Confluences of Anomaly Freedom Requirements
in $M$-theory

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Abstract
A topological fact about eleven dimensions is used to motivate a potential new duality in $M$-theory. We complete the discussion of consistent limits of $M$-theory raised in a previous paper, to include gravitational anomaly cancelation and four-form flux quantization in the context of the $M^{10} \times S^1/Z_2$ compactifications. A surprise is found: If one includes a $Z_2$ anomaly which exists only in $8k + 3$ dimensions then there are two distinct quantum limits, one related to each of two equivalency classes of the orbifolds $M^{10} \times S^1/Z_2$. 

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1 Introduction

In 8\(k+3\) dimensions the sign of path integrals over fermions coupled to gauge fields and gravity is potentially ambiguous. This has been used in [2] to prove a half-integer quantization on period integrals of the four-form \(G\) of \(M\)-theory based on path integrals over worldvolume fermions on the \(M\)-theory membrane (which has a three-dimensional worldvolume so that \(k = 0\)). The relevant ambiguity is resolved by the specific topology of spacetime. Interestingly, the next higher dimension where this ambiguity could occur is eleven, where \(k = 1\). In this paper we present an intriguing consequence of this potential ambiguity, making a case for yet a new duality in \(M\)-theory.

In the construction of the low-energy effective action for \(M\)-theory compactified on \(M^{10} \times S^1/Z_2\), there are two parameters which remain unspecified by the minimal implementation of supersymmetry (ie: without incorporating loop effects). In this compactification, the primary goal is to couple eleven-dimensional supergravity to ten-dimensional vector multiplets propagating on each of the two hyperplanes fixed by the \(Z_2\) projection. It is well known by now that the consistent coupling requires that each of these vector multiplets live in the adjoint representation of \(E_8\). This construction also represents the strong-coupling limit of the \(E_8 \times E_8\) heterotic string theory. The two parameters which remain unfixed in the minimal coupling are the dimensionless ratio \(\nu \propto \lambda^6/\kappa^4\), where \(\lambda\) is the \(E_8\) coupling constant and \(\kappa\) is the eleven dimensional gravitational coupling constant, and a particular angle which appears in the definition of the four-form field strength \(G\). The first parameter, and the puzzle regarding its resolution was discussed in [3]. The second parameter was described in [1, 6, 7], and relates to the necessary coupling of \(G\) to the Chern-Simons forms associated with the \(E_8\) gauge potentials. We call this second parameter \(b\). It reflects a freedom to partially manipulate the Chern-Simons coupling from components \(G_{(11)}^{ABC}\) to components \(G_{ABCD}\), where \(A, B, C\) and \(D\) do not take the value 11.

As explained in [1] both of the parameters \(\nu\) and \(b\) are required to cancel all anomalies in the theory, and both parameters are fixed by this requirement.

There are three types of anomalies which we need to cancel. The first of these are the gauge anomalies, under which the \(E_8 \times E_8\) symmetry could be destroyed by quantum effects. The second are gravitational anomalies, under which eleven-dimensional general coordinate invariance (or equivalently local Lorentz invariance) could likewise be destroyed. The third type of anomaly relates to a topological quantization of \(G\) necessary to ensure the consistency of membranes in the quantum theory. In some sense this third anomaly is related to the tensor symmetry under which the three-form gauge potential transforms as \(C \rightarrow C + d\Lambda\).

Each of the three anomalies described in the last paragraph imply an independent constraint on the parameters \(b\) and \(\nu\). As described and proved in this paper, each of these constraints takes the form of a cubic equation defining a locus of permitted values of the pair \((b, \nu)\). It is remarkable that these loci can intersect at a common point. It is even more remarkable that there are precisely two such intersections. There are two main purposes of this paper. The first is to derive the three cubic equations describing the three independent anomaly cancelation requirements, and the second is to explain the relationship between the two intersection points. The reader may benefit from taking a preliminary glance at the figure.

For the case of \(M\)-theory compactified on \(M^{10} \times S^1/Z_2\) the anomalies arise for the following reasons. First of all, because of the \(Z_2\) projection the spin 3/2 gravitino \(\psi\) satisfies a constraint \(\Gamma_{11}\psi| = \psi|\), valid precisely on the fixed points of the projection. From the point of view of the ten-dimensional hyperplanes defined by the \(Z_2\) fixed points, this amounts to a chirality constraint. As a consequence of this, the gravitino fields contribute to a gravitational anomaly which is localized on the two fixed hyperplanes. This can only be cancelled if vector multiplets propagate on the fixed hyperplanes. These supply spin 1/2 gauginos which, in turn, further
contribute to the gravitational anomaly. The net gravitational anomaly can be canceled only if the gauge group is $E_8$. But at the same time, the $E_8$ transformations are subject to a gauge anomaly, also due to both the gravitino and the gauginos, which must be simultaneously canceled. When we refer to the gravitational or gauge anomalies, we refer to the anomalous variation of the quantum effective action $\delta \Gamma$ attributable to one-loop. An important aspect of our analysis concerns the overall sign of this anomaly. Although the precise structure and the precise coefficients which define $\delta \Gamma$ are readily determined from established results, we indicate that the overall sign of the anomaly is not so readily obtained, particularly in eleven dimensions, for reasons having to do with the ambiguity described above in the introductory paragraph.

Although the overall sign of the gravitational and gauge anomalies is correlated with the chirality of the gauginos (which are in-turn correlated with the chirality of the gravitino projection at the orbifold fixed points), this concern remains independent of the ambiguity which we are discussing, although in the end there may be a connection.

A conservative way to view the results of this paper is as an exploration of the effects of the overall sign of the one-loop gauge and gravitational anomalies on the supergravity construction described in the second paragraph. This analysis was motivated by an observation that, curiously, there exists an elegant solution to the anomaly cancelation requirements via a generalized Green-Schwarz mechanism for either of the two naive choices for the overall sign of this anomaly. Since it is not a-priori clear how this sign is fixed for the case of $M$-theory, since at the very least it involves a variety of deep issues from Euclideanization to regularization we leave this sign as an arbitrary parameter, and are encouraged by the observation in [2] that such an ambiguity is expected for $8k + 3$ dimensional path integrals. We defer a more detailed examination of the issue to a future publication. However, the elegance of our results speaks for itself. With this caveat, we refer to a sign ambiguity on $\delta \Gamma$ as the $Z_2$ anomaly. The essence of this $Z_2$ anomaly is that the path integral obtains a factor $(-1)^{\mu}$, where $\mu$ is an integer determined by topological considerations on the manifold. For the case at hand, these would include the $Z_2$ nature of our orbifold as well as the topology of $M^{10}$. Since $\mu$ necessarily falls into one of two equivalency classes “even” or “odd”, and since $\mu$ only enters into our discussion as the factor $(-1)^{\mu}$, we can without loss of generality restrict $\mu$ to take one of two values, 0 or 1. Furthermore, the path integration enters our discussion via its role in the computation of the anomaly. The effect is that the expression $\delta \Gamma$ which we would naively use in determining the anomaly freedom of $M$-theory should be replaced with $(-1)^{\mu} \delta \Gamma$, where $\mu$ remains an unspecified parameter which takes values 0 or 1 depending on yet-to-be-determined topological properties of $M^{10} \times S^1/Z_2$.

The anomaly freedom requirements of $M$-theory were previously analyzed for the choice $\mu = 0$ by several authors [3, 7, 8, 10]. In this paper we leave $\mu$ unspecified and find a more general set of constraints. As described above these distill into a system of three cubic equations which permit exactly two simultaneous solutions, one corresponding to each of the two choices $\mu = 0$ or $\mu = 1$. The unique point corresponding to $\mu = 0$ describes to a previous result due to Lu in [3], which requires $\lambda^6/\kappa^4 = (4\pi)^5$. The unique point corresponding to $\mu = 1$ requires $\lambda^6/\kappa^4$ to be exactly $1/27$ of the $\mu = 0$ result. It is nontrivial that either one of these cases should admit a solution, and quite remarkable that there is exactly one solution for each case. The possibility of a $Z_2$ anomaly highlights a possible duality between compactifications with spacetime topology corresponding to one value $\mu$ with those corresponding to the opposite value of $\mu$. The analysis of this paper involves a satisfying picture which includes both possibilities in a common framework.

Another remarkable aspect of $M$-theory is that its precepts offer an a-priori explanation for all Green-Schwarz terms necessary for gauge and gravitational anomaly cancelation. By way of
contrast, in the case of weakly-coupled string theory, the effective ten-dimensional supergravity theory has only the anomalies themselves to justify some of the Green-Schwarz terms. In M-theory, the counter terms are given by

\[ S_{GS} = -\frac{\sqrt{2}}{\kappa^3} \int_{M^{11}} \left( C \wedge G \wedge G - \frac{1}{(4\pi)^3 T_5} C \wedge X_8 \right) . \]  

(1.1)

The first term is known from ancient history [11], as it is one of the minimal couplings required by supersymmetry in eleven dimensions. However, the fundamental role of this term has only become evident in the last few years. The second term is required since M-theory contains fivebranes. The absence of worldvolume anomalies for the fivebrane requires this coupling. The eight-form \( X_8 \) is determined by this restriction \([5, 4]\), and is given by

\[ X_8 = -\frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2 , \]  

(1.2)

where \( R \) is the eleven-dimensional Riemann tensor expressed as a Lorentz-valued two-form. The factor \( T_5 \) is the fivebrane tension which, for reasons explained in section 3, is subject to a quantization rule given by

\[ T_5 = \left( \frac{\pi l}{2\kappa^4} \right)^{1/3} , \]  

(1.3)

where \( l \) is an integer. This quantization is related to the flux quantization of \( G \).

In this paper we always take \( \Gamma_{11} \psi | = \psi | \) (as opposed to \( \Gamma_{11} \psi | = -\psi | \)) so that we do not include certain irrelevant sign ambiguities present in [1].

In this paper we work exclusively in the upstairs approach. Thus, we have a boundary-free eleven-dimensional orbifold \( M^{10} \times S^1 / Z_2 \), where the \( Z_2 \) projection acts on \( x^{11} \in (-\pi, \pi) \) as \( x^{11} \rightarrow -x^{11} \). An alternative “downstairs” point of view, in which the spacetime has boundaries has been discussed \([3, 7, 6]\). According to the proponents of that alternate approach, it is merely a matter of taste or convenience to employ one picture or the other. This being so, provided that all parameters are chosen properly, it is clear that nothing is lost while clarity is gained by simply sticking to one convention.

This paper is structured as follows.

Section 2 comprises the main thrust of this paper. Here we present a derivation of the two independent constraints imposed by gauge anomaly freedom and gravitational anomaly freedom. These as well as a third relation expressing the flux quantization of \( G \) (which is derived in section 3), comprise algebraic relations between the two parameters \( \nu \) and \( b \) described above, as well as on a pair of ostensibly independent topological numbers. One of these is the number \( l \) which defines the fivebrane tension, and the other is a parameter \( m \) related to the flux quantization of \( G \). The two conditions on \( \nu \) and \( b \) each consist of an equation cubic in the parameter \( b \) and linear in the parameter \( \nu \). We discuss the simultaneous solution of these constraints, showing that there are precisely two solutions; the fivebrane tension is fixed, but the quantized flux of \( G \) can take one of two opposing values which are correlated with the \( Z_2 \) anomaly described above.

Section 3 consists of a derivation of the flux quantization condition for \( G \) expressed as an algebraic relation between \( b \) and \( \nu \) in the form which was used in section 2. This derivation is presented independently so as not to disturb the linear presentation in section 2.

Finally, section 4 summarizes what we have learned in this analysis, and describes several directions of future research naturally indicated by our results.

2 M-theory on \( M^{10} \times S^1 / Z_2 \)

We are considering M-theory compactified on the orbifold \( M^{10} \times S^1 / Z_2 \). As is well known by now, this requires ten dimensional vector supermultiplets to propagate on each of the two
ten-dimensional hyperplanes fixed by the orbifold projection. Considerations based on gauging the superalgebra then imply the following modification to the definition of the four-form field strength,

\[ G = 6dC - \frac{1}{6}(2b + 3\sqrt{2}) \frac{\kappa^2}{\lambda^2} \sum_{i=1}^{2} \delta(x^{11} - a_i) \, dx^{11} \wedge Q^0_{3(i)} - \frac{1}{6} b \frac{\kappa^2}{\lambda^2} \sum_{i=1}^{2} \theta(x^{11} - a_i) \, I^4_{4(i)}. \]  \hspace{1cm} (2.4)

In this expression the sums include contributions from each of the two orbifold fixed points, which are defined by \( x^{11} = a_i \) where \( a_1 = 0 \) and \( a_2 = \pi \). Note that the one-form gauge potentials \( A_{(i)} \), and therefore the respective two-form field strengths \( F_{(i)} \), are constrained to propagate on the fixed hyperplanes. The Heavyside step function \( \theta(x^{11} - a_i) \) has the property that \( d\theta(x^{11} - a_i) = 2\delta(x^{11} - a_i) \, dx^{11} \). The parameter \( b \) is a real parameter which was not described in the original work \[3\]. As was described in \[1\], and also previously in \[8, 7, 6\], this parameter is necessary for full anomaly cancelation. Further, its value is only selected by the requirement that quantum anomalies be absent (which includes the flux quantization of \( G \)).

The four-form \( I^4_{4(i)} \) is defined by

\[ I^4_{4(i)} = \frac{1}{2} \text{tr} R \wedge R - \text{tr} F_{(i)} \wedge F_{(i)} = dQ^0_{3(i)}, \]  \hspace{1cm} (2.5)

where \( Q^0_{3(i)} \) contains the Yang-Mills Chern-Simons three-form associated with the gauge potential \( A_{(i)} \) and also the Lorentz Chern-Simons three-form associated with the spin connection. Note that the presence of the \( F_{(i)} \wedge F_{(i)} \) term in \( I^4_{4(i)} \) follows from minimally implementing supersymmetry. Nicely, this contribution is independently required for gauge anomaly cancelation. These terms are required even in a classical theory in order to resolve the gauge superalgebra between the bulk eleven-dimensional supergravity fields and the \( E_8 \) fields propagating on the fixed hyperplanes. The \( R \wedge R \) modifications are included to enable cancelation of gravitational anomalies. The factor of one half that multiplies the \( R \wedge R \) term arises, essentially, because the gravitational anomaly is equally distributed between the two fixed hyperplanes.

The modified definition for \( G \) given in (2.4) has as a consequence the following Bianchi identity,

\[ dG = \frac{1}{\sqrt{2}} \frac{\kappa^2}{\lambda^2} \sum_{i=1}^{2} \delta(x^{11} - a_i) \, dx^{11} \wedge I^4_{4(i)}. \]  \hspace{1cm} (2.6)

Note that the parameter \( b \) cancels when applying the exterior derivative to equation (2.4). Alternatively, one can view \( b \) as parameterizing a family of solutions, given by (2.4) to the bianchi identity (2.6). When we include gauge anomaly cancelation, the value of \( b \) determines the ratio \( \lambda^6 / \kappa^4 \). But one requires cancelation of gravitational anomalies as well as a flux quantization rule on \( G \) in order to pin down the value of \( b \) necessary for a consistent quantum theory.

In what follows it is useful to define the dimensionless order parameter

\[ \nu = \frac{54\sqrt{2} \lambda^6}{(4\pi)^5 \kappa^4}, \]  \hspace{1cm} (2.7)

The numerical factor in this definition is chosen to optimally simplify certain expression to follow, particularly (2.11)-(2.13) below.

\(^2\)The parameter \( b \) in this paper is related to the parameter \( \alpha \) in [4] via the following correspondence, \( b = -3(1 + \alpha) / \sqrt{2} \).

\(^3\)As is customary for \( E_8 \) valued matrices, \( \text{tr} \equiv \frac{1}{30} \text{Tr} \), where \( \text{Tr} \) is the trace in the adjoint representation.
Gauge invariance of $G$ requires the three-form potential to transform as

$$\delta C = -\frac{1}{48} (2b + 3\sqrt{2}) \frac{\kappa^2}{\lambda^2} \sum_{i=1}^{2} \delta(x^{11} - a_i) \, dx^{11} \wedge Q^1_{2(i)},$$

(2.8)

where $Q^1_{2(i)}$ is defined by the relation $\delta Q^1_3(i) = dQ^1_{2(i)}$. The two-form $Q^1_{2(i)}$ is linear both in the transformation parameter for the $i$th $E_8$ factor and in the transformation parameter for a local Lorentz transformation. Thus, $C$ has special properties under each of these transformations.

As a consequence of the above, the Green-Schwarz terms shown in (1.1) transform as follows,

$$\delta S_{GS} = \frac{1}{12} (2b + 3\sqrt{2}) \frac{\kappa^2}{\lambda^2} \sum_{i=1}^{2} \int_{M^i_{10}} Q^1_{2(i)} \wedge I^4_4(i) \wedge I^4_4,$$

(2.9)

In deriving this result we have applied (2.8) to (2.7), have used the property of the step function that $\theta(x^{11} - a_i)^2 = 1$, and have used the definitions (1.3) and (2.7). It is then straightforward to algebraically manipulate the coefficients into the form shown in (2.9).

The quantum effective action gives rise to another anomalous variation due to the presence of certain one-loop diagrams. These are given by

$$\delta \Gamma = \frac{(-1)^\mu}{96 (2\pi)^5} \sum_{i=1}^{2} \int_{M^i_{10}} Q^1_{2(i)} \wedge (\frac{1}{4} I^4_4(i) \wedge I^4_4(i) - X_8).$$

(2.10)

This result is readily assembled from results in [3, 12, 13, 14]. Commonly the anomaly is expressed in terms of a formal twelve-form, which is a polynomial in traces over wedge products of the curvature two-form $R$ and the $E_8$ field strength two-form $F_{(i)}$. This gives rise to the above expression when a relevant pair of descent equations are employed. The reason why the anomaly is encoded in a twelve-form rather than a thirteen-form (thirteen being $D + 2$ for $D = 11$) is that the anomaly has a fundamentally ten-dimensional character. This is because the anomaly is concentrated only on the ten-dimensional fixed hyperplanes, since the fermions contributing to the anomaly, consisting of the gauginos and notably the eleven-dimensional gravitino, are only chiral in a ten-dimensional sense. The descent equations themselves give rise to ambiguities when they are applied to the twelve-form, particularly if factorization is implemented before the descent equations are solved. These ambiguities are resolved by enforcing a total permutation symmetry on the gauge factors associated with the result. This reflects the Bose symmetry of the responsible loop diagrams. Equation (2.10) represents the consistent anomaly determined in this way. As explained in the introduction, the overall sign of the anomaly is the only undetermined factor, due to the potential for a $Z_2$ anomaly in eleven-dimensions. This is relevant to the case at hand since the path integral describing the quantum $M$-theory is eleven-dimensional.

The variation of the full quantum effective action is given by the sum of the quantum anomaly (2.10) and the variation of the Green-Schwarz terms (1.1). Amazingly, due to the factorization property evident in (2.10), these two types of contributions have precisely the same form; the fact that the quantum anomaly factorizes in the way shown in (2.10) is one of the miracles of $M$-theory. Thus, we may impose $\delta S_{GS} + \delta \Gamma = 0$ to enforce anomaly freedom. This relation implies two independent restrictions, one imposing the cancelation of the $Q^1_{2} \wedge I^4_4 \wedge I^4_4$ terms in each of (2.9) and (2.10) and the other from the similar cancelation of the $Q^1_{2} \wedge X_8$ terms.
After some simple and straightforward algebraic manipulations, these two conditions can be expressed, respectively, as

\[(2b + 3\sqrt{2})b^2 = -(-1)^\mu \nu \quad \leftarrow \text{Gauge Anomaly Cancelation} \quad (2.11)\]

\[(2b + 3\sqrt{2})^3 = -(-1)^\mu l \nu \quad \leftarrow \text{Gravitational Anomaly Cancelation} \quad (2.12)\]

The first of these requirements coincides with the elimination of gauge anomalies, and was described in \[1\]. The second equation is the additional requirement imposed by gravitational anomaly cancelation. Notice that this equation represents a family of loci parameterized by the integer \(l\). As we will demonstrate, flux quantization of \(G\) uniquely selects the curve described by \(l = 1\). Flux quantization requires

\[b^3 = m \nu \quad \leftarrow \text{Flux Quantization} \quad (2.13)\]

where \(m\) is another integer. The complete set of requirements necessary to fix \(b\) and \(\nu\) is given by the three cubic equations (2.11), (2.12) and (2.13).

It is simple to see that only \(l = 1\) is permitted. This is done by replacing the first factor on the left-hand-side of (2.11) with \(-(-1)^\mu (l \nu)^{1/3}\), which is implied by (2.12). We then replace the remaining \(b^2\) factor with \((m \nu)^{1/3}\), which is implied by (2.13). The factors of \(-(-1)^\mu \nu\) cancel out, leaving us with the condition that \(lm^2 = 1\). Since \(l\) and \(m\) are integers, the only possibilities are \((l, m) = (1, \pm 1)\). The three equations, (2.11), (2.12) and (2.13) are presented as curves in the \((b, \nu)\) plane in the figure. The two solutions are then easily visualized. Nevertheless, we will now derive the solutions algebraically.

With \(l = 1\), we can solve for \(b\) by equating the left hand sides of (2.11) and (2.12). The result of doing this is a cubic equation which factorizes as

\[(b + 3\sqrt{2})(2b + 3\sqrt{2})(b + \sqrt{2}) = 0. \quad (2.14)\]

The roots are then apparent. Thus, there are three possibilities, \(b = -3\sqrt{2}, \ b = -\frac{3}{2}\sqrt{2}\) and \(b = -\sqrt{2}\). The second of these possibilities, \(b = -\frac{3}{2}\sqrt{2}\) would imply that \(\nu = 0\), as is easily seen by plugging the result into (2.11) or (2.12). But this is disallowed, since \(\nu\) is nonvanishing in the quantum theory (reflecting the necessity for the fixed-point gauge multiplets). This leaves only two possibilities. We examine these in turn.

\(\mu = 0:\)

For the root \(b = -3\sqrt{2}\), we can see from (2.12) that \(\nu = (-1)^\mu 54\sqrt{2}\) and then from (2.13) that \(m = (-1)^\mu\). But \(\nu\) is positive by its definition (2.7), so this solution is only valid for the case \(\mu = 0\). This then implies that \(m = -1\) and, using (2.7) we determine that

\[\frac{\lambda^6}{\kappa^4} = (4\pi)^5. \quad (2.15)\]

This is the result obtained in \[3\], and is represented by the lower of the two marked intersection points shown in the figure. Note that unlike the counterpart solution for \(\mu = 1\), this solution does not occur at any special point (ie: turning points or inflection points) on any of the three curves representing the three anomaly freedom requirements. Without the possibility of the \(Z_2\) anomaly in \(8k + 3\) dimensions this would represent the sole possibility for a consistent quantum theory. We could then take (2.15) as a defining relation for the \(M^{10} \times S^1/Z_2\) orbifolding of \(M\)-theory. However, as we have discussed, there is another possibility.

\(\mu = 1:\)

For the root \(b = -\sqrt{2}\), we can see from (2.12) that \(\nu = (-1)^\mu 2\sqrt{2}\) and then from (2.13) that
\[
\nu = \frac{54\sqrt{2}}{(4\pi)^5} \frac{\lambda^6}{\kappa^4}
\]

Gravitational Anomaly Cancelation

\[\mu = 1\]

Gauge Anomaly Cancelation

\[\mu = 0\]

Flux Quantization

\[\mu = 0\]

\[\mu = 1\]

\[m = -1\]

\[m = +1\]

\[\nu = 54\sqrt{2}\]

\[\sqrt{2}\]

\[2\sqrt{2}\]

\[b\]

Figure: Loci of \((b, \nu)\) which permit cancelation of the indicated anomalies. There are exactly two possible simultaneous solutions. The upper one corresponds to the choice \(\mu = 1\), while the lower one corresponds to \(\mu = 0\). The \(\mu = 0\) choice was discovered by Lu \[6\]. The \(\mu = 1\) solution is new to this paper.

\[m = -(-1)^\mu\]. But, again, \(\nu\) is positive by its definition \((2.7)\), so this solution is only valid for the case \(\mu = 1\). This then implies that \(m = +1\) and, using \((2.7)\) we determine that

\[
\frac{\lambda^6}{\kappa^4} = \frac{1}{27} (4\pi)^5.
\]

This solution is new, and is represented by the upper of the two marked intersection points shown in the figure. Since we cannot ignore the \(Z_2\) anomaly in eleven dimensions this represents the lone alternative to the \(\mu = 0\) solution. On certain aesthetic grounds, this solution has an advantage over the alternative. This is because this solution exists at the only special point for \(\nu \neq 0\) on the curve representing the gauge anomaly condition.

3 Flux Quantization

The four-form \(G\) obeys flux quantization such that, in appropriate units, its periods are half-integer (ie: half of an odd integer). This has been demonstrated by Witten in \[3\], by using a little-known fact that there can be a topologically-resolved sign ambiguity for path integrals over fermions in \(8k+3\) dimensions. This applies to the \(M\)-theory membrane (which has a three-dimensional worldvolume, or \(k = 0\)), as was demonstrated by Witten. This discovery modified prior conventional wisdom that \(G\) has integer periods \[4\]. To derive the precise rule requires a

\[4\]Witten also offered a heuristic justification based on the observation that \(\int G \sim 16\pi^2 \int (F \wedge F - \frac{1}{4} R \wedge R)\) can be either integer or half-integer depending on global properties of spacetime. So, if one could find a region of space over which a cycle \(\int G\) were half-integer, one could then extrapolate this local knowledge to infer something about the topology of remote regions of spacetime. He argued that this would be counterintuitive.
membrane worldvolume action as a starting point. With $T_2$ defined as a relevant membrane tension parameter, this quantization reads

$$T_2 \int G = \pi n$$

(3.17)

where $n$ is an odd integer, and the integral is over any four-cycle in the universe. To resolve this constraint in terms of factors and parameters of the eleven-dimensional supergravity requires comparison with the supergravity field equations, obtained by the variation of the $D = 11$ supergravity action given in [1]. Enforcing consistency between these and the above flux quantization rule implies a related quantization of the membrane tension, given by

$$T_2 = \frac{1}{\sqrt{2}} \left( \frac{2 \pi^2}{\kappa^2 l} \right)^{1/3}$$

(3.18)

where $l$ is an integer.

The $M$-theory membrane is dual to the $M$-theory fivebrane. As a consequence, the respective brane tensions are subject to a Dirac quantization rule. For the conventions which we have chosen, particularly the choice of factors in the supergravity action, and the choice of membrane action, reflected in the quantization rule (3.17), this Dirac quantization requires that $\sqrt{2} \kappa^2 T_2 T_5 / \pi$ is an integer. We can then use this relation to rewrite (3.18) as

$$T_5 = \left( \frac{\pi l}{2 \kappa^4} \right)^{1/3},$$

(3.19)

which is the same as equation (1.3). The issue of brane quantization is discussed in more detail in [5, 6], and a relationship of these relations to $D$-branes is discussed in [1]. Necessarily the identical quantization is obtained from analysis of the consistent fivebrane worldvolume theory.

As discussed in [3], stability of membranes and fivebranes requires that the respective brane tensions (and therefore the integer $l$) be necessarily positive. It was pointed out by Lu in [3] that this is automatically ensured for the case $\mu = 0$. In section 2 we showed that $l = +1$ for any consistent theory regardless of the value of $\mu$. For this reason we do not concern ourselves further with this restriction.

We can rewrite (3.17) in a more useful form by substituting (3.18). The integer $l$ and the odd integer $n$ then combine as $m \equiv l n^3$, which is an integer. The result of this simple reorganizing is

$$\int G = \left( \sqrt{2} \pi \kappa^2 m \right)^{1/3}.$$  

(3.20)

It is in this form that the half-integral flux quantization rule is most useful to us. It is apparent, from the definition (2.4) that this provides another constraint on the parameters $b$ and $\nu$. We presently examine this in detail.

**Implications of Flux Quantization:**

We wish to derive the implications of the flux quantization (3.20) on the parameters described above, particularly the parameter $b$ and the two coupling constants $\kappa$ and $\lambda$. These latter two combine to the dimensionless parameter $\nu$ defined in (2.7).

We consider four-cycles which have no extension in the eleventh dimension. Subtleties regarding the irregular functions are avoided by considering cycles removed from the fixed hyperplanes. Over such cycles, only the final term in (2.4) contributes to integrals of $G$. The step function in this term factors out of the integration and contributes a sign which we safely suppress in this section (and only in this section). This is permitted because we are only interested in the integrality of factors, so overall signs are irrelevant. At these points, the $R \wedge R$ terms of $I_4(i)$, defined in (2.3), are nonvanishing, so that the integrand does not vanish identically. In fact, two
factors of $\frac{1}{2} \text{tr} R \wedge R$ contribute to the integral in question: one from each of the two terms in the sum $\text{R} \wedge \text{R}$.

Finally, $\int R \wedge R = 16\pi^2 a$, where $a$ is an integer. Assembling this information, we determine that, based on the definition of $G$ (ie: equation (2.4)), for the periods which we are considering,

$$\int G = \frac{8}{3} b \frac{\kappa^2}{\lambda^2} \pi^2 a.$$  

To obtain the new restriction on $b$, we equate the right hand sides of (3.21) and (3.22). After a small amount of algebra, and using again the definition of the parameter $\nu$ (ie: equation (2.7)), we find $a b = (\nu m)^{1/3}$. The strongest restriction comes with the case $a = 1$ since any higher integer choice for $a$ can be compensated by a scaling of $m$. In other words, for any value of $a$ there must exist some integer $m$ for which this restriction is valid. But, if there exists an appropriate $m$ for the choice $a = 1$ then there necessarily exists an analogous $m$ for any other choice of $a$. Thus, without loss of generality we take $a = 1$, so that

$$b^3 = \nu m$$

where $m$ is some integer. We reiterate that signs have been ignored in this discussion.

In principle, other restrictions would follow from applying the flux quantization to other four-cycles, for instance, cycles which live entirely on the fixed hyperplanes, or cycles which have some extent in the eleventh dimension. These do not pose additional restrictions, however.

This constitutes a proof of (2.13), which is the same as (3.22). This is Witten’s half-integral flux quantization expressed in terms of the parameters $b$ and $\nu$ discussed above. As for other anomaly freedom requirements, the requirement of flux quantization also translates to a restriction of $(b, \nu)$ to a locus defined by a cubic equation, (3.22). It is nontrivial, as well as intriguing, that this locus, and the two ostensibly independent loci corresponding to gauge anomaly cancelation and gravitational anomaly cancelation can be arranged to intersect at a common point. The fact that they actually have exactly two intersections begs the question of the relation between the two points. This issue seems resolved by the eleven-dimensional $Z_2$ anomaly as discussed above.

4 Conclusions

The potential for a $Z_2$ anomaly in eleven dimensions gives rise to a pair of distinct classes of consistent low-energy constructions of $M^{10} \times S^1/Z_2$ orbifolds of $M$-theory, with distinct restrictions on the respective coupling constants. In each case the gauge coupling constant is proportional to the gravitational coupling constant to the $2/3$ power, but with different coefficients. Only one of these two cases has been previously discovered. Topology necessarily distinguishes which one of these is appropriate for a given compactification.

The $Z_2$ anomaly was invoked in [2] in order to prove the half-integral flux quantization of the four-form $G$. In that case it was the three-dimensional worldvolume of the $M$-theory membrane which exhibited the anomaly in question, and the analog of the parameter $\mu$ described in this paper was related to a particular characteristic class associated with spin bundles on the worldvolume. It is necessary to analogously resolve the precise topological definition of $\mu$ in order to understand further the implications of the results described in this paper. That analysis represents the obvious extension to this paper.

There are three possibilities. In the first case, the topology of $M^{10} \times S^1/Z_2$ would permit either $\mu = 0$ or $\mu = 1$ depending on the characteristics of a given $M^{10}$. In such a scenario the relevant parameters $\nu$ defined in

\footnote{This is because in the definition of $I_4(i)$ (ie: equation (2.3)) only the $F_i \wedge F_i$ term is actually localized on the $i$th fixed hyperplane; the $R \wedge R$ term remains nonvanishing throughout the bulk.}
which defines the gauge coupling in terms of the gravitational coupling, and perhaps more relevantly \( b \) which defines the Chern-Simons couplings to the four-form \( G \), will differ depending on the value of \( \mu \). It would then become important to explore what differences this would imply for the compactified physics \[13, 16, 17, 18, 19\], and if these differences have some analog in the case of the weakly coupled heterotic string, perhaps even implications for mirror symmetry. This possibility is attractive because the same exotic phenomenon, that of a \( Z_2 \) anomaly which can only exist in \( 8k + 3 \) dimensions, would have implications to \( M \)-theory coming from both the \( k = 0 \) case and from the \( k = 1 \) case, the former related to the membrane worldvolume and the latter from the eleven-dimensional spacetime.

In the second case, the topology of \( M^{10} \times S^1/Z_2 \) would necessarily imply that \( \mu = 0 \). Equivalently, there may be an argument to suggest that the ambiguity at the heart of this analysis is not relevant to \( M \)-theory. This would clearly be the least interesting of the three possibilities. It would imply that the previous result of Lu \[16\] represents the final word on the issue of parameter identification in the context of anomaly cancelation for \( M \)-theory on \( M^{10} \times S^2/Z_2 \) orbifolds.

In the third case, the topology of \( M^{10} \times S^1/Z_2 \) would necessarily imply that \( \mu = 1 \). This would then require a revision in our understanding of parameter identification in the context of anomaly cancelation for \( M \)-theory on \( M^{10} \times S^1/Z_2 \), and would represent an interesting application of a somewhat exotic agent, the \( 8k + 3 \) dimensional \( Z_2 \) anomaly.

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