Additive spanners: A simple construction

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Abstract

We consider additive spanners of unweighted undirected graphs. Let $G$ be a graph and $H$ a subgraph of $G$. The most naïve way to construct an additive $k$-spanner of $G$ is the following: As long as $H$ is not an additive $k$-spanner repeat: Find a pair $(u, v) \in H$ that violates the spanner-condition and a shortest path from $u$ to $v$ in $G$. Add the edges of this path to $H$.

We show that, with a very simple initial graph $H$, this naïve method gives additive 6- and 2-spanners of sizes matching the best known upper bounds. For additive 2-spanners we start with $H = \emptyset$ and end with $O(n^{3/2})$ edges in the spanner. For additive 6-spanners we start with $H$ containing $\lfloor n^{1/3} \rfloor$ arbitrary edges incident to each node and end with a spanner of size $O(n^{4/3})$.

1 Introduction

Additive spanners are subgraphs that preserve the distances in the graph up to an additive positive constant. Given an unweighted undirected graph $G$, a subgraph $H$ is an additive $k$-spanner if for every pair of nodes $u, v$ it is true that

$$d_G(u, v) \leq d_H(u, v) \leq d_G(u, v) + k$$

In this paper we only consider purely additive spanners, which are $k$-spanners where $k = O(1)$. Throughout this paper every graph will be unweighted and undirected.

Many people have considered a variant of this problem, namely multiplicative spanners and even mixes between additive and multiplicative spanners [3,4,6]. The problem of finding a $k$-spanner of smallest size has received a lot of attention. Most notably, given a graph with $n$ nodes Dor et al. [3] prove that it has a 2-spanner of size $O(n^{3/2})$, Baswana et al. [1] prove that it has a 6-spanner of size $O(n^{4/3})$, and Chechik [2] proves that it has a 4-spanner of size $O(n^{7/5} \log^{1/5} n)$. Woodruff [7] shows that for every constant $k$ there exist graphs with $n$ nodes such that every $(2k - 1)$-spanner must have at least $\Omega(n^{1+1/k})$ edges. This implies that the construction of 2-spanners are optimal. Whether there exists an algorithm for constructing $O(1)$-spanners with $O(n^{1+\epsilon})$ edges for some $\epsilon < 1/3$ is unknown and is an important open problem.

Let $G$ be a graph and $H$ a subgraph of $G$. Consider the following algorithm: As long as there exists a pair of nodes $u, v$ such that $d_H(u, v) > d_G(u, v) + k$, find a shortest path from $u$ to $v$ in $G$ and add the edges on the path to $H$. This process will be referred to as $k$-spanner-completion. After $k$-spanner-completion, $H$ will be a $k$-spanner of $G$. Thus, given a graph $G$, a general way to construct a $k$-spanner for $G$ is the following: Firstly, find a simple subgraph of $G$. Secondly use $k$-spanner-completion on this subgraph. The main contribution of this paper is:

**Theorem 1.1.** Let $G$ be a graph with $n$ nodes and $H$ the subgraph containing all nodes but no edges of $G$. For each node add $\lfloor n^{1/3} \rfloor$ edges adjacent to that node to $H$ (or, if the degree is less, add all edges incident to that node). After 6-spanner-completion $H$ will have at most $O(n^{4/3})$ edges.

It is well-known that a graph with $n$ nodes has a 6-spanner of size $O(n^{4/3})$ [1]. The techniques employed in our proof of correctness are similar to those in [1]. The creation of the initial graph $H$ corresponds to the clustering in [1] and the 6-spanner-completion corresponds to their path-buying algorithm. For completeness we show that the same method gives a 2-spanner of size $O(n^{3/2})$. This fact is already known due to [3] and is matched by a lower bound from [7].

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Theorem 1.2. Let \( G \) be a graph with \( n \) nodes and \( H \) the subgraph where all edges are removed. Upon 2-spanner-completion \( H \) has at most \( O(n^{3/2}) \) edges.

2 Creating a 6-spanner

The algorithm for creating a 6-spanner was described in the abstract and the introduction.

For a given graph \( G \), a 6-spanner of \( G \) can be created by strating with some subgraph \( H \) of \( G \) and applying 6-spanner-completion to \( H \). Theorem [1] states that for a suitable starting choice of \( H \) we get a spanner of size \( O(n^{4/3}) \). The purpose of this section is to show that the 6-spanner created has no more than \( O(n^{4/3}) \) edges. This will imply that the construction (in terms of the size of the 6-spanner) matches the best known upper bound [1].

of Theorem [1] Inserting (at most) \( \left\lceil n^{1/3} \right\rceil \) edges per node will only add \( n \left\lceil n^{1/3} \right\rceil = O(n^{4/3}) \) edges to \( H \). Therefore it is only necessary to prove that 6-spanner-completion adds no more than \( O(n^{4/3}) \) edges.

Let \( v(H) \) and \( c(H) \) be defined by:

\[
v(H) = \sum_{u,v \in V(G)} \max \{0, d_G(u,v) - d_H(u,v) + 5\}, \quad c(H) = \# E(H)
\]

Say that a shortest path, \( p \), from \( u \) to \( v \) is added to \( H \), and let \( H_0 \) be the subgraph before the edges are added. Let the path consist of the nodes:

\[ u = w_0, w_1, \ldots, w_r = v, r \in \mathbb{N} \]

Let \( u' = w_i \) be the node \( w_i \) with the smallest \( i \) such that \( \deg_{H_0}(w_i) \geq \left\lceil n^{1/3} \right\rceil \). Likewise let \( v' = w_j \) be the node \( w_j \), the largest \( j \) such that \( \deg_{H_0}(w_j) \geq \left\lceil n^{1/3} \right\rceil \). Remember that if \( \deg_{H_0}(w_i) < \left\lceil n^{1/3} \right\rceil \) then all the edges adjacent to \( w_i \) are already in \( H_0 \). This implies that \( d_{H_0}(u',v') > d_G(u',v') + 6 \) since \( d_{H_0}(u,v) > d_G(u,v) + 6 \).

Say that \( t \) new edges are added to \( H \). Then there must be at least \( t \) nodes on \( p \) with degree > \( n^{1/3} \). Since every node can be adjacent to no more than 3 nodes on \( p \) (since it is a shortest path) there must be \( \Omega(n^{1/3}t) \) nodes adjacent to \( p \) in \( H \). Let \( z \) and \( w \) be neighbours to \( u' \) and \( v' \) in \( H \) respectively and let \( r \) be any node adjacent to \( p \) in \( H \). Let \( s \) be a node on \( p \) such that \( r \) and \( s \) are adjacent in \( H \). See Figure 1 for an illustration.

![Figure 1: The dashed line denotes the shortest path from \( u \) to \( v \). The solid lines denote edges.](image)

By the triangle inequality we see that:

\[
d_H(z,r) + d_H(r,w) \leq d_G(u',v') + 4
\]

But on the other hand:

\[
d_{H_0}(z,r) + d_{H_0}(r,w) \geq d_{H_0}(z,w) \geq d_{H_0}(u',v') - 2 > d_G(u',v') + 4
\]

This means that \( \min \{d_{H_0}(z,r),5\} > d_H(z,r) \) or \( \min \{d_{H_0}(r,w),5\} > d_H(r,w) \). Since \( u' \) and \( v' \) have at least \( n^{1/3} \) neighbours and there are \( \Omega(n^{1/3}t) \) nodes in \( H \) adjacent to \( p \), the definition of \( v(H) \) implies that:

\[
v(H) - v(H_0) \geq \Omega(t(n^{1/3})^2)
\]

And since \( c(H) - c(H_0) = t \):

\[
\frac{v(H) - v(H_0)}{c(H) - c(H_0)} \geq \Omega(n^{2/3})
\]

Since \( v(H) \leq O(n^2) \) this implies that \( c(H) \) increases with no more than \( O(n^2/n^{2/3}) = O(n^{4/3}) \) in total when all shortest paths are inserted. Hence \( c(H) = O(n^{4/3}) \) when the 6-spanner-completion is finished which yields the conclusion. \( \square \)
3 Creating a 2-spanner

For completeness we show that 2-spanner-completion gives spanners with $O(n^{3/2})$ edges. This matches the upper bound from [3] and the lower bound from [7].

of Theorem 1.2 Let $G$ be a graph with $n$ nodes. Whenever $H$ is a spanner of $G$, define $v(H)$ and $c(H)$ as:

$$v(H) = \sum_{u,v \in V(G)} \max\{0, d_G(u,v) - d_H(u,v) + 3\}, \quad c(H) = \sum_{v \in V(G)} (\deg_H(v))^2$$

It is easy to see that $0 \leq v(H) \leq 3n^2$ and by Cauchy-Schwartz’s inequality $\sqrt{c(H)} \cdot n \geq 2\#E(H)$. The goal will be to prove that when the algorithm terminates $c(H) = O(n^2)$, since this implies that $\#E(H) = O(n^{3/2})$. This is done by proving that in each step of the algorithm $c(H) - 12v(H)$ will not increase. Since $v(H) = O(n^2)$ this means that $c(H) = O(n^2)$ which ends the proof. Therefore it is sufficient to check that $c(H) - 12v(H)$ never increases.

Consider a step where new edges are added to $H$ on a shortest path from $u$ to $v$ of length $t$. Let $H_0$ be the subgraph before the edges are added. Assume that $u,v$ violates the 2-spanner condition in $H_0$, i.e. $d_{H_0}(u,v) > d_G(u,v) + 2$. Let the shortest path consist of the nodes:

$$u = w_0, w_1, \ldots, w_{t-1}, w_t = v$$

It is obvious that:

$$c(H) - c(H_0) \leq \sum_{i=0}^{t} (\deg_H(w_i))^2 - (\deg_H(w_i) - 2)^2 \leq 4 \sum_{i=0}^{t} \deg_H(w_i)$$

Every node cannot be adjacent to more than 3 nodes on the shortest path, since otherwise it would not be a shortest path. Using this insight we can bound the number of nodes which in $H$ are adjacent to or on the shortest path from below by:

$$\frac{1}{3} \sum_{i=0}^{t} \deg_H(w_i)$$

Now let $z$ be a node in $H$ adjacent or on to the shortest path. Obviously:

$$d_H(u,z) + d_H(z,v) \leq d_G(u,v) + 2$$

Furthermore $d_{H_0}(u,z) + d_{H_0}(z,v) > d_G(u,v) + 2$ since otherwise there would exist a path from $u$ to $v$ in $H_0$ of length $\leq d_G(u,v) + 2$. Hence:

$$d_H(u,z) + d_H(z,v) < d_{H_0}(u,z) + d_{H_0}(z,v)$$

Now let $z$ be a node on the shortest path which is adjacent to $w_i$ in $H$ (every node on the path will also be adjacent in $H$ to such a node). Then by the triangle inequality:

$$d_H(u,z) \leq d_H(u,w_i) + d_H(w_i,z) = d_G(u,w_i) + 1 \leq d_G(u,z) + d_G(z,w_i) + 1 = d_G(u,z) + 2$$

And likewise $d_H(z,v) \leq d_G(z,v) + 2$. Combining these two observations yields:

$$\sum_{w \in V} \max\{0, d_G(z,w) - d_H(z,w) + 3\} < \sum_{w \in V} \max\{0, d_G(z,w) - d_{H_0}(z,w) + 3\}$$

Since this holds for every node in $H$ adjacent to or on the shortest path this means that:

$$v(H) - v(H_0) \geq \frac{1}{3} \sum_{i=0}^{t} \deg_H(w_i)$$

Combining this with the bound on $c(H) - c(H_0)$ gives:

$$(c(H) - 12v(H)) - (c(H_0) - 12v(H_0)) \leq 0$$

which finishes the proof. □
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