Transfer Operators and Topological Field Theory

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Abstract

Here, the transfer operator (TO) formalism of the dynamical systems (DS) theory is reformulated in terms of the cohomological (or Witten-type) topological field theory (TFT) of DS. It turns out that the so-called generalized TO (GTO) of the DS theory is the finite-time Fokker-Planck evolution operator of the TFT. As a result comes the supersymmetric trivialization of the so-called sharp trace and sharp determinant of the GTO, with the former being the Witten index of the TFT. On the DS theory side, the Witten index is the stochastically averaged Lefschetz index so that both indexes always equal the Euler characteristic of the phase space for all flow fields and noises. Moreover, the enabled possibility to apply the spectral theorems of the DS theory to the TFT Fokker-Planck Hamiltonian allows to extend the previously proposed picture of the topological supersymmetry breaking on situations with negative ground state's attenuation rate. The later signifies the exponential growth of the number of periodic solutions in the large time limit, which is a unique feature of chaotic behavior proving that the spontaneous breakdown of topological supersymmetry is indeed the mathematical essence of the concept of deterministic chaos and its stochastic generalization often called complex dynamics. In addition, the previous low-temperature classification of DS’s (markovian/intermittent/chaotic) is complemented by the discussion of the high-temperature regime where the intermittent and chaotic phases first merge and then disappear.

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I. INTRODUCTION

In their path-integral representation, all (stochastic) continuous time dynamical systems (DS) are the Witten-type or cohomological topological field theories (TFT) [1, 2, 5, 6, 10, 13, 28, 29] as it was recently demonstrated [19–21] within the framework of the Parisi-Sourlas stochastic quantization procedure. [22] The so emerged TFT of DS is essentially an approximation-free systematic approach to (stochastic, partial, nonlinear) differential equations. Its applicability goes beyond the world of DS’s. For example, it has been very recently adopted for quantum spin systems. [23] The primary importance of the TFT of DS, however, is from the standpoint of the theory of DS’s.

Indeed, the history of deterministic chaos is over a century old and can be traced back to the work by Poincaré on the three-body celestial dynamics (see, e.g., Ref. [18] and Refs. therein). The most notorious property of deterministic chaos, in turn, is the so-called butterfly effect, i.e., the highest sensitivity to initial conditions that was established numerically over fifty years ago. [18] At this, the classical DS theory did not manage to provide yet a mathematically rigorous definition of chaos that could lead to the explanation of the butterfly effect.

A common way around this theoretical difficulty is to believe that the butterfly effect is an intrinsic part of the definition of chaos. Even if existed, such definition would be in the same category of definitions as the following one: a quantum Hall system is the one that ... exhibits the quantum Hall effect. The only fundamental difference here is that the quantum Hall effect was found by a natural experiment while the butterfly effect was established by a numerical experiment. Clearly, even though such definitions are correct they do not have much of the explanatory or predictive powers.

Furthermore, a definition of chaos through the butterfly effect, which does not admit stochastic generalization, [31] is not very practical from a physicist’s viewpoint because all natural DS’s are never completely isolated from their environments and thus are always stochastic.

The TFT of DS provides a rigorous definition of chaos that both explains the butterfly effect and works just as well for stochastic DS’s. Within the TFT of DS, all DS’s possess topological supersymmetry and chaotic behavior is the result of its spontaneous breakdown that according to the Goldstone theorem must always be accompanied by the emergence
of "chaotic" long-range correlations. In the deterministic limit, these correlations take the form of the butterfly effect. In Nature, they reveal themselves through such well-established phenomena as 1/f noise, algebraic power-spectrum in spatiotemporal chaotic systems such as turbulent water (Kolmogorov law), algebraic statistics of "events" in various slip-stop-type behaviors such as sandpiles, neurodynamical avalanches, earthquakes, solar flares, (spin) glasses, creeps, Barkhausen-like effects such as crumpling paper and many, many others.

In fact, that the onset of chaos is a phase transition has been well known before. This understanding was actually the basis for the concept of the "universality in chaos". Furthermore, it has also been known that transition into chaos is topological in nature. At the transition, the topologically well defined attractors of a (deterministic) DS are substituted by topologically ill-defined fractal attractors that in case of three-dimensional phase spaces are made out of unstable periodic orbits that have nontrivial mutual linking numbers. This picture forms the basis for the concept of "topology of chaos". In the light of the above, the topological supersymmetry breaking is actually a natural, long overdue mathematical explanation of the phenomenon of dynamical chaos and its peculiar properties.

The TFT of DS may also be interesting from the field theoretic point of view. Firstly, through the TFT of DS, the Witten-type TFT’s find their physical realizations. Secondary, to the best of our knowledge, the phenomenon of the spontaneous breakdown of a supersymmetry has never been rigorously established (even theoretically) in any of the high energy physics models, which is believed to have anything to do with reality. Actually, this was the primary reason for the introduction of the concept of soft and/or explicit supersymmetry breaking (see Ref. and Refs. therein). The TFTs of (chaotic) DS’s, on the other hand, are the approximation-free supersymmetric field theoretic descriptions of physical DS’s in which a supersymmetry is spontaneously broken with certainty. These physical DS’s are easily realizable under laboratory conditions, e.g., turbulent water, chaotic chemical reactions, glasses, chaotic behavior in nanometer-scale devices etc. In other words, the TFT of DS provides numerous, easily available experimental realizations of the spontaneous breakdown of a supersymmetry. This may help building a better intuition on its counterparts from the high-energy physics models.

Unlike the conventional Hermitian quantum models, however, the TFT’s of DS are non-Hermitian. Fortunately, non-Hermitian models have been a subject of active scientific investigation for over a decade now. At this, most of the efforts were devoted to the
so-called $\mathcal{P}\mathcal{T}$-symmetric non-Hermitian models that have real spectrum. A real spectrum makes a non-Hermitian model a close relative with the conventional Hermitian quantum models. This looks especially comforting when it comes to the physical interpretation of various constituents of the theory. The TFT of DS, on the other hand, are pseudo-Hermitian, i.e., their spectrum contains also complex conjugate pairs of eigenvalues. At this, there should be no concern about relevance of the non-real spectrum to reality - complex conjugate eigenvalues of transfer operators has been known in the DS theory for a while now as Ruelle-Pollicott resonances. Thus, the TFT of DS is an "interpretable" pseudo-Hermitian supersymmetric quantum theory with its applicability ranging from social sciences to Astrophysics. This makes the TFT of DS especially interesting in the light of the modern search for the physical realizations of pseudo-Hermitian models (see, e.g., Ref.[14] and Refs. therein).

From a more general point of view, the DS theory and the TFT are naturally synergetic within the TFT of DS. This synergy has a potential of a fruitful cross-fertilization between concepts and developments of these two major theoretical constructions. It is in this line of thinking that in this paper the transfer operator (TO) formalism of the DS theory is given a TFT representation. This approach can be looked upon as a bottom-up derivation of the TFT of DS with the starting point being the DS theory concept of the TO rather then the previously used Parisi-Sourlas stochastic quantization procedure, which is a field theoretic construction rather than an intrinsic part of the modern DS theory. This approach allows to establish a few novel links between the DS theory and the TFT, and leads to several new findings.

The structure of the paper is as follows. In Sec.II the formalism of the Ruelle-Frobenuis-Perron TO is introduced. In Sec.III the TFT’s pathintegral representation of the weighted traces of the TO is derived. In Sec.IV the operator representation of the theory is briefly discussed. In particular, it is shown that the Witten index equals the Euler character of the phase space for any flow fields and temperatures. In Sec.VI it is demonstrated that the generalized (stochastic) TO (GTO) formalism of the DS theory is intrinsically incorporated in the TFT of DS. In Sec. VI, the supersymmetric trivialization of the sharp traces and determinants of the GTO is discussed. The phenomenon of the spontaneous breakdown of the topological supersymmetry is revisited and the situation is analyzed when the ground state’s attenuation rate is negative. It is shown that in such situations the stochastically averaged
number of periodic solutions grows exponentially in the large time limit. This is a direct
indication on that the spontaneous breakdown of topological supersymmetry is the math-
ematical essence of (stochastic) chaotic dynamical behavior. We also discuss briefly that the
topological supersymmetry is unbroken in the high temperature limit. This allows to extend
the previously proposed three-phase phase diagram (Markovian/Intermittent/Chaotic) onto
the high temperature regime. Sec.VII concludes the paper.

II. TRANSFER OPERATOR

This paper is mostly devoted to the continuous-time DS’s that can also be identified as
physical DS’s since for physical objects time is always continuous. Thus, we begin with
the introduction of a continuous-time deterministic DS defined by the ordinary differential
equation:

$$\partial_t x(t) = F(x(t)).$$  \hspace{1cm} (1)

Here, $x \in X$ are the variables from the $D$-dimensional phase space of the DS, $X$, and
$F(x) \in T_x X$, is the flow vector field from the tangent space of $X$. Eq.(1) defines a one-
parameter group of diffeomorphisms of $X$, $M_t(x) : X \times \mathbb{R} \to X$, such that

$$\partial_t M_t(x) = F(M_t(x)),$$  \hspace{1cm} (2)

and $M_0 = \text{Id}_X$. The Ruelle-Frobenius-Perron TO for the evolution of duration $t$ has the
following form:

$$ (L^t \rho)(x) = \int L^t(x, x') \rho(x') dx' = \sum_{x', M_t(x') = x} \frac{\rho(x')}{||M_t(x')||}, $$  \hspace{1cm} (3a)

$$ L^t(x, x') = \delta^D(x - M_t(x')). $$  \hspace{1cm} (3b)

where notation $|| \cdot ||$ stands for the absolute value of a determinant, $| \cdot |$, and matrix

$$ M_t^i_j(x) = \partial M^i_t(x)/\partial x^j, $$  \hspace{1cm} (4)

is the coordinate representation of the tangent map induced by $M_t$:

$$ \dot{M}_t(x) : T_x X \to T_{M_t(x)} X. $$  \hspace{1cm} (5)
The flow is a group so that $M_t$ is invertible (for finite $t$), $M_t^{-1} = M_{-t}$, and any image, $x$, has only one preimage, $x' = M_{-t}(x)$. In this situation the summation sign in Eq.(3b) is not really necessary. We, however, keep the notation for the summation over the preimages to emphasize that the TFT of DS can be generalized to DS’s with step-like temporal evolution that in general are semigroups and each image can have more than one preimage. This situation is discussed briefly in Sec. IV C.

Matrix (4) satisfies the following equation:

$$\partial_t \hat{M}_t(x) = \hat{F}(M_t(x))\hat{M}_t(x),$$

(6)

$$\hat{F}(x) \equiv F^i_j(x) = \partial F^i(x)/\partial x^j,$$

(7)

as it follows from Eq.(2). The formal solution of this equation with the initial condition $\hat{M}_0 = \hat{1}$ is

$$\hat{M}_t(x) = e^{\int_0^t \hat{F}(M_{t'}(x))dt'},$$

(8)

where columns denote chronological ordering. Also

$$|\hat{M}_t(x)| = e^{\int_0^t \text{Tr}\hat{F}(M_{t'}(x))dt'} > 0,$$

(9)

i.e., the flow preserves the orientation on $X$. The group structure of the flow is seen in

$$M_{t_1}(M_{t_2}(x)) = M_{t_1+t_2}(x),$$

(10a)

$$\mathcal{L}_{t_1}\mathcal{L}_{t_2}\rho = \mathcal{L}_{t_1+t_2}\rho,$$

(10b)

$$\hat{M}_{t_1}(M_{t_2}(x))\hat{M}_{t_2}(x) = \hat{M}_{t_1+t_2}(x).$$

(10c)

Let us define the weighted traces of the TO as

$$\text{Tr}\mathcal{L}_t\phi = \int \mathcal{L}_t(x,x)|\hat{M}_t(x)|\phi(x)d^Dx = \int \mathcal{L}_{-t}(x,x)\phi(x)d^Dx$$

$$= \sum_{x \in \mathbb{R}xM_t} \frac{\phi(x)}{\|1 - \hat{M}_{-t}(x)\|}.$$  

(11)

Here $\hat{M}_{-t}(x) = \hat{M}_t^{-1}(x)$, Eq.(9) is utilized, and $\phi(x)$ is some function on $X$ called weight function. Eq.(11) has a seeming appearance of time-reversed evolution that will be discussed in Sec. VII B. The sum notation in Eq.(11) suggests the assumption of isolated fixed points of $M_t$. The pathintegral version of the theory, however, deals equally well with more general
situations. Therefore, this assumption is not necessary even though we will sloppily keep using the sum notation.

The following two weight functions are of specific interest for us:

\[ w(x) = |\hat{1} - \hat{M}_{-t}(x)| = \sum_{k=1}^{D} (-1)^k m_k(x), \]  
\[ z(x) = |\hat{1} + \hat{M}_{-t}(x)| = \sum_{k=1}^{D} m_k(x), \]  

where

\[ m_k(x) = \sum_{i_1 < \ldots < i_k} \begin{vmatrix} M_{-t_{i_1}}(x) & \cdots & M_{-t_{i_1}}(x) \\ \vdots & \ddots & \vdots \\ M_{-t_{i_k}}(x) & \cdots & M_{-t_{i_k}}(x) \end{vmatrix}, \]  

and we used the characteristic polynomial formula

\[ |\hat{1} + \lambda \hat{M}_{-t}(x)| = \sum_{k=0}^{D} \lambda^k m_k(x). \]

The weighted trace with weight function (12a) is of topological nature. It is the matter of Lefschetz theorem

\[ \text{Tr} \mathcal{L}_t w = \sum_{x \in \text{fix} M_t} \text{sign} |\hat{1} - \hat{M}_{-t}(x)| = \sum_{k=0}^{D} (-1)^k \text{Tr}_{H^k} M_{-t}^*, \]  

where \( \text{Tr}_{H^k} \) denotes the trace over the \( k \)-th-degree cohomology of \( X \) and \( M_{-t}^* \) is the pullback induced by \( M_{-t} \). In the following, the topological nature of Eq.(13) will be reassured by its identification with the Witten index, \( W \), of the corresponding TFT:

\[ W_{cl} = \text{Tr} \mathcal{L}_t w. \]

Here and in the following the subscript 'cl' emphasizes the deterministic or "classical" nature of an object as opposed to the stochastic objects with no subscripts.

The identification of Eq.(13) with the Witten index immediately suggests that it is independent of \( t \) and thus evaluates to the Euler characteristic of \( X \) as can be seen in the \( t \to 0 \) limit where \( M_{-t} \to \text{Id}_X \).

The meaning of the other weight function is best seen in the opposite limit of the infinitely long temporal evolution, \( t \to \infty \). Eq.(11) can be rewritten as

\[ \text{Tr} \mathcal{L}_t z = \sum_{x \in \text{fix} M_t} \frac{|\hat{M}_t(x) + \hat{1}|}{||\hat{M}_t(x) - \hat{1}||}. \]
Matrix $\hat{M}_t(x)$ is real and its eigenvalues can be represented as $e^{\lambda_a(t)t}$, $a = 1, \ldots, D$, where $\lambda(t)$'s are either real or come in complex conjugate pairs. By using the argumentation that stands behind the introduction of global Lyapunov exponents, one assumes that in a wide enough class of DS's there are well defined limits $\lim_{t \to \infty} \lambda_a(t) = \lambda_a$. If none of $\Re \lambda_a$ is exactly zero,

$$\lim_{t \to \infty} \|\hat{M}_t(x) + \hat{1}\| = \lim_{t \to \infty} \prod_{i=a}^D \frac{e^{\lambda_a t} + 1}{|e^{\lambda_a t} - 1|} = 1.$$  

Therefore, under the above assumptions and in the long-time limit Eq.(15) equals the number of periodic points of $M_t$ and consequently can be interpreted as the physical partition function of the DS:

$$Z_{cl} = \text{Tr} \mathcal{L}_t z.$$  

### III. PATHINTEGRAL REPRESENTATION

In this section, the transfer operator formalism is reformulated in terms of pathintegrals. The fields involved in this formulation are schematically presented in Fig.1

**A. Bosonic fields**

With the help of Eq.(10b), Eq.(11) can be rewritten as

$$\text{Tr} \mathcal{L}_t \phi = \int \phi(x(0)) \prod_{p=0}^{N-1} \mathcal{L}_{-\Delta t}(x(t_p), x(t_{p+1})) d^D x(t_p).$$

Here the time domain $t \in (0, t)$ is split into $N$ segments $(t_{p-1}, t_p)$, with $t_p = pt/N$ being equally separated time slices, $\Delta t = t/N$, $x(t_p)$ are the intermediate variables at the corresponding times, and the periodic boundary conditions are assumed, $x(t_N) \equiv x(t) = x(0)$.

Now we introduce an additional field $B(t_p) \in T_{x(t_p)}^* X$ called Lagrange multiplier that belongs to the cotangent space of $X$, in order to exponentiate the $\delta$-functions:

$$\mathcal{L}_{-\Delta t}(x(t_p), x(t_{p+1})) = \delta^D (x(t_p) - M_{-\Delta t}(x(t_{p+1})))$$

$$= \int e^{iB(t_p)(M_{-\Delta t}(x(t_{p+1})) - x(t_p))} d^D B(t_p)/(2\pi)^D.$$  

Taking now the continuous-time limit ($N \to \infty$), one finds that

$$\text{Tr} \mathcal{L}_t \phi = \int \int \phi(x(0)) e^{S_B} DxD\mathcal{B},$$  

(18a)
where the path integration is over the closed trajectories (periodic boundary conditions), the action is:

\[
S_B(x, B) = \lim_{N \to \infty} \int \sum_{p=0}^{N-1} B_i(t_p) \times (x^i(t_p) - M^i_{-\Delta t}(x(t_{p-1})))
\]

\[
= i \int dt B_i(t) \left( \partial_t x^i(t) - F^i(x(t)) \right), \quad (18b)
\]

and the differential of the pathintegral is

\[
DxDB = \lim_{N \to \infty} \prod_{p=0}^{N-1} d^D x(t_p) d^D B(t_p) / (2\pi)^D. \quad (18c)
\]

**B. Anticommuting fields**

Expression (18a) has the form of a field theoretic expectation value of some temporarily local observable. This, however, is not quite correct because the weight functions of interest in Eq.(12) are temporarily nonlocal function(al)s of \(x(t)\). A consistent pathintegral representation of these weight functions can be achieved with the help of anticommuting fields called Fadeev-Popov ghosts. These anticommuting fields obey Berezin rules of integration that, in particular, suggest the following useful properties:\[8\]

\[
\delta^D \left( \hat{A} \chi \right) = \int e^{\bar{\chi} \hat{A} \chi} d^D \bar{\chi}, \quad (19a)
\]

\[
\int \delta^D \left( \hat{A} \chi \right) d^D \chi = |\hat{A}|. \quad (19b)
\]

With the help of Eqs.(19b), the weight function from Eq.(12a) can be given as

\[
w(x(0)) = |\hat{1} - \hat{M}_{-t}(x(0))| = \int_{PBC} \delta^D \left( \chi(0) - \hat{M}_{-t}(x(0)) \chi(t) \right) d^D \chi(0), \quad (20)
\]

where \(\chi(0), \chi(t) \in T_{x(t)}X\) are the anticommuting ghosts at the corresponding times and the subscript \(PBC\) signifies periodic boundary conditions for the ghosts: \(\chi(t) \equiv \chi(0)\). Using Eq.(10c) one can now utilize again the time slices’ picture of the previous subsection and bring Eq.(20) to the following form:

\[
w(x(0)) = \int_{PBC} \prod_{p=0}^{N-1} \delta^D \left( \chi(t_p) - \hat{M}_{-\Delta t}(x(t_p)) \chi(t_{p+1}) \right) d^D \chi(t_p). \quad (21)
\]
FIG. 1: The fields of the pathintegral representation of the theory. The entire interval of temporal evolution is sliced into \( N \to \infty \) equal segments. At each time slice, \( t_p, 0 \leq p \leq N \), we have a field, \( x(t_p) \in X(t_p) \), from a copy of the phase space, \( X(t_p) \), and an anticommuting ghost field, \( \chi(t_p) \in T_{x(t_p)}X(t_p) \), from the tangent space of \( X(t_p) \). In between the time-slices, there are the Langrange multiplier and the antighost, \( B(t_p), \bar{\chi}(t_p) \in T_{x(t_p)}^*X(t_p) \), both from the cotangent space of \( X(t_p) \) (or \( X(t_{p+1}) \); in the \( t \to \infty \) limit this must make no difference). \( B(t_p) \) and \( \bar{\chi}(t_p) \) are needed for the exponentiation of the \( \delta \)-functions representing the infinitesimal temporal evolution of the bosonic fields and the ghosts. In case of stochastic models, there are also variables representing the noise, \( \xi(t_p) \). Integration of all the fields with the periodic boundary conditions, \( x(t) = x(0), \chi(t) = \chi(0) \), leads to the Witten index. The integration with the antiperiodic boundary conditions for the ghosts, \( x(t) = x(0), \chi(t) = -\chi(0) \), gives the physical partition function. Integrating out only the intermediate fields and leaving the initial, \( x(0), \chi(0) \), and final, \( x(t), \chi(t) \), fields unspecified, gives the finite-time Fokker-Planck evolution operator, i.e., the generalized transfer operator of the DS theory.

Just as in the case of bosonic fields, one can further exponentiate the integrand in the previous expression with the help of Eq. (19a) and by the introduction of yet another anticommuting ghost field from the cotangent space of \( X \), \( \bar{\chi}(t_p) \in T_{x(t_p)}^*X \). In the continuous time limit, \( N \to \infty \), one arrives at

\[
w(x(0)) = \int \int_{PBC} e^{S_F} D\bar{\chi} D\chi,
\]  

(22a)
where
\[
S_F(x, \chi, \bar{\chi}) = i \lim_{N \to \infty} \sum_{p=0}^{N-1} \bar{\chi}_i(t_p) \times \left( \chi^i(t_p) - M_{-\Delta j}(x(t_p)) \chi^j(t_{p+1}) \right)
\]
\[
= -i \int dt \bar{\chi}_i(t) \left( \partial_t \chi^i(t) - \hat{F}_j^i(x(t)) \chi^j(t) \right), \quad (22b)
\]
with \( \hat{F} \) from Eq.(17), and the integration measure being
\[
D\bar{\chi}D\chi = \lim_{N \to \infty} \prod_{p=0}^{N-1} d^D(i\bar{\chi}(t_p))d^D\chi(t_p). \quad (22c)
\]

C. Emergence of TFT

Combining Eqs. (18) and (22), one finds
\[
W_{cl} = \int \int_{BC} e^{S_{cl}(\Phi)} D\Phi, \quad (23)
\]
where \( \Phi = (x, B, \chi, \bar{\chi}) \) is the collection of all the fields, and the total action, \( S_{cl} = S_B + S_F \),
is the sum of the bosonic and fermionic parts from Eqs. (18b) and (22b).

As it can be straightforwardly verified, the action of the theory is \( Q \)-exact, \( i.e. \), it can be represented as
\[
S_{cl}(\Phi) = \{ Q, \Psi_{cl}(\Phi) \}, \quad (24)
\]
where \( Q \) is the topological supersymmetry and/or Betti-Route-Store-Tyutin symmetry
\[
\{ Q, \Psi_{cl}(\Phi) \} = \int dt \left( \chi^i(t) \frac{\delta}{\delta x^i(t)} + B_i(t) \frac{\delta}{\delta \bar{\chi}_i(t)} \right) \Psi_{cl}(\Phi),
\]
acting on the so-called gauge fermion,
\[
\Psi_{cl}(\Phi) = i \int dt \bar{\chi}_i(t) \left( \partial_t x^i(t) - F^i(x(t)) \right). \quad (25)
\]
Models with \( Q \)-exact actions, such as the one in Eq.(23), are TFTs. \[1, 2, 5, 6, 10, 13, 28, 29\]

D. Generalization to non-autonomous DS

The above derivations can be generalized to cases of nonautonomous DS’s, \( i.e. \), to DS’s with time dependent flow fields. Indeed, let us note that the derivations that led to from Eq.(11) to Eq.(22) did not rely on the assumption that the flow vector field has no explicit dependence on time. Thus, these steps can be repeated also for time dependent flows, \( F(x, t) \). The only difference this will lead to is the substitution \( F(x(t)) \to F(x(t), t) \) in the definition of the gauge fermion in Eq.(25).
E. Stochastic generalization

Moreover, it is possible to generalize further to the case of stochastic external influence, \textit{i.e.}, the noise. First, let us isolate the time dependent part of the flow in the following manner:

\[ F^i(x, \xi(t)) = F^i(x) + T^{1/2} \epsilon_a^i(x) \xi^a(t). \]  

(26)

Here functions \( \epsilon_a^i(x) \) can be interpreted as vielbeins, \( \xi^a(t) \) are parameters representing external influence, and \( T \) is its temperature/intensity. One considers now the stochastic expectation value of the Witten index:

\[ W = \langle W_{cl}(\xi) \rangle_{Ns} \equiv \iint W_{cl}(\xi) P(\xi) D\xi, \]  

(27)

where \( P(\xi) \) is the normalized probability density functional of the configurations of noise. \( P(\xi) \) can be of a very general form. For example, the noise can be nonlocal in time, nonlinear, and it can as well have a non-vanishing "classical" component, \( \langle \xi(t) \rangle_{Ns} \neq 0 \). In Eq.(27),

\[ W_{cl}(\xi) = \iint_{PBC} e^{S_{cl}(\Phi, \xi)} D\Psi, \]  

(28)

with the action \( S_{cl}(\Phi, \xi) = \{ Q, \Psi_{cl}(\Phi, \xi) \} \) defined by the gauge fermion

\[ \Psi_{cl}(\Phi, \xi) = i \int dt \dot{\bar{\chi}}_i(t) \left( \partial_t x^i(t) - F^i(x(t), \xi(t)) \right). \]  

(29)

Integrating out \( \xi \)'s, one arrives back at Eq.(23) with the new action

\[ S(\Phi) = S_{cl}(\Phi) + \Delta S(y), \]  

(30)

where \( S \) is from Eq.(24), while the part provided by the noise is

\[ \Delta S(y) = \log(\langle e^{\int y_a(t) \xi^a(t)} \rangle_{Ns}) = \sum_{k=1}^{\infty} C_{a_1...a_k}^{a_1...a_k} (t_1...t_k) \prod_{j=1}^{k} y_{a_j}(t_j), \]  

(31)

with \( C \)'s being the (irreducible) correlators of noise and

\[ y_a(t) = \{ Q, -iT^{1/2} \dot{\bar{\chi}}_i(t) \epsilon_a^i(x(t)) \}. \]  

(32)

Due to the nilpotentcy of the differentiation by \( Q \), \textit{i.e.}, \( \{ Q, \{ Q, \Phi \} \} = 0 \), the product of any number of \( Q \)-exact factors is \( Q \)-exact itself, \( \{ Q, X_1 \} \{ Q, X_2 \}... = \{ Q, X_1 \{ Q, X_2 \}... \}. \)
Therefore, $\Delta S$ in Eq. (31), which is a functional only of $Q$-exact $y(t)$’s defined in Eq. (32), is $Q$-exact together with the entire new action

$$S = \{ Q, \Psi(\Phi) \}, \quad (33a)$$

with the new gauge fermion given as

$$\Psi(\Phi) = \Psi_{cl}(\Phi) + \sum_{k=1}^{\infty} T^{k/2} (-i)^{k} C^{a_{1}...a_{k}}(t_{1}...t_{k})$$

$$\times \bar{\chi}_{i}(t_{1}) e_{a_{1}}^{i}(x(t_{1})) \prod_{j=2}^{k} \{ Q, \bar{\chi}_{i}(t_{1}) e_{a_{1}}^{j}(x(t_{1})) \}, \quad (33b)$$

and with $\Psi_{cl}(\Phi)$ from Eq. (25). Therefore, after the generalization to nonautonomous and/or stochastic DS’s, the theory is still a TFT:

$$W = \int \int_{PBC} e^{S(\Phi)} D\Phi. \quad (33c)$$

F. Physical partition function

The trace of TO with the weight function from Eqs. (12b) also admits a path-integral representation similar to Eq. (33). Using again (19b), the weight function from Eq. (12b) can be represented as

$$z(x(0)) = |\hat{1} + \hat{M}_{-t}(x(0))| = \int_{APBC} \delta^{D}\left(\chi(0) - \hat{M}_{-t}(x(0))\chi(t)\right) d^{D} \chi(t), \quad (34)$$

where the subscript indicates antiperiodic boundary conditions: $\chi(t) = -\chi(0)$. This is the analogue of Eq. (20) with the only difference in the boundary conditions for the ghosts. This difference, however, does not interfere with the steps that led from Eq. (20) to Eq. (33). Therefore:

$$Z = \langle Z_{cl}(\xi) \rangle_{Ns} = \int \int_{APBC} e^{S(\Phi)} D\Phi, \quad (35)$$

with $S(\Phi)$ defined in Eq. (30) and with $Z_{cl}$ being the deterministic partition function defined in Eq. (17). According to the discussion at the end of Sec. II in the $t \to \infty$ limit Eq. (35) represents roughly the stochastically averaged number of periodic solutions.
IV. OPERATOR REPRESENTATION

One can now pass to the Schrödinger representation where $x$’s and $\chi$’s are diagonal, while:

$$i\hat{B} = \partial/\partial x, \quad i\hat{\chi} = \partial/\partial \chi.$$  \hfill (36)

The Taylor expansion in $\chi$’s of a general wave function, $\psi(x, \chi)$, terminates at the $D$’th term:

$$\psi(x, \chi) = \sum_{k=0}^{D} \frac{1}{k!} \psi^{(k)}(x),$$  \hfill (37)

$$\psi^{(k)}(x) = \psi_{i_1...i_k}(x)\chi^{i_1}...\chi^{i_k},$$  \hfill (38)

because all combinations $\chi^{i_1}...\chi^{i_N}$ with $N > D$ vanish due to the anticommuting character of the ghosts.

The time evolution of a wavefunction is given by the TFT Fokker-Planck equation

$$\partial_t \psi = -\hat{H}\psi,$$  \hfill (39)

with the TFT Fokker-Planck Hamiltonian that can be found as an (anti-)symmetrized version of its pathintegral representation, $H(\Phi)$, in $S = \int (iB_i \partial x^i - i\bar{\chi}_i \partial \chi^i - H(\Phi))$. In case of the Gaussian white noise, the TFT Fokker-Planck Hamiltonian is:

$$\hat{H} = [\hat{d}, \hat{j}]_+,$$  \hfill (40)

where $\hat{d} = \chi^i \partial / \partial x^i$ is the conserved Nöther charge associated with the $Q$-symmetry so that,

$$[\hat{d}, \hat{H}]_- = 0,$$  \hfill (41)

operator $\hat{j} = T\hat{d}^\dagger / 2 - \partial / \partial \chi^i F^i$ can be interpreted as the probability current with $\hat{d}^\dagger = -\partial / \partial \chi^i g^{ij} \nabla_j$ being the adjoint of $\hat{d}$ with respect to the metric, $g^{ij} = e^i_a e^j_a$, provided by veilbeins from Eq. (26), and $\nabla_j = \partial / \partial x^j - \Gamma^l_{jk} \chi^k (\partial / \partial \chi^l)$ is the covariant derivative with $\Gamma$’s being the Christoffel symbols.

The Hamiltonian is a real operator so that its eigenvalues are either real or come in complex conjugate pairs called Ruelle-Pollicott resonances:

$$\mathcal{E}_r = \Gamma_r, \quad \mathcal{E}_p^\pm = \Gamma_p \pm iE_p.$$  \hfill (42)

Models with this form of spectrum can be identified as pseudo-Hermitian. As a pseudo-Hermitian model, the model under consideration must possess the so-called $\eta T$-symmetry, where $\eta$ stands for nontrivial metric of the Hilbert space. The states with complex
FIG. 2: (a)-(f) Possible forms of the spectrum of the TFT Fokker-Planck Hamiltonian in relation to the phenomenon of the spontaneous breakdown of $Q$-symmetry. Among all the physical states with $\Gamma_n = \Gamma_g = \min_n \Gamma_n$, the ground state (indicated as black circles) is the one with the lowest $E_n$. Figures (a) and (b) correspond to cases when $\Gamma_g = 0$. For all except (a) the topological supersymmetry is broken. The $\eta T$-time reversal symmetry (see discussion after Eq.(42)) is broken for (b), (d), (f). Some form of spectra may not be realizable, however, as is discussed in Sec.VI. (g) The phase diagram of a generic DS’s discussed in the end of Sec.[VI]. The axes are the temperature and the other ”bifurcation” parameters. The low-temperature limit corresponds to the previously proposed three-phase picture consisting of the Markovian phase (M) with unbroken $Q$-symmetry, the intermittent chaotic phase (I) with $Q$-symmetry broken by (anti-)instantons, and the conventional chaotic phase, $C$, with $Q$-symmetry broken by strange attractors of the flow vector field.

Above certain temperature, $T_x$, the sharp boundary between the I- and C-phases must get smeared into a crossover, while the two phases themselves must merge into a $Q$-symmetry broken phase indicated as $x$. At even higher temperatures the $Q$-symmetry must get restored.

Conjugate eigenvalues must be $\eta T$-partners. Therefore, if the ground state of the model is one of the Ruelle-Pollicott resonances (see Fig[2]), the $\eta T$-symmetry is spontaneously broken.

The eigenstates of the Hamiltonian constitute a bi-orthogonal basis in the Hilbert space, $\mathcal{H}$:

$$\hat{H}|n\rangle = \mathcal{E}_n|n\rangle, \langle\langle n|\hat{H} = \langle\langle n|\mathcal{E}_n, \tag{43}$$

$$\langle\langle n|k\rangle = \delta_{nk}, \hat{1}_H = \sum_n |n\rangle\langle\langle n|. \tag{44}$$
The eigenstates can be either $Q$-symmetric or not. By definition, $Q$-symmetric eigenstates, 
that we call $\theta$'s, are such that $\langle\langle \theta | [\hat{d}, \hat{X}] | \theta \rangle \rangle = 0$ for any $\hat{X}$. This requirement is equivalent to the following:

$$\hat{d} |\theta\rangle = 0, \langle\langle \theta | \hat{d} = 0. \quad (45)$$

Clearly, all $Q$-symmetric states have zero eigenvalues because the Hamiltonian is $d$-exact.

A non-$Q$-symmetric state does not satisfy at least one of the conditions (45). If, for example, $\hat{H} |\vartheta\rangle = \mathcal{E}_\vartheta |\vartheta\rangle$ and $|\vartheta'\rangle = \hat{d} |\vartheta\rangle \neq 0$, then $\hat{H} |\vartheta'\rangle = \mathcal{E}_\vartheta |\vartheta'\rangle$ because $\hat{d}$ is commutative with the Hamiltonian. Furthermore, $|\hat{d} |\vartheta'\rangle = \hat{d}^2 |\vartheta\rangle \equiv 0$ because of $\hat{d}$ is nilpotent.

The same reasoning applies to the opposite situation when $\langle\langle \vartheta | \hat{d} \rangle \rangle = \langle\langle \vartheta | \mathcal{E}_\vartheta \rangle \rangle$ and $\langle\langle \vartheta | \hat{d} \rangle \rangle \neq 0$. In this manner, all non-$Q$-symmetric states come in boson-fermion (B-F) pairs, $i.e.$, pairs of states with odd and even number of ghosts.

There is yet another way to see that all eigenstates with non-zero eigenvalues come in B-F pairs. Consider an eigenstate $\hat{H} |\vartheta\rangle = \mathcal{E}_\vartheta |\vartheta\rangle$, $\mathcal{E}_\vartheta \neq 0$. If $\hat{d} |\vartheta\rangle \neq 0$, then the pairing is obvious due to Eq.(41). If, in contrary, $\hat{d} |\vartheta\rangle = 0$, when it follows immediately that $|\vartheta\rangle = \hat{d} |\vartheta'\rangle$, where $|\vartheta'\rangle = \hat{j} |\vartheta\rangle / \mathcal{E}_\vartheta$, where we used Eq.(40). Thus, $|\vartheta\rangle$ is a member of a B-F pair in this situation too.

Each B-F pair can be parametrized by a single bra-ket pair, $\langle\langle \vartheta |$ and $|\vartheta\rangle \rangle$, in the following manner:

$$\langle\langle \vartheta | \hat{d} \rangle \rangle, \quad (46a)$$

$$\langle\langle \vartheta |, \hat{d} |\vartheta\rangle \rangle, \quad (46b)$$

with $\langle\langle \vartheta | \hat{d} |\vartheta\rangle \rangle = 1$.

The Schrödinger representation for the Witten index is obtained from Eq.(23) by integrating out the Lagrange multiplier, $B_i(t)$, and the antighost, $\bar{\chi}_i(t)$:

$$W = \sum_n \langle\langle n | (-1)^F e^{-t\hat{H}} | n \rangle \rangle = \sum_n (-1)^F e^{-t\mathcal{E}_n} = \text{Tr} (-1)^F e^{-t\hat{H}}. \quad (47)$$

Here

$$\hat{F} = \chi^i \partial / \partial \chi^i, \quad (48)$$

is the ghost number operator, which is commutative with $\hat{H}$ so that it is a good quantum number:

$$\hat{F} |n\rangle = F_n |n\rangle. \quad (49)$$
The inclusion \((-1)^{\hat{F}}\) appears in Eq.(47) due to the unconventional periodic boundary conditions for the anticommuting ghosts.

The B-F pairs of non-\(Q\)-symmetric states from (46) do not contribute into the Witten index

\[
W = \sum_{k=0}^{D} (-1)^k N_k,
\]

where \(N_k = \#\{\theta|F_\theta = k\}\) is the number of \(Q\)-symmetric states with \(k\) ghosts. This expression of the Witten index is the TFT version of Eq.(13).

Witten index is independent of time duration \(t\). Therefore, \(W\) can be evaluated, for instance, in the \(t \to 0\) limit. In this limit and for any configuration of the external influence, \(\xi\), in Eq.(27), \(M_t \to \text{Id}_X\). For the identity map, the Lefschetz theorem (13) says that \(W(\xi)\) equals the Euler character of \(X\). Stochastic averaging in Eq.(27) of a constant yields the same constant and we have arrived at the conclusion that Witten index always equals the Euler character of the phase space.

As to the physical partition function, its operator representation expression is:

\[
Z = \text{Tr} e^{-t\hat{H}}. \tag{51}
\]

As compared to the Witten index, Eq.(51) is missing the topological factor \((-1)^{\hat{F}}\) as a result of different (antiperiodic) boundary conditions for the ghosts in Eq.(35).

V. TFT OF GTO

In DS theory, there is a fundamental object of interest known as generalized TO (GTO). The GTO is a way to account for the effect of stochastic noise. In this section, it is demonstrated that the TFT of DS provides a systematic framework for the formalism of the GTO which turns out to be nothing more than the finite-time Fokker-Planck evolution operator of the TFT.

A. Hilbert space and exterior algebra

Following Ref.[27], we identify the ghost operators with the exterior and interior multiplications:

\[
\chi^i = dx^i \wedge, \partial/\partial \chi^i = i\partial/\partial x^i. \tag{52}
\]
Accordingly, the Hilbert space is the complex-valued exterior algebra, \( \mathcal{H} = \Omega(X, \mathbb{C}) \), and wavefunctions with \( k \) ghosts are the differential \( k \)-forms on \( X \), \( \psi^{(k)} \in \mathcal{H}^k \equiv \Omega^k(X, \mathbb{C}) \), so that Eq.(38) should read:

\[
\psi^{(k)}(x) = \psi_{i_1...i_k}(x) \wedge_{l=1}^k dx^i.
\] (53)

In this picture, \( \hat{d} = dx^i \wedge \partial/\partial x^i \) is the exterior derivative, and the Hamiltonian from Eq.(40) is:

\[
\hat{H} = -T\triangle/2 + \hat{L}_F,
\] (54)

where \( -\triangle = [\hat{d},\hat{d}^\dagger]_+ \) is Laplace-Beltrami operator and \( \hat{L}_F = [\hat{d},\iota_F]_+ \) is the Lie derivative along the flow.

**B. Definition of GTO**

The GTO, \( \hat{M} : \mathcal{H} \rightarrow \mathcal{H} \), can be defined as a stochastically averaged pullback induced by a stochastic map. In our case of a stochastic flow,

\[
\hat{M}_t = \langle M^\ast_{-t}(\xi) \rangle_{\text{Ns}},
\] (55)

where \( M^\ast_{-t}(\xi) \) is the pullback induced by \( M_{-t}(\xi) \) defined by a time-dependent flow field in Eq.(26), and the notation for the stochastic averaging is introduced in Eq.(27).

Let us clarify at this point why in formulas for traces the "inverse" map shows up, \( M_{-t} \), and not the forward map \( M_t \). The reason lays in the "physical" evolution of wavefunctions, that, by the way, have the meaning of generalized probability densities. By the physical evolution we mean the "flow with the flow". In other words, the changes in wavefunctions are solely due to the coordinate transformation provided by the flow. In order to get the expression for the final wavefunction, one has to take the expression for the initial wavefunction in coordinates, \( x_{\text{init}} \), and formally make the coordinate transformation to the "final" coordinates, \( x_{\text{fin}} = M_{-t}(x_{\text{init}}) \). The result is the pullback by the inverse map, \( M_{-t} \).

**C. Generalization to DS’s with step-like evolution**

A pullback is a linear operator on \( \mathcal{H} \) so that the stochastic averaging, which is essentially a weighted summation of pullbacks at different \( \xi \)'s, is well defined. Now, it immediately follows
that the GTO is commutative with the exterior derivative because the exterior derivative commutes with any pullback.

This is true not only for the flows. Thus, we arrived at the conclusion that DS’s with step-like temporal evolution may also possess topological supersymmetry. The step-like DS’s in general case are defined by non-invertible maps that can map different points of $X$ to the same point. In order to take this possibility into account, the definition of the action of the pullback must include the summation over all the preimages of a point. Even after this generalization, there must exist a considerably wide class of step-like DS that possess the topological supersymmetry.

D. Finite-time evolution operator

In the deterministic limit, the equivalence between the pullback, $M^*_{-t}$, and the finite-time TFT Fokker-Planck evolution operator is seen from the definition of the Lie derivative

$$\partial_t M^*_{-t}(\xi)\psi(0) = -\hat{L}_{F(\xi(t))} M^*_{-t}(\xi)\psi(0),$$

where the noise modified time-dependent flow vector field is from Eq.(26) and $\psi(0)$ is any initial wavefunction. Formal integration gives

$$M^*_{-t}(\xi) =: e^{-\int_0^t \hat{L}_{F(\xi(t'))} dt'},$$

with columns denoting chronological ordering. It is now clear that Eq.(57) is the finite time deterministic evolution operator:

$$\psi(t) =: e^{-\int_0^t \hat{L}_{F(\xi(t'))} dt'} \psi(0),$$

while Eq.(56) is the deterministic TFT Fokker-Planck equation (39):

$$\partial_t \psi(t) = -\hat{L}_{F(\xi(t))} \psi(t),$$

with the deterministic Fokker-Planck Hamiltonian obtained from Eq.(54) by setting $T = 0$ and by allowing the flow vector field to be time-dependent.

To see that this also holds for the GTO, i.e., after the stochastic averaging, let us turn back to the pathintegrals’ language. One goes back to the ghost picture of the wavefunctions
in Eq. (38). In this picture, a deterministic pullback (with a fixed noise configuration) can be given the following operator form

$$M_\tau^* (\xi, x(t), \chi(t)|x(0), \chi(0)) = \delta^D(M_{-\tau}(x(t)) - x(0))$$

$$\times \delta^D(M_{-\tau}(x(0))\chi(t) - \chi(0)),$$  \hspace{1cm} (60)

so that the finite time evolution is

$$\psi(t, x(t), \chi(t)) = \int M_\tau^* (\xi, x(t), \chi(t)|x(0), \chi(0))$$

$$\times \psi(0, x(0), \chi(0))d^Dx(0)d^D\chi(0).$$  \hspace{1cm} (61)

(this expression emphasizes once again the linearity of the pullback in $\mathcal{H}$).

Now, one goes back again to the time slices picture of Sec. III (see Fig. 1), introduces the Lagrange multiplier, $B$, and the antighost field, $\bar{\chi}$, and arrives at

$$M_\tau^* (\xi, x(t), \chi(t)|x(0), \chi(0)) = \int\int e^{S_{cl}(\Phi, \xi)} D\Phi.$$  \hspace{1cm} (62)

Here the deterministic action is from Eq. (28), and the pathintegration here is over paths that connect the "in", $x(0), \chi(0)$, and "out", $x(t), \chi(t)$, arguments of the evolution operator in Eq. (61).

The next step is to integrate out $\xi$'s in the same manner as it was done in Sec. III E. This will substitute $S_{cl}(\Phi, \xi)$ by the action $S(\Phi)$ defined in Eq. (30):

$$\left\langle \int\int e^{S_{cl}(\Phi, \xi)} D\Phi \right\rangle_{Ns} = \int\int e^{S(\Phi)} D\Phi.$$  \hspace{1cm} (63)

At last, one can get back to the operator representation by integrating out $B$'s and $\bar{\chi}$'s and arrive at

$$\int\int e^{S(\Phi)} D\Phi = e^{-t\hat{H}}.$$  \hspace{1cm} (64)

This proves that for continuous time DS, the GTO is nothing else but the finite-time TFT Fokker-Planck evolution operator of the corresponding TFT:

$$\hat{M}_t = e^{-t\hat{H}}.$$  \hspace{1cm} (65)
E. Flat traces of GTO

The fundamental objects of study in the DS theory are the so called sharp traces and determinants of the GTO. Those, in turn, are defined through the so called flat traces of the GTO. The flat trace of degree $k$ is the trace of $\hat{M}$ over $\mathcal{H}^k$:

$$\operatorname{Tr}^{\flat} \hat{M}^{(k)}_t \equiv \operatorname{Tr}_{\mathcal{H}^k} e^{-t\hat{H}^{(k)}} = \sum_{F_n=k} e^{-t\xi_n},$$

(66)

where $\hat{M}^{(k)}_t \equiv e^{-t\hat{H}^{(k)}}$ together with $\hat{H}^{(k)}$ are projections on $\mathcal{H}^k$. The relation of the flat trace to the Ruelle-Frobenius-Perron TO in Sec.II can be established by considering the coordinate version of the action of the pullback on a wavefunction (53)

$$M^*_t \psi^{(k)}(x) = \psi_{i_1...i_k}(x') \wedge_{l=1}^k (dx')^i_l,$$

(67)

where $x' = M_t(x)$ and $(dx')^{i_1} = M_{-t,i_1}(x)dx^j$ is the tangent map (5) of differentials induced by $M_t$ (functional dependence on the noise configuration, $\xi$, is tacitly assumed in the above formulas).

In the standard manner, the trace of Eq.(67) is:

$$\operatorname{Tr}^{\flat} \hat{M}^{(k)}_t = \left\langle \sum_{x=\text{fix}M_t} \frac{\operatorname{Tr}^{k} M_{-t}(x)}{\|1- M_{-t}(x)\|} \right\rangle_{\mathcal{N}_s},$$

(68)

where the sum and the denominator come from the ”deterministic” trace over bosonic fields, while the fermionic trace is over the extension of the tangent map (5) on the $k$th exterior power of the tangent space, $\wedge^k \tilde{M}_{-t}(x) : \wedge^k T_x X \to \wedge^k T_{M_{-t}(x)} X$,

$$\operatorname{Tr} \wedge^k \tilde{M}_{-t}(x) = \sum_{i_1<...<i_k} i_{\partial/\partial x^{i_1}}...i_{\partial/\partial x^{i_k}} \wedge_{l=1}^k (dx')^{i_l} = m_k(x),$$

with $m_k(x)$ from Eq.(12c). Thus

$$\operatorname{Tr}^{\flat} \hat{M}^{(k)}_t = \langle \operatorname{Tr} \mathcal{L}_t m_k \rangle_{\mathcal{N}_s}.$$

(69)

F. Trace and determinants of GTO

Finally, the so called sharp and counting (ordinary) traces of the GTO are recognized as the Witten index and the physical partition function:

$$\operatorname{Tr}^{\sharp} \hat{M}_t = \sum_{k=1}^D (-1)^k \operatorname{Tr}^{\flat} \hat{M}^{(k)}_t = W,$$

(70a)

$$\operatorname{Tr}^{c} \hat{M}_t = \sum_{k=1}^D \operatorname{Tr}^{\flat} \hat{M}^{(k)}_t = Z.$$  

(70b)
Complementary to these, the DS theory also deals with the sharp and counting determinants of the GTO:

\[
\text{det}^\sharp (\hat{1} - z\hat{M}_t)^{-1} = \prod_{k=0}^{D} \text{det}^\sharp_k (\hat{1} - z\hat{M}_t^{(k)})^{(-1)^{k+1}} \\
= \prod_{n} (1 - ze^{-\varepsilon_n})^{-(-1)^{F_n+1}}, \tag{71a}
\]

\[
\text{det}^c (\hat{1} - z\hat{M}_t)^{-1} = \prod_{k=0}^{D} \text{det}^c_k (\hat{1} - z\hat{M}_t^{(k)})^{-1} \\
= \prod_{n} (1 - ze^{-\varepsilon_n})^{-1}. \tag{71b}
\]

VI. DISCUSSION

In the previous section it was established that in many aspects the TFT of DS is equivalent to the GTO formalism of the DS theory. As compared to the DS theory, however, the TFT of DS provides one very important piece of understanding. This is the understanding that all the DS’s possess topological supersymmetry.

An immediate consequence of the topological supersymmetry of DS’s is the trivialization of Eqs.(70a) and (71a). Because of the B-F pairing of the non-Q-symmetric states, the sharp trace of the GTO is a topological invariant, \(W\). By the same token, the sharp determinant of the GTO simplifies as:

\[
\text{det}^\sharp (\hat{1} - z\hat{M}_t)^{-1} = (1 - z)^{-W}. \tag{72}
\]

Another utility of the existence of Q-symmetry is the first theoretical explanation of the emergence of ubiquitous long-range correlations (1/f noise) in chaotic DS’s. Those can be attributed to the spontaneous breakdown of Q-symmetry. \[19, 20\] This explanation together with the picture of the Q-symmetry breaking is of ultimate importance for applications. Related to the Q-symmetry breaking, in turn, is the form of the spectrum of \(\hat{H}\). Indeed, the Q-symmetry is definitely broken when the ground state has non-zero eigenvalue, while which of the eigenstates are the ground states of the model is uniquely determined by the spectrum of \(\hat{H}\) (see Fig.2). On the other hand, the DS theory provides numerous theorems that address the spectrum of GTO. Thus, the TFT of the GTO established in Sec.V may prove useful by shedding some additional light on the spectrum of \(\hat{H}\) and thus on the issue of Q-symmetry breaking.

One of the theorems from the DS theory \[24, 25\] assures that under certain conditions
\[ \det_k(1 - z \hat{\mathcal{M}}_t^{(k)}) \] for any \( k \) has no poles (is meromorphic) for \( |z| < e^{-P} \), where \( P \) is some model specific constant related to a parameter called pressure.

As is seen from Eq. (71b), the logarithms of the positions of the poles of the GTO's determinants correspond to the eigenvalues of \( \hat{H} \). Therefore, the theorem seemingly assures that the attenuation rates of the eigenvalues in Eq. (42) are bounded from below. \[ 32 \] In other words, for the ground state \( \Gamma_g = \min_n \Gamma_n > -\infty \). In many cases spectral theorems also suggest that at \( |z| = e^{-P} \) there is a single real pole at \( z = e^{-P} \). In terms of the spectrum of the Hamiltonian this means that the ground states eigenvalue is real, \( \mathcal{E}_g = \Gamma_n \). The same picture seems to appear from the physical arguments in the forthcoming discussion.

An additional requirement on \( \Gamma_g \) can be established from purely TFT considerations. Models with non-zero Witten index must possess \( \mathcal{Q} \)-symmetric state(s) of zero eigenvalue. Thus, \( \Gamma_g \leq 0 \) when \( W \neq 0 \). This may turn out to be true even when \( W = 0 \), even though we do not have an argument supporting this proposition at this moment. From now on, however, we only consider \(-\infty < \Gamma_g \leq 0\).

In Ref. [20], the \( \mathcal{Q} \)-symmetry breaking was discussed under the assumption that \( \Gamma_g \) can not be negative (see Fig. 2b). This assumption was based the following argument. The Witten index can be viewed as the partition function of the noise, and since the noise does not have instabilities, states with negative attenuation rates must not exist. This argument, however, must be discarded because the eigenstates with nonzero eigenvalues are non-\( \mathcal{Q} \)-symmetric and consequently they do not contribute to the Witten index and/or to the partition function of the noise.

This does not necessarily suggest, however, that reasons forcing \( \Gamma_g \) to be nonnegative do not exist in some classes of DS’s. In DS’s that do have such a reason, Fig. 2b is the only possible picture of the \( \mathcal{Q} \)-symmetry breaking. In general case, on the other hand, \( \Gamma_g \) may be negative. The three possible spectra corresponding to this situation are given in Figs. 2i, 2j, and 2k.

For the situation in Fig. 2i, the partition function can take on negative values in the long time limit, \( Z|_{t \to \infty} \sim e^{i|\Gamma_g|} \cos \mathcal{E}_g t \). The same is true for Fig. 2j. If we recall now that in the long time limit \( Z \) must have a meaning of averaged number of periodic solutions (see Secs. III F and II), the negativeness of \( Z \) looks suspicious. This may point on the possibility that such spectra are not realizable. This leads to the conclusion that Fig. 2c is the most likely
picture of $Q$-symmetry breaking when $\Gamma_g < 0$. In such cases

$$\lim_{t \to \infty} Z \sim e^{t|\Gamma_g|}. \quad (73)$$

In other words, the stochastically averaged number of periodic solutions grows exponentially in the large time limit with rate $|\Gamma_g|$. This exponential growth is a unique feature of deterministic chaos and Eq. (73) is the stochastic generalization of this situation. Therefore, the picture we just arrived at, seemingly constitutes the proof that the spontaneous breakdown of the topological supersymmetry is the mathematical essence of the concept of deterministic chaos and/or of its stochastic generalization that is often referred to as complex dynamics.

As a final remark, we would like to revisit the issue of the generic phase diagram of a DS. Previously, it was proposed that there must exist three major phases: the Markovian phase with unbroken $Q$-symmetry, the intermittent phase with $Q$-symmetry broken by the condensation of instantonic-antiinstantonic configurations, and the conventional chaotic phase, where the $Q$-symmetry is broken even in the deterministic limit by topologically ill-defined fractal invariant manifolds (e.g., attractors) of the flow vector field. It was also noticed that the intermittent phase must shrink into the 'edge of chaos' in the deterministic limit, where anti-instantons disappear.

This picture works only in the low temperature limit, where the dominant part of in the Hamiltonian is the drift term (the Lie derivative along the flow). In this situation, the perturbative ground states are localized on the unstable manifolds of the flow vector field, while the presence of noise somewhat smears the ground state. It is only in this regime that the concept of (anti-)instantons are physically sound, i.e., an external observer would be able to tell one process from another.

In the high temperature limit, the diffusive part of the Hamiltonian (the Laplacian) become more and more dominant as one rises the temperature. The spectrum of the Laplacian, in turn, is real and positive, which corresponds to the unbroken topological supersymmetry. This suggests that in any DS the topological supersymmetry eventually gets restored as one rises the temperature.

Furthermore, the boundary between intermittent and conventional chaos is not a topological supersymmetry breaking phase transition even in the low-temperature limit (one can not break topological supersymmetry twice). In the high-temperature limit, this boundary
must disappear completely together with the concept of (anti-)instantons. The two phases must merge into a new phase with a spontaneously broken $Q$-symmetry. We did not manage to find a suitable identification for this phase in the Literature, so that in the emerging phase diagram in Fig.2 we just call it the x-phase.

VII. CONCLUSION

In conclusion, in this paper the connection is established between the transfer operator formalism of the dynamical systems’ theory and the recently proposed topological field theory of dynamical systems. The established connection provides a potential for a fruitful cross-fertilization between the developments in the dynamical systems theory and cohomological topological field theories. Three distinct results enabled by this connection were presented in this paper.

First, it was shown that the so called sharp trace and sharp determinant of the generalized transfer operator of the dynamical systems theory is subject to the supersymmetric trivialization due to the mere existence of the topological supersymmetry.

Second, this connection enabled the utilization of the Lefschetz theorem for the proof of that the Witten index always equals the Euler character of the phase space. To the best of our knowledge this is the first proof of this statement suitable for any flow field and noise.

Third, due to this connection it became possible to apply the spectral theorems of the dynamical systems theory to the spectrum of the TFT Fokker-Planck Hamiltonian and refine the previously proposed picture of the spontaneous breakdown of the topological supersymmetry. Specifically, it allowed to extend the picture to situations when the ground state’s attenuation rate is negative. In these situations the stochastically averaged number of periodic solutions grows exponentially in the large time limit, which is a unique feature of chaotic behavior. Hence, this constitutes a firm evidence proving that the spontaneous breakdown of the topological supersymmetry is indeed the mathematical essence of the deterministic chaos and/or of its stochastic generalization often referred to as complex dynamics.

In addition, in this paper we revisited the question of the generic phase diagram of that was previously analyzed in the low-temperature limit where it consists of the three major phases: markovian, intermittent, and chaotic. This picture was generalized to the high temperature regime, where the intermittent and chaotic phases first merge and at even
higher temperatures must disappear.

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Indeed, the one and the same DS may or may not exhibit the butterfly effect depending on the configuration of the stochastic noise. Furthermore, there is yet another property that is believed to be necessary for a deterministic DS to be chaotic and that cannot be generalized to stochastic situations. This property is called “topological mixing” and its essence is that any two open sets in the phase space are connected by a trajectory. In stochastic situations, the configuration of the stochastic noise can always be chosen such that any two given points in the phase space are connected.

This must be true even in the deterministic limit, where the diffusive/Laplacian part of the Hamiltonian vanishes.

Parameter $|\Gamma_g|$ can be identified as the previously mentioned pressure of the DS. Yet another parameter to which $\Gamma_g$ can be related is the so-called topological entropy of a chaotic DS. The later is the rate of the exponential growth of the number of prime periodic solutions.