Precise calculation of $M_W$, $\sin^2 \hat{\theta}_W(M_Z)$, and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$

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Abstract

The two-loop $O(g^4 M_t^2 / M_W^2)$ corrections are incorporated in the theoretical calculation of $M_W$, $\sin^2 \hat{\theta}_W(M_Z)$, and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, as functions of $M_H$. The analysis is carried out in a previously proposed $\overline{\text{MS}}$ formulation and two novel on-shell resummation schemes. It is found that the inclusion of the new effects sharply decreases the scheme and residual scale dependence of the calculations. QCD corrections are incorporated in two different approaches. Comparison with the world average of $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ leads to $M_H = 127^{+143}_{-71} \text{ GeV}$ and $M_W = 80.367 \pm 0.048 \text{ GeV}$, with small variations among the six calculations.

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The theoretical calculation of the W-boson mass $M_w$, the $\overline{\text{MS}}$ parameter $\sin^2 \hat{\theta}_w(M_Z)$, and the effective mixing parameter $\sin^2 \theta_{\text{eff}}^{\text{lep}}$ represents one of the most important applications of the Standard Model (SM) at the level of its quantum corrections. We recall that $\sin^2 \theta_{\text{eff}}^{\text{lep}}$ is determined from on-resonance observables at LEP and SLC, while $\sin^2 \hat{\theta}_w(M_Z)$ is theoretically very important, particularly in the context of GUT studies. The aim of this paper is to present an accurate calculation of these parameters that, in the case of $M_w$ and $\sin^2 \hat{\theta}_w(M_Z)$, includes the recently evaluated two-loop contributions of $O(g^4M_t^2/M_w^2)$ [1]. The corresponding contributions for the evaluation of $\sin^2 \theta_{\text{eff}}^{\text{lep}}$ are provided in the present paper. Although nominally subleading relative to the two-loop terms of $O(g^4M_t^4/M_w^4)$, these effects have been found to be of comparable magnitude [1]. Our strategy is to study the incorporation of the $O(g^4M_t^2/M_w^2)$ contributions in different renormalization schemes, and to investigate the residual scale dependence. In fact, as it will be shown, the inclusion of the new results greatly decreases the scheme and residual scale dependence of important radiative corrections. QCD corrections are evaluated following two different approaches.

We recall that, using the accurately known parameters $\alpha$, $G_\mu$, and $M_Z$ as inputs, $M_w$ can be calculated from the expression [2]

$$s^2c^2 = A^2/M_Z^2(1 - \Delta r), \quad (1)$$

where $s^2 = 1 - c^2$ is an abbreviation for $\sin^2 \theta_w \equiv 1 - M_W^2/M_Z^2$ ($M_w$ and $M_Z$ are the physical masses of the intermediate bosons), $A^2 = \pi \alpha/\sqrt{2}G_\mu = (37.2802 \text{GeV})^2$, and $\Delta r$ is the relevant radiative correction [2]. In order to incorporate the $O(g^4M_t^2/M_w^2)$ effects, it is necessary to specify the renormalization scheme employed in the evaluation of the one-loop contributions. In Ref. [3] this was done in the $\overline{\text{MS}}$ framework of Ref. [3], where

$$1 - \Delta r = [1 - \Delta \hat{\theta}_w] \hat{s}^2/\hat{s}^2, \quad (2)$$

$$\hat{s}^2/\hat{s}^2 = 1 + (c^2/s^2)(\hat{c}^2/\hat{s}^2) \Delta \hat{\rho}, \quad (3)$$

$$\Delta \hat{\rho} = \text{Re} \left[ A_{ww}(M_W^2) - c^2A_{zz}(M_Z^2) \right]_{\overline{\text{MS}}} / M_W^2, \quad (4)$$

$$\Delta \hat{\theta}_w = -(2\delta e/e)_{\overline{\text{MS}}} + (e^2/\hat{s}^2)\hat{f}, \quad (5)$$

$$\hat{f} = [(\text{Re}A_{ww}(M_w^2) - A_{ww}(0)) / M_w^2 + V_w + M_w^2 B_w]_{\overline{\text{MS}}}. \quad (6)$$

In these expressions $\overline{\text{MS}}$ denotes both the $\overline{\text{MS}}$ renormalization (i.e. the pole subtraction) and the choice $\mu = M_Z$ for the ’t-Hooft scale, $\hat{s}^2 = 1 - \hat{c}^2$ is an abbreviation for $\sin^2 \hat{\theta}_w(M_Z)$, $2\delta e/e$ is the charge renormalization counterterm, $\hat{c}^2 = c^2/(1 + 2\delta e/e)_{\overline{\text{MS}}}$ the $\overline{\text{MS}}$ electromagnetic coupling at $\mu = M_Z$, and $A_{ww}(q^2)$ and $A_{zz}(q^2)$ the transverse $W$ and $Z^0$ self-energies [2, 4] with $\hat{q}^2 = \hat{c}^2/\hat{s}^2$ factored out. In this paper we also factor out $\hat{g}^2$ in the definition of $\Delta \hat{\rho}$. As $M_t$ has been measured, it is simplest to use a pure $\overline{\text{MS}}$ subtraction, without decoupling the top contribution to $\delta e/e$ and the mixing self-energy $A_{\gamma Z}(M_Z^2)$ [3]. The term $V_w + M_W^2 B_w$ in Eq. (3) represents vertex and box diagram contributions to $\mu$ decay, modulo a factor $\hat{c}^2/\hat{s}^2$ [2]. QCD corrections to the self-energies are given, for example, in Ref. [3, 4] and are updated in this paper.
As shown in Ref. [1], in the \(\overline{\text{MS}}\) framework of Eqs. (1-6) the \(O(g^4 M^2_t/M^2_w)\) corrections are incorporated by adding two-loop irreducible contributions, \(\Delta \hat{r}^{(2)}_W\) and \((\hat{e}^2/\hat{s}^2)\Delta \hat{\rho}^{(2)}\), to the one-loop expressions \(\Delta \hat{r}^{(1)}_W\) and \((\hat{e}^2/\hat{s}^2)\Delta \hat{\rho}^{(1)}\), respectively. The correction \((\hat{e}^2/\hat{s}^2)\Delta \hat{\rho}^{(2)}\) also includes the previously evaluated irreducible contributions of \(O(g^4 M^4_t/M^4_w)\) \([7]\).

In order to discuss the scheme dependence, we note that a resummation formula analogous to Eq. (2) can be obtained in the on-shell (OS) renormalization scheme of Ref. [2]. For brevity, we show how it follows from Eqs.(1-6). Combining Eqs. (2,3,5) we have

\[
1 - \Delta r = 1 + (2\delta e/e)_{\overline{\text{MS}}} + (\hat{e}^2/\hat{s}^2) \Delta \hat{\rho} - (\hat{e}^2/\hat{s}^2)\hat{f}.
\]

In order to express \(\hat{e}^2/\hat{s}^2\) in terms of \(G_\mu\), we note that Eqs. (1-3) lead to

\[
\hat{e}^2/\hat{s}^2 = G_\mu/\sqrt{2} \ 8 M^2_w \left[ 1 + (2\delta e/e)_{\overline{\text{MS}}} - (\hat{e}^2/\hat{s}^2)\hat{f} \right].
\]

Approximating \(\hat{e}^2/\hat{s}^2 \rightarrow e^2/s^2\) in the square bracket and inserting Eq. (8) in the \(\Delta \hat{\rho}\) term of Eq. (7), we have

\[
1 - \Delta r = \left[ 1 + (2\delta e/e)_{\overline{\text{MS}}} - \frac{e^2}{s^2}\hat{f} \right] \left( 1 + \frac{c^2}{s^2} 8 M^2_w G_\mu/\sqrt{2}\Delta \hat{\rho} \right).
\]

The leading \(M_t\) contribution to \(\Delta \hat{\rho}^{(1)}\) is \(3 M^2_t/(64\pi^2 M^2_w)\), which is independent of \(s^2\). At the one-loop level \((2\delta e/e)_{\overline{\text{MS}}}\) is also independent of \(s^2\) and \(\hat{e}\) \([3,4]\), and \((\hat{e}^2/\hat{s}^2)\hat{f}^{(1)}\) does not involve terms proportional to \(M^2_t\) or large logarithms \(\ln(M_Z/m_f)\), where \(m_f\) is a small fermion mass. However, non-leading one-loop contributions contained in \((\hat{e}^2/\hat{s}^2)\hat{f}^{(1)}\) and \((G_\mu/\sqrt{2}) 8 M^2_w \Delta \hat{\rho}^{(1)}\) do depend on \(s^2\). To obtain a resummation formula involving only on-shell parameters, we replace \(s^2 = s^2 [1 + (c^2/s^2) \ (\hat{e}^2/\hat{s}^2) \Delta \hat{\rho}]\) in \(\hat{f}(s^2)\) and \(\Delta \hat{\rho}(s^2)\). Calling \(\tilde{f}(s^2)\) and \(\Delta \tilde{\rho}(s^2)\) the resulting functions, we obtain

\[
1 - \Delta r = \left[ 1 + \frac{2\delta e}{e} \right]_{\overline{\text{MS}}} - \frac{e^2}{s^2}\tilde{f}(s^2) \left[ 1 + \frac{c^2}{s^2} 8 M^2_w G_\mu/\sqrt{2}\Delta \tilde{\rho}(s^2) \right]
\]

which is an on-shell counterpart of the \(\overline{\text{MS}}\) expression of Eq. (2). We note from Eq. (8) that \(\hat{s}^2 - s^2 = c^2(\hat{e}^2/\hat{s}^2)\Delta \hat{\rho} \approx 3 c^2 x_t + \ldots\) where \(x_t = G_\mu M^2_t/8\pi^2 \sqrt{2}\) and the ellipses represent subleading contributions. Thus the replacement \(s^2 = s^2 [1 + (c^2/s^2) 3 x_t + \ldots]\) in \(\tilde{f}^{(1)}(s^2)\) and \(\Delta \tilde{\rho}^{(1)}(s^2)\) induces additional contributions of \(O(g^4 M^2_t/M^2_w)\). The corresponding functions, called \(\tilde{f}_{add}^{(2)}\) and \(\Delta \tilde{\rho}_{add}^{(2)}\), respectively, are given in the Appendix. Therefore in the OS scheme we obtain

\[
\tilde{f}(s^2) = \tilde{f}^{(1)}(s^2) + \tilde{f}^{(2)}(s^2)
\]

\[
\Delta \tilde{\rho}(s^2) = \Delta \tilde{\rho}^{(1)}(s^2) + \Delta \tilde{\rho}^{(2)}(s^2),
\]

where \(\tilde{f}^{(2)}(s^2) = \tilde{f}^{(2)}(s^2) + \tilde{f}_{add}^{(2)}(s^2), \Delta \tilde{\rho}^{(2)}(s^2) = \Delta \tilde{\rho}^{(2)}(s^2) + \Delta \tilde{\rho}_{add}^{(2)}(s^2)\). The amplitude \((e^2/s^2) \tilde{f}^{(2)}\) is given by Eqs.(7a,b) of Ref. [1] multiplied by \((\alpha/\pi s^2)\mathcal{N}_c x_t\) \((\mathcal{N}_c = 3)\), while \((G_\mu/\sqrt{2}) 8 M^2_w \Delta \tilde{\rho}^{(2)}\) is given by Eqs.(10a,b) of the same reference multiplied by \(\mathcal{N}_c x_t^2\).
An alternative OS resummation can be obtained by combining \((e^2/s^2) \Delta \hat{\rho}\) in Eq. (7) with Eq. (3) and once more replacing \(\hat{f} \rightarrow \tilde{f}\), \(\Delta \hat{\rho} \rightarrow \Delta \tilde{\rho}\). This leads to

\[
\Delta r = \Delta r^{(1)} + \Delta r^{(2)} + \left(\frac{e^2}{s^2} G_\mu \frac{8 M_W^2}{\sqrt{2}} \Delta \tilde{\rho}(s^2)\right)^2 (1 - \Delta \alpha), \tag{13}
\]

where \(\Delta r^{(1)}\) is the original one-loop OS result of Refs. [2, 4], expressed in terms of \(\alpha\) and \(s^2\), \(\Delta r^{(2)} = (e^2/s^2) \tilde{f}^{(2)} - (e^2/s^2)(e^2/s^2) \Delta \rho^{(2)}\), and \(\Delta \alpha\) is the renormalized photon vacuum polarization function at \(q^2 = M_Z^2\). If in the last term we only retain the two-loop contributions proportional to \(M_t^2\) and \(M_t^4\), Eq. (13) reduces to

\[
\Delta r = \Delta r^{(1)} + \Delta r^{(2)} + \left(\frac{e^2}{s^2}\right)^2 N_c x_t \left(2 \frac{e^2}{s^2} \Delta \tilde{\rho}^{(1)} - N_c \frac{\alpha}{16\pi} \frac{M_t^2}{s^2 M_W^2}\right). \tag{14}
\]

The last terms in Eq. (13,14) represent higher order reducible contributions induced by resummation of one-loop corrections, while \(\Delta r^{(2)}\) contains the corresponding irreducible components. Eqs. (2,10,13,14) satisfy the important property that, when inserted in Eq. (1), the enhancement factors due to the fact that complete Eq. (14) is exactly reducible two-loop effects, \(\hat{f}^{(2)}\) and \(\Delta \rho^{(2)}\) in Eq. (2), and \(\tilde{f}^{(2)}\) and \(\Delta \tilde{\rho}^{(2)}\) in Eqs. (10,13), cancels the \(\mu\)-dependence through \(O(g^4 M_t^2/M_Z^2)\), the order of validity of the calculation. There remains a significantly smaller \(\mu\)-dependence of \(O(g^4)\) without \(M_t^2\) enhancement factors due to the fact that complete \(O(g^4)\) corrections have not yet been evaluated. On the other hand, Eq. (14) is exactly \(\mu\)-independent since it retains only the complete two-loop contributions proportional to \(M_t^4\) and \(M_t^2\). We refer to Eqs. (3,8), Eq. (11), and Eq. (14) as the \(\overline{\text{MS}},\) OSI, and OSII schemes, respectively. Their numerical difference will give us a measure of the scheme dependence of \(\Delta r\) and the corresponding \(M_W\) predictions.

The effective parameter \(\sin^2 \theta_{\text{eff}}^{\text{lept}}\) is obtained from \(s^2\) or \(s^2\) by means of the relations

\[
\sin^2 \theta_{\text{eff}}^{\text{lept}} = \hat{k}(M_Z^2) \sin^2 \theta_W(M_Z) = k(M_Z^2) s^2 \tag{15}
\]

where \(\hat{k}(q^2)\) and \(k(q^2)\) are the real parts of electroweak form factors evaluated at \(q^2 = M_Z^2\), and \(s^2\) and \(s^2\) are related by Eq. (3). The amplitude \(\hat{k}\), evaluated in the \(\overline{\text{MS}}\) scheme with \(\mu = M_Z\), can be expressed as \(\hat{k} = 1 + \left(e^2/s^2\right)(\Delta \hat{k}^{(1)} + \Delta \hat{k}^{(2)})\). The one-loop contribution \(\Delta \hat{k}^{(1)}\) is given in Ref. [3], with the understanding that the top contribution to the mixing self-energy \(A_{\gamma Z}(M_Z^2)\) is not decoupled and, following the discussion of that work, certain two-loop effects induced by the imaginary parts of the self-energies are included. We recall that, for large \(M_t,\)
∆k^{(1)} grows like ln(M_t^2/M_w^2). The O(g^4M_t^2/M_w^2) corrections are contained in (ℓ^2/ŝ^2) ∆k^{(2)}, which is given in Eq. (A11), multiplied by N_c(α/4πs^2)M_w^2/(c^2M_Z^2). The on-shell amplitude k is obtained from ̂k using Eq. (8),  ̂k = (G_M/√2) 8 M_w^2 [1 − (c^2/ŝ^2) ̂f] (which follows from Eq. (8)), approximating ( ̂c^2/ŝ^2) ̂f(s^2) ≈ (G_M/√2) 8 M_w^2 ̂f(s^2) in the square bracket, expressing  ̂s^2 in ∆k^{(1)}(s^2) + ∆k^{(2)}(s^2) in terms of s^2 and taking into account the additional O(g^4M_t^2/M_w^2) contribution ∆k^{(2)}_{add} induced by the latter shift. Neglecting two-loop terms without M_t^2 enhancement factors, this leads to

\[
k = \left(1 + \frac{8M_w^2 G_M}{\sqrt{2}} \Delta ̂k(s^2)\right) \left[1 + \frac{c^2}{s^2} G_M \left(1 - \frac{8M_w^2 G_M}{\sqrt{2}} ̂f^{(1)}(s^2)\right) \Delta ̂\rho(s^2)\right],
\]

where ∆ ̂\rho is defined in Eq. (12) and ∆ ̂k(s^2) = ∆ ̂k^{(1)}(s^2) + ∆ ̂k^{(2)}(s^2), ∆ ̂k^{(2)}(s^2) = ∆ ̂k^{(2)}(s^2) + ∆ ̂k^{(2)}_{add}(s^2). If we only retain two-loop effects with M_t^1 and M_t^2 enhancement factors, Eq. (10) reduces to

\[
k = 1 + \frac{8M_w^2 G_M}{\sqrt{2}} \left[\Delta ̂k(s^2) + \frac{c^2}{s^2} \Delta ̂\rho(s^2) + \frac{c^2}{s^2} N_c x_t \left(\Delta ̂k^{(1)}(s^2) - ̂f^{(1)}(s^2)\right)\right],
\]

which is exactly µ-independent.

We briefly explain the strategies followed in the evaluation of the QCD corrections. When electroweak amplitudes proportional to the squared top-quark mass are expressed in terms of the pole mass M_t, the coefficients of α^n(t) (a ≡ α_s/π, n = 1, 2) are quite large, a feature that is absent when they are expressed in terms of the running MS mass µ_t = ̂m_t(µ_t). Recent studies of the scale and scheme dependence of the leading M_t-dependent electroweak corrections also suggest that the unknown QCD corrections to the O(g^4M_t^4/M_w^4) contributions are significant when M_t is employed [10]. These observations suggest the strategy of parametrizing the electroweak amplitudes in terms of µ_t. We illustrate this approach in the most sensitive amplitude, namely the contributions to ∆ ̂\rho and ∆ ̂\rho proportional to powers of M_t. In the case of ∆ ̂\rho we have

\[
\frac{8M_w^2 G_M}{\sqrt{2}} \Delta ̂\rho = N_c \left[\frac{G_M}{\sqrt{2}} \frac{\mu_t^2}{8\pi^2} + \frac{G_M}{\sqrt{2}} \frac{M_t^2}{2\sqrt{2}\pi^2} R_{add}\right] + \ldots,
\]

where R_{add} = 16π^2 ∆ ̂\rho_{add}/(N_c x_t) and

\[
\delta_{QCD}^{MS} = -0.19325 a(µ_t) - 3.970 a^2(µ_t),
\]

r_µ ≡ M_H^2/µ_t^2, r_e ≡ M_e^2/µ_t^2, R(r_µ, r_e) is the function given in Eqs.(10a,b) of Ref. [4] with M_t → µ_t and c^2 → C^2, and the ellipses stand for contributions not proportional to powers of µ_t. The coefficients in Eq. (19) are much smaller than in [8]

\[
\delta_{QCD} = -2.8599 a(M_t) - 14.594 a^2(M_t),
\]

the QCD correction when ∆ ̂\rho is expressed in terms of M_t. In the case of ∆ ̂\rho we have an analogous formula to Eq. (18), except that one retains the s^2 dependence of R, R_{add} is absent
and the couplings are expressed in terms of $\hat{e}^2/\hat{s}^2$. This strategy is extended to the other electroweak amplitudes using the perturbative relation $M_t = [1 + (4/3) a(\mu_t) + 8.236 a^2(\mu_t)] \mu_t$ in the terms of $O(\alpha)$ and $O(\alpha \hat{\alpha}_s)$ of Ref. [3]. As $M_t$ is important in the interpretation of the experiments, we employ

$$\mu_t/M_t = [1 + (8/3) a^2(\mu_t) - 4.47a^3(\mu_t)]/[1 + (4/3) a(\mu_t) - 1.072a^2(\mu_t)]$$

(21)

where $\hat{\mu}_t = 0.252M_t$, $\mu_t^* = 0.0960M_t$. Eq. (21) avoids large coefficients and follows from the BLM optimizations of $\mu_t/\hat{m}_t(M_t)$ and $M_t/\hat{m}_t(M_t)$ [11], with $a(\mu_t)$ and $a(\mu_t^*)$ evaluated from $a(M_2)$ using five active flavors. The corresponding FAC and PMS optimizations lead to nearly identical results [11]. In this approach, $\mu_t$ is obtained from $M_t$ via Eq. (21) and inserted in the various amplitudes such as Eq. (18). The second strategy followed in the paper is based on the conventional $M_t$ parametrization and Eq. (20). In both approaches we include very small $O(\alpha \hat{\alpha}_s^2)$ effects arising from the light isodoublets [12] and from an inverse top mass expansion of the $(t - b)$ isodoublet contribution [13]. As inputs in the Tables we employ $M_z = 91.1863$ GeV, $\hat{\alpha}_s(M_z) = 0.118$ [14], $\Delta \alpha_{\text{had}} = 0.0280$ [15], $M_t = 175$ GeV

The $O(g^4)$ contribution to $\text{Re}A_{zz}(M_2^2)$ in Eq. (4) involving $(\text{Im} A_{\gamma z}(M_2^2))^2$ and effects of light fermion masses in the one-loop corrections are also taken into account.

Tables 1 and 2 compare the $M_w$ and $\sin^2 \theta_{\text{eff}}^{\text{lep}}$ predictions based on OSI (Eqs. (10, 16)), OSII (Eqs. (14, 17)), $\overline{\text{MS}}$ (Eqs. (23, 15)), with and without the complete $O(g^4M_t^2/M_W^2)$ corrections, respectively. Specifically, in Table 2 we neglect the terms of $O(g^4M_t^2/M_W^2)$ in the functions $f$ and $\Delta \rho$, as well as the contributions of the same order in the last terms of Eqs. (14) and (17) and the second factor of Eq. (16). In both cases, the QCD corrections are evaluated using the $\mu_t$-parametrization, explained in Eq. (18) et seq., and Eq. (21). Table 3 includes the $O(g^4M_t^2/M_W^2)$ corrections but evaluates the QCD corrections using the $M_t$-parametrization and Eq. (20). In all cases the last column gives the values of $\sin^2 \theta_{\text{eff}}(M_z)$ (MS evaluation).

A number of striking theoretical features are apparent: (a) The incorporation of the irreducible $O(g^4M_t^2/M_W^2)$ corrections greatly decreases the scheme dependence of the predictions, as measured by the differences encountered in OSI, OSII, and $\overline{\text{MS}}$. For instance, the maximum differences in Table 2 are 11 MeV in $M_w$ and $2.1 \times 10^{-4}$ in $\sin^2 \theta_{\text{eff}}^{\text{lep}}$, while in Table 1 they are reduced to 2 MeV and $3 \times 10^{-5}$, respectively (similar very small differences can be observed in Table 3). (b) The differences between Table 2 and Table 1 are quite small in OSI. This means that this scheme absorbs a large fraction of the $O(g^4M_t^2/M_W^2)$ corrections in the leading contributions. (c) Although the QCD approaches we have considered are quite different, Tables 1 and 3 show very close results. This is due to a curious cancellation of screening and anti-screening effects in the difference between the two formulations.

Comparing the current world average $\sin^2 \theta_{\text{eff}}^{\text{lep}} = 0.23165 \pm 0.00024$ with Tables 1 and 3, and taking into account in quadrature the errors associated with $\Delta \alpha_{\text{had}} (\pm 2.3 \times 10^{-4})$

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1Results for different input values can be very easily obtained using a Mathematica package available from the authors.
and $\delta M_t = \pm 6 \text{ GeV} \left( \pm 1.9 \times 10^{-4} \right)$, together with an estimate of missing higher order QCD corrections ($\pm 2.4 \times 10^{-5}$) we obtain a determination of $M_H$. As the six calculations of $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ in those tables are very close, it matters relatively little which one is employed. Averaging the six central values and choosing the errors to cover the range of the six calculations, we find $M_H = 127^{+143}_{-71}\text{ GeV}$, compatible with MSSM expectations. The strength of this determination rests on the fact that unaccounted two-loop effects are not enhanced by $(M_t^2/M_W^2)^n \ (n = 1, 2)$ factors or large logarithms $\ln(M_Z/m_f)$. They are expected to affect $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ in the fifth decimal, which is consistent with the variations observed in Tables 1 and 3. Because of the approximate exponential dependence of $M_H$ on $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, this precision is important in order to obtain a reasonably accurate determination of $M_H$. For instance, a 0.1% difference in the theoretical calculation of $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ induces a $\approx 55\%$ shift in the $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ determination of $M_H$ and its 1$\sigma$ bound. Corresponding to the above determination of $M_H$, one finds from Tables 1 and 3 the prediction $M_W = 80.367 \pm 0.048\text{GeV}$, in very good agreement with the current world average $M_W = 80.356 \pm 0.125\text{GeV}$. 

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Appendix

In the following we adopt the notation and conventions of Ref. [11]. The additional contributions to $\bar{f}$ and $\Delta \bar{\rho}$, in units $N_c x_t/(16\pi^2)$, are given by

$$
\bar{f}_{\text{add}}^{(2)} = \frac{10}{3} + \frac{1}{3c^2} + 4c^2B0[M_W^2, 0, M_W^2] - \left( \frac{11}{3} + \frac{1}{3c^2} + 4c^2 \right) B0[M_W^2, M_W^2, M_Z^2] \\
+ \left( \frac{11 - 8c^2}{6s^2} - \frac{11}{3} + \frac{1}{3c^2} \right) \ln zt,
$$

$$
\Delta \bar{\rho}_{\text{add}}^{(2)} = \frac{542}{27} - \frac{2}{3c^2} - \frac{800c^2}{27} + \frac{1}{3} \left( 1 + 26c^2 + 24c^4 \right) B0[M_Z^2, M_W^2, M_W^2] + 4c^2B0[M_W^2, 0, M_W^2] \\
- \left( \frac{11}{3} + \frac{1}{3c^2} + 4c^2 \right) B0[M_W^2, M_W^2, M_Z^2] - \left( \frac{2}{3} + \frac{4c^2}{3} - 8c^4 \right) \ln c^2 \\
+ \left( c^2 - \frac{38}{3} + \frac{34c^2}{3} \right) \ln \frac{M_t^2}{\mu^2} + \frac{2(3 - 62c^2 + 74c^4 + 36c^6)}{9c^2} \ln zt.
$$

Fixing the one-loop result to the $\overline{\text{MS}}$ calculation of Ref. [9], the $O(g^4M_t^2/M_W^2)$ two-loop
contribution to $\Delta \hat{k}^{(2)}(M_Z)$ in units $N_c/(16\pi^2) (\hat{\alpha}/(4\pi s^2)) M_t^2/(\hat{c}^2 M_Z^2)$ reads \[\Delta \hat{k}^{(2)} = -\frac{211 + 24ht + 462 s^2 - 64ht \hat{s}^2}{432} + \left(\frac{3}{8} - \frac{s^2}{3}\right) B0[M_Z^2, M^2_{W}, M^2_{W} - \frac{\hat{c}^2}{6} \ln \hat{c}^2 - $$
\frac{(ht - 4) \sqrt{ht}(8 \hat{s}^2 - 3)g(ht)}{108} + \left(\frac{6 + 27ht - 10ht^2 + ht^3}{108}\right)(8 \hat{s}^2 - 3) \ln ht
$$
- \left(\frac{1}{4} + \frac{2}{9} s^2\right) \ln \frac{M_t^2}{\mu^2} + \left(\frac{3s^2 - 2}{18}\right) \ln zt + \frac{(ht - 1)(8 \hat{s}^2 - 3) \phi(h/\hat{t})}{18(4 - ht)ht}, \] (A3)

while the term to be added in the OS framework is given, in units $N_c x_t/(16\pi^2)$, by

\[
\Delta \bar{k}^{(2)}_{\text{add}} = -\frac{238 c^2}{27} + 8c^4 - 2c^2 \sqrt{4c^2 - 1} \left(3 + 4c^2\right) \arctan\left(\frac{1}{\sqrt{4c^2 - 1}}\right) - \frac{16c^2}{9} \ln zt
- \left(\frac{3}{4c^2 - 2c^2}\right) f_V(1) + 4c^2 g_V(c^2) - 7c^2 \ln c^2 - \frac{17}{3} c^2 \ln \frac{\mu^2}{M_Z^2}, \] (A4)

where the functions $f_V(x)$ and $g_V(x)$ are defined in Eqs.(6d) and (6e) of \[\text{(I7).}\]

References

[1] G. Degrassi, P. Gambino, and A. Vicini, Phys. Lett. B383 (1996) 219.

[2] A. Sirlin, Phys. Rev. D22 (1980) 971.

[3] G. Degrassi, S. Fanchiotti, A. Sirlin, Nucl.Phys. B351 (1991) 49.

[4] W. Marciano and A. Sirlin, Phys. Rev. D22 (1980) 2695.

[5] S. Fanchiotti, B.A. Kniehl, A. Sirlin, Phys. Rev. D48 (1993) 307.

[6] K. Chetyrkin, J.H. Kühn, M. Steinhauser, Phys. Lett. B351, 331 (1995) and Phys. Rev. Lett. 75, 3394 (1995); L. Avdeev et al., Phys. Lett. B336, 560 (1994); B349, 597 (E) (1995).

[7] R. Barbieri et al., Phys. Lett. B288 (1992) 95 and Nucl. Phys. B409 (1993) 105; J. Fleischer, O.V. Tarasov, F. Jegerlehner, Phys. Lett. B319 (1993) 249; G. Degrassi, S. Fanchiotti, P. Gambino, Int. J. Mod. Phys. A10 (1995) 1337.

[8] A. Sirlin, Phys. Rev. D29 (1984) 89.

[9] P. Gambino and A. Sirlin, Phys. Rev. D49 (1994) R1160.

[10] B.A. Kniehl and A. Sirlin, Nucl. Phys. B458 (1996) 35.

[11] A. Sirlin, Phys. Lett. B348, 201 (1995), and in Reports of the Working Group on Precision Calculations for the Z resonance, CERN 95-03.
[12] K. Chetyrkin, J.H. Kühn, M. Steinhauser, hep-ph/9606230, and refs. therein.

[13] Private communication from K. Chetyrkin and M. Steinhauser.

[14] A. Blondel, plenary talk at ICHEP '96, Warsaw, July 1996.

[15] S. Eidelman, F. Jegerlehner, Z. Phys. C67 (1995) 585.

[16] G. Degrassi and P. Gambino, in preparation; P. Gambino, talk given at the XXXVI Cracow School of Theoretical Physics, Zakopane, Poland, June 1996, to appear in the proceedings, Acta Phys. Polonica, hep-ph/9611358.

[17] G. Degrassi and A. Sirlin, Nucl. Phys. B352 (1991) 342.
| $M_H$ (GeV) | $M_W$ (GeV) | $\sin^2 \theta_{\text{eff}}$ | $\hat{s}^2$ |
|-------------|-------------|------------------------------|--------------|
|              | OSI         | OSII                        | MS           | OSI   | OSII | MS  | MS |
| 65          | 80.404      | 80.404                      | 80.406       | 0.23132 | 0.23132 | 0.23130 | 0.23120 |
| 100         | 80.381      | 80.381                      | 80.383       | 0.23152 | 0.23154 | 0.23151 | 0.23142 |
| 300         | 80.308      | 80.307                      | 80.309       | 0.23209 | 0.23212 | 0.23209 | 0.23200 |
| 600         | 80.255      | 80.254                      | 80.256       | 0.23248 | 0.23250 | 0.23247 | 0.23239 |
| 1000        | 80.216      | 80.215                      | 80.216       | 0.23275 | 0.23277 | 0.23275 | 0.23267 |

Table 1: Predicted values of $M_W$, $\sin^2 \theta_{\text{eff}}$, and $\sin^2 \hat{\theta}_W(M_Z)$. OSI: Eqs. (10, 16); OSII: Eqs. (14, 17); MS: Eqs. (2, 3, 15). QCD corrections based on $\mu_t$-parametrization and Eq. (21). $M_t = 175$ GeV, $\hat{\alpha}_s(M_Z) = 0.118$, $\Delta\alpha_{\text{had}} = 0.0280$.

| $M_H$ (GeV) | $M_W$ (GeV) | $\sin^2 \theta_{\text{eff}}$ | $\hat{s}^2$ |
|-------------|-------------|------------------------------|--------------|
|              | OSI         | OSII                        | MS           | OSI   | OSII | MS  | MS |
| 65          | 80.409      | 80.419                      | 80.417       | 0.23132 | 0.23111 | 0.23124 | 0.23114 |
| 100         | 80.385      | 80.395                      | 80.393       | 0.23154 | 0.23135 | 0.23145 | 0.23135 |
| 300         | 80.311      | 80.316                      | 80.318       | 0.23212 | 0.23203 | 0.23203 | 0.23195 |
| 600         | 80.257      | 80.258                      | 80.263       | 0.23251 | 0.23247 | 0.23242 | 0.23235 |
| 1000        | 80.215      | 80.214                      | 80.222       | 0.23279 | 0.23280 | 0.23271 | 0.23264 |

Table 2: As in Table 1, but excluding $O(g^4M_t^2/M_W^2)$ corrections (see text).

| $M_H$ (GeV) | $M_W$ (GeV) | $\sin^2 \theta_{\text{eff}}$ | $\hat{s}^2$ |
|-------------|-------------|------------------------------|--------------|
|              | OSI         | OSII                        | MS           | OSI   | OSII | MS  | MS |
| 65          | 80.405      | 80.404                      | 80.406       | 0.23132 | 0.23134 | 0.23130 | 0.23121 |
| 100         | 80.382      | 80.381                      | 80.383       | 0.23153 | 0.23155 | 0.23152 | 0.23142 |
| 300         | 80.308      | 80.306                      | 80.308       | 0.23210 | 0.23214 | 0.23210 | 0.23201 |
| 600         | 80.254      | 80.252                      | 80.254       | 0.23249 | 0.23252 | 0.23249 | 0.23241 |
| 1000        | 80.214      | 80.213                      | 80.214       | 0.23277 | 0.23279 | 0.23277 | 0.23269 |

Table 3: As in Table 1, but with QCD corrections based on $M_t$-parametrization and Eq. (20).