Localization for Branching Random Walks in Random Environment

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Abstract

We consider branching random walks in $d$-dimensional integer lattice with time-space i.i.d. offspring distributions. This model is known to exhibit a phase transition: If $d \geq 3$ and the environment is “not too random”, then, the total population grows as fast as its expectation with strictly positive probability. If, on the other hand, $d \leq 2$, or the environment is “random enough”, then the total population grows strictly slower than its expectation almost surely. We show the equivalence between the slow population growth and a natural localization property in terms of “replica overlap”. We also prove a certain stronger localization property, whenever the total population grows strictly slower than its expectation almost surely.

Key words and phrases: branching random walk, random environment, localization, phase transition.

1 Introduction

1.1 Branching random walks in random environment (BRWRE)

We begin by introducing the model. We write $\mathbb{N} = \{0,1,2,...\}$, $\mathbb{N}^* = \{1,2,...\}$ and $\mathbb{Z} = \{\pm x ; x \in \mathbb{N}\}$ in the sequel. Let $p(\cdot, \cdot)$ be a transition probability for the symmetric simple random walk on $\mathbb{Z}^d$:

$$p(x, y) = \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1, \\ 0 & \text{if } |x-y| \neq 1, \end{cases}$$

where $|x| = (|x_1|^2 + .. + |x_d|^2)^{1/2}$ for $x \in \mathbb{Z}^d$. To each $(t, x) \in \mathbb{N} \times \mathbb{Z}^d$, we associate a distribution $q_{t,x} = (q_{t,x}(k))_{k \in \mathbb{N}} \in [0,1]^\mathbb{N}$, $\sum_{k \in \mathbb{N}} q_{t,x}(k) = 1$ on $\mathbb{N}$. Then, the branching random walk (BRW) with offspring distribution $q = (q_{t,x})_{(t,x) \in \mathbb{N} \times \mathbb{Z}^d}$ is described as the following dynamics:

- At time $t = 0$, there is one particle at the origin $x = 0$.
- Suppose that there are $N_{t,x}$ particles at each site $x \in \mathbb{Z}^d$ at time $t$. At time $t+1$, the $\nu$-th particle at a site $x$ ($\nu = 1, .., N_{t,x}$) jumps to a site $y = X_{t,x}^\nu$ with probability $p(x, y)$ independently of each other. At arrival, it dies, leaving $K_{t,x}^\nu$ new particles there.

We formulate the above description more precisely. The following formulation is an analogue of [10 section 4.2], where non-random offspring distributions are considered. See also [3 section 5] for the random offspring case.

- Spatial motion: A particle at time-space location $(t, x)$ is supposed to jump to some other location $(t+1, y)$ and is replaced by its children there. Therefore, the spatial motion should be described by assigning destination of the each particle at each time-space location $(t, x)$. 

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So, we are guided to the following definition. We define the measurable space \((\Omega_X, \mathcal{F}_X)\) as the set \((\mathbb{Z}^d)^{\mathbb{N} \times \mathbb{Z}^d \times \mathbb{N}^*}\) with the product \(\sigma\)-field, and \(\Omega_X \ni X \mapsto X_{t,x}^{\nu} \) for each \((t, x, \nu) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}^*\) as the projection. We define \(P_X \in \mathcal{P}(\Omega_X, \mathcal{F}_X)\) as the product measure such that

\[
P_X(X_{t,x}^{\nu} = y) = p(x, y) \quad \text{for all } (t, x, \nu) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{N}^* \text{ and } y \in \mathbb{Z}^d.
\]

(1.2)

Here, we interpret \(X_{t,x}^{\nu}\) as the position at time \(t+1\) of the children born from the \(\nu\)-th particle at time-space location \((t, x)\).

- **Offspring distribution:** We set \(\Omega_q = \mathcal{P}(\mathbb{N} \times \mathbb{Z}^d)\), where \(\mathcal{P}(\mathbb{N})\) denotes the set of probability measures on \(\mathbb{N}\):

\[
\mathcal{P}(\mathbb{N}) = \{ q = (q(k))_{k \in \mathbb{N}} \in [0, 1]^\mathbb{N} ; \sum_{k \in \mathbb{N}} q(k) = 1 \}.
\]

Thus, each \(q \in \Omega_q\) is a function \((t, x) \mapsto q_{t,x} = (q_{t,x}(k))_{k \in \mathbb{N}}\) from \(\mathbb{N} \times \mathbb{Z}^d\) to \(\mathcal{P}(\mathbb{N})\). We interpret \(q_{t,x}\) as the offspring distribution for each particle which occupies the time-space location \((t, x)\). The set \(\mathcal{P}(\mathbb{N})\) is equipped with the natural Borel \(\sigma\)-field induced from that of \([0, 1]^\mathbb{N}\). We denote by \(\mathcal{F}_q\) the product \(\sigma\)-field on \(\Omega_q\).

We define the measurable space \((\Omega_K, \mathcal{F}_K)\) as the set \(\mathbb{N}^{\mathbb{Z}^d \times \mathbb{N}^*}\) with the product \(\sigma\)-field, and \(\Omega_K \ni K \mapsto K_{t,x}^{\nu}\) for each \((t, x, \nu) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{N}^*\) as the projection. For each fixed \(q \in \Omega_q\), we define \(P_K^q \in \mathcal{P}(\Omega_K, \mathcal{F}_K)\) as the product measure such that

\[
P_K^q(K_{t,x}^{\nu} = k) = q_{t,x}(k) \quad \text{for all } (x, t, \nu) \in \mathbb{Z}^d \times \mathbb{N} \times \mathbb{N}^* \text{ and } k \in \mathbb{N}.
\]

(1.3)

We interpret \(K_{t,x}^{\nu}\) as the number of the children born from the \(\nu\)-th particle at time-space location \((t, x)\).

We now define the branching random walk in random environment. We fix a product measure \(Q \in \mathcal{P}(\Omega_q, \mathcal{F}_q)\), which describes the i.i.d. offspring distribution assigned to each time-space location. Finally, we define \((\Omega, \mathcal{F})\) by

\[
\Omega = \Omega_X \times \Omega_K \times \Omega_q, \quad \mathcal{F} = \mathcal{F}_X \otimes \mathcal{F}_K \otimes \mathcal{F}_q,
\]

and \(P^q, P \in \mathcal{P}(\Omega, \mathcal{F})\) by

\[
P^q = P_X \otimes P_K^q \otimes \delta_q, \quad P = \int Q(dq)P^q.
\]

We denote by \(N_{t,x}\) the population at time-space location \((t, x) \in \mathbb{N} \times \mathbb{Z}^d\), which is defined inductively by

\[
N_{0,x} = \delta_{0,x}, \quad N_{t,x} = \sum_{y \in \mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} \delta_x(X_{t-1,y}^{\nu}K_{t-1,y}^{\nu}), \quad t \geq 1.
\]

(1.4)

We consider the filtration:

\[
\mathcal{F}_0 = \{ \emptyset, \Omega \}, \quad \mathcal{F}_t = \sigma(X_{s,\cdot}^{\nu}, K_{s,\cdot}^{\nu}, q_{s,\cdot}; \quad s \leq t - 1 \} \quad t \geq 1,
\]

(1.5)

which the process \(t \mapsto (N_{t,x})_{x \in \mathbb{Z}^d}\) is adapted to. The total population at time \(t\) is then given by

\[
N_t = \sum_{x \in \mathbb{Z}^d} N_{t,x} = \sum_{y \in \mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^{\nu}.
\]

(1.6)
We remark that the total population is exactly the classical Galton-Watson process if \( q_{t,x} \equiv q \), where \( q \in \mathcal{P}(\mathbb{N}) \) is non-random. On the other hand, if \( \mathbb{Z}^d \) is replaced a singleton, then \( N_t \) is the population of the Smith-Wilkinson model [11].

For \( p > 0 \), we write

\[
m^{(p)} = Q[m^{(p)}_{t,x}] \quad \text{with} \quad m^{(p)}_{t,x} = \sum_{k \in \mathbb{N}} k^p q_{t,x}(k),
\]

\[\tag{1.7}\]

\[
m = m^{(1)}.
\]

\[\tag{1.8}\]

Note that for \( p \geq 1 \),

\[
m^p \leq Q[m^p_{t,x}] \leq m^{(p)}
\]

by Hölder’s inequality. We set

\[
\overline{N}_{t,x} = N_{t,x}/m^t \quad \text{and} \quad \overline{N}_t = N_t/m^t.
\]

\[\tag{1.9}\]

\( \overline{N}_t = N_t/m^t \) is a martingale, and therefore the following limit always exists:

\[
\overline{N}_\infty = \lim_{t} \overline{N}_t, \quad \text{P-a.s.}
\]

\[\tag{1.10}\]

We denote the density of the population by:

\[
\rho_{t,x} = \frac{N_{t,x}}{N_t} = \frac{\overline{N}_{t,x}}{\overline{N}_t}, \quad t \in \mathbb{N}, x \in \mathbb{Z}^d
\]

\[\tag{1.11}\]

Interesting objects related to the density would be

\[
\rho^*_t = \max_{x \in \mathbb{Z}^d} \rho_{t,x}, \quad \text{and} \quad R_t = \sum_{x \in \mathbb{Z}^d} \rho^2_{t,x}.
\]

\[\tag{1.12}\]

\( \rho^*_t \) is the density at the most populated site, while \( R_t \) is the probability that a given pair of particles at time \( t \) are at the same site. We call \( R_t \) the replica overlap, in analogy with the spin glass theory. Clearly, \((\rho^*_t)^2 \leq R_t \leq \rho^*_t \). These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of \( \rho^*_t \) or \( R_t \) indicates that the most of the particles are concentrated on small numbers of “favorite sites” (localization), whereas small values of them implies that the particles are spread out over large number of sites (delocalization).

### 1.2 The phase transition in terms of the population growth

Due to the random environment, the population \( N_t \) has much more fluctuation as compared with the non-random environment case, e.g.,[10, section 4.2]. This fluctuation results from “disastrous locations” in time-space, where the offspring distribution \( q_{t,x}(k) \) happens to assign extremely high probability to small \( k \)'s. Thanks to the random walk, on the other hand, some of the particles are lucky enough to avoid those disastrous locations. Therefore, the spatial motion component of the model has the effect to moderate the fluctuation, while the random environment intensifies it. These competing factors in the model give rise to a phase transition as we discuss below.

We first look at the case where the randomness of the offspring distribution is well moderated by that of the random walk.

Let \((S_t)\) be two a simple symmetric random walks on \( \mathbb{Z}^d \), starting from 0. We denote by \( \pi_d \) the probability of the event \( \bigcup_{t \geq 1} \{S_t = 0\} \). As is well known \( \pi_d < 1 \) if and only if \( d \geq 3 \).
Proposition 1.2.1 (a) There exists a constant $\alpha^{\ast} > \frac{1}{\pi d}$ such that, if 
\[ m > 1, \ m^{(2)} < \infty, \ d \geq 3, \ \text{and} \ \alpha \overset{\text{def.}}{=} \frac{Q[m_{t,x}^2]}{m^2} < \alpha^{\ast}, \]  \hspace{1cm} (1.13)
then, $P(N_{\infty} > 0) > 0$.

(b) If one assumes the stronger assumption
\[ m > 1, \ m^{(2)} < \infty, \ d \geq 3, \ \text{and} \ \alpha < \frac{1}{\pi d}, \]  \hspace{1cm} (1.14)
then
\[ R_T = O(T^{-d/2}) \ \text{in} \ P(\cdot|N_{\infty} > 0)-\text{probability,} \]
i.e., the laws $P(T^{d/2}R_T \in \cdot|N_{\infty} > 0), T \geq 1$ are tight.

Conditions (1.13) and (1.14) control the randomness of the environment in terms of the random walk. Proposition 1.2.1(a) says that, under (1.13), the total population grows as fast as its expectation with strictly positive probability. This was obtained in \[3, \text{Theorem 4}].
Proposition 1.2.1(b) is a quantitative statement for delocalization under (1.14) in terms of the replica overlap \[12, \text{Proposition 1.2.3}\].

Next, we turn to the case where the randomness of the environment dominates:

Proposition 1.2.2 Suppose one of the following conditions:

(a1) $d = 1$, $Q(m_{t,x} = m) \neq 1$.

(a2) $d = 2$, $Q(m_{t,x} = m) \neq 1$.

(a3) $d \geq 3$, $Q\left[\frac{m_{t,x}}{m} \ln \frac{m_{t,x}}{m}\right] > \ln(2d)$.

Then, $P(N_{\infty} = 0) = 1$. Moreover, in cases (a1) and (a3), there exists a non-random number $c > 0$ such that
\[ \lim_{t \to \infty} \ln N_{t} \leq -c, \ \text{a.s.} \]  \hspace{1cm} (1.15)

Proposition 1.2.2 says that the total population grows strictly slower than its expectation almost surely, in low dimensions or in “random enough” environment. The result is in contrast with the non-random environment case, where $P(N_{\infty} = 0) = 1$ only for offspring distributions with very heavy tail, more precisely, if and only if $P[K_{t,x}^\nu \ln K_{t,x}^\nu] = \infty$ \[1\] page 24, Theorem 1]. Here, we can have $P(N_{\infty} = 0) = 1$ even when $K_{t,x}^\nu$ is bounded. Also, (1.15) is in sharp contrast with the non-random environment case, where it is well known –see e.g., \[1\] page 30, Theorem 3–that
\[ \{N_{\infty} > 0\} \overset{\text{a.s.}}{=} \{\lim_{t \to \infty} \ln \frac{N_{t}}{t} = 0\} \ \text{whenever} \ m > 1. \]

Proposition 1.2.2 was obtained in \[3, \text{Theorem 4}\] without (1.15), and in \[12, \text{Corollary 3.3.2}\] with (1.15).
1.3 The results: the localization/delocalization transition

In this paper, we aim at the localization problem for the branching random walk in random environment. We shall prove that for $d = 1, 2$ and for “random enough environment” in $d \geq 3$, almost surely, there exists a sequence of time $t$’s such that both the maximal density $\rho^*_t$ and overlap $R_t$ are bigger than some positive constant.

We first characterize the event $\{N_{\infty} = 0\}$ in terms of the replica overlap. Thanks to this characterization, we can rigorously identify the phase transition in terms of population growth as discussed in section 1.2 with the localization/delocalization transition in terms of the replica overlap.

**Theorem 1.3.1** Suppose that

$$m^{(3)} < \infty, \quad Q(m_t \cdot x = m) \neq 1, \quad Q(q_t, x = 0) = 1.$$  \hspace{1cm} (1.16)

Then,

$$\{N_{\infty} = 0\} \overset{a.s.}{=} \{ \sum_{s=0}^{\infty} R_s = \infty \},$$  \hspace{1cm} (1.17)

where $(R_t)_{t \geq 0}$ is defined by (1.12). Moreover, there exist constants $c_1, c_2 \in (0, \infty)$ such that,

$$\{N_{\infty} = 0\} \overset{a.s.}{=} \{-c_1 \ln N_t \leq \sum_{s=0}^{t-1} R_s \leq -c_2 \ln N_t \text{ for large enough } t \text{'s.}\}.$$  \hspace{1cm} (1.18)

We will prove Theorem 1.3.1 in section 2.

As we referred to before, the large values of the replica overlap, or the maximal density, indicates the localization of the particles to a small number of sites. We have the following lower bound for the replica overlap and the maximal density:

**Theorem 1.3.2** Suppose (1.16) and that $P(N_{\infty} = 0) = 1$. Then, there exists a non-random number $c \in (0, 1)$ such that

$$\lim_{t \rightarrow \infty} \rho^*_t \geq \lim_{t \rightarrow \infty} R_t \geq c, \quad a.s.,$$  \hspace{1cm} (1.19)

where $(\rho^*_t)_{t \geq 0}$ and $(R_t)_{t \geq 0}$ are defined by (1.12). In particular, (1.19) holds true if we assume any one of (a1) – (a3) in Proposition 1.2.2.

(1.19) says that the replica overlap persists, in contrast with Proposition 1.2.1(b), where the replica overlap $R_T$ decays like $O(T^{-d/2})$. The proof of Theorem 1.3.2 will be presented in section 3. Some more remarks on Theorem 1.3.2 are in order:

1) In cases (a1) and (a3) in Proposition 1.2.2, (1.19) follows easily from (1.15) and (1.18). However, the proof we present does not rely on (1.15), so that we can cover two dimensional case (a2) as well.

2) We prove (1.19) by way of the following stronger estimate:

$$\lim_{t \rightarrow \infty} \frac{\sum_{s=0}^{t} R_s^{3/2}}{\sum_{s=0}^{t} R_s} \geq c, \quad a.s.$$  \hspace{1cm} (1.20)

for some non-random number $c > 0$. This in particular implies the following quantitative lower bound on the number of times, at which the replica overlap is larger than a certain positive number:

$$\lim_{t \rightarrow \infty} \frac{\sum_{s=0}^{t} 1(R_s \geq \epsilon)}{\sum_{s=0}^{t} R_s} \geq \epsilon, \quad a.s.
for small enough $\epsilon > 0$.

3) For both Theorem 1.3.1 and Theorem 1.3.2, similar results are known for the directed polymers in random environment (DPRE) [4, 6, 7]. In fact, we have used ideas and techniques from the DPRE case. However, the results for DPRE do not seem to directly imply our results.

2 Proof of Theorem 1.3.1

2.1 Lemmas

For sequences $(a_t)_{t \in \mathbb{N}}$ and $(b_t)_{t \in \mathbb{N}}$ (random or non-random), we write $a_t \leq b_t$ if there exists non-random constant $c \in (0, \infty)$ such that $a_t \leq cb_t$ for all $t \in \mathbb{N}$. We write $a_t \asymp b_t$ if $a_t \leq b_t$ and $b_t \leq a_t$.

Lemma 2.1.1 (a) If $m^{(2)} < \infty$ and $Q(m_{t,x} = m) \neq 1$, then $P[(N_t - m_{t-1})^2 | F_{t-1}] \asymp \sum_{x \in \mathbb{Z}^d} N_{t-1,x}^2$.

(b) If $m^{(3)} < \infty$, then $P[(N_t - m_{t-1})^3 | F_{t-1}] \asymp \sum_{x \in \mathbb{Z}^d} N_{t-1,x}^3$.

Proof: (a): Since

$$N_t - m_{t-1} = \sum_{x} \sum_{\nu=1}^{N_{t-1,x}} (K_{t-1,x}^{\nu} - m),$$

we have $(N_t - m_{t-1})^2 = \sum_{x_1 \neq x_2} F_{x_1,x_2}$, where

$$F_{x_1,x_2} = \sum_{\nu_1=1}^{N_{t-1,x}} \sum_{\nu_2=1}^{N_{t-1,x}} (K_{t-1,x_1}^{\nu_1} - m)(K_{t-1,x_2}^{\nu_2} - m).$$

If $x_1 \neq x_2$, then $K_{t-1,x_1}^{\nu_1}$ and $K_{t-1,x_2}^{\nu_2}$ are mean $m$ independent r.v.’s under $P(\cdot | F_{t-1})$, and hence

$$P[F_{x_1,x_2} | F_{t-1}] = 0.$$  

We may therefore focus on the expectation of $F_{x_1,x_2}$ with $x_1 = x_2 = x$. In this case, $(K_{t-1,x}^{\nu})_{\nu=1}^{N_{t-1,x}}$ are independent under $P(\cdot | F_{t-1})$, where

$$\tilde{F}_{t-1} = \sigma(F_{t-1}, (q_{t-1,x})_{x \in \mathbb{Z}^d}).$$

Thus,

$$P[F_{x,x} | \tilde{F}_{t-1}] = N_{t-1,x}(N_{t-1,x} - 1)(m_{t-1,x} - m)^2 + N_{t-1,x} P^q[(K_{t-1,x}^{\nu} - m)^2].$$

The first and second terms on the right-hand-side come respectively from off-diagonal and diagonal terms in $F_{x,x}$. We now set $\alpha \overset{\text{def}}{=} Q(m_{t,x}^2)/m^2$. Then, $\alpha > 1$ (since $Q(m_{t,x} = m) \neq 1$) and

$$P[F_{x,x} | F_{t-1}] = (\alpha - 1)m^2 N_{t-1,x}(N_{t-1,x} - 1) + (m^{(2)} - m^2)N_{t-1,x}$$

$$= (\alpha - 1)m^2 N_{t-1,x}^2 + (m^{(2)} - \alpha m^2)N_{t-1,x}.$$ 

Therefore,

$$P[(N_t - m_{t-1})^2 | F_{t-1}] = (\alpha - 1)m^2 \sum_{x} N_{t-1,x}^2 + (m^{(2)} - \alpha m^2)N_{t-1},$$
which implies the desired bound.

(b): We have \((N_t - mN_{t-1})^3 = \sum_{x_1,x_2,x_3} F_{x_1,x_2,x_3}\), where

\[
F_{x_1,x_2,x_3} = \sum_{\nu_2=1}^{N_{t-1,x_2}} \sum_{\nu_1=1}^{N_{t-1,x_1}} \sum_{\nu_3=1}^{N_{t-1,x_3}} (K_{t-1,x_1}^{\nu_1} - m)(K_{t-1,x_2}^{\nu_2} - m)(K_{t-1,x_3}^{\nu_3} - m).
\]

If, for example, \(x_1 \notin \{x_2, x_3\}\), then \(K_{t-1,x_1}^{\nu_1}\) is independent of \(\{K_{t-1,x_2}^{\nu_2}, K_{t-1,x_3}^{\nu_3}\}\) under \(P(\cdot | \mathcal{F}_{t-1})\), and hence \(P[F_{x_1,x_2,x_3} | \mathcal{F}_{t-1}] = 0\). This implies that

\[
P[(N_t - mN_{t-1})^3 | \mathcal{F}_{t-1}] = \sum_x P[F_{x,x,x} | \mathcal{F}_{t-1}].
\]

We have on the other hand that,

\[
P[F_{x,x,x} | \tilde{\mathcal{F}}_{t-1}] = N_{t-1,x} P[q[(K_{t-1,x}^{\nu} - m)^3]
\]

\[
+ 3N_{t-1,x}(N_{t-1,x} - 1)P[q[(K_{t-1,x}^{\nu} - m)^2]P[q[K_{t-1,x}^{\nu} - m]
\]

\[
+ N_{t-1,x}(N_{t-1,x} - 1)(N_{t-1,x} - 2)P[q[K_{t-1,x}^{\nu} - m]^3].
\]

and therefore that

\[
\left| P[F_{x,x,x} | \tilde{\mathcal{F}}_{t-1}] \right| \leq N_{t-1,x}^3 P[q[(K_{t-1,x}^{\nu} - m)^3].
\]

Putting things together, we obtain

\[
| P[(N_t - mN_{t-1})^3 | \mathcal{F}_{t-1}] | \leq c \sum_x N_{t-1,x}^3, \text{ with } c = Q[(K_{t-1,x}^{\nu} - m)^3].
\]

\[\square\]

Let us now recall Doob’s decomposition in our settings. An \((\mathcal{F}_t)\)-adapted process \(X = (X_t)_{t \geq 0} \subset L1(P)\) can be decomposed in a unique way as

\[
X_t = M_t(X) + A_t(X), \quad t \geq 1,
\]

where \(M(X)\) is an \((\mathcal{F}_t)\)-martingale and

\[
A_0 = 0, \quad \Delta A_t = P[\Delta X_t | \mathcal{F}_{t-1}], \quad t \geq 1.
\]

Here, and in what follows, we write \(\Delta a_t = a_t - a_{t-1}\) (\(t \geq 1\)) for a sequence \((a_t)_{t \in \mathbb{N}}\) (random or non-random). \(M_t(X)\) and \(A_t(X)\) are called respectively, the martingale part and compensator of the process \(X\). If \(X\) is a square integrable martingale, then the compensator \(A_t(X^2)\) of the process \(X^2 = (X_t^2)_{t \geq 0} \subset L1(P)\) is denoted by \(\langle X \rangle_t\), and is given by the following formula:

\[
\Delta \langle X \rangle_t = P[(\Delta X_t)^2 | \mathcal{F}_{t-1}].
\]

Now, we turn to the Doob’s decomposition of \(X_t = -\ln \bar{N}_t\), whose martingale part and the compensator will be henceforth denoted \(M_t\) and \(A_t\) respectively;

\[
- \ln \bar{N}_t = M_t + A_t, \quad \Delta A_t = -P[\Delta \ln \bar{N}_t | \mathcal{F}_{t-1}]
\]

(2.1)

**Lemma 2.1.2** Suppose \((1.16)\). Then, \(\Delta \langle M \rangle_t \leq \mathcal{R}_{t-1} \times \Delta A_t\).

Proof: We set \(U_t = \frac{\Delta \bar{N}}{\Delta M}\) to simplify the notation. We first note the following:
(1) \( U_t \geq \frac{1}{m} - 1 > -1 \).

(2) \(|\Delta \ln N_t| \leq m|U_t|\).

(3) \( P[U^2_t|\mathcal{F}_{t-1}] \asymp P[\varphi(U_t)|\mathcal{F}_{t-1}] \asymp R_{t-1}\), where \( \varphi(x) = x - \ln(1 + x) \).

In fact, \( N_{t-1} \leq N_t \) by (1.16), and hence \((1/m)\bar{N}_{t-1} \leq \bar{N}_t\). These imply (1). (2) follows directly from (1) since

\[ |\ln x - \ln y| \leq \frac{m|x - y|}{y} \quad \text{if} \quad x, y > 0 \text{ and } x/y \geq 1/m.\]

As for (3), we have by Lemma 2.1.1(a) that

\[ P[U^3_t|\mathcal{F}_{t-1}] = \frac{1}{N^3_{t-1}} \cdot P[(N_t - mN_t)^3|\mathcal{F}_{t-1}] \leq \frac{1}{N^3_{t-1}} \sum_{x \in \mathbb{Z}^d} N^3_{t-1,x} \leq R_{t-1}.\]

we now note that there exists \( c \in (0, \infty) \), which depends only on \( m \) such that

\[ \frac{x^2}{4(2 + x)} \leq \varphi(x) \leq cx^2 \quad \text{for all } x \geq \frac{1}{m} - 1.\]

This, together with (1) implies that

\[ P[\varphi(U_t)|\mathcal{F}_{t-1}] \leq cP[U^2_t|\mathcal{F}_{t-1}] \asymp R_{t-1}.\]

On the other hand, we have by Lemma 2.1.1(b) that

\[ |P[U^3_t|\mathcal{F}_{t-1}]| = \frac{1}{N^3_{t-1}} \cdot |P[(N_t - mN_t)^3|\mathcal{F}_{t-1}]| \leq \frac{1}{N^3_{t-1}} \sum_{x \in \mathbb{Z}^d} N^3_{t-1,x} \leq R_{t-1}.\]

Thus,

\[ R_{t-1} \asymp P[U^2_t|\mathcal{F}_{t-1}] = P \left[ \frac{U_t}{\sqrt{2 + U_t}} U_t \sqrt{2 + U_t} |\mathcal{F}_{t-1} \right] \]

\[ \leq P \left[ \frac{U^2_t}{2 + U_t} |\mathcal{F}_{t-1} \right]^{1/2} P[2U^2_t + U^3_t|\mathcal{F}_{t-1}]^{1/2} \leq P[\varphi(U_t)|\mathcal{F}_{t-1}]^{1/2} R_{t-1}^{1/2},\]

and hence \( R_{t-1} \leq P[\varphi(U_t)|\mathcal{F}_{t-1}] \).

The rest of the proof is easy. We have by (3) that

\[ \Delta A_t = -P[\Delta \ln \bar{N}_t|\mathcal{F}_{t-1}] = -P[\ln(1 + U_t)|\mathcal{F}_{t-1}] = P[\varphi(U_t)|\mathcal{F}_{t-1}] \asymp R_{t-1}.\]

Similarly, by (2) and Lemma 2.1.1

\[ P[(\Delta \ln \bar{N}_t)^2|\mathcal{F}_{t-1}] \leq P[U^2_t|\mathcal{F}_{t-1}] \asymp R_{t-1}.\]

This, together with \( \Delta A_t \asymp R_{t-1} \) implies that

\[ \Delta \langle \mathcal{M} \rangle_t = P[(\Delta \mathcal{M}_t)^2|\mathcal{F}_{t-1}] \leq 2P[(\Delta \ln \bar{N}_t)^2|\mathcal{F}_{t-1}] + 2(\Delta A_t)^2 \leq R_{t-1}.\]
2.2 Proof of Theorem 1.3.1

The proof is based on the decomposition (2.1). It is enough to prove the following:

1. \( \{ \mathcal{N}_\infty = 0 \} \subset \{ A_\infty = \infty \} = \{ \sum_{s=0}^\infty \mathcal{R}_s = \infty \} \).

2. \( \{ \sum_{s=0}^\infty \mathcal{R}_s = \infty \} \subset \{ -c_1 \ln N_t \leq \sum_{s=0}^{t-1} \mathcal{R}_s \leq -c_2 \ln N_t \text{ for large enough } t's. \} \).

To prove these, we recall the following general facts on square integrable martingales—see for example [9, page 252, (4.9) and page 253, (4.10)]:

3. \( \{ \langle M \rangle_\infty < \infty \} \subset \{ \lim_{t \to \infty} M_t \text{ converges.} \} \).

4. \( \{ \langle M \rangle_\infty = \infty \} \subset \{ \lim_{t \to \infty} \frac{M_t}{\langle M \rangle_t} = 0 \} \).

By (3) and Lemma 2.1.2, we get (1) as follows:

\[ \{ t \sum_{s=0}^{t-1} \mathcal{R}_s < \infty \} \subset \{ A_\infty < \infty, \lim_{t \to \infty} M_t \text{ converges.} \} \subset \{ \mathcal{N}_\infty > 0 \} \]

We now turn to (2). Since \( \{ A_\infty = \infty \} = \{ \sum_{s=0}^\infty \mathcal{R}_s = \infty \} \) and

\[ -\frac{\ln N_t}{\sum_{s=0}^{t-1} \mathcal{R}_s} \leq -\frac{\ln N_t}{A_t} = \frac{M_t}{A_t} + 1, \]

by Lemma 2.1.2 (2) is a consequence of:

5. \( \{ A_\infty = \infty \} \subset \{ \frac{M_t}{A_t} \to 0 \} \).

Let us suppose that \( A_\infty = \infty \). If \( \langle M \rangle_\infty < \infty \), then again by (3), \( \lim_{t \to \infty} M_t \) converges and therefore (5) holds. If, on the contrary, \( \langle M \rangle_\infty = \infty \), then by (4) and Lemma 2.1.2

\[ \frac{M_t}{A_t} = \frac{M_t}{\langle M \rangle_t} \to 0 \text{ a.s.} \]

Thus, (5) is true in this case as well. \( \square \)

3 Proof of Theorem 1.3.2

We shall prove Theorem 1.3.2 in the same spirit of that of [4]. In the following subsection, we give some preliminary estimates and the final proof is given in the last subsection.

3.1 Lemmas

A technical result at first:

**Lemma 3.1.1** Let \( \eta_i, 1 \leq i \leq n \ (n \geq 2) \) be positive independent random variables on a probability space with the probability measure \( \mathbb{P} \), such that \( \mathbb{P}[\eta_i^2] < \infty \) for \( i = 1, \ldots, n \). Then,

\[ \mathbb{P} \left[ \frac{\eta_1 \eta_2}{\left( \sum_{i=1}^n \eta_i \right)^2} \right] \geq \frac{m_1 m_2}{M^2} - 2 \frac{m_2 \text{var}(\eta_1) + m_1 \text{var}(\eta_2)}{M^3}, \]  (3.1)

\[ \mathbb{P} \left[ \frac{\eta_1^2}{\left( \sum_{i=1}^n \eta_i \right)^2} \right] \geq \mathbb{P}[\eta_1^2] \left( 1 + \frac{2m_1}{M} \right) - 2 \mathbb{P}[\eta_1^3] \frac{1}{M^3}, \]  (3.2)

where \( m_i = \mathbb{P}[\eta_i] \) and \( M = \sum_{i=1}^n m_i \).
Proof: We set 
\[ U = \sum_{i=1}^{n} (\eta_i - m_i) = \sum_{i=1}^{n} \eta_i - M > -M. \]

Note that \((u + M)^{-2} \geq M^{-2}(1 - \frac{2u}{M})\) for \(u \in (-M, \infty)\). Thus, we have that

\[
\mathbb{P}\left[ \frac{\eta_1 \eta_2}{(\sum_{i=1}^{n} \eta_i)^2} \right] = \mathbb{P}\left[ \frac{\eta_1 \eta_2}{(U + M)^2} \right] \geq M^{-2} \left( m_1 m_2 - \frac{2}{M} \mathbb{P}[\eta_1 \eta_2 U] \right)
\]

\[
\mathbb{P}[\eta_1 \eta_2 U] = \mathbb{P}[\eta_1 \eta_2 (\eta_1 - m_1)] + \mathbb{P}[\eta_1 \eta_2 (\eta_2 - m_2)] = m_2 \var(\eta_1) + m_1 \var(\eta_2).
\]

These prove (3.1). Similarly,

\[
\mathbb{P}\left[ \frac{\eta_1^2}{(\sum_{i=1}^{n} \eta_i)^2} \right] = \mathbb{P}\left[ \frac{\eta_1^2}{(U + M)^2} \right] \geq M^{-2} \left( \mathbb{P}[\eta_1^2] - \frac{2}{M} \mathbb{P}[\eta_1^2 U] \right),
\]

\[
\mathbb{P}[\eta_1^2 U] = \mathbb{P}[\eta_1^2 (\eta_1 - m_1)] = \mathbb{P}[\eta_1^2] - m_1 \mathbb{P}[\eta_1^2].
\]

These prove (3.2). \(\square\)

As an immediate consequence, we have (by applying Lemma 3.1.1 to \(\alpha_i \eta_i\) instead of \(\eta_i\)):

**Corollary 3.1.2** Let \(\eta_i, 1 \leq i \leq n (n \geq 2)\) be positive i.i.d.r.v.’s on a probability space with the probability measure \(\mathbb{P}\), such that \(\mathbb{P}[\eta_i^3] < \infty\) for \(i = 1, \ldots, n\). Then, for any \(\alpha_i \geq 0\) satisfying \(\sum_{i=1}^{n} \alpha_i = 1\), we have

\[
\mathbb{P}\left( \frac{\eta_1 \eta_2}{(\sum_{i=1}^{n} \alpha_i \eta_i)^2} \right) \geq 1 - (\mathbb{P}(\eta_1^2) - 1)(\alpha_1 + \alpha_2),
\]

(3.3)

\[
\mathbb{P}\left( \frac{\eta_1^2}{(\sum_{i=1}^{n} \alpha_i \eta_i)^2} \right) \geq (1 + 2\alpha_1)\mathbb{P}(\eta_1^2) - 2\alpha_1 \mathbb{P}(\eta_1^3),
\]

(3.4)

where \(\tilde{\eta}_1 := \eta_1/m_1\).

**Lemma 3.1.3** Assume \(Q(q_{t,x}(0) = 0) = 1\) and \(Q(q_{t,x}(1) = 1) < 1\). Then,

\[
\lim_{t \to \infty} \frac{1}{t} \ln N_t \geq c_0, \quad \text{a.s.}
\]

(3.5)

where \(c_0 = -\ln Q(\sum_{k \geq 1} k^{-1} q_{t,x}) > 0\).

**Proof:** For any \((t, y, \nu)\), \(K_{t,y}^\nu\) is independent of \(\mathcal{F}_t\), hence

\[
P((K_{t,y}^\nu)^{-1} | \mathcal{F}_t) = P((K_{t,y}^\nu)^{-1}) = e^{-c_0}.
\]

It follows by Jensen’s inequality that

\[
P\left( \frac{1}{N_t} | \mathcal{F}_{t-1} \right) = P\left( \left[ \sum_{y} \left( \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^\nu \right)^{-1} \right] | \mathcal{F}_{t-1} \right)
\]

\[
= \frac{1}{N_{t-1}} P\left( \left[ \frac{1}{N_{t-1}} \sum_{y} \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^\nu \right] \left( K_{t-1,y}^\nu \right)^{-1} | \mathcal{F}_{t-1} \right)
\]

\[
\leq \frac{1}{N_{t-1}} P\left( \frac{1}{N_{t-1}} \sum_{y} \sum_{\nu=1}^{N_{t-1,y}} \left( K_{t-1,y}^\nu \right)^{-1} | \mathcal{F}_{t-1} \right)
\]

\[
= e^{-c_0} \frac{1}{N_{t-1}}.
\]
Hence $P\left(\frac{1}{N_t}\right) \leq e^{-c(l^t)}$, and (3.3) follows from the Borel-Cantelli lemma.

We denote by $(P_n, n = 0, 1, \ldots)$ the semigroup of a simple symmetric random walk on $\mathbb{Z}^d$, namely, $P_n f(x) := \sum_y P_n(x, y) f(y)$ where $P_n(x, y)$ is the probability that the random walk starting from $x$ lives at $y$ on the $n$-th step. Plainly, $P_1(x, y) = p(x, y)$. We write $P = P_1$. Let for any $z \in \mathbb{Z}^d$, 
$$r_t := P_{2l}(z, z) = P_{2l}(0, 0) \sim c l^{-d/2}, \quad l \to \infty.$$ For the sake of notational convenience, we write $\rho_t(x) \equiv \rho_{t,x}$, so that $\rho_t$ stands for a function on $\mathbb{Z}^d$.

**Lemma 3.14** Assume (1.16). For any $(y_1, \nu_1)$ and $(y_2, \nu_2)$, $t \geq 1$, we have

$$P\left(\frac{K_{t,y_1}^{\nu_1} K_{t,y_2}^{\nu_2}}{N_{t+1}} \mid F_t\right) \geq \frac{1}{N_t^2} \left[\left(\alpha - 1\right)1_{(y_1 = y_2)} - c_1 \rho_t(y_1) - c_2 \rho_t(y_2) - \frac{c_2}{N_t}\right], \quad (3.6)$$

on the event $\{N_{t,y_1} \wedge N_{t,y_2} \geq 1\}$, where $\alpha = \frac{Q|m^2,s|}{m^2}$ and $c_1$ and $c_2$ are some constants. Consequently,

$$P\left(\rho_{t+1}(y_1) \rho_{t+1}(y_2) \mid F_t\right) \geq \left(1 - \frac{c_2}{N_t}\right) P\rho_t(y_1) P\rho_t(y_2) + \left(\alpha - 1\right) \sum_z \rho_t^2(z)p(z, y_1)p(z, y_2) - c_1 P\rho_t(y_1) P(\rho_t^2)(y_2) P\rho_t(y_2) P(\rho_t^2)(y_1) - \frac{\alpha}{N_t} \sum_z p(z, y_1)p(z, y_2) \rho_t(z). \quad (3.7)$$

**Proof:** Firstly, we consider (3.6) in the case $(y_1, \nu_1) \neq (y_2, \nu_2)$. Let $A \in F_t$ and $A \subset \{N_{t,y_1} \wedge N_{t,y_2} \geq 1\}$. Under $P^A$, $(K_{t,y}^{\nu})_{t,\nu}$ are independent (but not identically distributed) and independent of $1_A, N_t$. Write $M_t = \sum y \sum_{\nu=1}^{N_t} K_{t,y}^{\nu}$ and applying (3.1) to $\eta_1 = K_{t,y_1}^{\nu_1}$ and $\eta_2 = K_{t,y_2}^{\nu_2}$, we get

$$P^A\left(1_A \frac{K_{t,y_1}^{\nu_1} K_{t,y_2}^{\nu_2}}{N_{t+1}^2}\right) \geq \frac{1}{N_t^2} \left(1_A \frac{m_{t,y_1} m_{t,y_2}}{M_t^2}\right) - 2 P^A\left(1_A \frac{m_{t,y_2} m_{t,y_1} m_{t,y_2} (2)}{N_t^3}\right),$$

since $M_t \geq N_t$. Therefore, by taking $Q$-expectation,

$$P\left(1_A \frac{K_{t,y_1}^{\nu_1} K_{t,y_2}^{\nu_2}}{N_{t+1}^2}\right) \geq P\left(1_A \frac{m_{t,y_1} m_{t,y_2}}{M_t^2}\right) - 2 P\left(1_A \frac{m_{t,y_2} m_{t,y_1} m_{t,y_2} (2)}{N_t^3}\right).$$

Observe that under $P$, $m_t$ are i.i.d. and independent of $F_t$. It turns out from (3.3) and (3.1) that

$$P\left(\frac{m_{t,y_1} m_{t,y_2}}{M_t^2} \mid F_t\right) = \frac{1}{N_t^2} P\left(\frac{m_{t,y_1} m_{t,y_2}}{(\sum y \rho_t(y)m_{t,y})^2} \mid F_t\right) \geq \frac{1}{N_t^2} \left[1 + \left(\alpha - 1\right)1_{(y_1 = y_2)} - c_1 \rho_t(y_1) - c_2 \rho_t(y_2)\right].$$

On the other hand, we have

$$P\left(\frac{m_{t,y_2} m_{t,y_1} (2)}{N_t^3} \mid F_t\right) \leq 2m(3) < \infty.$$
by our integrability assumption. Hence, with $c_2 = 4m(3)$,
\[ P\left( A\frac{K_{t,y_1}^{\nu_1} K_{t,y_2}^{\nu_2}}{N_{t+1}^2} \right) \geq P\left( \frac{A}{N_t^2} \left[ 1 + (\alpha - 1)1_{(y_1 = y_2)} - c_1\rho_t(y_1) - c_2\rho_t(y_2) - \frac{c_2}{N_t} \right] \right), \]
yielding (3.6) in the case $(y_1, \nu_1) \neq (y_2, \nu_2)$. The case $(y_1, \nu_1) = (y_2, \nu_2)$ is obtained in the same way by applying (3.7) instead of (3.1) and by eventually modifying the constants.

To obtain (3.7), we have that
\[
P\left( \rho_{t+1}(y_1)\rho_{t+1}(y_2) \mid F_t \right) = \sum_{z_1, z_2} \sum_{\nu_1=1}^{N_{t,1}} \sum_{\nu_2=1}^{N_{t,2}} P\left( \frac{\delta_{y_1}(X_{t,z_1}^{\nu_1})\delta_{y_2}(X_{t,z_2}^{\nu_2}) K_{t,z_1}^{\nu_1} K_{t,z_2}^{\nu_2}}{N_{t+1}^2} \mid F_t \right) + h_{1,2} P\left( \frac{K_{t,z_1}^{\nu_1} K_{t,z_2}^{\nu_2}}{N_{t+1}^2} \mid F_t \right) \]
by means of the independence between $(X_{t,z_1}^{\nu_1}, X_{t,z_2}^{\nu_2})$ and $(F_t, N_{t+1}, K_{t,z_1}^{\nu_1}, K_{t,z_2}^{\nu_2})$, and the function $h_{1,2}$ is defined as follows:
\[
h_{1,2} := P\left( \delta_{y_1}(X_{t,z_1}^{\nu_1})\delta_{y_2}(X_{t,z_2}^{\nu_2}) \right) = 1_{((z_1, \nu_1) = (z_2, \nu_2))}P(z_1, y_1)1_{(y_1 = y_2)} + 1_{((z_1, \nu_1) \neq (z_2, \nu_2))}P(z_1, y_1)P(z_2, y_2) \geq 1_{((z_1, \nu_1) \neq (z_2, \nu_2))}P(z_1, y_1)P(z_2, y_2). \]

Applying (3.6) we get
\[
P\left( \rho_{t+1}(y_1)\rho_{t+1}(y_2) \mid F_t \right) \geq \sum_{z_1, z_2} h_{1,2} \frac{1}{N_t} \left[ 1 + (\alpha - 1)1_{(z_1 = z_2)} - c_1\rho_t(z_1) - c_2\rho_t(z_2) - \frac{c_2}{N_t} \right] \geq \sum_{(z_1, \nu_1) \neq (z_2, \nu_2)} p(z_1, y_1)p(z_2, y_2) \frac{1}{N_t^2} \left[ g_t(z_1, z_2) + (\alpha - 1)1_{(z_1 = z_2)} \right],
\]
with $g_t(z_1, z_2) = 1 - c_1\rho_t(z_1) - c_2\rho_t(z_2) - \frac{c_2}{N_t}$. Let us compute explicitly the above sum $\sum_{(z_1, \nu_1) \neq (z_2, \nu_2)} \cdots$:
\[
\sum_{(z_1, \nu_1) \neq (z_2, \nu_2)} = \sum_{z_1 \neq z_2} N_{t,z_1}N_{t,z_2}p(z_1, y_1)p(z_2, y_2) \frac{1}{N_t} g_t(z_1, z_2) + \sum_z (N_{t,z}^2 - N_{t,z})p(z, y_1)p(z, y_2) \frac{1}{N_t^2} [g_t(z, z) + \alpha - 1],
\]
by removing the diagonal terms. Using the definition of $\rho_t(z) = N_{t,z}/N_t$, we get
\[
\sum_{(z_1, \nu_1) \neq (z_2, \nu_2)} = \sum_{z_1, z_2} \rho_t(z_1)\rho_t(z_2)p(z_1, y_1)p(z_2, y_2) g_t(z_1, z_2)
\]
\[
+ (\alpha - 1) \sum_z \rho_t(z)p(z, y_1)p(z, y_2) - \sum_z p(z, y_1)p(z, y_2) \rho_t(z) \frac{1}{N_t} [g_t(z, z) + \alpha - 1] \geq (1 - \frac{c_2}{N_t})p\rho_t(y_1)p\rho_t(y_2) + (\alpha - 1) \sum_z \rho_t(z)p(z, y_1)p(z, y_2)
\]
\[
- c_1 \left[ p\rho_t(y_1)\rho^2(y_2) + p\rho_t(y_2)\rho^2(y_1) \right] - \frac{\alpha}{N_t} \sum_z p(z, y_1)p(z, y_2) \rho_t(z),
\]
as desired. \(\Box\).

Recall that $R_t = \sum_x \rho_t^2(x)$. Let $t \geq 2$. The following lemma shows the rôles de semigroup in $R_t$. 

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Lemma 3.1.5 Assume \((I.10)\). There exists a constant \(c_3 > 0\) such that for all \(1 \leq s \leq t - 1\),

\[
P \left( \sum_x (P_{t-s+1} \rho_s(x))^2 \mid F_s \right) \geq \sum_x (P_{t-s} \rho_s(x))^2 + (\alpha - 1) r_{t-s} R_s - 2 c_1 R_s^{3/2} - \frac{c_3}{N_s}.
\]

Proof: Observe that

\[
\sum_x (P_{t-s+1} \rho_s(x))^2 = \sum_x \sum_{y_1, y_2} P_{t-s+1}(x, y_1) P_{t-s+1}(x, y_2) \rho_s(y_1) \rho_s(y_2).
\]

Applying \((3.7)\) gives

\[
P \left( \sum_x (P_{t-s+1} \rho_s(x))^2 \mid F_s \right) \geq (1 - \frac{c_2}{N_s}) I_2 + (\alpha - 1) I_3 - c_1 I_4 - \frac{\alpha}{N_s} I_5,
\]

with

\[
I_2 := \sum_x \sum_{y_1, y_2} P_{t-s+1}(x, y_1) P_{t-s+1}(x, y_2) \rho_s(y_1) \rho_s(y_2),
\]

\[
I_3 := \sum_x \sum_{y_1, y_2} P_{t-s+1}(x, y_1) P_{t-s+1}(x, y_2) \sum_z \rho_s^2(z) p(z, y_1) p(z, y_2),
\]

\[
I_4 := \sum_x \sum_{y_1, y_2} P_{t-s+1}(x, y_1) P_{t-s+1}(x, y_2) \left[ \rho_s(y_1) \rho_s^2(y_2) + \rho_s(y_2) \rho_s^2(y_1) \right],
\]

\[
I_5 := \sum_x \sum_{y_1, y_2} P_{t-s+1}(x, y_1) P_{t-s+1}(x, y_2) \sum_z p(z, y_1) p(z, y_2) \rho_s(z).
\]

Using the semigroup property and noting that \(\sum_x (P_{t-s}(x, z))^2 = P_{2t-2s}(z, z) = r_{t-s}\) for any \(z\), we obtain

\[
I_2 = \sum_x (P_{t-s} \rho_s(x))^2,
\]

\[
I_3 = \sum_x \sum_z (P_{t-s}(x, z))^2 \rho_s^2(z) = \sum_x \sum_z (P_{t-s}(x, z))^2 \rho_s^2(z) = r_{t-s} \sum_z \rho_s^2(z),
\]

\[
I_4 = 2 \sum_x P_{t-s} \rho_s(x) P_{t-s} \rho_s^2(x),
\]

\[
I_5 = \sum_x \sum_z (P_{t-s}(x, z))^2 \rho_s(z) = \sum_z \rho_s(z) r_{t-s} = r_{t-s}.
\]

By the translation invariance and Cauchy-Schwarz’s inequality, we see that

\[
R_s = \sum_x P_{t-s} \rho_s^2(x) \geq \sum_x (P_{t-s} \rho_s(x))^2 \geq \max_x P_{t-s} \rho_s(x))^2,
\]

and hence that \(I_4 \leq 2 R_s^{3/2}\). This implies the lemma with \(c_3 = \alpha + c_2\).

Define

\[
V_t = \sum_{s=1}^t R_s, \quad t = 1, 2, ...
\]
Lemma 3.1.6 Assume \((\ref{t10})\). Fix \(j \geq 0\). The martingale \(Z_j(\cdot)\) defined by

\[
Z_j(t) := \sum_{s=1}^{t} \left( \sum_x \left( P_j \rho_s(x) \right)^2 - P \left( \sum_x \left( P_j \rho_s(x) \right)^2 \mid \mathcal{F}_{s-1} \right) \right), \quad t \geq 1.
\]

satisfies the following law of large numbers:

\[
\{ V_\infty = \infty \} \overset{a.s.}{\subset} \{ \frac{Z_j(t)}{V_t} \to 0, \quad t \to \infty, \}.
\]

Proof: Let us compute the increasing process \(\langle Z_j \rangle\) associated to \(Z_j\). By Cauchy-Schwarz’ inequality, \((\sum_x P_j \rho_s(x))^2 \leq \sum_x P_j \rho_s^2(x) = R_s \leq 1\). It follows that

\[
(Z_j(s) - Z_j(s-1))^2 \leq 2 \left( \sum_x (P_j \rho_s(x))^2 \right)^2 + 2 \left( P \left( \sum_x (P_j \rho_s(x))^2 \mid \mathcal{F}_{s-1} \right) \right)^2
\]

\[
\leq 2R_s^2 + 2P \left( R_s \mid \mathcal{F}_{s-1} \right)^2
\]

\[
\leq 2R_s + 2P \left( R_s \mid \mathcal{F}_{s-1} \right).
\]

Hence,

\[
\langle Z_j \rangle_s - \langle Z_j \rangle_{s-1} = P \left( (Z_j(s) - Z_j(s-1))^2 \mid \mathcal{F}_{s-1} \right) \leq 4P \left( R_s \mid \mathcal{F}_{s-1} \right).
\]

We will prove that

\[
P \left( R_s \mid \mathcal{F}_{s-1} \right) \leq 2m^{(2)}R_{s-1}. \tag{3.10}
\]

Then, \(\langle Z_j \rangle_t \leq 8m^{(2)}V_{t-1}\), and the lemma follows from the standard law of large numbers for a square-integrable martingale, c.f. section 2.2.(4).

It remains to show \((3.10)\). Using \((3.8)\) and \((3.9)\) to \(y_1 = y_2 = y\), we have

\[
P \left( R_s \mid \mathcal{F}_{s-1} \right) = \sum_y P \left( \rho_s^2(y) \mid \mathcal{F}_{s-1} \right)
\]

\[
= \sum_y \sum_{z_1, z_2} \sum_{\nu_1=1}^{N_{s-1,z_1}} \sum_{\nu_2=1}^{N_{s-1,z_2}} h_{12} P \left( \frac{K_{s-1,z_1}^{\nu_1} R_{s-1,z_2}^{\nu_2}}{N_s^2} \mid \mathcal{F}_{s-1} \right)
\]

\[
\leq \sum_y \sum_{z_1, z_2} \sum_{\nu_1=1}^{N_{s-1,z_1}} \sum_{\nu_2=1}^{N_{s-1,z_2}} h_{12} \frac{m^{(2)}}{N_{s-1}^2}.
\]

To obtain the last inequality, we used \(N_s \geq N_{s-1}\) and the independence between \(K_{s-1}\) and \(\mathcal{F}_{s-1}\). We divide the last summation into bound the summation over \((z_1, \nu_1) = (z_2, \nu_2)\) and that over \((z_1, \nu_1) \neq (z_2, \nu_2)\), to see that

\[
\sum_{y} \sum_{z_1, z_2} \sum_{\nu_1=1}^{N_{s-1,z_1}} \sum_{\nu_2=1}^{N_{s-1,z_2}} h_{12} \frac{m^{(2)}}{N_{s-1}^2} \leq \frac{1}{N_{s-1}} + \sum_x \left( P \rho_{s-1}(x) \right)^2 \leq \frac{1}{N_{s-1}} + R_{s-1} \leq 2R_{s-1}.
\]

Here, we used \(R_{s-1} = \sum_x N_{s-1,x}^2/N_{s-1}^2 \geq 1/N_{s-1}\) to see the last inequality. Putting things together, we have \((3.10)\) and the proof of the lemma is now complete. \(\square\)
3.2 Proof of Theorem 1.3.2:

We first note that there are \(\epsilon > 0\) and \(t_0 \in \mathbb{N}\) such that

\[
\sum_{s=1}^{t_0} r_s \geq \frac{1 + \epsilon}{\alpha - 1}, \tag{3.11}
\]

For \(d = 1, 2\), we take \(\epsilon = 1\). Then, (3.11) holds for \(t_0\) large enough, since \(\sum_{s=1}^{\infty} r_s = \infty\). For \(d \geq 3\), our assumption \(P(N_\infty = 0) = 1\) implies \(\alpha \geq \alpha^* > 1/\pi_d\) by Proposition 1.2.1. Since \(\sum_{s=1}^{\infty} r_s = \frac{\pi_d}{1 - \pi_d}\), as is well known, (3.11) holds for small enough \(\epsilon > 0\) and large enough \(t_0\).

Let \(t > t_0\). Applying Lemma 3.1.5 to \(s = t - 1, t - 2, \ldots, t - t_0\) and taking the sum on \(s\), we get

\[
\sum_{s=t-t_0}^{t-1} (2c_1 R_{s+1}^{3/2} + \frac{c_3}{N_{t-j}}) \
\geq \sum_{s=t-t_0}^{t-1} \left( \sum_x (P_{t-s} \rho_s(x))^2 - P\left(\sum_x (P_{t-(s+1)} \rho_{s+1}(x))^2 \mid \mathcal{F}_s\right) \right) + (\alpha - 1) \sum_{s=t-t_0}^{t-1} r_{t-s} R_s \\
= \sum_{s=t-t_0}^{t-1} \left( \sum_x (P_{t-(s+1)} \rho_{s+1}(x))^2 - P\left(\sum_x (P_{t-(s+1)} \rho_{s+1}(x))^2 \mid \mathcal{F}_s\right) \right) + (\alpha - 1) \sum_{s=t-t_0}^{t-1} r_{t-s} R_s \\
= \sum_{s=t-t_0}^{t-1} \left[ Z_{t-(s+1)}(s + 1) - Z_{t-(s+1)}(s) \right] + \sum_x (P_{t_0} \rho_{t-t_0}(x))^2 - R_t + (\alpha - 1) \sum_{s=t-t_0}^{t-1} r_{t-s} R_s,
\]

where we recall that the martingale \(Z_j(\cdot)\) are defined in Lemma 3.1.6. By change of variable \(s = t - j\), we have proven that

\[
\sum_{j=1}^{t_0} (2c_1 R_{t-j}^{3/2} + \frac{c_3}{N_{t-j}}) \geq \sum_{j=1}^{t_0} \left[ Z_{j-1}(t - j + 1) - Z_{j-1}(t - j) \right] - R_t + (\alpha - 1) \sum_{j=1}^{t_0} r_j R_{t-j}.
\]

Taking the sum of these inequalities for \(t = t_0 + 1, \ldots, T\), we obtain that

\[
\sum_{t=t_0+1}^{T} \sum_{j=1}^{t_0} (2c_1 R_{t-j}^{3/2} + \frac{c_3}{N_{t-j}}) \geq \sum_{j=1}^{t_0} \left[ Z_{j-1}(T - j + 1) - Z_{j-1}(t_0 - j + 1) \right] - (V_T - V_{t_0}) + (\alpha - 1) \sum_{j=1}^{t_0} r_j (V_{T-j} - V_{t_0-j}).
\]

Since \(R_s \leq 1\),

\[
V_{T-j} - V_{t_0-j} \geq V_T - j - (t_0 - j) = V_T - t_0,
\]

\[
(\alpha - 1) \sum_{j=1}^{t_0} r_j (V_{T-j} - V_{t_0-j}) \geq (\alpha - 1) \sum_{j=1}^{t_0} r_j V_T - c_9 \geq (1 + \epsilon)V_T - c_9,
\]

with constant \(c_9 = (\alpha - 1)t_0 \sum_{j=1}^{t_0} r_j\). Hence,

\[
\sum_{t=t_0+1}^{T} \sum_{j=1}^{t_0} (2c_1 R_{t-j}^{3/2} + \frac{c_3}{N_{t-j}}) \geq \sum_{j=1}^{t_0} \left[ Z_{j-1}(T - j + 1) - Z_{j-1}(t_0 - j + 1) \right] + \epsilon V_T - c_9. \tag{3.12}
\]
Recall from Lemma 3.1.3 that \( \sum_{t=1}^{\infty} \frac{1}{N_t} < \infty, \text{a.s.} \), which combined with Lemma 3.1.6 imply that the two sums involving respectively \( \frac{c}{N_{t-j}} \) and \( Z_{j-1}(T-j+1) \) in (3.12) are negligible, relative to \( V_T \). It follows that

\[
\liminf_{T \to \infty} \frac{1}{V_T} \sum_{t=t_0+1}^{T} \sum_{j=1}^{t_0} R_{t-j}^{3/2} \geq \frac{\epsilon}{2c_1}, \quad \text{a.s.}
\]

Consequently,

\[
\liminf_{T \to \infty} \frac{1}{V_T} \sum_{t=1}^{T} R_t^{3/2} \geq \frac{\epsilon}{2c_1 t_0}, \quad \text{a.s.},
\]

which implies that

\[
\limsup_{t \to \infty} R_t \geq \left( \frac{\epsilon}{2c_1 t_0} \right)^2, \quad \text{a.s.}
\]

This completes the proof of Theorem. \( \square \)

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