ON THE RENORMALIZATION PROCEDURE FOR QUANTUM FIELDS WITH MODIFIED DISPERSION RELATIONS IN CURVED SPACETIMES

D. LÓPEZ NACIR and F.D. MAZZITELLI
Departamento de Física J.J. Giambiagi, Facultad de Ciencias Exactas y Naturales, UBA, Ciudad Universitaria, Pabellón 1, 1428 Buenos Aires, Argentina.

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We review our recent results on the renormalization procedure for a free quantum scalar field with modified dispersion relations in curved spacetimes. For dispersion relations containing up to 2s powers of the spatial momentum, the subtraction necessary to renormalize $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ depends on s. We first describe our previous analysis for spatially flat Friedman-Robertson-Walker and Bianchi type I metrics. Then we present a new power counting analysis for general background metrics in the weak field approximation.

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It has been argued that trans-Planckian effects could be relevant in the early universe and in the context of black hole physics. As a phenomenological approach to investigate physics near the Planck scale (or near a critical scale for which new physics could show up), it is useful to analyze the consequences of assuming modified dispersion relations (MDR) for the quantum fields, in order to assess the robustness of the predictions obtained in semiclassical gravity. The MDR will of course affect the structure of the quantum field theory, in particular its renormalizability. In the semiclassical approximation, the renormalization of the stress tensor is crucial to evaluate the backreaction of quantum fields.

The renormalization procedure for quantum fields satisfying the standard dispersion relation in curved backgrounds is well established. Indeed, there are well known covariant methods of renormalization that can be implemented in principle in any spacetime metric. When applied to the expectation value of the square of the field $\langle \phi^2 \rangle$, or to the mean value of the stress tensor $\langle T_{\mu\nu} \rangle$, one can obtain the associated renormalized quantities by making the subtractions:

$$\langle \phi^2 \rangle_{\text{ren}} = \langle \phi^2 \rangle - \langle \phi^2 \rangle^{(0)} - \langle \phi^2 \rangle^{(2i_{\text{max}})}, \quad (1a)$$

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle^{(0)} - \langle T_{\mu\nu} \rangle^{(2j_{\text{max}})}, \quad (1b)$$
where a superscript $2l$ denotes the terms of adiabatic order $2l$ of the corresponding expectation value (i.e., the terms containing $2l$ derivatives of the metric). For the usual dispersion relation, it is well known that in $n$ dimensions the subtraction involves up to $2i_{\text{max}} = 2\text{int}(n/2 - 1)$ for $\langle \phi^2 \rangle$ and $2j_{\text{max}} = 2\text{int}(n/2)$ for the stress tensor, where int$(x)$ is the integer part of $x$.

The case of MDR can be consistently studied in the framework of the Einstein-Aether theory. In this theory, the general covariance is preserved by introducing a dynamical vector field $u^\mu$ called the aether field, which is constrained to take a non-zero timelike value, $u^\mu u_\mu = -1$. In the semiclassical approximation, both the aether field and the spacetime metric are assumed to be classical. The action for a massive quantum scalar field $\phi$ can be written as

$$S_{\phi} = -\frac{1}{2} \int d^n x \sqrt{-g} \left[ \partial^\mu \phi \partial_\mu \phi + (m^2 + \xi R) \phi^2 + 2 \sum_{s,p \leq s} b_{sp} (D^{2s} \phi)(D^{2p} \phi) \right], \quad (2)$$

where $g = \det(g_{\mu\nu})$, $R$ the Ricci scalar and $D^{2s} \phi \equiv \mathcal{L}_{\mu} \nabla_{\mu}(\mathcal{L}_{\gamma} \nabla_{\gamma} \phi)$ (with $\mathcal{L}_{\mu} \equiv g_{\mu\nu} + u_\mu u_\nu$ and $\nabla_{\mu}$ the derivative operator associated with $g_{\mu\nu}$). The last term in Eq. (2) gives rise to the MDR.

It has been realized that some non-trivial issues arise in the renormalization procedure. On the one hand, the structure of the counterterms could be different from the case of the standard dispersion relation. Indeed, as the scalar field couples not only to the metric but also to the aether field, from a general effective field theory perspective one can expect that new counterterms constructed with both the metric and the aether field will be required. On the other hand, the presence of higher spatial derivatives affects the singularity structure of the propagator, and one is led to the question of whether higher values of $s$ in Eq. (2) imply milder divergences in the unrenormalized quantities or not. In other words, given a MDR, we are interested in knowing up to which adiabatic order the subtractions in Eq. (1) have to be carried out to get finite, physically meaningful expectation values. In the case of interacting quantum fields in Minkowski spacetime, it has been shown that higher spatial derivatives improve the UV behavior of Feynman diagrams. Here, we will show that while such improvement also occurs for $\langle \phi^2 \rangle$, the opposite holds for $\langle T_{\mu\nu} \rangle$.

For scalar fields propagating in a spatially flat Friedman-Robertson-Walker (FRW) spacetime of $n$ dimensions, the extension of the adiabatic subtraction scheme based in a WKB expansion of the field modes has been considered in Ref. 4. The Fourier modes of the scaled field $\chi = C^{(n-2)/4}(\eta) \phi$ satisfy

$$\chi''_{k} + \left[ (\xi - \xi_n)RC(\eta) + \omega_k^2 \right] \chi_{k} = 0, \quad (3)$$

where $\sqrt{C(\eta)}$ is the scale factor, primes stand for derivatives with respect to the conformal time $\eta$, $\xi_n = (n-2)/(4n-4)$, and

$$\omega_k^2 = |\vec{k}|^2 + C(\eta) \left[ m^2 + 2 \sum_{s,p \leq s} (-1)^{s+p} b_{sp} \left( \frac{|\vec{k}|}{\sqrt{C(\eta)}} \right)^{2(s+p)} \right]. \quad (4)$$
To get the WKB expansion, we express $\chi_k$ as

$$\chi_k = \frac{1}{\sqrt{2W_k}} \exp\left(-i \int_\eta W_k(\tilde{\eta}) d\tilde{\eta}\right), \quad (5)$$

and substitute this into Eq. (3) to obtain a nonlinear differential equation for $W_k^2$. Solving this equation iteratively, it can be shown that the $2l$–adiabatic order of $W_k^2$ scales as $\omega_k^{2-2l}$. After substituting Eq. (5) into $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$, one can determine whether a given adiabatic order of these expectation values is finite or not. In this way, for a MDR such that the frequency behaves as $\omega \sim |\vec{k}|^s$ for large values of $|\vec{k}|$, one can show that divergences appear up to

$$2i_{max} = 2 \int \frac{n-1}{2} \frac{1}{s-1} \quad 2j_{max} = 2 \int \frac{1}{2} + \frac{n-1}{2s}.$$  

(6)

In Ref. 4 the WKB expansion of the stress tensor was computed up to the fourth adiabatic order for the class of MDR given in Eq. (4). It was shown that these adiabatic orders can be absorbed into a redefinition of the gravitational bare constants of the theory, as for the usual dispersion relation (i.e., only geometric counterterms are needed). However, this simple result is due to the symmetries of the spatially flat FRW metric.

In Bianchi type I spacetimes, the WKB expansion can be obtained in a completely analogous way, and Eq. (6) also applies in this case. However, for these anisotropic metrics, one can show that new counterterms are necessary, which involve the timelike vector field in addition to the metric. For instance, a term proportional to $(\nabla_\mu u^\mu)^2$ in the aether Lagrangian is needed to absorb the divergences in $\langle T_{\mu\nu} \rangle^{(2)}$ (in addition to the usual Einstein-Hilbert action). The point is that in a spatially flat FRW background these new counterterms are indistinguishable from the usual ones. Concretely, once evaluated in this background, the stress tensor obtained from the variation of the most general action for the aether field containing two derivatives, turns out to be proportional to the Einstein tensor.

Currently, there are strong constraints on the parameters associated to terms containing two derivatives of the aether field. Therefore, the new counterterms of second adiabatic order should be carefully chosen to make the theory consistent with observation.

The values of $2i_{max}$ and $2j_{max}$ in Eq. (6) are a peculiarity of the spatially homogeneous backgrounds considered so far. To see this, let us consider a general background in the weak field approximation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, u_\mu = \delta_\mu^0 + v_\mu$. By keeping only linear terms in $h_{\mu\nu}$ and $v_\mu$, an integral expression of the Feynman propagator $G_F(x,x')$ for the scalar field can be obtained perturbatively: $G_F =$

\[a\]

As Eq. (6) indicates, for $n = 4$ the fourth adiabatic order is convergent when $s \geq 2$, and the second order is convergent when $s \geq 4$. However, there are subtle points in the renormalization procedure related to the trace anomaly.
\[ G_0^F(x, x') = (2\pi)^{-n} \int d^n k e^{ik(x-x')} [-k_0^2 + \omega^2(|\vec{k}|^2)]^{-1}, \quad (7a) \]

\[ G_1^F(x, x') = -\int d^n y G_0^F(x, y) F(y) G_0^F(y, x'). \quad (7b) \]

Here \( F \) is an operator linear in the perturbation fields, and \( \omega^2(|\vec{k}|^2) = m^2 - i\epsilon + |\vec{k}|^2 + 2 \sum_{s,p \leq s} b_{sp} (-1)^{s+p} |\vec{k}|^{2(s+p)}. \)

The expectation value \( \langle \phi^2 \rangle \) is given by the coincidence limit of \( \text{Im} G_F \). Analogously, \( \langle T_{\mu\nu} \rangle \) can be expressed as the coincidence limit of a derivative operator applied to \( \text{Im} G_F \). In this way, one can obtain integral expressions for both \( \langle \phi^2 \rangle \) and \( \langle T_{\mu\nu} \rangle \). Using an expansion in derivatives of the perturbation fields, one can study up to which adiabatic order these quantities contain divergences. For a MDR such that \( \omega \sim |\vec{k}|^s \) for large values of \( |\vec{k}| \), a power counting analysis yields

\[ 2i_{\text{max}} = 2 \int \left( \frac{n - 1 - s}{2} \right), \quad 2j_{\text{max}} = 2 \int \left( \frac{n - 1 + s}{2} \right). \quad (8) \]

The value of \( 2i_{\text{max}} \) is now generally larger than the one given in Eq. (6), although it also decreases with \( s \). However, contrary to the previous case, \( 2j_{\text{max}} \) increases with \( s \). Therefore we conclude that, in the weak field approximation, for a general background the subtraction in Eq. (1) should be performed up the adiabatic orders \( 2i_{\text{max}} \) and \( 2j_{\text{max}} \) given in Eq. (8). In particular, in order to renormalize the semiclassical Einstein-Aether equations, it will be necessary to introduce all possible counterterms constructed with \( g_{\mu\nu} \) and \( u_\mu \), up to the \( 2j_{\text{max}} \)-adiabatic order. It would be interesting to check if these results remain valid beyond the weak field approximation.

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