In the context of the teleparallel equivalent of general relativity, we show that the energy-momentum density for the gravitational field can be described by a true spacetime tensor. It is also invariant under local (gauge) translations of the tangent space coordinates, but transforms covariantly only under global Lorentz transformations. When the gauge gravitational field equation is written in a purely spacetime form, it becomes the teleparallel equivalent of Einstein’s equation, and we recover Møller’s expression for the canonical gravitational energy-momentum pseudotensor.

1 Introduction

One of the oldest and most controversial problems of gravitation is the definition of an energy-momentum density for the gravitational field. As a true field, it would be natural to expect that gravity should have its own local energy-momentum density. However, it is commonly asserted that such a density can not be locally defined because of the equivalence principle. As a consequence, any attempt to identify an energy-momentum density for the gravitational field leads to complexes that are not true tensors. The first of such attempt was made by Einstein who proposed an expression for the energy-momentum density of the gravitational field which was nothing but the canonical expression obtained from Noether’s theorem. Indeed, this quantity is a pseudotensor, an object that depends on the coordinate system. Several other attempts have been made, leading to different expressions for the energy-momentum pseudotensor for the gravitational field.

It is usually accepted that in the context of general relativity, despite the existence of some controversial points related to the formulation of the equivalence principle, no tensorial expression for the gravitational energy-momentum density can exist. However, as our results show, in the gauge context, the existence of an expression for the gravitational energy-momentum density, which is a true spacetime and gauge tensor, turns out to be possible. Accordingly, the absence of such expression should be attributed to the general relativity description of gravitation, which seems to be not the appropriate framework to deal with this problem.

There has been along the years a continuous interest in this problem. In particular, a quasilocal approach has been proposed recently which is highly clarifying. According to this approach, for each gravitational energy-momentum pseudotensor, there is an associated superpotential which is a hamiltonian boundary term. The energy-momentum defined by such a pseudotensor does not really depend on the
local value of the reference frame, but only on the value of the reference frame on the boundary of a region — then its quasi-local character. As the relevant boundary conditions are physically acceptable, this approach validates the pseudotensor approach to the gravitational energy-momentum problem. It should be mentioned that these results were obtained in the context of the general relativity description of gravitation.

In the present work, the gravitational energy-momentum problem is re-examined in the context of teleparallel gravity. Due to the fundamental character of the geometric structure underlying gauge theories, the concept of currents, and in particular the concepts of energy and momentum, are much more transparent when considered from the gauge point of view. Accordingly, we are going to consider gravity as described by a gauge theory. Our basic interest will be concentrated on the gauge theories for the translation group and in particular on the so called teleparallel equivalent of general relativity.

2 The Gravitational Energy-Momentum Density as a Gauge Current

Let us consider the gauge gravitational field Lagrangian

\[ \mathcal{L}_G = \frac{hc^4}{16\pi G} S^{\rho\mu\nu} T_{\rho\mu\nu}, \]  

where \( h = \det(h^a_\mu) \), \( K_{\rho\mu\nu} \) is the contorsion tensor and

\[ S^{\rho\mu\nu} = -S^{\mu\rho\nu} = \frac{1}{2} \left[ K^{\mu\nu\rho} - g^{\mu\nu} T_{\theta\rho} + g^{\rho\mu} T_{\theta\nu} \right] \]

is a tensor written in terms of the Weitzenböck connection only. As usual in gauge theories, the lagrangian is quadratic in the field strength, represented here by the torsion tensor. By using the relation

\[ \Gamma^\rho_{\mu\nu} = \Gamma^{\circ\rho}_{\mu\nu} + K^\rho_{\mu\nu}, \]

the gauge lagrangian can be rewritten in terms of the Levi-Civita connection only. Up to a total divergence, the result is the Hilbert-Einstein Lagrangian of general relativity

\[ \mathcal{L} = -\frac{c^4}{16\pi G} \sqrt{-g} \tilde{R}, \]

where the identification \( h = \sqrt{-g} \) has been made.

By performing variations in relation to the gauge field \( A_a^\rho \), we obtain from \( \mathcal{L}_G \) the teleparallel version of the gravitational field equation,

\[ \partial_\sigma (h S^{\sigma}_{\alpha\rho}) - \frac{4\pi G}{c^4} (h j_a^\rho) = 0, \]

where \( S^{\alpha\rho}_{\alpha} \equiv h_\alpha^\lambda S_{\alpha\lambda}^\rho \). Analogously to the Yang-Mills theories

\[ h j_a^\rho \equiv \frac{\partial \mathcal{L}_G}{\partial h^a_\rho} = -\frac{c^4}{4\pi G} hh_\alpha^\lambda S^{\mu\nu}_{\rho\lambda} T_{\mu\nu} + h_a^\rho \mathcal{L}_G \]
stands for the gravitational gauge current, which in this case represents the energy and momentum of the gravitational field. The term \((hS_{\sigma\rho})\) is called superpotential in the sense that its ordinary derivative yields the gauge current \((hj_{\sigma})\). Because of the anti-symmetry of \(S_{\sigma\rho}\) in the last two indices, \((hj_{\sigma})\) is conserved as a consequence of the field equation:

\[
\partial_\rho (hj_{\sigma}) = 0. \tag{6}
\]

Making use of the identity

\[
\partial_\rho h \equiv h\Gamma^\sigma_{\rho\nu} = h(\Gamma^\nu_{\rho\nu} - K^\nu_{\rho\nu}) , \tag{7}
\]

this conservation law can be rewritten as

\[
D_\rho j_{\sigma\rho} = \partial_\rho j_{\sigma\rho} + (\Gamma^\rho_{\lambda\rho} - K^\rho_{\lambda\rho}) j^\lambda_{\sigma} = 0 , \tag{8}
\]

where \(D_\rho\) is the teleparallel version of the covariant derivative, which is nothing but the Levi-Civita covariant derivative of general relativity rephrased in terms of the Weitzenböck connection. As can be easily checked, \(j_{\sigma\rho}\) transforms covariantly under a general spacetime coordinate transformation, and is invariant under local (gauge) translation of the tangent-space coordinates. This means that \(j_{\sigma\rho}\) is a true spacetime and gauge tensor. However, it transforms covariantly only under a global tangent-space Lorentz transformation.

3 The Gravitational Energy-Momentum Pseudotensor

To find the relation between the gauge approach and general relativity, it is necessary to express the gauge field equation in a pure spacetime form. By considering the expression of the Weitzenböck connection in terms of the tetrad field,

\[
\Gamma^\rho_{\mu\nu} = h_{\alpha}\partial_\nu h_{\alpha\mu} , \tag{9}
\]

Eq. (4) can be rewritten in the form

\[
\partial_\sigma (hS_{\lambda\sigma\rho}) - \frac{4\pi G}{c^4} (ht_{\lambda\rho}) = 0 , \tag{10}
\]

where now

\[
ht_{\lambda\rho} = \frac{c^4}{4\pi G} h \Gamma^\mu_{\nu\lambda} S_{\mu\nu\rho} + \delta_{\lambda\rho} \mathcal{L}_G \tag{11}
\]

stands for the teleparallel version of the canonical energy-momentum pseudotensor of the gravitational field. Despite not explicitly apparent, as a consequence of the local Lorentz invariance of the gauge Lagrangian \(\mathcal{L}_G\), the field equation (10) is symmetric in \((\lambda\rho)\). Furthermore, by using the relation (2), it can be rewritten in terms of the Levi-Civita connection only. As expected, due to the equivalence between the corresponding lagrangians, it is the same as Einstein’s equation:

\[
\frac{h}{2} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] = 0 . \tag{12}
\]
The canonical energy-momentum pseudotensor \( t_{\lambda \rho} \) is not simply the gauge current \( j_{a \rho} \) with the algebraic index “\( a \)” changed to the spacetime index “\( \lambda \)”.

It incorporates also an extra term coming from the derivative term of Eq. (4):

\[
t_{\lambda \rho} = h^{a}_{\lambda} j_{a \rho} + \frac{c}{4\pi G} \Gamma^\mu_{\lambda\rho} S_{\mu \rho}.
\]

(13)

We see thus clearly the origin of the connection-term which transforms the gauge current \( j_{a \rho} \) into the energy-momentum pseudotensor \( t_{\lambda \rho} \). Through the same mechanism, it is possible to appropriately exchange further terms between the derivative and the current terms of the field equation \( (10) \), giving rise to different definitions for the energy-momentum pseudotensor, each one connected to a different superpotential \( (hS_{\lambda \rho}) \). Like the gauge current \( (hj_{a \rho}) \), the pseudotensor \( (ht_{\lambda \rho}) \) is conserved as a consequence of the field equation:

\[
\partial_{\rho}(ht_{\lambda \rho}) = 0 .
\]

(14)

However, in contrast to what occurs with \( j_{a \rho} \), due to the pseudotensor character of \( t_{\lambda \rho} \), this conservation law can not be rewritten with a covariant derivative.

Because of its simplicity and transparency, the teleparallel approach to gravitation seems to be much more appropriate than general relativity to deal with the energy problem of the gravitational field. In fact, Møller already noticed a long time ago that a satisfactory solution to the problem of the energy distribution in a gravitational field could be obtained in the framework of a tetrad theory. In our notation, his expression for the gravitational energy-momentum density is

\[
ht_{\lambda \rho} = \frac{\partial \mathcal{L}}{\partial \partial_{\rho} h^{a}_{\mu}} \partial_{\lambda} h^{a}_{\mu} + \delta_{\lambda \rho} \mathcal{L} ,
\]

(15)

which is nothing but the usual canonical energy-momentum density yielded by Noether’s theorem. Using for \( \mathcal{L} \) the gauge Lagrangian \( [1] \), it is an easy task to verify that Møller’s expression coincides exactly with the teleparallel energy-momentum density appearing in the field equation \( (10-11) \). Since \( j_{a \rho} \) is a true spacetime tensor, whereas \( t_{\lambda \rho} \) is not, we can say that the gauge current \( j_{a \rho} \) is an improved version of Møller’s energy-momentum density \( t_{\lambda \rho} \). Mathematically, they can be obtained from each other by Eq. \( (13) \). It should be remarked, however, that both of them transform covariantly only under global tangent-space Lorentz transformations. This is, we believe, the farthest one can go in the direction of a tensorial definition for the energy and momentum of the gravitational field. The lack of a local Lorentz covariance can be considered as the teleparallel manifestation of the pseudotensor character of the gravitational energy-momentum density in general relativity. Accordingly, we can say that, if it were possible to define a local Lorentz covariant gauge current in the teleparallel gravity, the corresponding general relativity energy-momentum density would be represented by a true spacetime tensor.

4 Conclusions

In the context of a gauge theory for the translation group we have obtained an energy-momentum gauge current \( j_{a \rho} \) for the gravitational field which transforms covariantly under spacetime general coordinate transformations, and is invariant
under local (gauge) translations of the tangent-space coordinates. This means essentially that $j_a^\rho$ is a true spacetime and gauge tensor. By rewriting the gauge field equation in a purely spacetime form, it becomes equivalent to Einstein’s equation, and the gauge current $j_a^\rho$ reduces to the canonical energy-momentum pseudotensor of the gravitational field, which coincides with Møller’s well-known expression. In the ordinary context of general relativity, therefore, the energy-momentum density for the gravitational field will always be represented by a pseudotensor.

By considering the quasilocal approach, we can say that to any energy-momentum pseudotensor there is an associated superpotential which is a hamiltonian boundary term. On the other hand, the teleparallel field equations explicitly exhibit both the superpotential and the gravitational energy-momentum complex. We see then that, in fact, by appropriately exchanging terms between the superpotential and the current terms of the field equation (10), it is possible to obtain different gravitational energy-momentum pseudotensors with their associated superpotentials. In this context, our results can be rephrased according to the following scheme. First, notice that the left-hand side of the field equation (10) as a whole is a true tensor, though each one of its two terms is not. Then if we extract the spurious part from the first term — so that it becomes a true spacetime and gauge tensor — and add this part to the second term — the energy-momentum density — it becomes also a true spacetime and gauge tensor. We thus arrive at the gauge-type field equation (4), with $(hS_a^\rho\sigma)$ as the superpotential, whose corresponding expression for the conserved energy-momentum density for the gravitational field, given by $j_a^\rho$, though transforming covariantly only under a global tangent-space Lorentz transformation, is a true spacetime and gauge tensor.

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References

1. C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
2. See, for example, A. Trautman, in *Gravitation: an Introduction to Current Research*, ed. by L. Witten (Wiley, New York, 1962).
3. A. Papapetrou, Proc. Roy. Irish Acad. A 52, 11 (1948); P. G. Bergmann and R. Thompson, Phys. Rev. 89, 400 (1953); C. Møller, Ann. Phys. 4, 347 (1958); L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975);
4. For a critical discussion of the equivalence principle, see the preface of J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).
5. V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, Phys. Rev. Lett. 84, 4533 (2000).
6. J. W. Maluf, J. Math. Phys. 36, 4242 (1995).
7. M. Dubois-Violette and J. Madore, Comm. Math. Phys. 108, 213 (1987); L. B. Szabados, Class. Quant. Grav. 9, 2521 (1992); J. M. Aguirregabiria, A. Chamorro and K. S. Virbhadra, Gen. Rel. Grav. 28, 1393 (1996); T. Shirafuji and G. L. Nashed, Prog. Theor. Phys. 98, 1355 (1997); S. Deser, J. S. Franklin and D. Seminara, Class. Quant. Grav. 16, 2815 (1999); S. V. Babak and L. P. Grishchuck, Phys. Rev. D 61 024038 (2000); T. Kawai, Phys. Rev. D 62 10401 (2000); M. Blagojević and M. Vasić, gr-qc/0008159; J. W. Maluf and J. F. da Rocha-Neto, gr-qc/0008073.

8. C. C. Chang and J. M. Nester, Phys. Rev. Lett. 83, 1897 (1999), and references therein.

9. F. Gronwald and F. W. Hehl, On the Gauge Aspects of Gravity, in Proceedings of the 14th School of Cosmology and Gravitation, Erice, Italy, ed. by P. G. Bergmann, V. de Sabbata and H.-J Treder (World Scientific, Singapore, 1996).

10. For a general review, see: F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne’emann, Phys. Rep. 258, 1 (1995).

11. C. Pelegrini and J. Plebanski, Mat. Fys. Skr. Dan. Vid. Selsk. 2, n. 4 (1963); K. Hayashi and T. Nakano, Prog. Theor. Phys. 38, 491 (1967); K. Hayashi and T. Shirafuji, Phys. Rev. D 19, 3524 (1979).

12. J. W. Maluf, J. Math. Phys. 35, 335 (1994).

13. For a general description of teleparallel gravity, see V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, Teleparallel Gravity: An Overview, in Proceedings of the IX Marcel Grossmann Meeting, Rome, Italy (2000).

14. See, for example, P. Ramond, Field Theory: A Modern Primer, 2nd edition (Addison-Wesley, Redwood, 1989).

15. V. C. de Andrade and J. G. Pereira, Int. J. Mod. Phys. D 8, 141 (1999).

16. See, for example, S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972), page 371.

17. C. Møller, Mat. Fys. Skr. Dan. Vid. Selsk. 1, n. 10 (1961).