PARABOLIC EQUATIONS WITH VMO COEFFICIENTS
IN SPACES WITH MIXED NORMS

N.V. KRYLOV

Abstract. An $L_q(L_p)$-theory of divergence and non-divergence form parabolic equations is presented. The main coefficients are supposed to belong to the class $VMO_x$, which, in particular, contains all measurable functions depending only on $t$. The method of proving simplifies the methods previously used in the case $p = q$.

1. Introduction

The goal of this article is to prove the solvability of parabolic second-order divergence and non-divergence type equations in Sobolev spaces with mixed norms.

More precisely, we are dealing with two types of parabolic operators:

\[ Lu(t, x) = u_t(t, x) + a^{ij}(t, x)u_{x_i x_j}(t, x) + b^i(t, x)u_{x_i}(t, x) + c(t, x)u(t, x), \]

\[ \mathcal{L}u(t, x) = u_t(t, x) + (a^{ij}(t, x)u_{x_i}(t, x) + \hat{b}^i(t, x)u(t, x))_{x_j} + b^j(t, x)u_{x_j}(t, x) + c(t, x)u(t, x) \]

acting on functions given on

\[ \mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}, \]

where $\mathbb{R}^d$ is a $d$-dimensional Euclidean space of points $x = (x^1, \ldots, x^d)$.

The interest in results concerning equations in spaces with mixed $L_q(L_p)$-norms arises, in particular, when one wants to have better regularity of traces of solutions for each $t$ while treating linear or nonlinear equations (see, for instance, [6] and [16] for applications to the Navier-Stokes equations).

Parabolic equations in $L_q(L_p)$-spaces have been investigated in many articles for at least forty years. The interested reader can find many references and discussions of methods and obtained results in [6], [8], and [13].

\[1991 \textit{Mathematics Subject Classification.} \ 35K10, 35J15. \]
\[ \textit{Key words and phrases.} \ Second-order equations, vanishing mean oscillation, mixed norms. \]

The work was partially supported by NSF Grant DMS-0140405.
However, it seems to the author that apart from [13] (also see the references therein) in most other papers concerning $L_q(L_p)$-spaces the methods heavily depend on the properties of the elliptic part in $L$ or $L^r$, which is supposed to be independent of $t$ and have well behaving resolvent or generate a “good” semigroup. However, in [11] (also see references therein) there is a general theorem allowing one to treat the case when the coefficients are continuous in $t$. These restrictions exclude parabolic equations with coefficients measurable or even VMO in $t$ (even if they are independent of $x$, the case considered in [13]). In particular, in [8] the authors only consider equations with VMO coefficients independent of time, although combining their results with [11] would include equations with coefficients continuous in $t$. By the way, in the particular case that $q = p$ this also does not allow one to cover the results of [2], where the coefficients are in $VMO(\mathbb{R}^{d+1})$. Speaking about the case $q = p$ it is worth saying that there is a quite extensive literature about equations and systems with VMO coefficients. The interested reader can consult [3], [4], [5], [7], [8], [17], [18], [19], [20], [21], [22], [23], [24], and the references therein.

Our approach is based on a method from [14] and further developed in [11] and [12], where everything hinges on a priori pointwise estimates of the sharp functions of the second-order spatial derivatives of solutions. This method allows one to avoid using generalizations of the Calderón-Zygmund theorem and the Coifman-Rochberg-Weis commutator theorem as is often done when VMO is involved (see, for instance, [2], [7], [8], [17], [18], [19], [20], [21], [22], [23], [24], and the references therein). However, it is worth noting that if $p = q$ there is an approach to the divergence type equation suggested in [3] and [4], which also does not use the above mentioned tools. The approach from the present article has been already used in a very interesting article [10] to prove the solvability in usual Sobolev spaces of parabolic equations with partially VMO coefficients when most of the coefficients are just measurable in time and one of space variables and VMO with respect to the others.

In [14], in each small cylinder, the solution is split into two parts: a function, that is “harmonic” with respect to the operator with “frozen” coefficients, and the remainder. In order to do this decomposition one has to know that the corresponding boundary-value problems are solvable. This is not very convenient if one has in mind higher-order equations.

It turns out that instead one can use splitting of the right-hand side of the equation and rely on solvability of equations in the whole space. This approach not only simplifies some proofs from [14] but also allows
one to make stronger main technical estimates (see Lemmas 3.1 and 4.1), which after being combined with an approach suggested in \[13\] leads to $L_q(L_p)$-theory. Although, we are dealing only with the Cauchy problem for second-order operators, it seems that the new technique, which we develop here, is applicable to higher-order equations, systems, and boundary-value problems for elliptic and parabolic equations with VMO coefficients.

We are assuming that the main coefficients are measurable in time and VMO in spatial variables and prove the solvability in $L_q(L_p)$ spaces for $L$ if $q \geq p$ (Theorem 2.1) and for $\mathcal{L}$ without this restriction (Theorem 2.3). Theorem 2.1 generalizes the corresponding result of \[8\] to cover time-dependent coefficients. However, note that the results in \[8\] are proved also for higher-order parabolic systems, arbitrary $p, q \in (1, \infty)$, and $L_p$-spaces with $A_p$ Muckenhoupt weights.

The paper is organized as follows. In Section 2 we state our main results. Theorem 2.1 and 2.3 are proved in Sections 3 and 4, respectively, on the basis of Lemmas 3.1 and 4.1, respectively, which are proved later. In Sections 5 and 7 we present our new approach to treating parabolic equations with VMO$_x$ coefficients. The main results of these two sections are Theorems 5.1 (non-divergence equations) and 7.1 (divergence equations) about equations in usual Sobolev spaces without mixed norms. If one takes functions independent of $t$, these two theorems yield the basic estimates for elliptic equations. Finally, in Sections 6 and 8 we prove Lemmas 3.1 and 4.1, respectively.

The work on this article was stimulated by discussions during the author’s stay at Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, and it is a great pleasure to bring my sincere gratitude to G. Da Prato and M. Giaquinta for the invitation and hospitality.

We finish the section introducing some notation. Note that we also use without mentioning some common notation from PDEs. For $\infty \leq S < T \leq \infty$, $1 < p, q < \infty$ we set

\[
\mathbb{R}_S = (S, \infty), \quad \mathbb{R}_S^{d+1} = \mathbb{R}_S \times \mathbb{R}^d, \quad L_p = L_p(\mathbb{R}^{d+1}),
\]

\[
L_{q,p}((S,T)) = L_q((S,T), L_p(\mathbb{R}^d)), \quad L_{q,p} = L_{q,p}(\mathbb{R}),
\]

\[
W^{1,2}_{q,p}((S,T)) = \{ u : u, u_t, u_x, u_{xx} \in L_{q,p}((S,T)) \},
\]

\[
W^{1,2}_{q,p} = W^{1,2}_{q,p}(\mathbb{R}), \quad W^{1,2}_{p,p}(\mathbb{R}^{d+1}) = W^{1,2}_{p,p}(\mathbb{R}_S), \quad W^{1,2}_p = W^{1,2}_p(\mathbb{R}^{d+1}),
\]

\[
\|u\|_{L_{q,p}((S,T))}^q = \int_S^T \left( \int_{\mathbb{R}^d} |u(t,x)|^p \, dx \right)^{q/p} \, dt,
\]
\[\|u\|_{W^{1,2}_{q,p}(S,T)} = \|u\|_{L_{q,p}(S,T)} + \|u_x\|_{L_{q,p}(a,b)} + \|u_{xx}\|_{L_{q,p}(S,T)} + \|u_t\|_{L_{q,p}(S,T)}.\]

By \(W^{1,2}_{q,p}(S,T)\) we mean the subspace of \(W^{1,2}_{q,p}(\mathbb{R}_S)\) consisting of functions \(u(t,x)\) vanishing for \(t > T\). Finally,

\[H^1_{q,p}(S,T) = (1 - \Delta)^{1/2}W^{1,2}_{q,p}(S,T), \quad H^1_{q,p} = H^1_{q,p}(\mathbb{R}),\]

where \(\Delta\) is the Laplacian in \(x\) variables. In the above notation we write \(p\) in place of \(q,p\) if \(q = p\). For instance,

\[H^1_{p}(S,T) = (1 - \Delta)^{1/2}W^{1,2}_{p}(S,T), \quad H^{-1}_{p} = H^{-1}_{p}(\mathbb{R}).\]

2. Main results

We assume that the coefficients of \(L\) and \(L\) are measurable and by magnitude are dominated by a constant \(K < \infty\). We also assume that the matrices \(a = (a^{ij})\) are, perhaps, nonsymmetric and satisfy

\[a^{ij} \lambda^i \lambda^j \geq \kappa |\lambda|^2 (2.1)\]

for all \(\lambda \in \mathbb{R}^d\) and all possible values of arguments. Here \(\kappa > 0\) is a fixed constant.

To state our main assumption we set \(B_r(x)\) to be the open ball in \(\mathbb{R}^d\) of radius \(r\) centered at \(x\), \(B_r = B_r(0)\), \(Q_r(t,x) = (t, t + r^2) \times B_r(x)\), \(Q_r = Q_r(0,0)\), \(\mathbb{B}\) the collection of open balls in \(\mathbb{R}^d\), and \(Q\) the collection of \(Q_r(t,x), (t,x) \in \mathbb{R}^{d+1}\), \(r \in (0, \infty)\). Denote

\[\text{osc}_x(a, Q_r(t,x)) = r^{-2}|B_r|^{-2} \int_t^{t+r^2} \int_{y,z \in B_r(x)} |a(s,y) - a(s,z)| dydzds,\]

\[a^\#(x) = \sup_{(t,x) \in \mathbb{R}^{d+1}} \sup_{r \leq R} \text{osc}_x(a, Q_r(t,x)).\]

We assume that \(a \in VMO_x\), that is

\[\lim_{R \to 0} a^\#(x) = 0. \quad (2.2)\]

For convenience of stating our results we take any increasing continuous function \(\omega(R)\) on \([0, \infty)\), such that \(\omega(0) = 0\) and \(a^\#(x) \leq \omega(R)\) for all \(R \in (0, \infty)\). Obviously, \(a \in VMO_x\) if \(a\) depends only on \(t\).

Needless to say all equations below are understood in the sense of generalized functions.
Now we fix $T \in (0, \infty)$ and $q, p \in (1, \infty)$, set
$$\Omega(T) = (0, T) \times \mathbb{R}^d$$
and state our main results.

**Theorem 2.1.** Let $q \geq p$. Then for any $f \in L_{q,p}((0, T))$ there exists a
unique $u \in W^{1,2}_{q,p}(0, T)$ such that $Lu = f$ in $\Omega(T)$. Furthermore, there
is a constant $N$, depending only on $d, T, K, \kappa, q, p$, and the function
$\omega$, such that for any $u \in W^{1,2}_{q,p}(0, T)$ we have
$$\|u\|_{W^{1,2}_{q,p}(0, T)} \leq N \|Lu\|_{L_{q,p}((0, T))}. \quad (2.3)$$

**Remark 2.2.** Theorem 2.1 is similar to some results from [13] and [8]
(also see the references therein). However, in both articles there is
no restriction on $p, q$. On the other hand, in [13] the coefficients are
independent of $x$ and in [8] they are independent of $t$. As we have
already pointed out in the Introduction, by relying on [1], some results
from [8] can be extended to cover the case of coefficients continuous in $t$.

**Theorem 2.3.** Let $f = (f^1, \ldots, f^d)$, $g, f^i \in L_{q,p}((0, T))$ for $i = 1, \ldots, d$.
Then there is a unique $u \in C_x^1(0, T)$ such that $Lu = \text{div } f + g$ in $\Omega(T)$.
Furthermore, there is a constant $N$, depending only on $d, T, K, 
\kappa, q, p$, and the function $\omega$, such that
$$\|u\|_{L_{q,p}((0, T))} + \|u_x\|_{L_{q,p}((0, T))} \leq N \left( \|f\|_{L_{q,p}((0, T))} + \|g\|_{L_{q,p}((0, T))} \right). \quad (2.4)$$

**Remark 2.4.** As usual in such situations, from our proofs one can see
that instead of the assumption that $a \in \text{VMO}_x$ we are, actually, using
that there exists an $R \in (0, \infty)$ such that $a_R^{\#}(x) \leq \varepsilon$, where $\varepsilon > 0$ is a
constant depending only on $d, p, \kappa, K$.

**Remark 2.5.** Denote
$$u_Q = \int_Q u(s, y) \, dy \, ds,$$
the average value of a function $u(s, y)$ over $Q \in \mathbb{Q}$ and
$$u_B(t) = \int_B u(t, y) \, dy$$
the average value of a function $u(t, y)$ over $B \in \mathbb{B}$.

Also introduce $A$ as the set of $d \times d$ matrix-valued measurable functions $a = a(t)$ depending only on $t$, satisfying conditions (2.1) and such that $|a^{ij}| \leq K$. 


A standard fact to remember is that for any \( \bar{a} \in A \)
\[
\text{osc}_x(a, Q_r) \leq 2 \int_{Q_r} |a(s, x) - \bar{a}(s)| \, dx \, ds
\]
and for \( \bar{a}(t) = a_{B_r}(t) \)
\[
\int_{Q_r} |a(s, x) - \bar{a}(s)| \, dx \, ds \leq \text{osc}_x(a, Q_r).
\]
This allows one to give obvious equivalent definitions of \( \text{VMO}_x \).

3. **Proof of Theorem 2.1**

The following fact, which we prove in Section 6, is a considerable improvement of the key inequality from the proof of Theorem 3.6 of [14]. It goes without saying that the assumptions under which Theorem 2.1 is stated are supposed to hold.

**Lemma 3.1.** Let \( b = 0 \) and \( c = 0 \). Then there exists a constant \( N = N(d, \kappa, K, p, \omega) \) such that for any \( u \in C_0^\infty(\mathbb{R}^{d+1}) \), \( \nu \geq 16 \), and \( r \in (0, 1/\nu] \) we have
\[
(\|u_{xx} - (u_{xx})_{Q_r}\|_p)^+ \leq N\nu^{d+2} A_{\nu r} + N(\nu^{-p} + \nu^{d+2} \hat{a}^{1/2}) B_{\nu r},
\]
where
\[
A_\rho = (\|f\|_p)^+, \quad B_\rho = (\|u_{xx}\|_p)^+, \quad \hat{a} = a^\#(x), \quad f = Lu.
\]

**Corollary 3.2.** Let \( b = 0 \) and \( c = 0 \). Then there exists a constant \( N \) depending only on \( d, p, \kappa, K, \) and \( \omega \), such that for any \( u \in C_0^\infty(\mathbb{R}^{d+1}) \), \( r > 0 \), and \( \nu \geq 16 \), satisfying \( \nu r \leq 1 \), we have
\[
\int_{(0,r^2)} \int_{(0,r^2)} \int_{\mathbb{R}^d} |\|u_{xx}(t, \cdot)\|_{L_p(\mathbb{R}^d)} - \|u_{xx}(s, \cdot)\|_{L_p(\mathbb{R}^d)}|^p \, dtds \leq N\nu^{-p} + N\nu^{d+2} \hat{a}^{1/2} \int_{(0,\nu r^2)} \|u_{xx}(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p \, dt
\]
\[
+ N\nu^{d+2} \int_{(0,\nu r^2)} \|Lu(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p \, dt.
\]
Indeed, by the triangle inequality
\[
\|u_{xx}(t, \cdot)\|_{L_p(\mathbb{R}^d)} - \|u_{xx}(s, \cdot)\|_{L_p(\mathbb{R}^d)} \leq \|u_{xx}(t, \cdot) - u_{xx}(s, \cdot)\|_{L_p(\mathbb{R}^d)}^p,
\]
so that the left-hand side of (3.2) is less than
\[
I := \int_{(0,r^2)} \int_{(0,r^2)} \int_{\mathbb{R}^d} |u_{xx}(t, x) - u_{xx}(s, x)|^p \, dx \, dt \, ds
\]
\[
= \int_{(0,r^2)} \int_{(0,r^2)} \int_{\mathbb{R}^d} |u_{xx}(t, x + y) - u_{xx}(s, x + y)|^p \, dx \, dt \, ds,
\]
where \( y \) is any point in \( \mathbb{R}^d \). By taking the average of the extreme terms over \( y \in B_r \) we see that

\[
I = \int_{(0,r^2)} \int_{(0,r^2)} \int_{\mathbb{R}^d} \left( \int_{B_r(x)} |u_{xx}(t, z) - u_{xx}(s, z)|^p dz \right) dxdtds. \quad (3.3)
\]

Next, since

\[
|u_{xx}(t, z) - u_{xx}(s, z)|^p \leq 2^{p-1} |u_{xx}(t, z) - (u_{xx})_{Q_r(0,x)}|^p + 2^{p-1} |u_{xx}(s, z) - (u_{xx})_{Q_r(0,x)}|^p,
\]

we have that

\[
I \leq 2^p \int_{\mathbb{R}^d} (|u_{xx} - (u_{xx})_{Q_r(0,x)}|^p)_{Q_r(0,x)} dx.
\]

By Lemma 3.1 applied to shifted cylinders the last expression is dominated by a constant times

\[
(\nu^{-p} + \nu^{d+2} \overline{a}^{1/2}) \int_{\mathbb{R}^d} (|u_{xx}|^p)_{Q_{\nu r}(0,x)} dx + \nu^{d+2} \int_{\mathbb{R}^d} (|Lu|^p)_{Q_{\nu r}(0,x)} dx,
\]

which similarly to (3.3) is shown to equal

\[
(\nu^{-p} + \nu^{d+2} \overline{a}^{1/2}) \int_{(0,\nu^2r^2)} \|u_{xx}(t, \cdot)\|^p_{L_p(\mathbb{R}^d)} dt
\]

\[
+ \nu^{d+1} \int_{(0,\nu^2r^2)} \|Lu(t, \cdot)\|^p_{L_p(\mathbb{R}^d)} dt
\]

and this yields (3.2).

To move further fix a \( u \in C_0^\infty(\mathbb{R}^{d+1}) \) and set

\[
\phi(t) = \|u_{xx}(t, \cdot)\|_{L_p(\mathbb{R}^d)}, \quad f = Lu, \quad \psi(t) = \|f(t, \cdot)\|_{L_p(\mathbb{R}^d)}
\]

and for any locally integrable function \( \tau(s) \) on \( \mathbb{R} \) denote by

\[
M_t \tau(s) \text{ and } \tau^{\#(t)}(s)
\]

the maximal and sharp functions of \( \tau \), respectively.

**Lemma 3.3.** Let \( r_0 \in (0, \infty) \), \( b = 0 \), \( c = 0 \). Assume that the above \( u(t, x) = 0 \) for \( t \not\in (0, r_0^2) \). Then for any \( \nu \geq 16 \) and \( R \in (0, 1] \), we have

\[
\phi^{\#(t)} \leq N\nu^{(d+2)/p} M_t^{1/p}(\psi^p) + N\left((\nu r_0/R)^{2-2/p} + \nu^{-1} + \nu^{(d+2)/p}(R) \omega^{1/(2p)}(R)\right) M_t^{1/p}(\phi^p), \quad (3.4)
\]

where \( N = N(\omega, d, \kappa, K, p) \).
Proof. Obviously, Corollary \[ \text{Lemma 3.4.} \] in terms of the functions $\phi$ and $\psi$ yields

$$
\int_{(0,r^2)} \int_{(0,r^2)} |\phi(t) - \phi(s)|^p \, dt \, ds \leq N \nu^{d+2} \int_{(0,\nu r^2)} \psi^p(t) \, dt
$$

if $r \leq R/\nu$ (when $a_{ur}^{\#(x)} \leq a_{R}^{\#(x)} \leq \omega(R)$ and $\nu r \leq 1$). This corollary allows shifting the origin. Therefore, for any $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and $\beta - \alpha = r^2 \leq R^2/\nu^2$ we have

$$
\int_{(\alpha, \beta)} \int_{(\alpha, \beta)} |\phi(t) - \phi(s)|^p \, dt \, ds \leq N \nu^{d+2} \int_{(\alpha, \alpha+\nu^2(\beta-\alpha))} \psi^p(t) \, dt
$$

Take a point $t_0 \in \mathbb{R}$ and $\alpha$ and $\beta$ as above and such that $t_0 \in (\alpha, \beta)$. Then $t_0 \in (\alpha, \alpha+\nu^2(\beta-\alpha))$ and by definition

$$
\int_{(\alpha, \alpha+\nu^2(\beta-\alpha))} \psi^p(t) \, dt \leq M_1(\psi^p)(t_0),
$$

$$
\int_{(\alpha, \alpha+\nu^2(\beta-\alpha))} \phi^p(t) \, dt \leq M_1(\phi^p)(t_0).
$$

By applying Hölder’s inequality we conclude that

$$
\int_{(\alpha, \beta)} \int_{(\alpha, \beta)} |\phi(t) - \phi(s)| \, dt \, ds \quad (3.5)
$$

is dominated by the value at $t_0$ of the right-hand side of $\[ \text{Lemma 3.4.} \]$, whenever $t_0 \in (\alpha, \beta)$ and $\beta - \alpha \leq R^2/\nu^2$. However, if $\beta - \alpha > R^2/\nu^2$, then $\[ \text{Lemma 3.4.} \]$ is dominated by

$$
2 \int_{(\alpha, \beta)} I_{(0,r^2)} \phi \, dt \leq 2 \left( \int_{(\alpha, \beta)} I_{(0,r^2)} dt \right)^{1-1/p} \left( \int_{(\alpha, \beta)} \phi^p \, dt \right)^{1/p}
$$

$$
\leq 2(r_0^2/(\beta - \alpha))^{1-1/p} M_1^{1/p}(\phi^p)(t_0) \leq 2(\nu r_0/R)^{2-2/p} M_1^{1/p}(\phi^p)(t_0).
$$

In this case $\[ \text{Lemma 3.4.} \]$ is again less than the value at $t_0$ of the right-hand side of $\[ \text{Lemma 3.4.} \]$. By taking the supremum of $\[ \text{Lemma 3.4.} \]$ over all $\alpha < \beta$ such that $t_0 \in (\alpha, \beta)$ we obtain $\[ \text{Lemma 3.4.} \]$ at $t_0$. Since $t_0$ is arbitrary, the lemma is proved.

**Lemma 3.4.** There exists a constant $N$ depending only on $p, q, d, \kappa, K$, and the function $\omega$, such that for any $u \in C_0^\infty(\mathbb{R}^{d+1})$,

$$
||u_{xx}||_{L_{q,p}} + ||u_t||_{L_{q,p}} \leq N(||L u||_{L_{q,p}} + ||u_x||_{L_{q,p}} + ||u||_{L_{q,p}}). \quad (3.6)
$$
Proof. Notice that we included $\|u_x\|_{L_{q,p}}$ and $\|u\|_{L_{q,p}}$ on the right.
Therefore, while proving (3.6) we may certainly assume that $b \equiv 0$ and $c \equiv 0$. Since $u_t = Lu - a^{ij}u_{x_i x_j}$, we only need to estimate $u_{xx}$. If $p = q$ so that $L_{q,p} = L_p$, the result is known from \[14\].

In case $q > p$ we fix a number $r_0$ and first assume that $u(t,x) = 0$ for $t \not\in (0,r_0^2)$. Then set $f = Lu$ and also use other objects introduced before Lemma 3.3. We raise both parts of (3.4) to the power $q$, integrate over $\mathbb{R}$, and observe that since $q/p > 1$, by the Hardy-Littlewood theorem we have

$$\int_{\mathbb{R}} M_{t}^{q/p}(\psi^{p})(t) \, dt \leq N \int_{\mathbb{R}} \psi^{q}(t) \, dt = N \|f\|_{L_{q,p}}^{p},$$

$$\int_{\mathbb{R}} M_{t}^{q/p}(\phi^{p})(t) \, dt \leq N \|u_{xx}\|_{L_{q,p}}^{q}.$$  

We also use the Fefferman-Stein theorem and conclude that

$$\|u_{xx}\|_{L_{q,p}} \leq N_1 \nu^{(d+2)/p} \|f\|_{L_{q,p}}$$

$$+ N_2 \left( (\nu r_0/R)^{2-2/p} + \nu^{-1} + \nu^{(d+2)/p} \omega^{1/(2p)}(R) \right) \|u_{xx}\|_{L_{q,p}},$$

whenever $\nu \geq 16$ and $R \leq 1$, where $N_i$ are determined by $p, q, d, \kappa, K$ and the function $\omega$. We choose a large $\nu = \nu(N_2, d)$ and a small $R = R(N_2, d, q, \omega)$ so that

$$N_2 \left( \nu^{-1} + \nu^{(d+2)/p} \omega^{1/(2p)}(R) \right) \leq 1/4.$$  

After $\nu$ and $R$ have been fixed, we chose a small $r_0 = r_0(N_2, d, q, \omega)$ so that

$$N_2 (\nu r_0/R)^{2-2/q} \leq 1/4.$$  

Then (3.7) implies that

$$\|u_{xx}\|_{L_{q,p}} \leq 2 N_1 \nu^{(d+2)/p} \|f\|_{L_{q,p}}$$

for any $u \in C_{0}^{\infty}(\mathbb{R}^{d+1})$ such that $u(t,x) = 0$ if $t \not\in (0,r_0^2)$. We thus have obtained (3.6) even without the terms $\|u_x\|_{L_{q,p}}$ and $\|u\|_{L_{q,p}}$ on the right of (3.6).

Now take a nonnegative $\zeta \in C_{0}^{\infty}(\mathbb{R})$ such that $\zeta(t) = 0$ if $t \not\in (0,r_0^2)$ and

$$\int_{\mathbb{R}} \zeta^{p}(t) \, dt = 1.$$  

Also take a $u \in C_{0}^{\infty}(\mathbb{R}^{d+1})$ and observe that (3.8) is also true if we shift the $t$ axis. In particular, (3.8) is applicable to $u(t,x)\zeta(t-t_0)$. Then we
get
\[
\int_{\mathbb{R}} \zeta^q(t-t_0)\|u_{xx}(t,\cdot)\|_{L_p(\mathbb{R}^d)}^q \, dt \leq N \int_{\mathbb{R}} \zeta^q(t-t_0)\|Lu(t,\cdot)\|_{L_p(\mathbb{R}^d)}^q \, dt \\
+ N \int_{\mathbb{R}} |\zeta'(t-t_0)|^q\|u(t,\cdot)\|_{L_p(\mathbb{R}^d)}^q \, dt
\]
Upon integrating through with respect to \( t_0 \) we come to (5.6). The lemma is proved.

On the basis of this lemma by repeating almost word for word the proof of Theorem 4.1 of [13] (or using the method of proving Theorem 4.4 or Lemma 5.9) we obtain the following result.

**Theorem 3.5.** There are constants \( \lambda_0 \) and \( N \), depending only on \( p, K, \kappa, d, \) and \( \omega \), such that for any \( \lambda \geq \lambda_0 \) and \( u \in W^{1,2}_{q,p} \) we have
\[
\lambda\|u\|_{L_{q,p}} + \sqrt{\lambda}\|u_x\|_{L_{q,p}} + \|u_{xx}\|_{L_{q,p}} + \|u_t\|_{L_{q,p}} \leq N\|(L - \lambda)u\|_{L_{q,p}}.
\]
Furthermore, for any \( \lambda \geq \lambda_0 \) and \( f \in L_{q,p} \) there exists a unique \( u \in W^{1,2}_{q,p} \) such that \((L - \lambda)u = f\).

Finally, Theorem 3.5 implies Theorem 2.1 in the same way as Theorem 4.1 of [14] implies Theorem 2.1 of [14].

4. **Proof of Theorem 2.3**

We start with the following result which will be proved in Section 8 and which is an improvement of the key estimate found in the proof of Theorem 5.3 of [14]. We work in the setting in which Theorem 2.3 is stated.

**Lemma 4.1.** Let \( b = \hat{b} = 0, c = 0, f = (f^1, \ldots, f^d) \in L_{p,loc} \). Then there exists a constant \( N = N(d, \kappa, p, K, \omega) \) such that for any \( u \in H^1_{p,loc} \), \( \nu \geq 16 \), and \( r \in (0, 1/\nu) \), such that \( Lu = \text{div} \, f \) in \( Q_{rr} \), we have
\[
(|u_x - (u_x)|_Q^p)_Q \leq N\nu^{d+2}\mathcal{A}_{\nu r} + N(\nu^{-p} + \nu^{d+2}a_{1/2}^{1/2})\mathcal{B}_{\nu r},
\]
where
\[
\mathcal{A}_p = (|f|_Q)_Q, \quad \mathcal{B}_p = (|u_x|_Q)_Q, \quad a_{\nu}(x).
\]

The following is proved in the same way as Corollary 3.2.

**Corollary 4.2.** Let \( b = \hat{b} = 0, c = 0, u \in H^1_p((S, T)) \) for any finite \( S < T, Lu = \text{div} \, f \), where \( f = (f^1, \ldots, f^d) \in L_p((S, T) \times \mathbb{R}^d) \) for any finite \( S < T \). Then there exists a constant \( N = N(d, \kappa, p, K, \omega) \) such that for any \( \nu \geq 16 \) and \( r \in (0, 1/\nu) \) we have
\[
\int_{(0,r^2)} \int_{(0,r^2)} \|u_x(t,\cdot)\|_{L_p(\mathbb{R}^d)} - \|u_x(s,\cdot)\|_{L_p(\mathbb{R}^d)} \, dt ds
\]
\[ \leq N(\nu^{-p} + \nu^{d+2} \hat{a}^{1/2}) \int_{(0, \nu^2 r^2)} \|u_x(t, \cdot)\|_{L^p(\mathbb{R}^d)}^p dt \]
\[ + N\nu^{d+2} \int_{(0, \nu^2 r^2)} \|f(t, \cdot)\|_{L^p(\mathbb{R}^d)}^p dt. \]

After that in the same way as Lemma 3.4 is proved one derives its counterpart for divergence equations from Corollary 4.2.

**Lemma 4.3.** Let \( q \geq p, u \in H^1_{q,p}, b = \hat{b} = 0, c = 0, \nabla u = \text{div} f \) with \( f \in L_{q,p} \). Then there exists a constant \( N \), depending only on \( q, p, d, \kappa, K, \) and \( \omega \) such that
\[ \|u_x\|_{L^q,p} \leq N(\|f\|_{L^q,p} + \|u\|_{L^q,p}). \] (4.3)

Next we state and prove an analog of Lemma 5.5 of [14] where \( q = p \) and \( u \) is supposed to have small support.

**Theorem 4.4.** Let \( q \geq p, f = (f^1, \ldots, f^d), f^i, g \in L_{q,p}, u \in H^1_{q,p}, \lambda \in \mathbb{R}, \) and
\[ \nabla u - \lambda u = \text{div} f + g. \]
We assert that there exist constants \( \lambda_0, N \in (0, \infty) \), depending only on \( p, q, d, \kappa, K, \) and \( \omega \), such that
\[ \|u_t\|_{H^1_{q,p}} + \sqrt{\lambda}\|u_x\|_{L^q,p} + \lambda\|u\|_{L^q,p} \leq N(\sqrt{\lambda}\|f\|_{L^q,p} + \|g\|_{L^q,p}), \] (4.4) provided that \( \lambda \geq \lambda_0 \).

Proof. We follow the same pattern as in the proof of Lemma 5.5 of [14]. First, we observe that the terms \((\hat{b}u)_{x^i}\) and \(b'u_{x^i} + cu\) in \( \nabla u \) can be included in \( \text{div} f \) and \( g \), respectively. This will introduce new terms in the right-hand side of (4.4) but on the account of perhaps increasing \( \lambda_0 \) they can be absorbed into the left-hand side of (4.4). For this reason in the rest of the proof we may and will assume that \( b = \hat{b} = 0, c = 0 \).

In this case we use a method introduced by Agmon. Consider the space \( \mathbb{R}^{d+2} = \{(t, z) = (t, x, y) : t, y \in \mathbb{R}, x \in \mathbb{R}^d\} \) and the function
\[ \tilde{u}(t, z) = u(t, x)\zeta(y)\cos(\mu y), \] (4.5)
where \( \mu = \sqrt{\lambda} \) and \( \zeta \) is an odd \( C^\infty_0(\mathbb{R}) \)-function, \( \zeta \neq 0 \). Also introduce the operator
\[ \tilde{\mathcal{L}}u(t, z) = \nabla(t, x)u(t, z) + u_{yy}(t, z). \] (4.6)
As in [14] one checks that the coefficients of \( \tilde{\mathcal{L}} \) are \( VMO_x \)-functions (with respect to \( (t, z) \)).

Set \( \tilde{f}^i(t, z) = f^i(t, x)\zeta(y)\cos(\mu y) \) for \( i = 1, \ldots, d \) and
\[ \tilde{f}^{d+1}(t, z) = g(t, x)\zeta_1(y) - 2u(t, x)\zeta_2(y) + u(t, x)\zeta_3(y), \]
where
\[ \zeta_1(y) = \int_{-\infty}^{y} \zeta(s) \cos(\mu s) \, ds, \quad \zeta_3(y) = \int_{-\infty}^{y} \zeta''(s) \cos(\mu s) \, ds \]
\[ \zeta_2(y) = \mu \int_{-\infty}^{y} \zeta'(s) \sin(\mu s) \, ds = -\zeta'(y) \cos(\mu y) + \zeta_3(y). \]

Observe that \( \zeta_i \in C^\infty_0(\mathbb{R}) \) since \( \zeta \) is odd and has compact support. Furthermore, as is easy to check,
\[ \tilde{\zeta}(t, z) = (\tilde{f}^1(t, z), \dots + (\tilde{f}^d(t, z))_x^d + (\tilde{f}^{d+1}(t, z))_y. \]

We denote by \( \tilde{L}_p \) the \( L_p \) space of functions of \( z = (x, y) \) (avoiding using a confusing notation \( L_p(\mathbb{R}^{d+1}) \)) and by Lemma 4.3 obtain
\[ \int_{\mathbb{R}} \|\tilde{u}(t, \cdot)\|_{L_p}^q \, dt \leq N \sum_{i=1}^{d+1} \int_{\mathbb{R}} \|\tilde{f}^i(t, \cdot)\|_{L_p}^q \, dt + N \int_{\mathbb{R}} \|\tilde{u}(t, \cdot)\|_{L_p}^q \, dt. \tag{4.7} \]

Since
\[ \delta_0 := \int_{\mathbb{R}^d} |\zeta(y) \sin(\mu y)|^p \, dy, \quad \delta_1 := \int_{\mathbb{R}^d} |\zeta(y) \cos(\mu y)|^p \, dy \]
are bounded away from zero for \( \mu \geq 1 \), we get for each \( t \) and \( \mu \geq 1 \)
\[ \|u_x(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p = \delta_1^{-1} \int_{\mathbb{R}^{d+1}} |u_x(t, x)\zeta(y) \cos(\mu y)|^p \, dz \leq \delta_1^{-1} \|\tilde{u}_z(t, \cdot)\|_{L_p}^p, \]
\[ \|u(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p = \delta_0^{-1} \mu^{-p} \int_{\mathbb{R}^{d+1}} |\tilde{u}_y(t, z) - u(t, x)\zeta'(y) \cos(\mu y)|^p \, dz \leq N \mu^{-p}(\|\tilde{u}_z(t, \cdot)\|_{L_p}^p + \|u(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p). \]

It follows that if \( \mu \) is large enough, then
\[ \mu^p \|u(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p \leq N \|\tilde{u}_z(t, \cdot)\|_{L_p}^p. \]

Hence, by (4.7) for large \( \mu \)
\[ \mu^q \|\tilde{u}\|_{L_{q, p}}^q + \|u_x\|_{L_{q, p}}^q \leq N \sum_{i=1}^{d+1} \int_{\mathbb{R}} \|\tilde{f}^i(t, \cdot)\|_{L_p}^q \, dt + N \int_{\mathbb{R}} \|\tilde{u}(t, \cdot)\|_{L_p}^q \, dt. \tag{4.8} \]

Now we estimate the right-hand side of (4.8). Obviously,
\[ \|\tilde{f}^i(t, \cdot)\|_{L_p}^q \leq N \|f^i(t, \cdot)\|_{L_p(\mathbb{R}^d)}^q, \quad i = 1, \ldots, d, \]
\[ \|\tilde{u}(t, \cdot)\|_{L_p}^q \leq N \|u(t, \cdot)\|_{L_p(\mathbb{R}^d)}^q. \]

Furthermore,
\[ \zeta_1 = \mu^{-1} \left[ \zeta(y) \sin(\mu y) - \int_{-\infty}^{y} \zeta'(s) \sin(\mu s) \, ds \right], \]
which shows that $\zeta_1$ equals $\mu^{-1}$ times a uniformly bounded function with support not wider than that of $\zeta$. Hence,
\[
\|g\zeta_1(t, \cdot)\|_{L^p}^q \leq N\mu^{-q}\|g(t, \cdot)\|_{L^p(\mathbb{R}^d)}^q.
\]
Also $\zeta_2$ and $\zeta_3$ are uniformly bounded with support not wider than that of $\zeta$. Therefore,
\[
\|(2u\zeta_2 - u\zeta_3)(t, \cdot)\|_{L^p}^q \leq N\|u(t, \cdot)\|_{L^p(\mathbb{R}^d)}^q,
\]
\[
\|\tilde{f}^{d+1}(t, \cdot)\|_{L^p}^q \leq N\mu^{-q}\|g(t, \cdot)\|_{L^p(\mathbb{R}^d)}^q + N\|u(t, \cdot)\|_{L^p(\mathbb{R}^d)}^q.
\]
This and (4.8) yield (4.4) without the term with $u_t$. To estimate this term it suffices to observe that
\[
(1 - \Delta)^{-1/2}u_t = -(1 - \Delta)^{-1/2}D_j(a^{ij}u_{x^j} - f^j) + (1 - \Delta)^{-1/2}(\lambda u + g),
\]
so that, by the boundedness of $(1 - \Delta)^{-1/2}$ and $(1 - \Delta)^{-1/2}D_j$, for each $t$
\[
\|(1 - \Delta)^{-1/2}u_t(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq N(\|u_{x^j}(t, \cdot)\|_{L^p(\mathbb{R}^d)} + \lambda\|u(t, \cdot)\|_{L^p(\mathbb{R}^d)}
\]
\[
+ \|f(t, \cdot)\|_{L^p(\mathbb{R}^d)} + \|g(t, \cdot)\|_{L^p(\mathbb{R}^d)}).
\]
Upon raising both parts to the power $q$ and integrating over $t \in \mathbb{R}$ we get the required estimate of $u_t$. The theorem is proved.

A simple argument in Section 6 of [14] shows that
\[
\|u\|_{L^q, p} + \|u_{x^j}\|_{L^q, p} + \|u_t\|_{H^{-1}_{q, p}} \quad \text{and} \quad \|(1 - \Delta)^{-1/2}u\|_{W^{1, 2}_{q, p}}
\]
define equivalent norms in $H^1_{q, p}$. This argument also shows that, for each fixed $\lambda > 0$, the right-hand side of (4.4) dominates
\[
\||\text{div } f + g\|_{H^{-1}_{q, p}}
\]
and in turn one can find $\tilde{f}$ and $\tilde{g}$ so that $\text{div } f + g = \text{div } \tilde{f} + \tilde{g}$ and the right-hand side of (4.4) is dominated by
\[
N\|\text{div } \tilde{f} + \tilde{g}\|_{H^{-1}_{q, p}}.
\]
Therefore, Theorem 1.2 implies assertion (i) for $q \geq p$ in the following result.

**Theorem 4.5.** There is a constant $\lambda_0$ depending only on $p, q, d, \kappa, K, \omega$ such that for any $\lambda \geq \lambda_0$
\[(i) \text{ for any } u \in H^1_{q, p} \text{ we have}
\[
\|u\|_{H^1_{q, p}} \leq N(\lambda, p, d, \kappa, K, \omega)\|\mathcal{L}u - \lambda u\|_{H^{-1}_{q, p}};
\]
\[(ii) \text{ for any } h \in H^{-1}_{q, p} \text{ there exists a unique } u \in H^1_{q, p} \text{ such that } \mathcal{L}u - \lambda u = h.
\]
Proof. It is a classical result that for any \( \lambda > 0 \) and \( g \in L_{q,p} \) there exists a (unique) solution \( w \in W^{1,2}_{q,p} \) of \( \Delta w + w_t - \lambda w = g \) and one even can give \( w \) by a formula (see, for instance, Theorem 4.2 of [13] and the references in [13]). Then \( u := (1 - \Delta)^{1/2} w \) is in \( H^{1}_{q,p} \) and satisfies \( \Delta u + u_t - \lambda u = h \) with \( h = (1 - \Delta)^{1/2} g \). As \( g \) runs through \( L_{q,p} \), \( h \) runs through \( H^{-1}_{q,p} \) by definition.

Hence, the present theorem holds if \( Lu = \Delta u + u_t \). By what has been said before the theorem the a priori estimate (4.9) holds if \( q \geq p \). Then by the method of continuity assertion (ii) also holds if \( q \geq p \).

The case \( 1 < q < p \) is considered in a standard way by duality owing to the fact that the formally adjoint operator to \( L \) has the same structure as \( L \) only with reversed time axis. The theorem is proved.

Finally, Theorem 2.1 is derived from Theorem 4.5 in the same way as in similar situations in [14].

5. New approach to the \( L_p \)-theory for equations with VMO coefficients

We take an \( a \in \mathbb{A} \) and set \( \tilde{L}u(t, x) = a^{ij}(t)u_{x_i x_j}(t, x) + u_t(t, x) \).

In this section \( p \in (1, \infty) \) and \( \lambda \geq 0 \) unless explicitly specified otherwise.

Here we give a new proof of the following result from [14], which is a simplified version of Lemma 3.1.

**Theorem 5.1.** There is a constant \( N \), depending only on \( d, p, K, \) and \( \kappa \), such that for any \( u \in W^{1,2}_{p,\text{loc}}, \ r \in (0, \infty), \) and \( \nu \geq 4 \)

\[
( |u_{xx}(t, x) - (u_{xx})_{Q_r}|^p)_{Q_r} \leq N \nu^{p+2} (|\tilde{L}u|^p)_{Q_r} + N \nu^{-p} (|u_{xx}|^p)_{Q_r}. \tag{5.1}
\]

In [14] Theorem 5.1 is proved on the basis of solving boundary-value problems for parabolic equations. The proof we give later in the section is based on solvability of equations in the whole space and extends to more general operators and systems of equations without much effort. In particular, we will see that, once the solvability theory for operators \( \tilde{L} \) is developed in \( W^{1,2}_p(\mathbb{R}^{d+1}) \) for a \( p > 1 \), Theorem 5.1 becomes available and, according to simple arguments from [14], the solvability theory in \( W^{1,2}_q \) with \( q > p \) for equations with VMO \(_x\) coefficients becomes available as well.

This fact has the following methodological implication. If \( p = 2 \) one can construct the solvability theory for \( \tilde{L} \) in \( W^{1,2}_2(\mathbb{R}^{d+1}) \) by using the Fourier transform. Then by the above (or by what is done in Remark 5.12 below), the solvability theory for \( \tilde{L} \) in \( W^{1,2}_p(\mathbb{R}^{d+1}) \) with \( p > 2 \) is
available. By duality one gets it for $p \in (1, 2)$ as well and as has been pointed out, this is the only thing one needs to construct the solvability theory for operators with $VMO_x$ coefficients in $W^{1,2}_p(\mathbb{R}^{d+1})$, $p \in (1, \infty)$.

As usual, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \ldots\}$, we set
\[
D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad D_i u = \frac{\partial u}{\partial x^i}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.
\]

**Lemma 5.2.** Take $p \in [1, \infty)$ and $N_0 \in (0, \infty)$ and assume that for any $u \in W^{1,2}_p(\mathbb{R}^{d+1})$ we have
\[
\|u_t\|_{L^p(\mathbb{R}_0^{d+1})} + \|u_{xx}\|_{L^p(\mathbb{R}_0^{d+1})} \leq N_0(\|Lu\|_{L^p(\mathbb{R}^{d+1})} + \|u\|_{L^p(\mathbb{R}^{d+1})}). \tag{5.2}
\]

Then for any $0 < r < R < \infty$ there exists a constant $N$, depending only on $N_0$, $d$, $p$, $K$, $r$, and $R$, such that for any $u \in W^{1,2}_p(Q_R)$ we have
\[
\|u_t\|_{L^p(Q_r)} + \|u_{xx}\|_{L^p(Q_r)} \leq N(\|Lu\|_{L^p(Q_R)} + \|u_x\|_{L^p(Q_R)} + \|u\|_{L^p(Q_R)}). \tag{5.3}
\]

This is a trivial result, which is obtained by taking an appropriate cut-off function $\zeta$ and applying (5.2) to $u \zeta$.

**Remark 5.3.** With a little extra work (see the proof of Lemma 4.2 of [12]) one shows that the term with $u_x$ on the right in (5.3) can be dropped.

Another general result we need is a parabolic analog of Poincaré’s inequality. It is proved in the same way as Lemma 3.2 of [14] (also see Lemma 4.2 of [18]). We generalize Lemma 5.4 in Lemmas 6.2 and 6.4.

**Lemma 5.4.** Let $p \in [1, \infty)$. Then there is a constant $N = N(d, p)$ such that for any $r \in (0, \infty)$ and $u \in C_{\text{loc}}^\infty(\mathbb{R}^{d+1})$ we have
\[
\int_{Q_r} |u_x(t, x) - (u_x)_{Q_r}|^p \, dx \, dt \leq Nr^p \int_{Q_r} (|u_{xx}|^p + |u_t|^p) \, dx \, dt, \tag{5.4}
\]
\[
\int_{Q_r} |u(t, x) - u_{Q_r} - x^i(u_x)_{Q_r}|^p \, dx \, dt \leq Nr^2p \int_{Q_r} (|u_{xx}|^p + |u_t|^p) \, dx \, dt. \tag{5.5}
\]

We need the following classical result (which can be obtained, for instance, along the lines discussed after Theorem 5.1).

**Theorem 5.5.** There is a constant $N = N(p, d, \kappa, K)$ such that for any $\lambda \geq 0$, $T \in [-\infty, \infty)$, and $u \in W^{1,2}_p(\mathbb{R}^{d+1})$ we have
\[
\lambda\|u\|_{L^p(\mathbb{R}_0^{d+1})} + \|u_{xx}\|_{L^p(\mathbb{R}_0^{d+1})} + \|u_t\|_{L^p(\mathbb{R}_0^{d+1})} \leq N\|Lu - \lambda u\|_{L^p(\mathbb{R}_0^{d+1})}.
\]
Furthermore, for any \( \lambda > 0 \) and \( f \in L_p(\mathbb{R}^{d+1}_T) \) there exists a unique \( u \in W^{1,2}_p(\mathbb{R}^{d+1}_T) \) such that \( \bar{L}u + u_t - \lambda u = f \).

**Remark 5.6.** Owing to Theorem 5.5 the assertion of Lemma 5.2 holds with \( \bar{L} \) in place of \( L \).

**Remark 5.7.** In the proof of Theorem 5.1 we will use the decomposition \( f := \bar{L}u - \lambda u = g + h, \) where roughly speaking \( g = f|_{Q_{\nu r}} \), and accordingly have \( u = v + w \), where \( v \) is defined by the equation \( Lv - \lambda v = h \). The function \( v \) is “harmonic” in \( Q_{\nu r} \) in the sense that \( h = 0 \) there. Then the oscillation of \( u \) will be estimated by using Theorem 5.5 and that of \( v \) will be derived from what follows. Observe that since we solve the equation \( \bar{L}v - \lambda v = h \) in the whole space \( \mathbb{R}^{d+1} \) we need \( \lambda > 0 \).

**Lemma 5.8.** Take \( 0 < r < R < \infty \) and let \( m \in \{0,1,2,\ldots\} \). Take a function \( u \in W^{1,2}_p(Q_R) \) and assume that \( \bar{L}u \) vanishes in \( Q_R \). Then for any multi-index \( \alpha \) the derivatives \( \bar{L}^\alpha u \) and \( D^{\alpha}u_t \) are bounded in \( Q_r \) and, with \( N = N(|\alpha|, d, \kappa, K, r, R, p) \),

\[
\sup_{Q_r} |D^\alpha u| \leq N(\|u\|_{L_p(Q_R)} + \|u\|_{L_p(Q_R)}^\beta) =: NI, \quad \sup_{Q_r} |D^\alpha u_t| \leq NI.
\]

(5.6)

Proof. Since the coefficients of \( \bar{L} \) are independent of \( x \) we can mollify the function \( u \) with respect to \( x \) and have equation \( \bar{L}u = 0 \) in slightly smaller domain than \( Q_R \) for \( \bar{u} \) being the mollified \( u \). Then, if the result is true for \( \bar{u} \), we can pass to the limit as the support of the mollification kernel shrinks to the origin. It follows that without losing generality we may assume that \( D^\beta u \in W^{1,2}_p(Q_R) \) for any \( \beta \). Then, since \( D^\beta u_t = -a^\beta D^\beta u_{x,\beta} \) in \( Q_R \), we also have \( D^\beta u_t \in W^{1,2}_p(Q_R) \) for any \( \beta \).

By Remark 5.6 applied to \( D^\beta u \), for each integer \( k \geq 0 \) and \( r < r_1 < r_2 < R \) we have

\[
\sum_{|\beta| \leq k} \|D^\beta u_t\|_{L_p(Q_{r_1})} + \sum_{|\beta| \leq k+2} \|D^\beta u\|_{L_p(Q_{r_1})} \leq N \sum_{|\beta| \leq k+1} \|D^\beta u\|_{L_p(Q_{r_2})},
\]

\[
\sum_{|\beta| \leq k+1} \|D^\beta u\|_{L_p(Q_{r_1})} \leq N\left( \sum_{|\beta| \leq k} \|D^\beta u\|_{L_p(Q_{r_2})} + \|u\|_{L_p(Q_{r_2})} \right).
\]

By iterating the last relation we see that

\[
\sum_{|\beta| \leq k+1} \|D^\beta u\|_{L_p(Q_{s_1})} \leq N(\|u\|_{L_p(Q_{s_2})} + \|u\|_{L_p(Q_{s_2})}) \leq NI,
\]

whenever \( r < s_1 < s_2 < R \). Hence,

\[
\sum_{|\beta| \leq k} \|D^\beta u_t\|_{L_p(Q_r)} + \sum_{|\beta| \leq k+2} \|D^\beta u\|_{L_p(Q_r)} \leq NI.
\]
Furthermore, obviously
\[ |D_t^\beta D^\alpha u(t, \cdot)|_{L_p(B_r)} \leq \|D^\beta u(t, \cdot)|_{L_p(B_r)}. \]

Therefore, for \( \phi^\beta(t) := \|D^\beta u(t, \cdot)|_{L_p(B_r)} \) by embedding theorems we have
\[
\sup_{[0,r^2]} \phi^\beta \leq N(\|\phi^\beta\|_{L_p(0,r^2)} + \|\phi_t^\beta\|_{L_p(0,r^2)}) \\
= N(\|D^\beta u\|_{L_p(Q_r)} + \|D^\beta u_t\|_{L_p(Q_r)}) \leq NI.
\]

Thus,
\[
\sup_{[0,r^2]} \sum_{|\beta| \leq k} \|D^\beta u(t, \cdot)|_{L_p(B_r)} \leq NI.
\]

By embedding theorems, if \( k \) is large enough, then
\[
\sup_{x \in B_r} |D^\alpha u(t, x)| \leq N \sum_{|\beta| \leq k} \|D^\beta u(t, \cdot)|_{L_p(B_r)}
\]
and this leads to the first estimate in (5.6). One gets the second one from the equation \( D^\alpha u_t = -a^\beta D^\alpha u_{x^\beta} \). The lemma is proved.

Below, for an integer \( m \geq 0 \), by \( D^m u(t, x) \) we mean the collection of all \( m \)th order derivatives of \( u \) with respect to \( x \). In the set of these collection we define a Euclidean norm \( |D^m u(t, x)| \).

**Lemma 5.9.** Let \( m \in \{0, 1, 2, \ldots\} \), \( \lambda \geq 0 \), and \( u \in C_0^\infty(\mathbb{R}^{d+1}) \). Assume that \( \bar{L}u - \lambda u \) vanishes in \( Q_2 \). Then, with \( N = N(d, m, \kappa, p, K) \),
\[
\max_{Q_2} (|D^m u_{xx}|^p + |D^m u_t|^p) \leq N \int_{Q_2} (|u_{xx}|^p + |u_t|^p + \lambda^p/2|u|^p) \, dx \, dt.
\]

(5.7)

**Proof.** By Lemma 5.8
\[
I := \max_{Q_1} (|D^m u_{xx}|^p + |D^m u_t|^p) \leq N(\|u_{xx}\|^p_{L_p(Q_{3/2})} + \|u_t\|^p_{L_p(Q_{3/2})}).
\]

If \( \lambda = 0 \) we can replace here \( u \) with \( v := u - u_{Q_2} - x^\beta(u_{x^\beta})_{Q_2} \) without violating the fact that \( \bar{L}u + u_t \) vanishes in \( Q_2 \) or changing the left-hand side. Therefore,
\[
I \leq N(\|v_x\|^p_{L_p(Q_2)} + \|v_t\|^p_{L_p(Q_2)}),
\]
and using Lemma 5.4 yields the desired result.

In the general case that \( \lambda \geq 0 \) we again use a method suggested by S. Agmon. Introduce the function \( \hat{u}(t, z) = \hat{u}(t, x, y) \) by
\[
\hat{u}(t, z) = u(t, x) \cos(\sqrt{\lambda}y)
\]
and set
\[
\hat{Q}_r = (0, r^2) \times \{|z| < r\}.
\]
Obviously,
\[ D^m u_{xx}(t, x) = D^m \hat{u}_{xx}(t, x, 0), \quad D^m u_t(t, x) = D^m \hat{u}_t(t, x, 0) \]
Therefore,
\[ I \leq \max_Q \left( |D^m \hat{u}_{xx}|^p + |D^m \hat{u}_t|^p \right). \]
However,
\[ \hat{L} \hat{u} + \hat{u}_{yy} = 0 \quad \text{in} \quad \hat{Q}_2, \]
so that we can apply the above result to \( \hat{u} \) and conclude
\[ I \leq N \int_{\hat{Q}_2} (|\hat{u}_{zz}|^p + |\hat{u}_t|^p) \, dz \, dt. \quad (5.8) \]
Here the term \( \hat{u}_{zz} \) is the collection consisting of
\[ u_{xx} \cos(\sqrt{\lambda} y), \quad -\sqrt{\lambda} u_x \sin(\sqrt{\lambda} y), \quad \text{and} \quad -\lambda u \cos(\sqrt{\lambda} y). \]
This fact allows us to estimate the right-hand side of (5.8) and yields
\[ I \leq N \int_{\hat{Q}_2} (|u_{xx}|^p + |u_t|^p + \lambda^{p/2} |u_x|^p + \lambda^p |u|^p) \, dx \, dt. \quad (5.9) \]
This is all we need since \( \lambda |u| = |\hat{L} u| \) in \( Q_2 \) and the term \( \lambda^p |u|^p \) can be absorbed in \( |u_{xx}|^p + |u_t|^p \). The lemma is proved.

Now comes the estimate of \( v \) we were talking about in Remark 5.7.

**Theorem 5.10.** Let \( \lambda \geq 0, \nu \geq 2, \) and \( r \in (0, \infty) \) be some constants. Let \( u \in C^\infty_{\text{loc}}(\mathbb{R}^{d+1}) \) be such that \( f := \hat{L} u - \lambda u \) vanishes in \( Q_{\nu r} \). Then there is a constant \( N = N(d, \kappa, K, p) \) such that
\[ (|u_{xx}(t, x) - (u_{xx}Q_r)|^p)_{Q_r} \leq N \nu^{-p}(|u_{xx}|^p + |u_t|^p + \lambda^{p/2} |u_x|^p)_{Q_{\nu r}}. \quad (5.10) \]

Proof. Notice that \( v(t, x) := u(tr^2, xr) \) satisfy
\[ (|u_{xx}(t, x) - (u_{xx}Q_r)|^p)_{Q_r} = r^{-2p}(|v_{xx}(t, x) - (v_{xx}Q_1)|^p)_{Q_1}, \]
\[ (|u_{xx}|^p + |u_t|^p + \lambda^{p/2} |u_x|^p)_{Q_{\nu r}} = r^{-2p}(|v_{xx}|^p + |v_t|^p + \lambda^{p/2} r^p |v_x|^p)_{Q_r}, \]
and
\[ \hat{L}(tr^2)v(t, x) = r^2 \nu v(t, x) = r^2 f(tr^2, xr) \]
which vanishes in \( Q_{\nu} \). It follows that if (5.10) holds for \( r = 1 \), then it holds for any \( r > 0 \).

Therefore, in the rest of the proof we assume that \( r = 1 \) and observe that the left-hand side of (5.10) with \( r = 1 \) is obviously less than a constant \( N = N(d) \) times
\[ \max_{Q_1} (|u_{xxx}|^p + |u_{txx}|^p). \]
Therefore, we need only prove that
\[
\max_{Q_1} (|u_{xxx}|^p + |u_{txx}|^p) \leq N \nu^{-p} (|u_{xx}|^p + |u_t|^p + \lambda^{p/2} |u_x|^p)_{Q_{\nu}}. \tag{5.11}
\]
Observe that the function \(w(t, x) = u(t^{\nu^2/4}x^{\nu^2/2})\) satisfies
\[
\bar{L}(t^{\nu^2/4})w(t, x) - w(t, x)\nu^2 \lambda/4 = 0
\]
in \(Q_2\) and
\[
\left(|u_{xx}|^p + |u_t|^p + \lambda^{p/2} |u_x|^p\right)_{Q_{\nu}} = (2/\nu)^{2p} (|w_{xx}|^p + |w_t|^p + (\nu^2 \lambda/4)^{p/2} |w_x|^p)_{Q_{2/\nu}},
\]
\[
\max_{Q_1} |u_{xxx}|^p = (2/\nu)^{3p} \max_{Q_{2/\nu}} |w_{xxx}|^p \leq (2/\nu)^{3p} \max_{Q_1} |w_{xxx}|^p,
\]
\[
\max_{Q_1} |u_{txx}|^p \leq (2/\nu)^{4p} \max_{Q_1} |w_{txx}|^p.
\]
It follows that if (5.11) is true with \(\nu = 2\), then
\[
\max_{Q_1} (|u_{xxx}|^p + |u_{txx}|^p) \leq N \nu^{-3p} \max_{Q_1} (|w_{xxx}|^p + |w_{txx}|^p)
\]
\[
\leq N \nu^{-3p} (|w_{xx}|^p + |w_t|^p + (\nu^2 \lambda/4)^{p/2} |w_x|^p)_{Q_{2}}
\]
\[
= N \nu^{-p} (|u_{xx}|^p + |u_t|^p + \lambda^{p/2} |u_x|^p)_{Q_{\nu}}.
\]
Finally, (5.11) with \(\nu = 2\) is indeed true by Lemma 5.9 and the theorem is proved.

**Remark 5.11.** According to Theorem 7.4, applied to \(u_x\) in place of \(u\), the term \(|u_t|^p\) in (5.10) can be dropped.

**Proof of Theorem 5.1** In Remark 5.7 we explained that we need \(\lambda > 0\) to guarantee that certain equations have solutions. Therefore we take a \(\lambda > 0\), which in the end will be sent to 0.

Fix \(r \in (0, \infty)\) and \(\nu \geq 4\). We may certainly assume that \(a^{ij}\) are infinitely differentiable and have bounded derivatives. Also changing \(u\) for large \(|t| + |x|\) does not affect (5.11). Therefore, we may assume that \(u \in W_p^{1,2}\) and moreover \(u \in C_0^\infty(\mathbb{R}^{d+1})\). In that case define
\[
f = f_\lambda = \bar{L}u - \lambda u.
\]
Observe that \(f \in C_0^\infty(\mathbb{R}^{d+1})\). Also take a \(\zeta \in C_0^\infty(\mathbb{R}^{d+1})\) such that \(\zeta = 1\) on \(Q_{\nu r/2}\) and \(\zeta = 0\) outside \(Q_{\nu r} - Q_{\nu r}\) and set
\[
g = f\zeta, \quad h = f(1 - \zeta).
\]
Finally define \(v\) as the unique solution in \(W_p^{1,2}\) of the equation
\[
\bar{L}v - \lambda v = h.
\]
Since \( \lambda > 0 \), by classical theory we know that such a \( v \) indeed exists and is unique and infinitely differentiable. Since \( h = 0 \) in \( Q_{\nu r/2} \) and \( \nu/2 \geq 2 \), by Theorem 5.10 we obtain
\[
(\|v_{xx} - (v_{xx})_{Q_r}|^p)_{Q_r} \leq N\nu^{-p}(|v_{xx}|^p + |v_t|^p + \lambda^{p/2}|v_x|^p)_{Q_{\nu r/2}}
\]
\[
\leq N\nu^{-p}(|v_{xx}|^p + |v_t|^p + \lambda^{p/2}|v_x|^p)_{Q_r}.
\]

(5.12)

On the other hand the function \( w := u - v \in W^{1,2}_p \) satisfies
\[
Lw - \lambda w = g
\]

and by Theorem 5.5
\[
\int_{\mathbb{R}^{d+1}} (|w_t|^p + |w_{xx}|^p + \lambda^{p/2}|w_x|^p) \, dx \, dt
\]
\[
\leq N \int_{\mathbb{R}^{d+1}} |g|^p \, dx \, dt \leq N \int_{Q_r} |f|^p \, dx \, dt,
\]

(5.13)

\[
\int_{Q_r} |w_{xx}|^p \, dx \, dt \leq N \int_{Q_r} |f|^p \, dx \, dt,
\]

(5.14)

By combining this with (5.12) and observing that \( u = v + w \) and
\[
I := (\|u_{xx} - (u_{xx})_{Q_r}|^p)_{Q_r} \leq 2^p(\|w_{xx} - (w_{xx})_{Q_r}|^p)_{Q_r}
\]
\[
+ 2^p(\|v_{xx} - (v_{xx})_{Q_r}|^p)_{Q_r} \leq N(\|w_{xx}|^p)_{Q_r} + 2^p(\|v_{xx} - (v_{xx})_{Q_r}|^p)_{Q_r},
\]

we get
\[
I \leq N\nu^{d+2}(|f|^p)_{Q_r} + N\nu^{-p}(\|v_{xx}|^p + |v_t|^p + \lambda^{p/2}|v_x|^p)_{Q_{\nu r}}
\]
\[
\leq N\nu^{d+2}(|f|^p)_{Q_r} + N\nu^{-p}(\|u_{xx}|^p + |u_t|^p + \lambda^{p/2}|u_x|^p)_{Q_{\nu r}}
\]
\[
+ N\nu^{-p}(\|w_{xx}|^p + |w_t|^p + \lambda^{p/2}|w_x|^p)_{Q_{\nu r}}.
\]

Here by (5.13)
\[
(\|w_{xx}|^p + |w_t|^p + \lambda^{p/2}|w_x|^p)_{Q_{\nu r}} \leq N(|f|^p)_{Q_{\nu r}}
\]

and since \( \nu \geq 1 \) we conclude
\[
I \leq N\nu^{d+2}(|f|^p)_{Q_{\nu r}} + N\nu^{-p}(\|u_{xx}|^p + |u_t|^p + \lambda^{p/2}|u_x|^p)_{Q_{\nu r}}.
\]

To get (5.11) it only remains to use that \( u_t = f_\lambda + \lambda u - a^{ij}u_{x^i x^j} \) and let \( \lambda \downarrow 0 \). The theorem is proved.
Remark 5.12. Recall that for \( \phi \in L_{1,\text{loc}} \) the sharp function \( \phi^\# \) and the maximal function \( M\phi \) are defined by
\[
\phi^\#(t, x) = \sup_{Q \ni (t, x)} (|\phi - \phi_Q|)_Q, \quad Mf(t, x) = \sup_{Q \ni (t, x)} \phi_Q.
\]
In this notation Theorem 5.1 and Hölder’s inequality imply that on \( \mathbb{R}^{d+1} \) we have
\[
(u_{xx})^\# \leq N\nu^{(d+2)/p} M^{1/p}(|\bar{L}u|^p) + N\nu^{-1} M^{1/p}(|u_{xx}|^p).
\]
Then by using the Fefferman-Stein theorem we obtain for any \( q > p \)
\[
\|u_{xx}\|_{L_q} \leq N\| (u_{xx})^\# \|_{L_q} \leq N\nu^{(d+2)/p}\|\bar{L}u\|_{L_q} + N\nu^{-1}\|u_{xx}\|_{L_q},
\]
where the second inequality holds since \( \|M^{1/p}\phi\|_{L_q} \leq N\|\phi^{1/p}\|_{L_q} \) by the Hardy-Littlewood theorem. For \( \nu \) large enough we absorb the last term into the left-hand side and get
\[
\|u_{xx}\|_{L_q} \leq N\|\bar{L}u\|_{L_q}. \quad (5.15)
\]
This and what is said after Theorem 5.1 allow us to give one more proof of Theorem 5.5.

To summarize, after having proved Theorem 5.1 one can follow the same way as in [14] and get the solvability of equations with \( VMO \) leading coefficients.

In particular, we have the following result.

**Theorem 5.13.** There are constants \( \lambda_0 \) and \( N \), depending only on \( p \), \( K \), \( \kappa \), \( d \), and \( \omega \), such that for any \( \lambda \geq \lambda_0 \) and \( u \in W^{1,2}_p \) we have
\[
\lambda\|u\|_{L_p} + \sqrt{\lambda}\|u_x\|_{L_p} + \|u_{xx}\|_{L_p} + \|u_t\|_{L_p} \leq N\|(L - \lambda)u\|_{L_p}. \quad (5.16)
\]
Furthermore, for any \( \lambda \geq \lambda_0 \) and \( f \in L_p \) there exists a unique \( u \in W^{1,2}_p \) such that \( (L - \lambda)u = f \).

**Corollary 5.14.** There is a constant \( N_0 \), depending only on \( p \), \( K \), \( \kappa \), \( d \), and \( \omega \), such that for any \( u \in W^{1,2}_p(\mathbb{R}^{d+1}_0) \) we have
\[
\|u_{xx}\|_{L_p(\mathbb{R}^{d+1}_0)} + \|u_t\|_{L_p(\mathbb{R}^{d+1}_0)} \leq N_0(\|Lu\|_{L_p(\mathbb{R}^{d+1}_0)} + \|u\|_{L_p(\mathbb{R}^{d+1}_0)}). \quad (5.17)
\]
To prove this we first claim that [5.16] with \( L_p(\mathbb{R}^{d+1}_0) \) in place of \( L_p \) holds for any \( u \in W^{1,2}_p(\mathbb{R}^{d+1}_0) \).

Indeed, for such a \( u \) set \( f(t, x) = I_{t>0}(L - \lambda)u(t, x) \), let \( v \in W^{1,2}_p \) be any function on \( \mathbb{R}^{d+1} \) coinciding with \( u \) for \( t > 0 \), and set \( g = (L - \lambda)v \). Then find \( w \in W^{1,2}_p \) such that \( (L - \lambda)w = f \) and observe that \( (L - \lambda)(v - w) = g - f \) vanishes for \( t > 0 \). One can solve the equation \( (L - \lambda)\phi = g - f \) by the method of continuity starting from \( L = \Delta + D_t \), for which the solutions vanish for \( t > 0 \) if the right-hand
side does that, and then one sees that \( v = w \) for \( t > 0 \). This means that \( u = w \) for \( t > 0 \). Since estimate (5.16) holds with \( w \) in place of \( u \) and \( f \) in place of \((L - \lambda)u\), we get our claim.

After that it suffices to take \( \lambda = \lambda_0 \) and observe that
\[
\|Lu - \lambda_0 u\|_{L^p(\mathbb{R}^{d+1}_0)} \leq \|Lu\|_{L^p(\mathbb{R}^{d+1}_0)} + \lambda_0 \|u\|_{L^p(\mathbb{R}^{d+1}_0)}. 
\]

6. Proof of Lemma 3.1

The program of proof is to use Theorem 5.1 but replace \( \bar{L}u \) in (5.1) with \( Lu \). The error term we estimate by using Hölder’s inequality and on the account of right choice of \( \bar{L} \) come to (3.1) with \( (|u_{xx}|^2)^{1/2} \) in place of \( B_\rho \). Then the main issue is how to reduce power \( 2p \) back to \( p \). It turns out that this is possible if \( u \) is “harmonic” in \( Q_2 \) (see Corollary 5.4). After that we use the same kind of decomposition of \( u \) as in Remark 5.7. As in Lemma 3.1 we assume that \( p \in (1, \infty) \), \( b = 0 \), and \( c = 0 \).

We need two versions of Lemma 5.4 when the powers of summability on the right are less than on the left. Similar estimate is known even with \( \nu = 1 \) for the elliptic case as Poincaré’s inequality.

**Lemma 6.1.** Let \( q \geq 1 \), \( \nu \in (1, \infty) \),
\[
\frac{1}{q} < \frac{2}{d+2} + \frac{1}{p}. \tag{6.1}
\]
Then there is a constant \( N = N(d, p, q, \nu) \) such that for any \( u \in W^{1,2}_{q,\text{loc}} \) and \( r \in (0, \infty) \) we have
\[
(|u(t, x) - u_{Q_\nu} - x^i(u_{x^i})_{Q_\nu}|^q)^{1/p} \leq Nr^2(|u_{xx}|^q + |u_t|^q)^{1/q} \tag{6.2}
\]

Proof. First, observe that an argument based on self-similarity reduces the case of general \( r \) to the case that \( r = 1 \), the one we confine ourselves to. Then by obvious reasons we may assume that \( u \in C^\infty_0(\mathbb{R}^{d+1}) \). Finally, if \( q \geq p \), the result follows from Lemma 5.4 and Hölder’s inequality. Therefore, we assume that \( q \leq p \).

Take an infinitely differentiable function \( \zeta \) on \( \mathbb{R}^{d+1} \) such that \( \zeta = 1 \) on \( Q_1 \) and \( \zeta = 0 \) on \( \mathbb{R}^{d+1}_0 \setminus Q_\nu \), and set
\[
f = \Delta u + u_t, \quad v = \zeta(u - u_{Q_\nu} - x^i(u_{x^i})_{Q_\nu}),
\]
so that
\[
\Delta v + v_t = \zeta f + (u - u_{Q_\nu} - x^i(u_{x^i})_{Q_\nu})(\Delta \zeta + \zeta_t) + 2\zeta x^i(u_{x^i} - (u_{x^i})_{Q_\nu}) =: -g.
\]
Since $v \in C_0^\infty(\mathbb{R}^{d+1})$, we have

$$v(t, x) = \int_0^\infty \int_{\mathbb{R}^d} g(t+s, x+y)p(s, y) \, dy \, ds,$$

$$p(s, y) = \frac{1}{(4\pi s)^{d/2}} e^{-|y|^2/(4s)}.$$

Here, if $0 \leq t \leq 1$, there is no need to integrate with respect to $s$ beyond $[0, \nu^2]$, since $g(r, z) = 0$ for $r \geq \nu^2$. Therefore, upon denoting

$$\bar{v}(t, x) = |v(t, x)|I_{t \in [0,1]}, \quad \bar{g}(s, y) = |g(s, y)|I_{s \in [0,\nu^2]},$$

$$\bar{p}(s, y) = p(s, y)I_{s \in [0,\nu^2]},$$

we find

$$\bar{v}(t, x) \leq \int_{\mathbb{R}^{d+1}} \bar{g}(t+s, x+y)\bar{p}(s, y) \, dx \, ds$$

$$= \int_{\mathbb{R}^{d+1}} \bar{g}(t-s, x-y)\bar{p}(-s, y) \, dx \, ds.$$

Now we apply Young’s inequality

$$\|\bar{g} \ast \bar{p}\|_{L_p} \leq \|\bar{g}\|_{L_q} \|\bar{p}\|_{L_r}, \quad (6.3)$$

where

$$r = \frac{pq}{q-p+pq},$$

$r \geq 1$ since $q \leq p$, and $p^{-1} + 1 = q^{-1} + r^{-1}$. Also

$$rd < d + 2 \quad (6.4)$$

due to (6.1). Then we find

$$\|v\|_{L_p(Q_1)} \leq \|\bar{v}\|_{L_p} \leq \|g\|_{L_q(Q_\nu)} \|p\|_{L_r([0,\nu^2] \times \mathbb{R}^d)}. \quad (6.5)$$

Here by the definition of $g$ and Lemma 5.2 (just in case, recall that $N$ in (6.2) is allowed to depend on $\nu$)

$$\|g\|_{L_q(Q_\nu)} \leq N(\|u_{xx}\|_{L_q(Q_\nu)} + \|u_t\|_{L_q(Q_\nu)}).$$

Furthermore, changing variables shows that the integral

$$\int_{\mathbb{R}^d} t^{-d/2} e^{-r|x|^2/(4t)} \, dx$$

is finite and independent of $t > 0$. Therefore,

$$\|p\|_{L_r([0,4] \times \mathbb{R}^d)} = N \int_0^4 t^{-rd/2+d/2} \int_{\mathbb{R}^d} t^{-d/2} e^{-r|x|^2/(4t)} \, dx \, dt$$

$$= N \int_0^4 t^{-rd/2+d/2} \, dt < \infty,$$

where the inequality holds since owing to (6.4) we have $-rd/2 + d/2 > -1$. 


Thus, (6.5) implies that
\[ \|v\|_{L^p(Q_1)} \leq N(\|u_{xx}\|_{L^q(Q_\nu)} + \|u_t\|_{L^q(Q_\nu)}) \]
and it only remains to observe that the left-hand side here coincides with the left-hand side of (6.2). The lemma is proved.

Similar estimate holds for \( u_x - (u_x)_{Q_\nu} \).

**Lemma 6.2.** Let \( q \geq 1, \nu \in (1, \infty) \),
\[ \frac{1}{q} \leq \frac{1}{d + 2} + \frac{1}{p}. \]  
(6.6)

Then there is a constant \( N = N(d, p, q, \nu) \) such that for any \( u \in W^{1,2}_{q,loc} \) and \( r \in (0, \infty) \) we have
\[ (|u_x(t, x) - (u_x)_{Q_\nu}|^p)^{1/p} \leq N r (|u_{xx}|^q + |u_t|^q)^{1/q}_{Q_{\nu r}}. \]  
(6.7)

**Proof.** As in the proof of Lemma 6.1 we may assume that \( r = 1, q \leq p \), and \( u \in C^\infty_0(\mathbb{R}^{d+1}) \). Then, again take an infinitely differentiable function \( \zeta \) on \( \mathbb{R}^{d+1} \) such that \( \zeta = 1 \) on \( Q_1 \) and \( \zeta = 0 \) on \( \mathbb{R}^{d+1} \setminus Q_\nu \), and use the notation from the proof of Lemma 6.1 to obtain
\[ v_x(t, x) = \int_0^\infty \int_{\mathbb{R}^d} g(t + s, x + y) p_y(s, y) \, dy \, ds \]

Next, we use an elementary inequality
\[ x^\alpha e^{-\beta x} \leq N e^{-\beta x/2}, \quad \forall x \geq 0, \]
where \( \alpha, \beta > 0 \) and \( N = N(\alpha, \beta) \). Then by observing that
\[ p_y(s, y) = -\frac{y^i}{2s (4\pi s)^{d/2}} e^{-|y|^2/(4s)} \]
we find
\[ |p_y(s, y)| \leq \frac{1}{\sqrt{s}} \frac{|y|}{\sqrt{4\pi s}^{d/2}} e^{-|y|^2/(4s)} \leq N s^{-1/2} p(s/2, y), \]
which implies that
\[ |v_x(t, x)| \leq N \int_0^\infty \int_{\mathbb{R}^d} |g(t + s, x + y)| s^{-1/2} p(s/2, y) \, dy \, ds. \]
As before, if \( 0 \leq t \leq 1 \), there is no need to integrate with respect to \( s \) beyond \( [0, \nu^2] \). Therefore, upon denoting
\[ w(t, x) = |v_x(t, x)|_{I_t \in [0,1]}, \quad \bar{g}(s, y) = |g(s, y)|_{I_s \in [0,\nu^2]}, \]
\[ h(s, y) = s^{-1/2} p(s/2, y) I_{s \in [0,\nu^2]}, \]
we find
\[ w(t, x) \leq N \int_{\mathbb{R}^{d+1}} \bar{g}(t + s, x + y) h(s, y) \, dx \, ds \]
\[ N \int_{\mathbb{R}^{d+1}} \bar{g}(t - s, x - y) h(-s, y) \, dxds. \]

Now we apply (6.3) with the same \( r \), which also satisfies
\[ r(d + 1) < d + 2 \quad (6.8) \]
due to (6.6). Then we find
\[ \|v_x\|_{L^p(Q_1)} \leq \|w\|_{L^p} \leq N\|g\|_{L^q(Q_\nu)} \|h\|_{L^r(0, 2\times \mathbb{R}^d)}. \quad (6.9) \]

Here by the definition of \( g \) and Lemma 5.4
\[ \|g\|_{L^q(Q_\nu)} \leq N(\|uxx\|_{L^q(Q_\nu)} + \|ut\|_{L^q(Q_\nu)}). \]

Furthermore,
\[ \|h\|_{L^r(0, 2\times \mathbb{R}^d)} = N \int_0^2 t^{-r(d+1)/2 + d/2} \int_{\mathbb{R}^d} t^{-d/2} e^{-r|x|^2/(2t)} \, dx \, dt \]
\[ = N \int_0^2 t^{-r(d+1)/2 + d/2} \, dt < \infty, \]
where the inequality holds since owing to (6.8) we have \(-r(d + 1)/2 + d/2 > -1\).

Now it only remains to observe that the left-hand sides of (6.9) and (6.7) coincide. The lemma is proved.

Lemma 6.3. Let \( r \in (0, 1], \nu \in (1, \infty) \), and \( u \in W^{1,2}_{p, loc} \). Set \( f := Lu \).

Then
\[ \|uxx\|_{L^p(Q_\nu)} \leq N(\|f\|_{L^p(Q_\nu + 1)} + r^{-1}\|ux\|_{L^p(Q_\nu + 1)} + r^{-2}\|u\|_{L^p(Q_\nu + 1)}), \quad (6.10) \]
where \( N \) depends only on \( \nu, d, K, p, \kappa, \) and the function \( \omega \).

Proof. Obviously we may concentrate on \( u \in W^{1,2}_{p, \mathbb{R}^d} \). By Corollary 5.14 the assumption of Lemma 5.2 is satisfied. Therefore, (6.10) holds with \( r = 1 \).

For \( r \in (0, 1] \) and \( u \in W^{1,2}_{p, \mathbb{R}^d} \) introduce \( v(t, x) = u(r^2t, rx) \) and observe that
\[ v(t, x) + a(t, x) v(t, x) + r^2b(t, rx) v(t, x) + r^2 c(t, rx) v(t, x) = r^2f(r^2t, rx) =: g(t, x), \quad (6.11) \]
where \( a(t, x) = a(r^2t, rx) \).

Furthermore, for any \( \rho > 0 \) and \( t, x \)
\[ \rho^{-2}|B_{\rho}| \int_{t}^{t+\rho^2} \int_{y,z \in B_{\rho}(x)} |\bar{a}(s, y) - \bar{a}(s, z)| \, dy \, ds \]
Therefore, \( d^p(x) \leq \omega(r^p) \leq \omega(p) \). Also \(|rb| \leq K\) and \( r^2|c| \leq K\). It follows that the above result is applicable to (6.11) and
\[
\|v_{xx}\|_{L_p(Q_1)} \leq N(\|g\|_{L_p(Q_2)} + \|v_x\|_{L_p(Q_2)} + \|v\|_{L_p(Q_2)}).
\]
Expressing all terms here by means of \( u \) and \( f \) leads to (6.10). The lemma is proved.

The following is a crucial point in proving Lemma 6.1.

**Corollary 6.4.** If \( r \in (0, 1] \), \( q \geq 1 \), and \( u \in W^{1,2}_{p,loc} \) are such that that in \( Q_{2r} \) we have \( Lu = 0, b = 0, \) and \( c = 0, \) then
\[
(|u_{xx}|^p)^{1/p}_{Q_r} \leq N_1(|u_{xx}|^q)_{Q_{2r}} \leq N_1(|u_{xx}|^q)_{Q_{2r}},
\]
where \( N_1 \) depends only on \( d, p, \kappa, K \), and the function \( \omega \).

Proof. The second inequality in (6.12) follows from Hölder’s inequality. It turns out that, to prove the first one, it suffices to prove that if (6.6) holds, \( q \leq p, \nu \in (1, \infty) \), and \( Lu = 0 \) in \( Q_{hr} \), then
\[
(|u_{xx}|^p)^{1/p}_{Q_r} \leq N(|u_{xx}|^q)_{Q_{hr}}^{1/q},
\]
where \( N = N(\nu, d, p, q, \omega, \kappa, K) \). Indeed, one can find a decreasing sequence \( q_i \in [1, p], i = 0, 1, \ldots, m, \) where \( m \) depends only on \( p \) and \( d \), such that \( q_0 = p, q_m = 1, \) and \( q_i^{-1} < (d + 2)^{-1} + q_i^{-1} \). Then if (6.13) is true under the additional assumptions, then the \( L_{q_i} \) average norm of \( u_{xx} \) is estimated by the \( L_{q_{i+1}} \) average norm of \( u_{xx} \) in an expanded domain of averaging. We can then iterate (6.13) going along the sequence \( q_i \) and we can choose \( \nu = \nu(p) \) so close to 1, that during these finitely many steps the expanding domains would always be in \( Q_{2r} \) and (6.12) would follow.

Therefore, we concentrate on proving (6.13) assuming that (6.6) holds, \( q \leq p, \nu \in (1, \infty) \), and \( Lu = 0 \) in \( Q_{hr} \). Since (6.13) only involves the values of \( u \) in \( Q_{hr} \), we may assume that \( u \in W^{1,2}_p \). In that case introduce
\[
\nu = u - u_{Q_{hr}} - x^i(u_{x^i})_{Q_{hr}}.
\]
Since by assumption \( Lv = 0 \) in \( Q_{hr} \) and \( r \leq 1 \), by Lemma 6.3 we have
\[
\int_{Q_r} |u_{xx}|^p dxdt = \int_{Q_{hr}} |v_{xx}|^p dxdt \leq N r^{-p} \int_{Q_{hr}} |u_x - (u)_{Q_{hr}}|^p dxdt
\]
\[
+ N r^{-2p} \int_{Q_{hr}} |u - u_{Q_{hr}} - x^i(u_{x^i})_{Q_{hr}}|^p dxdt.
\]
By Lemmas 6.1 and 6.2 the right-hand side in (6.14) is less than the $p$-th power of the right-hand side in (6.13). The corollary is proved.

**Proof of Lemma 3.1** According to Theorem 2.1 of [14], there is a function $v$ such that it belongs to $W^{1,2}_{p}(\mathbb{R}^{d+1})$ for any $S > -\infty$, satisfies

$$Lv = f_{Q_{\nu r}}$$

in $\mathbb{R}^{d+1}$, and is such that $v(t, x) = 0$ for $t > 4$ (observe that $\nu r \leq 1$). Furthermore, as usual, since $f_{Q_{\nu r}} \in L^q$ for any $q \in (1, \infty)$, we have that $v \in W^{1,2}_{q}(\mathbb{R}^{d+1})$ for all $q \in (1, \infty)$ and $S$.

After that we set

$$w = u - v$$

and note for the future that $w \in W^{1,2}_{q,loc}$ for all $q \in (1, \infty)$.

Again by Theorem 2.1 of [14] we have

$$\int_{(0,4) \times \mathbb{R}^d} |v_{xx}|^p \, dx \, dt \leq N \int_{Q_{\nu r}} |f|^p \, dx \, dt$$

implying that

$$|v_{xx}|^p_{Q_{\nu r}} \leq N A_{\nu r}, \quad (|v_{xx}|^p)_{Q_{\nu r}} \leq N \nu^{d+2} A_{\nu r}.$$  

(6.16)

Next, observe that

$$w \in W^{1,2}_{2p,loc} \subset W^{1,2}_{p,loc}$$

and $Lw = 0$ in $Q_{\nu r}$ and $\nu/4 \geq 4$.

Now we apply Theorem 5.1 with $\nu/4$ in place of $\nu$, $\tilde{L}w = w_t + \tilde{a}^{ij}w_{x_i x_j}$ and $\tilde{a} \in \bar{A}$. As an intermediate step we also use Hölder’s inequality and the fact that $\tilde{L}w = (\tilde{a} - a)^{ij}w_{x_i x_j}$ in $Q_{\nu r}$ to find that

$$\int_{Q_{\nu r/4}} |\tilde{L}w|^p \, dx \, dt \leq N \left( \int_{Q_{\nu r/4}} |w_{xx}|^{2p} \, dx \, dt \right)^{1/2} \left( \int_{Q_{\nu r/4}} |a - \tilde{a}|^{2p} \, dx \, dt \right)^{1/2},$$

where for for an appropriate $\tilde{a}$

$$\left( \int_{Q_{\nu r/4}} |a - \tilde{a}|^{2p} \, dx \, dt \right)^{1/2} \leq N \left( \int_{Q_{\nu r/4}} |a - \tilde{a}| \, dx \, dt \right)^{1/2} \leq N \tilde{a}^{1/2}.$$

Then we obtain

$$|w_{xx} - (w_{xx})_{Q_{\nu r}}|^p_{Q_{\nu r}} \leq N \nu^{-p} |w_{xx}|^p_{Q_{\nu r/4}} + N \nu^{d+2} \tilde{a}^{1/2} \left( |w_{xx}|^{2p} \right)_{Q_{\nu r/4}}^{1/2}.$$  

(6.17)

Owing to (6.16) and the definition of $w$,

$$|w_{xx}|^p_{Q_{\nu r/4}} \leq N \left( |w_{xx}|^p \right)_{Q_{\nu r}} \leq N \left( |w_{xx} + v_{xx}|^p \right)_{Q_{\nu r}}$$

$$+ N \left( |v_{xx}|^p \right)_{Q_{\nu r}} \leq NB_{\nu r} + N \mathcal{A}_{\nu r}.$$  

(6.18)
Now we apply Corollary 6.4 with $2p$ in place of $p$ noting that the fact that $Lw = 0$ in $Q_{νr}$ allows us to do that. Then we see that
\[
[(|w_{xx}|^{2p})_{Q_{νr/4}}]^{1/2} \leq N(|w_{xx}|^{p})_{Q_{νr}}.
\]
We estimate the last term using (6.18) and then infer from (6.17) that
\[
(|w_{xx} - (w_{xx})_{Q_{νr}}|^{p})_{Q_{νr}} \leq N(ν^{-p} + ν^{d+2/2})(B_{νr} + A_{νr}).
\]
To finish proving (3.1) it only remains to combine this with (6.16) and observe that
\[
(|u_{xx} - (u_{xx})_{Q_{νr}}|^{p})_{Q_{νr}} \leq N(|v_{xx} - (v_{xx})_{Q_{νr}}|^{p})_{Q_{νr}} + N(|w_{xx} - (w_{xx})_{Q_{νr}}|^{p})_{Q_{νr}},
\]
so
\[
(|v_{xx} - (v_{xx})_{Q_{νr}}|^{p})_{Q_{νr}} \leq N(|v_{xx}|^{p})_{Q_{νr}}.
\]
The lemma is proved.

7. New approach to the $L_{p}$-theory for divergence type equations with VMO coefficients

Take an $a \in A$ and set
\[
\bar{L}u(t, x) = a^{ij}(t)u_{x_{i}x_{j}}(t, x) + u_{t}(t, x).
\]
In this section we show how to use results on solvability of equations in the whole space and prove the following statement which is a weak version of Lemma 4.1 and for $p = 2$ is Lemma 5.2 of [14] proved there by using the solvability of equations in cylinders. Throughout the section $p \in (1, ∞)$ and $λ ≥ 0$ unless explicitly specified otherwise.

**Theorem 7.1.** Let $u \in H_{p, loc}^{1}$, $f = (f^{1}, ..., f^{d})$, $f^{i} \in L_{p, loc}$, $ν ≥ 4$, $r > 0$. Assume that $\bar{L}u = \text{div} f$ in $Q_{νr}$. Then there exists a constant $N = N(d, κ, K, p)$ such that
\[
(|u_{x} - (u_{x})_{Q_{νr}}|^{p})_{Q_{νr}} \leq Nν^{-p}(|u_{x}|^{p})_{Q_{νr}} + Nν^{d+2}(|f|^{p})_{Q_{νr}}. \quad (7.1)
\]

Our strategy is very similar to what is done in Section 5. We need few auxiliary results. The first one is used also later in the proof of Corollary 8.3.

**Lemma 7.2.** Let $p \in [1, ∞)$, $R \in (0, ∞)$, $u \in H_{p, loc}^{1}$,
\[
f = (f^{1}, ..., f^{d}), \quad f^{i} \in L_{p, loc},
\]
and $\bar{L}u = \text{div} f + g$ in $Q_{R}$. Then for a constant $N = N(d, K, p)$ we have
\[
\int_{Q_{R}} |u(t, x) - u_{Q_{R}}|^{p} dxdt \leq NR^{p} \int_{Q_{R}} (|u_{x}|^{p} + |f|^{p} + R^{p}|g|^{p}) dxdt. \quad (7.2)
\]
Proof. Denote by \( \phi(\varepsilon) \) the convolution of \( \varepsilon^{-d-2}\zeta(\varepsilon^{-2}t, \varepsilon^{-1}x) \) with \( \phi = \phi(t, x) \), where \( \zeta \in C_0^\infty(\mathbb{R}^{d+1}) \) and \( \zeta \) integrates to one. Let \( \bar{L}(\varepsilon) \) be the operator constructed from \( a(\varepsilon) \). Observe that the equation

\[
\bar{L}(\varepsilon)u(\varepsilon) = \text{div } f^\varepsilon + g^\varepsilon,
\]

holds in a somewhat smaller domain than \( Q_R \). If the assertion of the lemma were applicable to \( (7.3) \) and somewhat smaller domains, then, since \( u(\varepsilon), u_x(\varepsilon), f^\varepsilon, \) and \( g^\varepsilon \) converge in \( L_p \) as \( \varepsilon \to 0 \) to \( u, u_x, f, \) and \( g, \) respectively, we would get \( (7.2) \). This argument convinces us that without losing generality we may assume that \( a, u, f, \) and \( g \) are infinitely differentiable. In that case our assertion is known as is Lemma 3.1 of [14]. The lemma is proved.

**Lemma 7.3.** Let \( m \in \{0, 1, 2, \ldots\} \) and \( u \in C_0^\infty(\mathbb{R}^{d+1}) \). Assume that \( \bar{L}u - \lambda u \) vanishes in \( Q_2 \). Then

\[
\max_{Q_1} (|D^m u_x|^p + |D^m u|^p) \leq N \int_{Q_2} (|u_x|^p + \lambda^{p/2}|u|^p) \, dx \, dt,
\]

where \( N = N(d, m, \kappa, K, p) \).

Proof. If \( \lambda = 0 \), we obtain the estimate of \( |D^m u_x| \) by applying \( (5.6) \) with \( |\alpha| \geq 1 \) to \( u - u_{Q_r} \) in place of \( u \) and using Lemma 7.2. The estimate for \( D^\alpha u \) then follows from the equation \( \bar{L}u = 0 \) in \( Q_2 \).

For general \( \lambda \) we just inspect the proof of Lemma 7.2 and observe that it works in the present case as well. The lemma is proved.

Here is a counterpart of Theorem 5.10 which is proved in the same way.

**Theorem 7.4.** Let \( \lambda \geq 0, \nu \geq 2, \) and \( r \in (0, \infty) \) be some constants. Let \( u \in C_0^\infty(\mathbb{R}^{d+1}) \) be such that \( \bar{L}u - \lambda u \) vanishes in \( Q_{\nu r} \). Then there is a constant \( N = N(d, \kappa, K, p) \) such that

\[
(|u_x - (u_x)_{Q_r}|^p)_{Q_r} \leq N \nu^{-p} (|u_x|^p + \lambda^{p/2}|u|^p)_{Q_{\nu r}}.
\]

**Proof of Theorem 7.4.** We follow the general scheme of proving Theorem 5.1. We may certainly assume that \( u \) and \( f \) have compact supports. Then as in the proof of Lemma 7.2 we may assume that \( a, u, f \) are infinitely differentiable.

In that case take a \( \lambda > 0 \), which in the future will be sent to 0, take a \( \zeta \in C_0^\infty(\mathbb{R}^{d+1}) \) such that \( \zeta = 1 \) on \( Q_{\nu r/2} - Q_{\nu r/2} \) and \( \zeta = 0 \) outside \( Q_{\nu r} \). Then set

\[
g = \text{div } (f \zeta), \quad h = \bar{L}u - g.
\]
Next, we define \( v, w^i \), and \( \phi \) as the unique solutions in \( W_p^{1,2} \) of the equations
\[
\bar{L} v - \lambda v = h, \quad \bar{L} w^i - \lambda w^i = f^i \zeta, \quad \bar{L} \phi - \lambda \phi = -\lambda u.
\]
Since \( \lambda > 0 \), by classical theory we know that such \( v, w, \phi \) indeed exist, are unique, and infinitely differentiable.

Since \( h = 0 \) in \( Q_{\nu r/2} \) and \( \nu/2 \geq 2 \), by Theorem 7.4 we obtain
\[
\left( |v_x - (v_x)_q|^p \right)_q \leq N \nu^{-p} (|v_x|^p + \lambda^{p/2} |v|^p)_{Q_{\nu r/2}}
\]
\[
\leq N \nu^{-p} (|v_x|^p + \lambda^{p/2} |v|^p)_{Q_{\nu r}}. \tag{7.6}
\]
Furthermore,
\[
\lambda \|w\|_{L_p(\mathbb{R}^{d+1})} + \lambda^{1/2} \|w_x\|_{L_p(\mathbb{R}^{d+1})} + \|w_t\|_{L_p(\mathbb{R}^{d+1})} + \|w_{xx}\|_{L_p(\mathbb{R}^{d+1})} \leq N \|f\|_{L_p(\mathbb{R}^{d+1})}.
\]
In particular, for \( \psi := \text{div } w \) we have
\[
\lambda^{p/2} \int_{Q_{\nu r}} |\psi|^p \, dx \, dt + \int_{Q_{\nu r}} |\phi|^p \, dx \, dt \leq N \int_{Q_{\nu r}} |f|^p \, dx \, dt.
\]
\[
\left( |\phi|^p + \lambda^{p/2} |\phi|^p \right)_q \leq N \nu^{-2} (|f|^p)_{Q_{\nu r}}. \tag{7.7}
\]
Also,
\[
\lambda \|\phi\|_{L_p(\mathbb{R}^{d+1})} + \lambda^{1/2} \|\phi_x\|_{L_p(\mathbb{R}^{d+1})} + \|\phi_t\|_{L_p(\mathbb{R}^{d+1})} + \|\phi_{xx}\|_{L_p(\mathbb{R}^{d+1})} \leq N \|u\|_{L_p(\mathbb{R}^{d+1})}.
\]
\[
\left( |\phi_x|^p + \lambda^{p/2} |\phi|^p \right)_{Q_{\nu r}} \leq N \lambda^{p/2} \nu^{-d} \|u\|_{L_p(\mathbb{R}^{d+1})}^p. \tag{7.8}
\]
Finally, we claim that
\[
\bar{L} u - \lambda u = h + \text{div } (f \zeta) - \lambda u = \bar{L} u - \lambda u
\]
and using uniqueness we get that \( \bar{u} = u \), indeed.

Indeed, owing to the additional assumptions on \( f \), for any multi-index \( \alpha \) we have \( D^\alpha w \in W_p^{1,2} \). Hence, \( \bar{u} \in W_p^{1,2} \). Upon observing that
\[
\bar{L} u - \lambda u = h + \text{div } (f \zeta) - \lambda u = \bar{L} u - \lambda u
\]
and using uniqueness we get that \( \bar{u} = u \), indeed.

After that, by using \( \bar{L} \phi, \ (7.6) \), and \( \bar{L} \), we can dominate the left-hand side of \( (7.7) \) by a constant times
\[
\left( |v_x - (v_x)_q|^p \right)_q + \left( |\phi|^p \right)_q \leq N \nu^{-p} (|v_x|^p + \lambda^{p/2} |v|^p)_{Q_{\nu r}}
\]
\[
+ N \nu^{-2} (|f|^p)_{Q_{\nu r}} + N \lambda^{p/2} \nu^{-d} \|u\|_{L_p(\mathbb{R}^{d+1})}^p.
\]
where
\begin{align*}
(\|u_x\|^p + \lambda^{p/2}\|u\|^p)_{Q_{dr}} & \leq N(\|u_x\|^p + \lambda^{p/2}\|u\|^p)_{Q_{dr}} \\
+ N(\|\psi_x\|^p + \lambda^{p/2}\|\psi\|^p)_{Q_{dr}} + N(\|\phi_x\|^p + \lambda^{p/2}\|\phi\|^p)_{Q_{dr}} \\
& \leq N(\|u_x\|^p + \lambda^{p/2}\|u\|^p)_{Q_{dr}} + N(\|f\|^p)_{Q_{dr}} + N\lambda^{p/2}r^{-d-2}\|u\|_{L^p(R_0^{d+1})}^p.
\end{align*}

Thus, the left-hand side of (7.1) is less than
\begin{align*}
N\nu^{-p}(\|u_x\|^p + \lambda^{p/2}\|u\|^p)_{Q_{dr}} + N\nu^{d+2}(\|f\|^p)_{Q_{dr}} + N\lambda^{p/2}r^{-d-2}\|u\|_{L^p(R_0^{d+1})}^p
\end{align*}

and to obtain (7.1) it only remains to let \( \lambda \downarrow 0 \). The theorem is proved.

Now we can repeat what is said in [14] and get the solvability of equations involving \( \mathcal{L} \) in \( \mathcal{H}_p^1 \). In particular, we have the following result. Recall that \( \mathbb{R}_S \) and \( \mathbb{H}_{p^{-1}}((S, T)) \) are introduced in Section 4.

**Theorem 7.5.** Let \( S \in [-\infty, \infty) \). Then there exists \( \lambda_0 \), depending only on \( p, d, K, \kappa, \) and \( \omega \), such that, for any \( u \in \mathcal{H}_p^1(R_0^{d+1}) \) and \( \lambda \geq \lambda_0 \), we have
\begin{align}
\|u_t\|_{\mathbb{H}_{p^{-1}}(\mathbb{R}_S)} + \|u_x\|_{L^p(R_0^{d+1})} + \|u\|_{L^p(R_0^{d+1})} \leq N\|(\mathcal{L} - \lambda)u\|_{\mathbb{H}_{p^{-1}}(\mathbb{R}_S)}, \tag{7.9}
\end{align}

where \( N \) depends only on \( p, d, K, \kappa, \omega, \lambda \). Furthermore, for each \( \lambda \geq \lambda_0 \) and \( f \in \mathbb{H}_{p^{-1}}(\mathbb{R}_S) \) there is a unique \( u \in \mathcal{H}_p^1(\mathbb{R}_S) \) such that \( (\mathcal{L} - \lambda)u = f \).

This is a version of Theorem 6.2 of [14]. There is only one difference. Theorem 6.2 of [14] is stated with \( \mathbb{H}_{p^{-1}} \) and \( L^p \) in place of \( \mathbb{H}_{p^{-1}}(\mathbb{R}_S) \) and \( L^p(\mathbb{R}_S^{d+1}) \), respectively. Passing from the former spaces to the latter ones is performed as in [14] on the basis of the fact that the a priori estimate (7.4) allows one to solve the corresponding equations by the method of continuity (cf. Corollary 5.14).

8. **Proof of Lemma 4.1**

Although the way we proceed are similar to what is done in [6] the details are quite different and the main reason for that is that we cannot prove a natural counterpart of Lemma 6.3.

**Lemma 8.1.** Let \( r \in (0, \infty), q \in (1, p] \), and assume that
\begin{align*}
\frac{1}{q} - \frac{1}{p} & \leq \frac{1}{d + 2}.
\end{align*}

Let \( \zeta \in C_0^\infty(\mathbb{R}_0^{d+1}) \) be such that \( \zeta = 1 \) in \( Q_r \). Then for any function \( u \) such that \( u_\zeta \in \mathcal{H}_q^1(\mathbb{R}_0) \), we have \( u \in L^p(Q_r) \) and
\begin{align*}
\|u\|_{L^p(Q_r)} \leq N\|u_\zeta\|_{\mathcal{H}_q^1(\mathbb{R}_0)},
\end{align*}

where \( N = N(r, d, p, q, \zeta) \).
Proof. Take a \( \lambda_0 \) which suits \( \mathcal{L} = D_t + \Delta \) in Theorem \( \text{[15]} \). By definition \( u \zeta = (1 - \Delta)^{1/2}w \), where \( w \in W^{1,2}_q(\mathbb{R}^{d+1}_0) \) and

\[
w_t + \Delta w - \lambda_0 w =: \phi \in L_q(\mathbb{R}^{d+1}_0).
\]

By applying to both parts \((1 - \Delta)^{1/2}\) we see that

\[
h := \Delta(u \zeta) + (u \zeta)_t - \lambda_0 u \zeta = (1 - \Delta)^{1/2}\phi.
\]

Observe that

\[
\| h \|_{H^{-1}_q(\mathbb{R}^d)} = \| \phi \|_{L_q(\mathbb{R}^d)} \leq N \| w \|_{W^{1,2}_q(\mathbb{R}^{d+1}_0)} = N \| u \zeta \|_{H^1_0(\mathbb{R}^d)}.
\]  

(8.2)

Next, write

\[
h = \text{div} f + g, \quad g := (1 - \Delta)^{-1/2}\phi = (1 - \Delta)^{-1}h, \quad f := -g_x
\]

and notice that

\[
\| f \|_{L_q(\mathbb{R}^{d+1}_0)} + \| g \|_{L_q(\mathbb{R}^{d+1}_0)} \leq N \| h \|_{H^{-1}_q(\mathbb{R}^d)};
\]  

(8.3)

where \( N = N(d, q) \).

Now define \( v \) and \( w \) as the unique solutions from \( W^{1,2}_q(\mathbb{R}^{d+1}_0) \) of

\[
\Delta v + v_t - \lambda_0 v = g, \quad \Delta w + w_t - \lambda_0 w = f.
\]

Then we have \( \bar{u} := v + \text{div} w \in H^1_0(\mathbb{R}^d) \) since

\[
D_t W^{1,2}_q(\mathbb{R}^{d+1}_0) = (1 - \Delta)^{1/2}[1 - (1 - \Delta)^{-1/2}D_t] W^{1,2}_q(\mathbb{R}^{d+1}_0)
\]

\[
\subset (1 - \Delta)^{1/2}W^{1,2}_q(\mathbb{R}^{d+1}_0).
\]

Furthermore, obviously \( \Delta \bar{u} + \bar{u}_t - \lambda_0 \bar{u} = h \). Since \( u \zeta \) also satisfies this equation, by Theorem \( \text{[15]} \) we have \( \bar{u} = u \zeta \). In particular, \( u = v + \text{div} w \) in \( Q_r \).

Finally, by classical results and \( \text{(8.3)} \) and \( \text{(8.2)} \)

\[
\| v_{xx} \|_{L_q(\mathbb{R}^{d+1}_0)} + \| v_t \|_{L_q(\mathbb{R}^{d+1}_0)} + \| v \|_{L_q(\mathbb{R}^{d+1}_0)} \leq N\| g \|_{L_q(\mathbb{R}^{d+1}_0)} \leq N\| u \zeta \|_{H^1_0(\mathbb{R}^d)},
\]

\[
\| w_{xx} \|_{L_q(\mathbb{R}^{d+1}_0)} + \| w_t \|_{L_q(\mathbb{R}^{d+1}_0)} + \| w \|_{L_q(\mathbb{R}^{d+1}_0)} \leq N\| u \zeta \|_{H^1_0(\mathbb{R}^d)}.
\]

It only remains to notice that, owing to \( \text{(8.1)} \), by Lemma II.3.3 of \( \text{[15]} \)

\[
\| v \|_{L_p(\mathbb{R}^{d+1}_0)} \leq N(\| v_{xx} \|_{L_q(\mathbb{R}^{d+1}_0)} + \| v_t \|_{L_q(\mathbb{R}^{d+1}_0)} + \| v \|_{L_q(\mathbb{R}^{d+1}_0)}),
\]

\[
\| w \|_{L_p(\mathbb{R}^{d+1}_0)} \leq N(\| w_{xx} \|_{L_q(\mathbb{R}^{d+1}_0)} + \| w_t \|_{L_q(\mathbb{R}^{d+1}_0)} + \| w \|_{L_q(\mathbb{R}^{d+1}_0)}).
\]

The lemma is proved.

To move further to equations in \( H^1_{q,p} \) spaces we need the following counterpart of Lemma \( \text{[8.3]} \)
Lemma 8.2. Let \( r \in (0, 1], \nu \in (1, \infty), q \in (1, p), \) and assume (8.1). Let \( u \in \mathcal{H}_{q, \text{loc}}, f = (f^1, \ldots, f^d), f^i, g \in L_{q, \text{loc}} \) and assume that \( \mathcal{L} u = \text{div} f + g \) in \( Q_{\nu r} \). Then \( u \in L_p(Q_r) \) and

\[
r^{-1}(|u|^p)_{Q_r}^{1/p} \leq N(|f|^q + r|g|^q + |u_x|^q + r^{-1}|u|^q)_{Q_{\nu r}},
\]

where \( N = N(\nu, d, \kappa, p, q, K) \). Furthermore, if, additionally, \( f \in L_{p, \text{loc}}, \) then \( u_x \in L_p(Q_r) \) and

\[
(|u|^p)_{Q_r}^{1/p} \leq N[(|f|^p)_{Q_{\nu r}}^{1/p} + (r|g|^q + |u_x|^q + r^{-1}|u|^q)_{Q_{\nu r}}],
\]

where \( N = N(\nu, d, \kappa, p, q, K) \).

Proof. By self-similarity we may assume that \( r = 1 \) (cf. the proof of Lemma 6.3). In that case take \( \lambda = \lambda_0 \) which suits Theorem 7.5 for both \( p \) and \( q \) in place of \( p \) there. Also take a \( \zeta \in C_0^\infty(\mathbb{R}^{d+1}_0) \) such that \( \zeta = 1 \) on \( Q_1 \) and \( \zeta = 0 \) in \( \mathbb{R}^{d+1}_0 \setminus Q_\nu \). Observe that in \( \mathbb{R}^{d+1}_0 \) we have

\[
\mathcal{L}(u\zeta) - \lambda u\zeta = \text{div} (\zeta f + \tilde{f}) + \tilde{g},
\]

where

\[
\tilde{f}^i = u a^{ij} \zeta_x^j, \quad \tilde{g} = \zeta g - \tilde{f}^i \zeta_x^i + u \left[ \zeta_t + (b^i + \tilde{b}) \zeta_x^i - \lambda \zeta \right] + a^{ij} u_x^i \zeta_x^j.
\]

Since the \( H^{-1}(\mathbb{R}_0) \)-norm of the right-hand side of (8.6) is less than a constant times the right-hand side of (8.4) we get (8.4) (with \( r = 1 \)) by Theorem 7.5 and Lemma 8.1. Below we are going to use a trivial extension of this result that (8.4) also holds with \( Q_{\nu'} \) in place of \( Q_1 \), where \( \nu' = (1 + \nu)/2 \). We will also assume that \( \zeta = 0 \) in \( \mathbb{R}^{d+1}_0 \setminus Q_{\nu'} \).

To prove (8.5) we want to apply Theorem 7.5 again to (8.6). By the above the \( L_p(\mathbb{R}^{d+1}_0) \)-norm of \( \zeta f + \tilde{f} \) is under control. To deal with \( g \) we define \( v \) as the unique \( W_1^{1,2}(\mathbb{R}^{d+1}_0) \) solution of \( \Delta v + v_t - \lambda v = \tilde{g} \). Notice that, as in the proof of Lemma 8.1

\[
\|v\|_{L_p(\mathbb{R}^{d+1}_0)} + \|v_x\|_{L_p(\mathbb{R}^{d+1}_0)} \leq N \|\tilde{g}\|_{L_q(\mathbb{R}^{d+1}_0)} \leq NI_1,
\]

where \( I_r \) is the content of the brackets on the right in (8.5). Furthermore, \( w = u\zeta - v \) satisfies

\[
\mathcal{L} w - \lambda w = \text{div} (\zeta f + \tilde{f}) + \tilde{g},
\]

where

\[
\tilde{f}^i = (\sigma^{ij} - a^{ij}) v_x^i - \tilde{b}^i v, \quad \tilde{g} = -b^i v_x^i - cv.
\]

By the above the right-hand side of (8.8) is in \( H^{-1}(\mathbb{R}_0) \) with norm controlled by \( NI_1 \). By Theorem 7.5 equation (8.8) has a unique solution in \( \mathcal{H}^1_{q, \text{loc}}(\mathbb{R}_0) \) and, since \( w = u\zeta - v \in \mathcal{H}^1_{q, \text{loc}}(\mathbb{R}_0) \), this certainly implies that \( w \in \mathcal{H}^1_p(\mathbb{R}_0) \). Also by Theorem 7.5

\[
\|w_x\|_{L_p(\mathbb{R}^{d+1}_0)} \leq NI_1,
\]
which along with (8.7) leads to (8.9). The lemma is proved.

**Corollary 8.3.** Let $r \in (0, 1]$, $\nu \in (1, \infty)$, $q \in (1, p]$, $u \in H_{q,\text{loc}}^1$, $f = (f^1, \ldots, f^d)$, $f^i, g \in L_{p,\text{loc}}$ and assume that $Lu = \text{div} f + g$ in $Q_{\nu r}$. Then $u, u_x \in L_p(Q_r)$ and

$$r^{-1}(|u|^p)_{Q_r}^{1/p} + (|u_x|^p)_{Q_r}^{1/p} \leq N(|f|^p)_{Q_{\nu r}}^{1/p}$$

$$+ Nr(|g|^p)_{Q_{\nu r}}^{1/p} + N(|u_x|^q)_{Q_{\nu r}}^{1/q} + Nr^{-1}(|u|^q)_{Q_{\nu r}}^{1/q},$$

(8.9)

where $N = N(\nu, d, \kappa, p, q, K)$.

Indeed, if our $p$ satisfies (8.1), then we have the result by Lemma 8.2. If $p$ is bigger, then we use Lemma 8.2 with $p_1$ in place of $p$, where $p_1$ is defined by $q^{-1} - p_1^{-1} = (d + 2)^{-1}$. Once we have the result for $p_1$, we take $p_1$ as new $q$ and keep iterating as many times as needed, each time reducing $p_k^{-1}$ by $(d + 2)^{-1}$ until we reach first $k$ such that $p_k^{-1} - p^{-1} \leq (d + 2)^{-1}$.

**Corollary 8.4.** If $r \in (0, 1]$, $q > 1$, and $u \in H_{q,\text{loc}}^1$ are such that that in $Q_{2r}$ we have $Lu = 0$, $b = \hat{b} = 0$, and $c = 0$, then $u_x \in L_p(Q_r)$ and

$$(|u_x|^p)_{Q_r}^{1/p} \leq N(|u|^q)_{Q_{2r}}^{1/q},$$

(8.10)

where $N$ depends only on $d, p, q, K, \kappa$, and the function $\omega$.

For $q \geq p$ equation (8.10) is obvious. To prove it for $q \leq p$ it suffices to apply (8.9) to $v = u - u_{Q_{\nu r}}$ in place of $u$, observe that $Lv = 0$ in $Q_{2r}$, and finally use Lemma 4.2 with $L = \Delta + D_t$ for which

$$\tilde{L}u = \left(\left(\delta^{ij} - a^{ij}\right)u_{x^i}\right)_{x^j}.$$  

**Proof of Lemma 4.1** We may certainly assume that $u \in H_{p,\text{loc}}^1$. According to Theorem 2.4 of [14], applied to the domains $(S, 4) \times \mathbb{R}^d$ for $S < 4$, on $\mathbb{R}^{d+1}$ there is a function $v$ such that it belongs to $H_{p,\text{loc}}^1(\mathbb{R}_S)$ for any $S$, satisfies

$$Lv = \text{div} (f I_{Q_{\nu r}})$$

(8.11)

in $\mathbb{R}^{d+1}$, and is such that $v(t, x) = 0$ for $t > 4$ (observe that $\nu r \leq 1$).

After that we set

$$w = u - v$$

and note for the future that $w \in H_{p,\text{loc}}^1$.

Again by Theorem 2.4 of [14] we have

$$\int_{(0,4)\times \mathbb{R}^d} |v_x|^p \, dx \, dt \leq N \int_{Q_{\nu r}} |f|^p \, dx \, dt$$
implying that
\[(|v_x|^p)_{Q_r} \leq NA_{vr}, \quad (|v_x|^p)_{Q_r} \leq N\nu^{d+2}A_{vr}. \quad (8.12)\]

Next, since \(Lw = 0\) in \(Q_{\nu r}\) and \(\nu r/2 \leq 1\), Corollary 8.3 implies that \(w_x \in L_{2p}(Q_{\nu r/2})\) and
\[(|w_x|^{2p})^{1/2}_{Q_{r/2}} \leq N(|w_x|^p)_{Q_r}. \quad (8.13)\]

Upon taking an \(\bar{a} \in A\), setting \(\bar{L}\phi = \phi_t + \bar{a}^{ij} \phi_{x_i x_j}\) and noting that \(\bar{L}w = \text{div} \bar{f}, \quad \bar{f}^j = (\bar{a}^{ij} - a^{ij})w_{x_i}\),

by Theorem 7.1 we get
\[(|\bar{f}|^p)_{Q_{r/2}} \leq N\hat{a}^{1/2}(|w_x|^p)_{Q_r}, \quad (4.1)\]

where the last term by Hölder’s inequality and (8.13) is dominated by a constant times
\[(|\bar{a} - a|^{2p})^{1/2}_{Q_{r/2}}(|w_x|^{2p})^{1/2}_{Q_{r/2}} \leq (|\bar{a} - a|^{1/2}_{Q_{r/2}}(|w_x|^p)_{Q_{r/2}}. \quad (8.13)\]

By using an appropriate choice of \(\bar{a}\) we obtain
\[(|\bar{f}|^p)_{Q_{r/2}} \leq N\hat{a}^{1/2}(|w_x|^p)_{Q_r}. \quad (8.13)\]

Combining the above and observing that
\[(|w_x|^p)_{Q_{r/2}} \leq 2^{d+2}(|w_x|^p)_{Q_r} \leq N(|u_x|^p)_{Q_r} + N(|v_x|^p)_{Q_r}
\leq N(|u_x|^p)_{Q_r} + NA_{vr} \quad (4.1)\]
yield that the left-hand side of (4.1) is dominated by
\[N(|v_x|^p)_{Q_r} + N(|w_x - (w_x)_{Q_r}|^p)_{Q_r} \leq N\nu^{d+2}A_{vr} + N\nu^{-p}(|u_x|^p)_{Q_r} + |\bar{a}|^{1/2}(|w_x|^p)_{Q_r}.
\]

This is almost exactly what is asserted and the lemma is proved.

**References**

[1] Amann H., *Maximal regularity for nonautonomous evolution equations*, Adv. Nonlinear Stud., Vol. 4 (2004), No. 4, 417-430.
[2] Bramanti M. and Cerutti M.C., *W^{1,2}_{p,solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients*, Comm. Partial Differential Equations, Vol. 18 (1993), No. 9-10, 1735-1763.
[3] Byun Sun-Sig, *Elliptic equations with BMO coefficients in Lipschitz domains*, Trans. Amer. Math. Soc., Vol. 357 (2005), No. 3, 1025-1046.
[4] Byun Sun-Sig, *Parabolic equations with BMO coefficients in Lipschitz domains, J. Differential Equations, Vol. 209 (2005), No. 2, 229–265.
[5] Denk R., Hieber M., and Prüss J., *“R-boundedness, Fourier multipliers and problems of elliptic and parabolic type”, Mem. Amer. Math. Soc. 166 (2003), No. 788.*
[6] Giga Y. and Sohr H., Abstract $L^p$ estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, Journal of Functional Analysis, Vol. 102 (1991), 72-94.

[7] Guidetti D, General Linear Boundary Value Problems for Elliptic Operators with VMO Coefficients, Math. Nachr., Vol. 237 (2002), 62-88.

[8] Haller-Dintelmann R., Heck H., and Hieber M., $L^p-L^q$-estimates for parabolic systems in non-divergence form with VMO coefficients, J. London Math Soc., 2006.

[9] Kim Doyoon, Second order elliptic equations in $\mathbb{R}^d$ with piecewise continuous coefficients, submitted to Potential Analysis.

[10] Kim Doyoon, Parabolic equations with measurable coefficients, II, submitted to the Journal of Mathematical Analysis and Applications.

[11] Kim Doyoon and Krylov N.V., Elliptic differential equations with measurable coefficients, submitted to SIAM J. Math. Anal., [http://arxiv.org/pdf/math.AP/0512515](http://arxiv.org/pdf/math.AP/0512515)

[12] Kim Doyoon and Krylov N.V., Parabolic equations with measurable coefficients, submitted to Potential Anal., [http://arxiv.org/pdf/math.AP/0604124](http://arxiv.org/pdf/math.AP/0604124)

[13] Krylov N.V., Parabolic equations in $L_p$-spaces with mixed norms, Algebra i Analiz., Vol. 14 (2002), No. 4, 91-106 in Russian; English translation in St. Petersburg Math. J., Vol. 14 (2003), No. 4, 603-614.

[14] Krylov N.V., Parabolic and elliptic equations with VMO coefficients, to appear in Communications in PDEs, [http://arxiv.org/pdf/math.AP/0511731](http://arxiv.org/pdf/math.AP/0511731)

[15] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural’tceva, “Linear and quasilinear parabolic equations”, Nauka, Moscow, 1967 in Russian; English translation: American Math. Soc., Providence, 1968.

[16] Maremonti P. and Solonnikov V.A., On the estimates of solutions of evolution Stokes problem in anisotropic Sobolev spaces with mixed norm, Zapiski nauchnykh seminarov POMI, 222 (1995), 124-150 in Russian; English translation in J. Math. Sci. (New York) 87 (1997), no. 5, 3859-3877.

[17] A. Maugeri, D. K. Palagachev, and L. G. Softova, "Elliptic and Parabolic Equations with Discontinuous Coefficients" Mathematical Research, Vol. 109, Wiley, Berlin etc., 2000.

[18] D. Palagachev and L. Softova, A priori estimates and precise regularity for parabolic systems with discontinuous data, Discrete and continuous dynamical systems, Vol. 13 (2005), No. 3, 721-742.

[19] D. Palagachev and L. Softova, Characterization of the interior regularity for parabolic systems with discontinuous coefficients, Rend. Mat. Acc. Lincei, Serie 9, Vol. 16 (2005), 125-132.

[20] D. Palagachev and L. Softova, Fine regularity for elliptic systems with discontinuous ingredients, Arch. Math. (Basel), Vol. 86 (2006), No. 2, 145–153.

[21] Ragusa M. and Tachikawa A., Partial regularity of the minimizers of quadratic functionals with VMO coefficients, J. London Math. Soc. (2), Vol. 72 (2005), No. 3, 609-620.

[22] L. Softova, Quasilinear parabolic operators with discontinuous ingredients, Nonlinear Analysis, Vol. 52 (2003), No. 4, 1079-1093.

[23] L. Softova, $W^{2,1}_p$-solvability for parabolic Poincaré problem, Comm. in PDEs, Vol. 29 (2004), No. 11&12, 1783-1798.
L. Softova, *Singular integrals and commutators in generalized Morrey spaces*, Acta Math. Sin. (Engl. Ser.), Vol. 22 (2006), No. 3, 757-766.

*E-mail address*: krylov@math.umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455