Constructing $SL(2, \mathbb{Z})$ invariant amplitudes
in type IIB superstring theory

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Abstract

The construction of $SL(2, \mathbb{Z})$ invariant amplitudes that generalize the Virasoro amplitude is investigated in detail. We describe a number of mathematical properties that characterize the simplest example, and present pieces of evidence that it represents the tree-level four-graviton scattering amplitude in membrane theory on $\mathbb{R}^9 \times T^2$ in the limit that the torus area goes to zero. In particular, we show that the poles of the $S$-dual amplitude are in precise correspondence with the states of membrane theory that survive in the type IIB limit. These are shown to be the states that span the Cartan subspaces of area preserving diffeomorphisms of the 2-torus; all other states become infinitely massive, and membrane world-volume theory acquires the structure of a free theory.

February 1998

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1. Introduction

With the premise that there is little prospect to determine an exact scattering amplitude in M-theory, as in any non-trivial quantum theory (especially if, as in the present case, the theory is unknown), we start by describing the purpose of this work. The basic information about four-graviton amplitude in ten-dimensional type IIB superstring theory comes from three different sources, namely string perturbation theory, low-energy results that are exact in the string coupling, and the symmetry under $SL(2,\mathbb{Z})$ transformations. In addition, one must demand that any correction of perturbative origin should appear with an integer power of $g_B^2$ (type IIB string coupling), and non-perturbative corrections should be in correspondence with D-instanton contributions. Although these ingredients are certainly insufficient to anticipate the general structure of the exact scattering amplitude, simple examples that satisfy these requirements can be constructed [1]. Here we will continue with this program, and also illustrate how corrections can be systematically introduced by preserving $SL(2,\mathbb{Z})$ symmetry at each step [1]. In addition, we will find indications that the simplest amplitude represents a special limit of the tree-level diagram for the four-graviton amplitude in membrane theory compactified on a 2-torus; this is the “type IIB” limit of M-theory on $T^2$, where the torus area goes to zero at fixed moduli, so that M-theory becomes the ten-dimensional type IIB superstring theory.

The $SL(2,\mathbb{Z})$ symmetry of type IIB superstring theory [3-5] requires that the effective action must be invariant under $SL(2,\mathbb{Z})$ transformations to all orders in the $\alpha'$ expansion. In the Einstein frame, a term of given order in derivatives involving the metric must be multiplied by a modular function of the coupling. Since there is a one-to-one correspondence between certain ($SL(2,\mathbb{Z})$ invariant) terms in the effective action and the terms of the momentum expansion of an N-graviton amplitude, the same modular functions appear in the N-graviton amplitude, which must therefore be invariant under $SL(2,\mathbb{Z})$ transformations.

In section 2 we review the scattering amplitude proposed in ref. [1] and, in addition, we describe a simple way to obtain it by incorporating non-perturbative states of the spectrum. The perturbative part of the S-dual scattering amplitude can be resummed into a simple closed expression, which is studied in section 3. In section 4 the effective action that reproduces the S-dual amplitude is examined. In section 5 we show that this amplitude is uniquely determined by a simple extra condition, the free wave equation in a locally flat three-dimensional space time parametrized by the type IIB string coupling $\tau = \tau_1 + i\tau_2$ and the string tension. Section 6 is an analysis of more general $SL(2,\mathbb{Z})$ invariant amplitudes. In all cases, the (non-BPS) $(p,q)$ string states of ten-dimensional type IIB theory play a central role. The membrane configurations in eleven dimensions that give rise to these states upon dimensional reduction and T-duality are described in section 7, where we also make some remarks on a possible derivation of the S-dual amplitude starting from eleven dimensions. Finally, in section 8 we discuss the interpretation of the results.

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1 A possible reorganization of the perturbative expansion respecting $SL(2,\mathbb{Z})$ invariance was also suggested in ref. [2].
2. Simplest \( SL(2, \mathbb{Z}) \) invariant amplitude

2.1. Definition and properties

The scattering amplitude introduced in [1] is given by the following formula:

\[
A_4 = \kappa^2 K A_4^{s(2)}(s, t, u),
\]

\[
A_4^{s(2)}(s, t, u) = \frac{1}{stu} \prod_{(p,q)} \frac{\Gamma(1 - s_{pq})\Gamma(1 - t_{pq})\Gamma(1 - u_{pq})}{\Gamma(1 + s_{pq})\Gamma(1 + t_{pq})\Gamma(1 + u_{pq})},
\]

\[
s_{pq} = \frac{\alpha's}{4|p + q\tau|}, \quad t_{pq} = \frac{\alpha't}{4|p + q\tau|}, \quad u_{pq} = \frac{\alpha'u}{4|p + q\tau|}, \quad s_{pq} + t_{pq} + u_{pq} = 0,
\]

where \( p \) and \( q \) are relatively prime, \( \tau = C^{(0)} + ig_B^{-1} \) is the usual coupling of type IIB superstring theory, and \( K \) is the same kinematical factor depending on the momenta and polarization of the external states appearing in the tree-level Virasoro amplitude of the form (see e.g. [3])

\[
K = \zeta_1^{A'A'} \zeta_2^{BB'} \zeta_3^{CC'} \zeta_4^{DD'} K_{ABCD}(k_i) K_{A'B'C'D'}(k_i),
\]

\[
K_{ABCD} = -\frac{1}{4} st \eta_{AC} \eta_{BD} + ...
\]

This amplitude can be obtained by the simple replacement

\[
\sum_{m=1}^{\infty} \frac{1}{m^{2k+1}} \rightarrow \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{1}{|m + n\tau|^{2k+1}},
\]

in the Virasoro amplitude,

\[
A_4(s, t) = \kappa^2 K A_4^0(s, t), \quad A_4^0(s, t) = \frac{1}{stu} e^{\delta_0(s, t)},
\]

\[
\delta_0(s, t) = 2 \sum_{k=1}^{\infty} \frac{\zeta(2k + 1)}{2k + 1} (\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1}),
\]

\[
\bar{s} = \frac{i}{4} \alpha's, \quad \bar{t} = \frac{i}{4} \alpha't, \quad \bar{u} = \frac{i}{4} \alpha'u, \quad \bar{s} + \bar{t} + \bar{u} = 0,
\]

The prescription (2.4) is motivated by a number of facts:

a) it is the obvious generalization of the analogous replacement

\[
\sum_{m=1}^{\infty} \frac{1}{m^3} \rightarrow \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{1}{|m + n\tau|^3},
\]
in the first term of the sum (2.6), which is known to account for all perturbative and non-perturbative contributions to the $R^4$ term [7-9].

b) The same structure (2.4) produces the non-perturbative contributions to the one-loop amplitude coming from Kaluza-Klein gravitons (D0-branes) of $D = 11$ supergravity [10,11]. The D0-branes are related by duality to the D-instantons that are the origin of the non-perturbative effects in the ten-dimensional type IIB theory.

c) This prescription leads to correct perturbative $g_B^{2k}$ and non-perturbative $O(e^{-2\pi mn/g_B})$ dependence, with $k, m, n$ integer numbers. This is non-trivial, and it is crucial in order to have a one-to-one correspondence between these terms and instanton contributions. [For example, an ansatz giving rise to $O(e^{-4\pi mn/g_B})$ dependence, could not be correct, since it would miss some D-instanton configurations.] This property follows by first writing $A_{4}^{s t u}$ in terms of Eisenstein series,

$$E_r(\tau) = \sum_{(p,q)} \frac{\tau_2^r}{|p+q\tau|^{2r}}, \quad (2.7)$$

$$A_{4}^{s t u}(s,t,u) = \frac{1}{stu} e^{\delta(s,t)} , \quad (2.8)$$

$$\delta(s,t) = 2 \sum_{k=1}^{\infty} \frac{\zeta(2k+1)g_B^{k+1/2} E_{k+1/2}(\tau)}{2k+1} \left( s^{2k+1} + t^{2k+1} + u^{2k+1} \right) , \quad (2.9)$$

and then using the expansion at large $\tau_2$,

$$E_r(\tau) = \tau_2^r + \gamma_r \tau_2^{1-r} + \frac{4\tau_2^{1/2} \pi^r}{\zeta(2r) \Gamma(r)} \sum_{n,w=1}^{\infty} \left( \frac{w}{n} \right)^{r-1/2} \cos(2\pi wn \tau_1) K_{r-1/2}(2\pi wn \tau_2) , \quad (2.10)$$

$$\gamma_r = \frac{\sqrt{\pi} \Gamma(r-1/2) \zeta(2r-1)}{\Gamma(r) \zeta(2r)} .$$

Using the asymptotic expansion for the Bessel function $K_{r-1/2}$,

$$K_{r-1/2}(2\pi wn \tau_2) = \frac{1}{\sqrt{4wn \tau_2}} e^{-2\pi wn \tau_2} \sum_{m=0}^{\infty} \frac{1}{(4\pi wn \tau_2)^m} \frac{\Gamma(r+m)}{\Gamma(r-m)m!} ,$$

we see that the $E_{k+1/2}(\tau)$ terms in the amplitude are of the form

$$g_B^{k+1/2} E_{k+1/2}(\tau) = 1 + \gamma_{k+1/2} g_B^{2k} + O(e^{-2\pi/g_B}) . \quad (2.11)$$

d) It gives an $SL(2,\mathbb{Z})$ invariant amplitude that in the limit $g_B^2 \to 0$ reduces to the Virasoro amplitude ($SL(2,\mathbb{Z})$ invariance is explicit in the Einstein frame, $g_{\mu\nu}^E = g_B^{-1/2} g_{\mu\nu}$, so that $s_E = g_B^{1/2} s$, etc.).
e) The resulting amplitude has poles in the $s$-$t$-$u$ channels at $s_{pq} = -n$, $t_{pq} = -n$, $u_{pq} = -n$, $n = 0, 1, 2, ...$ corresponding to exchange of particles with masses

\[
\frac{1}{4} \alpha' M^2 = n|p + q\tau|,
\]

which is precisely the desired spectrum of $(p, q)$ string states:

\[
M^2 = 4\pi T_{pq}(N_R + N_L) = \frac{2}{\alpha'} |p + q\tau| (N_R + N_L), \quad N_R = N_L.
\]

This spectrum corresponds to the zero winding sector of the spectrum originally studied in [12] for the nine-dimensional type IIB string theory.

2.2. $SL(2, \mathbb{Z})$ symmetric expressions by including $(p, q)$ strings

We start with the Virasoro amplitude, with $\delta_0$ as given in eq. (2.6). By writing the $\zeta$-functions as series, it takes the form

\[
\delta_0 = 2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2k + 1} \left( \frac{\alpha' s}{4m} \right)^{2k+1} + (s \to t) + (s \to u)
\]

or

\[
\delta_0 = \sum_{m=1}^{\infty} \delta_{(m)} , \quad \delta_{(m)} = \log \frac{M_m^2 + s}{M_m^2 - s} + (s \to t) + (s \to u) , \quad \alpha' M_m^2 = 4m.
\]

Thus $\delta_0$ is a direct sum of $\delta_{(m)}$ associated with each mass level. This form is suitable for $SL(2, \mathbb{Z})$ symmetrization: we just need to include in the sum (2.15) all contributions $M_{mn}^2$ associated with the masses of $(p, q)$ string states,

\[
\delta = \frac{1}{2} \sum_{(m, n) \neq (0, 0)} \log \frac{M_{mn}^2 + s}{M_{mn}^2 - s} + (s \to t) + (s \to u)
\]

i.e.

\[
\delta = \frac{1}{2} \sum_{(m, n) \neq (0, 0)} \log \frac{4|m + n\tau| + \alpha' s}{4|m + n\tau| - \alpha' s} + (s \to t) + (s \to u).
\]

Expanding the logarithm, this becomes

\[
\delta = \sum_{(m, n) \neq (0, 0)} \sum_{k=1}^{\infty} \frac{1}{2k + 1} \left( \frac{\alpha' s}{4|m + n\tau|} \right)^{2k+1} + (s \to t) + (s \to u)
\]

\[
= 2 \sum_{(p, q)'} \sum_{k=1}^{\infty} \frac{\zeta(2k + 1)}{2k + 1} \left( s_{pq}^{2k+1} + t_{pq}^{2k+1} + u_{pq}^{2k+1} \right),
\]
where \( s_{pq}, t_{pq}, u_{pq} \) were introduced in eq. (2.3). It is perhaps not a surprise that we obtain just the same amplitude \( A_{4}^{\text{sl}(2)} \)

\[
A_{4}^{\text{sl}(2)}(s, t, u) = \frac{1}{stu} e^{\delta(s, t, u)}. \tag{2.18}
\]

Along with the fundamental property described in section 5, this shows that simple attempts of \( SL(2, \mathbb{Z}) \) symmetrization indeed lead to the amplitude \( A_{4}^{\text{sl}(2)} \). One might also attempt to construct an \( SL(2, \mathbb{Z}) \) invariant amplitude by replacing the product over \((p, q)\)' states in eq. (2.2) by a sum over \((p, q)\)' states. This does not lead to a sensible amplitude: first of all, the leading term in the expansion in \( \alpha' \) is divergent; in addition, such object contains perturbative dependence in odd powers of \( g_B \), which cannot arise in string theory. Let us also point out that obtaining \( SL(2, \mathbb{Z}) \) symmetric terms in the effective action by summing over \((p, q)\) was recently investigated in [13].

3. Resummation of perturbative part

As shown in [1], and it is clear from eqs. (2.8)-(2.10), the amplitude \( A_{4}^{\text{sl}(2)} \) can be written as

\[
A_{4}^{\text{sl}(2)}(s, t, u) = A_{4}^{\text{pert}}(s, t, u) + O\left(e^{-2\pi/g_B}\right), \tag{3.1}
\]

\[
A_{4}^{\text{pert}}(s, t, u) = A_{4}^{0}(s, t, u) e^{\tilde{A}_{4}(s, t, u)}, \tag{3.2}
\]

with

\[
\tilde{A}_{4}(s, t, u) = \sqrt{\pi} \sum_{k=1}^{\infty} \frac{(k-1)!\zeta(2k)}{\Gamma(k+3/2)} g_{B}^{2k} (s^{2k+1} + t^{2k+1} + u^{2k+1}). \tag{3.3}
\]

Writting \( \zeta(2k) = \sum_{m=1}^{\infty} m^{-2k} \), and using the formula

\[
\frac{\sqrt{\pi}}{4} \sum_{k=1}^{\infty} \frac{(k-1)!}{\Gamma(k+3/2)} x^{k} = 1 - \sqrt{\frac{1}{x} - 1 \arcsin \sqrt{x}}, \quad |x| < 1,
\]

\[\text{2 There is a curious representation of the Virasoro amplitude in terms of free fermion variables } d_{m}, d_{m}^{\dagger}. \text{ Using log } \frac{2}{\pi} = \int_{0}^{\infty} \frac{dt}{t} (e^{-at} - e^{-bt})' , \text{ one can write}
\]

\[
\delta_{0} = - \int_{0}^{\infty} \frac{dt'}{t'} \text{Tr} \left[ (-1)^{F} e^{-t'H} \right],
\]

where \( H = -\frac{1}{4} \alpha' p^{2} + 2 \sum_{m=1}^{\infty} m(d_{m}^{\dagger} d_{m} - \frac{1}{2}) \), \( F = \sum_{m=1}^{\infty} d_{m}^{\dagger} d_{m} \). It is clear that \( \delta \) defining \( A_{4}^{\text{sl}(2)} \) is given by the same expression with \( H = -\frac{1}{4} \alpha' p^{2} + \sum \omega_{mn} (d_{mn}^{\dagger} d_{mn} - \frac{1}{2}) \), \( \omega_{mn} = |m + n\tau| \). The frequency \( \omega_{mn} \) can be associated with the oscillations of membranes that move along a \((p, q)\) cycle of the 2-torus carrying zero total momentum (with \( p/q = m/n \), see sect. 7).
the perturbative part of $A_{4}^{sl(2)}$ can be resummed with the result

$$\tilde{A}_{4}(s, t, u) = -4 \sum_{m=1}^{\infty} \sqrt{\frac{m^2}{g_B^2} - s^2} \arcsin \frac{s g_B}{m} + (s \to t) + (s \to u) ,$$  \hspace{1cm} (3.4)

or

$$A_{4}^{pert}(s, t, u) = A_{4}^{0}(s, t, u) \prod_{m=1}^{\infty} \left[ \frac{ig_B}{m} + \sqrt{1 - \frac{s^2 g_B^2}{m^2}} \right]^{4i} \sqrt{\frac{m^2}{g_B^2} - s^2} \times [s \to t] \times [s \to u] ,$$  \hspace{1cm} (3.5)

where we have written $\arcsin z = -i \log (iz + \sqrt{1 - z^2})$.

Although eq. (3.3) may suggest the presence of cuts for $\bar{s} > g_B^{-1}$, the amplitude $A_{4}^{pert}$ cannot be extended to this regime; the terms $O\left(e^{-2\pi/g_B}\right)$ that so far have been neglected become important. The full expression (2.2) indicates that $A_{4}^{sl(2)}$ has no cuts. This can be proved with no need of understanding the convergence properties of the infinite product in (2.2). Indeed, the presence of a cut at $\alpha's_E > 4/g_B^{1/2}$ would imply, by S-duality, the presence of a cut at $\alpha's_E > 4g_B^{1/2}$. For sufficiently small $g_B$, and $s_E, t_E, u_E$ fixed, eq. (3.3) is applicable at $\alpha's_E > 4g_B^{1/2}$ (viz. $\bar{s} = 1$), and it has no discontinuity cut at that point. Therefore there cannot be any cut at any $SL(2, \mathbb{Z})$ rotation of this condition, in particular, at $\alpha's_E > 4/g_B^{1/2}$.

The original expression (3.3) already exhibits the fact that there is no absorption via opening of inelastic channels, since for sufficiently small $g_B$ it is convergent at $\bar{s} = 1$ and real; the above argument can then be applied for any $SL(2, \mathbb{Z})$ rotation of this point. It is interesting to note that in the physical region of the parameter space one has

$$s > 0 , \quad 0 < -\frac{t}{s} = \sin^2 \frac{\phi}{2} < 1 , \quad 0 < -\frac{u}{s} = \cos^2 \frac{\phi}{2} < 1 ,$$

and similar conditions in the regions $t > 0$ or $u > 0$. Hence

$$\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1} = \bar{s}^{2k+1} \left[ 1 - \left( \sin^2 \frac{\phi}{2} \right)^{2k+1} - \left( \cos^2 \frac{\phi}{2} \right)^{2k+1} \right] > 0 .$$

Thus (in the region the sum converges, $g_B \bar{s} < 1$) $\tilde{A}_{4}$ is a real number greater than zero, so that

$$|A_{4}^{pert}(s, t, u)| = |A_{4}^{0}(s, t, u)| e^{\tilde{A}_{4}(s, t, u)} > |A_{4}^{0}(s, t, u)|$$

In particular, this also indicates that including the contribution of $(p, q)$ states increases the probability amplitude of the process. For $\alpha's > 4/g_B$, this analysis is not applicable, and the general expression (2.2) must be used.
4. Effective action

We would like to explore the structure of the effective action that reproduces $A^{{\text{sl}}(2)}_4$. It is convenient to introduce the notation:

$$f_k(\tau) = \frac{\zeta(2k + 1)}{k + \frac{1}{2}} \frac{k + \frac{1}{2}}{E_{k+1/2}(\tau)} ,$$

in terms of which $\delta$ (defined in (2.9)) becomes ($\alpha' = 4$)

$$\delta = \sum_{k=1}^{\infty} f_k(\tau) \left( s^{2k+1} + t^{2k+1} + u^{2k+1} \right) .$$

Expanding $A^{{\text{sl}}(2)}_4(s,t)$ in eq. (2.8) in powers of $\delta$, the amplitude (2.1) exhibits the pole due to the exchange of the massless supergravity multiplet plus and infinite number of terms containing polynomials in $s,t$,

$$A_4(s,t) = \kappa^2 K \left[ \frac{1}{stu} + \sum_{k=1}^{\infty} f_k(\tau) P_k(s,t) + \ldots + \frac{1}{N!} \sum_{k_1 \ldots k_N} f_{k_1}(\tau) \ldots f_{k_N}(\tau) P_{k_1 \ldots k_N}(s,t) + \ldots \right] ,$$

with

$$P_k(s,t) = \frac{1}{stu} (s^{2k+1} + t^{2k+1} + u^{2k+1}) ,$$

$$P_{k_1 \ldots k_N}(s,t) = P_{k_1} \ldots P_{k_N} (stu)^{N-1} .$$

Note that $P_{k}(s,t)$ is an homogeneous polynomial of degree $2k - 2$ in $s,t$, as it is clear after using $u = -s - t,$

$$P_k(s,t) = \sum_{l=1}^{k} \frac{(2k + 1)!}{l!(2k + 1 - l)!} \sum_{n=0}^{2k-2l} (-1)^n s^{2k-1-l-n} t^{l+n-1} .$$

At $g_B \ll 1$, $f_k(\tau)$ has the expansion (see eq. (2.11))

$$f_k(\tau) = \frac{\zeta(2k + 1)}{k + \frac{1}{2}} + \frac{\sqrt{\pi} \Gamma(k) \zeta(2k)}{\Gamma(k + \frac{3}{2})} g_B^{2k} + O(e^{-2\pi/g_B})$$

Using eq. (4.7), we see that at weak coupling the generic $N$ term in eq. (4.3) takes the form

$$\frac{1}{N!stu} \delta^N = \sum_{k_1 \ldots k_N} \left[ c_1 + c_2 g_B^2 + \ldots + c_h g_B^{2h} + O(e^{-2\pi/g_B}) \right] P_{k_1 \ldots k_N}(s,t) ,$$

$$h = 2(k_1 + \ldots + k_N) .$$
The first term $N = 1$ has only tree-level and genus $k$ contributions, and corresponds to local terms in the effective action of the form

$$S_{A_4} \bigg|_{N=1} = \int d^{10} x \sqrt{-G} \ g_B^{-2} \sum_{k=1}^{\infty} f_k(\tau) \ "\nabla^{4k-4}R^4"$$

$$= \int d^{10} x \sqrt{-G} \ g_B^{-2} \sum_{k=1}^{\infty} f_k(\tau) \ P_k(s,t)R^4 \quad (4.9)$$

$P_{k_1...k_N}(s,t)$ is an homogeneous polynomial in $s,t$ of degree $2h + N - 3$. Therefore, there are also contributions to the order $"\nabla^{4k-4}R^4"$ in derivatives coming from terms with $N > 1$, $N$ odd, $2h = 2k + 1 - N$. It is clear from (4.3), (4.3) that the tensor structure of each of such terms is different; they give new terms to the effective action which do not mix with (4.3).

From eq. (4.8) we see that the perturbative contribution in a generic $N$ term $"\nabla^{4k-4}R^4"$ of highest order corresponds to genus $k - \frac{1}{2}(N - 1) \leq k$. Similarly, terms with $N$ even contribute to $"\nabla^{4k-2}R^4"$, $k = 2, 3, ..., N$, with perturbative contribution of highest order corresponding to genus $k - \frac{1}{2}(N - 2) \leq k$. Thus either $"\nabla^{4k-4}R^4"$ or $"\nabla^{4k-2}R^4"$ do not receive contributions beyond genus $k$.

In a recent paper, Berkovits and Vafa [14] conjectured that the exact function of the coupling for the term $H^{4k-4}R^4$ in the type IIB effective action on $\mathbb{R}^{10}$ is given by $f_k(\tau)$ (up to a numerical multiplicative constant). This was based on explicit genus $k$ results [13] (see also [16,17] for analogous results in four dimensions), and it is also supported by the fact that such terms can only receive tree-level and genus $k$ contributions. As observed in [14], this conjecture should be related to the amplitude $A^{s(2)}_4$ conjectured in [1] by virtue of supersymmetry. By a slight elaboration of the argument of ref. [14], let us now argue that the $H^{4k-4}R^4$ conjecture actually implies that the terms containing $P_k(s,t)R^4$ in the exact four-graviton amplitude must be multiplied by $f_k(\tau)$. Supersymmetry transformations are expected to relate the term $H^{4k-4}R^4$ to a term with no $H$ field but with the same number of derivatives, i.e. of the form $"\nabla^{4k-4}R^4"$. The tree-level contribution to $H^{4k-4}R^4$ contains a single factor $\zeta(2k + 1)$. Because supersymmetry transformations on a given term cannot generate $\zeta(2k + 1)$ factors (there are no zeta-functions in the supersymmetry transformation laws), the term $"\nabla^{4k-4}R^4"$ that is in the same supersymmetric invariant as $H^{4k-4}R^4$ must also contain a single factor $\zeta(2k + 1)$. There is only one tree-level term of order $"\nabla^{4k-4}R^4"$ that contains such single factor, namely $P_k(s,t)R^4$, which must therefore be in the same supersymmetric invariant as $H^{4k-4}R^4$. Thus, the exact function of the coupling multiplying $P_k(s,t)R^4$ must be proportional to the exact function of the coupling multiplying $H^{4k-4}R^4$.

In this way one derives eq. (4.9) from the $H^{4k-4}R^4$ conjecture of [14]. It is remarkable that these two completely independent approaches have led to the same result. Unexpectedly, this holds for terms of arbitrarily high orders in derivatives. If the conjecture of [14]
is true, then the exact four-graviton amplitude can differ from $A_{4}^{sl(2)}$ only in extra non-local pieces (which are certainly expected), and in higher genus contributions to the terms $P_{k_{1}...k_{N}}(s,t)$ with $N > 1$. Possible corrections to the local terms with $N > 1$ are however highly constrained by the fact that they must not give any tree-level contribution and they must be $SL(2, \mathbb{Z})$ invariant by themselves, in addition to having correct perturbative $g_{B}^{2k}$ and non-perturbative $O(e^{-2\pi mn/g_{B}})$ dependence on the coupling. It should be noted that the terms $P_{k_{1}...k_{N}}(s,t)\mathcal{R}^{4}$ with $N > 1$ cannot be related by supersymmetry to $H^{4k-4}\mathcal{R}^{4}$ for the reasons explained above, i.e. the ratio of the coefficients of the respective tree-level parts is irrational. Thus the knowledge of terms $H^{4k-4}\mathcal{R}^{4}$ does not help in determining the exact functions of the coupling that multiplies $P_{k_{1}...k_{N}}(s,t)\mathcal{R}^{4}$ terms with $N > 1$.

The function of the coupling in front of each term ($\nabla^{2}g_{B}^{2}$) will in general be given by a product of Eisenstein functions. In particular, the first terms $\mathcal{R}^{4}$, $\nabla^{4}\mathcal{R}^{4}$, $\nabla^{6}\mathcal{R}^{4}$ in the effective action are of the form

$$S' = \int d^{10}x \sqrt{-G} \left(2\zeta(3)E_{3/2}(\tau)\mathcal{R}^{4} + 2\zeta(5)E_{5/2}(\tau)\nabla^{4}\mathcal{R}^{4}\right) + 2\zeta(3)^{2}E_{3/2}(\tau)\nabla^{6}\mathcal{R}^{4}\right).$$

In the above notation, these terms correspond to $P_{1}(s,t)$, $P_{2}(s,t)$, $P_{11}(s,t)$, respectively. In the string frame, this takes the following form at $g_{B}^{2} \ll 1$

$$S' = \int d^{10}x \sqrt{-G} \left((a_{0}g_{B}^{-2} + a_{1})\mathcal{R}^{4} + (b_{0}g_{B}^{-2} + b_{2}g_{B}^{2})\nabla^{4}\mathcal{R}^{4}\right) + (c_{0}g_{B}^{-2} + c_{1} + c_{2}g_{B}^{2})\nabla^{6}\mathcal{R}^{4}\right) + O(e^{-2\pi/g_{B}}).$$

We observe that there is no genus-one contribution to the order $s^{2}$ ($\sim \nabla^{4}\mathcal{R}^{4}$). The genus-one contribution to $s^{3}$ is given by $4\zeta(3)\zeta(2) = 2\pi^{2}/3\zeta(3)$.

5. **Fundamental property of $A_{4}^{sl(2)}(s,t,u)$**

Another strategy to produce an $SL(2, \mathbb{Z})$ invariant amplitude is by generalizing, when possible, properties of the Virasoro amplitude (which applies at $\tau_{2} \rightarrow \infty$) to the full fundamental domain $\mathcal{F} = SL(2, \mathbb{Z}) \setminus H$. Let us write the Virasoro amplitude (2.0) in terms of Einstein variables, $s_{E} = g_{B}^{1/2}s$, etc.

$$\delta_{0}(\tau_{2}, \alpha') = 2 \sum_{k=1}^{\infty} c_{k}(s_{E}, t_{E})\tau_{2}^{k+1/2}(\alpha')^{2k+1},$$

(5.1)
with
\[ c_k(s_E, t_E) = \frac{\zeta(2k+1)}{2k+1} (s_E^{2k+1} + t_E^{2k+1} + u_E^{2k+1}) . \]

As a function of \( \tau \) and \( \alpha' \), \( \delta_0 \) obeys the simple relation
\[ \frac{\tau_2}{\tau_2^2} \frac{\partial^2}{\partial \tau_2^2} \delta_0 = \eta^2 \frac{\partial^2}{\partial \eta^2} \delta_0 , \quad \eta \equiv (\alpha')^2 . \]

The \( SL(2, \mathbb{Z}) \) invariant condition is thus the following one:
\[ \Delta \delta_0 = \eta^2 \frac{\partial^2}{\partial \eta^2} \delta_0 , \quad (5.2) \]

where \( \Delta \) is the Laplace operator in the fundamental domain,
\[ \Delta = \tau_2^2 \left( \frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) . \]

Any solution to eq. (5.2) in \( \mathcal{F} \) that asymptotically approaches \( \delta_0 \) will provide an \( SL(2, \mathbb{Z}) \) generalization of the Virasoro amplitude. Interestingly, this strategy leads to the same amplitude discussed in sect. 2, as stated by the following theorem.

**Theorem:** Let \( \delta(\tau; \eta) \) be a function on \( \mathcal{F} \) satisfying
\[ \lim_{\tau_2 \to \infty} \delta(\tau; \eta) = \delta_0 , \quad (5.3) \]
\[ \Delta \delta = \eta^2 \frac{\partial^2}{\partial \eta^2} \delta , \quad \eta = (\alpha')^2 , \quad (5.4) \]

where the limit (5.3) is understood with \( \eta \to 0 \) so that \( \tau_2 \eta \) is fixed (“string frame”). Then such function is unique and given by
\[ \frac{1}{stu} e^\delta = A_{(s,t,u)}^{sl(2)}(s,t,u) . \]

**Proof:** We solve eq. (5.4) by separation of variables:
\[ \delta = \sum_r f_r(\eta) \psi_r(\tau) . \quad (5.5) \]

It follows that
\[ f_r(\eta) = A_r \eta^r + B_r \eta^{1-r} , \quad (5.6) \]
\[ \Delta \psi_r = r(r-1) \psi_r . \quad (5.7) \]
Since \( \psi_{1-r} \) lies in the space of solutions \( \psi_r \) to (5.4), with no loss of generality we can set \( B_r = 0 \). Using eq. (5.3), we find that only \( \psi_r \) with \( r = k + 1/2 \) appear in the sum (5.3), and they have the asymptotic behavior

\[
\psi_r \underset{\tau_2 \to \infty}{\longrightarrow} \delta_{r,k+\frac{1}{2}}\tau_2^r, \quad k = 1, 2, \ldots
\]

This implies that \( \psi_r(\tau) \) is a Maass waveform. [A Maass waveform is a function on \( \mathcal{F} \) which is an eigenfunction of the laplacian and which has at most polynomial growth at infinity [18].] If \( \mathcal{N}(SL(2, \mathbb{Z}), r(r - 1)) \) denotes the vector space of such waveforms, it is a known result that

\[
\mathcal{N}(SL(2, \mathbb{Z}), r(r - 1)) = CE_r, \quad \text{for } \Re r > \frac{1}{2}, \ r \notin \left[\frac{1}{2}, 1\right].
\]

Indeed, if there was another function \( f \) with the same asymptotic behavior, it would imply that we can find a constant \( c \) such that \( g = f - cE_r \) is square-integrable over the fundamental domain, with the invariant area element \( d^2\tau/\tau_2^2 \) (since its expansion would start with \( b\tau_2^{1-r} \), see eq. (2.10)). But this contradicts the fact that the Laplace operator is negative on \( \mathcal{F} \) [18].

Using the boundary condition, we can now determine the coefficients \( A_r \):

\[
\delta = 2 \sum_{k=1}^{\infty} c_k(s, t)E_{k+1/2}(\tau)\eta^{k+1/2},
\]

or

\[
\frac{1}{stu}e^\delta = A_{sl(2)}(s, t, u), \quad \text{(5.9)}
\]

Q.E.D.

Thus the differential equation (5.4) can be used to provide an alternative definition for \( A_{sl(2)}(s, t, u) \), with no need to refer to the infinite product (2.2), whose convergence properties are unknown. What is special about this differential equation? In the three-dimensional “space-time” with coordinates \( \{\eta, \tau_1, \tau_2\} \) it takes the simple form

\[
\Delta_{(3)} \delta = 0, \quad \text{(5.10)}
\]

with

\[
ds_3^2 = -\eta^{-4}d\eta^2 + \frac{1}{\tau_2^2}\eta^2(d\tau_1^2 + d\tau_2^2),
\]

i.e.

\[
ds_3^2 = -d\tau_0^2 + \frac{\tau_0^2}{\tau_2^2}(d\tau_1^2 + d\tau_2^2), \quad \tau_0 = \eta^{-1} = (\alpha')^{-2}.
\]

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This geometry may be interpreted as an “expanding universe”, with the spatial section being the fundamental domain \( F \). It does not have an Euclidean counterpart. The time parameter, which provides the scale, is the string tension squared, \( \tau_0 = (2\pi T)^2 \).

Introducing \( U = \tau_0 / \tau_2 \), \( V = \tau_0 \tau_2 \), we obtain

\[
 ds_3^2 = -dUdV + U^2 d\tau_1^2 .
\]

(5.13)

In this form, the geometry exhibits an orbifold singularity at \( U = 0 \) moving at the speed of light. This metric (with the range of \( U, V \) unrestricted) was called the “null orbifold” geometry in [19], but the connection with the fundamental domain of \( SL(2, \mathbb{Z}) \) was not noticed. Here the topology is not the same, since there is a restriction in the range of \( V/U \), and the geometry contains singularities at \( |\tau| = 1, \tau_1 = \pm1/2 \). A new change of coordinate shows that the geometry is flat everywhere away from the orbifold points,

\[
 ds_3^2 = -dUd\tilde{V} + dy^2 , \quad y = U\tau_1 \quad , \quad \tilde{V} = V + U\tau_1^2 .
\]

(5.14)

In other words, this three-dimensional space-time is nothing but the (Minkowskian) embedding of the fundamental domain. The differential equation that defines the amplitude is thus the simplest invariant differential equation that one can write down involving \( \tau, \alpha' \), namely the free wave equation in a flat three-dimensional space. This differential equation involves R-R and dilaton couplings, and \( \alpha' \), i.e. the length scale of the target metric. It might originate from a Ward identity (or perhaps from some saddle point approximation to the scattering problem).

6. More general S-dual amplitudes

Since the exact amplitude is a function on the fundamental domain \( F = SL(2, \mathbb{Z}) \setminus H \), any correction to \( A_{sl}^{(2)} \) must be invariant under \( SL(2, \mathbb{Z}) \) transformations. For square-integrable functions on \( F \) (with the standard measure \( d\tau_2 / \tau_2^2 \)) there exists a spectral decomposition in terms of cusp forms and \( E_{1/2+i\phi} \), namely the Roelcke-Selberg formula [18]. Although it is not completely clear whether the exact amplitude must be in \( L^2(SL(2, \mathbb{Z}) \setminus H) \), it is interesting to note that a function with the asymptotic behavior of the Virasoro amplitude would be square-integrable, since the relevant asymptotic region is \( \tau_2 \to \infty \) at fixed \( s_E, t_E, u_E \), in other words, \( s, t, u \to \infty \), where one has the well-known exponential fall off of the high-energy fixed-angle limit (recall \( (s, t, u) = \tau_2^{1/2}(s_E, t_E, u_E) \)). However, the exact scattering amplitude must approach the Virasoro amplitude only at \( \tau_2 \to \infty \) and fixed string-frame variables \( s, t, u \). Thus, in the region of interest, \( \tau_2 \to \infty \) at fixed \( s_E, t_E, u_E \), the asymptotic behavior of the exact amplitude is unknown. This is a region of high-energy scattering at fixed angles, where, in addition, the coupling \( g_B \) is sent to zero. It seems legitimate to make use of some version of unitarity bounds, which could
indicate that the exact amplitude must be square-integrable (as far as square-integrability is concerned, even a field-theoretic behavior may be sufficient; in local field theory, high-energy fixed angle scattering amplitudes typically fall off according to a power law, fact directly associated with the power singularities of the products of local operators at short distances). In any case, it is unlikely that a treatment based on the Roelcke-Selberg expansion would be of any use, since an orthonormal basis of cusp forms for $SL(2, \mathbb{Z})$ is not explicitly known.

The general structure of the exact scattering amplitude can be quite complicated, but here we will consider a subclass of possible corrections. They exhibit an interesting feature: each new order is $SL(2, \mathbb{Z})$ invariant by itself and, when expanded at $g_2 \ll 1$, starts with an additional power of $g_B^2$. In addition, it provides an example of a more general $SL(2, \mathbb{Z})$ invariant amplitude that also reproduces the Virasoro amplitude in the weak coupling limit and has a correct perturbative and non-perturbative dependence on the coupling.

In the one-loop four-graviton amplitude of eleven-dimensional supergravity on the torus [11], as well as in $A_4^{sl(2)}$, only those $E_r(\tau)$, with $r = k + 1/2 = 3/2, 5/2, ...$ appear. Other $E_r$ are simply not allowed, because they would contain wrong perturbative dependence at small $g_B$ (see (2.10)). It is therefore of interest to investigate possible additional corrections to $\delta$ that can be expressed as a linear combination of $E_{k+1/2}(\tau)$. Let us stress once again that this does not represent the most general function on $\mathcal{F}$. We shall thus consider a function on $\mathcal{F}$ of the form

$$\log A_4^S(s, t) \equiv -\log stu + 2 \sum_{k=1}^{\infty} g_B^{k+1/2}(s^{2k+1} + t^{2k+1} + u^{2k+1}) \sum_{h=0}^{h_0} c_k^{(h)} E_{k+1/2-2h}(\tau), \ (6.1)$$

$$c_k^{(0)} = \frac{\zeta(2k + 1)}{2k + 1}, \quad h_0 = \left[\frac{1}{2}(k - 1)\right].$$

It is worth noting that the sum over $h$ contains a finite number of terms. Terms with $h < 0$ are excluded because $A_4^S(s, t)$ must reduce to the usual genus zero result at $g_B \to 0$; terms with $h > h_0$ are related to the other terms by the functional relation $E_r = \text{const.} E_{1-r}$. For $k$ odd, a given power of $s$ in eq. (6.1) has the structure

$$g_B^{k+1/2} s^{2k+1} [c_k^{(0)} E_{k+1/2}(\tau) + c_k^{(1)} E_{k-3/2}(\tau) + ... + c_k^{(h_0)} E_{3/2}(\tau)] \ (6.2)$$

$$s^{2k+1} [c_k^{(0)} (1 + \gamma_0 g_B^2) + c_k^{(1)} (g_B^2 + \gamma_1 g_B^{2k-2}) + ... + c_k^{(h_0)} (g_B^{k-1} + \gamma_{h_0} g_B^{k+1})] \ (6.3)$$

whereas for $k$ even the last term in the above equations are respectively $c_k^{(h_0)} E_{5/2}$, $c_k^{(h_0)} (g_B^{k-2} + \gamma_{h_0} g_B^{k+2})$. Thus we have $A_4 = A_4^{sl(2)} + A_4^{sl(2)} + ...$,

$$h = 0 : \quad A_4^{sl(2)} \sim 1 + g_B^2 + ... + O(e^{-1/g_B}) \ ,$$

$$h = 1 : \quad A_4^{sl(2)}(h=1) \sim g_B^2 + g_B^4 + ... + O(e^{-1/g_B}) \ , \ \text{etc.}$$
These expressions uncover an important property of the sum over $h$: it adds new corrections by preserving $SL(2,\mathbb{Z})$ invariance and without affecting the leading term of the previous order. By S-duality, the same property holds in an expansion at large $g_B$.

Defining $m = k - 2h$, and in terms of Einstein-frame variables, eq. (6.1) takes the form ($\alpha' = 4$)

$$\log A_4^S(s, t) = -\log stu + 2 \sum_{m=1}^{\infty} E_{m+1/2}(\tau)s^{m+1} \sum_{h=0}^{\infty} c_m^{(h)} s^{4h} + (s \to t, u).$$  \hspace{1cm} (6.4)

In string-frame variables the sum over $h$ is $c_m^{(0)} + c_m^{(1)} g_B^2 s^4 + c_m^{(2)} g_B^4 s^8 + \ldots$. From eq. (6.4) we see that the amplitude $A_4^S(s, t)$ generalizes $A_4^{sl(2)}(s, t)$ by replacing the constant coefficient $c_m^{(0)}$ multiplying $E_{m+1/2}(\tau)$ by an analytic function $f_m(z) = \sum c_m^{(h)} z^h$, $z = (\alpha's_E)^4$ (and the same function for the terms with $t$ and $u$). It is clear (and also implied by the theorem of sect. 5) that the differential equation (5.4) is not satisfied unless $f_m$ are constants (i.e. unless $A_4^S(s, t) = A_4^{sl(2)}(s, t)$).

Let us also note that if, as argued in [14] and in sect. 4, supersymmetry transformations relate the term $H^{4k-4} \mathcal{R}^4$ in the type IIB effective action to a term of the form $\nabla^{4k-4} \mathcal{R}^4$, then the constants $c_m^{(h)}$ in eq. (6.1) are uniquely determined to be given by $c_m^{(h)} = c_m^{(0)} \delta_{h0}$, i.e. $A_4^S(s, t) = A_4^{sl(2)}(s, t)$. Indeed, an amplitude of the form (6.1) with $c_m^{(h)} \neq 0$ for some $h > 0$ would imply, by supersymmetry, contributions to $H^{4k-4} \mathcal{R}^4$ of genus lower than $k$ (cf. (6.3)), in contradiction with the results of [13]. This suggests that generalizations of $A_4^{sl(2)}$ of the form (6.1) are not possible.

To illustrate how the function $A_4^S(s, t)$ may look like after resummation in $m$, we consider the following concrete example. Let

$$c_m^{(h)} = c_h \frac{\zeta(2m+1)}{2m+1}, \quad c_0 = 1.$$

Now the full series can be resummed with the result:

$$A_4^S(s, t) = \frac{1}{stu} \prod_{(p, q)'} e^{-2\gamma(s_{pq} f(s) + t_{pq} f(t) + u_{pq} f(u))} \times \frac{\Gamma(1-s_{pq})}{\Gamma(1+s_{pq})} f(s) \frac{\Gamma(1-t_{pq})}{\Gamma(1+t_{pq})} f(t) \frac{\Gamma(1-u_{pq})}{\Gamma(1+u_{pq})} f(u),$$  \hspace{1cm} (6.5)

where $\gamma$ is the Euler constant, $\gamma = 0.5772\ldots$ and

$$f(s) = 1 + c_1 g_B^2 s^4 + c_2 g_B^4 s^8 + \ldots$$

The analytic structure is now more complicated. Let us extract the complete genus one contribution (proportional to $g_B^2$). We find

$$A_4^{(1)} = g_B^2 A_4^0(s, t) \left[ \frac{2\pi^2}{3} \tilde{s} \tilde{u} + \left( -2c_1 \gamma \tilde{s}^5 + c_1 s^4 \log \frac{\Gamma(1-\tilde{s})}{\Gamma(1+\tilde{s})} + (s \to t, u) \right) \right].$$  \hspace{1cm} (6.6)

Although this exhibits the presence of cuts, it is different from the full genus one string amplitude (at most, it might be a part of the genus one string amplitude; this contains, in addition, double poles, a non-local part coming from the massless loop contribution, etc., which are absent in the above expression). Note that the first term in (6.6) comes from the order $h = 0$, whereas the remaining part comes from the order $h = 1$. 

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7. Eleven-dimensional origin of $A_4^{sl(2)}$

So far we have examined some properties of $SL(2, \mathbb{Z})$ invariant generalizations of the Virasoro amplitude. We will now propose an interpretation of $A_4^{sl(2)}$ in the context of superstring/M-theory.

As pointed out in sect. 2, $A_4^{sl(2)}$ has poles corresponding to the exchange of $(p,q)$ string states. With the exception of the massless state $N_R = N_L = 0$, such states (which include the usual $(1,0)$ string excitations) are expected to be unstable, since they do not correspond to supersymmetric classical solutions (we recall that the states that survive in ten dimensions have vanishing RR and NS-NS charges). Once all quantum corrections have been taken into account, poles corresponding to unstable particles should lie away from the real axes. The amplitude $A_4^{sl(2)}$, which has no discontinuity cuts and (for generic coupling) has simple poles lying on the real axes, has the structure of a tree amplitude, in which a certain collection of states are exchanged in the $s$-$t$-$u$ channels.

From the point of view of eleven dimensions –this will be shown below– these $(p,q)$ string states of mass $\alpha' M^2 = 4N_R |p + q\tau|$ correspond to membrane configurations with excitations $N_R = N_L$ moving in the direction $(p,q)$ and carrying zero total momentum. They are not protected by supersymmetry, but these are the basic configurations that survive in the ten-dimensional type IIB limit; in this limit the BPS $(p,q)$ strings become infinitely massive and they do not contribute to the four-graviton amplitude. When $R_{11} \ll R_{10}$, the scattering is dominated by exchange of $(1,0)$ strings, since all the $(p,q)$ strings with $q \neq 0$ are very heavy. For $R_{11} \sim R_{10}$, the contribution of the $(1,0)$ string is of the same order as, e.g. that of the $(0,1)$ string. It is no longer justified to construct a perturbation theory based on the standard $(1,0)$ string, rather than including all $(p,q)$ strings at the same time. It must be stressed that for $g_B = R_{11}/R_{10} = O(1)$ all massive string excitations become very unstable. What this means is that there are other effects –in addition to the tree diagrams– that are of the same order of magnitude (see sect. 8).

Although in general there is not a simple correspondence between eleven dimensional loops and string loops, for small torus area the correspondence is more direct. The example of one-loop four-graviton amplitude in eleven dimensional supergravity on the torus exhibits the basic fact that the eleven-dimensional supergravity amplitude is $SL(2, \mathbb{Z})$ invariant order by order in the loop expansion ($SL(2, \mathbb{Z})$ symmetry being just part of reparametrization invariance). But it also shows that the 1-loop contribution, when represented in terms of the string coupling, already contains contributions to every genus order. The reason is that at any given loop order in eleven dimensions, additional dependence on the string coupling appears through the masses of the Kaluza-Klein states running in the loops, in a way that also affects lower loop orders in the string perturbation theory. Technically, this calculation applies when $R_{10}, R_{11}$ are much greater than the cutoff, given

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3 Further discussions on this can be found in [10, 11, 20].
by the eleven-dimensional Planck length $l_p$. In the opposite limit, $R_{10}, R_{11} \ll l_p$, the contribution of Kaluza-Klein states is in fact suppressed by factors $O\left(\exp(-l_p/R)\right)$, so they do not give any perturbative contribution.

Consider, for example, the term $\zeta(3)R^4$ in the effective action. At $R_{10}R_{11} \gg l_p^2$, it is the one-loop four-graviton amplitude in eleven dimensional supergravity that provides this contribution to the $R^4$ term, whereas at $R_{10}R_{11} \ll l_p^2$ such term, and the full Virasoro amplitude, originate from the multiple exchange of the usual (non-BPS) string excitations in the tree diagram. From the point of view of the eleven dimensional theory on a torus with $R_{10}R_{11} \ll l_p^2$, in the Virasoro amplitude the external gravitons are exchanging small winding membranes that have wave modes moving only in one world-volume direction $\sigma$ (wound around $x_{10}$). The $SL(2, \mathbb{Z})$ symmetry is recovered upon the inclusion of the exchange of the physical states representing oscillations collectively moving in an arbitrary $(p, q)$ direction, after summing over $(p, q)$. It will be argued below that (after taking $R_{10}R_{11} \to 0$) the $SL(2, \mathbb{Z})$ symmetric result that arises in this ‘tree-level’ eleven-dimensional calculation is given by $A_{4}^{sl(2)}$.

7.1. M-theory configurations corresponding to the $(p, q)$ strings

Let us first recall the eleven-dimensional description of the BPS $(p, q)$ strings \cite{12}. A BPS $(p, q)$ string bound state with NS-NS and R-R charges $(l_p, l_q)$ with a momentum boost $w_0$ along the string becomes, after T-duality in the string direction $x_{10}$, a bound state of a 0-brane of charge $l_q$, a fundamental string of charge $w_0$ and a wave of momentum $l_p$. The corresponding solution in eleven dimensions was described in \cite{21} and represents an extremal 2-brane of charge $w_0$ superposed with a gravitational wave with momentum components $(l_p, l_q)$ in the directions $(x_{10}, x_{11})$ (or, equivalently, momentum flux along the $(p, q)$ cycle of the torus). In the presence of an extra translational isometry, there exists a dual eleven-dimensional description of the same BPS $(p, q)$ string solution, obtained by applying T-duality in the extra isometric direction $x_9$, and lifting to eleven dimensions, giving a 2-brane with one leg wrapped around a $(p, q)$ cycle of the torus $(x_9, x_{11})$, and the other leg winding $l$ times around $x_{10}$, superposed with a gravitational wave carrying momentum $w_0$ in the direction $x_{10}$ (for the explicit solution, see \cite{22}).

At microscopic level, there are right-moving waves moving in the BPS $(p, q)$ string satisfying $N_R = lw_0$. There are many inequivalent physical states with the same value of $N_R$. The corresponding classical geometry in eleven dimensions is the same for all of them and given by the fundamental membrane with the momentum boost. We will consider a rectangular ($\tau_1 = 0$) torus of radii $R_{10}, R_{11}$. The membrane coordinates can be written as follows

$$X^{10}(\sigma, \rho) = w_0 R_{10} \sigma + \tilde{X}^{10}(\sigma, \rho),$$

$$X^{11}(\sigma, \rho) = R_{11} \rho + \tilde{X}^{11}(\sigma, \rho),$$

(7.1)
where $\tilde{X}^{10}, \tilde{X}^{11}$ are single-valued functions of the membrane world-volume coordinates $\sigma, \rho \in [0, 2\pi)$. The transverse membrane coordinates $X^i(\sigma, \rho)$, $i = 1, \ldots, 8$ are all single-valued (we use the notation where the eleven bosonic coordinates are \{$X^0, X^i, X^{10}, X^{11}$\}), and they can be expanded in terms of Fourier modes,

$$X^i(\sigma, \rho) = \sqrt{\alpha'} \sum_{k,m} X^i_{(k,m)} e^{ik\sigma + im\rho}, \quad P^i(\sigma, \rho) = \frac{1}{(2\pi)^2 \sqrt{\alpha'}} \sum_{k,m} P^i_{(k,m)} e^{ik\sigma + im\rho}.$$

We will assume that the dynamics of the oscillations is governed by a relativistic membrane theory [23]. Separating the winding contributions and inserting the Fourier expansions as in ref. [24], the membrane light-cone Hamiltonian [25,26] takes the form $H = H_0 + H_{\text{int}}$, with [24]

$$\alpha' H_0 = 8\pi^4 \alpha'T_3^2 R_{10}^2 R_{11}^2 w_0^2 + \frac{1}{2} \sum_n \left[ P_n^a P_n^a - \omega_{km} X_n^a X_n^a \right]$$

$$\alpha' H_{\text{int}} = \frac{1}{4g_A^2} \sum(n_1 \times n_2)(n_3 \times n_4) X^a_{n_1} X^b_{n_2} X^a_{n_3} X^b_{n_4}$$

$$+ \frac{i}{g_A} \sum mk^2 X^{10}_{(0,m)} X^i_{(k,n)} X^i_{(-k,-n-m)},$$

$$X^+ = \frac{X^0 + \tilde{X}^{11}}{\sqrt{2}} = x^+ + \alpha' p^+ \tau, \quad n \equiv (k, m), \quad a, b = 1, \ldots, 8, 10,$$

where $T_3$ is the membrane tension ([$T_3 = cm^{-3}$]) and

$$n \times n' = km' - mk', \quad \alpha' = \left(4\pi^2 R_{11} T_3\right)^{-1},$$

$$g_A^{-2} \equiv \frac{R_{11}^2}{\alpha'} = 4\pi^2 R_{11}^3 T_3, \quad \omega_{km} = \sqrt{k^2 + w_0^2 m^2 \tau_2^2}, \quad \tau_2 = \frac{R_{10}}{R_{11}}.$$

Here only the bosonic modes have been written explicitly (the inclusion of fermion modes is straightforward). The constant $g_A$ represents the type IIA string coupling. The mass operator is given by

$$M^2 = 2p^+ p^- - (p^\alpha)^2 = 2H_0 + 2H_{\text{int}} - (p^\alpha)^2.$$

$H_{\text{int}}$ is positive definite, and any state $|\Psi\rangle$ with $\langle\Psi|H_{\text{int}}|\Psi\rangle \neq 0$ will have infinite mass in the zero area limit, where $g_A \to 0$ (with $T_3 \to \infty$, so that $T = 2\pi R_{11} T_3$ and $\tau_2$ remain fixed). The only states that survive are those containing excitations in a Cartan subspace of the area-preserving diffeomorphism algebra, so that $H_{\text{int}}$ drops out from $\langle\Psi|M^2|\Psi\rangle$, i.e.

$$\langle\Psi_{\text{cartan}}|H_{\text{int}}|\Psi_{\text{cartan}}\rangle = 0.$$
To be precise, let us introduce mode operators as follows:

$$X^a_{(k,m)} = \frac{i}{\sqrt{2w(k,m)}} \left[ \alpha^a_{(k,m)} + \tilde{\alpha}^a_{(-k,-m)} \right], \quad P^a_{(k,m)} = \frac{1}{\sqrt{2w(k,m)}} \left[ \alpha^a_{(k,m)} - \tilde{\alpha}^a_{(-k,-m)} \right], \quad (7.4)$$

$$\left( X^a_{(k,m)} \right)^\dagger = X^a_{(-k,-m)}, \quad \left( P^a_{(k,m)} \right)^\dagger = P^a_{(-k,-m)}, \quad w(k,m) \equiv \epsilon(k) \omega_{km},$$

where $\epsilon(k)$ is the sign function. The canonical commutation relations imply

$$[X^a_{(k,m)}, P^b_{(k',m')}!] = i\delta_{kk'}\delta_{mm'}\delta^{ab},$$

$$[\alpha^a_{(k,m)}, \alpha^b_{(k',m')}!] = w(k,m)\delta_{kk'}\delta_{mm'}\delta^{ab}, \quad (7.5)$$

and similar relations for the $\tilde{\alpha}^a_{(k,m)}$. Let us write $(k, m) = n(\tilde{p}, \tilde{q})$, with $(\tilde{p}, \tilde{q})$ relatively prime. A Cartan subspace is constituted of all states made of operators $\alpha^a_{n(\tilde{p}, \tilde{q})}$, $\tilde{\alpha}^a_{n(\tilde{p}, \tilde{q})}$ with the same value of $(\tilde{p}, \tilde{q})$. We will denote this subspace by $\mathcal{H}_{\tilde{p}\tilde{q}}$. The states that have finite mass in the zero area limit (corresponding to the 10D type IIB limit) live in the direct sum of $\mathcal{H}_{\tilde{p}\tilde{q}}$ over $\tilde{p}, \tilde{q}$ coprime.

In this subspace and in this limit the interaction term can be dropped and the world-volume theory can be described in terms of free variables. The solution to the membrane equations of motion is given by

$$X^a(\sigma, \rho, \tau) = x^a + \alpha'^a \rho^\sigma \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq (0,0)} w_n^{-1} \left[ \alpha^a_n e^{ik\sigma + im\rho} + \tilde{\alpha}^a_n e^{-ik\sigma - im\rho} \right] e^{iwn\tau}. \quad (7.6)$$

Let the momentum components in the directions $X^{10}$ and $X^{11}$ be given by

$$p_{10} = \frac{lp}{R_{10}}, \quad p_{11} = \frac{lq}{R_{11}}.$$  

The (nine-dimensional) mass operator takes the form

$$M^2 = \frac{l^2 \rho^2}{R_{10}^2} + \frac{l^2 \rho^2}{R_{11}^2} + \frac{w_0^2 R_{10}^2}{\alpha'^2} + \frac{2}{\alpha'} \mathcal{H}, \quad (7.7)$$

$$\mathcal{H} = \frac{1}{2} \sum_n \left( \alpha^a_n \alpha^a_{-n} + \tilde{\alpha}^a_n \tilde{\alpha}^a_{-n} \right), \quad n \equiv (k, m).$$

The level-matching conditions are given by

$$N^+ - N^- = w_0 lp, \quad N^+ - N^- = lq,$$

where

$$N^+_\sigma = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{km}} \alpha^i_{(-k,-m)} \alpha^i_{(k,m)},$$

$$N^-_{\rho} = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{km}} \tilde{\alpha}^i_{(-k,-m)} \tilde{\alpha}^i_{(k,m)},$$

$$N^-_{\rho} = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{km}} \alpha^i_{(-k,-m)} \tilde{\alpha}^i_{(k,m)}.$$
right-moving waves only along the momentum vector \( l \). The possible states that can microscopically describe the BPS solution. That is, they live in \( H \) for states living in a given subspace \( \tilde{\theta} \) oscillations travel in a direction \( \theta \) relative to the direction \( x_{10} \), with

\[
\tan \theta = \frac{\tilde{q}w_0\tau_2}{\tilde{p}}.
\]

For states living in a given subspace \( H_{\tilde{p}\tilde{q}} \), with \( (k, m) = n(\tilde{p}, \tilde{q}) \), the frequency of oscillations becomes \( w_{km} = n\sqrt{\tilde{p}^2 + \tilde{q}^2w_0^2\tau_2^2} \). In the target space \((x_{10}, x_{11})\), the corresponding oscillations travel in a direction \( \theta \) relative to the direction \( x_{10} \), with

\[
\tan \theta = \frac{\tilde{q}w_0\tau_2}{\tilde{p}}.
\]

For the BPS \((p, q)\) string states, supersymmetry of the classical solution allows to add right-moving waves only along the momentum vector \( l(p, q) \). This imposes a restriction on the possible states that can microscopically describe the BPS solution. That is, they live in \( H_{\tilde{p}\tilde{q}} \), with \( \frac{\tilde{p}}{qw_0} = \frac{p}{q} \), with the extra condition that there are only right-moving excitations, which sets \( \tilde{\alpha}_i(k, m) = 0 \). Because \((p, q)\) and \((\tilde{p}, \tilde{q})\) are pairs of relatively prime integers, the relation \( \frac{\tilde{p}}{qw_0} = \frac{p}{q} \) implies that one of the following two cases is true: \((\tilde{p}, \tilde{q}) = (pw_0, q)\) or \((\tilde{p}, \tilde{q}) = (p, q/w_0)\). To be specific, let us consider this last one (and \( p \neq 0 \)). The discussion for the other case is similar. In the subspace \( H_{\tilde{p}\tilde{q}} \) the mass operator (7.7) reduces to

\[
M^2 = \frac{l^2p^2}{R_{10}^2} + \frac{l^2q^2}{R_{11}^2} + \frac{w_0^2R_{10}^2}{\alpha'\ell^2} + \frac{2}{\alpha'} \sqrt{p^2 + q^2w_0^2\tau_2^2} (N_R + N_L), \tag{7.8}
\]

\[
N_R - N_L = lw_0, \quad N_R = \sum_{n=1}^{\infty} \alpha_n^{-i} \alpha_n^i, \quad N_L = \sum_{n=1}^{\infty} \tilde{\alpha}_n^{-i} \tilde{\alpha}_n^i, \tag{7.9}
\]

where the \( \alpha_n^i \) are defined by

\[
\alpha_n^i = (p^2 + q^2\tau_2)^{-1/4} \alpha_{(np, nq/w_0)}^i, \quad [\alpha_n^i, \alpha_{n'}^j] = n\delta_{n+n',\delta^{ij}}, \tag{7.10}
\]

and similarly for \( \tilde{\alpha}_n^i \). Setting \( \tilde{\alpha}_n(k, m) = 0 \), so that \( N_L = 0 \), one reproduces the standard mass formula for the BPS states, as discussed in [21].

It is remarkable that the mass spectrum (7.8), (7.9) exactly coincides with the \((p, q)\) string mass spectrum, even if it includes non-supersymmetric states with \( N_L \neq 0 \). The sector \( l = 0 \) (relevant to ten dimensions) is constituted of both right and left moving excitations satisfying \( N_R = N_L \), and they describe the membrane states that have finite mass in the limit that the area goes to zero. Since in eleven dimensions these are waves.
moving along a \((p, q)\) direction of a membrane with winding \(w_0\), which carry zero total momentum, the large distance geometry must approximate that of a non-extremal black 2-brane \([27]\) of charge \(w_0\), irrespective of the microscopic state, in particular, irrespective of the \((p, q)\) orientation of the oscillations.

In the type IIB language, the winding of the membrane \(w_0\) represents the momentum of the type IIB string. The above discussion is invalid in the sector \(w_0 = 0\), that we do not know how to treat. Nevertheless, in the zero area limit, understanding this sector may not be essential: for \(R_{10} \rightarrow 0\), one can recover all continuum values of \(w_0 R_{10}\) (including zero) by formally starting with \(w_0 \neq 0\).

7.2. Heuristic derivation of \(A_4^{sl(2)}\)

Having argued that membrane theory on a vanishing torus area is constituted by different decoupled sectors \((p, q)\), each one being described by a free string theory with tension \(T_{pq} = T \sqrt{p^2 + q^2 r_2^2}\), we now examine very schematically a possible way to derive an \(SL(2, \mathbb{Z})\)-invariant four-graviton scattering amplitude starting from membrane theory.

The four-graviton scattering amplitude in string theory is formally given by

\[
A_4 = \langle (k_3, \zeta_3); (k_4, \zeta_4) | (k_1, \zeta_1); (k_2, \zeta_2) \rangle = \int_B [DX] e^S = \int [DX] e^S V_1 V_1 V_3 V_4 , \quad (7.11)
\]

where \(B\) stands for the boundary carrying the information about the quantum numbers of ingoing and outgoing states. By conformal invariance, \(B\) can be supplanted by the insertion of suitable vertex operators \(V_i = V(k_i; \zeta_i)\). As in the previous subsection, fermion variables are omitted.

The explicit calculation of the path integral \((7.11)\) gives the Virasoro amplitude \((2.3)\). The contribution due to massless exchange can be obtained by taking the limit \(\alpha' \rightarrow 0\) in \((2.3)\). This gives

\[
A_4 \bigg|_{zero} = \kappa^2 K \frac{1}{stu} .
\]

From eleven-dimensional point of view, the amplitude \((7.11)\) is only accounting for the exchange of those membranes that do not oscillate in the eleventh dimension. If \(R_{10} \sim R_{11}\), the contribution of membrane exchange with oscillations in an arbitrary \((p, q)\) direction on \(T^2\) will be equally important. The full amplitude will be given by

\[
A_4 = \langle (k_3, \zeta_3); (k_4, \zeta_4) | (k_1, \zeta_1); (k_2, \zeta_2) \rangle = \int_B [DX(\sigma, \rho)] e^S . \quad (7.12)
\]

Let us now write

\[
X^i(\sigma, \rho) = \sum_{(p,q)'} X^i_{pq} , \quad X^i_{pq} = \sqrt{\alpha'} \sum_n X^i_{n(p,q)} e^{in(p\sigma + q\rho)} , \quad (7.13)
\]
so that

\[ [DX] = \prod_{(p,q)'} [DX_{pq}] . \]

In the zero-area limit, the dynamics resembles that described by a direct sum of free-string theory lagrangians \( \mathcal{L}_{pq} \), differing only in the tension \( T_{pq} \). The amplitude takes the form

\[ A_4 = \prod_{(p,q),}\int_{\hat{B}_{pq}} [DX_{pq}] e^{S_{pq}} \sim \prod_{(p,q)'} \int [DX_{pq}] e^{S_{pq}} V_1 V_1 V_3 V_4 . \quad (7.14) \]

In this way the amplitude looks like an infinite product of string-theory amplitudes. There is, however, an important difference: in eq. (7.14), there is only one variable describing the center-of-mass coordinate (the membrane has only one center-of-mass mode \( (k,m) = (0,0) \) in eq. (7.2)). A direct product of string amplitudes with independent center-of-mass coordinates would not give the correct answer; in particular, each factor would be independently accounting for the exchange of a string in the zero-mode state (i.e. the exchange of the massless multiplet). This would lead to the appearance of the factor \( K_{stu}^{1/2} \) an infinite number of times. But it is clear from the original membrane-theory formulation that there must be only one factor \( K_{stu}^{1/2} \), corresponding to the exchange of a membrane with no oscillations (representing the massless supergravity multiplet). The \( SL(2,\mathbb{Z}) \)-invariant amplitude schematically represented in (7.14) therefore seems to have the structure of a product over \((p,q)\)' with a single factor \( K_{stu}^{1/2} \), just as the S-dual amplitude given by eq. (2.1). Obtaining a complete proof along these lines –i.e. starting from eleven dimensions and then taking the zero-torus area limit– may be a very complicated way. In eleven dimensions membrane theory is non-linear, and it is only in the limit of vanishing torus area that the theory seems to simplify. A more convenient approach may already exist in the ten-dimensional type IIB theory, without any reference to eleven dimensions (perhaps in the spirit of ref. [2]).

8. Discussion

In section 7 the amplitude \( A_{4}^{s(2)} \) has been interpreted as a tree-level amplitude in M-theory compactified on a 2-torus of small area. It is a well-known fact that in eleven dimensions there is no extra parameter that one can use to control loop corrections. The model of sect. 6 may be regarded as an example on how other effects might be systematically organized. The physical idea of this organization is the following. Loop diagrams can be constructed from tree diagrams by using unitarity (this is the way string loops were originally constructed). This is not a straightforward calculation, in particular, one first needs to symmetrize other tree-level (N-graviton) amplitudes of string theory, etc. Using \( A_{4}^{s(2)} \) as starting (tree-level) amplitude and assuming that the \((p,q)\) string states constitute
a complete set of intermediate states, this procedure generates an $SL(2,\mathbb{Z})$ invariant loop expansion. In section 7 we have presented evidence that this is the natural organization that follows from eleven-dimensional membrane theory in an expansion in topologies. The full amplitude defined in this way may represent the four-graviton amplitude in M-theory compactified on a 2-torus in the limit the area goes to zero at fixed modulus $\tau$.

A natural question is whether in the limit of weak (or strong) coupling such $SL(2,\mathbb{Z})$-invariant organization (which should follow automatically from eleven dimensions by virtue of reparametrization symmetry) represents an improvement of perturbation theory. It is not obvious that this will be the case, since the new contributions seem to correspond to the exchange of very unstable objects, and some of them may not even exist for a given $g_B \ll 1$. Although improving perturbation theory is not the aim of this work, we nevertheless expect that for any coupling $g_B A^{sl(2)}_4$ is closer to the exact scattering amplitude than what the Virasoro amplitude is: at small coupling $A^{sl(2)}_4$ becomes the Virasoro amplitude plus additional corrections; some of them (momentum 8 term, and the higher derivative terms related by supersymmetry to $H^{4k-4}R^4$) are believed to contain the exact function of the coupling, and the remaining ones have the correct form to be interpreted as genuine perturbative and non-perturbative contributions in superstring theory.

The symmetrization based on $(p,q)$ ‘strings’ may as well be regarded as a trick to incorporate D-instantons in the perturbative series (in addition to incorporating certain parts of higher genus perturbative corrections). Mathematically, one is summing over all possible $SL(2,\mathbb{Z})$ rotations of the tree-level expressions. Physically, we seem to be accounting for the exchange of all possible membrane states surviving in the zero area limit (not just those with oscillations along $x_{10}$).

**Acknowledgements**

I would like to acknowledge the support of the European Commission TMR programme grant ERBFMBI-CT96-0982.
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