{1}{\sigma_{E,3}} \left[ dx \cdot dx + \frac{(x \cdot dx)^2}{l^2 \sigma_{E,3}} \right]$ | | (3, 1) | $c_r \to 0$ | $\sigma_{-E,3} < 0$ |
| | $h_{DTHN} = -\sigma_{E,3} \partial^2_t$ | | | | |
| | $\Gamma_{DTHN t0i}^0 = \Gamma_{DTHN i0}^0 = \frac{x^i}{l^2 \sigma_{-E,3}}$ | | | | |
| | $\Gamma_{DTHN ijk}^i = \mp \frac{\delta^i_j x^k + \delta^i_k x^j}{l^2 \sigma_{E,3}}$ | | | | |
| $c' E'$ | $g^{E'} = \frac{1}{\nu^2 t^2} \left[ \frac{l^2 \sigma^+_3}{t^2} dt^2 - dx \cdot dx \right]$ | | (4, 4) | $c_r \to \infty$ | $t^2 > 0$ |
| | | $\nu^2 = \frac{1}{2} x \cdot dx$ | | | |
| | | $g^{P'} = \frac{1}{\nu^2 t^2} \left[ \frac{l^2 \sigma^-_3}{t^2} dt^2 - dx \cdot dx \right]$ | | (4, 4) | $c_r \to \infty$ | $t^2 > 0$ |
TABLE II: Algebras and their corresponding geometries (Cont.).

| Alg name | Geometrical variables | (g, h) Ranks | Signature | Contraction | Domain |
|----------|-----------------------|--------------|-----------|-------------|---------|
| g G      | \( g^G = c^2 dt^2 \)  | (1, 3)       | (+; -, -) | \( l_r, c_r \to \infty \) \( \nu_r \to 0 \) arbitrary |
|          | \( h_G = -\delta^i_j \partial_i \partial_j \) | \( \Gamma^\lambda_{G\mu\nu} = 0 \) | \( g^E_G = c^2 dt^2 \) | | |
| c C      | \( h_C = \partial^2_{ct} \) | (3, 1)       | (-, -; +) | \( l_r \to \infty \) \( c_r \to 0 \) arbitrary |
|          | \( g^C = -dx \cdot dx \) | \( \Gamma^\lambda_{C\mu\nu} = 0 \) | | | |
| c EC     | \( h_{EC} = \partial^2_{ct} \) | (3, 1)       | (+, +; +) | \( l_r \to \infty \) \( c_r \to 0 \) arbitrary |
|          | \( g^{EC} = dx \cdot dx \) | \( \Gamma^\lambda_{C\mu\nu} = 0 \) | | | |
| c2 C2    | \( h_C^2 = l^{-2} \nu^2 t^2 x^\mu x^\nu \partial_\mu \partial_\nu \) | (3, 1)       | (-, -; +) | \( l_r \to 0 \) \( c_r \to \infty \) \( t^2 > 0 \) |
|          | \( \Gamma^i_{C200} = -\frac{2}{c2} \) | \( \Gamma^i_{C2j0} = \frac{1}{c2} \delta^i_j \) | | | |
|          | \( g^{EC2} = d(x/(\nu t)) \cdot d(x/(\nu t)) \) | | \( g^{C2} = -d(x/\nu t) \cdot d(x/\nu t)^a \) | | |
| c2 EC2   | \( h_{EC2} = l^{-4} c^2 t^2 x^\mu x^\nu \partial_\mu \partial_\nu \) | (3, 1)       | (+, +; +) | \( l_r \to 0 \) \( c_r \to \infty \) \( t^2 > 0 \) |
|          | \( \Gamma^i_{EC200} = -\frac{2}{c2} \) | \( \Gamma^i_{EC2j0} = -\frac{1}{c2} \delta^i_j \) | | | |
|          | \( g^{EG2} = \frac{l^2 (x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)}{(x \cdot x)^2} \) | | \( g^{E2} = \frac{l^2 (x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)}{(x \cdot x)^2} \) | | |
| \( g_2 \) | \( h_{E2} = -l^{-4} x \cdot x (x^\mu \partial_\mu)^2 \) | (2, 1)       | (-, -)    | \( l_r, c_r \to 0 \) \( \nu_r \to \infty \) \( |x| > 0 \) |
|          | \( \Gamma^0_{G20i} \) | \( \Gamma^i_{G2i0} \) | \( \Gamma^i_{G2ij} \) | | |
| \( g_2 \) | Free parameter: \( lx^0 / \sqrt{x \cdot x} \) | (2, 1)       | (+, +; -) | \( l_r, c_r \to 0 \) \( \nu_r \to \infty \) \( |x| > 0 \) |
|          | \( \Gamma^0_{G20i} \) | \( \Gamma^i_{G2i0} \) | \( \Gamma^i_{G2ij} \) | | |

\(^a\) An overall minus in both degenerate covariant metric and degenerate contravariant metric has been ignored.
The contraction of $LBdS$ and $AdS$ geometries is the para-Galilei geometry, denoted by $G'$. The contraction of $Riem$ and $BdSL$ geometries is the Euclidean version of $G'$ geometry, denoted by $EG'$. They can also be obtained by the contraction from $P'$ and $EP'$ geometries in the limit of $l_r \to \infty$, respectively. The curvature of the $G'$ and $EG'$ geometries are zero.

On the other hand, the inequalities $\sigma^+_{E,r} > 0$, $\sigma^-_{E,r} < 0$, $\sigma^+_{r} > 0$, and $\sigma^-_{r} < 0$ in the limit of $l_r$, $c_r$, $\nu_r \to 0$ require $\delta_{ij}x^i x^j > 0$. It means that the spatial point $x^i = 0$ ($i = 1, 2, 3$) should be removed from the manifold as for $G'_2$ manifold. Again, $c_r/l_r^2$ is supposed to be fixed without loss of

### TABLE II: Algebras and their corresponding geometries (Cont.)

| Alg. name | Geometrical variables | $(g, h)$ Ranks | Signature | Contraction | Domain |
|-----------|-----------------------|---------------|-----------|-------------|--------|
| $g'$ | $G'$ | $g^{G'} = -i^2(d^1_{ct})^2$ | $(1, 3)$ | $(-; +, +)$ | $c_r, l_r \to \infty$, $\nu_r \to \infty$ |
| $h_{G'} = (\nu t)^{-2}\delta_{ij}\partial_i \partial_j$ | | | | | $t^2 > 0$ |
| $\Gamma^{0}_{i0} = \Gamma^{0}_{0j} = 0$ | | | | | |
| $\Gamma^{0}_{ij} = \Gamma^{0}_{ij} = -\frac{1}{ct^2}\delta^i_j$ | | | | | |
| $g^{EG'} = l^2(d^1_{ct})^2$ | | | | | |
| $h_{EG'} = (\nu t)^{-2}\delta_{ij}\partial_i \partial_j$ | | | | | |
| $g^{G'_2} = l^2(x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)$ | | | | | |
| $h_{G'_2} = c^{-2}l^{-2}(x \cdot x)(\partial_i)^2$ | | | | | |
| $\Gamma^{0}_{i0} = \Gamma^{0}_{0j} = -\frac{x^i}{x \cdot x}$ | | | | | |
| $\Gamma^{i}_{ij} = -\frac{x \cdot x}{x \cdot x}$ | | | | | |
| $g^{EG'_2} = l^2(x \cdot dx)(dx \cdot dx) - (x \cdot dx)^2$ | | | | | |
| $h_{EG'_2} = c^{-2}l^{-2}(x \cdot x)(\partial_i)^2$ | | | | | |
| $\Gamma^{0}_{i0} = \Gamma^{0}_{0j} = -\frac{x^i}{x \cdot x}$ | | | | | |
| $\Gamma^{i}_{ij} = -\frac{\delta^i_j x^j + \delta^i_j x^i}{x \cdot x}$ | | | | | |

The contraction of $LBdS$ and $AdS$ geometries is the para-Galilei geometry, denoted by $G'$. The contraction of $Riem$ and $BdSL$ geometries is the Euclidean version of $G'$ geometry, denoted by $EG'$. They can also be obtained by the contraction from $P'$ and $EP'$ geometries in the limit of $l_r \to \infty$, respectively. The curvature of the $G'$ and $EG'$ geometries are zero.
generality, when $c_r, l_r \to 0$. Then, $Riem$, $LBdS$, $dS$ and $DTdS$, geometries contract to, respectively,

$$
\lim_{l_r/c_r^2 \to 0} \frac{l_r^2}{\sigma_E^2} \left( \delta_{\mu\nu} + \frac{\delta_{\mu r_0} \delta_{r_0 r_0} x_0^2}{l_r^2 \sigma_E^2} \right) d\sigma^\mu d\sigma^\nu = \sigma_{E_r} = \sigma_{E_r}^- \leadsto 2 \sigma_{E_r}^0 = 2 \sigma_{E_r}^0 = -g_{G_2} = -g^{NH_2}
$$

$$
\lim_{l_r/c_r^2 \to \infty} \frac{l_r^2}{\sigma_r^2} \left( \eta_{\mu\nu} + \frac{l_r^{-2} x_r^\mu x_r^\nu}{l_r^2 \sigma_r^2} \right) d\sigma^\mu d\sigma^\nu = \sigma_{r} = \sigma_{r}^- \leadsto 2 \sigma_{r}^0 = 2 \sigma_{r}^0 = g_{G_2}^r = g^{NH_2}
$$

$$
\lim_{l_r, c_r \to 0} \frac{l_r^2}{c_r^2 \sigma_{E_r}^2} (\delta_{\mu\nu} \pm l_r^{-2} x_r^\mu x_r^\nu) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = \Gamma_{G_{2\mu\nu}^r}^l
$$

Then, $\pm \lim_{l_r, c_r \to 0} \frac{l_r^2}{c_r^2 \sigma_{E_r}^2} (\delta_{\mu\nu} \pm l_r^{-2} x_r^\mu x_r^\nu) \frac{\partial}{\partial x_r^\mu} \otimes \frac{\partial}{\partial x_r^\nu} = \Gamma_{G_{2\mu\nu}^r}$.

**FIG. 3:** Contraction scheme for the geometries for the possible kinematics.
The curvature is the same as that for $NH_2$ geometry. The sum of ranks of $g^G_2$ and $h^G_2$ is again only 3 and they have the signature ($-$, $-$) and ($+$), respectively. Thus, the contractions of $Riem$, $LBdS$, $dS$, and $DTdS$ geometries in this limit have the signatures ($+$, $+$; $+$), ($+$, $+$; $+$), ($-$, $-$; $+$), ($-$, $-$; $+$), respectively. They have the free parameter $l^2/r = l^2(\delta_{ij}x^ix^j)^{-1/2}$.

All these results can be easily obtained from the contraction of $NH_2$ geometries.

G. Summary

The algebras and their corresponding 45 geometries are listed in TABLE II. All of them have the same vanishing Weyl projective curvature tensor [15]

$$W^\lambda_{\mu\nu} := R^\lambda_{\mu\nu} + \frac{1}{3}(\delta^\lambda_\nu R_{\mu\nu} - \delta^\lambda_\mu R_{\nu\mu})$$

are projective equivalent to each other. In fact, each geometry is defined on a portion of a 4d real projective manifold.

From the viewpoint of differential geometry, these geometries are not all independent. For example, $dS$ and $LBdS$ describe the same space-time in different coordinate systems. Therefore, the
number of independent geometries will be less than 45.

**IV. RELATIONS AMONG SPACE-TIMES**

In the previous section, the geometries for all possible kinematics except static ones are presented by contraction procedure. The contraction scheme for these geometries is shown in FIG 3. The domains for the geometries are shown in FIG. 4–FIG. 7.

In addition to the contraction relation, there exist more relations among the geometries. In this section, we shall discuss these relations.

**A. Duality between the present and the time infinity**

The geometries for the possible kinematics almost appear in pairs. Each pair are invariant under the transformation

$$
\frac{1}{\nu^2 t} \leftrightarrow t, \quad \frac{x^i}{\nu t} \leftrightarrow x^i.
$$

(4.1)
It may be interpreted as the duality between the “present time” and “the time infinity”. For example, $p'$ is isomorphic to the ordinary Poincaré algebra but is regarded as a physically different one in [2]. The analysis in the previous section shows that the two geometries describe the different portions in the $RP^4$ manifold and have the time duality. The duality relation among the geometries for the possible kinematics are summarized in TABLE III. It is remarkable that the last 4 geometries in the TABLE are all self dual. In particular, the $NH_-$ geometry is one of them. Thus, the physics in the $NH_-$ space-time should share the same property.

It should be noted that in the viewpoint of differential geometry, the Minkowski space-time and $P'$ geometry actually describe the same space-time because they have the same metric tensor and topology. If the transformation Eq.(4.1) is considered as a coordinate transformation, the independent geometries remain 26. They are $Riem$, $Lob$, $dS$, $AdS$, $DTdS$, $Euc$, $Min$, $EG$, $G$, $EC$, $C$, $ENH_\pm$, $NH_\pm$, $E_2$, $EP_2\pm$, $P_2\pm$, $E_{2\pm}$, $DTP_2\pm$, $ENH_2$, $NH_2$, $DTNH_2$, $EG_2$, and $G_2$.

**B. Time geometry versus space geometry**

The forms of the degenerate metrics present a striking contrast between $NH_\pm$ ($ENH_\pm$) geometries and $HN_\pm$ ($EHN_\pm$) geometries in addition to the algebra consideration in [5]. The TABLE IV shows
the contrast when the “metrics” are expressed in terms of ‘inertial’ coordinates.

Similarly, $G$ ($EG$) geometry and $C$ ($EC$) geometry also present a contrast between the time geometry and space geometry though they are both flat.

C. Relation among $(E)G_2$, $(E)NH_2$, and $(E)G'_2$ geometries

The algebras $n_{+2}$, $g_2$ and $g'_2$ share the same generators $P'$, $K^c$, and $J$. Their only difference is at the “time translation” generator. The sum of the ‘time translation’ generators for $g_2$ and for $g'_2$ gives rise to the “time translation” generator — Beltrami time translation — for $n_{+2}$.

Their corresponding geometries have the same covariant degenerate metric

$$\pm l^2 \frac{(x \cdot dx)^2 - (x \cdot x)(dx \cdot dx)}{(x \cdot x)^2},$$

which is the metric of 2d sphere, and the same connection and curvature tensors. Their contravariant degenerate metrics also possess the simple additivity. Namely, the algebraic sum of contravariant degenerate metrics of $G_2$ (or $EG_2$) and $G'_2$ (or $EG'_2$) gives the contravariant degenerate metrics of $NH_2$ (or $ENH_2$).
TABLE III: Duality of the present and the time infinity

| $t, x^i$ | $1/\nu^2 t, x^i/\nu t$ | $t, x^i$ | $1/\nu^2 t, x^i/\nu t$ |
|----------|------------------------|----------|------------------------|
| $Min$    | $P'$                   | $Euc$    | $E'$                   |
| $G$      | $G'$                   | $EG$     | $EG'$                  |
| $C$      | $C_2$                  | $EC$     | $EC_2$                 |
| $G_2$    | $G'_2$                 | $EG_2$   | $EG'_2$                |
| $HN_+$   | $E'_2$                 | $EHN_+$  | $E'_2$                 |
| $HN_-$   | $P_2$                 | $EHN_-$  | $EP_2$                 |
| $HN'_-$  | $P_2'$                | $DTHN_-$ | $DTP_2'$               |
| $NH_+$   | $NH'_+$               | $ENH_+$  | $ENH'_+$               |
| $NH_2$   | $NH'_2$               | $ENH_2$  | $ENH'_2$               |
| $NH_-$   | $NH'_-$               | $ENH_-$  | $ENH'_-$               |
| $ENH_2$  | $ENH'_2$              | $DTNH_2$ | $DTNH'_2$              |

TABLE IV: Time geometry versus space geometry

| Covariant degenerate metric | Contravariant degenerate metric | Conformal factor $(C^{-2})$ |
|-----------------------------|---------------------------------|-----------------------------|
| $NH_\pm$                    | Beltrami model for 1d time      | $\sigma_n^\pm = 1 \mp \nu^2 t^2$ |
| $(ENH_\pm)$                 | $g = \frac{1}{\sigma_n^2}(1 \pm \frac{2\nu^2 t^2}{l^2 \sigma_n^2})c^2 dt^2$ | Conformal to 3d flat space |
| $HN_\pm$                    | $\sigma_n^\pm = 1 \mp \nu^2 t^2$ | $\sigma_{E,3}^\pm (x^i) = 1 \pm l^{-2} \delta_{ij} x^i x^j$ |
| $(EHN_\pm)$                 | Beltrami model for 3d space     | 1d flat time                |
| $G$ (EG)                    | $\frac{1}{\sigma_{E,3}^2}(\delta_{ij} \pm \frac{\delta_{ik}(\delta_{ij} x^k x^l)}{l^2 \sigma_{E,3}^2})dx^i dx^j$ | 1d flat time |
| $C$ (EC)                    | 1d flat time                    | 1d flat time |
|                            | 3d flat space                   | 1d flat time                |

D. Geometries with spatial isotropy and Lorentz-like signature

Although all the 22 algebras possess $so(3)$ isotropy, the geometries $DTdS$, $P_2^+$, $DTP_2^+$, $ENH_2$, $NH_2$, $DTNH_2$, $G_2$, $EG_2$, and their time dualities if exist do not have the spatial isotropy with respect to any point on the manifolds. Therefore, they cannot serve as the geometries for the genuine possible kinematics.

In addition, the geometries for genuine possible kinematics should have the right signature. Then, only 9 geometries remains. They are 3 relativistic geometries, $dS$, $AdS$, and $Min$, 3 absolute-time geometries, $NH_\pm$ and $G$, and 3 absolute-space geometries $E_2^-$, $P_2^-$, and $C$.

In order to obtain possible kinematics, Bacry and Lévy-Leblond require that the transformations generated by boost in any given direction form a noncompact subgroup. However, the requirement cannot guarantee the geometry has suitable Lorentz-like signature. For example, the $E'$ geometry is diffeomorphic to $Euc$ geometry. On the contrary, even when the transformations generated by
TABLE V: Contravariant degenerate metrics of $NH_2$, $G_2$ and $G'_2$ geometries

|    | $h$                                                                 |    | $h$                                                                 |
|----|---------------------------------------------------------------------|----|---------------------------------------------------------------------|
| $G_2$ | $l^{-4}(\mathbf{x} \cdot \mathbf{x})(x^\mu \partial_\mu)^2$      | $EG_2$ | $l^{-4}(\mathbf{x} \cdot \mathbf{x})(x^\mu \partial_\mu)^2$ |
| $NH_2$ | $l^{-4}(\mathbf{x} \cdot \mathbf{x})\left[\nu^{-2} \partial_t^2 - (x^\mu \partial_\mu)^2\right]$ | $EH_2$ | $l^{-4}(\mathbf{x} \cdot \mathbf{x})\left[\nu^{-2} \partial_t^2 + (x^\mu \partial_\mu)^2\right]$ |
| $G'_2$ | $l^{-4}(\mathbf{x} \cdot \mathbf{x})(\nu^{-2} \partial_t^2)$          | $EG'_2$ | $l^{-4}(\mathbf{x} \cdot \mathbf{x})(\nu^{-2} \partial_t^2)$       |

boost form a compact subgroup, geometries still possibly have right signature. $E_{2-}$ geometry is one of examples. Therefore, the requirement to pick up the possible kinematics should be the right signature.

V. CONCLUDING REMARKS

Except for the static ones, there are 22 different possible kinematical and geometrical algebras with $so(3)$ isotropy and 10 parameters. Their generators belong to the 4d “inertial-motion algebra” $\text{im}(4)$. Among these algebras, $\mathfrak{r}$, $\mathfrak{l}$ and $\mathfrak{o}^\pm$ algebras are basic ones. The generators of others can be obtained by the linear combinations of the generators of the 4 algebras or by the contraction from the 4 algebras.

The geometries corresponding these algebras are all presented from the contraction of the Beltrami models of Riemann space, Lobachevsky space, (anti-)de Sitter space-times, and double time de Sitter space-time, Lobachevsky-Beltrami model of de Sitter space-time, and $BdS$ model of Lobachevsky space, in the similar way to obtain the Euclid space, Minkowski space-time, and (anti-)Newton-Hooke space-times from Riemann space, Lobachevsky space and ($A$)dS space-times. It should be emphasized that the conditions $\sigma^+_E > 0$, $\sigma^-_E \geq 0$, $\sigma^\pm \geq 0$ are important in the Beltrami models. They specify the domains of the geometries. In the limiting process, they should be always valid. The requirement implies that not all geometries are contractible.

The geometries can be classified in several ways. By the determinant of the metric, the geometries are classified into two categories. One is non-degenerate, and the other is degenerate. It is well known that for the non-degenerate geometries the (covariant) metric is enough to determine their local properties. For degenerate geometries the covariant degenerate metric is not enough to determine the local properties. One has to supplement the contravariant degenerate metric and the connection. Many new geometries belong to the second category. $G$, $C$, and $NH\pm$ space-times are all familiar examples of 4d degenerate space-times.

Each geometry is defined in a suitable portion in the $RP^4$ manifold. On the $RP^4$ there exist a set of the coordinate systems, which are called ‘inertial’ coordinate systems. For a given “inertial” coordinate system $x^\mu$, the geometries fall into 3 categories according to whether $x^0 = ct = 0$ and $x^0 = ct = \infty$ are in the geometries. For the first category, $t = 0$ belongs to the geometries, while $t = \infty$ does not belong to the geometries. $Min$ and $G$ space-times are the most familiar representatives of the categories. For the second one, $t = \infty$ belongs to the geometries while $t = 0$ does not. $P'$ space-times and $G'$ space-times belong to the categories, which are regarded as the
physically different ones even though their algebras are isomorphic to \( p \) and \( g \), respectively \[2\]. There exist the correspondences between the geometries of the first and the second categories. They are linked by Eq.(4.1). For example, \( Min \) and \( P' \) space-times, \( G \) and \( G' \) space-times, \( C \) and \( C_2 \) space-times, \( HN_- \) and \( P_{2-} \), are linked together, respectively. The relation can be interpreted as the time dualities of the present time and time infinity. This behavior might be useful in the study of the space-time structure near the time infinity for asymptotically flat space-times. For the third category, both \( t = 0 \) and \( t = \infty \) belong to the geometries. The \( NH_- \) space-time is such a geometry. It is self dual under Eq.(4.1). Thus, the physics in the \( NH_- \) space-time should share the same property. By the way, there is no space duality in these geometries. The reason is that the three “spatial” Beltrami coordinates \( x^i \) are required to be on equal footing so that \( \mathfrak{so}(3) \) algebra is preserved.

The geometries can be casted into three categories according to the signature of metric tensors. For the non-degenerate cases, the signature of the metric tensor \( g \) is well-defined. The non-degenerate geometries are immediately classified into Euclidean, Lorentzian, double-time geometries. For the degenerate cases, the definition is somewhat obscure. Now, a 4d geometry is split into one 3d and one 1d geometries, or two 2d geometries, or even one 2d, one 1d geometries plus one free parameter. Obviously, the signature is meaningless if only the covariant (or contravariant) degenerate metric of 1d geometry is concerned. However, since the degenerate geometries are obtained in the contraction approach, \( g \) and \( h \) have the imprints of the non-degenerate progenitor which has well-defined signature. Therefore, the signature of a degenerate geometry may be defined by the imprints of the non-degenerate progenitor in \( g \) and \( h \). In this way, the degenerate geometries can also be classified into Euclidean, Lorentzian, and double-time geometries.

The aim of the third requirement in \[2\], the transformations generated by boost in any given direction form a noncompact subgroup, is to rule out the pure geometrical kinematics. However, the geometries shows that the possible kinematics satisfying the requirement may define a pure geometry (i.e. \( E' \) space with \( e' \cong \mathfrak{so}(4) \)) on one hand, and that the possible kinematics violating the requirement may define a Lorentz-like-signature geometries (i.e. \( E_{2-} \) space with \( e_2 \)) on the other.

Similarly, the first requirement in \[2\] is “space is isotropic (rotation invariance)”. Unfortunately, the concept of the space has not been well established in \[2\]. The rotation invariance is actually replaced by an \( \mathfrak{so}(3) \) algebra, \([J, J] = J\). Obviously, this condition cannot guarantee that the geometry has spatial isotropic. In fact, many geometries which are invariant under the transformations generated by \( \mathfrak{so}(3) \) do not have spatial isotropic with respect to any point in the manifolds. The \( P_{2+} \) is one of examples \[12\], which is split into a 3d space-time and one 1d space. The \( NH_2 \) geometry is another example, which is split into 2d space and 2d space-time.

Therefore, the right requirements to pick up the genuine possible kinematics should be that

1. space is isotropic with respect to any point on the manifold;
2. parity and time-reversal are automorphisms of the kinematical groups;
3. the geometry has Lorentz-like signature.

Then, the geometries for genuine possible kinematics are only 3 relativistic geometries, \( dS \), \( AdS \), and \( Min \); 3 absolute-time geometries, \( NH_{\pm} \), \( G \); 3 absolute-space geometries \( E_{2-} \), \( P_{2-} \), \( C \); and their time dualities, \( P' \), \( NH'_{\pm} \), \( G' \), \( HN_{\pm} \) and \( C_2 \).

In the viewpoint of differential geometry, the Minkowski space-time and \( P' \) geometry actually
describe the same space-time because they have the same metric tensor, the same topology and are
diffeomorphic to each other. The same identification can be made for other pairs of geometries on
the same reason. Hence, the genuine possible kinematics from the viewpoint of differential geometry
are given in TABLE VI. Clearly, the geometries in the middle column have vanishing curvature. $dS$

| TABLE VI: Geometries for the genuine possible kinematics |
|---------------------------------|------|------|------|
|                                | > 0  | = 0  | < 0  |
| Relativistic                   | $dS$ | $\min$ | $AdS$ |
| Absolute-time                  | $NH_+$ | $G$  | $NH_-$ |
| Absolute-space                 | $E_{2-}$ | $C$  | $P_{2-}$ |

and $AdS$ have 4d positive and negative curvature. $NH_\pm$ have conformal flat spaces and 1d “curved”
time in terms of Beltrami time. $E_{2-}$ and $P_{2-}$ have 3d sphere and 3d hyperboloid space, and 1d time
is conformal flat in Beltrami coordinates.

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[1] E. Inönü and E. P. Wigner, On the contraction of groups and their representations, PNAS 39 510-524
(1953). See also: R. Silmore, Lie Group, Lie Algebras, and Some of Their Applications (J. Wiley, New
York, 1974).
[2] H. Bacry and J.-M. Lévy-Leblond, Possible kinematics, J. Math. Phys. 9 1605-1614 (1968).
[3] I. M. Yaglom, A Simple Non-Euclidean Geometry and Its Physical Basis, (Springer Verlag, Berlin,
1979).
[4] M. A. F. Sanjuan, Int. Journ. Theor. Phys. 23 1-14 (1984).
[5] H.-Y. Guo, C.-G. Huang, H.-T. Wu and B. Zhou, Sci. China Phys. Mech. Astro., 53 591-597 (2010),
arXiv:0812.0871.
[6] H.-Y. Guo, H.-T. Wu, and B. Zhou, Phys. Lett. B 670 437 (2009), arXiv:0809.3562.
[7] H.-Y. Guo, C.-G. Huang, Z. Xu, and B. Zhou, On Beltrami model of de Sitter spacetime, Mod Phys
Lett A 19 1701-1710 (2004); On special relativity with cosmological constant, Phys. Lett. A 331 1-7
(2004); Three kinds of special relativity via inverse Wick rotation, Chinese Phys Lett, 22 2477-2480
(2005); Temperature at horizon in de Sitter spacetime, Europhys Lett 72 1045-1051 (2005).
[8] N. A. Umow, Einheitliche Ableitung der Transformationen, dir mitdem Relativitätsprinzip verträglich
sind, Physikalische Zeitschrift 11 905-915 (1910); H. Weyl, Mathemathische Analyse des Raumproblems
(Springer, Berlin, 1923); V. Fock, The Theory of Space-Time and Gravitation (Pergamon Press, Uni-
versity of California, 1964), and references therein; L.-G. Hua, Starting with the Unit Circle (Springer, New York, 1982).

[9] H.-Y. Guo, C.-G. Huang, Y. Tian, Z. Xu, and B. Zhou, Snyder’s model — de Sitter special relativity duality and de Sitter gravity, Class Quant Grav 24 4009-4035 (2007), arXiv:gr-qc/0703078v2.

[10] R. Aldrovandi and J. G. Pereira, A second Poincaré Group, arXiv:gr-qc/9809061.

[11] H.-Y. Guo, Transformation group and invarinats on typical space-times, Bull. Sci. 22 487-490 (1977).

[12] C.-G. Huang, Principle of relativity, 24 possible kinematical algebras and new geometries with Poincaré symmetry, in Proceedings of the 9th Asia-Pacific International Conference, Eds. J. Luo, Z.-B. Zhou, H. C. Yeh and J.-P. Hsu, (World Scientific Publishing, Singapore, 2010) 118-129, arXiv:1004.1268; C.-G. Huang, Y. Tian, X.-N. Wu, Z. Xu, and B. Zhou, New geometry with the Poincaré symmetry, arXiv:0909.2773; Geometries with the second Poncaré symmetry, submitted.

[13] C.-G. Huang, H.-Y. Guo, Y. Tian, Z. Xu, and B. Zhou, Int. Journ. Mod. Phys. A 22 2535 (2007); Y. Tian, H.-Y. Guo, C.-G. Huang, Z. Xu, and B. Zhou, Phys. Rev. D 71 044030 (2005).

[14] J.-G. Derome and J.-G. Dubois, Nuo. Cim. 9, 351 (1972); R. Aldrovandi, A.L. Barbosa, L.C.B. Crispino, J.G. Pereira, Class.Quant.Grav. 16 495 (1999); Yi-Hong Gao, arXiv:hep-th/0107067; G.W. Gibbons and C. E. Patricot, Class. Quant. Gravity 20, 5225 (2003).

[15] G. W. Gibbons and C. M. Warnick, Dark Energy and Projective Symmetry, arXiv:1003.3845.