A Novel Covariance Matrix Reconstruction Method for Robust Adaptive Beamforming

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Abstract—The computational complexity of the conventional adaptive beamformer is relatively large, and the performance degrades significantly due to both the model mismatch errors and the unwanted signals in received data. In this paper, a novel robust adaptive beamforming technique which is based on the covariance matrix reconstruction is proposed. Different from the prior covariance matrix reconstruction methods, a projection matrix is constructed to remove the unwanted signal from the received data, which improves the reconstruction accuracy of the covariance matrix. Considering that the computational complexity of most matrix reconstruction algorithms are relatively large due to the integral operation, we propose a Gauss-Legendre quadrature-based method to approximate the integral operation while maintaining the accuracy. Moreover, to improve the robustness of the beamformer, the mismatch in the desired steering vector is corrected by maximizing the output power of the beamformer under a constraint that the corrected steering vector cannot converge to any interference steering vector. Simulation results and prototype experiment demonstrate that the performance of the proposed beamformer outperforms the compared methods and is much closer to the optimal beamformer in different scenarios.

Index Terms—Covariance matrix reconstruction, desired signal removal, robust adaptive beamforming, Gauss-Legendre quadrature, steering vector estimation.

I. INTRODUCTION

Adaptive beamforming is an array signal processing technology which has been widely applied in radar, sonar, wireless communication and many other fields [1,2]. It is a data-based beamformer which adjusts the weights adaptive according to the received data to extract the desired signal and suppress the interference and noise [3]. The minimum variance distortionless response (MVDR) is one of the most well-known adaptive beamforming algorithms with the assumptions that the desired signal and antenna array structure are known accurately [4]. MVDR has excellent resolution and interference suppression capability, but it is sensitive to the steering vector and covariance matrix mismatch. The effectiveness of the beamformer degrades severely especially when the desired signal is presented in the received data, which is inevitable in practice [5]. Therefore, a lot of work has been spent on how to improve the robustness of the adaptive beamformer during the last decades [6]–[8].

In a general way, the robust adaptive beamforming techniques can be divided into several types: diagonal loading technique, eigenspace-based technique and covariance matrix reconstruction [9]. The diagonal loading is a widely used method, which adds a scaled identity matrix to the covariance matrix to get robustness [10,11]. However, how to choose the optimal diagonal loading factor is a difficult problem and it decides the performance of the beamformer directly. A uncertainty set of the steering vector is proposed to calculated the diagonal loading factors precisely in [12]. A simple tridiagonal loading method which called automatic tridiagonal loading is proposed to enhance the robustness in [13]. The eigenspace-based technique uses the orthogonality between the steering vector and subspace to correct the nominal steering vector and estimate the covariance matrix [14]–[17]. However, it is hard to obtain an accuracy subspace and the signal subspace can be covered by the noise subspace when the SNR is low. An eigenvalue beamformers is proposed to resolve the unknown signal of interest whose spatial signature lies in a known subspace in [18]. Ref. [19] uses the subspace fitting and subspace orthogonality techniques to improve performance of the beamformer.

The covariance matrix reconstruction technique is a novel method, which separates the desired signal component away from the sample covariance matrix to enhance the robustness of the beamformer [20,21]. In [7], the interference-plus-noise covariance matrix (IPNCM) is firstly reconstructed by using the Capon spectrum to integrate the nominal steering vector over an angle range which does not contain the desired signal direction. Based on this, an annulus uncertainty set is proposed to replace the normal linear integral interval to improve the robustness in [22], which makes the algorithm have better performance but higher complexity. The covariance matrix reconstruction in [23] follows a similar method to [7,22], but the maximum entropy power spectrum (MEPS) is used to replace the Capon spectrum to reconstruct the matrix, and the performance is further improved. Estimating the interference steering vectors and power is also a widely used approach to reconstruct the covariance matrix. Ref. [24] proposes a matrix reconstruction method by searching the interference steering vectors which lie inside the intersection of the interference subspace. The interference steering vector is estimated based on ad-hoc parameters in [25]. Ref. [26] introduces a matrix reconstruction method based on subspace and gradient vector. The adaptive beamforming method under the colored noise is discussed in [27]. The impacts of interference power estimation on robust adaptive beamforming are firstly analyzed in [28], and a matrix reconstruction method via simplified interference power estimation is introduced then. Based on
it, two novel IPNCM reconstruction methods by estimating the power and steering vectors of interference are proposed in [29]. To reduce the algorithm complexity, computational efficient matrix reconstruction algorithms are proposed in [30–32]. A low computational complexity beamformer using the covariance matrix taper technique is proposed in [33].

In this paper, to further improve the beamformer performance and reduce the algorithm computational complexity, a novel adaptive beamforming IPNCM-DRGLQ based on desired signal removal covariance matrix reconstruction and Gauss-Legendre quadrature is proposed. In most existing algorithms, the IPNCM is reconstructed based on the received data, and it contains the desired signals, which will affect the accuracy of the reconstruction. Thus, a projection matrix is constructed to remove the desired signal information from the received data. The quasi matrix can be obtained by projecting the received data onto the projection matrix, which contains little desired information. The quasi matrix is then used to reconstruct IPNCM based on the Capon spectrum. Due to quasi matrix has less desired information than the received data, the reconstruction of IPNCM based it is more accuracy. Considering that the conventional method to calculate the integral is to approximate it by polynomial summation, which usually leads to high computational complexity, a low-complexity algorithm based on 3-order Gauss-Legendre quadrature (GLQ) is introduced to simplify the integral operation. It reduces the computational complexity of the algorithm while maintaining high algebraic precision. Furthermore, the presumed desired signal steering vector is corrected by maximizing the beamformer output power, which is solved under the constraint that the corrected steering vector can not converge to any interference. By combining the reconstructed IPNCM and the corrected desired signal steering vector, the proposed adaptive beamformer can be obtained. Simulations and experiment are done to explore the performance of the algorithm.

The rest of paper is organized as follows. The signal model for the adaptive beamforming technique is introduced in Section II. The preliminary for IPNCM estimation is discussed in Section III. In Section IV, the novel algorithm is proposed detailedly. After numerical simulations and experiment, the performances of the proposed beamformer under different scenarios are demonstrated in Section V. The conclusion is presented in Section VI finally.

Notations: Upper-case and lower-case boldface letters denote the matrices and column vectors, respectively. $(\cdot)^T$ denotes matrix transpose, while the Hermitian transpose is denoted as $(\cdot)^H$. $E\{\cdot\}$ stands for the expectation operator of stochastic variables. $\|\cdot\|_2$ denotes the $\ell_2$ norm. $\text{Tr}\{\cdot\}$ denotes the trace of a matrix, and it is equal to the sum of the diagonal elements of the matrix.

II. SIGNAL MODEL FOR ADAPTIVE BEAMFORMING

Without loss of generality, a uniform linear array (ULA) with $M$ sensors is considered in this paper, as shown in Fig. 1. The ULA receives $P+1$ far-field narrow-band signals composing of 1 desired signal $s_0(t)$ and $P$ interference signals $s_p(t)\,(p = 1, 2, \ldots, P)$. The received signal sampled at the $k$-th snapshot can be expressed as

$$
\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_M(k)]^T
$$

where $x_s(k) \in C^{M \times 1}$, $x_t(k) \in C^{M \times 1}$ and $x_n(k) \in C^{M \times 1}$ are the desired signal, interference and noise, respectively, and these signals are statistically independent. $x_m(k)\,(m = 1, 2, \ldots, M)$ represents the received data at the $m$-th sensor. $\theta_0$ denotes the direction of the desired signal and $\theta_p$ is that of the $p$-th interference. $\mathbf{a}(\theta)$ defines a steering vector and can be given as

$$
\mathbf{a}(\theta) = [a_1, a_2, \cdots, a_M]^T = [1, e^{j2\pi d\sin\theta}, \cdots, e^{j2(M-1)\pi d\sin\theta}]^T,
$$

where $d$ is the spacing between the adjacent sensors, and $\lambda$ denotes the wavelength.

An adaptive beamforming method can be adopted to eliminate the interference, with a beamforming weight being $\mathbf{w} = [w_1, w_2, \ldots, w_M]^T \in C^{M \times 1}$, the beamforming output can be obtained as

$$
y(k) = \mathbf{w}^H \mathbf{x}(k).
$$

Traditionally, the optimal weight $\mathbf{w}_{\text{opt}}$ is obtained by maximizing the output signal-to-interference-plus-noise ratio (SINR)

$$
\text{SINR} = \frac{E\{\mathbf{w}^H \mathbf{x}_s \mathbf{x}_s^H \mathbf{w}\}}{E\{\mathbf{w}^H (\mathbf{x}_t + \mathbf{x}_n) (\mathbf{x}_t + \mathbf{x}_n)^H \mathbf{w}\}} = \frac{\sigma_s^2 \| \mathbf{w}^H \mathbf{a}(\theta_0) \|^2}{\mathbf{w}^H \mathbf{R}_{\text{INF}} \mathbf{w}},
$$

where $\sigma_s^2 = E\{|s_0(k)|^2\}$ denotes the power of the desired signal and $\mathbf{R}_{\text{INF}} \in C^{M \times M}$ is the IPNCM. When the inter-
ference and noise are irrelevant, \( R_{\text{INF}} \) can be expressed as

\[
R_{\text{INF}} = \sum_{p=1}^{P} \sigma_{p}^{2}a(\theta_{p})a^{H}(\theta_{p}) + \sigma_{n}^{2}I_{M},
\]

where \( \sigma_{p}^{2} \) is the \( p \)-th interference power, \( \sigma_{n}^{2} \) is the noise power, and \( I_{M} \) is an identity matrix with the size \( M \times M \).

The problem of maximizing (4) is mathematically equivalent to the following minimum variance distortionless response (MVDR) beamforming problem

\[
\min_{w} w^{H}R_{\text{INF}}w \quad \text{s.t.} \quad w^{H}a(\theta_{0}) = 1.
\]

The objective function in (6) is to minimum the power of the interference and noise, and the constraint ensures that the power of desired signal is not affected. The solution to (6) as the interference and noise, and the constraint ensures that the power of desired signal is not affected. The solution to (6) as the MVDR weight is

\[
w_{\text{opt}} = \frac{R_{\text{INF}}^{-1}a(\theta_{0})}{a^{H}(\theta_{0})R_{\text{INF}}^{-1}a(\theta_{0})}.
\]

In the practical system, since the received signal \( x(k) \) contains both the desired signal and the interference, the matrix \( R_{\text{INF}} \) cannot be estimated accurately from \( x(k) \). Hence, we replace \( R_{\text{INF}} \) by the following sample covariance matrix (SCM)

\[
\hat{R} = \frac{1}{K} \sum_{k=1}^{K} x(k)x^{H}(k),
\]

where \( K \) is the number of the snapshots. Then, substitute (8) into (7), and the MVDR beamformer turns to the sample matrix inversion (SMI) beamformer [3]

\[
w_{\text{SMI}} = \frac{\hat{R}^{-1}a(\theta_{0})}{a^{H}(\theta_{0})\hat{R}^{-1}a(\theta_{0})}.
\]

However, there is a large gap between the SCM \( \hat{R} \) and the IPNCM \( R_{\text{INF}} \), and the steering vector errors in the practical array degrade the performance of the beamformer \( w \) [34]. Thus, in this paper, we are trying to estimate both the IPNCM \( R_{\text{INF}} \) and the desired signal steering vector \( a(\theta_{0}) \) accurately from the received signal \( x(k) \) to improve the robustness and the output SINR of the adaptive beamforming method.

III. PRELIMINARY FOR IPNCM ESTIMATION

To estimate the IPNCM precisely, an efficient reconstruction method based on the Capon spatial spectrum has been proposed in [35], where the IPNCM is estimated by integrating over a angle range separated from the desired signal direction [36].

First, the Capon spatial spectrum is obtained by substituting the MVDR beamformer (9) back into the objective function of (6)

\[
P(\theta) = \frac{1}{a^{H}(\theta)\hat{R}^{-1}a(\theta)}.
\]

Second, as shown in Fig. 2, with the curve representing the steering vector \( a(\theta) \), The IPNCM is then reconstructed by integrating the Capon spatial spectrum over the range \( \Theta_{i} \) between \( \theta_{a} \) and \( \theta_{b} \)

\[
\hat{R}_{\text{INF}} = \int_{\Theta_{a}} \int_{\Theta_{b}} P(\theta)a(\theta)a^{H}(\theta)d\theta
\]

\[
= \int_{\Theta_{a}} \int_{\Theta_{b}} a(\theta)a^{H}(\theta)\hat{R}^{-1}a(\theta)d\theta,
\]

where the range \( \Theta_{i} \) is the interference range, and the desired signal is not included. Hence, \( \hat{R}_{\text{INF}} \) collects all interference and noise information without the effect of the desired signal.

However, the computational complexity of the integral operation in (11) is high and cannot be solved efficiently. Third, the integral operation can be approximated by a summation as

\[
\hat{R}_{\text{INF}} \approx \sum_{l=1}^{L} \frac{a(\theta_{l})a^{H}(\theta_{l})}{a^{H}(\theta_{l})\hat{R}^{-1}a(\theta_{l})}d\theta,
\]

where the range \( \Theta_{i} \) is discretized into \( L \) angles, the \( l \)-th angle is denoted as \( \theta_{l} \), and \( a(\theta_{l}) \) is the corresponding steering vector. The approximation accuracy is dependent on the discretization number \( L \), and the computational complexity of this method is \( O(M^{2}L) \).

However, in this method, the IPNCM \( R_{\text{INF}} \) is estimated as \( \hat{R} \) as shown in the denominator of (12), where the desired signal component is included. Moreover, a large number of the discretized angles are needed to approximate the integration operation precisely.

IV. THE PROPOSED IPNCM-DRGLQ ALGORITHM

In this section, a novel adaptive beamforming algorithm based on IPNCM reconstruction is proposed, which contains 3 steps as shown in Fig. 3. First, a projection matrix is introduced to eliminate the desired signal from the received one. A quasi covariance matrix with less desired information is obtained after the projection and replaces the SCM in (12). Second, a matrix reconstruction method based on the Gauss-Legendre quadrature (GLQ) is proposed to replace the integral operation and reduce the computational complexity. At last, the steering vector is updated to reduce the model mismatch by maximizing the output power of the beamformer.
A. Step 1: Projection Matrix Construction

In this step, a projection matrix \( B \) is constructed to remove the desired signal from the received signal \( x(k) \), so we have \( Bx_n(k) \to 0 \). Then, the IPNCM can be estimated more accurately. First, we construct a covariance-like matrix as

\[
C = \alpha a_i(\theta_0)a_i^H(\theta_0) + I_M, \tag{13}
\]

where \( \alpha \gg 1 \) is a construction parameter. With the eigenvalue decomposition, we have

\[
C = \sum_{i=1}^{M} \mu_i p_i p_i^H, \tag{14}
\]

where \( \mu_i \) (\( i = 1, 2, \ldots, M \)) denotes the eigenvalue in descending order, and the eigenvalues can be obtained as \( \mu_1 = M\alpha + 1 \) and \( \mu_i = 1 \). \( p_i \) is the eigenvector corresponding to \( \mu_i \). Then, the inverse matrix can be expressed as

\[
C^{-1} = \sum_{i=1}^{M} \frac{p_i p_i^H}{\mu_i} = \frac{p_1 p_1^H}{M\alpha + 1} + \sum_{i=2}^{M} p_i p_i^H. \tag{15}
\]

Considering that \( \sum_{i=1}^{M} p_i p_i^H = I_M \) is fulfilled and \( \alpha \gg 1 \), \( C^{-1} \) can be further approximated as

\[
C^{-1} \approx \sum_{i=2}^{M} p_i p_i^H = I_M - p_1 p_1^H. \tag{16}
\]

Hence, the projection matrix \( B \) can be chosen as

\[
B = I_M - p_1 p_1^H. \tag{17}
\]

A function \( \|B^H a(\theta)\|^2 \) versus \( \theta \) is plotted in Fig. 4 to show that the desired signal can be removed efficiently with the projection matrix \( B \). In this example, 10 sensors are considered. 1 desired signal comes from the direction 10° and 2 interference is assumed to be -30° and 40°. The signal to noise ratio (SNR) and interference to noise ratio (INR) in each sensor are both set as 20 dB, the parameter \( \alpha = 10^2 \text{Tr}\{\hat{R}\} \). As shown in Fig. 4, \( \|B^H a(\theta)\|^2 \) is small when \( \theta \) is located in the direction of desired signal, which means that the desired signal is removed effectively.

B. Step 2: IPNCM Reconstruction Using Gauss-Legendre Quadrature

Since the complexity of (12) grows rapidly as the number of discretization \( L \) increases, an efficient matrix reconstruction method based on the Gauss-Legendre quadrature is proposed. Gauss-Legendre quadrature is a kind of Gauss interpolation integral formula, and is one of the highest algebraic precision method in the interpolation-type quadrature formulas [37,38]. A generalized interpolation-type Gaussian quadrature formula with \( N \) points can be written as

\[
\int_a^b f(z)dz = \sum_{n=0}^{N-1} A_n f(z_n) + E, \tag{20}
\]

where \( z_n \in [a,b] \) (\( n = 0, 1, \ldots, N - 1 \)) is the Gauss integral node, \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \) are the lower and upper bounds of the integral respectively, \( E \) is the integral remainder (residual error), and \( A_n \) is weights, which can be calculated as

\[
A_n = \int_a^b \frac{h(z)}{(z-z_n)} \frac{\partial h(z)}{\partial z} dz \tag{21}
\]

where \( h(z) = (z-z_0)(z-z_1)\cdots(z-z_n) \) is a polynomial in \( z \). It is worth mentioning that Gauss theorem states that \( h(z) \) must be orthogonal to any polynomials of less than power \( N \) [39].

Moreover, the \( N \)-th normal Legendre polynomial is written as

\[
P_N(z) = \frac{1}{2^N N!} \frac{\partial^N(z^2 - 1)^N}{\partial z^N}. \tag{22}
\]
We can choose the roots of the Legendre polynomial as the integral nodes in the Gaussian quadrature formula (20), i.e., letting \( z_n \) as the roots of \( P_N(z) = 0 \), and construct the GLQ. Taking the trade-off between computational complexity and algebraic precision into account, we can choose 3-order GLQ [40] to calculate the integral. Additionally, the integral interval is chosen as \([-1, 1]\) to facilitate calculation, i.e., \( a = -1 \) and \( b = 1 \). Then, Eq. (20) can be rewritten as

\[
\int_{-1}^{1} f(z)dz = A_0 f(z_0) + A_1 f(z_1) + A_2 f(z_2) + E, \tag{23}
\]

where \( z_0 = -\sqrt{15}/5 \), \( z_1 = 0 \), \( z_2 = \sqrt{15}/5 \) are the roots of the 3-order Legendre polynomial, i.e.,

\[
P_3(z) = \frac{1}{2} (5z^3 - 3z) = 0. \tag{24}
\]

Next, letting \( h(z) = P_N(z) \), the weights \( A_n \) in (21) can be further obtained as

\[
A_n = \int_{-1}^{1} \frac{P_N(z)}{z - z_n} dz = \frac{2}{(1 - z_n^2)^2} \frac{\frac{\partial P_N(z_n)}{\partial z}}. \tag{25}
\]

Thus, using 3-order Legendre polynomial, the weights can be obtained as \( A_0 = 5/9 \), \( A_1 = 8/9 \), \( A_2 = 5/9 \).

Finally, by combining the roots \( x_n \) and the weights \( A_n \), Eq. (23) can be approximated as

\[
\int_{-1}^{1} f(z)dz \approx \frac{5}{9} f\left(\frac{-\sqrt{15}}{5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{15}}{5}\right). \tag{26}
\]

Furthermore, using a simple linear transformation, a general integral can be approximated by a 3-order GLQ as

\[
\int_{a}^{b} f(z)dz \approx \frac{b - a}{2} \left[ \frac{5}{9} f(l_0) + \frac{8}{9} f(l_1) + \frac{5}{9} f(l_2) \right], \tag{27}
\]

where \( l_n = \frac{1}{2}(a + b) + \frac{1}{2} z_n (b - a) \).

Therefore, the IPNCM (19) can be calculated by the 3-order GLQ as

\[
\hat{R}_{\text{INF}} = \int_{\Theta_1} a(\theta) a^H(\theta) d\theta = \frac{\theta_b - \theta_a}{2} \left[ \frac{5}{9} f(l_0) + \frac{8}{9} f(l_1) + \frac{5}{9} f(l_2) \right]. \tag{28}
\]

By substituting both the Capon beamformer (9) and the reconstructed IPNCM \( \hat{R}_{\text{INF}} \) (28) into Eq. (6), the output power of the beamformer can be obtained as

\[
P(\theta_0) = \frac{1}{a(\theta_0) \hat{R}_{\text{INF}}^{-1} a(\theta_0)} = \frac{1}{a(\theta_0)^H \hat{R}_{\text{INF}}^{-1} a(\theta_0)} \tag{29}
\]

which is a function of the steering vector \( a(\theta_0) \), and \( a(\theta_0) \) can be estimated by maximizing \( P(\theta) \). Suppose that \( \hat{a}(\theta_0) = a(\theta_0) + e \), where \( e \) is the mismatch vector, and the optimization problem of estimating \( \hat{a}(\theta_0) \) can be expressed as

\[
\min_{e} \quad (a(\theta_0) + e)^H \hat{R}_{\text{INF}}^{-1} (a(\theta_0) + e) \tag{30}
\]

s.t. \( (a(\theta_0) + e)^H \hat{R}_{\text{INF}} (a(\theta_0) + e) \leq a^H(\theta_0) \hat{R}_{\text{INF}} a(\theta_0) \),

where the constraint prevents the estimated steering vector \( a \) from converging to the range \( \Theta_1 \).

The mismatch vector \( e \) can be further decomposed into \( e_\perp \) and \( e_\parallel \), where \( e_\perp \) is orthogonal to \( a(\theta_0) \), while \( e_\parallel \) is parallel to \( a(\theta_0) \). Consider that \( e_\parallel \) is a scaled copy of \( \hat{a}(\theta_0) \), so it does not affect the beamforming performance. Thus, (30) can be transformed into

\[
\min_{e_\perp} \quad (a(\theta_0) + e_\perp)^H \hat{R}_{\text{INF}}^{-1} (a(\theta_0) + e_\perp) \tag{31}
\]

s.t. \( a^H(\theta_0) e_\perp = 0 \)

\[
(a(\theta_0) + e_\perp)^H \hat{R}_{\text{INF}} (a(\theta_0) + e_\perp) \leq a^H(\theta_0) \hat{R}_{\text{INF}} a(\theta_0),
\]

where the equality constraint maintains the orthogonality between \( e_\perp \) and \( \hat{a} \). Since the optimization problem (31) is a feasible quadratically constrained quadratic programming (QCQP) problem, it can be efficiently solved by convex optimization toolbox.

Finally, substituting the reconstructed IPNCM \( \hat{R}_{\text{INF}} \) with the estimated steering vector \( \hat{a}(\theta_0) \) back into (7), the adaptive beampformer can be calculated as

\[
\mathbf{w}_{\text{opt}} = \frac{\hat{R}_{\text{INF}} \hat{a}(\theta_0)}{a^H(\theta_0) \hat{R}_{\text{INF}} \hat{a}(\theta_0)}. \tag{32}
\]

Based on the above discussion, the detailed procedures of proposed adaptive beamforming algorithm are described in Algorithm 1. In the proposed IPNCM-DRGLQ algorithm, matrix \( \hat{R} \) is obtained from received data by projecting SCM \( R \) onto \( C \). Then, a IPNCM reconstruction method based GLQ and \( \hat{R} \) is proposed to reduce the computational complexity and maintain excellent performance. Combining the reconstructed IPNCM \( \hat{R} \) and corrected desired steering vector \( \hat{a}(\theta_0) \), the proposed beampformer is obtained.

\section*{D. Computation Complexity Analysis}

Since the IPNCM reconstruction method based on the Gauss-Legendre quadrature only needs three addition operations, the computational complexity of the proposed algorithm mainly depends on the projection matrix construction and steering vector estimation. In projection matrix construction, the computational complexity is determined by the eigenvalue decomposition of the covariance-like matrix \( C \) (13), which is \( O(M^3) \). In steering vector estimation, a QCQP (31) is solved and the complexity is \( O(M^{1.5}) \).
The proposed algorithm (IPNCM-DRGLQ) is compared with the following 5 different matrix reconstruction methods:

- The optimal algorithm MVD [4], which assumes the desired signal steering vector and antenna array structure are known precisely;
- Interference covariance matrix reconstruction beamformer (LINEAR) [7], which is the first one to reconstruct the IPNCM by linear integration over the Capon spectrum, where the parameter is $\epsilon = \sqrt{0.1}$ in LINEAR;
- The covariance matrix reconstruction beamformer based on volume integration (VOLUME) [22], which replaces the linear integration with a volume integration based on LINEAR, where the parameters are $\epsilon = 0.3, \phi_m' \subseteq [0, \pi]$, and $\rho = 0.9$;
- The interference covariance matrix reconstruction beamformer based on annular uncertainty set (AUS) [41], which estimates the desired steering vector by vector space projection, and improves the robustness of the algorithm, where $\rho = 0.9, \phi_m' \subseteq [0, \pi]$;
- The covariance matrix reconstruction based on subspace projection (SUB) [24], which reconstructs the IPNCM based on subspace and eigenvalue decomposition;
- The covariance matrix reconstruction with maximum entropy spectrum (MEPS) [23], which replaces the Capon spectrum in [7] with maximum entropy spectrum to improve the performance of beamformer.

All the optimization problems in the simulation and experiment are solved by the convex optimization toolbox CXV [42]. 300 Monte Carlo simulations are performed in each scenario and the desired signal is always included in the received data of each snapshots. The number of snapshots is fixed to $K = 30$ when the SNR changes. Similarly, the SNR is fixed to 20 dB when the number of snapshots changes.

### A. Example 1: The Comparison Between the Polynomial Summation and GLQ.

First, to verify the effectiveness of GLQ algorithm, we compare the performance of GLQ with the polynomial summation algorithm versus the number of discretizations. In this example, the input SNR and INR are both set as 20 dB.

The performance comparison between the proposed method and polynomial summation is plotted in Fig. 5. As it shows, compared with the GLQ, the performance of polynomial summation method is consistently worse, while the complexity of it is much bigger than GLQ. Furthermore, with the increase of the number of discretizations, the performance of the polynomial summation method improves slowly, which means a sufficiently large number of discretizations is needed to achieve the same performance as GLQ. Thus, compared with the normal polynomial summation method with the computational complexity $O(M^2L)$, the proposed GLQ method only needs three addition operations but gets much more excellent performance. By using GLQ, the computational complexity of the proposed adaptive beamforming is greatly reduced while the algorithm maintains high performance.
B. Example 2: Random Signal and Interference DOA Mismatch

First, the effect of random DOA mismatch is considered. The mismatch of both the desired signal and interference is uniformly distributed in $[-4^\circ, 4^\circ]$. Thus, the desired signal is $\theta_0 \in [6^\circ, 14^\circ]$, two interference are $\theta_1 \in [-34^\circ, -26^\circ]$ and $\theta_2 \in [36^\circ, 44^\circ]$, respectively. The DOA changes from run to run but keep fixed from snapshot to snapshot.

The performance of the output SINR versus the input SNR is shown in Fig. 6. To better distinguish the differences among each algorithm, Fig. 7 demonstrates the deviation between the optimal and other beamformers. The two figures show that the proposed algorithm has better performance compared to other beamformers in the case of random DOA mismatch. Although at the low SNR, the MEPS performs little better than proposed, the proposed algorithm has excellent performance in most cases. The performance of SUB gets decline when the SNR is lower. It may be caused by the projection operation since subspace is constructed based on the eigenvalue decomposition, which is inaccuracy when the SNR is low. Fig. 8 depicts the output SINR when the input SNR is fixed at 20 dB and number of snapshots is varied. It is can be found that the performances of all the algorithms fluctuate slightly when the snapshots number changes, while the proposed algorithm always has the best performance and is more robust.

C. Example 3: Mismatch due to Gain and Phase Perturbations

In this example, the influence of gain and phase perturbations on array output is considered. The gain and phase mismatches on the $m$-th sensor can be described as

$$a_m(\theta) = (1 + \gamma_m)e^{j(\pi \sin \theta (m-1) + \delta_m)}, \quad (33)$$
where $\gamma_m \in \mathcal{N}(0, 0.05^2)$ denotes the zero-mean random gain perturbation of $m$-th sensor, and $\delta_m \in \mathcal{N}(0, (0.025\pi)^2)$ denotes the zero-mean random phase perturbation of $m$-th sensor.

Fig. 9 reveals the performance curves versus the input SNR, while Fig. 10 shows the deviation between optimal method and the tested algorithm, respectively. As them show, even if all the beamformers have great performances, the proposed algorithm gets the best in the case of gain and phase perturbations. The SUB algorithm has excellent performance only when the SNR is higher than 15 dB. Fig. 10 shows that the proposed algorithm has the performance which is closest the optimal. The performance versus the number of snapshots is demonstrated in the Fig. 11. The AUS and VOLUME has almost the same performance, while the proposed performs the best regardless of the number of snapshots.

**D. Example 4: Mismatch due to Steering Vector Random Error**

The effect of mismatch due to steering vector random error is investigated in this example. Considering that each actual steering vector is generated by adding a random error vector to the nominal steering vector as

$$\mathbf{a}_i = \mathbf{a}_i + \mathbf{\xi}_i,$$

where $\mathbf{\xi}_i$ is the random error corresponding to $\mathbf{a}_i$, and can be expressed as

$$\mathbf{\xi}_i = \frac{\rho_i}{\sqrt{M}} [e^{j\phi_0}, e^{j\phi_1}, \ldots, e^{j\phi_{M-1}}]^T,$$

where the Euclidean norm $\rho_i$ follows a uniform distribution in the interval $[0, \sqrt{0.3}]$, and the phases $\phi_m, m = 0, 1, \ldots, M-1$ are uniformly distributed in $[0, 2\pi]$ and are independent with each other.
Fig. 13. Deviation from optimal SINR versus SNR for example 3.

Fig. 14. Output SINR versus number of snapshots for example 3.

Fig. 12 and Fig. 13 display the performance of the tested algorithms versus the input SNR. It can be found that in the case of steering vector random error, even if the MEPS and other algorithms can effectively reduce the error, the proposed algorithm demonstrates an obvious improvement compared to other beamformers. The AUS performs a good effect which is basically same as VOLUME. Fig. 14 displays the output SINR versus the number of snapshots, which shows the proposed algorithm has excellent robustness.

E. Experimental Data

In this section, an experiment is carried out by a S-band ULA system. The system consists of 4 received sensors and 2 transmitting sensors, where transmitting sensors are both about 4 meters away from the receiving sensors, and the spacing between the two transmitting sensors is about 4.2 meters. The spacing of adjacent receiving sensors is half-wavelength. Signals are generated by a field-programmable gate array (FPGA) signal generator, and consist of Gaussian white noise with 10 MHz bandwidth. The received signals are transmitted to the computer through the network cable for subsequent processing. The experimental scene is presented in Fig. 15.

According to the experimental data, we get the two transmitting sensors are located at the $-54.8^\circ$ and $8.1^\circ$ of the receiving sensors by using Capon spectrum, respectively. The transmitting sensor at $-54.8^\circ$ is chosen as the interference and the other one is chosen as the target. The SNR and INR of the received signal are both 5 dB.

The performance of the proposed algorithm is evaluated and compared with AUS, VOLUME, LINEAR, SUB and MEPS method. The beampattern of all the tested algorithms with the experimental data is shown in Fig. 16. We can found that almost all the algorithms can suppress the interference efficiently except the SUB method, it is mainly caused by the low SNR, which is consistent with the previous simulation results. The proposed algorithm forms a deeper null around the location of the interference, which is basically the same as MEPS algorithm but more precisely than MEPS, since the null is closer to the direction of the interference. Furthermore, the
proposed algorithm maintains the desired signal well, it only forms little attenuation in the direction of the desired signal, while the VOLUME method forms a nearly 10 dB attenuation of the desired signal.

VI. CONCLUSION

In this work, a novel adaptive beamforming algorithm based on the IPNCM reconstructed is proposed. A projection matrix is constructed to remove the desired signal from the received data to reconstructed the IPNCM accurately. To reduce the algorithm complexity, the Gauss-Legendre quadrature is introduced. Based on the reconstructed covariance matrix, the presumed steering vector of the signal is corrected by maximizing the array output power. The computational complexity of the proposed algorithm is $O(M^{3.5})$, which is less than most robust adaptive beamformers. The simulation results show that compared to other beamformers, the proposed beamformer can achieve excellent performance with less computation and is always close the optimal. Meanwhile, the experimental data shows the proposed algorithm has superior performance in real environment. Future work will focus on exploring novel reconstruction methods of IPNCM to further improve the robustness of beamformer and reduce the computational complexity.

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