Uniform estimates for almost primes over finite fields

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Abstract

We establish a new asymptotic formula for the number of polynomials of degree \( n \) with \( k \) prime factors over a finite field \( \mathbb{F}_q \). The error term tends to 0 uniformly in \( n \) and in \( q \). Previously, asymptotic formulas were known either for fixed \( q \), through the works of Warlimont and Hwang, or for small \( k \), through the work of Arratia, Barbour and Tavaré.

As an application, we estimate the total variation distance between the number of cycles in a random permutation on \( n \) elements and the number of prime factors of a random polynomial of degree \( n \) over \( \mathbb{F}_q \). The distance tends to 0 at rate \( 1/(q \sqrt{\log n}) \). Previously, this was only understood when either \( q \) is fixed and \( n \) tends to \( \infty \), or \( n \) is fixed and \( q \) tends to \( \infty \), by results of Arratia, Barbour and Tavaré.

1 Introduction

Given a positive integer \( n \), let \( \pi_n \) be a permutation chosen uniformly at random from \( S_n \). Given a prime power \( q \), we let \( f_n = f_{n,q} \in \mathbb{F}_q[T] \) be a polynomial chosen uniformly at random from \( \mathcal{M}_{n,q} \subseteq \mathbb{F}_q[T] \), the set of monic polynomials of degree \( n \) over the finite field \( \mathbb{F}_q \).

We define the following function:

\[
  h_q(x) := \prod_{p \in \mathcal{P}} \left( 1 - \frac{x}{|p|} \right)^{-1} \left( 1 - \frac{1}{|f|} \right)^x,
\]

where \( \mathcal{P} = \mathcal{P}_q \) is the set of monic irreducible polynomials over \( \mathbb{F}_q \) and \( |f| = q^\deg(f) \). Note that \( h_q(x) \) blows up when \( x \to q^{-} \). Our main result, Theorem 1.3 below, compares \( \mathbb{P}(\Omega(f_n) = k) \) with \( \mathbb{P}(K(\pi_n) = k) \).

Throughout the paper, \( n \geq 2 \), \( 1 \leq k \leq n \) and

\[
  r := \frac{k - 1}{\log n}.
\]

Unless stated otherwise, constants, both implied and explicit, are absolute. As Theorem 1.3 is somewhat technical, we first state two corollaries. As \( n \to \infty \), both \( K(\pi_n) \) and \( \Omega(f_n) \) become concentrated around their mean, which is \( \log n + O(1) \). The next corollary shows that the ratio of \( \mathbb{P}(\Omega(f_n) = k) \) and \( \mathbb{P}(K(\pi_n) = k) \) is asymptotic to \( h_q(r) \), in the most general limit \( q^n \to \infty \), for \( k \) as large as \( C \log n \) for an explicit \( C > 1 \).

Corollary 1.1. For \( r \leq 3/2 \) we have

\[
  \left| \frac{\mathbb{P}(\Omega(f_n) = k)}{\mathbb{P}(K(\pi_n) = k)} - h_q(r) \right| \leq \frac{Ck}{q(\log n)^2}, \quad q^n \to \infty.
\]

As we shall see in Lemma 2.1, \( h_q(r) \geq c \), and so (1.2) gives an asymptotic result.

Both \( K(\pi_n) \) and \( \Omega(f_n) \) are supported on \( [n] := \{1, 2, \ldots, n\} \). Denote by \( \mu_{K,n} \) and \( \mu_{\Omega,n} \) the distributions of \( K(\pi_n) \) and \( \Omega(f_n) \), which are measures on this set. Another corollary of our main result is an estimate for the total variation distance of the two measures.

Corollary 1.2. As \( q^n \) tends to infinity, we have

\[
  d_{TV}(\mu_{K,n}, \mu_{\Omega,n}) := \frac{1}{2} \sum_{k \in [n]} |\mathbb{P}(K(\pi_n) = k) - \mathbb{P}(\Omega(f_n) = k)| = \Theta\left( \frac{1}{q \sqrt{\log n}} \right).
\]
The main contribution to the total variation comes from values near \( \log n \). As \( h_q(1) = 1 \), if follows from Corollary 1.1 that \( \mathbb{P}(\Omega(f_n) = k) \) and \( \mathbb{P}(K(\pi_n) = k) \) are close when \( k \) is near \( \log n \), which explains heuristically why the total variation tends to 0 despite the correction factor \( h_q(r) \).

We now state the main result. Let \( X = X_n \) be a Poisson random variable with mean \( \log n \).

**Theorem 1.3.** Fix \( \delta \in (0, 1) \). Suppose \( n \geq 4(1-\delta)/\delta^2 \) and \( q \geq 1/(1-\delta)^2 \). For \( r \leq q(1-\delta) \) we have

\[
|\mathbb{P}(\Omega(f_n) = k) - \mathbb{P}(K(\pi_n) = k)h_q(r)| \leq C_{\delta}(r + 1)^{C_{\delta}r} \mathbb{P}(X = k - 1)\frac{k}{q(\log n)^2},
\]

(1.3)

Our theorem reduces the asymptotic study of \( \mathbb{P}(\Omega(f_n) = k) \) to that of \( \mathbb{P}(K(\pi_n) = k) \), at least in a certain range (see Remark 1.4 for a discussion of the range). By definition, \( \mathbb{P}(K(\pi_n) = k) = |s(n,k)|/n! \) where \( s(n,k) \) are the Stirling numbers of the first kind. Asymptotics of these numbers were studied, in the entire range \( 1 \leq k \leq n \), by Moser and Wyman [MW58].

**Remark 1.4.** From the work of Moser and Wyman, one can show that \( \mathbb{P}(X = k-1) \leq C e^{Cr^2} \mathbb{P}(K(\pi_n) = k) \), so that Theorem 1.3 implies

\[
\left| \frac{\mathbb{P}(\Omega(f_n) = k)}{\mathbb{P}(K(\pi_n) = k)} - h_q(r) \right| \leq C_{\delta} e^{Cr^2} \frac{k}{q(\log n)^2}
\]

when \( r \leq q(1-\delta) \). Since \( h_q(r) \geq 1 \) for \( r \geq 1 \), it follows that we have an asymptotic result whenever \( r \leq C_{\delta} \sqrt{\log(q(\log n))} \). However, we do not attempt to determine the widest range where \( \mathbb{P}(\Omega(f_n) = k)/\mathbb{P}(K(\pi_n) = k) \sim h_q(r) \) holds, as the current result suffices for our corollaries.

### 1.1 Previous works on pointwise bounds

Given a positive integer \( n \), we denote by \( \Omega(n) \) the number of its prime factors, counted with multiplicity. For a real number \( x > 1 \), we denote by \( N_x \) an integer chosen uniformly at random from \([1, x] \cap \mathbb{Z}\). Landau proved that [Lan09]

\[
\mathbb{P}(\Omega(N_x) = k) \sim \frac{1}{\log x} \frac{\log \log x}{(k - 1)!},
\]

as \( x \to \infty \), for any fixed \( k \geq 1 \). For \( k = 1 \) this is the Prime Number Theorem. For \( k \) growing with \( x \), one has the following result, proved by Sathe [Sat53], whose proof was greatly simplified by Selberg [Sel54]. Fix \( \delta \in (0, 2) \). Uniformly for \( x \geq 3 \) and \( 1 \leq k \leq (2-\delta) \log \log x \), one has

\[
\mathbb{P}(\Omega(N_x) = k) = \frac{1}{\log x} \frac{\log \log x}{(k - 1)!} H \left( k - 1, \frac{k}{\log \log x} \right) + O_{\delta} \left( \frac{k}{\log \log x} \right)
\]

(1.4)

as \( x \to \infty \), where

\[
H(x) := \frac{1}{\Gamma(x + 1)} \prod_{p \text{ prime}} \left( 1 - \frac{1}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^x.
\]

The proof is now a part of the general Selberg-Delange-Tenenbaum method, which is explained in detail in [Ten13 Ch. II.5].

Moser and Wyman [MW58] gave a simple asymptotic formula for \( \mathbb{P}(K(\pi_n) = k) = |s(n,k)|/n! \) in the range \( k = o(\log n) \), and a more complicated one, involving some implicit constants, for the complimentary range. Since we are interested in the wider range \( k = O(\log n) \), we state the following result of Hwang [Hwa95], proved by adapting the Selberg-Delange-Tenenbaum method:

\[
\mathbb{P}(K(\pi_n) = k) = \frac{1}{n} \frac{(\log n)^{k-1}}{(k - 1)!} \frac{1}{\Gamma(r + 1)} \left( 1 + O_{A} \left( \frac{k}{(\log n)^2} \right) \right)
\]

(1.5)

as \( n \to \infty \), uniformly for \( 1 \leq k \leq A \log n \). For \( n \to \infty \) and fixed \( q \), Warlimont [War93] proved that if we fix \( \delta \in (0, q) \), then

\[
\mathbb{P}(\Omega(f_n) = k) = \frac{1}{n} \frac{(\log n)^{k-1}}{(k - 1)!} \frac{1}{\Gamma(r + 1)} \left( h_q(r) + O_{\delta,q} \left( \frac{1}{\log n} \right) \right),
\]

(1.6)
uniformly for $1 \leq k \leq (q - \delta) \log n$. This is an analogue of (1.4); see also Car [Car82] and Afshar and Porritt [AP19]. Our Theorem 1.3 implies (1.6) with the improved error term $k/(\log n)^2$. Indeed, for $n \to \infty$ and fixed $q$ and $\delta \in (0, 1)$, we have $\mathbb{P}(X_n = k - 1) = O_{\delta,q}(\mathbb{P}(K(\pi_n) = k))$ for $r \leq q(1 - \delta)$ by (1.5), so that (1.3) takes the form $\mathbb{P}((\alpha_n) = \mathbb{P}(K(\pi_n) = k)(h_q(r) + O_{\delta,q}(k/(\log n)^2))$. By (1.5), this implies (1.6).

In the opposite limit, where $q \to \infty$ while $1 \leq k \leq n$ are fixed, we have

$$\mathbb{P}(\Omega(f_n) = k) = \mathbb{P}(K(\pi_n) = k) \left(1 + O_{\delta,q}(\frac{1}{q})\right)$$

by a standard argument, see Remark 1.5 below. We achieve an asymptotic formula for $\mathbb{P}(\Omega(f_n) = k)$, which holds in the most general limit $q^n \to \infty$, by replacing the main term

$$\frac{1}{n} \frac{(\log n)^{k-1}}{(k-1)!} \frac{h_q(r)}{(r+1)}$$

found by Warlimont, by a different one.\footnote{See [Gor17] for another example where modifying the main term leads to results in the $q^n \to \infty$ limit.}

These terms are asymptotic, in the large-$n$ limit, by the work of Hwang.

An uniform estimate for $\mathbb{P}(\Omega(f_n) = k)$, in a limited range, was established previously by Arratia, Barbour and Tavaré [ABT93 Thm. 6.1], who proved that

$$\mathbb{P}(\Omega(f_n) = k) = \mathbb{P}(K(\pi_n) = k) \left(1 + O\left(\frac{k}{q(\log n - k)}\right)\right), \quad k < \log n,$$

for $n > 1$. Their proof is probabilistic and uses a coupling argument. Corollary 1.1 implies (1.8), since $h_q(r) = 1 + O(r/q)$ for $r \leq 1$, by Lemma 2.3.

A computation of Afshar and Porritt [AP19 §5] shows that

$$\mathbb{P}(\Omega(f_n) = k) = \mathbb{P}(K(\pi_n) = k) \left(1 + O\left(\frac{k n}{q}\right)\right), \quad kn = O(q).$$

This gives an asymptotic estimate whenever $q$ grows faster than $kn$.

Finally, we mention another work of Hwang [Hwa98], who studied $\mathbb{P}(\Omega(f_n) = k)$ in the entire range of $k$, in the setting where $q$ is fixed.

### 1.2 Previous works on total variation

We may interpret $\mu_{K,n}$ and $\mu_{\Omega,n}$ as follows. Let $S_n^\#$ be the space of conjugacy classes in $S_n$. We have a natural map $X: S_n \to S_n^\#$, as well as the map $F_r: M_{n,q} \to S_n^\#$ defined as follows: if $f \in M_{n,q}$ factors as $\prod_{i=1}^{d} P_i$, $Fr(f)$ is the conjugacy class with cycle lengths $(\deg(P_i))_{i=1}^{d}$. For squarefree $f$, this map arises by labelling the roots of $f$ in the algebraic closure of $\mathbb{F}_q$ and considering the permutation induced on them by the action of the Frobenius $x \mapsto x^q$. Letting $\mu_S$ be the uniform measure on a finite set $S$, we have two measures on $S_n^\#$: $\mu_n := X_* \mu_{S_n}$ and $\mu_{n,q} := Fr_* \mu_{M_{n,q}}$, where we use $A_* B$ to denote the pushforward of the measure $B$ under the map $A$. In this notation, $\mu_{K,n} = K_* \mu_n$ and $\mu_{\Omega,n} = K_* \mu_{n,q}$.

The total variation distance of $\mu_{n,q}$ and $\mu_n$ was studied by Arratia, Barbour and Tavaré [ABT93 Cor. 5.6], who showed that it is of order $\Theta(1/q)$; see [BSG18] for an alternative proof by Bary-Soroker and the second author. This implies that

$$d_{TV}(\mu_{K,n}, \mu_{\Omega,n}) = O\left(\frac{1}{q}\right).$$

Additionally, in [ABT93 Thm. 6.8] it is proved that

$$d_{TV}(\mu_{\Omega,n}, Po(H_n)) = O\left(\frac{1}{\sqrt{\log n}}\right).$$
where $H_n$ is the $n$th harmonic number and $\text{Po}(\lambda)$ is the Poisson distribution with mean $\lambda$. From (1.9) and (1.10) and the triangle inequality, it follows by taking $q$ to infinity that (1.10) holds with $\mu_{\Omega,n}$ replaced by $\mu_{K,n}$. An additional application of the triangle inequality yields

$$d_{TV}(\mu_{K,n}, \mu_{\Omega,n}) = O\left(\frac{1}{\sqrt{\log n}}\right).$$

(1.11)

Corollary 1.2 improves upon both (1.9) and (1.11), and is optimal.

Remark 1.5. From (1.9), $P(\Omega(f_n) = k) = P(K(\pi_n) = k) + O(1/q)$ and (1.7) follows. In fact, the much weaker estimate $d_{TV}(\mu_{K,n}, \mu_{\Omega,n}) = O(q^{1/q})$ suffices; see [Coh70, Eq. (2.3)] or [ABSR15, Lem. 2.1] for a proof of it.

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2 Preparation

In what follows, $C$ and $c$ are always absolute constants whose values might change from one occurrence to the next. When constants appear with a subscript, their value may depend on the parameters in the subscript.

2.1 Primes

We denote by $\pi_q(n) := |P \cap M_{n,q}|$ the number of primes of degree $n$. From Gauss’s identity $\sum_{d|n} d\pi_q(d) = q^n$ [HAT93, Eq. (1.3)] we have the estimates

$$n\pi_q(n) \leq q^n$$

and

$$n\pi_q(n) = q^n + O(q^{\lfloor n/2 \rfloor}),$$

(2.1)

which shall be used frequently.

2.2 Generating functions

We define the following power series:

$$F(u, z) = \sum_{n,k \geq 0} P(K(\pi_n) = k) u^n z^k,$$

$$F_q(u, z) = \sum_{n,k \geq 0} P(\Omega(f_n) = k) u^n z^k.$$

Since $P(K(\pi_n) = k)$ and $P(\Omega(f_n) = k)$ are between 0 and 1, these series converge absolutely in

$$A := \{(u, z) \in \mathbb{C} \times \mathbb{C} : |u| < 1, |z| < 1\}$$

and define analytic functions in that domain. We shall show that they can be analytically continued to a larger region. The logarithm function will always be used with its principal branch. Define the infinite product

$$H_q(u, z) := \prod_{P \in \mathbb{P}} \left(1 - \left(\frac{u}{q}\right)^{\deg(P)}\right)^{z\deg(P)} \left(1 - z\left(\frac{u}{q}\right)^{\deg(P)}\right),$$

so that $H_q(1, x) = h_q(x)$. Here $(1 - (u/q)^{\deg(P)})^z = \exp(z \log(1 - (u/q)^{\deg(P)}))$. In the next lemma we study the convergence of $H_q(u, z)$ in

$$B := \{(u, z) \in \mathbb{C} \times \mathbb{C} : |u| < \sqrt{q}, |uz| < q\}.$$
Lemma 2.1. $H_q(u, z)$ converges uniformly to an analytic function on every compact subset of $B$.

Proof. For any $P \in \mathcal{P}$, let

$$h_P(u, z) := \frac{(1 - \frac{u}{q})^{\deg(P)} z}{1 - z \left(\frac{u}{q}\right)^{\deg(P)}},$$

which is analytic in $B$. We have

$$\log h_P(u, z) = \sum_{i \geq 2} \frac{(\frac{u}{q})^{\deg(P)i}}{i} (z^i - z)$$

in $B$. Fix a real number $r \in (0, \sqrt{q})$, and consider the compact subset $B_r := \{(u, z) \in \mathbb{C} \times \mathbb{C} : |u| \leq r, |z| \leq (\sqrt{q} - r)^{-1}, |uz/q| \leq r/\sqrt{q}\}$ of $B$. Any compact subset of $B$ is contained in $B_r$ for some $r$. We have, by the triangle inequality,

$$\sum_{\deg(P) \leq N} |\log h_P(u, z)| \leq \sum_{\deg(P) \leq N} \sum_{i \geq 2} \frac{|u|^{\deg(P)i}}{i} (|z|^i + |z|)$$

$$= \sum_{n \geq 1} \frac{|u|^n}{q} \sum_{d \leq N} d \pi_q(d)(|z|^{n/d} + |z|)$$

for $(u, z) \in B_r$. Recalling $\pi_q(d) \leq q^d/d$. We may assume without loss of generality that $|z| \geq 1$ (by possibly increasing $r$), since the right-hand side of (2.2) is increasing in $|z|$. The function $s(t) = q^t|z|^{n/t}$ on $[1, \min\{N, n/2\}]$ attains its maximum on one of the endpoints (since $(\log s(t))^n \geq 0$). Hence we have in $B_r$

$$\sum_{\deg(P) \leq N} |\log h_P(u, z)| \leq \sum_{n \geq 1} \frac{|u|^n}{q} \max_{1 \leq t \leq \min\{N, n/2\}} (q^t|z|^{n/t}) + \sum_{n \geq 1} \frac{(r)^n q^{n/2}|z|}{q^{n/2}} =: S_1 + S_2.$$

We bound $S_1$:

$$S_1 \leq \sum_{n \geq 1} \frac{|u|^n}{q} (q^{|z|^n} + q^{\min\{N, n/2\} - \min\{N, n/2\}})$$

$$= q \sum_{n \geq 1} \frac{|u z|}{q^n} + |z|^2 \sum_{n \leq 2N} \frac{|u|}{q^n} + q^N \sum_{n > 2N} \left(\frac{|u||z|^{1/N}}{q^n}\right)^n.$$

The first sum is at most $q \sum_{n \geq 1} (r/\sqrt{q})^n = q r/\sqrt{q}$. The second sum is at most $|z|^2 \sum_{n \geq 1} (r/\sqrt{q})^n = r^2/(\sqrt{q} - r)$. If $|z| < 1$, the third sum is at most $q^N \sum_{n > 2N} (1/\sqrt{q})^n \leq 4/\sqrt{q}$. Otherwise, $|u z|/q^n \leq |u z|/q < 1$ and so the third sum is $q^N \sum_{n \geq 2N} (1/\sqrt{q})^n \leq q^{-1} |z|^2 |u| (u^2/q)^N / (1 - r/\sqrt{q}) \leq (r/(\sqrt{q}(\sqrt{q} - r)^4)) (r^2/q)^N$. We evaluate $S_2$:

$$S_2 = |z| \sum_{n \geq 1} \frac{(r)^n}{\sqrt{q}^n} = \frac{|z| r}{\sqrt{q} - r}.$$

All in all,

$$\sum_{\deg(P) \leq N} |\log h_P(u, z)| \leq \frac{(q + |z|^2 + |z|) r}{\sqrt{q} - r} + \frac{4}{\sqrt{q}} + \frac{r}{\sqrt{q}(\sqrt{q} - r)^4} \left(\frac{r^2}{q}\right)^N$$

for $(u, z) \in B_r$. Taking $N$ to infinity, we find that $\sum_{P \in \mathcal{P}} |\log h_P(u, z)|$ converges and is bounded by a constant independent of $(u, z) \in B_r$. This proves that $H_q(u, z)$ converges uniformly to an analytic function on $B_r$. \qed
Lemma 2.2. For \((u, z) \in A\) we have

\[
F(u, z) = (1 - u)^{-z},
\]
\[
F_q(u, z) = (1 - u)^{-z} H_q(u, z).
\]

Proof. By the exponential formula for permutations \([Sta99\text{ Cor. 5.1.9]}\), we have the equality

\[
\sum_{n \geq 0} \sum_{\pi \in S_n} \frac{z^k(n)}{n!} u^n = \exp \left( \sum_{i=0}^{\infty} \frac{z^i}{i!} \right) = \exp (-z \log(1 - u)) = (1 - u)^{-z},
\]

which should be interpreted as equality of formal power series. The left-hand side of (2.3) is \(F(u, z)\). Since both sides of (2.3) define analytic function in \(A\), the uniqueness principle implies \(F(u, z) = (1 - u)^{-z}\) in \(A\). We have

\[
\prod_{P \in \mathcal{P}} \left( 1 - \left( \frac{u}{q} \right)^{\deg(P)} \right)^{-1} = \prod_{P \in \mathcal{P}, \deg(P) \leq N} \left( 1 + z \left( \frac{u}{q} \right)^{\deg(P)} + z^2 \left( \frac{u}{q} \right)^{\deg(P^2)} + \ldots + z^N \left( \frac{u}{q} \right)^{\deg(P^N)} \right)
\]

Fix a positive integer \(N\). For real \(u, z \in (0, 1)\), we have, by unique factorization in \(\mathbb{F}_q[T]\),

\[
\sum_{f \in \mathbb{F}_q[T], \text{monic}} \left( \frac{u}{q} \right)^{\deg(f)} z^{\Omega(f)} \leq \prod_{P \in \mathcal{P}, \deg(P) \leq N} \left( 1 + z \left( \frac{u}{q} \right)^{\deg(P)} + z^2 \left( \frac{u}{q} \right)^{\deg(P^2)} + \ldots + z^N \left( \frac{u}{q} \right)^{\deg(P^N)} \right)
\]

Letting \(N \to \infty\), we obtain \(F_q(u, z) \leq (1 - u)^{-z} H_q(u, z)\). To prove the reverse inequality, fix positive integers \(N < M\) and note that, again by unique factorization,

\[
\prod_{\deg(P) \leq N} \left( 1 + z \left( \frac{u}{q} \right)^{\deg(P)} + z^2 \left( \frac{u}{q} \right)^{\deg(P^2)} + \ldots + z^M \left( \frac{u}{q} \right)^{\deg(P^M)} \right) \leq \sum_{f \in \mathbb{F}_q[T], \text{monic}} \left( \frac{u}{q} \right)^{\deg(f)} z^{\Omega(f)}.
\]

Letting \(M \to \infty\) we obtain \(\prod_{\deg(P) \leq N} \left( \sum_{n \geq 0} z^n \left( \frac{u}{q} \right)^{\deg(P^n)} \right) \leq F_q(u, z)\). Letting \(N \to \infty\) we obtain \((1 - u)^{-z} H_q(u, z) \leq F_q(u, z)\). Thus \((1 - u)^{-z} H_q(u, z)\) and \(F_q(u, z)\) agree on \((0, 1) \times (0, 1)\) and so by the uniqueness principle are equal.

From now on we consider the function \((1 - u)^{-z}\) as an analytic function in \(\mathbb{C} \times (\mathbb{C} \setminus [1, \infty))\), by using the definition \((1 - u)^{-z} = \exp(-z \log(1 - u))\).

Lemma 2.3. Fix \(\delta \in (0, 1)\). Suppose \(q \geq (1 - \delta)^{-2}\), \(|u_0| \leq (1 - \delta)^{-1/2}\) and \(|z_0| \leq (1 - \delta)q\). Then

\[
\left| \left( \frac{\partial}{\partial u} H_q \right)(u_0, z_0) \right|, \left| \left( \frac{\partial}{\partial z} H_q \right)(u_0, z_0) \right|, \left| \left( \frac{\partial^2}{\partial z^2} H_q \right)(u_0, z_0) \right| \leq C_\delta \frac{|z_0|^2}{q} \exp \left( C_\delta \frac{|z_0|^2}{q} \right).
\]

Proof. We have

\[
H_q(u, z) = \exp(\log H_q(u, z)) = \exp \left( \sum_{n \geq 1} \frac{(\frac{u}{q})^n}{n} \sum_{d | n, d \neq n} d \pi_q(d)(z^{n/d} - z) \right),
\]
where the sum converges absolutely and uniformly in some neighborhood of \((u_0, z_0)\) by Lemma 2.1 and its proof. For all \(i, j \geq 0\),
\[
(\frac{\partial^{i+j}}{\partial^i u \partial^j z} \log H_q)(u, z) = \sum_{n \geq 2} w^{n-i} q^{-n(n-1)\cdots(n-(i-1))} \frac{n}{n} \sum_{d|n, d \neq n} d \pi_q(d)(z^{\frac{n-1}{d-1}}(\frac{n}{d})^{-1} - (\frac{n}{d} - (j-1))^{-z/j}),
\]
where \(z^k\) should be interpreted as 0 for negative \(k\). Recall the bound \(\pi_q(d) \leq q^d/d\), and that the function \(s(t) = q^{|z_0|^n/t}\) on \([1, n/2]\) attains its maximum on one of the endpoints if \(|z_0| \geq 1\). Otherwise, \(s(t) \leq q^{n/2}\).
Hence
\[
\left|\frac{\partial^{i+j}}{\partial^i u \partial^j z} \log H_q)(u_0, z_0)\right| \leq C \sum_{n \geq 2} (1 - \delta)^{-n/2} q^{-n(i+j)(|z_0|^{n} + q^{n/2}(1 + |z_0|^2))}
\]
for all \(i, j \geq 0\). As \(\sum_{n \geq 1} x^n n^m \leq C_{k+m} x^k/(1-x)^{m+1}\) for \(x \in (0, 1)\), we find
\[
\left|\frac{\partial^{i+j}}{\partial^i u \partial^j z} \log H_q)(u_0, z_0)\right| \leq \frac{C_{i+j, \delta}(|z_0|^2 + 1)}{q}.
\]
(2.4)
Since \((\exp(g))' = g'\exp(g)\) and \((\exp(g))'' = (g'' + g^2)\exp(g)\) for any analytic function \(g\), we are done. □

Lemma 2.4. If \(q > x \geq 1\),
\[h_q(x) \geq 1 + \frac{x - 1}{2q} \geq 1.\]

If \(0 \leq x \leq 1\),
\[h_q(x) \geq c.\]  (2.5)
Proof. By Bernoulli’s inequality, \((1 - 1/|P|)^x \geq 1 - x/|P|\) for \(x \geq 1\), and so \(h_q(x) \geq 1\) for \(x \geq 1\). By considering the contribution of linear primes to \(h_q(x)\) in (1.1), we see that for \(x \geq 1\),
\[h_q(x) \geq \left(1 - \frac{1}{q}\right)^{xq} \left(1 - \frac{x}{q}\right)^{-q} = \exp\left(\sum_{i \geq 1} \frac{x^i - x}{iq^{i-1}}\right) \geq \exp\left(\frac{x^2 - x}{2q}\right) \geq 1 + \frac{x^2 - x}{2q} \geq 1 + \frac{x - 1}{2q}.
\]
For \(0 \leq x \leq 1\), we have \(\log h_q(x) = O(x/q)\) by (2.4), so that \(h_q(x) \geq \exp(-cx/q) \geq c.\) □

2.3 Poisson distribution

Lemma 2.5. \cite{MM03}, Thm. 5.4\] Let \(X\) be a Poisson random variable with mean \(\lambda > 0\). We have \(\Pr(X \geq x) \leq (e\lambda/x)^x e^{-\lambda}\) for \(x > \lambda\).

2.4 Integral estimates

Recall \(1/(z\Gamma(z))\) is an entire function.

Lemma 2.6. Let \(G(z) = 1/(z\Gamma(z))\). We have \(|G'(z)|, |G(z)| \leq C(A + 1)^{CA}\) for \(|z| \leq A\).
Proof. The bound for \(G\) is \cite{SS03} Ch. 6, Thm. 1.6\] and the bound for \(G'\) follows from the one for \(G\) by Cauchy’s integral formula. □

Lemma 2.7. Fix \(A > 0\). For all \(|z| \leq A\) and \(n \geq 1\) we have
\[\left|\binom{n+z-1}{n} - \frac{n^{z-1}}{\Gamma(z)}\right| \leq C(A + 1)^{CA} n^{\Re z - 2}.
\]
Proof. For \(z\) a non-positive integer, the left-hand side is 0 or sufficiently small. Otherwise, dividing by \(n^{\Re z - 2}\), it suffices to bound
\[\left|\frac{\Gamma(n+z)}{(n+1)\Gamma(z)n^{\Re z - 2}} - \frac{n}{\Gamma(z)}\right|,
\]
Lemma 2.8. Let \( n \geq 2A + 1, \Re(n + z) \geq n/2 \) and we may apply Stirling’s approximation to find \( \Gamma(n + z)/(\Gamma(n + 1)n^{z-2}) = n + O((A + 1)^{CA}) \) and the desired bound follows from Lemma 2.6. If \( n < 2A + 1 \), the terms \( 1/n^{z-2} \) and \( \Gamma(n + z)/\Gamma(z) = |(n + z - 1)(n + z - 2)\ldots(z)| \) are all bounded from above by \( O((A + 1)^{CA}) \), as well as \( |1/\Gamma(z)|, 1/\Gamma(n + 1) \) by Lemma 2.6 which finishes the proof.

For the rest of this section, let \( X = X_n \) be a Poisson random variable with mean \( \log n \).

**Lemma 2.8.** Let \( n \geq k > 1 \) and set \( r = (k-1)/\log n \). Let \( \beta \) be the circle \(|z| = r\) oriented counterclockwise. For \( j \geq 0 \) we have

\[
\int_{\beta} \left| \frac{(z-r)^jn^{z-1}}{z^k} \right| |dz| \leq C_j \mathbb{P}(X = k-1) \left( \frac{\sqrt{k}}{\log n} \right)^j, \tag{2.6}
\]

\[
\int_{\beta} \left| \frac{(z-r)^jn^{z-1}}{z^{k+1}\Gamma(z)} \right| |dz| \leq C_j \mathbb{P}(X = k-1) \left( \frac{\sqrt{k}}{\log n} \right)^j (r+1)^{Cr}, \tag{2.7}
\]

\[
\int_{\beta} \frac{(z-r)n^{z-1}}{z^k} |dz| = 0. \tag{2.8}
\]

**Proof.** Using the parametrization \( z = re^{it} \) and the estimate \( \cos t - 1 \leq -ct^2 \) for \( t \in [-\pi, \pi] \),

\[
\int_{\beta} \left| \frac{(z-r)^jn^{z-1}}{z^k} \right| |dz| \leq \frac{n^{r-1}r}{\pi^{k-1}} \int_{-\pi}^{\pi} |e^{it} - 1|^j n^{-rct^2} dt \leq \frac{n^{r-1}r}{\pi^{k-1}} \int_{-\pi}^{\pi} |t|^j n^{-rct^2} dt,
\]

and we conclude (2.6) by using the change of variables \((k-1)t^2 = s^2\) and Stirling’s approximation. To obtain (2.7) we repeat the computation and appeal to Lemma 2.6. To obtain (2.8), observe that the coefficient of \( z^{k-1} \) in \((z-r)n^{z-1}\) is

\[
n^{-1} \left( \frac{\log n}{(k-2)!} - r \frac{(\log n)^{k-1}}{(k-1)!} \right) = 0,
\]

as needed.

**Proposition 2.9.** Let \( n \geq k > 1 \). Let \( \beta \) be the circle \(|z| = r\) oriented counterclockwise in the \( z\)-plane. Let \( \gamma \) be the path in the \( u\)-plane depicted in Figure 1. In formulas, \( \gamma \) is oriented counterclockwise as well, and we write it as a union of two curves, \( \gamma_1 \) and \( \gamma_2 \). Let \( R = 1 + 1/\sqrt{n} \) and define \( \theta_1 \in (0, \pi) \) by \( R\sin(\theta_1) = 1/n \).
The curve $\gamma_1$ is $\gamma_1' + \gamma_1'' + \gamma_1'''$, with
\[
\gamma_1'(t) = -\frac{i}{n} - t, \quad t \in [-R \cos(\theta_1), -1],
\]
\[
\gamma_1''(\theta) = 1 + \frac{e^{(2\pi - \theta)}}{n}, \quad \theta \in [\pi/2, 3\pi/2],
\]
\[
\gamma_1'''(t) = \frac{i}{n} + t, \quad t \in [1, R \cos(\theta_1)],
\]
and $\gamma_2$ given by
\[
\gamma_2(\theta) = Re^{i\theta}, \quad \theta \in [\theta_1, 2\pi - \theta_1].
\]

We have
\[
\int_{\beta} \int_{\gamma} \frac{|(1 - u)^{-z}|}{|u|^{n+1}|z|^{k+1}}|u - 1||du||dz| \leq C \mathbb{P}(X = k - 1)(r + 1)^{C_r \log n} \frac{n}{nk}. \quad (2.9)
\]

Proof. Let $I_1$ and $I_2$ be the integrals over $\beta \times \gamma_1$ and $\beta \times \gamma_2$, respectively:
\[
I_i := \int_{\beta} \int_{\gamma_i} \frac{|(1 - u)^{-z}|}{|u|^{n+1}|z|^{k+1}}|u - 1||du||dz|, \quad i = 1, 2.
\]

By performing the change of variables $u = 1 + n^{-1}v$, we obtain
\[
I_1 = \frac{1}{n^2} \int_{\beta} \int_{\gamma_3} \frac{|(v)^{-z}|}{|1 + n^{-1}v|^{n+1}}|dv||dz|, \quad (2.10)
\]
where $\gamma_3$ is depicted in Figure 1. We continue by bounding the inner integral:
\[
\max_{|z| \leq r} \int_{\gamma_3} |(v)^{-z}| |v| |1 + n^{-1}v|^{-(n+1)} |dv| \leq e^{\pi r} \max_{|z| \leq r} \int_{\gamma_3} |v|^{C(r+1)} |1 + n^{-1}v|^{-(n+1)} |dv|
\]
\[
\leq e^{\pi r} (C + C \int_0^{\infty} t^{C(r+1)} e^{-ct} dt)
\]
\[
\leq e^{\pi r} \Gamma(C(r+1)) \leq C(r + 1)^{C_r}.
\]

We substitute the last bound in (2.10), parametrize $\beta$ as $z = re^{it}$ and use the inequality $\Re z \leq r(1 - c t^2)$, which leads to
\[
I_1 \leq \frac{C(r + 1)^{C_r}}{n^2 r^k} \int_{-\pi}^{\pi} n^{r(1 - cr^2)} dt = \frac{C(r + 1)^{C_r} n^{r-2}}{r^k \sqrt{\log n}} \int_{-\pi}^{\pi} \sqrt{\log n} e^{-cs^2} ds \leq \frac{C(r + 1)^{C_r} n^{r-2}}{r^k \sqrt{\log n}}.
\]

Thus, by (a weak version of) Stirling’s approximation we obtain
\[
I_1 \leq C(r + 1)^{C_r} \mathbb{P}(X = k - 1) \frac{\log n}{nk}.
\]

We turn to bound $I_2$. On $\beta \times \gamma_2$ we have $|(1 - u)^{-z}| \leq C \exp(\pi r + k/2)$, and so
\[
I_2 \leq C \frac{\exp(\pi r + \frac{k}{2})}{R^{n+1} r^k} \leq C(r + 1)^{C_r} \mathbb{P}(X = k - 1) \exp(-ck - c\sqrt{n}),
\]
where here again apply Stirling. As both $I_1$ and $I_2$ are bounded by the right-hand side of (2.9), we conclude the proof.
3 Proof of Theorem 1.3

For $k = 1$, the result follows from (2.1), so we may suppose $k > 1$. Fix $\delta \in (0, 1)$ and suppose $r \leq q(1 - \delta)$, $q \geq (1 - \delta)^{-2}$ and $n \geq 4(1 - \delta)/\delta^2$ (so that $1 + 1/\sqrt{n} \leq (1 - \delta)^{-1/2}$). By Cauchy’s integral formula, we have

$$P(K(\pi_n) = k) = \left(\frac{1}{2\pi i}\right)^2 \int_\beta \int_\gamma \frac{(1 - u)^{-z}}{u^{|n + 1|z^{k+1}}} du \, dz,$$

$$P(\Omega(f_n) = k) = \left(\frac{1}{2\pi i}\right)^2 \int_\beta \int_\gamma \frac{(1 - u)^{-z}}{u^n z^{k+1}} H_q(u, z) du \, dz,$$

where $\beta$ and $\gamma$ are as defined in Proposition 2.9. Recall that $h_q(\bullet) = H_q(1, \bullet)$. Thus,

$$P(\Omega(f_n) = k) - P(K(\pi_n) = k) = h_q(r) = \left(\frac{1}{2\pi i}\right)^2 \int_\beta \int_\gamma \frac{(1 - u)^{-z}}{u^n z^{k+1}} (H_q(u, z) - H_q(1, r)) du \, dz. \quad (3.1)$$

We have

$$H_q(u, z) - H_q(1, r) = (H_q(u, z) - H_q(1, z)) + (H_q(1, z) - H_q(1, r)) = O_{r, \delta, q}(\lvert u - 1 \rvert) + H_q(1, z) - H_q(1, r),$$

where the implied constant is, by Lemma 2.3

$$C_r \frac{r^2 + 1}{q} \exp \left(\frac{C_r}{q} \exp(C_r)\right). \quad (3.2)$$

Proposition 2.9 shows that the total contribution of the $O_{r, \delta, q}(\lvert u - 1 \rvert)$-term to the right-hand side of (3.1) is acceptable. Since the $n$th coefficient of $(1 - u)^{-z}$ is $(\binom{n + z - 1}{n})$, we can reduce the problem to a problem in the $z$-plane, namely bounding

$$\int_\beta \int_\gamma \frac{(1 - u)^{-z}}{u^n z^{k+1}} (H_q(1, z) - H_q(1, r)) du \, dz = \int_\beta \frac{(n + z - 1)(H_q(1, z) - H_q(1, r))}{z^{k+1}} dz.$$

By Lemmas 2.3, 2.4 and 2.8, we may replace $(\binom{n + z - 1}{n})$ with $n^{z-1}/\Gamma(z)$ and $H_q(1, z) - H_q(1, r)$ with $(z - r)(\frac{\partial}{\partial z} H_q(1, r) + O_{r, \delta, q}(z - r)^2)$ (the implied constant being again (3.2)), and the error terms will be acceptable. To bound the remaining integral, we use a first-order Taylor approximation for $G(z) = 1/(z\Gamma(z))$ to write

$$\int_\beta\frac{n^{z-1}(z - r)}{\Gamma(z) z^{k+1}} dz = \frac{1}{\Gamma(r)} \int_\beta \frac{n^{z-1}(z - r)}{z^k} dz + O \left(\max_{|t| \leq r} |G'(t)| \int_\beta \frac{n^{z-1}(z - r)^2}{z^k} |dz|\right).$$

The main term vanishes by (2.5), and the error term is small enough by (2.6) and Lemma 2.6. This finishes the proof.

4 Proof of Corollary 1.1

For $n \leq 100$, the result follows from (1.7) since $h_q(r) = 1 + O(1/q)$ for $r \leq 3/2$ by Lemma 2.8. Otherwise, let us take $\delta = 1/5$ in Theorem 1.3 and obtain

$$P(\Omega(f_n) = k) - P(K(\pi_n) = k) h_q(r) = O \left(\frac{P(X = k - 1)k}{q(\log n)^2}\right)$$

for all $n \geq 100$ and $q \geq 2$. The proof is finished by noting that $P(X = k - 1) = O \left(P(K(\pi_n) = k)\right)$ uniformly in the range $k \leq 3\log n/2$ by (1.5).
5 Proof of Corollary 1.2

We may assume $n \geq C$, since for any fixed $n$ the following argument works. An upper bound of $O_n(1/q)$ on the total variation follows from Remark 1.3 while a lower bound of order $1/q$ follows from considering the contribution of $k = n$:

$$|\mathbb{P}(\Omega(f_n) = n) - \mathbb{P}(K(\pi_n) = n)| = \frac{(q n^{-1})}{q^n} - 1 = \frac{1}{n!} \left( \prod_{i=1}^{n-1} \left( 1 + \frac{i}{q} \right) - 1 \right) \geq \frac{1}{q} \frac{1}{n!} \left( \frac{n}{2} \right).$$

Let $I_1 = [1, 3 \log n/2]$, $I_2 = (3 \log n/2, \sqrt{q} \log n]$, $I_3 = (\sqrt{q} \log n, n]$. For $1 \leq i \leq 3$, let $S_i$ be the contribution of $k \in I_i$ to the total variation:

$$S_i = \sum_{k \in I_i} |\mathbb{P}(\Omega(f_n) = k) - \mathbb{P}(K(\pi_n) = k)|.$$

We shall show that $S_3 = O(1/(q \sqrt{\log n}))$ for each $i$. Observe that $1 = h_q(1)$ and that $h_q'(z) = O(1/q)$ for $|z| \leq 3/2$ by Lemma 2.3. By Theorem 1.3 and the estimate $h_q(z) - h_q(1) = O((z - 1)/q)$,

$$S_1 = \sum_{k \in I_1} \mathbb{P}(K(\pi_n) = k) (h_q(r) - h_q(1)) + O \left( \frac{\mathbb{E}(X - k - 1)^2}{q} \right) \leq \frac{C}{q} \left( \sum_{k \in I_1} \mathbb{P}(K(\pi_n) = k) |r - 1| + \sum_{k \in I_1} \mathbb{P}(X = k - 1) \right).$$

From (5.1) we deduce the upper bound $\mathbb{P}(K(\pi_n) = k) \leq C \mathbb{P}(X = k - 1)$ for $k \leq 3 \log n/2$, so that

$$S_1 \leq \frac{C}{q} \sum_{k \in I_1} \mathbb{E}X \mathbb{E}X - \log n | \mathbb{E}X - \log n | \left( \frac{k - 1}{\log n} - 1 \right) \leq \frac{C}{q} \left( \frac{\mathbb{E}X - \log n}{\log n} + \mathbb{E}X + 1 \right) \leq \frac{C}{q} \sqrt{\log n},$$

where the last inequality uses Cauchy-Schwarz: $\mathbb{E}X - \log n | \leq \text{Var}(X)^{1/2} = \sqrt{\log n}$. For $k \in I_2$, we have $h_q(r) - 1 = O(r^3/q)$ by Lemma 2.3 with $\delta = 1/5$. By Theorem 1.3 with $\delta = 1/5$,

$$S_2 \leq \frac{C}{q} \left( \sum_{k \in I_2} \mathbb{P}(K(\pi_n) = k) r^3 + \sum_{k \in I_2} \mathbb{P}(X = k - 1) \right) \leq \frac{C}{q} \sqrt{\log n}.$$

We bound the first sum using Cauchy-Schwarz:

$$\sum_{k \in I_2} \mathbb{P}(K(\pi_n) = k) r^3 \leq \mathbb{E}K^3(\pi_n) \cdot \mathbb{E}K^3(\pi_n) \leq \frac{\mathbb{E}K^6(\pi_n)}{(\log n)^3} \leq \frac{\mathbb{E}K^6(\pi_n)}{(\log n)^3}.$$

By Markov’s inequality and $\mathbb{E}K^6(\pi_n) = n + 1$ [AW01 Thm. 1.3], we have

$$\mathbb{P}(K(\pi_n) > 3 \log n/2) = \mathbb{P}(2K(\pi_n) > n^{(log 8/2)}) \leq n^{-1} \mathbb{P}(2K(\pi_n) = (n + 1)n^{-1} \leq n^{-c}.$$$$A$$ similar argument shows $\mathbb{P}(K(\pi_n) > 10 \log n) = O(1/n^6)$, yielding $\mathbb{E}K^6(\pi_n) \leq C(\log n)^6$. Hence, the first sum in (5.1) is $O(n^{-c})$. To bound the second sum, we partition $I_2$ into intervals of length $\log n/2$:

$$\sum_{k \in I_2} \mathbb{P}(X = k - 1) \leq \sum_{j=3}^{2\sqrt{q}} \sum_{k \in I_{j,j+1}} \mathbb{P}(X = k - 1) \leq \sum_{j=3}^{2\sqrt{q}} \mathbb{P}(X \geq \frac{1}{2} \log n - 1) \leq \mathbb{P}(X \geq \frac{1}{2} \log n - 1) \leq \mathbb{P}(X \geq \frac{1}{2} \log n - 1) \leq \frac{1}{j+1} \log n (j+1)^{c_j}.$$
By Lemma 2.5 the probability in the right-hand side of (5.2) is bounded by

$$\mathbb{P}(X \geq \frac{j}{2} \log n - 1) \leq n^{\frac{1}{2}(1-\log \frac{j}{2})-1}e^{Cj} \leq (j + 1)^{-c} \log n e^{Cj},$$

where in the last inequality we use the fact that \((j/2)(1 - \log(j/2)) - 1\) is negative for all \(j \geq 3\). Hence,

$$\sum_{k \in I_n} \frac{\mathbb{P}(X = k - 1)k}{(\log n)^2} (r + 1)^C \leq n^{-c} \sum_{j \geq 3} (j + 1)^{-c} \log n (j + 1)^Cj \leq n^{-c}$$

for sufficiently large \(n\). Substituting this bound into (5.1) we conclude that \(S_2 \leq 1/(qn^c)\).

To bound \(S_3\), recall that \(\text{Var}(K(\pi_n)) = \log n + O(1)\) \([\text{Gon42}]\) and that \(\text{Var}(\Omega(f_n)) = \log n + O(1)\) (this is a function-field version of the main result of \([\text{Tur34}]\), and both implied constants are absolute. Applying Chebyshev’s inequality, we find \(\mathbb{P}(K(\pi_n) \geq \sqrt{q} \log n), \mathbb{P}(\Omega(f_n) \geq \sqrt{q} \log n) \leq C/(q \log n)\), and so \(S_3 = O(1/(q \log n))\).

We now turn to prove a matching lower bound. Recall we may assume \(n \geq C\). We consider the contribution to the total variation coming from \(k - \log n \in [1, \sqrt{\log n}]\), which, by Corollary 1.1 is

$$\sum_{k - \log n \in (0, \sqrt{\log n})} \mathbb{P}(K(\pi_n) = k) |h_q(r) - 1| + O \left( \frac{1}{q \log n} \right). \quad (5.3)$$

By (1.5), \(\mathbb{P}(K(\pi_n) = k) \geq c\mathbb{P}(X = k - 1)\) for \(r \leq 3/2\). Additionally, \(h_q(r) \geq 1 + (r - 1)/(2q)\) for \(r \geq 1\) by (2.5). Hence, the last sum is bounded from below by

$$\frac{c}{q \log n} \sum_{k - \log n \in [1, \sqrt{\log n}]} \mathbb{P}(X = k - 1) |k - 1 - \log n|.$$

By Stirling’s approximation, \(\mathbb{P}(X = i + \lfloor \sqrt{\log n} \rfloor) \geq c/\sqrt{\log n}\) for \(i = O(\sqrt{\log n})\), so that the last expression is bounded from below by

$$\frac{c}{q (\log n)^{3/2}} \sum_{3 \leq i \leq \sqrt{\log n} - 3} \frac{i}{q \sqrt{\log n}} \geq \frac{c}{q \sqrt{\log n}}.$$

If \(n\) is large enough, the error term in (5.3) is small compared to \(c/(q \sqrt{\log n})\), and the lower bound for the total variation follows.

References

[ABSR15] J. C. Andrade, L. Bary-Soroker, and Z. Rudnick. Shifted convolution and the Titchmarsh divisor problem over \(F_q[t]\). Philos. Trans. Roy. Soc. A, 373(2040):20140308, 18, 2015.

[ABT93] Richard Arratia, A. D. Barbour, and Simon Tavaré. On random polynomials over finite fields. Math. Proc. Cambridge Philos. Soc., 114(2):347–368, 1993.

[AP19] Ardavan Afshar and Sam Porritt. The function field Sathe-Selberg formula in arithmetic progressions and ‘short intervals’. Acta Arith., 187(2):101–124, 2019.

[BSG18] Lior Bary-Soroker and Ofir Gorodetsky. Roots of polynomials and the derangement problem. Amer. Math. Monthly, 125(10):934–938, 2018.

[Car82] Mireille Car. Factorisation dans \(F_q[X]\). C. R. Acad. Sci. Paris Sér. I Math., 294(4):147–150, 1982.

[Coh70] Stephen D. Cohen. The distribution of polynomials over finite fields. Acta Arith., 17:255–271, 1970.

[Gon42] W. Gontcharoff. Sur la distribution des cycles dans les permutations. C. R. (Doklady) Acad. Sci. URSS (N.S.), 35:267–269, 1942.
[Gor17] Ofir Gorodetsky. A polynomial analogue of Landau’s theorem and related problems. *Mathematika*, 63(2):622–665, 2017.

[Hwa95] Hsien-Kuei Hwang. Asymptotic expansions for the Stirling numbers of the first kind. *J. Combin. Theory Ser. A*, 71(2):343–351, 1995.

[Hwa98] Hsien-Kuei Hwang. A Poisson * negative binomial convolution law for random polynomials over finite fields. *Random Structures Algorithms*, 13(1):17–47, 1998.

[Lan09] E. Landau. Handbuch der Lehre von der Verteilung der Primzahlen. Erster Band. Leipzig u. Berlin: B. G. Teubner. X + 564 S. (1909)., 1909.

[MU05] Michael Mitzenmacher and Eli Upfal. *Probability and computing*. Cambridge University Press, Cambridge, 2005. Randomized algorithms and probabilistic analysis.

[MW58] L. Moser and M. Wyman. Asymptotic development of the Stirling numbers of the first kind. *J. London Math. Soc.*, 33:133–146, 1958.

[Sat53] L. G. Sathe. On a problem of Hardy on the distribution of integers having a given number of prime factors. I. *J. Indian Math. Soc. (N.S.)*, 17:63–82, 1953.

[Sel54] Atle Selberg. Note on a paper by L. G. Sathe. *J. Indian Math. Soc. (N.S.)*, 18:83–87, 1954.

[SS03] Elias M. Stein and Rami Shakarchi. *Complex analysis*, volume 2 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ten15] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015. Translated from the 2008 French edition by Patrick D. F. Ion.

[Tur34] Paul Turán. On a Theorem of Hardy and Ramanujan. *J. London Math. Soc.*, 9(4):274–276, 1934.

[vLW01] J. H. van Lint and R. M. Wilson. *A course in combinatorics*. Cambridge University Press, Cambridge, second edition, 2001.

[War93] R. Warlimont. Arithmetical semigroups. IV. Selberg’s analysis. *Arch. Math. (Basel)*, 60(1):58–72, 1993.

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