Mass Relation Between Top and Bottom Quarks

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Abstract

In the framework of the recently proposed electroweak theory on a Planck lattice, we are able to solve approximately the lattice Dyson equation for the fermion self-energy functions, and obtain the ratio between the masses of the $t-$ and $b-$ quarks in terms of the electroweak coupling constants. The predicted top mass agrees with recent determinations from electroweak observables.

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In recent papers [1][2][3] we have proposed to incorporate in the electroweak theory the possible effects of violent gravity quantum fluctuations at the Planck scale ($a_p \approx 10^{-33} \text{cm}$) by means of a (random) space-time lattice structure, whose lattice constant is just $a_p$. Remarking that such a lattice structure, a kind of “worm-hole” condensation in the ground state of the quantum gravitational field, is not a new proposal, but it is suggested by some analysis of the small scale quantum fluctuations of gravity [4]. We noticed that the well known “no-go” theorem of Nielsen and Ninomiya [5] would not allow a simple transcription of the electroweak lagrangian on such lattice - to be called Planck lattice -, and argued for the necessity to extend the usual electroweak Lagrangian of the Standard Model to include effective gauged Nambu-Jona Lasinio (NJL) [6] types of interactions, quadrilinear in the Fermi-field. In this way the gauge principle could be obeyed while avoiding the difficulties of the “no-go” theorem [1]. In addition, even though the NJL-terms are at this time simply added for consistency, it is however possible to envisage their origin as “effective” interactions induced by Quantum Gravity (QG), thus tying finally together in a fundamental way the physics of the Standard Model to QG.

As a first step in the development of our program we neglected the usual gauge-field interactions, and analyzed the solutions of the Dyson equations involving the NJL interactions only with the following results [2]:

1. The fundamental chiral symmetry of the full Lagrangian is spontaneously broken, and as a consequence only one quark family, which is identified with the top (t) quark and bottom (b) quark doublet, acquires a mass, that within the mentioned approximation is the same for both, i.e. $m_t = m_b$;

2. The consistent solution of the gap equations produces also non-zero Wilson-type parameters $r_q \approx 0.3$ and mass counterterms for each quark, thus solving “in practice” the difficulties of the “no-go” theorem, by removing through the Wilson-mechanism the unobserved “doublers”;

3. The composite Goldstone particles carry the quantum numbers of the gauge bosons $W^\pm$ and $Z^0$, and should end up as the longitudinal modes of these gauge bosons. The mechanism by which the Goldstone particles get “eaten up” by the gauge bosons, that in so doing become massive, has been discussed in the NJL-context in Refs. [7]. The composite scalar, on the other hand, that in the continuum theory replaces the Higgs meson [8], is seen to acquire a mass of the order of the Planck mass ($m_p \approx 10^{19} \text{GeV}$), thus becoming unobservable.

4. The effective action that is left after such massive rearrangement of the vacuum
is \((a = a_\mu)\)

\[
S = S_G + S_D + \sum_{xF} \bar{\psi}^F(x)m_F\psi(x) \\
-\frac{1}{2a} \sum_{Fx\mu} \left[ \bar{\psi}^F(x) \left( I^{F}_\mu(x) + R^{F}_\mu(x) \right) r_F U^{F}_\mu(x) \psi^F(x + a_\mu) + \text{h.c.} \right] + \cdots,
\]

where \(S_G\) is the usual Wilson gauge-action, \(S_D\) the usual Dirac action, \(F = l(q)\) denotes the lepton (quark) sector, \(m_F, r_F\) are matrices in flavour and weak isospin space, and \(\ldots\) denotes the necessary counterterms. Finally

\[
L^{F}_\mu(x) = U^{L}_\mu(x)V^{YF}_\mu(x); \quad G^{R}_\mu(x) = \begin{pmatrix} V^{YF}_{k1}(x) & 0 \\ 0 & V^{YF}_{k2}(x) \end{pmatrix},
\]

and

\[
U^{c}_\mu(x) \in SU(3), \quad U^{L}_\mu(x) \in SU(2) \text{ and } V^{c}_\mu(x) \in U_Y(1).
\]

In this note we go one step forward and study the Dyson equations for the massive quark doublet \((t, b)\) including the interactions with the gauge fields \(W^\pm, Z_0\) and \(\gamma\) (photon) based on the action \([1]\). Our aim is, of course, to see whether these latter interactions are capable to lift the identity of \(m_t\) and \(m_b\), which is experimentally known to be badly violated.

We take into consideration of the NJL interaction and gauge interactions and the Dyson equations will thus have the structure depicted diagrammatically in Fig.1. The Landau mean-field and the large-\(N_c\) approach have been adopted. For external momenta \(p_\mu a \ll 1\), we divide the integration domain over the variable \(q_\mu\) in two regions: the “continuum” region: \(0 \leq |q_\mu a| \leq \epsilon\) and the “lattice” region: \(\epsilon \leq |q_\mu a| \leq \pi\), where \(|ap_\mu| \ll \epsilon \ll \pi\). With this separation we can write the Dyson equation as:

\[
\Sigma^{t,b}_c(p) = m_t(NJL)^{t,b} + [C_{\gamma}(p) + C_{QCD}(p) + C_{Z_0}(p)]^{t,b} \\
+ [L_{\gamma}(p) + L_{QCD}(p) + L_{Z_0}(p) + L_{W}(p)]^{t,b},
\]

where \(\Sigma^{t,b}_c\) denotes the self-energy function of \(t\)– and \(b\)– quarks, respectively in the “continuum region”, and

\[
(NJL)^{t,b} = 2g_1 \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \frac{1}{\sin^2 q_\mu + [m_{t,b} a + rw(q)]^2},
\]

\[
C^{t,b}_g(p) = \frac{1}{4\pi^2} \int_{Ap} \lambda^{\gamma}_g(q) d^4q \frac{1}{(p - q)^2 + m_g^2} \frac{\sum^{t,b}_c(q)}{q^2 + m_{t,b}^2},
\]

where the subscript “\(g\)” denotes the relevant gauge-interaction, with the appropriate running gauge-coupling \(\lambda^{\gamma}_g(q) = \frac{3}{\pi} \frac{g^{2}_{\gamma}(q)}{4\pi}\) in the “continuum region”; \(g_1\) is the
dimensionless coupling introduced in [2], and $w(q) = \sum_\mu (1 - \cos q_\mu a)$. Note that no $W^\pm$-term exists in Eq. (9) in the “continuum” region due to the unique chirality of W-exchange in this region. Note also that the integrals over the “continuum” region run up to $\Lambda_p$ and not to $\Lambda = \epsilon \Lambda_p$ for, as shown in [3], the $\epsilon \Lambda_p$-term contained in $L_g$ (with the exception of $L_W$) has been added to the “continuum” region integral, thus leading to an overall $\epsilon$-independence of the terms appearing in Eq. (9), as it should happen. The $\epsilon$-independent parts $L_g$ of $L_g$ can in general be determined by numerical fitting:

$$\lambda_g^L m_{t,b}(\bar{L}_g^{t,b}(r) + \theta \ln \epsilon) = -\lambda_g^L \int_{[\epsilon, \pi]} \frac{d^4 l}{(2\pi)^4} \cdot \frac{\sum_{t,b}^b(l)}{4 \sin^2 \frac{l_\mu}{2}} \cdot \frac{-\cos^2 \frac{l_\mu}{2} + r^2 x \sin^2 \frac{l_\mu}{2}}{\sin^2 l_\mu + (r w(l))^2}, \quad (7)$$

where the approximation $\sum_{t,b}^b(l) \simeq m_{t,b}$ is made for momenta $l_\mu \in [\epsilon, \pi]$. $\bar{L}_g^{t,b}(r)$ is plotted as a function of the Wilson parameter $r$ in Fig.2. In Eq. (7) $x = 1$ for $L_\gamma$ and $L_{QCD}$, and

$$x = \begin{cases} 
\frac{3}{2} \sin^2 \theta_w + \frac{1}{2} \sin^2 \theta_w + \cos^2 \theta_w & \text{for the } b - \text{quark} \\
\frac{3}{4} \sin^2 \theta_w (\cos^2 \theta_w - \frac{1}{3} \sin^2 \theta_w) & \text{for the } t - \text{quark} 
\end{cases} \quad (8)$$

for the $L_{ZO}$ contribution. The $W^\pm$-contribution $L_W$ is given by

$$L_W^{t,b} = -\lambda_W^L \int_{-\pi}^\pi \frac{d^4 l}{16\pi^2} \frac{r^2}{4} \frac{\sum_{t,b}^b(l)}{\sin^2 l_\mu + [r w(l)]^2}, \quad (9)$$

and will play a very important rôle in the $t - b$ mass splitting. The gauge coupling $\lambda_g^L = \frac{3}{\pi} \frac{\alpha_g F}{4 \pi}$ and $\lambda_W^L = \frac{3}{\pi} \frac{\alpha_{QCD}^L}{\sin^2 \theta_w} (\sin^2 \theta_w \simeq 0.5)$, the “lattice” gauge couplings, are accordingly determined by the “bare” coupling constants. As for the “lattice” region terms, denoted in Eq. (9) by $L_i(p)$, we remark that the terms that diverge like $\frac{1}{a}$, contained in $L_g^{t,b}$, can all be consistently cancelled by mass counterterms [2].

The complicated system (9) can now be enormously simplified by setting $p_\mu a \simeq 0$ and by neglecting all momentum dependences in $\sum_{t,b}^b(l)$, thus replacing them by $m_t$ and $m_b$ respectively. We note right away that $m_t = m_b$ is no more a solution of the ensuing system, indeed the simplified Eq. (9) for the $t$-quark becomes:

$$m_t [1 - (NJL) - m_t \sum_g (\bar{C}_g^t + \bar{L}_g^t)] = m_b L_W, \quad (10)$$

where $\bar{C}$ and $L$ are the contributions discussed above divided by the “mean-field” value $\sum_{t,b}^b(q) = m_{t,b}$. Analogously for the $b$-quark we have

$$m_b [1 - (NJL) - \sum_g (\bar{C}_g^b + \bar{L}_g^b)] = m_t L_W, \quad (11)$$
where due to \( m_t a \approx m_b a \approx 0 \), (NJL), \( \bar{L}^{QCD} \) and \( \bar{L}^{W} \) are the same for both quarks. Our philosophy is that the world is so constructed as to yield small masses in a theory that starts out with only one mass scale, the Planck scale. In order for this to arise in the above two equations (10) and (11), one must “tune” the NJL-term (the coupling \( g_1 \) in (5)) so that the LHS of (10) is a small number. Now if we make the above “tuning”, the analogous term in (11) cannot be so small due to \( \bar{C}_b^b \neq \bar{C}_t^b \) and \( \bar{L}^{Z_0}_b \neq \bar{L}^{Z_0}_t \). It is clear that our “tuning” does not allow us to determine both \( m_t \) and \( m_b \), but only their ratio \( \frac{m_t}{m_b} \). Thus from Eqs. (10) and (11) we get trivially

\[
\bar{L}^{W} m_t^2 - 2 \Delta m_b m_t - \bar{L}^{W} m_b^2 = 0,
\]

where

\[
2 \Delta = (\bar{C}^t - \bar{C}^b) + (\bar{C}^{Z_0}_b - \bar{C}^{Z_0}_t) + (\bar{C}^{QCD}_b - \bar{C}^{QCD}_t),
\]

note that \( \bar{L}^{Z_0}_t \) is negligible with respect to \( \bar{C}^t \). Furthermore, the physical \( Z_0 \) boson and gluon masses being very heavy, \( \bar{C}^{QCD}_b - \bar{C}^{QCD}_t \) and \( \bar{C}^{Z_0}_b - \bar{C}^{Z_0}_t \) can also be neglected, we can approximate \( 2 \Delta \) as

\[
2 \Delta \simeq (\bar{C}^t - \bar{C}^b).
\]

In Eq. (8), the running coupling constant \( \lambda^{QED}(q^2) \) is introduced by

\[
\alpha^{QED}(q^2) = \left( 1 - \frac{\alpha^{QED}(q^2)}{3\pi} \sum_f Q_f^2 \frac{\alpha^L}{m_f^2} \right) \alpha^L,
\]

where all fermion loop contributions have been taken into account. Thus form Eq. (8) we obtain

\[
2 \Delta \simeq -0.0025 \frac{\pi \alpha^{QED}(r)}{24} + 0.0025 \frac{3 \alpha^{QED}(r)}{8\pi} \left[ \frac{4}{9} \left( \ln \frac{\Lambda_p}{m_t} \right)^2 - \frac{1}{9} \left( \ln \frac{\Lambda_p}{m_b} \right)^2 \right] + 0.644 \left( \frac{3 \alpha^{QED}(r)}{2\pi} \right) \left[ \frac{4}{9} \ln \frac{\Lambda_p}{m_t} - \frac{1}{9} \ln \frac{\Lambda_p}{m_b} \right].
\]

As for \( \bar{L}^{W} \) we have

\[
\bar{L}^{W} = 6\pi \alpha_L \frac{r^2}{4} G(r); \quad G(r) = \int_{-\pi}^{\pi} \, \frac{d\theta}{\sin^2 \theta + r^2 W(l)^2},
\]

where \( G(r) \) is plotted in Fig.3. For \( r = r_m \sim 0.3 \), the ground state

\[
G(r_m) \simeq 0.325.
\]

We are now in a position to solve Eq. (12)

\[
m_t = m_b \frac{\Delta + \sqrt{\Delta^2 + \bar{L}^{W}_W^2}}{\bar{L}^{W}_W},
\]
By substituting Eqs. (16) and (17) into Eq. (18), we find that a consistent solution to (18) is

\[ |m_t| \simeq 30|m_b|. \tag{19} \]

Setting \( m_b = 4.7 \text{ GeV} \), as given by a theoretical analysis of the \( q\bar{q} \)-spectrum \([9]\), we predict

\[ m_t \simeq 145 \text{ GeV}, \tag{20} \]

which appears to agree with indirect determinations from electroweak parameters \([10]\). Let us point out that we achieve here for the first time the goal of relating the very different masses of the top and bottom quarks to their different charges, to the Planck length, and lattice effects \( \bar{L}_w \), even though our analysis of fermion masses is still at a preliminary stage and the main uncertainty comes from our approximate way of determining the Wilson parameter \( r_q \).

The fact that we have been able to obtain a relation between the masses of the two quarks that become massive from the spontaneous breaking of the chiral symmetry, that occurs in the NJL-extensions of the electroweak theory on the Planck lattice, appears to us as a rather pleasing signal of the physical relevance of the ideas that we have been developing recently. To summarize, the picture that is emerging from our work is that

1. the standard low-energy electroweak theory is the low-energy (much smaller than the Planck mass) approximation of a chirally symmetric theory on a Planck lattice;

2. mass gets generated spontaneously, and to a first stage a quark doublet becomes massive, together with the gauge bosons \( W^\pm \) and \( Z^0 \);

3. the generation of the gauge bosons’ masses does not produce any additional massive particle, like the Higgs boson, for on a Planck lattice scalar states get lifted to the Planck mass \( \bar{L}_w \);

4. by taking due account of all the gauge interactions, in a reliable approximation to the gap equations, we are able to determine the mass ratio of the two quarks that become massive, that turns out to be in agreement with the indirect information that is now available.

So far so good; but what lies ahead? Clearly we must study the relation between the \( W^\pm \) and \( Z^0 \) masses and the \( t \)-quark mass. When this is understood we will be able to turn our attention to the possible origins of the masses of the other fermions, as well of the CKM mass-matrix; and the strategy followed in this paper appears rather promising. But these are only speculations: thus it is time to stop.
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Figure Captions

Figure 1: The Dyson equations for top and bottom quarks.
Figure 2: The function $\bar{L}(r)$ in terms of $r_q$ for $x = 1$.
Figure 3: The function $G(r)$ in terms of $r_q$. 
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