Residual error-disturbance uncertainties in successive spin-1/2 measurements tested in matter-wave optics

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(Dated: November 12, 2015)

The indeterminacy inherent in quantum measurement is an outstanding character of quantum theory, which manifests itself typically in Heisenberg’s error-disturbance uncertainty relation. In the last decade, Heisenberg’s relation has been generalized to hold for completely general quantum measurements. Nevertheless, the strength of those relations has not been clarified yet for mixed quantum states. Recently, a new error-disturbance uncertainty relation (EDUR), stringent for generalized input states, has been introduced by one of the present authors. A neutron-optical experiment is carried out to investigate this new relation: it is tested whether error and disturbance of quantum measurements disappear or persist in mixing up the measured ensemble. Our results exhibit that measurement error and disturbance remain constant independent of the degree of mixture. The tightness of the new EDUR is confirmed, thereby validating the theoretical prediction.

Quantum measurement, through which a value of a physical property is assigned, has still eluded our consistent, physical understanding\textsuperscript{11}. Born’s rule gives physical connection between the quantum-mechanical formalism and the prediction of probabilities of events occurring in a single quantum-system\textsuperscript{2}. Our studies, however, are not limited to measurements of physical quantities on a single quantum-system\textsuperscript{3}, but are rather concerned with statistical ensembles of a quantum system reflecting actual circumstances. All information of physical importance is, thus, attributed to a statistical state, represented by a so-called density matrix\textsuperscript{4}. Note that there is no uniqueness of the representation of a mixed state as a convex sum of pure states\textsuperscript{5}. That is, the same mixed-state density matrix can be obtained with different blends for that\textsuperscript{6,7}; experiments can distinguish the difference in mixture but no evidence can be found in different generation methods of mixture. All as-if realities consisting in blending is not accessible, turning to be virtual\textsuperscript{7}.

In this letter, we report on experimental investigations of the influence of the state mixture on error and disturbance uncertainties in successive spin-1/2 measurements. For this purpose, we generate mixed ensembles of the spin state of neutrons and tune the degree of mixture systematically. It is well-known that the occurrence of a dephasing in double-slit experiments leads to (phase) mixture, easily washing out interference fringes, i.e., quantum interference vanishes for mixed states and quasi-classical behavior can emerge in certain circumstances\textsuperscript{8,9}. Thus, it is an interesting problem worth investigating whether the mixture of the measured ensemble increases or decreases the measurement uncertainty. Since all the states of a quantum system, used in practical resources, are more or less statistically mixed ensembles, our results will help to classify the practical role of a quantum effect employed in quantum technology.

The uncertainty principle proposed by Heisenberg\textsuperscript{10} in 1927 states that it is impossible to simultaneously measure two conjugate observables with arbitrary precision. By the famous γ ray microscope thought experiment, Heisenberg showed the error-disturbance relation $q_1 p_1 \sim \hbar$ for the error $q_1$ of a position measurement and the disturbance $p_1$ thereby caused on the momentum. In his mathematical derivation of this relation, he introduced the famous preparational uncertainty relation $\Delta q \Delta p \geq \frac{\hbar}{2}$ for standard deviations $\Delta q$ and $\Delta p$ for position $q$ and momentum $p$; a general proof was given shortly afterward by Kennard\textsuperscript{11}. Robertson generalized this relation to an arbitrary pair of non-commuting observables $A, B$ for a given quantum state $|\psi\rangle$ replacing the lower bound $\hbar/2$ by the bound $C_{AB} = \frac{1}{2} \langle \psi | [A, B] | \psi \rangle$\textsuperscript{12}.

An error-disturbance uncertainty relation (EDUR) valid for an arbitrary pair of observables and for arbitrary generalized measurements was derived by Ozawa\textsuperscript{13,14} as

$$\epsilon(A) \eta(B) + \epsilon(A) \Delta B + \eta(B) \Delta A \geq C_{AB},$$

validity of which were experimentally tested with neutrons\textsuperscript{15,16} and with photons\textsuperscript{17,18}. Other approaches to measuremental uncertainty relations can be found for example in\textsuperscript{19,23}.

In pursuit of an improvement of relation\textsuperscript{11}, a stronger inequality

$$\epsilon(A)^2 \Delta B^2 + \eta(B)^2 \Delta A^2 + 2 \epsilon(A) \eta(B) \sqrt{\Delta A^2 \Delta B^2 - C_{AB}^2} \geq C_{AB}^2$$

was introduced by Branciard\textsuperscript{24}. Later, it was pointed out that the relation above is not stringent for mixed states in general, when the Robertson bound
$C_{AB}$ is simply extended to $C'_{AB} = \frac{1}{2} \text{Tr}([A, B] |\rho|)$, which decreases for mixed states and vanishes for totally mixed states. Further improvement of the bound was put forward by Ozawa [26] who showed that the constant $C_{AB}$ in Eq. (2) can be replaced by a stronger constant $\tilde{D}_{AB}$ defined by $\tilde{D}_{AB} = \frac{1}{2} \text{Tr}(|\sqrt{\rho}| [A, B] |\sqrt{\rho}|)$. This new parameter coincides with the Robertson bound $C_{AB}$ when $\rho$ is a pure state, but makes the EDUR in the form of Eq. (2) stronger for a mixed ensemble.

All these considerations so far have been valid for a general, arbitrary pair of non-commuting observables. As the simplest case, spin-$1/2$ observables, represented by a set of Pauli operators, have been a major focus of investigations of EDURs. Braunli [24] showed that for binary measurements with $A^2 = B^2 = 1$ and $\langle A \rangle = \langle B \rangle = 0$, where $\langle \cdots \rangle$ stands for the expectation value in the system state, Eq. (2) can be strengthened to a stronger EDUR. Ozawa demonstrated that it can be further strengthened by replacing again the bound $C_{AB}$ by $\tilde{D}_{AB}$ for mixed spin states [26].

We carried out a neutron polarimeter experiment at the research reactor in Vienna as depicted in FIG. 1. The incident neutrons with a wavelength $\lambda \approx 2.02$ Å, are polarized by the first supermirror. A guide field between Polarizer/Analyzer 1 and between Analyzer1/Analyzer 2 in $+-z$-direction is applied and determines the quantization axis. Building on the previous performance of the studies of the EDUR for pure states [15, 16, 27], we extend here the investigation by applying two procedures, i.e. the generation of mixed states and modification of the first measurement process in apparatus (M1) by unitary transforming the output states. The former allows the study of the EDUR for mixed states and the latter enables to tune the disturbance.

The polarimeter set-up consists of three stages: (1) the state preparation, (2) the apparatus M1 performing a projective $O_A$-measurement plus the correction procedure and (3) the apparatus M2 performing a $B$-measurement. Larmor precession induced by magnetic fields $B_x$ in the DC-coils allows to orient all required directions of the spin-measurements. The mixing of the state can be tuned by a noise magnetic field $B_{\text{noise}}$ [23]. In practice we realize $\pi/2$-rotations with noisy fields by one DC-coil (DC-1), where the required mixture can be adjusted by the amplitude of the noise signal.

In stage one, the input states were chosen to be $\rho_x(\alpha) = \frac{1}{2} (1 + \alpha \sigma_x)$ with five different mixtures $\alpha = \{1, 0.75, 0.5, 0.25, 0\}$. The degree of mixture was verified by measuring the expectation values of the Pauli-spin operators $\text{Tr}(\sigma_i \rho_x)$ for $i = x, y, z$ each. Typical fidelity $F = \text{Tr}(\sqrt{\rho_x \rho} \exp i \theta \sigma_z)$ of the pure input state $\rho_x$ was 0.982(5). The so-called ‘three-state-method’ [14] is applied to acquire the values of $\epsilon(A)$ and $\eta(B)$ (see Supplement for more details).

The second stage represents the apparatus M1 in which the coil DC-2/3 plus Analyzer 1 perform the projective measurement that actually measures the observable $O_A = \cos(\theta_O) \sigma_x + \sin(\theta_O) \sigma_y$; $\theta_O$ is the detuning angle of this measurement and leaves the neutron in the $|O_A = \pm 1\rangle$ states. In the correction stage, this output state of the $O_A$-measurement is transformed by a
for any mixed states $\rho$, the reduction and the enhancement of disturbance by angle ($\theta$) as a function of the polar and azimuthal states are generated and the detuning angle is studied. By applying the optimal correction procedure the minimal disturbances are measured tuning the input state $|\psi(\theta, \phi)\rangle = (\cos(\theta/2), e^{i\phi}\sin(\theta/2))^T$ and $|\psi(\theta, \phi)\rangle = |\psi(\pi - \theta, \phi + \pi)\rangle$ of the unitary operation. Blue and red arrow indicate the position of the minimal ($\frac{\pi}{4}, \frac{3\pi}{4}$) and maximal ($\frac{\pi}{4}, \frac{3\pi}{4}$) disturbance.

unitary operator $U^{\text{corr}}$. One can realize the optimal (and anti-optimal) correction by adjusting $U^{\text{corr}}$. Note that in our previous study [15, 16, 27] the unitary operation $U^{\text{corr}}$ is not applied and fixed as $U^{\text{corr}} = 1$ in practice. The last stage consists of the successive measurement of $B = \sigma_z$ in apparatus M2 which is accomplished again by a DC-coil (DC-4) plus analyzer (Analyzer 2) combination. Note that the final spin rotation is not applied, since it has no influence on measured intensities.

We investigate a neutron spin measurement in which $A = \sigma_z, B = \sigma_y$ and consider a general mixed ensemble represented by $\rho = \frac{1}{2}(I + r \cdot \sigma)$ satisfying $\langle A \rangle = \langle B \rangle = 0$; then, $\rho$ is generally parameterized as $\rho_x(\alpha) = \frac{1}{2}(I + \alpha \sigma_x)$. In this case, the parameter $D_{AB} = 1$ is constant and yields the tight relation [20]

$$\left(\epsilon(A)^2 - 2\right)^2 + \left(\eta(B)^2 - 2\right)^2 \leq 4 \quad (3)$$

for any mixed states $\rho_x(\alpha)$, while the parameter $C'_{AB}$ depends on the mixture, i.e. the length $|r|$ of the vector. Experimental tests of this relation for pure input states were carried out by using photonic systems [29, 30] and neutrons [27].

Our first study examines the influence of the unitary transformation in apparatus (M1). First, pure input states are generated and the detuning angle $\theta_{OA}$ is fixed at $5\pi/18$. Then, the eigenstate of $O_A$ after apparatus M1 is unitarily transformed to the state $|\psi(\theta, \phi)\rangle$, given by $|\psi(\theta, \phi)\rangle = (\cos(\theta/2), e^{i\phi}\sin(\theta/2))^T$. The measured disturbance as a function of the polar and azimuthal angle ($\theta, \phi$) is plotted in FIG. 2. This plot clearly exhibits the reduction and the enhancement of disturbance by the choice of $\theta$ and $\phi$. In addition, it is shown that the minimal and maximal disturbances, illustrated by blue and red arrows in FIG. 2, are achieved when the state after measurement is unitarily transformed into eigenstates of the observable $B = \sigma_y$ (see Supplement for theoretical details of the correction/anti-correction procedure).

After determination of maximum and minimum values of the disturbance, the EDUR given by Eq. 5 is analyzed. The experimentally determined error versus maximum and minimum disturbances are plotted in FIG. 3 for pure states together with the theoretically predicted bound. The red shaded area marks the forbidden region. The lower and upper bound was measured for angle $\theta_{OA} = [0, \pi]$ with a step of $\pi/18$. For $\theta_{OA} = 0$ we have $\epsilon(A) = 0$ at which point the disturbance is unique. When $\theta_{OA} = \pi/2$ ($O_A = B$), the disturbance reaches its maximum value, depending on the unitary [anti]-optimal correction transformation. When $\theta_{OA} = \pi$, $O_A = -A$ the error is maximal and disturbance is independent of the transformation once again.

Next the influence of the mixture of the input states is studied. By applying the optimal correction procedure the minimal disturbances are measured tuning the input.
FIG. 4. Error $\epsilon(A)$ versus disturbance $\eta(B)$ for the standard configuration ($A = \sigma_x, B = \sigma_y$) with four different mixtures of the state $\rho_x(\alpha) = \frac{1}{2}(1 + \alpha \sigma_x)$: (a) $\alpha = 0.75$, (b) $\alpha = 0.5$, (c) $\alpha = 0.25$ and (d) $\alpha = 0$. The red shaded areas are forbidden according to Eq. (3). The border indicates the lower bound $D_{AB} = 1$ which is saturated by our data points. The theoretical behavior of the bound $C'_{AB}$ is indicated by the colored dashed lines. A change of the mixture parameter $\alpha$ has no effect on the final error-disturbance relation in the standard configuration as initially predicted by the expectation value $C_{AB}$.

FIG. 5. Results of error and disturbance for a different observable $B$, parametrized by the angle $\theta_B$. The bound depends on the ‘degree of commutativity’ $[A, B]$ but not on the input states. Analogous to FIG. 4 the results were recorded with five different mixtures. (a) Result for $\theta_B = \pi/3$. (b) Result for $\theta_B = \pi/6$.

states $\rho_x(\alpha) = \frac{1}{2}(1 + \alpha \sigma_x)$. The results are plotted in FIG. 3. Each plot exhibits optimal EDUR for a particular mixture with theoretical predictions by $D_{AB}$ and $C'_{AB}$. It is immediately seen that the error-disturbance uncertainty is insensitive to dephasing or amplitude damping of the input states caused by the fluctuating magnetic field and that the bound is preserved perfectly. The measured values always saturate inequality Eq. (3), for mixed spin states no dependence on mixture appears. Only the bound given by $D_{AB}$ leads to saturation of the error-disturbance uncertainty relation. This statement is also true for different choices of the observables $A$ and $B$. If an extended configuration including non-maximally incompatible pairs of observables $A$ and $B$ is considered, e.g. $B = \cos(\theta_B)\sigma_z + \sin(\theta_B)\sigma_y$ then minimal disturbance is given by $\eta(B) = 2\sin(|\theta_{OA} - \theta_B|/2)$. The results for two different angles $\theta_B = \pi/3$ and $\theta_B = \pi/6$ are plotted in FIG. 5. For pure input-states, according to the change of the commutator $[A, B]$, both parameters $C_{AB}$ and $D_{AB}$ represent the lower bound of the EDUR. For mixed input-states, however, only the bound $D_{AB}$ explains the correct behavior.

The successive nature of the measurement made it obvious how the correction procedure, i.e., a unitary transformation, can be incorporated to the whole measurement. Disturbance is strongly affected by this correction and we have observed the maximum and the minimum disturbance by optimal and anti-optimal corrections. Our experiment successfully demonstrates the tightness of the bound $D_{AB}$ and the non-tightness of the simply extended Robertson bound $C'_{AB}$. We confirmed the independence of the EDUR on the mixture of the states for the case of dichotomic observables $A, B$ with $\langle A \rangle = \langle B \rangle = 0$. This is considered to be due to the fact that the observed uncertainty for Pauli operators is originated more in observables than in input states: this reminds us another state-independence appearing in quantum contextuality, which was confirmed in an
ion experiment [31]. Since quantum states, practically used in application such as quantum communication and computation, are more mixed ensemble due to (unavoidable) dephasing and decoherence than in a laboratory, our study shed a light on the new aspects of quantum measurements available for practical applications.

We acknowledge support by the Austrian Science Fund, FWF (Nos. P27666-N20 and P24973- N20), the European Research Council, ERC (No. MP1006), the John Templeton Foundations (ID 35771) and Japan Society for the Promotion of Science, JSPS KAKENHI (No. 26247016). We thank H. Rauch, (Vienna), M.J.W. Hall (Brisbane), F. Buscemi (Nagoya) and A. Hosoya (Tokyo) for their helpful comments.
SUPPLEMENTARY MATERIAL

Theoretical framework

Theory of error and disturbance. Any apparatus $M$ is described by an indirect measurement model $(K, |\xi\rangle, U, M)$, where $K$ is the apparatus state space, $|\xi\rangle$ is the initial apparatus state, $U$ is the unitary operator describing the object-apparatus interaction, and $M$ is the meter observable of the apparatus $|\Pi\rangle$. Let $\rho$ be the initial object state, the error $\epsilon(A)$ for measuring an observable $A$ and the disturbance $\eta(B)$ caused on an observable $B$ are defined as [2]

$$
\epsilon(A)^2 = \text{Tr} \left[ (U^+ (1 \otimes M) U - A \otimes 1)^2 \rho \otimes |\xi\rangle \langle \xi| \right],
$$

$$
\eta(B)^2 = \text{Tr} \left[ (U^+ (B \otimes 1) U - B \otimes 1)^2 \rho \otimes |\xi\rangle \langle \xi| \right].
$$

(S1)

To further evaluate error and disturbance, we suppose that the meter observable $M$ has non-degenerate eigenvalues $m$ with spectral decomposition $M = \sum m |m\rangle \langle m|$. Then, the apparatus $M$ is characterized by the family of measurement operators $\{ M_m \}$ defined by $M_m = \langle m| U |\xi\rangle$. The positive operator-valued measure (POVM) of $M$ is the family $\{ P_m \}$ of positive operators defined by $P_m = M_m^\dagger M_m$. We can rewrite error and disturbance, assuming $P_m$ are mutually orthogonal projectors, as

$$
\epsilon(A)^2 = \langle (O_A - A)^2 \rangle + \langle (O_A^{(2)} - O_A^2) \rangle,
$$

$$
\eta(B)^2 = \langle (O_B - B)^2 \rangle + \langle (O_B^{(2)} - O_B^2) \rangle,
$$

(S2)

where the output operators are given by $O_A^{(k)} = \sum m^k P_m$ and $O_B^{(k)} = \sum m M_m^\dagger B^k M_m$. As usual, we require the meter observable $M$ to have the same spectrum as the observable measured $A$. For binary observables $A^2 = B^2 = 1$, we have $O_A^{(2)} = O_B^{(2)} = 1$, and we obtain

$$
\epsilon(A)^2 = 2 - 2 \Re \langle AO_A \rangle, \quad \eta(B)^2 = 2 - 2 \Re \langle BO_B \rangle.
$$

(S3)

Optimization of disturbance. The measurement operators of the projective measurement of $O_A = \cos(\theta_O A) \sigma_z + \sin(\theta_O A) \sigma_y$ carried out by coil DC-2/3 plus Analyzer 1 are given by $\{ M_m \} = \{| m_{O_A} \rangle \langle m_{O_A}| \}$, where $| m_{O_A} \rangle = | O_A = m \rangle$ for $m = \pm 1$. The coil DC$_{corr}$ accounts for the unitary transformation $U_{corr}$ after the projective $O_A$ - measurement and before the $B$ - measurement, which modifies the output state of the projective $O_A$ - measurement to attain the optimal or anti-optimal bounds for the disturbance as suggested in [3] in the pure state case. Thus, the measurement operators of apparatus M1 are modified as $\{ M_m \} = \{ U_{corr} | m_{O_A} \rangle \langle m_{O_A}| \}$ without changing the POVM $\{ P_m \} = \{| m_{O_A} \rangle \langle m_{O_A}| \}$ and the output operator $O_A = \sum m^2 P_m$.

From Eq. (S3) apparatus M1 has the error $\epsilon(A) = 2 \sin^2(\theta_O/2)$. For the calculation of the disturbance $\eta(B)$ we consider the following observable: $B = \cos(\theta_B) \sigma_z + \sin(\theta_B) \sigma_y$, where $0 \leq \theta_B \leq \pi/2$. The angle $\theta_B$ quantifies the closeness of the observables $A$ and $B$, where maximal incompatibility is attained for the angle $\theta_B = \pi/2$. In this case we get $D_{AB} = \sin \theta_B$, optimal and anti-optimal corrections are carried out, minimal and maximal disturbance is given by

$$
2 \left| \sin \frac{\theta_O A - \theta_B}{2} \right| \leq \eta(B) \leq 2 \cos \frac{\theta_O A - \theta_B}{2}.
$$

(S4)

To show the above, let $B_m = \langle m_{O_A} | U_{corr \dagger} B U_{corr} | m_{O_A} \rangle$ for $m = \pm 1$. Then we have

$$
O_B = \sum_{m=\pm 1} | m_{O_A} \rangle \langle m_{O_A}| B_m | m_{O_A} \rangle = \frac{(B_+ + B_-)}{2} \Id + \frac{(B_+ - B_-)}{2} O_A,
$$

$$
\Re \langle BO_B \rangle = \frac{(B_+ - B_-)}{2} \cos(\theta_B - \theta_O A).
$$

(S5)

Since the extreme values are given by $\frac{(B_+ - B_-)}{2} = \pm 1$, the optimal and anti-optimal values of $\eta(B)$ are given by

$$
\eta(B)^2 = 2 \mp 2 \Re \langle BO_B \rangle = 2 \mp 2 \cos(\theta_B - \theta_O A).
$$

(S6)
FIG. S1. Detected count rates of the successive measurements carried out by apparatuses M1 and M2. The successive measurements of $O_A$ and $B$ have four outcomes, denoted as $I_+ + I_- I_+ I_- I_+ I_- I_+ I_-$. which are recorded for four input states, i.e. $\rho, ApA = B \rho B = \rho$, $P_A^+ \rho P_A^+$, and $\rho_B = P_B^+ \rho P_B^+$. The initial states are given by $\rho = \frac{1}{2} (\mathbb{1} + \alpha \sigma_x)$ with five different mixtures $\alpha = \{1, 0.75, 0.5, 0.25, 0\}$. The observable $A$ is set as $\sigma_z$ and the polar angle of the observable $B$ is tuned as $\theta_B = \frac{\pi}{2}$ (for the standard configuration) and $\frac{\pi}{3}$. The detuned observable $O_A$ are adjusted within the $zy$-plane with polar angles $(\theta_{OA}, \theta_B)$ given by $(0, \frac{\pi}{2})$ in (a) and $(\frac{2\pi}{9}, \frac{\pi}{3})$ in (b). Error bars represent one standard deviation of the normalized intensities. Some error bars are at the size of the markers.

Consequently

$$4 \sin^2 \frac{\theta_B - \theta_{OA}}{2} \leq \eta(B)^2 \leq 4 \cos^2 \frac{\theta_B - \theta_{OA}}{2}, \quad (S7)$$

and Eq. (S4) follows.

Data Treatment

Three state method. A re-ordering of the previous expressions (S3) of error and disturbance allows one to obtain them by measuring the mean values of $O_A$ and $O_B$ in three different states. We have

$$\epsilon(A)^2 = 2 - 4 \text{Tr}(P_A^+ \rho) \text{Tr}(\rho A O_A) + \text{Tr}(A \rho A O_A) + \text{Tr}(\rho O_A),$$
$$\eta(B)^2 = 2 - 4 \text{Tr}(P_B^+ \rho) \text{Tr}(\rho B O_B) + \text{Tr}(B \rho B O_B) + \text{Tr}(\rho O_B). \quad (S8)$$

For each projector combination of $O_A$ and $B$ an intensity output is acquired and the expectation value is calculated by combination of four intensities. We label these intensities as $I_{mb}$ where $m, b$ take values $\pm 1$. The expectation value of $O_A$ and $O_B$ for any state $\rho$ are obtained by

$$\text{Tr}(O_A \rho) = \frac{\sum_{m,b} m I_{mb}}{\sum_{m,b} I_{mb}}, \quad \text{Tr}(O_B \rho) = \frac{\sum_{m,b} b I_{mb}}{\sum_{m,b} I_{mb}}. \quad (S9)$$

To determine the error $\epsilon(A)$ these intensities have to be measured for the state $\rho$, the reflected state $ApA$ and the pure state $\rho_A = P_A^+ \rho P_A^+$. The same applies to the measurement of disturbance $\eta(B)$ where the input
FIG. S2. Squared disturbance $\eta(B)^2$ vs. squared error $\epsilon(A)^2$. For detuning angle of $\theta_{OA} = 0$ the output observable $O_A$ coincides with $A = \sigma_z$ with $(\epsilon(A), \eta(B)) = (0, 2)$. For increasing angles $\theta_{OA}$ the error increases, while disturbance either minimizes when corrected or maximizes when anti-corrected.

states are $\rho$, $B\rho B$ and the pure state $\rho_{B} = P_{B}^+ \rho P_{B}^+ / \text{Tr}(P_{B}^+ \rho)$. The prefactors $\text{Tr}(P_{A}^+ \rho)$ and $\text{Tr}(P_{B}^+ \rho)$ in Eq. S8 are obtained separately in the state preparation adjustment process. If $\rho = \frac{1}{2}(1 + \sum_j r_j \sigma_j)$ is a general mixed qubit state then the polarization of the state is given by $\text{Tr}(\sigma_i \rho) = r_i$. This relation allows to prepare and check the initial state’s direction and the degree of mixtures.

**Experimental determination of error and disturbance.** Here, we consider an extended configuration including non-maximally incompatible pairs of observables $A$ and $B$. The observable $A$ is left as $A = \sigma_z$ but $B$ is set as $B = \cos(\theta_B) \sigma_z + \sin(\theta_B) \sigma_y$, where $0 \leq (\theta_B) \leq \frac{\pi}{2}$. The measuring apparatus M1 performs a projective spin measurement of the observable $O_A = \cos(\theta_{OA}) \sigma_z + \sin(\theta_{OA}) \sigma_y$, where $0 \leq \theta_{OA} \leq \frac{\pi}{2}$, followed by the unitary operation $U_{\text{corr}}$ as in the main text, and the measurement operators of apparatus M1 are given by $\{M_m\} = \{U_{\text{corr}} | m_{OA}\} \{m_{OA}\}$. Furthermore, apparatus M2 carries out the projective measurement of $B$ immediately after the measurement carried out by apparatus M1. Let $O_B = \sum_m M_m B M_m$. Then, the error $\epsilon(A)$ and the disturbance $\eta(B)$ are given by Eq. S4. By the “three state method”, the error $\epsilon(A)$ and the disturbance $\eta(B)$ can be experimentally obtained as a sum of expectation values of the outputs from apparatus M1 and M2 in three different states as in Eq. S8. For the determination of error $\epsilon(A)$ and disturbance $\eta(B)$, the expectation values of $O_A$ and $O_B$ in a state $\rho$ in Eq. S8 are derived from the intensities of the four possible outputs of the measurement $O_A$ and $B$ denoted as $I_+ I_+ I_- I_-$ as given by Eq. S9.

In Fig. S1 typical sets of intensities for different values of $\theta_{OA}$ and $\theta_B$ for five different mixtures $\alpha = \{1, 0.75, 0.5, 0.25, 0\}$ of the input state $\rho_\alpha = \frac{1}{2}(1 + \alpha \sigma_z)$ are depicted. To determine the error $\epsilon(A)$, intensities have to be measured for the input state $\rho = \rho_\alpha$, the auxiliary state $\rho_A = \rho_{\sigma_z}(\alpha)$ and $\rho_A = P_{B}^+ \rho P_{B}^+ / \text{Tr}(P_{B}^+ \rho)$ (this is the pure state $|+z\rangle$). For the disturbance, $\eta(B)$ the input states of $\rho_B B \rho_B$ are $\rho_{\sigma_z}(\alpha)$ and the pure state $\rho_B = P_{B}^+ \rho P_{B}^+ / \text{Tr}(P_{B}^+ \rho)$ are prepared. In Fig. S1 $P_{B}^+ \rho P_{B}^+$ is $|+y\rangle$ in (a) and $\frac{\sqrt{2}}{2} |+z\rangle + \frac{1}{2} |z\rangle$ and $\frac{\sqrt{2}}{2} |+z\rangle$. The pre-factors $\text{Tr}(P_{A}^+ \rho)$ and $\text{Tr}(P_{B}^+ \rho)$ are measured separately, by applying only a single apparatus. The resulting values of the squared error $\epsilon(A)^2$ and the squared disturbance $\eta(B)^2$ are plotted in Fig. S2 for corrected and anti-corrected case, under variation of $\theta_{OA}$. 

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Mixed state generation

For a general mixed state given by \( \rho_i = \frac{1}{2}(1 + r_i \sigma_i) \) (\( \sigma_i \) represents the direction of the Bloch vector and is given by Pauli operators), the degree of polarization is given by \( P = \text{Tr}(r_i \sigma_i) \). In order to prepare the mixed states, required for the determination of error \( \epsilon(A) \) and disturbance \( \eta(B) \), this \( P \) has to be varied. This is achieved by applying a random noisy magnetic field in addition to the static one in DC1 (see FIG. 1 in the main manuscript). That is, neutrons with different arriving times at the coil DC1 experience different magnetic field strengths. This is equivalent to apply different unitary operators \( U_{\text{noise}} \), describing the noisy \( \pi/2 \)-rotation about the \( x \)-axes, at each time: this is written in a form of \( U_{\text{noise}}(\pi/2 + \Delta \xi(t) : 0, -\pi/2) = \tilde{U}_{\text{noise}}(\Delta \xi(t)) \) (the terms 0 and \(-\pi/2\) denote the polar and azimuthal angle of the rotation axis i.e. the \( x \)-axis in our case). For the whole ensemble we have to take the time integral. Although transformation at each time is unitary, this procedure as a whole ends up as a non-unitary operation due to due to the randomness of the noisy signal and prepares mixed states [4]. For the preparation of the input state \( \rho \) (and \( A\rho A \)), DC1 is positioned in such way that \( \rho = \rho_x(\alpha) = \frac{1}{2}(1 + \alpha \sigma_x) \) is generated at the end of the preparations section, depicted in FIG. 1 in the main manuscript.

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