Edge Antimagic Total Labeling on Two Copies of Path

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Abstract. A graph $G = (V(G), E(G))$ denotes the vertex set and the edge set, respectively. A $(p,q)$-graph $G$ is a graph such that $|V(G)| = p$ and $|E(G)| = q$. Graph of order $p$ and size $q$ is called $(a,d)$-edge-anti magic total if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ such that the edge weights $w(uv) = f(u) + f(uv) + f(v)$ form an arithmetic sequence $\{a, a + d, a + 2d, \ldots, a + (p - 1)d\}$ with the first term $a$ and common difference $d$. Two copies of path is disjoint union of two path graph with same order $(P_{2n}, P_{2n})$ denoted by $2P_{2n}$. In this paper we construct the $(a,d)$-edge-anti magic total labeling in two copies of path for some differences $d$.

1. Introduction

In 1967, the concept of graph labeling was introduced by Rosa [1]. Graph labeling is an assignment of integers to the edges or vertices or both subject to certain conditions. Labeled graphs serve as useful models in broad range of applications such as circuit design, communication network addressing, X-ray crystallography, radar, astronomy, data base management and coding theory. Graph labeling have found application in partitioning complete graphs into isomorphic subgraphs, in combinatorial and algebraic structures such as cyclic difference sets, characterizing neo-fields, generalized and near-complete mappings [2,3]. In most applications labels are positive (or nonnegative) integers, though in algebraic structures such as cyclic difference sets, characterizing neo-fields, generalized and near-complete mappings [2,3]. In most applications labels are positive (or nonnegative) integers, though in algebraic structures such as cyclic difference sets, characterizing neo-fields, generalized and near-complete mappings [2,3].

Simanjuntak, et al introduced $(a,d)$-edge-antimagic total labeling. They proved that $C_{2n}$, has $(2n+2,1)$- and $(3n+2,1)$-edge-antimagic total labeling; $C_{2n}$ has $(4n+2,2)$- and $(4n+3,2)$-edge-antimagic total labeling; $C_{2n+1}$ has $(3n+4,3)$- and $(3n+5,3)$-edge-antimagic total labelings; $P_{2n}$ has $(3n+4,2)$- and $(3n+5,2)$-edge-antimagic total labelings; $P_{2n}$ has $(6n,1)$- and $(6n+2,2)$-edge-antimagic total labelings [6].

A labeling of graph $G$ is any mapping that sends some set of graph elements to a set of non-negative integers. If the domain is the vertex set or the edge set, the labelings are called vertex labelings, edge labelings or total labelings, respectively. Moreover, if the domain is $V(G) \cup E(G)$ then the labeling is called a total labeling [7].

Let $f$ be a vertex labeling of a graph $G$. Then we define the edge-weight of $uv \in E(G)$ to be $w(uv) = f(u) + f(v)$. If $f$ is a total labeling then the edge-weight of $uv$ is $w(uv) = f(u) + f(uv) + f(v)$.

An $(a,d)$-edge-anti magic total labeling of a $(p,q)$-graph $G$ is a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v)$, $uv \in E(G)$, form an arithmetic sequence $\{a, a + d, a + 2d, \ldots, a + (p - 1)d\}$ with the first term $a$ and common difference $d$. Two copies of path is disjoint union of two path graph with same order $(P_{2n}, P_{2n})$ denoted by $2P_{2n}$. In this paper we construct the $(a,d)$-edge-anti magic total labeling in two copies of path for some differences $d$.

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An $(a,d)$-edge-anti magic total labeling of a $(p,q)$-graph $G$ is a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v)$, $uv \in E(G)$, form an arithmetic sequence $\{a, a + d, a + 2d, \ldots, a + (p - 1)d\}$ with the first term $a$ and common difference $d$. Two copies of path is disjoint union of two path graph with same order $(P_{2n}, P_{2n})$ denoted by $2P_{2n}$. In this paper we construct the $(a,d)$-edge-anti magic total labeling in two copies of path for some differences $d$.
arithmetic progression \( \{a, a + d, a + 2d, \ldots + a + (q - 1)d\} \), where \( a > 0 \) and \( d \geq 0 \) are two fixed integers. If such a labeling exists then \( G \) is said to be an \((a,d)\)-edge-antimagic total graph[7].

Two copies of path is disjoint union of two path graph with same order \((P_n \cup P_m)\) denoted by \(2P\). Though \((a,d)\)-edge anti magic total labeling have been studied for different kinds of graphs, \((a,d)\)-edge anti magic total labeling for two copies of path have not been investigated. In this paper we construct the \((a,d)\)-edge anti magic total labeling on two copies of path for some differences \(d\).

2. Main Result
Let \( G = G(V,E) \) be a finite, simple and undirected graph. By a labeling we mean one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels (usually the set of integers). In this paper we deal labeling with domain as the set of all vertices and edges \([1]\).

**Lemma 1.** Let \( G \) be a finite, simple, and undirected graph with \( p \) vertices and \( q \) edges. If \( G \) has \((a,d)\)-edge-antimagic total labeling then

\[
d \leq \frac{3p + 3q - 9}{q - 1}
\]

**Proof.**
Assume that \( G \) has \((a,d)\)-edge-antimagic total labeling \( f : V(G) \cup E(G) \rightarrow \{1, \ldots, p + q\} \) and the set of edge-weights \( w(u,v) = f(u) + f(uv) + f(v) \) form an arithmetic sequence \( \{a, a + d, a + 2d, \ldots, a + (q - 1)d\} \).

The minimal edge-weight is 6. Consequently \( a \geq 6 \). On the other hand, the maximum edge-weight is \( p + q - 2 + p + q - 1 + p + q = 3p + 3q - 3 \). Thus, \( a + (q - 1)d \leq 3p + 3q - 3 \).

Using the last inequality and fact that \( a \geq 6 \), we have

\[
d \leq \frac{3p + 3q - 9}{q - 1}
\]

From now, every edge \( v_{ij}, v_{i+1,j} \in E(2P_n) \) will be denoted by \( e_{ij} \). Let \( 2P_n \) be a graph with \( p = 2n \) and \( q = 2(n - 1) \). According to the inequality above, the following observation is hold.

**Observation 1.** If path \( 2P_2 \) has an \((a,d)\)-EAT labelling, then \( d \leq 9 \).

**Theorem 1.** For \( 2P_2 \), has \((10,1)\)-EAT, \((9,3)\)-EAT, \((8,5)\)-EAT, \((7,7)\)-EAT, and \((6,9)\)-EAT labelings.

**Proof.** Let \( 2P_2 \) be a graph with four vertices. Let \( \{v_{1,1}, v_{2,1}, v_{2,2}, v_{2,3}\} \) be a set of vertices of \( 2P_2 \). Label the vertices and the edges of \( 2P_2 \) as follow:

\[
f_j(v_{ij}) = \begin{cases} 
1 & \text{if } i,j = 1,1 \\
2 & \text{if } i,j = 1,2 \\
3 & \text{if } i,j = 2,1 \\
4 & \text{if } i,j = 2,2
\end{cases}
\]

\[
f_i(e_{ij}) = \begin{cases} 
1 & \text{if } i,j = 1,1 \\
2 & \text{if } i,j = 1,2
\end{cases}
\]

We can see that \( f_j \) is a bijection and the edge weights are:

\[
w(e_{1,1}) = f_j(v_{1,1}) + f_j(v_{2,1}) + f_j(e_{1,1}) = 10,
\]

\[
w(e_{1,2}) = f_j(v_{1,2}) + f_j(v_{2,2}) + f_j(e_{1,2}) = 11.
\]
The edge weights form an arithmetic sequence with difference 1. Thus, $f_1$ is a $(10,1)$-EAT labeling for $2P_2$.

Define a new labeling $f_2$ for the vertices and the edges of $2P_2$ as follow:

$$f_2(v_{i,j}) = \begin{cases} 
1 & \text{if } i,j=1,1 \\
2 & \text{if } i,j=1,2 \\
3 & \text{if } i,j=2,1 \\
4 & \text{if } i,j=2,2 
\end{cases}$$

$$f_2(e_{i,j}) = \begin{cases} 
5 & \text{if } i,j=1,1 \\
6 & \text{if } i,j=1,2 
\end{cases}$$

We can see that $f_2$ is a bijection and the edge weights are:

$$\text{wt}(e_{1,1}) = f_2(v_{1,1}) + f_2(v_{1,2}) + f_2(e_{1,1}) = 9,$$

$$\text{wt}(e_{1,2}) = f_2(v_{1,2}) + f_2(v_{2,2}) + f_2(e_{1,2}) = 12.$$ 

The edge weights form an arithmetic sequence with difference 3. Thus, $f_2$ is a $(9,3)$-EAT labeling for $2P_2$.

Define a new labeling $f_3$ for the vertices and the edges of $2P_2$ as follow:

$$f_3(v_{i,j}) = \begin{cases} 
1 & \text{if } i,j=1,1 \\
3 & \text{if } i,j=1,2 \\
4 & \text{if } i,j=2,1 \\
6 & \text{if } i,j=2,2 
\end{cases}$$

$$f_3(e_{i,j}) = \begin{cases} 
3 & \text{if } i,j=1,1 \\
5 & \text{if } i,j=1,2 
\end{cases}$$

We can see that $f_3$ is a bijection and the edge weights are:

$$\text{wt}(e_{1,1}) = f_3(v_{1,1}) + f_3(v_{2,1}) + f_3(e_{1,1}) = 8,$$

$$\text{wt}(e_{1,2}) = f_3(v_{1,2}) + f_3(v_{2,2}) + f_3(e_{1,2}) = 13.$$ 

The edge weights form an arithmetic sequence with difference 5. Thus, $f_3$ is a $(8,5)$-EAT labeling for $2P_2$.

Define a new labeling $f_4$ for the vertices and the edges of $2P_2$ as follow:

$$f_4(v_{i,j}) = \begin{cases} 
1 & \text{if } i,j=1,1 \\
3 & \text{if } i,j=1,2 \\
4 & \text{if } i,j=2,1 \\
6 & \text{if } i,j=2,2 
\end{cases}$$

$$f_4(e_{i,j}) = \begin{cases} 
2 & \text{if } i,j=1,1 \\
5 & \text{if } i,j=1,2 
\end{cases}$$

We can see that $f_4$ is a bijection and the edge weights are:
The edge weights form an arithmetic sequence with difference 7. Thus, \( f_4 \) is a \((7,7)\)-EAT labeling for \(2P_2\).

Define a new labeling \( f_5 \) for the vertices and the edges of \(2P_2\) as follow:

\[
\begin{align*}
    f_5(v_{i,j}) &= \begin{cases} 
        1 & \text{if } i,j=1,1 \\
        4 & \text{if } i,j=1,2 \\
        3 & \text{if } i,j=2,1 \\
        6 & \text{if } i,j=2,2
    \end{cases}, \\
    f_5(e_{i,j}) &= \begin{cases} 
        2 & \text{if } i,j=1,1 \\
        5 & \text{if } i,j=1,2
    \end{cases}.
\end{align*}
\]

We can see that \( f_5 \) is a bijection and the edge weights are:

\[
\begin{align*}
    \text{wt}(e_{1,1}) &= f_5(v_{1,1}) + f_5(v_{2,1}) + f_5(e_{1,1}) = 6, \\
    \text{wt}(e_{1,2}) &= f_5(v_{1,2}) + f_5(v_{2,2}) + f_5(e_{1,2}) = 15.
\end{align*}
\]

The edge weights form an arithmetic progression with difference 9. Thus, \( f_5 \) is a \((6,9)\)-EAT labeling for \(2P_2\).

**Observation 2.** If path \(2P_3\) has an \((a,d)\)-EAT labeling. Then \(d \leq 7\).

**Observation 3.** If path \(2P_n, n \geq 4\), has an \((a,d)\)-EAT labeling. Then \(d \leq 6\).

**Theorem 2.** For every \(2P_n, n \geq 3\) has \((4n+2,1)\)-EAT and \((6n+2,1)\) -EAT labelings.

**Proof.** Let us define the vertices and edges labelings of \(2P_n\) as follow:

\[
\begin{align*}
    f_6(v_{i,j}) &= \begin{cases} 
        2i-1 & \text{if } j=1, i=1,\ldots,n \\
        2i & \text{if } j=2, i=1,\ldots,n
    \end{cases}, \\
    f_6(e_{i,j}) &= \begin{cases} 
        4n-2i & \text{if } j=1, i=1,\ldots,n-1 \\
        4n-2i-1 & \text{if } j=2, i=1,\ldots,n-1.
    \end{cases}
\end{align*}
\]

It is easy to verify that \( f_6 \) is bijective. For all edge \(e_{i,j}, i = 1,\ldots,n-1, j = 1,2\) of \(2P_n\), the edge weights are:

\[
\begin{align*}
    \text{wt}(e_{i,1}) &= f_6(v_{i,1}) + f_6(e_{i,1}) + f_6(v_{i+1,1}) \\
    &= (2i-1) + (4n-2i) + (2(i+1)-1) \\
    &= 4n+2i, \\
    \text{wt}(e_{i,2}) &= f_6(v_{i,2}) + f_6(e_{i,2}) + f_6(v_{i+1,2}) \\
    &= 2i + (4n-2i-1) + (2i+2) \\
    &= 4n+2i+1.
\end{align*}
\]
We can see that $e_{i,j}$ has the smallest weight $4n+2$ and set of all edge weights form a consecutive sequence $4n+2,4n+3,\ldots,6n-1$ with $a=4n+2$ and difference $d=1$. Thus, $f_6$ is $(4n+2,1)$-EAT labeling of $2P_n, n \geq 3$.

Define new labeling $f_7$ as follow:

$$f_7(v_{i,j}) = \begin{cases} 
2i+2n-1, & j=1, i=1,\ldots,n \\
2i+2n, & j=2, i=1,\ldots,n 
\end{cases}$$

$$f_7(e_{i,j}) = \begin{cases} 
2n-2i, & j=1, i=1,\ldots,n-1 \\
2n-2i-1, & j=2, i=1,\ldots,n-1
\end{cases}$$

It is easy to verify that $f_7$ is bijective. For all edge $e_{i,j}$, $i = 1,\ldots,n-1, j = 1,2$ of $2P_n$, the edge weights are:

$$\text{wt}(e_{i,j}) = f_7(v_{i,j}) + f_7(e_{i,j}) + f_7(v_{i+1,j}) = (2i+2n-1) + (2n-2i) + (2n+2i+1) = 6n+2i.$$  

We can see that $e_{i,j}$ has the smallest weight $6n+2$ and set of all edge weights form a consecutive sequence $6n+2,6n+3,\ldots,8n-1$ with $a=6n+2$ and difference $d=1$. Thus, $f_7$ is $(6n+2,1)$-EAT labeling of $2P_n, n \geq 3$. 

**Theorem 3.** For every $2P_n, n \geq 3$ has a $(2n+5,3)$-EAT labeling

**Proof.** Let us define the vertices and edges labelings of $2P_n$ as following way:

$$f_8(v_{i,j}) = \begin{cases} 
2i-1, & j=1, i=1,\ldots,n \\
2i, & j=2, i=1,\ldots,n'
\end{cases}$$

$$f_8(e_{i,j}) = \begin{cases} 
2n+2i-1, & j=1, i=1,\ldots,n-1 \\
2n+2i, & j=2, i=1,\ldots,n-1
\end{cases}$$

It is easy to verify that $f_8$ is bijective. For all edge $e_{i,j}$, $i = 1,\ldots,n-1, j=1,2$ of $2P_n$, the edge weights are:

$$\text{wt}(e_{i,j}) = f_8(v_{i,j}) + f_8(e_{i,j}) + f_8(v_{i+1,j}) = (2i-1) + (2n+2i-1) + (2i+1) = 2n+6i-1.$$  

We can see that $e_{i,j}$ has the smallest weight $2n+5$ and set of all edge weights form a consecutive sequence $2n+5,2n+8,\ldots,8n-4$ with $a=2n+5$ and difference $d=3$. Thus, $f_8$ is $(2n+5,3)$-EAT labeling of $2P_n, n \geq 3$.

Define new labeling of $2P_n$ as follow:
It is easy to verify that \( f_9 \) is bijective. For all edge \( e_{ij}, i=1,...,n-1, j=1,2 \) of \( 2P_n \), the edge weights are:

\[
\begin{align*}
\text{wt}(e_{i,1}) &= f_9(v_{i,1}) + f_9(e_{i,1}) + f_9(v_{i+1,1}) \\
&= (2n+2i-1) + (2i-1) + (2n+2i+1) \\
&= 4n+6i-1, \\
\text{wt}(e_{i,2}) &= f_9(v_{i,2}) + f_9(e_{i,2}) + f_9(v_{i+1,2}) \\
&= (2n+2i) + 2i + (2n+2i+2) \\
&= 4n+6i+2.
\end{align*}
\]

We can see that \( e_{1,1} \) has the smallest weight \( 4n+5 \) and set of all edge weights form a consecutive sequence \( 4n+5, 4n+8, ..., 10n-4 \) with \( a=4n+5 \) and difference \( d=3 \). Thus, \( f_9 \) is \((4n+5,3)\)-EAT labeling of \( 2P_n, n\geq3 \).

### 3. Conclusion

By the research we have done obtained edge antimagic total labeling on two copies of path. we obtain the edge antimagic total labeling for \( n \geq 3 \) with common different are \( d = 1 \) and \( d = 3 \), and common different are \( d = 1, d = 3, d = 5, d = 7, \) and \( d = 9 \) for \( n > 2 \). This result can be an important reference for further research, especially in graph theory research and generally for other applied science researchers in order to produce useful applications in various areas of society.

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