Quasilocal Center-of-Mass

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Gravitating systems have no well-defined local energy-momentum density. Various quasilocal proposals have been made, however the center-of-mass moment (COM) has generally been overlooked. Asymptotically flat gravitating systems have 10 total conserved quantities associated with the Poincaré symmetry at infinity. In addition to energy-momentum and angular momentum (associated with translations and rotations) there is the boost quantity: the COM. A complete quasilocal formulation should include this quantity. Getting good values for the COM is a fairly strict requirement, imposing the most restrictive fall off conditions on the variables. We take a covariant Hamiltonian approach, associating Hamiltonian boundary terms with quasilocal quantities and boundary conditions. Unlike several others, our covariant symplectic quasilocal expressions do have the proper asymptotic form for all 10 quantities.

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I. INTRODUCTION

Associated with the flat spacetime geometric symmetries there are 10 conserved quantities: energy-momentum (EM) (translations), angular momentum (AM) (rotations) and its often overlooked covariant partner, the center-of-mass moment (COM) (boosts). Asymptotically flat gravitating systems have all of these quantities. For such spacetimes the total values exist globally but there are no well defined local densities.

The localization of energy-momentum for gravitating systems remains an outstanding problem. The source of gravity is the EM density. EM is exchanged locally between sources and gravity, hence we expect something like a local gravitational EM density. But standard techniques give only non-covariant (coordinate dependent) pseudotensors; gravity has no proper EM (nor AM/COM) density. This is consistent with the equivalence principle: gravity cannot be detected at a point. It is now believed that the proper idea is quasilocal quantities (i.e., associated with a closed 2-surface). We want good quasilocal expressions for EM and AM.

Most earlier quasilocal investigations focused on energy-momentum. Angular momentum also received some attention, however its 4-covariant associate, the center-of-mass moment, has been very much neglected. Obtaining the correct value for this quantity is actually quite a severe requirement: consequently it provides a very discriminating test for proposed expressions.

A good approach to energy-momentum is via the Hamiltonian. The Hamiltonian for a (finite or infinite) region necessarily includes a boundary term. The Hamiltonian boundary terms give both the quasilocal quantities and the boundary conditions. We have developed a covariant Hamiltonian formalism which has given certain special covariant-symplectic boundary terms. These expressions have already been well tested for EM and AM.

The COM test has recently been applied to our “covariant-symplectic” Hamiltonian-boundary-term quasilocal expressions. Here we briefly describe the results of that investigation, their correspondence with the asymptotically acceptable total expressions, outline how they give the quasilocal COM for Einstein’s GR, and compare them with other quasilocal expressions.

II. THE HAMILTONIAN APPROACH

The Hamiltonian depends on a spacetime displacement vector field; it includes both a volume term and a spatial boundary term:

$$H(N) = \int_{\Sigma} \mathcal{H}(N) = \int_{\Sigma} N^\mu \mathcal{H}_\mu + \oint_{S=\partial\Sigma} B(N).$$

The Hamiltonian boundary term has important roles. The value of the Hamiltonian gives the quasilocal quantities. This value is conserved; different displacements give the quasilocal EM and AM/COM. Since $\mathcal{H}_\mu$ vanishes “on shell” the value of the Hamiltonian is determined by the Hamiltonian boundary term (HBT): $B(N)$. The “natural” HBT inherited from the Lagrangian can—and should—be adjusted (as first clearly noted for GR by Regge and Teitelboim).

The general expression for $B$ depends on the choice of variables, a displacement vector field (e.g. translation...
for energy-momentum and rotation for angular momentum), a reference configuration and boundary conditions. Boundary conditions are determined by the Hamiltonian variation boundary principle. Formally there are an infinite number of possible choices for \( B \) corresponding to different reference configurations and boundary conditions, hence selection criteria are needed. Good asymptotics is one important condition. Another is covariance. We found that there are only two choices which give rise to a boundary term in the variation of the Hamiltonian which requires us to hold fixed (certain projected components of) covariant objects (essentially these two choices correspond to Dirichlet or Neumann boundary conditions). In particular for GR, associated with boundary conditions imposed on \( \pi^{\mu\nu} := (-g)^{1/2}g^{\mu\nu} \) we have

\[
B_g^{\mu\nu}(N) = N^\tau \pi^{\beta\lambda}\Delta^\alpha_{\gamma} \delta_{\alpha}^{\mu\nu} + \bar{D}_{\beta} N^\alpha \Delta^\beta_{\alpha\lambda} \delta_{\alpha\lambda}. \tag{2}
\]

Here \( \Delta^\gamma_{\alpha\beta} := \Gamma - \bar{\Gamma}, \Delta^\pi := \pi - \bar{\pi} \), with \( \Gamma, \bar{\pi} \) being reference values. (The reference values have a simple meaning: all quasilocal quantities vanish when the dynamic variable takes on the reference values: the standard reference choice is flat Minkowski spacetime). Technically we prefer the differential form version:

\[
B(N) = \Delta^\alpha_{\beta\gamma} \pi_{\alpha\beta} + \bar{D}_{\beta} N^\alpha \Delta^\beta_{\alpha\lambda} \eta_{\alpha\beta}, \tag{3}
\]

where \( \eta_{\alpha\beta} := \ast (\partial^\alpha \wedge \partial^\beta) \). For details see \[2, 3, 4, 5, 8\].

Note the form of these Hamiltonian boundary terms:

\[
B(N) = N^\mu P_{\mu} + D^\mu N^\nu S_{\mu\nu}, \tag{4}
\]

qualitatively: \( B = \text{“Freud” + “Komar”} \); for AM it is easy to see that this corresponds to “orbital” + “spin”. Let the displacement have the asymptotic Poincaré form \( N^\mu = Z^\mu + \omega^\mu x^\nu \), with \( \omega^\mu = \omega|_{\nu=0} \). Then

\[
B = Z^\mu P_{\mu} + \frac{1}{2} \omega^{\mu\nu} J_{\mu\nu}, \tag{5}
\]

where

\[
J_{\mu\nu} = x^\mu P_{\nu} - x^\nu P_{\mu} + S_{\mu\nu}. \tag{6}
\]

This includes angular momentum: \( x^i P_j - x^j P_i \), and the center-of-mass: \( x^0 p^k - x^k p^0 \).

### III. RESULTS

We have compared our expressions with various other proposals. We first consider total expressions at spatial infinity. MTW \[8\], Eq. (20.9), gives the necessary asymptotic form for all 10 Poincaré quasilocal quantities (in an essentially 4-covariant form). It is straightforward to verify that our expressions \[8, 9\] have that asymptotic limit; the DN contribution, \( S \), is distinctive \[8\]. We have, in the first part of Table I, summarized the degree of success for various pseudotensor expressions, including those of Einstein/Freud, Duan and Feng, Landau and Lifshitz, Weinberg/MTW, Papapetrou and Goldberg \[2, 3, 8\].

Turning to Hamiltonian approaches, we note that the famous ADM expressions \[11\] have no DN term, these investigators did not consider the COM. This was done later by Regge and Teitelboim (RT) \[7\]; their Eq. (5.13) has a DN term which plays an important role in determining the COM. Beig and Ó Murchadha (BóM) \[12\] have given a refinement of the RT work (see their Eq. (3.37)); they noted that an explicit reference configuration is needed. More recently Szabados \[10\] has given an even more careful discussion, further refining the BóM results. These investigations have shown the overall importance of the COM: in order for it to be well defined one must impose the most strict asymptotic conditions of the variables. The second part of Table I, provides a summary.

Turning to quasilocal proposals, the seminal Brown and York work \[13\] has no DN term—and no COM discussion (a serious shortcoming in our view). Both the Witten spinor Hamiltonian and Tung’s spin 3/2 Hamiltonian have no DN term, and apparently cannot give AM/COM \[13\]. The apparently necessary DN terms do appear in our covariant symplectic expressions \[10\]. (Note: one of our “covariant symplectic” expressions was found independently by Katz, Bičáč and Lynden-Bell \[10\].)

This brings us to our central question: is the DN term absolutely essential? We find that it is necessary not only for the Hamiltonian variation but also generally for the COM value. Indeed the COM value is the only case for which it plays an essential role. The reason can be understood by first noting that asymptotically, because of the fall off rates, this term can affect only AM and/or COM, not EM.

Consider first angular momentum. We found that the DN contribution can play an important role for angular momentum, but it is not essential. In particular Vu considered the teleparallel equivalent of GR, GR\[11\]. Its tetrad version GRtet, which lacks DN terms, can give

| TABLE I: Success of asymptotic total expressions |
|-----------------------------------------------|
| **Pseudotensor approaches:**                  |
| Einstein/Freud                               | ok   | no  | no |
| Duan & Feng                                  | ok   | special gauge |
| LL                                          | ok   | ok  | ok |
| Weinberg/MTW                                 | ok   | ok  | ok |
| Papapetrou                                   | ok   | ok  | ok |
| Goldberg                                    | ok   | ok  | best |
| **Hamiltonian formulations:**                 |
| ADM                                          | ok   | ok  | no |
| RT                                           | ok   | ok  | good |
| BóM                                         | ok   | ok  | better |
| Szabados                                     | ok   | ok  | best |

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Consider first angular momentum. We found that the DN contribution can play an important role for angular momentum, but it is not essential. In particular Vu considered the teleparallel equivalent of GR, GR\[11\]. Its tetrad version GRtet, which lacks DN terms, can give
TABLE II: Success of various quasilocal expressions

| Quasilocal expressions               | EM | AM | COM |
|--------------------------------------|----|----|-----|
| Witten spinor Hamiltonian            | ok | no | no  |
| Tung’s 3/2 QSL Hamiltonian          | ok | no | no  |
| Brown & York                         | ok | ok | ok  |
| GRtet                                | ok | special incorrect gauge |
| GR||                                      | ok | ok | ok  |
| covariant-symplectic GR             | ok | ok | ok  |

the AM—but only in a certain frame gauge, whereas the GR|| version (which has DN terms) succeeds in a general frame [10]. On the other hand, many investigations have obtained the correct angular momentum at spatial infinity without using such a term [7, 12, 13]; also one can likewise get good quasilocal angular momentum without such a contribution [14].

We find, however, that DN terms play an essential role in obtaining the COM. It is readily apparent from the investigations at spatial infinity [7, 12, 13] that DN contributes only to the COM. Ho found that the DN terms are essential to get the COM in the GR|| theory [17]. Here, for GR, we outline a calculation showing their important role in obtaining the correct COM.

We did a simple test on the eccentric Schwarzschild geometry. Take the isotropic Schwarzschild solution,

$$ds^2 = -N^2dt^2 + \varphi^4(dx^2 + dy^2 + dz^2), \quad (7)$$

where \(\varphi = 1 + m/2r\) and \(N\varphi = 1 - m/2r\), and displace the center:

$$\frac{1}{r} \rightarrow \frac{1}{|r - a|} \approx \frac{1}{r} + \frac{a \cdot r}{r^3}. \quad (8)$$

Now, using the obvious Minkowski reference, evaluate \(\mathbf{N}\) with \(N^0 = \mathbf{v} \cdot \mathbf{r}\), this is a “boost” in the v direction. The “Freud” term gives: \((2/3)m\mathbf{a} \cdot \mathbf{v}\) and the “Komar” term gives \((1/3)m\mathbf{a} \cdot \mathbf{v}\). Together they give the total center-of-mass moment: \(m\mathbf{a}\).

The popular Brown and York (BY) quasilocal formalism [14] appears to have a major shortcoming: according to our discussion, from their quasilocal energy expression:

$$N(k - k_0), \quad (9)$$

it seems like one cannot get the correct total COM, since there is no DN term. However Baskaran, Lau and Petrov [18] have demonstrated, via a remarkable elaborate calculation, that \(\varrho\) asymptotically agrees (up to a term with vanishing integral) with the BöM expression. Thus the BY expression, contrary to our original belief, does give the correct COM value. In Table II we summarize the success of the various quasilocal expressions (for a detailed discussion of the Witten spinor Hamiltonian and Tung’s spin 3/2 QSL Hamiltonian see [15]; for GRtet and GR||, see [10, 17, 19]).

IV. CONCLUSION

Asymptotically flat spaces have 10 conserved quantities. A good description of EM should also include AM/COM (note: relativistic invariance requires COM along with AM). Many proposals have, unfortunately, overlooked the COM. To get the correct COM is a strong requirement; the COM imposes the strictest fall off conditions. Considering the COM can be decisive. From COM consideration we find that tetrad GR, Witten spinor, and Tung’s spin 3/2 quasilocal proposals all have serious shortcomings. Aside from the BY expression, only our covariant-symplectic GR, and GR|| satisfy the good COM requirement. Their asymptotic spatial limit does give the correct value for the total COM. Moreover the asymptotic form of our covariant symplectic Hamiltonian boundary expressions agrees with accepted expressions [3, 6, 12, 13]. To our knowledge they are the best behaved expressions which have so far been identified.

In summary, our investigation considered an important neglected quantity: the quasilocal COM; the investigation provides additional support for the covariant symplectic quasilocal expression.

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[1] J. M. Nester, Class. Quantum Grav. 21, S261 (2004).
[2] C. C. Chang, J. M. Nester and C. M. Chen, “Energy-Momentum (Quasi-)Localization for Gravitating Systems,” in Gravitation and Astrophysics, ed L. Liu et al (World Scientific, Singapore, 2000) 163, arXiv:gr-qc/9912058.
[3] C. M. Chen, J. M. Nester and R. S. Tung, Phys. Lett. A 203, 5 (1995) arXiv:gr-qc/9411048.
[4] C. M. Chen and J. M. Nester, Class. Quant. Grav. 16, 1279 (1999) arXiv:gr-qc/9809020.
[5] C. M. Chen and J. M. Nester, Grav. Cosmol. 6, 257 (2000) arXiv:gr-qc/0001088.
[6] F. F. Meng, “Quasilocal Center of Mass Moment for GR (MSc. thesis, NCU, 2002)
[7] T. Regge and C. Teitelboim, Annals Phys. 88, 286 (1974).
[8] C. C. Chang, J. M. Nester and C. M. Chen, Phys. Rev. Lett. 83, 1897 (1999) [arXiv:gr-qc/9809040].
[9] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation, (Freeman, San Francisco, 1973).
[10] K. H. Vu, “Quasilocal Energy-Momentum and Angular Momentum for Teleparallel Gravity” (MSc. Thesis, NCU, 2000)
[11] A. Arnowit, S. Deser and C. W. Misner, The dynamics of general relativity, in Gravitation: An Introduction to Current Research, ed L. Witten (Wiley, New York, 1962) pp 227–65.
[12] R. Beig and N. ´O Murchadha, Annals Phys. 174, 463 (1987).
[13] L. B. Szabados, Class. Quant. Grav. 20, 2627 (2003) [arXiv:gr-qc/0302033].
[14] J. D. Brown and J. W. . York, Phys. Rev. D 47, 1407 (1993) [arXiv:gr-qc/9209012].
[15] C. M. Chen, J. M. Nester and R. S. Tung, “Spinor Formulations for Gravitational Energy-Momentum”, in Clifford Algebras: Applications to Mathematics, Physics and Engineering (Progress in Mathematical Physics vol 34), ed R. Ablamowicz (Birkhauser, Boston, 2003) pp 417–30 [arXiv:gr-qc/0209100].
[16] J. Katz, J. Biˇc´ak and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997).
[17] F. H. Ho, “Quasilocal center-of-mass for GR||” (MSc. thesis, NCU, 2003).
[18] D. Baskaran, S. R. Lau and A. N. Petrov, Annals Phys. 307, 90 (2003) [arXiv:gr-qc/0301069].
[19] J. M. Nester, F. H. Ho and C. M. Chen, “Quasilocal Center-of-Mass for Teleparallel Gravity”, to appear in Proceedings of the 10th Marcel Grossman meeting (Rio de Janeiro, 2003) [arXiv:gr-qc/0403101].