On the center of a compact group

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Abstract

We prove a conjecture due to Baumgärtel and Lledó [1] according to which for every compact group $G$ one has $\hat{Z}(G) \cong C(G)$, where the ‘chain group’ $C(G)$ is the free abelian group (written multiplicatively) generated by the set $\hat{G}$ of isomorphism classes of irreducible representations of $G$ modulo the relations $[Z] = [X] \cdot [Y]$ whenever $Z$ is contained in $X \otimes Y$. Thus the center $Z(G)$ depends only on the (ordered) representation ring of $G$. Furthermore, we prove that every ‘t-map’ $\varphi : \hat{G} \to A$ into an abelian group, i.e. every map satisfying $\varphi(Z) = \varphi(X) \varphi(Y)$ whenever $X, Y, Z \in \hat{G}$ and $Z \prec X \otimes Y$, factors through the restriction map $\hat{G} \to \hat{Z}(G)$. All these results generalize to pro-reductive groups over algebraically closed fields of characteristic zero.

1 Introduction

With every compact group $G$ one can associate two canonical compact abelian groups, to wit the center $Z(G)$ and the abelianization $G_{ab} = G/[G, G]$. Since every compact group can be recovered from its (abstract) category of finite dimensional unitary representations [2], it is natural to ask whether the said abelian groups can be recovered directly from Rep$G$ without appealing to a reconstruction theorem à la Tannaka-Krein-Doplicher-Roberts or Saavedra-Rivano-Deligne-Milne. Since Rep$G$ is a discrete structure it is clear that one will rather recover the duals $\hat{G}_{ab}$ and $\hat{Z}(G)$. In the case of $\hat{G}_{ab}$ it is well known how to proceed: Writing $\mathcal{C} = \text{Rep}G$, let $\mathcal{C}_1 \subset \mathcal{C}$ be the full subcategory of one dimensional representations. Then the set of isomorphism classes of objects in $\mathcal{C}_1$ is a (discrete) abelian group, and it is easy to see that

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it is isomorphic to $\widehat{G}_{ab}$. It is natural to ask whether also $\widehat{Z}(G)$ can be recovered directly from $\text{Rep } G$.

Motivated by certain operator algebraic considerations closely related to and inspired by [3], Baumgärtel and Lledó [1, Section 5] defined, for every compact group $G$, a discrete abelian group $C(G)$ in terms of the representation category $\text{Rep } G$. They identified a surjective homomorphism $C(G) \rightarrow \widehat{Z}(G)$ and conjectured the latter to be an isomorphism. They substantiated this conjecture by explicit verification for several finite and compact Lie groups. (According to [1], the case of $SU(N)$ was checked by C. Schweigert.) In this paper we prove $\widehat{Z}(G) \cong C(G)$ for all compact groups, exploiting a remark made in [4], and we derive two useful corollaries. Despite our general proof the examples in [1] remain quite instructive.

2 Definitions and Preparations

Throughout the paper, $G$ denotes any compact group and $\widehat{G}$ the set of equivalence classes of irreducible representations. We allow ourselves the usual harmless sloppiness of not always distinguishing between an irreducible representation $X$ and its equivalence class $[X] \in \widehat{G}$. (Thus ‘Let $X \in \widehat{G}$’ means ‘Let $X \in \widehat{G}$ and let $X \in \text{Rep } G$ be simple such that $[X] = X$’.) While $\widehat{G}$ is a group iff $G$ is abelian, there always is a notion of ‘homomorphism’ into an abelian group:

2.1 Definition Let $G$ be a compact group and $A$ an abelian group. A map $\varphi : \widehat{G} \rightarrow A$ is called a t-map (tensor product compatible) if we have $\varphi(Z) = \varphi(X) \varphi(Y)$ whenever $X, Y, Z \in \widehat{G}$ and $Z \preceq X \otimes Y$.

2.2 Lemma If $\varphi : \widehat{G} \rightarrow A$ is a t-map then $\varphi(1) = 1$, where the first 1 denotes the trivial representation of $G$, and $\varphi(X) = \varphi(X)^{-1}$ for every $X \in \widehat{G}$.

Proof. We have $\varphi(1) = \varphi(1 \otimes 1) = \varphi(1) \varphi(1)$, thus $\varphi(1) = 1$. For any $X \in \widehat{G}$, we have $1 \preceq X \otimes \overline{X}$, thus $1 = \varphi(1) = \varphi(X) \varphi(\overline{X})$, which proves the second claim. ■

The following proposition is essentially due to [1]:

2.3 Proposition For every compact group $G$ there is a universal t-map $p_G : \widehat{G} \rightarrow C(G)$. (Thus for every t-map $\varphi : \widehat{G} \rightarrow A$ there is a unique homomorphism $\beta : C(G) \rightarrow A$ of abelian groups such that

\[ \begin{array}{ccc} \widehat{G} & \xrightarrow{p_G} & C(G) \\ & \searrow \varphi & \\ & A & \end{array} \]
commutes.) Here the ‘chain group’ $C(G)$ is the free abelian group (written multiplicatively) generated by the set $\hat{G}$ of isomorphism classes of irreducible representations of $G$ modulo the relations $[Z] = [X] \cdot [Y]$ whenever $Z$ is contained in $X \otimes Y$. The obvious map $p_G : \hat{G} \to C(G)$ is a t-map.

Proof. We clearly must take $\beta$ to send the generator $[X]$ of $C(G)$ to $\varphi([X])$, proving uniqueness. In view of the definition of a t-map this is compatible with the relations imposed on $C(G)$, whence existence of $\beta$. 

2.4 Remark 1. The above elegant definition of $C(G)$ is due to J. Bernstein. In [1], $C(G)$ was defined as $\hat{G}/\sim$, where $\sim$ is the equivalence relation given by $X \sim Y$ whenever there exist $n \in \mathbb{N}$ and $Z_1, \ldots, Z_n \in \hat{G}$ such that both $X$ and $Y$ are contained in $Z_1 \otimes \cdots \otimes Z_n$. Denoting the $\sim$-equivalence class of $X$ is denoted by $\langle X \rangle$, $C(G)$ is an abelian group w.r.t. the operations $\langle X \rangle \langle Y \rangle = \langle Z \rangle$, where $Z$ is any irrep contained in $X \otimes Y$, and $\langle X \rangle^{-1} = \overline{\langle X \rangle}$. The easy verification of the equivalence of the two definitions is left to the reader.

2. A chain group $C(C)$, in general non-abelian, satisfying an analogous universal property can be defined for any fusion category $C$, but we need only the case $C = \text{Rep} G$ and write $C(G)$ rather than $C(\text{Rep} G)$. 

The following, proven in [1], is the most interesting example of a t-map:

2.5 Proposition The restriction of irreducible representations of $G$ to the center defines a surjective t-map $r_G : \hat{G} \to \hat{Z}(G)$. Thus also the homomorphism of abelian groups $\alpha_G : C(G) \to \hat{Z}(G)$ arising as above is surjective.

Proof. If $Z \in \hat{G}$ and $g \in Z(G)$ then $\pi_Z(g)$ commutes with $\pi_Z(G)$, thus by Schur’s lemma we have $\pi_Z(g) = \chi_Z(g)1_Z$, where $\chi_Z(g) \in U(1)$. Clearly, $g \mapsto \chi_Z(g)$ is a homomorphism, thus $\chi_Z \in \hat{Z}(G)$. This defines a map $r_G : \hat{G} \to \hat{Z}(G)$, which is easily seen to be a t-map. Since $Z(G)$ is a closed subgroup of $G$, [6] Theorem 27.46] says that for every irreducible representation (thus character) $\chi$ of $Z(G)$ there is a unitary representation $\pi$ of $G$ such that $\chi \prec \pi \upharpoonright Z(G)$. We thus have $r_G([\pi]) = \chi$, thus $r_G$ is surjective. Therefore also the map $\alpha_G : C(G) \to \hat{Z}(G)$ arising from Proposition 2.3 is surjective. 

For brevity we denote as fusion categories the semisimple $\mathbb{C}$-linear tensor categories with simple unit and two-sided duals, e.g. the $C^*$-tensor categories with conjugates, direct sums and subobjects of $\mathbb{K}$. (We do not assume finiteness.) All subcategories considered below are
full, monoidal, replete (closed under isomorphisms) and closed under direct sums, subobjects and duals, thus they are themselves fusion categories.

2.6 Definition Let $\mathcal{C}$ be a fusion category. Then $\mathcal{C}_0$ denotes the full tensor subcategory generated by the simple objects $X$ for which the exists a simple object $Y \in \mathcal{C}$ such that $X \preceq Y \otimes \overline{Y}$.

2.7 Remark The subcategory $\mathcal{C}_0$ of a fusion category seems to have first been considered by Etingof et al. [4, Section 8.5], where the following fact is remarked in parentheses. The proof might be well known, but we are not aware of a suitable reference. ◼

2.8 Proposition Let $G$ be a compact group and $\mathcal{C} = \text{Rep} G$. Then the category $\mathcal{C}_0$ coincides with the full subcategory $\mathcal{C}_Z \subset \mathcal{C}$ consisting of those representations that are trivial when restricted to $Z(G)$. Thus $\mathcal{C}_0 \cong \text{Rep}(G/Z(G))$.

Proof. If $X, Y \in \hat{G}$ are simple and $X \preceq Y \otimes \overline{Y}$ then the restriction of $X$ to $Z(G)$ is trivial, implying $\mathcal{C}_0 \subset \mathcal{C}_Z$. As to the converse, let $g \in G$ be such that $g \in \ker \pi_X$ for all $X \in \mathcal{C}_0$. This holds iff $(\pi_Y \otimes \pi_{\overline{Y}})(g) = 1$ for all simple $Y \in \text{Rep} G$. The latter means

$$\pi_Y(g) \otimes \pi_{\overline{Y}}(g^{-1})^t = 1,$$

which is true iff $\pi_Y(g) \in C_{1_Y}$. Now, if $g \in G$ is represented by scalars in all irreps $Y \in \hat{G}$ then $g \in Z(G)$. (This follows from the fact that the irreducible representations separate the elements of $G$.) In view of the Galois correspondence of full monoidal subcategories $D \subset \text{Rep} G$ and closed normal subgroups $H \subset G$ given by

\[
H_D = \{g \in G \mid \pi_X(g) = \text{id} \forall X \in D\}, \quad \text{Obj } D_H = \{X \in \text{Rep} G \mid \pi_X(g) = \text{id} \forall g \in H\}.
\]

we have $H_{\mathcal{C}_0} \subset Z(G) = H_{\mathcal{C}_Z}$, thus $\mathcal{C}_Z \subset \mathcal{C}_0$ and therefore $\mathcal{C}_0 = \mathcal{C}_Z$. ◼

2.9 Lemma Let $G$ be compact and $\mathcal{C} = \text{Rep} G$. For a simple object $X \in \mathcal{C}$ we have $p_G([X]) = 1$ iff $X \in \mathcal{C}_0$.

Proof. If $Z$ and $X_i, Y_i$, $i = 1, \ldots, n$ are simple with $X_i \preceq Y_i \otimes \overline{Y_i}$ and $Z \preceq X_1 \otimes \cdots \otimes X_n$ then $1, Z \preceq Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n \otimes \overline{Y_1} \otimes \cdots \otimes \overline{Y_n}$, thus $Z \sim 1$. This implies that $p_G([X_i]) = \langle X_i \rangle = 1$ for every simple $X \in \mathcal{C}_0$. Conversely, let $X \in \mathcal{C}$ be simple such that $p_G([X]) = 1$. This is equivalent to $X \sim 1$, thus there are simple $Y_1, \ldots, Y_n$ such that $1, X \preceq Y_1 \otimes \cdots \otimes Y_n$. Then $X \preceq Y_1 \otimes \cdots \otimes Y_n \otimes \overline{Y_1} \otimes \cdots \otimes \overline{Y_n}$, and therefore $X \in \mathcal{C}_0$. ◼
3 Results

3.1 Theorem The homomorphism $\alpha_G : C(G) \to \widehat{Z}(G)$ is an isomorphism for every compact group $G$.

Proof. Since all maps in the diagram

$$
\begin{array}{ccc}
\hat{G} & \xrightarrow{p_G} & C(G) \\
\downarrow{r_G} & & \downarrow{\alpha_G} \\
\hat{Z}(G) & & 
\end{array}
$$

are surjective, $\alpha_G$ is an isomorphism iff $\ker p_G = \ker r_G$. By Lemma 2.9 $[X] \in \ker p_G$ iff $X \in C_0$. On the other hand, $[X] \in \ker r_G$ iff $X \in C_Z$. By Proposition 2.8 we have $C_0 = C_Z$, thus we are done. ■

$C(G)$ is defined in terms of the set $\hat{G}$ and the multiplicities $N_{ij}^k = \dim \text{Hom}(\pi_k, \pi_i \otimes \pi_j)$, $i, j, k \in \hat{G}$ (the ‘fusion rules’ in physicist jargon). The same information is contained in the representation ring $R(G)$ provided we take its canonical $\mathbb{Z}$–basis or its order structure [5] into account. We thus have the following

3.2 Corollary The center of a compact group $G$ depends only on the (ordered) representation ring $R(G)$, not on the associativity constraint or the symmetry of the tensor category $\text{Rep} G$. (In general, both the latter are needed to determine $G$ up to isomorphism.)

3.3 Remark A considerably stronger result holds for connected compact groups: Every isomorphism of the (ordered) representation rings of two such groups is induced by an isomorphism of the groups, cf. [5]. For non-connected groups this is wrong: The finite groups $D_{8l}$ and $Q_{8l}$ are non-isomorphic but have isomorphic representation rings, cf. [5]. Yet, as remarked in [11] Section 5.1], the centers are isomorphic (to $\mathbb{Z}/2\mathbb{Z}$), as they must by Corollary 3.2.

As an obvious consequence of Proposition 2.8 and Theorem 3.1 we have:

3.4 Corollary Let $G$ be a compact group and $A$ an abelian group. Then every $t$–map $\varphi : \hat{G} \to A$ factors through $\widehat{Z}(G)$, i.e. there is a homomorphism $\beta : \widehat{Z}(G) \to A$ of abelian groups such that

$$
\begin{array}{ccc}
\hat{G} & \xrightarrow{r_G} & \hat{Z}(G) \\
\downarrow{\varphi} & & \downarrow{\beta} \\
A & & 
\end{array}
$$

3.5 Remark This result should be considered as dual to the well known (and much easier) fact that every homomorphism $G \to A$ from a group into an abelian group factors through the quotient map $G \to G_{ab}$. □

3.6 Remark The results of this note were formulated for compact groups mainly because of the author’s taste and background. In view of [2] all results of this paper generalize without change to pro-reductive algebraic groups over algebraically closed fields of characteristic zero. □

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