THE FEFFERMAN-STEIN TYPE INEQUALITIES FOR STRONG AND DIRECTIONAL MAXIMAL OPERATORS IN THE PLANE

HIROKI SAITO AND HITOSHI TANAKA

Abstract. The Fefferman-Stein type inequalities for strong maximal operator and directional maximal operator are verified with composition of the Hardy-Littlewood maximal operator in the plane.

1. Introduction

The purpose of this paper is to develop a theory of weights for strong maximal operator and directional maximal operator in the plane. We first fix some notations. By weights we will always mean non-negative and locally integrable functions on $\mathbb{R}^n$. Given a measurable set $E$ and a weight $w$, $w(E) = \int_E w(x) \, dx$, $|E|$ denotes the Lebesgue measure of $E$ and $1_E$ denotes the characteristic function of $E$. Let $0 < p \leq \infty$ and $w$ be a weight. We define the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(\mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p}.$$

For a locally integrable function $f$ on $\mathbb{R}^n$, we define the Hardy-Littlewood maximal operator $M_Q$ by

$$M_Q f(x) = \sup_{Q \in \mathcal{Q}} 1_Q(x) \bar{\int}_Q |f(y)| \, dy,$$

where $\mathcal{Q}$ is the set of all cubes in $\mathbb{R}^n$ (with sides not necessarily parallel to the axes) and the barred integral $\bar{\int}_Q f(y) \, dy$ stands for the usual integral average of $f$ over $Q$. For a locally integrable function $f$ on $\mathbb{R}^n$, we define the strong maximal operator $M_{\mathcal{R}}$ by

$$M_{\mathcal{R}} f(x) = \sup_{R \in \mathcal{R}} 1_R(x) \int_R |f(y)| \, dy,$$

where $\mathcal{R}$ is the set of all rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

2010 Mathematics Subject Classification. Primary 42B25; Secondary 42B35.

Key words and phrases. directional maximal operator; Fefferman-Stein type inequality; Hardy-Littlewood maximal operator; strong maximal operator.

The first author is supported by Grant-in-Aid for Young Scientists (B) (15K17551), the Japan Society for the Promotion of Science. The second author is supported by Grant-in-Aid for Scientific Research (C) (15K04918), the Japan Society for the Promotion of Science.
Let $\mathcal{I} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $p > 1$, be a sublinear operator. It is a fundamental problem of the weight theory that to determine some maximal operator $\mathcal{M}_\mathcal{I}$ capturing certain geometric characteristics of $\mathcal{I}$ such that

$$\int_{\mathbb{R}^n} |\mathcal{I} f(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \mathcal{M}_\mathcal{I} w(x) \, dx$$

holds for arbitrary weight $w$. It is well known that

$$\hat{\mathcal{R}}^n M_Q f(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_Q w(x) \, dx,$$

holds for arbitrary weight $w$ and $p > 1$, and further that

$$\sup_{t > 0} t w(\{ x \in \mathbb{R}^n : M_Q f(x) > t \}) \leq C \int_{\mathbb{R}^n} |f(x)| M_Q w(x) \, dx.$$ 

These are called the Fefferman-Stein inequality and are toy models of the problem (1.1) (see [3]).

There is a problem in the book [4, p472]:

**Problem 1.1.** Does the analogue of the Fefferman-Stein inequality hold for the strong maximal operator, i.e.,

$$\int_{\mathbb{R}^n} \mathcal{M}_R f(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \mathcal{M}_R w(x) \, dx, \quad p > 1,$$

for arbitrary $w(x) \geq 0$?

Concerning Problem (1.1) it is known that, see [8] (also [11, 12]), (1.3) holds for all $p > 1$ if $w \in A^*_\infty$.

We say that $w$ belongs to the class $A^*_p$ whenever

$$[w]_{A^*_p} = \sup_{R \subset \mathbb{R}^n} \int_R w(x) \, dx \left( \int_R w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty, \quad 1 < p < \infty,$$

$$[w]_{A^*_1} = \sup_{R \subset \mathbb{R}^n} \text{ess inf}_{x \in R} w(x) < \infty.$$

It follows by Hölder’s inequality that the $A^*_p$ classes are increasing, that is, for $1 \leq p \leq q < \infty$ we have $A^*_p \subseteq A^*_q$. Thus one defines

$$A^*_\infty = \bigcup_{p > 1} A^*_p.$$

The endpoint behavior of $\mathcal{M}_R$ close to $L^1$ is given by Mitsis [10] (for $n = 2$) and Luque and Parissis [9] (for $n > 2$). That is,

$$w(\{ x \in \mathbb{R}^n : \mathcal{M}_R f(x) > t \}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{t} \left( 1 + \left( \log^+ \frac{|f(x)|}{t} \right)^{n-1} \right) \mathcal{M}_R w(x) \, dx, \quad t > 0,$$

holds for any $w \in A^*_\infty$, where $\log^+ t = \max(0, \log t)$.

In this paper concerning Problem (1.1) we shall establish the following.
Theorem 1.2. Let $w$ be any weight on $\mathbb{R}^2$ and set $W = M_R M_Q w$. Then
\[
w\left( \{ x \in \mathbb{R}^2 : M_R f(x) > t \} \right) \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \left( 1 + \log^+ \left( \frac{|f(x)|}{t} \right) \right) W(x) \, dx, \quad t > 0,
\]
holds, where the constant $C > 0$ does not depend on $w$ and $f$.

By interpolation, we have the following corollary.

Corollary 1.3. Let $w$ be any weight on $\mathbb{R}^2$ and set $W = M_R M_Q w$. Then, for $p > 1$, there exists a constant $C_p > 0$ such that
\[
\| M_R f \|_{L^p(\mathbb{R}^2, w)} \leq C_p \| f \|_{L^p(\mathbb{R}^2, W)}
\]
holds for all $f \in L^p(\mathbb{R}^2, W)$.

Let $\Sigma$ be a set of unit vectors in $\mathbb{R}^2$, i.e., a subset of the unit circle $S^1$. Associated with $\Sigma$, we define $B_\Sigma$ to be the set of all rectangles in $\mathbb{R}^2$ whose longest side is parallel to some vector in $\Sigma$. For a locally integrable function $f$ on $\mathbb{R}^2$, we also define the directional maximal operator $M_\Sigma$ associated with $\Sigma$ as
\[
M_\Sigma f(x) = \sup_{R \in B_\Sigma} 1_R(x) \int_R |f(y)| \, dy.
\]
Many authors studied this operator, see [1, 2, 6, 7, 13, 14], and Katz showed that $M_\Sigma$ is bounded on $L^2(\mathbb{R}^2)$ with the operator norm $O(\log N)$ for any set $\Sigma$ with cardinality $N$.

For fixed sufficiently large integer $N$, let
\[
\Sigma_N = \left\{ \left( \cos \frac{2\pi j}{N}, \sin \frac{2\pi j}{N} \right) : j = 0, 1, \ldots, N - 1 \right\}
\]
be the set of $N$ uniformly spread directions on the circle $S^1$. In this paper we shall prove the following, which is a weighted version of the classical result due to Strömberg [14].

Theorem 1.4. Let $N > 10$ and $w$ be any weight on $\mathbb{R}^2$. Set $W = M_{\Sigma_N} M_Q w$. Then
\[
\sup_{t > 0} t w\left( \{ x \in \mathbb{R}^2 : M_{\Sigma_N} f(x) > t \} \right)^{1/2} \leq C (\log N)^{1/2} \| f \|_{L^2(\mathbb{R}^2, W)}
\]
holds for all $f \in L^2(\mathbb{R}^2, W)$, where the constant $C > 0$ does not depend on $w$ and $f$.

By interpolation, we have the following corollary.

Corollary 1.5. Let $N > 10$ and $w$ be any weight on $\mathbb{R}^2$. Set $W = M_{\Sigma_N} M_Q w$. Then, for $2 < p < \infty$, there exists a constant $C_p > 0$ such that
\[
\| M_{\Sigma_N} f \|_{L^p(\mathbb{R}^2, w)} \leq C_p (\log N)^{1/p} \| f \|_{L^p(\mathbb{R}^2, W)}
\]
holds for all $f \in L^p(\mathbb{R}^2, W)$.

The letter $C$ will be used for the positive finite constants that may change from one occurrence to another. Constants with subscripts, such as $C_1$, $C_2$, do not change in different occurrences.
2. Proof of Theorem 1.2

In what follows we shall prove Theorem 1.2. Our proof relies upon the refinement of the arguments in [10]. With a standard argument, we may assume that the basis \( \mathcal{R} \) is the set of all dyadic rectangles \( R \) (cartesian products of dyadic intervals) having long side pointing in the \( x_1 \)-direction. We denote by \( P_i, i = 1, 2 \), the projection on the \( x_i \)-axis. Fix \( t > 0 \) and given the finite collection of dyadic rectangles \( \{R_i\}_{i=1}^M \subset \mathcal{R} \) such that

\[
\int_{R_i} |f(y)| \, dy > t, \quad i = 1, 2, \ldots, M.
\]

It suffices to estimate \( w(\bigcup_{i=1}^M R_i) \) (see the next section for details).

First relabel if necessary so that the \( R_i \) are ordered so that their long sidelengths \( |P_1(R_i)| \) decrease. We now give a selection procedure to find subcollection \( \{\tilde{R}_i\}_{i=1}^N \subset \{R_i\}_{i=1}^M \).

Take \( \tilde{R}_1 = R_1 \) and let \( \tilde{R}_2 \) be the first rectangle \( R_j \) such that

\[
|R_j \cap \tilde{R}_1| < \frac{1}{3} |R_j|.
\]

Suppose have now chosen the rectangles \( \tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{i-1} \). We select \( \tilde{R}_i \) to be the first rectangle \( R_j \) occurring after \( \tilde{R}_{i-1} \) so that

\[
\bigg| \bigcup_{k=1}^{i-1} R_j \cap \tilde{R}_k \bigg| < \frac{1}{3} |R_j|.
\]

Thus, we see that

\[
(2.2) \quad \bigg| \bigcup_{j=1}^{i-1} \tilde{R}_i \cap \tilde{R}_j \bigg| < \frac{1}{3} |\tilde{R}_i|, \quad i = 2, 3, \ldots, N.
\]

We claim that

\[
(2.3) \quad \bigcup_{i=1}^M R_i \subset \left\{ x \in \mathbb{R}^2 : \mathcal{M}_Q[1_{\bigcup_{i=1}^N \tilde{R}_i}](x) \geq \frac{1}{3} \right\}.
\]

Indeed, choose any point \( x \) inside a rectangle \( R_j \) that is not one of the selected rectangles \( \tilde{R}_i \). Then, there exists a unique \( K \leq N \) such that

\[
\bigg| \bigcup_{i=1}^K R_j \cap \tilde{R}_i \bigg| \geq \frac{1}{3} |R_j|.
\]

Since, \( |P_1(R_j)| \leq |P_1(\tilde{R}_i)| \) for \( i = 1, 2, \ldots, K \), we have

\[
P_1(R_j) \cap P_1(\tilde{R}_i) = P_1(R_j) \text{ when } R_j \cap \tilde{R}_i \neq \emptyset,
\]

where we have used the dyadic structure;

\[
(2.4) \quad \text{If both } I \text{ and } J \text{ are the dyadic interval then } I \cap J \in \{I, J, \emptyset\}.
\]
Thus,
\[ \bigcup_{i=1}^{K} R_j \cap \tilde{R}_i = \bigcup_{i=1}^{K} P_1(R_j) \times (P_2(R_j) \cap P_2(\tilde{R}_i)) = P_1(R_j) \times \bigcup_{i=1}^{K} P_2(R_j) \cap P_2(\tilde{R}_i). \]

Hence,
\[ \left| \bigcup_{i=1}^{K} P_2(R_j) \cap P_2(\tilde{R}_i) \right| \geq \frac{1}{3} |P_2(R_j)|. \]

Thanks to the fact that \(|P_2(R_j)| \leq |P_1(R_j)| \leq |P_1(\tilde{R}_i)|\), this implies that
\[ \left| \bigcup_{i=1}^{K} Q \cap \tilde{R}_i \right| \geq \frac{1}{3} |Q|, \]

where \(Q\) is a unique dyadic cube containing \(x\) and having the side length \(|P_2(R_j)|\). This proves (2.3).

It follows from (2.3) and (1.2) that
\[
\begin{align*}
    w \left( \bigcup_{i=1}^{M} R_i \right) &\leq w \left( \left\{ x \in \mathbb{R}^2 : M_Q [\bigcup_{i=1}^{N} \tilde{R}_i] (x) \geq \frac{1}{3} \right\} \right) \\
    &\leq C U \left( \bigcup_{i=1}^{N} \tilde{R}_i \right) \leq C \sum_{i=1}^{N} U(\tilde{R}_i),
\end{align*}
\]

where \(U = M_Q w\). We shall evaluate the quantity
\[
(i) = \sum_{i=1}^{N} U(\tilde{R}_i).
\]

Let \(\mu_U(x)\) be the weighted multiplicity function associated to the family \(\{\tilde{R}_i\}\), that is,
\[ \mu_U(x) = \sum_{i=1}^{N} \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} 1_{\tilde{R}_i}(x). \]

By (2.1), choosing \(\delta_0\) small enough determined later,
\[
(i) \leq \sum_{i=1}^{N} \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \frac{|f(y)|}{t} dy \\
= \delta_0 \int_{\mathbb{R}^2} \mu_U(x) W(x)^{-1} \cdot \frac{|f(x)|}{\delta_0 t} \cdot W(x) dx.
\]

Using the elementary inequality
\[ ab \leq (e^a - 1) + b(1 + \log^+ b), \quad a, b \geq 0, \]
we get

\[(i) \leq \delta_0 \int_{\mathbb{R}^2} (\exp(\mu_U(x)W(x)^{-1}) - 1) W(x) \, dx \\
+ \delta_0 \int_{\mathbb{R}^2} \left| f(x) \right| \left( 1 + \log^{+} \frac{|f(x)|}{\delta_0 t} \right) W(x) \, dx \\
\leq \delta_0 \int_{\mathbb{R}^2} (\exp(\mu_U(x)W(x)^{-1}) - 1) W(x) \, dx \\
+ (1 - \log \delta_0) \int_{\mathbb{R}^2} \left| f(x) \right| \left( 1 + \log^{+} \frac{|f(x)|}{t} \right) W(x) \, dx.\]

We have to evaluate the quantity

\[(ii) = \int_{\mathbb{R}^2} (\exp(\mu_U(x)W(x)^{-1}) - 1) W(x) \, dx.\]

We expand the exponential in a Taylor series. Then

\[(ii) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^2} (\mu_U(x)W(x)^{-1})^k W(x) \, dx \\
= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^2} \mu_U(x)^k W(x)^{1-k} \, dx.\]

Fix \(k \geq 2\). We use an elementary inequality

\[\left( \sum_{i=1}^{\infty} a_i \right)^k \leq k \sum_{i=1}^{\infty} a_i \left( \sum_{j=1}^{i} a_j \right)^{k-1},\]

where \(\{a_i\}_{i=1}^{\infty}\) is a sequence of summable nonnegative reals. Then

\[\int_{\mathbb{R}^2} \mu_U(x)^k W(x)^{1-k} \, dx \\
\leq k \sum_{i=1}^{N} \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \left( \sum_{j=1}^{i} \frac{U(\tilde{R}_j)}{|\tilde{R}_j|} 1_{\tilde{R}_j}(x) \right)^{k-1} W(x)^{1-k} \, dx \\
\leq k \sum_{i=1}^{N} \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \left( \sum_{j=1}^{i} 1_{\tilde{R}_j}(x) \right)^{k-1} \, dx,\]

where we have used

\[\sum_{j=1}^{i} \frac{U(\tilde{R}_j)}{|\tilde{R}_j|} 1_{\tilde{R}_j}(x) \leq \left( \sum_{j=1}^{i} 1_{\tilde{R}_j}(x) \right) W(x).\]

We claim that, for \(n = 1, 2, \ldots, N\),

\[(2.5) \quad |X_{i,n}| \leq 3^{1-n}|\tilde{R}_i|,\]

where

\[X_{i,n} = \left\{ x \in \tilde{R}_i : \sum_{j=1}^{i} 1_{\tilde{R}_j}(x) \geq n \right\}.\]
Indeed, first we notice that, for any $k$ and $j$ with $N \geq k > j \geq 1$, if $\tilde{R}_k \cap \tilde{R}_j \neq \emptyset$, then
\[
\tilde{R}_k \cap \tilde{R}_j = P_1(\tilde{R}_k) \times P_2(\tilde{R}_j),
\]
because we have $P_1(\tilde{R}_k) \subset P_1(\tilde{R}_j)$ and, by (2.2), $|P_2(\tilde{R}_k) \cap P_2(\tilde{R}_j)| < \frac{1}{3}|P_2(\tilde{R}_k)|$. With this in mind, we can observe the following:

There exists a set of dyadic intervals $\{I_{jk}\}$ with $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, K_j$ that satisfies the following:

- The dyadic intervals $I_{jk}$ are pairwise disjoint for varying $k$;
- For each $I_{jk}$, $j > 1$, there exists a unique $I_{(j-1)i} \supseteq I_{jk}$;
- For each $I_{jk}$ there exists a unique number $i_{jk} \leq i$ such that $I_{jk} = P_2(\tilde{R}_{ijk})$;
- $P_2(X_{i,1}) = I_{11}$, $P_2(X_{i,2}) = \bigcup_{k=1}^{K_2} I_{2k}$, \ldots, $P_2(X_{i,j}) = \bigcup_{k=1}^{K_j} I_{jk}$, \ldots, $P_2(X_{i,n}) = \bigcup_{k=1}^{K_n} I_{nk}$;
- If $I_{jk} \subset I_{(j-1)i}$, then $i_{jk} < i_{(j-1)i}$ and $\tilde{R}_{i(j-1)i} \cap \tilde{R}_{ijk} \neq \emptyset$.

It follows from the last relation and (2.2) that
\[
3 \sum_{j=1}^{K_j} |I_{j1}| < \sum_{k=1}^{K_{j-1}} |I_{(j-1)k}|, \quad j = 2, 3, \ldots, n.
\]
This gives us that
\[
3^{n-1} \sum_{k=1}^{K_n} |I_{nk}| < |I_{11}|,
\]
which yields (2.5).

It follows from (2.5) that
\[
\frac{U(\tilde{R}_i)}{|R_i|} \int_{\tilde{R}_i} \left( \sum_{j=1}^{i} \tilde{R}_j(x) \right)^{k-1} dx \leq \frac{U(\tilde{R}_i)}{|R_i|} \sum_{n=1}^{N} n^{k-1} |X_{i,n}| \leq U(\tilde{R}_i) \sum_{n=1}^{N} n^{k-1} 3^{1-n}.
\]

Altogether, the quantity (ii) can be majorized by
\[
(i) \times \left[ 1 + \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{n=1}^{N} n^{k-1} 3^{1-n} \right].
\]

There holds
\[
\sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{n=1}^{N} n^{k-1} 3^{1-n} \leq 3 \sum_{n=1}^{\infty} \left( \frac{e}{3} \right)^n =: C_0.
\]
If we choose $\delta_0$ so that $\delta_0(1 + C_0) = \frac{1}{2}$, we obtain
\[ (i) \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \left( 1 + \log^+ \frac{|f(x)|}{t} \right) W(x) \, dx. \]

This completes the proof.

Remark. Since our proof relies only upon the dyadic structure (2.4), it can be applied the basis $\mathcal{R}$ of the form the set of all rectangles in $\mathbb{R}^n$ whose sides parallel to the coordinate axes and which are congruent to the rectangle $(0, a)^{n-1} \times (0, b)$ with varying $a, b > 0$.

3. Proof of Theorem 1.4

In what follows we shall prove Theorem 1.4. We follow the argument in [5] Chapter 10, Theorem 10.3.5. To avoid problems with antipodal points, it is convenient to split $\Sigma_N$ as the union of eight sets, in each of which the angle between any two vectors does not exceed $\pi/4$. It suffices therefore to obtain (1.4) for each such subset of $\Sigma_N$. Let us fix one such subset of $\Sigma_N$, which we call $\Sigma_N^1$. To prove (1.4), we fix a $t > 0$ and we start with a compact subset $K$ of the set $\{ x \in \mathbb{R}^2 : \mathcal{M}_{\Sigma_N^1} f(x) > t \}$. Then for every $x \in K$, there exists an open rectangle $R_x$ that contains $x$ and whose longest side is parallel to a vector in $\Sigma_N^1$. By compactness of $K$, there exists a finite subfamily $\{ R_\alpha \}_{\alpha \in A}$ of the family $\{ R_x \}_{x \in K}$ such that
\[ \int_{R_\alpha} |f(y)| \, dy > t \]
for all $\alpha \in A$ and such that the union of the $R_\alpha$’s covers $K$.

In the sequel we denote by $\theta_\alpha$ the angle between the $x_1$-axis and the vector pointing in the longer direction of $R_\alpha$ for any $\alpha \in A$. We also denote by $l_\alpha$ the shorter side of $R_\alpha$ and by $L_\alpha$ the longer side of $R_\alpha$ for any $\alpha \in A$.

We shall select the subfamily $\{ R_\beta \}_{\beta \in B}$ as follows:

Without loss of generality we may assume that $A = \{ 1, 2, \ldots, \ell \}$ with $L_j \geq L_{j+1}$ for all $j = 1, 2, \ldots, \ell - 1$. Let $\beta_1 = 1$ and choose $\beta_2$ to be the first number in $\{ \beta_1 + 1, \beta_1 + 2, \ldots, \ell \}$ such that
\[ |R_{\beta_1} \cap R_{\beta_2}| \leq \frac{1}{2} |R_{\beta_2}|. \]

We next choose $\beta_3$ to be the first number in $\{ \beta_2 + 1, \beta_2 + 2, \ldots, \ell \}$ such that
\[ |R_{\beta_1} \cap R_{\beta_3}| + |R_{\beta_2} \cap R_{\beta_3}| \leq \frac{1}{2} |R_{\beta_3}|. \]

Suppose we have chosen the numbers $\beta_1, \beta_2, \ldots, \beta_{j-1}$. Then we choose $\beta_j$ to be the first number in $\{ \beta_{j-1} + 1, \beta_{j-1} + 2, \ldots, \ell \}$ such that
\[ \sum_{k=1}^{j-1} |R_{\beta_k} \cap R_{\beta_j}| \leq \frac{1}{2} |R_{\beta_j}|. \]

Since the set $A$ is finite, this selection stops after $m$ steps.
Define $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and let
\[
Y(x) = \sum_{\beta \in \mathcal{B}} 1_{(R_\beta)^*}(x),
\]
where $(R_\beta)^*$ is the rectangle $R_\beta$ expanded 5 times in both directions.

We claim that
\[
(3.3) \quad w(K) \leq w\left( \bigcup_{\alpha \in \mathcal{A}} R_\alpha \right) \leq C(\log N) \int_{\mathbb{R}^2} Y(x) U(x) \, dx,
\]
where $U(x) = \mathfrak{M}_Q w(x)$. To verify this claim, we need the following lemma.

We set $\omega_k = \frac{2\pi k}{N}$ for $k \in \mathbb{Z}^+$ and $\omega_0 = 0$. We let $M = \left\lfloor \log(N/8) \right\rfloor / \log 2$.

**Lemma 3.1** ([5, Lemma 10.3.6]). Let $R_\alpha$ be a rectangle in the family $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ and let $0 \leq k < M$.

Suppose that $\beta \in \mathcal{B}$ is such that $\omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}$ and such that $L_\beta \geq L_\alpha$. Let
\[
s_\alpha = 8 \max(l_\alpha, \omega_k L_\alpha).
\]
Then we have
\[
(3.4) \quad \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} \leq 32 \frac{|(R_\beta)^* \cap Q_k|}{|Q_k|}.
\]

We shall prove (3.3). By (1.2) it suffices to show that
\[
(3.5) \quad \bigcup_{\alpha \in \mathcal{A}} R_\alpha \subset \left\{ x \in \mathbb{R}^2 : \mathfrak{M}_Q Y(x) > \frac{C}{\log N} \right\}.
\]
Since we may assume that $C/(\log N) < 1$, the set $\bigcup_{\beta \in \mathcal{B}} R_\beta$ is contained in the set of the right hand side of (3.5). So, we fix $\alpha \in \mathcal{A} \setminus \mathcal{B}$. Then the rectangle $R_\alpha$ was not selected in the selection procedure.

By the construction and (3.2), we see that there exists $j$ such that
\[
\sum_{k=1}^j |R_{\beta_k} \cap R_\alpha| > \frac{1}{2} |R_\alpha|
\]
and such that $L_{\beta_k} \geq L_\alpha$ for all $k = 1, 2, \ldots, j$.

Let $\mathcal{B}_j = \{\beta_1, \beta_2, \ldots, \beta_j\}$. It follows from Lemma 3.1 that
\[
\frac{1}{2} < \sum_{\beta \in \mathcal{B}_j} \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} = \sum_{k=0}^M \sum_{\beta \in \mathcal{B}_j : \omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}} \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|}
\]
\[
\leq 32 \sum_{k=0}^M \sum_{\beta \in \mathcal{B}_j : \omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}} \frac{|(R_\beta)^* \cap Q_k|}{|Q_k|},
\]
where $Q_k$ is a square determined by Lemma 3.1 with an arbitrary $x \in R_\alpha$. Since we have $M \leq C(\log N)$ and

$$\sum_{\beta \in B; \omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}} \frac{|(R_\beta)\ast \cap Q_k|}{|Q_k|} \leq C \mathcal{M}_Q Y(x)$$

for all $x \in R_\alpha$, we obtain

$$\mathcal{M}_Q Y(x) > \frac{C}{\log N}$$

for all $x \in R_\alpha$, which implies (3.5) and, hence, (3.3).

We now evaluate

$$(i) = \int_{R^2} Y(x) U(x) \, dx = \sum_{\beta \in B} U((R_\beta)\ast).$$

By (3.1) and Hölder’s inequality we have

$$(i) \leq \frac{1}{t} \sum_{\beta \in B} U((R_\beta)\ast) \int_{R_\beta} |f(y)| \, dy
= \frac{1}{t} \int_{R^2} \left( \sum_{\beta \in B} \frac{U((R_\beta)\ast)}{|R_\beta|} 1_{R_\beta}(y) \right) |f(y)| \, dy
\leq \frac{1}{t} \left( \int_{R^2} \left( \sum_{\beta \in B} \frac{U((R_\beta)\ast)}{|R_\beta|} 1_{R_\beta}(y) \right)^2 \right)^{1/2} \|f\|_{L^2(R^2, W)}.$$

We have further

$$(ii) = \int_{R^2} \left( \sum_{\beta \in B} \frac{U((R_\beta)\ast)}{|R_\beta|} 1_{R_\beta}(y) \right)^2 \, W(y)^{-1} \, dy
= \sum_{j=1}^m \left( \frac{U((R_{\beta_j})\ast)}{|R_{\beta_j}|} \right)^2 \int_{R_{\beta_j}} W(y)^{-1} \, dy
+ 2 \sum_{j=1}^m \frac{U((R_{\beta_j})\ast)}{|R_{\beta_j}|} \sum_{k=1}^{j-1} \frac{U((R_{\beta_k})\ast)}{|R_{\beta_k}|} \int_{R_{\beta_k} \cap R_{\beta_j}} W(y)^{-1} \, dy.$$

We notice that, for any $y \in R_{\beta_k} \cap R_{\beta_j}$

$$W(y) \geq \frac{U((R_{\beta_k})\ast)}{|(R_{\beta_k})\ast|} = \frac{U((R_{\beta_k})\ast)}{25|R_{\beta_k}|}.$$
This yields

\[(ii) \leq 25 \sum_{j=1}^{m} U((R_{\beta_j})^*)
\]

\[+ 50 \sum_{j=1}^{m} U((R_{\beta_j})^*) \frac{1}{|R_{\beta_j}|} \sum_{k=1}^{j-1} |R_{\beta_k} \cap R_{\beta_j}|
\]

\[\leq 50 \sum_{\beta \in B} U((R_{\beta})^*),
\]

where we have used (3.2). Altogether, we obtain

\[(i) \leq C \frac{t^2}{2} \|f\|^2_{L^2(R^2, W)},
\]

which yields by (3.3)

\[w(K) \leq C \frac{(\log N)}{t^2} \|f\|^2_{L^2(R^2, W)}.
\]

Since $K$ was an arbitrary compact subset of $\{x \in R^2 : M_{\Sigma_N} f(x) > t\}$, the same estimate is valid for the latter set and we finish the proof.

**REFERENCES**

[1] A. Alfonseca, F. Soria and A. Vargas, *A remark on maximal operators along directions in $\mathbb{R}^2$*, Math. Res. Lett., 10 (2003), no. 1, 41–49.

[2] , *An almost-orthogonality principle in $L^2$ for directional maximal functions*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 1–7, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003.

[3] C. Fefferman and E. M. Stein *Some Maximal Inequalities*, Amer. J. math., 93 (1971), no. 1, 107–115.

[4] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Math. Stud., 116 (1985).

[5] L. Grafakos, *Modern Fourier Analysis*, volume 250 of Graduate Texts in Mathematics, Springer, New York, 2nd edition, 2008.

[6] N. H. Katz, *Maximal operators over arbitrary sets of directions*, Duke Math. J., 97 (1999), no. 1, 67–79.

[7] , *Remarks on maximal operators over arbitrary sets of directions*, Bull. London Math. Soc., 31 (1999), no. 6, 700–710.

[8] Kai-Ching Lin, Ph.D. University of California, Los Angeles 1984 United States. Dissertation: Harmonic Analysis on the Bidisc.

[9] T. Luque and I. Parissis *The endpoint Fefferman-Stein inequality for the strong maximal function*, J. Funct. Anal. 266 (2014), no. 1, 199–212.

[10] T. Mitsis, *The weighted weak type inequality for the strong maximal function*, J. Fourier Anal. Appl. 12 (2006), no. 6, 645–652.

[11] C. Pérez, *Weighted norm inequalities for general maximal operators*, Conference on Mathematical Analysis (El Escorial, 1989), Publ. Mat. 35 (1991), no. 1, 169–186.

[12] , *A remark on weighted inequalities for general maximal operators*, Proc. Amer. Math. Soc., 119 (1993), no. 4, 1121–1126.

[13] H Saito and H. Tanaka, *Directional maximal operators and radial weights on the plane*, Bull. Austral. Math. Soc., 89 (2014), no 2, 397–414.

[14] J. O. Strömberg, *Maximal functions associated to rectangles with uniformly distributed directions*, Ann. Math. (2), 107 (1978), no. 2, 399–402.
Academic support center, Kogakuin University, 2665–1, Nakanomachi, Hachioji-shi Tokyo, 192–0015, Japan
E-mail address: j1107703@gmail.com

Research and Support Center on Higher Education for the Hearing and Visually Impaired, National University Corporation Tsukuba University of Technology, Kasuga 4-12-7, Tsukuba City, Ibaraki, 305-8521 Japan
E-mail address: htanaka@k.tsukuba-tech.ac.jp