Modelling High-Frequency Backscattering from a Mesh of Curved Surfaces Using Kirchhoff Approximation

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Abstract

The Kirchhoff approximation (K-A) to calculate the acoustic backscattering of a complex structure can be evaluated using a discretized version of its surface (i.e. a mesh). From the computational viewpoint, the most handy approach is the one based on flat facets. However, in the high frequency range, where the K-A provides good agreement and is therefore applicable, it requires a mesh with such a large number of facets that it turns impractical. To avoid these difficulties a mesh of curved triangles can be used to model the scatterer’s complex structure. Previous computational implementations reported in the literature did not accomplish satisfactory results for high frequency. In this work we propose a numerical model based upon an iterative integration using Gauss-Legendre rules. The model was validated against exact solutions and led us to achieve adequate results in the high-frequency range.

1 Introduction

The Kirchhoff approximation (K-A) is probably the quintessential tool to calculate approximate high-frequency scattering from impenetrable bodies since it provides not only a computationally easy implementation but even a faster one. The K-A allows for obtaining the far-field backscattering amplitude function $f_\infty$ through an integration over the insonified scatterer surface $S_i$ [1],

$$f_\infty = \frac{ik}{2\pi} \int_{S_i} e^{2\pi ik \cdot \hat{x}} \hat{k} \cdot \hat{n}(x) \ dS(x),$$  \hspace{1cm} (1)

where $k$ is the wavenumber of the incident field, $\hat{k}$ its incidence direction and $\hat{n}$ the exterior normal to the scatterer surface. When the object is convex, this approximation can be used to calculate the scattering provided the frequency is higher enough ($ka \gg 1$, being $a$ a characteristic length of the scatterer). For non convex bodies, multiple scattering must be taken into account in order to allow that the K-A still works. Furthermore, in that case shadowing algorithms are needed to determine which region of the object surface is insonified by the incoming wave and its successive reflections.

Several methods have been developed over the years to model more accurately the high-frequency scattering by supplementing the K-A with the accounting of edge diffraction [2], material properties through reflection coefficients [3] and multiple scattering contributions. These issues (non convexities, shadowing and diffraction corrections) are out of the scope of this work and consequently, only the convex case and the bare K-A will be considered here.

When the surface of a scatterer is easily described, as in the case of spheres, cylinders or parallelepipeds, the integration can be carried out by analytical methods and sometimes
it has a closed-form \cite{4}. On the other hand, when the scatterer has a complex geometry (e.g. fish, phytoplankton or submarine hulls), an usual approach consists of approximating its surface by a mesh composed of simpler elements as triangular or quadrilateral facets (planar or curved). Then, using the linearity of the integral, the K-A reduces to the sum of the individual integrals over each insonified element.

There are many works where the approach based on planar triangular facets is applied. For example, George \cite{5} has used analytical approximations that may introduce certain lack of precision, whereas Pignier et al.\cite{6} have used quadrature rules but they led to computationally expensive expressions. Other authors explicitly use the fact that the integral in Eq. (1) has an exact solution based on elementary functions when the integration surface (the facet) is a planar triangle. Therefore, the computational cost is reduced to a minimum when the integrations implied by Eq. (1) are carried out in this fashion\cite{7,8,9}.

However, since flat facets cannot exactly represent any curvature, the description of a curved surface with this kind of mesh is only a approximation, although it is a very good one if the number of facets is big enough. It can be shown that higher the frequency bigger the required mesh size. The advantage of the simple integration provided by the model of planar triangles is spoiled in the high frequency regime because of the required mesh size, which makes impractical the computational implementation of Eq. (1). For this reason, difficulties arise even for $ka > 20$, as stated in the literature \cite{9}. Other researchers provide numerical results that reach $ka \approx 42$ and express that the method potentiality reaches up to $ka \approx 200$, although results for this case are not shown \cite{8}.

Several authors have pointed out the issue that arises when attempting to represent a curved surface with planar facets (the so called curvature problem). With the aim of overcoming it, curvilinear facets may be employed, but in this case there are no longer analytical solutions thus quadrature rules should be used. Some authors who follow this approach have considered seven-point Gaussian quadrature in 2D \cite{10,11}, nevertheless this kind of quadrature rules are not appropriate at high frequencies because the oscillatory behavior of the integrand in Eq (1) causes numerical errors. Therefore, this method is unable to provide high $ka$ values.

In this work, a K-A model for a mesh of curved triangles is presented. Its two main features are: (a) it is based on curvilinear triangles to take into account the curvature problem, (b) integrals are computed through an adaptive quadrature rule as an alternative to settle the high frequency ranges.

This paper is organized as follows. Section 2 addresses the model: the curved triangle description and coordinatization as well as the resulting integrals and its numerical evaluation. In Section 3 the model is validated against certain known K-A exact solutions and in Section 4 a comparison with a modern representative planar triangle model \cite{9} (which uses exact triangle integration) is provided. In Section 5 an application to underwater acoustics is exhibited in order to show the relevance of the model. The conclusions of the work are summarized in Section 6.

### 2 Model description

The model calculates the far-field backscattering amplitude function $f_\infty$ considering the surface of the scattered body given by a curved-triangle mesh $\mathcal{M} = \bigcup_{l=1}^{N} \triangle_l$, through

$$
 f_\infty = \sum_{l=1}^{N} \frac{i k}{2 \pi} \int_{\triangle_l} \kappa_l(\mathbf{x}) e^{2 i k \hat{\mathbf{k}} \cdot \mathbf{x}} \hat{\mathbf{k}} \cdot \hat{n}(\mathbf{x}) \, dS(\mathbf{x}). \tag{2}
$$
The condition imposed by the K-A, that the integral must be exclusively calculated over the insonified surface, is accomplished introducing the function κ whose definition is

\[ \kappa_l(x) = \begin{cases} 
1 & \text{if } x \text{ is insonified} \\
0 & \text{if } x \text{ is in shadow} 
\end{cases} \]

With the help of this function the summatory in Eq. (2) only encompasses the contributions of the insonified areas in each l-th triangle.

2.1 Curved triangle description and integration

The integration over the individual triangles of the mesh is then carried out using a description of second order triangles based on the shape functions \( L_i \), defined by Carley [13] as

\[
\begin{align*}
L_1 &= 2(1 - \xi - \eta)(1/2 - \xi - \eta) \\
L_2 &= 2\xi(\xi - 1/2) \\
L_3 &= 2\eta(\eta - 1/2) \\
L_4 &= 4\xi(1 - \xi - \eta) \\
L_5 &= 4\xi\eta \\
L_6 &= 4\eta(1 - \xi - \eta)
\end{align*}
\]

These functions describe the reference triangle in terms of parameters \( \xi, \eta \) that verify \( 0 \leq \xi \leq 1, 0 \leq \eta \leq 1 - \xi \), as shown in the left side of Figure 1. Thus, a point \( T \) belonging to the curved triangle is given by

\[
T(\xi, \eta) = \sum_{i=1}^{6} L_i(\xi, \eta) V_i
\] (3)

where \( V_i \) are the curved triangle vertices, illustrated in Figure 1 (right).

![Reference triangle and curved triangle](image)

Figure 1: Reference triangle (left) and curved triangle (right).

The integration over a single triangle appearing in Eq. 2 in terms of the parameterization 3 reads

\[
\int_\Delta \kappa_l(x) e^{2ik \cdot x} \hat{k} \cdot \hat{n}(x) dS(x) = \int_0^1 \int_0^{1-\xi} \kappa_l(T) e^{2ik \cdot T} \hat{k} \cdot (T_\xi \times T_\eta) d\eta d\xi
\]

3
where

\[ \hat{k} \cdot T = C_1 \xi^2 + C_2 \xi + C_3 \eta \xi + C_4 \eta^2 + C_5 \eta + C_6 \]

\[ \hat{k} \cdot (T_\xi \times T_\eta) = D_1 \xi^2 + D_2 \xi + D_3 \xi \eta + D_4 \eta^2 + D_5 \eta + D_6, \]

being \( C_i \) and \( D_i \) constants defined in terms of the \( V_i \) \((i = 1, 2, \ldots, 6)\). These expressions are tabulated explicitly in the Appendix.

Now, the contribution \( f_\infty^{\Delta t} \) from the \( l \)-th triangle to the total \( f_\infty = \sum_I f_\infty^{\Delta t} \) is

\[ f_\infty^{\Delta t} = \frac{ik}{2\pi} e^{2ikC_6} F_l(\{C_m\}, \{D_n\}) \]

where

\[ F_l(\{C_m\}, \{D_n\}) = \int_0^1 \int_0^{1-\xi} \left( D_1 \xi^2 + D_2 \xi + D_6 \right) \left( D_3 \xi + D_5 \right) \left( D_4 \right) \kappa_l(T) e^{2ikQ(\xi,\eta)d\eta} d\xi \]

\[ + \left( D_3 \xi + D_5 \right) \int_0^{1-\xi} \kappa_l(T) e^{2ikQ(\xi,\eta)d\eta} d\xi + D_4 \int_0^{1-\xi} \kappa_l(T) e^{2ikQ(\xi,\eta)d\eta} d\xi \] (4)

and

\[ P(\xi) = C_1 \xi^2 + C_2 \xi, \quad Q(\xi, \eta) = C_3 \eta \xi + C_4 \eta^2 + C_5 \eta. \]

The function \( F(\{C_m\}, \{D_n\}) \) \((m = 1 \ldots 5, n = 1 \ldots 6)\) accounts for the three iterated integrals over \( \eta \) and \( \xi \). These integrals are calculated through a two nested quadrature rule implemented by a specialized computational routine QuadGK [14] available under the Julia language programming [15].

In summary, the model consists of the following three key elements: (1) a curved triangle mesh \( M \), (2) a function \( \kappa \) that determinates the insonified surface and (3) an integration algorithm over a curved triangle based on iterated quadrature rules.

### 3 Validation

We validate the model against the K-A for three simple shapes whose backscattering amplitude function \( f_\infty \) has an analytical closed-form solution: the sphere, the cylinder with flat end caps and the prolate spheroid. In each case we present plots of \( |f_\infty| \) in terms of an adimensional parameter \( k \ell \) (being \( \ell \) a characteristic length), which is the usual magnitude of interest in many applications.

#### 3.1 Sphere

For a sphere of radius \( r \), the K-A integral has a simple closed-form [9],

\[ f_\infty = \frac{i}{4k} e^{-2ikr} \left( e^{2ikr} - 2ikr - 1 \right). \] (5)

For the comparison with the exact solution, a mesh with 1278 curved triangles has been used. The resulting \( |f_\infty| \) is exhibited in Figure 2 for the interval \( 0.1 \leq kr \leq 100 \) where \( r = 1 \) is the sphere radius.
Figure 2: Results for the $|f_\infty|$ of the sphere. The mesh used for the curved triangle model has 1278 triangles.

3.2 Cylinder with flat endcaps

The geometry and the coordinates considered for the insonification of the following validation examples, namely, cylinder and spheroid, are shown in Figure 3.

![Figure 3: Schemes for the incidence over the cylinder with flat endcaps (left) and the spheroid (right). The dimensions used in all the tests were $a = 0.5$ and $b = 1$.](image)

The expression of the backscattering from a finite cylinder of radio $a$ and length $2b$ (without endcaps) for an incidence direction characterized by the $\theta$ angle is [4],

$$f_{cyl, \infty} = -\frac{a}{2} \tan(\theta) \sin(2kb \cos(\theta)) \left[ \frac{2}{\pi} - H_1(2ka \sin(\theta)) - iJ_1(2ka \sin(\theta)) \right],$$

(6)

where $H_1$ and $J_1$ are the first order Struve and cylindrical Bessel functions, respectively. The contribution of the top flat surface (circle of radius $a$) is

$$f_{\text{top}, \infty} = -ika^2 \cos(\theta) e^{-2ikb \cos(\theta)} \frac{J_1(2ka \sin(\theta))}{2ka \sin(\theta)}.$$

(7)

In order to obtain the exact K-A solution for the cylinder is enough to consider the sum of the expressions of Eqs. (6) and (7). These account for the backscattering in the
0 ≤ \( \theta \leq \pi/2 \) incidence while the range \( \pi/2 \leq \theta \leq \pi \) is easily built up from the former using symmetry considerations.

![Figure 4: Results for the \( |f_\infty| \) of the flat endcaps cylinder (radius \( a = 0.5 \), semilength \( b = 1 \)) at incidence angle \( \theta = \pi/4 \). The mesh used for the curved triangle model has 1238 triangles.](image)

The agreement between the exact solution and the curved triangle model for the test mesh (\( a = 0.5 \) and \( b = 1 \)) is illustrated in Figure 4. The mesh consists of 1238 curved triangles. The fixed incidence used was \( \theta = \pi/4 \).

3.3 Prolate spheroid

Considering a prolate spheroid of semiaxes \( a \) and \( b \) (\( b > a \)), the backscattering amplitude function \( f_\infty \) can be obtained in a closed-form for the cases \( \theta = 0 \), i.e. parallel to the foci line, and \( \theta = \pi/2 \), perpendicular to the foci line, (see Figure 3, right). In these cases we have

\[
f_\infty(\theta=0) = -\frac{i}{4k}\left(\frac{a}{b}\right)^2 e^{-2ika} (e^{i2ka} - 2ika - 1)
\]

and

\[
f_\infty(\theta=\pi/2) = -\frac{i}{4k}\left(\frac{b}{a}\right) e^{-2ika} (e^{i2ka} - 2ika - 1),
\]

both expressions having the same form that Eq. (5). In particular, at an incidence angle \( \theta = 0 \) the Eq. (8) verifies

\[
f_\text{spheroid} = (a/b)^2 f_\infty(r = b),
\]

i.e. the backscattering corresponds to a multiple of the backscattering from a sphere.

The corresponding \( |f_\infty| \) at the incidences \( \theta = 0 \) and \( \theta = \pi/2 \) are shown in Figure 5 and Figure 6 respectively. In both cases a spheroid mesh of 1386 triangles was used and the \( |f_\infty| \) is plotted against \( kb \), being \( b = 1 \) the major semiaxis.

Some errors are visually evident in the Figure 5 which corresponds to the \( \theta = 0 \) incidence. Here the incident wave faces the spheroid in the end-on aspect, so that curvature issues are more crucial than in beam aspect (\( \theta = \pi/2 \)). Evidently for \( kb > 90 \) a more refined mesh is neccessary.
4 Comparison with planar triangles

When using the flat triangles the integration results in a expression simpler than on any other type of curved surface but as drawback artificial edges are introduced and, as stated in Section 1, a big number of triangles are necessary for describing a general geometry with negligible error in curvature unless the obstacle is a polyhedron.

The planar triangle model consequently requires for the correct description of curved surfaces a much bigger mesh than in the curved case. In a recent work [9] the author recommends that the all triangle’s edge size $e$ must verify the relation $10e < \lambda$ (being $\lambda = 2\pi/k$ the wavelength) for an accurate backscattering evaluation (ten elements per wavelength). Moreover, because of the flatness of the planar triangles, the determination of the insonified area is also approximate. Each triangle has a unique normal $\hat{n}$ so that a given triangle can only be fully insonified or fully in shadow.

To evaluate the performance of both approaches we take a unit ($r = 1$ radius) sphere.
To avoid biased results caused by anisotropies of the mesh, the amplitude function \( f_\infty \) presented here is the average of three incidences along the coordinate axes. The relative error curves for this comparison are shown in Figure 7. It is evident that the curved triangle model obtains better fit (minor relative error) in the entire frequency range considered, even against a planar triangle mesh with a number of triangles \( \sim 500 \) times larger.

![Figure 7: Relative error in the \( f_\infty \) according to (10) in the backscattering calculation for a unit radius sphere with a planar triangle mesh (\( N = 42472, N = 171472 \) and \( N = 982874 \)) against a curved triangle mesh of \( N = 2016 \).](image)

In this section, aimed to applications, we use the curved triangle model to evaluate the \( \text{TS} \) for the backscattering of a submarine.
Figure 8: Relative error in the $f_\infty$ according to (10) in the backscattering calculation for a unitary sphere under two planar triangle meshes of $N = 5034$, 6081410 against a curved triangle mesh of $N = 5034$.

To test our curved triangle model we construct a generic simplified submarine (based on the simple BeTSSi model [17]), which is shown in Figure 9. From this submarine two meshes are built; a planar one with $N = 77220$ triangles and a curved one with $N = 18116$. The planar mesh is used to evaluate a reference solution for the backscattering problem which is based on a BEM method.

The length $\ell$ of the triangle’s side (edges) for the plane mesh has a mean value of 0.207 m and verifies the condition $\lambda \geq 6\ell$ for 93 % of the occurrences (the length distribution is approximately gaussian). This relation between edge length $\ell$ and wavelength $\lambda$ assures that the object is adequately represented by the mesh in terms of the backscattering calculation for the BEM reference solution.

The BEM method, under a conventional implementation, provides relatively accurate solutions only for the low or intermediate frequency regimes. Nevertheless it’s expected that the K-A coincides with this reference solution (for some incidences) even in the intermediate frequency regime because the conditions of the former are satisfied under several incidences in a object like the actual test submarine.

Figure 9: Generic submarine used for testing. The real part of a plane wave of $f = 1000$ Hz and incidence direction given by $-\hat{z}$ is showed on the submarine surface.

The generic submarine can be characterized by two lengths; those corresponding to
the length of the cylinder that forms the external hull and its radius. These lengths are \( a = 31 \) m and \( b = 3.75 \) m. For a incident wave of frequency \( f \) the corresponding wave number is \( k = 2\pi f/c \) where \( c \) is the sound speed in the water. For the case \( f = 1000 \) Hz and taking \( c = 1500 \) m/seg (an accepted average value) we have \( k = 4.188 \) and the scattering can be adimensionally characterized by \( ka \approx 130 \) and \( kb \approx 16 \). The Figure 9 shows the real part of an incident wave \( \exp(ikk \cdot x) \) (with \( k = (0, 0, -1) \)) evaluated on the submarine surface.

For the TS calculation we restrict ourselves to incidence directions with altitude 0° (i.e. incidence belonging to the \( yz \) plane), parameterized in terms of a \( \theta \) angle by \( \hat{k} = -(0, \sin \theta, \cos \theta) \). The results of the backscattering TS for the submarine model at 0° altitude is shown in Figure 10 for the K-A with the curved mesh and the BEM reference solution with the planar one for the frequency \( f = 1 \) KHz. It follows from the figure that the K-A provides a good agreement, specially up to 115°.

![Figure 10: TS for the submarine model for an incidence \( \hat{k} = -(0, \sin \theta, \cos \theta) \) and frequency \( f = 1000 \) Hz evaluated by the K-A with a curved triangle mesh and by a BEM solution (used as a reference).](image1)

![Figure 11: TS for the submarine model for an incidence \( \hat{k} = -(0, \sin \theta, \cos \theta) \) and frequencies \( f = 8 \) KHz and \( f = 12 \) KHz evaluated by the K-A with the curved triangle mesh.](image2)

It is expected that for \( \theta \) tending to 180° (stern direction) the K-A validity (as solution
of the backscattering problem) weakens because in these incidences the submarine presents a less smooth surface due to the presence of the edges that form the union of the aft (which in this simplified model is a cone) with the main body and those that constitutes the back of the veil. Any effects due to edge’s diffraction are, of course, not taken into account in the K-A.

The TS for $0^\circ$ altitude and the frequencies $f = 8$ KHz and $f = 12$ KHz which means for $ka \approx 1039, kb \approx 126$ and $ka \approx 1558, kb \approx 188$, respectively, is shown in the Figure 11.

6 Conclusions

The model presented in this work, based on numerical iterated integration over curved triangles and with exact accounting of the insonified surface of the scatterer, has been able to provide excellent agreement against known exact solutions of the K-A (the validation examples of section 3) up to high $k\ell$ values and requiring only small size meshes. The application’s range previously reported in the literature could be also extended.

The comparison with a model based on flat triangle integration, performed in section 4, showed that a small mesh of curved triangles can provide better fit than a much larger mesh of planar ones.

While the curved triangle model is somewhat complex because requires numerical integration techniques, it has the advantage of adequately modeling curved surfaces with a small mesh size, and also determine in a more precisely way the insonified area due to triangles that can be partially insonified, in contrast to the case of planar triangular facets. In the latter, although the integration is almost trivial the only way to overcome the curvature and partial insonification issues is through the increase in mesh size and that will lead, sooner or later, to a bottleneck in computational storage if the frequency is high enough.

On the other hand, the algorithm for the calculation of backscattering from curved triangles is easily parallelizable so that it can take advantage of the existence of multiple cores in the computer where it is executed. This fact can contribute significantly to decrease the computing time.

The model can provide an approximate technique for the determination of complex object backscattering in the high frequency regime as it has been shown in the section 5 where the backscattering of a generic submarine was calculated under this approach and good agreement (compared to a reference solution) was obtained for incidence directions over which the K-A applicability was guaranteed.

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\section*{A\hspace{1em}Constants $C_i, D_i$}

Expressions of the constants involved in the integration over the curved triangles in terms of the vertices $V_i$.

\begin{align*}
C_1 &= \hat{k} \cdot (V_1 + V_2 - 2V_4) \\
C_2 &= \hat{k} \cdot (-3V_1 - V_2 + 2V_4) \\
C_3 &= \hat{k} \cdot (V_1 - V_4 + V_5 - V_6) \\
C_4 &= \hat{k} \cdot (V_1 + V_3 - 2V_6) \\
C_5 &= \hat{k} \cdot (-3V_1 - V_3 + 4V_6) \\
C_6 &= \hat{k} \cdot V_1 \\
\end{align*}

\begin{align*}
D_1 &= 16 \left( \hat{k} \times V_4 \cdot [-V_1 + V_2 - 2V_5 + 2V_6] + \hat{k} \times V_2 \cdot [V_1 + V_5 - V_6] + \hat{k} \times V_1 \cdot [V_5 - V_6] \right) \\
D_2 &= -4 \left( \hat{k} \times V_1 \cdot [-4V_2 + V_3 + 7V_4 + 3V_5 - 7V_6] + \hat{k} \times V_2 \cdot [V_3 - V_4 + V_5 - 5V_6] + \hat{k} \times V_4 \cdot [-2V_3 - 4V_5 + 12V_6] \right) \\
D_3 &= 16 \left( \hat{k} \times V_1 \cdot [V_3 - V_2 + 2V_4 - 2V_6] + \hat{k} \times V_2 \cdot [V_3 - 2V_6] + \hat{k} \times V_1 \cdot [-2V_3 + 4V_6] \right) \\
D_4 &= -16 \left( \hat{k} \times V_1 \cdot [-V_3 - V_4 + V_5 + V_6] + \hat{k} \times V_2 \cdot [-V_4 + V_5] + \hat{k} \times V_6 \cdot [V_3 + 2V_4 - 2V_5] \right) \\
D_5 &= 4 \left( \hat{k} \times V_6 \cdot [-4V_5 + 12V_4 - 2V_2 - 7V_1] + \hat{k} \times V_1 \cdot [3V_5 - 7V_4 - 4V_3 + V_2] + \hat{k} \times V_3 \cdot [V_2 - 5V_4 + V_5 - V_6] \right) \\
D_6 &= \hat{k} \times V_6 \cdot [-16V_4 + 4V_2 + 12V_1] + \hat{k} \times V_1 \cdot [12V_4 - 3V_2] + \hat{k} \times V_3 \cdot [-3V_1 + 4V_4 - V_2] \\
\end{align*}