Abstract. This paper studies the scattering matrix $S(E; \hbar)$ of the problem
\[-\hbar^2 \psi''(x) + V(x)\psi(x) = E\psi(x)\]
for positive potentials $V \in C^\infty(\mathbb{R})$ with inverse square behavior as $x \to \pm\infty$. It is shown that each entry takes the form
\[S_{ij}(E; \hbar) = S_{ij}^0(E; \hbar)(1 + \hbar \sigma_{ij}(E; \hbar))\]
where $S_{ij}^0(E; \hbar)$ is the WKB approximation relative to the modified potential $V(x) + \frac{\hbar^2}{2\hbar^2}(x)^{-2}$ and the correction terms $\sigma_{ij}$ satisfy $|\partial_k \sigma_{ij}(E; \hbar)| \leq C_k E^{-h}$ for all $k \geq 0$ and uniformly in $(E, \hbar) \in (0, E_0) \times (0, \hbar_0)$ where $E_0, \hbar_0$ are small constants. This asymptotic behavior is not universal: if $-\hbar^2 \partial_x^2 + V$ has a zero energy resonance, then $S(E; \hbar)$ exhibits different asymptotic behavior as $E \to 0$. The resonant case is excluded here due to $V > 0$.

1. Introduction

This paper revisits the much studied problem of determining the reflection and transmission coefficients for semi-classical operators of the form
\[(1.1)\quad P(x, \hbar D) := -\hbar^2 \frac{d^2}{dx^2} + V(x)\]
where $V$ is real-valued and assumed to decay at infinity. There are two atypical features of this work, at least relative to the existing literature on this topic:

(i) we wish to understand the zero energy limit, in fact uniformly in small $\hbar$
(ii) the smooth potential $V$ decays like an inverse square at both ends.

We remark that (i) and (ii) are closely related. Indeed, the $(x)^{-2}$ decay is “critical” with respect to the zero energy limit in the sense that $(x)^{-2-\varepsilon}$ is easier and behaves very differently. In the semi-classical literature it is more customary to encounter the criticality of the Coulomb decay $(x)^{-1}$; the reason for this is that the Coulomb decay is critical for positive energies. Note that the numerology around these decay rates applies to all dimensions and not just to one dimension. The motivation...
for considering this particular problem comes from several sources. First, smooth potentials which behave as an inverse square at one or both ends arise in several contexts in physics and geometry, for example in general relativity in connection with Schwarzschild and de-Sitter spaces, see [6]. Second, this paper is part of the program initiated in [23] and [24]. In fact, the analysis carried out here is an essential part in the solution of the “large angular momentum” problem from [24].

Let us briefly review some elementary features of scattering, cf. [7] and [17]: For fixed \( \lambda > 0 \), consider the following bases of the space of solutions to the equation \( Hf = \lambda^2 f \):

\[
(f_+(\cdot, \lambda), f_-(-\cdot, \lambda)), \quad (f_+(-\cdot, -\lambda), f_-(-\cdot, -\lambda))
\]

The former is referred to as \textit{outgoing} and the latter as \textit{incoming}. In that case the matrix \( S(\lambda) \) which transforms the coefficients of a solution relative to these bases satisfies

\[
S(\lambda) = \begin{bmatrix} t(\lambda) & r_-(\lambda) \\ r_+(\lambda) & t(\lambda) \end{bmatrix}
\]

It is called the \textit{scattering matrix} and is unitary. Of special interest to us is the behavior as \( \lambda \to 0^+ \). Note that if

\[
\int_{-\infty}^{\infty} (x)|V(x)| \, dx < \infty
\]

then \( f_+(x, \lambda) \to f_+(x, 0) \) as \( \lambda \to 0 \) where the latter satisfies the limit equation of (1.2), viz.

\[
f_+(x, 0) = 1 + \int_{x}^{\infty} (y-x)V(y)f_+(y, 0) \, dy
\]

It is known that \( S(\lambda) \) is continuous in \( \lambda \geq 0 \) under this moment condition, see [12]. To describe the possible values of \( S(0) \), recall that \( H \) has a \textit{zero energy resonance} iff \( f_\pm(\cdot, 0) \) are linearly dependent or, equivalently, iff \( W(0) = 0 \). Furthermore, since
where the correction terms satisfy the bounds

\[ V \]

with a constant \( C \) if instead of (1.4)

\[ \text{and uniformly in } 0 \]

and let

\[ E \]

whereas in the resonant case

\[ S(0) = \begin{bmatrix} t & -r \\ r & t \end{bmatrix} \]

for some real \( r, t \in [-1, 1] \), \( t \neq 0 \).

If \( \langle x \rangle V(x) \notin L^1(\mathbb{R}) \), then the behavior of \( S(\lambda) \) as \( \lambda \to 0 \) is completely different. In this paper, we focus on the border line case of positive inverse square potentials for (1.1) and \( \hbar \) small (for the remainder of the paper, we now let \( \hbar \) be a small positive quantity). It is precisely this case which arises in the geometric problem considered in [23], [24]. Our main theorem is as follows. We denote the energy by \( E = \lambda^2 > 0 \), see above, and the scattering matrix of (1.1) by

\[ S(E; h) = \begin{bmatrix} t(E; h) & r_-(E; h) \\ r_+(E; h) & t(E; h) \end{bmatrix} = \begin{bmatrix} S_{11}(E; h) & S_{12}(E; h) \\ S_{21}(E; h) & S_{22}(E; h) \end{bmatrix} \]

In view of (1.3) it suffices to describe the first row of this matrix. In this paper, \( O(\cdot) \) terms will be differentiable functions and we will typically state bounds on their derivatives with regard to the relevant variables depending on the context.

**Theorem 1.** Let \( V \in C^\infty(\mathbb{R}) \) with \( V > 0 \) and \( V(x) = \mu_+^2 x^{-2} + O(x^{-3}) \) as \( x \to \pm \infty \) where \( \mu_+ \neq 0 \), \( \mu_- \neq 0 \) and \( \partial_x^k O(x^{-3}) = O(x^{-3-k}) \) for all \( k \geq 0 \). Denote

(1.4) \[ V_0(x; h) := V(x) + \frac{\hbar^2}{4}(x)^{-2} \]

and let \( E_0 > 0 \) be such that for all \( 0 < E < E_0 \) and \( 0 < h < 1 \), \( V_0(x; h) = E \) has a unique pair of solutions, which we denote by \( x_2(E; h) < x_1(E; h) \). Define

(1.5) \[ S(E; h) := \int_{x_2(E; h)}^{x_1(E; h)} \sqrt{V_0(y; h) - E} \, dy \]

and let \( E_0 > 0 \) be such that for all \( 0 < E < E_0 \) and \( 0 < h < h_0 \) where \( h_0 = h_0(V) > 0 \) is small and \( 0 < E < E_0 \)

(1.6) \[ S_{11}(E; h) = e^{-\frac{i}{\hbar}(S(E; h) + IT(E; h))} (1 + h \sigma_{11}(E; h)) \]

\[ S_{12}(E; h) = -ie^{-\frac{i}{\hbar}T_+(E; h)} (1 + h \sigma_{12}(E; h)) \]

where the correction terms satisfy the bounds

(1.7) \[ |\partial_E^k \sigma_{11}(E; h)| + |\partial_E^k \sigma_{12}(E; h)| \leq C_k E^{-k} \quad \forall k \geq 0, \]

with a constant \( C_k \) that only depends on \( k \) and \( V \). The same conclusion holds if instead of (1.4) we were to define \( V_0 \) as \( V_0 := V + h^2 V_1 \) with \( V_1 \in C^\infty(\mathbb{R}) \), \( V_1(x; h) = \frac{1}{\hbar}(x)^{-2} + O(x^{-3}) \) as \( x \to \pm \infty \) with \( \partial_x^k O(x^{-3}) = O(x^{-3-k}) \) for all \( k \geq 0 \) and uniformly in \( 0 \leq \hbar \ll 1 \).
The addition of $\frac{\hbar^2}{4} \langle x \rangle^2$ to $V(x)$ is crucial and similar to the “Langer modification”, see for example [9]. Indeed, if we were to use $V$ instead of $V_0$ in (1.5), then the bounds (1.7) would fail due to a factor of $\log E$ as $E \to 0$. This is in contrast to potentials decaying like $|x|^{-\alpha}$ with $0 < \alpha < 2$ for which the modification is not needed, i.e., the usual WKB ansatz works, see [25]. On the other hand, note that as long as $E_0 > E > \varepsilon > 0$ the turning points $x_j(E; \hbar)$ will remain bounded and the distinction between $V_0$ and $V$ is therefore moot. Indeed, the effect of passing from $V$ to $V_0$ and vice versa is merely a harmless factor of the form $1 + O(\hbar)$ where the $O(\cdot)$ term of course depends on $\varepsilon$. In the range $E_0 > E > \varepsilon > 0$ Theorem 1 is well-known and classical. See for example Chapter 13 of [19] as well as Ramond’s work [22] for a more recent reference (Ramond, however, is more concerned with the scattering problem for energies close to the maximum of a barrier and he also assumes that the potential is dilation analytic).

We remark that the infinite differentiability assumption on $V$ can be relaxed to some finite amount of smoothness (in which case we can only ask for correspondingly many derivatives with respect to $E$), but we do not elaborate on this issue here. A more substantial problem is that of relaxing the positivity assumption. We conjecture that $V > 0$ can be replaced by the strictly weaker assumption that zero energy is not a resonance of $P(x, \hbar D)$. Recall the definition of a zero energy resonance in this context, cf. [3], [25], and Section 3 of [24]: it means that the two subordinate zero-energy solutions at $\pm \infty$ are linearly dependent (a “subordinate solution” at either end refers to the nonzero solution of $P(x, \hbar D)f = 0$ with the slowest possible growth at that end; it is unique up to a nonzero scalar factor).

Note, however, that some condition is needed in Theorem 1; indeed, in [24] it was shown that for operators of the form considered in Theorem 1 with $\mu_+^2 = \mu_-^2 = \nu^2 - \frac{1}{4}$, $\nu > \frac{1}{2}$, and $\hbar = 1$

$$W(E; \hbar) \sim E^{\frac{1}{2} - \nu}(W_0 + O(\varepsilon)) \quad \text{as} \quad E \to 0+$$

for some $W_0 \neq 0$ and $\varepsilon > 0$ provided there is no zero energy resonance. In the resonant case, it was shown in [24] that $W_0 = 0$. The following relation between $S_{11}$ in (1.6) and the Wronskian $W(E; \hbar)$

$$W(E; \hbar) = \frac{-2i\sqrt{E}}{\hbar S_{11}(E; \hbar)}$$

allows one to deduce (1.8) with $W_0 \neq 0$ from Theorem 1 (note that for inverse square potentials $S(E; \hbar)$ behaves like $|\log E|$ so that the apparent exponential behavior in (1.6) turns into a power-law in $E$). This deduction also proves that Theorem 1 necessarily fails in the presence of a zero energy resonance. Another aspect of (1.8) concerns the case of large $\hbar$, say $\hbar = 1$. Indeed, it shows that Theorem 1 gives the correct behavior of the scattering matrix even in that case, but then the energy takes over as the small parameter.

This paper is organized as follows. Section 2 constructs a fundamental system of zero energy solutions to (1.1) via the usual WKB ansatz but for $V_0$ rather than for $V$. Since we require uniform bounds in Theorem 1 as $E \to 0$, the construction of Jost solutions for positive energies which is carried out in Section 3 needs to yield the zero energy solutions in the limit $E \to 0$. We choose to reverse this process and show that $V_0$ is precisely the right potential to use in the WKB method at zero
energy. The logic is simple: the WKB ansatz
\[ \psi_{0,\pm}(x) := V^{-\frac{1}{4}}(x) \exp \left( \pm \hbar^{-1} \int_{x_0}^{x} \sqrt{V(y)} \, dy \right) \]
satisfies an equation of the form
\[ (-\hbar^2 \frac{d^2}{dx^2} + V)\psi_{0,\pm} = \hbar^2 (-x^{-2}/4 + O(x^{-3}))\psi_{0,\pm} \]
where the \( x^{-2}/4 \) term on the right-hand side is universal for all potentials that have an inverse square decay as \( x \to \infty \) as specified in Theorem 1. Since this term has the same decay as \( V \) we need to bring it to the left-hand side leading to our choice of \( V_0 \).

The main technical work of this paper is carried out in Section 3. It is here that the (semi-classical) Jost solutions are constructed for all energies in the range \( 0 < E < E_0 \). We use Langer’s method which is based on the Liouville-Green transform, see Chapters 6 and 11 in [19]: switching to the new independent variable
\[ \zeta = \zeta(x, E; \hbar) := \text{sign}(x - x_1(E; \hbar)) \left| \frac{3}{2} \int_{x_1(E; \hbar)}^{x} \sqrt{|V_0(x; \hbar) - E|} \, dn \right|^{\frac{1}{2}}, \quad x \geq 0 \]
and to the new dependent variable \( w(\zeta) = \sqrt{\zeta} f \) reduces \( P(x, \hbar D)f = Ef \), see (1.1), to an Airy equation perturbed by a potential of size \( \hbar^2 \). It is here that \( V > 0 \) becomes relevant: it ensures that for all small \( E > 0 \) there is a unique turning point \( x_1(E) > 0 \) and that \( V_0(x; \hbar) > E \) for all \( 0 < x < x_1(E; \hbar) \). Hence we can cover \( x \geq 0 \) by the intervals \( \zeta(0, E; \hbar) < \zeta \leq 0 \) and \( \zeta \geq 0 \). In each of these intervals we solve the perturbed Airy equations up to multiplicative errors of the form \( 1 + O(\zeta) \) where the \( O(\cdot) \) term is uniform in \( E \). It is in the range \( \zeta(0, E; \hbar) < \zeta \leq 0 \) that the choice of \( V_0 \) (rather than \( V \)) becomes decisive; this of course is to be expected as this range turns into the whole interval \( x \geq 0 \) as \( E \to 0 \) and WKB applied to \( V \) instead of \( V_0 \) fails at \( E = 0 \), see Section 2. Theorem 1 is proved in Section 5 by evaluating the Wronskians
\[ W(f_+(\cdot, E), f_-(\cdot, E)), \quad W(f_+(\cdot, E), f_-(\cdot, E)) \]
at \( x = 0 \). Section 6 discusses the range of validity of Theorem 1 as the energy increases towards a unique non-degenerate maximum of a barrier potential. Finally, the appendix describes a certain “normal-form” reduction of (1.1) to a Bessel equation on a region containing the turning point. Even though we do not base our asymptotic analysis on this reduction (but rather the Airy equation), we still believe that this is of independent interest.

Needless to say, there is a vast literature related to the semi-classical analysis of the Schrödinger equation and it is impossible to do any justice to it here. Somewhat curiously, however, there does not seem to be any literature on potentials which are globally smooth on the line and which exhibit inverse square decay. On the other hand, potentials which are \textit{exactly} inverse square are of course ubiquitous, especially in the physics literature. For a recent paper in this direction involving WKB see [1] and for a time-dependent analysis see the recent papers [20], [21], as well as [2], [3] and the references cited there. Potentials which decay of the form \( |x|^{-\alpha}, 0 < \alpha < 2 \), have been studied with similar objectives as here, see [13], [18] and [23]. For other work on low energies see [2], [3], [8], and [26], as well as [10].
2. Zero energy solutions

In order to motivate the choice of $V_0$ in Theorem 1 we will now obtain a fundamental system for the equation

$$-\hbar^2 f''(x) + V(x)f(x) = 0 \quad \text{(2.1)}$$

on the half axis $x > x_0$. Here we assume that $V(x) = \mu^2 x^{-2} + O(x^{-3})$ with $\mu > 0$ as $x \to \infty$ and $x_0$ is chosen so large that $V(x) > 0$ for $x > x_0$. As before, we require $\partial_k^k O(x^{-3}) = O(x^{-3-k})$ for all $k \geq 0$. Ignoring the $O(\cdot)$ term, we have on the one hand

$$\left( -\hbar^2 \partial_x^2 + \mu^2 x^{-2} \right) x^{2\pm\alpha} = 0, \quad \alpha^2 = \frac{1}{4} + \mu^2 \hbar^{-2} \quad \text{(2.2)}$$

On the other hand, with $Q(x; \hbar) := \mu^2 x^{-2} + \hbar^2 x^{-2}/4$,

$$Q^{-\frac{1}{2}}(x; \hbar)e^{\pm \int_{x_0}^x \sqrt{Q(y; \hbar)} \, dy} = e^{\pm \alpha}$$

with some $c \neq 0$. This motivates the following result.

**Proposition 2.** On $x > x_0$ a fundamental system of solutions for (2.1) is given by

$$\psi_j(x; \hbar) = \tilde{\psi}_j(x; \hbar)(1 + h a_j(x; \hbar)), \quad j = 1, 2 \quad \text{(2.3)}$$

with

$$\tilde{\psi}_1(x; \hbar) = V_0(x; \hbar) \frac{1}{\sqrt{\hbar}} e^{\frac{1}{4} S(x; \hbar)},$$
$$\tilde{\psi}_2(x; \hbar) = V_0(x; \hbar) \frac{1}{\sqrt{\hbar}} e^{\frac{1}{4} S(x; \hbar)},$$

where $V_0(x; \hbar) = V(x) + \frac{\hbar^2}{4} (x^{-2} - 1)$, $S(x; \hbar) = \int_{x_0}^x \sqrt{V_0(t; \hbar)} \, dt$ and

$$\sup_{0 < \hbar < 1} |\partial_x^\ell a_j(x; \hbar)| \leq C_{\ell, \mu} x^{-\ell} \quad \text{(2.4)}$$

for $x > x_0$, $j = 1, 2$ and $\ell = 0, 1$. Their Wronskian satisfies

$$W(\psi_1, \psi_2) = \frac{-2}{\hbar}(1 + O(\hbar)) \quad \text{(2.5)}$$

as $\hbar \to 0$.

**Proof.** Let us consider the case of $\tilde{\psi}_1$. Hence, we need to find $a_1$ so that $\psi_1$ is a solution to the differential equation

$$-\hbar^2 u''(x) + V(x)u(x) = 0. \quad \text{(2.6)}$$

Substituting the first expression of (2.2) into the differential equation (2.6) yields

$$-\hbar^2 [\tilde{\psi}_1''(1 + h a_1) + 2\hbar \tilde{\psi}_1 a_1'' + h \tilde{\psi}_1 a_1'''] + V \tilde{\psi}_1(1 + h a_1) = 0 \quad \text{(2.7)}$$

Setting $V_2 := \frac{1}{4} (x^{-2} - 1) - \frac{1}{\hbar^2} V'_0 + \frac{5}{16} \frac{V''_0}{V_0}$ and observing that $-\hbar^2 \tilde{\psi}_1'' + V \tilde{\psi}_1 = -\hbar^2 V_2 \tilde{\psi}_1$, we deduce after dividing the equation by $\tilde{\psi}_1$

$$- (1 + h a_1) V_2 = h (a_1'' + 2 \frac{\tilde{\psi}_1''}{\tilde{\psi}_1} a_1'), \quad \text{(2.8)}$$

We now note the following essential feature of $V_2$ (which was the reason for defining $V_0$ as above):

$$|V_2(x)| \leq C x^{-3}, \quad |\partial_x^k V_2(x)| \leq C_k x^{-3-k} \quad \forall k \geq 0$$
To solve (2.8) we multiply both sides by $\bar{\psi}_1^2$ and obtain

$$\left( a'_1 \bar{\psi}_1^2 \right)' = \frac{-1}{\hbar} V_2 \bar{\psi}_1^2 - a_1 V_2 \bar{\psi}_1^2.$$  

Integration and using the definition of the $\bar{\psi}_1$ yield

$$a'_1(x) = \frac{1}{\hbar} \int_x^{x_0} V_2(y) \bar{\psi}_1^{-2}(x) \bar{\psi}_1^2(y) dy + \int_x^{x_0} a_1(y) V_2(y) \bar{\psi}_1^{-2}(x) \bar{\psi}_1^2(y) dy$$

$$= \frac{1}{\hbar} \int_x^{x_0} V_0(x) \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x) \bar{\psi}_1^2(y) dy + \int_x^{x_0} a_1(y) V_2(y) \bar{\psi}_1^{-2}(x) \bar{\psi}_1^2(y) dy$$

$$= \frac{1}{\hbar} \int_x^{x_0} V_0(x) \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x) \bar{\psi}_1^2(y) dy + \int_x^{x_0} a_1(y) V_2(y) \bar{\psi}_1^{-2}(x) \bar{\psi}_1^2(y) dy.$$  

Strictly speaking, $a_1 = a_1(x, \hbar)$ but we suppress the $\hbar$ from the notation here. After integration in (2.10) we obtain

$$a_1(x) = \frac{-1}{\hbar} \int_x^{x_0} \int_{x'}^{y} V_0(x') \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x') \bar{\psi}_1^2(y) dy dx'$$

$$- \int_x^{x_0} \int_{x'}^{y} V_0(x') \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x') \bar{\psi}_1^2(y) dy dx'$$

$$= \frac{-1}{\hbar} \int_x^{x_0} \int_{x'}^{y} V_0(x') \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x') \bar{\psi}_1^2(y) dy dx'$$

$$- \int_x^{x_0} \int_{x'}^{y} V_0(x') \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x') \bar{\psi}_1^2(y) dy dx'.$$

Furthermore,

$$\int_x^{y} V_0(x') \frac{1}{2} V_0(y) \bar{\psi}_1^{-2}(x') \bar{\psi}_1^2(y) dx' = \frac{-\hbar}{2} [e^{\tilde{S}(y)} - e^{\tilde{S}(x)}].$$

and

$$V_0(y) \frac{1}{2} e^{\tilde{S}(y)} \int_x^{y} V_0(x') \frac{1}{2} e^{\tilde{S}(x')} dx' = \frac{-\hbar}{2} V_0(y) \frac{1}{2} [1 - e^{\tilde{S}(y)} - e^{\tilde{S}(x)}].$$

From this it follows that

$$a_1(x) = \frac{1}{2} \int_x^{x_0} V_0(y) \frac{1}{2} [e^{\tilde{S}(y)} - e^{\tilde{S}(x)}] dy.$$  

This is a standard Volterra equation. To solve it, we first introduce a new function $\rho(x)$ given by

$$\rho(x) = \int_x^{x_0} |V_0(y) \frac{1}{2} V_2(y)| dy.$$  

In view of the decay of $V_2$ we see that the integrand here decays like $y^{-2}$ so that $\rho \in L^\infty(x_0, \infty)$. Then we define a sequence $a_s^1(x)$, $s = 0, 1, \ldots$, with $a_0^1(x) = 0$ and

$$a_s^1(x) = \frac{1}{2} \int_x^{x_0} V_0(y) \frac{1}{2} [e^{\tilde{S}(y)} - e^{\tilde{S}(x)}] V_2(y) |1 + \hbar a_{s-1}^1(y)| dy.$$  

We claim that

$$|a_s^1(x) - a_{s-1}^1(x)| \leq \frac{\rho_s(x) \hbar^{s-1}}{s!}.$$
To prove this we proceed by induction and observe that
\[ S(y) - S(x) = -\int_y^x V_0^{\frac{1}{2}}(t, h) \, dt \]
and hence for \( 0 < x_0 < y < x \), \(|e^{\hat{\psi}(S(x)-S(x))} - 1| \leq 2\). Therefore, (2.17) is valid for \( s = 1 \). Furthermore, if we assume the validity of (2.17) for \( s = k \) then since
\[ a_1^{k+1}(x) - a_1^k(x) = \frac{\hbar}{2} \int_x^{x_0} V_0(y) \frac{d}{dy} \left[ e^{\hat{\psi}(S(x)-S(x))} - 1 \right] V_2(y) (a_1^k(y) - a_1^{k-1}(y)) \, dy, \]
we have
\[ |a_1^{k+1}(x) - a_1^k(x)| \leq \frac{\hbar^k}{k!} \int_x^{x_0} V_0(y) \frac{d}{dy} V_2(y) \rho^k(y) \, dy \]
\[ = -\frac{\hbar^k}{k!} \int_x^{x_0} \rho'(y) \rho^k(y) \, dy = \frac{\rho^{k+1}(x) \hbar^k}{(k+1)!}. \]
We would like to have an estimate for the function \( \rho(x) \), therefore it suffices that we obtain an estimate for \( \int_{x_0}^{\infty} |V_0(y)| \frac{d}{dy} |V_2(y)| \, dy \). As already noted above, for \( x > x_0 \)
\[ \rho(x) \leq \int_{x_0}^{\infty} |V_0(y)| \frac{d}{dy} |V_2(y)| \, dy \leq C(\mu) < \infty. \]
Hence, the solution to the integral equation (2.13) is given by
\[ a_1(x) = \sum_{s=1}^{\infty} (a_1^s(x) - a_1^{s-1}(x)) \]
and satisfies \( \sup_{x > x_0} |a_1(x)| \leq C(\mu) < \infty \) uniformly in \( 0 < \hbar < 1 \). To derive the estimate for \( a_1'(x) \), we observe that
\[ \frac{1}{\hbar} e^{\hat{\psi}(S(x)-S(x))} = \frac{1}{2} V_0^{-\frac{1}{2}}(y) \partial_y e^{\hat{\psi}(S(x)-S(x))}. \]
Therefore, using this observation and integrating by parts in (2.10) yields
\[ a_1'(x) = \frac{1}{2} \int_{x_0}^{x} \left( \frac{V_{0}(x)}{V_{0}(y)} \right)^{\frac{1}{2}} \partial_y \left[ e^{\hat{\psi}(S(x)-S(x))} - 1 \right] V_0(y) \frac{d}{dy} V_2(y) \, dy \]
\[ + \int_{x_0}^{x} \left( \frac{V_{0}(x)}{V_{0}(y)} \right)^{\frac{1}{2}} V_2(y) a_1(y) \, dy \]
\[ = -\frac{1}{2} \int_{x_0}^{x} V_0(x) \frac{d}{dy} \left[ e^{\hat{\psi}(S(x)-S(x))} - 1 \right] V_2(x_0) \, dy \]
\[ - \frac{1}{2} \int_{x_0}^{x} V_0(x) \frac{d}{dy} \left[ e^{\hat{\psi}(S(x)-S(x))} - 1 \right] V_2(x_0) \, dy \]
\[ + \int_{x_0}^{x} \left( \frac{V_{0}(x)}{V_{0}(y)} \right)^{\frac{1}{2}} V_2(y) a_1(y) \, dy. \]
At this point we note that for \( x > x_0 \) (\( x_0 \) large enough), we have \( |\partial_y V_0(x, \hbar)| \leq c, \mu x^{-2-\ell}, \) uniformly in \( \hbar \). Hence
\[ |\partial_y [V_0(y)^{-1} V_2(y)]| \lesssim y^{-2}, \]
and using the boundedness of \( a_1 \), together with that of \( \rho(x) \), obviously implies that for \( x > x_0 \)
\[ |a_1'(x)| \leq C_\mu x^{-1}, \]
uniformly in $0 < \hbar < 1$ as desired. For the case of $a_2$ one proceeds in essentially the same way using, however, the forward Green function rather than the backward one. This yields

$$a_2(x) = \frac{1}{\pi} \int_{x}^{\infty} V_0(y)^{-1} \left[ e^{\frac{\pi}{2}(S(y) - S(x))} - 1 \right] V_2(y) \left[ 1 + h a_2(y) \right] dy.$$  

The same arguments as before now show that $a_2$ satisfies (2.4). The Wronskian $W(\psi_1, \psi_2)$ is obtain by evaluating at $x = \infty$. □

The same analysis of course yields zero energy solutions with the correct asymptotic behavior as $x \to -\infty$. Note that the solution $\psi_2$ is the subordinate one, i.e., it is the unique (up to a nonzero scalar multiple) solution with the slowest possible growth. Hence, a zero energy resonance in the context of Theorem 1 would mean the existence of a nonzero solution to $P(x, \hbar D)f = 0$ with $f(x) \sim c_{\pm}|x|^\frac{1}{2} - \alpha_{\pm}$ as $x \to \pm\infty$ and with

$$\alpha_{\pm} = \sqrt{\frac{1}{4} + \mu_{\pm}^2 \hbar^{-2}}.$$  

It is easy to see that such a solution cannot exist if $V > 0$. Indeed, let $\chi$ be a standard cut-off function with $\chi(0) = 1$ and set $\chi_R(x) := \chi(x/R)$. If a globally subordinate solution $f(x)$ did exist, then

$$0 = \limsup_{R \to \infty} \langle P(\cdot, \hbar D)f, \chi_R f \rangle = \int \left[ (f')^2(x) + V(x)f^2(x) \right] dx$$  

implies that $f = 0$, which is a contradiction.

3. The Liouville-Green transform for small energies

In this section, we consider the equation

$$(3.1) \quad -\hbar^2 f''_{\pm}(x) + V(x)f_{\pm}(x) = Ef_{\pm}(x)$$  

where $V$ is as in Theorem 1. As explained in the introduction, we will use the Liouville-Green transform to reduce (3.1) to a perturbed Airy equation. We begin with a statement of the formal aspects (i.e., not involving estimates) of this transform, cf. Chapter 6 in [19] and Langer’s papers [14]–[16]. Henceforth, $V, V_0$ are as in Theorem 1. Throughout this section $x \geq 0$.

**Lemma 3.** There exists $E_0 = E_0(V) > 0$ so that for all $0 < E < E_0$ one has the following properties: the equation $V_0(x; \hbar) - E = 0$ has a unique (simple) solution on $x > 0$ which we denote by $x_1 = x_1(E; \hbar)$. With $Q_0 := V_0 - E$

$$\zeta = \zeta(x, E; \hbar) := \text{sign}(x - x_1(E; \hbar)) \left[ \frac{3}{2} \int_{x_1(E; \hbar)}^{x} \sqrt{|Q_0(u, E; \hbar)|} du \right]^\frac{1}{2}$$  

defines a smooth change of variables $x \mapsto \zeta$ for all $x \geq 0$. Let $q := -\frac{Q_0}{\zeta}$. Then $q > 0$, $\frac{dq}{d\zeta} = \zeta' = \sqrt{q}$, and

$$-\hbar^2 f'' + (V - E)f = 0$$  

transforms into

$$(3.3) \quad -\hbar^2 \ddot{w}(\zeta) = (\zeta + \hbar^2\ddot{V}(\zeta, E; \hbar))w(\zeta)$$
under \( w = \sqrt{\gamma} f = q^\frac{1}{4} f \). Here \( \dot{\gamma} = \frac{d}{dx} \) and

\[
\tilde{V} := \frac{1}{4} q^{-1}(x)^{-2} - q^{-\frac{1}{4}} \frac{d^2 q^\frac{1}{4}}{dx^2}
\]

Proof. Let \( E_0 > 0 \) be such that \( V_0(x; h) = E \) has a unique pair of solutions denoted by \( x_2(E; h) < 0 < x_1(E; h) \). It is clear that \( (3.8) \) defines a smooth map away from \( x = x_1(E; h) \). Taylor-expanding \( Q_0(x, E; h) \) in a neighborhood of that point and using that \( \frac{d}{dx} \left( x_1(E; h) \right) < 0 \) implies that \( \zeta(x, E; h) \) is smooth around \( x = x_1 \) as well with \( \zeta'(x_1, E; h) > 0 \). Next, one checks that

\[
\dot{w} = q^{-\frac{1}{4}} f' + \frac{dq^\frac{1}{4}}{d\zeta} f, \quad \ddot{w} = q^{-\frac{3}{4}} f'' + \frac{d^2 q^\frac{1}{4}}{dx^2} f
\]

and thus, using \( -\hbar^2 f'' = (E - V)f \),

\[
-\hbar^2 \ddot{w} = q^{-1}(E - V)w - \hbar^2 q^{-\frac{1}{4}} \frac{d^2 q^\frac{1}{4}}{dx^2} w
\]

\[
= q^{-1}(-Q_0 + h^2(x)^{-2}/4)w - \hbar^2 q^{-\frac{1}{4}} \frac{d^2 q^\frac{1}{4}}{dx^2} w
\]

\[
= \zeta w(\zeta) + h^2 q^{-1}(x)^{-2}/4 - q^{-\frac{1}{4}} \frac{d^2 q^\frac{1}{4}}{dx^2} w
\]

as claimed. \( \square \)

We now analyze the properties of the change of variables introduced in the previous lemma. Recall that

\[ V_0(x; h) = (\mu_+^2 + h^2/4)x^{-2}(1 + O(x^{-1})) \]

which implies that

\[ x_1(E; h) = c(h)E^{-\frac{1}{2}}(1 + O(E^{\frac{1}{4}})), \quad c(h) = \sqrt{\mu_+^2 + h^2/4} \]

It will be convenient for us to normalize the constants here so that \( c(h) = \sqrt{2} \) and we shall assume that for the remainder of this section. Moreover, we shall mostly suppress the harmless \( h \) dependence of various functions in our notation. We begin with the following “normal form” lemma which will allow us to describe the function \( \zeta \) in any region of the form \( x \geq E^{1/2} \) (which, in particular, contains the turning point). Note that Lemma 4 normalizes the turning point \( x_1 \) to \( \zeta = 1 \) by scaling out the energy.

**Lemma 4.** Let \( \varepsilon > 0 \) but fixed. There exists a smooth map \( \xi = \xi(y, E) \) on \( (y, E) \in (\varepsilon, \infty) \times (0, E_0) \) with \( E_0 \) small so that \( \xi(E^{1/2}x_1(E), E) = 1 \) and for all \( (y, E) \) in this range,

\[ 1 - E^{-1}V_0(E^{1/2}y) = \left( \frac{d\xi}{dy} \right)^2 (1 - \zeta^{-2}) \]

and such that, with some constant \( \xi_0(E) \),

\[ \xi(y, E) = y + \xi_0(E) + y^{-1} \rho_0(y, E) \]

\[ |\partial^k_E \partial^\ell_y \xi(y, E)| \leq C_{k, \ell} E^{-k} y^{1-\ell} \]

for all \( k, \ell \geq 0 \). The functions \( \xi_0 \) and \( \rho_0 \) from \( \xi_0 \) satisfy

\[ |\partial^k_E \xi_0(E)| \leq C_k E^{-k}, \quad |\partial^k_E \partial^\ell_y \rho_0(y, E)| \leq C_{j,k} y^{-j} E^{-k} \]
for all \( k, j \geq 0 \) and uniformly in \((y, E) \in (1, \infty) \times (0, E_0)\). For fixed \( 0 < E < E_0 \) the map \( y \mapsto \xi(y, E) \) is a global diffeomorphism whose inverse \( y = y(\xi, E) \) satisfies the bounds

\[
y(\xi, E) = \xi + y_0(E) + \xi^{-1}\tilde{\rho}_0(\xi, E)
\]

(3.9)

\[
|\partial_{\xi}^k \partial_{E}^\ell y(\xi, E)| \leq C_{k, \ell} E^{-k-\ell}
\]

(3.10)

for all \( k, \ell \geq 0 \) and with functions \( y_0, \tilde{\rho}_0 \) satisfying \( 3.8 \) but relative to \( \xi \) rather than \( y \). All constants are allowed to depend on \( \varepsilon > 0 \).

**Proof.** Set \( y_1 = y_1(E) := E^\frac{1}{4} x_1(E) \). Then \( y_1 = \sqrt{2} + O(E^\frac{1}{4}) \) as \( E \to 0^+ \). Note that the \( O(\cdot) \) term here satisfies

\[
\partial_{E}^k O(E^\frac{1}{4}) = O(E^\frac{1}{4-k}) \quad \forall k \geq 0
\]

due to the corresponding assumption on the error term of \( V \). The same comment applies to every \( O(\cdot) \) term appearing in this proof, both with respect to derivatives in \( E \) and spatial variables. Define the change of variables \( \xi = \xi(y, E) \) via

\[
\int_{y_1}^{y} \sqrt{1 - E^{-1}V_0(E^{-\frac{1}{4}} u)} \, du = \int_{y_1}^{\xi} \sqrt{1 - t^{-2}} \, dt \quad y > y_1
\]

(3.11)

\[
\int_{y}^{y_1} \sqrt{1 - E^{-1}V_0(E^{-\frac{1}{4}} u) - 1} \, du = \int_{\xi}^{1} \sqrt{t^{-2} - 1} \, dt \quad \varepsilon < y < y_1
\]

(3.12)

By monotonicity, these identities define a unique correspondence between \( \xi \) and \( y \) on these ranges which is, moreover, smooth and strictly increasing on \( \varepsilon < y < y_1 \) and \( y_1 < y < \infty \). By inspection, they also satisfy \( 3.5 \). Since

\[
1 - E^{-1}V_0(E^{-\frac{1}{4}} y) = 1 - \frac{2}{u^2} + O(E^\frac{1}{4} u^{-3})
\]

it follows furthermore that the interval \( \varepsilon \leq y < \infty \) is transformed into one of the form \( 0 < \xi_1(E) < \xi < \infty \) where

\[
\xi_1(E) = \xi_1(0) + O(E^\frac{1}{4}) \quad \text{as} \quad E \to 0^+
\]

(3.13)

and \( \xi_1(0) > 0 \) is a constant. We first show that the map \( \xi = \xi(y, E) \) so defined, is smooth for all \((y, E) \in (\varepsilon, 2) \times (0, E_0)\) together with the desired estimates. To this end, write

\[
1 - E^{-1}V_0(E^{-\frac{1}{4}} y) = (y - y_1)U(y, E)
\]

\[
U(y, E) := -E^{-\frac{1}{4}} \int_{0}^{1} V_0'(E^{-\frac{1}{4}}(y_1 + t(y - y_1))) \, dt
\]

Then for all \( 0 < E \leq E_0 \) and all \( k, \ell \geq 0 \),

\[
\max_{1 \leq y \leq 2} |\partial_{E}^k \partial_{y}^\ell U(y, E)| \leq C_{k, \ell} E^{-k}, \quad \min_{1 \leq y \leq 2} U(y, E) \geq c_0 > 0
\]

(3.14)

For all \( \varepsilon < y < 2 \) we rewrite \( 3.11 \) and \( 3.12 \) in the form

\[
(y - y_1)Y(y, E) = (\xi - 1)X(\xi)
\]

(3.15)

---

\(^3\)We will say that a \( O(\cdot) \) term behaves like a symbol if its derivatives are governed by such power-laws.
where

\[ Y(y, E) := \left( \int_0^1 \sqrt{(1-t)U(y_1 + t(y - y_1), E)} \, dt \right)^{\frac{3}{2}} \]

\[ X(\xi) := \left( \int_0^1 \frac{\sqrt{s(2+s(\xi - 1))}}{1+s(\xi - 1)} \, ds \right)^{\frac{3}{2}} \]

By the preceding,

\[ |\partial_k^E \partial_\ell y Y(y, E)| \leq C_{k,\ell} E^{-k}, \quad |\partial_\ell^j X(\xi)| \leq C_j \quad \forall \ k, \ell, j \geq 0 \]

uniformly on the interval \( \varepsilon \leq y \leq 2 \) and the corresponding interval in \( \xi \). By the inverse function theorem, (3.15) defines a (unique) smooth map also locally around \( \xi = 1 \) and \( y = y_1 \); this agrees with the previous definition for \( y \neq y_1 \) and thus furnishes the desired smooth extension through the point \( y = y_1 \). Furthermore, from

\[ y_1 = \sqrt{2} + O(E^{\frac{3}{2}}), \quad \partial_\xi^k y_1 = O(E^{\frac{3}{2} - k}) \]

and (3.15), (3.14) we conclude that

\[ \max_{1 \leq y \leq 2} |\partial_k^E \partial_\ell^j Y(y, E)| \leq C_{k,\ell} E^{-k} \]

for all \( k, \ell \geq 0, 0 < E < E_0 \). For large \( y \), we write

\[ \int_{y_1}^y \left\{ 1 - \frac{2}{u^2} \left( 1 + O(E^{\frac{3}{2}} u^{-1}) \right) \right\}^{\frac{3}{2}} \, du = \int_1^\xi \sqrt{1-v^{-2}} \, dv \]

The integral on the right-hand side satisfies

\[ \int_1^\xi \sqrt{1-v^{-2}} \, dv = \xi + \kappa + O(\xi^{-1}) \]

with a constant \( \kappa \), whereas the one on the left-hand side is equal to

\[ y + y_0 + O(y^{-1}) + O(E^{\frac{3}{2}} y^{-2}) \]

with a constant \( y_0(E) \). It is easy to see that

\[ \xi + \kappa + O(\xi^{-1}) = y + y_0 + O(y^{-1}) + O(E^{\frac{3}{2}} y^{-2}) \]

implies (3.6) and we are done. The statements about the inverse follow easily. \( \square \)

We refer to Lemma 3 as a “normal form” since (3.1), on the interval \( x > E^{-\frac{1}{2}} \), turns into a suitably normalized (perturbed) Bessel equation in the variable \( \xi \), see the appendix. By means of the change of variables introduced in Lemma 4 it is now an easy matter to describe \( \zeta(x, E) \) from Lemma 3 on the interval \( x \geq E^{-\frac{1}{2}} \).

**Corollary 5.** For all \( 0 < E < E_0 \) the following holds: there exists a constant \( c_0 > \sqrt{2} \) so that on the interval \( \varepsilon < \sqrt{E} x < c_0 \)

\[ \zeta(x, E) = 2^\frac{3}{4} (\xi - 1) \left[ 1 + O(\xi - 1) \right] \]

with \( O(\cdot) \) analytic and \( \xi = \xi(\sqrt{E} x, E) \). For all \( x \geq E^{-\frac{1}{2}} c_0 \),

\[ \zeta(x, E) = \xi + \gamma + O(\xi^{-1}) \]

where \( \gamma \) is some constant and \( O(\cdot) \) is analytic. Neither of the \( O(\cdot) \) terms here depend on \( E \) (other than through \( \xi \)).
Proof. We begin with $\xi$ close to $\xi = 1$. The action $S(x, E)$ then satisfies, with $\xi$ as in Lemma 4

$$S(x, E) = \text{sign}(x - x_1(E)) \int_{x_1(E)}^{x} \sqrt{|E - Q_0(u)|} \, du$$

$$= \text{sign}(x - x_1(E)) \int_{\sqrt{E}x}^{\sqrt{E}x_1(E)} \sqrt{|1 - E^{-1}V_0(E^{-\frac{1}{2}}y)|} \, dy$$

$$= \text{sign}(\xi(\sqrt{E}x) - 1) \int_{1}^{\xi(\sqrt{E}x, E)} \sqrt{|1 - \eta^{-2}|} \, d\eta$$

$$= \sqrt{2} \text{sign}(\xi - 1)(\xi - 1)^{\frac{2}{3}} (1 + O(\xi - 1)) \int_{1}^{\xi(\sqrt{E}x, E)} \sqrt{1 - \eta^{-2}} \, d\eta$$

where the $O(\cdot)$ term is analytic in $|\xi - 1| < 1$ and $\xi = \xi(\sqrt{E}x, E)$. In terms of $\zeta$ this means that

$$\zeta(x, E) = 2^{\frac{2}{3}}(\xi - 1)[1 + O(\xi - 1)]$$

which is (3.16). The constant $c_0$ is chosen so that $1 < \xi(c_0E^{-\frac{1}{2}}, E) < 2$ for all $0 < E < E_0$. Since $x_1(E) = \sqrt{2/E} + o(1)$ as $E \to 0$ we see that $c_0 > \sqrt{2}$. As for (3.17),

$$S(x, E) = \int_{1}^{\xi(\sqrt{E}x, E)} \sqrt{1 - \eta^{-2}} \, d\eta$$

$$= \int_{1}^{\zeta} (1 + O(\eta^{-2})) \, d\eta = \xi + \gamma + O(\xi^{-1})$$

Since $\zeta = (\frac{2}{3}S)^{\frac{2}{3}}$, we are done.

$$\square$$

In the region $0 < x < \varepsilon x_1(E)$ we have the following description of $\zeta(x, E)$ with $\varepsilon$ the same as in Lemma 4. In fact, in the following lemma we will need $\varepsilon$ small and then use this choice in Lemma 4.

**Lemma 6.** For sufficiently small and fixed $\varepsilon > 0$ there exists a smooth function $\tilde{x}(x, E)$ on $0 \leq x \leq \varepsilon x_1(E)$ with

$$\tilde{x}(x, E) = x(1 + O(Ex^2))$$

and such that

$$\frac{2}{3} \zeta^2(x, E) = \int_{\tilde{x}(x, E)}^{x_1(E)} \sqrt{V_0(v)} \, dv + O(E \log E)$$

for all $0 \leq x \leq \varepsilon x_1(E)$ and $0 < E < E_0$. The $O(\cdot)$ here behave like symbols.

**Proof.** Define $\tilde{x}$ via

$$\int_{0}^{\tilde{x}(x, E)} \sqrt{V_0(v)} \, dv = \int_{0}^{x} \sqrt{V_0(u) - E} \, du = \int_{0}^{x} \sqrt{V_0(u)(1 + O(Eu^2))} \, du$$

$$= \int_{0}^{x} \sqrt{V_0(u) + O(Eu^2)}$$

Provided $\varepsilon > 0$ is sufficiently small (independently of $E$, of course), it follows from monotonicity considerations that $\tilde{x}(x, E)$ exists with the desired properties. Next,
note that for all $0 < E < E_0$,
\[
\int_0^{x_1(E)} \sqrt{V_0(u)} \, du - \int_0^{x_1(E)} \sqrt{V_0(u) - E} \, du = O(E \log E)
\]
and thus
\[
\frac{2}{3} \zeta^2 = \int_0^{x_1(E)} \sqrt{V_0(u)} \, du - \int_0^x \sqrt{V_0(u) - E} \, du
\]
\[
= \int_0^{x_1(E)} \sqrt{V_0(u)} \, du - \int_0^{\bar{x}_1(E)} \sqrt{V_0(v)} \, dv + O(E \log E)
\]
\[
= \int_0^{x_1(E)} \sqrt{V_0(v)} \, dv + O(E \log E)
\]
as claimed. \hfill \Box

The point of (3.18) is that $O(E \log E)$ is negligible as compared to the integral on the right-hand side which is on the order of $| \log(E(\tilde{x})^2)| \gtrsim 1$. Thus, $\zeta$ behaves to leading order like $| \log(E(x)^2)|^{1/2}$. We now turn to estimating the functions $q, \tilde{V}$ from Lemma [3]. In what follows, the notation $A \sim B$ will denote proportionality of $A, B > 0$ by some constants that are only allowed to depend on $V$. Also, $A \lesssim B$ will denote $A \leq CB$ where $C$ is a constant, and similarly for $A \gtrsim B$.

**Lemma 7.** Using the notations of Lemma [3] let $0 < E < E_0$. Then on the interval $\zeta \geq -1$ the functions $q = q(\zeta, E)$ and $\tilde{V} = \tilde{V}(\zeta, E)$ satisfy
\[
| \partial^k_{E} \partial^\ell \xi | \leq C_{k, \ell} E^{1-k} (\zeta)^{-\ell-1}
\]
\[
| \partial^k_{E} \partial^\ell \tilde{V}(\zeta, E) | \leq C_{k, \ell} E^{-k} (\zeta)^{-2-\ell} \quad \forall k, \ell \geq 0
\]
On the interval $\zeta(0, E) \leq \zeta \leq -1$ we view $q, \tilde{V}$ as functions of $x$ via (3.2). Then one has $q \sim |\zeta|^{-1} (x)^{-2}$ and there is the representation
\[
\tilde{V}(\zeta, E) = -\frac{5}{16\zeta^2} + q^{-1}(E \beta_0(x, E) + (x)^{-3} \beta_1(x, E))
\]
where $\beta_j$ satisfy the bounds
\[
| \partial^k_{E} \partial^\ell \beta_j(x, E) | \leq C_{k, \ell} E^{-k} (x)^{-\ell-1} \quad j = 0, 1
\]
\[
| \partial^k_{E} \partial^\ell \xi \tilde{V}(\zeta, E) | \leq C_{k, \ell} E^{-k} \zeta |\zeta|^{-1} (x)^{-2-\ell} \quad \forall k, \ell \geq 0
\]
All constants are independent of $E$.

**Proof.** The case $\zeta \geq -1$ corresponds to $x \geq \varepsilon x_1(E)$ by Lemma [4] and Corollary [5]. We now use that corollary to write
\[
\zeta = \zeta(\xi, E), \quad \xi = \xi(y, E), \quad y = E^{1/2} x
\]
Then
\[
q = (\zeta')^2 = E(\partial_\zeta \zeta(\xi, E))^2 (\partial_\xi \xi(y, E))^2 \sim E
\]
The derivative bounds on $q$ now follow from those obtained in Lemma [4] and Corollary [5]. As for $\tilde{V}$, we compute, with $\dot{\gamma} = \frac{4}{d\zeta}$,
\[
\tilde{V} = \frac{1}{4} q^{-1}(x)^{-2} - q^{-\frac{1}{4}} \frac{d^2 q^\theta}{d^2 \zeta} = \frac{1}{4} q^{-1}(x)^{-2} + \frac{3}{16} q^{-2} q^2 - \frac{1}{4} q^{-1} \ddot{q}
\]
From the bounds on $q$ which we just derived, the last two terms on the right-hand side of (3.22) are \( \lesssim \zeta^{-2} \) and behave as stated under differentiation. To treat the first term, we invoke (3.21), (3.16), and (3.17) to write
\[
q^{-1}(x)^{-2} = q^{-1}(E^{-\frac{1}{2}}y)^{-2} = q^{-1}(E^{-\frac{1}{2}}y(\xi, E))^{-2} \\
= q^{-1}(E^{-\frac{1}{2}}y(\xi(\zeta, E), E))^{-2}
\]
which implies the correct bounds. Indeed, if \( |\zeta| \lesssim 1 \), then \( q \sim E \) and the change of variables \( \zeta \mapsto \xi \mapsto y \) has derivatives of size \( \lesssim 1 \) relative to \( \zeta \) uniformly in \( E \).
This implies that \( q^{-1}(x)^{-2} \sim 1 \) with derivatives with respect to \( \zeta \) of size \( \lesssim 1 \); furthermore, each derivative in \( E \) costs one power of \( E \). Next, if \( \zeta \geq 1 \), then the change of variables \( \zeta \mapsto \xi \mapsto y \) acts like \( \zeta^{\frac{3}{2}} \) by Corollary 6. Thus,
\[
q^{-1}(x)^{-2} \sim E^{-1}(E^{-\frac{1}{2}}\zeta^{\frac{3}{2}})^{-2} \sim \zeta^{-2}
\]
with each \( \zeta \) derivative gaining one more power of decay in \( \zeta \).
In the remaining case \( \zeta \leq -1 \) one first calculates, on the one hand,
\[
q^{\frac{1}{2}}d^2q^{\frac{1}{2}} = \frac{5}{16\zeta^2} + \frac{1}{4}\hat{Q}_0 - \frac{1}{4\zeta Q_0} - \frac{3}{16} \left( \frac{\hat{Q}_0}{Q_0} \right)^2 \\
= \frac{5}{16\zeta^2} + q^{-1}\left[ \frac{1}{4}\hat{V}_0'' - \frac{5}{16}\left( \frac{V_0'}{Q_0} \right)^2 \right] 
\]
where \( ' = \frac{d}{dx} \). Thus, from (3.22),
\[
\tilde{V} = \frac{1}{4}q^{-1}(x)^{-2} - q^{\frac{1}{2}}d^2q^{\frac{1}{2}} \\
= -\frac{5}{16\zeta^2} + q^{-1}\left[ \frac{1}{4}(x)^{-2} - \frac{1}{4}\hat{V}_0'' + \frac{5}{16}\left( \frac{V_0'}{Q_0} \right)^2 \right] \\
= -\frac{5}{16\zeta^2} + q^{-1}\left[ E\beta_0(x, E) + (x)^{-3}\beta_1(x, E) \right] 
\]
where we have set
(3.23) \( \beta_0(x, E) := E^{-1}\left[ \frac{1}{4}\left( \frac{V_0''}{V_0} - \frac{V_0''}{Q_0} \right) + \frac{5}{16}\left( \frac{V_0'}{Q_0} \right)^2 - \left( \frac{V_0'}{V_0} \right)^2 \right] \)
(3.24) \( \beta_1(x, E) := (x)^{-3}\left[ \frac{1}{4}(x)^{-2} - \frac{1}{4}\hat{V}_0'' + \frac{5}{16}\left( \frac{V_0'}{V_0} \right)^2 \right] \)
As already noted in Section 2, the \( x^{-2} \) terms inside the brackets in (3.24) cancel so that the leading order is \( x^{-3} \). In fact, \( |\partial_x^2\beta_1(x, E)| \leq C_\ell(x)^{-\ell} \) in view of our assumptions on \( V \), see Theorem 4. As for \( \beta_0 \), we note that in the range \( \zeta \leq -1 \), one has \( Q_0 \sim V_0 \). Since \( Q_0 = V_0 - E \), this implies that the expression in brackets in (3.23) is \( \lesssim E \) together with the natural derivative bounds. The bounds on
\[
q = (V_0 - E)|\zeta|^{-1} \sim (x)^{-2}|\zeta|^{-1}, \quad |\zeta| \sim |\log(E(x)^2)|^{\frac{3}{2}}
\]
follow from (3.18) and we are done. \( \square \)

In Lemma 7 the modification of \( V \) to \( V_0 \) only played a role in the regime \( \zeta \ll -1 \) which is the same as \( x < \varepsilon x_1(E) \). This is natural, since we know from Section 2 that this modification really comes from the \( E = 0 \) case which corresponds to \( x_1 = +\infty \). We will see this mechanism at work in the following section, too.

\footnote{\( \beta_1 \) does not depend on \( E \), but this makes no difference.}
4. Solving the perturbed Airy equation

This section is devoted to solving (3.3), at least in the asymptotic sense relative to $\hbar$. We shall use the notations and results of the previous section. For the properties of the Airy functions $\text{Ai}, \text{Bi}$ listed below we refer the reader to Chapter 11 of [19].

Proposition 8. Let $\hbar_0 > 0$ be small. A fundamental system of solutions to (3.3) in the range $\zeta \leq 0$ is given by

$$
\phi_1(\zeta, E, h) = \text{Ai}(\tau)[1 + h a_1(\zeta, E, h)]
$$

$$
\phi_2(\zeta, E, h) = \text{Bi}(\tau)[1 + h a_2(\zeta, E, h)]
$$

with $\tau := -\hbar^{-\frac{2}{3}}\zeta$. Here $a_1, a_2$ are smooth, real-valued, and they satisfy the bounds, for all $k \geq 0$ and $j = 1, 2$, and with $\zeta_0 := \zeta(0, E)$,

$$
|\partial^k_E a_j(\zeta, E, h)| \lesssim E^{-k} \min \left[ \hbar^\frac{1}{2} (\hbar^{-\frac{2}{3}}\zeta)^\frac{1}{2}, 1 \right]
$$

$$
|\partial^j E \partial_\zeta a_j(\zeta, E, h)| \lesssim E^{-k} \left[ \hbar^{-\frac{1}{2}} (\hbar^{-\frac{2}{3}}\zeta)^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + \zeta^\frac{1}{2} \chi_{[\zeta_0 \leq \zeta \leq -1]} \right]
$$

uniformly in the parameters $0 < h < \hbar_0$, $0 < E < E_0$.

Proof. Let $\phi_{1,0}(\zeta, h) := \text{Ai}(\tau)$ and $\phi_{2,0}(\zeta, h) := \text{Bi}(\tau)$. We seek a basis of the form

$$
\phi_j(\zeta) = \phi(\zeta, h, E) = \phi_{j,0}(\zeta, h)(1 + h a_j(\zeta, h, E))
$$

for $\zeta \leq 0$. This representation is meaningless for $\zeta > 0$ since $\phi_{j,0}$ have real zeros there. On the other hand, on $\zeta \leq 0$ they do not vanish. We obtain the equation

$$
(\phi_j^2 \partial_\zeta) = -\frac{1}{\hbar} \tilde{V} \phi_j^2(1 + h a_j)
$$

for $j = 1, 2$ where $\partial_\zeta = \partial_\zeta$. A solution of (4.2) on $\zeta \leq 0$ is given by, with $a_2(\zeta) = a_2(\zeta, h, E)$,

$$
a_2(\zeta) := -\frac{1}{\hbar} \int_\zeta^0 \phi_{2,0}(\eta, h) \int^\eta \phi_{2,0}^2(\eta, h) d\eta \tilde{V}(\eta, E)(1 + h a_2(\eta)) d\eta
$$

$$
(4.3) = -\hbar^\frac{1}{2} \text{Bi}^2(w) \int_u^{-\hbar^\frac{1}{2}} \text{Bi}^{-2}(v) dv \tilde{V}(\hbar^\frac{1}{2} u, E)(1 + h a_2(\hbar^\frac{1}{2} u)) du
$$

This solution is unique with the property that $a_2(0) = \hat{a}_2(0) = 0$. Recall the asymptotic behavior, see [19],

$$
\text{Bi}(x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{\frac{2}{3} x^\frac{3}{2}} [1 + O(x^{-\frac{2}{3}})] \quad \text{as } x \to \infty
$$

$$
\text{Bi}(x) \geq \text{Bi}(0) > 0 \quad \forall x \geq 0
$$

$$
\text{Ai}(x) = \frac{1}{2} \pi x^{-\frac{1}{2}} e^{-\frac{2}{3} x^\frac{3}{2}} [1 + O(x^{-\frac{2}{3}})] \quad \text{as } x \to \infty
$$

$$
\text{Ai}(x) > 0 \quad \forall x \geq 0
$$

Also note the useful fact, valid for any $0 \leq x_0 < x_1$,

$$
(4.4) \int_{x_0}^{x_1} \text{Bi}^{-2}(y) dy = \pi^{-1} \left( \frac{\text{Ai}(x_0)}{\text{Bi}(x_0)} - \frac{\text{Ai}(x_1)}{\text{Bi}(x_1)} \right)
$$

which implies that

$$
\left| \text{Bi}^2(x_0) \int_{x_0}^{x_1} \text{Bi}^{-2}(y) dy \right| \lesssim (x_0)^{-\frac{1}{2}}$$
The leading term in $\textbf{(4.3)}$, i.e.,

$$a_{2,0}(\zeta, E, \hbar) := -\hbar^{\frac{1}{2}} \int_{0}^{\zeta} \sin^{2}u \left( \int_{u}^{\zeta} \sin^{-2}v \, dv \right) V(-\hbar^{\frac{2}{3}}u, E) \, du$$
	herefore satisfies the bound (dropping $E, \hbar$ from $a_{2,0}$ for simplicity)

$$|a_{2,0}(\zeta)| \lesssim \hbar^{\frac{1}{2}} \int_{0}^{\zeta} \langle u \rangle^{-\frac{1}{2}} |V(-\hbar^{\frac{2}{3}}u, E)| \, du$$

We now use the estimates from Lemma [7] to bound the right-hand side. If $-1 \leq \zeta \leq 0$, then this yields

$$|a_{2,0}(\zeta)| \lesssim \hbar^{\frac{1}{2}} \int_{0}^{\zeta} \langle u \rangle^{-\frac{1}{2}} \, du \lesssim \hbar^{\frac{1}{2}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}} \tag{4.5}$$

On the other hand, if $\zeta := \langle 0, E \rangle \leq \zeta \leq -1$, then we obtain

$$|a_{2,0}(\zeta)| \lesssim \hbar^{\frac{1}{2}} \int_{0}^{\zeta} \langle u \rangle^{-\frac{1}{2}} \, du \tag{4.6}$$

$$+ \hbar^{\frac{1}{2}} \int_{\zeta}^{0} \langle u \rangle^{-\frac{1}{2}} \, du$$

$$\tag{4.7}$$

The variable $z$ appearing in $\textbf{(4.7)}$ is tied to the integration variable $u$ via $-\hbar^{\frac{2}{3}}u = \zeta(z, E)$, see Lemma [8]. The integral in $\textbf{(4.6)}$ and the first term inside the brackets in $\textbf{(4.7)}$ contribute

$$\hbar^{\frac{1}{2}} \int_{0}^{\zeta} \langle u \rangle^{-\frac{1}{2}} \, du + \hbar^{\frac{1}{2}} \int_{\zeta}^{0} \langle u \rangle^{-\frac{1}{2}} \, du \lesssim 1$$

Next, with $\zeta(x_{2}, E) = -1$, and $v = \zeta(z, E)$, $dv = \sqrt{q} \, dz$,

$$\hbar^{\frac{1}{2}} \int_{\zeta}^{0} \langle u \rangle^{-\frac{1}{2}} \, du = E \int_{1}^{\zeta} v^{-\frac{1}{2}} q^{-1} \, dv$$

$$= E \int_{x}^{x_{2}} Q_{0}^{-\frac{1}{2}}(z, E) \, dz \lesssim E \int_{x}^{x_{1}} z \, dz$$

$$\lesssim E x_{1}(E)^{2} \lesssim 1$$

Finally, using that $\frac{dz}{dz} = \sqrt{q}$ once again one obtains

$$\hbar^{\frac{1}{2}} \int_{\zeta}^{0} \langle u \rangle^{-\frac{1}{2}} \, du = \int_{1}^{\zeta} v^{-\frac{1}{2}} q^{-1}(v) \, dv$$

$$= \int_{x}^{x_{2}} Q_{0}^{-\frac{1}{2}}(z, E) \, dz$$

$$\lesssim \int_{x}^{x_{1}} z^{-2} \, dz \lesssim 1$$

In summary$^5$

$$|a_{2,0}(\zeta, E, \hbar)| \lesssim \min(1, \hbar^{\frac{1}{2}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}})$$

$^5$Had we used $V$ instead of $V_{0}$ in our definition of $\zeta$, then we would be losing a factor of $\log E$ at this point. Indeed, for the case of $V$ we would need to replace $E + \langle z \rangle^{-3}$ by the strictly weaker $\langle z \rangle^{-2}$ in $\textbf{(4.7)}$ which then leads to the logarithmically divergent integral $\int_{x}^{x_{1}} \langle z \rangle^{-2} \, dz$. 

uniformly in \( \zeta \in [\zeta_0, 0] \), \( 0 < E < E_0 \), and \( 0 < h < h_0 \). Due to the linear nature of (4.3), a contraction argument now yields the same bound for \( a_2 \): in fact, due to the derivative bounds of Lemma 4, we obtain the more general estimate

\[
|\partial_{\zeta}^k a_2(\zeta, E, h)| \leq C_k E^{-k} \min(1, h^k \langle h^{-\frac{3}{10}} \zeta \rangle^{\frac{3}{2}}) \quad \forall k \geq 0
\]

uniformly in the parameters. As for the first derivative in \( \zeta \), observe that

(4.8) \[ \dot{a}_2(\zeta) = \frac{h^{-1}}{\phi_{2,0}(\zeta, h)} \int_{\zeta}^{0} \phi_{2,0}^2(\eta, h) \bar{V}(\eta, E)(1 + ha_2(\eta)) d\eta \]

whence, for all \(-1 \leq \zeta \leq 0\),

\[
|\dot{a}_2(\zeta, E, h)| \lesssim h^{-1} \phi_{2,0}^2(\zeta, h) \int_{\zeta}^{0} \phi_{2,0}^2(\eta, h) |\bar{V}(\eta, E)| d\eta
\]

\[
\lesssim h^{-\frac{1}{2}} Bi^{-2}(-h^{-\frac{3}{10}} \zeta) \int_0^{-h^{-\frac{3}{10}} \zeta} Bi^2(u) du
\]

\[
\lesssim h^{-\frac{1}{2}} \langle h^{-\frac{3}{10}} \zeta \rangle^{\frac{3}{2}} e^{-\frac{3}{10} h^{-1} |\zeta|^2} \int_0^{-h^{-\frac{3}{10}} \zeta} \langle u \rangle^{-\frac{1}{2}} e^{\frac{3}{2} u^2} du
\]

\[
\lesssim h^{-\frac{1}{2}} \langle h^{-\frac{3}{10}} \zeta \rangle^{\frac{3}{2}} e^{-\frac{3}{10} h^{-1} |\zeta|^2} \int_0^{-h^{-\frac{3}{10}} \zeta} \langle u \rangle^{-\frac{1}{2}} e^{\frac{3}{2} u^2} du
\]

If \( \zeta_0 \leq \zeta \leq -1 \), then

(4.9) \[ |\dot{a}_2(\zeta, E, h)| \lesssim h^{-1} \phi_{2,0}^2(\zeta, h) \int_{\zeta}^{0} \phi_{2,0}^2(\eta, h) |\bar{V}(\eta, E)| d\eta \]

\[
\lesssim h^{-\frac{1}{2}} Bi^{-2}(-h^{-\frac{3}{10}} \zeta) \int_0^{-h^{-\frac{3}{10}} \zeta} Bi^2(u) du
\]

(4.10) \[ + h^{-\frac{1}{2}} Bi^{-2}(-h^{-\frac{3}{10}} \zeta) \int_{h^{-\frac{1}{2}} \zeta}^{-h^{-\frac{3}{10}} \zeta} Bi^2(u) \left[ (h^3 u)^{-2} + \frac{E + (z)^{-3}}{q(-h^\frac{3}{2} u)} \right] du \]

where \( z \) has the same meaning as in (1.7). First, note that (4.9) is rapidly (in fact, super exponentially) decreasing as \( |\zeta| \) increases: \( |\dot{a}_2| \lesssim e^{-|\zeta|^\frac{3}{2}} \). Second, we bound the first part of (4.10) by

\[
h^{-\frac{1}{2}} Bi^{-2}(-h^{-\frac{3}{10}} \zeta) \int_{h^{-\frac{1}{2}} \zeta}^{-h^{-\frac{3}{10}} \zeta} Bi^2(u)(h^\frac{3}{2} u)^{-2} du
\]

\[
\lesssim h^{-\frac{3}{2}} Bi^{-2}(-h^{-\frac{3}{10}} \zeta) \int_{h^{-\frac{1}{2}} \zeta}^{-h^{-\frac{3}{10}} \zeta} u^{-\frac{3}{2}} e^{\frac{3}{2} u^2} du
\]

\[
\lesssim h^{-\frac{3}{2}} (h^{-\frac{3}{10}} |\zeta|)^{-\frac{3}{2}} \lesssim |\zeta|^{-\frac{3}{2}}
\]
The contribution to (4.10) involving $E$ is

$$h^{-\frac{1}{2}}\text{Bi}^{-2} (-h^{-\frac{1}{2}} \zeta) \int h^{-\frac{1}{2}} \text{Bi}^2 (u) (q(u h^\frac{1}{2} \zeta)\zeta -1 E du$$

(4.11)

$$\lesssim E h^{-\frac{1}{2}} (h^{-\frac{1}{2}} |\zeta|)^{\frac{1}{2}} e^{-\frac{\zeta}{\text{Bi}^2}} \int h^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-\frac{u^2}{2}} (q(u h^\frac{1}{2} \zeta))^{-1} du$$

(4.12)

$$\lesssim h^{-1} |\zeta|^\frac{1}{2} e^{-\frac{\zeta}{\text{Bi}^2}} \int_1^z \zeta e^{\frac{u^2}{2}} dz \lesssim |\zeta|^\frac{1}{2} E(x)^2$$

To pass from (4.11) to (4.12), we substituted $u = h^{-\frac{1}{2}} v$ and then changed variables $dv = \sqrt{q}-v dz$ followed by $q(v) = Q_0(z)$; in (4.12) the relation between $v$ and $z$, as well as $\zeta$ and $x$, is given by (4.2). i.e., $v = \zeta(z,E)$, $\zeta = \zeta(x,E)$. To pass to the final inequality in (4.12) we integrate by parts so as to gain a factor of $h$:

$$\int_1^z \zeta e^{\frac{u^2}{2}} dz \lesssim \int_z^{x^2} \zeta e^{\frac{v^2}{2}} \int_z^{x^1} \sqrt{Q_0(v)} dv dz$$

$$\lesssim h(x)^2 e^{\frac{v^2}{2}} \int_z^{x^1} \sqrt{Q_0(v)} dv = h(x)^2 e^{\frac{v^2}{2}} |\zeta|^\frac{1}{2}$$

where $\zeta(x_2,h) = -1$. Finally, we turn the contribution of $\langle z \rangle^{-3}$ in (4.10). Using the same conventions regarding the relation between the variables this contribution is of the form

$$h^{-\frac{1}{2}}\text{Bi}^{-2} (-h^{-\frac{1}{2}} \zeta) \int h^{-\frac{1}{2}} \text{Bi}^2 (u) (q(u h^\frac{1}{2} \zeta)\zeta -1 \langle z \rangle^{-3} du$$

$$\lesssim h^{-\frac{1}{2}} (h^{-\frac{1}{2}} |\zeta|)^{\frac{1}{2}} e^{-\frac{\zeta}{\text{Bi}^2}} \int h^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-\frac{u^2}{2}} (q(u h^\frac{1}{2} \zeta))^{-1} \langle z \rangle^{-3} du$$

$$\lesssim h^{-1} |\zeta|^\frac{1}{2} e^{-\frac{\zeta}{\text{Bi}^2}} \int_1^z \langle z \rangle^{-2} e^{\frac{u^2}{2}} dz \lesssim |\zeta|^\frac{1}{2} h^2(x)^{-2}$$

The final inequality here is based on the same kind of integration by parts as before:

$$\int_1^z \langle z \rangle^{-2} e^{\frac{u^2}{2}} dz \lesssim \int_z^{x^2} \langle z \rangle^{-2} e^{\frac{v^2}{2}} \int_z^{x^1} \sqrt{Q_0(v)} dv dz$$

$$\lesssim h(x)^{-2} e^{\frac{v^2}{2}} \int_z^{x^1} \sqrt{Q_0(v)} dv = h(x)^{-2} e^{\frac{v^2}{2}} |\zeta|^\frac{1}{2}$$

with $x_2$ as above. In conclusion, we estimate the contributions of (4.9) and (4.10) by

$$|a_2(\zeta,E,h)| \lesssim h^{-\frac{1}{2}} (h^{-\frac{1}{2}} \zeta)^{-\frac{1}{2}} |\zeta|^{-1} \chi_{[-1,0]} + |\zeta|^\frac{1}{2} \chi_{[0,1]}$$

(4.13) as claimed.

Next, we turn to $\phi_1(\zeta,E)$ (dropping $h$ for simplicity). As usual we make the reduction ansatz

$$\phi_1(\zeta,E) = g(\zeta,E) \phi_2(\zeta,E)$$

which leads to the equation $(\phi_2''/\phi_2) = 0$. At this point it is convenient to extend the solutions $\phi_2$, which are originally defined on the interval $\zeta(0,E) \leq \zeta \leq 0$, to all of $\zeta \leq 0$. This is done in such a way that the bounds (4.1) remain valid for $\zeta \leq \zeta_0$.
without, however, making any reference to the ODE \[6.3\] for those \( \zeta \). We can now solve for \( g \) in the form

\[
\phi_1(\zeta, E) = \pi \hbar^{\frac{3}{2}} \phi_2(\zeta, E) \int_{-\infty}^{\zeta} \phi_2(\eta, E)^{-2} \, d\eta
\]

Inserting our representation of \( \phi_2 \) into this formula yields

\[
\phi_1(\zeta, E) = \pi \hbar^{\frac{3}{2}} \text{Bi}(-\hbar^{\frac{3}{2}} \zeta)[1 + h a_2(\zeta, E)] \int_{-\infty}^{\zeta} \text{Bi}^{-2}(-\hbar^{\frac{3}{2}} \eta)[1 + h a_2(\eta, E)]^{-2} \, d\eta
\]

First, we note that from \[4.4\],

\[
\pi \hbar^{\frac{3}{2}} \text{Bi}(-\hbar^{\frac{3}{2}} \zeta) \int_{-\infty}^{\zeta} \text{Bi}^{-2}(-\hbar^{\frac{3}{2}} \eta) \, d\eta = \text{Ai}(-\hbar^{\frac{3}{2}} \zeta)
\]

Second, \([1 + h a_2]^{-2} = 1 + h \tilde{a}_2\) where \( \tilde{a}_2 \) satisfies the same bounds as \( a_2 \) (since \(|a_2| \lesssim 1\)). Thus, inspection of our formula for \( \phi_1 \) reveals that \( a_1 = \pi (a_2 + \tilde{a}_1) \) where

\[
a_1(\zeta) := \hbar^{\frac{3}{2}} \text{Bi}(-\hbar^{\frac{3}{2}} \zeta)[1 + h a_2(\zeta, E)] \int_{-\infty}^{\zeta} \text{Bi}^{-2}(-\hbar^{\frac{3}{2}} \eta) a_2(\eta, E) \, d\eta
\]

\[
= \frac{\text{Bi}}{\text{Ai}}(-\hbar^{\frac{3}{2}} \zeta)[1 + h a_2(\zeta, E)] \int_{-\hbar^{\frac{3}{2}} \zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{3}{2}} \eta, E) \, d\eta
\]

Furthermore, from \[4.11\],

\[
\pi \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{3}{2}} \eta, E) \, d\eta = -\int_{-\infty}^{-h^{\frac{3}{2}} \zeta} \tilde{a}_2(-h^{\frac{3}{2}} \eta, E) \left[ \frac{\text{Ai}}{\text{Bi}}(\eta) \right] \, d\eta
\]

\[
\pi \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{3}{2}} \eta, E) \, d\eta = \frac{\text{Ai}}{\text{Bi}}(-h^{\frac{3}{2}} \zeta) \tilde{a}_2(\zeta, E) - h^{\frac{3}{2}} \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\eta) (\partial_1 \tilde{a}_2)(-h^{\frac{3}{2}} \eta, E) \, d\eta
\]

(4.14)

where \( \partial_1 \) refers to the derivative in the first variable. The first term in \[4.14\] makes an admissible contribution to \( a_1 \) whereas the second one is controlled as follows:

\[
h^{\frac{3}{2}} \frac{\text{Bi}}{\text{Ai}}(-h^{\frac{3}{2}} \zeta) \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\eta) \left| \left( \partial_1 \tilde{a}_2 \right)(-h^{\frac{3}{2}} \eta, E) \right| \, d\eta
\]

\[
\lesssim h^{\frac{3}{2}} e^{-\frac{\pi}{\sqrt{6}}(\zeta)} \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \epsilon^{-\frac{\pi}{\sqrt{6}}(\eta)} \left[ h^{-\frac{3}{2}}(\eta)^{-\frac{1}{2}} \chi_{[-1 \leq \hbar^{\frac{3}{2}} \eta \leq 0]} + |\hbar^{\frac{3}{2}} \eta|^\frac{3}{2} \chi_{[\hbar^{\frac{3}{2}} \eta \leq -1]} \right] \, d\eta
\]

\[
\lesssim h^{\frac{3}{2}} (h^{\frac{3}{2}} \zeta)^{-1} \chi_{[-1 \leq \zeta \leq 0]} + h \chi_{[0 \leq \zeta \leq -1]} \lesssim h^{\frac{3}{2}} (h^{\frac{3}{2}} \zeta)^{-1} \chi_{[-1 \leq \zeta \leq 0]} + \chi_{[0 \leq \zeta \leq -1]}
\]

as desired. For the derivative in \( \zeta \),

\[
\partial_\zeta a_1(\zeta) = -\pi \hbar^{\frac{3}{2}} \text{Ai}^{-2}(-h^{\frac{3}{2}} \zeta)[1 + h a_2(\zeta, E)] \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{3}{2}} \eta, E) \, d\eta
\]

\[
+ \hbar^{\frac{3}{2}} (\text{AiBi})^{-1}(-h^{\frac{3}{2}} \zeta)[1 + h a_2(\zeta, E)] \tilde{a}_2(\zeta, E)
\]

\[
+ \hbar^{\frac{3}{2}} \frac{\text{Bi}}{\text{Ai}}(-h^{\frac{3}{2}} \zeta) \tilde{a}_2(\zeta, E) \int_{-h^{\frac{3}{2}} \zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{3}{2}} \eta, E) \, d\eta
\]
Let $\zeta$ it suffices to treat the first derivative in $\zeta$ cut-off at
and systematically to higher $\ell$ the calculations are quite involved and it is not clear how to extend this approach
principle, it is possible to treat relative to $E$ with
basis of the form (dropping $h$)
whence
Proof. Let $\zeta$ is given by
Using (3.3) we remove the dangerous $h^{-\frac{1}{2}}$ terms from the first two lines here
whence
\[ \partial \tilde{a}_1(\zeta) = A i^{-2}(-h^{-\frac{1}{2}} \zeta)[1 + h a_2(\zeta, E)] \int_{-h^{-\frac{1}{2}} \zeta}^{\infty} \frac{A i}{B i}(\eta)(\partial \tilde{a}_2)(-h^{-\frac{1}{2}} \eta, E) d\eta \]
(4.15)
\[ + h \frac{B i}{A i}(-h^{-\frac{1}{2}} \zeta) \partial \tilde{a}_2(\zeta, E) \int_{-h^{-\frac{1}{2}} \zeta}^{\infty} B i^{-2}(\eta) \tilde{a}_2(h^{-\frac{1}{2}} \eta, E) d\eta \]
The contribution by the first integral here is treated as the integral in (4.14) and
is bounded by
\[ \lesssim \langle h^{-\frac{1}{2}} \zeta \rangle^{\frac{1}{2}} \left[ \langle h^{-\frac{1}{2}} \zeta \rangle^{-1} \chi_{[-1 \leq \zeta \leq 0]} + h^{\frac{1}{2}} \chi_{[0 \leq \zeta \leq -1]} \right] \]
\[ \sim h^{\frac{1}{2}} \langle h^{-\frac{1}{2}} \zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^\frac{1}{2} \chi_{[0 \leq \zeta \leq -1]} \]
which is exactly as needed. Finally, the contribution of (4.15) is bounded by
\[ \lesssim h \left[ \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^\frac{1}{2} \chi_{[0 \leq \zeta \leq -1]} \right] \]
\[ \sim h(\zeta)^{\frac{1}{2}} \lesssim h^{\frac{1}{2}} \langle h^{-\frac{1}{2}} \zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^\frac{1}{2} \chi_{[0 \leq \zeta \leq -1]} \]
and we are done with the $k = 0$ case of (4.1) for $a_1$. However, since $E$ enters into
$a_1$ only through $a_2, \tilde{a}_2$ which do satisfy (4.1) for all $k \geq 0$, we see that the
previous estimates carry over unchanged and provide the stated estimates for $\partial^\ell_E a_1(\zeta, E)$
and $\partial^k_E \partial_\zeta a_1(\zeta, E)$.

We remark that the method employed in the previous proof does not extend easily
to derivatives $\partial^\ell_E a_j$ with $\ell \geq 2$ (that is, without losing excessive powers of $h^{-1}$). In
principle, it is possible to treat $\ell = 2$ by a similar method, but instead of a sharp
cut-off at $\zeta = 0$ one needs to use a smooth cut-off function in (4.13). However, the
calculations are quite involved and it is not clear how to extend this approach
systematically to higher $\ell$ (the same comment applies to Proposition 9 below). On
the other hand, for the purposes of Theorem 11 as well as for those of [23] and [24],
it suffices to treat the first derivative in $\zeta$ (however, we do need many derivatives
relative to $E$). Next, we turn to $\zeta \geq 0$ which requires an oscillatory basis.

**Proposition 9.** Let $h_0 > 0$ be small. In the range $\zeta \geq 0$ a basis of solutions
to (3.3) is given by
\[ \psi_1(\zeta, E; h) = (A i(\tau) + i B i(\tau))[1 + h b_1(\zeta, E; h)] \]
\[ \psi_2(\zeta, E; h) = (A i(\tau) - i B i(\tau))[1 + h b_2(\zeta, E; h)] \]
with $\tau := -h^{-\frac{1}{2}} \zeta$ and where $b_1, b_2$ are smooth, complex-valued, and satisfy the
bounds for all $k \geq 0$, and $j = 1, 2$
\[ |\partial^k_E b_j(\zeta, E; h)| \leq C_k E^{-k} \langle \zeta \rangle^{-\frac{1}{2}} \]
(4.16)
\[ |\partial_\zeta \partial^k_E b_j(\zeta, E)| \leq C_k E^{-k} h^{-\frac{1}{2}} \langle h^{-\frac{1}{2}} \zeta \rangle^{-\frac{1}{2}} (\zeta)^{-2} \]
ununiformly in the parameters $0 < h < h_0$, $0 < E < E_0$, $\zeta \geq 0$.

**Proof.** Let $\psi_{1,0}(\zeta; h) := (A i + i B i)(\tau)$ and $\psi_{2,0}(\zeta; h) := (A i - i B i)(\tau)$. We seek a
basis of the form (dropping $h$ as an independent variable from the notation)
\[ \psi_j(\zeta) = \psi_j(\zeta, E) = \psi_{j,0}(\zeta)(1 + h b_j(\zeta, E)) \]
for $\zeta \geq 0$. This representation is meaningful since $\mathrm{Ai}$ and $\mathrm{Bi}$ have no common zeros (as their Wronskian does not vanish). We obtain the equation

$$\left(\psi_{j,0}^2 \dot{b}_j\right) = -\frac{1}{h} \tilde{V} \psi_{j,0}^2 (1 + hb_j)$$

for $j = 1, 2$, c.f. (4.17). A solution of (4.17) on $\zeta \geq 0$ is given by, with $b_j(\zeta) = b_j(\zeta, E)$,

$$b_j(\zeta) := -\frac{1}{h} \int_{\zeta}^{\infty} \psi_{j,0}^2(\eta) \int_{\zeta}^{\eta} \psi_{j,0}^{-2}(\eta^\prime) \tilde{V}(\eta, E) (1 + hb_j(\eta)) d\eta$$

Recall the asymptotic behavior, see [19],

$$\mathrm{Ai}(-x) \pm i\mathrm{Bi}(-x) = \frac{1}{\sqrt{\pi} x^{\frac{3}{2}}} e^{\mp i\left(\frac{\pi}{4} - \frac{\xi}{x}\right)} (1 + O(\xi^{-1}))$$

as $x \to \infty$. Here $\xi = \frac{2}{3} x^{\frac{3}{2}}$ and the $O(\cdot)$ term is complex-valued and exhibits symbol behavior:

$$\partial_\xi^k O(\xi^{-1}) = O(\xi^{-1-k}) \quad \forall \, k \geq 0$$

Therefore, for any $0 > x_0 > x_1$,

$$\left| \int_{x_0}^{x_1} \left(\mathrm{Ai} + i\mathrm{Bi}\right)^2(\eta) \left(\mathrm{Ai} + i\mathrm{Bi}\right)^{-2}(y) \, dy \right| \lesssim \langle x_1 \rangle^{-\frac{1}{2}}$$

The leading term in (4.18), i.e.,

$$b_{1,0}(\zeta, h, E) := h^{\frac{1}{2}} \int_{h^{-\frac{2}{3}} \zeta}^{\infty} (\mathrm{Ai}+i\mathrm{Bi})^2(-u) \left[ \int_{h^{-\frac{2}{3}} \zeta}^{u} (\mathrm{Ai}+i\mathrm{Bi})^{-2}(-v) \, dv \right] \tilde{V}(-h^{\frac{2}{3}}u, E) \, du$$

therefore satisfies the bound, see Lemma 7

$$|b_{1,0}(\zeta)| \lesssim h^{\frac{1}{2}} \int_{h^{-\frac{2}{3}} \zeta}^{\infty} (u)^{-\frac{1}{2}} |\tilde{V}(-h^{\frac{2}{3}}u, E)| \, du \lesssim h^{\frac{1}{2}} \int_{h^{-\frac{2}{3}} \zeta}^{\infty} (u)^{-\frac{1}{4}} (h^{\frac{2}{3}}u)^{-2} \, du \lesssim \langle \zeta \rangle^{-\frac{1}{2}}$$

uniformly in $\zeta \geq 0$, $0 < E < E_0$, and $0 < h < h_0$. Due to the linear nature of (4.18), a contraction argument now yields the same bound for $b_1$; in fact, due to the derivative bounds of Lemma 7 relative to $E$, we obtain the more general estimate

$$|\partial_E^k b_j(\zeta, E)| \leq C_k E^{-k} \langle \zeta \rangle^{-\frac{2}{3}} \quad \forall \, k \geq 0$$

uniformly in the parameters for both $j = 1, 2$. As for the first derivative in $\zeta$, observe that

$$\dot{b}_j(\zeta) = \frac{h^{-1}}{\psi_{j,0}^2(\zeta)} \int_{\zeta}^{\infty} \psi_{j,0}^2(\eta) \tilde{V}(\eta, E) (1 + hb_j(\eta)) \, d\eta$$

In order to exploit the cancellation in this integral, one integrates by parts once. To this end, write for $u \geq 0$,

$$\left(\mathrm{Ai} + i\mathrm{Bi}\right)^2(-u) = \left(\frac{\omega}{u}\right)^{\frac{3}{2}} \omega(u), \quad |\omega(u)| \lesssim \langle u \rangle^{-\frac{1}{2}}, \quad |\omega'(u)| \lesssim \langle u \rangle^{-\frac{3}{2}},$$

$$\psi_{1,0}^2(\zeta; h) = \left(\frac{\omega}{(h^{\frac{2}{3}} \zeta)^{\frac{3}{2}}}\right)^{\frac{3}{2}} \omega(h^{\frac{2}{3}} \zeta)$$

Since

$$\psi_{1,0}^2(\zeta; h) \, d\zeta = \frac{1}{2i} h^{\frac{2}{3}} (h^{-\frac{2}{3}} \zeta)^{-\frac{3}{2}} \omega(h^{-\frac{2}{3}} \zeta) \, d\left[\left(\frac{\omega}{(h^{\frac{2}{3}} \zeta)^{\frac{3}{2}}}\right)^{\frac{3}{2}}\right]$$
integration by parts yields

\[
\hat{b}_1(\xi) = \frac{\hbar^{-\frac{3}{2}}}{2i\psi_{\xi,0}(\xi)} \int_\xi^\infty (\hbar^{\frac{3}{2}} \eta)^{-\frac{1}{2}} \omega(\hbar^{\frac{3}{2}} \eta) \tilde{V}(\eta, E)(1 + \hbar b_j(\eta)) d\left[e^{\frac{3}{2}(\hbar^{\frac{3}{2}} \eta)^{\frac{1}{2}}}\right]
\]

(4.21)

\[
= O(\hbar^{-\frac{3}{2}} (\hbar^{\frac{3}{2}} \eta)^{-\frac{1}{2}} (\xi)^{-2})-
\]

(4.22)

The leading order here is given by (4.21); indeed, if we estimate the \( \hat{b}_1(\eta) \) term in (4.22) by (4.21), then (4.22) \( \lesssim \hbar(\xi)^{-1} \), which is much better than (4.21). The conclusion is that

\[
|\partial_\xi \partial_E^k b_j(\xi, E)| \lesssim E^{-k} \hbar^{-\frac{3}{2}} (\hbar^{\frac{3}{2}} \xi)^{-\frac{1}{2}} (\xi)^{-2}
\]

as claimed. \( \square \)

5. The proof of Theorem 11

Let \( f_{\pm}(x, E; h) \) be the Jost solutions of \( P(x, hD) \) from (1.1). For ease of notation, we shall first assume the symmetry \( V(x) = V(-x) \) and later indicate how to treat the general case. Also, as usual, we drop \( h \) from the arguments of functions. Then \( f_-(x, E) = f_+(x, E) \) so that the Wronskian of \( f_+, f_- \) is

\[
W(E) = -2f_+(0, E)f_-^*(0, E)
\]

Next, from (4.19), and with \( \zeta = \zeta(x, E) \) as in (4.2) and \( T_+(E) \) as in (4.3),

\[
f_+(x, E) = \sqrt{\pi} E^{\frac{3}{4}} \hbar^{-\frac{3}{4}} e^{i\frac{T_+(E)}{\hbar} + E^{\frac{1}{2}} q^{-\frac{1}{2}}(\zeta)} \psi_2(\zeta, E)
\]

This is obtained by matching the asymptotic behavior of \( f_+ \) with that of \( \psi_2(\zeta) \) as \( x \to \infty \) and we used the relation \( w = q^{1/2} f \) from Lemma 3. We now connect \( \psi_2 \) to the basis \( \phi_j(\zeta, E) \) of Proposition 8

\[
\psi_2(\zeta, E) = c_1(E) \phi_1(\zeta, E) + c_2(E) \phi_2(\zeta, E)
\]

where

\[
c_1(E) = \frac{W(\psi_2(\cdot, E), \phi_2(\cdot, E))}{W(\phi_1(\cdot, E), \phi_2(\cdot, E))}, \quad c_2(E) = -\frac{W(\psi_2(\cdot, E), \phi_1(\cdot, E))}{W(\phi_1(\cdot, E), \phi_2(\cdot, E))}
\]

By Proposition 8

\[
W(\phi_1(\cdot, E), \phi_2(\cdot, E)) = -\hbar^{-\frac{3}{2}} W(Ai, Bi) + O(\hbar^{-\frac{3}{2}}) = -\pi^{-1} \hbar^{-\frac{3}{2}} (1 + O(h))
\]

where we evaluated the Wronskian on the left-hand side at \( \zeta = 0 \). Next, by Propositions 8 and 9

\[
W(\psi_2(\cdot, E), \phi_2(\cdot, E)) = -\hbar^{-\frac{3}{2}} [(Ai(0) - iBi(0))Bi'(0)] - (Ai'(0) - iBi'(0))Bi(0) + O(h)
\]

\[
= -\hbar^{-\frac{3}{2}} [W(Ai, Bi) + O(h)]
\]

(5.1)

\[
W(\psi_1(\cdot, E), \phi_1(\cdot, E)) = -\hbar^{-\frac{3}{2}} [(Ai(0) - iBi(0))Ai'(0)] - (Ai'(0) - iBi'(0))Ai(0) + O(h)
\]

\[
= -\hbar^{-\frac{3}{2}} [iW(Ai, Bi) + O(h)]
\]
so that
\[
(5.2) \quad c_1(E) = 1 + O(h), \quad c_2(E) = -i + O(h)
\]
where the \( O() \) terms satisfy \( |\partial_E^k O(h)| \leq C_k E^{-k} \). For the remainder of the proof, we set \( \zeta_0 := \zeta(0, E) \). Then
\[
f_+(0, E) = \sqrt{\pi} e^{i(T(E) + \zeta)} E^{\frac{1}{2} - \frac{1}{2}q} \left\{ \phi_0(0) q - \frac{1}{4}(\zeta_0) \left[ \psi'_0(\zeta_0, E) - \frac{1}{4q} \right] \right\}(\zeta_0, E)
\]
\[
f'_+(0, E) = \sqrt{\pi} e^{i(T(E) + \zeta)} E^{\frac{1}{2} - \frac{1}{2}q} \left\{ \phi_0'(0) q - \frac{1}{4}(\zeta_0) \left[ \psi''_0(\zeta_0, E) - \frac{1}{4q} \right] \right\}(\zeta_0, E)
\]
Recall from Lemma \( k \) that \( \zeta' = q^{\frac{1}{2}} \). From \( V'(0) = 0 \) we obtain
\[
\dot{q}(\zeta_0) = \frac{Q_0(0)}{\zeta_0^2} = -\frac{q(\zeta_0)}{\zeta_0}
\]
and thus
\[
f_+(0, E)f'_+(0, E) = i\pi E^{\frac{1}{2}} e^{2i(T(E) + \zeta)} h^{-\frac{1}{2}} \left[ c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E) \right] \times
\]
\[
\times [c_1(E)\phi_1'(\zeta_0, E) + c_2(E)\phi_2'(\zeta_0, E)]
\]
\[
+ \frac{i}{4} \pi E^{\frac{1}{2}} e^{2i(T(E) + \zeta)} h^{-\frac{1}{2}} \phi_0^{-1}(\zeta_0, E) [c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)]^2
\]
From Proposition \( k \)
\[
\phi_1(\zeta_0, E) = \text{Ai}(\zeta_0, E) \quad (1 + O(h))
\]
\[
\phi_2(\zeta_0, E) = \text{Bi}(\zeta_0, E) \quad (1 + O(h))
\]
\[
\phi_1'(\zeta_0, E) = -h^{-\frac{1}{2}} \text{Ai}'(\zeta_0, E) \quad (1 + O(h)) + O(h)|\zeta_0|^{\frac{1}{2}} \text{Bi}(\zeta_0, E)
\]
\[
\phi_2'(\zeta_0, E) = -h^{-\frac{1}{2}} \text{Bi}'(\zeta_0, E) \quad (1 + O(h)) + O(h)|\zeta_0|^{\frac{1}{2}} \text{Bi}(\zeta_0, E)
\]
which implies via the standard asymptotics of the Airy functions that
\[
\phi_1(\zeta_0, E) = (4\pi)^{\frac{1}{2}}(\zeta_0, E) - \frac{1}{2} e^{-\frac{1}{2}h^{-1}|\zeta_0|^{\frac{1}{2}}} (1 + O(h))
\]
\[
\phi_2(\zeta_0, E) = \pi e^{\frac{1}{2}h^{-1}|\zeta_0|^{\frac{1}{2}}} (1 + O(h))
\]
\[
\phi_1'(\zeta_0, E) = h^{-\frac{1}{2}} (4\pi)^{-\frac{1}{2}}(\zeta_0, E) - \frac{1}{2} e^{-\frac{1}{2}h^{-1}|\zeta_0|^{\frac{1}{2}}} (1 + O(h))
\]
\[
\phi_2'(\zeta_0, E) = -h^{-\frac{1}{2}} e^{\frac{1}{2}h^{-1}|\zeta_0|^{\frac{1}{2}}} (1 + O(h))
\]
Hence, using that \( e^{-h^{-1}|\zeta_0|^{\frac{1}{2}}} = O(h) \) where \( \partial_E^k O(h) = O(E^{-k}h) \), one concludes that
\[
h^{-\frac{1}{2}} [c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)] \times
\]
\[
\times [c_1(E)\phi_1'(\zeta_0, E) + c_2(E)\phi_2'(\zeta_0, E)]
\]
\[
= \pi^{-1} h^{-1} e^{\frac{1}{2}h^{-1}|\zeta_0|^{\frac{1}{2}}} (1 + O(h))
\]
as well as
\[ \hbar^{-\frac{2}{3}|\zeta_0|^2}[c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)]^2 = -\pi^{-1}|\zeta_0|^{-\frac{2}{3}} e^{\frac{2}{3}\hbar^{-1}|\zeta_0|^2}(1 + O(\hbar)) \]
Since \( T(E) = 2T_+(E) \) we finally arrive at
\[ W(E) = -2f_+(0, E)f'_+(0, E) = -2ie^{2i\frac{T_+(E)}{\hbar}}\hbar^{-\frac{1}{2}} e^{\frac{2}{3}\hbar^{-1}|\zeta_0|^2}(1 + O(\hbar)) \]
(5.3)
We used here that
\[ \frac{4}{3}|\zeta_0|^2 = 2\int_0^{\pm 1} \sqrt{V_0(\eta) - E} \, d\eta = S(E) \]
All the \( O(\hbar) \) appearing above behave as required under differentiation with respect to \( E \); indeed, this is both due to the bounds of Propositions [8] and [9] as well as the aforementioned fact that
\[ e^{-\frac{2}{3}\hbar^{-1}|\zeta_0|^2} = O(\hbar^{|\zeta_0|^{-\frac{2}{3}}}) = O(\hbar) \]
has the required behavior since \( |\zeta_0|^{-\frac{2}{3}} = O(|\log E|^{-\frac{2}{3}}) \) as \( E \to 0^+ \). In view of (5.3), (5.3) implies the sought after asymptotic relation for \( S_{12} \) in Theorem [1] see (1.6). In order to find \( S_{12} \) and \( S_{21} \) (i.e., the reflection coefficients), we need to also asymptotically evaluate the following Wronskians:
\[ W(f_+(\cdot, E), f_-(\cdot, E)) = W(f_+(\cdot, E), f_-(\cdot, E)) = -2\text{Re} \left[ f_+(0, E)f'_+ (0, E) \right] \]
Using the same notations as in the computation of \( W(E) \), we obtain
\[ -2\text{Re} \left[ f_+(0, E)f'_+ (0, E) \right] = -2\text{Re} \left\{ \pi E^{\frac{1}{2}}\hbar^{-\frac{1}{3}}[c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)] \times [c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)] \right\} \]
\[ -\frac{\pi}{2} E^{\frac{1}{2}}\hbar^{-\frac{1}{3}}|\zeta_0|^2 [c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)]^2 \]
Finally, evaluating this expression as above, we obtain
\[ W(f_+(\cdot, E), f_-(\cdot, E)) = -2\text{Re} \left[ f_+(0, E)f'_+ (0, E) \right] = \frac{2\sqrt{E}}{\hbar} e^{E\frac{S(E)}{\hbar}}(1 + O(\hbar)). \]
Forming the ratio between this formula and the one for \( W(E) \) yields the desired expression for \( S_{12} = S_{21} \), see (1.6). Indeed,
\[ r_-(E) = -\frac{W(f_+(\cdot, E), f_-(\cdot, E))}{W(f_+(\cdot, E), f_-(\cdot, E))} = -ie^{-i\hbar^{-1}T(E)}(1 + O(\hbar)) \]
where \( O(\hbar) \) behaves like a symbol with respect to \( E \), as usual. This concludes the proof of Theorem [1] in the symmetric case. If \( V(x) \neq V(-x) \), then only minor changes are needed. Indeed, on \( x \leq 0 \) we can still use the same bases \( \phi_j, \psi_j \) from Section [4] but with \( \zeta = \zeta(-x, E) \). This is due to the fact that the difference between the left-hand and right-hand branches of \( V \) does not affect the estimates from Section [4] (since we are assuming inverse square decay at both ends and the constants \( \mu_\pm \) have no effect on the leading order behavior). Let
\[ \tilde{\zeta}_0(E) := \frac{3}{2} \int_{x_2(E)}^{0} \sqrt{V_0(\eta) - E} \, d\eta \]
Thus, in addition to the expressions for $f_+(0, E)$ and $f'_+(0, E)$ from above we now also have
\[
 f_-(0, E) = \sqrt{\pi} e^{i \frac{\pi}{4} (1 - \chi)} E^{\frac{1}{2}} h^{\frac{1}{2} + \frac{1}{3} (\tilde{c}_0) [c_1(E) \phi_1(\tilde{c}_0, E) + c_2(E) \phi_2(\tilde{c}_0, E)]} \\
 f'_-(0, E) = -\sqrt{\pi} e^{i \frac{\pi}{4} (1 - \chi)} E^{\frac{1}{2}} h^{\frac{1}{2}} \frac{1}{4} q^{-1} (\tilde{c}_0) [c_1(E) \phi_1(\tilde{c}_0, E) + c_2(E) \phi_2(\tilde{c}_0, E)]
\]
Inserting these expressions into
\[
 W(E) = f_+(0, E)f'_-(0, E) - f'_+(0, E)f_-(0, E),
\]
and using that
\[
 \frac{2}{3} \left( \frac{\pi}{16} + \frac{\pi}{16} \right) = \int_{x_1(E)}^{x_2(E)} \sqrt{V_0(\eta) - E} d\eta = S(E)
\]
as well as $T(E) = T_+(E) + T_-(E)$, one again arrives at (5.3). The same comments apply to the off-diagonal terms of the scattering matrix and we are done. As for the very last claim of the theorem concerning $V_0 = V + h^2 V_1$, simply note that the main calculations entering into the above proof only make use of the leading order part of $V_1$, i.e., $\frac{1}{4}(x)^{-2}$ whereas the cubic piece gets absorbed into the error term.

6. FROM SMALL TO LARGE ENERGIES

In this section, we present an extension of Theorem [1] to the case of large energies. More specifically, suppose $V$ is as in Theorem [1] but with the following additional properties:

- $0 < V(x) \leq 1$ for all $x \in \mathbb{R}$, $V(0) = 1$, $V'(0) = 0$, $V''(0) = -1$
- $V'(x) < 0$ for all $x > 0$, $V'(x) > 0$ for all $x < 0$

Note that this precisely is the kind of barrier potential considered by Ramond [22] (but without any analyticity assumptions). For the purposes of this section we refer to it as a simple barrier potential. Even though Theorem [1] by design only considers small energies $0 < E < E_0$, it is natural to ask to what extent it remains correct as $E_0 \to 1$. As already remarked before, for energies $E > \varepsilon > 0$ there is no difference between $V$ and $V_0$ as far as Theorem [1] is concerned. Indeed, switching from $V$ to $V_0$ only affects the error term. Moreover, for the kind of $V$ we are considering here, the theorem remains valid in any range $0 < E < 1 - \varepsilon$ with $\varepsilon$ fixed. This is due to the fact that in this range there is a unique pair of turning points $x_2(E), x_1(E)$ as before. The action $S(E; h)$ lies between two positive constants (depending on $\varepsilon$) and the previous proof goes through without changes. Somewhat more interesting and very relevant for later applications, cf. [23, 24], is the case where $\varepsilon = h^\alpha$. The question is then how large $\alpha \geq 0$ can be allowed to be. First note that we can no longer expect the error term in (1.6) to be of the form $O(h)$ in that case. Rather, it will need to be $O(h^\delta)$ for some $\delta = \delta(\alpha) > 0$ and this condition will determine how large we can take $\alpha$. It turns out that the range $0 \leq \alpha < 1$ is admissible here.

In the following corollary, we use the notations introduced in Theorem [1]
Corollary 10. Let $V$ be a simple barrier potential. For every $0 < \alpha < 1$ there exists and $h_0 = h_0(\alpha)$ small such that for all $0 < h < h_0$ and $0 < E \leq 1 - h^\alpha$ 

$$S_{11}(E; h) = e^{-\frac{i}{\hbar}(S(E; h) + iT(E; h))} (1 + h(1 - E)^{-1} \sigma_{11}(E; h))$$

and the correction terms satisfy the bounds

$$|\partial^k_E \sigma_{11}(E; h)| + |\partial^k_E \sigma_{12}(E; h)| \leq C_k \max(E^{-k}, (1 - E)^{-k}) \quad \forall \ k \geq 0,$$

with a constant $C_k$ that only depends on $k$ and $V$.

Proof. We will only sketch the proof as there is no point in repeating all the details of the proof of Theorem 1. In fact, inspection of the previous section shows that the main issue is to prove that Propositions 8 and 9 remain valid albeit with errors of the form $\hbar^{1-\alpha}$ rather than $\hbar$ (we need to pay particular attention to the derivative $\partial^k_\tau$) when $E = 1 - h^\alpha$. We will freely use the notation from Section 3 and 4. By the preceding comments, it will suffice to consider the range $1 - \varepsilon < E \leq 1 - h^\alpha$. In fact, it will be enough to set $E = 1 - h^\alpha$ so that $x_1(E) \sim \hbar^{\frac{3}{2}}$. The range $0 < x < x_1(E)$ then corresponds to the region $-\hbar^{\frac{3}{2}} \lesssim \zeta \leq 0$. A simple calculation shows that $q \sim \hbar^{\frac{3}{2}}$ in that range, as well as $|V| \lesssim \hbar^{-\frac{3}{2}}$ with the usual behavior under differentiation in $E$. In fact, for all $0 \leq x \leq x_1(E)$ we have

$$V(x) = -E \int_x^{x_1} V'(y) dy \sim x_1^2 - x^2 \sim (x_1 - x)x_1$$

and thus

$$\zeta \sim -x^\frac{1}{3}(x_1 - x), \quad q = \frac{V - E}{-\zeta} \sim x_1(x_1 - x) = x_1^2 \sim \hbar^{\frac{2}{3}}$$

as claimed. Next, recall (3.22), viz.

$$\hat{V} = \frac{1}{4}q^{-1}(x')^{-2} + \frac{3}{10}q^{-2}q^2 - \frac{1}{4}q^{-1}q^3$$

Since $q = q^{-\frac{3}{2}}q'$ where $q' = \frac{4a}{\hbar^2} \sim x_1^{-\frac{2}{3}} \sim q^{-\frac{1}{3}}$, the second term here is of size

$$q^{-2}q^2 \lesssim q^{-3}(q')^2 \lesssim q^{-4} \sim \hbar^{-\frac{3}{2}}$$

The other two terms are smaller whence $|\hat{V}| \lesssim \hbar^{-\frac{3}{2}}$ as claimed. Turning to Proposition 3, we seek a basis of the form

$$\phi_1(\zeta, E, h) = \text{Ai}(\tau)(1 + \hbar^\delta a_1(\zeta, E, h))$$

$$\phi_2(\zeta, E, h) = \text{Bi}(\tau)(1 + \hbar^\delta a_2(\zeta, E, h))$$

with $\delta := 1 - \alpha$. Proceeding as in the proof of Theorem 1 we arrive at the following analogue of (1.5)

$$|a_{2,0}(\zeta)| \lesssim \hbar^{1-\delta} \int_0^{-\hbar^{-\frac{3}{2}}} q^{-\frac{3}{2}}(u)^{-\frac{3}{2}}|\hat{V}(-\hbar^{\frac{3}{2}}u, E)| \, du$$

which yields

$$|a_{2,0}(\zeta)| \lesssim \hbar^{1-\delta} (\hbar^{-\frac{3}{2}} \zeta)^{\frac{1}{2}} \hbar^{-\frac{3}{2}} \lesssim \hbar^{1-\alpha-\delta}$$

This shows that with our choice of $\delta$, we have

$$\sup_{\zeta(0, E) \leq \zeta \leq 0} |a_{2,0}(\zeta)| \lesssim 1$$
For the derivatives, the analogue of (4.8), viz.,

\[
\dot{a}_2(\zeta) = \frac{\hbar^{-\delta}}{\phi'_{2,0}(\zeta, \hbar)} \int_{0}^{0} \phi'_{2,0}(\eta, \hbar) V(\eta, E)(1 + \hbar a_2(\eta)) \, d\eta
\]
yields

\[
|\dot{a}_2(\zeta, E, \hbar)| \lesssim \hbar^{\frac{2}{3} - \delta} \Bi^{-2}(-\hbar^{\frac{2}{3}} \zeta) \int_{0}^{0} \Bi^2(u) h^{-\frac{2}{3}} \, du
\]

\[
\lesssim \hbar^{\frac{2}{3} - \delta - \frac{4}{3} \delta} (\hbar^{\frac{2}{3}} \zeta)^{-\frac{1}{2}} \lesssim \hbar^{-\frac{2}{3}}
\]

where we again used that \( \alpha < 1 \) in the final step. An analogous estimate holds for \( \dot{\phi}_1 \). We claim that these bounds are sufficient provided the same type of estimates hold for the analogue of Proposition 9 at \( \zeta = 0 \). Indeed, inspection of (5.1) shows that in that case

\[
W(\psi_2(\cdot, E), \phi_2(\cdot, E)) = -\hbar^{-\frac{2}{3}} [(\Ai'(0) - i\Bi(0))\Bi'(0) - (\Ai'(0) - i\Bi(0))\Bi(0) + O(\hbar^\delta)]
\]

\[
= -\hbar^{-\frac{2}{3}} [W(\Ai, \Bi) + O(\hbar^\delta)]
\]

\[
W(\psi_1(\cdot, E), \phi_1(\cdot, E)) = -\hbar^{-\frac{2}{3}} [(\Ai(0) - i\Bi(0))\Ai'(0) - (\Ai'(0) - i\Bi(0))\Ai(0) + O(\hbar^\delta)]
\]

\[
= -\hbar^{-\frac{2}{3}} [iW(\Ai, \Bi) + O(\hbar^\delta)]
\]

Note that there is an exact balance here between the \( \hbar^{-\frac{2}{3}} \) coming from the derivatives of the main contributions and the losses stemming from \( \dot{a}_j, \dot{b}_j \). Hence,

\[
c_1(E) = 1 + O(\hbar^\delta), \quad c_2(E) = -i + O(\hbar^\delta)
\]
as desired. Since \( \hbar^{-1}|\zeta_0|^{\frac{2}{3}} \sim \hbar^{\alpha - 1} \) and thus also

\[
e^{-\hbar^{-1}|\zeta_0|^{\frac{2}{3}}} = O(h^{1 - \alpha}) = O(\hbar^\delta)
\]

the reader will easily check that the remainder of the proof in Section 5 goes through.

It therefore remains to deal with the oscillatory regime. In analogy with Proposition 9 we seek a basis

\[
\psi_1(\zeta, E; \hbar) = (\Ai(\tau) + i\Bi(\tau))[1 + \hbar^\delta b_1(\zeta, E; \hbar)]
\]

\[
\psi_2(\zeta, E; \hbar) = (\Ai(\tau) - i\Bi(\tau))[1 + \hbar^\delta b_2(\zeta, E; \hbar)]
\]

For this we need to understand \( V \) on \( \zeta \geq 0 \). First one checks that for all \( x \geq x_1(E) \),

\[
\zeta \sim \begin{cases} 
  x^\frac{1}{2} (x - x_1) & x_1 \leq x \leq 2x_1 \\
  x^\frac{1}{2} & 2x_1 \leq x < 1 \\
  x^\frac{1}{2} & x \geq 1
\end{cases}
\]

and thus

\[
g \sim \begin{cases} 
  x^\frac{3}{2} & x_1 \leq x < 1 \\
  x^{-\frac{5}{2}} & x \geq 1
\end{cases}
\]

Hence, (5.22) implies that

\[
|\bar{V}| \lesssim x^{-\frac{5}{2}} \chi_{[x_1 \leq x \leq 1]} + \zeta^{-2} \chi_{[x \geq 1]}
\]
Going through the proof of Proposition \[9\] shows that
\[ |b_j(0)| \lesssim \hbar^{1-\alpha-\delta} \lesssim 1, \quad |\dot{b}_j(0)| \lesssim \hbar^{\frac{5}{2}-\delta-\frac{3\alpha}{2}} \lesssim \hbar^{-\frac{3}{2}} \]
as desired. The derivatives relative to \( E \) are left to the reader. □

Thus, the semi-classical approximation obtained in Theorem \[1\] breaks down precisely at \( E = 1 - \hbar \). As is well-known, the Airy equation is no longer the correct approximating equation for energies close to the unique maximum \( V(0) = 1 \) of a simple barrier potential. In fact, there exists an analytic change of variables which reduces the Schrödinger equation with such energies to the Weber equation locally around the origin. Alternatively, Ramond \[22\] invokes micro-local methods and the Helffer-Sjöstrand normal form in that case.

**Appendix A. A normal form reduction to Bessel’s equation**

In this section we sketch an alternative route for the asymptotic analysis of Section \[4\]. It is based on Lemma \[4\] and reduces equation \[4.1\] to a Bessel equation rather than an Airy equation. However, we emphasize that these approaches are in fact quite related as the Airy functions are used to describe Bessel functions \( J_n \) and \( Y_n \) in the large \( n = \hbar^{-1} \) asymptotics very much in the spirit of Section \[4\] see \[19\]. A possible advantage of working with the Bessel representation lies with the fact that they apply to all \( x \in [\varepsilon E^{-\frac{1}{2}}, \infty) \) which is a region containing the turning point \( x_1(E) \). On the other hand, since they cannot be used on the region \([0, \varepsilon E^{-\frac{1}{2}}] \), one is again faced with a connection problem as in Section \[4\]. Moreover, we have found that using distinct changes of variables in these two regions leads to a number of complications as compared to the global action-based coordinates introduced in Lemma \[3\]. For this, as well as other reasons, we ultimately found it technically advantageous to work with the Airy approximation directly, but we wish to sketch the Bessel method since it seems to be of independent interest. In this section, we shall use the notations of Lemma \[4\] and always work on \( y \geq 1 \) which transforms into \( \xi \geq \xi_1(E) \), see \[3.13\]. First, a preliminary technical lemma.

**Lemma 11.** The function \( \mu(\eta, E) := (\partial_x y(\eta, E))^2 (\partial_x y \xi)(y(\eta, E), E) \) satisfies
\[ |\partial_k \partial_j \mu(\eta, E)| \leq C_{kj} E^{-k} \eta^{-j-3} \tag{A.1} \]
and the positive smooth function
\[ \Omega(\eta, E) := \exp \left( - \int_\eta^\infty \mu(t, E) \, dt \right) \tag{A.2} \]
satisfies \( \frac{\partial \Omega}{\partial \eta} = \mu \) and \( \Omega = 1 + O(\eta^{-2}) \) (we write \( ' = \partial_\eta \)) with a symbol-type \( O(\eta^{-2}) \).

**Proof.** The \( \eta^{-3} \) decay in \( (A.1) \) is due to \( \|3.9\| \). Otherwise, the lemma is an immediate consequence of Lemma \[4\] □

Now for the transformation of the equation with \( V, V_0 \) as in Theorem \[1\]. To motivate our way of obtaining the Bessel equation as an approximating equation, consider the model operator
\[ P(x, hD) := -\hbar^2 \partial_x^2 + \langle x \rangle^{-2} \]
It is tempting to introduce the Bessel operator
\[ P_0(x, hD) := -\hbar^2 \partial_x^2 + x^{-2} \]
which should be a good approximation for large $x$. The problem here is that even though the error decays like $x^{-4}$ it is not small compared to $\hbar$ unless $x > \hbar^{-\frac{1}{4}}$. Since we need to be able to send $\hbar$ and $E$ to zero independently, such an approximation is useless for the case were $E$ is small but fixed and $\hbar \to 0$. To idea behind our reduction to the Bessel equation is essentially to let $\langle x \rangle$ be a new independent variable. The reader will easily see that this is precisely what Lemma 4 does (in addition, we scale out $E$ and fix the turning point to lie at 1).

**Lemma 12.** For any $0 < E < E_0$ the following holds: $f(x)$ is a smooth solution of

$$-\hbar^2 f''(x) + V(x)f(x) = Ef(x) \quad \text{on} \quad x > E^{-\frac{1}{4}}$$

iff $\phi(\xi) = \phi(\xi,E) := \Omega(\xi,E)^{\frac{1}{2}} f(E^{-\frac{1}{4}} y(\xi,E))$, with $\Omega$ as in (A.2), is a smooth solution of

(A.3) $-\hbar^2 \phi''(\xi) + [\xi^{-2}(1 - \hbar^2/4) - 1] \phi(\xi) = \hbar^2 W_0(\xi,E)\phi(\xi) \quad \text{on} \quad \xi > \xi_1(E)$

with a potential $W_0$ satisfying

(A.4) $|\partial_{\xi}^k \partial_{\xi}^\ell W_0(\xi,E)| \leq C_{k,\ell} E^{-k} \xi^{-3-\ell}$

for all $k, \ell \geq 0$.

**Proof.** Under the change of variables $g(y) = f(E^{-\frac{1}{4}} y)$ the following equations are equivalent, with $V_1(x) = -\frac{1}{4}(x)^{-2}$:

$$-\hbar^2 f''(x) + (V_0(x) + \hbar^2 V_1(x))f(x) = Ef(x)$$

$$-\hbar^2 g''(y) + (E^{-1} V_0(E^{-\frac{1}{4}} y) - 1)g(y) = -\hbar^2 E^{-1} V_1(E^{-\frac{1}{4}} y)g(y)$$

Now let $\xi = \xi(y,E)$ be as in Lemma 4 and set $\psi(\xi) = g(y(\xi,E))$, or equivalently, $\psi(\xi,y,E)) = g(y)$. Then, with $\mu$ as in Lemma 4

$$-\hbar^2 [\psi''(\xi) + (\partial_\xi y(\xi,E))^2 (\partial_{yy} \xi)(y(\xi,E),E)\psi'(\xi)] + (\xi^{-2} - 1)\psi(\xi)$$

(A.5) $= -\hbar^2 [\psi''(\xi) + \mu(\xi,E)\psi'(\xi)] + (\xi^{-2} - 1)\psi(\xi)$

$$= -\hbar^2 (\partial_\xi y(\xi,E))^2 E^{-1} V_1(E^{-\frac{1}{4}} y(\xi,E))\psi(\xi)$$

Let $\Omega$ be as in Lemma 4. In view of (A.2), $\phi := \Omega^{\frac{1}{2}} \psi$ satisfies the equation

(A.6) $-\hbar^2 \phi''(\xi) + (\xi^{-2} - 1) \phi(\xi) = \hbar^2 W(\xi,E)\phi(\xi)$

with

$$W(\xi,E) := -(\partial_\xi y(\xi,E))^2 E^{-1} V_1(E^{-\frac{1}{4}} y(\xi,E)) - \frac{1}{2} \frac{\Omega''(\xi,E)}{\Omega(\xi,E)} + \frac{1}{4} \left(\frac{\Omega'(\xi,E)}{\Omega(\xi,E)}\right)^2$$

The second part here involving $\Omega$ decays like $\xi^{-4}$, whereas the first only decays like $\xi^{-2}$. We need to extract this leading order decay: the asymptotic expansion

$$V_1(\xi) = -\frac{1}{4\xi^2} + O(\xi^{-3}) \quad \text{as} \quad \xi \to \infty$$

and Lemma 4 imply that

$$(\partial_\xi y(\xi,E))^2 E^{-1} V_1(E^{-\frac{1}{4}} y(\xi,E)) = -\frac{1}{4\xi^2} + \xi^{-3} V_r(\xi,E)$$

where

$$|\partial_{\xi}^k \partial_{\xi}^\ell V_r(\xi,E)| \leq C_{k,\ell} E^{-k} \xi^{-\ell}$$

In view of (A.6), this yields equation (A.3) and we are done. \qed
A fundamental system \( \{ \phi_{j,n}^{(0)} \}_{j=1}^{2} \) of the homogeneous form of (A.3), i.e.,
\[
-H^2 \phi''(\xi) + \left( \xi^{-2}(1 - \hbar^2/4) - 1 \right) \phi(\xi) = 0
\]
is given in terms of Hankel functions:
\[
\phi_{j,n}^{(0)}(\xi) = \xi^j H_n^{(j)}(n\xi), \quad j = 1, 2, \quad n := \hbar^{-1}
\]
or, equivalently, by the Bessel functions
\[
\tilde{\phi}_{1,n}^{(0)}(\xi) = \xi^{1/2} J_n(n\xi), \quad \tilde{\phi}_{2,n}^{(0)}(\xi) = \xi^{-1/2} Y_n(n\xi)
\]
with Wronskian
\[
W(\tilde{\phi}_{1,n}^{(0)}, \tilde{\phi}_{2,n}^{(0)}) = \frac{2}{\pi}, \quad W(\phi_{1,n}^{(0)}, \phi_{2,n}^{(0)}) = \frac{4i}{\pi}
\]
Hence, the forward Green function of (A.7) is
\[
G_n(\xi, \xi') := \frac{\pi}{4i} [\phi_{1,n}(\xi) \phi_{2,n}(\xi') - \phi_{1,n}(\xi') \phi_{2,n}(\xi)] \chi_{[\xi < \xi']}
\]
\[
= \frac{\pi}{2} [\tilde{\phi}_{1,n}^{(0)}(\xi) \tilde{\phi}_{2,n}^{(0)}(\xi') - \tilde{\phi}_{1,n}^{(0)}(\xi') \tilde{\phi}_{2,n}^{(0)}(\xi)] \chi_{[\xi < \xi']}
\]
\[
= \frac{\pi}{2} (\xi \xi')^{-1/2} [J_n(n\xi) Y_n(n\xi') - J_n(n\xi') Y_n(n\xi)] \chi_{[\xi < \xi']}
\]
and thus a basis \( \{ \phi_{j,n} \}_{j=1}^{2} \) of (A.3) is given by the Volterra equation
\[
\phi_{j,n}(\xi) = \tilde{\phi}_{j,n}(\xi) + \int_{\xi}^{\infty} G_n(\xi, \xi') W_0(\xi', E) \phi_{j,n}(\xi') d\xi'
\]
that one now needs to solve. This of course requires a thorough understanding of the behavior of \( J_n(n\xi) \) and \( Y_n(n\xi) \) for large \( n \) on intervals of the form \( \xi > \xi_0 > 0 \) where \( 0 < \xi_0 \ll 1 \) is fixed, see [1] and [19]. We leave it to the interested reader to pursue this direction.

References

1. Abramowitz, M., Stegun, I. Handbook of mathematical functions with formulas, graphs, and mathematical tables, Reprint of the 1972 edition. Wiley-Interscience Publication; National Bureau of Standards, Washington, DC, 1984.
2. Bollé, D., Gesztesy, F., Schweiger, W. Scattering theory for long-range systems at threshold. J. Math. Phys. 26 (1985), no. 7, 1661–1674.
3. Bollé, D., Gesztesy, F., Wilk, S. F. J. A complete treatment of low-energy scattering in one dimension. J. Operator Theory 13 (1985), no. 1, 3–31.
4. Burq, N., Planchon, F., Stalker, J., Tahvildar-Zadeh, A. Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential. J. Funct. Anal. 203 (2003), no. 2, 519–549.
5. Burq, N., Planchon, F., Stalker, J., Tahvildar-Zadeh, A. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay. Indiana Univ. Math. J. 53 (2004), no. 6, 1665–1680.
6. Chandrasekhar, S. The mathematical theory of black holes. Reprint of the 1992 edition. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 1998.
7. Deift, P., Trubowitz, E. Inverse scattering on the line. Comm. Pure Appl. Math. 32 (1979), no. 2, 121–251.
8. Derezhinski, J., Skibsted, E. Quantum scattering at low energies. preprint, 2007.
9. Friedrich, H., Troest, J. Accurate WKB functions for weakly attractive inverse-square potentials. Physical Review A, vol. 59, no. 2 (1999), 1683–1686.
10. Guillarmou, C., Hassell, A. Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. I., preprint 2007.
11. Goldberg, M., Schlag, W. *Dispersive estimates for Schrödinger operators in dimensions one and three*. Comm. Math. Phys. 251 (2004), no. 1, 157–178.

12. Klaus, M. *Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line*. Inverse Problems 4 (1988), 505–512.

13. Kvitsinski˘ı, A. A. *Scattering by long-range potentials at low energies*. Teoret. Mat. Fiz. 59 (1984), no. 3, 472–478.

14. Langer, R. E. *On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order*. Trans. Amer. Math. Soc. 33 (1931), no. 1, 23–64.

15. Langer, R. E. *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point*. Trans. Amer. Math. Soc. 67 (1949), 461–490.

16. Langer, R. E. *Asymptotic theories for linear ordinary differential equations depending upon a parameter*. J. Soc. Indust. Appl. Math. 7 (1959), 298–305.

17. Marchenko, V. A. *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.

18. Nakamura, S. *Low energy asymptotics for Schrödinger operators with slowly decreasing potentials*. Comm. Math. Phys. 161 (1994), no. 1, 63–76.

19. Olver, F. W. J. *Asymptotics and Special Functions*, A K Peters, Ltd., Wellesley, MA, 1997.

20. Planchon, F., Stalker, J., Tahvildar-Zadeh, A. *$L^p$ estimates for the wave equation with the inverse-square potential*. Discrete Contin. Dyn. Syst. 9 (2003), no. 2, 427–442.

21. Planchon, F., Stalker, J., Tahvildar-Zadeh, A. *Dispersive estimate for the wave equation with the inverse-square potential*. Discrete Contin. Dyn. Syst. 9 (2003), no. 6, 1387–1400.

22. Ramond, T. *Semiclassical study of quantum scattering on the line*. Comm. Math. Phys. 177 (1996), no. 1, 221–254.

23. Schlag, W., Soffer, A., Staubach, W. *Decay for the wave and Schrödinger evolutions on manifolds with conical ends, part I*, preprint 2006.

24. Schlag, W., Soffer, A., Staubach, W. *Decay for the wave and Schrödinger evolutions on manifolds with conical ends, part II*, preprint 2007.

25. Yafaev, D. R. *The low energy scattering for slowly decreasing potentials*. Comm. Math. Phys. 85 (1982), no. 2, 177–196.

26. Yafaev, D. R. *On the quasi-classical asymptotics of the forward scattering amplitude and of the total scattering cross-section*. Séminaire sur les Équations aux Dérivées Partielles, 1988–1989, Exp. No. VII, 10 pp., École Polytech., Palaiseau, 1989.

Costin, Tanveer: Department of Mathematics, The Ohio State University 100 Math Tower, 231 West 18th Avenue, Columbus, OH 43210-1174, U.S.A.

E-mail address: costin@math.ohio-state.edu, tanveer@math.ohio-state.edu

Schlag: The University of Chicago, 5734 South University Avenue, Chicago, IL 60637, U.S.A.

E-mail address: schlag@math.uchicago.edu

Staubach: Department of Mathematics, Colin Maclaurin Building, Heriot-Watt University, Edinburgh, EH14 4AS, U.K.

E-mail address: W.Staubach@hw.ac.uk