Let $F$ be a field and $m \geq 2$ be an integer not divisible by the characteristic of $F$. Consider the absolute Galois group $G_F = \text{Gal}(\overline{F}/F)$, where $\overline{F}$ denotes the (separable) algebraic closure of $F$. The famous Milnor–Kato conjecture [6, 5] claims that the natural homomorphism of graded rings

$$K^n_m(F) \otimes \mathbb{Z}/m \longrightarrow H^n(G_F, \mu_m \otimes \mathbb{Z}/m^m)$$

from the Milnor K-theory of the field $F$ modulo $m$ to the Galois cohomology of $F$ with cyclotomic coefficients is an isomorphism (here, as usually, $\mu_m$ denotes the group of $m$-roots of unity in $\overline{F}$). One can see that it suffices to verify this conjecture in the case when $m$ is a prime number.

V. Voevodsky [11] proved this conjecture for $m$ equal to a power of 2. In his paper he also outlined a general approach to the Milnor-Kato conjecture for arbitrary $m$. The first step of his argument [11, Sections 5–6] dealt with a prime number $\ell$, a field $F$ having no finite extensions of degree prime to $\ell$, and an integer $n \geq 1$ such that the group $K^{n+1}_m(F)$ is $\ell$-divisible. Assuming that the conjecture holds in degree less or equal to $n$ for $m = \ell$ and any field containing $F$, Voevodsky was proving that $H^{n+1}(G_F, \mathbb{Z}/\ell) = 0$. The main goal of this paper is to give a simplified elementary version of Voevodsky’s proof of this step. In particular, we do not need to consider fields transcendental over $F$ and we make no use of motivic cohomology at all. Our main result is formulated as follows.

**Theorem 1.** Let $\ell$ be a prime number, $n \geq 1$ be an integer, and $F$ be a field of characteristic not equal to $\ell$, having no finite extensions of degree prime to $\ell$. Suppose that the homomorphism $K^n_m(L)/\ell K^n_m(L) \longrightarrow H^n(G_L, \mathbb{Z}/\ell)$ is an isomorphism for any finite extension $L$ of the field $F$. Furthermore, assume that $K^{n+1}_m(F) = \ell K^{n+1}_m(F)$. Then one has $H^{n+1}(G_F, \mathbb{Z}/\ell) = 0$.

We deduce the above theorem from two results, one related to the Milnor K-theory of fields and the other to the cohomology of Galois groups. The first one is but a version of a lemma of Suslin [10], which was also used in [11]. We only rewrote it for the Milnor K-theory groups modulo $\ell$. The second statement is essentially the “relative conjecture” of Bloch and Kato [3], which was proven for $n = 2$ by Merkurjev and Suslin [7, §15] already. It is obvious for $m = 2$ (when it holds for any pro-finite group). For odd primes $\ell$ and composite numbers $m > 2$ we show that the desired cohomological sequence is exact provided that certain Bockstein homomorphisms vanish. This is the key point of this paper.
Proposition 2. Let \( m \geq 2 \) be an integer, \( G \) be a pro-finite group, and \( H \subset G \) be a normal subgroup such that \( \Sigma = G/H \) is a cyclic group of order \( m \). Let \( T_{m^2} \) be a free module over \( \mathbb{Z}/m^2 \) endowed with an action of \( G \), and let \( T_m = mT_{m^2} = T_{m^2}/mT_{m^2} \) be the corresponding \( G \)-module over \( \mathbb{Z}/m \). Assume that for a certain \( n \geq 0 \) both Bockstein homomorphisms

\[
H^n(G, T_m) \to H^{n+1}(G, T_m) \quad \text{and} \quad H^n(H, T_m) \to H^{n+1}(H, T_m)
\]

vanish. Then the following sequence of cohomology groups is exact

\[
H^n(G, T_m) \xrightarrow{\text{res}} H^n(H, T_m)_\Sigma \xrightarrow{\text{cor}} H^n(G, T_m)
\]

\[u \cup \] \[H^{n+1}(G, T_m) \xrightarrow{\text{res}} H^{n+1}(H, T_m)_\Sigma \xrightarrow{\text{cor}} H^{n+1}(G, T_m),\]

where \( u \in H^1(G, \mathbb{Z}/m) \) is the class corresponding to the character \( G \to \Sigma \simeq \mathbb{Z}/m \).

(The letter \( \Sigma \) in the upper or lower index denotes the invariants or the coinvariants with respect to a natural action of the group \( \Sigma \).)

In the second half of this paper we apply the same techniques to obtain further exact sequences of Galois cohomology for cyclic, biquadratic, and dihedral field extensions. In particular, our method proves the biquadratic exact sequences conjectured by Merkurjev–Tignol [8] and Kahn [4]. In addition, we introduce an extended version of the classical Bass–Tate lemma [1] and deduce some corollaries about generators and relations of annihilator ideals in Galois cohomology rings.

I would like to express my gratitude to Vladimir Voevodsky for posing the problem and for very helpful conversations and explanations. He also always urged me to write down this proof. I am grateful to Alexander Vishik for numerous very useful discussions and to Roman Bezrukavnikov for one helpful remark. Most of this work was done during my stay at the Institute for Advanced Study in Princeton and the rest at the Institut des Hautes Études Scientifiques in Paris, Max-Planck-Intitut für Mathematik in Bonn and the Independent University of Moscow. I wish to thank all these four institutions. Finally, I want to thank the referees for their valuable suggestions.

1. Suslin’s Lemma for Milnor K-theory modulo \( \ell \)

The goal of this section is to deduce Theorem 1 from Proposition 2. This may be the least elementary part of this paper, to the extent that the proof of Corollary 4 is based on the existence of transfer homomorphisms on the Milnor K-theory groups. We will only use transfer maps for extensions of degree \( \ell \) of fields having no extensions of degree prime to \( \ell \). The existence of transfers for such field extensions was established in [1]; their basic properties are the base change and the projection formula.
Let us fix a prime number \( \ell \), a number \( n \geq 1 \), and a field \( F \) with \( \text{char} F \neq \ell \), having no finite extensions of degree prime to \( \ell \). Introduce the notation \( k_m(K) = K_m^M(K)/\ell \) for the Milnor K-theory groups modulo \( \ell \) of a field \( K \). For a field extension \( K \subset L \) we will denote by

\[
i_{L/K}: k_m(K) \longrightarrow k_m(L) \quad \text{and} \quad tr_{L/K}: k_m(L) \longrightarrow k_m(K)
\]

the inclusion and transfer homomorphisms.

**Lemma 3.** Assume that for any finite field extensions \( F \subset K \subset L \) with \([L : K] = \ell\) the sequence of Milnor K-theory groups

\[
k_n(K) \xrightarrow{i} k_n(L) \xrightarrow{tr} k_n(K), \quad \Sigma = \text{Gal}(L/K)
\]

is exact. Further assume that \([E : F] = \ell\) and the map \( tr_{E/F}: k_n(E) \longrightarrow k_n(F) \) is surjective. Then the sequence

\[
k_{n+1}(F) \xrightarrow{i} k_{n+1}(E) \xrightarrow{tr} k_{n+1}(F), \quad \Sigma = \text{Gal}(E/F)
\]

is also exact.

**Proof.** In order to prove that the map \( tr_{E/F}: k_{n+1}(E)\Sigma/i_{E/F}k_{n+1}(F) \longrightarrow k_{n+1}(F) \) is injective, we will construct a surjective homomorphism

\[
\phi: k_{n+1}(F) \longrightarrow k_{n+1}(E)\Sigma/i_{E/F}(k_{n+1}(F))
\]

for which \( tr_{E/F} \circ \phi = \text{id} \). This is done as follows.

By our assumptions, the map \( tr_{E/F}: k_n(E)\Sigma/i_{E/F}k_n(F) \longrightarrow k_n(F) \) is an isomorphism. Therefore, we can define a homomorphism

\[
\Phi: k_1(F) \otimes k_n(F) \longrightarrow k_{n+1}(E)\Sigma/i_{E/F}(k_{n+1}(F))
\]

by the rule \( \Phi(b \otimes a) = \overline{b}\overline{a} \), where \( b \in k_n(E) \) is such that \( tr_{E/F} \beta = \alpha \). Let us verify that \( \Phi \) can be factorized through \( k_{n+1}(F) \). It suffices to show that \( \Phi((1-a) \otimes a) = 0 \) whenever \( a \in k_n(F) \) is divisible by the class \( (a) \in k_1(F) \) of an element \( a \in F \setminus \{0,1\} \).

Let us first assume that \( a \) is not an \( \ell \)-th power in \( E \). Consider the field \( K = F(\sqrt[\ell]{a}) \); let \( L = EK \) be the field generated by \( E \) and \( K \) over \( F \). Then we have \( tr_{L/K} i_{L/E} \beta = i_{K/F} tr_E tr_{E/F} \beta = 0 \), since \( tr_{E/F} \beta = \alpha \) is divisible by \( (a) \). By assumption, it follows that \( i_{L/E} \beta \in i_{L/K} k_n(K) + (1-\sigma)k_n(L) \), where \( \sigma \) is a generator of the group

\[
\text{Gal}(L/K) = \text{Gal}(E/F).
\]

Now we have

\[
(1-a) \cdot \beta = \ tr_{L/E}((1-\sqrt[\ell]{a})) \cdot \beta = \ tr_{L/E}((1-\sqrt[\ell]{a}) \cdot i_{L/E} \beta)
\]

\[
\subset \ tr_{L/E}(i_{L/K} k_{n+1}(K) + (1-\sigma)k_{n+1}(L)) \subset i_{E/F} k_{n+1}(F) + (1-\sigma)k_{n+1}(E),
\]

hence \( \Phi((1-a) \otimes a) = 0 \).
It remains to consider the case \( \sqrt[n]{a} \in E \). In this case we have \( \sum_{s=1}^{\ell} \sigma^s(\beta) = i_{E/F} \cdot tr_{E/F} \beta = i_{E/F} \alpha = 0 \) and therefore
\[
(1 - a) \cdot \beta = \sum_{s=1}^{\ell} \sigma^s(1 - \sqrt[n]{a}) \cdot \beta = \sum_{s=1}^{\ell} (1 - \sqrt[n]{a}) \cdot \sigma^{-s}(\beta) = 0
\mod (1 - \sigma)k_{n+1}(E).
\]

Thus we have constructed the desired map \( \phi \). Since the field \( F \) has no finite extensions of degree less than \( \ell \), we have \( k_{n+1}(E) = k_1(F) \cdot k_n(E) \) (see [1] and section 4); hence the map \( \phi \) is surjective. It is obvious that \( tr_{E/F} \circ \phi = id. \)

In the remaining part of this section (which is included for the sake of completeness of the exposition) we closely follow [11, Section 5].

**Corollary 4.** Assume that for any finite field extensions \( F \subset K \subset L \) with \( [L : K] = \ell \) the following sequence of Milnor K-theory groups is exact
\[
k_n(K) \xrightarrow{i} k_n(L)_{\Sigma} \xrightarrow{tr} k_n(K) \xrightarrow{u} k_{n+1}(K),
\]
where \( u \in k_1(K) \) is the element corresponding to the cyclic extension \( K \subset L \) and \( \Sigma = Gal(L/K) \). Further assume that \( k_{n+1}(F) = 0 \). Then one has \( k_{n+1}(E) = 0 \) for any finite field extension \( F \subset E \).

**Proof.** Proceeding by induction, we only have to consider the case when \( [E : F] = \ell \). Since \( k_{n+1}(F) = 0 \), it follows from our exact sequence that the transfer homomorphism \( tr_{E/F} : k_n(E) \rightarrow k_n(F) \) is surjective. Therefore, the conditions of Lemma 3 are satisfied and we can conclude that \( k_{n+1}(E)_{\Sigma} = 0 \). Thus \( k_{n+1}(E) = (1 - \sigma)k_{n+1}(E) = (1 - \sigma)^{\ell}k_{n+1}(E) = 0. \)

**Proof of Theorem 1.** Let \( F \subset K \subset L \) be finite field extensions with \( [L : K] = \ell \). We will apply Proposition 2 for \( m = \ell \), the pro-finite group \( G = G_K \), the subgroup \( H = G_L \), the modules \( T_\ell = \mu_{\ell^n} \) and \( T_{\ell} \simeq \mathbb{Z}/\ell \) over \( G \), and the degree \( n \). It follows from the assumptions of Theorem 1 that the required Bockstein homomorphisms vanish. From Proposition 2 we conclude that the conditions of Corollary 4 are satisfied for the field \( F \). Thus we have \( k_{n+1}(E) = 0 \) for any field \( E \) finite over \( F \).

Now assume that \( \alpha \) is a nonzero element of \( H^{n+1}(G_F, \mathbb{Z}/\ell) \). It is clear that there exist finite field extensions \( F \subset K \subset L \) such that \( res_{K/F} \alpha \neq 0 \), but \( res_{L/F}(\alpha) = 0 \) and \( [L : K] = \ell \). Using Proposition 2 again, we see that \( res_{K/F} \alpha = u_{L/K} \cup \beta \) for some \( \beta \in H^n(G_K, \mathbb{Z}/\ell) \). Since \( k_n(K) \simeq H^n(G_K, \mathbb{Z}/\ell) \) and \( k_{n+1}(K) = 0 \), it follows that \( res_{K/F} \alpha = 0 \). This contradiction proves that \( H^{n+1}(G_F, \mathbb{Z}/\ell) = 0. \)

\( \square \)
2. The Six-Term Cohomological Exact Sequence

In this section we develop a rather general setting for exact sequences of cohomology under the assumptions of vanishing of Bockstein homomorphisms. The results below are actually valid for an arbitrary cohomological functor on an abelian category endowed with a central element with zero square, etc. We restrict ourselves to the cohomology of a pro-finite group for the sake of simplicity of exposition only.

Let us fix a pro-finite group $G$ and a number $m \geq 2$. We will use the following notation: $A_2$, $B_2$, etc. will denote free modules over $\mathbb{Z}/m^2$ endowed with a discrete action of $G$ and $A_1$, $B_1$, etc. will be the corresponding $G$-modules over $\mathbb{Z}/m$, defined as $A_1 = mA_2 = A_2/mA_2$. For a $G$-module $M$, we will simply write $H^n(G, M)$.

The Bockstein homomorphisms corresponding to exact triples

$$0 \longrightarrow M_1 \overset{\tau}{\longrightarrow} M_2 \overset{\pi}{\longrightarrow} M_1 \longrightarrow 0, \quad \tau \pi = m$$

will be denoted by $\beta^n_M : H^n(M_1) \longrightarrow H^{n+1}(M_1)$.

**Lemma 5.** Let $0 \rightarrow X_2 \overset{i}{\longrightarrow} Y_2 \overset{p}{\longrightarrow} Z_2 \rightarrow 0$ be an exact triple of $G$-modules over $\mathbb{Z}/m^2$, $0 \rightarrow X_1 \longrightarrow Y_1 \longrightarrow Z_1 \rightarrow 0$ be the corresponding exact triple of $G$-modules over $\mathbb{Z}/m$, and $\partial_{XZ} : H^n(Z_1) \longrightarrow H^{n+1}(X_1)$ be the induced homomorphism of pro-finite group cohomology.

Let $h$ be a homotopy map on the complex $X_2 \rightarrow Y_2 \rightarrow Z_2$ such that $dh + hd = m \cdot \text{id}$, where $d = (i, p)$. Then the induced map $h_1$ is an endomorphism of degree $-1$ of the complex $X_1 \rightarrow Y_1 \rightarrow Z_1$. It follows that there is a map $h_{XZ} : Z_1 \rightarrow X_1$ such that $h_{XZ} \circ p_1 = h_1 : Y_1 \rightarrow X_1$ and $i_1 \circ h_{XZ} = -h_1 : Z_1 \rightarrow Y_1$.

Then the following equality holds: $\partial_{XZ} = \beta_X \circ H(h_{XZ}) - H(h_{XZ}) \circ \beta_Z$.

**Proof.** The desired equality of homomorphisms of cohomology is to be understood as the corollary to an identity in the $G$-module extension group $\text{Ext}^1_{\mathbb{Z}/m^2[G]}(Z_1, X_1)$, where $\partial_{XZ}$ and the betas represent certain extension classes and $h_{XZ}$ is just a map of modules. This identity in the group of extension classes is verified as follows.

The composition of the Bockstein extension $X_1 \rightarrow X_2 \rightarrow X_1$ with $h_{XZ} : Z_1 \rightarrow X_1$ is an extension of the form $X_1 \rightarrow X_2 \cap X_1, Z_1 \rightarrow Z_1$, while the composition of $Z_1 \rightarrow Z_2 \rightarrow Z_1$ with $h_{XZ}$ is $X_1 \rightarrow X_1 \cup Z_1, Z_2 \rightarrow Z_1$ (where $\cap$ and $\cup$ denote the relative product and coproduct, respectively). We have to check that the difference of these extensions is isomorphic to $X_1 \rightarrow Y_1 \rightarrow Z_1$. The middle term of the desired difference can be defined as the homology module of the sequence

$$X_1 \overset{(\tau, 0, \text{id}, 0)}{\longrightarrow} (X_2 \cap X_1, Z_1) \oplus (X_1 \cup Z_1, Z_2) \overset{(0, \text{id}, 0, -\pi)}{\longrightarrow} Z_1.$$

Then the arrows $j$ and $q$ forming this extension are induced by

$$X_1 \overset{(\tau, 0, 0, 0)}{\longrightarrow} (X_2 \cap X_1, Z_1) \oplus (X_1 \cup Z_1, Z_2) \quad \text{and} \quad (X_2 \cap X_1, Z_1) \oplus (X_1 \cup Z_1, Z_2) \overset{(0, 0, 0, \pi)}{\longrightarrow} Z_1,$$
respectively. Let us construct an homomorphism $f$ from $Y_1$ to the homology module in the middle. Given an element $y_1 \in Y_1$, we choose its preimage $y_2 \in Y_2$ and set $f(y_1) = (h(y_2), p_1(y_1)) \oplus (h_1(y_1), p(y_2))$. It is obvious that $(0, \text{id}, 0, -\pi) \circ f(y_1) = p_1(y_1) - \pi p(y_2) = 0$, since $p$ commutes with $\pi$. Now let us check that $f$ is well-defined. It suffices to restrict oneself to the case when $y_1 = 0$ and $y_2 = \tau(y'_1)$. By the above formula, we have $f(y_1) = (h\tau(y'_1), 0) \oplus (0, p\tau(y'_1)) = (\tau h_1(y'_1), 0) \oplus (0, \tau p_1(y'_1)) = (\tau h_1(y'_1), 0) \oplus (h_1(y'_1), 0) = (\tau, 0, \text{id}, 0)(h_1(y'_1))$. It remains to check commutativity of the triangles formed by the maps $f$, $i_1$, $j$ and $f$, $p_1$, $q$. For any $x_1 \in X_1$, choose a preimage $x_2 \in X_2$ and take $y_2 = i(x_2)$ for $y_1 = i_1(x_1)$. By the same formula, we have $f(i_1(x_1)) = (hi_2(x_2), 0) \oplus (h_1i_1(x_1), 0) = (mx_2, 0) \oplus (0, 0) = (\tau(x_1), 0) \oplus (0, 0) = j(x_1)$. Finally, for any $y_1 \in Y_1$ with a preimage $y_2 \in Y_2$, we have $(q \circ f)(y_1) = \pi p(y_2) = p_1(y_1).

Theorem 6. Let $G$ be a pro-finite group, $m \geq 2$ and $n \geq 0$ be some integers, and

$$0 \longrightarrow A_2 \longrightarrow B_2 \longrightarrow C_2 \longrightarrow D_2 \longrightarrow 0$$

be a 4-term exact sequence of free $\mathbb{Z}/m^2$-modules with a discrete action of $G$ in it. Suppose that we are given a homotopy map $h$ in this exact sequence such that $dh + hd = m$. Assume that the Bockstein maps

$$\beta^n_X : H^n(G, X_1) \longrightarrow H^{n+1}(G, X_1), \quad X_1 = mX_2 = X_2/m$$

vanish for all $X = A, B, C, or D$ and the given $n$. Then there is a 6-term exact sequence of the form

$$H^n(B_1 \oplus D_1) \xrightarrow{(d_1, h_1)} H^n(C_1) \xrightarrow{d_1} H^n(D_1) \xrightarrow{\nu} H^{n+1}(A_1) \xrightarrow{d_1} H^{n+1}(B_1) \xrightarrow{(h_1, d_1)} H^{n+1}(A_1 \oplus C_1).$$

The middle arrow $\nu$ comes from a certain extension

$$0 \longrightarrow A_1 \longrightarrow N \longrightarrow D_1 \longrightarrow 0$$

of discrete $G$-modules over $\mathbb{Z}/m$, whose compositions with the morphisms $A_1 \longrightarrow B_1$ and $C_1 \longrightarrow D_1$ are trivial. If the group $G$ acts on the original exact quadruple of modules through its finite quotient group $\Gamma$, then $N$ is a $\mathbb{Z}/m[\Gamma]$-module.

Proof. Let $L_2$ denote the image of the differential $B_2 \longrightarrow C_2$. It is clear that $L_2$ is a free module over $\mathbb{Z}/m^2$. Now our 4-term exact sequence of $G$-modules consists of two exact triples: $A_2 \longrightarrow B_2 \longrightarrow L_2$ and $L_2 \longrightarrow C_2 \longrightarrow D_2$. Reducing modulo $m$, we see that the map $h_1 : C_1 \longrightarrow B_1$ defines a homomorphism from the exact triple $L_1 \longrightarrow C_1 \longrightarrow D_1$ to $A_1 \longrightarrow B_1 \longrightarrow L_1$. The corresponding “intermediate” induced extension provides the desired exact triple $A_1 \longrightarrow N \longrightarrow D_1$.

Now the 6-term sequence is constructed; it remains to check that it’s exact. Let us introduce notation for maps $p: B_2 \longrightarrow L_2$ and $i : L_2 \longrightarrow C_2$; the same maps reduced
modulo $m$ will be denoted by $p_1$ and $i_1$. It is easy to see that the homotopy map $h$ induces analogous homotopies on the exact triples $A_2 \rightarrow B_2 \rightarrow L_2$ and $L_2 \rightarrow C_2 \rightarrow D_2$. Therefore, both triples satisfy the conditions of Lemma 5. We have connecting maps $\partial_{AL}: H^n(L_1) \rightarrow H^{n+1}(A_1)$ and $\partial_{LD}: H^n(D_1) \rightarrow H^{n+1}(L_1)$ in the pro-finite group cohomology; the cohomology maps induced by the module maps $h_{AL}: L_1 \rightarrow A_1$ and $h_{LD}: D_1 \rightarrow L_1$ will be denoted simply by $h_{AL}$ and $h_{LD}$. According to Lemma 5, there are two identities: $\partial_{LD} = \beta_L \circ h_{LD} - h_{LD} \circ \beta_D$ and $\partial_{AL} = \beta_A \circ h_{AL} - h_{AL} \circ \beta_L$. We also know that $\nu = \partial_{AL} \circ h_{LD} = -h_{AL} \circ \partial_{LD}$.

Assume that $x \in H^n(D_1)$ and $\nu(x) = 0$. Let us prove that $x \in d_1 H^n(C_1)$. We have $\partial_{AL} h_{LD}(x) = 0$, hence $h_{LD}(x) = p_1(z)$ for some $z \in H^n(B_1)$. Since $p_1$ is induced by a map of modules over $\mathbb{Z}/m^2$, it commutes with betas. So it follows from our assumptions about vanishing of Bockstein homomorphisms that $0 = p_1(\beta_B(z) = \beta_L h_{LD}(x) = \beta_L h_{LD}(x) - h_{LD} \nu(x) = \partial_{LD}(x)$. Thus $x \in d_1 H^n(C_1)$.

Next assume that $w \in H^{n+1}(A_1)$ and $d_1(w) = 0$. Then $w = \partial_{AL}(u)$ for some $u \in H^n(L_1)$. Hence $w = \beta_A h_{AL}(u) - h_{AL} \beta_L(u) = -h_{AL} \beta_L(u)$. The map $i_1$ commutes with betas for the same reason as above, so according to our assumption about Bockstein maps we have $i_1 \beta(u) = \beta_C i_1(u) = 0$. It follows that $\beta_L(u) = \partial_{LD}(x)$ for some $x \in H^n(D_1)$. Thus $w = -h_{AL} \beta_L(u) = -h_{AL} \partial_{LD}(x) = \nu(x)$.

Now assume that $y \in H^n(C_1)$ and $d_1(y) = 0$. Let us prove that $y \in d_1 H^n(B_1) + h_1 H^n(D_1)$. Since $d_1(y) = 0$, we have $y = i_1(u)$ with $u \in H^n(L_1)$. Then $i_1 \beta_L(u) = \beta_C i_1(u) = 0$, hence $\beta_L(u) = \partial_{LD}(x)$ for some $x \in H^n(D_1)$. Therefore, $\partial_{AL}(u) = \beta_A h_{AL}(u) - h_{AL} \beta_L(u) = -h_{AL} \beta_L(u) = -h_{AL} \partial_{LD}(x) = \partial_{AL} h_{LD}(x)$. It follows that $\partial_{AL}(u - h_{LD}(x)) = 0$ and $u - h_{LD}(x) = p_1(z)$ for some $z \in H^n(B_1)$. Applying $i_1$, we obtain $y = i_1 p_1(z) + i_1 h_{LD}(x) = d_1(z) - h_1(x)$.

Finally, assume that $t \in H^{n+1}(B_1)$ is such that $d_1(t) = 0$ and $h_1(t) = 0$. Then $i_1 p_1(t) = d_1(t) = 0$, hence $p_1(t) = \partial_{LD}(x)$ for some $x \in H^n(D_1)$. We have $\partial_{AL} h_{LD}(x) = -h_{AL} \partial_{LD}(x) = -h_{AL} p_1(t) = -h_1(t) = 0$, so $h_{LD}(x) = p_1(z)$ with $z \in H^n(B_1)$. It follows that $p_1(t) = \partial_{LD}(x) = \beta_L h_{LD}(x) - h_{LD} \beta_D(x) = \beta_L h_{LD}(x) = \beta_L p_1(z) = p_1 \beta_B(z) = 0$ and therefore $t \in d_1 H^{n+1}(A_1)$. 

Remark. The middle arrow $\nu$ does not depend on the choice of a homotopy map $h$, provided that $h$ satisfies the conditions of Theorem 6. Indeed, let us consider another homotopy map $h' = h + t$, where $dt + td = 0$. Then the induced maps $t_{AL}$ and $t_{LD}$ commute with Bockstein homomorphisms, and according to the above computations we have $\nu' - \nu = \partial_{AL} \circ t_{LD} = \beta_A \circ h_{AL} \circ t_{LD} - h_{AL} \circ \beta_L \circ t_{LD} = \beta_a \circ h_{AL} \circ t_{LD} - h_{AL} \circ t_{LD} \circ \beta_D = 0$ on $H^n(D_1)$. If $A_2$ and $D_2$ are permutational modules (as in all applications below), then $h_{AL} \circ t_{LD}$ can be lifted to a map $D_2 \rightarrow A_2$ and it follows that the extension $A_1 \rightarrow N \rightarrow D_1$ does not depend on $h$ either.
3. The Relative Conjecture of Bloch and Kato

In this section we will prove two theorems providing exact sequences for Galois cohomology of cyclic field extensions, both of them generalizations of Proposition 2. The first one will be essentially the “relative conjecture” of Bloch and Kato [3, Conjecture 3.1]. In particular, the proof of Theorem 1 will be completed.

Throughout this section, we will use the following notation: $m \geq 2$ is an integer, $G$ is a pro-finite group, $T_{m^2}$ is a free module over $\mathbb{Z}/m^2$ endowed with an action of $G$, and $T_m = mT_{m^2} = T_{m^2}/mT_{m^2}$ is the corresponding $G$-module over $\mathbb{Z}/m$. Furthermore, $n \geq 0$ is another integer, and it is assumed that the Bockstein homomorphisms $H^n(H, T_{m^2}) \xrightarrow{\text{res}} H^{n+1}(H, T_{m^2})$ vanish for all relevant open subgroups $H$ in $G$, including $G$ itself. In applications to Galois cohomology it will be always meant that $G = G_F$ is the absolute Galois group of a field $F$ and $T_{n^2} = \mu_{n^2}$ is the cyclotomic module.

**Proposition 7.** Let $H \subset G$ be a normal subgroup such that $\Sigma = G/H$ is a cyclic group of an order $k$ dividing $m$. Then the following sequence of cohomology groups is exact

$$H^n(H, T_m) \xrightarrow{m/k \cdot \text{res}} H^n(H, T_{m^2}) \xrightarrow{\text{cor}} H^n(G, T_m) \xrightarrow{u \cup} H^{n+1}(G, T_m) \xrightarrow{\text{res}} H^{n+1}(H, T_{m^2}) \xrightarrow{m/k \cdot \text{cor}} H^{n+1}(G, T_m),$$

where $u \in H^1(G, \mathbb{Z}/m)$ is the class corresponding to the character $G \rightarrow \Sigma \hookrightarrow \mathbb{Z}/m$.

**Proof.** Consider the following very well known 4-term exact sequence of modules over the cyclic group $\Sigma$

$$0 \xrightarrow{} \mathbb{Z} \xrightarrow{1 + \cdots + \sigma^{k-1}} \mathbb{Z}[\Sigma] \xrightarrow{1-\sigma} \mathbb{Z}[\Sigma] \xrightarrow{\sigma^i \rightarrow 1} \mathbb{Z} \xrightarrow{} 0,$$

where $\sigma$ is a generator of $\Sigma$. Let us construct a homotopy map $h_k$ such that $dh_k + h_k \cdot d = k$. Set $f(x)$ to be the polynomial for which $k - (1 + \cdots + x^{k-1}) = (1-x)f(x)$. The leftmost component of $h_k$ sends $\sigma^i$ to 1 for all $i$, the middle part is given by the multiplication with $f(\sigma)$ in the group ring $\mathbb{Z}[\Sigma]$, and the rightmost component sends 1 to $1 + \cdots + \sigma^{k-1}$. For the purposes of our proof it suffices to put $h = m/k \cdot h_k$, tensor the above exact quadruple with $T_{m^2}$, and apply Theorem 6.

The proof of Theorem 1 is now finished.

**Proposition 8.** Let $H \subset G$ be a normal subgroup such that $\Sigma = G/H$ is a cyclic group of an order $k$ divisible by $m$. Let $\Pi \subset \Sigma$ be the cyclic subgroup of order $m$ and $K \subset G$ be kernel of the projection $G \rightarrow \Sigma/\Pi$. Then the following sequences of cohomology groups are exact

$$H^n(H \Sigma) \xrightarrow{\text{cor}} H^n(K \Sigma/\Pi) \xrightarrow{\text{cor} \circ u \cup} H^{n+1}(G) \xrightarrow{\text{res}} H^{n+1}(H \Sigma) \xrightarrow{\text{cor}} H^{n+1}(K \Sigma/\Pi)$$
and
\[ H^n(K)_{\Sigma/P} \xrightarrow{\text{res}} H^n(H)_\Sigma \xrightarrow{\text{cor}} H^n(G) \xrightarrow{u \cup \circ \text{res}} H^{n+1}(K)_{\Sigma/P} \xrightarrow{\text{res}} H^{n+1}(H)_\Sigma, \]
where we write \( H^i(I) = H^i(I, T_m) \) for any subgroup \( I \subset G \), and \( u \in H^1(K, \mathbb{Z}/m) \) is the cohomology class corresponding to the character \( K \rightarrow \Pi \simeq \mathbb{Z}/m \).

**Proof.** The first of the desired exact sequences holds due to the following exact quadruple \( Q_\Sigma \) of permutational \( \Sigma \)-modules
\[ \mathbb{Z} \longrightarrow \mathbb{Z}[\Sigma] \oplus \mathbb{Z} \longrightarrow \mathbb{Z}[\Sigma] \oplus \mathbb{Z}[\Sigma/\Pi] \longrightarrow \mathbb{Z}[\Sigma/\Pi], \]
where the first map is \((1 + \cdots + \sigma^{k-1}, -m)\), the third map is \((\text{pr}_\Pi, -1 + \bar{\sigma})\), and the middle arrow is given by the matrix \( U \) with components \( U_{11} = 1 - \sigma \), \( U_{12} = 0 \), \( U_{21} = \text{pr}_\Pi \), and \( U_{22} = 1 + \cdots + \bar{\sigma}^{k/m-1} \). Here \( \sigma \) is a generator of \( \Sigma \), by \( \bar{\sigma} \) we denote its image in \( \Sigma/\Pi \), and \( \text{pr}_\Pi \) is the projection map \( \mathbb{Z}[\Sigma] \longrightarrow \mathbb{Z}[\Sigma/\Pi] \). Let us construct a homotopy map \( h \) with \( dh + hd = m \). Set \( f(x) \) to be the polynomial for which \( m - (1 + x^{k/m} + \cdots + x^{(m-1)k/m}) = (1 - x)f(x) \). Denote by \( \#\Pi \) the lifting map \( \mathbb{Z}[\Sigma/\Pi] \longrightarrow \mathbb{Z}[\Sigma] \) which sends \( \bar{\sigma}^i \) to \((1+\sigma^{k/m}+\cdots+\sigma^{(m-1)k/m})\bar{\sigma}^i \). Then the leftmost piece of \( h \) is \((0, -1)\), the rightmost piece is \((\#\Pi, -f(\bar{\sigma}))\), and the middle map is given by the matrix \( H \) with entries \( H_{11} = f(\sigma) \), \( H_{12} = \#\Pi \), \( H_{21} = 0 \), and \( H_{22} = 0 \). Using this exact quadruple, one argues as in the proof of Proposition 7 above. To obtain the second exact sequence of cohomology, one has to use the dual exact quadruple of \( \Sigma \)-modules \( \text{Hom}_{\mathbb{Z}}(Q_\Sigma, \mathbb{Z}) \). \( \square \)

4. The Bass–Tate Lemma and Applications to Annihilator Ideals

The next theorem provides a more sophisticated version of the classical technique known as the “Bass–Tate lemma” [1].

**Theorem 9.** Let \( E/F \) be a separable extension of fields of degree \( k \). Assume that the field \( F \) has no nontrivial finite extensions of degrees less or equal to \( k/2 \). Then \( K^M_\geq 1(E) = \bigoplus_{n=1}^\infty K^M_n(E) \) is a quadratic module over the ring \( K^M_n(F) \), i. e., a graded module generated in degree 1 with relations in degree 2.

**Lemma 10.** Let \( E = F[t] \) be a finite extension of fields of degree \( k \), where \( t \in E \) is a generator. Then any element of the field \( E \) can be represented as a fraction of polynomials \( f(t)/g(t) \), where \( f \) and \( g \) are two polynomials of degree \( \leq k/2 \) each.

**Proof.** Let \( V \subset E \) be the vector subspace of all elements of the form \( f(t) \), where \( \deg f \leq k/2 \). Then \( 2 \dim V \geq k + 1 \), thus for any element \( e \in E \) one has \( V \cap eV \neq 0 \). This simple argument was communicated to the author by Alexander Vishik. \( \square \)
Lemma 11. Let $A \rightarrow B$ be a surjective homomorphism of abelian groups. Then the kernel $I^*$ of the induced map of exterior rings $\Lambda^*(A) \rightarrow \Lambda^*(B)$ is generated in degree 1 as an ideal in $\Lambda^*(A)$.

Proof. Consider an exterior expression of elements of $A$ which becomes zero in $\Lambda^*(B)$. There is a chain of transformations (by the rules of exterior algebra) of exterior expressions of elements of $B$ connecting this image with the zero. It is clear that one would be able to lift this chain of transformations to the original expression in $A$ if one only were allowed to freely change the elements of $A$ in the expressions with other elements of $A$ with the same image in $B$. In other words, the ideal $I^*$ is additively generated by exterior monomials $a_1 \wedge \cdots \wedge a_n$ where one of the $a_i$’s belongs to the kernel of $A \rightarrow B$. But this is exactly what we wanted to prove. \qed

Proof of Theorem 9. Consider the field of rational functions $F(x)$ in a transcendental variable $x$. Let $D \subset F(x)^*$ be the multiplicative subgroup generated by polynomials of degree $\leq 1$. Let $R_\ast = \bigoplus_{n = 0}^\infty R_n$ be the graded anti-commutative ring generated by $R_1 = D$ with relations $\{f, 1 - f\} = 0$ for all $f, 1 - f \in D$ and $\{f, -f\} = 0$ for all $f \in D$. Arguing as in [1, Theorem I.5.1], it is not hard to show that $R_\ast$ is a free $K^M_\ast(F)$-module with a basis consisting of $1$ and $\{x - a\}$ for all $a \in F$. Hence the $K^M_\ast(F)$-module $R_{\geq 1}$ is the direct sum of $K^M_{\geq 1}(F)$ and a free module, so it is quadratic.

Now let us choose an element $t \in E$ such that $E = F[t]$. Let $\pi(x) \in F[x]$ be the irreducible equation of the element $t$. Obviously, there is a homomorphism $p_\pi : R_\ast \rightarrow K^M_\ast(E)$ assigning $\{f \text{ mod } \pi\}$ to $\{f\}$ for any $f \in D$. According to Lemma 10, this homomorphism is surjective. Moreover, the ideal $J_\ast = \ker p_\pi \subset R_\ast$ is generated by its first-degree component $J_1$. Indeed, the kernel of the map of exterior algebras $\Lambda^*(D) \rightarrow \Lambda^*(E^\ast)$ is generated in degree 1 by Lemma 11, and it remains to show that any Steinberg element in $\Lambda^2(E^\ast)$ can be lifted to one of the relations defining $R$. This follows from the fact, again established by Lemma 10, that for any $e \in E \setminus \{0, 1\}$ there exists $f \in D \subset F(x)^*$ such that $1 - f \in D$ and $\pi(f) = e$.

Since $R_\ast$ is anti-commutative, we have

$$J_\ast = R_\ast \cdot J_1 = (K^M_\ast(F) + K^M_\ast(F) \cdot R_1) \cdot J_1 = K^M_\ast(F) \cdot (J_1 + R_1 J_1).$$

Therefore, the $K^M_\ast(F)$-module $J_\ast$ is generated by $J_1$ and $J_2$, while the module $R_{\geq 1}$ is quadratic. Thus the quotient module $K^M_{\geq 1}(E) = R_{\geq 1}/J_\ast$ is also quadratic. \qed

For reader’s convenience, the statement of Bass–Tate lemma proper is formulated here, together with an improvement by Becher [2].

Proposition 12. Let $E/F$ be a separable field extension of degree $k = \ell^s$, where $\ell$ is a prime number. Assume that either (a) $\ell = 2$, or (b) the field $F$ has no finite extensions of degree different from powers of $\ell$ and less or equal to $k/2$, or (c) $E/F$ is a tower of extensions of degree $\ell$ such that the largest proper subfield of $E$ in
this tower has no nontrivial finite extensions of degree \( \leq \ell/2 \). Then the Milnor ring \( K_M^*(E) \) is generated in degree \( \leq s \) as a module over \( K_M^*(F) \).

Proof. The case (a) is covered by the result of [2]. The case (b) follows from Lemma 10 and [1, Lemma I.5.2]. In the case (c) one argues by induction in \( s \), applying (b) to extensions of degree \( \ell \). \( \square \)

From now on and for the rest of this section we will assume that the Milnor–Kato conjecture holds for all relevant fields, primes, and degrees. Under this assumption we will deduce corollaries from Proposition 7, Proposition 8, and Theorem 9.

Corollary 13. Let \( \ell \) be a prime number, \( m \) be a power of \( \ell \), and \( F \) be a field containing a primitive root of unity of degree \( m \). Consider the ring \( H^*(G_F, \mathbb{Z}/m) \). Let \( u \in H^1(G_F, \mathbb{Z}/m) \) be an element of additive order \( k = \ell^s \) and \( E/F \) be the corresponding cyclic field extension of degree \( k \). Assume that the maximal proper subfield of \( E \) over \( F \) has no nontrivial finite extensions of degree \( \leq \ell/2 \). Then the ideal \( \text{Ann}(u) \) in the ring \( H^*(G_F, \mathbb{Z}/m) \) is generated by elements of degree less or equal to \( s \). In particular, if \( m = \ell \) is a prime and \( F \) has no nontrivial extensions of degree \( \leq \ell/2 \), then \( \text{Ann}(u) \) is generated in degree 1.

Proof. According to Proposition 7, we have an exact sequence of \( H^*(G_F, \mathbb{Z}/m) \)-modules

\[
H^*(G_E, \mathbb{Z}/m) \xrightarrow{\text{cor}} H^*(G_F, \mathbb{Z}/m) \xrightarrow{u \cup} H^*(G_F, \mathbb{Z}/m).
\]

But \( H^*(G_E, \mathbb{Z}/m) \) is generated in degree \( \leq s \) as a module over \( H^*(G_F, \mathbb{Z}/m) \); this follows from the Milnor–Kato conjecture and Proposition 12. \( \square \)

Corollary 14. Let \( \ell \) be a prime number, \( m \) and \( k \) be powers of \( \ell \), and \( F \) be a field containing a primitive root of unity of degree \( m \). Let \( L/F \) be a cyclic extension of degree \( k \). Consider the kernel \( J \) of the ring homomorphism \( H^*(G_F, \mathbb{Z}/m) \to H^*(G_L, \mathbb{Z}/m) \). If \( k \) is divisible by \( m \), consider the intermediate extension \( F \subset E \subset L \) of degree \( k/m \) over \( F \) and assume that the maximal proper subfield of \( E \) over \( F \) has no nontrivial finite extensions of degree \( \leq \ell/2 \). Then the ideal \( J \) in the ring \( H^*(G_F, \mathbb{Z}/m) \) is generated by elements of degree less or equal to \( s+1 \), where \( k/m = \ell^s \) if \( k \) is divisible by \( m \) and \( s = 0 \) if \( m \) is divisible by \( k \).

Proof. The case when \( k \) is divisible by \( m \) follows from Proposition 8 in the way analogous to the proof of Corollary 13. Namely, one has to use the exact sequence

\[
H^n(G_E, \mathbb{Z}/m) \xrightarrow{\text{cor} \circ u \cup} H^{n+1}(G_F, \mathbb{Z}/m) \xrightarrow{\text{res}} H^{n+1}(G_L, \mathbb{Z}/m),
\]

where \( u \in H^1(G_E, \mathbb{Z}/m) \) is the class of the extension \( L/E \). The case when \( m \) is divisible by \( k \) follows directly from Proposition 7. \( \square \)

The next corollary is the main result of this section.
Corollary 15. Let $\ell$ be a prime number and $F$ be a field containing a primitive root of unity of degree $\ell$. Assume that $F$ has no nontrivial finite extensions of degree $\leq \ell/2$ (note that for $\ell = 2$ or $\ell = 3$ this always holds). Then for any element $u \in H^1(G_F, \mathbb{Z}/\ell)$ the ideal $\text{Ann}(u)$ in the cohomology ring $H^*(G_F, \mathbb{Z}/\ell)$ is a quadratic module over this ring, i.e., it is isomorphic to a graded module with generators and relations, where generators are in degree 1 and relations in degree 2.

Proof. Let $E/F$ be the cyclic extension of degree $\ell$ corresponding to the class $u$ and $\Sigma$ be its Galois group. Consider the spectral sequence related to the maximal ($\ell$-term) filtration of the $G_F$-module $\mathbb{Z}/\ell[\Sigma]$. It follows from the exact sequence

$$H^n(G_E, \mathbb{Z}/\ell) \xrightarrow{\text{cor}} H^n(G_F, \mathbb{Z}/\ell) \xrightarrow{u \cup} H^{n+1}(G_F, \mathbb{Z}/\ell) \xrightarrow{\text{res}} H^{n+1}(G_E, \mathbb{Z}/\ell)$$

that this spectral sequence degenerates at the term $E_2^{r,s}$. In other words, there is a filtration $V^1 \supset \cdots \supset V^\ell$ on the $H^*(G_F, \mathbb{Z}/\ell)$-module $H^*(G_E, \mathbb{Z}/\ell)$ with associated quotient modules of the form $V_i/V^i \simeq \ker(u \cup)/\text{im}(u \cup)$ for $i = 2, \ldots, \ell - 1$, and $V^\ell \simeq \text{coker}(u \cup)$. From this filtration it follows that the $H^*(G_F, \mathbb{Z}/\ell)$-modules $H^{>1}(G_E, \mathbb{Z}/\ell)$ and $\text{Ann}(u)$ are quadratic simultaneously. (They are also simultaneously Koszul, see [9].) Now it remains to apply Theorem 9. Alternatively, one can argue as in the proof of Corollary 20 below. \(\square\)

Conjecture 16. Let $\ell$ be a prime number and $F$ be a field containing a primitive root of unity of degree $\ell$. Then for any element $u \in H^1(G_F, \mathbb{Z}/\ell)$ the $H^*(G_F, \mathbb{Z}/\ell)$-module $\text{Ann}(u) \subset H^*(G_F, \mathbb{Z}/\ell)$ is Koszul. For any cyclic extension $E/F$ of degree $\ell$, the $H^*(G_F, \mathbb{Z}/\ell)$-module $H^{>1}(G_E, \mathbb{Z}/\ell)$ is Koszul (see [9] for the definition).

5. Dihedral Field Extensions

Using Theorem 6 and assuming the Milnor–Kato conjecture, one can construct some exact sequences of Galois cohomology for any finite field extension whose Galois group admits a “nontrivial enough” exact quadruple of permutational representations over it. Unfortunately, examples of finite groups with such exact quadruples of representations that I am aware of are very few. All of them turn out to be finite subgroups of $\text{PGL}_2(\mathbb{C})$. These include cyclic groups, dihedral groups, the biquadratic group $\mathbb{Z}/2 \times \mathbb{Z}/2$, and the symmetric group $S_4$. The goal of the next two sections is to construct those exact quadruples for biquadratic and dihedral groups and deduce some corollaries, including the conjectures of Merkurjev–Tignol [8] and Kahn [4] about biquadratic field extensions. Throughout these sections we will assume that the characteristic of our fields is not equal to 2 and the Milnor–Kato conjecture holds for $\ell = 2$. This is called the Milnor conjecture, and since it is proven by Voevodsky already [11], our results are in fact unconditional.

We will sometimes use the notation $H^i(F) = H^i(G_F, \mathbb{Z}/2)$. 

Let $\Delta = \Delta_k$ be a dihedron of order $k$, i. e., a regular polygon with $k$ vertices and $k$ edges. We will denote by $\Delta'$ the set of all vertices of $\Delta$ and by $\Delta''$ the set of all its edges. Let $\Gamma = \Gamma_k$ be the group of all automorphisms (symmetries) of $\Delta$ and $\Sigma \subset \Gamma$ be the subgroup of orientation-preserving automorphisms (rotations). Then $\Sigma$ is a cyclic group of order $k$ and a normal subgroup of order 2 in $\Gamma$.

Assume that $k$ is divisible by 4. Let $\Pi$ be the (only) subgroup of order 2 in $\Sigma$. Then the following quadruple of permutational representations of $\Gamma$ is exact

$$(Q_{\Gamma}) \quad \mathbb{Z} \longrightarrow \mathbb{Z}[\Delta'] \oplus \mathbb{Z} \longrightarrow \mathbb{Z}[\Delta'] \oplus \mathbb{Z}[\Delta''/\Pi] \longrightarrow \mathbb{Z}[\Delta''/\Pi],$$

where the arrows are defined by the formulas below. Choose an arbitrary generator $\sigma \in \Sigma$; let $\bar{\sigma}$ be its image in $\Sigma/\Pi$. Further, let $\sigma^{\pm 1/2}$ be maps $\Delta' \longrightarrow \Delta'' \longrightarrow \Delta'$, commuting with $\Sigma$ and inverted by the action of $\Gamma \setminus \Sigma$, whose squares are $\sigma^{\pm 1}$, and let $\bar{\sigma}^{\pm 1}$ be the induced maps $\Delta'/\Pi \longrightarrow \Delta''/\Pi \longrightarrow \Delta'/\Pi$. Denote by $pr_{\Pi}$ the projection maps modulo $\Pi$ and by $\# X$ the sum of all elements of a set $X$. Then the first map in this sequence is $(\# \Delta', -2)$, the third one is $(pr_{\Pi}, -\sigma^{-1/2} - \sigma^{1/2})$, and the middle arrow is given by the matrix $U$ with components $U_{11} = \sigma^{-1/2} + \sigma^{1/2}$, $U_{12} = \# \Delta''$, $U_{21} = pr_{\Pi}$, and $U_{22} = \# \Delta'/\Pi$.

Moreover, there exists a chain homotopy $h$ such that $dh + hd = 2$. Of the three maps constituting $h$, the leftmost one is given by the pair $(- pr_{\Pi}, - k/2 - 1)$, where $pr_{1}: \mathbb{Z}[X] \rightarrow \mathbb{Z}$ sends every $x \in X$ to 1. To define the remaining two maps, one needs some preparatory work. Let $f(x)$ be a Laurent polynomial solving the equation $(x^{-1/2} + x^{1/2})^2 f(x) = 1 + \sum_{i=1}^{k/2 - 1} x^i$. Set $\psi = (\sigma^{-1/2} + \sigma^{1/2}) f(\sigma) \in \sigma^{1/2} \mathbb{Z}[\Sigma]$ and $\bar{\psi} = \psi \mod \Pi \in \bar{\sigma}^{1/2} \mathbb{Z}[\Sigma/\Pi]$. The rightmost piece of $h$ is equal to $(1 + \pi, \bar{\sigma}^{1/2} \# \Sigma/\Pi - \bar{\psi})$, where $\pi$ is the only nontrivial element of $\Pi$. The middle arrow is given by the matrix $H$ with components $H_{11} = \psi$, $H_{12} = 1 + \pi$, $H_{21} = - pr_{\Pi}$, and $H_{22} = 0$.

Further useful exact quadruples can be obtained by restricting $Q_{\Gamma}$ to a dihedral subgroup of index 2. The group $\Gamma_{k/2}$ can be embedded into $\Gamma_k$ in two ways; namely, under the first embedding the action of $\Gamma_{k/2}$ on $\Delta'_k$ is transitive, while under the second embedding the action of $\Gamma_{k/2}$ on $\Delta''_k$ is. These embeddings correspond to the two essentially different ways of defining a regular polygon with $k/2$ vertices in terms of the given polygon $\Delta_k$: one can make edges of the new polygon out of every second edge of the original one, or one can take every second vertice of the original polygon to be the new polygon’s vertices. Taking the restriction of $Q_{\Gamma}$ with respect to the first of these embeddings, we obtain an exact quadruple of $\Gamma_{k/2}$-modules

$$(Q^+_{\Gamma}) \quad Z \longrightarrow Z[\Delta^+] \oplus Z \longrightarrow Z[\Delta'] \oplus Z[\Delta''/\Pi] \longrightarrow Z[\Delta'/\Pi] \oplus Z[\Delta''/\Pi],$$

where $\Delta'$ and $\Delta''$ denote the sets of all vertices and edges of the regular polygon $\Delta_{k/2}$, while $\Delta^+$ denotes a principal homogeneous space for the dihedral group $\Gamma_{k/2}$ which is defined as the set of all pairs (an edge of $\Delta_{k/2}$; one of its two vertices).
Now let $M/F$ be an extension of fields with a dihedral Galois group $\Gamma$. Then the category of intermediate field extensions $K/F$ embeddable into $M$ is equivalent to the category of sets endowed with a transitive action of $\Gamma$. Let $L'$ and $L''$ be the intermediate fields corresponding to the $\Gamma$-sets $\Delta'$ and $\Delta''$, let $K'$ and $K''$ be the fields corresponding to $\Delta'/\Pi$ and $\Delta''/\Pi$, and $E \supset F$ be the normal extension of $F$ corresponding to the subgroup $\Pi \subset \Gamma$.

**Proposition 17.** For any dihedral field extension $M/F$ of degree $[M:F]$ divisible by 8, there is an exact sequence of Galois cohomology of the form

$$H^n(L'') \oplus H^n(K') \xrightarrow{\text{cor}^{\text{cor}}_{L''/K'', \text{cor} E/K'' \circ \text{res}_{E/K'}}} H^n(K'') \xrightarrow{\text{cor}^{\text{cor}}_{K''/F \circ \text{res}_{L''/K''}} \cup} H^{n+1}(F) \xrightarrow{\text{res}_{L'/F}} H^{n+1}(L') \xrightarrow{\text{cor}^{\text{cor}}_{M/L'' \circ \text{res}_{M/L'}, \text{cor} L'/K'}} H^{n+1}(L'') \oplus H^{n+1}(K').$$

For any dihedral extension $M/F$ of degree divisible by 4, there is an exact sequence

$$H^n(L') \oplus H^n(L'') \oplus H^n(E) \xrightarrow{\text{cor}^3} H^n(K') \oplus H^n(K'') \xrightarrow{\text{cor}^{\text{cor}}_{K(i)/F \circ \text{res}_{L(i)/K(i)}} \cup} H^{n+1}(F) \xrightarrow{\text{res}} H^{n+1}(M) \xrightarrow{\text{cor}^3} H^{n+1}(L') \oplus H^{n+1}(L'') \oplus H^{n+1}(E).$$

**Proof.** The proposition follows from Theorem 6 applied to the above exact quadruples $Q_\Gamma$ and $Q_\Gamma^\times$, respectively. There are also dual exact sequences of cohomology, corresponding to the dual exact quadruples $\text{Hom}_{Z}(Q_\Gamma, Z)$ and $\text{Hom}_{Z}(Q_\Gamma^\times, Z)$, but we will not write them down here. □

**Remark.** There is yet another exact quadruple of permutational representations of a dihedral group $\Gamma$, whose restrictions to the subgroup of rotations and dihedral subgroups of index 2 are also of some interest. Let $\Delta_k$ be the dihedron of an arbitrary order $k$ and $\Pi \subset \Sigma \subset \Gamma_k$ be a cyclic subgroup of order $m$. Then there is a self-dual exact quadruple of the form

$$\mathbb{Z}[\Delta'/\Pi] \longrightarrow \mathbb{Z}[\Delta'] \oplus \mathbb{Z} \longrightarrow \mathbb{Z}[\Delta''] \oplus \mathbb{Z} \longrightarrow \mathbb{Z}[\Delta''/\Pi]$$

if $k/m$ is even (and $\Delta''$ must be replaced with $\Delta'$ if $k/m$ is odd). The first map in this sequence is $(\#\Pi, - \text{pr}_1)$, the third map is $(\text{pr}_1, - \#\Delta''/\Pi)$, and the middle arrow is given by the matrix $U$ with components $U_{11} = \sigma^{-(k/m-1)/2} + \cdots + \sigma^{(k/m-1)/2}$, $U_{12} = \#\Delta''$, $U_{21} = \text{pr}_1$, and $U_{22} = m$ in the above notation. It is not difficult to find a homotopy map $h$ with $dh + hd = m$.

6. **Biquadratic Field Extensions**

Let $\Theta = \{1, a, b, c\}$ denote an abelian group isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. We keep the notation of section 5 related to dihedral groups. The group $\Theta$ can be embedded into
the dihedral group $\Gamma_4$ in two ways: as above, under the first embedding the action of $\Theta$ on $\Delta'$ is transitive, while under the second embedding the action of $\Theta$ on $\Delta''$ is.

Taking the restrictions on $\Theta$ of the exact quadruple $Q_\Gamma$ for $\Gamma = \Gamma_4$, we obtain (after one simple cancellation in the second case) two exact quadruples of permutational representations of $\Theta$. They have the form

\[(Q_\Theta) \quad \mathbb{Z} \longrightarrow \mathbb{Z}[\Theta] \oplus \mathbb{Z} \longrightarrow \mathbb{Z}[\Theta/a] \oplus \mathbb{Z}[\Theta/b] \oplus \mathbb{Z}[\Theta/c] \longrightarrow \mathbb{Z}^3 / \mathbb{Z} \]

and

\[\mathbb{Z} \longrightarrow \mathbb{Z}[\Theta/a] \oplus \mathbb{Z}[\Theta/b] \longrightarrow \mathbb{Z}[\Theta] \oplus \mathbb{Z} \longrightarrow \mathbb{Z}[\Theta/c],\]

where we denote for simplicity by $a$ the subgroup $\{1, a\}$ etc. All maps in these sequences can be defined directly in a very simple way.

We are interested in the exact sequence of cohomology corresponding to the first of these quadruples. Let $L/F$ be a biquadratic field extension with Galois group $\Theta$. We will consider $\Theta$ as a two-dimensional vector space over the field $\mathbb{Z}/2$. Notice that there is a canonical isomorphism between $\Theta$ and the dual vector space $\Theta^*$, given by the only nonzero bilinear form on $\Theta$ with the property that $(\theta, \theta) = 0$ for all $\theta \in \Theta$. The three intermediate fields $F \subset K_\theta \subset L$ bijectively correspond to nonzero elements $a, b, c$ of the group $\Theta$ in a natural way. The six-term exact sequence of Galois cohomology corresponding to the exact quadruple $Q_\Theta$ has the form

\[H^n(L) \oplus H^n(F)^{\otimes 3} \xrightarrow{(\text{cor, res}^3)} \bigoplus_{\theta=a,b,c} H^n(K_\theta) \xrightarrow{\text{cor}^3} \Theta \otimes_{\mathbb{Z}/2} H^n(F) \]

where the middle arrow $\Theta \otimes H^n(F) \longrightarrow H^{n+1}(F)$ is defined in terms of the embedding $\Theta \simeq \text{Hom}(\Theta, \mathbb{Z}/2) \longrightarrow \text{Hom}(G_F, \mathbb{Z}/2) \simeq H^1(F)$. This 6-term exact sequence is a part of one of the 7-term exact sequences of Merkurjev–Tignol and one of the 8-term exact sequences of Kahn [8, 4] for a biquadratic extension of fields.

**Corollary 18.** Both 8-term sequences of Kahn [4] for Galois cohomology in a biquadratic field extension $L/F$ are exact, provided that the first Bockstein homomorphism $H^n(E, \mathbb{Z}/2) \longrightarrow H^{n+1}(E, \mathbb{Z}/2)$ vanishes in the relevant degree $n$ for the five fields $E$ between $F$ and $L$ (namely, $F$, $L$, and all three intermediate fields).

**Proof.** The middle 6-term part of the first of Kahn’s 8-term sequences is written down right above; its exactness follows from Theorem 6 applied to the exact quadruple $Q_\Theta$ tensored with $\mu_4^{\otimes n}$. The middle part of the second sequence can be obtained from the dual exact quadruple $\text{Hom}_\mathbb{Z}(Q_\Theta, \mathbb{Z})$. Exactness at the remaining terms follows from exactness at the middle terms due to Merkurjev–Tignol and Kahn’s results.

**Corollary 19.** Let $L/F$ be a field extension of degree 4 and $M/F$ be its Galois closure. Suppose that the degree $[M : F]$ is a power of 2. Then the kernel $J$ of the restriction map $H^*(G_F, \mathbb{Z}/2) \longrightarrow H^*(G_L, \mathbb{Z}/2)$ is generated in degrees 1 and 2 as an
ideal in $H^*(G_F, \mathbb{Z}/2)$. Moreover, the kernel of the restriction map $H^*(G_F, \mathbb{Z}/2) \rightarrow H^*(G_M, \mathbb{Z}/2)$ is also generated in degrees 1 and 2.

**Proof.** There are only three cases: either $L/F$ is a cyclic extension, or a biquadratic one, or it has the form $L'/F$ for a dihedral extension $M/F$ with Galois group $\Gamma_4$. The first case follows from Corollary 14, the second from Corollary 18, and the third from Proposition 17 (both statements). Note that in the biquadratic case the ideal $J$ is actually generated by a two-dimensional subspace in $H^1(G_F, \mathbb{Z}/2)$.

**Corollary 20.** For any field $F$ and any two-dimensional subspace $\Theta \subset H^1(G_F, \mathbb{Z}/2)$ the kernel of the multiplication map $\Theta \otimes H^*(G_F, \mathbb{Z}/2) \rightarrow H^{*+1}(G_F, \mathbb{Z}/2)$ is a quadratic module over the algebra $H^*(G_F, \mathbb{Z}/2)$.

**Proof.** Clearly, any two-dimensional subspace in $H^1(F)$ corresponds to a biquadratic field extension $L/F$, so we can apply the 6-term exact sequence of Corollary 18. The kernel in question is isomorphic to the quotient module of $\bigoplus_{\theta} H^{\geq 1}(K_{\theta})$ by the image of the module $H^{\geq 1}(L) \oplus H^{\geq 1}(F)^{\oplus 3}$. Since the first module is quadratic (by Theorem 9) and the second module is generated in degree 1 and 2 (by Proposition 12), the quotient module is quadratic.

**Conjecture 21.** For any field $F$, the ideal generated by any two-dimensional subspace $\Theta \subset H^1(G_F, \mathbb{Z}/2)$ in the cohomology algebra $H^*(G_F, \mathbb{Z}/2)$ is a Koszul module (see [9]) over that algebra.

**Remark.** It is possible to extend the first statement of Corollary 19 to arbitrary field extensions of degree 4 using the following exact quadruple of permutational representations of the symmetric group $S_4$:

$$
\begin{align*}
Z & \rightarrow Z[X_4] \oplus Z \rightarrow Z[X_6] \oplus Z \rightarrow Z[X_3],
\end{align*}
$$

where $X_4$ is a four-element set on which $S_4$ acts in the standard way, $X_6$ is the set of all two-element subsets of $X_4$, and $X_3$ is the quotient of $X_6$ modulo the involution $*$ sending a subset to its complement. The first map in this sequence is $(\#X_4, -2)$, the third map sends a two-element subset to its class modulo $*$ and 1 to $-\#X_3$, and the middle arrow is given by the matrix $U$ with component maps $U_{11} : Z[X_4] \rightarrow Z[X_6]$ sending each element of $X_4$ to the sum of three two-element subsets containing it, $U_{12} = \text{pr}_1$, $U_{21} = \#X_6$, and $U_{22} = 2$. It is easy to construct a homotopy map $h$ with $dh + hd = 2$. In fact, one can generalize the 6-term biquadratic exact sequences to arbitrary field extensions of degree 4 using this exact quadruple!
References

[1] H. Bass, J. Tate. The Milnor ring of a global field. In *K-theory II*, Lecture Notes in Math. 342, p. 349–446, 1973.

[2] K. J. Becher. Milnor K-groups and finite field extensions. *K-theory* 27, #3, p. 245–252, 2002.

[3] S. Bloch, K. Kato. *P-adic étale cohomology*. *Publ. Math. IHÉS* 63, p. 107–152, 1986.

[4] B. Kahn. On the cohomology of biquadratic extensions. *Comment. Math. Helvetici* 69, p. 120–136, 1994.

[5] K. Kato. Galois cohomology of complete discrete valuation fields. In *Algebraic K-theory, Part II (Oberwolfach, 1980)*, Lecture Notes in Math. 967, p. 215–238, 1982.

[6] J. Milnor. Algebraic K-theory and quadratic forms. *Inventiones Math.* 9, p. 318–344, 1970.

[7] A. S. Merkurjev, A. A. Suslin. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Math. USSR—Izvestiya* 21, #2, p. 307–340, 1983.

[8] A. S. Merkurjev, J.-P. Tignol. Galois cohomology of biquadratic extensions. *Comment. Math. Helvetici* 68, p. 138–169, 1993.

[9] L. Positselski. Koszul property and Bogomolov’s conjecture. *Intern. Math. Research Notices* 2005, #31, p. 1901–1936, 2005. arXiv:1405.0965 [math.KT]

[10] A. Suslin. Algebraic K-theory and the norm residue homomorphism. *Journ. Soviet Math.* 30, p. 2556–2611, 1985.

[11] V. Voevodsky. Motivic cohomology with Z/2-coefficients. *Publ. Math. IHES* 98, p. 59–104, 2003.

Independent University of Moscow

E-mail address: posic@mccme.ru