Riemann-Hilbert Approach of the Complex Sharma-Tasso-Olver Equation and its $N$-soliton solutions*

Sha Li, Tiecheng Xia,† Jian Li‡

$^a$ Department of Mathematics, Shanghai University, Shanghai 200444, China

Email: xiatc@shu.edu.cn; lijianstud@163.com; lishastud@163.com

Abstract: In this paper, we use Riemann-Hilbert method to study the $N$-soliton solutions of the complex Sharma-Tasso-Olver (cSTO) equation. And then, based on analyzing the spectral problem of the Lax pair, the matrix Riemann-Hilbert problem for this integrable equation can be constructed, the $N$-soliton solutions about this system are given explicitly under the relationship of scattering matrix. At last, under the condition that some specific parameter values are given, the three-dimensional diagrams of the 2-soliton solution and the trajectory of the soliton solution will be simulated.

Keywords: the complex Sharma-Tasso-Olver equation; the Riemann-Hilbert problem; the scattering relationship; $N$-soliton solutions.

MSC codes: 35Q51; 35Q15; 37K10

1 Introduction

As we all know, the Sharma-Tasso-Olver (STO) equation

$$u_y + \frac{1}{2}(u_{xxx} - 3(u_x u)_x - 3u^2 u_x) = 0,$$  \hfill (1.1)

was first proposed to an example of odd members of Burgers hierarchy, which extends the "linearization" achieved via the Cole-Hopf ansatz to equations consisting of highest derivatives odd space derivatives. Moreover, due to the application of this equation in mathematical physics, many physicists and mathematicians showed great interest in it [1]. In this paper, we consider the complex Sharma-Tasso-Olver (cSTO) equation that was derived by Fan [2]

$$u_y + \frac{1}{2}(u_{xxx} - 3i(u_x u)_x - 3u^2 u_x) = 0,$$  \hfill (1.2)

and given the initial value $u(x, 0) = u_0(x) \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R}) = \{ f(x) \mid \int_{-\infty}^{\infty} [1 + |x|^p f(x)] dx < \infty, \ p > 1 \}$ is the Schwartz space. What’s more, with the fast development of the integrable

---

*The work is in part supported by the Natural Science Foundation of China under Grant No. 11975145.
†Corresponding authors. E-mail: xiatc@shu.edu.cn
‡Corresponding authors. E-mail: lijianstud@163.com
systems, it plays an important role in mathematical physics in recent years. In fact, it is significant to find the solutions and symmetries of the cSTO equation.

Therefore, constructing the $N$-soliton solutions of the cSTO equation via Riemann-Hilbert approach is very meaningful. Such problem goes back to 1997, when Fokas proposed the matrix Riemann-Hilbert problem (RHp) [3], which can be used not only to solve the initial-boundary value problems [4–9] and the long-time asymptotic behavior of the solutions [10], but also to obtain the solutions of the integral equations by the theory of inverse scattering [11–14]. It is because the Riemann-Hilbert approach is so widely used that it has developed into an important tool for solving some problems in applied mathematics and other areas.

Moreover, in addition to the Riemann-Hilbert method, there are bilinear method, Darboux transformation, etc., which are frequently used to solve soliton solutions of nonlinear partial differential equations, such as the multi-component AKNS integrable hierarchies [15], the $N$-coupled Hirota equations [16], the coupled nonlinear Schrödinger equations [17], the nonlocal Fokas-Lenells equation [18], the generalized Sasa-Satsuma equation [19], the coupled derivative Schrödinger equations [20], the (4+1)-dimensional Fokas equation [21], the (3+1)-dimensional extended Jimbo-Miwa equation [22] and so on.

The content of this paper is divided into the following parts. In the Section 2, firstly, based on the Lax pair of the cSTO equation, we have the Riemann-Hilbert problem for the cSTO equation, meanwhile, the variable transformation and the spectrum analysis can be applied in this process. In the Section 3, through Riemann-Hilbert method and the scattering relationship, the $N$-soliton solutions of the cSTO equation can be obtained in some detail, then under the results obtained above, we can give some three-dimensional figures for the case of some specific parameters. A brief conclusion for this paper and some discussion about this equation can be given in the last Section.

2 Fundamentals of the cSTO equation

Firstly, let’s introduce some of the basic of RHp:

**Definition 1.** Let the contour $\Gamma$ be the union of a finite number of smooth and oriented curves on the Riemann sphere $\mathbb{C}$, such that $\mathbb{C} \setminus \Gamma$ has only a finite number of connected components. Let $J(\lambda)$ be a $2 \times 2$ matrix defined on the contour $\Gamma$, the RHP $(\Gamma, J)$ is the problem that devotes to find a $2 \times 2$ matrix-valued function $M(\lambda)$ that satisfies:

1. $M(\lambda)$ is analytic for all $k \in \mathbb{C} \setminus \Gamma$, and extends continuously to the contour $\Gamma$;
2. $M_+(\lambda) = M_-(\lambda)J(\lambda)$, $\lambda \in \Gamma$;
3. $M(\lambda) \to I$, as $\lambda \to \infty$.

2.1 Lax pair for the cSTO equation

The cSTO equation admits the Lax pair

\[
\begin{align*}
\varphi_x &= M \varphi = (-i\lambda^2 \sigma_3 - \frac{1}{2}i u \sigma_3 + \lambda R)\varphi, \\
\varphi_y &= N \varphi = (-2i\lambda^0 \sigma_3 + U + V \sigma_3)\varphi, 
\end{align*}
\]
where \( R = \begin{pmatrix} 0 & u \\ 2 & 0 \end{pmatrix} \), define \( R = R^H \), the superscript \( H \) represents conjugate transpose, \( \sigma_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_- = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \), \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( \sigma_3 \) is the third Pauli’s matrix, \( U = 2\lambda^5 R + \lambda^3(iR_x + uR) + \lambda(-\frac{1}{2}R_{xx} + \frac{3}{8}i(R^3)_x - \frac{1}{2}iRR_x\sigma_- + \frac{1}{2}u^2R), V = -2i\lambda^4 u + \lambda^2(ux - iu^2) + \frac{1}{4}iu_{xx} + \frac{3}{4}uu_x - \frac{1}{2}iu^3 \), and \( \lambda \in \mathbb{C} \) is the spectral parameter. Then we can check that the compatibility condition of the Lax pair in (2.1) about the cSTO equation generates the zero-curvature equation \( M_y - N_x + [M, N] = 0 \), and it can yield the cSTO equation (1.2), the square bracket is the usual matrix commutator defined as \( [M, N] = MN - NM \) in the equation we mentioned above. In what follows, we’re going to extend the column vector \( \varphi \) to a \( 2 \times 2 \) matrix and introduce a matrix \( J_{2 \times 2} \) for convenience, so the formula can be shown

\[
J = \varphi e^{i\lambda^2 x \sigma_3 + 2i\lambda^6 y \sigma_3}, x \to \pm \infty, \tag{2.2}
\]

according to the Equ.(2.2), the Lax pair (2.1) can be rewritten as follows:

\[
\begin{align*}
J_x + i\lambda^2 [\sigma_3, J] &= M'J, \\
J_y + 2i\lambda^6 [\sigma_3, J] &= N'J,
\end{align*} \tag{2.3}
\]

where \( M' = -\frac{1}{2}i\lambda\sigma_3 + \lambda R, N' = U + V\sigma_3, [\sigma_3, J] = \sigma_3 J - J\sigma_3. \) Indeed, it can be written in full derivative form:

\[
d(e^{i(\lambda^2 x + 2\lambda^6 y)\sigma_3} J) = W, \tag{2.4}
\]

where

\[
W = e^{i(\lambda^2 x + 2\lambda^6 y)\sigma_3} (M' dx + N' dy). \tag{2.5}
\]

Here it should be noted that \( \hat{\sigma}_3 \) represents the matrix commutator of \( \sigma_3 \), i.e., \( \hat{\sigma}_3 \kappa = [\sigma_3, \kappa] \), and \( e^{\sigma_3(\kappa)} = e^{\sigma_3} \kappa e^{-\sigma_3} (\kappa \) is a \( 2 \times 2 \) matrix).

### 2.2 Associated Riemann-Hilbert problems for the cSTO equation

Since the initial-value \( u(x, 0) \) belongs to the Schwartz space, we can then get the asymptotic condition on the basis of the matrix spectral problems of the Lax pair (2.1), and have the following equation:

\[
\varphi \sim e^{-i\lambda^2 x \sigma_3 - 2i\lambda^6 y \sigma_3}, \tag{2.6}
\]

where \( x, y \to \pm \infty \). According to the equivalent variable transformation and \( tr(M') = tr(N') = 0 \), we can obtain that

\[
\det J = 1, \tag{2.7}
\]

founded on the generalized Liouville’s formula. Due to the importance of the spectrum problem of the cSTO equation, the analysis and symmetry of the spectral function should be put on the first step, and then, we can introduce the following two related Volterra integral equations:

\[
J_L(x, \lambda) = I + \int_{-\infty}^{x} e^{i\lambda^2 (x - \tau)\hat{\sigma}_3 (M'(\tau)J_L(\tau, \lambda))} d\tau, \tag{2.8}
\]

\[
J_R(x, \lambda) = I - \int_{x}^{\infty} e^{i\lambda^2 (x - \tau)\hat{\sigma}_3 (M'(\tau)J_R(\tau, \lambda))} d\tau, \tag{2.9}
\]
where $I$ is an identity matrix. Define
\[ J_L = (J_L^1, J_L^2) = \left( \begin{array}{cc} J_L^{11} & J_L^{12} \\ J_L^{21} & J_L^{22} \end{array} \right), \quad J_R = (J_R^1, J_R^2) = \left( \begin{array}{cc} J_R^{11} & J_R^{12} \\ J_R^{21} & J_R^{22} \end{array} \right). \]

where the first column of $J_L$ and $J_R$ are $J_L^1$ and $J_R^1$, $J_L^2$, $J_R^2$ are the second column of $J_L$ and $J_R$, respectively. According to the Volterra integral equation (2.8), the formula $e^{i\lambda^2(x-r)\sigma_3} M' e^{-i\lambda^2(x-r)\sigma_3}$ can be calculated, and then the following formula is obtained
\[ e^{i\lambda^2(x-r)\sigma_3} M' e^{-i\lambda^2(x-r)\sigma_3} = \left( \begin{array}{cc} -\frac{1}{2} i u & \lambda u e^{2i\lambda^2(x-r)} \\ 2\lambda e^{-2i\lambda^2(x-r)} & \frac{1}{2} i u \end{array} \right). \]

Given the above calculation results and the equation (2.10) defines the formula, we can obtain the facts that $J_L^1$ is analytic in $D_1 = \{ \lambda^2 | \text{Im} \lambda^2 < 0 \}$, $J_L^2$ is analytic in $D_2 = \{ \lambda^2 | \text{Im} \lambda^2 > 0 \}$, $J_R^1$ is analytic in $D_2$, $J_R^2$ is analytic in $D_1$, respectively. Actually, due to the Abel’s formula and $tr(M') = 0$, it can derive that the determinants of $J_L$ and $J_R$ are independent for all $x$,
\[ \begin{cases} J_L \rightarrow I, x \rightarrow -\infty, \\ J_R \rightarrow I, x \rightarrow +\infty. \end{cases} \]

From two matrix solutions of (2.12), we can express them as
\[ \varphi_L = J_L E, \varphi_R = J_R E, E = e^{i\lambda^2 x \sigma_3}, \]

because of their linear correlation, the formula about them can be put as follows
\[ J_L E = J_R ER(\lambda), \]

where $R(\lambda) = (r_{ij})_{2 \times 2}$ is the scattering matrix. In fact, it is evident that $\det R(\lambda) = 1$ as $\det J_L = 1$ and $\det J_R = 1$. In what follows, assuming that
\[ J_L^{-1} = \left( \begin{array}{cc} (J_L^1)^{-1} & (J_L^2)^{-1} \\ (J_L^{21})^{-1} & (J_L^{22})^{-1} \end{array} \right), \quad J_R^{-1} = \left( \begin{array}{cc} (J_R^1)^{-1} & (J_R^2)^{-1} \\ (J_R^{21})^{-1} & (J_R^{22})^{-1} \end{array} \right), \]

analytical analysis of each column vector of $J_L$ and $J_R$ according to the above. It is accessible that $(J_L^1)^{-1}$ is analytic in $D_2$, $(J_L^2)^{-1}$ is analytic in $D_1$, $(J_R^2)^{-1}$ is analytic in $D_2$, $(J_R^1)^{-1}$ is analytic in $D_1$. In accordance with the equation of (2.14), we can obtain that the analyticity of $R(\lambda)$ in the following
\[ ER(\lambda) E^{-1} = J_R^{-1} J_L = \left( \begin{array}{cc} (J_R^1)^{-1} J_L^{1} & (J_R^1)^{-1} J_L^{2} \\ (J_R^1)^{-1} J_L^{1} & (J_R^1)^{-1} J_L^{2} \end{array} \right), \]

indeed, $r_{11}$ is analytic in $D_1$, $r_{22}$ is analytic in $D_2$. Using Riemann-Hilbert method is very extensive and important for solving a class of equations, so we will construct the RHp for the cSTO equation in this section. First of all, we can introduce two matrix functions that consisted of $J_L$, $J_R$, $(J_L)^{-1}$, $(J_R)^{-1}$ and letting
\[ P_L = (J_L^1, J_L^2) = J_L L_1 + J_R L_2, \]
\[ P_R = (J_R^1, J_R^2) = (J_R^1)^{-1} (J_R^2)^{-1} = L_1 (J_L)^{-1} + L_2 (J_R)^{-1}, \]
where \( L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \) \( L_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \) Furthermore, it implies that \( P_R \) is analytic in \( D_1, \) \( P_L \) is analytic in \( D_2 \) and 
\[
\begin{align*}
P_R \to I \\
P_L \to I
\end{align*},
\]
where \( \lambda \to \infty. \) In the context of the previous preparation knowledge of the cSTO equation, the related Riemann-Hilbert problem can be given as below:

1. \( P_R \) is analytic in \( D_1, \) \( P_L \) is analytic in \( D_2, \) respectively;
2. \( P_LP_R = G(x,y;\lambda); \)
3. \( P \to I, \) when \( \lambda \to \infty, \)

where \( G(x,y;\lambda) = e^{(i\lambda^2x+2i\lambda^6y)\sigma_3} \begin{pmatrix} 1 & h_{12} \\ r_{21} & 1 \end{pmatrix} e^{-(i\lambda^2x+2i\lambda^6y)\sigma_3} \) is the jump matrix and \( R^{-1} = H = (h_{ij})_{1 \leq i,j \leq 2}. \) On account of \( \det R(\lambda) = 1, \) we can get that \( h_{11}r_{11} + h_{12}r_{21} = 1. \)

### 3 N-soliton solutions for the cSTO equation

In this section, through the Riemann-Hilbert problem, the cSTO equation can be given specific soliton solutions. Now let’s assume that the Riemann-Hilbert is irregular, and then, we can obtain that \( \det P_R \) and \( \det P_L \) make sense in their own analytic areas except for a few zeros. On the basis of the definitions of \( P_R, P_L \) and the scattering relationship (2.14), we can get the following form

\[
\begin{align*}
\det P_L &= \det(L_1J_L^{-1} + L_2J_R^{-1}) = h_{11}(\lambda), \\
\det P_R &= \det(J_LL_1 + J_RL_2) = r_{11}(\lambda),
\end{align*}
\]

(3.1)
it is obvious that the zeros of \( \det P_L \) and \( \det P_R \) are the same as \( h_{11}(\lambda) \) and \( r_{11}(\lambda), \) respectively. Furthermore, since the relation

\[
J_{R,L}^H(\lambda^*) = J_{R,L}^{-1}(\lambda),
\]

(3.2)
the following facts can be obtained that

\[
P_R^H(\lambda^*) = P_L^{-1}(\lambda), R^H(\lambda^*) = R^{-1}(\lambda), \lambda \in D_2.
\]

(3.3)
Indeed, according to the equation of (3.3), the formula that we can obtain

\[
h_{11}^H(\lambda^*) = r_{11}(\lambda),
\]

(3.4)
and each zero \( \pm \lambda_k^* \) of \( h_{11} \) corresponds to each zero \( \pm \lambda_k \) of \( r_{11}. \) Suppose that \( \det(P_R) \) exists \( N \) simple zeros \( \tilde{\lambda}_1 \cdots \tilde{\lambda}_N \in D_1 \) and \( \det(P_L) \) exists \( N \) simple zeros \( \lambda_1^* \cdots \lambda_N^* \in D_2, \) then, it is shown that

\[
\lambda_k^* = \tilde{\lambda}_k, \ 1 \leq k \leq N.
\]

(3.5)
Now, assume that \( \eta_k \) is nonzero column vector and \( \eta_k^* \) is nonzero row vector. What follows presents the linear equations about \( \eta_k \) and \( \eta_k^*. \)

\[
\begin{align*}
P_R(\lambda_k)\eta_k(\lambda_k) &= 0, \\
\eta_k^*(\lambda_k^*)P_L(\lambda_k^*) &= 0.
\end{align*}
\]

(3.6)
From the formulas above, we can get the relation of the eigenvectors

$$\eta^*_k = \eta^H_k, \ 1 \leq k \leq N,$$

and

$$\begin{align*}
P_R(\lambda_k)\left(\frac{\partial \eta_k}{\partial x} - i\lambda^2 k \sigma_3 \eta_k\right) &= 0, \\
P_R(\lambda_k)\left(\frac{\partial \eta_k}{\partial y} - 2i\lambda^2 k \sigma_3 \eta_k\right) &= 0.
\end{align*}$$

(3.7)

Comparing with the Lax pair (2.3), (3.6) and (3.7), we can get the expression of $$\eta_k$$ and $$\eta^*_k$$

$$\begin{align*}
\eta_k &= e^{i\lambda^2 k x \sigma_3 + 2i\lambda^2 k y \sigma_3} \eta_{k,0}, \ 1 \leq k \leq N, \\
\eta^*_k &= \eta^H_{k,0} e^{-i(\lambda^*_k)^2 x \sigma_3 - 2i(\lambda^*_k)^2 y \sigma_3}, \ 1 \leq k \leq N,
\end{align*}$$

(3.9)

where $$\eta_{k,0} = (\alpha_k, \beta_k)^T (1 \leq k \leq N)$$ is two-dimension constant column vector. When the scattering data satisfies the condition $$r_{21} = 0$$, the soliton solutions can be presented explicitly. Hence, the solutions of the Riemann-Hilbert problem can be given in the following

$$\begin{align*}
P_R(\lambda) &= I - \sum_{k,j=1}^{N} \frac{\eta_k \eta^*_j (D^{-1})_{kj}}{\lambda - \lambda^*_j}, \\
P_L(\lambda) &= I + \sum_{k,j=1}^{N} \frac{\eta_k \eta^*_j (D^{-1})_{kj}}{\lambda - \lambda_k},
\end{align*}$$

(3.10)

where $$D$$ is a $$N \times N$$ matrix that can be defined by

$$d_{kj} = \frac{\eta_k \eta_j}{\lambda_j - \lambda^*_k}, \ 1 \leq k, j \leq N.$$  

(3.11)

In fact, we can express the asymptotic expansion of $$P_R(\lambda)$$ as

$$P_R = I + \frac{P^1_R}{\lambda} + \frac{P^2_R}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \lambda \in D_1 \to \infty.$$  

(3.12)

Moreover, the equation of (3.9) can be substituted into the first equation of (2.3), and then, the formula we can get as below

$$u = 2i(P^1_R)_{12},$$

(3.13)

where $$(P^1_R)_{12}$$ is the $$(1, 2)$$-element of $$P^1_R$$. Meanwhile, through the equation of (3.10), we can deduce the expression of $$P^1_R$$ as follows

$$P^1_R = -\sum_{k=1}^{N} \sum_{j=1}^{N} \eta_k \eta^*_j (D^{-1})_{kj}.$$  

(3.14)

And so we can obtain the $$N$$-soliton solutions of the cSTO equation (1.1)

$$u(x, y) = -2i \sum_{k=1}^{N} \sum_{j=1}^{N} \eta_k \eta^*_j (D^{-1})_{kj},$$

(3.15)
then, under the condition of the formula (3.15) established above, when \( N = 2 \), the 2-soliton solution of the cSTO equation (1.1) can be given by

\[
u(x, y) = -\frac{2i}{d_{11}d_{22} - d_{12}d_{21}}(\alpha_1\beta_2^*d_{12}e^{\vartheta_2 - \vartheta_1} - \alpha_1\beta_1^*d_{22}e^{\vartheta_1 - \vartheta_1} + \alpha_2\beta_1^*d_{21}e^{\vartheta_1 - \vartheta_2} - \alpha_2\beta_2^*d_{11}e^{\vartheta_2 - \vartheta_2}),
\]

where

\[
d_{11} = \frac{1}{\lambda_1 - \lambda_2^*}(|\alpha_1|^2e^{-\vartheta_1 - \vartheta_1} + |\beta_1|^2e^{\vartheta_1 + \vartheta_1}),
\]

\[
d_{12} = \frac{1}{\lambda_2 - \lambda_1^*}(\alpha_1^*\alpha_2e^{-\vartheta_1 - \vartheta_2} + \beta_2^*\beta_1e^{\vartheta_1 + \vartheta_2}),
\]

\[
d_{21} = \frac{1}{\lambda_1 - \lambda_2^*}(\alpha_2^*\alpha_1e^{-\vartheta_2 - \vartheta_1} + \beta_1^*\beta_2e^{\vartheta_2 + \vartheta_1}),
\]

\[
d_{22} = \frac{1}{\lambda_2 - \lambda_1^*}(\alpha_2^*\alpha_2e^{-\vartheta_2 - \vartheta_2} + |\beta_2|^2e^{\vartheta_2 + \vartheta_2})
\]

and

\[
\vartheta_1 = i\lambda_1^2x + 2i\lambda_1^6y,
\]

\[
\vartheta_2 = i\lambda_2^2x + 2i\lambda_2^6y.
\]

What’s more, under the condition that some specific parameter values are given, some three-dimensional plots of the 2-soliton solutions and the trajectory of the soliton solution can be shown in Figure 1 and Figure 2.

Figure 1: Evolution plot of the 2-soliton solutions to the cSTO equation with parameters by \( \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = -i, \alpha = 1, \lambda_1 = -\frac{1}{2} - \frac{1}{5}i, \lambda_2 = -\frac{3}{5} - \frac{2}{5}i \).

Figure 2: The soliton of \(|u|\) along x axis with different y.

7
4 Conclusion and Discussion

In this paper, according to the Riemann-Hilbert method, the $N$-soliton of the cSTO equation can be constructed. And the next step is to try to generalize the cSTO equation in multiple components, such as the multi-component generalization of the Kundu-NLS equation [23]. In fact, the cSTO equation is a kind of significant equation in mathematical physics, so it is very meaningful to do something similar to the STO equation. Moreover, we can use Riemann-Hilbert method to solve nonlocal integrable equations and integrable hierarchies. Therefore, it is important to continue to study it. Meanwhile, the mathematical structure and physical properties of the cSTO equation are also need to study.

5 Conflict of Interest

The authors declare that they have no conflict of interest.

6 Data availability statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

References

[1] LOU, S.Y.: Dark parameterizations, equivalent partner fields and integrable systems. Comm. Theor. Phys. 55, 743-746 (2011)

[2] Fan, E.G.: A family of completely integrable multi-Hamiltonian systems explicitly related to some celebrated equations. J. Math. Phys. 42, 4327-4344 (2001)

[3] Fokas, A.S.: A unified transform method for solving linear and certain nonlinear PDEs. Proc. R. Soc. Lond. A 453, 1411-1443 (1997)

[4] Zhang, N., Xia, T.C., Hu, B.B.: A Riemann-Hilbert approach to the complex Sharma-Tasso-Olver equation on the half-line. Commun. Theor. Phys. 68, 580-594 (2017)

[5] Fokas, A.S., Its, A.R., Sung, L.Y.: The nonlinear Schrödinger equation on the half-line. Nonlinearity 18, 1771-1822 (2005)

[6] Xu, J., Fan, E.G.: The three-wave equation on the half-line. Phys. Lett. A 378, 26-33 (2014)

[7] Deift, P., Zhou, X.: A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation. Ann. Math. 137, 295-368 (1993)

[8] Hu, B.B., Xia, T.C., Ma, W.X.: Riemann-Hilbert approach for an initial-boundary value problem of the two-component modified Korteweg-de Vries equations on the half-line. Appl. Math. Comput. 332, 148-159 (2018)
[9] Hu, B.B., Xia, T.C., Zhang, N., Wang, J.B.: Initial-boundary value problems for the coupled higher-order nonlinear Schrödinger equations on the half-line. Int. J. Nonlin. Sci. Num. 19, 83-92 (2018)

[10] Wang, D.S., Wang, X.L.: Long-time asymptotics and the bright N-soliton solutions of the Kundu-Eckhaus equation via the Riemann-Hilbert approach. Nonlinear Anal.-Real. 41, 334-361 (2018)

[11] Wazwaz, A.M.: Two-mode fifth-order KdV equations: necessary conditions for multiple-soliton solutions to exist. Nonlinear Dyn. 87, 1685-1691 (2017)

[12] Wazwaz, A.M.: Multiple soliton solutions and multiple complex soliton solutions for two distinct Boussinesq equations. Nonlinear Dyn. 85, 731-737 (2016)

[13] Guo, H.D., Xia, T.C.: Multi-soliton solutions for a higher-order coupled nonlinear Schrödinger system in an optical fiber via Riemann-Chi Hilbert approach. Nonlinear Dyn. 103, 1805-1816 (2021).

[14] Li, Y., Li, J., Wang, R.Q.: Multi-soliton solutions of the N-component nonlinear Schrödinger equations via Riemann-Hilbert approach. Nonlinear Dyn. 105, 1765-1772 (2021)

[15] Ma, W.X.: Application of the Riemann-Hilbert approach to the multicomponent AKNS integrable hierarchies. Nonlinear Anal. 47, 1-17 (2019)

[16] Kang, Z.Z., Xia, T.C.: Construction of multi-soliton solutions of the N-coupled Hirota equations in an optical fiber. Chin. Phys. Lett. 36, 110201 (2019)

[17] Wang, D.S., Yin, S.J., Tian, Y., Liu, Y.F.: Integrability and bright soliton solutions to the coupled nonlinear Schrödinger equations with higher-order effects. Appl. Math. Comput. 229, 296-309 (2014)

[18] Li, J., Xia, T.C.: N-soliton solutions for the nonlocal Fokas-Lenells equation via RHP. Appl. Math. Lett. 113, 106850 (2021)

[19] Geng, X.G., Wu, J.P.: Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation. Wave Motion 60, 62-72 (2016)

[20] Guo, B.L., Ling, L.M.: Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation. J. Math. Phys. 53, 073506 (2012)

[21] Li, W., Liu, Y.P.: To construct lumps, breathers and interaction solutions of arbitrary higher-order for a (4+1)-dimensional Fokas equation. Mod. Phys. Lett. B 34, 2050221 (2020)

[22] Cui, W.Y., Li, W., Liu, Y.P.: Multiwave interaction solutions for the (3+1)-dimensional extended Jimbo-Miwa equation. Mod. Phys. Lett. B 34, 2050405 (2020)

[23] Li, J., Xia, T.C.: A Riemann-Hilbert Approach to the Kundu-nonlinear Schrödinger equation and its multi-component generalization. J. Math. Anal. Appl. 500, 125109 (2021)