Quaternionic Representation of Snub 24-Cell and its Dual Polytope Derived From $E_8$ Root System

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Abstract

Vertices of the 4-dimensional semi-regular polytope, *snub 24-cell* and its symmetry group $W(D_4) : C_3$ of order 576 are represented in terms of quaternions with unit norm. It follows from the icosian representation of $E_8$ root system. A simple method is employed to construct the $E_8$ root system in terms of icosians which decomposes into two copies of the quaternionic root system of the Coxeter group $W(H_4)$, while one set is the elements of the binary icosahedral group the other set is a scaled copy of the first. The quaternionic root system of $H_4$ splits as the vertices of 24-cell and the *snub 24-cell* under the symmetry group of the snub 24-cell which is one of the maximal subgroups of the group $W(H_4)$ as well as $W(F_4)$. It is noted that the group is isomorphic to the semi-direct product of the Weyl group of $D_4$ with the cyclic group of order 3 denoted by $W(D_4) : C_3$, the Coxeter notation for which is $[3, 4, 3^+]$.

We analyze the vertex structure of the *snub 24-cell* and decompose the orbits of $W(H_4)$ under the orbits of $W(D_4) : C_3$. The cell structure of the snub 24-cell has been explicitly analyzed with quaternions by using the subgroups of the group $W(D_4) : C_3$. In particular, it has been shown that the dual polytopes 600-cell with 120 vertices and 120-cell with 600 vertices decompose as 120=24+96 and 600=24+96+192+288 respectively under the group $W(D_4) : C_3$. The dual polytope of the *snub 24-cell* is explicitly constructed. Decompositions of the Archimedean $W(H_4)$ polytopes under the symmetry of the group $W(D_4) : C_3$ are given in the appendix.

1 Introduction

The non-crystallographic Coxeter group $W(H_4)$ has some relevance to the quasicrystallography [1,2,3]. It is one of the maximal subgroups of $W(E_8)$ [4], the Weyl group of the exceptional Lie group $E_8$ which seems to be playing an important role in high energy physics [5]. The symmetries $A_4, B_4, F_4$ occur in high energy physics in model building either as a gauge symmetry like
SU(5) [6] or as a little group SO(9) [7] of M-theory. The exceptional Lie group $F_4$ is also in the domain of interest of high energy physicists [7,8]. The Coxeter-Weyl groups $W(H_4), W(A_4), W(B_4)$ and $W(F_4)$ describe the symmetries of the regular 4D polytopes. In particular, the Coxeter group $W(H_4)$ arises as the symmetry group of the polytope 600-cell, $\{3,3,5\}$ [9], vertices of which can be represented by 120 quaternions of the binary icosahedral group [10, 11]. The dual polytope 120-cell, $\{5,3,3\}$ with 600 vertices, can be constructed from 600-cell in terms of quaternions [11]. The symmetries of the the 4D polytopes can be nicely described using the finite subgroups of quaternions [12].

In this paper we study the symmetry group of the semi-regular 4D polytope snub 24-cell and construct its 96 vertices in terms of quaternions. It is a semi-regular polytope with 96 vertices, 432 edges, 480 faces of equilateral triangles and 144 cells of two types which was first discovered by Gosset [13]. We explicitly show how these vertices form 120 tetrahedral and 24 icosahedral cells which constitute the snub 24-cell. We organize the paper as follows. In Section 2 we construct the root system of $E_8$ using two sets of quaternionic representations of the roots as well as the weights of three 8-dimensional representations of $D_4$ [14]. This construction leads to two copies of quaternionic representations of the vertices of 600-cell. In Section 3 we introduce the quaternionic root system of $H_4$ decomposed in terms of its conjugacy classes where classes correspond to the orbits of the Coxeter group $W(H_3)$ in which the vertices of the icosidodecahedron are represented by imaginary quaternions [15]. We review the cell structures of the 600-cell and the 120-cell using the conjugacy classes of the binary icosahedral group. In Section 4 we construct the maximal subgroup $W(D_4) : C_3$ of $W(H_4)$ using the quaternionic vertices of 24-cell corresponding to the quaternionic elements of the binary tetrahedral group and work out explicitly the decomposition of the 600-cell as well as 120-cell under $W(D_4) : C_3$. Section 5 is devoted to the explicit study of the cell structures of the snub 24 cell. We construct the dual polytope of snub 24 cell in Section 6 in terms of quaternions. Remarks and discussions are given in the conclusion. In the appendix we give decompositions of the regular and semi-regular orbits of $W(H_4)$ under the group $W(D_4) : C_3$.

2 Construction of the root system of $E_8$ in terms of icosians
Let \( q = q_0 + q_i e_i, (i = 1, 2, 3) \) be a real quaternion with its conjugate defined by \( \bar{q} = q_0 - q_i e_i \) where the quaternionic imaginary units satisfy the relations

\[
e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k, (i, j, k = 1, 2, 3). \tag{1}
\]

Here \( \delta_{ij} \) and \( \varepsilon_{ijk} \) are the Kronecker and Levi-Civita symbols respectively and summation over the repeated indices is implicit. Quaternions generate the four dimensional Euclidean space where the quaternionic scalar product can be defined as

\[
(p, q) = \frac{1}{2} (\bar{p}q + \bar{q}p). \tag{2}
\]

The group of unit quaternions is isomorphic to \( SU(2) \) which is a double cover of the proper rotation group \( SO(3) \). The imaginary quaternionic units \( e_i \) can be related to the Pauli matrices \( \sigma_i \) by \( e_i = -i \sigma_i \) and the unit quaternion is represented by a \( 2 \times 2 \) unit matrix. The roots of \( D_4 \) can be obtained from the Coxeter-Dynkin diagram (Figure 1) where the scaled simple roots are denoted by quaternions of unit norm

\[
\alpha_1 = e_1, \ \alpha_2 = \frac{1}{2} (1 - e_1 - e_2 - e_3), \ \alpha_3 = e_2, \ \alpha_4 = e_3. \tag{3}
\]

![Coxeter-Dynkin diagram of \( D_4 \) with quaternionic simple roots.](image)

This will lead to the following orbits of weights of \( D_4 \) under the Weyl group \( W(D_4) \):

- \( O(1000) : V_1 = \{ \frac{1}{2} (\pm 1 \pm e_1), \frac{1}{2} (\pm e_2 \pm e_3) \} \)
- \( O(0010) : V_2 = \{ \frac{1}{2} (\pm 1 \pm e_2), \frac{1}{2} (\pm e_3 \pm e_1) \} \)
- \( O(0001) : V_3 = \{ \frac{1}{2} (\pm 1 \pm e_3), \frac{1}{2} (\pm e_1 \pm e_2) \} \)
- \( O(0100) : T = \{ \pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2} (\pm 1 \pm e_1 \pm e_2 \pm e_3) \} \).

Let us also define the set of quaternions \( T' = \{ \sqrt{2} V_1 \oplus \sqrt{2} V_2 \oplus \sqrt{2} V_3 \} \), dual of \( T \).
We will adopt the Lie algebraic notation $\Lambda = (a_1 a_2 ... a_n)$ for the highest weight [16] for a Lie group $G$ of rank $n$ but the notation $O(a_1 a_2 ... a_n) = W(G)(a_1 a_2 ... a_n)$ stands for the orbit deduced from the highest weight. The first three sets of quaternions in (4) are known as the weights of three 8-dimensional vector, spinor, and antispinor representations of $SO(8)$ and the last 24 quaternions are the non-zero roots of the same Lie algebra. Actually the set of quaternions in (4) constitute the non-zero roots of $F_4$. When the set of quaternions of $V_i (i = 1, 2, 3)$ are multiplied by $\sqrt{2}$ and taken together with the set $T$, they constitute the elements of the binary octahedral group of order 48. The set $T$ alone represents the binary tetrahedral group of order 24 and also they are the vertices of the 24-cell whose symmetry is the group $W(F_4)$ of order 1152.

The $E_8$ root system can be constructed in terms of two sets of quaternionic root systems of $F_4$ as follows [4]:

$$(T, 0) \oplus (0, T) \oplus (V_1, V_3) \oplus (V_2, V_1) \oplus (V_3, V_2)$$  \hspace{1cm} (5)

where the ordered pair $(A, B)$ means $A + \sigma B$ and $\sigma = \frac{1 + \sqrt{5}}{2}$ with the golden ratio $\tau = \frac{1 + \sqrt{5}}{2}$ they satisfying the relations $\tau \sigma = -1$, $\tau + \sigma = 1$, $\tau^2 = \tau + 1$ and $\sigma^2 = \sigma + 1$ and $\oplus$ represents the union of the sets. The 240 quaternions in (5) can be written as the union of $I \oplus \sigma I$ [4] where $I$ stands for the set of 120 quaternionic elements of the binary icosahedral group which also constitutes the root system of $H_4$ representing the vertices of 600-cell. Any two quaternions $p, q$ from (5) satisfy the scalar product $(p, q) = x + \sigma y$ where $x, y = 0, \pm \frac{1}{2}, \pm 1$. If one defines the Euclidean scalar product $(p, q)_E = x$ [17] then the set of quaternions in (5) represents the roots of $E_8$. This proves that the roots of $E_8$ which correspond to the vertices of the $E_8$-Gosset’s polytope decompose as two copies of 600-cells in 4 dimensional Euclidean space, one is a scaled copy of the other. If one adopts the Euclidean scalar product then $\sigma I$ and $I$ lie in two orthogonal 4D spaces.

3 Quaternions and $W(H_4)$

It is well known that the set of icosians $I$ constitute the roots of $H_4$ and can be generated from the Coxeter diagram where the simple roots are represented by unit quaternions [11]. In our analysis the subgroup $W(H_3)$ plays a crucial role. Therefore we want to display the elements of $I$ as the orbits of $W(H_3)$. They are tabulated in Table 1 in terms of its conjugacy classes of the binary icosahedral group, in other words as the orbits of $W(H_3)$. The conjugacy classes represent the vertices of four icosahedra, two dodecahedra and one
icosidodecahedron as well as two single points ±1.

Table 1: Conjugacy classes of the binary icosahedral group \( I \) represented by quaternions

| Conjugacy classes and orders of elements | The sets of the conjugacy classes denoted by the number of elements |
|-----------------------------------------|---------------------------------------------------------------|
| 1                                       | 1                                                             |
| 2                                       | −1                                                            |
| 10                                      | \( 12_+ : \frac{1}{2}(\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(\tau \pm e_2 \pm \sigma e_1), \frac{1}{2}(\tau \pm e_3 \pm \sigma e_2) \) |
| 5                                       | \( 12_- : \frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(-\tau \pm e_2 \pm \sigma e_1), \frac{1}{2}(-\tau \pm e_3 \pm \sigma e_2) \) |
| 10                                      | \( 12'_+ : \frac{1}{2}(\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(\sigma \pm e_2 \pm \tau e_3), \frac{1}{2}(\sigma \pm e_3 \pm \tau e_1) \) |
| 5                                       | \( 12'_- : \frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(-\sigma \pm e_2 \pm \tau e_3), \frac{1}{2}(-\sigma \pm e_3 \pm \tau e_1) \) |
| 6                                       | \( 20_+ : \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(1 \pm \tau e_1 \pm \tau e_2 \pm \tau e_3), \frac{1}{2}(1 \pm \tau e_3 \pm \sigma e_1) \) |
| 3                                       | \( 20_- : \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(-1 \pm \tau e_1 \pm \tau e_2 \pm \tau e_3), \frac{1}{2}(-1 \pm \tau e_3 \pm \sigma e_1) \) |
| 4                                       | \( 30 : \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm \sigma e_1 \pm \tau e_2 \pm e_3), \frac{1}{2}(\pm \sigma e_2 \pm \tau e_3 \pm e_1), \frac{1}{2}(\pm \sigma e_3 \pm \tau e_1 \pm e_2) \) |

Denote by \( p, q \in I \) any two unit quaternions. One can define a transformation on an arbitrary quaternion \( r \) by

\[
[p, q] : r \rightarrow r' = prq, \quad [p, q]^* : r \rightarrow r'' = \bar{p}rq.
\] (6)

The set of all these transformations constitute the Coxeter group

\[
W(H_4) = \{ [p, q] \oplus [p, q]^* ; \ p, q \in I \}
\] (7)

of order 14,400 [11]. It is clear that the set \( I \) itself is invariant under the group \( W(H_4) \). The binary tetrahedral group \( T \) is one of the maximal subgroups of \( I \). Let \( p \) be an arbitrary element of \( I \) which satisfies \( p^5 = \pm 1 \). Note that we have the relations

\[
\bar{p} = \pm p^4, \quad \bar{p}^2 = \pm p^3, \quad \bar{p}^3 = \pm p^2, \quad \bar{p}^4 = \pm p.
\] (8)

Then one can write the set of elements of the binary icosahedral group, in other words, the vertices of the 600-cell as [10, 11]
\[ I = \sum_{j=0}^{4} \oplus p^j T = T \oplus \sum_{j=1}^{4} \oplus p^j T = T \oplus \sum_{j=1}^{4} \oplus T \bar{p}^j. \]  \tag{9}

This decomposition is true when \( p^j \) is replaced by any conjugate \( p_c^j = qp^j \bar{q} \) with \( q \in I \).

## 4 Embedding of the group \( W(D_4) : C_3 \) in the Coxeter group \( W(H_4) \)

The five conjugate groups of the group \( T \) can be represented by the sets of elements \( p^i T \bar{p}^i (i = 0, 1, 2, 3, 4) \). The sets \( p^i T \) are the five copies of the 24-cell in the 600-cell. It is then natural to expect that the Coxeter group \( W(H_4) \) contains the group with the set of 576 elements \( \{[T, T] \oplus [T, T]^*\} \) as a maximal subgroup [18]. One can prove that it is the extension of the Coxeter-Weyl group \( W(D_4) \) of order 192 by the cyclic group \( C_3 \) which permutes, in the cyclic order, three outer simple roots \( e_1, e_2, e_3 \) of \( D_4 \) in Figure 1. In reference [18] it was shown that the group is isomorphic to the semi-direct product of these two groups, namely, the group \( W(D_4) : C_3 = \{[T, T] \oplus [T, T]^*\} \). The group \( W(D_4) : C_3 \) is also a maximal subgroup of the group \( W(F_4) \approx W(D_4) : S_3 \) of order 1152 which is the symmetry group of the 24-cell [8]. The 25 conjugate groups of the group \( W(D_4) : C_3 = \{[T, T] \oplus [T, T]^*\} \) in the group \( W(H_4) \) can be represented by the groups

\[
(W(D_4) : C_3)^{(i, j)} = \{[p^i T \bar{p}^j, p^i T \bar{p}^j] \oplus [p^i T \bar{p}^j, p^i T \bar{p}^j]^*\}; \tag{10}
\]

with \( p \) satisfying \( p^5 = \pm 1 \). Without loss of generality, we will work with the group \( (W(D_4) : C_3)^{(0,0)} = W(D_4) : C_3 = \{[T, T] \oplus [T, T]^*\} \).

Let us denote by

\[
S = I - T = \sum_{j=1}^{4} \oplus p^j T = \sum_{j=1}^{4} \oplus T \bar{p}^j \tag{11}
\]

the set of 96 quaternions representing the vertices of the snub 24-cell. The above discussion indicates that the snub 24-cell can be embedded in the 600-cell five different ways. Of course when two copies of 24-cell \( T \oplus \sigma T \) is removed from the root system of \( E_8 \), in other words from the Gosset’s polytope, what remain are two copies of the snub 24-cell.
\{O(1000) + \sigma O(0001)\} \oplus \{O(0001) + \sigma O(0010)\} \\
\oplus \{O(0010) + \sigma O(1000)\} = S \oplus \sigma S. \quad (12)

We will work with the set $S$ because the other is just a scaled copy of $S$. It is clear that the set $T$ representing the 24-cell is invariant under the group $W(D_4) : C_3$ which acts on the set $S$ as follows:

$$[T, T] : S \Rightarrow \sum_{j=1}^{4} \oplus T p^j T T = \sum_{j=1}^{4} \oplus p^j_c T = \sum_{j=1}^{4} \oplus T p^j_c \quad (13)$$

and

$$[T, T]^* : S \Rightarrow \sum_{j=1}^{4} \oplus T T \bar{p}^j T = \sum_{j=1}^{4} \oplus T \bar{p}^j = \sum_{j=1}^{4} \oplus p^j_c T = \sum_{j=1}^{4} \oplus T p^j_c \quad (14)$$

where $p_c$ is an arbitrary conjugate of $p$. It implies that the set $S$ is invariant under the group $W(D_4) : C_3$. In the Coxeter’s notation this group is denoted by $[3, 4, 3^+]$ [19].

It was shown in the references [10,11] that the set of 600 vertices of the 120-cell can be written as the set of quaternions $J = \sum_{i,j=0}^{4} \oplus p^i T' \bar{q}^j = \sum_{i=0}^{4} \oplus p^i T' \bar{q}^j = \sum_{i,j=0}^{4} \oplus p^i \bar{p}^j T'$, $(i, j = 0, 1, 2, 3, 4)$ where the quaternions $p$ and $q$ satisfy the relations $p^5 = q^5 = \pm 1$ and $p = \bar{c} q^i c$. Here $p^i$ is obtained from $p$ by replacing $\sigma \leftrightarrow \tau$ and $c$ is an arbitrary element of $T'$. It is not difficult to show that under the group $W(D_4) : C_3$ it decomposes as

$$J = \sum_{i,j=0}^{4} \oplus p^i \bar{p}^j T' = T' + \sum_{i=1}^{4} \oplus p^i \bar{p}^j T' + \sum_{i=1}^{4} (\oplus p^i T' \oplus \bar{p}^i T') \oplus \sum_{i \neq j=1}^{4} \oplus p^i \bar{p}^j T'. \quad (15)$$

These are the orbits of $W(D_4) : C_3$ of sizes 24, 96, 192, and 288. Let us denote the orbits of sizes 96, 192 and 288 in 120-cell by $S' = \sum_{i=1}^{4} \oplus p^i \bar{p}^j T'$, $M = \sum_{i=1}^{4} (\oplus p^i T' \oplus \bar{p}^j T')$, $N = \sum_{i \neq j=1}^{4} \oplus p^i \bar{p}^j T'$ respectively. We shall discuss later that the orbits $T, T', S'$ of sizes 24, 24, 96 respectively are important for the dual snub 24-cell.

We now discuss the cell structure of the snub 24-cell [20]. In reference [11]
we have given a detailed analysis of the cell structures of the 4D polytopes 600-cell and 120-cell which are orbits of the Coxeter group $W(H_4)$. There, we have proved that the conjugacy classes in those two examples denoted by $12_+$ can be decomposed as 20 sets of 3-quaternion system, each set is representing an equilateral triangle. The vertices of all equilateral triangles are equidistant from the unit quaternion 1, implying that one obtains a structure of 20 tetrahedra meeting at the vertex 1. The centers of the 20 tetrahedra constitute the vertices of a dodecahedron. The symmetry of a tetrahedron is the group $W(A_3) \approx S_4$ of order 24. Therefore the number of tetrahedra constituting the 600-cell is the index of $W(A_3)$ in $W(H_4)$ that is 600. The set $12_+$, representing an icosahedron in three dimensions, is indeed an orbit of the icosahedral subgroup $W(H_3)$ whose index in $W(H_4)$ is 120. Those 20 sets of quaternions in $12_+$ represent the faces of the icosahedron, five of which meet at one vertex. This shows that each quaternion of $12_+$ belongs to 5 of those 20 sets. The decomposition of quaternions in Table 1 is made with respect to a subgroup $W(H_3)$ which leaves the unit quaternion 1 invariant. The group $W(H_3)$ can be represented as the subset of elements of $W(H_4)$ by

$$W(H_3) = \{[I, \bar{I}] \oplus [I, \bar{I}]^*\}. \quad (16)$$

In fact, one of the maximal subgroup of $W(H_4)$ is $W(H_3) \times C_2$ of order 240 where $C_2$, in our notation, is generated by the group element $[1, -1]$. In the Coxeter group $W(H_4)$ there are 60 conjugates of the group $W(H_3) \times C_2$, each one is leaving one pair of elements $\pm q$ invariant. The conjugates of the group $W(H_3) \times C_2$ can be represented compactly by the set of group elements

$$\{W(H_3) \times C_2\}^q = \{[I, \pm q\bar{I}q] \oplus [I, \pm q\bar{I}q]^*\}, \quad q \in I. \quad (17)$$

The orbits of $W(H_3)^q$ in $I$ can be written as $\pm q, q(12_+) \equiv 12_+(q), \quad q(12'_+) \equiv 12'_+q, q(20_+) \equiv 20_+(q), q(30) \equiv 30(q)$. The set of quaternions in the conjugacy classes are multiplied by the quaternion $q$ on the left or on the right. Let $t \in T \subset I$, then we can form the set $t(12_+)$ in 24 different ways. Each set $t(12_+)$ together with one $t \in T$ represents 20 tetrahedra. Therefore, with the set of quaternions of $T$, each is sitting at one vertex; one obtains $24 \times 20 = 480$ tetrahedra. Actually the centers of these 480 tetrahedra lie on the 480=192+288 vertices of the 120-cell. Removing 480 tetrahedra from $I$ results in removal of the sets $M$ and $N$ from the set $J$. We know that the set $t(12_+)$ does not involve any quaternion from $T$. If that were the case then the scalar product of $t$ with this element would yield to $\frac{\tau}{2}$. But this is impossible for any two elements of the set $T$. When the vertices of $T$ are removed from the 600-cell the remaining 120 tetrahedral cells belong to the snub 24-cell. It
as clear then that when the quaternion \( t \in T \) is removed what is left in the void is the icosahedron represented by the vertices \( t(12_+) \). Therefore instead of 480 tetrahedra there has been created 24 icosahedra, the vertices of which are the sets \( t(12_+) \) with \( t \) taking 24 values in the set \( T \). Adding this to the remaining 120 tetrahedral cells then the number of cells of the snub 24-cell will be 144. Below we will work out the detailed cell structure of the polytope of concern.

5 Detailed analysis of the cell structure of the snub 24-cell

The conjugacy class \( 12_+(1) \) can be written as the products of three elements; denote the elements by

\[
p = \frac{1}{2}(\tau + e_1 + \sigma e_3), t_1 = \frac{1}{2}(1 + e_1 - e_2 - e_3),
\]

\[
t_2 = \frac{1}{2}(1 + e_1 + e_2 - e_3). \quad (18)
\]

The set of three elements where \( t_1, t_2 \) and \( t_1 t_2 = e_1 \) are elements of \( T \). Then we can represent the set of elements of \( 12_+(1) \) by

\[
12_+(1) = \{p, p^2(\bar{t}_2 \bar{t}_1), \bar{p}^2 t_2, \bar{p} t_2, p^{2} \bar{t}_2, p^{2} \bar{t}_1, \\
p^{2} t_1 t_2, \bar{p} t_1, \bar{p} t_1^2, \bar{p} t_2\}. \quad (19)
\]

As we noted earlier this set of elements represents an icosahedron. Now multiply this set either from left or right by \( t_1 \) and \( t_2 \) to obtain the sets

\[
12_+ (t_1) = \{pt_1, p^2 \bar{t}_2, \bar{p}^2 t_2 t_1, p \bar{t}_2, p \bar{t}_2 t_1, p^{2} t_2 t_1, p^{2}, \\
p^{2} t_1 t_2 t_1, p, \bar{p} t_1, \bar{p} t_1^2, \bar{p} t_2 t_1 \} \quad (20)
\]

\[
12_+ (t_2) = \{pt_2, p^2 (\bar{t}_2 \bar{t}_1 t_2), \bar{p}^2 t_1 t_2, p^{2} t_2, p, p^{2}, \\
p^{2} \bar{t}_1 t_2, p^{2} \bar{t}_1 t_2, p \bar{t}_1 t_2, p \bar{t}_1 t_2, p \bar{t}_1 t_2 \}. \quad (21)
\]

Each set represents an icosahedron. Here the sets in (20) and (21) are left invariant respectively by the conjugate groups of \( W(H_3) \), namely by, \( W(H_3)^{t_1} = \{[I, t_1 \bar{t}_1] + [I, t_1 \bar{t}_1]^* \} \) and \( W(H_3)^{t_2} = \{[I, \bar{t}_2 t_2] + [I, t_2 \bar{t}_2]^* \} \).
respectively. The centers of these three icosahedra are represented, up to a scale factor, by three quaternions $1, t_1$ and $t_2$ respectively. The crucial thing in (19-21) is that all three icosahedra have one common vertex represented by the quaternion $p$. This proves that three icosahedra meet at one vertex $p$. Now we show that there are exactly five tetrahedra meeting at the vertex $p$. When we multiply the set of elements in (19) by $p$ we obtain the set

$$12_+(p) = \{ p^2, p^3(\bar{t}_2 \bar{t}_1), \bar{p} t_1, \bar{p} t_2, p^2 \bar{t}_2, p^3 \bar{t}_2, p^3 \bar{t}_1, \bar{p} t_1 t_2, p^2 \bar{t}_1, 1, t_1, t_2 \}. \quad (22)$$

We have already explained that the set of elements in (22) form the set of 20 tetrahedra, all connected to the vertex represented by $p$. We also note that 15 of them have vertices involving the quaternions $1, t_1, t_2$ which are elements of 24-cell $T$. The remaining set of five tetrahedra involves the vertices belonging only to the set $S$. The above arguments indicate that, at the given vertex $p$, the snub 24-cell has five tetrahedral and three icosahedral cells. This is true for all elements of the set $S$. Therefore the snub 24-cell has $\frac{96 \times 5}{4} = 24$ icosahedral cells and $\frac{96 \times 5}{4} = 120$ tetrahedral cells as explained before with a total number of 144 cells.

One can look at the problem from the symmetry point of view of the cells. One of the maximal subgroups of $\{W(H_3)\}^q$ in (17) is obtained when $I$ is restricted to $T$ which can be written in the form

$$\{A_4 \times C_2\}^q = \{ [T, qTq] \oplus [T, qTq]^* \}. \quad (23)$$

Here $A_4$ stands for the even permutations of the four letters, a group of order 12, and together with the cyclic group $C_2$, it is a group of order 24 which permutes the vertices of icosahedron leaving its center represented by the quaternion $q \in T$ invariant. The group in (23) can be embedded in the symmetry group $W(D_4) : C_3$ of the snub 24-cell in 24 different ways, in each case leaving one quaternion $q \in T$ invariant. This shows that the number of icosahedral cells of the snub 24-cell is 24 and the centers of these icosahedra are represented, up to a scale factor, by the set of quaternions belonging to the set $T$. The group in (23) is not the full symmetry of an icosahedron but just a subgroup of it because the symmetry group of the snub 24-cell does not involve the whole symmetry group of the icosahedron. As we noted before in 600-cell, $I$, the number of vertices closest to a given vertex is 12 while this number in the snub 24-cell that is in the set $S$ is 9 as shown in (22). The 9 vertices nearest to the quaternion $p$ in (22) represent the vertices of five tetrahedra as well as the five nearest vertices of three icosahedra. Let us denote them by
\[
q_1 = \frac{1}{2}(-\sigma + \tau e_1 - e_3),
q_2 = \frac{1}{2}(\sigma - \tau e_1 - e_2),
q_3 = \frac{1}{2}(\tau - \sigma e_1 + e_2),
q_4 = \frac{1}{2}(\tau + \sigma e_2 - e_3),
q_5 = \frac{1}{2}(\tau - \sigma e_2 - e_3),
q_6 = \frac{1}{2}(1 - \sigma e_1 - \tau e_3),
q_7 = \frac{1}{2}(\tau + e_1 - \sigma e_3),
q_8 = \frac{1}{2}(1 + \tau e_1 - \sigma e_2),
q_9 = \frac{1}{2}(1 + \tau e_1 + \sigma e_2).
\]

The vertices of five tetrahedra \(P(i)\) \((i = 1, 2, 3, 4, 5)\) meeting at the point \(q_{10} \equiv p = \frac{1}{2}(\tau + e_1 + \sigma e_3)\) can be obtained from (24) and their corresponding centers \(c_i\) \((i = 1, \ldots, 5)\) can be written up to a scale factor as follows:

\[
\begin{align*}
P(1) &= \{q_{10}, q_7, q_8, q_9\}; \quad c_1 = \frac{1}{\sqrt{2}}(1 + e_1), \\
P(2) &= \{q_{10}, q_4, q_5, q_6\}; \quad c_2 = \frac{1}{2\sqrt{2}}((\tau - \sigma) - \sigma e_1 - \tau e_3), \\
P(3) &= \{q_{10}, q_3, q_7, q_8\}; \quad c_3 = \frac{1}{2\sqrt{2}}(\tau - \sigma + \tau e_1 - \tau e_2), \\
P(4) &= \{q_{10}, q_1, q_8, q_9\}; \quad c_4 = \frac{1}{2\sqrt{2}}(\tau + (\tau - \sigma)e_1 + \sigma e_3), \\
P(5) &= \{q_{10}, q_2, q_7, q_9\}; \quad c_5 = \frac{1}{2\sqrt{2}}(\tau - \sigma + \tau e_1 + \sigma e_2).
\end{align*}
\]

Now one can check that there exist 15 equilateral triangles having \(p\) as a common vertex. Then the number of faces of the snub 24-cell is \(\frac{15 \times 96}{3} = 480\).

Similarly the number of edges are \(\frac{2 \times 96}{2} = 432\).

The nine vertices \(q_i\) \((i = 1, 2, \ldots, 9)\) closest to the quaternion \(p\) represent the vertices of the vertex figure of the snub 24-cell as we will discuss later. It is interesting to note that the center of the tetrahedron \(P(1)\) belongs to the set \(T'\) which is invariant under the symmetry group of the snub 24-cell, \(W(D_4) : C_3\). Action of the group \(W(D_4) : C_3\) on the tetrahedron \(P(1)\) will generate 24 tetrahedra whose centers, up to a scale factor, lie on the vertices of the 24-cell, \(T'\). One can show that the centers of the tetrahedra \(P(i)\) \((i = 2, 3, 4, 5)\) belong to the set of 96 vertices of the set \(S'\). The subgroup of the group \(W(D_4) : C_3\) preserving the tetrahedron \(P(1)\) can be written in the form

\[
S_4 = \{[T, c_1 T c_1] \oplus [T, c_1 T c_1]^*\}, c_1 = \frac{1}{\sqrt{2}}(1 + e_1) \in T'.
\]

This is the tetrahedral subgroup of \(W(D_4) : C_3\) which is isomorphic to the symmetric group \(S_4\) of four letters. It is the group which permutes the vertices of the tetrahedron \(P(1)\) while fixing its center \(c_1 = \frac{1}{\sqrt{2}}(1 + e_1)\). The conjugates of the group \(S_4\) represented by (26) are the 24 different subgroups each fixing one element of \(T'\). This is another proof that there exist 24 tetrahedra of type \(P(1)\) whose centers lie on the orbit \(T'\). It is also interesting to note that the centers of the 24 icosahedral cells lie on the vertices of the
other 24-cell represented by the set $T$ which can be obtained from $T'$ by rotation around some axis.

The subgroup which preserves one of the remaining 96 tetrahedra is a group isomorphic to the symmetric group $S_3$. This is not a surprise because the index of $S_3$ in the group $W(D_4) : C_3$ is 96 corresponding to the 96 tetrahedra. Let us construct the subgroup of $W(D_4) : C_3$ which fixes the vertex $q_{10} \equiv p = \frac{1}{2}( \tau + e_1 + \sigma e_3) = \frac{1}{2}[(1 + e_1) - \sigma(1 - e_3)] = \frac{1}{2}[\tau(1 + e_1) + \sigma(e_1 + e_3)]$ implying that $p$ can be written as a linear combination of two elements of the set $T'$. This is evident because we have already noted this property in (12) which followed from (5). Another interesting relation similar to (12) can be written in the form

$$\{\tau O(1000) + \sigma O(0010)\} \oplus \{\tau O(0010) + \sigma O(0001)\} \oplus \{\tau O(0001) + \sigma O(1000)\} = S \oplus S'.$$ (27)

This shows that the elements of two sets $S$ and $S'$ can be written, up to a scale factor, in the form of $\tau a + \sigma b$ where $a$ and $b$ are elements of the set $T'$. We have already shown that the subgroup of the group $W(D_4) : C_3$ leaving one element of $T'$ invariant can be written as $S_4 = \{[T, \bar{a}T]a] \oplus [T, aT]\}$. A subgroup of this group which also fixes the element $b$ leaves the quaternion $\tau a + \sigma b$ invariant. To find the generators we apply the above group elements on $b$, namely,

$$[t, \bar{a}Ta] : b \rightarrow tba\bar{a} = b; \quad [t, a\bar{a}] : b \rightarrow tba\bar{a} = b.$$ (28)

where $t \in T$.

From the first relation it follows that $ba\bar{a} = \bar{t}ba$ implying that $t = ba$ is one of the solutions besides the quaternion 1. The second relation will lead to $b\bar{a}t = \bar{t}ba$. For the case of $p$ above one can choose $a = \frac{1}{\sqrt{2}}(1 + e_1), b = \frac{1}{\sqrt{2}}(e_1 + e_3)$ which leads to the element $ba \equiv s_1 = \frac{1}{2}(1 + e_1 - e_2 + e_3), \overline{ba} \equiv s_2 = \frac{1}{2}(1 - e_1 - e_2 - e_3)$ of $T$ and the group generator would be

$$[s_1, s_2].$$ (29)

The second relation implies that the group element of $T$ must be $e_2$ and the group generator reads

$$[e_2, -e_2].$$ (30)

One can prove that the generators in (29) and in (30) generate a group of order 6 isomorphic to the symmetric group $S_3$. It is also interesting to
note that the same group fixes the quaternions $c_1$ and $c_2$ corresponding to
the centers of two tetrahedra $P(1)$ and $P(2)$ which implies that these two
tetrahedra are left invariant by the group fixing the quaternion $p$. One can
show that the tetrahedra $P(3)$, $P(4)$, $P(5)$ are permuted by the group $S_3$.
Similarly the three icosahedra $12_+(1), 12_+(t_1), 12_+(t_2)$ given in (19-21) are
permuted among themselves. This group is also the symmetry of the vertex
figure of the snub 24-cell as we will discuss it now.
The vertex figure of any convex polytope is the convex solid for
the nearest vertices to it. In our case they are the quaternions $q_i$ ($i = 1, 2, ..., 9$).
Since all these quaternions have the same scalar product, $\tau/2$, with the
quaternion $p$ they all lie in the same hyperplane orthogonal to $p$. One should
then express them in an orthogonal basis involving $p$. The new set of basis
vectors can be obtained by multiplying the quaternionic units $1, e_1, e_2, e_3$
by $p$ on the
right or left and define the new basis as follows

\[
\begin{align*}
p_0 &= p = \frac{1}{2}(\tau + e_1 + \sigma e_3), \\
p_1 &= e_1 p = \frac{1}{2}(-1 + \tau e_1 - \sigma e_2), \\
p_2 &= e_2 p = \frac{1}{2}(\sigma e_1 + \tau e_2 - e_3), \\
p_3 &= e_3 p = \frac{1}{2}(-\sigma + e_2 + \tau e_3).
\end{align*}
\]  

(31)

When 9 quaternions $q_i$, $i = 1, 2, ..., 9$ are expressed in terms of the new basis
vectors and the first component multiplying $p_0$ is removed then the nine
quaternion with remaining components would read

\[
(\pm 1, 0, \sigma), (1, 0, -\sigma), (\sigma, \pm 1, 0), (0, \sigma, 1), (-\sigma, -1, 0), (0, -\sigma, \pm 1).
\]

(32)

An overall scale factor $\frac{1}{2}$ is omitted. These vertices represent the tridiminished icosahedron, a Johnson’s solid, $J_{63}$ [21] as shown in Figure 2(a) and
Figure 2(b). If three more vertices $(-1, 0, -\sigma), (-\sigma, 1, 0), (0, \sigma, -1)$ are added
to (32) we would have the vertices of an icosahedron as shown in Figure 2(c). A net of the tridiminished icosahedron is depicted in Figure 3 where the
vertices identified with those nine quaternions.
Removing those three vertices from an icosahedron reduces the symmetry of
the icosahedron of order 120 to the symmetry of order 6. The tridiminished icosahedron has three pentagonal and five triangular faces. Its symmetry is
the symmetric group $S_3$, generated by the generators given by (29-30), which
permutes the vertices of each set among themselves:

\[
(q_1, q_2, q_3), (q_4, q_5, q_6), (q_7, q_8, q_9).
\]

(33)
The symmetry axis goes through the centers of two opposite triangular faces
represented by $(q_4, q_5, q_6), (q_7, q_8, q_9)$. Three pentagonal faces and the remaining three triangular faces are permuted by $S_3$. 

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Dual polytope of a given regular or semi-regular polytope is constructed by taking the centers of its cells as the vertices of the dual polytope. Since snub-24 cell has 144 cells, the dual snub-24 cell will have 144 vertices. In order that the dual cell has the same symmetry group of its original polytope, its vertices should lie in the hyperplanes orthogonal to the vertices of the original polytope. We have studied the Catalan solids which are duals of the Archimedean solids in a different paper in the context of quaternions [22]. The vertices of the dual cell of a semi-regular polytope have, in general, different lengths lying on the concentric 3-spheres $S^3$ with different radii. Here, in our example, we should determine hyperplane orthogonal to the vertex, say again, $p$. The centers of five tetrahedra, up to a scale factor, are already determined by the unit quaternions $c_i$ ($i = 1, 2, 3, 4, 5$) given in (25). We recall that the centers of those three icosahedra lie on the unit vectors $1, t_1, t_2$ up to a scale factor. When we multiply the latter three unit quaternions by $\frac{1}{\sqrt{2}}$, then all eight quaternions $\frac{1}{\sqrt{2}}1, \frac{1}{\sqrt{2}}t_1, \frac{1}{\sqrt{2}}t_2, c_i$ ($i = \ldots$)
Figure 3: The net of tridiminished icosahedron where the vertices are identified with nine quaternions

1, 2, 3, 4, 5) lie in the same hyperplane determined by the equation $\tau q_0 + q_1 + \sigma q_3 = \frac{\tau^2}{\sqrt{2}}$ which is orthogonal to the vertex $p$. Now we use the basis defined in (31) to express above eight quaternions as follows:

\[
\begin{align*}
\tau \sqrt{2} & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 - \tau p_1 + p_3), \\
\tau \sqrt{2} t_1 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 - p_2 - \tau p_3), \\
\tau \sqrt{2} t_2 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 + p_1 + \tau p_2), \\
c_1 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 - \sigma p_1 + \sigma p_2 - \sigma p_3), \\
c_2 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 + \sigma p_1 - \sigma p_2 + \sigma p_3), \\
c_3 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 + \sigma^2 p_1 + p_3), \\
c_4 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 + p_1 - \sigma^2 p_2), \\
c_5 & = \frac{1}{2\sqrt{2}}(\tau^2 p_0 - p_2 + \sigma^2 p_3). \\
\end{align*}
\]

After removing the components along $p = p_0$ and omitting the factor $\frac{1}{2\sqrt{2}}$ one writes the 8 vertices in the above order as follows:

\[
(-\tau, 0, 1), (0, -1, -\tau), (1, \tau, 0), (-\sigma, \sigma, -\sigma), \\
(\sigma, -\sigma, \sigma), (\sigma^2, 0, 1), (1, -\sigma^2, 0), (0, -1, \sigma^2).
\]

These are the vertices of one cell of the dual polytope of the snub 24-cell shown in Figure 4(a) and Figure 4(b). It is clear from the Figures 4(a)-4(b) that the cell has three kite faces with edge lengths $\frac{1}{\sqrt{2}}$ and $\frac{\sigma^2}{\sqrt{2}}$ with the shorter diagonal of length $\frac{\sigma}{\sqrt{2}}$ and six isosceles triangles with edge lengths $\frac{1}{\sqrt{2}}$ and $\frac{\tau}{\sqrt{2}}$ as shown in the Figure 5(a) and Figure 5(b).

The symmetric group $S_3$, fixing the vertex $p$ also fixes the vertices $c_1$ and $c_2$ and permutes each set of three vertices $\frac{\tau}{\sqrt{2}}$1, $\frac{\tau}{\sqrt{2}} t_1$, $\frac{\tau}{\sqrt{2}} t_2$ and $(c_3, c_4, c_5)$. We
Figure 4: (a) Cell of the dual snub 24-cell. (b) Cell of the dual snub 24-cell (another view)

Figure 5: (a) Kite face of the cell of dual snub 24-cell. (b) Isosceles triangular face of the cell of dual snub 24-cell

have 96 cells of the same type constituting the dual snub-24 cell. It is clear that the dual snub 24-cell is cell transitive for the group $W(D_4) : C_3$ is transitive on the vertices of the snub 24-cell. The vertices of three kites are given as follows:

\[(c_1, c_3, c_5, \frac{\tau}{\sqrt{2}} 1); (c_1, c_4, c_5, \frac{\tau}{\sqrt{2}} t_1); (c_1, c_3, c_4, \frac{\tau}{\sqrt{2}} t_2).\] (35)

The vertices of the six isosceles triangles are represented by sets of three quaternions:

\[(c_2, \frac{\tau}{\sqrt{2}} 1, \frac{\tau}{\sqrt{2}} t_1); (c_2, \frac{\tau}{\sqrt{2}} t_1, \frac{\tau}{\sqrt{2}} t_2); (c_2, \frac{\tau}{\sqrt{2}} t_2, \frac{\tau}{\sqrt{2}} 1);\]

\[(c_5, \frac{\tau}{\sqrt{2}} 1, \frac{\tau}{\sqrt{2}} t_1); (c_4, \frac{\tau}{\sqrt{2}} t_1, \frac{\tau}{\sqrt{2}} t_2); (c_3, \frac{\tau}{\sqrt{2}} t_2, \frac{\tau}{\sqrt{2}} 1).\] (36)
The $S_4$ symmetry, fixing the vertex $c_1 = \frac{1}{\sqrt{2}}(1 + e_1)$, when applied on the vertex $c_2$, will generate four vertices surrounding the vertex $c_1 \in T'$. Since $c_2$ is fixed by the $S_3$ subgroup, one generator, say $[1, e_1]^*$ of $S_4$, not belonging to $S_3$, would suffice to determine the four quaternions when applied on $c_2$. They are given by the set of quaternions:

$$
\begin{align*}
c_2 &= \frac{1}{\sqrt{2}}(\sqrt{5} - \sigma e_1 - \tau e_3),
\quad c_2' = \frac{1}{\sqrt{2}}(-\sigma + \sqrt{5}e_1 + \tau e_2),
\quad c_2'' = \frac{1}{\sqrt{2}}(\sqrt{5} - \sigma e_1 + \tau e_3),
\quad c_2''' = \frac{1}{\sqrt{2}}(-\sigma + \sqrt{5}e_1 - \tau e_2).
\end{align*}
$$

(37)

This means there are four cells connected to the vertex $c_1 \in T'$ while only one cell is connected to any vertex of the set $S'$. These four vertices form a tetrahedron when $c_1$ is taken to be the origin. Similarly the group fixing $c_1$ will generate six vertices belonging to the set of quaternions $T$ which can be written as

$$
1, t_1, t_2, e_1, s_1, s_2.
$$

(38)

They form an octahedron around the vertex $c_1$. When we apply the group $S_4$ on three vertices $c_3, c_4, c_5$ we will have one more vertex $c_6 = \frac{1}{2\sqrt{2}}(\tau + \sqrt{5}e_1 - \sigma e_3)$. These four vertices also form a tetrahedron around the vertex $c_1$. One should note that the three sets of vertices $(c_3, c_4, c_5, c_6)$ and those given in (37) and (38) lie in three parallel hyperplanes orthogonal to $c_1$. The cell of the dual snub 24-cell which is rotated by the group generator $[1, e_1]^*$ can be displayed by the set of quaternions as follows:

$$
\begin{align*}
&\begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix} \rightarrow \begin{pmatrix} c_6 \\ c_3 \\ c_4 \end{pmatrix} \rightarrow \begin{pmatrix} c_5 \\ c_6 \\ c_3 \end{pmatrix} \rightarrow \begin{pmatrix} c_4 \\ c_5 \\ c_6 \end{pmatrix} \rightarrow \\
&\begin{pmatrix} \tau \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ t_1 \\ t_2 \end{pmatrix} \rightarrow \begin{pmatrix} \tau \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} e_1 \\ s_2 \\ s_1 \end{pmatrix} \rightarrow \begin{pmatrix} \tau \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ s_2 \\ s_1 \end{pmatrix} \rightarrow \begin{pmatrix} \tau \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} e_1 \\ s_1 \\ t_1 \end{pmatrix} \rightarrow \\
&c_2 \rightarrow c_2' \rightarrow c_2'' \rightarrow c_2'''
\end{align*}
$$

(39)

One can count the number of edges and faces of the dual snub 24-cell. The number of faces is 432 made of 144 kites and 288 isosceles triangles while the number of edges is 480.
7 Conclusion

The beauty of \textit{snub 24-cell} lies in the fact that it contains both types of cells, tetrahedral cells as well as icosahedral cells. While the first one is relevant to the crystallography, the second one describes the quasi crystallography. Such structures are rare. To obtain the \textit{snub-24 cell} from the root system of $E_8$ we followed a chain of symmetry $W(D_4) : C_3 \subset W(H_4) \subset W(E_8)$, each being a maximal subgroup in the larger group. This is a new approach which has not been discussed in the literature although it has been known that the removal of a 24-cell from 600-cell leads to the \textit{snub 24- cell}. Another novel approach in our discussion is that we use quaternions to describe the vertices as well as the group elements acting on the vertices. One does not need any computer calculation to obtain the vertices and the symmetries of the polytope of concern and its dual except in plotting of three dimensional cells. To the best of our knowledge the structure of the \textit{dual snub 24-cell} has not been discussed elsewhere. The decomposition of the Platonic as well as the Archimedean $H_4$ orbits under the symmetry group $W(D_4) : C_3$ has been given in the Appendix.

\textbf{Appendix: Decomposition of the Platonic and Archimedean orbits of $W(H_4)$ under the group $W(D_4) : C_3$}

Orbit 1:
$600 = 192 + 96 + 288 + 24$.

Orbit 2:
$1200 = 144 + 576 + 288 + 2(96)$.

Orbit 3:
$720 = 2(288) + 144$.

Orbit 4:
$120 = 24 + 96$.

Orbit 5:
$3600 = 5(576) + 4(288) + 144$.

Orbit 6:
$2400 = 2(576) + 2(96) + 3(288) + 192$.

Orbit 7:
$3600 = 144 + 4(576) + 4(288)$.

Orbit 8:
$1440 = 5(288)$. 

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Orbit 9:
2400 = 3(288)+ 2(576)+ 2(96)+ 192.

Orbit 10:
3600 = 4(288)+ 4(576)+ 144.

Orbit 11:
7200 = 10(576)+ 5(288).

Orbit 12:
7200 = 5(288)+ 10(576).

Orbit 13:
7200 = 5(288)+ 10(576).

Orbit 14:
7200 = 10(576)+ 5(288).

Orbit 15:
14400 = 25(576).

References

[1] V. Elsevier, N. J. A. Sloane, J. Phys. Math. Gen. A20 (1987) 6161.

[2] J. F. Sadoc, R. Mosseri, J. Non-cryst. Solids 153 (1993) 243.

[3] M. Koca, R. Koç, M. Al-Barwani, J. Phys. Math. Gen. A34 (2001) 11201.

[4] M. Koca, J. Phys. Math. Gen. A22 (1989) 1949; J. Phys. Math. Gen. A22 (1989) 4125.

[5] M.B. Green, J. Schwarz, E. Witten, Superstring Theory, in two volumes, Cambridge: Cambridge University Press, 1987; J. Polchinski, String Theory, in two volumes, Cambridge: Cambridge University Press, 1998; M. Kaku, Introduction to Superstrings and M-Theory, Springer-Verlag, 1999.

[6] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438.

[7] T. Pengpan and P. Ramond, Phys. Rep. C315 (1999) 137.

[8] M. Koca, R. Koç, M. Al-Barwani, J. M. Phys 47 (2006) 043507-1.

[9] H. S. M. Coxeter, Regular Complex Polytopes, Cambridge University Press, 1973; Regular Polytopes, Dover Publications Inc., 1973, New York, pp. 151–153.
[10] P. du Val, Homographies, Quaternions and Rotations, Oxford University Press, 1964.

[11] M. Koca, M. Al-Ajmi and R. Koç, J. Phys. A: Math. Theor. 40 (2007) 7633.

[12] M. Koca, M. Al-Ajmi and R. Koç, African Phys. Rev. (2008) 2:0009; (2008) 2:0010.

[13] T. Gosset, On the Regular and Semi-Regular Figures in Space of n Dimensions, Messenger Mathematics, Macmillan, 1900

[14] M. Koca and N. Ozdes, J. Phys. A: Math. Gen. A22 (1989) 1469.

[15] M. Koca, R. Koç and M. Al-Ajmi, J. Math. Phys. 48 (2007) 113514.

[16] R. Slansky, Phys. Rep. 79 (1981) 1

[17] J. H. Conway and N. J. A. Sloane, Sphere Packing, Lattices and Groups, Springer-Verlag, NY, 1998.

[18] M. Koca, R. Koç, M. Al-Barwani, S. Al-Farsi, Linear Algebra. Appl. 412 (2006) 441.

[19] J. H. Conway and D. A. Smith, On Quaternions and Octonions: their Geometry, Arithmetic, and Symmetry, A K Peters, Ltd. (2003).

[20] http://en.wikipedia.org/wiki/snub-24-cell

[21] N. Johnson, Canadian Journal of Mathematics, 18 (1966) 169.

[22] M. Koca and R. Koç, to be published.