\section{Introduction}

Alfvén cascade (AC) modes are a form of Alfvén eigenmode localized to the $q_{\text{min}}$ region of reverse shear plasmas. They were discovered in reversed shear discharges on the Joint European Torus,\textsuperscript{1–4} the Tokamak Fusion Test Reactor,\textsuperscript{5} Alcator C-Mod,\textsuperscript{6} and on JT-60U\textsuperscript{7} and DIII-D,\textsuperscript{8} where they are referred to as reversed-shear Alfvén eigenmodes (rsAE), and have been seen on many machines since. A principle characteristic of these modes is a slow (equilibrium timescale) sweep of the mode frequency, typically upward, but downward frequency sweeps have also been seen.\textsuperscript{4} The modes provide a useful check on the measurement of the evolution of the current profile in the plasma.

The addition of a motional Stark effect (MSE) diagnostic\textsuperscript{9} to the National Spherical Torus Experiment (NSTX) (Ref. 10) has made it possible to verify that NSTX has produced reversed shear discharges during the current ramp phase. Yet, Alfvén cascade (AC) modes were not observed in these plasmas. An explanation for their absence may be found in a recent theoretical model developed to explain the observed onset frequency of AC modes in JET as a coupling to the geodesic acoustic mode (GAM),\textsuperscript{11} or $\beta$-induced Alfvén eigenmode (BAE).\textsuperscript{12} The frequency sweeps saturate at the toroidal Alfvén eigenmode (TAE) frequency, a prediction supported by data from JET.\textsuperscript{11} Here, we show that this theory can also explain the absence of AC modes in the relatively high $\beta$ (thermal-to-magnetic energy ratio) of spherical tokamak plasmas where the GAM/BAE-onset frequency approaches or exceeds the TAE frequency, leaving no frequency band in which the AC modes might exist.\textsuperscript{13}

\section{Description of Experiment}

In order to test the hypothesis that the high $\beta$ of typical spherical tokamak plasmas suppressed the AC modes, a very low beta, neutral beam heated regime was explored in NSTX. The operational parameters used for these experiments were 0.8 MA of toroidal plasma current and $\approx$4.5 kG toroidal field. The plasmas were heated with 2 MW of deuterium neutral beam injection (NBI) power at a full energy of 90 keV. The vacuum vessel was well conditioned to minimize deuterium fueling from the protective carbon tiles. The major radius of the plasma magnetic axis was $\approx$1.0 m, and the plasma minor radius was $\approx$0.65 m. NBI heating was started early during the current ramp. In Figs. 1(b) and 1(c), time traces of the plasma current, NBI power, and toroidal beta are shown.

Under these conditions, it was possible to operate with peaked electron density profiles and central electron density as low as $\approx$1.0–1.2 $\times 10^{13}/$cm$^3$. For the data shown in Fig. 1, the electron density profile was nearly constant between 0.15 s and 0.35 s, with a peak density on axis of about 1.2 $\times 10^{13}/$cm$^3$. The electron temperature on axis increased from about 0.6 keV to 1.4 keV over this time (as well as for the two higher density, higher $\beta$ shots discussed below), and the ion temperature, measured by charge exchange recombination spectroscopy (CHERS),\textsuperscript{14} remained at about half the electron temperature. The plasma current was held constant after 250 ms and the plasma $\beta$ ($\beta$ is the ratio of thermal to magnetic energy) remained below $\approx$3% through the course of the discharge. At 0.25 s and at a major radius of $R = 1.2$ m, the local electron $\beta$ was $\approx$1.2%. At these very low densities, the frequency at the center of the TAE gap at the magnetic axis is given approximately by $f_{\text{TAE}} \approx V_{\text{Alfvén}}/(4\pi q R) = (160$ kHz/$q$), where $V_{\text{Alfvén}} = B^2/4\pi \rho$. The plasma rotation, measured by CHERS, remained low with the measured toroidal rotation frequency mostly $<2$ kHz except in a small region near the axis where it reached $\approx 5$ kHz.

The frequency sweeps in Fig. 1(a) begin above 50 kHz–80 kHz and saturate in a broad range near the center frequency of the TAE gap, as calculated using the $q_{\text{min}}$ deduced from AC modes and the density at the approximate location of $q_{\text{min}}$ (solid blue curve). Multiple modes appear sequen-
with an burst lasting 100 s. With Fourier transform, the modes are seen as a sequence of short bursts of higher time resolution, i.e., shorter sample time for the fast Fourier transforms. The evolution of the mode is resolved to be a sequence of short bursts.

The toroidal mode numbers, $n$, in the sequence of 2, 3, 2, 3, 4, with an $n=1$ mode coincident with the second $n=2$ mode.

In Fig. 1, the modes appear continuous in time, due to the low time resolution used for the spectrogram. With higher time resolution, i.e., shorter sample time for the fast Fourier transform, the modes are seen as a sequence of short bursts lasting 100 µs–200 µs, and with the bursts separated by a similar interval as shown in Fig. 2. Occasionally the bursts also show a downward frequency chirp. The last burst visible in Fig. 2 is accompanied by a lower frequency, strongly chirping mode. Coincident with the chirping mode, the neutron rate fell by ≈6%. Following this burst, the AC modes are suppressed, suggesting that the strongly chirping mode affected the fast ion population driving the AC modes.

For comparison with the low $\beta$ discharge of Fig. 1, Fig. 3(a) shows a spectrogram of a second, higher $\beta$ discharge. Helium gas puffing was used to raise the central density to ≈$3 \times 10^{13}/$cm$^3$ by 0.25 s and the local electron $\beta$ to ≈5%. The range of the upward frequency sweeps is reduced and the mode has become continuous in time. For example, the $n=3$ mode (green) appearing at ≈0.185 s shows two clear upward frequency sweeps, the first around 0.225 s and the second around 0.280 s. The $n=2$ (red), $n=4$ (blue), $n=5$ (cyan), and $n=6$ (magenta) modes show similar behavior. The plasma toroidal rotation frequency is increasing nearly linearly during this time, reaching ≈5 kHz at 250 ms at the approximate location of the modes. Doppler-shift corrections to the mode frequency evolution are necessary to interpret the data, and the mode frequency, corrected for the Doppler shift at the approximate mode location of $R\approx1.25$ m, is shown in Fig. 3(b). The modes are assumed localized near the $q_{\text{min}}$ location, which is at a major radius of about 1.25 m in similar plasmas. The reflectometer measurements of mode amplitude are consistent with mode amplitude peaking near this radius. However, the modes are clearly global, and as such will likely be affected by the shape of the rotation profile as well.

At even higher density and higher electron $\beta_e\approx7\%$ (at $R=1.2$ m), the slow upward sweeps disappear altogether, and only bursting modes identified as TAE are seen. An example just at this threshold is shown in Fig. 4. The $n=3$ (green) and $n=4$ (blue) TAE appear and disappear as, presumably, the $q_{\text{min}}$ passes through and between low order rationals. The frequency sweeps which had previously connected the periods of TAE activity in Fig. 3 are now absent.

The internal amplitude and radial structure of the modes are measured with three fixed-frequency quadrature reflectometers. In Fig. 5(a) it is shown that the amplitude of the $n=3$, 4 and $n=5$ modes at 0.271 s peaks near the $q_{\text{min}}$
Here the $n=3$ mode is near a minimum of the frequency, the $n=4$ mode is sweeping down in frequency, and the $n=5$ mode is near frequency saturation. The localization is strongest for the highest $n$ mode as would be expected for higher poloidal mode numbers. The reflectometer data are from the shot shown in Fig. 3 and the $q$ profile is from a higher density shot without AC modes.

The toroidal mode numbers were measured with an array of 12 coils measuring poloidal magnetic field fluctuations. The data were acquired at a 4 MHz sampling rate. An additional array of coils, sampled at 5 MHz, measures the amplitude and relative phase of the toroidal and poloidal magnetic fluctuations, from which the mode polarization is deduced. Here, polarization refers to shear (perpendicular to the equilibrium field) vs compressional (parallel to the equilibrium field) magnetic fluctuations. Figure 6(a) shows the polarization of the $n=4$ mode magnetic field fluctuations, determined from this array, as an ellipse, with the vertical axis showing poloidal magnetic fluctuation and the horizontal axis showing toroidal fluctuation. The waves qualitatively have a shear polarization, that is, the magnetic fluctuations are transverse to the equilibrium field and the ellipse is nearly (within uncertainty) degenerate, forming a line perpendicular to the equilibrium field. This is consistent with the expectation that AC modes and TAE have “shear” polarization.

The pitch of the equilibrium field varies across the plasma radius as shown in Fig. 6(b). The radial eigenfunction of AC modes should be dominated by the poloidal component localized near the minimum in $q$. Assuming that the polarization of the mode magnetic fluctuations, even as measured at the vacuum vessel wall, reflects the polarization of the dominant poloidal harmonic, the pitch of this ellipse would put the magnetic fluctuations perpendicular to the equilibrium magnetic field at a major radius of about 1.18 m, as shown in Fig. 6(b). This is $\approx 5$ cm inside of the radius of $q_{\text{min}}$ determined from MSE constrained equilibrium fits. It is
also inside the radius of the peak in mode amplitude suggested by reflectometer measurements, although these measurements have, at present, rather limited radial resolution. It remains to be seen whether theoretical modeling can explain this apparent, albeit small, discrepancy.

III. ANALYSIS OF DATA

The modes with upward frequency sweeps in Figs. 1 and 3 have many of the characteristics of Alfvén cascade (AC) modes. The mode frequency evolution shown in these figures will next be analyzed in the context of the Alfvén cascade mode dispersion relation and shown to be consistent with the interpretation of these modes as Alfvén cascades.

Cascade modes are found in reverse-shear plasmas, localized near the radial minimum in \( q \) (1/\( q \) is a normalized measure of magnetic field twist). The simple dispersion relation for AC modes at low \( \beta_e \) is \( \omega = k_i V_{\text{Alfvén}} \) where \( k_i = (m - n q_{\text{min}})/q_{\text{min}} R \), and unlike TAE, the modes are assumed to have a single dominant poloidal mode number, \( m \). When \( q_{\text{min}} = m/n, k_i = 0 \) and as \( q_{\text{min}} \) drops, the mode frequency, \( \omega_{\text{cascade}} \) starts near zero frequency and sweeps upwards. When the \( q_{\text{min}} \) drops approximately halfway to the next lower rational value for a given \( n \), the mode is believed to convert to a TAE and the mode frequency saturates at the TAE frequency. As the \( q_{\text{min}} \) continues to fall, the mode may transform back into a cascade mode and the frequency will sweep downwards.

A more complete dispersion relation\(^{11}\) for higher \( \beta_e \) plasmas includes coupling to the GAM or BAE and is given by

\[
\omega_{\text{AC}}^2 = \left[ k_i^2 V_{\text{Alfvén}}^2 + \frac{2C_s^2}{R^2} \left( 1 + \frac{7}{4} \frac{T_i}{T_e} \right) \right], \quad k_i = \frac{m - n q_{\text{min}}}{q_{\text{min}} R}.
\]  

(1)

In this dispersion relation, when \( q_{\text{min}} = m/n, \) i.e., \( k_i = 0 \), there is a local minimum in mode frequency,

\[
\omega_{\text{min}}^2 = \frac{2C_s^2}{R^2} \left( 1 + \frac{7}{4} \frac{T_i}{T_e} \right).
\]  

(2)

As the temperature and density rise, the TAE frequency drops and the GAM frequency increases. When \( \beta_e \) is high enough so that \( \omega_{\text{min}} \approx \omega_{\text{TAE}} \approx V_{\text{Alfvén}}(2qR)^{-1} \), the range of the frequency sweep is zero.

The evolution of \( q_{\text{min}} \) can be deduced from the frequency sweep of the modes in Fig. 1, assuming that the modes follow the cascade dispersion relation [Eq. (1)]. First the poloidal mode number corresponding to each toroidal mode number must be determined. The second \( n=2 \) mode appearing concurrently with the \( n=1 \) mode suggest a \( q_{\text{min}} = 2 \) crossing at about 0.185 s, and these two modes would have \( (m/n) = (4,2) \) and \( (2,1) \). The assumption that \( q_{\text{min}} \) is decreasing monotonically then implies the following time sequence for the cascade mode numbers: \( (m/n) = (5,2), (4,2), (2,1), (5,3), (3,2), (4,3), (5,4) \).

The frequency evolution of the seven distinct upward frequency sweeps in Fig. 1 is used to calculate \( q_{\text{min}}(t) \). Equation (1) is solved for \( q_{\text{min}} \), using the \( (m,n) \) from above for each mode,

\[
q_{\text{min}} = \frac{mV_{\text{Alfvén}}}{nV_{\text{Alfvén}} + R(\omega_{\text{mode}}^2 - \omega_{\text{min}}^2)^{1/2}}.
\]  

(3)

The \( \omega_{\text{min}} \) term, small for this shot, is calculated from Eq. (2). In Fig. 7 the \( q_{\text{min}}(t) \) for each of the modes seen in Fig. 1 is shown, color-coded as in the spectrograms. Of course, when the frequency saturates, the frequency evolution would no longer be described by this simple dispersion relation, and the \( q_{\text{min}}(t) \) calculated by this method also saturates.

The mode behavior in Fig. 3(b) resembles the cascade mode behavior seen, e.g., in Fig. 3 of Breizman et al.\(^{11}\) (The continuum damping is thought to be larger during the downward frequency sweeps, thus explaining the usual absence of the downward sweeping portion of the mode evolution.) For these data we can identify the times of the rational \( q_{\text{min}} \).
crossings as the times at which the frequency evolution hits local minima and the \( q_{\text{min}} \) derived by this method are indicated by the black squares in Fig. 8.

Near the frequency minima, the \( q_{\text{min}}(t) \) evolution can be calculated from the frequency evolution using Eq. (3), for the \( n=2, n=3, \) and \( n=4 \) mode frequency evolutions (where the assumed poloidal mode number decreases by one at each frequency peak). The poloidal mode numbers used for each toroidal mode number are indicated in the figure. For the analysis of these data, the \( \omega_{\text{min}} \) is found directly from Fig. 3(b). The \( \omega_{\text{min}} \) is assumed to vary linearly over the time interval between the two minima for each frequency curve [e.g., red line in Fig. 3(b)]. This linear approximation to the \( \omega_{\text{min}} \) evolution is \( \approx 0.9 \) times the value predicted by Eq. (2) for the \( n=2 \) mode [black line in Fig. 3(b)]. The local (in time) evolution of \( q_{\text{min}} \) from this method agrees well with the \( q_{\text{min}} \) rational crossings identified from the frequency minima.

Direct measurements of the \( q \) profile evolution were not available for the examples shown in Figs. 1 and 3. However, MSE data starting at \( \approx 0.27 \) s is available for a similar shot with slightly higher density \( (=16\% \) higher), to that shown in Fig. 3. The \( q \) profile in this later shot clearly had reversed shear with the outboard \( q_{\text{min}} \) at \( R \approx 1.2 \) to \( 1.25 \) m. In Fig. 8, from \( 0.27 \) s onward, the black curve shows the time evolution of \( q_{\text{min}} \) deduced from MSE data for this slightly higher density shot. The MSE system presently measures pitch angles at 12 spatial locations approximately equally spaced from about the plasma axis to the edge. The evolution of \( q_{\text{min}} \) inferred from multiple frequency sweeps consistent with the measured \( q_{\text{min}} \) evolution, albeit from a different but similar shot, provides a strong argument that these modes are Alfvén cascades.

The threshold \( \beta \) for Alfvén cascade suppression can be estimated by setting the AC minimum frequency equal to the TAE frequency,

\[
\frac{\omega_{\text{min}}^2}{\omega_{\text{TAE}}^2} = \frac{4q^2R^2}{V_A^2} \left( \frac{2C_s}{R^2} \left( 1 + \frac{7}{4} T_i/T_e \right) \right) \approx 1.
\]

However, these dispersion relations for the TAE and GAM frequencies are approximate. We can arrive at a better estimate of the \( \beta_{\text{crit}} \) by taking into account the empirical correction factors for \( \omega_{\text{min}} \) and \( \omega_{\text{TAE}} \) from Fig. 3(b). A scale factor of \( \approx 0.9 \) was determined for \( \omega_{\text{min}} \) from the data shown in Fig. 3(b), and the \( \omega_{\text{TAE}} \) is underestimated by a factor of \( \approx 1.25 \), which together yields an empirical correction factor of \( \approx 1.4 \) for the frequency ratio, or when squared \( \approx 2 \) for \( \beta_{\text{crit}} \). We can then estimate from Eq. (4) the critical \( \beta \) as \( \beta_{\text{crit}} \approx \left( 20\% - 25\% \right)/q^2 \) for the range of ion and electron temperature ratios in these data. The \( \beta_{i}/\beta_{\text{crit}} \) may be estimated, including the ion temperature term, as \( \beta_{i}/\beta_{\text{crit}} \approx 0.06 - 0.2 \) for the data in Fig. 1, \( \beta_{i}/\beta_{\text{crit}} \approx 0.5 - 0.7 \) for the data shown in Fig. 3, and \( \beta_{i}/\beta_{\text{crit}} \approx 1 \) for the shot with MSE data where AC were not seen (Fig. 4).

IV. SUMMARY

In summary, the absence of Alfvén cascades in reversed-shear NSTX discharges was particularly striking given the otherwise rich spectrum of fast-ion driven instabilities in NSTX. The results of the experiment reported here demonstrate that this unusual absence of AC modes in spherical tokamaks is a direct consequence of their typically higher beta and is not attributable to a peculiar evolution of the \( q \) profile.
or a deficiency in Alfvén cascade theory. In fact, the clear observation of Alfvén cascades in relatively low $\beta$ NSTX discharges is strong confirmation of the validity of the established theory. The $q_{\text{min}}$ evolution deduced from the Alfvén cascades is in reasonable agreement with the $q_{\text{min}}$ inferred from MSE measurements in a similar, but slightly higher density shot. The modes are further shown to be localized to the $q_{\text{min}}$ region, as predicted by theory. These observations suggest that AC modes are likely to not be a problem during the normal operation of spherical tokamaks, nor, however, will they provide a measurement of the $q_{\text{min}}$ evolution.

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