FIBRANT RESOLUTIONS FOR MOTIVIC THOM SPECTRA

GRIGORY GARKUSHA AND ALEXANDER NESHITOV

ABSTRACT. Using the theory of framed correspondences developed by Voevodsky [32] and the machinery of framed motives introduced and developed in [13], various explicit fibrant resolutions for a motivic Thom spectrum $E$ are constructed in this paper. It is shown that the bispectrum

$$M^G_E(X) = (M_E(X), M_E(X)(1), M_E(X)(2), \ldots),$$

each term of which is a twisted $E$-framed motive of $X$, introduced in the paper, represents $X_+ \wedge E$ in the category of bispectra. As a topological application, it is proved that the $E$-framed motive with finite coefficients $M_E(pt)(pt)/N$, $N > 0$, of the point $pt = \text{Spec } k$ evaluated at $pt$ is a quasi-fibrant model of the topological $S^2$-spectrum $Re^E/E)/N$ whenever the base field $k$ is algebraically closed of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Furthermore, the algebraic cobordism spectrum $MGL$ is computed in terms of $\Omega$-correspondences in the sense of [21]. It is also proved that $MGL$ is represented by a bispectrum each term of which is a sequential colimit of simplicial smooth quasi-projective varieties.

CONTENTS

1. Introduction 2
2. Preliminaries 6
3. The functor $\Theta^\infty$ and the layer filtration 10
4. The Mayer–Vietoris sequence 11
5. Fibrant replacements of Thom spectra 18
6. The functor $\Theta^\infty_{sym}$ 19
7. The spectrum $C, Fr^E_1(S_T)$ 22
8. Fibrant resolutions of symmetric Thom spectra 26
9. $E$-framed motives and bispectra 27
10. Topological Thom spectra with finite coefficients 33
11. Normally $E$-framed motives 35
12. Computing the algebraic cobordism spectrum $MGL$ 45
Appendix A. Technical lemmas 51
Acknowledgements 54
References 54

2010 Mathematics Subject Classification. 14F42, 55P42.
Key words and phrases. Motivic homotopy theory, motivic Thom spectra, $E$-framed motives.
1. Introduction

Voevodsky [32] introduced framed correspondences in order to suggest a new approach to stable motivic homotopy theory which will be more amenable to explicit computations. In [13] the machinery of (big) framed motives is developed converting the classical Morel–Voevodsky stable motivic homotopy theory into a local theory of framed bispectra and yielding a new model for $SH(k)$ in [14]. A key computation of [13] is to give explicit fibrant resolutions of the suspension spectra/bispectra of smooth algebraic varieties.

The main results of this paper are concentrated around explicit computations of motivic Thom spectra, described below, which play a central role in stable motivic homotopy theory. We use computational miracles of Voevodsky’s framed correspondences to extend the machinery of framed motives that are of crucial importance in [13] to “$E$-framed motives”, where $E$ is a motivic Thom $T$-spectrum.

By definition, $E$ is called a Thom spectrum if every space $E_n$ has the form

$$E_n = \colim E_{n,i}, \quad E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i}),$$

where $V_{n,i} \to V_{n,i+1}$ is a directed sequence of smooth varieties, $Z_{n,i} \to Z_{n,i+1}$ is a directed system of smooth closed subschemes in $V_{n,i}$. We say that a Thom spectrum $E$ has the bounding constant $d$ if $d$ is the minimal integer such that codimension of $Z_{n,i}$ in $V_{n,i}$ is strictly greater than $n - d$ for all $i, n$. If $E$ is also symmetric then it is said to be a spectrum with contractible alternating group action, if for any $n$ and any even permutation $\tau \in \Sigma_n$ there is an $\mathbb{A}^1$-homotopy $E_n \to \Hom(\mathbb{A}^1, E_n)$ between the action of $\tau$ and the identity map. In other words, $E$ neglects the action of even permutations up to $\mathbb{A}^1$-homotopy. The most interesting examples of such symmetric Thom spectra, all of which have the bounding constant $d = 1$, are given by the spectra $MGL$, $MSL$ or $MSp$ (the latter two are regarded as $T^2$-symmetric spectra for which the above definitions remain the same). These Thom spectra are of fundamental importance. If we regard $E$ as a $\mathbb{P}^1$-spectrum, denote by $\Theta^\infty(E)$ the standard stabilization $\colim \Hom(\mathbb{P}^\infty, E[n])$ of $E$. Taking the Suslin complex at each level, we get a $\mathbb{P}^1$-spectrum $C_1 \Theta^\infty(E)$.

Our first computation (see Theorem 5.4) is as follows.

1.1. Theorem. Let $E$ be a Thom spectrum with the bounding constant $d$. Let $C_1 \Theta^\infty(E)^f$ be a spectrum obtained from $C_1 \Theta^\infty(E)$ by taking a level Nisnevich local fibrant replacement. Then the spectrum $C_1 \Theta^\infty(E)^f$ is motivically fibrant starting from level $\max(0, d)$ and is stably equivalent to $E$.

If $E$ is a symmetric $T$-spectrum, then there is another natural stabilization functor $\Theta^\infty_{\text{sym}}(E)$ (see Definition 6.2). It is different from $\Theta^\infty(E)$ and involves actions of certain permutations on $E$. As above, we can take the Suslin complex at each level and form a $\mathbb{P}^1$-spectrum $C_1 \Theta^\infty_{\text{sym}}(E)$.

Given a Thom $T$-spectrum $E$, denote by $\Fr^F_n(X)$ the space $\Fr^F_n(X) = \Hom(\mathbb{P}^\infty, X_+ \wedge E_n)$ and $\Fr^F(X) := \colim \Fr^F_n(X) = X^\infty(X_+ \wedge E)_0$. By the Voevodsky lemma 2.12 $\Fr^F_n(X)$ and $\Fr^F(X)$ have an explicit geometric description. We can similarly define the sheaves $\Fr^F(T^i), i \geq 0$. Altogether they form a $\mathbb{P}^1$-spectrum $\Fr^F(S_T) := (\Fr^F(S^0), \Fr^F(T), \Fr^F(T^2), \ldots)$. As usual, denote by $C_1 \Fr^F(S_T)$ the $\mathbb{P}^1$-spectrum obtained from $\Fr^F(S_T)$ by taking the Suslin complex levelwise.

The next computation (see Theorem 8.1) gives the following fibrant resolutions of $E$ (starting from some level).
1.2. **Theorem.** For a symmetric Thom $T$-spectrum $E$ with the bounding constant $d$ and contractible alternating group action the following $\mathbb{P}^1$-spectra are isomorphic to $E$ in $SH(k)$ and motivically fibrant starting from level $\max(0,d)$:

- $C_\ast \text{Fr}^T(S_T)^f$
- $C_\ast \Theta^m(E)^f$
- $C_\ast \Theta^m_{\text{sym}}(E)^f$,

where “$f$” refers to levelwise Nisnevich local fibrant replacements of the corresponding spectra.

Our next goal is to represent a Thom spectrum $E$ in the category of $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectra and construct an explicit fibrant resolution for it. To this end, we introduce and study in Section 9 $E$-framed motives of smooth algebraic varieties $M_E(X)$, $X \in \text{Sm}_k$. They are defined similarly to framed motives introduced in [13] and are explicit sheaves of $S^1$-spectra.

The main result here (see Theorem 9.13) is as follows.

1.3. **Theorem.** Suppose $X \in \text{Sm}_k$ and $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d$ and contractible alternating group action.

(1) If $d = 1$ then the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum

$$M^E_0(X)_{f} := (M_E(X)_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots)$$

is motivically fibrant and represents the $T$-spectrum $X_+ \wedge E$ in the category of bispectra, where “$f$” refers to stable local fibrant replacements of $S^1$-spectra.

(2) If $d < 1$ then the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum

$$M^E_0(X)_f := (M_E(X)_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots)$$

is motivically fibrant and represents the $T$-spectrum $X_+ \wedge E$ in the category of bispectra, where “$f$” refers to level local fibrant replacements of $S^1$-spectra.

(3) If $d > 1$ then the $(S^1, \mathbb{G}_m^{\wedge 1})$-bispectrum

$$\Omega^{d-1}_{S^1 \wedge \mathbb{G}_m^{\wedge 1}}((M_E[d-1](X)_f, M_E[d-1](X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_E[d-1](X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots))$$

is motivically fibrant and represents the $T$-spectrum $X_+ \wedge E$ in the category of bispectra, where “$f$” refers to stable local fibrant replacements of $S^1$-spectra. Here $E[d-1]$ stands for the $(d-1)$-th shift of $E$ in the sense of Definition 2.6. Another equivalent model for the $T$-spectrum $X_+ \wedge E$ in the category of bispectra is given by

$$\Omega^{d-1}_{S^1 \wedge \mathbb{G}_m^{\wedge 1}}((M_{T^d-1}E(X)_f, M_{T^d-1}E(X_+ \wedge \mathbb{G}_m^{\wedge 1})_f, M_{T^d-1}E(X_+ \wedge \mathbb{G}_m^{\wedge 2})_f, \ldots)).$$

This bispectrum is motivically fibrant and “$f$” refers to stable local fibrant replacements of $S^1$-spectra.

One of the most impressive applications of the theory of framed correspondences and the machinery of framed motives is that they lead to computing explicit fibrant resolutions of classical topological objects in terms of algebraic varieties. These computations are far relatives for the celebrated constructions of Pontrjagin [25] who interpreted homotopy groups of spheres in terms of smooth manifolds. For example, the classical topological sphere spectrum is computed in [13] as the framed motive $M_{f_0}(pt)(pt)$ of the point $pt = \text{Spec} k$ evaluated at the point whenever the base field $k$ is algebraically closed of characteristic zero. We use the preceding theorem.
to get a similar topological application in Theorem 10.3. The main example here concerns the motivic cobordism spectrum $MGL$ whose realization is isomorphic to the topological complex cobordism spectrum $MU$.

1.4. Theorem. Let $k$ be an algebraically closed field of characteristic zero with an embedding $ε : k \hookrightarrow \mathbb{C}$. Suppose $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d \leq 1$ and contractible alternating group action. Then for all integers $N > 1$ and $n \in \mathbb{Z}$, the natural realisation functor $Re^E : SH(k) \to SH$ in the sense of [23] induces an isomorphism

$$\pi_n(ME(pt)(pt); \mathbb{Z}/N) \cong \pi_n(Re^E(E); \mathbb{Z}/N)$$

between stable homotopy groups with mod $N$ coefficients.

Given a motivic Thom spectrum $E$ and $X \in Sm/k$, we fix any group completion $Fr^E(\Delta^*_k, X)^{gp}$ of the space $Fr^E(\Delta^*_k, X)$, which is functorial in $X$. For instance, one can take $Fr^E(\Delta^*_k, X)^{gp} = \Omega_{Fr}Fr^E(\Delta^*_k, X \otimes S^1)$. Put

$$\pi^E_n(X) := \pi_n(Fr^E(\Delta^*_k, X)^{gp})$$

and call $\pi^E_n(X)$ the $n$-th singular algebraic $E$-homotopy group of $X$.

The following result on the singular algebraic $E$-homotopy is an analogue of the celebrated theorem of Suslin and Voevodsky [30] on the singular algebraic homology (see Theorem 10.5):

1.5. Theorem. Suppose $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d \leq 1$ and contractible alternating group action. The assignment $X \mapsto \pi^E_n(X)$ is a generalized homotopy theory on $Sm/\mathbb{C}$. Moreover, passing to homotopy groups with finite coefficients, we get equalities

$$\pi^E_n(X; \mathbb{Z}/m) = \pi_n(X(\mathbb{C})_+ \wedge Re^E(E); \mathbb{Z}/m)$$

for all integers $n \geq 0$ and $m \neq 0$.

Also, the first part of this theorem is true over any perfect field $k$. Namely, the assignment $X \mapsto \pi^E_n(X)$ is a generalized homotopy theory on the category $Sm/k$.

We can simplify $E$-framed motives further by removing a bit of information in the definition of $E$-framed correspondences. In this way we arrive at “normally framed motives $\tilde{M}_E(X)$” (see Definition 11.24). They play a pivotal role in our analysis and – most importantly – lead to explicit computations of the algebraic cobordism spectrum $MGL$ (see below).

We prove the following result (see Theorem 11.26) computing $E$ in terms of normally framed motives.

1.6. Theorem. Suppose $X \in Sm_k$ and $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d = 1$ and contractible alternating group action. Then we have a $(S^1, \mathbb{G}_{m}^{\wedge 1})$-bispectrum

$$\tilde{M}_E^G(X)_f := (\tilde{M}_E(X)_f, \tilde{M}_E(X_+ \wedge \mathbb{G}_{m}^{\wedge 1})_f, \tilde{M}_E(X_+ \wedge \mathbb{G}_{m}^{\wedge 2})_f, \ldots),$$

which is motivically fibrant and represents the $T$-spectrum $X_+ \wedge E$ in the category of bispectra, where “$f$” refers to stable local fibrant replacements of $S^1$-spectra.

The last section is dedicated to further explicit models representing the algebraic cobordism spectrum $MGL$ in the category of bispectra. We first introduce Nisnevich sheaves $Emb_n(-, X) = \text{colim}_n Emb_n(\mathbb{C}_m(X), X) \in Sm_k$, where $Emb_n(U, X)$ is the set of couples $(Z, f)$ such that $Z$ is
we compute \( \pi_0 \mathcal{A} \) of \( \mathcal{A} \). Here "N" refers to the nerve of isomorphisms. In particular, the \((\mathcal{A}, \mathcal{G}_m^{\wedge 1})\)-bispectra

\[
(C_n \text{Emb}(X_+ \wedge \mathbb{S}), C_n \text{Emb}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}), \ldots)
\]

or

\[
(C_n \text{NCor}^\Omega (X_+ \wedge \mathbb{S}), C_n \text{NCor}^\Omega (X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}), \ldots).
\]

Here "N" refers to the nerve of isomorphisms. In particular, the \((\mathcal{A}, \mathcal{G}_m^{\wedge 1})\)-bispectra

\[
(C_n \text{Emb}(X_+ \wedge \mathbb{S})_f, C_n \text{Emb}(X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S})_f, \ldots)
\]

and

\[
(C_n \text{NCor}^\Omega (X_+ \wedge \mathbb{S})_f, C_n \text{NCor}^\Omega (X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S})_f, \ldots)
\]

are motivically fibrant and represent the \( T \)-spectrum \( X_+ \wedge \mathbb{L} \) in the category of bispectra, where "f" refers to stable local fibrant replacements of \( \mathcal{A} \)-spectra.

We finish the paper by the following important computation (see Theorem 12.16) of the algebraic cobordism in terms of smooth quasi-projective varieties. This computation is an application of the preceding theorem.

1.8. Theorem. The \((\mathcal{A}, \mathcal{G}_m^{\wedge 1})\)-bispectrum \( M^G_{\mathcal{M}GL}(X) \) is isomorphic in \( SH(k) \) to a bispectrum

\[
(E^X_+ \wedge \mathbb{S}, E^X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}, \ldots),
\]

each term of which is given by a sequential colimit of simplicial smooth quasi-projective varieties \( E^X_+ \wedge \mathbb{G}_m^{\wedge 1} \wedge \mathbb{S}^i, \ldots \), \( i, j \geq 0 \).

Throughout the paper we denote by \( \text{Sm}_k \) the category of smooth separated schemes of finite type over the base field \( k \). We shall assume that \( k \) is perfect for the reason that the main result of [12] (complemented by [6] in characteristic 2 and by [5, A.27] for finite fields) says that over such fields for any \( \mathbb{A}^1 \)-invariant quasi-stable radditive framed presheaf of Abelian groups \( \mathcal{F} \), the associated Nisnevich sheaf \( \mathcal{F}_{nis} \) is strictly \( \mathbb{A}^1 \)-invariant. By a motivic space we shall mean a pointed simplicial Nisnevich sheaf on \( \text{Sm}_k \). If \( \mathcal{C} \) is a category cotensored over the category of pointed motivic spaces \( \mathcal{M} \), we shall write \( \text{Hom}(A, C) \in \mathcal{C} \) for the cotensor object associated with \( A \in \mathcal{M} \) and \( C \in \mathcal{C} \) unless it is specified otherwise. We choose the flasque local/motivic model structures on motivic spaces (respectively \( \mathcal{S}^1 \)- or \( \mathbb{P}^1 \)-spectra of motivic spaces) in the sense of [17].
Relations to other works. This paper (first appeared in the archive in April 2018) depends on a series of papers on framed motives [3, 10, 12, 13]. Computations of motivic Thom spectra like those of Theorem 9.13 in terms of tangentially framed correspondences as defined in [7] were later obtained in [8]. Our approach is based on Voevodsky’s framed correspondences [32]. Technique developed in Sections 6 and 7 is crucial for the theory of motivic Γ-spaces [15], an extension of the celebrated Segal machine of Γ-spaces [28] to the world of motivic homotopy theory. A systematic study of normally framed correspondences associated with Thom spectra is given in Section 11. This type of correspondences associated with the motivic sphere spectrum is of great utility in [2, 7]. Normally framed correspondences lead to representability of some important motivic Thom spectra like $MGL$ by schemes (see Theorem 12.16). The representability theorem is also proven in [8].

2. Preliminaries

In this section we collect basic facts about spectra and motivic spaces with framed correspondences.

Spectra of Thom type.

2.1. Definition. For every space $X$ denote by $C_X$ its Suslin complex. It is the diagonal of the bisimplicial sheaf $(n, m) \mapsto \text{Hom}(\Delta^m, X)$ where $X_m$ is the sheaf of $m$-simplices of $X$.

2.2. Definition. Given two spaces $X,Y$ and maps $f,g: X \to Y$,

- a simplicial homotopy between $f$ and $g$ is a map $H: X \wedge \Delta[1]_+ \to Y$ such that the composition $H i_0 = f$ and $H i_1 = g$, where $i_0, i_1: X \to X \wedge \Delta[1]_+$ are the face maps;
- an $\mathbb{A}^1$-homotopy between $f$ and $g$ is a map $H: X \to \text{Hom}(\mathbb{A}^1, Y)$ such that $i_0 H = f$ and $i_1 H = g$, where $i_0, i_1: \text{Hom}(\mathbb{A}^1, Y) \to Y$ are maps induced by zero and unit embeddings of $\text{pt}$ into $\mathbb{A}^1$.

2.3. Remark. Every $\mathbb{A}^1$-homotopy $H: X \to \text{Hom}(\mathbb{A}^1, Y)$ between $f$ and $g$ gives rise to a simplicial homotopy $H': C_X \wedge \Delta[1]_+ \to C_Y$ between $f$ and $g$.

2.4. Convention. We shall use the following notation:

- Given two motivic spaces $B$ and $C$, we denote by $\text{tw}$ the twist isomorphism $C \wedge B \xrightarrow{\cong} B \wedge C$.
- For brevity, we shall sometimes write $(A, B)$ to denote $\text{Hom}(A, B)$, where $A$ and $B$ are motivic spaces. We shall use the canonical map
  
  $$(A, B) \wedge C \to (A, C \wedge B)$$

  which is adjoint to

  $$(A, B) \xrightarrow{\text{Can} \wedge \text{id}} \left( C \wedge A, C \wedge B \right) \cong (C, (A, C \wedge B)).$$

  When $C$ and $B$ are distinct spaces we shall often compose the previous map with the twist isomorphism $\text{tw}: C \wedge B \to B \wedge C$ to get the map

  $$(A, B) \wedge C \to (A, B \wedge C).$$


If there is no likelihood of confusion, we shall use the equality sign $P^{\wedge m} \wedge P^{\wedge n} = P^{\wedge n} \wedge P^{\wedge m}$ for the associativity isomorphism

$$P^{\wedge m} \wedge P^{\wedge n} \cong P^{\wedge m+n} \cong P^{\wedge n} \wedge P^{\wedge m}.$$  

For any $m, n$ we shall identify the spaces (via associativity isomorphisms)

$$(P^{\wedge m}, (P^{\wedge n}, X)) = (P^{\wedge m} \wedge P^{\wedge n}, X) = (P^{\wedge n} \wedge P^{\wedge m}, X) = (P^{\wedge n}, (P^{\wedge m}, X)).$$

Let $T$ be the pointed Nisnevich sheaf $A^1/(A^1 - 0)$. A $T$-spectrum is a sequence of spaces $E_n$ together with bonding maps, denoted by $u$. In what follows we work with right spectra, and so each bonding map is a map $u: E_n \wedge T \to E_{n+1}$. Denote by $\Sigma_n$ the $n$th symmetric group. A symmetric $T$-spectrum is a spectrum $E$ together with a left action of $\Sigma_n$ on $E_n$ such that the bonding maps satisfy the relevant equivariance properties.

2.5. **Definition.** Given $\tau \in \Sigma_n$ we shall write $\tau = (\tau(1), \ldots, \tau(n))$. The reader should not confuse this notation with cyclic permutations. For any $n, m$, denote by $\chi_{t, m} \in \Sigma_{n+m}$ the obvious shuffle permutation $\chi_{t, m} = (n + 1, \ldots, n+m, 1, \ldots, n)$.

If we denote by $S_T$ the symmetric motivic sphere $T$-spectrum $(S^0, T, T^2, \ldots)$, then any symmetric $T$-spectrum is a right module over the monoid $S_T$ in the category of symmetric sequences [16, 7.2].

2.6. **Definition.** Given a symmetric $T$-spectrum $E$ and $n \geq 0$, denote by $u_t: T \wedge E_n \to E_{n+1}$ the composition

$$T \wedge E_n \mathrel{\overset{iw}{\to}} E_n \wedge T \mathrel{\overset{u}{\to}} E_{n+1} \mathrel{\overset{\chi_{t, n}}{\to}} E_{1+n}.$$  

Observe that the maps give a map of symmetric $T$-spectra $u_t: T \wedge E \to E[1]$. Here $E[1]$ is the shift symmetric spectrum whose spaces are given by $E[1]_n = E_{1+n}$ with action of $\Sigma_n$ by restriction of the $\Sigma_{1+n}$ action on $E_{1+n}$ along the obvious embedding $\Sigma_n \hookrightarrow \Sigma_{1+n}$ taking $\tau \in \Sigma_n$ to $1 \oplus \tau \in \Sigma_{1+n}$. The structure maps of $E[1]$ are the reindexed structure maps for $E$. In turn, $T \wedge E$ is the suspension spectrum of $E$ whose spaces are defined as $(T \wedge E)_n = T \wedge E_n$. The symmetric group $\Sigma_n$ acts on $T \wedge E_n$ through the given action on $E_n$ and trivially on $T$. Each structure map is the composite

$$(T \wedge E)_n \wedge T \cong T \wedge (E_n \wedge T) \mathrel{\overset{id \wedge u_t}{\longrightarrow}} (T \wedge E)_{n+1}.$$  

2.7. **Definition.** A symmetric spectrum $E$ is said to be a spectrum with contractible alternating group action, if for any $n$ and any even permutation $\tau \in \Sigma_n$ there is an $A^1$-homotopy $E_n \to \text{Hom}(A^1, E_n)$ between the action of $\tau$ and the identity map. In other words, $E$ neglects the action of even permutations up to $A^1$-homotopy.

2.8. **Definition.** A $T$-spectrum $E$ is called a Thom spectrum if every space $E_n$ has the form

$$E_n = \text{colim}_i E_{n,i}, E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i}),$$  

where $V_{n,i} \to V_{n,i+1}$ is a directed sequence of smooth varieties, $Z_{n,i} \to Z_{n,i+1}$ is a directed system of smooth closed subschemes in $V_{n,i}$.

We shall say that a Thom spectrum $E$ has the bounding constant $d$ if $d$ is the minimal integer such that codimension of $Z_{n,i}$ in $V_{n,i}$ is strictly greater than $n - d$ for all $i, n$.  

7
2.9. Example. The suspension spectrum $\Sigma^\infty X_+ = X_+ \wedge S_T$ and the algebraic cobordism spectrum $MGL$ of [31] (see also [22, 24]) are examples of symmetric Thom spectra with the bounding constant 1 and contractible alternating group action.

If $E$ is a Thom $T$-spectrum with the bounding constant $d \geq 0$, then its $n$th shift $E[n]$ (see Definition 2.6) as well as the spectrum $T^n \wedge E$, $n \geq 0$, is a Thom spectrum with the bounding constant $d - n$. In turn, its negative shift $E[-n] = (\ast, \ast, E_0, E_1, \ldots)$ having $E_0$ in the $n$th entry is a Thom $T$-spectrum with the bounding constant $n + d$. By definition, the trivial Thom spectrum $\ast = (\ast, \ast, \ldots)$ has the bounding constant $+\infty$.

In practice we also deal with symmetric Thom $T^2$-spectra like $MSL$ or $MSp$ (see [24] for definitions). We also say that a Thom $T^2$-spectrum $E$ has the bounding constant $d$ if $d$ is the minimal integer such that codimension of $\mathbb{Z}_{n,i}$ in $V_{n,j}$ is strictly greater than $2n - d$ for all $i, n$. The Thom $T^2$-spectra $MSL$ and $MSp$ have the bounding constant $d = 1$.

By construction (see [24]), the action of the symmetric group $\Sigma_n$ on the spaces $MSL_{2n}$ and $MSp_{2n}$ factors through the action of $SL_{2n}$ and $Sp_{2n}$ respectively. Since $SL_{2n}$ and $Sp_{2n}$ are semisimple simply-connected groups, the sets of $k$-points $SL_{2n}(k)$ and $Sp_{2n}(k)$ are generated by the root subgroups $U_{\alpha}(k)$ (see [26]). Since every root subgroup $U_{\lambda}$ is isomorphic to the affine line $A^1_k$, we have that for every element $A$ of $G = SL_{2n}$ or $G = Sp_{2n}$ there exists a map $h: A^1_k \to G$ such that $h(0) = I, h(1) = A$. It follows that $MSL$ and $MSp$ are $T^2$-spectra with contractible alternating group action as well.

2.10. Lemma. Let $\Sigma_n \to GL_n(k)$ be the standard inclusion and let $\tau$ be an even permutation. Then there is an $A^1$-curve $L: A^1_k \to GL_n$ such that $L(0) = I$ is the identity matrix and $L(1) = \tau$.

Proof. Since $\tau$ is even, its image belongs to $SL_n(k)$. Thus it can be written as a product of elementary matrices:

$$\tau = \prod_{l=1}^m e_{i_l,j_l}(\lambda_l), \quad \text{where } 1 \leq i_l, j_l \leq n, \lambda_l \in k, i_l \neq j_l.$$  

Here an elementary matrix $e_{i,j}(\lambda)$ is a matrix with all its diagonal elements equal to 1, $\lambda$ being placed in the $(i,j)$-th entry and zero elsewhere. Then $L(t) = \prod_{l=1}^m e_{i_l,j_l}(t\lambda_l)$ defines a regular map $A^1_k \to GL_n(k)$ with $L(0) = I$ and $L(1) = \tau$. \qed

In order to avoid a heavy presentation, from now on we shall deal with Thom $T$-spectra only. The interested reader will be able to prove the relevant results for Thom $T^2$-spectra as well.

2.11. Definition. There is a functorial fibrant replacement of motivic spaces $X \to X^f$ in the flasque Nisnevich local model structure (e.g. given by controlled fibrant models in the sense of [18, Section 1.2]) such that for any $\mathbb{P}^1$- or $T$- or $S^1$-spectrum $E = (E_0, E_1, \ldots)$ the sequence $E^f = (E_0^f, E_1^f, \ldots)$ can be canonically equipped with a structure of a spectrum and $E \to E^f$ is a map of spectra.

The Voevodsky Lemma. One of the key facts in the theory of framed correspondences is the following lemma of Voevodsky that computes Hom-sets between certain Nisnevich sheaves. Its proof can be found in [13, Section 3].
2.12. Lemma (Voevodsky’s Lemma). For \(X, Y \in \text{Sm}_k\) and a closed subset \(X'\) of \(X\) and open subset \(V\) of \(Y\) the set

\[
\text{Hom}_{\text{Shv}}(X/X', Y/V)
\]

is in a natural bijection with the set of equivalence classes of triples \((U, Z, \phi)\), where \(Z\) is a closed subset of \(X\) disjoint with \(X'\), \(U\) is an étale neighborhood of \(Z\) in \(X\) and \(\phi : U \to Y\) is a regular map such that \(\phi^{-1}(Y - V) = Z\). By definition, two triples \((U, Z, \phi)\) and \((U', Z', \phi')\) are equivalent if \(Z = Z'\) and \(\phi, \phi'\) coincide on some common étale neighbourhood of \(Z\) in \(X\).

2.13. Corollary. For any Thom spectrum \(E\) there is a natural isomorphism of motivic spaces

\[
\text{Hom}(\mathbb{P}^m, E) \cong \text{Hom}(\mathbb{P}^m/\mathbb{P}^m-1, E).
\]

As a consequence, the \(\Sigma_m\)-action on \(\text{Hom}(\mathbb{P}^m, E)\) permuting factors of \(\mathbb{P}^m\) can be extended to an action of \(\text{GL}_m(k)\) (it naturally acts on \(\mathbb{P}^m/\mathbb{P}^m-1\)), and thus for any even permutation \(\tau \in \Sigma_m\) there is an \(\AA^1\)-homotopy between the action of \(\tau\) and the identity map of \(\text{Hom}(\mathbb{P}^m, E)\) by Lemma 2.10.

Proof. By Definition 2.8 \(E_n = \text{colim}_i E_{n,i}\), where \(E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i})\). For any \(X \in \text{Sm}_k\), Lemma 2.12 implies both sets \(\text{Hom}(X_+ \wedge \mathbb{P}^m, E_{n,i})\) and \(\text{Hom}(X_+ \wedge \mathbb{P}^m/\mathbb{P}^m-1, E_{n,i})\) are naturally isomorphic. They are described up to isomorphism of sets (see [13, Section 3]) as the equivalence classes of triples \((U, Z, \phi)\), where \(Z\) is a closed subset of \(\AA^m_X\), finite over \(X\), \(U\) is its étale neighborhood and \(\phi : U \to V_{n,i}\) is such that \(\phi^{-1}(Z_{n,i}) = Z\).

2.14. Definition. Following [13] for \(X, Y \in \text{Sm}_k\) and an open subscheme \(U\) of \(Y\), we set

\[
\text{Fr}_n(X, Y/U) := \text{Hom}_{\text{Shv}}(X_+ \wedge \mathbb{P}^m, (Y/U) \wedge T^n).
\]

\(\text{Fr}_n(X, Y/U)\) is pointed at the empty correspondence or, equivalently, at the zero map. By smashing the elements of \(\text{Fr}_n(X, Y/U)\) with the canonical motivic equivalence \(\sigma : \mathbb{P}^1 \to T\), we get a map of pointed sets \(\text{Fr}_n(X, Y/U) \to \text{Fr}_{n+1}(X, Y/U)\). Denote by

\[
\text{Fr}(X, Y/U) := \text{colim}_n \cdots \to \text{Fr}_n(X, Y/U) \to \text{Fr}_{n+1}(X, Y/U) \to \cdots.
\]

We shall also write \(C, \text{Fr}(Y/U)\) to denote the Suslin complex associated to the Nisnevich sheaf \(X \mapsto \text{Fr}(X, Y/U)\) (see Definition 2.1).

More generally, we can define the sets \(\text{Fr}_n(X, \mathcal{G}) := \text{Hom}_{\text{Shv}}(X_+ \wedge \mathbb{P}^m, \mathcal{G} \wedge T^n)\), \(\text{Fr}(X, \mathcal{G})\) for every pointed Nisnevich sheaf \(\mathcal{G}\) as well as the Suslin complex \(C, \text{Fr}(\mathcal{G})\) associated to the Nisnevich sheaf \(X \mapsto \text{Fr}(X, \mathcal{G})\).

Below we shall often deal with sheaves of the form

\[
\text{Hom}(\mathbb{P}^m, \mathcal{F}_n(Y/U)), \quad i, n \geq 0.
\]

By Voevodsky’s Lemma its value at \(X \in \text{Sm}_k\) consists of the triples \((W, Z, \phi)\), where \(Z\) is a closed subset of \(\AA^{m+n}_X\), finite over \(X\), \(W\) is its étale neighborhood and \(\phi : W \to Y\) is such that \(\phi^{-1}(Y - U) = Z\).

2.15. Proposition (Additivity Theorem). Suppose \(X, X' \in \text{Sm}_k\) and \(Z, Z'\) are closed subsets of \(X\) and \(X'\) respectively. Denote \(Y = X/(X - Z), Y' = X'/(X' - Z')\). Then for every \(i \geq 0\) the canonical map

\[
\text{Hom}(\mathbb{P}^m, C, \text{Fr}(Y \cup Y')) \to \text{Hom}(\mathbb{P}^m, C, \text{Fr}(Y)) \times \text{Hom}(\mathbb{P}^m, C, \text{Fr}(Y'))
\]
is a schemewise weak equivalence.

Proof. The proof is like that of the Additivity Theorem of [13]. \qed

2.16. Corollary. Let $\Gamma^\mathbb{P}$ be the category of finite pointed sets and pointed maps. Under the notation of Proposition 2.15 the association

$$K \in \Gamma^\mathbb{P} \mapsto \text{Hom}(\mathbb{P}^\wedge_i, \text{C}_s \text{Fr}(Y \wedge K))$$

is a special $\Gamma$-space in the sense of Segal [28]. As a result, the Segal $S^1$-spectrum

$$\text{Hom}(\mathbb{P}^\wedge_i, M_{fr}(Y)) = \text{Hom}(\mathbb{P}^\wedge_i, \text{C}_s \text{Fr}(Y \wedge \mathbb{S}))$$

is sectionwise positively fibrant. Here $\mathbb{S} = (S^0, S^1, S^2, \ldots)$ is the sphere spectrum and $M_{fr}(Y) := C_s \text{Fr}(Y \wedge \mathbb{S})$ is the framed motive of $Y$ in the sense of [13].

3. The functor $\Theta^\omega$ and the layer filtration

If there is no likelihood of confusion, we shall often regard $T$-spectra as $\mathbb{P}^1$-spectra by means of the canonical motivic equivalence $\sigma : \mathbb{P}^\wedge \rightarrow T$. Given a $T$-spectrum $E$, denote by $u^* : E_i \rightarrow \text{Hom}(\mathbb{P}^\wedge_i, E_{i+1})$ the adjoint to the bonding map. Following Jardine [18, §2], we give the following definition.

3.1. Definition. Denote by $E \wedge T$ the fake $T$-suspension spectrum with terms $(E \wedge T)_i = E_i \wedge T$ and bonding maps given by

$$(E_i \wedge T) \wedge T \xrightarrow{u \wedge T} E_{i+1} \wedge T.$$  

It is important to note that the bonding maps do not permute two copies of $T$ on the left. Denote by $\Omega^E_i$ the fake loop $\mathbb{P}^1$-spectrum with terms $\Omega^E_i = \text{Hom}(\mathbb{P}^1, E_i)$ and bonding maps adjoint to

$$\text{Hom}(\mathbb{P}^\wedge_i, E_i) \xrightarrow{\text{Hom}(\mathbb{P}^\wedge_i, \text{Hom}(\mathbb{P}^\wedge_i, E_{i+1}))).$$

We notice again that two copies of $\mathbb{P}^1$ on the right are not permuted.

3.2. Definition. We denote by $E[1]$ the shifted $T$-spectrum $E[1]_i = E_{i+1}$. Its bonding maps $u: E_i \wedge T \rightarrow E_{i+1}$ induce a map of $T$-spectra $u: E \wedge T \rightarrow E[1].$

3.3. Definition. Denote by $\Theta^1(E) = \Omega^1(E[1])$. By adjointness there is a canonical map of $\mathbb{P}^1$-spectra

$$E \rightarrow \Omega^E \rightarrow \Omega^1(E[1]) = \Theta^1(E).$$

Denote by $\Theta^E_i$ the $n$-fold composition $\Theta^1(\Theta^1 \ldots (E))$. There are natural stabilization maps $\Theta^E \rightarrow \Theta^{n+1}_E$ and $\Theta^\omega(n)$ denotes the colimit

$$\Theta^\omega(E) = \text{colim}_n \Theta^E.$$

3.4. Remark. We shall need the following explicit description of spaces of the $\mathbb{P}^1$-spectrum $\Theta^\omega E$ and its bonding maps. The $j$th space equals

$$\Theta^\omega_j(E) = \text{Hom}(\mathbb{P}^\wedge_{n}, E_{j+n}).$$

The bonding maps of $\Theta^\omega(E)$ are adjoint to

$$(\mathbb{P}^\wedge_n, E_{n+j}) \xrightarrow{u_*} (\mathbb{P}^\wedge_n, \mathbb{P}^\wedge_i, E_{n+j+1})) = (\mathbb{P}^\wedge_{n+1}, E_{n+j+1}) = (\mathbb{P}^\wedge_{n+1}, E_{n+j+1}).$$
One should note that we do not permute copies of $\mathbb{P}^{\wedge 1}$ here. The stabilization map $\Theta^\ell(E)_j \to \Theta^{\ell+1}(E)_j$ can be described as the composite map

$$\operatorname{Hom}(\mathbb{P}^{\wedge n}, E_{j+n}) \xrightarrow{\sim \cup} \operatorname{Hom}(\mathbb{P}^{\wedge n} \wedge \mathbb{P}^{\wedge 1}, E_{j+n} \wedge T) \xrightarrow{\cup} \operatorname{Hom}(\mathbb{P}^{\wedge n+1}, E_{j+n+1}).$$

Here the left arrow smashes the simplices of the left space with $\sigma : \mathbb{P}^{\wedge 1} \to T$.

3.5. Lemma. For any spectrum $E$ the adjoint of each bonding map

$$\Theta^\ell(E)_i \to \operatorname{Hom}(\mathbb{P}^{\wedge 1}, \Theta^\ell(E)_{i+1})$$

in the spectrum $\Theta^\ell E$ is an isomorphism.

Proof. For every $n$ and $i$ the adjoint of the bonding map

$$\Theta^\ell(E)_i \to \operatorname{Hom}(\mathbb{P}^{\wedge 1}, \Theta^\ell(E)_{i+1}) = \operatorname{Hom}(\mathbb{P}^{\wedge n+1}, E_{i+n+1}) = \Theta^{\ell+1}(E)_i$$

coincides with the stabilization map $\Theta^\ell(E)_i \to \Theta^{\ell+1}(E)_i$. Thus we get an isomorphism of sequences $\Theta^{\ell+1}(E)_i$ and $\operatorname{Hom}(\mathbb{P}^{\wedge 1}, \Theta^\ell(E)_{i+1})$. \hfill $\square$

3.6. Definition. Given a $T$-spectrum $E$, define its $n$-th layer $L_n(E)$ as the $T$-spectrum

$$L_nE = (E_0, E_1, \ldots, E_n, E_n \wedge T, E_n \wedge T^2, \ldots).$$

The maps of spaces $E_n \wedge T^i = E_n \wedge T \wedge T^{i-1}$ induce maps of spectra $L_nE \to L_{n+1}E$ and an obvious isomorphism of spectra $E \cong \operatorname{colim}_n L_nE$.

We recall the following lemma from [13, Section 13]. It says that $\Theta^\ell$ converts $T$-spectra into framed $\mathbb{P}^{\wedge}$-spectra, i.e. spectra whose spaces are spaces with framed correspondences.

3.7. Lemma. For any $T$-spectrum $E$ there is a canonical isomorphism of $\mathbb{P}^{\wedge}$-spectra

$$\Theta^\ell E = \operatorname{colim}_n \Theta^\ell(L_nE).$$

Moreover, there is a canonical isomorphism of spaces $\Theta^\ell(L_nE)_i = \operatorname{Hom}(\mathbb{P}^{\wedge n}, \operatorname{Fr}(E_n \wedge T^i))$.

Proof. Note that $\Theta^\ell(E) = \operatorname{colim}_n \Theta^\ell(L_nE)$ for every $m$. Thus passing to the colimit over $m$, we get the first isomorphism. For the second statement note that the left hand side is the colimit over $m$ of the sequence $(\mathbb{P}^{\wedge m+n}, E_n \wedge T^{m+i}) \to (\mathbb{P}^{\wedge m+n} \wedge \mathbb{P}^{\wedge 1}, E_n \wedge T^{m+i} \wedge T)$. Thus the identification $(\mathbb{P}^{\wedge m+n}, E_n \wedge T^{m+i}) = (\mathbb{P}^{\wedge n}, (\mathbb{P}^{\wedge m}, E_n \wedge T^i \wedge T))$ (we do not permute copies of $T$ here) provides its isomorphism with the sequence $(\mathbb{P}^{\wedge n}, \operatorname{Fr}(E_n \wedge T^i))$. \hfill $\square$

4. The Mayer–Vietoris sequence

4.1. Definition. Suppose $X \in \mathbf{Sm}_k$ and $Z$ is a smooth closed subvariety of codimension $d$. We say that the embedding $Z \to X$ is trivial if there is an étale map $\alpha : X \to \mathbb{A}^{n+d}$ such that $Z = X \times_{\alpha^{-1}d} \mathbb{A}^n$, where $\mathbb{A}^n \to \mathbb{A}^{n+d}$ is the standard linear embedding.

4.2. Lemma. For every closed embedding of smooth varieties $Z \to X$ of codimension $d$ there is an open cover of $X$ by $X_i$ such that the inclusion $X_i \cap Z \to X_i$ is trivial.
Lemma. 4.4. Suppose \( G \) is a strictly homotopy invariant Nisnevich sheaf, then for any bounded chain complex of presheaves \( X \) there is an isomorphism of presheaves \( \text{Hom}_{D_{\text{Nis}}}((\text{Tot}(C_*(X)))_{\text{Nis}},G[n]) \cong \text{Hom}_{D_{\text{Nis}}}((X)_{\text{Nis}},G[n]) \).

Proof. Consider the stupid truncation \( \sigma_{\geq i}X \). Then there is a short exact sequence of complexes of presheaves

\[
0 \to C_*X_i[-i] \to \text{Tot}(C_*(\sigma_{\geq i}X)) \to \text{Tot}(C_*(\sigma_{\geq i+1}X)) \to 0
\]

Note that \( \text{Hom}_{D_{\text{Nis}}}((C_*X_i)_{\text{Nis}},G[n]) \cong \text{Hom}_{D_{\text{Nis}}}((X_i)_{\text{Nis}},G[n]) \) by [20, Prop. 12.19], and \( \sigma_{\geq N_0}X = 0 \) and \( \sigma_{\geq N_1}X = X \) for some \( N_0,N_1 \). Then the statement follows by induction.

4.5. Lemma. Suppose \( F \) is a bounded complex of \( \mathbb{Z}F_\ast \)-presheaves such that \( F_{\text{Nis}} \) is quasi-isomorphic to zero and the homology presheaves \( H_i(\text{Tot}(C_*(F))) \) are quasi-stable. Then the complex of sheaves \( (\text{Tot}(C_*(F)))_{\text{Nis}} \) is locally quasi-isomorphic to zero.

Proof. The presheaves \( H_i(\text{Tot}(C_*(F))) \) are quasi-stable and homotopy invariant. By [12] the associated sheaves \( H_i = H_i(\text{Tot}(C_*(F)))_{\text{Nis}} \) are strictly homotopy invariant, and hence by Lemma 4.4 there is an isomorphism

\[
\text{Hom}_{D_{\text{Nis}}}((\text{Tot}(C_*(F)))_{\text{Nis}},H_i[n]) \cong \text{Hom}_{D_{\text{Nis}}}((F_{\text{Nis}},H_i[n]) = 0.
\]

The inductive argument as in the proof of [20, 13.12] gives a map \( (\text{Tot}(C_*(F)))_{\text{Nis}} \to H_i[i] \) inducing an isomorphism on homology sheaves. It is zero by the above arguments, hence \( H_i = 0 \).
4.6. Lemma. The maps $Y_{12} \to Y_1, Y_{12} \to Y_2$ are injective and the sheaf $Y$ is the pushout of the diagram $Y_1 \leftarrow Y_{12} \to Y_2$.

Proof. For any Henselian local scheme $U$ the map

$$Y_{12}(U) = X_{12}(U)/(X_1 - Z_{12})(U) \to X_1(U)/(X_1 - Z_1)(U) = Y_1(U)$$

is injective, because $(X_1 - Z_{12})(U) = X_{12}(U) \cap (X_1 - Z_1)(U)$. Similarly, the map $Y_{12}(U) \to Y_2(U)$ is injective. Note that $Y(U) = X(U)/(X - Z)(U)$, $X(U) = X_1(U) \cup_{X_{12}(U)} X_2(U)$ and $(X - Z)(U) = (X_1 - Z_1)(U) \cup_{(X_1 - Z_{12})(U)} (X_2 - Z_2)(U)$. Hence $Y(U) = Y_1(U) \cup_{Y_{12}(U)} Y_2(U)$. □

4.7. Definition. Let $F^{P^{i,j}}(U, Y), U \in \text{Sm}_k$, be the set of $x \in \text{Hom}(\mathbb{P}^{i,j}, \text{Fr}(Y))(U)$ such that the support of $x$ is connected. The free abelian group generated by $F^{P^{i,j}}(U, Y)$ is denoted by $\mathbb{Z}F^{P^{i,j}}(U, Y)$. Then $\mathbb{Z}F^{P^{i,j}}(U, Y)$ is functorial in $U$. Moreover, $\mathbb{Z}F^{P^{i,j}}(-, Y)$ is a Nisnevich sheaf.

The following result gives an explicit computation of homology of the motivic $S^1$-spectrum $\text{Hom}(\mathbb{P}^{i,j}, M_{fr}(Y))$.

4.8. Lemma. There are isomorphisms of graded presheaves

$$\pi_*((\mathbb{Z}\text{Hom}(\mathbb{P}^{i,j}, M_{fr}(Y)))) = H_*(C_*\mathbb{Z}F^{P^{i,j}}(Y))$$

for all $i \geq 0$.

Proof. The proof repeats the proof in [10, 1.2] word for word. □

4.9. Lemma. For any $i \geq 0$, the natural maps $Y_{12} \to Y_2, Y_{12} \to Y_1, Y_1 \to Y, Y_2 \to Y$ give rise to a short exact sequence of Nisnevich sheaves

$$0 \to \mathbb{Z}F^{P^{i,j}}(Y_{12}) \to \mathbb{Z}F^{P^{i,j}}(Y_1) \oplus \mathbb{Z}F^{P^{i,j}}(Y_2) \to \mathbb{Z}F^{P^{i,j}}(Y) \to 0.$$

Proof. Let $U$ be a local Henselian scheme. There is a coequalizer diagram of pointed sets

$$F^{P^{i,j}}(U, Y_{12}) \rightrightarrows F^{P^{i,j}}(U, Y_2) \cup F^{P^{i,j}}(U, Y_1) \to F^{P^{i,j}}(U, Y).$$

Thus it gives rise to a right exact sequence

$$\mathbb{Z}F^{P^{i,j}}(Y_{12}) \to \mathbb{Z}F^{P^{i,j}}(Y_1) \oplus \mathbb{Z}F^{P^{i,j}}(Y_2) \to \mathbb{Z}F^{P^{i,j}}(Y) \to 0.$$

It remains to note that the latter sequence is also exact on the left. □

4.10. Corollary. The cone of the morphism of complexes

$$C_*\mathbb{Z}F^{P^{i,j}}(Y_{12}) \to C_*\mathbb{Z}F^{P^{i,j}}(Y_1) \oplus C_*\mathbb{Z}F^{P^{i,j}}(Y_2)$$

is locally quasi-isomorphic to the complex $C_*\mathbb{Z}F^{P^{i,j}}(Y))$. In particular, we have a triangle in the derived category of complexes of sheaves

$$C_*\mathbb{Z}F^{P^{i,j}}(Y_{12}) \to C_*\mathbb{Z}F^{P^{i,j}}(Y_1) \oplus C_*\mathbb{Z}F^{P^{i,j}}(Y_2) \to C_*\mathbb{Z}F^{P^{i,j}}(Y).$$

Proof. Note that homology presheaves of $C_*\mathbb{Z}F^{P^{i,j}}(Y)$ are quasi-stable. Then by Lemmas 4.9 and 4.5 the totalization of the bicomplex

$$0 \to C_*\mathbb{Z}F^{P^{i,j}}(Y_{12}) \to C_*\mathbb{Z}F^{P^{i,j}}(Y_1) \oplus C_*\mathbb{Z}F^{P^{i,j}}(Y_2) \to C_*\mathbb{Z}F^{P^{i,j}}(Y) \to 0$$

is locally quasi-isomorphic to zero. □
4.11. **Proposition** (The Mayer–Vietoris sequence). For every \( i \geq 0 \) the square of \( S^1 \)-spectra

\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^\Lambda, M_{fr}(Y_{12})) & \longrightarrow & \text{Hom}(\mathbb{P}^\Lambda, M_{fr}(Y_1)) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^\Lambda, M_{fr}(Y_2)) & \longrightarrow & \text{Hom}(\mathbb{P}^\Lambda, M_{fr}(Y))
\end{array}
\]

is a homotopy pushout square in the local stable model structure of \( S^1 \)-spectra.

**Proof.** The natural map from the cone of the morphism

\[
\text{Hom}(\mathbb{P}^\Lambda, M_{fr}(Y_{12})) \to \text{Hom}(\mathbb{P}^\Lambda, M_{fr}(Y_1) \vee M_{fr}(Y_2))
\]

becomes locally an equivalence on homology between connective spectra by Corollary 4.10. Then it is a local stable equivalence. \(\square\)

For any space \( A \), there is an obvious map \( \text{Fr}_n(A) \to \text{Hom}(B, \text{Fr}_n(A \wedge B)) \) defined by \( (\mathbb{P}^\Lambda, A \wedge T^n) \to (\mathbb{P}^\Lambda, A \wedge B \wedge T^n) \). It gives rise to a map of spectra \( M_{fr}(A) \to \text{Hom}(B, M_{fr}(A \wedge B)) \).

4.12. **Lemma.** For any \( X \in \text{Sm}_k \) for \( j \geq 1 \) the map

\[
C_* \text{Fr}_n(X_+ \wedge T^j) \to \text{Hom}(\mathbb{P}^\Lambda, C_* \text{Fr}_n(X_+ \wedge T^j \wedge T^i))
\]

is a local weak equivalence for any \( i \geq 0 \).

**Proof.** The map in question is obtained as the colimit of the maps

\[
C_* \text{Fr}_n(X_+ \wedge T^j) \to \text{Hom}(\mathbb{P}^\Lambda, C_* \text{Fr}_n(X_+ \wedge T^j \wedge T^i)). \tag{4.13}
\]

Consider the triangle

\[
\begin{array}{ccc}
C_* \text{Hom}(\mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda, X_+ \wedge T^j \wedge T^i) & \to & C_* \text{Hom}(\mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda, X_+ \wedge T^j \wedge T^i) \\
\downarrow & & \downarrow \cong \\
C_* \text{Hom}(\mathbb{P}^\Lambda, X_+ \wedge T^j \wedge T^i)
\end{array}
\]

where the vertical map is the map (4.13), the skew map is the isomorphism given by identification \( \mathbb{P}^\Lambda = \mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda \) and \( T^n = T^j \wedge T^i \wedge T^i \), and the horizontal map is induced by the stabilization map \( (\mathbb{P}^\Lambda, C_* \text{Fr}_n(X_+ \wedge T^j)) \to (\mathbb{P}^\Lambda, C_* \text{Fr}_n(X_+ \wedge T^i)) \). The composite map of the triangle differs from the left vertical map by the shuffle permutation action \( \chi_{n,i,j} \) on \( \mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda \wedge \mathbb{P}^\Lambda \) and on \( T^j \) respectively. Thus if \( n \) is even then the triangle is commutative up to a simplicial homotopy by Corollary 2.13 and Remark 2.3. Note that the horizontal map induces an isomorphism on the colimit over \( n \). Thus the vertical map induces a bijection on the colimits of sheaves \( \pi_n \). For \( j \geq 1 \) the space \( C_* \text{Fr}(X_+ \wedge T^j) \) is locally connected by [10, 8.1]. The space \( \text{Hom}(\mathbb{P}^\Lambda, C_* (\text{Fr}(X_+ \wedge T^j \wedge T^i))) \) is isomorphic to \( C_* (\text{Fr}(X_+ \wedge T^j)) \) by means of the horizontal map, and hence it is locally connected as well. We see that the vertical map induces a local weak equivalence. \(\square\)

4.14. **Lemma.** Suppose \( Z \to X \) is a closed embedding of smooth varieties of codimension \( d \). Then for \( i < d \) the space \( \text{Hom}(\mathbb{P}^\Lambda, (C_* \text{Fr}(X / X - Z))) \) is locally connected.
Proof. If $U$ is a local Henselian scheme, then every correspondence $c$ in $\text{Hom}(\mathbb{P}^n, \text{Fr}(X/X - Z)) = \text{colim}_n \text{Hom}(\mathbb{P}^n, (X/X - Z) \wedge T^n)$ can be described by triples $c = (S, U, \phi)$, where the support $S$ is a closed subset of $\mathbb{A}^{n+1}_U$, finite over $U$, and $\phi : U \to X \times \mathbb{A}^{n}$ is a regular map from an étale neighborhood of $S$ such that $\phi = (Z \times 0)$ (see Voevodsky’s Lemma 2.12 and [13, Section 3] for details). Since $S$ is finite over Henselian $U$, it is a disjoint union of local schemes $S_j$, finite over $U$, for $j = 1, \ldots, l$. Each map $S_j \to Z$ factors through $S_j \to Z_j$, where $Z_j = X_j \cap Z$ for some open $X_j$ in $X$ and such that $Z_j \to X_j$ is the trivial embedding. Thus the correspondence $c$ lies in the image of $\text{Hom}(\mathbb{P}^n, \text{Fr}(\bigwedge_j (X_j/X_j - Z_j))) = \text{Hom}(\mathbb{P}^n, \text{Fr}(\bigwedge_j (Z_j \wedge T^n))) = \text{Fr}(\bigwedge_j (Z_j \wedge T^n))$, and $\pi_0^{\text{nis}}(\mathbb{P}^n, \text{Fr}(\bigwedge_j (Z_j \wedge T^n))) = \pi_i^{\text{nis}}(\mathbb{P}^n, \text{Fr}(\bigwedge_j (Z_j \wedge T^n))) = 0$ for $i < d$ by [10, A.1]. Since the class of $c \in \pi_0^{\text{nis}}(\text{Hom}(\mathbb{P}^n, (C, \text{Fr}(X/X - Z))))$ belongs to the image of $\pi_0^{\text{nis}}(C, \text{Fr}(\bigwedge_j (Z_j \wedge T^n))) = 0$, then $c$ equals the class of the basepoint of $\text{Hom}(\mathbb{P}^n, (C, \text{Fr}(X/X - Z)))$. We conclude that $\pi_i^{\text{nis}}(\text{Hom}(\mathbb{P}^n, (C, \text{Fr}(X/X - Z)))) = 0$.

4.15. Lemma. Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension $d$. Then for $i < d$ the $S^1$-spectrum $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$ is locally an $\Omega$-spectrum and the $S^1$-spectrum $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$, obtained from $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$, is a local fibrant replacement levelwise, is motivically fibrant. In particular, the motivic space $\text{Hom}(\mathbb{P}^n, (C, \text{Fr}(X/X - Z)))$ is motivically fibrant.

Proof. It follows from Additivity Theorem 2.15, Corollary 2.16 and Lemma 4.14 that the $\Gamma$-space taking a finite pointed set $K$ to $\text{Hom}(\mathbb{P}^n, (C, \text{Fr}(X/X - Z) \wedge K))$ is locally very special. By the Segal machine [28] $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z)) = \text{Hom}(\mathbb{P}^n, (C, \text{Fr}(X/X - Z)) \wedge S^n)$ is locally an $\Omega$-spectrum, and hence so is $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$. Since all spaces of $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$ are locally fibrant, we see that $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$ is sectionwise an $\Omega$-spectrum. Since the sheaves of homotopy groups of $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$ are strictly homotopy invariant by [12, 1.1], $\text{Hom}(\mathbb{P}^n, M_{fr}(X/X - Z))$ is motivically fibrant by [13, 7.1].

4.16. Corollary. For a Thom spectrum $E$ with the bounding constant $d$, $\text{Hom}(\mathbb{P}^n, (C, \text{Fr}(E_\infty \wedge T^n)))$ is a motivically fibrant space for $i \geq 0, d$.

4.17. Lemma. Given $i, n \geq 0$, the natural map of $S^1$-spectra

$$M_{fr}(X^+ \wedge T^n) \to \text{Hom}(\mathbb{P}^n, M_{fr}(X^+ \wedge T^n \wedge T^i))$$

is a levelwise local weak equivalence in positive degrees, where “$f$” refers to a levelwise local fibrant replacement. In particular, the map is a stable local weak equivalence. If $n > 0$ then this map is a levelwise local weak equivalence of spectra in all degrees.

Proof. The statement of the lemma can be reformulated as follows for $n \geq 0$: the map of $S^1$-spectra

$$M_{fr}(X^+ \wedge S^1 \wedge T^n) \to \text{Hom}(\mathbb{P}^n, M_{fr}(X^+ \wedge S^1 \wedge T^n \wedge T^i))$$

is a levelwise local weak equivalence. The spectra $M_{fr}(X^+ \wedge S^1 \wedge T^n), M_{fr}(X^+ \wedge S^1 \wedge T^n \wedge T^i)$ are both motivically fibrant by [13, 7.5].

The proof of [13, 4.1(2)] shows that the map in question is a levelwise local weak equivalence if so is the map

$$M_{fr}(X^+ \wedge S^1 \wedge T^n) \to \text{Hom}(\mathbb{G}_m^{\wedge n} \wedge S^i, M_{fr}(X^+ \wedge S^1 \wedge T^n \wedge \mathbb{G}_m^{\wedge n} \wedge S^i)),$$
where $G_m^{\wedge 1}$ is the mapping cone of $pt_+ \to (G_m)_+$ sending $pt$ to $1 \in G_m$ and $G_m^{\wedge 1}$ is the $i$th smash product of $G_m^{\wedge 1}$. Our assertion now follows from the Cancellation Theorem for framed motives [3]. The same arguments apply to show that the map

$$M_{fr}(X_+ \wedge T^n)_f \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(X_+ \wedge T^n \wedge T^i)_f)$$

is a levelwise local weak equivalence in all degrees for $n > 0$. □

4.18. Proposition. Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension $d$ and $M_{fr}(X/(X-Z))_f$ is obtained from $M_{fr}(X/(X-Z))$ by taking a level local fibrant replacement. Then

$$\text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(X/(X-Z)))_f \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(X/(X-Z))_f)$$

is a levelwise local weak equivalence of $S^1$-spectra for $i < d$. In particular, the right spectrum is a fibrant replacement of the left spectrum in the stable motivic model structure of $S^1$-spectra whenever $i < d$. If $i = d$ then the map is a levelwise local weak equivalence in positive degrees. In particular, the map is a stable local weak equivalence for $i = d$.

Proof. Suppose $i < d$. By Lemma 4.2 there is a cover of $X$ by open subsets $X_j$ such that $X_j \cap Z \to X_j$ is a trivial embedding. We proceed by induction on $n$, the number of elements in the cover. For $n = 1$ we have $X/(X-Z) \cong Z_+ \wedge T^d$ by Lemma 4.3. Then the map in question fits into a commutative square

$$\text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Z_+ \wedge T^d)) \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Z_+ \wedge T^d)_f)$$

$$\simeq \quad \simeq$$

$$M_{fr}(Z_+ \wedge T^{d-i}) \to M_{fr}(Z_+ \wedge T^{d-i})_f$$

The left arrow is a levelwise local weak equivalence by Lemma 4.12, and the right arrow is a levelwise local weak equivalence by Lemma 4.17. Thus the upper map is a levelwise local weak equivalence.

For the induction step present $X$ as the union of $X_1$ and $X_2$ such that $X_1$ can be covered by $n - 1$ trivial open pieces, and $Z \cap X_2 \to X_2$ is a trivial embedding. Then for $X_{12} = X_1 \cap X_2$ the embedding $Z \cap X_{12} \to X_{12}$ is trivial. Denote by $Y$ the sheaf $X/X-Z$ and by $Y_i$ the sheaf $X_i/(X_i - (X_i \cap Z))$. Consider a commutative diagram of $S^1$-spectra

$$\text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y_{12})) \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y_1) \vee \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y_2))) \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y))$$

$$\text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y_{12})_f) \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y_1)_f \vee \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y_2)_f)) \to \text{Hom}(\mathbb{P}^{\wedge 1}, M_{fr}(Y)_f)$$

The upper row is a homotopy cofiber sequence in the local stable model structure of $S^1$-spectra by Proposition 4.11. By Lemma 4.15 $M_{fr}(Y_{12})_f, M_{fr}(Y_1)_f, M_{fr}(Y_2)_f, M_{fr}(Y)_f$ are motivically fibrant. It follows from Proposition 4.11 that the sequence

$$M_{fr}(Y_{12})_f \to M_{fr}(Y_1)_f \vee M_{fr}(Y_2)_f \to M_{fr}(Y)_f$$

is a homotopy cofiber of motivically fibrant spectra in the local stable model structure, and hence so is the lower sequence of the commutative diagram above, because $\mathbb{P}^{\wedge 1}$ is a flasque.
is also satisfied for spaces of the form \( \mathbb{P}^d \). Then the space \( S \) is a stable local weak equivalence of spectra. By Corollary 2.16 the left spectrum is sectionwise an \( \Omega \)-spectrum. Since a stable equivalence between \( \Omega \)-spectra is a levelwise weak equivalence, it follows that the map of spectra is a levelwise local weak equivalence. Therefore, the map

\[
\text{Hom}(\mathbb{P}^d, M_f)(Y) \to \text{Hom}(\mathbb{P}^d, M_f)(Y)_f
\]

is a levelwise weak equivalence of spectra. By Corollary 2.16 the left spectrum is sectionwise an \( \Omega \)-spectrum. Since a stable equivalence between \( \Omega \)-spectra is a levelwise weak equivalence, it follows that the map of spectra is a levelwise local weak equivalence. Therefore, the map

\[
\text{Hom}(\mathbb{P}^d, M_f)(Y) \to \text{Hom}(\mathbb{P}^d, M_f)(Y)_f
\]

is a levelwise weak equivalence in positive degrees. □

4.19. Theorem. Suppose \( Z \to X \) is a closed embedding of smooth varieties of codimension \( d \). Then the space \( \text{Hom}(\mathbb{P}^d, C, \text{Fr}(X \times Z)_f) \) is motivically fibrant and

\[
\text{Hom}(\mathbb{P}^d, C, \text{Fr}(X \times Z)) \to \text{Hom}(\mathbb{P}^d, C, \text{Fr}(X \times Z)_f)
\]

is a local weak equivalence for \( i < d \).

Proof. The statement follows from Proposition 4.18. □

4.20. Corollary. If \( E \) is a Thom spectrum with the bounding constant \( d \), then the motivic space \( \text{Hom}(\mathbb{P}^m, C, \text{Fr}(E_n \times T^i)_f) \) is motivically fibrant and

\[
\text{Hom}(\mathbb{P}^m, C, \text{Fr}(E_n \times T^i)) \to \text{Hom}(\mathbb{P}^m, C, \text{Fr}(E_n \times T^i)_f)
\]

is a local weak equivalence for \( m \leq n + i - d \).

Proof. We have \( E_n \times T^i = \colim V_{n,j} \times \mathbb{A}^{j}/(V_{n,j} \times \mathbb{A}^{j} - Z_{n,j} \times 0) \), where codimension of \( Z_{n,j} \) in \( V_{n,j} \) is strictly greater than \( n - d \). Then codimension of \( Z_{n,j} \times 0 \) in \( V_{n,j} \times \mathbb{A}^{j} \) is strictly greater than \( n + i - d \). Then for \( E_{n,j} = V_{n,j}/(V_{n,j} - Z_{n,j}) \) we get that \( \text{Hom}(\mathbb{P}^m, C, \text{Fr}(E_{n,j} \times T^i)_f) \) is motivically fibrant and the map \( \text{Hom}(\mathbb{P}^m, C, \text{Fr}(E_{n,j} \times T^i)) \to \text{Hom}(\mathbb{P}^m, C, \text{Fr}(E_{n,j} \times T^i)_f) \) is a local weak equivalence for every \( j \) by Theorem 4.19. By passing to the colimit and using the fact that a directed colimit of flasque motivically fibrant spaces (respectively a directed colimit of local weak equivalences) is flasque motivically fibrant, we get the statement of the lemma. □
5. Fibrant replacements of Thom spectra

In this section we give a model for a fibrant replacement of a Thom spectrum $E$. First we need the following.

5.1. **Lemma.** Suppose $E$ is a Thom spectrum with the bounding constant $d$. Then for $i \geq \max(0, d)$ and $n \geq 0$ the map of spaces $\text{Hom}(\mathbb{P}^\wedge 1, C_i \Theta^\omega(L_n E)_{i+1}) \to \text{Hom}(\mathbb{P}^\wedge 1, C_i \Theta^\omega(L_n E)_{i+1}^f)$ is a local weak equivalence, where $L_n E$ is the $n$-th layer of $E$ and $(L_n E)_{i+1}^f$ is a local fibrant replacement of the space $(L_n E)_{i+1}$.

**Proof.** By Corollary 4.20 the space $\text{Hom}(\mathbb{P}^\wedge n, C_i Fr(E_n \wedge T^{i+1})^f)$ is motivically fibrant. By Lemma 3.7 the map in question coincides with the horizontal map of the diagram

$$\text{Hom}(\mathbb{P}^\wedge 1, \text{Hom}(\mathbb{P}^\wedge n, C_i Fr(E_n \wedge T^{i+1})^f)) \to \text{Hom}(\mathbb{P}^\wedge 1, \text{Hom}(\mathbb{P}^\wedge n, C_i Fr(E_n \wedge T^{i+1})(^f)))$$

(5.2)

The diagram (5.2) is obtained by applying $\text{Hom}(\mathbb{P}^\wedge 1, -)$ to the diagram

$$\text{Hom}(\mathbb{P}^\wedge n, C_i Fr(E_n \wedge T^{i+1})) \to \text{Hom}(\mathbb{P}^\wedge n, C_i Fr(E_n \wedge T^{i+1}))^f$$

(5.3)

The slanted arrow exists by the right lifting property for fibrant spaces. The horizontal arrow of (5.3) is a local weak equivalence, and the vertical arrow of (5.3) is a local weak equivalence by Corollary 4.20. It follows that the slanted arrow of (5.3) is a local weak equivalence between fibrant spaces, and hence so is the slanted arrow of (5.2) since $\mathbb{P}^\wedge 1$ is a cofibrant space. The vertical arrow of (5.2) is a local weak equivalence by Corollary 4.20. We see that the horizontal map of (5.2) is a local weak equivalence. \qed

The following theorem says that a fibrant replacement of a Thom spectrum $E$ can be computed (starting at some level depending on its bounding constant) by first applying the $\Theta^\omega$-functor to $E$, then by taking the Suslin complex of each space of $\Theta^\omega(E)$ and finally by taking local fibrant replacements for $C_i \Theta^\omega(E)$.

5.4. **Theorem.** Let $E$ be a Thom spectrum with the bounding constant $d$. Let $C_i \Theta^\omega(E)^f$ be a spectrum obtained from $C_i \Theta^\omega(E)$ by taking a level local fibrant replacement. Then the spectrum $C_i \Theta^\omega(E)^f$ is motivically fibrant starting from level $\max(0, d)$ and is stably equivalent to $E$.

**Proof.** Since a directed colimit of flasque locally fibrant spaces is flasque locally fibrant, it follows that $C_i \Theta^\omega(E)^f = \text{colim}_n C_i \Theta^\omega(L_n E)^f$. Hence it is sufficient to prove that for every $n$ the spectrum $C_i \Theta^\omega(L_n E)^f$ is motivically fibrant starting from level $d$. For $i \geq d$ the space $C_i \Theta^\omega(L_n E)^f$ equals $\text{Hom}(\mathbb{P}^\wedge n, C_i Fr(E_n \wedge T^i))^f$ by Lemma 3.7. Moreover, it is motivically fibrant by Corollary 4.16. Thus it remains to prove that each bonding map

$$C_i \Theta^\omega(L_n E)^f_{i+1} \to \text{Hom}(\mathbb{P}^\wedge 1, C_i \Theta^\omega(L_n E)_{i+1}^f)$$
is a local weak equivalence. It fits into the following commutative diagram:

\[
\begin{array}{c}
C_n \Theta^o(L_n E)_i \quad \xrightarrow{\text{Hom}(\mathbb{P}^\land, C_n \Theta^o(L_n E)_i)} \quad C_n \Theta^o(L_n E)_i \\
\| \| \\
\| \|
\end{array}
\]

where the right vertical arrow is a local weak equivalence by Lemma 5.1 and the lower arrow is an isomorphism by Lemma 3.5. Since the left vertical arrow is a local weak equivalence, then so is the upper arrow, as required. \( \square \)

6. THE FUNCTOR \( \Theta_{\text{sym}}^n \)

Whenever a Thom \( T \)-spectrum \( E \) is symmetric, we can also construct further fibrant replacements for it. To this end, we introduce another stabilization functor \( \Theta_{\text{sym}}^n \) on the level of symmetric \( T \)-spectra, which is slightly different from \( \Theta^o \). The spaces of \( \Theta_{\text{sym}}^n(E) \) and \( \Theta^o(E) \) are in fact isomorphic, but the bonding maps are different: the bonding maps of \( \Theta_{\text{sym}}^n(E) \) require the structure of a symmetric spectrum on \( E \), whereas the bonding maps of \( \Theta^o(E) \) do not.

Given a \( T \)-spectrum \( E \), let \( T \land E \) be the suspension spectrum of \( E \) (see Definition 2.6). The functor \( E \mapsto T \land E \) has a right adjoint loop functor \( E \rightarrow \Omega_T E \), where \( \Omega_T E \) has the spaces \( (\Omega_T E)_i = \text{Hom}(T, E)_i \). If there is no likelihood of confusion, we denote by \( \Omega E \) the \( \mathbb{P}^\land \)-spectrum with \( (\Omega E)_i = \text{Hom}(\mathbb{P}^\land, E)_i \) and the bonding maps are given by

\[
\text{Hom}(\mathbb{P}^\land, E)_i \land \mathbb{P}^\land \rightarrow \text{Hom}(\mathbb{P}^\land, E_i \land \mathbb{P}^\land) \xrightarrow{f} \text{Hom}(\mathbb{P}^\land, E_i \land T) \xrightarrow{u} (\mathbb{P}^\land, E_{i+1}),
\]

where \( \mathbb{P}^\land \rightarrow T \) is a canonical motivic equivalence.

6.1. Definition. Define the functor \( \Theta_{\text{sym}}^1(E) = \Omega(E[1]) \), where \( E[1] \) is the shift spectrum (see Definition 2.6), and

\[
\Theta_{\text{sym}}^n(E) := \Theta_{\text{sym}}^1(\Theta_{\text{sym}}^1(\ldots(\Theta_{\text{sym}}^1(E)))) \quad (n \text{ times}).
\]

6.2. Definition. If \( E \) is a symmetric \( T \)-spectrum, then there is a canonical map of \( T \)-spectra \( T \land E \rightarrow E[1] \) (see Definition 2.6). Notice that this map requires the symmetric spectrum structure of \( E \). By adjointness we have a map \( E \rightarrow \Omega(T \land E) \rightarrow \Omega(E[1]) = \Theta_{\text{sym}}^1(E) \). Iterating the latter map, we get a sequence of maps of spectra

\[
E \rightarrow \Theta_{\text{sym}}^1(E) \rightarrow \Theta_{\text{sym}}^2(E) \rightarrow \ldots
\]

Denote by \( \Theta_{\text{sym}}^n(E) \) the colimit of this sequence. Then for every symmetric \( T \)-spectrum \( E \) there is a natural map of \( \mathbb{P}^\land \)-spectra

\[
\varepsilon : E \rightarrow \Theta_{\text{sym}}^n(E).
\]

6.3. Remark. We need to describe bonding maps of \( \Theta_{\text{sym}}^n(E) \) and stabilization maps \( \Theta_{\text{sym}}^n(E) \rightarrow \Theta_{\text{sym}}^{n+1}(E) \) explicitly. One has,

\[
\Theta_{\text{sym}}^n(E)_i = \text{Hom}(\mathbb{P}^\land, E_{n+i}).
\]

Each bonding map equals the composition

\[
\text{Hom}(\mathbb{P}^\land, E_{n+i}) \land \mathbb{P}^\land \rightarrow \text{Hom}(\mathbb{P}^\land, E_{n+i} \land \mathbb{P}^\land) \xrightarrow{\sigma} \text{Hom}(\mathbb{P}^\land, E_{n+i} \land T) \xrightarrow{u} \text{Hom}(\mathbb{P}^\land, E_{n+i+1})
\]
and the stabilization map $\Theta_n^\mu(E)_i \to \Theta_{\text{sym}}^{n+1}(E)_i$ equals the composition
\[
\text{Hom}(\mathbb{P}^\wedge n, E_{n+i}) \to (\mathbb{P}^\wedge n, \text{Hom}(\mathbb{P}^\wedge n, E_{n+i} \wedge T)) \xrightarrow{\zeta_n} \text{Hom}(\mathbb{P}^\wedge n, E_{n+i+1}) \xrightarrow{\chi_{n+1}} \text{Hom}(\mathbb{P}^\wedge n, E_{n+1+i}),
\]
where the left arrow is induced by the external smash product with $\sigma : \mathbb{P}^\wedge 1 \to T$ and $\chi_{i+1}$ is the shuffle permutation in $\Sigma_{n+i+1}$ permuting the last element with preceding $i$ elements and preserves the first $n$ elements.

6.4. Lemma. For any symmetric $T$-spectrum $E$ for any $i$ there is an isomorphism of motivic spaces $\Theta^n(E)_i \cong \Theta_{\text{sym}}^\mu(E)_i$.

Proof. Define a map $f_n : \Theta^n(E)_i \to \Theta_{\text{sym}}^\mu(E)_i$ by the formula
\[
f_n : \text{Hom}(\mathbb{P}^\wedge n, E_{n+i}) \xrightarrow{\chi_n} \text{Hom}(\mathbb{P}^\wedge n, E_{n+i+1}),
\]
where $\chi_n$ is the shuffle permutation that permutes the last $n$ elements with the first $i$ elements. Then the following diagram is commutative:
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^\wedge n, E_{n+i}) & \xrightarrow{\chi_n} & \text{Hom}(\mathbb{P}^\wedge n, E_{n+i+1}) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^\wedge n, E_{n+1+i+1}) & \xrightarrow{\chi_{n+1}} & \text{Hom}(\mathbb{P}^\wedge n, E_{n+1+i}).
\end{array}
\]
Here the left vertical arrow is the stabilization map $\Theta^n(E)_i \to \Theta^{n+1}(E)_i$ and the right vertical map is the stabilization map $\Theta_{\text{sym}}^\mu(E)_i \to \Theta_{\text{sym}}^{n+1}(E)_i$ of Remark 6.3. So the maps $f_n$ induce a morphism of sequences. Then the maps $f_n$ induce the desired isomorphism on colimits $f : \Theta^n(E)_i \cong \Theta_{\text{sym}}^\mu(E)_i$. \qed

6.5. Lemma. For any symmetric $T$-spectrum $E$ there are isomorphisms of spaces
\[
\Theta^n(\Theta_{\text{sym}}^\mu(E))_i \cong \Theta_{\text{sym}}^\mu(\Theta^n(E))_i, \quad \Theta^n(\Theta_{\text{sym}}^\mu(E))_i \cong \Theta_{\text{sym}}^\mu(\Theta^n(E))_i.
\]

Proof. Applying Lemma 6.4 to the symmetric spectrum $\Theta_{\text{sym}}^\mu(E)$, we have
\[
\Theta^n(\Theta_{\text{sym}}^\mu(E))_i \cong \Theta_{\text{sym}}^\mu(\Theta^n(E))_i = \Theta_{\text{sym}}^\mu(E)_i.
\]
Also,
\[
\Theta^n(E)_i = \text{Hom}(\mathbb{P}^\wedge n, \Theta_{\text{sym}}^\mu(E))_i \cong \text{Hom}(\mathbb{P}^\wedge n, \Theta^n(E))_i = \Theta^n(\Theta_{\text{sym}}^\mu(E))_i = \Theta^n(E)_i,
\]
as required. \qed

6.6. Lemma. Suppose $E$ is a Thom $T$-spectrum with the bounding constant $d$. Then the space $C_*\Theta^n(E)_i$ is locally connected for $i \geq \max(0,d)$.

Proof. By Lemma 4.14 the space $C_*\text{Hom}(\mathbb{P}^\wedge n, \text{Fr}(E_n \wedge T^i)) = C_*\Theta^n(L_n(E))_i$ is locally connected for every $n$. Then $C_*\Theta^n(E)_i = \text{colim}_n C_*\Theta^n(L_n(E))_i$ is a locally connected space. \qed

6.7. Proposition. Suppose $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d$ and contractible alternating group action. Then the natural maps of spectra
\[
\xi : C_*\Theta^n_{\text{sym}}(E) \to C_*\Theta^n_{\text{sym}}(\Theta^n_{\text{sym}}(E))
\]
and
\[ C_\ast \Theta^\infty : C_\ast \Theta^\infty (E) \to C_\ast \Theta^\infty (\Theta^\infty_{\text{sym}}(E)), \]
obtained from the map \( \varepsilon : E \to \Theta^\infty_{\text{sym}}(E) \) by applying \( C_\ast \Theta^\infty \) to it, induce local weak equivalences of spaces starting from level \( \max(0,d) \). In particular, there is a commutative diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\eta} & C_\ast \Theta^\infty (E) \\
\downarrow{\varepsilon} & & \downarrow{C_\ast \Theta^\infty (\varepsilon)} \\
C_\ast \Theta^\infty_{\text{sym}}(E) & \xrightarrow{\xi} & C_\ast \Theta^\infty (\Theta^\infty_{\text{sym}}(E))
\end{array}
\]
of \( \mathbb{P}^1 \)-spectra, in which all arrows are stable motivic equivalences.

**Proof.** Fix a number \( i \geq d \). Consider a two-dimensional sequence
\[ A_{n,m} = C_\ast \Theta^\infty (\Theta^m_{\text{sym}}(E))_i \]
with horizontal maps \( A_{n,m} \to A_{n+1,m} \) induced by \( \Theta^\infty \to \Theta^{n+1} \) and vertical maps \( A_{n,m} \to A_{n,m+1} \) induced by \( \Theta^m_{\text{sym}} \to \Theta^{m+1}_{\text{sym}} \). To prove the statement, we need to show that the maps \( \text{colim}_m A_{0,m} \to \text{colim}_m A_{n,m} \) and \( \text{colim}_n A_{n,0} \to \text{colim}_m A_{n,m} \) are local weak equivalences.

Without loss of generality it is sufficient to prove that for every \( m,n \) the maps
\[ \text{colim}_m A_{2n,2m} \to \text{colim}_m A_{2n,2m+2} \tag{6.8} \]
and
\[ \text{colim}_m A_{2n,2m} \to \text{colim}_m A_{2n+2,2m} \tag{6.9} \]
are local weak equivalences.

Note that the spaces \( C_\ast \Theta^\infty (\Theta^m_{\text{sym}}(E))_i \) and \( C_\ast \Theta^\infty (\Theta^m_{\text{sym}}(E))_i \) are isomorphic to \( C_\ast \Theta^\infty (E)_i \) by Lemmas 6.5 and 6.4. Hence they are locally connected by Lemma 6.6.

To prove that (6.8) is a local weak equivalence, we apply Lemma 6.10 below for the case \( A_n = A_{2n,2m}, B_n = A_{2n,2m+2} \) and the maps \( \iota_n^A : A_{2n,2m} \to A_{2n,2m+2}, g_n : A_{2n,2m} \to A_{2n+2,2m+2}, f_n : A_{2n,2m} \to A_{2n,2m+2} \) are given by maps of the two dimensional sequences above. Define a map \( g_n : A_{2n,2m+2} \to A_{2n+2,2m+2} \) as an identification via associativity isomorphism
\[
A_{2n,2m+2} = C_\ast \text{Hom}(\mathbb{P}^{2n+2}, \text{Hom}(\mathbb{P}^{2m+2}, E_{2n+2m+2+i}))) = \\
C_\ast \text{Hom}(\mathbb{P}^{2n+2}, \text{Hom}(\mathbb{P}^{2m}, E_{2n+2m+2+i}))) = A_{2n,2m+2}.
\]
Then \( g_n f_n \) differs from \( \iota_n^A \) by the action of an even permutation on \( \mathbb{P}^{2n+2} \) and an even permutation on \( E_{2n+2m+2+i} \). Thus \( g_n f_n \) and \( \iota_n^A \) are simplicially homotopic by Corollary 2.13 and our assumption that \( E \) is a spectrum with contractible alternating group action as well as the fact that \( A_1 \)-homotopies become the usual ones after applying Suslin’s complex \( C_\ast \). Also, \( f_n g_n \) differs from \( \iota_n^B \) by the action of an even permutation on \( \mathbb{P}^{2n+2m+4} \) and an even permutation on \( E_{2n+2m+4} \). Therefore \( f_n g_n \) is simplicially homotopic to \( \iota_n^B \) for the same reasons as above. Thus the map on the colimits is a local weak equivalence by Lemma 6.10. The proof for the map (6.9) is analogous.

Finally, the map \( \eta \) of the commutative square of the proposition is a stable motivic equivalence by [16, 4.11], because the flasque motivic model structure on spaces is almost finitely
generated in the sense of [16]. By the first part of the proof \( \xi, C, \Theta^x (\varepsilon) \) are stable motivic equivalences, and hence so is \( \varepsilon \) by the two-out-of-three property for weak equivalences. \( \square \)

6.10. **Lemma.** Suppose \( i_n^A: A_n \to A_{n+1}, \quad i_n^B: B_n \to B_{n+1} \) are directed systems of spaces, and \( f_n: A_n \to B_n \) is a map of directed sequences. Suppose that there are maps \( g_n: B_n \to A_{n+1} \) such that \( g_n f_n \) is simplicially homotopic to \( i_n^A \) and \( f_{n+1} g_n \) is simplicially homotopic to \( i_n^B \). Also, suppose that the spaces \( A = \colim A_n \) and \( B = \colim B_n \) are locally connected. Then the map \( f = \colim f_n: A \to B \) is a local weak equivalence.

**Proof.** Given a local Henselian scheme \( U \), the map \( \pi_i(f)(U): \pi_i(A(U)) \to \pi_i(B(U)) \) equals the colimit of the system \( \pi_i(f_n(U)) \). Note that the maps \( \pi_i(g_n)(U) \) form a map of sequences \( \pi_i(B_n(U)) \to \pi_i(A_{n+1}(U)) \), which are inverse to \( \pi_i(f_n)(U) \). Therefore the colimit \( \pi_i(f)(U) = \colim \pi_i(f_n)(U) \) is bijective, and hence the map \( f(U) \) induces a weak equivalence of connected simplicial sets \( A(U) \to B(U) \). \( \square \)

7. **THE SPECTRUM \( C \cdot \text{Fr}^E(S_T) \)**

The purpose of this section is to introduce another spectrum \( C \cdot \text{Fr}^E(S_T) \) associated with a symmetric \( T \)-spectrum \( E \). We show that it is stably equivalent to the spectrum \( C \cdot \Theta^x_\text{sym}(E) \) whenever \( E \) is a Thom spectrum with the bounding constant \( d \) and contractible alternating group action (see Proposition 7.7).

7.1. **Definition.** Given a \( T \)-spectrum \( E \) and \( X \in \text{Sm}_k \), denote by \( \text{Fr}^E_n(X) \) the space \( \Theta^x(X_+ \wedge E_0) \):

\[
\text{Fr}^E_n(X) = \text{Hom}(\mathbb{P}^n \wedge X_+, E_n)
\]

and \( \text{Fr}^E(X) := \colim_n \text{Fr}^E_n(X) = \Theta^x(X_+ \wedge E_0) \).

7.2. **Definition.** For any symmetric \( T \)-spectrum \( E \) and any \( m, n \geq 0 \), define a pairing

\[
\text{Fr}_n(X, Y) \times \text{Fr}_m^E(Y, Z) \to \text{Fr}_{n+m}^E(X, Z)
\]

as follows. Let \( a \in \text{Fr}_n(X, Y) \) be given by a map \( a: X_+ \wedge \mathbb{P}^n \to Y_+ \wedge T^n \) and let \( b \in \text{Fr}_m^E(Y, Z) \) be given by \( b: Y_+ \wedge \mathbb{P}^m \to Z_+ \wedge E_m \). Define \( b \circ a \) as the composition

\[
\begin{align*}
X_+ \wedge \mathbb{P}^n \wedge \mathbb{P}^m &\quad \xrightarrow{a \wedge 1} Y_+ \wedge T^n \wedge \mathbb{P}^m \quad \xrightarrow{1 \wedge T^n} Y_+ \wedge \mathbb{P}^m \wedge T^n \\
&\quad \xrightarrow{b \wedge T^n} Z_+ \wedge E_m \wedge T^n \\
&\quad \xrightarrow{\text{tw}} Z_+ \wedge T^n \wedge E_m \wedge T^n \\
&\quad \xrightarrow{u} Z_+ \wedge E_{n+m},
\end{align*}
\]

where \( u \) is the map of Definition 2.6.

Note that if \( E = S_T \), this definition coincides with the definition of the composition of framed correspondences defined in [32, 13].

7.3. **Lemma.** The pairing above endows \( \text{Fr}^E(X) \) with a structure of a presheaf with framed correspondences.

**Proof.** This is straightforward. \( \square \)

7.4. **Definition.** Given a \( T \)-spectrum \( E \) and \( X \in \text{Sm}_k \), denote by \( \text{Fr}^E_n(X_+ \wedge S_T) \) the \( T \)-spectrum with the spaces

\[
\text{Fr}^E_n(X_+ \wedge S_T)_i := \text{Fr}^E_n(X_+ \wedge T^i) = \text{Fr}_n^{T^i \wedge E}(X).
\]
The bonding maps \( \text{Fr}_n^E(X_+ \land T^i) \land T \to \text{Fr}_n^E(X_+ \land T^{i+1}) \) are defined as the composite maps
\[
\text{Hom}(\mathbb{P}^{\land n}, X_+ \land T^i \land E_n) \land T \to \text{Hom}(\mathbb{P}^{\land n}, X_+ \land T^i \land E_n \land T) \xrightarrow{\text{tw}} \text{Hom}(\mathbb{P}^{\land n}, X_+ \land T^i \land T \land E_n).
\]
In what follows we normally regard \( \text{Fr}_n^E(X_+ \land S_T) \) as a \( \mathbb{P}^1 \)-spectrum. The stabilization maps \( \text{Fr}_n^E(X_+ \land T^i) \to \text{Fr}_{n+1}^E(X_+ \land T^i) \), given by the compositions
\[
\text{Hom}(\mathbb{P}^{\land n}, X_+ \land T^i \land E_n) \xrightarrow{- \land \sigma} \text{Hom}(\mathbb{P}^{\land n} \land \mathbb{P}^{\land 1}, X_+ \land T^i \land E_n \land T) \xrightarrow{u_n} \text{Hom}(\mathbb{P}^{\land n+1}, X_+ \land T^{i+1} \land E_{n+1}),
\]
define a map of \( \mathbb{P}^1 \)-spectra \( \text{Fr}_n^E(X_+ \land S_T) \to \text{Fr}_{n+1}^E(X_+ \land S_T) \). Denote by
\[
\text{Fr}^E(X_+ \land S_T) := \text{colim}_n \text{Fr}_n^E(X_+ \land S_T).
\]
If \( E = S_T \) the spectrum \( \text{Fr}^E(X_+ \land S_T) \) coincides with the spectrum \( \text{Fr}_{\mathbb{P}^1 T}(X) \) defined in [13].

Note that for \( X \in \text{Sm}_k \) the spectrum \( \text{Fr}^E(X_+ \land S_T) \) is isomorphic to the spectrum \( \text{Fr}^{X_+ \land E}(S_T) \).

If \( E \) is a symmetric Thom spectrum with the bounding constant \( d \), then so is \( X_+ \land E \). Thus we shall consider spectra of the form \( \text{Fr}^E(S_T) \) in what follows.

For any \( n \geq 0 \) and any symmetric \( T \)-spectrum \( E \), construct a map of \( \mathbb{P}^1 \)-spectra \( f_n : \text{Fr}_n^E(S_T) \to \Theta_{s_{ym}}^n(E) \) as the composition at each level \( i \geq 0 \)
\[
f_{n,i} : \text{Hom}(\mathbb{P}^{\land n}, T^i \land E_n) \xrightarrow{\text{tw}} \text{Hom}(\mathbb{P}^{\land n}, E_n \land T^i) \xrightarrow{u} \text{Hom}(\mathbb{P}^{\land n}, E_{n+i}),
\]
where the first map is induced by twist \( T^i \land E_n \to E_n \land T^i \).

7.5. Lemma. Each map \( f_n, n \geq 0 \), is a morphism of spectra commuting with stabilization maps \( \text{Fr}_n^E(S_T) \to \text{Fr}_{n+1}^E(S_T) \) and \( \Theta_{s_{ym}}^n(E) \to \Theta_{s_{ym}}^{n+1}(E) \). In particular, they induce a map of spectra
\[
f : \text{Fr}^E(S_T) \to \Theta_{s_{ym}}^\infty(E).
\]

Proof. The following diagram commutes:
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^{\land n}, T^i \land E_n) \land T & \xrightarrow{\text{tw}} & \text{Hom}(\mathbb{P}^{\land n}, E_n \land T^i) \land T \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^{\land n}, T^{i+1} \land E_n) & \xrightarrow{\text{tw}} & \text{Hom}(\mathbb{P}^{\land n}, E_n \land T^{i+1})
\end{array}
\]
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^{\land n}, E_n \land T^i) & \xrightarrow{u} & \text{Hom}(\mathbb{P}^{\land n}, E_{n+i}) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^{\land n+1}, T^i \land E_{n+1}) & \xrightarrow{\text{tw}} & \text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1} \land T^i)
\end{array}
\]
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1} \land T^i) & \xrightarrow{u} & \text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1+i})
\end{array}
\]
where the left vertical arrow is the \( i \)th bonding map of the spectrum \( \text{Fr}_n^E(S_T) \), and the right vertical map is the \( i \)th bonding map of \( \Theta_{s_{ym}}^n(E) \). We see that each map \( f_n \) is a morphism of spectra. Consider a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^{\land n}, T^i \land E_n) & \xrightarrow{\text{tw}} & \text{Hom}(\mathbb{P}^{\land n}, E_n \land T^i) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^{\land n+1}, T^i \land E_{n+1}) & \xrightarrow{\text{tw}} & \text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1} \land T^i)
\end{array}
\]
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1} \land T^i) & \xrightarrow{u} & \text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1+i}) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1+i})
\end{array}
\]
in which the left vertical map is the stabilization \( \text{Fr}_n^E(S_T)_i \to \text{Fr}_{n+1}^E(S_T)_i \) from Definition 7.4, and the right vertical map is the stabilization map \( \Theta_{s_{ym}}^n(E)_i \to \Theta_{s_{ym}}^{n+1}(E)_i \) (see Remark 6.3). The middle vertical arrow equals the composite map
\[
\text{Hom}(\mathbb{P}^{\land n}, E_n \land T^i) \xrightarrow{- \land \sigma} \text{Hom}(\mathbb{P}^{\land n+1}, E_n \land T^{i+1}) \xrightarrow{u} \text{Hom}(\mathbb{P}^{\land n+1}, E_n \land T^{i+1} + 1) \xrightarrow{u} \text{Hom}(\mathbb{P}^{\land n+1}, E_{n+1} \land T^i),
\]
where \((\chi_{i,1})_n\) is induced by the shuffle map \(\chi_{i,1} : T^{i+1} \rightarrow T^{1+i}\). For commutativity of the right square we also use here the fact that the diagram

\[
\begin{array}{ccc}
E_n \wedge T^{i+1} & \xrightarrow{\mu} & E_{n+i} \wedge T \\
\downarrow{id \wedge \chi_{i,1}} & & \downarrow{1 \circ \chi_{i,1}} \\
E_n \wedge T^{1+i} & \xrightarrow{\mu} & E_{n+1} \wedge T^i
\end{array}
\]

is commutative because the compositions of horizontal maps are \(\Sigma_n \times \Sigma_{i+1}\)-equivariant maps. Thus the maps \(f_{n,i}\) are compatible with stabilization.

\[7.6.\text{Corollary.}\] If \(E = X_+ \wedge S_T\) then the map \(f\) of Lemma 7.5 gives an isomorphism of spectra

\[
\text{Fr}_{\mathbb{P}^1,T}(X) = \text{Fr}_{X_+ \wedge E}(S_T) \xrightarrow{\sim} \Theta_{\text{sym}}^n(X_+ \wedge S_T).
\]

\[\text{Proof.}\] It suffices to note that the bonding maps of \(X_+ \wedge S_T\) are isomorphisms.

\[7.7.\text{Proposition.}\] For a symmetric Thom \(T\)-spectrum \(E\) with the bounding constant \(d\) and contractible alternating group action, the map \(f\) induces a local weak equivalence for any \(i \geq \max(0,d)\):

\[f_i : C_s \text{Fr}^E(S_T)_i \rightarrow C_s \Theta_{\text{sym}}^n(E)_i.\]

\[\text{Proof.}\] The map \(f_{n,i} : \text{Fr}^E_n(T^i) \rightarrow \Theta_{\text{sym}}^n(E)_i\) fits into the following commutative diagram

\[
\begin{array}{ccc}
\text{Fr}^E_n(T^i) & \xrightarrow{f_{n,i}} & \Theta_{\text{sym}}^n(E)_i \\
\downarrow{\chi_n} & & \downarrow{\chi_n} \\
\Theta^n(T^i \wedge E)_0 & \xrightarrow{\mu_i} & \Theta^n(E[i])_0 = \Theta^n(E)_i
\end{array}
\]

where \(\chi_{i,n}\) is the map of Lemma 6.4 and \(\mu_i\) is the left bonding map from Definition 2.6. Note that the maps of the diagram are compatible with stabilization maps, and hence we can pass to the colimit over \(n\). Thus our assertion follows from Lemma 7.9 below.

\[7.8.\text{Lemma.}\] The natural map of \(T\)-spectra \(u : E \wedge T^i \rightarrow E[i]\) induces a levelwise isomorphism of spaces \(\Theta^n(E \wedge T^i) \xrightarrow{\sim} \Theta^n(E[i])\) for any \(T\)-spectrum \(E\).

\[\text{Proof.}\] For any \(m\) the map \(\text{Hom}(\mathbb{P}^n,E_{n+m} \wedge T^i) \xrightarrow{\mu} \text{Hom}(\mathbb{P}^{2n},E_{n+m+i})\) commutes with stabilization by \(n\) and induces an isomorphism on colimits \(\Theta^n(E \wedge T^i)_m \rightarrow \Theta^n(E[i])_m\).

\[7.9.\text{Lemma.}\] For a symmetric Thom \(T\)-spectrum \(E\) with the bounding constant \(d\) and contractible alternating group action, the map of spectra \(u_i : T^i \wedge E \rightarrow E[i]\) induces a local weak equivalence of spaces

\[C_s \Theta^n(T^i \wedge E)_0 \rightarrow C_s \Theta^n(E[i])_0\]

for any \(i \geq \max(0,d)\).

\[\text{Proof.}\] For \(i \geq \max(0,d)\) the space \(C_s \Theta^n(E)_i\) is locally connected by Lemma 6.6. We apply Lemma 7.10 below to spaces

\[A_n = C_s \Theta^{2n}(T^i \wedge E)_0 = C_s \text{Hom}(\mathbb{P}^{2n},T^i \wedge E_{2n}),\]
where the maps $i^A_n$ and $i^B_n$ from that lemma are induced by stabilization maps $\Theta^{2n} \to \Theta^{2n+2}$ and $C = C_\ast \Theta^n(E[i])_0$. Define $f_n : A_n \to B_n$ to be the map induced by the twist $\tau \colon T^i \to E_{2n} \to E_{2n} \wedge T^i$. Then the composition $f_{n+1} \circ i^A_n$ coincides with the composition $f_{n+1} \circ i^A_n : C_\ast \text{Hom}(\mathbb{P}^{2n}, T^i \wedge E_{2n}) \to C_\ast \text{Hom}(\mathbb{P}^{2n+2}, E_{2n} \wedge T^2 \wedge T^i) \to C_\ast \text{Hom}(\mathbb{P}^{2n+2}, E_{2n+2} \wedge T^i)$. It differs from the composition $i^B_n \circ f_n$ by the permutation $E_{2n} \wedge T^2 \wedge T^i \to E_{2n} \wedge T^i \wedge T^{2\text{assoc}} \to E_{2n} \wedge T^2 \wedge T^i$. Since it is an even permutation and $E$ is a spectrum with contractible alternating group action, we have that $f_{n+1} \circ i^A_n$ and $i^B_n \circ f_n$ are simplicially homotopic. Similarly, in the triangle

$$
\begin{array}{ccc}
C_\ast(\mathbb{P}^{2n}, T^i \wedge E_{2n}) & \xrightarrow{\mu} & C_\ast(\mathbb{P}^{2n}, E_{2n+i}) \\
\downarrow \tau & & \downarrow u \\
C_\ast(\mathbb{P}^{2n}, E_{2n} \wedge T^i) & & \\
\end{array}
$$

the composition $u \circ \tau$ differs from $\mu$ by the action of the shuffle permutation $\chi_{2n,i}$ on $E_{2n+i}$, which is an even permutation. Thus the triangle commutes up to simplicial homotopy, because $E$ is a spectrum with contractible alternating group action. Then for any $i \geq d$ the space $C_\ast \Theta^n(E[i])_0$ is connected and our statement follows from Lemmas 7.10 and 7.8. □

7.10. Lemma. Suppose $i^A_n : A_n \to A_{n+1}$ and $i^B_n : B_n \to B_{n+1}$ are directed sequences of spaces, $C$ is a locally connected space and there are maps of sequences $A_n \to C$ and $B_n \to C$. Suppose that for any $n$ there is a local weak equivalence $f_n : A_n \to B_n$ such that the diagrams

$$
\begin{array}{ccc}
A_n & \xrightarrow{f_n} & A_{n+1} \\
\downarrow & & \downarrow \uparrow \\
B_n & \xrightarrow{f_{n+1}} & B_{n+1}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
A_n & \xrightarrow{f_n} & C \\
\downarrow & & \downarrow \\
B_n & & 
\end{array}
$$

commute up to a simplicial homotopy. Let $A = \text{colim} A_n, B = \text{colim} B_n$. Then the map $B \to C$ is a local weak equivalence if and only if so is the map $A \to C$.

Proof. Given local Henselian scheme $U$ and $i \geq 0$, the maps $\pi_i(f_n(U))$ form a map of sequences $\pi_i(f_n(U)) : \pi_i(A_n(U)) \to \pi_i(B_n(U))$, and the map colim$_n \pi_i(f_n(U))$ fits into the commutative diagram

$$
\begin{array}{ccc}
\text{colim} \pi_i(f_n(U)) & \xrightarrow{\pi_i} & \pi_i(C(U)) \\
\downarrow & & \downarrow \\
\pi_i(B(U)) & & 
\end{array}
$$

Every $\pi_i(f_n(U))$ is a bijection, and hence so is $\text{colim} \pi_i(f_n(U))$. If the map $B \to C$ is a local weak equivalence, then $B(U)$ is connected and all maps $\pi_i(B(U)) \to \pi_i(C(U))$ are bijective. Then $A(U)$ is connected, and all the maps $\pi_i(A(U)) \to \pi_i(C(U))$ are bijective. Therefore the map $A \to C$ is a local weak equivalence. Similarly, if $A \to C$ is a local weak equivalence, then so is $B \to C$. □
8. Fibrant resolutions of symmetric Thom spectra

We have discussed three types of spectra associated with a symmetric Thom $T$-spectrum $E$ each of which is obtained from $E$ by a certain stabilization and taking the Suslin complex at each level: $C_*\text{Fr}^E(S_T)$, $C_*\Theta^m(E)$ and $C_*\Theta^m_{\text{sym}}(E)$. Moreover, by Propositions 6.7 and 7.7 they are isomorphic to each other in $\text{SH}(k)$ under certain reasonable assumptions on $E$. The next theorem says that if we take local fibrant replacements at each level in these spectra, they become motivically fibrant starting from some level $d$ onwards. More precisely, the following result is true:

8.1. Theorem. For a symmetric Thom $T$-spectrum $E$ with the bounding constant $d$ and contractible alternating group action the following $\mathbb{P}^1$-spectra are isomorphic to $E$ in $\text{SH}(k)$ and motivically fibrant starting from level $\max(0,d)$:

\begin{itemize}
  \item $C_*\text{Fr}^E(S_T)^f$
  \item $C_*\Theta^m(E)^f$
  \item $C_*\Theta^m_{\text{sym}}(E)^f$,
\end{itemize}

where “$f$” refers to levelwise local fibrant replacements of the corresponding spectra.

Proof. By Propositions 6.7 and 7.7 we have the following levelwise local weak equivalences of $\mathbb{P}^1$-spectra starting from level $d$:

$$C_*\text{Fr}^E(S_T) \rightarrow C_*\Theta^m_{\text{sym}}(E) \rightarrow C_*\Theta^m(\Theta^m_{\text{sym}}(E)) \leftarrow C_*\Theta^m(E).$$

Since the canonical map $E \rightarrow C_*\Theta^m(E)$ is a stable motivic equivalence by Theorem 5.4, we see that $E$ is isomorphic in $\text{SH}(k)$ to each of the spectrum of the theorem.

**Sublemma.** Suppose a map of $\mathbb{P}^1$-spectra $f: E \rightarrow E'$ is a levelwise local weak equivalence and all spaces $E_i, E'_i, i \geq 0$, are fibrant in the flasque local model structure. Then $E$ is motivically fibrant if and only if so is $E'$.

**Proof.** Since each map $f_i: E_i \rightarrow E'_i, i \geq 0$, is a local weak equivalence between locally fibrant spaces, it is a sectionwise weak equivalence. Therefore if $E_i$ is motivically fibrant, then $E'_i$ is $\mathbb{A}^1$-invariant, and hence motivically fibrant as well. Since $E_i, E'_i$ are flasque fibrant by assumption, then the map $\text{Hom}(\mathbb{P}^1, E_{i+1}) \rightarrow \text{Hom}(\mathbb{P}^1, E'_{i+1})$ is a sectionwise weak equivalence, because $\mathbb{P}^1$ is flasque cofibrant. Hence the adjoint to the bonding map $E_i \rightarrow \text{Hom}(\mathbb{P}^1, E_{i+1})$ is a sectionwise weak equivalence if and only if so is the map $E'_i \rightarrow \text{Hom}(\mathbb{P}^1, E'_{i+1})$. \qed

Since the spectrum $C_*\Theta^m(E)^f$ is motivically fibrant starting from level $d$ by Theorem 5.4, then so are $C_*\Theta^m_{\text{sym}}(E)^f$ and $C_*\Theta^m(\Theta^m_{\text{sym}}(E))^f$ by Proposition 6.7 and the sublemma above. Likewise, $C_*\text{Fr}^E(S_T)^f$ is motivically fibrant starting from level $d$ by Proposition 7.7 and the sublemma above. This completes the proof of the theorem. \qed

The spectrum $C_*\text{Fr}^E(S_T)^f$, which is isomorphic to $E$ in $\text{SH}(k)$ by Theorem 8.1, is of particular interest, because it will lead to an equivalent model of $E$ in the category of $(S^1, \mathbb{G}^1_m)$-bispectra (see Theorem 9.13).
9. E-FRAMED MOTIVES AND BISPECTRA

Following Definition 7.1, for any space \( \mathcal{X} \) and any \( T \)-spectrum \( E \) denote by \( \text{Fr}_n^E(\mathcal{X}) = \text{Hom}(\mathbb{P}^n, \mathcal{X} \wedge E_n) \), \( n \geq 0 \). Also, set \( \text{Fr}^E(\mathcal{X}) = \text{colim}_n(\text{Hom}(\mathbb{P}^n, \mathcal{X} \wedge E_n)) = \Theta^o(\mathcal{X} \wedge E)_0 \). If \( \mathcal{X} = X_+ \), \( X \in \text{Sm}_k \), then we shall write \( \text{Fr}^E(X) \) dropping + from notation.

9.1. Lemma. \( \text{Fr}^E(\mathcal{X}) \) is functorial in \( \mathcal{X} \) and \( E \). If \( E \) is a directed colimit of \( T \)-spectra \( \colim_k E_k \), then \( \text{Fr}^E(\mathcal{X}) = \text{colim}_k \text{Fr}^{E_k}(\mathcal{X}) \). In particular, \( \text{Fr}^E(\mathcal{X}) = \text{colim}_k \text{Fr}^{E_k}(\mathcal{X}) \), where \( L_k E \) is the \( k \)-th layer of \( E \).

9.2. Definition. Given a \( T \)-spectrum \( E \), the assignment \( K \mapsto C_* \text{Fr}^E(\mathcal{X} \wedge K) \) is plainly a \( \Gamma \)-space. The \( E \)-framed motive \( M_E(\mathcal{X}) \) of \( \mathcal{X} \) is the Segal symmetric \( S^1 \)-spectrum \( C_* \text{Fr}^E(\mathcal{X} \wedge S) \). If \( E = S_T \) then \( M_E(\mathcal{X}) \) is the framed motive \( M_{fr}(\mathcal{X}) \) of \( \mathcal{X} \) in the sense of [13].

Lemma 9.1 implies the following

9.3. Corollary. \( M_E(\mathcal{X}) \) is functorial in \( \mathcal{X} \) and \( E \). If \( E \) is a directed colimit of \( T \)-spectra \( \colim_k E_k \), then \( M_E(\mathcal{X}) = \text{colim}_k M_{E_k}(\mathcal{X}) \). In particular, \( M_E(\mathcal{X}) = \text{colim}_k M_{L_k E}(\mathcal{X}) \), where \( L_k E \) is the \( k \)-th layer of \( E \).

The next statement is straightforward.

9.4. Lemma. \( M_{L_k E}(\mathcal{X}) = \text{Hom}(\mathbb{P}^k, M_{fr}(\mathcal{X} \wedge E_k)) \) for any \( k \geq 0 \).

9.5. Definition. Given a Thom \( T \)-spectrum \( E \), \( U \in \text{Sm}_k \) and \( Y = X/(X - Z) \), where \( X \in \text{Sm}_k \) and \( Z \) is a closed subset in \( X \), denote by \( \mathbb{Z}F^E_n(U, Y) \) the free Abelian group generated by the elements of \( \text{Fr}^E_n(U, Y) = \text{Hom}(\mathbb{P}^n, Y \wedge E_n) \) with connected support (recall that the elements of \( \text{Fr}^E_n(U, Y) \) have an explicit geometric description using Voevodsky’s Lemma 2.12). We also set \( \mathbb{Z}F^E(U, Y) := \text{colim}_n \mathbb{Z}F^E_n(U, Y) \), where the colimit maps are defined in the same fashion with those of \( \text{Fr}^E(U, Y) \).

The assignment \( K \mapsto C_* \mathbb{Z}F^E(U, Y \wedge K) \) is plainly a \( \Gamma \)-space. The linear \( E \)-framed motive \( LM_E(Y) \) of \( Y \) is the Segal symmetric \( S^1 \)-spectrum \( C_* \mathbb{Z}F^E(Y \wedge S) \). If \( E = S_T \) then \( LM_E(Y) \) is the linear framed motive \( LM_{fr}(Y) \) of \( Y \) in the sense of [13]. Note that the presheaves of stable homotopy groups \( \pi_*(LM_E(Y)) \) are computed as the presheaves of homology groups of the complex \( C_*\mathbb{Z}F^E(Y) \) (we freely use the Dold–Kan correspondence here).

As above we have the following

9.6. Lemma. \( LM_E(Y) = \text{colim}_k LM_{L_k E}(Y) \) and \( LM_{L_k E}(Y) = LM_{fr}^{E_k}(Y \wedge E_k) \), where \( LM_{fr}^{E_k}(Y \wedge E_k) \) is Segal’s spectrum associated with the \( \Gamma \)-space \( K \mapsto C_*\mathbb{Z}F^{E_k}(U, Y \wedge K) \) (see Definition 4.7).

The following lemma says that \( LM_E(Y) \) computes homology of the \( E \)-framed motive of \( Y \).

9.7. Lemma. Given a Thom \( T \)-spectrum \( E \) and \( Y \) as above, there is an isomorphism of graded presheaves \( \pi_*(\mathbb{Z}M_E(Y)) = \pi_*(LM_E(Y)) \).

Proof. We have that \( \pi_*(\mathbb{Z}M_E(Y)) = \text{colim}_k \pi_*(\mathbb{Z}M_{L_k E}(Y)) = \pi_*(\mathbb{Z}(\text{Hom}(\mathbb{P}^k, M_{fr}(Y \wedge E_k)))) = \pi_*(LM_{fr}^{E_k}(Y \wedge E_k)) = \pi_*(LM_E(Y)) \). We have used here Corollary 9.3, Lemmas 4.8, 9.4 and 9.6. □
Following notation of [10], denote by $A^1/\mathbb{G}_m$ the mapping cone of the natural embedding $(\mathbb{G}_m)_+ \hookrightarrow A^1_\ast$. It is represented by a simplicial scheme from $\text{Fr}_0(k)$.

9.8. Lemma. For a Thom $T$-spectrum $E$ with the bounding constant $d$, the natural map

$$M_E(T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i}) \to M_E(T^\ell \wedge T^i), \quad \ell := \max(0,d-1),$$

is a local stable weak equivalence for any $i > 0$.

Proof. By Corollary 9.3 and Lemma 9.4 $M_E(T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i}) = \text{colim}_n \text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i}))$ and $M_E(T^\ell \wedge T^i) = \text{colim}_n \text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i))$. Since a directed colimit of stable local weak equivalences is a stable local weak equivalence, it is sufficient to check that the natural map

$$\text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i})) \to \text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i))$$

is a local stable weak equivalence of spectra. By definition of the bounding constant $d$, the space $E_n \wedge T^\ell$ is a colimit of spaces of the form $X/X - Z$ where $Z$ has codimension greater than or equal to $n$. Consider a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i})) & \longrightarrow & \text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i)) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i}))_f & \longrightarrow & \text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge T^\ell \wedge T^i))_f,
\end{array}
\]

where “$f$” refers to a level local fibrant replacement. Then by Proposition 4.18 the vertical arrows are local stable weak equivalences, and the bottom arrow is a stable weak equivalence between motivically fibrant $S^1$-spectra by [10, 1.1; A.1]. By the two-out-of-three-property the upper arrow is a local stable weak equivalence. \qed

9.9. Proposition. Let $E$ be a Thom $T$-spectrum with the bounding constant $d$ and $\ell = \max(0,d-1)$. Then for every $i > 0$ the $S^1$-spectra $M_E(T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i})_f, M_E(T^\ell \wedge T^i)_f$ and $M_E(T^\ell \wedge (S^1 \wedge \mathbb{G}_m)^{\wedge i})_f$, where “$f$” refers to a level local fibrant replacement, are motivically fibrant.

Proof. Without loss of generality we assume $d \leq 1$. Indeed, if $d > 1$ we replace $E$ by $T^{d-1} \wedge E$, which has the bounding constant 1, observing that $M_E(T^{d-1} \wedge -) \cong M_{T^{d-1} \wedge E}(-)$. It suffices to prove the statement for the spectrum $M_E((A^1/\mathbb{G}_m)^{\wedge i})_f$, because our arguments will be the same for the other two spectra.

By Corollary 9.3 and Lemma 9.4 one has $M_E((A^1/\mathbb{G}_m)^{\wedge i}) = \text{colim}_n \text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge (A^1/\mathbb{G}_m)^{\wedge i}))$. Since flasque motivically fibrant spectra are closed under filtered colimits, it is enough to show that each spectrum $\text{Hom}(\mathbb{P}^{\wedge n}, M_{fr}(E_n \wedge (A^1/\mathbb{G}_m)^{\wedge i}))_f$ is motivically fibrant. This can shown similarly to Lemma 4.15 if we note that the space $C, \text{Fr}(X_+ \wedge (A^1/\mathbb{G}_m)^{\wedge i}), X \in \text{Sm}_k$, is locally connected by [10, A.1] and if we apply the proof of Lemma 4.14 to show that the space $\text{Hom}(\mathbb{P}^{\wedge n}, C, \text{Fr}(E_n \wedge (A^1/\mathbb{G}_m)^{\wedge i}))$ is locally connected. \qed

9.10. Lemma. Under the assumptions of Proposition 9.9 the map $M_E(T^\ell \wedge (A^1/\mathbb{G}_m)^{\wedge i})_f \to M_E(T^\ell \wedge (S^1 \wedge \mathbb{G}_m)^{\wedge i})_f$ is a sectionwise level equivalence for $\ell = \max(0,d-1)$.  

28
Proof. The proof of Proposition 9.9 shows that it suffices to prove the assertion for \( d \leq 1 \) and that the map of \( \text{Hom}(\mathbb{P}^1, M_f_r(E_n \wedge (\mathbb{A}^1/\mathbb{G}_m)^\wedge)) \to \text{Hom}(\mathbb{P}^1, M_f_r(E_n \wedge (\mathbb{A}^1/\mathbb{G}_m)^\wedge)) \) is a local stable weak equivalence for any \( n \geq 0 \). Since a map of locally connected spectra is a stable local equivalence if and only if so is the map on homology, then using Lemma 4.8 our assertion reduces to showing that the map of complexes \( C_*\mathbb{Z}F^p_{/\mathcal{X}}(E_n \wedge (\mathbb{A}^1/\mathbb{G}_m)^\wedge) \to C_*\mathbb{Z}F^p_{/\mathcal{X}}(E_n \wedge (\mathbb{A}^1/\mathbb{G}_m)^\wedge) \) is locally a quasi-isomorphism. The latter fact is proved similar to [13, 8.2].

We shall need the following fact which was proven in [13, Section 12].

9.11. Lemma. Let \( \mathcal{X} \) be an \( \mathbb{A}^1 \)-local motivic \( S^1 \)-spectrum whose presheaves of stable homotopy groups are homotopy invariant quasi-stable additive presheaves with framed correspondences. Suppose \( \mathcal{X}^f \) is a local stable fibrant replacement of \( \mathcal{X} \). Then the map of spectra \( \text{Hom}(\mathbb{G}_m^1, \mathcal{X}) \to \text{Hom}(\mathbb{G}_m^1, \mathcal{X}^f) \) is a local stable equivalence.

9.12. Proposition. The following statements are true:

1. Suppose \( Z \to X \) is a closed embedding of smooth varieties of codimension \( d \). Then the natural map

\[
\alpha : \text{Hom}(\mathbb{P}^1, M_f_r(X/X-Z)) \to \text{Hom}(\mathbb{P}^1, \mathbb{G}_m^1, M_f_r((X/X-Z) \wedge \mathbb{G}_m^1))
\]

is a stable local weak equivalence of \( S^1 \)-spectra for all \( i \leq d \).

2. If \( E \) is a Thom \( T \)-spectrum with the bounding constant \( d \leq 1 \), then the natural map

\[
\beta : M_E(X) \to \text{Hom}(\mathbb{G}_m^1, M_E(X_+ \wedge \mathbb{G}_m^1)), \quad X \in \text{Sm}_k,
\]

is a stable local weak equivalence of \( S^1 \)-spectra.

3. If \( E \) is a Thom \( T \)-spectrum with the bounding constant \( d \leq 1 \) and \( M_E(X)_f \) is a stable local fibrant replacement of \( M_E(X)_f \), then \( M_E(X)_f \) is a motivically fibrant \( S^1 \)-spectrum.

Proof. (1). First suppose \( i = 0 \). Without loss of generality we assume that \( X/X-Z = Z_+ \wedge T^d \), because we can apply the Mayer–Vietoris sequence of Proposition 4.11 to reduce the general case to this particular one. Indeed, \( M_f_r(X/X-Z) \) and \( M_f_r((X/X-Z) \wedge \mathbb{G}_m^1) \) are homotopy pushouts of framed motives of the form \( M_f_r(Z_+ \wedge T) \) and \( M_f_r(Z_+ \wedge T^d) \) respectively. Using Lemma 9.11 the functor \( \text{Hom}(\mathbb{G}_m^1, -) \) respects homotopy pullbacks of framed motives in question. Therefore our assertion reduces to showing that the natural map of \( S^1 \)-spectra

\[
M_f_r(Z_+ \wedge T^d) \to \text{Hom}(\mathbb{G}_m^1, M_f_r(Z_+ \wedge T^d \wedge \mathbb{G}_m^1))
\]

is a stable local weak equivalence. It fits into a commutative diagram

\[
\begin{array}{ccc}
M_f_r(Z_+ \wedge T^d) & \longrightarrow & \text{Hom}(\mathbb{G}_m^1, M_f_r(Z_+ \wedge T^d \wedge \mathbb{G}_m^1)) \\
\downarrow & & \downarrow \\
M_f_r(Z_+ \wedge (\mathbb{A}^1/\mathbb{G}_m)^\wedge) & \longrightarrow & \text{Hom}(\mathbb{G}_m^1, M_f_r(Z_+ \wedge (\mathbb{A}^1/\mathbb{G}_m)^\wedge \wedge \mathbb{G}_m^1))
\end{array}
\]

in which the lower arrow is a stable local weak equivalence by the Cancellation Theorem for framed motives of [3]. The vertical arrows are stable local weak equivalences by [10, 1.1] and Lemma 9.11. We see that the upper arrow is a stable local weak equivalence, as required.
Suppose now \( i \leq d \). Consider a commutative diagram of \( S^1 \)-spectra

\[
\begin{array}{ccc}
\text{Hom}(\mathbb{P}^A, M_f(X/X - Z)) & \longrightarrow & \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f((X/X - Z) \wedge \mathbb{G}^A_1)) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{P}^A, M_f(X/X - Z)_f) & \longrightarrow & \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f((X/X - Z) \wedge \mathbb{G}^A_1)_f)
\end{array}
\]

where “\( f \)” refers to a level local fibrant replacement. By Proposition 4.18 the left vertical map is a levelwise local weak equivalence in positive degrees, and hence a stable local weak equivalence. Since \( M_f(X/X - Z)_f, M_f((X/X - Z) \wedge \mathbb{G}^A_1)_f \) are motivically fibrant spectra in positive degrees, the lower arrow is a sectionwise weak equivalence in positive degrees by the first assertion and Proposition 4.18, and hence a sectionwise stable weak equivalence. The right vertical map is a levelwise local weak equivalence in positive degrees by Lemma 9.11 and Proposition 4.18, hence it is a stable local weak equivalence. We see that the upper arrow is a stable local weak equivalence.

(2) If \( E \) is a Thom \( T \)-spectrum with the bounding constant \( d \leq 1 \), then the natural map

\[ \beta : M_f(X) \to \text{Hom}(\mathbb{G}^A_1, M_f(X_+ \wedge \mathbb{G}^A_1)) \]

is isomorphic to the sequential colimit of maps

\[ \beta_k : M_{L_k E}(X) \to \text{Hom}(\mathbb{G}^A_1, M_{L_k E}(X_+ \wedge \mathbb{G}^A_1)) \]

Every such map is isomorphic to

\[ \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f(X_+ \wedge E_k)) \to \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A_1, M_f(X_+ \wedge E_k \wedge \mathbb{G}^A_1)) \]

It follows from assertion (1) that each \( \beta_k \) is a stable local weak equivalence of \( S^1 \)-spectra, and hence so is \( \beta \) as a sequential colimit of stable local weak equivalences of \( S^1 \)-spectra.

(3) We can compute a stable local fibrant replacement \( M_f(X)_f \) of \( M_f(X) \) as the spectrum\n
\[ \text{colim}_k M_{L_k E}(X)_f = \text{colim}_k \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f(X_+ \wedge E_k)) \]

because a sequential colimit of fibrant spectra is fibrant in the flasque local stable model model structure of \( S^1 \)-spectra. So it suffices to show that each spectrum \( \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f(X_+ \wedge E_k))_f \) is \( A^1 \)-invariant. By the Mayer–Vietoris sequence of Proposition 4.11 this reduces to showing that every spectrum of the form \( \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f(X_+ \wedge T^k)) \) is \( A^1 \)-invariant. But the latter is obvious because \( \text{Hom}(\mathbb{P}^A \wedge \mathbb{G}^A, M_f(X_+ \wedge T^k))_f \cong M_f(X)_f \) and \( M_f(X)_f \) is \( A^1 \)-invariant by [13, 7.1].

In what follows by bispectra we shall mean \( (S^1, \mathbb{G}^A) \)-bispectra in the category of motivic spaces. We are now in a position to prove the main result of the section. It gives an explicit fibrant resolution of a symmetric Thom spectrum in the category of bispectra.

9.13. **Theorem.** Suppose \( X \in \text{Sm}_m \) and \( E \) is a symmetric Thom \( T \)-spectrum with the bounding constant \( d \) and contractible alternating group action.

(1) If \( d = 1 \) then the \( (S^1, \mathbb{G}^A_1) \)-bispectrum

\[ M^E_f(X)_f := (M_f(X)_f, M_f(X_+ \wedge \mathbb{G}^A_1)_f, M_f(X_+ \wedge \mathbb{G}^A_2)_f, \ldots) \]

is motivically fibrant and represents the \( T \)-spectrum \( X_+ \wedge E \) in the category of bispectra, where “\( f \)” refers to stable local fibrant replacements of \( S^1 \)-spectra.
(2) If \( d < 1 \) then the \((S^1, \mathbb{G}_m^1)\)-bispectrum

\[
M^E_f(X) := (M_E(X), M_E(X_+ \wedge \mathbb{G}_m^1), M_E(X_+ \wedge \mathbb{G}_m^2), \ldots)
\]

is motivically fibrant and represents the \( T \)-spectrum \( X_+ \wedge E \) in the category of bispectra, where "\( f \)" refers to level local fibrant replacements of \( S^1 \)-spectra.

(3) If \( d > 1 \) then the \((S^1, \mathbb{G}_m^1)\)-bispectrum

\[
\Omega^{d-1}_{S^1 \wedge \mathbb{G}_m^1}((M_E[d-1](X)_f , M_E[d-1](X_+ \wedge \mathbb{G}_m^1)_f , M_E[d-1](X_+ \wedge \mathbb{G}_m^2)_f , \ldots))
\]

is motivically fibrant and represents the \( T \)-spectrum \( X_+ \wedge E \) in the category of bispectra, where "\( f \)" refers to stable local fibrant replacements of \( S^1 \)-spectra. Here \( E[d-1] \) stands for the \((d-1)\)-th shift of \( E \) in the sense of Definition 2.6. Another equivalent model for the \( T \)-spectrum \( X_+ \wedge E \) in the category of bispectra is given by

\[
\Omega^{d-1}_{S^1 \wedge \mathbb{G}_m^1}((M_{T^d-1 \wedge E}(X)_f , M_{T^d-1 \wedge E}(X_+ \wedge \mathbb{G}_m^1)_f , M_{T^d-1 \wedge E}(X_+ \wedge \mathbb{G}_m^2)_f , \ldots))
\]

This bispectrum is motivically fibrant and "\( f \)" refers to stable local fibrant replacements of \( S^1 \)-spectra.

**Proof.** (1). The fact that the bispectrum \( M^E_f(X) \) is motivically fibrant follows from Proposition 9.12 and Lemma 9.11. It remains to show that it represent \( E \) in the category of bispectra.

Without loss of generality we may assume \( X = pt \), because we can replace \( E \) with \( X_+ \wedge E \). It follows from Theorem 8.1 that \( E \) is isomorphic in \( SH_T(k) \) to \( C_\ast Fr^E(S_T)^f \). The latter is a \( T \)-spectrum (see Definition 7.4). It is positively fibrant by Theorem 8.1. The motivic equivalence \( T := \mathbb{A}^1/\mathbb{G}_m \to T \) induces an equivalence of categories \( SH_T(k) \cong SH_{\overline{T}}(k) \). It takes \( E \) to a \( \overline{T} \)-spectrum isomorphic to \( C_\ast Fr^E(S_T)^f \). By Lemma 9.8 and Proposition 9.9 the natural map \( C_\ast Fr^E(S_T)^f \to C_\ast Fr^E(S_T)^f \) is a sectionwise weak equivalence in positive degrees, where \( S_{\overline{T}} = (S^0, \overline{T}, \overline{T}^\wedge 2, \ldots) \). By the sublemma on p. 26 \( C_\ast Fr^E(S_T)^f \) is a motivically fibrant \( \overline{T} \)-spectrum in positive degrees (notice that each space \( C_\ast Fr^E(S_T)^f_{E\in 0} \) is motivically fibrant by Proposition 9.9). We see that \( C_\ast Fr^E(S_T)^f \) is a positively fibrant \( \overline{T} \)-spectrum representing \( E \) in \( SH_{\overline{T}}(k) \). Consider now the canonical motivic weak equivalence \( \overline{T} \to S^1 \wedge \mathbb{G}_m^1 \). For the same reasons \( C_\ast Fr^E(S_{S^1 \wedge \mathbb{G}_m^1})^f \) is a positively fibrant \( \overline{T} \)-spectrum which is sectionwise weakly equivalent to \( C_\ast Fr^E(S_T)^f \). We use here Lemma 9.10 as well. It follows that \( C_\ast Fr^E(S_{S^1 \wedge \mathbb{G}_m^1})^f \) is a positively fibrant \( S^1 \wedge \mathbb{G}_m^1 \)-spectrum representing \( E \) in \( SH_{S^1 \wedge \mathbb{G}_m^1}(k) \). It remains to observe that this spectrum is equivalent to the diagonal spectrum for the bispectrum

\[
M^E_f(X) := (M_E(X), M_E(X_+ \wedge \mathbb{G}_m^1), M_E(X_+ \wedge \mathbb{G}_m^2), \ldots).
\]

(2). This immediately follows from (1) if we observe that a levelwise local fibrant replacement of each weighted \( E \)-framed motive \( M_E(X_+ \wedge \mathbb{G}_m^n)_f, n \geq 0 \), is a motivically fibrant \( S^1 \)-spectrum. To see the latter, we repeat the proof of Proposition 9.12(3) and apply Lemma 4.15.

(3). This follows from (1) if we observe that \( E[d-1] \) and \( T^{d-1} \wedge E \) are symmetric Thom \( T \)-spectra with the bounding constant \( d = 1 \).

We finish the section by proving the following useful result.
9.14. **Theorem.** Suppose $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d \leq 1$ and contractible alternating group action.

(1) For every elementary Nisnevich square

\[
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
\]

the square of $S^1$-spectra

\[
\begin{array}{ccc}
M_E(U') & \longrightarrow & M_E(X') \\
\downarrow & & \downarrow \\
M_E(U) & \longrightarrow & M_E(X)
\end{array}
\]

is homotopy cartesian locally in the Nisnevich topology.

(2) The natural map $M_E(X \times \mathbb{A}^1) \to M_E(X)$ is a stable local weak equivalence of $S^1$-spectra. The same is also true for linear $E$-framed motives.

**Proof.** (1). The square of motivic $T$-spectra

\[
\begin{array}{ccc}
U'_+ \wedge E & \longrightarrow & X'_+ \wedge E \\
\downarrow & & \downarrow \\
U_+ \wedge E & \longrightarrow & X_+ \wedge E
\end{array}
\]

is homotopy cartesian in the stable motivic model structure. By Theorem 9.13 it induces a homotopy cartesian square of motivically fibrant bispectra

\[
\begin{array}{ccc}
M^G_E(U'_f) & \longrightarrow & M^G_E(X'_f) \\
\downarrow & & \downarrow \\
M^G_E(U_f) & \longrightarrow & M^G_E(X_f)
\end{array}
\]

Here ‘$f$’ refers to local replacements in each weight (see Theorem 9.13). Passing to weight zero motivic $S^1$-spectra, one gets a homotopy cartesian square of motivically fibrant $S^1$-spectra

\[
\begin{array}{ccc}
M_E(U'_f) & \longrightarrow & M_E(X'_f) \\
\downarrow & & \downarrow \\
M_E(U_f) & \longrightarrow & M_E(X_f)
\end{array}
\]

Our statement now follows.

(2). It is proven similarly to (1) if we start with the stable motivic equivalence of $T$-spectra $(X \times \mathbb{A}^1)_+ \wedge E \to X_+ \wedge E$.

The same statements for linear $E$-framed motives follow from (1), (2) and Lemma 9.7. \(\square\)
10. **Topological Thom spectra with finite coefficients**

In this section we give a topological application of Theorem 9.13. Namely, many important topological Thom spectra like $MU$ can be obtained as the realization of their motivic counterparts if the base field is $\mathbb{C}$. We shall prove below that the stable homotopy groups of such topological Thom spectra with finite coefficients can be computed by means of the stable homotopy groups with finite coefficients of weight zero of the associated $E$-framed motive, which is an explicit positively fibrant $S^1$-spectrum by the very construction. We first need a couple of useful lemmas.

By $f_0(SH(k))$ we shall mean the full triangulated subcategory of effective $T$-spectra, i.e., the subcategory which is compactly generated by the suspension $T$-spectra of the smooth algebraic varieties. We shall also write $f_i(SH(k))$, $\ell \in \mathbb{Z}$, to denote $f_0(SH(k)) \wedge T^\ell$.

10.1. **Lemma.** Suppose $Z \to X$ is a closed embedding of smooth varieties of codimension $\ell$. Then the suspension $T$-spectrum $\Sigma_T(X/X-Z)$ of the sheaf $X/X-Z$ belongs to $f_1(SH(k))$.

**Proof.** If $X/X-Z = Z_+ \wedge T^\ell$ then our assertion is trivial. By using induction, we can cover $X$ by open subsets $X_1, X_2$ such that $Z_2 = Z \cap X_2 \to X_2$ is a trivial embedding and $X_1$ is covered by $n-1$ open trivial pieces. Then for $X_{12} = X_1 \cap X_2$ the embedding $Z_{12} := Z \cap X_{12} \to X_{12}$ is trivial. Denote by $Y := X/X-Z$ and by $Y_i := X_i/X_i-Z_i$. Then $\Sigma^n Y_{12}, \Sigma^n Y_2 \in f_1(SH(k))$ and $\Sigma^n Y_1 \in f_1(SH(k))$ by induction hypothesis. By Lemma 4.6 $Y$ is a pushout of sheaves embeddings $Y_1 \hookrightarrow Y_{12} \hookrightarrow Y_2$. Therefore we have a triangle in $SH(k)$

$$
\Sigma^n Y_{12} \to \Sigma^n Y_1 \oplus \Sigma^n Y_2 \to \Sigma^n Y \to
$$

in which the left two entries belong to $f_1(SH(k))$. It follows that $\Sigma^n Y \in f_1(SH(k))$. \qed

10.2. **Lemma.** Let $E$ be a Thom $T$-spectrum with the bounding constant $d$. Then $E$ belongs to $f_{1-d}(SH(k))$. In particular, $E$ is an effective $T$-spectrum if $d \leq 1$.

**Proof.** Since $f_d(SH(k)) = f_0(SH(k)) \wedge T^d$ for any integer $d$, we may assume $d = 1$ and show that $E$ is an effective $T$-spectrum in this case.

We have $E = \colim_k L_k E$, where each layer has stable homotopy type of $\Omega^k_1((\Sigma^n E_k)^/)$. Here $(\Sigma^n E_k)^/$ stands for a stable motivic fibrant replacement of $\Sigma^n E_k$. Then $E$ is isomorphic in $SH(k)$ to $\hocolim_k \Omega^k_1((\Sigma^n E_k)^/)$. Each $E_k = \colim_j (V_{k,j}/V_{k,j}-Z_{k,j})$, where codimension of $Z_{k,j}$ in $V_{k,j}$ is larger than or equal to $k$. By Lemma 10.1 the flasque cofibrant $T$-spectrum $\Sigma^n(\Sigma^n V_{k,j}/\Sigma^n V_{k,j}-Z_{k,j})$ is in $f_k(SH(k))$. Since $\Sigma^n E_k$ is isomorphic in $SH(k)$ to $\hocolim_j (\Sigma^n(\Sigma^n V_{k,j}/\Sigma^n V_{k,j}-Z_{k,j}))$ and $f_k(SH(k))$ is closed under homotopy colimits, it follows that $\Sigma^n E_k \in f_k(SH(k))$, and hence $\Omega^k_1((\Sigma^n E_k)^/) \in f_0(SH(k))$. The isomorphism $E \cong \hocolim_k \Omega^k_1((\Sigma^n E_k)^/)$ in $SH(k)$ now implies $E \in f_0(SH(k))$. \qed

Suppose that the base field $k$ has an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Following Panin–Pimenov–Röndigs [23, §A4] there is a natural realization functor

$$Re^\varepsilon : SH(k) \to SH,$$

where $SH$ is the homotopy category of the stable model category of classical $S^2$-spectra of topological spaces (it is canonically equivalent to the homotopy category of the stable model
category of classical $S^1$-spectra as well). $Re^e$ is an extension of the functor

$$\text{An} : \text{Sm}_k \to \text{Top}$$

sending a $k$-smooth variety $X$ to $X^\text{an} := X(\mathbb{C})$ with the classical topology.

Following Levine’s indexing [19], denote by $\pi^A_0(E)$, where $E \in SH(k)$, the Nisnevich sheaf on $\text{Sm}_k$ associated to the presheaf $U \mapsto \text{Hom}_{SH(k)}(\Sigma^b_a \Sigma^b_\mathbb{Z}[U_+], E)$. For $E \in SH(k)$ (respectively $E \in SH$) and a positive integer $N$, we let $E/N$ denote an object of $SH(k)$ (respectively $E/N \in SH$) that fits into a triangle $E \xrightarrow{N \cdot \text{id}} E \to E/N \to E[1]$. By definition, $\pi^A_0(E; \mathbb{Z}/N) := \pi^A_0(E/N)$ (respectively $\pi_n(E; \mathbb{Z}/N) := \pi_n(E/N)$).

We are now in a position to prove the main result of the section.

10.3. **Theorem.** Let $k$ be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Suppose $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d \leq 1$ and contractible alternating group action. Then for all integers $N > 1$ and $n \in \mathbb{Z}$, the realization functor $Re^\varepsilon$ induces an isomorphism

$$\pi_n(ME(pt); \mathbb{Z}/N) \cong \pi_n(Re^\varepsilon(E); \mathbb{Z}/N)$$

between stable homotopy groups with mod $N$ coefficients.

**Proof.** By Lemma 10.2 $E$ is an effective $T$-spectrum. It follows from [19, 7.1] that the map

$$\pi^A_0(E; \mathbb{Z}/N)(pt) \to \pi_n(Re^\varepsilon(E); \mathbb{Z}/N)$$

is an isomorphism for all $n \in \mathbb{Z}$. Theorem 9.13 implies that $\pi^A_0(E; \mathbb{Z}/N)$ is computed as the sheaf $\pi^N_n(M_E(pt); \mathbb{Z}/N)$. It remains to observe that

$$\pi^N_n(M_E(pt); \mathbb{Z}/N)(pt) = \pi_n(M_E(pt); \mathbb{Z}/N)(pt) = \pi_n(M_E(pt); \mathbb{Z}/N),$$

what completes the proof.

As the realization of $\text{MGL}$ is isomorphic to $\text{MU}$ in $SH$, the complex cobordism $S^2$-spectrum, and, by Quillen’s Theorem [27], $\pi_n(\text{MU})$ is isomorphic to the Lazard ring $\text{Laz} = \mathbb{Z}[x_1, x_2, \ldots]$, $\deg(x_i) = 2i$, the preceding theorem implies the following

10.4. **Corollary.** Let $k$ be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. For all $n > 1$ and $i \in \mathbb{Z}$, there is an isomorphism $\pi_n(M_{\text{MGL}}(pt); \mathbb{Z}/n) \cong \text{Laz}/n\text{Laz}$, where $M_{\text{MGL}}(pt)$ is the MGL-motive of the point $pt = \text{Spec}(k)$.

We finish the section by the following result about the singular algebraic $E$-homotopy defined in the introduction. It is an analogue of the celebrated theorem of Suslin and Voevodsky [30] on singular algebraic homology.

10.5. **Theorem.** Let $k$ be an algebraically closed field of characteristic zero with an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Suppose $E$ is a symmetric Thom $T$-spectrum with the bounding constant $d \leq 1$ and contractible alternating group action and $X \in \text{Sm}/k$. There are canonical isomorphisms of Abelian groups

$$\pi^E_n(X; \mathbb{Z}/m) = \pi_n(X; \mathbb{C}) \wedge Re^\varepsilon(E; \mathbb{Z}/m)$$

for all integers $n \geq 0$ and $m \neq 0$. 

34
Moreover, if $k$ is any perfect field, then the assignment
\[ X \mapsto \pi^E_*(X) = \pi_*(Fr^E(\Delta^*_k, X)^{gp}) \]
is a generalized homology theory on $\text{Sm}/k$.

**Proof.** Since $M_E(X)$ can be identified with $M_{X, \wedge E}(pt)$ and $X(\mathbb{C})_+ \wedge Re^E(E) \cong Re^E(X_+ \wedge E)$ by [23, A.23], Theorem 10.3 implies
\[ \pi_n(M_E(X)(pt); \mathbb{Z}/m) \cong \pi_n(X(\mathbb{C})_+ \wedge Re^E(E); \mathbb{Z}/m), \quad n \geq 0. \]
We have that
\[ \pi_n(M_E(X)(pt); \mathbb{Z}/m) \cong \pi_n(\Omega^E(\Delta^*_k, X \otimes S^1); \mathbb{Z}/m) \cong \pi_n(Fr^E(\Delta^*_k, X)^{gp}; \mathbb{Z}/m) = \pi^E_*(X; \mathbb{Z}/m). \]
Now the fact that the assignment
\[ X \mapsto \pi^E_*(X) = \pi_*(Fr^E(\Delta^*_k, X)^{gp}) \]
is a generalized homology theory on $\text{Sm}/k$ with $k$ perfect immediately follows from Theorem 9.14 (verifying the excision property and the homotopy invariance property for homology theories). $\square$

### 11. Normally $E$-framed motives

Suppose $E$ is a symmetric Thom $T$-spectrum with contractible alternating group action and the bounding constant $d = 1$. In Theorem 9.13 we have constructed an explicit fibrant bispectrum representing $E$ in terms of $E$-framed motives. We can simplify $E$-framed motives further by forgetting a bit of information and construct, up to a local equivalence of $S^1$-spectra, an equivalent model for them, called normally $E$-framed motives. Then we construct in Theorem 11.26 an explicit fibrant bispectrum representing $E$ whose entries are expressed in terms of weighted normally $E$-framed motives. Another advantage of normally $E$-framed motives is that they lead to representability of important Thom spectra like $MGL$ by schemes (this material is treated in the next section in details).

#### 11.1. Convention

From now on we shall assume that a symmetric Thom spectrum $E$ with the bounding constant $d = 1$ and contractible alternating group action is of the form:

- for any $n \geq 0$, $E_n = Th(V_n)$ with $V_n \to Z_n$ a $\Sigma_n$-equivariant vector bundle of rank $n$ over $Z_n \in \text{Sm}_k$;
- The bonding maps $E_n \wedge T^m \to E_{n+m}$ are induced by closed embeddings $i_{n,m} : Z_n \to Z_{n+m}$ such that we have a Cartesian square

\[
\begin{array}{ccc}
V_n \times T^m & \xrightarrow{i_{n,m}} & V_{n+m} \\
\downarrow & & \downarrow \\
Z_n & \xrightarrow{i_{n,m}} & Z_{n+m}
\end{array}
\]

and $i_{n+m,r} \circ i_{n,m} = i_{n+m+r}$. Applying the shuffle permutation $\chi_{n,m} \in \Sigma_{n+m}$, define the left inclusion maps $i^l_{n,m} := \chi_{n,m} \circ i_{n,m}$. We require the left inclusion maps $i^l_{n,m}$ to fit into
Cartesian squares

\[ \mathbb{A} V, n \times V_n \xrightarrow{i_{V,m}} V_{m+n} \]

where \( i_{V,m} \) is the composition \( \mathbb{A} V, n \times V_n \xrightarrow{i_{V,m}} V_n \times \mathbb{A} V, n \xrightarrow{i_{m,n}} V_{n+m} \xrightarrow{(i_{m,n})} V_{m+n} \). Observe that the maps \( i_{V,m} \) induce the left bonding maps \( u_l : T^m \to E_{m+n} \) in the sense of Definition 2.6.

11.2. **Remark.** The spectrum \( \Sigma^n X_e \) satisfies conditions of 11.1 with \( Z_n = X, V_n = X \times \mathbb{A} n \). The spectrum \( MGL \) is a directed colimit of spectra of the form 11.1. Indeed, for any \( i \geq 0 \) there is a spectrum \( E^{(i)} \), where \( E^{(i)}_n = Th(V^{(i)}_n) \) and \( V^{(i)}_n = Gr(n) \times \mathbb{A} n \) is the tautological vector bundle over the Grassmannian \( Z^{(i)} = Gr(n, n) \). Then the spectra \( E^{(i)} \) satisfy conditions of 11.1 and \( MGL = colim_i E^{(i)} [22, \S 2.1] \).

When \( E \) is a symmetric Thom \( T^2 \)-spectrum with the bounding constant \( d = 1 \) and contractible alternating group action, we impose analogous conditions:

- \( E_n = Th(V_n) \), where \( V_n \to Z_n \) is a \( \Sigma \)-equivariant vector bundle of rank \( 2n \) over \( Z_n \in Sm_k \) for any \( n \geq 0 \);
- The bonding maps \( E_n \wedge T^{2m} \to E_{n+m} \) are induced by closed embeddings \( i_{n,m} : Z_n \to Z_{n+m} \) such that we have a Cartesian square

\[ V_n \times \mathbb{A} V, n \xrightarrow{i_{V,m}} V_{m+n} \]

and \( i_{n+m,n} \circ i_{n,m} = i_{n,m+n} \). Applying the shuffle permutation \( \chi_{n,m} \in \Sigma_{n+m} \), one sets \( i^\prime_{n,m} = \chi_{n,m} \circ i_{n,m} \). The maps \( i_{n,m} \) are required to fit into Cartesian squares

\[ \mathbb{A} V, m \times V_n \xrightarrow{i_{m,n}} V_{m+n} \]

where \( i_{m,n} \) is the composition \( \mathbb{A} V, m \times V_n \xrightarrow{i_{m,n}} V_n \times \mathbb{A} V, m \xrightarrow{i_{n,m}} V_{n+m} \xrightarrow{(i_{n,m})} V_{m+n} \). Observe that the maps \( i_{m,n} \) induce the left bonding maps \( u_l : T^{2m} \to E_{m+n} \) in the sense of Definition 2.6.

11.3. **Remark.** The \( T^2 \)-spectra \( MLS \) and \( MSp \) are directed colimits of spectra that satisfy the above assumptions. Namely, \( MLS = colim_i E^{(i)} \) where \( E^{(i)}_n = Th(V^{(i)}_n), V^{(i)}_n = Gr(n, n) \) is the tautological special bundle over the special Grassmannian \( Z^{(i)} = SGr(n, n) [24, \S 4] \). The spectrum
\[ MSp = \text{colim} E^{(i)} \text{ where } E^{(i)}_n = Th(V^{(i)}_n), V^{(i)}_n = \mathcal{F}Sp_{n,i} \] is the tautological symplectic bundle over the symplectic Grassmannian \( Z^{(i)}_n = HGr(n,ni) \) [24, §6].

As in the previous sections we shall only consider the case of a \( T \)-spectrum \( E \). The interested reader will easily do the same constructions for \( T^2 \)-spectra in a similar fashion.

11.4. **Definition.** (Cf. [9, B.7.1]) Suppose \( X \to Y \) is a closed embedding. We call \( X \) a **locally complete intersection (l.c.i.) subscheme** of \( Y \) if for every point of \( X \) there is an affine neighborhood in \( Y \) such that ideal of definition of \( X \) is generated by a regular sequence.

11.5. **Remark.** ([1, Corollary 4.5]) If \( Y \) is regular and \( X \) is a closed subscheme of codimension \( d \), then \( X \) is an l.c.i. subscheme if and only if the ideal of definition of \( X \) is locally generated by \( d \) elements.

11.6. **Lemma.** For \( X,Y \in SM_k \), there is a natural bijection between the set \( Fr^F_n(X,Y) \) and the set of equivalence classes of quadruples \( (U,Z,\phi,f) \), where

- \( Z \) is a closed l.c.i. subscheme of \( \mathbb{A}^n_X \), finite and flat over \( X \);
- \( U \) is an étale neighborhood of \( Z \) in \( \mathbb{A}^n_X \);
- \( \phi : U \to V_n \) is a regular map, called a framing, such that \( Z = U \times_{V_n} Z_0 \);
- \( f : U \to Y \) is a regular map.

Two quadruples \( (U,Z,\phi,f) \) and \( (U',Z',\phi',f') \) are equivalent if \( Z = Z' \) and there is an open neighborhood \( U_0 \) of \( Z \) in \( U \times_{\mathbb{A}^n_X} U' \) such that the framings \( \phi,\phi' \) as well as regular maps \( f,f' \) coincide on \( U_0 \).

**Proof.** By Voevodsky’s Lemma 2.12, the elements of \( Fr^F_n(X,Y) \) can be described as the sets of equivalence classes of quadruples \( (U,Z,\phi,f) \), where \( Z \) is a closed subset of \( \mathbb{A}^n_U \), finite over \( X \), \( U \) is its étale neighborhood, and \( \phi : U \to V_n \) is a regular map such that \( Z = \phi^{-1}(Z_0) \), and \( f : U \to Y \) is a regular map.

Two quadruples \( (U,Z,\phi,f) \) and \( (U',Z',\phi',f') \) are equivalent if \( Z = Z' \) and there is an open neighborhood \( U_0 \) of \( Z \) in \( U \times_{\mathbb{A}^n_X} U' \), where \( \phi \) coincides with \( \phi' \), and \( f \) coincides with \( f' \).

For any such quadruple the framing \( \phi : U \to E_n \) defines a closed subscheme \( Z' = U \times_{V_n} Z_0 \). Then \( (Z')_{red} = Z \), hence \( Z \) has codimension \( n \) in \( U \), and is locally defined by \( n \) equations. Then it is an l.c.i. subscheme of \( U \) by Remark 11.5. Since \( (Z')_{red} = Z \) is finite over \( X \), \( Z \) is finite over \( X \) as well, and the composition \( Z' \to U \to \mathbb{A}^n_X \) is a closed embedding by [29, Tag 04XV]. Since \( U \to \mathbb{A}^n_X \) is étale, it induces an isomorphism between conormal sheaves of \( Z' \) in \( U \) and \( Z' \) in \( \mathbb{A}^n_X \) by [29, Tag 0635]. Then by Nakayama’s lemma \( Z' \) locally is defined in \( \mathbb{A}^n_X \) by \( n \) equations. Thus \( Z ' \) is an l.c.i. subscheme in \( \mathbb{A}^n_X \), finite over \( X \). It is flat over \( X \) by [29, Tag 00R3]. Then the assignment \( (U,Z,\phi,f) \mapsto (U',Z',\phi',f') \) defines the desired bijection between \( Fr^F_n(X,Y) \) and the set of the statement of the lemma. \( \square \)

11.7. **Definition.** For \( X,Y \in SM_k \) the set of normally framed correspondences \( Fr(E^n)_n(X,Y) \) is the set of equivalence classes of quintuples \( (U,Z,\phi,\psi,f) \), where

- \( Z \) is an l.c.i. subscheme of \( \mathbb{A}^n_X \), finite and flat over \( X \);
- \( U \) is an étale neighborhood of \( Z \) in \( \mathbb{A}^n_X \);
- \( \psi : U \to Z_0 \) is a regular map and \( \phi : N_{Z/\mathbb{A}^n_X} \cong (\psi^*)^{-1} U \) is an isomorphism of vector bundles, where \( i \) is the inclusion \( i : Z \to U \);
• \( f: Z \to Y \) is a regular map.

Two quintuples \( (U, Z, \phi, \psi, f) \) and \( (U', Z', \phi', \psi', f') \) are equivalent if \( Z = Z' \) as subschemes of \( \mathbb{A}^n_X \) and there is an open neighborhood \( U'' \) of \( Z \) in \( U \times_{\mathbb{A}^n_X} U' \) such that \( \psi = \psi' \) on \( U'' \), \( \phi = \phi' \), and \( f = f' \).

11.8. Definition. For an affine scheme \( X = \text{Spec} A \) and its closed subscheme \( Z = \text{Spec} A/I \) of \( X \) denote by \( X^h := \text{Spec} A^h \), where \( (A^h, I^h) \) is the Hensel pair associated to \( (A, I) \)[29, Tag 09XD]. We call \( X^h \) the Henselization of \( X \) in \( Z \). If \( (A, I) \) is a Henselian pair, we will call \( (X, Z) = (\text{Spec} A, \text{Spec} A/I) \) a Henselian pair of schemes.

11.9. Remark. Suppose \( i: Z \to U \) is a closed l.c.i. subscheme and \( \phi: U \to V_n \) is a regular map such that \( \phi \cdot i \subseteq I \), where \( I \) is the sheaf of ideals defining \( Z \) in \( U \) and \( J \) is the sheaf of ideals defining \( Z_n \) in \( V_n \). Then it defines a morphism of vector bundles
\[
N(\phi): N_{Z/U} \to (\pi i)^* V_n,
\]
(11.10)
where \( \pi \) is the projection \( \pi: V_n \to Z_n \), which is dual to the morphism of sheaves
\[
J/J^2 \otimes_{\mathcal{O}_Z} \mathcal{O}_Z \to I/I^2.
\]

11.11. Lemma. In the notation of Remark 11.9 one has:

- (1) if \( \phi: U \to V_n \) is a framing of \( Z \), then \( N(\phi) \) is an isomorphism;
- (2) if \( U \) is affine, \( (Z, U) \) is a Henselian pair, and the morphism \( N(\phi) \) is an isomorphism, then \( \phi \) is a framing of \( Z \) in \( U \).

Proof. (1). Note that when \( \phi \) is a framing, \( I \) is generated by the image of \( J \). Hence the map \( J/J^2 \otimes_{\mathcal{O}_Z} \mathcal{O}_Z \to I/I^2 \) induced by \( \phi \) is a surjection of locally free sheaves of rank \( n \), and so it is an isomorphism. Thus \( N(\phi) \) is an isomorphism.

(2). Let \( U = \text{Spec} R \) and let \( I' \subseteq I \) denote the ideal generated by the image of \( J \). Since \( N(\phi) \) is an isomorphism, the dual map \( J/J^2 \otimes_{\mathcal{O}_Z} \mathcal{O}_Z \to I/I^2 \) is surjective, hence \( I = I' + I^2 \). Since \( I \subseteq \text{Jac}(R) \), then \( I = I' \) by Nakayama’s lemma. We see that \( I \) is generated by the image of \( J \), and hence \( \phi \) is a framing. \( \square \)

There is a forgetful map \( \text{fr}_n^E(X, Y) \to \text{Fr}_{n+1}^E(X, Y) \), \( (U, Z, \phi, f) \mapsto (U, Z, N(\phi), \pi \phi, f) \).

There is also a stabilization map \( \text{Fr}_n^E(X, Y) \to \text{Fr}_{n+1}^E(X, Y) \), \( (U, Z, \psi, \phi, f) \mapsto (U \times \mathbb{A}^1, Z \times 0, \psi', \phi', f) \),

where \( \psi' \) is the composition \( \psi': U \times \mathbb{A}^1 \to U \xrightarrow{\psi} Z_n \to Z_{n+1} \) and \( \phi' \) is the composition
\[
\phi': N_{Z \times 0/\mathbb{A}^n_X} = N_{Z/\mathbb{A}^n_X} \oplus 1 \xrightarrow{\phi \oplus 1} (\psi i)^* (V_n \oplus 1) = (\psi' i)^* (V_{n+1}).
\]

Denote by \( \text{Fr}^E(-, Y) \) the colimit of the presheaves \( \text{Fr}_n^E(-, Y) \) with respect to the stabilization maps
\[
\text{Fr}^E(-, Y) := \text{colim}_n \text{Fr}_n^E(-, Y).
\]
11.12. Lemma. The presheaf \( \tilde{\text{Fr}}^E(Y) \) admits framed transfers and the forgetful map induces a map \( \text{fog}: \text{Fr}^E(Y) \to \tilde{\text{Fr}}^E(Y) \) of presheaves with framed transfers.

Proof. We shall construct a pairing

\[
\text{Fr}_n(X, Y) \times \text{Fr}_m(Y, W) \to \tilde{\text{Fr}}^E_n(X, W), (b, a) \mapsto b^*(a)
\]

as follows. If \( a = (U, Z, \phi, \psi, f) \in \text{Fr}_m^E(Y, W) \) and \( b = (U', Z', \phi', f') \in \text{Fr}_n(X, Y) \), define \( b^*(a) = (U'', Z'', \phi'', \psi'', f'') \), where \( U'' = U' \times_Y U, Z'' = Z' \times_Y Z \), \( \psi'' \) is the composition

\[
\psi'': U' \times_Y U \to U \to Z_m \xrightarrow{\delta_m} Z_{n+m}.
\]

Since the canonical map \( \tau: (N_{Z'/U'})|_{Z''} \oplus (N_{Z'/U})|_{Z''} \to N_{Z'/U''} \) is a surjection of vector bundles of the same rank, it is an isomorphism. Define the isomorphism \( \phi'' \) as the composition

\[
\phi'': N_{Z'/U''} \xrightarrow{\tau^{-1}} (N_{Z'/U'})|_{Z''} \oplus (N_{Z'/U})|_{Z''} \xrightarrow{N(\phi) \oplus \phi} 1^n \oplus (i\psi)^*V_m \to (i\psi'')^*V_{n+m},
\]

where \( N(\phi) \) stands for the isomorphism of formula (11.10) for \( E = S_T \). The function \( f'' \) is, by definition, the composition \( U'' \to U \to W \). This pairing is plainly compatible with stabilization by \( m \) and endows \( \tilde{\text{Fr}}^E(Y) \) with the structure of a framed presheaf such that the forgetful map \( \text{Fr}^E(Y) \to \tilde{\text{Fr}}^E(Y) \) is a morphism of framed presheaves. \( \square \)

11.13. Remark. If \( X \) is an affine smooth variety, the set \( \text{Fr}_n^E(X, Y) \) (respectively \( \tilde{\text{Fr}}_n^E(X, Y) \)) is in bijective correspondence with the set of triples \( (Z, \phi, f) \), where \( Z \) is an l.c.i. closed subscheme in \( A^n \), finite and flat over \( X \), such that \( Z = (A^n |_X)^h \to V_n \), where \( Z \) is a closed subscheme in \( A^n \), finite and flat over \( X \), \( \psi: (A^n |_X)^h \to Z_n \), and \( \phi: N_{Z/A^n} \to (\psi)^*V_n, f: (A^n |_X)^h \to Y \).

11.14. Lemma. Suppose there is an étale map \( Y \to \mathbb{A}^d \). Then the forgetful map of presheaves \( \text{fog}: \text{Fr}_n^E(Y) \to \tilde{\text{Fr}}_n^E(Y) \) is locally surjective in the Nisnevich topology.

Proof. Suppose \( X \) is a local Henselian scheme and \( (Z, \phi, \psi, f) \in \text{Fr}_n^E(X, Y) \). Then \( Z \) is semi-local Henselian and the map \( \psi: U \to Z_n \), where \( U \) is the Henselization of \( Z \) in \( A^n \), factors as \( U \xrightarrow{\tilde{\psi}} Z_n' \subseteq Z_n \), where \( Z_n' \) is an open subset of \( Z_n \) such that the fiber \( V_n' \) over \( Z_n' \) is a trivial vector bundle. Let \( i \) denote the inclusion \( Z \hookrightarrow U \). Fix a trivialization \( V_n|_{Z_n'} \cong Z_n' \times A^h \). It gives a trivialization of \( (\psi|_n)^*V_n \). Composing the latter trivialization with \( \phi \), one gets a trivialization \( \tilde{\phi} \) of the bundle \( N_{Z/A^n} = N_{Z/U} \). The trivialization \( \tilde{\phi} \) provides a basis of the \( k[Z]\)-module \( I/I^2 \), where \( I \) is the ideal of definition of \( Z \) in \( U \). The basis of \( I/I^2 \) lifts to a set of generators \( \gamma = (\gamma_1, \ldots, \gamma_n) \) of \( I \). They define a map \( \phi': (\tilde{\psi}, \gamma): U \to Z_n' \times A^h \to V_n \) such that \( \phi = N(\phi') \) in the sense of formula (11.10).

Now let us extend the regular map \( f: Z \to Y \) to \( f': U \to Y \). By assumption, there is an étale map \( g: Y \to \mathbb{A}^d \). There is also a map \( h: U \to \mathbb{A}^d \) that extends the composition \( gf: Z \to \mathbb{A}^d \). Then \( W = U \times_D Y \) will give an étale neighbourhood of \( Z \) in \( U \), hence there is a section \( U \to W \). Then the composition \( f': U \to W \to Y \) extends \( f \). We see that the triple \( (Z, \phi', f') \in \text{Fr}_n^E(X, Y) \) is a preimage of \( (Z, \phi, \psi, f) \in \text{Fr}_n^E(X, Y) \). \( \square \)
11.15. **Definition.** For $Y \in \text{Sm}_{k}$ define a presheaf of $S^{1}$-spectra $\tilde{\text{Fr}}^{E}(Y \otimes \mathbb{S})$ associated to the presheaf of $\Gamma$-spaces $K \mapsto \text{Fr}^{E}(Y \otimes K)$ (cf. [13, Section 5])

$$\tilde{\text{Fr}}^{E}(Y \otimes \mathbb{S}) = (\text{Fr}^{E}(Y), \text{Fr}^{E}(Y \otimes S^{1}), \text{Fr}^{E}(Y \otimes S^{2}), \ldots).$$

The normally $E$-framed motive of $Y$ is the presheaf of $S^{1}$-spectra

$$\tilde{M}_{E}(Y) = C_{\ast}\text{Fr}^{E}(Y \otimes \mathbb{S}) = (C_{\ast}\text{Fr}^{E}(Y), C_{\ast}\text{Fr}^{E}(Y \otimes S^{1}), C_{\ast}\text{Fr}^{E}(Y \otimes S^{2}), \ldots).$$

It follows from Lemma 11.12 that both $\text{Fr}^{E}(Y \otimes \mathbb{S})$ and $\tilde{M}_{E}(Y)$ are presheaves of $S^{1}$-spectra with framed transfers.

11.16. **Lemma.** The presheaves of stable homotopy groups $\pi_{i}(\text{Fr}^{E}(Y \otimes \mathbb{S}))$ have $\mathbb{Z}F_{s}$-transfers and the presheaves of stable homotopy groups $\pi_{i}(\tilde{M}_{E}(Y))$ are $\mathbb{A}^{1}$-invariant stable $\mathbb{Z}F_{s}$-presheaves.

**Proof.** For $X_{1}, X_{2}$ there is a natural bijection $\tilde{\text{Fr}}^{E}_{n}(X_{1} \sqcup X_{2}, Y) \rightarrow \tilde{\text{Fr}}^{E}_{n}(X_{1}, Y) \times \tilde{\text{Fr}}^{E}_{n}(X_{2}, Y)$, hence there is an isomorphism $\tilde{\text{Fr}}^{E}(X_{1} \sqcup X_{2}, Y \otimes \mathbb{S}) \rightarrow \tilde{\text{Fr}}^{E}_{n}(X_{1}, Y \otimes \mathbb{S}) \times \tilde{\text{Fr}}^{E}_{n}(X_{2}, Y \otimes \mathbb{S})$ of $S^{1}$-spectra. Then the presheaves $\pi_{i}(\text{Fr}^{E}(Y \otimes \mathbb{S}))$ are additive with framed transfers, and hence these are $\mathbb{Z}F_{s}$-presheaves.

Recall that $\sigma_{X} \in \text{Fr}_{1}(X, X)$ uniquely corresponds to the canonical motivic equivalence $X_{+} \wedge \mathbb{P}^{n} \rightarrow X_{+} \wedge T$ and is given by the quadruple $(X \times 0, X \times \mathbb{A}^{1}, pr_{\mathbb{A}^{1}}, pr_{X})$. Then $\sigma_{X}^{E} : \tilde{\text{Fr}}^{E}_{n}(X, Y) \rightarrow \tilde{\text{Fr}}^{E}_{n+1}(X, Y)$ differs from the stabilization map $\tilde{\text{Fr}}^{E}_{n}(X, Y) \rightarrow \tilde{\text{Fr}}^{E}_{n+1}(X, Y)$ by the action of the shuffle permutation $\chi_{1,n}$ on $\mathbb{A}^{1+k}$ and on the vector bundle $V_{n+1}$. As usual, when $n$ is even, they differ by an $\mathbb{A}^{1}$-homotopy, hence induce homotopic maps $C_{\ast}\tilde{\text{Fr}}^{E}_{n}(X, Y) \rightarrow C_{\ast}\tilde{\text{Fr}}^{E}_{n+1}(X, Y)$. Thus $\sigma_{X}^{E}$ induces the identity map on presheaves of homotopy groups.

11.17. **Definition.** For a map of simplicial presheaves $f : X \rightarrow Y$ denote by $\tilde{C}(f)$ the diagonal of the Čech bisimplicial presheaf with $n$-simplices given by simplicial presheaf

$$\tilde{C}(f)_{n} = X \times_{Y} \ldots \times_{Y} X \ (n + 1 \text{ times})$$

with the usual face and degeneracy maps. Then $f$ factors as a composition

$$X \xrightarrow{d(f)} \tilde{C}(f) \xrightarrow{p(f)} Y$$

where $d(f)$ is the diagonal map

$$d(f)_{n} : X_{n} \rightarrow X_{n} \times_{Y_{n}} \ldots \times_{Y_{n}} X_{n}$$

and $p(f)$ is the projection

$$p(f)_{n} : X_{n} \times_{Y_{n}} \ldots \times_{Y_{n}} X_{n} \rightarrow Y_{n}.$$

Note that if $X(U) \rightarrow Y(U)$ is surjective for $U \in \text{Sm}_{k}$, then $p(f) : \tilde{C}(f)(U) \rightarrow Y(U)$ is a weak equivalence of simplicial sets.
11.18. Definition. For every simplicial presheaf $X$ denote by $C_\ast X$ the diagonal of the bisimplicial presheaf $n \mapsto X(\Delta^n_k)$. Then there is a canonical inclusion map $c_0 : X \to C_\ast(X)$, and for every $U \in \text{Sm}_k$ the map

$$C_\ast(c_0) : C_\ast X(U) \to C_\ast C_\ast X(U)$$

is a weak equivalence of simplicial sets.

11.19. Lemma. Suppose there is an étale map $g : Y \to \mathbb{A}^d$ and $f \circ g : \text{Fr}_n^E(Y) \to \tilde{\text{Fr}}_n^E(Y)$ is the forgetful map. Then there exists a map of simplicial presheaves $H_Y : \tilde{C}(f \circ g)_Y \to C_\ast \text{Fr}_n^E(Y)$ on the category of smooth affine varieties, compatible with stabilization by $n$, and that fits into the commutative diagram

$$\begin{array}{ccc}
\text{Fr}_n^E(Y) & \xrightarrow{c_0} & C_\ast \text{Fr}_n^E(Y) \\
d(f \circ g) \downarrow & & \downarrow \sigma \\
\tilde{C}(f \circ g)_Y & \xrightarrow{C(f \circ g)_Y} & C_\ast(\text{Fr}_n^E(Y)) \\
\end{array}$$

Moreover, the map $H_Y$ is functorial in $Y$ in the following sense: if $g : Y \to \mathbb{A}^d$ is étale, $g' : Y' \to \mathbb{A}^d$ is étale, and $q : Y \to Y'$ is a map such that $g' \circ q = g$, then the diagram

$$\begin{array}{ccc}
\tilde{C}(f \circ g)_Y & \xrightarrow{H_Y} & C_\ast \text{Fr}_n^E(Y) \\
\downarrow & & \downarrow \\
\tilde{C}(f \circ g')_{Y'} & \xrightarrow{H_{Y'}} & C_\ast \text{Fr}_n^E(Y') \\
\end{array}$$

is commutative. Here the vertical arrows are induced by $q$.

Proof. For brevity we sometimes write $f \circ g$ instead of $f \circ g$. For an affine smooth $X$ the set of $m$ simplices $\tilde{C}(f \circ g)_m(X)$ consists of $(m + 1)$-triples of correspondences $(Z, \phi_0, f_0), \ldots, (Z, \phi_m, f_m)$ in $\text{Fr}_n^E(X, Y)$ such that the maps $\pi \phi_0, \ldots, \pi \phi_m : U \to Z_n$ are equal, isomorphisms on normal bundles $N(\phi_i) : N_{Z/\mathbb{A}^d_{\text{X}}} \to (\pi \phi_i)^* V_n$ are equal for $i = 0, \ldots, m$, and the regular maps $f_i : U \to Y$ coincide on $Z$. Here $U$ denotes the Henselization of $Z$ in $\mathbb{A}^d_{\text{X}}$ and we use Remark 11.13 here. The addition map $V_n \times Z_n V_n \to V_n$ and scalar multiplication map $\mathbb{A}^1 \times V_n \to V_n$ give rise to the linear combination map

$$V_n \times Z_n V_n \times \ldots \times Z_n V_n \times \mathbb{A}^{m+1} \to V_n,$$

$$(v_0, \ldots, v_m) \mapsto t_0 v_0 + \ldots + t_m v_m.$$

For a $(m + 1)$-tuple $(Z, \phi_0, f_0), \ldots, (Z, \phi_m, f_m)$ in $\tilde{C}(f \circ g)_m(X)$ the maps $\phi_0, \ldots, \phi_m$ coincide after composing them with $\pi : V_n \to Z_n$, hence they define a map $\phi : U \to V_n \times Z_n \times \ldots \times Z_n V_n$. Taking composition with the linear combination map, we get a map

$$\Phi = t_0 \phi_0 + \ldots + t_m \phi_m : U \times \Delta^n_k \to V_n,$$

where $t_0, \ldots, t_m$ denote the barycentric coordinates on $\Delta^n_k$. Let $J$ denote the sheaf of ideals defining $Z_n$ in $V_n$. For every $\phi_i$ we have that $\phi_i^\ast(J)$ lies inside the ideal $I$ defining $Z$ in $U$. Then $\Phi^\ast(J)$ lies inside the ideal $I \otimes k[\Delta^n] \subseteq k[U] \otimes k[\Delta^n]$ which defines $Z \times \Delta^n_k$ inside $U \times \Delta^n_k$.

Let $\Phi^h : (U \times \Delta^n_k)^h \to V_n$ denote the map on Henselization induced by $\Phi$. Since $\Phi^\ast(J) \subseteq I \otimes k[\Delta^n]$, then $(\Phi^h)^\ast(J) \subseteq (I \otimes k[\Delta^n])^h$, where the latter denotes the corresponding ideal in
the Henselization ring $k(U \times \Delta^m)^h$. The normal bundles $N_{Z \times \Delta^m \to U \times \Delta^m}$ and $N_{Z \times \Delta^m \to (U \times \Delta^m)^h}$ are canonically isomorphic. We denote them by $N_{Z \times \Delta^m}$ for brevity. By 11.9 $\Phi^h$ defines a morphism of vector bundles

$$N(\Phi^h) : N_{Z \times \Delta^m} \to (\pi \Phi^h)^* V_n,$$

where $i_\Delta : Z \times \Delta^m \to (U \times \Delta^m)^h$ denotes the inclusion. Let $i : Z \to U$ denote the inclusion, and $p : Z \times \Delta^m \to Z$ denote the projection. Then $\pi \Phi^h i_\Delta = \pi \Phi^h i p$ for every $j = 0, \ldots, m$. In particular, $\pi \Phi^h i_\Delta = \pi \phi ip$. The normal bundle $N_{Z \times \Delta^m}$ is canonically isomorphic to the pullback $p^* N_{Z/U}$.

By construction, the morphism of bundles $N(\Phi^h)$ equals the sum

$$N(\Phi^h) = t_0 p^* N(\phi_0) + \ldots + t_m p^* N(\phi_m) : p^* N_{Z/U} \to p^* (\pi \phi_0 i)^* V_n.$$

Since $N(\phi_i) = N(\phi_0)$ for all $i = 0, \ldots, m$, then $N(\Phi^h) = p^* N(\phi_0)$ is an isomorphism, because so is $N(\phi_0)$. Then $\Phi^h$ is a framing of $Z \times \Delta^m$ in $(U \times \Delta^m)^h$ by Lemma 11.11(2).

The maps $f_0, \ldots, f_m : U \to Y$ coincide on $Z$. Consider the map

$$t_0 g f_0 + \ldots + t_m g f_m : U \times \Delta^m \to A^d.$$

Then the fiber product $U' = (U \times \Delta^m) \times_{A^d} Y$ is an étale neighborhood of $Z \times \Delta^m$ in $U \times \Delta^m$, hence there is a unique section $s : (U \times \Delta^m)^h \to U'$, where $(U \times \Delta^m)^h$ is the Henselization of $Z \times \Delta^m$ in $U \times \Delta^m$. Denote by $t_0 f_0 + \ldots + t_m f_m$ the composition

$$(U \times \Delta^m)^h \xrightarrow{s} U' \to Y.$$

Then for every $m \geq 0$ one gets a map $H_m : \hat{C}(f \circ g)_m \to C_m F_{\phi n} E(Y)$ defined as

$$H_m : (Z, \phi_0, f_0, \ldots, Z, \phi_m, f_m) \mapsto (Z \times \Delta^m, \Phi^h, t_0 f_0 + \ldots + t_m f_m)$$

in the notation of Remark 11.13. Clearly, the maps $H_m$ are compatible with the face and degeneracy maps and yield the desired morphism of simplicial presheaves $H_Y : \hat{C}(f \circ g)_Y \to C_* F_{\phi n} E(Y)$ on the category of smooth affine varieties.

If a $m$-tuple $(Z, \phi_0, f_0, \ldots, Z, \phi_m, f_m)$ in $\hat{C}(f \circ g)_m$ is in the image of $d(f \circ g)_m$, then $(Z, \phi_i, f_i) = (Z, \phi_0, f_0)$ for $i = 0, \ldots, m$. Thus $(Z \times \Delta^m, \Phi^h, t_0 f_0 + \ldots + t_m f_m) = (Z \times \Delta^m, \Phi^h, p r, f_0 \circ p r)$, where $pr : U \times \Delta^m \to U$ is the projection. Then the left triangle in the diagram (11.20) is commutative.

As we have already proved, if $\Phi = t_0 \phi_0 + \ldots + t_m \phi_m$ then $N(\Phi^h) = p^* N(\phi_0)$. It follows that the right triangle in the diagram (11.20) is commutative as well.

To see that the diagram (11.21) commutes when $q : Y \to Y'$ is a map over $A^d$, we note that the following maps coincide:

$$t_0 g f_0 + \ldots + t_m g f_m = t_0 g' f_0 + \ldots + t_m g' f_m : U \times \Delta^m \to A^d.$$

Then the diagram

$$(U \times \Delta^m)^h \xrightarrow{(U \times \Delta^m) \times_{A^d} Y} Y \xrightarrow{q} Y'$$

is commutative. Thus we get that

$$q(t_0 f_0 + \ldots + t_m f_m) = t_0 q f_0 + \ldots + t_m q f_m : (U \times \Delta^m)^h \to Y'.$$
and hence (11.21) commutes.

\[ \square \]

11.22. **Lemma.*** Suppose there is an étale map \( g : Y \to \mathbb{A}^d \). Then the natural map \( f \circ g : \text{Fr}^E(Y) \to \tilde{\text{Fr}}^E(Y) \) induces a local stable weak equivalence of \( S^1 \)-spectra \( \text{M}_E(Y) \to \tilde{\text{M}}_E(Y) \).

**Proof.** For every \( n \geq 0 \) the map \( f \circ g : \text{Fr}^E_n(Y) \to \tilde{\text{Fr}}^E_n(Y) \) is locally surjective by Lemma 11.14. It follows that the induced map \( \tilde{C}(f \circ g) \to \tilde{\text{Fr}}^E_n(Y) \) is a local weak equivalence. Let \( \tilde{C}(f \circ g) \) denote the presheaf of Segal \( S^1 \)-spectra associated to the presheaf of \( \Gamma \)-spaces \( K \mapsto \tilde{C}(f \circ g) \), where \( f \circ g \) is the forgetful map \( f \circ g : \text{Fr}^E_n(Y \otimes K) \to \tilde{\text{Fr}}^E_n(Y \otimes K) \). Then the induced map

\[
\tilde{C}(f \circ g) \to \tilde{\text{Fr}}^E_n(Y \otimes \mathbb{S})
\]

is a levelwise local weak equivalence of \( S^1 \)-spectra.

For any finite pointed set \( K \) we have that \( Y \otimes K \) is an étale over \( \mathbb{A}^d \) via the natural composition \( Y \otimes K \to Y \xrightarrow{\Delta} \mathbb{A}^d \) and for any map \( K \to K' \) of pointed sets the induced map \( Y \otimes K \to Y \otimes K' \) is a map of varieties over \( \mathbb{A}^d \). Then the maps \( H_{Y \otimes K} \) of Lemma 11.19 induces a map of presheaves of \( S^1 \)-spectra \( H : \tilde{C}(f \circ g) \to C \circ \text{Fr}^E_n(Y \otimes \mathbb{S}) \). Applying \( C_* \) we get a commutative diagram

\[
\begin{array}{ccc}
C_* \text{Fr}^E_n(Y \otimes \mathbb{S}) & \xrightarrow{C_* f \circ g} & C_* \text{Fr}^E_n(Y \otimes \mathbb{S}) \\
C_* \hskip 1cm & \downarrow C_* H & \downarrow C_* (f \circ g) \\
C_* \tilde{C}(f \circ g) & \xrightarrow{C_* f \circ g} & C_* \tilde{\text{Fr}}^E_n(Y \otimes \mathbb{S})
\end{array}
\]

The horizontal arrows in the diagram are motivic stable weak equivalences. Then \( C_* H \) has both a left and a right inverse in \( SH_{S^1}(k) \). So \( C_* H \) is a motivic stable weak equivalence as well, and hence so are the vertical arrows. Since \( C_* \) is an idempotent operation up to motivic equivalence and sequential colimits preserve stable motivic equivalences, it follows that \( C_* (f \circ g) : \text{M}_E(Y) \to \tilde{\text{M}}_E(Y) \) is a motivic stable weak equivalence. It follows from Lemmas 9.12(3), 11.16, [13, 7.1] and [12, 1.1] that local stable fibrant replacements \( \text{M}_E(Y)_f, \text{M}_E(Y)_f \) of \( \text{M}_E(Y), \text{M}_E(Y) \) are motivically fibrant \( S^1 \)-spectra. Therefore the induced map \( C_* (f \circ g)_f : \text{M}_E(Y)_f \to \tilde{\text{M}}_E(Y)_f \) is a sectionwise level weak equivalence of spectra, and hence \( C_* (f \circ g) : \text{M}_E(Y) \to \tilde{\text{M}}_E(Y) \) is a stable local weak equivalence, as required. \[ \square \]

11.23. **Lemma.*** Suppose \( Y \in \text{Sm}_k \) equals the union of two open subschemes \( Y_1 \) and \( Y_2 \). Let \( Y_{12} = Y_1 \cap Y_2 \). Then

\[
\begin{array}{ccc}
\tilde{\text{M}}_E(Y_{12}) & \xrightarrow{} & \tilde{\text{M}}_E(Y_1) \\
\downarrow & & \downarrow \\
\tilde{\text{M}}_E(Y_2) & \xrightarrow{} & \tilde{\text{M}}_E(Y)
\end{array}
\quad
\begin{array}{ccc}
\text{M}_E(Y_{12}) & \xrightarrow{} & \text{M}_E(Y_1) \\
\downarrow & & \downarrow \\
\text{M}_E(Y_2) & \xrightarrow{} & \text{M}_E(Y)
\end{array}
\]

are homotopy pushout squares in the local stable model structure of \( S^1 \)-spectra.

**Proof.** Similarly to [13, Definition 8.3] one can introduce the presheaves of abelian groups \( \mathbb{Z}\text{Fr}^E(Y) \) imposing the additivity relation on supports in \( \mathbb{Z}\tilde{\text{Fr}}^E(Y) \). The same reasons as in [10,
Theorem 1.2] show that homology of the complex \( C_\ast \tilde{Z}F(Y)(X) \) computes homology of the \( S^1 \)-spectrum \( \tilde{M}_E(Y)(X) \) for any \( X \in \mathbf{Sm}_k \). Repeating Lemma 4.9, Corollary 4.10 and Proposition 4.11 literally, one gets that the sequence

\[
0 \to \tilde{Z}F(Y_{12}) \to \tilde{Z}F(Y_1) \oplus \tilde{Z}F(Y_2) \to \tilde{Z}F(Y) \to 0
\]

is locally exact, hence there is a triangle in the derived category

\[
C_\ast \tilde{Z}F(Y_{12}) \to C_\ast \tilde{Z}F(Y_1) \oplus C_\ast \tilde{Z}F(Y_2) \to C_\ast \tilde{Z}F(Y)
\]

of complexes of sheaves. So the first square in the statement of the lemma is homotopy pushout. The same proof applies to showing that the second square is homotopy pushout. \( \square \)

11.24. Definition. Suppose \( E \) is a directed colimit of spectra \( E^{(i)} \) satisfying Condition 11.1. Define the presheaf \( \widetilde{Fr}^E(Y) \) as the directed colimit

\[
\widetilde{Fr}^E(Y) = \text{colim} \widetilde{Fr}^{E^{(i)}}(Y),
\]

and the normally \( E \)-framed motive \( \widetilde{M}_E(Y) \) as the directed colimit \( \widetilde{M}_E(Y) = \text{colim} \widetilde{M}_{E^{(i)}}(Y) \).

11.25. Proposition. Suppose \( Y \in \mathbf{Sm}_k \) and \( E \) is a directed colimit of spectra \( E^{(i)} \) satisfying Condition 11.1. Then the natural forgetful map \( f \circ g : \widetilde{Fr}^E(Y) \to \widetilde{Fr}^E(Y) \) induces a local stable weak equivalence of \( S^1 \)-spectra \( M_E(Y) \to \widetilde{M}_E(Y) \).

Proof. Every smooth variety \( Y \) of dimension \( d \) has a Zariski cover by varieties \( Y_i \) that admit étale maps \( Y_i \to \mathbb{A}^d \). Then the statements follow by induction on the number of varieties in the cover of \( Y \) if we apply Lemmas 11.22 and 11.23 as well as the fact that \( M_E(Y) = \text{colim} M_{E^{(i)}}(Y) \).

Similarly to framed correspondences there is a natural action of the category \( \text{Fr}_0(k) \) on Nisnevich sheaves \( \widetilde{Fr}^E(-,X) \) that takes \( U \in \text{Fr}_0(k) \) to the sheaf \( \widetilde{Fr}^E(- \times U,X \times U) \). The action gives rise to maps of \( S^1 \)-spectra

\[
a_n : \widetilde{M}_E(X_+ \wedge G_m^1, X_+ \wedge \mathbb{G}^{n+1}_m) \to \text{Hom}(K_0, \tilde{M}_E(X_+ \wedge G_m^{n+1})), \quad n \geq 0,
\]

literally repeating the construction of the same maps for weighted \( K \)-motives in [11, Section 3].

We finish the section by the following computation.

11.26. Theorem. Suppose \( X \in \mathbf{Sm}_k \) and \( E \) is a symmetric Thom \( T \)-spectrum with the bounding constant \( d = 1 \) and contractible alternating group action. Then the \( (S^1, G_m^1) \)-bispectrum

\[
\tilde{M}_E^G(X)_f := (\tilde{M}_E(X)_f, \tilde{M}_E(X_+ \wedge G_m^1)_f, \tilde{M}_E(X_+ \wedge \mathbb{G}^2_m)_f, \ldots)
\]

with bonding maps induced by \( a_n \)-s above is motivically fibrant and represents the \( T \)-spectrum \( X_+ \wedge E \) in the category of bispectra, where “\( f \)” refers to stable local fibrant replacements of \( S^1 \)-spectra.

Proof. By Lemma 11.16 the sheaves of stable homotopy groups of each \( S^1 \)-spectrum in \( \tilde{M}_E(X) \) are \( \mathbb{A}^1 \)-invariant, stable with framed transfers. It follows from [12] that they are strictly \( \mathbb{A}^1 \)-invariant. By [13, 7.1] all \( S^1 \)-spectra of the bispectrum are motivically fibrant. Observe that the natural map of bispectra

\[
M_E^G(X)_f \to \tilde{M}_E^G(X)_f,
\]

44
induced by the forgetful map, is a level equivalence by Proposition 11.25. It follows from Theorem 9.13 that $M_E^T(X)_{/f}$ is motivically fibrant and represents the $T$-spectrum $X_+ \wedge E$ in the category of bispectra. □

12. COMPUTING THE ALGEBRAIC COBORDISM SPECTRUM $MGL$

In this section we give another description of the bispectrum $M_E^T(X)$ for the case $E = MGL$ in terms of Hilbert schemes and $\Omega$-correspondences.

12.1. Definition. Given a ring $R$, we call a submodule $M$ of $R^N$ admissible if the quotient $R^N/M$ is projective. If $M$ is admissible, then it is also projective. We say that a map $f : M \rightarrow R^N$ is an admissible embedding if $f$ is injective and $f(M)$ is an admissible submodule of $R^N$.

12.2. Definition. Given a ring $R$, denote by $R[\Delta^n] = R[t_0, \ldots, t_n]/(t_0 + \ldots + t_n - 1)$ the coordinate ring on $\Delta^n$. Also, $R[\partial \Delta^n] := R[\Delta^n]/(t_0 t_1 \ldots t_n)$ and for every $0 \leq i \leq n$, $R[\partial_i \Delta^n] := R[\Delta^n]/t_i$ denotes the ring of functions on the $i$-th face. We also set $R[\partial_i j \Delta^n] := R[\Delta^n]/(t_i t_j)$.

For every $R[\Delta^n]$-module $M$ denote by $\partial M = M \otimes_{R[\Delta^n]} R[\partial \Delta^n]$, $\partial_i M = M \otimes_{R[\Delta^n]} R[\partial_i \Delta^n]$, $\partial_{ij} M = M \otimes_{R[\Delta^n]} R[\partial_i j \Delta^n]$.

12.3. Lemma. For any affine $X$ there is a bijection between $\widetilde{Fr}_{n,X}^{MGL}(X, Y)$ and the set of quadruples $(Z, \phi, \psi, f)$, where $Z$ is a closed l.c.i. subscheme of $k^n_X$, finite and flat over $X$, $R$ is the Henselization ring of $Z$ in $k^n_X$, $i : Z \rightarrow \text{Spec} R$ is the embedding, $\psi : \text{Spec} R \rightarrow Gr(n)$, $\phi : N_Z/k^n_X \rightarrow (\psi)^* \tau_n$ is an isomorphism of $k[Z]$-modules, and $f : Z \rightarrow Y$ is a regular map.

Proof. This follows from Remark 11.13, Definition 11.24 and the fact that $\text{Gr}(n)(R)$ equals $\text{colim} \text{Gr}(n, N)(R)$ for any $k$-algebra $R$. □

12.4. Definition. For $X, Y \in \text{Sm}_k$ denote by $\text{Emb}_k(X, Y)$ the set of couples $(Z, f)$, where $Z$ is a closed l.c.i. subscheme in $\mathbb{A}^n_X$, finite and flat over $X$, and $f$ is a regular map $f : Z \rightarrow Y$. Note that $\text{Emb}_k(X, Y)$ is pointed at the couple $(\emptyset, \emptyset \rightarrow Y)$.

We need the following intermediate object:

12.5. Definition. For $X, Y \in \text{Sm}_k$ denote by $B_n(X, Y)$ the set of quadruples $(Z, \phi, \psi, f)$, where $Z$ is a closed l.c.i. subscheme of $k^n_X$, finite and flat over $X$, $\psi : Z \rightarrow Gr(n)$, $\phi : N_Z/k^n_X \rightarrow \psi^* \tau_n$ is an isomorphism of vector bundles over $Z$, and $f : Z \rightarrow Y$ is a regular map.

12.6. Remark. The motivic space $Gr(n) = \text{colim}_N Gr(n, N)$ is a directed colimit of closed embeddings of smooth varieties. For a closed $k$-scheme $Z$ by a regular map $\psi : Z \rightarrow Gr(n)$ we mean an element of $\text{colim}_N \text{Hom}(Z, Gr(n, N))$. Then every regular map $\psi : Z \rightarrow Gr(n)$ induces a vector bundle $\psi^* \tau(n)$ over $Z$. Note that for a $k$-algebra $R$ the set $Gr(n, N)(R)$ is in bijective correspondence with the set of rank $n$ admissible submodules of $R^N$ (see [29, Tag 089R]).

Note that $B_n(-, -), \text{Emb}_k(-, -)$ are presheaves on $\text{Sm}_k$. There are natural forgetful maps $\widetilde{Fr}_n^{MGL}(-, Y) \rightarrow B_n(-, Y) \rightarrow \text{Emb}_k(-, Y)$. We shall prove that for any smooth affine $X$ these maps induce weak equivalences of simplicial sets

$$C_* \widetilde{Fr}_n^{MGL}(X, Y) \rightarrow C_* B_n(X, Y) \rightarrow C_* \text{Emb}_n(X, Y).$$
12.7. Lemma. For every affine smooth \( X \) the map \( C_n Fr_n^{MGL}(X, Y) \rightarrow C_n B_n(X, Y) \) is a trivial Kan fibration of simplicial sets.

Proof. The map on zero simplices \( Fr_n^{MGL}(X, Y) \rightarrow B_n(X, Y) \) is surjective by Lemma \( A.5 \). Suppose \( \sigma: \Delta[m] \rightarrow C_n(B_n(X, Y)) \) is a \( m \)-simplex and there is a lift of the boundary \( \gamma: \partial \Delta[m] \rightarrow C_n Fr_n^{MGL}(X, Y) \). Then \( \gamma \) is represented by a collection \( \gamma_i: \partial \Delta[m] \rightarrow C_n Fr_n^{MGL}(X, Y) \), such that \( \gamma_i \) and \( \gamma_j \) agree on the intersection \( \partial \Delta[m] \cap \partial \Delta[m] \).

Suppose \( \sigma \) is represented by a quadruple \( (Z, \phi, \psi, f) \in B_n(\Delta^m_n, Y) \). Let \( \partial \Delta^m_n \) be the variety \( \text{Spec} k[\Delta^m_n]/(t_0 t_1 \ldots t_m) \), where \( t_0, \ldots, t_m \) are the barycentric coordinates of the algebraic simplex \( \Delta^m_n \). Let \( \partial Z = Z \times_{\Delta^m_n} \partial \Delta^m_n \) denote the fiber of \( Z \) over \( \partial \Delta^m_n \).

Let \( R \) denote the Henselization ring of \( Z \) inside \( \Delta^m_n \). Note that by [29, Tag09XXK] the ring \( R' = R \otimes_{k[\Delta^m_n]} k[\partial \Delta^m_n] \), is the Henselization ring of \( \partial Z \), and \( R' = R \otimes_{k[\Delta^m_n]} k[\partial \Delta^m_n] \) is the Henselization of \( \partial Z \), and \( R' \otimes_R R' \) is the Henselization of \( \partial Z \).

Then each \( \gamma_i \) is represented by a quadruple \( (\partial Z, \phi|_{\partial Z}, \psi_i'|_{\partial Z}) \) as in Lemma 12.3, where \( \psi_i': \text{Spec} R' \rightarrow Gr(n) \) extends the map \( \psi_i: \partial Z \rightarrow Gr(n) \). For any \( i, j \) the maps \( \psi_i' \) and \( \psi_j' \) agree on \( \text{Spec} R' \otimes_R R' \). Then they descend to a map \( \psi': \text{Spec} R' \rightarrow Gr(n) \).

Then by Lemma \( A.5 \) there exists a map \( \psi': \text{Spec} R \rightarrow Gr(n) \) that extends \( \psi' \) and \( \psi \). Clearly, the quadruple \( (Z, \psi', \phi, f) \) in \( Fr_n^{MGL}(\Delta^m_n, Y) \) is the desired lift of \( \sigma \) that extends \( \gamma \). \( \square \)

12.8. Lemma. For any \( n \) and any affine smooth \( X \) the forgetful map \( f: B_n(\sigma, Y) \rightarrow C_n(\sigma, Y) \) induces a trivial Kan fibration of simplicial sets \( C_n f: C_n B_n(X, Y) \rightarrow C_n C_n B_n(X, Y) \).

Proof. Suppose \( m \geq 0 \), \( \sigma: \Delta[m] \rightarrow C_n B_n(X, Y) \) is a \( m \)-simplex and \( \gamma: \partial \Delta[m] \rightarrow C_n B_n(X, Y) \) is a lift of its boundary. Let us prove that there is a \( m \)-simplex \( \sigma': \Delta[m] \rightarrow C_n B_n(X, Y) \) making the diagram

\[
\begin{array}{ccc}
\partial \Delta[m] & \xrightarrow{\gamma} & C_n B_n(X, Y) \\
\downarrow & & \downarrow f \\
\Delta[m] & \xrightarrow{\sigma'} & C_n B_n(X, Y)
\end{array}
\]

commutative.

Suppose \( \sigma \) is given by a couple \( (Z, f) \in B_n(\Delta^m_n, Y) \). The map \( \gamma \) is given by a collection of quadruples \( \gamma_i = (\partial Z, \phi_i, \psi_i, f|_{\partial Z}) \in B_n(\partial \Delta^m_n, Y) \) as in Definition 12.5, where \( \partial \Delta^m_n \) denotes the \( i \)-th face of the algebraic simplex \( \Delta^m_n \) and \( \partial Z \) is the fiber of \( Z \over \partial \Delta^m_n \). The elements \( \gamma_i \) coincide on the intersections \( \partial i \Delta^m_n \cap \partial j \Delta^m_n \), and hence the regular maps \( \psi_i: \partial Z \rightarrow Gr(n) \) coincide on the intersections \( \partial i Z \). So they descend to a regular map \( \psi: \partial Z \rightarrow Gr(n, N) \) for some number \( N \) by Remark 12.6. The map \( \psi \) defines an admissible submodule \( j: P = \psi^* \tau(n, N) \subseteq k[\partial Z]^N \). The isomorphisms \( \phi_i: N_{\partial Z} \rightarrow \partial P \) coincide on intersections \( N_{\partial Z} \), and then by Lemma \( A.6 \) there is a unique isomorphism \( \phi: N_{\partial Z} \rightarrow P \) that extends \( \phi_i \).

Then \( j \circ \phi: N_{\partial Z} \rightarrow k[\partial Z]^N \) is an admissible embedding. By Lemma \( A.8 \) it can be extended to an admissible embedding \( \Phi: N_Z \rightarrow k[Z] \oplus k[Z]^d \) such that \( \partial \Phi \) equals the composition of \( j \circ \phi \) and the standard embedding \( k[\partial Z]^N \rightarrow k[\partial Z]^N \oplus k[\partial Z]^d \). It follows that the image \( \Phi(N_Z) \subseteq k[Z]^N \oplus k[Z]^d \) is a rank \( n \) admissible submodule, and so it corresponds to a regular map \( \Psi: Z \rightarrow 46 \).
\[ \Gamma \]  

Proof. For \( Y \in \text{Sm}_k \) the sheaf \( \text{Emb}_n(-, Y) \) is representable by a countable disjoint union \( E^Y_n := \bigsqcup_{d \geq 0} E^Y_{n,d} \) of smooth quasi-projective varieties.

Proof. Denote by \( \text{Emb}_n(U, Y)_d \) the set of couples \((Z, f)\) where \( Z \) is a closed l.c.i. subscheme of \( \mathbb{A}^n \), finite of degree \( d \) and flat over \( U \). Then \( \text{Emb}_n(-, Y)_d \) is a subsheaf of \( \text{Emb}_n(-, Y) \), and \( \text{Emb}_n(U, Y)_d \) is the disjoint union of \( \text{Emb}_n(U, Y)_d, d \geq 0 \), for any connected \( U \in \text{Sm}_k \).

By [7, Lemma 5.1.3] the presheaf \( \text{Emb}_n(-, k)_d \) is represented by a smooth quasi-projective scheme \( \text{Hilb}^d(-, k) \). There is the universal finite flat map \( W_d \to \text{Hilb}^d_{k}(-, k) \). Then the Weil restriction functor \( R_{W_d/\text{Hilb}^d_{k}(-, k)}(W_d \times k Y) \) coincides with \( \text{Emb}_n(-, Y)_d \) and is represented by a quasi-projective smooth scheme \( E^Y_{n,d} \) over \( k \) by [4, 7.6.4-7.6.5].

The natural inclusions of affine spaces \( \mathbb{A}^n \to \mathbb{A}^{n+1} \) induce stabilization maps of pointed sheaves \( \text{Emb}_n(-, Y) \to \text{Emb}_{n+1}(-, Y) \). Denote by \( \text{Emb}(-, Y) \) the pointed sheaf \( \text{Emb}(-, Y) = \text{colim}_n \text{Emb}_n(-, Y) \). Note that forgetful maps \( \Phi^{\text{MGL}}_n (-, Y) \to \text{Emb}_n(-, Y) \) are consistent with the stabilization maps.

12.10. Corollary. The sheaf \( \text{Emb}_n(-, Y) \) is isomorphic to a sequential colimit \( E^Y \) of smooth quasi-projective varieties.

Proof. This follows from Proposition 12.9 and the fact that \( \bigsqcup_{d \geq 0} E^Y_{n,d} \) is \( \text{colim}_{k \geq 0} (E^Y_{n,d_k} \sqcup \cdots \sqcup E^Y_{n,d_k}) \). Hence \( \text{Emb}(-, Y) \) is isomorphic to \( E^Y := \text{colim}_{n,k \geq 0} (E^Y_{n,d_k} \sqcup \cdots \sqcup E^Y_{n,d_k}) \).

We shall give an alternative description of the space \( C_s \text{Emb}(-, Y) \) in terms of \( \Omega \)-correspondences studied in [21].

12.11. Definition. For \( X, Y \in \text{Sm}_k \) denote by \( \text{Cor}^{\Omega}_n(X, Y) \) the groupoid with objects given by the set \( \text{Emb}_n(X, Y) \) whose morphisms between \((Z_1, f_1) \) and \((Z_2, f_2) \) are isomorphisms \( \alpha : Z_1 \to Z_2 \) such that \( \pi_{Z_2} \alpha = \pi_{Z_1} \) and \( f_2 \alpha = f_1 \), where \( \pi_{Z_2} \) denotes the projection \( \pi_{Z_2} : Z_1 \to \mathbb{A}^n \to X \). The assignment \( X \mapsto \text{Cor}^{\Omega}_n(X, Y) \) defines a presheaf of groupoids on \( \text{Sm}_k \). There are natural stabilization maps \( \text{Cor}^{\Omega}_n(-, Y) \to \text{Cor}^{\Omega}_{n+1}(-, Y) \) induced by the natural inclusions \( \mathbb{A}^n \to \mathbb{A}^{n+1} \). Denote by \( \text{Cor}^{\Omega}(X, Y) \) the colimit \( \text{Cor}^{\Omega}(-, Y) = \text{colim}_n \text{Cor}^{\Omega}_n(X, Y) \).

12.12. Lemma. Suppose \( f : X \to Y \) is a l.c.i. embedding, \( g : X \to W \) is any regular map and \( W \) is regular. Then the map \( (f, g) : X \to Y \times W \) is an l.c.i. embedding.

Proof. The map \( (f, g) \) is the composition \( X \xrightarrow{\Gamma_{\pi}} X \times W \xrightarrow{f \times id} Y \times W \). The map \( f \times id \) is a l.c.i. embedding. The graph inclusion \( \Gamma_{\pi} : X \to X \times W \) fits into the pullback diagram
where the right arrow is the diagonal embedding. Since $W$ is regular, the diagonal map $W \to W \times W$ is a l.c.i. embedding, so the ideal defining $X$ in $X \times W$ is locally generated by $n$ elements, where $n = \dim W$. Then $\Gamma_n: X \to X \times W$ is a l.c.i. embedding by Remark 11.5. Then the composition of $\Gamma_n$ and $f \times id$ is an l.c.i. embedding by [9, B.7.4].

12.13. Lemma. Let $NCor^{\Omega}(X, Y)$ be the nerve of the groupoid $Cor^{\Omega}(X, Y)$. Then for a smooth affine $X$ and $Y \in Sm_{k}$ the natural map $f: Emb(X, Y) \to NCor^{\Omega}(X, Y)$ induces a weak equivalence of simplicial sets

$$C_{*}f: C_{*}Emb(X, Y) \to C_{*}NCor^{\Omega}(X, Y).$$

Proof. Note that $C_{*}NCor^{\Omega}(X, Y)$ is a bisimplicial set with $m$-simplices given by $C_{n}NCor^{\Omega}(X, Y)$. Thus it is sufficient to prove that for any $m$ the map

$$C_{*}f: C_{*}Emb(X, Y) \to C_{*}NCor^{\Omega}(X, Y)$$

is a weak equivalence of simplicial sets. Note that the map of presheaves $f: Emb_{n}(-, Y) \to NCor^{\Omega}_{n}(-, Y)$ is an inclusion admitting a retraction

$$p: NCor^{\Omega}_{n}(-, Y) \to Emb_{n}(-, Y)$$

that sends $((Z_{0}, f_{0}) \overset{\alpha_{0}}{\to} (Z_{1}, f_{1}) \to \cdots \overset{\alpha_{m-1}}{\to} (Z_{m}, f_{m})) \in NCor^{\Omega}_{n}(X, Y)$ to $(Z_{0}, f_{0}) \in Emb_{n}(X, Y)$.

For every smooth affine $X$ and $((Z_{0}, f_{0}) \overset{\alpha_{0}}{\to} (Z_{1}, f_{1}) \to \cdots \to (Z_{m}, f_{m})) \in NCor^{\Omega}_{n}(X, Y)$ consider the map

$$r_{i}: Z_{i} \times \mathbb{A}^{1} \to \mathbb{A}^{n}_{X} \times_{X} \mathbb{A}^{n}_{X} \times \mathbb{A}^{1}, \quad (z, t) \mapsto ((1-t)z+t\beta_{i}(z), t(t-1)z, t).$$

Here $t$ denotes the coordinate on $\mathbb{A}^{1}$ and $\beta_{i}: Z_{i} \to Z_{0}$ is the isomorphism of $X$-schemes $\beta_{i} = \alpha_{0}^{-1} \circ \cdots \circ \alpha_{i-1}^{-1}$.

Note that $r_{i}$ is a map of schemes over $X \times \mathbb{A}^{1}$. The map $r_{i}$, restricted to the fiber over $X \times (\mathbb{A}^{1} - \{0, 1\})$, fits into the diagram

$$Z_{i} \times (\mathbb{A}^{1} - \{0, 1\}) \xrightarrow{r_{i}} \mathbb{A}^{n}_{X} \times_{X} \mathbb{A}^{n}_{X} \times (\mathbb{A}^{1} - \{0, 1\}) \xrightarrow{r'_{i}} \mathbb{A}^{n}_{X} \times_{X} \mathbb{A}^{n}_{X} \times (\mathbb{A}^{1} - \{0, 1\}),$$

where $r'_{i}: (z, t) \mapsto ((1-t)z+t\beta_{i}(z), z, t)$ is a l.c.i. embedding by Lemma 12.12. The fiber of $r_{i}$ over $X \times 0$ and $X \times 1$ is a l.c.i. embedding. Then the map $r_{i}$ is a l.c.i. embedding by Lemma A.9. Let us denote by $Z'_{i}$ the image of $r_{i}$. Note that the composition $Z_{i} \times \mathbb{A}^{1} \to Z'_{i} \subset \mathbb{A}^{n}_{X} \times_{X} \mathbb{A}^{n}_{X} \times \mathbb{A}^{1} \to X \times \mathbb{A}^{1}$ coincides with $\pi \times id_{\mathbb{A}^{1}}$, where $\pi$ is the projection $\pi: Z_{i} \subset \mathbb{A}^{n}_{X} \to X$. Therefore $Z'_{i}$ is finite and flat over $X \times \mathbb{A}^{1}$.

We construct an $\mathbb{A}^{1}$-homotopy

$$H: NCor^{\Omega}_{n}(-, Y) \to NCor^{\Omega}_{2n}(\mathbb{A}^{1} \times -\times Y)$$

as follows. We set

$$H: ((Z_{0}, f_{0}) \overset{\alpha_{0}}{\to} (Z_{1}, f_{1}) \overset{\alpha_{1}}{\to} \cdots \overset{\alpha_{m-1}}{\to} (Z_{m}, f_{m})) \mapsto ((Z'_{0}, f'_{0}) \overset{\gamma_{0}}{\to} (Z'_{1}, f'_{1}) \overset{\gamma_{1}}{\to} \cdots \overset{\gamma_{m-1}}{\to} (Z'_{m}, f'_{m})), $$

48
where each isomorphism $\gamma_i$ is given by the composition
\[
\gamma_i: Z'_i \xrightarrow{r_i^{-1}} Z_i \times A^1 \xrightarrow{\alpha \times id} Z_{i+1} \times A^1 \xrightarrow{r_{i+1}} Z'_{i+1},
\]
and each map $f'_i: Z'_i \to Y$ is given by the composition
\[
f'_i: Z'_i \xrightarrow{r_i^{-1}} Z_i \times A^1 \xrightarrow{\pi_{Z_i}} Z_i \xrightarrow{f_i} Y.
\]

Then $H_0: N_mCor^\Omega_\ast(-,Y) \to N_mCor^\Omega_\ast(-,Y)$ is the stabilization map and $H_1: N_mCor^\Omega_\ast(-,Y) \to N_mCor^\Omega_2(-,Y)$ equals the composition
\[
H_1: N_mCor^\Omega_\ast(-,Y) \xrightarrow{p} \text{Emb}_n(-,Y) \xrightarrow{f} N_mCor^\Omega_n(-,Y) \xrightarrow{\text{stab}} N_mCor^\Omega_2(-,Y),
\]
where the last arrow is the stabilization map. The $A^1$-homotopy $H$ gives rise to a simplicial homotopy
\[
H: C_nN_mCor^\Omega_\ast(-,Y) \times \Delta[1] \to C_nN_mCor^\Omega_2(-,Y).
\]
By construction, for any $X$ we have $H(C_n\text{Emb}_n(X,Y) \times \Delta[1]) \subseteq C_n\text{Emb}_{2n}(X,Y)$. Then by Lemma 12.14 the map $f: C_n\text{Emb}(X,Y) \to C_n\text{Emb}^\Omega(X,Y)$ is a weak equivalence.

**12.14. Lemma.** Suppose $X_n \subseteq X_{n+1}$ is a directed system of inclusions of simplicial sets, $Y_n \subseteq Y_{n+1}$ is a directed system of simplicial subsets $Y_n \subseteq X_n$, and $p_n: X_n \to Y_n$ is a sequence of retractions that agree with inclusions $X_n \subseteq X_{n+1}$ and $Y_n \subseteq Y_{n+1}$. Assume that for every $n$ there is a homotopy $H(n): X_n \times \Delta[1] \to X_{2n}$ such that $H(n)_0: X_n \to X_{2n}$ is the inclusion map, $H(n)(Y_n \times \Delta[1]) \subseteq Y_{2n}$, and the map $H(n)_1: X_n \to X_{2n}$ equals the composition
\[
P_n \circ Y_n \subseteq Y_{2n} \subseteq X_{2n}.
\]
Then the inclusion $Y \to X$ is a weak equivalence, where $Y = \text{colim}_n Y_n, X = \text{colim}_n X_n$.

**Proof.** Consider a point $y \in Y_n$. The inclusion map $j: X_n \to X_{2n}$ and the composition $f: X_n \xrightarrow{p_n} Y_n \subseteq X_{2n}$ are homotopy by means of the free homotopy $H(n)$. Then the two induced maps
\[
p_i(j), p_i(f): p_i(X_n, y) \to p_i(X_{2n}, y)
\]
differ by the action $[\gamma]_s$ of the class $[\gamma] \in p_i(X_{2n}, y)$ on $p_i(X_{2n}, y)$, where $\gamma: \Delta[1] \to Y_{2n}$, $\gamma(t) = H(n)(y,t)$ is the loop given by the image of the base point $y$ under the homotopy $H(n)$. Since the loop $\gamma$ lies inside $Y_n$, the action of $[\gamma]$ on $p_i(Y_{2n}, y)$ preserves the image of $p_i(Y_n, y)$ under the inclusion map $Y_n \to X_{2n}$. Then the image
\[
p_i(j)(p_i(X_n, y)) = [\gamma]_s p_i(f)(p_i(X_n, y))
\]
lies inside the image of $p_i(Y_n, y)$. Then $p_i(Y, y) \to p_i(X, y)$ is surjective for any point $y \in Y$. The existence of retractions $p_n$ implies that the map $p_i(Y, y) \to p_i(X, y)$ is also injective and for every point $x \in X_n$ the map $t \mapsto H(n)(x,t)$ gives a path between $x$ and the point of $Y_n$. We see that $p_i(Y) \to p_i(X)$ is surjective, and hence $Y \to X$ is a weak equivalence.

Note that for any pointed finite set $K$ the assignment
\[
K \mapsto \text{Emb}(X_+ \wedge K)
\]
defines a sheaf of $\Gamma$-spaces. Denote by $\text{Emb}(X_+ \wedge S)$ the corresponding $S^1$-spectrum. For any $W \in \text{Sm}_k$ there is a canonical map

$$\text{Emb}(-, X) \rightarrow \text{Emb}(- \times W, X \times W),$$

functorial in $W \in \text{Fr}_0(k)$. These two constructions give rise to a $(S^1, G^\wedge m)$-bispectrum

$$(C_+ \text{Emb}(X_+ \wedge S), C_+ \text{Emb}(X_+ \wedge S \wedge G^\wedge m_1), \ldots).$$

Its structure maps literally repeat the construction of the structure maps for $K$-motives [11, Section 3]. We also define a $(S^1, G^\wedge m)$-bispectrum

$$(C_+ \text{NCor}^\Omega(X_+ \wedge S), C_+ \text{NCor}^\Omega(X_+ \wedge G^\wedge m_1 \wedge S), \ldots)$$
in a similar fashion.

The following theorem computes $M^G_{MGL}(X)$ as the above two bispectra.

12.15. **Theorem.** For $X \in \text{Sm}_k$ there is a natural levelwise stable local equivalence between $(S^1, G^\wedge m)$-bispectra $M^G_{MGL}(X)$ and

$$(C_+ \text{Emb}(X_+ \wedge S), C_+ \text{Emb}(X_+ \wedge G^\wedge m_1 \wedge S), \ldots)$$
or

$$(C_+ \text{NCor}^\Omega(X_+ \wedge S), C_+ \text{NCor}^\Omega(X_+ \wedge G^\wedge m_1 \wedge S), \ldots).$$

In particular, the $(S^1, G^\wedge m)$-bispectra

$$(C_+ \text{Emb}(X_+ \wedge S))_f, C_+ \text{Emb}(X_+ \wedge G^\wedge m_1 \wedge S)_f, \ldots)$$

and

$$(C_+ \text{NCor}^\Omega(X_+ \wedge S))_f, C_+ \text{NCor}^\Omega(X_+ \wedge G^\wedge m_1 \wedge S)_f, \ldots)$$

are motivically fibrant and represent the $T$-spectrum $X_+ \wedge MGL$ in the category of bispectra, where “$f$” refers to stable local fibrant replacements of $S^1$-spectra.

**Proof.** The first claim follows from Proposition 11.25 and Lemmas 12.3,12.7,12.8,12.13. The proof of Theorem 11.26 shows that the bispectra

$$(C_+ \text{Emb}(X_+ \wedge S))_f, C_+ \text{Emb}(X_+ \wedge G^\wedge m_1 \wedge S)_f, \ldots)$$

and

$$(C_+ \text{NCor}^\Omega(X_+ \wedge S))_f, C_+ \text{NCor}^\Omega(X_+ \wedge G^\wedge m_1 \wedge S)_f, \ldots)$$

are motivically fibrant and represent the $T$-spectrum $X_+ \wedge MGL$ in the category of bispectra. □

We already know from Corollary 12.10 that the sheaf $\text{Emb}(-, Y)$ is isomorphic to a sequential colimit $E^Y$ of smooth quasi-projective varieties. Thus the $(S^1, G^\wedge m)$-bispectrum

$$(\text{Emb}(X_+ \wedge S), \text{Emb}(X_+ \wedge G^\wedge m_1 \wedge S), \ldots), \quad X \in \text{Sm}_k,$$

can be presented as the $(S^1, G^\wedge m)$-bispectrum $(E^{X_+ \wedge S}, E^{X_+ \wedge G^\wedge m_1 \wedge S}, \ldots)$. By construction, the $(i, j)$-th term of the latter bispectrum is a sequential colimit of simplicial smooth quasi-projective varieties $E^{X_+ \wedge G^\wedge m_i \wedge S^j}$.

By using the preceding theorem, we therefore get the following result:
12.16. **Theorem.** The \((S^1, \mathbb{G}_m^1)\)-bispectrum \(M_{\text{MGL}}^G(X)\) is isomorphic in \(\text{SH}(k)\) to the bispectrum 
\((E^X, \mathbb{G}_m^j \wedge S^0, \ldots)\), each term of which is given by a sequential colimit of simplicial smooth quasi-projective varieties \(E^X, \mathbb{G}_m^j \wedge S^0, i, j \geq 0\).

**APPENDIX A. TECHNICAL LEMMAS**

In this section we recall standard facts about projective modules over Henselian pairs. Throughout this section \(R\) denotes a Noetherian \(k\)-algebra. By 12.6 the set \(\text{Gr}(n,N)(R)\) equals the set of rank \(n\) admissible submodules of \(R^N\). If \(R \to S\) is a map of \(k\)-algebras, by \(P \otimes_R S\) we shall mean the image of \(P\) in \(S^N\). It gives an element of \(\text{Gr}(n,N)(S)\). It is important to recall from [29, Tag 089R] that \(\text{Gr}(n,N)(R)\) is functorial in \(R\).

**A.1. Lemma.** Suppose \((R, I)\) is a Henselian pair, and \(J\) is an ideal in \(R\). Suppose \(B\) is an integral \(R\)-algebra, and \(e' \in B/IB\), \(e'' \in B/JB\) are two idempotents that coincide in \(B/(I+J)B\). Then there is an idempotent \(e \in B\) such that \(e+IB = e'\) and \(e+JB = e''\).

**Proof.** By [29, Tag 09XI] there is a bijections between idempotents in \(B\) and \(B/IB\) as well as there is a bijection between idempotents in \(B/J\) and \(B/(I+J)B\). If \(e\) is an idempotent in \(B\) such that \(e+IB = e'\), then \(e+JB = e''\). \(\Box\)

Denote by \(\text{Idemp}_n(R)\) the set of idempotents of the matrix ring \(M_n(R)\).

**A.2. Lemma.** Suppose \((R, I)\) is a Henselian pair, \(J\) is an ideal in \(R\). Consider a diagram of sets:

\[
\text{Idemp}_n(R) \to \text{Idemp}_n(R/I) \times \text{Idemp}_n(R/J) \Rightarrow \text{Idemp}_n(R/(I+J)).
\]

Suppose \((x', x'') \in \text{Idemp}_n(R/I) \times \text{Idemp}_n(R/J)\) and the images of \(x', x''\) coincide in \(\text{Idemp}_n(R/(I+J))\). Then there is \(x \in \text{Idemp}_n(R)\) such that the image of \(x\) in \(\text{Idemp}_n(R/I)\) equals \(x'\), and the image of \(x\) in \(\text{Idemp}(R/J)\) equals \(x''\).

**Proof.** There is a right exact sequence of \(R\)-modules

\[
M_n(R) \to M_n(R/I) \oplus M_n(R/J) \to M_n(R/(I+J)) \to 0.
\]

Take a matrix \(y \in M_n(R)\) to be a preimage of \((x', x'')\). Let \(f(t) \in R[t]\) be the characteristic polynomial of the matrix \(y\). Denote by \(B = R[t]/f(t)\). We follow the proof of [29, Tag 07M5]. Note that \(B\) is integral over \(R\) and there is a ring map \(g: B \to M_n(R)\) that sends \(t\) to \(y\). For any prime ideal \(p\) containing \(J\) the image of \(f(t)\) in \(k(p)[t]\) is the characteristic polynomial of an idempotent matrix, hence it divides \(t^n(t-1)^n\). Then \(t^n(1-t)^n \in \sqrt{JB}\) and there exists a constant \(N_0\) such that for any \(N \geq N_0\) the element \(t^N + (1-t)^N\) is invertible in \(B/JB\). It follows that \(e'' := t^n/(t^N+(1-t)^N)\) in \(B/JB\) is an idempotent and a preimage of \(x''\) in \(M_n(R/J)\). Likewise there is a constant \(N_1\) such that for any \(N > N_1\) the element \(e' = t^n/(t^N+(1-t)^N)\) in \(B/IB\) is an idempotent and a preimage of \(x'\) in \(M_n(R/I)\). Then for \(N > \max(N_0, N_1)\) the images of \(e'\) and \(e''\) coincide in \(B/(I+J)B\) and by the previous lemma there is an idempotent \(e\) in \(B\) lifting \(e'\) and \(e''\). Then \(x = g(e)\) is an idempotent matrix in \(M_n(R)\) such that the image of \(x\) in \(M_n(R/I)\) equals \(x'\) and the image of \(x\) in \(M_n(R/J)\) equals \(x''\). \(\Box\)
A.3. **Lemma.** Suppose \((R, I)\) is a Henselian pair, \(i: P \subseteq R^n\) is an admissible submodule. Then \(i': P \otimes_R R/I \subseteq (R/I)^n\) is an admissible submodule of \((R/I)^n\). Assume that there is projection \(\pi': (R/I)^n \to P \otimes_R R/I\) such that \(\pi' i' = id\). Then there is a projection \(\pi: R^n \to P\) such that \(\pi i = id\) and \(\pi \otimes R/I = \pi'\).

**Proof.** Let \(e_i, i = 1, \ldots, n\), denote the standard basis of \(R^n\) and let \(\tilde{e}_i\) be the standard basis of \((R/I)^n\). Take a map \(p: R^n \to P\) sending \(e_i\) to some preimage of \(\pi' (\tilde{e}_i)\). Then \(p \circ i\) is an endomorphism of \(P\) such that \((p \circ i) \otimes_R R/I\) is the identity endomorphism of \(P \otimes_R R/I\). Then \(p \circ i\) is invertible by Nakayama’s lemma. It follows that \(\pi = (p \circ i)^{-1} p: R^n \to P\) is a projection onto \(P\)

lifting \(\pi'\) and \(\pi \circ i = id\). \(\Box\)

A.4. **Lemma.** Suppose \((R, I)\) is a Henselian pair, \(J\) is an ideal in \(R\). Suppose \(P_1 \in Gr(n, N)/(R/I)\) and \(P_2 \in Gr(n, N)/(R/J)\) are such that \(\pi_1 \otimes_R R/I + J = \pi_2 \otimes_R R/J\) in \(Gr(n, N)/(R/I + J)\). Then there is \(P \in Gr(n, N)/(R)\) such that \(\pi \otimes R/I = P_1\) in \(Gr(n, N)/(R/I)\) and \(\pi \otimes R/J = P_2\) in \(Gr(n, N)/(R/J)\).

**Proof.** Choose a projection \(p_1: (R/I)^N \to P_1\). Then \(p_1 \otimes_R R/I + J = \pi_2 \otimes_R R/J\) is a projection onto \(P_1 \otimes_R R/I + J = P_2 \otimes_R R/J\). The pair \((R/I, I/J)\) is Henselian by [29, Tag 09XK], then by Lemma A.3 there is a projection \(p_2: (R/J)^N \to P_2\) that lifts \(p_1 \otimes_R R/I + J\). Then \(A_1 = i_1 p_1\) and \(A_2 = i_2 p_2\) are idempotents that coincide in \(Idemp_{R}(R/I + J)\). By Lemma A.2 there is an idempotent \(A \in Idemp_{R}(R)\) such that \(A \otimes R/I = A_1\) and \(A \otimes R/J = A_2\). Then \(P = A(R^N)\) is an element of \(Gr(n, N)/(R)\) such that \(\pi \otimes R/I = P_1\) and \(\pi \otimes R/J = P_2\). \(\Box\)

A.5. **Lemma.** Suppose \((R, I)\) is a Henselian pair and \(J\) is an ideal in \(R\). Suppose \(f_1: \text{Spec} R/I \to Gr(n)\) and \(f_2: \text{Spec} R/J \to Gr(n)\) coincide on \(\text{Spec} R/(I+J)\). Then there is \(f: \text{Spec} R \to Gr(n)\) that extends \(f_1\) and \(f_2\).

**Proof.** This follows from the previous lemma and the fact that \(Gr(n)/(R) = \colim_N Gr(n, N)/(R)\). \(\Box\)

In the ring \(R[\Delta^n]\) denote by \(t\) the product \(t = t_0 \cdots t_n\) of barycentric coordinates in \(R[\Delta^n]\). For any \(R[\Delta^n]-\text{module}\) we denote by \(M_i = M \otimes_{R[\Delta^n]} R[\Delta^n][1/t_i]\).

A.6. **Lemma.** Suppose \(\partial B\) is a finite flat \(R[\partial \Delta^n]\)-algebra, \(M\) is a finitely generated projective \(\partial B\)-module, \(P \subseteq (\partial B)^N\) is an admissible submodule, and for every \(i = 0, \ldots, n\) there is an isomorphism \(f_i: \partial_i M \to \partial_i P\), and for every \(i, j\) the maps \(f_i \otimes \partial_j B\) and \(f_j \otimes \partial_i B\) coincide on \(\partial_i M\) (see Definition 12.2). Then there is an isomorphism \(f: M \to P\) such that \(f \otimes \partial B = f_i\).

**Proof.** There is a left exact sequence of \(R[\partial \Delta^n]\)-modules

\[
0 \to R[\partial \Delta^n] \to \bigoplus_{i=0}^n R[\partial_i \Delta^n] \to \bigoplus_{i<j} R[\partial_{ij} \Delta^n].
\]

Tensoring it with \(\partial B\) over \(R[\partial \Delta^n]\), we get a left exact sequence for every projective \(\partial B\)-module. The maps \(f_i\) induce a commutative square in the following diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & M \\
\downarrow f_i & & \downarrow f_{ij} \\
0 & \longrightarrow & P \\
\end{array}
\]

\[
\begin{array}{ccc}
\bigoplus_i \partial_i M & \longrightarrow & \bigoplus_{i<j} \partial_{ij} M \\
\bigoplus_i \partial_i P & \longrightarrow & \bigoplus_{i<j} \partial_{ij} P.
\end{array}
\]
Since $\partial B$ is flat $R[\partial \Delta^n]$, both $M$ and $P$ are flat as $R[\partial \Delta^n]$-modules, then the rows in the diagram are exact, as they are obtained by tensoring with the left exact sequence above. Then there is a unique isomorphism $f : M \to P$ that makes the diagram commutative.  

A.7. Lemma. Suppose $M$ is a finitely generated projective $R[\Delta^n]$-module. A map

$$f : M \to R[\Delta^n]^N$$

is an admissible embedding if and only if its restriction to the boundary

$$f \otimes R[\partial \Delta^n] : M \otimes_{R[\Delta^n]} R[\partial \Delta^n] \to R[\partial \Delta^n]^N$$

is an admissible embedding and the localized map

$$\bar{f}_t : M_t \to R[\Delta^n]^N_t$$

is an admissible embedding.

Proof. Let $K$ and $C$ denote the kernel and cokernel of $f$ respectively. We need to check that $K = 0$ and $C$ is projective. Note that $K$ is a submodule of a free finite rank $R[\Delta^n]$-module. Since $t$ is not a zero divisor in $R[\Delta^n]$, then the localization map $K \to K_t$ is injective and $K_t = \ker(f_t) = 0$, hence $K = 0$. Let $r$ denote the rank of $M$. For every maximal ideal $m$ of $R[\Delta^n]$ if $t \notin m$ then $C_m$ is a localization of $C$, hence it is a free module of rank $N - r$. If $t \in m$, then $C/mC$ is a free module of rank $N - r$. By Nakayama’s lemma there is a surjection $g : R[\Delta^n]_{\bar{m}} \to C_m$ of modules over the local ring $R[\Delta^n]_m$. Then localization $(C_m)_t$ is a localization of the projective module $C_t$ of rank $N - r$. Then $(C_m)_t$ is projective of rank $N - r$. Since $g_t$ is a surjective map between projective modules of the same rank, then it is an isomorphism, and so $\ker(g)_t = 0$. Then $\ker(f_t) = 0$, since $\ker(f)$ is a submodule of the free module $R[\Delta^n]_m$, and $t$ is not a zero divisor of $R[\Delta^n]$.  

A.8. Lemma. Suppose $M$ is a finitely generated projective $R[\Delta^n]$-module. Assume that there is an admissible embedding $f' : M \otimes_{R[\Delta^n]} R[\partial \Delta^n] \to R[\partial \Delta^n]^N$. Then there is a number $d$ and an admissible embedding $f : M \to R[\Delta^n]^N \oplus R[\Delta^n]^d$ such that the map

$$f \otimes R[\partial \Delta^n] : M \otimes_{R[\Delta^n]} R[\partial \Delta^n] \to R[\partial \Delta^n]^N \oplus R[\partial \Delta^n]^d$$

equals the composition of $f'$ and the standard embedding $R[\partial \Delta^n]^N \to R[\partial \Delta^n]^N \oplus R[\partial \Delta^n]^d$.

Proof. Consider some admissible embedding $j : M \to R[\Delta^n]^d$ and some projection $p : R[\Delta^n]^d \to M$ such that $p \circ j = id_M$. Let $e_1, \ldots, e_d$ denote the standard basis of $R[\Delta^n]^d$ and let $\bar{e}_1, \ldots, \bar{e}_d$ be the standard basis of $R[\partial \Delta^n]^d$. Consider the composition

$$R[\partial \Delta^n]^d \xrightarrow{p \otimes id} M \otimes_{R[\Delta^n]} R[\partial \Delta^n] \xrightarrow{f'} R[\partial \Delta^n]^N.$$

For $i = 1, \ldots, d$ take $x_i \in R[\Delta^n]^N$ to be any preimage of $f'((p \otimes id)(\bar{e}_i)) \in R[\partial \Delta^n]^N$. Then there is a homomorphism

$$F : R[\Delta^n]^d \to R[\Delta^n]^N \oplus R[\Delta^n]^d, \quad e_i \mapsto (x_i, (t_0 t_1 \ldots t_n) e_i),$$

where $t_0, \ldots, t_n$ denote the coordinates in the ring $R[\Delta^n]$. Take $f : M \to R[\Delta^n]^N \oplus R[\Delta^n]^d$ to be the composition $f = F \circ j$. Let us check that $f$ is an admissible embedding.
Note that $f \otimes_{R[\Delta^n]} R[\partial \Delta^n]$ is the composition of $f'$ and the standard embedding $R[\partial \Delta^n]^N \to R[\partial \Delta^n]^N \oplus R[\partial \Delta^n]^d$. In particular, $f \otimes_{R[\Delta^n]} R[\partial \Delta^n]$ is an admissible embedding.

The localization $f_i : M_i \to R[\Delta^n]_{j_i}^N$ is the composition of $f_i \circ j_i$ and $f_i$ fits into a commutative triangle

$$
\begin{array}{c}
R[\Delta^n]_{j_i}^d \\
\downarrow g \oplus id \\
R[\Delta^n]_{j_i}^N \oplus R[\Delta^n]^d
\end{array}
\xrightarrow{id \oplus t} 
\begin{array}{c}
R[\Delta^n]_{j_i}^N \\
\downarrow t \oplus id \\
R[\Delta^n]_{j_i}^N \oplus R[\Delta^n]^d
\end{array}
$$

where $g : R[\Delta^n]_{j_i}^d \to R[\Delta^n]_{j_i}^N$ is the map that sends $e_i$ to $x_i$. The right arrow of the triangle is an isomorphism and $g \oplus id$ is an admissible embedding. Then $f_i$ is an admissible embedding, and hence so is $f_i$. By Lemma A.7 $f$ is an admissible embedding.

**A.9. Lemma.** Suppose $X$ is an affine variety over $k$, $A$ and $Y$ are equidimensional flat affine $X$-schemes, $A \to X$ is finite, $Y$ is Cohen–Macaulay, and $f : A \to Y$ is a morphism over $X$. Suppose $Z$ is a closed subset of $X$ and the map on the fiber products $f_{Z} : A \times_X Z \to Y \times_X Z$ and $A \times_X (X - Z) \to Y \times_X (X - Z)$ are l.c.i. embeddings. Then $f$ is an l.c.i. embedding.

**Proof.** Denote by $n = \dim Y - \dim A$ and let $A_Z$ (resp. $Y_Z$, $A_{X - Z}$, $Y_{X - Z}$) be the fiber product $A \times_X Z$ (respectively $Y \times_X Z$, $A \times_X (X - Z)$, $Y \times_X (X - Z)$). Let us check that $k[Y] \to k[A]$ is surjective. For every point $x \in X$ if $x \in X - Z$, then the localization map $k[Y]_x \to k[A]_x$ is surjective. If $x \in Z$, then the map $k[Y] \otimes_{k[X]} k(x) \to k[A] \otimes_{k[X]} k(x)$ is surjective. It follows from Nakayama’s lemma that the map on localizations $k[Y]_x \to k[A]_x$ is surjective. Then $k[Y] \to k[A]$ is surjective, hence $A \to Y$ is a closed embedding. Let $I$ denote the kernel of $k[Y] \to k[A]$. For every point $y \in Y$ if $y$ is in $Y_{X - Z}$, then $I_y$ is generated by a regular sequence of length $n$. If $y$ is in $Y_Z$, the sequence

$$0 \to I_y \otimes_{k[X]} k[Z] \to k[Z]_y \to k[A]_y \to 0$$

is exact, because $k[A]$ is flat over $k[X]$. Then $I_y \otimes_{k[X]} k[Z]$ is generated by $n$ elements over $k[Y]_y$, hence $I_y \otimes_{k[Y]} k(y)$ is generated by $n$ elements. By Nakayama’s lemma $I_y$ is generated by $n$ elements. Since $A$ has codimension $n$ in $Y$, these elements form a regular sequence [1, III.4.5]. Then $A$ is an l.c.i. subscheme in $Y$.

**Acknowledgements.** The authors thank Marc Hoyois and Ivan Panin for helpful discussions.

**References**

[1] A. Altman, S. Kleiman, *Introduction to Grothendieck Duality Theory*, Lecture Notes in Mathematics, Vol. 146, Springer-Verlag, Berlin–New York, 1970.
[2] A. Ananyevskiy, A. Neshitov, Framed and MW-transfers for homotopy modules, arXiv:1710.07412, *Sel. Math. New Ser*. 25 (2019), article 26.
[3] A. Ananyevskiy, G. Garkusha, I. Panin, Cancellation theorem for framed motives of algebraic varieties, arXiv:1601.06642, *Adv. Math.*, 383 (2021), article 107681.
[4] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 21, Springer-Verlag, Berlin, 1990.
[5] A. Druzhinin, H. Kolderup, P. A. Østvær, Strict $A^1$-invariance over the integers, preprint arXiv:2012.07365.
[6] A. Druzhinin, I. Panin, Surjectivity of the etale excision map for homotopy invariant framed presheaves, arXiv:1808.07765, Proc. Steklov Inst. Math. 320 (2023), 91–114.
[7] E. Elmanto, M. Hoyois, A. Khan, V. Sosnilo, M. Yakerson, Motivic infinite loop spaces, arXiv:1711.05248, Cambridge J. Math. 9(2) (2021), 431–549.
[8] E. Elmanto, M. Hoyois, A. Khan, V. Sosnilo, M. Yakerson, Modules over algebraic cobordism, arXiv:1908.02162, Forum Math. Pi 8:e14 (2020), 1–44.
[9] W. Fulton, Intersection theory, Springer-Verlag, Berlin-Heidelberg, 1984.
[10] G. Garkusha, A. Neshitov, I. Panin, Framed motives of relative motivic spheres, arXiv:1604.02732, Trans. Amer. Math. Soc. 374(7) (2021), 5131–5161.
[11] G. Garkusha, I. Panin, On the motivic spectral sequence, arXiv:1210.2242, J. Inst. Math. Jussieu 17(1) (2018), 137–170.
[12] G. Garkusha, I. Panin, Homotopy invariant presheaves with framed transfers, arXiv:1504.00884, Cambridge J. Math. 8(1) (2020), 1–94.
[13] G. Garkusha, I. Panin, Framed motives of algebraic varieties (after V. Voevodsky), arXiv:1409.4372, J. Amer. Math. Soc. 34(1) (2021), 261–313.
[14] G. Garkusha, I. Panin, The triangulated categories of framed bispectra and framed motives, arXiv:1809.08006, Algebra i Analiz 34(6) (2022), 135–169.
[15] G. Garkusha, I. Panin, P. A. Østvær, Framed motivic Γ-spaces, arXiv:1907.00433, Izv. Math. 87(1) (2023), 3–32.
[16] M. Hovey, Spectra and symmetric spectra in general model categories, J. Pure Appl. Algebra 165(1) (2001), 63–127.
[17] D. Isaksen, Flasque model structures for simplicial presheaves, K-Theory 36 (2005), 371–395.
[18] J.F. Jardine, Motivic symmetric spectra, Doc. Math. 5 (2000), 445–552.
[19] M. Levine, A comparison of motivic and classical stable homotopy theories, J. Topology 7 (2014), 327–362.
[20] G. Mazza, V. Voevodsky, C. Weibel, Lecture notes on motivic cohomology. Clay Mathematics Monographs, 2. American Mathematical Society, Providence, Cambridge, MA, 2006.
[21] A. Neshitov, Rigidity theorem for presheaves with Ω-transfers, Algebra i Analiz 26(6) (2014), 78–98. English transl. in St. Petersburg Math. J. 26(6) (2015), 919–932.
[22] I. Panin, K. Pimenov, O. Röndigs, A universality theorem for Voevodsky’s algebraic cobordism spectrum, Homology, Homotopy Appl. 10(2) (2008), 211–226.
[23] I. Panin, K. Pimenov, O. Röndigs, On Voevodsky’s algebraic K-theory spectrum, Abel. Symp. Proc. 4 (2009), 279–330.
[24] I. Panin, C. Walter, On the algebraic cobordism spectra MSL and MSpi, arXiv:1011.0651, Algebra i Analiz 34(1) (2022), 144–187.
[25] L. S. Pontrjagin, Smooth manifolds and their applications in homotopy theory, Tr. Mat. Inst. Steklova 45 (1955), 1–139. (Russian). English transl. in AMS translations, ser. 2, 11, 1–114, AMS, Providence, RI, 1959.
[26] G. Prasad, M. S. Raghunathan, On the Kneser–Tits problem, Comment. Math. Helv. 60 (1985), 107–121.
[27] D. Quillen, On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. 75(6) (1969), 1293–1298.
[28] G. Segal, Categories and cohomology theories, Topology 13 (1974), 293–312.
[29] The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu, 2018.
[30] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic varieties, Invent. Math. 123 (1996), 61–94.
[31] V. Voevodsky, A¹-homotopy theory, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), Doc. Math. 1998, Extra Vol. I, 579–604.
[32] V. Voevodsky, Notes on framed correspondences, unpublished, 2001. Also available at math.ias.edu/vladimir/files/framed.pdf
DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, FABIAN WAY, SWANSEA SA1 8EN, UK

Email address: g.garkusha@swansea.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO N6A 5B7, CANADA

Email address: alexander.neshitov@gmail.com