Tidal effects in Schwarzschild black hole in massive gravity

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In this paper, we investigate tidal effects produced in the spacetime of the Schwarzschild black hole in massive gravity, which has two additional mass parameters due to massive gravitons. As a result, we have obtained that massive gravitons affect the angular component of the tidal force, while the radial component is the same with the one in massless gravity. On the other hand, by solving the geodesic deviation equations according to appropriate boundary conditions, we have found that radial components of two nearby geodesics keep tightening while falling into the black hole and after passing the event horizons get abruptly infinitely stretched due to massive gravitons. And angular components of two nearby geodesics get stretched firstly, reach a peak and then get compressed while falling into the black hole. Moreover, we have also shown that the angular components are more easily deformed near the departure position as the mass of a black hole is smaller for a fixed graviton mass.

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I. INTRODUCTION

It is well-known that a body in free fall toward the center of another body gets stretched in the radial direction and compressed in the angular one. The stretching and compression arise from a tidal effect of gravity, which is given by a difference in the strength of gravity between two points \[1\]. Tidal phenomena are common in the universe from our solar system to stars in binary systems, to galaxies, to cluster of galaxies, and even to gravitational waves \[2\]. In particular, Wheeler \[3\] proposed the possible break-up of a star in the ergosphere of the Kerr black hole due to tidal interaction and subsequently emitting a jet composed of the debris from the star as a mechanism for the jets production. Since then, the investigation of tidal effects in astrophysical context has been devoted to tidal disruption of stars deeply plunging into black holes \[4\]. Many other studies in theoretical context have also been preformed for the extensive description of tidal effects in various black holes. Authors in Refs. \[5\] studied tidal force for a particle freely falling in the Reissner-Nordström (RN) black hole and found that radial and angular components of the tidal force change their signs either outside or inside of the event horizon unlike in the Schwarzschild black hole. Gad \[6\] investigated tidal force in stringy charged black hole by comparing it with the Schwarzschild and RN black holes. Shahzad and Jawad \[7\] showed that the radial and angular components of tidal force in Kiselev black hole surrounded by radiation and dust fluids also change the sign between event and Cauchy horizons as in the RN black hole. Investigation of tidal effects has been further extended to other various black holes \[8\].

On the other hand, Einstein’s theory of general relativity is a theory of a massless spin-2 graviton. In order to accommodate a massive graviton, Fierz and Pauli \[9\] developed a massive spin-2 theory propagating on a flat spacetime background. However, it was later known that the massive gravity suffered from the Boulware-Deser ghost

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problem \cite{28} and the van Dam, Veltman and Zakharov (vDVZ) discontinuity \cite{29, 30} in the massless graviton limit. A decade ago, de Rham, Gabadadze and Trolley (dRGT) \cite{31, 32} obtained a ghost free massive gravity, which has nonlinearly interacting mass terms constructed from the metric coupled with a symmetric background tensor, called the reference metric. In the dRGT massive gravity, the nondynamical reference metric is set to be the Minkowskian one. In order to preserve diffeomorphism invariance, Hassan et al. \cite{33, 34} developed the ghost free massive gravity with a general reference metric. For more details, the readers can refer to the reviews \cite{35, 36}. Later, Vegh \cite{37} studied a nonlinear massive gravity with a special singular reference metric in the scenario of the gauge/gravity duality \cite{38–40}, which is used to study momentum dissipation for describing the electric and heat conductivity for normal conductors. Massive gravity theories with a singular reference metric have been exploited to investigate many black hole models \cite{41–50}. In this framework, we have recently studied a charged Bañados-Teitelboim-Zanelli (BTZ) black hole in massive gravity \cite{51} and the Schwarzschild-anti de Sitter (SAdS) black hole in massive gravity \cite{52}. As a result, we have explicitly shown that global flat embedding dimensions are differently given according to mass parameters. We have also obtained the generalized Hawking, Unruh, and freely falling temperatures of the charged BTZ and the SAdS black holes with massive graviton effects.

In this paper, we will study tidal effects produced in the spacetime of the Schwarzschild black hole in massive gravity, which has two additional mass parameters due to massive gravitons, comparing with the Schwarzschild black hole in massless gravity. In Sec. II and III, we briefly recapitulate features of the geodesic equations and tidal forces for the Schwarzschild black hole in massless gravity \cite{13–15}, and newly generalize them to massive gravity in order to study massive graviton effects. In Sec. IV, we find solutions of the geodesic deviation equations for radially falling bodies toward the Schwarzschild black hole in massive gravity and analyze the results having massive gravitons comparing with the ones in massless gravity. Conclusions are drawn in Sec. V.

II. GEODESICS IN THE SCHWARZSCHILD BLACK HOLE

A. Geodesic equations in the Schwarzschild black hole in massless gravity

In this subsection, we briefly recapitulate features of geodesic equations in the Schwarzschild black hole in massless gravity which line element is given by

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.1}
\]

Then, the geodesic equations of

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \tag{2.2}
\]

are obtained from the metric where \(x^\mu = (t, r, \theta, \phi)\). Here, the Christoffel symbol is given by

\[
\Gamma^\mu_{\rho\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_{\rho} g_{\nu\sigma} + \partial_{\nu} g_{\rho\sigma} - \partial_{\sigma} g_{\rho\nu}). \tag{2.3}
\]

This gives us explicitly

\[
\frac{dv^0}{d\tau} + \frac{2m}{r^2 - 2mr} v^0 v^1 = 0, \tag{2.4}
\]

\[
\frac{dv^1}{d\tau} + \frac{m}{r} (r^2 - 2mr)(v^0)^2 - \frac{m}{r^2 - 2mr} (v^1)^2 - (r - 2m) \left[ (v^2)^2 + \sin^2 \theta (v^3)^2 \right] = 0, \tag{2.5}
\]

\[
\frac{dv^2}{d\tau} + \frac{2}{r} v^1 v^2 - \sin \theta \cos \theta (v^3)^2 = 0, \tag{2.6}
\]

\[
\frac{dv^3}{d\tau} + \frac{2}{r} v^1 v^3 + \cot \theta v^2 v^3 = 0, \tag{2.7}
\]

in terms of the four velocity vector \(v^\mu = dx^\mu / d\tau\).

Now, without loss of generality, one can consider the geodesics on the equatorial plane \(\theta = \pi/2\) for all \(\tau\). Then, one has \(v^2 = \theta = 0\) and the geodesic equations are reduced to

\[
\frac{dv^0}{d\tau} + \frac{2m}{r^2 - 2mr} v^0 v^1 = 0, \tag{2.8}
\]
\[
\frac{dv^1}{d\tau} + \frac{m}{r^4}(r - 2mr)(v^0)^2 - \frac{m}{r^2 - 2mr}(v^1)^2 - (r - 2m)(v^3)^2 = 0, \tag{2.9}
\]
\[
\frac{dv^2}{d\tau} = 0, \tag{2.10}
\]
\[
\frac{dv^3}{d\tau} + \frac{2}{r}v^1v^3 = 0. \tag{2.11}
\]

By considering \(v^1 = \dot{r}\), one can integrate Eqs. (2.8) and (2.11) as
\[
v^0 = \frac{c_1 r}{r - 2m}, \tag{2.12}
\]
\[
v^3 = \frac{c_2}{r^2}, \tag{2.13}
\]
respectively, where \(c_1\) and \(c_2\) are integration constants. At large \(r\) the constant \(c_1\) becomes energy per unit mass \(E\) and \(c_2\) angular momentum per unit mass \(L\). Finally, by letting \(ds^2 = -k d\tau^2\) in Eq. (2.1) and making use of the geodesic equations (2.12) and (2.13), one can obtain
\[
v^1 = \pm \left[ E^2 - \left( k + L^2 \frac{1}{r^2} \right) \left( 1 - \frac{2m}{r} \right) \right]^{1/2}, \tag{2.14}
\]
where the \((-)\) sign means \(dr/d\tau < 0\) so that it is for inward motion falling to the black hole. Likewise, the \((+)\) sign is for outward motion. Note that timelike (nulllike) geodesic is for \(k = 1\) (0).

### B. Geodesic equations in the Schwarzschild black hole in massive gravity

In this subsection, we will newly study the geodesic equations of the Schwarzschild black hole in massive gravity, which is described by the action
\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ \mathcal{R} + \tilde{m}^2 \sum_{i=1}^{4} a_i \mathcal{U}_i(g_{\mu\nu},f_{\mu\nu}) \right], \tag{2.15}
\]
where \(\mathcal{R}\) is the scalar curvature, \(\tilde{m}\) is the graviton mass\(^1\), \(a_i\) are constants and \(\mathcal{U}_i\) are symmetric polynomials of the eigenvalue of the matrix \(K^\mu_{\nu} = \sqrt{g^{\mu\alpha}}f_{\alpha\nu}\) given by
\[
\begin{align*}
\mathcal{U}_1 &= [K], \\
\mathcal{U}_2 &= [K]^2 - [K^2], \\
\mathcal{U}_3 &= [K]^3 - 3[K][K^2] + 2[K^3], \\
\mathcal{U}_4 &= [K]^4 - 6[K^2][K^2] + 8[K^3][K] + 3[K^4]^2 - 6[K^4].
\end{align*}
\tag{2.16}
\]
The square root in \(K\) means \((\sqrt{A})^{\mu}_{\nu}(\sqrt{A})^\nu_{\mu} = A^0_{\nu}\) and \([K]\) denotes the trace \(K^\mu_{\mu}\). Finally, \(f_{\mu\nu}\), called the reference metric, is a non-dynamical, fixed symmetric tensor introduced to construct nontrivial interaction terms in massive gravity. Then, with a gauge-fixed ansatz for the reference metric as
\[
f_{\mu\nu} = \text{diag}(0,0,a_0^2,a_0^2\sin^2 \theta), \tag{2.17}
\]
where \(a_0\) is a positive constant\(^3\)\(^4\)\(^5\)\(^6\), one can find the spherically symmetric black hole solution as
\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.18}
\]
with
\[
f(r) = 1 - \frac{2m}{r} + 2Rr + C. \tag{2.19}
\]

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\(^1\) In particular, we will call it massless when \(\tilde{m}\) is zero in this work.
Generality, one can find that the geodesic equations are reduced to

\[ \text{massive Sch. (C=0.3)} \]
\[ \text{massive Sch. (C=0, R=0)} \]
\[ \text{massive Sch. (C=−1)} \]
\[ \text{massive Sch. (C=−1.3)} \]

Then, as in the previous subsection, by considering the geodesics on the equatorial plane according to the values of \( C \) with \( C = 0 \) and \( C = 0 \). From Eqs. (2.2) and (2.21), one can easily obtain the geodesic equations as

\[ \Gamma^0_{01} = \frac{2m + R}{2(1 - \frac{2m}{r} + Rr + C)}, \quad \Gamma^1_{00} = \frac{1}{2} \left( \frac{2m}{r^2} + R \right) \left( 1 - \frac{2m}{r} + Rr + C \right), \]
\[ \Gamma^1_{11} = \frac{2m + R}{2(1 - \frac{2m}{r} + Rr + C)}, \quad \Gamma^1_{22} = -r \left( 1 - \frac{2m}{r} + Rr + C \right), \]
\[ \Gamma^1_{33} = -r \left( 1 - \frac{2m}{r} + Rr + C \right) \sin^2 \theta, \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \]
\[ \Gamma^2_{33} = -\sin \theta \cos \theta, \quad \Gamma^3_{23} = \cot \theta, \]

which are reduced to the well-known results in the Schwarzschild black hole in massless gravity in the limit of \( R = 0 \) and \( C = 0 \). From Eqs. (2.22) and (2.23), one can easily obtain the geodesic equations as

\[ \frac{dv^0}{d\tau} + \frac{2(m + Rr^2)}{r^2 - 2mr + 2Rr^3 + Cr^2} v^0 v^1 = 0, \]
\[ \frac{dv^1}{d\tau} + \frac{(m + Rr^2)}{r^4} (r^2 - 2mr + 2Rr^3 + Cr^2)(v^0)^2 - \frac{m + Rr^2}{r^2 - 2mr + 2Rr^3 + Cr^2}(v^1)^2 \]
\[ - \frac{r^2 - 2mr + 2Rr^3 + Cr^2}{r} [(v^2)^2 + \sin^2 \theta (v^3)^2] = 0, \]
\[ \frac{dv^2}{d\tau} + \frac{2}{r} v^1 v^2 - \sin \theta \cos \theta (v^3)^2 = 0, \]
\[ \frac{dv^3}{d\tau} + \frac{2}{r} v^1 v^3 + \cot \theta v^2 v^3 = 0. \]

Then, as in the previous subsection, by considering the geodesics on the equatorial plane \( \theta = \pi/2 \) without loss of generality, one can find that the geodesic equations are reduced to

\[ \frac{dv^0}{d\tau} + \frac{2(m + Rr^2)}{r^2 - 2mr + 2Rr^3 + Cr^2} v^0 v^1 = 0, \]
\[
\frac{dv^1}{d\tau} + \left(\frac{m + Rr^2}{r^4}\right) \left(v^2 - 2mr + 2Rr^3 + Cr^2\right)(v^0)^2 - \frac{m + Rr^2}{r^2 - 2mr + 2Rr^3 + Cr^2}(v^1)^2
- \frac{r^2 - 2mr + 2Rr^3 + Cr^2}{r} (v^3)^2 = 0,
\]
(2.27)
\[
\frac{dv^2}{d\tau} = 0,
\]
(2.28)
\[
\frac{dv^3}{d\tau} + \frac{2}{r} v^1 v^3 = 0.
\]
(2.29)

Making use of \(v^1 = \dot{r}\) again, one can integrate Eqs. (2.26) and (2.29) as
\[
v^0 = \frac{c_1 r^2}{r^2 - 2mr + 2Rr^3 + Cr^2},
\]
(2.30)
\[
v^3 = \frac{c_2}{r^2},
\]
(2.31)
respectively, where \(c_1\) and \(c_2\) are integration constants. It seems appropriate to comment that for the Killing vectors \(\xi^\mu = (1, 0, 0, 0)\) and \(\psi^\mu = (0, 0, 0, 1)\), two conserved quantities of \(E\) and \(L\) are given by
\[
E = -g_{\mu\nu} \xi^\mu v^\nu = f(r)v^0,
\]
(2.32)
\[
L = g_{\mu\nu} \psi^\mu v^\nu = r^2 v^3.
\]
(2.33)
Comparing these relations with Eqs. (2.30) and (2.31), we can fix the integration constants as
\[
c_1 = E, \quad c_2 = L
\]
(2.34)
in terms of the conserved quantities \(E\) and \(L\).

Finally, by letting \(ds^2 = -kd\tau^2\) in Eq. (2.18), one can obtain from Eqs. (2.30) and (2.31)
\[
v^1 = \pm \left[ E^2 - \left( k + \frac{L^2}{r^2} \right) \left( 1 - \frac{2m}{r} + 2Rr + C \right) \right]^{1/2},
\]
(2.35)
where the \((-/+\)) sign is for inward/outward motion as before. Moreover, timelike (nulllike) geodesic is for \(k = 1\) (0).

Note that in the massless limit of both \(R = 0\) and \(C = 0\), one can easily obtain the radial velocity (2.14) of the Schwarzschild black hole in massless gravity. We end up with finding geodesic equations in the Schwarzschild black hole in massive gravity.

III. TIDAL FORCE IN THE SCHWARZSCHILD BLACK HOLE

A. Tidal force in the Schwarzschild black hole in massless gravity

Now, let us consider the tidal force acting in the Schwarzschild black hole in massless gravity. First of all, let us define the geodesic deviation, or separation four-vectors \(\eta^\mu\) which denote the infinitesimal displacement between two nearby particles in free fall. Then, the equations of the geodesic deviation [1–4] are given by
\[
\frac{D^2 \eta^\mu}{D\tau^2} + R_{\nu\rho\sigma}^{\mu} v^\nu \eta^\rho v^\sigma = 0,
\]
(3.1)
where \(R_{\nu\rho\sigma}^{\mu}\) is the Riemann curvature and \(v^\mu\) is the unit tangent vector to the geodesic line.

In order to study the behavior of the separation vector in detail, we consider the timelike geodesic equation with \(L = 0\) for simplicity. We also introduce the tetrad basis describing a freely falling frame given by
\[
e_0^\mu = \left( \frac{E}{r - 2m}, -\sqrt{E^2 - 1 + \frac{2m}{r}}, 0, 0 \right),
\]
\[
e_1^\mu = \left( -\sqrt{E^2 - 1 + \frac{2m}{r}}, \frac{r}{r - 2m} E, 0, 0 \right),
\]
\[
= \left( r, r, 0, 0 \right).
\]
FIG. 2: (a) Radial tidal effect \( T_R = \frac{1}{\eta^0} \partial^0 \frac{\partial^1}{\partial t} \) and (b) angular tidal effect \( T_A = \frac{1}{\eta^i} \partial^i \frac{\partial^1}{\partial \theta} \) for the Schwarzschild black hole in massless gravity with \( m = 10 \). The dotted lines denote positions of the event horizon.

\[
e_2^\mu = \left( 0, 0, \frac{1}{r}, 0 \right),
\]
\[
e_3^\mu = \left( 0, 0, 0, \frac{1}{r \sin \theta} \right),
\]

(3.2)

satisfying the orthonormality relation

\[
e_\alpha^\mu e_\beta^\nu = \eta_\alpha^\beta,
\]

(3.3)

with \( \eta_\alpha^\beta = \text{diag}(-1, 1, 1, 1) \). The separation vector can also be expanded as

\[
\eta^\mu = e_\alpha^\mu \eta^\alpha
\]

(3.4)

with a fixed temporal component of \( \eta^0 = 0 \) [2, 4].

In the tetrad basis, the Riemann tensor can be written as

\[
R_\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta} = e_\mu^\alpha e_\nu^\beta e_\rho^\gamma e_\sigma^\delta R_\mu\nu\rho\sigma,
\]

(3.5)

so one can obtain the independent non-vanishing components of the Riemann tensor as

\[
R_{101}^0 = \frac{2m}{r^3}, \quad R_{202}^0 = \frac{m}{r^3},
\]
\[
R_{303}^1 = \frac{m}{r^3}, \quad R_{212}^1 = \frac{m}{r^3},
\]
\[
R_{313}^2 = \frac{m}{r^3}, \quad R_{323}^2 = \frac{m}{r^3}.
\]

(3.6)

Then, one can find the equations for tidal forces in the radially free fall frame as

\[
\frac{d^2 \eta_i}{d\tau^2} = \frac{2m}{r^3} \eta_i,
\]
\[
\frac{d^2 \eta_i}{d\tau^2} = -\frac{m}{r^3} \eta_i,
\]

(3.7)

(3.8)

where \( i = 2, 3 \). Radial and angular tidal effects are depicted in Fig.2 One can see that nothing happens for an observer in the radially freely falling when she is crossing the event horizon. The radial tidal effect in Fig.2(a) is always positive and as the observer approaches the singularity it diverges, or it is infinitely stretched. On the other hand, the angular tidal effect in Fig.2(b) approaches negative infinity, or the observer is infinitely compressing in the angular direction.

B. Tidal force in the Schwarzschild black hole in massive gravity

Now, let us newly consider the tidal force acting in the Schwarzschild black hole in massive gravity.
FIG. 3: (a) Radial tidal effect $T_R$ and (b) angular tidal effect $T_A$ for the Schwarzschild black hole in massive gravity with $m = 10$ and $R = 0.1$. The dotted curve is for the case of the Schwarzschild black hole in massless gravity. The dotted lines denote positions of the event horizon.

First of all, for simplicity, we consider the timelike geodesic equation with $L = 0$ as before. We also introduce the tetrad basis as

\[
e^\mu_0 = \left(\frac{E}{f(r)}, -\sqrt{E^2 - f(r)}, 0, 0\right),
\]
\[
e^\mu_1 = \left(-\sqrt{E^2 - f(r)}/f(r), E, 0, 0\right),
\]
\[
e^\mu_2 = \left(0, 0, 1/r, 0\right),
\]
\[
e^\mu_3 = \left(0, 0, 0, 1/r \sin \theta\right),
\]

(3.9)
satisfying the orthonormality relation

\[
e^\mu_\alpha e^\mu_\beta = \eta_{\dot\alpha \dot\beta},
\]

(3.10)
with $\eta_{\dot\alpha \dot\beta} = \text{diag}(-1, 1, 1, 1)$. As before, the separation vector can also be expanded as

\[
\eta^\mu = e^\mu_\dot\alpha \eta^{\dot\alpha}
\]

(3.11)
with a fixed temporal component of $\eta^0 = 0$ [2, 4].

In the tetrad basis, the independent non-vanishing components of the Riemann tensor in massive gravity can be obtained as

\[
R^\hat{\dot{\alpha}}_{101} = -\frac{f''(r)}{2}, \quad R^\hat{\dot{\alpha}}_{202} = -\frac{f'(r)}{2r},
\]
\[
R^\hat{\dot{\alpha}}_{303} = -\frac{f'(r)}{2r}, \quad R^\hat{\dot{\alpha}}_{212} = \frac{f'(r)}{2r},
\]
\[
R^\hat{\dot{\alpha}}_{313} = \frac{f'(r)}{2r}, \quad R^\hat{\dot{\alpha}}_{323} = \frac{1 - f(r)}{r^2}.
\]

(3.12)
Then, one can obtain the desired tidal forces in the radially freely falling frame as

\[
\frac{d^2 \eta^1}{d\tau^2} = -\frac{f''(r)}{2} \eta^1 = \frac{2m}{r^3} \eta^1,
\]
\[
\frac{d^2 \eta^i}{d\tau^2} = -\frac{f'(r)}{2r} \eta^i = -\left(\frac{m}{r^3} + \frac{R}{r}\right) \eta^i,
\]

(3.13)
(3.14)
where $i = 2, 3$. Thus, one can find that comparing with the case of the Schwarzschild black hole in massless gravity, the tidal effect in the radial direction is exactly the same. However, the tidal effect in the angular direction has
FIG. 4: For the Schwarzschild black hole in massless gravity, radial solutions of the geodesic deviation equation: (a) for \( \eta^1(b) = 1, \frac{d\eta^1(b)}{dr} = 0 \) with \( m = 10 \) and (b) for \( \eta^1(b) = 1, \frac{d\eta^1(b)}{dr} = 1 \), \( m = 5, 10, 15, 20 \) from bottom to top.

additional term proportional to \( R/r \) in the Schwarzschild black hole in massive gravity. Note that the tidal forces are independent of the constant term \( C \) in \( f(r) \) because these forces are obtained from \( f'(r) \) and \( f''(r) \). Radial and angular tidal forces are depicted in Fig. 3. As before, one can see that nothing happens for an observer in the radially freely falling when she is crossing the event horizon in the figure. As a result, we have shown that the massive graviton effect seems to appear only in the tidal force in the angular direction. In the next section, we will see further the massive graviton effect on the radial and angular solutions of the geodesic deviation equation which would show a variety of different behaviors of the separation vectors.

IV. GEODESIC DEVIATION EQUATIONS OF THE SCHWARZSCHILD BLACK HOLE

A. Solutions of the geodesic deviation equations of the Schwarzschild black hole in massless gravity

In this subsection, we solve the geodesic deviation equations of \( f \) and \( g \), and find the behavior of the geodesic deviation vectors of test particles freely falling into the Schwarzschild black hole in massless gravity. Then, we will compare this with the case of the Schwarzschild black hole in massive gravity in the next subsection.

First of all, making use of \( v^1 = \dot{r} = -\sqrt{E^2 - 1 + 2m/r} \), the tidal forces in Eqs. (3.7) and (3.8) can be rewritten in terms of \( r \)-derivative as

\[
2m \left( \frac{1}{r} - \frac{1}{b} \right) \frac{d^2\eta^i}{dr^2} - \frac{m}{r^2} \frac{d\eta^i}{dr} - \frac{2m}{r^3} \eta^i = 0, \tag{4.1}
\]

\[
2m \left( \frac{1}{r} - \frac{1}{b} \right) \frac{d^2\eta^i}{dr^2} - \frac{m}{r^2} \frac{d\eta^i}{dr} + \frac{m}{r^2} \eta^i = 0. \tag{4.2}
\]

Here, we are considering a body released from rest at \( r = b \) so we have \( E = (1 - 2m/b)^{1/2} \). The solution of the radial component (4.1) is given by

\[
\eta^1(r) = \sqrt{2m \left( \frac{1}{r} - \frac{1}{b} \right)} \left( d_1 + d_2 \int \frac{dr}{[2m (\frac{1}{r} - \frac{1}{b})]^{3/2}} \right), \tag{4.3}
\]

and the angular component (4.2) is

\[
\eta^i(r) = r \left( d_3 + d_4 \int \frac{dr}{r^2 \sqrt{2m (\frac{1}{r} - \frac{1}{b})}} \right), \tag{4.4}
\]

where \( d_1, \ldots, d_4 \) are constants of integration. Since we are considering a body falling from rest at \( r = b \), we find the constants of integration as

\[
d_1 = \frac{b^2}{m} \frac{d\eta^1(b)}{d\tau}, \quad d_2 = \frac{m}{b^2} \eta^1(b),
\]

\[
d_3 = \frac{1}{b} \eta^i(b), \quad d_4 = -b \frac{d\eta^1(b)}{d\tau}. \tag{4.5}
\]
FIG. 5: For the Schwarzschild black hole in massless gravity, angular solutions of the geodesic deviation equation: (a) for \( \eta^i(b) = 1, \frac{d\eta^1(b)}{dr} = 0 \) with \( m = 10 \) and (b) for \( \eta^i(b) = 1, \frac{d\eta^i(b)}{dr} = 1 \), \( m = 5, 10, 15, 20 \) from top to bottom.

FIG. 6: For the Schwarzschild black hole in massless gravity: (a) radial component for \( \eta^1(b) = 1, 50, 100, 150 \) from bottom to top with \( \frac{d\eta^1(b)}{dr} = 1, m = 10 \) and (b) angular component for \( \eta^i(b) = 1, 50, 100, 150 \) from bottom to top with \( \frac{d\eta^i(b)}{dr} = 1, m = 10 \).

Thus, the solutions are finally written as

\[
\eta^1(r) = b \sqrt{\frac{2b}{m}} \frac{d\eta^1(b)}{dr} \left( \frac{b}{r} - 1 \right)^{1/2} + \eta^1(b) \left[ \frac{3}{2} - \frac{r}{2b} + \frac{3}{4} \left( \frac{b}{r} - 1 \right)^{1/2} \cos^{-1} \left( \frac{2r}{b} - 1 \right) \right],
\]  

\( \text{(4.6)} \)

\[
\eta^i(r) = r \left[ \frac{\eta^i(b)}{b} + \sqrt{\frac{2b}{m}} \frac{d\eta^i(b)}{dr} \left( \frac{b}{r} - 1 \right)^{1/2} \right].
\]  

\( \text{(4.7)} \)

These solutions of \( \eta^1(r) \) and \( \eta^i(r) \) show how the initial radial and angular separations of two nearby geodesics are changed while falling to the Schwarzschild black hole in massless gravity. It is appropriate to comment that \( \eta^1(b) \) and \( \eta^i(b) \) are initial separation distances between two nearby geodesics at \( r = b \) to the radial and angular directions, respectively. Moreover, \( \frac{d\eta^1(b)}{dr} \) and \( \frac{d\eta^i(b)}{dr} \) are initial velocities at \( r = b \) to the radial and angular directions, respectively, which mean either exploding when they are positive, or imploding when they are negative. Here we consider two cases where one is \( \eta^i(b) \neq 0 \), \( \frac{d\eta^i(b)}{dr} = 0 \) and the other is \( \eta^i(b) \neq 0 \), \( \frac{d\eta^i(b)}{dr} \neq 0 \). In Figs. 4 and 5 (a) corresponds to the former case, and (b) to the latter case. Note that when \( \frac{d\eta^i(b)}{dr} = 0 \), \( \eta^i(r) \) become

\[
\eta^1(r) = \eta^1(b) \left[ \frac{3}{2} - \frac{r}{2b} + \frac{3}{4} \left( \frac{b}{r} - 1 \right)^{1/2} \cos^{-1} \left( \frac{2r}{b} - 1 \right) \right],
\]  

\( \text{(4.8)} \)

\[
\eta^i(r) = \frac{\eta^i(b)}{b} r,
\]  

\( \text{(4.9)} \)

respectively, which mean that the separation distance goes to infinity for the radial component and goes to zero for the angular one as the body is falling to the black hole, a process known well as spaghettification.

On the other hand, when \( \frac{d\eta^i(b)}{dr} \neq 0 \), the radial components show the similar behaviors as in Fig. 4(b), however the separation distances of the angular components start to increase, reach a peak, then decrease to zero, as the
body falls to the black hole, which is shown in Fig. 5(b). It is interesting to note that the separation distance of the angular component is smaller as the mass of the black hole is larger. In Fig. 6, we have drawn the radial and angular components by varying the initial distances of $\eta^i(b)$ at $r = b$. It is also interesting to note that the angular component is more deformed as $\eta^i(b)$ is larger.

B. Solutions of the geodesic deviation equations of the Schwarzschild black hole in massive gravity

For the geodesic deviation equations (3.13) and (3.14) of the Schwarzschild black hole in massive gravity, the tidal forces can be rewritten in terms of $r$-derivative as

$$[E^2 - f(r)] \frac{d^2 \eta^i}{dr^2} - \frac{f'(r)}{2} \frac{d\eta^i}{dr} + \frac{f''(r)}{2} \eta^i = 0, \quad (4.10)$$

$$[E^2 - f(r)] \frac{d^2 \eta^i}{dr^2} - \frac{f'(r)}{2} \frac{d\eta^i}{dr} + \frac{f''(r)}{2r} \eta^i = 0. \quad (4.11)$$

The solution of the radial component (4.10) is given by

$$\eta^1(r) = d_1 \sqrt{E^2 - f(r)} + d_2 \sqrt{E^2 - f(r)} \int \frac{dr}{[E^2 - f(r)]^{3/2}}, \quad (4.12)$$

and the angular component (4.11) is

$$\eta^i(r) = r \left( d_3 + d_4 \sqrt{ \frac{2m}{b} - \frac{bR}{m} } \right), \quad (4.13)$$

where $d_1, \ldots, d_4$ are constants of integration.

However, at this stage, one finds that the integrals cannot be analytically carried out with the metric (2.19) so that we expand the integrand to the power of $R$ by noting that the constant terms of $C$ are cancelled out. Then, by integrating it out term by term, we have obtained up to the $R^4$ terms as follows

$$\eta^1(r) = d_1 \sqrt{ \frac{2m}{b} - \frac{bR}{m} } \left( \frac{b}{r} - 1 \right)^{1/2} \left( \frac{bR}{m} \right)^{1/2}$$

$$+ d_2 \frac{b}{2m} \left( \frac{b}{r} - 1 \right)^{1/2} \left[ 2bg_1(r) + \frac{3b}{2}g_2(r)(\frac{b}{r} - 1)^{1/2} \cos^{-1}\left( \frac{2r}{b} - 1 \right) \right], \quad (4.14)$$

$$\eta^i(r) = d_3 r - d_4 \sqrt{ \frac{2m}{b} - \frac{bR}{m} } \left[ g_3(r) \left( \frac{b}{r} - 1 \right)^{1/2} - g_4(r) \cos^{-1}\left( \frac{2r}{b} - 1 \right) \right], \quad (4.15)$$

where

$$g_1(r) = \frac{3}{2} \frac{r}{2b} + \left( \frac{3r^2}{2b} + \frac{15r}{16} - \frac{45b}{16} \right) \left( \frac{bR}{m} \right) - \left( \frac{5r^3}{16b} + \frac{35r^2}{64} + \frac{175b}{128} - \frac{525b^2}{128} \right) \left( \frac{bR}{m} \right)^2$$

$$+ \left( \frac{105r^4}{256} + \frac{735br^2}{1024} + \frac{3675b^2r}{2048} - \frac{11025b^3}{2048} \right) \left( \frac{bR}{m} \right)^3$$

$$- \left( \frac{63r^5}{256} + \frac{693br^3}{1024} + \frac{2079br^2}{4096} + \frac{14553b^2r^2}{16384} + \frac{72765b^3r}{32768} - \frac{218295b^4}{32768} \right) \left( \frac{bR}{m} \right)^4 + \mathcal{O}(R^5),$$

$$g_2(r) = 1 - \frac{15b}{8} \left( \frac{bR}{m} \right) + \frac{175b^2}{64} \left( \frac{bR}{m} \right)^2 - \frac{3675b^3}{1024} \left( \frac{bR}{m} \right)^3 + \frac{72765b^4}{16384} \left( \frac{bR}{m} \right)^4 + \mathcal{O}(R^5),$$

$$g_3(r) = 1 + \frac{3br}{16} \left( \frac{bR}{m} \right)^2 - \left( \frac{5br^2}{64} + \frac{15b^2r}{128} \right) \left( \frac{bR}{m} \right)^3 + \left( \frac{3br^3}{768} + \frac{175b^2r^2}{3072} + \frac{175b^3r}{2048} \right) \left( \frac{bR}{m} \right)^4 + \mathcal{O}(R^5),$$

$$g_4(r) = \frac{b}{4} \left( \frac{bR}{m} \right) - \frac{3b^2}{32} \left( \frac{bR}{m} \right)^2 + \frac{15b^3}{256} \left( \frac{bR}{m} \right)^3 - \frac{175b^4}{4096} \left( \frac{bR}{m} \right)^4 + \mathcal{O}(R^5). \quad (4.16)$$
is smaller for a given $m = 5$. Here, the constants of integration are given by $d_1 = \frac{b^2}{m} \left( 1 + \frac{b^2 R}{m} \right)^{-1} \frac{d\eta^1(b)}{d\tau}$, $d_2 = \frac{m}{b^2} \left( 1 + \frac{b^2 R}{m} \right)^{-1/2} g_1^{-1}(b)\eta^1(b)$, $d_3 = \frac{1}{b}\eta^1(b)$, $d_4 = -b\frac{d\eta^1(b)}{d\tau}$.

Thus, the solutions are finally written as

\[
\begin{align*}
\eta^1(r) &= b\sqrt{\frac{2b}{m}} \left( 1 + \frac{b^2 R}{m} \right)^{-1} \frac{d\eta^1(b)}{d\tau} \left( 1 + \frac{b R r}{m} \right)^{1/2} \left( \frac{b}{r} - 1 \right)^{1/2} \\
& \quad + \left( 1 + \frac{b^2 R}{m} \right)^{-1/2} g_1^{-1}(b)\eta^1(b) \left( 1 + \frac{b R r}{m} \right)^{1/2} \left[ g_1(r) + \frac{3}{4} g_2(r) \left( \frac{b}{r} - 1 \right)^{1/2} \cos^{-1} \left( \frac{2r}{b} - 1 \right) \right], \\
\eta^2(r) &= r \left[ \frac{\eta^1(b)}{b} + \sqrt{\frac{2b}{m}} \frac{d\eta^1(b)}{d\tau} \left( g_3(r) \left( \frac{b}{r} - 1 \right)^{1/2} - g_4(r) \cos^{-1} \left( \frac{2r}{b} - 1 \right) \right) \right].
\end{align*}
\]

In Fig. 7 we have drawn the radial components of the separation vectors by varying $R$ and the mass of the black holes, respectively, when $\eta^1(b) \neq 0$ and $\frac{d\eta^1(b)}{d\tau} = 0$. As can be seen in the figure, massive gravitons keep tightening two nearby geodesics either as the graviton’s mass $\tilde{m}$ embodied in $R$ is bigger in Fig. 7(a) or as the black hole’s mass $m$ is smaller for a given $R$ in Fig. 7(b). It is also interesting to see that when $\frac{d\eta^1(b)}{d\tau} = 0$ the angular component is...
FIG. 9: For the Schwarzschild black hole in massive gravity: (a) radial solutions of the geodesic deviation equation for \( \eta^1(b) = 1, 50, 100, 150 \) from bottom to top with \( \frac{d\eta^1(b)}{dr} = 1 \), \( m = 10 \) and \( R = 0.1 \) and (b) angular solutions of the geodesic deviation equation for \( \eta^i(b) = 1, 50, 100, 150 \) from bottom to top with \( \frac{d\eta^i(b)}{dr} = 1 \), \( m = 10 \) and \( R = 0.001 \).

FIG. 10: For the Schwarzschild black hole in massive gravity, angular solution of the geodesic deviation equation by varying \( R \) with \( \eta^i(b) = 1, \frac{d\eta^i(b)}{dr} = 1 \) and \( m = 10 \).

linearly shrink to be zero as the body approaches the singularity by

\[
\eta^i(r) = \frac{\eta^i(b)}{b} r, \tag{4.20}
\]

regardless of the black hole’s mass as shown in Fig. 8(a). Moreover, when \( \frac{d\eta^i(b)}{dr} \neq 0 \) with a fixed \( R \), two nearby geodesics are more easily separated as the mass of the black hole is smaller, then reach a peak, decrease to zero, as the body falls to the black hole, as seen in Fig. 8(b). In Fig. 8 we have drawn the radial and angular solutions of the geodesic equations by varying the initial separations \( \eta^i(b) \) with fixed other parameters, which are similar to the figures in Fig. 6. In Fig. 10 we have drawn the angular solution of the geodesic deviation equation by varying the massive graviton parameter \( R \) with fixed other parameters. This shows that as \( R \) is increased, the distance between the nearby geodesics in the angular direction is increased near the departure point of \( r = b \).
V. DISCUSSION

In summary, we have studied the geodesics and tidal effects produced in the spacetime of the Schwarzschild black hole in massive gravity, which has two additional mass parameters of $R$ and $C$ due to massive gravitons, comparing with the known results in massless gravity. As a result, we have newly found that massive gravitons affect the angular component of the tidal force giving an additional term proportional to only $R/r$ coming from $f''(r)$, while the radial component of the tidal force in massive gravity is the same with the one in massless gravity, which is derived from $f''(r)$.

In order to see its implication, we have further investigated the solutions of the geodesic deviation equations which shows the behavior of two nearby geodesics freely falling into the Schwarzschild black hole. As for the Schwarzschild black hole in massless gravity, with boundary conditions of $\eta^\alpha(b) \neq 0$ and $\eta^\alpha(b) d\eta^\alpha = 0$ ($\alpha = 1, 2, 3$), the radial component gets infinitely stretched and the angular component is compressed to zero by the tidal force as shown in Fig. 4(a) and 5(a). However, with exploding boundary conditions of $\eta^\alpha(b) \neq 0$ and $\eta^\alpha(b) d\eta^\alpha \neq 0$, the separation distances of the angular component start to get stretched, reach a peak and then get compressed as shown in Fig. 5(b), while the radial components behave similarly to the case of $\eta^\alpha(b) \neq 0$ and $\eta^\alpha(b) d\eta^\alpha = 0$ as in Fig. 4(b).

On the other hand, as for the Schwarzschild black hole in massive gravity, with boundary conditions of $\eta^\alpha(b) \neq 0$ and $\eta^\alpha(b) d\eta^\alpha = 0$, the radial components keep tightening and after passing the event horizons get abruptly infinitely stretched due to the massive gravitons, as shown in Fig. 7(a), while the angular components are unaffectedly compressed as in Fig. 8(a). With exploding boundary conditions of $\eta^\alpha(b) \neq 0$ and $\eta^\alpha(b) d\eta^\alpha \neq 0$, the radial components also show the abrupt stretch after passing the event horizons, while the angular components are more deformed as the mass of a black hole is smaller. As a result, in the Schwarzschild black hole in massive gravity, we have newly shown that the massive gravitons keep tightening radial components of two nearby geodesics as $R$ is bigger. Moreover, the massive gravitons make angular components of two nearby geodesics more deformed as the mass of the black holes $m$ is smaller.

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