1. Introduction

Let $X$ be a complete metric space with metric denoted by $d$. Albeit such abstract scenery, best motivated examples come from and lie in Euclidean spaces with a possible prospect for Hilbert and Banach spaces.

Recall from the fractal geometry that a system $\Phi = (X; f_i, i = 1, \ldots, N)$ of maps $f_i : X \to X$ is called an iterated function system, shortly IFS \cite{IFS}. We assume that the maps $f_i$ are nonexpansive:

$$\forall x_1, x_2 \in X \quad d(f_i(x_1), f_i(x_2)) \leq d(x_1, x_2).$$

This allows for situations where neither a strict attractor \cite{8} nor a Lasota-Myjak semiattractor \cite{29} exists, despite the fact that such attractors may be present in permanently noncontractive cases (cf. \cite{7}). However the nature of basic examples which took our attention and sparkled this research justifies the assumptions we make here and add further. To avoid a mystery: we are interested in invariant sets rather than attractors of IFSs. Concerning the existence of invariant sets one can assure it under very mild dissipativity conditions even in the absence of continuity of actions, see e.g. the references in \cite{30} or \cite{23}. So let us refine our goal.

The study of systems of contractions (and relatives) became a standard topic of several books, e.g., \cite{4, 18, 32, 31, 28} to mention only a small portion of literature. It is known also quite a lot about the dynamics of a single nonexpansive map and omega-limit sets, see \cite{16, 37, 11, 21, 36, 38, 17}. However it is still visibly less known about iterated systems of nonexpansive maps. Frankly the idea of a nonexpansive IFS appears in different disguise shortly before 1939 in the context of linear algebra and functional analysis. This is the method of projections (introduced by von Neumann and Kaczmarz). This topic also received a great attention \cite{9, 11, 20, 19, 13, 12}, again, as with the fractal geometry, due to its applicability potential. Yet for the full understanding and synthesis of the projection method some gaps await its explanation. One, towards which we aspire, reads as follows: what happens when we use projections onto sets with empty intersection? In the case of two sets one easily finds out that the iteration recovers the two nearest points in the sets realizing the (infimum) distance between
these sets (11.4.3). Our goal is to show, under very weak contractivity conditions (allowing for orthogonal projections among others), a general principle that the iterated maps recover a minimal invariant set in the sense that the omega-limit set of the generated orbit constitutes an invariant set. (Note that for \( N = 1 \), a single map, this is an elementary exercise). Related results together with a panorama of examples can be found in [3].

Given an IFS \((X; f_i, i = 1, \ldots, N)\) we define the Hutchinson operator \( \Phi : 2^X \setminus \{\emptyset\} \to 2^X \setminus \{\emptyset\} \) via

\[
\forall S \in 2^X \setminus \{\emptyset\} \quad \Phi(S) := \bigcup_{i=1}^N f_i(S),
\]

where \( 2^X \setminus \{\emptyset\} \) stands for the family of nonempty subsets of \( X \). (Note that we do not take the closure of the union in this variant of the definition).

A nonempty \( S \subset X \) is said to be an invariant set for the system of maps \( \{f_1, \ldots, f_N\} \) when \( \Phi(S) = S \), and a subinvariant set when \( \Phi(S) \subset S \). Traditional dynamics set focus on compact invariant sets (see however [32] for a reasonable deviation from this rule). This is the case here too, but keeping a general definition will prove handy in the next Section.

By an orbit \((x_n)_{n=0}^\infty\) starting at \( x_0 \in X \) with a driving sequence of symbols \((i_n)_{n=1}^\infty \in \{1, \ldots, N\}^\infty \) we understand

\[
x_n := f_{i_n} \circ \ldots \circ f_{i_1}(x_0).
\]

Such iterations are fundamental for some numerical methods in fractal geometry (chaos game algorithm [4, 28]) and convex geometry (cyclic projection algorithm [11, 19]). The driving sequence may be called a driver ([33]) or a control sequence ([9]); by a slight abuse, against the direction of compositions of maps, a code or an address ([4, 23]) might be accepted too. If the process generating symbols is stochastic, then the terms ‘random driver’ and ‘random orbit’ are justified. However random driver can mean a sequence where each symbol repeats infinitely often ([9]; repetitive below) and ‘random orbit’ can stand for an orbit driven by sufficiently complex deterministic sequence of symbols; with this respect also ‘chaotic orbit’ is in use ([4, 6]).

We list below various types of drivers ([9, 6, 15]). A sequence \((i_n)_{n=1}^\infty \in \{1, \ldots, N\}^\infty \) is called

- **cyclic**, if \( i_n = \pi((n-1) \mod N + 1) \), for all \( n \geq 1 \) under some fixed permutation \( \pi \) of \( \{1, \ldots, N\} \),
- **repetitive**, if for every \( \sigma \in \{1, \ldots, N\} \) the set \( \{n \geq 1 : i_n = \sigma\} \) is infinite,
- **disjunctive**, if it contains every possible finite word as its subword, namely for all \( m \geq 1 \) and every word \((\sigma_1, \ldots, \sigma_m) \in \).
$\{1, \ldots, N\}^m$ there exists $n_0 \geq 1$ s.t. $i_{n_0-1} + j = \sigma_j$ for $j = 1, \ldots, m$.

A disjunctive driver and a cyclic driver are necessarily repetitive but the reverse implications are obviously false. Also neither a cyclic sequence is disjunctive nor vice-versa.

Further we shall employ a disjunctive driver in the main theorem. To understand why this feature fits well a standard cyclic projection onto two sets, one should recognize that the (linear or metric nearest point) projection map $P$ is idempotent, $P \circ P = P$ (a retraction put in a nonlinear topology framework). The cancellations in any orbit build from $N = 2$ projections show that the result is similar (modulo repetitions) to an alternating projection orbit regardless of how complex disjunctive driver was applied, see Examples 1 and 2.

We define the **omega-limit set** of $(x_n)_{n=0}^{\infty}$ in the usual way by a descending intersection of the closures of tails of an orbit

$$\omega((x_n)) := \bigcap_{m=0}^{\infty} \{x_n : n \geq m\}.$$  

One should be aware that in the case of IFSs and multivalued dynamical systems various kinds of omega-limit sets can be defined, consult e.g. [34, 2, 23, 25]. Comparing the nonautonomous discrete dynamical systems ([27]) with the framework of IFSs (and also multivalued systems, cf. [22, 29]), one sees that the orbit of the nonautonomous system is determined by the starting point though the dynamics changes over time and to determine the orbit of the IFS one needs additionally to specify the driving sequence; loosely speaking in the theory of IFSs we deal with the infinite number of nonautonomous systems upon a finite (sometimes countable [32] or compact [33, 23, 30]) set of generating maps. Yet one can cast the IFS as a skew-product system, e.g., [25, Example 4.3].

We finish this Section by giving motivating examples where the projection method invites a nonexpansive IFSs viewpoint (cf. [3]). Specifically Example 1 suggests the way we should interpret the result of the projection algorithm in general. We use a common notation $H_i \subset X := \mathbb{R}^2$ for lines (hyperplanes) and $P_i : X \to H_i$ for orthogonal projections onto $H_i$ constituting a nonexpansive IFS $(X; f_1, f_2, \ldots)$, $f_i := P_i$.

**Example 1** Given two lines intersecting at $x_*$ one projects alternately onto them to recover solution $x_*$ of the linear system. The picture in Figure 1 is the hallmark of the projection method. We have the orbit $P_2 \circ P_1 \circ P_2 \circ P_1 \circ P_2(x_0)$ converging to $x_*$. Note that the composition $P_1 \circ P_2$ is contractive.
Example 2 Given two parallel lines one projects alternately onto them to get a pair of minimally distanced points. The visualization provides Figure 2. Here $P_1 \circ P_2$ is not contractive, yet it behaves contractively on the orbit, see the last Section for precise formulation of this phenomenon. Usually the case of parallel lines exhibits instability upon parameters for the solution problem of linear systems.

Example 3 Suppose we have the following configuration: four lines so that each one is orthogonal to two others and parallel to the third one. Points of intersection, denoted $y_{12}, y_{13}, y_{23}, y_{24}$, span a rectangle, see Figure 3. Then projecting onto these lines in a sufficiently “random” manner, e.g., $P_2 \circ P_3 \circ P_1 \circ P_4 \circ P_2 \circ P_1 \circ P_2 \circ P_1(x_0)$, one quickly recovers the four corner points $C := \{y_{12}, y_{13}, y_{23}, y_{24}\}$. Note that analogously to Example 1, the composition $P_3 \circ P_1$ ($H_1$ and $H_3$ orthogonal) is contractive; hence $P_4 \circ P_2 \circ P_3 \circ P_1$ is contractive. The minimal closed set invariant on the joint action of all projections $P_1, P_2, P_3, P_4$ is exactly the set $C$; see the next Section for the precise definition of an invariant set.

Since the orthogonal projection onto a hyperplane is nonexpansive w.r.t. the taxi-cab $\ell^1$-norm in the Euclidean space one might hope (in
Figure 3. Projections onto four pairwise orthogonal or parallel lines.

\[ H_1 \]
\[ x_0 = x_4 \]
\[ y_{13} = x_7 \]
\[ y_{23} = x_8 \]

Figure 4. Randomly applied projections onto three lines.

\[ H_1 \]
\[ x_0 = x_3 \]
\[ x_2 = x_4 \]
\[ y_{12} = x_6 \]
\[ y_{24} = x_5 \]

according with the Examples so far) that omega-limit sets of systems consisting of orthogonal projections are finite sets ([1, 36, 17]). This is not true as shown below.

**Example 4** If we project onto three lines each two of which are intersecting and the choice of projections follows a disjunctive driver, then the omega-limit set of such iteration constitutes a triangle with vertices at intersection points. This phenomenon was observed quite long time ago for drivers generated via discrete stochastic processes (‘chaos game algorithm’ [5]).

We sketch the orbit \( P_1 \circ P_3 \circ P_1 \circ P_2 \circ P_3 \circ P_1 \circ P_2(x_0) \) in Figure 4. Note that any two projections \( P_i \circ P_j \), \( i \neq j \), compose to contractions; in particular \( P_3 \circ P_2 \circ P_1 \) is a contraction.

\[ \diamond \]

The reader is encouraged to delve in [8] for more examples with detailed analyses of polygonal omega-limit sets.

2. Generalities

We shall present here a general relationship between invariant sets and omega-limit sets. Throughout let \( C \subset X \) denote a nonempty closed
bounded subinvariant set of the nonexpansive IFS \((X; f_i, i = 1, \ldots, N)\), \(\Phi\) the Hutchinson operator and \((x_n)_{n=0}^\infty\) the orbit.

Given a nonempty set \(S \subset X\), we employ also the notation:

- \(d(p, S) := \inf_{s \in S} d(p, s)\) for the distance from the point \(p \in X\) to \(S\);
- \(N_\varepsilon S := \{x \in X : d(x, S) < \varepsilon\}\) for the \(\varepsilon\)-neighbourhood of \(S\).

It turns out to be convenient to use a known sequential characterization of the omega-limit set:

\[ x_* \in \omega((x_n)) \text{ iff } x_{k_n} \to x_* \text{ for some subsequence } k_n \nearrow \infty. \]

Whenever we make statements about omega-limit sets we assume that they are nonempty; for empty omega-limit sets the statements are void.

In this respect we have the following basic criterion.

**Lemma 1** ([34, 2, 27]). If the orbit \((x_n)_{n=0}^\infty\) is precompact, then \(\omega((x_n))\) is nonempty and compact.

Recall that the orbit \(\{x_n\}_{n=0}^\infty\) in a complete space \(X\) is precompact if it has compact closure.

The simple observations below are crucial for basic relationship between invariant sets and omega-limit sets.

**Lemma 2.** Let \((x_n)_{n=0}^\infty\) be the orbit and suppose that there exists a nonempty closed bounded set \(C\) which is subinvariant, \(\Phi(C) \subset C\). Then

(i) \(d(x_{n+1}, C) \leq d(x_n, C) \leq d(x_0, C)\),

(ii) \((x_n)_{n=0}^\infty\) is bounded,

(iii) \(d(y, C) = \inf_n d(x_n, C) \equiv \text{const for } y \in \omega((x_n))\).

**Proof.** Fix \(\varepsilon > 0.\) Find appropriate \(c_0 \in C\) to write

\[ d(x_0, C) + \varepsilon \geq d(x_0, c_0) \geq d(f_{i_1}(x_0), f_{i_1}(c_0)) = d(x_1, f_{i_1}(c_0)) \geq d(x_1, C); \]

the last inequality relies on \(f_{i_1}(c_0) \in \Phi(C) \subset C\). By induction (i) follows.

Moreover \(x_n \in N_{2d(x_0, C)} C\) for \(n \geq 1\), where the latter set is bounded. Hence (ii) follows.

The sequence \(d(x_n, C)\) is monotonely decreasing and bounded from below by 0, thus it is convergent to const = \(\inf_n d(x_n, C)\). Let \(x_{k_n} \to y\).

By monotonicity again

\[ d(x_{k_n}, C) \to \inf_n d(x_{k_n}, C) = \text{const}, \]

but continuity of the distance yields \(d(x_{k_n}, C) \to d(y, C)\). Therefore we have (iii). \(\square\)

**Proposition 1.** If a closed bounded subinvariant set \(C\) intersects the omega-limit set, \(C \cap \omega((x_n)) \neq \emptyset\), then it contains that omega-limit set, \(C \supset \omega((x_n))\). In particular, the omega-limit set is the minimal invariant set, provided it is invariant.
Proof. Let \( y_0 \in C \cap \omega((x_n)) \). From Lemma 2 we know that any \( y \in \omega((x_n)) \) necessarily obeys \( d(y, C) = d(y_0, C) = 0 \). \( \square \)

Note that the set \( C \) in the above proposition need not be bounded (as careful analysis of the proof of Lemma 2 (i), (iii) shows). Although the omega-limit set need not be invariant there holds

**Proposition 2** ([5]). The omega-limit set is superinvariant, i.e.,

\[
\Phi(\omega((x_n))) \supset \omega((x_n)).
\]

Proof. Let \( x_\star \in \omega((x_n)) \), \( x_{k_n} \to x_\star \). Then there exists a symbol \( \sigma \in \{1, \ldots, N\} \) s.t. \( x_{k_n+1} = f_\sigma(x_{k_n}) \) for infinitely many \( k_n \)'s, say \( k_{i_n} \). Hence

\[
x_{k_{i_n}+1} = f_\sigma(x_{k_{i_n}}) \to f_\sigma(x_\star) \in \Phi(\omega((x_n))).
\]

\( \square \)

For good properties of orbits and omega-limit sets one may need the notion of a proper space. A metric space \((X, d)\) is called proper (cf. [7]) if every ball not equal to the whole space has compact closure; the terminology is not standardized, e.g., Beer uses the term ‘space with nice closed balls’, see [14] 5.1.8 p.142. Necessarily any such space is complete and locally compact. Conversely, a locally compact space is proper after suitable remetrization ([14] 5.1.12 p.143). The main advantage of \( X \) being proper is that any bounded orbit is precompact and consequently admits nonempty compact omega-limit set. We would like to stress out that our further considerations do not rely on the properness.

### 3. MAIN THEOREM

Let \( \Phi = (X; f_1, \ldots, f_N) \) be a nonexpansive IFS on a complete space \( X \). We constantly assume that \( \Phi \) possesses a nonempty closed bounded subinvariant set. Hence by observations made in the previous Section all orbits and omega-limit sets are warranted to be bounded. This is not very restrictive hypothesis, because if the omega-limit set recovers an invariant set, then there must be present at least one (sub)invariant set.

Suppose \((x_n)_{n=0}^\infty, (y_n)_{n=0}^\infty\) are two orbits generated by the driving sequences \((i_n)_{n=1}^\infty, (j_n)_{n=1}^\infty\) and starting at \( x_0, y_0 \in X \), respectively.

**Proposition 3.** Let us assume that

- (D) the driving sequences for orbits \((x_n)_{n=0}^\infty\) and \((y_n)_{n=0}^\infty\) are disjunctive,
- (C) there exists a sequence of symbols \((u_1, \ldots, u_l) \in \{1, \ldots, N\}^l\) s.t. the composition \( f_{u_l} \circ \ldots \circ f_{u_1} \) is a Lipschitz contraction.

Then the omega-limit set does not depend on the initial point

\[
\omega((x_n)) = \omega((y_n)).
\]
Proof. We additionally assume that the driving sequences are the same for both orbits, \( i_n = j_n \). The general case will follow from Lemma 3 below.

Denote by \( L < 1 \) the Lipschitz constant of \( f_{u_l} \circ \ldots \circ f_{u_1} \). By disjunctivity of the driver \( (i_n)_{n=1}^{\infty} \), given any subsequence \( k_n \) there exists a deeper subsequence \( k_{k_n} \) s.t. \( (i_{k_n}, i_{k_n-1}, \ldots, i_1) \) contains as a subword the sequence \( (u_l, \ldots, u_1, \ldots, u_l, \ldots, u_1) \in \{1, \ldots, N\}^{m_n-1} \), i.e., \( (u_l, \ldots, u_1) \) repeated \( m_n \)-times, and additionally \( m_n \nearrow \infty \). Therefore
\[
d(x_{k_n}, y_{k_n}) \leq L^{m_n} \cdot d(x_0, y_0) \to 0.
\]
This means that both orbits admit the same limit points. \( \square \)

We are going to strengthen this result, by showing under contractivity condition substantially weaker than (C), that the omega-limit sets of orbits starting at the same point are identical provided the sequences driving these orbits are disjunctive.

Lemma 3. Let us assume about orbits \( (x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty} \) that they start from the same point \( y_0 = x_0 \), and obey (D) and

(CO) there exists a sequence of symbols \( (u_1, \ldots, u_l) \in \{1, \ldots, N\}^l \) s.t. the composition \( f_{u_l} \circ \ldots \circ f_{u_1} \) is a Lipschitz contraction when restricted to the set
\[
\bigcup_{n=0}^{\infty} \Phi^n(\{x_0\}) = \{x_0, f_1(x_0), \ldots, f_N(x_0)\},
\]
\[
f_1 \circ f_1(x_0), f_1 \circ f_2(x_0), \ldots, f_1 \circ f_N(x_0), \ldots,
\]
\[
f_N \circ f_1(x_0), \ldots, f_N \circ f_N(x_0), \ldots
\]
(a branching tree with root at \( x_0 \)).

Then the omega-limit sets coincide
\[
\omega((x_n)) = \omega((y_n)).
\]

Proof. Denote by \( L \) the Lipschitz constant of \( f_{u_l} \circ \ldots \circ f_{u_1} \). Fix \( x_* \in \omega((x_n)) \), \( x_{k_n} \to x_* \). Similarly as in the proof of Proposition 3 there exists \( k_{k_n} \) s.t. \( (i_{k_n}, \ldots, i_1) \) contains \( (u_l, \ldots, u_1) \) repeated consecutively \( m_n \)-times, \( m_n \nearrow \infty \). Then, due to disjunctivity of \( j_n \), we can find a subsequence \( j_{k_n} \) s.t.
\[
f_{j_{k_n}} = f_{i_{k_n}},
\]
\[
f_{j_{(j_{k_n}-1)}} = f_{i_{(k_n-1)}},
\]
\[
\ldots, f_{j_{(j_{(k_n-1)}-1)}} = f_{i_1}.
\]
Hence we arrive at
\[
d(x_{k_n}, y_{r_n}) \leq L^{m_n} \cdot d(x_0, y_{r_n-k_n}) \to 0.
\]
Therefore \( x_* \in \omega((y_n)) \). \( \square \)
Now we establish the main result of the whole article.

**Theorem 1.** If the orbit \((x_n)_{n=0}^\infty\) of the nonexpansive IFS \((X; f_1, \ldots, f_N)\) is bounded and driven by a disjunctive sequence of symbols and the system has the property (CO) of contractivity on orbits, then \(\omega((x_n))\) is a minimal closed invariant set. Under stronger condition (C) of contractivity, any two omega-limit sets generated by a disjunctive choice of maps coincide.

**Proof.** By Proposition 1 (or by Proposition 2 if one prefers) we only need to prove subinvariance:

\[ f_\sigma(\omega((x_n))) \subset \omega((x_n)) \quad \text{for every} \quad \sigma \in \{1, \ldots, N\}. \]

Let \(f_\sigma(x_*) \in f_\sigma(\omega((x_n)))\), \(x_k \rightarrow x_*\). Take a finer subsequence \(k_{in}\) according to the following rules: the sequence \((i_{k_{(n-1)}}+2, \ldots, i_{k_{in}})\) contains

(a) all finite words of length \(n\) build over the alphabet \(\{1, \ldots, N\}\),
(b) \(m_n\)-times repeated word \((u_1, \ldots, u_l)\) appearing in the condition (CO), \(m_n \nearrow \infty\).

Redefine \(i_n\) in such a way that \(\tilde{i}_{k_{in}+1} = \sigma\) and \(\tilde{i}_n = i_n\) otherwise. The sequence \(\tilde{i}_n\) is disjunctive due to (a). Hence the orbit \(y_n\) starting at \(y_0 := x_0\) with the driver \(j_n := \tilde{i}_n\) has the property that \(d(y_{k_{in}}, x_{k_{in}}) \rightarrow 0\) as warranted by (b). So

\[ y_{k_{in}+1} = f_\sigma(y_{k_{in}}) \rightarrow f_\sigma(x_*) \in \omega((y_n)). \]

Therefore

\[ \omega((x_n)) = \omega((y_n)) \ni f_\sigma(x_*) \]

via Lemma 3.

**Example 5** Consider a system \((X; f_i, i = 1, \ldots, N)\) comprising of orthogonal projections \(f_i := P_i\) onto affine subspaces \(H_i \subset X\) in the Euclidean space \(X\). Due to [35] we know that any orbit produced by projections is bounded. Therefore in the standard situation we do not need to assume that there exists a bounded subinvariant set. Moreover Theorem 1 warrants the existence of a bounded invariant set.

The most important fact about orthogonal projections is that the composition \(P_N \circ \ldots \circ P_1\) is contractive on orbits, obeys condition (CO). Several results in this direction have been obtained throughout the years: [26, 10, 24].

This settles the case of the Kaczmarz algorithm when projecting onto multiple hyperplanes with empty intersection (cf. [3]).

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