Hedging the Drift: Learning to Optimize under Non-Stationarity

Wang Chi Cheung  
Department of Industrial Systems Engineering and Management, National University of Singapore isecwc@nus.edu.sg

David Simchi-Levi  
Institute for Data, Systems, and Society, Massachusetts Institute of Technology, Cambridge, MA 02139, dslevi@mit.edu

Ruihao Zhu  
Statistics and Data Science Center, Massachusetts Institute of Technology, Cambridge, MA 02139, rzhu@mit.edu

We introduce general data-driven decision-making algorithms that achieve state-of-the-art dynamic regret bounds for non-stationary bandit settings. It captures applications such as advertisement allocation and dynamic pricing in changing environments. We show how the difficulty posed by the (unknown a priori and possibly adversarial) non-stationarity can be overcome by an unconventional marriage between stochastic and adversarial bandit learning algorithms. Our main contribution is a general algorithmic recipe that first converts the rate-optimal Upper-Confidence-Bound (UCB) algorithm for stationary bandit settings into a tuned Sliding Window-UCB algorithm with optimal dynamic regret for the corresponding non-stationary counterpart. Boosted by the novel bandit-over-bandit framework with automatic adaptation to the unknown changing environment, it even permits us to enjoy, in a (surprisingly) parameter-free manner, this optimal dynamic regret if the amount of non-stationarity is moderate to large or an improved (compared to existing literature) dynamic regret otherwise. In addition to the classical exploration-exploitation trade-off, our algorithms leverage the power of the “forgetting principle” in their online learning processes, which is vital in changing environments. We further conduct extensive numerical experiments on both synthetic data and the CPRM-12-001: On-Line Auto Lending dataset provided by the Center for Pricing and Revenue Management at Columbia University to show that our proposed algorithms achieve superior dynamic regret performances.

Key words: data-driven decision-making, non-stationary bandit optimization, parameter-free algorithm

1. Introduction

Consider the following general decision-making framework: a decision-maker (DM) interacts with an environment by picking actions one at a time sequentially. Upon selecting an action, it instantly receives a reward drawn randomly from a probability distribution tied to this action. The goal of the DM is to maximize its cumulative rewards. However, it also faces the following challenges:

- **Uncertainty**: the reward distribution of each action is initially unknown to the DM, and it thus has to estimate the underlying distributions via interacting with the environment.

- **Non-Stationarity**: moreover, the environment is non-stationary, and the reward distributions can evolve over time.
• **Partial/Bandit Feedback:** finally, the DM can only observe the random reward of the selected action each time; while the rewards of the un-chosen actions remain unknown.

Evidently, the DM faces a trilemma among exploration, exploitation as well as adaptation to changes. On one hand, the DM wishes to exploit, and to select the action with the best historical performances to earn as much rewards as possible; while on the other, it also wants to explore other actions to get a more accurate estimation of the reward distributions. The changing environment makes the exploration-exploitation trade-off even more delicate. Indeed, past observations could become obsolete due to the changes in the environment, and the DM needs to explore for changes and refrain from exploiting possibly outdated observations. It turns out that many applications fall naturally into this general framework:

1. **Advertisement Allocation:** an online platform allocates advertisements (ads) to a sequence of users. For each arriving user, the platform has to deliver an ad to her, and only observe the users’ responses to the displayed ads. The platform has full access to the features of the ads and the users. Following (Agrawal and Goyal 2013), we assume that a user’s click behavior towards an ad, or simply the click through rate (CTR) of this ad by a particular user, follows a probability distribution governed by a common, but initially unknown response function that is linear in the features of the ad and the user. The platform’s goal is to maximize the total profit. However, the unknown response function can change over time. For instance, if it is around the time that Apple releases a new iPhone, one can expect that the popularity of an Apple’s ad grows.

2. **Dynamic Pricing:** a seller decides the (personalized) price dynamically (Keskin and Zeevi 2014, 2016, Besbes and Zeevi 2015, Ban and Keskin 2018) for each of the incoming customers with the hope to maximize sales profit. Begin with an unknown demand function, the DM only observes the purchase decision of a customer under the posted price, but not any other price. In addition, the customers’ reaction towards the same price can vary across time due to the product reviews, the emergence of competitive products, etc.

Our framework is closely related to the multi-armed bandit (MAB) problems. MAB problems are online problems with partial feedback, when the DM is subject to uncertainty in his/her learning environment. Traditionally, most MAB problems are studied in the stochastic (Auer et al. 2002b) and adversarial (Auer et al. 2002a) environments. In the former, the model uncertainty is static and the partial feedback is corrupted by a mean zero random noise. The DM aims at estimating the latent static environment using historical data, and converging to a static optimal decision. In the latter, the model is dynamically changed by an adversary. The DM strives to hedge against the changes, and compete favorably in comparison to certain benchmark policies.

Unfortunately, strategies for the stochastic environments can quickly deteriorate under non-stationarity as historical data might “expire”; while the permission of an confronting adversary in
the adversarial settings could be too pessimistic. Recently, a stream of research works (see Related Works) focuses on MAB problems in a drifting environment, which is a hybrid of a stochastic and an adversarial environment. Although the environment can be dynamically and adversarially changed, the total change (quantified by a suitable metric) in a $T$ step problem is upper bounded by $B_T (= \Theta(T^p)$ for some $p \in (0,1)$), the variation budget. The feedback is corrupted by a mean zero random noise. The aim is to minimize the dynamic regret, which is the optimality gap compared to the sequence of (possibly dynamically changing) optimal decisions, by simultaneously estimating the current environment and hedging against future changes every time step. Most of the existing works for non-stationary bandits have focused on the the somewhat ideal case in which $B_T$ is known. In practice, however, $B_T$ is often not available ahead. Though some efforts have been made towards this direction ([Karnin and Anava 2016] [Luo et al. 2018]), how to design algorithms with low dynamic regret when $B_T$ is unknown remains largely as a challenging problem.

In this paper, we design and analyze novel algorithms for bandit problems in a drifting environment. Our main contributions are listed as follows.

- When the variation budget $B_T$ is known, we characterize the lower bound of dynamic regret, and develop a tuned Sliding Window Upper-Confidence-Bound (SW-UCB) algorithm with matched dynamic regret upper bound up to logarithmic factors.

- When $B_T$ is unknown, we propose a novel Bandit-over-Bandit (BOB) framework that tunes SW-UCB adaptively. When the amount of non-stationarity is moderate to large, the application of BOB on SW-UCB algorithm recovers the optimal dynamic regret; otherwise, it obtains a dynamic regret bound with best dependence on $T$ compared to existing literature.

- Our results are general in the sense that given any Upper-Confidence-Bound (UCB) algorithm with optimal regret for the stationary bandit settings, we are able to convert it into the corresponding SW-UCB algorithm with optimal dynamic regret (when $B_T$ is known) and the BOB algorithm with nearly optimal dynamic regret (when $B_T$ is unknown), respectively.

- Our algorithm design and analysis shed light on the fine balance between exploration, exploitation and adaptation to changes in dynamic learning environments. We rigorously incorporates the “forgetting principle” into the Optimism-in-Face-of-Uncertainty principle, by demonstrating that the DM should dispose of observations which are sufficiently old. We provide precise criteria for the disposal, and rigorously show the convergence to optimality under these criteria.

- Finally, we point out that a preliminary version of this paper will appear in the 22nd International Conference on Artificial Intelligence and Statistics (AISTATS 2019) ([Cheung et al. 2019]), and the current paper is a significantly extended version of it. Specifically, the current version provides a substantially refined analysis and thus a greatly improved regret bound for the BOB algorithm when
compared to (Cheung et al. 2019). More importantly, we demonstrate the generality of the established results in the current paper, as exemplified by the extensions to the prevalent $d$-armed bandit and generalized linear bandit settings.

The rest of the paper is organized as follows. In Section 2, we review existing works in stationary and non-stationary environments. In Section 3, we formulate the non-stationary linear bandit model. We note that the choice of linear model is purely for the purpose of illustration, and all the results derived shall be applicable to other settings (Please see Section 8). In Section 4, we establish a minimax lower bound for our problem. In Section 5, we describe the sliding window estimator for parameter estimation under non-stationarity. In Section 6, we develop the sliding window-upper confidence bound algorithm with optimal dynamic regret (when the amount of non-stationarity is known ahead). In Section 7, we introduce the novel bandit-over-bandit framework with nearly optimal regret. In Section 8, we demonstrate the generality of the established results by applying them to other popular bandit settings, such as the multi-armed bandit and generalized linear bandit settings. In Section 9, we conduct extensive numerical experiments with both synthetic and CPRM-12-001: on-line auto lending datasets to show the superior empirical performances of our algorithms. In Section 10, we conclude our paper.

2. Related Works

MAB problems with stochastic and adversarial environments are extensively studied, as surveyed in (Bubeck and Cesa-Bianchi 2012, Lattimore and Szepesvári 2018). To model inter-dependence relationships among different arms, models for linear bandits in stochastic environments have been studied. In (Auer 2002, Dani et al. 2008, Rusmevichientong and Tsitsiklis 2010, Chu et al. 2011, Abbasi-Yadkori et al. 2011), UCB type algorithms for stochastic linear bandits were studied, and Abbasi-Yadkori et al. (Abbasi-Yadkori et al. 2011) possessed the state-of-the-art algorithm for the problem. Thompson sampling algorithms proposed in (Russo and Roy 2014, Agrawal and Goyal 2013, Abeille and Lazaric 2017) are able to bypass the high computational complexities provided that one can efficiently sample from the posterior on the parameters and optimize the reward function accordingly. Unfortunately, achieving optimal regret bound via TS algorithms is possible only if the true prior over the reward vector is known.

(Besbes et al. 2014, 2018) considered the $K$-armed bandit in a drifting environment. They achieved the tight dynamic regret bound $\hat{O}((KB_T)^{1/3}T^{2/3})$ when $B_T$ is known. Wei et al. (Wei et al. 2016) provided refined regret bounds based on empirical variance estimation, assuming the knowledge of $B_T$. Subsequently, Karnin et al. (Karnin and Anava 2016) considered the setting without knowing $B_T$ and $K = 2$, and achieved a dynamic regret bound of $\hat{O}(B_T^{0.18}T^{0.83} + T^{0.77})$. In a recent work, (Luo et al. 2018) considered $K$-armed contextual bandits in drifting environments,
and in particular demonstrated an improved bound $\tilde{O}(KB_1^{1/5}T^{4/5})$ for the $K$-armed bandit problem in drifting environments when $B_T$ is not known, among other results. (Keskin and Zeevi 2016) considered a dynamic pricing problem in a drifting environment with linear demands. Assuming a known variation budget $B_T$, they proved an $\Omega(B_1^{1/3}T^{2/3})$ dynamic regret lower bound and proposed a matching algorithm. When $B_T$ is not known, they designed an algorithm with $\tilde{O}(B_1T^{2/3})$ dynamic regret. In (Besbes et al. 2015), a general problem of stochastic optimization under the known budgeted variation environment was studied. The authors presented various upper and lower bound in the full feedback settings. Finally, various online problems with full information feedback and drifting environments are studied in the literature (Chiang et al. 2012, Jadababaie et al. 2015).

Apart from drifting environment, numerous research works consider the \textit{switching environment}, where the time horizon is partitioned into at most $S$ intervals, and it switches from one stochastic environment to another across different intervals. The partition is not known to the DM. Algorithms are designed for various bandits, assuming a known $S$ (Auer et al. 2002a, Garivier and Moulines 2011, Luo et al. 2018), or assuming an unknown $S$ (Karnin and Anava 2016, Luo et al. 2018). Notably, the Sliding Window-UCB and the “forgetting principle” is first proposed by Garivier et al. (Garivier and Moulines 2011), while it is only analyzed under $K$-armed switching environments.

Finally, it is worth pointing out that our Bandit-over-Bandit framework has connections with algorithms for online model selection and bandit corralling, see \textit{e.g.}, (Agarwal et al. 2017, Besbes et al. 2018) and references therein. This and similar techniques have been investigated under the context of non-stationary bandits in (Luo et al. 2018, Besbes et al. 2018). Notwithstanding, existing works either obtain sub-optimal dynamic regret bounds or only empirical performance guarantees.

3. Problem Formulation

In this section, we introduce the notations that will be used throughout the discussions and the model formulation.

3.1. Notation

Throughout the paper, all vectors are column vectors, unless specified otherwise. We define $[n]$ to be the set $\{1, 2, \ldots, n\}$ for any positive integer $n$. The notation $a:b$ is the abbreviation of consecutive indexes $a, a+1, \ldots, b$. We use $\|x\|$ to denote the Euclidean norm of a vector $x \in \mathbb{R}^d$. For a positive definite matrix $A \in \mathbb{R}^{d \times d}$, we use $\|x\|_A$ to denote the matrix norm $\sqrt{x^\top A x}$ of a vector $x \in \mathbb{R}^d$. We also denote $x \lor y$ and $x \land y$ as the maximum and minimum between $x, y \in \mathbb{R}$, respectively. When logarithmic factors are omitted, we use $\tilde{O}(\cdot)$ to denote function growth.
3.2. Learning Protocol

In each round $t \in [T]$, a decision set $D_t \subseteq \mathbb{R}^d$ is presented to the DM, and it has to choose an action $X_t \in D_t$. Afterwards, the reward

$$Y_t = \langle X_t, \theta_t \rangle + \eta_t$$

is revealed. Here, we allow $D_t$ to be chosen by an oblivious adversary whose actions are independent of those of the DM, and can be determined before the protocol starts (Cesa-Bianchi and Lugosi 2006). The vector of parameter $\theta_t \in \mathbb{R}^d$ is an unknown $d$-dimensional vector, and $\eta_t$ is a random noise drawn i.i.d. from an unknown sub-Gaussian distribution with variance proxy $R$. This implies $E[\eta_t] = 0$, and $\forall \lambda \in \mathbb{R}$ we have

$$E[\exp(\lambda \eta_t)] \leq \exp\left(\frac{\lambda^2 R^2}{2}\right).$$

Following the convention of existing bandits literature (Abbasi-Yadkori et al. 2011, Agrawal and Goyal 2013), we assume there exist positive constants $L$ and $S$, such that $\|X\| \leq L$ and $\|\theta_t\| \leq S$ holds for all $X \in D_t$ and all $t \in [T]$, and the problem instance is normalized so that $|\langle X, \theta_t \rangle| \leq 1$ for all $X \in D_t$ and $t \in [T]$.

Instead of assuming the stochastic environment, where the reward function remains stationary across the time horizon, we allow it to change over time. Specifically, we consider the general drifting environment: the sum of $\ell_2$ differences of consecutive $\theta_t$’s should be bounded by some variation budget $B_T = \Theta(T^\rho)$ for some $\rho \in (0, 1)$, i.e.,

$$\sum_{t=1}^{T-1} \|\theta_{t+1} - \theta_t\| \leq B_T. \quad (1)$$

We again allow the $\theta_t$’s to be chosen adversarially by an oblivious adversary. We also denote the set of all possible obliviously selected sequences of $\theta_t$’s that satisfies inequality (1) as $\Theta(B_T)$.

The DM’s goal is to design a policy $\pi$ to maximize the cumulative reward, or equivalently to minimize the worst case cumulative regret against the optimal policy $\pi^*$, that has full knowledge of $\theta_t$’s. Denoting $x_t^* = \arg\max_{x \in D_t} \langle x, \theta_t \rangle$, the dynamic regret of a given policy $\pi$ is defined as

$$R_T(\pi) = \sup_{\theta_1:T \in \Theta(B_T)} E[\text{Regret}_T(\pi)] = \sup_{\theta_1:T \in \Theta(B_T)} E\left[\sum_{t=1}^{T} \langle x_t^* - X_t, \theta_t \rangle\right],$$

where the expectation is taken with respect to the (possible) randomness of the policy.

4. Lower Bound

We first provide a lower bound on the the regret to characterize the best achievable regret.

**Theorem 1.** For any $T \geq d$, the dynamic regret of any policy $\pi$ satisfies

$$R_T(\pi) = \Omega\left(d^\frac{3}{2} B_T^\frac{3}{4} T^\frac{1}{4}\right).$$
Proof Sketch. The construction of the lower bound instance is similar to the approach of (Besbes et al. 2014): nature divides the whole time horizon into $[T/H]$ blocks of equal length $H$ rounds (the last block can possibly have less than $H$ rounds). In each block, the nature initiates a new stationary linear bandits instance with parameters from the set $\{ \pm \sqrt{d/4H} \}^d$. Nature also chooses the parameter for a block in a way that depends only on the DM’s policy, and the worst case regret is $\Omega(d\sqrt{H})$. Since there is at least $[T/H]$ number of blocks, the total regret is $\Omega(dT/\sqrt{H})$. By examining the variation budget constraint, we have that the smallest possible $H$ one can take is $\lceil (dT)^{\frac{2}{3}} B_T^{-\frac{2}{3}} \rceil$. The statement then follows. Please refer to Section A for the complete proof.

5. Sliding Window Regularized Least Squares Estimator

As a preliminary, we introduce the sliding window regularized least squares estimator, which is the key tool in estimating the unknown parameters $\{ \theta_t \}_{t=1}^T$. Despite the underlying non-stationarity, we show that the estimation error of this estimator can gracefully adapt to the parameter changes.

Consider a sliding window of length $w$, and consider the observation history $\{(X_s, Y_s)\}_{s=1\vee(t-w)}^{t-1}$ during the time window $(1 \vee (t-w)) : (t-1)$. The ridge regression problem with regularization parameter $\lambda > 0$ is stated below:

$$\min_{\theta : \theta \in \mathbb{R}^d} \lambda \|\theta\|^2 + \sum_{s=1\vee(t-w)}^{t-1} (X_s^T \theta - Y_s)^2. \quad (2)$$

Denote $\hat{\theta}_t$ as a solution to the regularized ridge regression problem, and define matrix $V_{t-1} := \lambda I + \sum_{s=1\vee(t-w)}^{t-1} X_s X_s^T$. The solution $\hat{\theta}_t$ has the following explicit expression:

$$\hat{\theta}_t = V_{t-1}^{-1} \left( \sum_{s=1\vee(t-w)}^{t-1} X_s Y_s \right) = V_{t-1}^{-1} \left( \sum_{s=1\vee(t-w)}^{t-1} X_s X_s^T \theta_s + \sum_{s=1\vee(t-w)}^{t-1} \eta_s X_s \right). \quad (3)$$

The difference $\hat{\theta}_t - \theta_t$ has the following expression:

$$V_{t-1}^{-1} \left( \sum_{s=1\vee(t-w)}^{t-1} X_s X_s^T \theta_s + \sum_{s=1\vee(t-w)}^{t-1} \eta_s X_s \right) - \theta_t$$

$$= V_{t-1}^{-1} \left( \sum_{s=1\vee(t-w)}^{t-1} X_s X_s^T (\theta_s - \theta_t) + V_{t-1}^{-1} \left( \sum_{s=1\vee(t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right) \right), \quad (4)$$

The first term on the right hand side of eq. (4) is the estimation inaccuracy due to the non-stationarity; while the second term is the estimation error due to random noise. We now upper bound the two terms separately. We upper bound the first term in the $\ell_2$ sense.

Lemma 1. For any $t \in [T]$, we have

$$\left\| V_{t-1}^{-1} \sum_{s=1\vee(t-w)}^{t-1} X_s X_s^T (\theta_s - \theta_t) \right\| \leq \sum_{s=1\vee(t-w)}^{t-1} \|\theta_s - \theta_{s+1}\|.$$
Proof Sketch. Our analysis relies on bounding the maximum eigenvalue of
\[ V_{t-1}^{-1} \sum_{s=1}^{p} X_s X_s^\top \]
for each \( p \in \{1 \lor (t-w), \ldots, t-1\} \). Please refer to Section [3] of the appendix for the complete proof.

By applying [Abbasi-Yadkori et al. 2011], we upper bound the second term in the matrix norm sense.

**Lemma 2 ([Abbasi-Yadkori et al. 2011]).** For any \( t \in [T] \) and any \( \delta \in [0, 1] \), we have
\[
\left\| \sum_{s=1 \lor (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right\|_{V_{t-1}^{-1}} \leq R \sqrt{d \ln \left( \frac{1 + w L^2 / \lambda}{\delta} \right)} + \sqrt{\lambda S}
\]
holds with probability at least \( 1 - \delta \).

From now on, we shall denote
\[
\beta := R \sqrt{d \ln \left( \frac{1 + w L^2 / \lambda}{\delta} \right)} + \sqrt{\lambda S} \tag{5}
\]
for the ease of presentation. With these two lemmas, we have the following deviation inequality type bound for the latent expected reward of any action \( x \in D_t \) in any round \( t \).

**Theorem 2.** For any \( t \in [T] \) and any \( \delta \in [0, 1] \), we have with probability at least \( 1 - \delta \),
\[
\left| x^\top (\hat{\theta}_t - \theta_t) \right| \leq L \sum_{s=1 \lor (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \|_{V_{t-1}^{-1}} + \beta \| x \|_{V_{t-1}^{-1}}
\]
holds for all \( x \in D_t \).

**Proof Sketch.** The proof is a direct application of Lemmas [1] and [2]. Please refer to Section [3] of the appendix for the complete proof.

6. **Sliding Window-Upper Confidence Bound (SW-UCB) Algorithm: A First Order Optimal Strategy**

In this section, we describe the Sliding Window Upper Confidence Bound (SW-UCB) algorithm. When the variation budget \( B_T \) is known, we show that SW-UCB algorithm with a tuned window size achieves a dynamic regret bound which is optimal up to a multiplicative logarithmic factor. When the variation budget \( B_T \) is unknown, we show that SW-UCB algorithm can still be implemented with a suitably chosen window size so that the regret dependency on \( T \) is optimal, which still results in first order optimality in this case ([Keskin and Zeevi, 2016]).
6.1. Design Intuition

In the stochastic environment where the reward function is stationary, the well known UCB algorithm follows the principle of optimism in face of uncertainty (Auer et al. 2002b, Abbasi-Yadkori et al. 2011). Under this principle, the DM selects the action that maximizes the UCB, or the value of “mean plus confidence radius” (Auer et al. 2002b). We follow the principle by choosing in each round the action $X_t$ with the highest UCB, i.e.,

$$X_t = \arg \max_{x \in D_t} \left\{ \langle x, \hat{\theta}_t \rangle + L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| x \|_{V_{t-1}} \right\} = \arg \max_{x \in D_t} \left\{ \langle x, \hat{\theta}_t \rangle + \beta \| x \|_{V_{t-1}} \right\}. \tag{6}$$

When the number of actions is moderate, the optimization problem (6) can be solved by an enumeration over all $x \in D_t$. Upon selecting $X_t$, we have

$$\langle x^*_t, \hat{\theta}_t \rangle + L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| x^*_t \|_{V_{t-1}} \leq \langle X_t, \hat{\theta}_t \rangle + L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| X_t \|_{V_{t-1}}, \tag{7}$$

by virtue of UCB. From Theorem 2, we further have with probability at least $1 - \delta$,

$$\langle x^*_t, \theta_t - \hat{\theta}_t \rangle \leq L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| x^*_t \|_{V_{t-1}}, \tag{8}$$

and

$$\langle X_t, \hat{\theta}_t \rangle + L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| X_t \|_{V_{t-1}} \leq \langle X_t, \theta_t \rangle + 2L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + 2\beta \| X_t \|_{V_{t-1}}. \tag{9}$$

Combining inequalities (7), (8), and (9), we establish the following high probability upper bound for the expected per round regret, i.e., with probability $1 - \delta$,

$$\langle x^*_t - X_t, \theta_t \rangle \leq 2L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + 2\beta \| X_t \|_{V_{t-1}}. \tag{10}$$

The regret upper bound of the SW-UCB algorithm (to be formalized in Theorem 3) is thus

$$2 \sum_{t \in [T]} L \sum_{s=1 \vee (t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| X_t \|_{V_{t-1}} = \widetilde{O} \left( wB_T + \frac{dT}{\sqrt{w}} \right). \tag{11}$$

If $B_T$ is known, the DM can set $w = \lfloor d^{2/3}T^{2/3}B_T^{-2/3} \rfloor$ and achieve a regret upper bound $\widetilde{O}(d^{2/3}B_T^{1/3}T^{2/3})$. If $B_T$ is not known, which is often the case in practice, the DM can set $w = \lfloor (dT)^{2/3} \rfloor$ to obtain a regret upper bound $\widetilde{O}(d^{2/3}(B_T + 1)T^{2/3})$. 
6.2. Design Details

In this section, we describe the details of the SW-UCB algorithm. Following its design guideline, the SW-UCB algorithm selects a positive regularization parameter $\lambda > 0$, and initializes $V_0 = \lambda I$. In each round $t$, the SW-UCB algorithm first computes the estimate $\hat{\theta}_t$ for $\theta_t$ according to eq. 3, and then finds the action $X_t$ with largest UCB by solving the optimization problem (6). Afterwards, the corresponding reward $Y_t$ is observed. The pseudo-code of the SW-UCB algorithm is shown in Algorithm 1.

**Algorithm 1** SW-UCB algorithm

1: **Input:** Sliding window size $w$, dimension $d$, variance proxy of the noise terms $R$, upper bound of all the actions’ $\ell_2$ norms $L$, upper bound of all the $\theta_t$’s $\ell_2$ norms $S$, and regularization constant $\lambda$.
2: **Initialization:** $V_0 \leftarrow \lambda I$.
3: for $t = 1, \ldots, T$ do
4: Update $\hat{\theta}_t \leftarrow V_{t-1}^{-1} \left( \sum_{s=1}^{t-1} X_s Y_s \right)$.
5: $X_t \leftarrow \arg \max_{x \in D_t} \left\{ x^\top \hat{\theta}_t + \|x\|_V^{-1} \left[ R \sqrt{d \ln \left( \frac{1 + wL^2/\lambda}{\delta} \right)} + \sqrt{\lambda S} \right] \right\}$.
6: Observe $Y_t = \langle X_t, \theta_t \rangle + \eta_t$.
7: Update $V_t \leftarrow \lambda I + \sum_{s=1}^{t} X_s X_s^\top$.
8: end for

6.3. Regret Analysis

We are now ready to formally state a regret upper bound of the SW-UCB algorithm.

**Theorem 3.** The dynamic regret of the SW-UCB algorithm is upper bounded as

$$ R_T (\text{SW-UCB algorithm}) = \tilde{O} \left( wB_T + \frac{dT}{\sqrt{w}} \right). $$

When $B_t$ ($> 0$) is known, by taking $w = O \left( (dT)^{2/3}B_T^{-2/3} \right)$, the dynamic regret of the SW-UCB algorithm is

$$ R_T (\text{SW-UCB algorithm}) = \tilde{O} \left( d^{3/2}B_T^{5/3}T^{2/3} \right). $$

When $B_t$ is unknown, by taking $w = O \left( (dT)^{2/3} \right)$, the dynamic regret of the SW-UCB algorithm is

$$ R_T (\text{SW-UCB algorithm}) = \tilde{O} \left( d^{3/2} (B_T + 1) T^{2/3} \right). $$
Proof Sketch. The proof utilizes the fact that the per round regret of the SW-UCB algorithm is upper bounded by the UCB of the chosen action, and decomposes the UCB into two separated terms according to Lemmas 1 and 2, i.e., regret in round $t$ is equal to

\[
\text{regret due to non-stationarity in round } t + \text{regret due to estimation error in round } t.
\]

The first term can be upper bounded by a intuitive telescoping sum; while for the second term, although a similar quantity is analyzed by the authors of (Abbasi-Yadkori et al. 2011) using a (beautiful) matrix telescoping technique under the stationary environment, we note that due to the “forgetting principle” of the SW-UCB algorithm, we cannot directly adopt the technique. Our proof thus makes a novel use of the Sherman-Morrison formula to overcome the barrier. Please refer to Section D of the appendix for the complete proof. □

Remark 1. When the variation budget $B_T$ is known, Theorem 3 recommends choosing the length $w$ of the sliding window to be decreasing with $B_T$. The recommendation is in agreement with the intuition that, when the learning environment becomes more volatile, the DM should focus on more recent observations. Indeed, if the underlying learning environment is changing at a higher rate, then the DM’s past observations become obsolete faster. Theorem 3 pins down the intuition of forgetting past observation in face of drifting environments, by providing the mathematical definition of the sliding window length $w$ that yields the optimal regret bound.

7. Bandit-over-Bandits (BOB) Algorithm: Automatically Adapting to the Unknown Variation Budget

In Section 6, we have seen that, by properly tuning $w$, the DM can achieve a first order optimal $O(d^{2/3}(B_T + 1)T^{2/3})$ regret bound even if the knowledge of $B_T$ is not available. However, in the case of an unknown and large $B_T$, i.e., $B_T = \Omega(T^{1/3})$, the bound becomes meaningless as it is linear in $T$. To handle this case, we wish to design an online algorithm that incurs a dynamic regret of order $O(d^\nu B_T^{1/4}T^\sigma)$ for some $\nu \in [0, 1]$ and $\sigma \in (0, 1)$, without knowing $B_T$. Note from Theorem 1, no algorithm can achieve a dynamic regret of order $o(d^{2/3}B_T^{1/3}T^{2/3})$, so we must have $\sigma \geq \frac{2}{3}$. In this section, we develop a novel Bandit-over-Bandits (BOB) algorithm that achieves a regret of $O(d^{2/3}B_T^{1/4}T^{3/4})$. Hence, (BOB) still has a dynamic regret sublinear in $T$ when $B_T = \Theta(T^\rho)$ for any $\rho \in (0, 1)$ and $B_T$ is not known, unlike the SW-UCB algorithm.

7.1. Design Challenges

Reviewing Theorem 3, we know that setting the window length $w$ to a fixed value

\[
w^* = \left[(dT)^{2/3}(B_T + 1)^{-2/3}\right]
\]
can give us a $\tilde{O}\left(d^{2/3}(B_T + 1)^{1/3}T^{2/3}\right)$ regret bound. But when $B_T$ is not provided a priori, we need to also “learn” the unknown $B_T$ in order to properly tune $w$. In a more restrictive setting in which the differences between consecutive $\theta_t$’s follow some underlying stochastic process, one possible approach is applying a suitable machine learning technique to learn the underlying stochastic process at the beginning, and tune the parameter $w$ accordingly. In the more general setting, however, this strategy cannot work as the change between consecutive $\theta_t$’s can be arbitrary (or even adversarially) as long as the total variation is bounded by $B_T$.

7.2. Design Intuition

The above mentioned observations as well as the established results motivate us to make use of the SW-UCB algorithm as a sub-routine, and “hedge” (Auer et al. 2002a, Audibert and Bubeck 2009) against the (possibly adversarial) changes of $\theta_t$’s to identify a reasonable fixed window length.

To this end, we describe the main idea of the Bandit-over-Bandits (BOB) algorithm. As illustrated in Fig. 1, the BOB algorithm divides the whole time horizon into $\lceil T/H \rceil$ blocks of equal length $H$ rounds (the last block can possibly have less than $H$ rounds), and specifies a set $J$ from which each $w_i$ is drawn from. For each block $i \in \lceil T/H \rceil$, the BOB algorithm first selects a window length $w_i \in J$, and restarts the SW-UCB algorithm with the selected window length as a sub-routine to choose actions for this block. On top of this, the BOB algorithm also maintains a separate algorithm for adversarial multi-armed bandits, e.g., the EXP3 algorithm for adversarial multi-armed bandits against an oblivious adversary (Auer et al. 2002a, Audibert and Bubeck 2009), to govern the selection of window length for each block, and thus the name Bandit-over-Bandits. Here, the total reward of each block is used as feedback for the EXP3 algorithm. It is worth emphasizing that

1. For validity of the EXP3 algorithm, the SW-UCB algorithm for each block $i$ does not use any data collected in previous blocks $1, \ldots, i-1$ except for the choice of window length $w_i$ (please see Remark 2 for a more involved discussion).

2. Due to the design of restarting, any instance of the SW-UCB algorithm cannot last for more than $H$ rounds. As a consequence, even if the EXP3 selects a window length $w_i > H$ for some block $i$, the effective window length is $H$. It is thus reasonable to enforce that $J$ is a subset of $[H]$, i.e., $J \subseteq [H]$.

To determine $H$ and $J$, we first consider the regret of the BOB algorithm. As mentioned above, since $w^*$ is not necessarily attainable, i.e., by definition in eq. (12), $w^* = \lfloor (dT)^{2/3}(B_T + 1)^{-2/3} \rfloor$ might be larger than $H$ when $B_T$ is small, we hence denote the optimally (over $J$) tuned window length as $w^*$.

To determine $H$ and $J$, we first consider the regret of the BOB algorithm. As mentioned above, since $w^*$ is not necessarily attainable, i.e., by definition in eq. (12), $w^* = \lfloor (dT)^{2/3}(B_T + 1)^{-2/3} \rfloor$ might be larger than $H$ when $B_T$ is small, we hence denote the optimally (over $J$) tuned window length as $w^*$. By design of the BOB algorithm, its regret can be decomposed as the regret of the SW-UCB algorithm with the optimally tuned window length $w_i = w^*$ for each block $i$ plus the loss due to learning the value $w^*$ with the EXP3 algorithm, i.e.,

$$
E[\text{Regret}_T(\text{BOB algorithm})]
$$
Here, eq. (13) holds as the BOB algorithm restarts the SW-UCB algorithm in each block, and for a round $t$ in block $i$, $X_t(w)$ refers to the action selected in round $t$ by the SW-UCB algorithm with window length $w \land (t - (i-1)H - 1)$ initiated at the beginning of block $i$.

By Theorem 3, the first expectation in eq. (13) can be upper bounded as

$$
E\left[\sum_{i=1}^{T} \langle x_t^*, \theta_t \rangle - \sum_{i=1}^{\lceil T/H \rceil} \sum_{t=(i-1)H+1}^{iH \land T} \langle X_t(w^i), \theta_t \rangle\right] = \sum_{i=1}^{\lceil T/H \rceil} \tilde{O}\left( w^iB_T(i) + \frac{dH}{\sqrt{w^i}} \right)
$$

where

$$B_T(i) = \sum_{t=(i-1)H+1}^{(iH \land T)-1} \|\theta_t - \theta_{t+1}\|$$

is the total variation in block $i$.

We then turn to the second expectation in eq. (13). We can easily see that the number of rounds for the EXP3 algorithm is $\lceil T/H \rceil$ and the number of possible values of $w_i$'s is $|J|$. Denoting the maximum absolute sum of rewards of any block as random variable $Q$, the authors of [Auer et al. 2002a] give the following regret bound.

$$
E\left[\sum_{i=1}^{\lceil T/H \rceil} \sum_{t=(i-1)H+1}^{iH \land T} \langle X_t(w^i), \theta_t \rangle - \sum_{i=1}^{\lceil T/H \rceil} \sum_{t=(i-1)H+1}^{iH \land T} \langle X_t(w_i), \theta_t \rangle\right] \leq \tilde{O}\left( Q\sqrt{\frac{|J|}{H}} \right).
$$

(15)
To proceed, we have to give a high probability upper bound for $Q$.

**Lemma 3.** With probability at least $1 - \frac{2}{T}$, $Q$ does not exceed $H + 2R\sqrt{\frac{H \ln(T/\sqrt{H})}{\sqrt{H}}}$, i.e.,

$$\Pr\left( Q \leq H + 2R\sqrt{\frac{H \ln(T/\sqrt{H})}{\sqrt{H}}} \right) \geq 1 - \frac{2}{T}. $$

**Proof Sketch.** The proof makes use of the $R$-sub-Gaussian property of the noise terms as well as the union bound. Please refer to Section E of the for the complete proof. □

Note that the regret of our problem is at most $T$, eq. (15) can be further upper bounded as

$$E \left[ \sum_{t=1}^{[T/H]} \sum_{i=(t-1)H+1}^{i \wedge T} \langle X_t(w^t), \theta_i \rangle - \sum_{t=1}^{[T/H]} \sum_{i=(t-1)H+1}^{[T/H]} \langle X_t(w_i), \theta_i \rangle \right]$$

$$\leq \tilde{O} \left( Q \sqrt{\frac{|J|}{H}} \right) Q \leq H + 2R\sqrt{\ln T} \times \Pr \left( Q \leq H + 2R\sqrt{\ln \frac{T}{\sqrt{H}}} \right)$$

$$+ E \left[ \tilde{O} \left( Q \sqrt{\frac{|J|}{H}} \right) Q \geq H + 2R\sqrt{\ln T} \times \Pr \left( Q \geq H + 2R\sqrt{\ln \frac{T}{\sqrt{H}}} \right) \right]$$

$$= \tilde{O} \left( \sqrt{H|J|T} \right) + \frac{2}{T} + \tilde{O} \left( \sqrt{H|J|T} \right).$$

Combining eq. (13), (14), and (16), the regret of the BOB algorithm is

$$\mathcal{R}_T(\text{BOB algorithm}) = \tilde{O} \left( w^T B_T + \frac{dT}{\sqrt{w^T}} + \sqrt{H|J|T} \right).$$

Eq. (17) exhibits a similar structure to the regret of the SW-UCB algorithm as stated in Theorem 3, and this immediately indicates a clear trade-off in the design of the block length $H$:

- On one hand, $H$ should be small to control the regret incurred by the EXP3 algorithm in identifying $w^t$, i.e., the third term in eq. (17).
- On the others, $H$ should also be large enough to allow $w^t$ to get close to $w^* = \lfloor (dT)^{2/3} (B_T + 1)^{-2/3} \rfloor$ so that the sum of the first two terms in eq. (17) is minimized.

A more careful inspection also reveals the tension in the design of $J$. Obviously, we hope that $|J|$ is small to minimize the third term in eq. (17), but we also wish $J$ to be dense enough so that it forms a cover to the set $H$. Otherwise, even if $H$ is large enough that $w^t$ can approach $w^*$, approximating $w^*$ with any element in $J$ can cause a major loss.

These observations suggest the following choice of $J$.

$$J = \left\{ H^0, \lfloor H^{1/3} \rfloor, \ldots, H \right\}$$

(18)

for some positive integer $\Delta$, and since the choice of $H$ should not depend on $B_T$, we can set $H = \lfloor d^\alpha T^\alpha \rfloor$ with some $\alpha \in [0, 1]$ and $\epsilon > 0$ to be determined. We then distinguish two cases depending on whether $w^*$ is smaller than $H$ or not (or alternatively, whether $(B_T + 1)$ is larger than $d^{(2-3\epsilon)/2} T^{(2-3\alpha)/2}$ or not).
Case 1: \( w^* \leq H \) or \((B_T + 1) \geq d^{(2-3\epsilon)/2}T^{(2-3\alpha)/2}\). Under this situation, \( w^\dagger \) can automatically adapt to the nearly optimal window length \( \text{clip}_J(w^*) \), where \( \text{clip}_J(x) \) finds the largest element in \( J \) that does not exceed \( x \). Notice that \(|J| = \Delta + 1\), the regret of the BOB algorithm then becomes

\[
R_T(\text{BOB algorithm}) = \tilde{O} \left( d^\frac{3}{2} (B_T + 1)^{\frac{1}{2}} T^\frac{3}{2} H^\frac{3}{2} + d^\frac{3}{2} T^{\frac{3}{2} \frac{\Delta + 1}{2}} \Delta^\frac{1}{2} \right). \tag{19}
\]

Case 2: \( w^* > H \) or \((B_T + 1) < d^{(2-3\epsilon)/2}T^{(2-3\alpha)/2}\). Under this situation, \( w^\dagger \) equals to \( H \), which is the window length closest to \( w^* \), the regret of the BOB algorithm then becomes

\[
R_T(\text{BOB algorithm}) = \tilde{O} \left( d^{1-\frac{\Delta}{2}} T^{\frac{3}{2} \epsilon} + d^{\frac{3}{2} T^{\frac{3}{2} \frac{\Delta + 1}{2}}} \Delta^\frac{1}{2} \right), \tag{20}
\]

where we have make use of the fact that \((B_T + 1) < d^{(2-3\epsilon)/2}T^{(2-3\alpha)/2}\).

Now both eq. (19) and eq. (20) suggests that we should set \( \Delta = \lceil \ln H \rceil \), and eq. (20) further reveals that we should take \( \alpha = 1/2 \) and \( \epsilon = 1 \). Plugging these choices of parameters back to case 1 and eq. (19), we have when \( B_T + 1 \geq d^{-1/2}T^{1/4} \), the dynamic regret of the BOB algorithm is upper bounded as

\[
R_T(\text{BOB algorithm}) = \tilde{O} \left( d^\frac{3}{2} (B_T + 1)^{\frac{1}{2}} T^\frac{3}{2} \right); \tag{21}
\]

while if \( B_T + 1 < d^{-1/2}T^{1/4} \) (or case 2), the dynamic regret of the BOB algorithm is upper bounded as

\[
R_T(\text{BOB algorithm}) = \tilde{O} \left( d^\frac{3}{2} T^\frac{3}{2} \right) \tag{22}
\]

according to eq. (20). Here we have to emphasize that the choice of \( w^\dagger, \alpha, \) and \( \epsilon \) are purely for the purpose of analysis, while the only parameters that we need to decide are

\[
H = \left\lfloor dT^{\frac{1}{2}} \right\rfloor, \Delta = \lceil \ln H \rceil, J = \left\{ H^0, \lceil H^{\frac{1}{2}} \rceil, \ldots, H \right\}, \tag{23}
\]

which clearly do not depend on \( B_T \).

7.3. Design Details

We are now ready to describe the details of the BOB algorithm. With \( H, \Delta \) and \( J \) defined as eq. (23), the BOB algorithm additionally initiates the parameter

\[
\gamma = \min \left\{ 1, \sqrt{\frac{\Delta + 1}{(\Delta + 1) \ln B_T}} \right\}, s_{j,1} = 1 \quad \forall j = 0, 1, \ldots, \Delta. \tag{24}
\]
for the EXP3 algorithm (Auer et al. 2002a). The BOB algorithm then divides the time horizon $T$ into $[T/H]$ blocks of length $H$ rounds (except for the last block, which can be less than $H$ rounds). At the beginning of each block $i \in [T/H]$, the BOB algorithm first sets

$$p_{j,i} = (1 - \gamma) \frac{s_{j,i}}{\sum_{u=0}^{\Delta} s_{u,i}} + \frac{\gamma}{\Delta + 1} \quad \forall j = 0, 1, \ldots, \Delta,$$

and then sets $j_i = j$ with probability $p_{j,i} \forall j = 0, 1, \ldots, \Delta$. The selected window length is thus $w_i = \lfloor H^{j_i/\Delta} \rfloor$. Afterwards, the BOB algorithm selects actions $X_t$ by running the SW-UCB algorithm with window length $w_i$ for each round $t$ in block $i$, and the total collected reward is

$$\sum_{t=(i-1)H+1}^{iH+1} Y_t = \sum_{t=(i-1)H+1}^{iH+1} \langle X_t, \theta_t \rangle + \eta_t.$$

Finally, the rewards are rescaled by dividing $2H + 4R\sqrt{H \ln(T/\sqrt{H})}$, and then added by $1/2$ so that it lies within $[0, 1]$ with high probability, and the parameter $s_{j,i,i+1}$ is set to

$$s_{j,i,i+1} = \exp\left( \frac{\gamma}{(\Delta + 1)p_{j,i}} \left( \frac{1}{2} + \frac{\sum_{t=(i-1)H+1}^{iH+1} Y_t}{2H + 4R\sqrt{H \ln(T/\sqrt{H})}} \right) \right);$$

while $s_{u,i+1}$ is the same as $s_{u,i}$ for all $u \neq j_i$. The pseudo-code of the BOB algorithm is shown in Algorithm 2.

**Algorithm 2 BOB algorithm**

1. **Input**: Time horizon $T$, the dimension $d$, variance proxy of the noise terms $R$, upper bound of all the actions’ $\ell_2$ norms $L$, upper bound of all the $\theta_t$’s $\ell_2$ norms $S$, and a constant $\lambda$.
2. **Initialize** $H, \Delta, J$ by eq. (23), $\gamma, \{s_{j,1}\}_{j=0}^{\Delta}$ by eq. (24).
3. **for** $i = 1, 2, \ldots, [T/H] \textbf{ do}$
   4. Define distribution $(p_{j,i})_{j=0}^{\Delta}$ by eq. (25), and set $j_i \leftarrow j$ with probability $p_{j,i}$.
   5. $w_i \leftarrow \lfloor H^{j_i/\Delta} \rfloor$, $V_{(i-1)H} \leftarrow \lambda I$.
   6. **for** $t = (i-1)H + 1, \ldots, i \cdot H \textbf{ do}$
      7. Update $\hat{\theta}_t \leftarrow V_{t-1}^{-1} \left( \sum_{s=(i-1)H+1}^{iH+1} X_s Y_s \right)$.
      8. Pick action $X_t \leftarrow \arg \max_{x \in D_i} \left\{ x^\top \hat{\theta}_t + \|x\|_{V_{t-1}} \left[ R \sqrt{d \ln(T(1 + w_iL^2/\lambda)) + \lambda S} \right] \right\}$.
      9. Observe $Y_t = \langle X_t, \theta_t \rangle + \eta_t$.
      10. Update $V_t \leftarrow \lambda I + \sum_{s=(i-1)H+1}^{iH+1} X_s X_s^\top$.
   11. **end for**
5. **end for**
6. Update $s_{j_i,i+1}$ according to eq. (26), and $s_{u,i+1} \leftarrow s_{u,i} \forall u \neq j_i$.
7. **end for**
7.4. Regret Analysis

We are now ready to present the regret analysis of the BOB algorithm.

**Theorem 4.** The dynamic regret of the BOB algorithm is upper bounded as follows.

- **When** $B_T + 1 \geq d^{-1/2} T^{1/4}$,
  \[
  R_T(BOB\ \text{algorithm}) = \tilde{O}\left(d^2 (B_T + 1)^{1/4} T^{3/4}\right).
  \]

- **When** $B_T + 1 < d^{-1/2} T^{1/4}$,
  \[
  R_T(BOB\ \text{algorithm}) = \tilde{O}\left(d T^{3/4}\right).
  \]

**Proof Sketch.** The proof of the theorem essentially follows Section 7.2, and please refer to Section F of the appendix for the complete proof.

**Remark 2.** The block structure and restarting the SW-UCB algorithm with a single window length for each block are essential for the correctness of the BOB algorithm. Otherwise, suppose the DM utilizes the EXP3 algorithm to select the window length $w_t$ for each round $t$, and implements the SW-UCB algorithm with the selected window length without ever restarting it. Instead of eq. (13), the regret of the BOB algorithm is then decomposed as

\[
\sum_{t=1}^{T} \left( \text{Reward of SW-UCB}\left(\{w^t\}_{\tau=1}^{t-1}\right) \text{ in round } t - \text{Reward of SW-UCB}\left(\{w_{\tau}\}_{\tau=1}^{t-1}\right) \text{ in round } t \right)
\]

\[
+ \sum_{t=1}^{T} \left( \text{Optimal reward in round } t - \text{Reward of SW-UCB}\left(\{w^t\}_{\tau=1}^{t-1}\right) \text{ in round } t \right)
\]

Here, with some abuse of notations, $\text{SW-UCB}\left(\{w^t\}_{\tau=1}^{t-1}\right)$ refers to in round $t$, the DM runs the SW-UCB algorithm with window length $w^t$ and historical data, e.g., (action, reward) pairs, generated by running the SW-UCB algorithm with window length $w^t$ for rounds $\tau = 1, \ldots, t - 1$. Same as before, the second term of eq. (27) can be upper bounded as a result of Theorem 3. It is also tempting to apply results from the EXP3 algorithm to upper bound the first term. Unfortunately, this is incorrect as it is required by the adversarial bandits protocol (Auer et al. 2002a) that the DM and its competitor should receive the same reward if they select the same action, i.e., the reward of $\text{SW-UCB}\left(\{w^t\}_{\tau=1}^{t-1}, w_t = w \right)$ in round $t$ and the reward of $\text{SW-UCB}\left(\{w_{\tau}\}_{\tau=1}^{t-1}, w_t = w^t \right)$ in round $t$ should be the same for every $w$. Nevertheless, this is violated as running the SW-UCB algorithm with different window length for previous rounds can generate different (action, reward) pairs, and this results in possibly different estimated $\hat{\theta}_t$’s for the two SW-UCB algorithms even if both of them use the same window length in round $t$. Hence, the selected actions and the corresponding rewards by these two instances might also be different.
By the careful design of blocks as well as the restarting scheme, the \textbf{BOB} algorithm decouples the \texttt{SW-UCB} algorithm for a block from previous blocks, and thus fixes the above mentioned problem, \textit{i.e.}, the regret of the \textbf{BOB} algorithm is decomposed as eq. (13).

\textbf{Remark 3.} The assumptions that the decision sets \(D_t\)'s and the underlying parameters \(\theta_t\)'s are chosen by an oblivious adversary as well as the i.i.d. noise terms are important. With these, the \texttt{EXP3} algorithm is also facing an obliviously adversarial environment, and thus satisfies the adversarial bandits protocol \cite{Auer2002a, Audibert2009}. To see this, assuming additional access to all the i.i.d. (and thus independent of the DMs' actions) noise terms in advance, the adversary can determine the total rewards of each block with respect to each window length in \(J\), independently of the \texttt{EXP3} algorithm, by running the \texttt{SW-UCB} algorithm with that window length as well as the obliviously chosen \(D_t\)'s and \(\theta_t\)'s,

\textbf{Remark 4.} The structure of the \textbf{BOB} algorithm can be roughly seen as using a \texttt{EXP3} algorithm to govern the behaviors of many copies of \texttt{SW-UCB} algorithms with different window lengths. As mentioned in Section \ref{sec:appendix} it has a flavor similar to the technique of bandits coralling/aggregation, see \textit{e.g.}, \cite{Agarwal2017, Besbes2018} and references therein. Existing works, such as \cite{Luo2018, Besbes2018}, have tried to apply bandits coralling/aggregation to the non-stationary bandits settings, but when \(B_T\) is unknown, they can only obtain either sub-optimal dynamic regret bounds \cite{Luo2018} or empirical performance guarantees \cite{Besbes2018}.

8. Applications

In this section, we demonstrate the generality of our established results. As illustrative examples, we apply the sliding window UCB algorithm as well as the bandit-over-bandit framework to the prevalent multi-armed bandits \cite{Auer2002b} and generalized linear bandit settings. In what follows, we shall derive the \texttt{SW-UCB} algorithm as well as the parameters required by the \textbf{BOB} algorithm, \textit{i.e.}, similar to those defined in eq. (23), for both cases. We note that same steps can be applied to other settings, such as Gaussian process bandits \cite{Srinivas2010} and combinatorial bandits with sem-bandit feedback \cite{Kveton2015}, to derive similar results.

8.1. Multi-Armed Bandits

In the \(d\)-armed bandits setting, every action set \(D_t\) is comprised of \(d\) actions \(e_1, \ldots, e_d\). The \(i\)th action \(e_i\) has coordinate \(i\) equals to 1 and all other coordinates equal to 0. Therefore, the reward of choosing action \(X_t = e_{t_i}\) in round \(t\) is

\[ (X_t, \theta_t) + \eta_t = \theta_{t_i} + \eta_t, \]
where $\theta_{t,I}$ is the $I$th coordinate of $\theta_t$. For a window length $w$, we also define $N_{i,t-1}$ as the number of times that action $i$ is chosen within rounds. $(t-w), \ldots, (t-1)$, i.e., for all $i \in [d]$,

$$N_{i,t-1} = \sum_{s=1}^{t-1} 1[X_s = e_i].$$

Here $1[\cdot]$ is the indicator function.

Similar to the procedure in Section 5, we set the regularization parameter $\lambda = 0$, and compute the sliding window least squares estimate $\hat{\theta}_t$ for $\theta_t$ in each round, i.e.,

$$\hat{\theta}_t = V_{t-1}^* \left( \sum_{s=1}^{t-1} X_s Y_s \right),$$  \hspace{1cm} (28)

where $V_{t-1}^*$ is Moore-Penrose pseudo-inverse of $V_{t-1}$. We can also derive the deviation inequality type bound for the latent expected reward of every action $x \in D_t$ in any round $t$.

**Theorem 5.** For any $t \in [T]$ and any $i \in [d]$, we have with probability at least $1 - 1/T$,

$$\left| e_i^T (\hat{\theta}_t - \theta_t) \right| \leq \sum_{s=1}^{t-1} \| \theta_s - \theta_{s+1} \| + R \sqrt{2 \ln (2dT^2)} \| e_i \|_{V_{t-1}^*},$$

holds for all $x \in D_t$.

**Proof Sketch.** The proof essentially follows that of Theorem 2 but it further takes advantage of the fact that all the actions are “1-hot” vectors. Please refer to Section G of the appendix for the complete proof. □

We can now follow the same principle in Section 6 by choosing in each round the action $X_t$ with the highest UCB, i.e.,

$$X_t = \arg \max_{x \in D_t} \left\{ \langle x, \hat{\theta}_t \rangle + R \sqrt{2 \ln (2dT^2)} \| x \|_{V_{t-1}^*} \right\},$$  \hspace{1cm} (29)

and arrive at the following regret upper bound for the SW-UCB algorithm.

**Theorem 6.** The dynamic regret of the SW-UCB algorithm is upper bounded as

$$R_T (SW-UCB \text{ algorithm}) = \tilde{O} \left( wB_T + \frac{\sqrt{dT}}{\sqrt{w}} \right).$$

When $B_t$ ($>0$) is known, by taking $w = O \left( d^{1/3}T^{2/3}B_T^{-2/3} \right)$, the dynamic regret of the SW-UCB algorithm is

$$R_T (SW-UCB \text{ algorithm}) = \tilde{O} \left( d^{1/3}B_T^{1/3}T^{4/3} \right).$$

When $B_t$ is unknown, by taking $w = O \left( d^{1/3}T^{2/3} \right)$, the dynamic regret of the SW-UCB algorithm is

$$R_T (SW-UCB \text{ algorithm}) = \tilde{O} \left( d^{1/3} (B_T + 1) T^{4/3} \right).$$
Proof Sketch. The proof of this theorem is very similar to that of Theorem 3 and is thus omitted. The key difference is that $\beta$ (defined in eq. (5) for the linear bandit setting) is now set to $R\sqrt{2\ln(2dT^2)}$, and this saves the extra $\sqrt{d}$ factor presented in eq. (60). Hence the dynamic regret bounds can be obtained accordingly. □

Comparing the results obtained in Theorem 6 to the lower bound presented in (Besbes et al. 2015), we can easily see that the dynamic regret bound is optimal when $B_T$ is known. When $B_T$ is unknown, we can implement the BOB algorithm with the following parameters:

$$H = \left\lfloor (dT)^{\frac{1}{2}} \right\rfloor, \Delta = \lfloor \ln H \rfloor, J = \left\{ H^0, \lfloor H^{\frac{1}{2}} \rfloor, \ldots, H \right\}.$$  (30)

The regret of the BOB algorithm for the MAB setting is characterized as follows.

**Theorem 7.** The dynamic regret of the BOB algorithm is upper bounded as follows.

- When $B_T + 1 \geq d^{-1/4}T^{1/4}$,

$$R_T(BOB \text{ algorithm}) = \tilde{O}\left(d^{\frac{1}{2}}(B_T + 1)^{\frac{1}{4}}T^{\frac{3}{4}}\right).$$

- When $B_T + 1 < d^{-1/4}T^{1/4}$,

$$R_T(BOB \text{ algorithm}) = \tilde{O}\left(d^{\frac{1}{2}}T^{\frac{3}{4}}\right).$$

Proof Sketch. The proof of the theorem uses the same regret decomposition and a similar cutoff for $B_T$ (except that the exponent of $d$ is halved) as those of Theorem 4 and it is thus omitted. □

### 8.2. Generalized Linear Bandits

For the generalized linear bandits model, we adopt the setup in Filippi et al. (2010), Li et al. (2017): it is essentially the same as the linear bandit setting except that the decision set is time invariant, i.e., $D_t = D$ for all $t \in [T]$, and the reward of choosing action $X_t \in D$ is

$$Y_t = \mu(\langle X_t, \theta \rangle) + \eta_t.$$  

Let $\hat{\mu}(\cdot)$ and $\check{\mu}(\cdot)$ denote the first derivative and second derivative of $\mu(\cdot)$, respectively, we follow (Filippi et al. 2010) to make the following assumptions.

**Assumption 1.** There exists a set of $d$ actions $a_1, \ldots, a_d \in D$ such that the minimal eigenvalue of $\sum_{i=1}^d a_i a_i^\top$ is $\lambda_0$ ($> 0$).

**Assumption 2.** The link function $\mu(\cdot): \mathbb{R} \to \mathbb{R}$ is strictly increasing, continuously differentiable, Lipschitz with constant $k_\mu$, and we define $c_\mu = \inf_{x \in D, \theta \in \mathbb{R}^d, \| \theta \| \leq S} \hat{\mu}(\langle x, \theta \rangle)$.

**Assumption 3.** There exists $Y_{\text{max}} > 0$ such that for any $t \in [T]$, $Y_t \in [0, Y_{\text{max}}]$.  

Similar to the procedure in Section 5, we compute the maximum quasi-likelihood estimate \( \hat{\theta}_t \) for \( \theta_t \) in each round \( t \in [T] \) by solving the equation

\[
\sum_{s=1\setminus(t-w)}^{t-1} \left( Y_s - \mu \left( \langle X_s, \hat{\theta}_t \rangle \right) \right) X_s = 0.
\] (31)

Defining

\[
\beta = \frac{2k_{\mu} Y_{\max} \sqrt{2d \ln(w) \ln(2dT^2) (3 + 2 \ln(1 + 2L^2/\lambda_0))}}{c_{\mu}}
\]

we can also derive the deviation inequality type bound for the latent expected reward of every action \( x \in D_t \) in any round \( t \).

**Theorem 8.** For any \( t \in [T] \), we have with probability at least \( 1 - 1/T \),

\[
\left| \mu \left( x^\top \hat{\theta}_t \right) - \mu \left( x^\top \theta_t \right) \right| \leq \frac{k_{\mu}^2 L}{c_{\mu}} \sum_{s=1\setminus(t-w)}^{t-1} \| \theta_s - \theta_{s+1} \| + \beta \| x \|_{V_{t-1}^*}
\]

holds for all \( x \in D_t \).

**Proof Sketch.** The proof is a consequence of Proposition 1 of [Filippi et al. 2010] and Theorem 2. Please refer to Section H of the appendix for the complete proof. \( \square \)

We can now follow the same principle in Section 6 to design the SW-UCB algorithm. Note that in order for \( V_{t-1} \) to be invertible for all \( t \), our algorithm should select the actions \( a_1, \ldots, a_d \) every \( w \) rounds for some window length \( w \). For the rest of the rounds, it chooses in each round the action \( X_t \) with the highest UCB, i.e.,

\[
X_t = \arg \max_{x \in D_t} \left\{ \langle x, \hat{\theta}_t \rangle + \beta \| x \|_{V_{t-1}^*} \right\},
\] (32)

and arrive at the following regret upper bound.

**Theorem 9.** The dynamic regret of the SW-UCB algorithm is upper bounded as

\[
R_T (SW-UCB \text{ algorithm}) = \tilde{O} \left( wB_T + \frac{dT}{\sqrt{w}} \right).
\]

When \( B_t \ (> 0) \) is known, by taking \( w = O \left( (dT)^{2/3} B_T^{-2/3} \right) \), the dynamic regret of the SW-UCB algorithm is

\[
R_T (SW-UCB \text{ algorithm}) = \tilde{O} \left( d^{3/2} B_T^{3/2} T^{2} \right).
\]

When \( B_t \) is unknown, by taking \( w = O \left( (dT)^{2/3} \right) \), the dynamic regret of the SW-UCB algorithm is

\[
R_T (SW-UCB \text{ algorithm}) = \tilde{O} \left( d^{3} (B_T + 1) T^{2} \right).
\]
Proof Sketch. The proof of this theorem is similar to that of Theorem 3 and is thus omitted. The only difference is that we need to include the regret contributed by selecting actions \(a_1, \ldots, a_d\) every \(w\) rounds. But these sums to \(\tilde{O}(dT/w)\), which is dominated by the term \(\tilde{O}(dT/\sqrt{w})\). Hence the dynamic regret bounds can be obtained similarly as the linear bandit setting.

Theorem 10. The dynamic regret of the BOB algorithm is upper bounded as follows.

- When \(B_T + 1 \geq d^{-1/2} T^{1/4}\),
  \[
  R_T(\text{BOB algorithm}) = \tilde{O} \left( d^{3/2} (B_T + 1)^{1/4} T^{3/4} \right).
  \]
- When \(B_T + 1 < d^{-1/2} T^{1/4}\),
  \[
  R_T(\text{BOB algorithm}) = \tilde{O} \left( d^{3/2} T^{3/4} \right).
  \]

9. Numerical Experiments

As a complement to our theoretical results, we conduct numerical experiments on synthetic dataset and the CPRM-12-001: On-Line Auto Lending dataset provided by the Center for Pricing and Revenue Management at Columbia University to compare the regret performances of the SW-UCB algorithm and the BOB algorithm with a modified EXP3.S algorithm analyzed in (Besbes et al. 2018) as well as the UCB type algorithms for stationary \(K\)-armed and linear bandit settings proposed in (Auer et al. 2002b, Abbasi-Yadkori et al. 2011).

9.1. Experiments on Synthetic Dataset

For synthetic dataset, we first evaluate the growth of regret when \(T\) increases, under each of the above-mentioned algorithms in Section 9.1.1. We follow the setup of (Besbes et al. 2018) for fair comparisons. Then, in Section 9.1.2, we fix \(T = 10^5\), and evaluate the behavior of these algorithms across time steps.

9.1.1. The Trend of Regret with Varying \(T\)

We consider a 2-armed bandit setting, and we vary \(T\) from \(3 \times 10^4\) to \(2.4 \times 10^5\) with a step size of \(3 \times 10^4\). We set \(\theta_t\) to be the following sinusoidal process, \(\text{i.e.}, \forall t \in [T],\)

\[
\theta_t = \begin{pmatrix}
0.5 + 0.3 \sin (5B_T \pi t/T) \\
0.5 + 0.3 \sin (\pi + 5B_T \pi t/T)
\end{pmatrix}.
\]

The total variation of the \(\theta_t\)'s across the whole time horizon is upper bounded by \(\sqrt{2}B_T = O(B_T)\). We also use i.i.d. normal distribution with \(R = 0.1\) for the noise terms.
Known Constant Variation Budget. We start from the known constant variation budget case, i.e., $B_T = 1$, to measure the regret growth of the two optimal algorithms, i.e., the SW-UCB algorithm and the modified EXP3.S algorithm, with respect to the total number of rounds. The log-log plot is shown in Fig. 2(a). From the plot, we can see that the regret of SW-UCB algorithm is only about 20% of the regret of EXP3.S algorithm.

Unknown Time-Dependent Variation Budget. We then turn to the more realistic time-dependent variation budget case, i.e., $B_T = T^{1/3}$. As the modified EXP3.S algorithm does not apply to this setting, we compare the performances of the SW-UCB algorithm and the BOB algorithm. The log-log plot is shown in Fig. 2(b). From the results, we verify that the slope of the regret growth of both algorithms roughly match the established results, and the regret of BOB algorithm’s is much smaller than that of the SW-UCB algorithm’s.

![Log-log plot for $B_T = O(1)$](image1)

![Log-log plot for $B_T = O(T^{1/3})$](image2)

Figure 2 Results for gradually change environment with 2 arms

9.1.2. A Further Study on the Algorithms’ Behavior We provide additional numerical evaluation, by considering piecewise linear instances, where the reward vector $\theta_t \in \mathbb{R}^d$ is a randomly generated piecewise linear function of $t$. To generate such an instance, we first set $T = 10^5$, and then we randomly sample 30 time points in $\tau_1, \tau_2, \ldots, \tau_{30} \in \{2, \ldots, T - 1\}$ without replacement. We further denote $\tau_0 = 1, \tau_{31} = T$. After that, we randomly sample 32 random unit length vectors $v_0, \ldots, v_{31} \in \mathbb{R}^d$. Finally, for each $t \in [T]$, we define $\theta_t$ as the linear interpolation between $v_s, v_{s+1}$, where $\tau_s \leq t \tau_{s+1}$. More precisely, we have $\theta_t = ((\tau_{s+1} - t)v_s + (t - \tau_s)v_{s+1})/(\tau_{s+1} - \tau_s)$. Note that the random reward in each period can be negative.

In what follows, we first evaluate the performance of the algorithms by (Besbes et al. 2018) as well as our algorithms in a 2-armed bandit piece-wise linear instance. Then, we evaluate the
performance of our algorithms in a linear bandit piece-wise linear instance, where $d = 5$, and each $D_t$ is a random subset of 40 unit length vectors in $\mathbb{R}^d$. We do not evaluate the algorithms by (Besbes et al. 2018) in the second instance, since the algorithms by (Besbes et al. 2018) are only designed for the non-stationary $K$-armed bandit setting. For each instance, each algorithm is evaluated 50 times.

**Two armed bandits.** We first evaluate the performance of the modified EXP.3S in (Besbes et al. 2018) as well as the performance of the SW-UCB algorithm, BOB algorithm in a randomly generated 2-armed bandit instance. Fig 3(a) illustrates the average cumulative reward earned by each algorithm in the 50 trials, and Fig 3(b) depicts the average dynamic regret incurred by each algorithm in the 50 trials. In Figs 3(a), 3(b), shorthand SW-UCB-opt is the SW-UCB algorithm, where $B_T$ is known and $w = w^{opt}$ is set to be regret-optimal (see Appendix I for the expression of $w^{opt}$). Shorthand EXP3.S stands for the modified EXP3.S algorithm by (Besbes et al. 2018), where $B_T$ is known and the learning rate is set to be regret-optimal. Shorthand BOB stands for the BOB algorithm. Shorthand SW-UCB-obl is the SW-UCB algorithm, where $B_T$ is not known, and $w = w^{obl}$ is obliviously set (see Appendix I for the expression of $w^{obl}$). Finally, shorthand UCB stands for the UCB algorithm by (Abbasi-Yadkori et al. 2011), which is designed for the stationary linear bandit problem. Note that $B_T$ is known to SW-UCB-opt, EXP3.S, but not to BOB, SW-UCB-obl, UCB.

Overall, we observe that SW-UCB-opt is the better performing algorithm when $B_T$ is known, and BOB is the best performing when $B_T$ is not known. It is evident from Fig 3(a) that SW-UCB-opt, EXP3.S and BOB are able to adapt to the change in the reward vector $\theta_t$ across time $t$. We remark that BOB, which does not know $B_T$, achieves a comparable amount of cumulative reward to EXP3.S, which does know $B_T$, across time. It is also interesting to note that UCB, which is designed for the stationary setting, fails to converge (or even to achieve a non-negative total reward) in the long run, signifying the need of an adaptive UCB algorithm in a non-stationary setting.

**Linear bandits.** Next, we move to the linear bandit case, and we consider the performance of SW-UCB-opt, SW-UCB-obl, BOB and UCB, as illustrated in Figs 4(a), 4(b). While the performance of the algorithms ranks similarly to the previous 2-armed bandit case, we witness that UCB, which is designed for the stationary setting, has a much better performance in the current case than the 2-armed case. We surmise that the relatively larger size of the arm space $D_t$ here allows UCB to choose an arm that performs well even when the reward vector is changing.

9.2. **Experiments on Online Auto-Lending Dataset**

We now conduct experiments on the on-line auto lending dataset, which was first studied by (Phillips et al. 2015), and subsequently used to evaluate dynamic pricing algorithms by (Ban and
Similar to \cite{Ban and Keskin 2018}, we use the first $T = 5 \times 10^4$ arrivals that span 276 days for this experiment, and impute the price of a loan as the net present value of future payments (a function of the monthly payment, customer rate, and term approved, please refer to the cited references for more details). The range of price in our experiment is set to $[0, 500]$ with a step size of 10, and we use the feature selection results in \cite{Ban and Keskin 2018} to pick FICO score, the term of contract, the loan amount approved, prime rate, the type of car, and the competitor’s
rate as the feature vector for each customer. We adopt the commonly used \cite{Li2010,Besbes2015} linear model to interpolate the response of each customer: for a customer with feature \( x \), if price \( p \) is offered, she accepts the offer with “probability” \( \langle \theta, [x; px] \rangle \). Although the customers’ responses are binary, \( i.e., \) whether or not she accepts the loan, \cite{Besbes2015} theoretically justified that the revenue loss caused by using this misspecified model is negligible. For the changing environment, we assume that the \( \theta_t \) remains stationary in a single day period, but can change across days, and use linear regression method to recover the underlying \( \theta_t \)'s. The resulted \( B_T \) is \( 1.9 \times 10^2 \approx 10^{0.48} \), which means we are in the moderately non-stationary environment.

We measure the dynamic regrets of the SW-UCB algorithm \( ( \text{known } B_T \text{ and unknown } B_T) \), the BOB algorithm, the UCB algorithm, and the company’s original decisions. The results are shown in Fig. 5. The plot shows that the SW-UCB algorithm with known \( B_T \) \( (\text{SW-UCB-opt}) \) and the BOB algorithm have the lowest dynamic regrets. Besides, the dynamic regret of the parameter-free BOB algorithm is \( \geq 24\% \) less than those of the obliviously tuned SW-UCB algorithm \( (\text{SW-UCB-obl}) \) and the UCB algorithm. It also saves \( \geq 32\% \) dynamic regret when compared to the company’s original decisions. The results clearly indicate that the SW-UCB algorithm and the BOB algorithm can deal with the drift while the UCB algorithm fails to keep track of the dynamic environment. More importantly, the results validate our theoretical findings regarding the parameter-free adaptation of the BOB algorithm.

![Figure 5](image_url)  
*Figure 5  Results for the on-line auto lending dataset.*

10. Conclusion

In this paper, we developed general data-driven decision-making algorithms with state-of-the-art dynamic regret bounds for non-stationary bandit settings. We characterized the minimax dynamic
regret lower bound and presented a tuned Sliding Window Upper-Confidence-Bound algorithm with matching dynamic regret. We further proposed the parameter-free bandit-over-bandit framework that automatically adapts to the unknown non-stationarity. Finally, we conducted extensive numerical experiments on both synthetic and real-world data to validate our theoretical results.

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Appendix. Proofs

A. Proof of Theorem 1

First, let’s review the lower bound of the linear bandits setting. The linear bandits setting is almost identical to ours except that the \( \theta_t \)'s do not vary across rounds, and are equal to the same (unknown) \( \theta \), i.e., \( \forall t \in [T] \; \theta_t = \theta \).

**Lemma 4** ([Lattimore and Szepesvári 2018]). For any \( T_0 \geq \sqrt{d}/2 \) and let \( D = \{ x \in \mathbb{R}^d : \|x\| \leq 1 \} \), then there exists a \( \theta \in \{ \pm \sqrt{d/4T_0} \}^d \), such that the worst case regret of any algorithm for linear bandits with unknown parameter \( \theta \) is \( \Omega(\sqrt{dT_0}) \).

Going back to the non-stationary environment, suppose nature divides the whole time horizon into \( \lceil T/H \rceil \) blocks of equal length \( H \) rounds (the last block can possibly have less than \( H \) rounds), and each block is a decoupled linear bandit instance so that the knowledge of previous blocks cannot help the decision within the current block. Following Lemma 4, we restrict the sequence of \( \theta_t \)'s are drawn from the set \( \{ \pm \sqrt{d/4H} \}^d \). Moreover, \( \theta_t \)'s remain fixed within a block, and can vary across different blocks, i.e.,

\[
\forall i \in \left\lfloor \frac{T}{H} \right\rfloor \forall t_1, t_2 \in [(i-1)H+1, i \cdot H \land T] \quad \theta_{t_1} = \theta_{t_2}.
\]

(33)

We argue that even if the DM knows this additional information, it still incur a regret \( \Omega(d^{2/3}B_1^{1/3}T^{2/3}) \). Note that different blocks are completely decoupled, and information is thus not passed across blocks. Therefore, the regret of each block is \( \Omega \left( d\sqrt{H} \right) \), and the total regret is at least

\[
\left( \left\lfloor \frac{T}{H} \right\rfloor - 1 \right) \Omega \left( d\sqrt{H} \right) = \Omega \left( dTH^{-\frac{1}{2}} \right).
\]

(34)

Intuitively, if \( H \), the number of length of each block, is smaller, the worst case regret lower bound becomes larger. But too small a block length can result in a violation of the variation budget. So we work on the total variation of \( \theta_t \)'s to see how small can \( H \) be. The total variation of the \( \theta_t \)'s can be seen as the total variation across consecutive blocks as \( \theta_t \) remains unchanged within a single block. Observe that for any pair of \( \theta, \theta' \in \{ \pm \sqrt{d/4H} \}^d \), the \( \ell_2 \) difference between \( \theta \) and \( \theta' \) is upper bounded as

\[
\sqrt{\sum_{i=1}^{d} \frac{4d}{AH}} = \frac{d}{\sqrt{H}}
\]

(35)

and there are at most \( \lceil T/H \rceil \) changes across the whole time horizon, the total variation is at most

\[
B = \frac{T}{H} \cdot \frac{d}{\sqrt{H}} = dTH^{-\frac{1}{2}}.
\]

(36)

By definition, we require that \( B \leq B_T \), and this indicates that

\[
H \geq (dT)^{\frac{1}{2}} B_T^{-\frac{1}{2}}.
\]

(37)

Taking \( H = \lceil (dT)^{\frac{1}{2}} B_T^{-\frac{1}{2}} \rceil \), the worst case regret is

\[
\Omega \left( dT \left( (dT)^{\frac{1}{2}} B_T^{-\frac{1}{2}} \right)^{-\frac{1}{2}} \right) = \Omega \left( d^{\frac{1}{2}} B_T^{\frac{1}{2}} T^{\frac{1}{2}} \right).
\]

(38)
B. Proof of Lemma 1

In the proof, we denote $B(1)$ as the unit Euclidean ball, and $\lambda_{\max}(M)$ as the maximum eigenvalue of a square matrix $M$. By folklore, we know that $\lambda_{\max}(M) = \max_{z \in B(1)} z^\top M z$. In addition, recall the definition that $V_{t-1} = \lambda I + \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top$. We prove the Lemma as follows:

$$\left\| V_{t-1}^{-1} \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right\| = \left\| V_{t-1}^{-1} \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top \left( \sum_{p=s}^{t-1} (\theta_p - \theta_{p+1}) \right) \right\|$$

$$\leq \sum_{p=1 \sqrt{(t-w)}}^{t-1} \left\| \left( \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top \right) (\theta_p - \theta_{p+1}) \right\|$$

$$\leq \sum_{p=1 \sqrt{(t-w)}}^{t-1} \lambda_{\max} \left( \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top \right) \| \theta_p - \theta_{p+1} \|.$$  

Equality (39) is by the observation that both sides of the equation is summing over the terms $X_s X_s^\top (\theta_p - \theta_{p+1})$ with indexes $(s, p)$ ranging over $\{(s, p) : 1 \sqrt{(t-w)} \leq s \leq p \leq t-1\}$. Inequality (40) is by the triangle inequality.

To proceed with the remaining steps, we argue that, for any index subset $S \subseteq \{1 \sqrt{(t-w)}, \ldots, t-1\}$, the matrix $V_{t-1}^{-1} \sum_{s \in S} X_s X_s^\top$ is positive semi-definite (PSD). Now, let’s denote $A = \sum_{s \in S} X_s X_s^\top$. Evidently, matrix $A$ is PSD, while matrix $V_{t-1}^{-1}$ is positive definite, and both matrices $V_{t-1}^{-1} A$ and $V_{t-1}^{-1/2} A V_{t-1}^{-1/2}$ have the same sets of eigenvalues, since these matrices have the same characteristics polynomial (with the variable denoted as $\eta$ below):

$$\det(\eta I - V_{t-1}^{-1} A) = \det(V_{t-1}^{-1/2} \det(\eta V_{t-1}^{1/2} - V_{t-1}^{-1/2} A)$$

$$= \det(\eta V_{t-1}^{1/2} - V_{t-1}^{-1/2} A) \det(V_{t-1}^{-1/2}) = \det(\eta I - V_{t-1}^{-1/2} A V_{t-1}^{-1/2}).$$

Evidently, $V_{t-1}^{-1/2} A V_{t-1}^{-1/2}$ is PSD, since for any $y \in \mathbb{R}^d$ we clearly have $y^\top V_{t-1}^{-1/2} A V_{t-1}^{-1/2} y = \|A^{1/2} V_{t-1}^{-1/2} y\|^2 \geq 0$ (Matrices $A^{1/2}, V_{t-1}^{-1/2}$ are symmetric). Altogether, we have shown that

$$V_{t-1}^{-1} \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top$$

is PSD.

Inequality (41) is by the fact that, for any matrix $M \in \mathbb{R}^{d \times d}$ with $\lambda_{\max}(M) \geq 0$ and any vector $y \in \mathbb{R}^d$, we have $\| M y \| \leq \lambda_{\max}(M) \| y \|$. Without loss of generality, assume $y \neq 0$. Now, it is evident that

$$\| M y \| = \left\| M \frac{y}{\| y \|} \right\| \leq \left\| \max_{z \in B(1)} M z \right\| \cdot \| y \| = \| \lambda_{\max}(M) \| \cdot \| y \| = \lambda_{\max}(M) \| y \|.$$

Applying the above claim with $M = V_{t-1}^{-1} \left( \sum_{s=1 \sqrt{(t-w)}}^{t-1} X_s X_s^\top \right)$, which is PSD, and $y = \theta_p - \theta_{p+1}$ demonstrates inequality (41).
Finally, inequality (42) is by the inequality \( \lambda_{\max} \left( V_{t-1}^{-1} \left( \sum_{s=1}^{p} V_{s}^{-1} X_{s}X_{s}^{T} \right) \right) \leq 1 \). Indeed,

\[
\lambda_{\max} \left( V_{t-1}^{-1} \left( \sum_{s=1}^{p} X_{s}X_{s}^{T} \right) \right) = \max_{z \in \Theta(1)} z^{T} V_{t-1}^{-1} \left( \sum_{s=1}^{p} X_{s}X_{s}^{T} \right) z
\]

\[
\leq \max_{z \in \Theta(1)} \left\{ z^{T} V_{t-1}^{-1} \left( \sum_{s=1}^{p} X_{s}X_{s}^{T} \right) z + z^{T} V_{t-1}^{-1} \left( \sum_{s=p+1}^{t-1} X_{s}X_{s}^{T} \right) z + \lambda z^{T} V_{t-1}^{-1} z \right\}
\]

(43)

where inequality (43) is by the property that both matrices \( V_{t-1}^{-1} \left( \sum_{s=p+1}^{t-1} X_{s}X_{s}^{T} \right) \), \( V_{t-1}^{-1} \) are PSD, as we establish previously. Altogether, the Lemma is proved.

C. Proof of Theorem 2

Fixed any \( \delta \in [0, 1] \), we have that for any \( t \in [T] \) and any \( x \in D_t \),

\[
\left| x^{T} (\hat{\theta}_t - \theta_t) \right| = x^{T} \left( V_{t-1}^{-1} \sum_{s=1}^{t} X_{s}X_{s}^{T} (\theta_s - \theta_t) \right) + x^{T} V_{t-1}^{-1} \left( \sum_{s=1}^{t} \eta_s X_s - \lambda \theta_t \right)
\]

\[
\leq \left| x^{T} \left( V_{t-1}^{-1} \sum_{s=1}^{t} X_{s}X_{s}^{T} (\theta_s - \theta_t) \right) \right| + \left| x^{T} V_{t-1}^{-1} \left( \sum_{s=1}^{t} \eta_s X_s - \lambda \theta_t \right) \right|
\]

(44)

\[
\leq \|x\| \left| V_{t-1}^{-1} \sum_{s=1}^{t} X_{s}X_{s}^{T} (\theta_s - \theta_t) \right| + \|x\|_{V_{t-1}^{-1}} \left| \sum_{s=1}^{t} \eta_s X_s - \lambda \theta_t \right|
\]

(45)

\[
\leq L \sum_{s=1}^{t} \|\theta_s - \theta_{s+1}\| + \|x\|_{V_{t-1}^{-1}} \left[ R \sqrt{d \ln \left( 1 + \frac{wL^2}{\delta} \right)} + \sqrt{S} \right],
\]

(46)

where inequality (44) uses triangle inequality, inequality (45) follows from Cauchy-Schwarz inequality, and inequality (46) are consequences of Lemmas 1, 2.

D. Proof of Theorem 3

In the proof, we choose \( \lambda \) so that \( \beta \geq 1 \), for example by choosing \( \lambda \geq 1/S^2 \). By virtue of UCB, the regret in any round \( t \in [T] \) is

\[
\langle x_t^* - X_t, \theta_t \rangle \leq L \sum_{s=1}^{t} \|\theta_s - \theta_{s+1}\| + \langle X_t, \hat{\theta}_t \rangle + \beta \|X_t\|_{V_{t-1}^{-1}} - \langle X_t, \theta_t \rangle
\]

(47)

\[
\leq 2L \sum_{s=1}^{t} \|\theta_s - \theta_{s+1}\| + 2\beta \|X_t\|_{V_{t-1}^{-1}}.
\]

(48)

Inequality (47) is by an application of our SW-UCB algorithm established in equation (10). Inequality (48) is by an application of inequality (46), which bounds the difference \( |\langle X_t, \hat{\theta}_t - \theta_t \rangle| \) from above. By the evident fact that \( \langle X_t, \hat{\theta}_t - \theta_t \rangle \leq 2 \), we have

\[
\langle x_t^* - X_t, \theta_t \rangle \leq 2L \sum_{s=1}^{t} \|\theta_s - \theta_{s+1}\| + 2\beta \left( \|X_t\|_{V_{t-1}^{-1}} \right) \cdot (1).
\]

(49)

Summing equation (49) over \( 1 \leq t \leq T \), the regret of the SW-UCB algorithm is upper bounded as

\[
E[\text{Regret}_T (\text{SW-UCB algorithm})] = \sum_{t=1}^{T} \langle x_t^* - X_t, \theta_t \rangle.
\]
where we state a technical lemma based on a novel use of the Sherman-Morrison formula. Consider any $\theta_t = \theta_{t-1} + \gamma_t w$, whose components are accumulated according to the sliding window principle, the key eq. (6) in Lemma 11’s proof breaks, and thus the analysis of (Abbasi-Yadkori et al. 2011) cannot be applied here. To this end, we state a technical lemma based on a novel use of the Sherman-Morrison formula.

**Lemma 5.** For any $i \leq \lceil T/w \rceil - 1$,

$$
\sum_{t=i \cdot w+1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2 \leq \sum_{t=i \cdot w+1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2,
$$

where

$$
V_{t-1} = \sum_{s=i \cdot w+1}^{t-1} X_s X_s^\top + \lambda I.
$$

**Proof of Lemma 5.** For a fixed $i \leq \lceil T/w \rceil - 1$,

$$
\sum_{t=i \cdot w+1}^{(i+1)w} 1 \wedge \|X_t\|_{V_{t-1}}^2 = \sum_{t=i \cdot w+1}^{(i+1)w} 1 \wedge X_t^\top V_{t-1}^{-1} X_t = \sum_{t=i \cdot w+1}^{(i+1)w} 1 \wedge X_t^\top \left( \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top + \lambda I \right)^{-1} X_t.
$$

Note that $i \cdot w + 1 \geq 1$ and $i \cdot w + 1 \geq t - w \forall t \leq (i+1)w$, we have

$$
i \cdot w + 1 \geq 1 \vee (t - w).
$$

Consider any $d$-by-$d$ positive definite matrix $A$ and $d$-dimensional vector $y$, then by the Sherman-Morrison formula, the matrix

$$
B = A^{-1} - \left( A + yy^\top \right)^{-1} = A^{-1} - A^{-1} + A^{-1}yy^\top A^{-1} \left( \frac{1}{1 + y^\top A^{-1}y} \right) = A^{-1}yy^\top A^{-1} \left( \frac{1}{1 + y^\top A^{-1}y} \right)
$$

(56)
is positive semi-definite. Therefore, for a given $t$, we can iteratively apply this fact to obtain

$$X_t^T \left( \sum_{s=i-w+1}^{t-1} X_s X_s^T + \lambda I \right)^{-1} X_t$$

$$= X_t^T \left( \sum_{s=i-w}^{t-1} X_s X_s^T + \lambda I \right)^{-1} X_t + X_t^T \left( \sum_{s=i-w+1}^{t-1} X_s X_s^T + \lambda I \right)^{-1} \left( \sum_{s=i-w}^{t-1} X_s X_s^T + \lambda I \right) X_t$$

$$= X_t^T \left( \sum_{s=i-w}^{t-1} X_s X_s^T + \lambda I \right)^{-1} X_t + X_t^T \left( \sum_{s=i-w+1}^{t-1} X_s X_s^T + \lambda I \right)^{-1} \left( X_{i-w} X_{i-w}^T + \sum_{s=i-w+1}^{t-1} X_s X_s^T + \lambda I \right)^{-1} X_t$$

$$\geq X_t^T \left( \sum_{s=i-w}^{t-1} X_s X_s^T + \lambda I \right)^{-1} X_t$$

$$\vdots$$

$$\geq X_t^T \left( \sum_{s=i-w}^{t-1} X_s X_s^T + \lambda I \right)^{-1} X_t, \quad (57)$$

Plugging inequality $(57)$ to $(54)$, we have

$$\sum_{t=(i+1)w}^{(i+1)w} \sum_{t=(i+1)w+1}^{(i+1)w} 1 \land \|X_t\|_{V_{i-1}}^2 \leq \sum_{t=(i+1)w+1}^{(i+1)w} 1 \land \|X_t\|_{V_{i-1}}^2$$

$$\leq \sum_{t=(i+1)w+1}^{(i+1)w} 1 \land \|X_t\|_{V_{i-1}}^2, \quad (58)$$

which concludes the proof. $\square$

From Lemma 5 and eq. (52), we know that

$$2 \beta \sum_{i \in [T]} 1 \land \|X_i\|_{V_{i-1}}^2 \leq 2 \beta \sqrt{T} \cdot \frac{\sum_{i \in [T]} 1 \land \|X_i\|_{V_{i-1}}^2}{T}$$

$$\leq 2 \beta \sqrt{T} \cdot \frac{\sum_{i \in [T]} 2 \ln \left( \frac{d \lambda + w L^2}{d \lambda} \right)}{w \ln \left( \frac{d \lambda + w L^2}{d \lambda} \right)} \leq 2 \beta T \sqrt{\frac{2d \ln \left( \frac{d \lambda + w L^2}{d \lambda} \right)}{w \ln \left( \frac{d \lambda + w L^2}{d \lambda} \right)}. \quad (59)$$

Here, eq. (59) follows from Lemma 11 of [Abbasi-Yadkori et al. 2011].

Now putting these two parts to eq. (50), we have

$$\mathbb{E} \left[ \text{Regret}_T \left( \text{SW-UCB algorithm} \right) \right] \leq 2 L w B_T + 2 \beta T \sqrt{\frac{2d}{w} \ln \left( \frac{d \lambda + w L^2}{d \lambda} \right) + 2 T \delta}$$

$$= 2 L w B_T + \frac{2T}{\sqrt{w}} \left( R \sqrt{\ln \left( \frac{1 + w L^2}{\lambda} / \delta \right) + \sqrt{\lambda} S} \right) \sqrt{2 \ln \left( \frac{d \lambda + w L^2}{d \lambda} \right) + 2 T \delta}. \quad (60)$$

Now if $B_T$ is known, we can take $w = O \left( (dT)^{2/3} B_T^{-2/3} \right)$ and $\delta = 1/T$, we have

$$\mathbb{E} \left[ \text{Regret}_T \left( \text{SW-UCB algorithm} \right) \right] = O \left( d^{3/2} B_T^{3/2} T^{3/2} \right);$$
while if $B_T$ is not unknown taking $w = O\left((dT)^{2/3}\right)$ and $\delta = 1/T$, we have

$$E[\text{Regret}_T(\text{SW-UCB algorithm})] = \tilde{O}\left(d^2 (B_T + 1) T^{\frac{2}{3}}\right).$$

### E. Proof of Lemma 3

For any block $i$, the absolute sum of rewards can be written as

$$\left| \sum_{t=(i-1)H+1}^{iH} \langle X_t, \theta_t \rangle + \eta_t \right| \leq \sum_{t=(i-1)H+1}^{iH} |\langle X_t, \theta_t \rangle| + \sum_{t=(i-1)H+1}^{iH} \eta_t \leq H + \sum_{t=(i-1)H+1}^{iH} \eta_t,$$

where we have iteratively applied the triangle inequality as well as the fact that $|\langle X_t, \theta_t \rangle| \leq 1$ for all $t$.

Now by property of the $R$-sub-Gaussian \cite{rigollet2018statistics}, we have the absolute value of the noise term $\eta_t$ exceeds $2R\sqrt{\ln T}$ for a fixed $t$ with probability at most $1/T^2$, i.e.,

$$\Pr\left( \left| \sum_{t=(i-1)H+1}^{iH} \eta_t \right| \geq 2R\sqrt{\ln \frac{T}{H}} \right) \leq \frac{2H}{T^2}. \quad (61)$$

Applying a simple union bound, we have

$$\Pr\left( \exists i \in \left[\frac{T}{H}\right] : \sum_{t=(i-1)H+1}^{iH} \eta_t \geq 2R\sqrt{\ln \frac{T}{H}} \right) \leq \sum_{i=1}^{\lceil T/H \rceil} \Pr\left( \left| \sum_{t=(i-1)H+1}^{iH} \eta_t \right| \geq 2R\sqrt{\ln \frac{T}{H}} \right) \leq \frac{2}{T}. \quad (62)$$

Therefore, we have

$$\Pr\left( Q \geq H + 2R\sqrt{\ln \frac{T}{H}} \right) \leq \Pr\left( \exists i \in \left[\frac{T}{H}\right] : \sum_{t=(i-1)H+1}^{iH} \eta_t \geq 2R\sqrt{\ln \frac{T}{H}} \right) \leq \frac{2}{T}. \quad (63)$$

The statement then follows.

### F. Proof of Theorem 4

Following the discussions in Section 7.2, the regret of the BOB algorithm can be decomposed as the regret of an algorithm that optimally (over $J$) tunes the window length $w_i = w^i$ for each block $i$ plus the loss due to learning the value $w^i$ with the EXP3 algorithm,

$$E[\text{Regret}_T(\text{BOB algorithm})] = E\left[ \sum_{t=1}^{T} \langle X^*_t, \theta_t \rangle - \sum_{t=1}^{T} \langle X_t, \theta_t \rangle \right] = E\left[ \sum_{t=1}^{T} \langle x^*_t, \theta_t \rangle - \sum_{t=1}^{T} \sum_{i=1}^{\lceil T/H \rceil} \langle X_i(w^i), \theta_t \rangle \right] + E\left[ \sum_{i=1}^{\lceil T/H \rceil} \sum_{t=(i-1)H+1}^{iH} \langle X_i(w^i), \theta_t \rangle - \sum_{i=1}^{\lceil T/H \rceil} \sum_{t=(i-1)H+1}^{iH} \langle x^*_i - X_i(w^i), \theta_t \rangle \right]. \quad (64)$$

Here, eq. (64) holds as the BOB algorithm restarts the SW-UCB algorithm in each block, and for a round $t$ in block $i$, $X_i(w)$ refers to the action selected in round $t$ by the SW-UCB algorithm with window length $w \wedge (t - (i-1)H - 1)$ initiated at the beginning of block $i$.

By Theorem 3, the first expectation in eq. (64) can be upper bounded as

$$E\left[ \sum_{t=1}^{T} \langle x^*_t, \theta_t \rangle - \sum_{t=1}^{T} \sum_{i=1}^{\lceil T/H \rceil} \langle X_i(w^i), \theta_t \rangle \right] = E\left[ \sum_{i=1}^{\lceil T/H \rceil} \sum_{t=(i-1)H+1}^{iH} \langle x^*_i - X_i(w^i), \theta_t \rangle \right]$$
regret is window length absolute sum of rewards of any block, the authors of (Auer et al. 2002a) gives the following regret bound.

\[ EXP3 \text{ algorithm is } \left\lceil \frac{T}{H} \right\rceil \]

while if \( B_T = 1 \geq d^{-1/2}T^{1/4} \), the \( \text{BOB} \) algorithm is able to converge to the optimal window length \( i.e., w^* = w^* \), and the dynamic regret of the \( \text{BOB} \) algorithm is upper bounded as

\[ R_T(\text{BOB algorithm}) = \tilde{O} \left( d^2 (B_T + 1) \frac{H}{T} + d \frac{H}{T} \right) = \tilde{O} \left( d^2 (B_T + 1) \frac{H}{T} + d \frac{H}{T} \right); \]

while if \( B_T + 1 < d^{-1/2}T^{1/4} \), the \( \text{BOB} \) algorithm converges to the window length \( w^* = H \), and the dynamic regret is

\[ R_T(\text{BOB algorithm}) = \tilde{O} \left( d B_T \frac{H}{T} + d \frac{H}{T} \right) = \tilde{O} \left( d^2 T \frac{H}{T} \right). \]

G. Proof of Theorem 5

Similar to eq. (4), we can rewrite the difference \( \hat{\theta}_t - \theta_t \) as

\[ V_{t-1}^* \left( \sum_{s=1}^{t-1} X_s^\top X_s \theta_s + \sum_{s=1}^{t-1} \eta_s X_s \right) - \theta_t \]
$$V^*_{t-1} = \sum_{s=1}^{t-1} X_s X_s^T (\theta_s - \theta_t) + V^*_{t-1} \left( \sum_{s=1}^{t-1} \eta_s X_s \right).$$  \tag{71}

We then bound the two terms in eq. (71) separately. For the first term,

$$\left\Vert V^*_{t-1} \sum_{s=1}^{t-1} X_s X_s^T (\theta_s - \theta_t) \right\Vert = \left\Vert V^*_{t-1} \sum_{s=1}^{t-1} X_s X_s^T \left( \sum_{p=s}^{t-1} (\theta_p - \theta_{p+1}) \right) \right\Vert$$

$$= \left\Vert \sum_{p=1}^{t-1} V^*_{t-1} \sum_{s=1}^{p} X_s X_s^T (\theta_p - \theta_{p+1}) \right\Vert$$

$$\leq \sum_{p=1}^{t-1} \lambda_{\max} \left( V^*_{t-1} \sum_{s=1}^{p} X_s X_s^T \right) \left\Vert \theta_p - \theta_{p+1} \right\Vert$$

$$\leq \sum_{p=1}^{t-1} \left\Vert \theta_s - \theta_{s+1} \right\Vert.$$  \tag{72}

Here, almost all the steps follow exactly the same arguments as those of eq. (39)-(42), except that in inequality (72), we make the direct observation that

$$V^*_{t-1} = \begin{pmatrix} \frac{1[N_{1,t-1} > 0]}{N_{1,t-1}} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1[N_{2,t-1} > 0]}{N_{2,t-1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1[N_{d,t-1} > 0]}{N_{d,t-1}} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1[N_{d,t-1} > 0]}{N_{d,t-1}} \end{pmatrix}$$  \tag{74}

and

$$\sum_{s=1}^{p} X_s X_s^T = \begin{pmatrix} N^*_1, p & 0 & \cdots & \cdots & 0 \\ 0 & N^*_2, p & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & N^*_d, p & 0 \\ 0 & 0 & \cdots & 0 & N^*_d, p \end{pmatrix},$$  \tag{75}

where $N^*_i, p$ is the number of times that action $e_i$ is selected during rounds $1 \vee (t-w), \ldots, p$ for all $i \in [d]$. As $p \leq t-1$, we have $N^*_i, p \leq N_{i,t-1}$ for all $i \in [d]$. Hence, $V^*_{t-1} \sum_{s=1}^{p} X_s X_s^T$ is a diagonal matrix with all entries less than 1, and its maximal eigenvalue is thus upper bounded by 1.

For the second term of eq. (71), we consider for any fixed $i \in [d],$

$$e_i^T V^*_{t-1} \left( \sum_{s=1}^{t-1} \eta_s X_s \right) = \frac{1[N_{i,t-1} > 0]}{N_{i,t-1}} e_i^T \left( \sum_{s=1}^{t-1} \eta_s X_s \right)$$

$$= \frac{1[N_{i,t-1} > 0]}{N_{i,t-1}} \frac{1[N_i = i]}{N_{i,t-1}} \eta_i.$$  \tag{76}
where the first step again use the definition of $V_{t-1}^*$ in eq. (74). Now if $N_{i,t-1} = 0$, eq. (76) equals to 0; while if $N_{i,t-1} > 0$, we can apply the Corollary 1.7 of [Rigollet and Hutter 2018] to obtain that

$$\Pr \left( \left| \frac{1}{N_{i,t-1}} \left( \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right) \right| \geq \eta \right) \leq \frac{2 \ln (2dT^2)}{N_{i,t-1}} \geq 1 - \frac{1}{dT^2}. \quad (77)$$

Hence, with probability at least $1 - 1/dT^2$, for any fixed $t \in [T]$ and any fixed $i \in [d]$,

$$\left| e_i^T (\hat{\theta}_t - \theta_t) \right| \leq e_i^T V_{t-1}^* \left( \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right) + e_i^T V_{t-1}^* \left( \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s - \lambda \theta_t \right) \leq \left\| e_i \right\| \cdot \left| V_{t-1}^* \left( \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top (\theta_s - \theta_t) \right) \right| + R \sqrt{\frac{2 \ln (2dT^2)}{N_{i,t-1}}} \quad (78)$$

$$\leq \sum_{s=1 \vee (t-w)}^{t-1} \left| \theta_s - \theta_{s+1} \right| + R \sqrt{\frac{2 \ln (2dT^2)}{N_{i,t-1}}}. \quad (80)$$

where inequality (78) applies the triangle inequality, inequality (79) follows from the Cauchy-Schwarz inequality and inequality (76) and (77), and inequality (80) follows from inequality (73).

The statement of the theorem now follows immediately by applying union bound over the decision set and the time horizon as well as the simple observation $\left| e_i \right|_{V_{t-1}^*} = \sqrt{1/N_{i,t-1}}$.

H. Proof of Theorem 8

From the proof of Proposition 1 in [Filippi et al. 2010], we know that for all $x \in D$

$$\left| \mu (\langle x, \theta_t \rangle) - \mu (\langle x, \hat{\theta}_t \rangle) \right| \leq k_\mu \left| x^\top G_{t-1}^{-1} \left[ \sum_{s=1 \vee (t-w)}^{t-1} \left( \mu (\langle X_s, \theta_t \rangle) - \mu (\langle X_s, \hat{\theta}_t \rangle) \right) X_s \right] \right|, \quad (81)$$

where

$$G_{t-1} = \int_0^1 \left[ \sum_{s=1 \vee (t-w)}^{t-1} X_s X_s^\top \mu (\langle X_s, s_0 \theta_t + (1 - s_0) \hat{\theta}_t \rangle) \right] ds_0$$

By virtue of the maximum quasi-likelihood estimation, i.e., eq. (31) we have

$$\sum_{s=1 \vee (t-w)}^{t-1} \mu (\langle X_s, \hat{\theta}_t \rangle) X_s = \sum_{s=1 \vee (t-w)}^{t-1} Y_s X_s = \sum_{s=1 \vee (t-w)}^{t-1} (\mu (\langle X_s, \theta_s \rangle) + \eta_s) X_s, \quad (82)$$

and (81) is

$$k_\mu \left| x^\top G_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} (\mu (\langle X_s, \theta_t \rangle) - \mu (\langle X_s, \theta_s \rangle) - \eta_s) X_s \right|$$

$$= k_\mu \left| x^\top G_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} (\mu (\langle X_s, \theta_t \rangle) - \mu (\langle X_s, \theta_s \rangle)) X_s - x^\top G_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s \right|$$

$$\leq k_\mu \left| x^\top G_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} (\mu (\langle X_s, \theta_t \rangle) - \mu (\langle X_s, \theta_s \rangle)) X_s \right| + k_\mu \left| x^\top G_{t-1}^{-1} \sum_{s=1 \vee (t-w)}^{t-1} \eta_s X_s \right| \quad (83)$$
\begin{equation}
\leq k_\mu \left\| x^T G_{t-1}^{-1} \sum_{s=1}^{t-1} (\langle X_s, \theta_t \rangle - \langle X_s, \theta_s \rangle) X_s \right\|_V + \beta \|x\|_V \tag{84}
\end{equation}

\begin{equation}
\leq k_\mu \|x\| \left\| G_{t-1}^{-1} \sum_{s=1}^{t-1} (\langle X_s, \theta_t \rangle - \langle X_s, \theta_s \rangle) X_s \right\| + \beta \|x\|_V \tag{85}
\end{equation}

\begin{equation}
\leq \frac{k_\mu L}{c_\mu} \left\| V_{t-1}^{-1} \sum_{s=1}^{t-1} (\langle X_s, \theta_t \rangle - \langle X_s, \theta_s \rangle) X_s \right\| + \beta \|x\|_V \tag{86}
\end{equation}

Here, inequality (83) is a consequence of the triangle inequality, inequality (84) again follows from Proposition 1 of Filippi et al. (2010), inequality (85) is the Cauchy-Schwarz inequality, and the last step uses the fact that $G_{t-1} \geq c_\mu V_{t-1}$. For the first quantity, we have

\begin{equation}
\leq \sum_{p=1}^{t-1} \left\| V_{t-1}^{-1} \sum_{s=1}^{p} X_s (\mu (\langle X_s, \theta_{p+1} \rangle - \langle X_s, \theta_p \rangle)) \right\|
\end{equation}

\begin{equation}
= \sum_{p=1}^{t-1} \left\| V_{t-1}^{-1} \sum_{s=1}^{p} X_s \mu (\langle X_s, \theta_{p+1} \rangle - \langle X_s, \theta_p \rangle) \right\|
\end{equation}

\begin{equation}
= \lambda_{\max} \left( \sum_{s=1}^{t-1} \sum_{p=1}^{t} \mu (\langle X_s, \theta_p \rangle) X_s X_s^\top (\theta_{p+1} - \theta_p) \right)
\end{equation}

\begin{equation}
\leq k_\mu \sum_{p=1}^{t-1} \lambda_{\max} \left( V_{t-1}^{-1} \sum_{s=1}^{p} X_s X_s^\top \right) \| (\theta_{p+1} - \theta_p) \|
\end{equation}

where inequality (86) is an immediate consequence of the triangle inequality, eq. (87) utilizes the mean value theorem (with $\tilde{\theta}_p$ being some certain linear combination of $\theta_p$ and $\theta_{p+1}$ for all $p$), and inequalities (88) and (89) follow from the same steps as the proof of Lemma 1 in Section 3.

I. Supplementary Details for Section 9

When $B_T$ is known, we select $w^{opt}$ that minimizes the explicit regret bound in (60), resulting in

\begin{equation}
w^{opt} = \left[ \frac{\tilde{w}}{B_T^{2/3}} \right], \text{ where } \tilde{w} = \frac{d^{1/3}T^{2/3} \left(R \sqrt{d \ln (T + T^2 L^2 / \lambda)} + \sqrt{\tilde{X} S} \right)^{2/3} \log^{1/3} \left(1 + \frac{TL^2}{d\lambda^2} \right)}{B_T^{2/3}}.
\end{equation}

When $B_T$ is not known, we select $w^{opt} = [\tilde{w}]$, which is independent of $B_T$. 
