On the sum of fourth powers of numbers in arithmetic progression

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Abstract

We prove that the equation \((x - y)^4 + x^4 + (x + y)^4 = z^n\) has no integer solutions \(x, y, z\) with \(\gcd(x, y) = 1\) for all integers \(n > 1\). We mainly use a modular approach with two Frey \(\mathbb{Q}\)-curves defined over the field \(\mathbb{Q}(\sqrt{30})\).

1 Introduction

In this paper we will study an equation of the form

\[(x - y)^k + x^k + (x + y)^k = z^n, \quad x, y, z \in \mathbb{Z},\]

i.e. the sum of three \(k\)-th powers in arithmetic progression being a perfect power. Such equations have been intensively studied in the case \(y = 1\), i.e. consecutive \(k\)-th powers. The earliest results in that case were already formulated by Euler in the case \(k = n = 3\). Zhang [30] gave a complete solution for consecutive integers for \(k = 2, 3, 4\) and this was extended by Bennett, Patel and Siksek [6] for \(k = 5, 6\). In both cases the modular method was used with Frey curves defined over the rationals.

Also the more general case of equation (11) has been studied before. For the case \(k = 2\) and \(\gcd(x, z) = 1\) Koutsianas and Patel [19] used prime divisors of Lehmer sequences to determine all solutions when \(1 \leq y \leq 5000\). Koutsianas [18] further studied this case when \(y\) is a prime power \(p^m\) for specific prime numbers \(p\). The case \(k = 3\) was partially solved by Argáez-García and Patel [2] giving all solutions in case \(1 \leq y \leq 10^6\) using different techniques including the modular method for some Frey curves over the rationals. In [19] and [2] the bounds on \(y\) are merely for computational purposes, whilst the techniques would generalize to larger bounds.
In this paper we look at the case \( k = 4 \). Zhang \([31]\) proved a partial result in this case. By considering \( y \) as a parameter and using the modular method with Frey curves over \( \mathbb{Q} \), he managed to prove the non-existence of solutions for certain families of values for \( y \). Although his approach could be pushed to include more families of values for \( y \) it appears this method cannot be generalized to treat all values of \( y \) simultaneously.

In this paper we will give a complete solution for the case \( k = 4 \) in the case \( \gcd(x, y) = 1 \). Essential in the proof is the construction of the two Frey \( \mathbb{Q} \)-curves defined over \( \mathbb{Q}(\sqrt{30}) \). Using the modular method on these curves overcomes the limitations in \([31]\), allowing us to prove the following main result.

**Theorem 1.1.** The sum of three coprime fourth powers in arithmetic progression is not a perfect power, i.e. the equation

\[
(x - y)^4 + x^4 + (x + y)^4 = z^l
\]

has no solutions \( x, y, z \in \mathbb{Z} \) with \( \gcd(x, y) = 1 \) for integers \( l > 1 \).

Note that any solution to equation (2) gives rise to a solution for \( l \) a prime number. For our proof it thus suffices to prove Theorem 1.1 for \( l \) prime as we shall do throughout this paper.

As mentioned, the construction of two Frey curves over the field \( \mathbb{Q}(\sqrt{30}) \) will be essential in the proof. The construction of these curves can be found in Section 4.

Since \( \mathbb{Q}(\sqrt{30}) \) is a real quadratic field the most direct approach to apply the modular method is to use Hilbert modularity of curves defined over real quadratic fields. We will perform the initial steps to this approach in Section 5. However we will also argue that the computation of the corresponding spaces of Hilbert modular forms is out of reach for the current computational power, making this approach unfeasible.

Instead we will use that the curves in this paper are by construction also \( \mathbb{Q} \)-curves for which a separate modularity result is known \([25]\). A \( \mathbb{Q} \)-curve approach to solving Diophantine equations has already been used in articles such as the ones by Ellenberg \([13]\), Dieulefait and Freitas \([11]\), Dieulefait and Urroz \([12]\), Chen \([8, 9]\), Bennett and Chen \([4]\), and Bennett, Chen, Dahmen and Yazdani \([5]\). We will discuss this approach in Section 6.

As in the mentioned articles we will follow \([21]\) for general results about \( \mathbb{Q} \)-curves. The main differences lie in that the restrictions of scalars of our curves are not abelian varieties of GL\(_2\)-type themselves, but will decompose as a product of such varieties. This also happens in \([11]\) and \([8]\). In the first
this issue is dealt with by studying the relation between the corresponding Galois representations. We will rather study the relation between the corresponding newforms as was done in [8] and will be more specific about computing the character that defines this relation.

Another difference is in the way we compute the elliptic curve of which one should take the restriction of scalars. Most mentioned articles simply refer to [22] to prove the existence of a twist of the original curve that will suffice and perform a small search to find this twist. In [4] a more direct approach is given in case one can find a map \( \alpha : G_\mathbb{Q} \to \mathcal{O}_K^* \) with a certain coboundary. We show that also in our case we can find such a map if the sought twist exist and that in general one can find a map \( G_\mathbb{Q} \to \mathcal{O}_{K,S}^* \) for a finite set \( S \) of primes of \( K \) if \( K \) does not have class number 1.

Furthermore the approach we took should generalize to other Frey \( \mathbb{Q} \)-curves and is mostly algorithmic. The author has written code [1] for SAGE [29] that can perform the algorithmic part of this approach, which can also be used for other Frey \( \mathbb{Q} \)-curves. The code also supports MAGMA [7] integration for faster computation of newforms. Throughout the paper we will also use MAGMA to do some other calculations.

Since the modular method approach only works for primes \( l > 5 \), Section 3 is dedicated to proving the cases for small \( l \). The case \( l = 2 \) follows immediately from a local obstruction, whereas the cases \( l = 3 \) and \( l = 5 \) require the computation of some points on hyperelliptic curves to prove the non-existence of solutions.

Section 2 introduces some preliminary results about equation (2) and introduces some notation that will be used throughout the paper.

## 2 Preliminaries

In this section we will prove some general results about integer solutions to equation (2) with \( \gcd(x, y) = 1 \). Throughout the paper \((a, b, c)\) will denote an arbitrary such solution. Note that we have

\[
c^l = (a - b)^4 + a^4 + (a + b)^4 = 3a^4 + 12a^2b^2 + 2b^4,
\]

which leads to the following result.

**Proposition 2.1.** The integer \( c \) is not divisible by 2, 3 or 5.

**Proof.** This follows immediately by considering equation (3) modulo 4, 9 and 5. \( \square \)
Let \( f(x, y) \) be the left hand side of equation (2). The most general factorization of \( f \) is obtained by factoring over the splitting field \( L \) of \( f(x, 1) \). Since \( f(x, 1) \) is irreducible, the polynomial \( f(x, y) \) factors as a product of a constant and Galois conjugates of

\[
h(x, y) = x + vy, \tag{4}
\]

where \( v \) is a root of \( f(x, 1) \). To be precise we have

\[
f(x, y) = 3(x + vy)(x - vy)(x + \sqrt{(-v^2 - 4)y})(x - \sqrt{(-v^2 - 4)y}). \tag{5}
\]

We can say a lot about the factor \( h(a, b) \) and its Galois conjugates.

**Lemma 2.2.** The distinct Galois conjugates of \( h(a, b) \) are coprime outside primes above 3. Furthermore, the valuation of \( h(a, b) \) at all primes above 2 and 5 is zero and its valuation at the unique prime \( p_3 \) in \( L \) above 3 is \(-1\).

**Proof.** Since any field that contains \( h(a, b) \) and its Galois conjugates contains \( L \) we can safely do all computations in \( L \). Note that \( a \) and \( b \) are integers and that \( v \) is only not integral at the unique prime \( p_3 \) above 3, so the only prime at which \( h(a, b) \) and its Galois conjugates are not integral is \( p_3 \). For two distinct Galois conjugates \( \sigma(h(a, b)) \) and \( \tau(h(a, b)) \) their difference is equal to \((\sigma v - \tau v)b\). Any prime dividing both can not divide \( b \) as it then also divides \( a \) contradicting their coprimality. Therefore the only common primes are in the differences \( \sigma v - \tau v \). Computing the primes dividing these differences in SAGE [29] we find the first conclusion.

For the second statement, we note that \( a \) and \( b \) are integral and \( v \) only has negative valuation at \( p_3 \) with \( \text{ord}_{p_3}(v) = -1 \). This implies that the valuation of \( h(a, b) \) and its conjugates is at least 0 at all primes above 2 and 5 and at least \(-1\) at \( p_3 \). Applying this information to equation (5) and using that \( c \) has valuation 0 at all these primes by Proposition 2.1, the second result immediately follows.

Lemma 2.2 and equation (5) tell us that

\[(h(a, b)) = p_3^{-1}I^l\]

for some integral ideal \( I \) of \( L \). We will need this general result to solve the case \( l = 5 \).

For the other cases, we can limit ourselves to the subfield \( K = \mathbb{Q}(\sqrt{30}) \) of \( L \). In this case we have two factors

\[
g_1(x, y) := x^2 + \left(2 + \frac{1}{3}\sqrt{30}\right)y^2 \tag{6}
\]

\[
g_2(x, y) := x^2 + \left(2 - \frac{1}{3}\sqrt{30}\right)y^2, \tag{7}
\]
and the factorization is

\[ z^l = f(x, y) = 3g_1(x, y)g_2(x, y). \] (8)

Note that \( g_1 \) and \( g_2 \) are both the product of two Galois conjugates of \( h \) and since these are all distinct we can conclude that \( g_1(a, b) \) and \( g_2(a, b) \) are coprime outside primes above 3. Also the result about valuations carries over, so both have valuation 0 at primes above 2 and 5 and since the unique prime \( q_3 \) of \( K \) above 3 factors as \( p_3^2 \) in \( L \) they have valuation \(-1\) at \( q_3 \). Furthermore we find that

\[
\begin{align*}
(g_1(a, b)) &= q_3^{-1}I_1^l \\
(g_2(a, b)) &= q_3^{-1}I_2^l,
\end{align*}
\]

with \( I_1 \) and \( I_2 \) coprime integral ideals of \( K \).

Throughout this paper \( K \) and \( L \) will be the same as in this section, as will \( g_1, g_2 \) and \( h \).

3 Cases for small \( l \)

In this section we will solve equation (2) for small prime exponents \( l \) as the modular method used in the next sections only works for \( l > 5 \). All small cases have a slightly different approach.

3.1 Case \( l = 2 \)

In case \( l = 2 \) equation (2) has a local obstruction at 3, which can be seen by considering equation (3) modulo 9. This proves the non-existence of solutions in this case. A similar obstruction can be found modulo 5.

3.2 Case \( l = 3 \)

Suppose \( (a, b, c) \) is a solution to equation (2) for \( l = 3 \) with \( \gcd(a, b) = 1 \). From Section 2 we know that

\[
(g_1(a, b)) = q_3I_1^3 = q_3^{-4}(q_3I_1)^3
\]

as fractional ideals in \( K \). Since \( K \) has class number 2 and \( q_3^{-4} = \left(\frac{1}{9}\right) \) we find that \( q_3I_1 \) must be a principal ideal. Hence we conclude that

\[
g_1(a, b) = \frac{1}{9}u^3,
\]
for $u \in \mathcal{O}_K^\ast$ and $\gamma \in q_3$. Note that $\mathcal{O}_K^\ast$ is generated by $u_0 = (-1)^3$ and an element $u_1$ of infinite order, hence we can take $u = u_j^i$ for $j = 0, 1, 2$. By parameterizing the elements of $q_3$ with integral parameters $s$ and $t$ we get that

$$
\begin{align*}
\gamma &= 3g_1(\sqrt{s}, \sqrt{t}) \\
a^2 &= F_{3,j}(s, t) \\
b^2 &= G_{3,j}(s, t) \\
c &= 3s^2 + 12st + 2t^2
\end{align*}$$

for $F_{3,j}(s, t)$ and $G_{3}(s, t)$ some homogeneous polynomials over $\mathbb{Q}$ of degree 3.

Note that $t = 0$ corresponds to solutions in which $c$ would be divisible by 3. So by Proposition 2.1 we find that $t \neq 0$. By multiplying the middle two equations in (9) and dividing by $t^6$ we find hyperelliptic curves $C_j$ in the variables $X = \frac{a}{t}$ and $Y = \frac{b}{t^2}$. Explicitly these curves are given by

\begin{align*}
C_0 : Y^2 &= 27X^5 + 108X^4 + 84X^3 - 288X^2 - 564X - 368 \\
C_1 : Y^2 &= -1242X^6 - 1269X^5 - 432X^4 + 84X^3 + 72X^2 + 12X \\
C_2 : Y^2 &= -599940X^6 - 627237X^5 - 273132X^4 - 63276X^3 \\
&- 8208X^2 - 564X - 16.
\end{align*}

As $t \neq 0$ every solution $(a, b, c)$ corresponds to a point on such a curve. Using MAGMA [7] we see that the curve $C_2$ has no solution in $\mathbb{Q}_3$, hence none in $\mathbb{Q}$. Also using MAGMA we can compute that the Jacobian of $C_0$ has only two-torsion points as rational points. Note that such points correspond to factors of the defining polynomial, i.e. of $F_{3,0}(s, t)G_{3,0}(s, t)$. Since the only linear factor in $F_{3,0}G_{3,0}$ is $t$, the only rational point on $C_0$ corresponds to the case $t = 0$ which we already excluded from corresponding to a solution.

The curve $C_j$ the corresponding curve has no local obstruction. Furthermore the rank of its Jacobian is bounded above by 1 and its L-function suggests the rank is 1. However no point of infinite order on the Jacobian can be found within a small bound. We shall therefore apply a different approach.

For the case $j = 1$ the equations in (9) become explicitly

$$
\begin{align*}
a^2 &= -3s(23s^2 + 12st + 2t^2) \\
b^2 &= 18s^3 + 9st^2 - 2t^3 \\
c &= 3s^2 + 12st + 2t^2
\end{align*}$$

Note that in the first equation $23s^2 + 12st + 2t^2$ is congruent to $c$ modulo 20. By Proposition 2.1 this implies $23s^2 + 12st + 2t^2$ is not divisible
by 2 or 5. Note that \( s \) and \( t \) must be coprime in order for \( a \) and \( b \) to be coprime, hence \( s \) and \( 23s^2 + 12st + 2t^2 \) must be coprime outside 2. Since 2 does not divide \( 23s^2 + 12st + 2t^2 \) the two must be coprime and we find that

\[
a = 3a_1 a_2 \\
s = (-1)^{e_1} 3^{e_2} a_1^2 \\
23s^2 + 12st + 2t^2 = (-1)^{1-e_1} 3^{1-e_2} a_2^2,
\]

with \( e_1, e_2 \in \{0, 1\} \). Now note that

\[
23s^2 + 12st + 2t^2 = 2(\beta s + t) (\overline{\beta} s + t) = 2N_{Q(\sqrt{-10})}^{Q(\sqrt{-10})} (\beta s + t),
\]

where \( \beta = 3 + \sqrt{10}/2 \), \( \overline{\beta} \) is its Galois conjugate and \( N_{Q(\sqrt{-10})}^{Q(\sqrt{-10})} \) is the norm of \( Q(\sqrt{-10}) \). Since \( Q(\sqrt{-10}) \) is an imaginary field its norm is positive, hence \( e_1 = 1 \). Furthermore the unique prime above 3 in \( Q(\sqrt{-10}) \) has norm 9, so \( e_2 = 1 \). We thus have that

\[
2(\beta s + t) (\overline{\beta} s + t) = a_2^2.
\]

Note that the factors \( \beta s + t \) and \( \overline{\beta} s + t \) are coprime outside primes dividing \( \beta - \overline{\beta} = \sqrt{-10} \). Since \( a_2 \) is an integer not divisible by 2 and 5 which ramify in \( Q(\sqrt{-10}) \) these factors must have valuation \(-1\) at the unique prime \( p_2 \) above 2 and valuation 0 at the unique prime above 5. This implies that

\[
(\beta s + t) = p_2^{-1} I^2
\]

for some ideal \( I \) of \( O_{Q(\sqrt{-10})} \). Since the class number of \( Q(\sqrt{-10}) \) is 2 this would imply that \( p_2 \) is principal, which is not the case. Therefore no solution can correspond to the case \( j = 1 \) and thereby no solution to equation \( (2) \) with \( \gcd(x, y) = 1 \) exists for \( l = 3 \).

Remark. Geometrically, equations \( (3) \) define a curve with a degree 4 map to \( \mathbb{P}^1 \). The hyperelliptic curves we constructed in these sections are quotients of these curves through which this degree 4 map factors as two degree 2 maps. The only other geometric quotients with this same property are defined by taking the equation for \( a^2 \) and setting \( t = 1 \) or taking the equation for \( b^2 \) and setting \( t = 1 \). The quotient we considered is the only quotient for which rational points correspond to rational solutions of the corresponding equations, making this the natural choice. The same will be true for the equations \( (11) \) in the next section.
3.3 Case $l = 5$

Now suppose we have a solution $(a, b, c)$ to equation (2) with $\gcd(a, b) = 1$ and $l = 5$. Recall from Section [2] that

$$(h(a, b)) = p_3^{-1}I^5 = p_3^4(p_3^{-1}I)^5$$

as fractional ideals in $L$. Since $p_3^4 = (3)$ and 5 does not divide the order of the class group of $L$ we find that $p_3^{-1}I$ must be a principal ideal, hence

$$h(a, b) = 3u\gamma^5$$

for some unit $u \in O_L^*$ and $\gamma \in L$. Furthermore the valuation of $\gamma$ is only negative at $p_3$ where it is $-1$, hence $\gamma \in 1/3O_L$. Note that in these arguments we may also replace $L$ with $\mathbb{Q}(v)$. The field $\mathbb{Q}(v)$ is a number field of degree 4 and hence $O_{\mathbb{Q}(v)}$ can be parameterized by four integer coefficients.

Using this description we obtain for each choice of $u$ a parameterization of $a$ and $b$ together with two equations in the four indeterminates. Since $O_{\mathbb{Q}(v)}^*$ is generated by $u_0 = (-1)^5$ and an element $u_1$ of infinite order it is sufficient to consider $u = u_i$ for $i \in \{0, \ldots, 4\}$. Considering the equations obtained for each of these $i$ modulo 5, we see that only two of them can parameterize coprime $a$ and $b$, leaving only the cases $i = 0, 4$.

Without loss of generality we may assume that $g_1$ is the product of $h$ with $\sigma h$, where $\sigma$ is the automorphism on $\mathbb{Q}(v)$ mapping $v$ to $-v$. This implies that we have

$$g_1(a, b) = 9(u^\sigma u)(\gamma^\sigma \gamma)^5.$$ 

Since $\mathbb{Q}(v^2) = \mathbb{Q}(\sqrt{30}) = K$ we find that $u' := u^\sigma u \in O_K^*$ and $\gamma' := \gamma^\sigma \gamma \in K$. Again $\gamma'$ is only not integral at $q_3$ and furthermore $\text{ord}_{q_3}(\gamma) = -1$, so we have that $\gamma' \in q_3^{-1} = (\frac{1}{3}) q_3$. From the case $l = 3$ we know that $q_3$ has an integral basis formed by 3 times the coefficients of $g_1$, hence $q_3^{-1}$ has an integral basis formed by the coefficients of $g_1$. In particular it has an integral basis of the form $\left\{1, \frac{\sqrt{30}}{3}\right\}$ which gives us a parameterization of the form

$$\gamma' = g_1(\sqrt{s}, \sqrt{t})$$
$$a^2 = F_{5,u'}(s, t)$$
$$b^2 = G_{5,u'}(s, t),$$
$$c = 3s^2 - 10t^2$$

for each choice of unit $u'$. Here $F_{5,u'}$ and $G_{5,u'}$ are homogeneous polynomials over $\mathbb{Q}$ of degree 5.
We now study these equations for the remaining cases $u' = 1^\sigma 1 = 1$ and $u' = u_1^4 \sigma u_4^4 = u_8^4$. As in the case $l = 3$ we can see that $t \neq 0$ and hence we can construct hyperelliptic curves by multiplying the equations for $a^2$ and $b^2$ and dividing the result by $t^{10}$. These give us the hyperelliptic curves

$$C_1 : Y^2 = 405 X^9 - 4050 X^8 + 16200 X^7 - 54000 X^6 + 113400 X^5 - 198000 X^4 + 180000 X^3 - 120000 X^2 + 50000 X - 20000$$

$$C_{u_8} : Y^2 = -3083 903 014 930 297 409 520 X^{10} - 56 304 108 214 517 165 808 555 X^9 - 462 585 452 239 544 611 432 050 X^8 - 2 252 164 328 580 632 342 200 X^7 - 7 195 773 701 504 027 888 934 000 X^6 - 15 765 150 300 064 806 426 395 400 X^5 - 23 985 912 338 346 757 629 798 000 X^4 - 25 024 048 095 340 962 581 580 000 X^3 - 17 132 794 527 390 541 164 120 000 X^2 - 6 951 124 470 928 045 161 550 000 X - 1 269 095 890 917 817 864 020 000$$

in the variables $X = \frac{x}{t}$ and $Y = \frac{ab}{t^5}$. Studying the Jacobians of these curves in MAGMA [7] we find that both Jacobians only contain two-torsion points. Since $F_{5,1}G_{5,1}$ and $F_{5,u_8}G_{5,u_8}$ only contain one linear factor we conclude as in the case $l = 3$ that both curves only have one rational point. Note that these rational points correspond to values for $s$ and $t$ for which either $2 \mid c$ or $3 \mid c$ which is impossible by Proposition 2.1. This proves that no solution $(a, b, c)$ to equation (2) with $\gcd(a, b) = 1$ and $l = 5$ can exist.

Remark. It is necessary to first look at the factorization in $\mathbb{Q}(v)$, since some of the hyperelliptic curves that come from units we have not considered over $K$ have Jacobians with a rank bound that is not zero. Furthermore these curves also don’t have a local obstruction.

4 The Frey curves

In this section we construct Frey curves for our problem. A Frey curve is an elliptic curve that depends on the solution $(a, b, c)$, which has a mod $l$ Galois representation unramified outside some fixed set of primes $S$. The set $S$ should be independent of the chosen solution $(a, b, c)$. Such curves
can be realized by a curve which has a minimal discriminant that is an \( l \)-th power outside \( S \) and which has potentially good reduction outside \( S \).

For our cases we construct such curves using the following fact. Given two arbitrary elements \( B_1 \) and \( B_2 \) of a field \( k \) of which their sum is a square, i.e. \( B_1 + B_2 = A^2 \), we can look at the elliptic curve

\[
E : y^2 = x^3 + 2Ax^2 + B_1x
\]

defined over \( k \), for which this model has discriminant \( \Delta = 64B_1^2B_2 \). Furthermore this curve can only have additive reduction at primes above 2 and primes that divide both \( B_1 \) and \( B_2 \). This is easily verified by looking at the invariant \( c_4 = 16(B_1 + 4B_2) \), which is coprime to \( \Delta \) outside such primes.

Note that this recipe will give us a Frey curve if \( B_1 \) and \( B_2 \) are coprime \( l \)-th powers outside the fixed set \( S \). In fact this is the same Frey curve considered for the generalized Fermat equation with signature \((l, l, 2)\).

Now over the field \( K = \mathbb{Q}(\sqrt{30}) \) we know that we have two factors \( g_1(a, b) \) and \( g_2(a, b) \) which are coprime \( l \)-th powers outside the set of primes above 2, 3 and 5. Furthermore we have that

\[
\left( \frac{1}{2} - \frac{1}{10}\sqrt{30} \right) g_1(a, b) + \left( \frac{1}{2} + \frac{1}{10}\sqrt{30} \right) g_2(a, b) = a^2
\]

\[
\frac{1}{20}\sqrt{30} g_1(a, b) - \frac{1}{20}\sqrt{30} g_2(a, b) = b^2,
\]

hence we can apply the construction given above substituting for \( B_1 \) and \( B_2 \) the right multiples of \( g_1(a, b) \) and \( g_2(a, b) \). We will construct the Frey curves we will use from the four resulting Frey curves

\[
E'_1 : y^2 = x^3 + 2ax^2 + \left( \frac{1}{2} - \frac{1}{10}\sqrt{30} \right) g_1(a, b)x,
\]

\[
E''_1 : y^2 = x^3 + 2ax^2 + \left( \frac{1}{2} + \frac{1}{10}\sqrt{30} \right) g_2(a, b)x,
\]

\[
E'_2 : y^2 = x^3 + 2bx^2 + \frac{1}{20}\sqrt{30} g_1(a, b)x, \text{ and}
\]

\[
E''_2 : y^2 = x^3 + 2bx^2 - \frac{1}{20}\sqrt{30} g_2(a, b)x.
\]

Note that \( E''_1 \) and \( E''_2 \) are Galois conjugates of \( E_1 \) and \( E_2 \) respectively, so it suffices to consider only one of each pair. We pick \( E''_1 \) and \( E'_2 \) and twist these curves by 30 and 20 respectively to obtain two Frey curves with an
integral model

\[ E_1 : y^2 = x^3 + 60ax^2 + 30\left(15 + 3\sqrt{30}\right) a^2 + \sqrt{30} b^2 \] x, and
\[ E_2 : y^2 = x^3 + 40bx^2 + 20\left(\sqrt{30} a^2 + (10 + 2\sqrt{30}) b^2\right) x. \]

These models have respective discriminants

\[ \Delta_1 = -2^9 \cdot 3^6 \cdot 5^4 \left(5 + \sqrt{30}\right) \cdot g_1(a, b) \cdot g_2(a, b)^2 \] and
\[ \Delta_2 = -2^{13} \cdot 3 \cdot 5^4 \sqrt{30} \cdot g_1(a, b)^2 \cdot g_2(a, b), \]

c\text{4}-invariants

\[ c_{4,1} = -2^5 \cdot 3^2 \cdot 5 \left(5 + \sqrt{30}\right) \cdot \left((43 - 8\sqrt{30}) a^2 + (6 - \sqrt{30}) b^2\right) \] and
\[ c_{4,2} = -2^6 \cdot 3^{-1} \cdot 5 \sqrt{30} \cdot \left(9a^2 + (18 - 5\sqrt{30}) b^2\right), \]

and j-invariants

\[ j_1(a, b) = \left(11 + 2\sqrt{30}\right) \cdot 2^6 \cdot \frac{\left((43 - 8\sqrt{30}) a^2 + (6 - \sqrt{30}) b^2\right)^3}{g_1(a, b) \cdot g_2(a, b)^2} \] and
\[ j_2(a, b) = 2^6 \cdot 3^{-3} \cdot \frac{\left(9a^2 + (18 - 5\sqrt{30}) b^2\right)^3}{g_1(a, b)^2 \cdot g_2(a, b)}. \]

The j-invariants of these elliptic curves are not integral. We will prove this here as we will need this later on. In particular this implies that these curves do not have complex multiplication.

**Lemma 4.1.** The j-invariants \( j_1(a, b) \) and \( j_2(a, b) \) are not integral. Furthermore there exists a prime of characteristic \( > 5 \) such that \( j_1(a, b) \) and \( j_2(a, b) \) are not integral at that prime.

**Proof.** Note that since \( \gcd(a, b) = 1 \) the left hand side of equation (2) is the sum of at least two non-zero fourth powers, hence \( c > 1 \). By Proposition 2.1 there must be a prime number \( p > 5 \) dividing \( c \). This implies that either \( g_1(a, b) \) or \( g_2(a, b) \) is divisible by a prime above \( p \). It thus suffices to prove that the numerators of \( j_1(a, b) \) and \( j_2(a, b) \) are not divisible by the same prime.

Note that the factors

\[ \left(43 - 8\sqrt{30}\right) a^2 + \left(6 - \sqrt{30}\right) b^2, \]
and
\[ 9a^2 + \left(18 - 5\sqrt{30}\right)b^2, \]
in the numerators of \(j_1(a, b)\) and \(j_2(a, b)\) are coprime with \(g_1(a, b)\) and \(g_2(a, b)\) outside primes of characteristic 2, 3 and 5. This can be easily seen by computing the resultants of those polynomials with \(g_1\) and \(g_2\).

\[ \square \]

**Corollary 4.2.** The curves \(E_1\) and \(E_2\) do not have complex multiplication.

**Proof.** This follows directly from [27, II, Theorem 6.1] as the \(j\)-invariants are not integral. \[ \square \]

\section*{5 A Hilbert modular approach}

A natural way of using the Frey curves would be to use the modularity of elliptic curves over real quadratic fields to prove that there are Hilbert modular forms which have the same mod \(l\) Galois representation as \(E_1\) or \(E_2\). The level of these newforms will only depend on certain congruence classes of the chosen solution and hence all possible candidates can be explicitly computed. It turns out that the dimension of the corresponding spaces is too high to perform these computations in a reasonable time. Nevertheless we here describe the start of this approach.

We first need to compute the conductor of \(E_1\) and \(E_2\) as the level of the Hilbert modular forms associated to them depends on it.

**Proposition 5.1.** The conductor of \(E_1\) is
\[ N_1 = \begin{cases} p_2^{12}p_3^2p_5^2 \text{Rad}_{30}(g_1(a, b)g_2(a, b)^2) & \text{if } 2 \mid b \\ p_2^{10}p_3^2p_5^2 \text{Rad}_{30}(g_1(a, b)g_2(a, b)^2) & \text{if } 2 \nmid b, \end{cases} \]
and the conductor of \(E_2\) is
\[ N_2 = p_2^{14}p_3^3 \text{Rad}_{30}(g_1(a, b)^2g_2(a, b)), \]
where \(p_2, p_3\) and \(p_5\) are the unique primes above 2, 3 and 5 respectively and where \(\text{Rad}_{30}(N)\) is the product of all primes that divide \(N\) and do not divide 30.

**Proof.** This is a computation performed by the code [1] of the author, which makes use of Tate’s algorithm. \[ \square \]
It has been proven by Freitas, Le Hung and Siksek [14] that elliptic curves over real quadratic fields are modular. In particular the curves $E_1$ and $E_2$ are modular. According to [14] this means there are Hilbert cuspidal eigenforms $f_1$ and $f_2$ over $K = \mathbb{Q}(\sqrt{30})$ of parallel weight 2 with rational Hecke eigenvalues such that for all prime numbers $p$ we have

$$\rho_{E_i,p} \cong \rho_{f_i,p} : G_K \to \text{GL}_2(\mathbb{Q}_p), \quad i \in \{1, 2\}.$$  

Here $\rho_{E_i,p}$ is the $p$-adic Galois representation of $E_i$ induced by the Galois action on the Tate module $T_p(E)$ and $\rho_{f_i,p}$ is the $p$-adic Galois representation associated to $f_i$ by Carayol, Blasius, Rogawski, Wiles and Taylor.

Note that the conductor of $\rho_{E_i,p}$ is precisely the conductor of $E_i$ and the conductor of $\rho_{f_i,p}$ is precisely the level of $f_i$. Therefore we know from Proposition 5.1 that $f_1$ and $f_2$ have respective levels $N_1$ and $N_2$.

The levels $N_1$ and $N_2$ are not explicit as they depend on the chosen solution $(a, b, c)$. However if we take $p = l$ and look at the mod $l$ representations $\bar{\rho}_{E_i,l} : G_K \to \text{End}(E[l]) \cong \text{GL}_2(\mathbb{F}_l)$, rather than the $p$-adic representation, we find that these representations are irreducible and finite at all primes not dividing 30. This allows us to lower the level to a level only divisible by those primes dividing 30. We prove that the representation is finite here, but irreducibility needs more results later on and will be proven in Theorem 6.7.

**Proposition 5.2.** The mod $l$ Galois representations

$$\bar{\rho}_{E_i,l} : G_K \to \text{End}(E[l]) \cong \text{GL}_2(\mathbb{F}_l)$$

are finite outside all primes dividing 30. In particular they are unramified outside all primes dividing $30l$.

**Proof.** Note that for each finite prime $p$ of $K$ that does not divide 30 the order of $p$ in $N_1$ or $N_2$ is at most one. In case it is zero the curve has good reduction at $p$, hence the mod $l$ Galois representation is finite at $p$. We are left with the case the order is one, in which case $p$ must divide $g_1(a, b)$ or $g_2(a, b)$. Since $g_1(a, b)$ and $g_2(a, b)$ are $l$-th powers outside primes dividing 30, this implies that the order of $p$ in the corresponding discriminant $\Delta_i$ is a multiple of $l$. Since the corresponding curve $E_i$ has multiplicative reduction at $p$ the mod $l$ Galois representation is finite at $p$. This is a standard result that can be easily proved using the Tate curve if $p \mid l$.

Using the level lowering result found in [15] Theorem 7 for $l > 5$ we now find that there must be Hilbert cuspidal eigenforms $f_1'$ and $f_2'$ over $K$
of parallel weight 2 such that $\mathfrak{p}_{E_i,l} \cong \mathfrak{p}_{f^*_i,\lambda}$ for $i \in \{1, 2\}$, where $\lambda | l$ is a prime in the coefficient field of $f_i'$. Here the levels of these newforms are respectively

$$\tilde{\mathcal{N}}_1 = \begin{cases} p_2^{12} p_3^2 p_5^2 & \text{if } 2 | b \\ p_2^{10} p_3^2 p_5^2 & \text{if } 2 \nmid b \end{cases}$$

$$\tilde{\mathcal{N}}_2 = p_2^{14} p_5^2.$$

The strategy would now be to compute all cuspidal Hecke eigenforms over $K$ of parallel weight 2 and levels $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$ and prove that none of them can have a mod $\lambda | l$ representation isomorphic to the mod $l$ representation of the corresponding $E_i$. This would prove a contradiction, hence the implicit assumption that a primitive solution $(a, b, c)$ to equation (2) would be false as we want.

**Remark.** Besides working with the curves $E_1$ and $E_2$ one might also want to work with curves that are isomorphic over $\mathbb{Q}$. In case we do not want to change the field $K$ over which the curves are defined, these isomorphic curves would be twists of our original curves. Note that twisting by an element $\gamma \in K^*$ can only change the conductor of the curve at a prime $p$ if the corresponding field extension $K(\sqrt{\gamma})$ is ramified at $p$. The values $\gamma \in K^*$ for which $K(\sqrt{\gamma})$ is unramified at a fixed prime $p$ form a subgroup $H_p \subseteq K^*$ of which the quotient $K^*/H_p$ is finite. Limiting ourselves to the primes dividing $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$ we thus only have a finite computation to see if these levels can be made any smaller.

It turns out that by twisting we can get the lowest level

$$\tilde{\mathcal{N}}_1 = \begin{cases} p_2^{12} p_3^2 p_5^2 & \text{if } 2 | b \\ p_2^4 p_3^2 p_5^2 & \text{if } 2 \nmid b \text{ and } a \equiv 1 \pmod{4} \\ p_2^3 p_3^2 p_5^2 & \text{if } 2 \nmid b \text{ and } a \equiv 3 \pmod{4} \end{cases},$$

in case we twist $E_1$ with $6 + \sqrt{30}$. If we twist the curve $E_1$ by $-6 - \sqrt{30}$ we get the same level, but with the latter two conditions interchanged.

Using Magma we quickly find that the dimension of some of the sought spaces of newforms would be way too large to compute in. For example using the levels of the untwisted curves the dimension of the smallest space is 206720, which is way beyond the largest computational examples done in the literature. We can only do better in case $2 \nmid b$ where the twisted curve in the remark gives us a space of dimension 542 for the newforms of level $p_2^3 p_5^2$. A lower level for the case $2 | b$ lacks though, making this insufficient to prove Theorem 1.1 completely.
6 \( \mathbb{Q} \)-curves

In this section we use the modularity of \( \mathbb{Q} \)-curves to prove the non-existence of solutions. This technique has been applied to other Diophantine equations in works such as \([11]\), \([12]\), \([1]\), \([9]\), \([8]\), \([5]\), and \([13]\). The approach here is similar to the one in the mentioned articles, leaning heavily on the work by Quer \([21]\). It differs in some crucial points, where we will give an algorithmic approach that works in a general context.

Look back at our original curve

\[ E : y^2 = x^3 + 2Ax^2 + B_1x, \]

where \( A^2 = B_1 + B_2 \). Note that by construction this curve has a 2-torsion point and hence an obvious 2-isogeny defined over \( \mathbb{Q} \). From \([28, \text{III, example 4.5}]\) we deduce that the image of this 2-isogeny is

\[ \tilde{E} : y^2 = x^3 - 4Ax^2 + (4A^2 - 4B_1)x = x^3 - 4Ax^2 + 4B_2x, \]

which is a twist by \(-2\) of the complementary curve

\[ E' : y^2 = x^3 + 2Ax^2 + B_2x. \]

In particular such a curve is thus 2-isogenous over a field extension containing \( \sqrt{-2} \) to the curve in which the roles of \( B_1 \) and \( B_2 \) are swapped.

Note that for the Frey curves \( E_1 \) and \( E_2 \) we constructed, the chosen \( B_1 \) and \( B_2 \) were Galois conjugates of one another, whilst \( A \) was rational. This implies that the curves \( E_1 \) and \( E_2 \) are 2-isogenous to their Galois conjugates over \( K(\sqrt{-2}) \). This means that even though the curves are not defined over \( \mathbb{Q} \), their isogeny class over \( \overline{\mathbb{Q}} \) is. Curves over \( \overline{\mathbb{Q}} \) for which their isogeny class is defined over \( \mathbb{Q} \) are called \( \mathbb{Q} \)-curves.

Ribet \([25]\) proved, using the Serre conjectures, that \( \mathbb{Q} \)-curves, which do not have complex multiplication, are modular in the sense that they are isogenous to a quotient of \( J_1(N) \) for some integer level \( N > 0 \). This form of modularity gives us classical modular forms associated to our elliptic curves.

The proof of Ribet makes use of abelian varieties of GL\(_2\)-type. An abelian variety \( A \) over \( \mathbb{Q} \) is called of GL\(_2\)-type if \( \text{End} A \otimes \mathbb{Q} \) contains a number field of degree equal to \( \dim A \). Such a variety is also \( \mathbb{Q} \)-simple if and only if \( \text{End} A \otimes \mathbb{Q} \) is such a number field (\([25, \text{Theorem 2.1}]\)).

\( \mathbb{Q} \)-simple abelian varieties of GL\(_2\)-type naturally arise as the quotients of \( J_1(N) \) associated to Hecke eigenforms of weight 2. One dimensional quotients over \( \overline{\mathbb{Q}} \) of such varieties are naturally \( \mathbb{Q} \)-curves. Ribet proved that \( \mathbb{Q} \)-curves without complex multiplication are always isogenous to such
a quotient ([25, Theorem 6.1]). Furthermore he shows that \( \mathbb{Q} \)-simple abelian varieties of \( \text{GL}_2 \)-type are isogenous to a quotient of \( J_1(N) \) for some \( N > 0 \) if the Serre modularity conjectures hold ([25, Theorem 4.4]).

We want to apply this theory to our \( \mathbb{Q} \)-curves \( E_1 \) and \( E_2 \). In particular we want to explicitly compute levels \( N_1 \) and \( N_2 \) for which \( E_1 \) and \( E_2 \) are isogenous to quotients of \( J_1(N_1) \) and \( J_1(N_2) \) respectively, such that we can compute the associated modular forms. For this we will need to work more explicitly with the abelian varieties of \( \text{GL}_2 \)-type of which our curves will be quotients. We will use results of Quer [21] to compute these.

**Remark.** In the papers of Ribet and Quer [25, 21] all \( \mathbb{Q} \)-curves considered are without complex multiplication. For ease of writing we shall also assume this for all \( \mathbb{Q} \)-curves mentioned. This is fine since by Corollary 4.2 the curves \( E_1 \) and \( E_2 \) do not have complex multiplication.

### 6.1 Basic invariants

We will use the following main result by Quer.

**Proposition 6.1.** [21, Proposition 5.1] Let \( E \) be a \( \mathbb{Q} \) curve for which both \( E \) and the isogenies defining the \( \mathbb{Q} \)-curve structure are defined over some Galois number field \( F \). Let \( B = \text{Res}^F_{\mathbb{Q}} E \) be the restriction of scalars. If \( \text{End}_\mathbb{Q}(B) \) is a commutative algebra, then \( B \) factors over \( \mathbb{Q} \) as a product of \( \mathbb{Q} \)-simple mutually non \( \mathbb{Q} \)-isogenous abelian varieties of \( \text{GL}_2 \)-type.

In order to apply this result to our curves \( E_1 \) and \( E_2 \) we will first recap some general theory.

Let \( E \) be \( \mathbb{Q} \)-curve defined by isogenies \( \phi_\sigma : E^\sigma \to E \). To such a curve we can associate the *degree map* 

\[
d : G_\mathbb{Q} \to \mathbb{Q}^*, \quad \sigma \mapsto \deg \phi_\sigma,
\]

and a 2-cocycle \( c_E : G_\mathbb{Q}^2 \to \mathbb{Q}^* \) given by

\[
c_E(\sigma, \tau) = \phi_\sigma^{-1} \phi_\tau \phi^{-1}_{\sigma\tau}, \tag{11}
\]

where \( \phi^{-1}_{\sigma\tau} \) is the dual of \( \phi_{\sigma\tau} \) divided by its degree. Since \( E \) has no complex multiplication the element on the right can be considered as a non-zero element of \( \text{End}(E) \otimes \mathbb{Q} \cong \mathbb{Q} \), hence the equality makes sense. By taking degrees in equation (11) it is obvious that \( c_E^2 \) is the coboundary of \( d \). Furthermore Proposition 2.1 of [21] tells us that \( d : G_\mathbb{Q} \to \mathbb{Q}^*/(\mathbb{Q}^*)^2 \) and \( \xi(E) = [c_E] \in H^2(G_\mathbb{Q}, \mathbb{Q}^*) \) are invariants of the isogeny class of \( E \).
Note that \(d : G_{\mathbb{Q}} \to \mathbb{Q}^*/(\mathbb{Q}^*)^2\) has trivial coboundary, hence is a homomorphism. Its fixed field \(K_d\) is by Remark 2.6.b of [21] the smallest field over which a curve isogenous to \(E\) can be defined.

We say that \(E\) is completely defined over some field \(F\) if it is defined over \(F\) and all \(\phi_\sigma\) are also defined over \(F\). According to Remark 2.6.a of [21] there are curves isogenous to \(E\) that are completely defined over the field \(K_d(\sqrt{\pm d_1}, \ldots, \sqrt{\pm d_n})\) for \(\{d_1, \ldots, d_n\}\) a minimal set of generators of the image of \(d\) in \(\mathbb{Q}^*/(\mathbb{Q}^*)^2\).

A splitting map is a continuous map \(\beta : G_{\mathbb{Q}} \to \mathbb{Q}^*\) of which the coboundary with respect to the trivial action on \(\mathbb{Q}^*\) is precisely \(c_E\). If we consider such a \(\beta\) as a map to \(\mathbb{Q}^*/\mathbb{Q}^*\) it has trivial coboundary, hence it is a homomorphism. We will call the fixed field \(K_\beta\) a splitting field of \(E\).

A dual basis for the degree map \(d\) is a set \(\{(a_1, d_1), \ldots, (a_n, d_n)\} \subseteq \mathbb{Z}^2\) such that

- \(K_d = \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})\) and \([K_d : \mathbb{Q}] = 2^n\), and
- Each \(d_i\) is the image of \(d\) at some \(\sigma_i \in G_{K_d}\), where \(\sigma_i\) satisfies
\[
\sigma_i \sqrt{a_j} = (-1)^{\delta_{ij}} \sqrt{a_j}
\]

Theorem 3.1 of [21] tells us that the sign component \(\xi_{\pm}(E)\) of \(\xi(E)\), i.e. the part generated by the sign of \(c_E\), can be described as
\[
\xi_{\pm}(E) = \prod_{i=1}^n (a_i, d_i),
\]
where \((a, d)\) is the quaternion algebra over \(\mathbb{Q}\) defined by \(i^2 = a, j^2 = d\) and \(ij = k = -ji\) inside \(\text{Br}_2(\mathbb{Q}) = H^2(G_{\mathbb{Q}}, \{\pm 1\})\).

This description of \(\xi_{\pm}(E)\) in terms of a dual basis gives us a condition to see if a curve isogenous to \(E\) can be completely defined over \(K_d\) [21 Corollary 3.3]. Furthermore combined with the discussion on page 302 of [21] we know that a character \(\varepsilon : G_{\mathbb{Q}} \to \mathbb{Q}^*\) is a splitting character if and only if
\[
\varepsilon_p(-1) = \prod_{i=1}^n (a_i, d_i)_p,
\]
for every prime number \(p\). Here \(\varepsilon_p\) is the \(p\)-part of \(\varepsilon\) considered as a Dirichlet character and \((a_i, d_i)_p\) is a Hilbert symbol.
A decomposition field for $E$ is an abelian number field that contains both a field over which $E$ is completely defined and a splitting field of $E$. Proposition 5.2 of [21] tells us that the condition that $\text{End}_\mathbb{Q}(B)$ is commutative in Proposition 6.1 can be replaced by the condition that $F$ is a decomposition field and the existence of some splitting map defined over $G_F^\mathbb{Q}$. Furthermore [21, Proposition 5.2] states that if $F$ is a decomposition field, the latter is always true for some curve isogenous to $E$ over $\mathbb{Q}$.

We now explicitly compute all these quantities for our $\mathbb{Q}$-curves $E_1$ and $E_2$.

**Proposition 6.2.** For both $E_1$ and $E_2$ we have the same data listed below.

- The degree map $d : G_\mathbb{Q} \rightarrow \mathbb{Q}^*$ given by
  
  $$d(\sigma) = \begin{cases} 1 & \text{if } \sigma \in G_K \\ 2 & \text{if } \sigma \notin G_K. \end{cases}$$

- The 2-cocycle $c : G_\mathbb{Q}^2 \rightarrow \mathbb{Q}^*$ given by
  
  $$c(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma \in G_K(\sqrt{-2}) \text{ or } \tau \in G_K \\ 1 & \text{if } \sigma \notin G_K, \quad \sigma \sqrt{-2} = -\sqrt{-2} \text{ and } \sigma \sqrt{30} = \sqrt{30}, \\ -1 & \text{if } \tau \notin G_K, \quad \sigma \sqrt{-2} = -\sqrt{-2} \text{ and } \sigma \sqrt{30} = -\sqrt{30}, \\ 2 & \text{if } \tau \notin G_K, \quad \sigma \sqrt{-2} = \sqrt{-2} \text{ and } \sigma \sqrt{30} = -\sqrt{30}, \\ -2 & \text{if } \tau \notin G_K, \quad \sigma \sqrt{-2} = \sqrt{-2} \text{ and } \sigma \sqrt{30} = \sqrt{30}, \\ 2 & \text{if } \tau \notin G_K, \quad \sigma \sqrt{-2} = -\sqrt{-2} \text{ and } \sigma \sqrt{30} = \sqrt{30}, \end{cases}$$

- The field $K_d = K = \mathbb{Q}(\sqrt{30})$ over which the curves are defined.
- The field $K(\sqrt{-2}) = K_d(\sqrt{-2})$ over which the curves are completely defined.
- A dual basis $\{(30, 2)\}$.
- A splitting character $\varepsilon : G_\mathbb{Q} \rightarrow \overline{\mathbb{Q}}^*$, that as a Dirichlet character is one of the characters of conductor 15 and order 4, with corresponding fixed field $K_\varepsilon = \mathbb{Q}(\zeta_{15} + \zeta_{15}^{-1})$ of degree 4.
- A splitting field $K_\beta = K(\zeta_{15} + \zeta_{15}^{-1}) = \mathbb{Q}(\sqrt{6}, \zeta_{15} + \zeta_{15}^{-1})$ of degree 8.
- A decomposition field $K_{\text{dec}} = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}, \zeta_{15} + \zeta_{15}^{-1})$ of degree 16.

**Proof.** All this data can be computed from the isogenies $\phi_\sigma : ^\sigma E_i \rightarrow E_i$, which we can take to be the identity if $\sigma \in G_K$ and the 2-isogeny over $K(\sqrt{-2})$. 

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described before otherwise. Note that the latter can be explicitly described using the formula in [28, III, example 4.5] and the map scaling with $\sqrt{-2}$. In fact we compute from this by hand the degrees of these isogenies and the one-cocycle discussed on pages 288 and 289 of [21] from which the code [1] can compute all the other data.

**Remark.** Note that the field $K(\sqrt{-2})$ is a minimal field over which a curve isogenous to $E_1$ or $E_2$ can be completely defined. This is easily verified through Corollary 3.3 in [21], as it excludes the case that an isogenous curve can be defined over the field $K_{d} = K$ which is the only possible smaller field. For this one checks that

$$(30, 2) \neq 1 \quad \text{and} \quad (30, 2) \neq (-1, 30),$$

inside $\text{Br}_2(\mathbb{Q})$, which can be verified by checking the corresponding Hilbert symbols at 5.

### 6.2 A decomposable twist

As mentioned Proposition 5.2 in [21] tells us that if $F$ is a decomposition field, then Proposition 6.1 applies to some curve isogenous to $E_i$. We thus know that there are curves isogenous to $E_1$ and $E_2$ of which the restrictions of scalars over $K_{\text{dec}}$ decompose as the product of $\mathbb{Q}$-simple abelian varieties of $\text{GL}_2$-type.

Our goal now is to find these curves isogenous to $E_1$ and $E_2$ for which the result of Proposition 6.1 applies. According to Proposition 5.2 of [21] these should be curves for which the class of the 2-cocycle in $H^2(G_{\mathbb{Q}}^{K_{\text{dec}}}, \mathbb{Q}^*)$ is the same as the coboundary of the chosen splitting map. It is already discussed on page 297 of [21] that these curves can be obtained as twists of the original curves by some element $\gamma \in K_{\text{dec}}$. This result depends on a certain embedding problem being unobstructed as proven in [22].

In most articles in which Frey $\mathbb{Q}$-curves are used a correct twist is found by making an educated guess or a small search using results from [22], for example in [11], [12], [8], and [9]. We present here an algorithmic approach that always works based on the approach in [4].

We first explain the approach for a general $\mathbb{Q}$-curve $E$. Let notation be as before and let $F$ be a decomposition field of $E$ with corresponding splitting map $\beta$. If $c_\beta$ is the coboundary of $\beta$ with respect to trivial action on $\mathbb{Q}^*$, we can look at the 2-cocycle $c_\beta/c_E$. By correctly rescaling $\beta$ if necessary this can be interpreted as an element of $H^2(G_{\mathbb{Q}}^F, \{\pm 1\})$, hence it defines an embedding problem of Galois groups as noted in [21, page 297].
The solutions $F(\sqrt{\gamma})$ to this embedding problem give $\gamma \in F^*$ that perform the sought twist. Furthermore these are the $\gamma$ that satisfy

$$\sigma \gamma = \alpha(\sigma)^2 \gamma \text{ for all } \sigma \in G^F_Q,$$

for some $\alpha : G^F_Q \to F^*$ of which the coboundary is $c_\beta / c_E$. By finding an explicit $\alpha : G^F_Q \to F^*$ of which the coboundary is $c_\beta / c_E$ we can find such a $\gamma \in F^*$ by applying Hilbert 90 to $\alpha^2$.

The trick used in [4] is to find such $\alpha$ that take values in the finitely generated group $O^*_F$. This makes finding $\alpha$ a linear problem in the exponents with the equations

$$\alpha(\sigma) \sigma \alpha(\tau) \alpha(\sigma \tau)^{-1} = \frac{c_\beta(\sigma, \tau)}{c_E(\sigma, \tau)} \quad \sigma, \tau \in G^F_Q.$$

In general such an $\alpha$ does not exist, but we can make a more precise statement. Since any solution $\gamma$ in the embedding problem may be changed by a square, we can limit the primes appearing in $\sigma \gamma^{-1}$ to those generating the class group and their conjugates. This implies we can find a finite set $S$ of primes of $F$ for which an $\alpha$ with values in the $S$-units $O^*_F, S$ should exist if any solution exists. In particular if the class number is one, such an $\alpha$ exists with values in $O^*_F$. In any case $O^*_F, S$ is finitely generated so the problem is solvable using linear algebra.

**Remark.** We can be very explicit about how unique the twist $\gamma \in F^*$ is. Combining the long exact sequences over $G^F_Q$ for the short exact sequences

$$1 \longrightarrow \{\pm 1\} \longrightarrow F^* \longrightarrow (F^*)^2 \longrightarrow 1,$$

and

$$1 \longrightarrow (F^*)^2 \longrightarrow F^* \longrightarrow F^*/(F^*)^2 \longrightarrow 1,$$

we get a diagram of the form

$$H^1 \left( G^F_Q, F^* \right) = 1 \quad \downarrow \quad \mathbb{Q}^* \to \left( F^*/(F^*)^2 \right)^{G^F_Q} \to H^1 \left( G^F_Q, (F^*)^2 \right) \to H^1 \left( G^F_Q, F^* \right) = 1 \quad \downarrow \quad H^2 \left( G^F_Q, \{\pm 1\} \right).$$
Here $H^1 \left( G^F_{Q}, F^* \right) = 1$ follows from Hilbert 90. Now the class of the sought $\gamma$ lives in $\left( F^*/(F^*)^2 \right)^{G^F_{Q}}$, that is all those $\gamma \in F^*$ for which $\sigma \gamma \gamma^{-1} \in (F^*)^2$ for all $\sigma \in G^F_{Q}$. Furthermore we see that two such $\gamma$ define the same resulting class in $H^2 \left( G^F_{Q}, \{ \pm 1 \} \right)$ if and only if they differ by an element of $Q^* (F^*)^2$.

We now apply this theory to $E_1$ and $E_2$. Since $K_{dec}$ has class number 1 we can search for an $\alpha : G^{K_{dec}}_{Q} \to O^*_{K_{dec}}$. The code \cite{1} does this to find suitable twists of $E_1$ and $E_2$. As remarked above we can change the twist by a square, which we do to find one for which the twist parameter has a smaller minimal polynomial. In fact we find that for $\gamma \in K_{dec}$ any root of the polynomial

$$x^8 - 40x^7 - 550x^6 - 1840x^5 - 285x^4 + 3600x^3 - 1950x^2 + 200x + 25,$$

the twists $E_{1,\gamma}$ and $E_{2,\gamma}$ of $E_1$ and $E_2$ by $\gamma$ satisfy the conditions of Proposition 6.1.

Note that by twisting the curves, we must also change the isogenies that define the $\mathbb{Q}$-curve structure. On page 291 of \cite{21} it is made explicit how these isogenies change under twists. The code \cite{1} computes these new isogenies automatically. As the isogenies differ and $\mathbb{Q}(\gamma) = K_\beta$, the twisted curves $E_{1,\gamma}$ and $E_{2,\gamma}$ are completely defined over $K_\beta$. Since the same splitting maps also work for the twisted curves, this means $K_\beta$ is a decomposition field for $E_{1,\gamma}$ and $E_{2,\gamma}$.

Applying Proposition 6.1 with the facts above we get the following proposition.

**Proposition 6.3.** For each $i \in \{ 1, 2 \}$ the abelian variety $\text{Res}^{K_\beta}_{Q} E_{i,\gamma}$ is $\mathbb{Q}$-isogenous to a product of $\mathbb{Q}$-simple, mutually non $\mathbb{Q}$-isogenous abelian varieties of $GL_2$-type.

In order to work with the abelian varieties of $GL_2$-type that arise in Proposition 6.3 we need to compute some explicit data about these varieties. In particular we will need the dimension and endomorphism ring of each factor. For this we need some more theory which we again first discuss for a general $\mathbb{Q}$-curve $E$.

Let $F$ be a decomposition field for $E$. As discussed in Section 5 of \cite{21}, the isogenies $\phi_\sigma : \sigma E \to E$ induce endomorphisms $\Phi_\sigma$ on $\text{Res}^F_{Q} E$ that are defined over $\mathbb{Q}$. If $\text{Res}^F_{Q} E$ also decomposes into abelian varieties $A$ of $GL_2$-type, these $\Phi_\sigma$ induce non-zero elements $\beta(\sigma) \in \text{End} A \otimes \mathbb{Q}$ on each $A$. Since
the ring \( \text{End} A \otimes \mathbb{Q} \) is a number field, these give rise to maps \( \beta : G^F_{Q} \rightarrow \mathbb{Q}^* \). Since

\[
\phi_\sigma \phi_\tau \phi_{\sigma \tau}^{-1} = c_E(\sigma, \tau) \text{ for all } \sigma, \tau \in G^F_{Q},
\]

we find that

\[
\beta(\sigma) \beta(\tau) \beta(\sigma \tau)^{-1} = c_E(\sigma, \tau) \text{ for all } \sigma, \tau \in G^F_{Q},
\]

so each \( \beta \) is a splitting map.

In this setting we also have that \( R = \text{End}_Q(\text{Res}^F_{Q} E) \otimes \mathbb{Q} \) is generated by \( \Phi_\sigma \) as a \( \mathbb{Q} \)-vector space. Given a splitting map \( \beta : G^F_{Q} \rightarrow \mathbb{Q}^* \) we get a ring homomorphism

\[
R \rightarrow \mathbb{Q}, \quad \Phi_\sigma \mapsto \beta(\sigma),
\]

of which the kernel defines a \( \mathbb{Q} \)-simple factor \( A \) of \( \text{Res}^F_{Q} E \). This and the fact above gives a correspondence between splitting maps \( G^F_{Q} \rightarrow \mathbb{Q}^* \) and the factors of \( \text{Res}^F_{Q} E \).

Note that any two splitting maps \( \beta_1, \beta_2 : G^F_{Q} \rightarrow \mathbb{Q}^* \) have the same coboundary, hence the difference \( \chi = \beta_2 \beta_1^{-1} \) is a character. Since there are only finitely many characters on \( F \) we can compute all such splitting maps by computing a single one, which we already did to compute a decomposition field \( F \).

Two distinct splitting maps might however correspond to \( \mathbb{Q} \)-simple factors of \( \text{Res}^F_{Q} E \) that are isogenous over \( \mathbb{Q} \). According to Lemma 5.3 of [21] this is the case if and only if the two splitting maps are Galois conjugates of one another. To compute the \( \mathbb{Q} \)-simple factors of \( \text{GL}_2 \)-type it thus suffices to compute all Galois conjugacy classes of splitting maps \( G^F_{Q} \rightarrow \mathbb{Q}^* \).

Note that by the construction of an abelian variety \( A \) from a splitting map \( \beta \), the ring \( \text{End} A \otimes \mathbb{Q} \) must be the smallest number field containing the values of \( \beta \). According to Proposition 4.1 in [21] this field can be explicitly computed from the corresponding splitting character and a dual basis of the degree map. Since \( A \) is \( \mathbb{Q} \)-simple and of \( \text{GL}_2 \)-type the degree of this field is also the dimension of \( A \).

Performing the necessary computations we can improve Proposition 6.3 as follows.

**Theorem 6.4.** Let \( i \in \{1, 2\} \). We have that

\[
\text{Res}^{K_\beta}_{Q} E_{i, \gamma} \text{ is } \mathbb{Q} \text{-isogenous to } A_{i, 1} \times A_{i, 2},
\]

where
• each $A_{i,j}$ is a $\mathbb{Q}$-simple abelian variety of $GL_2$-type over $\mathbb{Q}$ of dimension 4 with $\text{End} A_{i,j} \otimes \mathbb{Q} \cong L_\beta = \mathbb{Q}(\zeta_8)$, and
• the varieties $A_{i,1}$ and $A_{i,2}$ are not isogenous over $\mathbb{Q}$.

6.3 Modularity of $\mathbb{Q}$-curves

Since the Serre conjectures have been proven by Khare and Wintenberger in [16] and [17] we can now use the modularity result proven by Ribet in [25]. In order to be explicit about the level and character of the corresponding newforms we need to discuss some more theory.

Let $E$ be a $\mathbb{Q}$-curve as before and assume $B := \text{Res}_E^F \mathbb{Q}$ decomposes into abelian varieties of $GL_2$-type. If $A$ is a $\mathbb{Q}$-simple factor of $B$ with corresponding splitting map $\beta : G_Q^F \rightarrow \overline{\mathbb{Q}}^*$, we have a commutative diagram

$$
\begin{array}{ccc}
E \stackrel{\sigma}{\rightarrow} \sigma E & \rightarrow \phi_\sigma & \rightarrow E \\
\uparrow & \uparrow & \uparrow \\
B \stackrel{\sigma}{\rightarrow} B & \rightarrow \Phi_\sigma & \rightarrow B \\
\downarrow & \downarrow & \downarrow \\
A \stackrel{\sigma}{\rightarrow} A & \rightarrow \beta(\sigma) & \rightarrow A
\end{array}
$$

for each $\sigma \in G_Q$. Since the map $\Phi_\sigma \circ \sigma$ on $B$ is completely determined by $\phi_\sigma \circ \sigma$ on $E$, we find that the map $\beta(\sigma) \circ \sigma$ on $A$ is also completely fixed by this. In particular $\beta(\sigma) \circ \sigma$ acts on the Tate module $V_p(A)$ as $\phi_\sigma \circ \sigma$ acts on $V_p(E)$ for each prime number $p$.

Let $L_A := \text{End} A \otimes \mathbb{Q}$. As argued at the start of Section 2 in [25] the Tate module $V_p(A)$ is a free module over $L_A \otimes \mathbb{Q}_p$ of rank 2 and hence the action of $G_Q^F$ on it decomposes as a product of 2-dimensional $p$-adic representations $\rho_{A,p} : G_Q \rightarrow \text{GL}_2(L_{A,p})$ for $p \mid p$ a prime of $L_A$. By the argument above the action of $\sigma$ on $V_p(A) \cong (L_A \otimes \mathbb{Q}_p)^2$ is also $\beta(\sigma)^{-1}$ times the action of $\phi_\sigma \circ \sigma$ on $V_p(E)$, so $\rho_{A,p}(\sigma) \sim \beta(\sigma)^{-1} \rho_{E,p}(\sigma)$ if $\phi_\sigma = 1$. Here $\sim$ means the matrices are conjugate to each other. This is in particular the case for all $\sigma \in G_F$.

Remark. If there is an isogeny $\phi : E \rightarrow E'$ defined over $\overline{\mathbb{Q}}$, then by choosing specific isogenies $\phi'_\alpha : \sigma E' \rightarrow E$ we have a commutative diagram

$$
\begin{array}{ccc}
E \stackrel{\sigma}{\rightarrow} \sigma E & \rightarrow \phi_\sigma & \rightarrow E \\
\downarrow \phi & \downarrow \phi & \downarrow \phi \\
E' \stackrel{\sigma}{\rightarrow} \sigma E' & \rightarrow \phi'_\sigma & \rightarrow E'
\end{array}
$$

23
for all \( \sigma \in G_\mathbb{Q} \). This implies that \( \beta(\sigma) \circ \sigma \) acts on \( V_p(A) \) as \( \phi_\sigma \circ \sigma \) acts on \( V_p(E') \). We thus find that \( \rho_{A,p} \sim \beta(\sigma)^{-1} \rho_{E',p}(\sigma) \) for all \( \sigma \in G_\mathbb{Q} \) with \( \phi_\sigma = 1 \).

For each \( \sigma \in G_\mathbb{Q} \) we note that

\[
\det \rho_{A,p}(\sigma) = \det (\beta(\sigma)^{-1} \phi_\sigma \circ \sigma) = \beta(\sigma)^{-2} \det (\phi_\sigma \circ \sigma) = \beta(\sigma)^{-2} d(\sigma) \chi_p(\sigma) = \varepsilon(\sigma)^{-1} \chi_p(\sigma),
\]

where \( \varepsilon \) is the splitting character of the splitting map \( \beta \), \( d \) is the degree map and \( \chi_p \) is the \( p \)-cyclotomic character. The fact that \( \det (\phi_\sigma \circ \sigma) = d(\sigma) \chi_p(\sigma) \) can be easily shown using the Weil pairing.

Theorem 4.4 in [25] tells us that \( A \) is a quotient of \( J_1(N) \) for some \( N > 0 \). Looking at the proof we in fact have that \( A \) is isogenous to the quotient \( A_f \) of \( J_1(N) \) associated to a particular weight 2 newform \( f \) of level \( N \). The character of this newform is according to Lemma 4.3 and Lemma 3.1 in [25] equal to \( \chi_p^{-1} \cdot \det \rho_{A,p} = \varepsilon^{-1} \). Furthermore \( A \) and \( A_f \) being isogenous implies that \( N^{\dim A} \) is equal to the conductor of \( A \).

Applying the above to all factors of \( B = \text{End}_\mathbb{Q}^F E \) we find that

\[
B \text{ is } \mathbb{Q}\text{-isogenous to } \prod_i A_{f_i},
\]

for each \( f_i \) a newform of weight 2, level \( N_i \) and character \( \varepsilon_i^{-1} \). Here \( \varepsilon_i \) is the splitting character corresponding to \( A_{f_i} \). Using the formula to compute the conductor of the restriction of scalars from [20 Proposition 1] we now have two ways to compute the conductor \( N_B \) of \( B \), hence

\[
\Delta_F^2 N_{\mathbb{Q}}^E N_E = N_B = \prod_i N_{i}^{\dim A_{f_i}}, \quad (12)
\]

where \( N_E \) is the conductor of \( E \) over \( F \), \( \Delta_F \) is the discriminant of \( F \), and \( N_{\mathbb{Q}}^E \) is the norm of \( F \).

Note that the \( p \)-adic Galois representation \( \rho_{f_i,p} : G_{\mathbb{Q}} \to \text{GL}_2(K_{f_i,p}) \) is in fact \( \rho_{A_{f_i},p} : G_{\mathbb{Q}} \to GL_2(K_{f_i,p}) \) by definition. Here \( K_{f_i} \) is the coefficient field of \( f_i \) and we have \( K_{f_i} = \text{End} A_{f_i} \otimes \mathbb{Q} \). If \( \beta_i \) is the splitting map corresponding to \( A_{f_i} \) then each \( \rho_{f_i,p} \) is in fact defined by \( \beta_i^{-1} \) times the same action on \( V_p(E) \) as described before. This implies that for \( i \neq j \) we have \( \rho_{f_i,p} = \beta_i \beta_j^{-1} \rho_{f_j,p} \), where \( \chi = \beta_i \beta_j^{-1} \) is a character. Therefore the coefficients of \( f_j \) are a twist of the coefficients of \( f_i \), hence \( f_j \) is a twist of \( f_i \) by \( \chi \).

Remark. In the case there is only one factor \( A_{f_i} \) equation (12) is already sufficient to determine the level of the corresponding newform \( f_i \). This is
the case in [13], [12], [9], and [4]. In [11] there are two factors, but this
problem is solved by studying the relation between the conductors of the
Galois representations $\rho_{A_i,p}$ rather than the relation between the levels of
the $f_i$. In [8] there is also two factors and the same approach as the one here
is used.

In [3] Atkin and Li give results on how the level of a newform can change
under a twist. This allows us to relate the levels $N_i$ of the newforms related
to $E$. Combined with the formula in [12] this gives a way to compute a
finite list of candidates for the levels. For completeness we formulate the
result by Atkin and Li we use here.

**Proposition 6.5.** Let $f$ be a newform of level $N$, weight $k$ and character $\varepsilon$
and let $\chi$ be a Dirichlet character of level $p^\beta$ for some prime number $p$.
Let $\alpha$ and $\gamma$ be the valuation of $p$ in the conductor of $\varepsilon$ and $\varepsilon\chi$ respectively.
Let $\delta = \text{ord}_p N$ and $\delta' = \max\{\delta, \beta + 1, \beta + \gamma\}$, then for the newform $g$ that
is $f$ twisted by $\chi$ we have

- $g$ is a cusp form of level $p^{\delta'-\delta}N$, character $\varepsilon\chi^2$ and weight $k$.
- $g$ is not a cusp form of a level $N'$ such that $\text{ord}_q(N') < \text{ord}_q(N)$ for
  any prime $q \neq p$.
- $g$ is not a cusp form of a level $N'$ with $\text{ord}_p(N') < \delta'$ if either
  1. $\delta > \max\{\beta + 1, \beta + \gamma\}$,
  2. $\delta < \max\{\beta + 1, \beta + \gamma\}$ and $\gamma \geq 2$, or
  3. $\alpha = \beta = \gamma = \delta = 1$.

**Remark.** Proposition 6.5 is an analogue of Theorem 3.1 in [3]. The formulation
is different and stronger as the reasoning given in [3] can be used to
prove this stronger result. However the arguments in [3] fail for case 3 in
the last part, which was noted in [26]. A proof for this case was given by Li
and can be found in [26] for the case of Hilbert modular forms. This proof
also applies to modular forms.

We now apply this theory to our curves $E_1$ and $E_2$ to obtain the following
result.

**Theorem 6.6.** For each $i \in \{1,2\}$ there exists a factor $A_{i,j}$ such that $A_{i,j}$
is $\mathbb{Q}$-isogenous to the abelian variety $A_f$ of a newform

$$
\begin{align*}
&f \in S_2 \left( \Gamma_1 \left( 2^9 \cdot 3^2 \cdot 5 \text{ Rad}_{30} c \right), \varepsilon \right) & \text{if } i = 1, b \text{ even}, \\
&f \in S_2 \left( \Gamma_1 \left( 2^8 \cdot 3^2 \cdot 5 \text{ Rad}_{30} c \right), \varepsilon \right) & \text{if } i = 1, b \text{ odd}, \\
&f \in S_2 \left( \Gamma_1 \left( 2^{10} \cdot 3 \cdot 5 \text{ Rad}_{30} c \right), \varepsilon \right) & \text{if } i = 2.
\end{align*}
$$
Here $\text{Rad}_{30} c$ is the product of all primes $p \mid c$ with $p \nmid 30$ and $\varepsilon$ is one of the two Dirichlet characters of conductor 15 and order 4 for which the choice does not matter.

**Proof.** By the theory above each factor $A_{i,j}$ is isogenous to some abelian variety $A_f$ of some newform $f$. The character of these newforms is by the same theory equal to the inverse of a corresponding splitting character. The code [1] computes such a splitting character for each $A_{i,j}$, which are all one of the two characters mentioned. Since the mentioned characters are Galois conjugates of each other and Galois conjugates of splitting maps correspond to the same factor $A_{i,j}$ the choice indeed does not matter.

For the levels of these newforms we first compute the left hand side of equation (12) to find that

$$N_i = \begin{cases} 2^{72} 3^{16} 5^{12} (\text{Rad}_{30} c)^8 & \text{if } i = 1 \text{ and } 2 \mid b \\ 2^{64} 3^{16} 5^{12} (\text{Rad}_{30} c)^8 & \text{if } i = 1 \text{ and } 2 \nmid b \\ 2^{80} 3^8 5^{12} (\text{Rad}_{30} c)^8 & \text{if } i = 2, \end{cases}$$

where $N_i$ is the conductor of $\text{Res}_{K/\mathbb{Q}}^{K_\beta} E_{1,\gamma}$. The conductor of the elliptic curves was computed using a version of Tate’s algorithm implemented in the code [1] that works for Frey curves.

For each $i$ the newforms $f_{i,1}$ and $f_{i,2}$ corresponding to $A_{i,1}$ and $A_{i,2}$ are twists of one another by the character $\chi = \epsilon_8 \epsilon_5$ and its inverse. Here $\epsilon_8$ is the character of conductor 8 with $\epsilon_8(-1) = -1$ and $\epsilon_5$ is a character with conductor 5 and order 4. We can thus apply Proposition 6.5 by first twisting with $\epsilon_8$ and then with $\epsilon_5$ or $\epsilon_5^{-1}$. We immediately see that the levels should be the same for all primes $p \neq 2, 5$.

Note that for the order of 2 in the levels $N_{i,1}$ and $N_{i,2}$ of $f_{i,1}$ and $f_{i,2}$ respectively we know that

$$4 \text{ord}_2 N_{i,1} + 4 \text{ord}_2 N_{i,2} \geq 64,$$

by equation (12), hence the order of 2 in one of the two is at least 8. Since all splitting characters and twist characters can be defined modulo 120 the $\beta$ and $\gamma$ in Proposition 6.5 can never exceed $\text{ord}_2 120 = 3$. This implies that we are in case 1 in the last list of Proposition 6.5 and the order of 2 in both levels must be the same.

Now for the order of 5 we have

$$4 \text{ord}_5 N_{i,1} + 4 \text{ord}_5 N_{i,2} = 12.$$
Since the characters of the corresponding newforms have a conductor divisible by 5 at least one factor 5 has to appear in both levels. This implies that one of the levels has a single factor 5 and the other has a factor $5^2$. By picking the first we get the result as stated in this theorem.

Remark. The reasoning done in the proof to determine the order of a prime $p$ in the level of a newform can be generalized to an arbitrary case, however it might result in multiple candidates for levels. The code \cite{1} automates this procedure to compute the part of the level consisting of bad primes, i.e. primes which ramify in the decomposition field, divide the conductor of one of the relevant characters, or are divisible by a prime of additive reduction for the elliptic curve.

6.4 Level lowering

The levels appearing in Theorem\cite{6.6} still depend on the specific solution $(a, b, c)$ of equation \cite{2}. To get rid of the additional primes we will need to look at the mod $l$ Galois representations and use some level lowering results.

Let us first introduce what we consider to be the mod $p$ Galois representation. For an elliptic curve $E$ defined over $K$ and a prime number $p$, the mod $p$ Galois representation is simply

$$\overline{\rho}_{E,p} : G_K \rightarrow \text{End} E[p] \cong \text{GL}_2(\mathbb{F}_p),$$

induced by the Galois action on $p$-torsion points. This naturally is the reduction of the $p$-adic Galois representation

$$\rho_{E,p} : G_K \rightarrow \text{End} V_p(E) \cong \text{GL}_2(\mathbb{Q}_p),$$

induced by the Galois action on the Tate module $V_p(E)$. For a $\mathbb{Q}$-simple abelian variety $A$ of GL$_2$-type with $F = \text{End} A \otimes \mathbb{Q}$, we have seen that the Galois action on the Tate module $V_p(A)$ factors into multiple $p$-adic Galois representations $\rho_{A,p} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(F_p)$, as $V_p(A)$ is a 2-dimensional vector space over $\text{End} A \otimes \mathbb{Q}_p = \prod_{p|l} \mathbb{F}_p$. Similarly the action of $G_{\mathbb{Q}}$ on the $p$-torsion points factors into multiple mod $p$ Galois representations

$$\overline{\rho}_{A,p} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}_p),$$

as $A[l]$ is a 2-dimensional vector space over $\text{End} A \otimes \mathbb{F}_p = \prod_{p|l} \mathbb{F}_p$. Again these can be considered as reductions of their $p$-adic counterparts. For newforms the mod $p$ Galois representations are defined as those of the corresponding abelian variety.
To apply level lowering results we first need to know that the mod $l$ Galois representations of $E_1$ and $E_2$ are absolutely irreducible and unramified at the primes that should be eliminated from the level.

**Theorem 6.7.** For any $i = 1, 2$ and $l > 5$ the mod $l$ Galois representation $\overline{\rho}_{E_i,l}: G_K \to \text{End} E_i[l] \cong \text{GL}_2(F_l)$ is irreducible.

**Proof.** Suppose that the mod $l$ Galois representation of one of these curves is reducible for some prime $l > 5$. If $l = 11$ or $l > 13$ Proposition 3.2 of [13] tells us that the corresponding curve has potentially good reduction at all primes of characteristic $p > 3$. This would imply that the $j$-invariant is integral at all those primes ([25, VII.5.5]), which contradicts Lemma 4.1. Therefore $l \in \{7, 13\}$.

Now note that the mod $l$ representation being reducible means that the corresponding curve has an $l$-isogeny. Both $E_1$ and $E_2$ also have a $2$-isogeny and are defined over $K$, hence the curve must correspond to a $K$-point on the curve $X_0(2l)$. They are however not $\mathbb{Q}$-points as we have seen that the minimal field over which a curve in their isogeny class can be defined is $K$. We do however know that both curves are $2$-isogenous to their Galois conjugate, hence would correspond to a $\mathbb{Q}$-point of the quotient $C_l = X_0(2l)/w_2$, where $w_2$ is the Atkin-Lehner-involution. We study such points using MAGMA [7] for the remaining cases $l = 7, 13$.

In the cases $l = 7, 13$ the curve $C_l$ is an elliptic curve with only finitely many rational points. Note that these points correspond to $\mathbb{Q}(\sqrt{-7})$ points on $X_0(2l)$ for $l = 7$ and to $\mathbb{Q}(\sqrt{13})$ points on $X_0(2l)$ for $l = 13$. Since $E_1$ and $E_2$ do not correspond to rational points this gives a contradiction. □

**Remark.** Note that the proof above can also be extended to the cases $l = 3, 5$. In this case $X_0(2l)$ is a genus 0 curve, but by explicitly writing out the quotient map $X_0(2l) \to C_l$ we can see that no $K$-points on $X_0(2l)$ map to $\mathbb{Q}$-points on $C_l$. Since we will not need these cases, the proof has been left out.

**Corollary 6.8.** Let $f$ be a newform as in Theorem 6.6 and $l > 5$ be a prime number, then for each prime $\lambda | l$ in the coefficient field of $f$ the mod $\lambda$ Galois representation $\overline{\rho}_{f,\lambda}: G_{\mathbb{Q}} \to \text{GL}_2(F_{\lambda})$ is absolutely irreducible.

**Proof.** First of all we note that $\overline{\rho}_{f,\lambda} \sim \overline{\rho}_{A_{i,j},\lambda}$ for the corresponding factor $A_{i,j}$ in Theorem 6.6. By the discussion on page 23 we know that $\overline{\rho}_{A_{i,j},\lambda}(\sigma)$ is completely determined by the action of $\beta(\sigma) \circ \sigma$ on $A_{i,j}$. This is the same as the action of $\phi_{\sigma,\gamma} \circ \sigma$ on $E_{i,\gamma}$. As remarked on page 23 this is by construction
the same as the action of $\phi_{\sigma} \circ \sigma$ on $E_i$. Since $\phi_{\sigma} = 1$ for all $\sigma \in \mathcal{G}_K$, we have that
\[ \overline{\rho}_{A_{i,j},\lambda}|_{\mathcal{G}_K} = \beta^{-1}|_{\mathcal{G}_K} \cdot \overline{\rho}_{E_i,l}. \]
Since $\overline{\rho}_{E_i,l}$ is irreducible by Theorem 6.7, this implies that $\overline{\rho}_{A_{i,j},\lambda}$ is irreducible.

To get that $\overline{\rho}_{f,\lambda} \sim \overline{\rho}_{A_{i,j},\lambda}$ is absolutely irreducible, we note that this representation is odd by Lemma 3.2 of [25] and that $l$ is odd, in which case irreducibility and absolute irreducibility are equivalent.

**Proposition 6.9.** Let $f$ be a newform as in Theorem 6.6 and $l > 5$ be a prime number, then for each prime $\lambda \mid l$ in the coefficient field of $f$ the mod $\lambda$ Galois representation $\overline{\rho}_{f,\lambda}: \mathcal{G}_\mathbb{Q} \to \text{GL}_2(\mathbb{F}_\lambda)$ is finite outside primes dividing 30. In particular it is unramified outside primes dividing $30l$.

**Proof.** As in Corollary 6.8 we know that
\[ \overline{\rho}_{f,\lambda}|_{\mathcal{G}_K} \sim \beta^{-1}|_{\mathcal{G}_K} \cdot \overline{\rho}_{E_i,l} \]
for some $i \in \{1, 2\}$. Since $\beta$ is trivial on $\mathcal{G}_{K_\beta}$ we find that
\[ \overline{\rho}_{f,\lambda}|_{\mathcal{G}_{K_\beta}} \sim \overline{\rho}_{E_i,l}|_{\mathcal{G}_{K_\beta}}. \]
Since the discriminant of $K_\beta$ is only divisible by the prime numbers 2, 3 and 5, we find that the ramification subgroup $I_p$ of a prime number $p \mid 30$ is contained in $\mathcal{G}_{K_\beta}$. Note that $\overline{\rho}_{f,\lambda}$ being finite at $p \mid 30$ only depends on $\overline{\rho}_{f,\lambda}|_{I_p} \sim \overline{\rho}_{E_i,l}|_{I_p}$, for which this was already proven in Proposition 5.2. \(\square\)

We can now use level lowering results proven by Diamond in [10] based on work by Ribet [23] to lower the level to something independent of the chosen solution $(a, b, c)$.

**Theorem 6.10.** For each elliptic curve $E_{i,\gamma}$ and prime number $l > 5$ there exists a factor $A_{i,j}$ as in Proposition 6.6 such that for each prime ideal $\lambda \mid l$ of $\text{End} A_{i,j} \otimes \mathbb{Q} = \mathbb{Q}(\zeta_8)$ we have $\overline{\rho}_{A_{i,j},\lambda} \sim \overline{\rho}_{g,\lambda'}$ for some prime ideal $\lambda' \mid l$ in the appropriate field and a newform $g$ satisfying
\[
g \in S_2(\Gamma_1(23040), \varepsilon) \quad \text{if } i = 1, b \text{ even},
g \in S_2(\Gamma_1(11520), \varepsilon) \quad \text{if } i = 1, b \text{ odd},
g \in S_2(\Gamma_1(15360), \varepsilon) \quad \text{if } i = 2.
\]
Here $\varepsilon$ is one of the two Dirichlet characters of conductor 15 and order 4. The choice does not matter.
Proof. We start by picking some \( f \) as in Theorem 6.6. Let \( A_{i,j} \) be the corresponding factor. We already have that \( \bar{\rho}_{A_{i,j},\lambda} \sim \bar{\rho}_{f,\lambda} \) for an arbitrary prime \( \lambda \mid l \). We will show that we can find a newform of the level as in this theorem which still has an isomorphic Galois representation.

Note that \( \bar{\rho}_{f,\lambda} \) is irreducible by Corollary 6.8 and odd as it is the Galois representation of a newform.

We apply Theorem 4.1 in [10], which tells us we can find a newform \( g \) of weight 2 with an isomorphic Galois representation \( \bar{\rho}_{g,\lambda} \). The level of \( g \) is the level of \( f \) divided by all prime numbers \( p \) that appear only once in the level, do not divide \( l \) or the conductor of the character of \( f \), and at which the Galois representation is unramified. The levels and the explicit character in Proposition 6.6 and the result from Proposition 6.9 tell us that all those prime numbers \( p \) not dividing \( 30l \) satisfy these conditions and can thus be removed from the level.

Lastly we use Theorem 2.1 in [24] which shows that the same result holds for a newform \( g \) of weight 2 and a level in which all powers of \( l \) are also removed. The weight remains 2 as the Galois representation is finite at \( l \) by Proposition 6.9 and the fact that \( l > 5 \). The resulting level is the one given in this theorem.

Remark. Note that the newforms in Theorem 6.10 are in fact those that have the Serre level and weight of the corresponding irreducible representation \( \bar{\rho}_{A_{i,j},\lambda} \). The result of this theorem would therefore also directly follow from the Serre conjectures.

6.5 Newform elimination

The strategy to complete the proof of Theorem 1.1 is to show that the conclusion of Theorem 6.10 will give a contradiction, implying that the implicit assumption of a solution \((a, b, c)\) to equation (2) existing with \( l > 5 \) and \( \gcd(a, b) = 1 \) must be false. We derive this contradiction by comparing traces of Frobenius of \( \bar{\rho}_{E_{i,j},l} : G_{K_\beta} \to \text{GL}_2(\mathbb{F}_l) \) and \( \bar{\rho}_{g,\lambda} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_\lambda) \) for \( g \) in one of the given spaces. Note that since both representations are defined over different Galois groups, we need a small result.

Lemma 6.11. For any prime \( p \) of \( K_\beta \) of characteristic \( p \nmid 30l \) we have that \( \bar{\rho}_{g,\lambda}(\text{Frob}_p) \sim \bar{\rho}_{g,\lambda}(\text{Frob}_p^d) \), where \( d \) is the degree of the residue field of \( p \) and \( \sim \) denotes the two are conjugates.

Proof. Let \( p \) be an arbitrary prime of \( K_\beta \) of characteristic \( p \nmid 30l \). Note that a Frobenius element \( \text{Frob}_p \in G_{K_{\text{dec},p}} \) maps to the homomorphism \( x \mapsto x^{#\mathbb{F}_p} \)...
inside $G_{F_p}$, just as does $\text{Frob}_p^d$ for a Frobenius element $\text{Frob}_p \in G_{\mathbb{Q}_p}$. This means their difference lies in the ramification subgroup of $G_{\mathbb{Q}_p}$. Since $\overline{\rho}_{g,\lambda'}$ is unramified at $p$ by Proposition 6.9 we find that $\overline{\rho}_{g,\lambda'}(\text{Frob}_p) \sim \overline{\rho}_{g,\lambda'}(\text{Frob}_p^d)$.

Now the rest of the proof becomes a computation.

First we compute the newforms in the spaces mentioned in Proposition 6.10. These computations take quite some time, especially the computation for the newforms of level 15360, which took approximately 5 days of computation time in MAGMA [7] using a desktop computer (Intel core i5-6600 CPU, 3.3 GHz). For comparison computing the space of newforms of level 11520 took just under 8 minutes on the same machine and the space of newforms of level 23040 took just over an hour. For this reason all newforms were pre-computed and then stored by saving the Fourier coefficients for all primes smaller than 500, as this data is sufficient to compute the sought traces of Frobenius for those primes.

The table below gives some general data about each space of newforms. It lists from left to right the level of the newforms, the dimension of the corresponding newspace, the number of Galois conjugacy classes of newforms, the possible sizes of the Galois conjugacy classes, and the total number of newforms among all conjugacy classes. Note that the last is always twice the dimension mentioned before, since the Galois conjugacy class of the character consists of two characters.

| level  | dim. | # conj. classes | size of conj. classes | # newforms |
|--------|------|----------------|----------------------|------------|
| 11520  | 192  | 30             | 4, 8, 16, 24, 32, 48 | 384        |
| 23040  | 384  | 20             | 8, 40, 48            | 768        |
| 15360  | 752  | 14             | 16, 64, 80, 96, 128, 176, 192 | 1504       |

Table 1: Data of the computed newforms

For some small primes $p \nmid 30l$ of $K_{\beta}$ we compute the set of possible values of $\text{Tr} \overline{\rho}_{E_{i,\gamma,l}}(\text{Frob}_p)$. A standard result is that

$$\text{Tr} \overline{\rho}_{E_{i,\gamma,l}}(\text{Frob}_p) = \begin{cases} 
\#F_p + 1 - \#E_{i,\gamma}(\mathbb{F}_p) \mod l \\
\#F_p + 1 \mod l \\
-\#F_p - 1 \mod l 
\end{cases}$$

where the cases correspond to $E$ having good, split multiplicative and non-split multiplicative reduction at $p$ respectively. Note that the right hand side is each time the reduction of an integer that does not depend on $l$, but
does depend on the chosen solution \((a, b, c)\) used to form \(E_{i,\gamma}\). In fact it only depends on the value of \(a\) and \(b\) modulo \(p\), the characteristic of \(p\). Ignoring the case \(a \equiv b \equiv 0 \pmod{p}\) as \(a\) and \(b\) are coprime, we compute the set \(S_{i,p}\) of possible integers on the right hand side from all possible values of \(a\) and \(b\) modulo \(p\). The reduction type in these cases can be computed using the version of Tate’s algorithm implemented in the code [1].

On the other hand we compute the values of \(\text{Tr} \overline{\rho}_{g,\lambda'}(\text{Frob}_p)\) for each newform \(g\) found before, where \(p\) is the characteristic of \(p\), \(d = [\mathbb{F}_p : \mathbb{F}_p]\), and \(\lambda' \mid l\) is the prime ideal corresponding to a fixed \(\lambda \mid l\) in Proposition 6.10. This trace can be computed from \(\text{Tr} \overline{\rho}_{g,\lambda'}(\text{Frob}_p)\) and \(\text{det} \overline{\rho}_{g,\lambda'}(\text{Frob}_p)\) by the fact that for a 2-by-2 matrix \(A\) the value of \(\text{Tr} A^d\) can be expressed as a polynomial in \(\text{Tr} A\) and \(\text{det} A\). Since \(p\) does not divide the level we have

\[
\text{Tr} \overline{\rho}_{g,\lambda'}(\text{Frob}_p) = a_p(g) \pmod{\lambda'}
\]

\[
\text{det} \overline{\rho}_{g,\lambda'}(\text{Frob}_p) = \varepsilon(p) p \pmod{\lambda'},
\]

where \(a_p(g)\) is the \(p\)-th coefficient in the Fourier expansion of \(g\) and \(\varepsilon\) is the character of \(g\). Note again that the right hand side for both these values is the reduction of an algebraic integer that is independent of \(\lambda'\). Using these algebraic integers in the formula for \(\text{Tr} A^d\) we get an algebraic integer \(t_{g,p}\) independent of \(\lambda'\) of which the reduction \(\pmod{\lambda'}\) is the value of \(\text{Tr} \overline{\rho}_{g,\lambda'}(\text{Frob}_p)\).

Now for each newform \(g\) with corresponding curve \(E_{i,\gamma}\) and a prime ideal \(p \nmid 30\) we can compute the integer

\[
M_{g,p} = p \prod_{s \in S_{i,p}} N(s - t_{g,p}),
\]

where \(N\) denotes the norm of the appropriate number field. By Theorem 6.10 and Lemma 6.11 this algebraic integer is divisible by \(l\) since either \(l = p\) or one of the factors \(s - t_{g,p}\) reduces to

\[
\text{Tr} \overline{\rho}_{E_{i,t}}(\text{Frob}_p) - \text{Tr} \overline{\rho}_{g,\lambda'}(\text{Frob}_p) = 0
\]

modulo \(\lambda'\). Therefore if \(g\) would be the newform obtained in Theorem 6.10 corresponding to \(E_{i,\gamma}\), then \(l > 5\) should divide the norm of \(M_g\).

We now apply this result. For each newform \(g\) in the spaces of Theorem 6.10 we compute the integer

\[
M_g = \gcd\{M_{g,p} : p \nmid 30 \text{ of characteristic } p < 40\}.
\]

Furthermore we remove all factors 2, 3 and 5 from \(M_g\), giving us a number divisible only by all \(l > 5\) for which the newform \(g\) can still be the one in
Theorem 6.10. We eliminate all newforms for which \( M_g = 1 \) which leaves us with 14 newforms of level 11520, 12 newforms of level 23040 and 7 newforms of level 15360.

The last step is to use both Frey curves simultaneously. This is known as a multi-Frey approach and was also used in [11], [4] and [9]. Instead of computing the sets \( S_1, p \) and \( S_2, p \) independently we now compute one set \( S_p \subset \mathbb{Z}^2 \). This set contains for each congruence class of \((a, b)\) modulo \( p \) a pair \((s_1, s_2)\) such that \( s_i \) is the integer that reduces to \( \text{Tr}_{E_i, \gamma, l} (\text{Frob}_p) \) for that congruence class.

Now for each prime ideal \( p \nmid 30 \) and each pair of newforms \((g_1, g_2)\) remaining with \( g_i \) corresponding to \( E_{i, \gamma} \) we can compute

\[
M_{g_1, g_2, p} = \prod_{(s_1, s_2) \in S_p} \gcd(N(s_1 - t_{g_1, p}), N(s_2 - t_{g_2, p}))
\]

where \( N \) again denotes the norm in the appropriate field. As before this integer is divisible by all \( l > 5 \) for which \( g_1 \) and \( g_2 \) are newforms for which Theorem 6.10 holds.

We compute

\[
M_{g_1, g_2} = \gcd\{N M_{g_1, g_2, p} : p \nmid 30 \text{ of characteristic } p < 50\}
\]

for each pair of newforms \((g_1, g_2)\) and see that none of them are divisible by primes \( l > 5 \). If a solution \((a, b, c)\) with \( \gcd(a, b) = 1 \) to equation (2) would exist for \( l > 5 \), then this would contradict Theorem 6.10. Therefore no such solution to equation (2) can exist, proving Theorem 1.1 for \( l > 5 \) prime.

Remark. Most prime exponents \( l > 5 \) can already be eliminated by only looking at the curve \( E_{2, \gamma} \) at more primes than considered here and using more restrictions on \( a \) and \( b \). However it seems impossible to eliminate the case \( l = 7 \) in this way, hence the use of the multi-Frey curve approach.

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