Obliquely Reflected BSDEs

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Abstract

In this paper, we study existence and uniqueness to multidimensional Reflected Backward Stochastic Differential Equation in an open convex domain, allowing for oblique directions of reflection. In a Markovian framework, combining a priori estimates for penalised equations and compactness arguments, we obtain existence results under quite weak assumptions on the driver of the BSDEs and the direction of reflection, which is allowed to depend on both $Y$ and $Z$. In a non Markovian framework, we obtain existence and uniqueness result for direction of reflection depending on time and $Y$. We make use in this case of stability estimates that require some smoothness condition on the domain and the direction of reflection. In a last Section, we illustrate the application of our theoretical results by introducing randomised switching problems.

Key words: BSDE with oblique reflections, Switching problems.

MSC Classification (2000): 93E20, 65C99, 60H30.
1 Introduction

In this paper, we study a class of BSDE whose solution is constrained to stay in an open convex domain, hereafter denoted $\mathcal{D}$. The “reflection” at the boundary of the domain is made along an oblique direction. Such equations are known as Obliquely Reflected BSDEs and they allow to represent the solution of some stochastic control problems. Precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t)_{t \in [0,T]}$ a $k$-dimensional Brownian motion, defined on this space, whose natural filtration is denoted $(\mathcal{F}_t)_{t \in [0,T]}$. $\mathcal{P}$ is the $\sigma$-algebra generated by the progressively measurable processes on $[0,T] \times \Omega$. In this paper, we are interested in the study of existence and uniqueness of a $\mathcal{P}$- measurable solution $(Y, Z, \Phi)$ to the following equation

$$
\begin{align*}
(i) \quad & Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T H(t, Y_t, Z_t)\Phi_t ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \\
(ii) \quad & Y_t \in \mathcal{D}, \quad \Phi_t \in \partial \varphi(Y_t), \quad \int_0^T |\Phi_t| 1_{\{ Y_t \notin \mathcal{D} \}} dt = 0, 
\end{align*}
$$

where $\partial \mathcal{D}$ is the boundary of $\mathcal{D}$, $\varphi$ its (convex) indicator function, $\partial \varphi$ the subdifferential of $\varphi$ and $(f, H): \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to (\mathbb{R}^d, \mathbb{R}^{d \times d})$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d \times k})$-measurable function. The terminal value $\xi$ is given as a parameter and belongs to $\mathcal{L}^2(F_T)$, where for $p > 0$ and a $\sigma$-algebra $\mathcal{B}$, $\mathcal{L}^p(\mathcal{B})$ is the space of $\mathcal{B}$-measurable random variable $R$ satisfying $\mathbb{E}[|R|^p] < +\infty$. Of course, we shall require some extra conditions. Classically, we will look for solution with the following integrability property: $(Y, Z, \Phi) \in \mathcal{L}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$, where, for $p \in [1, \infty]$, $\mathcal{H}^p$ is the set of progressively measurable process $V$ such that $\mathbb{E} \left[ \left( \int_0^T |V_t|^2 dt \right)^{p/2} \right] < +\infty$, and $\mathcal{H}^p$ is the set of continuous and adapted processes $U$ satisfying $\mathbb{E} \left[ \sup_{t \in [0,T]} |U_t|^p \right]$. The main constraints are given in (1.1)(ii) on the couple $(Y, \Phi)$. As already mentioned, the first one is that $Y$ takes its value in $\mathcal{D}$, where $\mathcal{D}$ is a non-empty open convex subset of $\mathbb{R}^d$. The fact that $\Phi_t \in \partial \varphi(Y_t)$ imposes that $\Phi$ is directed along the outward normal of the convex domain, the important point being that in (i) this direction is perturbed by the operator $H$ and we are thus generally dealing with an oblique direction of reflection. When (1.1)(i) is viewed backward in time, the process $\Phi$ or, more precisely $\Psi := H(\cdot)\Phi$, is the process allowing $Y$ to stay in $\mathcal{D}$. The condition $\int_0^T |\Phi_t| 1_{\{ Y_t \notin \mathcal{D} \}} dt = 0$ is then interpreted as a minimality condition, in the sense that $\Psi$ will be active only when $Y$ touches the boundary of the domain. This is of course one of the main ingredients to get an uniqueness result for this kind of equation.

These equations appear in the study of stochastic control problem, as illustrated in Section 5. Moreover, they allow for a probabilistic interpretation of a large class of quasivariational PDEs. Let us now mention some known results about these equations. In the one dimensional case, they have been first studied in [1] for the so-called simply reflected case and in [2] for the doubly reflected case. The literature on this specific form of equation has then grown very importantly due to their range of application, in particular in mathematical finance. The multidimensional case is only well understood.
in the case of normal reflection i.e. when the matrix-valued random function $H$ is equal to the identity, see [6]. The case of oblique direction of reflection has been only partially studied. Up until recently, only very specific cases have been considered for the couple $(H, \mathcal{D})$. In [19], the author studies the case of the reflection in an orthant with some restriction on the direction of oblique reflection and the driver $f$. Another case that has received a lot of attention is the setting of RBSDEs associated to switching problem, see e.g. [10, 9, 1] and the references therein: the multidimensional domain has a specific form and the direction is along the axis, see also Section 4.1 for more details. In this case, structural conditions on $f$ are required to retrieve existence and uniqueness results also. This restriction are based on the technique of proof used to obtain the results and which is mainly based on a monotonic limit theorem à la Peng [18], in a multidimensional setting. It seems that the first attempt to treat the question of BSDEs with oblique reflection in full generality can be found in [5]. Unfortunately, their setting is still quite restrictive concerning $H$ and $f$.

To the best of our knowledge, there is no, up to now, satisfying global approach for the question of well-posedness of Obliquely Reflected BSDEs, especially when compared to the case of forward SDEs, where existence and uniqueness results are obtained for quite general oblique reflection and domain, see e.g. the seminal paper [12].

Our goal in this paper is thus to provide existence and uniqueness results for the RBSDEs (1.1) for generic $H$ and convex domain $\mathcal{D}$ without imposing any structural dependence condition on the driver $f$ of the equation. In this direction, we are able to obtain very general existence result in a Markovian setting, assuming some very weak domination property of the forward process, see Section 4.1 and we also discuss the non-uniqueness issue. In the general case of $\mathcal{P}$-measurable random coefficients $f$ and terminal condition $\xi$, we need to impose some smoothness assumptions on the domain and $H$, which depends then only on time and $y$. In this case, we obtain both existence and uniqueness for the solution of (1.1).

The main tool to obtain the existence result is to consider a sequence of penalised equation: for $n \in \mathbb{N}$, $t \in [0, T]$,

$$Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds - \int_t^T Z^n_sdW_s - \int_t^T H(s, Y^n_s, Z^n_s)\nabla \varphi^n(y^n)ds,$$

(1.2)

where, for $y \in \mathbb{R}^d$, and some $M > 0$,

$$\varphi^n(y) := n \inf_{x \in \mathcal{D}} \theta_M(y - x) \quad \text{with} \quad \theta_M(h) = \begin{cases} M|h| - \frac{M^2}{2} & \text{if } |h| > M, \\ \frac{1}{2} |h|^2 & \text{if } |h| \leq M. \end{cases}$$

(1.3)

The key point is to obtain the convergence in a strong sense of $(Y^n)_n$ to some process $Y$ along with some $a priori$ estimates on $(Z^n, \Phi^n)$. This will then allow to obtain the existence of some limiting process $(Z, \Phi)$ as well. Of course, one has then to prove that the limit $(Y, Z, \Phi)$ satisfies indeed the obliquely reflected BSDEs (1.1). In the setting of oblique direction of reflection, the question of uniqueness has generally to be investigated separately.
The first possible argument to obtain the convergence of \((Y^n)\) is to prove some monotonicity on the sequence to apply Peng’s monotonic limit theorem \[18\]. In a multidimensional setting, this monotonicity is obtained under very restrictive structural condition on the coefficient. Nevertheless, it has been successfully used for the study of RBSDE associated to switching problem. Another possible argument is to invoke some fine compactness arguments and this is the approach followed in \[5\]. But, again some strong structural conditions are required to obtain convergence results in a weak setting. In this paper, we follow a similar approach in the Markovian setting, see Section \[4\]. At the heart of our proof, we use the paper \[8\], which was concerned with multidimensional (non-reflected) BSDEs with continuous only driver \(f\). With this approach, in the Markovian setting, we are able to obtain existence result for \(H\) that can depend on \(Z\) and even be discontinuous. To the best of our knowledge, this is the first time such general setting is considered successfully. It has been brought to our attention that independently from us, \[3\] has followed a similar approach to treat BSDEs associated to the classical switching problem in a more restrictive setting.

The last approach to obtain convergence of the sequence \((Y^n)\) is to show classically that it is a Cauchy sequence. This approach has been used in the case of multidimensional RBSDE when there is no perturbation \(H\) of the direction of reflection, namely \(H\) is the identity matrix of \(\mathbb{R}^d\), in the seminal paper \[6\]. To obtain this result and a key stability estimate, \[6\] uses dramatically the convexity property of the domain linked with the normal reflection by applying Itô’s formula to the Euclidean norm of the difference of two solutions. In our setting of general perturbation \(H\), we cannot follow their proof. In order to retrieve the stability estimate, we modify the Euclidean norm to take into account the oblique reflection inspired by \[12\]. Unfortunately, this produces new terms that have to be controlled. The most difficult one is certainly the term linked to the quadratic variation of the martingale term in \((1.1)\) or \((1.2)\). Let us emphasize that this term cannot be dealt with as one would do in the forward SDE case. Nevertheless, we are able to control this term with the use of BMO tools. To the best of our knowledge, this approach is completely new in the setting of Reflected BSDEs. We are then able to obtain in the non-Markovian setting existence results when \((D, H)\) satisfies some \(C^2\) smoothness condition, with \(H\) depending only on time and \(y\). Let us note also that in this case the uniqueness result is obtained as an easy consequence of the stability estimate.

The rest of the paper is organised as follows. In the next Section, we present precisely our framework and the assumptions made on the coefficients along with some discussions on these assumptions. We also prove the key \textit{a priori} and stability estimates, that will be used later on. In Section \[3\] we present our first novel result on existence and uniqueness of Obliquely Reflected BSDEs in a regular setting for \((D, H)\). In Section \[4\], restricting to a Markovian framework, we extend our previous existence result assuming no regularity on \((D, H)\) and allowing a dependence in \(Z\) for the operator \(H\). Finally, in Section \[5\] we illustrate our results by applying them to a new class of randomised switching problems.
Notations: We denote by \( \varphi \) the indicator function of \( D \)

\[
\varphi(y) = \begin{cases} 
0 & \text{if } y \in \bar{D}, \\
+\infty & \text{otherwise},
\end{cases}
\]

and \( \partial \varphi \) its subdifferential operator:

\[
\partial \varphi(y) = \begin{cases} 
\{ \hat{y} \in \mathbb{R}^d : \hat{y} \cdot (z - y) \leq 0, \forall z \in \bar{D} \} & \text{if } y \in \bar{D} \\
\emptyset & \text{if } y \notin \bar{D}.
\end{cases}
\]

In particular, \( \partial \varphi(y) \) is the closed cone of outward normal to \( D \) at \( y \) when \( y \in \partial D \) and \( \partial \varphi(y) = \{0\} \) when \( y \in \bar{D} \). Finally, we denote by \( \mathcal{P} \) the projection onto \( \bar{D} \) and by \( n(y) \) the set of unit outward normal at \( y \in \partial D \).

For a matrix \( M \), we denote \( M^T \) its transpose.

We denote \( \mathcal{B}^2 \), the set of processes \( V \in \mathcal{H}^2 \), such that

\[
\|V\|_{\mathcal{B}^2} := \left\| \sup_{t \in [0,T]} \mathbb{E} \left[ \int_t^T |V_s|^2 \, ds | \mathcal{F}_t \right] \right\|_{L^\infty}^{\frac{1}{2}} < +\infty.
\]

Let us remark that \( V \in \mathcal{B}^2 \) means that the martingale \( \int_0^t V_s \, dW_s \) is a BMO martingale and \( \|V\|_{\mathcal{B}^2} \) is the BMO-2 norm of \( \int_0^T V_s \, dW_s \). We refer to [11] for further details about BMO martingales.

Finally, the set of continuous function from \([0, T]\) to \( \mathbb{R}^n \) is denoted \( C([0, T], \mathbb{R}^n) \). For \( x \in C([0, T], \mathbb{R}^n) \), we denote by \( \|x\|_\infty := \sup_{t \in [0,T]} |x_t| \), the sup-norm on this space.

2 Setting and preliminary estimates

In this section, we first introduce and discuss the main assumptions that will be used to obtain our existence and uniqueness results. In a second part, we give important \textit{a priori} estimates and prove a key stability result, which is one of the novelty in our approach in solving Obliquely Reflected BSDEs.

2.1 Framework

The first minimal set of assumption that we consider here is the following.

Assumption (A)

i) \( \xi \) is an \( \mathcal{F}_T \)-measurable random variable, \( \mathbb{R}^d \)-valued such that \( \mathbb{E} \left[ |\xi|^2 \right] < +\infty. \)

ii) \( f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^d \) is a \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d \times k}) \)-measurable function and there exists a non negative progressively measurable process \( \alpha \in \mathcal{H}^2(\mathbb{R}) \) and a constant \( L \) such that

\[
|f(t, y, z)| \leq \alpha_t + L(|y| + |z|), \quad \forall (t, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k}.
\]
iii) $H : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^{d \times d}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d \times k})$-measurable function and there exist constants $b > 0$, $L > 0$ such that, for any $(t, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k}$

$$H(t, \mathcal{P}(y), z)v \cdot v \geq \eta, \quad v \in \mathcal{N}(\mathcal{P}(y)), \quad (2.2)$$

$$|H(t, \mathcal{P}(y), z)| \leq L. \quad (2.3)$$

The above assumptions are too weak to obtain existence and uniqueness result in a general random framework. They will be used in Section 4 in a Markovian framework with their Markovian counterpart (AM). Nevertheless, it is possible to derive very useful a priori estimates in the general setting of (A).

**Remark 2.1.** In applications, $H(t, \cdot, z)$ is usually specified only on the boundary $\partial D$. The extension to $\mathbb{R}^d \backslash D$ is done easily by setting $H(t, y, z) := H(t, \mathcal{P}(y), z)$. Moreover, if $H(t, \cdot, z)$ is a continuous and bounded function on $\partial D$ it is possible to extend it to a continuous and bounded function on $D$. Indeed, $\partial D$ is homeomorphic to a set $S$ which is a half plane of $\mathbb{R}^d$ or $\mathbb{R}^r \times B^{d-r}$, with $0 \leq r \leq d$. Moreover, the boundary of $D$ is sent to the boundary of $S$. Then we remark that the extension of $H(t, \cdot, z)$ is straightforward when $D = S$.

In the non-Markovian setting, our results require more smoothness and control on the parameters of the BSDE. We will then work under the following assumption.

**Assumption (SB)**

i) $\xi$ is an $\mathcal{F}_T$-measurable $D$-valued random variable and the martingale $\mathcal{Y}_t^\xi := \mathbb{E}[\xi] = \xi - \int_0^T Z_s^\xi dW_s$, $t \leq T$, is BMO (see [11] for further details on BMO martingales).

ii) $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^d$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d \times k})$-measurable function, there exists a constant $L$ such that, for all $(t, y, y', z, z') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \times \mathbb{R}^{d \times k}$,

$$|f(t, y, z) - f(t, y', z')| \leq L \left(|y - y'| + |z - z'|\right), \quad (2.4)$$

$$|f(t, y, z)| \leq L(1 + |z|).$$

Moreover, the process $\theta^\xi := f(\cdot, \mathcal{Y}_\cdot^\xi, Z_\cdot^\xi)$ belongs to $\mathcal{B}^2$.

iii) The open convex domain $D$ is bounded and given by a $C^2_b(\mathbb{R}^d, \mathbb{R})$ function $\phi$ namely $D = \{ \phi < 0 \}$ and $\partial D = \{ \phi = 0 \}$. This function satisfies moreover

$$|\phi(x)| = d(x, \partial D) \quad \text{for} \quad x \in \mathcal{V} \cup \mathbb{R}^d \backslash D, \quad \text{where} \ \mathcal{V} \ \text{is a neighbourhood of} \ \partial D. \quad (2.5)$$

iv) $H : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is valued in the set of matrices $Q$ satisfying

$$|Q| \leq L, \quad L|v|^2 \geq v^\dagger (Q^{-1})^\dagger v \geq \frac{1}{L}|v|^2, \quad \forall v \in \mathbb{R}^d, \quad (2.6)$$
and for some $\eta > 0$,
\[
H(t, \mathcal{P}(y))v \cdot v \geq \eta, \quad \forall y \in \mathbb{R}^d, t \in [0, T], v \in n(\mathcal{P}(y)). \tag{2.7}
\]
Both $H$ and the mapping $(t, y) \mapsto (H^{-1})^\dagger(t, y)$ are $C^2$ functions satisfying
\[
|\partial_y H| + |(H^{-1})^\dagger| + |\partial_t (H^{-1})^\dagger| + |\partial_{yy} (H^{-1})^\dagger| + |\partial_{yy}^2 (H^{-1})^\dagger| \leq \Lambda, \tag{2.8}
\]
for some positive $\Lambda$. We first comment the assumption made on the parameter of the BSDE.

**Remark 2.2.**

i) Let us observe that under the BMO condition, there exists $\mu^\xi > 0$, such that $E\left[e^{\mu^\xi \sup_{t \in [0, T]} |Y^\xi_t|}\right] < \infty$ and that $\|Z^\xi\|_{L^2} < \infty$. For later use, we define
\[
\sigma^\xi := E\left[e^{\mu^\xi \sup_{t \in [0, T]} |Y^\xi_t|}\right] + \|Z^\xi\|_{L^2} + \|\theta^\xi\|_{L^2} < \infty. \tag{2.9}
\]

ii) Condition $(SB)(ii)$ is a mix of the property of $\xi$, $f$ and the domain $D$. In many applications, it will be straight forward to check. For example, it is trivially satisfied in the following cases:

(a) $\sup_{y \in D} f(s, y, z) \leq C$;

(b) $\xi \in L^\infty$;

(c) $D$ is a bounded domain.

iii) If $(SB)$ holds, then $(A)$ holds as well. Indeed, one can set $\alpha := L\left(|Y^\xi| + |Z^\xi| + |\theta^\xi|\right)$.

We now discuss the various assumptions made on $H$ and the domain $D$.

**Remark 2.3.**

i) The function $\phi$ can be constructed as in e.g. [6] Section 2.4 if the convex domain $D$ is $C^2$. From $(SB)(iv)$, it follows that $\mathcal{P}(x)$ (resp. $n(x)$) is the outward normal (resp. unit outward normal) of $D$ at a point $x \in \partial D$. Moreover, since $D$ is convex, $\phi$ is convex on $\mathbb{R}^d \setminus D$ and thus, $\mathcal{P}^2 \phi$ is a semi-definite matrix on this domain. Let us also observe that the application $\mathcal{P} : \mathbb{R}^d \setminus D \to \partial D$ is $C^2$.

ii) The matrix $H$ defines on $\partial D$ a unit vector field $\nu$ in the following way
\[
\tilde{\nu}(t, y) := H(t, y)n(y) \quad \text{and} \quad \nu(t, y) := \frac{\tilde{\nu}(t, y)}{|\tilde{\nu}(t, y)|}, \quad \text{for} \ y \in \partial D,
\]
which represents the oblique direction of reflection. Then, [27] rewrites as
\[
\langle \tilde{\nu}(t, y), n(y) \rangle \geq \eta, \quad \text{for} \ y \in \partial D. \tag{2.10}
\]
In applications, it is generally the case that only the smooth vector field $\nu$ is given on $\partial D$. Following [22], it is possible to construct $H$ satisfying $(SB)(iv)$ on $\partial D$ and then to extend it on $\bar{D}$ under $(SB)(iii)$ using classical extension results, see e.g. [7].
We now introduce a class of terminal conditions that are admissible for the purpose of work, in the sense that we can obtain an existence and uniqueness result for this class.

**Definition 2.1.** For $\beta > 0$, the class $\mathfrak{T}_\beta$ is the subset of $\xi \in L^2(\mathcal{F}_T)$ satisfying: there exists $\lambda_\xi > \beta$, such that

$$
\mathbb{E}
\left[
\exp \lambda_\xi \int_0^T |\mathcal{Z}_s^\xi|^2 ds
\right]< \infty,
$$

(2.11)

where $\mathcal{Z}_s^\xi$ is given by the martingale representation theorem applied to $\mathcal{Y}_t^\xi := \mathbb{E}_t[\xi] = \xi - \int_t^T \mathcal{Z}_s^\xi dW_s$, $t \leq T$.

We study the class $\mathfrak{T}_\beta$ in Section 2.4. Especially, we exhibit some specific elements of this class that are quite useful for applications.

**Remark 2.4.** In the following, we will use in the proofs the notation $C$ to denote a generic constant that may change from line to line and that depends in an implicit way on $T, L, \eta$. We shall denote it $C_\theta$, if it depends on an extra parameter $\theta$. In the statement of the results, the dependence upon any extra parameters of the constant involved will also be made clear.

### 2.2 A priori estimates

In this section, we prove some a priori control on the solution $(Y, Z, \Phi) \in \mathcal{S}_2^2 \times \mathcal{H}_2^2 \times \mathcal{H}_2^2$ to the following generalised BSDE

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T H(s, Y_s, Z_s)\Phi_sds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{2.12}
$$

Importantly, we assume that $(Y, Z, \Phi)$ satisfies the following structural condition:

$$
\mathbb{E}_t \left[ \int_t^T |\Phi_s|^2 ds \right] \leq K \mathbb{E}_t \left[ \int_t^T |f(s, Y_s, Z_s)|^2 ds \right], \text{ for some } K > 0. \tag{2.13}
$$

Equation (2.12) encompasses both the obliquely reflected BSDE (1.1) and its penalised approximation given in equation (1.2). The key point for these two equations will then be to prove that their solutions satisfy condition (2.13).

Our first estimate is quite classical.

**Lemma 2.1.** Assume (A). Let $(Y, Z, \Phi) \in \mathcal{S}_2^2 \times \mathcal{H}_2^2 \times \mathcal{H}_2^2$ be a solution to (2.12) with condition (2.13) holding true. Then, for some $c := c(K, L)$,

$$
\sup_{t \in [0, T]} \mathbb{E}[|Y_t|^2] + \mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right] \leq c \mathbb{E}[|\xi|^2 + \int_0^T |\alpha_s|^2 ds].
$$
Proof. We apply Itô’s formula to $|Y|^2$ to obtain

$$
|Y_t|^2 + \int_t^T |Z_s|^2 ds = 2 \int_t^T Y_s f(s, Y_s, Z_s) ds - 2 \int_t^T Y_s H(s, Y_s, Z_s) \Phi_s ds - 2 \int_t^T Y_s Z_s dW_s.
$$

(2.14)

We observe, thanks to the square integrability of $Y$ and $Z$ that $\tilde{s}\int_0^T Y_s Z_s dW_s$ is a true martingale. This yields,

$$
E[|Y_t|^2 + \int_t^T |Z_s|^2 ds] = E[|\xi|^2 + 2 \int_t^T Y_s f(s, Y_s, Z_s) ds - 2 \int_t^T Y_s H(s, Y_s, Z_s) \Phi_s ds].
$$

We thus compute, using (2.13) and Assumption (A)(ii), the boundedness of $H$ and Young’s inequality, for some $\epsilon \in (0, 1),

$$
E\left[\int_t^T Y_s H(s, Y_s, Z_s) \Phi_s ds\right] \leq C_{K, L} E\left[\int_t^T \left(\frac{1}{\epsilon} |Y_s|^2 + \epsilon |\Phi_s|^2\right) ds\right] \leq C_{K, L} E\left[\int_t^T \left(\frac{1}{\epsilon} |Y_s|^2 + \epsilon |Z_s|^2 + |\alpha_s|^2\right) ds\right].
$$

Similarly, we get

$$
E\left[\int_t^T Y_s f(s, Y_s, Z_s) ds\right] \leq C_{L} E\left[\int_t^T \left(\frac{1}{\epsilon} |Y_s|^2 + \epsilon |Z_s|^2 + |\alpha_s|^2\right) ds\right].
$$

For $\epsilon$ small enough and using Gronwall Lemma, we deduce

$$
E\left[|Y_t|^2 + \int_t^T |Z_s|^2 ds\right] \leq C_{K, L} E\left[|\xi|^2 + \int_t^T |\alpha_s|^2 ds\right].
$$

The following Proposition refines the previous estimates in the smooth setting of Assumption (SB). It will also allow to use the stability result proved in the next section. Interestingly, it shows that most of the property of the martingale $Y^\xi$ are transferred to the non linear process given in equation (2.12).

**Proposition 2.1.** Assume that (SB) holds. Let $(Y, Z, \Phi) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ be a solution to (2.12) with condition (2.13) in force. Then, the followings holds

i) $(Y, Z, \Phi) \in \mathcal{S}^2 \times \mathcal{B}^2 \times \mathcal{B}^2$ with for some $c := c(L, K, \sigma^\xi)$

$$
E\left[e^{\lambda \sup_{[0,T]} |Y_t|}\right] + \|\Phi\|_{\mathcal{H}^2} + \|Z\|_{\mathcal{B}^2} \leq c,
$$

and, for all $b > 0$ and some $c' := c'(b, L, K, \sigma^\xi)$

$$
E\left[e^{b \int_0^T \left(|\Phi_s| + |Z_t| + \sigma^\xi_s|\right) ds}\right] \leq c'.
$$

(2.15)
Moreover, if \( \xi \in \mathbb{S}_\beta \), for some \( \beta > 0 \). Then, there exists \( \Theta^2 \in \mathcal{H}^2 \) such that, for all increasing process \( \gamma \) satisfying \( \mathbb{E}[|\gamma_T|^p] < \infty \) for some \( p > 1 \) (depending on \( \gamma \)), we have for all \( t \in [0, T] \)

\[
\mathbb{E}_t \left[ \int_t^T \gamma_s Z_s^2 ds \right] \leq \mathbb{E}_t \left[ \int_t^T \gamma_s \Theta_s^2 ds \right] < +\infty
\]

and for some \( \lambda \in (\beta, \lambda^\xi) \) and \( c := c(L, K, \sigma^\xi, \lambda) \),

\[
\mathbb{E} \left[ e^{\lambda T} |\Theta_T^2| \right] \leq c.
\]

**Proof.** An important step to obtain our estimates below is to compare the BSDE \((Y, Z)\) with the martingale \( \mathcal{M}^\xi \). To this end, we introduce for this proof \( \Delta Y := Y - \mathcal{M}^\xi \) and \( \Delta Z = Z - \mathcal{M}^\xi \).

1.a We apply Itô’s formula to \(|\Delta Y|^2\) to obtain

\[
|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds = 2 \int_t^T \Delta Y_s f(s, Y_s, Z_s) ds - 2 \int_t^T \Delta Y_s H(s, Y_s) \Phi_s ds - 2 \int_t^T \Delta Y_s \Delta Z_s dW_s.
\]

We observe, thanks to the square integrability of \( \Delta Y \) and \( \Delta Z \) that \( \int_0^T \Delta Y_s \Delta Z_s dW_s \) is a true martingale. This yields, for all \( r \leq t \),

\[
\mathbb{E}_r \left[ |\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds \right] = 2 \mathbb{E}_r \left[ \int_t^T \Delta Y_s f(s, Y_s, Z_s) ds - \int_t^T \Delta Y_s H(s, Y_s) \Phi_s ds \right] \tag{2.17}
\]

We thus compute, using (2.13), the Lipschitz continuity of \( f \), the boundedness of \( H \) and Young’s inequality, for all \( r \leq t \) and some \( \epsilon \in (0, 1) \)

\[
\mathbb{E}_r \left[ \int_t^T \Delta Y_s H(s, Y_s) \Phi_s ds \right] \leq C_{L, K} \mathbb{E}_r \left[ \int_t^T \left( \epsilon |\Delta Y_s|^2 + |\Phi_s|^2 \right) ds \right],
\]

\[
\leq C_{L, K} \mathbb{E}_r \left[ \int_t^T \left( \epsilon |\Delta Y_s|^2 + |\Delta Z_s|^2 + |\Phi_s|^2 \right) ds \right].
\]

Similarly, we obtain

\[
\mathbb{E}_r \left[ \int_t^T \Delta Y_s f(s, Y_s, Z_s) ds \right] \leq C_L \mathbb{E}_r \left[ \int_t^T \left( \frac{1}{\epsilon} |\Delta Y_s|^2 + |\Delta Z_s|^2 + |\Phi_s|^2 \right) ds \right].
\]

Combining the last two estimates with (2.17), setting \( \epsilon \) small enough and using Gronwall Lemma, we get

\[
\mathbb{E}_r \left[ |\Delta Y_t|^2 + \frac{1}{2} \int_t^T |\Delta Z_s|^2 ds \right] \leq C_{L, K} \mathbb{E}_r \left[ \int_t^T |\Phi_s|^2 ds \right]. \tag{2.18}
\]

1.b Setting \( r = t \) in the previous inequality, we have

\[
\sup_{t \in [0, T]} |\Delta Y_t|^2 + \|\Delta Z\|_{\text{lip}}^2 \leq C_{L, K, \sigma^\xi} \tag{2.19}
\]
from which we straightforwardly deduce
\[ E[e^{\mu \sup_{t \in [0,T]} |Y_t|}] \leq E[e^{\mu \sup_{t \in [0,T]} (|\Delta Y_t| + |\Delta Y_t^2|)}] \leq C_{L,K,\sigma} \mu \] (2.20)
and \[ \|Z\|_{\mathcal{L}^2} \leq \|Z\|_{\mathcal{L}^2} + \|\Delta Z\|_{\mathcal{L}^2} \leq C_{L,K,\sigma} \mu . \] (2.21)

Combining (2.19) with (2.13), we obtain
\[ \|\gamma\|_{\mathcal{L}^2} \leq C_{L,K,\sigma} \mu . \] (2.22)

2.a We denote \( \Gamma := |\Phi| + |\Delta Z| + |\theta|^2 \). For all \( b > 0 \), we use Young inequality to get
\[ \mathbb{E}\left[e^{b \int_0^T \Gamma_s ds}\right] \leq e^{b^2 T} \mathbb{E}\left[e^{b \int_0^T |\Gamma_s|^2 ds}\right] \] (2.23)
for all \( \varepsilon > 0 \). Then, by taking \( \varepsilon = \left(1 + 4 \|\Phi\|_{\mathcal{L}^2}^2 + 4 \|\Delta Z\|_{\mathcal{L}^2}^2 + 4 \|\theta\|_{\mathcal{L}^2}^2\right)^{-1} \) we compute, for all \( r \in [0,T] \),
\[ \mathbb{E}\left[\int_r^T \varepsilon |\Gamma_s|^2 ds\right] \leq 2 \mathbb{E}\left[\int_r^T |\Phi_s|^2 + |\Delta Z_s|^2 + |\theta_s|^2 ds\right] \leq \frac{1}{2}. \]

Going back to (2.23) and applying the John-Nirenberg formula, see Theorem 2.2 in [11], we obtain
\[ \mathbb{E}\left[e^{b \int_0^T \Gamma_s ds}\right] \leq C_{L,K,\sigma} \mu , \] (2.24)

2.b Applying Itô's formula to \( \gamma_t \Delta Y_t^2 \), on \( [t,T] \), we compute,
\[ \gamma_t \Delta Y_t^2 + \int_t^T \gamma_s |\Delta Z_s|^2 ds + \int_t^T |\Delta Y_s|^2 d\gamma_s = 2 \int_t^T \gamma_s \Delta Y_s f(s,Y_s,Z_s) ds \]
\[ - 2 \int_t^T \gamma_s \Delta Y_s H(s,Y_s) \Phi_s ds - 2 \int_t^T \gamma_s \Delta Y_s \Delta Z_s dW_s. \] (2.25)

Let us observe that the local martingale \( \int_0^T \gamma_t \Delta Y_t \Delta Z_t dW_t \) is a true martingale. Indeed, we compute, using Burkholder-Davis-Gundy inequality,
\[ \mathbb{E}\left[\sup_{s \in [0,T]} \left| \int_0^s \gamma_t \Delta Y_t \Delta Z_t dW_t \right| \right] \leq C \mathbb{E}\left[\left( \int_0^T |\gamma_t \Delta Y_t \Delta Z_t|^2 \right)^{\frac{1}{2}} \right] \]
\[ \leq C_{L,K,\sigma} \mu \mathbb{E}\left[|\gamma_T| \left( \int_0^T |\Delta Z_t|^2 dt \right)^{\frac{1}{2}} \right]. \]

where we used (2.19) for the last inequality. Using Hölder inequality, denoting \( q \) the conjugate exponent of \( p \), we get
\[ \mathbb{E}\left[\sup_{s \in [0,T]} \left| \int_0^s \gamma_t \Delta Y_t \Delta Z_t dW_t \right| \right] \leq C_{L,K,\sigma} \mu \mathbb{E}[|\gamma_T|^{\frac{1}{2}}] \mathbb{E}\left[\left( \int_0^T |\Delta Z_t|^2 dt \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} . \]
From the energy inequality, we have that
\[ \mathbb{E} \left[ \left( \int_0^T |\Delta Z_t|^2 dt \right)^{\frac{q}{2}} \right] \leq C_q \|\Delta Z\|_{\mathcal{G}^2}^q . \]

We thus deduce
\[ \mathbb{E} \left[ \sup_{s \in [0,T]} \left| \int_0^s \gamma_t \Delta Y_t \Delta Z_t dW_t \right| \right] < \infty . \]

Since \( \gamma \) is non decreasing, we then compute, using (2.25), (2.13) and the Lipschitz continuity of \( f \),
\[ \mathbb{E} \left[ \int_t^T \gamma_s |\Delta Z_s|^2 ds \right] \leq \mathbb{E} \left[ \int_t^T \gamma_s \Xi Z_s ds \right] , \tag{2.26} \]
where we set \( \Xi Z := C_{L,K,\sigma}(1 + \mathfrak{R}) \) recalling that \( \Delta Y \) is bounded by (2.19) and \( \mathfrak{R} \) is defined in step 2.a. Using (2.24) we compute
\[ \mathbb{E} \left[ e^{b \int_t^T |\Xi Z|^2 ds} \right] \leq C_{L,K,\sigma,b} , \tag{2.27} \]
for all \( b > 0 \).

2.c We set \( \lambda = (1 + \epsilon)\beta \) with \( \epsilon > 0 \) such that \( (1 + \epsilon)^2 \beta \leq \lambda \epsilon \), recalling Definition 2.1.

Now we define
\[ \Theta Z := (1 + \epsilon)|Z|^2 + (1 + \frac{1}{\epsilon})\Xi Z . \]

We observe that
\[ \mathbb{E} \left[ \int_t^T \gamma_s |\Xi Z_s|^2 ds \right] < \infty , \]
as applying Itô’s formula to \( \gamma |Y|^2 \), we obtain
\[ \mathbb{E} \left[ \int_0^T \gamma_s |\Xi Z_s|^2 ds \right] \leq \mathbb{E} [\gamma_T |\Xi|^2 ] \leq \|\gamma_T\|_{L^p} \|\Xi\|_{L^q} < \infty . \]

From the definition of \( \Theta Z \), we have that, for \( t \leq T \),
\[ \mathbb{E} \left[ \int_t^T \gamma_s \Theta_s^Z ds \right] \geq (1 + \epsilon) \mathbb{E} \left[ \int_t^T \gamma_s |\Xi Z_s|^2 ds \right] + (1 + \frac{1}{\epsilon}) \mathbb{E} \left[ \int_t^T \gamma_s |\Delta Z_s|^2 ds \right] \]
where we used (2.26). Then it follows from Young’s inequality,
\[ \mathbb{E} \left[ \int_t^T \gamma_s \Theta_s^Z ds \right] \geq \mathbb{E} \left[ \int_t^T \gamma_s |Z_s|^2 ds \right] . \]
Proposition 2.2. Assume that $\Theta$ with $\delta f$ and $A$ associated to parameters and $\delta Y$ and $\Phi$ with condition (2.13) in force and assume moreover that $\delta Y \leq \sup_{t \in [0,T]} |Y_t| = c$.

Proof. We observe that $|Y_t^\xi| \leq \|\xi\|_{L^\infty}$ and then conclude using (2.19). \qed

2.3 A stability result

In this section, we prove the key estimate for the difference of two solutions of the generalised BSDE (2.12) satisfying (2.13). For some $c := c(L, K, \sigma^\xi, \|\xi\|_{L^\infty})$, we have

$$\sup_{t \in [0,T]} |Y_t| \leq c.$$ 

and using the fact that $\xi \in \mathcal{F}_\beta$ and (2.27), we obtain

$$\mathbb{E}\left[e^{(1+\epsilon)\beta\int_0^T |\Theta^\xi|^2 ds}\right] \leq C L, K, \sigma^\xi, (1+\epsilon)\beta.$$

Let us remark the following result, that will be useful in the next section.

Corollary 2.1. Assume that (SB) holds. Let $(Y, Z, \Phi) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ be a solution to (2.12) with condition (2.13) in force and assume moreover that $\delta Y \leq \sup_{t \in [0,T]} |Y_t| = c$. We now define $\Phi_t \in \partial \varphi(\mathbb{A}(Y_t))$ and $\partial \varphi(\mathbb{A}(Y_t)) dP \otimes dt - a.e.$ and (2.29)

$$\int_0^T |\Phi_t| 1_{\{Y_t \in \mathcal{E}\}} dt = 0.$$

Remark 2.5. The above assumption allows us to cover both cases of equation (1.1) and equation (1.2).

We now define $\delta Y = 1_Y - 2Y, \delta Z = 1_Z - 2Z, \delta \Phi = \delta Y - \delta \Phi, \delta f = \delta f(\cdot, 1_Y, \Phi)$ and $\delta f = \delta f(\cdot, 1_Y, 1_Z) - 2f(\cdot, 1_Y, 2Z)$. We have the following key result for our work.

Proposition 2.2. Assume that (SB) holds. There exist two increasing functions $\mathbb{B}(\cdot)$ and $\mathbb{A}(\cdot)$ from $(0, \infty)$ to $(0, \infty)$, such that for all $\xi \in \mathcal{F}_\beta(L)$, setting

$$\Gamma_t := e^{\mathbb{A}(t) + \mathbb{B}(t)}(\partial \varphi(\mathbb{A}(Y)) \partial \varphi(\mathbb{A}(Y)) dP \otimes dt - a.e.$$

with $\Theta^{\mathcal{E}} := |1_Y| + |2\Phi|, \Theta^\xi := |1_Z - \mathcal{E}| + |\delta \Phi|, \Theta^\xi := |\delta Y| + |\delta Z|$, we have,

i) $\mathbb{E}[\Gamma_T |P|] < c$ for some $p := p(\Lambda) > 1$ and $c := c(L, K, \Lambda, \sigma^\xi, \sigma^\xi)$;

ii) for some $c' := c'(L, K, \Lambda, \eta, \sigma^\xi, \sigma^\xi)$, and for all $t \leq T$,

$$|\delta Y_t|^2 + \mathbb{E}_t \left[ \int_t^T |\delta Y_s|^2 ds \right] \leq c' \mathbb{E}_t \left[ \Gamma_t |\delta Z|^2 + \Gamma_s (|1_Y - 2Y|^2 (s, 1_Y, 1_Z))^2 ds \right]$$

$$+ \int_t^T \Gamma_s (|1_Y - 2Y|^2 (s, 1_Y, 1_Z))^2 ds.$$ 

Finally, we compute using Hölder’s inequality,

$$\mathbb{E}\left[e^{(1+\epsilon)\beta\int_0^T |\Theta^\xi|^2 ds}\right] \leq C \mathbb{E}\left[e^{(1+\epsilon)\beta\int_0^T |\Theta^\xi|^2 ds}\right] \leq C \mathbb{E}\left[e^{(1+\epsilon)\beta\int_0^T |\Theta^\xi|^2 ds}\right] \leq C L, K, \sigma^\xi, (1+\epsilon)\beta.$$
2.1, for all

We first show the integrability property of \( \Gamma \). We first recall that from Proposition 2.1, for all \( b > 0 \), we have

\[
\mathbb{E}\left[e^{b \int_0^T |\Theta_t^y| + |\Theta_t^z|} dt\right] \leq C_{L,K,a,b}.
\]  

(2.31)

Setting \( p := p(\Lambda) > 1 \) such that \( p^2 \mathfrak{B}(\Lambda) \leq \lambda \xi \), recall Definition 2.1, we obtain using Hölder inequality,

\[
\mathbb{E}[|\Gamma_T|^p] \leq C_{\Lambda} \mathbb{E}\left[e^{p \mathfrak{B}(\Lambda) \int_0^T |\Theta_t^y| + |\Theta_t^z|} dt\right]^{\frac{1}{p}} \leq C_{L,K,L,a} \mathbb{E}\left[e^{C_{L,K,L,a} \int_0^T |\Theta_t^y| + |\Theta_t^z|} dt\right]^{\frac{1}{p}}
\]

(2.32)

where we used (2.31) and Proposition 2.1 (ii).

2.a To obtain the stability result, we first expand the product \( (\Gamma_t \delta Y_t^i A(t, Y_t) \delta Y_t^j)_{0 \leq t \leq T} \).

Applying Itô's formula, we then compute, for \( i, j \leq d \),

\[
d\Gamma_t a_t^{ij} \delta Y_t^i \delta Y_t^j | \Gamma_t = \delta Y_t^i \delta Y_t^j \left(a_t^{ij} \alpha + \delta_t a_t^{ij}\right) dt =: (\mathcal{E}^{ij}_t) dt
\]

(2.35)

\[
+ a_t^{ij} \delta Y_t^i \delta Y_t^j \left(\mathfrak{B}(\Lambda) \Theta_t^Z + \frac{1}{2} \text{Tr}[\delta_{ij}^{2} a_t^{ij} Z_t^i Z_t^j]\right) dt =: (\mathcal{E}^{ij}_Z) dt
\]

(2.36)

\[
+ \left\{a_t^{ij} \left(-\delta Y_t^i \delta f_t^i - \delta Y_t^i \delta f_t^i + \mathfrak{B}(\Lambda) \Theta_t^Z \delta Y_t^i \delta Y_t^j - \delta_y a_t^{ij} f_t \delta Y_t^i \delta Y_t^j\right) + \mathfrak{B}(\Lambda) \Theta_t^Z \delta Y_t^i \delta Y_t^j\right\} dt =: (\mathcal{E}^{ij}_l) dt
\]

(2.37)

\[
+ \left\{a_t^{ij} \delta Y_t^i \delta Z_t^j + \delta Y_t^i \delta Z_t^j \right\} + \delta Y_t^i \delta Y_t^j \delta_y a_t^{ij} Z_t + \mathfrak{B}(\Lambda) \mathfrak{A}^{ij} \delta Y_t^i \delta Y_t^j\right\} dt =: (\mathcal{E}^{ij}_R) dt
\]

(2.38)

\[
+ \left\{a_t^{ij} \delta Y_t^i \delta \Psi_t + \delta Y_t^i \delta \Psi_t + \mathfrak{B}(\Lambda) a_t^{ij} \delta \Psi_t + \mathfrak{B}(\Lambda) \mathfrak{A}^{ij} \Theta_t^Z \delta Y_t^i \delta Y_t^j\right\} dt =: (\mathcal{E}^{ij}_R) dt
\]

(2.39)

We now study each terms separately.

2.b We start by the reflection terms in (2.41). We first observe that

\[
\sum_{1 \leq i,j \leq d} (\mathcal{E}^{ij}_R) = A(t, 1 Y_t) \delta Y_t \cdot \delta \Psi_t + \sum_{1 \leq i,j \leq d} \delta_y a_t^{ij} \delta \Psi_t + \mathfrak{B}(\Lambda) \mathfrak{A}^{ij} \delta Y_t \cdot A(t, 1 Y_t) \delta Y_t.
\]

(2.41)

Recalling (2.6) and (2.8), we compute

\[
\sum_{1 \leq i,j \leq d} (\mathcal{E}^{ij}_R) \geq A(t, 1 Y_t) \delta Y_t \cdot \delta \Psi_t + \left(\frac{\mathfrak{B}(\Lambda)}{L} \Theta_t^Z - C_{\Lambda} \delta \Psi_t\right) |\delta Y_t|^2.
\]

(2.42)
For the first term in the right hand side of (2.42), we compute
\[ A(t, 1Y_t)\delta Y_t \cdot \delta \Psi_t = A(t, 1Y_t)\delta Y_t \cdot 1\Psi_t - A(t, 2Y_t)\delta Y_t \cdot 2\Psi_t - \{A(t, 1Y_t) - A(t, 2Y_t)\}\delta Y_t \cdot 2\Psi_t. \]  
(2.43)

We now observe that,
\[ A(t, 1Y_t)\delta Y_t \cdot 1\Psi_t = \delta Y_t \cdot A^\dagger (t, 1Y_t)\frac{1}{2}\Psi_t = \delta Y_t \cdot 1\Phi_t \]
\[ \geq \left(1Y_t - \mathcal{B}(1Y_t) + \mathcal{B}(2Y_t) - 2Y_t\right) \cdot 1\Phi_t \]
\[ \geq - (|\mathcal{B}(1Y_t) - 1Y_t| + |\mathcal{B}(2Y_t) - 2Y_t|) |1\Phi_t| \]  
(2.44)

where we used (2.29) and the convexity property of \(\mathcal{D}\). Similarly, we compute
\[ -A(t, 2Y_t)\delta Y_t \cdot 2\Psi_t \geq - (|\mathcal{B}(1Y_t) - 1Y_t| + |\mathcal{B}(2Y_t) - 2Y_t|) |2\Phi_t|. \]  
(2.45)

For the last term in the right-hand side of (2.42), we get, using the Lipschitz property of \(A\) that
\[ \{A(t, 1Y_t) - A(t, 2Y_t)\}\delta Y_t \cdot 2\Psi_t \geq -C_A |\delta Y_t|^2 |2\Phi_t|. \]  
(2.46)

Combining (2.44)-(2.45)-(2.46) with (2.42), we obtain, for \(\mathcal{B}(A)\) large enough,
\[ \mathbb{E}_t \left[ \Gamma_s \sum_{1 \leq i,j \leq d} (\mathcal{E}^R_s)^{ij} ds \right] \geq -C \mathbb{E}_t \left[ \int_t^T (|\mathcal{B}(2Y_s) - 2Y_s| + |\mathcal{B}(1Y_s) - 1Y_s|)(|1\Phi_s| + |2\Phi_s|) ds \right]. \]  
(2.47)

2c Using Young’s inequality, we compute, recalling (2.6) and (2.8),
\[ \sum_{1 \leq i,j \leq d} (\mathcal{E}^Z_t)^{ij} \geq \frac{1}{2L} |\delta Z_t|^2 - C_L |\delta Y_t|^2 |Z_t|^2. \]  
(2.48)

The terms \(\mathcal{E}^I\) in (2.38) can be lower bounded, using Young’s inequality, by
\[ \sum_{1 \leq i,j \leq d} (\mathcal{E}^I_t)^{ij} \geq -\frac{1}{3L} |\delta Z_t|^2 + \left(\frac{\mathcal{B}(A)}{L} \Theta_t - C_{L,A} \sigma_t (1 + |\Theta_t| + |Z_t - Z_t^\xi|) \right) |\delta Y_t|^2 \]
\[ -C \left(|\Theta - 2\Theta| (t, 1Y_t, 1Z_t) \right)^2 \]  
(2.49)

recalling (2.19). We also have that
\[ \sum_{1 \leq i,j \leq d} (\mathcal{E}^T_t)^{ij} \geq \left(\frac{\mathcal{B}(A)}{L} \Theta_t - C_A |Z_t|^2 \right) |\delta Y_t|^2 \]
\[ \text{and } \sum_{1 \leq i,j \leq d} (\mathcal{E}^T)^{ij} \geq \left(\frac{\mathcal{B}(A)}{L} - C_A \right) |\delta Y_t|^2. \]  
(2.50)
2.d Combining the above results we get, for \( \mathfrak{B}(\Lambda) \) and \( \mathfrak{A}(\Lambda) \) large enough,

\[
\Gamma_t \delta Y_t \cdot A(t, Y_t) \delta Y_t + \mathbb{E}_t \left[ \int_t^T \Gamma_s \frac{1}{6L} |\delta Z_s|^2 \, ds + \mathcal{M}_t - \mathcal{M}_T \right] \leq \mathbb{E}_t[\Gamma_T \delta \xi \cdot A(T, Y_T) \delta \xi] + C_{\Lambda} \mathbb{E}_t \int_t^T \Gamma_s \left( |(\frac{1}{2}f(s, Y_s, Z_s)^2) + (\mathfrak{P}(Y_s^2) - 2Y_s) + |\mathfrak{P}(Y_s)| |\Phi_s| + |\Phi_s| \right) \, ds,
\]

where \( \mathcal{M} \) is a local martingale term given by

\[
\mathcal{M}_t := \sum_{1 \leq i, j \leq d} \int_0^t \Gamma_s dM_s^{ij}. \tag{2.51}
\]

Moreover, we have, recalling (2.6),

\[
\mathbb{E}_t[\Gamma_T \delta \xi \cdot A(T, Y_T) \delta \xi] \leq L \mathbb{E}_t[|\delta \xi|^2].
\]

3. We now show that \( \mathcal{M} \) is a martingale.

Applying Burkholder-Davis-Gundy inequality and observing that \( \delta Y \) is bounded, we compute, using (2.8),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \Gamma_t \delta Y_t \cdot \partial_y q_t \frac{1}{2} Z_t \, dW_t \right| \right] \leq C_{\Lambda} \mathbb{E} \left[ \left( \int_0^T \Gamma_t^2 |\delta Y_t|^2 |Z_t|^2 \, dt \right)^{\frac{1}{2}} \right] \leq C_{\Lambda} \mathbb{E} \left[ |\Gamma_T|^p \right]^{\frac{1}{p}} \mathbb{E} \left[ \sup_{t \in [0,T]} |\delta Y_t|^q \left( \int_0^T |Z_t|^2 \, dt \right)^{q} \right]^{\frac{1}{2}} \leq C_p \mathbb{E} \left[ |\Gamma_T|^p \right] \mathbb{E} \left[ |\delta Y_t|^q \left( \int_0^T |Z_t|^2 \, dt \right)^{q} \right]^{\frac{1}{2}}
\]

where we used Cauchy-Schwarz inequality, the energy inequality, with the fact that \( \sup_t |\delta Y_t| \) is bounded in any \( \mathcal{L}^p \) and \( 1Z \in \mathcal{B}^2 \). From step 1. we deduce then that the supremum of the local martingale term is integrable, which conclude the proof for this term. The other terms in (2.40) are treated similarly.

\[\square\]

**Remark 2.6.** The dependence upon \( \Lambda \) is a key fact that will restrain us to extend straightforwardly to rougher coefficients our main existence and uniqueness results in the non-Markovian case, recall assumption (SB).

### 2.4 Some interesting facts about the class \( \mathcal{T}_\beta \)

We first make the following observation.

**Proposition 2.3.** Let \( \xi \in \mathcal{L}^2(\mathcal{F}_T) \) satisfying (SB)(i). If we have, for some \( \beta > 0 \),

\[
d_{\mathcal{L}^2}(\mathcal{Z}^\xi, \mathcal{H}^\infty) < \sqrt{\beta}, \tag{2.52}
\]

then \( \xi \in \mathcal{T}_\beta \).
Proof. We can find $V \in \mathcal{H}^\infty$, s.t. $\|Z^\xi - V\|_{\mathcal{F}^2} = \frac{1}{(1+\eta)^{1/2}}$, for some $\eta > 0$ small enough. We now set $\lambda := \left(\frac{1+\frac{2}{3}}{1+\frac{3}{2}}\right)^2 \beta$ and we compute, using Young’s inequality,

$$|Z^\xi|^2 \leq (1 + \frac{\eta}{3})|Z^\xi - V|^2 + (1 + \frac{3}{\eta})|V|^2.$$  

This leads, using Hölder inequality, to

$$\mathbb{E}\left[e^{\lambda \int_0^T |Z_t^\xi|^2 \, dt}\right] \leq C \mathbb{E}\left[e^{(1+\frac{2}{3})^2 \lambda \int_0^T |Z_t^\xi - V_t|^2 \, dt}\right]^{\frac{1}{1+\frac{3}{2}}}$$  

where we used the fact that $V \in \mathcal{H}^\infty$. Since $\left(1 + \frac{3}{2}\right) \sqrt{\lambda} (Z^\xi - V) \|_{\mathcal{F}^2} = \frac{1+\frac{2}{3}}{1+\eta} < 1$, we can apply the John-Nirenberg inequality, see Theorem 2.2 in [11], to obtain

$$\mathbb{E}\left[e^{(1+\frac{2}{3})^2 \lambda \int_0^T |Z_t^\xi - V_t|^2 \, dt}\right] < \infty,$$

which concludes the proof. \quad \square

The next result shows that a class of path-dependent function of some smooth processes are naturally contained in $\mathcal{F}_\beta$ and actually for all $\beta > 0$. This class is quite important for applications.

**Proposition 2.4.** Let $(X_s)_{s \in [0,T]}$ be a continuous and adapted process such that for all $t, s \leq T$, the Malliavin derivatives of $X_s$ denoted $D_tX_s$ is well defined and satisfies $\|\sup_t \mathbb{E}[D_tX_s]\|_{\mathcal{F}^\infty} < \infty$. Let $g : C^0([0,T], \mathbb{R}^n) \rightarrow \mathbb{R}^d$ be a uniformly continuous function, then denoting $\xi = g \left((X_s)_{s \in [0,T]}\right)$, we have that $Z^\xi \in \mathcal{H}^\infty_{\mathcal{F}^2}$.

**Proof.** 1.a We first start by considering a sequence $(g_N)$ of $N$-Lipschitz regularisation of $g = (g^1, \ldots, g^d)$ given by

$$g_N^i(x) = \inf_{u \in C^0([0,T], \mathbb{R}^n)} \{g^i(u) + N|u - x|_x\}, \quad \text{for all } x \in C^0([0,T], \mathbb{R}^n), \quad 1 \leq i \leq n.$$  

Let us observe that $g_N$ is finite for $N$ large enough due to the linear growth of $g$. Then we have, for all $x \in C^0([0,T], \mathbb{R}^n)$ and $1 \leq i \leq n$,

$$g^i(x) \geq g_N^i(x) \geq \inf_{u \in C^0([0,T], \mathbb{R}^n)} \{g^i(x) - \omega_{g^i}(|u - x|_x) + N|u - x|_x\}$$  

$$\geq g^i(x) + \inf_{u \in C^0([0,T], \mathbb{R}^n)} \{N|u|_x - \omega_{g^i}(|u|_x)\}$$

where $\omega_{g^i}$ is a concave modulus of continuity for the uniformly continuous component $g^i$ of $g$. Thus we get

$$|g_N - g|_\infty \leq C \sum_{i=1}^d \sup_{u \in C^0([0,T], \mathbb{R}^n)} \{\omega_{g^i}(|u|_x) - N|u|_x\} := c(N). \quad (2.53)$$  

Since $\omega_{g^i}(h) = o(1)$ when $h \rightarrow 0^+$ then $c(N) = o(1)$ when $N \rightarrow +\infty$.  

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1.b Defining $Y_t^N := \mathbb{E}[g_N(X_t)] = g_N(X_t) - \int_t^T Z_s^N dW_s$ and applying Itô’s formula to $|Y_t^N - Y_t^\xi|^2$, we compute

$$
|Y_t^N - Y_t^\xi|^2 + \mathbb{E}\left[ \int_t^T |Z_s^N - Z_s^\xi|^2 dt \right] = \mathbb{E}[|g_N(X_t) - g(X_t)|^2] \leq c(N)^2
$$

recall (2.53). From this, we deduce that for all $\epsilon > 0$, there exists $N_\epsilon$, s.t. for all $N \geq N_\epsilon$,

$$
\|Z^N - Z^\xi\|_{\mathcal{G}^2} \leq \epsilon . \tag{2.54}
$$

2. We now show that $Z^N$ introduced above, belongs to $\mathcal{H}^\infty$. This fact combined with (2.54) proves the statement of the proposition.

Following Lemma 4.1 in [13] there exists a family $\Pi = \{\pi\}$ of partitions of $[0, T]$ and a family of discrete functionals $(g_{N,\pi})$ such that

- for each $\pi \in \Pi$, with $\pi : 0 = t_0 < \ldots < t_m = T$, we have that $g_{N,\pi} \in C^\infty([0, T], \mathbb{R}^n)$, and satisfies
  $$
  \sum_{i=0}^m |\partial_{x_i} g_{N,\pi}(x)| \leq N, \quad \forall x \in C^0([0, T], \mathbb{R}^n), \tag{2.55}
  $$

  where $g_{N,\pi}(x) := g_N(x(t_0), \ldots, x(t_m))$.

- for any $x \in C^0([0, T], \mathbb{R}^n)$ it holds that
  $$
  \lim_{|\pi| \to 0} |g_{N,\pi}(x) - g_N(x)| = 0. \tag{2.56}
  $$

We naturally consider $(Y_t^{N,\pi}, Z_t^{N,\pi})$ given by

$$
Y_t^{N,\pi} := \mathbb{E}[g_{N,\pi}(X_t)] = g_{N,\pi}(X_t) - \int_t^T Z_s^{N,\pi} dW_s .
$$

2.a By the Clark-Ocone formula, we have that

$$
Z_t^{N,\pi} = \mathbb{E}[D_t g_{N,\pi}(X)]
$$

$$
= \sum_{i=1}^m \partial_{x_i} g_{N,\pi}(X) \mathbb{E}[D_t X_i].
$$

Now, using (2.55) and the assumption on $DX$, we obtain

$$
\|Z^{N,\pi}\|_{\mathcal{G}^2} \leq C_N . \tag{2.57}
$$

2.b Combining (2.56) and the dominated convergence theorem, we get

$$
\lim_{|\pi| \to 0} \mathbb{E}[(g_N(X) - g_{N,\pi}(X))^2] = 0 ,
$$
which leads to
\[
\lim_{|\pi| \to 0} \mathbb{E} \left[ \int_0^T |Z^{N,\pi} - Z^N|^2 dt \right] = 0 .
\]
Up to a subsequence, we have $Z^{N,\pi} \to Z^N$ $d\mathbb{P} \otimes dt$-a.e. and, moreover, $|Z^{N,\pi}| \leq C_N$. We thus obtain for (a version of) the limit process
\[
\int_0^T |Z^N_t|^2 dt \leq TC^2_N, \mathbb{P} - a.s.
\]
which conclude the proof of this step.

3. Finally, we remark that $\xi^\beta \in \mathcal{B}^2$ since $\xi^\beta - Z^N \in \mathcal{B}^2$ and $Z^N \in \mathcal{H}^\infty \subset \mathcal{B}^2$. We conclude the proof by using (2.54).

\[ \square \]

**Corollary 2.2.** Let $X$ be solution of the Lipschitz SDE
\[
X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,
\]
where $\sigma$ and $b$ are Lipschitz continuous functions such that $\sigma$ is bounded, then $\xi := g((X_s)_{s \in [0,T]})$ belongs to $\Sigma_\beta$, for all $\beta > 0$, when $g$ is a uniformly continuous function. Moreover, if $\xi \in \mathcal{L}^\infty$, then $\xi + \tilde{\xi}$ belongs to $\Sigma_\beta$ for all $\beta \leq \|\xi\|_{\mathcal{L}^\infty}$.

**Proof.** When $\sigma$ and $b$ are smooth enough, it is well known, see e.g. [13], that $X$ is Malliavin differentiable and, for all $1 \leq i \leq k$, $(D_i^\alpha X_s)_{s \in [t,T]}$ is solution of the linear SDE given by
\[
D_i^\alpha X_s = \sigma^i(X_t) + \int_t^s \nabla b(X_r)D_i^\alpha X_r dr + \int_t^s \sum_{j=1}^k \nabla \sigma^j(X_s)dW^j.
\]

Then, we easily get that $|\mathbb{E}[D_i^\alpha X_s]| \leq e^{K_\beta T} M$ with $K_\beta$ the Lipschitz constant of $b$ and $M$ a bound of $\sigma$. Then we can apply Proposition 2.4 to get the first part of the result. When coefficients are not smooth enough, a standard approximation gives us the result, pointing out the fact that $\sup_t \mathbb{E}[D_i^\alpha X_s]_{\mathcal{L}^\infty}$ can be uniformly bounded with respect to the approximation. For the second part of the corollary, we just have to remark that
\[
d_{\mathcal{B}^2}(\xi^{\epsilon + \hat{\xi}}, \mathcal{H}^\infty) \leq \left\| \xi^{\epsilon + \hat{\xi}} - \xi^{\epsilon} \right\|_{\mathcal{B}^2} + d_{\mathcal{B}^2}(\xi^{\epsilon}, \mathcal{H}^\infty) = \left\| \xi^{\epsilon} \right\|_{\mathcal{B}^2} .
\]

Moreover, applying Itô’s formula to $|\xi^{\epsilon}|^2$ gives us that
\[
\left\| \xi^{\epsilon} \right\|_{\mathcal{B}^2} \leq \left\| \xi \right\|_{\mathcal{L}^\infty},
\]
which implies
\[
d_{\mathcal{B}^2}(\xi^{\epsilon + \hat{\xi}}, \mathcal{H}^\infty) \leq \left\| \xi \right\|_{\mathcal{L}^\infty} .
\]

Thus, we just have to apply Proposition 2.3 to conclude.

\[ \square \]
3 Existence and uniqueness in a regular setting

In this section, we obtain an existence and uniqueness result in a non Markovian setting, working under assumption \((SB)\) and considering terminal condition in the class \(T_\beta\), for some \(\beta > 0\). This \(\beta\), as shown in the previous section, depends dramatically on the smoothness of the coefficients. Our proof is done in two main steps. In the first step, we restrict to the case of a bounded terminal condition. We study the wellposedness of the penalised equations, and prove their convergence to an obliquely reflected BSDEs. In a second step, we extend our result to all terminal condition in the class \(T_\beta\).

3.1 Bounded terminal condition

We first obtain some results on the penalised BSDE that will be used later in this section and also in Section 4 in the Markovian case. We thus essentially work here under the assumption \((A)\).

We start with the following lemma that verifies the well-posedness of equation (1.2) under some classical conditions.

**Lemma 3.1.** We assume that \((A)\) is in force and that \(f\) and \(H\) are Lipschitz continuous with respect to \(p, z\). Then there exists a unique solution to (1.2) in \(S^2 \hat{H}^2\).

**Proof.** Since \(D\) is convex, \(\varphi_n^M\) is convex and \(nM\)-Lipschitz continuous. Indeed, denoting \(D_M := \{y \in \mathbb{R}^d | d(y, D) \leq M\}\), we have that

\[
\varphi_n^M(h) = \begin{cases} 
\frac{1}{2}d^2(h, D) & \text{if } h \in D_M \\
M^2(h, D) - \frac{nM^2}{2} & \text{if } h \notin D_M
\end{cases}
\]  

(3.1)

and

\[
\nabla \varphi_n^M(y) = \begin{cases} 
0 & \text{if } y \in D \\
nd(y, D) \frac{y - \hat{\varphi}(y)}{|y - \hat{\varphi}(y)|} & \text{if } y \in D_M \backslash D \\
nM \frac{y - \hat{\varphi}(y)}{|y - \hat{\varphi}(y)|} & \text{if } y \notin D_M
\end{cases}
\]  

(3.2)

Finally \(H\) and \(\nabla \varphi_n^M\) are two Lipschitz bounded functions which proves that the penalised BSDE (1.2) has a Lipschitz driver: the classical result of [16] then applies to get the existence and uniqueness result. \(\square\)

**Lemma 3.2.** Assume that \((A)\) holds and that there exists a solution to (1.2) in \(\mathcal{H}^2 \times \mathcal{H}^2\). Then, \((Y^n, Z^n, \Phi^n)\) satisfies Condition (2.13) with \(K := K(\eta)\) and for some \(c := c(\eta, L)\) we have

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \varphi_n^M (Y^n_t) \right] + \mathbb{E} \left[ \int_0^T |\nabla \varphi_n^M (Y^n_s)|^2 ds \right] \leq c \mathbb{E} \left[ |\xi|^2 + \int_0^T |\alpha_s|^2 ds \right] \quad (3.3)
\]

Moreover, if \((SB)\) holds, then, there exists \(c' := c'(\eta, L, \sigma^2)\) such that

\[
\sup_{t \in [0, T]} \varphi_n^M (Y^n_t) + \|\nabla \varphi_n^M (Y^n)\|_{\mathcal{H}^2}^2 \leq c'. \quad (3.4)
\]

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Proof. Since \( \varphi^M_n \) is a \( C^1 \) convex function, we have the following inequality (see Lemma 2.38 in [17]): for \( s \in [t, T] \),

\[
\varphi^M_n(Y^n_s) + \int_s^T \nabla \varphi^M_n(Y^n_u) \cdot H(u, Y^n_u, Z^n_u)\nabla \varphi^M_n(Y^n_u) du \\
\leq \varphi^M_n(\xi) + \int_s^T \nabla \varphi^M_n(Y^n_u) \cdot f(u, Y^n_u, Z^n_u) du - \int_s^T \nabla \varphi^M_n(Y^n_u) \cdot Z^n_u dW_u,
\]

and we recall that \( \varphi^M_n(\xi) = 0 \). We observe, using (2.2) that

\[
\nabla \varphi^M_n(Y^n_u) \cdot H(u, Y^n_u, Z^n_u)\nabla \varphi^M_n(Y^n_u) \geq \eta |\nabla \varphi^M_n(Y^n_u)|^2
\]

and combining Cauchy-Schwarz inequality with Young’s inequality

\[
\int_s^T \nabla \varphi^M_n(Y^n_u) \cdot f(u, Y^n_u, Z^n_u) du \leq \frac{\eta}{2} \int_s^T |\nabla \varphi^M_n(Y^n_u)|^2 du + \frac{2}{\eta} \int_s^T |f(u, Y^n_u, Z^n_u)|^2 du.
\]

From this, we deduce

\[
\varphi^M_n(Y^n_t) + \mathbb{E} \left[ \int_t^T |\nabla \varphi^M_n(Y^n_u)|^2 du \right] \leq \frac{4}{\eta} \mathbb{E} \left[ \int_t^T |f(u, Y^n_u, Z^n_u)|^2 du \right],
\]

which proves (2.13) for \((Y^n, Z^n, \Phi^n)\). This allows then to invoke Lemma 2.1 to obtain (3.3) under (A). Under (SB), (3.7) allows also to conclude recalling that \( f \) is Lipschitz continuous, \( \theta^k \in \mathcal{B}^2 \) and (2.19).

We now prove our first existence result for the obliquely reflected BSDE

\[
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T H(s, Y_s) \Phi_s ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \\
Y_t \in \overline{D}, \quad \Phi_t \in \partial \varphi(Y_t), \quad \int_0^T 1_{\{Y_t \notin \overline{D}\}} |\Phi_t| dt = 0.
\end{aligned}
\]

Proposition 3.1. Assume that (SB) holds and that \( \xi \in \mathcal{L}^\infty \cap \mathcal{F}_0(\Lambda) \). Then, there exists a solution in \( \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2 \) to the obliquely reflected BSDE (3.8).

Proof. To obtain the existence result, we consider a sequence of penalised BSDEs given by equation (1.1) for which we have existence and uniqueness from Lemma 3.1. In the definition of \( \varphi^M_n \), recall (1.3), we set \( M = 2c \) where \( c \) is given in Corollary 2.1. In particular, we observe that for this choice of \( M \), for \( 0 \leq t \leq T \),

\[
\Phi^n_t := \nabla \varphi^M_n(Y^n_t) = n (Y^n_t - \Psi(Y^n_t)) \quad \text{and} \quad \frac{1}{n} \varphi_n(Y^n_t) = \frac{1}{2} |\Psi(Y^n_t) - Y^n_t|^2,
\]

recall (3.1) and (3.2). We will use this fact later on.

1.a. We now prove that \((Y^n, Z^n)\) is a Cauchy sequence in \( \mathcal{S}^2 \times \mathcal{H}^2 \). Indeed, let \( m \geq 0 \) and \( n \geq 0 \), thanks to Lemma 3.2 we can apply Proposition 2.2 to obtain

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ |Y^n_t - Y^m_t|^2 \right] + \|Z^n - Z^m\|_{\mathcal{H}^2}^2 \leq C_A \mathbb{E} \left[ \int_0^T \Gamma_{s,m} (|\Psi(Y^n_s) - Y^n_s|^2 + |\Psi(Y^m_s) - Y^m_s|^2) (|\Phi^n_s|^2 + |\Phi^m_s|^2) ds \right] =: A^{n,m}.
\]
Let us notice that, from the same proposition, there exist \( p > 1 \) and a constant \( C \) such that
\[
\mathbb{E}[|\Gamma_{T}^{n,m}|^p] \leq C ,
\tag{3.11}
\]
where, importantly, \( p \) and \( C \) do not depend on \((n, m)\). Applying Itô’s formula to \( |Y^n - Y^m|^2 \) on \([0, T]\), we compute, using usual arguments,
\[
|Y^n - Y^m|^2 \leq C \mathbb{E} \left[ \int_0^T |Y^n_t - Y^m_t| (|\Phi^n_t| + |\Phi^m_t|) \, dt \right] + C \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t (Y^n_s - Y^m_s)(Z^n_s - Z^m_s) \, dW_s \right| \right].
\]
Using Burkholder-Davis-Gundy inequality and Young’s inequality, we obtain
\[
|Y^n - Y^m|^2 \leq C \mathbb{E} \left[ \int_0^T |Y^n_t - Y^m_t| (|\Phi^n_t| + |\Phi^m_t|) \, dt \right] + C \|Z^n - Z^m\|_{\mathcal{F}^2}^2.
\] Applying Cauchy-Schwarz inequality, and using Lemma 3.2, we get
\[
|Y^n - Y^m|^2 \leq C \left( \|Y^n - Y^m\|_{\mathcal{F}^2} + \|Z^n - Z^m\|_{\mathcal{F}^2}^2 \right) .
\tag{3.12}
\]
Combining the previous inequality with (3.10), we have
\[
|Y^n - Y^m|^2_{\mathcal{F}^2} \leq C \left( A_{n,m} + \sqrt{A_{n,m}} \right) .
\tag{3.13}
\]
1.b We now study the \( A_{n,m} \) term. We first observe, recalling Lemma 3.2 and (3.9),
\[
\mathbb{E} \left[ \int_0^T \Gamma_{t}^{n,m} |\mathcal{P}(Y^n_t) - Y^n_t||\Phi^m_s| \, ds \right] \leq \sup_t \left| \mathcal{P}(Y^n_t) - Y^n_t \right| \mathbb{E} \left[ \Gamma_{T}^{n,m} \int_0^T |\Phi^m_s| \, ds \right] \leq \frac{C}{\sqrt{n}} \mathbb{E} \left[ \Gamma_{T}^{n,m} \int_0^T |\Phi^m_s| \, ds \right] .
\]
Applying Hölder inequality, denoting \( q \) the conjugate exponent of \( p \) introduced in (3.11), we deduce from the previous inequality
\[
\mathbb{E} \left[ \int_0^T \Gamma_{t}^{n,m} |\mathcal{P}(Y^n_t) - Y^n_t||\Phi^m_s| \, ds \right] \leq \frac{C}{\sqrt{n}} \mathbb{E} \left[ \left( \int_0^T |\Phi^m_s|^2 \, ds \right)^{\frac{q}{2}} \right] .
\]
Then, combining the energy inequality with (3.4), we conclude
\[
\mathbb{E} \left[ \int_0^T \Gamma_{t}^{n,m} |\mathcal{P}(Y^n_t) - Y^n_t||\Phi^m_s| \, ds \right] \leq \frac{C}{\sqrt{n}} .
\tag{3.14}
\]
Similarly we obtain,
\[
\mathbb{E} \left[ \int_0^T \Gamma_{t}^{n,m} |\mathcal{P}(Y^m_t) - Y^m_t||\Phi^n_s| \, ds \right] \leq \frac{C}{\sqrt{m}} .
\]
Combining the previous inequalities with (3.13), we compute that
\[ \|Y^n - Y^m\|_{\mathcal{F}^2}^2 + \|Z^n - Z^m\|_{\mathcal{H}^2}^2 \leq C \left( n^{-\frac{1}{4}} + m^{-\frac{1}{4}} \right) \]
which proves that \((Y^n, Z^n)_n\) is a Cauchy sequence in \(\mathcal{F}^2 \times \mathcal{H}^2\) and we denote \((Y, Z)\) its limit.

2. We now prove that \((Y, Z)\) is solution to an obliquely reflected BSDE, namely we pass to the limit in (1.2). Let us first observe that, passing to the limit in (3.4) yields that \(\Phi \in \bar{D}\) as expected.

2.a We now study the reflecting term. Since, by Lemma 3.2, we have, up to a subsequence, the following weak \(L^2([0, T] \times \Omega)\)-convergence:
\[ \nabla \varphi_n(Y^n) \rightharpoonup \Phi, \quad \text{when } n \to +\infty. \]
Let \((V_t)_{t \in [0, T]}\) be a continuous adapted process valued in \(\bar{D}\). From the convexity property of \(D\) and the fact that \(\nabla \varphi_n(Y^n) = n(Y^n - \Phi(Y^n))\), recall (3.9), we have
\[ \int_0^T (Y^n_t - V_t)^\dagger \nabla \varphi_n(Y^n_t) dt \leq 0. \]
By strong convergence of \((Y^n)_{n \geq 0}\) to \(Y\), weak convergence of \((\nabla \varphi_n(Y^n))_{n \geq 0}\) and the uniform \(L^2\)-bound on \(\nabla \varphi_n(Y^n)\), recall Lemma 3.2 we obtain
\[ \mathbb{E} \left[ \int_0^T (Y_t - V_t)^\dagger \Phi_t dt 1_A \right] \leq 0, \]
for all \(A \in \mathcal{F}_T\). This leads to \(\int_0^T (Y_t - V_t)^\dagger \Phi_t dt \leq 0\). Using Lemma 2.1 in \(\mathbb{R}\) \(\omega\)-wise, we obtain that
\[ \Phi \in \partial \varphi(Y) \quad \text{and} \quad \int_0^T 1_{\{V_t \notin D\}} |\Phi_t| dt = 0, \]
which fully characterise \(\Phi\).

2.b Now we want to show that \((Y, Z, \Phi)\) is solution of (3.8). By strong convergence of \((Y^n, Z^n)\) to \((Y, Z)\) and the Lipschitz-continuity of \(f\), we have
\[ f(\cdot, Y^n, Z^n) \xrightarrow{\mathcal{H}^2} f(\cdot, Y, Z) \quad \text{and} \quad \int_0^t Z^n_s dW_s \xrightarrow{\mathcal{L}^2} \int_0^t Z_s dW_s, \]
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for all $t \leq T$. Moreover, $\Phi^n \rightarrow \Phi$ in $L^2([0,T] \times \Omega)$, when $n \rightarrow +\infty$. Using Mazur’s Lemma, we know that there exists a convex combination of the above converging strongly in $L^2([0,T] \times \Omega)$, namely

$$\tilde{\Phi} := \sum_{r=p}^{N_p} \lambda_p^r \Phi^r \xrightarrow{p \rightarrow \infty} \Phi,$$

where $\lambda_p^r \geq 0$ for all $p \in \mathbb{N}$ and $p \leq r \leq N_p$, and $\sum_{r=p}^{N_p} \lambda_p^r = 1$. Let us observe that by strong convergence, the following combination

$$(pY, pZ) := \sum_{r=p}^{N_p} \lambda_p^r (Y^r, Z^r)$$

still converges to $(Y, Z)$ in $\mathcal{S}^2 \times \mathcal{H}^2$ and, by strong convergence,

$$\sum_{r=p}^{N_p} \lambda_p^r f(\cdot, Y^r, Z^r) \xrightarrow{\mathcal{H}^2} f(\cdot, Y, Z) \text{ and } \int_t^T pZ_s dW_s \xrightarrow{\mathcal{L}^2} \int_t^T Z_s dW_s.$$ 

Moreover, we remark that

$$\sum_{r=p}^{N_p} \lambda_p^r H(\cdot, Y^r) \Phi^r = \sum_{r=p}^{N_p} \lambda_p^r [H(\cdot, Y^r) - H(\cdot, Y)] \Phi^r + H(\cdot, Y) \Phi.$$ 

Using the Lipschitz property of $H$ and the uniform $L^2$-bound on $\nabla \varphi_n(Y^n)$, the first term in the right hand side of the previous equation tends to zero in $\mathcal{H}^2$. Then we get

$$\sum_{r=p}^{N_p} \lambda_p^r H(\cdot, Y^r) \Phi^r \xrightarrow{\mathcal{H}^2} H(\cdot, Y) \Phi.$$ 

Finally, we just have to pass to the limit into

$$pY_t = \xi + \int_t^T \sum_{r=p}^{N_p} \lambda_p^r f(s, Y^r_s, Z^r_s) ds - \int_t^T pZ_s dW_s - \int_t^T \sum_{r=p}^{N_p} \lambda_p^r H(s, Y^r_s) \Phi^r_s ds$$

and thus conclude the proof of the theorem.

\[ \square \]

### 3.2 General case

**Theorem 3.1.** Assume that (SB) holds and $\xi \in \mathfrak{F}_B(\Lambda)$. There exists a unique solution $(Y, Z, \Phi) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{B}^2$ to (1.1).

Before proving our main result, we consider the following lemma which is a key result for the study of Obliquely Reflected BSDEs, as it proves, among other things, the structural condition (2.13). It is the counterpart of Lemma 3.2 introduced for the penalised BSDE.
Lemma 3.3. Assume that (SB) holds. Let \((Y, Z, \Phi) \in \mathcal{F}^2 \times \mathcal{H}^2 \times \mathcal{H}^2\) be a solution to the Obliquely Reflected BSDE \((1.1)\). Then, the structural condition \((2.13)\) holds true for \((Y, Z, \Phi)\) with \(K := K(\eta, L)\). Moreover, there exists \(c' := c'(\eta, L, \sigma^\xi)\) such that

\[
\|\Phi\|_{\mathcal{F}^2} \leq c'.
\]  

(3.15)

Proof. Applying Itô’s formula to \(U_t := \varphi(Y_t)\), recall assumption (SB), we compute that \(dU_t = a_t dt + b_t dW_t\) with

\[
a_t := \partial \varphi(Y_t)\{-f(t, Y_t, Z_t) + H(t, Y_t)\Phi_t\} + \frac{1}{2} \text{Tr}[\partial^2 \varphi(Y_t)Z_t Z_t^\ast] \quad \text{and} \quad b_t := \partial \varphi(Y_t)Z_t dW_t.
\]

Using Itô-Tanaka formula, we obtain

\[
d[-U_t^\ast] = -a_t 1_{\{U_t < 0\}} dt - b_t 1_{\{U_t < 0\}} dW_t + dL_t^0
\]

where \(L_t^0\) is the local time at 0 of the semi-martingale \(U\). Taking the difference of the two previous equation, we obtain

\[
0 = a_t 1_{\{U_t = 0\}} dt + b_t 1_{\{U_t = 0\}} dW_t + dL_t^0
\]

which leads to \(a_t 1_{\{U_t = 0\}} dt \leq 0\). We then deduce

\[
|\Phi_t| dt \leq \frac{1}{\eta} [\partial \varphi(Y_t) f(t, Y_t, Z_t)]^\ast dt,
\]

(3.16)

recall \((2.10)\). From this, we deduce that \(a \text{ fortiiori} \ (2.13)\) holds true. \(\square\)

We now turn to the proof of our main result for this section.

Proof of Theorem 3.1

1. We first prove uniqueness of the solution. Let \((1Y, 1Z, 1\Phi)\) and \((2Y, 2Z, 2\Phi)\) be two solutions of \((3.8)\) in \(\mathcal{F}^2 \times \mathcal{H}^2 \times \mathcal{H}^2\). We first observe that both solutions satisfies \((2.13)\) by application of Lemma 3.3 which allows us to invoke Proposition 2.1. Moreover, both solutions satisfy \((2.29)\) by definition. Then, a straightforward application of Proposition 2.2 concludes the proof of this step, noticing that all the terms in the right hand side of \((2.30)\) are null.

2. We now turn to the existence question.

2.a We first approximate \(\xi\) by a sequence of bounded random variables \((\xi_N)_{N\geq 1}\). Let \((\tau_N)_{N\geq 1}\) be the sequence of stopping time defined by

\[
\tau_N := \inf\{t \geq 0 | |Y_t^\xi| \geq N\} \wedge T,
\]

and we set \(\xi_N := Y_{\tau_N}^\xi\). Importantly, we observe that \(\xi_N\) satisfies (SB)(i) and it belongs also to the class \(\mathcal{F}_{[0 \wedge \tau_N]}\), indeed \(\int_0^T |Z_t^\xi|^2 ds \leq \int_0^{\tau_N} |Z_t^\xi|^2 ds\). For later use, let us also remark that

\[
\sigma^\xi_N \leq \sigma^\xi, \quad \text{for all} \quad N \geq 1,
\]

(3.17)
recall (2.9). Moreover, since
\[ \xi_N \to \xi \text{ a.s. and } |\xi_N - \xi| \leq 2 \sup_{t \in [0,T]} |\xi_t|, \]
we have that by the dominated convergence theorem, recall Remark 2.2(i), \( \xi_N \to \xi \) in \( L^q \), for any \( q \geq 1 \).

2.b Applying Proposition 3.1, we introduce a sequence of Obliquely RBSDEs, \((Y^N, Z^N, \Phi^N)\) with terminal condition \( \xi_N \). We now show that \((Y^N, Z^N)\) is a Cauchy sequence in \( \mathcal{L}^2 \times \mathcal{H}^2 \). First, we apply the stability estimate given in Proposition 2.2 for \( N, P \geq 1 \), we have
\[ \sup_{t \in [0,T]} \mathbb{E}[|Y^N_t - Y^P_t|^2] + \|Z^N - Z^P\|_{\mathcal{H}^2}^2 \leq C \mathbb{E}\left[ \Gamma^{N,P}_T |\xi_N - \xi^P|^2 \right], \]
with \( \Gamma^{N,P} \) such that for some \( p > 1 \) and \( C > 0 \),
\[ \mathbb{E}\left[ |\Gamma^{N,P}_T|^p \right] \leq C, \]
where importantly \( p \) and \( C \) do not depend on \((N, P)\), recall (3.17). Using Hölder inequality, we then obtain
\[ \sup_{t \in [0,T]} \mathbb{E}[|Y^N_t - Y^P_t|^2] + \|Z^N - Z^P\|_{\mathcal{H}^2}^2 \leq C \|\xi_N - \xi^P\|_{\mathcal{L}^2q}^2. \]
Following classical arguments, see Step 2.a in the proof of Proposition 3.1 we compute also
\[ \|Y^N - Y^P\|_{\mathcal{H}^2}^2 \leq C \mathbb{E}\left[ \int_0^T |Y^N_t - Y^P_t|^2 (|\Phi^N_t| + |\Phi^P_t|) \, dt \right] + C \|Z^N - Z^P\|_{\mathcal{H}^2}^2. \]
Applying Cauchy-Schwarz inequality, and combining Lemma 3.3 and (3.17), we get
\[ \|Y^N - Y^P\|_{\mathcal{H}^2}^2 \leq C \left( \|Y^N - Y^P\|_{\mathcal{H}^2}^2 + \|Z^N - Z^P\|_{\mathcal{H}^2}^2 \right). \]
Eventually, we obtain
\[ \|Y^N - Y^P\|_{\mathcal{H}^2}^2 + \|Z^N - Z^P\|_{\mathcal{H}^2}^2 \leq C \left( \|\xi_N - \xi^P\|_{\mathcal{L}^2q}^2 + \|\xi_N - \xi^P\|_{\mathcal{L}^2q}^2 \right). \]
From the conclusion of Step 1. we deduce the Cauchy property of the sequence \((Y^N, Z^N)\) and we denote \((Y, Z)\) its limit. The proof is then concluded following the same arguments as in step 2 of Proposition 3.1 once observed that by Lemma 3.3,
\[ \mathbb{E}\left[ \int_0^T |\Phi^N_s|^2 \, ds \right] \leq C, \]
where again \( C \) does not depend on \( N \) from (3.17). \( \square \)
4 A general existence result in the Markovian framework

In this section, we introduce a Markovian framework: for all \( p, t, x, q \), \( (X^{t,x})_{s \in [0,T]} \) is the solution of the SDE

\[
\begin{align*}
dX_s &= b(s, X_s)ds + \sigma(s, X_s)dW_s, \quad s \in [t, T], \\
X_s &= x, \quad s \in [0, t].
\end{align*}
\]

We consider the following Markovian reflected BSDE

\[
\begin{align*}
Y_t &= g(X_T^{0,a}) + \int_t^T f(s, X_s^{0,a}, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T H(s, X_s^{0,a}, Y_s, Z_s)\Phi_s ds, \\
Y_t &\in D, \quad \Phi_t \in \partial \varphi(Y_t), \quad 0 \leq t \leq T, \quad \int_0^T 1_{\{Y_t \notin D\}}|\Phi_t|dt = 0.
\end{align*}
\]

The main goal here is to prove an existence result for the above reflected BSDEs when \( H \) is only continuous, compare with assumption (SB). We also discuss the case of discontinuous \( H \) and the difficulty arising for uniqueness in this setting.

4.1 Continuous oblique direction of reflection

We now introduce the main setting for this part. The set of assumption below echoes assumption (A) introduced in Section 2.1 but in a Markovian setting.

**Assumption (AM)**

i) \( b : [0, T] \times \mathbb{R}^q \to \mathbb{R}^q \) and \( \sigma : [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q \times k} \) are measurable functions satisfying linear growth condition and uniform Lipschitz condition with respect to \( x \).

ii) \( g : \mathbb{R}^q \to \mathbb{R}^d \) is a measurable function and there exists \( p \in \mathbb{R}^+ \) such that for any \( x \in \mathbb{R}^q \),

\[
|g(x)| \leq L(1 + |x|^p).
\]

iii) \( f : [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to \mathbb{R}^d \) is a measurable function continuous in its \((y, z)\) variable, there exists \( p \in \mathbb{R}^+ \) such that, for any \((t, x, y, z) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k}\), we have

\[
|f(t, x, y, z)| \leq L(1 + |x|^p + |y| + |z|),
\]

and, for all \((t, x) \in [0, T] \times \mathbb{R}^d\), \( f(t, x, \ldots) \) is continuous on \( \mathbb{R}^d \times \mathbb{R}^{d \times k} \).

iv) \( H : [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to \mathbb{R}^{d \times d} \) is a measurable function, there exist \( \eta > 0 \) such that, for any \((t, x, y, z) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k}\)

\[
H(t, x, \mathcal{P}(y), z) v \cdot v \geq \eta, \quad v \in n(\mathcal{P}(y)),
\]

and \( |H(t, x, \mathcal{P}(y), z)| \leq L. \)
v) Let \( X \) \( = \{ \mu(t, x; s, dy), x \in \mathbb{R}^q \} \) be the family of laws of \( X^{t,x} \) on \( \mathbb{R}^q \), i.e., the measures such that \( \forall A \in \mathcal{B}(\mathbb{R}^q), \mu(t, x; s, A) = \mathbb{P}(X^{t,x}_s \in A) \). For any \( t \in [0, T] \), for any \( \mu(0, a; t, dy) \)-almost every \( x \in \mathbb{R}^q \), and any \( \delta \in [0, T - t] \), there exists an application \( \phi_{t,x} : [t, T] \times \mathbb{R}^d \to \mathbb{R}^+ \) such that:

\[
\begin{align*}
(a) & \quad \forall k \geq 1, \phi_{t,x} \in L^2([t + \delta, T] \times [-k, k]^q; \mu(0, a; s, dy)ds), \\
(b) & \quad \mu(t, x; s, dy)ds = \phi_{t,x}(s, y)\mu(0, a; s, dy)ds \text{ on } [t + \delta, T] \times \mathbb{R}^q.
\end{align*}
\]

vi) For \( (t, x) \in [0, T] \times \mathbb{R}^d \), \( H(t, x, \cdot) \) is continuous on \( \mathbb{R}^d \times \overline{\mathcal{D}} \).

**Remark 4.1.**

i) We observe that \( H(t, X, \cdot) \) and \( f(t, X, \cdot) \) satisfy assumption \((A)\) and we will thus use in the sequel the a priori estimates obtained in Section 2.2.

ii) Remark 2.1 apply for \( H \) which is continuous in this context.

iii) The \( L^2 \)-domination condition \((AM)(v)\) was already introduced in [8]. We refer to [8, 9] for examples of assumptions on coefficients of the SDE \((4.1)\) under which \((AM)(v)\) is true.

**Theorem 4.1.** Assume \((AM)\). Then, there exists a solution \((Y, Z) \in \mathcal{H}^2 \times \mathcal{H}^2 \) to \((4.2)\). Moreover we have the following Markovian representation: there exist \( u : [0, T] \times \mathbb{R}^q \to \mathbb{R}^d \) and \( v : [0, T] \times \mathbb{R}^q \to \mathbb{R}^dkt \) some measurable functions such that

\[
Y_t = u(t, X^{0,x}_t) \quad \text{and} \quad Z_t = v(t, X^{0,x}_t),
\]

and, for all \((t, x) \in [0, T] \times \mathbb{R}^q\),

\[
|u(t, x)| \leq L(1 + |x|^p).
\]

By choosing properly the function \( H \) we can obtain the following corollary.

**Corollary 4.1.** Let us consider the following obliquely reflected Markovian BSDE

\[
\begin{aligned}
Y_t &= g(X^{0,a}_t) + \int_t^T f(s, X^{0,a}_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + \int_t^T \Psi_s ds, \quad 0 \leq t \leq T, \\
Y_t^\ell &\geq \max_{j \in \mathcal{I}} \{ Y_t^j - c^{ij} \}, \quad 0 \leq t \leq T, \quad \ell \in \mathcal{I}, \\
\int_0^T [Y_t^\ell - \max_{j \in \mathcal{I}(\ell)} \{ Y_t^j - c^{ij} \}] \Psi_t^\ell dt &= 0, \quad \ell \in \mathcal{I},
\end{aligned}
\]

where \( \mathcal{I} := \{1, \ldots, d\} \) and the switching costs \((c^{ij})_{i,j \in \mathcal{I}}\) satisfy the following structure condition

\[
\begin{cases}
&c^{ii} = 0, \quad \text{for } 1 \leq i \leq d; \\
&\{c^{ij} + c^{ji} - c^{il} \} > 0, \quad \text{for } 1 \leq i, j \leq d \text{ with } i \neq j, j \neq l.
\end{cases}
\]

We assume that assumption \((AM)\) is in force. Then there exists a solution \((Y, Z, \Phi) \in \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^2 \) to \((4.4)\). Moreover we have the following Markovian representation:
there exist \( u : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^d \) and \( v : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^{d \times k} \) some measurable functions such that
\[
Y_t = u(t, X_t^{0,x}) \quad \text{and} \quad Z_t = v(t, X_t^{0,x}),
\]
and, for all \( (t, x) \in [0, T] \times \mathbb{R}^q \),
\[
|u(t, x)| \leq C(1 + |x|^p).
\]

**Remark 4.2.** The main novelty here is the dependence of the generator on the whole \( z \) (as in the concomitant article [3]) which extend the result of [10, 9, 1] and the possibility to consider negative switching costs. We refer to [14] and references inside for a recent work dealing with switching problems with signed switching costs. Our result only cover to the case of constant switching costs due to a priori estimates obtained previously in the framework of a deterministic domain \( D \). Nevertheless our approach could be adapted to treat random domains and then tackle the problem of switched BSDEs with random signed switching costs.

Before giving the proof of Theorem 4.1 and Corollary 4.1, we start by considering an approximation of (4.2). Let \( \theta \) be an element of \( C^\infty(\mathbb{R}^{d+2k}, \mathbb{R}) \) with compact support and satisfying
\[
\int_{\mathbb{R}^{d+2k}} \theta(y, z)dydz = 1.
\]

For \( n \in \mathbb{N} \) and \( (t, x, y, z) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \) we set
\[
f_n(t, x, y, z) = \int_{\mathbb{R}^{d+2k}} n^2 f(t, x, y, z, \theta(n(y - u), n(z - v)))dudv
\]
\[
H_n(t, x, y, z) = \int_{\mathbb{R}^{d+2k}} n^2 H(t, x, y, z, \theta(n(y - u), n(z - v)))dudv.
\]

By classical convolution arguments functions \( (f_n)_{n \in \mathbb{N}} \) and \( (H_n)_{n \in \mathbb{N}} \) satisfy following properties.

**Lemma 4.1.**

i) \( f_n : [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^d \) and \( H_n : [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^{d \times d} \) are measurable and uniformly Lipschitz with respect to \( (y, z) \).

ii) \( |f_n(t, x, y, z)| \leq C(1 + |x|^p + |y| + |z|) \) and \( |H_n(t, x, y, z)| \leq C \) for all \( (t, x, y, z) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \).

iii) For any \( (t, x) \in [0, T] \times \mathbb{R}^q \) and \( K \) a compact subset of \( \mathbb{R}^d \times \mathbb{R}^{d \times k} \)
\[
\sup_{(y, z) \in K} |f_n(t, x, y, z) - f(t, x, y, z)| + \sup_{(y, z) \in K} |H_n(t, x, y, z) - H(t, x, y, z)| \xrightarrow{n \rightarrow +\infty} 0.
\]

For \( n \in \mathbb{N} \), we consider the following BSDE
\[
Y^n_t = g(X_t^{n,0,a}) + \int_t^T f_n(s, X_s^{n,0,a}, Y^n_s, Z^n_s)ds
\]
\[
- \int_t^T Z^n_s dW_s - \int_t^T H_n(s, X_s^{n,0,a}, Y^n_s, Z^n_s)\nabla \phi_n(Y^n_s)ds, \quad t \in [0, T] \quad (4.6)
\]
where \( \varphi_n \) is defined in (1.3). Note that, in this section, for the reader’s convenience, we write simply \( \varphi_n \) instead of \( \varphi_n^M \).

**Lemma 4.2.** There exists a unique solution to (4.6) in \( \mathcal{S}^2 \times \mathcal{H}^2 \). Moreover, we have a Markovian representation for this solution: for all \( n \in \mathbb{N} \), there exist \( u_n : [0, T] \times \mathbb{R}^q \to \mathbb{R}^d \) and \( v_n : [0, T] \times \mathbb{R}^q \to \mathbb{R}^{d \times k} \) some measurable functions such that

\[
Y^n_t = u_n(t, X^n_T) \quad \text{and} \quad Z^n_t = v_n(t, X^n_T).
\]

Moreover, for all \( (t, x) \in [0, T] \times \mathbb{R}^q \), \( (u_n(s, X^n_s), v_n(s, X^n_s))_{s \in [t, T]} \) is the unique solution in \( \mathcal{S}^2 \times \mathcal{H}^2 \) of the equation

\[
Y^{n,t,x}_s = g(X^{t,x}_T) + \int_s^T f_n(r, X^{t,x}_r, Y^{n,t,x}_r, Z^{n,t,x}_r) \, dr - \int_s^T Z^{n,t,x}_r \, dW_r
\]

\[
- \int_s^T H_n(r, X^{t,x}_r, Y^{n,t,x}_r, Z^{n,t,x}_r) \nabla \varphi_n(Y^{n,t,x}_r) \, dr \quad s \in [t, T]. \tag{4.8}
\]

**Proof.** This the same proof as for Lemma 3.1: since \( H_n \) and \( \nabla \varphi_n \) are two Lipschitz bounded functions (with respect to \( y \) and \( z \)), this proves that the penalised BSDE (4.6) has a Lipschitz driver: classical theory then applies to get the existence, uniqueness and representation result.

By applying Lemma 2.1 and Lemma 3.2, we obtain the following estimates for \( Y^{n,t,x}_s, Z^{n,t,x}_s \).

**Proposition 4.1.** For all \( (t, x) \in [0, T] \times \mathbb{R}^q \), we have

\[
\sup_{t \leq s \leq T} \mathbb{E} \left[ |Y^{n,t,x}_s|^2 + \varphi_n(Y^{n,t,x}_s) \right] + \mathbb{E} \left[ \int_t^T |Z^{n,t,x}_s|^2 \, ds + \int_t^T |\nabla \varphi_n(Y^{n,t,x}_s)|^2 \, ds \right] \leq C(1 + |x|^{2p}).
\]

In particular, Proposition 4.1 yields that, for all \( n \in \mathbb{N} \) and \( (t, x) \in [0, T] \times \mathbb{R}^q \),

\[
|u_n(t, x)| \leq C(1 + |x|^p).
\]

We now turn to the proof of the main result for this section.

**Proof of Theorem 4.1**

The proof follows mainly from arguments in [S]. Some extra work is required to identify the reflecting process properly.

1. Define,

\[
F^n(t, x) = f_n(t, x, u_n(t, x), v_n(t, x)), \quad G^n(t, x) = H_n(t, x, u_n(t, x), v_n(t, x)) \nabla \varphi_n(u_n(t, x)),
\]

and

\[
\mathfrak{F}_n := F_n - G_n.
\]
Now, we compute
\[ \int_{\mathbb{R}^d} \int_0^T |\mathfrak{F}_n(s, y)|^2 \mu(0, a; s, dy) ds = \mathbb{E} \left[ \int_0^T \mathfrak{F}_n(s, X_s^0, a)|^2 ds \right] \]
\[ \leq \mathbb{E} \left[ \int_0^T C(1 + |X_s^0, a|^2p + |Z_s^0|^2 + |\nabla \varphi_n(Y_s^n)|^2) ds \right] \]
\[ \leq C, \]
by using Proposition 4.1. Thus we get \( \mathfrak{F}_n \to \mathfrak{F} \) in \( L^2([0, T] \times \mathbb{R}^d; \mu(0, a; s, dx) ds) \).

2. We now show that \( (u_n(t, x))_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathbb{R}^d \) for all \( t \in [0, T] \) and for \( \mu(0, a; t, dx) \)-almost every \( x \in \mathbb{R}^d \). When \( t = T \) the sequence is constant and the result is obvious. When \( t < T \), \( x \in \mathbb{R}^d \) and \( \delta \in (0, T-t] \), we compute
\[ |u_n(t, x) - u_m(t, x)| = \left| \mathbb{E} \left[ \int_t^{t+\delta} \left( \mathfrak{F}_n(s, X_s^t, x) - \mathfrak{F}_m(s, X_s^t, x) \right) ds \right] \right| \]
\[ \leq \mathbb{E} \left[ \int_t^{t+\delta} |\mathfrak{F}_n(s, X_s^t, x) - \mathfrak{F}_m(s, X_s^t, x)| ds \right] =: A_1 \]
\[ + \mathbb{E} \left[ \int_t^{t+\delta} |\mathfrak{F}_n(s, X_s^t, x) - \mathfrak{F}_m(s, X_s^t, x)| \mathbbm{1}_{|X_s^t, x| \geq k} ds \right] =: A_2 \]
\[ + \mathbb{E} \left[ \int_t^{t+\delta} \left( \mathfrak{F}_n(s, X_s^t, x) - \mathfrak{F}_m(s, X_s^t, x) \right) \mathbbm{1}_{|X_s^t, x| < k} ds \right] =: A_3 \]

For the first two terms, we easily get
\[ A_1 \leq \delta \frac{1}{2} \mathbb{E} \left[ \int_t^{t+\delta} |\mathfrak{F}_n(s, X_s^t, x) - \mathfrak{F}_m(s, X_s^t, x)|^2 ds \right] \]
\[ \leq C(1 + |x|^p) \delta \frac{1}{2}, \]
\[ A_2 \leq Ck^{-\frac{1}{2}} \mathbb{E} \left[ \int_t^{t+\delta} |X_s^t, x| ds \right] \mathbb{E} \left[ \int_t^{t+\delta} |\mathfrak{F}_n(s, X_s^t, x) - \mathfrak{F}_m(s, X_s^t, x)|^2 ds \right] \]
\[ \leq C(1 + |x|^{p+1}) k^{-\frac{1}{2}}, \]
where \( C \) is a constant that does not depend on \( n \) nor \( m \). For the third term, we have
\[ A_3 = \left| \int_{\mathbb{R}^d} \int_{t+\delta}^T \left( \mathfrak{F}_n(s, y) - \mathfrak{F}_m(s, y) \right) \mu(t, x; s, dy) ds \right| \]
\[ = \left| \int_{\mathbb{R}^d} \int_{t+\delta}^T \left( \mathfrak{F}_n(s, y) - \mathfrak{F}_m(s, y) \right) \mathbbm{1}_{|y| \leq k} \phi_{t, x}(s, y) \mu(0, a; s, dy) ds \right| \]
for \( \mu(0, a; s, dx) \)-almost every \( x \in \mathbb{R}^d \), where we used the \( L^2 \)-domination assumption. By weak convergence, \( A_3 \to 0 \) when \( n, m \to \infty \). Thus, for all \( t \in [0, T] \) and for \( \mu(0, a; t, dx) \)-almost every \( x \in \mathbb{R}^d \) is a Cauchy sequence. So, there exists a Borelian application \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) such that for all \( t \in [0, T] \), for \( \mu(0, a; t, dx) \)-almost every \( x \in \mathbb{R}^d \),
\[ u(t, x) = \lim_{n \to \infty} u_n(t, x). \] (4.9)
Since we start by studying the convergence of the generator. Firstly, we compute
\[ p \]  
where we used Proposition 4.1, the fact that \( \sup_{0 \leq s \leq T} E_{\Omega} (|p|) \leq C \)  
and \( \sup_{0 \leq s \leq T} E_{\Omega} (|q|) \leq C \).

3. We can also prove that the process \( Y \) lives in the convex set \( D \). Indeed, we have, recalling (3.1),
\[ E_{\Omega} \left[ \sup_{0 \leq s \leq T} E_{\Omega} (|\varphi_1(Y_s)|) \right] \leq M \sup_{0 \leq s \leq T} E_{\Omega} (|Y_s - Y^n_s|) + \frac{1}{n} \rightarrow \infty 0, \]
where we used Proposition 4.1 the fact that \( \varphi_1 \) is a \( M \)-Lipschitz function and the convergence of \( (Y^n_{\infty})_{n \in \mathbb{N}} \). Then, for all \( s \in [0, T], d(Y_s, D) = 0 \) a.s. and so \( Y_s \in D \) a.s.

4. We now show that \( Z^n = (v_n(t, X^n_{t^0}), t \in [0, T]) \) is a Cauchy sequence in \( L^2([0, T] \times \Omega, dt \otimes d\mathbb{P}) \). For \( n, m \geq 1 \), we compute, applying Itô’s formula,
\[ \mathbb{E} \left[ \int_0^T |f_n(s, X^n_{s^0, Y^n_s, Z^n_s}) - f(s, X^n_{s^0, Y^n_s, Z^n_s})|^2 ds \right] \leq C \mathbb{E} \left[ \int_0^T |Y^n_s - Y^n_t|^2 dt \right] \]
which goes to 0 as \( n, m \to \infty \).

We denote by \( Z \) the limit. Until the end we chose the progressively measurable version of \((Y, Z)\).

5.a In the last step we have to prove that \( (Y, Z) \) is a solution to BSDE (4.2). We start by studying the convergence of the generator. Firstly, we compute
\[ B_1 \]
\[ B_2 \]
\[ B_3 \]
Since \( f \) and \( f_n \) have a linear growth that does not depend on \( n \), and \((Y^n, Z^n)\) are uniformly bounded in \( L^2([0, T] \times \Omega, dt \otimes d\mathbb{P}) \), we get
\[ B_2 \leq \frac{C}{k} \]
Moreover, we also get

\[ B_3 \leq \mathbb{E}\left[ \int_0^T \frac{|f(s, X_s^{0,a}, Y_s^n, Z_s^n) - f(s, X_s^{0,a}, Y_s, Z_s)|^2}{(1 + |Y_s^n| + |Z_s^n|)^2} \, ds \right]^{1/2} \mathbb{E}\left[ \int_0^T (1 + |Y_s^n| + |Z_s^n|)^2 \, ds \right]^{1/2} \]

and thus, the dominated convergence theorem gives us that \( B_3 \) converges to 0 as \( n \to +\infty \).

Now, let us treat the first term \( B_1 \). We have

\[ |f_n(s, X_s^{0,a}, Y_s^n, Z_s^n) - f(s, X_s^{0,a}, Y_s^n, Z_s^n)| \leq C(1 + 2k + |X_s^{0,a}|), \]

and

\[ |f_n(s, X_s^{0,a}, Y_s^n, Z_s^n) - f(s, X_s^{0,a}, Y_s, Z_s^n)| \leq \sup_{(y,z), |y|+|z|\leq k} |f_n(s, X_s^{0,a}, y, z) - f(s, X_s^{0,a}, y, z)|. \]

Thanks to Lemma 4.1(iii) we can assert that the second term of the last inequality converges to 0 and then, by applying the dominated convergence theorem, \( B_1 \) converges also to 0. It follows that \( f_n(t, X_t^{0,a}, Y_t^n, Z_t^n) |_{t \in [0,T]} \) converges to \( f(t, X_t^{0,a}, Y_t, Z_t) |_{t \in [0,T]} \) in \( L^1([0,T] \times \Omega, dt \otimes dP) \).

5.b Finally we study the reflecting term. Since

\[ \mathbb{E}\left[ \int_0^T |\nabla \varphi_n(Y_s^n)|^2 \, ds \right] \leq C, \]

we have, up to a subsequence, the following weak \( L^2([0,T] \times \Omega) \)-convergence:

\[ \nabla \varphi_n(Y_s^n) \rightharpoonup \Phi, \quad \text{when} \ n \to +\infty, \]

and we can follow step 2.a in the proof of Proposition 3.1 to obtain

\[ \Phi \in \partial \varphi(Y) \quad \text{and} \quad \int_0^T 1_{\{Y_t \notin \mathcal{D}\}} |\Phi_t| \, dt = 0, \]

which fully characterize \( \Phi \). We now follow step 2.b in the proof of Proposition 3.1. Using Mazur’s Lemma, we know that there exists a convex combination of \( (\Phi_n)_{n \in \mathbb{N}} := (\nabla \varphi_n(Y^n))_{n \in \mathbb{N}} \) converging strongly in \( L^2([0,T] \times \Omega) \), namely

\[ \Phi^p := \sum_{r=p}^{N_p} \lambda^p_r \Phi^r \xrightarrow{p \to \infty} \Phi, \]

where \( \lambda^p_r \geq 0 \) for all \( p \in \mathbb{N} \) and \( p \leq r \leq N_p \), and \( \sum_{r=p}^{N_p} \lambda^p_r = 1 \). Let us observe that by strong convergence, the following combination

\[ (\Phi^p, (Y^r, Z^r)) := \sum_{r=p}^{N_p} \lambda^p_r (Y^r, Z^r) \]
still converges to \((Y, Z)\) in \(\mathcal{S}^2 \times \mathcal{H}^2\) and, by strong convergence,

\[
\sum_{r=p}^{N_p} \lambda_r^p f_r(\cdot, X_s^{0,a}, Y_s^r, Z_s^r) \sim L_1([0,T] \times \Omega, dt \otimes dP) f(\cdot, X_s^{0,a}, Y, Z) \quad \text{and} \quad \int_0^t p Z_s dW_s \sim \int_0^t Z_s dW_s.
\]

Moreover, we remark that, for all \(t \leq T\),

\[
\mathcal{E}^p := \int_0^t \lambda_r^p H_r(s, X_s^{0,a}, Y_s^r, Z_s^r) \Phi^r ds - \int_0^t H(s, X_s^{0,a}, Y_s, Z_s) \Phi ds
\]

is bounded, recall Lemma 4.1(iii), with the dominated convergence theorem, since \(H_r\) and \(H\) are bounded, to get that for all \(\epsilon > 0\) there exists \(N_{k, \epsilon}\) such that

\[
B_1^p \leq \epsilon \quad \text{for all} \quad p \geq N_{k, \epsilon}.
\]

Combining Cauchy-Schwartz inequality with the uniform square integrability of \(Y^n\) and \(Z^n\), we easily obtain that

\[
B_2^p \leq \frac{C}{k}.
\]

For the first term we use the uniform convergence (on compact set) of \(H_r\) to \(H\), recall Lemma 4.1(iii), with the dominated convergence theorem, since \(H_r\) and \(H\) are bounded, to get that for all \(\epsilon > 0\) there exists \(N_{k, \epsilon}\) such that

\[
B_1^p \leq \epsilon \quad \text{for all} \quad p \geq N_{k, \epsilon}.
\]

Combining (4.12) and (4.13), we then get

\[
\lim_p \mathbb{E}[A_1^p] = 0.
\]
Next, we compute, using Cauchy-Schwartz inequality and the uniform bound on $\|\Phi^n\|_{\mathcal{W}^2}$,

$$
\mathbb{E}[|A^n_s|] \leq C \sum_{r=p}^{Nn} \lambda^n_r \mathbb{E} \left[ \int_0^t |H(s, X_s^{0,a}, Y_s, Z_s) - H(s, X_s^{0,a}, Y_s, Z_s)|^2 ds \right]^{\frac{1}{2}}
$$

and we deduce

$$
\lim_{p} \mathbb{E}[|A^n_s|] = 0,
$$

(4.15)

from the continuity of $H$ and the strong convergence of $(Y', Z')$ to $(Y, Z)$. Finally we use the boundedness of $H$ and the strong convergence of $\Phi^n$ to $\Phi$ to get

$$
\lim_{p} \mathbb{E}[|A^n_s|] = 0,
$$

(4.16)

Combining (4.14), (4.15) and (4.16) with (4.10) yields

$$
\lim_{n} \mathbb{E}[|E^n_s - E^n_t|] = 0,
$$

(4.17)

which concludes the proof of the theorem. Let us remark that the previous equation allows us to consider a continuous version of the process $Y$.

We conclude this section by giving the Proof of Corollary 4.1 which is an interesting application of Theorem 4.1 to the well studied case of BSDEs for switching problems. Following our approach, the main question reduces now to find an appropriate continuous $H$ to describe the direction of reflection such that $H(\cdot)\Phi = \Psi$, compare (4.2) and (4.4).

**Proof of Corollary 4.1**

We define a continuous function $H$ on $\partial \mathcal{D}$, recall Remark 2.1. We have

$$
\mathcal{D} = \{y \in \mathbb{R}^d : y^l = \max_{j \in I} (y^j - c^j), l \in I\},
$$

(4.17)

Thus, $\bar{\mathcal{D}}$ is a non-compact convex polyhedron. We can remark that

$$
\mathcal{D}^0 := \bar{\mathcal{D}} \cap \{y^d = 0\}
$$

is, by abuse of notation, a compact convex polyhedron of $\mathbb{R}^{d-1}$ and so it is a convex polytope. Indeed, we have

$$
\mathcal{D}^0 \subset \{(y^1, \ldots, y^{d-1})| y^i \in [-c^d, c^d], \forall i \in \{1, \ldots, d-1\}\}.
$$

We just have to define $H$ on $\partial \mathcal{D}^0$ and then extend $H$ to $\partial \mathcal{D}$ in this way: for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^k \times \mathcal{D} \times \mathbb{R}^{d \times k}$, we define

$$
H(t, x, y, z) := H(t, x, (y^1 - y^d, \ldots, y^{d-1} - y^d, 0), z).
$$
Since $\mathcal{D}^0$ is a convex polytope, then, by Krein-Milman theorem, it is the convex hull of its extremal points. We will define $H$ on all extremal points and then the value of $H$ on all facets

$$\mathcal{C}^ij = \{ y \in \partial \mathcal{D}^0 : y^l = y^j - c^j \}, \quad l, j \in I, \quad l \neq j,$$

will be defined by linear interpolations. Let us consider an extremal point $(\bar{y}^1, \ldots, \bar{y}^{d-1})$: we know that there exist $(l_i, j_i) \in \{1, \ldots, d-1\} \times \{1, \ldots, d\}$ such that

- $(l_i, j_i) \neq (l_k, j_k)$ when $i \neq k$,
- for all $i \in \{1, \ldots, d-1\}$, $\bar{y}^{l_i} = \bar{y}^{j_i} - c^{j_i}$ where $\bar{y}^d = 0$.

Then, we set $H(t, x, (\bar{y}^1, \ldots, \bar{y}^{d-1}, 0), z)$ as the orthogonal projection onto $\text{span}\{e^{l_1}, \ldots, e^{l_{d-1}}\}$. Now we have to check that $H(t, x, (y^1, \ldots, y^{d-1}, 0), z)$ send the vector $e^l - e^j$ to the vector $e^l - e^j$ when $(y^1, \ldots, y^{d-1}) \in \mathcal{C}^ij$. To do such a thing, it is sufficient to show the result only for extremal points. In order to do it, let us assume that $(\bar{y}^1, \ldots, \bar{y}^{d-1}) \in \mathcal{C}^ij$ is an extremal point and let us reuse previous notations: we just have to show that $e^j \notin \{e^{l_1}, \ldots, e^{l_{d-1}}\}$.

Let us prove it by contradiction: we assume that there exists $i \in \{1, \ldots, d-1\}$ such that

$$j = l_i \quad \text{and} \quad \bar{y}^{j_i} = \bar{y}^{l_i} - c^{j_i}. \quad (4.18)$$

Moreover, we have $(\bar{y}^1, \ldots, \bar{y}^{d-1}) \in \mathcal{C}^ij$ so

$$\bar{y}^j = \bar{y}^l - c^j, \quad (4.19)$$

By combining (4.18), (4.19) and the structure condition (4.5), we obtain

$$\bar{y}^j = \bar{y}^l - c^j = \bar{y}^{l_i} - (c^{j_i} + c^{j_i}) < \bar{y}^{j_i} - c^{j_i},$$

which is in contradiction with the definition of $\mathcal{D}$ given by (4.17).

### 4.2 The case of discontinuous $H$

In this section, we consider the case of discontinuous direction of reflection on the boundary $\partial D$. We obtain an existence result for an obliquely reflected BSDE but the characterisation of the reflecting part is somehow more involved, specially at the discontinuity point of $H$, where many directions of reflection are allowed at the limit. This too weak characterisation, by nature itself of the problem, leads to non-uniqueness result as illustrated in the next paragraph. The limiting equation we are studying here is then

$$Y_t = g(X^0_t, a) + \int_t^T f(s, X^0_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T \Psi_s ds, \quad t \in [0, T] \quad (4.20)$$

$$\Psi_s \in E(s, X^0_s, Y_s, Z_s) \quad \text{and} \quad Y_s \in D \quad \text{dP} \otimes \text{d}s \quad \text{a.e.},$$

with $a \in \mathbb{R}^q$ and, for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^q \times \mathcal{D} \times \mathbb{R}^{d \times k}$,

$$E(t, x, y, z) := \begin{cases} \cap_{c \geq 0} \text{pos}(\{H(t, x, \bar{y}, \bar{z})u \mid (\bar{y}, \bar{z}) \in B((y, z), \varepsilon), u \in \partial \varphi(y)\}) & \text{if} \ y \in \partial D, \\ \{0\} & \text{if} \ y \in D, \end{cases}$$

where $\text{pos}(\{v_i\})$ is the closure of the positive linear span of the family $\{v_i\}$, and $B(x, \varepsilon)$ is the closed Euclidean ball of center $x$ and radius $\varepsilon$. 

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Theorem 4.2. Assume that assumptions (AM)(i)-(v) hold. Then, there exists a solution in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ to (4.20).

**Remark 4.3.** When $H$ is continuous, we can easily show that
\[ E(t, x, y, z) = H(t, x, y, z)\partial_y \varphi(y) \]
which is consistent with Theorem 4.1.

**Proof.** The proof of Theorem 4.2 strongly follows the proof of Theorem 4.1 since it stays the same from step 1 to step 5.a. Let us start directly at step 5.b by studying the reflecting term. Since
\[ E_p(t, x, y, z) = H_p(t, x, y, z)\partial_y \varphi_n(Y_n) \]
we have, up to a subsequence, the following weak $L^2([0, T] \times \Omega)$-convergence:
\[ \Psi_n := H_p(t, x, y, z)\partial_y \varphi_n(Y_n) \rightharpoonup \Psi, \quad \text{when } n \to +\infty. \]

Using once again Mazur’s Lemma, we know that there exists a convex combination of $(\Psi_n)_{n \in \mathbb{N}}$ converging strongly in $L^2([0, T] \times \Omega)$, namely
\[ \Psi := \sum_{r=p}^{N_p} \lambda_r^p \Psi_r P \rightharpoonup \Psi, \]
where $\lambda_r^p \geq 0$ for all $p \in \mathbb{N}$ and $p \leq r \leq N_p$, and $\sum_{r=p}^{N_p} \lambda_r^p = 1$. As usual, the following combination
\[ \Psi = \sum_{r=p}^{N_p} \lambda_r^p (Y_r, Z_r) \]
still converges to $(Y, Z)$ in $\mathcal{S}^2 \times \mathcal{H}^2$ and, by strong convergence,
\[ \sum_{r=p}^{N_p} \lambda_r^p f_r(\cdot, X_0^0, Y^r, Z^r) \overset{L^1([0, T] \times \Omega; dt \otimes dP)}{\longrightarrow} f(\cdot, X_0^0, Y, Z) \quad \text{and} \quad \int_0^t \nu Z_s dW_s \overset{L^2}{\longrightarrow} \int_0^t Z_s dW_s. \]

So we can pass to the limit into
\[ \Psi_t = g(X_t^0) + \int_t^T \sum_{r=p}^{N_p} \lambda_r^p f_r(s, X_s^0, Y_s^r, Z_s^r) ds - \int_t^T \nu Z_s dW_s - \int_t^T \Psi_s ds \]
to obtain that
\[ Y_t = g(X_t^0) + \int_t^T f(s, X_s^0, Y_s, Z_s) ds - \int_t^T Z_s dW_s - \int_t^T \Psi_s ds, \quad dt \otimes dP \text{ a.e.} \]
Since we have
\[
\Psi^n_t := H(t, X_t^{0,a}, Y^n_t, Z^n_t) \nabla \varphi_n(Y^n_t) \in H(t, X_t^{0,a}, Y^n_t, Z^n_t) \partial \varphi(Q(Y^n_t))
\]
and \((Y^n_t, Z^n_t) \rightharpoonup (Y, Z)\) \(\text{dt} \otimes \text{d}P\) a.e. then, for all \(\varepsilon_1 > 0\) and \(\varepsilon_2 > 0\), there exists \(N\) (that depends on \(\omega\)) such that, for all \(n \geq N\),
\[
\Psi^n_t \in \text{pos}(\{H(t, X_t^{0,a}, \tilde{y}, \tilde{z})u[(\tilde{y}, \tilde{z})] \in B([Y_t, Z_t], \varepsilon_1), \tilde{y} \in B(Y_t, \varepsilon_2) \cap \bar{D}, u \in \partial \varphi(\tilde{y})\}),
\]
dt \(\otimes\) dP a.e. It implies that, for all \(p > N\),
\[
\Psi_t \in \text{pos}(\{H(t, X_t^{0,a}, \tilde{y}, \tilde{z})u[(\tilde{y}, \tilde{z})] \in B([Y_t, Z_t], \varepsilon_1), \tilde{y} \in B(Y_t, \varepsilon_2) \cap \bar{D}, u \in \partial \varphi(\tilde{y})\})
\]
dt \(\otimes\) dP a.e. Finally we get that
\[
\Psi_t \in E(t, X_t^{0,a}, Y_t, Z_t)
\]
dt \(\otimes\) dP a.e.
where
\[
\tilde{E}(t, x, y, z) := \bigcap_{\varepsilon_1 > 0, \varepsilon_2 > 0} \text{pos}(\{H(t, x, \tilde{y}, \tilde{z})u([\tilde{y}, \tilde{z}] \in B([y, z], \varepsilon_1), \tilde{y} \in B(y, \varepsilon_2) \cap \bar{D}, u \in \partial \varphi(\tilde{y})\}).
\]

When \(y \in D\) we can remark that \(\partial \varphi(\tilde{y}) = 0\) when \(\tilde{y} \in B(y, \varepsilon_2) \cap \bar{D}\) with \(\varepsilon_2\) small enough and thus \(E(t, x, y, z) = 0\). When \(y \notin D\), we have that, for any \((\tilde{y}, \tilde{z}) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}\),
\[
\bigcap_{\varepsilon_2 > 0} \text{pos}(\{H(t, x, \tilde{y}, \tilde{z})u[\tilde{y}] \in B(y, \varepsilon_2) \cap \bar{D}, u \in \partial \varphi(\tilde{y})\}) = \text{pos}(\{H(t, x, \tilde{y}, \tilde{z})u[\tilde{y}] \in \partial \varphi(\tilde{y})\}),
\]
and so \(\tilde{E} = E\) which concludes the proof.

\[\square\]

**A counter-example to uniqueness** Inspired by Remark 4.4 in [12], we suggest the following counter-example to uniqueness in a non-smooth setting. The domain \(D\) is given by
\[
D = \{y \in \mathbb{R}^3 \mid y_1 \geq 0 \text{ and } y_2 + y_1 \geq 0\}
\]
Observe that \(\partial D = F_1 \cup F_2\), where \(F_1\) and \(F_2\) are given by
\[
F_1 = \{y \in \mathbb{R}^3 \mid y_1 = 0 \text{ and } y_2 \geq 0\}, \quad F_2 = \{y \in \mathbb{R}^3 \mid y_1 \geq 0 \text{ and } y_2 + y_2 \geq 0\}
\]
and we denote by \(G = F_1 \cap F_2\), the corner of the domain. On \(F_1\) we assume that the reflection is normal so that \(H = I_d\), including points on \(G\) where the outward cone of reflection if given by
\[
\mathcal{K} = \{y \in \mathbb{R}^d \mid y_1 \geq 0, y_2 \geq 0 \text{ and } y_2 \geq y_1\}.
\]
The direction of reflection is along the \(y_1\) axis on \(F_2 \setminus G\) and is thus oblique, \(H\) is constant but not equal to \(I_d\). \(H\) is thus discontinuous at the corner.

We consider a BSDE with the following data: \(X = W, \xi = (0, 0, X_T)\), \(f(t, x, y, z) = -(z^3, z^3, 0)^\top\) is constant. Note that it satisfies the assumption \((\text{AM})\)(i)-(v), and we give now two distinct solutions:

1. The first solution is given by \(Y_t = (0, 0, W_t)\), \(Z_t = (0, 0, 1)\) and \(K_t = (t, t, 0)\).
2. The second solution is given by \(Y'_t = (t, -t, W_t)\), \(Z'_t = (0, 0, 1)\) and \(K'_t = (2t, 0, 0)\).
5 Applications to randomised switching problems

Let us consider a generalisation of the classical switching problem. An agent can decide when she wants to switch a system from one state to another but, contrarily to the classical switching problem framework, the state where the system will end up to is random. Precisely, we consider that it is determined by a Markov chain with an exogenous randomness. The agent has an a priori knowledge on the probability transition of the Markov chain and switching costs. Moreover, we assume that she can have a control on this probability transition and these switching costs. Her goal is to maximise some expected reward. In practice, the strategy of an agent will be a sequence of stopping times and controls on the probability transition of the Markov chain and switching costs. Then, at each switching date the state of the system is chosen randomly and exogenously, using the controlled probability transition.

Let us now describe more precisely the mathematical framework. Let $K$ be a compact subset of a metric space $U$. We denote $(p^n)_{u \in K}$ a family of probability transitions on the state space $\{1, \ldots, d\}$ such that

- $p^n_{ii} = 0$, for all $u \in K$, $i \in \{1, \ldots, d\}$,
- $u \mapsto p^n$ from $K$ to $\mathbb{R}^{d \times d}$ is continuous.

We also denote $(c^n)_{u \in K}$ a family of switching costs such that

- $c^n_{ii} = 0$, for all $u \in K$, $i \in \{1, \ldots, d\}$,
- $u \mapsto c^n$ from $K$ to $\mathbb{R}^{d \times d}$ is continuous,
- there exists $\varepsilon > 0$ such that $\inf_{u \in K} c^n_{ij} \geq \varepsilon$, for all $i, j \in \{1, \ldots, d\}$ with $i \neq j$,
- there exists $(\tilde{c}^n_{ij})_{i,j \in \{1, \ldots, d\}}$ such that, for all $0 \leq i, j \leq d$ and $u \in K$, $\sum_{j=1}^{d} p^n_{ij} \tilde{c}^n_{ij} = 0$ and, for all $i, j, l \in \{1, \ldots, d\}$ with $i \neq j$, $j \neq l$,

$$
\inf_{u \in K} (c^n_{ij} + \tilde{c}^n_{ij}) + \inf_{u \in K} (c^n_{jl} + \tilde{c}^n_{jl}) - \inf_{u \in K} (c^n_{il} + \tilde{c}^n_{il}) > 0.
$$

(5.1)

For any sequence $(u_n)_{n \in \mathbb{N}}$ of $K$, we denote $(\zeta^n_{u_n})_{n \geq 0}$ a non homogeneous Markov chain such that

- $\zeta^n_{u_0}$ is deterministic,
- the transition probability at time $n \in \mathbb{N}$ is given by $p^{u_{n+1}}$:

$$
\mathbb{P}(\zeta^n_{s+1} = j | \zeta^n_{s} = i) = p^{u_{n+1}}_{ij}.
$$

We denote by $(\mathcal{G}_n)_{n \in \mathbb{N}}$ the associated filtration and assume that $\mathcal{G}_\infty$ is independent of $\mathcal{F}_\infty$. For later use, we also introduce the filtrations for $n \geq 0$, $(\mathcal{H}_{n+s}^0)_{s \in [0,T]} = (\mathcal{G}_n \vee \mathcal{F}_s)_{s \in [0,T]}$. Observe that $\mathcal{H}_0^0 = \mathcal{F}$. We denote by $T^n$ the set of respective stopping times.

A switching strategy $a$ is a non-decreasing sequence of stopping times $(\theta^n_a)_{n \in \mathbb{N}}$ combined with a sequence of random variables $(\alpha^n_a)_{n \in \mathbb{N}}$ valued in $K$ such that $\theta_0 \in \mathcal{F}_0$, $\alpha_0$ is $\mathcal{H}_0^0$-measurable and, for all $n \in \mathbb{N}$,
\[ \theta_n \in T^{n-1}, \]
\[ \alpha_n \text{ is } \mathcal{H}^{n-1}_{\theta_n} \text{-measurable.} \]

We denote by \( \mathcal{A} \) the set of such strategies. For \( a = (\theta_j, \alpha_j)_{j \in \mathbb{N}} \in \mathcal{A} \), we introduce \( N^a_1 \) the (random) number of switches before \( t \in [0,T] \), we associate the current state process \( X_t \) where

\[ N^a_1 = \#\{k \in \mathbb{N}^* : \theta_k \leq t \}, \]

and \( N^a := N^a_T \) the total number of switches. To any switching strategy \( a = (\theta_j, \alpha_j)_{j \in \mathbb{N}} \in \mathcal{A} \), we associate the current state process \( (a_t)_{t \in [0,T]} \) and the cumulative cost process \( (A^a_t)_{t \in [0,T]} \) defined respectively by

\[ a_t := \zeta_0^a 1_{0 \leq t < \theta_0} + \sum_{j=1}^{N^a} \zeta_{j-1}^a \alpha_j 1_{t < \theta_j} \quad \text{and} \quad A^a_t := \sum_{j=1}^{N^a} c_{\zeta_{j-1}^a} \alpha_j \theta_j 1_{t \leq T} \]

for \( 0 \leq t \leq T \). We also introduce a truncated version of these two processes: for all \( k \in \mathbb{N}^* \) and \( 0 \leq t \leq T \),

\[ a^k_t := \zeta_0^a 1_{0 \leq t < \theta_0} + \sum_{j=1}^{k} \zeta_{j-1}^a \alpha_j 1_{t < \theta_j} + \zeta_k^a \alpha_k 1_{t \leq \theta_k} \quad \text{and} \quad A^k_t := \sum_{j=1}^{k} c_{\zeta_{j-1}^a} \alpha_j \theta_j 1_{t \leq T} \]

For \((t,i) \in [0,T] \times \{1, \ldots, d\}\), the set \( \mathcal{A}_{t,i} \) of admissible strategies starting from \( i \) at time \( t \) is defined by

\[ \mathcal{A}_{t,i} = \{ a = (\theta_j, \alpha_j)_{j \in \mathbb{N}} \in \mathcal{A} | t = \theta_0, \zeta_0^a = i, \mathbb{E}[A^a_T] < \infty \}. \]

**Remark 5.1.** If \( a \in \mathcal{A}_{t,i} \) then \( 0 \leq A^a_T < +\infty \) a.s. and \( N^a \leq \varepsilon^{-1} A^a_T < +\infty \) a.s.

Let us define the reward function by

\[ J(a, t) = g^{a_T}(X_T) + \int_t^T f^{a_s}(X_s) ds - A^a_T \]

where \( X \) is solution to (1.1) and \( g : \mathbb{R}^q \to \mathbb{R}^d \) and \( f : \mathbb{R}^q \to \mathbb{R}^d \) are measurable functions and there exists \( p \in \mathbb{R}^+ \) such that for any \( x \in \mathbb{R}^q \),

\[ |g(x)| + |f(x)| \leq C(1 + |x|^p). \]

As in the previous section, we also assume that the family of laws associated to \( X \) satisfies a \( \mathcal{L}^2 \)-domination namely \((\text{AM})(v)\).

For all \( i \in \{1, \ldots, d\}, t \in [0,T] \), we set

\[ V^i_t := \text{esssup}_{a \in A_i} \mathbb{E}[J(a,t) | \mathcal{F}_t] \]

Our goal here is to study \((V^i_t)_{t \in [0,T]}\) and, in particular, to show that it is linked to an obliquely reflected BSDE. Then we will be able to determine an optimal strategy and as a byproduct the uniqueness for the corresponding obliquely reflected BSDEs.
To this end, let us consider the following convex domain

\[ D := \{ y \in \mathbb{R}^d \mid y^i \geq \sup_{u \in K} \sum_{j=1}^{d} p_{ij}^u (y^j - c_{ij}^u) \} \]

and the following obliquely reflected BSDE

\[
\begin{align*}
Y_t &= g(X_T) + \int_t^T f(X_s) \, ds - \int_t^T Z_s \, dW_s + \int_t^T \Psi_s \, ds, \quad 0 \leq t \leq T, \\
Y_t^\ell &\geq \sup_{u \in K} \left( \sum_{j=1}^{d} p_{ij}^u (Y_t^j - c_{ij}^u) \right), \quad 0 \leq t \leq T, \; \ell \in \mathcal{I}, \\
\int_0^T \left[ Y_t^\ell - \sup_{u \in K} \left( \sum_{j=1}^{d} p_{ij}^u (Y_t^j - c_{ij}^u) \right) \right] \Psi_t^\ell \, dt &= 0, \quad \ell \in \mathcal{I},
\end{align*}
\]

(5.2)

where \( \mathcal{I} := \{1, \ldots, d\} \). In particular, for all \( \ell \in \mathcal{I} \), \( \Psi^\ell \) is an increasing process. Let us remark that the domain

\[ \mathcal{D}' := \left\{ y \in \mathbb{R}^d \mid y^i \geq \max_{1 \leq k \leq d} \left( y^k - \inf_{u \in K} (c_{ik}^u + \tilde{c}_{ik}^u) \right), \; 1 \leq i \leq d \right\} \]

is included in \( \mathcal{D} \) since we have, for all \( 1 \leq i \leq d \),

\[
\max_{1 \leq k \leq d} \left( y^k - \inf_{u \in K} (c_{ik}^u + \tilde{c}_{ik}^u) \right) = \sup_{u \in K} \max_{1 \leq k \leq d} \left( y^k - (c_{ik}^u + \tilde{c}_{ik}^u) \right) \\
\geq \sup_{u \in K} \sum_{j=1}^{d} p_{ij}^u (y^j - c_{ij}^u) \\
\geq \sup_{u \in K} \sum_{j=1}^{d} p_{ij}^u (y^j - c_{ij}^\ell),
\]

Since we have assumed that (5.1) is fulfilled, then \( \mathcal{D}' \) is not empty (see the proof of Corollary 4.1) and so \( \mathcal{D} \) is not empty.

If we want to show that the BSDE (5.2) has at least a solution we cannot apply Corollary 4.1 since the domain is not the classical one: we have to prove it by using Theorem 4.1 and, in order to do it, we just have to construct a function \( H \) that satisfies Assumption (AM) and that will give \( H(\cdot)\Phi = \Psi \), compare (4.2) and (5.2). The construction of such a function in this general framework is not straightforward and fall out of the scope of this article. Nevertheless some particular cases will be treated at the end of this subsection. Thus, to establish the link between the BSDE (5.2) and the randomised switching problem we will assume in the remaining of this part that there exists a function \( H \) that satisfies (AM), which implies that the BSDE (5.2) has at least a solution \( (Y, Z, \Psi) \in \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^2 \). We recall that a priori this solution is not unique in the Markovian framework.

The following proposition is important as it shows that \( Y \) is linked to a dynamic programming principle, in the context of our randomised switching problem.
Proposition 5.1. For any \( n \geq 1 \), \( \tau \in T^{n-1} \) and \( u_n \) a \( \mathcal{H}^{n-1}_\tau \)-measurable random variable with values in \( K \), the following holds

\[
Y^{\zeta_n}_\tau \equiv \mathbb{E}\left[\hat{\zeta}^{\zeta_n} (X_T) 1_{\theta > T} + \int^\theta_T f^{\zeta_n} (X_s) \, ds + (Y^{\zeta_n}_{\theta} - c^{\zeta_n}_{\theta} \zeta^{\zeta_n}_{\theta+1})1_{\theta \leq T} | \mathcal{H}^{n-1}_\tau \right]
\]  

(5.3)

for all \( \theta \in T(n) \) with \( \tau \leq \theta \) and \( u_{n+1} \) a \( \mathcal{H}^{n+1}_\theta \)-measurable random variable with values in \( K \). Moreover, we have

\[
Y^{\zeta_n}_\tau = \mathbb{E}\left[\hat{\zeta}^{\zeta_n} (X_T) 1_{\theta > T} + \int^\theta_T f^{\zeta_n} (X_s) \, ds + (Y^{\zeta_n}_{\theta} - c^{\zeta_n}_{\theta} \zeta^{\zeta_n}_{\theta+1})1_{\theta \leq T} | \mathcal{H}^{n-1}_\tau \right]
\]  

(5.4)

where

\[
\theta^* = \inf \left\{ t \geq \tau \mid Y^{\zeta_n}_t = \sup_{u \in K} \left( \sum_{j=1}^d p^{u}_{\zeta_n} j (Y^j_t - c^{u}_{\zeta_n} j) \right) \right\}
\]

and, if \( \theta^* \leq T \), \( u^* \) is a \( \mathcal{H}^{n+1}_{\theta^*} \)-measurable random variable with values in \( K \) such that

\[
Y^{\zeta_n}_{\theta^*} = \sum_{j=1}^d p^{u^*}_{\zeta_n} j (Y^j_{\theta^*} - c^{u^*}_{\zeta_n} j).
\]

Proof. 1. We observe that, for all \( i \in \{1, \ldots, d\} \),

\[
Y^{i}_{\theta \wedge T} \geq \sup_{u \in K} \left( \sum_{j=1}^d p^{u}_{ij} (Y^j_{\theta \wedge T} - c^{u}_{ij}) \right).
\]  

(5.5)

From the definition of \( Y \) we deduce easily

\[
Y^{\zeta_n}_\tau \geq Y^{\zeta_n}_{\theta \wedge T} + \int^\theta_T f^{\zeta_n} (X_s) \, ds - \int^\theta_T Z^{\zeta_n}_s \, dW_s
\]

\[
\geq Y^{\zeta_n}_{\tau} 1_{\theta > T} + \int^\theta_T f^{\zeta_n} (X_s) \, ds + \sum_{j=1}^d p^{u}_{\zeta_n} j (Y^j_{\theta} - c^{u}_{\zeta_n} j)1_{\theta \leq T} - \int^\theta_T Z^{\zeta_n}_s \, dW_s.
\]

Taking on both sides the conditional expectation with respect to \( \mathcal{H}^{n-1}_\tau \), recalling the definition of \( \zeta^{n+1}_{n+1} \), we obtain (5.3).

2. From the definition of \( \theta^* \), we have equality in (5.3) on the event \( \{ \theta \leq T \} \). Moreover, \( \Phi^{\zeta_n} = 0 \) \( \mathrm{dP} \otimes \mathrm{dt} \) a.e. on the interval \( [\tau, \theta \wedge T] \). Then the result follows simply by using same arguments as in step 1, replacing inequalities by equalities. \( \square \)

Proposition 5.2. For all \( \ell \in \mathcal{I} \) and \( t \in [0, T] \), the following holds.

(i) 

\[
Y^\ell_t \geq \mathbb{E}[J(a, t) | \mathcal{F}_t], \quad \text{a.s. for any } a \in \mathcal{A}_{\ell,t}.
\]
(ii) Define the strategy \( a^* = (\theta_j^*, \alpha_j^*)_{j \in \mathbb{N}} \) recursively by \( \theta_0^* = t \), \( \zeta_0^* = i \) and, for \( j \geq 1 \),

\[
\theta_j^* = \inf \left\{ t : \sup_{u \in K} \left( \sum_{k=1}^{d} p_{\zeta_{j-1}^*}^{u} (Y_t^k - c_{\zeta_{j-1}^*}^u) \right) \right\},
\]

where \( \zeta_{j-1}^* := \zeta_{j-1}^i \), and \( \alpha_j^* \) is a \( \mathcal{H}_t^{j-1} \)-measurable random variable with values in \( K \) given by

\[
\alpha_j^* = \inf \left\{ u \in K \left| \sum_{k=1}^{d} p_{\zeta_{j-1}^*}^{u} (Y_t^k - c_{\zeta_{j-1}^*}^u) = \sup_{u \in K} \left( \sum_{k=1}^{d} p_{\zeta_{j-1}^*}^{u} (Y_t^k - c_{\zeta_{j-1}^*}^u) \right) \right. \right\}.
\]

Then, we have \( a^* \in \mathcal{A}_t \) and

\[
Y_t^\ell = \mathbb{E}[J(a^*, t) | \mathcal{F}_t], \quad a.s.
\]

(iii) The following “Snell envelope” representation holds:

\[
Y_t^\ell = \text{esssup}_{a \in \mathcal{A}_t} \mathbb{E}[J(a, t) | \mathcal{F}_t], \quad a.s.
\]

**Proof.** We observe first that assertion (iii) is a direct consequence of (i) and (ii).

1. We first prove (ii).

   From the definition of \( \theta_t^* \) and Proposition 5.1, we have

   \[
   Y_t^\ell = \mathbb{E} \left[ g_t^\ell (X_T) 1_{\theta_1^* > T} + \int_{t}^{\theta_1^* \wedge T} f_t^\ell (X_s) ds + (Y_{\zeta_1^*} - c_{\zeta_1^*}^\ell) 1_{\theta_1^* \leq T} | \mathcal{F}_t \right].
   \]

   Similarly, we have

   \[
   Y_{\zeta_1^*} = \mathbb{E} \left[ g_{\zeta_1^*}^\ell (X_T) 1_{\theta_2^* > T} + \int_{\theta_1^* \wedge T}^{\theta_2^* \wedge T} f_{\zeta_1^*}^\ell (X_s) ds + (Y_{\zeta_2^*} - c_{\zeta_2^*}^\ell) 1_{\theta_2^* \leq T} | \mathcal{H}_1^{\theta_1^*} \right],
   \]

   which inserted into the previous equality leads to

   \[
   Y_t^\ell = \mathbb{E} \left[ g_t^\ell (X_T) 1_{\theta_1^* > T} + g_{\zeta_1^*}^\ell (X_T) 1_{\theta_1^* \leq \theta_2^* < T} + Y_{\zeta_2^*} 1_{\theta_2^* \leq T} | \mathcal{F}_t \right]
   \]

   \[
   + \mathbb{E} \left[ \int_{t}^{\theta_1^* \wedge T} f_t^\ell (X_s) ds + \int_{\theta_1^* \wedge T}^{\theta_2^* \wedge T} f_{\zeta_1^*}^\ell (X_s) ds - c_{\zeta_1^*}^{\ell} 1_{\theta_1^* \leq T} - c_{\zeta_2^*}^{\ell} 1_{\theta_2^* \leq T} | \mathcal{F}_t \right].
   \]

   By an easy induction argument, we have for any \( n \in \mathbb{N}^* \),

   \[
   Y_t^\ell = \mathbb{E} \left[ g_{t,n}^\ell (X_T) 1_{\theta_n^* > T} + Y_{\theta_n^*}^{t,n} 1_{\theta_n^* \leq T} + \int_{t}^{\theta_n^* \wedge T} f_{t,n}^\ell (X_s) ds - A_{\theta_n^* \wedge T} | \mathcal{F}_t \right]. \tag{5.6}
   \]
In particular, we have that
\[ E\left[A_{\theta_n^*}^{a,n,T}\right] \leq C \|Y\|_{L^2} + CT(1 + \|X\|_{L^2}) \leq C \]
where the constant $C$ does not depend on $n$. Since $(A_{\theta_n^*}^{a,n,T})_{n\in\mathbb{N}^*}$ is a non decreasing sequence of non negative r.v., we can use the monotone convergence theorem to obtain that $E\left[A_{\theta_n^*}^{a,n,T}\right] < +\infty$ which implies $a^* \in \mathcal{A}_{\mathcal{F}_t}$. Moreover, we also get that $N_{a^*} < +\infty$ a.s., so, when $n \to +\infty$ we have $\theta_n^* \wedge T \to T$ a.s., $1_{\theta_n^* \wedge T} \to 1$ a.s. and $a_n^{a,n} \to a^*_t$ for all $s \in [t,T]$. Thus, we can apply the dominated convergence theorem to take the limit (in $n$) in (5.6) and obtain
\[ Y_{\ell}^t \doteq \mathbb{E}\left[J(a^*, t) \mid \mathcal{F}_t\right], \text{ a.s.} \]

2. Now we prove (i). Let $a$ be a strategy in $\mathcal{A}_{\mathcal{F}_t}$. Using Proposition 5.1 with $\tau = t$ and $\theta = \theta_1$, we obtain
\[ Y_{\ell}^t \geq \mathbb{E}\left[g_{\ell}(X_T)1_{\theta_1 > T} + \int_t^{\theta_1 \wedge T} f_{\ell}(X_s)ds + (Y^{\alpha_{\ell_1}}_{\theta_1} - c_{\ell_1 \alpha_1}^{\alpha_{\ell_1}})1_{\theta_1 \leq T} \mid \mathcal{F}_t\right]. \]
Then, by same arguments as in the proof of (ii), we get that
\[ Y_{\ell}^t \geq \mathbb{E}\left[g_{\ell}(X_T)1_{\theta_1 > T} + Y_{\theta_1}^{\alpha_{\ell_1}}1_{\theta_1 \leq T} + \int_t^{\theta_1 \wedge T} f_{\ell}(X_s)ds - A_{\theta_1 \wedge T}^\alpha \mid \mathcal{F}_t\right]. \]
and we can take the limit (in $n$) to finally get
\[ Y_{\ell}^t \geq \mathbb{E}\left[J(a^*, t) \mid \mathcal{F}_t\right], \text{ a.s.} \]
which concludes the proof.

Corollary 5.1. There exists a unique solution $(Y, Z, \Psi) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ to the obliquely reflected BSDE (5.2).

**Proof.** From the previous results, we observe that two solution $Y$ and $Y'$ will be equal to $V$. Thus $Y$ is unique. The uniqueness of $Z$ follows classically by applying Itô’s formula to $|Y - Y'|^2$ and then the uniqueness of $\Psi$ is obvious.

Remark 5.2. 1. It is possible to obtain a uniqueness result for BSDEs with a more general generator than (5.2) by mimicking [10] and [11]. Roughly speaking, the idea is to consider a formal controlled problem where the reward function is replaced by a switched BSDE. Nevertheless, since this approach is based on comparison results for BSDEs, the generator should have a weak dependence with respect to $z$, i.e. the $i$th component of the generator only depend on $z^i$.

2. If we also add a control on the drift $b$ of the SDE as in [10] then the obliquely reflected BSDE associated to the optimal switching control problem has a generator that depends on $z$.
3. Contrarily to the classical switching problem, the optimal switching strategy can switch several times at the same time. Let us e.g. consider the simple example where there is no control on the Markov chain \((K = \{0\}), c_{ij} = 1 \text{ for all } i \neq j \in \mathbb{I}\) and

\[
p = \begin{pmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{pmatrix}.
\]

Then we easily show that there exist some points \(y \in \mathbb{R}^3\) such that

\[
y^1 = y^2 - 1 \quad \text{and} \quad y^2 = y^3 - 1.
\]

**Remark 5.3.** The classical optimal switching problem is a particular case of the randomized switching problem defined in this subsection. Indeed, it is sufficient to set \(K = \{1, \ldots, d\}, \) and, for all \(\ell \in K,\)

\[
p^{(\ell)}_{ij} = \begin{cases}
\delta_{\ell j} & \text{when } i \neq \ell, \\
\delta_{\ell \ell+1} & \text{when } i = \ell < d, \\
\delta_{1j} & \text{when } i = \ell = d,
\end{cases}
\]

and then we get

\[
\overline{D} := \{y \in \mathbb{R}^d \mid y^i \geq \sup_{\ell \in \{1, \ldots, d\}} \sum_{j=1}^{d} p^{(\ell)}_{ij}(y^j - c^{ij})\} = \{y \in \mathbb{R}^d \mid y^i \geq \max_{i \neq j} (y^j - c^{ij})\}.
\]

Let us also remark that, if we restrict the size of \(K\) then all transitions are not allowed, which is equivalent to take an infinite value for associated switching costs.

To exemplify the kind of domains we can obtain, when we consider these randomized optimal switching problems, we set \(d = 3\) and we study three possible randomisations. For each of them, we draw \(\overline{\mathcal{D}}^0\) in Figure 1, where \(\mathcal{D}^0 := \overline{D} \cap \{y^3 = 0\}\) and \(D\) is the associated domain, and we construct \(H\) if needs be. Let us remark that, starting from \(\overline{D}^0\), it is easy to recover \(\overline{D}\) by translating \(\overline{D}^0\) along the line with direction vector \((1, 1, 1).\)

1. The first example is the classical optimal switching problem obtained by defining \(K\) and \(p\) as in Remark 5.3 and by assuming that \(c^{ij} = 1\) for all \(i \neq j\). The construction of \(H\) is done in the proof of Corollary 4.1.

2. We can consider the case where the Markov chain and costs are not controled \((K = \{0\}).\) For example, we can see what happens when

\[
p = \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\]

For the construction of \(H,\) we just have to define it on the three vertices of the triangle, then extend it on all the triangle by linear interpolation and finally extend it on all \(\mathbb{R}^3.\)
by a translation along the line with direction vector $(1,1,1)$. It is easy to check that we can take $H$ that send
\[
\left( \begin{array}{c}
1 \\
-1/2 \\
-1/2 \\
\end{array} \right)
\left( \begin{array}{c}
-1/2 \\
-1/2 \\
-1/2 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \right)
to
\left( \begin{array}{c}
1 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \right)
\] at point $\left( \begin{array}{c}
0 \\
2 \\
2 \\
\end{array} \right)$,
\[
\left( \begin{array}{c}
-1/2 \\
-1/2 \\
-1/2 \\
\end{array} \right)
\left( \begin{array}{c}
-1/2 \\
1 \\
1 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \right)
to
\left( \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
1 \\
1 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \right)
\] at point $\left( \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right)$,
\[
\left( \begin{array}{c}
1 \\
-1/2 \\
-1/2 \\
\end{array} \right)
\left( \begin{array}{c}
-1/2 \\
1 \\
1 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \right)
to
\left( \begin{array}{c}
1 \\
0 \\
0 \\
\end{array} \right)
\left( \begin{array}{c}
0 \\
1 \\
1 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \right)
\] at point $\left( \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right)$.

3. By defining a more complex $K$ it is possible to round some angles. For example, we can take $K = [0,1],
\[
p^u = \left( \begin{array}{ccc}
0 & u & 1-u \\
u & 0 & 1-u \\
u & 1-u & 0 \\
\end{array} \right),
\]
and
\[
e^u = \left( \begin{array}{ccc}
0 & 1-u(1-u) & 1-u(1-u) \\
1-u(1-u) & 0 & 1-u(1-u) \\
1-u(1-u) & 1-u(1-u) & 0 \\
\end{array} \right).
\]
Let us remark that, if we restrict $K$ to $\{0,1\}$ then we get back to the first example. In particular it implies that the domain $\overline{D}_0$ obtained is included into the domain of the classical switching problem given by the first example. Let us also remark that if we consider the same framework with $c^u_{ij} = 1$ for all $i \neq j$ and $u \in K$, then we get back once again to the first example. For the construction of $H$, we can use the same approach than for the second example: we just start by defining correctly $H$ on the three corners and then extend it on the whole rounded triangle, but contrarily to the second example, we cannot use directly a linear interpolation since we have to take into account the curvature of edges. Nevertheless, it is not difficult to tackle this last problem by considering a parametrisation of edges.

References

[1] J.-F. Chassagneux, R. Elie, and I. Kharroubi. A note on existence and uniqueness for solutions of multidimensional reflected BSDEs. *Electron. Commun. Probab.*, 16:120–128, 2011.

[2] J. Cvitanic and I. Karatzas. Backward stochastic differential equations with reflection and Dynkin games. *The Annals of Probability*, pages 2024–2056, 1996.

[3] T. De Angelis, Ferrari G., and S. Hamadène. A note on a new existence result for reflected BSDEs with interconnected obstacles. [arXiv:1710.02389v1](https://arxiv.org/abs/1710.02389v1).
Figure 1: Examples of domains $\mathbb{D}^0$
[4] N. El Karoui, C. Kapoudjian, É. Pardoux, S. Peng, and M.-C. Quenez. Reflected solutions of backward SDE’s, and related obstacle problems for PDE’s. the Annals of Probability, pages 702–737, 1997.

[5] A. M. Gassous, A. Răşcanu, and E. Rotenstein. Multivalued backward stochastic differential equations with oblique subgradients. Stochastic Processes and their Applications, 125(8):3170–3195, 2015.

[6] A. Gégout-Petit and É. Pardoux. Équations différentielles stochastiques rétrogrades réfléchies dans un convexe. Stochastics Stochastics Rep., 57(1-2):111–128, 1996.

[7] D Gildbarg and NS Trudinger. Elliptic Partial Differential Equations, volume 1. 1977.

[8] S. Hamadène, J.-P. Lepeltier, and S. Peng. BSDEs with continuous coefficients and stochastic differential games. In Backward stochastic differential equations, pages 115–128. Harlow: Longman, 1997.

[9] S. Hamadène and J. Zhang. Switching problem and related system of reflected backward SDEs. Stochastic Process. Appl., 120(4):403–426, 2010.

[10] Y. Hu and S. Tang. Multi-dimensional BSDE with oblique reflection and optimal switching. Probab. Theory Related Fields, 147(1-2):89–121, 2010.

[11] N. Kazamaki. Continuous exponential martingales and BMO, volume 1579 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1994.

[12] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. Comm. Pure Appl. Math., 37(4):511–537, 1984.

[13] J. Ma and J. Zhang. Path regularity for solutions of backward stochastic differential equations. Probab. Theory Relat. Fields, 122(2):163–190, 2002.

[14] R. Martyr. Finite-horizon optimal multiple switching with signed switching costs. Math. Oper. Res., 41(4):1432–1447, 2016.

[15] D. Nualart. The Malliavin calculus and related topics. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[16] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. Systems Control Lett., 14(1):55–61, 1990.

[17] É. Pardoux and A. Răşcanu. Stochastic differential equations, Backward SDEs, Partial Differential Equations, volume 69. Springer, 2014.

[18] S. Peng. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob–Meyers type. Probability theory and related fields, 113(4):473–499, 1999.

[19] S. Ramasubramanian. Reflected backward stochastic differential equations in an orthant. Proc. Indian Acad. Sci. Math. Sci, 112(2):347–360, 2002.