DYNAMICS OF A FOOD CHAIN MODEL WITH RATIO-DEPENDENT AND MODIFIED LESLIE-GOWER FUNCTIONAL RESPONSES

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Abstract. The paper is concerned with a diffusive food chain model subject to homogeneous Robin boundary conditions, which models the trophic interactions of three levels. Using the fixed point index theory, we obtain the existence and uniqueness for coexistence states. Moreover, the existence of the global attractor and the extinction for the time-dependent model are established under certain assumptions. Some numerical simulations are done to complement the analytical results.

1. Introduction. Understanding the spatial and temporal behaviors of interaction species has become a central issue in population ecology, where one aspect of great interest for a multi-species model is whether the involved species can persist or have a coexistence steady state. At the beginning of 20th century, it is indicated by a constant positive solution of a ordinary differential equation system in the case where the species are homogeneously distributed; in recent years, stationary pattern induced by diffusion has been studied extensively, and many important phenomena have been observed.

In population dynamics, the traditional Lotka-Voltera type predator-prey systems are very important in the models of multi-species population dynamics. Yet the traditional prey-dependent predator-prey models have been challenged by several biological and physiological evidence that in many situations, especially, predators have to search, share or compete for food. A more suitable general predator-prey model appeared based on the so-called ratio-dependent theory, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance, and so there appears the so-called

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ratio-dependent function response. These hypotheses are strongly supported by numerous fields and laboratory experiments and observations. A general form of a ratio-dependent model is:

\[
\begin{align*}
\dot{x} &= rx(1 - \frac{x}{K}) - \phi(\frac{x}{y})y, \\
\dot{y} &= y(\mu\phi(\frac{x}{y}) - d).
\end{align*}
\]

There are many authors who have studied the above system based on various response functions. In [2], the permanence and global stability of positive equilibria were studied for a delayed ratio-dependent predator-prey system; in [16, 17], the long time behavior and stability of equilibria and stationary pattern formation were investigated to a diffusive one-prey and two-竞争-predator system; in [19], the authors showed the global asymptotic stability of possible three steady states and the nonexistence of nontrivial positive periodic solutions. More details, see [13, 6, 10] and references therein.

More realistic models are derived in view of laboratory experiments and observations, the “carrying capacity” of the predator’s environment is proportional to the population size of the prey, Leslie [20, 21] introduced the following predator-prey model (also see [22, 18, 4, 5]):

\[
\begin{align*}
\dot{x} &= r_1 x(1 - \frac{x}{K}) - \phi(\frac{x}{y})y, \\
\dot{y} &= y(\mu_2\phi(\frac{x}{y}) - d_2).
\end{align*}
\]

Furthermore, many kinds of modified Leslie-Gower models have been discussed, such as [28, 14, 12, 15, 11, 1] and references therein. For example, Ji and Jiang proved the existence of the stationary distribution for a predator-prey model with modified Leslie-Gower and Holling-type II response functions in [14]: Aziz-Alaoui and Daher Okiye proved the boundedness of solutions and global asymptotic stability of the interior equilibrium for a predator-prey model with modified Leslie-Gower and Holling-type II response function in [1].

Recently, Upadhyay et al [29] have proposed the following model:

\[
\begin{align*}
\frac{dX}{dt} &= a_1 X(1 - \frac{X}{K}) - \frac{\omega_1 X Y}{X + D_1}, \\
\frac{dY}{dt} &= -\omega_1 X Y - \frac{\omega_2 Y Z}{Y + D_2} + \frac{\omega_1 X Y}{X + D_1}, \\
\frac{dZ}{dt} &= c Z^2 - \frac{\omega_3 Z^2}{Z + D_3}.
\end{align*}
\]

For the detailed background of the above model, one can see [29], where the boundedness of the system, existence of an attracting set, local and global stability of non-negative equilibrium points are established there. Particularly, we point out that the square term \(cZ^2\) signifies the fact that mating frequency is directly proportional to the number of males as well as that of females present at any instant of time. 

In this paper, taking account of the ratio-dependent theory, we consider the following diffusive food chain system:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= a_1 u(1 - \frac{u}{K}) - \frac{\alpha_{1uv} u v}{u + D_v}, \\
\frac{\partial v}{\partial t} - \Delta v &= -a_2 v + \frac{\alpha_{1uv} u v}{u + D_v} - \frac{\beta_1 v w}{v + D_w}, \\
\frac{\partial w}{\partial t} - \Delta w &= c w - \frac{\beta_2 w^2}{v + D_3},
\end{align*}
\quad t > 0, x \in \Omega, (1.1)
\]

subject to the boundary conditions

\[
k_1 \frac{\partial u}{\partial n} + u = k_2 \frac{\partial v}{\partial n} + v = k_3 \frac{\partial w}{\partial n} + w = 0, \quad t > 0, x \in \partial \Omega (1.2)
\]
and the initial conditions
\[ u = u_0 \geq 0(\neq 0), \quad v = v_0 \geq 0(\neq 0), \quad w = w_0 \geq 0(\neq 0), \quad t = 0, x \in \Omega. \] (1.3)

In this model, \( \Omega \) denotes a bounded domain in \( \mathbb{R}^N \) \( (N \geq 1) \) with smooth boundary \( \partial \Omega \), \( n \) is the outward unit normal over \( \partial \Omega \), \( k_i > 0 \) \( (i = 1, 2, 3) \) are constants and for simplicity, we may take \( k_i = 1 \) \( (i = 1, 2, 3) \) throughout the whole paper. A prey population of size \( u \) is the only food for the predator population of size \( v \), which, in turn, is the food for the predator population of size \( w \). \( a_1 \) is the intrinsic growth rate of the prey population \( u \); \( a_2 \) is the intrinsic death rate of the predator \( v \) in the absence of the only food \( u \); \( c \) measures the rate of self-reproduction of predator \( w \). \( k \) is the carrying capacity, \( D \) and \( D_1 \) quantify the extent to which environment provides protection to the prey \( u \) and can be regarded as a measure of the effectiveness of the prey in evading a predator’s attack. \( D_2 \) is the value at which per capita removal rate of \( v \) becomes \( \beta_1/2, \alpha_1, \alpha_2, \beta_1, \beta_2 \) are the maximum values which per capita growth rate can attain. And, we always assume \( \alpha_2 > \alpha_1 > 0 \) throughout the whole paper, so that the predator population of size \( v \) can persist. The interaction between the prey and the predator \( v \) is the ratio-dependent Holling type II functional response \( q(u,v) = \frac{uv}{u+D_1v} \). The interaction between the predator \( v \) and the predator \( w \) is the Beddington-DeAngelis type functional response \( q(v,w) = \frac{vw}{D_2w+vw} \) and the modified Leslie-Gower type functional response \( q(v,w) = \frac{w^2}{v+D_3w} \), where \( D_3 \) is added in the denominator to normalize the residual reduction in predator \( w \) due to severe scarcity of the predator \( v \).

The corresponding steady state system with (1.1)-(1.3) is as follows:
\[
\begin{align*}
-\Delta u &= a_1u(1 - \frac{u}{k}) - \frac{a_1uv}{u+D_1v}, \quad x \in \Omega, \\
-\Delta v &= -a_2v + \frac{\alpha_2uv}{u+D_1v} - \frac{\beta_1vw}{D_2w+vw}, \quad x \in \Omega, \\
-\Delta w &= cw - \frac{\beta_2w^2}{v+D_3w}, \quad x \in \Omega,
\end{align*}
\] (1.4)

subject to the boundary conditions
\[
\frac{\partial u}{\partial n} + u = \frac{\partial v}{\partial n} + v = \frac{\partial w}{\partial n} + w = 0, \quad x \in \partial \Omega. \tag{1.5}
\]

Since \((0,0,0)\) is the singular point of (1.4)-(1.5), we first extend the domain of the response functions \( f(u,v,w) \) and \( g(u,v,w) \) to \( \{(u,v,w) : u \geq 0, v \geq 0, w \geq 0\} \), such that \((0,0,0)\) becomes a trivial solution of the revised form of (1.4)-(1.5). But the extended response functions are not differentiable at \((u,v,w) = (0,0,0)\) yet, which makes the calculation of the index \( P(\mathcal{A},(0,0,0)) \) more complicated as shown in the proof of Lemma 3.4.

The present paper is built up as follows. In section 2, we give some known results about the eigenvalue problem and the fixed point index on positive cone; in section 3, we get the sufficient and necessary condition for the existence of positive steady states by using monotone method and the fixed point index theory on a positive cone, furthermore, the uniqueness is shown. The main result in this section is given in Theorems 3.2, 3.12 and 3.14; section 4 is devoted to consider global attractor and the extinction for the model; we illustrate our results with numerical simulations in section 5.

2. Preliminaries. In this section, we shall give some fundamental lemmas, which will be used in the subsequent paper.
Lemma 2.1 ([26, 32, 33]). Let \( q(x) \in C^\alpha(\overline{\Omega}) \) \((0 < \alpha < 1)\), \( \lambda_1(q) \) be the principal eigenvalue of the following eigenvalue problem
\[
\begin{align*}
-\Delta \varphi + q(x) \varphi &= \lambda \varphi, & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial n} + \varphi &= 0, & \text{on } \partial \Omega.
\end{align*}
\]
Then \( \lambda_1(q) \) depends continuously on \( q \), \( \lambda_1(q) \) is a simple eigenvalue and the corresponding eigenfunctions do not change sign on \( \Omega \). Furthermore, \( q_1 \leq q_2, q_1 \neq q_2 \) imply \( \lambda_1(q_1) < \lambda_1(q_2) \). We denote \( \lambda_1(0) \) by \( \lambda_1 \) for simplicity and the eigenfunction corresponding to \( \lambda_1 \) is denoted by \( \varphi_1 \) and \( \max_{x \in \varphi_1} = 1 \).

Consider the following equation
\[
\begin{align*}
-u_t - \Delta u &= uf(x, u), & \text{in } \Omega \times \mathbb{R}^+,
\end{align*}
\]
and its steady state equation, the elliptic equation
\[
\begin{align*}
-\Delta u &= uf(x, u), & \text{in } \Omega,
\end{align*}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). Assume that function \( f(x, u) : \Omega \times [0, \infty) \rightarrow \mathbb{R} \) satisfies the following hypotheses:

(H1) \( f(\cdot, u) \in C^\alpha(\overline{\Omega}) \) \((0 < \alpha < 1)\) for all \( u \in [0, \infty) \);

(H2) \( f(x, \cdot) \in C^1([0, \infty)) \) for all \( x \in \overline{\Omega} \), \( f_u(x, u) < 0 \) for all \( (x, u) \in \Omega \times (0, \infty) \);

(H3) \( f(x, u) \leq 0 \) for all \( (x, u) \in \overline{\Omega} \times [C, \infty) \) for some positive constant \( C \).

Lemma 2.2 ([33, 3]). Suppose \( f(x, u) \) satisfies conditions (H1), (H2) and (H3). Then

(i) The nonnegative solution \( u(x) \) of (2.2) satisfies \( u(x) \leq C \) for all \( x \in \overline{\Omega} \);

(ii) If \( \lambda_1(-f(x, 0)) \geq 0 \), then (2.2) has no positive solutions. Moreover, the trivial solution \( u(x) = 0 \) is globally asymptotically stable;

(iii) If \( \lambda_1(-f(x, 0)) < 0 \), then (2.2) has a unique positive solution which is globally asymptotically stable. In this case, the trivial solution \( u(x) = 0 \) is unstable.

Let \( E \) be a real Banach space and \( P \subset E \) be the natural positive cone of \( E \). For \( y \in P \), define \( P_y = \{ x \in E : y + rx \in P \text{ for some } r > 0 \} \) and \( S_y = \{ x \in \overline{P_y} : -x \in \overline{P_y} \} \). Then \( \overline{P_y} \) is a wedge containing \( P_y \) and \( -y \), while \( S_y \) is a closed subset of \( E \) containing \( y \). Let \( T \) be a compact linear operator on \( E \), which satisfies \( T(\overline{P_y}) \subset \overline{P_y} \). We say that \( T \) has property \( \alpha \) on \( P_y \) if there is a \( t \in (0, 1) \) and an element \( y_0 \in \overline{P_y} \) such that \( (I - tT)y_0 \in S_y \). Assume \( \mathcal{A} : P \rightarrow P \) is a compact operator with a fixed point \( y \in P \). Let \( \mathcal{L} = \mathcal{A}'(y) \) be the Fréchet derivative of \( \mathcal{A} \) at \( y \). Then \( \mathcal{L} \) maps \( \overline{P_y} \) into itself. Dancer’s theorem can be stated as follows.

Lemma 2.3 ([23, 31]). Assume that \( I - \mathcal{L} \) is invertible on \( \overline{P_y} \).

(i) If \( \mathcal{L} \) has property \( \alpha \) on \( P_y \), then \( \text{index}_P(\mathcal{A}, y) = 0 \);

(ii) If \( \mathcal{L} \) does not have property \( \alpha \) on \( P_y \), then \( \text{index}_P(\mathcal{A}, y) = (-1)^\delta \), where \( \delta \) is the sum of multiplicities of all the eigenvalue of \( \mathcal{L} \) which are greater than 1.

Lemma 2.4 ([32, 23]). Assume \( h(x) \in C^\alpha(\overline{\Omega}) \) \((0 < \alpha < 1)\) and \( M \) is a sufficiently large number such that \( M > h(x) \) for all \( x \in \overline{\Omega} \). Define a positive and compact operator \( \mathcal{L} := (-\Delta + M)^{-1}(M - h(x)) : C^1_B(\overline{\Omega}) \rightarrow C^1_B(\overline{\Omega}) = \{ u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} + u = 0 \text{ on } \partial \Omega \} \). Denote the spectral radius of \( \mathcal{L} \) by \( r(\mathcal{L}) \). Then

(i) \( \lambda_1(h) > 0 \) if and only if \( r(\mathcal{L}) < 1 \);

(ii) \( \lambda_1(h) < 0 \) if and only if \( r(\mathcal{L}) > 1 \);
(iii) \( \lambda_1(h) = 0 \) if and only if \( r(\mathcal{L}) = 1 \).

Suppose \( E_1 \) and \( E_2 \) are ordered Banach spaces with positive cones \( C_1 \) and \( C_2 \), respectively. Let \( E = E_1 \times E_2 \) and \( C = C_1 \times C_2 \). Then clearly \( E \) is an ordered Banach space with positive cone \( C \). Let \( \Gamma \) be an open set in \( C \) containing 0 and \( \mathcal{A}_i : \Gamma \to C_i \) be completely continuous operators, \( i = 1, 2 \). Denote by \((u, v)\) a general element in \( C \) with \( u \in C_1 \) and \( v \in C_2 \). Let \( \mathcal{A} : \Gamma \to C \) be defined by

\[
\mathcal{A}(u, v) = (\mathcal{A}_1(u, v), \mathcal{A}_2(u, v)).
\]

We also define

\[
C_2(\delta) = \{ v \in C_2 : \|v\|_{E_2} < \delta \}.
\]

**Lemma 2.5** ([7, 8]). Suppose \( U \subset C_1 \cap \Gamma \) is relatively open and bounded, and

\[
\mathcal{A}_1(u, 0) \neq u, \quad \text{for} \; u \in \partial U, \quad \mathcal{A}_2(u, 0) \equiv 0, \quad \text{for} \; u \in \overline{U}.
\]

Suppose \( \mathcal{A}_2 : \Gamma \to C_2 \) extends to a continuously differentiable mapping of a neighborhood of \( \Gamma \) into \( E_2 \), \( C_2 - C_2 \) is dense in \( E_2 \) and \( T = \{ u \in U : u = \mathcal{A}_1(u, 0) \} \). Then the following conclusions are true:

(i) \( \deg_C(I - \mathcal{A}, U \times C_2(\delta), 0) = 0 \) for \( \delta > 0 \) small if for any \( u \in T \), the spectral radius \( r(\mathcal{A}_2'(u, 0)|_{C_2}) > 1 \) and 1 is not an eigenvalue of \( \mathcal{A}_2'(u, 0)|_{C_2} \) corresponding to a positive eigenvector;

(ii) \( \deg_C(I - \mathcal{A}, U \times C_2(\delta), 0) = \deg_C(I - \mathcal{A}_1|_{C_1}, U, 0) \) for \( \delta > 0 \) small if for any \( u \in T \), \( r(\mathcal{A}_2'(u, 0)|_{C_2}) < 1 \).

### 3. Existence and uniqueness.

Thanks to Lemma 2.2, we give the following notations. Suppose \( a, m, d \) are positive constants, \( A(x) \in C^\alpha(\overline{\Omega}) \) (0 < \( \alpha < 1 \)) and \( A(x) > 0 \) on \( \Omega \). If \( \alpha > \lambda_1 \), we denote by \( \theta[a, A] \) the unique positive solution of (3.1). If \( m - d > \lambda_1 \), we denote by \( \zeta[m, d, A] \) the unique positive solution of (3.2).

\[
\begin{cases}
-\Delta \chi = \chi[a - A(x)\chi], & \text{in } \Omega, \\
\frac{\partial \chi}{\partial n} + \chi = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and

\[
\begin{cases}
-\Delta \chi = \chi[m \frac{A(x)}{\chi} - d], & \text{in } \Omega, \\
\frac{\partial \chi}{\partial n} + \chi = 0, & \text{on } \partial \Omega.
\end{cases}
\]

**Theorem 3.1.** Assume \( \alpha_2 > a_2 \) holds. Then any nonnegative solution \((u, v, w)\) of (1.4)-(1.5) has a priori bounds:

\[
u \leq M_1, v \leq M_2, w \leq M_3,
\]

where \( M_1 = k, M_2 = \frac{k(a_2 - a_2)}{a_2 D_1}, M_3 = \frac{ck(a_2 - a_2) + a_2 D_1 D_2}{a_2 D_1} \).

The proof is standard. Since \( u \) satisfies

\[
\begin{cases}
-\Delta u \leq a_1 u(1 - u/k), & \text{in } \Omega, \\
\frac{\partial u}{\partial n} + u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

it follows from the maximum principle and the Hopf lemma that \( u \leq k = M_1 \). The other two inequalities can be proved similarly.

Now we give the sufficient and necessary conditions for the existence of positive solution to system (1.4)-(1.5) by means of the upper-lower solution method.
Theorem 3.2. Assume that \( a_1 - \frac{a_2}{D} > \lambda_1, \alpha_2 - \frac{a_2}{k} > \lambda_1 \) and \( c > \lambda_1 \) hold.

(i) If (1.4)-(1.5) has a positive solution \((u, v, w)\), then
\[
(u_*, v_*, w_*) \leq (u, v, w) \leq (u^*, v^*, w^*),
\]
where \( u^* = \theta[a_1, \frac{a_1}{D}], u_* = \theta[a_1 - \frac{a_1}{D}, \frac{a_1}{D}], v^* = \zeta[a_2, a_2 + \frac{a_2}{D}], v_* = \zeta[a_2, a_2 + \frac{a_2}{D}], w^* = \theta[c, v_*/D], \) and \( w_* = \theta[c, v_*/D] \).

(ii) (1.4)-(1.5) has at least one positive solution satisfying \( U_* \leq U \leq U^* \), where \( U = (u, v, w), U^* = (u^*, v^*, w^*) \) and \( U_* = (u_*, v_*, w_*) \).

Proof. (i) If (1.4)-(1.5) has a positive solution \((u, v, w)\), then
\[
\Delta u + u[(a_1 - \frac{a_1}{D}) - \frac{a_1}{D}] - \frac{a_1}{k} \leq \Delta u + a_1 u(1 - u/k) - \frac{a_1}{D} \leq \Delta u + a_1 u(1 - u/k).
\]
By Lemma 2.2 and the upper-lower solution method for elliptic equation, we conclude that \( u_* = \theta[a_1 - \frac{a_1}{D}, \frac{a_1}{D}] \leq u \leq \theta[a_1, \frac{a_1}{D}] = u^* \) due to the assumption \( a_1 - \frac{a_2}{k} > \lambda_1 \).

Similarly, we have
\[
\Delta v + v[\frac{\alpha_2 u_*}{u_* + D_1 v} - (a_2 + \frac{\beta_1}{b})] \leq \Delta v - a_2 v + \frac{\alpha_2 uv}{u + D_1 v} - \frac{\beta_1 uv}{D_2 + v + bw} \leq \Delta v + v[(\frac{\alpha_2 u^*}{u^* + D_1 v} - a_2),
\]
\[
\Delta w + w(c - \frac{\beta_2 w}{v + D_3}) \leq \Delta w + cw - \frac{\beta_2 w^2}{v + D_3} \leq \Delta w + w(c - \frac{\beta_2 w}{v^* + D_3}).
\]
The existence of \( v^*, v_*, w^*, w_* \) is obvious. Hence, we have
\[
v_* = \zeta[a_2, a_2 + \frac{\beta_1}{D}], \leq v \leq \zeta[a_2, a_2 + \frac{u^*}{D}] = v^*,
\]
\[
w_* = \theta[c, v_*/D] \leq w \leq \theta[c, v_*/D] = w^*.
\]

(ii) Set \( f(u, v, w) = a_1 u(1 - u/k) - \frac{a_1}{D}, g(u, v, w) = -a_2 v + \frac{\alpha_2 uv}{u + D_1 v} - \frac{\beta_1 uv}{D_2 + v + bw}, \)
\( h(u, v, w) = cw - \frac{\beta_2 w}{v + D_3} \). Then \((f, g, h)\) has a mixed quasimonotone form. It is easy to check that \((u^*, v^*, w^*)\), \((u_*, v_*, w_*)\) are the couple upper and lower solutions for (1.4)-(1.5), i.e., \( u^* \geq u_*, v^* \geq v_*, w^* \geq w_* \) on \( \Omega \), and they satisfy the following inequalities:
\[
\Delta u^* + a_1 u^*(1 - u^*/k) - \frac{\alpha_1 u^*}{D_1} \leq 0, \quad \Delta u_* + a_1 u_*(1 - u_*/k) - \frac{\alpha_1 u_*}{D_1} \leq 0, \quad \Delta v^* + a_2 v^* - \frac{\alpha_2 u^* v^*}{u^* + D_1 v} \leq 0, \quad \Delta v_* - a_2 v_* + \frac{\alpha_2 u_* v_*}{u_* + D_1 v} \leq 0, \quad \Delta w^* + cw^* - \frac{\beta_2 w^*}{v^*/D} \leq 0 \leq \Delta w_* + cw_* - \frac{\beta_2 w_*}{v_*/D} \leq 0.
\]
\[
\frac{\partial u^*}{\partial n} + u^* \geq 0 \geq \frac{\partial u_*}{\partial n} + u_*, \quad \frac{\partial v^*}{\partial n} + v^* \geq 0 \geq \frac{\partial v_*}{\partial n} + v_*, \quad \frac{\partial w^*}{\partial n} + w^* \geq 0 \geq \frac{\partial w_*}{\partial n} + w_*, \quad \text{on } \partial \Omega.
\]
Evidently, \((f, g, h)\) satisfies the Lipschitz condition on \([u_*, u^*] \times [v_*, v^*] \times [w_*, w^*] \).
Therefore, the existence of positive solution follows from the existence-comparison argument [24] for the elliptic system with a mixed quasimonotone form. This finishes the proof of the theorem. \( \square \)

**Theorem 3.3.** If (1.4)-(1.5) has a positive solution, then \( a_1 > \lambda_1, a_2 - a_2 > \lambda_1 \) and \( c > \lambda_1 \).
The proof of Theorem 3.3 is fundamental, we omit it here.

In the rest of the section, we give the existence and uniqueness of positive solution by using the fixed point index theory on a positive cone. It is obvious that 

$$(0, 0, 0)$$

is the singular point of (1.4)-(1.5), we follow the idea of Kuang and Beretta in [19] and assume $f(0, 0, \cdot) = 0$ and $g(0, 0, 0) = 0$, since the limit of $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = 0$ at the trivial solution $(0, 0, 0)$ to (1.4)-(1.5). Then we extend the domain of $f(u, v, w)$ and $g(u, v, w)$ to $(u, v, w)$ such that $(0, 0, 0)$ becomes a trivial solution of (1.4)-(1.5), where $f(u, v, w)$ and $g(u, v, w)$ are redefined just before Lemma 3.4. In addition, (1.4)-(1.5) may have the following forms of solutions:

(i) Nonnegative solutions with exactly two components identically zero

$$(u_1, 0, 0), (v_1, 0, 0), (w_1, 0, 0);$$

(ii) Nonnegative solutions with exactly one component identically zero

$$(u_2, v_2, 0), (u_3, 0, w_2), (v_3, v_3);$$

(iii) Nonnegative solutions with no component identically zero $(u_4, v_4, w_4)$. If $a_1, c > \lambda_1$, then $(\theta(a_1, \frac{c}{M}), 0, 0)$ and $(0, 0, \theta(c, \frac{c}{M}))$ are the only solutions of form (i).

Set $E = C_B[0] \times C_B[0] \times C_B[0]$, where $C_B[0] = \{ \varphi \in C^1[0] : \frac{\partial \varphi}{\partial n} + \varphi = 0 \text{ on } \partial \Omega \}$. $P = K \times K \times K$, a natural positive cone of $E$, where $K = \{ \varphi \in C_B[0] : \varphi \geq 0 \text{ on } \partial \Omega \}$. Define $\Xi = \{(u, v, w) \in P : u < M_1 + 1, v < M_2 + 1, w < M_3 + 1 \}$, where $M_i(i = 1, 2, 3)$ are defined in Theorem 3.1.

It is easy to see that all the nonnegative solutions of (1.4)-(1.5) are in $\Xi$ by Theorem 3.1. Choosing a positive constant $M$ sufficiently large with $M > \max\{a_1 + \frac{\beta_1 v w}{2 \beta_2}, a_2 + \frac{\beta_1 v w}{2 \beta_2}, \frac{\beta_1 v w}{2 \beta_2} \}$, we know that $a_1 u/(1 - u/k) - \frac{\alpha_1 v}{u + D u} + M u, -a_2 v + \frac{\alpha_3 w}{u + D v} - \frac{\beta_1 v w}{D_2 + v + b w} + M v$ and $c w - \frac{\beta_2 v w}{D_2 + v + b w} + M w$ are monotone increasing with respect to $u, v, w$ for all $(u, v, w) \in [0, M_1] \times [0, M_2] \times [0, M_3]$. We define functions $\tilde{f}, \tilde{g}, \tilde{h}$ in $(u, v, w) : u \geq 0, v \geq 0, w \geq 0$ by

$$\tilde{f}(u, v, w) = \begin{cases} u(a_1 - \frac{\alpha_1 u}{k}) - \frac{\alpha_1 v}{u + D u}, & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}$$

$$\tilde{g}(u, v, w) = \begin{cases} v(-a_2 + \frac{\alpha_3 w}{u + D v} - \frac{\beta_1 v w}{D_2 + v + b w}), & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}$$

$$\tilde{h}(u, v, w) = h(u, v, w),$$

and also an operator $A$ by

$$A(u, v, w) = (\Delta + M)^{-1} \begin{pmatrix} \tilde{f}(u, v, w) + M u \\ \tilde{g}(u, v, w) + M v \\ \tilde{h}(u, v, w) + M w \end{pmatrix}.$$ 

By the definition of $M$, $A$ is a positive operator, which maps $\Xi$ to $P$. It follows from the standard elliptic regularity theory that $A$ is a completely continuous operator. (1.4)-(1.5) has a positive solution if and only if $A$ has a positive fixed point in interior of $\Xi$.

Now we give the degree of $I - A$ in $\Xi$ relative to $P$ and the fixed point index of $A$ at the trivial solution $(0, 0, 0)$ of (1.4)-(1.5) relative to $P$.

**Lemma 3.4.** Assume $a_1 - \frac{\alpha_1}{\beta_2} > \lambda_1$, $a_2 > a_2$ and $c \neq \lambda_1$. Then

(i) $\text{index}_P(A, \Xi) = 1$, (ii) $\text{index}_P(A, (0, 0, 0)) = 0$. 

Proof. (i) Define an operator $A_\theta : E \to E$ by
\[
A_\theta(u, v, w) = (-\Delta + M)^{-1} \left( \begin{array}{c} \frac{\partial f(u, v, w) + Mu}{\partial u} \\ \frac{\partial f(u, v, w) + Mu}{\partial v} \\ \frac{\partial h(u, v, w) + Mw}{\partial w} \end{array} \right),
\]
where $\theta \in [0, 1]$, $A_\theta$ is also a positive, completely continuous operator. If $(u, v, w)$ is a nonnegative fixed point of $A_\theta$, then one can show that $(u, v, w)$ must satisfy (3.3). Hence, $A_\theta$ has a unique fixed point $(0, 0)$ in $\Xi$. We see that $A_0$ has a unique fixed point $(0, 0, 0)$ in $\Xi$ and $\text{index}_P(A_0, (0, 0, 0)) = 1$ by Lemma 2.3. Hence,
\[
\text{index}_P(A, \Xi) = \text{index}_P(A_1, \Xi) = \text{index}_P(A_0, \Xi) = 1.
\]
(ii) Since $A(u, v, w)$ is not Fréchet differentiable at $(0, 0, 0)$, we cannot directly calculate $\text{index}_P(A, (0, 0, 0))$. In order to calculate $\text{index}_P(A, (0, 0, 0))$, we redefine functions $f, g, h$ in $(\epsilon, u, v, w) : \epsilon \geq 0, u \geq 0, v \geq 0, w \geq 0$ by
\[
\tilde{f}(\epsilon, u, v, w) = \begin{cases} u(a_1 - \frac{a_1 u}{u + v + w}) - \frac{a_1 v}{u + v + w}, & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}
\]
\[
\tilde{g}(\epsilon, u, v, w) = \begin{cases} v(-a_2 + \frac{a_2 u}{u + v + w} - \frac{a_2 w}{u + v + w}), & (u, v) \neq (0, 0), \\ 0, & (u, v) = (0, 0), \end{cases}
\]
and also an operator $A_\epsilon$ by
\[
A_\epsilon(u, v, w) = (-\Delta + M)^{-1} \left( \begin{array}{c} \tilde{f}(\epsilon, u, v, w) + Mu \\ \tilde{g}(\epsilon, u, v, w) + Mv \\ \tilde{h}(\epsilon, u, v, w) + Mw \end{array} \right).
\]
Since the second part of the proof is long, we divide it into two propositions. □

**Proposition 1.** If $a_1 > \lambda_1$ and $c \neq \lambda_1$ or $c > \lambda_1$ and $a_1 \neq \lambda_1$, then for all $\epsilon > 0$, $\text{index}_P(A_\epsilon, (0, 0, 0)) = 0$.

Proof. Observe that $\overline{P}_{(0,0,0)} = K \times K \times K$, $S_{(0,0,0)} = \{0\} \times \{0\} \times \{0\}$ and
\[
\mathcal{L} = A_\epsilon'(0, 0, 0) = (-\Delta + M)^{-1} \left( \begin{array}{ccc} a_1 + M & 0 & 0 \\ 0 & -a_2 + M & 0 \\ 0 & 0 & c + M \end{array} \right).
\]
Assume that $\mathcal{L}(\varphi, \psi, \eta) = (\varphi, \psi, \eta)$, where $(\varphi, \psi, \eta) \in \overline{P}_{(0,0,0)}$. Then $\varphi$, $\psi$ and $\eta$ respectively satisfy
\[
\left\{ \begin{array}{l} -\Delta \varphi = a_1 \varphi, \quad \text{in} \ \Omega, \\ \frac{\partial \varphi}{\partial n} + \varphi = 0, \quad \text{on} \ \partial \Omega \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} -\Delta \psi = -a_2 \psi, \quad \text{in} \ \Omega, \\ \frac{\partial \psi}{\partial n} + \psi = 0, \quad \text{on} \ \partial \Omega \end{array} \right.
\]
and
\[
\left\{ \begin{array}{l} -\Delta \eta = c \eta, \quad \text{in} \ \Omega, \\ \frac{\partial \eta}{\partial n} + \eta = 0, \quad \text{on} \ \partial \Omega \end{array} \right.
\]
By Lemma 2.1, we have $\psi = 0$. If $\varphi \in K, \varphi \neq 0$, then $a_1 = \lambda_1$, a contradiction, so $\varphi = 0$. Similarly, $\eta = 0$. We prove that $I - \mathcal{L}$ is invertible on $\overline{P}_{(0,0,0)}$.

Since $a_1 > \lambda_1$, $\lambda_1 (-a_1) = \lambda_1 - a_1 < 0$, we know $r[(\Delta + M)^{-1}(a_1 + M)] > 1$ by Lemma 2.4. Then by Krein-Rutman theorem, $r[(\Delta + M)^{-1}(a_1 + M)]$ is the principal eigenvalue of $(\Delta + M)^{-1}(a_1 + M)$ with a corresponding eigenfunction $\varphi \in K \setminus \{0\}$. Let $t_0 = \frac{1}{r[(\Delta + M)^{-1}(a_1 + M)]}$. It is obvious that $(I - t_0 \mathcal{L})(\varphi, 0, 0) = (0, 0, 0) \in S_{(0,0,0)}$, i.e., $\mathcal{L}$ has property $a$. Consequently, $\text{index}_P(A_\epsilon, (0, 0, 0)) = 0$ by Lemma 2.3.

Similarly, if $c > \lambda_1$ and $a_1 \neq \lambda_1$, then $\text{index}_P(A_\epsilon, (0, 0, 0)) = 0$ for all $\epsilon > 0$. □
Proposition 2. If $a_1 - \frac{\alpha_1}{D} > \lambda_1$ and $c \neq \lambda_1$, then $\text{index}_P(A, (0, 0, 0)) = 0$.

Proof. From Proposition 1, we know $(0, 0, 0)$ is an isolated fixed point of $A_e$ in $P$ for all $\epsilon > 0$ if $a_1 - \frac{\alpha_1}{D} > \lambda_1$.

We claim that $(0, 0, 0)$ is an isolated fixed point of $A$ in $P$ if $a_1 - \frac{\alpha_1}{D} > \lambda_1$. Suppose that the statement was not true. Then there nonnegative nontrivial solution $U_n = (u_n, v_n, w_n)$ such that $U_n \to (0, 0, 0)$ as $n \to \infty$ and $(I - A)U_n = 0$. Then $U_n$ must be one of the following three cases for large $n$: $U_n$ with $w_n \neq 0$; $U_n$ with $u_n \neq 0$; $U_n$ with $v_n \neq 0$.

For the case $U_n$ with $w_n \neq 0$, by the maximum principle, $w_n > 0$. Let $\overline{w}_n = w_n/\|w_n\|_{\infty}$. Then $\overline{w}_n$ satisfies

$$-\Delta \overline{w}_n = c\overline{w}_n - \frac{\beta_2 w_n}{v_n + D_3} \overline{w}_n.$$ 

Passing to a subsequence if necessary, by $L^p$ estimates and Sobolev embedding theorems, there exists $\overline{w}$, such that $\overline{w}_n \to \overline{w}$ in $C^1$ as $n \to \infty$. Taking the limit of the equation of $\overline{w}_n$, we obtain

$$-\Delta \overline{w} = c\overline{w},$$

then $c = \lambda_1$, a contradiction. Hence, $w_n \equiv 0$ and then $u_n, v_n$ satisfy

$$\left\{\begin{array}{l}
-\Delta u_n = a_1 u_n (1 - \frac{u_n}{k}) - \frac{\alpha_1 u_n v_n}{u_n + D_3 v_n}, \\
-\Delta v_n = -a_2 v_n + \frac{\alpha_2 u_n}{u_n + D_3 v_n}.
\end{array}\right.$$ 

It is obvious that $v_n \equiv 0$ if $u_n \equiv 0$; $u_n = \theta[a_1 - \frac{\alpha_1}{D}, \frac{\alpha_1}{D}]$ if $v_n \equiv 0$. So, we only need to consider the case $U_n = (u_n, v_n, 0)$ with $u_n \geq 0$, $v_n \geq 0$. Due to the maximum principle, it is sufficient to consider the case $U_n = (u_n, v_n, 0)$ with $u_n > 0$, $v_n > 0$. From Theorem 3.2, $u_n \geq \theta[a_1 - \frac{\alpha_1}{D}, \frac{\alpha_1}{D}]$ for $a_1 - \frac{\alpha_1}{D} > \lambda_1$, where $\theta[a_1 - \frac{\alpha_1}{D}, \frac{\alpha_1}{D}]$ is defined in Theorem 3.2. So, $u_n \to 0$ can not hold forever, a contradiction with $U_n \to (0, 0, 0)$ as $n \to \infty$. The claim has been proved.

We claim that for all $\epsilon > 0$, there exists $\delta_0 > 0$ such that for all $U = (u, v, w) \in \overline{B}((0, 0, 0), \delta_0) \cap P \setminus \{(0, 0, 0)\}$, $(I - A)U \neq 0$. If the statement was not true, then there exists $\delta_0 > 0$ and $U_n = (u_n, v_n, w_n) \in \overline{B}((0, 0, 0), 1/n) \cap P \setminus \{(0, 0, 0)\}$, $(I - A)U_n = 0$. It is obvious that $U_n = (u_n, v_n, w_n) \to (0, 0, 0)$ as $n \to \infty$. Similarly to the above discussion, passing to a subsequence if necessary, $w_n \equiv 0$ and $u_n, v_n$ satisfy

$$\left\{\begin{array}{l}
-\Delta u_n = a_1 u_n (1 - \frac{u_n}{k}) - \frac{\alpha_1 u_n v_n}{c_0 + u_n + D_3 v_n}, \\
-\Delta v_n = -a_2 v_n + \frac{\alpha_2 u_n}{c_0 + u_n + D_3 v_n}.
\end{array}\right.$$ 

And we need only consider the case $U_n = (u_n, v_n, 0)$ with $u_n > 0, v_n > 0$. Let $\overline{u}_n = u_n/\|u_n\|_{\infty}$ and $\overline{v}_n = v_n/\|v_n\|_{\infty}$. Then $\overline{u}_n$ and $\overline{v}_n$ satisfy

$$\left\{\begin{array}{l}
-\Delta \overline{u}_n = a_1 \overline{u}_n (1 - \frac{u_n}{k}) - \frac{\alpha_1 \overline{u}_n \overline{v}_n}{c_0 + \overline{u}_n + D_3 \overline{v}_n}, \\
-\Delta \overline{v}_n = -a_2 \overline{v}_n + \frac{\alpha_2 \overline{u}_n}{c_0 + \overline{u}_n + D_3 \overline{v}_n}.
\end{array}\right.$$ 

Passing to a subsequence if necessary, by $L^p$ estimates and Sobolev embedding theorems, there exists $\overline{u}$ and $\overline{v}$, such that $\overline{u}_n \to \overline{u}$ and $\overline{v}_n \to \overline{v}$ in $C^1$ and $\overline{u} \neq 0, \overline{v} \neq 0$. So, $\overline{u} > 0, \overline{v} > 0$. Taking limit of the equations of $\overline{u}_n, \overline{v}_n$, we obtain

$$\left\{\begin{array}{l}
-\Delta \overline{u} = a_1 \overline{u}, \\
-\Delta \overline{v} = -a_2 \overline{v}.
\end{array}\right.$$
Lemma 3.5. Assume that \( \alpha_2 > a_2, a_1 > \lambda_1, c \neq \lambda_1 \) and \( \alpha_2 - a_2 \neq \lambda_1 \) hold. Then

(i) \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{a_1}{k}], 0, 0)) = 0 \), if \( \alpha_2 - a_2 > \lambda_1 \) or \( c > \lambda_1 \);

(ii) \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{a_1}{k}], 0, 0)) = 1 \), if \( \alpha_2 - a_2 < \lambda_1 \) and \( c < \lambda_1 \).

Proof. Observe that \( \mathcal{L} = \mathcal{A}'(\theta[a_1, \frac{a_1}{k}], 0, 0) \)

\[
L = (-\Delta + M)^{-1} \begin{pmatrix}
1 & 0 & -\alpha_1 \\
0 & -a_2 + a_2 + M & 0 \\
0 & 0 & c + M
\end{pmatrix}.
\]

Assume that \( \mathcal{L}(\varphi, \psi, \eta) = (\varphi, \psi, \eta) \), where \( (\varphi, \psi, \eta) \in \mathcal{F}(\theta[a_1, \frac{a_1}{k}], 0, 0) \). Then

\[
\begin{align*}
-\Delta \varphi + (a_1 - \frac{a_2}{k})\varphi &= \alpha_1 \psi, & \text{in } \Omega, & \frac{\partial \varphi}{\partial n} + \varphi &= 0, & \text{on } \partial \Omega; \\
-\Delta \psi &= (a_2 - a_2)\psi, & \text{in } \Omega, & \frac{\partial \psi}{\partial n} + \psi &= 0, & \text{on } \partial \Omega; \\
-\Delta \eta &= c\eta, & \text{in } \Omega, & \frac{\partial \eta}{\partial n} + \eta &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

Because \( \psi, \eta \in \mathcal{K}, c \neq \lambda_1 \) and \( \alpha_2 - a_2 \neq \lambda_1 \), \( \psi, \eta \equiv 0 \) by Lemma 2.1, we have \( \varphi \equiv 0 \), i.e., \( I - \mathcal{L} \) is invertible on \( \mathcal{F}(\theta[a_1, \frac{a_1}{k}], 0, 0) \).

(i) If \( \alpha_2 - a_2 > \lambda_1 \), then \( \lambda_1(a_2 - a_2) = \lambda_1 + a_2 - a_2 < 0 \). So, \( r[(-\Delta + M)^{-1}(-a_2 + a_2 + M)] > 1 \), where \( r[(-\Delta + M)^{-1}(-a_2 + a_2 + M)] \) is the principal eigenvalue of \( (-\Delta + M)^{-1}(-a_2 + a_2 + M) \) with a corresponding eigenfunction \( \psi \in \mathcal{K} \). Let \( t_0 = \frac{1}{r[(-\Delta + M)^{-1}(-a_2 + a_2 + M)]} \). It is obvious that

\[
(I - t_0 \mathcal{L}) \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -t_0(-\Delta + M)^{-1}(-\alpha_1 \psi) \end{pmatrix} \in S(\theta[a_1, \frac{a_1}{k}], 0, 0),
\]

i.e., \( \mathcal{L} \) has property \( \alpha \), therefore, \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{a_1}{k}], 0, 0)) = 0 \) by Lemma 2.3. Similarly, if \( c > \lambda_1 \), we have \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{a_1}{k}], 0, 0)) = 0 \).

(ii) If \( \alpha_2 - a_2 < \lambda_1 \), then \( \lambda_1(a_2 - a_2) = \lambda_1 + a_2 - a_2 > 0 \). Hence, \( r[(-\Delta + M)^{-1}(-a_2 + a_2 + M)] < 1 \). Let \( t_0 = \frac{1}{r[(-\Delta + M)^{-1}(-a_2 + a_2 + M)]} \). It is obvious that \( t_0 > 1 \). Now we prove \( \mathcal{L} \) does not have property \( \alpha \). On the contrary, we suppose that \( \mathcal{L} \) has property \( \alpha \) on \( \mathcal{F}(\theta[a_1, \frac{a_1}{k}], 0, 0) \). Then there exist \( t \in (0, 1) \) and \( (\varphi, \psi, \eta) \in \mathcal{F}(\theta[a_1, \frac{a_1}{k}], 0, 0) \) such that \( (I - t \mathcal{L})(\varphi, \psi, \eta) \in S(\theta[a_1, \frac{a_1}{k}], 0, 0) \). If \( \psi \neq 0 \), then \( \frac{1}{t} \) is eigenvalue of \( (-\Delta + M)^{-1}(-a_2 + a_2 + M) \), i.e., \( (-\Delta + M)^{-1}(-a_2 + a_2 + M) \psi = \frac{1}{t} \psi \). Hence, \( \frac{1}{t} > 1 \) is eigenvalue of \( (-\Delta + M)^{-1}(-a_2 + a_2 + M) \), a contradiction with \( r[(-\Delta + M)^{-1}(-a_2 + a_2 + M)] < 1 \). Similarly, we have \( \eta \equiv 0 \).
for \( c < \lambda_1 \), which implies that \( \mathcal{L} \) does not have property \( \alpha \) on \( P_{(\theta[a_1, \frac{A_1}{k}], 0, 0)} \) and \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{A_1}{k}], 0, 0)) = (-1)^{\delta} \) where \( \delta \) is the sum of the multiplicities of all eigenvalues of \( \mathcal{L} \) which are greater than 1.

Now we assume that \( \mu \) is an eigenvalue of \( \mathcal{L} \) which is greater than 1. Let \( \mathcal{L}(\varphi, \psi, \eta) = \mu(\varphi, \psi, \eta) \), where \( (\varphi, \psi, \eta) \in E \), i.e.,

\[
\begin{cases}
-\mu \Delta \varphi + M(\mu - 1) \varphi = (a_1 - \frac{2a_1 \theta(a_1, \frac{A_1}{k})}{k}) \varphi - \alpha_1 \psi, \quad \text{in } \Omega, \\
-\mu \Delta \psi + M(\mu - 1) \psi = (-a_2 + \alpha_2) \psi, \quad \text{in } \Omega, \\
-\mu \Delta \eta + M(\mu - 1) \eta = c \eta, \quad \text{in } \Omega,
\end{cases}
\]

Since \( \alpha_2 - a_2 < \lambda_1 \) and \( c < \lambda_1 \), \( \varphi, \psi, \eta \equiv 0 \) by Lemma 2.1. It follows that \( \mathcal{L} \) has no eigenvalues which are greater than 1 and \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{A_1}{k}], 0, 0)) = (-1)^0 = 1 \).

The next proposition gives the fixed point index of \( \mathcal{A}_e \) at the trivial solution \((0, 0, \theta[c, \frac{B_2}{D_3}]) \) relative to \( P \), where \( \mathcal{A}_e \) is defined in Lemma 3.4.

**Proposition 3.** Assume \( c > \lambda_1 \) and \( a_1 \neq \lambda_1 \). Then

(i) \( \text{index}_P(\mathcal{A}_e, (0, 0, \theta[c, \frac{B_2}{D_3}])) = 0 \), if \( a_1 > \lambda_1 \);

(ii) \( \text{index}_P(\mathcal{A}_e, (0, 0, \theta[c, \frac{B_2}{D_3}])) = 1 \), if \( a_1 < \lambda_1 \) and \( c \leq \lambda_1 \left(\frac{3}{D_3}\right) \).

**Proof.** Observe that \( P_{(0, 0, \theta[c, \frac{B_2}{D_3}])} = K \times K \times \mathcal{C}(\Omega), \mathcal{N}(0, 0, 0) = \{0\} \times \{0\} \times \mathcal{C}(\Omega) \) and

\[
\mathcal{L} = \mathcal{A}_e'(0, 0, \theta[c, \frac{B_2}{D_3}]) = (-\Delta + M)^{-1} \begin{pmatrix} a_1 + M & 0 & 0 \\ 0 & -a_2 - \frac{\beta_1 \theta[c, \frac{B_2}{D_3}]}{D_2 + \beta_1 \theta[c, \frac{B_2}{D_3}]} + M & 0 \\ 0 & c - \frac{2\beta_2 \theta[c, \frac{B_2}{D_3}]}{D_3} + M \end{pmatrix}.
\]

Assume that \( \mathcal{L}(\varphi, \psi, \eta) = (\varphi, \psi, \eta) \), where \( (\varphi, \psi, \eta) \in P_{(0, 0, \theta[c, \frac{B_2}{D_3}])} \). Then

\[
\begin{cases}
-\Delta \varphi = a_1 \varphi, \quad \text{in } \Omega, \\
-\Delta \psi = (-a_2 - \frac{\beta_1 \theta[c, \frac{B_2}{D_3}]}{D_2 + \beta_1 \theta[c, \frac{B_2}{D_3}]}) \psi, \quad \text{in } \Omega, \\
-\Delta \eta = (c - \frac{2\beta_2 \theta[c, \frac{B_2}{D_3}]}{D_3}) \eta + \frac{2\beta_2 \theta[c, \frac{B_2}{D_3}]}{D_3} \psi, \quad \text{in } \Omega,
\end{cases}
\]

By Lemma 2.1, one can see that \( \varphi, \psi, \eta \equiv 0 \). So, \( I - \mathcal{L} \) is invertible on \( P_{(0, 0, \theta[c, \frac{B_2}{D_3}])} \).

(i) Since \( a_1 > \lambda_1 \), \( \lambda_1 (-a_1) = \lambda_1 - a_1 < 0 \), we have \( r[(-\Delta + M)^{-1}(a_1 + M)] > 1 \), where \( r[(-\Delta + M)^{-1}(a_1 + M)] \) is the principal eigenvalue of \( (-\Delta + M)^{-1}(a_1 + M) \) with a corresponding eigenfunction \( \varphi \in K \setminus \{0\} \). Let \( t_0 = \frac{1}{r[(-\Delta + M)^{-1}(a_1 + M)]} \). It is obvious that \( (I - t_0 \mathcal{L})(\varphi, 0, 0) = (0, 0, 0) \in \mathcal{N}(0, 0, 0, \theta[c, \frac{B_2}{D_3}]) \), i.e., \( \mathcal{L} \) has property \( \alpha \), therefore, \( \text{index}_P(\mathcal{A}_e, (0, 0, \theta[c, \frac{B_2}{D_3}])) = 0 \) by Lemma 2.3.

(ii) Since \( a_1 < \lambda_1 \), \( r[(-\Delta + M)^{-1}(a_1 + M)] < 1 \). Let \( t_0 = \frac{1}{r[(-\Delta + M)^{-1}(a_1 + M)]} \). It is obvious that \( t_0 > 1 \). On the contrary, suppose that \( \mathcal{L} \) has property \( \alpha \) on \( P_{(0, 0, \theta[c, \frac{B_2}{D_3}])} \). Then there exists \( t \in (0, 1) \) and \( (\varphi, \psi, \eta) \in \mathcal{N}(0, 0, 0, \theta[c, \frac{B_2}{D_3}]) \) such that \( (I - t \mathcal{L})(\varphi, \psi, \eta) \in \mathcal{N}(0, 0, 0, \theta[c, \frac{B_2}{D_3}]) \). If \( \varphi \neq 0 \), then \( \frac{1}{r} \) is eigenvalue of \( (-\Delta + M)^{-1}(a_1 + M) \), i.e., \( (-\Delta + M)^{-1}(a_1 + M) \psi = \frac{1}{r} \psi \), hence, \( \frac{1}{r} > 1 \) is eigenvalue of
Let $\mu > 1$, a contradiction with $r[(\Delta + M)^{-1}(a_1 + M)] < 1$, which implies that $\mathcal{L}$ does not have property $\alpha$ on $\mathcal{P}_{(0, 0, \theta_c, 0)}$, and $\text{index}_P(A, (0, 0, \theta_c, 0)) = (-1)^\delta$, where $\delta$ is the sum of the multiplicities of all eigenvalues of $\mathcal{L}$ which are greater than 1.

Now we assume that $\mu$ is an eigenvalue of $\mathcal{L}$ which are greater than 1. Then for all $(\varphi, \psi, \eta) \in E$, we have $\mathcal{L}(\varphi, \psi, \eta) = \mu(\varphi, \psi, \eta)$, i.e.,

$$
\begin{cases}
-m\Delta \varphi + M(\mu - 1)\varphi = a_1 \varphi, & \text{in } \Omega, \\
-m\Delta \psi + M(\mu - 1)\psi = -a_2 - \frac{\beta_a \theta_c \beta_v}{D_2 + b_0 \theta_c \beta_v}, & \text{in } \Omega, \\
-m\Delta \eta + M(\mu - 1)\eta = c - \frac{2\beta_a \theta_c \beta_v}{D_3}, & \text{in } \Omega, \\
\partial \varphi + \varphi = 0, \partial \psi + \psi = 0, \partial \eta + \eta = 0, & \text{on } \partial \Omega.
\end{cases}
$$

Since $c \leq \lambda_1^2(\frac{\beta_a \theta_c \beta_v}{D_3}) + \varphi, \psi, \eta \equiv 0$ by Lemma 2.1. It follows that $\mathcal{L}$ has no eigenvalues of which are greater than 1 and $\text{index}_P(A, (0, 0, \theta_c, 0)) = (-1)^0 = 1$.

By the idea of Proposition 2, we give the index of $A$ at the semi-trivial solution $(0, 0, \theta_c, 0)$ of (1.4)-(1.5) relative to $P$.

**Lemma 3.6.** Assume $a_1 - \frac{\beta_a}{D_3} > \lambda_1$ and $c > \lambda_1$. Then $\text{index}_P(A, (0, 0, \theta_c, 0)) = 0$.

We now turn to consider the following two sub-systems, which are very important for the existence of semi-trivial solutions.

$$
\begin{cases}
-m\Delta u = a_1 u(1 - u/k) - \frac{a_2 u v}{u + D_2}, & \text{in } \Omega, \\
-m\Delta v = -a_2 v + \frac{a_2 u v}{u + D_2}, & \text{in } \Omega, \\
\partial \varphi + \varphi = 0, \partial \psi + \psi = 0, & \text{on } \partial \Omega.
\end{cases}
$$

(3.4)

and

$$
\begin{cases}
-m\Delta u = a_1 u(1 - u/k), & \text{in } \Omega, \\
-m\Delta w = c w - \frac{\beta_a w^2}{D_3}, & \text{in } \Omega, \\
\partial \varphi + \varphi = 0, \partial \psi + \psi = 0, & \text{on } \partial \Omega.
\end{cases}
$$

(3.5)

Obviously, when $a_1, c > \lambda_1$, (3.5) has a unique positive solution $(\theta[a_1, \frac{a_2}{D_3}], \theta[c, \frac{\beta_a}{D_3}])$, i.e., $(\theta[a_1, \frac{a_2}{D_3}], 0, \theta[c, \frac{\beta_a}{D_3}])$. The next lemma gives the index of $A$ at $(\theta[a_1, \frac{a_2}{D_3}], 0, \theta[c, \frac{\beta_a}{D_3}])$ relative to $P$. The proof of this result is quite similar to that of Lemma 3.5 and so is omitted.

**Lemma 3.7.** Assume $a_2 > a_1 > \lambda_1, c > \lambda_1$ and $a_2 - a_1 / \lambda_1 = 1$. Then

(i) $\text{index}_P(A, (\theta[a_1, \frac{a_2}{D_3}], 0, \theta[c, \frac{\beta_a}{D_3}])) = 0$, if $a_2 - a_1 > \lambda_1(\frac{\beta_a \theta_c \beta_v}{D_2 + b_0 \theta_c \beta_v})$;

(ii) $\text{index}_P(A, (\theta[a_1, \frac{a_2}{D_3}], 0, \theta[c, \frac{\beta_a}{D_3}])) = 1$, if $a_2 - a_1 < \lambda_1(\frac{\beta_a \theta_c \beta_v}{D_2 + b_0 \theta_c \beta_v})$.

Now, we consider the existence of positive solution of (3.4). Let $\hat{E} = C_B^1(\overline{\Omega}) \times C_B^1(\overline{\Omega})$, $\hat{P} = K \times K$, $\hat{\Xi} = \{(u, v) \in \hat{P} : u < M_1 + 1, v < M_2 + 1\}$, where $M_i (i = 1, 2)$
are defined in Theorem 3.1. Define an operator \( \hat{A} : \hat{E} \to \hat{E} \) by
\[
\hat{A}(u,v) = (-\Delta + M)^{-1} \left( \overline{f}(u,v,0) + Mu \overline{g}(u,v,0) + Mv \right).
\]
By the denotation of \( M \), \( \hat{A} \) is a positive operator, which maps \( \hat{\Xi} \) to \( \hat{P} \). Similarly to the proof before, we only give Lemmas 3.8 and 3.9 and omit the proof here. The proof of Lemma 3.10, which uses the same arguments as those of Theorem 3.12, will be proved later.

**Lemma 3.8.** Assume \( a_1 - \frac{\alpha_1}{D} > \lambda_1 \). Then

(i) \( \text{index}_P(\hat{A}, \hat{\Xi}) = 1 \), (ii) \( \text{index}_P(\hat{A}, (0,0)) = 0 \).

**Lemma 3.9.** Assume \( \alpha_2 > a_2, a_1 > \lambda_1 \) and \( \alpha_2 - a_2 \neq \lambda_1 \). Then

(i) \( \text{index}_P(\hat{A}, (\theta[a_1, \frac{\alpha_1}{D}], 0)) = 0 \), if \( \alpha_2 - a_2 > \lambda_1 \);
(ii) \( \text{index}_P(\hat{A}, (\theta[a_1, \frac{\alpha_1}{D}], 0)) = 1 \), if \( \alpha_2 - a_2 < \lambda_1 \).

**Lemma 3.10.** Assume \( a_1 - \frac{\alpha_1}{D} > \lambda_1 \) and \( \alpha_2 - a_2 > \lambda_1 \). Then (3.4) has at least one positive solution.

Let us first define \( \Phi = \{(u,v) : (u,v) \) is a positive solution of (3.4)\} and \( \Psi = \{(u,v,0) : (u,v,0) \in \Phi \} \). Next, we calculate the degree of \( I - \mathcal{A} \) in \( \Psi \) relative to \( P \).

Set \( E_1 = C_B^1(\overline{\Omega}) \times C_B^1(\overline{\Omega}), E_2 = C_B^1(\overline{\Omega}), P_1 = K \times K, P_2 = K, P_2(\epsilon) = \{w \in P_2 : ||w||_{E_2} < \epsilon \} \) and
\[
\mathcal{A}_1(u,v,w) = (-\Delta + M)^{-1} \left( \overline{f}(u,v,w) + Mu \overline{g}(u,v,w) + Mw \right),
\]
\[
\mathcal{A}_2(u,v,w) = (-\Delta + M)^{-1}[\overline{f}(u,v,w) + Mw].
\]
Then \( P_2 - P_2 = E_2 \) and \( \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) \). In order to use Lemma 2.5, we choose a relatively open and bounded set \( \mathcal{U} \subset P_1 \cap \Xi, \) such that \( (0,0), (\theta[a_1, \frac{\alpha_1}{D}], 0) \notin \mathcal{U}, \) then by \( \alpha_2 - a_2 > \lambda_1, \)
\[
\mathcal{A}_1(u,v,0) = (u,v) \) with \( (u,v) \in \partial \mathcal{U} \Leftrightarrow (u,v,0) \in \Psi, \)

Denote \( \mathcal{L}w = \mathcal{A}_2(u,v,0)|_{P_2}w = (-\Delta + M)^{-1}(c + M)w, \) \( (u,v) \in \Phi. \) Then if \( c > \lambda_1, \lambda_1(c) = \lambda_1 - c < 0, \) and then \( r[(-\Delta + M)^{-1}(c + M)] > 1. \) If \( c < \lambda_1, \)
\[
\mathcal{L}w = \mathcal{A}_2(u,v,0)|_{P_2}w = (-\Delta + M)^{-1}(c + M)w\]
\[
\text{deg}_P(I - \mathcal{A}, \mathcal{U} \times P_2(\epsilon), 0) = \begin{cases} 0, & \text{if } c > \lambda_1, \\ \text{deg}_{P_1}(I - \mathcal{A}_1|_{P_1}, \mathcal{U}, 0), & \text{if } c < \lambda_1. \end{cases}
\]

Thanks to Lemma 3.10 and the above conclusion, we have the following lemma.

**Lemma 3.11.** If \( a_1 - \frac{\alpha_1}{D} > \lambda_1, \alpha_2 - a_2 > \lambda_1 \) and \( c > \lambda_1, \) then \( \text{deg}_P(I - \mathcal{A}, \Psi) = \text{deg}_P(I - \mathcal{A}, \mathcal{U} \times P_2(\epsilon), 0) = 0. \)

**Theorem 3.12.** Assume \( a_1 - \frac{\alpha_1}{D} > \lambda_1, c > \lambda_1 \) and \( \alpha_2 - a_2 > \lambda_1(\frac{\beta_1\theta[c, \frac{\alpha_1}{D}]}{D_2 + \theta[c, \frac{\alpha_1}{D}]}) \).

Then (1.4)-(1.5) has at least one positive solution.

**Proof.** Since \( a_1 > \lambda_1, \text{index}_P(\mathcal{A}, \Xi) = 1 \) and \( \text{index}_P(\mathcal{A}, (0,0,0)) = 0 \) by Lemma 3.4. Combined with Lemma 3.5, 3.6, 3.7 and 3.11, we know \( \text{index}_P(\mathcal{A}, (\theta[a_1, \frac{\alpha_1}{D}], 0, 0)) + \text{index}_P(\mathcal{A}, (0,0, \theta[c, \frac{\beta_1}{D_2}])), (\theta[a_1, \frac{\alpha_1}{D}], 0, \theta[c, \frac{\beta_1}{D_2}])) + \text{deg}_P(I - \mathcal{A}, \Psi) = 0 \neq 1. \) The proof is completed. \( \square \)
Theorem 3.13. Assume \( a_1 > \lambda_1, c > \lambda_1 \) and \( a_2 - a_1 > \lambda_1 \). Then any positive solution \((u, v, w)\) of (1.4)-(1.5) (if it exists) tends to \((u^*, v^*, w^*)\) as \( \alpha_1 \to 0, \beta_1 \to 0 \), where \((u^*, v^*, w^*)\) are defined in Theorem 3.2.

According to Theorem 3.1, it follows that \( u, v, w \) are uniformly bounded on \( \overline{\Omega} \) independent of \( \alpha_1 \) and \( \beta_1 \). By the standard regularity theory, one can get the result of Theorem 3.13.

Theorem 3.14. Assume \( a_1 - \frac{\beta_2}{\beta_1} > \lambda_1, c > \lambda_1 \) and \( a_2 - a_1 > \lambda_1 \). Then there exists a small, such that (1.4)-(1.5) has a unique positive solution when \( \alpha_1, \beta_1 < \epsilon \). Moreover, it is non-degenerate and asymptotically stable.

Proof. The existence result is a direct consequence of what we have proved in Theorem 3.12. Now, we prove all the positive solutions are non-degenerate and linearly stable first. By virtue of [9], it suffices to prove that the corresponding linearized eigenvalue problem of (1.4)-(1.5) has no eigenvalue with Re\( \mu \) \( \leq \) 0. To do this, we proceed with a contradiction argument and assume that (1.4)-(1.5) has a positive solution \((\alpha_{1,i}, \beta_{1,i}, \tau_{i})\) with \( \alpha_{1,i} \to 0, \beta_{1,i} \to 0 \) where \( i \geq 1 \). Thus there exist \( \mu_i \) with Re\( (\mu_i) \leq 0 \) and \((\xi_i, \eta_i, \tau_i) \neq (0,0,0)\) such that

\[
\begin{aligned}
-\Delta \xi_i + \left(-a_1 + \frac{2a_1 u_i}{k} + \frac{\lambda_{a_1}(D^2)}{(u_i + D_1 v_i)^2}\right) \xi_i + \frac{\lambda_{a_1}(u_i)}{(u_i + D_1 v_i)^2} \eta_i = \mu_i \xi_i, & \quad \text{in } \Omega, \\
-\Delta \eta_i + \left(a_2 - \frac{\lambda_{a_2}(D^2)}{(u_i + D_1 v_i)^2} + \frac{\lambda_{a_2}(u_i)(D_2 + b w_i)}{(D_2 + v_i + b w_i)^2}\right) \eta_i = 0, & \quad \text{in } \Omega, \\
-\Delta \tau_i + \left(-c + \frac{2b w_i}{v_i + D_3}\right) \tau_i = \mu_i \tau_i, & \quad \text{in } \Omega, \\
\frac{\partial \xi_i}{\partial n} + \xi_i = 0, & \quad \frac{\partial \eta_i}{\partial n} + \eta_i = 0, & \quad \frac{\partial \tau_i}{\partial n} + \tau_i = 0, & \quad \text{on } \partial \Omega.
\end{aligned}
\]

Recall that \((u, v, w) \to (u^*, v^*, w^*)\) as \( \alpha_1 \to 0, \beta_1 \to 0 \) by Theorem 3.13. Assume \( \| \xi_i \|_{L^2} + \| \eta_i \|_{L^2} + \| \tau_i \|_{L^2} = 1 \). From (3.7) we have

\[
\begin{aligned}
\mu_i = \int_{\Omega} |\nabla \xi_i|^2 + \int_{\Omega} \left(-a_1 + \frac{2a_1 u_i}{k} + \frac{\lambda_{a_1}(D^2)}{(u_i + D_1 v_i)^2}\right) |\xi_i|^2 + \int_{\Omega} \frac{\lambda_{a_1}(u_i)}{(u_i + D_1 v_i)^2} |\xi_i|^2 + \int_{\Omega} \frac{\lambda_{a_1}(u_i)}{(u_i + D_1 v_i)^2} |\eta_i|^2 + \int_{\Omega} |\tau_i|^2 & \\
+ \int_{\Omega} |\nabla \eta_i|^2 + \int_{\Omega} \left(a_2 - \frac{\lambda_{a_2}(D^2)}{(u_i + D_1 v_i)^2} + \frac{\lambda_{a_2}(u_i)(D_2 + b w_i)}{(D_2 + v_i + b w_i)^2}\right) |\eta_i|^2 & \\
- \int_{\Omega} \frac{\lambda_{a_2}(D^2)}{(u_i + D_1 v_i)^2} |\xi_i|^2 + \int_{\Omega} \frac{\lambda_{a_2}(u_i)(D_2 + b w_i)}{(D_2 + v_i + b w_i)^2} |\tau_i|^2 + \int_{\Omega} \left(-c + \frac{2b w_i}{v_i + D_3}\right) |\tau_i|^2 & \\
+ \int_{\Omega} \left(-c + \frac{2b w_i}{v_i + D_3}\right) |\tau_i|^2 & - \int_{\Omega} \frac{\lambda_{a_2}(u_i)}{(u_i + D_1 v_i)^2} |\xi_i|^2 - \int_{\Omega} \frac{\lambda_{a_2}(u_i)}{(u_i + D_1 v_i)^2} |\eta_i|^2,
\end{aligned}
\]

where \( \overline{\xi_i}, \overline{\eta_i} \) and \( \overline{\tau_i} \) are the complex conjugate of \( \xi_i, \eta_i \) and \( \tau_i \). Now that \( u_i, v_i, w_i \) are uniformly bounded on \( \overline{\Omega} \) independent of \( \alpha_{1,i} \) and \( \beta_{1,i} \), it follows that \( \{Im(\mu_i)\} \) and \( \{\text{Re}(\mu_i)\} \) are bounded, and so \( \{\mu_i\} \) is bounded. By passing to a subsequence of \( i \) if necessary, we assume \( \mu_i \to \mu \), then \( \text{Re}(\mu) \leq 0 \). By the standard regularity theory, \( \{\xi_i\}, \{\eta_i\} \) and \( \{\tau_i\} \) are bounded in \( H^2(\Omega) \), hence, we may assume that \( \xi_i \to \xi, \eta_i \to \eta \) and \( \tau_i \to \tau \) in \( H^1(\Omega) \) up to a subsequence. Taking the limit in (3.7), it
follows that \((\xi, \eta, \tau)\) weakly (then strongly) satisfies

\[
\begin{align*}
-\Delta \xi + (-a_1 + \frac{2a_1 u^*}{k})\xi &= \mu \xi, & \text{in } \Omega, \\
-\Delta \eta + (a_2 - \frac{\alpha_2(u^*)^2}{v^*+D_1 v^*}) \eta &= \mu \eta, & \text{in } \Omega, \\
-\Delta \tau + (c + \frac{2\beta_2 w^*}{v^*+D_3 v^*}) \tau &= \mu \tau, & \text{in } \Omega, \\
\frac{\partial \xi}{\partial n} + \xi &= 0, \quad \frac{\partial \eta}{\partial n} + \eta = 0, \quad \frac{\partial \tau}{\partial n} + \tau = 0, & \text{on } \partial \Omega.
\end{align*}
\]  

(3.8)

First, we claim that \(\xi = 0\). Otherwise \(\xi \neq 0\), and \(0 \geq \mu \geq \lambda_1(-a_1 + \frac{2a_1 u^*}{k}) > \lambda_1(-a_1 + \frac{2a_1 u^*}{k}) = 0\), which gives a contradiction. Next, we claim that \(\eta = 0\), since otherwise \(0 \geq \mu \geq \lambda_1(a_2 - \frac{\alpha_2(u^*)^2}{v^*+D_1 v^*}) > \lambda_1(a_2 - \frac{\alpha_2 u^*}{v^*+D_1 v^*}) = 0\). Finally, \(\tau = 0\), since otherwise \(0 \geq \mu \geq \lambda_1(-c + \frac{2\beta_2 w^*}{v^*+D_3 v^*}) > \lambda_1(-c + \frac{2\beta_2 w^*}{v^*+D_3 v^*}) = 0\), also a contradiction. Hence, we have \((\xi, \eta, \tau) = (0, 0, 0)\), which leads to a contradiction with \(\|\xi, \eta\|_{2,2}^2 + \|\eta\|_{2,2}^2 + \|\tau\|_{2,2}^2 = 1\). Therefore, we have shown that any positive solution of \((4.4)-(4.5)\) is non-degenerate and linearly stable whenever \(\alpha_1, \beta_1\) small.

Next, we show the uniqueness of positive solution. According to Theorem 3.13, it follows that the trivial and semi-trivial solutions are bounded away from the positive solutions. Hence, it follows from a simple compactness argument that \((4.4)-(4.5)\) has at most finitely many positive solutions. Let them be \((u_i, v_i, w_i)\), \(1 \leq i \leq l\). Using the linear stability of \((u_i, v_i, w_i)\), it is easy to check that index\(_P(\mathcal{A}, (u_i, v_i, w_i)) = 1\). Combined with Lemma 3.4-3.7 and 3.11, we know

\[
1 = \text{index}\(_P(\mathcal{A}, \Xi) = \sum_{i=1}^l \text{index}\(_P(\mathcal{A}, (u_i, v_i, w_i)) + \text{index}\(_P(\mathcal{A}, (0, 0, 0)) \\
+ \text{index}\(_P(\mathcal{A}, (0, c, \frac{\beta_2}{D_3})) + \text{index}\(_P(\mathcal{A}, (0, c, \frac{\beta_2}{D_3})) + \text{index}\(_P(\mathcal{A}, (\theta[0, a_1, \frac{a_1}{k}], 0, 0)) \]

Hence \(l = 1\) and we have thus proved the theorem. \(\Box\)

4. Asymptotic behavior: Global attractor and extinction. In this section, we shall consider the asymptotic behavior of the solutions of \((1.1)-(1.3)\), i.e., the sufficient conditions for the permanence and extinction to system \((1.1)-(1.3)\) are investigated.

It follows from the proof of Theorem 3.2 that the quasisolutions \((\pi, \tau, \omega)\) and \((\eta, \eta, \omega)\) exist, which satisfy

\[
\begin{align*}
-\Delta \pi &= a_1 \pi(1 - \pi/k) - \frac{\alpha_1 \tau \pi}{\pi + D_1 \pi}, & \text{in } \Omega, \\
-\Delta \tau &= a_1 \tau(1 - \tau/k) - \frac{\alpha_1 \pi \tau}{\pi + D_1 \pi}, & \text{in } \Omega, \\
-\Delta \omega &= a_2 \omega + \frac{\alpha_2 \pi \omega}{\pi + D_1 \pi} - \frac{\beta \tau \omega}{D_2 + \omega + b \omega}, & \text{in } \Omega, \\
-\Delta \omega &= a_2 \omega + \frac{\alpha_2 \pi \omega}{\pi + D_1 \pi} - \frac{\beta \tau \omega}{D_2 + \omega + b \omega}, & \text{in } \Omega, \\
\frac{\partial \pi}{\partial n} + \pi &= \frac{\partial u}{\partial n} + u = 0, \quad \frac{\partial \tau}{\partial n} + \tau = \frac{\partial v}{\partial n} + v = 0, & \text{on } \partial \Omega. \\
\frac{\partial \omega}{\partial n} + \omega &= \frac{\partial w}{\partial n} + w = 0, & \text{on } \partial \Omega.
\end{align*}
\]  

(4.1)
Theorem 4.1. Assume the conditions in Theorem 3.2 hold. Then $[u, \overline{u}] \times [v, \overline{v}] \times [w, \overline{w}]$ is a global attractor of (1.1)-(1.3), i.e., independently of initial data, we have

$$
\mathcal{U}(x) \leq \liminf_{t \to \infty} U(t, x) \leq \limsup_{t \to \infty} U(t, x) \leq \mathcal{U}(x),
$$

where $\mathcal{U}(x) = (\overline{u}, \overline{v}, \overline{w})$, $U(x) = (u, v, w)$ and $U(x, t) = (u(x, t), v(x, t), (x, t))$.

Proof. In view of the forms in (1.1)-(1.3), it is easy to see from the comparison theorem that any solution $(u, v, w)$ of (1.1)-(1.3) satisfies

$$
u \geq 0, v \geq 0, w \geq 0,$$

as long as the solution $(u, v, w)$ exists. Recall the denotations of $(u^*, v^*, w^*)$ and $(u_*, v_*, w_*)$, which are defined in Theorem 3.2.

First, assume that $a_1 - \frac{\alpha_1}{b} > \lambda_1$. We have

$$
\frac{\partial u}{\partial t} - \Delta u = a_1 u(1 - u/k) - \frac{\alpha_1 uv}{u + Dv} \leq a_1 u(1 - u/k), \quad t > 0, x \in \Omega.
$$

Thus $\limsup_{t \to \infty} u(x, t) \leq u^*$ uniformly on $\overline{\Omega}$, i.e., for all $\epsilon > 0$, there exists $T > 0$ such that $u(x, t) \leq u^* + \epsilon$, when $t > T$. And we also have

$$
\frac{\partial u}{\partial t} - \Delta u = a_1 u(1 - u/k) - \frac{\alpha_1 uv}{u + Dv} \geq u[(a_1 - \frac{\alpha_1}{D}) - a_1 u/k], \quad t > 0, x \in \Omega.
$$

Hence, $\liminf_{t \to \infty} u(x, t) \geq u_*$ uniformly on $\overline{\Omega}$, i.e., for all $\epsilon > 0$, there exists $T_1 > 0$ such that $u(x, t) \geq u_* - \epsilon$, when $t > T_1$.

Second, when $t > T_1$,

$$
\frac{\partial v}{\partial t} - \Delta v = -a_2 v + \frac{\alpha_2 uv}{u + D_1 v} - \frac{\beta_1 vw}{D_2 + v + bw} \leq -a_2 v + \frac{\alpha_2(u^* + \epsilon)v}{(u^* + \epsilon) + D_1 v} \leq v(-a_2 + \frac{\alpha_2 u^*}{u^* + D_1 v} + \frac{\alpha_2 \epsilon}{u^* + \epsilon}), \quad t > T_1, x \in \Omega.
$$

Since $a_2 - a_2 > \lambda_1$, $\lambda_1(a_2 - \frac{\alpha_2}{u^* + \epsilon} - a_2) \leq \lambda_1(a_2 - a_2) = \lambda_1 + a_2 - a_2 < 0$. Combined with the perturbation theory, we get $\lim_{t \to \infty} v(x, t) \leq v^*$ uniformly on $\overline{\Omega}$, i.e., for all $\epsilon > 0$, there exists $T_2 > 0$ such that $v(x, t) \leq v^* + \epsilon$, when $t > T_2$.

When $t > T_1$,

$$
\frac{\partial v}{\partial t} - \Delta v = -a_2 v + \frac{\alpha_2 uv}{u + D_1 v} - \frac{\beta_1 vw}{D_2 + v + bw} \geq -a_2 v + \frac{\alpha_2 uv}{u + D_1 v + 2 \epsilon} - \frac{\beta_1 v}{b} \geq v(-a_2 + \frac{\alpha_2(u_\epsilon + \epsilon)}{u_\epsilon + \epsilon + D_1 v} - \frac{2 \alpha_2 \epsilon}{u_\epsilon + \epsilon + D_1 v} - \frac{\beta_1}{b}), \quad t > T_1, x \in \Omega.
$$

Since $a_2 - a_2 - \frac{\beta_1}{b} > \lambda_1$, $\lambda_1(a_2 - a_2 - \frac{\beta_1}{b} - \frac{\beta_1}{b}) = \lambda_1 + a_2 - a_2 - \frac{\beta_1}{b} < 0$. Take $0 < \epsilon < \frac{\alpha_2 - a_2 - \lambda_1 - \frac{\beta_1}{b}}{a_2 + a_2 + \frac{\beta_1}{b}}$, then $\lambda_1(a_2 - a_2 + \frac{2 \alpha_2 \epsilon}{u_\epsilon + \epsilon} + \frac{\beta_1}{b} - \frac{\beta_1}{b} < 0$. Similarly, we have $\liminf_{t \to \infty} v(x, t) \geq v_\epsilon$ uniformly on $\overline{\Omega}$. So, for all $\epsilon > 0$, there exists $T_2 > 0$ such that $v(x, t) \geq v_\epsilon - \epsilon$, when $t > T_2$. 
At last, when \( t > T^2 \),
\[
\frac{\partial w}{\partial t} - \Delta w = cw - \frac{\beta_2 w^2}{v + D_3} \leq w(c - \frac{\beta_2 w}{v^* + \epsilon + D_3}), \quad t > T^2, \quad x \in \Omega,
\]
when \( t > T_2 \), taking \( \epsilon < \inf_{\Omega} v_* \),
\[
\frac{\partial w}{\partial t} - \Delta w = cw - \frac{\beta_2 w^2}{v + D_3} \geq w(c - \frac{\beta_2 w}{v_* - \epsilon + D_3}), \quad t > T_2, \quad x \in \Omega.
\]
Since \( c > \lambda_1 \), we have \( \limsup_{t \to \infty} w(x,t) \leq w^* \), \( \limsup_{t \to \infty} w(x,t) \geq w_* \) uniformly on \( \Omega \). It follows that, for all \( \epsilon > 0 \), there exist \( T^3 > 0 \) (\( T_3 > 0 \)) such that \( w(x,t) \leq w^* + \epsilon \) (\( w(x,t) \geq w_* - \epsilon \)), when \( t > T^3 \) (\( t > T_3 \)).

Let \( T = \max\{T^1, T^2, T_2, T_3, T_3\} \). Then for all \( t > T \),
\[
(u,v,w) \in [u_* - \epsilon, w^* + \epsilon] \times [v_* - \epsilon, v^* + \epsilon] \times [w_* - \epsilon, w^* + \epsilon], \quad \text{for all } \epsilon > 0.
\]

Taking into account the arbitrariness of \( \epsilon \), our result follows by Theorem 2.1 and Corollary 2.1 of [25].

**Theorem 4.2.** Let \( (u(x,t), v(x,t), w(x,t)) \) be a positive solution of (1.1)-(1.3).

(i) If \( a_1 \leq \lambda_1 \), \( \alpha_2 - a_2 \leq \lambda_1 \) and \( c \leq \lambda_1 \), then \( (u,v,w) \to (0,0,0) \) as \( t \to \infty \);

(ii) If \( a_1 > \lambda_1 \), \( \alpha_2 - a_2 \leq \lambda_1 \) and \( c \leq \lambda_1 \), then \( (u,v,w) \to (\theta[a_1, \frac{a_1}{\theta}], 0, 0) \) as \( t \to \infty \);

(iii) If \( a_1 \leq \lambda_1 \), \( \alpha_2 - a_2 \leq \lambda_1 \) and \( c > \lambda_1 \), then \( (u,v,w) \to (0, 0, \theta[c, \frac{a_2}{D_3}]) \) as \( t \to \infty \);

(iv) If \( a_1 > \lambda_1 \), \( \alpha_2 - a_2 \leq \lambda_1 \) and \( c > \lambda_1 \), then \( (u,v,w) \to (\theta[a_1, \frac{a_1}{\theta}], 0, \theta[c, \frac{a_2}{D_3}]) \) as \( t \to \infty \).

**Proof.** Here we only show the proof (ii). The proof of the others are similar as (ii). Since \( a_1 > \lambda_1 \), it can easily be verified that \( \limsup_{t \to \infty} u(x,t) \leq \theta[a_1, \frac{a_1}{\theta}] \) uniformly on \( \Omega \) by Lemma 2.2. Let \( \epsilon \) be a sufficiently small positive constant such that \( a_1 - \alpha_1 \epsilon > \lambda_1 \). Then there exists \( T > 0 \), such that \( u(x,t) \leq \theta[a_1, \frac{a_1}{\theta}] + \epsilon \), for all \( t > T \). Hence, we have
\[
\frac{\partial v}{\partial t} - \Delta v \leq v\left(\frac{\alpha_2(\theta[a_1, \frac{a_1}{\theta}] + \epsilon)}{\theta[a_1, \frac{a_1}{\theta}] + \epsilon} + D_1 v\right) - a_2), \quad t > T, \quad x \in \Omega,
\]
which concludes that \( v(x,t) \to 0 \) uniformly on \( \Omega \) as \( t \to \infty \) for \( a_2 - a_2 \leq \lambda_1 \), i.e., there exists \( T_1 > 0 \), such that \( v(x,t) \leq \epsilon \), for all \( t > T_1 \).
\[
\frac{\partial u}{\partial t} - \Delta u \geq a_1 u(1 - u/k) - \frac{a_1 u}{u + \epsilon D} \geq u(a_1 - \alpha_1 \epsilon - \frac{a_1 u}{k}), \quad t > T_1, \quad x \in \Omega,
\]
which concludes that \( \liminf_{t \to \infty} u(x,t) \geq \theta[a_1 - \alpha_1 \epsilon, \frac{a_1}{k}] \) uniformly on \( \Omega \) for \( a_1 - \alpha_1 \epsilon > \lambda_1 \) by Lemma 2.2. So, we have
\[
\theta[a_1 - \alpha_1 \epsilon, \frac{a_1}{k}] \leq \liminf_{t \to \infty} u(x,t) \leq \limsup_{t \to \infty} u(x,t) \leq \theta[a_1, \frac{a_1}{\theta}].
\]
Note that when \( \epsilon \to 0^+ \), \( \theta[a_1 - \alpha_1 \epsilon, \frac{a_1}{k}] \to \theta[a_1, \frac{a_1}{\theta}] \) uniformly on \( \Omega \), then \( u(x,t) \to \theta[a_1, \frac{a_1}{\theta}] \) uniformly on \( \Omega \) as \( t \to \infty \). And also we have
\[
\frac{\partial w}{\partial t} - \Delta w \leq w(c - \frac{\beta_2 w}{\epsilon + D_3}), \quad t > T_1, \quad x \in \Omega,
\]
which concludes that \( w(x,t) \to 0 \) uniformly on \( \Omega \) as \( t \to \infty \) for \( c \leq \lambda_1 \) by Lemma 2.2. □
Therefore, (1.1)-(1.3) become

\[\begin{align*}
u_t - \nu_{xx} &= \alpha_1 u(1 - u/k) - \frac{\alpha_1 uv}{u + D_1 v}, \\
v_t - v_{xx} &= -\alpha_2 v + \frac{\alpha_2 uv}{u + D_1 v} - \beta_1 vw, \\
w_t - w_{xx} &= cw - \beta_2 w, (x, t) \in (0, l) \times (0, \infty), \\
-u_x(0, t) + u(0, t) &= -v_x(0, t) + v(0, t) = -w_x(0, t) + w(0, t) = 0, \\
v_x(l, t) + u(l, t) &= w_x(l, t) + v(l, t) = w(l, t) = 0, t \in (0, \infty), \\
u(x, 0) &= u_0 \geq 0(\neq 0), v(x, 0) = v_0 \geq 0(\neq 0), w(x, 0) = w_0 \geq 0(\neq 0), x \in (0, l).
\end{align*}\]

We perform the initial-boundary-value problem numerically based on the Crank-Nicholson scheme and let \(l = 10\). In each simulation, the figures are plotted at sufficiently final time (here, we take \(t = 100\)), which allow us to regard the solutions as steady states. In the finite difference scheme, we take the temporal axis with grid spacing \(\Delta t = 0.01\) and the spatial axis with grid spacing \(\Delta x = 0.1\). In addition, \(\sqrt{\lambda_1}\) satisfies \(\cot(10s) = \frac{1}{2}s - \frac{1}{\pi}, 0 < s < \pi/10\). Through MATLAB software, we can easily get the approximate number \(\lambda_1 \approx 0.0691\). Our numerical simulations indicate the following outcomes.

(i) In Figures 1-2, we are concerned with the effects of the parameter \(c\) on the existence of positive steady state solutions. Here, we fix \(k = 1.0, \alpha_1 = 1.2, \alpha_2 = 1.0, \alpha_1 = 1.0, \alpha_2 = 0.1, \beta_1 = 2.0, \beta_2 = 1.0, D_1 = 1.0, D_2 = 2.0, D_3 = 3.0, b = 1.0\), but let \(c\) change. Theorem 3.12 gives us a fact: if \(a_1 - \alpha_1/D > \lambda_1, c > \lambda_1, \alpha_2 - \alpha_2 > \lambda_1(\frac{\beta_1 \theta_c}{D_2 + b \theta_c[\theta_c]})\), then (1.4)-(1.5) has at least one coexistence solution.

Here, we consider the weaker condition \(a_1 > \lambda_1\) replaced the one in the previous hypotheses, under which the positive steady states maybe exist numerically, and proceed with the similar argument in outcome (ii). We know that if \(\alpha_2 - \alpha_2 > \lambda_1 + (\beta_1 c D_3)/(D_2 \beta_2 + bc D_3)\), which is denoted by \((H)\), then \(\alpha_2 - \alpha_2 > \lambda_1(\frac{\beta_1 \theta_c}{D_2 + b \theta_c[\theta_c]})\).

The density of steady states for different values of the parameters, shown in Figure 1(c)-(d) and Figure 2(e)-(f), are in good agreement with the results of Theorem 3.12.

(ii) In Figure 3, we are concerned with the effects of the parameter \(\alpha_1, \beta_1\) on the density of positive steady state solutions. Here, we fix \(k = 1.0, \alpha_2 = 1.0, a_1 = 1.0, a_2 = 0.1, \beta_2 = 1.0, D = 1.0, D_1 = 1.0, D_2 = 2.0, D_3 = 3.0, b = 1.0, c = 0.15\), and fix \(a_1\), let \(\beta_1\) change or fix \(\beta_1\), let \(a_1\) change. Figure 3 (a)-(d) present a phenomenon that the population density of \(u\) is increasing as \(\beta_1\) increases, but the population density of \(v\) and \(w\) is in the opposite direction; Figure 3 (e)-(h) present a phenomenon that the population density of \(u, v, w\) decreases with increasing of the parameter \(\alpha_1\). In addition, for some parameter region we have chosen, there appears a small bump in the distribution of predator population of size \(v\) near the boundary, but the prey population of size \(u\) and the predation population of size \(w\) distribute by a tendency of “a high and flat central portion, but two low end portions”.

Remark 1. In the above simulations, we consider the weaker condition \(a_1 > \lambda_1\) replaced the one in the hypotheses of Theorem 3.12, under which the positive steady
states maybe exist numerically. However, we can not prove it strictly mathematically this case, since $(0, 0, 0)$ is the singular point of (1.4)-(1.5), and point out that $a_1 - a_1/D > \lambda_1$ may be only a technique condition in the proof of the existence of positive steady states.

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Figure 2. Different values of the parameter $c$. $c = 0.25, 0.45, 0.6, 1.0$ respectively in (e) – (h). In (e)-(f), $a_1 > \lambda_1$, $c > \lambda_1$ and $(H)$ holds; in (g)-(h), $a_1 > \lambda_1$ and $c > \lambda_1$, but $(H)$ doesn’t hold.

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Figure 3. Different values of the parameter $\alpha_1, \beta_1$. In (a)-(d), $\alpha_1 = 1.2, \beta_1 = 2.0, 3.0, 4.0, 5.0$ respectively, in (e)-(h), $\beta_1 = 2.0, \alpha_1 = 0.3, 0.55, 0.8, 1.2$ respectively.

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