Abstract

In this paper we study the large $N_c$ limit of $SO(N_c)$ gauge theory coupled to a real scalar field following ideas of Rajeev\cite{1}. We will see that the phase space of this resulting classical theory is $Sp_1(H)/U(H_+)$ which is the analog of the Siegel disc in infinite dimensions. The linearized equations of motion give us a version of the well-known 't Hooft equation of two dimensional QCD.

1 Introduction

Gauge theories are the fundamental theories that describe nature: Quantum Chromodynamics (QCD) is the gauge theory of hadrons, and it is believed that one can compute the masses and excitations of all these hadrons from QCD. As yet there is no satisfactory understanding of these bound states. All the hadrons are color singlets, in fact we never see the underlying quarks as asymptotic states. That means QCD should be a confining theory and there should be an independent formulation of it which is expressed completely in terms of these color invariant states.

In \cite{1} Rajeev has constructed such a theory of mesons in two dimensions in the limit $N_c$, the number of colors in $SU(N_c)$, goes to infinity. The idea that QCD should simplify while keeping all its essential features in this limit goes back to 't Hooft\cite{2,3} and that this limit should be a kind of classical mechanics to Migdal and Witten. Even this large-$N_c$ theory is quite complicated and 't Hooft looked at a two dimensional model to understand the meson spectrum in this limit and obtained his bound state equation\cite{3}. It is not so clear how to treat the baryons in the large-$N_c$ limit. Witten suggested that baryons could also be understood (as solitonic excitations) in his by now classic papers\cite{4,5}. A much more ambitious program was presented by Lee and Rajeev\cite{6} for the large-$N_c$ limit of more complicated gauge theories.
In this paper we study the large $N_c$ limit of $SO(N_c)$ gauge theory coupled to a real scalar field. This theory is not physical, but it is a good model to test some of the ideas about gauge theories. We will apply the methods developed by Rajeev to this toy model. We recommend the lectures of Rajeev for a more detailed exposition of the underlying ideas \[7\]. Rajeev in his work \[1\] has shown that the phase space of the two dimensional QCD is an infinite dimensional Grassmannian, well-known from the theory of integrable systems and loop groups \[8\]. Scalar QCD was worked out using the same methods in \[9\], where it was shown that the phase space of the theory is an infinite dimensional disc. Originally scalar two dimensional QCD was worked out by Shei and Tsao in \[10\] following ‘t Hooft, and later by Tomaras using Hamiltonian methods in \[11\]. These works obtained the analog of the ‘t Hooft equation for this case. A natural extension of these would be to look at combined (fermionic) QCD and scalar QCD, this is done in a paper of Aoki\[12\] where it is shown that the various types of mesons are possible and they all obey ‘t Hooft equations. About the same time following a path integral approach and bilocal fields, coupled fermions and bosons as well as some other models in two dimensions are worked out in a beautiful article by Cavicchi \[13\]. Recently Konechny and the second author have extended the methods of \[1\] to the above case and showed that the underlying large-$N_c$ phase space is a certain kind of super-Grassmannian. The linearized equations agree with the ones found in \[12\]. The correct equations are nonlinear, and there is a baryon number operator which corresponds to the supertrace of the basic variable\[14\].

The real scalar field is an interesting testing ground. There are some ideas in the literature which suggest that gauge theories in two dimensions all behave in a very similar way\[15\]. In this work we show that the phase space of the resulting classical theory is $Sp_1(\mathcal{H})/U(\mathcal{H}_+)$ which is the analog of the Siegel disc in infinite dimensions. The linearized equations of motion give us a version of the well-known ‘t Hooft equation of two dimensional QCD, and this new one is the same equation found in \[10\] apart from some numerical factors. That means that we have the same spectral behaviour for the mesons of the theory. Since most of the details are very similar to the ones in Rajeev’s lectures\[7\] and various aspects of the geometry of the phase space was given in a few other places \[16, 17, 18\] our treatment will be brief.

2 The scalar $SO(N_c)$ model in the light cone

Since the basic philosophy was explained in \[1\] we will state our conventions and write down the Lagrangian of the theory. We will use the light cone coordinates $x^+ = \frac{1}{\sqrt{2}}(t + x)$, $x^- = \frac{1}{\sqrt{2}}(t - x)$ and choose $A_- = 0$ gauge.

\[
S = \int dx^+ dx^- \left[ \frac{1}{2} \text{Tr} F_+^- F_+^- + \frac{1}{2} \partial_- \phi^\alpha \partial_+ \phi_\alpha + \frac{g}{2} (\partial_- \phi^\alpha A_+^\alpha \phi_\beta - \phi^\alpha A_+^\alpha \partial_- \phi_\beta) - \frac{m^2}{2} \phi^\alpha \phi_\alpha \right].
\]  

(1)

Here we have $SO(N_c)$ gauge theory for which the matter fields are in the fundamental representation and $\text{Tr}$ denotes an invariant inner product in the Lie algebra. The Lie algebra condition implies that $A_+^\alpha = -A_+^\alpha$. To compute the variations we need the independent degrees of freedom, we can expand $A_+ = A_+^a T^a$ where $T^a$ are the generators of $SO(N_c)$.
Lie algebra. We can choose them such that \( \text{Tr} T^a T^b = -\frac{1}{2} \delta^{ab} \). When we use the light cone approach in \( 1+1 \) dimensions, the gauge fields do not carry dynamical degrees of freedom. We first eliminate the gauge fields and then write the resulting action. Let us find the equation of motion for the gauge field once we rewrite the action.

\[
S = \int dx^+ dx^- \left[ \frac{1}{2} \phi^\alpha (2 \partial_-) \phi_\alpha + \frac{g^2}{2} A^a_+ (\partial_- \phi^\alpha T^a_\alpha - \partial_+ \phi^\alpha T^a_\alpha) + \frac{1}{2} (\partial_- A^a_+)^2 - \frac{m^2}{2} \phi^\alpha \phi_\alpha \right].
\]  

(2)

If we define the current \( J^a = \frac{1}{2} (\phi^\alpha T^a_\alpha \partial_- \phi_\alpha - \partial_- \phi^\alpha T^a_\alpha \phi_\alpha) \), we get

\[- \partial_-^2 A^a_+ = g J^a, \]

(3)

which can be solved formally (an actual solution can be found if we specify some boundary conditions) and by substituting this into our Lagrangian again,

\[
S = \int dx^+ dx^- \left( \frac{1}{2} \phi^\alpha (2 \partial_-) \phi_\alpha + [\frac{g^2}{2} J^a_+ \frac{1}{\partial_-} J^a_+ + \frac{m^2}{2} \phi^\alpha \phi_\alpha] \right).
\]

(4)

Written in this form we immediately see that we have the following symplectic form

\[
\omega^{-1}(x^+, y^-) = \langle x^- | (2 \partial_-)^{-1} | y^- \rangle = -\frac{1}{4} \text{sgn}(x^- - y^-)
\]

(5)

and the Hamiltonian,

\[
H = \int dx^- \left[ - \frac{g^2}{2} \frac{1}{\partial_-} J^a_+ J^a_+ + \frac{m^2}{2} \phi^\alpha \phi_\alpha \right].
\]

(6)

The same boundary conditions as the one used to find the symplectic form gives us the more explicit expression,

\[
H = \frac{m^2}{2} \int dx^- \phi^\alpha (x^-) \phi_\alpha (x^-) - \frac{g^2}{4} \int dx^- dy^- J^a_+(x^-) |x^- - y^-| J^a_-(y^-)
\]

(7)

We note that one needs the properties of the group and its representation to compute the interaction term. This can be achieved due to the identity \( \sum_a (T^a_\alpha (T^a_\lambda))^\gamma = -\frac{1}{2} (\delta_{\alpha \gamma} \delta_{\lambda \beta} - \delta_{\alpha \lambda} \delta_{\beta \gamma}) \).

It is now possible to compute the equations of motion for the classical variable \( \phi(x^-, x^+) \) using

\[
\frac{\partial \phi}{\partial x^+} = \{ \phi, H \}.
\]

(8)

It is a useful exercise to find the equations of motion even for the free field theory (see the beautiful lectures by Heinzl [19]). Another important exercise is to check that the theory is Poincare invariant written in this new way, by finding the generators.

We will follow [1] (or [7]) and rewrite the theory in terms of the color invariant bilinears of the field variable \( \phi \) after canonical quantization. In the large \( N_c \) limit these will be the only dynamical variables, and the theory has a completely classical formulation in terms of these bilinears. We will see that the remaining global \( SO(N_c) \) symmetry we have imposes
a constraint for these variables and that means the phase space of the theory is a curved manifold in infinite dimensions.

Canonical quantization is standard, since the theory is super-renormalizable the result is the same as free field theory and the choice of vacuum is exactly the same. The equal “time” commutator is given by

$$[\hat{\phi}_\alpha(x^-, x^+), \hat{\phi}_\beta(y^-, x^+)] = -\frac{i}{4}\delta_{\alpha\beta}\text{sgn}(x^- - y^-).$$

(9)

This means that it is simpler to introduce creation and annihilation operators, satisfying,

$$[a_\alpha(p), a_\beta(q)] = 2\pi\delta(p + q)\delta_{\alpha\beta}\text{sgn}(p)$$

(10)

such that

$$\hat{\phi}_\alpha(x^-) = \int\frac{dp}{2\pi}\frac{1}{\sqrt{2|p|}}e^{-ipx^-}.$$  

(11)

For quantum theory we introduce the Fock vacuum $|0>: a_\alpha(p)|0 > = 0$ when $0 \leq p$

(12)

To get well-defined expressions for various operators—such as the Hamiltonian—we need a normal ordering prescription:

$$: a_\alpha(p)a_\beta(q) := \begin{cases} a_\beta(q)a_\alpha(p) & \text{if } q < 0, p > 0 \\ a_\alpha(p)a_\beta(q) & \text{otherwise} \end{cases}$$

(13)

We note that it is also possible to think about the creation and annihilation operators via a Fourier expansion,

$$\hat{\phi}_\alpha(x^-) = \int_0^\infty \frac{dp}{2\pi\sqrt{2|p|}}(a_\alpha(p)e^{-ipx^-} + a_\alpha^\dagger(p)e^{ipx^-}),$$

(14)

manifesting the real valuedness of the field $\hat{\phi}_\alpha^\dagger = \hat{\phi}_\alpha$. This automatically implies that $a_\alpha^\dagger(p) = a_\alpha(-p)$ for $p > 0$. In the next section we will see that this is a more appropriate way to think about quantization, yet from a calculational point of view the other is better. One notes that this is consistent with the commutation relations of $a_\alpha$’s. (See [19] and [20] for more details about the light-cone vacuum structure of the real scalar field).

The normal ordering can be written in terms of the ordinary products of the operators and a vacuum subtraction,

$$: a_\alpha(p)a_\beta(q) := a_\alpha(p)a_\beta(q) - \frac{1}{2}(\text{sgn}(p) + 1)2\pi\delta(p + q)\delta_{\alpha\beta}. $$

(15)

We will make use of this relation quite frequently in the next section.

3 Algebra of Color Invariant Operators

In this section we will discuss the class of operators we will use to reformulate the gauge theory in the large-$N_c$ limit. Since we have fixed the gauge as $A_- = 0$ we are not allowed
to make any more space dependent gauge transformations. (The equations of motion at the
quantum level imply that the “time” dependent transformations cannot be made arbitrarily
but given by the evolution of the scalar field. We do not need to look at these in any case
since in the Hamiltonian formalism observables at a fixed “time” slice are enough). Yet there
is still a global $SO(N_c)$ symmetry which is left over. To emphasize the contraction we write
down the color invariant bilinears with one index up the other index down,

$$N(x^-, y^-) = \frac{1}{N_c} : \phi^\alpha(x^-) \phi^\alpha(y^-) : \quad (16)$$

The set of these equal time bilinears constitute the set of all possible color invariant operators
for this theory. One may equally look at the Fourier transform of these operators, so the
basic bilinears in this case become,

$$\hat{T}(p, q) = \frac{2}{N_c} \sum_\alpha : a_\alpha(p) a_\alpha(q) : \quad (17)$$

As we will see in the next section that conceptually it is more natural to use the variables

$$\hat{K}(p, q) = -\frac{2}{N_c} \text{sgn}(p) \sum_\alpha : a_\alpha^+(p) a_\alpha(q) : \quad (18)$$

but for calculations it is easier to keep the above variables. The basic idea of the large-$N_c$
theory is to write everything in terms of these color invariant bilinears. In the limit $N_c$
becomes large only the color invariant operators survive and furthermore the expectation
values of color invariant operators split as a product upto $1/N_c$ corrections. This implies that
the set of color invariant operators becomes classical, all color invariant operators should
be representable as classical observables. The resulting theory, restricted to the space of
color invariant states therefore becomes a classical theory \[, 3, 21\]. To define this classical
theory we compute the commutator of two such color invariant operators and then take the
appropriate large-$N_c$ limit. The result will be postulated as a classical Poisson bracket of
these classical variables. We will see later on that this Poisson bracket actually comes from
a symplectic form on a very natural infinite dimensional homogeneous symplectic manifold
\[7\].

When we compute the commutator of such bilinears we get,

$$[\hat{T}(p, q), \hat{T}(s, t)] = \frac{2}{N_c} \left( \text{sgn}(p) \delta[p + s] \hat{T}(q, t) + \text{sgn}(q) \delta[q + s] \hat{T}(p, t) + \text{sgn}(p) \delta[p + t] \hat{T}(s, q) + \text{sgn}(q) \delta[q + t] \hat{T}(s, p) \\
+ (\text{sgn}(p) + \text{sgn}(q)) (\delta[p + s] \delta[q + t] + \delta[p + t] \delta[s + q]) \right),$$

where we defined $\delta[p + q] = 2\pi \delta(p + q)$ for convenience.

If we take the limit $N_c \to \infty$ we assume that there are corresponding classical observables
and the commutators go to Poisson brackets of these observables. We still denote them by
the same letter except dropping the hat on the top. Applying the rule $-\frac{i}{\hbar} [A, B] \mapsto \{A, B\},$
as $\hbar = N_c^{-1} \to 0$, we get\[.\]

\[\footnote{There is really no way to determine the correct quantization parameter in this approach. We can only find this when we quantize the theory back again. The most natural method to employ is geometric quantization, due to the natural geometry of the phase space. We will come back to these issues in a separate publication.}
\{T(p, q), T(s, t)\} = 
-2i \left( \text{sgn}(p)\delta[p + s]T(q, t) + \text{sgn}(q)\delta[q + s]T(p, t) + \text{sgn}(p)\delta[p + t]T(s, q) + \text{sgn}(q)\delta[q + t]T(s, p) 
+ (\text{sgn}(p) + \text{sgn}(q))(\delta[p + s]\delta[q + t] + \delta[p + t]\delta[s + q]) \right). 

We will postulate these to be the basic Poisson brackets of our dynamical variables. It is a good exercise to compute the equations of motion for the free field in this language and write down the solution.

These variables acting on the color invariant sector are not completely independent, there is a constraint coming from the global color invariance. Recall that the global \(SO(N_c)\) is generated by the operators acting on the Fock space,

\[ \hat{Q}_{\alpha\beta} = \int_{-\infty}^{\infty} dp \frac{1}{2\pi} a_{\alpha}^\dagger(p)a_{\beta}(p) - \int_{-\infty}^{\infty} dp \frac{1}{2\pi} a_{\beta}^\dagger(p)a_{\alpha}(p). \]  

(19)

These operators satisfy \(\hat{Q}_{\alpha\beta}|0\rangle = 0\) and

\[ [\hat{Q}_{\alpha\beta}, \hat{Q}_{\lambda\gamma}] = \hat{Q}_{\lambda\alpha}\delta_{\beta\gamma} + \hat{Q}_{\beta\lambda}\delta_{\alpha\gamma} + \hat{Q}_{\gamma\beta}\delta_{\lambda\alpha} + \hat{Q}_{\lambda\alpha}\delta_{\gamma\beta}. \]  

(20)

One can see that \(\hat{Q}_{\alpha\beta} = -\hat{Q}_{\beta\alpha}\). Recall the related set of bilinear variables,

\[ \hat{K}(p, q) = -\frac{2}{N_c} \text{sgn}(p) :a_{\alpha}^\dagger(p)a_{\alpha}(q):, \]  

(21)

a careful computation shows that when we restrict these variables to the color invariant sector of the Fock space in the large-\(N_c\) limit we get,

\[ \int_{-\infty}^{\infty} K(p, s)K(s, q)[ds] - \text{sgn}(p)K(p, q) - K(p, q)\text{sgn}(q) = 0. \]  

(22)

This operator equation is interpreted as now an equation for the kernel of an integral operator acting on the one-particle Hilbert space. We can write the same constraint in a more succinct manner as,

\[ (K + \epsilon)^2 = I, \]  

(23)

where \(\epsilon(p, q) = -\text{sgn}(p)\delta[p - q]\) and we interpret this as an operator equation again. We will talk about the meaning of this equation from a more geometric point of view in the next section. The important assumption is that when we let \(N_c \rightarrow \infty\), the above constraint translated into a constraint for the classical variables \(K\). So the dynamical variables \(K\) satisfy this constraint, which implies a constraint for \(T(p, q)\) trivially.

We rewrite the Hamiltonian by redefining the coupling constant as \(g^2N_c \rightarrow g^2\) and dividing the Hamiltonian by an overall factor of \(N_c\). Thus the Hamiltonian becomes, after mass renormalization,

\[ H_0 = \frac{1}{8}(m_R^2 - \frac{g^2}{2\pi})P \int \frac{[dp]}{|p|}T(-p, p), \]  

(24)

where the renormalized mass is given by \(m^2 = m_R^2 + \frac{g^2}{4\pi} \ln \frac{\Lambda_U}{\Lambda_I}, \Lambda_U, \Lambda_I\) referring to the ultraviolet and infrared cut-offs respectively, also we used the shorthand \([dp] = \frac{dp}{2\pi}\), and \(P\) denotes the
principal value prescription. This is not a simple computation but the essential steps are given in \[7\], and the interaction part

\[
H_I = \frac{g^2}{64} \mathcal{FP} \int \frac{[dpdqdsdt]}{\sqrt{|pqst|}} \delta[p + q + s + t] \frac{sq - st + pt - pq}{(p + s)^2} T(p, q)T(s, t),
\]

where \(\mathcal{FP}\) denotes the finite part, as explained in \[7\]. For simplicity of notation from now on we will drop the symbols, \(\mathcal{P}\) and \(\mathcal{FP}\), but the calculations should be performed keeping these in mind. At this point we have the complete formulation of our theory, one can compute the equations of motion using the above form of the Hamiltonian and the Poisson brackets of the variables \(T(u, v)\). At this stage we will not be able to give an analysis of these nonlinear equations and instead confine ourselves to the linear approximation.

For the linear approximation we follow \[7\], we will write the above constraint in terms of the variables \(T\) for a bound state solution. In the following we will keep all the equations of motion to this approximation and search for a bound state solution.

We can compute the equations of motion in the linear approximation: this means we look at \(T(u, v)\) for \(u, v > 0\) or \(u, v < 0\), the other cases imply \(T(u, v) = 0\) from the constraint equation. Let us look at \(u, v < 0\) case and define \(P = -(u + v)\) and \(x = -u/P\). This means \(u = -Px, v = -P(1 - x)\) and \(0 < x < 1\). If we actually compute the equations of motion \(\partial_+ T(u, v; x^+) = \{T(u, v; x^+), H\}\), and make an ansatz, \(T(u, v) = e^{ip_+x^+}\zeta(x)\), we get

\[
\mu^2 \zeta(x) = (m_R^2 - \frac{g^2}{2\pi})[\frac{1}{x} + \frac{1}{1 - x}]\zeta(x) - \frac{g^2}{8\pi} \int_0^1 \frac{y(1 - x) + x(1 - y) + y(1 - y) + x(1 - x)}{(x - y)^2} \zeta(y)dy,
\]

where \(\mu^2 = 2P_+P\) is the invariant mass of this excitation. We should solve this eigenvalue equation to find the allowed values of \(\mu^2\) and the function \(\zeta\). This will determine the spectrum of the theory. One notes that the equation is symmetric under \(x \mapsto 1 - x\) and \(y \mapsto 1 - y\), that means we may choose \(\zeta(x) = \zeta(1 - x)\). This simplifies our equation to

\[
\mu^2 \zeta(x) = (m_R^2 - \frac{g^2}{2\pi})[\frac{1}{x} + \frac{1}{1 - x}]\zeta(x) - \frac{g^2}{4\pi} \int_0^1 \frac{(x + y)(2 - x - y)}{(x - y)^2} \zeta(y)dy.
\]

(27)

The above form is in fact identical to the bound state equation found in reference \[10\] and later on by Tomaras using Hamiltonian methods apart from the numerical factors (this approach is closer to the one in \[1\]). It is known that this theory has only discrete states, that is we only have bound state solutions and no scattering states.

We may search for the baryons in this theory (from a more standard point of view, we do not have any \(U(1)\) symmetry in the classical action, this suggests that there should not be
baryon number conservation and no baryons, we will see that the baryon number is indeed not conserved for the gauge theory).

Note that there is no anti-baryon. Let us write down a typical baryonic operator;

\[ B(p_1, p_2, ..., p_{N_c}) = \frac{1}{Z} \epsilon_{\alpha_1 \alpha_2 ... \alpha_{N_c}} a^\dagger_{\alpha_1}(p_1)a^\dagger_{\alpha_2}(p_2)...a^\dagger_{\alpha_{N_c}}(p_{N_c}), \]

(28)

where \( Z \) is an appropriate normalization factor. When we take the large-\( N_c \) limit these operators become infinite strings which are not representable in a simple way. But we can still detect them if they are present in a physical state. We write a one-baryon state as \( B(p_1, p_2, ..., p_{N_c}) |0 \rangle \), and define the baryon operator,

\[ \hat{B} = \frac{1}{N_c} \int_0^\infty [dp] : a^\dagger_{\alpha}(p) a_{\alpha}(p) :. \]

(29)

In general we have the action of the baryon operator on many baryon states,

\[ \frac{1}{N_c} \int_0^\infty [dp] : a^\dagger_{\alpha}(p) a_{\alpha}(p) : B(p_1, p_2, ..., p_{N_c}) B(q_1, q_2, ..., q_{N_c}) ... B(s_1, s_2, ..., s_{N_c}) |0 \rangle = \]

\[ \left( \text{number of baryons} \right) B(p_1, p_2, ..., p_{N_c}) B(q_1, q_2, ..., q_{N_c}) ... B(s_1, s_2, ..., s_{N_c}) |0 \rangle . \]

We may have mesonic parts in general, but in this picture they seem to be of smaller order. Note that this operator will survive the large-\( N_c \) limit and can be represented as the half trace of the variable \( T(p, q) \) evaluated only for the positive momenta. A natural question is if the baryon number operator is conserved under the evolution of our system— it does not follow from a simple symmetry principle—A direct method is to see if this operator Poisson commutes with a quadratic Hamiltonian. Let us write down a general quadratic Hamiltonian as

\[ H = \int [dp] h(p) T(-p, p) + \int [dpdqdsdt] G(p, q; s, t) T(p, q) T(s, t). \]

(30)

The Poincare invariance will impose certain restrictions on the choice of functions \( h, G \). There are a few obvious symmetries coming from the properties of the variable \( T \), the considerations of the next section shows all the symmetries required on \( G(p, q; s, t) \). If we compute now

\[ \{H, \int_{-\infty}^{\infty} T(-u, u)[du]\} = 2i \int [\text{sgn}(p) + \text{sgn}(q)] G(p, q; s, t) T(p, q) T(s, t)[dpdqdsdt], \]

(31)

the use of the symmetries in general will not give zero: this means that the baryon number is not conserved in general! In our case the computation gives a nonzero result, thus in the conventional sense we do not have baryons, yet we may have nonzero values of the trace implying possible baryonic states. We will see more comments on this from the geometry in the next section.

4 Geometry of the Phase Space

In this section we present a somewhat more rigorous approach and provide an interpretation of the underlying phase space of the theory. To do this let us discuss quantization again, for
this we closely follow the ideas in the article by Bowick and Rajeev[17] and for a more detailed presentation we refer to the beautiful article by Gracia-Bondía and Varilly[16]. There is also a nice representation theoretic presentation in [22].

When we look at a real scalar field in two dimensions in the light cone formalism, we may formally quantize the system by declaring existence of operators corresponding to the fields and we replace the Poisson bracket relations of these fields by commutators with an additional factor of $i$. Of course we assume that there is an underlying complex Hilbert space, on which these operators act! In this formal process we do not see where the complex structure comes from. In fact there is a natural complex structure: let us assume that the free hamiltonian is formally written as a quadratic form in the fields, $H = \frac{1}{2} \phi_\alpha Q_{\alpha \beta} \phi_\beta$, and we have a symplectic structure, $\omega$, $\int \frac{1}{2} \phi_\alpha \omega_{\alpha \beta} \partial_x \phi_\beta$. This symplectic structure defines a skew form on the space of solutions to the classical field equations. The natural operator to introduce is $\tilde{\omega} = \omega^{-1}Q$, this is a real antisymmetric operator(matrix) of type $(1,1)$, and comes from the equations of motion. We use its polar decomposition, $\tilde{\omega} = J S$, where $J^T J = 1$ and $S^T = S$ with $S > 0$. Now using the antisymmetry of $\tilde{\omega}$ we see that $J^2 = -1$. This defines a complex structure which we can use to complexify our real Hilbert space. If we apply this to our case, the metric coming from the free Hamiltonian, $H_0 = \frac{m^2}{2} \int dx - \phi_\alpha(x^-) \phi_\alpha(x^-)$, becomes, $Q_{\alpha \beta}(x^-, y^-) = m^2 \delta(x^- - y^-) \delta_{\alpha \beta}$, and the symplectic form (see the previous section)

$$\omega_{\alpha \beta}(x^-, y^-) = <x^-| - 2 \partial_-|y^- > \delta_{\alpha \beta}.$$  

If we write down the polar decomposition, we have,

$$J_{\alpha \beta}(x^-, y^-) = <x^-| - (\partial_7 \partial_-)^{1/2} \partial_-^{-1}|y^- > \delta_{\alpha \beta} = <x^-| - (\partial_7^2)^{1/2} \partial_-^{-1}|y^- > \delta_{\alpha \beta}. \tag{32}$$

Written in this form this is a real operator acting on the $L^2$ space of initial data on the light cone. We can extend this operator to a complex Hilbert space and it is then possible to diagonalize the above $J$ in this complexified space. So we think of a complex $L^2$ space, $V^C = V \otimes C = W \oplus \bar{W}$, where $W$ is isomorphic to $\bar{W}$, in the infinite dimensional case they are both separable. The decomposition we use corresponds to the eigenspaces of $J$. If we write $J$ as a block diagonal on such a decomposition we get $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. We know from our experience in physics that this is the form we use. If we decompose the fields into Fourier modes at the initial data surface $x^2 = 0$,

$$\phi_\alpha(x^-) = \int_0^\infty \frac{dp}{\sqrt{2p}} (\bar{z}_\alpha(p)e^{-ipx^-} + z_\alpha(p)e^{ipx^-}) \tag{33}$$

and act upon it by $J$, we see that we get

$$(J \phi_\alpha)(x^-) = \int_0^\infty \frac{dp}{\sqrt{2p}} (-iz_\alpha(p)e^{ipx^-} + i\bar{z}_\alpha(p)e^{-ipx^-}). \tag{34}$$

So we see that the decomposition of the field into its positive and negative frequency modes is the same as using the eigenvalue decomposition of the underlying complex structure. (We note that this decomposition is relativistically invariant, and the division by momentum variable $\sqrt{2p}$ is for convenience). Now we can also see that the inverse of our skew form transforms under such a change of basis as $R^{-1}\omega^{-1}(R^{-1})^T$, where we represent the Fourier transform as $R$, here $T$ refers to the ordinary transpose. Thus we evaluate,

$$\int dx^-dy^- e^{ipx^-} \sqrt{2|q|} (\frac{-1}{2\theta_-})e^{ipy^-} \sqrt{2|p|} = i\text{sgn}(p)\delta[p + q], \tag{35}$$
which shows that the symplectic form transforms to the standard form now defined on a complex Hilbert space.

The correct quantization in the infinite dimensional case requires this complex structure, the formal quantization rule,

$$\left[\hat{\phi}_\alpha(x^-), \hat{\phi}_\beta(y^-)\right] = -\frac{i}{4}\delta_{\alpha\beta}\text{sgn}(x^- - y^-), \quad (36)$$

clearly requires a complex space, we assume the real field is a self-adjoint operator, $\hat{\phi}^\dagger(x^-) = \hat{\phi}(x^-)$. In fact we really think of this system in terms of creation and annihilation operators acting on a complex Hilbert space. This is best done by going into a Fourier decomposition and introducing the creation and annihilation operators corresponding to positive and negative frequency components. Such a decomposition is necessary to make the commutation relations meaningful, a glance at them shows that $[a_\alpha(p), a_\beta(q)] = \text{sgn}(p)\delta[p + q]$ is consistent with the creation and annihilation operator interpretation if we define $a_\alpha(p)$ to be the annihilation and $a_\alpha(-p)$ to be the creation operators for $p > 0$. Now we see that what determines this is precisely the complex structure, $J = -i\text{sgn}(p)$. This form of the complex structure reveals another important aspect of this problem: there is no dependence on the mass. If the bare mass changes due to the interactions, this does not change the quasi-free representation of our commutation relations that were chosen at the start using the free part only. The frequencies obviously change but that does not affect the representation. To make the Hamiltonian and various other operators of physical interest well-defined in this Fock space we must introduce a normal ordering prescription.

If we compute the commutator of two normal ordered bilinears of the field operators, that provides a realization of the real Symplectic Lie algebra in its standard form. When we switch to the Fourier modes, and use the corresponding creation and annihilation operators we use the embedding of the real symplectic Lie algebra into the complex symplectic Lie algebra. In fact our operators $K(p, q)$, in the large-$N_c$ limit, correspond to the Lie algebra generators with respect to this embedding, we will discuss this below. If we define our symplectic form as a matrix $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the complex structure as $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we can diagonalize our complex structure in a complex Hilbert space by $R = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, then $R^{-1}JR = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, whereas $R^T\omega R = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In such an embedding the real symplectic group defined by $\omega$ becomes,

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad (37)$$

and naturally still preserves the transformed form of $\omega$, but that is the same as the complex symplectic group, since $\omega$ as a matrix preserves its form. A general complex symplectic matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, satisfies,

$$a^Tc = c^Ta \quad b^Td = d^Tb \quad a^Td - c^Tb = 1. \quad (38)$$
In our example we see that the Fourier transform does this transformation: it brings $J$ into diagonal form and $\omega$ to the standard form.

The real Lie algebra can be written as

$$1 + i \begin{pmatrix} F & G \\ -\bar{G} & -\bar{F} \end{pmatrix},$$

where $F^\dagger = F$ and $G^T = G$. In fact one can check that the large-$N_c$ limit operators $K(p,q)$ obey these conditions. Furthermore there will be convergence conditions coming from the super-renormalizability of our theory. This corresponds to the fact that we require normal ordered bilinears to create finite norm states when they act on any other state constructed from the vacuum by the action of creation operators–of course strictly speaking we should think about smeared out operators but we will ignore this technical part for this work. We can simply say that the off-diagonal components of these operators, that is $b$ parts, should be Hilbert-Schmidt operators. In the same way we demand the same for the Lie algebra elements. (In higher dimensional theories this requirement is not satisfied and one needs a much more sophisticated not completely understood approach. One possibility was proposed by Mickelsson and Rajeev\cite{24,23}).

In this language the constraint should be written as $(iK + i\epsilon)^2 = -1$, and $i\epsilon = J$, i.e. it is the diagonal form of the complex structure we were to begin with. There is the skew form which has a matrix form in this basis $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which defines the symplectic group on $W \oplus \bar{W}$. We will see that the constraint actually defines a homogenous manifold of the underlying real symplectic group. If we introduce a variable $\Phi = K + \epsilon$, the constraint becomes,

$$\Phi^2 = 1.$$

One can also verify the following condition,

$$\Phi^T = \omega^{-1} \Phi \omega.$$  \hspace{1cm} (41)

This is nothing but the Lie algebra condition. In this basis there is no difference as matrices between $\omega$ and $\omega^{-1}$ but we should remember that they transform differently. Furthermore the convergence condition becomes,

$$[\epsilon, \Phi] \in \text{Hilbert – Schmidt}.$$  \hspace{1cm} (42)

As we will see in the following part these conditions correspond to the infinite dimensional version of the Siegel disc.

We now define a homogeneous manifold which will be denoted by $D^R_1$. It is essentially a real version of the disc which corresponds to the pseudo-unitary group. Let us define a Hilbert space $\mathcal{H}_+$, which refers to the positive frequency modes of the theory. We can also say that these are the functions which have only positive modes in their Fourier decomposition. We introduce a set of operators $Z : \mathcal{H}_+ \to \mathcal{H}_-$, where $\mathcal{H}_-$ is $\mathcal{H}_+$ in the above language. (If we use the full complex Hilbert space, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$). We impose $Z^T = Z$. We have a complex conjugation $\sigma$, this intertwines between $\mathcal{H}_+$ and $\mathcal{H}_-$, we define $Z^T = \sigma Z^\dagger \sigma$, note that $Z^\dagger : \mathcal{H}_- \to \mathcal{H}_+$, thus $Z^T : \mathcal{H}_+ \to \mathcal{H}_-$. Furthermore $Z = \sigma Z \sigma : \mathcal{H}_+ \to \mathcal{H}_-$. There is
an extra condition on $Z$: $1 - Z^\dagger Z > 0$. We also need a convergence condition which comes from the infinite dimensionality of the theory: $Z \in I_2$, where $I_2$ denotes the Hilbert-Schmidt ideal $[23, 1, 23]$.

We introduce a real restricted symplectic group, $Sp_1$ embedded into the above mentioned complex symplectic group, which we precisely define below:

$$Sp_1^c(\mathcal{H}) = \{ g : \mathcal{H} \to \mathcal{H}| g^{-1} \text{ exists, } g^T \omega g = \omega \text{ and } [\epsilon, g] \in I_2 \},$$

(43)

here we are using ordinary matrix transpose to be able to write explicit matrix elements. We can see that this is a group and we call it the restricted complex symplectic group, and its subgroup of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad b \in I_2$$

(44)

corresponds to the restricted real symplectic group $Sp_1(\mathcal{H})$. $J$ itself is a real symplectic matrix and we are using a basis for the complexified Hilbert space in which $J$ becomes diagonal.

The real symplectic group has an action on the space of operators $Z$, given by

$$g \circ Z = (aZ + b)(\bar{b}Z + \bar{a})^{-1}.$$  

(45)

We can check that the action obeys the usual rule $g_1 \circ (g_2 \circ Z) = (g_1 g_2) \circ Z$. To prove that the action preserves all the conditions we look at the orbit of $Z = 0$, which is obviously in this set $D_1^R$. We see that the the resulting element $\bar{b}a^{-1}$ satisfies all the properties, hence the orbit remains inside the disc. (We should of course see that the inverse of $\bar{a}$ exists, but that is easy using the properties of the group). Let us assume that a $Z$ is given, we claim that any such element lies in the orbit of $Z = 0$. To show this we explicitly construct a group element which does this:

$$g(Z) = \begin{pmatrix} (1 - \bar{Z}Z)^{-1/2} & Z(1 - \bar{Z}Z)^{-1/2} \\ \bar{Z}(1 - Z\bar{Z})^{-1/2} & (1 - \bar{Z}Z)^{-1/2} \end{pmatrix},$$

(46)

note that everything here is well-defined. We leave it to the reader to check that $g(Z)$ is an element of the real group. This shows that the disc is actually a homogeneous space: take any element $Z$, pull it back to $Z = 0$, by $g^{-1}(Z)$ and to reach any element $\tilde{Z}$ use the group element corresponding to this for the orbit of $Z = 0$, $g(\tilde{Z})$ and use the compatibility condition, $Z = (g(\tilde{Z})g^{-1}(Z)) \circ Z$. It is clear that the action then remains inside the disc.

We see that the disc is actually a complex homogeneous space, the stability subgroup corresponding to $Z = 0$ is given by

$$U(\mathcal{H}_+) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}.$$  

(47)

If we use symplectic condition we get $a^\dagger a = aa^\dagger = 1$, which means $a$ is an element of the unitary group of $\mathcal{H}_+$. This means we have

$$D_1^R = \frac{Sp_1(\mathcal{H})}{U(\mathcal{H}_+)}.$$  

(48)
We will in fact see that the above space is a complex homogeneous symplectic manifold, but before this it is useful to introduce a variable \( \Phi(Z) \):

\[
\Phi(Z) = -1 + 2 \left( \begin{pmatrix} (1 - ZZ)^{-1} & -(1 - ZZ)^{-1}Z \\ Z(1 - ZZ)^{-1} & -Z(1 - ZZ)^{-1}Z \end{pmatrix} \right). \quad (49)
\]

Using the defining properties of \( Z \) we can check that

\[
\Phi(Z)^2 = 1 \quad \Phi(Z)^T = \omega^{-1} \Phi(Z) \omega \quad [\epsilon, \Phi(Z)] \in \mathcal{I}_2, \quad (50)
\]

where we used the explicit standard matrix form of \( \omega \). Note that these are the same conditions on our physical variable \( \Phi \). We claim that all such \( \Phi(Z) \) lie on the orbit of \( \epsilon = \Phi(Z = 0) \).

This is easy to see using

\[
\Phi(Z) = -g(Z) \omega^{-1} g(Z)^T \omega, \quad (51)
\]

which also verifies the above conditions once more. One can see using the above identification that the action of the group on \( Z \) becomes quite simple in terms of \( \Phi \),

\[
g \circ Z \mapsto g \Phi g^{-1}. \quad (52)
\]

We can check that this action preserves all the conditions on \( \Phi \).

The manifold we have found is actually symplectic. We may define a natural two form,

\[
\Omega = \frac{i}{4} \text{Tr} \Phi d\Phi \wedge d\Phi. \quad (53)
\]

This formal expression should be understood as follows, we look at vector fields at a point \( \Phi \), any such thing can be expressed in terms of the Lie algebra elements, \( V_u(\Phi) = [u, \Phi] \), where \( u \) is an element of the Lie algebra. then the two form becomes,

\[
\Omega(V_u, V_v) = \frac{i}{8} \text{Tr} \Phi[[u, \Phi], [v, \Phi]] = \frac{i}{8} \text{Tr} \epsilon[[\epsilon, U^{-1}ug], [\epsilon, U^{-1}vg]]. \quad (54)
\]

The above form shows that the trace is well-defined due to the Hilbert-Schmidt conditions \[1, 9, 7\]. From this point of view it is easy to see that the above form is homogeneous, and it is closed (see \[1, 3\]). Nondegeneracy can be proved at \( \Phi = \epsilon \) and using homogeneity this is true over the manifold. If we look at the symplectic form at \( \epsilon \) by using the \( Z \) coordinates, we get

\[
\Omega|_{\epsilon} = 2i \text{Tr} d\bar{Z} \wedge dZ. \quad (55)
\]

A short computation reveals that when we write \( g^{-1}ug = i \begin{pmatrix} F_1 & G_1 \\ -\bar{G}_1 & -\bar{F}_1 \end{pmatrix} \) and same for \( v \),

\[
g^{-1}vg = i \begin{pmatrix} F_2 & G_2 \\ -\bar{G}_2 & -\bar{F}_2 \end{pmatrix}
\]

we get

\[
\Omega(V_u, V_v) = -\frac{i}{2} \text{Tr}(G_1 \bar{G}_2 - G_2 \bar{G}_1) = -i \text{Im} \text{Tr} G_1 \bar{G}_2. \quad (56)
\]

In fact the previous Poisson brackets come from this symplectic form, as can be checked. We will leave the details to the reader. We note an important point about \( \Phi \), the reader can verify that

\[
\Phi - \epsilon = \begin{pmatrix} 2Z(1 - ZZ)^{-1} & -(1 - ZZ)^{-1}Z \\ 2\bar{Z}(1 - ZZ)^{-1} & -2\bar{Z}(1 - ZZ)Z \end{pmatrix} \in \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 \end{pmatrix}, \quad (57)
\]

13
where $I_1$ denotes the ideal of trace class operators, hence a conditional trace for the variable $\Phi - \epsilon$ exists. We may therefore find moment maps which generate the underlying symmetry of the theory. We write down the answer but do not spend much time on it since we will not make use of these maps: $F_u = -\frac{1}{2} \text{Tr}_\epsilon u(\Phi - \epsilon)$, here $\text{Tr}_\epsilon A = \frac{1}{2} \text{Tr}(A + \epsilon A \epsilon)$. These provide a Poisson realization of the Lie algebra.

There could be baryonic states in the finite $N_c$ theory given by,

$$\frac{1}{Z} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_{N_c}} a_{\alpha_1}^\dagger (p_1) a_{\alpha_2}^\dagger (p_2) \ldots a_{\alpha_{N_c}}^\dagger (p_{N_c}) |0>, \quad (58)$$

where all the momenta are positive (see the previous section). We can measure this baryonic content by the half-trace of the operator $K$. We iterate again that this is not a conserved quantity, hence there is no baryon in the usual sense or a baryon number. The full trace gives zero since there is no anti-baryon. Let us see this by looking at the operator $\Phi$. If we evaluate the trace $\text{Tr}_\epsilon (\Phi - \epsilon) = 2(\text{Tr} b b^\dagger - \text{Tr} \overline{b} b^T)$, where we used the appropriate group element $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ to write $\Phi$. One can see that $\text{Tr} b b^T = \overline{\text{Tr} b b^\dagger} = \text{Tr} b b^\dagger$ since $bb^\dagger$ is positive hermitian. This shows that the trace is zero. In fact physically the correct one to take is half of this trace as we have seen in the previous section, so we define

$$B = \frac{1}{2} \text{Tr}[\left(\frac{1 + \epsilon}{2}\right)(\Phi - \epsilon)]. \quad (59)$$

We see that this is a positive number, which in the large-$N_c$ limit corresponds to the some type of baryonic content. The authors are unable to find a reason for this to be an integer, unlike the case discussed by Rajeev in [1], where the trace is related to the Fredholm index of the operators, thus is automatically an integer. We face another puzzle, not only the baryonic content is non-integer, it is always non-zero, that is when there are mesons there are also baryons! The limit we use seems to suggest that the baryon content and mesonic states start to mix up, since the above trace is zero only for the vacuum $\epsilon$. We are unable to resolve this issue at the moment. Another perspective on baryons is to think of the solitonic excitations of the gauge theory, and in our case a nonzero trace perhaps implies these type of excitations. The reader may question then the validity of the linear approximation, since we claim that the baryon number is always nonzero. In the linear approximation the above trace should be taken zero, since it corresponds to a quadratic.

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