Normal form of $m$-by-$n$-by-2 matrices for equivalence

Genrich Belitskii∗
Dept. of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel, genrich@cs.bgu.ac.il

Maxim Bershadsky,
Dept. of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel, maximb@math.bgu.ac.il

Vladimir V. Sergeichuk†
Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine sergeich@imath.kiev.ua

Abstract

We study $m \times n \times 2$ matrices up to equivalence and give a canonical form of $m \times 2 \times 2$ matrices over any field.

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1 Introduction and the main results

Complex $2 \times 2 \times 2$ matrices up to equivalence were classified by Schwartz [9] and Duschek [3]. Canonical forms of complex and real $2 \times 2 \times 2$ matrices for equivalence were given by Oldenburger [6–8]; they are presented in [10, Section IV, Theorem 1.1]. Ehrenborg [4] also got a canonical form of complex

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2 × 2 × 2 matrices for equivalence basing on a collection of covariants that separates the canonical matrices.

In this paper we give a canonical form of $m \times 2 \times 2$ matrices for equivalence over any field $F$, but first we establish when $m \times n \times 2$ matrices, whose two $m \times n \times 1$ submatrices are in the Kronecker canonical form for matrix pencils, are equivalent over $F$. Using an alternative method, the authors recently obtained in [1] a canonical form of $m \times 2 \times 2$ matrices for equivalence over a field of characteristic different from 2.

Note that the canonical form problem for $m \times n \times 3$ matrices for equivalence is wild; this means that it contains the problem of classifying pairs of linear operators and therefore it contains the problem of classifying an arbitrary system of linear operators (see, for example, [2, Theorems 4.5 and 2.1]).

All matrices and spatial matrices in this article are considered over an arbitrary field $F$. By an $m \times n \times q$ spatial matrix over $F$ we mean an array

$$A = [a_{ijk}]_{i=1}^{m} \mid_{j=1}^{n} \mid_{k=1}^{q}, \quad a_{ijk} \in F. \quad (1)$$

Two $m \times n \times q$ matrices $A = [a_{ijk}]$ and $B = [b_{ijk}]$ are equivalent if there exist nonsingular $m \times m$, $n \times n$, and $q \times q$ matrices

$$R = [r_{ii'}], \quad S = [s_{jj'}], \quad T = [t_{kk'}] \quad (2)$$
such that

$$b_{i'j'k'} := \sum_{ijk} a_{ijk} r_{ii'} s_{jj'} t_{kk'}. \quad (3)$$

This notion arises in the theory of forms: each trilinear form $f : U \times V \times W \to F$ on vector spaces with bases $\{u_i\}_{i=1}^{m}$, $\{v_j\}_{j=1}^{n}$, and $\{w_k\}_{k=1}^{q}$ is given by the spatial matrix (1) with $a_{ijk} := f(u_i, v_j, w_k)$. Its entries change by (3) if we go to other bases with the transition matrices (2).

We will give the spatial matrix (1) by the $q$-tuple of $m \times n$ matrices

$$A = \|A_1 \mid \ldots \mid A_q\|, \quad A_k = [a_{ijk}]_{ij}, \quad (that \ is, \ by \ the \ list \ of \ its \ horizontal \ slices).$$

The transfer from $A$ to $B$ given by (3) can be realized in two steps: by the simultaneous equivalence transformation with the horizontal slices

$$\|C_1, \ldots \mid C_q\| := \|R^T A_1 S \mid \ldots \mid R^T A_q S\|, \quad (4)$$
and then by the nonsingular linear substitution

\[
B_1 = C_{1t_1} + \cdots + C_{q t_q}, \quad \ldots, \quad B_q = C_{1 t_q} + \cdots + C_{q t_q},
\]

(5)

where \( R, S, \) and \( T \) are the matrices \([2] \). The last transformation can be made by \textit{elementary operations} on the set \( \{ C_1, \ldots, C_q \} \) of horizontal slices: interchange any two slices, multiply one slice by a non-zero scalar, and add a scalar multiple of one slice to another one. This implies the following lemma.

\textbf{Lemma 1.} Two spatial matrices are equivalent if and only if one can be transformed to the other by a sequence of

(i) simultaneous equivalence transformations with all horizontal slices, and

(ii) elementary operations on the set of horizontal slices. \( \square \)

We denote the \( m \)-by-\( n \) zero matrix by \( 0_{mn} \). The numbers \( m \) and \( n \) may be zero: the matrices \( 0_{m0} \) and \( 0_{0n} \) represent the linear mappings \( 0 \to \mathbb{F}^m \) and \( \mathbb{F}^n \to 0 \). For every \( p \times q \) matrix \( M_{pq} \) we have

\[
M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0_{mq} \end{bmatrix}, \quad M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}.
\]

For each natural number \( r \), we define the \((r-1) \times r\) matrices

\[
F_r := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{bmatrix}, \quad G_r := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
\]

(6)

For each polynomial

\[
\chi(x) = x^l - u_1 x^{l-1} - \cdots - u_l \in \mathbb{F}[x], \quad l \geq 1,
\]

we define the \( l \times l \) matrix

\[
\Phi_\chi := \begin{bmatrix} 0 & 0 & u_l \\ 1 & \cdots & \vdots \\ \vdots & \cdots & 0 \\ 0 & 1 & u_1 \end{bmatrix},
\]

(7)

whose characteristic polynomial is \( \chi(x) \).
We also define the direct sum of matrix pairs:

\[(A, B) \oplus (A', B') := (A \oplus A', B \oplus B').\]

The next theorem will be proved in Section 2; it extends Theorem 4.4 of [2] dealing with spatial matrices over an algebraically closed field.

**Theorem 1.** Over any field \(\mathbb{F}\), every \(m \times n \times 2\) matrix \(A = [A_1 | A_2]\), in which \(\min(m, n)\) is less than or equal to the number of elements of \(\mathbb{F}\), is equivalent to some \(B = [B_1 | B_2]\), in which

\[
(B_1, B_2) = \bigoplus_{i=1}^{p_1} (F_{r_i}, G_{r_i}) \oplus \bigoplus_{j=1}^{p_2} (F^T_{s_j}, G^T_{s_j}) \oplus \bigoplus_{k=1}^{q} (I_{l_k}, \Phi_{\chi_k}),
\]

\(p_1, p_2, q\) are nonnegative integers, all \(r_i, s_j, l_k\) are natural numbers, and each polynomial \(\chi_k\) has degree \(l_k\) and is a power of an irreducible polynomial. This sum is determined by \(A\) uniquely, up to permutation of summands and up to simultaneous replacement of all \(\Phi_{\chi_k}\) by \(\Phi_{\eta_k}\) with

\[
\eta_k(x) := \varepsilon_k (d - xb)^{l_k} \chi_k \left( \frac{xa - c}{d - xb} \right),
\]

where

- \(a, b, c, d\) are arbitrary elements of \(\mathbb{F}\) satisfying \(ad - bc \neq 0\) and
  \[a + b\lambda_k \neq 0 \quad \text{if} \quad \chi_k(x) = (x - \lambda_k)^{l_k},\]
- each \(\varepsilon_k\) is a nonzero element of \(\mathbb{F}\) that makes the coefficient of the highest order term of \(\eta_k(x)\) equalling 1 (the characteristic polynomial \(\eta_k(x)\) must be monic).

Let \(A = [a_{ijk}]_{i=1}^{m} \times \{1, \ldots, q\} = [a_{ijk}]_{i=1}^{m} \times \{1, \ldots, q\}\) be a spatial matrix. Consider the sets

\[
\mathcal{S} = \{A_1, \ldots, A_q\}, \quad \tilde{\mathcal{S}} = \{\tilde{A}_1, \ldots, \tilde{A}_n\}, \quad \tilde{\mathcal{S}} = \{\tilde{A}_1, \ldots, \tilde{A}_m\}
\]

of its \(m \times n, m \times q,\) and \(n \times q\) submatrices

\[
A_k := [a_{ijk}]_{ij}, \quad \tilde{A}_j := [a_{ijk}]_{ik}, \quad \tilde{A}_i := [a_{ijk}]_{jk}.
\]
We say that $\mathcal{A}$ is *regular* if each of the sets (11) is linearly independent.

Suppose $\mathcal{A}$ is non-regular and let $q'$, $n'$, $m'$ be the ranks of the sets (11). Make the first $q'$ matrices in $\mathcal{S}$ linearly independent and the others zero by elementary operations on the set $\mathcal{S}$. Reduce the “new” $\tilde{\mathcal{S}}$ and then the “new” $\tilde{\tilde{\mathcal{S}}}$ in the same way. We obtain a spatial matrix $\mathcal{B} = [b_{ijk}]$, whose $m' \times n' \times q'$ submatrix

$$\mathcal{B}' = [b_{ijk}]_{i=1,j=1,k=1}^{m',n',q'}$$

is regular, and whose entries outside of $\mathcal{B}'$ are zero; $\mathcal{B}'$ is called a *regular part* of $\mathcal{A}$. Two spatial matrices of the same size are equivalent if and only if their regular parts are equivalent [2, Lemma 4.7]. Hence, it suffices to give canonical forms of regular spatial matrices. The following theorem will be proved in Section 3.

**Theorem 2.** Over any field $\mathbb{F}$, each regular $m \times n \times q$ matrix $\mathcal{A}$ with $n \leq 2$ and $q \leq 2$ is equivalent to one of the spatial matrices:

$$\begin{vmatrix} 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} (1 \times 1 \times 1), \quad (12)$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 \end{vmatrix} (2 \times 2 \times 1), \quad (13)$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} (2 \times 1 \times 2), \quad (14)$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \end{vmatrix} (1 \times 2 \times 2), \quad (15)$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} (3 \times 2 \times 2), \quad (16)$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} (3 \times 2 \times 2), \quad (17)$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} (4 \times 2 \times 2), \quad (18)$$

$$\mathcal{A}(v) := \begin{vmatrix} 1 & 0 & 0 & v \\ 0 & 1 & 1 & 0 \end{vmatrix} (v \in \mathbb{F}, \ 2 \times 2 \times 2), \quad (19)$$
These spatial matrices are pairwise inequivalent except for the following cases:

- If \( \text{char} \, \mathbb{F} \neq 2 \), then \( \mathcal{A}(v) \) is equivalent to each \( \mathcal{A}(v') \) with
  \[
  v' = vz, \quad 0 \neq z \in \mathbb{F}^2 := \{ a^2 \mid a \in \mathbb{F} \}.
  \]  
  (21)

- If \( \text{char} \, \mathbb{F} = 2 \), then \( \mathcal{A}(v) \) is equivalent to each \( \mathcal{A}(v') \) with
  \[
  v' = \frac{\alpha v + \beta}{\gamma v + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{F}^2, \quad \alpha \delta + \beta \gamma \neq 0, \quad \gamma v + \delta \neq 0,
  \]  
  (22)
  and \( \mathcal{B}(v) \) is equivalent to each \( \mathcal{B}(v') \) with
  \[
  v' = v + \beta + \beta^2, \quad \beta \in \mathbb{F}.
  \]  
  (23)

In particular, if \( \mathbb{F} \) is algebraically closed, then each regular \( m \times n \times q \) matrix \( \mathcal{A} \) with \( n \leq 2 \) and \( q \leq 2 \) is equivalent to exactly one of the following spatial matrices: \((12)-(18), \mathcal{A}(0), \) and

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (2 \times 2 \times 2).
\]  
(24)

2 Proof of Theorem 1

We say that two pairs of matrices of the same size are equivalent if the matrices of the first pair are simultaneously equivalent to the matrices of the second pair.

**Lemma 2.** Let \( (I_i, \Phi_{\chi}) \) and \( (I_i, \Phi_{\eta}) \) be two matrix pairs given by arbitrary monic polynomials \( \chi \) and \( \eta \) of degree \( l \). Let

\[
T := \begin{bmatrix}
a & c \\
b & d
\end{bmatrix}, \quad ad - bc \neq 0,
\]

be a nonsingular matrix.

(a) If the pair

\[
(aI_l + b\Phi_{\chi}, cI_l + d\Phi_{\chi})
\]  
(25)
is equivalent to \((I, \Phi_\eta)\), then
\[
\eta(x) = \varepsilon(d - xb)^\ell \chi_k \left( \frac{xa - c}{d - xb} \right)
\] (26)
for some \(\varepsilon \in \mathbb{F}\).

(b) If (26) holds then the characteristic polynomials of
\[
(cI_l + d\Phi_\chi) \cdot (aI_l + b\Phi_\chi)^{-1}
\] (27)
and \(\Phi_\eta\) are equal.

**Proof.** (a) Since the pair (25) is equivalent to \((I, \Phi_\eta)\), \(aI_l + b\Phi_\chi\) is nonsingular, and so the pair (28) is equivalent to
\[
(I, (cI_l + d\Phi_\chi) \cdot (aI_l + b\Phi_\chi)^{-1}).
\] (28)

Hence (27) is similar to \(\Phi_\eta\) and their characteristic polynomials are equal:
\[
\eta(x) = \det \left[ xI_l - (cI_l + d\Phi_\chi) \cdot (aI_l + b\Phi_\chi)^{-1} \right]
\]
\[
= \det \left[ (xa - c)I_l - (d - xb)\Phi_\chi \right] \cdot \det(aI_l + b\Phi_\chi)^{-1}
\]
\[
= (d - xb)^\ell \det \left( \frac{xa - c}{d - xb} I_l - \Phi_\chi \right) \cdot \det(aI_l + b\Phi_\chi)^{-1}
\]
\[
= (d - xb)^\ell \chi_k \left( \frac{xa - c}{d - xb} \right) \cdot \det(aI_l + b\Phi_\chi)^{-1}.
\] (29)

This proves (26).

(b) This statement follows from (29). \(\square\)

Recall [11] that each square matrix \(A\) over an arbitrary field \(\mathbb{F}\) is similar to a matrix of the form \(\Phi = \Phi_{\chi_1} \oplus \cdots \oplus \Phi_{\chi_q}\), where \(\chi_1, \ldots, \chi_q\) are powers of an irreducible polynomials and \(\Phi_{\chi_k}\) are defined in (7). The matrix \(\Phi\) is called the **Frobenius canonical form** of \(A\) and is determined by \(A\) uniquely up to permutations of summands.

Each pair \((A_1, A_2)\) of matrices of the same size is equivalent to a pair of the form
\[
(B_1, B_2) = \bigoplus_{i=1}^{p_1} (F_{r_i}, G_{r_i}) \oplus \bigoplus_{j=1}^{p_2} (F_{s_j}^T, G_{s_j}^T)
\]
\[\oplus \bigoplus_{k=1}^{q_1} (I_{l_k}, \Phi_{\chi_k}) \oplus \bigoplus_{k=q_1+1}^{q} (J_{l_k}(0), I_{l_k}), \] (30)
where \( p_1, p_2, q_1, q_2 \) are nonnegative integers, \( F_r \) and \( G_r \) are defined in (6), each polynomial \( \chi_k \) has degree \( l_k \) and is a power of an irreducible polynomial, and

\[
J_l(\lambda) = \begin{bmatrix}
\lambda & 0 \\
1 & \lambda \\
\ddots & \ddots \\
0 & 1 & \lambda
\end{bmatrix} \quad (l \text{-by}-l).
\]

The pair (30) is determined by \((A_1, A_2)\) uniquely up to permutation of summands and is called the Kronecker canonical form of \((A_1, A_2)\) (see, for example, [5, Section 1.8]).

Proof of Theorem 1. Step 1. Let \((A_1, A_2)\) be a pair of matrices of the same size and let (30) be its Kronecker canonical form. In this step, we prove that for each nonsingular matrix

\[
T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{F}^{2 \times 2}, \quad ad - bc \neq 0,
\]

the Kronecker canonical form of the pair

\[
(C_1, C_2) = (aB_1 + bB_2, cB_1 + dB_2)
\]

has the same number \( p_1 + p_2 + q \) of direct summands as (30) and, after a suitable permutation of its summands, it has the same first \( p_1 + p_2 \) summands as (30) and the same sizes \( l_1 \times l_1, \ldots, l_q \times l_q \) of the remaining \( q \) summands as (30).

A matrix pair is decomposable if it is equivalent to a direct sum of pairs of smaller sizes. All direct summands in (30) are indecomposable. The transformation (31) takes them into indecomposable matrix pairs. Indeed, if it takes a summand \( \mathcal{P} \) into a decomposable \( \mathcal{R} \), then the inverse transformation (given by the matrix \( T^{-1} \)) takes \( \mathcal{R} \) into a decomposable one, which is equivalent to \( \mathcal{P} \), contrary to the indecomposability of all direct summands of (30).

All indecomposable pairs of \((r-1) \times r\) or \(r \times (r-1)\) matrices are equivalent to \((F_r, G_r)\) or, respectively, \((F_r^T, G_r^T)\). Hence, though transformations (31) may spoil the direct summands \((F_{r_1}, G_{r_1})\) and \((F_{s_j}^T, G_{s_j}^T)\) in (30), but they are restored by equivalence transformations.

| \( \lambda \) | \( \chi_k \) |
|----------------|----------------|
| \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \\ \ddots & \ddots \\ 0 & 1 & \lambda \end{bmatrix} | (l \text{-by}-l) |
Step 2. Suppose \( A = \|A_1|A_2\| \) satisfies the hypotheses of Theorem 1. In this step, we reduce \( A \) by equivalence transformations to some \( B = \|B_1|B_2\| \) with \((B_1,B_2)\) of the form (8).

From the start, we reduce \((A_1,A_2)\) to the form (30).

Thereupon in the case \( q_1 < q \) we reduce the pair (30) to a pair of the form (8) (with other \( \chi_1, \ldots, \chi_{q_1} \)) as follows. The transformation (31) with (30) given by

\[
T = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad b \neq 0,
\]
takes the direct sum of the last \( q \) summands into

\[
\bigoplus_{k=1}^{q_1} (I_{l_k} + b\Phi_{\chi_k}, \Phi_{\chi_k}) \oplus \bigoplus_{k=q_1+1}^{q} (J_{l_k}(b), I_{l_k}). \tag{32}
\]

If some \( I_{l_k} + b\Phi_{\chi_k} \) is singular, then \( \chi_k(x) = (x - b^{-1})^{l_k} \). Indeed, 0 is an eigenvalue of \( I_{l_k} + b\Phi_{\chi_k} \), hence \( I_{l_k} + b\Phi_{\chi_k} \) has an eigenvalue in \( \mathbb{F} \), and so \( \Phi_{\chi_k} \) is similar to a Jordan block. Further, this Jordan block must be \( J_{l_k}(-b^{-1}) \).

In view of the hypotheses of Theorem 1, \( \min(m,n) \) is less than or equal to the number of elements of \( \mathbb{F} \). Since \( q_1 < q \leq \min(m,n) \), the number \( q_1 \) of the summands \((I_{l_k}, \Phi_{\chi_k})\) in (30) is less than or equal to the number of nonzero elements of \( \mathbb{F} \).

First suppose that one of these summands is \((I_{l_k}, J_{l_k}(0))\). Then there exists a nonzero \( b \in \mathbb{F} \) such that \( \chi_k(x) \neq (x - b^{-1})^{l_k} \) for all \( k \leq q_1 \), this means that all \( I_{l_k} + b\Phi_{\chi_k} \) are nonsingular. We take such \( b \) and reduce (32) to the form

\[
(I_{l_1}, \Phi_{\eta_1}) \oplus \cdots \oplus (I_{l_q}, \Phi_{\eta_q}) \tag{33}
\]
by equivalence transformations.

Now suppose that there are no summands \((I_{l_k}, J_{l_k}(0))\). Then the second matrix in each of the last \( q \) summands of (30) is nonsingular. We interchange the matrices \( B_1 \) and \( B_2 \) in the pair (30) and reduce its last \( q \) summands to the form (33).

Step 3. Suppose \( A = \|A_1|A_2\| \) is equivalent both to \( B = \|B_1|B_2\| \) with \((B_1,B_2)\) of the form (8) and to another \( B' = \|B_1'|B_2'|\| \) with

\[
(B_1', B_2') = \bigoplus_{i=1}^{p_1'} (F_{r'_i}, G_{r'_i}) \oplus \bigoplus_{j=1}^{p_2'} (F_{s'_j}, G_{s'_j}) \oplus \bigoplus_{k=1}^{q'} (I_{l'_k}, \Phi_{\eta_k}) \tag{34}
\]
Let us prove that (34) coincides, after a suitable permutation of its summands, with (8) except for $\chi_k$ and $\eta_k$, and that (9) is fulfilled.

Since $\mathcal{B}$ and $\mathcal{B}'$ are equivalent, by Lemma 1 ($B'_1, B'_2$) is the Kronecker canonical form of some pair $(C_1, C_2)$ of the form (31) with $ad - bc \neq 0$. In view of Step 1, $p'_1, p'_2, q'$ and all $r'_i, s'_j, l'_k$ coincide with $p_1, p_2, q$ and all $r_i, s_j, l_k$ after a suitable permutation of the summands of (34).

The transformation (31) converts each summand $(I_{lk}, \Phi_{\chi_k})$ of (8) to the matrix pair

$$ (aI_{lk} + b\Phi_{\chi_k}, cI_{lk} + d\Phi_{\chi_k}), \quad (35) $$

which is equivalent to $(I_{lk}, \Phi_{\eta_k})$. The matrix $aI_{lk} + b\Phi_{\chi_k}$ is nonsingular; this means that if $\Phi_{\chi_k}$ is similar to some Jordan block $J_l(\lambda_k)$, then $a + b\lambda_k \neq 0$; we have the condition (10). Due to Lemma 2(a), $\eta_k(x)$ is represented in the form (9).

Conversely, let $(B_1, B_2)$ of the form (8) and (34) coincide with except for $\chi_k$ and $\eta_k$ that satisfy (9). By Lemma 2(b), the characteristic polynomials of the matrices

$$ (cI_{lk} + d\Phi_{\chi_k}) \cdot (aI_{lk} + b\Phi_{\chi_k})^{-1} \quad (36) $$

and $\Phi_{\eta_k}$ are equal for each $k$. Since $(I_{lk}, \Phi_{\chi_k})$ is indecomposable, in view of Step 1 the matrix pair (35) is indecomposable too, hence the matrix (36) is indecomposable with respect to similarity and its Frobenius canonical form is $\Phi_{\eta_k}$. Therefore, each $\|I_{lk}\|\Phi_{\chi_k}\|$ is equivalent to $\|I_{lk}\|\Phi_{\eta_k}\|$, and so $\mathcal{A}$ is equivalent to $\mathcal{B}$.

3 Proof of Theorem 2

Lemma 3. A spatial matrix

$$ D(u, v) := \begin{vmatrix} 1 & 0 & 0 & v \\ 0 & 1 & 1 & u \\ \end{vmatrix}, \quad u, v \in \mathbb{F}, \quad (37) $$

is equivalent to $D(u', v')$ if and only if there exist $a, b, c, d \in \mathbb{F}$ such that

$$ ad - bc \neq 0, \quad a^2 + uab - vb^2 \neq 0 \quad (38) $$

and

$$ u' = \frac{2ac + uad + ucb - 2vbd}{a^2 + uab - vb^2}, \quad v' = \frac{-c^2 - ucd + vd^2}{a^2 + uab - vb^2}. \quad (39) $$
**Proof.** Notice that
\[ D(u, v) = \|I_2|\Phi_\chi\|, \quad D(u', v') = \|I_2|\Phi_\eta\|, \]
where
\[ \chi(x) := x^2 - ux - v, \quad \eta(x) := x^2 - u'x - v'. \]

\[ \Rightarrow \]. Let \( \|I_2|\Phi_\chi\| \) and \( \|I_2|\Phi_\eta\| \) be equivalent. By Lemma 1, there exists a nonsingular matrix
\[ \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad ad - bc \neq 0, \]
such that the pairs
\[ (aI_2 + b\Phi_\chi, cI_2 + d\Phi_\chi), \quad (I_2, \Phi_\eta) \quad (40) \]
are equivalent. Then \( aI_2 + b\Phi_\chi \) is nonsingular; i.e.,
\[ \det(aI_2 + b\Phi_\chi) = a^2 + uab - vb^2 \neq 0. \]

By Lemma 2(a), \( \eta(x) \) satisfies (26), this means that for some nonzero \( \varepsilon \)
\[ \eta(x) = \varepsilon \left[ (xa - c)^2 - u(xa - c)(d - xb) - v(d - xb)^2 \right] \]
\[ = \varepsilon \left[ x^2(a^2 + uab - vb^2) + x(-2ac - uad - uc - 2vbd) \right] \]
\[ + (c^2 + ucd - vd^2) = x^2 - u'x - v'. \quad (41) \]
Therefore, \( \varepsilon = (a^2 + uab - vb^2)^{-1} \) and the conditions (38) and (39) hold true.

\[ \Leftarrow \]. Conversely, let (38) and (39) hold. Then (41) is fulfilled and we have (26). By Lemma 2(b), the characteristic polynomials of
\[ (cI_2 + d\Phi_\chi) \cdot (aI_2 + b\Phi_\chi)^{-1} \quad (42) \]
and \( \Phi_\eta \) are equal. Since (42) is 2-by-2, this implies that its Frobenius canonical form is either \( \Phi_\eta \), or a direct sum of two 1-by-1 Frobenius blocks \( \lambda I_1 \oplus \mu I_1 \) for some \( \lambda, \mu \in \mathbb{F} \).

In the last case, \( \eta(x) = (x - \lambda)(x - \mu) \). But \( \eta(x) \) is a power of an irreducible polynomial. Hence, \( \lambda = \mu \) and (42) is \( \lambda I_2 \). We get consecutively
\[ cI_2 + d\Phi_\chi = \lambda(aI_2 + b\Phi_\chi), \quad (c - \lambda a)I_2 = (\lambda b - d)\Phi_\chi, \]
\[ c - \lambda a = \lambda b - d = 0, \quad (c, d) = \lambda(a, b), \]

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Therefore, (42) is similar to \( \Phi_\eta \), the pairs (40) are equivalent, and so \( \|I_2|\Phi_\chi\| \) is equivalent to \( \|I_2|\Phi_\eta\| \).

\[ \square \]

**Proof of Theorem 2.** Let \( A \) be a regular \( m \times n \times q \) matrix with \( n \leq 2 \) and \( q \leq 2 \).

**Step 1.** Let us prove that \( A \) is equivalent to at least one of the spatial matrices (12)–(20). This is clear if \( A \) is \( m \times n \times 1 \) with \( n \leq 2 \): indeed, since \( A = \|A\| \) is regular, it reduces by elementary transformations (4) to (12) or (13).

So we suppose that \( A \) is \( m \times n \times 2 \) with \( n \leq 2 \). By Theorem 1, \( A \) is equivalent to some \( B = \|B_1|B_2\| \) with \((B_1, B_2)\) of the form (8). Since \( A \) is regular, (8) does not have the summands \((F_1, G_1)\) and \((F_T^1, G_T^1)\). If \( m = 1 \) or \( n = 1 \), then \((B_1, B_2)\) is \((F_2, G_2)\) or \((F_T^2, G_T^2)\), we have (15) or (14).

It remains to consider \( A \) of size \( m \times 2 \times 2 \) with \( m \geq 2 \). Then \((B_1, B_2)\) is one of the pairs:

\[
(F_T^3, G_T^3), \quad (F_T^2, G_T^2) \oplus (F_T^2, G_T^2), \quad (F_T^2, G_T^2) \oplus (I_1, J_1(\lambda)), \quad (I_1, J_1(\lambda)) \oplus (I_1, J_1(\mu)), \quad (I_2, \Phi_\chi). (43)
\]

The first and the second pairs give (17) and (18). In the third pair we take \( \lambda = 0 \) (because \( 1|\lambda| \) and \( 1|0| \) are equivalent) and obtain (16). In the fourth pair, \( \lambda \neq \mu \) since \( A \) is regular, and so it is equivalent to \((I_2, \Phi_\chi)\) with \( \chi(x) = (x - \lambda)(x - \mu) \).

Hence, the spatial matrices that are given by (44) are equivalent to \( D(u, v) \) defined in (37).

If \( \text{char} \mathbb{F} \neq 2 \), then each \( D(u, v) \) is equivalent to \( D(0, v') \) for some \( v' \) due to Lemma 3 substituting

\[ (a, b, c, d) := (1, 0, -u/2, 1) \]

in (38) and (39), we obtain \( u' = 0 \). This gives (19).

If \( \text{char} \mathbb{F} = 2 \), then each \( D(u, v) \) is equivalent to \( D(0, v') \) or \( D(1, v') \): for each \( u \neq 0 \) we get \( u' = 1 \) putting

\[ (a, b, c, d) := (1, 0, 0, u^{-1}) \]

in (38) and (39). This gives (19) and (20).
Step 2. Let us prove that $A$ is equivalent to exactly one of the spatial matrices (12)–(20) up to replacements (21)–(23).

Let two distinct spatial matrices among (12)–(20) be equivalent. Then they have the same size, and so they are $3 \times 2 \times 2$ or $2 \times 2 \times 2$. The spatial matrices (16) and (17) are inequivalent in view of Theorem 1 since the corresponding decompositions (8) are $(F_2 \oplus (I_1, J_1(0)))$ and $(F_3 \oplus (I_3, J_3))$. Hence, they are (19) or (20).

Let char $F \neq 2$, and let $A(v)$ be equivalent to $A(v')$. By Lemma 3 there exist $a, b, c, d$ satisfying (38) and (39) with $u = u' = 0$. Then the equalities (39) ensure
\[ ac - vbd = 0, \quad v' = \frac{-c^2 + vd^2}{a^2 - vb^2}, \]
and so
\[ v'(a^2 - vb^2)^2 = (a^2 - vb^2)(-c^2 + vd^2) = -a^2c^2 + a^2vd^2 + vb^2c^2 - a^2d^2 \]
\[ = -(ac - vbd)^2 + v(ad - bc)^2 = v(ad - bc)^2. \]
We have (21) with
\[ z = \left( \frac{a^2 - vb^2}{ad - bc} \right)^2. \]

Conversely, if (21) holds, then $A(v)$ is equivalent to $A(v')$ due to Lemma 3 since the conditions (38) and (39) are fulfilled with $u = u' = 0, (a, b, c, d) := (1, 0, 0, z^{-1/2}).$

Let char $F = 2$. If $D(0, v)$ is equivalent to $D(u', v')$, then by (39) $u' = 0$. Hence $A(v)$ and $B(v')$ (defined in (19) and (20)) are inequivalent for all $v$ and $v'$.

Due to Lemma 3, $A(v)$ and $A(v')$ are equivalent if and only if the conditions (38) and (39) with $u = u' = 0$ hold for some $a, b, c, d \in F$. The first condition in (39) is the identity, putting
\[ (\alpha, \beta, \gamma, \delta) := (d^2, c^2, b^2, a^2) \]
in the other conditions gives the conditions (22).

Let $B(v)$ be equivalent to $B(v')$. Then there exist $a, b, c, d$ such that the conditions (38) and (39) hold for $u = u' = 1.$
We first suppose that \( b = 0 \). The conditions (39) take the form \( 1 = ad/a^2 \) (and so \( a = d \neq 0 \)) and
\[
v' = \frac{c^2 + ca + v^2}{a^2} = v + \frac{c}{a} + \frac{c^2}{a^2};
\]
this gives (23) with \( \beta = c/a \).

Let now \( b \neq 0 \). Denote
\[
\alpha := a/b, \quad \gamma := c/b, \quad \delta := d/b.
\]
Remembering that \( \text{char} \mathbb{F} = 0 \) and \( u = u' = 1 \), rewrite (39) in the form
\[
1 = \frac{\alpha \delta + \gamma}{\alpha^2 + \alpha + v}, \quad v' = \frac{\gamma^2 + \gamma \delta + v \delta^2}{\alpha^2 + \alpha + v}.
\]
From the first equality determine
\[
\gamma = v + \alpha + \alpha^2 + \alpha \delta,
\]
substitute it to the second:
\[
v' = \frac{(v + \alpha + \alpha^2)^2 + (\alpha \delta)^2 + (v + \alpha + \alpha^2) \delta + \alpha \delta^2 + v \delta^2}{\alpha^2 + \alpha + v}
= v + \alpha + \alpha^2 + \delta + \delta^2 = v + (\alpha + \delta) + (\alpha + \delta)^2,
\]
and obtain (23).

Conversely, let \( v' = v + \beta + \beta^2 \) for some nonzero \( \beta \in \mathbb{F} \). The conditions (38) and (39) hold for \( u = u' = 1 \) and
\[
(a, b, c, d) := \begin{cases} (\beta, 1, v', 0) & \text{if } v' \neq 0, \\ (1, 0, \beta, 1) & \text{if } v' = 0; \end{cases}
\]
hence \( \mathcal{B}(v) \) and \( \mathcal{B}(v') \) are equivalent by Lemma 3.

**Step 3.** Let \( \mathbb{F} \) be algebraically closed. Then \( \mathbb{F}^2 = \mathbb{F} \). If \( \text{char} \mathbb{F} \neq 2 \), each \( \mathcal{A}(v) \) is equivalent to \( \mathcal{A}(0) \) or \( \mathcal{A}(1) \) (if \( v \neq 0 \), we put \( z = 1/v \) in (21)). The spatial matrix \( \mathcal{A}(1) \) is equivalent to (24) since it reduces to (24) by the following transformations: add the first slice \( I_2 \) to the second, reduce the second to the form \( J_1(0) \oplus J_1(2) \) by simultaneous similarity transformations with the slices, divide the second by 2, and subtract the second slice from the first:
\[
\mathcal{A}(1) \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]
Suppose \( \text{char } \mathbb{F} = 2 \). Then all \( A(v) \) are equivalent to \( A(0) \) since the conditions (22) hold for \( v' = 0 \) and \( (\alpha, \beta, \gamma, \delta) := (1, v, 0, 1) \). All \( B(v) \) are equivalent to \( B(0) \) since the equation (23) with \( v' = 0 \) is solvable for \( \beta \). We reduce the second slide of \( B(0) \) to the form \( J_1(0) \oplus J_1(1) \) by simultaneous similarity transformations with the slices, and then subtract the second slice from the first.

References

[1] G. Belitskii, M. Bershadsky, V. V. Sergeichuk, Canonical form of \( m \)-by-2-by-2 matrices over a field of characteristic other than two, *Linear Algebra Appl.* 418 (2006) 15-19.

[2] G. R. Belitskii, V. V. Sergeichuk, Complexity of matrix problems, *Linear Algebra Appl.* 361 (2003) 203-222.

[3] A. Duschek, Eine abbildung der binarn trilinearform, *Jahresber. Deutsch. Math. Verein.* 32 (1923) 234-239.

[4] R. Ehrenborg, Canonical forms of two by two by two matrices, *J. Algebra* 213 (1999) 195-244.

[5] P. Gabriel, A. V. Roiter, *Representations of Finite-Dimensional Algebras*, Springer-Verlag, Berlin, 1997.

[6] R. Oldenburger, On canonical binary trilinear forms, *Bull. Amer. Math. Soc.* 38 (1932) 385-387.

[7] R. Oldenburger, On arithmetic invariants of binary cubic and binary trilinear forms, *Bull. Amer. Math. Soc.* 42 (1936) 871-873.

[8] R. Oldenburger, Real canonical binary trilinear forms, *Amer. J. Math.* 59 (1937) 427-435.

[9] E. Schwartz, Ueber binre trilineare formen, *Math. Z.* 12 (1922) 18-35.

[10] N. P. Sokolov, *Spatial matrices and their applications*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960 (in Russian).

[11] B. L. van der Waerden, *Algebra*. New York, NY: Springer-Verlag, 1991.