N=2 supersymmetric QCD and Heun equation

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Abstract

We investigate the relation between the four dimensional N=2 SU(2) super Yang-Mills theory with four fundamental flavors and the quantum mechanics model with Treibich-Verdier potential described by the Heun equation in the elliptic form. We study the precise correspondence of quantities in the gauge theory and the quantum mechanics model. An iterative method is used to obtain the asymptotic expansion of the spectrum for the quantum model, we are able to fix the precise relation between the energy spectrum and the instanton partition function of the gauge theory. We also study asymptotic expansions for the Heun equation which correspond to the strong coupling regions of the Seiberg-Witten theory.
1 Introduction

Recently we have witnessed some surprising connection between supersymmetric gauge theories, conformal field theories and integrable theories. Among the many implications of physical and mathematical interests, we gain some new understanding about the nonperturbative dynamics of quantum gauge theories. The four dimensional $N = 2$ super Yang-Mills theory provides a major playground for these connections since the development of the Seiberg-Witten theory\cite{1,2}. Its connections to string theory, classical integrable system were among the major concerns for physics interest\cite{3,4,5}. The instanton counting\cite{6} provides an essential quantum field theory explanation for the Seiberg-Witten solution, and as recently revealed it also provides a quantum generalization for the corresponding integrable system\cite{7}. The relation of 4D gauge theory/quantum integrable model is part of the program
recently outlined by Nekrasov and Shatashvili, relating the vacuum space of various gauge theories to the Bethe states of some quantum integrable systems[8].

In some very simple cases, the integrable system reduces to some simple quantum mechanical problem. Apply the conventional methods of quantum physics, we can compute the quantum corrections, through this kind of quantization we gain a deformed version of the Seiberg-Witten theory, therefore in this context the deformation in the Nekrasov partition function has a clear physical meaning.

The relation between gauge theories and integrable systems has various interesting implications, it reveals some structures not explicitly manifested in the original formulation of the two kinds of theories. However, for the moment our understanding of their relation is based on some sporadic examples we have studied, there is not a dictionary allowing us to build the precise relation between a given gauge theory and an integrable system, or vice versa. Therefore a closer study of some particular examples is still worthwhile to better understand how the correspondence works.

In this paper we investigate one example of the gauge theory/integrable model correspondence, namely the N=2 SU(2) gauge theory with four flavors in the fundamental representation, i.e. SU(2) N_f = 4 QCD, and the relation to the BC1 Calogero-Inozemtsev (CI) model, i.e. the quantum mechanical model described by the Heun differential equation[9, 10]. Our goal of the investigation is to find the precise relation between certain quantities on the two sides, especially to match the infrared dynamics of the gauge theory and the eigenvalue spectrum of the Heun equation.

The direct connection between the two theories was not noticed during the study of classical integrability of Seiberg-Witten theory[11], recently it appears to be clear partly due to the relation between gauge theories and CFT which provides another perspective to the gauge theory/integrable model subject[12]. The Heun equation in the elliptic form describes a quantum particle in the Treibich-Verdier potential[13], which is an interesting topic in its own right. The potential, with its first appearance in Darboux’s work[14], is a notable object in modern study of dynamical system. It belongs to the so called integrable finite-gap potential, has close relation to KdV soliton theory and algebraic-integrability theory[15]. We hope the study about its relation to SYM theory would enrich the subject.

The quantum mechanical model, as we call it, is not Hermitian, however it fairly makes sense in the context of algebraic integrable system. The methods of the Hermitian quantum mechanics apply as well, actually the tools of complex analysis are very useful, as we show in the subsequent sections. We use “quantum correction” to refer to the ε-corrections to both the quantum CI model and the gauge theory.

The plan of the paper is the following.
The Section 2 is devoted to some necessary background. We briefly explain how the gauge theory is related to the Heun equation through the AGT correspondence, hence we can identify the relation between the mass parameters of gauge theory and the coupling strength of the Treibich-Verdier potential, given in (7). The couplings take the form of finite-gap potential. We explain an exact WKB method for some linear spectrum problems. For the Heun equation, a shift of the parameters, manifested as a relation between the WKB expansion of two functions \( \Theta \) and \( \Xi \) in (19), is useful in our discussion.

In the Section 3 we first apply the exact WKB quantization method to obtain the spectrum of the Treibich-Verdier quantum model for the case when the kinetic energy is very large compared to the potential. The contour integral of the WKB perturbation is carried out by applying an iterative method, we obtain the perturbative expansion for the integral for general mass parameters. We then relate the perturbative spectrum of the Treibich-Verdier potential and the low energy solution of the SU(2) \( N_f = 4 \) QCD. The relation between the energy eigenvalue and the prepotential of gauge theory is given in (37), by matching low order perturbations on the two sides. A consistent shift argument allows us to extend the match to higher order perturbations, therefore predict the full quantum spectrum for the Heun equation from the Nekrasov partition function of the super QCD, given in (42).

In the Section 4 we study another perturbation expansion for the spectrum where the kinetic energy is small compared to the potential. It corresponds to the strong coupling expansion of the gauge theory, the procedure of performing the WKB integral follows the spirit of the Seiberg-Witten theory. We present the simpler case of equal mass parameter in this section and leave the general case to the Appendix 8.

In the Section 5 we discuss various limit cases, here we can relate our results to some earlier literature.

In the Appendix 7 we give the Heun equation in different forms, in accordance with the convention we use, they are suitable to explain different aspects about its relation to the gauge theory and the quantum mechanical model. In the Appendix 8 we explain the iterative method to solve the polynomial equation \( P_4(z) = 0 \) in order to carry out the elliptic integral. The iterative method gives a workable way to compute the strong coupling expansions of the Seiberg-Witten theory, including the \( \epsilon \)-correction if taking the Nekrasov-Shatashvili(NS) limit of the Omega background. Applying this method we obtain the expansion for generic mass, given in (90). The Appendix 9 includes a bit more explanation about the shift arguments in the Section 3.
2 SU(2) $N_f = 4$ super QCD and the Heun equation

2.1 The equation from AGT correspondence

There is not an obvious way to derive the Heun equation from any aspects of the gauge theory, it appears through a recently discovered relation between $N=2$ SU(2) gauge theory and Liouville CFT, the Alday-Gaiotto-Tachikawa correspondence\[12\]. The AGT states a relation between Nekrasov partition function of $N = 2$ gauge theory and the chiral part of the conformal block of Liouville CFT, both associated to certain punctured Riemann surfaces. For SU(2) $N_f = 4$ supersymmetric QCD, the relevant instanton partition function for gauge theory in the $\Omega$ background is related to the 4-point conformal block on the sphere.

In CFT theory, the degenerate operators satisfy constraint conditions of Virasoro generators. Inserting a degenerate operator in the correlator results in certain differential equation. Sometimes these differential equations are very useful, for example, along this line in the minimal models of CFT the 4-point correlator including one degenerate operator can be solved through the hypergeometric equation\[16\]. In the AGT context, we can do the similar thing for Liouville CFT, and the corresponding gauge theory is also affected. It is argued that the CFT correlator with an additional degenerate operator inserted is related to the $N=2$ gauge theory with surface operator\[17\], and the resulting conformal block/Nekrasov partition function in the NS limit is related to the eigenfunction of the corresponding quantum integrable system\[18\].

In the case of 4-point correlator with an extra degenerate operator constrained by level two generators, the procedure results in a second order differential equation for the 5-point conformal block, in the NS limit it is the normal form of the Heun equation\[19\],

\[( - \epsilon^2 \partial_z^2 + U(z, m, \epsilon, \Theta) ) \Phi(z) = 0. \tag{1} \]

where

\[
U(z) = \frac{\tilde{m}_1^2 - \frac{\epsilon^2}{4} + m_1 (m_1 - \epsilon)}{z^2} + \frac{m_0 (m_0 - \epsilon)}{(z - q)^2} + \frac{m_0 (m_0 - \epsilon)}{(z - 1)^2} - \frac{m_0 (m_0 - \epsilon) + m_1 (m_1 - \epsilon) - \tilde{m}_0^2 + \tilde{m}_1^2}{z(z - 1)} - \frac{(1 - q) \Theta}{z(z - q)(z - 1)}. \tag{2}
\]

in this equation $z$ is the position of the degenerate operator and it takes complex value, and $\Theta$ is the accessory parameter. All the parameters are related to gauge theory quantities through the AGT correspondence. $m_0, \tilde{m}_0, m_1, \tilde{m}_1$ are parameters determining the conformal weight of the four primary operators of CFT, they are related to the physical mass of the
four flavors \(\mu_i\) in gauge theory as

\[
m_0 = \frac{1}{2}(\mu_1 + \mu_2), \quad \tilde{m}_0 = \frac{1}{2}(\mu_1 - \mu_2),
\]

\[
m_1 = \frac{1}{2}(\mu_3 + \mu_4), \quad \tilde{m}_1 = \frac{1}{2}(\mu_3 - \mu_4).
\]

The singularity parameter \(q\) is the cross ratio of the position of the four non-degenerate primary operators in CFT, identical to the UV coupling of the super QCD theory. We have set the \(\Omega\) background deformation to the NS limit: \(\epsilon_1 = \epsilon, \epsilon_2 = 0\), which is relevant for the quantum integrability.

We can further rewrite the equation (1) in the elliptic form. Define the new coordinate

\[
z = \frac{\wp(x) - e_2}{e_1 - e_2},
\]

and the new function \(W(x)\) by

\[
\Phi(z) = (\wp(x) - e_1)^{-\frac{m_0}{2}}(\wp(x) - e_2)^{-\frac{\tilde{m}_0}{2}}(\wp(x) - e_3)^{-\frac{m_1}{2}}(\wp(x) - e_4)^{-\frac{\tilde{m}_1}{2}}W(x).
\]

where \(\wp(x)\) is the double periodic Weierstrass elliptic function, \(2\omega_1, 2\omega_2\) are the periods of \(\wp(x)\), and \(\omega_0 = 0, \omega_3 = \omega_1 + \omega_2\). The periods determine the module by \(\tau = \frac{\omega_2}{\omega_1}\), and the modulus \(q = \exp(2\pi i \tau)\) is given by \(q = \frac{e_3 - e_2}{e_1 - e_2}\) where \(e_1, e_2, e_3 = \wp(\omega_1, \omega_2)\). Then we get the following form of the equation,

\[
\frac{d^2}{dx^2}W + \left(E - \sum_{i=0}^{3} b_i\wp(x + \omega_i)\right)W = 0.
\]

It takes the form a linear differential equation for an eigenvalue problem, the multi-component elliptic potential is the Treibich-Verdier potential. The Heun equation is equivalent to the quantum BC1 Calogero-Inozemtsev model which is the simplest case of integrable models of Calogero type associated to the \(BC_N\) Lie algebra. The coupling strengths are related to the gauge theory mass parameters,

\[
b_0 = (2\tilde{m}_0 - \frac{1}{2})(2\tilde{m}_0 + \frac{1}{2}), \quad b_1 = (2m_0 - \frac{3}{2})(2m_0 - \frac{1}{2}),
\]

\[
b_2 = (2\tilde{m}_1 - \frac{1}{2})(2\tilde{m}_1 + \frac{1}{2}), \quad b_3 = (2m_1 - \frac{3}{2})(2m_1 - \frac{1}{2}).
\]

In the AGT paper [12] the flavors \(\mu_1, \mu_2\) are in the antifundamental representation and \(\mu_3, \mu_4\) are in the fundamental representation. In the Nekrasov partition function, the fundamental matter of mass \(\mu\) contributes the same factor as the antifundamental matter with mass \(\epsilon_1 + \epsilon_2 - \mu\) contributes. Therefore the parameters in [12], in the NS limit, are related to ours by \(m_0 = \epsilon - m_{agt}, \tilde{m}_0 = -m_{agt}, m_1 = m_{agt}, \tilde{m}_1 = \tilde{m}_{agt}\). But note that they appear in the equation as \(m_0(m_0 - \epsilon)\) and \(\tilde{m}_0^2 - \frac{\epsilon^2}{4}\) unaffected.

The identification of mass parameters to the \(\gamma, \eta, \lambda\) in the equation is not unique, see the Appendix [17]. The coefficients for the equation in the elliptic form depends on the identification. We choose the first column identification in [17].
The eigenvalue $E$ is related to the accessory parameter by

$$E = 4(e_1 - e_3)\frac{\Theta}{\epsilon^2} + f\left(\frac{m_0}{\epsilon}, \frac{m_1}{\epsilon}, \frac{\tilde{m}_0}{\epsilon}, \frac{\tilde{m}_1}{\epsilon}, e_{1,2,3}\right).$$

(8)

where the precise form of the function $f$ can be recovered from (68), (77) in Appendix 7. Therefore we also call $\Theta$ as energy eigenvalue although the normal form of the Heun equation does not display it as eigenvalue. $\Theta$ is a function of $m_2^0, m_2^1, \tilde{m}_2^0, \tilde{m}_2^1, q, \epsilon$, and the quantum number we call it $\nu$, its precise form can be computed by the WKB method and related to gauge theory partition function.

Note that the map (4) is not one-to-one, in the complex plane every fundamental region of the $x$-plane is mapped to the whole complex plane of $z$. The points of $x = \omega_1, \omega_2, \omega_3, 0$, module periods, are mapped to $z = 1, 0, q, \infty$, respectively.

Therefore we get the direct relation between the supersymmetric QCD theory and the quantum model of Treibich-Verdier potential. Actually, the Heun equation satisfied by the 5-point conformal block with one degenerate operator was already discovered for a different purpose, in the Weierstrass’s form [20]. A previous study relevant to the Heun equation is in [21], with input from the AGT correspondence.

### 2.2 Integrable finite-gap potentials

The following linear spectrum problem for the potential $u(x)$,

$$-\psi'' + u(x)\psi = E\psi,$$

(9)

is an outstanding source of knowledge for subjects from classical functional analysis to quantum physics and integrable theory. In real analysis, a particular interesting case is when then potential is periodic, $u(x) = u(x + T)$, then the solution of the equation takes the form of Floquet-Bloch wave function,

$$\psi(x + T) = e^{i\nu T}\psi(x).$$

(10)

For every quantum number $\nu$ the spectrum has a solution $E(\nu)$, however the energy must locates in the allowed zones, with the boundaries determined by the periodic and anti-periodic solution $E(\nu_*)$, with $\nu_*$ determined by

$$e^{i\nu_* T} = \pm 1.$$

(11)

Other regions are the forbidden zones where there are no stable solutions. As one dimensional model for electrons in a lattice, the gap structure is responsible for the electroconductivity of materials.
Generally, the width of energy gaps decreases as the energy increases, it is possible somewhere in the spectrum the gap disappears. If moreover the spectrum is bounded from below, then the number of forbidden zones is a finite number, and the potential is called finite-gap potential.

Later the finite-gap potentials are studied in the context of algebraic integrability theory where analytical tools of Riemann surface are available\[15\]. The Hamiltonian system of the KdV hierarchy generates a family of elliptic finite-gap potentials by the fact that (quasi)periodic solutions of the stationary higher KdV equations are finite-gap potentials\[22, 23\]. The $g$-gap potentials can be written in term of the Riemann Theta function on the genus $g$ surface\[24\], and they are isospectral under the KdV flows.

Interestingly, the elliptic potentials related to 4D SU(2) SYM theories are among the most studied finite-gap potentials. The Lamé potential, acquiring the name form the Lamé equation,

$$u(x) = g(g + 1)\wp(x), \quad (12)$$

is a finite-gap potential with $g$ forbidden zones in the spectrum when $g$ is an integer, a result due to E. L. Ince. The Lamé equation is related to the mass deformed SU(2) $N = 4$ Yang-Mills theory($N = 2^*$ theory), we have analysed their relation in \[31\].

The Treibich-Verdier potential\[13\], a generalization of the Lamé potential, is another finite-gap potential when the coupling coefficients $b_i$ take the following form,

$$u(x) = \sum_{i=0}^{3} g_i(g_i + 1)\wp(x + \omega_i), \quad (13)$$

and $g_i$ are integers. When the couplings are identified with the mass parameters of the supersymmetric QCD, the coupling constants $b_i$ in (7) indeed can be written in the form above, but we have no reason to demand $g_i$ to be integers. It is interesting to consider if there is a deeper relation between the gauge theory in the Omega background and the integrable finite-gap potential concerned about this point.

There are few ways to reduce the Treibich-Verdier potential to the Lamé potential.

- The obvious one is by turning any three parameters among $b_0, b_1, b_2, b_3$ to zero.
- We can also turn them to $b_0 = b_1 = b_2 = b_3 = b$ and use the duplication formula for the elliptic function,

$$4\wp(2x) = \wp(x) + \wp(x + \omega_1) + \wp(x + \omega_2) + \wp(x + \omega_3). \quad (14)$$

- The less obvious case is when two of the four coefficients are zero and two others are the same, such as $b_0 = b_1 = b, b_2 = b_3 = 0$(but except $b_0 = b_3, b_1 = b_2 = 0$ and $b_0 = b_3 = 0, b_1 =$
We can reduce the potential in this class to the Lamé potential because we have the following relation by the Landen transformation,

\[
\wp(x; \omega_1, 2\omega_2) = \wp(x; 2\omega_1, 2\omega_2) + \wp(x + \omega_1; 2\omega_2, 2\omega_2) - \wp(\omega_1; 2\omega_1, 2\omega_2),
\]

\[
\wp(x; 2\omega_1, \omega_2) = \wp(x; 2\omega_1, 2\omega_2) + \wp(x + \omega_2; 2\omega_2, 2\omega_2) - \wp(\omega_2; 2\omega_1, 2\omega_2),
\]

we keep the dependence on periods explicit to emphasize the change of the modulus \( \tau = \omega_2/\omega_1 \). A further scaling limit reduces the Lamé equation the Mathieu equation, related to the \( N = 2 \) SU(2) pure Yang-Mills theory.

At this point it is also worth to mention that the \( BC_1 \) CI model is related to the sixth Painlevé equation. The sixth Painlevé equation can be written in the form of a time dependent Hamiltonian system whose potential is the Treibich-Verdier potential[25]. Accordingly, the sixth Painlevé equation is a non-autonomous version of the classical \( BC_1 \) CI model, and in fact the relation to classical Hamiltonian system can be generalised to all other Painlevé equations[26].

### 2.3 The linear spectrum problem and exact WKB method

For a linear spectrum problem, a standard method to obtain the perturbative solution is the WKB method when there is a expansion parameter \( \epsilon \). For a potential with turning points \( x_1, x_2 \), the quantization condition for the exact wave function is

\[
\frac{1}{\pi} \int_{x_1}^{x_2} \sum_{n=0}^{\infty} \epsilon^n p_n(E, u(x)) dx = \nu,
\]

it gives the relation between the eigenvalue and the quantum number \( \nu \) and other parameters. However, for a general potential on the real axis the higher WKB components \( p_n \) contain nonintegrable singularities at the turning points. Some efforts were devoted to overcome this difficult to achieve the exact WKB quantization method [27, 28]. Two useful tricks are introduced:

1. We should extend the integrals to the complex plane, with branch cuts determined by the equation itself. So the quantization condition becomes a contour integral along the branch cut.

2. In the contour integral, we can trade the higher order integrands \( p_n \) by functions of derivatives with respect to the energy. The integral and derivative operations are commutative. Therefore the integration can be computed from the less singular integrands whose singularities become integrable.

This kind of exact WKB method works effectively for the periodic potential \( u(x) = \cos x \) related to SU(2) pure SYM theory, as demonstrated in [29]. The contour integral of the
WKB component $p_0$ is directly related to the contour integral of the Seiberg-Witten form in gauge theory, the integrals along different branch cuts correspond to integrals along the conjugate contours of the Seiberg-Witten curve. The higher order WKB corrections are identified with the Omega background deformation of the gauge theory in the NS limit [6].

The exact WKB method is used to investigate the Mathieu and the Lamé equation which are related to two kinds of gauge theory models [30, 31]. We learn that for a quantum particle moving in an elliptic potential $u(x) = g(g + 1)\wp(x)$, at every stationary point of the potential there is an asymptotic expansion for the eigenvalue and eigenfunction. Correspondingly, the stationary points of the potential are related to the weak and strong coupling regions of the gauge theory where asymptotic expansions for the effective action is available and they are related by electric-magnetic duality. This paper is a natural continuation of our previous work, here we will analyse the eigenvalue expansion of the Heun equation, expanded as the WKB series.

**Application to the Heun equation**

We start the WKB analysis from the normal form the Heun equation (1), because the integral turns out to be simpler than other forms of the equation. However, the potential $U(z, m, \epsilon, \Xi)$ involving the Plank constant $\epsilon$ makes the WKB analysis more complicated. There is a simple way to take into account of this effect. We do not need to work on the Schrödinger equation (1), instead we start from the Schrödinger equation with the following potential $V(z, m, \Xi)$,

$$(-\epsilon^2 \partial_z^2 + V(z, m, \Xi))\Psi(z) = 0,$$

where

$$V(z) = \frac{\tilde{m}_1^2}{z^2} + \frac{m_1^2}{(z-q)^2} + \frac{m_0^2}{(z-1)^2} - \frac{m_0^2 + m_1^2 - \tilde{m}_0^2 + \tilde{m}_1^2}{z(z-1)} - \frac{(1-q)\Xi}{z(z-q)(z-1)}.$$

The masses $m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2$ are free parameters, they appear in the eigenvalue $\Xi$ and the corresponding eigenfunction. We can shift them by any quantity $\delta m_0^2, \delta m_1^2, \delta \tilde{m}_0^2, \delta \tilde{m}_1^2$, then the eigenvalue and eigenfunction with the new parameters $m_0^2 + \delta m_0^2, m_1^2 + \delta m_1^2, \tilde{m}_0^2 + \delta \tilde{m}_0^2, \tilde{m}_1^2 + \delta \tilde{m}_1^2$ still satisfy the Schrödinger equation (17). This is of course true when the shifts take the special values: $\delta m_0^2 = -\epsilon m_0, \delta m_1^2 = -\epsilon m_1, \delta \tilde{m}_0^2 = -\epsilon^2/4, \delta \tilde{m}_1^2 = -\epsilon^2/4$. This fact is already clear in our study of N=2* gauge theory [31] where the adjoint mass appears in the shifted form as $m(m - \epsilon)$.

Therefore, we can solve the eigenvalue problem of the Schrödinger equation (17) with potential (18), we get the eigenvalue as function of a quantum number $\nu$ and all other parameters, $\Xi(\nu, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon)$. Then the eigenvalue $\Theta$ of equation (1) with the potential
takes the same functional form as $\Xi$ but with the mass parameters shifted,
\[
\Theta(\nu, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon) = \Xi \left( \nu, m_0(m_0 - \epsilon), m_1(m_1 - \epsilon), \tilde{m}_0^2 - \frac{\epsilon^2}{4}, \tilde{m}_1^2 - \frac{\epsilon^2}{4}, q, \epsilon \right). \tag{19}
\]
This fact not only simplify the computation, we also use this fact to fix the precise relation between the spectrum expansion and the gauge theory partition function, as we explain later. We emphasize it is the equation (1) directly related to gauge theory and CFT.

The WKB form of the wave function is expanded by the Plank constant $\epsilon$,
\[
\Psi(z) = \exp i \int^z dx \left( \frac{p_0(x)}{\epsilon} + p_1(x) + \epsilon p_2(x) + \cdots \right), \tag{20}
\]

The Schrödinger equation gives $p_n(z)$ order by order,
\[
p_0(z) = i \sqrt{V(z)}, \quad p_1(z) = \frac{i}{2} (\ln p_0)' , \quad p_2(z) = \frac{-p_1^2 + ip_1'}{2p_0}, \quad \cdots \tag{21}
\]

The potential contains four turning points determined by $p_0(z) = 0$, therefore in the complex plane there are two branch cuts $\alpha$ and $\beta$.

The monodromy of the wave function along the contour $\alpha$ or $\beta$, associated to the branch cuts, is
\[
\nu = \frac{1}{2\pi \epsilon} \oint_{\alpha,\beta} p(z) dz. \tag{22}
\]
$\nu$ can be expanded in accordance with $p(z)$, $\nu = \epsilon^{-1} \nu_0 + \epsilon \nu_2 + \epsilon^3 \nu_4 + \cdots$, and we have $\nu_n = (2\pi)^{-1} \oint p_n(z) dz$.

First, let us work out the leading order monodromy, it is
\[
\nu_0 = \frac{1}{2\pi} \oint_{\alpha,\beta} p_0(z) dz = \oint_{\alpha,\beta} \frac{i}{2\pi} \frac{\sqrt{P_4(z)}}{z(z-q)(z-1)} dz, \tag{23}
\]
where $P_4(z)$ is a quartic polynomial of $z$ whose explicit form is in Appendix 8. As shown in [32], it is simpler to compute the contour integral for $\partial_{\Xi} \nu_0$ first because $\partial_{\Xi} P_4(z) = -(1-q)z(z-q)(z-1)$, therefore we have
\[
\frac{\partial \nu_0}{\partial \Xi} = \frac{1-q}{4\pi i} \oint_{\alpha,\beta} \frac{dz}{\sqrt{P_4(z)}}, \tag{24}
\]
As $P_4(z)$ is a quartic polynomial, the integral is complete elliptic integral. The result depends on the four roots of the equation $P_4(z) = 0$. Suppose we have four roots $z_i, i = 1, 2, 3, 4$, we can factorize $P_4(z)$ as
\[
P_4(z) = \tilde{m}_0^2 (z - z_1)(z - z_2)(z - z_3)(z - z_4), \tag{25}
\]
where the factor $\tilde{m}_0^2$ is the coefficient of $z^4$. Then the integrals are
\begin{align*}
\int_{z_1}^{z_2} \frac{dx}{\sqrt{(x-z_1)(x-z_2)(x-z_3)(x-z_4)}} &= \frac{2i}{\sqrt{(z_1-z_3)(z_2-z_4)}} K(k), \\
\int_{z_2}^{z_3} \frac{dx}{\sqrt{(x-z_1)(x-z_2)(x-z_3)(x-z_4)}} &= \frac{-2}{\sqrt{(z_1-z_3)(z_2-z_4)}} K(k'),
\end{align*}
for contour $\alpha$ and $\beta$ respectively. $K(k)$ is the complete elliptic integrals of the first kind, $k$ is the modulus given by the cross ratio of the four roots, and $k'$ is the complementary modulus.

\begin{align*}
k^2 &= \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}, & k'^2 &= 1 - k^2 = \frac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)}.
\end{align*}

The modulus $k^2$ and $q$ are different quantities, $k^2$ (or $k'^2$) describes the IR coupling of gauge theory while $q$ describes the UV coupling of gauge theory.

In order to get an asymptotic expansion, we need the four roots with a hierarchical structure that makes either $k^2$ or $k'^2$ small. This happens when two of the roots collide while other two remain at finite distance, for the case of generic parameters, there are six possible ways. This can be achieved by turning the parameters in the polynomial $P_4(z)$, in N=2 gauge theory this is controlled by moving in the moduli space. In the following, we will discuss the expansions for the Treibich-Verdier potential, which are related to massless particles of different U(1) charge, therefore we use the gauge theory terminology referring them as “electric/magnetic/dyonic” expansions.

The first order perturbation given by the contour integral $\oint p_1(z)dz$ does not contribute because the integrand is a total derivative. Actually, all odd order contour integral are zero because the integrands $p_{2i+1}(z)$ are all total derivatives. Higher order contour integral $\oint p_{2i}(z)dz$ can be generated from the leading order monodromy, by the action of certain differential operators with respect to the energy and mass parameters, as demonstrated in \cite{29,30,31} for other simpler potentials. However, for the Treibich-Verdier potential we have more mass parameters hence the higher order differential operators are hard to obtain, we do not proceed further in this direction. But there is a consistency condition allowing us to obtain higher order expansion from the gauge theory side, see the next section.

3 The perturbative spectrum of Treibich-Verdier potential

3.1 The expansion for large $\Xi$

We start from the expansion region that corresponds to the electric expansion of gauge theory where the effective coupling is weak. The first input from the gauge theory is the
identification of parameter Θ, or Ξ after the masses shifted, with the moduli space which is a large quantity in the electric region, \( \Xi \gg m_0^2, \bar{m}_1^2, \bar{\tilde{m}}_1^2 \). Then the contour integral is along the \( \alpha \)-cycle where roots \( z_1 \sim 0, z_2 \sim q \) collide with each other. The leading order monodromy \( \nu_0 \) is identified with the v.e.v of the scalar field in the undeformed gauge theory \( a_0 \), therefore we interpret \( \nu_n \) as the quantum corrections \( a_n \) due to the Omega background, and relate \( \nu \) with the deformed v.e.v of the adjoint scalar \( a = a_0 + \epsilon^2 a_2 + \epsilon^4 a_4 + \cdots \) by \( \nu = \frac{2}{\epsilon} \).

First, let us concentrate on the leading order WKB integral for \( \nu_0 = a_0 \). For large \( \Xi \), the equation \( P_4(z) = 0 \) can be iteratively solved order by order in the \( \Xi \) expansion if we correctly choose the leading order solution. For details of the method see Appendix 8. The result is

\[
\begin{align*}
    z_1 &= \frac{q\bar{m}_1^2}{(1-q)\Xi} + \frac{f_1^2}{\Xi^2} + \frac{f_1^3}{\Xi^3} + \cdots, \\
    z_2 &= q - \frac{q\bar{m}_1^2}{\Xi} + \frac{f_2^2}{\Xi^2} + \frac{f_2^3}{\Xi^3} + \cdots, \\
    z_3 &= 1 + \frac{m_0^2}{\Xi} + \frac{f_0^2}{\Xi^2} + \frac{f_3^3}{\Xi^3} + \cdots, \\
    z_4 &= \frac{(1-q)\Xi}{m_0^2} + \frac{f_4^0}{\Xi} + \frac{f_4^2}{\Xi^2} + \cdots. \\
\end{align*}
\]

(28)

the coefficients \( f_i^j \) with the subscript denotes roots \( i = 1, 2, 3, 4 \), the superscript denotes the (minus of) power of \( \Xi \). The roots have the right hierarchical pattern, \( z_1 \sim 0 \ll z_2 \sim q \ll z_3 \sim 1 \ll z_4 \sim \infty \), then the modulus \( k \) is

\[
k^2 = q - \frac{m_0^2 + \bar{m}_1^2 + m_1^2 + \bar{\tilde{m}}_1^2}{\Xi} q + O(\Xi^{-2}) + O(q^2).
\]

(29)

It remains small because \( q \) is small. Then the \( \partial \Xi \nu_0 \) and its \( \Xi \) expansion is given by

\[
\frac{\partial \Xi}{\partial \Xi} = \frac{(1-q)}{\pi m_0} \frac{K(k)}{\sqrt{(z_1 - z_3)(z_2 - z_4)}} = \frac{1}{2} \Xi^{-\frac{3}{4}} (h_0 + h_1 \Xi^{-1} + h_2 \Xi^{-2} + \cdots),
\]

(30)

where \( h_i \) are functions of \( m_0, m_1, \bar{m}_0, \bar{m}_1, q \). Integrate \( \Xi \), and reverse the series, we get

\[
\Xi = \frac{a_0^2}{h_0^2} [1 + 2h_0h_1a_0^{-2} + \left( \frac{2}{3}h_0^3h_2 - h_0^2h_1^2 \right)a_0^{-4}
+ \left( \frac{2}{3}h_0^5h_3 - 2h_0^4h_1h_2 + 2h_0^3h_1^3 \right)a_0^{-6} + \cdots].
\]

(31)

Substituting the expressions of \( h_i \), we have

\[
\Xi = a_0^2 - m_1^2 - \bar{m}_1^2 + \frac{(a_0^2 + m_0^2 - \bar{m}_0^2)(a_0^2 + m_1^2 - \bar{m}_1^2)}{2a_0^4} q + O(q^2).
\]

(32)
The first non-zero quantum correction to $\nu$ is $\nu_2 = a_2$. Following the method of [29], it would be given by a differential operator acting on the leading order result. We do not proceed here because the coefficients are very lengthy if the masses are of generic value. Including all quantum effects, the monodromy can be expanded as $\epsilon^{-1} a = \epsilon^{-1} a_0 + \epsilon a_2 + \epsilon^3 a_4 + \cdots$, $a_n$ are function of $\Xi$. The inverse gives the expansion of $\Xi$ in the form $\Xi_0 + \epsilon^2 \Xi_2 + \epsilon^4 \Xi_4 + \cdots$ with $\Xi_{2n}$ functions of $a$, instead of $a_0[30]$. In this form we can relate the function $\Xi(a, m, q, \epsilon)$ to the Nekrasov partition function of the gauge theory, through the Matone’s relation[33, 34]. In the following section, we will show how to match with the gauge theory without performing higher order WKB computation.

3.2 Match with the instanton counting

According to the gauge theory/integrable model relation, the moduli parameter of gauge theory is proportional to the energy eigenvalue of integrable model, the v.e.v of adjoint scalar is identified with the momentum of quasi-particles. In order to establish the precise relation between the two models, we need to compute the expansions on the two sides, at least for the first few orders. For N=2 gauge theory, the leading order solution can be computed from the Seiberg-Witten curve, but this mechanism is unable to get information about quantum corrections.

There is a particular region in the moduli space where the gauge theory is formulated by a Lagrangian and the coupling is weak, therefore the theory is in good control by QFT method. The exponentially suppressed instanton contribution is given by a counting algorithm[6], incorporating the $\epsilon$-deformation by the Omega background, the $\epsilon$-corrected effective action obtained is directly related to the quantum spectrum of the model.

The effective action of the $N = 2$ gauge theory in the $\Omega$ background contains the perturbative part and the instanton part. The perturbative part of the SU(2) theory with four flavors is[35]

$$Z^{\text{pert}} = \exp(a^2 \ln q - \gamma_{\epsilon_1, \epsilon_2}(2a - \epsilon_1) - \gamma_{\epsilon_1, \epsilon_2}(2a - \epsilon_2) + \sum_{i=1}^{4} \gamma_{\epsilon_1, \epsilon_2}(a - \mu_i) + \gamma_{\epsilon_1, \epsilon_2}(-a - \mu_i)).$$ (33)

where $\gamma_{\epsilon_1, \epsilon_2}(x)$ is the logarithm of the double gamma function. Only the first term play a role in the identification (37). The instanton contribution can be computed from the Nekrasov partition function[6]. The $k$ instanton partition function can be computed by the following counting formula,

$$Z_k^{\text{inst}} = \sum_{\sum |\alpha| = k} z_{\text{vec}} z_{\text{matter}}.$$ (34)
with contributions from the vector multiplets,

$$z_{vec} = \left( \prod_{\alpha, \beta = 1, 2} \prod_{s \in Y_\alpha} \prod_{s' \in Y_\beta} E_{\alpha \beta}(s)(\epsilon_+ - E_{\beta \alpha}(s')) \right)^{-1},$$

(35)

where \( E_{\alpha \beta}(s) = a_\alpha - a_\beta - h_\beta(s) \epsilon_1 + (v_\alpha(s) + 1) \epsilon_2, \epsilon_+ = \epsilon_1 + \epsilon_2, a_1 = -a_2 = a \). And the contribution from the hypermultiplets all in the fundamental representation,

$$z_{matter} = \prod_{s \in \tilde{Y}} \prod_{i=1}^{N_f} (\varphi(s) + \epsilon_+ - \mu_i).$$

(36)

where \( \tilde{Y} = (Y_1, Y_2) \) is the Young diagram doublet used in the localization, and for a box \( s \in Y_\lambda \) we have \( \varphi(s) = a_\lambda + (i_\lambda - 1) \epsilon_1 + (j_\lambda - 1) \epsilon_2 \). We use the computing formula of \cite{36, 37}. After deriving the prepotential from the instanton partition function from \( \mathcal{F} = -\epsilon_1 \epsilon_2 \ln(Z_{pert}Z_{inst}) \), we take the Nekrasov-Shatashvili limit \( \epsilon_1 = \epsilon, \epsilon_2 = 0 \), and we can expand the prepotential as \( \mathcal{F} = \mathcal{F}_{(0)} + \epsilon \mathcal{F}_{(1)} + \cdots \) where at each order \( \mathcal{F}_{(n)} \) contains the perturbative part and the instanton part.

It is the function \( \Theta \) associated to the Heun equation \cite{11} directly related to the gauge theory, therefore we need to shift the masses parameters of the function \( \Xi \) for the equation \cite{17} obtained in the previous section to get \( \Theta \), then relate it to gauge theory. With the data from the first two order \( \epsilon \)-expansion we can fix the relation between the function \( \Theta \) and the prepotential of SU(2) \( N_f = 4 \) supersymmetric QCD to be

$$\Theta(a, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon) + m_1(m_1 - \epsilon) + (\tilde{m}_1^2 - \frac{\epsilon^2}{4}) + \frac{2q}{1-q}(m_0-\epsilon)(m_1-\epsilon) = q \frac{\partial}{\partial q} \mathcal{F}(a, \mu_i, q, \epsilon).$$

(37)

where in the left hand side the mass parameters are \( m_0, m_1, \tilde{m}_0, \tilde{m}_1 \), on the right hand side the mass parameters are \( \mu_1, \mu_2, \mu_3, \mu_4 \), they are related through \( \Xi \). This relation is consistent with the conjecture that in the semiclassical limit the CFT conformal block gives the spectrum of the quantised integrable model, the term \( 2q(1-q)^{-1}(m_0-\epsilon)(m_1-\epsilon) \) corresponds the stripped U(1) factor in the AGT \cite{12} in the NS limit.

From this relation we can derive the instant part of the prepotential of the gauge theory \( \mathcal{F}_{(0)}^{inst} \) and \( \mathcal{F}_{(1)}^{inst} \) from the leading order result \cite{32}, and the relation \cite{19} and \cite{37}. The first few order results, now written in terms of the physical mass \( \mu_i \) used in gauge theory, are

$$\mathcal{F}_{(0)}^{inst} = \frac{1}{2a^2} (a^4 + a^2 \sum_{i < j} \mu_i \mu_j + \prod_i \mu_i) q$$

$$+ \frac{1}{64a^6} [13a^8 + a^6(\sum_i \mu_i^2 + 16 \sum_i \mu_i \mu_j) + a^4(\sum_i \mu_i^2 \mu_j^2 + 16 \prod_i \mu_i)]$$

$$- 3a^2 \sum_{i < j < k} \mu_i^2 \mu_j^2 \mu_k^2 + 5 \prod_i \mu_i^2 |q|^2 + \mathcal{O}(q^3).$$

(38)
and
\[ F_{(1)}^{\text{inst}} = -\frac{1}{4a^2} \left( 5a^2 \sum_i \mu_i + \sum_{i<j<k} \mu_i \mu_j \mu_k \right) q \]
\[ -\frac{1}{64a^6} \left[ 41a^6 \sum_i \mu_i + a^4 \left( \sum_{i<j} \mu_i^2 \mu_j + \sum_{i<j<k} \mu_i \mu_j \mu_k \right) + 8 \sum_{i<j<k} \mu_i \mu_j \mu_k \right] \]
\[ -3a^2 \sum_{i<j<k} \left( \mu_i^2 \mu_j^2 \mu_k + \mu_i \mu_j^2 \mu_k^2 + \mu_i \mu_j \mu_k^2 \right) + 5 \prod_i \mu_i \sum_{i<j<k} \mu_i \mu_j \mu_k \right] q^2 \]
\[ + O(q^3). \] (39)

where in indices \( i, j, k \in \{1, 2, 3, 4\} \). The \( F_{(0)} \) is the Seiberg-Witten solution. They precisely agree with the results of Nekrasov instanton partition function for SU(2) \( N_f = 4 \) theory with generic masses, hence confirm the relation (37) for low order of \( \epsilon \)-expansion.

In order to validate the relation (37), we should work out few higher order WKB analysis of the Heun equation. In principle this can be done. However, even without the higher order WKB results there is a nontrivial consistency condition which allows us to proceed further. The functions \( \Xi \) and \( \Theta \), expanded as series of \( \epsilon \), have different forms. Because in the equation (1) the potential \( V(z) \) does not contain \( \epsilon \), the Schrödinger operator is invariant under the change \( \epsilon \to -\epsilon \), therefore the spectrum function \( \Xi \) contains only the even order of \( \epsilon \). Indeed, for a potential independent of \( \epsilon \) the contour integrals for odd order WKB component \( p_{2n+1} \) vanish. Therefore we have
\[ \Xi = \sum_{2n} \epsilon^{2n} \xi_{2n}(a, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q), \quad n \geq 0. \] (40)

But in the equation (11) the potential \( U(z, \epsilon) \) involves \( \epsilon \), the integrals of \( p_{2n+1} \) are nonzero, the function \( \Theta \) should contain all order of \( \epsilon \).
\[ \Theta = \sum_n \epsilon^{n} \theta_n(a, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q), \quad n \geq 0 \]

As discussed previously, if we shift the arguments of masses in the function \( \Theta \) in a proper way, then formally we can write it as a series with only even power of \( \epsilon \),
\[ \Theta = \sum_n \epsilon^{2n} \tilde{\theta}_{2n} \left( a, m_0(m_0 - \epsilon), m_1(m_1 - \epsilon), \tilde{m}_0^2 - \frac{\epsilon^2}{4}, \tilde{m}_1^2 - \frac{\epsilon^2}{4}, q \right), \quad n \geq 0. \] (41)

now with \( \theta_{2n+1} \) vanish, and the new functions \( \tilde{\theta}_{2n} \) take the same functional form as \( \xi_{2n} \), merely with arguments shifted. Notice that if we suppose the relation (37) correct, then the function \( \Theta \) can be derived from the prepotential, indeed it contains terms of all order of \( \epsilon \).
It is a nontrivial requirement that we can rearrange the expansion in such a way where all odd order \( \epsilon \) terms vanish by shifting the mass parameters.
The rearrangement indeed works. As the instanton counting can be easily computed in a programmed way, therefore we can use the relation (37) to predict the higher WKB expansion for the function $\Theta$. Then we shift the masses as explained and get the expansion (41), from the functional form of $\tilde{\theta}_{2n}$ we obtain the function $\xi_{2n}$. For example, in the first few order of $\epsilon$-expansion for $\Xi$ all the odd order vanish,

$$
\Xi = a^2 - m_1^2 - \tilde{m}_1^2 + \frac{(a^2 + m_0^2 - \tilde{m}_0^2)(a^2 + m_1^2 - \tilde{m}_1^2)}{2a^2}q + O(q^2)
+ \epsilon^2 \left( \frac{1}{4} + \frac{-a^4 + (m_0^2 - \tilde{m}_0^2)(m_1^2 - \tilde{m}_1^2)}{8a^4}q + O(q^2) \right)
+ \epsilon^4 \left( \frac{(m_0^2 - \tilde{m}_0^2)(m_1^2 - \tilde{m}_1^2)}{32a^6}q + O(q^2) \right)
+ O(\epsilon^6).
$$

(42)

However this does not mean we can shift the masses in the same way to make the odd order $F_{(2n+1)}$ vanish, because in the relation (37) the term $(m_0 - \epsilon)(m_1 - \epsilon)$ is not in the shifted form, it plays a special role in the rearrangement.

4 Dual expansion for small $\Delta$

According to the duality of the gauge theory, there are other asymptotic expansions given by contour integrals along the dual cycles, such as $\beta$ and $\alpha + \beta$. The dual expansion for the case of generic masses is too complicated for presentation, therefore in this section we only present the case for four equal masses with $m_0 = m_1 = m$, $\tilde{m}_0 = \tilde{m}_1 = 0$. i.e. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = m$. We also only compute the leading order $\epsilon$ expansion, although the first order $\epsilon$ correction can be easily obtained by shifting $m \to m - \frac{\epsilon}{2}$. The computation itself is straightforward, it is useful because this special case clearly illustrates the structure of the dual expansion, gives us hint about the expansion pattern for the case of generic mass. In the Appendix 8 we give some details for the case of generic masses to show how the dual expansions can be computed.

4.1 Magnetic expansion

First we need to find the expansion point in the $\Xi$ parameter space where two other roots of $P_4(z) = 0$ approach to each other. This can be determined form the discriminant of the polynomial $P_4(z)$. In the equal mass limit, the discriminant is factorised as

$$
\mathcal{D}(P_4) = q^2(1 - q)^2(\Xi - q\Xi - 2qm^2)^4[(1 - q)^2\Xi^2 + 2m^2(1 - q)^2\Xi + m^4(1 - 6q + q^2)].
$$

(43)
The are three solution for $D(P_4) = 0$ for $\Xi$ at some finite value, of weight $(4, 1, 1)$ representing the number of massless particles at each singularity [2]. The weight 4 singularity is at the semiclassical region, its expansion is (42) with all masses taking the same value.

We are interested in the singularity that survives under the decoupling limit $q \to 0, m \to \infty$, $q^{1/4}m = \Lambda$ as the magnetic singularity of pure SYM. Our previous work on N=2 theory with an adjoint mass is helpful, where we observed the dual expansion parameter is the special combination $q^{1/4}m$ [31], representing the one instanton factor. We expect the expansion for the gauge theory with equal fundamental mass should also present $q^{1/4}m$ behavior, and indeed it does.

It turns out that the last factor of (43) gives two degenerate points of the curve, one is the magnetic expansion point at

$$\Xi_{\text{mag}} = -\frac{1 - 2\sqrt{q} - q}{1 - q}m^2.$$  \hfill (44)

The shifted coordinate $\tilde{\Xi}_{\text{mag}} = m^2 + \Xi_{\text{mag}} \to 2\Lambda^2$ in the decoupling limit. So we set

$$\Xi = -\frac{1 - 2\sqrt{q} - q}{1 - q}m^2 + \Delta,$$  \hfill (45)

with $\Delta$ a small quantity as compared to the dimensional parameter $\Delta \ll \sqrt{qm^2}$. The polynomial $P_4(z)$ now reduces to cubic form $P_3(z)$ as

$$P_3(z) = (m^2 - 2q^{1/2}m^2 + qm^2 - \Delta + q\Delta)z^3 + (2q^{1/2}m^2 - 4qm^2 + 2q^{3/2}m^2 + \Delta - q^2\Delta)z^2 + (qm^2 - 2q^{3/2}m^2 + q^2m^2 - q\Delta + q^2\Delta)z.$$  \hfill (46)

The roots of the equation $P_3(z)$ are at $z_2, z_3$ and $z_4 = 0$, they can be expanded as

$$z_2 = -q^{1/2} - \frac{(1 + q^{1/2})^{3/2}q^{1/4}\Delta^{1/2}}{m} - \frac{(1 + q^{1/2})^3 \Delta}{2(1 - q^{1/2})^2 m^2} - \frac{(1 + q^{1/2})^{5/2}(1 + 6q^{1/2} + q)\Delta^{3/2}}{8(1 - q^{1/2})^3 q^{1/4} m^3} - \frac{(1 + q^{1/2})^4 \Delta^2}{2(1 - q^{1/2})^2 m^4} + \cdots,$$

$$z_3 = -q^{1/2} + \frac{(1 + q^{1/2})^{3/2}q^{1/4}\Delta^{1/2}}{m} - \frac{(1 + q^{1/2})^3 \Delta}{2(1 - q^{1/2})^2 m^2} + \frac{(1 + q^{1/2})^{5/2}(1 + 6q^{1/2} + q)\Delta^{3/2}}{8(1 - q^{1/2})^3 q^{1/4} m^3} - \frac{(1 + q^{1/2})^4 \Delta^2}{2(1 - q^{1/2})^2 m^4} + \cdots.$$  \hfill (47)

The roots present a hierarchical structure as $z_4 \ll |z_2| \sim |z_3|(\ll |z_1| \sim \infty)$ and $z_2 - z_3 \sim$
The last factor of (43) gives another degenerate point, it is the dyonic expansion point at $\Xi_{dy} = \frac{-1 + 2\sqrt{q} - q}{1-q} m^2$. Therefore it gives the modulus

$$k^2 = 1 - \frac{2(1 + q^{1/2})^{3/2}}{(1 - q^{1/2})^{1/2} q^{1/4} m} \Delta^{1/2} - \frac{2(1 + q^{1/2})^{3} \Delta}{(1 - q^{1/2})^{1/2} q^{1/2} m^2}$$

$$+ \frac{(1 + q^{1/2})^{5/2}(1 + 14 q^{1/2} + 5q) \Delta^{3/2}}{4(1 - q^{1/2})^{3/2} q^{3/4} m^3} + \frac{(1 + q^{1/2})^{4}(1 + 6q^{1/2} + q) \Delta^{2}}{2(1 - q^{1/2})^{2} q^{1/2} m^4} + O\left(\frac{\Delta^{5/2}}{q^{5/4} m^5}\right), \quad (48)$$

and hence the complementary modulus $k^2 = 1 - k^2$ is a small quantity, suitable for the dual expansion. Substituting all these into the following formula, where we denote the dual v.e.v. of $a$ by $a_D$, now related to the monodromy by the relation $\nu = \frac{a_D}{a}$,

$$\frac{\partial a_D}{\partial \Delta} \frac{1 - q}{4\pi i} \iint d\beta \sqrt{P_3(z)} = \frac{(1-q)}{\pi} \frac{\Delta}{ \sqrt{z_2(m^2 - 2q^{1/2}m^2 + qm^2 - \Delta + q\Delta)}}. \quad (49)$$

The $z_2$ in the square root gives a factor $\sqrt{1}$, therefore we define $a_D = a_D$ as in $[30]$. The procedure is similar to the previous section, then we get the dual expansion for $a_D$,

$$\hat{a}_D = \frac{(1 + q^{1/2}) \Delta}{2q^{1/4} m} + \frac{(1 + q^{1/2})^2(1 + 6q^{1/2} + q) \Delta^2}{64(1 - q^{1/2}) q^{3/4} m^3}$$

$$- \frac{(1 - q^{1/2})^3(3 - 20q^{1/2} + 8q - 20q^{3/2} + 3q^2) \Delta^3}{2^{11}(1 - q^{1/2})^{2} q^{5/4} m^5} + O\left(\frac{\Delta^4}{q^{7/4} m^7}\right). \quad (50)$$

Note the disappearance of all terms of $\Delta^n/2$ for $n \in \mathbb{Z}$. Reversing the series gives

$$\Delta = -m\hat{a}_D \frac{2q^{1/4}}{1 + q^{1/2}} + \hat{a}_D^2 \frac{1 - 6q^{1/2} + q}{2^{5}(1 - q)} + \frac{\hat{a}_D^3}{m} \frac{(1 + q^{1/2})^3}{2^{7}(1 - q^{1/2})^{2} q^{1/4}}$$

$$+ \frac{\hat{a}_D^4}{m^2} \frac{5(1 + q^{1/2})^3(1 - 6q^{1/2} + q)}{2^{12}(1 - q^{1/2})^{3} q^{1/2}} + O\left(\frac{\hat{a}_D^5}{m^3 q^{3/4}}\right). \quad (51)$$

In the next section we will discuss the mass decoupling limit under which the $N_f = 4$ gauge theory reduces to the $N=2$ pure Yang-Mills theory, the coincidence of the first few coefficients with the pure gauge theory expansion provides nontrivial justification for our results.

### 4.2 Dyonic expansion

The last factor of (13) gives another degenerate point, it is the dyonic expansion point at

$$\Xi_{dy} = \frac{-1 + 2\sqrt{q} - q}{1-q} m^2. \quad (52)$$
Similarly here we should set
\[ \Xi = -\frac{1 + 2\sqrt{q - q}}{1 - q} m^2 + \Delta. \] (53)
again with \( \Delta \ll q^{1/2}m^2 \).

The dyonic expansion comes from a rotation of the phase of \( q \) by \( 2\pi \). From the relation \( q = \exp(2\pi i \tau) \) with \( \tau = \frac{4\pi i}{g_{sv}} + \frac{\theta}{2\pi} \), the rotation \( q \to e^{2\pi i}q \) induces the shift of the theta angle by \( 2\pi \), this would shift the electric charge of a magnetic particle in the low energy gauge theory according to the Witten effect [38], resulting a dyon of charge (1,1).

## 5 Various limit cases

We provide few limit cases, where we can compare our results to previous study.

(I). Partial equal masses limit

In the last section we have set the mass of four flavors the same, this makes the computation simple enough for presentation. We can relax the condition a bit. When \( \mu_1 = \mu_2, \mu_3 \neq \mu_4 \) we have \( \tilde{m}_0 = 0 \), the polynomial \( P_4(z) \) degenerate to third order, the computation can be easily carried out as in the previous section. Similarly, it is also simple to study the case for \( \mu_1 \neq \mu_2, \mu_3 = \mu_4 \) we have \( \tilde{m}_1 = 0 \).

(II). Massless limit

We can turn some flavors massless, while keep other flavors massive. This is allowed in the electric expansion but not allowed in the dual expansions. A particular interesting case is the full massless limit, \( m_0 = m_1 = \tilde{m}_0 = \tilde{m}_1 = 0 \), where the gauge theory becomes conformal. Let us look at the Seiberg-Witten solution,

\[ a_0 = -\frac{1}{2\pi} \int_0 \frac{dz}{\sqrt{z(q - z)(z - 1)}} = -\frac{2K(q)}{\pi} \sqrt{(1 - q)^\Xi}. \] (54)

From \( \Xi = q \frac{\partial F}{\partial q} \) we get the prepotential, therefore the effective coupling is

\[ 4\pi i \tau_{ir} = \frac{\partial^2 F}{\partial a_0^2} = \frac{\pi^2}{2} \int \frac{dq}{q (1 - q)K^2(q)} = -2\pi \frac{K(1 - q)}{K(q)}. \] (55)

If we set \( q = \exp(2\pi i \tau_{uv}), p = \exp(4\pi i \tau_{ir}) \), then for weak coupling \( |q| \ll 1 \) we have the relation

\[ 2\pi i \tau_{ir} = 2\pi i \tau_{uv} - 4 \ln 2 + \frac{1}{2} q + \frac{13}{64} q^2 + \frac{23}{192} q^3 + \frac{2701}{32768} q^4 + \cdots, \] (56)

or the inverse relation

\[ q = \frac{\theta_3(p^{1/2})}{\theta_3(p^{1/2})}. \] (57)
which is exactly the relation of [39]. The duality of the massless gauge theory is encoded in
the modular transformation of the Theta functions.

(III). Mass decoupling limit
In the infinite mass limit, by turning the UV coupling properly, the resulting theory
is a gauge theory with less flavors. In both electric and magnetic expansions, in the final
results the mass parameters always appear in a symmetric way as algebraic polynomial of
\( \mu_i^{n_i}, \mu_i^{n_i} \mu_j^{n_j}, \mu_i^{n_i} \mu_j^{n_j} \mu_k^{n_k}, \mu_1^{n_1} \mu_2^{n_2} \mu_3^{n_3} \mu_4^{n_4} \) with \( n_i \in \mathbb{Z}^+ \). This feature makes the mass decoupling
procedure straightforward, we can decouple any one, any number, of the four flavors.

For example, in the electric expansion, if we keep \( \mu_1, \mu_2, \mu_3 \) finite and turn \( \mu_4 \to \infty, q \to 0 \) while make \( q \mu_4 = \Lambda_3 \) finite, we get the N=2 gauge theory with \( N_f = 3 \). By turning
\( \mu_3, \mu_4 \to \infty, q \to 0 \) with \( q \mu_3 \mu_4 = \Lambda_2^2 \) we get the N=2 gauge theory with \( N_f = 2 \). And
\( \mu_2, \mu_3, \mu_4 \to \infty, q \to 0 \) with \( q \mu_2 \mu_3 \mu_4 = \Lambda_1^3 \) gives the N=2 gauge theory with \( N_f = 1 \). Finally
we get the N=2 pure Yang-Mills theory by decoupling all flavors as \( \mu_1, \mu_2, \mu_3 \to \infty, q \to 0 \) with
\( q \mu_1 \mu_2 \mu_3 \mu_4 = \Lambda^4 \). We can compare the decoupling results of formula (38) to the ones in for example [40].

We can also do the decoupling limit for the magnetic(and dyonic) expansion, in the equal
mass case, by turning \( m \to \infty, q \to 0 \) and keep \( q^{1/4} m = \Lambda \) finite. In the formula (51), only the following terms survive in the limit,
\[
\Delta_{N_f=0} = -2 \hat{a}_D \Lambda + \frac{1}{2^3} \hat{a}_D^2 + \frac{1}{2^7} \hat{a}_D^3 + \frac{5}{2^{12}} \hat{a}_D^4 + \frac{33}{2^{17}} \hat{a}_D^5 + \frac{63}{2^{20}} \hat{a}_D^6 + \cdots .
\] (58)
In agree with the result of pure gauge theory[30], if we scale our \( \hat{a}_D \to 2 \hat{a}_D \) because in our
parameterization the singularities in the moduli space are at \( \pm 2 \Lambda^2 \). The case for generic
masses is presented in Appendix 8.

Taking the mass decoupling limits of the gauge theory corresponds to taking scaling
limits of the Heun equation, including proper scaling of the the coordinate \( z \), which gives
the Confluent Heun equations. We do not discuss the details in this paper.

(IV). Limits related to \( N = 2^* \) SYM
According the AGT correspondence, the partition function of the N=2 \( N_f = 4 \) QCD
is related to the 4-point conformal block on the sphere, and the partition function of the
\( N = 2^* \) SYM is related to the 1-point conformal block on the torus. It turns out that there
is a surprising relation between the two pairs. We will show that, from the perspective of
relation between gauge theory/CFT/integrable potential, these facts are in consist with that
the Treibich-Verdier potential reduces to the Lamé potential in particular limits.

The relation on the CFT side is found in[20]. The 1-point correlation for a generic primary
operator on the torus is related to 4-point correlator on the sphere with a special choice of
conformal weight for the primary operators. With a proper identification of parameters as
in (7), in the NS limit, this choice makes $m_0 = m_1 = \tilde{m}_1 = \frac{\epsilon}{4}$ and $\tilde{m}_0$ remains free, therefore we have

$$b_0 = \left( \frac{2\tilde{m}_0}{\epsilon} - \frac{1}{2} \right) \left( \frac{2\tilde{m}_0}{\epsilon} + \frac{1}{2} \right), \quad b_1 = b_2 = b_3 = 0.$$  

(59)

The Treibich-Verdier potential in this limit is the Lamé potential with a single $\wp(x)$ function. It implies a partial massless limit for the corresponding gauge theory, $\mu_1 = \tilde{m}_0 + \frac{\epsilon}{4}, \mu_2 = -\tilde{m}_0 + \frac{\epsilon}{4}, \mu_3 = \frac{\epsilon}{4}, \mu_4 = 0$.

Meanwhile, a relation between the Nekrasov partition functions of the $N_f = 4$ gauge theory and the $N = 2^*$ gauge theory is also found in [41]. The choice of the flavor masses is

$$\mu_1 = \frac{1}{2}M, \quad \mu_2 = \frac{1}{2}(M + \epsilon_1), \quad \mu_3 = \frac{1}{2}(M + \epsilon_2), \quad \mu_4 = \frac{1}{2}(M + \epsilon_1 + \epsilon_2).$$  

(60)

In the NS limit, it implies the parameters for the elliptic potential are

$$b_0 = b_2 = 0, \quad b_1 = b_3 = \frac{M}{\epsilon} \left( \frac{M}{\epsilon} - 1 \right).$$  

(61)

The Treibich-Verdier potential in this limit is actually also the Lamé potential, because we have the relation (15) for the elliptic function.

6 Conclusion

In this paper we investigate a particular case of the $N=2$ gauge theory/integrable model correspondence, i.e. the SU(2) gauge theory with four fundamental flavors and the quantum mechanics of the $BC_1$ Calogero-Inozemtsev model, i.e. a quantum particle in the Treibich-Verdier potential. This relation implies a few other cases of gauge theory/integrable potential correspondence, by taking various limits on both sides.

For the Treibich-Verdier potential, there are six stationary points, they correspond to six singularities in the moduli of the gauge theory. We analyse the asymptotic expansions at these singularities which can be compared to known results in various mass decoupling limits. An iterative method is used to factorize the polynomial $P_4(z)$ in a proper way to find the asymptotic expansions of the elliptic integral, applicable to all stationary/singularity points.

The study supports the fact that the new parameterization of the Seiberg-Witten curve, manifested as the potential $V(z)$ in (18), indeed captures all the ingredients of the low energy gauge theory as originally formulated in[2]. We demonstrate the Coulomb branch low energy dynamics of these SU(2) gauge theories are equivalent to the quantum spectrum of some integrable finite-gap potentials, by comparing various facts on the two sides.

There is an obvious question related to the problem we have studied, for general $N=2$ gauge theory with SU($N_c$) gauge group and $N_f = 2N_c$ flavor, is there a corresponding integrable model with the potential of elliptic form? If it exists, is it a finite-gap potential?
Appendix: The Heun equation

We used the convention a bit different from [10] to relate the equation to gauge theory quantities, therefore in order to avoid confusion we go through some details on different forms of the Heun equation.

The Heun equation is the general second order Fuchsian linear differential equation with four regular singularities, it is an extension of the hypergeometric equation. It can be written in several different forms, the most familiar form is

\[
\frac{d^2}{dz^2} w(z) + \left( \frac{\gamma}{z} + \frac{\eta}{z-1} + \frac{\lambda}{z-q} \right) \frac{d}{dz} w(z) + \frac{\alpha \beta z - Q}{z(z-1)(z-q)} w(z) = 0. \]

with \( \alpha + \beta + 1 = \gamma + \eta + \lambda \), \( q \) is called the singularity parameter, \( \alpha, \beta, \gamma, \eta, \lambda \) are called the exponent parameters, and \( Q \) is called the accessory parameter. There are four regular singularities at \( 0, q, 1, \infty \), the merging of regular singularities gives the Confluent Heun equations with irregular singularities.

Define a new function \( \tilde{w}(z) \) by

\[
w(z) = z^{-\gamma/2}(z-1)^{-\eta/2}(z-q)^{-\lambda/2} \tilde{w}(z),
\]

then we can write the Heun equation in the normal form,

\[
\frac{d^2}{dz^2} \tilde{w}(z) - \left[ \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-q} + \frac{D}{z^2} + \frac{E}{(z-1)^2} + \frac{F}{(z-q)^2} \right] \tilde{w}(z) = 0.
\]

with \( A + B + C = 0 \). The parameters \( A, B, C, D, E, F \) are related to \( \alpha, \beta, \gamma, \eta, \lambda \) by

\[
A = \frac{2Q - q \gamma \eta - \gamma \lambda}{2q}, \quad B = \frac{2Q - 2\alpha \beta + \gamma \eta + \lambda \eta - q \gamma \eta}{2(1-q)},
\]

\[
D = \frac{1}{2} \gamma (\frac{1}{2} \gamma - 1), \quad E = \frac{1}{2} \eta (\frac{1}{2} \eta - 1), \quad F = \frac{1}{2} \lambda (\frac{1}{2} \lambda - 1).
\]

In a slightly different form it is

\[
\frac{d^2}{dz^2} \tilde{w}(z) - \left[ \frac{D}{z^2} + \frac{F}{(z-q)^2} + \frac{E}{(z-1)^2} + \frac{(1-q)B - qA}{z(z-1)} + \frac{q(1-q)(A+B)}{z(z-1)(z-q)} \right] \tilde{w}(z) = 0.
\]

Compare with the equation (1) obtained from CFT/gauge theory, we can identify the parameters \( \gamma, \eta, \lambda, \alpha, \beta, Q \) with \( m_0, m_1, \tilde{m}_0, \tilde{m}_1, q, \epsilon \) of the SU(2) \( N_f = 4 \) theory. The identification is not unique,

\[
\gamma = \frac{2\tilde{m}_1}{\epsilon} + 1, \quad \text{or} \quad -\frac{2\tilde{m}_1}{\epsilon} + 1,
\]

\[
\eta = \frac{2m_0}{\epsilon}, \quad \text{or} \quad -\frac{2m_0}{\epsilon} + 2,
\]

\[
\lambda = \frac{2m_1}{\epsilon}, \quad \text{or} \quad -\frac{2m_1}{\epsilon} + 2.
\]
Then $\alpha, \beta$ and $Q$ are accordingly determined by
\[
\begin{align*}
\alpha + \beta &= \gamma + \eta + \lambda - 1, \\
\alpha \beta &= \frac{\gamma \eta + \gamma \lambda + \lambda \eta}{2} + \frac{m_0 (m_0 \epsilon - 1)}{\epsilon} + \frac{m_1 (m_1 \epsilon - 1)}{\epsilon} - \frac{\tilde{m}_0^2}{\epsilon^2} + \frac{\tilde{m}_1^2}{\epsilon^2}, \\
Q &= - (1 - q) \Theta \frac{\epsilon^2}{\epsilon} + q \frac{\gamma \eta + \gamma \lambda}{2} + q \left[ \frac{m_0 (m_0 \epsilon - 1)}{\epsilon} + \frac{m_1 (m_1 \epsilon - 1)}{\epsilon} - \frac{\tilde{m}_0^2}{\epsilon^2} + \frac{\tilde{m}_1^2}{\epsilon^2} \right].
\end{align*}
\] (68)

On the other hand, transform the equation to the elliptic form is useful for the study of the Treibich-Verdier potential. Define the coordinate $\zeta$ by
\[
z = q \times \text{sn}^2(\zeta, q),
\] (69)
the Jacobi elliptic function $\text{sn}(\zeta, q)$ depends on the elliptic modulus $q$. Then the Heun equation can be written in the Jacobi elliptic form as
\[
\frac{d^2}{d\zeta^2} w(\zeta) + \left( \frac{2 \gamma - 1}{\text{sn} \zeta} \frac{\text{cn} \zeta \text{dn} \zeta}{\text{sn} \zeta} - \frac{2 \eta - 1}{\text{cn} \zeta} \frac{\text{sn} \zeta \text{cn} \zeta}{\text{dn} \zeta} - (2 \lambda - 1) \frac{\text{sn} \zeta \text{dn} \zeta}{\text{cn} \zeta} \right) \frac{d}{d\zeta} w(\zeta) + 4(q \alpha \beta \text{sn}^2 \zeta - Q) w(\zeta) = 0.
\] (70)

The term of $d\zeta w$ can be canceled if we define
\[
w(\zeta) = \left( \text{sn} \zeta \right)^{(1-2\gamma)/2} \left( \text{dn} \zeta \right)^{(1-2\eta)/2} \left( \text{cn} \zeta \right)^{(1-2\lambda)/2} \tilde{w}(\zeta),
\] (71)
and transform the equation to
\[
\frac{d^2}{d\zeta^2} \tilde{w}(\zeta) + \left( h - b_0 q \text{sn}^2 \zeta - b_1 q \frac{\text{cn}^2 \zeta}{\text{dn}^2 \zeta} - b_2 \frac{1}{\text{sn}^2 \zeta} - b_3 \frac{\text{dn}^2 \zeta}{\text{cn}^2 \zeta} \right) \tilde{w}(\zeta) = 0.
\] (72)
where
\[
\begin{align*}
h &= -4Q + (\gamma + \lambda - 1)^2 + q(\gamma + \eta - 1)^2, \\
b_0 &= -4\alpha \beta + (\gamma + \eta + \lambda - 1)^2(\gamma + \eta + \lambda - \frac{3}{2}), \\
b_1 &= (\eta - \frac{1}{2})(\eta - \frac{3}{2}), \\
b_2 &= (\gamma - \frac{1}{2})(\gamma - \frac{3}{2}), \\
b_3 &= (\lambda - \frac{1}{2})(\lambda - \frac{3}{2}).
\end{align*}
\] (73)
This is the equation appears in Darboux’s work[14]. In the limit $q \to 0$ and other quantities remain finite, the potential reduces to the Pöschl-Teller potential.

In order to further transform the equation to the Weierstrass elliptic form, we define variables $x$ by
\[
\frac{\zeta + i K'}{(e_1 - e_2)^{1/2}} = x, \quad \text{i.e.} \quad \text{sn}^2 \zeta = \frac{\wp(x) - e_2}{e_3 - e_2},
\] (74)
and the function $W(x)$ by
\[
w(\zeta) = \left( \text{sn} \zeta \right)^{(1-2\gamma)/2} \left( \text{dn} \zeta \right)^{(1-2\eta)/2} \left( \text{cn} \zeta \right)^{(1-2\lambda)/2} W(x),
\] (75)
then we get the following form of equation,

$$\frac{d^2}{dx^2} W(x) + \left( H - b_0 \varphi(x) - b_1 \varphi(x + \omega_1) - b_2 \varphi(x + \omega_2) - b_3 \varphi(x + \omega_3) \right) W(x) = 0. \quad (76)$$

where

$$H = (e_1 - e_2)h + e_2 \sum_i b_i$$
$$= 4(e_2 - e_1)Q - 4e_2\alpha\beta + e_1(\gamma + \lambda - 1)^2 + e_2(\eta + \lambda - 1)^2 + e_3(\gamma + \eta - 1)^2. \quad (77)$$

The stationary points of the potential are determined by the condition $\sum_i b_i \partial_x \varphi(x + \omega_i) = 0$, if rewritten in the variable $z$ it is a polynomial equation of degree six $Q_6(z) = 0$. The six solutions are in correspondence to the six solution for the discriminant equation $D(P_4)(\Xi) = 0$, discussed in the next Appendix 8.

8 Appendix: Iterative solution for $P_4(z) = 0$

Let us start from some basic facts about quartic polynomial. For a quartic polynomial defined in the complex domain

$$P_4(z) = az^4 + bz^3 + cz^2 + dz + e, \quad (78)$$

we can associated an elliptic curve to the polynomial, $y^2 = P_4(z)$. The shape of the curve is controlled by the modulus parameter $[27]$, which is given by the cross ratio of the roots of the equation $P_4(z) = 0$.

These elliptic curves are used in the Seiberg-Witten theory to determine the dynamics of the gauge theory, when the curve degenerates the gauge theory has a weak coupling description. It happens when two of the roots collide, making either the modulus $k^2$ or its complementary modulus $k'^2$ small. $k^2$ and $k'^2$ are related to the gauge coupling and the dual gauge coupling, respectively. In this paper we need to deal with an integral that is similar to the integration of the Seiberg-Witten form, the spirit is the same: when the curve degenerate the integral can be written as an asymptotic expansion. The condition for the degeneration is given by the vanishing of the discriminant of the polynomial,

$$D(P_4) = b^2c^2d^2 - 4ac^3d^2 - 4b^3d^3 + 18abcd^3 - 27a^2d^4 - 4b^2c^3e + 16ac^4e$$
$$+ 18b^3cde - 80abc^2de - 6ab^2d^2e + 144a^2cd^2e - 27b^4e^2 + 144ab^2ce^2$$
$$- 128a^2c^2e^2 - 192a^2bde^2 + 256a^3e^3. \quad (79)$$
Now we specific to the quartic polynomial used in our story, it is

\[
P_4(z) = m_0^2 z^4 + (-\Xi + q\Xi - 2q\tilde{m}_0^2 + 2qm_1^2 + m_0^2 - \tilde{m}_0^2 - m_1^2 - \tilde{m}_1^2)z^3
\]

\[
+ (\Xi - q^2\Xi + q^2\tilde{m}_0^2 - q^2m_1^2 - 2qm_0^2 + 2q\tilde{m}_0^2 + 2qm_1^2 + m_1^2 + \tilde{m}_1^2)z^2
\]

\[
+ (-q\Xi + q^2\Xi + q^2m_0^2 - q^2\tilde{m}_0^2 + q^2m_1^2 - q^2\tilde{m}_1^2 - 2q\tilde{m}_1^2)z + q^2\tilde{m}_1^2.
\] (80)

The polynomial does not define the Seiberg-Witten curve of the super QCD, however, the spirit of finding the degenerate points of the associated curve to carry out the asymptotic expansion is similar. In general the solution to the equation \(P_4(z) = 0\) can not be written in a closed form, but here we can turn some parameters very small and asymptotic solutions can be found. Starting from a weak coupling UV theory, we have \(q\) always small. We can also turn the scalar v.e.v which controls the magnitude of \(\Xi\), therefore we can make \(\Delta = \Xi - \Xi_*\) either very large or very small, where \(\Xi_*\) is a particular expansion point.

The asymptotic solution can be expanded by small parameters, the corresponding expansion coefficients can be solve by an iterative method, if we correctly choose the leading order solution and the subsequent expansion pattern. The iterative method works as a self-checking process, it only works well when we choose the right expansion pattern, otherwise it always ends in an obstruction.

The location of the degeneration point \(\Xi_*\) where \(D(P_4)(\Xi_*) = 0\), determines the nature of the asymptotic solution, therefore we need to find its value and do asymptotic expansion near this point. For generic masses, the discriminant of \(P_4(z)\) is a large polynomial of degree six for \(\Xi\). Therefore there are six singularities in the \(\Xi\) space at finite position, each of them is of weight one. Their locations are

\[
\Xi_* \sim -m_1^2 - \tilde{m}_1^2 + (m_0 \pm \tilde{m}_0)^2 + \mathcal{O}(m^2q),
\]

\[
\pm 2m_1\tilde{m}_1 + \mathcal{O}(m^2q),
\]

\[
-m_1^2 - \tilde{m}_1^2 \pm 2(m_0^2 - \tilde{m}_0^2)^{1/2}(m_1^2 - \tilde{m}_1^2)^{1/2}q^{1/2} + \mathcal{O}(m^2q). \quad (81)
\]

Then the shifted coordinates defined by,

\[
\tilde{\Xi} = m_1^2 + \tilde{m}_1^2 + \frac{2q}{1-q}m_0m_1 + \Xi,
\] (82)

has singularities at

\[
\tilde{\Xi}_* \sim (m_0 \pm \tilde{m}_0)^2 + \mathcal{O}(m^2q),
\]

\[
(m_1 \pm \tilde{m}_1)^2 + \mathcal{O}(m^2q),
\]

\[
\pm 2(m_0^2 - \tilde{m}_0^2)^{1/2}(m_1^2 - \tilde{m}_1^2)^{1/2}q^{1/2} + \mathcal{O}(m^2q). \quad (83)
\]
Note that \((m_0 \pm \tilde{m}_0), (m_1 \pm \tilde{m}_1)\) are \(\mu_1, \mu_2, \mu_3, \mu_4\), hence we conclude the first four singularities, denoted as \(\Xi_{1,2,3,4}\), are associated with the flavor \(\mu_i\) which becomes massless at the singularity \(\Xi_{*i}\) of the Coulomb moduli, they are in the semiclassical region in the moduli space. The asymptotic expansion in this region is formula (42), when \(a \sim \mu_i\) then \(\Xi\) is at the corresponding singularity. The four singularities would be pushed to infinity under the decoupling limit for the flavor \(\mu_i\), and remain finite for other mass decoupling limits.

In the semiclassical region, \(\Xi\) is large, we should have set \(\Xi = -m_1^2 - \tilde{m}_1^2 + \Delta\) with \(\Delta\) large, but in section 3 we treat \(\Xi\) itself as a large quantity. We can divide all the coefficients of the polynomial (80) by \(\Xi\), and the degeneration at \(\Xi \sim \infty\) is obvious.

Other two singularities in (81) are associated with magnetic and dyonic particles, in the mass decoupling limit they survive as the corresponding strong coupling singularities for the gauge theory with less flavors.

Before we proceed to solve, it is helpful to notice the following properties the solutions should satisfy for generic cases of parameters:

1. When \(\tilde{m}_0 \to 0\), the equation reduces to third order, it means a root is pushed to infinity.
2. When \(\tilde{m}_1 \to 0\), the equation has a root \(z = 0\).
3. When \(m_0 \to 0\), the equation has a root \(z = 1\)
4. When \(m_1 \to 0\), the equation has a root \(z = q\).

**Large \(\Xi\) solution**

For large \(\Xi\), the leading order of the polynomial reduces to a cubic one, \(P_4(z) = \Xi(q - 1)z(z - q)(z - 1) + \cdots\), therefore the leading order solution is given by the equation \(z(z - q)(z - 1) = 0\) which gives three roots at \(0, q, 1\), the forth root \(z_4\) is near infinity. We use small quantity \(\frac{1}{\Xi}\) to control the expansion, with a little of guess, we have the large expansion of all four roots as presented in (88). The subleading coefficients are iteratively solved. Take \(z_2\) as the example, if we already got the coefficient \(f_j^2, 2 \leq j \leq n\), substitute the trial solution

\[
\tilde{z}_2 = q + \sum_{j=1}^{n} \frac{f_j^2}{\Xi^j} + \frac{f_2^{n+1}}{\Xi^{n+1}},
\]

into the polynomial \(P_4(z)\), then the leading order non-zero coefficient in the \(\frac{1}{\Xi}\) expansion of \(P_4(z)\) is a function of the form \(c_1 + c_2 f_2^{n+1}\) with \(c_{1,2}\) functions of masses and \(q\). We can solve the equation at this order by setting \(c_1 + c_2 f_2^{n+1} = 0\) therefore obtain the coefficient \(f_2^{n+1}\).
In this way we can iteratively solve all $f_i$. We give the first few of them.

\[
\begin{align*}
    f_1^2 &= \frac{qm_1^2(qm_0^2 + qm_1^2 - q\tilde{m}_0^2 - \tilde{m}_1^2)}{(1 - q)^2}, \\
    f_1^3 &= -(1 - q)^{-3}qm_1^2[q^2(m_0^4 + 2m_0^2m_1^2 + m_1^4 - 2m_0\tilde{m}_0^2 - 2m_1\tilde{m}_1^2 + \tilde{m}_0^4 + \tilde{m}_1^4)] \\
    &\quad - 3qm_1^2(m_0^2 + m_1^2 - \tilde{m}_0^2 + \tilde{m}_1^2) / (1 - q)^2, \\
    f_2^2 &= \frac{qm_1^4(1 - 2q)}{1 - q}, \\
    f_2^3 &= -\frac{qm_1^4[q^2(4m_1^2 + \tilde{m}_0^2) + q(m_0^2 - 4m_1^2 - \tilde{m}_0^2 - \tilde{m}_1^2) + m_1^2 + \tilde{m}_1^2]}{(1 - q)^2}, \\
    f_3^2 &= \frac{m_0^2[q(m_1^2 - \tilde{m}_0^2 + \tilde{m}_1^2) + m_0^2 - m_1^2 + \tilde{m}_0^2 - \tilde{m}_1^2]}{1 - q}, \\
    f_3^0 &= \frac{q\tilde{m}_0^2 - 2q\tilde{m}_1^2 - m_0^2 + m_1^2 + \tilde{m}_1^2}{\tilde{m}_0^2}, \\
    f_4^1 &= -\frac{q^2m_1^2 - qm_0^2 - qm_1^2 + q\tilde{m}_1^2 + m_0^2}{1 - q}.
\end{align*}
\]

(85)

Note that $q$ is automatically incorporated into the expansion in a reasonable form. These solution satisfy the properties of limits (1.)-(4.).

**Small $\Delta$ solution**

In order to determine the location for other expansions in the strong coupling regions in the moduli space, we need to find finite value solutions for the six degree equation $\mathcal{D}(P_4)(\Xi) = 0$. In the case of generic mass, the solutions are complicated, however, asymptotic solutions can be found by the iterative method as explained above.

One of the solution takes the form

\[
\Xi_{mag} = -(m_1^2 + \tilde{m}_1^2) + g_n q^{n/2}, \quad n = 1, 2, 3, 4, \ldots
\]

(86)

The first few $g_n$ are

\[
\begin{align*}
    g_1 &= 2((m_0^2 - \tilde{m}_0^2)(m_1^2 - \tilde{m}_1^2))^{1/2}, \\
    g_2 &= \frac{m_1^2\tilde{m}_0^2 + m_0^2\tilde{m}_1^2 - 2m_0^2\tilde{m}_0^2\tilde{m}_1^2 - 2m_1^2\tilde{m}_0^2\tilde{m}_1^2 + m_0^4\tilde{m}_1^2 + m_0^2\tilde{m}_1^4}{(m_0^2 - \tilde{m}_0^2)(m_1^2 - \tilde{m}_1^2)}, \\
    \ldots
\end{align*}
\]

(87)

In the mass decoupling limit only the term $O(q^{1/2})$ survives, it is symmetric w.r.t the masses, therefore the shifted coordinate $\tilde{\Xi}_{mag}$ is finite under all steps of the successive decoupling limits.

Then we set $\Xi = \Xi_{mag} + \Delta$, with $\Delta$ a small quantity compare to $q^{1/2}(m_0^2 - \tilde{m}_0^2)^{1/2}(m_1^2 - \tilde{m}_1^2)^{1/2} = q^{1/2}(\mu_1\mu_2\mu_3\mu_4)^{1/2}$. Substitute $\Xi = \Xi_{mag} + \Delta$ into $P_4(z)$, the polynomial now involves
the small quantity \( \Delta \), then we can continue to iteratively solve the equation \( P_4(z) = 0 \). For
this case, the solution should be written as a double expansion with respect to \( \Delta \) and \( q \),
the iterative method works as well. We have the following leading order solution, and the
general expansion pattern,

\[
z_1 = -\frac{m_0^2 - \tilde{m}_0^2}{m_0^2} + \frac{2m_0^2(m_1^2 - \tilde{m}_1^2)^{1/2}}{m_0^2(m_1^2 - \tilde{m}_1^2)^{1/2}} q^{1/2} + \sum_{n=2}^{\infty} c_n^* q^{n/2} \\
+ \left( \frac{m_0^2}{m_0^2 - \tilde{m}_0^2} + \frac{4m_0^2(m_1^2 - \tilde{m}_1^2)^{1/2}}{(m_0^2 - \tilde{m}_0^2)^{3/2}} q^{1/2} + \sum_{n=2}^{\infty} c_n^* q^{n/2} \right) \Delta \\
+ \sum_{m=2}^{\infty} (\sum_{n=0}^{\infty} c_n^* q^{n/2}) \Delta^m,
\]

\[
z_2 = -\frac{(m_0^2 - \tilde{m}_1^2)^{1/2}}{(m_0^2 - \tilde{m}_0^2)^{1/2}} q^{1/2} + \frac{m_1^2(m_0^2 - \tilde{m}_0^2)^2 - m_0^2(m_1^2 - \tilde{m}_1^2)^2}{(m_0^2 - \tilde{m}_0^2)^2(m_1^2 - \tilde{m}_1^2)^2} q + \sum_{n=3}^{\infty} c_n^* q^{n/2} \\
+ \left( \frac{(m_0^2 - \tilde{m}_1^2)^{1/4}}{(m_0^2 - \tilde{m}_0^2)^{3/4}} q^{1/4} - \frac{2m_0^2(m_1^2 - \tilde{m}_1^2)^{3/4}}{(m_0^2 - \tilde{m}_0^2)^{9/4}} q^{3/4} + \sum_{n=2}^{\infty} c_n^* q^{(n+2)/4} \right) \Delta^{1/2} \\
+ \sum_{m=1}^{\infty} (\sum_{n=0}^{\infty} c_n^* q^{n/2}) \Delta^m + \sum_{m=1}^{\infty} (\sum_{n=0}^{\infty} c_n^* q^{1/4-m/2+n/2}) \Delta^{m+1/2},
\]

\[
z_3 = \frac{(m_0^2 - \tilde{m}_1^2)^{1/2}}{(m_0^2 - \tilde{m}_0^2)^{1/2}} q^{1/2} + \frac{m_1^2(m_0^2 - \tilde{m}_0^2)^2 - m_0^2(m_1^2 - \tilde{m}_1^2)^2}{(m_0^2 - \tilde{m}_0^2)^2(m_1^2 - \tilde{m}_1^2)^2} q + \sum_{n=3}^{\infty} c_n^* q^{n/2} \\
- \left( \frac{(m_0^2 - \tilde{m}_1^2)^{1/4}}{(m_0^2 - \tilde{m}_0^2)^{3/4}} q^{1/4} - \frac{2m_0^2(m_1^2 - \tilde{m}_1^2)^{3/4}}{(m_0^2 - \tilde{m}_0^2)^{9/4}} q^{3/4} + \sum_{n=2}^{\infty} c_n^* q^{(n+2)/4} \right) \Delta^{1/2} \\
+ \sum_{m=1}^{\infty} (\sum_{n=0}^{\infty} c_n^* q^{n/2}) \Delta^m - \sum_{m=1}^{\infty} (\sum_{n=0}^{\infty} c_n^* q^{1/4-m/2+n/2}) \Delta^{m+1/2},
\]

\[
z_4 = -\frac{\tilde{m}_1^2}{m_1^2 - \tilde{m}_1^2} q - \frac{2m_1^2\tilde{m}_1^2(m_0^2 - \tilde{m}_0^2)^{1/2}}{(m_1^2 - \tilde{m}_1^2)^{3/2}} q^{3/2} + \sum_{n=4}^{\infty} c_n^* q^{n/2} \\
- \left( \frac{m_1^2\tilde{m}_1^2}{(m_1^2 - \tilde{m}_1^2)^3} q + \frac{4m_1^2\tilde{m}_1^2(m_0^2 - \tilde{m}_0^2)^{1/2}(m_1^2 + \tilde{m}_1^2)^{3/2}}{(m_1^2 - \tilde{m}_1^2)^{9/2}} q^{3/2} + \sum_{n=4}^{\infty} c_n^* q^{n/2} \right) \Delta \\
+ \sum_{m=2}^{\infty} (\sum_{n=1}^{\infty} c_n^* q^{n}) \Delta^m.
\]

where we use \( c_* \) to denote any coefficients which are functions of only mass parameters.
These solution also satisfy the properties of limits (1.)-(4.). In principle the roots can be
solved order by order, but the coefficients becomes lengthy. The roots indeed give us the small complementary modulus as

$$k^{'2} = \frac{2\Delta^{1/2}}{(m_0^2 - \tilde{m}_0^2)^{1/4}(m_i^2 - \tilde{m}_i^2)^{1/4}q^{1/4}} - \frac{2\Delta}{(m_0^2 - \tilde{m}_0^2)^{1/2}(m_i^2 - \tilde{m}_i^2)^{1/2}q^{1/2}} + \cdots \ll 1. \quad (89)$$

In $k^{'}^2$ we only present the leading order terms that survive in the full mass decoupling limit which scale as $(q\mu_1\mu_2\mu_3\mu_4)^{-n/4}$.

Substitute these data to the integral (24), we finally get the reverse expansion which would be compact if written in terms of physical masses, it begins as

$$\Delta = \left( -2(\mu_1\mu_2\mu_3\mu_4)^{1/4}q^{1/4} + \frac{\sum_{i<j<k} \mu_i^2\mu_j^2\mu_k^2}{2(\mu_1\mu_2\mu_3\mu_4)^{6/4}}q^{3/4} \right) \hat{a}_D$$

$$+ \left( \frac{1}{8} - \frac{3\sum_{i<j<k} \mu_i^2\mu_j^2\mu_k^2}{16(\mu_1\mu_2\mu_3\mu_4)^{3/2}}q^{1/2} \right)$$

$$+ \frac{39\sum_{i<j<k} \mu_i^4\mu_j^4\mu_k^4 - 30(\mu_1\mu_2\mu_3\mu_4)^2\sum_{i<j} \mu_i^2\mu_j^2 - 8(\mu_1\mu_2\mu_3\mu_4)^3}{128(\mu_1\mu_2\mu_3\mu_4)^3}q + \mathcal{O}(q^{3/2}) \right) \hat{a}_D$$

$$+ \mathcal{O}(\hat{a}_D^3), \quad (90)$$

where we have defined $\hat{a}_D = ia_D$, as in the Section 4. The masses appear in a symmetric way as expected. From the leading order expansion obtained above, which is the first order in $\Xi = \Xi_0 + \epsilon^2\Xi_2 + \cdots$ if we recover the abbreviated subscript, we can derive the first order quantum correction to $\Theta = \Theta_0 + \epsilon\Theta_1 + \cdots$ which comes from the shift of $m_0^2, m_1^2$,

$$\Theta_0 = 0, \quad \Theta_1 = \sum_{i=1}^4 \frac{\partial}{\partial \mu_i} \Theta_0 = - \sum_{i=1}^4 \frac{\partial}{\partial \mu_i} \Theta_0, \quad (91)$$

where $\Theta_0 = \Theta_0(\mu_i, \tilde{a}_D, q)$ for the dual expansion.

The mass decoupling procedure is obvious, especially we can decouple any number of flavor. For example decoupling of one flavor is $\mu_4 \to \infty, q \to 0, q\mu_4 = \Lambda_3$; decoupling of two flavors is $\mu_3, \mu_4 \to \infty, q \to 0, q\mu_3\mu_4 = \Lambda_2^2$; etc. In this way we can obtain the dual expansion of theories with $N_f = 0, 1, 2, 3$. Notice that the subleading terms in the $q$-expansion disappear only in the last step when decouple all four flavors.

9 Appendix: Solving $U(z, m, \epsilon)$

We shown here how to solve the Schrödinger equation (11) with the potential (2), to better understand the shift argument in the Section 2 and Section 3. From the computation of the
first order quantum correction it will be clear why the odd order \( \epsilon \) corrections are caused by the shift of the mass parameters.

The higher order WKB components of the wave function \( \Phi(z) \) in (11) get some extra terms,

\[
p_0(z) = i\sqrt{V(z)},
\]

\[
p_1(z) = \frac{\sqrt{\epsilon}}{2}(\ln p_0)' + \frac{q^2(m_0 + m_1) - (q^2m_1 + 2qm_0 + 2qm_1 - m_1)z + (2qm_1 + m_0 - m_1)z^2}{2p_0z(z - q)^2(z - 1)^2},
\]

\[
p_2(z) = \frac{-p_1^2 + ip_1'}{2p_0} + \frac{1}{8p_0z^2}, \quad \ldots \tag{92}
\]

Now in the second order contour integral \( \oint p_1(z)dz \), the total derivative term does not contribute, and we need to evaluate

\[
2\pi i a_1 = \oint_a q^2(m_0 + m_1) - (q^2m_1 + 2qm_0 + 2qm_1 - m_1)z + (2qm_1 + m_0 - m_1)z^2

\frac{2p_0z(z - q)^2(z - 1)^2}{2p_0z(z - q)^2(z - 1)^2}. \tag{93}
\]

It can be shown that \( a_1 \) satisfies

\[
a_1 = -\frac{1}{2}(\frac{\partial}{\partial m_0} + \frac{\partial}{\partial m_1})a_0, \tag{94}
\]

and

\[
\frac{\partial a_1}{\partial \Theta} = -\frac{1}{2}(\frac{\partial}{\partial m_0} + \frac{\partial}{\partial m_1})\frac{\partial a_0}{\partial \Theta}. \tag{95}
\]

Therefore the first order quantum correction to \( h_i \) can be obtained by acting the operator on \( h_i \),

\[
h_i \to h_i - \frac{\epsilon}{2}(\frac{\partial}{\partial m_0} + \frac{\partial}{\partial m_1})h_i + O(\epsilon^2). \tag{96}
\]

Then in the reversed function \( \Xi(a_0, h_i) \) of (11) we can substitute \( a_0 \) by \( a \) and \( h_i \) by the quantum corrected ones, up to the first order quantum correction we get

\[
\Theta = a^2 - m_i^2 - \tilde{m}_1^2 + \frac{(a^2 + m_0^2 - \tilde{m}_0^2)(a^2 + m_1^2 - \tilde{m}_1^2)}{2a^2}q + O(q^2)
\]

\[
+\epsilon \left( m_1 - \frac{a^2m_0 + a^2m_1 + m_0^2m_1 + m_0m_1^2 - m_1\tilde{m}_0^2 - m_0\tilde{m}_1^2}{2a^2}q + O(q^2) \right)
\]

\[
+O(\epsilon^2) \tag{97}
\]

Indeed, it is easy to see that the order \( \epsilon \) correction can be obtained by acting \(-\frac{1}{2}(\frac{\partial}{\partial m_0} + \frac{\partial}{\partial m_1})\) on the leading order result (19). All these shifts of order \( \epsilon \) entirely come from the shift of the mass parameters \( m_0^2, m_1^2 \), but start from order \( \epsilon^2 \) the quantum corrections come from both shift all of the masses and integrals \( \oint p_{2n}dz \) for the potential \( V(z, m) \).

We can proceed to obtain the differential operators for higher order integrals \( a_n \), however, the extra terms of order \( \epsilon \) and \( \epsilon^2 \) in the potential \( U(z, m, \epsilon) \) make higher order WKB analysis much more complicated. Therefore, practically working on the potential \( V(z, m) \) is simpler, then we make the shifts in the final formula.
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