Effective inhomogeneous inflation: curvature inhomogeneities of the Einstein vacuum

Thomas Buchert and Nathaniel Obadia

1 Université Lyon 1, Centre de Recherche Astrophysique de Lyon, 9 Avenue Charles André, F-69230 Saint-Genis-Laval, France
2 École Normale Supérieure de Lyon, Centre de Recherche Astrophysique de Lyon, 46 Allée d’Italie, F-69364 Lyon Cedex 07, France

E-mail: buchert@obs.univ-lyon1.fr and nathaniel.obadia@ens-lyon.fr

Received 5 February 2011, in final form 11 May 2011
Published 21 July 2011
Online at stacks.iop.org/CQG/28/162002

Abstract
We consider spatially averaged inhomogeneous universe models and argue that, already in the absence of sources, an effective scalar field arises through foliating and spatially averaging inhomogeneous geometrical curvature invariants of the Einstein vacuum. This scalar field (the ‘morphon’) acts as an inflaton, if we prescribe a potential of some generic form. We show that, for any initially negative average spatial curvature, the morphon is driven through an inflationary phase and leads—on average—to a spatially flat, homogeneous and isotropic universe model, providing initial conditions for pre-heating and, by the same mechanism, a possibly natural self-exit.

PACS numbers: 98.80.—k, 98.80.Cq, 95.36.+x, 98.80.Es, 98.80.Jk, 04.20.—q, 04.20.Cv

(Some figures in this article are in colour only in the electronic version)

1. Introduction

For about 30 years now, inflation [1] is the main paradigm used to explain the various caveats of the Hot Big Bang scenario. In particular, the flatness, smoothness and horizon issues [2] are often advocated to be solved by a long enough period of de Sitter-like expansion, provoked by a slow-rolling scalar (multi-)field that dominates over the other components of the energy budget of the Universe [3]. The success of the concept of inflation stems not only from its simplicity but also from the consequences this framework bears, i.e. conservative predictions concerning scale invariance, the tensor-to-scalar ratio, Gaussianity of fluctuations, etc. However, as many paradigms praised for their consequences, inflation lacks a physical cause. Indeed, despite several attempts to justify the global predominance of a yet unobserved fundamental scalar field, the existence of the inflaton remains a conundrum to theorists.

Confronted with this issue, physicists might roughly choose between three distinct philosophies. Either a non-standard part of the particle physics spectrum, for instance,
the Higgs boson [4], or a pseudo Nambu–Goldstone boson [5] could be candidates for the fundamental field whose potential part dominated the Early Universe. Or a known part of the physical fields could emulate an effective scalar field via a known process like boson or fermion condensation [6, 7]. Or a neglected part in the underlying theories, due to their special nature or just due to idealizing assumptions, prevented us so far to take into account physical effects that could lead to an inflationary era. The idea we present here lies within this last line of thought by applying to inflation what has been proposed previously for a conservative explanation of the dark energy problem through inhomogeneities [8]. In concrete terms we shall provide a physical cause to inflation by identifying the inflaton with an effective classical field that describes the averaged bulk effect on the dynamics of inhomogeneities (also dubbed backreaction) prior to, during and eventually after the inflationary era.

It is important to stress that the model expounded here pertains to the dynamics of backreaction—described by a conservative set of equations and a minimal number of physical assumptions—and not to a suitably chosen set of initial conditions. Furthermore, as a natural starting point, the inhomogeneities we shall describe hereby are due to geometry and a priori not related to the baryonic/dark matter, dark energy or radiation content of the Universe [9]; we shall confine ourselves to inhomogeneities in the gravitational field only. In other words, this model will demonstrate in simple terms how inflation can naturally emerge out of vacuum inhomogeneities. This last term does not refer to quantum vacuum fluctuations on a FLRW background (as practically always used in the cosmological context), but to the average of classical fluctuations of the gravitational field only.

2. Inhomogeneous universe models

Einstein’s equations are assumed to hold for a four-dimensional tube of the Early Universe, featuring the 4-Ricci tensor \( R_{\mu\nu} = 0 \). We suggest further to foliate this tube into three-dimensional space-like hypersurfaces according to the ADM formalism. We here choose a comoving-synchronous line element (in forthcoming papers the embedding issue will be thoroughly addressed) \( ds^2 = -dt^2 + g_{ij}dX^idX^j \), where the proper time \( t \) labels the hypersurfaces and \( X^i \) are Gaussian normal coordinates; \( g_{ij} \) are the components of the full inhomogeneous 3-metric of the hypersurfaces of constant proper time. In this metric, the components of the 4-Ricci tensor can be expressed geometrically through the three-dimensional extrinsic curvature \( K_{ij} \) of the embedding into spacetime, and intrinsic curvature \( R_{ij} \) of the hypersurfaces at constant \( t \) with the following well-known Gauß–Codazzi–Mainardi relations:

\[
\begin{align*}
R_{00} &= K - K^{ij}K_{ij}, \\
R_{ik} &= K_{il} - K_{lj}K_l^j, \\
R_{ij} &= R_{ij} - K_{ij} - 2K_{ik}K^k_j + K_{ij},
\end{align*}
\]

(1)

where a vertical slash denotes partial spatial derivative with respect to \( X^i \), a double vertical slash covariant spatial differentiation with respect to the 3-metric, and an overdot the covariant time derivative. We end up with the following set of equations (supplemented by the defining equation for \( K_{ij} \)):

\[
\begin{align*}
g_{ij} &= -2K_{ij}, & -\dot{K} + K^{ij}K_{ij} &= 0; & K_{ij} - K_{ij}|_{ik} &= 0; & -\dot{K}_{ij} + KK_{ij} - 2K_{ik}K^k_j &= R_{ij}.
\end{align*}
\]

(2)

The second equation is the vacuum version of Raychaudhuri’s equation, and the third set of equations are the momentum constraints. Forming the trace of the last equation and inserting it into Raychaudhuri’s equation, we obtain the vacuum version of the Hamiltonian constraint:

\[
c = 8\pi G = 1.
\]

3 Greek indices run through 0, 1, 2, 3, while latin indices run through 1, 2, 3, and we set \( c = 8\pi G = 1 \).
\[ R + K^2 - K^i_j K^i_j = 0, \]
where \( R = R^k_k \) is the scalar curvature with respect to the 3-metric, and
\[ K := K^k_k = -\Theta \]
can be interpreted as (minus) the local expansion rate of the hypersurfaces.
We further propose to spatially average the scalar parts of the above equations, defined for any scalar function \( \Psi(t, X) \) as
\[ \langle \Psi(t, X') \rangle_D := \frac{1}{V_D} \int_D \Psi(t, X') \sqrt{\det(g_{ij})} \, d^3X, \]
where the volume of an arbitrary compact domain is \( V_D(t) := \int_D \sqrt{\det(g_{ij})} \, d^3X \). Defining a volume scale factor by \( a_D(t) := (V_D(t))/V_D(t_0)^{1/3} \), and averaging Raychaudhuri’s equation and the Hamiltonian constraint using the non-commutativity relation (true for any scalar \( S \)),
\[ \langle S \rangle_D - \langle \dot{S} \rangle_D = \langle \Theta S \rangle_D - \langle \Theta \rangle_D \langle S \rangle_D, \]
we obtain the following well-known equations [10]:
\[ \frac{\dot{a}_D}{a_D} = \frac{Q_D}{3}; \quad H_D^2 = -\frac{k_D}{a_D^2} - \frac{1}{6} (W_D + Q_D), \]
where \( H_D \) denotes the (domain-dependent) Hubble rate \( H_D = \dot{a}_D/a_D = -1/3 \langle K \rangle_D \), and \( k_D \) a (domain-dependent) constant of integration. The kinematical backreaction \( Q_D \) is composed of extrinsic curvature invariants, while \( W_D \) is an intrinsic curvature invariant and describes the deviation from constant curvature,
\[ Q_D := \{ K^2 - K^i_j K^i_j \}_D - \frac{2}{3} \langle K \rangle_D^2; \quad W_D := \langle \mathcal{R} - \frac{6k_D}{a_D^2} \rangle_D. \]
For a homogeneous domain, the above backreaction terms, being covariantly defined with respect to a given spatial embedding, are zero. Therefore, they encode the departure from spatial homogeneity. The integrability condition connecting the two equations (4) reads
\[ a_D^{-2} \langle a_D^6 W_D \rangle = -a_D^6 \langle a_D^6 Q_D \rangle, \]
expressing a combined conservation law for intrinsic curvature and extrinsic fluctuations.

3. Effective scalar field: the morphon

We rewrite the set of spatially averaged equations (4) and (6) and cast it into a Friedmannian form with a purely geometrical effective energy–momentum tensor [11, 12]:
\[ 3 \frac{\dot{a}_D}{a_D} = -\frac{\dot{\rho}_D}{2} + \frac{3 \rho_D}{a_D^2}; \quad 3H_D^2 + \frac{3k_D}{a_D^2} = \varrho_{\text{eff}}; \quad \dot{\varrho}_{\text{eff}} + 3H_D (\varrho_{\text{eff}} + p_{\text{eff}}) = 0, \]
where the effective densities are ‘morphed’ by a (minimally coupled) scalar field, the morphon \( \Phi_D \), thoroughly discussed in [13] and here defined through
\[ \rho_{\text{eff}}^D := -\frac{Q_D + W_D}{2} = \frac{1}{2} \Phi_D^2 + U_D; \quad p_{\text{eff}}^D := -\frac{3Q_D - W_D}{6} = \frac{1}{2} \Phi_D^2 - U_D, \]
where \( \xi = \pm \) for a standard/phantom scalar field, and \( U_D = U_D(\Phi_D) \) is the self-interaction potential. From the above equations, we obtain the following correspondence:
\[ Q_D = U_D - \xi \Phi_D^2; \quad W_D = -3U_D. \]
We appreciate that the deviation of the averaged scalar curvature from a constant-curvature model \( \mathcal{V}_D \) is directly proportional to the potential energy; an expanding (or contracting) classical vacuum with on average negative scalar curvature deviation (a positive potential \( U_D \)) can be attributed to a negative potential energy of a morphon field. The homogeneous case, \( Q_D = 0 \), corresponds to a virial equilibrium of the scalar field energies. Inserting (9)
into the integrability condition (6) implies that $\Phi_D$, for $\Phi_D \neq 0$, obeys the (scale-dependent) Klein–Gordon equation:

$$\ddot{\Phi}_D + 3H_0 \dot{\Phi}_D + \xi \frac{\partial}{\partial \Phi_D} U_D = 0.$$  \hspace{1cm} (10)

Averaged universe models obeying the previous set of equations follow, thus, Friedmannian kinematics with no fundamental sources, but with an effective scalar field perfect fluid that reflects the shape of spatial hypersurfaces and their embedding into spacetime. Given a choice of $\xi$ and of the potential, the evolution of these models is fixed (the governing equations are closed).

In order to model inflation, we impose the constraints $(U_D > 0, \xi = +1) \Rightarrow (W_D < 0, W_D + 3Q_D < 0)$: the morphon is canonical and its potential competes with its kinetic energy. These conditions still allow for any sign of the kinematical backreaction term; rewriting the extrinsic curvature invariants in (5) in terms of the local expansion and shear [8], we get that geometries dominated by their expansion fluctuations have $Q_D > 0 \Leftrightarrow U_D > \Phi_D^2$ (hence, $\ddot{a}_D > 0$, see (4)), those dominated by their shear fluctuations, $Q_D < 0$, and homogeneous spacetimes obey $Q_D = 0$ on all scales.

4. A morphonic inflaton

We offer the idea that the morphon can formally play the role of the inflaton, and that (unavoidable) curvature inhomogeneities occurring at one point of the Universe’s history could be the actual cause of a de Sitter-like era prior to the baryon nucleosynthesis era, provided they are governed by an appropriate potential. In the average quantities’ dictionary, inflation can be read off in terms of the effective first slow-roll parameter

$$\ddot{a}_D > 0 \Leftrightarrow \epsilon := -\frac{\dot{H}_D}{H_0^2} < 1.$$  \hspace{1cm} (11)

Some remarks are already necessary at this stage in order to assess our paradigm and to distinguish it from the usual ‘FLRW+fundamental sources’ inflationary picture. First, since equations (4) only functionally depend on a metric that need not be specified, requiring $\ddot{a}_D > 0$ does not imply that the comoving Hubble distance $1/aH$ decreases; however, such is the essence of the fitting problem to find the best FLRW fitting model to a lumpy Universe [14, 15]. Second, a fair solution of the flatness and smoothness problems would necessitate to implement averaging on the light cone, or would require an explicit inhomogeneous metric. Lacking the latter in the present context, we claim that realizing $\epsilon < 1$ during a sufficient number of e-folds is the best condition to address the dynamics of inflation. Third, so as to legitimate the neglect of any averaged matter, we refer to [26], where it is shown with the same class of assumptions (a flat enough curvature invariant $W_D$, see (12) below) that the corresponding cosmological parameter $\Omega_m^D = \langle \rho \rangle_D / 3H_0^2$ dives under a percent within a few $H_0^{-1}(t_i)$ times. Therefore, a large enough initial ‘Hubble parameter’ $H_D(t_i)$ consigns the matter content to oblivion.

A suitable potential $U_D$ has to be chosen in order to identify the morphon as a trustworthy inflaton. This identification implies that the cause (the inhomogeneities) and the consequence (the smoothness) of the present model are at odds. Therefore, a meaningful potential needs not only to describe how inhomogeneities can induce inflation but also how they can disappear throughout this process. Translated to the standard inflationary paradigm, this condition can be easily implemented by any potential that possesses a minimum downward which the inflaton rolls. One of the simplest examples, which has been extensively studied in the context of chaotic inflation [17], is a potential of the Ginzburg–Landau form:

$$U_D^{GL} = U_0 (\Phi_D^2 - \Phi_D^2_0)^2 / \Phi_D^4.$$  \hspace{1cm} (12)
This quartic potential can also be related to a fundamental Higgs field. However, even if such a scalar field is fundamental, there is the possibility that no Higgs particle is involved—as in our case—e.g. [18]. Contrary to standard inflation one must here raise the issue of the ‘reality’ of such a potential, since we postulate that $U_D$ is actually given by the averaged scalar curvature (9). More precisely, one should question the assumption of a slow-roll period, which corresponds to a nearly constant value of the averaged curvature. At this stage we cannot provide a proof for this possibility, e.g. through a suitable exact solution or through a well-defined approximation for the local evolution of curvature, but we can summarize plausible hints that the actual physical properties of an intrinsic curvature distribution could comply with the expected: first, there are indications from perturbation theory that the perturbative expansion is led by a term $\mathcal{W}_D \propto a_P^{-1}$, when putting the Friedmannian zero-order curvature to zero (so that the scenario is exclusively governed by backreaction, i.e. the curvature deviation from a flat model), and this perturbative expansion can be extended up to including even a constant term [19]; second, a concrete modeling of the non-perturbative aspects of the curvature distribution in a multiscale analysis reveals that, even if subdomains are evolving according to the term $\mathcal{W}_D \propto a_P^{-1}$, the variance between subdomains can lead to a de Sitter-like behavior on large scales [16]. These results, though not being a proof of the actual choice we made (say, we employ (12) as an illustrative example), open the possibility that there exist some configurations of the geometrical inhomogeneities that enable a slowly varying curvature, that is, a sufficiently flat potential. Obviously, here we are in the same situation as for an explanation of the dark energy phenomenon through inhomogeneities, an ongoing research field that is qualitatively well understood as for the mechanism but is not yet quantitatively conclusive.

Once the minimum $\Phi_0$ is fixed, the evolution of the morphon, given the integrability condition (6), is practically independent of the initial conditions$^4$ [20]. In figure 1, we show how all acceptable initial conditions are reinterpreted in terms of the curvature and expansion/shear fluctuations. For any of these types, we find the behavior shown in the inset of figure 2: instead of simple inflation $\epsilon \ll 1$, the choice of $U_D^{1/2}$ gives us slow-roll inflation $\epsilon \ll 1$ during which the energy density budget is dominated by dust $\rho_D = -\mathcal{W}_D/6H_D^2 \simeq 3/2$, and a negative kinematical backreaction density $\Omega_D^D = -Q_D/6H_D^2 \simeq -1/2$, i.e. by expansion variance, whatever the initial value of the homogeneous part of the curvature, $\Omega_Q^D = -k_D/3H_D^2 H_D^2 = 1 - \Omega_D^D - \Omega_Q^Q$, is. At the end of the inflationary era, the morphon oscillates down its potential and, because of the order of the polynomial (12), the effective scale factor behaves as if it were dominated by dust matter, $a_P \propto t^{2/3}$, and the effective deceleration parameter $q^D = -\ddot{a}_D(a_PH_D^2)^{-1} = 2\Omega_Q^D$ oscillates around $q^D \equiv 1/2$. While $\Omega_Q^Q$ oscillates around $3/4$ and $\Omega_Q^Q$ around $1/4$, $\mathcal{W}_D$ and $Q_D$ tend to vanish, i.e. our model generically tends to be—on average—homogeneous and quasi-isotropic.

Let us sum up the evolution scenario driven by kinematical backreaction $Q_D$: thanks to a favorable configuration of the latter and of the average curvature, (a) a period of accelerated expansion kicks out ($Q_D$ rises more rapidly than $H_D^2$), (b) an inflationary era follows during which the expansion variance stays positive and almost constant ($Q_D \simeq 3H_D^2 \simeq const.$), (c) the outcome is a globally inflated Universe, hence exponentially smooth, where gravity and

$^4$ $U_0$ determines the inflation’s onset time and also influences its duration. Due to the necessity to recover the CMB temperature fluctuations, ‘classical’ inflationary models suffer from fine-tuning issues, once perturbation theory is performed on the inflaton. Such is not the case here, since our model tackles the bulk effect of inflation within a highly nonlinear regime, without invoking perturbation theory. Hence, though a lack, the absence of perturbative constraints allows us to get rid of fine-tuning.
shear pull back, causing on time average a negative kinematical backreaction \( \dot{Q}_D \simeq -2/3t^2 \). Hence, the inhomogeneities wash out at the end of the process, so does the acceleration, hereby providing a natural graceful exit, univocally based, should we stress, on mere gravitation.
5. Discussion

Looking at a portion of the classical vacuum within Einstein’s theory, we obtained, by foliating spacetime and spatially averaging the scalar parts of the Einstein vacuum equations, a Friedmann-like evolution of the scale factor, however, driven by an effective scalar field. Starting with a sufficiently flat potential, the backreaction mechanism initiates a metamorphosis of the geometrical properties of space that goes along with an exponential inflationary phase. This dynamics tends to flatten the averaged scalar curvature and suppresses fluctuations. Although the present scenario just prescribes the initial conditions for a follow-up pre-heating, the same mechanism sets the conditions to provoke an intrinsic exit scenario. Note also that while acceleration was associated with the formation of inhomogeneities in the Late Universe, e.g. [21], the application of the same mechanism to the Early Universe tends to homogenize spatial hypersurfaces.

Let us list some of the immediate consequences of this mechanism. First, due to its classical nature, the inflaton’s mass is no longer limited by the Planck/SUSY scale (see e.g. [22]). Second, inflation is already possible for the unmodified Hilbert action, which of course does not exclude the need for improvements of Einstein’s theory. Third, inflation, often dubbed to be unsustainable in inhomogeneous spacetimes [23] (though see e.g. [24]), could occur despite the natural chaotic (inhomogeneous) initial conditions invoked by the theory [25] (the reason for this behavior is essentially that we do not study the stability of the fluctuations on a homogeneous reference background but that of a background-free general average).

Once this average is correctly computed, one can address the issue of fluctuations which one expects to be the seeds of large-scale structure and of the CMB spectrum; the absence of such a prediction is certainly a dearth that we should overcome (by formulating a fluctuation theory about the average). In a joined paper [26], we improved the present model by proving that a foliation into flat space sections is unstable and is attracted by an inhomogeneous negative-curvature state, and also by proving that the ‘empty’ model we considered is a generic attractor of ‘filled’ models under some conditions. In a forthcoming paper, we shall address a more general model that includes matter, radiation and fundamental scalar field inhomogeneities. If sources are present, there are interesting interactions with a morphon field, the latter being always present in the case of an inhomogeneous cosmology.

The presented scenario points to a huge potential of studying scalar field models in the Early Universe. We gave the simplest conceivable model that generates inflation out of the classical inhomogeneous vacuum. Contrary to the quasi-Newtonian standard picture, we render the bulk effect of curvature responsible for inflation. In other words, more than being acquitted of preventing inflation, inhomogeneities, when treated non-perturbatively, could be the actual cause of it.

Acknowledgments

The authors thank A Arbey, B Bassett, H van Elst, S Räsänen, X Roy and D Schwarz for valuable discussions, and X Roy for checking the results. This work is supported by ‘Fédération de Physique André-Marie Ampère’ of Université Lyon 1 and École Normale Supérieure de Lyon.

References

[1] Starobinsky A A 1980 A new type of isotropic cosmological models without singularity Phys. Lett. B 91 99
[2] Guth A H 1981 The Inflationary Universe: a possible solution to the Horizon and Flatness problems Phys. Rev. D 23 347
[2] Bassett B A, Tsujikawa S and Wands D 2006 Inflation dynamics and reheating Rev. Mod. Phys. 78 537
[3] Liddle A R, Parsons P and Barrow J D 1994 Formalizing the slow roll approximation in inflation Phys. Rev. D 50 7222

[4] Barbon J L F and Espinosa J R 2009 On the naturalness of Higgs inflation Phys. Rev. D 79 081302

[5] Dolgov A and Freese K 1995 Calculation of particle production by Nambu Goldstone bosons with application to inflation reheating and baryogenesis Phys. Rev. D 51 2693

[6] Parker L and Zhang Y 1993 Relativistic condensate as a source for inflation Phys. Rev. D 47 416

Brout R 2003 Condensation of Planckian modes and the inflaton arXiv:gr-qc/0305054

[7] Shankaranarayanan 2009 What if inflation is a spinor condensate? Int. J. Mod. Phys. D 18 2173

[8] Buchert T 2006 Dark Energy from structure—a status report Gen. Rel. Grav. 40 467

[9] Lifshitz E M and Khalatnikov I M 1963 Investigations in relativistic cosmology Adv. Phys. 12 185

[10] Buchert T 2000 On average properties of inhomogeneous fluids in general relativity: dust cosmologies Gen. Rel. Grav. 32 105

[11] Buchert T 2001 On average properties of inhomogeneous fluids in general relativity: perfect fluid cosmologies Gen. Rel. Grav. 33 1381

[12] Buchert T 2006 On globally static and stationary cosmologies with or without a cosmological constant and the dark energy problem Class. Quantum Grav. 23 817

[13] Buchert T, Larena J and Alimi J M 2006 Correspondence between kinematical backreaction and scalar field cosmologies: the ‘morphon field’ Class. Quantum Grav. 23 6379

[14] Ellis G R F and Stoeger W 1987 The ‘fitting problem’ in cosmology Class. Quantum Grav. 4 1697

[15] Ellis G R F and Buchert T 2005 The universe seen at different scales Phys. Lett. A. 347 38

[16] Wiegand A and Buchert T 2010 Multiscale cosmology and structure–emerging dark energy: a plausibility analysis Phys. Rev. D 82 023523

[17] Linde A D 1983 Chaotic inflation Phys. Lett. B 129 177

[18] Dehnen H and Frommert H 1993 Higgs mechanism without Higgs particle Int. J. Theor. Phys. 32 1135

[19] Li N and Schwarz D J 2007 On the onset of cosmological backreaction Phys. Rev. D 76 083011

[20] Linde A D 1985 Initial conditions for inflation Phys. Lett. B 162 281

[21] Rüscher S 2006 Backreaction and spatial curvature in a dust universe Class. Quantum Grav. 23 1823

[22] McDonald J 2000 Reheating temperature and inflaton mass bounds from thermalization after inflation Phys. Rev. D 61 083513

[23] Goldwirth D S and Piran T 1992 Initial conditions for inflation Phys. Rep. 214 223

Feinstein A and Ibañez J 1993 Exact inhomogeneous scalar field universes Class. Quantum Grav. 10 L227

[24] Iguchi O and Ishihara H 1997 Onset of inflation in inhomogeneous cosmology Phys. Rev. D 56 3216

Bergliaffa S E P and Hibberd K E 1999 Inhomogeneous scalar field solutions and inflation Int. J. Mod. Phys. D 8 705

[25] Turok N 2002 A critical review of inflation Class. Quantum Grav. 19 3449

[26] Roy X, Buchert T, Carloni S and Obadia N 2011 Global gravitational instability of FLRW backgrounds—interpreting the dark sectors Class. Quantum Grav. 28 165004 arXiv:1103.1146