THE NONSTATIONARY FLOWS OF MICROPOLAR FLUIDS
WITH THERMAL CONVECTION: AN ITERATIVE APPROACH

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Abstract. We consider a problem that describes the motion of a viscous
incompressible and heat-conducting micropolar fluids in a bounded domain
Ω ⊂ R³. We use an iterative method to analyze the existence, uniqueness, and
regularity of the solutions. We also determine the convergence rates in several
norms.

1. Introduction. Let Ω ⊂ R³ be a smooth bounded domain with boundary ∂Ω
and T > 0. We consider a problem that describes the motion of a viscous incom-
pressible and heat-conducting micropolar fluids in the region Qₜ = Ω × (0, T):

\[
\begin{align*}
    u_t - (\mu + \mu_r) \Delta u + u \cdot \nabla u + \nabla p &= 2\mu_r \text{rot} w + f(\theta) \quad \text{in } Q_T, \\
    w_t + Lw + u \cdot \nabla w + 4\mu_r w &= 2\mu_r \text{rot} u + g(\theta) \quad \text{in } Q_T, \\
    \theta_t + u \cdot \nabla \theta - \kappa \Delta \theta &= \Phi(u, w) + h \quad \text{in } Q_T, \\
    \text{div } u &= 0 \quad \text{in } Q_T, \\
    u = w = \theta &= 0 \quad \text{on } S_T = \partial \Omega \times (0, T), \\
    u(0) = u_0, \quad w(0) = w_0, \quad \theta(0) &= \theta_0, \quad \text{in } \Omega,
\end{align*}
\]  

where L is the Lamé operator given by

\[
    Lw = -(c_a + c_d) \Delta w - (c_0 + c_d - c_a) \nabla \text{div } w. 
\]

The vector functions \( u = (u_1, u_2, u_3) \), \( w = (w_1, w_2, w_3) \) and the scalar functions \( p \) and \( \theta \) denote the velocity, the angular velocity vector of rotation of particles, the
pressure, and the temperature respectively. The functions \( f, g \) denote the external
sources of linear and angular momentum, and \( h \) the heat source. The positive
constants \( \mu, \mu_r, c_0, c_a \) and \( c_d \) verify \( c_0 + c_d > c_a \) and the positive constant \( \kappa \) is the
heat conductivity. The quadratic nonlinear term \( \Phi \) denotes the dissipation function.

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This function is derived from the energy balance, and it is given by \( \Phi = \sum_{i=1}^{5} \Phi_i \), where

\[
\Phi_1(\mathbf{u}) = \frac{1}{2} \mu \sum_{i,j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 ,
\]

\[
\Phi_2(\mathbf{u}, \mathbf{w}) = 4 \mu_r \left[ \frac{1}{2} \text{rot} \mathbf{u} - \mathbf{w} \right]^2 ,
\]

\[
\Phi_3(\mathbf{w}) = c_0 (\text{div} \mathbf{w})^2 ,
\]

\[
\Phi_4(\mathbf{w}) = (c_a + c_d) \sum_{i,j=1}^{3} \left( \frac{\partial w_i}{\partial x_j} \right)^2 ,
\]

\[
\Phi_5(\mathbf{w}) = (c_d - c_a) \sum_{i,j=1}^{3} \frac{\partial w_i}{\partial x_j} \frac{\partial w_j}{\partial x_i} .
\]

Problem (1) includes some well known problems: if the energy balance equation is not considered we have the micropolar fluid equations, which have been extensively studied since its formulation given by Eringen in [3], see for instance [4], [10], [12], [13] and the references therein. If we omit the equation for the angular momentum balance and \( \Phi = 0 \) we have the heat convection equation which is also not considered we have the micropolar fluid equations, which have been extensively studied since its formulation given by Eringen in [3], see for instance [4], [10], [12], [13] and the references therein. If we omit the energy and the angular momentum balance equations we have the well known classical Navier-Stokes equations, see [7], [9] and [16]. Now, if we omit the energy and the angular momentum balance equations we have the well known classical Navier-Stokes equations, see [7], [9] and [16]. Throughout this work, we assume that the functions \( f, g \) and \( h \) verify the following condition: \( f(0) = g(0) = 0 \), there exist constants \( M_f, M_g > 0 \) such that

\[
|f(s) - f(t)| \leq M_f |t - s|, \quad |g(s) - g(t)| \leq M_g |t - s|
\]

(3) for \( s, t \in \mathbb{R} \) and \( h \in L^2(0,T;L^2(\Omega)) \).

Let \( H \) and \( V \) be the closure of the space \( C_{0,\alpha}^\infty = \{ \mathbf{u} \in C_{0,\alpha}^\infty(\Omega)^3; \text{div} \mathbf{u} = 0 \} \) in \( L^2(\Omega) \) and \( H_0^1(\Omega) \) respectively. A solution of problem (1) is understood in the following sense:

**Definition 1.1.** We say that a triple of functions \( (\mathbf{u}, \mathbf{w}, \theta) \) is a solution of problem (1) if

\[
\begin{align*}
\mathbf{u} &\in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \\
\mathbf{w} &\in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\
\theta &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)),
\end{align*}
\]

verifies the identities

\[
\begin{align*}
\int_0^T (\mathbf{u}_t - (\mu + \mu_r) \Delta \mathbf{u}, \varphi) dt &= \int_0^T \left( (\mathbf{u} + \mathbf{w}) - (\mu + \mu_r) \Delta \mathbf{u}, \varphi \right) dt, \\
\int_0^T \left( \mathbf{w}_t + \nabla \varphi, \mathbf{w} \right) dt &= \int_0^T \left( \mathbf{w} - (\mu + \mu_r) \Delta \mathbf{u}, \psi \right) dt, \\
\int_0^T (\theta_t, \phi) + \kappa \int_0^T (\nabla \theta, \nabla \phi) dt &= \int_0^T (\mathbf{u} \cdot \nabla \theta, \phi) dt, \\
\int_0^T (\mathbf{u}, \nabla \mathbf{w} + \mathbf{w}, \mathbf{w}) dt &= \int_0^T (\mathbf{u}, \nabla \theta, \phi) dt,
\end{align*}
\]

for \( \varphi \in L^2(0,T;H), \psi \in L^2(0,T;L^2(\Omega)), \phi \in L^2(0,T;H^1_0(\Omega)) \), and the initial conditions \( \mathbf{u}(0) = \mathbf{u}_0, \mathbf{w}(0) = \mathbf{w}_0 \) and \( \theta(0) = \theta_0 \).

As usual we use \( \| \cdot \| \) and \( (\cdot, \cdot) \) to denote the norm and the inner product in \( L^2(\Omega) \). Here, \( b : H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R} \) is a trilinear continuous form
defined by \( b(u, w, z) = (u \cdot \nabla w, z) = \sum_{i,j=1}^{3} \int_{\Omega} u_{i} \frac{\partial w_{j}}{\partial x_{j}} z_{j} dx \), where \( u = (u_{1}, u_{2}, u_{3}) \), \( w = (w_{1}, w_{2}, w_{3}) \) and \( z = (z_{1}, z_{2}, z_{3}) \).

The stationary problem associated to (1) was studied in [11]. It was shown that if \( f, g \) verify condition (3) and \( h \in L^{2}(0, T; L^{2}(\Omega)) \) then there exists a unique solution
\[
(u, w, \theta) \in H^{2}(\Omega) \times H^{2}(\Omega) \times H^{2}(\Omega),
\]
for \( \mu \) and \( c_{u} + c_{d} \) sufficiently large.

The existence and uniqueness of solutions of problem (1) was established in [6, Theorem 1.1] using the Banach fixed point argument. More precisely, we have the following result.

**Theorem 1.2** ([6]). Assume that \( f, g \) verify condition (3), \( h \in L^{2}(0, T; L^{2}(\Omega)) \), \( u_{0} \in V, w_{0} \in H^{2}_{0}(\Omega) \) and \( \theta_{0} \in L^{2}(\Omega) \). There exists a positive number \( T^{*} \leq T \) and a unique solution \((u, w, \theta)\) of problem (1) on \([0, T^{*}]\).

Moreover, if \( \mu_{r}, M_{f}, M_{g}, u_{0}, w_{0} \) and \( h \) are sufficiently small, then the solution can be extended in a unique way to the whole interval \([0, T]\).

Let \( P \) be the projection from \( L^{2}(\Omega) \) onto \( H \). The operator \( A = -P\Delta \) with \( D(A) = H^{2}(\Omega) \cap V \) is the Stokes operator and \( B = -\Delta \) with \( D(B) = H^{2}(\Omega) \cap H^{1}_{0}(\Omega) \) is the Laplace operator.

For every \( u_{0} \in V \), \( w_{0} \in H^{2}_{0}(\Omega) \) and \( \theta_{0} \in L^{2}(\Omega) \), we consider the sequence \( \{(u^{n}, w^{n}, \theta^{n})\}_{n \geq 1} \) defined by
\[
\begin{align*}
(u^{1}(t) = e^{-t(\mu+c_{d})B}u_{0}, \quad w^{1}(t) = e^{-t(\mu+c_{d})B}w_{0}, \quad \theta^{1}(t) = e^{t\Delta}\theta_{0},
\end{align*}
\]
where \( e^{-t(\mu+c_{d})B} \) and \( e^{-t(\mu+c_{d})B} \) denote the semigroups generated by the Stokes and Laplacian operators, respectively. For \( n \geq 2 \), \( (u^{n}, w^{n}, \theta^{n}) \) is given by the solution of the following linearized problem:
\[
\begin{align*}
\begin{cases}
\frac{u^{n+1}}{\mu} + \mu_{r}Au^{n+1} + P(u^{n} \cdot \nabla u^{n+1}) = 2\mu_{r}P(\text{rot} u^{n}) + Pf(\theta^{n}), & \text{in } Q_{T},
\frac{w^{n+1}}{\mu} + Lw^{n+1} + u^{n} \cdot \nabla w^{n+1} + 4\mu_{r}w^{n+1} = 2\mu_{r}\text{rot } u^{n} + g(\theta^{n}), & \text{in } Q_{T},
\theta^{n+1} = u^{n} \cdot \nabla \theta^{n+1} - \kappa \Delta \theta^{n+1} = \Phi(u^{n}, w^{n}) + h, & \text{in } Q_{T},
\text{div } u^{n+1} = 0, & \text{in } Q_{T},
\text{div } w^{n+1} = 0, & \text{in } S_{T},
w^{n+1}(0) = u_{0}, \quad w^{n+1}(0) = w_{0}, \quad \theta^{n+1}(0) = \theta_{0} & \text{in } \Omega.
\end{cases}
\end{align*}
\]
(4)

See Theorem 1.3 for details about the existence and properties of this sequence.

The main objective of this work is to analyze the convergence and rate of convergence of the sequence \( \{(u^{n}, w^{n}, \theta^{n})\}_{n \geq 1} \). As consequence, we establish existence, uniqueness and regularity results for problem (1). In particular, we recover Theorem 1.2, for initial data verifying conditions (5) and (6). Moreover, assuming additional hypotheses on the functions \( f \) and \( g \), we show that
\[
(u \in L^{\infty}(0, T; H^{2}(\Omega) \cap V), w \in L^{\infty}(0, T; H^{2}(\Omega) \cap H^{1}_{0}(\Omega)), \theta \in L^{\infty}(0, T; H^{1}_{0}(\Omega)),
\]
see Theorem 1.5 for details. We hope that the technique here used can be applied for the numerical analysis of the problem, where the full discrete scheme is considered.

It is important to point out that this method was proposed by Zarubin in [17] for finding the solution of the heat convection equation in the Boussinesq approximation, and was also used to treat the micropolar fluid problem in [14].

In our first result, we establish the existence and uniform estimates for the sequence \( \{(u^{n}, w^{n}, \theta^{n})\}_{n \geq 1} \).
Theorem 1.3. Let $f, g$ in $L^2(0,T;L^2(\Omega))$, $h$ in $L^2(0,T;L^2(\Omega))$, $u_0 \in V, w_0 \in H^1_0(\Omega)$ and $\theta_0 \in L^2(\Omega)$. Assume that
\[ \|\nabla u_0\|^2 + \frac{8\mu_r(\|u_0\|^2 + \|w_0\|^2)}{\mu + \mu_r}(c_a + c_d) < \delta^2 \] (5)
with
\[ \delta < 2^{-3/2} \lambda_1^{1/8} M_1^{-1} M_2^{-2} (\mu + \mu_r), \] (6)
where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$, $M_1$ and $M_2$ are constants given in (20) and (21).

Then, for every $n \geq 2$ there exist an unique solution $(u^n, w^n, \theta^n)$ of problem (4), defined on the interval $[0,T_1]$, with $0 < T_1 \leq T$, such that
\[ u^n \in L^\infty(0,T_1;V) \cap L^2(0,T_1;D(A)), \]
\[ w^n \in L^\infty(0,T_1;H^1_0(\Omega)) \cap L^2(0,T_1;D(B)), \]
\[ \theta^n \in L^\infty(0,T_1;L^2(\Omega)) \cap L^2(0,T_1;H^1_0(\Omega)), \]
\[ u^n_t \in L^2(0,T_1;H), \quad w^n_t \in L^2(0,T_1;L^2(\Omega)), \theta^n_t \in L^2(0,T_1;H^{-1}(\Omega)). \] (7)

In addition, there exits a constant $M_0 > 0$, independent of $n$, such that
\[ \int_0^t \|\nabla u^n(\tau)\|^2 d\tau + \int_0^t \|\nabla w^n(\tau)\|^2 d\tau + \int_0^t \|\nabla \theta^n(\tau)\|^2 d\tau \leq M_0, \]
\[ \sup_{t \in (0,T_1)} \{\|\nabla u^n(t)\|^2 + \|\nabla w^n(t)\|^2 + \|\theta^n(t)\|^2\} \leq M_0, \]
\[ \int_0^t \|Au^n(\tau)\|^2 d\tau + \int_0^t \|Bu^n(\tau)\|^2 d\tau \leq M_0, \]
\[ \int_0^t \|u^n_\tau(\tau)\|^2 d\tau + \int_0^t \|w^n_\tau(\tau)\|^2 d\tau + \int_0^t \|\theta^n_\tau(\tau)\|^2 d\tau \leq M_0, \]
for every $t \in (0,T_1)$.

The value of $T_1$ depends only on $\mu, \mu_r, c_a, c_d, \kappa, M_f, M_g, \Omega$ and $h$. Moreover, if $\mu, M_f, M_g$ and $h$ are sufficiently small, then it is possible to choose $T_1 = T$.

Remark 1. For every $n \in \mathbb{N}$, there exists $p^{n+1} \in L^2(0,T_1;H^1(\Omega)/\mathbb{R})$ such that
\[ u^{n+1}_t + u^n \cdot \nabla u^{n+1} - (\mu + \mu_r)\Delta u^{n+1} + \nabla p^{n+1} = 2\mu_r \text{rot } w^n + f(\theta^n) \]
in $Q_{T_1}$, and $\int_0^t \|p^{n+1}(\tau)\|^2_{H^1(\Omega)/\mathbb{R}} d\tau \leq C_0$, for every $t \in [0,T_1]$ and some constant $C_0 > 0$. Indeed, from (4) we have $(\mu + \mu_r)Au^{n+1} = P(F^{n+1})$, where $F^{n+1} = 2\mu_r \text{rot } w^n + f(\theta^n) - u^n \cdot \nabla u^{n+1} - u^{n+1}_t$. Thus,
\[ \|F^{n+1}\|^2 \leq C \|\nabla w^n\|^2 + C \|\theta^n\|^2 + C \|u^n\|_{L^4} \|\nabla u^{n+1}\|_{L^4} + C \|u^{n+1}_t\|^2 \]
\[ \leq C \|\nabla w^n\|^2 + C \|\theta^n\|^2 + C \|\nabla u^n\|^2 + \|Au^{n+1}\| + C \|u^{n+1}_t\|^2. \]

From Theorem 1.3, we conclude that $\int_0^T \|P^{n+1}(\tau)\|^2 d\tau \leq C$, that is, $F^{n+1} \in L^2(0,T_1;L^2(\Omega))$. Hence, by Theorem 3 of Amrouche and Girault [1], there exists a unique function $p^{n+1} \in L^2(0,T_1;H^1(\Omega)/\mathbb{R})$ so that $-(\mu + \mu_r)\Delta u^{n+1} + \nabla p^{n+1} = F^{n+1}$ in $Q_{T_1}$, and $\|p^{n+1}\|^2_{H^1(\Omega)/\mathbb{R}} \leq C \|F^{n+1}\|^2$.

In the next result we establish the existence and uniqueness of solutions of problem (1).
Theorem 1.4. Assume the hypothesis of Theorem 1.3. The sequence \( \{(u^n, w^n, \theta^n)\}_{n \geq 1} \) converges to a function \( (u, w, \theta) \) in \([0, T_1]\), as \( n \to \infty \). Moreover, \( (u, w, \theta) \) is the unique solution of problem (1), and the rate of convergence satisfies the following inequalities:

\[
\begin{align*}
\| (u^n - u, w^n - w, \theta^n - \theta)(t) \|_{V^* \times H^1_0 \times L^2} & \leq C \Lambda_{n-1}(t), \\
\int_0^t \| (u^n - u, w^n - w, \theta^n - \theta)(\tau) \|_{H^2 \times H^1 \times H^{-1}}^2 d\tau & \leq C \Lambda_{n-1}(t), \\
\int_0^t \| (u^n - u, w^n - w, \theta^n - \theta)(\tau) \|_{H^1 \times H^0 \times L^2}^2 d\tau & \leq C \Lambda_{n}(t), \\
\int_0^t \| (u^n_t - u_t, w^n_t - w_t, \theta^n_t - \theta_t)(\tau) \|_{L^2 \times L^2 \times H^{-1}}^2 d\tau & \leq C \sum_{i=0}^2 \Lambda_{n-i}(t),
\end{align*}
\]

for every \( t \in (0, T_1) \), where \( \Lambda_n(t) = (t^n/n!)^{1/2} \). Furthermore, if \( \mu_r, M_f, M_g, \) and \( h \) are sufficiently small, the solution exists on \([0, T]\).

In the next result we obtain more regularity for the solution of problem (1).

Theorem 1.5. Assume the hypothesis of Theorem 1.3, \( u_0 \in V \cap H^2(\Omega), w_0 \in H_0^1(\Omega) \cap H^2(\Omega), \theta_0 \in H_0^1(\Omega) \), \( f' \) and \( g' \) are bounded functions verifying condition (3) for constants \( M_F \) and \( M_G \), respectively. If \( (u, w, \theta) \) is the solution of problem (1) given by Theorem 1.4. Then, there exists \( \theta < T_2 \leq T_1 \) so that

\( u \in L^\infty(0, T_2; D(A)), w \in L^\infty(0, T_2; D(B)), \theta \in L^\infty(0, T_2; H_0^1(\Omega)), \)

and we have the following rates of convergence:

\[
\begin{align*}
\| (u^n_t - u_t, w^n_t - w_t)(t) \|_{L^2 \times L^2} & \leq C \sum_{i=1}^3 \Lambda_{n-i}(t) + CA^{1/2}_{n-2}(t), \\
\int_0^t \| (u^n_t - u_t, w^n_t - w_t, \theta^n_t - \theta_t)(\tau) \|_{H^1 \times H^0 \times H^{-1}}^2 d\tau & \leq C \sum_{i=1}^3 \Lambda_{n-i}(t) + CA^{1/2}_{n-2}(t), \\
\| (u^n - u, w^n - w, \theta^n - \theta)(t) \|_{H^2 \times H^0 \times H^0}^2 & \leq C \sum_{i=1}^2 \Lambda_{n-i}(t) + CA^{1/2}_{n-2}(t),
\end{align*}
\]

for every \( t \in (0, T_2) \), where \( \Lambda_n(t) = (t^n/n!)^{1/2} \).

Remark 2. Since \( f', g' \) are bounded, by the mean value theorem, we see that \( f, g \) verify condition (3).

2. Proof of the results. As usual, to simplify the notation, we denote by \( C \) a generic positive constant depending only on \( \Omega \) and the other fixed parameters of the problem. It may have different values in different expressions. When necessary, we emphasize that the constants may have different values using the notation \( C_1, C_2, \) and so on.

2.1. Proof of Theorem 1.3. The existence and regularity of the solution of problem (4) is proved using the spectral Galerkin method, see for instance [15]. Specifically, let \( \{\varphi_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}} \) and \( \{\theta_k\}_{k \in \mathbb{N}} \) be the set of the eigenfunctions of the Stokes operator \( A \), the operator \( B \), and the Laplacian operator \( -\Delta \), respectively. For every \( k \in \mathbb{N} \), we denote \( P_k, R_k \) and \( S_k \) the orthogonal projections from \( L^2(\Omega) \) onto \( V_k = \text{span}[\varphi^1, ..., \varphi^k] \subset V \cap H^2(\Omega), H_k = \text{span}[\psi^1, ..., \psi^k] \subset H_0^1(\Omega) \cap H^2(\Omega), W_k = \text{span}[\theta^1, ..., \theta^k] \subset H_0^1(\Omega) \cap H^2(\Omega) \), respectively.
Fix an integer \( n \geq 1 \). Given \( (u^n, w^n, \theta^n) \), verifying the regularity conditions (7), we consider the following discrete variational formulation of (4),

\[
\begin{align*}
(u_{i,t}^{n+1}, \varphi) + (\mu + \mu_r)(Au_i^{n+1}, \varphi) + (u^n \cdot \nabla u_i^{n+1}, \varphi) &= 2\mu_r(\text{rot } w^n, \varphi) + (f(\theta^n), \varphi), \\
(w_{i,t}^{n+1}, \psi) + (Lw_i^{n+1} + 4\mu_r, w_i^{n+1}, \psi) + (u^n \cdot \nabla w_i^{n+1}, \psi) &= 2\mu_r(\text{rot } u^n, \psi) + (g(\theta^n), \psi), \\
(\theta_{i,t}^{n+1}, \phi) + (u^n \cdot \nabla \theta_i^{n+1}, \phi) - \kappa(\Delta \theta_i^{n+1}, \phi) &= (\Phi(u^n), w^n), \phi + (h, \phi), \\
u_i^{n+1}(0) &= P_i u_0, w_i^{n+1}(0) = R_i w_0, \quad \theta_i^{n+1}(0) = S_i \theta_0,
\end{align*}
\]

for all \( \varphi \in V_i, \psi \in H_i, \phi \in W_i, \) where \( u_i^{n+1}, w_i^{n+1} \) and \( \theta_i^{n+1} \) have the form:

\[
\begin{align*}
u_i^{n+1}(\cdot, t) &= \sum_{k=1}^i a_{k,i}^{n+1}(t) \varphi^k(\cdot) \in V_i, \\
w_i^{n+1}(\cdot, t) &= \sum_{k=1}^i b_{k,i}^{n+1}(t) \psi^k(\cdot) \in H_i, \\
\theta_i^{n+1}(\cdot, t) &= \sum_{k=1}^i c_{k,i}^{n+1}(t) \theta^k(\cdot) \in W_i.
\end{align*}
\]

From (8) we have a linear system of ordinary differential equations for the coefficients \( a_{k,i}^{n+1}, b_{k,i}^{n+1}, c_{k,i}^{n+1} \). Under our conditions the existence of solutions for this system is guaranteed on the interval \([0,T]\). Moreover, due the regularity of \( u^{n-1}, w^{n-1}, \theta^{n-1} \), the solution \( u_i^{n+1}, w_i^{n+1}, \theta_i^{n+1} \) of system (8) have at least the same regularity. Thus, the computation that we will to perform to determine the estimates are justified.

The uniform estimates on \( n \) are prove in some steps.

**Step 1.** Estimate of \( \theta^n \) in the space \( L^\infty(0,T_1;L^2(\Omega)) \). We use the second principle of induction on \( n \). Assume that

\[
\sup_{t \in (0,T_1)} \| \theta^n(t) \| \leq R + 1
\]

holds for \( 1 \leq j \leq n \), where \( R > 0 \) is given by (38) and \( T_1 > 0 \) is sufficiently small such that verifies conditions (25) and (39).

We will show that this estimate also holds for \( n + 1 \),

(i) **Estimates for \( u^{j+1} \) and \( w^{j+1} \) in**

\[
L^\infty(0,T_1;L^2(\Omega)) \cap L^2(0,T_1;V) \text{ and } L^\infty(0,T_1;L^2(\Omega)) \cap L^2(0,T_1;H^1_0(\Omega))
\]

respectively.

Multiplying in \( L^2(\Omega) \) the first and the second equation of (4) (for \( n = j \)) by \( u^{j+1} \) and \( w^{j+1} \) respectively we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u^{j+1} \|^2 + (\mu + \mu_r) \| \nabla u^{j+1} \|^2 = 2\mu_r(\text{rot } w^j, u^{j+1}) + (f(\theta^j), u^{j+1})
\]

\[
\frac{1}{2} \frac{d}{dt} \| w^{j+1} \|^2 + (c_a + c_d) \| \nabla w^{j+1} \|^2 + (c_0 + c_d - c_a) \| \text{div } w^{j+1} \|^2 + 4\mu_r \| w^{j+1} \|^2 = 2\mu_r(\text{rot } u^j, w^{j+1}) + (g(\theta^j), w^{j+1}).
\]
Since \( \|u\|^2 \leq \lambda_1^{-1} \|\nabla u\|^2 \), and \( \|\text{rot} \ u\| = \|\nabla u\| \) for all \( u \in V \), using Hölder’s and Young’s inequalities we get

\[
|2\mu_r(\text{rot} \ w^j, u^{j+1})| = |2\mu_r(\text{rot} \ w^j, \text{rot} \ u^{j+1})| \\
\leq \frac{4\mu_r^2}{\mu + \mu_r}\|w^j\|^2 + \frac{\mu + \mu_r}{4}\|\nabla u^{j+1}\|^2,
\]

\[
|(f(\theta^j), u^{j+1})| \leq \frac{\lambda_1^{-1} M_f^2}{\mu + \mu_r}\|\theta^j\|^2 + \frac{\mu + \mu_r}{4}\|\nabla u^{j+1}\|^2,
\]

\[
|2\mu_r(\text{rot} \ w^j, u^{j+1})| = |2\mu_r(\text{rot} \ w^j, \text{rot} \ u^{j+1})| \\
\leq \frac{2\mu_r^2}{c_a + c_d} \|w^j\|^2 + \frac{c_a + c_d}{2} \|\nabla u^{j+1}\|^2,
\]

\[
|(g(\theta^j), u^{j+1})| \leq \frac{M_g^2}{16\mu_r}\|\theta^j\|^2 + 4\mu_r\|u^{j+1}\|^2.
\]

From these estimates, (10) and (11) we have

\[
\frac{d}{dt} \|u^{j+1}\|^2 + (\mu + \mu_r)\|\nabla u^{j+1}\|^2 \leq \frac{8\mu_r^2}{\mu + \mu_r}\|w^j\|^2 + \frac{2\lambda_1^{-1} M_f^2}{\mu + \mu_r}\|\theta^j\|^2 \leq \frac{8\mu_r^2}{\mu + \mu_r}\|w^j\|^2 + \frac{2\lambda_1^{-1} M_f^2}{\mu + \mu_r}\|\theta^j\|^2 (12)
\]

\[
\frac{d}{dt} \|w^{j+1}\|^2 + (c_a + c_d)\|\nabla w^{j+1}\|^2 \leq \frac{4\mu_r^2}{c_a + c_d}\|w^j\|^2 + \frac{M_g^2}{8\mu_r}\|\theta^j\|^2. (13)
\]

Thus, by (9) we have

\[
\|u^{j+1}(t)\|^2 + \|w^{j+1}(t)\|^2 + (\mu + \mu_r) \int_0^t \|\nabla u^{j+1}(s)\|^2 ds + (c_a + c_d) \int_0^t \|\nabla w^{j+1}(s)\|^2 ds + 2(c_0 + c_d - c_a) \int_0^t \|\text{div} \ w^{j+1}(s)\|^2 ds \\
\leq C_1 + C_2 \int_0^t (\|w^j(s)\|^2 + \|w^j(s)\|^2) ds,
\]

where

\[
C_1 = C_1(T_1) = \|u_0\|^2 + \|w_0\|^2 + (R + 1)T_1 \left( \frac{2 M_f^2}{\lambda_1 (\mu + \mu_r)} + \frac{M_g^2}{8\mu_r} \right),
\]

\[
C_2 = \max \left\{ \frac{8\mu_r^2}{\mu + \mu_r}, \frac{4\mu_r^2}{c_a + c_d} \right\}.
\]

Now, we define \( \phi_k(t) = \|u^k(t)\|^2 + \|w^k(t)\|^2 \) for \( k \geq 1 \) and \( t \in [0, T_1] \). Thus, estimate (14) implies

\[
\phi_{j+1}(t) \leq C_1 + C_2 \int_0^t \phi_j(t_1) dt_1,
\]

for every \( j = 1, \ldots, n \). Note that

\[
\phi_{j+1}(t) \leq C_1 + C_2 \int_0^t \left[ C_1 + C_2 \int_0^{t_1} \phi_{j-1}(t_2) dt_2 \right] dt_1 \\
\leq C_1 + C_1 C_2 t + C_2 \int_0^t \left[ C_1 + C_2 \int_0^{t_2} \phi_{j-2}(t_3) dt_3 \right] dt_2 dt_1 \\
\leq C_1 + C_1 C_2 t + C_1 \frac{C_2^2 t^2}{2} + C_2 \int_0^t \int_0^{t_2} \phi_{j-2}(t_3) dt_3 dt_2 dt_1
\]
\[
\leq C_1 \sum_{k=0}^{j-1} \frac{(C_2t)^k}{k!} + C_2 \int_0^t \cdots \int_0^{t_{j-1}} \phi_{j-(j-1)}(t) dt \cdots dt_1.
\]

Since \( \phi_{j-(j-1)}(t) = \phi_1(t) = 0 \) and \( \sum_{k=0}^{j-1} \frac{(C_2t)^k}{k!} \leq \sum_{k=0}^{\infty} \frac{(C_2t)^k}{k!} = \exp(C_2t) \), we conclude that \( \phi_{j+1}(t) \leq C_1 \exp(C_2t) \). Hence,

\[
\|u^{j+1}(t)\|^2 + \|w^{j+1}(t)\|^2 \leq C_1 \exp(C_2T_1) \equiv C_3(T_1) = C_3,
\]

for \( j = 1, \ldots, n \) and \( t \in [0, T_1] \).

Inserting estimate (15) into the right hand side of (14) we get

\[
\begin{align*}
(\mu + \mu_r) \int_0^t \|\nabla u^{j+1}(s)\|^2 ds + (c_a + c_d) \int_0^t \|\nabla w^{j+1}(s)\|^2 ds \\
\leq C_1 + C_2 \int_0^t C_3 ds \leq C_1 + C_2 C_3 T_1 \equiv C_4(T_1) = C_4.
\end{align*}
\]

Thus,

\[
\int_0^t \|u^{j+1}(s)\|^2 ds \leq \frac{C_4}{\mu + \mu_r}, \quad \int_0^t \|w^{j+1}(s)\|^2 ds \leq \frac{C_4}{c_a + c_d},
\]

for \( j = 1, \ldots, n \), \( t \in (0, T_1) \).

(ii) Estimate for \( u^{j+1} \) and \( w^{j+1} \) in

\[
L^\infty(0, T_1; V) \cap L^2(0, T_1 ; D(A)) \text{ and } L^\infty(0, T_1; H^1_0(\Omega)) \cap L^2(0, T_1 ; H^2(\Omega))
\]

respectively.

Multiplying in \( L^2(\Omega) \) the first equation of (4) (with \( n = j \)) by \( Au^{j+1} \) we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\nabla u^{j+1}\|^2 + (\mu + \mu_r)\|Au^{j+1}\|^2 = 2\mu_r (\text{rot } w^j, Au^{j+1}) + (f(\theta^j), Au^{j+1}) \\
- (u^j \cdot \nabla u^{j+1}, Au^{j+1}),
\end{align*}
\]

for \( j = 1, \ldots, n \). From the Sobolev's inequality

\[
\|\varphi\|_{L^4} \leq 2^{1/2}\|\varphi\|^{1/4}\|\nabla \varphi\|^{3/4}
\]

which holds for \( \varphi \in H^1_0(\Omega) \), we obtain

\[
\|(u^j \cdot \nabla u^{j+1}, Au^{j+1})\| \leq \|u^j\|_{L^4} \|\nabla u^{j+1}\|_{L^4} \|Au^{j+1}\| \leq 2^{1/2}\|u^j\|^{1/4}\|u^{j+1}\|^{3/4}\|\nabla u^{j+1}\|_{L^4} \|Au^{j+1}\| \leq 2^{1/2} \lambda_1^{-1/8} \|u^j\| \|\nabla u^{j+1}\|_{L^4} \|Au^{j+1}\|.
\]

On the other hand, since \( H^2(\Omega) \hookrightarrow W^{1,4}(\Omega) \), we have

\[
\|\nabla \varphi\|_{L^4} \leq M_1 \|\varphi\|_{H^2}
\]

for all \( \varphi \in H^2(\Omega) \) and for some constant \( M_1 > 0 \). Moreover, by Cattabriga's inequality

\[
\|u\|_{H^2} \leq M_2 \|Au\|
\]

for \( u \in D(A) \), and some constant \( M_2 > 0 \) we conclude that \( \|\nabla u\|_{L^4} \leq M_1 M_2 \|Au\| \), for all \( u \in D(A) \). Thus, from (19) we obtain

\[
\|(u^j \cdot \nabla u^{j+1}, Au^{j+1})\| \leq 2^{1/2} \lambda_1^{-1/8} M_1 M_2 \|\nabla u^j\| \|Au^{j+1}\|.
\]
To estimate the others terms of the right hand side of (17) we argue as follow:

\[|2\mu_r(\text{rot } w^j, A w^{j+1})| \leq \frac{4\mu_r^2}{\mu + \mu_r} \|\nabla w^j\|^2 + \frac{\mu + \mu_r}{4} \|A w^{j+1}\|^2, \tag{23}\]

\[|(f(\theta^j), A w^{j+1})| \leq \frac{M_f^2}{\mu + \mu_r} \|\theta^j\|^2 + \frac{\mu + \mu_r}{4} \|A w^{j+1}\|^2. \tag{24}\]

Inserting estimates (22)-(24) in (17), and integrating from 0 to \(t\), we have

\[\|\nabla u^{j+1}(t)\|^2 + \int_0^t (\mu + \mu_r - 2^{3/2} \lambda_1^{-1/8} M_1 M_2 \|\nabla w^j(s)\|) \|A w^{j+1}(s)\|^2 ds \leq \|\nabla u_0\|^2 + \int_0^t \|\nabla w^j(s)\|^2 ds + \frac{2M_f^2}{\mu + \mu_r} \int_0^t \|\theta^j(s)\|^2 ds \leq \|\nabla u_0\|^2 + \frac{8\mu_r^2}{(\mu + \mu_r)(c_a + c_d)} + \frac{2(R + 1)M_f^2T_1}{\mu + \mu_r} < \delta^2, \tag{25}\]

for \(j = 1, \ldots, n\) and \(T_1\), sufficiently small (see condition (5)).

Since \(u^1 = e^{-t(\mu + \mu_r)A} u_0\), we have that \(\|\nabla u^1(t)\| \leq \|\nabla u_0\| < \delta\). From condition (6) we have

\[\mu + \mu_r - 2^{3/2} \lambda_1^{-1/8} M_1 M_2 \|\nabla u^1(t)\| \geq \mu + \mu_r - 2^{3/2} \lambda_1^{-1/8} M_1 M_2 \delta > 0,\]

and by (25) we conclude that

\[\|\nabla u^2\|^2 + (\mu + \mu_r - 2^{3/2} \lambda_1^{-1/8} M_1 M_2 \delta) \int_0^t \|A u^2(s)\|^2 ds \leq \delta^2,\]

for \(t \in (0, T_1)\). In particular, \(\|\nabla u^2\| \leq \delta\). Repeating this argument we obtain that

\[\sup_{t \in [0, T_1]} \|\nabla u^{j+1}(t)\| \leq \delta, \tag{26}\]

and

\[\int_0^{T_1} \|A u^{j+1}(s)\|^2 ds \leq \delta^2/(\mu + \mu_r - 2^{3/2} \lambda_1^{-1/8} M_1 M_2 \delta) = C_5, \tag{27}\]

for \(j = 1, \ldots, n\).

On the other hand, since the operator \(L\), defined by (2), is a strongly elliptic operator, we have

\[(L w, B w) \geq (c_a + c_d) \|B w\|^2 - N_0 \|\nabla w\|^2, \tag{28}\]

where \(N_0 > 0\) is a constant that depends only on \(c_a + c_d, c_0 + c_d - c_a\) and \(\partial\Omega\) (see [8] p. 70). Thus, from the second equation of (4) we have

\[w^{j+1} + L w^{j+1} + 4\mu_r w^{j+1} = 2\mu_r \text{rot } w^j - w^j \cdot \nabla w^{j+1} + g(\theta^j),\]

for \(j = 1, \ldots, n\). Multiplying this equality by \(B w^{j+1}\) and using (28) we obtain

\[\frac{1}{2} \frac{d}{dt} \|\nabla w^{j+1}\|^2 + (c_a + c_d) \|B w^{j+1}\|^2 + 4\mu_r \|\nabla w^{j+1}\|^2 \leq N_0 \|\nabla w^{j+1}\|^2 - (w^j \cdot \nabla w^{j+1}, B w^{j+1}) + 2\mu_r (\text{rot } w^j, B w^{j+1}) + (g(\theta^j), B w^{j+1}) \tag{29}\]

We now estimate the terms of the right-hand side of (29). By Poincare’s inequality, estimate (26), the inequality

\[\|\nabla \varphi\|_{L^4} \leq M_3 \|\nabla \varphi\|^{1/4} \|B \varphi\|^{3/4},\]
which holds for all \( \varphi \in H^2(\Omega) \), and some constant \( M_3 > 0 \), and Young’s inequality we have
\[
\| (u^j \cdot \nabla w^{j+1}, Bw^{j+1}) \| \leq 2^{1/2}\lambda_1^{-1/8} \| \nabla u^j \| \| \nabla w^{j+1} \|_{L^4} \| Bw^{j+1} \|
\leq 2^{1/2}\lambda_1^{-1/8} M_3 \| \nabla w^{j+1} \|_{H^{1/4}} \| Bw^{j+1} \|^{7/4}
\leq \alpha \| \nabla w^{j+1} \|^2 + \frac{c_a + c_d}{6} \| Bw^{j+1} \|^2,
\]
where \( \alpha = (1/7)(8/7)^{-8}(2^{1/2}\lambda_1^{-1/8}M_3)^8 ((c_a + c_d)/6)^{-7} \),
\[
\| [2\mu, (\text{rot }u^j, Bw^{j+1})] \| \leq \frac{6\mu^2}{c_a + c_d} \| \nabla u^j \|^2 + \frac{c_a + c_d}{6} \| Bw^{j+1} \|^2,
\]
\[
\| (g(\theta^j), Bw^{j+1}) \| \leq \frac{3M_g^2}{2(c_a + c_d)} \| \theta^j \|^2 + \frac{c_a + c_d}{6} \| Bw^{j+1} \|^2.
\]
Using these estimates in (29), integrating from 0 to \( t \), and using (16) we have
\[
\| \nabla w^{j+1}(t) \|^2 + (c_a + c_d) \int_0^t \| Bw^{j+1}(s) \|^2 ds + 8\mu \int_0^t \| \nabla w^{j+1}(s) \|^2 ds
\leq \| \nabla w_0 \|^2 + 2(N_0 + \alpha) \int_0^t \| \nabla w^{j+1}(s) \|^2 ds + \frac{12\mu^2}{c_a + c_d} \int_0^t \| \nabla u^j(s) \|^2 ds
+ \frac{6M_g^2}{2(c_a + c_d)} \int_0^t \| \theta^j(s) \|^2 ds
\leq \| \nabla w_0 \|^2 + \frac{2C_2(N_0 + \alpha)}{c_a + c_d} c_a + c_d + \frac{12C_2\mu}{c_a + c_d} + \frac{6M_g^2(R + 1)T_1}{2(c_a + c_d)}
= C_6(T_1) = C_6,
\]
for \( j = 1, \ldots, n \) and \( t \in [0, T_1] \).
\( \text{(iii) Estimates for } \theta^{n+1} \text{ in } L^\infty(0, T_1; L^2(\Omega)) \cap L^2(0, T_1; H_0^1(\Omega)) \).

To obtain the estimates for \( \theta^{n+1} \) we multiplying the third equation of (4) by \( \theta^{n+1} \) and integrating the third equation of (4) by \( \Phi_i \), 1 \( \leq i \leq 5 \). Using the inequalities
\[
\| \varphi \|_{L^3} \leq M_4 \| \varphi \|^{1/2} \| \nabla \varphi \|^{1/2}, \text{ for } \varphi \in H_0^1(\Omega),
\]
\[
\| \nabla u \|_{L^3} \leq M_5 \| \nabla u \|^{1/2} \| A u \|^{1/2}, \text{ for } u \in D(A),
\]
\[
\| \nabla w \|_{L^3} \leq M_6 \| \nabla w \|^{1/2} \| Bw \|^{1/2}, \text{ for } w \in D(B),
\]
and
\[
\| \varphi \|_{L^6} \leq M_6 \| \nabla \varphi \|, \text{ for } \varphi \in H_0^1(\Omega)
\]
we have
\[
\| (\Phi_i(u^n), \theta^{n+1}) \| \leq 2\mu \int (| \nabla u^n |^2 | \theta^{n+1} | dx
\leq 2\mu \| \nabla u^n \| \| \nabla u^n \|_{L^3} \| \theta^{n+1} \|_{L^6}
\leq 2\mu M_5 M_6 \| \nabla u^n \|^{3/2} \| A u^n \|^{1/2} \| \nabla \theta^{n+1} \|
\leq \frac{\kappa}{20} \| \nabla \theta^{n+1} \|^2 + \frac{5(2\mu M_5 M_6)^2}{\kappa} \| \nabla u^n \|^3 \| A u^n \|,
\]
which establishe
On the other hand, by Poincaré inequality

\[
|\langle \Phi_2(u^n, w^n), \theta^{n+1} \rangle| \leq 4\mu_r \int_{\Omega} |\nabla u^n|^2|\theta^{n+1}| \, dx + \int_{\Omega} |w^n|^2|\theta^{n+1}| \, dx
\]

Combining these inequalities we obtain

\[
|\langle \Phi_3(w^n), \theta^{n+1} \rangle| \leq c_0 \int_{\Omega} |\text{div} w^n|^2|\theta^{n+1}| \, dx
\]

From (31), using estimates (34), (35) follows that

\[
\|\theta^{n+1}\|^2 + \kappa \int_0^t \|\nabla \theta^{n+1}(s)\|^2 \, ds
\]

\[
\leq \|\theta_0\|^2 + C_7 T_1 + \int_0^t \|A u^n(s)\|^2 \, ds + \int_0^t \|B w^n(s)\|^2 \, ds + \frac{4}{\kappa \lambda_1} \int_0^t \|h\|^2 \, ds,
\]
where

\[ C_7 = C_7(T_1) = \left[ \frac{20}{\kappa} (M_5 M_0)^2 (\mu^2 + 8 \mu_r^2) \right]^2 \delta^6 + \frac{20}{\kappa} (8 \mu_r M_4 M_0)^2 C_3^{3/2} C_0^{1/2} \]

\[ + \left\{ \frac{5}{\kappa} (M_5 M_0)^2 (9 \omega_0^2 + (c_a + c_d)^2 + (c_d - c_a)^2) \right\}^2 C_3^2. \]

Therefore,

\[ \| \theta^{n+1}(t) \|^2 \leq \| \theta_0 \|^2 + C_7 T + C_5 + \frac{C_6}{c_a + c_d} + \frac{4}{\kappa \lambda_1} \| h \|^2_{L^2(0,T;L^2)}, \]  

(37)

for \( t \in (0,T_1) \). So, considering

\[ R = \| \theta_0 \|^2 + C_5 \]

\[ + \frac{1}{c_a + c_d} \left[ \| \nabla w_0 \|^2 + (\| u_0 \|^2 + \| w_0 \|^2) \left( \frac{N_0 + \alpha}{c_a + c_d} + \frac{6 \mu_r}{(c_a + c_d)(\mu + \mu_r)} \right) \right] \]

\[ + \frac{4}{\kappa \lambda_1} \| h \|^2_{L^2(0,T;L^2)}, \]

from (30) and (37), we see that

\[ \| \theta^{n+1}(t) \|^2 \leq \| \theta_0 \|^2 + C_7 T + C_5 + \frac{1}{c_a + c_d} \left[ \| \nabla w_0 \|^2 + C_4 \left( \frac{N_0 + \alpha}{c_a + c_d} + \frac{6 \mu_r}{(c_a + c_d)(\mu + \mu_r)} \right) + \frac{3 M_2^2 T_1 (R + 1)}{2(c_a + c_d)} \right] \]

\[ + \frac{4}{\kappa \lambda_1} \| h \|^2_{L^2(0,T;L^2)}, \]

and \( C_4 = \left[ \| u_0 \|^2 + \| w_0 \|^2 \left( \frac{2 M_f}{\lambda_1 (\mu + \mu_r)} + \frac{M_2^2}{8 \mu_r} \right) T(R + 1) \right] [1 + C_2 T_1 \exp(C_2 T_1)]. \)

Therefore, there exists \( T_1 \) possibly smaller such that

\[ \| \theta^{n+1}(t) \| \leq \| \theta_0 \|^2 + C_7 T + C_5 + \frac{C_6}{c_a + c_d} + \frac{4}{\kappa \lambda_1} \| h \|^2_{L^2(0,T;L^2)} \leq R + 1. \]  

(39)

for \( t \in [0,T_1] \). So the induction process is finished.

**Step 2.** Estimate of \( \theta^n \) in \( L^2(0,T;H_0^1(\Omega)) \). For \( n = 1 \) this is clear. Moreover, from (36) and (37) we have that \( \int_0^{T_1} \| \nabla \theta^n(s) \|^2 ds \leq 1/\kappa \), for \( n \geq 2 \). Thus, the result follows.

**Step 3.** Estimates for \( u^n_t, w^n_t \) and \( \theta^n_t \) in

\[ L^2(0,T_1;H), \ L^2(0,T_1;L^2(\Omega)) \text{ and } L^2(0,T_1;H^{-1}(\Omega)), \]

respectively.

Multiplying in \( L^2(\Omega) \) the first equation of (4) by \( u^{n+1}_t \) and using H"older’s inequality we obtain

\[ \| u^{n+1}_t \|^2 \leq (\mu + \mu_r) \| Au^{n+1} \| \| u^{n+1}_t \| + \| \nabla u^{n+1} \| \| u^{n+1}_t \| \]

\[ + 2 \mu_r \| \nabla w^n \| \| u^{n+1}_t \| + \| f(\theta^n) \| \| u^{n+1}_t \|. \]  

(40)
To estimate the second term of the right-hand side of (40) we use estimates (32) and (33) as follows
\[
\|u^n \cdot \nabla u^{n+1}\|\|u_t^{n+1}\| \leq \|u^n\|_{L^\infty} \|\nabla u^{n+1}\|_{L^2} \|u_t^{n+1}\| \\
\leq C \|\nabla u^n\| \|\nabla u^{n+1}\|^{1/2} \|Au^{n+1}\|^{1/2} \|u_t^{n+1}\| \\
\leq C \|\nabla u^n\|^2 \|\nabla u^{n+1}\| \|Au^{n+1}\| + \frac{1}{8} \|u_t^{n+1}\|^2 \\
\leq C \|\nabla u^n\|^4 \|\nabla u^{n+1}\|^2 + C \|Au^{n+1}\|^2 + \frac{1}{8} \|u_t^{n+1}\|^2.
\]

Estimating the other terms of the right-hand side of (40) in a usual way, and using estimates by (26), (27), (30) and (9) we get
\[
\int_0^t \|u_t^{n+1}(\tau)\|^2 d\tau \\
\leq C \int_0^t \|Au^{n+1}(\tau)\|^2 d\tau + C \int_0^t \|\nabla u^n(\tau)\|^2 \|\nabla u^{n+1}(\tau)\|^2 d\tau \\
+ C \int_0^t \|\nabla w^n(\tau)\|^2 d\tau + C \int_0^t \|\theta^n(\tau)\|^2 d\tau \leq C.
\]

Now, multiplying the second equation of (4) by \(w_t^{n+1}\) we have
\[
\|w_t^{n+1}\|^2 + \frac{1}{2} (c_0 + c_d - c_0) \frac{d}{dt} \|\text{div} \, w^{n+1}\| \leq -(c_n + c_d) (B w^{n+1}, w_t^{n+1}) - (u^n \cdot \nabla w^{n+1}, w_t^{n+1}) \\
- 4\mu_0 (w^{n+1}, w_t^{n+1}) + 2\mu_r (\text{rot} \, u^n, w_t^{n+1}) + (g(\theta^n), w_t^{n+1}).
\]

Since \(H^2(\Omega) \hookrightarrow L^\infty(\Omega)\), by estimate (30) and Cattabriga’s inequality (21) we have
\[
\|u^n \cdot \nabla w^{n+1}\| \|w_t^{n+1}\| \leq \|u^n\|_{L^\infty} \|\nabla w^{n+1}\| \|w_t^{n+1}\| \\
\leq C \|Au^n\|^2 + \frac{1}{10} \|w_t^{n+1}\|^2.
\]

Estimating, in a usual way, the other terms of (41) it is possible to conclude
\[
\int_0^t \|w_t^{n+1}(\tau)\|^2 d\tau \leq C \int_0^t \|B w^{n+1}(\tau)\|^2 d\tau + C \int_0^t \|w^{n+1}(\tau)\|^2 d\tau \\
+ C \int_0^t \|\nabla u^n(\tau)\|^2 d\tau + C \int_0^t \|Au^n(\tau)\|^2 d\tau \\
+ C \int_0^t \|\theta^n(\tau)\|^2 d\tau \leq C,
\]
from (27), (30) and (37).

Finally, from the third equation of (4) we have
\[
\|\theta_t^{n+1}\|_{H^{-1}} \leq \|u^n \cdot \nabla \theta^{n+1}\|_{H^{-1}} + \kappa \|\Delta \theta^{n+1}\|_{H^{-1}} \\
+ \|\Phi(u^n, \theta^n)\|_{H^{-1}} + \|h\|_{H^{-1}}.
\]

Using the bounds for \(u\) and \(w\) given by (26) and (30) we have
\[
\|u^n \cdot \nabla \theta^{n+1}\|_{H^{-1}} \leq \sup_{\|\varphi\|_{H^1} \leq 1} \|(u^n \cdot \nabla \theta^{n+1}, \varphi)\| \\
\leq C \sup_{\|\varphi\|_{H^1} \leq 1} \|u^n\|_{L^2} \|\nabla \theta^{n+1}\| \|\nabla \varphi\| \\
\leq C \|\nabla \theta^{n+1}\|,
\]

where \(H^1(\Omega) \hookrightarrow L^\infty(\Omega)\).
\[ \| \Delta \theta^{n+1} \|_{H^{-1}} = \sup_{\| \varphi \|_{H^1} \leq 1} |(\Delta \theta^{n+1}, \varphi)| \leq C \sup_{\| \varphi \|_{H^1} \leq 1} \| \nabla \theta^{n+1} \| \| \nabla \varphi \| \overset{(44)}{\leq} C \| \nabla \theta^{n+1} \|, \]

\[ \| \Phi_1(u^n) \|_{H^{-1}} \leq C \sup_{\| \varphi \|_{H^1} \leq 1} \| \nabla u^n \| \| \nabla u^n \|_{L^3} \| \varphi \|_{L^6} \overset{(45)}{\leq} C \| A u^n \|, \]

\[ \| \Phi_2(u^n, w^n) \|_{H^{-1}} \leq C \sup_{\| \varphi \|_{H^1} \leq 1} \| \nabla u^n \| \| \nabla w^n \|_{L^3} \| \varphi \|_{L^6} + C \sup_{\| \varphi \|_{H^1} \leq 1} \| w^n \| \| w^n \|_{L^3} \| \varphi \|_{L^6} \]

and for \( i = 3, 4, 5 \) we have

\[ \| \Phi_i(w^n) \|_{H^{-1}} \leq C \sup_{\| \varphi \|_{H^1} \leq 1} \| \nabla w^n \| \| \nabla u^n \|_{L^3} \| \varphi \|_{L^6} \overset{(45)}{\leq} C \| B w^n \|. \]

From (27), (30), (37) and (42) we have

\[ \int_0^t \| \theta^{n+1}_t(\tau) \|_{H^{-1}} d\tau \leq C \int_0^t (\| \nabla \theta^{n+1}(\tau) \|^2 + \| A u^n(\tau) \|^2 + \| B w^n(\tau) \|^2 + \| h(\tau) \|^2) d\tau \leq C, \]

for \( t \in (0, T_1) \).

**Step 4.** If \( \mu, M_f, M_g \) and \( \| h \|_{L^2(0,T;L^2(\Omega))} \) are sufficiently small, we see that condition (6) is verified and the constants \( C_5, C_6 \) and \( C_7 \) can be choosing small enough such that inequality (38) is also verified for \( T = T_1 \).

### 2.2. Proof of Theorem 1.4.

Let \((u^n, w^n, \theta^n)_{n \geq 1}\) the sequence defined on \((0, T_1)\) and given by Theorem 1.3. To show the convergence of this sequence we will prove that it is a Cauchy sequence. To do this, let

\[ u^{n,s}(t) = u^{n+s}(t) - u^n(t), \]

\[ w^{n,s}(t) = w^{n+s}(t) - w^n(t) \]

and \( \theta^{n,s}(t) = \theta^{n+s}(t) - \theta^n(t) \), for \( n, s \in \mathbb{N} \) and \( t \in (0, T_1) \).

From (4) we have

\begin{align*}
\theta_t^{n,s} + (\mu + \mu_r) A u^{n,s} + P(u^{n-1,s} \cdot \nabla u^{n,s}) &= 2\mu_r P(\text{rot} w^{n-1,s} - \mu_r u^{n-1,s} \cdot \nabla u^n) + P f(\theta^{n-1,s}) - P f(\theta^{n-1}), \\
\text{rot} w^{n,s} + (c_0 + c_2) B w^{n,s} + (u^{n-1,s} \cdot \nabla w^{n,s}) &= (c_0 + c_2 - c_0) \nabla \text{div} w^{n,s} + 4\mu_r w^{n,s}
\end{align*}

\[ = 2\mu_r \text{rot} u^{n-1,s} - (u^{n-1,s} \cdot \nabla w^n) + g(\theta^{n-1,s}) - g(\theta^{n-1}) \]

\[ \theta_t^{n,s} - \kappa \Delta \theta^{n,s} + u^{n-1,s} \cdot \nabla \theta^{n,s} = \Phi(u^{n-1,s}, w^{n-1,s}) - \Phi(u^{n-1}, \theta^{n-1}) - u^{n-1,s} \cdot \nabla \theta. \]

We need some preliminaries results.
Lemma 2.2. Assume that (49) holds. Then

\[ 0 \leq \phi_n(t) \leq C_0 \int_0^t \phi_{n-1}(s) ds, \]

for every \( n \in [0, T] \) and some constant \( C_0 > 0 \). Then

\[ \phi_n(t) \leq M \frac{(C_0 t)^{n-1}}{(n-1)!}, \quad (49) \]

for all \( t \in [0, T] \) and \( n \geq 1 \).

Proof. We use the first principle of induction. For \( n = 1 \), it is clear that (49) holds. Assume that (49) holds. Then

\[ \phi_{n+1}(t) \leq C_0 \int_0^t \phi_n(s) ds \leq M C_0 \frac{C_0}{(n-1)!} t \int_0^t s^{n-1} ds \leq M \frac{(C_0 t)^{n}}{(n)!}. \]

Therefore, estimate (49) holds for \( n + 1 \).

Lemma 2.2. For \( t \in [0, T_1] \), \( n, s \in \mathbb{N} \), let \( \Psi^{n,s}(t) = \|
abla u^{n,s}(t)\|^2 + \|w^{n,s}(t)\|^2 \). There exits a constant \( C_8 > 0 \), which is independent of \( n \) and \( s \), so that

\[ \Psi^{n,s}(t) + (\mu + \mu_r) \int_0^t \| Au^{n,s}(\tau) \|^2 d\tau + (c_a + c_d) \int_0^t \| Bu^{n,s}(\tau) \|^2 d\tau \]

\[ \leq C_8 \int_0^t (\| Au^n(\tau) \| + \| Bu^n(\tau) \| + 1) \left[ \Psi^{n-1,s}(\tau) + \|\theta^{n-1,s}(\tau)\|^2 \right] d\tau. \quad (50) \]

Proof. Multiplying (46) by \( Au^{n,s} \) and integrating on \( \Omega \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \| \nabla u^{n,s} \|^2 + (\mu + \mu_r) \| Au^{n,s} \|^2 \]

\[ = -\langle u^{n-1+s} \cdot \nabla u^{n,s}, Au^{n,s} \rangle + 2 \mu_r (\operatorname{rot} w^{n-1,s}, Au^{n,s}) \]

\[ + \langle u^{n-1,s} \cdot \nabla u^n, Au^{n,s} \rangle + \langle f(\theta^{n-1+s}) - f(\theta^{n-1}), Au^{n,s} \rangle. \quad (51) \]

From estimates (32), (33) and the estimates of Theorem 1.3 we have

\[ |u^{n-1+s} \cdot \nabla u^{n,s}, Au^{n,s})| \]

\[ \leq \| u^{n-1+s} \|_{L^p} \| \nabla u^{n,s} \| L^q \| Au^{n,s} \| \]

\[ \leq C \| \nabla u^{n-1+s} \| \| \nabla u^{n,s} \|^2 \| Au^{n,s} \|^{3/2} \]

\[ \leq C \| \nabla u^{n,s} \|^2 + \frac{\mu + \mu_r}{8} \| Au^{n,s} \|^2 \]

\[ |2 \mu_r (\operatorname{rot} w^{n-1,s}, Au^{n,s})| \]

\[ \leq C \| \nabla w^{n-1,s} \|^2 + \frac{\mu + \mu_r}{8} \| Au^{n,s} \|^2 \]

\[ |(u^{n-1,s} \cdot \nabla u^n, Au^{n,s})| \]

\[ \leq \| u^{n-1+s} \|_{L^p} \| \nabla u^n \| L^q \| Au^{n,s} \| \]

\[ \leq C \| \nabla u^{n-1,s} \| \| \nabla u^n \|^2 \| Au^{n,s} \|^2 \]

\[ \leq C \| \nabla u^{n-1,s} \| \| Au^n \|^{1/2} \| Au^{n,s} \|^{1/2} \]

\[ \leq C \| Au^n \|^2 \| Au^{n-1,s} \|^2 + \frac{\mu + \mu_r}{8} \| Au^{n,s} \|^2 \]

\[ |(f(\theta^{n-1+s}) - f(\theta^{n-1}), Au^{n,s})| \]

\[ \leq \| f(\theta^{n-1+s}) - f(\theta^{n-1}) \| \| Au^{n,s} \| \]

\[ \leq C \| \theta^{n-1,s} \| \| Au^{n,s} \|^2 \]

\[ \leq C \| \theta^{n-1,s} \|^2 + \frac{\mu + \mu_r}{8} \| Au^{n,s} \|^2. \]
From these estimates and (51) we get
\[ \|\nabla u^{n,s}(t)\|^2 + (\mu + \mu_r) \int_0^t \| Au^{n,s}(\tau)\|^2 d\tau \]
\[ \leq C \int_0^t (\|\nabla u^{n,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|) d\tau \]
\[ + C \int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 \| Au^{n}(\tau)\| d\tau. \]

Similarly, multiplying (47) by \( B w^{n,s} \) and using (28), we obtain
\[ \|\nabla w^{n,s}(t)\|^2 + (c_a + C_d) \int_0^t \| B w^{n,s}(\tau)\|^2 d\tau \]
\[ \leq C \int_0^t (\|\nabla w^{n,s}(\tau)\|^2 + \|\nabla u^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|)^2 d\tau \]
\[ + C \int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 \| B w^{n}(\tau)\| d\tau. \]

Adding inequalities (52), (53), and using Gronwall’s inequality we get the result. □

**Lemma 2.3.** There exists a constant \( C_9 > 0 \), independent of \( n \) and \( s \), such that
\[ \int_0^t \| u^{n,s}_t(\tau)\|^2 d\tau + \int_0^t \| w^{n,s}_t(\tau)\|^2 d\tau \]
\[ \leq C_9 \int_0^t \|\nabla w^{n,s}(\tau)\|^2 d\tau \]
\[ + C_9 \int_0^t (\| Au^{n}(\tau)\| + \| B w^{n}(\tau)\| + 1) \left[ \Psi^{n-1,s}(\tau) + \|\theta^{n-1,s}(\tau)\|^2 \right] d\tau, \]
where \( \Psi^{n,s}(t) = \|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \), for every \( n, s \in \mathbb{N}, t \in [0, T_1] \).

**Proof.** Multiplying (46) by \( u^{n,s}_t \), and integrating on \( \Omega \) we have
\[ \| u^{n,s}_t \|^2 \]
\[ = - (\mu + \mu_r) (Au^{n,s}_t, u^{n,s}_t) - (u^{n-1,s} \cdot \nabla u^{n,s}_t, u^{n,s}_t) + 2\mu_r (\text{rot} w^{n-1,s}, u^{n,s}_t) \]
\[ - (u^{n-1,s} \cdot \nabla u^{n}_t, u^{n,s}_t) + (f(\theta^{n-1}), u^{n,s}_t) \]
\[ \leq C \| Au^{n,s}_t \| \| u^{n,s}_t \| + C \| u^{n-1,s} \| \| \nabla u^{n,s}_t \| \| u^{n,s}_t \| + C \| \nabla w^{n-1,s} \| \| u^{n,s}_t \| \]
\[ + C \| u^{n-1,s} \| \| \nabla u^{n}_t \| \| u^{n,s}_t \| + C \| \theta^{n-1,s} \| \| u^{n,s}_t \| \]
\[ \leq C \| Au^{n,s}_t \|^2 + C \| \nabla w^{n-1,s} \|^2 + C \| \nabla u^{n-1,s} \|^2 \| Au^{n} \| \]
\[ + C \| \theta^{n-1,s} \|^2 + \frac{1}{2} \| u^{n,s}_t \|^2. \]

Here, we use (18), (20), (21) and the bounds of Theorem 1.3 as follows
\[ \| u^{n-1+s} \|_{L^6} \| \nabla u^{n,s} \|_{L^3} \| u^{n,s}_t \| \]
\[ \leq C \| u^{n-1+s} \|_{L^6}^{1/4} \| \nabla u^{n-1+s} \|_{L^3}^{3/4} \| Au^{n,s} \| \| u^{n,s}_t \|. \]

We use also (32), (33) and estimates of Theorem 1.3 to conclude
\[ \| u^{n-1,s} \|_{L^6} \| \nabla u^{n} \|_{L^3} \| u^{n,s}_t \| \]
\[ \leq C \| \nabla u^{n-1,s} \| \| \nabla u^{n} \|^{1/2} \| Au^{n} \|^{1/2} \| u^{n,s}_t \| \]
\[ \leq C \| \nabla u^{n-1,s} \| \| Au^{n} \|^{1/2} \| u^{n,s}_t \|. \]
Analogously, we obtain
\[
\|w^n_s(t)\|^2 + \frac{1}{2}(c_0 + c_d - c_a) \frac{d}{dt}\|\text{div} w^n_s\|^2 = -(c_a + c_d)(B w^n_s, w^n_s(t)) - (u^{n-1,s} \cdot \nabla w^{n,s}, w^n_s(t)) - 4\mu(t)(w^{n,s}, w^n_s(t)) + 2\mu(\text{rot} u^{n-1,s}, w^{n,s}_t) - (u^{n-1,s} \cdot \nabla w^{n,s}, w^n_s(t)) + (g(\theta^{n-1,s}) - g(\theta^{n-1,s}), w^n_s(t)) \\
\leq C\|B w^n_s\|\|w^n_s\| + C\|u^{n-1,s}\|\|\nabla w^n_s\|\|w^n_s\| + C\|w^{n,s}\|\|w^n_s\| + C\|\nabla w^{n,s}\|\|w^n_s\| + C\|\theta^{n-1,s}\|\|w^n_s\| \\
\leq C\|B w^n_s\|^2 + C\|\nabla w^{n,s}\|^2 + C\|\nabla u^{n-1,s}\|^2 + C\|\nabla w^{n-1,s}\|^2 + C\|\theta^{n-1,s}\|^2 + C\|w^n_s\|^2.
\]
Adding (54) - (55) we get
\[
\int_0^t \|u^{n,s}_t(\tau)\|^2 d\tau + \int_0^t \|w^n_s(\tau)\|^2 d\tau \\
\leq C\int_0^t (\|A u^{n,s}(\tau)\|^2 + \|B w^n_s\|^2) d\tau \\
+ C\int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|^2) d\tau \\
+ C\int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 (\|A u^n(\tau)\| + \|B w^n(\tau)\|) d\tau + C\int_0^t \|\nabla w^{n,s}(\tau)\|^2 d\tau,
\]
where \(C > 0\) does not depend on \(n\) and \(s\). The conclusion follows from Lemma 2.2.

**Proof of Theorem 1.4.** In Lemmas 2.2 and 2.3, we obtain some preliminaries estimates for \(\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2\) and \(\int_0^t (\|u^{n,s}_t(\tau)\|^2 + \|w^n_s(\tau)\|^2) d\tau\). We use those to estimates \(\theta^{n,s}\). We consider some steps.

**Step 1.** Multiplying (48) by \(\theta^{n,s}\) and integrating on \(\Omega\) we obtain
\[
\frac{1}{2} \frac{d}{dt}\|\theta^{n,s}\|^2 + \kappa \|\nabla \theta^{n,s}\|^2 = (\Phi(u^{n-1,s}, w^{n-1,s}) - \Phi(u^{n-1}, w^{n-1}), \theta^{n,s}) - (u^{n-1,s} \cdot \nabla \theta^n, \theta^{n,s}).
\]
By (32), (33) and the estimates of Theorem 1.3 we have
\[
\|(u^{n-1,s} \cdot \nabla \theta^n, \theta^{n,s})\| = \|u^{n-1,s} \cdot \nabla \theta^n, \theta^{n,s}\| \\
\leq \|\theta^n\|_{L^3} \|u^{n-1,s}\|_{L^6} \|\nabla \theta^{n,s}\| \\
\leq M_0^{1/2} M_0 \|\nabla \theta^n\|^{1/2} \|\nabla u^{n-1,s}\| \|\theta^{n,s}\| \\
\leq \frac{3}{\kappa} M_0 M_0^2 \|\nabla u^{n-1,s}\|^2 \|\nabla \theta^n\| + \frac{\kappa}{12} \|\theta^{n,s}\|^2.
\]
To estimate the terms of function \(\Phi\) we use the following inequalities:
\[
\begin{align*}
|\Phi_1(u_1) - \Phi_1(u_2)| &\leq 18\mu(|\nabla u_1| + |\nabla u_2|)(|\nabla u_1 - u_1|), \\
|\Phi_2(u_1, w_1) - \Phi_2(u_2, w_2)| &\leq 4\mu_r(|\nabla u_1| + |\nabla u_2| + |w_1| + |w_2|)(\|\nabla u_1 - u_2\| + |w_1 - w_2|), \\
|\Phi_3(w_1) - \Phi_3(w_2)| &\leq 9\alpha(|\nabla w_1 + |\nabla w_2\|)|\nabla (w_1 - w_2)|, \\
|\Phi_4(w_1) - \Phi_4(w_2)| &\leq |c_0 + c_d|(|\nabla w_1| + |\nabla w_2|)|\nabla (w_1 - w_2)|, \\
|\Phi_5(w_1) - \Phi_5(w_2)| &\leq 18|c_1 - c_2|(|\nabla w_1| + |\nabla w_2|)|\nabla (w_1 - w_2)|.
\end{align*}
\]
Arguing as in the derivation of (56) we have

\[ |(\Phi_1(u^{n-1+s}) - \Phi_1(u^{n-1}), \theta^{n,s})| \]
\[ \leq 18\mu(||\nabla u^{n-1+s}||_{L^2} + ||\nabla u^{n-1}||_{L^2})||\nabla u^{n-1,s}||_{L^6} \]
\[ \leq 6M_0M_5M_6M_1^{1/2}(||Au^{n-1+s}||_{1/2} + ||Au^{n-1}||_{1/2})||\nabla u^{n-1,s}||_{L^6} ||\nabla \theta^{n,s}|| \]
\[ \leq \frac{6}{\kappa}M_0(18\mu M_5M_6)^2(||Au^{n-1+s}|| + ||Au^{n-1}||)||\nabla u^{n-1,s}||^2 + \frac{\kappa}{12}||\nabla \theta^{n,s}||^2, \]

and for \( i = 3, 4, 5 \)

\[ |(\Phi_2(u^{n-1+s}) - \Phi_2(u^{n-1}), \theta^{n,s})| \]
\[ \leq 4\mu_r(||\nabla u^{n-1+s}||_{L^2} + ||\nabla u^{n-1}||_{L^2} + ||u^{n-1+s}||_{L^3} + ||u^{n-1}||_{L^3}) \]
\[ \leq 4\mu_rM_5M_6M_1^{1/2}(||Au^{n-1+s}||_{1/2} + ||Au^{n-1}||_{1/2})||\nabla u^{n-1,s}||_{L^6} ||\nabla \theta^{n,s}|| \]
\[ \leq \frac{18}{\kappa}(4\mu_r M_5M_6)^{1/2}(1 + 2M_1^{1/2})(||Au^{n-1+s}|| + ||Au^{n-1}|| + 1)||\nabla u^{n-1,s}||^2 + \frac{\kappa}{12}||\nabla \theta^{n,s}||^2, \]

where \( L_3 = 9c_0, L_4 = 9(c_a + c_d) \) and \( L_5 = 18|c_d - c_a| \).

Set
\[ M_{n,s}(t) = ||Au^{n-1+s}|| + ||Au^{n-1}|| + ||Bw^{n-1+s}|| + ||Bw^{n-1}|| + ||\theta^n|| + 1. \]

From the anterior estimates we conclude that

\[ ||\theta^{n,s}(t)||^2 + \kappa \int_0^t ||\nabla \theta^{n,s}(\tau)||d\tau \]
\[ \leq L_0 \int_0^t M_{n,s}(\tau)(||\nabla u^{n-1,s}(\tau)||^2 + ||\nabla \theta^{n-1,s}(\tau)||^2) d\tau, \]

where \( L_0 = \frac{6}{\kappa}M_0(M_5M_6)^2 + \frac{12}{\kappa}M_0(18\mu M_5M_6)^2 + \frac{36M_0}{\kappa}(4\mu_r M_5M_6)^{1/2}(1 + 2M_1^{1/2})^2 + \frac{12}{\kappa}M_0 \sum_{i=1}^3 (M_5M_6L_i)^2. \)

**Step 2.** Let \( \phi_{n,s}(t) = ||\nabla u^{n,s}(t)||^2 + ||\nabla w^{n,s}(t)||^2 + ||\theta^{n,s}(t)||^2 \). Adding inequalities (50), (58), we obtain

\[ \phi_{n,s}(t) + (\mu + \mu_r) \int_0^t ||Au^{n,s}(\tau)||^2 d\tau + (c_a + c_d) \int_0^t ||Bw^{n,s}(\tau)||^2 d\tau \]
\[ + \kappa \int_0^t ||\nabla \theta^{n,s}(\tau)||d\tau \leq C_{10} \int_0^t M_{n,s}(\tau) \phi_{n-1,s}(\tau) d\tau, \]

where \( C_{10} = C_8 + L_0. \)

By Theorem 1.3, \( M_{n,s} \in L^2(0, T_1) \). Thus by Hölder’s inequality we get

\[ \phi_{n,s}(t) \leq C_{10} \left( \int_0^t M_{n,s}(\tau)^2 d\tau \right)^{1/2} \left( \int_0^t \phi_{n-1,s}(\tau)^2 d\tau \right)^{1/2}. \]
Thus,
\[ \phi_{n,s}^2(t) \leq C_{11} \int_0^t \phi_{n-1,s}^2(\tau) d\tau, \]
where \( C_{11} = C_{10}^2 \int_0^{T_1} M_{n,s}^2(\tau) d\tau < \infty. \)

On the other hand, by Theorem 1.3, we have \( \phi_{1,s}(t) = \|\nabla u^s(t)\|^2 + \|\nabla w^s(t)\|^2 + \|\theta^s(t)\|^2 \leq M_0 \) for all \( t \in [0, T_1] \). Therefore, by Lemma 2.1 we conclude that
\[ \|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 + \|\theta^{n,s}(t)\|^2 = \phi_{n,s} \leq M_0^{1/2} \left[ \frac{(C_{11} t)^{n-1}}{(n-1)!} \right]^{1/2} = M_0^{1/2} \Lambda_{n-1}(t), \]
for all \( t \in [0, T_1] \), \( n \geq 1 \). Inserting this estimate in (59) it follows that
\[ (\mu + \mu_r) \int_0^t \| A u^{n,s}(\tau) \|^2 d\tau + (c_a + c_d) \int_0^t \| B w^{n,s}(\tau) \|^2 d\tau \]
\[ + \kappa \int_0^t \| \nabla \theta^{n,s}(\tau) \|^2 d\tau \leq C_{11}^{1/2} \left( \int_0^t \phi_{n-1,s}^2(\tau) d\tau \right)^{1/2} \]
\[ \leq M_0^{1/2} \left[ \frac{(C_{11} t)^{n-1}}{(n-1)!} \right]^{1/2} = M_0^{1/2} \Lambda_{n-1}(t). \]

Integrating the inequality (60), from 0 to \( t \), and using Hölder inequality, we obtain
\[ \int_0^t \|\nabla u^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\nabla w^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\theta^{n,s}(\tau)\|^2 d\tau \]
\[ \leq M_0^{1/2} \int_0^t \left[ \frac{(C_{11} \tau)^{n-1}}{(n-1)!} \right]^{1/2} \]
\[ \leq C_{12} \left[ \frac{(C_{11} t)^n}{n!} \right]^{1/2} = C_{12} \Lambda_n(t) \]
where \( C_{12} = (M_0 T_1)^{1/2} \). Hence, by Lemma 2.3, Theorem 1.3 and (60)
\[ \int_0^t \|u^{n,s}(\tau)\|^2 d\tau + \int_0^t \|w^{n,s}(\tau)\|^2 d\tau \]
\[ \leq C_9 \int_0^t \|\nabla u^{n,s}(\tau)\|^2 d\tau + C_9 \int_0^t \phi_{n-1,s}(\tau) (\|A u^n\| + \|B w^n\| + 1) d\tau \]
\[ \leq C_9 C_{12} \left[ \frac{(C_{11} t^n)}{n!} \right]^{1/2} \]
\[ + \frac{C_9}{4} \left( \int_0^t \phi_{n-1,s}(\tau) d\tau \right)^{1/2} \left[ \int_0^t (\|A u^n\|^2 + \|B w^n\|^2 + 1) d\tau \right]^{1/2} \]
\[ \leq C_{13} \left[ \frac{(C_{11} t^n)}{n!} \right]^{1/2} + C_{14} \left[ \frac{(C_{11} t)^{n-1}}{(n-1)!} \right]^{1/2} \]
\[ \leq C_{13} \Lambda_n(t) + C_{14} \Lambda_{n-1}(t) \]
where \( C_{13} = C_9 C_{12} \) and \( C_{14} = \frac{C_9 (M_0 + T_1)^{1/2} M_0}{4 C_{11}}. \)
On the other hand, from (48) we have
\[
\|\theta^n_{n,s}\|_{H^{-1}}^2 + \|\nabla u^n_{n,s} \cdot \nabla \theta^n_{n,s}\|_{H^{-1}}^2 + \|\nabla \theta^n_{n,s}\|_{H^{-1}}^2 + \|\nabla \theta^n_{n,s}\|_{H^{-1}}^2 = (64)
\]

Now, we estimate the terms in the right-hand side of (64) as follows.
\[
\kappa \|\Delta \theta^n_{n,s}\|_{H^{-1}} \leq \kappa \sup_{\|\psi\|_{H^1}^2 \leq 1} |(\nabla \theta^n_{n,s}, \nabla \varphi)| \\
\leq \kappa \|\nabla \theta^n_{n,s}\|_L^2.
\]

From Poincare's inequality, (33) and Theorem 1.3 we have
\[
\|u^{n-1+s} \cdot \nabla \theta^n_{n,s}\|_{H^{-1}} \leq \sup_{\|\varphi\|_{H^1}^2 \leq 1} |(u^{n-1+s} \cdot \nabla \theta^n_{n,s}, \varphi)| \\
\leq \sup_{\|\varphi\|_{H^1}^2 \leq 1} \|u^{n-1+s}\|_L^3 \|\nabla \theta^n_{n,s}\|_L \|\varphi\|_L^6 \\
\leq M_0 M_0 / \lambda_1 \|\nabla \theta^n_{n,s}\|_L.
\]

Using the Sobolev's embeddings $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, that is, $\|\psi\|_{L^\infty} \leq M_8 \|\psi\|_{H^2}$ for all $\psi \in H^2(\Omega)$, and by (21) we have
\[
\|u^{n-1,s} \cdot \nabla \theta^n\|_{H^{-1}} \leq \sup_{\|\varphi\|_{H^1}^2 \leq 1} |(u^{n-1,s} \cdot \nabla \theta^n, \varphi)| \\
\leq \sup_{\|\varphi\|_{H^1}^2 \leq 1} \|u^{n-1,s}\|_{L^\infty} \|\theta^n\| \|\nabla \varphi\| \\
\leq M_2 M_8 \|Au^{n-1,s}\|.
\]

Similarly, using (57) we obtain
\[
\|\Phi_1(u^{n-1,s}) - \Phi_1(u^{n-1})\|_{H^{-1}} \\
\leq 18 \mu \sup_{\|\varphi\|_{H^1}^2 \leq 1} (\|\nabla u^{n-1+s}\| + \|\nabla u^{n-1}\|) \|\nabla \theta^n_{n,s}\|_L \|\varphi\|_L^6 \\
\leq 18 \mu M_0 M_2 M_0'(\|\sqrt{n+1}\| + \|\nabla u^{n-1}\|) \|Au^{n-1,s}\| \\
\leq 36 \mu M_0 M_0 M_2 M_0'(\|Au^{n-1,s}\|),
\]
\[
\|\Phi_2(u^{n-1+s}, u^{n-1+s}) - \Phi_2(u^{n-1}, u^{n-1+s})\|_{H^{-1}} \\
\leq 4 \mu \sup_{\|\varphi\|_{H^1}^2 \leq 1} (\|\nabla u^{n-1+s}\| + \|\nabla u^{n-1}\| + \|w^{n-1+s}\| + \|w^{n-1}\|) \\
\leq C_{15} \|Au^{n-1,s}\| + \|\nabla w^{n-1,s}\|_L^3),
\]
where $C_{15} = 16 \mu M_0 M_0 M_2 M_0 M_2 M_0' + M_4 \lambda_1^{1/2}$. Moreover, for $i = 3, 4, 5$
\[
\|\Phi_3(u^{n-1+s}) - \Phi_3(u^{n-1})\|_{H^{-1}} \\
\leq C_{16} M_0 (\|\nabla u^{n-1+s}\| + \|\nabla u^{n-1}\|) \|\nabla w^{n-1,s}\|_L^3 \\
\leq 2 \mu M_0 M_0 M_0 M_4 \lambda_1^{1/2} \|\nabla w^{n-1,s}\|,
\]
where $C_{16} = \max\{c_a + c_d, c_d, 9c_0, 18|c_d - c_a|\}$. Therefore, from (61) and (62)
\[
\int_0^t \|\theta^n_{n,s}(\tau)\|_{H^{-1}}^2 d\tau \leq C_{17} \int_0^t (\|\nabla \theta^n_{n,s}\|^2 + \|Au^{n-1,s}\|^2 + \|\nabla w^{n-1,s}\|^2) d\tau \\
\leq C_{18} [\lambda_n(t) + \lambda_{n-2}(t) + \lambda_{n-1}(t)]
\]
where
\[
C_{17} = \max\{\kappa + M_0 M_0 / \lambda_1, M_2 M_8 + 36 \mu M_0 M_2 M_0', C_{15} / 2, C_{15} / 2, 6C_{16} M_0 M_0 M_4 \lambda_1^{1/2}, \}
\]
and $C_{18} = C_{17} \max \{M_0^{1/2} / \kappa, M_0^{1/2} / (\mu + \mu_r), C_{12} \}$.

**Step 3.** Since the spaces $L^\infty(0, T; V)$, $L^\infty(0, T; H_0^0(\Omega))$, $L^\infty(0, T; L^2(\Omega))$, $L^2(0, T; D(A))$, $L^2(0, T; D(B))$, $L^2(0, T; H_0^1(\Omega))$, $L^2(0, T; H)$, $L^2(0, T; L^2(\Omega))$ and $L^2(0, T; H^{-1}(\Omega))$ are Banach spaces, from estimates (60)-(63) and (65) we conclude that there exists functions $u, w$ and $\theta$ such that

\[
\begin{align*}
&u^n \to u \text{ strongly in } L^\infty(0, T; V) \cap L^2(0, T; D(A)), \\
&w^n \to w \text{ strongly in } L^\infty(0, T; H_0^0(\Omega)) \cap L^2(0, T; D(B)), \\
&\theta^n \to \theta \text{ strongly in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)),
\end{align*}
\]

as $n \to \infty$.

Since $(u^n, w^n, \theta^n)$ verifies (4) we have that the limit $(u, w, \theta)$, as $n \to \infty$, is the solution of (1). Indeed, if $\varphi \in L^2(0, T; H)$

\[
\begin{align*}
&\left| \int_0^{T_1} (u^{n+1}_t - u_t(t), \varphi) \, dt \right| \leq \|u^{n+1}_t - u_t\|_{L^2(0, T; H)} \|\varphi\|_{L^2(0, T; H)}, \\
&\left| \int_0^{T_1} (Au^{n+1} - Au(t), \varphi) \, dt \right| \leq C\|u^{n+1} - u\|_{L^2(0, T; D(A))} \|\varphi\|_{L^2(0, T; H)}, \\
&\left| \int_0^{T_1} (\text{rot } w^{n+1} - \text{rot } w, \varphi) \, dt \right| \leq T_1^{1/2} \|w^{n+1} - w\|_{L^\infty(0, T; H_0^1)} \|\varphi\|_{L^2(0, T; H)}, \\
&\left| \int_0^{T_1} (f(\theta^n) - f(\theta), \varphi) \, dt \right| \leq M_1 T_1^{1/2} \|\theta^n - \theta\|_{L^\infty(0, T; L^2)} \|\varphi\|_{L^2(0, T; H)}.
\end{align*}
\]

On the other hand, using the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and the bound (25) we have

\[
\begin{align*}
&\left| \int_0^{T_1} (u^n \cdot \nabla u^{n+1} - u \cdot \nabla u, \varphi) \, dt \right| \\
&\leq \left| \int_0^{T_1} ((u^n - u \cdot \nabla u^{n+1}), \varphi) \, dt \right| + \left| \int_0^{T_1} (u \cdot \nabla (u^{n+1} - u), \varphi) \, dt \right| \\
&\leq \int_0^{T_1} \|u^n - u\|_{L^\infty} \|\nabla u^{n+1}\| \|\varphi\| \, dt + \int_0^{T_1} \|u\|_{L^4} \|\nabla (u^{n+1} - u)\| \|\varphi\| \, dt \\
&\leq C\|u^n - u\|_{L^2(0, T; D(A))} \|\varphi\|_{L^2(0, T; H)} + C\|u\|_{L^\infty(0, T; V)} \|u^{n+1} - u\|_{L^2(0, T; H)} \|\varphi\|_{L^2(0, T; H)}.
\end{align*}
\]

Therefore, letting $n \to \infty$ we see that $u$ verifies the first identity of Definition 1.1. The second identity is obtained similarly, and the third one is obtained using the estimates of $\Phi$, given by (57).

**Step 4.** Uniqueness. Let $(u_1, w_1, \theta_1)$ and $(u_2, w_2, \theta_2)$ be two solutions of (1), and let $u = u_1 - u_2$, $w = w_1 - w_2$ and $\theta = \theta_1 - \theta_2$. Then

\[
\begin{align*}
&(u_t - (\mu + \mu_r)Au, \varphi) + b(u, u_1, \varphi) + b(u_2, u, \varphi) \\
&= (2\mu, \text{rot } w + f(\theta_1) - f(\theta_2), \varphi), \\
&(w_t + Lu + 4\mu, w, \psi) + b(u_1, w, \psi) + b(u, w_2, \psi) \\
&= (2\mu, \text{rot } u + g(\theta_1) - g(\theta_2), \psi),
\end{align*}
\]

and

\[
\begin{align*}
&\langle \theta_1, \phi \rangle + \kappa \langle \nabla \theta, \nabla \phi \rangle + \langle u_1 - u_2 - u + u_2, \nabla \phi \rangle \\
&= \langle \Phi(u_1, w_1) - \Phi(u_2, w_2), \phi \rangle,
\end{align*}
\]

as desired.
for $\varphi \in H$, $\psi \in L^2(\Omega)$ and $\phi \in H^1_0(\Omega)$. Setting $\varphi = Au$ we can argue as in the derivations of (22)-(24) to conclude
\[
12 \frac{d}{dt} \|\nabla u(t)\|^2 + (\mu + \mu_r) \|Au(t)\|^2
= -(u \cdot \nabla u_1, Au) + (\text{rot } w, Au) + (f(\theta_1) - f(\theta_2), Au),
\]
where $\phi$ and $\psi$ are given in (66). We have
\[
\|u\|_{L^4} \|\nabla u_1\|_{L^4} \|\phi\| \leq C \|\nabla u_1\| \|Au\| \leq C \|\nabla u\|^2 + \frac{\mu + \mu_r}{6} \|Au\|^2,
\]
\[
\|\text{rot } w, Au\| \leq \|\nabla w\| \|Au\| \leq C \|\nabla w\|^2 + \frac{\mu + \mu_r}{6} \|Au\|^2,
\]
\[
\|f(\theta_1) - f(\theta_2), Au\| \leq C \|\theta\| \|Au\| \leq C \|\theta\|^2 + \frac{\mu + \mu_r}{6} \|Au\|^2.
\]
Hence, we have
\[
12 \frac{d}{dt} \|\nabla u(t)\|^2 + (\mu + \mu_r) \|Au(t)\|^2 \leq C(\|Au_1\|^2 \|\nabla u\|^2 + \|\nabla w\|^2 + \|\theta\|^2). \quad (67)
\]
Similarly, using (28) we can conclude that
\[
12 \frac{d}{dt} \|\nabla w\|^2 + \mu + \mu_r \|Au\|^2 \leq C(\|Bw_2\|^2 + \|\nabla u\|^2 + \|\nabla w\|^2 + \|\theta\|^2). \quad (68)
\]
Letting $\phi = \theta$ in (66) we have
\[
12 \frac{d}{dt} \|\theta\|^2 + \kappa \|\nabla \theta\|^2 = -(u \cdot \nabla \theta_2, \theta) + (\Phi(u_1, \omega_1) - \Phi(u_2, \omega_2), \theta). \quad (69)
\]
By the embedding $H^2(\Omega) \to L^\infty(\Omega)$, (21), (18), (20) and Young’s inequality we obtain
\[
\|\Phi_1(u_1) - \Phi_1(u_2), \theta\| \leq C(\|\nabla u_1\|^4 + \|\nabla u_2\|^4) \|u\|^4 \|\theta\|
\leq C(\|\nabla u_1\|^4 + \|\nabla u_2\|^4) \|Au\| \|\theta\|
\leq C(\|\nabla u_1\|^4 + \|\nabla u_2\|^4) \|\theta\|^2 + \frac{\kappa}{12} \|Au\|^2.
\]
Similarly, by (21), (18), (20) and (57)
\[
\|\Phi_2(u_1, \omega_1) - \Phi_2(u_2, \omega_2), \theta\| \leq C(\|\nabla u_1\|^4 + \|\nabla u_2\|^4) \|u_1\|^4 \|\theta\|
\leq C(\|\nabla u_1\|^4 + \|\nabla u_2\|^4) \|Au\| \|\theta\|
\leq C(\|\nabla u_1\|^4 + \|\nabla u_2\|^4) \|\theta\|^2 + \frac{\kappa}{12} \|Au\|^2
\]
\[
\|\Phi_3(w_1) - \Phi_3(w_2), \theta\| \leq C(\|\nabla w_1\|^4 + \|\nabla w_2\|^4) \|\nabla w_1\|^4 \|\theta\|
\leq C(\|\nabla w_1\|^4 + \|\nabla w_2\|^4) \|\theta\|^2 + \frac{\kappa}{12} \|Bw\|^2,
\]
for $i = 3, 4, 5$.

From the above estimates and (68)-(69) we obtain
\[
\Psi(t) \leq C(\|Au_1\|^2 + \|Au_2\|^2 + \|Bw_1\|^2 + \|Bw_2\|^2 + \|\nabla \theta_2\|^2 + 1) \Psi,
\]
where $\Psi(t) = \|\nabla u(t)\|^2 + \|\nabla w(t)\|^2 + \|\theta(t)\|^2$. Thus, by Gronwall inequality, it follows that $\Psi = 0$. Therefore, $u_1 = u_2, w_1 = w_2$, and $\theta_1 = \theta_2$ on $[0, T]$. 

2.3. Proof of Theorem 1.5. The regularity of the approximate solution of problem (4), given by \((u^n, w^n, \theta^n)\), can be proved using the spectral Galerkin’s method as in the proof of Theorem 1.3. The regularity of its limit \((u, w, \theta)\), given by Theorem 1.4, is shown in some steps.

**Step 1.** We show that there exists \(T_2 > 0\) so that
\[
\int_0^{T_2} \|\theta^n(\tau)\|^2 d\tau \leq 1,
\] (70)
for all \(n \in \mathbb{N}\). We use the second principle of induction on \(n\), that is, we assume that (70) holds for \(1 \leq j \leq n\).

Differentiating with respect to \(t\) the first equation of (4) (with \(n = j\)) we have
\[
\frac{d}{dt}\|u^{j+1}_t\|^2 + (\mu + \mu_r)\|\nabla u^{j+1}_t\|^2
= 2\mu_r P(\text{rot} u^j_t) + P f'(\theta^j)\theta^j_t.
\]
Multiplying by \(u^{j+1}_t\), and integrating on \(\Omega\), we obtain
\[
\frac{1}{2} \frac{d}{dt}\|u^{j+1}_t\|^2 + (\mu + \mu_r)\|\nabla u^{j+1}_t\|^2
= -(u^j_t \cdot \nabla u^{j+1}_t, u^{j+1}_t) + 2\mu_r (\text{rot} u^j_t, u^{j+1}_t) + (f'(\theta^j)\theta^j_t, u^{j+1}_t).
\] (71)
We now estimate the terms of the right-hand side of (71).
\[
2\mu_r|\langle \text{rot} u^j_t, u^{j+1}_t \rangle| \leq C\|u^j_t\|^2 + \frac{\mu + \mu_r}{6}\|\nabla u^{j+1}_t\|^2
\]
\[
|\langle f'(\theta^j)\theta^j_t, u^{j+1}_t \rangle| \leq C|\theta^j_t, u^{j+1}_t| \leq C\|\theta^j_t\|^2 + \frac{\mu + \mu_r}{6}\|\nabla u^{j+1}_t\|^2
\]
\[
|\langle u^j_t \cdot \nabla u^{j+1}_t, u^{j+1}_t \rangle| \leq C\|u^j_t\|\|Au^{j+1}_t\|\|\nabla u^{j+1}_t\|
\]
\[
\leq C\|u^j_t\|^2 \|Au^{j+1}_t\|^2 + \frac{\mu + \mu_r}{6}\|\nabla u^{j+1}_t\|^2.
\]
Using these estimates in (71) we get
\[
\frac{d}{dt}\|u^{j+1}_t\|^2 + (\mu + \mu_r)\|\nabla u^{j+1}_t\|^2 \leq C\|u^j_t\|^2 + C\|\theta^j_t\|^2 + C\|u^j_t\|^2 \|Au^{j+1}\|^2,
\] (72)
for \(1 \leq j \leq n\).

To estimate \(\|Au^{j+1}\|\) we multiply the first equation of (4) (for \(n = j\)) by \(Au^{j+1}\).

We obtain
\[
(\mu + \mu_r)\|Au^{j+1}\|^2 = -\langle u^{j+1}_t, Au^{j+1} \rangle - \langle u^j_t \cdot \nabla u^{j+1}, Au^{j+1} \rangle
+ 2\mu_r \langle \text{rot} w^j, Au^{j+1} \rangle + \langle f(\theta^j), Au^{j+1} \rangle.
\] (73)
We now estimate the terms of the right-hand side of (73)
\[
|\langle u^{j+1}_t, Au^{j+1} \rangle| \leq C\|u^{j+1}_t\|^2 + \frac{\mu + \mu_r}{8}\|Au^{j+1}\|^2,
\]
\[
|\langle u^j_t \cdot \nabla u^{j+1}, Au^{j+1} \rangle| \leq \|u^j_t\|_L^2 \|\nabla u^{j+1}\|_L^2 \|Au^{j+1}\|
\]
\[
\leq C\|\nabla u^j_t\|^2 \|\nabla u^{j+1}\|^2 + \frac{\mu + \mu_r}{8}\|Au^{j+1}\|^2,
\]
\[
|\langle 2\mu_r \text{rot} w^j, Au^{j+1} \rangle| \leq C\|\nabla w^j\|^2 + \frac{\mu + \mu_r}{8}\|Au^{j+1}\|^2,
\]
\[
|\langle f(\theta^j), Au^{j+1} \rangle| \leq C\|\theta^j_t\|^2 + \frac{\mu + \mu_r}{8}\|Au^{j+1}\|^2.
\]
Hence and the estimates given by Theorem 1.3 we conclude
\[
\|Au^{j+1}\|^2 \leq C(1 + \|u^{j+1}_t\|^2).
\] (74)
From (72) we get
\[
\frac{d}{dt} \|u_t^{j+1}\|^2 + (\mu + \mu_r)\|\nabla u_t^{j+1}\|^2 \leq C\|u_t^j\|^2 + C\|\theta_t^j\|^2 + C\|u_t^j\|^2(1 + \|u_t^{j+1}\|^2).
\]
By (70) and estimates of Theorem 1.3 we conclude
\[
\|u_t^{j+1}\|^2 + (\mu + \mu_r)\int_0^t \|\nabla u_t^{j+1}(\tau)\|^2 d\tau \leq C + C\int_0^t \|u_t^j(\tau)\|^2 \|u_t^{j+1}(\tau)\|^2 d\tau.
\]
Applying Gronwall’s inequality we conclude
\[
\|u_t^{j+1}\|^2 + (\mu + \mu_r)\int_0^t \|\nabla u_t^{j+1}(\tau)\|^2 d\tau \leq C \exp\left(C\int_0^t \|u_t^j(\tau)\|^2 d\tau\right) \leq C
\]
for \(1 \leq j \leq n\) and \(t \in (0, T_1)\). Thus, from (74) we get
\[
\|A u^{j+1}\|^2 \leq C,
\]
for \(1 \leq j \leq n\) and \(t \in (0, T_1)\). Analogously, it is possible to conclude that
\[
\|B u^{j+1}(t)\|^2 \leq C,
\]
\[
\|u_t^{j+1}(t)\|^2 + (c_n + c_d)\int_0^t \|\nabla u_t^{j+1}(\tau)\|^2 d\tau \leq C,
\]
for \(1 \leq j \leq n\) and \(t \in (0, T_1)\). Finally, we multiply the third equation of (4) by \(\theta_t^{n+1}\). Hence,
\[
\|\theta_t^{n+1}\|^2 + \kappa \frac{d}{dt}\|\nabla\theta_t^{n+1}\|^2 = -(u^n \cdot \nabla\theta_t^{n+1}, \theta_t^{n+1}) + (\Phi(u^n, w^n), \theta_t^{n+1}).
\]
Using the estimates of Theorem 1.3 and (74)-(78) we have
\[
\|u^n \cdot \nabla\theta_t^{n+1}, \theta_t^{n+1}\| \leq \|u^n\|_{L^\infty} \|\nabla\theta_t^{n+1}\| \|\theta_t^{n+1}\| \leq C\|Au^n\|\|\nabla\theta_t^{n+1}\| \|\theta_t^{n+1}\| \leq C\|\nabla\theta_t^{n+1}\|^2 + \frac{1}{12}\|\theta_t^{n+1}\|^2
\]
\[
|\Phi_1(u^n), \theta_t^{n+1}| \leq C\|\nabla u^n\|^2 \|\theta_t^{n+1}\|^2 \leq C\|\nabla u^n\|^2 \|\theta_t^{n+1}\|^2 \leq C\|Au^n\|^4 + \frac{1}{12}\|\theta_t^{n+1}\|^2
\]
\[
|\Phi_2(u^n, w^n)| \leq C\|\nabla u^n\|^2 + |w^n|^2 \|\theta_t^{n+1}\| \leq C\|\nabla u^n\|^2 + \|w^n\|_{L^4} \|\theta_t^{n+1}\|^2 \leq C\|Au^n\|^4 + \|Bw^n\| + \|\theta_t^{n+1}\|^2
\]
\[
|\Phi_3(w^n, \theta_t^{n+1})| \leq C\|\nabla w^n\|^2 \|\theta_t^{n+1}\|^2 \leq \|\nabla w^n\|^2 \|\theta_t^{n+1}\|^2 \leq C\|Bw^n\|^4 + \frac{1}{12}\|\theta_t^{n+1}\|^2,
\]
for \(i = 3, 4, 5\). Therefore, from (79), (76), (77) and the anterior estimates we have
\[
2\kappa\|\nabla\theta_t^{n+1}(t)\|^2 + \int_0^t \|\theta_t^{n+1}\|^2 d\tau \leq C + C\int_0^t \theta_t^{n+1}\|^2 d\tau,
\]
for \(t \in (0, T_1)\). By Gronwall’s inequality we have
\[
2\kappa\|\nabla\theta_t^{n+1}(t)\|^2 + \int_0^t \|\theta_t^{n+1}(\tau)\|^2 d\tau \leq C(e^{Ct} - 1),
\]
for \( t \in (0, T_1) \). Hence, there exists \( T_2 \in (0, T_1) \) so that
\[
2\kappa \|\nabla \theta^{n+1}(t)\|^2 + \int_0^t \|\theta_i^{n+1}(\tau)\|^2 d\tau \leq 1,
\]
for \( t \in (0, T_2) \). Therefore, \((70)\) holds for \( n + 1 \) and the induction argument is concluded.

**Step 2.** We show estimates for \( u^{n,s}, w^{n,s} \) and \( \theta^{n,s} \) defined by \((46)-(48)\). To do this, we differentiate equation \((46)\), with respect to \( t \), and multiply by \( u_i^{n,s} \). So,
\[
\frac{1}{2} \frac{d}{dt} \left\| u_i^{n,s} \right\|^2 + (\mu + \mu_r) \left\| \nabla u_i^{n,s} \right\|^2
\]
\[
= -(u_i^{n-1,s} \nabla u_i^{n,s}, u_i^{n,s}) + 2\mu_r (\text{rot} w_i^{n-1,s}, u_i^{n,s}) - (u_i^{n-1,s} \cdot \nabla u_i^{n,s}, u_i^{n,s})
\]
\[
- (u_i^{n-1,s} \nabla u_i^{n,s}, u_i^{n,s}) + \left( f'((\theta^{n-1})^2) \theta_i^{n-1,s} + \left| f'((\theta^{n-1})^2 - f'((\theta^{n-1})) \theta_i^{n-1,s} + u_i^{n,s} \right|.
\]

Estimating the right-hand side in a usual way we obtain
\[
\frac{d}{dt} \left\| u_i^{n,s} \right\|^2 + (\mu + \mu_r) \left\| \nabla u_i^{n,s} \right\|^2
\]
\[
\leq C \left\| u_i^{n-1,s} \right\|^2 + C \left\| A u_i^{n,s} \right\|^2 + C \left\| w_i^{n-1,s} \right\|^2 + C \left\| \nabla u_i^{n-1,s} \right\|^2 \left\| \nabla u_i^{n,s} \right\|^2
\]
\[
+ C \left\| \theta_i^{n-1,s} \right\|^2 + C \left\| \theta_i^{n-1,s} \right\|^2 \left\| \nabla \theta^{n-1,s} \right\|.
\]

where the last term was estimated using \((80)\) as follows
\[
\left| \left( f'(\theta^{n-1}) - f'(\theta^{n-1}) \theta_i^{n-1,s} , u_i^{n,s} \right) \right|
\]
\[
\leq L_f \left( \left\| \theta_i^{n-1,s} \right\| \left\| \theta_i^{n-1} \right\| \left\| \nabla u_i^{n,s} \right\| \right)
\]
\[
\leq L_f \left( \left\| \theta_i^{n-1,s} \right\|^\frac{1}{2} \left\| \nabla \theta^{n-1,s} \right\|^\frac{1}{2} \left\| \theta_i^{n-1} \right\| \left\| \nabla u_i^{n,s} \right\| \right)
\]
\[
\leq C \left( \left\| \theta_i^{n-1,s} \right\|^\frac{1}{2} \left\| \nabla \theta^{n-1,s} \right\|^\frac{1}{2} \left\| \theta_i^{n-1} \right\| \left\| \nabla u_i^{n,s} \right\| \right)
\]
\[
\leq C \left\| \theta_i^{n-1,s} \right\|^\frac{1}{2} \left\| \theta_i^{n-1} \right\|^\frac{1}{2} \left\| \nabla u_i^{n,s} \right\|.
\]

Similarly, we have
\[
\frac{d}{dt} \left\| w_i^{n,s} \right\|^2 + (c_a + c_d) \left\| \nabla w_i^{n,s} \right\|^2
\]
\[
\leq C \left\| w_i^{n-1,s} \right\|^2 + C \left\| B w_i^{n,s} \right\|^2 + C \left\| u_i^{n-1,s} \right\|^2
\]
\[
+ C \left\| \nabla u_i^{n-1,s} \right\|^2 \left\| \nabla w_i^{n,s} \right\|^2 + C \left\| \theta_i^{n-1,s} \right\|^2 + C \left\| \theta_i^{n-1,s} \right\|^2 \left\| \nabla \theta^{n-1,s} \right\|.
\]

Adding the last two inequalities, integrating from 0 to \( t \), using \((60), (63), (61), (75), (78), \) and (80) we have
\[
\left\| u_i^{n,s} \right\|^2 + \left\| w_i^{n,s} \right\|^2 + (\mu + \mu_r) \int_0^t \left\| \nabla u_i^{n,s}(\tau) \right\|^2 d\tau + (c_a + c_d) \int_0^t \left\| \nabla w_i^{n,s}(\tau) \right\|^2 d\tau
\]
\[
\leq C \int_0^t \left( \left\| u_i^{n-1,s}(\tau) \right\|^2 + \left\| w_i^{n-1,s}(\tau) \right\|^2 \right) d\tau
\]
\[
+ C \int_0^t \left( \left\| A u_i^{n,s}(\tau) \right\|^2 + \left\| B w_i^{n,s}(\tau) \right\|^2 \right) d\tau
\]
\[
+ C \int_0^t \left\| \nabla u_i^{n-1,s}(\tau) \right\|^2 \left( \left\| \nabla u_i^{n,s}(\tau) \right\|^2 + \left\| \nabla w_i^{n,s}(\tau) \right\|^2 \right) d\tau
\]
\[
+ C \int_0^t \left\| \theta_i^{n-1,s}(\tau) \right\|^2 d\tau + C \int_0^t \left\| \theta_i^{n-1,s}(\tau) \right\|^2 \left\| \theta_i^{n-1,s}(\tau) \right\| d\tau
\]
\[
\leq C \left[ \Lambda_{n-1}(t) + \Lambda_{n-2}(t) + \int_0^t \left\| \theta_i^{n-1,s}(\tau) \right\|^2 d\tau + \Lambda_{n-2}(t) \right].
\]
Thus, from (46), (47) and (60) it is possible to show that
\[
\|Au^{n,s}\|^2 + \|Bw^{n,s}\|^2 \\
\leq C(\|\nabla u^{n,s}\|^2 + \|\nabla w^{n,s}\|^2) + C(\|u^{n,s}\|^2 + \|w^{n,s}\|^2) \\
+ C(\|\nabla u^{n-1,s}\|^2 + \|\nabla w^{n-1,s}\|^2) + C\|\theta^{n-1,s}\|^2 \\
\leq C[A_{n-1}(t) + A_{n-2} + A_{n-3}(t) + \int_0^t \|\theta^{n-1,s}(\tau)\|^2 d\tau + A_{n-3}(t)]
\]

(82)

Multiplying (48) by \(\theta^{n,s}_t\) we have
\[
\|\theta^{n,s}_t\|^2 + \frac{\kappa}{2} \frac{d}{dt} \|\nabla \theta^{n,s}\|^2 = -(u^{n-1,s}\nabla \theta^{n,s}, \theta^{n,s}_t) - (u^{n-1,s}\nabla \theta^{n,s}, \theta^{n,s}_t) \\
+ (\Phi(u^{n-1,s}, w^{n-1,s}) - \Phi(u^{n-1}, w^{n-1}), \theta^{n,s}_t)
\]

We estimate the terms of the right-hand side by using (76), (77), (80)
\[
\|(u^{n-1,s}\nabla \theta^{n,s}, \theta^{n,s}_t)\| \leq \|u^{n-1,s}\|_{L^\infty} \|\nabla \theta^{n,s}\| \|\theta^{n,s}_t\| \\
\leq C \|Au^{n-1,s}\| \|\nabla \theta^{n,s}\| \|\theta^{n,s}_t\| \\
\leq C \|\nabla \theta^{n,s}\|^2 + \frac{\kappa}{14} \|\theta^{n,s}_t\|^2
\]

\[
\|(u^{n-1,s}\nabla \theta^{n,s}, \theta^{n,s}_t)\| \leq \|u^{n-1,s}\|_{L^\infty} \|\nabla \theta^{n,s}\| \|\theta^{n,s}_t\| \\
\leq C \|Au^{n-1,s}\| \|\nabla \theta^{n,s}\| \|\theta^{n,s}_t\| \\
\leq C \|Au^{n-1,s}\| \|\nabla \theta^{n,s}\|^2 + \frac{\kappa}{14} \|\theta^{n,s}_t\|^2
\]

\[
\|(\Phi_1(u^{n-1,s}), \Phi_1(u^{n-1,s}, \theta^{n,s}_t)) - (\Phi_2(u^{n-1}, w^{n-1}), \theta^{n,s}_t)\| \\
\leq C(\|\nabla u^{n-1,s}\|_{L^4} + \|\nabla w^{n-1}\|_{L^4}) \|\nabla u^{n-1,s}\|_{L^4} \|\theta^{n,s}_t\| \\
\leq C(\|Au^{n-1,s}\| + \|Bu^{n-1,s}\| + \|Bu^{n-1}||A\theta^{n-1,s}| |\|\theta^{n,s}_t\|^2 \\
\leq C \|Au^{n-1,s}\|^2 + \frac{\kappa}{14} \|\theta^{n,s}_t\|^2
\]

\[
\|(\Phi_1(u^{n-1,s}), \Phi_1(u^{n-1,s}, \theta^{n,s}_t)) - (\Phi_2(u^{n-1}, w^{n-1}), \theta^{n,s}_t)\| \\
\leq C(\|\nabla w^{n-1,s}\|_{L^4} + \|\nabla w^{n-1}\|_{L^4}) \|\nabla w^{n-1,s}\|_{L^4} \|\theta^{n,s}_t\| \\
\leq C(\|Bu^{n-1,s}\| + \|Bu^{n-1}||B\theta^{n-1,s}| |\|\theta^{n,s}_t\|^2 \\
\leq C \|Bu^{n-1,s}\|^2 + \frac{\kappa}{14} \|\theta^{n,s}_t\|^2
\]

Thus, we have
\[
\|\nabla \theta^{n,s}(\tau)\|^2 + \int_0^t \|\theta^{n,s}_t(\tau)\|^2 d\tau \\
\leq C \int_0^t \|\nabla \theta^{n,s}(\tau)\|^2 d\tau + C \int_0^t (\|Au^{n-1,s}(\tau)\|^2 + \|Bu^{n-1,s}(\tau)\|^2) d\tau.
\]

By Gronwall’s inequality and (61) we conclude
\[
\|\nabla \theta^{n,s}(\tau)\|^2 + \int_0^t \|\theta^{n,s}_t(\tau)\|^2 d\tau \leq C \int_0^t (\|Au^{n-1,s}(\tau)\|^2 + \|Bu^{n-1,s}(\tau)\|^2) d\tau \\
\leq C A_{n-3}(t).
\]

(83)

**Step 3.** Passage to limit \(s \to \infty\). Since the spaces \(L^2(0,T_2; V), L^\infty(0,T_2; H), L^\infty(0,T_2; B_0(\Omega)), L^\infty(0,T_2; L^2(\Omega)), L^\infty(0,T_2; D(A)), L^\infty(0,T_2; D(B)), \)
$L^\infty(0, T_2; H^1_0(\Omega))$ and $L^2(0, T_2; L^2(\Omega))$ are Banach spaces, from estimates (81)-(83) we conclude that there exists functions $u$, $w$ and $\theta$ such that

\begin{align*}
    u^n &\rightharpoonup u \text{ strongly in } L^\infty(0, T_2; D(A)), \\
    u^n_t &\rightharpoonup u_t \text{ strongly in } L^\infty(0, T_2; H) \cap L^2(0, T_2; V), \\
    w^n &\rightharpoonup w \text{ strongly in } L^\infty(0, T_2; D(B)), \\
    w^n_t &\rightharpoonup w_t \text{ strongly in } L^\infty(0, T_2; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \\
    \theta^n &\rightharpoonup \theta \text{ strongly in } L^\infty(0, T_2; H^1_0(\Omega)), \\
    \theta^n_t &\rightharpoonup \theta_t \text{ strongly in } L^2(0, T_2; L^2(\Omega)),
\end{align*}

as $n \to \infty$. The estimates follow from (81), (82) and (83) letting $s \to \infty$.

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