On Bosonic and Supersymmetric Current Algebras for Non-semi-simple Groups

Noureddine Mohammedi

Physikalisches Institut
der Universität Bonn
Nussallee 12
D-53115 Bonn, Germany

Abstract

We present a systematic approach to constructing current algebras based on non-semi-simple groups. The Virasoro central charges corresponding to these current algebras are not, in general, given by integer numbers. The key point in this construction is that the bilinear form appearing in the current algebra can be different from the bilinear form used to raise and lower group indices. The action which realises this current algebra as its symmetry is also found.
1 Introduction

Recently, a Wess-Zumino-Novikov-Witten (WZNW) model describing string propagation on a four dimensional Lorentz-signature space-time was built on the centrally extended Euclidean group in two dimensions [1]. Further studies of this model [2,3] and generalisations to higher dimensions were presented in [4,5].

All these models describe string backgrounds with a target space metric having a covariantly constant null Killing vector [6-10]. Another feature of the WZNW models constructed so far [1,4,5], is that they all have integer central charges. The central charges are equal to the dimensions of the group manifold built on the non-semi-simple groups.

In this note, we systematically describe both the bosonic and the supersymmetric current algebras based on non-semi-simple groups. The Virasoro central charges corresponding to these current algebras are not necessarily integer numbers. We take the bilinear form entering the operator product expansions of the current algebra to be different from the bilinear form which raises and lowers group indices. We then construct a Virasoro generator via the Sugawara method. The generators of the current algebra are primary fields of conformal weight (1,1) with respect to this energy-momentum tensor. Finally, we present an action which has the current algebra as its symmetry.

2 The Bosonic Current Algebra

Let $G$ be a non-semi-simple group whose Lie algebra is given by

$$\left[ T^a, T^b \right] = f^{ab}_{\ c} T^c .\quad (2.1)$$

The Cartan bilinear form, $\gamma^{ab}$, defined by

$$\gamma^{ab} = f^{ae}_{\ c} f^{eb}_{\ e} \quad (2.2)$$

is degenerate (not invertible) for non-semi-simple groups and cannot, therefore, be used to raise and lower group indices. However, $\gamma^{ab}$ is an invariant of the group, namely

$$f^{ab}_{\ c} \gamma^{cd} + f^{ad}_{\ c} \gamma^{eb} = 0 .\quad (2.3)$$

In order to be able to raise and lower group indices, we need to look for another bilinear form, denoted here by $\Omega^{ab}$, which is symmetric, invertible and is an invariant of the group. In other words, we must have

$$f^{ab}_{\ c} \Omega^{cd} + f^{ad}_{\ c} \Omega^{eb} = 0$$

$$\Omega^{ab}\Omega_{bc} = \delta^a_c , \quad \Omega^{ab} = \Omega^{ba} ,\quad (2.4)$$
where $\Omega_{ab}$ is the inverse matrix of $\Omega^{ab}$. In what follows, we will always assume that such a bilinear form does exist.

The bosonic current algebra built on this Lie algebra is generated by the holomorphic currents $J^a(z)$ having the operator product expansion

$$J^a(z)J^b(w) = -k \frac{g^{ab}}{(z-w)^2} + f^{ab}_{\ c} \frac{J^c(w)}{(z-w)} .$$

(2.5)

The new symmetric matrix $g^{ab} = g^{ba}$ is a new bilinear form that we are going to determine. In fact, the associativity of the above operator product expansions shows that $g^{ab}$ is an invariant of the group obeying

$$f^{ab}_{\ cd}g^{cd} + f^{ad}_{\ bc}g^{cb} = 0 .$$

(2.6)

Therefore $g_{ab}$ must be a linear combination of $\gamma_{ab}$ and $\Omega_{ab}$

$$g^{ab} = m\gamma^{ab} + n\Omega^{ab} ,$$

(2.7)

where $m$ and $n$ are two constants that we are going to find. Here $g^{ab}$ need not be invertible.

The energy-momentum tensor is then written as a general quadratic sum of the currents $J^a(z)$

$$T(z) = L_{ab} : J^a J^b : (z) ,$$

(2.8)

with $L_{ab}$ a symmetric matrix. We will determine $L_{ab}$ by requiring that the currents $J^a(z)$ are primary fields of conformal weight 1 with respect to the stress tensor $T(z)$. Therefore by requiring that

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}$$

(2.9)

we get the following equations for the matrices $L_{ab}$ and $g^{ab}$

$$L_{eb}f^{ba}_{\ e} + L_{eb}f^{ba}_{\ c} = 0$$

$$-2kL_{eb}g^{ba} + L_{bd}f^{ab}_{\ e}f^{ed}_{\ c} = \delta^a_{\ c} .$$

(2.10)

The first equation of this set is equivalent to the first equation in (2.4) and is uniquely solved by

$$L_{ab} = l\Omega_{ab} ,$$

(2.11)

where $l$ is any arbitrary number. Using this solution, the second equation of the set leads to

$$g^{ab} = -\frac{1}{2kl} \left( \Omega^{ab} - l\gamma^{ab} \right) .$$

(2.12)

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$^1$ $L_{ab}$ would have to satisfy a master equation if we did not require $J^a(z)$ to be of conformal weight equal to one [11,12].
With these expressions for $L_{ab}$ and $g^{ab}$, the above energy-momentum tensor satisfies the Virasoro algebra

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} .$$

(2.13)

The central charge of this algebra is given by

$$c = \dim(G) - l_{\gamma}^{ab} \Omega_{ab},$$

(2.14)

In performing the above operator product expansion, we have defined the normal ordered product of two operators $A(z)$ and $B(z)$ at coincident points as [13]

$$: AB : (z) = \frac{1}{2\pi i} \oint_{C_z} \frac{dx}{x-z} A(x)B(z) ,$$

(2.15)

where the countour of integration, $C_z$, surrounds the point $z$. This definition leads to the following form of Wick’s theorem for calculating the product expansion of $A(z)$ with a composite field $: BC : (w)$

$$A(z): BC : (w) = \frac{1}{2\pi i} \oint_{C_w} \frac{dx}{x-w} \{ A(z)B(x)C(w) + (-1)^{BC} A(z)C(w)B(x) \} ,$$

(2.16)

where $(-1)^{BC} = -1$ iff both $B$ and $C$ are fermionic fields and the contraction $(\underline{\quad})$ stands for the singular part in the expansion of the product of two operators at distinct points.

## 3 The Supersymmetric Current Algebra

Let us now turn our attention to the supersymmetric case of the above current algebra. The supercurrent algebra is given by the operator product expansions of the supercurrents $J^a(Z)$ [14,15,16]

$$J^a(Z_1)J^b(Z_2) = -kh^{ab}Z_{12}^{-1} + f^{ab}_{\ c}Z_{12}^{-1/2}J^c(Z_2) .$$

(3.1)

Here $h^{ab}$ is a symmetric matrix and, by associativity of the super operator product expansions, is an invariant of the group $\mathcal{G}$. In these expressions $Z = (z, \theta)$ denotes the holomorphic coordinate of two-dimensional superspace and the symbol $Z_{ij}^M$, $M \in \mathcal{Z}$, is defined by

$$Z_{ij}^M = \begin{cases} (z_i - z_j - \theta_i \theta_j)^M, & M \in \mathcal{Z} \\ (\theta_i - \theta_j)(z_i - z_j - \theta_i \theta_j)^{M-1/2}, & M \in \mathcal{Z} + \frac{1}{2} \end{cases} .$$

(3.2)

We postulate the super energy-momentum tensor to have the form

$$T(Z) = N_{ab} : DJ^aJ^b : (Z) + M_{abc} : J^a : J^bJ^c :: (Z) ,$$

(3.3)
where \( N_{ab} \) is symmetric and \( M_{abc} \) is totally antisymmetric. The super covariant derivatives is \( D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \) obeying \( D^2 = \partial \).

These two tensors are then determined by requiring that the supercurrents \( J^a(Z) \) are primary operators of dimension \( \frac{1}{2} \) with respect to the super stress tensor and that \( T(Z) \) satisfies the super Virasoro algebra. Demanding that

\[
T(Z_1)J^a(Z_2) = \frac{1}{2} Z_{12}^{-3/2} J^a(Z_2) + \frac{1}{2} Z_{12}^{-1} D J^a(Z_2) + Z_{12}^{-1/2} \partial J^a(Z_2)
\]

leads to the following six equations

\[
\begin{align*}
-k N_{bc} h^{ac} &= \frac{1}{2} \delta^a_c \\
N_{bc} f^{ab}_d - 3 M_{bcd} h^{ab} &= 0 \\
N_{bc} f^{ab}_d + N_{bd} f^{ac}_e &= 0 \\
-k h^{de} f^{ab}_d N_{cb} &= 0 \\
M_{bcd} f^{ab}_e + M_{ebd} f^{ac}_c + M_{ecb} f^{ab}_d &= 0 \\
-k N_{bc} h^{ab} + N_{bc} f^{ab}_d f^{de}_e - 3 k M_{bcd} f^{ab}_e h^{ed} &= \frac{1}{2} \delta^a_c
\end{align*}
\]

The first three equations of this set are uniquely solved by

\[
N_{ab} = l \Omega_{ab} \quad h^{ab} = -\frac{1}{2kl} \Omega^{ab} \quad M_{abc} = \frac{2l^2}{3} f_{abc} \equiv \frac{2l^2}{3} \Omega_{ar} \Omega_{bs} f^{rs}_{\ c} .
\]

Here \( l \) is again an arbitrary number. The last three equations in the set are then automatically satisfied once the Jacobi identities for the structure constants \( f^{ab}_c \) are used.

Using these expressions for \( N_{ab} \), \( M_{abc} \) and \( h^{ab} \), the super energy-momentum tensor does indeed satisfy the super Virasoro algebra

\[
T(Z_1)T(Z_2) = \frac{c}{6} Z_{12}^{-3} + \left[ \frac{3}{2} Z_{12}^{-3/2} + \frac{1}{2} Z_{12}^{-1} D + Z_{12}^{-1/2} \partial \right] T(Z_2) .
\]

The central charge of this algebra is given by

\[
c = \frac{3}{2} \dim(G) - l \gamma^{ab} \Omega_{ab} .
\]

In calculating the operator product expansions, we have used the the following definition of normal ordering of superfields [13,15]

\[
: A(Z_1) B(Z_1) := \frac{1}{2\pi i} \oint_{C_1} dz \oint d\theta Z_{12}^{-1/2} A(Z_2) B(Z_1) .
\]
with the $z_2$-integration contour $C_1$ encircling the point $z_1$. This normal ordering obeys also the analogue of Wick’s theorem for calculating the operator product expansion of $A(Z)$ with a composite operator $:BC:(W)$ [13,15]

$$
: A(Z_1) : BC : (Z_2) : = \frac{1}{2\pi} \oint_{C_1} dz \oint d\theta Z_{23}^{-1/2} \times \left\{ A(Z_1)B(Z_3)C(Z_2) + (-1)^{BC} A(Z_1)C(Z_2) B(Z_3) \right\},
$$

where $(-1)^{BC} = -1$ iff both $B$ and $C$ are fermion fields.

Let us now examine the supercurrent algebra in components. The superfields $J^a(Z)$ and $T(Z)$ can be decomposed into components as

$$
J^a(Z) = \psi^a(z) + \theta J^a(z),
$$

$$
T(Z) = \frac{1}{2} G(z) + \theta T(z).
$$

In terms of these components, the supercurrent algebra in (3.1) yields

$$
\psi^a(z_1)\psi^b(z_2) = -k \frac{h^{ab}}{(z_1 - z_2)},
$$

$$
J^a(z_1)\psi^b(z_2) = f^{ab}_{\ c} \frac{\psi^c(z_2)}{(z_1 - z_2)},
$$

$$
J^a(z_1)J^b(z_2) = -k \frac{h^{ab}}{(z_1 - z_2)^2} + f^{ab}_{\ c} \frac{J^c(z_2)}{(z_1 - z_2)^2}.
$$

The supercurrent algebra can be made to look like a direct sum of a bosonic current algebra and an algebra of free Majorana fermions. This can be achieved by introducing a modified current $\hat{J}^a(z)$ defined by

$$
\hat{J}^a(z) = J^a(z) + R^a_{\ bc} : \psi^b\psi^c : (z),
$$

where $R^a_{\ bc}$ is antisymmetric in the indices $b$ and $c$, and is determined by requiring that the fermions $\psi^a(z)$ and the bosonic currents $\hat{J}^a$ decouple. Indeed, the following operator product expansions

$$
\psi^a(z_1)\psi^b(z_2) = -k \frac{h^{ab}}{(z_1 - z_2)},
$$

$$
\hat{J}^a(z_1)\psi^b(z_2) = 0,
$$

$$
\hat{J}^a(z_1)\hat{J}^b(z_2) = -k \frac{h^{ab} + \frac{1}{2k} \gamma^{ab}}{(z_1 - z_2)^2} + f^{ab}_{\ c} \frac{\hat{J}^c(z_2)}{(z_1 - z_2)^2},
$$

hold only if the tensor $R^a_{\ bc}$ satisfies

$$
h^{bc}R^a_{\ be} = -\frac{1}{2k} f^{ac}_{\ e}, \quad h^{ab} = -\frac{1}{2kl} \Omega^{ab}.
$$
Notice that we have the relation
\[ h^{ab} + \frac{1}{2k} \gamma^{ab} = g^{ab} , \] (3.16)
where \( g^{ab} \) is the bilinear form previously found for the bosonic current algebra. The components of the super stress tensor are written in terms of these modified currents in the form
\[
T(z) = l \Omega_{ab} : \tilde{J}^a : (z) - l \Omega_{ab} : \psi^a \partial \psi^b : (z) \\
G(z) = 2l \Omega_{ab} : \psi^a \tilde{J}^b : (z) - \frac{2l^2}{3} f_{abc} : \psi^a : \psi^b \psi^c :: (z) .
\] (3.17)
The components \( T(z) \) and \( G(z) \) satisfy the usual \( N = 1 \) superconformal algebra. The contribution to the central charge due to the stress tensor of the bosonic currents \( \tilde{J}^a(z) \) is \( \left[ \dim(G) - \gamma^{ab} \Omega_{ab} \right] \) while that of the fermions \( \psi^a(z) \) is given by \( \frac{1}{2} \dim(G) \). These two contributions add up to give, as expected, the central charge for the \( N = 1 \) algebra calculated in (3.8).

4 The WZNW Action

Let us now find a bosonic action (the supersymmetric action is then straightforward) having the bosonic current algebra defined in (2.5) as a symmetry. This action is constructed out of the “gauge fields” \( A_\mu^a \), defined for an element \( g \) in the Lie group via
\[
A_\mu^a = g^{-1} \partial_\mu g .
\] (4.1)
Under an infinitesimal transformation of the form
\[
g \rightarrow g + \bar{\omega} g + g \omega ,
\] (4.2)
where \( \omega = \omega_a T^a \) and \( \bar{\omega} = \bar{\omega}_a T^a \), the variation of the gauge field \( A_\mu^a \) is given by
\[
A_{\mu a} \rightarrow A_{\mu a} + \partial_\mu \lambda + A_{\nu r} \lambda_s f_r^s a ,
\] (4.3)
with
\[
\lambda_a = \omega_a + \theta_a , \quad \theta_a T^a = g^{-1} \bar{\omega} g .
\] (4.4)
A crucial property of the gauge field \( A_{\mu a} \) is that its field strength vanishes
\[
F_{\mu \nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + A_{\mu r} A_{\nu s} f_r^s a = 0 .
\] (4.5)
The gauge invariant action is then given by

\[ I(g) = -\frac{k}{8\pi} \int_{\partial B} d^2 x \sqrt{-\eta} \eta_{\mu\nu} g^{ab} A_{\mu a} A_{\nu b} + \frac{ik}{12\pi} \int_B d^3 y \epsilon_{\mu\nu\rho} g^{ab} (\partial_{\mu} A_{\nu a}) A_{\rho b}, \]  

(4.6)

where \( B \) is a three-dimensional manifold whose boundary is the two-dimensional surface \( \partial B \).

The only requirement that we have on the symmetric matrix \( g^{ab} \) is that it is an invariant of the group \( \left( f^{ab}_{\phantom{ab}cd} + f^{ad}_{\phantom{ad}cb} = 0 \right) \). Therefore, the above action is valid for both semi-simple and non-semi-simple groups. Notice that the Wess-Zumino term in this action can be transformed into the usual form by virtue of equation (4.5).

Using the fact that \( F_{\mu\nu a} \) vanishes and \( g^{ab} \) is an invariant of the group, the variation of the action is

\[ \delta I(g) = -\frac{k}{4\pi} \int_{\partial B} d^2 x \left( \sqrt{-\eta} \eta_{\mu\nu} - i \epsilon_{\mu\nu} \right) g^{ab} [A_{\mu a} \partial_{\nu} \omega_b + A_{\nu a} V^c_b \partial_{\mu} \bar{\omega}^c], \]  

(4.7)

where \( V^a_b \) is defined through

\[ V^a_b T^b = g^{-1} T^a g \]  

(4.8)

and satisfies the following equation (which was also used to derive \( \delta I \))

\[ \partial_{\mu} V^a_b = V^a_r A_{\mu s} f^s_b. \]  

(4.9)

In complex coordinates \( (z, \bar{z}) \) such that \( \eta^{z\bar{z}} = 1 \) and \( \epsilon^{z\bar{z}} = i \), we see that this variation of the action vanishes if \( \omega_a = \omega_a(z) \) and \( \bar{\omega}_a = \bar{\omega}_a(\bar{z}) \). The Noether current associated to some Lie algebra element, \( T^a \), are given by

\[ J^a_z = -\frac{k}{4\pi} g^{ab} A_{zb}, \quad J^a_{\bar{z}} = -\frac{k}{4\pi} g^{bc} A_{zb} V^a_c. \]  

(4.10)

The currents \( J^a_z \) and \( J^a_{\bar{z}} \) are, by virtue of the equations of motion, holomorphic and antiholomorphic currents, respectively. These currents do satisfy two copies of the current algebra (2.5) as shown in [18].

To summarise, we have constructed bosonic and supersymmetric current algebras based on non-semi-simple groups. The central charge corresponding to the Sugawara construction is not, in general, an integer number. In fact it depends on a free parameter. The key observation in our construction is that the bilinear form used to define the current algebra is taken to be different from the bilinear form which raises and lowers group indices. We have also constructed a WZNW action using the bilinear form entering the operator product expansions of the current algebra.
It is worth mentioning that in the case of the centrally extended two dimensional Euclidean group studied in \cite{1}, the difference between the two bilinear form (the invertible one and the one used in the operator product expansions) amounts to a difference in the parameter $b$ defined in \cite{1}. It is therefore of interest to explore our construction for other non-semi-groups.

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