HOMOLOGY SPHERES WITH $E_8$-FILLINGS AND ARBITRARILY LARGE CORRECTION TERMS

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Abstract. In this paper we construct families of homology spheres which bound 4-manifolds with intersection forms isomorphic to $-E_8$. We show that these families have arbitrary large correction terms. This result says that among homology spheres, the difference of the maximal rank of minimal sub-lattice of definite filling and the maximal rank of even definite filling is arbitrarily large.

1. Introduction

1.1. Definite fillings and homology cobordism invariants. If a 3-manifold $Y$ bounds $X$, then we call $X$ a filling of $Y$. If a filling of $Y$ has a definite (even, or spin) intersection form, then the filling is called definite filling (even filling or spin filling respectively). Under the assumption that the homology of a filling has no 2-torsions, an even filling is equivalent to spin filling. If a definite filling has a positive (or negative) definite intersection, then we call the filling positive-definite filling (or negative-definite filling respectively).

Let $Y$ be an integer homology sphere. Rohlin invariant $\mu(Y)$ is defined to be $\sigma(W)/8 \in \mathbb{Z}/2\mathbb{Z}$ for a spin filling $W$ of $Y$. We can assume that the spin filling $W$ is $H_1(W,\mathbb{Z}) = \{0\}$ (we say homologically 1-connected). In this article we mainly consider homologically 1-connected definite fillings.

Ozsváth and Szabó defined a homology cobordism invariant $d$ in [9]. If a 3-manifold has a negative-definite filling of $Y$, then the $d$-invariant has the following restriction.

Theorem 1.1 ([9]). Let $Y$ be an integer homology three-sphere, then for each negative-definite four-manifold $X$ which bounds $Y$, we have the inequality

$$\xi^2 + \text{rk}(H_2(X,\mathbb{Z})) \leq 4d(Y)$$

for each characteristic vector $\xi$. 

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Furthermore, if a homology sphere $Y$ has an even negative-definite filling $W$, then $b_2(W) \leq 4d(Y)$ is satisfied. For example $\Sigma(2,3,5)$ is the boundary of the $-E_8$-plumbing. Here $-E_8$ is the unique unimodular, even, negative-definite, rank 8 quadratic form. The computation $d(\Sigma(2,3,5)) = 2$ means that $b_2$ of any even negative-definite filling is at most 8. The plumbing realized a filling with $b_2 = 8$. If $Y$ has a definite filling with intersection form $nE_8$ for some integer $n$, then the filling is called $nE_8$-filling.

On the other hand, the $d$-invariant of $\Sigma(2,3,7)$ is 0. Thus, if there exists an even negative-definite filling, then the $b_2$ has to be 0. Hence, the $b_2$ of any positive-definite filling is also zero. Since $\mu(\Sigma(2,3,7)) = 1$, it has no homology 1-connected even definite filling. The plumbing of $\Sigma(2,3,7)$ with all weights $-2$ can give an even filling with even intersection form $(-E_8) \oplus H$, where $H$ is the hyperbolic intersection form. This filling is a homologically 1-connected even (equivalently spin) indefinite filling.

In [14] the author defined the following invariants. If $Y$ has an $nE_8$-filling, then we define $g_8$ (or $\underline{g_8}$) to be

$$g_8(Y) = \max\{b_2(W)/8|W: \text{nE}_8\text{-filling of } Y, H_1(W) = \{0\}\},$$

$$\underline{g_8}(Y) = \min\{b_2(W)/8|W: \text{nE}_8\text{-filling of } Y, H_1(W) = \{0\}\}.$$ 

If $Y$ has no $nE_8$-fillings, then $g_8(Y) = -\infty$.

We call the invariant $g_8$ $E_8$-genus. If $Y$ has an $nE_8$-filling, then we can immediately see the following bound

$$2g_8(Y) \leq |d(Y)|.$$

For example, for any integer $n$, $d(\Sigma(2,3,12n + 5)) = 2$ holds. The author in [14] showed $g_8(\Sigma(2,3,12n + 5)) = 1$ when $0 \leq n \leq 13$ or $n = 15$. In [14] we gave the examples with $2g_8(Y) = |d(Y)|$. The simple question is the following:

**Question 1.2.** Among homology spheres $Y$ with non-negative $E_8$-genus is $|d(Y)| - 2g_8(Y)$ bounded?

We give families of Brieskorn homology spheres to obtain negative answers for this question.

### 1.2. Main results

Here we give the main result:

**Theorem 1.3.** For any integer $n$, Brieskorn homology spheres $\Sigma([p],[q],[r])$ for a pairwise coprime positive integer triple $(p,q,r)$ below have homologically 1-connected $-E_8$-fillings with $g_8 = -\bar{\mu} = 1$.

- (i) $(2,8n - 3,14n - 5)$, (ii) $(2,14n + 3,24n + 5)$
- (iii) $(2,16n + 3,26n + 5)$, (iv) $(2,10n - 3,16n - 5)$
- (v) $(5,35n - 2,50n - 3)$, (vi) $(5,25n - 2,40n - 3)$
- (vii) $(3,15n - 2,36n - 5)$, (viii) $(3,9n - 2,24n - 5)$
- (ix) $(3,21n - 4,36n - 7)$, (x) $(3,27n - 4,48n - 7)$
- (xi) $(4,28n - 3,64n - 7)$, (xii) $(4,32n - 3,76n - 7)$
The invariant $\bar{\mu}$ is the Neumann-Siebenmann invariant, which will be defined in Section 2.3. These examples can be useful for realizing desired fillings restricted by gauge theory. For example, see recent Scaduto’s study [12].

**Theorem 1.4.** For positive integer $n$ the correction terms of Brieskorn homology spheres (i), (ii), (iii) and (iv) in Theorem 1.3 have the following inequalities:

\[
2 \left\lceil \frac{n}{2} \right\rceil \leq d(\Sigma(2, 8n - 3, 14n - 5)), \quad 2 \left\lceil \frac{n+1}{2} \right\rceil \leq d(\Sigma(2, 14n+3, 24n+5)), \\
2 \left\lceil \frac{n+1}{2} \right\rceil \leq d(\Sigma(2, 16n+3, 26n+5)), \quad 2 \left\lceil \frac{n}{2} \right\rceil \leq \Sigma(2, 10n - 3, 16n - 5).
\]

These theorems say that for any positive integer $n$, the Brieskorn homology spheres (i), (ii), (iii), and (iv) have $-E_8$-fillings and $d(Y) - 2g_8(Y) = d(Y) + 2\bar{\mu}(Y)$ are arbitrarily large.

**Remark 1.5.** Let $(Y, c)$ be a pair of Seifert rational homology sphere $Y$ and a spin structure $c$. According to [18] the $\bar{\mu}(Y, c)$ is the equivalent to the Fukumoto-Furuta invariant $w(Y, c)$.

Manolescu in [6] defined homology cobordism invariants $\alpha$, $\beta$, and $\gamma$ in the framework of Pin(2) Seiberg-Witten Floer homology. A result in [13] says that for any Brieskorn homology sphere $Y$ (with usual orientation) $\beta(Y) = \gamma(Y) = -\bar{\mu}(Y)$ and $\alpha(Y) = d(Y)/2$ or $d(Y)/2 + 1$. Hence, our result means the existence of homology spheres that $\beta(Y) = 1$ and $\alpha(Y)$ is arbitrarily large.

**Remark 1.6.** As a conjecture, the inequalities in Theorem 1.4 would become the equalities actually for any positive integer $n$. The evidence is due to Karakurt’s program [5]. Similarly, for positive integer $n$ we predict the following equalities for other Brieskorn homology spheres in Theorem 1.3

- For $(p, q, r) = (5, 35n - 2, 50n - 3), (5, 25n - 2, 40n - 3)$, we have $d(\Sigma(p, q, r)) = 6n$.
- For $(p, q, r) = (3, 15n - 2, 36n - 5), (3, 9n - 2, 24n - 5), (3, 21n - 4, 36n - 7), (3, 27n - 4, 48n - 7)$, we have $d(\Sigma(p, q, r)) = 2n$.
- For $(p, q, r) = (4, 28n - 3, 64n - 7), (4, 32n - 3, 76n - 7)$, we have $d(\Sigma(p, q, r)) = 4 \left( \frac{n}{2} + \left\lceil \frac{n}{2} \right\rceil \right)$.

For non-positive integer $n$ we conjecture that for any homology sphere above $d(\Sigma(|p|, |q|, |r|))$ are all 2.

Here we compare the following result by Ue [17] with the result above.

**Theorem 1.7** ([17]). Let $(S, c)$ be a pair of a spherical 3-manifold and a spin structure on it. Then $d(S, c) = -2\bar{\mu}(S, c)$. 

In fact, in [17] it is shown that the general correction term $d(S, t)$ of spin$^c$ spherical 3-manifold coincides with the Fukumoto-Furuta invariant, which is defined by using the index of the Dirac operator of a line bundle over a 4-orbifold.

We remark $\bar{\mu}$ here is defined to be the $\bar{\mu}$ divided by 8 in [17]. The examples in Theorem 1.3 are homology spheres that $d(\Sigma) + 2\bar{\mu}(\Sigma)$ are arbitrarily large. The relationship between correction term $d$ and Neumann-Siebenmann invariant $\bar{\mu}$ for non-spherical 3-manifolds is known not so many things even in the case of Seifert 3-manifolds.

1.3. Other invariants related definite fillings.

1.3.1. Homologically 1-connected fillings. In [14] homology cobordism invariant $\mathfrak{d}_8(Y)$ is defined to be

$$\text{the maximal } b_2(W)/8 \text{ among homologically 1-connected, even definite fillings } W$$

of $Y$. A homologically 1-connected even filling gives a spin filling. Any invariant related to a kind of filling is defined to be $-\infty$, if there exists no such a kind of fillings. We define $\sigma(Y)$ to be

$$\text{the maximal rank of the minimal definite lattice } L \text{ that } L \oplus (\pm 1)^n \text{ is the intersection form of a homologically 1-connected definite filling } W$$

of $Y$. Here ‘minimal’ means that any square $\pm 1$ element is not included in the lattice and $n$ is some non-negative integer. By the definition, we have

$$8g_8(Y) \leq \mathfrak{d}_8(Y) \leq \sigma(Y).$$

For example, in the case of $Y = \Sigma(2, 5, 9)$, according to Corollary 1.2 in [12], $g_8(Y) = \mathfrak{d}_8(Y) = 1$ and $\sigma(Y) = 12$ holds. We have the following question.

**Question 1.8. Differences of these invariants are bounded or unbounded?**

We can show that homology spheres in Theorem 1.4 satisfy the following unbounded property.

**Corollary 1.9. Among homology spheres $Y$, $\sigma(Y) - 8\mathfrak{d}_8(Y)$ is unbounded.**

These integer homology spheres satisfy $\mathfrak{d}_8(Y) = g_8(Y)$. It is not known whether for some integer homology spheres $Y$ the differences $\mathfrak{d}_8(Y) - g_8(Y)$ are positive or unbounded.

1.3.2. General definite fillings. Define $E(Y)$ to be

$$\text{the maximal among } b_2(W)/8 \text{ even definite filling } W$$

of $Y$. Note the filling is possibly non-spin. In the same way, we define $O(Y)$ to be

$$\text{the maximal rank of minimal sub-lattice of definite fillings } W$$

of $Y$ and $G_8(Y)$ to be

$$\text{the maximal } |n| \text{ among } nE_8\text{-fillings}$$
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of $Y$. Since fillings for invariants $G_8(Y), E(Y), O(Y)$ do not assume homologically 1-connected, we have $g_8(Y) \leq G_8(Y), \mathfrak{d}_8(Y) \leq E(Y)$, and $\mathfrak{o}(Y) \leq O(Y)$. We have the similar inequality to (2):

(3) $8G_8(Y) \leq 8E(Y) \leq O(Y)$.

For example, consider the case of $Y = \Sigma(2, 3, 7)$. As we mentioned above $Y$ has no even definite fillings with homologically 1-connected, i.e., $g_8(Y) = \mathfrak{d}_8(Y) = -\infty$. On the other hand, Fintushel and Stern constructed a rational ball that bounds $Y$ in [2], i.e., $E(Y) = G_8(Y) = 0$. Let $W$ be a negative-definite filling of $Y$ and $\Xi$ the set of characteristic elements in $H_2(W)$. $d(Y) = 0$ means $\max_{c \in \Xi}(c^2 + b_2(W)) \leq 0$. The Elkies theorem in [1] concludes the inequality means the equality and the negative-definite lattice must be diagonalized. For the positive-definite filling of $Y$ one has only to consider negative-definite fillings of $-Y$. In fact the plumbing lattice of $\Sigma(2, 3, 7)$ is diagonalized, therefore, $O(Y) = \mathfrak{o}(Y) = 0$.

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2. Notations and Preliminaries

2.1. Plumbing diagram. For $i = 1, 2$ let $V_i \to S^2$ be two $D^2$-bundles over $S^2$ or let $V \to S^2$ be a $D^2$-bundle. For the $D^2$-bundles $V_1$ and $V_2$ we take sub-$D^2$-bundles over each disk in two base spaces $S^2$. For $V \to S^2$ we take sub-$D^2$-bundles over two disjoint disks in $S^2$. Plumbing process is a surgery obtained by identifying two $D^2$-bundles in such a way that one exchanges the roles of their sections and fibers. We call the plumbing of $V \to S^2$ self-plumbing. Actually, to define the plumbing process we need choose one of the two possibilities of the orientation of the identification as in p.201 in [4]. Since we deal with the tree-type graph only later, then we do not explain the choices.

We define a plumbing diagram (or graph) as explained in [11]. Let $V$ be the set of vertices with a weight function $m : V \to \mathbb{Z}$. We assign for $v \in V$ the $D^2$-bundle over $S^2$ with the Euler number $m(v)$. Let $E$ be the set of edges. Each edge $\{v, w\} \in E$ of a plumbing diagram means the plumbing process between the $D^2$-bundles over $S^2$. If $v = w$, then the edge means the self-plumbing. Hence, if $\mathbb{Z}$-weighted graph $(V, E, m)$ is called plumbing graph or plumbing diagram.

Let $(V, E, m)$ be a plumbing diagram. The plumbing process along a plumbing graph $\Gamma = (V, E, m)$ gives a 4-manifold $P(\Gamma)$ and we call $P(\Gamma)$
a plumbed 4-manifold. The boundary $\partial P(\Gamma)$ of $P(\Gamma)$ is called a plumbed 3-manifold. Here $[v]$ is the class represented by the core sphere of the $D^2$-bundle corresponding to the vertex $v$. The intersection form $(\cdot, \cdot) : H_2(P(\Gamma)) \times H_2(P(\Gamma)) \to \mathbb{Z}$ on $P(\Gamma)$ is computed from the linear extension of the following definition.

$$([v], [w]) = \begin{cases} m(v) & v = w \\ 1 & v \neq w \text{ and } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

A tree-type graph with at most one vertex of degree larger than two is called a star-shaped graph. A plumbed 3-manifold with a star-shaped graph is called a Seifert manifold. For $i = 1, 2, \cdots, n$, let $(\alpha^{(i)}_1, \alpha^{(i)}_2, \cdots, \alpha^{(i)}_{r_i})$ be a sequence of weights of vertices of the $i$-th branch of a star-shaped graph. We compute the continued fraction for the sequence as follows:

$$(4) \quad \frac{a_i}{b_i} = [\alpha^{(i)}_1, \alpha^{(i)}_2, \cdots, \alpha^{(i)}_{r_i}],$$

where $\alpha^{(i)}_j$ is some integer and $(a_i, b_i)$ are coprime integers. Here the continued fraction is defined to be

$$[c_1, c_2, \cdots, c_m] = c_1 - \frac{1}{c_2 - \cdots - \frac{1}{c_m}}.$$}

The Seifert manifold with the rational numbers $a_i/b_i (i = 1, 2, \cdots, n)$ and the central weight $e$. We present such a Seifert manifold as

$$S(e; (a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)).$$

We call these integers the Seifert invariant. Instead of $(a_i, b_i)$, we also present it as $\alpha^{(i)}_1 \cdot \alpha^{(i)}_2 \cdots \alpha^{(i)}_{r_i}$. Thus, we denote the plumbing process by dot ·. We present several consecutive integers by the power as follows:

$$\cdots 2 \cdot 2 \cdot \cdots = \cdots 2^{[m]} \cdots .$$

Brieskorn homology sphere $\Sigma(p, q, r)$ is defined to be

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^p + z_2^q + z_3^r = 0\} \cap S^5,$$

where $p, q,$ and $r$ are pairwise coprime positive integers. The manifold is a plumbed 3-manifold with a branch number three star-shaped graph. The Seifert invariant is

$$S(e; (p, p'), (q, q'), (r, r')),$$

where $e - (p'/p + q'/q + r'/r) = -1/\text{pqr}$. For example, the plumbing diagram of $\Sigma(2, 3, 5)$ is described as follows:

$$S(-2; (-2)^{[4]}, (-2)^{[2]}, -2).$$

In [14], the author showed that the Brieskorn homology sphere $\Sigma(p, q, r)$ whose intersection matrix of minimal plumbed 4-manifold is isomorphic to $-E_8$ is $\Sigma(2, 3, 5)$ or $\Sigma(3, 4, 7)$.  

2.2. Notations. We explain two new notations as below. The first notation is the following:

\[ L(a_1 \cdot a_2 \cdots a_n \cdot (k) b_m \cdot b_{m-1} \cdots b_1), \]

It presents the plumbing as in Figure 1. Here the integers \( a_i, b_j \) and \( k \) are integers. A dot with \((k)\) means the two components with framing \( a_n \) and \( b_m \) geometrically link \( k \)-times. Here, the box with the \( k \) means the full \( k \)-twist.

As an example, we consider a linear diagram of \( \Sigma(2, 3, 5) \). Sliding the first branch of \( S(-2; (-2)^[4], (-2)^[2], -2) \) to the third branch, we have the following linear diagram:

\[ S(-2; (-2)^[4], (-2)^[2], -2) = L((-2)^[2] \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2)^[3]) = L((-2)^[4] \cdot (-2) \cdot (-2)^[3]). \]

The 4-manifold having the framed link as in Figure 1 is called 4-manifold having linear diagram (5).

The second notation is a linear diagram with torus knot component. Consider a linear diagram that the framing of the nearest component to \((k)\) is zero. Then, the surgery diagram can be deformed as in Figure 2. We present the deformation of the linear diagram as follows:

\[ Y = L(\cdots q \cdot n \cdot 0^{(k)} p \cdots) = L(\cdots q^{(k)} p + nk^2)_{(k, nk+1)} \cdots. \]

The component with the underline having index \((k, nk + 1)\) stands for the \((k, nk + 1)\)-torus knot with framing \( p + nk^2 \). The last surgery diagram gives a 4-manifold \( X \) bounded by \( Y \). Clearly, the intersection form of \( X \) is isomorphic to the intersection form of the 4-manifold having the linear diagram as follows:

\[ L(\cdots q^{(k)} (p + nk^2) \cdots). \]

2.3. An estimate of \( \bar{\mu} \)-invariant. Neumann-Siebenmann’s \( \bar{\mu} \)-invariant in [7] is defined for any plumbed 3-manifold \( M = \partial P(\Gamma) \). We set \( \Gamma = (V, E, m) \). We assume that the plumbing graph is tree and \( \partial P(\Gamma) \) is a homology sphere. We define the Wu class \( w(\Gamma) \in H_2(P(\Gamma), \mathbb{Z}) \) as follows:

1. The class \( w(\Gamma) \) is written by \( w(\Gamma) = \sum_{v \in V} \epsilon_v [v] \) for \( \epsilon_v = 0 \) or 1.
2. For any \( v \in V \) we have \((w(\Gamma), [v]) \equiv ([v], [v]) \mod 2\).
Let $\sigma(\Gamma)$ be the signature of the intersection form $(\cdot, \cdot)$ associated with $\Gamma$. Then we define the $\bar{\mu}$-invariant of $M$ to be

$$\bar{\mu}(M) = \frac{\sigma(\Gamma) - w(\Gamma)^2}{8}.$$ 

The invariant $\bar{\mu}$ can be extended to any rational plumbed spin 3-manifold $(M, c)$ naturally as in [7].

**Theorem 2.1 ([7]).** Suppose that a Seifert rational homology 3-sphere $M$ with spin structure $c$ bounds a negative-definite spin 4-manifold $Y$ with spin structure $c_Y$. Then

$$b_2(Y) \equiv -8\bar{\mu}(M, c) \mod 16$$
$$\bar{\mu}(M, c) = \frac{\sigma(\Gamma) - w(\Gamma, c)^2}{8}$$
$$-\frac{8\bar{\mu}(M, c)}{9} \leq b_2(Y) \leq -8\bar{\mu}(M, c).$$

In particular, if a Seifert homology sphere $M$ has a spin negative-definite filling $Y$, then $b_2(Y) \leq -8\bar{\mu}(M)$ holds.

3. **The families of Brieskorn homology spheres in Theorem 1.1.**

3.1. **Proof of Theorem 1.3**. We prove Theorem 1.3

**Proof.** The Seifert invariants of

- $\Sigma(2, 14n - 5, 8n - 3), \Sigma(2, 24n + 5, 14n + 3)$
- $\Sigma(2, 26n + 5, 16n + 3), \Sigma(2, 10n - 3, 16n - 5)$
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- $\Sigma(5, 35n - 2, 50n - 3), \Sigma(5, 25n - 2, 40n - 3)$
- $\Sigma(3, 15n - 2, 36n - 5), \Sigma(3, 9n - 2, 24n - 5)$
- $\Sigma(2, 14n - 5, 8n - 3), \Sigma(2, 16n - 5, 10n - 3)$
- $\Sigma(3, 21n - 4, 36n - 7), \Sigma(3, 27n - 4, 48n - 7)$
- $\Sigma(4, 28n - 3, 64n - 7), \Sigma(4, 32n - 3, 76n - 7)$

with reverse orientation are as follows: $S(1; (p_1, 1), (p_2, q_2), (p_3, q_3))$.

| $p_1$ | $(p_2, q_2)$ | $(p_3, q_3)$ | $p_1$ | $(p_2, q_2)$ | $(p_3, q_3)$ |
|-------|--------------|--------------|-------|--------------|--------------|
| 2     | $(14n - 5, 7n - 6)$ | $(8n - 3, 2)$ | 2     | $(14n + 3, 7n - 2)$ | $(24n + 5, 6)$ |
| 2     | $(26n + 5, 13n - 4)$ | $(16n + 3, 4)$ | 2     | $(10n - 3, 5n - 4)$ | $(16n - 5, 4)$ |
| 5     | $(35n - 2, 28n - 3)$ | $(50n - 3, 2)$ | 5     | $(40n - 3, 32n - 4)$ | $(25n - 2, 1)$ |
| 3     | $(15n - 2, 10n - 3)$ | $(36n - 5, 4)$ | 3     | $(24n - 5, 16n - 6)$ | $(9n - 2, 1)$ |
| 3     | $(21n - 4, 14n - 5)$ | $(36n - 7, 4)$ | 3     | $(48n - 7, 32n - 10)$ | $(27n - 4, 3)$ |
| 4     | $(28n - 3, 21n - 4)$ | $(64n - 7, 4)$ | 4     | $(76n - 7, 57n - 10)$ | $(32n - 3, 2)$ |

We deform the presentation as follows. The notation $\sim$ stands for a deformation of presentation preserving the intersection form.

$$S(1; 2, 2 \cdot (n + 1) \cdot 2^{[6]}, (2 - 4n) \cdot 2) = S(0, 2, (-2) \cdot n \cdot 2^{[6]}, (2 - 4n) \cdot 2)$$

$$= L(2^{[6]} \cdot n \cdot 0.2) (4 - 4n) = L(2^{[6]} \cdot 4 \cdot 2_{(2,2n+1)} \cdot 2) \sim L(2^{[6]} \cdot 2 \cdot 2)$$

$$S(1; 2, 2 \cdot (n + 1) \cdot 4 \cdot 2, (-4n) \cdot 2^{[5]}) = S(0; 2, (-2) \cdot n \cdot 4 \cdot 2, (-4n) \cdot 2^{[5]})$$

$$= L(2^{[5]} \cdot 2 \cdot 4 \cdot 2 \cdot 0 \cdot n \cdot 4 \cdot 2 = L(2^{[5]} \cdot 4 \cdot 2_{(2,2n+1)} \cdot 2) \sim L(2^{[5]} \cdot 2 \cdot 2)$$

$$S(1; 2, 2 \cdot (n + 1) \cdot 4 \cdot 2^{[3]}, (-4n) \cdot 2^{[3]}) = S(0; 2, (-2) \cdot n \cdot 4 \cdot 2^{[3]}, (-4n) \cdot 2^{[3]})$$

$$= L(2^{[3]} \cdot 4 \cdot n \cdot 4 \cdot 2^{[3]} = L(2^{[3]} \cdot 4 \cdot 2_{(2,2n+1)} \cdot 2^{[3]})$$

$$\sim L(2^{[3]} \cdot 4 \cdot 2)$$

$$S(1; 2, 2 \cdot (n + 1) \cdot 2^{[4]}, (2 - 4n) \cdot 2^{[3]}) = S(0; 2, (-2) \cdot n \cdot 2^{[4]}, (2 - 4n) \cdot 2^{[3]})$$

$$= L(2^{[4]} \cdot n \cdot 4 \cdot 2^{[3]} = L(2^{[4]} \cdot 4 \cdot 2_{(2,2n+1)} \cdot 2^{[3]}) \sim L(2^{[4]} \cdot 4 \cdot 2^{[3])}$$

$$S(0; -5, 5 \cdot n \cdot (-7), (2 - 25n) \cdot 2) = L((-7) \cdot n \cdot 0 \cdot (-5) \cdot (-3 - 25n) \cdot 2)$$

$$= L((-7) \cdot (5, -5, -5n + 1) \cdot 2) = L(2 \cdot 1 \cdot (5, -5, -5n + 1) \cdot 2)$$

Here we slide the $-3$-framed $(-5, -5n + 1)$-torus knot component to the component with $(-5)$-framed component. Then we have a 4-manifold with intersection form of the plumbed 4-manifold for $S(1; 2, 2^{[2]}, -5)$. By doing four blow-ups and one blow-down, we have the intersection $E_8$.

In the same way, the 4-manifolds that the last diagrams in the following equalities present can be deformed into 4-manifolds with intersection form
$E_8$. The results of the latter four equalities are a plumbed 4-manifold that reduces to the Seifert invariant $S(2; 2^{[2]}, 2^{[4]}, 2 \cdot 4) = -\Sigma(3, 4, 7)$.

$$S(0; -5, 5 \cdot n \cdot (-8), (2 - 25n)) = L((-3 - 25n) \cdot (-5) \cdot n \cdot (-5) \cdot 1 \cdot 2^{[2]}$$

$$= L((-3, -5n + 1) \cdot (-5) \cdot n \cdot (-5) \cdot 1 \cdot 2^{[2]})$$

$$S(0; -3, 3 \cdot n \cdot (-5), (2 - 9n) \cdot 2^{[3]}) = L((-5) \cdot n \cdot 0 \cdot (-3) \cdot (-1 - 9n) \cdot 2^{[3]})$$

$$= L(2 \cdot 1 \cdot (-3) \cdot (-3) \cdot (-3) \cdot (1) \cdot (-3, -3n + 1) \cdot 2^{[3]})$$

$$S(0; -3, 3 \cdot n \cdot (-8), (2 - 9n)) = L(2^{[4]} \cdot 1 \cdot (-3) \cdot n \cdot 0 \cdot (-3) \cdot (-1 - 9n))$$

$$= L(2^{[4]} \cdot 1 \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-1, -3n + 1))$$

$$S(1; 2, 2 \cdot n \cdot (-7), (2 - 4n) \cdot 2^{[2]}) = S(0; -2, 2 \cdot n \cdot (-7), (2 - 4n) \cdot 2^{[2]})$$

$$= L(2^{[4]} \cdot 1 \cdot (-2) \cdot n \cdot 0 \cdot (-2) \cdot (2 - 4n) \cdot 2^{[2]})$$

$$= L(2^{[4]} \cdot 1 \cdot (-2) \cdot (-2) \cdot 2 \cdot (-2, -2n + 1) \cdot 2^{[2]})$$

$$S(1; 2, 2 \cdot n \cdot (-6), (2 - 4n) \cdot 2^{[3]}) = S(0; -2, 2 \cdot n \cdot (-6), (2 - 4n) \cdot 2^{[3]})$$

$$= L(2^{[3]} \cdot 1 \cdot (-2) \cdot n \cdot 0 \cdot (-2) \cdot (-4n) \cdot 2^{[3]})$$

$$= L(2^{[3]} \cdot 1 \cdot (-2) \cdot (-2) \cdot 0 \cdot (-2, -2n + 1) \cdot 2^{[3]})$$

$$S(0; -3, 3 \cdot n \cdot (-7), (2 - 9n) \cdot 4) = L(2^{[3]} \cdot 1 \cdot (-3) \cdot n \cdot 0 \cdot (-3) \cdot (-1 - 9n) \cdot 4)$$

$$= L(2^{[3]} \cdot 1 \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-1, -3n + 1) \cdot 4)$$

$$S(0; -3, 3 \cdot n \cdot (-5) \cdot 3, (2 - 9n) \cdot 2^{[2]})$$

$$= L(2^{[2]} \cdot (-1 - 9n) \cdot (-3) \cdot n \cdot (-3) \cdot 1 \cdot 2 \cdot 4)$$

$$= L(2^{[2]} \cdot (-1, -3n + 1) \cdot (-3) \cdot (-3) \cdot 1 \cdot 2 \cdot 4)$$

$$S(0; -4, 4 \cdot n \cdot (-7), (2 - 16n) \cdot 4)$$

$$= L(4 \cdot (-2 - 16n) \cdot (-4) \cdot n \cdot (-4) \cdot 1 \cdot 2^{[2]})$$

$$= L(4 \cdot (-2 - 16n) \cdot (-4) \cdot (-4) \cdot 1 \cdot 2^{[2]})$$

$$S(0; -4, 4 \cdot n \cdot (-6) \cdot 3, (-2 - 16n) \cdot 2)$$

$$= L(2 \cdot (-2 - 16n) \cdot (-4) \cdot n \cdot (-4) \cdot 1 \cdot 2 \cdot 4)$$

$$= L(2 \cdot (-2 - 16n) \cdot (-4) \cdot (4) \cdot 1 \cdot 2 \cdot 4)$$

According to the definition of $\mu$ as above, computing the $\mu$-invariants for these Brieskorn homology spheres, we can see $\mu = -1$ easily. From the description under Theorem 2.1 we obtain $g_8 \leq 1$. Namely, the homology spheres have all $g_8 = 1$. □
4. Brieskorn homology spheres with $E_8$-filling and arbitrarily large correction terms.

4.1. Heegaard Floer homology and one preparation. In [9] for any spin$^c$ rational homology sphere $(Y, s)$ the Heegaard Floer homology $HF^+(Y, s)$ has the following exact sequence:

$$0 \to T^+_{d(Y,s)} \to HF^+(Y, s) \to HF_{\text{red}}(Y, s) \to 0.$$

$T^+_s$ is isomorphic to $T^+ := \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$ with the minimal degree $s$. $HF_{\text{red}}(Y, s)$ is a finite dimensional torsion $\mathbb{F}[U]$-module. $d(Y, s)$ is called the correction term of $(Y, s)$. We call the submodule $T^+_{d(Y,s)}$ in $HF^+(Y, s)$ the $T^+$-part of $HF^+(Y, s)$.

Here we prepare a lemma to prove Theorem [12]. We abbreviate $d(L(p, q), i)$ by $d(p, q, i)$. Here we define the lens space $L(p, q)$ to be the $p/q$-surgery of the unknot. The identification of spin$^c$ structures with $\mathbb{Z}/p\mathbb{Z}$ is due to Fig.2 in [9]. Here the $p$-Dehn surgery of a knot $K$ in a homology sphere $Y$ is the surgery $(Y \setminus S^1 \times D^2) \cup V$, where $V \cong S^1 \times D^2$. Here the attaching meridian of the new solid torus $V$ is mapped to $p \cdot [m] + [l] \in H_1(\partial(S^1 \times D^2))$ where $m$ is the meridian of $K$ and $l$ is the homologically trivial longitude of $K$. We denote the $p$-Dehn surgery of a knot in $\Sigma$ by $\Sigma(p)$.

**Lemma 4.1.** Let $\Sigma$ be a homology sphere and $K \subset \Sigma$ a knot. For some positive integer $p$, if $\Sigma(p-1)$ is an L-space and $\Sigma(p)$ is a lens space $L(p, q)$, then the correction term $d(\Sigma)$ is computed as follows.

$$(\text{6}) \quad d(\Sigma) = \max \{ d(p, q, ki + c) - d(p, 1, i) | 0 \leq i < p \},$$

where $k$ is the dual class of $[\tilde{K}] \in H_1(L(p, q), \mathbb{Z})$ and $c = (k + 1 + p)(k - 1)/2$, where $\tilde{K}$ is the surgery dual of the lens space surgery.

Let $C$ be a core circle of the genus one Heegaard decomposition. Then the dual class $k$ is defined $k[C] = [\tilde{K}] \in H_1(L(p, q), \mathbb{Z})$. The dual class is used in the situation of the integral lens space surgery of a homology sphere in [16].

**Proof.** We use the following surgery exact sequence (Corollary 9.13 in [9]):

$$\cdots \to HF^+(\Sigma) \to HF^+(\Sigma(p-1)) \xrightarrow{G^+} HF^+(\Sigma(p)) \xrightarrow{F^+} HF^+(\Sigma) \to \cdots.$$

Since $\Sigma(p-1)$ and $\Sigma(p)$ are L-spaces and the corresponding map $F^\infty$ on $HF^\infty$ is surjective, $F^+$ is also surjective onto the $T^+$-part in $HF^+(\Sigma)$. The map $F^+$ is induced from the cobordism $\Sigma(p)$ to $\Sigma$ obtained by attaching a 0-framed 2-handle along the meridian of $\tilde{K}$. The spin$^c$ structures on $\Sigma(p)$ are identified with $\mathbb{Z}/p\mathbb{Z}$ due to the description in p.213 in [9]. For any integer $j$ with $0 \leq j < p$ consider the surgery exact sequence in Theorem 9.19 in [8]:

$$\cdots \to HF^+(\Sigma) \to HF^+(\Sigma(0), [j]) \to HF^+(\Sigma(p), j) \xrightarrow{F^+} HF^+(\Sigma) \to \cdots.$$
$F_j^+$ is a component of $F^+$ restricted to the spin$^c$ structure $j$. It is also a sum of homogeneous maps $f_i^+$ with respect to the spin$^c$ cobordism from $(\Sigma(p), j)$ to the unique spin$^c$ manifold on $\Sigma$. Namely, $F_j^+$ is described by the sum $F_j^+ = \sum_{j=1}^{p} f_i^+$. The degree shift of $f_i^+$ is $(4p-(2i-p)^2)/(4p)$ due to \cite{9}. The maximal degree shift among $\{f_i^+|j \equiv i \mod p\}$ is $(4p-(2j-p)^2)/(4p) = -d(p,1,j)$. Since $F_j^+$ is a surjective $U$-equivariant map, for $0 \leq j < p$ we have
\[ d(\Sigma) \geq d(p,q,kj+c) - d(p,1,j). \]

The 1-1 correspondence $\mathbb{Z}/p\mathbb{Z} \to \text{Spin}^c(L(p,q))$ in Corollary 7.5 in \cite{9} is described by $ki+c$. See \cite{15}.

Suppose that $d(\Sigma) > d(p,q,kj+c) - d(p,1,j)$ for any integer $j$ with $0 \leq j < p$. Then any element with the minimal degree in $HF^+(\Sigma(p))$ is included in the kernel of $F^+ = \sum_{0 \leq j < p} F_j^+$. Thus the kernel of $F^+$ includes at least $p$ components. On the other hand, for a sufficient large number $N$, $\ker(F^+)/(U^N = 0)$ is $(p-1)$-fold direct sum of $T^+$ from the exact sequence of the version of $HF^\infty$. Hence, this implies that in the image of $G^+$ there is a torsion $\mathbb{F}[U]$-module by at least one component. However, since $\Sigma(p-1)$ is an L-space, the image of $G^+$ has no torsion $\mathbb{F}[U]$-module. This is a contradiction. Therefore for some $j$, $d(\Sigma) = d(p,q,kj+c) - d(p,1,j)$ holds. \hfill \Box

4.2. The $d$-invariants for the four families of Brieskorn homology spheres. We prove Theorem 1.4

**Proof.** The Seifert presentations of Brieskorn homology spheres from (i) to (iv) in Theorem 1.3 are the below:

|   |   |
|---|---|
| (i) | $S(1; 2, 2 \cdot (-n + 1) \cdot 7, (4n - 1) \cdot 2)$ |
| (ii) | $S(1; 2, 2 \cdot (-n) \cdot (-3) \cdot 2, (4n + 1) \cdot 6)$ |
| (iii) | $S(1; 2, 2 \cdot (-n) \cdot (-3) \cdot 4, (4n + 1) \cdot 4)$ |
| (iv) | $S(1; 2, 2 \cdot (-n + 1) \cdot 5, (4n - 1) \cdot 4)$ |

Let $\Sigma_n$ be one of Brieskorn homology spheres parametrized by $n$ in the list above. We do 0-surgery and +1-surgery of the homology sphere along the meridian of the singular fiber of multiplicity 2. We call the meridian $K_n$. Note the coefficients 0 and 1 are the framing of the unknot $K_n$ in the diagram. The 0-surgery and 1-surgery give lens spaces $L(r_n, s_n)$ and $L(p_n, q_n)$. The results are the lens spaces in the list below.

|   |   |
|---|---|
| 0-surgery $(r_n, s_n)$ | 1-surgery $(p_n, q_n)$ |
| (i) | $56n^2 - 41n + 7, 8n^2 - 7n + 2$ | $56n^2 - 41n + 8, 8n^2 - 7n + 1$ |
| (ii) | $65n^2 + 71n + 7, 72n^2 + 27n + 4$ | $65n^2 + 71n + 8, 72n^2 + 27n + 1$ |
| (iii) | $208n^2 + 79n + 7, 48n^2 + 17n + 2$ | $208n^2 + 79n + 8, 48n^2 + 17n + 1$ |
| (iv) | $80n^2 - 49n + 7, 16n^2 - 13n + 4$ | $80n^2 - 49n + 8, 16n^2 - 13n + 1$ |

These examples satisfy $p_n = r_n + 1$. As a result, the 0-surgery means a positive $r_n$-Dehn surgery along $K_n$.

We set $d_n := d(\Sigma_n)$. Here using Lemma 4.1 we compute the lower bound of $d_n$. We argue the case of (i) only. Other cases are able to be proven by
similar arguments. Let \( \Sigma_n \) be the Brieskorn homology sphere in the type (i). We set \( p_n = 56n^2 - 41n + 8 \), \( q_n = 8n^2 - 7n + 1 \), \( k_n = 14n - 5 \) and \( c_n \equiv 42n^2 - 29n + 4 \mod p_n \). The \( k_n \) is the dual class in the lens space \( L(p_n, q_n) \) which presents \( K_n \).

Here we set \( i = \lfloor \frac{q_n+1}{2} \rfloor - n \). Then by modulo \( p_n \) we have

\[
k_ni + c_n \equiv \begin{cases} \frac{7n-5}{2} & \text{n: odd} \\ -\frac{7n}{2} & \text{n: even}. \end{cases}
\]

If \( n \) is an odd number, then by using the reciprocity formula in \([9]\), we have

\[
d(L(p_n, q_n), \frac{7n-5}{2}) = \frac{224n^3 + 8n^2 - 95n + 25}{4p_n}
\]

and

\[
d(L(p_n, 1), i) = -\frac{52n^2 - 37n + 7}{4p_n}
\]

Thus we have

\[
d(L(p_n, q_n), k_ni + c_n) - d(L(p_n, 1), i) = n + 1.
\]

If \( n \) is an even number, then we have

\[
d(L(p_n, q_n), -\frac{7n}{2}) = \frac{224n^3 - 216n^2 + 73n - 8}{4p_n}
\]

and

\[
d(L(p_n, 1), i) = -\frac{52n^2 - 41n + 8}{4p_n}
\]

Thus we have

\[
d(L(p_n, q_n), k_ni + c_n) - d(L(p_n, 1), i) = n.
\]

Therefore we have \( d_n \geq 2\left\lceil \frac{n}{2} \right\rceil \).

4.3. **Proof of Corollary 1.9** Here we prove Corollary 1.9

**Proof.** Let \( \{\Sigma_n\} \) be one family of homology spheres in Theorem 1.4. Since the correction term is positive, \( \Sigma_n \) is no even positive-definite filling of \( \Sigma_n \). A Seifert homology sphere \( \Sigma_n \) has a negative-definite plumbing \( \Sigma_n = \partial P(\Gamma_n) \). Due to Corollary 1.5 in \([10]\), \( 4d(\Sigma_n) = \max_{c \in \Xi} (c^2 + \text{rank}(\Gamma_n)) \) holds, where \( \Xi_n \) is the set of characteristic classes in \( \Gamma_n \). If for a non-negative integer \( N \), \( \Gamma_n = \Gamma_n' \oplus (-1)^N \) and \( \Gamma_n' \) is a minimal sub-lattice, then \( \max_{c \in \Xi} (c^2 + \text{rank}(\Gamma_n)) \) is decomposed into the sum of the two maximal values according to the direct sum. Thus, we have

\[
\max_{c \in \Xi} (c^2 + \text{rank}(\Gamma_n)) = \max_{c \in \Xi'} (c^2 + \text{rank}(\Gamma_n')) \leq \text{rank}(\Gamma_n'),
\]

where \( \Xi' \) is the set of characteristic elements of \( L' \). Since \( d(\Sigma_n) \) has no upper bound, the maximal of the rank of minimal lattice \( \Gamma_n' \) is also unbounded. On the other hands, \( \bar{\mu}(Y_n) = 1 \) and our construction of filling of \( \Sigma_n \) implies that \( \delta s(Y_n) = 1 \). Therefore \( \delta s(\Sigma_n) - \delta s(Y_n) \) is unbounded. \( \square \)
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