Subordinated Langevin equations for anomalous diffusion in external potentials — Biasing and decoupled external forces

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Abstract – The role of external forces in systems exhibiting anomalous diffusion is discussed on the basis of the describing Langevin equations. Since there exist different possibilities to include the effect of an external field, the concept of biasing and decoupled external fields is introduced. Complementary to the recently established Langevin equations for anomalous diffusion in a time-dependent external force field (Magdziarz M. et al., Phys. Rev. Lett., 101 (2008) 210601), the Langevin formulation of anomalous diffusion in a decoupled time-dependent force field is derived.

Introduction. – Over the last two decades it has become apparent that many complex systems exhibit a phenomenon that has been termed anomalous diffusion [1–3]. On account of this, there has been an increasing interest in stochastic processes deviating basically from standard diffusion processes characterized by a Gaussian behavior. Systems exhibiting anomalous diffusion differ from the linear time dependence of the second moment and rather show $\langle x^2 \rangle \sim t^\alpha$, where $0 \leq \alpha \leq 2$. In this context, processes with $\alpha > 1$ dispersing faster than standard diffusion processes are called superdiffusive while $\alpha < 1$ means that a system displays subdiffusive behavior.

In the realm of anomalous diffusion, the classical diffusion equation has to be replaced by the so-called generalized diffusion equations [4]. The most prominent representatives of this class of equations are probably the fractional diffusion equations [3], where the derivatives with respect to time or to space or both are replaced by non-integer order derivatives. A more fundamental account to anomalous diffusion is provided by a stochastic process called continuous time random walk (CTRW). This process generalizes the standard random walk and allows for random jump length and random waiting periods between the jumps [5]. It is well known that the generalized diffusion equation can be derived from the governing equations of the CTRW. Another approach to anomalous diffusion has been put forward by Fogedby who proposed a coupled system of Langevin equations leading to the generalized diffusion equations [6]. In a sense, this approach can be considered as a continuous realization of the CTRW.

In the present paper, we consider the effect of external forces onto processes exhibiting anomalous diffusion. Although the incorporation of external forces is straightforward in classical diffusion theory, leading to the well-known Fokker-Planck equations, this task appears to be rather involved when anomalous diffusion is considered. The arising difficulties are due to the long jumps and the long waiting times that can occur. Throughout this paper we distinguish between biasing and decoupled external forces. This notation shall indicate that there are two different possibilities of the action of the force. When we speak of a biasing field, we mean that the external field acts as a bias only at the time of the actual jump. In contrast to this, we speak of a decoupled field if the diffusing particle is affected permanently during the waiting time periods and hence the diffusion process is decoupled from the effect of the field. Note that this distinction is not necessary for classical diffusion processes.

While the inclusion of external potentials is relatively well understood on the level of the generalized diffusion equations and the generalized Fokker-Planck equations, respectively, there are still some open questions as long as the corresponding Langevin equations are considered. However, an exhaustive comprehension of the Langevin equations is inevitable to investigate the properties of sample paths of such processes.

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The aim of this paper is to clarify the different possibilities of including an external force into the framework of anomalous diffusion, namely the difference between biasing and decoupled external forces, by considering the corresponding Langevin equations. It is organized as follows. First we state some fundamentals concerning the theory of anomalous diffusion and thereby shortly review Fogedby’s continuous formulation of CTRWs and the concept of subordination. After shortly reviewing some well-known and some very recent results on the Langevin formulations of the generalized Fokker-Planck equations for biasing external fields, we establish the Langevin equations for generalized Fokker-Planck equations for decoupled external potentials that have not been considered so far. We conclude with a discussion on the role of external forces in anomalous diffusion.

**CTRWs and generalized diffusion equations.** – A suited stochastic process to describe discrete sample realizations of many microscopic processes leading to anomalous diffusion is provided by the continuous time random walk (CTRW). This process is an extension of the standard random walk and allows for random waiting times between jumps of random length. In the decoupled case, the CTRW is characterized by a waiting time distribution \( W(t) \) and a jump length distribution \( F(\Delta x) \). Depending on the properties of these distributions, the CTRW can lead to an anomalous behavior of the mean-squared-displacement [4]. The governing equation for the probability distribution (pdf) of the position of the walker is the Montroll-Shlesinger master equation [7]

\[
\frac{\partial}{\partial t} f(x,t) = \int \! dx' F(x; x') \left( \int_0^t \! dt' \Phi(t - t') f(x', t') - \int_0^t \! dt' \Phi(t - t') f(x, t') \right),
\]

where the time-kernel \( \Phi(t - t') \) is related to the waiting time distribution\(^1\). The master equation (1) has a straightforward interpretation. It states that the density of particles at position \( x \) and time \( t \) is increased by particles that have been at \( x' \) at time \( t' \) and perform a jump from \( x' \) to \( x \) at time \( t \). On the other hand, the density is decreased by particles that have been at \( x \) and jump away at time \( t \) to some other position. The resulting process is non-Markovian for waiting time distributions with a power law tail.

Another account to describe the evolution of pdfs in the context of anomalous diffusion are the generalized diffusion or fractional diffusion equations.

For a subdiffusive process, the generalized diffusion equation can be cast into the form

\[
\frac{\partial}{\partial t} f(x,t) = \int_0^t \! dt' \phi(t - t') D \frac{\partial^2}{\partial x^2} f(x,t'),
\]

where the kernel \( \phi(t - t') \) is closely related to the kernel \( \Phi(t - t') \) of the CTRW. A fractional diffusion equation is obtained from eq. (2) by the formal substitution

\[
\int_0^t \! dt' \phi(t - t') f(x, t') \rightarrow D^{1-\alpha}_{t} f(x, t),
\]

where the so-called Riemann-Liouville fractional derivative \( D^{1-\alpha}_{t} \) is defined via

\[
D^{1-\alpha}_{t} f(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \! \frac{dt'}{(t-t')^{1-\alpha}} f(x, t'),
\]

for \( 0 < \alpha \leq 1 \) and \( \Gamma \) denotes the well-known \( \Gamma \) function (see, e.g., [3]). The underlying physical reason for the formal substitution is the assumed limiting behavior

\[
\phi(t - t') \propto \frac{-1}{(t-t')^{2-\alpha}}
\]

of the time-kernel \( \phi(t - t') \). The formal substitution (3) can be considered as a regularization of the kernel-function (5) that leads to divergent terms for \( (t-t') \rightarrow 0 \). For a detailed description of this procedure, (see [4]).

It is well known that generalized diffusion equations can be derived from the Montroll-Shlesinger equation (see, e.g., [8]). The respective fractional diffusion equations correspond in this case to Mittag-Leffler-type waiting time distributions \( W(t) \sim t^{-1-\alpha} \).

In order to describe superdiffusive processes the so-called space-fractional diffusion equations have to be taken into account. These equations describe Markovian processes with power law distributed jump length and are often referred to as Lévy flights.

**Fogedby’s approach and subordination.** – A continuous realization of the CTRW has been considered by Fogedby in [6]. His formulation is based on a system of coupled Langevin equations for the position \( x \) and time \( t \):

\[
\dot{x}(s) = \Gamma(s), \quad \dot{t}(s) = \eta(s),
\]

where \( \Gamma(s) \) and \( \eta(s) \) are random noise sources that are assumed to be independent. In this context, \( \eta(s) \) has to be positive due to causality. The system (6) can be interpreted as a standard Langevin equation in an internal time \( s \) that is subjected to a random time change. This random time change to the physical time \( t \) is described by the second equation. The combined process in physical time is then given according to \( x(t) = x[s(t)] \), where \( s(t) \) is the inverse process to \( t(s) \) defined as

\[
s(\tilde{t}) = \inf\{s: t(s) > \tilde{t}\}.
\]

Closely related to this concept of Fogedby is the mathematical method of subordination. Using the (not to formal) notation of Fogedby, one calls the process \( x(s) \) a parent process and \( s \) its operational time. The random time-transformation function \( t(s) \) has to be a non-decreasing right-continuous function with an inverse function \( s(t) \).

\[^1\text{In Laplace space the time-evolution kernel is related to the waiting time by } \Phi(\lambda) = \frac{\lambda W(\lambda)}{1 - \lambda W(\lambda)}.\]
The resulting process in physical time $t$ is then obtained by $x(t) = x[s(t)]$ and is referred to as subordinated to the parent process. Consequently, the processes $t(s)$ and $s(t)$ are named subordinator and inverse subordinator, respectively.

In [6] it was shown that the Langevin equations (6) lead to a time-fractional diffusion equation if the $\eta(s)$ are governed by a generic one-sided $\alpha$-stable distribution. Generally, the pdf of the subordinated process can be stated in the form

$$ f(x, t) = \int_0^\infty ds \, p(s, t) \, f_0(x, s), \quad (8) $$

where $p(s, t)$ is the pdf of the inverse subordinator and $f_0(x, s)$ is the solution of the parent process [9,10].

**Biasing external force fields.** Throughout this paper we will restrict to anomalous diffusion processes governed by waiting time distributions, i.e., we will consider equations of the form (2). Hence we consider processes that are, e.g., ruled by Lévy-stable subordinators and time-fractional equations. The role of external potentials for Lévy flights is discussed in [11].

Let us first clarify what we mean by biasing external forces. Therefore, consider the generic scenario of a subdiffusive CTRW governed by power law distributed waiting times. A biasing external potential or force shall not affect the dynamics of the diffusing particle during the waiting periods but only provide it with a bias at the instance of a jump. Note, however, that the residence times at the respective sites can be affected by the external force even if the dynamical properties are unchanged [12,13].

If the considered force is time-independent, it is well known that anomalous diffusion in biasing fields can be described by the generalized Fokker-Planck equation

$$ \frac{\partial}{\partial t} f(x, t) = \int_0^t dt' \phi(t - t') \left[ -\frac{\partial}{\partial x} F_x(t) + \frac{\partial^2}{\partial x^2} \right] f(x, t'), \quad (9) $$

where $F_x(t)$ is the external force [3,8]. The time evolution of the pdf of such a process with a constant external force is displayed in fig. 1. The equivalent description based on Langevin equations is provided by the coupled system

$$ \dot{x}(s) = F_x(x) + \Gamma(s), \quad \dot{t}(s) = \eta(s), \quad \text{ (10)} $$

where $\Gamma(s)$ is a Gaussian and $\eta(s)$ is a fully skewed $\alpha$-stable Lévy noise source [6].

If a time-dependent external force $F(t)$ is considered, it turns out that the situation is by far more involved. There exist different alternatives to include the force. One can, for example, consider the generalized Fokker-Planck equation

$$ \frac{\partial}{\partial t} f(x, t) = \int_0^t dt' \phi(t - t') \left[ -\frac{\partial}{\partial x} F(t') + \frac{\partial^2}{\partial x^2} \right] f(x, t'). \quad (11) $$

However, such a generalized Fokker-Planck equation turns out to be physically meaningless.

Subordinated Langevin equations with external forces

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Fig. 1: (Color online) Time evolution of the probability density for a constant external biasing force $F = v$ for Mittag-Leffler-type waiting time distributions with $\alpha = 0.5$ for the consecutive times $t = 1–5$. One can observe the persistence of the maximum at the origin indicating that there is no internal dynamics between the waiting times. The external biasing force acts only at the time of the displacements resulting in an asymmetry of the pdf. The plume stretches more and more into the direction of the force.

The general answer to this difficult problem is still lacking. But for special types of forcing, the correct equation has been found recently [14–16]:

$$ \frac{\partial}{\partial t} f(x, t) = \left[ -\frac{\partial}{\partial x} F(t) + \frac{\partial^2}{\partial x^2} \right] \int_0^t dt' \phi(t - t') f(x, t'), \quad (12) $$

yielding for Mittag-Leffler-type waiting time distributions the fractional Fokker-Planck equation (FFPE) (cf. eq. (3)):

$$ \frac{\partial}{\partial t} f(x, t) = \left[ -\frac{\partial}{\partial x} F_x(t) + \frac{\partial^2}{\partial x^2} \right] D_{1 - \alpha} f(x, t). \quad (13) $$

However, one should note at this point that eqs. (12) and (13) have only been proven to be valid for special types of time-dependent external forces and it is not possible to validate this concept for ad hoc included arbitrary time-dependent external forces. In fact, the class of external fields to which eqs. (12) and (13) apply could be very restrictive. For a discussion of this issue, see [15].

Notice that the difficulty of time-dependent external forces stems from the fact that in this case the fractional derivative and the Fokker-Planck drift-term do not commute anymore. On the basis of the generalized Fokker-Planck equation, the difficulty is due to the fact that it is not clear whether the external force has to depend on $t'$ or $t$. For a detailed treatment of this issue, we refer the reader to the original papers. At this point, we want to confine ourselves to a simple plausibility argument to account for the correct operator ordering that naturally does not replace a rigorous derivation.

As we have already mentioned, when time-dependent transition amplitudes $F(x; x')$ are considered, the question arises whether this amplitude has to depend on $t'$ or $t$, which is equivalent to operator ordering problem in eq. (12). To answer this question, let us consider the corresponding CTRW governed by the Montroll-Shlesinger equation (1). According to the interpretation of this
equation, the probability to be at position $x$ at time $t$ is increased by the particles that jump at time $t$ from some $x'$ to $x$. Since this jump that occurs at time $t$ is governed by the transition amplitude $F(x; x')$, it is clear that the transition amplitude has to depend on the time of the jump, i.e., $F(x; x', t)$. Performing the appropriate limit procedure, one obtains eq. (12) as the correct FFPE for time-dependent Fokker-Planck operators.

Consequently, the corresponding Langevin equation for a time-dependent forcing is not straightforward to derive [17]. If the force is assumed to depend on the internal time, i.e.,

$$\dot{x}(s) = F(s) + \Gamma(s), \quad \dot{l}(s) = \eta(s)$$

(14)

which corresponds to a completely subordinated force, the corresponding generalized Fokker-Planck equation would be eq. (11) and thus eq. (14) lacks a physical interpretation.

The appropriate Langevin system has been found recently by Magdziarz and co-workers [18]. They argued that a deterministic force should not be modified by the subordination procedure and should depend on the physical time $t$ and hence proposed the Langevin equations

$$\dot{x}(s) = F(t(s)) + \Gamma(s), \quad \dot{l}(s) = \eta(s)$$

(15)

One recognizes that the force depends on the subordination process. Subordination of the process $x(s)$ then yields for the force term $F[t(s[t])]=F(t)$ since $t(s[t])=t$ and hence the desired dependence on the physical time. It has been proven in [18] that the Langevin equations (15) yields the same probability distributions as eq. (12) and hence that they describe the same process. However, it should be emphasized that the Langevin equation (15) has only the same area of application as the corresponding Fokker-Planck equations (12) and (13).

**Decoupled external force fields.** — If the dynamics of a particle is assumed to be affected by an external potential throughout the whole waiting time period and the anomalous diffusion process is independent of this potential, we speak of a *decoupled* potential.

It is instructive to consider a simple example where a particle is advected by a constant external velocity field $v_{ext}$ during the waiting periods and performs jumps after the waiting periods. The pdf of such a process has been proven to be governed by

$$\left\lbrack \frac{\partial}{\partial t} + v_{ext}\frac{\partial}{\partial x} \right\rbrack f(x, t) = \int_{0}^{t} \left( \frac{\partial}{\partial t'} \phi(t-t') \right) \frac{\partial^2}{\partial x^2} f\left( x-v_{ext}(t-t'), t' \right), \quad \text{(16)}$$

which can be considered as a generalized advection-diffusion equation, where the advection is normal while the diffusion is anomalous [19]. The time evolution of the pdf of this process is shown in fig. 2. Observe the retardation of the pdf on the right-hand side, which renders the equation non-local in space. Like for the usual advection-diffusion equation, a solution of this equation can be found after passing into a co-moving reference frame. The ansatz $f(x, t) = F(\xi, t)$ with the shifted variable $\xi = x - v_{ext}t$ yields a generalized diffusion equation for $\xi$:

$$\frac{\partial}{\partial t} F(\xi, t) = \int_{0}^{t} \left( \frac{\partial}{\partial t'} \phi(t-t') \right) \frac{\partial^2}{\partial \xi^2} F(\xi, t')$$

(17)

whose solution is given by (see eq. (8))

$$F(\xi, t) = \int_{0}^{\infty} ds \, p(s, t) \, F_{0}(\xi, s)$$

(18)

where $F_{0}$ is the solution of the standard diffusion equation. Hence this solution can be interpreted as a force-free anomalous diffusion process in the co-moving reference frame.

In order to establish the corresponding set of Langevin equations, we have to be aware of the decoupled character of the advective field. That means, the advection has to be completely independent of the internal time $s$. Let us therefore consider the following set of Langevin equations:

$$\dot{x}(s) = v_{ext} \eta(s) + \Gamma(s), \quad \dot{l}(s) = \eta(s)$$

(19)

The solution of the subordinated process $x[s(t)]$ can be found by integration:

$$x(t) = x[s(t)] = \int_{0}^{s(t)} v_{ext} \eta(s') ds'[s'(t')] + B[s(t)]$$

(20)

where $B[s(t)]$ means subordinated Brownian motion, that is the force-free pure subdiffusive part of the process [10, 18,20,21]. The integral can be rewritten as

$$\int_{0}^{s(t)} v_{ext} \eta(s') ds'[s'(t')] = \int_{0}^{s(t)} v_{ext} \frac{dt'}{ds'} ds'[s'(t')] = \int_{0}^{t} v_{ext} dt' = v_{ext} t$$

(21)
yielding for the subordinated process
\[ x(t) = v_{\text{ext}} t + B[s(t)]. \] (22)

Introducing the variable \( \xi = x - v_{\text{ext}} t \) again, this equation can be written as
\[ \xi(t) = B[s(t)]. \] (23)

Thus the variable \( \xi \) performs a force-free subdiffusive process and therefore yields the probability distributions given by eq. (18), which proves that the Langevin equations (19) actually correspond to the generalized Fokker-Planck equation (16).

The case of time-dependent external field is only slightly more difficult. Consider a process where the particle (of unit mass) performs during the waiting periods an overdamped, space-independent motion according to the equation of motion
\[ \dot{x}(t) = F(t), \] (24)
where \( F(t) \) is some time-dependent force field. The corresponding generalized Fokker-Planck equation reads [19]
\[
\left[ \frac{\partial}{\partial t} + F(t) \frac{\partial}{\partial x} \right] f(x,t) = \int_0^t dt' \phi(t - t') \frac{\partial^2}{\partial x^2} e^{-\int_0^{t'} F(t'')dt''} \frac{\partial}{\partial x} f(x,t').
\] (25)

The exponential function on the right-hand side is the so-called Frobenius-Perron operator of the equation of motion for the deterministic part of \( x(t) \). This operator ensures the proper retardation of the probability distribution during the waiting period [22,23].

Since eq. (24) describes space-independent dynamics, eq. (25) can be expressed as (see [22])
\[
\left[ \frac{\partial}{\partial t} + F(t) \frac{\partial}{\partial x} \right] f(x,t) = \int_0^t dt' \phi(t - t') \frac{\partial^2}{\partial x^2} f(x,t') - \int_0^{t'} F(t')dt' \frac{\partial}{\partial x} f(x,t') + \Gamma(t) \frac{\partial}{\partial x} f(x,t').
\] (26)

Performing the ansatz \( f(x,t) = F(\xi,t) \) with \( \xi = x - \int_0^t F(t')dt' \), the pdf of \( \xi \) is governed by the generalized diffusion equation (17).

The corresponding Langevin equation reads
\[ \dot{x}(s) = F(s) \eta(s) + \Gamma(s), \quad \dot{\xi}(s) = \eta(s). \] (27)

Integration of this equation yields for the subordinated process (cf. eq. (21)):
\[
x(t) = x[s(t)] = \int_0^{s(t)} F(s') \eta(s')ds' + B[s(t)]
= \int_0^t F(t')dt' + B[s(t)].
\] (28)

Evidently, \( \xi = x - \int_0^t F(t')dt' \) performs for this case a force-free subdiffusive process that proves that \( x(t) \) is a solution of eq. (26).

Note, however, at this point, that for general space-dependent dynamics, the Frobenius-Perron operator cannot be expressed by a substitution operator as in eq. (26). Even the simple case of a linearly damped motion between the random kicks, i.e., \( \dot{x} = -\gamma x \) leads to a generalized Fokker-Planck equation whose solution cannot be expressed in a closed form [19]. Hence, the proof used here is not applicable anymore. Similarly, a closed form solution of the Langevin equation cannot be stated for this case.

Comparing the Langevin equation for a biasing time-dependent force eq. (15) and the Langevin equation for the decoupled case, one realizes the difference between these equations. For the case of a biasing force, the force has to depend on the subordination process in the parent process. Then the force term yields the contribution \( \int_0^t F(t')ds(t') \) to the process. Observe that the force depends indeed on the physical time \( t \) but is integrated over the subordinated measure. In the decoupled case, however, the force is integrated in physical time and thus is completely independent of the diffusion process.

Of course it is possible to state the Langevin equation for a process where a biasing and decoupled force are acting independently. If \( F_B \) denotes the biasing and \( F_D \) the decoupled force, the corresponding Langevin equation reads
\[ \dot{x}(s) = F_D(s) \eta(s) + F_B(t(s)) + \Gamma(s), \quad \dot{\xi}(s) = \eta(s). \] (29)

The time evolution of the pdf of such a process is displayed in fig. 3 for a constant biasing force and a constant decoupled force with same amplitude but opposite sign. Note, however, that in many settings the two contributions are not independent and thus can display dependences.

Conclusions. – Concluding, in this paper we have discussed the effect of external forces on anomalous diffusion processes on the basis of their corresponding
Langevin equations. We have introduced the concept of a biasing and a decoupled external field, which has no classical analogue. Corresponding to the recently established Langevin formulation of biased diffusion in a time-dependent external field [18], we have rigorously derived the Langevin equations for decoupled forces. To clarify the concept of biasing a decoupled external force in systems exhibiting anomalous diffusion, we have presented the time evolution of probability densities for the different considered cases. We have shown that the established Langevin equation for decoupled force fields can be solved exactly for space-independent dynamics.

The presented work has aimed at a clarification of the role of external forces in complex systems that are characterized by subdiffusion and long waiting times, respectively. The approach based on the Langevin equation has provided thereby deep insight into the physical nature of the processes.

Concluding, we shall exemplify the concept by two simple applications each with a constant force. First consider the diffusion of tracer particles in an advective flow, which has frequent obstacles such as, e.g., sediments. In this case, the external force, i.e., the advective flow, is decoupled from the diffusion process. Second, if the diffusion of groundwater through porous media is examined, the gravity field provides a bias on the anomalous diffusion process.

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