Small-time expansions for the transition distributions of Lévy processes

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Abstract: Let \( X = (X_t)_{t \geq 0} \) be a Lévy process with absolutely continuous Lévy measure \( \nu \). Small time polynomial expansions of order \( n \) in \( t \) are obtained for the tails \( P(X_t \geq y) \) of the process, assuming smoothness conditions on the Lévy density away from the origin. By imposing additional regularity conditions on the transition density \( p_t \) of \( X_t \), an explicit expression for the remainder of the approximation is also given. As a byproduct, polynomial expansions of order \( n \) in \( t \) are derived for the transition densities of the process. The conditions imposed on \( p_t \) require that its derivatives remain uniformly bounded away from the origin, as \( t \to 0 \); such conditions are shown to be satisfied for symmetric stable Lévy processes as well as for other related Lévy processes of relevance in mathematical finance. The expansions seem to correct asymptotics previously reported in the literature.

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1. Introduction

Lévy processes are important building blocks in stochastic models whose evolution in time might exhibit sudden changes in value. Such models can be constructed in rather general ways, such as stochastic differential equations driven by Lévy processes or time-changes of Lévy processes. Many of these models have been suggested and heavily studied in the area of mathematical finance (see [3] for an introduction to some of these applications).

A Lévy process \( X = (X_t)_{t \geq 0} \) is typically described in terms of a triplet \((\sigma^2, b, \nu)\) such that the process can be understood as the superposition of a Brownian motion with drift, say \( \sigma W_t + bt \), and a pure-jump component, whose discontinuities are determined by \( \nu \) in that, the average intensity (per unit time) of jumps whose size fall in a given set of values \( A \) is \( \nu(A) \). Thus, for instance, if \( \nu((-\infty,0]) = 0 \), then \( X \) will exhibit only positive jumps. A common assumption

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in many applications is that \( \nu \) is determined by a function \( s : \mathbb{R} \setminus \{0\} \to [0, \infty) \), called the \textit{Lévy density}, in the sense that
\[
\nu(A) := \int_A s(x) \, dx, \quad \forall A \in \mathcal{B}(\mathbb{R} \setminus \{0\}).
\]

Intuitively, the value of \( s \) at \( x_0 \) provides information on the frequency of jumps with sizes “close” to \( x_0 \).

Still, Lévy models have some important shortcomings for certain applications. For instance, given that typically the law of \( X_t \) is specified via its characteristic function \( \varphi_t(u) := \mathbb{E} e^{iuX_t} \), neither its density function \( p_t \) nor its distribution function \( P(X_t \leq y) \) are explicitly given in many cases. Therefore, the computation of such quantities necessitates numerical or analytical approximation methods. In this paper we study, short time, analytical approximations for the tail distributions \( P(X_t \geq y) \). This type of asymptotic results plays an important role in the non-parametric estimation of the Lévy measure based on high-frequency sampling observations of the process as carefully reported in [5] (see also [17], [4], and [20]). In Section 2, we present some of the ideas behind this important application of our results.

It is a well-known fact that the first order approximation is given by \( t \nu([y, \infty)) \), in the sense that
\[
\lim_{t \to 0} \frac{1}{t} P(X_t \geq y) = \nu([y, \infty)),
\]
provided that \( y \) is a point of continuity of \( \nu \) (see, e.g., Chapter 1 of Bertoin [1]). A natural question is then to determine the rate of convergence in (1.1). In case of a compound Poisson process, this rate is \( O(t) \), and it is then natural to ask whether or not the limit below exists for general Lévy processes:
\[
\lim_{t \to 0} \frac{1}{t} \left\{ \frac{1}{t} P(X_t \geq y) - \nu([y, \infty)) \right\}.
\]

In this paper, we study the validity of the more general polynomial expansion:
\[
P(X_t \geq y) = \sum_{k=1}^{n} d_k \frac{t^k}{k!} + \frac{t^{n+1}}{n!} R_n(t),
\]
for certain constants \( d_k \) and a remainder term \( R_n(t) \) bounded for \( t \) small enough. Note that in terms of the coefficients of (1.3), the limit (1.2) is given by
\[
d_2 \frac{2}{2} = \lim_{t \to 0} \frac{1}{t} \left\{ \frac{1}{t} P(X_t \geq y) - \nu([y, \infty)) \right\}.
\]

For a compound Poisson process, the expansion (1.3) results easily from conditioning on the number of jumps on \([0, t]\). Thus, infinite-jump activity processes
are the interesting cases. Ruschendorf and Woerner [18] (see Theorem 2 in Section 3) report that for a fixed \( N \geq 1 \) and \( \eta > 0 \), there exists a \( \varepsilon'(N) > 0 \) and \( t_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon'(N)) \) and \( t \in (0, t_0) \),

\[
P(X_t \geq y) = \sum_{i=1}^{N-1} \frac{t^i}{i!} \nu_{\varepsilon^i}([y, \infty)) + O_{\varepsilon, \eta}(t^N), \quad \text{for } y > \eta, \tag{1.5}
\]

where \( \nu_{\varepsilon}(dx) = 1_{|x| \geq \varepsilon} \nu(dx) \). When \( N = 3 \), this result would imply that, for \( 0 < \varepsilon < y/2 \wedge \varepsilon'(N) \),

\[
P(X_t \geq y) = t \nu([y, \infty)) + \frac{t^2}{2} \int_{|u| \geq \varepsilon} \int_{|v| \geq \varepsilon} 1_{u+v \geq y} \nu(du)\nu(du) + O_{\varepsilon, \eta}(t^3).
\]

Thus, (1.5) would imply that

\[
\lim_{t \to 0} \frac{1}{t} \left\{ \frac{1}{t} P(X_t \geq y) - \nu([y, \infty)) \right\} = \frac{1}{2} \int_{|u| \geq \varepsilon} \int_{|v| \geq \varepsilon} 1_{u+v \geq y} \nu(du)\nu(du),
\]

which is independent of the Brownian component \( \sigma W_t \) and of the “drift” \( bt \). We actually found that this limiting value is not the correct one and provide below the correction using two different approaches. Let us point out where we believe the arguments of [18] are lacking. The main problem arises from the application of their Lemma 3 in Theorem 2 (see also Lemma 1 in Theorem 1). In those lemmas, the value of \( t_0 \) actually depends on \( \delta \). Later on in the proofs, \( \delta \) is taken arbitrarily small, which is likely to result in \( t_0 \to 0 \) (unless otherwise proved).

We prove (1.3) using two approaches. The first approach is similar in spirit to that in [18]. It consists in decomposing the Lévy process \( X \) into two processes, one compound Poisson process \( \bar{X} \) collecting the “big” jumps and another process \( \bar{X}_e \) accounting for the “small” jumps. By conditioning on the number of big jumps during the time interval \([0, t]\), it yields an expression of the form

\[
P(X_t \geq y) = e^{-\lambda t} \sum_{k=1}^{n} \frac{(\lambda t)^k}{k!} \mathbb{P}(\bar{X} + \sum_{i=1}^{k} \xi_i \geq y) + O(t^{n+1}).
\]

By taking a compound Poisson process \( \bar{X} \) with jumps \( \{\xi_i\}_{i \geq 1} \) having a smooth density, one can expand further each term on the right-hand side using the following power series expansion:

\[
\mathbb{E} g(X_t) = g(0) + \sum_{k=1}^{n} \frac{t^k}{k!} L^k g(0) + \frac{t^{n+1}}{n!} \int_0^1 (1 - \alpha)^n \mathbb{E} \{L^{n+1} g(X_{\alpha t})\} d\alpha, \tag{1.6}
\]

valid for any \( n \geq 0 \) and \( g \in C_b^{2n+2} \), the class of functions having continuous and bounded derivatives of order \( 0 \leq k \leq 2n + 2 \). Above, \( L \) is the infinitesimal generator of the Lévy process, i.e.,

\[
(Lg)(x) := \frac{\sigma^2}{2} g''(x) + bg'(x) + \int (g(u+x) - g(x) - ug'(x)1_{|u| \leq 1}) \nu(du), \tag{1.7}
\]
for any function \( g \in C_b^2 \).

For \( n = 0 \), (1.6) takes a familiar form (see e.g. Lemma 19.21 in [11]):

\[
E g(X_t) = g(0) + t \int_0^1 E \{ Lg(X_{\alpha t}) \} d\alpha = g(0) + \int_0^t E \{ Lg(X_u) \} du,
\]

(1.8)

which is an easy consequence of Itô’s formula. The general case follows easily by induction in \( n \). Indeed, if (1.6) is valid for \( n \), applying (1.8),

\[
\int_0^1 (1 - \alpha)^n \{ L^{n+1}g(0) + \alpha t \int_0^1 E \{ L^{n+2}g(X_{\alpha' t}) \} d\alpha' \} d\alpha
\]

\[
= \frac{1}{n+1} L^{n+1}g(0) + \frac{t}{n+1} \int_0^1 (1 - \hat{\alpha})^{n+1} E \{ L^{n+1}g(X_{\hat{\alpha} t}) \} d\hat{\alpha},
\]

where we changed variables \( \hat{\alpha} := \alpha \alpha' \) and applied Fubini’s Theorem. Another proof of (1.6) is given in [9] based on Fourier approximations of \( g \) (Proposition 1 and 4 in there).

In the second order approximation case \( (n = 2) \), we give another proof for (1.3) which relaxes the assumptions on the Lévy density \( s \), by requiring only smoothness in a neighborhood of \( y \) and local boundedness away from the origin. This approach is based on the following recent asymptotic result by Jacod [10]:

\[
\lim_{t \to 0} \frac{1}{t} E g(X_t) = \sigma^2 + \int g(x) \nu(dx),
\]

(1.9)

valid for a \( \nu \)-continuous bounded function \( g \) such that \( g(x) \sim x^2 \), as \( x \to 0 \). In case the process is of finite variation and has no diffusion term, we prove the second order expansion as long as \( s \) is continuous at \( y \) and locally bounded away from 0. We also present a counterexample, originally suggested by Philippe Marchal, which shows that the result is not valid if \( s \) is not continuous (see [13] for further developments).

In order to provide explicit formulas for the coefficients \( d_k \) in (1.3), in Section 4 we consider a second approach whose basic first step is to approximate the indicator function \( 1_{[y, \infty)} \) by smooth functions \( f_m \) in such a way that

\[
\lim_{m \to \infty} E f_m(X_t) = \mathbb{P}(X_t \geq y).
\]

The idea to derive (1.3) is to apply (1.6) to each smooth approximation \( f_m \) and show that the limit of each term in the power expansion converges as \( m \to \infty \). We emphasize that this approach is carried out without additional assumption on \( s \), except smoothness and local boundedness away from the origin. In Section 5 we exploit further the approximation of \( f \) by the smooth functions \( f_m \) to provide an explicit formula for the remainder \( R_n(t) \) in (1.3). To carry out this plan, we impose more stringent conditions on \( X \) than those required in the first approach. In particular, we require that \( X_t \) has a \( C^\infty \)-transition density \( p_t \), whose derivatives remain uniformly bounded away from the origin, as \( t \to 0 \).
As a byproduct of the explicit remainder, polynomial expansions of order \( n \) in \( t \) are derived for the transition densities of the process extending a result in [18].

In Section 6, the boundedness conditions on the derivatives of the transition densities are shown to hold for symmetric stable Lévy processes. The validity of this uniform boundedness for general tempered stable processes is also considered in Section 7 via a recursive formula for the derivatives of the transition density. Tempered stable processes have received a great dealt of attention in the last decade due to their applications in mathematical finance. Among their members, we can list the CGMY model of [2]. See Rosiński [16] for a detailed study of this class of processes.

We note finally, that throughout the paper we only consider asymptotics for \( \mathbb{P}(X_t \geq y), y > 0 \), but that our methodology also gives results for \( \mathbb{P}(X_t \leq -y), y > 0 \), replacing \( \nu((y, +\infty)) \) by \( \nu((-\infty, -y]) \).

2. An application: nonparametric estimation of the Lévy density

In this part we present an application of the small-time asymptotics considered in this work as a matter of motivation. One problem that has received attention in recent years is that of estimating the Lévy density \( s \) of the process in a non-parametric fashion. This means that, by only imposing qualitative constraints on the Lévy density (e.g. smoothness, monotonicity, etc.), we aim at constructing a function \( \hat{s} \) that is consistent with the available observations of the process \( X \).

The minimal desirable requirement of our estimator \( \hat{s} \) is consistency; namely, the convergence \( \hat{s} \to s \), say in a mean-square error sense, must be ensured when the available sample of the process increases.

When the data available consists of the whole trajectory of the process during a time interval \( [0, T] \), the problem is equivalent to estimating the intensity function of an inhomogeneous Poisson process (see e.g. [15] for the case of finite intensity functions and [17] for the case of Lévy processes, where the intensity function could be infinite). However, a continuous-time sampling is not feasible in reality, and thus, the relevant problem is that of estimating \( s \) based on discrete sample data \( X_{t_0}, \ldots, X_{t_n} \) during a time interval \( [0, T] \). In that case, the jumps are latent variables whose statistical properties can in principle be assessed if the frequency and time horizon of observations increase to infinity.

It turns out that asymptotic results such as (1.2) and (1.3) play important roles in determining how frequently one should sample (given the time horizon \( T \) at hand) such that the resulting discrete sample contains sufficient information about the whole path. We can say that a given discrete sample scheme is good enough if we can devise a discrete-based estimator for the parameter of interest that enjoys a rate of convergence comparable to that of a good continuous-based estimator. Let us explain this point with a concrete example. Consider the estimation of the following functional of \( s \):

\[
\beta(\varphi) := \int \varphi(x)s(x)dx,
\]
where \( \varphi \) is a function that is smooth on its support. Assume also that the support of \( \varphi \) is an interval \([c, d]\) so that the indicator \( 1_{[c, d]} \) vanishes in a neighborhood of the origin. A natural continuous-based estimator of \( \beta(\varphi) \) is given by

\[
\beta^c_T(\varphi) := \frac{1}{T} \sum_{s \leq T} \varphi(\Delta X_s).
\]

Using the well-known formulas for the mean and variance of Poisson integrals (see e.g. [19, Proposition 19.5]), the above estimator can be seen to converge to \( \beta(\varphi) \), and moreover,

\[
\mathbb{E}(\beta^c_T(\varphi) - \beta(\varphi))^2 = \frac{1}{T} \beta(\varphi^2).
\]

We can thus say that \( \beta^c_T(\varphi) \) converges to \( \beta(\varphi) \) at the rate of \( O(T^{-1/2}) \), in the mean-square sense.

Suppose that instead we use a reasonable discrete-based proxy of \( \beta^c_T(\varphi) \), using the increments \( X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \) of the process instead of the jumps \( \Delta X_t \):

\[
\beta^n_T(\varphi) := \frac{1}{T} \sum_{i=1}^n \varphi(X_{t_i} - X_{t_{i-1}}),
\]

where \( \pi : t_0 < \cdots < t_n = T \). A natural question is then the following: How frequently should the process be sampled so that \( \beta^n_T(\varphi) \rightarrow \beta(\varphi) \) at a rate of \( O(T^{-1/2}) \)? To show in a simple manner the connection between the previous question and the asymptotics \([1.2]\), suppose that the sampling is “regular” in time with fixed time span \( \Delta_n := T/n \) between consecutive observations. In that case, we have

\[
\mathbb{E}(\beta^n_T(\varphi) - \beta(\varphi))^2 \leq \frac{1}{T} \beta(\varphi^2) + \frac{1}{T} \left\{ \frac{1}{\Delta_n} \mathbb{E} \varphi^2(X_{\Delta_n}) - \beta(\varphi^2) \right\}^2 + \left\{ \frac{1}{\Delta_n} \mathbb{E} \varphi(X_{\Delta_n}) - \beta(\varphi) \right\}^2.
\]

From the previous inequality we see that the rate of convergence in the limit

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{E} \varphi(X_{\Delta}) = \beta(\varphi), \tag{2.1}
\]

will determine the rate of convergence of \( \beta^n_T(\varphi) \) towards \( \beta(\varphi) \). To determine the rate of convergence in \([2.1]\), one can simply link \( \mathbb{E} \varphi(X_{\Delta}) \) to \( \mathbb{P}(X_{\Delta} \geq y) \), and link \( \beta(\varphi) \) to \( \nu([y, \infty)) \). This is easy if \( \varphi \) is smooth on its support \([c, d]\). Indeed, we have that

\[
\left| \frac{\mathbb{E} \varphi(X_{\Delta})}{\Delta} - \beta(\varphi) \right| \leq (\|\varphi\|_\infty + \|\varphi'\|_1) \sup_{y \in [c, d]} \left| \frac{1}{\Delta} \mathbb{P}(X_{\Delta} \geq y) - \nu([y, \infty)) \right|.
\]

Hence, the rate of convergence of \( \Delta^{-1} \mathbb{P}(X_{\Delta} \geq y) \) towards \( \nu([y, \infty)) \) determines the rate of convergence of \( \Delta^{-1} \mathbb{E} \varphi(X_{\Delta}) \) towards \( \beta(\varphi) \). In particular, the result
will tell us that, for $\beta^\pi_T(\varphi)$ to converge to $\beta(\varphi)$ at a rate of $O(T^{-1/2})$, in the mean-square sense, it suffices that the time span between consecutive observations $\Delta$ is $o(T^{-1/2})$. It is important to remark that (1.2) can be seen to hold uniformly in $y > y_0$ for an arbitrary $y > 0$.

The ideas outlined in this section, as well as the asymptotic result (1.2), are heavily exploited in [5] and [6], where the general problem of nonparametric estimation of the Lévy density $s$ is studied using Grenander’s method of sieves.

3. Expansions for the transition distribution

As often, e.g. see [18], the general strategy is to decompose the Lévy process into two processes: one accounting for the “small” jumps and a compound Poisson process collecting the “big” jumps. Concretely, suppose that $X$ has Lévy triplet $(\sigma^2, b, \nu)$; that is, $X$ admits the decomposition

$$X_t = bt + \sigma W_t + \int_0^t \int_{|x| \leq 1} x (\mu - \bar{\mu})(dx, ds) + \int_0^t \int_{|x| > 1} x \mu(dx, ds),$$

where $W$ is a standard Brownian motion and $\mu$ is an independent Poisson measure on $\mathbb{R}_+ \times \mathbb{R}\{0\}$ with mean measure $\bar{\mu}(dx, dt) := \nu(dx)dt$. Note that $\mu$ is the random measure associated to the jumps of $X$. Given a smooth truncation function $c_\varepsilon \in C^\infty$ such that $1_{[\varepsilon/2,\varepsilon/2]}(x) \leq c_\varepsilon(x) \leq 1_{[\varepsilon,\varepsilon]}(x)$, set

$$\tilde{X}_t := \int_0^t \int_{\mathbb{R}} x \tilde{c}_\varepsilon(x) \mu(dx, ds),$$
$$X_t := X_t - \tilde{X}_t,$$

where $\tilde{c}_\varepsilon(x) := 1 - c_\varepsilon$. It is well-known that $\tilde{X}_t$ is a compound Poisson process with intensity of jumps $\lambda_\varepsilon := \int \tilde{c}_\varepsilon(x) \nu(dx)$, and jumps distribution $\tilde{c}_\varepsilon(x) \nu(dx)/\lambda_\varepsilon$. The remaining process $X_\varepsilon$ is then a Lévy process with jumps bounded by $\varepsilon$ and Lévy triplet $(\sigma^2, b_\varepsilon, c_\varepsilon(x) \nu(dx))$, where

$$b_\varepsilon := b - \int_{|x| \leq 1} x \tilde{c}_\varepsilon(x) \nu(dx).$$

There are two key results that will be used to arrive to (1.3). The first is the expansion (1.6). The following tail estimate will also play an important role in the sequel:

$$P(|X_t| \geq y) \leq \exp\{ay_0 \log y_0\} \exp\{ay - ay \log y\} t^{ya},$$

valid for an arbitrary, but fixed, positive real $a$ in $(0, \varepsilon^{-1})$, and for any $t, y > 0$ such that $t < y_0^{-1}y$, where $y_0$ depends only upon $a$ (see [18 Lemma 3.2] or [19 Section 26] for a proof).
Remark 3.1. For an alternative proof of (3.4), use a generic concentration inequality such as [8, Corollary 1] to get (when $\sigma = 0$):

$$\mathbb{P}(X_t^\varepsilon \geq y) = \mathbb{P}(X_t^\varepsilon - \mathbb{E}X_t^\varepsilon \geq x)$$

$$\leq e^{-\frac{x}{2} + \left(\frac{x}{\varepsilon} + \frac{\varepsilon^2}{2}\right)\log(1 + \frac{1}{\varepsilon^2})} \leq \left(\frac{eV^2}{\varepsilon x}\right)^{\frac{1}{2}} t^{\frac{1}{2}},$$

whenever $x := y - \mathbb{E}X_t^\varepsilon > 0$, and with $V^2 := \int_{|y| \leq \varepsilon} u^2 \nu(du)$. Now $\mathbb{E}X_t^\varepsilon = t(b\varepsilon + \int_{1 < |x| \leq \varepsilon} x \nu(dx))$, and as $t \to 0$, $x \to y$ and $(eV^2/\varepsilon x)^{x/\varepsilon} \to (eV^2/\varepsilon y)^{y/\varepsilon}$, with moreover $t^{2\varepsilon}/t^2 = \exp((y - \mathbb{E}X_t^\varepsilon - 2\varepsilon) \log t/\varepsilon) \to 0$, as long as $y > 2\varepsilon$. Finally, since as $t \to 0$, $\mathbb{P}(\sigma W_t \geq y/2)/t^2 \to 0$, the general case follows.

We are ready to show (1.3). Below, $L_\varepsilon$ is the infinitesimal generator of $X^\varepsilon$ and we use the following notation:

$$s_\varepsilon := c_\varepsilon s, \quad \bar{s}_\varepsilon := 1 - s_\varepsilon, \quad L^0_\varepsilon g = g, \quad \bar{s}^1_\varepsilon = \bar{s}_\varepsilon$$

$$\bar{s}^i_\varepsilon(x) = \int \bar{s}^{(i-1)}(x - u)\bar{s}_\varepsilon(u)du, \quad (i \geq 2), \quad \bar{s}^0_\varepsilon * g = g.$$

Theorem 3.2. Let $y > 0$, $n \geq 1$, and $0 < \varepsilon < y/(n+1) \land 1$. Assume that $\nu$ has a density $s$ such that for any $0 \leq k \leq 2n + 1$ and any $\delta > 0$,

$$a_{k,\delta} := \sup_{|x| > \delta} |s^{(k)}(x)| < \infty.$$  

Then, there exists a $t_0 > 0$ such that, for any $y \geq y$ and $0 < t < t_0$,

$$\mathbb{P}(X_t \geq y) = e^{-\lambda_\varepsilon t} \sum_{j=0}^{n} c_j \frac{t^j}{j!} + O_{\varepsilon,y}(t^{n+1}),$$

(3.5)

where

$$c_j := \sum_{i=1}^{j} \binom{j}{i} L^j_\varepsilon - i \hat{f}_i(0),$$

with $\hat{f}_i(x) := \int_{y-x}^{\infty} \bar{s}^i_\varepsilon(u)du$.

Proof. Throughout this part, we write $f(x) := 1_{\{x \geq y\}}$. In terms of the decomposition $X := X^\varepsilon + \bar{X}^\varepsilon$ described at the beginning of this section, by conditioning on the number of jumps of $\bar{X}^\varepsilon$ during the interval $[0,t]$, we have that

$$\mathbb{E}f(X_t) = \mathbb{E}f(X_t^\varepsilon) e^{-\lambda_\varepsilon t} + \mathbb{E}f(X_t^\varepsilon) \sum_{k=n+1}^{\infty} \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}f\left(X_t^\varepsilon + \sum_{i=1}^{k} \xi_i\right)$$

(3.6)

$$+ e^{-\lambda_\varepsilon t} \sum_{k=1}^{n} \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}f\left(X_t^\varepsilon + \sum_{i=1}^{k} \xi_i\right)$$

(3.7)
where \( \xi_i \overset{\text{id}}{\sim} \xi(x)s(x)dx/\lambda \). Taking \( a := (n + 1)/y \), (3.4) and \( 0 \leq f \leq 1 \) imply that the two terms on the right hand side of (3.6) are \( O_{c, y}(t^{n+1}) \) as \( t \to 0 \), provided that \( t < t_0 := y_0^{-1}y \). Next, for each \( k \geq 1 \),

\[
\mathbb{E} f \left( X^\varepsilon_t + \sum_{i=1}^{k} \xi_i \right) = \mathbb{E} \tilde{f}_k \left( X^\varepsilon_t \right),
\]

where

\[
\tilde{f}_k(x) := \mathbb{E} f \left( x + \sum_{i=1}^{k} \xi_i \right) = \mathbb{P} \left( x + \sum_{i=1}^{k} \xi_i \geq y \right),
\]

which is \( C^{2n+2}_b \), since the density of \( \xi_i \) is \( C^{2n+1}_b \). Then, one can apply (1.6) to get

\[
\mathbb{E} \tilde{f}_k \left( X^\varepsilon_t \right) = \sum_{i=0}^{n-k} \int_{t}^{t^{n+1-k}} \frac{t'}{(n-k)!} \int_{0}^{1} (1 - \alpha)^{n-k} \mathbb{E} \left( L^{n+1-k}_c \tilde{f}_k \left( X^\varepsilon_t \right) \right) d\alpha.
\]

(3.8)

Let \( L_c \) be the infinitesimal generator of \( X^\varepsilon \), given by

\[
(L_c g)(x) = b_x g'(x) + \frac{\sigma^2}{2} g''(x) + \int \int_{0}^{1} g''(x + \beta w)(1 - \beta) d\beta w^2 c_x(w) s(w) dw,
\]

for \( g \in C^2_b \), and for \( k \geq 1 \), let

\[
d\pi^k_c := \prod_{\ell=1}^{k} (1 - \beta_k) d\beta_k w^2 c_x(w) s(w) dw,
\]

which clearly a finite measure on \([0,1]^k \times \mathbb{R}^k\). Then, note that

\[
(L^*_c g)(x) = \sum_{k \in K_i} c_k \left( i \atop k \right) A^*_k g(x),
\]

(3.9)

where \( K_i := \{ k := (k_1, k_2, k_3) : k_1 + k_2 + k_3 = i \} \),

\[
c_k := b_k^{k_2} \left\{ \sigma^2/2 \right\}^{k_2},
\]

\[
A^*_k g(x) := \int g^{(k_1 + 2k_2 + 2k_3)} \left( x + \sum_{\ell=1}^{k_3} \beta_\ell w_\ell \right) d\pi^k_c,
\]

if \( k_3 \geq 1 \) and \( A^*_k g(x) := g^{(k_1 + 2k_2)}(x) \), if \( k_3 = 0 \). Since

\[
\tilde{f}^{(\ell)}_k(x) = \lambda_{c}^{-k} (-1)^{\ell-1} s_{c}^{(k-1)} \ast s_{c}^{(\ell-1)}(y - x),
\]

and \( s_{c}(\cdot) \in C^{2n+1}_b \), there exists a constant \( b_{n,c} < \infty \) (independent of \( y \)), such that

\[
\|L^{n+1-k}_c \tilde{f}_k\|_{\infty} \leq b_{n,c}(a_{2n+1,c}/2)
\]
and so, the last term in (3.8) is $O(t^{n+1-k})$. Plugging (3.8) into (3.6) and rearranging terms, we get

$$E f(X_t) = e^{-\lambda t} \sum_{j=1}^{n} \sum_{k=1}^{m} \binom{m}{k}^j \lambda^{k} \bar{L}_{\varepsilon}^{m-k} \bar{f}_k(0) + O_{\varepsilon,y}(t^{n+1}),$$

which is exactly (3.5), because $\lambda^{k} \bar{L}_{\varepsilon}^{m-k} \bar{f}_k = \bar{L}_{\varepsilon}^{m-k} \hat{f}_k$.

\[\square\]

**Remark 3.3.**

(i) The expansion (1.3) follows from (3.5). Indeed, expanding $e^{-\lambda t}$, we get

$$P(X_t \geq y) = \sum_{k=1}^{n} d_k \frac{\lambda^k}{k!} + O_{\varepsilon,y}(t^{n+1}), \quad (3.10)$$

with

$$d_k = \sum_{j=1}^{k} \binom{k}{j} c_j (-\lambda)^{k-j}. \quad (3.11)$$

In the next section we give a more explicit expression for $d_k$.

(ii) The first two terms in (3.10) can be easily computed:

$$d_1 = \int_{y}^{\infty} s(u) du = \nu([y, \infty))$$

$$d_2 = -2\lambda \nu([y, \infty)) + \int \int 1_{(u_1 + u_2 \geq y)} \bar{s}_{\varepsilon}(u_1) \bar{s}_{\varepsilon}(u_2) du_1 du_2$$

$$-\sigma^2 s'(y) + 2b_\varepsilon s(y) - 2 \int \int s'(y - \beta w)(1 - \beta)d\beta w^2 \bar{s}_{\varepsilon}(w)dw.$$

(iii) The coefficients $d_k$ in (3.10) are independent of $\varepsilon$ since they can be defined iteratively as limits of $P(X_t \geq y)$. For instance,

$$\lim_{t \to 0} \frac{1}{t} P(X_t \geq y) = d_1, \quad \lim_{t \to 0} \frac{1}{t} \left\{ \frac{1}{t} P(X_t \geq y) - d_1 \right\} = d_2.$$

One can obtain an expression for $d_2$ that is independent of $\varepsilon$ by taking the limit as $\varepsilon \to 0$. For instance, if $X$ is of bounded variation with drift $b_0 := b - \int_{|x| \leq 1} \nu(dx)$ and volatility $\sigma$, then $d_2$ becomes

$$d_2 = -\sigma^2 s'(y) + 2b_0 s(y) - (\nu([y, \infty)))^2$$

$$+ \int_{y}^{0} \int_{y-x}^{0} s(u) du s(x) dx + 2 \int_{y}^{0} \int_{y-x}^{\infty} s(u) du s(x) dx.$$
In general, it turns out (see the Appendix) that \( d^2 \) “simplifies” to the following expression when \( \varepsilon \to 0 \):

\[
\begin{align*}
\frac{d^2}{2} &= -\sigma^2 s'(y) + 2bs(y) - \nu((y, \infty))^2 + \nu((y/2, y))^2 \\
&\quad + 2 \int_{-\infty}^{y/2} s(u)du(s(x)dx - 2s(y) \int_{y/2 < |x| \leq 1} xs(x)dx \\
&\quad + 2 \int_{-y/2}^{y/2} \int_{y-x}^{y} \{s(u) - s(y)\} dus(x)dx.
\end{align*}
\]

We now present an alternative proof for the expansion (1.4) that requires less stringent assumptions. The following asymptotic result due to Jacod [10] will be of importance:

\[
\lim_{t \to 0} \frac{1}{t} \mathbb{E}g(X_t) = \sigma^2 + \int g(x)\nu(dx),
\]

valid if \( g \) is \( \nu \)-continuous, bounded, and such that \( g(x) \sim x^2 \), as \( x \to 0 \).

**Proposition 3.4.** Let \( y > 0 \) and \( 0 < \varepsilon < y/2 \wedge 1 \). Assume that \( \nu \) has a density \( s \) which is bounded outside of the interval \([-\varepsilon, \varepsilon]\), and that is \( C^1 \) in a neighborhood of \( y \). Then, the limit (1.4) exists and can be written as:

\[
\frac{d^2}{2} = -\sigma^2 s'(y) + b_\varepsilon s(y) + \int s(y-u) - s(y) dus_\varepsilon(x)dx
\]

\[
+ \frac{1}{2} \int \int 1_{\{x+u \geq y\}} \tilde{s}_\varepsilon(u)\tilde{s}_\varepsilon(x)du dx - \lambda_\varepsilon \nu((y, \infty)).
\]

**Proof.** Let \( f(x) := 1_{\{x \geq y\}} \) and let

\[
A(t) := \frac{1}{t} \left\{ \frac{1}{t} \mathbb{E} f(X_t) - \int f(x)\nu(dx) \right\}.
\]

With the notation of Theorem 3.2, we have

\[
A(t) = \frac{1}{t^2} \mathbb{E} f(X_t^\varepsilon) e^{-\lambda_\varepsilon t} + e^{-\lambda_\varepsilon t} \int \frac{1}{t} \left\{ \mathbb{E} f(X_t^\varepsilon + x) - f(x) \right\} \tilde{s}_\varepsilon(x)dx
\]

\[
- \frac{1}{t} e^{-\lambda_\varepsilon t} \int f(x)\tilde{s}_\varepsilon(x)dx + e^{-\lambda_\varepsilon t} \sum_{n=2}^{\infty} \frac{(\lambda_\varepsilon)^n t^{n-2}}{n!} \mathbb{E} f \left( X_t^\varepsilon + \sum_{i=1}^{n} \xi_i \right),
\]

since \( \varepsilon < y/2 \). In view of (3.4), the first term on the right hand side vanishes when \( t \to 0 \). Then, except for the second term, all the other terms are easily seen to be convergent. Let us thus analyze the second term. Let

\[
B(t) := \int \left\{ \mathbb{E} f(X_t^\varepsilon + x) - f(x) \right\} \tilde{s}_\varepsilon(x)dx.
\]
Since $0 < \varepsilon < y/2$ and the support of $c_\varepsilon$ is $[\varepsilon, \varepsilon]$, $B(t)$ can be decomposed as

$$B(t) := \int_{y-\varepsilon}^{y} \mathbb{P}(X_t^\varepsilon \geq y - x) s(x)dx - \int_{y}^{y+\varepsilon} \mathbb{P}(X_t^\varepsilon < y - x) s(x)dx$$

$$+ \int_{x<y-\varepsilon} \mathbb{P}(X_t^\varepsilon \geq y - x) \bar{s}_\varepsilon(x)dx - \int_{y+\varepsilon}^{\infty} \mathbb{P}\{X_t^\varepsilon < y - x\} \bar{s}_\varepsilon(x)dx.$$

Since $s$ is bounded and integrable away from the origin, the last two terms can be upper bounded by $\lambda \varepsilon \mathbb{P}\{|X_t^\varepsilon| > \varepsilon\}$, which divided by $t$, converges to 0 in view of (3.11). After changing variables to $u = y - x$ and applying Fubini’s Theorem, the first term above becomes:

$$\int_{y-\varepsilon}^{y} \mathbb{P}(X_t^\varepsilon \geq y - x) s(x)dx = \int_{0}^{\varepsilon} \mathbb{P}(X_t^\varepsilon \geq u) s(y - u)du = \mathbb{E} f_+(X_t^\varepsilon),$$

where $f_+(x) := \int_{0}^{(x\wedge\varepsilon)\vee 0} s(y - u)du$. Similarly,

$$\int_{y}^{y+\varepsilon} \mathbb{P}(X_t^\varepsilon < y - x) s(x)dx = \int_{0}^{\varepsilon} \mathbb{P}(X_t^\varepsilon < u) s(y + u)du = \mathbb{E} f_-(X_t^\varepsilon),$$

where $f_-(x) := \int_{0}^{(-x\wedge\varepsilon)\vee 0} s(y + u)du$. Next, consider the function

$$\bar{f}(x) := \begin{cases} 
  f_+(x) - s(y) (x \wedge \varepsilon), & x > 0 \\
  -f_-(x) + s(y) (-x \wedge \varepsilon), & x < 0,
\end{cases}$$

and note that $\lim_{x \to 0} \bar{f}(x)/x^2 = -s'(y)/2$. In view of (3.12), we conclude that

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E} \bar{f}(X_t^\varepsilon) = -\frac{s'(y)}{2} \sigma^2 + \int f(x)c_\varepsilon(x)s(x)dx.$$

Thus, the sum of the first two terms in the decomposition of $B(t)$ are

$$\mathbb{E} f_+(X_t^\varepsilon) - \mathbb{E} f_-(X_t^\varepsilon) = \mathbb{E} \bar{f}(X_t^\varepsilon) + s(y) \mathbb{E} h(X_t^\varepsilon). \quad (3.14)$$

where $h(x) = x1_{|x| \leq \varepsilon} - \varepsilon x1_{x < -\varepsilon} + \varepsilon x1_{x > \varepsilon}$. Let us analyze the last term in (3.14):

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E} h(X_t^\varepsilon) = \lim_{t \to 0} \frac{1}{t} \mathbb{E} X_t^\varepsilon - \lim_{t \to 0} \frac{1}{t} \mathbb{E} X_t^\varepsilon 1_{|X_t^\varepsilon| > \varepsilon}$$

$$+ \varepsilon \lim_{t \to 0} \frac{1}{t} \mathbb{P}\{X_t^\varepsilon > \varepsilon\} - \varepsilon \lim_{t \to 0} \frac{1}{t} \mathbb{P}\{X_t^\varepsilon < -\varepsilon\} = b_\varepsilon.$$

We are finally able to give the limit of $B(t)/t$:

$$\lim_{t \to 0} \frac{1}{t} B(t) = -\frac{s'(y)}{2} \sigma^2 + s(y) b_\varepsilon + \int \bar{f}(x)c_\varepsilon(x)s(x)dx.$$

A little extra work leads to the expression in the statement of the result. □
It is not clear whether or not Proposition 3.4 remains true when $\sigma \neq 0$ and the density of $\nu$ is not differential in a neighborhood of $y$. If $\sigma = 0$, one can relax the differentiability condition as follows.

**Proposition 3.5.** Let $y > 0$ and $0 < \varepsilon < y/2 \land 1$. Assume that $\nu$ has a density $s$ which is bounded outside of the interval $[-\varepsilon, \varepsilon]$ and that is continuous in a neighborhood of $y$. Assume also that $\sigma = 0$ and that

$$\int_{|x| \leq 1} |x| \nu(dx) < \infty. \tag{3.15}$$

Then, the limit (1.2) exists and is given by

$$\frac{d^2}{2} := b_0 s(y) + \int \int_{|x| \leq \varepsilon} s(y - u) s(u) s(x) du dx + \frac{1}{2} \int \int_{x + u \geq y} \bar{s}_{e}(u) \bar{s}_{e}(x) dudx - \lambda_{e}(|y, \infty)), \tag{3.16}$$

where $b_0 := b - \int_{|x| \leq 1} x \nu(dx)$.

**Proof.** The proof is very similar to that of Proposition 3.4. However, instead of (3.12), we use the following asymptotic result

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E} g(X_t) = |b_0| + \int g(x) \nu(dx), \tag{3.16}$$

valid for any continuous bounded function $g$ such that $g(x) \sim |x|$, as $x \to 0$ (see e.g. Jacod [10]). Define the function

$$\hat{f}(x) := \begin{cases} f_+(x), & x > 0, \\ -f_-(x), & x < 0, \end{cases}$$

and note that $\lim_{x \to 0} \hat{f}(x)/x = s(y)$. By (3.16),

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E} \hat{f}(X_t^t) = s(y)b_0 + \int_{|x| \leq \varepsilon} \hat{f}(x) \nu(dx). \tag{3.17}$$

Using the arguments of the proof of Proposition 3.4, (3.17) implies that

$$\lim_{t \to 0} \frac{1}{t} B(t) = b_0 s(y) + \int \int_{|x| \leq \varepsilon} s(y - u) s_{e}(x) du dx,$$

and this gives the value of $d_2$ stated in the statement of the result.

**Remark 3.6.** The continuity of $s$ is needed in Proposition 3.5 as the following example suggested by Philippe Marchal shows (see [13], for further developments). Let $X_t := S_t + Y_t$, where $S$ is a strictly $\alpha$-stable Lévy process such that $S_t \overset{D}{=} t^{1/\alpha} S_1$ and $Y$ is an independent compound Poisson with jumps
\{\xi_i\}, and jump intensity 1. Suppose that 1 < \alpha < 2 and that the density \( p \) of \( \xi \) is such that \( p(y^-) \neq p(y^+) \). Notice that the Lévy density of the process is \( s(x) = x^{-\alpha - 1} + p(x) \). Then, as shown next,
\[
C(t) := \frac{1}{t^{1/\alpha}} \left\{ \frac{1}{t} P(X_t \geq y) - \nu([y, \infty)) \right\},
\]
converges to a non-zero limit as \( t \to 0 \), and so, (1.2) is infinite. Indeed, conditioning in the number of jumps of the compound Poisson component \( Y_t \),
\[
P(X_t \geq y) = P(S_t \geq y) e^{-t} + e^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} P \left( S_t + \sum_{i=1}^{n} \xi_i \geq y \right).
\]
Then, one easily writes
\[
C(t) = t^{1-\frac{1}{\alpha}} e^{-t} \left\{ \frac{1}{t} P(S_t \geq y) - \int_{y}^{\infty} x^{-\alpha - 1} dx \right\}
+ e^{-t} t^{\frac{1}{\alpha}} \left\{ P(S_t + \xi_1 \geq y) - P(\xi_1 \geq y) \right\} + O(t^{1-\frac{1}{\alpha}}).
\]
Using the self-similarity of \( S \), the second term on the right hand side converges to \((p(y^-) - p(y^+)) E S_1^+ \) and so, for \( 1 < \alpha < 2 \),
\[
\lim_{t \to 0} C(t) = (p(y^-) - p(y^+)) E S_1^+ \neq 0.
\]

4. Expansions via approximations by smooth functions

The identity (1.6) suggests the possibility of achieving power expansions for \( P(X_t \geq y) \) by approximating \( f(x) = 1_{\{x \geq y\}} \) using functions \( f_m \in C^b \). To this end, let us introduce mollifiers \( \varphi_m \in C^\infty \) with compact support contained in \([-1, 1]\) that converges to the Dirac delta function in the space of Schwartz distribution. For concreteness, we take \( \varphi_m(x) := m \varphi(mx) \), where \( \varphi \) is a symmetric bump like function integrable to 1. Notice that
\[
f_m(x) := f * \varphi_m(x) = \int_{-\infty}^{x-y} \varphi_m(u) du,
\]
converges to \( f(x) \), for any \( x \neq y \). Clearly, applying (1.6) to each \( f_m \),
\[
\mathbb{E} f_m(X_t) = \sum_{k=1}^{n} \frac{t^k}{k!} L^k f_m(0) + \frac{t^{n+1}}{n!} \int_{0}^{1} (1 - \alpha)^n \mathbb{E} \left\{ L^{n+1} f_m(X_{\alpha t}) \right\} d\alpha,
\]
and by the dominated convergence theorem,
\[
\lim_{m \to \infty} \mathbb{E} f_m(X_t) = \mathbb{P}(X_t \geq y).
\]
Thus, the problem is to identify conditions for the limit of each term on the right-hand side to converge as \( m \to \infty \) and to identify the corresponding limiting
value. The advantage of working with (4.2) instead of the decomposition of \(X\) of the previous section is that the coefficients \(d_k\) of (1.3) can be identified more explicitly.

As before, \(c_\varepsilon \in C^\infty\) denotes a smooth truncation function such that
\[
1_{[-\varepsilon/2,\varepsilon/2]}(x) \leq c_\varepsilon(x) \leq 1_{[-\varepsilon,\varepsilon]}(x).
\]
The following operators will be useful in the sequel
\[
L_i g(x) := b_i g^{(i)}(x), \quad i = 0, 1, 2
\]
\[
L_3 g(x) := \int g(x + u)\bar{c}_\varepsilon(u)\nu(du)
\]
\[
L_4 g(x) := \int \int_0^1 g''(x + \beta u)(1 - \beta)d\beta w^2c_\varepsilon(w)\nu(du),
\]
where \(\bar{c}_\varepsilon(u) := 1 - c_\varepsilon(u), \quad b_0 := -\int \bar{c}_\varepsilon(u)\nu(du), \quad b_1 := b - \int u(1_{\{|u|\leq 1\}} - c_\varepsilon(u))\nu(du)\) and \(b_2 := \sigma^2/2\). Note that
\[
L g = \sum_{i=1}^5 L_i g,
\]
for any bounded \(g \in C^2_b\). Moreover, it turns out that the following commuting properties hold true for any \(g \in C^2_b\):
\[
L_i L_j g = L_j L_i g.
\]

**Remark 4.1.** Under additional assumptions on the Lévy triplet \((\sigma^2, b, \nu)\), we can choose more parsimonious decompositions of the infinitesimal generator. For instance, if one of the \(b_i\)’s is zero, then the corresponding operator is superfluous and can be omitted in the analysis below. Also, if \(\int_{|u|\leq 1} |u|\nu(du) < +\infty\) (in which case the Lévy process has bounded variation), then \(L_4\) can be defined as:
\[
L_4 g := \int (g(x + w) - g(x)) c_\varepsilon(w)\nu(du) = \int \int_0^1 g''(x + \beta u)d\beta wc_\varepsilon(w)\nu(du),
\]
provided that \(b_1\) is adjusted accordingly. If \(\nu(\mathbb{R}\setminus\{0\}) < \infty, L_4\) can be omitted, provided that we define \(L_3, b_0\) and \(b_1\) via: \(L_3 g(x) = \int g(x + u)\nu(du), \quad b_0 = \nu(\mathbb{R}\setminus\{0\})\) and \(b_1 = b - \int_{|u|\leq 1} u\nu(du)\).

Let us introduce some more notation. For \(k := (k_0, \ldots, k_4)\) with \(k_0, \ldots, k_4 \geq 0, \ u := (u_1, \ldots, u_{k_4}), \ w := (w_1, \ldots, w_{k_4}), \) and \(\beta := (\beta_1, \ldots, \beta_{k_4})\), define the finite measure
\[
d\pi_k(u, w, \beta) = \prod_{i=1}^{k_3} \bar{c}_\varepsilon(u_i)\nu(du_i) \prod_{j=1}^{k_4} c_\varepsilon(w_j)w_j^2\nu(dw_j) \prod_{j=1}^{k_4} (1 - \beta_j)d\beta_j,
\]
on the space $\tilde{E}_k := \mathbb{R}^{k_3+k_4} \times [0,1]^{k_4}$. Consider also the following related finite measure

$$d\tilde{\pi}^\varepsilon_k(u_2, \ldots, u_{k_3}, w, \beta) = \prod_{i=2}^{k_3} c_i(u_i)\nu(du_i) \prod_{j=1}^{k_4} c_j(w_j)w_j^2\nu(dw_j) \prod_{j=1}^{k_4} (1 - \beta_j) d\beta_j.$$  

We sometimes drop the subscript $k$ and superscript $\varepsilon$ in the measures defined above. Also, the integral of a function $g$ with respect to a measure $\pi^\varepsilon_k$ is denoted by $\pi^\varepsilon_k(g)$ and we assume, by convention, that $\pi^\varepsilon_k(g) = g$, when $k_3 = k_4 = 0$.

Similarly, $\tilde{\pi}^\varepsilon_k(g) = g$ if $k_4 = 0$ and $k_3 = 1$ or 0.

Let $K_0$ be the class of all $k = (k_0, \ldots, k_4)$ such that $k_i \geq 0$ and $k_0 + \cdots + k_4 = k$. Note that, for any $k \geq 1$,

$$L^k g(x) = \sum_{k \in K_k} b_0^{k_0} b_1^{k_1} b_2^{k_2} \binom{k}{k} B_k g(x),$$  

(4.4)

where $\binom{k}{k} = k!/(k_0! \cdots k_4!)$ is the multinomial coefficient and

$$B_k g(x) := \int g^{(k_3+2k_2+2k_4)} \left( x + \sum_{i=1}^{k_3} u_i + \sum_{j=1}^{k_4} \beta_j w_j \right) d\pi_k.$$  

We first show that all terms in the right-hand side of (4.2) converges.

**Proposition 4.2.** Let $y > 0$, $n \geq 1$, and $0 < \varepsilon < y/(n+1) \wedge 1$. Assume that $\nu$ has a density $s$ such that for any $0 \leq k \leq 2n + 1$ and any $\delta > 0$,

$$a_{k,\delta} := \sup_{|x| > \delta} |s^{(k)}(x)| < \infty.$$  

(4.5)

Then, for any $1 \leq k \leq n$,

$$\hat{d}_k(y) := \lim_{m \to \infty} L^k f_m(0) = \sum_{k \in K_k} \hat{c}_k \binom{k}{k} a_k,$$  

(4.6)

where, for $k = (k_0, \ldots, k_4)$ and $\ell_k := k_1 + 2k_2 + 2k_4$, $\hat{c}_k$ and $a_k := a_k(y)$ are given by

$$\hat{c}_k := b_0^{k_0} b_1^{k_1} b_2^{k_2} (-1)^{(k_1-1)1\{\ell_k > 0\}}$$

$$a_k := \begin{cases} \int (\tilde{c}_\varepsilon s)^{(\ell_k-1)} \left( y - \sum_{i=2}^{k_3} u_i - \sum_{j=1}^{k_4} \beta_j w_j \right) d\tilde{\pi}_k, & k_3 > 0, \ \ell_k > 0, \\ \int 1\{\sum_{i=1}^{k_3} u_i \geq y\} d\pi_k, & k_3 > 0, \ \ell_k = 0, \\ 0, & \text{otherwise.} \end{cases}$$  

(4.7)
In particular, the limit
\[ R_n(t, y) := \lim_{m \to \infty} \int_0^1 (1 - \alpha)^n \mathbb{E} L^{n+1} f_m(X_{\alpha t}) d\alpha, \] (4.8)
exists, and moreover,
\[ \mathbb{P}(X_t \geq y) = \sum_{k=1}^n \hat{d}_k(y) \frac{t^k}{k!} + \frac{t^{n+1}}{n!} R_n(t, y). \] (4.9)

Proof. We write \( \hat{d}_k \) for \( \hat{d}_k(y) \) and \( R_n(t, y) \) for \( R_n(t, y) \). From (4.4), it suffices to show that for any \( k = (k_0, \ldots, k_4) \in K_k \),
\[ \lim_{m \to \infty} B_k f_m(x) = (-1)^{k_1 - 1} \mathbb{1}_{\ell > 0} a_k, \]
with \( \ell := k_1 + 2k_2 + 2k_4 \) (recalling that by convention \( \int g d\mathbb{P}_k = g \), if \( k_3 + k_4 = 0 \)).

In case \( \ell = 0 \),
\[ B_k f_m(x) = \int f_m \left( x + \sum_{i=1}^{k_3} u_i \right) d\mathbb{P}_k(u), \]
which clearly converges to \( \int 1 \{ \sum_{i=1}^{k_3} u_i \geq y \} d\mathbb{P}_k \), since \( f_m(x) \xrightarrow{m \to \infty} \mathbb{1}_{\ell \geq y} \) for any \( x \neq y \), \( |f_m| \leq 1 \), and \( \pi_k \) is a non-atomic finite measure.

Consider the case \( k_3, \ell > 0 \). Writing \( z = \sum_{i=2}^{k_3} u_i + \sum_{j=1}^{k_4} \beta_j w_j \) and integrating by parts,
\[ \int f^\ell_m \left( x + u_1 + z \right) \tilde{c} \cdot s(u_1) du_1 = \int \varphi^{\ell-1}_m (x + u_1 + z - y) \tilde{c} \cdot s(u_1) du_1 \]
\[ = (-1)^{k_1-1} \int \varphi_m (x + u_1 + z - y) (\tilde{c} s)^{(\ell-1)} (u_1) du_1 \]
\[ \xrightarrow{m \to \infty} (-1)^{k_1-1} (\tilde{c} s)^{(\ell-1)} (y - x - z), \]
provided that \( c \in C^{\ell-1}(\mathbb{R} \setminus \{0\}) \). Moreover, we have that
\[ \left| \int f^\ell_m \left( x + u_1 + z \right) \tilde{c} \cdot s(u_1) du_1 \right| \leq 2^{\ell-1} \max_{k \leq \ell} \sup_{z > 0} |s^{(k)}(x)| < \infty. \]

Thus, applying first Fubini’s theorem and then the dominated convergence theorem give:
\[ \lim_{m \to \infty} B_k f_m(x) = (-1)^{k_1-1} \int (\tilde{c} s)^{(\ell-1)} (y - x - \sum_{i=2}^{k_3} u_i - \sum_{j=1}^{k_4} \beta_j w_j) d\tilde{\pi}. \]

In case \( k_3 = 0 \) and \( \ell > 0 \),
\[ B_k f_m(0) = \int f^\ell_m \left( \sum_{j=1}^{k_4} \beta_j w_j \right) d\pi_k = \int \varphi^{\ell-1}_m \left( \sum_{j=1}^{k_4} \beta_j w_j - y \right) d\pi_k = 0, \]
for \( m \) large enough since, by construction, \( \varepsilon \) is chosen so that \( y - (n + 1)\varepsilon > 0 \), and \( \beta_j w_j \) takes values in \([-\varepsilon, \varepsilon]\) on the support of \( \pi_n \). Then, the existence of (4.8) and the identity (4.9) follow from (4.2) and (4.3).

Notice that we cannot yet conclude that \( \hat{d}_k := \hat{d}_k(y) \) are the same constants as the \( d_k \)s in equations (3.10) and (3.11), since we have not shown that

\[
\limsup_{t \to 0} |R(t, y)| < \infty. \tag{4.10}
\]

Actually, in view of Theorem 3.2, these two conditions are equivalent; namely, \( d_k = \hat{d}_k \) for all \( k \leq n \) if and only if (4.10) holds. We show these facts in the following result.

**Theorem 4.3.** Under the conditions of Theorem 3.2, (4.10) holds and moreover, \( \hat{d}_k = d_k \) in (4.6), which are independent of \( \varepsilon \) and given by

\[
\lim_{t \to 0} \frac{1}{t^n} \left\{ \mathbb{P} (X_t \geq y) - \sum_{k=1}^{n-1} \hat{d}_k \frac{t^k}{k!} \right\} = \frac{\hat{d}_n}{n!},
\]

for \( k \leq n \).

**Proof.** Using a proof as in Theorem 3.2, we conclude that

\[
\mathbb{E} f_m(X_t) = e^{-\lambda t} \sum_{j=1}^{n} c_{j,m} \frac{t^j}{j!} + O_{\varepsilon}(t^{n+1}), \tag{4.11}
\]

where \( c_{j,m} := \sum_{i=1}^{j} \left( \frac{j}{i} \right) L_i^{j-i} \hat{f}_{i,m}(0) \), with \( \hat{f}_{i,m}(x) := \int f_m(x + u) \tilde{s}_x^i(u) du \). As in Remark 3.3 (i), (4.11) leads to

\[
\mathbb{E} f_m(X_t) = \sum_{k=1}^{n} d_{k,m} \frac{t^k}{k!} + O_{\varepsilon}(t^{n+1}), \tag{4.12}
\]

with

\[
d_{k,m} = \sum_{j=1}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) c_{j,m} (-\lambda)^{k-j}. \tag{4.13}
\]

Since the last term in (4.2) is \( O_{\varepsilon}(t^{n+1}) \), we have

\[
d_{k,m} = L^k f_m(0).
\]

To show that \( \hat{d}_k := \lim_{m \to \infty} d_{k,m} \) is identical to \( d_k \) of (3.10)-(3.11), it suffices that \( \lim_{m \to \infty} c_{j,m} = c_j \), or equivalently, that

\[
\lim_{m \to \infty} L^k \hat{f}_{i,m}(0) = L^k \hat{f}_{i}(0),
\]

for all \( k \geq 0 \) and \( i \geq 1 \). The case \( k = 0 \) is clear. For \( k \geq 1 \), from (3.9), we only need to have

\[
\lim_{m \to \infty} \int \hat{f}^{(p)}_{i,m}(x + \sum_{\ell=1}^{k} \beta_{\ell} w_{\ell}) d\pi^x_k = \int \hat{f}^{(p)}_{i}(x + \sum_{\ell=1}^{k} \beta_{\ell} w_{\ell}) d\pi^x_k, \tag{4.14}
\]
for any $k \geq 0$ and $p \geq 1$. Since $\varphi_m$ is symmetric,

$$\hat{f}_{i,m}(x) = \int_{y-x}^{\infty} \varphi_m * \bar{s}_{\varepsilon}^\ell(u)du,$$

and thus, $\hat{f}_{i,m}(x) = \varphi_m * \bar{s}_{\varepsilon}^\ell(y - x)$, which converges to $\hat{f}_{i}(x) = \bar{s}_{\varepsilon}^\ell(y - x)$, as $m \to \infty$, uniformly in $x$. Then,

$$\hat{f}'_{i,m}(x) = \varphi_m * \bar{s}_{\varepsilon}^\ell(y - x),$$

which converges to $\hat{f}'_{i}(x) = \bar{s}_{\varepsilon}^\ell(y - x)$, as $m \to \infty$, uniformly in $x$. Since $\hat{\pi}_k$ is a finite measure, (4.14) holds. We have just proved that $d_k = \hat{d}_k$, for $k \leq n$, and by matching (3.10) and (4.9), it follows that $R_n(t) = O(1)$, as $t \to 0$. The last two statements of the result are easily proved by induction.

5. The remainder and expansions for the transition densities

In this section we give a more explicit expression for the remainder $R_n$ in (4.8), whose existence was proved in Theorem 4.2. In order to do this, we expand $L_{n+1}f_m(X_{\alpha t})$ using (4.4) and show that the limit of the resulting terms exists. The hardest case to tackle corresponds to $k_3 = 0$ and $\ell > 0$, where we will need to impose the following condition on the transition density $p_{\varepsilon}(x)$:

$$c_{k,\delta} := \sup_{0 < u \leq t_0} \sup_{|x| > \delta} |p_{\varepsilon}^{(k)}(u)| < \infty, \quad (5.1)$$

for any $\delta > 0$ and for some $t_0 > 0$. Condition (5.1) is reasonable since it is known that

$$\lim_{t \to 0} \sup_{|x| > \varepsilon} \left| \frac{1}{t} p_t(x) - s(x) \right| = 0, \quad (5.2)$$

(see Proposition III.6 in [12] and Corollary 1.1 in [18]). We confess however that, in general, this condition might be hard to verify since the transition densities $p_{\varepsilon}$ of a Lévy model are not explicitly given in many cases. Let us point out that, under certain conditions, Picard [14] proves that

$$\sup_{x} |p_{\varepsilon}^{(k)}(x)| \leq t^{-(k+1)/\beta}, \quad (5.3)$$

where $\beta$ is the Blumenthal-Getoor index of $X$. The approach in [14] was built on earlier methods and results of Léandre [12], who proves (5.2) and (5.3) for $k = 0$ using Malliavin calculus. In view of (5.3), for values of $t$ away from 0, the derivatives of $p_{\varepsilon}$ are uniformly bounded, and condition (5.1) is then related to the behavior of $p_{\varepsilon}^{(k)}$ when $t \to 0$. In Sections 6 and 7, we prove that the
condition \(5.1\) holds for symmetric stable Lévy processes and other related processes, rising the hope to use similar methods in other cases. As in the previous section, we take \(y > 0, n \geq 1, \varepsilon > 0\) such that

\[
0 < \varepsilon < y/(n + 1) \wedge 1.
\]

**Theorem 5.1.** Assume that \(\nu\) has a density \(s\) such that \(4.3\) holds for any \(0 \leq k \leq 2n + 1\) and any \(\delta > 0\). Also, assume that there exists a \(t_0 > 0\) such that for all \(0 < t < t_0\), \(X_t\) has a \(C^{2n+1}\) density \(p_t\) satisfying \(5.1\) for any \(0 \leq k \leq 2n + 1\) and any \(\delta > 0\). Then, the remainder

\[
\mathcal{R}_n(t, y) := \lim_{m \to \infty} \int_0^1 (1 - \alpha)^n E L^{n+1} f_m(X_{\alpha t}) d\alpha,
\]

is given by

\[
\mathcal{R}_n(t, y) = \sum_{k \in \mathcal{K}_{n+1}} c_k \binom{n+1}{k} \int_0^1 (1 - \alpha)^n a_k(t; \alpha, y) d\alpha,
\]

where, for \(k = (k_0, \ldots, k_4) \in \mathcal{K}_{n+1}\), \(c_k\) and \(a_k(t; \alpha)\) are defined via:

\[
c_k := b_0 b_1 b_2 (-1)^{(k_1-1)1(\varepsilon > 0)},
\]

\[
a_k(t; \alpha, y) := \begin{cases} 
\int \mathbb{P} \left( X_{\alpha t} + \sum_{i=1}^{k_3} u_i \geq y \right) d\pi_k, & \ell = 0 \\
\int \mathbb{E} (\tilde{\xi}^{(\ell-1)}) \left( y - X_{\alpha t} - \sum_{i=2}^{k_3} u_i - \sum_{j=1}^{k_4} \beta_j w_j \right) d\pi_k, & k_3 > 0, \ell > 0 \\
\int p_{\alpha t}^{(\ell-1)} \left( y - \sum_{j=1}^{k_4} \beta_j w_j \right) d\pi_k, & k_3 = 0, \ell > 0,
\end{cases}
\]

with \(\ell := k_1 + 2k_2 + 2k_4\).

**Proof.** From \(4.4\), it suffices to show that for any \(k = (k_0, \ldots, k_4) \in \mathcal{K}_k\),

\[
\lim_{m \to \infty} \int_0^1 (1 - \alpha)^n \mathbb{E} B_{\alpha t} f_m(X_{\alpha t}) d\alpha = (-1)^{(k_1-1)1(\varepsilon > 0)} \int_0^1 (1 - \alpha)^n a_k(t; \alpha) d\alpha.
\]

Below, \(T_k g(x; \cdot)\) is the function defined on \(E_k := \mathbb{R}^{k_1+k_4} \times [0, 1]^{k_4}\) via

\[
T_k g(x; u_1, \ldots, u_{k_3}, w_1, \ldots, w_{k_4}, \beta_1, \ldots, \beta_{k_4}) = g^{(k_1+2k_2+2k_4)} \left( x + \sum_{i=1}^{k_3} u_i + \sum_{j=1}^{k_4} \beta_j w_j \right).
\]

We break our proof in different cases. Suppose first that \(\ell := k_1 + 2k_2 + 2k_4 = 0\). Since \(0 \leq f_m \leq 1\), apply Fubini’s theorem to get:

\[
\mathbb{E} B_{\alpha t} f_m(X_{\alpha t}) = b_0 b_1 b_2 \pi_k \left( \mathbb{E} T_k f_m(X_{\alpha t}, \cdot) \right).
\]
Since $X_{\alpha t}$ is a continuous random variable, the dominated convergence theorem implies that

$$
\mathbb{E} f_m \left( X_{\alpha t} + \sum_{i=1}^{k_3} u_i \right) \xrightarrow{m \to \infty} \mathbb{P} \left( X_{\alpha t} \geq y - \sum_{i=1}^{k_3} u_i \right).
$$

Again, by dominated convergence, $\int_0^1 (1 - \alpha)^n \mathbb{E} B_k f_m (X_{\alpha t}) \, d\alpha$ converges to $\int_0^1 (1 - \alpha)^n \mathbb{P} \left( X_{\alpha t} \geq y - \sum_{i=1}^{k_3} u_i \right) \, d\pi \, d\alpha$.

Next, we consider the case $\ell > 0$ and $k_3 = 0$. Again by Fubini’s theorem,

$$\mathbb{E} B_k f_m (X_{\alpha t}) = \pi (\mathbb{E} T_k f_m (X_{\alpha t}, \cdot)) .$$

Writing $z = \sum_{j=1}^{k_4} \beta_j w_j$, integrating by parts, and changing variables, we have

$$
\mathbb{E} f_m (X_{\alpha t} + z) = \int \varphi_m^{(\ell-1)} (x + z - y) \, p_{\alpha t} (x) \, dx \\
= (-1)^{k_1 - 1} \int \varphi_m (x) \, p_{\alpha t}^{(\ell-1)} (x + y - z) \, dx,
$$

which converges to $(-1)^{k_1 - 1} \int \varphi_m^{(\ell-1)} (y - z) \, d\pi$ as $m \to \infty$, if $p_{\alpha t}^{(\ell-1)}$ is continuous. Moreover, under (5.1) and with the help of (5.4), for $m$ large enough,

$$
\sup_{0 < \alpha \leq 1} |\mathbb{E} f_m (X_{\alpha t} + \sum_{j=1}^{k_4} \beta_j w_j)| \leq \sup_{0 < \alpha \leq 1} \sup_{|z| \geq \delta} |p_{\alpha t}^{(\ell-1)} (x)| < \infty,
$$

taking $\delta := (y - (n + 1)\varepsilon)/2$. Then, by dominated convergence:

$$
\lim_{m \to \infty} \int_0^1 (1 - \alpha)^n \mathbb{E} B_k f_m (X_{\alpha t}) \, d\alpha = \\
(-1)^{k_1 - 1} \int_0^1 (1 - \alpha)^n \int p_{\alpha t}^{(\ell-1)} (y - \sum_{j=1}^{k_4} \beta_j w_j) \, d\pi (w_1, \ldots, w_{k_4}, \beta_1, \ldots, \beta_{k_4}) \, d\alpha.
$$

Note that the previous limiting value is uniformly bounded in $t$ and $y$ by

$$
\frac{1}{n + 1} \pi (\mathbb{R}^{2k_4}) \sup_{0 < u \leq \ell} \sup_{|x| \geq \delta} |p_u^{(\ell-1)} (x)| < \infty.
$$

The only remaining case to tackle is when $\ell > 0$ and $k_3 > 0$. Writing $z = \sum_{i=2}^{k_3} u_i + \sum_{j=1}^{k_4} \beta_j w_j$, we have that

$$
\int f_m^{(\ell)} (X_{\alpha t} + u_1 + z) \, (\bar{c} \bar{s}) (u_1) \, du_1 = \int \varphi_m^{(\ell-1)} (X_{\alpha t} + u_1 + z - y) \, (\bar{c} \bar{s}) (u_1) \, du_1 \\
= (-1)^{k_1 - 1} \int \varphi_m (X_{\alpha t} + u_1 + z - y) \, (\bar{c} \bar{s})^{(\ell-1)} (u_1) \, du_1,
$$

where $\bar{c}$ and $\bar{s}$ are constants depending only on $\alpha$.
which converges to \((-1)^{k_1-1}(cs)^{(\ell-1)}(y - X_{\alpha t} - z)\) as \(m \to \infty\), provided that \(c \in C^{\ell-1}(\mathbb{R}\setminus\{0\})\). Moreover, under (4.5), we have that

\[
\left| \int f_m^{(\ell)}(X_{\alpha t} + u_1 + z)(\bar{c}s)(u_1)du_1 \right| \leq 2^{\ell-1} \max_{k \leq \ell-1} \sup_{|x| > \varepsilon} |s^{(k)}(x)| < \infty.
\]

Thus, applying first Fubini’s theorem and then the dominated convergence theorem give:

\[
\lim_{m \to \infty} \int_0^1 (1 - \alpha)^n E_B k f_m(X_{\alpha t})d\alpha = (-1)^{k_1-1} \int_0^1 (1 - \alpha)^n \int \mathbb{E}(\bar{c}s)^{(\ell-1)}(y - X_{\alpha t} - \sum_{i=2}^{k_3} u_i - \sum_{j=1}^{k_4} \beta_j w_j)d\bar{c} d\alpha.
\]

This last case achieves the proof of the theorem. □

**Remark 5.2.** From the proof is clear that

\[
|R_n(t, y)| \leq \frac{\alpha_n}{n+1} \max \left\{ \max_{k \leq 2n+1} c_{k, \varepsilon}, \max_{k \leq 2n+1} a_{k, \varepsilon}, 1 \right\},
\]

for any \(t \in (0, t_0)\), where

\[
\alpha_n := \sum_{k \in \mathbb{N} \setminus \{1\}} |\mathcal{N}| \binom{n+1}{k} \max \{ \pi_k(1), \tilde{\pi}_k(1), \hat{\pi}_k(1) \}.
\]

This bound on \(R(t, y)\) is valid for any \(y \geq y\), taking also \(\varepsilon\) such that \(0 < \varepsilon < y/(n+1) \wedge 1\).

One of the advantages of an explicit expression for the remainder \(R_n(t, y)\) of (4.9) is that we can obtain small time expansions in \(t\) for the transition density \(p_t(y)\) of \(X_t\), by a formal differentiation of (4.9). With this application in mind we need to show that the coefficients \(a_k(y) := a_k\) of (4.7) and

\[
A_k(t, y) := \int_0^1 (1 - \alpha)^n a_k(t; \alpha, y)d\alpha,
\]

of (5.5) are differentiable in \(y\). The following result corrects Theorem 1 from [IS].

**Proposition 5.3.** Let \(y > 0, n \geq 1,\) and \(0 < \varepsilon < y/(n+1) \wedge 1\). Assume that \(\nu\) has a density \(s\) such that (4.5) holds for any \(0 \leq k \leq 2n+2\) and any \(\delta > 0\). Also, assume that there exists a \(t_0 > 0\) such that for all \(0 < t < t_0\), \(X_t\) has a \(C^{2n+2}\) density \(p_t\) satisfying (5.1) for any \(0 \leq k \leq 2n+2\) and any \(\delta > 0\). Then,

\[
p_t(y) = \sum_{k=1}^{n} \hat{d}_k(y) \frac{t^k}{k!} + O_\varepsilon(t^{n+1}),
\]

(5.6)
where for any $1 \leq k \leq n$,
\[
\hat{d}_k'(y) := \sum_{k \in K_k} \hat{c}_k \hat{d}_k'(y),
\]
(5.7)
and, for $k = (k_0, \ldots, k_4)$ and $\ell_k := k_1 + 2k_2 + 2k_4$,
\[
\hat{c}_k := b_0 b_1 b_2 (-1)^{(k_1-1)1(\epsilon_k > 0)},
\]
\[
a_k'(y) := \begin{cases} 
\int (\tilde{c}_s)(\ell_k) \left( y - \sum_{i=2}^{k_3} u_i - \sum_{j=1}^{k_4} \beta_j w_j \right) d\tilde{\pi}_k, & k_3 > 0, \ell_k > 0, \\
\int (\tilde{c}_s) \left( y - \sum_{i=2}^{k_3} u_i \right) d\tilde{\pi}_k, & k_3 > 0, \ell_k = 0, \\
0, & \text{otherwise.}
\end{cases}
\]
(5.8)

Proof. In view of (4.9) and (5.5), we first have to show that the derivative of (4.7), for each case, exists and is given by (5.8). This will follow from the fact that $\tilde{\pi}_k$ and $\pi_k$ are finite measures and the integrands have continuous uniformly bounded derivatives with respect to $y$. Also, we need to show that the derivatives, with respect to $y$, of $a_k(t; \alpha, y)$ exist and are continuous as well as bounded for $t$ small. When $\ell = 0$ and $k_3 = 0$,
\[
A_k(t, y) = \int_0^1 (1 - \alpha)^n a_k(t; \alpha, y) = \int_0^1 (1 - \alpha)^n \int_y^\infty p_{\alpha t}(z) dz d\alpha.
\]
From (5.2), there exist $K > 0$ and $t_0 > 0$ such that $\sup_{0 < u < t_0} \sup_{|x| > \delta} p_u(x) < K$. Hence, one can interchange derivation and integration:
\[
\frac{\partial A_k(t, y)}{\partial y} = \int_0^1 (1 - \alpha)^n p_{\alpha t}(y) d\alpha,
\]
(5.9)
and moreover, the supremum of (5.9) over $0 < t < t_0$ is finite. In case of $\ell = 0$ and $k_3 > 0$, one can write
\[
A_k(t, y) = \int_0^1 (1 - \alpha)^n \int_y^\infty \int p_{\alpha t} \left( z - \sum_{i=1}^{k_3} u_i \right) d\pi_k dz d\alpha.
\]
The inner integral is continuous in $z$ and is such that
\[
0 \leq \bar{p}_{\alpha t}(z) := \int \ldots \int p_{\alpha t} \left( z - \sum_{i=1}^{k_3} u_i \right) (\tilde{c}_s)(u_1) d\tilde{\pi}_k \leq \sup_{u_1} \bar{s}_c(u_1) \lambda_{k_3}^{k_3-1}.
\]
Thus, one can interchange derivation and integration to get:
\[
\frac{\partial A_k(t, y)}{\partial y} = \int_0^1 (1 - \alpha)^n \bar{p}_{\alpha t}(y) d\alpha,
\]
and moreover, the supremum over \( t > 0 \) is finite. In case \( k_3 > 0 \) and \( \ell > 0 \),

\[
A_k(t, y) = \int_0^1 (1 - \alpha)^n \int E(\xi s)^{(\ell - 1)} \left( y - X_{\alpha t} - \sum_{i=2}^{k_3} u_i - \sum_{j=1}^{k_4} \beta_j w_j \right) d\pi_k d\alpha.
\]

Assuming that \( \xi s \in C^\ell_b \), one can interchange the derivative with respect to \( y \) and the integral, and the resulting term will be uniformly bounded in \( t \) since \( \pi_k \) is a finite measure. Finally, when \( k_3 = 0 \) and \( \ell > 0 \),

\[
A_k(t, y) = \int_0^1 (1 - \alpha)^n \int p^{(\ell - 1)} \left( y - \sum_{j=1}^{k_4} \beta_j w_j \right) d\pi_k d\alpha.
\]

Assuming that \( p \in C^\ell \) and satisfies \((5.1)\) with \( k = \ell \), one can interchange the derivative with respect to \( y \) and the integral, and the resulting term will be uniformly bounded in \( 0 < t < t_0 \) since \( \pi_k \) is a finite measure. All previous cases will imply that

\[
\frac{\partial A_k(t, y)}{\partial y}
\]

exists and is uniformly bounded in \( 0 < t < t_0 \). Hence, the derivative of the last term in \((4.9)\) is \( O(\varepsilon^{\alpha n+1}) \).

6. Symmetric stable Lévy processes

In this section, we analyze the assumption \((5.1)\), needed for the validity of Theorem 5.1 and 5.3, in the case of symmetric stable Lévy processes.

Let us assume that the Lévy triplet \((\sigma^2, b, \nu)\) is such that \( b = 0 \) and that \( \nu \) is symmetric. Furthermore, let us assume that

\[
\liminf_{\varepsilon \to 0} \int_{|x| > \varepsilon} x^2 \nu(dx) > 0,
\]

for \( 0 < \alpha < 2 \). Condition \((6.1)\) is equivalent to

\[
\lim_{\varepsilon \to 0} \varepsilon^{\alpha} \int_{|x| > \varepsilon} \nu(dx) > 0.
\]

Condition \((6.1)\) is known to be sufficient for \( X_t \) to have a \( C^\infty \)-density \( p_t \) (see e.g. [12, Theorem I.1] or [19, Proposition 28.3]). It will be useful to outline the proof of this result. The first step is to bound the characteristic function \( \psi_t(u) = \mathbb{E} e^{iuX_t} \) as follows:

\[
|\psi_t(u)| \leq e^{-ct|u|^{\alpha}},
\]

which is valid for \( u \) large enough (cf. page 190 in [19]). Note that the right hand side of \((6.2)\) is the characteristic function of a symmetric \( \alpha \)-stable Lévy process. In particular,

\[
\int |\psi_t(u)| |u|^n du < \infty,
\]
for any $n = 0, \ldots, $ and the following inversion formula for $p_t^{(n)}$ holds:

$$p_t^{(n)}(x) = \frac{(-i)^n}{2\pi} \int e^{-iux} u^n \psi_t(u) du,$$  \hspace{1cm} (6.3)

see [19] Proposition 2.5. Finally, the Riemann-Lebesgue lemma implies that

$$\lim_{|x| \to \infty} p_t^{(n)}(x) = 0.$$

Let us try to modify the above argument for our purposes. In the case that $b = 0$ and $\nu$ is symmetric, $\psi_t(u)$ is positive real and even, and thus,

$$p_t^{(n)}(x) = \begin{cases} 
\frac{(-1)^{n/2}}{\pi} \int_0^\infty \cos(ux) u^n \psi_t(u) du, & \text{if } n \text{ is even}, \\
\frac{(-1)^{(n+1)/2}}{\pi} \int_0^\infty \sin(ux) u^n \psi_t(u) du, & \text{if } n \text{ is odd}.
\end{cases} \hspace{1cm} (6.4)
$$

In light of (6.2), it is important to analyze the case of a symmetric $\alpha-$stable Lévy process. It is not surprising that a great deal is known for this class (see e.g. Section 14 in [19]). For instance, from the self-similarity property $X_t \overset{D}{=} t^{1/\alpha} X_1,$

$$p_t(x) = t^{-1/\alpha} p_1 \left( t^{-1/\alpha} x \right).$$

Asymptotic power series in $x$ are available for $p_1(x),$ from which one can also obtain the following asymptotic behavior of $p_1(x)$ when $x \to \infty$:

$$p_1(x) \sim x^{-\alpha-1}. \hspace{1cm} (6.5)$$

Note that (6.5) is consistent with the well-known asymptotic result that

$$\lim_{t \to 0} \frac{1}{t} p_t(x) = s(x) = x^{-\alpha-1},$$

for any $x \neq 0$ (see e.g. [18 Corollary 1]).

We want to show that the condition (5.1) holds for symmetric stable distributions (and possibly for more general symmetric distributions satisfying (6.1)). With this goal in mind, we give a method to bound $x^{\alpha+1} p_1(x).$ First, we need the following lemma:

**Lemma 6.1.** Let $\phi : (0, \infty) \to \mathbb{R}_+$ be an integrable function. Then, the following statements hold:

(i) If $\phi$ is monotone decreasing and there exists $\beta \in [0, 1]$ such that

$$\limsup_{u \to 0} \frac{\phi(u) u^\beta}{u^\beta} < \infty, \hspace{1cm} (6.6)$$

then there exists a constant $c < \infty,$ independent of $x,$ such that

$$\left| \int_0^\infty \kappa(ux) \phi(u) du \right| \leq \frac{c}{x^{1-\beta}}, \hspace{1cm} (6.7)$$

where $\kappa$ can be either $\cos$ or $\sin.$
(ii) If \( \phi \) is unimodal with mode \( u^* \), then
\[
\left| \int_0^\infty \kappa(ux)\phi(u)du \right| \leq \frac{\pi \phi(u^*)}{x},
\]
for all \( x > 0 \), where \( \kappa \) can be either the function \( \cos \) or \( \sin \). Moreover, if \( \phi \) is continuous, then
\[
\lim_{x \to \infty} x \int_0^\infty \kappa(ux)\phi(u)du = \pi \phi(u^*).
\]

Proof. To show (i), we first note the following two easy inequalities:
\[
0 \leq \int_0^\infty \sin(ux)\phi(u)du \leq \int_0^{\frac{\pi}{x}} \phi(u)du < \infty,
\]
valid for any nonnegative function \( \phi \) that is decreasing and integrable. Therefore, if \( \kappa \) is either \( \cos \) or \( \sin \), then
\[
\left| \int_0^\infty \kappa(ux)\phi(u)du \right| \leq \int_0^{\frac{2\pi}{x}} \phi(u)du.
\]
In view of the condition (6.6), there exists a \( c' > 0 \) and \( x_0 > 0 \) such that for all \( x > x_0 \), \( \phi(u) \leq c'u^{-\beta} \), in \( (0, 2\pi/x] \), for any \( x > x_0 \). Then, (6.7) is clear for \( x > x_0 \). The values \( x \leq x_0 \) can be taken care of easily since
\[
\int_0^{2\pi/x} \phi(u)du \uparrow \int_0^\infty \phi(u)du,
\]
when \( x \downarrow 0 \). Let us now show (ii). First, set
\[
q(x) := \int_0^\infty \kappa(ux)\phi(u)du.
\]
By assumption, \( \phi \) is increasing on \( [0, u^*] \), and decreasing on \( [u^*, \infty) \). It can be shown that for any \( x > 0 \), there exists a positive number \( u(x) \) such that
\[
\kappa(xu(x)) = 0, \quad |u^* - u(x)| \leq \frac{2\pi}{x}, \quad \text{and}
\]
\[
\int_{u(x) - \pi/x}^{u(x)} \kappa(ux)\phi(u)du \leq q(x) \leq \int_{u(x)}^{u(x) + \pi/x} \kappa(ux)\phi(u)du,
\]
(see e.g. Figure 1 where the choice of \( u(x) \) is illustrated when \( \kappa(u) = \cos(u) \)). Next, the upper and lower bounds on \( q \) are such that:
\[
\int_{u(x)}^{u(x) + \pi/x} \kappa(ux)\phi(u)du \leq \frac{\pi}{x} \phi(\bar{u}(x)) \leq \frac{\pi}{x} \phi(u^*),
\]
\[
\int_{u(x) - \pi/x}^{u(x)} \kappa(ux)\phi(u)du \geq -\frac{\pi}{x} \phi(\underline{u}(x)) \geq -\frac{\pi}{x} \phi(u^*),
\]
where \( \bar{u}(x) \in [u(x), u(x) + \pi/x] \) and \( \underline{u}(x) \in [u(x) - \pi/x, u(x)] \). The inequality (6.8) is thus clear, while (6.9) results from the fact that both \( \bar{u}(x) \) and \( \underline{u}(x) \) converges to \( u^* \) as \( x \to \infty \).

---

**Proposition 6.2.** Let \( X \) be a symmetric \( \alpha \)-stable Lévy process, and let \( p_t \) be the density of the marginal \( X_t \). The following two statements hold:

(a) If \( 0 < \alpha \leq 1 \), then there exists an absolute constant \( c \) such that

\[
\sup_x |x|^{\alpha+1} p_1(x) \leq c.
\]

(b) If \( 1 < \alpha \leq 2 \), then for any \( \varepsilon > 0 \), there exists a constant \( 0 < c(\varepsilon) < \infty \) such that

\[
\sup_{|x|>\varepsilon} |x|^{\alpha+1} p_1(x) \leq c(\varepsilon).
\]

**Proof.** Without loss of generality suppose that \( x > 0 \). By (6.4), the well-known representation of the characteristic function of \( X_t \), and an integration by parts,

\[
p_1(x) = \frac{1}{\pi} \int_0^\infty \cos(ux)e^{-u} du = \frac{\alpha}{x} \int_0^\infty \sin(ux)u^{\alpha-1}e^{-u} du. \tag{6.13}
\]

If \( 0 < \alpha \leq 1 \), then we can apply (6.7) with \( \beta = 1 - \alpha \), and hence,

\[
|p_1(x)| \leq \frac{\alpha}{x} \cdot \frac{c'}{x^{1-\beta}} = \frac{c}{x^{\alpha+1}},
\]
for a constant $c$. Now, let $1 < \alpha \leq 2$. Applying another integration by parts in (6.13), we have
\begin{align}
p_1(x) &= \frac{\alpha(\alpha - 1)}{x^2} \int_0^\infty \cos(ux)u^{\alpha-2}e^{-u^\alpha} du \quad (6.14) \\
&\quad - \frac{\alpha^2}{x^2} \int_0^\infty \cos(ux)u^{2(\alpha-1)}e^{-u^\alpha} du. \quad (6.15)
\end{align}

The first term in the previous inequality can be bounded using (6.7) with $\beta = 2 - \alpha$:
\[ \left| \frac{1}{x^2} \int_0^\infty \cos(ux)u^{\alpha-2}e^{-u^\alpha} du \right| \leq \frac{c}{x^\alpha} \leq \frac{c}{x^{\alpha+1}}. \]

The term in (6.15) can be bounded using (6.8) since $\phi(u) = u^{2(\alpha-1)}e^{-u^\alpha}$ is unimodal and thus,
\[ \left| \frac{\alpha^2}{x^2} \int_0^\infty \cos(ux)u^{2(\alpha-1)}e^{-u^\alpha} du \right| \leq \frac{1}{x^3} \leq \frac{c'}{x^{3\alpha+1}}, \]
for all $x > \varepsilon$, where $c, c' < \infty$ are constants depending only on $\varepsilon$. Plugging in the above bounds in (6.15), we obtain the second statement in the proposition. \(\square\)

**Remark 6.3.** In view of the above proposition, we obtain the following bound for the transition density $p_t$ of a symmetric $\alpha$-stable Lévy process:
\[ p_t(x) \leq \frac{c t}{x^{\alpha+1}}, \]
valid for all $t > 0$ and $|x| > \varepsilon$, and where $c$ is a constant depending only on $\varepsilon$.

We can now generalize the ideas of Proposition 6.2 to deal with the derivatives of the transition density.

**Theorem 6.4.** Under the conditions of Proposition 6.2, for any $\varepsilon > 0$, there exists a constant $c_n(\varepsilon)$ such that
\[ \sup_{|x| > \varepsilon} \left| x^{\alpha+1+n} \left| p_1^{(n)}(x) \right| \right| \leq c_n(\varepsilon). \quad (6.16) \]

**Proof.** We prove the following more general bound:
\[ \sup_{|x| > \varepsilon} \left| x^{\alpha+1+n} \int_0^\infty \kappa(ux)u^\alpha e^{-u^\alpha} du \right| \leq d_n(\varepsilon) < \infty, \quad (6.17) \]
where $\kappa$ can be either cos or sin. Without loss of generality, let us assume that $x > 0$. Our proof is then performed by induction on $n$. Proposition 6.2 yields (6.17) for $n = 0$ and $\kappa(x) = \cos(x)$. The case $\kappa(x) = \sin(x)$ can be dealt with in an analogous way; namely, we first integrate by parts, once when $\alpha \leq 1$, or twice when $1 < \alpha \leq 2$, and secondly, we use (6.7) if $\alpha \leq 1$, or (6.8) if $1 < \alpha \leq 2$.\[ \square \]
Now, assume that \(6.17\) holds for \(n = 0, \ldots, m - 1\). We want to prove the case \(n = m > 1\). Set
\[
q_m(x) := \int_0^\infty \kappa(ux)u^m e^{-u^\alpha} du.
\]
Applying consecutive integrations by parts, one can find constants \(b_j\) (depending only on \(\alpha\) and \(m\)) such that
\[
q_m(x) = -\frac{1}{x} \int_0^\infty \hat{\kappa}(ux)u^{m-1} e^{-u^\alpha} du + \frac{1}{x^m} \sum_{j=1}^m b_j \int_0^\infty \bar{\kappa}(ux)u^{i\alpha} e^{-u^\alpha} du,
\]
where \(\hat{\kappa}, \bar{\kappa}\) are either cos or sin. By the induction hypothesis, the first term in \(6.18\) is such that
\[
\sup_{|x| > \varepsilon} \left| \frac{1}{x} \int_0^\infty \hat{\kappa}(ux)u^{m-1} e^{-u^\alpha} du \right| |x|^{\alpha+1+m} \leq d_{m-1}(\varepsilon), \tag{6.19}
\]
as we wanted to show.

Now, for the second term, let us consider first \(\alpha < 1\). Let \(k \geq 1\) be such that
\[
\frac{k-1}{k-1+m} < \alpha \leq \frac{k}{k+m},
\]
Also, for each \(1 \leq j \leq m+k\), let \(1 \leq r_j \leq j\) be such that
\[
\frac{r_j-1}{j} < \alpha \leq \frac{r_j}{j}.
\]
Setting \(S(x) := \sum_{j=1}^m b_j \int_0^\infty \kappa(ux)u^{j\alpha} e^{-u^\alpha} du\), and applying successive integrations by parts to each of the terms of \(S(x)\), it follows that
\[
S(x) = \sum_{j=1}^{m+k} a_j x^{r_j} \int_0^\infty \kappa_j(ux)u^{j\alpha-r_j} e^{-u^\alpha} du \tag{6.20}
\]
for some constants \(a_j\), and where \(\kappa_j\) is either cos or sin. By the way \(r_j\) is chosen, the inequality \(6.7\) can be applied to estimate the absolute value of each term in \(6.20\). Then,
\[
|S(x)| = \sum_{j=1}^{m+k} a_j x^{r_j} \left| \int_0^\infty \kappa_j(ux)u^{j\alpha-r_j} e^{-u^\alpha} du \right| \leq \sum_{j=1}^{m+k} \hat{a}_j x^{j\alpha+1}, \tag{6.21}
\]
for some \(\hat{a}_j \geq 0\). Combining \(6.18\)-\(6.21\), there exists a constant \(c_m(\varepsilon)\) such that
\[
\sup_{|x| > \varepsilon} \left| p_1^{(m)}(x) \right| |x|^{\alpha+1+m} \leq c_m(\varepsilon). \tag{6.22}
\]
Next, we consider the case of $1 < \alpha \leq 2$. Note that, for some constants $a_0, a_1, a_2$ depending only on $\alpha$ and $j$, the term $C_j(x) := \int_0^\infty \kappa(ux)u^{j\alpha-2}e^{-u^\alpha} \, du$ can be broken into three pieces:

\[
C_j(x) = \frac{a_0}{x^2} \int_0^\infty \kappa(ux)u^{j\alpha-2}e^{-u^\alpha} \, du + \frac{a_1}{x^2} \int_0^\infty \kappa(ux)u^{(j+1)\alpha-2}e^{-u^\alpha} \, du + \frac{a_2}{x^2} \int_0^\infty \kappa(ux)u^{(j+2)\alpha-2}e^{-u^\alpha} \, du.
\] (6.23)

If $j = 1$, then the first term in (6.23) can be bounded using (6.7) with $\beta = 2 - \alpha$:

\[
\left| \frac{a_0}{x^2} \int_0^\infty \kappa(ux)u^{\alpha-2}e^{-u^\alpha} \, du \right| \leq \frac{c}{x^{2+1-\beta}} = \frac{c}{x^{\alpha+1}},
\]

for some $c < \infty$. For the other two terms of the case $j = 1$ or any other $2 \leq j \leq m$, we can apply (6.8) since then the function multiplying $\kappa$ is unimodal.

Then, for any $\varepsilon > 0$, we can bound $S(x) := \sum_{j=1}^m b_j \int_0^\infty \kappa(ux)u^{j\alpha}e^{-u^\alpha} \, du$ in the following way:

\[
\sup_{x > \varepsilon} |S(x)| \leq \frac{c}{x^{\alpha+1}} + \frac{c'}{x^3} \leq \frac{c''}{x^{\alpha+1}},
\]

for a constant $c''$ depending only on $\varepsilon$.

Finally, let us verify the case $\alpha = 1$. Without loss of generality, assume that $\kappa(x) = \cos(x)$. After two integrations by parts, we have that

\[
\int_0^\infty \cos(ux)u^m e^{-u} \, du = -\frac{m}{x} \int_0^\infty \sin(ux)u^{m-1} e^{-u} \, du + \frac{1}{x} \int_0^\infty \sin(ux)u^m e^{-u} \, du
\]
\[
= -\frac{m}{x} \int_0^\infty \sin(ux)u^{m-1} e^{-u} \, du - \frac{1}{x^2} \int_0^\infty \cos(ux)u^m e^{-u} \, du
\]
\[
+ \frac{m}{x^2} \int_0^\infty \cos(ux)u^{m-1} e^{-u} \, du.
\]

We can then write the above equality in the following manner:

\[
\left(1 + \frac{1}{x^2}\right) \int_0^\infty \cos(ux)u^m e^{-u} \, du = -\frac{m}{x} \int_0^\infty \sin(ux)u^{m-1} e^{-u} \, du
\]
\[
+ \frac{m}{x^2} \int_0^\infty \cos(ux)u^{m-1} e^{-u} \, du.
\]

The result follows by applying our induction hypothesis to bound each of the two terms in the right-hand side of the last equality.

\[\square\]

**Corollary 6.5.** With the notation of Proposition 6.4, for any $0 < \alpha \leq 2$, $\varepsilon > 0$, and $n \geq 0$, there exist a constant $c_{n,\varepsilon}$ such that

\[
\sup_{|x| > \varepsilon} \left| \mathbb{P}_t^{(n)}(x) \right| \leq c_{n,\varepsilon} t, \quad \text{(6.24)}
\]

for any $0 < t \leq 1$. 

7. General Lévy processes

In this part, we examine the validity of the assumption (5.1) for general Lévy processes, whose Lévy density \( s \) is stable like around the origin.

The main tool will be a recursive relations between the derivatives of a density \( p \). Consider a distribution \( \mu \) such that its characteristic function \( \psi(u) := \hat{\mu}(u) \) is \( C^\infty \) with also
\[
\left| u^m \right| \psi^{(r)}(u) \, du < \infty, \tag{7.1}
\]
for all \( r \geq 0 \) and \( m \geq 0 \). Recall that in that case \( \mu \) admits a \( C^\infty \)-density \( p \) and moreover,
\[
p^{(m)}(x) = \frac{(-i)^m}{2\pi} \int e^{-ixu} u^m \psi(u) du. \tag{7.2}
\]
By applying two consecutive integration by parts, we can derive the following formulas
\[
p^{(m)}(x) = -\frac{m}{x} p^{(m-1)} - \frac{(-i)^{m-1}}{2\pi x} \int e^{-ixu} u^m \frac{d\psi(u)}{du} du,
\]
\[
p^{(m)}(x) = -2\frac{m}{x} p^{(m-1)} - \frac{m (m - 1)}{x^2} p^{(m-2)} + \frac{(-i)^{m-2}}{2\pi x^2} \int e^{-ixu} u^m \frac{d^2\psi(u)}{du^2} du,
\]
where we are assuming that \( m \geq 2 \). However, even if \( m < 2 \), we can deduce a recursive formula for \( p^{(m)} \) in terms of all its lower order derivatives and the integral of the function \( e^{-ixu} u^m \frac{d^r\psi(u)}{du^r} \).

Indeed, we have:

**Theorem 7.1.** Let \( r \geq 0 \) and \( m \geq 0 \). Then, for all \( x \), \( p^{(m)}(x) \) can be written as
\[
\sum_{j=1}^{\lfloor m \rfloor} \sum_{i=0}^{j-1} c_{r,j}^m \frac{(m - j - 1)!}{x^j} p^{(m-j)}(x) + (-1)^r \frac{(-i)^{m-r}}{2\pi x^r} \int e^{-ixu} u^m \frac{d^r\psi(u)}{du^r} du,
\]
where \( c_{r,j}^m \) are given by the following recursive formulas:
\[
c_{r,0}^m = -1, \quad c_{r,j}^m = 0, \quad (j > r), \quad c_{r+1,j}^m = c_{r,j}^m + c_{r,j-1}^m. \tag{7.3}
\]

**Proof.** We prove the formula by induction in \( m \). Consider the case \( m = 0 \). We want to prove that
\[
p(x) = (-1)^r \frac{(-i)^{-r}}{2\pi x^r} \int e^{-ixu} \frac{d^r\psi(u)}{du^r} du,
\]
for any \( r \geq 0 \). This can be done by induction on \( r \) and integration by parts. Suppose that the formula is valid for \( m = k \) and all \( r \geq 0 \). We want to show the
formula for $m = k + 1$ and all $r \geq 0$. Now, we use induction on $r$. The case $r = 0$ is just (7.2) with $m = k + 1$. Suppose the result holds for $r = \ell$ and $m = k + 1$:

$$p^{(k+1)}(x) = \sum_{j=1}^{\ell \wedge (k+1)} c_{\ell,j}^{k+1} \prod_{i=0}^{j-1} (k + 1 - i) \frac{1}{x^i} p^{(k+1-j)}(x)$$

(7.4)

$$+ (-1)^{\ell} \frac{(-i)^{k+1-\ell}}{2\pi x^{\ell}} \int e^{-iux} x^{k+1} \frac{dt}{du} \psi(u) du.$$

Next, with an integration by parts in the last term,

$$p^{(k+1)}(x) = \sum_{j=1}^{\ell \wedge (k+1)} c_{\ell,j}^{k+1} \prod_{i=0}^{j-1} (k + 1 - i) \frac{1}{x^i} p^{(k+1-j)}(x)$$

(7.5)

$$+ (-1)^{\ell+1} \frac{(-i)^{k+1-\ell-1}}{2\pi x^{\ell+1}} \int e^{-iux} x^{k+1} \frac{dt}{du} \psi(u) du$$

$$+ (-1)^{\ell+1} \frac{(-i)^{k+1-\ell-1}}{2\pi x^{\ell+1}} \cdot (k + 1) \int e^{-iux} x^{k} \frac{dt}{du} \psi(u) du.$$

Then, writing (7.3) for $m = k$ and $r = \ell$ and solving for the last term gives

$$(-1)^{\ell+1} \frac{(-i)^{k-\ell}}{2\pi x^{\ell}} \int e^{-iux} x^{k} \frac{dt}{du} \psi(u) du =$$

$$- p^{(k)}(x) + \sum_{j=1}^{\ell \wedge k} c_{\ell,j}^{k} \prod_{i=0}^{j-1} (k - i) \frac{1}{x^i} p^{(k-j)}(x)$$

Plugging in (7.5), we get (7.3) with $r = \ell + 1$ and $m = k + 1$ provided that we define the coefficients $c_{\ell+1,j}$ as follows:

$$c_{\ell+1,1} := c_{\ell,1}^{k+1} - 1, \quad c_{\ell+1,j} := c_{\ell,j}^{k+1} + c_{\ell,j-1}^{k}.$$

This proves the case of $r = \ell + 1$ and so, the result holds for all $r$ and all $m$. \(\Box\)

The following corollary gives further information when working with the transition distributions of a Lévy process.

**Corollary 7.2.** Let $(X_t)_{t \geq 0}$ be a Lévy process such that $\mu$, the distribution of $X_1$, satisfies (7.1). Let $\gamma$ be such that $\psi_t(x) := e^{t \gamma(u)}$, where $\psi_t$ is the characteristic function of $X_t$. Then, the density $p_t$ of $X_t$ admits the representation:

$$p_t^{(m)}(x) = \sum_{j=1}^{r} \sum_{i=0}^{m} c_{r,i}^{m} \prod_{i=0}^{j-1} (m - i) \frac{1}{x^i} p_t^{(m-j)}(x) + (-1)^{r} \frac{(-i)^{m-r}}{2\pi x^{r}} \mathcal{I}_r^m(t, x),$$

(7.6)

where

$$\mathcal{I}_r^m(t, x) := \sum_{(i_1, i_2, j_1, j_2)} d_{i_1,j_1}^{i_2,j_2} \cdot i_1 + j_2 \int e^{-iux} \left( \gamma^{(i_1)}(u) \right)^{j_1} \left( \gamma^{(i_2)}(u) \right)^{j_2} e^{t \gamma(u)} du,$$

for some constants $d_{i_1,j_1}^{i_2,j_2}$. The above summation is over all non-negative integers $i_1, i_2, j_1, j_2$ such that $0 < i_2 \leq i_1$ and $i_1 j_1 + i_2 j_2 = r$. 
As an application let us consider a Lévy process as in Corollary 7.2 such that for each \( i \geq 1 \), there exists \( c_i < \infty \) and \( u_{0,i} > 0 \) such that

\[
|\gamma^{(i)}(u)| \leq c_i |u|^\alpha - i,
\]

for all \( |u| > u_{0,i} \). Also, assume that there exists \( u_0 > 0 \) and \( c_0 < \infty \) such that

\[
|\psi_1(u)| \leq e^{-c_0 |u|^{\alpha}},
\]

for all \( u > u_0 \). Remember that \( \|r\|_{\alpha} \) implies the above condition (cf. Sato [19, Proposition 28.3]). Then, we have the following result:

**Proposition 7.3.** Let \( \|r\|_{\alpha} \) and \( \psi_1 \) be true for \( 0 < \alpha \leq 2 \). Then, for any \( m \geq 0 \), any \( \varepsilon > 0 \), and any \( t_0 > 0 \),

\[
\sup_{0 < t \leq t_0} \sup_{|x| > \varepsilon} |p_t^{(m)}(x)| < \infty.
\]

**Proof.** The proof is by induction on \( m \geq 0 \). The recursive formula (7.6) with \( r = 1 \) and \( m = 0 \) leads to

\[
|p_t(x)| \leq t \int e^{-iux_0} e^{\gamma(x_0)u} \, dx_0.
\]

Note that we can assume that there exist constants \( u_0 > 0 \), \( b_0 \), and \( b_1 \) such that

\[
\sup_{|u| \leq u_0} |\gamma'(u)| e^{\gamma(u)} \leq b_0, \quad |\gamma'(u) - e^{\gamma(u)}| \leq b_1 |u|^{\alpha-1} e^{-c_0 |u|^{\alpha}},
\]

for all \( |u| > u_0 \) and \( 0 < t \leq t_0 \). Then, for all \( t \leq t_0 \),

\[
|p_t(x)| \leq b_0 u_0^{\alpha-1} e^{-c_0 u_0^{\alpha}}.
\]

Next, let the statement of the proposition hold true for \( m = 0, \ldots, k \), and let us show it for \( m = k + 1 \). In view of (7.6), it suffices to show that

\[
\sup_{0 < t \leq t_0} \sup_{|x| > \varepsilon} |T_t^m(t, x)| < \infty,
\]

for some \( r \geq 0 \). Moreover, it suffices to show that

\[
\sup_{0 < t \leq t_0} \sup_{|x| > \varepsilon} t^{j_1 + j_2} \int e^{-iux} u^{\alpha} \gamma_{(j_1)}(u) \gamma_{(j_2)}(u) e^{\gamma(u)} \, du < \infty,
\]

for any \( i_1 \geq i_2 > 0 \) and \( j_1, j_2 \geq 0 \) such that \( i_1 j_1 + i_2 j_2 = r \). As before, we can assume that there exist constants \( u_0 > 0 \), \( b_0 \), and \( b_1 \) such that

\[
\sup_{|u| \leq u_0} |\gamma_{(j_1)}(u)\gamma_{(j_2)}(u)| e^{\gamma(u)} \leq b_0 \quad \text{and} \quad |\gamma_{(j_1)}(u)\gamma_{(j_2)}(u)| e^{\gamma(u)} \leq b_1 |u|^{j_1 + j_2} e^{-c_0 t |u|^{\alpha}},
\]

for all \( |u| > u_0 \). We need to show that there exists an \( r \) such that the supremum on \( 0 < t < t_0 \) of

\[
t^{j_1 + j_2} \int_0^\infty u^{j_1 + j_2} e^{-c_0 t u^{\alpha}} \, du = t^{\frac{1}{2} (r-m-1)} \int_0^\infty v^{(j_1 + j_2)\alpha + m - r} e^{-c_0 u^{\alpha}} \, dv,
\]

is finite. The supremum above will be finite if \( r = m + 1 \).
Example 7.4. Consider the CGMY Lévy model introduced in [2] and of great popularity in the area of mathematical finance. This process is a tempered stable one in the sense of Rosiński [16]. Its characteristic function is given by
\[
\psi_t(u) = \exp \left\{ iC \Gamma(-\alpha) \left( (M - iu)^\alpha - M^\alpha + (G + iu)^\alpha - G^\alpha \right) \right\} 
\] (see Theorem 1 in [2]). Then,
\[
\gamma(u) := C \Gamma(-\alpha) \left( (M - iu)^\alpha - M^\alpha + (G + iu)^\alpha - G^\alpha \right).
\]
We can then verify that \( \gamma \) satisfies (7.7) and (7.8).

The next result generalizes the conclusions in the above example to more general tempered stable processes. For simplicity, we take symmetric processes, even though the proof can be extended to the general case.

Proposition 7.5. Let \( X \) be a Lévy process with Lévy triplet \((0, 0, \nu)\). Assume that \( \nu \) is of the form \( \nu(ds) = |s|^{1-\alpha} q(|s|)ds \), where \( 0 < \alpha < 2 \) and \( q \) is a completely monotone function on \( \mathbb{R}_+ \) such that
\[
\int_1^\infty s^{j-\alpha-1} q(s)ds < \infty, \quad (7.9)
\]
for all \( j \geq 1 \). Assume also that the measure \( F \) for which \( q(s) = \int_0^\infty e^{-\lambda s} F(d\lambda) \) is such that
\[
\int_0^{\infty} \lambda^j F(d\lambda) < \infty, \quad (7.10)
\]
for all \( j \geq 0 \). Then, the function \( \gamma \) associated with the characteristic function of \( X \) via \( \psi_t(x) := e^{t\gamma(u)} \) satisfies the conditions (7.7) and (7.8).

Proof. Clearly,
\[
\lim_{\varepsilon \to 0} \inf \frac{\int_0^\varepsilon s^{1-\alpha} q(s)ds}{\varepsilon^{2-\alpha}} > 0, \quad (7.11)
\]
and thus, condition (7.8) will follow. Now, we claim that there exists a constant \( C \) such that
\[
\left| \int_0^\infty \sin(us)s^{-\alpha} e^{-\lambda s} ds \right| \leq Cu^{\alpha-1}, \quad (7.12)
\]
for all \( \lambda, u > 0 \) and \( 0 < \alpha < 2 \). Indeed, if \( 0 < \alpha \leq 1 \), (7.12) results from (6.7). If \( 1 \leq \alpha < 2 \), then changing variables and using \( \sin v \leq v \),
\[
\left| \int_0^\infty \sin(us)s^{-\alpha} e^{-\lambda s} ds \right| \leq u^{\alpha-1} \left| \int_0^\infty \sin(v) v^{-\alpha} e^{-\lambda v/u} dv \right| \leq u^{\alpha-1} \int_0^\pi v^{-\alpha} dv + u^{\alpha-1} \int_\pi^\infty v^{-\alpha} dv \leq Cu^{\alpha-1},
\]
for a constant \( C \) independent of \( u \) and \( \lambda \). Moreover, it can be proved that there exists a constant \( C_j \) such that
\[
\left| \int_0^\infty \kappa(us)s^{j-\alpha} e^{-\lambda s} ds \right| \leq C_j (1 + \lambda)^j u^{\alpha-1-j}, \quad (7.13)
\]
for \( j \geq 1, \lambda, u > 0 \), and \( 0 < \alpha < 2 \), and where \( \kappa \) can be either \( \cos \) or \( \sin \). Indeed, the case \( j = 1 \) can be proved as follows. If \( 0 < \alpha \leq 1 \), then we apply two times integration by parts (similar to the case \( \alpha = 1 \) in the proof of Theorem 4.4). Then, we can apply part (i) of Lemma 6.1. If \( 1 < \alpha \leq 2 \), then one can apply directly part (i) of Lemma 6.1. The case \( j \geq 1 \) can be proved using induction on \( j \) with the help of two integration by parts. From the previous estimates, we have that, for \( j \geq 0 \),

\[
\int_0^\infty \kappa_j(us)s^{j-\alpha}e^{-\lambda s}ds \leq C_j(1 + \lambda)^j u^{\alpha-j}, \quad (7.14)
\]

where \( C_j \) is a constant independent of \( \lambda \) and \( u \) and \( \kappa_j(u) = \cos(u) \) if \( j \) is odd, and \( \kappa_j(u) = \sin(u) \) if \( j \) is even. Next, from the conditions on \( X \), the function \( \gamma \) is given by \( \gamma(u) = 2 \int_0^\infty (1 - \cos us)s^{-\alpha-1}q(s)ds \). Condition (7.9) implies that

\[
\|\gamma^{(j)}(u)\| = 2 \int_0^\infty \kappa_{j-1}(us)s^{j-\alpha-1}q(s)ds, \quad j \geq 1,
\]

where \( \kappa_j \) is as above. In that case, using (7.10), applying Fubini’s Theorem, and (7.14), we have

\[
\|\gamma^{(j)}(u)\| \leq 2 \int_0^\infty \left| \int_0^\infty \kappa_{j-1}(us)s^{j-\alpha-1}e^{-\lambda s}ds \right| F(d\lambda) \leq 2C_ju^{\alpha-j} \int_0^\infty (1 + \lambda)^{j-1}F(d\lambda) \leq c_ju^{\alpha-j},
\]

for a constant \( c_j \) independent of \( u \).

**Remark 7.6.** Rosiński [10] (see Proposition 2.7) gives conditions for (7.9) to hold. In terms of the notation of Proposition 7.5, (7.9) holds with \( j > 1 \) if and only if \( \int_0^1 \lambda^{j-1}F(d\lambda) < \infty \), which is also necessary and sufficient for \( j = 1 \) provided that \( \alpha < 1 \). If \( \alpha > 1 \), then (7.9) always hold for \( j = 1 \), while when \( \alpha = 1 \), (7.9) hold with \( j = 1 \) if and only if \( \int_1^1 \lambda^{-1}\log(\lambda^{-1})F(d\lambda) < \infty \).

**Appendix A: Verification of the claim in Remark 3.3 (iii).**

Note that the expression for \( d_2 \) in Remark 3.3 (ii) can be modified so that one can replace \( \tilde{s}(x) \) by \( s(x)1_{\{|x| \geq \varepsilon\}} \). We will get:

\[
d_2 = -\sigma^2 s'(y) + 2bs(y) - 2 \int_{|w| \leq \varepsilon} \int_0^1 s'(y - \beta w)(1 - \beta)d\beta w^2s(w)dw \quad (A.1)
\]

\[
+ \int_{|x| \geq \varepsilon} \int_{|u| \geq \varepsilon} 1_{\{x+u \geq y\}}s(x)s(u)dxdw \quad (A.2)
\]

\[
- 2 \int_{|x| \geq \varepsilon} s(x)dx \int_y^\infty s(x)dx - 2 \int_{\varepsilon \leq |x| \leq 1} xs(x)dxs(y). \quad (A.3)
\]
The last term in (A.1) converges to 0 as \( \varepsilon \to 0 \). The term in (A.2) can be written as follows (omitting the integrand \( s(x)s(u) \) and using symmetry of this about \( x = u \)):

\[
2 \int_{-\infty}^{-\varepsilon} dx \int_{y-x}^{\infty} du + 2 \int_{-\varepsilon}^{0} dx \int_{y}^{\infty} du - \int_{y}^{\infty} dx \int_{y}^{\infty} du
\]

\[
+ \int_{y/2}^{y} dx \int_{y/2}^{y} du + 2 \int_{\varepsilon}^{y} dx \int_{y-x}^{\infty} du.
\]

Similarly, we can decompose the terms in line (A.3) as

\[
-2 \int_{-1}^{-\varepsilon} s(x)dx \int_{y}^{\infty} s(u)du - 2 \int_{-\varepsilon}^{-1} s(x)dx \int_{y}^{\infty} s(u)du
\]

\[
-2 \int_{\varepsilon}^{y} s(x)dx \int_{\varepsilon}^{\infty} s(u)du - 2s(y) \int_{-1}^{-\varepsilon} xs(x)dx - 2s(y) \int_{\varepsilon}^{1} xs(x)dx
\]

Now, \( A_2 + B_3 = 0 \), \( A_1 + B_1 + B_2 + B_4 \) becomes

\[
2 \int_{-\varepsilon}^{0} \int_{y-x}^{y} \{s(u) - s(y)\} dus(x)dx + 2 \int_{-\varepsilon}^{0} \int_{y-x}^{y} s(u) dus(x)dx,
\]

and \( A_5 + B_5 \) becomes

\[
2 \int_{\varepsilon}^{y/2} \int_{y-x}^{y} \{s(u) - s(y)\} dus(x)dx - 2s(y) \int_{y/2}^{1} xs(x)dx.
\]

Then, after changing variables, \( d_2 \) becomes:

\[
- \sigma^2 s'(y) + 2s(y)b + 2 \int_{-\varepsilon}^{x} \{s(y-u) - s(y)\} dus(x)dx - \nu([y, \infty))^2
\]

\[
+ \int_{y/2}^{y} s(x)dx \int_{y/2}^{y} s(u)du + 2 \int_{-\infty}^{-y} \int_{y-x}^{y} s(u) dus(x)dx - 2s(y) \int_{y/2}^{1} xs(x)dx
\]

\[
+ 2 \int_{-1}^{0} \{s(y-u) - s(y)\} dus(x)dx + 2 \int_{\varepsilon}^{y/2} \int_{0}^{x} \{s(y-u) - s(y)\} dus(x)dx.
\]

Now, taking \( \varepsilon \to 0 \) in the above and further manipulation, gives the expression in Remark 3.3 (iii).

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