Abstract

Vertex operators associated with level two $U_q(\widehat{sl}_2)$ modules are constructed explicitly using bosons and fermions. An integral formula is derived for the trace of products of vertex operators. These results are applied to give $n$-point spin correlation functions of an integrable $S = 1$ quantum spin chain, extending an earlier work of Jimbo et al for the case $S = 1/2$.
1 Introduction

Since Drinfeld [1] and Jimbo [2] introduced quantum groups, which are one parameter deformation of the universal enveloping algebras of Kac-Moody Lie algebras with the Hopf algebra structure, it has been recognized that, besides purely theoretical interests from mathematics, the quantum groups have many applications and play important roles in various fields in mathematics and physics.

Finding out explicit constructions of fundamental representations of the algebra is a natural and important subject in the study. Level one irreducible highest weight modules were constructed by Frenkel et al [3] for simply laced quantum affine algebras, the construction being called bosonization; the structure of the modules was entirely analogous to the Frenkel-Kac construction for affine Lie algebras. Constructions of irreducible modules of higher levels are, however, quite involved and there is no general recipe. An exception is the construction via Wakimoto modules for generic complex value of level [4, 5]; but it seems that the construction for integral level is not complete yet, in which one needs to take quotient to get the irreducible one. Bernard [6] found construction of level one modules over $U_q(B^{(1)}_r)$; his results for $U_q(B^{(1)}_1)$ can be translated into those for level two $U_q(sl_2)$-modules. One of the aims of this paper is to recall a part of his results and to give explicit constructions of level two irreducible highest weight modules over the quantum affine algebra $U_q(sl_2)$, which we denote $V(2\Lambda_0)$, $V(2\Lambda_1)$ and $V(\Lambda_0 + \Lambda_1)$ ($\Lambda_i$ being the fundamental weights of $\hat{sl}_2$); the modules are constructed in terms of a single boson, a fermion of Neveu-Schwarz or Ramond type (according to the highest weights $2\Lambda_{0,1}$ or $\Lambda_0 + \Lambda_1$, respectively), and the weight lattice of $sl_2$. The result is analogous to that for $\hat{sl}_2$ by Lepowsky et al [7].

Our second aim is to give explicit forms of $q$-deformed vertex operators of type I (in the sense of Ref.[8]) associated with the level two modules defined as homomorphisms of $U_q(sl_2)$-modules

$$\Phi(z) : V(\lambda_m) \longrightarrow V(\lambda_{2-m}) \otimes V_z$$

where $\lambda_m = (2 - m)\Lambda_0 + m\Lambda_1$, $m = 0, 1, 2$, $V_z = V \otimes \mathbb{C}[z, z^{-1}]$, $V \cong \mathbb{C}^3$. In conformal field theories (CFTs) vertex operators of this type associated with the affine Lie algebras are called primary fields, and their $n$-point functions

$$\langle \lambda | \Phi(z_1) \cdots \Phi(z_n) | \lambda \rangle \quad (|\lambda\rangle : \text{highest weight vector})$$
describe correlations of primary fields. The \( n \)-point functions \( (1) \) satisfy differential equations called the Knizhnik-Zamolodchikov (KZ) equations. Analogously, Frenkel et al \([9]\) considered the \( n \)-point functions \( (1) \) for the \( q \)-deformed vertex operators, and showed that \( (1) \) satisfy difference equations called the \( q \)-deformed Knizhnik-Zamolodchikov (\( q \)-KZ) equations. We generalize \( (1) \) and consider

\[
\text{tr } V_\lambda(z)q^{-2d}y^\alpha \Phi_{1}(z_1)\cdots \Phi_{n}(z_n)
\]

for the vertex operators associated with level two irreducible modules over \( U_q(\hat{sl}_2) \), where \( \xi \) and \( y \) are free parameters, \( d \) the grading operator in \( U_q(\hat{sl}_2) \), and \( \alpha \) the simple root of \( sl_2 \); the previous ones \( (1) \) are obtained from ours \( (2) \) by specializing \( \xi \to 0 \) as coefficients of appropriate powers of \( y \). We note that our \( n \)-point functions \( (2) \), too, satisfy \( q \)-difference equations of \( q \)-KZ type (cf. Refs.\([10, 11, 12]\); we will not study the difference equations in this paper). The goal of this paper is to give an integral formula for the \( n \)-point functions \( (2) \); the final formula is eq.(52) or eq.(53). We shall observe that the integral formula gets simplified at a special value of the parameter \( \xi = q^2 \), which is due to delta functions arising from the fermion trace; this special case is important because the formula just at \( \xi = q^2 \) has an application in the study of correlations in integrable spin systems and lattice models.

In recent progress in the study of integrable quantum spin systems and lattice models it has been recognized that the \( q \)-deformed vertex operators play important roles in the models that have quantum affine algebra symmetry \([8, 13, 10]\). Davies et al \([8]\) recognized that the XXZ model, a spin \( 1/2 \) quantum spin chain, has an exact symmetry \([H, U_q'(\hat{sl}_2)] = 0\), \( H \) being the Hamiltonian, \( U_q'(\hat{sl}_2) \) a subalgebra of \( U_q(\hat{sl}_2) \) without the grading operator \( d \); they conjectured that the space of states is a level zero \( U_q(\hat{sl}_2) \) module \( \mathcal{F} \equiv \bigoplus_{\lambda,\lambda'} V(\lambda) \otimes V(\lambda')^{\text{na}} \simeq \bigoplus_{\lambda,\lambda'} \text{Hom}_C(V(\lambda), V(\lambda)) \), \( \lambda, \lambda' \) being level one highest weights in their case; a physical state in \((C^2)^{\otimes \infty}\) is identified with a vector in \( \mathcal{F} \) via \( q \)-deformed vertex operators; they succeeded to construct the creation/annihilation operators in the model and diagonalize the Hamiltonian. Following this work Jimbo et al \([13]\) showed that an arbitrary zero-temperature spin correlation function of the XXZ model is formulated as a trace of the product of vertex operators over an irreducible highest weight \( U_q(\hat{sl}_2) \)-module of level one. They obtained an integral formula for the \( n \)-point function using the Frenkel-Jing construction of level one modules \([3]\).
Extensions to higher levels (spins) and incorporations of the formulation in \cite{8} were done by Idzumi \textit{et al} \cite{10}.

A motivation of the present paper was to extend the work of Jimbo \textit{et al} \cite{13} for the case $S = 1/2$ to $S = 1$. We will apply our integral formula for the level two vertex-operator $2n$-point functions to zero-temperature spin $n$-point correlation functions of an integrable $S = 1$ spin chain which is an integrable extension of the XXZ model to spin one: $H = \sum_{i \in \mathbb{Z}} \cdots \otimes 1 \otimes h \otimes 1 \otimes \cdots$, where

$$h = s^x \otimes s^x + s^y \otimes s^y + \text{ch}(2\eta) \cdot s^z \otimes s^z - \left(\sum_{j=1}^{3} s^j \otimes s^j\right)^2$$

$$+ 2s^x(\eta) \cdot \left[(s^z)^2 \otimes \text{id} + \text{id} \otimes (s^z)^2 - (s^z \otimes s^z)^2 - 2 \cdot \text{id} \otimes \text{id}\right]$$

$$+ (2 + 4e^{-\eta}\text{ch}(\eta)) \cdot (s^x \otimes s^x + s^y \otimes s^y)s^z \otimes s^z$$

$$+ (2 + e^{\eta}) \cdot s^z \otimes s^z(s^x \otimes s^x + s^y \otimes s^y);$$

(3)

where we have set $q = -e^{-\eta}$. We note that they can be regarded as the thermal average of a product of in-row $n$ variables for an integrable nineteen-vertex model, whose Boltzmann weights being given by

$$\tilde{R}(z, w) = \begin{bmatrix}
1 & c_1 & p & g_1 & b \\
& c_1 & p & g_1 & b \\
& & e_2 & h_1 & o & h_2 \\
& & & e_1 & p \\
& & & & e_2 \\
& & & & & 1
\end{bmatrix}$$

(4)

where $b = q^2(w - z)(q^2w - z)/d_2d_4$, $c_1 = z^2(1 - q^2)(1 - q^4)/d_2d_4$, $c_2 = w^2(1 - q^2)(1 - q^4)/d_2d_4$, $e_1 = z(1 - q^4)/d_4$, $e_2 = w(1 - q^4)/d_4$, $p = q^2(w - z)/d_4$, $g_1 = z(w - z)(q^2w - z)q(1 - q^2)/d_2d_4$, $g_2 = w(w - z)(q^2w - z)q(1 - q^2)/d_2d_4$, $h_1 = z(w - z)(q^2w - z)q(1 + q^2)(1 - q^4)/d_2d_4$, $h_2 = w(w - z)(q^2w - z)q(1 + q^2)(1 - q^4)/d_2d_4$, $o = q^2(w - z)(w - q^2)z + (1 - q^2)(1 - q^4)zw/d_2d_4$ with $d_2 = w - zq^2$, $d_4 = w - zq^4$.

Finally we give a remark. The vertex-operator correlation functions (3) for general level can be calculated through the Wakimoto constructions of
highest weight modules which are not irreducible by themselves; cf. [4, 11].
The final formula includes the Jackson-type integrals. Compared with their
approach through the Wakimoto constructions, our bosonizations are more
direct and our final formula (52) or (53) for (2) gives rich information on
the properties of the correlations (2), although we are restricted to level two
only.

The paper is organized as follows. In Section 2 we give several defini-
tions needed in the subsequent sections: the quantum affine algebra
$U_q(\widehat{sl}_2)$, the Drinfeld realization of the subalgebra $U'_q(\widehat{sl}_2)$, irreducible highest weight
modules of level $k$, associated $q$-deformed vertex operators, their $n$-point
functions (2), and spin correlation functions for the spin-$k/2$ analog of the
XXZ model; there we will not specialize the level and we will give definitions
for general positive integer $k$. From Section 3 to the end we consider $k = 2$
only. In Section 3 we give explicit constructions (bosonizations) of level $k = 2$
irreducible highest weight modules over $U_q(\widehat{sl}_2)$; besides a boson contained
in the Drinfeld realization of $U'_q(\widehat{sl}_2)$, we need a Neveu-Schwarz or Ramond
fermion as well as the weight lattice of $sl_2$. In Section 4 we present explicit
forms of vertex operators (VOs) (Eq.(40)) and prove intertwining relations
with Chevalley generators. In Section 5 we give an integral formula for the
VO $n$-point function; it is an $n$-multiple contour integral of a meromorphic
function (Eqs.(52) and (53): the main result of this paper). We illustrate
it for the VO two-point functions in Section 5.4. In Section 6 the simplifi-
cation at $\xi = q^2$ mentioned earlier is explained. As an example we show
all VO two-point functions simplified at $\xi = q^2$ (Section 6.1). Finally we
give an application to the spin correlation functions for the spin-1 analog of the
XXZ model, which corresponds to a further specialization of the general
formula (Section 6.3). In Appendix we assemble some formulae concerning
boson fermion calculus needed for the proof in Section 4 and calculations in
Section 5.

2 Definitions for general level $k$

We give here definitions needed in this paper, which are the algebra
$U_q(\widehat{sl}_2)$, irreducible highest weight modules, vertex operators and their $n$-point
functions. Spin correlation functions are also given. We shall follow the notations
of Ref.[10].
In this section we do not specialize a level \( k \in \mathbb{Z}_{\geq 0} \). In the subsequent sections where we shall give explicit constructions of modules and vertex operators, we shall fix the level \( k = 2 \).

Throughout this paper we assume \( q^n \neq 1 \) for any \( n \in \mathbb{Z} \neq 0 \). In an application to a spin chain or a vertex model we further assume \(|q| < 1 \).

### 2.1 Quantum affine algebra \( U_q(\hat{sl}_2) \)

We give a definition of the quantum affine algebra \( U_q(\hat{sl}_2) \) which is a deformation with a parameter \( q \) of the universal enveloping algebra of an affine Lie algebra \( \hat{sl}_2 \).

We first fix the notations for the affine Lie algebra \( \hat{sl}_2 \). Let \( \Lambda_0, \Lambda_1 \) be the fundamental weights and let \( \delta \) be the null root. They span the weight lattice \( P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta \). The simple roots are given by \( \alpha_1 = 2\Lambda_1 - 2\Lambda_0 \), \( \alpha_0 = \delta - \alpha_1 \). We normalize the invariant symmetric bilinear form on \( P \) by \( \langle \alpha_i, \alpha_i \rangle = 2 \). Let \( P^* \) be the dual lattice to \( P \) and let \( \{ h_0, h_1, d \} \) be the basis of \( P^* \) dual to \( \{ \Lambda_0, \Lambda_1, \delta \} \) with respect to the natural pairing \( \langle \ , \ \rangle : P \times P^* \rightarrow \mathbb{Z} \). Therefore

\[
(h_i, h_j) = \begin{cases} 
2(i = j) \\
-2(i \neq j)
\end{cases}, \quad (h_i, d) = \delta_{i,0}, \quad (d, d) = 0;
\]

\[
(L_i, L_j) = \frac{1}{2} \delta_{i,1} \delta_{j,1}, \quad (L_i, \delta) = 1, \quad (\delta, \delta) = 0,
\]

\[
(\alpha_i, \alpha_j) = \begin{cases} 
2(i = j) \\
-2(i \neq j)
\end{cases}, \quad (\alpha_i, \Lambda_0) = \delta_{i,0}.
\]

We use the standard notation \( \rho = \Lambda_0 + \Lambda_1 \).

The quantum affine algebra \( U_q(\hat{sl}_2) \) is an associative algebra over a field \( F = \mathbb{C} \) with unit 1 generated by \( e_i, f_i(i = 0, 1) \), \( q^h(h \in P^*) \) with relations

\[
q^h q^{h'} = q^{h+h'}, \quad q^0 = 1,
\]

\[
q^h e_i q^{-h} = q^{(\alpha_i, h)} e_i, \quad q^h f_i q^{-h} = q^{-(\alpha_i, h)} f_i,
\]

\[
e_i f_j - f_j e_i = \delta_{ij} t_i - t_i^{-1} \frac{q - q^{-1}}{q^h (t_i = q^h)},
\]

\[
e_i^2 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j),
\]

...
let $h, h' \in P^*, i, j = 0, 1$. We use the notations
\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [n]! = [n][n-1] \cdots [2][1], \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.
\]

The coproduct and the antipode are given by the formulae
\[
\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h,
\]
\[
a(e_i) = -t_i^{-1}e_i, \quad a(f_i) = -f_it_i, \quad a(q^h) = q^{-h} \quad (h \in P^*)
\]
which equip $U_q(\hat{sl}_2)$ with a Hopf algebra structure.

The generators \{e_i, f_i, t_i = q^h| i = 0, 1\} generate a subalgebra $U'_q(\hat{sl}_2)$ of $U_q(\hat{sl}_2)$. Drinfeld [14] introduced a new realization of $U'_q(\hat{sl}_2)$ which is convenient for our aim: let $A$ be an algebra generated by $x^\pm_m (m \in \mathbb{Z}), a_m (m \in \mathbb{Z}_{\neq 0}), \gamma$ and $K$ (which we refer to as Drinfeld generators) with relations
\[
\gamma: \text{central}
\]
\[
[a_m, a_n] = \delta_{m+n,0} \frac{[2m]}{m} \gamma^m - \gamma^{-m},
\]
\[
[a_m, K] = 0,
\]
\[
K x^\pm_m K^{-1} = q^{\pm 2} x^\pm_m,
\]
\[
[a_m, x^\pm_n] = \pm \frac{[2m]}{m} \gamma^{\mp|m|} x^\pm_{m+n},
\]
\[
x^\pm_{m+1} x^\pm_n - q^{\pm 2} x^\pm_n x^\pm_{m+1} = q^{\pm 2} x^\pm_m x^\pm_{n+1} - x^\pm_n x^\pm_{m+1},
\]
\[
[x^+_m, x^-_n] = \frac{1}{q - q^{-1}} (\gamma^{\frac{1}{2}(m-n)} \psi_{m+n} - \gamma^{-\frac{1}{2}(m-n)} \varphi_{m+n}),
\]
where
\[
\sum_{m=0}^\infty \psi_m z^{-m} = K \exp\left( (q - q^{-1}) \sum_{m=1}^\infty a_m z^{-m} \right),
\]
\[
\sum_{m=0}^\infty \varphi_m z^m = K^{-1} \exp\left( -(q - q^{-1}) \sum_{m=1}^\infty a_{-m} z^m \right)
\]
and $\psi_{-m} = \varphi_m = 0$ for $m > 0$; here the bracket $[x, y]$ means $xy - yx$.

The equations for the formal Laurent series are to be interpreted as a set
of equations for coefficients. The essential theorem is that the algebra $A$ is isomorphic to $U_q'(\mathfrak{sl}_2)$; the Chevalley generators are given by the identification

$$t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = x_0^+, \quad f_1 = x_0^-, \quad e_0 = x_1^- K^{-1}, \quad f_0 = K x^{-1}_1.$$  

(7)

We identify the algebra $A$ with $U_q'(\mathfrak{sl}_2)$. The coproduct of the Drinfeld generators is known partially: for $n \geq 0$ and $l > 0$ we have

$$\Delta(x_n^+) = x_n^+ \otimes \gamma^n + \gamma^{2n} K \otimes x_n^+ + \sum_{i=1}^{n-1} \gamma^{(n+3i)/2} \psi_{n-i} \otimes \gamma^{-i} x_i^+ \mod N_- \otimes N_+^2,$$

$$\Delta(x_n^-) = x_n^- \otimes \gamma^{-l} + K^{-1} \otimes x_n^- + \sum_{i=1}^{l-1} \gamma^{(l-i)/2} \varphi_{l-i} \otimes \gamma^{-l+i} x_i^+ \mod N_- \otimes N_+^2,$$

$$\Delta(x_i^+) = \gamma^i \otimes x_i^- + x_i^- \otimes \gamma^{i/2} \psi_i \mod N_+ \otimes N_-,$$

$$\Delta(x_i^-) = \gamma^{-i} x_i^- + x_i^- \otimes \gamma^{-i/2} \psi_i \mod N_- \otimes N_+^2,$$

$$\Delta(a_l) = a_l \otimes \gamma^{l/2} + \gamma^{3l/2} \otimes a_l \mod N_- \otimes N_+,$$

$$\Delta(a_{-l}) = a_{-l} \otimes \gamma^{-3l/2} + \gamma^{-l/2} \otimes a_{-l} \mod N_- \otimes N_+;$$  

(8)

here $N_\pm$ and $N_\pm^2$ are left $F[\gamma^\pm, \psi_\pm, \varphi_\pm | r, -s \in \mathbb{Z}_{\geq 0}]$-modules generated by $\{x_m^+ | m \in \mathbb{Z}\}$ and $\{x_m^+ x_n^- | m, n \in \mathbb{Z}\}$ respectively. It gives sufficient information for our study (we do not use $\Delta(x_m^-), m \in \mathbb{Z}$, in this paper).

We will use the coproduct and the antipode to define a tensor product and dual representations, respectively. Let $(\pi_V, V)$ be a representation. The tensor product representation $(\pi_V \otimes W, V \otimes W)$ is defined to be $\pi_V \otimes \pi_W \circ \Delta$. The dual representations $(\pi_V^{a \pm 1}, V^*)$ are defined to be $\pi_V^{a \pm 1} = \pi_V \circ a_{\pm 1}$; we also refer to them as modules $V^{a \pm 1}$. Note that the $V^{a \pm 1}$ are left modules.
2.2 Finite-dimensional representations

For a positive integer $k$ let $V(k) = \bigoplus_{j=0}^{k} Fu_j$ be a $(k+1)$-dimensional vector space. The following gives the irreducible $(k+1)$-dimensional representation $(\pi, V(k))$ of the algebra $U_q(\hat{sl}_2)$:

$$\pi(t_1)u_j = q^{k-2j}u_j, \quad \pi(e_1)u_j = [j]u_{j-1}, \quad \pi(f_1)u_j = [k-j]u_{j+1},$$

$$\pi(t_0) = \pi(t_1)^{-1}, \quad \pi(e_0) = \pi(f_1), \quad \pi(f_0) = \pi(e_1)$$

or

$$\pi(\gamma)u_j = u_j, \quad \pi(K)u_j = q^{k-2j}u_j, \quad \pi(a_m)u_j = a^j_m u_j,$$

$$\pi(x^+_m)u_j = q^{m(k-2j)}[j]u_{j-1}, \quad \pi(x^-_m)u_j = q^{m(k-2j)}[k-j]u_{j+1}$$

where

$$a^j_m = \frac{2m! [jm]}{m!} q^{m(k-j+1)} + \frac{[km]}{m}.$$

Put $V^{(k)}_z = V(k) \otimes F[z, z^{-1}]$. Then, the following gives a representation $(\pi_z, V^{(k)}_z)$ of $U_q(\hat{sl}_2)$:

$$\pi_z(x) = \pi(x) \otimes \text{id} \quad \text{for} \quad x = e_1, f_1, t_1, t_0,$$

$$\pi_z(e_0) = \pi(f_1) \otimes z, \quad \pi_z(f_0) = \pi(e_1) \otimes z^{-1},$$

$$\pi_z(q^d)(u_j \otimes z^n) = q^n u_j \otimes z^n$$

or

$$\pi_z(\gamma) = \pi(\gamma) \otimes \text{id}, \quad \pi_z(K) = \pi(K) \otimes \text{id}, \quad \pi_z(a_m) = \pi(a_m) \otimes z^m,$$

$$\pi_z(x^+_m) = \pi(x^-_m) \otimes z^m.$$

Later, when considering an application to a spin chain or a vertex model, we will need the following isomorphisms between left $U_q(\hat{sl}_2)$-modules:

$$V^{(k)}_{zq^{-2}} \cong V^{(k)*a^\pm 1}_z$$

$$u_j \mapsto c^+_j u^*_k \quad (j = 0, 1, \ldots, k)$$

where

$$c^\pm_j = (-1)^j q^{j^2-(k+1)j} \frac{1}{\begin{bmatrix} k \n \end{bmatrix}}.$$

We shall denote $c_j = c^+_j$ (we do not use $c^-_j$ in this paper).
2.3 Irreducible highest weight modules

Set $P_+ = \mathbb{Z}_{\geq 0}\Lambda_0 + \mathbb{Z}_{\geq 0}\Lambda_1$. For $\lambda \in P_+$, a $U_q(\hat{sl}_2)$-module $V(\lambda)$ is called an irreducible highest weight module with highest weight $\lambda$ if the following conditions are satisfied: there is a vector $|\lambda\rangle \in V(\lambda)$, called the highest weight vector, such that $q^h|\lambda\rangle = q^{(\lambda,h)}|\lambda\rangle (h \in P^*)$, $e_i|\lambda\rangle = f_i^{(\lambda,h_i)+1}|\lambda\rangle = 0 (i = 0, 1)$, and $V(\lambda) = U_q(\hat{sl}_2)|\lambda\rangle$. We say that $V(\lambda)$ has level $k$ if $\gamma|\lambda\rangle = q^k|\lambda\rangle$. The $V(\lambda)$ has a weight-space decomposition $V(\lambda) = \bigoplus_{\mu} V(\lambda)_{\mu}$. We write the weight of a weight vector $v$ as $\text{wt}(v)$. We use the notation $\lambda_m = (k-m)\Lambda_0 + m\Lambda_1$, $m = 0, 1, \ldots, k$, for level $k$ highest weights.

2.4 Vertex operators

Vertex operators of type I are mappings
$$\Phi_{\lambda_m}^{\lambda_{k-m}}(z) : V(\lambda_m) \rightarrow V(\lambda_{k-m}) \otimes V_z^{(k)}$$
that commute with the action of the algebra $U_q(\hat{sl}_2)$ (intertwiners), i.e.,
$$\Delta(x) \circ \Phi(z) = \Phi(z) \circ x \quad \text{for } x \in U_q(\hat{sl}_2).$$
(11)

A convenient normalization is
$$\bar{\Phi}_{\lambda_m}^{\lambda_{k-m}}(z)(|\lambda_m\rangle) = |\lambda_{k-m}\rangle \otimes u_{k-m} + \ldots.$$  
(12)

Let us write
$$\Phi(z) = \sum_{j=0}^{k} \Phi_j(z) \otimes u_j$$
and call $\Phi_j(z)$ the $j$-th component of $\Phi(z)$.

It was proved that vertex operators $\Phi_{\lambda_m}^{\lambda_{k-m}}$ ($m = 0, 1, \ldots, k$) which we defined above exist uniquely up to normalization [16].

Precisely speaking, the right hand side of (10) is to be interpreted as $\bigoplus_{\mu} \Pi_{\nu} V(\lambda_{k-m})_{\nu} \otimes (V_z^{(k)})_{\mu-\nu}$; but we will not go into such details on topology.

2.5 Vertex-operator $n$-point functions

The aim of this paper is to get an integral formula for the $n$-point function of vertex operators
$$\tilde{F}_{i_1,\ldots,i_n}^{(\lambda)}(z_1,\ldots,z_n|x,y) \equiv \text{tr} V(\lambda)(\xi^{-2d}y^\alpha \bar{\Phi}_{i_1}(z_1) \cdots \bar{\Phi}_{i_n}(z_n))$$
(13)
where $y^\alpha$ acts as $y^\alpha v = y^{(\alpha, \text{wt}(v))} v$ on $v \in V(\lambda)$; $\Phi_{i_1}(z_1) \cdots \Phi_{i_n}(z_n)$ being the $(i_1, \ldots, i_n)$-component of a composite mapping

$$(\Phi(z_1) \otimes \text{id}_{V \otimes \cdots \otimes V}) \circ \cdots \circ (\Phi(z_{n-1}) \otimes \text{id}_V) \circ \Phi(z_n) : V(\lambda) \longrightarrow V(\lambda) \otimes V_{z_1}^{(k)} \otimes \cdots \otimes V_{z_n}^{(k)};$$

it is understood that $\tilde{\Phi}_{i_n}(z_n) : V(\lambda) \to V(\sigma(\lambda)), \, \tilde{\Phi}_{i_{n-1}}(z_{n-1}) : V(\sigma(\lambda)) \to V(\lambda)$, etc., where $\sigma(l\Lambda_0 + m\Lambda_1) \equiv m\Lambda_0 + l\Lambda_1$; note that $n$ must be even unless $k = \text{even}$ and $\lambda = (k/2)(\Lambda_0 + \Lambda_1)$. $\xi$ and $y$ are free parameters. We assume $|\xi| < 1$ to make the trace convergent.

2.6 Spin correlation functions

We first introduce the inverse of the vertex operator. Then we give a formula for a spin correlation function of an $S = k/2$ quantum spin chain or an integrable vertex model (which is defined to be the vacuum expectation value of an $n$-point local operator in the spin chain, or the thermal average of an in-row product of $n$ variables in the vertex model). As we shall see the spin $n$-point correlation function is obtained from the vertex operator $2n$-point functions (13) by specializing parameters.

2.6.1 Inverse vertex operators

Let $\tilde{\Phi}_{\lambda_{k-m}}^\lambda V(z)$ be vertex operators

$$\tilde{\Phi}_{\lambda_{k-m}}^\lambda V(z) : V(\lambda_{k-m}) \otimes V_z^{(k)} \longrightarrow V(\lambda_m)$$

which commute with the action of $U_q(\hat{sl}_2)$ (intertwiners) with normalization

$$\tilde{\Phi}_{\lambda_{k-m}}^\lambda V(z)(|\lambda_{k-m}\rangle \otimes u_{k-m}) = |\lambda_m\rangle + \ldots$$

(the omitted terms ... have weight not equal to $\lambda_m$). A natural identification of $V(\mu) \otimes V_z^{(k)} \to V(\lambda)$ with $V(\mu) \to V(\lambda) \otimes V_z^{(k)*a}$ (as $U_q(\hat{sl}_2)$-intertwiners) and the isomorphism $V_z^{(k)} \simeq V_z^{(k)*a}$ yield

$$\tilde{\Phi}_{\lambda_{k-m}}^\lambda V_{\cdot,j}(z) = \frac{c_{k-j}}{c_m} \tilde{\Phi}_{\lambda_{k-m},k-j}(zq^{-2}), \quad m = 0, 1, \ldots, k; \quad (14)$$
being given by (9). They are regarded as the inverses (up to constant factors) of the previous ones in the following sense (for the proof see [10]):

\[
\tilde{\Phi}^\lambda_{k-m} V(z) \circ \tilde{\Phi}^\lambda_{k-m} V(z) = g_{\lambda_m} \times \text{id}_{V(\lambda_m)};
\]

\[
\tilde{\Phi}^\lambda_{k-m} V(z) \circ \tilde{\Phi}^\lambda_{k-m} V(z) = g_{\lambda_m} \times \text{id}_{V(\lambda_k-m) \otimes V}
\]

where

\[
g_{\lambda_m} = q^{(k-m)m} \frac{k}{m} \binom{(q^{2(k+1)}; q^4)_{\infty}}{(q^2; q^4)_{\infty}}.
\] (15)

We shall frequently use the notation

\[
(z; x)_{\infty} = \prod_{n=1}^{\infty} (1 - zx^{n-1}).
\]

2.6.2 Spin correlations

The Hamiltonian of the integrable extension to spin \( k/2 \) of the XXZ model (we call the spin \( k/2 \) analog of the XXZ model) is defined by

\[
H = \sum_{l \in \mathbb{Z}} \cdots \otimes I^{l+2} \otimes h^l \otimes I^{l-1} \otimes \cdots,
\]

\[
h = (-1)^k (q^k - q^{-k}) \left[ \frac{d}{dz} \tilde{R}(z, 1) \right]_{z=1};
\] (16)

here \( \tilde{R} \) is a \( U'_q(\hat{sl}_2) \)-homomorphism, called the \( R \)-matrix,

\[
\tilde{R}(z_1/z_2) : V^{(k)}_{z_1} \otimes V^{(k)}_{z_2} \longrightarrow V^{(k)}_{z_2} \otimes V^{(k)}_{z_1}.
\]

The \( R \)-matrix itself defines an integrable \( k + 1 \)-state vertex model.

In the formulation of Refs. [8, 13, 10] the vacuum expectation value of a local operator \( L \in \text{End}(V^{(k) \otimes n}) \) (where \( V^{(k) \otimes n} \) is understood as \( n \)th to 1st components in \( V^{(k) \otimes \infty} \)) is given by [10]

\[
\langle L \rangle_{z_n, \ldots, z_1}^{(\lambda)} = \frac{\text{tr}_{V^{(\lambda)}} (q^{-2\rho} g_{z_n, \ldots, z_1}^{(\lambda)} (L))}{\text{tr}_{V^{(\lambda)}} (q^{-2\rho})}
\] (17)
where \( \rho = \Lambda_0 + \Lambda_1 \) and

\[
\phi^{(\lambda)}_{z_n, \ldots, z_1} (L) = (\Phi^{(n)}_\lambda (z_n, \ldots, z_1))^{-1} \circ (\text{id}_{V^{(\lambda)}} \otimes L) \circ \Phi^{(n)}_\lambda (z_n, \ldots, z_1)),
\]

\[
\Phi^{(n)}_\lambda (z_n, \ldots, z_1)
\]

\[
= (\Phi^{(n)}_{\lambda(n-1)} (z_n) \otimes \text{id}_{V^{(\lambda)}}) \circ \ldots \circ (\Phi^{(n)}_{\lambda(2)} (z_2) \otimes \text{id}_{V^{(\lambda)}}) \circ \Phi^{(n)}_{\lambda(1)} (z_1),
\]

\[
\Phi^{(n)}_\lambda (z_n, \ldots, z_1)^{-1}
\]

\[
= \Phi^{(1)}_{\lambda(n)} (z_1) \circ (\Phi^{(n-1)}_{\lambda(n)} (z_n) \otimes \text{id}_{V^{(\lambda)}}) \circ \ldots \circ (\Phi^{(2)}_{\lambda(n)} (z_2) \otimes \text{id}_{V^{(\lambda)}}) \circ (\Phi^{(1)}_{\lambda(n)} (z_1))
\]

\[
\times (g_\lambda g_{\lambda(1)} \cdots g_{\lambda(n-1)})^{-1}.
\]

A general local operator is expressed as a linear combination of \( E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \); so it is sufficient to give a formula for this basis operator (below we shall set \( \lambda = \lambda_m \)):

\[
\langle E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \rangle^{(\lambda_m)} = \frac{\text{tr} V^{(\lambda_m)} (q^{-2\rho} \Phi^{\lambda_m}_{\lambda(1)} V_{i_1} (z_1) \cdots \Phi^{\lambda(n)}_{\lambda(n-1)} V_{i_n} (z_n) \cdots \Phi^{\lambda(1)}_{\lambda(1)} V (z_1))}{g_{\lambda_m} g_{\lambda(1)} \cdots g_{\lambda(n-1)} \text{tr} V^{(\lambda_m)} (q^{-2\rho})};
\]

\((E_{ij})_{i'j'} = \delta_{ii'} \delta_{jj'}; \lambda^{(l)} = \sigma (\lambda^{(l-1)}), \lambda^{(0)} = \lambda_m.\) Substituting (14) we see that the spin \( n \)-point correlation function is obtained from a vertex operator \( 2n \)-point function by specializing the parameters \( \xi = q^2, y = q^{-1} \) (cf. \( \rho = 2d + \frac{1}{2} \alpha \)):

\[
\langle E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \rangle^{(\lambda_m)}_{z_n, \ldots, z_1} = (-1)^{mn+k(\frac{d}{2})+n} \sum _{l=1} ^n (k-i) \cdots \frac{(q^2; q^4) _n}{(q^{2(k+1)}; q^{4}) _n} \prod _{l=1} ^n \left[ k \right]_{i_l} ^{-1}
\]

\[
\times \Phi^{(\lambda)}_{\lambda(n)} (z_1 q^{-2}, \ldots, z_n q^{-2}, z_n, \ldots, z_1 | q^2, q^{-1}) \frac{\text{tr} V^{(\lambda_m)} (q^{-2\rho})}{\text{tr} V^{(\lambda_m)} (q^{-2\rho})};
\]

(18)

we have used (14), (15) and introduced \([x] \equiv \min \{ n \in \mathbb{Z} | n \geq x \}, \lfloor x \rfloor \equiv \max \{ n \in \mathbb{Z} | n \leq x \} \).

We note that \( z_j \) are regarded as the (trigonometric) spectral parameters in the lattice model language. For spin \( k/2 \) analog of the XXZ model we
specialize the spectral parameters to $z_1 = \cdots = z_n$. For the vertex model we are not necessarily to specialize them; in this case $z_j$ is the spectral parameter of the $j$th vertical line and the correlator is for the inhomogeneous $k+1$-state vertex model.

Finally we must mention the range of the parameter $q$. In the application to spin correlation functions we must assume that $|q| < 1$ in order to make traces convergent. Furthermore, we assume $-1 < q < 0$; this assumption seems necessary to verify our identification of the vector $|\text{vac}\rangle = \sum v_i \otimes v_i^* \in V(\lambda) \otimes V(\lambda)^{\ast}$ with a physical vacuum (ground state) via a discussion using the crystal base (cf. [3, 10]). We shall see, however, the final formula still has a meaning even at positive $q$; i.e., for $|q| < 1$.

3 Constructions of level two irreducible highest weight modules over $U_q(\hat{sl}_2)$

We shall give explicit constructions of three level two irreducible highest weight modules, $V(\lambda)$, $\lambda = 2\Lambda_0, 2\Lambda_1, \Lambda_0 + \Lambda_1$. We use the Drinfeld realization of $U'_q(\hat{sl}_2)$ in which the operators $a_m$ generate a boson algebra $\mathcal{A}$. To construct boson vacuum spaces we have to introduce a fermion: a Neveu-Schwarz fermion for $V(2\Lambda_i)$, $i = 0, 1$, and a Ramond fermion for $V(\Lambda_0 + \Lambda_1)$; just as in the constructions of level two modules over the affine Lie algebra $\hat{sl}_2$.

We first define a total Fock space $\mathcal{F}$ in which the irreducible modules are to be embedded. The structure of the total Fock space is as follows: $\mathcal{F} = \mathcal{A}\Omega$, where $\Omega$ is a boson vacuum space (i.e., each vector in $\Omega$ generates a boson Fock space); the $\Omega$ is realized by a product of the group algebra of the weight lattice of $sl_2$ and a fermion Fock space. The aim is to single out the highest weight vector $|\lambda\rangle$ in $\mathcal{F}$ and to find explicit forms of the currents $x^\pm(z) \equiv \sum_{m \in \mathbb{Z}} x_m^\pm z^{-m}$ as operators on $\mathcal{F}$ so that the $|\lambda\rangle$ and the $x^\pm(z)$ generate the irreducible module $V(\lambda)$.

It should be noted that the constructions were essentially obtained by Bernard who gave in [3] the level one modules over $U_q(B^{(1)}_r)$, the case $r = 1$ corresponding to ours; but it seems that an explicit exposition in the $U_q(\hat{sl}_2)$ case is not found in literatures; so we decided to describe it in details without proof.
From here we fix the level $k = 2$.

### 3.1 Fock spaces

We first prepare necessary operators and Fock spaces.

Drinfeld generators $a_m, \gamma$ form a Heisenberg subalgebra $A$ of the algebra $U_q'(\hat{sl}_2)$. Let us put $\gamma = q^2$ since we want to construct level two modules; then

$$[a_m, a_n] = \delta_{m+n,0} \frac{[2m]^2}{m}, \quad m, n \in \mathbb{Z} \neq 0.$$ 

We refer to the generators $a_m$ as bosons; $a_{-m}, m > 0$, are creation operators and $a_m, m > 0$, annihilation operators. A Fock space of the boson is defined as usual which we denote $F^a = F[a_{-1}, a_{-2}, \ldots]$: $F^a \equiv \bigoplus_{1 \leq i_1 < \cdots < i_s} F^{a_{i_1} \cdots a_{i_s}} |\rangle$, $|\rangle$ (which will be denoted 1 in the following) being a vacuum vector such that $a_m |\rangle = 0$ for $m > 0$, with the natural left action of the boson algebra.

Let $P = \mathbb{Z}^2_{\alpha}$ and $Q = \mathbb{Z} \alpha$ be the weight lattice and the root lattice of the Lie algebra $sl_2$, respectively; the $\alpha$ being the simple root of $sl_2$, identified with $\alpha_1$. Let $F[P], F[Q]$ be their group algebras. Elements $e^{n \alpha}, n \in \mathbb{Z}$ span the vector space $F[P]$. $F[Q]$ is a subspace of $F[P]$, and $F[P] = F[Q] \oplus e^\alpha F[Q]$. We introduce two linear operators on $F[P]$ which are $e^\beta (\beta \in \mathbb{Z} \alpha)$ and $\partial_{\alpha}$ defined by

$$e^{\beta_1} \cdot e^{\beta_2} = e^{\beta_1 + \beta_2} (\beta_1, \beta_2 \in P), \quad \partial_{\alpha} \cdot e^\beta = (\alpha, \beta) e^\beta (\beta \in P).$$

Let us introduce the Neveu-Schwarz fermion $\{\phi_n^{NS} | n \in \mathbb{Z} + \frac{1}{2}\}$, and the Ramond fermion $\{\phi_n^R | n \in \mathbb{Z}\}$; both satisfy the anti-commutation relation

$$[\phi_m, \phi_n]_+ = \delta_{m+n,0} \frac{q^{2m} + q^{-2m}}{q^2 + q^{-2}}.$$ 

We have dropped the indices $NS, R$. Elements $\phi_{-n}, n > 0$, are creation operators and $\phi_n, n > 0$, annihilation operators. Construct Fock spaces for these fermions as usual and denote them $F^{\phi^{NS}} = F[\phi_{-1/2}, \phi_{-3/2}, \ldots], F^{\phi^R} = F[\phi_{-1}, \phi_{-2}, \ldots]$. 

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One should notice that $\phi_0$ of the Ramond is special since it anti-commutes with $\phi_n$ ($n \in \mathbb{Z}_{\neq 0}$) and $\phi_0^2 = 1/[2]$; we must treat the Ramond fermion carefully. It will be convenient to write the Ramond fermion as

$$\phi_n = \psi_n \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (n \neq 0), \quad \phi_0 = \psi_0 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$[\psi_m, \psi_n]_+ = \delta_{m+n,0} \frac{q^{2m} + q^{-2m}}{q^2 + q^{-2}} \quad (m, n \in \mathbb{Z}_{\neq 0}), \quad \psi_0 = [2]^{-\frac{1}{2}} \in F.$$

Then

$$\mathcal{F}^{\phi_R} = \mathcal{F}^\psi \otimes \mathbb{C}^2 = F[\psi_{-1}, \psi_{-2}, \ldots] \otimes \mathbb{C}^2.$$

A convenient basis of $\mathbb{C}^2$ is $\{ \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} | \epsilon = \pm 1 \}$; the vectors are eigenvectors of $\phi_0$ and flipped to each other by $\phi_n$ ($n \neq 0$) since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} = \epsilon \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} = \begin{pmatrix} 1 \\ -\epsilon \end{pmatrix}.$$

Define two vector spaces by

$$\mathcal{F}^{(0)} \equiv \mathcal{F}^a \otimes \mathcal{F}^{\phi_{NS}} \otimes F[Q],$$

$$\mathcal{F}^{(1)} \equiv \mathcal{F}^a \otimes \mathcal{F}^{\phi_R} \otimes e^x F[Q] \quad (20)$$

which we shall refer to as total Fock spaces. The boson, fermion, and lattice operators naturally act on the spaces (the boson acts on the first component, etc.). As we shall see modules $V(2\Lambda_0)$ and $V(2\Lambda_1)$ are contained in $\mathcal{F}^{(0)}$ and a module $V(\Lambda_0 + \Lambda_1)$ is contained in $\mathcal{F}^{(1)}$.

### 3.2 Irreducible highest weight modules $V(2\Lambda_i)$, $i = 0, 1$

We first consider an irreducible highest weight module with highest weight $2\Lambda_0$ or $2\Lambda_1$.

We define the action of Drinfeld generators $\gamma$ and $K$ as

$$\gamma = q^2, \quad K = q^{\beta_a}.$$

Then, from the Drinfeld relations (6), explicit forms of currents (or generating functions)

$$x^\pm(z) = \sum_{m \in \mathbb{Z}} x_m z^{-m}$$

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as operators on the total Fock space $F^{(0)}$ are determined to be
\[ x^{\pm}(z) = [2]^{\frac{1}{2}}E^{\pm}_{<}(z)E^{\pm}_{>}(z)\phi(z)e^{\pm \alpha z^{\frac{1}{2}} \frac{\partial}{\partial \alpha}}; \]  
(22)

\[ E^{\pm}_{<}(z) = \exp(\pm \sum_{m=1}^{\infty} \frac{a_{-m}}{[2m]}q^{m} z^{m}); \quad E^{\pm}_{>}(z) = \exp(\mp \sum_{m=1}^{\infty} \frac{a_{m}}{[2m]}q^{m} z^{-m}); \]

$\phi(z)$ being the Neveu-Schwarz fermion field
\[ \phi(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \phi_{n} z^{-n}; \]
equations (22) are to be understood as a set of equations for coefficients. We have defined the action of the algebra $U'_{q}(\hat{sl}_{2})$ on the total Fock space $F^{(0)}$; i.e., $F^{(0)}$ is a $U'_{q}(\hat{sl}_{2})$-module.

A vector $1$ with weight $2\Lambda_{0}$ (cf. $\gamma \cdot 1 = q^{2}1$, $K \cdot 1 = 1$) generates a $U'_{q}(\hat{sl}_{2})$-submodule
\[ F^{(0)}_{+} = F^{a} \otimes \left\{ (F^{\phi}_{even} \otimes F[2Q]) \oplus (F^{\phi}_{odd} \otimes e^{a} F[2Q]) \right\}; \]
this subspace of $F^{(0)}$ is identified with $V(2\Lambda_{0})$, the highest weight vector being $1$.

A vector $e^{a}$ with weight $2\Lambda_{1}$ (cf. $\gamma \cdot e^{a} = q^{2}e^{a}$, $K \cdot e^{a} = q^{2}e^{a}$) generates a $U'_{q}(\hat{sl}_{2})$-submodule
\[ F^{(0)}_{-} = F^{a} \otimes \left\{ (F^{\phi}_{even} \otimes q^{a} F[2Q]) \oplus (F^{\phi}_{odd} \otimes F[2Q]) \right\}; \]
this subspace is identified with $V(2\Lambda_{1})$, the highest weight vector being $e^{a}$.

Note that the total Fock space is a direct sum of the two subspaces
\[ F^{(0)} = F^{(0)}_{+} \oplus F^{(0)}_{-}. \]

Define the operator $d$ by
\[ d = -\sum_{m=1}^{\infty} mN^{a}_{m} - \sum_{k>0} kN^{\phi}_{k} - \frac{1}{8} \frac{\partial^{2}}{\partial \alpha} + \frac{(\lambda, \lambda)}{4}; \]  
(23)
where
\[ N^{a}_{m} = \frac{m}{[2m]^{2}} a_{-m} a_{m}, \quad N^{\phi}_{m} = \frac{q + q^{-1}}{q^{2m} + q^{-2m}} \phi_{-m} \phi_{m} \quad (m > 0). \]  
(24)
This operator is identified with the grading operator $d$ in the algebra $U_{q}(\hat{sl}_{2})$; with this $d$ and setting $\lambda = 2\Lambda_{i}$ we have irreducible highest weight modules $V(2\Lambda_{i}) \simeq F^{(0)}_{\pm} (\pm$ according to $i = 0, 1$) over $U_{q}(\hat{sl}_{2})$. 

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3.3 Irreducible highest weight module $V(\Lambda_0 + \Lambda_1)$

Now we consider an irreducible highest weight module with highest weight $\Lambda_0 + \Lambda_1$.

The action of Drinfeld generators $\gamma$ and $K$ is defined as before. Explicit forms of currents as operators on the total Fock space $\mathcal{F}^{(1)}$ are the same as (22) except for the fermion field to be replaced by the Ramond one

$$\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n}.$$ 

Thus we have got a $U'_q(\hat{sl}_2)$-module $\mathcal{F}^{(1)}$.

Fix a one of the vectors $\left(\frac{1}{\epsilon}\right)$, $\epsilon = \pm 1$. A vector $\left(\frac{1}{\epsilon}\right) \otimes e^{\frac{\phi}{2}}$ with weight $\Lambda_0 + \Lambda_1$ (cf. values of $\gamma$ and $K$ on this vector are $q^2$ and $q$ respectively) generates a $U'_q(\hat{sl}_2)$-submodule

$$\mathcal{F}^{(1)}_\epsilon = F^a \otimes \left\{ (F_{\text{even}}^R \otimes\left(\frac{1}{\epsilon}\right)) \oplus (F_{\text{odd}}^R \otimes\left(-\frac{1}{\epsilon}\right)) \right\} \otimes e^{\frac{\phi}{2}} F[Q]; \quad (25)$$

this subspace of $\mathcal{F}^{(1)}$ is identified with $V(\Lambda_0 + \Lambda_1)$ with the highest weight vector $\left(\frac{1}{\epsilon}\right) \otimes e^{\frac{\phi}{2}}$.

The total Fock space is a direct sum of the two subspaces $\mathcal{F}^{(1)}_\epsilon$

$$\mathcal{F}^{(1)} = \mathcal{F}^{(1)}_+ \oplus \mathcal{F}^{(1)}_-.$$ 

Introducing the operator $d$ defined by (23) with $\lambda = \Lambda_0 + \Lambda_1$ and the fermion being the Ramond one we have an irreducible highest weight modules $V(\Lambda_0 + \Lambda_1) \simeq \mathcal{F}^{(1)}_\epsilon$ over $U'_q(\hat{sl}_2)$.

3.4 Characters

Characters

$$\text{tr}_{V(\lambda)}(\xi^{-2d}y^{\beta_\alpha})$$

for irreducible highest weight modules $V(\lambda)$, $\lambda = 2\Lambda_0, 2\Lambda_1, \Lambda_0 + \Lambda_1$, can be computed directly:

$$\text{tr}_{V(2\Lambda_1)}(\xi^{-2d}y^{\beta_\alpha}) = \xi^{-\frac{1}{2}(2\Lambda_1,2\Lambda_1)} \times \left[ \text{tr}_{\mathcal{F}^{(0)}}(\xi^{-2d}y^{\beta_\alpha}) \right]^{a_i}$$
\[
\xi^{-\frac{d}{2}(2\Lambda_i,2\Lambda_i)} \times \left[ \frac{1}{(\xi^2;\xi^2)_{\infty}} (-\xi; -\xi)_{\infty} (-\xi y^2;\xi^2)_{\infty} (-\xi y^{-2};\xi^2)_{\infty} \right]^{\sigma_i},
\]

\[
\text{tr}_{V(\Lambda_0+\Lambda_1)}(\xi^{-2d}y^{d_0}) = y(-\xi^2;\xi^2)_{\infty} (-\xi^2 y^2;\xi^2)_{\infty} (-y^{-2};\xi^2)_{\infty}
\]

where \(d' = d - (\lambda,\lambda)/4\) in \(V(\lambda)\), and \(\sigma_i\) indicates operations on arbitrary functions of \(\xi\)

\[
f(\xi)^{\sigma_i} \equiv \frac{1}{2}(f(\xi) \pm f(-\xi)) \quad (\pm \text{according to } i = 0, 1).
\]

In particular, a specialization \(\xi = q^2, y = q^{-1}\) yields

\[
\text{tr}_{V(2\Lambda_i)}(q^{-2d}) = q^{-2d}(-q^2; q^2)_{\infty} (-q^4; q^4)_{\infty}, \quad \text{tr}_{V(\Lambda_0+\Lambda_1)}(q^{-2d}) = q^{-1}(-q^2; q^2)_{\infty} (-q^2; q^4)_{\infty}.
\]

### 3.5 Boson calculus, fermion calculus

Later, when we prove intertwining relations for vertex operators and calculate \(n\)-point functions, we shall need several formulae concerning calculus on \(a_m\) and \(\phi_m\). We have assembled some of them in Appendix: formulae for operator product expansions, normal ordering, and traces.

### 4 Vertex operators

We give expressions of level two vertex operators associated with our explicit constructions of level two irreducible highest weight modules over \(U_q(\hat{sl}_2)\); see \(\{\{\},\{\}\}\) with \(k = 2\). First we present explicit forms of components \(\Phi_j(z)\); then we give the proof.

#### 4.1 Intertwining relations

By definition the vertex operator must satisfy intertwining relations \(\{\{\}\}\). Let us write down the relations for the Chevalley generators (we write them in terms of \(x_n^\pm\); cf.\(\{\}\)):

\[
f_1: \quad [\Phi_2(z), x^-_0]_{q^2} = \Phi_1(z)
\]

\[
[\Phi_1(z), x^-_0] = [2]\Phi_0(z)
\]

\[
[\Phi_0(z), x^-_0]_{q^{-2}} = 0
\]

\[
f_0: \quad [\Phi_2(z), x^+_1] = 0
\]

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\[ \Phi(z) = \exp\left( \sum_{m=1}^{\infty} \frac{a_m}{2m} q^{5m-z^m} \right), \quad \Phi(z) = \exp\left( -\sum_{m=1}^{\infty} \frac{a_m}{2m} q^{-3m-z^m} \right), \]

\[ D_1(z, w_1) = \frac{1 - q^4}{-q^4 z} \frac{1}{w_1(1 - q^{-2} w_1 z)/(1 - q^6 z/w_1)}, \quad \left| \frac{w_1}{q^2 z} \right|, \left| \frac{q^6 z}{w_1} \right| < 1, \]

\[ D_0(z, w_1, w_2) = \frac{(1 - q^4)^2}{q^7 z^2} \frac{1 - q^4 w_2 w_1}{w_2 \prod_{j=1,2} (1 - q^{-2} w_1 z)(1 - q^6 z/w_j)}, \quad \left| \frac{w_j}{q^2 z} \right|, \left| \frac{q^6 z}{w_j} \right| < 1; \]

the contours for the \( w \)-integrals are anti-clockwise in the regions indicated after the definitions of \( D_0 \) and \( D_1 \); the \( \hat{x}^\pm (w) \) are currents without fermion:

\[ \hat{x}^\pm (z) = [2]^{\frac{1}{2}} E^\pm_\infty(z) E^\pm(z) e^{\pm \alpha z^\pm \frac{1}{2} \theta_\alpha}. \]
The expressions for $\Phi_1$, $\Phi_0$ were obtained from $\Phi_2$ using (27) and (28).

$\Phi_j(z)$ are regarded as mappings $\mathcal{F}^{(i)} \rightarrow \mathcal{F}^{(i)}$ (we must choose the NS, R fermion for $i = 0, 1$, respectively) as well as $V(\lambda) \rightarrow V(\sigma(\lambda))$; $\sigma$ being defined by $\sigma((2 - m)\Lambda_0 + m\Lambda_1) = m\Lambda_0 + (2 - m)\Lambda_1$, $m = 0, 1, 2$.

The vertex operators with normalizations (12) are given by

\[
\tilde{\Phi}_{2\lambda_0}^{2\lambda_1 V}(z) = \Phi(z),
\tilde{\Phi}_{\lambda_0 + \lambda_1}^{\lambda_0 + \lambda_1 V}(z) = \epsilon(-q^4 z^{\frac{3}{2}})\Phi(z),
\tilde{\Phi}_{2\lambda_1}^{2\lambda_0 V}(z) = (-q^4 z)\Phi(z);
\]  

(41)

at the second identification we have chosen the realization $V(\lambda + \Lambda_1) = \mathcal{F}_e^{(1)}$. The factors were derived by applying our $\Phi(z)$ to the highest weight vectors in our representations.

The equation for $\Phi_2$ is derived (guessed) as follows: recall the coproducts for the Drinfeld generators (8) and write down intertwining relations for $a_m$, $x_n$ and $K$; then we have equations to be satisfied by $\Phi_2$:

\[
\begin{align*}
a_m \Phi_k(z) - \Phi_k(z)a_m &= q^{(\frac{m}{2} + 2)m}\left[\frac{km}{m}\right]z^m\Phi_k(z) \quad \text{for } m > 0, \\
a_{-m} \Phi_k(z) - \Phi_k(z)a_{-m} &= q^{-\left(\frac{m}{2} + 2\right)m}\left[\frac{km}{m}\right]z^{-m}\Phi_k(z) \quad \text{for } m > 0, \\
\Phi_k(z)x^+(w) - x^+(w)\Phi_k(z) &= 0, \\
K\Phi_k(z)K^{-1} &= q^k\Phi_k(z);
\end{align*}
\]

put $k = 2$ (level) for our aim; first two equations determine the boson part of $\Phi_2(z)$ and the last one requires $e^\alpha$; the third equation is satisfied by our $\Phi_2(z)$, so we expect that our $\Phi_2(z)$ is probably correct.

In fact the vertex operator we proposed does actually satisfy all the intertwining relations, which we will prove now.

### 4.3 Proof of the intertwining relations for the Chevalley generators

We prove here that our vertex operator does satisfy intertwining relations for the Chevalley generators. We use the operator product expansion

\[
\phi(w_1)\phi(w_2) = \langle \phi(w_1)\phi(w_2) \rangle + :\phi(w_1)\phi(w_2):; \quad \text{etc}.
\]

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see Appendix A.2, A.3 for definitions and notations.

Clearly the relations (33) are satisfied by our $\Phi_j$. Among the rest relations we shall prove here four typical ones which are (33), (38), (34) and (35); one can prove other relations in almost similar ways.

**Proof.** (33): it holds trivially since $[\Phi_2(z), x^+(w)] = 0$.

(38):

$$[\Phi_0(z), x_1^-]_{q^2} = \oint \frac{dw}{2\pi i} [\Phi_0(z), x^-(w)]_{q^2}$$

$$= \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \oint \frac{dw_3}{2\pi i} (1 - q^2)^2$$

$$\times (1 - q^2 \frac{w_3}{w_1})_{\infty} (1 - q^2 \frac{w_3}{w_2})_{\infty}$$

$$\times q^2 \frac{w_1}{w_2} (1 - q^2 \frac{w_1}{w_3}) + q^2 \frac{w_1}{w_2} (1 - q^2 \frac{w_1}{w_2})$$

$$\times \phi(w_1)\phi(w_2)\phi(w_3) : \Phi_2(z)\Phi^-(w_1)\Phi^+(w_2)\Phi^-(w_3) :$$

Expand the $\{\cdots\}$ and first examine the first term; it is an odd function with respect to $w_3 \leftrightarrow w_2$ and the $w_2$, $w_3$-integrals are independent [note that the factor $(1 - q^2 \frac{w_3}{w_1})(1 - q^2 \frac{w_3}{w_2})$ cancels the poles at $\frac{w_3}{w_2} = q^{\pm 2}$ of $\langle \phi(w_2)\phi(w_3) \rangle$]; therefore this term vanishes after $w_2$, $w_3$-integrations. Similarly the second term vanishes after $w_1$, $w_2$-integrations. Therefore we get $[\Phi_0(z), x_1^-]_{q^2} = 0$.

(34): we will need the operator product expansion (OPE) of $\phi(w)\phi(w_1)$.

$$[\Phi_1(z), x_0^+] = \oint \frac{dw}{2\pi i} \Phi(w) [\Phi_1(z), x^+(w)]$$

$$= \oint \frac{dw}{2\pi i} F(w) \frac{\phi(w_1)\phi(w)}{w_1 - w} - \oint \frac{dw}{2\pi i} F(w) \frac{\phi(w)\phi(w_1)}{w - w_1}$$

$$= \oint \frac{dw}{2\pi i} F(w) \frac{\phi(w_1)\phi(w)}{w_1 - w} - \oint \frac{dw}{2\pi i} F(w) \frac{\phi(w)\phi(w_1)}{w - w_1}$$

$$\quad + \oint \frac{dw}{2\pi i} F(w) : \phi(w_1)\phi(w) : - \oint \frac{dw}{2\pi i} F(w) : \phi(w)\phi(w_1) :$$

$C_1$ being a contour in a region $|w| < |q^{\pm 2}w_1|$, $C_2$ in $|w| > |q^{\pm 2}w_1|$ and

$$F(w) = \oint \frac{dw}{2\pi i} \frac{1 - q^4 (w - q^4 z)}{q^4 z - ww_1 (1 - q^{-2} \frac{w}{w_1}) (1 - q^6 \frac{w}{w_1})}.$$
The last integral can be replaced by an integral along $C_1$ plus a residue at $w = w_1$; the former together with the third term, and the residue, vanish respectively because of the anti-symmetry of $:\phi(w_1)\phi(w)$ : In the same way, deforming the contour $C_2$ of the second integral and combining with the first, we have

$$
[\Phi_1(z), x_0^+] = -\operatorname{Res}_{w=q^2w_1} \left\{ F(w) \frac{\phi(w)\phi(w_1)}{w-w_1} \right\} - \operatorname{Res}_{w=q^{-2}w_1} \left\{ F(w) \frac{\phi(w)\phi(w_1)}{w-w_1} \right\}
$$

$$
= \oint \frac{dw_1}{2\pi i} \frac{q}{w_1(w_1-q^6z)} : \Phi_2(z) \hat{\phi}^{-}(w_1) \hat{\phi}^{+}(q^2w_1) :
$$

$$
+ \oint \frac{dw_1}{2\pi i} \frac{1}{w_1(q^{-4}w_1-q^2z)} : \Phi_2(z) \hat{\phi}^{-}(w_1) \hat{\phi}^{+}(q^{-2}w_1) :
$$

$$
= [2]q^2 F^-_{\phi}(z) F^+_{\phi}(z) q^\alpha (-q^6z)^{-d_\alpha} = [2]t_1 \Phi_2(z);
$$

The first integral gives the residue at $\infty$ which is 1, and the second the residue at $w_1 = 0$ which is 0; hence

$$
[\Phi_1(z), x_0^+] = [2]q^2 F^-_{\phi}(z) F^+_{\phi}(z) q^\alpha (-q^6z)^{-d_\alpha} = [2]t_1 \Phi_2(z);
$$

thus we have proved the relation (34). Keys are

$$
: \hat{\phi}^{-}(w_1) \hat{\phi}^{+}(q^\pm 2w_1) : \propto \exp(\pm \sum_{m>0} \frac{a_{m\pm}}{2m} (q^m-q^{-3m})w_1^{\pm m})
$$

does not contain boson creation, annihilation operators, respectively) which reduce the contour integrals to simple residue calculations.

(35): the proof will go along the same way as the one for (34); but in this case we will need the OPE of $\phi(w)\phi(w_1)\phi(w_2)$.

$$
[\Phi_0(z), x_0^+] = \oint \frac{dw}{2\pi i} \frac{1}{w} [\Phi_0(z), x^+(w)]
$$

$$
= \oint_{C_1} \frac{dw}{2\pi i} F(w) \frac{\phi(w_1)\phi(w_2)\phi(w)}{(w_1-w)(w_2-w)} - \oint_{C_2} \frac{dw}{2\pi i} F(w) \frac{\phi(w)\phi(w_1)\phi(w_2)}{(w-w_1)(w-w_2)}
$$
where

\[ C_1 : |w| < |w_1|, |w_2| \quad \text{and} \quad |w| < |q^{\pm 2}w_2|; \]
\[ C_2 : |w| > |w_1|, |w_2| \quad \text{and} \quad |w| > |q^{\pm 2}w_1| \]

(the composition of fermions \( \phi(w_1)\phi(w_2)\phi(w) \) requires the second condition for \( C_1 \); and similarly we need the one for \( C_2 \)) and

\[
F(w) = \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \left( \frac{1 - q^2}{q^2 z^2} \right) \times \left( \frac{(w - q^4 z)(w_1 - q^2 w_2)}{ww_1w_2 \prod_{j=1,2}^{q^0 \frac{w_j}{z}}(1 - q^{-2 \frac{w_j}{z}})(1 - q^{6 \frac{w_j}{z}})} \right).
\]

Substituting the OPE

\[
\phi(w_1)\phi(w_2)\phi(w_3) = \langle \phi(w_1)\phi(w_2)\rangle \phi(w_3) - \langle \phi(w_1)\phi(w_3)\rangle \phi(w_2)
+ \langle \phi(w_2)\phi(w_3)\rangle \phi(w_1) + \phi(w_1)\phi(w_2)\phi(w_3) ;
\]

deforming the contour \( C_2 \) to \( C_1 \) plus ones around poles, as in the proof of \[14\], we get

\[
[\Phi_0(z), x_0^+] = \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \left( \frac{1 - q^2}{q^2 z^2} \right) \prod_{j=1,2} \left( \frac{1}{(1 - q^{-2 \frac{w_j}{z}})(1 - q^{6 \frac{w_j}{z}})} \right)
\times \left\{- \left[ \frac{2}{1 - q^4} \right] q^4 z \phi(w_2) : \Phi_2(z) \hat{x}^-(w_2) : q^{\partial_\alpha} \right. 
\times \left( \frac{1 - q^{-2 \frac{w_1}{z}}}{w_1w_2(w_1 - q^2 w_2)} \exp \left( \sum_{m>0} \frac{a_m}{2m} (q^m - q^{-3m})w_1^{-m} \right) \right)
- \left[ \frac{2}{1 - q^4} \right] q^4 \frac{1 - q^6 \frac{z}{w_1}}{w_2} \exp \left( - \sum_{m>0} \frac{a_{-m}}{2m} (q^m - q^{-3m})w_1^m \right)
\times \phi(w_2) : \Phi_2(z) \hat{x}^-(w_2) : q^{-\partial_\alpha} 
- \left[ \frac{2}{1 - q^4} \right] q^6 z \phi(w_1) : \Phi_2(z) \hat{x}^-(w_1) : q^{\partial_\alpha} 
\times \left( \frac{1 - q^{-2 \frac{w_2}{z}}}{w_1w_2} \exp \left( \sum_{m>0} \frac{a_m}{2m} (q^m - q^{-3m})w_2^{-m} \right) \right)
- \left[ \frac{2}{1 - q^4} \right] \frac{1 - q^6 \frac{z}{w_1}}{w_1(w_1 - q^2 w_2)} \exp \left( - \sum_{m>0} \frac{a_{-m}}{2m} (q^m - q^{-3m})w_2^m \right) \right\}.
\]
\times \phi(w_1) : \Phi_2(z) : \Phi_2^-(w_1) : q^{-\partial_\alpha} \}

where contours should be $|q^a z| < |w_j| < |q^2 z|$ and $|w_2/w_1| < |q^1/2|$. The $w_1$-integral in the first term gives the residue at $w_1 = \infty$; the one in the second the residue at $w_1 = 0$ (which is 0); the $w_2$-integral in the third term gives the residue at $w_2 = \infty$; the one in the last the residue at $w_2 = 0$ (which is 0). Thus, the first and the third terms remain non-zero and give the right hand side of the equation (35).

As was noted, the other relations are proved similarly.

We have completed the proof of the intertwining relations; hence we have proved that our expression of the vertex operator is correct.

5 Integral formula for $n$-point functions of vertex operators

In this section we derive an integral formula for the $n$-point function of vertex operators (VO $n$-point function) defined in 5.1 (the trace of the product of vertex operators). The final formula is given in (52) and (53). All the two-point functions are shown in 5.4 as an example.

5.1 VO $n$-point functions (definition)

Introduce more notations

\[ d' = d^a + d^\phi - \frac{1}{8} \partial_\alpha^2, \quad d^a = - \sum_{m=1}^{\infty} m N^a_m, \quad d^\phi = - \sum_{m>0} m N^\phi_m; \]

the $d'$ is related to the grading operator $d$ by $d = d' + \frac{1}{4}(\lambda, \lambda)$ in the representation $V(\lambda)$ (cf. (23); see (24) for definitions of $N^a_m, N^\phi_m$).

We shall give an integral formula for the following traces over Fock spaces (19) and (25):

\[
F_{i_1, \ldots, i_n}^{(n)} (z_1, \ldots, z_n|\xi, y) \equiv \text{tr}_F(\xi^{-2d'} y^{\partial_\alpha} \Phi_{i_1}(z_1) \cdots \Phi_{i_n}(z_n)), \\
F_{i_1, \ldots, i_n}^{(1, e)} (z_1, \ldots, z_n|\xi, y) \equiv \text{tr}_{F_1}(\xi^{-2d'} y^{\partial_\alpha} \Phi_{i_1}(z_1) \cdots \Phi_{i_n}(z_n)) 
\]

(42)
where the vertex operators are unnormalized ones \( \text{[10]} \). Since \( d' \) is bounded from above in Fock spaces we assume \( |\xi| < 1 \) to make the traces convergent.

The \( n \)-point functions of the normalized vertex operators \( \text{[11]} \) are defined by

\[
\tilde{F}^{(\lambda)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y) \equiv \text{tr}_{\nu(\lambda)}(\xi^{-2d} \phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n)). \tag{43}
\]

They are obtained from \( F \) as:

\[
\begin{align*}
\tilde{F}^{(2\Lambda_0)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y) &= \xi^{-\frac{1}{2}(2\Lambda_0, 2\Lambda_0)} \cdot (-q^4 z_1) (-q^4 z_3) \cdots (-q^4 z_{n-1}) \left[ F^{(0)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y) \right]^\sigma_0, \\
\tilde{F}^{(2\Lambda_1)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y) &= \xi^{-\frac{1}{2}(2\Lambda_1, 2\Lambda_1)} \cdot (-q^4 z_2) (-q^4 z_4) \cdots (-q^4 z_n) \left[ F^{(0)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y) \right]^\sigma_1, \\
\tilde{F}^{(\Lambda_0 + \Lambda_1)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y) &= \xi^{-\frac{1}{2}(\Lambda_0 + \Lambda_1, \Lambda_0 + \Lambda_1)} \cdot \epsilon^n \prod_{l=1}^n (-q^4 z_l)^{\frac{1}{2}} \cdot F^{(1, \epsilon)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n \mid \xi, y); \\
\end{align*}
\]

\( \sigma_i \) being defined in \( \text{[26]} \).

If the trace is non-zero, \( n \) must be even for \( F^{(0)} \) and \( \tilde{F}^{(2\Lambda_i)} \); it is not necessarily so for \( F^{(1, \epsilon)} \) and \( \tilde{F}^{(\Lambda_0 + \Lambda_1)} \).

### 5.2 Traces

Before taking the trace in the \( n \)-point function \( \text{[12]} \) it is better to make everything in normal order. Then we compute the trace separately for boson, fermion, and lattice part (cf. Appendix for boson and fermion). The computations are straightforward (it is not necessary to introduce an auxiliary boson as in Ref.\[13\]). Here we only quote results for fermion in order to fix the notations.

Fermion traces are taken over the Fock space \( F^{\phi_{NS}} \) or

\[
F^{\phi_R}_\epsilon \equiv (F^{\phi_{\text{even}}} \otimes \left( \frac{1}{\epsilon} \right)) \oplus (F^{\phi_{\text{odd}}} \otimes \left( \frac{1}{-\epsilon} \right)) \quad (\epsilon = \pm 1) \tag{44}
\]

according to Neveu-Schwarz or Ramond fermion. A fermion trace is expressed by a Pfaffian of a matrix \( G \).

\[
\text{tr} \left( \xi^{-2d} \phi(w_1) \cdots \phi(w_n) \right)
\]
\[ G_N^S(w_1, w_2) = \frac{1}{q + q^{-1}} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{(w_2/w_1)^m q^{2m} + q^{-2m}}{1 + \xi^{2m}} \]

\[ G_R(w_1, w_2) = \frac{1}{q + q^{-1}} \sum_{m \in \mathbb{Z}} \frac{(w_2/w_1)^m q^{2m} + q^{-2m}}{1 + \xi^{2m}} \]

\[ = \frac{1}{q + q^{-1}} \frac{(\xi^2; \xi^2)^3 \Theta_{\xi^{2}}(q^4)}{\Theta_{\xi^{2}}(q^2) \Theta_{\xi^{2}}(-q^2)} \cdot \frac{\Theta_{\xi^{2}}(w_2 w_1) \Theta_{\xi^{2}}(-w_2 w_1)}{\Theta_{\xi^{2}}(q^2 w_2 w_1) \Theta_{\xi^{2}}(q^{-2} w_2 w_1)} \]

\[ = \frac{1}{q + q^{-1}} \frac{(\xi^2; \xi^2)^3 \Theta_{\xi^{2}}(q^4)}{\Theta_{\xi^{2}}(q^2) \Theta_{\xi^{2}}(-q^2)} \cdot \frac{\Theta_{\xi^{2}}((w_2 w_1)^2)}{\Theta_{\xi^{2}}(q^2 w_2 w_1) \Theta_{\xi^{2}}(q^{-2} w_2 w_1)} \]

(46)

The matrix \( G(\{w\}) \) is defined as follows. If \( n = \text{even} \) \( G(\{w\}) \) is an anti-symmetric \( n \times n \) matrix with entries being fermion two-point functions:

\[ G_{ij}(\{w\}) = G(w_i, w_j) \equiv \frac{\text{tr} (\xi^{-2d\phi} \phi(w_i) \phi(w_j))}{\text{tr} (\xi^{-2d\phi})}, \quad 1 \leq i < j \leq n \]

where

\[ G_N^S(w_1, w_2) = \frac{1}{q + q^{-1}} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{(w_2/w_1)^m q^{2m} + q^{-2m}}{1 + \xi^{2m}} \]

\[ G_R(w_1, w_2) = \frac{1}{q + q^{-1}} \sum_{m \in \mathbb{Z}} \frac{(w_2/w_1)^m q^{2m} + q^{-2m}}{1 + \xi^{2m}} \]

\[ = \frac{1}{q + q^{-1}} \frac{(\xi^2; \xi^2)^3 \Theta_{\xi^{2}}(q^4)}{\Theta_{\xi^{2}}(q^2) \Theta_{\xi^{2}}(-q^2)} \cdot \frac{\Theta_{\xi^{2}}(w_2 w_1) \Theta_{\xi^{2}}(-w_2 w_1)}{\Theta_{\xi^{2}}(q^2 w_2 w_1) \Theta_{\xi^{2}}(q^{-2} w_2 w_1)} \]

\[ = \frac{1}{q + q^{-1}} \frac{(\xi^2; \xi^2)^3 \Theta_{\xi^{2}}(q^4)}{\Theta_{\xi^{2}}(q^2) \Theta_{\xi^{2}}(-q^2)} \cdot \frac{\Theta_{\xi^{2}}((w_2 w_1)^2)}{\Theta_{\xi^{2}}(q^2 w_2 w_1) \Theta_{\xi^{2}}(q^{-2} w_2 w_1)} \]

for Neveu-Schwarz, Ramond fermion, respectively; the expressions are defined in a region \( |q^{-2} \xi^2| < |w_2/w_1| < |q^2| \); here we have introduced a theta function

\[ \Theta_x(z) \equiv (z; x)_{\infty} (xz^{-1}; x)_{\infty} (x; x)_{\infty} = \sum_{n \in \mathbb{Z}} (-1)^n x^{\frac{1}{2} n(n-1)} z^n. \]

If \( n = \text{odd} \) and \( \phi = \phi^R \) the matrix \( G(\{w\}) \) is an anti-symmetric \( (n+1) \times (n+1) \) defined by

\[ G(\{w\}) = \left( G_N^-(\{w\}) \right)_{i,j} \begin{cases} 0, 1, \ldots, n \\ \downarrow \\ 0, 1, \ldots, n \end{cases} \]

\[ G_N^-_{ij}(\{w\}) = G^{-}(w_i, w_j) \]

26
\[
\sum_{m \in Z \setminus \{0\}} \left( \frac{w_j}{w_i} \right)^m q^{2m} + q^{-2m} \frac{1}{1 - \xi^{2m}} \right), \quad 1 \leq i < j \leq n;
\]

\[
G_{0j}(\{w\}) = \frac{\text{tr}_{\mathcal{F}^{(1)}(\xi^{-2d\phi})}}{\text{tr}_{\mathcal{F}^{(1)}(\xi^{-2d\phi})}} \frac{\epsilon}{\sqrt{2}} \frac{(\xi^2; \xi^2)}{(-\xi^2; \xi^2)_\infty}, \quad 1 \leq j \leq n.
\]

For derivations see Appendix.

Note that the two-point function \( G(w_1, w_2) \) becomes the delta function \( \delta(z) = \sum_{m \in Z} z^m \) at \( \xi = q^2 \):

\[
G^{NS,R}(w_1, w_2) \big|_{\xi = q^2} = \frac{1}{q + q^{-1}} \left( q^{-2} \frac{w_2}{w_1} \right)^{i/2} \delta \left( q^{-2} \frac{w_2}{w_1} \right). \tag{47}
\]

\((i = 1, 0 \text{ according to } NS, R)\)

5.3 Integral formula

In order to describe formulae concisely let us introduce some notations and an order ‘\(<\)’ for the set of indices \( I \equiv \{i_1, \ldots, i_n\} \), which indicate the components of vertex operators in the product \( \Phi_{i_1}(z_1) \cdots \Phi_{i_n}(z_n) \), let

\[ I_j = \{l(1 \leq l \leq n)|i_l = j\}, \quad s_j = \#I_j \quad (j = 0, 1, 2); \tag{48} \]

note that by definition \( s_0 + s_1 + s_2 = n \); let

\[ Z = \{z_l(1 \leq l \leq n)\}, \quad W = \{w^{(l)}_1(l \in I_1), w^{(l)}_2(l \in I_0)\} \tag{49} \]

where \( w^{(l)}_1 \) is the integral variable (dummy) of \( \Phi_1(z_l), l \in I_1 \), and \( w^{(l)}_1, w^{(l)}_2 \) the integral variables of \( \Phi_0(z_l), l \in I_0 \) (cf. (34)); in a set of variables \( Z \cup W \) define an order ‘\(<\)’ by:

(i) if \( l < l' \) then

\[ z_l < z_{l'}, \quad w^{(l)}_1 < w^{(l')}_1, \quad z_l < w^{(l)}_1, \quad w^{(l)}_1 < z_{l'}; \tag{50} \]

(ii) if \( l = l' \) then

\[ z_l < w^{(l)}_1, \quad w^{(l)}_1 < w^{(l)}_2 (\text{if } l \in I_0); \tag{51} \]

we further define \( W_l \equiv \{w \in W|z_l < w < z_{l+1}\} \).

Remark. The number of the fermion fields \( \phi(w) \) and the number of the \( w \)-integrals in the product \( \Phi_{i_1}(z_1) \cdots \Phi_{i_n}(z_n) \) are \( 2s_0 + s_1 \). Furthermore, for the nontrivial (nonzero) trace we have \( s_0 = s_2 \) and, therefore, \( n = 2s_0 + s_1 \).
After tedious but straightforward computations we obtain an integral formula for the $n$-point function of vertex operators (42):

\begin{equation}
F^{(0)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n | \xi, y)
= a_I(Z | \xi, y) \prod_{w \in W} \oint \frac{dw}{2\pi i} B_I(W, Z | y) C_I^{(0)}(W, Z | \xi, y),
\end{equation}

\begin{equation}
F^{(1, \epsilon)}_{i_1, \ldots, i_n}(z_1, \ldots, z_n | \xi, y)
= a_I(Z | \xi, y) \prod_{w \in W} \oint \frac{dw}{2\pi i} B_I(W, Z | y) C_I^{(1, \epsilon)}(W, Z | \xi, y),
\end{equation}

where

\begin{align*}
a_I(Z | \xi, y) & = (-1)^{s_1 + \sum_{l=2}^{l-1} (l-1)(i_l - 1)} [2]^{s_1 + \frac{l}{2}} q^{-7s_0 - 3s_1 + 4 \sum_{l=2}^{l-1} (l-1)(i_l - 1)} \frac{f(1 | \xi^2)^n}{(\xi^2; \xi^2)_\infty} \\
& \times \prod_{l=1}^{n-1} \left\{ \prod_{w} w^{-\sum_{l'>(i_l - 1)} (l')} \prod_{l \in I_1} \frac{1}{w_1^{(l)}} \prod_{l \in I_0} \frac{1}{w_2^{(l)}} \right\} \prod_{w \in w'} f(\frac{w}{w'} | \xi^2) \\
& \times \frac{\prod_{w < w'} f(\frac{w}{w'} | \xi^2) \prod_{z < z'} f(\frac{z}{z'} | \xi^2)}{\prod_{z < w} f(\frac{w}{w} | \xi^2)},
\end{align*}

\begin{align*}
B_I(W, Z | \xi) & = \prod_{l=1}^{n-1} \left\{ \prod_{w \in W_l} w^{-\sum_{l'>(i_l - 1)} (l')} \prod_{l \in I_1} \frac{1}{w_1^{(l)}} \prod_{l \in I_0} \frac{1}{w_2^{(l)}} \right\} \prod_{w \in w'} f(\frac{w}{w'} | \xi^2) \\
& \prod_{z < w} f(\frac{w}{w} | \xi^2) \prod_{z < z'} f(\frac{z}{z'} | \xi^2),
\end{align*}

\begin{align*}
C_I^{(0)}(W, Z | \xi, y) & = \text{tr} \left( \xi^{-2d} \prod_{w} \phi^{NS} (w) \right) \cdot \prod_{w} w^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} \xi^n \left( y^2 \frac{\pi \xi(-q^4 z)}{\pi w w} \right)^n, \\
C_I^{(1, \epsilon)}(W, Z | \xi, y) & = \text{tr} \left( \xi^{-2d} \prod_{w} \phi^{R} (w) \right) \cdot \prod_{w} w^{\frac{1}{2}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \xi^n \left( y^2 \frac{\pi \xi(-q^4 z)}{\pi w w} \right)^n,
\end{align*}

and

\begin{equation*}
f(z | x) \equiv (q^2 z; x)_\infty (q^2 xz^{-1}; x)_\infty.
\end{equation*}
The contours of $w$-integrals are anti-clockwise around the origin such that
\[ |\xi^2 q^6 z| < |w| < |q^2 z| \quad \text{for} \quad z < w, \quad |q^6 z| < |w| < |\xi^{-2} q^2 z| \quad \text{for} \quad z > w; \]
and, additionally,
\[ |\xi^2 q^{-2}| < |\frac{w'}{w}| < |q^2| \]

must be forced if the integrand contains fermion two-point functions $G(w, w')$ ($w < w'$) when expanding the fermion Pfaffian $G(\{w\})$. The fermion traces are given in 5.2 and depend on the order in the set $W$.

Many of the factors $f(\frac{w'}{w}|\xi^2)$ in the numerator of the integrand are cancelled by those in the denominators of the fermion two-point function $G(w, w')$ (cf. Eq. (13)). Observe that the contours get pinched when letting $\xi \to q^2$ with $|\xi| < |q^2|$. We regard the formulae as analytic functions of $\xi$, and define them outside the region $|\xi| < |q^2|$ by analytic continuation; we shall demonstrate it in the next subsection 5.4. As noted before, just at $\xi = q^2$, the fermion trace yields the delta function (see (47)), and the formula gets simplified; this point is further discussed in 6.2. A specialized formula at $\xi = q^2$ is important in an application to spin correlation functions of an $S = 1$ quantum spin chain, which will be discussed in 6.3.

Finally we note that $a_1(Z|\xi, y)$ and $B_1(W, Z|\xi)$ are even functions of $\xi$.

5.4 Example: VO two-point functions

To illustrate the content of the formula (52), (53), we write down all two-point functions of vertex operators:

\[ F_{20}^{(0)}(z_1, z_2|\xi, y) = -[2]q^{-11} f(1|\xi^2)^2 \frac{f(\frac{w}{w_1}|\xi^2)}{(\xi^2, \xi^2)} \sum_{n \in \mathbb{Z}} \frac{z_1^2 z_2^2}{z_1 z_2} \prod_{i,j=1,2} f(\frac{w_i}{w_{1,2}}|\xi^2) G_{NS}(w_1, w_2)(-\xi; \xi^2)^{n} \]

\[ F_{02}^{(0)}(z_1, z_2|\xi, y) = -[2]q^{-3} f(1|\xi^2)^2 \frac{f(\frac{w}{w_1}|\xi^2)}{(\xi^2, \xi^2)} \sum_{n \in \mathbb{Z}} \frac{z_1^2 z_2^2}{z_1 z_2} \prod_{i=1,2} \frac{f(\frac{w_i}{w_{1,2}}|\xi^2)}{f(\frac{w_i}{w_{1,2}}|\xi^2)} \frac{f(\frac{w_1}{w_2}|\xi^2)}{f(\frac{w_1}{w_2}|\xi^2)} \frac{f(\frac{w_2}{w_1}|\xi^2)}{f(\frac{w_2}{w_1}|\xi^2)} \frac{f(\frac{w_1}{w_2}|\xi^2)}{f(\frac{w_1}{w_2}|\xi^2)}, \]

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\[ F^{(0)}_{11}(z_1, z_2|\xi, y) = [2]^3 q^{-6} \frac{f(1|\xi^2)^2 f(\frac{z_1}{z_2}|\xi^2)}{(\xi^2; \xi^2)_\infty} \frac{z_1 z_2}{w_1 w_2} \]

\[ \times \oint dw_1 \oint dw_2 w_1^{-\frac{1}{2}} w_2^{-\frac{1}{2}} f(\frac{w_2}{w_1}|\xi^2) G^{NS}(w_1, w_2)(-\xi; \xi^2)_\infty \sum_{n\in\mathbb{Z}} \xi^{n^2} (y^2 \frac{q^n z_1 z_2}{w_1 w_2})^n \]

\[ G^{NS}(w_1, w_2)(-\xi; \xi^2)_\infty \sum_{n\in\mathbb{Z}} \rightarrow G^{R}(w_1, w_2)(-\xi; \xi^2)_\infty \sum_{n\in\mathbb{Z} + \frac{1}{2}}. \]

The contours are restricted in the regions described below Eqs. (52) and (53); for example, \(|\xi^2 q^6 z_i| < |w_j| < |q^2 z_i| (i, j = 1, 2), |\xi^2 q^{-2}| < |w_2/w_1| < |q^2| \) for \( F^{(0)}_{20} \), etc.

Let us examine the two-point function \( F^{(0)}_{20}(z_1, z_2|\xi, y) \) as a function of \( \xi \) more closely and show how to obtain its analytic continuations. The factor \( f(\frac{w_2}{w_1}|\xi^2) \) in the integrand is cancelled by that from \( G^{NS} \). Apart from poles at \( \xi^{2n} q^6 z_i, \xi^{-2n} q^2 z_i \) \((n = 0, 1, \ldots; i = 1, 2)\), the integrand as a function of \( w_2 \) has poles at

\[ \ldots, \xi^6 q^{-2} w_1, \xi^4 q^{-2} w_1, \xi^2 q^{-2} w_1; q^2 w_1, \xi^{-2} q^2 w_1, \xi^{-4} q^2 w_1, \ldots \]

The original contour of the \( w_2 \)-integral is confined in a region between the poles \( \xi^2 q^{-2} w_1 \) and \( q^2 w_1 \). We deform it to a sum of a contour anti-clockwise around the origin in a region \(|q^2 w_1| < |w_2| < |\xi^{-2} q^2 w_1| \) and a one clockwise around the pole at \( w_2 = q^2 w_1 \) (which counts minus a residue), so as to get an expression of \( F^{(0)}_{20} \) as a function of \( \xi \) in a neighbourhood of \( \xi = q^2 \). Noting that

\[ \left[ f(\frac{w_2}{w_1}|\xi^2) G^{NS,R}(w_1, w_2) \cdot (w_2 - q^2 w_1) \right]_{w_2 \rightarrow q^2 w_1} \]

\[ = -q^2 \frac{[2]}{(q^4; \xi^2)_\infty (\xi^2; \xi^2)_\infty} w_1 \]

the residue at \( w_2 = q^2 w_1 \) is evaluated simply; the result is

\[ F^{(0)}_{20}(z_1, z_2|\xi, y) = -q^{-10}(q^4; \xi^2)_\infty f(1|\xi^2)^2 \frac{f(\frac{z_1}{z_2}|\xi^2)}{z_1 z_2} \]

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This is an analytic continuation of the original formula to a neighborhood of \( \xi = q^2 \). Observe that the second term vanishes at \( \xi = q^2 \), since it includes a factor \( \Theta_{\xi^2}(q^4) \) which arose from the fermion two-point function; this fact follows directly from (47).

6 Simplification at \( \xi = q^2 \) and an application to a spin chain

We have seen that the formula for the VO two-point functions gets simplified when specialized at \( \xi = q^2 \) (Section 5.4). In this section we first write down all simplified integral formula for the VO two-point functions at \( \xi = q^2 \). Then we describe how to obtain a simplified integral formula for the VO n-point function at \( \xi = q^2 \) from the general formulae (52) and (53). In this simplified formula the number of contour integrals reduces from \( n \) to \( n/2 \). It is due to a property of the fermion two-point function which becomes the delta function at \( \xi = q^2 \). An application to the \( S = 1 \) spin chain is given in 6.3.

We restrict ourselves to \( n = \text{even}\).

6.1 VO two-point functions at \( \xi = q^2 \)

From the first term of the equation (54) we get an expression of \( F^{(0)}_{20} \) at \( \xi = q^2 \); similarly we get other VO two-point functions at \( \xi = q^2 \):

\[
F^{(0)}_{20}(z_1, z_2|q^2, y) = -q^{-10}(q^4; q^4)_\infty f(1|q^4)_2^2 \frac{f(\frac{z_1}{z_2}|q^4)}{z_1 z_2^2} \\
\times \oint \frac{dw_1}{2\pi i} \frac{w_1(-\xi; q^4)_\infty \sum_{n \in \mathbb{Z}} \xi^n (y^2 q^{2z_1 z_2})^n}{(w_1 - w_2)^2 (w_1 - q^{2w_1})^2 (w_1 - q^{4w_1})^2 (w_1 - q^{8w_1})^2},
\]

\[
F^{(0)}_{02}(z_1, z_2|q^2, y) = -q^{-4}(q^4; q^4)_\infty f(1|q^4)_2 \frac{f(\frac{z_1}{z_2}|q^4)}{z_1}.
\]
\[ \times \oint \frac{dw_1}{2\pi i} \left( -\xi; q^4 \right)_\infty \sum_{n \in \mathbb{Z}} \xi^n \left( y^2 \frac{q_{w_1}}{w_1} \right)^n \]

\[ F_{11}^{(0)}(z_1, z_2 | q^2, y) = [2]^{2}q^{-5}(q^4; q^4)_\infty f(1|q^4)^2 \frac{f(z_2 | q^4)}{z_1 z_2} \]

\[ F_{ij}^{(1, e)} \] are obtained from \( F_{ij}^{(0)} \) by the replacement

\[ (-\xi; q^4)_\infty \sum_{n \in \mathbb{Z}} \rightarrow (-\xi^2; q^4)_\infty \sum_{n \in \mathbb{Z} + \frac{1}{2}}. \]

We have reserved odd \( \xi \) for convenience of (anti-)symmetrization with respect to \( \xi \), which is necessary to obtain expressions of \( \tilde{F}_{ij}^{(1)} \) (cf. Section 5.3).

### 6.2 Simplified formula for the VO \( n \)-point function at \( \xi = q^2 \)

We have seen that the formula for the VO two-point functions gets greatly simplified when specialized at \( \xi = q^2 \). This feature holds as well for the VO \( n \)-point functions; the number of integrals reduces to \( \frac{1}{2}n \) at \( \xi = q^2 \).

A simplified formula for the VO \( n \)-point function at \( \xi = q^2 \) is obtained as follows: first we expand the Pfaffian in the formula (52) or (53) into \( (n-1) \) terms, each of which being a product of \( n/2 \) fermion two-point functions, say,

\[ G(w_{r_1}, w_{r_2})G(w_{r_3}, w_{r_4}) \cdots G(w_{r_{n-1}}, w_{r_n}); \]

deform the contours of \( w_{r_2}, w_{r_4}, \ldots, w_{r_n} \) and evaluate minus the residue at \( q^2 w_{r_1}, q^2 w_{r_3}, \ldots, q^2 w_{r_{n-1}} \), respectively, as in the case of two-point functions in the previous subsection; let \( \xi \rightarrow q^2 \); then we get an formula containing only \( w_{r_1}, w_{r_3}, \ldots, w_{r_{n-1}} \) contour integrals (the rest terms with more integrals vanish).

Practically we are to replace

\[ f \left( \frac{w_{r_{2j}}}{w_{r_{2j-1}}} \ | \xi^2 \right) G(w_{r_{2j-1}}, w_{r_{2j}}) \mapsto \frac{q^2}{[2]}(q^4; q^4)_\infty (\xi^2; \xi^2)_\infty w_{r_{2j-1}} \]
for all $j = 1, 2, \ldots, \frac{n}{2}$, remove $\frac{\delta w}{\delta x}$, $r = r_2, r_4, \ldots, r_n$, and put $\xi = q^2$, $w_{r_2} = q^2 w_{r_2-1}$, $j = 1, 2, \ldots, \frac{n}{2}$. Then we obtain a simplified formula at $\xi = q^2$.

6.3 Application to an integrable $S = 1$ spin chain

The spin $n$-point function for the integrable $S = 1$ spin chain (the spin-1 analog of the XXZ model) is given by the formula (55) with $k = 2$; note that $(q^2; q^4)\infty/(q^2; q^4)\infty = 1 - q^2$.

The formula can be written in terms of $F$:

$$\langle E_{i_0 n_0} \otimes \cdots \otimes E_{i_k n_k} \rangle^{(\lambda_m)}_{z_{n_0}, \ldots, z_1} = e^{mn}(-1)^{mn+\sum_i (i_i-1)}$$

$$\times q^{2n+\sum_{i=1}^n (2-i_i)(1-i_i)} (1 - q^2)^n [2]^{-\# \{1 \leq i \leq n \mid i_i = 1\}} \frac{(q^2)^{-\frac{1}{2}}(\lambda_m, \lambda_m)}{\text{tr} V(\lambda_n)(q^{-2\rho})}$$

$$\times \prod_{l=1}^n z_l \cdot F^{(\epsilon)}_{k-i_1, \ldots, k-i_n, j_1, j_2}(z_1 q^{-2}, \ldots, z_n q^{-2}, z_n, \ldots, z_1 | q^2, q^{-1})$$

(55)

where $F^{(\epsilon)}$ stands for $[F^{(0)}]^{\sigma_0}$, $[F^{(0)}]^{\sigma_1}$, $F^{(1, \epsilon)}$ for $m = 0, 2, 1$, respectively. $\rho = \Lambda_0 + \Lambda_1 = 2d + \frac{1}{2} \alpha$ (in our representation), $\lambda_m = (2 - m)\Lambda_0 + m\Lambda_1$, $\Lambda_0$ being given in (3). (This formula for $m = 1$ appears to depend on our choice of realization $V(\Lambda_0 + \Lambda_1) = F_{\epsilon}^{(1)}$, $\epsilon = \pm 1$; but it does not actually as can be seen if we substitute (55).)

For the spin chain correlations we set $z_1 = \cdots = z_n$; equation (55) itself with general $(z_1, \ldots, z_n)$ is the correlation function of the inhomogeneous vertex model. The range of the parameter $q$ is $-1 < q < 0$; but it might be extended to $|q| < 1$ (cf. Section 2.6.2).

As an example we consider the spin one-point functions, which are related to VO two-point functions, and represented by a single contour integral respectively. It is easily seen that identities $\langle E_{j, j} \rangle^{(\lambda_m)} = \langle E_{2-j, 2-j} \rangle^{(\lambda_2-m)}$, $j = 0, 1, 2$, hold. If we expand the expressions in $q$ by hand we get

$$\langle E_{22} \rangle^{(2\Lambda_0)} = 1 - 2q^2 + 5q^6 - 2q^8 - 9q^{10} + 3q^{12} + 19q^{14} + O(q^{16})$$

$$\langle E_{00} \rangle^{(2\Lambda_0)} = 2q^4 + q^6 - 4q^8 - 9q^{10} + 7q^{12} + 19q^{14} + O(q^{16})$$

$$\langle E_{11} \rangle^{(2\Lambda_0)} = 2q^2 - 2q^4 - 6q^6 + 6q^8 + 18q^{10} - 10q^{12} - 38q^{14} + O(q^{16})$$

$$\langle E_{00} \rangle^{(\Lambda_0 + \Lambda_1)} = 2q^2 - 4q^4 + 2q^6 + 6q^{10} - 4q^{12} - 6q^{14} + O(q^{16})$$

$$\langle E_{11} \rangle^{(\Lambda_0 + \Lambda_1)} = 1 - 4q^2 + 8q^4 - 4q^6 - 12q^{10} + 8q^{12} + 12q^{14} + O(q^{16})$$

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From this list we have

\[ \langle s^z \rangle_{z}^{(2\Lambda_1)} = 1 - 2q^2 - 2q^4 + 4q^6 + 2q^8 - 4q^{12} + O(q^{16}) \]

and \( \langle s^z \rangle_{z}^{(\Lambda_0 + \Lambda_1)} = 0; \) the both agree with the Bethe Ansatz results in Ref.[17]. We note that the equation for \( \langle s^z \rangle_{z}^{(2\Lambda_1)} \) in Ref.[17] can be factorized as

\[ \langle s^z \rangle_{z}^{(2\Lambda_1)} = (q^2; q^2)_{\infty}/(-q^4; q^4)_{\infty}. \]

Acknowledgement

This paper is a thoroughly revised and rewritten version of my previous manuscript [18] and the thesis [19] in which, compared with the present one, I emphasized an application to the spin chain correlations, in particular, spin one-point functions, and did not write down the integral formula for \( n \)-point functions. I wish to thank Michio Jimbo for suggesting me that my previous results can be simplified more, and for discussions from which I learned much. I also thank T. Miwa and T. Tokihiro for discussions and assistances in completing the previous work.

After completing the previous version [18, 19] there appeared a preprint [20] in which they considered the same spin chain correlations and gave an integral formula for them; but they did not give the construction of \( V(\Lambda_0 + \Lambda_1) \) and their formula in [20] seems more complicated than ours in the present paper; I am grateful to Robert A. Weston for communications and his interest in my work [18, 19].

Appendix: Boson fermion calculus

In this Appendix we assemble formulae of normal ordering and traces needed for deriving or proving formulae in the text. Among other things we must be careful with the Ramond fermion.

A.1 Boson calculus

Normal ordering

A normal order product : \( A \) : of an element \( A \) of the boson algebra is defined as usual: annihilation operators are placed at the right to the creation
operators; for example, : $a_m a_{-n} := a_{-n} a_m \ (m, n > 0)$.

We collect here formulae for the normal ordering of exponentials of boson operators which are used in this paper; recall that

$$\Phi(z) = F_<(z) F_>(z) e^a (-q^4 z)^{\frac{1}{4} \partial a},$$

$$x^+(z) = [2]^{\frac{1}{2}} E^+((z) E_<(z) \phi(z) e^{a} z^{\frac{1}{4} + \frac{1}{4} \partial a},$$

$$x^-(z) = [2]^{\frac{1}{2}} E^-(z) E_<(z) \phi(z) e^{-a} z^{\frac{1}{4} - \frac{1}{4} \partial a},$$

where

$$F_<(z) = \exp \left( \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2m}} q^{5m} z^m \right), \quad F_>(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2m}} q^{-3m} z^{-m} \right),$$

$$E^+_<(z) = \exp \left( \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2m}} q^{-m} z^m \right), \quad E^+_<(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2m}} q^{-m} z^{-m} \right),$$

$$E^-_(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2m}} q^{m} z^{-m} \right), \quad E^-_(z) = \exp \left( \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2m}} q^{m} z^{-m} \right);$$

we have the following:

$$F_>(z_1) F_<(z_2) = (1 - q^2 z_2 / z_1) F_<(z_2) F_>(z_1),$$

$$F_>(z_1) E^-_(z_2) = \frac{1}{1 - q^2 z_2 / z_1} E^-_(z_2) F_>(z_1),$$

$$E^-_(z_1) F_<(z_2) = \frac{1}{1 - q^6 z_2 / z_1} F_<(z_2) E^-_(z_1),$$

$$E^-_(z_1) E^-_(z_2) = (1 - q^2 z_2 / z_1) E^-_(z_2) E^-_(z_1),$$

$$F_>(z_1) E^+_<(z_2) = (1 - q^{-4} z_2 / z_1) E^+_<(z_2) F_>(z_1),$$

$$E^+_<(z_1) F_<(z_2) = (1 - q^4 z_2 / z_1) F_<(z_2) E^+_<(z_1),$$

$$E^+_<(z_1) E^+_<(z_2) = (1 - q^{-2} z_2 / z_1) E^+_<(z_2) E^+_<(z_1),$$

$$E^+_<(z_1) E^-_(z_2) = \frac{1}{1 - z_2 / z_1} E^-_(z_2) E^+_<(z_1),$$

$$E^-_(z_1) E^+_<(z_2) = \frac{1}{1 - z_2 / z_1} E^+_<(z_2) E^-_(z_1);$$

the right hand sides are normal order products.
Traces of boson exponentials

\[
\text{tr}_{\mathcal{F}} \left( x \sum_{m > 0} m N_a \exp\left( - \sum_{m > 0} a_m f_m \exp\left( \sum_{m > 0} a_m f_m \right) \right) \right) = \prod_{m=1}^{\infty} \frac{1}{1 - x^m} \times \prod_{m=1}^{\infty} \exp \left( - [a_m, a_{-m}] f_m f_{-m} \right) \frac{x^m}{1 - x^m}.
\]

We have used

\[
\sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} X^n = \frac{1}{1 - X^k} \left( \frac{X}{1 - X^k} \right)^k.
\]

Further specialization \([a_m, a_{-m}] = [2m]^2/m, A_m = \frac{q^m}{[2m]} f_m, A_{-m} = \frac{q^m}{[2m]} f_{-m}\) and \(f_m f_{-m} = \sum_y \epsilon_y y^m\) yields

\[
\text{tr}_{\mathcal{F}} \left( x \sum_{m > 0} m N_a \exp\left( - \sum_{m > 0} a_m f_m \exp\left( \sum_{m > 0} a_m f_m \right) \right) \right) = \frac{1}{(x; x)_{\infty}} \times \prod_y (q^2 xy; x)_{\infty}^y.
\]

A.2 Neveu-Schwarz fermion

Operator product expansions

Operator product expansions of two and three Neveu-Schwarz fermion fields are

\[
\phi(w_1) \phi(w_2) = \langle \phi(w_1) \phi(w_2) \rangle + \phi(w_1) \phi(w_2) :,
\]

\[
\phi(w_1) \phi(w_2) \phi(w_3) = \langle \phi(w_1) \phi(w_2) \rangle \phi(w_3) - \langle \phi(w_1) \phi(w_3) \rangle \phi(w_2) \\
+ \langle \phi(w_2) \phi(w_3) \rangle \phi(w_1) + : \phi(w_1) \phi(w_2) \phi(w_3) :,
\]

where the singular part is

\[
\langle \phi(w_1) \phi(w_2) \rangle = \frac{\left( \frac{w_2}{w_1} \right)^{1/2} \left( 1 - \frac{w_2}{w_1} \right)}{\left( 1 - q^2 \frac{w_2}{w_1} \right) \left( 1 - q^{-2} \frac{w_2}{w_1} \right)}
\]

and the product \(\phi(w_1) \phi(w_2)\) is defined in a region \(q^2 \frac{w_2}{w_1}, |q^{-2} \frac{w_2}{w_1}| < 1\).
The normal order products are defined as usual by
\[ :\phi(w_1)\phi(w_2) : = \sum_{m,n} \mathcal{N}[^\phi_m^\phi_n] w_1^{-m} w_2^{-n} , \]
\[ :\phi(w_1)\phi(w_2)\phi(w_3) : = \sum_{m,n,l} \mathcal{N}[^\phi_m^\phi_n^\phi_l] w_1^{-m} w_2^{-n} w_3^{-l} \]
where \( \mathcal{N} \) is defined for two-products by
\[ \mathcal{N}[^\phi_m^\phi_n] = -^\phi_n^\phi_m \text{ if } m > 0, n < 0 \]
\[ = ^\phi_m^\phi_n \text{ otherwise} \]
and similarly for three-products; note that they are totally anti-symmetric; for example, \( :\phi(w_1)\phi(w_2) := -:\phi(w_2)\phi(w_1) :. \)

**Traces: Wick’s theorem**

The \( n \)-point function of fermion fields is written by a Pfaffian of a matrix whose entries are two-point functions (Wick’s theorem).

Introduce a notation
\[ \langle\langle A \rangle\rangle \equiv \frac{\text{tr}_{\mathcal{F}^\phi}(\xi^{-2d^\phi} A)}{\text{tr}_{\mathcal{F}^\phi}(\xi^{-2d^\phi})} . \]

Noting that \( [^\phi_m^\phi_n] = \pm^\phi_{m+n}(m > 0), \) \( \xi^{2d^\phi} ^\phi_m^\phi_{m+n} \xi^{-2d^\phi} = \xi^{\pm 2m} ^\phi_{m+n}(m > 0), \) we can derive the following:
\[ \langle\langle \phi(w_1)\phi(w_2) \rangle\rangle = \frac{1}{q + q^{-1}} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \frac{(w_2 - w_1)^m}{w_1} \frac{q^{2m} + q^{-2m}}{1 + \xi^{2m}} , \]
\[ \langle\langle \phi(w_1)\cdots\phi(w_n) \rangle\rangle = \begin{cases} \text{Pf } G & \text{if } n = \text{even;} \\ 0 & \text{if } n = \text{odd,} \end{cases} \]
where a \( n \times n \) anti-symmetric matrix \( G = (G_{ij}) \) is defined by \( G_{ij} = G(w_i, w_j) \equiv \langle\langle \phi(w_i)\phi(w_j) \rangle\rangle \) \( (i < j) \).
A.3 Ramond fermion

Operator product expansions

It seems convenient to define the singular part and the normal order product (artificially) by

\[ \langle \phi(w_1) \phi(w_2) \rangle = \frac{1}{2} \left( \frac{1 + \frac{w_2}{w_1}}{1 - q^2 \frac{w_2}{w_1}} \right), \]

so that the normal order products are totally anti-symmetric: for example, we have:

\[ \phi(w_1) \phi(w_2) : = - : \phi(w_2) \phi(w_1) :. \]

Operator product expansions are the same as for the NS fermion, given by (56), (57).

Traces: Wick’s theorem

We can write the n-point function of the Ramond fermion fields in terms of a Pfaffian; in this case the function for odd \( n \) does not vanish.

Define

\[ \langle \langle A \rangle \rangle = \frac{\text{tr} \mathcal{F}^{\phi_R}(\xi^{-2d^R} A)}{\text{tr} \mathcal{F}^{\phi_R}(\xi^{-2d^R})} \mathcal{F}^{\phi_R}_{\lambda}, \]

given in (14).

Direct computation yields \( \text{tr} \mathcal{F}^{\phi_R}(\xi^{-2d^R}) = (-\xi^2; \xi^2)_{\infty} \). We have

\[ \langle \langle \phi(w_1) \rangle \rangle = \langle \langle \phi_0 \rangle \rangle = \psi_0 \langle 1 \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \rangle = \epsilon \psi_0 \frac{(\xi^2; \xi^2)_{\infty}}{(-\xi^2; \xi^2)_{\infty}}, \]

\[ \langle \langle \phi(w_1) \phi(w_2) \rangle \rangle = \frac{1}{q + q^{-1}} \sum_{m \in \mathbb{Z}} \left( \frac{w_2}{w_1} \right)^m q^{2m} + q^{-2m} \frac{1 + \xi^{2m}}{1 + \xi^{2m}}, \]

and in general

\[ \langle \langle \phi(w_1) \cdots \phi(w_n) \rangle \rangle = \text{Pf} \ G^\pm \ \text{according to} \ n = \text{even, odd} \]
where if \( n \) = even \( G^+ \) is an anti-symmetric \( n \times n \) matrix with entries \( G^+_{ij} = \langle \langle \phi(w_i)\phi(w_j) \rangle \rangle \) for \( i < j \); if \( n \) = odd then \( G^- \) is an anti-symmetric \((n+1) \times (n+1)\) matrix

\[
G^- = \begin{pmatrix}
G^-_{ij} & i & \downarrow & 0,1,\ldots,n \\
\uparrow & j & \rightarrow & 0,1,\ldots,n
\end{pmatrix};
\]

\[
G^-_{ij} = \frac{1}{q+q^{-1}} \left( 1 + \sum_{m \in \mathbb{Z}_{\neq 0}} \left( \frac{w_j}{w_i} \right)^m q^{2m} + q^{-2m} \right), \quad 1 \leq i < j \leq n;
\]

\[
G^-_{0j} = \langle \langle \phi_0 \rangle \rangle, \quad 1 \leq j \leq n.
\]

We note that \([\phi_0, N^\phi_m] = 0, \xi^{2\phi^\phi} \phi_0 \xi^{-2\phi^\phi} = \phi_0\), and their consequence

\[
(1+(-1)^n\xi^{2m_1})\langle \langle \phi_{m_1} \cdots \phi_{m_n} \rangle \rangle = \sum_{j=2}^{n} (-1)^j \delta_{m_1+m_j,0}[\phi_{m_1}, \phi_{-m_1}] + \langle \langle \phi_{m_2} \cdots \phi_{m_n} \rangle \rangle;
\]

from this equation we can derive the last result.

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