Union bound for quantum information processing

Samad Khabbazi Oskouei∗ Stefano Mancini† Mark M. Wilde‡

November 20, 2018

Abstract

In this paper, we prove a quantum union bound that is relevant when performing a sequence of binary-outcome quantum measurements on a quantum state. The quantum union bound proved here involves a tunable parameter that can be optimized, and this tunable parameter plays a similar role to a parameter involved in the Hayashi-Nagaoka inequality [IEEE Trans. Inf. Theory, 49(7):1753 (2003)], used often in quantum information theory when analyzing the error probability of a square-root measurement. An advantage of the proof delivered here is that it is elementary, relying only on basic properties of projectors, the Pythagorean theorem, and the Cauchy–Schwarz inequality. As a non-trivial application of our quantum union bound, we prove that a sequential decoding strategy for classical communication over a quantum channel achieves a lower bound on the channel’s second-order coding rate. This demonstrates the advantage of our quantum union bound in the non-asymptotic regime, in which a communication channel is called a finite number of times. We expect that the bound will find a range of applications in quantum communication theory, quantum algorithms, and quantum complexity theory.

1 Introduction

The union bound, alternatively known as Boole’s inequality, represents one of the simplest yet non-trivial methods for bounding the probability that either one event or another occurs, in terms of the probabilities of the individual events (see, e.g., [1]). By induction, the bound applies to the union of multiple events, and it often provides a good enough bound in a variety of applications whenever the probabilities of the individual events are small relative to the number of events. Concretely, given a finite set \( \{A_i\}_{i=1}^L \) of events, the union bound is the following inequality:

\[
\Pr\left\{ \bigcup_{i=1}^L A_i \right\} \leq \sum_{i=1}^L \Pr\{A_i\}. \tag{1.1}
\]

By applying DeMorgan’s law and basic rules of probability theory, we can rewrite the union bound such that it applies to the probability that an intersection of events does not occur

\[
1 - \Pr\left\{ \bigcap_{i=1}^L A_i \right\} \leq \sum_{i=1}^L \Pr\{A_i^c\}, \tag{1.2}
\]

∗Department of Mathematics, Islamic Azad University, Varamin-Pishva Branch, 33817-7489, Iran
†School of Science and Technology, University of Camerino, Via M. delle Carceri 9, I-62032 Camerino, Italy & INFN–Sezione Perugia, Via A. Pascoli, I-06123 Perugia, Italy
‡Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA
and this is the form in which it is typically employed in applications. Recently, the union bound has been listed as the second step to try when attempting to “upper-bound the probability of something bad,” with the first step being to determine if the trivial bound of one is reasonable in a given application [2].

Generalizing the union bound to a quantum-mechanical setup is non-trivial. A natural setting in which we would consider this generalization is when the goal is to bound the probability that two or more successive measurement outcomes do not occur. Concretely, suppose that the state of a quantum system is given by a density operator $\rho$. Suppose that there are $L$ projective quantum measurements $\{P_i, I - P_i\}$ for $i \in \{1, \ldots, L\}$, where $P_i$ is a projector, thus satisfying $P_i = P_i^\dagger$ and $P_i = P_i^2$ by definition. Suppose that the first measurement is performed, followed by the second measurement, and so on. If the projectors $P_1, \ldots, P_L$ commute, then the probability that the outcomes $P_1, \ldots, P_L$ do not occur is calculated by applying the Born rule and can be bounded as

$$1 - \text{Tr}\{P_L P_{L-1} \cdots P_1 \rho P_1 \cdots P_{L-1}\} \leq \sum_{i=1}^{L} \text{Tr}\{(I - P_i)\rho\},$$

(1.3)

with the bound following essentially from an application of the union bound. However, if the projectors $P_1, \ldots, P_L$ do not commute, then classical reasoning does not apply and alternative methods are required.

Recently, Gao proved a quantum union bound [3] that has been useful in a variety of applications, including quantum communication theory [4, 5, 6, 7], quantum algorithms [8, 9, 10], quantum complexity theory [9, 11], and Hamiltonian complexity theory [12, 13]. Given an arbitrary set of projectors $\{P_i\}_{i=1}^L$, each corresponding to one outcome of a binary-valued measurement, Gao’s quantum union bound is the following inequality [3, Theorem 1]:

$$1 - \text{Tr}\{P_L P_{L-1} \cdots P_1 \rho P_1 \cdots P_{L-1}\} \leq 4 \sum_{i=1}^{L} \text{Tr}\{(I - P_i)\rho\}. \quad (1.4)$$

By comparing (1.4) with (1.3), we notice that the only difference is the factor of four in (1.4). The factor of four is inconsequential for many applications, but nevertheless, it is natural to wonder whether this bound can be improved. Furthermore, at least one application in which improving the factor of four does make a difference is in the context of whether a sequential decoding strategy can be used to achieve the second-order coding rate for classical communication—we discuss this application in more detail later.

2 Summary of results

In this paper, we prove the following quantum union bound:

**Theorem 1 (Quantum union bound)** Let $\rho$ be a density operator acting on a separable Hilbert space $\mathcal{H}$, let $\{P_i\}_{i=1}^L$ be an arbitrary set of projectors, each acting on $\mathcal{H}$, and let $c > 0$ be an arbitrary positive constant. Then

$$1 - \text{Tr}\{P_L P_{L-1} \cdots P_1 \rho P_1 \cdots P_{L-1}\} \leq (1 + c) \text{Tr}\{(I - P_L)\rho\} + \left(2 + c + c^{-1}\right) \sum_{i=2}^{L-1} \text{Tr}\{(I - P_i)\rho\} + \left(2 + c^{-1}\right) \text{Tr}\{(I - P_1)\rho\}. \quad (2.1)$$
Our proof of the above theorem is elementary, relying only on basic properties of projectors, the Pythagorean theorem, and the Cauchy–Schwarz inequality. Furthermore, the theorem directly applies to states of infinite-dimensional quantum systems and can thus be employed to analyze practical situations involving not only qubits but also bosonic quantum systems [14]. Similar to the classical case discussed in the introduction, the quantum union bound of Theorem 1 provides a useful bound when the individual probabilities Tr{(I − P)ρ} are small relative to the number L of them, and this scenario occurs, for example, in the application to communication presented in Section 5. Furthermore, the tunable parameter c > 0 is a significant advantage of our quantum union bound, and it is essential in the application mentioned above, in which it really is necessary for c > 0 to be decreasing with the number of channel uses so that the prefactor in front of the term Tr{(I − PL)ρ} is as close to one as possible. More generally, one could certainly take an infimum over the parameter c > 0 in any given application in order to have the upper bound be as tight as possible.

Our quantum union bound represents a strict improvement over that of Gao’s in (1.4). Indeed, by setting c = 1 and then loosening the above bound further, we recover Gao’s. Our quantum union bound can also be compared with the Hayashi–Nagaoka (HN) inequality from [15, Lemma 2], which is often used to analyze the error probability of the square-root measurement. The HN inequality also features a tunable parameter c > 0, and this is one of the main reasons why quantum information theory has recently advanced in the direction of characterizing second-order asymptotics for communication tasks [16, 17, 18, 19, 20, 21, 22, 23, 24]. Our quantum union bound provides essentially the same trade-off given by the HN inequality, but just slightly improved, in the sense that the prefactor for the term Tr{(I − PL)ρ} is 1 + c, while the prefactor for L − 2 other terms is 2 + c + c−1 and the prefactor for the term Tr{(I − P1)ρ} is 2 + c−1, the last prefactor representing the improvement.

In the previous paragraphs, we focused exclusively on the comparison of Theorem 1 with Gao’s bound in (1.4). However, there were other works that preceded Gao’s, which we recall now. [25] established a quantum union bound, with applications in quantum complexity theory. [26] analyzed the error probability of a sequential decoding strategy and proved that it can achieve the Holevo information of a quantum channel for classical communication. The work of [26] then inspired [27], who established another quantum union bound (also called “non-commutative union bound”) of the following form:

\begin{equation}
1 − \text{Tr}\{PLPL−1 · · · P1ρP1 · · · PL−1\} \leq 2\sqrt{\sum_{i=1}^{L} \text{Tr}\{(I − Pi)ρ\}}.
\end{equation}

[28] subsequently generalized the result of [27] beyond projectors, such that it would hold for a set of operators \(\{Λ_i\}_{i=1}^{L}\), each of which satisfies 0 ≤ Λ_i ≤ I. Then Gao’s bound in (1.4) appeared after [28]. Clearly, Gao’s bound was a significant improvement over (2.2), eliminating the square root at the cost of a doubling of the prefactor.

To demonstrate an application in which Theorem 1 is useful, we show how a sequential decoding strategy achieves a lower bound on the second-order coding rate for classical communication over a quantum channel. We consider the cases in which there is entanglement assistance as well as no assistance, and our result here also covers the important case when the channel takes input density operators acting on a separable Hilbert space to output density operators acting on a separable Hilbert space. An advantage of our proof is that it is arguably simpler than other approaches that
could be taken to solve this problem, relying on a method called position-based coding \cite{29}, as well as sequential decoding \cite{26, 27, 28}, and an error analysis that uses Theorem 1. Our proof can be compared with the proof from \cite{30, 31}, in which it was shown how to achieve the capacity for energy-constrained classical communication (i.e., the first-order coding rate), and we advocate here that our proof is considerably simpler.

We organize the rest of our paper as follows. In Section 3, we provide a proof of Theorem 1. In Section 4, we consider the generalization of Theorem 1 to positive operator-valued measures (POVMs). Section 5 discusses the application to obtaining a lower bound on the second-order coding rate for classical communication. In Section 6, we conclude with a summary and discuss some open directions for future research.

3 Proof of Theorem 1

We prove our main result, Theorem 1, by establishing the following more general result:

**Theorem 2** Let $\mathcal{H}$ be a separable Hilbert space, let $|\psi\rangle \in \mathcal{H}$, let $\{P_i\}_{i=1}^L$ be a finite set of projectors acting on $\mathcal{H}$, and let $c > 0$. Then

$$
\|\psi\|_2^2 - \|P_LP_{L-1}\cdots P_1|\psi\rangle\|_2^2 \leq (1 + c) \|(I - P_L)|\psi\rangle\|_2^2 + (2 + c + c^{-1}) \sum_{i=2}^{L-1} \|(I - P_i)|\psi\rangle\|_2^2 + (2 + c^{-1}) \|(I - P_1)|\psi\rangle\|_2^2.
$$

Theorem 1 is a direct consequence of Theorem 2. Indeed, a density operator $\rho$ acting on a separable Hilbert space has a spectral decomposition as follows:

$$
\rho = \sum_{j \in J} p_j |\psi_j\rangle \langle \psi_j|,
$$

where the index set $J$ is countable, $\{p_j\}_{j \in J}$ is a probability distribution, and $\{|\psi_j\rangle\}_{j \in J}$ is an orthonormal set of eigenvectors \cite{32}. Applying Theorem 2 we find that

$$
1 - \text{Tr}\{P_LP_{L-1}\cdots P_1|\psi_j\rangle \langle \psi_j|P_1\cdots P_{L-1}\}\nonumber = \|\psi_j\|_2^2 - \|P_LP_{L-1}\cdots P_1|\psi_j\rangle\|_2^2
$$

$$
\leq (1 + c) \|(I - P_L)|\psi_j\rangle\|_2^2 + (2 + c + c^{-1}) \sum_{i=2}^{L-1} \|(I - P_i)|\psi_j\rangle\|_2^2 + (2 + c^{-1}) \|(I - P_1)|\psi_j\rangle\|_2^2,
$$

$$(2 + c^{-1}) + \text{Tr}\{(I - P_1)|\psi_j\rangle \langle \psi_j|\}.
$$

The reduction from Theorem 1 to Theorem 2 follows by averaging over the distribution $\{p_j\}_{j \in J}$.
So now we shift our focus to proving Theorem 2, and we do so with the aid of several lemmas. To simplify the notation, hereafter we employ the following shorthand:

\[ \| \cdots \| \equiv \| \cdots \|_2, \]  
\[ \langle \cdots \rangle \equiv \langle \cdots | \psi \rangle, \]  
\[ Q_i \equiv I - P_i. \]  

The convention we take with the shorthand \( \langle A \rangle \) for a non-Hermitian operator \( A \) is that \( \langle A \rangle = \langle \psi | \varphi \rangle \) where \( | \varphi \rangle = A | \psi \rangle \). Furthermore, we also assume without loss of generality that the vector \( | \psi \rangle \) in Theorem 2 is a unit vector. Clearly, this assumption can be easily released by scaling the resulting inequality by an arbitrary positive number.

First recall that, due to the idempotence of projectors, we have the following identities holding for all \( i \in \{1, 2, \ldots, L\} \):

\[ \langle Q_i P_{i-1} \cdots P_1 \rangle = \langle Q_i Q_i P_{i-1} \cdots P_1 \rangle, \quad \langle P_1 \cdots P_i \rangle = \langle P_1 \cdots P_i P_i \rangle, \]  
under the convention that \( P_{i-1} \cdots P_1 = P_1 \cdots P_{i-1} = I \) for \( i = 1 \).

**Lemma 3** For a set \( \{P_i\}_{i=1}^L \) of projectors acting on a separable Hilbert space \( \mathcal{H} \), a unit vector \( | \psi \rangle \in \mathcal{H} \), and employing the shorthand in (3.6)-(3.8), we have the following identities:

\[ \sum_{i=1}^L \langle Q_i P_{i-1} \cdots P_1 \rangle = 1 - \langle P_L \cdots P_1 \rangle, \]  
\[ \sum_{i=1}^L \langle P_1 \cdots P_{i-1} Q_i \rangle = 1 - \langle P_1 \cdots P_L \rangle, \]  
\[ \sum_{i=1}^L \langle P_1 \cdots P_{i-1} Q_i P_{i-1} \cdots P_1 \rangle = 1 - \langle P_1 \cdots P_L \cdots P_1 \rangle, \]  
\[ 1 - \sqrt{\langle P_L \rangle} \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \leq \sum_{i=1}^L \sqrt{\langle Q_i \rangle} \sqrt{\langle P_i \cdots P_{i-1} Q_i P_{i-1} \cdots P_1 \rangle}, \]

under the convention that \( P_{i-1} \cdots P_1 = P_1 \cdots P_{i-1} = I \) for \( i = 1 \).

**Proof.** The following identities are straightforward:

\[ 1 = \langle Q_1 \rangle + \langle Q_2 P_1 \rangle + \cdots + \langle Q_L P_{L-2} \cdots P_1 \rangle + \langle Q_L P_{L-1} \cdots P_1 \rangle + \langle P_L P_{L-1} \cdots P_1 \rangle, \]  
\[ 1 = \langle Q_1 \rangle + \langle P_1 Q_2 \rangle + \cdots + \langle P_1 \cdots P_{L-2} Q_{L-1} \rangle + \langle P_1 \cdots P_{L-1} Q_L \rangle + \langle P_1 \cdots P_{L-1} P_L \rangle, \]  
\[ 1 = \langle Q_1 \rangle + \langle P_1 Q_2 P_1 \rangle + \cdots + \langle P_1 \cdots P_{L-2} Q_{L-1} P_{L-2} \cdots P_1 \rangle \]
\[ + \langle P_1 \cdots P_{L-1} Q_L P_{L-1} \cdots P_1 \rangle + \langle P_1 \cdots P_{L-1} P_L P_{L-1} \cdots P_1 \rangle. \]  

Consequently, from the equalities in (3.14), (3.15), and (3.16), we obtain (3.10), (3.11), and (3.12), respectively. The following equality is a direct consequence of (3.14) and (3.9):

\[ 1 = \langle Q_1 \rangle + \langle Q_2 Q_2 P_1 \rangle + \cdots + \langle Q_L Q_{L-1} P_{L-2} \cdots P_1 \rangle + \langle Q_L Q_L P_{L-1} \cdots P_1 \rangle \]
\[ + \langle P_L P_L P_{L-1} \cdots P_1 \rangle. \]  

\( (3.17) \)}
By applying the Cauchy-Schwarz inequality to (3.17), we find that
\[
1 \leq \langle Q_1 \rangle + \sqrt{\langle Q_2 \rangle \langle P_1 Q_2 P_1 \rangle} + \cdots + \sqrt{\langle Q_L \rangle \langle P_1 \cdots P_{L-1} Q_L P_{L-1} \cdots P_1 \rangle} + \sqrt{\langle P_L \rangle \langle P_1 \cdots P_{L-1} P_L P_{L-1} \cdots P_1 \rangle}, \tag{3.18}
\]
from which (3.13) immediately follows. □

**Lemma 4** For a set \( \{P_i\}_{i=1}^L \) of projectors acting on a separable Hilbert space \( \mathcal{H} \), a unit vector \( |\psi\rangle \in \mathcal{H} \), and employing the shorthand in (3.6)–(3.8), the following inequality holds for \( L \geq 2 \):
\[
\sum_{i=1}^L \| Q_i (I - P_{i-1} \cdots P_1) \|^2 \leq \sum_{i=1}^{L-1} \| Q_i \|^2, \tag{3.19}
\]
under the convention that \( P_{i-1} \cdots P_1 = P_1 \cdots P_{i-1} = I \) for \( i = 1 \). Equivalently,
\[
\sum_{i=2}^L \| Q_i (I - P_{i-1} \cdots P_1) \|^2 \leq \sum_{i=1}^{L-1} \| Q_i \|^2, \tag{3.20}
\]
*due to the aforementioned convention.*

**Proof.** Consider the following chain of equalities:
\[
\sum_{i=1}^L \| Q_i (I - P_{i-1} \cdots P_1) \|^2 = \sum_{i=1}^L \| Q_i - Q_i P_{i-1} \cdots P_1 \|^2 \\
= \sum_{i=1}^L \left( \| Q_i \|^2 - \langle Q_i P_{i-1} \cdots P_1 \rangle - \langle P_1 \cdots P_{i-1} Q_i \rangle + \langle P_1 \cdots P_{i-1} Q_i P_{i-1} \cdots P_1 \rangle \right) \tag{3.21}
= \left( \sum_{i=1}^L \| Q_i \|^2 \right) - 1 + \langle P_L \cdots P_1 \rangle - 1 + \langle P_1 \cdots P_L \rangle + 1 - \langle P_1 \cdots P_L \cdots P_1 \rangle \tag{3.22}
= \left( \sum_{i=1}^L \| Q_i \|^2 \right) - 1 + \langle P_L P_L P_{L-1} \cdots P_1 \rangle + \langle P_1 \cdots P_L P_L P_L \rangle - \langle P_1 \cdots P_L \cdots P_1 \rangle. \tag{3.23}
\]
To obtain (3.21), we used the identities in (3.9). Next, to get (3.22), the identities in (3.10), (3.11), and (3.12) of Lemma 3 were used. Continuing, we have that
\[
\text{Eq. (3.23)} \leq \left( \sum_{i=1}^L \| Q_i \|^2 \right) - 1 + \langle P_1 \cdots P_L \cdots P_1 \rangle + 2 \sqrt{\langle P_L \rangle} \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \tag{3.24}
= \left( \sum_{i=1}^L \| Q_i \|^2 \right) - 1 + \langle P_L \rangle - \left( \sqrt{\langle P_L \rangle} - \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)^2 \tag{3.25}
\leq \left( \sum_{i=1}^L \| Q_i \|^2 \right) - \| Q_L \|^2 = \sum_{i=1}^{L-1} \| Q_i \|^2. \tag{3.26}
\]
To obtain (3.24), the Cauchy-Schwarz inequality was employed. □
We are now in a position to prove Theorem 2.

**Proof of Theorem 2.** Consider that

\[
1 - \|P_L \cdots P_1\|^2 = 1 - \langle P_1 \cdots P_L \cdots P_1 \rangle + 2 \left( 1 - \sqrt{\langle P_L \rangle} \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)
\]

\[
- 2 \left( 1 - \sqrt{\langle P_L \rangle} \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)
\]

\[
= 2 \left( 1 - \sqrt{\langle P_L \rangle} \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)
\]

\[
- \left( \sqrt{\langle P_L \rangle} - \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)^2 - 1 + \langle P_L \rangle.
\]

Continuing, we have that

\[
\text{Eq. (3.28)} \leq - \|Q_L\|^2 + 2 \left( 1 - \sqrt{\langle P_L \rangle} \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)
\]

\[
\leq - \|Q_L\|^2 + 2 \sum_{i=1}^{L} \sqrt{\langle Q_i \rangle} \sqrt{\langle P_1 \cdots P_{i-1}Q_iP_{i-1} \cdots P_1 \rangle}
\]

\[
\leq - \|Q_L\|^2 + 2 \sum_{i=1}^{L} \sqrt{\langle Q_i \rangle} (\|Q_i\| + \|Q_i(I - P_{i-1} \cdots P_1)\|).
\]

First, (3.29) is obtained by observing that

\[
- \left( \sqrt{\langle P_L \rangle} - \sqrt{\langle P_1 \cdots P_L \cdots P_1 \rangle} \right)^2 - 1 + \langle P_L \rangle \leq -1 + \langle P_L \rangle = -\|Q_L\|^2.
\]

Next, (3.30) follows from (3.13) of Lemma 3. Then, (3.31) is a consequence of the triangle inequality:

\[
\sqrt{\langle P_1 \cdots P_{i-1}Q_iP_{i-1} \cdots P_1 \rangle} = \|Q_iP_{i-1} \cdots P_1\|
\]

\[
= \|Q_i(-I + I - P_{i-1} \cdots P_1)\|
\]

\[
\leq \|Q_i\| + \|Q_i(I - P_{i-1} \cdots P_1)\|,
\]

under the convention that \(P_{i-1} \cdots P_1 = I\) for \(i = 1\). Continuing, we have that

\[
\text{Eq. (3.31)} = - \|Q_L\|^2 + 2 \sum_{i=1}^{L} \|Q_i\|^2 + 2 \sum_{i=1}^{L} (\|Q_i\| \|Q_i(I - P_{i-1} \cdots P_1)\|)
\]

\[
= - \|Q_L\|^2 + 2 \sum_{i=1}^{L} \|Q_i\|^2 + 2 \sum_{i=2}^{L} (\|Q_i\| \|Q_i(I - P_{i-1} \cdots P_1)\|)
\]

\[
\leq - \|Q_L\|^2 + 2 \sum_{i=1}^{L} \|Q_i\|^2 + \sum_{i=2}^{L} \left( c \|Q_i\|^2 + c^{-1} \|Q_i(I - P_{i-1} \cdots P_1)\|^2 \right)
\]

\[
\leq - \|Q_L\|^2 + 2 \sum_{i=1}^{L} \|Q_i\|^2 + c \sum_{i=2}^{L} \|Q_i\|^2 + c^{-1} \sum_{i=1}^{L-1} \|Q_i\|^2
\]

\[
\leq (1 + c) \|Q_L\|^2 + (2 + c^{-1}) \|Q_1\|^2 + (2 + c + c^{-1}) \sum_{i=2}^{L-1} \|Q_i\|^2.
\]

Eq. (3.37) follows from the convention that \(P_{i-1} \cdots P_1 = I\) for \(i = 1\). Eq. (3.38) is a consequence of the inequality \(2xy \leq cx^2 + c^{-1}y^2\), holding for \(x, y \in \mathbb{R}\) and \(c > 0\). Finally, (3.39) is obtained by using Lemma 4. □
4 Generalization to POVMs

Just as the bound from [27] was generalized in [28, Section 3] from projectors to positive semi-definite operators having eigenvalues between zero and one, we can do the same here. This generalization is useful for applications, and we discuss one such application in the next section.

We now give an extension of the quantum union bound in Theorem 1 that applies for general measurements. The main idea behind it is the well known Naimark extension theorem, following the approach from [28, Section 3].

\[ \text{Lemma 5} \]

Let \( \rho \) be a positive semi-definite operator acting on a separable Hilbert space \( H \), let \( \{ \Lambda_i \}_{i=1}^L \) denote a set of positive semi-definite operators such that \( 0 \leq \Lambda_i \leq I \) for all \( i \in \{ 1, \ldots, L \} \), and let \( c > 0 \). Then the following quantum union bound holds

\[
\text{Tr} \{ \rho - \Pi_{\Lambda_L} \cdots \Pi_{\Lambda_1} (\rho \otimes |0\rangle\langle 0|_{p_L}) \Pi_{\Lambda_1} \cdots \Pi_{\Lambda_L} \} \leq (1 + c) \text{Tr} \{ (I - \Lambda_L)\rho \} \\
+ (2 + c + c^{-1}) \sum_{i=2}^{L-1} \text{Tr} \{ (I - \Lambda_i)\sigma \} + (2 + c^{-1}) \text{Tr} \{ (I - \Lambda_1)\rho \},
\]

where \( |0\rangle_{p_L} \equiv |0\rangle_{P_1} \otimes \cdots \otimes |0\rangle_{P_L} \) is an auxiliary state of \( L \) qubit probe systems and \( \Pi_{\Lambda_i} \) is a projector defined as \( \Pi_{\Lambda_i} \equiv U_{SP_i}^\dagger (I \otimes |1\rangle\langle 1|_{P_i}) U_{SP_i} \), for some unitary \( U_i \) and projector \( P_i \) such that

\[
\text{Tr} \{ \Pi_{\Lambda_i} (\rho \otimes |0\rangle\langle 0|_{p_L}) \} = \text{Tr} \{ \Lambda_i \rho \}.
\]

\[ \text{Proof.} \] This extension of Theorem 1 follows easily by employing the Naimark extension theorem. Concretely, to each operator \( \Lambda_i \), we associate the following unitary:

\[
U_{SP_i} \equiv \sqrt{I - (\Lambda_i)_{S} \otimes (I \otimes |0\rangle\langle 0|_{P_i})} + \sqrt{(\Lambda_i)_{S} \otimes (|1\rangle\langle 1|_{P_i} - |0\rangle\langle 0|_{P_i})}.
\]

Then defining the projectors \( \Pi_{\Lambda_i} \equiv U_{SP_i}^\dagger (I \otimes |1\rangle\langle 1|_{P_i}) U_{SP_i} \), a straightforward calculation gives that \( \text{Tr} \{ \Pi_{\Lambda_i} (\rho \otimes |0\rangle\langle 0|_{p_L}) \} = \text{Tr} \{ \Lambda_i \rho_S \} \). Observe that the operator \( \Pi_{\Lambda_i} \) is an orthogonal projector (because it is Hermitian and idempotent), so that Theorem 1 applies to each of these operators. Then (4.1) follows.

5 Lower bound on the second-order coding rate for classical communication

One application of our main result, Theorem 1, is in achieving the second-order coding rate for classical communication. As we stated earlier, this area of quantum information theory has advanced in recent years [16, 17, 18, 19, 20, 21, 22, 23, 24], with one of the main reasons being the availability of the tunable parameter \( c > 0 \) in the Hayashi–Nagaoka inequality [15, Lemma 2]. That is, one can let \( c > 0 \) vary, to become closer to zero, as the number of channel uses increases.

An advantage of our Theorem 1 is that it applies directly to the case of states and projectors that act on an infinite-dimensional, separable Hilbert space. Thus, the theorem can be applied directly in order to achieve a lower bound on the second-order coding rate for classical communication. To our knowledge, prior to our work here, [20] presented the only case in which lower bounds on the second-order coding rates have been considered in this general case, and there, the analysis was limited to channels that accept a classical input and output a pure quantum state. The situation that we analyze here is thus more general.
5.1 Information quantities

Before we begin with the application, let us recall some information quantities that are essential in the analysis. Let \( \mathcal{H} \) denote a separable Hilbert space, and let \( \mathcal{D}(\mathcal{H}) \) denote the set of density operators acting on \( \mathcal{H} \) (positive, semi-definite operators with trace equal to one). Let spectral decompositions of \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) be given as

\[
\rho = \sum_{x \in \mathcal{X}} \lambda_x P_x, \quad \sigma = \sum_{y \in \mathcal{Y}} \mu_y Q_y,
\]

(5.1)

where \( \mathcal{X} \) and \( \mathcal{Y} \) are countable index sets, \( \{\lambda_x\}_{x \in \mathcal{X}} \) and \( \{\mu_y\}_{y \in \mathcal{Y}} \) are probability distributions with \( \lambda_x, \mu_y \geq 0 \) for all \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \) and \( \sum_{x \in \mathcal{X}} \lambda_x = \sum_{y \in \mathcal{Y}} \mu_y = 1 \), and \( \{P_x\}_{x \in \mathcal{X}} \) and \( \{Q_y\}_{y \in \mathcal{Y}} \) are sets of projections such that \( \sum_{x \in \mathcal{X}} P_x = \sum_{y \in \mathcal{Y}} Q_y = I \).

The hypothesis testing relative entropy \( D^\varepsilon_H(\rho\|\sigma) \) is defined for \( \varepsilon \in [0, 1] \) as

\[
D^\varepsilon_H(\rho\|\sigma) \equiv -\log_2 \inf_\Lambda \{\text{Tr}\{\Lambda \sigma\} : \text{Tr}\{\Lambda \rho\} \geq 1 - \varepsilon \land 0 \leq \Lambda \leq I\}.
\]

(5.2)

The quantum relative entropy [35], the quantum relative entropy variance [16, 36, 37], and the \( T \) quantity [16, 36, 37] are defined as

\[
D(\rho\|\sigma) \equiv \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \lambda_x \text{Tr}\{P_x Q_y\} \log_2 \left( \frac{\lambda_x}{\mu_y} \right),
\]

(5.3)

\[
V(\rho\|\sigma) \equiv \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \lambda_x \text{Tr}\{P_x Q_y\} \left[ \log_2 \left( \frac{\lambda_x}{\mu_y} \right) - D(\rho\|\sigma) \right]^2,
\]

(5.4)

\[
T(\rho\|\sigma) \equiv \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \lambda_x \text{Tr}\{P_x Q_y\} \left| \log_2 \left( \frac{\lambda_x}{\mu_y} \right) - D(\rho\|\sigma) \right|^3.
\]

(5.5)

For states \( \rho \) and \( \sigma \) satisfying

\[
D(\rho\|\sigma), V(\rho\|\sigma), T(\rho\|\sigma) < \infty, \quad V(\rho\|\sigma) > 0,
\]

(5.6)

the following expansion holds for the hypothesis testing relative entropy for \( \varepsilon \in (0, 1) \) and a sufficiently large positive integer \( n \):

\[
D^{\varepsilon}_H(\rho^\otimes n\|\sigma^\otimes n) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n),
\]

(5.7)

where

\[
\Phi(a) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} dx \exp \left(-x^2/2\right), \quad \Phi^{-1}(\varepsilon) \equiv \sup \{a \in \mathbb{R} \mid \Phi(a) \leq \varepsilon\}.
\]

(5.8)

The equality in (5.7) was proven for finite-dimensional states \( \rho \) and \( \sigma \) in [16, 36]. For the case of states acting on infinite-dimensional, separable Hilbert spaces, the inequality \( \leq \) in (5.7) was proven in [38] and [37, Appendix C]. In Appendix A we prove the inequality \( \geq \) in (5.7). The proof that we detail follows the development in [38, Appendix C] very closely, which is in turn based on [36, Section 3.2].
5.2 Communication codes

We now recall what we mean by a code for classical communication and one for entanglement-assisted classical communication, starting with the former. Note that classical communication was considered for the asymptotic case in [39, 40]. Suppose that a channel $\mathcal{N}_{A \rightarrow B}$ connects a sender Alice to a receiver Bob. For positive integers $n$ and $M$, and $\varepsilon \in [0,1]$, an $(n,M,\varepsilon)$ code for classical communication consists of a set $\{\rho_{A^n}^m\}_{m \in \mathcal{M}}$ of quantum states, which are called quantum codewords, and where $|\mathcal{M}| = M$. It also consists of a decoding POVM $\{\Lambda_{B^n}^m\}_{m \in \mathcal{M}}$ and satisfies the following condition:

$$\frac{1}{M} \sum_{m \in \mathcal{M}} \text{Tr}\{(I_{B^n} - \Lambda_{B^n}^m)\mathcal{N}_{A \rightarrow B}^\otimes n(\rho_{A^n}^m)\} \leq \varepsilon,$$  \hspace{1cm} (5.9)

which we interpret as saying that the average error probability is no larger than $\varepsilon$, when using the quantum codewords and decoding POVM described above. The non-asymptotic classical capacity of $\mathcal{N}_{A \rightarrow B}$, denoted by $C(\mathcal{N}_{A \rightarrow B},n,\varepsilon)$ is equal to the largest value of $\frac{1}{n} \log_2 M$ (bits per channel use) for which there exists an $(n,M,\varepsilon)$ code as described above.

Entanglement-assisted classical communication is defined similarly, but one allows for Alice and Bob to share an arbitrary quantum state $\Psi_{A'B'}$ before communication begins. Note that entanglement-assisted classical communication was considered for the asymptotic case in [41, 42, 43]. For positive integers $n$ and $M$, and $\varepsilon \in [0,1]$, an $(n,M,\varepsilon)$ code for entanglement-assisted classical communication consists of the resource state $\Psi_{A'B'}$, a set $\{\mathcal{E}_{A'n \rightarrow A'n}^m\}_{m \in \mathcal{M}}$ of encoding channels, where $|\mathcal{M}| = M$. It also consists of a decoding POVM $\{\Lambda_{B^n'B'}^m\}_{m \in \mathcal{M}}$ and satisfies the following condition:

$$\frac{1}{M} \sum_{m \in \mathcal{M}} \text{Tr}\{(I_{B^n'B'} - \Lambda_{B^n'B'}^m)\mathcal{N}_{A'n \rightarrow B'}(\mathcal{E}_{A'n \rightarrow A'n}^m(\Psi_{A'B'}))\} \leq \varepsilon,$$  \hspace{1cm} (5.10)

which we interpret as saying that the average error probability is no larger than $\varepsilon$, when using the entanglement-assisted code described above. The non-asymptotic entanglement-assisted classical capacity of $\mathcal{N}_{A \rightarrow B}$, denoted by $C_{EA}(\mathcal{N}_{A \rightarrow B},n,\varepsilon)$ is equal to the largest value of $\frac{1}{n} \log_2 M$ (bits per channel use) for which there exists an $(n,M,\varepsilon)$ entanglement-assisted code as described above.

5.3 Lower bound on second-order coding rate

Defining the $\varepsilon$-mutual information of a bipartite state $\tau_{CD}$ as

$$I^{\varepsilon}_{H}(C;D)_{\tau} \equiv D^{\varepsilon}_{H}(\tau_{CD}\|\tau_{C} \otimes \tau_{D}),$$

the following inequality was proven recently in [44, Theorem 8] for the finite-dimensional case, improving upon a prior result from [19]:

$$C_{EA}(\mathcal{N}_{A \rightarrow B},1,\varepsilon) \geq I^{\varepsilon - \eta}_{H}(R;B)_{\zeta} - \log_2(4\varepsilon/\eta^2),$$  \hspace{1cm} (5.12)

where $\varepsilon \in (0,1)$, $\eta \in (0,\varepsilon)$, $\zeta_{RB} \equiv \mathcal{N}_{A' \rightarrow B}(\rho_{RA})$, and $\rho_{RA}$ is a bipartite state. The techniques employed in the proof of [44, Theorem 8] were position-based coding [29] and the Hayashi–Nagaoka inequality [15, Lemma 2]. Note that the position-based coding method can be understood as a variation of the well known and studied coding technique called pulse position modulation [15, 46]. We now generalize the inequality in (5.12) to the infinite-dimensional case by applying Theorem 4 along with position-based coding [29] and the sequential decoding strategy from [28].
Theorem 6 Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_R$ be separable Hilbert spaces. Let $\mathcal{N}_{A\rightarrow B}$ be a quantum channel, taking $\mathcal{D}(\mathcal{H}_A)$ to $\mathcal{D}(\mathcal{H}_B)$. Then the following bound holds:

$$C_{\text{EA}}(\mathcal{N}_{A\rightarrow B}, 1, \varepsilon) \geq I^{\varepsilon-\eta}_H(R; B)_{\zeta} - \log_2(4\varepsilon/\eta^2),$$

(5.13)

where $\varepsilon \in (0, 1)$, $\eta \in (0, \varepsilon)$, $\zeta_{RB} \equiv \mathcal{N}_{A\rightarrow B}(\rho_{RA})$, and $\rho_{RA} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$ is a bipartite state.

Proof. Let $\Lambda_{RB}$ be a measurement operator (i.e., $0 \leq \Lambda_{RB} \leq I_{RB}$) satisfying

$$\text{Tr}\{(I_{RB} - \Lambda_{RB})\mathcal{N}_{A\rightarrow B}(\rho_{RA})\} \leq \varepsilon - \eta.$$

(5.14)

To this operator $\Lambda_{RB}$ is associated a unitary $U_{RB}$, defined as

$$U_{RB} \equiv \sqrt{I_{RB} - \Lambda_{RB} \otimes [0\langle 0|_p + |1\rangle\langle 1|_p] + \sqrt{\Lambda_{RB} \otimes [1\langle 1|_p - |0\rangle\langle 0|_p]}.}$$

(5.15)

Then defining the projectors

$$\Pi_{RB} \equiv U_{RB}^\dagger (I_{RB} \otimes |1\rangle\langle 1|_p) U_{RB},$$

$$\hat{\Pi}_{RB} \equiv I_{RB} - \Pi_{RB} = U_{RB}^\dagger (I_{RB} \otimes |0\rangle\langle 0|_p) U_{RB},$$

(5.16) (5.17)

the inequality in (5.13) and a simple calculation imply that

$$\text{Tr}\{(I_{RB} - \Pi_{RB})\mathcal{N}_{A\rightarrow B}(\rho_{RA}) \otimes |0\rangle\langle 0|_p\} \leq \varepsilon - \eta.$$

(5.18)

This kind of construction and equality is known as the Naimark extension theorem.

The position-based coding strategy then proceeds as follows. We let Alice and Bob share $M$ copies of the resource state $\rho_{RA}$, where Bob has the $R$ systems and Alice the $A$ systems. If Alice would like to transmit message $m \in M$, then she simply selects the $m$th $A$ system, and sends it through the channel $\mathcal{N}_{A\rightarrow B}$. The marginal state of Bob’s systems is then as follows:

$$\rho_{R_1 \otimes \cdots \otimes R_{m-1} \otimes A_{m \rightarrow B}}(\rho_{R_m A_m}) \otimes \rho_{R_{m+1} \otimes \cdots \otimes R_M}.$$

(5.19)

Bob then uses a sequential decoding strategy to determine which message Alice transmitted. He introduces $M$ auxiliary probe systems in the state $|0\rangle\langle 0|$, so that Bob’s overall state is now

$$\omega_{RMP} \equiv \rho_{R_1 \otimes \cdots \otimes R_{m-1} \otimes A_{m \rightarrow B}(\rho_{R_m A_m}) \otimes \rho_{R_{m+1} \otimes \cdots \otimes R_M} \otimes |0\rangle\langle 0|_P \otimes \cdots \otimes |0\rangle\langle 0|_P_M}.$$ (5.20)

He then performs the binary measurements $\{\Pi_{RB}, \hat{\Pi}_{RB}\}$ sequentially, in the order $i = 1, i = 2$, etc. With this strategy, the probability that he decodes the $m$th message correctly is given by

$$\text{Tr}\{\Pi_{R_m B_k} \hat{\Pi}_{R_{m-1} B_{m-1}} \cdots \hat{\Pi}_{R_1 B_1} \omega_{RMP} \hat{\Pi}_{R_1 B_1} \cdots \hat{\Pi}_{R_{m-1} B_{m-1}}\}.$$ (5.21)

Applying Theorem 1 we can bound the complementary (error) probability as

$$p_e(m) \equiv 1 - \text{Tr}\{\Pi_{R_m B_k} \hat{\Pi}_{R_{m-1} B_{m-1}} \cdots \hat{\Pi}_{R_1 B_1} \omega_{RMP} \hat{\Pi}_{R_1 B_1} \cdots \hat{\Pi}_{R_{m-1} B_{m-1}}\}$$

(5.22)

$$\leq (1 + c) \text{Tr}\{\hat{\Pi}_{R_m B_k} \omega_{RMP} \hat{\Pi}_{R_1 B_1} \cdots \hat{\Pi}_{R_{m-1} B_{m-1}}\} + (2 + c + c^{-1}) \sum_{i=1}^{m-1} \text{Tr}\{\Pi_{R_i B_i} \omega_{RMP}\}$$

(5.23)

$$= (1 + c) \text{Tr}\{(I_{RB} - \Lambda_{RB})\mathcal{N}_{A\rightarrow B}(\rho_{RA})\} + (2 + c + c^{-1}) (m - 1) \text{Tr}\{\Lambda_{RB} [\rho_R \otimes \mathcal{N}_{A\rightarrow B}(\rho_A)]\}$$

(5.24)

$$\leq (1 + c) (\varepsilon - \eta) + (2 + c + c^{-1}) M \text{Tr}\{\Lambda_{RB} [\rho_R \otimes \mathcal{N}_{A\rightarrow B}(\rho_A)]\},$$

(5.25)
where \( c > 0 \). Since the whole development above holds for all measurement operators \( \Lambda_{RB} \) satisfying (5.14), we can take an infimum over all of them to obtain the following uniform bound on the error probability when sending an arbitrary message \( m \in M \):

\[
P_e(m) \leq (1 + c)(\varepsilon - \eta) + (2 + c + c^{-1})M2^{-I_H^{\varepsilon-\eta}(R;B)\zeta},
\]

(5.26)

Picking \( c = \eta/(2\varepsilon - \eta) \) and taking \( M \) such that

\[
\log_2 M = I_H^{\varepsilon-\eta}(R;B)\zeta - \log_2(4\varepsilon/\eta^2)
\]

(5.27)

then implies that \( p_e(m) \leq \varepsilon \) for all \( m \in M \). Since we have shown the existence of a \((1, M, \varepsilon)\) entanglement-assisted code, where \( M \) satisfies (5.27), this concludes the proof.

Remark 7 It is worthwhile to note that the code above has an error probability less than \( \varepsilon \) for every message \( m \in M \), and so the error criterion is maximal error probability and not just average error probability.

The above result also implies rates that are achievable for unassisted classical communication, by combining Theorem 6 with an analysis nearly identical to that given in [24, Section 3.3]. In particular, we could allow Alice and Bob to share many copies of the following classical–quantum state before communication begins:

\[
\rho_{XA} \equiv \sum_{x \in \mathcal{X}} p(x)|x\rangle\langle x|_X \otimes \rho^x_A,
\]

(5.28)

where \( \mathcal{H}_X \) and \( \mathcal{H}_A \) are separable Hilbert spaces, \( \mathcal{X} \) is a countable index set, \( \{p(x)\}_{x \in \mathcal{X}} \) is a probability distribution, \( \{|x\rangle_X\}_{x \in \mathcal{X}} \) is a set of orthonormal states, and \( \{\rho^x_A\}_{x \in \mathcal{X}} \) is a set of states. This state then plays the role of the resource state \( \rho_{RA} \) in the proof of Theorem 4. However, the above state is a classical–quantum state, and as such, the code can be derandomized. Specifically, to do so, we can employ the analysis given in [24, Section 3.3], but replacing the square-root measurement there with the sequential decoding strategy. This leads to the following result, which generalizes one of the main results of [34] to the infinite-dimensional case:

Corollary 8 Let \( \mathcal{H}_A, \mathcal{H}_B, \) and \( \mathcal{H}_X \) be separable Hilbert spaces. Let \( N_{A \rightarrow B} \) be a quantum channel, taking \( \mathcal{D}(\mathcal{H}_A) \) to \( \mathcal{D}(\mathcal{H}_B) \). Then the following bound holds:

\[
C(N_{A \rightarrow B}, 1, \varepsilon) \geq I_H^{\varepsilon-\eta}(X;B)\zeta - \log_2(4\varepsilon/\eta^2),
\]

(5.29)

where \( \varepsilon \in (0, 1), \eta \in (0, \varepsilon), \zeta_{XB} \equiv N_{A \rightarrow B}(\rho_{XA}), \) and \( \rho_{XA} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_A) \) is a bipartite, classical–quantum state of the form in (5.28).

5.4 Energy constraints

It is common in the theory of communication over infinite-dimensional channels [30, 31, 47] to impose energy constraints on the space of inputs. If we do not so, then the capacities can be infinite. The definitions of these energy-constrained non-asymptotic capacities are the same as we discussed previously, except that we impose energy constraints on the channel input states.

Before defining them, let us first recall the notion of an energy observable [47, 48].
Definition 9 (Energy Observable) For a Hilbert space $\mathcal{H}$, let $G \in L_+(\mathcal{H})$ denote a positive semi-definite operator, defined in terms of its action on a vector $|\psi\rangle$ as

$$G|\psi\rangle = \sum_{j=1}^{\infty} g_j |e_j\rangle \langle e_j| \psi\rangle,$$

(5.30)

for $|\psi\rangle$ such that $\sum_{j=1}^{\infty} g_j |\langle e_j| \psi\rangle|^2 < \infty$. In the above, $\{|e_j\rangle\}$ is an orthonormal basis and $\{g_j\}$ is a sequence of non-negative, real numbers. Then $\{|e_j\rangle\}$ is an eigenbasis for $G$ with corresponding eigenvalues $\{g_j\}$.

Definition 10 The $n$th extension $G_n$ of an energy observable $G$ is defined as

$$G_n \equiv \frac{1}{n} [G \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes G],$$

(5.31)

where $n$ is the number of factors in each tensor product above.

Then the non-asymptotic, energy-constrained classical capacity $C(N_{A\rightarrow B}, G, P, n, \varepsilon)$ is defined exactly as it was previously in Section 5.2 except that we demand that

$$\frac{1}{M} \sum_{m \in M} \text{Tr}\{G_n^{m\rho^m_{A^m}}\} \leq P,$$

(5.32)

for a real $P \in [0, \infty)$. Similarly, the non-asymptotic, energy-constrained entanglement-assisted classical capacity $C_{EA}(N_{A\rightarrow B}, G, P, n, \varepsilon)$ is defined exactly as it was previously in Section 5.2 except that we demand that

$$\frac{1}{M} \sum_{m \in M} \text{Tr}\{(G_n \otimes I_{B'}) G_{m\rho^m_{A^m}}(\Psi_{A'B'})\} \leq P.$$

(5.33)

One could alternatively demand that the energy constraint hold for every codeword, not just on average. Note that we recover the usual notion of capacity (unconstrained) by taking $G = I$ and setting $P = 1$.

An advantage of the approach given in the proof of Theorem 6 is that we easily obtain a lower bound on the second-order coding rate for energy-constrained entanglement-assisted classical communication over a quantum channel:

Theorem 11 Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_R$ be separable Hilbert spaces. Let $\varepsilon \in (0, 1)$. Let $G$ be an energy observable, and let $P \in [0, \infty)$. Let $N_{A\rightarrow B}$ be a quantum channel, taking $\mathcal{D}(\mathcal{H}_A)$ to $\mathcal{D}(\mathcal{H}_B)$. Then the following bound holds:

$$C_{EA}(N_{A\rightarrow B}, G, P, n, \varepsilon) \geq I(R; B)_{\zeta} + \sqrt{\frac{1}{n} V(R; B)_{\zeta}} \Phi^{-1}(\varepsilon) + O\left(\frac{1}{n} \log n\right),$$

(5.34)

where $\zeta_{RB} \equiv N_{A\rightarrow B}(\rho_{RA})$ and $\rho_{RA} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$ is a bipartite state such that

$$I(R; B)_{\zeta}, V(R; B)_{\zeta}, T(R; B)_{\zeta} < \infty, \quad V(R; B)_{\zeta} > 0,$$

(5.35)

and $\text{Tr}\{G_{\rho_{A}}\} \leq P$. In the above, we have the mutual information, mutual information variance, and another quantity:

$$I(R; B)_{\zeta} \equiv D(\zeta_{RB}||\zeta_{R\otimes B}), \quad V(R; B)_{\zeta} \equiv V(\zeta_{RB}||\zeta_{R\otimes B}), \quad T(R; B)_{\zeta} \equiv T(\zeta_{RB}||\zeta_{R\otimes B}).$$

(5.36)
Proof. Let $\zeta_{RA}$ be a state satisfying the conditions stated above. Then the claim follows by applying Theorem 6, picking $\eta = 1/\sqrt{n}$, and invoking the expansion in (5.7).

The proof given above is quite simple once all of the relevant components are in place (namely, the quantum union bound from Theorem 11, position-based coding [29], and the expansion in (5.7)). This is to be contrasted with the approach taken in [30, 31], in which the energy-constrained entanglement-assisted classical capacity was identified. Not only can we argue to have a simpler approach for the achievability part, but our method also delivers a lower bound on the second-order coding rate. An important open question remaining however is to determine whether this lower bound on the second-order coding rate is tight. To our knowledge, this tightness has only been shown for finite-dimensional channels that are covariant [19].

We note here that the bound in Theorem 11 applies to the practically relevant case of bosonic Gaussian channels [14]. The energy-constrained entanglement-assisted classical capacity of these channels was identified in [49, 30, 50, 31]. The additive-noise, thermal, and amplifier channels are of major interest for applications, as stressed in [31, 47]. It is known that the energy-constrained, entanglement-assisted capacity formula for these channels is achieved by a two-mode squeezed vacuum state, whose reduction to the channel input system has an average photon number meeting the desired photon number constraint. Thus, we could evaluate the lower bound in Theorem 11 by taking $\rho_{RA}$ therein to be the two-mode squeezed vacuum and then applying the formula from [52] to evaluate the mutual information variance $V(R; B)_\zeta$, while noting that the quantity $T(R; B)_\zeta$ is finite for any finite-energy state, as proven in [37, Appendix D].

We end this section on a different note, by remarking that the same argument as above gives a non-trivial lower bound on the second-order coding rate for energy-constrained classical communication with randomness assistance. However, it remains open to determine whether this rate is achievable without the assistance of randomness. It is also open to extend the result to a continuous (uncountable) index set $X$. We suspect that these extensions should be possible but leave it for future work.

6 Conclusion

In this paper, we proved Theorem 11 which improves Gao’s quantum union bound to include a tunable parameter $c > 0$ that plays a role similar to the tunable parameter available in the Hayashi–Nagaoka inequality from [15, Lemma 2]. An advantage of the proof of Theorem 11 is that it is elementary, relying only on basic properties of projectors, the Pythagorean theorem, and the Cauchy–Schwarz inequality. Due to our improvement, the quantum union bound can now be employed in a wider variety of scenarios. As an example application, we showed how to achieve a lower bound on the second-order coding rate for classical communication over a quantum channel by employing a sequential decoding strategy.

For future directions, we think it would be interesting to determine whether the improved bound in Theorem 11 would find application in areas including quantum algorithms [8, 9, 10], quantum complexity theory [9, 11], and Hamiltonian complexity theory [12, 13]. We also wonder whether Theorem 11 could be useful outside of quantum information, for example in the analysis of projection algorithms.
A Proof of the inequality $\geq$ in Eq. (5.1)

The goal of this appendix is to prove the inequality $\geq$ in (5.1). The proof follows the development in Appendix C of [35] very closely, which is in turn based on [36, Section 3.2].

Consider quantum states $\rho$ and $\sigma$ acting on a separable Hilbert space $\mathcal{H}$, with spectral decompositions as given in (5.1). Observe that each $P_x$ is finite-dimensional, as a consequence of the fact that $\rho$ is trace class. Indeed, were it not the case, then $\text{Tr}\{P_x\}$ would be infinite, and $\rho$ could thus not be trace class. By the same reasoning, each $Q_y$ is finite-dimensional.

By defining a random variable $Z$ taking values $\log_2(\lambda_x/\mu_y)$ with probability $p(x,y) \equiv \lambda_x \text{Tr}\{P_x Q_y\}$, observe that

$$D(\rho\|\sigma) = \mathbb{E}\{Z\}, \quad V(\rho\|\sigma) = \text{Var}\{Z\}, \quad T(\rho\|\sigma) = \mathbb{E}\left\{ |Z - \mathbb{E}\{Z\}|^3 \right\},$$

where $D(\rho\|\sigma)$, $V(\rho\|\sigma)$, and $T(\rho\|\sigma)$ are defined in (5.3)-(5.5).

**Lemma 12** Let $\rho$ and $\sigma$ denote states acting on a separable Hilbert space $\mathcal{H}$. Let $L > 0$. Then there exists a measurement operator $T_L$ (i.e., $0 \leq T_L \leq I$) such that

$$\text{Tr}\{T_L \rho\} \geq \text{Pr}\{Z \geq \log_2 L\}, \quad \text{Tr}\{T_L \sigma\} \leq \frac{1}{L},$$

where $Z$ is the random variable defined just before (A.1).

**Proof.** Let us define the positive semi-definite operator $\tilde{T}_L$ as

$$\tilde{T}_L \equiv \sum_{x,y: L \leq \lambda_x/\mu_y} Q_y P_x Q_y.$$  \hspace{1cm} (A.3)

By inspection, this operator is positive semi-definite. The measurement operator $T_L$ is then defined to be the projection onto the support of $\tilde{T}_L$. Let $|\psi\rangle$ be a unit vector such that $P_x |\psi\rangle = |\psi\rangle$ for some $x$. It follows that $|\langle \psi |\psi\rangle| \leq P_x$. Then, for any $\mu_y$ such that $L \leq \lambda_x/\mu_y$, we have, from the definition of $\tilde{T}_L$, that $|\langle \psi |\psi\rangle| \leq \tilde{T}_L$. This in turn implies that $Q_y |\psi\rangle \in \text{supp}(\tilde{T}_L)$. From this, we then conclude that

$$\frac{Q_y |\langle \psi |\psi\rangle| Q_y}{\|Q_y |\psi\rangle\|^2} \leq T_L.$$  \hspace{1cm} (A.4)

Furthermore, we have that $\{Q_y |\psi\rangle\}_y$ forms a family of orthogonal vectors. Then the following inequality holds

$$\sum_{y: L \leq \lambda_x/\mu_y} \frac{Q_y |\langle \psi |\psi\rangle| Q_y}{\|Q_y |\psi\rangle\|^2} \leq T_L.$$  \hspace{1cm} (A.5)

From this, we conclude that

$$\text{Tr}\{T_L |\psi\rangle \langle \psi|\} \geq \text{Tr}\left\{ \sum_{y: L \leq \lambda_x/\mu_y} \frac{Q_y |\langle \psi |\psi\rangle| Q_y}{\|Q_y |\psi\rangle\|^2} |\psi\rangle \langle \psi| \right\} = \sum_{y: L \leq \lambda_x/\mu_y} \text{Tr}\left\{ \frac{Q_y |\langle \psi |\psi\rangle| Q_y}{\|Q_y |\psi\rangle\|^2} |\psi\rangle \langle \psi| \right\}$$

$$= \sum_{y: L \leq \lambda_x/\mu_y} \frac{\|Q_y |\psi\rangle\|^4}{\|Q_y |\psi\rangle\|^2} = \sum_{y: L \leq \lambda_x/\mu_y} \|Q_y |\psi\rangle\|^2 = \sum_{y: L \leq \lambda_x/\mu_y} \text{Tr}\{Q_y |\psi\rangle \langle \psi| Q_y\}.$$  \hspace{1cm} (A.6)

15
Now, recall that $P_x$ is a finite-dimensional projector for each $x$. (As stated above, if $P_x$ were not finite-dimensional, then this would contradict the assumption that $\rho$ is trace class.) Furthermore, we can write it as $P_x = \sum_{i=1}^{\text{Tr}(P_x)} |\psi_{x,i}\rangle\langle \psi_{x,i}|$, for some orthonormal set $\{|\psi_{x,i}\rangle\}_{i=1}^{\text{Tr}(P_x)}$. Then the development in \[(A.6)\]–\[(A.7)\] implies that
\[
\text{Tr}\{T_L P_x\} \geq \sum_{y: L \leq \lambda_x / \mu_y} \text{Tr}\{Q_y P_x Q_y\} = \sum_{y: L \leq \lambda_x / \mu_y} \text{Tr}\{Q_y P_x\}. \quad (A.8)
\]
We can then use this to conclude that
\[
\text{Tr}\{T_L \rho\} = \sum_x \lambda_x \text{Tr}\{T_L P_x\} \geq \sum_{x: y: L \leq \lambda_x / \mu_y} \lambda_x \text{Tr}\{Q_y P_x\} = \sum_{x: y: \log_2 L \leq \log_2(\lambda_x / \mu_y)} \lambda_x \text{Tr}\{Q_y P_x\} = \text{Pr}\{Z \geq \log_2 L\}, \quad (A.9)
\]
where the second equality uses the fact that $\log_2 : (0, \infty) \to (-\infty, \infty)$ is invertible, and the last line follows from the definition of the random variable $Z$, given just before \[(A.1)\].

What remains is to place an upper bound on $\text{Tr}\{T_L \sigma\}$. Observe that for all $x$ and $y$, the following equivalence holds
\[
\text{ran}(Q_y P_x) = \text{supp}(Q_y P_x), \quad (A.11)
\]
where ran denotes the range of an operator. For some $x$, define the following subspace:
\[
\tilde{S}_x \equiv \bigvee_{y: L \leq \lambda_x / \mu_y} \text{ran}(Q_y P_x) = \bigvee_{y: L \leq \lambda_x / \mu_y} \text{supp}(Q_y P_x), \quad (A.12)
\]
where the operation $\vee$ realizes a space formed as the union of subspaces. Due to the fact that $P_x$ is finite-dimensional, it follows that the subspace $\tilde{S}_x$ is finite-dimensional. We now employ a Gram–Schmidt orthogonalization procedure for these subspaces. First order the eigenvalues of $\rho$ as $\lambda_1 > \lambda_2 > \ldots$. Now define a family $\{S_x\}_x$ of subspaces of the whole Hilbert space $\mathcal{H}$ as
\[
S_1 = \tilde{S}_1, \quad S_x = \left( \bigvee_{i=1}^{x-1} \tilde{S}_i \right) \land \left( \bigvee_{i=1}^{x-1} \tilde{S}_i \right)^\perp \text{ for } x \geq 2, \quad (A.13)
\]
where the operation $\land$ corresponds to the intersection of subspaces. The subspaces $S_x$ are mutually orthogonal by construction. Furthermore, by the procedure given above, the following holds for any positive integer $w \geq 1$:
\[
\bigvee_{x=1}^{w} \tilde{S}_x = \bigvee_{x=1}^{w} S_x. \quad (A.14)
\]
By definition, $T_L$ is the projection onto the following subspace:
\[
\bigvee_{x, y: L \leq \lambda_x / \mu_y} \text{supp}(Q_y P_x Q_y) = \bigvee_{y: L \leq \lambda_x / \mu_y} \text{ran}(Q_y P_x) = \bigvee_{x} \tilde{S}_x = \bigvee_{x} S_x = \bigoplus_{x} S_x. \quad (A.15)
\]
Thus, it follows that $T_L$ can be written as $T_L = \sum_x P_{S_x}$, where $P_{S_x}$ is the projection onto $S_x$. By the procedure given above, we have that $S_x \subseteq \tilde{S}_x$, and from the definition of $\tilde{S}_x$, we find that
\[
\text{Tr}\{P_{S_x}\} \leq \text{Tr}\{P_{\tilde{S}_x}\} \leq \text{Tr}\{P_x\}. \quad (A.16)
\]
We then find that
\[
\text{Tr}\{T_L\sigma\} = \sum_{y,z} \mu_y \text{Tr}\{Q_y P_{xz}\} = \sum_{y,z} \mu_y \text{Tr}\{Q_y P_{xz}\} \quad (A.17)
\]
\[
\leq \frac{1}{L} \sum_{y,z} \lambda_x \text{Tr}\{Q_y P_{xz}\} \leq \frac{1}{L} \sum_{y,z} \lambda_x \text{Tr}\{Q_y P_{xz}\} \quad (A.18)
\]
\[
= \frac{1}{L} \sum_{x} \lambda_x \text{Tr}\{P_x\} = \frac{1}{L} \sum_{x} \lambda_x \text{Tr}\{P_x\} = \frac{1}{L} \text{Tr}\{\rho\} = \frac{1}{L}. \quad (A.19)
\]

In the above, the second equality follows because \(Q_y P_{xz} = 0\) unless \(L \leq \lambda_x / \mu_y\) (from the definition of the space \(S_x\) and the fact that \(S_x \subseteq \overline{S}_x\)). The third equality follows from \(\sum_y Q_y = I\), and the third inequality follows from (A.16). \(\blacksquare\)

We now apply the above lemma to the i.i.d. states \(\rho^{\otimes n}\) and \(\sigma^{\otimes n}\), with spectral decompositions
\[
\rho^{\otimes n} = \sum_{x^n} \lambda_{x^n} P_{x^n}, \quad \sigma^{\otimes n} = \sum_{y^n} \mu_{y^n} Q_{y^n}, \quad (A.20)
\]
where \(x^n = (x_1, \ldots, x_n)\), \(y^n = (y_1, \ldots, y_n)\), \(\lambda_{x^n} = \lambda_{x_1} \times \cdots \times \lambda_{x_n}\), \(\mu_{y^n} = \mu_{y_1} \times \cdots \times \mu_{y_n}\), \(P_{x^n} = P_{x_1} \otimes \cdots \otimes P_{x_n}\), and \(Q_{y^n} = Q_{y_1} \otimes \cdots \otimes Q_{y_n}\). Then the i.i.d. random sequence \(Z^n \equiv (Z_1, \ldots, Z_n)\) takes the values
\[
\log_2 \left( \frac{\lambda_{x^n}}{\mu_{y^n}} \right) = \sum_{i=1}^{n} \log_2 \left( \frac{\lambda_{x_i}}{\mu_{y_i}} \right), \quad (A.21)
\]
with probability
\[
\rho(x^n, y^n) = \lambda_{x^n} \text{Tr}\{P_{x^n}Q_{y^n}\} = \prod_{i=1}^{n} \lambda_{x_i} \text{Tr}\{P_{x_i}Q_{y_i}\}. \quad (A.22)
\]

The Berry–Esseen theorem \([53, 54]\) states that if \(A_1, \ldots, A_n\) are i.i.d. random variables such that \(\mathbb{E}\{A_1\} = 0\), \(\mathbb{E}\{|A_1|^2\} \equiv \tau^2 \in (0, \infty)\), and \(\mathbb{E}\{|A_1|^3\} \equiv \omega < \infty\), then
\[
|\text{Pr}\{B_n \sqrt{n}/\tau \leq x\} - \Phi(x)| \leq \frac{C \omega}{\tau^3 \sqrt{n}}, \quad (A.23)
\]
where \(x \in \mathbb{R}\), \(\Phi(x) \equiv [2\pi]^{-1/2} \int_{-\infty}^{x} dy \exp(-y^2/2)\), \(B_n \equiv \frac{1}{n} \sum_{i=1}^{n} A_i\), and \(C\) is the Berry–Esseen constant satisfying \(0.40973 \leq C \leq 0.4784\).

**Proposition 13** Let \(\rho\) and \(\sigma\) denote states acting on a separable Hilbert space \(\mathcal{H}\). Suppose that \(D(\rho||\sigma), V(\rho||\sigma), T(\rho||\sigma) < \infty\) and \(V(\rho||\sigma) > 0\). Suppose \(n\) is sufficiently large such that \(\varepsilon - C \cdot T(\rho||\sigma)/\sqrt{n [V(\rho||\sigma)]^3} > 0\). Then
\[
D_{H}^{\otimes n}(\rho^{\otimes n}||\sigma^{\otimes n}) \geq nD(\rho||\sigma) + \sqrt{nV(\rho||\sigma)}\Phi^{-1}\left(\varepsilon - C \cdot T(\rho||\sigma)/\sqrt{n [V(\rho||\sigma)]^3}\right) \quad (A.24)
\]
\[
= nD(\rho||\sigma) + \sqrt{nV(\rho||\sigma)}\Phi^{-1}(\varepsilon) + O(1). \quad (A.25)
\]
Proof. Applying the Berry–Esseen theorem to the random sequence \( Z_1 - D(\rho\|\sigma), \ldots, Z_n - D(\rho\|\sigma) \), with \( Z_i \) defined in (A.21)–(A.22), we find that

\[
\left| \Pr \left\{ \frac{\sum_{i=1}^{n} Z_i}{\sqrt{V(\rho\|\sigma)}} \leq x \right\} - \Phi(x) \right| \leq C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3},
\]

(A.26)

where \( Z^n \equiv \frac{1}{n} \sum_{i=1}^{n} \left[ Z_i - D(\rho\|\sigma) \right] \), which implies that

\[
\Pr \left\{ \sum_{i=1}^{n} Z_i \leq nD(\rho\|\sigma) + x\sqrt{nV(\rho\|\sigma)} \right\} \leq \Phi(x) + C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3}.
\]

(A.27)

Picking \( x = \Phi^{-1} \left( \varepsilon - C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3} \right) \), this becomes

\[
\Pr \left\{ \sum_{i=1}^{n} Z_i \leq nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)} \Phi^{-1} \left( \varepsilon - C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3} \right) \right\} \leq \varepsilon.
\]

(A.28)

Choosing \( L \) such that

\[
\log_2 L = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)} \Phi^{-1} \left( \varepsilon - C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3} \right)
\]

(A.29)

and applying Lemma 12, we find that

\[
\text{Tr}\{T^n \rho \otimes^n\} \geq \Pr \left\{ \sum_{i=1}^{n} Z_i \geq \log_2 L \right\} = 1 - \Pr \left\{ \sum_{i=1}^{n} Z_i \leq \log_2 L \right\} \geq 1 - \varepsilon,
\]

while

\[
\text{Tr}\{T^n \sigma \otimes^n\} \leq \frac{1}{L} = e^{-\left[nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)} \Phi^{-1} \left( \varepsilon - C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3} \right)\right]}.
\]

(A.30)

This implies that

\[
- \log_2 \text{Tr}\{T^n \sigma \otimes^n\} \geq nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)} \Phi^{-1} \left( \varepsilon - C \cdot T(\rho\|\sigma)/\sqrt{n \cdot V(\rho\|\sigma)^3} \right).
\]

(A.31)

Since \( D_H(\rho \otimes^n \| \sigma \otimes^n) \) involves an optimization over all possible measurement operators \( T^n \) satisfying \( \text{Tr}\{T^n \rho \otimes^n\} \geq 1 - \varepsilon \), we conclude the bound in (A.24). The equality after (A.24) follows from expanding \( \Phi^{-1} \) at the point \( \varepsilon \) using Lagrange’s mean value theorem.

Ethics statement. This work did not involve any active collection of human data, and it did not involve animals.

Data accessibility statement. This work does not have any experimental data.

Competing interests statement. We have no competing interests.

Authors’ contributions. All authors contributed equally toward proving the main results and writing the paper.
Acknowledgements. MMW is grateful to SM for hosting him for a research visit to University of Camerino in June and July, 2016, during which this research project was initiated. SM is grateful to MMW for hosting him for a research visit to Louisiana State University at Baton Rouge in March, 2018, during which this research project was completed.

Funding. MMW acknowledges support from the US National Science Foundation under grant no. 1714215.

References

[1] Y. A. Rozanov. Probability Theory: A Concise Course. Dover Books on Mathematics. Dover Publications, 1977.

[2] Scott Aaronson. How to upper-bound the probability of something bad. https://www.scottaaronson.com/blog/?p=3712, April 2018.

[3] Jingliang Gao. Quantum union bounds for sequential projective measurements. Physical Review A, 92(5):052331, November 2015. arXiv:1410.5688.

[4] Christoph Hirche. Polar codes in quantum information theory. Master’s thesis, Leibniz Universität Hannover, January 2015. arXiv:1501.03737.

[5] Holger Boche, Ning Cai, and Janis Nötzel. The classical-quantum channel with random state parameters known to the sender. Journal of Physics A: Mathematical and Theoretical, 49(19):195302, May 2016. arXiv:1506.06479.

[6] Joseph M. Renes. Belief propagation decoding of quantum channels by passing quantum messages. New Journal of Physics, 19(7):072001, July 2017. arXiv:1607.04833.

[7] Hayata Yamasaki, Akihito Soeda, and Mio Murao. Graph-associated entanglement cost of a multipartite state in exact and finite-block-length approximate constructions. Physical Review A, 96(3):032330, September 2017. arXiv:1705.00006.

[8] Ashley Montanaro and Ronald de Wolf. A Survey of Quantum Property Testing. Number 7 in Graduate Surveys. Theory of Computing Library, July 2016. arXiv:1310.2035.

[9] Scott Aaronson. The complexity of quantum states and transformations: From quantum money to black holes. July 2016. arXiv:1607.05256.

[10] Aram W. Harrow, Cedric Yen-Yu Lin, and Ashley Montanaro. Sequential measurements, disturbance and property testing. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1598–1611, 2017. arXiv:1607.03236.

[11] Adam Bene Watts, Aram W. Harrow, Gurtej Kanwar, and Anand Natarajan. Algorithms, bounds, and strategies for entangled XOR games. January 2018. arXiv:1801.00821.

[12] Anurag Anshu, Itai Arad, and Thomas Vidick. Simple proof of the detectability lemma and spectral gap amplification. Physical Review B, 93(20):205142, May 2016. arXiv:1602.01210.
[13] Michael J. Kastoryano and Angelo Lucia. Divide and conquer method for proving gaps of frustration free Hamiltonians. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(3):033105, March 2018. arXiv:1705.09491.

[14] Alessio Serafini. *Quantum Continuous Variables*. CRC Press, 2017.

[15] Masahito Hayashi and Hiroshi Nagaoka. General formulas for capacity of classical-quantum channels. *IEEE Transactions on Information Theory*, 49(7):1753–1768, July 2003. arXiv:quant-ph/0206186.

[16] Marco Tomamichel and Masahito Hayashi. A hierarchy of information quantities for finite block length analysis of quantum tasks. *IEEE Transactions on Information Theory*, 59(11):7693–7710, November 2013. arXiv:1208.1478.

[17] Marco Tomamichel and Vincent Y. F. Tan. Second-order asymptotics for the classical capacity of image-additive quantum channels. *Communications in Mathematical Physics*, 338(1):103–137, August 2015. arXiv:1308.6503.

[18] Nilanjana Datta and Felix Leditzky. Second-order asymptotics for source coding, dense coding, and pure-state entanglement conversions. *IEEE Transactions on Information Theory*, 61(1):582–608, January 2015. arXiv:1403.2543.

[19] Nilanjana Datta, Marco Tomamichel, and Mark M. Wilde. On the second-order asymptotics for entanglement-assisted communication. *Quantum Information Processing*, 15(6):2569–2591, June 2016. arXiv:1405.1797.

[20] Mark M. Wilde, Joseph M. Renes, and Saikat Guha. Second-order coding rates for pure-loss bosonic channels. *Quantum Information Processing*, 15(3):1289–1308, March 2016. arXiv:1408.5328.

[21] Marco Tomamichel, Mario Berta, and Joseph M. Renes. Quantum coding with finite resources. *Nature Communications*, 7:11419, May 2016. arXiv:1504.04617.

[22] Felix Leditzky. *Relative entropies and their use in quantum information theory*. PhD thesis, University of Cambridge, Girton College, November 2016. arXiv:1611.08802.

[23] Mark M. Wilde, Marco Tomamichel, and Mario Berta. Converse bounds for private communication over quantum channels. *IEEE Transactions on Information Theory*, 63(3):1792–1817, March 2017. arXiv:1602.08898.

[24] Mark M. Wilde. Position-based coding and convex splitting for private communication over quantum channels. *Quantum Information Processing*, 16(10):264, October 2017. arXiv:1703.01733.

[25] Scott Aaronson. QMA/qpoly ⊆ PSPACE/poly: de-Merlinizing quantum protocols. In *Twenty-First Annual IEEE Conference on Computational Complexity*, page 261273, Prague, Czech Republic, July 2006. IEEE. arXiv:quant-ph/0510230.

[26] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Achieving the Holevo bound via sequential measurements. *Physical Review A*, 85:012302, January 2012. arXiv:1012.0386.
[27] Pranab Sen. Achieving the Han-Kobayashi inner bound for the quantum interference channel by sequential decoding. September 2011. arXiv:1109.0802.

[28] Mark M. Wilde. Sequential decoding of a general classical-quantum channel. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 469(2157), September 2013. arXiv:1303.0808.

[29] Anurag Anshu, Rahul Jain, and Naqueeb Ahmad Warsi. One shot entanglement assisted classical and quantum communication over noisy quantum channels: A hypothesis testing and convex split approach. February 2017. arXiv:1702.01940.

[30] Alexander S. Holevo. Entanglement-assisted capacity of constrained channels. Proceedings of SPIE, First International Symposium on Quantum Informatics, 5128:62–69, July 2003. arXiv:quant-ph/0211170.

[31] Alexander S. Holevo. Entanglement-assisted capacities of constrained quantum channels. Theory of Probability & Its Applications, 48(2):243–255, July 2004. arXiv:quant-ph/0211170.

[32] Alexander S. Holevo. Probabilistic and Statistical Aspects of Quantum Theory. Scuola Normale Superiore Pisa, 2011.

[33] Francesco Buscemi and Nilanjana Datta. The quantum capacity of channels with arbitrarily correlated noise. IEEE Transactions on Information Theory, 56(3):1447–1460, March 2010. arXiv:0902.0158.

[34] Ligong Wang and Renato Renner. One-shot classical-quantum capacity and hypothesis testing. Physical Review Letters, 108(20):200501, May 2012. arXiv:1007.5456.

[35] Göran Lindblad. Entropy, information and quantum measurements. Communications in Mathematical Physics, 33(4):305–322, December 1973.

[36] Ke Li. Second order asymptotics for quantum hypothesis testing. Annals of Statistics, 42(1):171–189, February 2014. arXiv:1208.1400.

[37] Eneet Kaur and Mark M. Wilde. Upper bounds on secret key agreement over lossy thermal bosonic channels. Physical Review A, 96(6):062318, December 2017. arXiv:1706.04590.

[38] Nilanjana Datta, Yan Pautrat, and Cambyse Rouzé. Second-order asymptotics for quantum hypothesis testing in settings beyond i.i.d. - quantum lattice systems and more. Journal of Mathematical Physics, 57(6):062207, June 2016. arXiv:1510.04682.

[39] Alexander S. Holevo. The capacity of the quantum channel with general signal states. IEEE Transactions on Information Theory, 44(1):269–273, January 1998. arXiv:quant-ph/9611023.

[40] Benjamin Schumacher and Michael D. Westmoreland. Sending classical information via noisy quantum channels. Physical Review A, 56(1):131–138, July 1997.

[41] Charles H. Bennett, Peter W. Shor, John A. Smolin, and Ashish V. Thapliyal. Entanglement-assisted classical capacity of noisy quantum channels. Physical Review Letters, 83(15):3081–3084, October 1999. arXiv:quant-ph/9904023.
[42] Charles H. Bennett, Peter W. Shor, John A. Smolin, and Ashish V. Thapliyal. Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem. *IEEE Transactions on Information Theory*, 48(10):2637–2655, October 2002. arXiv:quant-ph/0106052.

[43] Alexander S. Holevo. On entanglement assisted classical capacity. *Journal of Mathematical Physics*, 43(9):4326–4333, September 2002. arXiv:quant-ph/0106075.

[44] Haoyu Qi, Qingle Wang, and Mark M. Wilde. Applications of position-based coding to classical communication over quantum channels. *Journal of Physics A*, 51(44):444002, November 2018. arXiv:1704.01361.

[45] Sergio Verdu. On channel capacity per unit cost. *IEEE Transactions on Information Theory*, 36(5):1019–1030, 1990.

[46] G. Cariolaro and T. Erseghe. *Pulse Position Modulation*. John Wiley & Sons, Inc., 2003.

[47] Alexander S. Holevo. *Quantum systems, channels, information: a mathematical introduction*, volume 16. Walter de Gruyter, 2012.

[48] Alexander S. Holevo and Maksim E. Shirokov. On the entanglement-assisted classical capacity of infinite-dimensional quantum channels. *Problems of Information Transmission*, 49(1):15–31, January 2013. arXiv:1210.6926.

[49] Alexander S. Holevo and Reinhard F. Werner. Evaluating capacities of bosonic Gaussian channels. *Physical Review A*, 63(3):032312, February 2001. arXiv:quant-ph/9912067.

[50] Vittorio Giovannetti, Seth Lloyd, Lorenzo Maccone, and Peter W. Shor. Entanglement assisted capacity of the broadband lossy channel. *Physical Review Letters*, 91(4):047901, July 2003. arXiv:quant-ph/0304020.

[51] Alexander S. Holevo and Vittorio Giovannetti. Quantum channels and their entropic characteristics. *Reports on Progress in Physics*, 75(4):046001, April 2012. arXiv:1202.6480.

[52] Mark M. Wilde, Marco Tomamichel, Seth Lloyd, and Mario Berta. Gaussian hypothesis testing and quantum illumination. *Physical Review Letters*, 119(12):120501, September 2017. arXiv:1608.06991.

[53] V. Yu. Korolev and Irina G. Shevtsova. On the upper bound for the absolute constant in the Berry-Esseen inequality. *Theory of Probability & Its Applications*, 54(4):638–658, November 2010.

[54] Irina Shevtsova. On the absolute constants in the Berry-Esseen type inequalities for identically distributed summands. November 2011. arXiv:1111.6554.