Dynamics of a charged particle in a dissipative Fermi-Ulam model

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The dynamics of a metallic particle confined between charged walls is studied. One wall is fixed and the other moves smoothly and periodically in time. Dissipation is considered by assuming a friction produced by the contact between the particle and a rough surface. We investigate the phase space of the simplified and complete versions of the model. Our results include (i) coexistence of islands of regular motion with an attractor located at the low energy portion of phase space in the complete model; and (ii) coexistence of attractors with trajectories that present unlimited energy growth in the simplified model.

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I. INTRODUCTION

Processes of accelerating particles have been a subject of large interest in recent years. Billiards are systems widely used to study these processes due to their adaptability to several situations by exploiting different mechanisms. A billiard is a system where a particle (or more) collides with boundaries while it interacts, or not, with some potentials. These boundaries can be static or time dependent, as well as the potentials. Billiards systems usually present, mathematically speaking, a nonlinear term and their phase spaces exhibit a rich variety of behaviours including, but not limited to, chaotic trajectories and quasi-periodic orbits. The studies include: (i) the non-dissipative case, where dissipation due to friction, drag or inelastic collisions is absent, and; (ii) the dissipative case, where usually the islands of regular motion give place to attractors and, depending on parameters, the chaotic sea becomes a chaotic attractor or eventually, a chaotic transient. Moreover, depending on perturbation, the particle can present a power-law energy growth, that is suppressed by weak linear dissipation, or a robust exponential energy growth.

The phenomena where a classical particle acquires unbounded energy when it is excited by an external driving structure is called Fermi acceleration (FA). The studies of FA are risen since the first half of 20th century, when high energy cosmic rays were observed. The basic mechanism of acceleration was proposed by Enrico Fermi\textsuperscript{1} and it consists of the interaction of the cosmic ray with time dependent magnetic fields generated by the activity of galaxies nuclei. Such interaction was claimed to be the responsible for giving the high energy to the particle.

Fermi’s idea can be placed in terms of a classical model. Indeed it is similar to the problem of a particle colliding against a rigid and a moving walls. The particle corresponds to the cosmic ray while the moving wall represents the driving provided by the time dependent magnetic fields. Ulam\textsuperscript{2} was who proposed this model, where a particle moves in absence of any external field and dissipation between collisions with the walls. The fixed wall works as a mechanism to re-inject the particle back for further collisions with the moving wall. Such a model, known as Fermi-Ulam model (FUM) is described by a two-dimensional and area preserving mapping. The amplitude of oscillation of the moving wall defines the strength of the nonlinear term. For non null amplitude of oscillation, the phase space of the FUM presents mixed structure, where regions of chaotic motion coexist with Kolmogorov-Arnold-Moser (KAM) islands and invariant tori, also called as invariant spanning curves or invariant rotational curves. Moreover, these spanning curves limit the energy gain for a chaotic orbit, prohibiting an unlimited diffusion in the velocity axis. Therefore, FA is not achieved in FUM, as it was discussed by Lichtenberg et al.\textsuperscript{3}.

A variation of FUM called as bouncer regards the dynamics of a classical particle moving in the presence of a constant gravitational field and suffering elastic collisions with a periodically moving wall. The re-injection mechanism is now produced by the gravitational field. For specific sets of control parameter and initial conditions, the bouncer model\textsuperscript{4,5} indeed presents FA. For small non null amplitude of oscillation of the platform the phase space presents small regions of chaotic motion, islands of regular motion and invariant spanning curves that prevent the particle to gain unlimited energy. Consistently with KAM theorem, invariant spanning curves are continually destroyed as the amplitude of oscillation increases until all of them disappear. As a consequence, the chaotic component of phase space becomes open and the average velocity of the particle grows unlimitedly leading to FA. Lichtenberg et al.\textsuperscript{2,6} demonstrated that the bouncer model is globally equivalent to the Chirikov’s standard map, explained the origin of FA and, moreover, obtained the combinations of initial conditions and values of parameter where the accelerating modes are observed in the bouncer model.
Concerning FA in two-dimensional time dependent billiards, it was conjectured by Loskutov, Ryabov and Akinshin that a sufficient condition for the observation of FA is the presence of chaotic motion of the particle in the static boundaries version of such a billiard (existence of positive Lyapunov exponents on a nontrivial region of phase space) \[7,8\]. This conjecture, known as LRA conjecture, was confirmed in different billiards \[4,12\]. Recently it has been reported FA in an harmonically driven version of the integrable static elliptical billiard \[13\]. The invariant manifolds observed for the static version become penetrable in the driven model, generating a laminar dynamics. Moreover, the driving replaces the separatrix by a chaotic layer where the motion is stochastic, introducing a diffusion process to the velocity and leading the particle to present FA. After that, the LRA conjecture was amended \[14\], where the existence of a chaotic component was replaced by the existence of a heteroclinic orbit in phase space.

Recently, it has been studied the aspects of a non-standard exponential energy growth \[15,16\]. This phenomena was numerically observed in a slitted rectangular billiard with a bar oscillating smoothly, where it was demonstrated that the origin of the exponential FA is due to the transition of the particle between the invariant spanning curves observed in phase space of the FUM \[15\]. Also, the aspects of a robust exponential FA were obtained analytically and numerically in a class of systems that consists of billiards where a particle collides against time dependent boundaries that are deformed, separated and then reconnected \[17\]. Moreover, it has been mathematically demonstrated the existence of orbits with energy growth rate ranging from power law to exponential FA in non-autonomous billiards \[17,18\].

Other important observation on the state of the art of the problem is that FA has demonstrated to be not a robust phenomena under the presence of inelastic collisions \[20,21\] and under viscous drag force \[22\]. However few studies are known about consequences of friction due to the slip of a body on a surface on FA process. When friction acts on a particle in a FUM where a wall moves according to saw-tooth function, FA can either persist or not depending on initial conditions and parameters \[22\].

Before we start to talk about this work, it is interesting to say that the study of the billiards class is of interest specially because the versatility of practical description to a wide variety of systems, such as waveguide and rippled channel \[24,25\], magnetic field lines in toroidal plasma devices \[26\], cold atoms \[27\], channel flows \[28,29\], escape processes \[30,31\] and transport properties \[32,33\].

In this work we study a modified version of Fermi-Ulam model where a charged particle moves under dissipation due to friction and it is accelerated by an electric field between metallic walls. We are seeking to understand and describe some dynamical properties of the system either as function of the time as well as according to the control parameters. We regard that the oscillating wall moves smoothly and we study both the simplified and complete models. Under specific situations the model under study is a prototype for FA and in the scenario of null electric field the model corresponds to a FUM with friction. Depending on parameters, the simplified model presents a coexistence of attractors with trajectories that present FA. Moreover, in the complete model we observe numerically the coexistence of KAM islands with trajectories that evolve to null velocity, even for null electric field.

The paper is organized as follow. In the next section we present the model and we discuss the results of both simplified and complete versions. In section III we provide the conclusions and final remarks.

II. THE MODEL

It is well known that charged particles are accelerated by electric fields. In this paper we propose a modification of the FUM where the walls are assumed to be metallic and sufficiently large and massive. They are charged with opposite charges generating an electric field $\vec{E}$ that acts on a charged particle between them. For instance, let us say that $\vec{E}$ is in the direction $x$, $\vec{E} = E\hat{i}$. At each collision with the left wall the particle acquires charge $+q$, $q \geq 0$, and an electric force $F_e = qE$ acts on the particle forcing it to move to the right. Similarly, the particle acquires charge $-q$ when it collides against the right wall and it becomes under action of an electric force $F_e = -qE$. The left wall oscillates smoothly with amplitude $\epsilon$ and frequency $\omega$ according to the expression $x = \epsilon \cos(\omega t + \phi_0)$, where $t$ is time and $\phi_0$ is an initial phase. The right wall is fixed at position $x = l$. We regard the situation where collisions are elastic, so the absolute value of the velocity of the particle relative to the walls is not affected by collisions.

Additionally, we consider that the particle moves under action of a friction force $F_f = \pm \sigma N$, due to the friction of it on a rough surface. The quantity $\sigma \geq 0$ corresponds to the kinetic coefficient friction and $N \geq 0$ is the magnitude of a normal force. Because $F_f$ opposes the motion (velocity) of the particle, $F_f \geq 0$ when the particle moves to the left (velocity $v < 0$) and vice-versa.

It is also important to define $\sigma_s > \sigma$ as the static friction coefficient. So $|F_s| = \sigma_s N$ is the maximum absolute value of the static friction force that acts on the particle when it is stopped. The signal of $F_s$ depends on tendency of motion. So if the particle stops after colliding the right wall, then $F_s \geq 0$. Similar situation occurs when the particle stops after colliding the moving wall.

Therefore, when the particle is moving between the walls, it is under action of a force $F = F_e + F_f$. When the particle stops it is under a force $F = F_e - F_f$. So, depending on signal and strength of electric and friction forces the particle can gain or loose energy while it moves between two collisions. In terms of electric, static and dynamical friction forces we define the fol-
ollowing set of parameters: \( A = (qE - \sigma_N)/\omega^2m \), \( B = (qE + \sigma_N)/\omega^2m \), \( C = (qE - \sigma_N)/\omega^2m \), where \( m \) is the mass of the particle. Moreover, it is appropriate to define the following set of dimensionless variables:

\[
X = x/l, \quad V = v/\omega, \quad t = \omega t' \quad \text{and} \quad \phi = \omega t' + \phi_0. 
\]

We define the dimensionless amplitude of oscillation by the expression \( \varepsilon = \varepsilon/l \).

From the above definitions, we have that the parameter \( A \) represents the competition between the electric and dynamic friction forces. If \( A > 0 \), then the electric force is greater than the dynamical friction force (and vice-versa). Similarly, the parameter \( C \) represents the competition between the electric and static friction forces. Therefore if the particle reaches the rest and \( C < 0 \), then the maximum static friction force is greater than the electric force. If this situation occurs in the region where the moving wall oscillates, then the particle remains in rest and waiting a collision with the moving wall. However if the particle stops somewhere outside the region of oscillation of the moving wall, then its dynamics is over. If \( C > 0 \), then the particle: (i) does not stop after colliding the fixed wall or; (ii) it stops instantaneously if its velocity is negative after a collision with the moving wall.

It is interesting to observe that the situations where \( A < 0 \) and \( C < 0 \) can be interpreted as null electric force, \( qE = 0 \), and the model corresponds to a FUM with dissipation due to friction. The non null electric field is important to produce non-negative values of \( A \) and \( C \).

Let us discuss initially the dynamics of the particle regarding a simplification of the model and, after that, we present the results of the complete model.

### A. The simplified map

In the simplified version of the model the oscillating wall imparts momentum to the particle. However, it is assumed that it takes a stationary position. So the time interval spent by the particle in the travel between collisions does not depend on the phase of the oscillating wall. The simplified versions are widely studied because they are useful under some aspects. For example, they allow to obtain analytical results that can be extended to the complete versions and, usually, they speed up greatly the simulations when compared to the corresponding complete versions.

If at an instant \( t_n \) the particle hits the moving wall and acquires velocity \( V_n \), then the dynamics of the system after subsequent collisions is obtained by the following two-dimensional map

\[
T : \begin{cases} 
V_{n+1} = | - \sqrt{V_n^2 + 4A + 2\varepsilon \sin \phi_n + 1} | \\ 
\phi_{n+1} = (\phi_n + \Delta t_{n+1}) \mod 2\pi 
\end{cases} 
\]

where \( \phi_n = t_n + \phi_0 \) is the phase of the wall at instant \( t_n \), and

\[
\Delta t_{n+1} = \frac{1}{A} \sqrt{V_n^2 + 4A - V_n}.
\]

is the interval of time between two collisions with the wall. The absolute value bars are required to avoid the particle to leave the region between the walls. If the inequality \( V_n^2 + 4A < 0 \) is satisfied, then the particle stops between the walls and the simulation for this trajectory is terminated.

Let us now proceed with the investigation of the fixed points and their stability, which will be important latter on along the paper. The fixed points are obtained from map \( T \) by applying the conditions \( V_{n+1} = V_n = V^* \) and \( \phi_{n+1} = \phi_n = \phi^* \). So we obtain the coordinates \( (\phi^*, V^*) \) of the fixed points in phase space as

\[
\begin{align*}
V^* &= \frac{1}{\pi} - \pi i A \\
\phi^* &= \arcsin \left( \frac{\pi A}{2} \right),
\end{align*}
\]

where \( i \) is a non null positive integer. For each value of \( i \) and \( \varepsilon \) the above expression furnishes, in the interval \( -\pi/(\pi i) < A < \pi/(\pi i) \), a pair of fixed points with the same velocity and different values of phase. Let us nominate \( \phi_1^* \) and \( \phi_2^* \) the two possible values of \( \phi^* \) associated to each value of \( V^* \) given by Eq. (3). If the absolute value of \( A \) increases, or \( \varepsilon \) decreases, then the maximum value of \( i \) decreases and, therefore, it decreases the maximum number of fixed points. So, for \( |A|/\varepsilon > \pi^{-1} \) the phase space does not present any fixed point.

To classify the fixed points according to their stability, we must obtain the eigenvalues of the Jacobian matrix

\[
J = \left( \frac{\partial V_{n+1}}{\partial V_n} \frac{\partial V_{n+1}}{\partial \phi_n} \right)
\]

where

\[
\begin{align*}
\frac{\partial V_{n+1}}{\partial V_n} &= \frac{V_n}{\sqrt{V_n^2 + 4A}} - 2\varepsilon \cos \phi_{n+1} \frac{\partial \phi_{n+1}}{\partial V_n}, \\
\frac{\partial V_{n+1}}{\partial \phi_n} &= -2\varepsilon \cos \phi_{n+1}, \\
\frac{\partial \phi_{n+1}}{\partial V_n} &= \frac{1}{\pi} \left( \frac{V_n}{\sqrt{V_n^2 + 4A}} - 1 \right), \\
\frac{\partial \phi_{n+1}}{\partial \phi_n} &= 1.
\end{align*}
\]

The eigenvalues \( \Lambda \) are obtained from equation \( \det(J - \Lambda I) = 0 \). The procedure furnishes, for each fixed point, a pair of eigenvalues \( \Lambda_{1,2} = \frac{1}{2} \left( \text{Tr}J \pm \sqrt{\text{Tr}J^2 - 4 \det J} \right) \),

where \( \text{Tr}J = \frac{V_n}{\sqrt{V_n^2 + 4A}} - 2\varepsilon \cos \phi_{n+1} \frac{\partial \phi_{n+1}}{\partial V_n} + 1 \) and

\[
\det J = \frac{V_n}{\sqrt{V_n^2 + 4A}}.
\]

We evaluated the eigenvalues of \( J \) regarding the fixed points given by Eq. (3) as a function of \( A \) for \( \varepsilon = 10^{-3} \). Figure 1 illustrates the eigenvalues for the fixed points \( (\phi^*_1, V^*_1) \). The smaller the \( |A| \) the greater the number of fixed points. To simplify the visualization we include the eigenvalues of fixed points with \( i = 1, 2, 3 \). But decreasing the absolute value of \( A \) we observe similar curves for other values of \( i \). Both \( A_1 \) and \( A_2 \) are real quantities, and
because $\lambda_1 \geq 1$ and $\lambda_2 \leq 1$, the fixed points $(\phi_1^+, v^+)$ are classified as saddle.

The fixed points $(\phi_2^+, v^+)$ have complex eigenvalues. Therefore these fixed points are classified as spiral focus (attracting or repelling, as we shall see). Fig. 2b) illustrates the real part of $\lambda_1, \lambda_2$, while Fig. 2b) illustrates the imaginary part of $\lambda_1$ ($\text{Im}(\lambda_1) = -\text{Im}(\lambda_2)$). The absolute value of both $\lambda_{1,2}$ is displayed at Fig. 2c). We observe that these fixed points are repelling focus for $A < 0$ (|$\lambda_{1,2}$| > 1) and they are attracting focus for $A > 0$ (|$\lambda_{1,2}$| < 1).

From Figs. 1 and 2 and the above discussion we conclude that the phase space presents saddle-node bifurcations at $A = \pm \varepsilon/(\pi i)$. The bullets in such figures indicate the values of $A$ where some of these bifurcations occur.

Let us now discuss the dynamical aspects of trajectories in phase space and how these trajectories are organized in terms of the manifolds of the fixed points of the system under study. Each saddle fixed point presents four manifolds. Two of them are stable (attractive), in sense that trajectories on them converge asymptotically to the saddle point, and two are unstable (repulsive), where ICs generate trajectories that diverge from the saddle point. Each manifold is constructed using the direction of the eigenvectors $\eta$ at the saddle point, obtained from the expression $J\eta = \Lambda \eta$.

The unstable manifolds are constructed by iterating a set of ICs defined along a line with orientation of the corresponding eigenvector, near the saddle point. The construction of the stable manifolds is slightly more complicated because it requires the construction of the inverse of the map, which furnishes the values of velocity, $V_n$, and phase of the wall $\phi_n$ from the next collision, when velocity and phase are $V_{n+1}$ and $\phi_{n+1}$,

$$T^{-1}: \begin{cases} V_n = \sqrt{(V_{n+1} + 2\varepsilon \sin \phi_{n+1})^2 - 4A} \\ \phi_n = [\phi_{n+1} - \Delta t_{n+1}] \mod 2\pi \end{cases},$$

where $\Delta t_{n+1}$ is given by Eq. (2). Near the saddle point, we define a set of points orientated with the stable eigenvectors as the ICs of the inverse map and obtain the stable manifolds.

Let us discuss at first the situation where $A < 0$. For $\varepsilon = 10^{-3}$ and $A = -1.32 \times 10^{-4}$ the phase space presents two pairs of fixed points (see Eq. (3)). As we observe in Figs. 1 and 2, for such a combination of parameters, two of these fixed points are saddle and two are repelling focus.
The unstable manifolds produce trajectories that evolve to the low energy region of the phase space. For these manifolds, the particle reaches the rest after some collisions.

The behaviour discussed above for the manifolds of saddle fixed point with \( i = 1 \) applies also for the manifolds of the saddle point with \( i = 2 \). An amplification of these manifolds near the saddle point with \( i = 2 \) is presented in Fig. 3b).

For \( A < 0 \) all trajectories evolve to the low energy portion of phase space leading the particle to reach the rest in the region between the walls. Figure 3a illustrates this behavior for the trajectories of three ICs. Two ICs were chosen near the repelling fixed points, named \( R1 \) and \( R2 \), associated to \( i = 1 \) and \( i = 2 \), respectively. Each trajectory turns around the corresponding repelling fixed point while the velocity oscillation increases. After a number of collisions the trajectories reach the region below the saddle point and evolve to values of velocity that lead the particle to reach the rest before a new collision. The third IC is \((\phi_0, V_0) = (0, 0.4)\), located above the fixed points. Above \( R1 \) the trajectory evolves decreasing the value of \( \phi_t \); below \( R1 \) the trajectory evolves increasing the value of phase. The inset in Fig. 3a illustrates the variation of velocity, \( \Delta V_\phi \), during the incursion of the trajectory around \( R1 \). A similar behaviour occurs when the trajectory passes around \( R2 \). After some collisions, the value of velocity reaches the minimum value and the time interval to the next collision diverges. Different ICs generate trajectories with the same qualitative behaviours described above. For \( A < 0 \) we have a situation where the balance between the contribution of the electric field, dissipation and amplitude of oscillation leads the particle to loose all its kinetic energy. The exceptions to this rule are, obviously, the trajectories in the stable manifolds and the fixed points.

For \( A > 0 \), however, the manifolds organize the trajectories in phase space in a different way. For \( \varepsilon = 10^{-3} \) and \( A = 1.1 \times 10^{-4} \), Eq. (3) furnishes two pairs of fixed points. Let us call the saddle and the attractor points for \( i = 1 \) as \( S1 \) and \( A1 \), respectively. Similarly \( S2 \) and \( A2 \) are the saddle and attractor points for \( i = 2 \). Figure 3a) illustrates the stable and unstable manifolds of the saddle points. We used \( \varepsilon = 10^{-3} \), \( A = 1.1 \times 10^{-4} \), \( B = 1.9 \times 10^{-4} \) and \( C = 1.09 \times 10^{-4} \). To simplify the notation, let us call \( U11 \) and \( U12 \) the unstable manifolds of saddle point \( S1 \), and \( E11, E12 \) the stable manifolds of \( S1 \). Similarly, \( U21 \) and \( U22 \) are the unstable manifolds of \( S2 \), and \( E21, E22 \) are the stable manifolds of \( S2 \). Figures 3b) and c) illustrate these manifolds near \( S1, A1 \) and \( S2, A2 \), respectively.

The manifold \( U11 \) evolves to the high energy portion of phase space and leads the particle to experience FA. The manifolds \( U12 \) and \( U22 \) evolve towards the attractors \( A1 \) and \( A2 \), respectively. The stable manifolds of both saddle points come from the low energy region of phase space. We observe that \( E11 \) and \( E12 \) are close to each other below the saddle point \( S1 \). Similarly, \( E21 \) is near \( E22 \) below \( S2 \). We observe that the region of phase space located above \( E12 \) and below \( E11 \) forms a thin channel where trajectories evolve until they pass below/near \( S1 \) and, after that, converge asymptotically to the attractor \( A1 \).

The region of phase space located above \( E11 \) and below \( E12 \) forms a large channel where trajectories pass around \( A1 \) outside \( E12 \). These trajectories access the region of high energy of the phase space and present unlimited energy gain (FA).

To illustrate the behaviour discussed above, we define a set of \( 10^4 \) ICs with \( V_0 = 0.1735 \), \( \phi_0 \) randomly chosen.
in interval \((0, 2\pi]\) and evolve them in time. Figure 6a) illustrates these ICs and the manifolds \(E_{11}, E_{12}\) and \(U_{21}\). The black points at \(V_0 = 0.1735\) correspond to ICs located in the thin channel above \(E_{12}\) and below \(E_{11}\) and they evolve to \(A_1\). The gray (green) points at \(V_0 = 0.1735\) are ICs in the large channel above \(E_{11}\) and below \(E_{12}\) and present FA. We chose this value of initial velocity because it is located above the fixed points with \(i = 2\) and below the fixed points with \(i = 1\), a region where we observe heteroclinic intersections of the unstable manifold \(U_{21}\) with the stable manifolds \(E_{11}\) and \(E_{12}\). Some points of \(U_{21}\) converge to the attractor \(A_1\) while most of them present FA, depending on their locations in phase space with relation to \(E_{11}\) and \(E_{12}\).

Similar asymptotic behaviors are observed for the regions of phase space limited by the stable manifolds \(E_{22}\) and \(E_{21}\). Figure 6b) illustrates such manifolds and the manifolds \(E_{11}, E_{12}\) of Fig. 6a). We observe in Fig. 6b) several incursions of \(E_{11}\) and \(E_{12}\) above and below the manifolds \(E_{21}\) and \(E_{22}\). As made before, we defined a set of ICs with random values of \(\phi_0\) with \(V_0 = 0.1435\) and evolved them. This value of \(V_0\) was chosen because we are interested in the behaviour of trajectories in the region below the fixed points with \(i = 2\), where the stable manifolds \(E_{21}\) and \(E_{22}\) are located. The gray (cyan) bullets correspond to ICs in the thin channel between \(E_{12}\) and \(E_{11}\). As discussed before, the asymptotic behaviour of these ICs converges to \(A_1\). The black bullets denote ICs located in the thin channel above \(E_{22}\) and below \(E_{21}\). All trajectories in this thin channel evolve to the attractor \(A_2\). The small gray (green) points are in the large channel between \(E_{11}\) and \(E_{12}\) discussed above. Therefore, these ICs present FA. We must say that both the thin and large channels formed by \(E_{11}\) and \(E_{12}\) assume a very stretched and bended shape below the fixed points \(A_1\) and \(S_2\). Figure 7 illustrates these asymptotic behaviours for three trajectories with \(V_0 = 10^{-3}\) and different values of \(\phi_0\). Figure 7b) includes the best fit to the numerical data of the energy growth associated to FA. The procedure furnishes that \(V \propto n^\gamma\) with \(\gamma \approx 1/2\) with good accuracy. Figure 7a) is an amplification of the portion corresponding to small values of \(n\) and \(V\) of Fig. 7b). In this figure we observe two trajectories evolving to the spiral attractors \(A_1\) and \(A_2\).

As the reasoning presented above for \(V_0 = 0.1435\) and \(V_0 = 0.1735\) can be extended for all the phase space, we defined a \(10^3 \times 10^3\) grid of ICs in phase space uniformly distributed in the intervals \(0 < \phi_0 \leq 2\pi\) and \(0 < V_0 \leq 0.36\) and let each initial condition to evolve in time seeking for their final state. We present these ICs in Fig. 8 where the different colors indicate the three possible asymptotic behaviours of the trajectories. The black region corresponds to ICs that converge to the attractor \(A_1\). Similarly, the gray (red) region corresponds to the ICs that evolve to \(A_2\). The light gray (yellow) region corresponds to ICs leading to FA. The previous discussion about the asymptotic behaviour in terms of the manifolds is consistent with the shapes of the basins of attraction of \(A_1\) and \(A_2\) and the channels associated to FA.

Let us present some technical information about the construction of the manifolds. We used \(10^4\) ICs in a maximum distance from the saddle points of \(10^{-3}\). The number of iterations changed depending on needs, sometimes we used less than 300 and sometimes \(10^4\) iterations.

The classification of the asymptotic behaviour of trajectories discussed about in Figs. 6 and 7 followed the procedure. We iterated the map for each IC until one of the conditions was satisfied: (i) the trajectory converged
to the attractor A1, (ii) the trajectory converged to A2 or (iii) the value of velocity reached the value \( V = 0.36 \).
We chose the value \( V = 0.36 \) because it guarantees the trajectory surrounded A1 at outside the manifold E12, had access to the high energy region of phase space and, therefore, it presents FA.

The situations where all trajectories evolve to periodic and/or chaotic attractors located at some specific regions of phase space are usual. The main point we are reporting here is the coexistence of attractors and trajectories that present FA in the phase space of a dissipative system. In what follow, we present the discussion on the complete model.

### B. The complete map

The complete version takes into account the movement of the wall in the region \([-\varepsilon, \varepsilon]\). The map of the complete version is described by the two-dimensional map of the type

\[
T: \begin{cases} 
V_{n+1} = -V^{(n+1)} - 2\varepsilon\sin\phi_{n+1} \\
\phi_{n+1} = (\phi_n + \Delta t_{n+1}) \mod 2\pi 
\end{cases}
\]

where \( V^{(n+1)} \) is the velocity of particle immediately before it collides with the moving wall at instant \( t_{n+1} = t_n + \Delta t_{n+1} \). The quantity \( t_{n+1} \) is the instant of collision \((n + 1)\). The term \( \Delta t_{n+1} \) is the smallest solution of equation \( f(\Delta t_{n+1}) = 0 \). The expressions of \( V^{(n+1)} \) and \( f(\Delta t_{n+1}) \) assume different forms depending on each situation. Because there are several details to be regarded, we describe now just some of them.

Let us consider, for example, the situation where \( A < 0 \) and \( V_n > 0 \). The quantity \( X_\Delta \) defines the position where the particle stops. If \( X_n = X_n - V_n^2/(2A) \leq \varepsilon \), then we must determine if a collision occurs (i) before or (ii) after the particle reaches the rest. In the case (i) we have \( V^{(n+1)} = V_n + A\Delta t_{n+1} \) and \( f(\Delta t_{n+1}) = X_n + V_n^2(\Delta t_{n+1}) + (A\Delta t_{n+1}^2)/2 - \varepsilon\cos(\phi_n + \Delta t_{n+1}) \), while for the case (ii) we have \( V^{(n+1)} = 0 \) and \( f(\Delta t_{n+1}) = X_n - \varepsilon\cos(\phi_n + \Delta t_{n+1}) \).

If the quantity \( X_n > \varepsilon \), then we must determine if a collision occurs before the particle leaves the region \([-\varepsilon, \varepsilon]\) using the equations of case (i) above. Note that these situations correspond to direct collisions, when the particle suffers successive collisions with the moving wall without leaving the collision zone. If a collision does not occur and if \( \varepsilon < X_n < 1 \), then the particle reaches the rest and its dynamics is over. If \( X_n \geq 1 \), then the particle hits the fixed wall and we have two possibilities: (I) the particle stops at some position \( X \in (\varepsilon, 1) \) and its dynamics dies or (II) the particle reaches the region \([-\varepsilon, \varepsilon]\).

In the last case we must determine if (a) the particle stops before colliding the moving wall or (b) a collision occurs before the particle reaches the rest. For the case (a) we have \( V^{(n+1)} = 0 \) and \( f(\Delta t_{n+1}) = X_n - \varepsilon\cos(\phi_{n+1}) \), where now \( X_n = 2 - \varepsilon\cos(\phi_n + V_n^2/(2A)) \). And for case (b) we have \( V^{(n+1)} = V_n^2 - A(t_{n+1} - t_n) \) and \( f(\Delta t_{n+1}) = \varepsilon + V_n^4 - (t_{n+1} - t_n) - (t_{n+1} - t_n)^2A/2 - \varepsilon\cos(\phi_n + \Delta t_{n+1}) \), where \( V_n^4 = \sqrt{V_n^2 - 2A(\varepsilon\cos(\phi_n + 2\pi - \varepsilon))} \) and \( t_n = t_n - (V_n + V_n^2)/A \). If \( X_n < -\varepsilon \) we must use the equations of case (b). Note that the situations (a) and (b) correspond to indirect collisions, because the particle hits the fixed wall before it collides the moving wall.

The other situations to be regarded in the complete version of the model include \( V_n < 0 \) and all the possible situations for \( A > 0 \). We must also weigh up the competition between the electric and the static/dynamic friction forces. Depending on situation, the particle does not stop, but there are situations where the particle stops only instantaneously and the signal of velocity reverses. There are also situations where the particle remains in rest for a finite time interval waiting for a collision with the moving wall and situations where the time interval between two collisions diverges, when the particle stops between the walls forever. Our complete version includes also locking phenomena, when both wall and particle move together until the instant when the particle is launched. We do not explain in details all of the cases here although the computational code takes into account all of the situations.

Figure 9 displays the phase space of complete model for both \( A > 0 \) and \( A < 0 \). Figure 9b) illustrates the phase space of the complete model for \( \varepsilon = 10^{-3}, A = 1.1 \times 10^{-4}, B = 1.9 \times 10^{-4} \) and \( C = 1.09 \times 10^{-4} \). These values of parameters are the same as those of Fig. 3. We used a set of \( 5 \times 10^3 \) ICs with \( V_0 = 10^{-3} \) and different values of \( \phi_0 \) uniformly distributed in the interval \( 0 < \phi_0 \leq 2\pi \). The trajectories of all these ICs evolve to the high energy portion of phase space and present FA. We defined two other sets of initial conditions, whose trajectories re-
result in quasi-periodic orbits and generate the islands of regular motion observed in Fig. 9a). The region near the island at $V \approx 0.16$ is shown in Fig. 9b). This result contrasts with the obtained for the simplified model, which presents attracting focus for such a combination of parameters. Moreover, it is important to observe that the phase space presents a mixed structure where regions of regular motion coexist with regions where trajectories evolve to quasi-periodic orbits and generate the islands of regular motion observed in Fig. 9a). The region near the smallest island located at $V = 0.16$. For $A < 0$ the complete version displays, therefore, a coexistence of islands of regular motion with regions where trajectories evolve to $V = 0$. The × symbols and the circles denote the fixed points obtained from simplified map 1. As before, the former symbols correspond to saddle points. The last ones correspond to elliptic points in the complete model, instead the repelling or attracting nodes of the simplified map.

The plot of the velocity as function of $n$ for $(\phi_0, V_0) = (0, 10^{-3})$ and the parameters used in Figs. 9a,b) is very similar to the curve that presents FA in Fig. 7a) obtained for the simplified model. And the plot of a single trajectory with $V_0 = 0.4$ and the parameters of Figs. 9c,d) is essentially the same observed in Fig. 1. Therefore we do not include these plots of the complete model.

Let us now discuss the above results. At first glance we may have a strange feeling when we observe the coexistence of conservative and dissipative behaviours in Figs. 9b,d). However, the definition of a 'dissipative system' is not so clear [34]. Some people can say that dissipation is associated to friction, which results in energy dissipation and corresponds to non-modelled degrees of freedom. Other people can say that dissipation corresponds to the situations where the volume of phase space is not preserved or, in other words, the system is not described by the Hamilton's equations.

The recurrence theorem, which is a consequence of Liouville's theorem, states that, in a Hamiltonian system where the phase space is bounded, exists a finite neighbourhood of a point in phase space where trajectories emanates from and eventually return to this neighbourhood.

In context of the results reported here, for the situations where the particle reaches the rest between the walls, the time until the next collision diverges. Therefore, in contrast to the hypothesis of the recurrence theorem, the phase space of the system is not a bounded domain. The velocity axis is also unbounded in vertical. The important point is that the theory protects itself from an apparent paradox [34]; our results do not violate the recurrence theorem.

A similar coexistence of trajectories that present FA with KAM islands (Figs. 9b,b)) is observed for certain values of parameter in the non-dissipative Fermi-Pustylnikov model [3]. Regions of regular dynamics were also reported in a simplified FUM with a drag force proportional to the velocity of the particle [35].

As discussed before, the situations $A < 0$ and $C < 0$ can be interpreted as null electric force ($qE = 0$). In this case the parameters are given by $A = -\sigma N/\omega^2 lm$, $B = \sigma N/\omega^2 lm$ and $C = -\omega^2 lm$. Therefore, the discussed coexistence of islands of regular motion and an attractor occurs in a FUM where a particle moves under friction in absence of electric field. It is an interesting observation because it makes us to believe that such a coexistence occurs in other systems as, for example, in time dependent two-dimensional billiards.

The main result we report here is the coexistence of an attractor located at $V = 0$ with islands of regular motion.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{These plots illustrate the phase space of the complete model for two combinations of control parameters: a) $A = 1.1 \times 10^{-3} \times 10^{-4}$, $B = 1.9 \times 10^{-4}$ and $C = 1.09 \times 10^{-4}$ and c) $A = -1.32 \times 10^{-4}$, $B = 6.68 \times 10^{-4}$ and $C = -1.72 \times 10^{-4}$. We used $\varepsilon = 10^{-3}$ in both cases. The plots b) and d) correspond to magnifications of the regions of the small islands in plots a) and c), respectively.}
\end{figure}
C. Decay of energy: an analytical description for the case \(|A|/V^2 \ll 1\)

As presented in previous sections, for \(A < 0\) both simplified and complete models present trajectories that evolve to the low energy portion of the phase space. We now present an analytical approximation to describe this velocity decay.

From the map of the simplified model, Eq. (11), we have for \(V \gg 2 \varepsilon\) that \(V_{n+1} \approx V_n \sqrt{1 + \frac{4A}{V_n^2}}\). Given an initial condition \(V_0\) we have, after the first collision, \(V_1 \approx V_0 \sqrt{1 + \frac{4A}{V_0^2}}\). If \(4|A|/V_0^2 < 1\) we evaluate a Taylor expansion until fifth order in \(A/V_0^2\) and obtain

\[
V_1 \approx V_0 \left[ 1 + 2 \frac{A}{V_0^2} - 2 \left( \frac{A}{V_0^2} \right)^2 + 4 \left( \frac{A}{V_0^2} \right)^3 - 10 \left( \frac{A}{V_0^2} \right)^4 + 28 \left( \frac{A}{V_0^2} \right)^5 + O \left( \left( \frac{A}{V_0^2} \right)^6 \right) \right].
\]

When evaluating the expression of \(V_2\) the quantity in the square brackets in above equation appears as power with exponents \(-1, -3, -5, -7, -9\). Applying again the Taylor expansion until 5th order in \(A/V_0^2\) we obtain

\[
V_2 \approx V_0 \left[ 1 + 4 \frac{A}{V_0^2} - 8 \left( \frac{A}{V_0^2} \right)^2 + 32 \left( \frac{A}{V_0^2} \right)^3 - 160 \left( \frac{A}{V_0^2} \right)^4 + 896 \left( \frac{A}{V_0^2} \right)^5 + O \left( \left( \frac{A}{V_0^2} \right)^6 \right) \right].
\]

Applying the same reasoning some times more, we obtain the following approximation for \(V_n\)

\[
V_n \approx V_0 \left[ 1 + \frac{4n^2}{V_0^2} - \frac{2n^2}{V_0^2} \left( \frac{A}{V_0^2} \right)^2 + 4n^3 \left( \frac{A}{V_0^2} \right)^3 - 10n^4 \left( \frac{A}{V_0^2} \right)^4 + 28n^5 \left( \frac{A}{V_0^2} \right)^5 + O \left( \left( \frac{A}{V_0^2} \right)^6 \right) \right]. (8)
\]

Figure 10 displays the numerical data obtained by iterating an IC for both simplified and complete versions. Figure 10 includes also the plot of the values of \(V\), obtained from the above approximation. For the simplified version we used \(V_0 = 31.8\) and for the complete version we used \(V_0 = 60\). The parameters used were \(\varepsilon = 10^{-3}\), \(A = -10^{-2}\), \(B = 1.2 \times 10^{-2}\) and \(C = -1.11 \times 10^{-2}\) for both models. For small values of \(n\) we observe a good agreement between numerical data and the approximation given by Eq. (8) for both simplified and complete versions. The greater is the number of terms in Taylor expansion the better is the agreement between numerical data and the approximation for large values of \(n\).

Before the conclusions, let us discuss the results presented in the previous sections. The energy of the particle is affected by three accelerating mechanisms: i) the electric field, that most of time furnishes energy to the particle, ii) the dissipation on the surface, that drains the energy of the particle, and iii) the oscillating wall, that furnishes or absorbs energy from the particle depending on the phase. The friction force acts in the trajectory of the particle continuously. The same rule applies for the electric force. The motion of the wall affects the energy of the particle at discrete time instants, when the collisions occur.

There are regions in phase space where the energy lost or received during the travel between the walls comes into a dynamical equilibrium with the amount of energy provided at the collisions instants with the oscillating wall in a such a way that islands of regular motion are formed in the complete model. In other words, there is a compensation between losing and gaining energy. These islands are observed also for null electric field, where the energy dissipated by the friction is balanced by the contribution of the oscillating wall.

The simplified model furnishes a good approximation to the location of the fixed points, given by Eq. (3). The dependence on parameter \(A\) in \(V^*\) and \(\phi^*\) expressions corresponds to the correction due to the contributions of the electric and friction forces. For \(A = 0\) the expression furnishes the location of the fixed points in FUM.

Regarding the results of the simplified map, we observe in Fig. 10 that elliptic islands are observed for \(A = 0\), when \(|A_1| = |A_2| = 1\). Taking the limit \(A \to 0\) in the map of simplified model we recover the situation where the particle moves inertially between elastic collisions with the boundaries and we obtain the map of FUM, as it is expected. The simplified FUM retains the nonlinearity and several characteristics of the complete FUM, such as the mixed structure of phase space, where KAM islands coexist with chaotic portions and invariant spanning curves prevent trajectories of acquiring unlimited energy growth. The simplified version we study here preserves the occurrence of trajectories that present FA, for \(A > 0\), or trajectories that evolve to \(V = 0\), for \(A < 0\). However, although we regard small values of \(\varepsilon\), when compared to the distance between the walls, the
simplified model does not preserve the islands of regular motion observed in phase space of the complete model. Disregarding the small displacement of the particle inside the collision zone, defined by the interval $X \in [-\varepsilon, \varepsilon]$, affects the qualitative behavior of the trajectories near the fixed points.

The stability of the islands of the complete model, Fig 2, was confirmed numerically for initial conditions iterated up to $10^9$ iterations. Because islands of regular motion are observed in this dissipative system even for null electric field, it is quite possible to one to find such structures in the phase space of two-dimensional time dependent billiards, when the particle slips on a rough surface.

III. CONCLUSIONS

We studied a version of Fermi-Ulam model where a metallic particle interacts with charged walls and with the field generated between them. Moreover, we regarded the parameter $\epsilon$ defines the strength of nonlinearity and the parameters $A$, $B$ and $C$ represent combinations of the electric and friction forces. We studied the dynamics of the particle regarding a simplified and the complete versions of the model, which present phase spaces with different structures.

The phase space of the simplified version presents spiral repelling fixed points and trajectories that evolve to $V = 0$, for $A < 0$. In the other hand, the phase space presents a coexistence of trajectories that evolve asymptotically to attractor fixed points and trajectories that present Fermi acceleration for $A > 0$. The asymptotic behaviour of the trajectories is described in terms of the location of initial points with relation to the channels formed by the stable manifolds of the saddle points.

The phase space of complete model presents KAM islands coexisting with trajectories that i) present FA, when $A > 0$, or ii) evolve to $V = 0$, when $A < 0$. We discuss that the coexistence of conservative and dissipative behaviours observed for $A < 0$ does not violate the recurrence theorem, because the phase space is not a bounded domain. Moreover, we discuss that such behaviour occurs even for null electric force and, therefore, this result gives a hint of observation of this coexistence in other dynamical systems where the particle moves under action of the friction force, including the class of time dependent two-dimensional billiards. However, numerical confirmation is needed.

Finally, we obtained an analytical approximation to the velocity by evaluating a Taylor expansion until 5th order in $A/V^2$ regarding the map of the simplified model. So, we described the velocity decay observed in both simplified and complete versions for $A < 0$. The approximation is good for not very long values of time, where the condition $|A|/V^2 \ll 1$ is satisfied.

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