Rank of a tensor and quantum entanglement

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\textbf{ABSTRACT}

The rank of a tensor is analysed in the context of quantum entanglement. A pure quantum state $v$ of a composite system consisting of $d$ subsystems with $n$ levels each is viewed as a vector in the $d$-fold tensor product of $n$-dimensional Hilbert space and can be identified with a tensor with $d$ indices, each running from 1 to $n$. We discuss the notions of the generic rank and the maximal rank of a tensor and review results known for the low dimensions. Another variant of this notion, called the border rank of a tensor, is shown to be relevant for the characterization of orbits of quantum states generated by the group of special linear transformations. A quantum state $v$ is called entangled, if it cannot be written in the product form, $v \neq v_1 \otimes v_2 \otimes \cdots \otimes v_d$, what implies correlations between physical subsystems. A relation between various ranks and norms of a tensor and the entanglement of the corresponding quantum state is revealed.

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\section{Introduction}

\subsection{Quantum mechanics and tensors}

Quantum mechanics was born in the beginning of twentieth century. One way of its description was the ‘matrix mechanics’ created by Heisenberg–Born–Jordan in 1925 \cite{1}. The matrix formalism of quantum mechanics was created by von Neumann \cite{2} and Landau \cite{3}. The probabilistic nature of quantum mechanics is puzzling, which led Einstein–Podolsky–Rosen to the notion of entanglement \cite{4}. Entanglement deals with quantum systems, for which there exists a physically distinguished partition of the entire system into $d$ parts with $d \geq 2$. The mathematical description of $d$-partite systems involves the theory of tensors. The common physics notation for tensors was introduced by Dirac in 1939 \cite{5}. It is very concise, but it is unfamiliar to a good portion of mathematical community, and in several cases it is not adequate.

There are several ways to measure the entanglement of a $d$-partite system described by a $d$-tensor $T$, which in quantum mechanics can be called a state. It is a nonzero tensor, which
can be thought as a $d$-array with entries $T_{i_1,...,i_d}$. The bi-partite case, $d = 2$, corresponds to matrices, so a 2-tensor will sometimes be denoted by $T$. The case $d \geq 3$ corresponds to tensors with $d$ indices. For simplicity of exposition, we will call $T$ any $d$-array for $d \geq 2$.

The simplest integer invariant of $T$ is its rank, denoted as $r(T)$. Rank-1 tensor corresponds to a tensor product of $d$-vectors, which is called a product state. The rank of $T$ is the minimum number of summands in the decomposition of $T$ to a sum of rank-1 tensor. Thus the higher the rank of a tensor, the more entangled the corresponding quantum state. A different way to decompose $T$ into a sum of rank-1 tensors is to have a decomposition with the minimal norm, which in quantum physics might be related to energy. This brings us to the notion of the nuclear norm (87). The minimal number of rank-1 terms in the nuclear norm decomposition is called the nuclear rank and denoted as $r_{\text{nuc}}(T)$. For matrices the rank and the nuclear rank are equal to the standard rank of matrices. The nuclear norm of a matrix is a sum of its singular values. For tensors $d \geq 3$, while the description of the rank and nuclear rank are relatively easy, the numerical computation can be quite extensive. In computer science terminology, the computation of all the above-mentioned quantities in NP-hard.

The aim of this paper is to give a survey of many results on the rank of tensors and to emphasize the connection to quantum mechanics and quantum information theory. This work is addressed to a broad audience consisting of computer scientists, mathematicians and physicists.

### 1.2. Quantum information and related fields

In this section, we discuss in more detail Quantum information theory that our paper deal with. Since the publication of the pioneering paper of Ingarden [6], there was an explosion in the theory of quantum information and related fields in the last 25 years. On the one hand, one could witness enormous success related to quantum cryptography [7] and practical implementations of quantum communication [8]. On the other hand, the progress in quantum computation is still moderate [9] and some experts raise doubts [10], whether an operating quantum computer will be ever constructed.

A quantum computer would allow us to solve some problems that are not known to be solvable in polynomial time on a classical computer, for instance factorization of large integers [11]. The practical impossibility of factoring integers to primes is the basis for the widely used RSA cryptographic scheme. One of the key advantages of processing of information by a quantum computer relies in the possibility of going beyond the standard set of digits ‘0’ and ‘1’, and using nonclassical states, represented by normalized vectors in a $n$ -dimensional, complex Hilbert space $\mathcal{H}_n$. They include the superposition of classical states, which after a measurement yield the result ‘0’ with a certain probability $p$ and ‘1’ with probability $1-p$, often written as $\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$. We used here the notation of Dirac, explained in detail in Section 2.1, in which $|0\rangle$ and $|1\rangle$ denote any two normalized orthogonal vectors – the elements of an orthonormal basis in $\mathcal{H}_n$. Assumed normalization of vectors has a simple interpretation: The squared modulus of each component represents the probability of the state to be measured in the corresponding basis state, and the sum of probabilities is equal to 1.

Non-classical properties characterize also the notable Bell state [12], written $(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)/\sqrt{2}$, which exhibits quantum entanglement: the effect of non-classical
correlations between subsystems, predicted already by Einstein and Schrödinger [4,13,14]. Note that the outcomes of a measurement performed on both subsystems are perfectly correlated: if '0' is registered in the subsystem A, the same result will be obtained in the subsystem B.

To define quantum entanglement, one requires a physical system consisting of two (or more) distinguished subsystems [15,16], the state of which is described by a vector from a complex Hilbert space with a tensor product structure, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. A special role in such a space is played by the product states, $|\psi_{AB}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, where $|\phi_A\rangle \in \mathcal{H}_A$ and $|\phi_B\rangle \in \mathcal{H}_B$. A state $|\psi\rangle \in \mathcal{H}_{AB}$ is called entangled if it is not of the product form, as it carries some quantum correlations between subsystems. For the definitions of necessary notions used in quantum mechanics, see Appendix.

The motivation for the definition of entanglement between two predefined systems stems from the classical probability theory: A product quantum state can be compared to a product probability vector, $p(x, y) = p_1(x)p_2(y)$, which describes independent variables $x$ and $y$, while entangled state corresponds to correlated events. It is thus important to characterize degree of entanglement, interpreted as a degree of quantum correlations between subsystems.

Product states allow one to construct a complete orthonormal product basis, which reads $|i\rangle_A \otimes |j\rangle_B$, with $i, j = 0, \ldots n - 1$ for a system consisting of two subsystems $A$ and $B$ with $n$ levels each. Any pure state $|\psi_T\rangle$ in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ of a bipartite system (i.e. consisting of two subsystems) can be represented in a product basis by a matrix $T$ of expansion coefficients, $|\psi_T\rangle = \sum_{i,j=0}^{n-1} T_{ij} |i\rangle \otimes |j\rangle$.

Note that the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are identified with two vector spaces, while $\mathcal{H}_A \otimes \mathcal{H}_B$ is identified with the space of matrices. Similarly, the tensor product of $d \geq 3$ vector spaces $\bigotimes_{j=1}^{d} \mathcal{H}_j$ is identified with the space of $d$-mode multiarrays (tensors). A rank-1 tensor (matrix) corresponds to a product state.

### 1.3. Matrices and tensors

To characterize the entanglement of the state $|\psi_T\rangle$ it is sufficient to perform the singular value decomposition (SVD) of the matrix $T$: if there is more than one nonzero singular value then the corresponding quantum state is not of the product form, so it is entangled. In other words, the bipartite pure state is entangled if the rank of the corresponding matrix $T$ is larger than one [16].

In a similar way any pure state of a multipartite system, consisting of $d > 2$ subsystems, can be represented by a tensor $T$ with $d$ indices. A standard definition of a rank of $T \neq 0$ is the minimum number of product states whose sum is $T$. However, the description of entanglement in such a multipartite case is more difficult than for bipartite systems, as algebraic transformations on tensors are much more involved than the operations on matrices [17,18]. Tensor product and entanglement optimization are used in modern quantum chemistry [19].

For matrices there exists a single notion of the rank of a matrix, as various ways to introduce this quantity lead to the same definition. This is not the case for tensors with $d \geq 3$ modes, for which different approaches lead to different definitions of the rank of a tensor (with various names). Furthermore, most interesting definitions of the rank of tensors, as
the above definition of the rank, are hard to compute [20–24]. In the present paper by ‘hard’ we mean that the complexity of their computations is suspected to be at least NP-complete.

A special phenomenon occurs for tensors that does not hold for matrices. There are tensors of rank greater than 1 such that their rank decomposition is unique (up to permutation of order of rank-1 summands). Such tensors are called identifiable. It is conjectured that general tensors of rank less than the generic rank are identifiable [25,26].

Another example is a generalization notion of SVD for the space of tensors [27,28]. However in general, these decompositions of a tensor as a sum of rank-1 factors do not have the property that in each mode the vectors are orthogonal as in the matrix case.

An analysis of tensors as multiarrays can be traced to the two papers of Hitchcock [29,30] which introduce the notion of $d$-mode tensor and its rank. In algebraic geometry, the study of symmetric tensors, which is equivalent to homogeneous polynomials, was started by Sylvester [31]. Sylvester discussed the decomposition of a homogeneous polynomial of degree $d$ in 2-variables as a minimal sum of linear forms of degree $d$. In modern terminology, a minimal decomposition of a homogeneous polynomial of degree $d$ as a minimal sum of $d$-powers of linear terms is called the Waring decomposition [32]. Waring raised the general problem of minimum decomposition of an integer as a sum of $d$-powers of integers [33].

The purpose of this review is twofold: first, we aim to introduce the problems of pure state entanglement to the community of researchers working in applied and pure mathematics, and computer science. Second, we wish to present a survey of recent mathematical results concerning the rank of a tensor and its various generalizations that are relevant in studies of quantum entanglement.

Note that the rank of a tensor $r(T)$ with $d$ modes over the complex numbers, can be viewed as a simple integer quantity characterizing the entanglement of a pure state of a quantum system composed out of $d$ subsystems. The rank of a nonzero tensor ranges from 1, which characterizes a product (separable) state, to the maximum rank $r_{\text{max}}$, defined for a given size of the tensor. It is convenient to use the notion ‘rank of a pure quantum state’ $|\psi\rangle$, which means the rank of the corresponding tensor $T$.

As most of the problems in tensors, the problem of finding the rank of a given tensor is NP-hard [21]. A $d$-mode complex-valued tensor of dimensions $n = (n_1, \ldots, n_d)$ with its entries chosen independently at random (a generic tensor) has a fixed rank $r_{\text{gen}}(n)$ with probability one. The value of $r_{\text{gen}}(n)$ can be computed in randomized polynomial time using Terracini’s lemma [34]. The original Terracini’s lemma characterizes the generic Waring rank of a homogeneous polynomial of degree $d$ of $n$-variables over the complex numbers as discussed above.

There is a general conjecture about the value of the generic rank of 3-mode tensors [35], see Conjecture 4.12, which is known to hold in special cases [36]. A generalization of this conjecture to $d$-mode tensors is stated as an open problem below (47). A related problem is the value of the tensor rank of the tensor product of two tensors, linked to the issue of characterization of entanglement and finding an optimal decomposition of several copies of a given entangled state [37–39].

The rank of the tensor product of tensors is submultiplicative (the rank of the tensor product is not greater than the product of ranks) and can be strictly submultiplicative [40]. It is then important to ask whether the rank of the tensor product of two generic tensors of the same dimension is equal to the product of their ranks. The rank of the direct sum
of two tensors is subadditive. Strassen’s direct sum conjecture stated that the rank of the
direct sum of two tensors is the sum of the ranks [41]. This conjecture was disproved by
Shitov [42]. A special case of Strassen’s conjecture can be related to the above product rank
conjecture for the generic case, i.e. the rank of the product of \( k \) -copies of a generic tensor
is the \( k \) -power of the rank of this tensor [43]. Yet another related notion is the border rank
of a tensor: It is the minimum \( r \) such that a given tensor is a limit of tensors of rank \( r \).
This notion is fundamental in algebraic geometry and numerical analysis, in particular for
problems of approximation of tensors by lower rank tensors.

A special interesting case is the case of symmetric tensors, which correspond in quantum
physics to problems involving bosons: particles with an integer spin following the Bose–Einstein
statistics. Symmetric tensors can be viewed as homogeneous polynomials [32,44]. The relevant notion of the rank of a symmetric tensor, is the symmetric rank, which
is the Waring rank of the homogeneous polynomial discussed above. Comon’s conjecture claimed that the symmetric rank and the rank of a symmetric tensor are equal [45].
Recently Shitov gave a counterexample to this conjecture [42]. The generic symmetric rank
of a symmetric tensor is given in [46], which is the analog of Conjecture 4.12.

Another way to measure the entanglement of a quantum pure state is the spectral or
nuclear norm (see definitions (79) and (87)) of the corresponding tensor [47,48], which
are dual norms. The spectral norm, also known as the injective norm [49] of a ten-
sor, determines the geometric measure of entanglement [50] of the corresponding pure
quantum state. The nuclear norm, also known as projective norm [49], is the minimum
value of the ‘energy’ of a decomposition of a tensor as sum of rank-1 tensors (see the
sentence below (87)). The minimum number of rank-1 tensors in the minimal decom-
position of a given tensor (88) is called the nuclear rank, and denoted \( r^{\text{nuc}}(T) \). Hence
\( r(T) \leq r^{\text{nuc}}(T) \). For matrices the nuclear rank is equal to the rank. Nuclear rank of ten-
sors possesses similar properties as matrix rank, unlike the tensor rank [20]. That is, if
\( \lim_{k \to \infty} T_k = T \) then \( \liminf_{k \to \infty} r^{\text{nuc}}(T_k) \geq r^{\text{nuc}}(T) \). Assume that there exists \( T \) such that
\( r(T) \) is greater than the generic rank \( r_{\text{gen}} \). Then \( r^{\text{nuc}}(T) \geq r(T) > r_{\text{gen}} \), and the nuclear
rank of every tensor in a small enough neighbourhood of \( T \) is not less than \( r^{\text{nuc}}(T) \). There-
fore there exists an open set of tensors whose nuclear rank is greater than their ranks. The
nuclear rank of the quantum state provides another simple integer measure of quantum
entanglement.

The application part of our paper is how to compute (or estimate) various ranks we sur-
vey. Apart of computing the generic ranks, which can be done in randomized polynomial
time, all other quantities seem to be NP-hard to compute. A good way to find or estimate
from below the rank and symmetric rank is to use polynomial equations, in particular the
effective Nullstellensatz [51]: If \( r \) is less than the rank of a tensor then the corresponding
system of polynomials equations is not solvable (see Section 5.5). Hence the Bertini soft-
ware [52] is an appropriate tool. To compute the nuclear rank, we can use the numerical
methods and the software suggested in [47].

1.4. Notation

We use several notions of rank in various context, hence we present a table that collects all
symbols used in the text with reference to the page on which the objects appear for the first
time:
1.5. Summary of the paper

Section 2 discusses basic notions in quantum information theory (QIT) and tensors that will be used in this paper. Section 2.1 discusses briefly the Dirac notation and the notion of a simple physical system. A simple system is represented by a vector, which is usually of length one (normalized) and corresponds to a single party system. Section 2.2 discusses complex systems and the notion of entanglement. A composite system is represented by tensor product of $d$-simple components. The simplest composite system consists of two parties ($d = 2$) and usually referred as a bipartite system. It is represented by a matrix, which usually has Frobenius norm one (normalized). The bipartite state is entangled if and only if the corresponding matrix has rank greater than 1. For $d \geq 3$, a $d$-partite system is represented by a $d$-mode tensor, which usually has Hilbert–Schmidt norm one (normalized). It is entangled if and only if the corresponding tensor has rank greater than 1. Section 2.3 discusses the Kronecker product of $d$-vector spaces and their interpretation in QIT.

Section 3 discusses properties and results on tensor rank of general tensors. Section 3.1 recalls the standard properties of matrix rank. Section 3.2 discusses the well-known properties of Singular Value Decomposition (SVD), which is known in physics community as Schmidt decomposition. In Section 3.3, we discuss the notions of tensor rank, generic rank and border rank, and some of their properties.

In Section 4, we discuss in detail the rank properties of 3-mode tensors. Section 4.1 gives basic results on the rank of 3-tensors. In particular, we discuss Kruskal’s uniqueness theorem [53]. Section 4.2 reviews the formula for the rank of $m \times n \times 2$ tensors discovered in [54]. In Section 4.3, we discuss the validity of Strassen’s direct sum conjecture for certain 3-tensors. Section 4.4 discusses known results and a conjecture on the formula for the generic rank of 3-tensors. In Section 4.5, we give a randomized polynomial-time algorithm to compute the generic rank of 3-tensor. Section 4.6 discusses known results on maximal ranks of 3-tensors.

In Section 5, we discuss ranks of $d$-tensors for $d \geq 4$. In Section 5.1, we bring known generalizations of results from Section 4.1 to $d$-mode tensors. In Section 5.2, we recall Terracini’s lemma, which is a basic tool in understanding and analysing the set of tensors of rank $r$. Section 5.3 gives an upper bound on the generic rank of tensors using pure combinatorial methods. In Section 5.4, we discuss the generic rank of $d$-qunits ($d$-mode tensors

| Description                                      | Notation   | Defined at page |
|--------------------------------------------------|------------|-----------------|
| Rank of a matrix $A$                              | $r(A)$     | 8               |
| Rank of a tensor $T$                              | $r(T)$     | 10              |
| Generic rank of $T \in \mathbb{C}^n$             | $r_{\text{gen}}(n)$ | 11              |
| Maximum rank of $T \in \mathbb{C}^n$             | $r_{\text{max}}(n)$ | 11              |
| Border rank of $T \in \mathbb{C}^n$              | $r_b(T)$   | 11              |
| Kruskal’s rank of vectors $x_1, \ldots, x_l \in \mathbb{C}^m$ | $r_K(x_1, \ldots, x_l)$ | 12              |
| Symmetric rank of $S$                            | $r_s(S)$   | 27              |
| Generic rank of symmetric tensor                  | $r_{\text{gen}}(d, n)$ | 27              |
| Maximum rank of symmetric tensor                  | $r_{\text{max}}(d, n)$ | 27              |
| Nuclear rank of $T$                              | $r_{\text{nucl}}(T)$ | 33              |
| Generic nuclear rank of $T \in \mathbb{C}^n$    | $r_{\text{nucl}}(n)$ | 33              |
| Maximum nuclear rank of $T \in \mathbb{C}^n$    | $r_{\text{max}}(n)$ | 33              |
| Symmetric nuclear rank of $S$                     | $r_{\text{nucl}}^s(S)$ | 35              |
whose each mode is \( \mathbb{C}^n \). Section 5.5 discusses an algorithmic way to find the rank of a tensor using the solvability of system of linear equations with many variables. In Section 5.6, we discuss the problem of generic identifiability of tensors. Namely, assuming an integer \( r \) is less than a generic rank, when a generic tensor of rank \( r \) has a unique decomposition as a sum of \( r \) rank one tensors?

In Section 6, we discuss symmetric tensors – bosons in physics. Section 6.1 discusses basic properties of symmetric tensors and their relation to homogeneous polynomials. In Section 6.2, we give a known general upper bound for the maximum symmetric rank in terms of generic symmetric rank, and some known values of maximum symmetric rank. Section 6.3 shows that the rank of symmetric tensor \(| W_d \rangle \) is \( d \), while its border rank is 2. Here \(| W_d \rangle \in S^d \mathbb{C}^2 \) corresponds to the polynomial \( dx_1^{d-1}x_2 \). For instance, the three-qubit \(| W \rangle \) state reads \(| W_3 \rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \). In Section 6.4, we show explicitly that the rank of Kronecker and tensor products can be strictly submultiplicative by considering the ranks of \(| W \rangle \otimes_k | W \rangle \) and \(| W \rangle \otimes | W \rangle \), which are 7 and 8 respectively, while the square of the rank of \(| W \rangle \) is 9. In a short, Section 6.5, we discuss briefly computational methods for symmetric rank of symmetric tensors. Section 6.6 gives a short account of the results in [55], which show that the generic identifiability property of symmetric tensors holds for a rank less than the symmetric generic rank, except a number of known cases.

Section 7 is mainly devoted to the notion of the nuclear rank of a tensor. In Section 7.1, we discuss the spectral norm and geometric measure of entanglement. Section 7.2 introduces the nuclear norm and nuclear rank. In Section 7.3, we discuss the faces of the unit ball of the nuclear norm. Section 7.4 discusses the exposed faces and facets of the unit balls of matrix nuclear and spectral norms. In Section 7.5, we show that the nuclear rank of the state \(| GHZ \rangle = |000\rangle + |111\rangle / \sqrt{2} \) is 2. Section 7.6 discusses the generic and maximum nuclear rank of symmetric 3-qubits.

In Appendix, we present definitions of some notions used in quantum theory and discussed in this work.

2. Preliminary results

This section is organized as follows: In Section 2.1, we introduce the Dirac notation and the notion of a simple quantum system aimed at mathematicians. In Section 2.2, we describe a composite systems consisting of \( d \)-simple systems. Mathematically this is equivalent to the introduction of \( d \)-tensor product of the corresponding vector spaces. Next we introduce the notion of entanglement, which corresponds to a tensor of rank greater than 1. We discuss the differences between the bipartite case that corresponds to matrices and \( d \)-partite case for \( d \geq 3 \). In Section 2.3, we discuss Kronecker tensor product of tensor spaces and their quantum interpretation.

2.1. Dirac notation and simple systems

Let \( \mathcal{H} \) be a finite-dimensional Hilbert space with an inner product \( \langle x, y \rangle \). (So \( \mathcal{H} \) is a vector space over the field of complex numbers \( \mathbb{C} \). We assume that the inner product is linear in \( x \) and bar linear in \( y \), see below.) We denote by \( \mathcal{H}_n \) an \( n \)-dimensional Hilbert space. Let \( e_1, \ldots, e_n \) be an orthonormal basis in \( \mathcal{H}_n \). We identify \( \mathcal{H}_n \) with the column space \( \mathbb{C}^n = \{ x = (x_1, \ldots, x_n)^\top \} \), where \( x = \sum_{i=1}^{n} x_i e_i \), \( y^* = (\bar{y}_1, \ldots, \bar{y}_n) \) and \( \langle x, y \rangle = y^*x \). Here
\[ \langle ax + bz, y \rangle = a \langle x, y \rangle + b \langle z, y \rangle, \quad \langle x, ay + bz \rangle = \bar{a} \langle x, y \rangle + \bar{b} \langle x, z \rangle. \]

The notation of Dirac, which is routinely used in the literature on quantum physics, can be summarized as follows. The symbol $|i\rangle$ represents a basis vector in $\mathcal{H}_n$ and usually stands for $e_{i+1}$ for $i \in \{0, \ldots, n-1\}$, or for $e_i$, where $i \in [n] = \{1, \ldots, n\}$. For simplicity of our exposition, the latter convention will be used.

A vector in $\mathcal{H}_n$ is often denoted as $|\psi\rangle = \sum_{i=1}^n x_i |i\rangle$. In quantum theory, such a vector describes a physical system and it is called a pure quantum state, but for brevity the shorter versions as a ‘pure state’ or a ‘state’ are also used. Thus $|\psi\rangle$ corresponds to $x$ and $\langle \psi |$ corresponds to $x^\star$. Furthermore $\langle \phi | \psi \rangle$ is the inner product $\langle \psi, \phi \rangle$. For physical applications, one assumes that $|\psi\rangle \neq 0$ and the states are normalized, $\langle \phi | \phi \rangle = \|\phi\|^2 = 1$, but sometimes non-normalized states will also be used here. A given quantum system is described by a state in a $n$ dimensional space, $|\psi\rangle \in \mathcal{H}_n$, also denoted as $\mathbb{C}^n$. In physics parlance the spaces $\mathbb{C}^2$, $\mathbb{C}^3$ and $\mathbb{C}^n$ are often called the space of qubits, qudits and qudits, respectively. As the overall phase is physically not relevant, a quantum pure state refers to entire equivalence class, $|\psi\rangle \sim e^{ia} |\psi\rangle$, with $a \in \mathbb{R}$. Thus the space of normalized pure quantum states forms a complex projective manifold, $\mathbb{C}P^{n-1}$, which is obtained from $\mathbb{C}^n$ by identifying each complex line through the origin as a point in $\mathbb{C}P^{n-1}$. It has complex dimension $n-1$ and real dimension $2(n-1)$. In the simple case of a single qubit, $n = 2$, the space of pure states is called the Bloch sphere, $\mathbb{C}P^1 = S^2$.

Observe that $|\psi\rangle \langle \phi |$ stands for the corresponding matrix $xy^\star$ of rank 1. In particular $|\psi\rangle \langle \psi | = P_\psi$ represents a projector onto a normalized state $|\psi\rangle$, and it is easy to see that $P_\psi^2 = |\psi\rangle \langle \psi | P_\psi = P_\psi$, where we used the fact that $\langle \psi | \psi \rangle = 1$. Besides pure states, in quantum physics one works with convex combinations of projectors onto pure states, $\rho = \sum_i a_i |\psi_i\rangle \langle \psi_i |$ with nonnegative weights, $a_i \geq 0$; $\sum_i a_i = 1$, called mixed states. They are represented by positive semi-definite density matrices, $\rho^\star = \rho \geq 0$ – see Appendix 1 – but in this work they appear only sporadically.

### 2.2. Composite systems and quantum entanglement

A key axiom of quantum theory states that a quantum system composed of two subsystems $A$ and $B$, of dimension $n_1$ and $n_2$, respectively, is described in a tensor product Hilbert space, $\mathcal{H}_{n_1n_2} = \mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2}$, sometimes written $\mathcal{H}_A \otimes \mathcal{H}_B$. For instance, if the system consists of two particles $A$ and $B$ each represented as vectors $x \in \mathcal{H}_A$, $y \in \mathcal{H}_B$ then the joint system $AB$ is represented by their tensor product or a matrix $X \in \mathcal{H}_A \otimes \mathcal{H}_A$. Note that $X = xy^\top$ if and only if both subsystems are not entangled, see below. The composite system $AB$ is said to consist of two parts $A$ and $B$. In a similar way, a physical system composed of $d$ parts is represented in a tensor product Hilbert space with $d$ factors.

The shortness of Dirac notation is transparent when considering tensor product of $d$ Hilbert spaces: $\mathcal{H} = \bigotimes_{k=1}^d \mathcal{H}_{n_k} = \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_d}$ of product dimension, $\prod_{j=1}^d n_j$. This tensor product is viewed in quantum physics as the $d$-partite space. Assume that $|\psi_k\rangle \in$
$\mathcal{H}_{n_k}$ for $k \in [d]$. The product vector (state) is denoted as

$$|\psi_1\rangle|\psi_2\rangle\cdots|\psi_d\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_d\rangle = \bigotimes_{k=1}^{d} |\psi_k\rangle.$$  

Assume that $\mathbf{e}_{1,k}, \ldots, \mathbf{e}_{n_k,k}$ is an orthonormal basis of $\mathcal{H}_{n_k}$. Then $\bigotimes_{k=1}^{d} \mathbf{e}_{i_k,k}$ for $i_1 \in [n_1], \ldots, i_d \in [n_d]$ is an orthonormal basis in $\mathcal{H}$. In Dirac notation, the vector $\bigotimes_{k=1}^{d} \mathbf{e}_{i_k,k}$ is denoted as $|i_1 \cdots i_d\rangle$ or $|i_1 \rangle \otimes |i_2 \rangle \otimes \cdots \otimes |i_d\rangle$.

Any pure quantum state $|\psi\rangle$ in $\mathcal{H}$ (a vector of length one) corresponding to a physical system composed of $d$ subsystems with $n$ levels each and described in a vector space of dimension $n_1 n_2 \cdots n_d$, can be written in a product basis,

$$|\psi\rangle = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} T_{i_1,i_2,\ldots,i_d} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_d\rangle. \quad (1)$$  

Thus, for a multipartite system, $d > 2$, the state $|\psi\rangle$ corresponds to a tensor $T \in \bigotimes_{k=1}^{d} \mathbb{C}^{n_k}$, whose coordinates in the standard basis are $T_{i_1,\ldots,i_d} \in \mathbb{C}$. In the special cases of simple systems, $d = 1$, or bipartite systems, $d = 2$, the tensor $T$ reduces to a vector or to a matrix, respectively. The normalization condition, $\langle \psi | \psi \rangle = \| \psi \|^2 = 1$, implies that the Hilbert–Schmidt norm (also known as Frobenius norm for matrices)

$$\|T\|_2 = \sqrt{\sum_{i_1,\ldots,i_d=1}^{n_1 \cdots n_d} |T_{i_1,i_2,\ldots,i_d}|^2} \quad (2)$$

is fixed to unity, $\|T\|_2 = 1$.

In many parts of this paper, we will restrict ourselves to the most interesting case in quantum setting: $n_1 = \cdots = n_d = n$. This means that each subsystem lives in the Hilbert space $\mathcal{H}_n$ of the same size $n$. Then $\mathcal{H}$ is denoted either $\mathcal{H}^\otimes d$ (physical notation) or $\bigotimes_{k=1}^{d} \mathcal{H}_n$ (mathematical notation). Furthermore, assume that $|\psi_1\rangle = \cdots = |\psi_d\rangle = |\psi\rangle$ then $\bigotimes_{k=1}^{d} |\psi_k\rangle$ is denoted either $|\psi\rangle^\otimes d$ or $\bigotimes_{k=1}^{d} |\psi\rangle$.

In an experimental setting, one may change the physical partition of the entire system into a different composition of subsystems. This change corresponds to a reshaping tensor in such a way that the total number of elements is preserved. For instance, a matrix $6 \times 6$ describes a bipartite $6 \times 6$ system, while a four-index tensor $\sum_{i_1,i_2,i_3,i_4} T_{i_1,i_2,i_3,i_4} |i_1 i_2 i_3 i_4\rangle$ with $i_1, i_2 \in \{1, 2\}$ and $i_3, i_4 \in \{1, 2, 3\}$ represents a $2 \times 2 \times 3 \times 3$ system composed of two qubits and two qutrits.

A quantum state $|\psi\rangle \in \mathcal{H}$ is called separable (and hence non-entangled) if the state has a product form, $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_d\rangle$, so that the rank of the corresponding tensor $T$ is equal to 1. In all other cases, the state is called entangled, as it has no product form – see Appendix.

Note that the term entanglement has a meaning if the tensor product structure is specified. Then the physical partition of the entire system into $d$ subsystems is fixed, and the product basis $|i_1 i_2 \cdots i_d\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_d\rangle$ in which the state (1) is represented is well defined.
The characterization of the degree of entanglement is easier in the case of bipartite systems. A normalized pure state of an \( n \times n \) system can be represented in a product basis by a complex square matrix, \( |\psi_{AB}\rangle = \sum_{i,j=1}^{n} T_{ij} |i,j\rangle \), where states \( |i\rangle \) with \( i \in [n] \) form an orthonormal basis in the first subsystem \( A \), while an analogous basis \( |j\rangle \) refers to the second subsystem \( B \). Complex expansion coefficients \( T_{ij} \) form a vector of length \( n^2 \), but it is often convenient to treat them as a square matrix of size \( n \). The quantity \( |T_{ij}|^2 \) can be viewed as a probability that the system \( AB \) is measured in the state \( |ij\rangle \).

The state \( |\psi_{AB}\rangle \) is called separable if and only if the rank of the matrix \( T \) is one, so the state has the product form, \( |\psi_{AB}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \). Separability refers to the fact that the joint physical system \( AB \) can be then divided into two separate parts, \( A \) and \( B \), and the results of measurements performed separately on both of them are not correlated. Note that the space of separable pure states is equivalent to the Cartesian product of two complex projective manifolds [17], which forms a Segre embedding, \( \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \subset \mathbb{C}P^{n^2-1} \). In the simplest case of two-qubit system, \( n = 2 \), the set of separable states forms the Cartesian product of two Bloch spheres, \( S^2 \times S^2 \subset \mathbb{C}P^3 \).

A pure state which is not separable is called entangled and one may introduce several measures of quantum entanglement [16,17], which aim to quantify, to what extent a given bipartite state \( |\psi_{AB}\rangle \) is not of the product form. In general, entanglement measures are not equivalent, in the sense that their maximal values are attained for different states. The simplest quantity is given by the rank \( r \) of the corresponding matrix \( T \) of size \( n \), but it is not a smooth function of the state. Another possibility is to deal with various norms of \( T \). Assumed normalization of the state, \( \langle \psi_{AB} | \psi_{AB} \rangle = 1 \), implies the following constraint for the Frobenius (Hilbert–Schmidt) norm, \( \| T \|_2^2 = \text{Tr} \ T^* T = 1 = \sum_{i=1}^{n} \lambda_i \). Here \( \lambda_i \) denotes the eigenvalues of the positive semidefinite matrix \( T^* T \), while \( \sigma_i = \sqrt{\lambda_i} \) represents the singular values of \( T \). They arise by the singular value decomposition, \( T = UDV^* \), where \( U \) and \( V \) are unitary, while \( D \) is a diagonal matrix with non-negative entries \( \sigma_i \) at the diagonal – see Section 3.2. Then the corresponding bipartite state can be written using the Schmidt decomposition, \( |\psi\rangle = \sum_{j=1}^{n} \sigma_j |j\rangle \otimes |j'\rangle \), equivalent to Equation (1), where unitaries \( U \) and \( V \) determine the basis \( |j\rangle \) and \( |j'\rangle \), respectively.

Let \( \lambda_{\max} \) denote the largest eigenvalue of \( T^* T \), so that the spectral norm of \( T \) is given by the largest singular value, \( \| T \|_\infty = \sigma_{\max} = \sqrt{\lambda_{\max}} \). Then the state \( |\psi_{AB}\rangle \) is separable if and only if \( \lambda_{\max} = \sigma_{\max} = 1 \), so that, the smaller the norm \( \| T \|_\infty \) (under the restriction that the Frobenius norm of \( T \) is fixed), the larger entanglement. Thus as a measure of entanglement one can take a suitable smooth function of \( \lambda_{\max} \) (or \( \sigma_{\max} \)), for instance \( 1 - \lambda_{\max} \) or \( -\log(\lambda_{\max}) \) advocated in [50]. The Schmidt decomposition implies that the maximal overlap of the analysed state with any product state reads, \( \max_{|\psi_{sep}\rangle} |\langle \psi_{AB} | \psi_{sep} \rangle|^2 = \lambda_{\max} \), where \( |\psi_{sep}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \). This scalar product determines the minimal distance of the analysed state to the manifold of separable (product) states, a quantity called geometric measure of entanglement [50,56,57].

In the case of pure states, the natural geodesic distance on \( \mathbb{C}P^{n^2-1} \) is equivalent to the Fubini–Study distance, \( D_{FS}^{\min} = \arccos(|\langle \psi_{AB} | \psi_{CD} \rangle|) \) also called the quantum angle, as it corresponds to the angle between \( |\psi_{AB}\rangle \) and \( |\psi_{CD}\rangle \). One may also analyse various distances in the space of mixed states (density matrices) between the projector on the analysed state, \( \rho_{\psi_{AB}} = |\psi_{AB}\rangle \langle \psi_{AB}| \), and the projector on the closest product state. Depending on the distance selected [57,58], one obtains \( D_{HS}^{\min} = \sqrt{2(1 - \lambda_{\max})} \) for the smallest Hilbert–Schmidt distance between two states \( \rho \) and \( \omega \), namely \( D_{HS}(\rho, \omega) = [\text{Tr}(\rho - \omega)^2]^{1/2} \). The smallest
value with respect to the trace distance, \( D_1(\rho, \omega) = \text{Tr} |\rho - \omega| \) with \(|X| = \sqrt{X^*X}\) reads \( D_1^{\text{min}} = 2\sqrt{1 - \lambda_{\text{max}}} \), while the Bures distance, \( D_B(\rho, \omega) = [2 - 2 \text{Tr} |\sqrt{\rho} \sqrt{\omega}|]^{1/2} \), leads to \( D_B^{\text{min}} = [2(1 - \sigma_{\text{max}})]^{1/2} \) — see also Section 7.1.

Another, more precise entanglement measure can be obtained from the entire vector of squared singular values of the corresponding tensor, \( \lambda_i \), representing the analysed state, allows one to compute its trace norm, \( \|T\|_1 = \text{Tr} \sqrt{T^*T} = \sum_{i=1}^n \sigma_i \). Since the bipartite state \( |\psi_{AB}\rangle \) is separable if and only if \( \sigma_{\text{max}} = 1 \) so that \( \|T\|_1 = 1 \), to construct an alternative measure of bipartite entanglement one can consider the quantity \( \|T\|_1 - 1 \). In short, under the normalization assumption, \( \|T\|_2 = 1 \), the larger trace norm of \( T \), the more entangled state. In the case of a two-qubit system the maximal value of the trace norm, \( \|T\|_1^{\text{max}} = \sqrt{2} \), is achieved for the Bell entangled state \( |\phi_{AB}^+\rangle = (|00\rangle + |11\rangle) / \sqrt{2} \), for which \( \lambda_1 = \lambda_2 = 1/2 \), so that \( S(|\psi_{AB}\rangle) = \log 2 \). Recall that \( |00\rangle \) is a useful shorthand for the product state, often written in various ways, \( |0\rangle_A \otimes |0\rangle_B = |0\rangle \otimes |0\rangle = |0,0\rangle = |00\rangle \).

The vector of singular values of the matrix \( T \), representing the analysed state, allows one to compute its trace norm, \( \|T\|_1 = \text{Tr} \sqrt{T^*T} = \sum_{i=1}^n \sigma_i \). Since the bipartite state \( |\psi_{AB}\rangle \) is separable if and only if \( \sigma_{\text{max}} = 1 \) so that \( \|T\|_1 = 1 \), to construct an alternative measure of bipartite entanglement one can consider the quantity \( \|T\|_1 - 1 \). In short, under the normalization assumption, \( \|T\|_2 = 1 \), the larger trace norm of \( T \), the more entangled state. In the case of a two-qubit system the maximal value of the trace norm, \( \|T\|_1^{\text{max}} = \sqrt{2} \), is achieved for the Bell entangled state \( |\phi_{AB}^+\rangle = (|00\rangle + |11\rangle) / \sqrt{2} \).

For composite systems with \( d \geq 3 \) parts, a similar strategy does not work as the singular value decomposition of a matrix has no direct generalization for a tensor with \( d \) indices \([59,60]\). However, the rank of the tensor can still serve as one of the simplest measures of quantum entanglement, as in this setting the rank \( r(T) \) is given by the minimal natural number \( r \) such that the corresponding state (1) can be represented as a superposition of \( r \) product states,

\[
|\psi\rangle = \sum_{i=1}^r a_i |\phi_1^{(i)}\rangle \otimes |\phi_2^{(i)}\rangle \otimes \cdots \otimes |\phi_d^{(i)}\rangle,
\]

with arbitrary complex coefficients \( a_i \). Note that the states \( |\phi_j^{(i)}\rangle \) related to the subsystem number \( j \), with \( 1 \leq j \leq d \), need not be orthogonal. In physics literature one uses the term rank of a composite pure state, which is equal to the rank of the corresponding tensor \( T \). Furthermore, to quantify the entanglement of a multipartite state one uses the Schmidt measure \([18,61]\), equal to the log of the rank of the corresponding tensor, \( E_S(|\psi\rangle) = \log r(T) \). More precise description of multipartite entanglement can be obtained by studying the spectral norm \( \|T\|_\infty \) of a tensor (79), under the assumption that the Hilbert–Schmidt norm (2) is fixed to unity. An alternative approach is based on the nuclear norm \( \|T\|_1 \) of a tensor (87), which can be considered as a generalization of the matrix trace norm for tensors.

### 2.3. Kronecker tensor product

As discussed in Section 2.1, the tensor product \( \mathcal{H}_A \otimes \mathcal{H}_B \) corresponds to a bipartite space, while \( \mathcal{H} = \bigotimes_{k=1}^d \mathcal{H}_{n_k} \) corresponds to a \( d \)-partite space. Denote by \( \mathcal{H}^\vee_B = \{|\psi\rangle^\vee, |\psi\rangle \in \mathcal{H}^\vee_B \} \).
The dual space to $\mathcal{H}_B$ is denoted as $\mathcal{H}_B$. That is, $|\psi\rangle^\vee$ is viewed as a linear function on $\mathcal{H}_B$: $|\psi\rangle^\vee(|\phi\rangle) = \langle\psi|\phi\rangle$. Then bipartite product $\mathcal{H}_A \otimes \mathcal{H}_B^\vee$, is identified in a classical way with the space of linear operators $L(\mathcal{H}_A, \mathcal{H}_B) = \{ L, L : \mathcal{H}_B \rightarrow \mathcal{H}_A \}$. Namely, $|\theta\rangle \otimes |\psi\rangle^\vee(|\phi\rangle) = \langle\psi|\phi\rangle|\theta\rangle$.

It is customary to abuse the notation by identifying $\mathcal{H}_B$ with the space of $m \times n$ matrices $\mathbb{C}^{m \times n}$. Let $A = [A_{ij}] \in \mathbb{C}^{m \times n}$ and $B = [B_{kl}] \in \mathbb{C}^{p \times q}$. Then the Kronecker tensor product $C = A \otimes_K B$ is the block matrix $C = [A_{ij}B_{kl}] \in \mathbb{C}^{(mp) \times (nq)}$. (Note that $A \otimes B$ is a $4$-tensor in $\mathbb{C}^m \otimes \mathbb{C}^p \otimes \mathbb{C}^n \otimes \mathbb{C}^q$ with $(A \otimes B)_{ijkl} = A_{ij}B_{kl}$.) This is equivalent to $(\mathcal{H}_m \otimes \mathcal{H}_n) \otimes_K (\mathcal{H}_p \otimes \mathcal{H}_q) = \mathcal{H}_{mp} \otimes \mathcal{H}_{nq}$, where $\mathcal{H}_m \otimes \mathcal{H}_p$ and $\mathcal{H}_n \otimes \mathcal{H}_q$ are viewed as Hilbert spaces $\mathcal{H}_{mp}$ and $\mathcal{H}_{nq}$ respectively. More generally, given $l$ tensor product spaces $\mathcal{H}_{nj_1 \ldots nj_l}$, for $i \in [l]$, the Kronecker tensor product is defined as the following $d$-tensor product:

$$
\bigotimes_K^{i \in [l]} \mathcal{H}_{nj_1 \ldots nj_l} = \bigotimes_K^{j_1 \ldots j_l} \mathcal{H}_{nj_1} = \bigotimes_{j=1}^{d} \bigotimes_{i=1}^{l} \mathcal{H}_{nj_i}.
$$

Note that for $d = 2$, the above definition reduces to the standard fact that Kronecker product of $l$ matrices is a matrix. In [40], the Kronecker product $\otimes_K$ is denoted by $\otimes$.

The Kronecker product (4) has the following quantum interpretation [43]. Consider a group of $d$ people that share a state with $d$-subsystems each. This means that the person $j$ controls $l$ subsystems, corresponding to the subspaces $\mathcal{H}_{nj_1}, \ldots, \mathcal{H}_{nj_l}$. These subsystems are described by vectors in the product Hilbert space $\mathcal{H}_{nj_1 \ldots nj_l} = \bigotimes_{i=1}^{l} \mathcal{H}_{nj_i}$. Thus we can interpret the total system shared by $d$ of users, each controls $l$ subsystems, as a $d$-partite system on the corresponding Hilbert spaces.

It is possible to define the Kronecker tensor product of $l$ tensor product spaces $\mathcal{H}_{nj_1 \ldots nj_l}$ for $i \in [l]$, where $d_1, \ldots, d_l$ are different. Set $d = \max\{d_1, \ldots, d_l\}$. For $d_i < d$ set $\mathcal{H}_{d_i+k_i} = \mathbb{C}, n_{d_i+k_i} = 1$ for $k \in [d - d_i]$. Define $\mathcal{H}_{nj_1 \ldots nj_{d_i}} = \bigotimes_{j=1}^{d_i} \mathcal{H}_{nj_i}$. Let $\mathcal{H}'_i$ be a tensor product obtained from $\mathcal{H}_{nj_1 \ldots nj_{d_i}}$ by permuting the factors $\mathcal{H}_{d_i+1,j}, \ldots, \mathcal{H}_{d_i,l}$ with other factors. For $d_i = d$ let $\mathcal{H}'_i = \mathcal{H}_{nj_1 \ldots nj_{d_i}}$. Then

$$
\bigotimes_K^{i \in [l]} \mathcal{H}_{nj_1 \ldots nj_{d_i}} = \bigotimes_K^{i \in [l]} \mathcal{H}'_i.
$$

### 3. Tensor rank

In this section, we discuss basic notions and results on matrices and tensors, with the emphasis on the notion of tensor rank. In Section 3.1, we discuss the well-known characterizations and properties of matrix rank. In Section 3.2, we recall the properties of Singular Value Decomposition (SVD), which is known in the physics community as Schmidt decomposition. We also discuss the geometric measure of entanglement of a bipartite state, which has a simple formula in terms of the operator norm ($\sigma_{\text{max}}$) of the corresponding matrix. The maximally entangled bi-partite Bell state has the minimum $\sigma_{\text{max}}(A)$. In Section 3.3, we discuss the notion of the rank of tensor. We point out the submultiplicativity of the rank of tensors under the tensor and Kronecker tensor product. An example of strict submultiplicativity is given in
Lemma 6.1. A similar result holds for the subadditivity of tensor rank under the direct sum. In some cases the strict inequality holds, that is the Strassen’s direct sum conjecture is false. The notions of generic, maximum and border ranks of tensors are also reviewed.

3.1. Matrix rank

Let $A \in \mathbb{C}^{m \times n}$ be a nonzero matrix. Then rank $A$ also written $r(A)$ is the minimum $k \in \mathbb{N}$ such that $A = \sum_{i=1}^{k} x_i y_i^*$, where $x_i \in \mathbb{C}^m, y_i \in \mathbb{C}^n$ for $i \in [k]$. Equivalently, for a bipartite state $A \in \mathcal{H}_m \otimes \mathcal{H}_n$ the rank of $A$ is the minimum number of summands in the decomposition of $A$ as a sum of the product states. The rank of zero matrix is 0. The rank of a product state is 1, and the matrix $x y^*$, for $x \in \mathbb{C}^m \setminus \{0\}, y \in \mathbb{C}^n \setminus \{0\}$, is called rank-1 matrix.

It is quite simple to find the rank of a matrix $A$ using Gauss elimination. Hence the complexity of finding the rank of $A$ is $O(\min(m, n)^2 \max(m, n))$ in exact arithmetic. Better complexity results can be found in [62]. There are many equivalent ways to define the rank of a matrix. We bring together a few of the equivalent definitions and some related inequalities [63,64]:

**Lemma 3.1:** Let $A \in \mathbb{C}^{m \times n}$. Then each of the integers below is $r(A)$:

1. The dimension of the row space (subspace spanned by the rows of $A$).
2. The dimension of the column space, (subspace spanned by the columns of $A$).
3. The dimension of a maximal nonzero minor of $A$. (Minor of $A$: a determinant of a square submatrix of $A$.)
4. The rank of $PAQ$ for two invertible matrices $P$ and $Q$ of dimension $m$ and $n$ respectively.

Furthermore,

(a) Assume that $B$ is a submatrix of $A$ (obtained by deleting some rows and columns of $A$). Then $r(B) \leq r(A)$.

(b) Assume that $P \in \mathbb{C}^{m \times m}, Q \in \mathbb{C}^{n \times n}$. Then $r(PAQ) \leq r(A)$.

(c) Assume that $A_k \in \mathbb{C}^{m \times n}$ for $k \in \mathbb{N}$, and $\lim_{k \to \infty} A_k = A$.

Then $\lim \inf_{k \to \infty} r(A_k) \geq r(A)$.

The last statement of Lemma 3.1 is the lower semicontinuity of the matrix rank.

Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q}$. Then $A \oplus B$ is the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{C}^{(m+p) \times (n+q)}$. Hence the dimension of the column space of $A \oplus B$ is the sum of the dimensions of the column space of $A$ and $B$, which yields the equality, $r(A \oplus B) = r(A) + r(B)$.

A decomposition of $A$ as a sum of $r(A)$ rank-1 matrices is called a rank decomposition. If $r = r(A) > 1$, then this decomposition is not unique. (We ignore the order of the summands.) For example, if we choose a basis $x_1, \ldots, x_r \in \mathbb{C}^m$ in the column space of $A$, then there exists unique basis $y_1, \ldots, y_r \in \mathbb{C}^n$ of the column space of $A^* = \bar{A}^\top$ such that $A = \sum_{i=1}^{r} x_i y_i^*$. 
Let $\text{GL}_n \subset \mathbb{C}^{n \times n}$ be the group of invertible matrices. Denote by

$$\text{orb}(A) = \{ B = PAQ, P \in \text{GL}_m, Q \in \text{GL}_n \},$$

the orbit of $A$ under the action of $\text{GL}_m \times \text{GL}_n$. Since any basis $x_1, \ldots, x_m$ in $\mathbb{C}^m$ is of the form $P e_1, \ldots, P e_m$ for a unique $P \in \text{GL}_m$, it follows that $\text{orb}(A)$ is the set (quasi algebraic) variety of all matrices of rank $r(A)$. Furthermore, the closure of $\text{orb}(A)$ is the (algebraic) variety of all matrices of at most $r(A)$. In terms of quantum physics this statement is written that the SLOCC transformations of a given bipartite state of rank $r$ do not increase its rank. The above abbreviation stands for stochastic local operations and classical communication, as these experimentally realizable transformations play a key role in the theory of quantum information [17].

Observe next that if we choose at random a matrix $A$ in $\mathbb{C}^{m \times n}$ (where each entry has a standard Gaussian distribution) then $r(A)$ is $\min(m, n)$ with probability 1. That is, the value of the maximal minor of $A$ of order $\min(m, n)$ is nonzero with probability 1. In the language of algebraic geometry, the generic rank of $\mathbb{C}^{m \times n}$ is $\min(m, n)$. Note that $\min(m, n)$ is also the maximal possible rank of matrices in $\mathbb{C}^{m \times n}$. That is a generic bipartite state is maximally entangled, if the rank of the tensor is considered as a simple discrete measure of quantum entanglement. There exist also continuous measures of entanglement including the geometric measure – see the next subsection.

The rank of a matrix behaves nicely under the Kronecker tensor product [63]:

$$r \left( \bigotimes_{K} A_i \right) = \prod_{i=1}^{l} r(A_i).$$

The reason for that is very simple. Observe that column space of $B = \bigotimes_{K}^{i \in [l]} A_i$ is the tensor product of the column spaces of $A_1, \ldots, A_l$. Hence the dimension of the column space of $B$ is the product of the dimension of the column spaces of $A_1, \ldots, A_l$.

### 3.2. SVD or Schmidt decomposition

There is a standard way to make a minimal rank decomposition unique in a generic case. This is the Singular Value Decomposition (SVD) (in mathematics) or Schmidt decomposition (in physics): For a given $A \in \mathbb{C}^{m \times n}$ there exists a decomposition

$$A = \sum_{i=1}^{r(A)} \sigma_i(A) u_i v_i^*,$$

where $u_1, \ldots, u_{r(A)}$ and $v_1, \ldots, v_{r(A)}$ are orthonormal bases of the column spaces of $A$ and $A^*$ respectively. Furthermore $\sigma_1(A) \geq \cdots \geq \sigma_{r(A)}(A) > 0$ are the positive singular values of $A$. Note that

$$AA^* u_i = \sigma_i(A)^2 u_i, \quad A^* A v_i = \sigma_i(A)^2 v_i, \quad i \in [r(A)].$$

That is, the square of the positive singular values of $A$ are the positive eigenvalues of $AA^*$ and $A^* A$. In particular, the decomposition (7) is unique if and only if $\sigma_1(A) > \cdots > \sigma_{r(A)}(A)$.
\( \sigma_r(A)(A) \) [63]. By uniqueness we mean here that each rank-1 matrix \( u_i v_i^* \) is unique, but \( u_i \) and \( v_i \) need not be unique.

We now recall various approximation properties of the SVD decomposition of \( A \). Recall that \( \mathbb{C}^{m \times n} \) is a Hilbert space with the inner product \( \langle A, B \rangle = \text{Tr} B^* A \), where the trace of a square matrix \( C = [C_{ij}] \in \mathbb{C}^{m \times m} \) is given as \( \text{Tr} C = \sum_{i=1}^m C_{ii} \). Then the Frobenius norm (also called Hilbert–Schmidt norm) of \( A \) is given by \( \|A\|_F = \sqrt{\text{Tr} A^* A} = \sqrt{\sum_{i=1}^r \sigma_i(A)^2} \).

The operator norm of \( A \) is given by
\[
\|A\| = \sigma_1(A) = \max \{ \|A x\|, \|x\| = 1 \}
= \max \{ \|y^* A x\|, \|x\| = 1 \} = \max \{ \|y^* A x\|, \|x\| = 1 \}.
\]

The real part of a complex number \( z \) is denoted by \( \Re z \).

Denote by \( \Pi(m, n) \subset \mathcal{H}_m \otimes \mathcal{H}_n \) all normalized product states with the norm set to 1. Assume that \( |\psi\rangle \in \mathcal{H}_m \otimes \mathcal{H}_n \) is a normalized state. The geometric measure of entanglement can be described [50,57] by the Hilbert–Schmidt distance of \( |\psi\rangle \) to \( \Pi(m, n) \):
\[
\min \{ \| |\psi\rangle - |\xi\rangle \otimes |\eta\rangle \|, \| |\xi\rangle \| = \| |\eta\rangle \| = 1 \}
= \min \{ \sqrt{2 - 2 \Re \langle \xi |\psi\rangle \langle \eta |\psi\rangle}, \| |\xi\rangle \| = \| |\eta\rangle \| = 1 \} = \sqrt{2(1 - \sigma_1(|\psi\rangle))}.
\]

Hence the maximally entangled states with respect to the geometric measure of entanglement are the Bell states \( |\psi\rangle \), which are characterized by \( \sigma_i(|\psi\rangle) = \frac{1}{\sqrt{\min(m,n)}} \) for \( i \in [\min(m,n)] \).

Furthermore, for each \( k \in [r(A)] \) let \( B_k \in \mathbb{C}^{m \times n} \) be any element such that \( r(B_k) = k \). Then the distance of \( A \) from the orbit orb(\( B_k \)), or its closure, is \( \sigma_{k+1}(A) \), and is achieved at \( A_k := \sum_{i=1}^k \sigma_i(A) u_i v_i^* \). Recall that \( \sigma_j(A) = 0 \) for \( j > r(A) \). See for example [63].

### 3.3. Definition of a rank of a tensor

Let \( d > 2 \) be a positive integer. Assume that \( n_1, \ldots, n_d \) are positive integers. Let \( \mathbf{n} = (n_1, \ldots, n_d) \) and denote \( \mathcal{H}_\mathbf{n} = \bigotimes_{i=1}^d \mathcal{H}_{n_i} \) and \( \mathbb{C}^\mathbf{n} = \bigotimes_{i=1}^d \mathbb{C}^{n_i} \). The dimension of these vector spaces is \( N(\mathbf{n}) = \prod_{i=1}^d n_i \). A non-normalized \( d \)-product state \( \bigotimes_{i=1}^d |\psi_i\rangle \in \mathcal{H}_\mathbf{n} \) corresponds to a rank-one tensor \( \otimes_{i=1}^d x_i \in \mathbb{C}^\mathbf{n} \setminus \{0\} \). Note that \( \otimes_{i=1}^d x_i \in \mathbb{C}^\mathbf{n} \) is the zero tensor if and only if at least one of \( x_i \) is a zero vector. Assume that \( T \in \mathbb{C}^\mathbf{n} \) is the pure state \( |\psi\rangle \in \mathcal{H}_\mathbf{n} \). Then \( T \) has a representation (1). The rank of a nonzero tensor \( T \in \mathbb{C}^\mathbf{n} \), denoted as \( r(T) \), is the minimal number of summands in the representation of \( T \) as a sum of rank-1 tensors. Equivalently, the rank of the state \( |\psi\rangle \in \mathcal{H}_\mathbf{n} \) is the minimal dimension of a subspace spanned by normalized product states that contains \( |\psi\rangle \). The equality (1) yields that \( r(T) \leq N(\mathbf{n}) \). Actually, a stronger inequality is known – see Section 5.1:
\[
r(T) \leq \frac{N(\mathbf{n})}{\max(n_1, \ldots, n_d)}, \quad T \in \mathbb{C}^\mathbf{n}.
\] (9)

While the definition of the rank of the tensor is in principle the same as for matrices, the calculation of the rank of a given tensor can be NP-hard even for 3-tensors [21].

Let \( \mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d \). Assume that \( \mathcal{U} = [\mathcal{U}_{1,\ldots,\mathbf{p}}] \subset \mathbb{C}^\mathbf{p} \). Recall that \( T \otimes \mathcal{U} \in \mathbb{C}^\mathbf{q} \), where \( \mathbf{q} = (\mathbf{n}, \mathbf{p}) \). On the other hand the tensor \( T \otimes_K \mathcal{U} \in \mathbb{C}^{\mathbf{n}\mathbf{p}} \), where \( \mathbf{n} \cdot \mathbf{p} =
(n_1p_1, \ldots, n_dp_d). From the rank minimal decomposition of \( T \) and \( U \) we deduce the obvious inequalities [43]
\[
    r(T \otimes_K U) \leq r(T \otimes U) \leq r(T)r(U). \tag{10}
\]
Recall that \( V = T \oplus U \) is a tensor in \( \mathbb{C}^{n+p} \), such that \((T \oplus U)_{i_1, \ldots, i_d} = T_{i_1, \ldots, i_d} \) for \( i_k \in [n_k], k \in [d], (T \oplus U)_{n_1+j_1, \ldots, n_d+j_d} = U_{j_1, \ldots, j_d} \) for \( j_k \in [p_k], k \in [d], \) and all other entries are zero. It is straightforward to show that \( r(T \oplus U) \leq r(T) + r(U) \). Recall that for \( d = 2 \) we have equality in the above inequality. In [41], Strassen asked if \( r(T \oplus U) = r(T) + r(U) \) for 3-tensors. For general \( d \) this problem is sometimes called Strassen’s direct sum conjecture. For \( d = 3 \), this is true if \( \min(n_1, n_2, n_3, p_1, p_2, p_3) \leq 2 \), see Section 4.3. Some additional cases where Strassen’s direct sum conjecture holds are discussed in [65, 66].

For border rank this conjecture was known to be wrong [67], see below. Recently Shitov showed that even for \( d = 3 \) the direct sum conjecture of Strassen is in general false [68]. Let \( \bigoplus^k T \) be the direct sum of \( k \) copies of \( T \). By definition \( r(\bigoplus^k T) \leq kr(T) \) and the restricted Strassen’s conjecture [43] is asking, whether equality holds, \( r(\bigoplus^k T) = kr(T) \)?

It was shown in [43, 67] that this equality can be stated in the following form. Let \( \mathcal{I}(k,d) \in \mathbb{C}^{d} \) be the identity tensor: \( \mathcal{I}(k,d) = \sum_{i=1}^{k} |i\rangle \otimes d \). One can show that \( \mathcal{I}(k,d) \otimes_K T \) is \( \bigoplus^k T \), if we use the lexicographical order on the standard bases \( \mathbb{C}^{d} \otimes_K \mathbb{C}^n \) and \( r(\mathcal{I}(k,d)) = k \). (It follows from the observation that if we view \( \mathcal{I}(k,d) \) as a matrix in \( \mathbb{C}^{k \times k^d} \), then this matrix has rank \( k \)). Hence the restricted Strassen conjecture (which is still open) is equivalent to
\[
    r(\mathcal{I}(k,d) \otimes_K T) = r(\mathcal{I}(k,d))r(T) = kr(T). \tag{11}
\]
Assume that the above equality holds for some \( T \). Observe that (10) and (11) imply that equalities in (10) hold. This implies [43] that \( r(\mathcal{I}(k,d) \otimes T) = kr(T) \).

Consider the following simple example. Let \( T \) be \( 2 \times 2 \times 2 \) with entries \( T_{i,j,k} \) where \( i,j,k \in [2] \). Then the tensor \( U = T \oplus T \) is \( 4 \times 4 \times 4 \) tensor. To give the exact formula for the entries of \( U \) we relabel \( \{1,2,3,4\} \) as the pairs as \( \{(1,1), (1,2), (2,1), (2,2)\} \). Then the entries of \( U \) are \( U_{(i_1,j_1), (j_1,k_2), (k_1,k_2)} \). These entries are zero unless \( i_1 = j_1 = k_1 = 1 \) \( i \in [2] \). The formula for possible nonzero entries of \( U \) is \( U_{(i_1,j_2), (j_1,k_2), (k_1,k_2)} = T_{i_2,j_2,k_2} \). The nonzero entries of \( 2 \times 2 \times 2 \) tensor \( \mathcal{I}(2, 3) \) are the entries \( (1,1,1) \) and \( (2,2,2) \) which are equal to 1. It is straightforward to see that \( \mathcal{I}(2, 3) \otimes T = U \). Corollary 4.9 claims that in this example the restricted Strassen conjecture holds.

The generic rank of a tensor in \( \mathbb{C}^n \), denoted as \( r_{\text{gen}}(n) \), is the rank of a tensor \( T \in \mathbb{C}^n \) whose entries are chosen at random, assuming that the entries of tensors in \( \mathbb{C}^n \) are \( N(n) \) independent Gaussian random variables. We will justify later (Section 4.4) the existence of generic rank and discuss briefly how to compute efficiently this rank using Terracini’s lemma [34]. For example the generic rank of an \( m \times n \) matrix is \( \min(m, n) = r_{\text{gen}}(m, n) \). It is well known that \( r_{\text{gen}}(2, 2, 2) = 2 \), and the state \( |\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{\sqrt{2}} \mathcal{I}(2, 3) \) serves as an example of a state with such a rank [35] – see the discussion after Lemma 4.5. The maximum rank of a tensor in \( \mathbb{C}^n \), denoted as \( r_{\text{max}}(n) \), is the maximum possible rank of tensors in \( T \in \mathbb{C}^n \). By definition, for tensors \( r_{\text{gen}}(n) \leq r_{\text{max}}(n) \), while for matrices equality holds. Furthermore, the maximal rank for a three-qubit state
reads \( r_{\text{max}}(2, 2, 2) = 3 \) and the state \(|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \) saturates the bound -- see discussion following Lemma 4.5. Nice results in [69] state that

\[
\begin{equation}
\label{eq:12}
\sigma_{\text{gen}}(n) \leq 2r_{\text{gen}}(n) - 1.
\end{equation}
\]

See example 16 (2) on page 7 in [69]. For \( n = (2, 2, 2) \) the generic rank is 2 and \( \sigma_{\text{gen}} - 1 \) is the variety of rank-1 tensors. This variety has projective dimension 3 in the space of projective dimension 7. Note that for \( n = (2, 2, 2) \) this inequality boils down to \( 3 < 4 \). We will outline a short proof of the weaker inequality \( r_{\text{max}}(n) \leq 2r_{\text{gen}}(n) \) [70] later.

We now discuss the \emph{border} rank of \( T \in \mathbb{C}^n \), denoted as \( r_b(T) \), which was discovered in [71]. It is the smallest \( k \in \mathbb{N} \) with the following properties: There exists a sequence \( T_j, j \in \mathbb{N} \) such that \( r(T_j) = k \) for all \( j \in \mathbb{N} \), and \( \lim_{j \to \infty} T_j = T \). By definition, inequality \( r_b(T) \leq r(T) \) holds, which is always saturated for matrices. Rank-1 tensor satisfies the equality \( r_b(T) = r(T) \). That is, the set of all tensors of rank 1 and norm one is closed. We will show later that \( r_b(T) \leq r_{\text{gen}}(n) \) for any \( T \in \mathbb{C}^n \). It is known that \( r_b|W\rangle = 2 \), see §4.1. Actually, this result follows from the above remarks. The border rank is subadditive: \( r_b(T \oplus U) \leq r_b(T) + r_b(U) \), and the inequality may be strict [62,67]. Thus the conjecture of Strassen for border rank is false. More about algebraical methods and criteria of determining the border rank of tensors with border rank not greater than two, can be found in [72].

Denote by \( \text{GL}(n) \) the product group \( \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_d} \). Then \( \text{GL}(n) \) acts on \( \mathbb{C}^n \) as follows. Let \( (A_1, \ldots, A_d) \in \text{GL}(n) \). Assume that \( T = \sum_{i=1}^{r'} \bigotimes_{j=1}^d x_{j,i} \). Then \( (A_1, \ldots, A_d)(T) = \sum_{i=1}^{r'} \bigotimes_{j=1}^d (A_j x_{j,i}) \). The orbit of \( T \) under this action reads

\[
\begin{equation}
\label{eq:13}
\text{orb}(T, \text{GL}(n)) = \left\{ \sum_{i=1}^{r'} \bigotimes_{j=1}^d (A_j x_{j,i}), (A_1, \ldots, A_d) \in \text{GL}(n) \right\}.
\end{equation}
\]

This orbit corresponds to all states equivalent to \( T \) under the SLOCC operations [17]. Note that each tensor in \( \text{orb}(T, \text{GL}(n)) \) has rank \( r(T) \). The closure of the orbit of \( T \) is denoted by \( \text{Closure}(\text{orb}(T, \text{GL}(n))) \), it contains the set of the states that can be obtained from \( T \) by SLOCC. It can happen that this closed set may contain tensors of rank greater than \( r(T) \). For instance, the closure of the set of 2 \( \times 2 \times 2 \) tensors obtained from the \( |\text{GHZ}\rangle \) state by changing bases in each copy of \( \mathbb{C}^2 \) forms the entire set of all \( 2 \times 2 \times 2 \) tensors, see Section 4.1.

To illustrate the challenges of finding rank of \( d \)-mode tensors and other ranks of tensors we present a small survey on the ranks of 3-tensors.

### 4. Ranks of 3-tensors

In this section, we survey several known results on the rank of 3-tensors of dimension \( (m, n, p) \) related to quantum entanglement. In Section 4.1, we bring the celebrated Kruskal’s theorem that gives a necessary condition that a decomposition of a tensor into a sum of rank-1 tensors is unique up to a permutation of the summands. Section 4.2 covers the results of JàJá [54,73], which completely characterize the rank of \( m \times n \times 2 \) tensors. Similar results were obtained by Grigoriev [74,75]. In Section 4.3, we recall the results of
JáJá–Takche [76] and Buczyński–Postinghel–Rupniewski [65], which give sufficient conditions under which the direct sum conjecture of Strassen holds. Section 4.4 discusses the values of generic rank for 3-tensors. Theorem 4.10 combined with (25) implies that the generic rank of such a three-tensor is equal to \( p \) if \((m - 1)(n - 1) + 1 \leq p \leq mn\). For \( 2 \leq m \leq n \leq p \leq (m - 1)(n - 1) \) we state a well-known conjecture concerning the value of generic rank. This conjecture holds in some cases. In Section 4.5, we describe a known algorithm to find the generic rank. Section 4.6 surveys briefly some known results on maximal ranks of 3-tensors.

4.1. Basic results on rank of 3-tensors

Assume that \( d = 3 \) and \( n = (m, n, p) \). Since 3-tensors represent three-partite system, the order of the parties: Alice, Bob and Charlie is arbitrary. In some cases, we are going to assume

\[
2 \leq m \leq n \leq p. \tag{14}
\]

(The reason for the assumption that \( m \geq 2 \) is that for \( m = 1 \) a 3-tensor is a matrix.) Given a 3-tensor \( T = [T_{i_1, i_2, i_3}] \in \mathbb{C}^n \) we can associate with it four types of ranks. The first rank is \( r(T) \), while the other three ranks \( r_A(T), r_B(T) \) and \( r_C(T) \) are corresponding matrix ranks. Let us first consider \( r_C(T) \). View the two parties \( \{A, B\} \) (Alice and Bob) as one party, which corresponds to the Hilbert space \( \mathcal{H}_{mn} \). Then \( T \) is viewed as a bipartite state \( T_C \in \mathcal{H}_{mn} \otimes \mathcal{H}_p \). It has \( p \) columns \( T_k = [T_{i_1, i_2, i_3}] \in \mathbb{C}^{m \times n} \) for \( k \in [p] \). Each column is a matrix, and \( T_k \) is called a frontal slice. The collection of the \( p \) columns \( \{T_1, \ldots, T_p\} \) can be viewed as an album of \( p \) photos, where the matrix \( T_k \) is \( k \)th photo. Then \( r_C(T) \) is the dimension of the subspace in \( \mathbb{C}^{m \times n} \) spanned by \( T_1, \ldots, T_p \). We next observe that \( r_C(T) \leq r(T) \). Indeed, a singular value decomposition of \( T_C \) is

\[
T_C = \sum_{k=1}^{r_C(T)} \sigma_k(T_C) U_k \otimes z_k, \quad \text{Tr } U_j^* U_k = z_j^* z_k = \delta_{jk}, \quad j, k \in [r_C(T)]. \tag{15}
\]

Note that here \( U_j \) does not have to be a rank-1 matrix. Observe next that a rank decomposition \( T = \sum_{i=1}^{r(T)} x_i \otimes y_i \otimes z_i \) induces a decomposition \( T_C = \sum_{i=1}^{r(T)} (x_i \otimes y_i) \otimes z_i \) to rank-1 vectors in \( \mathcal{H}_{mn} \otimes \mathcal{H}_p \).

The ranks \( r_A(T) \) and \( r_B(T) \) are defined similarly. Hence

\[
\max(r_A(T), r_B(T), r_C(T)) \leq r(T). \tag{16}
\]

Assume that

\[
T = \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i. \tag{17}
\]

Under what conditions \( r = r(T) \) ? A simple sufficient condition is: the set of the matrices \( x_1 \otimes y_1, \ldots, x_r \otimes y_r \) and the set of vectors \( z_1, \ldots, z_r \) are linearly independent. Indeed, this condition insures that \( r_C(T) = r \leq r(T) \).

Kruskal’s conditions [53] give sufficient conditions for \( r = r(T) \), and that the above rank decomposition of \( T \) is unique: Any rank decomposition of \( T \) is a sum of rank-one
tensors $x_1 \otimes y_1 \otimes z_1, \ldots, x_r \otimes y_r \otimes z_r$ in any order. To state Kruskal’s condition we need to define Kruskal’s rank of $l$ nonzero vectors $x_1, \ldots, x_l \in \mathbb{C}^m$, denoted as $r_K(x_1, \ldots, x_l)$. Namely, $r = r_K(x_1, \ldots, x_l)$ if and only if any $r$ vectors in $\{x_1, \ldots, x_l\}$ are linearly independent, and there are $r+1$ vectors in $\{x_1, \ldots, x_l\}$ that are linearly dependent. For example, if $x_1, \ldots, x_l \in \mathbb{C}^m$ are chosen at random, then $r_K(x_1, \ldots, x_l) = \min(l, m)$. We call a set $\{x_1, \ldots, x_l\} \subset \mathbb{C}^m$ generic, or in general position, if $r_K(x_1, \ldots, x_l) = \min(l, m)$. We call a decomposition (17) generic if the three sets of vectors $\{x_1, \ldots, x_l\}, \{y_1, \ldots, y_r\}, \{z_1, \ldots, z_r\}$ are generic. Theorem of Kruskal [53] yields:

**Theorem 4.1:** Let $T \in \mathbb{C}^{m \times n \times p}$ have a decomposition (17), where each $x_i, y_j, z_i$ is nonzero. If

$$r_K(x_1, \ldots, x_r) + r_K(y_1, \ldots, y_r) + r_K(z_1, \ldots, z_r) \geq 2r + 2$$

then $r = r(T)$ and the decomposition (17) is unique.

Note that for $m = n > 1$, $p = 2$ and $r = m$ this result is sharp. Indeed, assume that $r_K(x_1, \ldots, x_m) = r_K(y_1, \ldots, y_m) = m$. Note that $r_K(z_1, \ldots, z_m) = l$ where $l \in [2]$. If $l = 2$, that is any pair $z_i, z_j$ is linearly independent then Kruskal’s theorem claims that $r(T) = m$ and the decomposition (17) is unique. Assume now that $z_1, \ldots, z_l$ are nonzero colinear vectors. Then $T$ is a matrix of the form $T = \sum_{i=1}^{m} a_i x_i \otimes y_i$ where each $a_i \neq 0$. Thus $r(T) = r(T) = m$ but the rank decomposition of $T$ is not unique, since the rank decomposition of a matrix of rank greater than one is not unique. Hence the rank decomposition of $T$ is not unique. See [77] for more examples showing that Kruskal’s theorem is sharp. See [78] for a simple proof of Kruskal’s theorem. We will discuss Kruskal’s theorem for $d$-mode tensors, where $d > 3$, later.

The following corollary follows from Kruskal’s theorem:

**Corollary 4.2:** Let $T \in \mathbb{C}^{m \times n \times p}$. Assume that a decomposition of $T$ is generic. If

$$\min(r, m) + \min(r, n) + \min(r, p) \geq 2r + 2,$$

then $r = r(T)$ and the rank decomposition of $T$ is unique.

We call a 3-tensor $T$ that satisfies the conditions of the above corollary a rank-$r$ tensor with generic decomposition.

The following theorem [35] gives a characterization of the rank of 3-tensor:

**Theorem 4.3:** Let $n = (m, n, p)$. Assume that $T \in \mathbb{C}^n$, and let $T_1, \ldots, T_p \in \mathbb{C}^{m \times n}$ be the $p$ frontal slices of $T$. Then $r(T)$ is the minimum dimension of a subspace of $\mathbb{C}^{m \times n}$ spanned by rank-one matrices that contains the subspace spanned by $T_1, \ldots, T_p$.

A composite space $\mathbb{C}^{m \times n}$ is spanned by $mn$ linearly independent tensors of rank one. Hence $r(T) \leq mn$. This yields the inequality (9) for $d = 3$, as we can assume (14).

Denote by $r_{\text{max}}(m, n, p)$ the maximum possible rank of tensors in $\mathbb{C}^{m \times n \times p}$. The inequality (9) yields that $r_{\text{max}}(m, n, p) \leq \frac{mn}{\max(m, n, p)}$.

**Proposition 4.4:** Let $m, n, p \in \mathbb{N}$. For each $k \in \{1, \ldots, r_{\text{max}}(m, n, p)\}$ there exists a tensor $T \in \mathbb{C}^{m \times n \times p}$ such that $r(T) = k$. 
Proof: Assume that \( A \in \mathbb{C}^{m \times n \times p} \), and \( r(A) = r = r_{\max}(m, n, p) > 1 \). Write \( A = x_1 \otimes y_1 \otimes z_1 + \cdots + x_r \otimes y_r \otimes z_r \) and \( B_k = x_1 \otimes y_1 \otimes z_1 + \cdots + x_k \otimes y_k \otimes z_k \) for \( k \in [r-1] \). Clearly, \( r(B_k) \leq k \). Use the rank decomposition of \( B_k \), and the fact that \( A = B_k + \sum_{j=k+1} x_j \otimes y_j \otimes z_j \) to deduce that \( r = r(A) \leq r(B_k) + r - k \) which yields that \( r(B_k) \geq k \). Hence \( r(B_k) = k \).

Let us assume that \( p = 2 \). So \( T \) has two frontal slices \( T_1, T_2 \in \mathbb{C}^{m \times n} \). Let us first examine all possible nonzero ranks of \( 2 \times 2 \times 2 \) tensors in terms of the matrices in the subspace spanned by the frontal sections:

Lemma 4.5: Let \( n = (2, 2, 2) \) and assume that \( T \in \mathbb{C}^n \setminus \{0\} \). Suppose that \( T_1, T_2 \in \mathbb{C}^{2 \times 2} \) are the two frontal slices of \( T \). Then

1. \( r(T) = 1 \) if and only if \( T_1 \) and \( T_2 \) are linearly dependent, and one of the slices is rank-1 matrix.
2. \( r(T) = 2 \) if and only if one of the following conditions hold:
   a. The matrices \( T_1 \) and \( T_2 \) are linearly dependent, and one of the slices is rank-2 matrix.
   b. The matrices \( T_1 \) and \( T_2 \) are linearly independent, and each matrix in \( \text{span}(T_1, T_2) \) is singular.
   c. The subspace \( \text{span}(T_1, T_2) \) contains two linearly independent matrices \( X, Y \) such that \( X \) is invertible and \( X^{-1}Y \) is diagonalizable.
3. \( r(T) = 3 \) if and only if \( T_1, T_2 \) are linearly independent, and the \( \text{span}(T_1, T_2) \) contains two matrices \( X, Y \) such that \( X \) is invertible and \( X^{-1}Y \) is not diagonalizable.

Consider a rank-2 tensor \( T = x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_2 \otimes z_2 \) with the generic decomposition. Corollary 4.2 yields that the rank decomposition of \( T \) is unique. The orbit of the tensor \( T \) with respect to the general linear transformations, written \( \text{orb}(T, \text{GL}) \), consists of all rank-two tensors with generic decomposition. In particular, the GHZ (non-normalized) state \( |GHZ\rangle = |111\rangle + |222\rangle \) is a rank-2 state with generic decomposition. Furthermore, the closure of \( \text{orb}(|GHZ\rangle, \text{GL}) \) is \( \mathbb{C}^n \). Let \( W \) be the (non-normalized) state

\[ |W\rangle = |112\rangle + |121\rangle + |211\rangle = M_1 \otimes e_1 + M_2 \otimes e_2 \]

with

\[ M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]

As \( M_1 \) is invertible and \( M_1^{-1}M_2 \) is not diagonalizable, it follows that \( r(|W\rangle) = 3 \). This is essentially [54, Corollary 3.2.1] after a change of variables in the first factor \( \mathbb{C}^2 \). Note that \( r_K(\{1\}, \{1\}, \{2\}) = 1 \). Hence the above decomposition of \( |W\rangle \) fails to satisfy the conditions of Kruskal’s theorem: The sum of Kruskal ranks reads \( 1 + 1 + 1 = 3 \) and \( 2 \times 3 + 2 = 8 \). It is easy to show that the above rank decomposition of \( |W\rangle \) is not unique. It is also known that \( r(T) = 3 \) if and only if \( T \in \text{orb}(|W\rangle, \text{GL}) \). (It is enough to show that if \( r(T) = 3 \) then \( (A_1, A_2, A_3)(T) = |W\rangle \) for some \( (A_1, A_2, A_3) \in \text{GL}(n) \).)

We give a short proof of this claim. Assume that \( T = T_1 \otimes e_1 + T_2 \otimes e_2 \) for some \( T_1, T_2 \in \mathbb{C}^{2 \times 2} \). First note that the action of \( A_3 \) on \( T \) is equivalent to choose a different basis \( T_1', T_2' \) in \( \text{span}(T_1, T_2) \). Choose \( X = aT_1 + bT_2 \) to be invertible, \( Y = cT_1 + dT_2 \) be
such that $X^{-1}Y$ is not diagonalizable. Then $X^{-1}Y$ has a double eigenvalue $\lambda$. Set $Z = Y - \lambda X$. Then $X^{-1}Z$ is a rank-1 nondiagonalizable matrix. Hence $
abla_1 = (I_2, I_2, A_3)(T) = X \otimes e_1 + Z \otimes e_2$ for some $A_3 \in \text{GL}_2$. Observe next that for $B_1, B_2 \in \text{GL}_2$ we obtain $
abla_2 = (B_1, B_2, I_2)(T) = B_1 X B_2^T \otimes e_1 + B_1 Z B_2^T \otimes e_2$. Choose $B_1, B_2$ such that $B_1 X B_2^T = I_2$. Then $\nabla_2 = I_2 \otimes e_1 + C \otimes e_2$. Observe that $C$ is similar to $X^{-1}Z$. As $X^{-1}Z$ is a rank-1 non-diagonalizable matrix it follows that $C$ is similar to the Jordan block $M_1^{-1}M_2$. That is $C = Q^{-1}(M_1^{-1}M_2)Q$. Let $\nabla_3 = (Q^{-1}, Q^T, I_2)(\nabla_2) = I_2 \otimes e_1 + (M_1^{-1}M_2) \otimes e_2$. Finally, $(M_1, I_2, I_2)(\nabla_3) = |W|$. The equality

$$|W| = \lim_{t \to 0} \frac{1}{t} \left( (|1| + t|2|)^{\otimes 3} - |1|^{\otimes 3} \right)$$

(20)

shows that the border rank of $|W|$ is 2.

4.2. The rank of $m \times n \times 2$ tensors

In this section, we describe the complete solution to the problem of computing the rank of $\nabla = [T_{i,j,k}] \in \mathbb{C}^{m \times n \times 2}$. Denote by $\nabla_1 = [T_{i,j,1}]$, $\nabla_2 = [T_{i,j,2}] \in \mathbb{C}^{m \times n}$ the frontal slices of $\nabla$. It was shown by Jäälä [54] how to apply the Kronecker theory of the canonical form of a pencil of matrices [79] to determine the rank of a $m \times n \times 2$ tensor. We present some results of [54] using the notions and the results discussed above.

Theorem 4.3 states that $r(\nabla)$ is the minimal dimension of a subspace spanned by rank-1 matrices in $\mathbb{C}^{m \times n}$, which contains the subspace $V = \text{span}(\nabla_1, \nabla_2)$. We can assume that $\nabla_1$ and $\nabla_2$ are linearly independent, otherwise the rank of $\nabla$ is $\text{max}(r(\nabla_1), r(\nabla_2))$. Then one can change a basis in $\text{span}(\nabla_1, \nabla_2)$ to $\nabla_1', \nabla_2'$. This is equivalent to considering $\nabla' = (I_m, I_m, A_3)(\nabla)$ corresponding to some $A_3 \in \text{GL}(2)$. Next we consider $\nabla_1 = (P, Q^T, I_2)(\nabla')$, where $P \in \text{GL}(m)$, $Q \in \text{GL}(n)$. This corresponds to replacing the pair $(\nabla_1', \nabla_2')$ by $(PT_1'Q, PT_2'Q)$. It is a classical problem to find the canonical form of a pair of matrices $(A, B) \in \mathbb{C}^{m \times n} \times \mathbb{C}^{m \times n}$ under the simultaneous equivalence: $(A, B) \leftrightarrow P(A, B)Q$, where $P \in \text{GL}(m)$, $Q \in \text{GL}(n)$. This problem was solved completely by Kronecker [79]. See the classical exposition in [80], or a short exposition in [63, Problems, §2.1]. It is common to consider the matrix $A + tB$ with a complex parameter $t$, which is usually called a pencil [63].

Let us first consider the case where $m = n$ and $\text{span}(\nabla_1, \nabla_2)$ contains an invertible matrix. In this case $(\nabla_1, \nabla_2)$ is called a regular pair. Equivalently, the pencil $\nabla_2 + t\nabla_1$ is called a regular pencil. So we can assume that $\nabla_1' \in \text{span}(\nabla_1, \nabla_2)$ is invertible. Now choose $P = \nabla_1'^{-1}$, $Q = I_m$ to obtain that the pair $(\nabla_1', \nabla_2')$ is equivalent to the pair $(I_m, A)$. Note that $r_A(\nabla) = m$. Hence $r(\nabla) \geq m$.

All other pairs of the form $(I_m, B)$ are equivalent to $(I_m, A)$ if $B = QAQ^{-1}$ for some $Q \in \text{GL}(m)$. We can choose $B$ to be the Jordan canonical form of $A$, or to be the rational canonical form of $A$ [63].

We first discuss the case where $A$ is a diagonalizable matrix. That is, we can choose $B$ to be the diagonal matrix diag($\lambda_1, \ldots, \lambda_n$). Then $\text{span}(I_m, B)$ is contained in the span of $m$ linearly independent rank-1 diagonal matrices. Hence $r(\nabla) \leq m$. Whence $r(\nabla) = m$. Vice versa, assume that $r(\nabla) = m$. Then $\text{span}(x_1y_1^T, \ldots, x_my_m^T)$ that contains $\nabla_1, \nabla_2$ must contain an invertible matrix. So $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are linearly independent. Hence
there exist unique $P, Q^\top \in \mathbf{GL}(m)$ such that $P\mathbf{x}_i = Q^\top \mathbf{y}_i = \mathbf{e}_i$ for $i \in [m]$. Thus $PT_1Q$ and $PT_2Q$ are diagonal matrices. In particular, $B$ is a diagonal matrix, hence $A$ is diagonalizable.

Assume now that $A$ is not diagonalizable. Hence $r(T) > m$. We now discuss the case where $r(T) = m + 1$.

Recall the notion of the companion matrix [63] which corresponds to monic polynomial $p(t) = t^m - p_1t^{m-1} - \cdots - p_m$:

$$C(p) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
p_m & p_{m-1} & p_{m-2} & \cdots & p_2 & p_1
\end{bmatrix}.$$  \hspace{1cm} (21)

Then $\det(tI_m - C(p)) = p(t)$. Assume that $p(t) = \prod_{i=1}^k (t - \lambda_i)^{m_i}$, where $k \in [m]$, each $m_i$ is a positive integer, $\sum_{i=1}^k m_i = m$, and $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the Jordan canonical form of $C(p)$ has exactly one Jordan block of order $m_i$ corresponding to the eigenvalue $\lambda_i$ for $i \in [k]$. Assume that $k < m$. Hence $C(p)$ is not diagonalizable.

Suppose that $B = C(p)$. Let $\mathbf{x} = \mathbf{e}_n, \mathbf{y} = (-1 + p_m, p_{m-1}, \ldots, p_1)^\top$. Then $C(p) - \mathbf{y}(\mathbf{y}^\top = C(q)$ where $q(t) = t^m - 1$. Hence $C(q)$ is diagonalized, and there exists $m$ rank-one matrices $\mathbf{x}_1\mathbf{y}_1^\top, \ldots, \mathbf{x}_m\mathbf{y}_m^\top$ whose span contains $I_m, C(q)$. Therefore the span of $\mathbf{x}_1\mathbf{y}_1^\top, \ldots, \mathbf{x}_m\mathbf{y}_m^\top$ contains $I, C(p)$. Hence $r(T) \leq m + 1$ and therefore $r(T) = m + 1$.

Recall next that a matrix $A \in \mathbb{C}^{m \times m}$ is similar to the unique matrix $B = \bigoplus_{i=1}^l C(p_i)$ where $p_{i+1}(t)$ divides $p_i(t)$ for $i \in [l - 1]$. $B$ is called the rational canonical form of $A$ [63]. $A$ is similar to $C(p)$ if and only if $l = 1$. The polynomials $p_1(t), \ldots, p_l(t)$ are called the invariant polynomials of $tI_m - A$ (or simply of $A$.) Thus $(I_m, B) = \bigoplus_{i=1}^l (I_n, C(p_i))$, where $n_i$ is the degree of $p_i$ for $i \in [l]$.

The result of JäJá [54] can be summarized in the following lemma:

**Lemma 4.6:** Let $T \in \mathbb{C}^{m \times m \times 2}$. Let $T_1, T_2$ be two frontal slices of $T$. Suppose that $\text{span}(T_1, T_2)$ has dimension 2 and contains an invertible matrix $X$. Let $X, Y$ be a basis in $\text{span}(T_1, T_2)$, and assume that $X^{-1}Y$ has the rational canonical form $\bigoplus_{i=1}^l C(p_i)$. If $p_1$ has simple roots then $r(T) = m$. Suppose that $p_1, \ldots, p_k$ have multiple roots, and $p_{k+1}$ has simple roots if $k < l$. Then $r(T) = m + k$.

Indeed, observe first that if $p_1$ has simple roots, then all other $p_i$ have also simple roots, as each $p_i$ divides $p_1$. Hence each $C(p_i)$ is diagonalizable, and whence $X^{-1}Y$ is diagonalizable. Therefore $r(T) = m$. Suppose now that $p_1$ does have multiple roots. Then $C(p_1)$ is not diagonalizable. Hence there exists $(A_1, A_2, A_3) \in \mathbf{GL}(m, m, 2)$ such that $(A_1, A_2, A_3)(T) = \bigoplus_{i=1}^l T_i$, where $T_i \in \mathbb{C}^{m \times m \times 2}$, where the two frontal slices of $T_i$ are $I_m, C(p_i)$. If $p_1$ have simple roots then $r(T_i) = m_i$. Otherwise $r(T_i) = m_i + 1$. Therefore $r(T) \leq m + k$. It is shown in [54] that $r(T) = m + k$.

The rank of tensors $T \in \mathbb{C}^{m \times n \times 2}$ which do not satisfy conditions of Lemma 4.6 is determined by the following theorem [54]:

**Theorem 4.7:** Assume that $T \in \mathbb{C}^{m \times n \times 2}$ where the two frontal slices $T_1, T_2$ are linearly independent. Suppose furthermore that either $m \neq n$ or $m = n$ and the $\text{span}(T_1, T_2)$ does
not contain an invertible matrix. Then there exists \((A_1, A_2, A_3) \in \text{GL}(m, n, 2)\) such that \((A_1, A_2, A_3)(T) = \bigoplus_{i=1}^p T_i, T_i \in \mathbb{C}^{m_i \times n_i \times 2}\) for \(i \in [p]\). Either \(p > 1\), \(m_p = n_p\) and the subspace spanned by the two frontal slices of \(T_p\) contains an invertible matrix, or \(|m_p - n_p| = 1\). If \(p > 1\) then for all other \(i \in [p - 1]\) one has the equality \(|m_i - n_i| = 1\).

1. Suppose that \(n_1 = m_1 + 1\). Then the two frontal slices of \(T_j\) are \([I_{m_j}0], [0I_{m_j}]\) \(\in \mathbb{C}^{m_j \times n_j}\).

In this case \(r(T_j) = n_j\).

2. Suppose that \(m_1 = n_1 + 1\). Then the two frontal slices of \(T_j\) are \([I_{n_j}0]^{\top}, [0J_{n_j}]^{\top}\) \(\in \mathbb{C}^{m_j \times n_j}\).

In this case \(r(T_j) = m_j\).

Finally, \(r(T) = \sum_{i=1}^p r(T_i)\).

To see that the \(r(T_j)\) in the case (1) is \(n_j\) we do as follows: We extend \(T_j \in \mathbb{C}^{m_j \times n_j \times 2}\) to \(\tilde{T}_j \in \mathbb{C}^{n_j \times n_j \times 2}\) by adding a row \(n_j\) to the two frontal sections. To the first section, we add the row \(e_{n_j}^{\top}\) to obtain \(I_{n_j}e_{n_j}\) and to the second section we add the zero row to obtain the Jordan block \(I_{n_j}\) corresponding to the eigenvalue 0. \(J_{n_j}\) is the companion matrix of \(p(t) = t^{n_j}\). Hence \(r(\tilde{T}_j) = n_j + 1\). We showed above that there are \(n_j + 1\) rank-one matrices whose linear combinations span \(I_{n_j}\) and \(J_{n_j}\). As in the case \(p = 1, m_p = n_p\) and rank \(T = n_1 + 1\) discussed in the beginning of this section, we can assume that one of these rank-one matrices is of the form \(e_n e_i^{\top}\). Hence, if we delete the last row of the other \(n_j\) rank-one matrices, they will span the two frontal slices of \(T_j\). Thus \(r(T_j) \leq n_j\). It is straightforward to show that \(r(\tilde{T}_j) > m_j\). The case (2) can be shown similarly. The main result of this theorem is its last part.

We now bring one application of this theorem [81]:

\[
\max_r(m, n, 2) = \begin{cases} 
    m + \left\lfloor\frac{n}{2}\right\rfloor & \text{for } 2 \leq m \leq n \leq 2m, \\
    2m & \text{for } 2 \leq m, 2m < n. 
\end{cases}
\]  

(22)

First observe that the second case and the first case with \(n = 2m\) is a simple consequence of Theorem 4.3 when applied to horizontal sections \(H_1, \ldots, H_n \in \mathbb{C}^{m \times 2}\) of \(T\). In that case \(r(T) \leq 2m\) because the whole space \(\mathbb{C}^{m \times 2}\) is spanned by \(2m\) rank-one matrices. Assume now that \(A_1, \ldots, A_{2m}\) are linearly independent. Then these matrices span \(\mathbb{C}^{m \times 2}\) and \(r(T) \geq 2m\).

We now discuss the first case of (22). Let us consider first the case \(m = n\). For \(m = 2\) we know that the maximal rank is \(3 = 2 + \left\lfloor2/2\right\rfloor\). Furthermore, equality holds if and only if \(T \in \text{orb}(W)\).

Let us consider the case \(m = 3\). Then we have two choices in Theorem 4.7. First, \(T\) is a direct sum of two singular pencils of dimensions \(1 \times 2\) and \(2 \times 1\). In this case, the rank of \(T\) is 4. The other choice is that the two frontal sections form a regular pair. Then Lemma 4.6 yields that the maximal rank is 4.

Next consider the case \(m = 4\). If the two sections form a nonsingular pencil, then \(r(T) \leq 6\). Equality is achieved if the Jordan canonical form \(X^{-1}Y\) in Lemma 4.6 forms two nilpotent Jordan blocks of order 2. Other choices have smaller rank.

We now deduce the general formula for the case \(m = n\) as follows. For \(m\) even, we have \(\max_r(m, m, 2) = 3m/2\) which is achieved for a nonsingular two frontal slices, which are equivalent to \((I, C)\), where \(C\) is a sum of \(m/2\) nilpotent Jordan blocks. If \(m \geq 3\) is odd, we
have two possible ways to achieve the maximum rank \((3m - 1)/2\). First, a nonsingular pair \((I, C)\) where \(C\) is a sum of \((m - 1)/2\) nilpotent Jordan blocks and one Jordan block. Second, a direct sum of \(3 \times 3\) singular pair of rank 4, and a regular pair \((I, C)\) of order \(m-3\) with the maximal rank \(3(m-3)/2\).

For the case \(m < n \leq 2m\) the maximum possible rank is obtained as follows. First, we consider the sum of \(n-m\) copies of singular pairs of \(1 \times 2\). This part contributes \(2(n-m)\) to the rank of \(T\). If \(n = 2m\) we are done. Otherwise we are left with a regular pencil of order \(m - (n-m) = 2m - n\) with the maximal rank \(2m - n + \lfloor(2m - n)/2\rfloor\). Hence the maximal rank is

\[
2(n-m) + 2m - n + \lfloor(2m - 2n)/2 + n/2\rfloor = m + \lfloor n/2\rfloor. \tag{23}
\]

### 4.3. Validity of Strassen’s direct sum conjecture for certain 3-tensors

The results of JáJá and Takche [76] yield:

**Theorem 4.8:** Let \(T \in \mathbb{C}^n, U \in \mathbb{C}^p\), where \(\mathbf{n} = (n_1, n_2, n_3), \mathbf{p} = (p_1, p_2, p_3)\). Then \(r(T \oplus U) = r(T) + r(U)\) if one of the following conditions holds:

1. \(2 \in \{n_1, n_2, n_3, p_1, p_2, p_3\}\).
2. \(2 \in \{n_i - n_k, p_ip_j - p_k\}\) for some \(i, j, k\) satisfying \(\{i, j, k\} = \{1, 2, 3\}\).

Use induction on \(k\) to deduce the following result:

**Corollary 4.9:** Let \(T \in \mathbb{C}^n\), where \(\mathbf{n} = (n_1, n_2, n_3)\). Then

\[
r(I(k, 3) \otimes_K T) = r(I(k, 3) \otimes T) = kr(T) \tag{24}
\]

if one of the following conditions holds:

1. \(2 \in \{n_1, n_2, n_2\}\).
2. \(2 \in \{n_i - n_k\}\) for some \(i, j, k\) satisfying \(\{i, j, k\} = \{1, 2, 3\}\).

A recent paper [65] gives additional conditions where Strassen’s additivity conjecture holds. Namely, Theorem 4.8 holds if one of the following conditions is satisfied:

(a) \((p_1, p_2, p_3) = (p_1, 3, 3)\).
(b) \(r(U) \leq 6\).
(c) \(\max(r_A(U), r_B(U), r_C(U)) + 2 \geq r(U)\).

### 4.4. Generic rank of 3-tensors

We first observe that \(r_{\text{gen}}(m, n, p)\) is a symmetric function in the positive integer variables \(m, n, p\). Let us fix \(m, n \in \mathbb{N}\) and assume that \(2 \leq m \leq n\). We first observe the simple equality

\[
r_{\text{gen}}(m, n, p) = r_{\text{max}}(m, n, p) = mn \text{ for } p \geq mn. \tag{25}
\]
Indeed, let $T_1, \ldots, T_p \in \mathbb{C}^{m \times n}$ be the $p$ frontal sections of $\mathcal{T} \in \mathbb{C}^{m \times n \times p}$. Hence $T_1, \ldots, T_p$ are chosen at random, where each entry of each $T_k, k \in [p]$ is independent Gaussian random variable. As $p \geq mn$, every set of $mn$ matrices out of $T_1, \ldots, T_p \in \mathbb{C}^{m \times n}$ is linearly independent. Hence the subspace spanned by $T_1, \ldots, T_p$ is $\mathbb{C}^{m \times n}$. Theorem 4.3 yields that $r(\mathcal{T}) = mn$. Apply Theorem 4.3 to deduce that for $p \geq mn$ one has $r_{\max}(m, n, p) = mn$.

It is left to discuss the case where $p < mn$. We now bring the following well known result, see [35] and references therein:

**Theorem 4.10:** Assume that $2 \leq m \leq n$ and $(m - 1)(n - 1) + 1 \leq p \leq mn - 1$. Then $r_{\text{gen}}(m, n, p) = p$.

We now outline briefly the proof of this theorem which will need some basic notions and results in algebraic geometry that we will be using in this paper. A good reference on a basic algebraic geometry is [82]. A set $V \subset \mathbb{C}^N$ is called a *variety* if it is zero of a finite number of polynomials in $N$ complex variables. The algebra of polynomials in $N$ complex variables is denoted by $\mathbb{C}[x], x = (x_1, \ldots, x_N)^T \in \mathbb{C}^N$. $V$ is called *irreducible* if it is not a union of two varieties, each strictly contained in $V$. Assume that $V$ is an irreducible variety. There is a strict subvariety of $V$, called Sing $V$, which consists of singular points of $V$, such that $M = V \setminus \text{Sing} V$ is a connected complex manifold. The complex dimension of $M$ is called the dimension of $V$, and denoted by $\dim V$. In general, a point $z \in \mathbb{C}^N$ is called *generic*, or in *general position* if $z \in \mathbb{C}^N \setminus V$. Usually, $V$ will depend on the property that one studies.

A variety $V \neq \{0\}$ is called a *projective* if for each $t \in \mathbb{C} \setminus \{0\}$ we have that $tV = V$. Note that a projective irreducible variety $V$ satisfies $\dim V \geq 1$. A simplest irreducible projective variety of dimension $d$ will be a subspace $L \subset \mathbb{C}^N$ of dimension $d$. A basic result in algebraic geometry says that given a projective irreducible variety $V$, $d = \dim V \in [N - 1]$ then for each subspace $L$ of dimension $N - d + 1$ the intersection $V \cap L$ contains at least one line, i.e. a subspace of dimension 1. Furthermore, there exist a subvariety $W(V)$ on the ‘space’ of all vector spaces in $\mathbb{C}^N$ of dimension $N - d + 1$, such that $V \cap L$ has a constant number of lines for each $L \not\in W(V)$, which is denoted as $\deg V$. Moreover, for $L \not\in W(V)$, each set of $\min(N, \deg V)$ lines in $V \cap L$ is linearly independent. Note that if $V$ is also a subspace, then $\deg V = 1$.

We now consider the variety of rank-one matrices plus zero matrix in $\mathbb{C}^{m \times n}$. We view $\mathbb{C}^{m \times n}$ as $\mathbb{C}^{mn}$. This variety is called the Segre variety $\text{Seg}(\mathbb{C}^{m \times n})$. This variety has dimension $m + n - 1$ and has one singular point $A = 0$. Let us take a vector space $L$ of dimension $mn - (m + n - 1) + 1 = (m - 1)(n - 1) + 1$. The above results yield that each such subspace contains a rank-one matrix. One can compute the degree of $\text{Seg}(\mathbb{C}^{m \times n})$ and it is not less than $(m - 1)(n - 1) + 1$ [35]. Hence a generic $(m - 1)(n - 1) + 1$ dimensional subspace of $\mathbb{C}^{m \times n}$ is spanned by $(m - 1)(n - 1) + 1$ rank-one matrices. Consider now a generic tensor $\mathcal{T} \in \mathbb{C}^{m \times n \times (m - 1)(n - 1) + 1}$. Let $T_{1}, \ldots, T_{(m - 1)(n - 1) + 1}$ be its frontal sections. Hence $\text{span}(T_{1}, \ldots, T_{(m - 1)(n - 1) + 1})$ is a generic subspace of dimension $(m - 1)(n - 1) + 1$, which has a basis consisting of rank-one matrices. Theorem 4.3 yields that $r(\mathcal{T}) = (m - 1)(n - 1) + 1$. Similar arguments yield Theorem 4.10 for $p$ that satisfies the inequalities $(m - 1)(n - 1) + 1 < p < mn$.

Letting $m = 2$ in Theorem 4.10 we deduce that

$$r_{\text{gen}}(2, n, p) = p \quad \text{for } n \leq p \leq 2n. \quad (26)$$
Thus it is left to determine the generic rank in the critical range

\[ 3 \leq m \leq n \leq p \leq (m - 1)(n - 1). \quad (27) \]

For any subset \( X \) of \( \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_d} \) we denote the closure of \( X \) in the standard Euclidean topology on \( \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_d} \) by \( \text{Closure}(X) \). Let us now recall the lemma of Terracini’s [34] – see [35].

**Lemma 4.11:** Let \( m, n, p \) be positive integer greater than 1. Fix \( r \in \mathbb{N} \) and consider the polynomial map \( F_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r \to \mathbb{C}^{m \times n \times p} \) given as follows:

\[
F_r(x_1, y_1, z_1, \ldots, x_r, y_r, z_r) = \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i.
\]

Then

1. The set \( F_r((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r) \) is the set of all tensors in \( \mathbb{C}^{m \times n \times p} \) of rank at most \( r \).
2. The set \( \text{Closure}(F_r((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r)) \) is the set of all tensors in \( \mathbb{C}^{m \times n \times p} \) of border rank at most \( r \).
3. The set \( \text{Closure}(F_r((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r)) \) is an irreducible variety \( V_r \) in \( \mathbb{C}^{m \times n \times p} \).
4. There exists a subvariety \( W_r \subset V_r \) such that \( F_r((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r) \supseteq V_r \setminus W_r \).
5. The dimension of \( V_r \) is the maximal rank of the Jacobian of \( F_r \).
6. There exists a subvariety \( U_r \subset (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r \) such that the rank of the Jacobian of \( F_r \) for each point not in \( U_r \) is \( \dim V_r \).
7. The generic rank \( r_{gen}(m, n, p) \) is the minimal \( r \) such that \( \dim V_r = mnp \).
8. \( F_r((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r) \subseteq F_{r+1}((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^{r+1}) \). Equality holds if and only if \( r \geq r_{max}(m, n, p) \). In particular \( F_r((\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p)^r) = \mathbb{C}^{m \times n \times p} \) if and only if \( r \geq r_{max}(m, n, p) \).

We now give a lower bound for \( r_{gen}(m, n, p) \). Denote by

\[
\text{Seg}(\mathbb{C}^{m \times n \times p}) = \{ x \otimes y \otimes z; x \in \mathbb{C}^m, y \in \mathbb{C}^n, z \in \mathbb{C}^p \},
\]

the variety of all tensors of rank at most 1. So \( \text{Seg}(\mathbb{C}^{m \times n \times p}) \), the Segre variety, is a projective variety of dimension \( m + n + p - 2 \), with one singular point \( \mathbf{0} \). Observe that the polynomial map \( F_r \) can be viewed as an \( r \)-secant map \( F_r : (\text{Seg}(\mathbb{C}^{m \times n \times p}))^r \to \mathbb{C}^{m \times n \times p} \). Note that the dimension of the variety \( (\text{Seg}(\mathbb{C}^{m \times n \times p}))^r \) is \( r(m + n + p - 2) \). Hence Lemma 4.11 yields that \( r_{gen}(m, n, p)(m + n + p - 2) \geq mnp \). Introducing a new quantity \( r_0(m, n, p) \) we obtain a lower bound:

\[
r_0(m, n, p) := \left\lceil \frac{mnp}{m + n + p - 2} \right\rceil \leq r_{gen}(m, n, p). \quad (28)
\]

Note that for the case \( p = (m - 1)(n - 1) + 1 \) Theorem 4.10 yields equality. For \( p > (m - 1)(n - 1) + 1 \) one can have strict inequality in (28). For example for \( m = n = 3 \) and \( p = 5 \), 6 we have equality in the above inequality, while for \( p = 7 \) we have a strict inequality.

We now state the conjecture on the value of \( r_{gen}(m, n, p) \) in the critical range [35]:
Conjecture 4.12: Assume that $m$, $n$, $p$ are integers satisfying (27). Then equality in (28) holds unless $(m, n, p) = (3, 2k + 1, 2k + 1)$ for $k \in \mathbb{N}$. In this exceptional case, it is known [83] that $r_{\text{gen}}(3, 2k + 1, 2k + 1) = r_0(3, 2k + 1, 2k + 1) + 1$.

It was shown in [83] that for $(m, n, p) = (3, 2k + 1, 2k + 1)$ the tensors of border rank at most $r_0(3, 2k + 1, 2k + 1)$ forms a hypersurface in $(3, 2k+1,2k+1)$. The conjecture holds for $(3, n, n)$ and $n \geq 3$ [83], for $(4, n, n)$ and $n \geq 3$ [36], and $(n, n, n)$ for $n \geq 4$ [36, 84].

We conclude this section with a short outline of a weaker version of the inequality (12) [70] for 3-tensors. Set $r = r_{\text{gen}}(m, n, p)$. It is enough to consider the case where $r < r_{\text{max}}(m, n, p)$. Observe that a finite union of subvarieties of $\mathbb{C}^N$ is a subvariety of some hypersurface $H(p) = \{ x \in \mathbb{C}^N, p(x) = 0 \}$. Identify $\mathbb{C}^{m \times n \times p}$ with $\mathbb{C}^{mnp}$. Theorem 4.11 implies that $V_r = \mathbb{C}^{m \times n \times p}$ and $V_{r-1}$ is a strict subvariety of $\mathbb{C}^{m \times n \times p}$. There exists a polynomial $p \in \mathbb{C}^{mnp}[x]$ such that $H(p) \supseteq V_{r-1} \cup W_r$. Hence all tensors in $\mathbb{C}^{m \times n \times p} \setminus H(p)$ have rank $r$. Let $T \in \mathbb{C}^{m \times n \times p}$ such that $r(T) = r_{\text{max}}(m, n, p)$. So $T \in H(p)$. Recall that there exists a line through $T$ that intersects $H(f)$ at finite number of points. Choose two points $T_1, T_2$ on this line which do not lie in $H(p)$. Hence $r(T_1) = r(T_2) = r$ and $T$ is a linear combination of $T_1$ and $T_2$. Hence $r(T) \leq 2r$.

The inequality (12) can be improved to $r(T) \leq 2r - 2$ if the closure of tensors of rank $r-1$ is a hypersurface [69].

4.5. A numerical way to compute $r_{\text{gen}}(m, n, p)$

View $f(x, y, z) = x \otimes y \otimes z$, as $f : \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^p \rightarrow \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p$. Then the Jacobian of $f(x, y, z)$ is given by a rectangular block matrix [35]:

$$Df(x, y, z) = [A_x(y, z), A_y(x, z), A_z(x, y)],$$

where

$$A_x(y, z) = [e_{1,1} \otimes y \otimes z | \cdots | e_{m,1} \otimes y \otimes z]$$

$$A_y(x, z) = [x \otimes e_{1,2} \otimes z | \cdots | x \otimes e_{n,2} \otimes z]$$

$$A_z(x, y) = [x \otimes y \otimes e_{1,3} | \cdots | x \otimes y \otimes e_{p,3}]$$

(30)

We assume that we are in the critical range (27). For a positive integer $r$ we define $F_r$ as in Lemma 4.11. Then the Jacobian $DF_r$ is given by

$$DF_r(x_1, y_1, z_1, \ldots, x_r, y_r, z_r) = \begin{bmatrix}
Df(x_1, y_1, z_1) \\
\vdots \\
Df(x_r, y_r, z_r)
\end{bmatrix}.$$

(31)

We fix a positive integer $N$. We start our procedure with $r = r_0(m, n, p)$ (28) and $j = 1$. Next we select $r$ triplets $x_i \in \mathbb{C}^m, y_i \in \mathbb{C}^n, z_i \in \mathbb{C}^p$ at random for $i \in [r]$. It is enough to assume that components of each vector are drawn from the standard Gaussian distribution. We compute the rank of $DF_r(x_1, y_1, z_1, \ldots, x_r, y_r, z_r)$, denoted as $R$. If $R = mnp$, then $r_{\text{gen}}(m, n, p) = r$, and we stop our procedure. If $R < mnp$ and $j < N$ we set $j = j + 1$ and repeat the above procedure. If $j = N$ and $R < mnp$ we conclude that $r < r_{\text{gen}}(m, n, p)$. We set $r = r + 1$ and repeat until the procedure stops.
One may assume that the generic rank $r_{\text{gen}}(m, n, p)$ is equal to $r_0(m, n, p)$, which is often the case. For $3 \leq n \leq p \leq 20$ in the critical range (27), numerical results show that $r_0(m, n, p)$ is the generic rank, except the cases of $(3, 2k + 1, 2k + 1)$ with $k \in [9]$. In these exceptional cases $r_{\text{gen}}(3, 2k + 1, 2k + 1) = r_0(3, 2k + 1, 2k + 1) + 1$. That is, Conjecture 4.12 holds for $3 \leq n \leq p \leq 20$ in the critical range (27). Such anomalies for generic rank are analogous to those reported earlier for $(3, 3)$ and $(3, 5, 5)$.

It was shown in [83] that for all positive integers $k$

$$r_{\text{gen}}(3, 2(k + 1), 2(k + 1)) = r_0(3, 2(k + 1), 2(k + 1))$$  \hspace{1cm} (32)

and

$$r_{\text{gen}}(3, 2k + 1, 2k + 1) = r_0(3, 2k + 1, 2k + 1) + 1.$$  \hspace{1cm} (33)

### 4.6. Known results on maximal ranks of 3-tensors

Besides the exact values (22), we know the following results. The table in [81] gives all the values of the maximal rank $r_{\text{max}}(3, 3, p)$ for $n = (3, 3, p)$ for $p \in [9] \setminus \{5\}$. It is known that $r_{\text{max}}(3, 3, 5) \in \{6, 7\}$. We give their table:

$$
\begin{array}{cccccccc}
p & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
r_{\text{max}}(3, 3, p) & 3 & 4 & 5 & 6 & \{6, 7\} & 7 & 8 & 8 & 9 \\
\end{array}
\hspace{1cm} (34)
$$

Recall the table of the generic rank of $3 \times 3 \times p$ tensor.

$$
\begin{array}{cccccccc}
p & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
r_{\text{gen}}(3, 3, p) & 3 & 3 & 5 & 5 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\hspace{1cm} (35)
$$

We now explain briefly this formula. Recall (25) and Theorem 4.10. First, $r_{\text{gen}}(3, 3, 1)$ is the maximal possible rank of $3 \times 3$ matrix which is 3. Now $r_{\text{gen}}(3, 3, 2) = r(2, 3, 3)$. As $3 = (2 - 1)(3 - 1) + 1$ it follows from Theorem 4.10 that $r_{\text{gen}}(3, 3, 2) = 3$. The equality $r_{\text{gen}}(3, 3, 3) = 5$ is well known and is stated in the big table of generic rank in Section 5.4. Again for $p \geq (3 - 1)(3 - 1) + 1 = 5$ we get that $r_{\text{gen}}(3, 3, p) = p$ for $5 \leq p \leq 9$. As $r_{\text{gen}}(3, 3, 3) \leq r_{\text{gen}}(3, 3, 4) \leq r_{\text{gen}}(3, 3, 5)$ it follows that $r_{\text{gen}}(3, 3, 4) = 5$.

The papers [81,85] give the following upper bounds on the rank of 3-tensors

$$r_{\text{max}}(m, n, n) \leq \left\lfloor \frac{(m + 1)n}{2} \right\rfloor, \hspace{1cm} 3 \leq m, n,$$  \hspace{1cm} (36)

$$r_{\text{max}}(m, n, p) \leq m + \left\lfloor \frac{p}{2} \right\rfloor n, \hspace{1cm} 3 \leq m \leq n, 3 \leq p$$  \hspace{1cm} (37)

$$r_{\text{max}}(m, n, mn - u) = mn - \left\lfloor \frac{u}{2} \right\rfloor,$$

$$3 \leq m \leq n, u \leq \min(4, m, n).$$ \hspace{1cm} (38)

These results and the known results that $r(m, m, m) = \left\lfloor \frac{m^3}{3m - 2} \right\rfloor$ for $m > 3$ [84] yield:

$$r_{\text{max}}(4, 4, 4) \leq 10, \hspace{1cm} r_{\text{gen}}(4, 4, 4) = 7,$$  \hspace{1cm} (39)

$$r_{\text{max}}(5, 5, 5) \leq 15, \hspace{1cm} r_{\text{gen}}(5, 5, 5) = 10,$$  \hspace{1cm} (40)
\[ r_{\text{max}}(6, 6, 6) \leq 21, \quad r_{\text{gen}}(6, 6, 6) = 14, \quad (41) \]
\[ r_{\text{max}}(7, 7, 7) \leq 28, \quad r_{\text{gen}}(7, 7, 7) = 19. \quad (42) \]

5. Ranks of \(d\)-tensors for \(d \geq 4\)

This section discusses ranks of \(d\)-tensors for \(d \geq 4\), where \(n = (n_1, \ldots, n_d), 2 \leq n_1 \leq \cdots \leq n_d\). In Section 5.1, we give a well-known characterization of the tensor rank in terms of the dimension of the minimal subspace spanned by rank-1 \((d-1)\)-mode tensors that contains all frontal sections of the given tensor. Next we bring a generalization of the Kruskal uniqueness theorem by Sidiropoulos–Bro [86]. Theorem 5.6 states the known result, under which conditions the generic rank \(r_{\text{gen}}(n)\) of a tensor is equal to \(n_d\). Section 5.2 states the lemma of Terracini. In Section 5.3, we give an upper bound on the generic rank of a tensor using purely combinatorial methods. This upper bound is sharp for the case of \(d\) subsystems with \(n\) levels each: \(n_1 = \cdots = n_d = n\), for which perfect codes exist [87]. Section 5.4 concentrates on the generic rank of \(d\)-qunits. In particular, we provide a table of generic ranks of tensors with \(d\) indices running from 1 to \(n\), for which the values are known. Section 5.5 discusses an algorithmic way to find the rank of a tensor using solvability of a system of linear equations with several variables. In Section 5.6, we discuss the problem of generic identifiability of tensors. Namely, assuming an integer \(r\) is less than a generic rank, when a generic tensor of rank \(r\) has a unique decomposition as a sum of \(r\) rank-1 tensors? The results of Chiantini–Ottaviani–Vannieuwenhoven [25] show that if \(\prod_{i=1}^{d}(n_i - 1) \leq 15,000\) then the generic identifiability property holds except in a number of known cases.

5.1. General case

We now bring the analog of Theorem 4.3 for a general \(d \geq 3\).

Theorem 5.1: Let \(n = (n_1, \ldots, n_d), \) where \(d \geq 3\) and \(n_i \geq 2\) for \(i \in [d]\). Assume that \(T = [T_{i_1, \ldots, i_d}] \in \mathbb{C}^n\), and let \(T_1 = [T_{i_1, \ldots, i_d, 1}], \ldots, T_{n_d} = [T_{i_1, \ldots, i_{d-1}, n_d}] \in \mathbb{C}^{(n_1, \ldots, n_{d-1})}\) be the \(n_d\) frontal slices of \(T\). Then \(r(T)\) is the minimum dimension of a subspace of \(\mathbb{C}^{(n_1, \ldots, n_{d-1})}\) spanned by rank-1 tensors that contains the subspace spanned by \(T_1, \ldots, T_{n_d}\).

In particular, we deduce that \(r(T) \leq N(n)/n_d\). Apply the above theorem to a mode \(k \in [d]\) to deduce that \(r(T) \leq N(n)/n_k\). This proves (9).

The following generalization of Kruskal’s theorem for \(d\)-tensors is due to Sidiropoulos–Bro [86]:

Lemma 5.2: Assume that \(3 \leq d\), \(2 \leq n_1 \leq \cdots \leq n_d\) are integers. Assume that

\[ T = \sum_{i=1}^{r} \bigotimes_{j=1}^{d} x_{i,j}, \quad x_{i,j} \in \mathbb{C}^{n_j} \setminus \{0\}, i \in [r], j \in [d]. \quad (43) \]

If

\[ \sum_{j=1}^{d} r_K(x_{1,j}, \ldots, x_{r,j}) \geq 2r + (d - 1) \quad (44) \]
then \( r = r(T) \) and the decomposition (43) is unique.

To establish the uniqueness of a rank decomposition using the above lemma, it seems that in many cases it is beneficial to view a \( d \)-tensor as a 3-tensor in \( \mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2} \otimes \mathcal{H}_{n_3}, \ldots, n_d \) [88]:

**Lemma 5.3:** Assume that \( 3 \leq d, 2 \leq n_1 \leq \cdots \leq n_d \) are integers. Decompose the multiset \( \{n_1, \ldots, n_d\} \) to a union of three nonempty disjoint multisets \( S_1 \cup S_2 \cup S_3 \), which induce three vectors \( n_1 \in \mathbb{N}[S_1], n_2 \in \mathbb{N}[S_2], n_3 \in \mathbb{N}[S_3] \). Then \( \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \) is obtained from \( \otimes_{j=1}^d \mathbb{C}^{n_j} \) by permuting factors \( \mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_d} \). Thus each \( T \in \mathbb{C}^n \) induces \( \hat{T} \in \otimes_{k=1}^3 \mathbb{C}^{n_k} \). Assume that \( T \) has a decomposition (43). Let \( \hat{T} = \sum_{i=1}^r \otimes_{k=1}^3 T_{i,k} \), where each \( T_{i,k} \in \mathbb{C}^{n_k} \) is a rank-one tensor induced decomposition of \( T \). View each \( T_{i,k} \) as a vector in \( \mathbb{C}^{N(n_k)} \). If

\[
    r_{k}(T_{1,1}, \ldots, T_{r,1}) + r_{k}(T_{1,2}, \ldots, T_{r,2}) + r_{k}(T_{1,3}, \ldots, T_{r,3}) \geq 2r + 2
\]

then \( r(T) = r(\hat{T}) = r \) and the above decomposition of \( \hat{T} \) and the corresponding decomposition of \( T \) is unique up to a permutation of rank-one tensors in the decomposition.

**Proposition 5.4:** Let the assumptions of Lemma 5.3 hold. Moreover, assume that \( N(n_1) \leq N(n_2) \leq N(n_3) \). Suppose that \( T \) has a decomposition (43), where all \( x_{i,j} \) are in general position. (The entries of each \( x_{i,j} \) are chosen from independent \( N(0,1) \) Gaussian distribution.) Then \( r = r(T) \) and the decomposition (43) of \( T \) is unique up to a permutation of summands for the following values of \( r \):

1. If \( r \leq N(n_1) + N(n_2) - 2 \) and \( N(n_3) \geq N(n_1) + N(n_2) - 2 \).
2. If \( r \leq N(n_1) + N(n_2) - 3, N(n_1) \geq 3 \) and \( N(n_3) = N(n_1) + N(n_2) - 3 \).
3. If \( r \leq 1/2(N(n_1) + N(n_2) + N(n_3) - 2), N(n_1) \geq 4 \) and \( N(n_1) + N(n_2) - 4 \geq N(n_3) \).

**Proof:** The results in [88] yield that \( r_{k}(T_{1,k}, \ldots, T_{r,k}) = \min(r, N(n_k)) \) for \( k \in [3] \). Suppose first that \( 2 \leq r \leq N(n_1) \). Then the left hand side of (45) is \( 3r \). As \( r \geq 2 \) the inequality (45) holds. Hence \( r(T) = r \).

Assume now that \( N(n_1) < r \leq N(n_2) \). Then \( r_{k}(T_{1,1}, \ldots, T_{r,1}) = N(n_1) \) and \( r_{k}(T_{1,k}, \ldots, T_{r,k}) = r \) for \( k \in \{2, 3\} \). Then the inequality (45) holds. Hence \( r(T) = r \). Assume that \( N(n_2) \leq r \leq N(n_3) \). Then \( r_{k}(T_{1,k}, \ldots, T_{r,k}) = N(n_k) \) for \( k \in [2] \) and \( r_{k}(T_{1,3}, \ldots, T_{r,3}) = r \). Then the inequality (45) is equivalent to \( r \leq N(n_1) + N(n_2) - 2 \). Therefore (1) holds.

Suppose that \( N(n_3) = N(n_1) + N(n_2) - 3 \). As \( N(n_3) \geq N(n_2) \) we deduce that \( N(n_1) \geq 3 \). Suppose that \( N(n_2) \leq r \leq N(n_3) = N(n_1) + N(n_2) - 3 \). Then the above arguments show that (45) holds. For \( r = N(n_1) + N(n_2) - 2 \) we obtain that \( r_{k}(T_{1,k}, \ldots, T_{r,k}) = N(n_k) \) for \( k \in [3] \). For this value of \( r \) the inequality (45) does not hold. Thus (2) is the best one can obtain for the case \( N(n_3) = N(n_1) + N(n_2) - 3 \).

Assume that \( N(n_1) + N(n_2) - 4 \geq N(n_3) \). As \( N(n_3) \geq N(n_2) \) we deduce that \( N(n_1) \geq 4 \). Suppose that \( r \leq N(n_3) \). Then the above arguments show that (45) holds. Assume that \( r > N(n_3) \). Then \( r_{k}(T_{1,k}, \ldots, T_{r,k}) = N(n_k) \) for \( k \in [3] \). Hence (45) is equivalent to \( r \leq 1/2(N(n_1) + N(n_2) + N(n_3) - 2) \). This establishes (3).
It seems that the best way to group the multiset \{n_1, \ldots, n_d\} to \(S_1 \cup S_2 \cup S_3\) is in such a way that \(n_1 = (n_1), N(n_2) \leq N(n_3)\) and \(N(n_3) - N(n_2)\) is smallest possible. (Note that \(N(n_1)N(n_2)N(n_3) = N(n)\).) The following Corollary reveals the advantage of our decomposition of a tensor \(T\) as a three tensor:

**Corollary 5.5:** Assume that \(d = 2p + 1, p \geq 2, n_1 = \cdots = n_d = n \geq 2\) are integers. Suppose that \(T\) has a decomposition \(\sum_{i=1}^{r} \otimes_{j=1}^{d} x_{i,j}\), where all \(x_{i,j}\) are in general position. (The entries of each \(x_{i,j}\) are chosen from independent \(N(0, 1)\) Gaussian distribution.) If \(r\) satisfies the following inequalities then \(r = r(T)\):

\[
r \leq \begin{cases} 
np & \text{if } n = 2, 3, \\
np - 1 + \frac{n}{2} & \text{if } n \geq 4,
\end{cases}
\]

and the above decomposition of \(T\) is unique.

We now bring the known analog of Theorem 4.10:

**Theorem 5.6:** Assume that \(3 \leq d, 2 \leq n_1 \leq \cdots \leq n_d\) are integers. Then

\[
r_{\text{gen}}(n) = n_d \quad \text{for } \prod_{j=1}^{d-1} n_j + d - 1 - \sum_{j=1}^{d-1} n_j \leq n_d \leq N(n).
\]

The proof is similar to the proof of Theorem 4.10. Denote by \(M(n) := 1 - d + \sum_{j=1}^{d} n_j\) the dimension of the Segre variety in \(\mathbb{C}^n\). Let \(n' = (n_1, \ldots, n_{d-1})\). Hence a generic subspace in \(\mathbb{C}^{n'}\) of dimension \(N(n') - M(n') + 1\) intersects the Segre variety in \(\mathbb{C}^{n'}\) in a finite number of points, whose linear span is this subspace. Use Theorem 5.1 to deduce Theorem 5.6 for \(n_d = N(n') - M(n') + 1\). Similar arguments yield the theorem for \(n_d > N(n') - M(n') + 1\).

Introducing the generalized version of the lower bound given in (28) we obtain the following lower bound for the generic rank

\[
r_0(n) := \left\lceil \frac{N(n)}{M(n)} \right\rceil \leq r_{\text{gen}}(n).
\]

Assume that \(2 \leq n_1 \leq \cdots \leq n_d\). Is the above inequality optimal for \(n_d \leq N(n') - M(n')\)?

In Section 5.4 we show some affirmative results for the case \(n = n_1 = \cdots = n_d\), which we call \(d\)-qunit states, or simply \(d\)-qunits. We now discuss in detail Terracini’s lemma in the general setting.

### 5.2. Terracini’s lemma

We recall the results in [35]. For a fixed \(r \in \mathbb{N}\) consider the map

\[
F_r : (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})^r \to \mathbb{C}^n,
\]

\[
F_r(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) = \sum_{i=1}^{r} \otimes_{j=1}^{d} x_{i,j}.
\]

\[
F_r : (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})^r \to \mathbb{C}^n,
\]

\[
F_r(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) = \sum_{i=1}^{r} \otimes_{j=1}^{d} x_{i,j}.
\]
The set $F_r((\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})^r)$ is a constructible set, of dimension $d(n, r)$, whose closure is an irreducible variety in $\mathbb{C}^n$. (A constructible set of dimension $k$ in $\mathbb{C}^m$ is a finite union of irreducible varieties whose maximal dimension is $k$ minus a union of a finite number of constructible sets of dimension at most $k-1$ [82].) The dimension $d(n, r)$ is the rank of the Jacobian matrix of $F_r$ at a generic point

$$(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) \in (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})^r.$$

The following results are known [35]:

1. $d(n, r_{\text{gen}}(n)) = N(n)$.
2. The sequence $d(n, r)$ is strictly increasing for $r \in [r_{\text{gen}}(n)]$.
3. $d(n, r) = N(n)$ for each integer $r > r_{\text{gen}}(n)$.

The rank of the Jacobian $DF_r$ at the point $(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r})$ is the dimension of the subspaces spanned by the following vectors:

$$\left( \bigotimes_{j=1}^{k-1} x_{j,i} \right) \otimes e_{l_k,i} \otimes \left( \bigotimes_{j=k+1}^d x_{j,i} \right), \quad l_k \in [n_k], i \in [r].$$

Here $e_{1,k,i}, \ldots, e_{n_k,k,i}$ is a basis in $\mathbb{C}^{n_k}$ for $k \in [d]$ and $i \in [r]$, since for each rank-one component $\bigotimes_{j=1}^d x_{j,i}$ one can have a different basis in each component $\mathbb{C}^{n_j}$.

### 5.3. An upper bound on $r_{\text{gen}}(n)$

We now give an upper bound on the generic rank using pure combinatorial methods. Consider the standard basis in $\mathbb{C}^n$:

$$\bigotimes_{j=1}^d e_{l,j}, \quad e_{l,j} = (\delta_{l,j_1}, \ldots, \delta_{l,j_d})^T, \quad l_j \in [n_j], j \in [d].$$

That is, each element in the basis corresponds to a $d$-tuple $(l_1, \ldots, l_d)$, where $l_j \in [n_j]$ for $j \in [d]$. Denote by $[n]$ the set of such of such $d$-tuples:

$$[n] := [n_1] \times \cdots \times [n_d] = \{l = (l_1, \ldots, l_d), \quad l_j \in [n_j], j \in [d]\}.$$

Recall the Hamming distance on $[n]$ is given by the formula:

$$\text{dist}((l_1, \ldots, l_d), (m_1, \ldots, m_d)) = p,$$

if $m_j \neq l_j$ for exactly $p$ indices. Denote by $O(I)$ the set of all points in $[n]$ whose distance from $I$ is at most 1. Note that the cardinality of $O(I)$, denoted as $|O(I)|$, is $M(n)$.

A subset $A \subseteq [n]$ is called a dominating set of $[n]$ if $\cup_{I \in A} O(I) = [n]$. The cardinality of each dominating set $A$ satisfies the inequality $|A|M(n) \geq N(n)$. Denote by $A(n)$ the set of dominating sets. Let $\gamma(n) := \min\{|A|, A \in A(n)\}$ be the minimum cardinality of the dominating set.
A subset $B$ of $[n]$ is called $3$-separated set if the Hamming distance between any two elements of $B$ is at least $3$. Note that if $B$ is $3$-separated then $|B|M(n) \leq N(n)$. Denote by $B(n)$ the set of $3$-separated sets of $[n]$. Let $\kappa(n) := \max\{|B|, B \in B(n)\}$ be the maximum cardinality of $3$-separable set. Observe that $\gamma(n) \geq \kappa(n)$. A maximum $3$-separated set $B$ is called a $1$-perfect code if $\gamma(n) = \kappa(n)$. That is, the Hamming distance between every two elements of $B$ is at least $3$, and for each $p \in [n]$ there exists $q \in B$ such that $\text{dist}(p, q) \leq 1$. The following result is due to [87]:

**Lemma 5.7:** Assume that $3 \leq d$ and $2 \leq n_1 \leq \cdots \leq n_d$ be integers. Then the following assertions hold:

1. The inequality $r_{\text{gen}}(n) \leq \gamma(n)$ holds.
2. For each $r \in [k(n)]$ the closure of $F_r(C^{n_1} \times \cdots \times C^{n_d})$ is an irreducible variety of dimension at most $rM(n)$. In particular, if the dimension of $F_r(C^{n_1} \times \cdots \times C^{n_d})$ is $rM(n)$ then most of tensors of rank $r$ have exactly $\deg F_r$ different rank decomposition.

The above inequalities for dominating and $3$-separated sets yield that $|B| = \frac{N(n)}{M(n)}$. In particular $\frac{N(n)}{M(n)}$ is an integer. Furthermore, the inequality (47) and Lemma 5.7 yield that $r_0(n) = r_{\text{gen}}(n)$.

It is known [89] that $1$-perfect code exists if

$$n_1 = \cdots = n_d = n = q^l, \quad d = \frac{n^{a+1} - 1}{n - 1}, \quad q \text{ is prime,} \quad l, a \in \mathbb{N}, a \geq 2.$$  

Use Lemma 5.7 to deduce that in this case $r_{\text{gen}}(n) = n^{d-a-1}$.

Let $G = (V, E)$ be a simple graph on the set of vertices $V$ and edges $E$. Recall that $A \subseteq V$ is a dominating set if each vertex $v$ not in $A$ is adjacent to some vertex in $A$. Then $\gamma(G)$ is called the domination number of $G$, if $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. We will show below that $\gamma(n) = \gamma(G(n))$, where $G(n) = ([n], E(n))$ is the induced graph on $[n]$ by the Hamming distance.

The domination number of $G$ is a solution to the following minimum problem in $|V|$ variables $x_v, v \in V$ whose values are in $\{0, 1\}$. For each $x = (x_v)_{v \in V} \in \{0, 1\}^V$ we denote by $\text{supp } x$ the subset $\{v \in V, x_v = 1\}$. Then $\text{supp } x$ is a dominating set in $V$ if and only if the following inequalities hold:

$$x_v + \sum_{u,(u,v) \in E} x_u \geq 1 \quad \text{for all } v \in V. \quad (50)$$  

Hence $\gamma(G)$ is the minimum of $\sum_{v \in V} x_v$ on $x \in \{0, 1\}^V$ subject to (50). It is known that computing $\gamma(G)$ for general graphs is an NP-complete problem [90].

A greedy algorithm to find an upper bound for $\gamma(G)$ is as follows: Let $G_1 = G$. Suppose that at the stage $k \in [V]$ we have the graph $G_k = (V_k, E_k)$, where $V_k$ is a nonempty subset of $V$, and $G_k$ is the induced subgraph of $G$ by the set $V_k$. We choose a vertex $v_k \in V_k$ of a maximum degree in $G_k$. Let $O_k \subseteq V_k$ be the neighbours of $v_k$ in $G_k$. Then $V_{k+1} = V_k \setminus \{v_k \cup O_k\}$. If $V_{k+1} = \emptyset$ then $A = \{v_1, \ldots, v_k\}$ is the dominating set. Otherwise set $k$ as $k + 1$. 

Recall the standard linear programming (LP) relaxation of the above minimal problem on \{0, 1\}^V [91]. Namely, we replace the condition \(x_v \in \{0, 1\}, \forall v \in V\) by the condition \(0 \leq x_v \leq 1, \forall v \in V\). Thus we consider the minimum \(\sum_{v \in V} x_v\) satisfying the inequalities (50) for \(x \in [0, 1]^V\). Denote this minimum by \(\beta(G)\).

The following result is well known [92]: Let \(G = (V, E)\) be a simple graph with maximal degree \(\Delta(G)\). Denote by \(A(G) \subseteq V\), a dominating set obtained by the above greedy algorithm. Then

\[
\beta(G) \leq \gamma(G) \leq |A(G)| \leq O(\log \Delta(G))\beta(G). \tag{51}
\]

Recall that \(G\) is called regular, if the degree of each vertex is \(\Delta(G)\). Then

\[
\beta(G) \leq \frac{\#V}{\Delta(G) + 1}. \tag{52}
\]

Indeed, define \(x_v = \frac{1}{\Delta(G) + 1}\) for each \(v \in V\). Then the conditions (50) are satisfied. As the following equality is true, \(\sum_{v \in V} x_v = \frac{\#V}{\Delta(G) + 1}\), the above inequality holds. Thus we showed that for regular graph \(G\) we have the following inequalities:

\[
\frac{\#V}{\Delta(G) + 1} \leq \gamma(G) \leq |A(G)| \leq O(\log \Delta(G))\frac{\#V}{\Delta(G) + 1}. \tag{53}
\]

(Recall the notation \(O(m)\) for some function \(f : \mathbb{N} \to [0, \infty)\). Namely, there exists a universal \(K > 0\) so that \(f(m) \leq Km\) for all \(m \in \mathbb{N}\).)

We now apply these results to estimate from above the generic rank. Let \(G(n) = ([n], E(n))\). Two vertices \(l, m \in [n]\) are adjacent if \(\text{dist}(l, m) = 1\). Observe that \(G(n)\) is a regular graph with \(\Delta(G(n)) = M(n) - 1\). It is easy to show that \(A \subseteq [n]\) is a dominating set if and only if \(A\) is a dominating set in \(G(n)\). That is, \(\gamma(n) = \gamma(G(n))\). Thus (53) applies to \(G(n)\). We do not know how good is the upper bound on \(\gamma(n)\) given in (53) in the general case. Apply Lemma (5.7) to deduce sandwich bound:

\[
r_0(n) \leq r_{\text{gen}}(n) \leq O\left(\log \left(\sum_{i=1}^{d} (n_i - 1)\right)\right) r_0(n). \tag{54}
\]

Let us consider the following simple examples for \(d = 3\) and \(n_1 = n_2 = n_3 = 3\). Choose

\[
A = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (2, 2, 3), (3, 3, 1)\}
\]

end

\[
B = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}.
\]

Then \(A\) is a dominating set and \(B\) is 3-separated set. Recall that \(r_{\text{gen}}(3, 3, 3) = 5 < |A| = 6\). It is straightforward to show that \(B\) is a maximal 3-separated set. So \(\kappa(3, 3, 3) = 3\).
5.4. The generic rank of d-qubits

Let $n \times d = (n, \ldots, n) \in \mathbb{N}^d$. Then $r_{\text{gen}}(n \times d)$ is the generic rank of $d$-qubits. Inequality (47) yields

$$r_{\text{gen}}(n \times d) \geq \lfloor \theta(n \times d) \rfloor, \quad \text{where } \theta(n \times d) = \frac{n^d}{d(n-1)+1}. \quad (55)$$

In previous section, we showed that equality holds if $[n]^d$ has 1-perfect code [87].

It was shown in [93] that equality holds in (55) for $n = 2$ and any $d \geq 2$. That is, the generic rank of $d$-qubits is $[2^d/(d+1)]$.

We now recall some results in [36] for $r_{\text{gen}}(n \times d)$. First, assume that $\theta(n \times d)$ is integer. (Thus $d = \frac{n^{a+1}-1}{n-1}$ for $a \in \mathbb{N}$.) Then $r_{\text{gen}}(n \times d) = \theta(n \times d)$. Second, assume that $\theta(n \times d)$ is not an integer. Let $\lfloor \theta(n \times d) \rfloor = n \mod n \delta(n \times d) \in \{0, \ldots, n-1\}$. Then

$$r_{\text{gen}}(n \times d) = \begin{cases} \frac{n^d}{d(n-1)+1} & \text{if } \delta(n \times d) = n-1, \\ r_{\text{gen}}(n \times d) \leq \frac{n^d}{d(n-1)+1} + n-1 - \delta(n \times d). \end{cases} \quad (56)$$

We now provide a few examples of the above equalities and inequalities. According to [93] for $n = 2$ and the known table of the values of $r_{\text{gen}}(n \times d)$, which is given later, in all below cases the upper bound on $r_{\text{gen}}(n \times d)$ is a strict inequality.

\[
\begin{align*}
\theta(2^x) &= 16/5, \quad \lfloor \gamma(2^x) \rfloor = 3, \quad \delta(2^x) = 1, \quad r_{\text{gen}}(2^x) = 4, \\
\theta(2^y) &= 32/6, \quad \lfloor \gamma(2^y) \rfloor = 5, \quad \delta(2^y) = 1, \quad r_{\text{gen}}(2^y) = 6, \\
\theta(2^z) &= 256/9, \quad \lfloor \gamma(2^z) \rfloor = 28, \quad \delta(2^z) = 0, \quad r_{\text{gen}}(2^z) = 29 < 30, \\
\theta(3^w) &= 27/7, \quad \lfloor \gamma(3^w) \rfloor = 3, \quad \delta(3^w) = 0, \quad r_{\text{gen}}(3^w) = 5 < 6, \\
\theta(3^v) &= 243/11, \quad \lfloor \gamma(3^v) \rfloor = 22, \quad \delta(3^v) = 1, \quad r_{\text{gen}}(3^v) = 23 < 24, \\
\theta(3^u) &= 729/13, \quad \lfloor \gamma(3^u) \rfloor = 56, \quad \delta(3^u) = 2, \quad r_{\text{gen}}(3^u) = 57.
\end{align*}
\]

Known values of generic rank, $r_{\text{gen}}(n \times d)$, for the system of $d$-qubits are listed in Table 1.

Table 2 provides a comparison between the generic rank $r_{\text{gen}}$, the maximal ranks $r_{\text{max}}$ and the maximal number $R_U$ of terms in the shortest representation of a pure state of $d$ subsystems with $n$ levels each in an orthogonal product basis in $\mathcal{H}_n^\otimes d$. The upper bound $R_U = n^d - d(n-1)/2$ follows directly from the work of Carteret, Higuchi and Sudbery [59]. They demonstrated that out of $n^d$ entries of any tensor $T$ one can set to zero $n(n-1)/2$ entries by performing a single unitary rotation which affects a single index. As there are $d$ independent indices, for which such a transformation can be applied, the total number of entries which can be set to zero is $dn(n-1)/2$. This explains the bound $R_U$ stated above.
Table 1. Values of the generic rank calculated for several $n$-level systems.

| $d \setminus n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|----|
| 2              | $[a,b]_2$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3              | $[a,c,d]_2$ | $[a,c,d]_5$ | $[a,c,d]_7$ | $[a,c,d]_{10}$ | $[a,c,d]_{14}$ | $[a,c,d]_{19}$ | $[a,c,d]_{24}$ | $[a,c,d]_{30}$ | $[a,c,d]_{36}$ |
| 4              | $[a,b,d]_4$ | $[a,d]_{9}$ | $[a,d]_{20}$ | $[a,d]_{37}$ | $[a,d]_{62}$ | $[a,d]_{97}$ | $[a,d]_{142}$ | $[a,d]_{199}$ | $[a,d]_{271}$ |
| 5              | $[a,b]_6$ | 23 | $[a,b]_6$ | 149 | $[a,b]_8$ | 300 | 543 | . | . |
| 6              | $[a]_10$ | $[a]_{16}$ | $[a]_{57}$ | $[a]_2$ | $[a]_{625}$ | $[a]_{156}$ | . | . | . |
| 7              | $[a]_16$ | $[a]_16$ | $[a]_6$ | 4194 | $[a]_{156}$ | 250 | . | . | . |
| 8              | $[a]_29$ | 386 | . | . | $[a]_y^6$ | . | . | . | . |
| 9              | $[a]_62$ | 1036 | . | . | . | . | . | . | . |
| 10             | $[a]_{194}$ | $[a]_{194}$ | $[a]_{1085}$ | $[a]_{1085}$ | $[a]_{6896}$ | . | . | . | . |
| 11             | $[a]_{171}$ | $[a]_{1258}$ | $[a]_{230}$ | $[a]_{1103}$ | 202 | $[a]_{1085}$ | $[a]_{1085}$ | $[a]_{6896}$ | . |
| 12             | $[a]_{316}$ | $[a]_{2125}$ | $[a]_{30}$ | $[a]_{1103}$ | 202 | $[a]_{1085}$ | $[a]_{1085}$ | $[a]_{6896}$ | . |
| 13             | $[a]_56$ | $[a]_{1093}$ | $[a]_{2125}$ | $[a]_{1103}$ | 202 | $[a]_{1085}$ | $[a]_{1085}$ | $[a]_{6896}$ | . |
| 14             | $[a]_{1093}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ |
| 15             | $[a]_{1093}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ | $[a]_{2490}$ |

Notes: Polygonal chain defines two areas in the array. Numbers in the upper-left part corresponding to the right-hand side of the formula (55) are confirmed numerically. Numbers decorated with $[a,b,c,d]$ on the left represent known results according to the reference in [36,62,93,94], respectively, while numbers in bold are results obtained in this work by numerical calculations.

Table 2. Generic ranks $r_{\text{gen}}$, maximal ranks $r_{\text{max}}$ and the maximal number $R_U$ of terms in the shortest representation of a pure state of $d$ subsystems with $n$ levels each in an orthogonal product basis in $H_n \otimes \mathbb{C}^d$.

| $n$ | $2$ | $3$ | $4$ | $5$ |
|-----|-----|-----|-----|-----|
| $d$ | $r_{\text{gen}}$ | $r_{\text{max}}$ | $R_U$ | $r_{\text{gen}}$ | $r_{\text{max}}$ | $R_U$ | $r_{\text{gen}}$ | $r_{\text{max}}$ | $R_U$ | $r_{\text{gen}}$ | $r_{\text{max}}$ | $R_U$ |
| 2   | 2   | 2   | 2   | 3   | 3   | 3   | 4   | 4   | 4   | 5   | 5   | 5   |
| 3   | 3   | 3   | 3   | 4   | 4   | 4   | 5   | 5   | 5   | 6   | 6   | 6   |
| 4   | 4   | 4   | 12  | 9   | 18  | 69  | 20  | 40  | 232 | 37  | 74  | 585 |

Notes: Numbers in bold denote exact results, other numbers denote upper bounds obtained in [94] and in [35,70].

5.5. Estimating and computing the rank of a tensor using polynomial equations

It is clear to many researchers in the field that the rank of a tensor over a given field $\mathbb{F}$ is equivalent to solvability of corresponding system of polynomial equations over $\mathbb{F}$. See for example [95], where the author deals with ranks of tensors over the real numbers. In this subsection we report briefly on the approach outlined in [51] to estimate and to compute the rank of a tensor using polynomial equations. We start with the following lemma:

**Lemma 5.8:** Assume that $d \geq 3$ and $T \in \mathbb{C}^n \setminus \{0\}$ is given. Fix $r \in \mathbb{N}$ and consider the equality (43) as a system of polynomial equations in the entries of unknown vectors $x_{ij}$ for $i \in [r], j \in [d]$:

$$f_{k_1,\ldots,k_d}(x_{1,1},\ldots,x_{d,1},\ldots,x_{1,r},\ldots,x_{d,r})$$

$$\equiv \left( \sum_{i=1}^{r} \otimes_{j=1}^{d} x_{ij} - T \right)_{k_1,\ldots,k_d} = 0, \quad k_j \in [n], j \in [d].$$

Then $r(T) > r$ if and only if the following equivalent conditions hold:
(a) The above system of polynomial equations is not solvable.
(b) There exist \( N(n) \) polynomials \( g_{k_1,\ldots,k_d}(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) \) for \( k_j \in [n_j], j \in [d] \) of degree at most

\[
L(r,n) = d^{-1} + \min(N(n), r \sum_{i=1}^{d} n_i).
\]

such that the following identity holds:

\[
\sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} g_{k_1,\ldots,k_d} f_{k_1,\ldots,k_d} = 1. \tag{59}
\]

Furthermore, \( r(T) \leq r \) if and only if the following equivalent conditions hold:

(i) The system of polynomial Equation (58) is solvable.
(ii) There are no \( N(n) \) polynomials \( g_{k_1,\ldots,k_d}(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) \) for \( k_j \in [n_j], j \in [d] \) of degree at most \( L(r,n) \) such that the identity (59) holds.

**Proof:** Clearly, \( r < r(T) \) if and only if the system (58) is not solvable. Hilbert’s Nullstellensatz states that non-solvability of (58) is equivalent to the existence of polynomials \( g_{k_1,\ldots,k_d}(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) \) that satisfy the identity (59). The claim that the degree of each \( g_{k_1,\ldots,k_d} \) is at most \( L(r,n) \) is due to Kől [96].

Suppose that \( r(T) \leq r \). Then the system (58) is solvable. (Some of \( x_{i,j} \) can be zero.) The identity (59) cannot hold. Part (b) yields that there are no polynomials \( g_{k_1,\ldots,k_d}(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r}) \) for \( k_j \in [n_j], j \in [d] \) of degree at most \( M(r,n) \) that satisfy (59).

We now explain briefly how to use this lemma effectively of estimate or to compute the rank of \( T \). For \( i \in [d] \) let \( T_i \) be the \( n_i \times (N(n)/n_i) \) matrix obtained from \( T \) by viewing \([n_1] \times \cdots \times [n_d]\) as \([n_1] \times ([n_1] \times \cdots [n_{i-1}] \times [n_{i+1}] \cdots \times [n_d])\). (We partition \( T \) to a bipartite state.) Let \( r_i(T) \) be the matrix rank \( r(T_i) \), which is easy to compute. (For \( d = 3 \) those are ranks \( r_A(T), r_B(T), r_C(T) \) introduced in Section 4.1.) As in Section 4.1 we have \( r(T) \geq rm = \max(r_1(T), \ldots, r_d(T)) \). Fix \( r \geq rm \). Write each \( g_{k_1,\ldots,k_d} \) as a polynomial of degree \( L(r,n) \) with unknown monomial coefficients. Then view the identity (59) as a huge system of linear equations in the unknown coefficients of monomials of \( g_{k_1,\ldots,k_d} \) for \( k_j \in [n_j], j \in [d] \). If this system of linear equations is solvable we deduce that \( r < r(T) \). If this system is not solvable then \( r \geq r(T) \). To find \( r(T) \) we start an algorithm with the above procedure for \( r = rm \). If this system of linear equations corresponding to (59) is not solvable then \( r(T) = r \). Otherwise set \( r = r + 1 \) and repeat the above procedure. The main drawback of this algorithm for finding \( r(T) \) is an exponential number of variables and equations in \( d \).

### 5.6. Generic identifiability of tensors

**Definition 5.9:** Assume that \( d \geq 3 \). A tensor \( T \in \mathbb{C}^n \) is identifiable if its rank decomposition as a sum of rank-1 decomposition is unique up to the order of summation.
Note that for $d = 2$ any matrix $T$ with $r(T) > 1$ is not identifiable. Lemma 5.3 gives sufficient conditions on the identifiability of a tensor. The obvious question arises, what happens if the inequality (45) does not hold. A simple example is the case of $T \in \otimes^3 \mathbb{C}^2$ discussed after Lemma 4.5. Namely a generic decomposition of $T$ as a sum of two rank-1 tensors is a unique rank decomposition of $T$, since it satisfies the condition (45). If $r(T) = 3$ then any rank decomposition of $T$ does not satisfy (45), and its rank decomposition is not unique. In particular a symmetric decomposition of $|W\rangle = \sum_{i=1}^{3} x_i \otimes x_i$ is not unique. It is shown in [77] that Kruskal’s theorem fails if we replace $2r + 2$ in the right-hand side of (45) by $2r + 1$.

Consider the space of tensors $\mathbb{C}^n$ where $d \geq 3$ and $2 \leq n_1 \leq \cdots \leq n_d$. Fix $r > 1$ and assume that there exists $T \in \mathbb{C}^n$ such that rank $T = r$. Let $F_r$ be the map defined by (48). A tensor $T$ is called a random tensor in $F_r(\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})$ if $T = F_r(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r})$, and the coordinates of vectors $x_{1,1}, \ldots, x_{d,r}$ are sampled from independent Gaussian complex-valued distribution. We say that the identifiability property holds if for a random $T$ of the above form $r = r(T)$, and the decomposition $T = F_r(x_{1,1}, \ldots, x_{d,1}, \ldots, x_{1,r}, \ldots, x_{d,r})$ is unique up to a permutation of the summands $\bigotimes_{j=1}^{d} x_{j,d}$. Recall the inequality (47). By counting the parameters we deduce that if $\frac{N(n)}{M(n)}$ is not an integer then for $r = r_{\text{gen}}(n)$ identifiability property fails. Thus it makes sense to consider the identifiability property for $r < r_{\text{gen}}(n)$. Theorem 5.6 states that $r_{\text{gen}}(n) = n_d$ if

$$\left( \prod_{j=1}^{d-1} n_j \right) + d - 1 - \sum_{j=1}^{d-1} n_j \leq n_d. \tag{60}$$

If $n_d$ satisfies the above inequality then the identifiability property holds if and only if [97]:

$$r \leq \left( \prod_{j=1}^{d-1} n_j \right) + d - 2 - \sum_{j=1}^{d-1} n_j. \tag{61}$$

Thus it is enough to consider the identifiability property for

$$n_1 \leq \cdots \leq n_{d-1} \leq n_d \leq \left( \prod_{j=1}^{d-1} n_j \right) + d - 2 - \sum_{j=1}^{d-1} n_j. \tag{62}$$

It is shown in [25] that if the above inequalities hold, and $\prod_{i=1}^{d} (n_i - 1) \leq 15,000$ then the identifiability property holds for $r < r_{\text{gen}}(n)$ except the following cases of $n = (n_1, \ldots, n_d)$ and the corresponding $r$:

$$(4, 5, 5) \quad \text{and} \quad r = 5,$$

$$(5, 5, 5) \quad \text{and} \quad r = 6,$$

$$(4, 7, 7) \quad \text{and} \quad r = 8,$$

$$(3, 3, n, n) \quad \text{and} \quad r = 2n - 1,$$

$$(3, 3, 3, 3) \quad \text{and} \quad r = 5. \tag{63}$$

Domanov and de Lathauwer studied the identifiability property for 3-mode tensors using mostly matrix methods in [98–101].
6. Symmetric tensors

This section is devoted to symmetric tensors, which can be applied in quantum physics to describe systems of bosons. In Section 6.1, we recall the known one-to-one correspondence between symmetric $d$-mode tensors and homogeneous polynomials of degree $d$. In particular, the symmetric rank of a symmetric tensor is the Waring rank of a homogeneous polynomial. Next we bring the celebrated result of Alexander–Hirschowitz [46] that gives the formula for the generic symmetric rank except a number of known cases. Section 6.2 discusses the recent upper bound of Buczynski–Han–Mella–Teitler on the maximum symmetric rank in terms of the generic symmetric rank [69]. We also provide some known values of the maximum symmetric rank. Section 6.3 shows that the rank of the symmetric tensor representing the state $|W_d⟩$ is equal to $d$, while its border rank is 2. In Section 6.4, we show explicitly that the rank of Kronecker and tensor products of quantum states can be strictly submultiplicative. This is achieved by considering the ranks of $|W⟩ ⊗_K |W⟩$ and $|W⟩ ⊗ |W⟩$, which read 7 and 8 respectively, while the square of the rank of $|W⟩$ is 9.

In a short Section 6.5, we discuss briefly computational methods for symmetric rank of symmetric tensors. Section 6.6 gives a short account of the results in [55], which show that the generic identifiability property of symmetric tensors holds for a rank less than the symmetric generic rank, except a number of known cases.

6.1. Basic properties and relation to homogeneous polynomials

A tensor $S = [S_{i_1,...,i_d}] ∈ \bigotimes^d \mathbb{C}^n$ is called symmetric if the value of the coordinates $S_{i_1,...,i_d}$ does not change under the permutation of indices. We denote by $S^d\mathbb{C}^n ⊂ \bigotimes^d \mathbb{C}^n$ the subspaces of all $d$-mode symmetric tensors over $\mathbb{C}^n$. In physics this space is called the $(d,n)$ boson space. A symmetric $S$ is rank-1 tensor if and only if $S = \bigotimes^d x$, where $x ∈ \mathbb{C}^n \setminus \{0\}$. There exists one-to-one correspondence between symmetric tensors and the space of all homogeneous polynomials of degree $d$ in $n$ complex variables denoted as $P(d,n)$. Indeed, let $f(x) = ⟨S, \bigotimes^d x⟩$, where $⟨·,·⟩$ is the standard inner product in $\bigotimes^d \mathbb{C}^n$. Then $f(x) ∈ P(d,n)$. Conversely, each polynomial $f(x) ∈ P(d,n)$ induces a unique $S ∈ S^d\mathbb{C}^n$ as we explain below.

We now introduce the standard multinomial notation. Let $\mathbb{Z}_+$ be the set of all nonnegative integers. Denote by $J(d,n)$ the set $J(d,n) = \{j = (j_1,...,j_n) ∈ \mathbb{Z}^n_+, j_1 + ⋯ + j_n = d\}$. Recall that $|J(d,n)|$, the cardinality of the set $J(d,n)$, is $\binom{n+d-1}{d}$. For $x = (x_1,...,x_n)^\top ∈ \mathbb{C}^n$ and $j = (j_1,...,j_n) ∈ J(d,n)$ let $x^j$ be the monomial $x_1^{j_1} ⋯ x_n^{j_n}$. Define $c(j) = \frac{d!}{j_1! ⋯ j_n!}$. Then $f(x) ∈ P(n,d)$ expressed as a sum of monomials is given by

$$f(x) = \sum_{j∈ J(d,n)} c(j)f_j x^j. \quad (64)$$

Suppose that $f(x) = ⟨S, \bigotimes^d x⟩$. Then the correspondence between $f_j$ and the entries of $S = [S_{i_1,...,i_d}]$ is as follows. Assume that $(i_1,...,i_d) ∈ [n]^d$ is fixed. For each $l ∈ [n]$ let $j_l$ be the number of times that $l$ appears in the sequence $i_1,...,i_d$. Set $j = (j_1,...,j_n)$. Then $f_j = S_{i_1,...,i_d}$.

Thus $\dim S^d\mathbb{C}^n = \binom{n+d-1}{d} = \binom{n+d-1}{n-1}$. This dimension is usually significantly smaller than $\dim \bigotimes^d \mathbb{C}^n = n^d$. For example for $n = 2$, the space $S^d\mathbb{C}^2$, the boson $d$-qubit space,
has dimension \(d + 1\), while the space \(\otimes^d \mathbb{C}^2\), of \(d\)-mode qubits is \(2^d\). Thus \((d, n)\) bosons are much less entangled than then \(d\)-qutrits.

A symmetric rank decomposition of \(S \in S^d \mathbb{C}^n\) is a decomposition of \(S\) to a sum of rank-1 symmetric tensors. This is analogous to the Waring decomposition of \(f \in P(d, n)\) to a sum of linear terms to the power \(d: f(x) = \sum_{i=1}^r \langle x, a_i \rangle^d\), where \(a_i \in \mathbb{C}^n \setminus \{0\}\). The minimal number of summands in symmetric rank decomposition of symmetric \(S\) is called the symmetric rank of \(S\), and denoted as \(r_S(S)\). This is equivalent to the Waring rank of \(f(x) = \langle S, \otimes^d \bar{x} \rangle\). The following inequality holds by definition, \(r(S) \leq r_S(S)\).

We now recall two positive results when \(r(S) = r_S(S)\). For a \((d, n)\) symmetric tensor \(S\) denote by \(r_A(S)\) the matrix rank of \(S\) viewed as a bipartite state in \(\mathbb{C}^n \otimes (\otimes^{d-1} \mathbb{C}^n)\). As in Section 4.1, we deduce that \(r_A(S) \leq r(S)\). In [88], it is shown that if \(r(S) \in \{r_A(S), r_A(S) + 1\}\), then \(r(S) = r_S(S)\). It is shown in [102] that if \(S \in S^d \mathbb{C}^n\) and \(r(S) \leq d\), then \(r(S) = r_S(S)\). However, even for general 3-mode symmetric tensors one has a strict inequality \(r(S) < r_S(S)\) [42].

As for general tensors, one can define a generic rank of \((d, n)\) symmetric tensor \(S\) as the symmetric rank of a random \(S \in S^d \mathbb{C}^n\). Denote by \(r_{\text{gen}}(d, n)\) the generic rank of \((d, n)\) symmetric tensor. Note that \(\sum_{i=1}^r \langle x, a_i \rangle^d\) has \(rn\) complex parameters. The dimension count yields the inequality

\[
    r_{\text{gen}}(d, n) \geq \left\lceil \frac{n + d - 1}{d} \right\rceil.
\]

(65)

The celebrated Alexander–Hirschowitz result [46] claims that equality holds in the above inequality except the following cases [103]:

\[
\begin{align*}
    n &= 3, \quad d = 4, \\
    n &= 4, \quad d = 4, \\
    n &= 5, \quad d = 3, \\
    n &= 5, \quad d = 4.
\end{align*}
\]

In all the exceptional cases, the value of generic rank is \(\left\lceil \frac{n + d - 1}{d} \right\rceil + 1\). Furthermore, in these exceptional cases, all tensors of border rank at most \(\left\lceil \frac{n + d - 1}{d} \right\rceil\) form a hypersurface in \(S^d \mathbb{C}^n\). (We thank G. Ottaviani for pointing out this fact to us.)

### 6.2. Maximum symmetric rank

Denote by \(r_{\text{max}}(d, n)\) the maximum rank of \((d, n)\) symmetric tensors. The following analogue of (12) is proved in [69]:

\[
    r_{\text{max}}(d, n) \leq 2r_{\text{gen}}(d, n) - 1.
\]

(66)

This bound can be further improved [69] to \(r_{\text{max}}(d, n) \leq 2r_{\text{gen}}(d, n) - 2\) if the variety of all symmetric tensors of border symmetric rank at most \(r_{\text{gen}}(d, n) - 1\) is a hypersurface. This assumption holds in all the above exceptional cases.

We now discuss briefly the known maximum ranks. The first nontrivial case is \(r_{\text{max}}(2, 3)\). As \(r_{\text{max}}(2, 2, 2) = 3\) and \(r(|W|) = 3\) we deduce that \(r_{\text{max}}(3, 2) = 3\). Observe
that the relation $r_{\text{gen}}(3, 2) = r_{\text{gen}}(2, 2, 2) = 2$ implies that inequality (66) is not sharp in this case.

The following maximum ranks are known. We also display the value of the generic symmetric rank in these cases:

$$r_{\text{max}}(d, 2) = d \quad [104] \text{, } [70, \text{Section } 3.1], \quad r_{\text{gen}}(d, 2) = \left\lceil \frac{d + 1}{2} \right\rceil,$$

$$r_{\text{max}}(3, 3) = 5 \quad [105, \text{Section } 96], [106], [107], \quad r_{\text{gen}}(3, 3) = 4,$$

$$r_{\text{max}}(4, 3) = 7 \quad [105, \S 97], [106], [107], \quad r_{\text{gen}}(4, 3) = 6,$$

$$r_{\text{max}}(5, 3) = 10 \quad [109], [110], \quad r_{\text{gen}}(5, 3) = 7.$$

(67)

In Section 6.4, we show that $r_{\text{max}}(3, 4) \geq 7$. See also [110].

### 6.3. The rank of $|W_d|$}

Denote by $|W_d| \in S^{d}C^2$ the symmetric tensor corresponding to polynomial $dx_1^{d-1}x_2$:

$$|W_d| = \sum_{j=0}^{d-1} \sum_{i=0}^{d-j-1} \binom{d-j}{i} \otimes e_1 \otimes e_2 \otimes (\otimes e_1).$$

(68)

Hence $r(|W_d|) \leq d$. We claim that $r(|W_d|) = r_s(|W_d|) = d$. As $|W| = |W_3|$, we know that $r(|W_3|) = 3$. We first claim that $r(|W_d|) = d$ [111]. We prove that by induction on $d = k \geq 3$. Suppose that $r(|W_k|) = k$ for $k \geq 3$. Assume to the contrary that

$$|W_{k+1}| = \sum_{i=1}^{r} \prod_{j=1}^{k+1} x_{ij}, \quad x_{ij} \in C^2, j \in [k+1], i \in [r], r < k + 1.$$

Observe that $|W_{k+1}| = |W_k| \otimes e_1 + e_1^{\otimes k} \otimes e_2$. Hence, span($x_{1,k+1}, \ldots, x_{r,k+1}$) $\subseteq C^2$. For $y \in C^2$ let $|W_{k+1}| \times y = \sum_{i=1}^{r} (y^T x_{i,k+1}) \otimes_{j=1}^{k} x_{ij}$ be the contraction with respect to the last coordinate. Choose $x_{i,k+1}$ which is linearly independent to $e_1$. Let $y \in C^2 \setminus \{0\}$ satisfy $y^T x_{i,k+1} = 0$. Hence $y^T e_1 \neq 0$ and we fix $y$ by letting $y^T e_1 = 1$. Thus $T = |W_{k+1}| \times y$ equals to $\sum_{i\in[r]\{l\}} (y^T x_{i,k+1}) \otimes_{j=1}^{k} x_{ij}$. Therefore $r(T) \leq k - 1$. Observe next that $T = |W_k| + (y^T e_2) e_1^{\otimes k}$. Furthermore, $T$ is a symmetric tensor which corresponds to the polynomial

$$f(x) = dx_1^{d-1}x_2 + (y^T e_2)x_1^d = dx_1^{d-1}(x_2 + (y^T e_2)/d)x_1).$$

Change coordinates $(x_1, x_2)$ to $(x_1, x_2 + ((y^T e_2)/d)x_1)$ to deduce that $T$ is in the orbit of $|W_k|$. Hence $r(T) = k$ which contradicts our assumption that $r(|W_{k+1}|) < k + 1$. Thus $r(|W_{k+1}|) = k + 1$.

Observe finally that $d = r(|W_d|) \leq r_s(|W_d|) \leq r_{\text{max}}(d, 2) = d$.

We close this subsection with the well known fact that $r_b(|W_d|) = 2$. As the set of rank-1 states is closed it follows that $r_b(|W_d|) \geq 2$. On the other hand, we have the equality

$$|W_d| = \lim_{t \to 0} \frac{1}{t} \left( (e_1 + te_2)^{\otimes d} - e_1^{\otimes d} \right).$$

(69)
6.4. Tensor rank of product of tensors

**Lemma 6.1:** Let \( \mathcal{U} \in \mathbb{C}^m, \mathcal{V} \in \mathbb{C}^n \) be two tensors, where \( \mathbf{m} = (m_1, \ldots, m_p), \mathbf{n} = (n_1, \ldots, n_q) \). Then

\[
 r(\mathcal{U} \otimes \mathcal{V}) \leq r(\mathcal{U} \otimes \mathcal{V}) \leq r(\mathcal{U})r(\mathcal{V}).
\]  

(70)

Furthermore

(a) If \( \mathcal{U} \in \mathbb{C} \otimes \mathbb{C}^d \otimes \mathbb{C}^d \) and \( \mathcal{V} \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m \) then equalities hold in (70).

(b) Suppose that \( \mathcal{U} = \mathcal{V} = |W\). Then strict inequalities hold in (70). More precisely

\[
r(|W| \otimes_K |W\)) = 7, \quad r(|W| \otimes |W\)) = 8.
\]

(71)

**Proof:** If \( p = q \), then we have the inequalities (10). The general case follows from the same arguments. The equality \( r(\mathcal{U} \otimes \mathcal{V}) = r(\mathcal{U})r(\mathcal{V}) \) yields the equality \( r(\mathcal{U} \otimes \mathcal{V}) = r(\mathcal{U})r(\mathcal{V}) \). It is easy to show that if either \( p = 1 \) or \( q = 1 \), then equality holds in (70). Indeed, it is enough to assume that \( q = 1 \) and \( \mathcal{V} \neq 0 \). As in the proof that \( r(|W_d\)) = d \) we deduce equality by contracting the last index in \( \mathcal{U} \otimes \mathcal{V} \). Since for matrices \( r(\mathcal{U} \otimes_K \mathcal{V}) = r(\mathcal{U})r(\mathcal{V}) \), it follows that for \( p = q = 2 \) equality holds in (70). Corollary 4.9 gives an example when one has equalities in (10) for special two 3-tensors.

We now show part (a) of the Lemma. Proposition 22 in [40] gives the following application of Theorem 4.7:

\[
r(\mathcal{X} \otimes_K \mathcal{Y}) = r(\mathcal{X} \otimes \mathcal{Y}) = r(\mathcal{X})r(\mathcal{Y}), \quad \mathcal{X} \in \mathbb{C} \otimes \mathbb{C}^d \otimes \mathbb{C}^d, \mathcal{Y} \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m.
\]

Note that \( \mathcal{X} \) can be viewed as a matrix \( X \in \mathbb{C}^d \otimes \mathbb{C}^d \). Any matrix \( T \in \mathbb{C}^d \times \mathbb{C}^d \) can be trivially extended to a bigger matrix \( X \in \mathbb{C}^d \times \mathbb{C}^d \) for \( d = \max(p, q) \) by adding additional zero rows or columns. It is straightforward to show that \( r(T \otimes \mathcal{Y}) = r(X \otimes \mathcal{Y}) \). Hence

\[
r(X \otimes \mathcal{Y}) = r(\mathcal{X})r(\mathcal{Y}), \quad X \in \mathbb{C}^p \otimes \mathbb{C}^q, \mathcal{Y} \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m.
\]

(72)

A special case of this equality is proved independently in the first part of Proposition 9 in [43].

We now show part (b) of the Lemma. First we show that one can have strict inequalities in (72) for \( \mathcal{U} = \mathcal{V} = |W_3\). Assume that \( \mathcal{X} = |W_3\otimes_K |W_3\) \( \in \mathbb{C}^3 \otimes \mathbb{C}^4 \). It will be convenient to use Dirac notation, where

\[
|00\rangle = |0\rangle, \quad |01\rangle = |1\rangle, \quad |10\rangle = |2\rangle, \quad |11\rangle = |3\rangle.
\]

(73)

Then

\[
\mathcal{X} = (|001\rangle + |010\rangle + |100\rangle) \otimes_K (|001\rangle + |010\rangle + |100\rangle)
\]

\[
= |003\rangle + |012\rangle + |102\rangle + |021\rangle + |030\rangle + |120\rangle + |201\rangle + |210\rangle + |300\rangle.
\]

The above three tensors are symmetric on \( \mathbb{C}^4 \) and it correspond to the following polynomial of degree 3, \( f(x_1, x_2, x_3, x_4) = 3x_1^2x_4 + 6x_1x_2x_3 \). Observe next that [37,43]:

\[
6x_1^2x_4 = (x_1 + x_4)^3 - (x_1 - x_4)^3 - 2x_4^3, \quad 24x_1x_2x_3
\]
\[(x_1 + x_2 + x_3)^3 - (-x_1 + x_2 + x_3)^3 - (x_1 - x_2 + x_3)^3 - (x_1 + x_2 - x_3)^3.\]

This implies that \(r_s(\mathcal{X}) \leq 7.\)

We now follow the arguments of [39] to show that \(r(\mathcal{X}) \geq 7.\) First observe that the four frontal sections of \(\mathcal{X}\) form the following four matrices:

\[
A_1 = |00\rangle\langle 00| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \quad A_2 = |01\rangle\langle 01| + |10\rangle\langle 10| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix},
\]

\[
A_3 = |02\rangle\langle 02| + |20\rangle\langle 20| = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \quad A_4 = |03\rangle\langle 03| + |12\rangle\langle 12| + |21\rangle\langle 21| + |30\rangle\langle 30| = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0\end{bmatrix}.
\]

Matrices \(A_1, A_2, A_3, A_4\) are linearly independent. Furthermore \(\det(A_4 + a_1A_1 + a_2A_2 + a_3A_3) = 1,\) which is obtained by expanding this determinant by the rows \(4, 3, 2, 1.\)

Assume to the contrary that \(r(\mathcal{X}) = r < 7.\) As \(r_3(\mathcal{X}) = 4\) we have that \(r \geq 4.\) Let \(B_1, \ldots, B_r\) be \(r\) linearly independent rank-one matrices so that they span the subspace \(V \subset \mathbb{C}^{4 \times 4},\) which contains \(A_1, \ldots, A_4.\) As \(A_1, A_2\) and \(A_3\) are independent we have a basis in \(V\) consisting of \(A_1, A_2, A_3\) and \(C_1, \ldots, C_{r-3} \in \{B_1, \ldots, B_r\}.\) Express \(A_4\) in this basis to deduce that \(A_4 + \sum_{i=1}^{3} a_iA_i = \sum_{j=1}^{r-3} c_jC_j.\) That is \(r(A_4 + \sum_{i=1}^{3} a_iA_i) \leq r - 3 \leq 3.\) This contradicts the equality \(\det(A_4 + a_1A_1 + a_2A_2 + a_3A_3) = 1.\)

We now show that the rank of \(\mathcal{Y} = |W_3\rangle\langle W_3| \otimes^2 \in \otimes^6 \mathbb{C}^2\) is \(8.\) We first give a simple decomposition of \(\mathcal{Y}\) as a sum of \(8\) rank-1 tensors as in [40]. Recall that the generic rank of tensors in \(\otimes^3 \mathbb{C}^2\) is \(2.\) Hence most rank-1 perturbation of \(|W_3\rangle\langle W_3|\) have rank \(2.\) For example, for \(z = |0\rangle\) the two tensors \(|W_3\rangle + z \otimes^3\) and \(|W_3\rangle + \frac{1}{2} z \otimes^3\) have rank 2. Next observe that

\[
|W_3\rangle \otimes^2 = (|W_3\rangle \otimes^2 + z \otimes^3 \otimes z \otimes^3 - z \otimes^3 \otimes (|W_3\rangle + \frac{1}{2} z \otimes^3).\]

Use the inequality (70) for each tensor product appearing in the right-hand side of the above identity to deduce that \(r(|W_3\rangle \otimes^2) \leq 4 + 2 + 2 = 8.\)

We now outline briefly the main arguments in [43] to show that \(r(|W_3\rangle \otimes^2) \geq 8.\) Recall that \(r(|W_3\rangle \otimes^2) \geq r(\mathcal{X}) = 7.\) Assume to the contrary

\[
|W_3\rangle \otimes^2 = \sum_{i=1}^{7} \otimes_{j=1}^{6} a_{j,i}.
\]

We claim that for each \(i \in [7]\) either \(a_{1,i}, a_{2,i}, a_{3,i} \in \text{span}(|0\rangle\langle 0|)\) or \(a_{4,i}, a_{5,i}, a_{6,i} \in \text{span}(|0\rangle\langle 0|).\) Suppose the opposite case. Then we may assume that this dichotomy does not hold for \(i = 7.\) Since each copy of \(|W_3\rangle\langle W_3|\) is symmetric, by permuting the first and the last 3 components of \(|W_3\rangle \otimes^2\), we can assume that \(a_{1,7}\) and \(a_{6,7}\) are not in \(\text{span}(|0\rangle\langle 0|).\) Contract \(|W_3\rangle \otimes^2\)
with respect to the first coordinate using a vector $x$ orthogonal to $a_{1,7}$:

$$x \times |W_3|^{\otimes 2} = (x \times |W_3|) \otimes |W_3| = \sum_{i=1}^{6} (x^T a_{1,i}) \otimes a_{j,i}. \quad (74)$$

Observe next that since $x \neq c|1)$ it follows that the rank of the $2 \times 2$ matrix $x \times |W_3|$ is 2. Use (72) to deduce that $r((x \times |W_3|) \otimes |W_3|) = 6$.

The second part of Proposition 9 in [43] states that the following six 3-mode rank-1 tensors are linearly dependent:

$$a_{2,i} \otimes a_{3,i} \otimes a_{p,i}, \quad i \in [6] \text{ for } p \in \{4, 5, 6\}, \quad (75)$$

and the following six 3-mode rank-1 tensors are linearly independent:

$$a_{p,i} \otimes a_{q,i} \otimes a_{r,i}, \quad i \in [6] \text{ for } p \in \{2, 3\}, 4 \leq q < r \leq 6. \quad (76)$$

Next contract $|W_3|^{\otimes 2}$ on the last mode with respect to $y$ orthogonal to $a_{6,7}$ and use [43, Proposition 9] to deduce that the following six vectors are linearly independent: $a_{2,i} \otimes a_{3,i} \otimes a_{4,6,7}, i \in [6]$. This contradicts to the previous statement that these six tensors are linearly dependent.

Thus we showed that for each $i \in [7]$ either $a_{1,i}, a_{2,i}, a_{3,i} \in \text{span}(|0\rangle)$ or $a_{4,i}, a_{5,i}, a_{6,i} \in \text{span}(|0\rangle)$. We now contradict this statement. Assume first that $a_{1,i}, a_{2,i}, a_{3,i} \in \text{span}(|0\rangle)$ for each $i \in [7]$. Then $|W_3|^{\otimes 2} = |0\rangle \otimes Z$ for some $Z \in \otimes^3 \mathbb{C}^2$. Thus $r(Z) \leq 3$ and $r(|W_3|^{\otimes 2}) \leq 3$ which is impossible. Similarly, one cannot have $a_{4,i}, a_{5,i}, a_{6,i} \in \text{span}(|0\rangle)$ for $i \in [7]$. Hence $r(|W_3|^{\otimes 2}) \geq 8$. ■

We now discuss briefly the ranks of $|W_3|^{\otimes k} \in \otimes^k \mathbb{C}^8$ and $\otimes^k (|W_3|) \in \otimes^k \mathbb{C}^{2^k}$. It is shown in [39] that $r(\otimes^k |W_3|) \geq 2^k - 1$, similar to the arguments we gave for the case $k = 2$. Hence $r(|W_3|^{\otimes k}) \geq 2^k - 1$. In particular, $r(|W_3|^{\otimes 3}) \geq 15$. It is known [112, Theorem] that $r(\otimes^3 |W_3|) = 16$. Combine this result with [43] to obtain that $16 \leq r(|W_3|^{\otimes 3}) \leq 20$. In [37] it is shown that

$$r(|W_d|^{\otimes k}) \geq r(|W_3|^{\otimes k}) + (d - 3)(2^k - 1). \quad (77)$$

A real sequence $\{a_k\}, k \in \mathbb{N}$ is called subadditive if $a_{p+q} \leq a_p + a_q$ for every $p, q \in \mathbb{N}$. Fekete’s subadditive lemma claims that for any subadditive sequence $a_k$ with $k \in \mathbb{N}$, the modified sequence converges $\lim_{k \to \infty} \frac{a_k}{k} = a$ with $a \in [-\infty, \infty]$.

Let $T \in \mathbb{C}^n$. The inequality (10) yields that the two sequences $\log r(\otimes^k \mathbb{T})$ and $\log r(T^{\otimes k})$ are subadditive. Let

$$r_{lim_k} (T) = \lim_{k \to \infty} \left( r\left( \otimes^k \mathbb{T} \right) \right)^{\frac{1}{k}} \quad \text{and} \quad r_{lim} (T) = \lim_{k \to \infty} \left( r(T^{\otimes k}) \right)^{\frac{1}{k}}.$$  

By definition $r_{lim_k} (T) \leq r_{lim} (T)$, while Corollary 12 in [40] claims that $r_{lim} (T) \leq r_b (T)$. Hence, the above results for $|W_d|$ yield the equalities

$$r_{lim_k} (|W_d|) = r_{lim} (|W_d|) = 2. \quad (78)$$
6.5. **Computational methods for symmetric rank of symmetric tensors**

Recall that the symmetric rank of a symmetric tensor is the Waring rank of the corresponding homogeneous polynomial. Hence one can use theoretical methods of algebraic geometry and the available software as Bertini [52]. The Waring rank of a homogeneous polynomial of degree $d$ in two variables can be determined very efficiently using Sylvester’s algorithm [113]. The paper [114] discusses Sylvester’s algorithm in the modern language of algebraic geometry. The authors discuss also algorithms to find small border ranks of symmetric tensors. In [51], the authors provide methods of linear algebra to find the rank of symmetric tensors similar to the algorithm for determining the rank of a general tensor discussed in Section 5.5.

6.6. **Generic identifiability of symmetric tensors**

In this section, we discuss the identifiability property for symmetric tensors, which is similar to our discussion of the identifiability property for general tensors in Section 5.6. Assume that $d \geq 3$ and $n \geq 2$. Suppose that $r > 1$ is an integer, and there exists a tensor $S \in S^d \mathbb{C}^n$ of symmetric rank $r$. Denote

$$G_r : (\mathbb{C}^n)^r \rightarrow S^d \mathbb{C}^n, \quad G_r(x_1, \ldots, x_r) = \sum_{i=1}^{r} x_i^d. \quad (79)$$

A tensor $S$ is called a random tensor in $G_r(\mathbb{C}^n)^r$ if $S = G_r(x_1, \ldots, x_r)$, and the coordinates of vectors $x_1, \ldots, x_r$ are sampled from independent Gaussian complex valued distribution. We say that the identifiability property holds if for a random $S$ of the above form $r = r_{\text{gen}}(S)$, and the decomposition $S = G_r(x_1, \ldots, x_r)$ is unique up to a permutation of the summands $\otimes^d x_i$.

Recall the identity (65):

$$r_{\text{gen}}(d, n) \geq \left\lceil \frac{(n+d-1)}{d} \right\rceil n.$$

Observe that if

$$\left\lfloor \frac{(n+d-1)}{d} \right\rfloor$$

is not an integer then by counting the number of parameters we deduce that identifiability property fails for $r = r_{\text{gen}}(d, n)$. Similarly, the identifiability property fails for $r > r_{\text{gen}}(d, n)$. The fundamental result in [55] states that the identifiability property holds for $r < r_{\text{gen}}(d, n)$ except the following cases:

- $d = 6, \quad n = 3, \quad \text{and} \quad r = 9$;
- $d = 4, \quad n = 4, \quad \text{and} \quad r = 8$;
- $d = 3, \quad n = 6, \quad \text{and} \quad r = 9$.

In the above exceptional cases a generic tensor of the corresponding rank has two distinct Waring decompositions. We list additional references on the identifiability property: [115–117].
7. Nuclear rank of a tensor

Section 7 is mainly devoted to the notion of the nuclear rank of a tensor. In Section 7.1, we discuss the spectral norm and the geometric measure of entanglement. We point out the concentration law concerning the geometric measure of entanglement. It states that this measure of entanglement of a random symmetric quantum state generated with respect to the Haar measure is close to the maximal possible value. Section 7.2 introduces the nuclear norm and nuclear rank. The minimal nuclear decomposition of a tensor plays the role analogous to the singular value decomposition of a matrix. Hence, from the point of view of applications in quantum physics, the nuclear rank of a tensor seems to be the right analog of the rank of a matrix.

In Section 7.3, we discuss the faces of the unit ball with respect to the nuclear norm. Lemma 7.1 characterizes the exposed faces of such a unit ball. We consider also the restriction of the nuclear norm to symmetric tensors, which gives rise to the definition of a symmetric nuclear rank. Section 7.4 concerns the exposed faces and facets of unit balls with respect to matrix nuclear norm and spectral norm. Theorem 7.3 characterizes the exposed faces of these unit balls. In Section 7.5, we show that the nuclear rank of $|\text{GHZ}\rangle$ state is 2. Section 7.6 discusses the generic and maximum nuclear rank of symmetric states of a three-qubit system. Theorem 7.4 characterizes the face nuclear norm in $2 \times 2 \times 2$ symmetric tensors which contains the state $|\text{W}\rangle$.

7.1. Geometric measure of entanglement and spectral norm

Denote the space of all product states in $\mathbb{C}^n$ by

$$\Pi(n) = \left\{ \mathcal{P} \in \mathbb{C}^n, \mathcal{P} = \bigotimes_{j=1}^d x_j, x_j \in \mathbb{C}^{n_j}, j \in [d], \|\mathcal{P}\|_2 = 1 \right\}. \quad (80)$$

For any multipartite state $|\psi\rangle$ represented by a tensor $\mathcal{T} \in \mathbb{C}^n$ normalized by a fixed Hilbert–Schmidt norm, $\|\mathcal{T}\|_2 = 1$, its entanglement can be characterized by the Fubini–Study distance of $|\psi\rangle$ to the set of product states [50,57]. This quantity can be related to the spectral norm of $\mathcal{T}$,

$$\|\mathcal{T}\|_\infty = \max \{|\langle \mathcal{T}, \mathcal{P} \rangle|, \mathcal{P} \in \Pi(n)\}. \quad (81)$$

In analogy to the bipartite case, corresponding to matrices, one defines the geometric measure of entanglement of the state $\mathcal{T}$ by $\sqrt{2(1 - \|\mathcal{T}\|_\infty)}$ which corresponds to the minimal Hilbert–Schmidt distance between the projector $\rho_\psi = |\psi\rangle\langle \psi|$ and the projector on a separable state – see Section 2.2.

We now make a few comments on the spectral norm of $\mathcal{T}$. First, note that $\|\mathcal{T}\|_\infty = \max\{|\mathbb{R}(\langle \mathcal{T}, \mathcal{P} \rangle), \mathcal{P} \in \Pi(n)\}$). Next, observe that for $d = 2$, i.e. matrices, $\|\mathcal{T}\|_\infty$ is the leading singular value $\sigma_{\text{max}}$ of the matrix $\mathcal{T}$, which is also the spectral norm of $\mathcal{T}$, viewed as a linear transformation from $\mathbb{C}^{n_2}$ to $\mathbb{C}^{n_1}$. Note that $\|\mathcal{T}\|_\infty$ sometimes denotes the operator norm of a matrix $\mathcal{T}$, where $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ are endowed with $\infty$-norms, which is different from $\sigma_1(\mathcal{T})$.

Assume that $\|\mathcal{T}\|_\infty = \mathbb{R}(\langle \mathcal{T}, \mathcal{P} \rangle)$ for some $\mathcal{P} \in \Pi(n)$. Then $\|\mathcal{T}\|_\infty \mathcal{P}$ is the best rank-1 approximation of $\mathcal{T}$. If $\|\mathcal{T} - \|\mathcal{T}\|_\infty \mathcal{P}\| \leq \|\mathcal{T} - \mathcal{X}\|$, where $\mathcal{X}$ is rank-1 tensor [27]. For a
matrix $A \neq 0$ a best rank-1 approximation $\|A\|_\infty \mathcal{P}$ is the term $\sigma_1 u_1 v_1^*$ in the SVD decomposition (7) [118] or [63, Corollary 4.13.2]. Furthermore, $r(A - \|A\|_\infty \mathcal{P}) = r(A) - 1$. This equality is not true for tensors with $d \geq 3$ indices [118].

Assume that $T$ has real entries. Then we can define the real spectral norm as $\|T\|_{\infty,\mathbb{R}} = \max\{\|\langle T, \mathcal{P} \rangle\|, \mathcal{P} \in \Pi(n) \cap \mathbb{R}^n\}$. By definition, the following inequality holds, $\|T\|_{\infty,\mathbb{R}} \leq \|T\|_\infty$ which is saturated for bipartite states, $d = 2$, represented by matrices $T$. However, for $d \geq 3$ one can have a strict inequality already for the space of 3-qubits [48]. That is, the closest product state to a real state may be complex-valued. The computation of spectral norms of $\|T\|_{\infty,\mathbb{R}}$ and $\|T\|_\infty$ for $d \geq 3$ is NP-hard [22,48].

In the case of a bipartite state represented by a matrix with the Hilbert–Schmidt norm fixed, the smaller spectral norm, the larger quantum entanglement – see Section 2.2. The physical interpretation is that most of the entanglement given by the generic rank of a tensor. Recall (55) and the result by Alexander–Hirschowitz [46],

$$r_{\text{gen}}(n^{\times d}) \geq \frac{n^d}{d(n - 1) + 1} > \frac{n^{d-1}}{d},$$

(85)
Thus $r_{\text{gen}}$ has an exponential growth in $d$ in contrast to the polynomial growth of $r_{\text{gen}}(d, n)$. This fact can be explained by observing that the dimension of $\otimes^d \mathbb{C}^n$ is exponential in $d$, while the dimension of $S^d \mathbb{C}^n$ is polynomial in $d$.

It is shown in [124] that

$$0 \leq \eta(S) \leq \log_2 \left(\frac{n + d - 1}{d}\right) = \log_2 \left(\frac{n + d - 1}{n - 1}\right).$$

Thus, for $n = 2$ we have that $\eta(S) \leq \log_2(d + 1)$. There is still the concentration law which shows that most of symmetric tensor for fixed $n$ and $d \gg 1$ concentrate at the upper bound given above [124]. In particular, for symmetric $d$-qubits one has the inequality:

$$P\left[\eta(T) \geq \log_2 d - \log_2(\log_2 d) - 3\right] \geq 1 - \frac{1}{2d^{5/2}}, \quad \text{for } d \geq 42. \quad (88)$$

Above results show that for a fixed $n \geq 2$ and $d \gg 1$ a symmetric state is typically much less entangled with respect to the geometric measure of entanglement than a generic states of the same dimension.

The computation of the spectral norm of $S \in S^d \mathbb{C}^n$ is NP-hard in $n$ for $d = 3$ [44]. However, for a fixed $S$, the computation of $\|S\|_{\infty}$ is polynomial in $d$ [44]. This result is obtained by showing that the computation of $\|S\|_{\infty}$ can be done by solving polynomial equations for the critical points of the function $\Re\langle S, \otimes^d x\rangle$ restricted to the unit sphere $\|x\| = 1$.

### 7.2. Nuclear norm and nuclear rank

Denote by the nuclear norm $\| \cdot \|_1$ the dual norm to the spectral one on $\mathbb{C}^n$. From the definition of the spectral norm it follows that the unit ball of the nuclear norm is the convex hull of $\Pi(n)$. As each $P \in \Pi(n)$ is the extreme point on the unit sphere of the Hilbert–Schmidt norm, we deduce that each $P$ is an extreme point on the unit sphere of the nuclear form. One can show that the nuclear norm has the following minimum characterization [48]:

$$\|T\|_1 = \min \left\{ \sum_{i=1}^r \prod_{j=1}^d \|x_{ij}\|, \ T = \sum_{i=1}^r \otimes_{j=1}^d x_{ij} \right\}. \quad (89)$$

Viewing $\sum_{i=1}^r \prod_{j=1}^d \|x_{ij}\|$ as energy of the expression of $T = \sum_{i=1}^r \otimes_{j=1}^d x_{ij}$, then $\|T\|_1$ is the minimal energy to decompose the tensor $T$ into a sum of rank-one tensors.

It is well known that for $d = 2$, the nuclear norm reduces to the trace norm, $\|T\|_1 = \text{Tr} \sqrt{TT^*}$, which is equal to the sum of singular values of the matrix $T \in \mathbb{C}^{n_1 \times n_2}$ [48]. For $d \geq 3$, the computation of the nuclear norm is NP-hard, since the computation of the (dual) spectral norm is NP-hard [48]. An interesting formula for the nuclear norms of special type tensors is given in [125, Theorem 3].
One can find numerically $\|T\|_1$ for $T \in \mathbb{C}^2 \times \mathbb{C}^m \times \mathbb{C}^n$ as follows: the two first mode sections of $T$ are $T_1, T_2 \in \mathbb{C}^{m \times n}$. Let $x = (x_1, x_2) \in \mathbb{C}^2$ be a vector of length 1: $|x_1|^2 + |x_2|^2 = 1$. Then $x \times T = T(x) = x_1 T_1 + x_2 T_2$ and $\|T\|_\infty = \max\{\|T(x)\|_\infty, \|x\| = 1\}$. Note that $T(x)$ is a matrix, so we can use software to find the singular value of $T(x)$. Due to numerical errors one needs to find all $x$ where $\|T(x)\|_1$ is a local maximum for $x$ of norm 1.

The minimal decomposition of $T$ with respect to the nuclear norm reads:

$$T = \sum_{i=1}^{r} \bigotimes_{j=1}^{d} x_{ij}, \quad \|T\|_1 = \sum_{i=1}^{r} \prod_{j=1}^{d} \|x_{ij}\|. \quad (90)$$

The nuclear rank of $T \neq 0$, denoted as $r^\text{nucl}(T)$, is the minimal $r$ in the above minimal decomposition. It is assumed that $r^\text{nucl}(0) = 0$. By definition one has $r(T) \leq r^\text{nucl}(T)$, hence $r^\text{nucl}(T)$ can be interpreted as yet another measure of the entanglement of any $d$-partite quantum pure state represented by tensor $T$. In the particular case $d = 2$, corresponding to bipartite systems, one arrives at the standard matrix rank, $r^\text{nucl}(T) = r(T)$.

Thus we can discuss similar notions for nuclear rank as for the regular rank:

1. What is the value of the maximum nuclear rank, denoted as $r^\text{nucl}_{\max}(n)$, and a good upper bound on its value?
2. What is a generic nuclear rank, denoted $r^\text{nucl}_{\text{gen}}(n)$ and what is its value?
3. Does the border rank notion exist for nuclear norm?
4. Are there efficient algorithms to compute the nuclear rank?

We now discuss some answers to these problems. In order to do this we need to recall some notions of convex sets in $\mathbb{R}^N$.

### 7.3. Faces of unit balls in $\mathbb{C}^n$

We now recall several standard notions of convex sets applied to a unit ball of any complex norm $\nu : \mathbb{C}^n \to [0, \infty) : B_\nu = \{T \in \mathbb{C}^n, \nu(T) \leq 1\}$. It is convenient to view $\mathbb{C}^n$ as a real space $\mathbb{R}^n \times \mathbb{R}^n$ of dimension $2N(n)$. That is $T = (\Re T, \Im T)$. Then, any real functional $\phi : \mathbb{C}^n \to \mathbb{R}$ is induced by $\mathcal{X} \subset \mathbb{C}^n : \phi(T) = \Re \langle T, \mathcal{X} \rangle$. We denote this linear functional by $\phi_{\mathcal{X}}$. For $\mathcal{X}, \mathcal{Y} \in B_\nu$, the set $[\mathcal{X}, \mathcal{Y}] = \{t \mathcal{X} + (1 - t) \mathcal{Y}, t \in [0, 1]\}$ is called a closed interval in $B_\nu$. A closed convex subset $F \subset B_\nu$ is called a face if any open interval, $(\mathcal{X}, \mathcal{Y}) = \{t \mathcal{X} + (1 - t) \mathcal{Y}, t \in (0, 1)\}$, that lies in $B_\nu$, and intersects $F$ lies completely in $F$. We denote that $F$ is a face of $B_\nu$ by $F \prec B_\nu$. Note that $\emptyset$ and $B_\nu$ are faces of $B_\nu$. Other faces of $B_\nu$ are called proper faces. A proper face $F$ lies on the boundary of $B_\nu$, the unit sphere with respect to the nuclear norm $S_\nu = \{T \in \mathbb{C}^n, \nu(T) = 1\}$. For example, any extreme point of $B_\nu$ is a zero dimensional face. A dimension of a given convex set $C \subset \mathbb{R}^N$, is the dimension of the linear subspace spanned by affine combinations of the elements in $C$. As $B_\nu$ is a norm ball, for each tensor $T \in S_\nu$, one has a supporting hyperplane at $T$. This supporting hyperplane can be neatly given by the dual norm $\nu^\vee(T) = \max \Re \langle (T, \mathcal{X}) \rangle$, $\mathcal{X} \in B_\nu$. Then for a given $T \in S_\nu$, each supporting hyperplane of $B_\nu$ at $T$ is $\phi_{\mathcal{X}}$ such that $\Re \langle (T, \mathcal{X}) \rangle = \nu^\vee(\mathcal{X})$. 

A proper face $F \triangleleft B_\nu$ is called an exposed face if it is an intersection of $B_\nu$ with a supporting hyperplane. That is, each $\mathcal{X} \in \mathbb{C}^n \setminus \{0\}$ induces an exposed face

$$F(\mathcal{X}) = \{ \mathcal{Y} \in B_\nu, \; \nu^\vee(\mathcal{X}) = \mathcal{N}(\mathcal{Y}, \mathcal{X}) \}. \tag{91}$$

It is known that there exist compact closed convex sets which have nonexposed faces. For example, take the standard real Hilbert norm in $B_\parallel \| \cdot \| \subset \mathbb{R}^N$, and a point $x \in \mathbb{R}^N$ outside this ball. Now take the Minkowski sum of $B_\parallel \| \cdot \|$ and the interval $[-x, x]$. Then there exist extreme points of this balanced convex set, corresponding to the norm $\nu$, which are not exposed. (In $\mathbb{R}^2$ there are 4 nonexposed extreme points.)

A facet of $B_\nu$ is a maximal set-theoretic proper face of $B_\nu$. By separation, every face is contained in an exposed face and thus facets are automatically exposed [126].

Let $B_1(n) \subset \mathbb{C}^n$ be the unit ball of the nuclear norm, which is the convex set spanned by $/\Pi_1(n)$. Since $/\Pi_1(n)$ is closed if follows from Carathéodory’s theorem that this convex set is closed.) Denote by $U(n) \subset \mathbb{C}^{n \times n}$ the unitary group acting on $\mathbb{C}^n$. Let $U(n)$ be the product group $U(n_1) \times \cdots \times U(n_d)$ which acts on $\mathbb{C}^n$. First observe that $/\Pi_1(n)$ is the orbit of one product state $\mathcal{P} \in /\Pi_1(n)$ under the action of $U(n)$: $/\Pi_1(n) = U(n) \mathcal{P}$. Hence $B_1(n)$ is an orbitope [126]. Since the nuclear norm is the dual norm of the spectral norm it follows that

$$\| \mathcal{X} \|_\infty = \max \{ \mathcal{N}(\mathcal{X}, \mathcal{Y}), \mathcal{Y} \in B_1(n) \}, \; \forall \mathcal{X} \in \mathbb{C}^n.$$ 

Thus we obtain the description of exposed faces of $B_1(n)$:

**Lemma 7.1:** Fix a state $\mathcal{X} \in \mathbb{C}^n$. Let

$$\Pi(\mathcal{X}) = \{ \mathcal{P} \in \Pi(n), \mathcal{N}(\mathcal{X}, \mathcal{P}) = \| \mathcal{X} \|_\infty \}. \tag{92}$$

Then $\Pi(\mathcal{X})$ is a closed set, and its convex hull is the exposed face $F(\mathcal{X})$ given by (91). Vice versa, every exposed face of $B_1(n)$ is of the form $F(\mathcal{X})$.

**Proof:** Every exposed face is of the form $F(\mathcal{X})$. Without loss of generality, we can assume that $\mathcal{X}$ is a state. Assume now that $\mathcal{X}$ is a state and consider $F(\mathcal{X})$. Since the linear functional $\xi(T) = \mathcal{N}(T, \mathcal{X})$ is a supporting hyperplane of $B_1(n)$ it follows that $F(\mathcal{X})$ is a facet. Let $\Pi(\mathcal{X})$ be defined as above. Then $\Pi(\mathcal{X})$ is a closed subset of $\Pi(n)$. Assume that $\mathcal{Y}$ is in a convex hull of $\Pi(\mathcal{X})$:

$$\mathcal{Y} = \sum_{i=1}^r \alpha_i \mathcal{P}_i, \; \mathcal{P}_i \in \Pi(\mathcal{X}), \; \alpha_i > 0, \; \sum_{i=1}^r \alpha_i = 1.$$ 

Then $\| \mathcal{Y} \|_1 \leq \sum_{i=1}^r \alpha_i \| \mathcal{P}_i \|_1 = \sum_{i=1}^r \alpha_i = 1$. Furthermore

$$\mathcal{N}(\mathcal{X}, \mathcal{Y}) = \sum_{i=1}^r \alpha_i \mathcal{P}_i \mathcal{X} = \| \mathcal{X} \|_1.$$ 

Thus $\mathcal{Y} \in F(\mathcal{X})$. 

Assume that $Y \in F(\mathcal{X})$. As $Y \in B_1(n)$, $Y$ is a convex combination of the extreme points of $B_1(n)$:

$$Y = \sum_{i=1}^{r} \alpha_i \mathcal{P}_i, \quad \mathcal{P}_i \in \Pi(n), \quad \alpha_i > 0, \quad \sum_{i=1}^{r} \alpha_i = 1.$$ 

Hence $\Re \langle \mathcal{X}, Y \rangle = \sum_{i=1}^{r} \alpha_i \Re \langle \mathcal{X}, \mathcal{P}_i \rangle \leq \| \mathcal{X} \|_1$. Since $Y \in F(\mathcal{X})$ it follows that $\mathcal{P}_i \in \Pi(\mathcal{X})$ for $i \in [r]$. 

As in [20, Proposition 4.3], one can generalize Lemma 7.1 to an exposed face of $B_{\nu}$, where $\Pi(n)$ is replaced by the set of the extreme points of $B_{\nu}$. The following corollary of Lemma 7.1 is given by [48, Lemma 4.1]:

**Corollary 7.2:** Let $T \in \mathbb{C}^n \setminus \{0\}$ and assume that $T = \sum_{i=1}^{r} \bigotimes_{j=1}^{d} x_{ij}$, where $\bigotimes_{j=1}^{d} x_{ij} \neq 0$ for $i \in [r]$. Then $\| T \|_1 \leq \sum_{i=1}^{r} \prod_{j=1}^{d} \| x_{ij} \|$. Equality holds if and only if there exists $B \in \mathbb{C}^n \setminus \{0\}$ such that $\Re \langle B, \bigotimes_{j=1}^{d} x_{ij} \rangle = \| B \|_1 \prod_{j=1}^{d} \| x_{ij} \|$ for $i \in [r]$.

It is plausible to assume that the generic nuclear rank corresponds to a generic facet of $B_1(n)$. More precisely, $r_{\text{gen}}^\nucl(n)$ is 1 plus the dimension of the generic facet of $B_1(n)$. By definition we know that $r_{\text{gen}}^\nucl(n) \geq r_{\text{gen}}(n)$. Caratheodory’s theorem implies that $r_{\text{max}}^\nucl(n)$ is at most 1 plus the dimension of the facet of $B_1(n)$ with maximum dimension. This implies that $r_{\text{max}}^\nucl(n) \geq r_{\text{max}}(n)$.

To find the generic nuclear rank one can do as follows: Choose at random state $T \in \mathbb{C}^n$. Then $Y = \frac{1}{\| T \|_1} T$ will be an interior point of a generic facet $F$ of $B_1(n)$. Let $r$ be the number of rank-1 components in a numerical minimal decomposition of $T$ as in Corollary 7.2. Then $r$ is the value of $r_{\text{gen}}^\nucl(n)$. One can find numerically the nuclear norm of $T$ using an algorithm suggested in [47].

One of the main advantages of the nuclear rank of a tensor is that it behaves as the rank of a matrix, in the sense that, the nuclear rank is a lower semicontinuous function [48]. Hence in the case of the nuclear rank there is no need to introduce its border rank.

Consider the subspace of symmetric tensors $S^d \mathbb{C}^n \subset \mathbb{C}^{n \times d}$. Then the dual version of the theorem of Banach [122] claims [48]:

$$\| S \|_1 = \min \left\{ \sum_{i=1}^{r} \| x_i \|^{d}, \quad S = \sum_{i=1}^{r} x_i \otimes^{d} \right\}, \quad S \in S^d \mathbb{C}^n. \tag{93}$$

The minimal $r$ in the above minimal decomposition of $S \in S^d \mathbb{C}^n$ is called the symmetric nuclear rank and is denoted as $r_{\text{sym}}^\nucl(S)$. Observe that $r_{\text{sym}}^\nucl(S) \geq r^\nucl(S)$, since in the definition of the former quantity we restrict the decomposition of $S$ to any combination of symmetric tensors of rank 1.

Denote $B_{1,s}(n^{\times d}) = B_1(n^{\times d}) \cap S^d \mathbb{C}^n$. Then $B_{1,s}(n^{\times d})$ represents the unit ball in sense of the nuclear norm restricted to symmetric tensors. The above characterization of $\| S \|_1$ yields that the extreme points of $B_{1,s}(n^{\times d})$ are $\Pi_{\text{s}}(n^{\times d}) = \Pi(n^{\times d}) \cap S^d \mathbb{C}^n$.

To have a better understanding of how generic and maximum nuclear ranks are related to facets of unit balls we discuss the matrix case.
7.4. Exposed faces and facets of matrix nuclear and spectral norms

In this section, we consider the case of \( m \times n \) matrices, where \( 2 \leq m \leq n \). Recall that the inner product in \( \mathbb{C}^{m \times n} \) is \( \text{Tr} AB^* \), for \( A, B \in \mathbb{C}^{m \times n} \). For \( A \in \mathbb{C}^{m \times n} \setminus \{0\} \) the singular value decomposition reads

\[
A = \sum_{i=1}^{r} \sigma_i(A)u_i v_i^*, \quad r = r(A), \quad \sigma_1(A) \geq \cdots \geq \sigma_r(A) > 0 = \sigma_{r+1}(A) = \cdots,
\]

\[
A v_i = \sigma_i(A) u_i, \quad A^* u_i = \sigma_i(A) v_i, \quad u_i \in \mathbb{C}^m, \quad v_i \in \mathbb{C}^n, \quad u_i^* u_j = v_i^* v_j = \delta_{ij}, i, j \in [r].
\]

The vectors \( u_i, v_i \) are called the left and right singular vectors of \( A \) corresponding to the singular value \( \sigma_i(A) \). Furthermore, \( \|A\|_\infty = \sigma_1(A) \) and \( \|A\|_1 = \sum_{i=1}^{m} \sigma_i(A) \).

The following results are likely to be known, but we prove them for completeness:

**Theorem 7.3:** Assume that \( 2 \leq m \leq n \). Denote by \( B_\infty(m, n), B_1(m, n) \subset \mathbb{C}^{m \times n} \) the unit balls with respect to the spectral and nuclear norm respectively. Then

1. Every exposed face of \( B_1(m, n) \) has dimension \( k^2 - 1 \), for \( k \in [m] \). It is given by \( \mathbf{X} \in \mathbb{C}^{m \times n} \) normalized by the condition \( 1 = \sigma_1(\mathbf{X}) = \cdots = \sigma_k(\mathbf{X}) > \sigma_{k+1}(\mathbf{X}) \). Let \( u_1, \ldots, u_k \in \mathbb{C}^m \) and \( v_1, \ldots, v_k \in \mathbb{C}^n \) be two orthonormal systems corresponding to the left and the right singular eigenvectors corresponding to the singular value \( 1 \) of \( \mathbf{X} \). Then the face \( \mathbf{F}(\mathbf{X}) \) is a convex combination of rank-1 matrices of the form \( uv^* \), where \( u \) is a unit vector in \( \text{span}(u_1, \ldots, u_k) \) and \( v = X^* u \). For \( k = 1 \), \( \mathbf{F}(\mathbf{X}) = \{ u_1 v_1^* \} \) is an extreme point of \( B_1(m, n) \). The face \( \mathbf{F}(\mathbf{X}) \) is a facet if and only if \( k = m \).

2. An exposed face of \( \mathbf{F} \subset B_\infty(m, n) \) is of dimension \( 2(m - k)(n - k) \) for \( k \in [m] \). It is of the following form: Fix orthonormal systems \( u_1, \ldots, u_k \in \mathbb{C}^m \) and \( v_1, \ldots, v_k \in \mathbb{C}^n \). Then

\[
\mathbf{F} = \{ X \in \mathbb{C}^{m \times n}, \|X\|_\infty = 1, X v_i = u_i^* v_i = u_i, i \in [k] \}.
\]

\( \mathbf{F} \) is a facet if and only if \( k = 1 \) and \( \mathbf{F} \) contains an extreme point if and only if \( k = m \). Every extreme point \( \mathbf{X} \in B_\infty(n) \) is an exposed face and is satisfies \( XX^* = I_m \).

**Proof:** (1) Let us consider an exposed face of \( B_1(m, n) \). By Lemma 7.1 it is of the form \( \mathbf{F}(\mathbf{X}) = \{ A \in \mathbb{C}^{m \times n}, \|A\|_1 = 1, \text{Tr} AX^* = \sigma_1(\mathbf{X}) \} \) for some \( \mathbf{X} \in \mathbb{C}^{m \times n} \setminus \{0\} \). Without loss of generality, we can assume that \( \sigma_1(\mathbf{X}) = 1 \). Suppose that \( 1 = \sigma_1(\mathbf{X}) = \cdots = \sigma_k(\mathbf{X}) > \sigma_{k+1}(\mathbf{X}) \). Let \( u_1, \ldots, u_k \in \mathbb{C}^m \) and \( v_1, \ldots, v_k \in \mathbb{C}^n \) are two sets of orthonormal left and right singular vectors of \( \mathbf{X} \) corresponding to the first singular value of \( \mathbf{X} \). It is straightforward to show [63] that \( \forall \mathbf{P} \ni \text{Tr} \mathbf{P}^* = \sigma_1(\mathbf{X}) \) for \( \mathbf{P} = uv^* \in \Pi(m, n) \) if and only if \( u \) is a unit vector in \( \text{span}(u_1, \ldots, u_k) \) and \( v = X^* u \). Suppose that \( Z \in \mathbf{F}(\mathbf{X}) \). Then the singular value decomposition of \( Z \) is \( Z = \sum_{j=1}^{r} \sigma_j(Z)x_j y_j^* \), where \( r = r(Z) \). Recall that \( \|Z\|_1 = \sum_{j=1}^{r} \sigma_j(Z) = 1 \). Lemma 7.1 yields that \( x_i \in \mathbb{C}^m \) has unit length, \( x_i \in \text{span}(u_1, \ldots, u_k) \) and \( y_i = X^* x_i \) for \( i \in [r] \). That is \( \mathbf{F}(\mathbf{X}) \) is a convex hull of \( uv^* \), where \( u \) of length one is in \( \text{span}(u_1, \ldots, u_k) \) and \( v = X^* u \). We claim that the dimension of this face is \( k^2 - 1 \). Indeed, without loss of generality we may assume that \( m = n = k \) and \( \mathbf{X} = I_k \). Then the face corresponds to all density matrices \( \rho \) of order \( k \), which are hermitian, positive semidefinite, \( \rho = \rho^* \geq 0 \), and normalized, \( \text{Tr} \rho = 1 \). The real dimension of this convex set is the real dimension of all \( k \times k \) hermitian matrices of trace 1, which is \( k^2 - 1 \).
The face $F(X)$ is maximally exposed if and only if $k = m$, i.e. $XX^* = I_m$. Indeed, if $k < m$ then extend the orthonormal systems $\{u_1, \ldots, u_k\}$, $\{v_1, \ldots, v_k\}$ to an orthonormal system $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_m\}$ and set $C = \sum_{i=1}^m u_i v_i^\dagger$. It now follows that $F(X) \not\subseteq F(C)$.

(2) Recall that an exposed face of $B_\infty(n)$ is

$$F(Y) = \{X \in S_\infty(n), \text{Tr} XY^* = \|Y\|_1\}.$$ 

Without loss of generality we can assume that $\|Y\|_1 = 1$. Assume that the SVD decomposition reads $Y = \sum_{i=1}^r \sigma_i(Y) u_i v_i^\dagger$, where $r = r(Y)$, $\sigma_1(Y) \geq \cdots \geq \sigma_r(Y) > 0$ and $\sum_{i=1}^r \sigma_i(Y) = 1$. Assume that $X \in S_\infty(n)$. Then $\text{Tr} X uv^\dagger \leq \|X\|_\infty = 1$. Equality holds if and only if $Xv_i = u_i$ and $X^* u_i = v_i$ for $i \in [r]$. Thus $F(Y)$ consists of all $X \in B_\infty(n)$ satisfying $Xv_i = u_i$ and $X^* u_i = v_i$ for $i \in [r]$. In particular, $\sigma_1(X) = \cdots = \sigma_r(X) = 1 \geq \sigma_{r+1}(X)$. By choosing an orthonormal bases $u_1, \ldots, u_m \in \mathbb{C}^m$ and $v_1, \ldots, v_n \in \mathbb{C}^n$ we see that $X$ is a direct sum $I_k \oplus X'$, where $X' \in \mathbb{C}^{(m-k) \times (n-k)}$ and $\|X'\|_{\infty} \leq 1$. Hence the real dimension of the face $F(Y)$ is $(m-k)(n-k)$.

Observe first that $F(Y)$ is a facet if $Y$ is a rank-1 matrix. In this case, the dimension of the facet is $2(m-1)(n-1)$. The face $F(Y)$ is zero dimensional if and only if $r(Y) = m$. In this case $F(Y) = \{X\}$, where $X = \sum_{i=1}^m u_i v_i^\dagger$. Thus $X$ is an extreme point of $B_\infty$. It is left to show that every extreme point $X$ of $B_\infty$ is of this form. Let $X \in B_1(n)$ and consider the full SVD decomposition of $X = \sum_{i=1}^m \sigma_i(X) u_i v_i^\dagger$, where $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ are orthonormal vectors. Furthermore, $1 = \sigma_1(X) \geq \cdots \geq \sigma_m(X) \geq 0$. Assume to the contrary that $1 > \sigma_m(X)$. Choose $\varepsilon > 0$ such that $1 > \sigma_m(X) + \varepsilon$. Then

$$X(\pm \varepsilon) = (\sigma_m(X) \pm \varepsilon) u_m v_m^\dagger + \sum_{i=1}^{m-1} \sigma_i(X) u_i v_i^\dagger \in B_1(n).$$

Since $X = \frac{1}{2}(X(\varepsilon) + X(-\varepsilon))$ it is not an extreme point.

**7.5. The nuclear rank of the $|GHZ\rangle$ state**

We now discuss the nuclear rank of 3-qubits. First observe that the state $|GHZ\rangle$ has nuclear rank 2. Indeed, up to a normalization constant we have

$$|GHZ\rangle = e_1^{\otimes 3} + e_2^{\otimes 3} = e_1 \otimes e_1^{\otimes 2} + e_2 \otimes e_2^{\otimes 2}.$$ 

(94)

Written in physics notation, $|GHZ\rangle = |000\rangle + |111\rangle$, this state has a two-term representation, so its rank is not more than two, $\| |GHZ\rangle \|_1 \leq 2$.

On the other hand, when we unfold $|GHZ\rangle$ in mode 1 we obtain $T \in \mathbb{C}^{2 \times 4}$, for which $\|T\|_1 = 2$. Hence the above decomposition of $|GHZ\rangle$ is a minimal decomposition. As $r(|GHZ\rangle) = 2$ we deduce that $r^{\text{nucl}}(|GHZ\rangle) = 2$. Let us try to find the facet that contains $|GHZ\rangle$. We claim that we can choose the supporting plane $\xi(T) = \text{null}(|GHZ\rangle, T)$. Indeed, $|GHZ\rangle$ is a symmetric tensor corresponding to symmetric polynomial $x_1^3 + x_2^3$. Then $\max\{8|x_1|^2 + x_2^3, |x_1|^2 + |x_2|^2 = 1\}$ is achieved at the vectors $\xi e_1, \xi e_2$ where $\xi^3 = 1$. By Corollary 7.2, we see that $\frac{1}{2}(|GHZ\rangle) \in F(|GHZ\rangle)$. 


7.6. Generic and maximum nuclear rank of symmetric 3-qubit states

We now discuss in details the exposed facets and faces of the nuclear norm ball in $S^3\mathbb{C}^2$. Denote by $B_1(3, 2) \subset S^3\mathbb{C}^2$ the unit ball of the nuclear norm. Denote by $\partial B_1(3, 2)$ the boundary of $B_1(2, 3)$, which is the unit sphere of the nuclear ball. Recall that the complex dimension of $S^3\mathbb{C}^2$ is 4. Hence, its real dimension is 8. Any supporting hyperplane of $B_1(3, 2)$ is of the form $\{ S \in B_1(3, 2), \Re \langle S, T \rangle = ||T||_\infty \}$ for some $T \in S^3\mathbb{C}^2 \setminus \{0\}$. For simplicity of notation we will identify $B$ points that correspond to 2 points of maximum will yield a supporting hyperplane of $B$ of unity. Hence $\Re \langle S, T \rangle$ is the unit sphere of the nuclear ball. Recall that the complex space $\mathbb{C}^2$, quotient by the action of multiplication by third roots of unity. The supporting hyperplane is a corresponding nonzero homogeneous polynomial $(ax_1 + bx_2)^3$. Thus the nuclear rank of an extreme point of $B_1(3, 2)$ is 1, and the dimension of the exposed face is 0.

We next discuss exposed face of dimension 1. Let us first consider the example of the $|GHZ\rangle$ state. It corresponds to the polynomial $x_1^3 + x_2^3$, up to multiplication by a constant. Hence $|||GHZ\rangle||_\infty = 1$ and the maximum is achieved for the two extreme points of $B_1(3, 2)$: $e_1^{\otimes 3}, e_2^{\otimes 3}$, and any convex combination of these two points. Thus the exposed face corresponding to $|GHZ\rangle$ is a convex combination of two unit orthogonal vectors in $\mathbb{C}^2$. The variety of all these one-dimensional exposed faces is of real dimension 4 – three for $x$ of length one, and an extra dimension for an orthogonal vector $y$ of length one. It is not known to the authors whether there exist additional exposed faces of dimension 1.

To find the spectral norm of a tensor one may follow the approach which is described in [44]. For a generic tensor in $S^3\mathbb{C}^2$ one can find all critical points for $\max \{ \Re \langle S, u^{\otimes 3} \rangle, ||u|| = 1 \}$. It will have usually at most 5 critical $u$ viewed as a point on the Riemann sphere. Those points that correspond to 2 points of maximum will yield a supporting hyperplane of $B_1(3, 2)$ which supports an exposed face of dimension 1. There are exceptional cases that are discussed in [44].

Let us now consider the three-qubit state $|W_3\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. Recall that $|W_3\rangle$ is a symmetric tensor corresponding to the polynomial $f = \sqrt{3}x_1^2x_2$. Hence

$$|||W_3\rangle||_\infty = \max \{|f(x_1, x_2)| = \sqrt{3}|x_1|^2|x_2|, |x_1|^2 + |x_2|^2 = 1\} = \frac{2}{3}, \quad (95)$$

as $|x_1| = \frac{\sqrt{3}}{\sqrt{3}}, |x_2| = \frac{1}{\sqrt{3}}$. Recall that among all 3-qubit states $|W_3\rangle$ has the minimal spectral norm, i.e. it has the highest geometrical measure of entanglement [127].

In [48, §6], we gave a nuclear decomposition of $|W_3\rangle$ with four terms. We showed that $|||W_3\rangle||_1 = 3/2 = |||W_3\rangle||_\infty = \frac{3}{2}$. The last equality follows from [47, Theorem 2.2], as $|W_3\rangle$ has the minimal spectral norm. In particular, $|W_3\rangle$ has the maximum nuclear norm among the 3-qubit states.

The four rank-one symmetric tensors of norm one that appear in the nuclear decomposition of $|W_3\rangle$, given in [48, §6] are all rank-one symmetric states for which $|W_3\rangle$ achieves its spectral norms. Thus $|W_3\rangle \in F(|W_3\rangle)$. 


Let us now consider $F(\{|W_3\}|)$ in the ball of the nuclear norm in $S^3\mathbb{C}^2$. Extreme points are all tensors $\mathbf{u}^{\otimes 3}$, $\|\mathbf{u}\| = 1$, such that $\Re \langle |W_3\rangle, \mathbf{u}^{\otimes 3} \rangle = \| |W_3\| \|_{\infty}$. It is straightforward to show that $\mathbf{u} = (\xi \sqrt{\frac{2}{3}}, \zeta^2 \frac{1}{\sqrt{3}})^T$, where $|\xi| = 1$.

Indeed, each rank-one tensor $\mathbf{u}^{\otimes 3}$, $\mathbf{u} = (a, b)^T$, corresponds to the polynomial

$$(ax_1 + bx_2)^3 = a^3 x_1^3 + a^2 b(3x_1^2x_2) + ab^2(3x_1x_2^2) + b^3(x_2^3).$$

Hence in the basis $\{x_1^3, 3x_1^2x_2, 3x_1x_2^2, x_2^3\}$ of homogeneous polynomials of degree 3 in $x_1, x_2$ tensor $\mathbf{u}^{\otimes 3}$ is represented by a vector $(a^3, a^2b, ab^2, b^3)$. Assume that $a \neq 0$. Then this vector is $a^3(1, z, z^2, z^3)$ and $z = b/a$. For the extreme points of the exposed face corresponding to $F(\{|W_3\}|)$ one has $z = \eta \frac{1}{\sqrt{3}}$, where $\eta = \tilde{\xi}^3$. So $\eta$ has an arbitrary value on the unit circle in $\mathbb{C}$. In particular, if we choose four pairwise distinct points on the unit circle $\eta_1, \ldots, \eta_4$, the four points $(1, \eta_i, \eta_i^2, \eta_i^3), i \in [4]$ are linearly independent, as their determinant has Vandermonde form and does not vanish.

Since we consider the convex hull of the extreme points of $F(\{|W_3\}|)$, we need to know what is the real dimension of this convex set. We claim that the dimension is 4:

**Theorem 7.4:** Let

$$F(\{|W_3\}|) = \{S \in B_1(3, 2), \Re \langle S, |W_3\rangle \rangle = \| |W_3\| \|_{\infty} \}$$

be the face of the ball corresponding to the real functional $S \mapsto \Re \langle S, |W_3\rangle \rangle$. Then

1. The real dimension of $F(\{|W_3\}|)$ is 4.
2. The state $|W_3\rangle$ has the following nuclear decomposition:

$$|W_3\rangle = \frac{\sqrt{3}}{2} \left( \left( \sqrt{\frac{2}{3}} e_1 + \frac{1}{\sqrt{3}} e_2 \right)^{\otimes 3} + \left( \sqrt{\frac{2}{3}} \tilde{\xi} e_1 + \frac{1}{\sqrt{3}} \tilde{\xi}^2 e_2 \right)^{\otimes 3} + \left( \sqrt{\frac{2}{3}} \xi e_1 + \frac{1}{\sqrt{3}} \xi^2 e_2 \right)^{\otimes 3} \right), \quad \xi = e^{2\pi i/9}.$$

As the above nuclear decomposition is a sum of three symmetric tensors of rank-one, and rank $|W_3\rangle = 3$, we deduce: the nuclear rank of $|W_3\rangle$ is 3, which is equal to its rank.

3. The nuclear rank of any $S \in F(\{|W_3\}|)$ is at most 4.
4. The subgroup of $G$ of the two dimensional unitary group $U(2)$ that fixes either $|W_3\rangle$ or $F(|W_3\rangle)$ is one dimensional subgroup of the form

$$G = \left\{ U \in \mathbb{C}^{2 \times 2}, U = \begin{bmatrix} \xi & 0 \\ 0 & \tilde{\xi} \end{bmatrix}, \xi \in \mathbb{C}, |\xi| = 1 \right\}.$$

5. The semialgebraic set of faces of the form $UF(|W_3\rangle), U \in U(2)$ has dimension 7.

**Proof:** (1) Recall that our three symmetric tensors, which are the extreme points of $F(|W_3\rangle)$ correspond to linear forms $(ax_1 + bx_2)^3$, where $a = \xi s$ and $b = \tilde{\xi}^2 t$ where $s = \sqrt{2}/\sqrt{3}$ and $t = 1/\sqrt{3}$. By considering new variables $y_1 = sx_1$ and $y_2 = tx_2$, we have that
the cubic forms are \((\zeta y_1 + \bar{\zeta}^2 y_2)^3\). Hence the Veronese coordinates of these cubic forms are \((\zeta^3, 1, \bar{\zeta}^3, \bar{\zeta}^6)\). Letting \(\xi = \bar{\zeta}^3\) and rearranging the coordinates we have that the coordinates are \((1, \xi, \bar{\xi}, \xi^2)\), where \(|\xi| = 1\). Since we consider convex (or affine) combination we can drop the first coordinate. Thus we are looking at convex combinations of the vectors of \((\xi, \bar{\xi}, \xi^2)\), where \(\xi\) is on the unit circle in \(\mathbb{C}\). Let us try to find a basis in the space of all real linear combinations whose sum is 1. The vector \((-\xi, \bar{\xi}, (-\xi)^2) = (-\xi, -\bar{\xi}, \xi^2)\) is such a point. Thus

\[
\left((\xi, \bar{\xi}, \xi^2) + (-\xi, -\bar{\xi}, \xi^2)\right)/2 = (0, 0, \xi^2), \quad |\xi| = 1. \tag{96}
\]

The set of all convex combinations of unimodular points, \(|\eta| = 1\), forms the unit disk, \(|z| \leq 1\), of real dimension 2. Thus the convex combination of all vectors of the form \((0, 0, \xi^2)\) has a real dimension 2. Note that \((0, 0, -\xi^2)\) is also in the convex set. Hence

\[
\frac{1}{2}(\xi, \bar{\xi}, 0) = \left((\xi, \bar{\xi}, \xi^2) + (0, 0, -\xi^2)\right)/2. \tag{97}
\]

Next, observe that the convex hull of the above vectors is \(1/2(z, \bar{z}, 0)\) where \(|z| \leq 1\). Thus its real dimension is 2. Hence the convex combinations of vectors \(1/2(z, \bar{z}, 0)\) and \((0, 0, w)\), where \(z\) and \(w\) are in a unit disk in \(\mathbb{C}\) has real dimension 4. Therefore the real dimension of \(F(|W_3|)\) is 4.

(2) Straightforward.

(3) We claim that any point in \(F(|W_3|)\) is a convex combination of at most 4 extreme points. Note that any convex combination of the extreme points is of the form \((z, \bar{z}, w)\). Hence it is enough to consider the convex combinations of vectors \((\xi, \xi^2)\) for \(|\xi| = 1\). As the ambient subspace is 4-real dimensional, Caratheodory’s theorem shows that every point is a convex combination of at most 5 extreme points. Assume that \((z, w)\) is in this convex set and

\[
(z, w) = \sum_{i=1}^{5} a_i(\xi_i, \xi_i^2), \quad a_i > 0, \sum_{i=1}^{5} a_i = 1.
\]

We assume that \(\xi_i \neq \xi_j\) for \(i \neq j\). If the real span of \((\xi_1, \xi_1^2), \ldots, (\xi_5, \xi_5^2)\) is three-dimensional we are done. Then, there must be a real nonzero linear combination \(\sum_{i=1}^{5} b_i(\xi_i, \xi_i^2) = 0\) such that \(\sum_{i=1}^{5} b_i = 0\). By considering the linear combination \(\sum_{i=1}^{5} (a_i + tb_i)(\xi_i, \xi_i^2)\) we can choose a positive \(t > 0\) such that \(a_i + tb_i \geq 0\) for all \(i \in [5]\) such that at least one \(a_i + tb_i = 0\).

Thus we are left with the case that every minimal convex combination that contains \((w, z)\) has exactly five extreme points with positive coefficients and which have four linear independent extreme points. Assume for simplicity that these linearly independent (over real domain) elements are \((\xi_1, \xi_1^2), \ldots, (\xi_4, \xi_4^2)\). Then \((\xi_5, \xi_5^2) = \sum_{i=1}^{4} c_i(\xi_i, \xi_i^2)\). Suppose first that \(\sum_{i=1}^{4} c_i = 1\). Since \((\xi_5, \xi_5^2)\) is an extreme point we must have that \(c_i < 0\) for some \(i \in [4]\). Observe next that

\[
(z, w) = \sum_{i=1}^{5} a_i(\xi_i, \xi_i^2) = \sum_{i=1}^{4} (a_i + a_5c_i)(\xi_i, \xi_i^2).
\]
Let $a_i(t) = a_i + tc_i$ for $i \in [4]$ and $a_5(t) = a_5 - t$. Then

$$\sum_{i=1}^{5} a_i(t) = 1, \quad (z, w) = \sum_{i=1}^{5} a_i(t)(\xi_i, \xi_i^2).$$

Start to increase $t$ from 0 to $a_5$ until $a_i(t)$ is zero. In this case we obtained that $(z, w)$ is a convex combination of at most 4 extreme points in $F(|W_3|)$.

Thus it is left to discuss the case where $(\xi_1, \xi_1^2), \ldots, (\xi_4, \xi_4^2)$ are linearly independent and $(\xi_5, \xi_5^2)$ is not an affine combination of $(\xi_1, \xi_1^2), \ldots, (\xi_4, \xi_4^2)$. Note that the convex span of $(\xi_1, \xi_1^2), \ldots, (\xi_4, \xi_4^2)$ has real dimension 3. Furthermore, the convex span of $(\xi_1, \xi_1^2), \ldots, (\xi_5, \xi_5^2)$ has dimension 4.

Let us denote by $\Delta = \Delta(w, z)$ these 5 tuples of points $((\xi_{i_1}, \xi_{i_1}1^2), \ldots, (\xi_{i_5}, \xi_{i_5}^2))$, where $\{i_1, \ldots, i_5\} = [5]$. This is an open set in $C^5$, where $C$ is a complex closed curve, $C := \{(\zeta, \xi^2) \in C^2, |\zeta| = 1\}$. That is $\Delta \subset C^5$ is an open set, which is not $C^5$. Now consider a boundary point of $\Delta$ in $C^5$. This boundary point consists of 5 tuples $\eta = ((\eta_1, \eta_1^2), \ldots, (\eta_5, \eta_5^2)), |\eta_i| = 1, i \in [5]$. As this boundary point is a limit of points in $\Delta$ it follows that $\eta \in \Delta$. But then, either the five points $(\eta_i, \eta_i^2), i \in [5]$ span at most three-dimensional subspace, or there are four extreme points which are linearly independent and the fifth point is an affine combination of the remaining four. In these cases, we obtain that $(z, w)$ is a convex combination of 4 extreme points.

(4) We now do the dimension count: How big is the subgroup of $U(2)$ that fixes $F(|W_3|)$.

It is equivalent to the subgroup that fixes $|W_3|$. It has a real dimension at least 1: Assume that $UE_{e_1} = \zeta e_1, UE_{e_2} = \zeta^2 e_2$ for $|\zeta| = 1$, which implies $U^{\otimes 3}|W_3| = |W_3|$. So for sure we have a one-parameter group that fixes $F(|W_3|)$.

We now show that the above subgroup is the only subgroup that fixes $|W_3|$. Indeed, suppose that

$$S = U^{\otimes 3}|W_3|$$

$$= x_1 \otimes x_1 \otimes x_2 + x_1 \otimes x_2 \otimes x_1 + x_2 \otimes x_1 \otimes x_1, \quad x_i^*x_j = \delta_{ij}, 1, j \in [2],$$

where the two pairs of vectors $e_1, x_1$ and $e_2, x_2$ are linearly independent. Consider the rank-1 matrix $S \times \bar{x}_2 = x_1 \otimes x_1$, where the contraction is on the third mode. Assume that $S = |W_3|$. Since $e_1$ and $x_1$ are linearly independent it follows that $x_1^*e_1 \neq 0$. A straightforward computation implies that the rank of the matrix $|W_3| \times \bar{x}_2$ is 2. This contradicts the assumption that $S = |W_3|$. Hence the subgroup $G$ is of the form given above.

(5) Observe that $UF(|W_3|), U \in U(2)$ must be also an exposed face of dimension 4. As the subgroup of $U(2)$ that fixes $F(|W_3|)$ is one dimensional, it follows that the dimension of the union of all faces of the form $UF(|W_3|)$ is $3 + 4 = 7$. Recall that 7 is the dimension of $\partial B(3, 2)$.

Let us conclude the work with a short list of open questions:

1. Is the nuclear rank of $S \in F(|W_3|)$ at most 3?
2. Is $F(|W_3|)$ a face of maximum real dimension?
3. Is $UF(|W_3|), U \in U(2)$ the set of all faces of $B_1(3, 2)$?
(4) Is the generic face of $B_1(3,2)$ of dimension 3, such that the subgroup of $U(2)$ that fixes generic faces is a finite group? In such a case the semialgebraic set of $UF, U \in U(2)$ has dimension $3 + 4 = 7$.

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**References**

[1] Wiki. Matrix mechanics. Available from: https://en.wikipedia.org/wiki/Matrix_mechanics.

[2] von Neumann J. Wahrscheinlichkeitstheoretischer aufbau der quantenmechanik. Göttinger Nachrich. 1927;1:245–272.

[3] Landau L. Das Dämpfungsproblem in der wellenmechanik. Z Phys. 1927;45:430.

[4] Einstein A, Podolsky B, Rosen N. Can quantum-mechanical description of physical reality be considered complete?. Phys Rev. 1935;47:777–780.

[5] Dirac PAM. A new notation for quantum mechanics. Proc Cambridge Philos Soc. 1939;35:416–418.

[6] Ingarden RS. Quantum information theory. Rep Math Phys. 1976;10:43–72.

[7] Bennett C.H, Brassard G. Quantum cryptography: public key distribution and coin tossing. Theor Comput Sci. 2014;560:7–11.

[8] Gisin N, Thew R. Quantum communication. Nat Photon. 2007;1:165–171.

[9] Preskill J. Quantum computing in the NISQ era and beyond. Quantum. 2018;2:79.

[10] Kalai G. The quantum computer puzzle. Not AMS. 2016;63:508–516.

[11] Shor PW. Polynomial-Time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM J Comput. 1997;26:1484–1509.

[12] Bell JS. On the einstein-podolsky-rosen paradox. Physics. 1964;1:195–290.

[13] Schrödinger E. Discussion of probability relations between separated systems. Math Proc Cambridge Philos Soc. 1935;31:555–563.

[14] Schrödinger E. Probability relations between separated systems. Math Proc Cambridge Philos Soc. 1936;32:446–452.

[15] Brylinski J-L. Algebraic measures of entanglement. In: Brylinski RK and Chen G, editors. Mathematics of quantum computation. Boca Raton: Chapman & Hall/CRC; 2002.

[16] Horodecki R, Horodecki P, Horodecki M, et al. Quantum entanglement. Rev Mod Phys. 2009;81:865–942.

[17] Bengtsson I, Życzkowski K. Geometry of quantum states: an introduction to quantum entanglement. II ed. Cambridge (UK): Cambridge University Press; 2017.

[18] Walter M, Gross D, Eisert J. Multi-partite entanglement. preprint arXiv:1612.02437. 2017.

[19] Szalay S, Pfeffer M, Murg V, et al. Tensor product methods and entanglement optimization for ab initio quantum chemistry. Int J Quant Chem. 2015;115:1342–1391.
[20] Friedland S, Lim L-H. The computational complexity of duality. SIAM J Optim. 2016;26:2378–2393.
[21] Håstad J. Tensor rank is NP-complete. J Algorithms. 1990;11:644–654.
[22] Hillar CJ, Lim L-H. Most tensor problems are NP-hard. J ACM. 2013;60:Art. 45, 39 pp.
[23] Shitov Y. How hard is the tensor rank? arXiv:1611.01559. 2016.
[24] Schaefer M, Stefanović D. The complexity of tensor rank. Theory Comput Syst. 2018;62(5):1161–1174.
[25] Chiantini L, Ottaviani G, Vannieuwenhoven N. An algorithm for generic and low-rank specific identifiability of complex tensors. SIAM J Matrix Anal Appl. 2014;35(4):1265–1287.
[26] Chiantini L, Ottaviani G, Vannieuwenhoven N. Effective criteria for specific identifiability of tensors and forms. SIAM J Matrix Anal Appl. 2017;38(2):656–681.
[27] de Lathauwer L, de Moor B, Vandewalle J. On the best rank-1 and rank-(R_1,R_2,...,R_N) approximation of higher-order tensors. SIAM J Matrix Anal Appl. 2000;21:1324–1342.
[28] Nion D, De Lathauwer L. An enhanced line search scheme for complex-valued tensor decompositions. Signal Process. 2008;88:749–755.
[29] Hitchcock FL. The expression of a tensor or a polyadic as a sum of products. J Math Phys. 1927;6:164–189.
[30] Hitchcock FL. Multiple invariants and generalized rank of p-way matrix or tensor. J Math Phys. 1927;7:39–79.
[31] Sylvester JJ. On a remarkable discovery in the theory of canonical forms and of hyperdeterminants. Originally Philos Mag. 1851;II:265–283. in Paper 41 in Mathematical Papers, Vol. 1, Chelsea, New York, 1973. Originally published by Cambridge University Press in 1904.
[32] Landsberg JM. Tensors: geometry and applications. Providence (RI): American Mathematical Society; 2012.
[33] Nathanson MB. Additive number theory: the classical bases. Springer-Verlag; 1996. (Graduate texts in mathematics; vol. 164).
[34] Terracini A. Sulla rappresentazione delle forme quaternarie mediante somme di potenze di forme lineari. Ann Mat Pura Appl. 1915;24:1–10.
[35] Friedland S. On the generic and typical ranks of 3-tensors. Linear Algebra Appl. 2012;436:478–497.
[36] Abo H, Ottaviani G, Peterson C. Induction for secant varieties of Segre varieties. Trans Amer Math Soc. 2009;361:767–792.
[37] Chen L, Chitambar E, Duan R, et al. Tensor rank and stochastic entanglement catalysis for multipartite pure states. Phys Rev Lett. 2010;105:200501.
[38] Chitambar E, Duan R, Shi Y. Tripartite entanglement transformations and tensor rank. Phys Rev Lett. 2008;101:140502.
[39] Yu Y, Chitambar E, Guo C, et al. Tensor rank of the tripartite state |W⟩⊗n. Phys Rev A. 2010;81:014301.
[40] Christandl M, Jensen AK, Zuiddam J. Tensor rank is not multiplicative under the tensor product. Linear Algebra Appl. 2018;543:125–139.
[41] Strassen V. Vermeidung von divisionen. J Reine Angew Math. 1973;264:184–202.
[42] Shitov Y. A counterexample to Comon’s conjecture. SIAM J Appl Algebra Geom. 2017;2:428–443.
[43] Chen L, Friedland S. The tensor rank of tensor product of two three-qubit W states is eight. Linear Algebra Appl. 2018;543:1–16.
[44] Friedland S, Wang L. Spectral norm of a symmetric tensor and its computation. Math Comput. 2020;89:2175–2215.
[45] Comon P, Golub G, Lim L-H, et al. Symmetric tensors and symmetric tensor rank. SIAM J Matrix Anal Appl. 2008;30:1254–1279.
[46] Alexander J, Hirschowitz A. Polynomial interpolation in several variables. J Algebraic Geom. 1995;4:201–222.
[47] Derksen H, Friedland S, Lim L-H, et al. Theoretical and computational aspects of entanglement. arXiv:1705.07160. 2017.
Friedland S, Lim L-H. Nuclear norm of higher order tensors. Math Comput. 2018;87:1255–1281.

Ryan RA. Introduction to tensor products of Banach spaces. London: Springer-Verlag; 2002.

Wei T-C, Goldbart PM. Geometric measure of entanglement and applications to bipartite and multipartite quantum states. Phys Rev A. 2003;68:042307.

Aliabadi M, Friedland S. On the complexity of finding tensor ranks. Commun Appl Math Comput. arXiv:2002.07151. 2020.

Bates DJ, Hauenstein JD, Sommese AJ, et al. Bertini: software for numerical algebraic geometry. Available from: bertini.nd.edu. Doi:10.7274/R0H41P5B.

Kruskal JB. Three-way arrays: rank and uniqueness of trilinear decompositions, with applications to arithmetic complexity. Lin Alg Appl. 1977;18:95–138.

JáJá J. Optimal evaluation of pairs of bilinear forms. SIAM J Comput. 1979;8:443–461.

Chiantini L, Ottaviani G, Vannieuwenhoven N. On generic identifiability of symmetric tensors of subgeneric rank. Trans Amer Math Soc. 2017;369:4021–4042.

Shimony A. Degree of entanglement. Ann NY Acad Sci. 1995;755:675–679.

Życzkowski K, Bengtsson I. Relativity of pure states entanglement. Ann Phys (NY). 2002;295:115–135.

Lockhart RB, Steiner MJ. Preserving entanglement under decoherence and sandwiching all separable states. Phys Rev A. 2002;65:022107.

Carteret HA, Higuchi A, Sudbery A. Multipartite generalisation of the Schmidt decomposition. J Math Phys. 2000;41:7932–7939.

De Lathauwer L, De Moor B, Vandewalle J. A multilinear singular value decomposition. SIAM J Matrix Anal Appl. 2000;21:1253–1278.

Eisert J, Briegel H-J. Schmidt measure as a tool for quantifying multipartite entanglement. Phys Rev A. 2001;64:022306.

Bürgisser P, Clausen M, Shokrollahi A. Algebraic complexity theory. Berlin: Springer-Verlag; 1997.

Friedland S. Matrices: algebra, analysis and applications. Singapore: World Scientific; 2015. 596 pp, Available from: http://www2.math.uic.edu/~friedlan/bookm.pdf.

Hiriart-Urruty J-B, Le HY. A variational approach of the rank function. TOP. 2013;21(2):207–240.

Buczyński J, Postninghel E, Rupniewski F. On Strassen’s rank additivity for small three-way tensors. SIAM J Matrix Anal Appl. 2020;41(1):106–133.

Teitler Z. Sufficient conditions for Strassen’s additivity conjecture. Illinois J Math. 2015;59:1071–1085.

Schönhage A. Partial and total matrix multiplication. SIAM J Comp. 1981;10(3):434–455.

Shitov Y. A counterexample to Strassen’s direct sum conjecture. Acta Math. 2019;222:363–379.

Buczyński J, Han K, Mella M, et al. On the locus of points of hight rank. Eur J Math. 2018;4(1):113–136.

Blekherman G, Teitler Z. On maximum, typical and generic ranks. Math Ann. 2015;362:1021–1031.

Bini D, Lotti G, Romani F. Approximate solutions for the bilinear form computational problem. SIAM J Comput. 1980;9(4):692–697.

Raicu C. Secant varieties of Segre-Veronese varieties. Algebra Number Theory. 2012;6:1817–1868.

JáJá J. Optimal evaluation of pairs of bilinear forms. In: Tenth Annual ACM Symposium on Theory of Computing (San Diego, Calif., 1978). New York: ACM; 1978. p. 173–183.

Grigoriev DY. Some new bounds on tensor rank. Preprint LOMI, E-2-78, 1978, 12 pp. Available from: https://logic.pdmi.ras.ru/grigorev/pub/pair.pdf.

Grigoriev DY. Multiplicativity of a pair of bilinear forms and of the polynomial multiplication. In: Mathematical foundations of computer science, 1978 (Proc. Seventh Sympos., Zakopane, 1978). Berlin-New York: Springer; 1978. p. 250–256. (Lecture Notes in Comput. Sci. 64).
[76] JáJá J, Takche J. On the validity of the direct sum conjecture. SIAM J Comput. 1986;15:1004–1020.
[77] Derksen H. Kruskal’s uniqueness inequality is sharp. Linear Algebra Appl. 2013;438:708–712.
[78] Rhodes JA. A concise proof of Kruskal’s theorem on tensor decomposition. Linear Algebra Appl. 2010;432:1818–1824.
[79] Kronecker L. Algebraische reduction der Scharen bilinearer formen. Berl Ber. 1890;1890:1225–1237.
[80] Gantmacher FR. The theory of matrices. Vol II. New York: Chelsea Publ. CO.; 1959.
[81] Atkinson MD, Stephens NM. On the maximal multiplicative complexity of a family of bilinear forms. Linear Algebra Appl. 1979;27:1–8.
[82] Harris J. Algebraic geometry: a first course. New York: Springer; 1992.
[83] Strassen V. Rank and optimal computation of generic tensors. Lin Alg Appl. 1983;52:645–685.
[84] Lickteig T. Typical tensor rank. Linear Algebra Appl. 1985;69:95–120.
[85] Atkinson MD, Lloyd S. Bounds on the ranks of some 3-tensors. Linear Algebra Appl. 1980;31:19–31.
[86] Sidiropoulos ND, Bro R. On the uniqueness of multilinear decomposition of N-way arrays. J Chemom. 2000;14:229–239.
[87] Catalisano MV, Geramita A, Gimigliano A. Ranks of tensors, secant varieties of Segre varieties and fat points. Linear Algebra Appl. 2002;355:263–285.
[88] Friedland S. Remarks on the symmetric rank of symmetric tensors. SIAM J Matrix Anal Appl. 2016;37:320–337.
[89] Thompson TM. Error correcting codes through sphere packings to simple groups. Washington (DC): MAA Press; 1983.
[90] Korte B,Vygen J. Combinatorial optimization: theory and algorithms. 5nd ed. Bonn: Springer; 2012.
[91] Cook WJ, Cunningham WH, Pulleyblank WR, et al. Combinatorial optimization. New York: Wiley; 1998.
[92] Petr S. A tight analysis of the greedy algorithm for set cover, STOC’96. p. 435–441, doi:10.1145/237814.23799
[93] Catalisano MV, Geramita A, Gimigliano A. Secant varieties of $(\mathbb{P}^1)^n$ are NOT defective for $n \geq 5$. J Algebraic Geom. 2011;20:295–327.
[94] Choulakian V. Some numerical results on the rank of generic three-way arrays over $\mathbb{R}$. SIAM J Matrix Anal Appl. 2009;30:2997–3007.
[95] Domanov I, De Lathauwer L. On the uniqueness of the canonical polyadic decomposition of third-order tensors–Part I: basic results and uniqueness of one factor matrix. SIAM J Matrix Anal Appl. 2013;34(3):855–875.
[96] Domanov I, De Lathauwer L. On the uniqueness of the canonical polyadic decomposition of third-order tensors–Part II: uniqueness of the overall decomposition. SIAM J Matrix Anal Appl. 2013;34(3):876–903.
[97] Domanov I, De Lathauwer L. Generic uniqueness conditions for the canonical polyadic decomposition and INDSCAL. SIAM J Matrix Anal Appl. 2015;36(4):1567–1589.
[98] Zhang X, Huang ZH, Qi L. Comon’s conjecture, rank decomposition, and symmetric rank decomposition of symmetric tensors. SIAM J Matrix Anal A. 2016;37:1719–1728.
[99] Brambilla MC, Ottaviani G. On the Alexander–Hirschowitz theorem. J Pure Appl Algebra. 2008;212:1229–1251.
[100] Comas G, Seiguer M. On the rank of a binary form. Found Comput Math. 2011;11:65–78.
Appendix. Basic notions of quantum theory

In this Appendix, we present definitions of some notions used in quantum theory and discussed in this work. To make it easier for the reader to study the literature of the subject we are going to use the Dirac notation presented in Section 2.1. In short, the ‘ket’ $|\psi\rangle$ denotes a complex vector of a fixed size $n$ represented by a column, the ‘bra’ $\langle \psi |$ is a conjugated (dual) vector forming a row, the scalar product is written as a bra–ket, $\langle \psi | \phi \rangle \in \mathbb{C}$, while $|\phi\rangle \langle \psi |$ forms an operator, represented by a rank-1 square matrix of size $n$.  

[105] Segre B. The non-singular cubic surfaces. Oxford: Oxford University Press; 1942.
[106] Comon P, Mourrain B. Decomposition of quincics in sums of powers of linear forms. Signal Process. 1996;53:93–107.
[107] Landsberg JM, Teitler Z. On the ranks and border ranks of symmetric tensors. Found Comput Math. 2010;10:339–366.
[108] Kleppe J. Representing a homogenous polynomial as a sum of powers of linear forms [Masters thesis]. Oslo: University of Oslo; 1999. https://www.mn.uio.no/math/personer/vit/ranestad/papers-preprints-scripts-students.
[109] De Paris A. A proof that the maximum rank for ternary quartics is seven. Matematiche (Catania). 2015;70:3–18.
[110] Buczyński J, Teitler Z. Some examples of forms of high rank. Collect Math. 2016;67:431–441.
[111] Bläser M. Fast matrix multiplication. Grad Surv Theory Comput Libr. 2013;5:1–60.
[112] Zuiddam J. A note on the gap between rank and border rank. Linear Algebra Appl. 2017;525:33–44.
[113] Sylvester JJ. Sur une extension d’un théorème de Clebsh relatif aux courbes du quatrième degré. Comptes rendus. Math Acad Sci Paris. 1886;102:1532–1534.
[114] Bernardi A, Gimigliano A, Idà M. Computing symmetric rank for symmetric tensors. J Symbol Comput. 2011;46(1):34–53.
[115] Angelini E, Chiantini L, Mazzon A. Identifiability for a class of symmetric tensors. Mediterr J Math. 2019;16(4):Paper No. 97, 14 pp.
[116] Angelini E, Chiantini L, Vannieuwenhoven N. Identifiability beyond Kruskal’s bound for symmetric tensors of degree 4. Atti Accad Naz Lincei Rend Lincei Mat Appl. 2018;29(3):465–485.
[117] Ballico E, Chiantini L. Sets computing the symmetric tensor rank. Mediterr J Math. 2013;10(2):643–654.
[118] Stegeman A, Comon P. Subtracting a best rank-1 approximation may increase tensor rank. Linear Algebra Appl. 2010;433(7):1276–1300.
[119] Gross D, Flammia ST, Eisert J. Most quantum states are too entangled to be useful as computational resources. Phys Rev Lett. 2009;102:190501.
[120] Li Z, Nakatsukasa Y, Soma T, et al. On orthogonal tensors and best rank-one approximation ratio. SIAM J Matrix Anal Appl. 2018;39:400–425.
[121] Derksen H, Makam V. Highly entangled tensors. Linear Multilinear Algebra. 2022;70(2):380–393.
[122] Banach S. Über homogene polynome in ($L^2$). Studia Math. 1938;7:36–44.
[123] Hübener R, Kleiman M, Wei T-C, et al. The geometric measure of entanglement for symmetric states. Phys Rev A. 2009;80:032324.
[124] Friedland S, Kemp T. Most Boson quantum states are almost maximally entangled. Proc Amer Math Soc. 2018;146:5035–5049.
[125] Sokoli F, Alber G. Generalized Schmidt decomposability and its relation to projective norms in multipartite entanglement. J Phys A Math Theor. 2014;47:325301.
[126] Sanyal R, Sottile F, Sturmfels B. Orbitopes. Mathematika. 2011;57:275–314.
[127] Tamaryan S, Wei TC, Park D. Maximally entangled three-qubit states via geometric measure of entanglement. Phys. Rev. A. 2009;80:052315.
[128] Werner RF. Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model. Phys Rev A. 1989;40:4277.
The term state is understood as a mathematical tool used to calculate the probability of a given outcome of any measurement. In the classical probability theory, one uses probability vectors, \( p = \{p_1, \ldots , p_n\} \), such that \( p_i \geq 0 \) and \( \sum_{i=1}^{n} p_i = 1 \). The natural number \( n \) describes the number of distinguishable events and is fixed. The set of classical states forms the probability simplex of dimension \( n-1 \). A probability vector \( p \) with a single component equal to unity, representing a vertex of the simplex, is called an extremal point or a classical pure state, and corresponds to a certain event. Any point inside the simplex can be considered as a classical mixed state.

One of the key notions of quantum theory is the quantum state, which characterizes the way a physical system was prepared and allows one to compute the probability of an outcome of any quantum measurement. Let us fix a natural number \( n \), assumed here to be finite, and discuss first a special class of states. A pure quantum state is represented by a ray in an \( n \)-dimensional complex Hilbert space \( \mathcal{H}_n \).

**Definition A.1:** Consider a complex vector \( |\psi\rangle \in \mathcal{H}_n \), normalized as \( \| \psi \|^2 = \langle \psi | \psi \rangle = 1 \) and an arbitrary complex phase, \( e^{i\alpha} \), with \( \alpha \in [0, 2\pi] \). A pure quantum state denotes the equivalence class, \( |\psi\rangle \sim e^{i\alpha} |\psi\rangle \).

The space of all pure states forms a complex projective space, \( \mathbb{C}P^{n-1} \), of \( 2(n-1) \) real dimensions. In the simplest case, \( n = 2 \), often called a single qubit (i.e. a quantum bit) system, this space forms a sphere, \( \mathbb{C}P^1 = S^2 \), in physics called the Bloch sphere. The set of pure quantum states is continuous, in contrast to the discrete set of classical pure states, which are the extreme points of the probability simplex.

A hermitian operator \( P_\psi = |\psi\rangle \langle \psi| = P_\psi^2 \) is a projection operator onto a pure state \( |\psi\rangle \). In physics literature, the term ‘pure state’ may denote a state \( |\psi\rangle \) or the corresponding projector \( P_\psi \) and the meaning depends on the context. This difference is not relevant in physics, since the vectors representing rays are in one-to-one correspondence with the projectors.

Any convex combination of such projectors, \( \rho = \sum_{j=1}^{k} q_j |\psi_j\rangle \langle \psi_j| \), forms a mixture of pure states, where \( q \) represents a probability vector of an arbitrary length \( k \). Such a mixture, called a density matrix or a mixed state or just a quantum state, can be introduced in a more formal way.

**Definition A.2:** A square matrix \( \rho \) of order \( n \) is called a density matrix if it is hermitian, \( \rho = \rho^* \), positive semi-definite, \( \rho \geq 0 \), and normalized, \( \text{Tr} \rho = 1 \).

Any convex mixture of one-dimensional projection operators \( P_\psi \) satisfies above properties. Note that the diagonal entries of a density operator represented in any basis are real and form a classical state - the probability vector \( p \) of length \( n \) with components \( p_i = \rho_{ii} \), \( i = 1, \ldots , n \). Let \( \rho = \sum_{j=1}^{n} \lambda_j |\chi_j\rangle \langle \chi_j| \) be the eigen-decomposition of the state \( \rho \), where the eigenvalues \( \lambda \) form a probability vector. The rank \( r \) of the state \( \rho \) is equal to the number of its positive eigenvalues. The case \( r = 1 \) corresponds to the projector, \( \rho = |\chi_1\rangle \langle \chi_1| = \rho^2 \). It forms an extremal point of the set of all density matrices of order \( n \), which explains the term ‘pure state’. In Euclidean geometry, induced by the Hilbert–Schmidt distance, \( d_{\text{HS}}(X, Y) = \|X - Y\| \), the space of all mixed states for \( n = 2 \) forms a solid Bloch ball with the Bloch sphere consisting of pure states at its boundary. In higher dimensions \( n \geq 3 \), there exist points from the boundary of the set of mixed states which are not pure – see e.g. [17].

In some physical systems, one can observe an internal structure and can identify its subsystems. Assume first, that the system is bipartite, so two parties, called \( A \) and \( B \) are distinguished. Let \( \mathcal{H}_A \) and \( \mathcal{H}_B \) denote Hilbert spaces used to describe subsystems \( A \) and \( B \), respectively. Then the bipartite system \( AB \) is described by a quantum state from the composite Hilbert space with a tensor product structure, \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \). Once both subsystems \( A \) and \( B \) are well defined and the above splitting of \( \mathcal{H}_{AB} \) is fixed, one can introduce the notion of separable and entangled states [12,16],

**Definition A.3:** A bipartite pure quantum state \( |\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) is called separable, if it has the product form, \( |\psi_{AB}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \), where \( |\phi_A\rangle \in \mathcal{H}_A \) and \( |\phi_B\rangle \in \mathcal{H}_B \).
**Definition A.4:** A bipartite pure quantum state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled if it is not separable, so it is not of the product form.

The above definitions do not depend on the choice of the local bases in both subspaces, but they do depend on the splitting of $\mathcal{H}_{AB}$ into $\mathcal{H}_A$ and $\mathcal{H}_B$. Let $n$ and $m$ denote the dimensions of these two subspaces, respectively. The entanglement is then invariant with respect to the local unitary transformations from the group $U(n) \otimes U(m)$, but it can change under global unitary transformations, $|\psi\rangle \rightarrow V|\psi\rangle$, with $V \in U(nm)$.

Note that a separable pure state represents independent events, for which the joint probability of events measured separately in both subsystems has the product form, while correlated events correspond to entangled states. In the general case of density matrices, the definition of separability is slightly more involved [128].

**Definition A.5:** A bipartite quantum (mixed) state $\rho^{AB}$ acting on the composite space $\mathcal{H}_A \otimes \mathcal{H}_B$ is called separable, if it can be represented as a convex combination of product states,

$$\rho^{AB}_{\text{sep}} = \sum_{j=1}^{k} q_j \rho^A_j \otimes \rho^B_j, \quad (A1)$$

where $q$ is a probability vector of length $k$, while $\{\rho^A_j\}$ and $\{\rho^B_j\}$ denote collections of $k$ quantum states acting on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

**Definition A.6:** A bipartite quantum (mixed) state $\rho^{AB}$ acting on the composite space $\mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled if it is not separable.

It is easy to see that in the case of pure states, which are extremal and cannot be represented by a mixture of other pure states, both definitions of entanglement are consistent. Furthermore, entanglement does not depend on the choice of the local bases also for mixed states, but it depends on the partition of the total system into two parts. In the case of a two-qubit system, $n = m = 2$, the necessary and sufficient conditions for separability of a given density matrix $\rho$ of size 4 are known [16,17], but already for $n = m = 3$ the separability problem remains open.

The above notions are easy to generalize for multipartite systems which consist of $d \geq 3$ subsystems. For instance, a separable pure state of a $d$-partite system has the product form, $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_d\rangle$. 