Atom–photon momentum entanglement with quantum interference

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With quantum interference of two-path spontaneous emissions, we propose a novel scheme to coherently control the atom–photon momentum entanglement through atomic internal coherence. A novel phenomenon called “momentum phase entanglement” is reported, and we found, under certain conditions, that more controllable entangled state can be produced with super–high degree of entanglement.

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Introduction.— Entanglement with continuous variable has fundamental importance in quantum nonlocality [1] and in quantum information [2]. Being one of the ways of physical realizations, momentum entanglement plays a unique role in recent studies [3, 4, 5, 6, 7, 8, 9, 10]. In spontaneous emission process, well localized atom–photon entangled wavepacket [3] can be produced due to the momentum conservation, with the degree of entanglement inversely proportional to the linewidth of the transition [3, 4, 5, 6]. Therefore, it is believed, by squeezing the effective transition linewidth, that highly entangled EPR–like state [1] could be produced in free space [3, 6].

Insofar experiments [7], the entanglement information can be extracted by correlated momentum measurements, and it is found that the degree of entanglement is completely detectable with the conditional–unconditional variance ratio [R–ratio in Eq. (7)] for a large variety of physical processes [3, 4, 5, 6, 7, 8]. Therefore, it is straightforward to ask if it is physically possible to produce entanglement beyond this momentum detection, the present work will answer this question. In a typical nearly–degenerated three–level atom, which will be discussed in the following, we find that the interference between different quantum pathways [11, 12] produces significantly the so–called “phase–entangled” state, in which the entanglement information can not be evaluated properly by solely using the momentum detection measure, i.e., the R–ratio. Within this model, we find that not only the degree but also the modes of the entanglement can be effectively manipulated by controlling the atomic internal coherence, and the entanglement degree exhibits, “anomalously”, to be proportional to the atomic linewidth of the excited energy levels. Therefore, it is possible to use this proposed scheme to produce a novel and more controllable highly entangled atom–photon system in realistic applications.

Theoretical model.— As shown in Fig. 1 (a), the three–level atom has two transition pathways “a” and “b” to induce the momentum entanglement with the emitted photon due to momentum conservation. We will, in the following, only consider the strong interference conditions, which assumes that the dipoles μa,b are parallel with each other [11], i.e., ε ≡ μa ⋅ μb/|μa| ⋅ |μb| = 1, and the upper–levels are nearly-degenerated: ω12 ≡ ωa − ωb < γa,b, where γa,b and ωa,b are the linewidths and central frequencies of the two transitions, respectively. The Hamiltonian of this system with the rotating wave approximation is:

\[ \hat{H} = \frac{(\hbar \hat{q})^2}{2m} + \sum_k \hbar \omega_k \hat{a}_k \hat{a}_k^\dagger + \hbar \omega_a \hat{\sigma}_{11} + \hbar \omega_b \hat{\sigma}_{22} \]

\[ + \hbar \sum_k \left[ g_a(k) \hat{\sigma}_{31} \hat{a}_k^\dagger e^{-i \vec{k} \cdot \vec{r}} + g_b(k) \hat{\sigma}_{32} \hat{a}_k e^{-i \vec{k} \cdot \vec{r}} + \text{H.c.} \right] , \]

where \( \hbar \hat{q} \) and \( \vec{r} \) are the atomic center–of–mass momentum and position operators, \( m \) is the atomic mass, \( \hat{\sigma}_{ij} \) is the atomic operator, \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) are the annihilation (creation) operator for the \( k \)th vacuum mode with wave vector \( \vec{k} \) and frequency \( \omega_k = ck \), and \( g_{a,b}(k) \) are the coupling co-

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are restricted as in Fig. 1 (b), then the one–dimensional
perposed internal state equation. Suppose the atom is initially prepared in a su-
internal states.
wave vector of the atom, the photon, and the atomic
where the arguments in the kets denote, respectively, the
wave vector of the atom, the photon, and the atomic
internal states.
Using the Born–Markov approximation, the evolution of
atom–photon state can be solved from Schrödinger
equation. Suppose the atom is initially prepared in a su-
perposed internal state $A_1(q, t \to \infty) = A_2(q, t \to \infty) = 0$,

$$B(q, k, t \to \infty) \propto \exp[-(\Delta q/\eta)^2] \times \left[ C_1(2gb_{s1}/\varepsilon \sqrt{\gamma_a \gamma_b} - g_a) + C_2(2gb_{s2}/\varepsilon \sqrt{\gamma_a \gamma_b} - g_b) \right] \times \left[ i(\Delta q + \Delta k) + (s_1/\gamma_a - \frac{1}{2}) + i(\Delta q + \Delta k) + (s_2/\gamma_a - \frac{1}{2}) \right]$$

where the parameters are defined as:

$$s_{1,2} = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 + \varepsilon^2 \gamma_a \gamma_b})$$

$$C_{1,2} = \pm \frac{s_{2,1} A_{10} + \frac{1}{2} \varepsilon \sqrt{\gamma_a \gamma_b} A_{20}}{s_2 - s_1}$$

and the effective wave vectors are defined by:

$$\Delta k = \frac{k - k_0}{\gamma_a/c}, \ \Delta q = \frac{h k_0}{m \gamma_a}(q - k_0), \ \ k_0 = \frac{\omega_a}{c}.$$

**Entanglement detection**—The nonfactorization of the wavefunction in Eq. (4) reveals the atom–photon en-
tanglement. In both theoretical [4, 9] and experimental studies [7], the ratio of the conditional and unconditional
variances [i.e., the $R$–ratio defined in Eq. (7)] plays a cen-
tral role, since it is a direct experimental measure of the
nonseparability (entanglement) of the system.

With the single–particle measurement, the uncondi-
tional variance for the effective atomic momentum is determined as $\delta q^{\text{single}} = \langle \Delta q^2 \rangle = \langle \Delta q^2 \rangle = \int d\Delta q \ d\Delta k \Delta q^2 |B(q, k)|^2$, where the average $\langle \cdot \rangle$ is taken
over the whole ensemble. Meanwhile, the coincidence
measurement gives the conditional variance as $\delta q^{\text{cond}} = \langle \Delta q^2 \rangle |\Delta k_0 - \langle \Delta q^2 \rangle | \Delta k_0 \times \int d\Delta q \ d\Delta q^2 |B(q, k)|^2$, where the photon is previously detected at some known $\Delta k_0$. With these two variances, the entanglement is evaluated by:

$$R \equiv \frac{\delta q^{\text{single}}}{\delta q^{\text{cond}}} \geq 1.$$

**FIG. 2:** (a) The “amplitude entanglement” $R$–ratio is plotted in dependence on the atomic coherence $r$ and $\theta$ with $\delta = 0.02$, $\eta = 0.1$. (b) The contour plot of Fig. 2 (a). The circular contours indicate the symmetric roles played by $\theta$ and $\eta$ in controlling the $R$–ratio. The FWHM is denoted by $\delta_0$ and $\delta_r$ as in the figure. (c) The FWHM of $R(r)$ is plotted in solid lines in dependence of $\delta$ with $\eta = 0.05, 0.1, 0.2$ from the top to the bottom. Dashed lines are the fitted function $2\delta/\eta$.

Due to the interference of two transition pathways,
the $R$–ratio highly depends on the initial coherence of
the two upper atomic levels, which is evaluated by
$A_{10}/A_{20} = \exp(r + i\theta)$, where $r$ depicts the relative
occupation probabilities of the two upper levels, and $\theta$
determines their coherence phase. In further discussions,
we assume $\gamma_a = \gamma_b = \gamma$, and define $\delta = \omega_1/\gamma < 1$ for
simplicity.

Following the above analysis, the dependence of $R$ on $r$
and $\theta$ is illustrated in Fig. 2. Under the conditions $\eta \ll 1$ and $\delta^2/\eta \ll 1$ [13], we find that $R(r, \theta)$ can well be
approximated by Lorentzian function, and the parameters $r$ and $\theta$, in spite of their quite different physical essences, play very symmetric roles in controlling the
detectable $R$–ratio [Fig. 2 (b)]. Under these conditions, the $R$–ratio is maximized at the dark state coherence $[(|1\rangle - |2\rangle)/\sqrt{2}]$:

$$R_{\max} = R(r = 0, \theta = \pi) \approx \sqrt{2\pi \eta/\delta^2},$$

the full width at half maximum (FWHM) of which can be
well approximated by $\delta_0 \approx \delta_\theta \approx 2\delta/\eta$, as shown in Figs.
2 (b) and (c). Therefore, with properly chosen atomic
parameters $\eta$ and $\delta$, this scheme could be used to produce
significant detectable entanglement in a relatively large
range of the initial atomic coherence. For example, with
$\eta = 0.01$, the system with $R > 100$ can be produced
within the range of $0.018 < |A_{10}/A_{20}|^2 < 0.05$, and
$0.367 < \theta < 1.67\pi$.

**Phase entanglement** — For a bipartite pure–state sys-
tem, the degree of entanglement can be completely eval-
uated by the Schmidt number \[ K \equiv 1/\sum_n \lambda_n^2 \geq 1, \]

where \( \lambda_n \)'s are eigenvalues for the entangled atomic modes \( \psi_n(q) \) and photonic modes \( \phi_n(k) \) in the Schmidt decomposition \[ B(q, k) = \sum_n \sqrt{\lambda_n} \psi_n(q) \phi_n(k). \]

In previous studies on atom–photon momentum entanglement \[ 3, 4, 5, 6, \] the \( R \)-ratio well measures the entanglement since one has \( R \propto K \). However, this is not true when the quantum interference is strong as shown in this model: since the \( R \)-ratio is constructed from the module part of the wavefunction, it reveals only the amplitude correlation between two particles’ momentum; therefore, when the phase is critical for the nature of the entanglement due to the interference, the traditional \( R \)-ratio measurement becomes inadequate for detecting the full entanglement information. Actually, by controlling the interference with the atomic internal coherence, one may produce two systems with \( K > K' \) whereas \( R < R' \), which indicates that significant entanglement information may be lost by the momentum detection with only the \( R \)-ratio.

We compare \( K \) and \( R \) in Fig. 3, from which one finds that both of them are maximized at the dark-state coherence \( (r = 0, \theta = \pi) \). However, compared with \( R(r, \theta) \), \( K(r, \theta) \) exhibits a much slower decay in the vicinity of the maximum, which indicates that, with different initial atomic coherences, some entanglement information may be transferred into the “phase” and can not be measured only by the amplitude-based detections. This phenomenon is particularly important for some highly entangled states, e.g., under the condition \( \delta = 2 \times 10^{-3}, \eta = 0.1, \ r = 0, \ \theta = \pi \), one may prepare a highly entangled state with \( K \approx 2.8 \times 10^4 \) and \( R \approx 6.2 \times 10^4 \); however, when the initial atomic momentum and coherence change to \( \eta' = 2\eta, r' = 0.13 \), the entanglement of the system does not change, i.e., \( K' = K \), but the \( R \)-ratio detection shows that \( R' = 0.05 R \). This shows that some entanglement information of the system is transferred into the phase.

Similar phenomenon of the so-called “phase entanglement” has been reported recently in the position space \[ 4, 9, 10 \]: due to the spreading of the wavepacket, it appears instantly and must be detected by a series of spatial measurements in time \[ 9 \]. For the momentum “phase entanglement” in this scheme, however, since it is caused by the quantum interference and is not affected by the wavepacket’s spreading, this phenomenon keeps steady in time and could be much easier to be directly observed in experiments \[ 7 \].

It is possible to evaluate the “phase entanglement” for the highly entangled states in this scheme. For the entanglement maximized at the dark-state coherence, the wavefunction takes a similar form as if no interference occurs \[ 3, 4, 5, 6 \]:

\[
B(q, k, t \to \infty) \propto \frac{\exp[-(\Delta q/\eta)^2]}{i(\Delta q + \Delta k) - \delta^2/4},
\]

and then the Schmidt number yields:

\[
K_{\text{max}} \approx 1 + 0.28(4\eta/\delta^2 - 1).
\]

Together with Eq. 18, one yields the relation

\[
K_{\text{max}} = K(r = 0, \theta = \pi) \approx \frac{R_{\text{max}}}{2.2} \approx \frac{1.12\hbar\kappa\delta\eta}{m\omega_1^2},
\]

well fulfilled for \( \eta/\delta^2 \gg 1 \) and \( \eta \ll 1 \) \[ 13 \], as shown in Fig. 3 (c). The linear relation between \( K_{\text{max}} \) and \( R_{\text{max}} \) shown in Eq. 12 indicates that the entanglement is completely detectable with fixed dark-state coherence \( r = 0 \) and \( \theta = \pi \). For general conditions (where \( r \) and \( \theta \) take arbitrary values, see Fig. 3), we have \( K \geq R/2.2 \). Therefore, the degree of “phase entanglement” can be evaluated by the following parameter:

\[
PE \equiv 2.2K/R \geq 1,
\]

which is valid for the states produced with different control parameters \( \eta, \delta, r \) and \( \theta \).

A traditional idea to enhance the entanglement of momentum is to correlate the atom and photon momentum by squeezing the transition linewidth since \( K \propto 1/\gamma \) \[ 3, 4, 5, 6 \]. However, in our proposed scheme, which employs an essentially different mechanism for producing the entanglement through quantum interference, we have, anomalously, that \( K_{\text{max}} \propto \gamma \) as shown in Eq. 12. With broader linewidth of the two upper energy levels, the interference will be enhanced and, as a result, increases the momentum entanglement. Therefore,
and $B$ position. According to Eq. (9), intuitively, be understood in terms of Schmidt decom-
position. Even for the atomic system, it is possible to use this mechanism to produce super-atomic modes and remain the Gaussian localization properties [3, 5] in spite of the shape distortions caused by the interference. Therefore, it is possible to apply this scheme to efficiently control the entangled modes under a certain degree of entanglement.

**Entangled modes.**—The phase entanglement can, more intuitively, be understood in terms of Schmidt decomposition. According to Eq. (9), $K$ is a measure for the number of the important Schmidt modes, while $R$-ratio is related to the coherence between these modes $|\psi_n(q)|^2$, which can be seen more clearly by rewriting the unconditioned and conditional variances as: $\delta q^{\text{single}} = \int d\Delta q \, \Delta q^2 \sum_n \lambda_n |\psi_n(q)|^2 = (\Delta q^2)_E$, $\delta q^{\text{coin}} = \int d\Delta q \, \Delta q^2 \sum_n \lambda_n |\psi_n(q)|^2 = (\Delta q^2)_E$. These formulæ indicate that the unconditioned variance $\delta q^{\text{single}}$ is the variance taken by an “incoherent” superposition of different Schmidt modes weighed by $\lambda_n$, i.e., $E_\psi(q) \equiv \sum_n \lambda_n |\psi_n(q)|^2$, while the conditional variance $\delta q^{\text{coin}}$ is taken over a “coherent superposition” of different modes, i.e., $E_\psi(q) \equiv |\sum_n \lambda_n |\psi_n(q)|^2$. Therefore, the $R$–ratio defined in Eq. (7) actually represents the wavepacket narrowing caused by the coherence between different Schmidt modes.

In Fig. 4, we compare the atomic Schmidt modes between the states $B(q,k)$ and $B'(q,k)$ with $K' = K$ and $R' \approx 0.38R$. It can be seen that the phase entanglement significantly broadens the first few Schmidt modes and decreases the number of peaks for the rest ones; moreover, the coherence between different Schmidt modes diminishes and, as a result, decreases the $R$–ratio. The photonic Schmidt modes exhibit similar properties as atomic modes and remain the Gaussian localization properties [3, 5] in spite of the shape distortions caused by the interference. Therefore, it is possible to apply this scheme to efficiently control the entangled modes under a certain degree of entanglement.

**Conclusion.**—In summary, we investigate the atom–photon momentum entanglement caused by the quantum interference in a three–level atom. The novel phenomenon of “momentum phase entanglement” is shown and evaluated quantitatively. Using this scheme, a novel atom–photon entangled state can be produced with super–high degree of entanglement and controllable entangled modes. Since the proposed configuration has been extensively studied both theoretically and experimentally [11, 12], and can be realized by mixing different parity levels or by using dressed-state ideas, these new features are most probable to be examined in experiments and used in realistic applications [2].

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major concern.

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