Non-Bayesian particle filters

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Abstract

Particle filters for data assimilation in nonlinear problems use “particles” (replicas of the underlying system) to generate a sequence of probability density functions (pdfs) through a Bayesian process. This can be expensive because a significant number of particles has to be used to maintain accuracy. We offer here an alternative, in which the relevant pdfs are sampled directly by an iteration. An example is discussed in detail.

Keywords particle filter, chainless sampling, normalization factor, iteration, non-Bayesian

1 Introduction.

There are many problems in science in which the state of a system must be identified from an uncertain equation supplemented by a stream of noisy data (see e.g. [1]). A natural model of this situation consists of a stochastic differential equation (SDE):

\[ \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) dt + \mathbf{g}(\mathbf{x}, t) d\mathbf{w}, \]

where \( \mathbf{x} = (x_1, x_2, \ldots, x_m) \) is an \( m \)-dimensional vector, \( d\mathbf{w} \) is \( m \)-dimensional Brownian motion, \( \mathbf{f} \) is an \( m \)-dimensional vector function, and \( \mathbf{g} \) is a scalar (i.e., an \( m \) by \( m \) diagonal matrix of the form \( \mathbf{gI} \), where \( \mathbf{g} \) is a scalar and \( \mathbf{I} \) is the identity matrix). The Brownian motion encapsulates all the uncertainty in this equation. The initial state \( \mathbf{x}(0) \) is assumed given and may be random as well.

As the experiment unfolds, it is observed, and the values \( \mathbf{b}^n \) of a measurement process are recorded at times \( t^n \); for simplicity assume \( t^n = n\delta \),
where $\delta$ is a fixed time interval and $n$ is an integer. The measurements are related to the evolving state $x(t)$ by

$$b^n = h(x^n) + GW^n,$$

where $h$ is a $k$-dimensional, generally nonlinear, vector function with $k \leq m$, $G$ is a diagonal matrix, $x^n = x(n\delta)$, and $W^n$ is a vector whose components are independent Gaussian variables of mean 0 and variance 1, independent also of the Brownian motion in equation (1). The task is to estimate $x$ on the basis of equation (1) and the observations (2).

If the system (1) is linear and the data are Gaussian, the solution can be found via the Kalman-Bucy filter. In the general case, it is natural to try to estimate $x$ as the mean of its evolving probability density. The initial state $x$ is known and so is its probability density; all one has to do is evaluate sequentially the density $P_{n+1}$ of $x^{n+1}$ given the probability density $P_n$ of $x^n$ and the data $b^{n+1}$. This can be done by following “particles” (replicas of the system) whose empirical distribution approximates $P_n$. In a Bayesian filter (see e.g. [2, 3, 4, 5, 6, 7, 8, 9], one uses the pdf $P_n$ and equation (1) to generate a prior density, and then one uses the new data $b^{n+1}$ to generate a posterior density $P_{n+1}$. In addition, one may have to sample backward to take into account the information each measurement provides about the past and avoid having too many identical particles. Evolving particles is typically expensive, and the backward sampling, usually done by Markov chain Monte Carlo (MCMC), can be expensive as well, because the number of particles needed can grow catastrophically (see e.g. [10]).

In this paper we offer an alternative to the standard approach, in which $P_{n+1}$ is sampled directly without recourse to Bayes’ theorem and backward sampling, if needed, is done by chainless Monte Carlo [11]. Our direct sampling is based on a representation of a variable with density $P_{n+1}$ by a collection of functions of Gaussian variables parametrized by the support of $P_n$, with parameters found by iteration. The construction is related to chainless sampling as described in [11]. The idea in chainless sampling is to produce a sample of a large set of variables by sequentially sampling a growing sequence of nested conditionally independent subsets. As observed in [12, 13], chainless sampling for a SDE reduces to interpolatory sampling, as explained below. Our construction will be explained in the following sections through an example where the position of a ship is deduced from the measurements of an azimuth, already used as a test bed in [6, 14, 15].
2 Sampling by interpolation and iteration.

First we explain how to sample via interpolation and iteration in a simple example, related to the example and the construction in [12]. Consider the scalar SDE
\[ dx = f(x,t)dt + \sqrt{\sigma}dw; \]  
we want to find sample paths \( x = x(t), 0 \leq t \leq 1 \), subject to the conditions \( x(0) = 0, x(1) = X \).

Let \( N(a, v) \) denote a Gaussian variable with mean \( a \) and variance \( v \). We first discretize equation (3) on a regular mesh \( t_0, t_1, \ldots, t_N \), where \( t_n = n\delta, \delta = 1/N, 0 \leq n \leq N \), with \( x^n = x(t^n) \), and, following [12], use a balanced implicit discretization [16, 17]:
\[ x^{n+1} = x^n + f(x^n, t^n)\delta + (x^{n+1} - x^n)f'(x^n)\delta + W^{n+1}, \]
where \( f'(x^n, t^n) = \frac{\partial f}{\partial x}(x^n, t^n) \) and \( W^{n+1} \) is \( N(0, \sigma/N) \). The joint probability density of the variables \( x^1, \ldots, x^{N-1} \) is \( Z^{-1} \exp(-\sum_0^N V^i) \), where \( Z \) is the normalization constant and
\[ V_i = \frac{((1 - \delta f')(x^{n+1} - x^n) - \delta f)^2}{2\sigma\delta} = \frac{(x^{n+1} - x^n - \delta f/(1 - \delta f'))^2}{2\sigma_n}, \]
where \( f, f' \) are functions of the \( x^j \), and \( \sigma_n = \sigma\delta/(1 - \delta f')^2 \) (see [18]). One can obtain sample solutions by sampling this density, e.g. by MCMC, or one can obtain them by interpolation (chainless sampling), as follows.

Consider first the special case \( f(x, t) = f(t) \), so that in particular \( f' = 0 \). Each increment \( x^{n+1} - x^n \) is now a \( N(a_n, \sigma/N) \) variable, with the \( a_n = f(t^n)\delta \) known explicitly. Let \( N \) be a power of 2. Consider the variable \( x^{N/2} \). On one hand,
\[ x^{N/2} = \sum_1^{N/2} (x^n - x^{n-1}) = N(A_1, V_1), \]
where \( A_1 = \sum_1^{N/2} a_n, V_1 = \sigma/2 \). On the other hand,
\[ X = x^{N/2} + \sum_{N/2+1}^N (x^n - x^{n-1}), \]
so that
\[ x^{N/2} = N(A_2, V_2), \]
with
\[ A_2 = X - \sum_{N/2+1}^{N-1} a_n, \quad V_2 = V_1. \]

The pdf of \( x^{N/2} \) is the product of the two pdfs; one can check that
\[
\begin{align*}
\exp \left( -\frac{(x - A_1)^2}{2V_1} \right) & \exp \left( -\frac{(x - A_2)^2}{2V_2} \right) \\
& = \exp \left( -\frac{(x - \bar{a})^2}{2\bar{v}} \right) \exp(-\phi),
\end{align*}
\]
where \( \bar{v} = \frac{V_1 V_2}{V_1 + V_2}, \quad \bar{a} = \frac{V_1 A_1 + V_2 A_2}{V_1 + V_2}, \quad \phi = \frac{(A_2 - A_1)^2}{2(V_1 + V_2)}, \) \( e^{-\phi} \) is the probability of getting from the origin to \( X \), up to a normalization constant.

Pick a sample \( \xi_1 \) from the \( N(0, 1) \) density; one obtains a sample of \( x^{N/2} \) by setting
\[ x^{N/2} = \bar{a} + \sqrt{\bar{v}} \xi_1. \]
Given a sample of \( x^{N/2} \) one can similarly sample \( x^{N/4}, x^{3N/4} \), then \( x^{N/8}, x^{3N/8} \), etc., until all the \( x^j \) have been sampled. If we define \( \xi = (\xi_1, \xi_2, \ldots, \xi_{N-1}) \), then for each choice of \( \xi \) we find a sample \((x^1, \ldots, x^{N-1})\) such that
\[
\begin{align*}
\exp \left( -\frac{\xi_1^2 + \cdots + \xi_{N-1}^2}{2} \right) & \exp \left( -\frac{(X - \sum_n a_n)^2}{2\sigma} \right) \\
& = \exp \left( -\frac{(x^1 - x^0 - a_0)^2}{2\sigma/N} - \frac{(x^2 - x^1 - a_1)^2}{2\sigma/N} \\
& \quad - \cdots - \frac{(x^N - x^{N-1} - a_{N-1})^2}{2\sigma/N} \right),
\end{align*}
\]
where the factor \( \exp \left( -\frac{(X - \sum_n a_n)^2}{2\sigma} \right) \) on the left is the probability of the fixed end value \( X \) up to a normalization constant. In this linear problem, this factor is the same for all the samples and therefore harmless. One can repeat this sampling process for multiple choices of the variables \( \xi_j \); each sample of the corresponding set of \( x^j \) is independent of any previous samples of this set.

Now return to the general case. The functions \( f, f' \) are now functions of the \( x^j \). We obtain a sample of the probability density we want by iteration. First pick \( \Xi = (\xi_1, \xi_2, \ldots, \xi_{N-1}) \), where each \( \xi_j \) is drawn independently from
the $N(0,1)$ density (this vector remains fixed during the iteration). Make a first guess $x^0 = (x_0^1, x_0^2, \ldots, x_0^{N-1})$ (for example, if $X \neq 0$, pick $x = 0$). Evaluate the functions $f, f'$ at $x^j$ (note that now $f' \neq 0$, and therefore the variances of the various increments are no longer constants). We are back in previous case, and can find values of the increments $x_{j+1}^n - x_j^n$ corresponding to the values of $f, f'$ we have. Repeat the process starting with the new iterate. If the vectors $x^j$ converge to a vector $x = (x^1, \ldots, x^{N-1})$, we obtain, in the limit, equation (4), where now on the right side $\sigma$ depends on $n$ so that $\sigma = \sigma_n$, and both $a_n, \sigma_n$ are functions of the final $x$. The left hand side of (4) becomes:

$$\exp\left(-\frac{\xi_1^2 + \cdots + \xi_{N-1}^2}{2}\right) \exp\left(-\frac{(X - \sum_n a_n)^2}{2 \sum_n \sigma_n}\right).$$

Note that now the factor $\exp\left(-\frac{(X - \sum_n a_n)^2}{2 \sum_n \sigma_n}\right)$ is different from sample to sample, and changes the relative weights of the different samples. In averaging, one should take this factor as weight, or resample as described at the end of the following section. In order to obtain more uniform weights, one also can use the strategies in [11, 12].

One can readily see that the iteration converges if $KTM < 1$, where $K$ is the Lipshitz constant of $f$, $T$ is the length of the interval on which one works (here $T = 1$), and $M$ is the maximum norm of the vectors $x^{j+1} - x^j$. If this inequality is not satisfied for the iteration above, it can be re-established by a suitable underrelaxation. One should course choose $N$ large enough so that the results are converged in $N$. We do not provide more details here because they are extraneous to our purpose, which is to explain chainless/interpolatory sampling and the use of reference variables in a simple context.

3 The ship azimuth problem.

The problem we focus on is discussed in [6, 14, 15], where it is used to demonstrate the capabilities of particular Bayesian filters. A ship sets out from a point $(x_0, y_0)$ in the plane and undergoes a random walk,

$$x^{n+1} = x^n + dx^{n+1},$$
$$y^{n+1} = y^n + dy^{n+1},$$

(5)
for \( n \geq 0 \), and with \( x^0 = y^0 \) given, and \( dx^{n+1} = N(dx^n, \sigma) \), \( dy^{n+1} = N(dy^n, \sigma) \), i.e., each displacement is a sample of a Gaussian random variable whose variance \( \sigma \) does not change from step to step and whose mean is the value of the previous displacement. An observer makes noisy measurements of the azimuth \( \arctan(y^n/x^n) \), recording

\[
b^n = \arctan \frac{y^n}{x^n} + N(0, s).
\]

(6)

where the variance \( s \) is also fixed; here the observed quantity \( b \) is scalar and is not be denoted by a boldfaced letter. The problem is to reconstruct the positions \( x^n = (x^n, y^n) \) from equations (5,6). We take the same parameters as [6]: \( x_0 = 0.01, y_0 = 20, dx^1 = 0.002, dy^1 = -0.06, \sigma = 1 \cdot 10^{-6}, s = 25 \cdot 10^{-6} \).

We follow numerically \( M \) particles, all starting from \( X_0^i = x_0, Y_0^i = y_0 \), as described in the following sections, and we estimate the ship’s position at time \( n\delta \) as the mean of the locations \( X_n^i = (X_n^i, Y_n^i), i = 1, \ldots, M \) of the particles at that time. The authors of [6] also show numerical results for runs with varying data and constants; we discuss those refinements in section 6 below.

### 4 Forward step.

Assume we have a collection of \( M \) particles \( X^n \) at time \( t^n = n\delta \) whose empirical density approximates \( P_n \); now we find increments \( dX^{n+1} \) such that the empirical density of \( X^{n+1} = X^n + dX^{n+1} \) approximates \( P_{n+1} \). \( P_{n+1} \) is known implicitly: it is the product of the density that can be deduced from the SDE and the one that comes from the observations, with the appropriate normalization. If the increments were known, their probability \( p \) (the density \( P_{n+1} \) evaluated at the resulting positions \( X^{n+1} \)) would be known, so \( p \) is a function of \( dX^{n+1} \), \( p = p(dX^{n+1}) \). For each particle \( i \), we are going to sample a Gaussian reference density, obtain a sample of probability \( \rho \), then solve (by iteration) the equation

\[
\rho = p(dX^{n+1})
\]

(7)

to obtain \( dX^{n+1} \).

Define \( f(x, y) = \arctan(y/x) \) and \( f^n = f(X^n, Y^n) \). We are working on one particle at a time, so the index \( i \) can be temporarily suppressed. Pick two independent samples \( \xi_x, \xi_y \) from a \( N(0, 1) \) density (the reference density
in the present calculation), and set \( \rho = \frac{1}{2\pi} \exp \left( -\frac{\xi_x^2}{2} - \frac{\xi_y^2}{2} \right) \); the variables \( \xi_x, \xi_y \) remain unchanged until the end of the iteration. We are looking for displacements \( dX^{n+1}, dY^{n+1} \), and parameters \( a_x, a_y, v_x, v_y, \phi \), such that:

\[
2\pi \rho = \exp \left( -\frac{(dX^{n+1} - dX^n)^2}{2\sigma} - \frac{(dY^{n+1} - dY^n)^2}{2\sigma} - \frac{(f^{n+1} - b^{n+1})^2}{2s} \right) \exp(\phi) = \exp \left( -\frac{(dX^{n+1} - a_x)^2}{2v_x} - \frac{(dY^{n+1} - a_y)^2}{2v_y} \right) \quad (8)
\]

The first equality states what we wish to accomplish: find increments \( dX^{n+1}, dY^{n+1} \), functions respectively of \( \xi_x, \xi_y \), whose probability with respect to \( P_{n+1} \) is \( \rho \). The factor \( e^\phi \) is needed to normalize this term (\( \phi \) is called below a "phase"). The second equality says how the goal is reached: we are looking for parameters \( a_x, a_y, v_x, v_y \), (all functions of \( X^n \)) such that the increments are samples of Gaussian variables with these parameters, with the assumed probability. One should remember that in our example the mean of \( dX^{n+1} \) is \( dX^n \), and similarly for \( dY^{n+1} \). We are not representing \( P_{n+1} \) as a function of a single Gaussian- there is a different Gaussian for every value of \( X^n \).

To satisfy the second equality we set up an iteration for vectors \( dX^{n+1,j} (= dX^j \) for brevity) that converges to \( dX^{n+1} \). Start with \( dX^0 = 0 \). We now explain how to compute \( dX^{j+1} \) given \( dX^j \).

Approximate the observation equation (6) by

\[
f(X^j) + f_x \cdot (dX^{j+1} - dX^j) + f_y \cdot (dY^{j+1} - dY^j) = b^{n+1} + N(0, s), \quad (9)
\]

where the derivatives \( f_x, f_y \) are, like \( f \), evaluated at \( X^j = X^n + dX^j \), i.e., approximate the observation equation by its Taylor series expansion around the previous iterate. Define a variable \( \eta^{j+1} = (f_x \cdot dX^{j+1} + f_y \cdot dY^{j+1})/\sqrt{f_x^2 + f_y^2} \). The approximate observation equation says that \( \eta^{j+1} \) is a \( N(a_1, v_1) \) variable, with

\[
a_1 = -\frac{f - f_x \cdot dX^j - f_y \cdot dY^j - b^{n+1}}{\sqrt{f_x^2 + f_y^2}},
\]

\[
v_1 = \frac{s}{f_x^2 + f_y^2}. \quad (10)
\]
On the other hand, from the equations of motion one finds that \( \eta^{j+1} \) is \( N(a_2, \nu_2) \), with \( a_2 = (f_x \cdot dX^n + f_y \cdot dY^n)/\sqrt{f_x^2 + f_y^2} \) and \( \nu_2 = \sigma \). Hence the pdf of \( \eta^{j+1} \) is, up to normalization factors,

\[
\exp \left( -\frac{(x - a_1)^2}{2\nu_1} - \frac{(x - a_2)^2}{2\nu_2} \right) = \exp \left( -\frac{(x - \bar{a})^2}{2\bar{\nu}} \right) \exp(-\phi),
\]

where \( \bar{v} = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \), \( \bar{a} = \frac{a_1 \nu_1 + a_2 \nu_2}{\nu_1 + \nu_2} \), \( \phi = \frac{(a_1 - a_2)^2}{2(\nu_1 + \nu_2)} = \phi^{j+1} \).

We can also define a variable \( \eta^*_+^{j+1} \) that is a linear combination of \( dX^{j+1} \), \( dY^{j+1} \) and is uncorrelated with \( \eta^{j+1} \):

\[
\eta^*_+^{j+1} = -f_y \cdot dY^{j+1} + f_x \cdot dX^{j+1} \sqrt{f_x^2 + f_y^2}.
\]

The observations do not affect \( \eta^*_+^{j+1} \), so its mean and variance are known. Given the means and variances of \( \eta^{j+1}, \eta^*_+^{j+1} \) one can easily invert the orthogonal matrix that connects them to \( dX^{j+1}, dY^{j+1} \) and find the means and variances \( a_x, \nu_x \) of \( dX^{j+1} \) and \( a_y, \nu_y \) of \( dY^{j+1} \) after their modification by the observation (the subscripts on \( a, \nu \) are labels, not differentiations). Now one can produce values for \( dX^{j+1}, dY^{j+1} \):

\[
dX^{j+1} = a_x + \sqrt{\nu_x} \xi_x, \quad dY^{j+1} = a_y + \sqrt{\nu_y} \xi_y,
\]

where \( \xi_x, \xi_y \) are the samples from \( N(0, 1) \) chosen at the beginning of the iteration. This completes the iteration.

This iteration converges to \( X^{n+1} \) such that \( f(X^{n+1}) = b^{n+1} + N(0, s) \), and the phases \( \phi^j \) converge to a limit \( \phi = \phi_i \), where the particle index \( i \) has been restored. The time interval over which the solution is updated in each step is short, and we do not expect any problem with convergence, either here or in the next section, and indeed there is none; in all cases the iteration converges in a small number of steps. Note that after the iteration the variables \( X_i^{n+1}, Y_i^{n+1} \) are no longer independent- the observation creates a relation between them.

Do this for all the particles. The particles are now samples of \( P_{n+1} \), but they have been obtained by sampling different densities (remember that the parameters in the Gaussians in equation (8) vary). One can get rid of this heterogeneity by viewing the factors \( \exp(-\phi) \) as weights and resampling, i.e., for each of \( M \) random numbers \( \theta_k, k = 1, \ldots, M \) drawn from the uniform distribution on \( [0, 1] \), choose a new \( \hat{X}_{k}^{n+1} = X_i^{n+1} \) such that \( Z^{-1} \sum_{j=1}^{i-1} \exp(-\phi_j) < \)
\[ \theta_k \leq Z^{-1} \sum_{j=1}^{i} \exp(-\phi_j) \] (where \( Z = \sum_{j=1}^{M} \exp(-\phi_j) \)), and then suppress the hat. We have traded the resampling of Bayesian filters for a resampling based on the normalizing factors of the several Gaussian densities; this is a worthwhile trade because in a Bayesian filter one gets a set of samples many of which may have low probability with respect to \( P_{n+1} \), and here we have a set of samples each one of which has high probability with respect to a pdf close to \( P_{n+1} \).

Note also that the resampling does not have to be done at every step—for example, one can add up the phases for a given particle and resample only when the ratio of the largest cumulative weight \( \exp(-\sum \phi_i) \) to the smallest such weight exceeds some limit \( L \) (the summation is over the weights accrued to a particular particle \( i \) since the last resampling). If one is worried by too many particles being close to each other (“depletion” in the Bayesian terminology), one can divide the set of particles into subsets of small size and resample only inside those subsets, creating a greater diversity. As will be seen in section 6, none of these strategies will be used here and we will resample fully at every step.

### 5 Backward sampling.

The algorithm of the previous section is sufficient to create a filter, but accuracy may require an additional refinement. Every observation provides information not only about the future but also about the past—it may, for example, tag as improbable earlier states that had seemed probable before the observation was made; one may have to go back and correct the past after every observation (this backward sampling is often misleadingly motivated solely by the need to create greater diversity among the particles in a Bayesian filter). As will be seen below, this backward sampling does not provide a significant boost to accuracy in the present problem, but it is described here for the sake of completeness.

Given a set of particles at time \((n+1)\delta\), after a forward step and maybe a subsequent resampling, one can figure out where each particle \( i \) was in the previous two steps, and have a partial history for each particle \( i \): \( X_{n-1}^i, X_n^i, X_{n+1}^i \) (if resamples had occurred, some parts of that history may be shared among several current particles). Knowing the first and the last member of this sequence, one can interpolate for the middle term as in section 2, thus projecting information backward. This requires that one recompute \( dX^n \).
Let \( dX^{\text{tot}} = dX^n + dX^{n+1} \); in the present section this quantity is assumed known and remains fixed. In the azimuth problem discussed here, one has to deal with the slight complication due to the fact that the mean of each increment is the value of the previous one, so that two successive increments are related in a slightly more complicated way than usual. The displacement \( dX^n \) is a \( N(dX^{n-1}, \sigma) \) variable, and \( dX^{n+1} \) is a \( N(dX^n, \sigma) \) variable, so that one goes from \( X^{n-1} \) to \( X^{n+1} \) by sampling first a \( (2dX^{n-1}, 4\sigma) \) variable that takes us from \( X^{n-1} \) to an intermediate point \( P \), with a correction by the observation half way up this first leg, and then one samples a \( N(dX^{\text{tot}}, \sigma) \) variable to reach \( X^{n+1} \), and similarly for \( Y \). Let the variable that connects \( X^{n-1} \) to \( P \) be \( dX^{\text{new}} \), so that what replaces \( dX^n \) is \( dX^{\text{new}}/2 \). Accordingly, \( dX^{\text{new}} = (dX^{\text{new}}, dY^{\text{new}}) \), and for parameters \( a_x^{\text{new}}, a_y^{\text{new}}, v_x^{\text{new}}, v_y^{\text{new}} \) such that

\[
\exp \left( -\frac{\xi_x^2 + \xi_y^2}{2} \right) = \exp \left( -\frac{(dX^{\text{new}} - 2dX^{n-1})^2}{8\sigma} - \frac{(dY^{\text{new}} - 2dY^{n-1})^2}{8\sigma} \right) 
\times \exp \left( -\frac{(f^n - b^n)^2}{2s} \right) 
\times \exp \left( -\frac{(dX^{\text{new}} - dX^{\text{tot}})^2}{2\sigma} - \frac{(dY^{\text{new}} - dX^{\text{tot}})^2}{2\sigma} \right) \exp(\phi) 
\exp \left( -\frac{(dX^{\text{new}} - \bar{a}_x)^2}{2v_x^{\text{new}}} - \frac{(dY^{\text{new}} - \bar{a}_y)^2}{2v_y^{\text{new}}} \right),
\]

where \( f^n = f(X^{n-1} + dX^{\text{new}}/2, Y^{n-1} + dY^{\text{new}}/2) \) and \( \xi_x, \xi_y \) are independent \( N(0, 1) \) Gaussian variables. As in equation (8), the first equality embodies what we wish to accomplish- find increments, functions of the reference variables, that sample the new pdf at time \( n\delta \) defined by the forward motion, the constraint imposed by the observation, and by knowledge of the position at time \((n + 1)\delta t\). The second equality states that this is done by finding particle-dependent parameters for a Gaussian density.

We again find these parameters as well as the increments by iteration. Much of the work is separate for the \( X \) and \( Y \) components of the equations of motion, so we write some of the equations for the \( X \) component only. Again set up an iteration for variables \( dX^{\text{new},j} = dX^j \) which converge to \( dX^{\text{new}} \). Start with \( dX^0 = 0 \). To find \( dX^{j+1} \) given \( dX^j \), approximate the observation
equation (6), as before, by equation (9); define again variables \( \eta_{j+1} \), \( \eta_{j+1}^{++} \), one in the direction of the approximate constraint and one orthogonal to it; in the direction of the constraint multiply the pdfs as in the previous section; construct new means \( a_{x,j}^{1} \), \( a_{y,j}^{1} \) and new variances \( v_{x,j}^{1} \), \( v_{y,j}^{1} \) for \( dX, dY \) at time \( n \), taking into account the observation at time \( n \), again as before. This also produces a phase \( \phi = \phi_{0} \).

Now take into account that the location of the boat at time \( n+1 \) is known; this creates a new mean \( \bar{a}_{x} \), a new variance \( \bar{v}_{x} \), and a new phase \( \phi_{x} \), by \( \bar{v} = \frac{v_{1}v_{2}}{v_{1}+v_{2}} \), \( \bar{a}_{x} = \frac{a_{1}v_{1}+a_{2}v_{2}}{v_{1}+v_{2}} \), \( \phi_{x} = \frac{(a_{1}-a_{2})^{2}}{v_{1}+v_{2}} \), where \( a_{1} = 2a^{1} \), \( v_{1} = 4v_{1}^{1} \), \( a_{2} = X_{\text{tot}}^{1} \), \( v_{2} = \sigma \). Finally, find a new interpolated position \( dX_{j+1}^{\text{new}} = a_{x}^{\text{new}}/2 + \sqrt{v_{x}^{\text{new}}} \xi_{x} \) (the calculation for \( dY_{j+1}^{\text{new}} \) is similar, with a phase \( \phi_{y} \)), and we are done. The total phase for in this iteration is \( \phi = \phi_{0} + \phi_{x} + \phi_{y} \). As the iterates \( dX^{j} \) converge to \( dX^{\text{new}} \), the phases converge to a limit \( \phi = \phi_{i} \). The probability of a particle arriving at the given position at time \( (n+1)\delta t \) having been determined in the forward step, there is no need to resample before comparing samples. Once one has the values of \( X^{\text{new}} \), a forward step gives corrected values of \( X^{n+1} \); one can use this interpolation process to correct estimates of \( X^{k} \) by subsequent observations for \( k = n-1, k = n-2, \ldots \), as many as are useful.

6 Numerical results.

Before presenting examples of numerical results for the azimuth problem, we discuss the accuracy one can expect. A single set of observations for our problem relies on 160 samples of a \( N(0, \sigma) \) variable. The maximum likelihood estimate of \( \sigma \) given these samples is a random variable with mean \( \sigma \) and standard deviation \( .11\sigma \). We estimate the uncertainty in the position of the boat by picking a set of observations, then making multiple runs of the boat where the random components of the motion in the direction of the constraint are frozen while the ones orthogonal to it are sampled over and over from the suitable Gaussian density, then computing the distances to the fixed observations, estimating the standard deviation of these differences, and accepting the trajectory if the estimated standard deviation is within one standard deviation of the nominal value of \( s \). This process generates a family of boat trajectories compatible with the given observations. In Table I we display the standard deviations of the differences between the resulting paths and the original path that produced the observations after the number of steps indicated there (the means of these differences are statistically

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indistinguishable from zero). This Table provides an estimate of the accuracy we can expect. It is fair to assume that these standard deviations are underestimates of the uncertainty- a variation of a single standard deviation in $s$ is a strict constraint, and we allowed no variation in $\sigma$.

Table I
Intrinsic uncertainty in the azimuth problem

| step | $x$ component | $y$ component |
|------|---------------|---------------|
| 40   | .0005         | .21           |
| 80   | .004          | .58           |
| 120  | .010          | .88           |
| 160  | .017          | .95           |

If one wants reliable information about the performance of the filter, it is not sufficient to run the boat once, record observations, and then use the filter to reconstruct the boat’s path, because the difference between the true path and the reconstruction is a random variable which may be accidentally atypically small or atypically large. We have therefore run a large number of such reconstructions and computed the means and standard deviations of the discrepancies between path and reconstruction as a function of the number of steps and of other parameters. In Tables II and III we display the means and standard deviations of these discrepancies (not of their mean!) in the $x$ and $y$ components of the paths with 2000 runs, at the steps and numbers of particles indicated, with no backward sampling. (Ref. [6] used 100 particles). On the average the error is zero, and the error that can be expected in any one run is of the order of magnitude of the unavoidable error. The standard deviation of the discrepancy is not significantly smaller with 2 particles that with 100- the main source of the discrepancy is the uncertainty in the data. Most of time one single particle (no resampling) is enough; however, a single particle may temporarily stray into low-probability areas and creates large arguments and numerical difficulties in the various functions used in the program. Two particles with resampling keep each other within bounds, because if one of them strays it gets replaced by a replica of the other. The various more sophisticated resampling strategies at the end of section 4 make no discernible difference here, and backward sampling does not help much either, because they too are unable to remedy the limitation of the data set.
Table IIa
Mean and standard variation of the discrepancy between synthetic data and their reconstruction, 2000 runs, no back step, 100 particles

| n. of steps | x component | y component |
|-------------|-------------|-------------|
|             | mean        | s.d.        | mean        | s.d.        |
| 40          | .0004       | .04         | .0001       | .17         |
| 80          | -.001       | .04         | -.01        | .54         |
| 120         | -.0008      | .07         | -.03        | 1.02        |
| 160         | -.002       | .18         | -.05        | 1.56        |

Table IIb
Mean and standard variation of the discrepancy between synthetic data and their reconstruction, 2000 runs, no back step, 2 particles

| n. of steps | x component | y component |
|-------------|-------------|-------------|
|             | mean        | s.d.        | mean        | s.d.        |
| 40          | .002        | .17         | -.0004      | .20         |
| 80          | .01         | .43         | -.0006      | .58         |
| 120         | .01         | .57         | .009        | 1.08        |
| 160         | .006        | .54         | .01         | 1.67        |

In Figure 1 we plot a sample boat path, its reconstruction, and the reconstructions obtained (i) when the initial data for the reconstruction are strongly perturbed (here, the initial data for $x, y$ were perturbed initially by, respectively, .1 and .4), and (ii) when the value of $\sigma$ assumed in the reconstruction is random: $\sigma = N(\sigma_0, \epsilon \sigma_0)$, where $\sigma_0$ is the constant value used until now and $\epsilon = 0.4$ but the calculation is otherwise identical. This produces variations in $\sigma$ of the order of 40%; any larger variance in the perturbations produced negative value of $\sigma$. The differences between the reconstructions and the true path remain within the acceptable range of errors. These graphs show that the filter has little sensitivity to perturbations (we did not calculate statistics here because the insensitivity holds for each individual run).

We now estimate the parameter $\sigma$ from data. The filter needs an estimate of $\sigma$ to function, call this estimate $\sigma_{\text{assumed}}$. If $\sigma_{\text{assumed}} \neq \sigma$, the other assumptions used to produce the data set (e.g. independence of the displacements and of the observations) are also false, and all one has to do is detect
the fallacy. We do it by picking a trajectory of a particle and computing the quantity

$$D = \frac{\left( \sum_{j=2}^{J} (dX^{j+1} - dX^j)^2 + \sum_{j=2}^{J} (dY^{j+1} - dY^j)^2 \right)^2}{\sum_{j=2}^{J} (dX^{j+1} - dX^j)^2 + \sum_{j=2}^{J} (dY^{j+1} - dY^j)^2}. $$

If the increments are independent then on the average $D = 1$; we will try to find the real $\sigma$ by finding a value of $\sigma_{\text{assumed}}$ for which this happens. We chose $J = 40$ (the early part of a trajectory is less noisy than the later parts).

As we already know, a single run cannot provide an accurate estimate of $\sigma$, and accuracy in the reconstruction depends on how many runs are used. In Table III we display some values of $D$ averaged over 200 and over 5000 runs as a function of the ratio of $\sigma_{\text{assumed}}$ to the value of $\sigma$ used to generate the data. From the longer computation one can find the correct value of $\sigma$ with an error of about 3%, while with 200 runs the uncertainty is about 10%.

| Table III |
|---|
| The mean of the discriminant $D$ as a function of $\sigma_{\text{assumed}}/\sigma$, 30 particles |
| $\sigma_{\text{assumed}}/\sigma$ | 5000 runs | 200 runs |
|-------------------------------|-----------|----------|
| .5                            | 1.14 ± .01 | 1.21 ± .08 |
| .6                            | 1.08 ± .01 | 1.14 ± .07 |
| .7                            | 1.05 ± .01 | 1.10 ± .07 |
| .8                            | 1.04 ± .01 | 1.14 ± .07 |
| .9                            | 1.00 ± .01 | 1.01 ± .07 |
| 1.0                           | 1.00 ± .01 | .96 ± .07 |
| 1.1                           | .97 ± .01  | 1.01 ± .07 |
| 1.2                           | .94 ± .01  | .99 ± .07 |
| 1.3                           | .93 ± .01  | 1.02 ± .07 |
| 1.4                           | .90 ± .01  | .85 ± .06 |
| 1.5                           | .89 ± .01  | .93 ± .07 |
| 2.0                           | .86 ± .01  | .78 ± .05 |

7 Conclusions.

We have exhibited a non-Bayesian filtering method, related to recent work on chaintless sampling, designed to focus particle paths more sharply and thus require fewer of them, at the cost of an added complexity in the evaluation of each path. The main features of the algorithm are a representation of a new pdf by means of a set of functions of Gaussian variables and a resampling based on normalization factors. The construction was demonstrated on a standard ill-conditioned test problem. Further applications will be published elsewhere.

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