COMPUTING THE GROUP OF MINIMAL NON-DEGENERATE EXTENSIONS OF A SUPER-TANNAKIAN CATEGORY

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Abstract. We prove an analog of the Künneth formula for the groups of minimal non-degenerate extensions of symmetric fusion categories. We describe in detail the structure of the group of minimal extensions of a pointed super-Tannakian fusion category. This description resembles that of the third cohomology group of a finite abelian group. We explicitly compute this group in several concrete examples.

1. Introduction

There is a notion of categorical “orthogonality” in a braided fusion category \( \mathcal{C} \). Namely, objects \( X, Y \) of \( \mathcal{C} \) centralize each other if the squared braiding between them is identity, i.e., \( c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y} \), where \( c \) denotes the braiding of \( \mathcal{C} \). For a fusion subcategory \( \mathcal{B} \subset \mathcal{C} \), its centralizer in \( \mathcal{B} \) is the fusion subcategory \( \mathcal{B}' \subset \mathcal{C} \) consisting of all objects \( X \) centralizing every object \( Y \) in \( \mathcal{B} \). When \( \mathcal{C} \) is pointed, i.e., corresponds to a pre-metric group \( (A, q) \), where \( q \) is a quadratic form on a finite Abelian group \( A \), fusion subcategories of \( \mathcal{B} \) are in bijection with subgroups of \( A \) and the centralizers are identified with orthogonal complements. This construction allows interpreting many aspects of the theory of braided fusion categories in terms of “categorical linear algebra”.

For example, a symmetric fusion subcategory \( \mathcal{E} \subset \mathcal{C} \) satisfies \( \mathcal{E} \subseteq \mathcal{E}' \) and so can be thought of as a categorical analog of a coisotropic subspace. When \( \mathcal{C} \) is non-degenerate, one has \( \mathcal{E} = \mathcal{E}' \), i.e., \( \mathcal{E} \) is a Lagrangian subcategory, if and only if \( \text{FPdim}(\mathcal{E})^2 = \text{FPdim}(\mathcal{C}) \), where \( \text{FPdim} \) denotes the Frobenius-Perron dimension. An embedding \( \mathcal{E} \hookrightarrow \mathcal{C} \) with this property will be called a minimal non-degenerate extension (or simply a minimal extension) of \( \mathcal{E} \). Lan, Kong, and Wen observed in \([26]\) that there is a natural product of minimal extensions of \( \mathcal{E} \), so that the set of their equivalence classes is an abelian group \( Mext(\mathcal{E}) \) (in fact, minimal extensions of \( \mathcal{E} \) form a symmetric 2-categorical group \( Mext(\mathcal{E}) \)).

Groups of minimal extensions were studied by many authors, including \([1, 2, 3, 11, 18, 21, 26, 27, 30]\). From the physics point of view, minimal extensions appear in the description of 2+1D topological orders and symmetry-protected trivial (SPT) orders \([26]\), and symmetric invertible fermionic phases \([1, 2]\). It is known that for a Tannakian category \( \mathcal{E} = \text{Rep}(G) \), where \( G \) is a finite group, the group \( Mext(\mathcal{E}) \) is isomorphic to \( H^3(G, \mathbb{K}^\times) \),
the third cohomology group of $G$. For $\mathcal{E} = s\mathbf{Vect}$, the category of super-vector spaces, it is isomorphic to $\mathbb{Z}/16\mathbb{Z}$ (this statement is known in physics as Kitaev’s 16-fold way \cite{25}).

The goal of this paper is to describe the group of minimal extensions of a super-Tannakian fusion category $\mathcal{E}$ (with an emphasis on the case when $\mathcal{E}$ is pointed) and compute it in several concrete examples.

The main results of the present paper are the following.

In Section 3, we prove a version of the Künneth formula for $\text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E})$, where $G$ is a finite group and $\mathcal{E}$ is a symmetric fusion category. Theorem 3.8 establishes a group isomorphism

\begin{equation}
\text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E}) \cong \text{Mext}(\mathcal{E}) \times 2\text{-Fun}(G, \text{Pic}(\mathcal{E})),
\end{equation}

where $2\text{-Fun}(G, \text{Pic}(\mathcal{E}))$ is the group of monoidal 2-functors from $G$ to the 2-categorical Picard group of $\mathcal{E}$. For $\mathcal{E} = \text{Rep}(L)$ this recovers the familiar Künneth formula computing the third cohomology of the product $G \times L$.

In Section 4, we analyze the structure of the group $\text{Mext}(\mathcal{E})$ for a pointed symmetric category $\mathcal{E}$. We consider a filtration

\begin{equation}
\text{Mext}_{\text{triv}}(\mathcal{E}) \subset \text{Mext}_{\text{pt}}(\mathcal{E}) \subset \text{Mext}_{\text{int}}(\mathcal{E}) \subset \text{Mext}(\mathcal{E}),
\end{equation}

consisting, respectively, of the subgroups of trivial, pointed, and integral minimal extensions of $\mathcal{E}$, and compute its composition factors in Theorem 4.18. This description of $\text{Mext}(\mathcal{E})$ generalizes that of the third cohomology group $H^3(A, \mathbb{Z}^\times)$ of a finite abelian group $A$ \cite{8, 28}. A new feature is the appearance of cohomological obstructions from the theory of graded extensions \cite{6, 17}.

Finally, in Section 5, we apply our results to compute the group of minimal extensions of concrete examples of super-Tannakian categories, namely, $\text{Rep}(\mathbb{Z}_2^n)$ and $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$.

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2. Preliminaries

In this paper, we work over an algebraically closed field $\mathbb{k}$ of characteristic 0. We adapt the following font convention for higher categorical groups: we use italics ($G$) to denote ordinary groups, calligraphic ($\mathcal{G}$) for categorical groups, and boldface $\mathbf{G}$ for 2-categorical groups. We use the same name in different fonts to denote the truncations of a given 2-categorical group. For example, we write $\text{Pic}(\mathcal{E}) = \pi_0(\text{Pic}(\mathcal{E}))$ and $\text{Pic}(\mathcal{E}) = \pi_{\leq 1}(\text{Pic}(\mathcal{E}))$ for the truncations of the 2-categorical Picard group $\text{Pic}(\mathcal{E})$. 
2.1. Symmetric fusion categories and their Picard groups. We refer the reader to [13, 16] for the basics of the theory of braided fusion categories.

By Deligne’s theorem [10], symmetric fusion categories are parameterized by pairs $(G, t)$, where $G$ is a finite group and $t \in Z(G)$ is a central element such that $t^2 = 1$. The corresponding category $\text{Rep}(G, t)$ consists of finite-dimensional representations of $G$, with the usual tensor product and braiding given by

$$c_{V,W}(v \otimes w) = \begin{cases} -w \otimes v, & \text{if } t|_V = -1 \text{ and } t|_W = -1, \\ w \otimes v, & \text{otherwise,} \end{cases}$$

for all irreducible representations $V, W$ of $G$, where $v \in V, w \in W$. This category is called Tannakian if $t = 1$ (so it is simply $\text{Rep}(G)$) and super-Tannakian if $t \neq 1$. When $G$ contains a unique automorphism orbit of central elements of order 2 we will use notation $\text{Rep}(G_f)$ for $\text{Rep}(G, t)$ which is common in physics. Here $f$ stands for “fermionic”. For example, $s\text{Vect} = \text{Rep}(\mathbb{Z}_2^f)$.

Let $\mathcal{B}$ be a braided fusion category. The 2-categorical Picard group $\text{Pic}(\mathcal{B})$ [17] is formed by invertible $\mathcal{B}$-module categories with the tensor product $\boxtimes$. Its 1-categorical truncation $\text{Pic}(\mathcal{B}) := \pi_{\leq 1}(\text{Pic}(\mathcal{B}))$ is equivalent to the categorical group $\text{Aut}(Z(\mathcal{B}); \mathcal{B})$ of braided tensor autoequivalences of $Z(\mathcal{B})$ trivializable on $\mathcal{B}$ [5].

The Picard group of a symmetric fusion category was determined by Carnovale in [4]:

$$\text{Pic}(\text{Rep}(G, t)) = \begin{cases} H^2(G, \mathbb{K}^\times) & \text{if } t = 1, \\ H^2(G, t, \mathbb{K}^\times) & \text{if } t \neq 1 \text{ and } \langle t \rangle \text{ is not a direct summand of } G, \\ H^2(G, t, \mathbb{K}^\times) \times \mathbb{Z}_2 & \text{if } t \neq 1 \text{ and } \langle t \rangle \text{ is a direct summand of } G. \end{cases}$$

Here the elements of $H^2(G, \mathbb{K}^\times)$ correspond to module categories $\text{Rep}(k_\mu[G])$ of projective representations of $G$ with a fixed 2-cocycle $\mu \in Z^2(G, \mathbb{K}^\times)$ and the generator of $\mathbb{Z}_2$ is $\text{Rep}(G_0)$, where $G = G_0 \times \langle t \rangle$.

The group $H^2(G, t, \mathbb{K}^\times)$ is defined as follows (see [4] and also [6]). There is a canonical bilinear map

$$H^2(G, \mathbb{K}^\times) \times G \to \mathbb{Z}/2\mathbb{Z} : (\mu, x) \mapsto \xi_\mu(x),$$

where $\xi_\mu(x)$ is defined by the condition

$$(-1)^{\xi_\mu(x)} = \frac{\mu(x, t)}{\mu(t, x)}, \quad \mu \in H^2(G, \mathbb{K}^\times), x \in G.$$

Introduce a new multiplication on $H^2(G, \mathbb{K}^\times)$ by

$$\mu \ast \nu(x, y) = \mu(x, y)\nu(x, y) (-1)^{\xi_\mu(x)\xi_\nu(y)}, \quad x, y \in G,$$
on representatives $\mu, \nu$ of cohomology classes in $H^2(G, k^\times)$. The resulting group will be denoted $H^2(G, t, k^\times)$. It is non-canonically isomorphic to $H^2(G, k^\times)$, see [4] for details.

There is a subgroup $\text{Pic}(\text{Rep}(G, t))_{\text{int}} \subset \text{Pic}(\text{Rep}(G, t))$ consisting of integral module categories, i.e., those in which all objects have integral Frobenius-Perron dimension. We have

$$\text{Pic}(\text{Rep}(G, t))_{\text{int}} = H^2(G, t, k^\times).$$

The braided 2-categorical Picard group $\text{Pic}_{\text{br}}(\mathcal{E})$ [6] of a symmetric fusion category $\mathcal{E}$ consists of invertible objects in the 2-center of a monoidal 2-category of $\mathcal{E}$-module categories. The underlying braided categorical group $\text{Pic}_{\text{br}}(\mathcal{E})$ was described in [6, Section 6]. One has $\text{Pic}_{\text{br}}(\mathcal{E}) \cong \text{Pic}(\mathcal{E}) \times \text{Aut}_{\otimes}(\text{id}_{\mathcal{E}})$ and the corresponding quadratic form

$$Q_{\mathcal{E}} : \text{Pic}_{\text{br}}(\mathcal{E}) \to \text{Inv}(\mathcal{E})$$

is explicitly computed in [6, Proposition 6.11]. In particular, for $\mathcal{E} = \text{Rep}(G, t)$ the integral part of $\text{Pic}_{\text{br}}(\mathcal{E})$ is $H^2(G, t, k^\times) \times Z(G)$, where $Z(G)$ denotes the center of $G$. In this case, $\text{Aut}_{\otimes}(\text{id}_{\mathcal{E}}) = \text{Hom}(G, k^\times)$ and the restriction of the quadratic form (8) on the integral part of $\text{Pic}_{\text{br}}(\mathcal{E})$ is given by

$$Q_{\mathcal{E}}(\mu, z) = \frac{\mu(z t_{\mu}^Z(z)+1, -)}{\mu(-, z t_{\mu}^Z(z)+1)}$$

for all $\mu \in H^2(G, t, k^\times)$ and $z \in Z(G)$.

2.2. The 2-categorical group of minimal extensions of a symmetric category.

We recall the definition given in [26]. Let $\mathcal{E}$ be a symmetric fusion category. An embedding $\mathcal{E} \hookrightarrow \mathcal{C}$ into a non-degenerate braided fusion category $\mathcal{C}$ is called a minimal non-degenerate extension (or, simply, a minimal extension) of $\mathcal{E}$ if the latter coincides with its centralizer in $\mathcal{C}$, i.e., $\mathcal{E} = \mathcal{E}'$, where

$$\mathcal{E}' = \{ X \in \mathcal{C} \mid c_{YX} \circ c_{XY} = \text{id}_{X \otimes Y} \text{ for all } Y \in \mathcal{E} \}.$$ 

This condition is equivalent to the equality $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{E})^2$, where $\text{FPdim}$ denotes the Frobenius-Perron dimension of the category. Minimal extensions of $\mathcal{E}$ form a 2-groupoid $\text{Mext}(\mathcal{E})$. An isomorphism between minimal extensions $\mathcal{E} \hookrightarrow \mathcal{C}_1$ and $\mathcal{E} \hookrightarrow \mathcal{C}_2$ is a braided equivalence $\mathcal{C}_1 \simeq \mathcal{C}_2$ that restricts to the identity on $\mathcal{E}$. A 2-isomorphism is a natural isomorphism of equivalences, again identical on $\mathcal{E}$.

There is a natural tensor product of minimal extensions of $\mathcal{E}$. Namely, let $\mathcal{E} \hookrightarrow \mathcal{C}_1$ and $\mathcal{E} \hookrightarrow \mathcal{C}_2$ be minimal extensions. Then $\mathcal{E} \boxtimes \mathcal{E}$ embeds into $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Since $\mathcal{E}$ is symmetric, the tensor product $\boxtimes : \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$ is a braided tensor functor. Its adjoint sends the unit object $1$ to an étale (i.e., separable commutative) algebra $A \in \mathcal{E} \boxtimes \mathcal{E}$. The fusion category $(\mathcal{C}_1 \boxtimes \mathcal{C}_2)_A$ of $A$-modules in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ contains a braided fusion subcategory

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2 := (\mathcal{C}_1 \boxtimes \mathcal{C}_2)_A^0$$
of local modules, i.e., of $A$-modules $(V, \rho : A \otimes V \to V)$ with $c_{X,A}c_{A,X}\rho = \rho$. Note that $E \cong (E \boxtimes E)_A$ is embedded into $C_1 \boxplus C_2$. The resulting embedding $E \hookrightarrow C_1 \boxplus C_2$ is, by definition, the tensor product of $E \hookrightarrow C_1$ and $E \hookrightarrow C_2$. The unit object for this tensor product is $E \hookrightarrow \mathcal{Z}(E)$, the embedding of $E$ into its Drinfeld center. The inverse of the embedding $E \hookrightarrow C$ is $E \hookrightarrow C^{\text{rev}}$, where $C^{\text{rev}}$ coincides with $C$ as a fusion category with a braiding obtained by reversing the braiding of $C$, namely $c^{\text{rev}}_{X,Y} = c^{-1}_{Y,X}, X, Y \in C$. We refer the reader to [26, Section 4.2] for details.

As agreed above, we denote $M_{\text{ext}}(E) = \pi_0(M_{\text{ext}}(E))$ and $M_{\text{ext}}(E) = \pi_{\leq 1}(M_{\text{ext}}(E))$, the truncations of $M_{\text{ext}}(E)$.

**Example 2.1.** Let $G$ be a finite group. The group $M_{\text{ext}}(\text{Rep}(G))$ was computed in [26, Section 4.3]. Namely, a typical element of this group is a twisted Drinfeld double of $G$: \[ \text{Rep}(G) \hookrightarrow \mathcal{Z}(\text{Vec}_G^\omega), \quad \omega \in Z^3(G, \mathbb{k}^\times). \]

The product of these extensions corresponds to the product of 3-cocycles. Furthermore, extensions corresponding to 3-cocycles $\omega_1, \omega_2$ are isomorphic if and only if $\omega_1, \omega_2$ are cohomologous. Thus,

$$M_{\text{ext}}(\text{Rep}(G)) \cong H^3(G, \mathbb{k}^\times).$$

This result can also be deduced from [12, Section 4.4.10] since for any minimal extension $\text{Rep}(G) \hookrightarrow C$, the image of $\text{Rep}(G)$ is a Lagrangian subcategory of $C$.

**Example 2.2.** It was shown independently in [3], [7, Proposition 5.14], [25], and [26, Theorem 4.25] that

$$M_{\text{ext}}(s\text{Vect}) \cong \mathbb{Z}_{16}.$$  

This statement is known as Kitaev’s 16-fold way [25]. Any Ising category, i.e., a non-pointed braided fusion category of dimension 4 [13, Appendix B], is a generator of this group. Other elements of $M_{\text{ext}}(s\text{Vect})$ are pointed braided fusion categories coming from metric groups $(A, q)$ of order 4 such that there exists $u \in A$ with $q(u) = -1$.

The isomorphism (11) can be identified with

$$M_{\text{ext}}(s\text{Vect}) \xrightarrow{\tilde{\sim}} \{ \xi \in \mathbb{k}^\times \mid \xi^{16} = 1 \} : C \mapsto \xi(C),$$

where $\xi(C)$ is the central charge of the category $C$. Thus, the class of a minimal non-degenerate extension of $s\text{Vect}$ is completely determined by its central charge.

Let $\text{Rep}(G, t)$ be a super-Tannakian category. It contains a unique maximal Tannakian subcategory $\mathcal{T} = \text{Rep}(G/(t))$. If $\text{Rep}(G, t) \hookrightarrow C$ is a minimal extension then the de-equivariantization $C^0 := \mathcal{T}' \boxtimes_{\mathcal{T}} \text{Vec}$ is a minimal extension of $s\text{Vect}$. The assignment

$$w_{(G,t)} : M_{\text{ext}}(\text{Rep}(G, t)) \to M_{\text{ext}}(s\text{Vect}) : C \mapsto C^0$$
is a group homomorphism \cite[Section 5.2]{26}. As in \cite{12}, this homomorphism is identified with taking the central charge.

By \cite[Corollary 4.9]{18}, $w_{(G,t)}$ is surjective if and only if $(t)$ is a direct summand of $G$. In this case, this homomorphism splits, i.e., $\text{Mext}(\text{Vect})$ is a direct summand of $\text{Mext}(\text{Rep}(G, t))$.

2.3. Central and Braided Graded Extensions. Let $B$ be a braided fusion category. Let $G$ be a finite group. A central $G$-graded extension of $B$ is a $G$-graded fusion category

$$C = \bigoplus_g C_g, \quad C_e = B,$$

along with an embedding $B \hookrightarrow Z(C)$. It was shown in \cite{22} that a central $G$-graded extension is the same thing as a $G$-crossed braided extension. By \cite{6, 17} the 2-groupoid of such extensions is equivalent to the 2-groupoid of monoidal 2-functors $G \rightarrow \text{Pic}(B)$. For such an extension there is canonical action of $G$ on the trivial component given by the composition

$$G \rightarrow \text{Pic}(B) \rightarrow \text{Aut}^\text{br}(B),$$

where the first functor corresponds to the graded extension \cite{14} and the second one is the canonical monoidal functor associated to $B$ \cite{5}.

Now let $A$ be a finite Abelian group. Braided $A$-graded extensions of a braided fusion category $B$ were classified in \cite{6}. The 2-groupoid of such extensions is equivalent to the 2-groupoid of braided monoidal 2-functors $A \rightarrow \text{Pic}_{\text{br}}(B)$, where the latter is the braided 2-categorical group of invertible braided $B$-module categories.

3. Central Graded Extensions and the Künneth Formula

3.1. The Group of Monoidal 2-Functors to a Braided 2-Categorical Group. Let $G$ be a group and let $\mathcal{G}$ be a braided categorical group. Let $C, C' : G \rightarrow \mathcal{G}$ be monoidal functors, where $C$ is given by $x \mapsto C_x$ with the monoidal structure $M_{x,y} : C_x \otimes C_y \xrightarrow{\sim} C_{xy}$ and $C'$ is given by $x \mapsto C'_x$ with the monoidal structure $M'_{x,y} : C'_x \otimes C'_y \xrightarrow{\sim} C'_{xy}$, $x, y \in G$. Clearly, such functors must factor through the commutator subgroup of $G$.

Define a monoidal functor

$$\tilde{C} := C \otimes C' : G \rightarrow \mathcal{G} : \quad x \mapsto C_x \otimes C'_x,$$

with the monoidal structure

$$\tilde{M}_{x,y} : C_x \otimes C'_x \otimes C_y \otimes C'_y \xrightarrow{B_{x',y}} C_x \otimes C_y \otimes C'_x \otimes C'_y \xrightarrow{M_{x,y} \otimes M'_{x,y}} C_{xy} \otimes C'_{xy}, \quad x, y \in G.$$
$C : x \mapsto C_x$ is $C^{-1} : x \mapsto C_x^{-1}$. If $\mathcal{G}$ is symmetric, the underlying group $\text{Fun}(G, \mathcal{G})$ is Abelian.

There is an obvious short exact sequence

$0 \to H^2(G, \pi_1(\mathcal{G})) \to \text{Fun}(G, \mathcal{G}) \to \text{Hom}(G, \pi_0(\mathcal{G})) \to 0.$

Here $H^2(G, \pi_1(\mathcal{G}))$ is isomorphic to the group of monoidal functor structures on the trivial functor. This sequence does not split in general because the braiding of $\mathcal{G}$ may be non-trivial.

Now let $G$ be a braided 2-categorical group. Let $\pi_0(G), \pi_1(G), \pi_2(G)$ denote, respectively, the group of invertible objects of $G$, the group of automorphisms of the unit object $1_G$, and the group of automorphisms of $\text{id}_{1_G}$. Let $\pi_{\leq 1}(G)$ denote the braided categorical group obtained by truncating $G$, whose objects are objects of $G$ and morphisms are isomorphism classes of 1-cells in $G$. Let $\pi_{\geq 1}(G)$ denote the symmetric categorical group of 1-automorphisms of the unit object of $G$.

There is a 2-categorical analog of the above construction of a categorical group of monoidal functors. It was explained in [6, Section 2.8] that isomorphisms classes of monoidal 2-functors from $G$ to $G$ also form a categorical group denoted $2\text{-Fun}(G, G)$. Namely, if $C, C' : G \to G$ are such 2-functors, then the monoidal structure of the product $C \otimes C'$ is defined as above and the structural associativity 2-cells are given by [6, diagram (2.78)]. These cells involve the associativity 2-cells of $C$ and $C'$ and the structure 2-cells of $G$.

The group $2\text{-Fun}(G, G)$ fits into the following exact sequence [6, Theorem 2.38]:

$H^1(G, \pi_1(G)) \xrightarrow{\alpha} H^3(G, \pi_2(G)) \xrightarrow{\beta} 2\text{-Fun}(G, G) \xrightarrow{\gamma} \text{Fun}(G, \pi_{\leq 1}(G)) \xrightarrow{\delta} H^4(G, \pi_2(G)).$

Here $\alpha$ assigns to a homomorphism $G \to \pi_1(G)$ the corresponding pullback of the associator of the categorical group $\pi_{\geq 1}(G)$ (the latter is an element of $H^3(\pi_1(G), \pi_2(G))$), $\beta$ assigns to a third cohomology class the monoidal 2-functor structure on the trivial 2-functor, $\gamma$ assigns to a monoidal 2-functor $G \to G$ the underlying 1-functor to the truncation of $G$, and $\delta$ gives the obstruction for a given monoidal functor to extend to a monoidal 2-functor.

**Example 3.1.** Let $\mathcal{E}$ be a symmetric fusion category and let $G = \text{Pic}(\mathcal{E})$ be the symmetric 2-categorical Picard group of $\mathcal{E}$. We have $\pi_{\leq 1}(\text{Pic}(\mathcal{E})) = \text{Pic}(\mathcal{E})$, the symmetric categorical Picard group of $\mathcal{E}$, and $\pi_{\geq 1}(\text{Pic}(\mathcal{E})) = \text{Inv}(\mathcal{E})$, the symmetric categorical group of invertible objects of $\mathcal{E}$. Since the associator of the latter is trivial, the exact sequence [19] becomes

$0 \xrightarrow{\alpha} H^3(G, \mathbb{K}^\times) \xrightarrow{\beta} 2\text{-Fun}(G, \text{Pic}(\mathcal{E})) \xrightarrow{\gamma} \text{Fun}(G, \text{Pic}(\mathcal{E})) \xrightarrow{\delta} H^4(G, \mathbb{K}^\times).$
Example 3.2. The restriction of the obstruction map $\delta$ from Example 3.1 to the subgroup $H^2(G, \text{Inv}(\mathcal{E}))$ of $\text{Fun}(G, \text{Pic}(\mathcal{E}))$ is given by the composition

\[
\delta(L)(x, y, z, w) = c_{L(x,y),L(z,w)}, \quad x, y, z, w \in G,
\]

where $c$ is the braiding in $\text{Inv}(\mathcal{E})$. Since $c_{X,Y} = (-1)^{q(X)q(Y)}$ for all $X, Y \in \text{Inv}(\mathcal{E})$, the formula (22) translates to (21).

For $\mathcal{E} = \mathbf{sVect}$ formula (21) describing the obstruction map

\[
H^2(G, \text{Inv}(\mathcal{E})) = H^2(G, \mathbb{Z}/2\mathbb{Z}) \to H^1(G, \mathbb{Z}/2\mathbb{Z})
\]

in terms of the cup product appeared in [1, Section VI.B].

3.2. The embedding $\text{Pic}(\mathcal{E}) \hookrightarrow \text{Pic}(\mathcal{Z}(\mathcal{E}))$. Let $\mathcal{E}$ be a symmetric fusion category. The induction 2-functor

\[
\text{Ind} : \text{Pic}(\mathcal{E}) \to \text{Pic}(\mathcal{Z}(\mathcal{E})) : \mathcal{M} \mapsto \mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{M}
\]

is a monoidal 2-embedding of categorical groups. On the level of 1-cells this functor embeds $\text{Inv}(\mathcal{E})$ into $\text{Inv}(\mathcal{Z}(\mathcal{E}))$.

There is an equivalence of categorical groups [6, 17]

\[
\partial : \pi_{\leq 1}(\text{Pic}(\mathcal{Z}(\mathcal{E}))) \to \text{Aut}^{br}(\mathcal{Z}(\mathcal{E})) : \mathcal{M} \mapsto \partial(\mathcal{M}),
\]

such that $\alpha^+_{\mathcal{M}} \cong \alpha^-_{\mathcal{M}} \circ \partial(\mathcal{M})$, where $\alpha^\pm_{\mathcal{M}} : \mathcal{Z}(\mathcal{E}) \cong \mathcal{Z}(\mathcal{E}) \to \mathcal{Fun}_{\mathcal{Z}(\mathcal{E})}(\mathcal{M}, \mathcal{M})$ are two equivalences defined, respectively, using the braiding of $\mathcal{Z}(\mathcal{E})$ and its reverse. This equivalence $\partial$ sends $Z \in \text{Inv}(\mathcal{Z}(\mathcal{E}))$ to $\delta(Z) \in \text{id}_{\mathcal{Z}(\mathcal{E})}$ defined by $\delta(Z)_X = c_{Z,X}c_{X,Z}$ for all $X \in \mathcal{Z}(\mathcal{E})$.

It was shown in [6] that under the equivalence (24), in the image of $\text{Ind}$ the objects are autoequivalences $\alpha \in \text{Aut}^{br}(\mathcal{Z}(\mathcal{E}))$ such that $\alpha|_{\mathcal{E}} \cong \text{id}_{\mathcal{E}}$ and automorphisms of the unit object are $\nu \in \text{id}_{\mathcal{Z}(\mathcal{E})}$ such that $\nu_X = 1$ for all $X \in \mathcal{E}$.

For a monoidal 2-functor

\[
M : G \to \text{Pic}(\mathcal{Z}(\mathcal{E})) : g \mapsto \mathcal{M}(g)
\]

we denote $\partial(g) = \partial(\mathcal{M}(g)) \in \text{Aut}^{br}(\mathcal{Z}(\mathcal{E}))$ and $\partial_{g,h} : \partial(g) \circ \partial(h) \to \partial(gh)$, $g, h \in G$, the monoidal structure on the composition functor $G \to \text{Aut}^{br}(\mathcal{Z}(\mathcal{E}))$. Note that $\partial_{g,h}$ is canonically identified with an element of $\text{Aut}(\text{id}_{\mathcal{Z}(\mathcal{E})}) = \text{Aut}(\partial(gh))$. 

Lemma 3.3. Suppose that the monoidal 2-functor \((25)\) is such that \(\partial(g)|_E \cong id_E\) and \((\partial_{g,h})|_X = id_X\) for all \(X \in E\). Then \((25)\) factors through the embedding \((23)\):

\[
G \to \text{Pic}(E) \xrightarrow{\text{Ind}} \text{Pic}(\mathcal{Z}(E)).
\]

Proof. This follows from the above description of the image of \text{Ind}. \(\square\)

3.3. The group of central extensions of a symmetric fusion category. Recall that for an étale algebra \(A\) in \(\mathcal{Z}(C)\), where \(C\) is a fusion category, the category \(C_A\) of \(A\)-modules in \(C\) is a fusion category. If \(A\) is an étale algebra in \(\mathcal{Z}_{sym}(B)\), where \(B\) is a braided fusion category, then \(B_A\) is a braided fusion category. The tensor product over a symmetric fusion category \(E\) is a special case of this construction. Indeed, the tensor product \(\otimes : E \otimes E \to E\) is a braided tensor functor. Let \(I : E \to E \boxtimes E\) be its adjoint, then \(A := I(1)\) is a canonical étale algebra in \(E \boxtimes E\). If \(E \hookrightarrow C_1, E \hookrightarrow C_2\) are central inclusions of \(E\) into fusion categories \(C_1, C_2\) then \(C_1 \boxtimes_E C_2 = (C_1 \boxtimes C_2)_A\) is a fusion category. If \(E \hookrightarrow B_1, E \hookrightarrow B_2\) are inclusions into symmetric centers of braided fusion categories \(B_1, B_2\) then \(B_1 \boxtimes_E B_2 = (B_1 \boxtimes B_2)_A\) is a braided fusion category.

Lemma 3.4. Let \(B \subset C\) be a central extension of a braided fusion category \(B\) and let \(A\) be an étale algebra in \(\mathcal{Z}_{sym}(B)\). Then \(B_A \subset C_A\) is a central extension.

Proof. This is straightforward since the half-braiding between objects of \(B\) and \(C\) induces the one between objects of \(B_A\) and \(C_A\). \(\square\)

Let \(\text{Pic}(E)\) denote the braided 2-categorical Picard group of \(E\) and let \(\textbf{2-Fun}(G, \text{Pic}(E))\) denote the 2-categorical group of monoidal functors from \(G\) to \(\text{Pic}(E)\).

It was observed in [6] that the 2-groupoid \(\text{Ex}_{ctr}(G, E)\) of central \(G\)-graded extensions of \(E\) is a braided 2-categorical group. The tensor product of \(G\)-graded central extensions \(E \hookrightarrow C^1\) and \(E \hookrightarrow C^2\) is defined as follows. Let \(A\) be the canonical étale algebra in \(E \boxtimes E\) defined above. Lemma 3.4 applies to the central \(G\)-graded extension \(E \boxtimes E \hookrightarrow \bigoplus_{g \in G} C^1_g \boxtimes C^2_g\), so we obtain a central \(G\)-graded extension

\[
(26) \quad E \cong (E \boxtimes E)_A \hookrightarrow \bigoplus_{g \in G} (C^1_g \boxtimes C^2_g)_A = \bigoplus_{g \in G} C^1_g \boxtimes_E C^2_g
\]

which is the product of extensions \(E \hookrightarrow C^1\) and \(E \hookrightarrow C^2\). We will denote this product by \(E \hookrightarrow C_1 \boxtimes C_2\).

Theorem 3.5. There is a monoidal 2-equivalence of 2-categorical groups

\[
(27) \quad \textbf{2-Fun}(G, \text{Pic}(E)) \cong \text{Ex}_{ctr}(G, E).
\]

Proof. The 2-equivalence is established for braided \(E\) in [17, Theorem 7.12] (see also [6, Theorem 8.13]). Namely, a monoidal 2-functor \(G \to \text{Pic}(E) : g \mapsto C_g\) gives rise to a
$G$-graded central extension $\bigoplus_{g \in G} C_g$. For symmetric $\mathcal{E}$, the monoidal structure of this 2-equivalence is evident since the tensor products in $\textbf{2-Fun}(G, \text{Pic}(\mathcal{E}))$ and $\text{Ex}_{\text{ctr}}(G, \mathcal{E})$ are defined by the very same formulas (cf. (16) and (26)) and have the same associativity 2-cells.

3.4. The center of a central extension of a symmetric fusion category. Let $\mathcal{E}$ be a symmetric fusion category and let

$$C = \bigoplus_{g \in G} C_g, \quad C_e = \mathcal{E},$$

be its central $G$-graded extension. It was shown in [19] that $Z(C)$ is equivalent to a $G$-equivariantization of the relative center $Z_{\mathcal{E}}(C)$. The latter is equivalent to the fusion category $Z(\mathcal{E}) \boxtimes_{\mathcal{E}} C$. It is a central $G$-graded extension of $Z(\mathcal{E})$ corresponding to the following monoidal 2-functor

$$G \to \text{Pic}(\mathcal{E}) \xrightarrow{\text{Ind}} \text{Pic}(Z(\mathcal{E})), \quad G \to \mathcal{E} \xrightarrow{\text{Ind}} \mathcal{E},$$

where the first functor corresponds to the central extension (28) and the second is the induction (23). The action of $G$ on $Z(\mathcal{E})$ is obtained by composing (29) with the canonical monoidal equivalence $\text{Pic}(Z(\mathcal{E})) \cong \text{Aut}^b(\mathcal{E})$. It follows from [6] (see Section 2.1) that this action restricts to the trivial action on $\mathcal{E} \subset Z(\mathcal{E})$. Therefore, $Z(C) = (Z(\mathcal{E}) \boxtimes_{\mathcal{E}} C)^G$ is a minimal extension of the symmetric fusion category $\mathcal{E}^G = \text{Rep}(G) \boxtimes \mathcal{E}$.

**Proposition 3.6.** The assignment

$$\text{Ex}_{\text{ctr}}(G, \mathcal{E}) \to \text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E}) : C \mapsto Z(C)$$

is a monoidal 2-functor.

**Proof.** For a braided fusion category $\mathcal{B}$ containing a subcategory equivalent to $\text{Rep}(G)$, let $\mathcal{B}_G = \mathcal{B} \boxtimes_{\text{Rep}(G)} \text{Vect}$ denote the corresponding de-equivariantization. Using definitions of the tensor products in $\text{Ex}_{\text{ctr}}(G, \mathcal{E})$ and $\text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E})$ we obtain equivalences

$$Z(C^1 \boxtimes C^2)_G \cong \bigoplus_{g \in G} Z(\mathcal{E}) \boxtimes_{\mathcal{E}} C^1_g \boxtimes_{\mathcal{E}} C^2_g \cong \bigoplus_{g \in G} (Z(\mathcal{E}) \boxtimes (Z(\mathcal{E})))^0_A \boxtimes_{\mathcal{E}} C^1_g \boxtimes_{\mathcal{E}} C^2_g \cong \bigoplus_{g \in G} ((Z(\mathcal{E}) \boxtimes_{\mathcal{E}} C^1_g) \boxtimes (Z(\mathcal{E}) \boxtimes_{\mathcal{E}} C^2_g))^0_A \cong (Z(C^1) \boxtimes Z(C^2))^G$$

for all $G$-graded central extensions $C^1, C^2$ of $\mathcal{E}$. Here $B^0_A$ denotes the category of local $A$-modules in $\mathcal{B}$. The symbol $\boxtimes$ stands for the tensor product in both $\text{Ex}_{\text{ctr}}$ and $\text{Mext}$.
Taking equivariantizations we get a canonical equivalence
\[ Z(C^1 \boxtimes C^2) \cong Z(C^1) \boxtimes Z(C^2) \]
in \text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E}) that equips the 2-functor \text{\textbf{(30)}} with a canonical monoidal structure. \hfill \square

3.5. The Künneth formula. Let \( G, L \) be finite groups.

**Proposition 3.7.** There is a split short exact sequence
\[
0 \rightarrow H^3(G, \mathbb{C}^\infty) \oplus H^3(L, \mathbb{C}^\infty) \oplus \left( \text{Hom}(G, \mathbb{C}^\infty) \otimes \text{Hom}(L, \mathbb{C}^\infty) \right) \rightarrow H^3(G \times L, \mathbb{C}^\infty) \rightarrow \text{Hom}(L, H^2(G, \mathbb{C}^\infty)) \oplus \text{Hom}(G, H^2(L, \mathbb{C}^\infty)) \rightarrow 0.
\]

**Proof.** Using the short exact sequence \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0 \), we obtain isomorphisms
\[ H^i(G, \mathbb{C}^\infty) \cong H^i(G, \mathbb{Q}/\mathbb{Z}) \cong H^{i+1}(G, \mathbb{Z}) \]
for any finite group \( G \) and \( i \geq 1 \). Also, \( H^1(G, \mathbb{Z}) = 0 \). Therefore, the sequence in question is obtained from the usual Künneth formula for integral cohomology. \hfill \square

Below we generalize the Künneth formula \text{\textbf{(31)}}. Namely, for a finite group \( G \) and a symmetric fusion category \( \mathcal{E} \), we explain how to compute the group \text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E})).

**Theorem 3.8.** There is a group isomorphism
\[
\text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E}) \cong \text{Mext}(\mathcal{E}) \times \text{2-Fun}(G, \text{Pic}(\mathcal{E}))
\]

**Proof.** Let \( C \) be a minimal extension of \( \text{Rep}(G) \boxtimes \mathcal{E} \). Let \( \tilde{C} \) denote the centralizer of \( \text{Rep}(G) \) in \( C \). The de-equivariantization \( \tilde{C} \boxtimes_{\text{Rep}(G)} \text{Vect} \) is a minimal extension of \( \mathcal{E} \) and the assignment
\[
\text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E}) \rightarrow \text{Mext}(\mathcal{E}) : C \mapsto \tilde{C} \boxtimes_{\text{Rep}(G)} \text{Vect}
\]
is a monoidal 2-functor between categorical groups.

The associated group homomorphism
\[
\text{Mext}(\text{Rep}(G) \boxtimes \mathcal{E}) \rightarrow \text{Mext}(\mathcal{E})
\]
is split surjective, since for any \( D \in \text{Mext}(\mathcal{E}) \) we have a minimal extension \( \text{Rep}(G) \boxtimes \mathcal{E} \hookrightarrow Z(\text{Rep}(G)) \boxtimes D \).

Let \( K(G, \mathcal{E}) \) denote the kernel of \text{\textbf{(34)}}. It remains to show that \( \text{2-Fun}(G, \text{Pic}(\mathcal{E})) = K(G, \mathcal{E}) \). Theorem 3.5 combined with Proposition 3.6 gives an inclusion
\[ \text{2-Fun}(G, \text{Pic}(\mathcal{E})) \cong \text{Ext}_{\text{ctr}}(G, \mathcal{E}) \rightarrow K(G, \mathcal{E}). \]

Let us show that it is surjective. A minimal extension \( \text{Rep}(G) \boxtimes \mathcal{E} \hookrightarrow \mathcal{C} \) is in \( K(G, \mathcal{E}) \) if and only if its de-equivariantization \( \mathcal{C}_G = \mathcal{C} \boxtimes_{\text{Rep}(G)} \text{Vect} \) is a \( G \)-graded central extension
of $\mathcal{Z}(\mathcal{E})$ such that the action of $G$ restricts trivially to the subcategory $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$. By \cite{5}, this means that the corresponding monoidal 2-functor $G \to \text{Pic}(\mathcal{Z}(\mathcal{E}))$ is the composition of a monoidal 2-functor $F : G \to \text{Pic}(\mathcal{E})$ and the induction 2-functor \cite{23}. As explained in Section 3.4.1 this means that $\mathcal{C} \cong \mathcal{Z}(\mathcal{A}_F)$, where $\mathcal{A}_F$ is the central extension of $\mathcal{E}$ corresponding to $F$. Thus, $\mathcal{C} \in K(G, \mathcal{E})$, as required. \hfill $\square$

Remark 3.9. Recall that a $G$-gauging of a braided fusion category $\mathcal{B}$ is the equivariantization of a faithful $G$-crossed braided (i.e., $G$-graded central) extension of $\mathcal{B}$. Theorem 3.8, in particular, characterizes the centers of central $G$-extensions of a symmetric fusion category $\mathcal{E}$ as $G$-gaugings of $\mathcal{Z}(\mathcal{E})$ in which the associated action of $G$ on $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$ is trivial. For $\mathcal{E} = \text{Vect}$, one recovers from this result the classification of twisted group doubles from \cite[Theorem 4.64]{12}.

Example 3.10. Let $L$ be a finite group and set $\mathcal{E} = \mathcal{R}ep(L)$ in Theorem 3.8. We recover the Künneth formula \cite{31} as follows. In this case, $\text{Pic}(\mathcal{E}) = H^2(L, \mathbb{k}^\times)$ and sequence \cite{18} (with $\mathcal{G} = \text{Pic}(\mathcal{E})$) splits:

\begin{equation}
\text{Fun}(G, \mathcal{P}ic(\mathcal{E})) \cong \text{Hom}(G, H^2(L, \mathbb{k}^\times)) \oplus H^2(G, \widehat{L}),
\end{equation}

where $\widehat{L} = \text{Hom}(L, \mathbb{k}^\times)$. We claim that the obstruction $\delta$ in \cite{19} vanishes. Indeed, it suffices to check that it vanishes on both summands of \cite{35}. Vanishing on the first one follows for the existence of a Schur covering group $L^*$ of $L$ \cite[Section 2.1]{24}, since $\mathcal{R}ep(L^*)$ is a faithful $H^2(L, \mathbb{k}^\times)$-graded extension of $\mathcal{R}ep(L)$. Vanishing on the second one follows from Example 3.2 since for the Tannakian category $\mathcal{R}ep(L)$ the quadratic homomorphism $q : \widehat{L} \to \mathbb{Z}/2\mathbb{Z}$ is trivial.

Therefore, the sequence \cite{20} becomes the following short exact sequence:

\[ 0 \to H^3(G, \mathbb{k}^\times) \to 2\text{-Fun}(G, \mathcal{P}ic(\mathcal{E})) \to \text{Hom}(G, H^2(L, \mathbb{k}^\times)) \oplus H^2(G, \widehat{L}) \to 0. \]

By the universal coefficient theorem, the last summand is further decomposed as

\[ H^2(G, \widehat{L}) \cong \text{Ext}(G/G', \widehat{L}) \oplus \text{Hom}(L, H^2(G, \mathbb{k}^\times)) \cong \left( \hat{G} \otimes \widehat{L} \right) \oplus \text{Hom}(L, H^2(G, \mathbb{k}^\times)), \]

where $G'$ is the commutator subgroup of $G$. Combining this with Theorem 3.8 we recover all the summands of $H^3(G \times L) \cong \text{Mext}(\mathcal{R}ep(G) \boxtimes \mathcal{R}ep(L))$ in the Künneth formula \cite{31}.

Example 3.11. We will deal with minimal extensions of pointed symmetric fusion categories in Section 4. For now, let us note that Theorem 3.8 allows calculating of their groups of minimal extensions inductively as follows. Let $r(\mathcal{E})$ denote the finite rank of the group $\text{Inv}(\mathcal{E})$, i.e., the minimal number of its generators. For $r(\mathcal{E}) = 1$, the group $\text{Mext}(\mathcal{E})$ will be computed in Section 5.1. When $r(\mathcal{E}) > 1$ we can write

\begin{equation}
\mathcal{E} \cong \mathcal{R}ep(\mathbb{Z}_N) \boxtimes \mathcal{E}_1,
\end{equation}
where \( r(\mathcal{E}_1) = r(\mathcal{E}) - 1 \). By Theorem 3.8,
\[
\text{Mext}(\mathcal{E}) \cong \text{Mext}(\mathcal{E}_1) \oplus 2\text{-Fun}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)).
\]
The last summand is computed as follows. Since \( H^4(\mathbb{Z}_N, \mathbb{k}^\times) = 0 \) and \( H^2(\mathbb{Z}_N, \text{Inv}(\mathcal{E}_1)) = \text{Ext}(\mathbb{Z}_N, \text{Inv}(\mathcal{E}_1)) \), sequences (18) and (20) become
\[
0 \to \text{Ext}(\mathbb{Z}_N, \text{Inv}(\mathcal{E}_1)) \to \text{Fun}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)) \to \text{Hom}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)) \to 0
\]
and
\[
0 \to H^3(\mathbb{Z}_N, \mathbb{k}^\times) \to 2\text{-Fun}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)) \to \text{Fun}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)) \to 0.
\]
Thus, the direct complement of \( \text{Mext}(\mathcal{E}_1) \) in \( \text{Mext}(\mathcal{E}) \) has a filtration with factors
\[
(37) \quad H^3(\mathbb{Z}_N, \mathbb{k}^\times) \cong \mathbb{Z}_N, \quad \text{Ext}(\mathbb{Z}_N, \text{Inv}(\mathcal{E}_1)), \quad \text{and} \quad \text{Hom}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)).
\]
We will see in Section 5.2 that the group \( 2\text{-Fun}(\mathbb{Z}_N, \text{Pic}(\mathcal{E}_1)) \) is not, in general, a direct sum of factors (37).

**Remark 3.12.** Let \( \mathcal{E} = \text{Rep}(G) \boxtimes s\text{Vect} \). Since \( \text{Mext}(s\text{Vect}) \cong \mathbb{Z}_{16} \), Theorem 3.8 gives
\[
\text{Mext}(\text{Rep}(G) \boxtimes s\text{Vect}) \cong \mathbb{Z}_{16} \oplus 2\text{-Fun}(G, \text{Pic}(s\text{Vect})).
\]
We have \( \text{Inv}(s\text{Vect}) \cong \mathbb{Z}_2 \) and \( \text{Pic}(s\text{Vect}) \cong \mathbb{Z}_2 \), so it follows from exact sequences (18) and (20) that the group \( 2\text{-Fun}(G, \text{Pic}(s\text{Vect})) \) has a filtration with factors
\[
(39) \quad H^3(G, \mathbb{k}^\times), \quad \text{Ker} \left( H^2(G, \mathbb{Z}_2) \xrightarrow{\delta} H^4(G, \mathbb{k}^\times) \right), \quad \text{and} \quad H^1(G, \mathbb{Z}_2)
\]
where \( \delta \) is the obstruction map (21). Explicit formulas describing the product in this group are given in [1].

**Remark 3.13.** Let \( \mathcal{E} = \text{Rep}(\tilde{G}, t) \) be a general symmetric fusion category, where the group \( \tilde{G} \) fits into a (not necessarily split) short exact sequence
\[
1 \to \mathbb{Z}_2 = \langle t \rangle \to \tilde{G} \to G \to 1.
\]
In this case, there is a parameterization of \( \text{Mext}(\mathcal{E}) \) by torsors over the cohomology groups listed in (39), see [1, Section VII.C] and [2, Table I]. These papers also contain explicit formulas for products of minimal extensions (i.e., symmetric invertible fermionic topological phases) in terms of this parameterization.

4. **The group of minimal extensions of a pointed symmetric fusion category**

4.1. **A canonical grading on a minimal extension.** Let \( A \) be a finite group and let \( \mathcal{E} = \text{Rep}(A, t) \) be a symmetric fusion category.
Proposition 4.1. Let \( \mathcal{E} \rightarrow \mathcal{C} \) be a minimal non-degenerate extension. Then \( \mathcal{C} \) is faithfully \( A \)-graded:

\[
\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \quad \mathcal{C}_e = \mathcal{E},
\]

where \( \mathcal{C}_x = \{ X \in \mathcal{C} \mid c_{X,V}c_{V,X} = x|_V \otimes \text{id}_X \text{ for all } V \in \mathcal{E} \} \), \( x \in A \).

Conversely, any \( A \)-graded braided extension of this form is a minimal non-degenerate extension of \( \mathcal{E} \). Two such minimal extensions are equivalent if and only if they are equivalent as \( A \)-graded braided extensions of \( \mathcal{E} \).

Proof. It follows from [13, Section 3.4] that \( \mathcal{E} \)-module components of \( \mathcal{C} \) are parameterized by characters of \( K_0(\mathcal{E}) \), i.e., by elements of \( A \). Namely, the squared braiding of a simple object \( X \in \mathcal{C} \) with simple objects of \( \phi \in \mathcal{E} \), where \( \phi \in \hat{\mathbb{A}} = \text{Hom}(A, \mathbb{K}^\times) \) determines a character on \( \hat{\mathbb{A}} \), i.e., an element \( a_X \in A = \hat{\mathbb{A}} \) such that

\[
c_{\phi,X}c_{x,\phi} = \phi(a_X) \text{id}_{\phi \otimes X}.
\]

It follows from the hexagon axioms that the assignment \( X \mapsto a_X \) determines a grading on \( \mathcal{C} \). Since \( \mathcal{C} \) is non-degenerate, we must have \( \mathcal{C}_x \neq 0 \) for all \( x \in A \), i.e., the above grading is faithful.

Conversely, we claim that an \( A \)-graded braided extension \((40)\) is non-degenerate. Let \( Z_{\text{sym}}(\mathcal{C}) \) denote the symmetric center of \( \mathcal{C} \). For \( x \neq e \), we have \( Z_{\text{sym}}(\mathcal{C}) \cap \mathcal{C}_x = 0 \), since there is \( V \in \mathcal{E} \) such that \( x|_V \neq \text{id}_V \), i.e., \( c_{X,V}c_{V,X} \neq \text{id}_{V \otimes X} \) for all non-zero \( X \in \mathcal{C}_x \).

If \( V \in Z_{\text{sym}}(\mathcal{C}) \cap \mathcal{E} \) is a non-trivial representation of \( A \), then there is \( x \in A \) such that \( x|_V \neq \text{id}_V \), i.e., \( V \) does not centralize \( \mathcal{C}_x \). Hence, \( Z_{\text{sym}}(\mathcal{C}) = \text{Vect} \), i.e., \( \mathcal{C} \) is non-degenerate.

It is a minimal extension since \( \text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{E})^2 \).

An equivalence of minimal extensions preserves the squared braiding and restricts to the identity on \( \mathcal{E} \). Therefore it must preserve the grading \((40)\). \( \square \)

Recall [20] that a fusion category is \textit{nilpotent} if it is obtained from \( \text{Vect} \) by a sequence of graded extensions.

Corollary 4.2. Let \( \mathcal{E} \) be a pointed symmetric fusion category and let \( p_1, \ldots, p_n \) be distinct primes dividing \( \text{FPdim}(\mathcal{E}) \). Let \( \mathcal{E} = \mathcal{E}_1 \boxtimes \cdots \boxtimes \mathcal{E}_n \) be the Sylow decomposition of \( \mathcal{E} \), where \( \text{FPdim}(\mathcal{E}_i) \) is a power of \( p_i \), \( i = 1, \ldots, n \). Then any minimal extension of \( \mathcal{E} \) is nilpotent and

\[
\text{Mext}(\mathcal{E}) = \text{Mext}(\mathcal{E}_1) \times \cdots \times \text{Mext}(\mathcal{E}_n).
\]

Proof. By Proposition 4.1, a minimal extension of \( \mathcal{E} \) is a graded extension of a pointed fusion category, so it is nilpotent of nilpotency class at most 2. It is shown in [12, Theorem 6.12] that any nilpotent braided fusion category \( \mathcal{C} \) admits a Sylow decomposition
\( \mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \). So if \( \mathcal{E} \hookrightarrow \mathcal{C} \) is a minimal extension then \( \mathcal{E}_i \hookrightarrow \mathcal{C}_i \) is a minimal extension for all \( i = 1, \ldots, n \). This implies the statement. \( \square \)

**Remark 4.3.** Minimal extensions of Tannakian fusion categories were classified in [26] where it was shown that \( \text{Mext}(\text{Rep}(G)) = H^3(G, \mathbb{K}^\times) \) for any finite group \( G \). In view of Corollary 4.2, to classify extension of pointed symmetric fusion categories it remains to determine \( \text{Mext}(\text{Rep}(A, t)) \), where \( A \) is an Abelian 2-group.

**Remark 4.4.** It follows from the description of the homogeneous components of the grading (40) that the groupoid \( \text{Mext}(\text{Rep}(A, t)) \) is equivalent to the groupoid of braided monoidal 2-functors \( F : A \to \text{Pic}_{\text{br}}(\text{Rep}(A, t)) \) such that the composition of group homomorphisms

\[
A \xrightarrow{\pi_0(F)} \text{Pic}_{\text{br}}(\text{Rep}(A, t)) = \text{Pic}(\text{Rep}(A, t)) \times A \xrightarrow{p_A} A,
\]

equals \( \text{id}_A \). Here \( p_A \) denotes the projection on \( A \).

### 4.2. A canonical filtration of \( \text{Mext}(\mathcal{E}) \)

We say that a minimal extension \( \mathcal{E} \hookrightarrow \mathcal{C} \) is **integral** if \( \mathcal{C} \) is an integral fusion category, i.e., \( \text{FPdim}(X) \) is an integer for all objects \( X \in \mathcal{C} \). Such extensions are characterized by the following property: the image of the corresponding homomorphism composition

\[
A \to \text{Pic}_{\text{br}}(\mathcal{E}) \to \text{Pic}(\mathcal{E})
\]

lies in \( \text{Pic}_{\text{int}}(\mathcal{E}) \), see (7).

An integral minimal extension \( \mathcal{E} \hookrightarrow \mathcal{C} \) is **pointed** if \( \mathcal{C} \) is a pointed category. Equivalently, this extension is **quasi-trivial** in the sense of [6, Section 8.7], i.e., the homomorphism (43) is trivial. In this case, the homomorphism \( A \to \text{Pic}_{\text{br}}(\mathcal{E}) \) is identified with the identity map \( A \to \text{Aut}_{\otimes}(\text{id}_\mathcal{E}) = A \). The braided monoidal functor

\[
A \to \text{Pic}_{\text{br}}(\mathcal{E})
\]

is determined by an element \( L \in H^2_{\text{ab}}(A, \hat{A}) = \text{Ext}(A, \hat{A}) \).

Here and below \( H^*_\text{ab}(A, M) = H^{*+1}(K(A, 2), M) \) denotes the abelian Eilenberg-Mac Lane cohomology group of \( A \) with coefficients in \( M \) [14]. A description of low dimensional abelian cohomology groups \( H^n_{\text{ab}}(A, M) \), \( n \leq 4 \) can be found, e.g., in [6, Section 2.1], where the term “braided cohomology” was used.

Finally, a pointed minimal extension \( \mathcal{E} \hookrightarrow \mathcal{C} \) is **trivial** if the monoidal functor (44) is trivial. Such extensions are easy to describe explicitly as follows. We have \( \mathcal{E} = \text{Rep}(A, t) = \mathcal{C}(\hat{A}, t) \), where \( t \) is viewed as a quadratic character on \( \hat{A} \). For any quadratic form \( q : A \to \mathbb{K}^\times \) define a quadratic form \( h_q : A \times \hat{A} \to \mathbb{K}^\times \) by

\[
h_q(a, \phi) = \langle at, \phi \rangle q(a), \quad a \in A, \phi \in \hat{A}.
\]
It is easy to see that this form is non-degenerate and
\[ \mathcal{E} = \mathcal{C}(\widehat{A}, t) \hookrightarrow \mathcal{M}_q := \mathcal{C}(A \times \widehat{A}, h_q). \]
is a typical trivial minimal extension of \( \mathcal{E} \).

**Lemma 4.5.** The set of trivial (respectively, pointed, integral) minimal extensions of \( \mathcal{E} \) is closed under the tensor product.

**Proof.** The statement about trivial extensions follows from their explicit description (46). Indeed, one can directly check that the assignment
\[ H^3_{ab}(A, \mathbb{k}^\times) \to Mext(\text{Rep}(A, t)) : q \mapsto \mathcal{M}_q \]
is a group homomorphism.

Since \( \mathcal{E} \) is pointed, the tensor product of its minimal extensions is obtained by taking a de-equivariantization with respect to the diagonal Tannakian subcategory in \( \mathcal{E} \otimes \mathcal{E} \). Clearly, a de-equivariantization of a pointed (respectively, integral) fusion category is pointed (respectively, integral). This proves the statement about pointed and integral extensions. □

**Remark 4.6.** Homomorphism (47) is, in general, not injective.

Thus, we have a filtration
\[ Mext_{\text{triv}}(\mathcal{E}) \subset Mext_{\text{pt}}(\mathcal{E}) \subset Mext_{\text{int}}(\mathcal{E}) \subset Mext(\mathcal{E}), \]
where \( Mext_{\text{triv}}(\mathcal{E}), Mext_{\text{pt}}(\mathcal{E}), Mext_{\text{int}}(\mathcal{E}) \) denote the categorical groups of trivial, pointed, and integral minimal extensions of \( \mathcal{E} \). There is a corresponding filtration of Abelian groups
\[ Mext_{\text{triv}}(\mathcal{E}) \subset Mext_{\text{pt}}(\mathcal{E}) \subset Mext_{\text{int}}(\mathcal{E}) \subset Mext(\mathcal{E}), \]
which we are going to study next. Our goal is to determine the factors of this filtration.

### 4.3. Trivial minimal extensions.
Recall that the third abelian cohomology group \( H^3_{ab}(A, \mathbb{k}^\times) \) is isomorphic to the group \( \text{Quad}(A, \mathbb{k}^\times) \) of quadratic forms on \( A \) with values in \( \mathbb{k}^\times \).

**Proposition 4.7.** \( Mext_{\text{triv}}(\mathcal{E}) \cong \text{Coker} \left( H^1(A, \widehat{A}) \xrightarrow{\kappa^t} H^3_{ab}(A, \mathbb{k}^\times) \right) \), where
\[ \kappa^t : H^1(A, \widehat{A}) \to \text{Quad}(A, \mathbb{k}^\times) : Z \mapsto q_Z, \quad q_Z(x) = \langle xt, Z(x) \rangle, \quad x \in A. \]

**Proof.** A trivial extension of \( \mathcal{E} = \text{Rep}(A, t) \) is obtained by deforming the structure constraints of the identity extension
\[ \mathcal{E} \hookrightarrow \mathcal{Z}(\mathcal{E}) = \bigoplus_{a \in A} \mathcal{Z}(\mathcal{E})_a \]
by means of an abelian 3-cocycle \((\omega, c) \in Z^3_{ab}(A, \mathbb{K}^\times)\). This is a special case of a \emph{zesting} procedure studied in \cite{[9]}. Namely, let \(a_{W,X,Y}\) and \(c_{X,Y}\) denote the associativity and braiding isomorphisms in \(Z(\mathcal{E})\). The deformed extension \(Z(\mathcal{E})^{(\omega, c)}\) coincides with \(Z(\mathcal{E})\) as an abelian category and has the same tensor product, while its associativity and braiding isomorphisms are given by

\[
\begin{align*}
\hat{a}_{W,X,Y} &= \omega(\deg(X), \deg(Y), \deg(Z))a_{W,X,Y}, \\
\hat{c}_{X,Y} &= \omega(\deg(X), \deg(Y))c_{X,Y},
\end{align*}
\]

for all homogeneous objects \(W, X, Y\).

Isomorphisms between trivial braided extensions were classified in \cite{[6], Section 8.7}. In particular, formula \((50)\) follows from \cite{[6], formula (8.52)], since the self braiding \(c_{X,X}\) of a simple object \(X \in \text{Rep}(A, t)\) is given by the evaluation \(\langle X, t \rangle\).

\(\square\)

**Proposition 4.8.** Let \(A(2) = \mathbb{Z}_{2^{n_1}} \times \cdots \times \mathbb{Z}_{2^{n_r}}\) be the Sylow 2-subgroup of \(A\). Then

\[(51)\quad M_{\text{ext}_{\text{triv}}} (\mathcal{E}) \cong \begin{cases} 
\mathbb{Z}_4 \times \mathbb{Z}_2^{-1} & \text{if } \langle t \rangle \text{ is a direct summand of } A, \\
\mathbb{Z}_2^r & \text{otherwise.}
\end{cases}\]

**Proof.** We write \(A = A(2) \times A(\text{odd})\) and note that the map \(\kappa^t\) from \((50)\) respects this decomposition and is an isomorphism on \(A(\text{odd})\). So we may assume that \(A\) is a 2-group. Isomorphism \((51)\) is easy to check when \(A = \mathbb{Z}_{2^n}\) is cyclic. In this case, we have

\[H^1(\mathbb{Z}_{2^n}, \mathbb{Z}_{2^n}) = \mathbb{Z}_{2^n}, \quad \text{Quad}(\mathbb{Z}_{2^n}, k^\times) \cong \mathbb{Z}_{2^{n+1}},\]

and the homomorphism \(\kappa^t : \mathbb{Z}_{2^n} \to \mathbb{Z}_{2^{n+1}}\) is injective if \(n > 1\) and is zero for \(n = 1\).

Let \(A = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle\), where \(|x_i| = 2^{n_i}\). We may assume that \(t \in \langle x_1 \rangle\). A quadratic form \(q : A \to \mathbb{K}^\times\) is uniquely determined by the values

\[q(x_i), i = 1, \ldots, r \quad \text{and} \quad b(x_j, x_l) := \frac{q(x_jx_l)}{q(x_j)q(x_l)}, \quad 1 \leq j < l \leq r.\]

Here \(q(x_i)\) is a \(2^{n_i+1}\)th root of unity and \(b(x_j, x_l)\) is a \(\text{min}\{2^{n_j}, 2^{n_l}\}\)th root of unity. Any choice of such roots of unity will give a quadratic form. Let us identify \(Z \in H^1(A, \hat{A})\) with a bilinear form on \(A\). The symmetric bilinear form associated to \(q_Z\) is

\[b_Z(x, y) := Z(x, y)Z(y, x), \quad x, y \in A,\]

which can realize all possible values of \(b(x_j, x_l)\). On the other hand, \(q_Z(x_i) = Z(x_i, tx_i)\) can be any \(2^{n_i}\)th root of unity if \(i > 1\) and \(q_Z(x_1) = Z(x_1, t)Z(x_1, x_1)\). The latter can be any \(2^{n_1}\)th root of unity if \(t \neq x_1\) (i.e., \(n_1 > 1\)) and equals 1 otherwise. From this, the cokernel of \((50)\) is easily determined. \(\square\)
4.4. Pointed minimal extensions. The group $\text{Mext}_{pt}(\text{Rep}(A, t))$ can be computed using the classification of quasi-trivial graded extensions from \cite[Section 8.7]{6}. For an abelian group $A$ we identify $\text{Ext}(A, \hat{A})$ with $H^2_{ab}(A, \hat{A})$. There is a natural involution

$$\varepsilon : H^2_{ab}(A, \hat{A}) \to H^2_{ab}(A, \hat{A})$$

that sends the class of an abelian extension

$$0 \to \hat{A} \to C \to A \to 0 \quad (52)$$

to the class of the dual extension obtained by applying the functor $\text{Hom}(-, \mathbb{k}^\times)$ to $\text{Ext}(A, \hat{C})$ obtained by applying the functor $\text{Hom}(-, \mathbb{k}^\times)$ to $\text{Ext}(A, \hat{C})$:

$$0 \to \hat{A} \to \hat{C} \to \hat{A} \to A \to 0. \quad (53)$$

This $\varepsilon$ was explicitly described in \cite[Section 8]{28}. Namely, let $L = \{L_{x,y}\}_{x,y \in A}$ be a normalized 2-cocycle in $H^2_{ab}(A, \hat{A})$ corresponding to the extension $\text{Ext}(A, \hat{C})$. For any $z \in A$ there is a normalized 1-cochain $a_z \in C^1(A, \mathbb{k}^\times)$ such that

$$L_{x,y}(z) = \frac{a_z(x)a_z(y)}{a_z(xy)}, \quad x, y, z \in A, \quad (54)$$

and $\varepsilon(L)$ is determined by

$$\varepsilon(L)_{x,y}(z) = \frac{a_z(x)a_y(z)}{a_{xy}(z)}, \quad x, y, z \in A. \quad (55)$$

Let us denote $H^2_{ab}(A, \hat{A})^\varepsilon = \{L \in H^2_{ab}(A, \hat{A}) \mid \varepsilon(L) = L\}$.

By Proposition \[4.1\] an extension $\mathcal{C} \in M\text{ext}_{pt}(\mathcal{E})$ defines a quasi-trivial $A$-graded braided extension of $\mathcal{E}$. The corresponding braided monoidal functor $F_C : A \to \text{Pic}_{br}(\mathcal{E})$ is completely determined by an abelian 2-cocycle $L_C \in H^2_{ab}(A, \hat{A})$ defining the monoidal structure of $F_C$. This $L_C$ is precisely the 2-cocycle corresponding to the central extension

$$0 \to \hat{A} \to \text{Inv}(\mathcal{C}) \to A \to 0.$$ 

Thus, there is a homomorphism

$$\lambda : M\text{ext}_{pt}(\mathcal{E}) \to H^2_{ab}(A, \hat{A}) : \mathcal{C} \mapsto L_C \quad (56)$$

whose kernel is $M\text{ext}_{triv}(\mathcal{E})$. The image of $\lambda$ consists of all $L \in H^2_{ab}(A, \hat{A})$ such that the corresponding braided monoidal functor $A \to \text{Pic}_{br}(\mathcal{E})$ admits an extension to a braided monoidal 2-functor. By \cite[Section 8.7]{6} this image is the kernel of the Pontryagin-Whitehead homomorphism

$$PW^2 : H^2_{ab}(A, \hat{A}) \to H^4_{ab}(A, \mathbb{k}^\times)$$

whose components are given by formulas \cite[(8.53)-(8.55)]{6}:

$$PW^2(L)(x, y, z, w) = c_{L_{x,y}, L_{z,w}}, \quad PW^2(L)(x, y, z) = 1, \quad PW^2(L)(x, y, z) = L_{y,z}(x), \quad (57)$$
for all \( x, y, z, w \in A \). Thus, there is an exact sequence of group homomorphisms
\[
0 \to M_{\text{ext\,triv}}(\mathcal{E}) \to M_{\text{ext\,pt}}(\mathcal{E}) \xrightarrow{\lambda} H^2_{ab}(A, \widehat{A}) \xrightarrow{\text{PW}^2} H^4_{ab}(A, \mathbb{k}^\times).
\]

For an Abelian group \( A \) let us denote \( A_2 = \{ x \in A \mid x^2 = e \} \).

**Proposition 4.9.** There is a group isomorphism
\[
M_{\text{ext\,pt}}(\mathcal{E})/M_{\text{ext\,triv}}(\mathcal{E}) \cong \text{Ker} \left( H^2_{ab}(A, \widehat{A})^\varepsilon \xrightarrow{\theta^t} \text{Hom}(A_2/\langle t \rangle, \mathbb{k}^\times) \right),
\]
where
\[
\theta^t : H^2_{ab}(A, \widehat{A})^\varepsilon \to \text{Hom}(A_2/\langle t \rangle, \mathbb{k}^\times) : L \mapsto \theta^t_L, \quad \theta^t_L(x) = L_{x,x}(xt), \quad x \in A.
\]

**Proof.** Given the exact sequence (58), all we need to show is that the kernels of \( \theta^t \) and \( \text{PW}^2 \) coincide. It was shown in [15] (see also [6, (2.19)]) that there is an exact sequence
\[
0 \to \text{Hom}(A_2, \mathbb{k}^\times) \to H^4_{ab}(A, \mathbb{k}^\times) \xrightarrow{h_A} H^2_{ab}(A, \widehat{A}).
\]

It follows from the construction described at the end of [6, Section 2.1] and formulas (54), (55) that
\[
h_A(PW^2(L)) = L \varepsilon(L)^{-1}, \quad L \in H^2_{ab}(A, \widehat{A}).
\]

There is a canonical isomorphism
\[
\iota_A : \text{Ker}(h_A) \xrightarrow{\sim} \text{Hom}(A_2, \mathbb{k}^\times) : \alpha \mapsto \iota_A(\alpha),
\]
defined by
\[
\iota_A(\alpha)(x) = \frac{\alpha(x, x, x, x)\alpha(x|x, x)}{\alpha(x, x|x)}, \quad x \in A_2.
\]

Combining formulas (57) and (62) we obtain
\[
\iota_A(PW^2(L))(x) = L_{x,x}(xt), \quad x \in A.
\]

Thus, \( PW^2(L) = 0 \) if and only if \( L \in H^2_{ab}(A, \widehat{A})^\varepsilon \) and \( L_{x,x}(xt) = 1 \) for all \( x \in A \), i.e., \( L \in \text{Ker}(\theta^t) \). Since the right hand side of (63) vanishes on \( t \), we conclude that \( PW^2 \) descends to a homomorphism
\[
\theta^t : H^2_{ab}(A, \widehat{A})^\varepsilon \to \text{Hom}(A_2/\langle t \rangle, \mathbb{k}^\times)
\]
defined in (59). \( \square \)

**Remark 4.10.** The group \( H^2_{ab}(A, \widehat{A}) \) is (non-canonically) isomorphic to \( \text{Hom}(\widehat{A} \otimes \widehat{A}, \mathbb{k}^\times) \). It was explained in [28, Lemma 8.2] that, upon this isomorphism, \( \varepsilon \) is identified with the transposition map, so
\[
H^2_{ab}(A, \widehat{A})^\varepsilon \cong \text{Sym}^2(\widehat{A}).
\]
In particular, if \( A \) is cyclic, then \( \varepsilon \) is the identity map.
Proposition 4.11. Homomorphism \( (59) \) is surjective.

Proof. We may assume that \( A = C_1 \times \cdots \times C_r \), where each \( C_i \) is a cyclic 2-group and \( t \in C_1 \). It suffices to check that for \( C_i = \langle x \mid x^{2N_i} = 1 \rangle \), \( i = 2, \ldots, r \), the homomorphism
\[
H^2_{ab}(C_i, \hat{C}_i) \to \text{Hom}(\mathbb{Z}_2, \mathbb{k}^\times) \cong \{\pm 1\} : L \mapsto L_{x^{N_i}, x^{N_i}}(x^{N_i})
\]
is surjective. To see that, let \( \xi \) be a primitive 2\( N_i \)-th root of 1 in \( k \) and let \( L \) be a generator of \( H^2_{ab}(C_i, \hat{C}_i) \cong \mathbb{Z}_{2N_i} \) explicitly defined by
\[
L_{x^k, x^l}(x^m) = \begin{cases} 
1 & \text{if } k + l < 2N_i, \\
\xi^m & \text{if } k + l \geq 2N_i, \quad m = 1, \ldots, 2N_i.
\end{cases}
\]
Then \( (64) \) sends \( L \) to \(-1\), as required. \( \square \)

4.5. Integral minimal extensions. We continue to denote \( \mathcal{E} = \mathcal{R}ep(A, t) \).

Lemma 4.12. We have
\[
[M_{\text{ext int}}(\mathcal{E}) : M_{\text{ext pt}}(\mathcal{E})] = |^3 A|.
\]

Proof. We will prove this by induction on \( r(\mathcal{E}) \), the finite rank of the Abelian group \( \text{Inv}(\mathcal{E}) \). When \( r(\mathcal{E}) = 1 \), i.e., when \( A \) is cyclic, we have \( \text{Pic}_{\text{int}}(\mathcal{E}) = 0 \) and, hence, all minimal extensions of \( \mathcal{E} \) are pointed, so both sides of \( (65) \) are equal to 0.

As in Example 3.11, any \( \mathcal{E} = \mathcal{R}ep(A, t) \) with \( r(\mathcal{E}) = r + 1 > 1 \), can be written as \( \mathcal{E} = \mathcal{R}ep(\mathbb{Z}_N) \boxtimes \mathcal{E}_1 \), where \( r(\mathcal{E}_1) = r \) and \( \mathcal{E} = \mathcal{R}ep(A_1, t) \). We have
\[
[M_{\text{ext int}}(\mathcal{E}) : M_{\text{ext pt}}(\mathcal{E})] = [M_{\text{ext int}}(\mathcal{E}_1) : M_{\text{ext pt}}(\mathcal{E}_1)] \times \frac{|M_{\text{ext int}}(\mathcal{E})|}{|M_{\text{ext int}}(\mathcal{E}_1)|} \times \frac{|M_{\text{ext pt}}(\mathcal{E}_1)|}{|M_{\text{ext pt}}(\mathcal{E})|}.
\]
The first factor in \( (66) \) is equal to \( |^3 A| \) by the inductive assumption. By Example 3.11 the second factor is equal to
\[
|H^3(\mathbb{Z}_N, \mathbb{k}^\times)| \times |\text{Ext}(\mathbb{Z}_N, \hat{A}_1)| \times |\text{Hom}(\mathbb{Z}_N, \wedge^2 A_1)| = |\mathbb{Z}_N| \times |\mathbb{Z}_N \otimes A_1| \times |\mathbb{Z}_N \otimes \wedge^2 A_1|.
\]
Using Propositions 4.8, 4.9, and 4.11 we compute the last factor in \( (66) \) as
\[
\frac{|M_{\text{ext int}}(\mathcal{E}_1)|}{|M_{\text{ext int}}(\mathcal{E})|} \times \frac{|M_{\text{ext pt}}(\mathcal{E}_1) : M_{\text{ext int}}(\mathcal{E}_1)|}{|M_{\text{ext pt}}(\mathcal{E}) : M_{\text{ext int}}(\mathcal{E})|} = 2 \times \frac{H^2_{ab}(A, \hat{A}_1)^c}{2H^2_{ab}(A, \hat{A}_1)^c} = \frac{1}{|\text{Sym}^2(A_1)|} = \frac{1}{|\mathbb{Z}_N| \times |\mathbb{Z}_N \otimes A_1|}.
\]
Substituting these quantities into \( (66) \) we obtain
\[
[M_{\text{ext int}}(\mathcal{E}) : M_{\text{ext pt}}(\mathcal{E})] = |\mathbb{Z}_N| \times |\mathbb{Z}_N \otimes A_1| \times |^3 A| = |^3 A|,
\]
as required. \( \square \)
Recall from (40) that a minimal extension \( E \hookrightarrow C \) admits a canonical faithful \( A \)-grading
\[
C = \bigoplus_{x \in A} C_x, \quad C_e = \mathcal{E}.
\]
When this extension is integral, its components are of the form \( C_x = \text{Rep}(k_{\mu_x}[G]) \), where
\[
A \to H^2(A, t, k^{\times}) = \text{Pic}_{\text{int}}(\mathcal{E}) : x \mapsto \mu_x
\]
is a group homomorphism. It follows from the description of the group \( \text{Pic}_{\text{br}}(\mathcal{E}) \) in Section 2.1 and Remark 4.4 that the corresponding homomorphism \( A \to \text{Pic}_{\text{br}}(\mathcal{E}) \) is
\[
A \to H^2(A, t, k^{\times}) \times A : x \mapsto (\mu_x, x).
\]
Since (70) comes from a braided monoidal 2-functor \( A \to \text{Pic}_{\text{br}}(\mathcal{E}) \), it must satisfy
\[
Q_E(\mu_x, x) = 1, \quad \text{for all } x \in A,
\]
where \( Q_{\mathcal{E}} : H^2(A, t, k^{\times}) \times A \to \hat{A} \) is the quadratic form (9).

Define a map \( \tau_C : A^3 \to k^{\times} \) by
\[
\tau_C(x, y, z) = (-1)^{\xi_{\mu x}(y)\xi_{\mu x}(z)} \frac{\mu_x(y, z)}{\mu_x(z, y)},
\]
where the bilinear map \( \xi_{\mu} : H^2(A, t, k^{\times}) \times A \to \mathbb{Z}/2\mathbb{Z} \), was introduced in (5).

**Proposition 4.13.** The map (72) is a trilinear form on \( A \) satisfying
\[
\tau_C(x, tx, y) = 1 \quad \text{and} \quad \tau_C(x, ty, y) = 1.
\]
for all \( x, y \in A \).

**Proof.** The linearity of \( \tau_C \) in the second and third arguments is clear since for each \( \mu \in H^2(A, k^{\times}) \) the map
\[
A^2 \to k^{\times} : (x, y) \mapsto \frac{\mu(x, y)}{\mu(y, x)}
\]
is an alternating bilinear form. To check the linearity in the first argument, we compute, using the definition of the product \(*\) from (6):
\[
\tau_C(wx, y, z) = (-1)^{\xi_{\mu wx}(y)\xi_{\mu wx}(z)} \frac{\mu_{wx}(y, z)}{\mu_{wx}(z, y)}
\]
\[
= (-1)^{\xi_{\mu w*\mu x}(y)\xi_{\mu w*\mu x}(z)} \frac{(\mu_w * \mu_x)(y, z)}{(\mu_w * \mu_x)(z, y)}
\]
\[
= (-1)^{\xi_{\mu w*\mu x}(y)\xi_{\mu w*\mu x}(z)} (-1)^{\xi_{\mu w}(y)\xi_{\mu w}(z) + \xi_{\mu w}(y)\xi_{\mu w}(z) \frac{\mu_w(y, z)}{\mu_w(z, y)} \frac{\mu_x(y, z)}{\mu_x(z, y)}
\]
\[
= (-1)^{\xi_{\mu x}(y)\xi_{\mu x}(z) + \xi_{\mu w}(y)\xi_{\mu w}(z)} \frac{\mu_w(y, z)}{\mu_w(z, y)} \frac{\mu_x(y, z)}{\mu_x(z, y)}
\]
\[
= \tau_C(w, y, z) \tau_C(x, y, z),
\]
for all $x, w, y, z \in A$, where we used that $\xi_{\mu \nu} = \xi_\mu + \xi_\nu$ for all $\mu, \nu \in H^2(A, t, \mathbb{k}^\times)$.

The condition (71) along with the formula (9) imply that $\xi_{\mu x}(x) = 0$, i.e., $\frac{\mu_x(x, t)}{\mu_x(x, x)} = 1$, and

$$1 = Q_\mathcal{C}(\mu_x, x)(y) = \frac{\mu_x(x, t, y)}{\mu_x(x, x, t)} = (-1)^{\xi_{\mu x}(x)\xi_{\mu x}(y)} \frac{\mu_x(x, t, y)}{\mu_x(y, x, t)} = \tau_\mathcal{C}(x, xt, y),$$

for all $x, y \in A$, which is the first identity in (73). Finally,

$$\tau_\mathcal{C}(x, yt, y) = \frac{\mu_x(yt, y)}{\mu_x(y, yt)} = (-1)^{\xi_{\mu x}(yt)\xi_{\mu x}(y)} = \frac{\mu_x(t, y)}{\mu_x(y, t)} (-1)^{\xi_{\mu x}(y)^2} = (-1)^{\xi_{\mu x}(y)+\xi_{\mu x}(y)^2} = 1,$$

which is the second identity in (73). □

**Proposition 4.14.** There is a (non-canonical) group isomorphism

$$M_{\text{ext int}}(\mathcal{E})/M_{\text{ext pl}}(\mathcal{E}) \cong \text{Hom}(\wedge^3 A, \mathbb{k}^\times)$$

**Proof.** The map (72) defines a group homomorphism

$$\tau : M_{\text{ext int}}(\mathcal{E}) \to \text{Hom}(A^3, \mathbb{k}^\times) : C \mapsto \tau_\mathcal{C}.$$

A minimal extension $\mathcal{C}$ is pointed if and only if the corresponding homomorphism (69) is trivial, hence pointed extensions belong to the kernel of $\tau$. Conversely, if $\tau_\mathcal{C} = 1$ then

$$1 = \tau_\mathcal{C}(x, y, y) = (-1)^{\xi_{\mu x}(y)},$$

so that $\xi_{\mu x}(y) = 0$ for all $x, y \in A$. This implies that $\tau_\mathcal{C}(x, y, z) = \frac{\mu_x(y, z)}{\mu_x(z, y)} = 1$ for all $x, y, z \in A$, so that $\mu_x = 0$ in $H^2(A, t, \mathbb{k}^\times)$ and $\mathcal{C}$ is pointed.

Thus, $M_{\text{ext int}}(\mathcal{E})/M_{\text{ext pl}}(\mathcal{E})$ is isomorphic to the image of $\tau$ in $\text{Hom}(A^3, \mathbb{k}^\times)$. We can choose a presentation $A = \langle e_1 \rangle \times \cdots \times \langle e_r \rangle$ such that we also have $A = \langle te_1 \rangle \times \cdots \times \langle te_r \rangle$. By Proposition 4.13 the trilinear form $\tau_\mathcal{C}$ is completely determined by its values $\tau_\mathcal{C}(e_i, te_j, e_k)$ when $i, j, k$ are distinct. Hence, the group of such forms is embedded into $\text{Hom}(\wedge^3 A, \mathbb{k}^\times)$. By Lemma 4.12 this embedding must be an isomorphism. □

**Remark 4.15.** For a Tannakian category $\mathcal{E} = \mathcal{Rep}(A)$, Proposition 4.14 implies a well-known fact that the alternator homomorphism $\text{alt} : H^3(A, \mathbb{k}^\times) \to \text{Hom}(\wedge^3 A, \mathbb{k}^\times)$ defined by

$$\text{alt}(\omega)(x, y, z) = \prod_{\sigma \in S_3} \omega(\sigma(x), \sigma(y), \sigma(z))^{\text{sign}(\sigma)}$$

for all $x, y, z \in A$, is surjective. Indeed, a typical minimal extension of $\mathcal{Rep}(A)$ is $\mathcal{Rep}(A) \hookrightarrow \mathcal{Z}(\text{Vec}_A^\times)$ for some $\omega \in H^3(A, \mathbb{k}^\times)$. In this case, the 2-cocycles $\mu_x, x \in A$, in (69) are given by

$$\mu_x(y, z) = \frac{\omega(x, y, z)\omega(y, z, x)}{\omega(y, x, z)}, \quad y, z \in G,$$
and
\[ \tau_{\text{Vec}_A}(x, y, z) = \frac{\mu_x(y, z)}{\mu_x(z, y)} = \text{alt}(\omega)(x, y, z). \]

**Remark 4.16.** An integral minimal extension of a pointed super-Tannakian category \( \mathcal{E} = \text{Rep}(A, t) \) of central charge 1 must be equivalent to \( \mathcal{E} \hookrightarrow \mathcal{Z}(\text{Vec}_A^\omega) \) for some \( \omega \) in \( H^3(G, \k^\times) \) [12]. But it is not always possible to choose a group \( G \) to be Abelian.

To see that twisted Drinfeld doubles of \( A \) are not sufficient, note that \( \mathcal{Z}(\text{Vec}_A^\omega) \), where \( \omega \in H^3(A, \k^\times) \), contains \( \mathcal{E} \) as a fusion category only if (76) factors through \( \text{Hom}(\bigwedge^3(A/\langle t \rangle), \k^\times) \rightarrow \text{Hom}(\bigwedge^3A, \k^\times) \).

Indeed, suppose there is an embedding \( \mathcal{E} \hookrightarrow \mathcal{Z}(\text{Vec}_A^\omega) \). Let \( \mathcal{T} = \text{Rep}(A/\langle t \rangle) \subset \mathcal{E} \). Then \( \mathcal{T}' \) contains pointed fusion subcategories \( \mathcal{E} \) and \( \text{Rep}(A) \). So \( \mathcal{T}' \) must be pointed. But this means that the \( t \)-component of the canonical braiding (40) is trivial, so \( \mu_t = 1 \) and \( \text{alt}(\omega)(t, -, -) = 1 \).

An example of a minimal extension of \( \mathcal{E} \) involving the twisted Drinfeld double of a non-Abelian group can be constructed as follows. Let \( I \) be an Ising category. Consider \( \mathcal{Z}(\mathcal{Z}(I)) = \mathcal{I} \boxtimes \mathcal{I} \boxtimes \mathcal{I}^\text{rev} \boxtimes \mathcal{I}^\text{rev} \). Let \( \mathcal{C} \) be the de-equivariantization of the maximal integral subcategory \( \mathcal{Z}(\mathcal{Z}(I))_{\text{int}} \) by its symmetric center (the latter is equivalent to \( \text{Rep}(\mathbb{Z}_2) \)). Then \( \mathcal{C} \) is a minimal extension of \( \mathcal{C}_{pt} \cong \text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^f) \). So \( \mathcal{C} \) does not contain any pointed Lagrangian Tannakian subcategories and so is not equivalent to the center of a pointed fusion category.

4.6. **General minimal extensions.**

**Proposition 4.17.** (a) Let \( \mathcal{E} \) be a Tannakian or non-split super-Tannakian category. Then all minimal extensions of \( \mathcal{E} \) are integral, i.e., \( M\text{ext}(\mathcal{E}) = M\text{ext}_{\text{int}}(\mathcal{E}) \).

(b) Let \( \mathcal{E} = \text{Rep}(A, t) \) be a split super-Tannakian category. Then
\[ M\text{ext}(\mathcal{E})/M\text{ext}_{\text{int}}(\mathcal{E}) \cong \text{Hom}(A, \mathbb{Z}_2). \]

**Proof.** Part (a) is clear since in this case \( \text{Pic}(\mathcal{E}) = \text{Pic}_{\text{int}}(\mathcal{E}) \) and so all components of the grading (10) are integral.

For the part (b), let \( A = A_0 \times \langle t \rangle \) so that \( \mathcal{E} = \text{Rep}(A_0) \boxtimes \text{s\text{\-vect}} \). By Theorem 3.8,
\[ M\text{ext}(\mathcal{E}) = M\text{ext}(\text{s\text{\-vect}}) \times E_{\text{ctr}}(A_0, \text{s\text{\-vect}}). \]

Combining this with homomorphisms \( M\text{ext}(\text{s\text{\-vect}}) \rightarrow \text{Pic}(\text{s\text{\-vect}}) \) and \( E_{\text{ctr}}(A_0, \text{s\text{\-vect}}) \rightarrow \text{Hom}(A_0, \text{Pic}(\text{s\text{\-vect}})) \) and using that \( \text{Pic}(\text{s\text{\-vect}}) \cong \mathbb{Z}_2 \), we obtain a group homomorphism \( A \rightarrow \mathbb{Z}_2 \). So we have a homomorphism
\[ M\text{ext}_{\text{int}}(\mathcal{E}) \rightarrow \text{Hom}(A, \mathbb{Z}_2). \]
Equivalently, this can also be described as follows. Let $F_C : A \to \text{Pic}_{br}(\mathcal{E})$ be a braided monoidal 2-functor corresponding to a minimal extension $\mathcal{E} \hookrightarrow \mathcal{C}$. The image of this extension under (78) is

\[(79) \quad A \xrightarrow{\pi_0(F_C)} \text{Pic}_{br}(\mathcal{E}) = \text{Pic}(\mathcal{E}) \times A \xrightarrow{p} \text{Pic}(\mathcal{E}) \to \text{Pic}(\mathcal{E})/\text{Pic}_{int}(\mathcal{E}) \cong \mathbb{Z}_2,
\]

where $p$ is the projection on $\text{Pic}(\mathcal{E})$. Its kernel consists of integral extensions, since $\text{Pic}_{int}(s\text{Vect}) \cong \{1\}$. It remains to check that (78) is surjective. For this, it suffices to check that any homomorphism $\phi : A_0 \to \text{Pic}(s\text{Vect}) \cong \mathbb{Z}_2$ gives rise to a central $A_0$-extension of $s\text{Vect}$. Note that any Ising category is a central $\mathbb{Z}_2$-extension of $s\text{Vect}$ and so gives a monoidal 2-functor $\mathbb{Z}_2 \to \text{Pic}(s\text{Vect})$. Composing this with $\phi$, we get a monoidal 2-functor $A_0 \to \text{Pic}(s\text{Vect})$ and, hence, a central $A_0$-extension $s\text{Vect} \hookrightarrow D$. As explained in Section 3.5, this gives rise to a minimal extension of $\mathcal{E}$ by taking the center of $D$. □

**Theorem 4.18.** Let $\mathcal{E} = \text{Rep}(A, t)$ be a super-Tannakian category. The filtration (49) of $\text{Mext}(\mathcal{E})$ has factors

\[\text{Mext}_{\text{triv}}(\mathcal{E}) \cong \text{Coker} \left( H^1(A, \hat{A}) \xrightarrow{\epsilon^t} H^3_{ab}(A, \mathbb{K}^\times) \right),
\]

\[\text{Mext}_{\text{pt}}(\mathcal{E})/\text{Mext}_{\text{triv}}(\mathcal{E}) \cong \text{Ker} \left( H^2_{ab}(A, \hat{A}) \xrightarrow{\theta^t} \text{Hom}(A_2/\langle t \rangle, \mathbb{K}^\times) \right),
\]

\[\text{Mext}_{\text{int}}(\mathcal{E})/\text{Mext}_{\text{pt}}(\mathcal{E}) \cong \text{Hom}(\wedge^3 A, \mathbb{K}^\times),
\]

\[\text{Mext}(\mathcal{E})/\text{Mext}_{\text{int}}(\mathcal{E}) \cong \begin{cases} \text{Hom}(A, \mathbb{Z}_2) & \text{if } \mathcal{E} \text{ is split,} \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** This follows from Propositions 4.7, 4.9, 4.14 and 4.17. □

**Remark 4.19.** For $t = 1$, i.e., when $\mathcal{E}$ is Tannakian, the factors in Theorem 4.18 were computed in [28] and [8].

5. Examples

Recall that when $t$ is a unique up to an automorphism central element of order 2 of a group $G$, we use notation $\text{Rep}(G^t)$ instead of $\text{Rep}(G, t)$.

5.1. $\text{Mext}(\text{Rep}(\mathbb{Z}^n_2))$.

**Proposition 5.1.** Let $A = \mathbb{Z}_m \times \mathbb{Z}_n$ and let $t \in A$ be such that $\langle t \rangle$ is not a direct factor. Then any minimal non-degenerate extension of $\text{Rep}(A, t)$ is pointed.

**Proof.** In this case, $\wedge^3 A = 0$, so the result follows from Proposition 4.14. □

For $n \geq 2$ let $\mathcal{E}_n = \text{Rep}(\mathbb{Z}^n_2)$. By Proposition 5.1, $\text{Mext}(\mathcal{E}_n) = \text{Mext}_{\text{pt}}(\mathcal{E}_n)$. It follows from Propositions 4.7 and 4.9 that there is an exact sequence

\[(80) \quad 0 \to H^1(\mathbb{Z}^n_2, \mathbb{Z}^n_2) \to H^3_{ab}(\mathbb{Z}^n_2, \mathbb{K}^\times) \to \text{Mext}(\mathcal{E}_n) \xrightarrow{\lambda} H^2_{ab}(\mathbb{Z}^n_2, \mathbb{Z}^n_2) \to 0,
\]
where \( \lambda \) assigns to the minimal extension \( \text{Rep}(\mathcal{E}_n) \hookrightarrow \mathcal{C} \) the cohomology class of the extension \( 0 \to \mathbb{Z}_{2^n} \to \text{Inv}(\mathcal{C}) \to \mathbb{Z}_{2^n} \to 0 \). Since \( H^3_{ab}(\mathbb{Z}_{2^n}, \mathbb{Z}) \cong \mathbb{Z}_{2^{n+1}} \) and \( H^3_{ab}(\mathbb{Z}_{2^n}, \mathbb{Z}_{2^n}) = \mathbb{Z}_{2^n} \), the above sequence becomes

\[
0 \to \mathbb{Z}_2 \xrightarrow{\alpha} \text{Mext}(\mathcal{E}_n) \xrightarrow{\lambda} \mathbb{Z}_{2^n} \to 0.
\]

Minimal non-degenerate extensions of \( \mathcal{E}_n \) can be explicitly described as follows. Recall \([16, 23]\) that a pointed braided fusion category \( \mathcal{C} \) with the group \( \text{Inv}(\mathcal{C}) = A \) of isomorphism classes of invertible objects is determined up to an equivalence by the quadratic form \( q : A \to \mathbb{R}^\times \), where \( q(X) = c_{X,Y} \). In this case, we denote \( \mathcal{C} = \mathcal{C}(A, q) \).

For each \( m \geq 0 \) and a primitive \( 2^{m+1} \)th root of unity \( \xi \) we define a non-degenerate quadratic form

\[
q_\xi : \mathbb{Z}_{2^m} \to \mathbb{R}^\times, \quad q_\xi(j) = \xi^{2^m} \quad \text{for all} \quad j \in \mathbb{Z}_{2^m}.
\]

The non-degenerate pointed braided fusion category \( \mathcal{C}(\mathbb{Z}_{2^m}, q_\xi) \) is a minimal extension of \( \mathcal{E}_n \).

For each \( k = 0, 1, \ldots, n \) and a \( 2^{n-k} \)th root of unity \( \zeta \) let

\[
\mathcal{M}_{k,\zeta} \cong \mathcal{C}(\mathbb{Z}_{2^k}, q_{-\zeta^{2^{(n-k)}}}) \boxtimes \mathcal{C}(\mathbb{Z}_{2^{2n-k}}, q_\zeta)
\]

Again, this is a pointed non-degenerate braided fusion category.

**Proposition 5.2.** For all \( k \) and \( \zeta \) as above, there is a non-degenerate minimal extension \( \mathcal{E}_n \hookrightarrow \mathcal{M}_{k,\zeta} \) given by the group homomorphism

\[
\iota_k : \mathbb{Z}_{2^n} \to \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{2n-k}} : j \mapsto (j, j^{2^n-k}).
\]

**Proof.** Clearly, \( \iota_k \) is an injective group homomorphism. Let \( q_{k,\zeta} : \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{2n-k}} \to \mathbb{R}^\times \) denote the quadratic form corresponding to \( \mathcal{M}_{k,\zeta} \). We have

\[
q_{k,\zeta}(\iota_k(1)) = -\zeta^{2^{(n-k)}} \cdot \zeta^{2^{(n-k)}} = -1.
\]

Viewing \( \iota_k \) as a homomorphism of metric groups, where \( \mathbb{Z}_{2^n} \) is equipped with a quadratic character \( q(l) = (-1)^l \), we obtain a braided tensor embedding \( \mathcal{E}_n \hookrightarrow \mathcal{M}_{k,\zeta} \), i.e., a minimal extension of \( \mathcal{E}_n \).

**Remark 5.3.**

1. By definition, \( \mathcal{M}_{0,\zeta} = \mathcal{C}(\mathbb{Z}_{2^{2n}}, q_\zeta) \) is a cyclic minimal extension of \( \mathcal{E}_n \).

2. For all \( k \) and \( \zeta \), the largest order of the root of unity that occurs as a value of \( q_{k,\zeta} \) is \( 2^{2n-k+1} \).

3. For all \( k = 0, \ldots, n-1 \), the square of \( \mathcal{M}_{k,\zeta} \) in \( \text{Mext}(\mathcal{E}_n) \) is \( \mathcal{M}_{k+1,\zeta^2} \) (this is a straightforward computation using the definition of the product of minimal extensions).

**Proposition 5.4.** \( \text{Mext}(\text{Rep}(\mathbb{Z}_{2^n}^f)) \cong \mathbb{Z}_{2^{n+1}} \) with any \( \mathcal{C}(\mathbb{Z}_{2^{2n}}, q_\zeta) \) as a generator.
Lemma. We need to show that the exact sequence (81) does not split. Observe that, for any primitive \(2^n+1\)th root of unity \(\zeta\), the minimal extension

\[E_n \hookrightarrow M_{n,\zeta} = \mathcal{C}(\mathbb{Z}_{2^n}, q_{-\zeta^{-1}}) \boxtimes \mathcal{C}(\mathbb{Z}_{2^n}, q_{\zeta}),\]

where \(E_n\) is embedded diagonally, is the generator of \(\text{Ker}(\lambda) \cong \mathbb{Z}_2\) in (81) (in particular, its class in \(\text{Mext}(E_n)\) does not depend on the choice of \(\zeta\)). Thus, it suffices to check that this minimal extension has a square root. But this follows from Remark 5.3(3). \(\square\)

Corollary 5.5. The kernel of the homomorphism \(\text{Mext}(\mathbb{Rep}(\mathbb{Z}_2^f)) \to \text{Mext}(s\text{Vect})\) is isomorphic to \(\mathbb{Z}_{2^n-2}\).

Remark 5.6. The minimal extensions of \(\mathbb{Rep}(\mathbb{Z}_4^f)\) and \(\mathbb{Rep}(\mathbb{Z}_8^f)\) were listed in [27, Tables XIV and XV]. Our description of their groups of minimal extensions is consistent with these tables and with the results of [1] and [31].

For \(n = 2\) Proposition 5.4 says that \(\text{Mext}(\mathbb{Rep}(\mathbb{Z}_4^f)) \cong \mathbb{Z}_8\). This disagrees with [30, Example 7.17], where it is claimed that \(|\text{Mext}(\mathbb{Rep}(\mathbb{Z}_4^f))| = 32\). Our explanation of this discrepancy is that [30] counts equivalence classes of \(\mathbb{Z}_2\)-crossed braided extensions of \(\mathbb{Z}(\mathbb{Rep}(s\text{Vect}))\) whose equivariantization is a minimal non-degenerate extension of \(\mathbb{Rep}(\mathbb{Z}_4^f)\). However, all such \(\mathbb{Z}_2\)-crossed braided extensions lead to the same element of \(\text{Mext}(\mathbb{Rep}(\mathbb{Z}_4^f))\), namely, to the identity extension \(\mathbb{Rep}(\mathbb{Z}_4^f) \hookrightarrow \mathbb{Z}(\mathbb{Rep}(\mathbb{Z}_4^f))\).

5.2. \(\text{Mext}(\mathbb{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_4^f))\). Let \(\mathcal{E} = \mathbb{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_4^f)\). Using Theorem 4.18 we see that the factors of the canonical filtration (49) of \(\text{Mext}(\mathcal{E})\) are

\[
\begin{align*}
\text{Mext}_{\text{triv}}(\mathcal{E}) &\cong \mathbb{Z}_4 \times \mathbb{Z}_2, \\
\text{Mext}_{\text{pt}}(\mathcal{E})/\text{Mext}_{\text{triv}}(\mathcal{E}) &\cong \mathbb{Z}_2^3, \\
\text{Mext}_{\text{int}}(\mathcal{E})/\text{Mext}_{\text{pt}}(\mathcal{E}) &\cong 0, \\
\text{Mext}(\mathcal{E})/\text{Mext}_{\text{int}}(\mathcal{E}) &\cong \mathbb{Z}_2^2.
\end{align*}
\]

Therefore, \(|\text{Mext}(\mathcal{E})| = 128\). The canonical homomorphism

\[w : \text{Mext}(\mathcal{E}) \rightarrow \text{Mext}(s\text{Vect}) \cong \mathbb{Z}_{16},\]

defined in (13), is split surjective and, hence,

\[\text{Mext}(\mathcal{E}) \cong \mathbb{Z}_{16} \times \text{Ker}(w),\]

where \(|\text{Ker}(w)| = 8\).

Example 5.7. The following categories are non-degenerate minimal extensions of \(\mathcal{E} = s\text{Vect} \boxtimes s\text{Vect}\) lying in \(\text{Ker}(w)\):

\[
\begin{align*}
\mathcal{M}_1(i) &\cong \mathcal{Z}(\mathcal{C}(\mathbb{Z}_2, q_i) \boxtimes \mathcal{C}(\mathbb{Z}_2, q_i)), \\
\mathcal{M}_2(\xi) &\cong \mathcal{Z}(\mathcal{C}(\mathbb{Z}_4, q_i)), \\
\mathcal{M}_3(\mathcal{I}) &\cong \mathcal{Z}((\mathcal{I}),
\end{align*}
\]
where $i$ and $\xi$ are primitive 4th and 8th roots of unity, respectively, and $\mathcal{I}$ is an Ising braided fusion category. We use the notation introduced in (82). In each of these three cases, there is a unique embedding of $\mathcal{E}$ (note that $\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ for any non-degenerate braided fusion category $\mathcal{C}$).

We have the following equalities in $\text{Mext}(\mathcal{E})$:

\begin{align}
(87) \quad M_1(i) &= M_1(-i) \quad \text{and} \quad M_2(\xi) = M_2(\xi') \quad \text{if and only if} \quad \xi' = \pm \xi.
\end{align}

**Lemma 5.8.** $\text{Ker}(w) \cap \text{Mext}_{\text{pt}}(\mathcal{E}) \cong \mathbb{Z}_4$ with a generator $M_2(\xi)$.

**Proof.** Since $\text{Ker}(w)$ contains a non-integral extension $M_3(\mathcal{I})$, we see that $|\text{Ker}(w) \cap \text{Mext}_{\text{pt}}(\mathcal{E})| = 4$.

The identity minimal extension $\mathcal{Z}(\mathcal{E})$ (respectively, $M_1(i)$ and $M_2(\xi)$) is a pointed minimal extension of $\mathcal{E}$ characterized by the property that the largest order of the root of unity that occurs as a value of the corresponding quadratic form is 2 (respectively, 4 and 8). Since for the square of $M_2(\xi)$ in $\text{Mext}(\mathcal{E})$ the quadratic form has values that are 4th roots of unity, it follows that $M_1(i)$ is a square in $\text{Ker}(w)$ and the statement follows. $\square$

**Lemma 5.9.** $\text{Ker}(w) \cong \mathbb{Z}_8$ with a generator $M_3(\mathcal{I})$ for any Ising category $\mathcal{I}$.

**Proof.** For a braided Ising fusion category $\mathcal{I}$ the values of the canonical twist on its simple objects are 1, $-1$, and a primitive 16th root of unity $\zeta$, see [13, Appendix B]. Therefore, the values of the twist on the simple objects of the integral part $(\mathcal{I} \boxtimes \mathcal{I})_{\text{pt}}$ of $\mathcal{I} \boxtimes \mathcal{I}$ are 1, 1, $-1$, $-1$, and $\zeta^2$. By definition of the tensor product of minimal extensions, the tensor square of $\mathcal{Z}(\mathcal{I})$ in $\text{Mext}(\mathcal{E})$ is the de-equivariantization of $(\mathcal{I} \boxtimes \mathcal{I})_{\text{pt}} \boxtimes (\mathcal{I}^{\text{rev}} \boxtimes \mathcal{I}^{\text{rev}})_{\text{pt}}$ (viewed as a subcategory of $\mathcal{Z}(\mathcal{I})^{\boxtimes 2}$) by the diagonal Tannakian subcategory of $\mathcal{E} \boxtimes \mathcal{E}$. The result contains simple objects with the twist (quadratic form) values being primitive 8th roots of unity. This means that

\[ \mathcal{Z}(\mathcal{I})^{\boxtimes 2} = \mathcal{Z}(\mathcal{C}(\mathbb{Z}_4, q^{\zeta^2})) = M(\zeta^2), \]

so that $M_3(\mathcal{I})$ has order 8 in $\text{Mext}(\mathcal{E})$ by Lemma 5.8. $\square$

**Corollary 5.10.** $\text{Mext}(\mathcal{R}ep(\mathbb{Z}_2 \times \mathbb{Z}_2^f)) \cong \mathbb{Z}_{16} \times \mathbb{Z}_8$. The extensions $\mathcal{Z}(\mathcal{R}ep(\mathbb{Z}_2)) \boxtimes \mathcal{I}_1$ and $\mathcal{Z}(\mathcal{I}_2)$, where $\mathcal{I}_1, \mathcal{I}_2$ are any Ising braided fusion categories, can be taken as generators of the cyclic factors.

The central charge homomorphism [13] is identified with the projection on the first factor.

**Proof.** This follows from [85] and Lemma 5.9. $\square$

**Corollary 5.11.** $\text{Mext}_{\text{pt}}(\mathcal{R}ep(\mathbb{Z}_2 \times \mathbb{Z}_2^f)) \cong \mathbb{Z}_8 \times \mathbb{Z}_4$. 


Proof. We have seen that $|\text{Mext}_{pt}(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f))| = 32$. Since the minimal extensions that project on Ising categories in $\text{Mext}(s\text{Vect})$ are non-integral, we conclude that $\text{Mext}_{pt}(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f))$ is a subgroup of $\mathbb{Z}_8 \times \mathbb{Z}_8$, which implies the result. \hfill \Box

Remark 5.12. All 128 minimal extensions of $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$ were listed in [27, Tables XVI-XIX]. Our contribution is the computation of the group structure of $\text{Mext}(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2^f))$. The filtration of this group and its integral part found in this Section are consistent with these tables. A description of this group was also given recently in [1, Section V.B and Appendix I] and in [2, Table V], where the eight elements of the group $\text{Ker}(w)$ were explicitly listed.

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