We propose to consider quark degrees of freedom as projections of an interior dynamics of baryons. We assume a hamiltonian structure on the Lie group $u(3)$ to describe the baryon spectrum. The ground state is identified with the proton. From this we calculate approximately the relative neutron to proton mass shift to within half a percentage of the experimental value. We calculate the nucleon and delta resonance spectrum with correct grouping and only one resonance missing when compared with the certain ones. We have no ad hoc masses nor other fitting parameters except the scale. For specific spin eigenfunctions we calculate the delta to nucleon mass ratio to within one percent. Finally we derive parton distribution functions that compare well with those for the proton valence quarks. Conceptually the Hamiltonian may represent an effective phenomenology or more radically describe the baryon itself as a fundamental entity and quarks and gluons as mere scattering structures.
the group to yield quark fields and parton distributions with the usual characteristics. The stationary states of the hamiltonian structure reproduce the neutral flavour baryon spectrum in both number and grouping of resonances [13]. In the present letter, however, we shall focus on the projection phenomenae.

2 Interior dynamics

We want to project out fields that transform in laboratory space under SU(3) rotations according to the fundamental representation (quarks) or the adjoint representation (gluons). Further we want to be able to include electric charge. Finally we want the dynamics to be confined per construction. We therefore suggest to formulate the dynamics directly in the compact Lie group u(3) which may factorize into a product of u(1) and su(3). The model is defined by the following Schrödinger equation

\[ \frac{\hbar c}{a} \left[ - \frac{1}{2} \Delta + \frac{1}{2} d^2(e, u) \right] \Psi(u) = E \Psi(u). \] (1)

Here \( u \) is our configuration variable and \( e \) is the neutral element of the group, i.e. our origo. The scale \( a \) is the only ad hoc constant in our model. Our hypothesis is that the stationary states of (1) reproduce the baryon spectrum, see fig. 1. It should be stressed that \( u(3) \) is our configuration space and neither a priori a symmetry group of the stationary states of (1) reproduce the baryon spectrum in both number and grouping of resonances [13]. In the present letter, however, we shall focus on the projection phenomenae.

The Laplacian in (1) can be parametrized in a polar decomposition where the three toroidal angles \( \theta_j \) take the roles of radii [14]. The potential can be expressed in the same parametrization of the manifold so as to yield a parametrized edition of (1)

\[ \left[ - \frac{1}{2} \left( \sum_{j=1}^{3} \frac{1}{r_j^2} \frac{\partial}{\partial r_j} r_j^2 \frac{\partial}{\partial r_j} - \sum_{i<j}^{3} \frac{K_i^2 + M_k^2}{8 \sin^2 \frac{1}{2}(\theta_i - \theta_j)} \right) \right] + \sum_{j=1}^{3} v(\theta_j) \Psi(u) = E \Psi(u), \] (5)

where \( E = E/\Lambda \) is the dimensionless eigenvalue corresponding to \( E \) and \( \Lambda = \hbar c/\alpha \) turns out to be 215 MeV, when the lowest neutral state of (1) is fitted to the neutron. We see that (5) is analogous to the Hamiltonian for the hydrogen atom in polar coordinates

\[ \left[ - \frac{\hbar}{2m} \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} L^2 \right) \right] + V(r) \Psi(r, \theta, \phi) = E \Psi(r, \theta, \phi). \] (6)

The potential in (1) is the square of the shortest geodetic [15]. It splits up as a sum over the square of the eigenangles \( \theta_j \) and maps out in parameter space as periodic functions (fig. 2)

\[ v(\theta) = \frac{1}{2} (\theta - n2\pi)^2, \quad \theta \in [(2n - 1)\pi, (2n + 1)\pi], \] (7)

where \( n \in \mathbb{Z} \). The Van de Monde determinant, the ‘Ja-
We can use $D$-functions, $D_{km}^{J}$, to construct symmetrized functions $T_{j}^{J}$ for the toroidal part of the wavefunction $\Psi$. For instance we get a promising ratio of the expectation value of $H$ for such spin eigenstates

$$
\frac{<JT_{j}^{3/2}|H|JT_{j}^{3/2}>}{<JT_{j}^{1/2}|H|JT_{j}^{1/2}>} = 1.32 \approx \frac{m_{A}(1232)}{m_{N}(939)} = 1.31.
$$

Here

$$
T_{j}^{1/2}(\theta_{1}, \theta_{2}, \theta_{3}) = (t_{j}^{1/2}(\theta_{1}, \theta_{2}, \theta_{3}) + t_{j}^{1/2}(\theta_{2}, \theta_{3}, \theta_{1}) + t_{j}^{1/2}(\theta_{3}, \theta_{1}, \theta_{2})/N_{j}^{1/2},
$$

where $N_{j}^{1/2}$ is a normalization factor and $t_{j}^{1/2}$ is a sum over the two $D$-functions for the two possible values $m = \frac{1}{2}, -\frac{1}{2}$ corresponding to total angular momentum $j = \frac{3}{2}$

$$
t_{j}^{1/2} = D_{1/2,1/2}^{j} + D_{1/2,-1/2}^{j} \propto e^{-i\theta_{j}/2}cos(\frac{\theta_{2}}{2})e^{-i\theta_{3}/2} - e^{-i\theta_{1}/2}sin(\frac{\theta_{2}}{2})e^{i\theta_{3}/2}.
$$

For total angular momentum $j = \frac{3}{2}$ there will be four terms corresponding to the allowed values of spin projection $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

The components of $M = (M_{1}, M_{2}, M_{3})$ connect the algebra by commuting into the subspace of the $K$’s

$$
[M_{k}, M_{l}] = -i\hbar K_{m}.
$$

They tie up with the interior spin quantum number $K$ and the conventional flavour degrees of freedom in a spectrum for the nominator in the centrifugal part of the Laplacian when this is integrated over the off-toroidal dimensions

$$
K(K + 1) + M^{2} = \frac{4}{3}(n + \frac{3}{2})^{2} - 3 - \frac{1}{3}y^{2} - 4i^{2}.
$$

Here $n$ is a non-negative integer, $y$ is hypercharge and $i_{3}$ is the isospin three-component.

The hamiltonian structure in (1) is historically related to work on lattice gauge theory [18] where a Kogut-Susskind Hamiltonian [19] for a link variable reads with Manton’s action [20] in stead of the usual Wilson action

$$
H_{M} = \frac{1}{a} \left( \frac{1}{2} g^2 E^\alpha E^\alpha - \frac{1}{2} \frac{1}{g^2} Tr \chi^2 \right).
$$

Here $a$ is the lattice spacing, $g$ is the dimensionless strong coupling, $U = e^{i\chi}$ is a link variable and $E^\alpha$ are ’electric’ field operators which also act as derivatives. The basic idea behind (1) lies in a shift of interpretation away from links. In stead of $U$ being an SU(3) algebra variable representing links in space we introduce a variable $u$ in interior space to serve as configuration variable for the dynamics in baryons. Thus the Hamiltonian now acts on wavefunctions supposed to describe entire baryons. The two Hamiltonians in (1) and (18) are both structurally related to skyrmion descriptions [22, 23] where one has rotational Hamiltonians like

\[ \text{Figure 2: Periodic parametric potential (7) originating from the squared geodetic distance in (1) from } u \text{ to the origo } e \text{ in the Lie group (Milnor [15])}. \text{ The dashed curve corresponds to the Wilson analogue [21] of the Manton potential [20] and is not considered in the present work.} \]
\[
H_{\text{rot}} = \frac{1}{2}\sum_{i=1}^{3} J_i^2 + \frac{1}{2}\sum_{k=4}^{7} J_k^2.
\]

The moments of inertia \(I_1\) and \(I_2\) of the skyrmions determine the scale and \(J_i\) and \(J_k\) are generators. The role of the potential in both (1) and (18) is covered in the skyrmion picture by introducing a certain profile function for the radial extent of the skyrmion.

### 3 Projection to space

From the parametrization of the manifold \(u(3)\) we have toroidal coordinate fields \(\partial_j\) implied by the toroidal generators

\[
\partial_j = \frac{\partial}{\partial u}\big|_{\theta=0} = u_i T_j.
\]

Related to these we have the torus forms \(d\theta_j\) such that

\[
d\theta_i(\partial_j) = \delta_{ij} \Leftrightarrow [\partial_j, \theta_i] = \delta_{ij}.
\]

We see (21) as origin of first quantization. The state \(\Psi\) projects out in laboratory space via the exterior derivative for the radial extent of the skyrmion.

\[
d\Phi = \bar{\psi}_j d\theta_j
\]

form the colour components of a quark field

\[
\bar{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.
\]

We see (22) as origin of second quantization. Exploiting the left invariance of the coordinate fields seen in (20) it can be shown that \(\bar{\psi}\) transforms as a member of the fundamental representation under \(SU(3)\) rotations in space. Similarly we can generate adjoint representations [24].

### 4 Parton distribution functions

The charge operator in a matrix representation analogue of our coordinate representation is the well-known combination of hypercharge and isospin three-component

\[
Q_2/\hbar = \frac{\sqrt{3}}{2}\lambda_8 + \frac{1}{2}\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

We can generate parton distribution functions by applying the momentum form \(d\Phi\) to 'skew' charge operators

\[
T_u = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T_d = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

To scatter on a parton \(xP\) in a baryon carrying four-momentum \(P\) corresponds to boost the baryon from rest at \(P_0 = (0, E_0)\) to \(P = (q, E)\) by impacting upon it a massless four-momentum \((q, E - E_0)\) to be taken up by the baryon. The boost parameter

\[
\xi \equiv \frac{E - E_0}{E} = \frac{2 - 2x}{2 - x}
\]

can then be used for projection along toroidal angle tracks \(\theta = \pi \xi\). With a probability amplitude interpretation for the wavefunction \(\Phi\) we sum over all three colour components to get the projection along a generator \(T\)

\[
\int f_T(x) \cdot dx = \left( \sum_{j=1}^{3} d\Phi_{\text{u}} = \exp(\theta_j T_j) (\partial_j) \right)^2 \cdot d\theta,
\]

where \(f_T(x)\) is the sought for distribution function and the generator may be expanded on the toroidal generators

\[
T = a_1 T_1 + a_2 T_2 + a_3 T_3.
\]

Our specific example in fig. 4 is generated from a first order approximation to the toroidal part of the proton state

\[
R(\theta_1, \theta_2, \theta_3) = \frac{1}{N} \begin{vmatrix} 1 & \sin \frac{1}{2} \theta_1 & \sin \frac{1}{2} \theta_2 & \sin \frac{1}{2} \theta_3 \\ \cos \theta_1 & \cos \theta_2 & \cos \theta_3 \end{vmatrix},
\]

where \(N\) is a normalization constant and we sum over all three colours for a specific flavour track generated by \(T\)

\[
\sum_{j=1}^{3} dR_{\text{u}} = \exp(\theta_j T_j) (\partial_j) =
\]

\[
\equiv D(\theta \cdot a_1, \theta \cdot a_2, \theta \cdot a_3) \equiv D_T(\theta),
\]

A possible reference for the first step in (30) is [25]. Note that \(R\) is antisymmetric under interchange of the colour.
variables \( \theta_j \). Note also that \( R \) has a double period in the second row. States with only single periods are interpreted as neutral, i.e. electric charge is considered as of topological origin. Comparing corresponding approximate eigenvalues gives a relative mass difference between the neutron and the proton of 0.138473 % which is close to the experimental value 0.137842 %.

For distributions along the track generated by \( T_u \) we get from (30)

\[
x f_{T_u}(x) = x \left[ D \left( \frac{2 - 2x}{2 - x}, \frac{2 - 2x}{2 - x}, 0, \frac{2 - 2x}{2 - x}, (-1) \right) \right]^2 \cdot \frac{\pi \cdot 2}{(2 - x)^2},
\]

where in its full glory

\[
ND(\theta_1, \theta_2, \theta_3) = \\
\frac{1}{2} \cos \frac{\theta_1}{2} \cdot (\cos \theta_3 - \cos \theta_2) - \sin \theta_1 \cdot (\sin \frac{\theta_3}{2} - \sin \frac{\theta_2}{2}) \\
\frac{1}{2} \cos \frac{\theta_2}{2} \cdot (\cos \theta_3 - \cos \theta_1) + \sin \theta_2 \cdot (\sin \frac{\theta_3}{2} - \sin \frac{\theta_1}{2}) \\
\frac{1}{2} \cos \frac{\theta_3}{2} \cdot (\cos \theta_2 - \cos \theta_1) - \sin \theta_3 \cdot (\sin \frac{\theta_2}{2} - \sin \frac{\theta_1}{2}).
\]

The third expression is the Compton wavelength and the last expression indicates an orbital mapping from interior space to laboratory space. We may thus imagine the extension of a baryon as a compromise between simultaneous localization (at rest) and definite introtangled energy-momentum (massiveness), i.e. mass is considered as introtangled energy. Parametrizations by a space coordinate \( x \) and a time coordinate \( t \) will both meet in an ultrarelativistic regime by

\[
a \theta = x = ct.
\]
In these projections we recognize the energy scale of our model

\[ \Lambda = \hbar \omega = \hbar d = \frac{\hbar c}{a}, \]  

(39)

To determine the scale we fit the eigenvalue of a solution of (1) to the nucleon ground state. This is discussed at length in [13] and the result is \( \Lambda = 215 \text{ MeV} \) for a fit to the neutron.

6 Open questions and perspectives

If the Ansatz (1) and its interpretations is to be accepted as more than just a phenomenological model we need to include the electron and its anti-neutrino in the neutron decay and we need to unfold the 'Yukawa' potential (4) into quark-antiquark projections in laboratory space.

Further we need to investigate to which degree the flavours above strangeness are already implied in the model by choosing higher values for \( n + y \) in (17) than the minimum value 2 which leads to an Okubo mass relation. In particular the top quark mass seems too high and disparate from the other five flavours to be related to the square dependence on \( n \). When created the top quark \( t \), like the electroweak interaction bosons \( W^\pm \) and \( Z^0 \) - not to mention the possible Higgs - only live over one baryonic Compton wavelength or less, so they might be related to the topological changes that arise when the dynamics shifts back and forth between laboratory space and the \( \mathfrak{u}(3) \) configuration space which we have called allospace [13].

In short we see the mass content of the energy-momentum tensor \( T_{\mu \nu} \) in Einstein's general theory of relativity as representing introtangled energy-momentum from an 'orthogonal' configuration space of quantum interactions and therefore we see no need for a particular quantization of spacetime in order to unify with the description of quantum phenomenae. In a certain sense conceptually the unification is already there in (33) and the intertwining of the two spaces manifests itself in (37) as a Compton wavelength - the size of the knot when energy is caught topologically as mass. However to accept this as a general conception of mass it is not enough to be able to calculate the neutron to proton mass shift. We would need e.g. to be able to calculate the electron to proton mass ratio. We have no idea on how to set up a framework for such a calculation, let alone to work it out in detail. We therefore suggest that the next step along the allospatial path be an attempt to figure out if mesons can be seen as a result of projections to space of the allospatial potential (4). A less ambitious but computationally demanding task would be to try to calculate in (3) the deuteron ground state.

7 Conclusions

We have investigated a model for interior dynamics of baryons and found that quarks may be conceived of as mere scattering structures that arise when the baryonic entity is disturbed by impacting momentum. From a hamiltonian structure on \( \mathfrak{u}(3) \) we have derived parameter free parton distribution functions that compare well in shape and peak value with the up and down valence quarks of the proton. For improved results we need a suitable set of base functions on which to expand the wavefunctions of charged states. For improved understanding we need to investigate the meson sector and the electroweak elements.

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