GROUP UNITARY REPRESENTATIONS EXTENDING THE LEFT REGULAR REPRESENTATION OF AN ALMOST NORMAL SUBGROUP AND EQUIVARIANT SPLITTING OF THE ASSOCIATED VON NEUMANN ALGEBRA

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ABSTRACT. Let $G$ be a discrete countable group, and let $\Gamma$ be an almost normal subgroup. In this paper we analyze the classification of (projective) unitary representations $\pi$ of $G$ on $l^2(\Gamma)$, extending the left regular representation of $\Gamma$. Let $K$ be the canonical profinite completion of the subgroup $\Gamma$ and let $S$ be the corresponding profinite extension of $G$. Let $M$ be the reduced, von Neumann algebra, crossed product $L(G \rtimes L^\infty(S, \nu))$, with respect Haar measure on $S$. Let $G^{op}$ be the group $G$ with opposite multiplication acting canonically on $M$. We establish a correspondence between unitary representations as above and $G^{op}$-equivariant splittings of the crossed product von Neumann algebra $M$.

Let $P = L(\Gamma \rtimes L^\infty(K, \mu))$ be the von Neumann algebra, reduced crossed product. Let $\rho_{\Gamma\backslash G}$ be the right quasi-regular representation of $G^{op}$ on $l^2(\Gamma\backslash G)$. Then, given a representation $\pi$ as above, there exists a representation $\alpha_g$ of $G$ into the automorphism group of $P$ (unique up to cocycle perturbation), such that, $G^{op}$ equivariantly, we have that $M \cong P \otimes B(l^2(\Gamma\backslash G))$, where on the left side of the above isomorphism, the group $G^{op}$ acts trivially on $L(G)$ and it acts by right translations, on $S$. On the right side we have the action of $G^{op}$ defined by $g \to \alpha_g \otimes \text{Ad} \rho_{\Gamma\backslash G}(g)$, $g \in G$.

INTRODUCTION AND DEFINITIONS

Let $G$ be a discrete group and let $\Gamma$ be an almost normal subgroup. In [Ra1], [Ra2] we considered unitary representations $\pi$ of $G$ with the property that $\pi$ restricted to $\Gamma$ is unitarily equivalent (respectively a finite multiple, see

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[Ra1]) to the left regular representation of \( \Gamma \). Such unitary (projective) representations arise usually from the square summable unitary representations of a semisimple Lie group \( \overline{G} \), containing \( G \) as a dense subgroup, and so that \( \Gamma \) is a lattice (see [GHJ]).

In the paper [Ra1] (see also [Ra3]) we proved that the matrix coefficients of the representation of \( \pi \), with respect to a vector that is also a trace vector for the representation \( \pi|\Gamma \), have remarkable algebraic properties. More precisely, this properties correspond to the construction of a canonical representation of the Hecke algebra \( H_0 = \mathbb{C}(\Gamma/G\backslash \Gamma) \) of double cosets of \( \Gamma \) in \( G \).

This representation in turn may be used to give an equivalent representation for the Hecke operators corresponding to \( \Gamma \)-invariant vectors for the diagonal unitary representation \( \pi \otimes \pi \) of \( G \). In turn, by Berezin’s quantization, this last representation corresponds to classical Hecke operators, in specific cases (see [Ra1], [Ra3]).

In this paper we are analyzing the classification problem of such representations. We prove that the Hecke algebra representations constructed using the family of matrix coefficients are in one to one correspondence with the splitting of the canonical action of \( G \) onto the crossed product von Neumann algebra \( \mathcal{M} \) corresponding to the action of \( G \) on a canonical profinite extension \( S \) of \( G \) (that is \( \mathcal{M} = L(G \rtimes L^\infty(S, \nu)) \)). For \( G = PGL_2(\mathbb{Z}_{[p]}), \Gamma = PSL_2(\mathbb{Z}) \) this profinite extension is \( S = PGL_2(\mathbb{Q}_p) \).

In particular, one obtains that the unitary representation \( \alpha \) of \( G \), associated to the Hecke operators for the \( \Gamma \)-invariant vectors for the diagonal, unitary representation \( \pi \otimes \pi \), is a factor of the representation of \( G \) on \( \mathcal{M} \). Moreover the representations \( \pi \) are classified by the 1-cohomology group of \( G \) with respect to the representation \( \alpha \).

We describe bellow this construction in more precise terms. For \( \sigma \) in \( G \) let \( \Gamma_\sigma = \sigma \Gamma \sigma^{-1} \cap \Gamma \) be the subgroup of finite index of \( \Gamma \) associated to \( \sigma \). Let \( \mathcal{G} \) be the downward directed lattice of finite index subgroups of \( \Gamma \), generated by the subgroups \( \Gamma_\sigma, \sigma \in G \).

Let \( K \) be the profinite completion of \( \Gamma \), and let \( S \) be the corresponding locally profinite extension of \( G \), obtained as the disjoint union of \( K \sigma K \), where \( \sigma \in G \) runs over a system of representatives for double cosets of \( \Gamma \) in \( G \).

We assume for simplicity that the corresponding Haar measures \( \mu, \nu \) on \( K \) and respectively \( S \) are bivariant with respect the left and right convolutions (thus we are assuming that for all \( \sigma \in G \), the subgroups \( \Gamma_\sigma \) and \( \Gamma_{\sigma^{-1}} \) have equal finite indices).
Using the canonical von Neumann algebra construction, of the reduced crossed product with respect an invariant measure, we construct the von Neumann algebras \( M = L(G \rtimes L^\infty(S, \nu)) \) and \( P = L(\Gamma \rtimes L^\infty(K, \mu)) \). If a group cocycle is present in the unitary representation considered below, (as in [Ra1], see also [Ra3]), then we will consider the skew version of the crossed product.

For simplicity we assume that the groups \( G, \Gamma \) have infinite, non-trivial, conjugacy classes, so that the corresponding reduced, von Neumann, crossed product algebras, are factors with (eventually infinite) traces (see e.g. [Sa]). We let \( G^{op} \) be the group corresponding to \( G \), with the opposite multiplication.

We consider the right product action of \( G^{op} \) on \( S \). The Koopmann unitary representation of \( G^{op} \) on \( L^\infty(S, \nu) \), induces an action \( (\beta_g)_{g \in G} \) defined on \( G^{op} \) with values into the automorphism group \( \text{Aut}(L(G \rtimes L^\infty(S, \nu))) \) of the von Neumann algebra \( M \). Here \( G^{op} \) acts trivially (identical) on \( L(G) \).

We also assume that the actions of \( G \), respectively \( \Gamma \), on \( S \) and respectively on \( K = \overline{\Gamma} \), are ergodical. Consequently the corresponding actions are outer ([Co]) and hence the von Neumann algebras \( M \) and \( P \) are semifinite factors.

Our main assumption is the existence of a (eventually projective) unitary representation \( \pi : G \to U(H_\pi) \), such that \( \pi |_\Gamma \) is unitarily equivalent to the left regular representation of \( \Gamma \).

Note that the case of a projective, unitary representation, will require to take the skewed, left regular representation (see e.g. [Ra1], or the N. Ozawa’s notes [Ra3]).

The more general case when \( \dim_{\pi(\Gamma)} H_\pi \) is just finite, instead of being equal to 1 as above, (see [(GHJ)] for the definition of the corresponding dimension function) may be treated similarly using the results [Ra 2]; this case will be deferred to a subsequent paper.

Let \( H_0 = C(\Gamma \setminus G/\Gamma) \) be the Hecke \( C^* \)-algebra of double cosets (see e.g. [BC]). We let \( H_0 = C(\Gamma \setminus G/\Gamma) \) act canonically on left and right cosets by right (and respectively left multiplication). Let \( H \subseteq B(l^2(\Gamma \setminus G)) \), the uniform norm closure of \( H_0 \), be the reduced Hecke von Neumann algebra associated to \( \Gamma \subseteq G \) (see e.g. [BC]). In [Ra1] we constructed, using the matrix coefficients of the representation \( \pi \), a representation \( t : H \to L(G) \), such that \( t([\Gamma \sigma \Gamma]) \) belongs to \( l^2(\Gamma \sigma \Gamma) \cap L(G) \). Thus, the group algebra support of \( t([\Gamma \sigma \Gamma]) \) is contained in the closure of the span of the group elements belonging to \( \Gamma \sigma \Gamma \), for any double coset \( [\Gamma \sigma \Gamma] \).
We proved in [Ra1] that the representation $t$ extends to a larger $\ast$-representation of the operator system

$$\mathcal{SO} = \mathcal{SO}(\Gamma, G) = \mathbb{C}(G/\Gamma) \otimes_{\mathbb{C}(\Gamma \backslash G/\Gamma)} \mathbb{C}(\Gamma \backslash G).$$

Here $\mathcal{SO}$ is canonically identified to the vector space

$$\mathbb{C}(\sigma_1 \Gamma \sigma_2 \mid \sigma_1, \sigma_2 \in G) / \mathcal{J},$$

where $\mathcal{J}$ is the subspace generated by the identities of the form

$$\sum_j [\sigma_j^1 \Gamma \sigma_j^2] = \sum_s [\mu_s^1 \Gamma \mu_s^2],$$

that hold true, whenever the following equality of disjoint unions holds true:

$$\bigcup_j \sigma_j^1 \Gamma \sigma_j^2 = \bigcup_s \mu_s^1 \Gamma \mu_s^2,$$

for all $\sigma_j^1, \sigma_j^2, \mu_s^1, \mu_s^2 \in G$.

In the paper [Ra1] (see also [Ra3]) we proved that the above representation extends to a representation $t : \mathcal{SO} \to L(G)$, associated to the matrix coefficients of the representation $\pi$. We identify $[\sigma \Gamma] \otimes [\Gamma]$ with $[\sigma \Gamma]$; then

$$t(\sigma \Gamma)^* = t(\Gamma \sigma^{-1}), \sigma \in G,$$

and

$$t(\sigma_1 \Gamma) t(\Gamma \sigma_2) = t(\sigma_1 \Gamma \sigma_2), \sigma_1, \sigma_2 \in G.$$

We proved in [Ra1], see also [Ra2], [Ra3] that the existence of the unitary representation $\pi$ is equivalent to the existence of a representation $t$, with the above properties, of the operator system $\mathcal{SO}$. The precise formula, for the representation $t$, is as follows: let 1 be a trace vector in $H_\pi$ for $\Gamma$. For $A$ a subset of $G$, one defines

$$t^A = t(A) = \sum_{\theta \in A} \langle \pi(\theta) 1, 1 \rangle \theta.$$

The reason for taking the conjugate is that (as shown in [Ra3]) the representation $t$ naturally lives in an extension of the commutant $\pi(\Gamma)'$ which is in turn isomorphic to $R(G)$, the von Neumann algebra of right convolutors. To identify that expression with $L(G)$ one has to take the conjugate. This also works when the representation is projective ([Ra1], [Ra3]).

In this paper we prove an equivalent characterization of the previous representation $t$. We identify the cosets of $K$ with the cosets of $\Gamma$, via the obvious correspondence, obtained by taking the closure in the profinite completion.
Let \( L^\infty(\Gamma \backslash G) \), \( L^\infty(G/\Gamma) \) be the algebras of bounded, left, and respectively right \( \Gamma \)-invariant functions, that is the algebras generated by the characteristic functions of left (respectively right) cosets of \( G \) by \( \Gamma \). Then, these algebras may be identified with the subalgebras of \( L^\infty(S, \nu) \), generated by translations by elements in \( G \), of the characteristic function of the cosets of \( K \), viewed as closed subsets of \( S \). We denote by \( L^2(\Gamma \backslash G) \), \( L^2(G/\Gamma) \) the corresponding Hilbert spaces, and denote by \( \rho\Gamma/G \), \( \lambda_{G/\Gamma} \), the corresponding quasi-regular, unitary representations of \( G \).

We prove that an alternative way to characterize the representations \( \tau \) is the fact that the following family, belonging to the crossed product algebra \( \mathcal{M} \), is a \( \Gamma_{op} \)-equivariant matrix unit:

\[
\{ v_{\Gamma_{\sigma_1}, \Gamma_{\sigma_2}} \Gamma_{\sigma_1}, \Gamma_{\sigma_2} \in \Gamma \backslash G \} = \{ \chi_{\Gamma_{\sigma_1}} L^2(\Gamma \backslash G) \Gamma_{\sigma_2} \chi_{\Gamma_{\sigma_2}} \Gamma_{\sigma_1}, \Gamma_{\sigma_2} \in \Gamma \backslash G \}.
\]

The \( \Gamma_{op} \)-equivariance should hold true with respect to the \( \Gamma_{op} \) right unitary representation \( \rho\Gamma_{\Gamma} \) of \( \Gamma_{op} \) into \( L^2(\Gamma \backslash G) \).

Using the existence of such a matrix unit, we prove that the von Neumann algebra \( \mathcal{M} \) is \( \Gamma_{op} \)-equivariantly isomorphic to \( \chi_K \mathcal{M} \chi_K \otimes B(l^2(\Gamma \backslash G)) \) with respect to an induced action of \( \Gamma_{op} \), of the form \( \alpha_g \otimes \text{Ad} \rho\Gamma_{\Gamma}(g) \), \( g \in \Gamma_{op} \). Moreover it is clear that \( \chi_K \langle L(G \rtimes L^\infty(S, \nu)) \rangle \chi_K \) is in fact isomorphic to \( L(\Gamma \rtimes L^\infty(K, \mu)) \) (the unit of the last algebra is identified to \( \chi_K = \chi_{\Gamma} \)).

On the other hand, the unitary action \( \text{Ad} \pi(\sigma), \sigma \in \Gamma \), on \( B(H_\pi) \), clearly restricts to the upward directed, union of commutants \( \bigcup_{\Gamma_{\sigma_1}} \pi(\Gamma_{\sigma_1})' \). Since all these commutant von Neumann algebras algebras are II\(_1\), factors, it follows that, after renormalizing the trace, and taking the inductive limit of the corresponding type II\(_1\) factors, that the actions \( \text{Ad} \pi(\sigma), \sigma \in \Gamma \), induce an action of \( \Gamma \) onto the II\(_1\) factor completion of the upward directed, inductive limit,

\[
\bigcup_{\{e\} \in \Gamma_{\sigma_1}, \Gamma_{\sigma_1} \in \mathcal{U}} \pi(\Gamma_{\sigma_1})'.
\]

This latest algebra is obviously isomorphic to the simultaneous Jones’ basic construction for all the inclusions \( \Gamma_\sigma \subseteq \Gamma \), \( \sigma \in \Gamma \).

The simultaneous Jones’ basic construction is clearly isomorphic to the type II\(_1\) factor

\[
R(\Gamma \rtimes L^\infty(K, \mu)),
\]

which, by taking the conjugate, is isomorphic to

\[
L(\Gamma \rtimes L^\infty(K, \mu)).
\]
Note that this is the same von Neumann algebra, as the one considered in ([OP]), where in the end of the paper, the unicity of the Cartan subalgebra \( L_\infty(K, \mu) \) was proved.

We will prove that the action \( \alpha_g \) into the automorphism group of the type \( \text{II}_1 \) factor \( L(\Gamma \rtimes L_\infty(K, \mu)) \), that we are constructing in Theorem 1 (and which gives the factorization), coincides with the action described above, induced by the inductive limit of the actions \( \text{Ad} \pi(\sigma), \sigma \in G \). Consequently, it will follow that the Hecke operators representation induced by the representation \( \alpha_g \) on \( L(\Gamma \rtimes L_\infty(K, \mu)) \) (having as \( \Gamma \)-fixed vectors the algebra \( L(\Gamma) \)) are the same as the Hecke operators as the Hecke operators in Hecke algebra representation constructed in [Ra1].

More precisely if we use the canonical, von Neumann conditional expectation (see e.g. [Sa]), \( E_{L_\infty(K, \mu)}^{L(\Gamma \rtimes L_\infty(K, \mu))} \) from \( L(\Gamma \rtimes L_\infty(K, \mu)) \) onto \( L(\Gamma) \); it follows that the Hecke algebra representation on \( L(\Gamma) \) associated to \( \pi, G, \Gamma \) as in [Ra1] is the same, up to unitary equivalence, as the representation associating to a double coset \( \Gamma \sigma \Gamma \) in \( G \), the completely positive map

\[
\Psi_{\Gamma \sigma \Gamma}(x) = E_{L(\Gamma)}^{L(\Gamma \rtimes L_\infty(K, \mu))}(\alpha_{\sigma}(x)), \quad x \in L(\Gamma), \sigma \in G.
\]

Note that the representation \( \alpha_g, g \in G \) of \( G \) (with \( \alpha_\gamma \) leaving \( L(\Gamma) \) fixed, for \( \gamma \) in \( \Gamma \)), into the automorphism group of \( L(\Gamma \rtimes L_\infty(K, \mu)) \), which determines the Hecke algebra representation, is a factor of the representation of \( G^{\text{op}} \) on \( L(G \rtimes L_\infty(\mathcal{S}, \nu)) \).

Thus the Hecke algebra representation constructed in [Ra1], is in fact induced by an action of \( G^{\text{op}} \) on \( L(\Gamma \rtimes L_\infty(K, \mu)) \), which in turn is a tensor product factor action in the action of \( G^{\text{op}} \) on \( L(G \rtimes L_\infty(\mathcal{S}, \nu)) \).

Moreover, unicity of the \( G \)-equivariant matrix units up to unitary perturbation by a 1-group cocycle proves that the action \( (\alpha_g)_{g \in G} \) is unique up to cocycle perturbation.

1. Main results

Using the previous definitions, we prove the main results of the paper: we prove the tensor splitting of the representation of \( G^{\text{op}} \) on \( L(G \rtimes L_\infty(\mathcal{S}, \nu)) \). This is realized in the following theorem, in which we present four equivalent statements, which can be used to classify the unitary representations \( \pi \) with the properties described in the introduction.
In particular, as we remark in the notes before the proof of the theorem, this shows that the Hecke operators acting on $\Gamma$ invariant vectors for the diagonal representation $\pi \otimes \pi$ are obtained from a representation $\alpha$ of $G$ into the automorphism group of $P = L(\Gamma \ltimes L^\infty(K, \mu))$. In particular cases, these are the classical Hecke operators on Maass forms (see [Ra1]). The representation $\alpha$ is a factor in a canonical representation of $G$ (the other factor being the right quasi-regular representation of $G$ on the space of left cosets of $\Gamma$ in $G$.

**Theorem.** Assume that $\Gamma \subseteq G$ is a pair consisting of a discrete group $G$ and an almost normal subgroup $\Gamma$, both assumed to have infinite, non-trivial, conjugacy classes. Let $G$ be the downward directed class of subgroups of $\Gamma$, generated by $\Gamma_\sigma = \sigma \Gamma \sigma^{-1} \cap \Gamma$, $\sigma \in G$, and let $(K, \mu)$ be the corresponding profinite completion of $\Gamma$. Let $(S, \nu)$ be the corresponding profinite extension of $\Gamma$, obtained as the disjoint union of $K \sigma K$, where $\sigma$ runs through a system of representatives for double cosets of $\Gamma$ in $G$. We denote by respectively $\nu$, $\mu$ the corresponding Haar measures on $K$ and respectively $S$. We also assume that $G$ acts ergodically on $S$.

We assume that for all $\sigma \in G$ the subgroups $\Gamma_\sigma$, $\Gamma_{\sigma^{-1}}$ have equal indices. Hence the Haar measures on $S$, $K$ are bivariant. Consequently, the reduced, von Neumann algebra crossed product factors $\mathcal{M} = L(G \ltimes L^\infty(S, \nu))$ and $P = L(\Gamma \ltimes L^\infty(K, \mu))$ are type $II_\infty$ (respectively $II_1$) factors.

The following statements are then equivalent:

1) There exists a (projective) unitary representation $\pi : G \to H$ such that the restriction of $\pi$ to $\Gamma$ is unitarily equivalent to the left regular representation. In the case when a cocycle is present in the unitary representation, then in the definitions of $\mathcal{M}$ and $P$ we take the skewed, crossed product von Neumann algebras. The hypothesis also implies (see [Ra1]) that $[\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma_{\sigma^{-1}}]$ for all $\sigma \in G$.

2) There exists a $*$-algebra representation of the operator system $SO$ as described in the introduction, $t : SO \to L(G)$ such that $t^\Gamma = t([\Gamma]) = \text{Id}$, $t^\Gamma\sigma = t(\Gamma\sigma) \in L(G) \cap L^2(\Gamma\sigma)$.

3) There exists a unitary representation, denoted by $\sigma \to u(\sigma), \sigma \in G$, of the group $G^{op}$ into the reduced, crossed product von Neumann algebra $\chi_K L((G \times G^{op}) \ltimes L^\infty(S, \mu)) \chi_K$, such that $u(\sigma)$ is of the form

$$\chi_K(t^{\Gamma\sigma\Gamma} \otimes \sigma^{-1})\chi_K,$$

for a selfadjoint element $t^{\Gamma\sigma\Gamma}$ in $L(G) \cap L^2(\Gamma\sigma\Gamma)$, which only depends on the coset $\Gamma\sigma\Gamma$. It will automatically follow that the map $[\Gamma\sigma\Gamma] \to t^{\Gamma\sigma\Gamma}$ extends to a $*$ representation of the Hecke algebra $C(\Gamma \backslash G/\Gamma)$. 

4) Let $\beta_g : G \to \text{Aut}(M)$ be the canonical representation of $G^{op}$ into the automorphism group of $M$, which acts by leaving $L(G)$ invariant and acts by composition with right translation on $L^\infty(S, \nu)$. There exists a $G^{op}$-equivariant matrix unit $(v_{\Gamma \sigma_1, \Gamma \sigma_2})_{\Gamma \sigma_1, \Gamma \sigma_2 \in \Gamma \setminus G}$ such that $v_{\Gamma \sigma_1, \Gamma \sigma_2} = \chi_{\Gamma \sigma_1} \in L^\infty(S, \nu)$, for $\sigma \in G$. Here, the $G^{op}$ equivariance condition means that for all $g \in G$ and all $\Gamma \sigma_1, \Gamma \sigma_2 \in \Gamma \setminus G$ we have that

$$\beta_g(v_{\Gamma \sigma_1, \Gamma \sigma_2}) = v_{\Gamma \sigma_1 g, \Gamma \sigma_2 g}.$$ 

In particular, if condition 4 holds true, then the factor $M$ is $G^{op}$ equivariantly isomorphic to $\chi_K \mathcal{M}_{\chi_K} \otimes B(l^2(\Gamma \setminus G))$.

Then $G^{op}$ acts on $\chi_K \mathcal{M}_{\chi_K} \otimes B(l^2(\Gamma \setminus G))$ by the automorphism $\alpha_g \otimes \text{Ad}_r_{\Gamma \setminus G}(g)$, $g \in G^{op}$.

Before proving the theorem we make a few notes. The $II_1$ factor $P = \chi_K \mathcal{M}_{\chi_K}$ is isomorphic to $L(\Gamma \rtimes L^\infty(K, \mu))$. Also note that $\alpha|_{\Gamma^{op}}$ acts as the identity on $L(\Gamma) \subseteq P$. Thus, in this representation the subfactor $L(\Gamma)$ is the set of fixed points for the action $\alpha|_{\Gamma}$ of $\Gamma^{op}$ on $P$.

If the above properties hold true then the $*$ algebra representation

$$\mathbb{C}(\Gamma \setminus G/\Gamma) \ni [\Gamma \sigma \Gamma] \to \Psi_{[\Gamma \sigma \Gamma]},$$

with values in to the completely positive maps on $\pi(\Gamma)' \cong R(\Gamma) \cong L(\Gamma)$, constructed in [Ra1], is unitarily equivalent to the completely positive maps $\Psi_{\Gamma \sigma \Gamma}$ on $L(\Gamma)$ defined by the formula

$$L(\Gamma) \ni x \to E^L_{L(\Gamma)}(\Gamma \times L^\infty(K, \mu))(\alpha_g(x)),$$

for $g \in \Gamma \sigma \Gamma$ (since $\alpha_\nu$ is the identity on $L(\Gamma)$ it follows that the definition of $\Psi_{\Gamma \sigma \Gamma}$ is independent of the choice of $g$ in $\Gamma \sigma \Gamma$).

Consequently, the properties of the Hecke operators associated to the (projective) unitary representation $\pi$ and $\pi \otimes \overline{\pi}$, constructed in [Ra1], depend on the ergodic properties of the $G \times G^{op}$ action on $S$.

**Proof.** First note that by construction, the diagonal algebra $l^\infty(\Gamma \setminus G)$ in the type I algebra $B(l^2(\Gamma \setminus G))$ associated to the $G^{op}$ equivariant matrix unit, coincides with the diagonal algebra $l^\infty(\Gamma \setminus G)$ on $L^\infty(S, \nu)$ spanned by the cosets $\chi_{\Gamma \sigma}$, $\sigma \in G$. 

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The equivalence of (1) and (2) has been proved in [Ra1].
To prove that (4) implies (2) we proceed as follows. For $\sigma$ in $G$ let

$$X^{\Gamma_\sigma}\Gamma = \sum_{\Gamma_1, \Gamma_2 \in \Gamma \setminus G} v_{\Gamma_1, \Gamma_2},$$

where the sum runs over all cosets $\Gamma_1, \Gamma_2$ in $\Gamma \setminus G$ such that $\Gamma_1 \Gamma_2^{-1} \Gamma = [\Gamma \sigma \Gamma]$.

By the $G^{\text{op}}$ equivariance of the matrix unit $(v_{\Gamma_1, \Gamma_2})_{\Gamma_1, \Gamma_2}$ we have that $\beta_g(v_{\Gamma_1, \Gamma_2}) = v_{\Gamma_1 g, \Gamma_2 g}$, $g \in G$. Hence $\beta_g(X^{\Gamma_\sigma}\Gamma) = X^{\Gamma_\sigma}\Gamma$ for all $g \in G$, $\sigma$ in $G$.

Since $G^{\text{op}}$ acts ergodically on $S$, it follows that $X^{\Gamma_\sigma}\Gamma$ belongs to $L(G)$ which is the algebra $\mathcal{M}^G$, of fixed points of the action of $G$ on $\mathcal{M}$.

Moreover, by definition, and since $v_{\Gamma, \sigma}$ is equal to $\chi_{\Gamma_\sigma}$, it follows that $v_{\Gamma_1, \Gamma_2}$ is a partial isometry from $\chi_{\Gamma_2}$ to $\chi_{\Gamma_1}$.

The formula defining $X^{\Gamma_\sigma}\Gamma$ implies that if $\Gamma_1, \Gamma_2$ are so that $\Gamma_1 \Gamma_2^{-1} \Gamma = [\Gamma \sigma \Gamma]$ then $\chi_{\Gamma_1} \chi_{\Gamma_2} = \chi_{\Gamma_1 \Gamma_2}$. Also, we have that:

$$\chi_{\Gamma_\sigma} X^{\Gamma_\sigma}\Gamma \chi_{\Gamma_\sigma} = \delta_{[\Gamma \sigma^{-1} \Gamma], [\Gamma \sigma \Gamma]} B_{\Gamma_\sigma} B_{\Gamma_\sigma},$$

for $\alpha, \beta, \sigma \in G$. Here, $\delta$ is the Kronecker symbol.

Obviously for $\theta \in G$, $\chi_{\Gamma_\sigma} \chi_{\Gamma_\tau} \neq 0$ if and only if there exist $\gamma_1, \gamma_2$ in $\Gamma$ such that $\theta \gamma_2 \sigma_2 = \gamma_1 \sigma_1$ and thus if and only if $\theta$ belongs to $\Gamma \sigma_1 \sigma_2^{-1} \Gamma$.

Thus necessary $X^{\Gamma_\sigma}\Gamma$ belongs to $L(G) \cap \mathcal{P}(\Gamma \sigma \Gamma)$. We note that in order to see that $X^{\Gamma_\sigma}\Gamma$ is bounded, we observe that formally, the operators $X^{\Gamma_\sigma}\Gamma$, $\sigma \in G$, which are a priori affiliated to $L(G)$, have also the property that the mapping

$$[\Gamma \sigma \Gamma] \rightarrow X^{\Gamma_\sigma}\Gamma$$

extends to a $*$-algebra representation of $C(\Gamma \setminus G/\Gamma)$. This last statement is a consequence of the way the cosets are combining themselves in the products. Moreover, this representation is trace preserving (with respect to the trace $\tau$ on $L(G)$). But then the above map extends to $\mathcal{H}_{\text{red}} = C^{\text{red}}(\Gamma \setminus G/\Gamma)$, the reduced $C^*$-Hecke algebra, and hence the elements $t^{\Gamma_\sigma}\Gamma$ are bounded.

The property that

$$\chi_{\Gamma_\alpha} t^{\Gamma_\alpha \Gamma_\beta^{-1} \Gamma} \chi_{\Gamma_\beta} = \chi_{\Gamma_\alpha} t^{\Gamma_\alpha \Gamma_\gamma^{-1} \Gamma} \chi_{\Gamma_\gamma}$$

implies, when moving in the left side member, the characteristic function $\chi_{\Gamma_\beta}$ to the right, a series of identities $\sum t^{A_i} = \sum t^{C_j}$ (see [Ra4]). Here, for $A$
a coset (of some subgroup group \( \Gamma_\sigma \) on \( \Gamma \)) we denote by \( t^A \) the sum \( \sum_{\theta \in A} t(\theta)\theta \), where \( t(\theta) \) are the coefficients in \( t^{\Gamma_\sigma \Gamma} = X^{\Gamma_\sigma \Gamma} \).

But these identities, when summed up, are the identities
\[
t^{\sigma_1 \Gamma} t^{\sigma_2} = t^{\sigma_1 \sigma_2}, \quad \sigma_1, \sigma_2 \in G,
\]
which are exactly the sufficient conditions that imply (see [Ra1]) that the map \([\sigma \Gamma] \rightarrow t^{\sigma \Gamma}, \sigma \in G\) extends to a representation of \( \mathcal{SO} \) (as in property (2) in the statement).

To prove that (1) implies (3) we use the following. We identify the commutant algebra \( \{ \pi(\Gamma) \}' \) with \( R(\Gamma) \), and the Jones simultaneous basic construction for the subfactors \( \{ \pi(\Gamma_\sigma) \}'' \subseteq \{ \pi(\Gamma) \}'' \) has the term at \( \infty \) the type \( \text{II}_1 \) factor \( A_\infty \), the inductive limit of the \( \text{II}_1 \) factors \( \{ \pi(\Gamma_\sigma) \}' = \{ \pi(\Gamma) \}' \bigvee \{ e_{\Gamma_\sigma} \}'' \), where \( e_{\Gamma_\sigma} \) are the Jones projection corresponding to the subfactors \( \{ \pi(\Gamma_\sigma) \}'' \subseteq \{ \pi(\Gamma) \}'' \), \( \sigma \in G \). Then \( A_\infty \) is clearly identified with \( R(\Gamma \rtimes K) \), where \( K \) is the profinite completion of \( \Gamma \), with respect to the subgroups \( \Gamma_\sigma \). The isomorphism is realized by identifying the Jones’s projection \( e_{\Gamma_\sigma} \) with the characteristic function \( \chi_{\Gamma_\sigma} \in C(K) \), (here the closure of \( \Gamma_\sigma \) is in the profinite completion).

Note that the Jones’s construction can be extended the the more general situation of a pair of subfactors of same index \( N_0, N_1 \subseteq M \), in the presence of a given isomorphism \( \theta : N_0 \rightarrow N_1 \).

In this case we define \( W_\theta : L^2(N_0) \rightarrow L^2(N_1) \) by \( W_\theta(n_0) = \theta(n_0) \), \( n_0 \in N_0 \). Viewed as an element of \( B(L^2(M, \tau)) \), \( W_\theta \) is a partial isometry from the image of the projection \( e_{N_0} \) onto the image of the projection \( e_{N_1} \), where \( e_{N_0}, e_{N_1} \) are the corresponding Jones projections onto the subfactors \( N_0, N_1 \).

We do a construction which is analogous with the Jones construction of the first step in the basic construction ([Jo]): in that case the first term of the basic construction is the algebra
\[
\langle M, e_N \rangle = Me_N M \cong N' \subseteq B(L^2(M, \tau)),
\]
for a subfactor \( N \subseteq M \). We consider instead of the commutant algebra \( N' \), the space of intertwiners defined by
\[
\text{Int}_\theta(N_0, N_1) = \{ \alpha \in B(L^2(M, \tau)) \mid \alpha n_0 = \theta(n_0)\alpha, \text{for } n_0 \in N_0 \}.
\]
Then \( \text{Int}_\theta \) is isomorphic to the space \( MW_\theta M \) and the multiplication corresponds to a product formula mapping the bimodule product \( MW_\theta M \times MW_\theta M \) into \( MW_\theta MW_\theta M \).
More generally, we consider a family of such subfactors \( N_0^S \rightarrow N_1^S \). By performing the above construction, we obtain a von Neumann algebra (viewed as a space of intertwiners), containing all the corresponding intertwiners \( W_{\theta_\sigma} \), and the algebra \( M \).

In the case of the subgroups \( \Gamma_\sigma \subseteq \Gamma, \sigma \in G \), letting the isomorphism \( \theta_\sigma \) from \( L(\Gamma_{\sigma^{-1}}) \rightarrow L(\Gamma_\sigma) \) be the conjugation by \( \sigma \), we obtain by taking inductive limit, a type II\(_1\) factor \( B_\infty \) which is clearly isomorphic to the reduced von Neumann algebra crossed product

\[
\chi_\Gamma(L((G \times G^{\text{op}}) \rtimes L^\infty(S, \nu))) \chi_\Gamma,
\]

which is further reduced by the projection \( \chi_\Gamma \).

Moreover, we switch in the above representation, from right convolutors to left convolutors, by taking the canonical conjugate linear map. We denote \( W_{\theta_\sigma} \) by \( W_\sigma \). In the above representation for crossed product algebra \( B_\infty \), the partial isometry \( W_\sigma \) is further identified with the partial isometry

\[
(\sigma \otimes \sigma^{-1})e_{\Gamma_{\sigma^{-1}}} = e_{\Gamma_{\sigma}}(\sigma \otimes \sigma^{-1}), \sigma \in G.
\]

By letting \( \sigma \otimes \sigma^{-1} \) act as a partial isomorphism, we have a groupoid action of \( G \times G^{\text{op}} \) on \( K \). Using this partial (groupoid action) we may also identify \( B_\infty \) with reduced, groupoid von Neumann algebra, cross product \( L((G \times G^{\text{op}}) \rtimes L^\infty(K, \mu)) \) with unit identified to \( \chi_\Gamma \). Here we are using the \( G \times G^{\text{op}} \) groupoid invariant measure on \( K \).

Then \( \pi(\sigma) \in \text{Int}_{\theta_\sigma}(\pi(\Gamma_{\sigma^{-1}}), \pi(\Gamma_\sigma)) \) and through the above identification of the space of intertwiners, the intertwiner \( \pi(\sigma), \sigma \in G \), is identified to the partial isometry

\[
\sum_i t_i^* W_{\theta_\sigma} s_i,
\]

where \( s_i \in L(\Gamma) = \{\pi(\Gamma)\}'' \) is a Pimsner Popa basis for \( L(\Gamma_\sigma) \subseteq L(\Gamma) \), that is a system of \( \Gamma_\sigma \) coset representatives, and \( t_i = \pi(\sigma)s_i \).

Thus we get that the unitary \( \pi(\sigma) \), in the identification of \( B_\infty \) with \( L((G \times G^{\text{op}}) \rtimes K) \), is the element

\[
u(\sigma) = \sum_i (t_i^* \otimes 1)(\sigma \otimes \sigma^{-1})e_{\Gamma_{\sigma^{-1}}} (s_i \otimes 1),
\]

which is clearly equal to

\[
\sum_i \chi_\Gamma(t_i^* \otimes 1)(\sigma \otimes \sigma^{-1})(s_i \otimes 1) \chi_\Gamma = \sum_i \chi_\Gamma(t_i^* s_i) \otimes \sigma^{-1} \chi_\Gamma.
\]
In this presentation, we denote for \( \sigma \in G \), by \( t(\Gamma \sigma \Gamma) \in L(G) \), the sum \( \sum_{i} t_{i} \sigma s_{i} \). Note that this is exactly the Hecke algebra representation considered in [Ra 1], and note that we may write

\[ u(\sigma) = \chi_{T}(t^{1} \sigma \Gamma \otimes 1)(1 \otimes \sigma^{-1})\chi_{T}, \quad \sigma \in G. \]

This is a representation of \( G^{\text{op}} \) into

\[ L((G \times G) \times L^{\infty}(K, \mu)) \times L^{\infty}(S, \nu) \chi_{T}, \]

since \( \pi(\sigma), \sigma \in G \), was a unitary (eventually projective) representation of \( G \). To prove that (3) implies (4) we note that we may write

\[ u(\sigma) = (v_{\Gamma, \Gamma \sigma} \otimes 1) \otimes (1 \otimes \sigma^{-1}), \sigma \in G. \]

Hence \( v_{\Gamma, \Gamma \sigma} = \chi_{T}(t^{1} \Gamma \sigma \Gamma) \chi_{T}^{-1} \) is an isometry, in the von Neumann algebra \( L(G \times L^{\infty}(S, \nu)) \), from \( \chi_{T} \) onto \( \chi_{T}^{-1} \).

We define \( v_{\Gamma_{1}, \Gamma_{2}} = \beta_{1}(v_{\Gamma_{1}, \Gamma_{2} \Gamma_{1}^{-1}}) \). This expression is thus equal to

\[ \chi_{\Gamma_{1}} t_{1} \Gamma_{1} \Gamma_{2}^{-1} \chi_{\Gamma_{2}} \]

which is a partial isometry from \( \chi_{\Gamma_{2}} \) onto \( \chi_{\Gamma_{1}} \).

The property that the family of unitaries \( u(\sigma), \sigma \in G \), is a representation of \( G \), translates into the fact that \( (v_{\Gamma_{1}, \Gamma_{2} \Gamma_{1}^{-1}})_{\Gamma_{1} \in \Gamma} \) is a matrix unit. By construction, the matrix unit is \( G^{\text{op}} \)-equivariant (recall that \( G^{\text{op}} \) acts on \( L(G \times L^{\infty}(S, \nu)) \) by leaving \( L(G) \) invariant, and by acting by the Koopmann representations of the right translations, on \( L^{\infty}(S, \nu) \).

The construction of the splitting

\[ L(G \times L^{\infty}(S, \nu)) = P \otimes B(l^{2}(\Gamma \setminus G)), \]

in a \( G^{\text{op}} \) equivariant way, is now straightforward. We let

\[ P = \chi_{T}(L(G \times L^{\infty}(S, \nu))) \chi_{T} = L(\Gamma \times L^{\infty}(K, \mu)), \]

(with unit identified to \( \chi_{T} \)) and, for \( p \in P \) we define,

\[ \alpha_{p}(p) = v_{\Gamma, \Gamma \sigma} \beta_{p}(p) v_{\Gamma \sigma, \Gamma}, \]

for \( p \in \chi_{T} M_{\Gamma \Sigma} \), so that \( \beta_{p}(p) \) belongs to \( \chi_{\Gamma \sigma} M_{\chi_{\Gamma \sigma}} \), for all \( p \) in \( P \).

Since \( G^{\text{op}} \) acts trivially on \( G \), it follows that \( \beta_{p} \) acts trivially on \( L(\Gamma) \) for every \( g \in G \). It follows that \( \alpha_{\gamma} \) acts trivially on \( L(\Gamma) \) for \( \gamma \) in \( \Gamma \). Indeed, in this case, for every

\[ x \in L(\Gamma) \subseteq \chi_{T} L(\Gamma \times L^{\infty}(K, \mu)) \chi_{T}, \]

and for every \( \gamma \in \Gamma^{\text{op}} \), we obtain that \( \alpha_{\gamma}(x) = v_{\Gamma, \Gamma} \beta_{\gamma}(x) v_{\Gamma, \Gamma} = x. \)
Thus \( \alpha_g \) is a \( G \)-automorphism representation into the automorphism group of the type \( II_1 \) factor \( L(\Gamma \rtimes L^\infty(K, \mu)) \) such that \( \alpha|_\Gamma \) acts identically on \( L(\Gamma) \).

Note that in this case the formula for \( \alpha_g(p) \) is \( u_{\Gamma, \Gamma g} \beta_g(p) u_{\Gamma, \Gamma} \). We recall that the formula for \( u(\sigma) \) is

\[
\chi_T(t^{\Gamma \sigma} \otimes 1)(1 \otimes \sigma^{-1}) \chi_T = \chi_T(t^{\Gamma \sigma} \otimes 1) \chi_T(1 \otimes \sigma^{-1}) = (u_{\Gamma, \Gamma \sigma} \otimes 1)(1 \otimes \sigma^{-1}).
\]

Thus we see that for \( x = \chi_T x \chi_T \) in \( L(\Gamma \rtimes L^\infty(K, \mu)) \) which is identified to \( L(\Gamma \rtimes L^\infty(K, \mu)) \otimes 1 \), we have that \( u(g) x u(g)^* \) is equal to

\[
u(g)(x \otimes 1) u(g)^* = (v_{\Gamma, \Gamma \sigma} \otimes 1)(1 \otimes \sigma^{-1})[\chi_T x \chi_T](1 \otimes \sigma)(v_{\Gamma, \Gamma} \otimes 1),
\]

and this is thus equal to

\[
(v_{\Gamma, \Gamma \sigma} \beta_g(x) v_{\Gamma, \Gamma}) \otimes 1.
\]

Thus \( u(\sigma) \in \chi_T(L(G \times G \rtimes L^\infty(S))) \chi_T \) normalizes \( P \) (which is identified to \( P \otimes 1 \)) and \( \alpha_\sigma(x) = Ad u(\sigma)(x) \), \( x \in P, \sigma \in G \).

For \( x \) in \( L(\Gamma) \subseteq \chi_T L(\Gamma \rtimes L^\infty(K, \mu)) \chi_T \), the formula for \( \alpha_\sigma(x) \) becomes

\[
\alpha_\sigma(x) = \chi_T t^{\Gamma \sigma} \chi_\Gamma x \chi_\Gamma t^{\Gamma \sigma} \chi_T
\]

Note that last expression depends, in fact, only on the coset \( \Gamma \sigma \in \Gamma \setminus G \).

In particular, for

\[
x \in L(\Gamma) = \chi_T L(\Gamma) \chi_T \subseteq \chi_T L(\Gamma \rtimes L^\infty(K, \mu)) \chi_T = L(\Gamma \rtimes L^\infty(K, \mu)),
\]

we obtain that

\[
E_{L(\Gamma)}(\alpha_\sigma(x)) = \sum \chi_T t^{\Gamma \sigma} \chi_\Gamma x \chi_\Gamma \sigma \chi_\Gamma t^{\Gamma \sigma} \chi_T.
\]

Here the family \( s_i \) are the corresponding coset representatives considered also above. The right hand term is clearly \( \chi_T t^{\Gamma \sigma} x t^{\Gamma \sigma} \chi_T \) which is further equal to the map \( \Psi_{\Gamma, \Gamma \sigma}(x), x \in L(\Gamma), \) constructed in [Ra1], in correspondence to the representation \( \pi \) in statement (1) of the equivalences.

Although the following implication is not needed we note that to prove that (3) implies (1) it is sufficient to use the formula

\[
\pi(\sigma)(x) = E_{L(\Gamma)}(t^{\Gamma \sigma} x \sigma^{-1}),
\]

for \( x \) in \( L^2(\Gamma) \).

\[\square\]

**Remark.** With the notation from the statement of the theorem, for \( g \in G \), the automorphism \( \alpha_g \) will map \( s_i \chi_{\Gamma g^{-1}} L^\infty(K, \mu) \) into \( t_i \chi_T L^\infty(K, \mu) \), for a
suitable choice of left coset representatives $s_i$ for $\Gamma_{g^{-1}}$ in $\Gamma$. Here, the elements $t_i$ are a Pimsner Popa basis ([PP]) for $L(\Gamma_g)$ in $L(G)$: they are the images of the elements $s_i$ through the representation $\pi$. The automorphism $\alpha_g$ acts as conjugation by $g$ from $\chi_{\Gamma_g^{-1}}L^\infty(K, \mu)$ onto $\chi_{\Gamma_g}L^\infty(K, \mu)$. We conjecture that this conditions uniquely determine the representation $\alpha_g, g \in G^{op}$, up to conjugation by an element in the unitary group of $L(\Gamma)$, by analogy with the results in the last chapter of the paper by N. Ozawa and S. Popa ([OP]).

We address now the question of unicity. We prove that up to unitary conjugacy, the group $(\alpha_g)_{g \in G}$ of automorphisms of $P = L(\Gamma \rtimes L^\infty(K, \mu))$ is uniquely determined. More precisely, we have:

**Corollary.** Given the action $(\alpha_g)_{g \in G}$ of the group $G$, constructed in the previous theorem, in the presence of the equivalent properties (1) - (4), any other such representation $(\tilde{\alpha}_g)_{g \in G}$ obtained from similar splitting data, is of the form $\tilde{\alpha}_g = \text{Ad}u_g\alpha_g$, where $u_g \in U(P)$ is a 1-cocycle of $G$, with respect to $\alpha_g, g \in G$ with values in the unitaries of $P$.

The 1-cocycle property means that

$$u_{g_1g_2} = u_{g_1}\alpha_g(u_{g_2}), g_1, g_2 \in G.$$  

In particular the representations $\pi$ are classified by the 1-cohomology group $H^1_\alpha(G, U(P))$, where $P = L(\Gamma \rtimes L^\infty(K, \mu))$ and $U(P)$ is the unitary group of $P$.

If the unitary representation $\pi$ is projective having a two cocycle $\varepsilon$, then the crossed product defining $P$ will be a skewed crossed product, with respect to the restriction of the cocycle $\varepsilon$ to $\Gamma$.

**Proof.** Consider $(\alpha_g)_{g \in G}$ as in the statement of the previous Theorem. Then $M = L(G \rtimes L^\infty(S, \nu))$ has the structure $P \otimes B(L(\Gamma / G))$ with

$$P = L(\Gamma \rtimes L^\infty(K, \mu)),$$

and the canonical action of $G^{op}$ onto $M$ becomes

$$\alpha_g \otimes \text{Ad}\rho_{\Gamma/G}(g), g \in G.$$  

Elements in $M$ are consequently identified with infinite matrices

$$(p_{\Gamma_{\sigma_1,\Gamma_{\sigma_2}}}g_{\sigma_1,\Gamma_{\sigma_2}}), \Gamma_{\sigma_1,\Gamma_{\sigma_2}} \in \Gamma \setminus G;$$

where the entries are in the algebra $P$.  

Any other $G^{\text{op}}$-equivariant matrix unit will be of the form

$$(u(\Gamma_1)u(\Gamma_2)^*)_{\Gamma_1, \Gamma_2 \in \Gamma \setminus G},$$

where $u(\Gamma)$, $\Gamma \in \Gamma \setminus G$ are unitaries in $P$.

Note that we are using here the fact that the diagonal algebra $D = L^\infty(\Gamma \setminus G) \subseteq B(L^2(\Gamma \setminus G)) \subseteq B(L^2(G \rtimes L^\infty(S, \nu)))$, is independent of the choice of equivariant matrix unit.

We may also describe $(u(\Gamma))_{\Gamma \in \Gamma}$ as a unitary

$$w = \sum u(\Gamma) \otimes \chi_{\Gamma}$$

in $D' \cap L(G \rtimes L^\infty(S, \nu))$. The new matrix unit

$$\tilde{v}_{\Gamma_1, \Gamma_2} = u(\Gamma_1)^* u(\Gamma_2) \otimes v_{\Gamma_1, \Gamma_2}$$

(where $v_{\Gamma_1, \Gamma_2}$ is the original matrix unit for $\alpha_g$) has also to be $G$-equivariant.

Denote $B = \{v_{\Gamma_1, \Gamma_2}\}_\nu$, $\tilde{B} = \{\tilde{v}_{\Gamma_1, \Gamma_2}\}_\nu$ and let

$$X^{\Gamma \sigma \Gamma} = \sum_{\Gamma_1, \Gamma_2, \Gamma \in \Gamma} v_{\Gamma_1, \Gamma_2, \Gamma'} \tilde{X}^{\Gamma \sigma \Gamma} = \sum_{\Gamma_1, \Gamma_2, \Gamma \in \Gamma} \tilde{v}_{\Gamma_1, \Gamma_2, \Gamma'}$$

for every double coset $\Gamma \sigma \Gamma$ in $G$. These are the elements constructed in the proof of the implication $(3) \Rightarrow (1)$ in the proof of the theorem.

Then both applications mapping $[\Gamma \sigma \Gamma] \to X^{\Gamma \sigma \Gamma}$, $[\Gamma \sigma \Gamma] \to \tilde{X}^{\Gamma \sigma \Gamma}$, (where $[\Gamma \sigma \Gamma]$ runs over double cosets), extend to representations of the algebra $\mathbb{C}(\Gamma \setminus G/\Gamma)$ (and in fact they extend to the reduced Hecke $C^*$-algebra with values into the algebra $L(G)$).

We then have $wX^{\Gamma \sigma \Gamma}w^* = \tilde{X}^{\Gamma \sigma \Gamma}$, for all $[\Gamma \sigma \Gamma]$, and $wBw^* = \tilde{B}$. By the $G^{\text{op}}$-invariance of the new matrix unit, it follows that $\beta_g(w)$ has the same properties as $w$, and thus $\beta_g(w)^* w$ belongs to $B' = P \otimes I$.

But then $c(g) = \beta_g(w)^* w$ is a 1-cocycle with respect to $\alpha_g$ with values in $U(P)$, for the group $G$. Moreover the action $\tilde{\alpha}_g$ of $G$ into the automorphism of $P$, associated to the matrix unit $(\tilde{v}_{\Gamma_1, \Gamma_2})$ is then $\tilde{\alpha}_g = \text{Ad}(c(g))\alpha_g$, $g \in G$.

If, in the 1-group cohomology $H^1_\alpha(G, P)$, the 1-cocycle $c$ with values in $P \cong P \otimes 1 = B' \subseteq D'$, vanishes then it follows that there exists a unitary $p$ in $P$, such that $c(g) = \beta_g(p)p$, for $g \in G$.

Thus $\beta_g(w)w = \beta_g(p)p$ for $g \in G$ and hence $\beta_g(wp^*) = wp^*$ for all $g$ in $G$. 
Thus \( wp^* \) belongs to \( L(G) \). Since both \( w, p \) belongs to \( D' \) it follows that \( wp^* \in L(G) \cap (L^\infty(\Gamma \setminus G))^* \) which is \( L(\Gamma) \).

Consequently \( w = xp \) for some unitary \( x \) in \( L(\Gamma) \). Hence \( wBw^* = x\tilde{B}x^* \), and hence we may assume that \( w \) belongs to \( L(\Gamma) \). Thus \( \tilde{X}^\Gamma_{\sigma p} = x X^\Gamma_{\sigma p} x^* \) and hence \( \alpha_g \) depends from \( \alpha_g \) by a perturbation with an element in \( L(\Gamma) \). But then the corresponding unitary representation of \( G \), corresponding through the equivalences in the theorem to the new matrix unit, is the same as the initial one (the representation \( \pi \)), the only difference being that we replace the original trace vector by the vector given by the unitary \( x \in L(\Gamma) \).

\[ \square \]

**Remark.** In the case \( G = PGL_2(\mathbb{Z}[\frac{1}{p}]), \Gamma = PSL_2(\mathbb{Z}), \) \( p \) a prime number, the space of cosets has a homogeneous tree structure. Then to get a \( G \)-invariant matrix unit in \( L(G \rtimes L^\infty(S, \nu)) \) as in the Theorem 1 it is sufficient to find \( X = X^* \) in \( L(G) \cap l^2(\Gamma_{\sigma p}, \Gamma) \), such that \( \chi_\Gamma X \chi_\Gamma^{\sigma p} \) is an isometry from \( \chi_\Gamma \) onto \( \chi_\Gamma^{\sigma p} \). Here, for \( n \in \mathbb{N} \), we define

\[
\sigma_{pn} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}.
\]

In addition (to have compatibility with the adjoint operation) we impose that \( \chi_\Gamma X \chi_\Gamma^{\sigma p} \) to be the adjoint of \( \beta_{(\sigma p, s_i)}(X) \) (if \( \Gamma_{\sigma p}^{-1} = \Gamma_{\sigma p, s_i} \)). Also \( X \) has to be invariant under the action of \( \beta_{\gamma}, \gamma \in \Gamma_{\sigma p} \).

**Proof.** We may take \( v_{\Gamma_{\sigma_1}, \Gamma_{\sigma_2}} = \beta_{\gamma_1}(\chi_\Gamma X \chi_\Gamma^{\sigma_2}) \) if \( \Gamma_{\sigma_1} \sigma_2^{-1} \Gamma = [\Gamma_{\sigma_2}, \Gamma] \), and the tree structure implies that we may define a \( G \)-equivariant matrix unit by defining for \( \Gamma_{\sigma_1} \sigma_n^{-1} \Gamma = [\Gamma_{\sigma_n}, \Gamma] \)

\[ v_{\Gamma_{\sigma_0}, \Gamma_{\sigma_n}} = \prod_i v_{\Gamma_{\sigma_i}, \Gamma_{\sigma_{i+1}}}, \]

where \( \Gamma_{\sigma_i} \Gamma_{\sigma_i+1}^{-1} \Gamma = [\Gamma_{\sigma_i}, \Gamma_{\sigma_i}^{-1} \Gamma] \), and \( (\Gamma_{\sigma_i}) \) are the edges in the path on the tree connecting \( \Gamma_{\sigma_0} \) to \( \Gamma_{\sigma_n} \). \[ \square \]

The factorization result in Theorem 1 eventually could make it possible to find directly the action \( \alpha_g \) from the ergodical action of \( G \times G^{op} \) on \( L^\infty(S, \nu) \). Note that in the new representation \( L(G \rtimes L^\infty(S, \nu)) \) as the tensor \( P \otimes B(l^2(\Gamma \setminus G)) \), the initial algebra \( L(\Gamma) \subseteq L(G) \), is in the new description is represented as the diagonal algebra \( \sum_{\gamma \in \Gamma \setminus G} (\alpha_g(x) \otimes \chi_{\Gamma \gamma})_{\gamma \in \Gamma \setminus G}, x \in L(\Gamma) \). (Recall that \( \alpha_g(x) \) for \( x \) in \( L(\Gamma) \) depends only on the coset \( \Gamma g \)).
The Hecke algebra operators compute the difference between the two representations of the algebra $L(\Gamma)$.

**Remark.** With the above notations, let $\widetilde{L}(\Gamma)$ be the new copy of $L(\Gamma)$ in the representation $\mathcal{M} = L(\Gamma \rtimes L^\infty(K, \mu)) \otimes B(L^2(\Gamma \setminus G))$.

The fixed points under the action of $\Gamma^\text{op}$ under the new representation are the crossed product of $L(\Gamma \rtimes \left[ C(K, B(L^2(\Gamma \setminus G)))^{\Gamma^\text{op}} \right])$.

Here $\Gamma$ acts on $[C(K, B(L^2(\Gamma \setminus G)))^{\Gamma^\text{op}}]$ by right translations on $K$, and by $\text{Ad}_{\rho_G}$ on $B(L^2(\Gamma / G))$. In particular the crossed product algebra $L(\Gamma \rtimes \left[ C(K, B(L^2(\Gamma \setminus G)))^{\Gamma^\text{op}} \right])$ contains $L(\Gamma) \otimes B(L^2(\Gamma \setminus G))^{\text{Ad}_{\rho_G}}$. It is obviously that $B(L^2(\Gamma \setminus G))^{\text{Ad}_{\rho_G}}$ is the algebra generated by the Hecke algebra acting on the right and by the diagonal algebra $L^\infty(\Gamma \setminus G / \Gamma)$ identified with the algebra generated by the characteristic functions of the corresponding cosets.

In the original representation of $\mathcal{M} = L(G \rtimes L^\infty(S, \nu))$, the $\Gamma^\text{op}$ fixed points are $L(G \rtimes L^\infty(G \setminus \Gamma))$ (where $L^\infty(\Gamma / G)$ is identified with the subalgebra of $L^\infty(S, \nu)$, by identifying the coset $\chi \Gamma \sigma$ with the characteristic function of $\chi_{K \sigma}$, $\sigma \in G$).

Then $\widetilde{L}(\Gamma)$, the new representation of $L(\Gamma)$, sits in $\{H, L^\infty(\Gamma \setminus G)\} \subseteq L^\infty(G \rtimes L^\infty(\Gamma \setminus G))$, where $H$ is identified with the subalgebra of $L(G)$ generated by the elements $t^{\Gamma \sigma} = \sum_{\Gamma \sigma_1, \Gamma \sigma_2} v_{\sigma_1, \Gamma \sigma_2}$, for a double coset $[\Gamma \sigma \Gamma]$, $\sigma \in G$, which determine a representation of the Hecke algebra.

**Remark.** To determine the behavior of Hecke operators we have to determine $E_{\mathcal{L}(\Gamma)}(x_{\sigma}(x))$, $x \in L(\Gamma)$, which means we have to compute the matrix coefficients $\tau(x_{\sigma}(x))$ for $x \in \mathcal{L}(\Gamma)$, where $\tau$ is the trace on the type $II_1$ factor $L(\Gamma \rtimes L^\infty(K, \mu))$, identified as a subalgebra of the type $II_\infty$ factor $L(G \rtimes L^\infty(S, \nu))$.

**APPENDIX**

Given a structure of Hilbert spaces (see Section 1 in [Ra1] for notations) of $\Gamma_\sigma$-invariant vectors $\sigma \in G$, for an action of $G$ on a vector space $V$, we assumed in [Ra1] that there exists well defined subspaces of $\Gamma_\sigma$-invariant
vectors $H^\Gamma_\sigma \subseteq V^{\Gamma_\sigma}$. In addition to the assumptions are that $\pi(\sigma)$, which originally maps $H^{\Gamma_{\sigma}}_{\sigma^{-1}}$ into $H^{\Gamma_{\sigma}}_\sigma$, extends to a unitary representation $\pi^{\text{ad}}$ of $G$ on the isometric, inductive limit $H^{\text{ad}}$ of the Hilbert spaces $H^{\Gamma_{\sigma}}$, as $\Gamma_{\sigma}$ shrinks to the identity element.

The Hecke operators $T^{\Gamma_{\sigma}\Gamma}$ were proved to be described by the diagram

$$
\begin{array}{ccc}
H^{\Gamma_{\sigma}}_{\sigma^{-1}} & \xrightarrow{\sigma} & H^{\Gamma_{\sigma}}_\sigma \\
\left< \sigma \right.& & \sqrt{[\Gamma : \Gamma\sigma]} \cdot \text{proj}_{H^{\Gamma_{\sigma}}} \\
H^\Gamma & \end{array}
$$

Then for $v \in H^\Gamma$, $\langle T^{\Gamma_{\sigma}\Gamma} v, v \rangle$ is $[\Gamma : \Gamma\sigma]\langle \pi^{\text{ad}}(\sigma)v, v \rangle$, $\sigma \in G$, and hence the matrix coefficients of the Hecke operators are computed by rescaling, from the matrix coefficients $\langle \pi^{\text{ad}}(g)v, v \rangle$. If $v \in H^\Gamma$, these matrix coefficients are constant on double cosets.

Thus to determine the continuity of the Hecke algebra representations given by the Hecke operators, with respect to the $H_{\text{red}}$ topology, it is sufficient to determine if the generic matrix coefficients

$$
\langle \pi^{\text{ad}}(g)v, v \rangle,
$$

are weakly contained in the corresponding unitary representations of $G$, associated to the $C^\ast$-reduced Hecke algebra.

In the standard representation we have for $\sigma \in G$ that $H^{\Gamma_{\sigma}}_{\text{st}}$ is the Hilbert space $L^2(\Gamma_{\sigma}/G)$ and hence we have clearly that $H^{\text{ad}}_{\text{st}} = L^2(S, \nu)$, and $\pi^{\text{ad}}_{\text{st}}$ is nothing else then the regular representation of $G$ on $H^{\text{ad}}_{\text{st}}$.

Thus continuity of the Hecke algebra representation on $H^\Gamma$, is equivalent to the fact that $\pi^{\text{ad}}$ is contained in the Koopman representation of $G$ on the $L^2$ space of its profinite completion $S$.

Note that in the case $G = PGL_2(\mathbb{Q}\{1\}/\mathbb{Q}_p)$, (see [Lu]), all irreducible representation, having $\Gamma$-invariant vector extend to $PGL_2(\mathbb{Q}_p)$, and the representation in the left regular representation are the principal series, while all others (corresponding to discrete laplacian eigenvalues $(2\sqrt{p}, p + 1)$) are in the complementary series.

**Observation.** The unitary representation $Ad\rho_{\text{G}}$ on the Hilbert Schmidt operators $C_2(B(l^2(\Gamma \setminus G))$ is weakly contained in the representations of the principal series.
Proof. Indeed the matrix coefficients in this case are linear combinations of the form \( \sum \lambda_i\lambda_j \chi_{\sigma_i^{-1}\Gamma_{\sigma_j}}(g) \), as a functions on \( G \). But these are weak limits of positive definite function from the left regular representation of \( G \) on \( \lim_{\Gamma_{\sigma}}^{\leftarrow} l^2(\Gamma_{\sigma} \setminus G) \), which in turn are limits of positive definite functions of the form
\[
\sum \lambda_i\lambda_j \chi_{\sigma_i^{-1}\Gamma_{\sigma_j}}
\]
for a fixed subgroup \( \Gamma_{\sigma} \) of \( G \). \hfill \Box

We conclude by noticing that the action of \( G^{op} \) on
\[
L^2(\mathcal{M}) = L^2(L(G \rtimes L^\infty(S, \nu))),
\]
is clearly weakly contained in the left regular representation.

Then we have a splitting, with \( P = L(\Gamma \rtimes L^\infty(K, \mu)) \)
\[
L^2(\mathcal{M}) = L^2(P) \otimes C_2(l^2(\Gamma \setminus G))
\]
in which the action of \( G^{op} \) is \( \alpha_g \otimes \text{Ad}_{\rho_{\Gamma \setminus G}} \). Because of the previous observation, and since the left regular representation absorbs any other representation, more specific information, coming from the specific tensor algebra decomposition in Theorem 1, is needed about the splitting, to conclude that the matrix coefficients \( \tau_P(\alpha_g(x)x^*) \), for \( x \) in \( L^2(P) \setminus C_1 \), are weakly contained in the Koopman representation of \( G \) on the \( L^2 \) space of its profinite completion \( S \).

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