Towards a Physical Oracle for the
\textsc{Partition} Problem using Analogue Computing

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Abstract

Despite remarkable achievements in its practical tractability, the notorious class of NP-complete problems has been escaping all attempts to find a worst-case polynomial time-bound solution algorithms for any of them. The vast majority of work relies on Turing machines or equivalent models, all of which relate to digital computing. This raises the question of whether a computer that is (partly) non-digital could offer a new door towards an efficient solution. And indeed, the \textsc{Partition} problem, which is another NP-complete sibling of the famous Boolean satisfiability problem \textsc{Sat}, might be open to efficient solutions using analogue computing. We investigate this hypothesis here, providing experimental evidence that \textsc{Partition}, and in turn also \textsc{Sat}, may become tractable on a combined digital and analogue computing machine. This work provides mostly theoretical and based on simulations, and as such does not exhibit a polynomial time algorithm to solve NP-complete problems. Instead, it is intended as a pointer to new directions of research on special-purpose computing architectures that may help handling the class NP efficiently.

1 Introduction

Among the vast number of problems known to computer scientists, the structurally simplest ones seem to notoriously escape attempts towards efficient solutions. The most prominent class of such presumably intractable problems are NP-complete, with the classical reference of [3] listing hundreds of well-known examples. Since all NP-complete problems are computationally equivalent, it suffices to solve any problem to get the entire class tackled in one single blow.

An oracle is a complexity-theoretic concept to represent some assumption about solvability of certain problems, so that the difficulty of one problem can be measured relative to the difficulty of another problem (polynomial reductions serve a similar purpose). Abstractly, an oracle is a set (language) \( A \subseteq \Sigma^* \) over some finite alphabet \( \Sigma \), for which the question “\( w \in A \)?” can be answered in

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Taking \( A \) as any \( \text{NP} \)-complete problem, say assuming that we can answer satisfiability of a Boolean formula in conjunctive normal form in constant time, reduces the effort for any other problem in \( \text{NP} \) to the labour of converting the given problem instance into an instance of \( A \). In brief: any oracle \( A \) that is \( \text{NP} \)-complete equates \( \text{P}^A = \text{NP}^A \). It has also been shown, however, that there are other oracles \( B \) that separate the classes as \( \text{P}^B \neq \text{NP}^B \), so that the famous \( \text{P} \)-vs-\( \text{NP} \) question cannot be settled by any argument that uses oracles \(^2\).

Still, all classical results in this area rely on the conventional Turing machine model or an equivalent thereof, which is intrinsically discrete (and hence suitable for digital computing). The hypothesis put forth and investigated in this report concerns the use of analog computing to physically build an oracle that may be able to solve an \( \text{NP} \)-complete problem, thus technically equalizing \( \text{P} \) and \( \text{NP} \) under this oracle, even though doing this outside the Turing machine model.

The problem of choice is \textsc{Cosine-Product-Integration} (problem [AN14] in \(^6\)):

**Instance:** A sequence \((a_1, a_2, \ldots, a_n)\) of integers

**Question:** With

\[
s_n(t) := \prod_{i=1}^{n} \cos(a_i \cdot t) \tag{1}
\]

does,

\[
\int_0^{2\pi} s_n(t) dt = 0 \tag{2}
\]

hold?

The analysis of this problem in terms of complexity and solvability is most instantly achieved by taking the Fourier transform of \( s \), which is

\[
\mathcal{F}(s_n)(\omega) = \frac{\sqrt{\pi/2}}{2^{n-1}} \sum_{M \subseteq N} \left[ \delta \left( \sum_{i \in M} a_i - \sum_{i \in N \setminus M} a_i + \omega \right) \right. \\
\left. + \delta \left( \sum_{i \in M \setminus N} a_i - \sum_{i \in M} a_i + \omega \right) \right], \tag{3}
\]

where \( \delta \) is the Dirac distribution, and \( N = \{1, 2, \ldots, n\} \). Equation \(^3\) is easily derived from the convolution theorem and induction on \( n \): For \( n = 1 \), we have \( \mathcal{F}(\cos(a_1 \cdot t))(\omega) = \sqrt{\pi/2} \delta(\omega - a_1) + \sqrt{\pi/2} \delta(a_1 + \omega) \), and \(^3\) holds. Now, assume \(^3\) to hold up to \( n-1 \), and consider \( \mathcal{F}(s_{n-1} \cdot \cos(a_n \cdot t)) = \mathcal{F}(s_{n-1}) \ast \mathcal{F}(\cos(a_n \cdot t)) \) by

\(^1\)The particular style of oracle access can create subtle differences here as discussed in \(^2\), but this is of no particular relevance here.
the convolution theorem. Since $\ast$ is distributive w.r.t. addition and associative with scalar multiplication, the induction hypothesis gives

$$(\mathcal{F}(s_{n-2}) \ast \mathcal{F}(\cos(a_n \cdot t)))(\omega) = \sqrt{\frac{\pi}{2}} \sum_{M \subseteq N} [\delta(A + \omega) + \delta(-A + \omega)] * \left(\sqrt{\frac{\pi}{2}} \delta(\omega - a_1) + \sqrt{\frac{\pi}{2}} \delta(a_1 + \omega)\right),$$

with the terms $A = \sum_{i \in M} a_i - \sum_{i \in N \setminus M} a_i$ to abbreviate the expression. The inner convolution evaluates to

$$\frac{\pi}{2} \delta(a_n + A - \omega) + \frac{\pi}{2} \delta(a_n - A + \omega) + \frac{\pi}{2} \delta(-a_n + A + \omega) + \frac{\pi}{2} \delta(a_n + A + \omega),$$

from which (3) directly follows after rearranging terms.

The Fourier spectrum provides a variety of useful insights: first, note that the zero-th harmonic $\delta(\omega)$ appears in the spectrum if and only if there is some subset $M_0 \subseteq N$ for which we have the identity

$$\sum_{i \in M_0} a_i = \sum_{j \in N \setminus M_0} a_j. \quad (4)$$

This is indeed a well-known $\text{NP}$-complete problem, known as PARTITION:

**Instance:** A finite set $A$ and a “size” $s(a) \in \mathbb{Z}^+$ for each $a \in A$.

**Question:** Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A \setminus A'} s(a)?$$

The formulation (4) is easily recognized as a special case of the PARTITION problem by taking the size of the element $a_i$ equal to the (positive) integer $a_i$ itself. A slight issue remains with the signs of the integers, since COSINE-PRODUCT-INTEGRATION allows negative integers in the list, but the cos-function is symmetric around 0 anyway, so the signs do not matter here.

Going back to the Fourier spectrum, we can thus solve the PARTITION problem if we could decide whether there is a zeroth harmonic in the Fourier spectrum. Doing this numerically would work by straightforward evaluation of the function $s$ defined by (1). Shannon’s sampling theorem tells us that we require at most twice as many points as the maximal frequency in the spectrum is. Here, we get another insight from (3), since the maximal frequency is precisely $f_{\text{max}} = a_1 + a_2 + \ldots + a_n$. Hence, we require $2 \cdot f_{\text{max}}$ points on the curve $s(t)$ within $t \in [0, 2\pi]$, to run a (fast) Fourier transform to get the zero frequency. The latter can be done in polynomial time in the number of points, but since this number depends on the “magnitude” of the problem (i.e., the range of the numbers that describe it), the overall running time of such a solution attempt would be pseudo-polynomial (as it is polynomial in both, the number of points...
and the magnitude of the problem). This confirms another well known statement made in [6], who mention that Cosine-Product-Integration is indeed solvable in pseudo-polynomial time. As such, it is one of the “easiest” among all \( \text{NP} \)-complete problems.

So, with a digital computer being presumably bound to an undesirable runtime (unless \( \text{P} = \text{NP} \)), our next goal is looking for what analogue computing can do here. The crux of analog computing lies in the effect of all (exponentially many) spectral components arising simultaneously in parallel, rather than sequentially as would be the case on a conventional computing architecture. After a final physical cut-off (damping) of high frequencies, we can sample the resulting signal at a constant rate, allowing us to extract the DC part as a reliable indicator towards a YES- or NO-answer to the initial problem. The overall procedure to solve PARTITION is the following:

1. \textbf{(analogue computing)}: Generate the signal \( s(t) \) (by analogue computation along a cascade of 4-quadrant multipliers; see Section 2).

2. \textbf{(analogue computing)}: Apply a low-pass filter with a constant cut-off frequency \( f_0 \), to enable sampling the signal efficiently without altering its zeroth harmonic.

3. \textbf{(analogue-digital transition)}: Record a sample at equidistant time steps \( t_1, \ldots, t_m \) over at least one full period of the output signal \( s(t) \) leaving out error terms here for simplicity. The result is a series of pairs \( (t_i, y_i) \) with \( y_i = s(t_i) \), for \( i = 1, 2, \ldots \). To avoid unwanted phase shifts in the recorded signal, the sampling must start at a time when all signals are aligned, which besides time \( T_0 = 0 \) happens first at time \( T_1 = \text{lcm} \{ a_i^{-1} | a_i \in V \} \) (leaving frequency error terms out here for simplicity). The number \( m \) of points sampled crucially influences the result of the subsequent Fourier analysis (via inducing noise visible the Fourier spectrum, exposed in simulations). That number depends on the sampling time step \( \tau \), which must satisfy \( \tau \leq 1/(2f_0) \) to enable a unique reconstruction of the signal (sampling theorem).

4. \textbf{(digital computing)}: Fourier-transform the data (say, on a normal computer) to distill the amplitude of the component \( \delta(\omega) \) in the signal, i.e., the direct current (DC) term, in the sampled signal. Call the result \( \text{DC} \).

Then, the underlying instance of PARTITION, has the following answer, based on (3):

\[
\text{if } \text{DC} = 0 \text{ then output "YES" else output "NO" } \quad (5)
\]

Without step 2, the procedure would require pseudo-polynomial time on a conventional computer, but computing step 1 is as well a matter of pseudo-polynomial complexity, since the convolution with the filter transfer function
depends on the magnitude of the Partition-instance. Therefore, the computation of the signal \( s(t) \) and its low-pass filtering are shifted to an analogue computing circuit described next. This escape of complexity is bought at the cost of an imperfect indicator \( DC \), which, even for YES-instances, may be nonzero, but still could be separated from its (expected) value for NO-instances. Thus, we will simulate a physical circuit to get a feeling on where the boundary between YES- and NO-instances is in terms of \( DC \).

2 Analog Multipliers and Low-Pass Filtering

Evaluating the product \( s \) is straightforward by cascading 4-quadrant multipliers with impedance converters and amplifiers in between. The integration could – in theory – also be done by an analogue circuit, but is most easily done by a low-pass filter. The crucial point here is to escape Shannon’s sampling condition by cutting off high frequency parts from the spectrum without touching the zeroth harmonic (i.e., the direct current part in the signal).

Consider an ideal low-pass filter, whose Fourier spectrum is a Heaviside function that jumps somewhere in the region \((0, \min_i a_i)\). This filter would leave only the DC part of the signal intact, thus directly delivering the sought output. Alas, the best that we can do in practice is using filters that strongly damp high frequencies, but we cannot annihilate them ultimately. However, for our purposes, this is not even necessary, since all we need to do is damping frequencies above a certain fixed limit \( f_0 \), since the low-pass filter leaves the DC part in any case intact, irrespectively of its cutoff frequency. However, if we can “disregard” harmonics \( \geq f_0 \), then we can sample a fixed number of \( 2 \cdot f_0 \in O(1) \) points on the signal to recover an approximate version thereof that contains the sought zeroth harmonic. At the same time, the complexity of this procedure would be no longer pseudo-polynomial, since the sampling has become independent of the problem’s magnitude (controlling the high frequency parts of the signal).

This is the trick that we seek to put to practice in the following, however, practical matters put us back to pseudo-polynomial complexity again, since no physical component has unlimited and perfect behavior over the entire bandwidth. That is, if we use an analog multiplier that works nicely up to frequencies of, say, \( f^* \), then problem instances for which \( \sum a_i > f^* \) holds will no longer be correctly evaluated on our analogue computer. Neither can any physical low-pass filter do a perfect cut-off of high-frequencies (we can only approximate the Heaviside function in the spectrum). Still, we can circumvent the former issue by “squeezing” the frequencies into the region less than \( f^* \), simply by downscaling the problem instance’s magnitude accordingly. To see why this works, note that we can multiply \( \Box \) with any \( \lambda > 0 \) without altering the problem’s answer. So, by using a sufficiently small \( \lambda \), we can downscale all integers \( a_i \) up to a magnitude where \( \sum_{i=1}^{n} \lambda \cdot a_i < f^* \), so that our analogue multipliers can work within their admissible bandwidth regions. This change puts the harmonics closer together so that the separation of the DC part becomes
more difficult, since the “squeezed” version \((\lambda \cdot a_1, \ldots, \lambda \cdot a_n)\) of the COSINE-PRODUCT-INTEGRATION-instance \((a_1, \ldots, a_n)\) has its closest pair of frequencies at distance \(\geq \lambda\).

The inevitably imperfect low-pass filtering in any case leaves frequencies \(> f^*\) in the spectrum, so that sampling at a rate of \(2 \cdot f^*\) will create alias bands overlapping the spectral region \([0, f^*]\). In turn, we will thus see an error in the DC part that depends on the damping of high frequencies. This error can be made small by a proper construction of the filter, and by allowing for a higher sampling rate, so that the alias frequencies have small amplitudes.

2.1 Cascading Multipliers

Figure 1 displays a simple circuit that multiplies three cosine-waves using the AD633 four quadrant multiplier by Analog Devices (drawn in LTSpice [3]). This component takes two signals \(x(t) = X1 - X2, y(t) = Y1 - Y2\) and outputs a signal \(\frac{x(t) \cdot y(t)}{10} + Z\), where \(Z\) is an auxiliary additive input (the symbols in this description correspond to the PIN configurations displayed in Figure 1). To compensate the downscaling by the factor 10 (upon cascading the multipliers) and to have approximately ideal input and output impedances, we add a non-inverting operational amplifier circuit in between two multipliers. The resulting structure can then straightforwadly cascaded to multiply several signals, before feeding the final signal into a passive low-pass filter and a last amplifier.

The model description of the AD633 component here directly gives the frequency window in which the component works correctly, which ranges roughly up to \(f^* = 120\text{kHz}\), which is our practical limit for \(\sum a_i\). The same consideration must be made for the intermediate multipliers, as frequencies accumulate along the multiplication chain, thus increasing the likelihood to get outside the bandwidth of the multipliers/amplifiers.

Observe that errors in the gain are of no significant relevance to our problem, since it is a simple matter of algebra to pull out all factors \(r_1 \cdot \cos(a_1 \cdot t)\) in the front of the integral before doing the analysis. So, we merely end up with a factor \(r = \prod_{i=1}^{n} r_i\) in front of the spectrum, but we are only interested in whether or not the integral vanishes. Likewise, it is immediate that a multiplication of \([4]\) leaves the solvability of the problem instance unchanged. The main reason to care about amplitude errors is to avoid steering the circuit eventually into
saturation at later stages, when the limit incurred by the supply voltage is hit. For this reason, we will work with amplitudes of 1V per signal and 10V supply voltage, so that the amplitude errors should not accumulate too much along the chain.

2.2 Frequencies and Phase-Shifts

Errors in the frequencies are trickier to handle, and induce a necessary tolerance domain around the origin when the final decision is made. Let us consider a distorted version \((a_i + \varepsilon_i)_{i=1}^n\) of the original problem instance \((a_1, \ldots, a_n)\), in which \(\varepsilon_i\) are the frequency errors. Obviously, the answer that we get is whether or not any subset of \(\{a_i + \varepsilon_i : i = 1, \ldots, n\}\) sums up to equality of the sum over the complement set. Put \(\varepsilon := \sum_{i=1}^n \varepsilon_i\). In the spectrum, we thus get a component \(\delta(\omega + \varepsilon')\) with \(\varepsilon' \leq \varepsilon\) for a YES-instance of the problem, and \(\delta(\omega + \varepsilon'')\) with \(\varepsilon'' \geq \min_{i,j}(a_i + \varepsilon_i - a_j - \varepsilon_j)\) for a NO-instance. Those two are distinguishable only if \(\varepsilon\) and \(\min_{i,j}(a_i + \varepsilon_i - a_j - \varepsilon_j)\) are separated far enough. The difficulty of this increases with the number of signals to be multiplied (bringing pseudopolynomial complexity back into the game here), since squeezing the problem instance reduces the separation of spectral frequencies and thus calls for more accurate frequencies.

The synchronization of oscillators gains even more importance if we take phase errors into account. Actually, the use of cosine waves in the multiplication is crucial, as the whole idea fails if we integrate a product of sine-waves instead (respective (counter)examples are very easy to find). Hence, synchronization and phase shift are vital aspects in a practical implementation, and could be done in several ways:

- Synchronization of coupled analogue oscillators: this is a well-studied problem in the literature (see, e.g., [7] to make a start), but induces the issue of time until the synchronization kicks in at sufficient accuracy. Synchronization arguments are typically based on Lyapunov functions, which tell that a system will become stable (i.e., synchronized), but usually remain silent on the speed at which this happens.

- Low-pass filtering of digital counters: consider a global clock simultaneously feeding \(n\) counters, each of which triggers a T-Flip-Flop at its output. That is, whenever the \(i\)-th counter reaches the value \(a_i\), it gets a reset signal and triggers an output T-Flip-Flop to produce an (approximately) rectangular pulse whose frequency corresponds to \(a_i\). Smoothing each of these by a low-pass filter, and integrating them to convert the sine- into a cosine-waves, we end up with the desired set of synchronized signals.

This approach, however, is not appealing for the reason of taking again literally pseudopolynomial time, since the counter would have to count up to \(\text{lcm}(a_1, \ldots, a_n)\) in order to have a fully synchronized set of signals for the first time.
For a lab experiment using only a few signals of low frequencies, software like LabView can equally well do the job. For our experimental evaluation to follow, we will put everything into a SPICE simulation, thus avoiding all matters of imperfection in the frequencies or phase shifts.

3 Experimental Evaluation in LTspice

The LTspice netlist can directly be written based on the circuit schematic in Figure 1. The analogue multiplier is Analog Devices’ AD633/AD. The operational amplifier is the (classical) ua741. We ran a transient analysis, we chose the sampling interval, starting and stopping time for the analysis as follows: Given frequencies \( a_1, a_2, \ldots, a_n \), the Nyquist frequency for the sampling is \( f_{Nyquist} = 2 \cdot \sum_{i=1}^{n} a_i \), so that the maximal time between taking samples is bound as \( \tau_{sampling} < 1/f_{Nyquist} \). Towards synchronizing the recording of values with the phase shift, and to allow for a burn in phase, the sampling can start at any integer multiple of \( \text{lcm}(T_1, \ldots, T_n) \), where \( T_i = 1/a_i \) is the period time of the \( i \)-th input signal at frequency \( a_i \). This value can be computed as

\[
\text{lcm}(T_1, \ldots, T_n) = \text{lcm}\left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right) = \frac{1}{\text{gcd}(a_1, \ldots, a_n)},
\]

by virtue of the general rule \( \text{lcm}(a/b, c/d) = \text{lcm}(a, c)/\text{gcd}(b, d) \). For our transient analysis, we used a (maximal) time step of \( 2 \mu s \) and started recording after 1.2ms (the 12-fold of the lcm of periods, being 1 in all our cases) until 3ms. The data for the Fourier analysis is thus sampled every \( 2 \mu s \).

For illustration, we simulate YES and NO-instances with 3 and 4 frequencies (for which the LTspice simulation runs reasonably fast\(^2\)).

Figure 2 shows the simulated vs. the ideal wave-forms, for a rough visual validation. Figure 3 shows the respective Fourier analyses vs. what is expected analytically from (3). The example used for illustration is the YES-instance \( (a_1, a_2, a_3) = (3, 2, 5) \).

As the simulations indicate, the DC parts found in the Fourier analysis are never fully vanishing for the YES-instances of the problem, although their magnitude is clearly different between YES- and NO-instances. This effect can be attributed to the additive offset voltages of the multiplier (typically ±5mV up to ±50mV for the output, and typically ±5mV up to ±30mV for the inputs),

\(^2\)To fix a convergence issue in the transient analysis, we added a shunt capacity of 0.5 fF by issuing `option cshunt=2e-15` in the LTSpice netlist. Given the frequencies that we use, the shunt capacity creates an impedance in the giga Ohm range, so we do not need to expect too much change in the circuit behavior. For validation of this conjecture, i.e., a verification that the circuit behavior is not dramatically altered by the shunt capacitance, we conducted an independent analysis using Spice Opus \(^8\), which roughly gave the same results.

\(^3\)The spectra obtained from the simulation are shown with linear axis scaling; displaying the same data with logarithmic scale (in dB) reveals a considerable lot of noise in the spectrum, which – for the simulation – is partly due to the discrete sampling done during the transient analysis. Reducing the time-step in turn reduces the computed noise, but a real circuit will necessarily show a much more complex spectrum due to thermal and other unavoidable noise.
(a) Simulated Signal $\cos(a_1 t) \cos(a_2 t)$ with $a_1 = 2\pi \cdot 20\, \text{kHz}$, $a_2 = 2\pi \cdot 30\, \text{kHz}$.

(b) Computed Curve $\cos(2t) \cos(3t)$

(c) Simulated Curve $\cos(a_1 t) \cos(a_2 t) \cos(a_3 t)$ with $a_1 = 2\pi \cdot 20\, \text{kHz}$, $a_2 = 2\pi \cdot 30\, \text{kHz}$ and $a_3 = 2\pi \cdot 50\, \text{kHz}$.

(d) Computed Curve $\cos(2t) \cos(3t) \cos(5t)$

Figure 2: Simulated vs. Computed Signals
(a) Spectrum of Simulated Signal $\cos(a_1 t) \cos(a_2 t)$ with $a_1 = 2\pi \cdot 20kHz$, $a_2 = 2\pi \cdot 30kHz$. Theoretical Spectrum: $\mathcal{F}(\cos(2t) \cos(3t))(\omega)$ has harmonics at $\omega \in \{-5, -1, +1, +5\}$

(b) Spectrum of Simulated Signal $\cos(a_1 t) \cos(a_2 t) \cos(a_3 t)$ with $a_1 = 2\pi \cdot 20kHz$, $a_2 = 2\pi \cdot 30kHz$ and $a_3 = 2\pi \cdot 50kHz$. Theoretical Spectrum: $\mathcal{F}(\cos(2t) \cos(3t) \cos(5t))(\omega)$ has harmonics at $\omega \in \{-10, -6, -4, 0, +4, +6, +10\}$

Figure 3: Simulated vs. Computed Spectra
Towards compensating these, we ran a simulation using the circuit as shown in Figure 1, with subsequent Fourier analysis (.four Spice directive) on the input and output nodes of each amplifier. The results found experimentally for the NO-instance \((3, 6, 4)\) (corresponding to 30kHz, 60kHz and 40kHz), was \(\approx 4.22\text{mV}\) at the output of the first multiplier, and \(\approx 4.31\text{mV}\) at the output of the second multiplier. Since we have a NO-instance, both should be zero, and we can use the Z-input of the AD633 component to compensate this offset by connecting it to the respective negative potential. Figure 4a shows how this is done in our simulation. In doing so by adding voltage sources to the Z-pins of the two multipliers, we obtained significantly better results. Table 1 compares the results without and with offset correction.

We emphasize that the fine-tuning, i.e., offset compensation, of the multipliers should be done by supplying NO-instances, since in these cases, the zeroth harmonic should vanish, and whatever remains in the DC part should be compensated. Also, the offsets were measured slightly different between different NO-instances, which suggests that the best pick would be the “closest” NO-instance to the given problem \(P\). The distance between \(P\) and the approximate NO-instance can, for example, be measured in any norm on \(\mathbb{R}^n\), but a systematic search for it appears infeasible without knowing that \(P\) is actually a YES- or NO-instance (in the latter case, the sought proximum is \(P\) itself).

A feasible probabilistic approach is adding random frequency distortions to the input signals, i.e., instantiating the circuit with frequencies \((a_i + \varepsilon_i)_{i=1}^n\), where \(\varepsilon_i \sim \mathcal{N}(0, \sigma^2)\) are Gaussian error terms. Then, the random variable \(X_i := a_i + \varepsilon_i \sim \mathcal{N}(a_i, \sigma^2)\) is as well Gaussian. Now, let an arbitrary subset of these

\[ X = \left\{ X_i \right\} \]

Practically, the compensation should be done as suggested in the datasheet and displayed in Figure 4b.
Table 2: Effects of Offset-Correction ( Instances of size 4)

| Instance (kHz) | Answer | DC part without offset correction | DC part with offset correction |
|---------------|--------|----------------------------------|-------------------------------|
| (10,90,10,40) | NO     | 0.514673V                        | 0.0675873                     |
| (30,90,20,40) | YES    | 0.794704                          | 0.356715                      |

be given, then (4) fails with probability 1 (since the event of true equality has measure zero), and the likelihood to find a spectral line within a neighborhood $[-\Delta, +\Delta]$ of 0V (i.e., a DC-part within a given tolerance) is quantified by the distribution of the random variable with distribution $\mathcal{N}(0, n \cdot \sigma^2)$ to measures the event $|\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n| \leq \Delta$. In choosing $\sigma^2$ properly, we can make this likelihood as small as we desire, and get a probable NO-instance that we can use to calibrate our circuit towards eliminating offsets.

For our experimental validation, things are simpler, as modifying the given instances into NO-instances works efficiently. Let us give another example of size $n = 4$ to measure the offset voltages, compensate them, and then see what we get. To this end, we added a third multiplier-and-amplifier stage to the circuit (cf. Figure 1), and did a Fourier transform at the respective input nodes of each of the three multiplier, giving $\approx 4.5mV$ for the first, $\approx 4.76mV$ for the second, and $\approx 4.54mV$ for the third multiplier (note that this is indeed close to the typical offset as told by the data sheet for the AD633 component [1]). Table 2 shows the results, where a similar improvement as in the previous case can be noticed.

In neither experiment, we corrected any offsets induced by the non-inverting amplifiers, but the simulations revealed that the offset contributed by these blocks in the circuit are quite low and not of substantial magnitude (still, they should be corrected when the circuitry is to be scaled towards large instances). Also, some component tolerances (i.e., resistances that blur the intended amplification factor) go into (1) as multiplicative factors that can be pulled in front of the integral (2) and leave the condition unchanged.

The more important observation is the degradation of the DC-part for the YES-instances, as is already foretold by the spectrum (3) when we consider the exponentially decaying amplitude

$$\frac{\sqrt{\pi/2}}{2^n}$$

of all spectral components, including the DC part in particular. This decay is necessarily exponential in the problem size, since the overall energy in the signal gets scattered over an number of frequencies that is exponential in $n$ (cf. Parseval’s theorem).

To compensate this effect, we need to exponentially amplify the zeroth harmonic, while applying the same exponential damping to the remaining harmonics (otherwise, we would require an exponential lot of energy for the amplification). The resulting filter is thus an $n$-th order active low-pass. Such a circuit...
can be constructed by a chain of \( n \) first order active low-pass filters, each of which has an amplification of 2, to ultimately cancel the denominator \( 2^n \) appearing in (6). Whether or not this works practically, or whether there are other ways to make the DC part measurable is up to experimental verification on a real circuit, and as such beyond this report in its current form.

**Bootstrapping:** To reliably use condition 5 in practice, it is advisable to calibrate the analogue circuit with a set of training YES- and NO-instances, towards identifying the voltage ranges where the respective DC parts can be expected to be. Once this data is available, the instance under question can be put through the circuit. Note that this method is strikingly similar to what statisticians call *bootstrapping*; cf. [4]. It appears reasonable to use the same term to describe the calibration and “learning” how the DC parts between YES- and NO-instances are separated. The application of statistical tests is left unexplored here, but will be part of future investigations.

4 Conclusion

The concept works in simulations, but these already indicate/confirm the difficulties expected from aliasing, noise, imperfect amplitude and phase gains, etc. It remains to experimentally verify whether careful measurements and calibration of the circuit components can increase the magnitude difference of the DC parts in YES- and NO-instances. Furthermore, since the gap seems to shrink the more multipliers are in the chain, scalability of the system is the second crucial aspect to test under lab conditions on a physical circuit.

An independent (yet minor) theoretical caveat is the indication of the circuit being a purely existential assertion; that is, we get only the answer to the decision problem, but no witness of it. Finding the partition that satisfies equation (4) is an independent problem and can be solved indirectly by exploiting the computational equivalence of \textsc{Partition} and Boolean satisfiability \textsc{Sat}. The latter is perhaps the most important practical application of our proposed concept, so let an instance \( \psi \) of \textsc{Sat} with literals \( X_1, \ldots, X_n \) be given. Since \textsc{Sat} and \textsc{Partition} are both \textsc{NP}-complete, the given \textsc{Sat}-instance can be converted into a \textsc{Partition}-instance \( P \) of size \( m = \text{poly}(n) \) (in polynomial time in \( n \)). Since \( \psi \) is satisfiable if and only if (4) holds for \( P \), we can rule out the existence of a satisfying assignment for \( \psi \) if the DC-part coming out of our circuit is negligible. The opposite tells us \( \psi \) *is* satisfiable, but we are usually interested in a satisfying assignment too.

The latter is left untold by our analogue computer, but can be figured out by an easy procedure as follows: put \( X_1 \leftarrow 1 \) (true), substitute this value into \( \psi \) and call the resulting (typically simpler) formula \( \psi|_{X_1=1} \). Using our analogue multiplier circuit, we can decide whether or not \( \psi|_{X_1=1} \) is satisfiable by converting the formula into an instance of \textsc{Partition}. If it is, then we have found \( X_1 = 1 \) as the first variable in the satisfying assignment. Otherwise, we have \( X_1 = 0 \) (false) and \( \psi|_{X_1=0} \) must be satisfiable. Let \( x_1 \) be the so-far
correct value for $X_1$. Now, we can repeat the same steps by guessing $X_2 = 1$ and checking satisfiability of $\psi|_{X_1=x_1, X_2=1}$, and so on, until all variables have been determined.

In the $k$-th step for $k = 1, 2, \ldots, n$, the decision was done at the cost of one conversion of $\psi|_{X_1=x_1, \ldots, X_k=x_k}$ into a PARTITION-instance, and one use of our analogue multiplier chain. The overall effort is thus no more than polynomial in $n$, plus $n$ uses of our analogue computer. This would – in theory – deliver a solution to our SAT-instance $\psi$ in feasible time, given a constant computing time on the analogue circuit.

As a matter of independent research interest, observe that the analogue circuit description is constructible by a Turing machine in polynomial time (as this merely means chaining copies of a fixed multiplier and amplifier circuit, with changes only in the particular component properties like voltages or resistances; disregarding offset compensations here). Still, this is not a uniform circuit in the usual sense of complexity theory, and as such provides an new object to be fitted into the complexity-theoretic landscape perhaps.

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