NSVZ scheme with the higher derivative regularization for $\mathcal{N} = 1$ SQED

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Abstract

The exact NSVZ relation between a $\beta$-function of $\mathcal{N} = 1$ SQED and an anomalous dimension of the matter superfields is studied within the Slavnov higher derivative regularization approach. It is shown that if the renormalization group functions are defined in terms of the bare coupling constant, this relation is always valid. In the renormalized theory the NSVZ relation is obtained in the momentum subtraction scheme supplemented by a special finite renormalization. Unlike the dimensional reduction, the higher derivative regularization allows to fix this finite renormalization. This is made by imposing the conditions $Z_3(\alpha, \mu = \Lambda) = 1$ and $Z(\alpha, \mu = \Lambda) = 1$ on the renormalization constants of $\mathcal{N} = 1$ SQED, where $\Lambda$ is a parameter in the higher derivative term. The results are verified by the explicit three-loop calculation. In this approximation we relate the DR scheme and the NSVZ scheme defined within the higher derivative approach by the finite renormalization.

Keywords: higher covariant derivative regularization, supersymmetry, $\beta$-function, subtraction scheme.

1 Introduction

Supersymmetry is a beautiful concept of quantum field theory [1], which is studied both by theoreticians and by experimentalists. Investigations of SUSY models reveal interesting theoretical features (see e.g. [2, 3, 4]). Among them is the finiteness of the $\mathcal{N} = 4$ supersymmetric Yang Mills (SYM) theory. As the consequence, this theory obeys the conformal symmetry even at the quantum level due to vanishing of the renormalization group (RG) $\beta$-function. This was proved in all orders of the perturbation theory in [5, 6, 7, 8] after the explicit analytical three-loop calculation of Ref. [9], made with the dimensional reduction (DRED) regularization [10] (this calculation was also confirmed in [11, 12]). DRED is a modification of the dimensional regularization method [13, 14, 15, 16]. It is applied in SUSY theories since the dimensional regularization explicitly breaks SUSY [17]. Renormalization of SUSY theories is usually made using a variant of the modified minimal subtraction scheme $\overline{\text{MS}}$ [18], namely in the $\text{DR}$ scheme. However, it is well known that DRED is not mathematically consistent [20]. Its inconsistency

\footnote{The detailed definition of the $\overline{\text{MS}}$ scheme is given in Ref. [19].}
leads to the loss of explicit SUSY \cite{21}, which will be broken by quantum corrections in higher loops \cite{22, 23}. Nevertheless, a recent analytical calculation \cite{24} demonstrates that the DRED regularization and the DR scheme are working in the $\mathcal{N} = 4$ SYM theory up to the four-loop approximation and give the vanishing result for its $\beta$-function.

SUSY also leads to the absence of radiative corrections to the $\beta$-function of $\mathcal{N} = 2$ SYM theories starting from the two-loop approximation \cite{25}. In this case vanishing of the two- and three-loop contributions to the $\beta$-function was explicitly demonstrated in \cite{26} using the DR scheme. However, the calculations made in Ref. \cite{22}, even after correcting them in Refs. \cite{27, 28}, show that the $\beta$-function obtained from the fermion-fermion-vector, scalar-scalar-vector and ghost-ghost-vector vertexes vanishes, but obtaining the $\beta$-function from the fermion-fermion-scalar vertex gives a non-vanishing result in the three-loop approximation \cite{28}. This result reveals unsolved theoretical problems which appear if DRED is used for the regularization of SUSY theories.

$\mathcal{N} = 1$ SUSY leads to the absence of divergent quantum corrections to the superpotential \cite{29}. Another interesting feature of $\mathcal{N} = 1$ SYM models is existence of the exact relation between the $\beta$-function and the anomalous dimension of the matter superfields, derived in Refs. \cite{30, 31, 32}. This relation is usually called ”the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) $\beta$-function”. It was first obtained in terms of the non-renormalized bare coupling constant $\alpha_0$ for SYM models in Refs. \cite{30, 31} and for the $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) in Refs. \cite{33, 34}. For $\mathcal{N} = 1$ SQED the NSVZ $\beta$-function is

$$
\beta(\alpha_0) = \frac{\alpha_0^3}{\pi} \left( 1 - \gamma(\alpha_0) \right).
$$

(1)

However, a general problem arises, whether it is possible to specify a concrete subtraction scheme, which leads to the exact NSVZ $\beta$-function, if the RG functions are defined through the renormalized coupling constant. With DRED, supplemented by the DR subtraction scheme, this problem was studied in Refs. \cite{35, 36, 37, 27}. The results of the one- \cite{38} and two-loop \cite{39} calculations agree with the NSVZ $\beta$-function, because a two-loop $\beta$-function and a one-loop anomalous dimension are scheme independent in theories with a single coupling constant. In higher orders \cite{36, 27, 40} the exact NSVZ $\beta$-function for the RG functions defined in terms of the renormalized coupling constant can be obtained with the DR scheme after an additional finite renormalization. This finite renormalization should be fixed in each order of the perturbation theory, starting from the three-loop approximation \cite{35, 36, 37, 27}. However, there is no general prescription, how one should construct this finite renormalization using the DR scheme. Investigations of quantum corrections using other regularizations \cite{41, 42} are usually made in one- and two-loop approximations.

In this paper we study in details, how the exact relation (1) can be obtained using the Slavnov higher derivative regularization \cite{43, 44}, which is mathematically consistent and does not break the supersymmetry in all orders \cite{45, 46}. Therefore, this regularization has theoretical advantages over the mathematically inconsistent DRED. Application of the higher derivative regularization to the evaluation of quantum corrections in $\mathcal{N} = 1$ SUSY theories reveals one more interesting feature: integrals needed for obtaining a $\beta$-function defined in terms of the bare coupling constant are integrals of total derivatives \cite{47, 48, 49, 50} and even double total derivatives \cite{51, 52, 53, 54}. This implies that one of the loop integrals can be calculated analytically, and a $\beta$-function in a $L$-loop approximation can be related with an anomalous dimension of the matter superfields in the ($L - 1$)-loop approximation \cite{55}. As a consequence, the NSVZ $\beta$-function can be naturally obtained for the RG functions defined in terms of the bare coupling constant $\alpha_0$ in the case of using the higher derivative regularization. However, if the RG functions are defined in terms of the renormalized coupling constant, the NSVZ $\beta$-function is obtained only in a special subtraction scheme, which is constructed in this paper for $\mathcal{N} = 1$ SQED regularized
by higher derivatives. Unlike DRED, the higher derivative regularization allows to construct the NSVZ scheme by imposing the boundary conditions on the renormalization constants. The results are verified by an explicit three-loop calculation. In this approximation we also relate the DR scheme with the NSVZ scheme obtained with the higher derivatives regularization.

The paper is organized as follows: In Sect. 2 we introduce the higher derivative regularization and remind how the NSVZ $\beta$-function can be obtained using this regularization, if the RG functions are defined in terms of the bare coupling constant and the Pauli–Villars masses are proportional to the parameter $\Lambda$ in the higher derivative term. However, after a rescaling of the Pauli–Villars masses which depends on the bare coupling constant the NSVZ relation for these RG functions is no longer valid. This is discussed in Sect. 3. A standard definition of the RG functions is recalled in Sect. 4. In general, these RG functions (defined in terms of the renormalized coupling constant) do not satisfy the NSVZ relation. However, there is an NSVZ scheme in which the NSVZ relation is valid. In Sect. 4 we demonstrate that this scheme is obtained by imposing a special boundary conditions on the renormalization constants, and the RG functions defined by two different ways coincide in this scheme. Changing the boundary conditions it is possible to change the relation between the $\beta$-function and the anomalous dimension. All these results are verified by an explicit three-loop calculation with the higher derivative regularization in Sect. 5. In Sect. 6 we compare the results obtained in the DR-scheme with the results obtained with the higher derivative regularization in the NSVZ scheme. Some technical details are discussed in the Appendices.

2 Derivation of the NSVZ $\beta$-function for $\mathcal{N} = 1$ SQED

In this section we recall how the NSVZ $\beta$-function can be obtained with the higher derivative regularization for $\mathcal{N} = 1$ SQED. It is convenient to describe this theory in terms of $\mathcal{N} = 1$ superfields [2, 3]. Then in the massless limit the action of $\mathcal{N} = 1$ SQED is written as

$$S = \frac{1}{4e_0^2} \text{Re} \int d^4x \, d^2\theta \, W^a W_a + \frac{1}{4} \int d^4x \, d^4\theta \left( \phi^* e^{2V} \phi + \tilde{\phi}^* e^{-2V} \tilde{\phi} \right),$$

(2)

where $e_0$ is a bare coupling constant, $\phi$ and $\tilde{\phi}$ are (bare) chiral superfields, and $V$ is a real gauge superfield.

In order to regularize this theory by higher derivatives, it is necessary to insert into the first term of Eq. (2) a regularizing function $R$, such that $R(0) = 1$ and $R(\infty) = \infty$, which contains higher derivatives [43, 44]:

$$\frac{1}{4e_0^2} \text{Re} \int d^4x \, d^2\theta \, W^a W_a \rightarrow \frac{1}{4e_0^2} \text{Re} \int d^4x \, d^2\theta \, W^a R(\partial^2 / \Lambda^2) W_a.$$  

(3)

For example, it is convenient to choose this function as $R = 1 + \partial^{2n} / \Lambda^{2n}$, where $\Lambda$ is a dimensionful parameter. Then the divergences remain only in the one-loop approximation [56]. According to the standard prescription they should be regularized by inserting the Pauli–Villars determinants into the generating functional [57]:

$$Z[J,j,\tilde{j}] = \int DV \, D\phi \, D\tilde{\phi} \prod_{I=1}^n (\text{det}(V,M_I))^{c_I} \exp \left( iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{source}} \right),$$

(4)

where $J$, $j$, and $\tilde{j}$ are the sources, and $M_I$ are masses of the Pauli–Villars fields.

Let us describe all parts of this expression:
\[
S_{\text{reg}} = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a R(\partial^2/\Lambda^2)W_a + \frac{1}{4} \int d^4x d^4\theta \left( \phi^* e^{2V} \phi + \tilde{\phi}^* e^{-2V} \tilde{\phi} \right); \\
S_{\text{gl}} = -\frac{1}{64e_0^2} \int d^4x d^4\theta \left( V R(\partial^2/\Lambda^2)D^2 \tilde{D}^2 V + VR(\partial^2/\Lambda^2)\tilde{D}^2 D^2 V \right); \\
S_{\text{source}} = \int d^4x d^4\theta VJ + \left( \int d^4x d^2\theta (\phi j + \tilde{\phi} \tilde{j}) + \text{c.c.} \right). 
\] (5)

(6)

(7)

In the Abelian case it is not necessary to introduce ghost (super)fields. The Pauli–Villars determinants can be presented as functional integrals over the corresponding (chiral) Pauli–Villars superfields \( \Phi \) and \( \tilde{\Phi} \):

\[
\left( \det(V, M) \right)^{-1} = \int D\Phi D\tilde{\Phi} \exp(iS_{\text{PV}}), 
\] (8)

where the action for the Pauli–Villars superfields is

\[
S_{\text{PV}} = \frac{1}{4} \int d^4x \left( \Phi^* e^{2V} \Phi + \tilde{\Phi}^* e^{-2V} \tilde{\Phi} \right) + \left( \frac{1}{2} \int d^4x d^2\theta M\Phi\tilde{\Phi} + \text{c.c.} \right). 
\] (9)

For cancellation of remaining one-loop divergences the coefficients \( c_I \) should satisfy the conditions

\[
\sum_I c_I = 1; \quad \sum_I c_I M_I^2 = 0. 
\] (10)

We choose masses of the Pauli–Villars fields so that they will be proportional to the parameter \( \Lambda \) in the higher derivative term \([50, 55]\):

\[
M_I = a_I \Lambda, 
\] (11)

where \( a_I \) are arbitrary real constants which do not depend of the bare charge. Then the NSVZ relation is obtained exactly in all orders for the RG functions defined in terms of the bare coupling constant independently of a concrete renormalization prescription. Let us briefly describe, how this can be shown.

Due to the Ward identity the two-point Green function of the gauge superfield is transverse:

\[
\Gamma^{(2)}_{V} - S_{\text{gl}} = -\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} d^2\theta V(\theta, -p) \partial^2 \Pi_{1/2} V(\theta, p) d^{-1}(\alpha_0, \Lambda/p), 
\] (12)

where the supersymmetric transversal projector is given by

\[
\partial^2 \Pi_{1/2} = -\frac{1}{8} D^a \tilde{D}^a. 
\] (13)

The function \( d \) defined by Eq. (12) coincides with the invariant charge \( \alpha_{\text{inv}} \) \([58]\), related with the polarization operator \( \Pi(\alpha_0, \Lambda/p) \) by the equation

\[
d^{-1}(\alpha_0, \Lambda/p) = \frac{1}{\alpha_0} \left( 1 + \frac{\alpha_0}{\pi} \Pi_0(\alpha_0, \Lambda/p) \right) = \frac{1}{\alpha_{\text{inv}}(\alpha_0, \Lambda/p)}. 
\] (14)

Similarly, a part of the effective action corresponding to the two-point Green function of the matter superfields can be written in the form

\[
\Gamma^{(2)}_{\phi} = \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} d^2\theta \left( \phi^*(\theta, -p) \phi(\theta, p) + \tilde{\phi}^*(\theta, -p) \tilde{\phi}(\theta, p) \right) G(\alpha_0, \Lambda/p). 
\] (15)
Here we use the following renormalization procedure [58, 59]: we define a renormalized coupling constant $\alpha(\alpha_0, \Lambda/\mu)$, where $\mu$ is a renormalization parameter, requiring that the inverse invariant charge $d^{-1}(\alpha_0(\alpha, \Lambda/\mu), \Lambda/p)$ is finite in the limit $\Lambda \to \infty$ (and, therefore, in this limit depends only on $\alpha$ and $\mu/p$). Then the renormalization constant $Z_3$ is defined according to the equation

$$\frac{1}{\alpha_0} \equiv \frac{Z_3(\alpha, \Lambda/\mu)}{\alpha}. \quad (16)$$

In order to renormalize the two-point Green function of the matter superfields, we construct the renormalization constant $Z$, requiring that the renormalized two-point Green function $ZG$ is finite in the limit $\Lambda \to \infty$:

$$G_{\text{ren}}(\alpha, \mu/p) = \lim_{\Lambda \to \infty} Z(\alpha, \Lambda/\mu)G(\alpha_0, \Lambda/p). \quad (17)$$

Certainly, the renormalized coupling constant $\alpha$ and the renormalization constant $Z$ are not uniquely defined and depend on a choice of a renormalization scheme [60].

The RG functions are usually defined in terms of the renormalized coupling constant. This definition is presented in Sect. 4. However, it is also possible to define the RG functions in terms of the bare coupling constant $\alpha_0$ according to the following prescription:

$$\beta(\alpha_0(\alpha, \Lambda/\mu)) \equiv \frac{d\alpha_0(\alpha, \Lambda/\mu)}{d\ln \Lambda} \bigg|_{\alpha=\text{const}}; \quad (18)$$

$$\gamma(\alpha_0(\alpha, \Lambda/\mu)) \equiv -\frac{d\ln Z(\alpha, \Lambda/\mu)}{d\ln \Lambda} \bigg|_{\alpha=\text{const}}, \quad (19)$$

where $\alpha$ and $\Lambda$ are considered as independent variables. Nevertheless, for finding these functions it is necessary to use the relation between the bare coupling constant $\alpha_0$ and the renormalized coupling constant $\alpha$. Really, the differentiation with respect to $\Lambda$ is made at a fixed value of $\alpha$. Therefore, before this differentiating we should express $\alpha_0$ in terms of $\alpha$ and $\Lambda/\mu$. This implies that expressions dependent on the renormalization scheme appear at intermediate steps of the calculation. However, final expressions for the RG functions (18) and (19) are independent of a renormalization prescription. In order to see this, let us consider the function

$$d^{-1}(\alpha_0, \Lambda/p) = d^{-1}(\alpha, \mu/p) + (\text{terms vanishing in the limit } p \to 0) \quad (20)$$

and differentiate it with respect to $\ln \Lambda$ (at a fixed value of $\alpha$). The derivative of the right hand side evidently vanishes in the limit $p \to 0$. Therefore,

$$0 = \lim_{p \to 0} \frac{dd^{-1}(\alpha_0, \Lambda/p)}{d\ln \Lambda} \bigg|_{\alpha=\text{const}} = \lim_{p \to 0} \left( \frac{\partial d^{-1}(\alpha_0, \Lambda/p)}{\partial \alpha_0} \beta(\alpha_0) - \frac{\partial d^{-1}(\alpha_0, \Lambda/p)}{\partial \ln p} \right), \quad (21)$$

where in the last equality $\alpha_0$ and $p$ are considered as independent variables. From this equation the $\beta$-function (18) is expressed through the expressions which evidently do not depend on a renormalization prescription. Therefore, this function does not depend on a choice of a renormalization scheme. Similarly, differentiating the equality

$$\ln G(\alpha_0, \Lambda/q) = \ln G_{\text{ren}}(\alpha, \mu/q) - Z(\alpha, \Lambda/\mu) + (\text{terms vanishing in the limit } q \to 0) \quad (22)$$

with respect to $\ln \Lambda$ at a fixed value of $\alpha$, in the limit $q \to 0$ we obtain

$$\gamma(\alpha_0) = \lim_{q \to 0} \left( \frac{\partial \ln G(\alpha_0, \Lambda/q)}{\partial \alpha_0} \beta(\alpha_0) - \frac{\partial \ln G(\alpha_0, \Lambda/q)}{\partial \ln q} \right). \quad (23)$$
Therefore, the anomalous dimension (19) does not depend on a choice of the function \( \alpha_0(\alpha, \Lambda/\mu) \) as well.

In order to find an expression for the \( \beta \)-function (18) in the case of \( \mathcal{N} = 1 \) SQED regularized by higher derivatives, it is possible to make the substitution

\[
V \rightarrow \bar{\theta}^a \theta^b \equiv \theta^4
\]

in the expression

\[
\Delta \Gamma^{(2)}_V = \Gamma^{(2)}_V - S_{gf} - S.
\]

Then using Eq. (12) we obtain

\[
\left. \frac{d(\Delta \Gamma^{(2)}_V)}{d \ln \Lambda} \right|_{V(x, \theta) = \theta^4} = (2\pi)^3 \delta^4(p) \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right)
\]

\[
= -(2\pi)^3 \delta^4(p) \frac{d\alpha_0^{-1}}{d \ln \Lambda} = (2\pi)^3 \delta^4(p) \frac{\beta(\alpha_0)}{\alpha_0^2},
\]

where (exactly as in Eqs. (18) and (19)) \( \alpha \) and \( \Lambda \) are considered as independent variables, and we take into account the first equality in Eq. (21).

The crucial observation is that if the higher derivative method is used for a regularization, the integrals obtained by calculating the left hand side of Eq. (26) are integrals of (double) total derivatives [47, 51, 55]. Therefore, one of the loop integrals can be calculated analytically. Using a rather complicated technique [55], it is possible to obtain

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \bigg|_{p=0} = \frac{1}{\pi} \left( 1 - \frac{d}{d \ln \Lambda} \ln G(\alpha_0, \Lambda/q) \bigg|_{q=0} \right)
\]

exactly in all orders. The right hand side of this equation can be expressed through the anomalous dimension (19):

\[
\frac{1}{\pi} \left( 1 - \frac{d}{d \ln \Lambda} \ln G(\alpha_0, \Lambda/q) \bigg|_{q=0} \right) = \frac{1}{\pi} - \frac{1}{\pi} \frac{d}{d \ln \Lambda} \left( \ln G_{\text{ren}}(\alpha, \mu/q) - \ln Z(\alpha, \Lambda/\mu) \bigg|_{q=0} \right)
\]

\[
= \frac{1}{\pi} \left( 1 - \gamma(\alpha_0(\alpha, \Lambda/\mu)) \right),
\]

where we take into account that the function \( ZG \) does not depend on \( \Lambda \) in the limit \( q \rightarrow 0 \). Thus, we obtain

\[
\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{1}{\pi} \left( 1 - \gamma(\alpha_0(\alpha, \Lambda/\mu)) \right).
\]

This relation gives the NSVZ \( \beta \)-function (1), which was first obtained in [33, 34] using a different method.

It is important that this relation between the RG functions defined in terms of the bare coupling constant is valid for an arbitrary choice of the functions \( \alpha_0(\alpha, \Lambda/\mu) \) and \( Z(\alpha, \Lambda/\mu) \), which specify a subtraction scheme. (Note that at intermediate steps these functions are needed for calculating the RG functions (18) and (19).) Thus, with the higher derivative regularization the NSVZ \( \beta \)-function is naturally obtained in terms of the bare coupling constant.

\[\text{2The limit } p \rightarrow 0, \text{ in which the derivative of the function } d^{-1} \text{ vanishes, is obtained due to the factor } \delta^4(p).\]
3 Regularization dependence of the RG functions defined in terms of the bare charge

The RG functions (18) and (19) are related by Eq. (1) independently of a particular choice of counterterms, if the theory is regularized according to the prescription described in Sect. 2. However, it is possible to change the higher derivative regularization so that to modify the NSVZ relation. This can be done, for example, by a bare coupling constant dependent rescaling of the Pauli–Villars masses. Let us investigate, how the NSVZ relation for bare quantities is modified after this rescaling.

Actually, the regularization for which the NSVZ relation is obtained for the RG functions (18) and (19) is singled out by the condition (11). More exactly, the coefficients $a_I$ should not depend on $\alpha_0$. However, one-loop divergences can be also cancelled, if we choose

$$M_I = a_I \Lambda / z_0(\alpha_0),$$

where $z(\alpha_0)$ is an arbitrary finite function of the bare coupling constant $\alpha_0 = e_0^2/4\pi$. This is equivalent to the redefinition of the $S_{\text{reg}}$ and $S_{\text{PV}}$:

$$S_{\text{reg}} = \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a + \frac{1}{4} z_0(\alpha_0) \int d^4x d^4\theta \left( \phi^* e^{2V} \phi + \bar{\phi}^* e^{-2V} \bar{\phi} \right);$$

$$S_{\text{PV}} = \frac{1}{4} z_0(\alpha_0) \int d^8x \left( \Phi^* e^{2V} \Phi + \bar{\Phi}^* e^{-2V} \bar{\Phi} \right) + \left( \frac{1}{2} \int d^4x d^2\theta M \Phi \bar{\Phi} + \text{c.c.} \right).$$

These actions are obtained after rescaling of the integration variables

$$\phi \to \sqrt{z_0(\alpha_0)} \phi; \quad \bar{\phi} \to \sqrt{z_0(\alpha_0)} \bar{\phi}; \quad \Phi_I \to \sqrt{z_0(\alpha_0)} \Phi_I; \quad \bar{\Phi}_I \to \sqrt{z_0(\alpha_0)} \bar{\Phi}_I.$$ 

in the generating functional (4). In order to avoid anomalous contributions of the integration measure, discussed in [61], the Pauli–Villars fields should be rescaled in the same way as the usual fields. Then $S_{\text{reg}}$ given by Eq. (31) and $S_{\text{PV}}$ given by Eq. (32) produce diagrams with insertions of

$$\frac{1}{4} (z_0(\alpha_0) - 1) \int d^4x d^4\theta \left( \phi^* e^{2V} \phi + \bar{\phi}^* e^{-2V} \bar{\phi} \right)$$

on the matter lines and similar diagrams with closed loops of the Pauli–Villars fields. In order to calculate a contribution of these diagrams, we note that Eq. (27) is also valid for all diagrams beyond the one-loop approximation and for all values of $z_0$. This can be proved repeating the calculations made in [55] taking into account that dependence on the Pauli–Villars masses is the same in both sides of this equation. However, a one-loop contribution to the right hand side of Eq. (27) is

$$\frac{1}{\pi} \cdot \frac{d}{d \ln \Lambda} \sum_{I=1}^{n} c_I \ln \frac{M_I}{p} \bigg|_{p=0}. \quad (35)$$

If the Pauli–Villars masses are given by Eq. (30), instead of Eq. (35) we obtain

$$\frac{1}{\pi} \sum_{I=1}^{n} c_I \left( 1 - \frac{d \ln z_0}{d \ln \Lambda} \right) = \frac{1}{\pi} \left( 1 - \frac{d \ln z_0}{d \ln \Lambda} \right) = \frac{1}{\pi} \left( 1 - \beta(\alpha_0) \frac{d \ln z_0}{d \alpha_0} \right). \quad (36)$$
For $z_0 = 1$ this gives $1/\pi$ in the right hand side of Eq. (27). Now it is evident that for an arbitrary function $z_0$ the generalization of Eq. (27) can be written as

$$\frac{d}{d \ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \bigg|_{p=0} = \frac{1}{\pi} \left( 1 - \frac{d}{d \ln \Lambda} \ln G(\alpha_0, \Lambda/q) \bigg|_{q=0} - \frac{d \ln z_0(\alpha_0)}{d \ln \Lambda} \right).$$

(37)

From this equation we easily obtain that the $\beta$-function (18), which depends on the bare coupling constant $\alpha_0$, has the form

$$\beta(\alpha_0) = \frac{\alpha_0^2}{\pi} \frac{1 - \gamma(\alpha_0)}{1 + \alpha_0^2 (d \ln z_0/d \alpha_0)/\pi}. \quad (38)$$

Therefore, changing the Pauli–Villars masses according to Eq. (30) we modify the expression for the exact $\beta$-function expressed in terms of the bare coupling constant.

Also the NSVZ relation can be modified by a finite tuning of the bare charge $\alpha_0 \to \alpha_0'$. Combining this finite tuning with the rescaling of the Pauli–Villars masses, it is possible to obtain the RG functions dependent on the bare charge with coefficients equal to those obtained within the $\overline{\text{DR}}$ approach.

4 NSVZ scheme

Although the above prescriptions allow to obtain the NSVZ $\beta$-function in terms of the bare coupling constant, it is possible to study a problem whether the NSVZ relation is valid for the RG functions defined in terms of the renormalized coupling constant, if the theory is regularized by higher derivatives.

Let us consider the RG functions, defined in terms of the renormalized coupling constant:

$$\tilde{\beta}(\alpha(\alpha_0, \Lambda/\mu)) \equiv \frac{d\alpha(\alpha_0, \Lambda/\mu)}{d \ln \mu} \bigg|_{\alpha_0=\text{const}}; \quad (39)$$

$$\tilde{\gamma}(\alpha(\alpha_0, \Lambda/\mu)) \equiv \frac{d}{d \ln \mu} \ln ZG(\alpha(\alpha_0, \Lambda/\mu)) \bigg|_{\alpha_0=\text{const}} = \frac{d \ln Z(\alpha(\alpha_0, \Lambda/\mu), \Lambda/\mu)}{d \ln \mu} \bigg|_{\alpha_0=\text{const}}, \quad (40)$$

where $\alpha_0$ and $\mu$ are considered as independent variables. These functions depend on the arbitrariness of choosing $\alpha$ and $Z$. As a consequence, in general, these functions do not satisfy the NSVZ relation, which was originally derived for the bare quantities.

The arbitrariness of choosing $\alpha$ and $Z$ can be fixed by imposing certain boundary conditions. For example, the momentum subtraction (MOM) scheme is defined by the following requirements:

$$Z_{\text{MOM}} G(\alpha_{\text{MOM}}, p = \mu) = 1; \quad d^{-1}(\alpha_{\text{MOM}}, p = \mu) = \alpha_{\text{MOM}}^{-1}. \quad (41)$$

In terms of the polarization operator the second condition can be rewritten as $\Pi(\alpha_{\text{MOM}}, p = \mu) = 0$. In the MOM scheme the $\beta$-function (39) coincides with the Gell-Mann–Low function $\Psi(\alpha)$ (see [62] for the detailed explanation).

The other schemes can be obtained by making finite renormalizations:

$$\frac{1}{\alpha} = \frac{z_3(\alpha_{\text{MOM}})}{\alpha_{\text{MOM}}}; \quad Z(\alpha, \Lambda/\mu) = z(\alpha_{\text{MOM}}) Z_{\text{MOM}}(\alpha_{\text{MOM}}, \Lambda/\mu),$$

(42)

where $z_3$ and $z$ are finite functions. In the three-loop approximation such a renormalization relating the NSVZ scheme and the MOM scheme is constructed in Appendix [A]
In this section we construct a scheme of the ultraviolet renormalization for which the NSVZ relation is also satisfied for the renormalized functions in case of using the higher derivative regularization. This scheme is related with the MOM scheme by the finite renormalization (42). It is constructed by imposing an additional condition, which can be formulated as follows: There should be a point \( x_0 = \ln \Lambda / \mu_0 \) such that

\[
\alpha_0(\alpha_{NSVZ}, x_0) = \alpha_{NSVZ}; \quad Z_{NSVZ}(\alpha_{NSVZ}, x_0) = 1 \tag{43}
\]

for all values of \( \alpha_{NSVZ} \). Equivalently, there is a point \( x_0 \) such that

\[
(Z_3)_{NSVZ}(\alpha_{NSVZ}, x_0) = 1; \quad Z_{NSVZ}(\alpha_{NSVZ}, x_0) = 1. \tag{44}
\]

Although these conditions are similar to the requirement \( Z_3 = 1 \), which was used in Ref. [63] to define the conformally invariant limit of perturbative QED [64], Eqs. (43) and (44) are imposed in a single point \( x_0 \) only. In Appendix B we prove that the scheme defined by Eq. (43) exists and without loss of generality it is possible to choose \( x_0 = 0 \). This result is also verified by an explicit calculation in the three-loop approximation.

Let us prove this statement. We assume that Eq. (43) is satisfied. It is convenient to define a variable \( x = \ln \Lambda / \mu \). Then the RG functions (39) and (40) can be written as

\[
\tilde{\beta}(\alpha(\alpha_0, x)) = -\frac{\partial \alpha(\alpha_0, x)}{\partial x}; \quad \tilde{\gamma}(\alpha(\alpha_0, x)) = -\frac{d \ln Z(\alpha(\alpha_0, x), x)}{dx} - \frac{\partial \ln Z(\alpha(\alpha_0, x), x)}{\partial x} \tag{45}
\]

where the total derivative with respect to \( x \) also acts on \( x \) inside \( \alpha \). Calculating these expressions at the point \( x = x_0 \) and taking into account Eq. (43) we obtain

\[
\tilde{\beta}(\alpha_0) = \beta(\alpha_0); \quad \tilde{\gamma}(\alpha_0) = \gamma(\alpha_0). \tag{46}
\]

Really, in order to prove the first equality we note that

\[
\alpha_0(\alpha, x) = \alpha + \beta(\alpha_0)(x - x_0) + O((x - x_0)^2); \quad \alpha(\alpha_0, x) = \alpha_0 - \tilde{\beta}(\alpha)(x - x_0) + O((x - x_0)^2)
\]

\[
= \alpha_0 - \tilde{\beta}(\alpha_0)(x - x_0) + O((x - x_0)^2). \tag{47}
\]

Similarly, the second equation in (46) can be obtained taking into account that

\[
\frac{\partial \ln Z(\alpha, x_0)}{\partial \alpha} = 0 \tag{48}
\]

due to Eq. (43). According to [65] the functions (13) and (19) satisfy the equation

\[
\beta(\alpha_0) = \frac{\alpha_0^2}{\pi} \left( 1 - \gamma(\alpha_0) \right), \tag{49}
\]

if one uses the regularization described in Sect. 2. Therefore, from Eq. (46) we conclude that

\[
\tilde{\beta}(\alpha) = \frac{\alpha^2}{\pi} \left( 1 - \tilde{\gamma}(\alpha) \right). \tag{50}
\]

if the boundary conditions (13) are imposed. In the three-loop approximation this is verified by an explicit calculation in Sect. 5.
5 The scheme dependence in the three-loop approximation

Let us reconsider the results of the three-loop calculation \[47\] for \( \mathcal{N} = 1 \) SQED and investigate the scheme dependence of the RG functions in this approximation. (The details of the three-loop calculation are described in Appendix \[41\].) We will start with the two-point Green function of the matter superfields. If the Pauli–Villars masses are chosen according to the prescription \[11\], in the two-loop approximation this function is given by the following expression \[65\]:

\[
G(\alpha_0, \Lambda/p) = 1 - \int \frac{d^4 k}{(2\pi)^4} \frac{2 \epsilon_0^2}{k^2 R_k(k+p)^2} + \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{4 \epsilon_0^4}{k^2 R_k^2 R_l^2} \left( \frac{1}{(k+p)^2(l+p)^2} \right) + \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{4 \epsilon_0^4}{k^2 R_k^2 (k+p)^2} \times \frac{1}{l^2(k+l)^2} - \sum_{l=1}^n c_l \left( \frac{1}{l^2(M_f^2)^2} \right) + O(\epsilon_0^6),
\]

where \( R_k \equiv R(k^2/\Lambda^2) \). (This expression is written in the Euclidean space after the Wick rotation.)

We split the bare coupling constant \( \alpha_0 = \epsilon_0^2/4\pi \) into the renormalized coupling constant \( \alpha \) and a counterterm. In the lowest approximation

\[
\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{1}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) + O(\alpha), \tag{52}
\]

where the logarithm compensates a one-loop divergence and a finite parameter \( b_1 \) is not fixed. Different choices of \( b_1 \) correspond to different subtraction schemes. Equivalently, Eq. \( (52) \) can be rewritten in the form

\[
\alpha_0 = \alpha \left( 1 + \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) + O(\alpha^2) \right), \tag{53}
\]

In order to cancel divergences in the two-point Green function of the matter superfields, the function \( G(\alpha, \Lambda/p) \) should be multiplied by the renormalization constant \( Z \). This corresponds to the following redefinition of the matter superfields:

\[
\phi = Z^{1/2}(\alpha, \Lambda/\mu) \phi_{\text{ren}}; \quad \tilde{\phi} = Z^{1/2}(\alpha, \Lambda/\mu) \tilde{\phi}_{\text{ren}}, \tag{54}
\]

where \( \phi \) and \( \tilde{\phi} \) are the bare fields, and \( \phi_{\text{ren}} \) and \( \tilde{\phi}_{\text{ren}} \) are the renormalized fields. It is easy to see that in the one-loop approximation the renormalization constant \( Z \), which is chosen so that the renormalized Green function \( ZG \) should be finite, is

\[
Z = 1 + \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) + O(\alpha^2), \tag{55}
\]

where \( g_1 \) is an arbitrary finite constant. In principle, without loss of generality the constant \( g_1 \) can be excluded by a redefinition of the parameter \( \mu \). However, we will not do this, because in the next orders similar coefficients cannot be excluded by this way. Choosing a higher derivative term in the momentum space as

\[\text{[The expression presented in [65] is slightly different, because it includes diagrams with one-loop counterterm insertions on the matter lines and, therefore, corresponds to } Z_{1\text{-loop}} \tilde{G}_{2\text{-loop}}.\]
it is possible to find the following explicit expression for the constant $Z$ in the next order \cite{65}:

$$Z = 1 + \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) + \frac{\alpha^2}{2\pi^2} \ln^2 \frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} \left( \sum_{l=1}^{n} c_l \ln a_l - b_1 + \frac{3}{2} \right) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3),$$

where

$$a_l = \frac{M_l}{\Lambda}$$

and $g_2$ is another finite constant. Choosing $g_2$ we partially fix the subtraction scheme in the considered approximation. The subtraction scheme in the considered approximation is completely fixed by choosing $g_1$, $g_2$, and the finite constants in the renormalization of the coupling constant, $b_1$ and $b_2$ (defined below).

Now, let us calculate the anomalous dimension according to Eqs. \cite{19} and \cite{10}.

In order to find the anomalous dimension according to the prescription \cite{19}, we should differentiate

$$\ln Z(\alpha, \Lambda/\mu) = \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) + \frac{\alpha^2}{2\pi^2} \ln^2 \frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} \left( \sum_{l=1}^{n} c_l \ln a_l - b_1 + \frac{3}{2} \right) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3)$$

with respect to $\ln \Lambda$ and write the result in terms of $\alpha_0$. Then we obtain

$$\gamma(\alpha_0) = -\frac{d \ln Z}{d \ln \Lambda} = -\frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2} \left( \ln \frac{\Lambda}{\mu} - \sum_{l=1}^{n} c_l \ln a_l + b_1 - \frac{3}{2} \right) + O(\alpha^3)$$

$$= -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2} \left( \frac{3}{2} + \sum_{l=1}^{n} c_l \ln a_l \right) + O(\alpha_0^3).$$

This expression is independent of the finite constants $g_i$ and $b_i$, which fix the subtraction scheme.

The anomalous dimension $\tilde{\gamma}(\alpha)$ defined by Eq. \cite{10} can be constructed similarly. For this purpose we rewrite $\ln Z$ in terms of $\alpha_0$ using Eq. \cite{52}:

$$\ln Z(\alpha(\alpha_0, \Lambda/\mu)/\lambda/\mu) = \frac{\alpha_0}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) - \frac{\alpha_0^2}{2\pi^2} \ln^2 \frac{\Lambda}{\mu}$$

$$- \frac{\alpha_0^2}{\pi^2} \ln \frac{\Lambda}{\mu} \left( \sum_{l=1}^{n} c_l \ln a_l + g_1 + \frac{3}{2} \right) + \frac{\alpha_0^2}{\pi^2} \left( g_2 - \frac{1}{2} g_1 \right) + O(\alpha_0^3).$$

Differentiating this expression with respect to $\ln \mu$ and writing the result in terms of $\alpha$ we obtain

$$\tilde{\gamma}(\alpha) = \frac{d \ln Z}{d \ln \mu} = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2} \left( \frac{3}{2} + \sum_{l=1}^{n} c_l \ln a_l - b_1 + g_1 \right) + O(\alpha^3).$$
Unlike Eq. (60), this expression depends on the constants $g_1$ and $b_1$.

Let us study, under what conditions the NSVZ relation is valid. For this purpose we write the explicit expression for the three-loop renormalized coupling constant. The integrals defining the three-loop $\beta$-function are presented, for example, in Ref. [55]. They were obtained in [47], but a more compact form for writing them was proposed later. If the higher derivative term is given by Eq. (56), these integrals can be calculated explicitly using the results of [65]. After this calculation we split the bare coupling constant into the renormalized coupling constant and a counterterm in such a way that the function $d^{-1}$ is finite. It is possible to find (see Appendix A) that the result can be written in the following form:

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{1}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) - \frac{\alpha}{\pi^2} \left( \ln \frac{\Lambda}{\mu} + b_2 \right)$$

$$- \frac{\alpha^2}{\pi^3} \left( \frac{1}{2} \ln^2 \frac{\Lambda}{\mu} - \ln \frac{\Lambda}{\mu} \sum_{I=1}^n c_I \ln a_I - \frac{3}{2} \ln \frac{\Lambda}{\mu} + b_1 \ln \frac{\Lambda}{\mu} + b_3 \right) + O(\alpha^3),$$

(63)

where $b_1$, $b_2$, and $b_3$ are finite constants, which define the subtraction scheme. (This three-loop expression coincides with Eq. (52) at the one-loop level.)

Differentiating Eq. (63) with respect to $\ln \Lambda$ and writing the result in terms of $\alpha_0$ we obtain the $\beta$-function defined by Eq. (18):

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{1}{\pi} + \frac{\alpha_0}{\pi^2} \left( \ln \frac{\Lambda}{\mu} - \sum_{I=1}^n c_I \ln a_I - \frac{3}{2} \ln \frac{\Lambda}{\mu} + b_1 \right) + O(\alpha^3)$$

$$= \frac{1}{\pi} + \frac{\alpha_0}{\pi^2} + \frac{\alpha_0}{\pi^2} \left( \sum_{I=1}^n c_I \ln a_I + \frac{3}{2} \right) + O(\alpha^3).$$

(64)

Thus, this $\beta$-function does not depend on the finite constants $g_i$ and $b_i$, as was proved in Eq. (21). Moreover, comparing this expression with Eq. (60) we see that for all values of $g_i$ and $b_i$ the NSVZ relation is valid.

In order to calculate the $\beta$-function defined by Eq. (59), we re-express $\alpha$ in terms of $\alpha_0$:

$$\frac{1}{\alpha} = \frac{1}{\alpha_0} + \frac{1}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) + \frac{\alpha_0}{\pi^2} \left( \ln \frac{\Lambda}{\mu} + b_2 \right)$$

$$+ \frac{\alpha_0}{\pi^3} \left( - \frac{1}{2} \ln^2 \frac{\Lambda}{\mu} - \ln \frac{\Lambda}{\mu} \sum_{I=1}^n c_I \ln a_I - \frac{3}{2} \ln \frac{\Lambda}{\mu} - b_2 \ln \frac{\Lambda}{\mu} - b_1 b_2 + b_3 \right) + O(\alpha_0^3).$$

(65)

Differentiating this expression with respect to $\ln \mu$ and writing the result in terms of $\alpha$, we obtain

$$\frac{\tilde{\beta}(\alpha)}{\alpha^2} = \frac{1}{\pi} + \frac{\alpha_0}{\pi^2} + \frac{\alpha_0}{\pi^3} \left( - \ln \frac{\Lambda}{\mu} - \sum_{I=1}^n c_I \ln a_I - \frac{3}{2} - b_2 \right) + O(\alpha_0^3)$$

$$= \frac{1}{\pi} + \frac{\alpha}{\pi^2} - \frac{\alpha^2}{\pi^3} \left( \sum_{I=1}^n c_I \ln a_I + \frac{3}{2} - b_1 + b_2 \right) + O(\alpha^3).$$

(66)

In this case the $\beta$-function is related with the anomalous dimension of the matter superfields (62) by Eq. (11) only if finite constants defining a subtraction scheme satisfy the condition

$$b_2 = g_1.$$

(67)
It is easy to see that this identity follows from the system of equations
\[ \alpha_0 = \alpha(\alpha_0, \Lambda/\mu_0) + O(\alpha_0^4); \quad 1 = Z(\alpha_0, \Lambda/\mu_0) + O(\alpha_0^2), \] (68)
which in the considered approximation coincides with Eq. (63). Indeed, from the second equation we get
\[ \ln \frac{\Lambda}{\mu_0} = -g_1. \] (69)
Substituting this relation to the first equation of (68) one can see that \( b_1 = g_1 \) and \( b_2 = g_1 \).

In higher orders it is also necessary to consider terms with larger degrees of \( \alpha_0 \). For example, in the four-loop approximation we should also consider terms proportional to \( \alpha_0^4 \) in the first equation of (43) and terms proportional to \( \alpha_0^2 \) in the second equation of (43). This allows to obtain the next coefficients \( b_3 \) and \( g_2 \). If we set \( g_1 = 0 \), then the NSVZ scheme corresponds to \( g_2 = b_1 = b_2 = b_3 = 0 \).

6 Relation between the NSVZ and \( \overline{\text{DR}} \) schemes in the three-loop approximation

The three-loop \( \beta \)-function and the two-loop anomalous dimension for an arbitrary \( \mathcal{N} = 1 \) SYM theory with matter in the \( \overline{\text{DR}} \) scheme have been calculated in [35]. The result does not satisfy the NSVZ relation. However, in [35] the authors obtained a special redefinition of the coupling constant after which the NSVZ relation is satisfied. For \( \mathcal{N} = 1 \) SQED, considered here, this redefinition is written as
\[ \alpha \equiv \alpha_{\text{NSVZ}} \rightarrow \alpha' \equiv \alpha_{\overline{\text{DR}}} = \alpha - \frac{\alpha^3}{4\pi^2} + O(\alpha^4). \] (70)

In general, under a finite renormalization
\[ \alpha \rightarrow \alpha' = \alpha; \quad Z'(\alpha', \Lambda/\mu) = z(\alpha)Z(\alpha, \Lambda/\mu) \] (71)
the \( \beta \)-function (30) and the anomalous dimension (20) are changed according to the following rules:
\[ \tilde{\beta}'(\alpha') = \left. \frac{d\alpha'}{d\ln \mu} \right|_{\alpha_0=\text{const}} = \frac{d\alpha'}{d\alpha}\tilde{\beta}(\alpha); \] (72)
\[ \tilde{\gamma}'(\alpha') = \left. \frac{d\ln Z'}{d\ln \mu} \right|_{\alpha_0=\text{const}} = \frac{d\ln z}{d\alpha} \cdot \tilde{\beta}(\alpha) + \tilde{\gamma}(\alpha). \] (73)

Using these equations it is easy to see that if \( \tilde{\beta}(\alpha) \) and \( \tilde{\gamma}(\alpha) \) satisfy the NSVZ relation, then
\[ \tilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha} \left. \frac{\alpha^2}{\pi} \frac{1 - \tilde{\gamma}'(\alpha')}{1 - \alpha^2(d\ln z/d\alpha)/\pi} \right|_{\alpha=\alpha(\alpha')}. \] (74)
From Eq. (72) we see that a shift \( \alpha \rightarrow \alpha + \delta \alpha \) leads to the following change of the \( \beta \)-function:
\[ \delta \alpha' = \frac{d\alpha'}{d\alpha} \left. \frac{\alpha^2}{\pi} \frac{1 - \tilde{\gamma}'(\alpha')}{1 - \alpha^2(d\ln z/d\alpha)/\pi} \right|_{\alpha=\alpha(\alpha')} \]...

\[ \gamma(\alpha) = 2\gamma(g), \quad \beta(\alpha) = g\beta(g)/2\pi. \]
\[ \delta \tilde{\beta}(\alpha) = \tilde{\beta}(\alpha) \frac{\partial \delta \alpha}{\partial \alpha} - \delta \alpha \frac{\partial \tilde{\beta}}{\partial \alpha}. \]  

(75)

Substituting the explicit expression for \( \delta \alpha \) from Eq. (70) we reproduce the result of [35] for \( N = 1 \) SQED:

\[ \delta \tilde{\beta}(\alpha) = \tilde{\beta}_{\text{DR}} - \tilde{\beta}_{\text{NSVZ}} = -\frac{\alpha^2}{\pi} \cdot \frac{3 \alpha^2}{4 \pi^2} + \frac{\alpha^3}{\pi} \cdot \frac{2 \alpha}{\pi} + O(\alpha^5) = -\frac{\alpha^4}{4 \pi^3} + O(\alpha^5). \]  

(76)

In this paper we relate the results obtained with the higher derivative regularization with the results obtained in [35]. More exactly, we construct a finite renormalization relating the NSVZ scheme with the DR scheme and boundary conditions for which the functions \( \tilde{\beta}(\alpha) \) and \( \tilde{\gamma}(\alpha) \), depending on the renormalized coupling constant, coincide with the results obtained in the DR scheme.

The coefficient \( b_1 \) is found by comparing the two-loop expression for the anomalous dimension of the matter superfield

\[ \tilde{\gamma}_{\text{DR}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} + O(\alpha^3) \]  

(77)

with Eq. (62). Two these expressions coincide, if

\[ b_1 - g_1 = \sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2}. \]  

(78)

The next coefficient \( b_2 \) can be found by comparing the results for the three-loop \( \beta \)-function. The result of [35] in our notation can be written as

\[ \tilde{\beta}_{\text{DR}}(\alpha) = \frac{\alpha^2}{\pi} + \frac{\alpha^3}{\pi^2} - \frac{5 \alpha^4}{4 \pi^3} + O(\alpha^5). \]  

(79)

Comparing this equation with (66) we obtain

\[ b_2 - g_1 = \frac{1}{4}. \]  

(80)

Therefore, for \( x_0 = \ln \Lambda/\mu_0 = -g_1 \)

\[ Z_{\text{DR}}(\alpha_0, x_0) = 1 + O(\alpha_0^2); \]

\[ \frac{1}{\alpha_{\text{DR}}(\alpha_0, x_0)} = \frac{1}{\alpha_0} + \frac{1}{\pi} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2} \right) + \frac{\alpha_0}{4 \pi^2} + O(\alpha_0^2). \]  

(81)

In the considered approximation these equations define a scheme in which the RG functions [39] and [40] coincide with the corresponding RG functions obtained with the DR scheme. (It is expedient to compare these conditions with Eq. (43) defining the NSVZ scheme.)

It is easy to see that \( Z_{\text{DR}} \) and \( \alpha_{\text{DR}} \) can be obtained from \( Z_{\text{NSVZ}} \) and \( \alpha_{\text{NSVZ}} \) (constructed with the higher derivative regularization) by the following finite renormalization:

\[ Z_{\text{DR}}(\alpha_{\text{DR}}, \Lambda/\mu) = \left( 1 - \frac{\alpha_{\text{NSVZ}}}{\pi} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2} - b_1 \right) + O(\alpha_{\text{NSVZ}}^2) \right) Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, \Lambda/\mu); \]

\[ \frac{1}{\alpha_{\text{DR}}} = \frac{1}{\alpha_{\text{NSVZ}}} + \frac{b_1}{\pi} + \frac{\alpha_{\text{NSVZ}}}{4 \pi^2} - \frac{\alpha_{\text{NSVZ}}}{\pi^2} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2} - b_1 \right) + O(\alpha_{\text{NSVZ}}^2), \]  

(82)
where $b_1$ is an arbitrary finite constant. For

$$b_1 = \sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2}$$  \hspace{1cm} (83)$$

Eq. (82) corresponds to the result of Ref. [35], which is given by Eq. (70), because in [35] the authors construct the NSVZ scheme in which the anomalous dimension is fixed by the condition

$$Z_{\text{DR}}(\alpha_{\text{DR}}, \Lambda/\mu) = Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, \Lambda/\mu).$$  \hspace{1cm} (84)$$

In the NSVZ scheme obtained in this paper at the one-loop level this condition can be satisfied by a special choice of the constant $b_1$. Fixing this additional constant we construct a finite renormalization after which not only the $\beta$-functions, but also the anomalous dimensions coincide. In the next orders the situation is the same: it is possible to redefine the coupling constant so that the NSVZ relation is valid for the RG functions defined in terms of the renormalized coupling constant. The finite renormalization of the matter superfields is not fixed by this requirement. This renormalization can be found requiring coincidence of the anomalous dimensions.

7 Conclusion

For $\mathcal{N} = 1$ SQED regularized by higher derivatives the NSVZ relation is automatically satisfied, if a $\beta$-function and an anomalous dimension are defined in terms of the bare coupling constant according to Eqs. (18) and (19). If these definitions are used, then the regularization described in Sect. 2 always gives the exact NSVZ $\beta$-function. In order to obtain a result different from the exact NSVZ $\beta$-function in this case, it is necessary to make a finite rescaling of the Pauli–Villars masses and a finite tuning of the bare charge $\alpha_0$. In particular, using this procedure it is possible to make coefficients of the RG functions defined in terms of the bare charge equal to the ones of the RG functions defined in terms of the renormalized charge in the $\overline{\text{DR}}$ scheme.

If a $\beta$-function and an anomalous dimension are defined in terms of the renormalized coupling constant, the NSVZ $\beta$-function is obtained only in a special (NSVZ) scheme, which is related with the MOM scheme by a finite renormalization. In case of using the DRED regularization the only way to construct this finite renormalization is to use the definition of the NSVZ scheme. However, if the theory is regularized by higher derivatives there is a concrete extra prescription: there should be a point $x_0$ in which

$$Z_3(\alpha, x_0) = 1; \quad Z(\alpha, x_0) = 1.$$  \hspace{1cm} (85)$$

In order to obtain a result different from the NSVZ $\beta$-function, it is possible to make a finite renormalization, which, in turn, changes the boundary conditions.

These results are verified by an explicit three-loop calculation. In particular, we relate the $\overline{\text{DR}}$ scheme and the NSVZ scheme obtained with the higher derivative regularization by a finite renormalization of both the coupling constant and the matter superfields similar to the result of [35]. However, an additional finite constant which gives the required value of the two-loop anomalous dimension should be fixed.

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Appendixes

A  Three-loop renormalization

In this appendix we describe the renormalization of $\mathcal{N} = 1$ SQED, regularized by higher derivatives, at the three-loop level following Refs. [65, 47]. We start with the expression (51) for the two-point Green function of the matter superfields. We split the bare coupling constant $\alpha_0 = e_0^2/4\pi$ into the renormalized coupling constant $\alpha$ and a counterterm. In the lowest approximation this can be done according to Eq. (52), in which the logarithm compensate a one-loop divergence, and a finite parameter $b_1$ is not fixed. The expression (51) was calculated in Ref. [65] for $R_k = 1 + k^{2n}/\Lambda^{2n}$. The result can be written in the following form:

$$G(\alpha_0, \Lambda/p) = 1 - \frac{\alpha_0}{\pi} \ln \frac{\Lambda}{p} - \frac{\alpha_0}{2\pi} \ln \frac{\Lambda}{p} + \frac{\alpha_0^2}{\pi^2} \ln \frac{\Lambda}{p} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{5}{2} \right) + \frac{\alpha_0^2}{\pi^2} c_2$$

$$+ \left( \text{terms vanishing in the limit } \Lambda \to \infty \right) + O(\alpha_0^3),$$

where the constant $c_2$ was not found in Ref. [65]. (A finite part of this function in the one-loop approximation has been explicitly calculated.) The finite constants $a_I$ are related with the Pauli–Villars masses by Eq. (11). In terms of the renormalized coupling constant the function $G$ is given by

$$G(\alpha, \Lambda/\mu, \Lambda/p) = 1 - \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + \frac{1}{2} \right) + \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} - b_1 \ln \frac{\Lambda}{\mu} + \frac{1}{2} \ln \frac{\Lambda}{\mu} - \frac{1}{2} b_1$$

$$+ \ln \frac{\Lambda}{p} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{5}{2} \right) + c_2 \right) + \left( \text{terms vanishing in the limit } \Lambda \to \infty \right) + O(\alpha^3).$$

This expression should be multiplied by a factor $Z$ such that the expression $ZG$ is finite. This is true, if the renomalization constant $Z$ is given by Eq. (57):

$$Z = 1 + \frac{\alpha}{\pi} \left( \ln \frac{\Lambda}{\mu} + g_1 \right) + \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda}{\mu} \left( \sum_{I=1}^{n} c_I \ln a_I - b_1 + \frac{3}{2} - g_1 \right) + \frac{\alpha^2}{\pi^2} g_2 + O(\alpha^3),$$

where $g_1$ and $g_2$ are undefined finite constants. Really, calculating the product $ZG$ in the limit $\Lambda \to \infty$, it is easy to see that all terms containing $\ln \Lambda$ cancel each other and

$$G_{\text{ren}}(\alpha, \mu/p) = \lim_{\Lambda \to \infty} ZG = 1 - \frac{\alpha}{\pi} \left( \ln \frac{\mu}{p} + g_1 + \frac{1}{2} \right) + \frac{\alpha^2}{\pi^2} \ln \frac{\mu}{p} + \ln \frac{\mu}{p} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{5}{2} \right)$$

$$- (b_1 + g_1) \ln \frac{\mu}{p} + c_2 - \frac{1}{2} g_1 + g_2 - \frac{1}{2} b_1 \right) + O(\alpha^3).$$

This expression is finite for all finite values of $b_1$, $g_1$, and $g_2$. Therefore, they can be chosen in an arbitrary way (unlike the constant $c_2$, which is fixed, but unknown).
Values of the constants $b_i$ and $g_i$ can be found from boundary conditions, which specify the scheme. For example, the MOM scheme is defined by the boundary conditions (41):

$$Z_{\text{MOM}}(\alpha_{\text{MOM}}, p = \mu) = 1; \quad d^{-1}(\alpha_{\text{MOM}}, p = \mu) = \alpha_{\text{MOM}}^{-1}.$$ 

In the considered case the first condition gives

$$g_1 = \frac{1}{2}; \quad g_2 = -c_2 + \frac{1}{2} b_1 = \frac{1}{4} + \frac{1}{2} b_1. \quad (89)$$

(The constant $b_1$ is defined from the other boundary condition.) Therefore, in the subtraction scheme defined by the boundary conditions (41) the constants $b_i$ and $g_i$ are related with the finite parts of the Green function ($c_1 = -1/2$ and $c_2$). However, it is possible to impose the different boundary conditions (44):

$$Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, \mu = \Lambda) = 1; \quad (Z_3)_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, \mu = \Lambda) = 1.$$ 

In this case

$$1 = Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, \Lambda = \mu) = 1 + \frac{\alpha_{\text{NSVZ}}}{\pi} g_1 + \frac{\alpha_{\text{NSVZ}}^2}{\pi^2} g_2 + O(\alpha_{\text{NSVZ}}^3). \quad (90)$$

so that

$$g_1 = 0; \quad g_2 = 0. \quad (91)$$

In this case the constants $g_i$ (and $b_i$, see below) are independent of the finite parts of the effective action (i.e. the constants $-1/2$ and $c_2$).

Let us proceed to the renormalization of the coupling constant. For this purpose we should write the function $d^{-1}$. The integrals defining this function in the three-loop approximation have been found in Ref. [47]. Although it is very difficult to calculate their finite parts, we know [55] that this function satisfies Eq. (27):

$$\left. \frac{d}{d\ln \Lambda} \left( d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \right|_{p=0} = \frac{1}{\pi} \left( 1 - \frac{d}{d\ln \Lambda} \ln G(\alpha_0, \Lambda/q) \right)_{q=0}. \quad (92)$$

This equation gives the NSVZ relation in terms of the bare charge, i.e. for the RG functions (18) and (19). The first term in the left hand side vanishes, because the function $d^{-1}$ depends only on $\alpha$ and $\mu/p$ and (expressed in terms of the renormalized coupling constant) is independent of $\Lambda$ in the limit $p \to 0$. The right hand side of this equation can be calculated differentiating the logarithm of Eq. (87), taking into account that the limit $\Lambda \to \infty$ in the massless case is equivalent to the limit $q \to 0$ due to the dependence on $q/\Lambda$. Then it is easy to see that all terms containing the dependence on the momentum $q$ vanish. Expressing the result in terms of the bare coupling constant we obtain Eq. (63):

$$-\frac{d}{d\ln \Lambda} (\alpha_0^{-1}) = \frac{1}{\pi} + \frac{\alpha_0}{\pi^2} - \frac{\alpha_0^2}{\pi^3} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{3}{2} \right) + O(\alpha_0^3).$$

A renormalization of the coupling constant can be found by integrating this equation, which should be written in terms of the renormalized coupling constant:

$$-\frac{d}{d\ln \Lambda} (\alpha_0^{-1}) = \frac{1}{\pi} + \frac{\alpha}{\pi^2} + \frac{\alpha^2}{\pi^3} \ln \frac{\Lambda}{\mu} + \frac{\alpha^2}{\pi^3} b_1 - \frac{\alpha^2}{\pi^3} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{3}{2} \right) + O(\alpha^3). \quad (92)$$

The result is given by Eq. (63):
\[ \frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{1}{\pi} \left( \ln \frac{\Lambda}{\mu} + b_1 \right) - \frac{\alpha}{\pi^2} \left( \ln \frac{\Lambda}{\mu} + b_2 \right) - \frac{\alpha^2}{\pi^2} \left( \frac{1}{2} \ln^2 \frac{\Lambda}{\mu} - \ln \frac{\Lambda}{\mu} \sum_{I=1}^{n} c_I \ln a_I - \frac{3}{2} \ln \frac{\Lambda}{\mu} + b_1 \ln \frac{\Lambda}{\mu} + b_3 \right) + O(\alpha^2), \]

where \( b_1, b_2, \) and \( b_3 \) are integration constants in each order.

Although we have not calculated the inverse invariant charge \( d^{-1} \), starting from this equation it is possible to restore its divergent part by requiring its finiteness after the substitution \( \alpha_0 \to \alpha_0(\alpha, \Lambda/\mu) \):

\[
\begin{align*}
d^{-1}(\alpha_0, \Lambda/p) &= \frac{1}{\alpha_0} + \frac{1}{\pi} \left( \ln \frac{\mu}{p} + d_1 \right) + \frac{\alpha_0}{\pi^2} \left( \ln \frac{\mu}{p} + d_2 \right) + \frac{\alpha_0^2}{\pi^3} \left( - \frac{1}{2} \ln^2 \frac{\mu}{p} - \ln \frac{\mu}{p} \sum_{I=1}^{n} c_I \ln a_I - \frac{3}{2} \ln \frac{\mu}{p} + (b_1 - d_2) \ln \frac{\mu}{p} + d_3 - b_3 + b_1 d_2 \right) + O(\alpha^3),
\end{align*}
\]

(94)

where \( d_1, d_2, \) and \( d_3 \) are finite constants, which should be found by calculating Feynman graphs. It is easy to see that for any values of \( d_i \) and for an arbitrary choice of \( b_i \) the above expression for \( d^{-1} \) is a finite function of \( \alpha \) and \( \mu/p \):

\[
\begin{align*}
d^{-1}(\alpha(\alpha, \Lambda/\mu), \Lambda/p) &= \frac{1}{\alpha} + \frac{1}{\pi} \left( \ln \frac{\mu}{p} + d_1 - b_1 \right) + \frac{\alpha}{\pi^2} \left( \ln \frac{\mu}{p} + d_2 - b_2 \right) + \frac{\alpha^2}{\pi^3} \left( - \frac{1}{2} \ln^2 \frac{\mu}{p} - \ln \frac{\mu}{p} \sum_{I=1}^{n} c_I \ln a_I - \frac{3}{2} \ln \frac{\mu}{p} + (b_1 - d_2) \ln \frac{\mu}{p} + d_3 - b_3 + b_1 d_2 \right) + O(\alpha^3).
\end{align*}
\]

(95)

The coefficients \( b_i \) can be found from boundary conditions. If the boundary conditions \( [41] \), defining the MOM scheme, are imposed, we obtain:

\[
\begin{align*}
b_1 &= d_1; \quad b_2 = d_2; \quad b_3 = d_3 + b_1 d_2 = d_3 + d_1 d_2.
\end{align*}
\]

(96)

Therefore, in this case the coefficients defining the scheme are expressed in terms of the finite parts of the Green function. In the limit \( \Lambda \to \infty \) it is easy to see that in the MOM scheme

\[
Z_{\text{MOM}} G(\alpha_{\text{MOM}}, \mu/p) = 1 - \frac{\alpha_{\text{MOM}}}{\pi} \ln \frac{\mu}{p} + \frac{\alpha_{\text{MOM}}^2}{\pi^2} \left( \frac{1}{2} \ln^2 \frac{\mu}{p} + \ln \frac{\mu}{p} \left( \sum_{I=1}^{n} c_I \ln a_I + 2 - d_1 \right) \right) + O(\alpha_{\text{MOM}}^3);
\]

(97)

\[
\begin{align*}
d^{-1}(\alpha_{\text{MOM}}, \mu/p) &= \frac{1}{\alpha_{\text{MOM}}} + \frac{1}{\pi} \frac{\ln \frac{\mu}{p}}{p} + \frac{\alpha_{\text{MOM}}}{\pi^2} \frac{\ln \frac{\mu}{p}}{p} + \frac{\alpha_{\text{MOM}}^2}{\pi^3} \left( - \frac{1}{2} \ln^2 \frac{\mu}{p} - \ln \frac{\mu}{p} \sum_{I=1}^{n} c_I \ln a_I - \frac{3}{2} \ln \frac{\mu}{p} + (d_1 - d_2) \ln \frac{\mu}{p} \right) + O(\alpha_{\text{MOM}}^3).
\end{align*}
\]

(98)

If we impose the boundary conditions \( [41] \), defining the NSVZ scheme, then the coefficients \( b_i \) do not depend on the finite parts of the Green function:
\[
\frac{1}{\alpha_{NSVZ}} = \frac{1}{\alpha_0} = \frac{1}{\alpha_{NSVZ}} - \frac{1}{\pi} b_1 - \frac{\alpha_{NSVZ}}{\pi^2} b_2 - \frac{\alpha_{NSVZ}^2}{\pi^3} b_3 + O(\alpha_{NSVZ}^3),
\]
so that
\[
b_1 = 0; \quad b_2 = 0; \quad b_3 = 0.
\]
In this case the renormalized Green functions in the limit \(\Lambda \to \infty\) are written as
\[
Z_{NSVZ}G(\alpha_{NSVZ}, \mu/p) = 1 - \frac{\alpha_{NSVZ}}{\pi} \left( \ln \frac{\mu}{p} + \frac{1}{2} \right)
+ \frac{\alpha_{NSVZ}^2}{\pi^2} \left[ \ln \frac{2\mu}{p} + \ln \frac{\mu}{p} \left( \sum_{I=1}^{n} c_I \ln a_I + \frac{5}{2} \right) + c_2 \right] + O(\alpha_{NSVZ}^3);
\]
\[
d^{-1}(\alpha_{NSVZ}, \mu/p) = \frac{1}{\alpha_{NSVZ}} + \frac{1}{\pi} \left( \ln \frac{\mu}{p} + d_1 \right) + \frac{\alpha_{NSVZ}}{\pi^2} \left( \ln \frac{\mu}{p} + d_2 \right)
+ \frac{\alpha_{NSVZ}^2}{\pi^3} \left( \frac{1}{2} \ln \frac{2\mu}{p} - \ln \frac{\mu}{p} \sum_{I=1}^{n} c_I \ln a_I - \frac{3}{2} \ln \frac{\mu}{p} - d_2 \ln \frac{\mu}{p} + d_3 \right) + O(\alpha_{NSVZ}^3).
\]

It is easy to verify that the MOM scheme defined by the boundary conditions (41) and the NSVZ scheme defined by the boundary conditions (44) are related by the following finite renormalization:
\[
\frac{1}{\alpha_{NSVZ}} = z_3(\alpha_{MOM}) = \frac{1}{\alpha_{MOM}} - \frac{1}{\pi} d_1 - \frac{\alpha_{MOM}}{\pi^2} d_2 - \frac{\alpha_{MOM}^2}{\pi^3} (d_3 + d_1 d_2) + O(\alpha_{MOM}^3);
\]
\[
Z_{NSVZ}(\alpha_{NSVZ}, \Lambda/\mu) = z(\alpha_{MOM}) Z_{MOM}(\alpha_{MOM}, \Lambda/\mu)
= \left( 1 - \frac{\alpha_{MOM}}{2\pi} - \frac{\alpha_{MOM}^2}{\pi^2} \left( \frac{1}{2} d_1 - c_2 \right) + O(\alpha_{MOM}^3) \right) Z_{MOM}(\alpha_{MOM}, \Lambda/\mu),
\]
where the remaining coefficients can be found by calculating finite parts of the Feynman diagrams.

**B Existence of the NSVZ scheme**

In this section we prove that the scheme defined by Eq. (43) exists and without loss of generality it is possible to set \(x_0 = 0\). For this purpose we note that the renormalized coupling constant \(\alpha\) is defined so that the function \(d^{-1}(\alpha(\alpha, \Lambda/\mu), \Lambda/p)\) does not depend on \(\Lambda\) in the limit \(\Lambda \to \infty\). Certainly, this condition does not uniquely define \(\alpha\). In particular, it is possible to perform a finite redefinition of the coupling constant \(\alpha = \alpha(\alpha')\). After this redefinition
\[
\alpha(\alpha, \Lambda/\mu) \to \alpha(\alpha(\alpha'), \Lambda/\mu).
\]
Because the function \(\alpha(\alpha')\) is finite, the expression
\[
d^{-1}(\alpha(\alpha(\alpha'), \Lambda/\mu), \Lambda/p)
\]
is also finite. Therefore, \(\alpha'\) can be also chosen as a renormalized coupling constant.
Let us choose the renormalized coupling constant in an arbitrary way. It is defined by a function $\alpha_0(\alpha, \Lambda/\mu)$. Then we construct a new function

$$\alpha_0(\alpha, x = 0) \equiv b(\alpha),$$

where

$$x \equiv \ln \frac{\Lambda}{\mu}. \quad (106)$$

Evidently, the function $b(\alpha)$ does not depend on $\Lambda$ and is finite in the limit $\Lambda \to \infty$. Then we define a new renormalized coupling constant according to the prescription

$$\alpha'(\alpha) \equiv b(\alpha). \quad (107)$$

Then, from Eq. (105) we obtain

$$\alpha_0(\alpha', x = 0) = \alpha'. \quad (108)$$

Therefore, choosing $\alpha'$ as a renormalized coupling constant we satisfy the first condition in Eq. (43) with $x_0 = 0$.

Similarly, by definition, the renormalization constant $Z$ is chosen in such a way that the expression

$$Z(\alpha, \Lambda/\mu)G(\alpha_0(\alpha, \Lambda/\mu), \Lambda/p) \quad (109)$$

is finite. Again, this condition does not uniquely define the function $Z$. In particular, it is possible to multiply $Z$ by an arbitrary finite function of the renormalized coupling constant $\alpha$. This possibility can be used for constructing the renormalization constant which satisfies the second condition in Eq. (43). For this purpose we consider an arbitrary renormalization constant $Z$ and construct the function

$$Z(\alpha, x = 0) \equiv g(\alpha). \quad (110)$$

Then the renormalization constant $Z'$ is defined according to the prescription

$$Z'(\alpha', \Lambda/\mu) \equiv \frac{1}{g(\alpha(\alpha'))} Z(\alpha(\alpha'), \Lambda/\mu). \quad (111)$$

It is evident that the expression

$$Z'(\alpha', \Lambda/\mu)G(\alpha_0(\alpha', \Lambda/\mu), \Lambda/p) \quad (112)$$

does not depend on $\Lambda$ in the limit $\Lambda \to \infty$. Therefore the function $Z'(\alpha', \Lambda/\mu)$ can be chosen as a renormalization constant. Moreover, due to Eqs. (110) and (111) this function satisfies the condition

$$Z'(\alpha', x = 0) = 1. \quad (113)$$

Therefore, we have constructed the renormalization constants $Z_3'(\alpha', \Lambda/\mu)$ and $Z'(\alpha', \Lambda/\mu)$ which satisfy the conditions (44) with $x_0 = 0$.

Let us verify these results by an explicit three-loop calculation. From Eq. (63) we obtain

$$\frac{1}{\alpha_0(\alpha, x = 0)} = \frac{1}{\alpha} - \frac{1}{\pi} b_1 - \frac{\alpha}{\pi^2} b_2 - \frac{\alpha^2}{\pi^3} b_3 + O(\alpha^3). \quad (114)$$
As a consequence, the function \( b(\alpha) \) defined by Eq. (105) is given by

\[
b(\alpha) = \alpha \left( 1 - \frac{\alpha}{\pi} b_1 - \frac{\alpha^2}{\pi^2} b_2 - \frac{\alpha^3}{\pi^3} b_3 \right) + O(\alpha^5) \equiv \alpha'(\alpha). \tag{115}
\]

Rewriting the bare coupling constant \( \alpha_0 \) in terms of \( \alpha' \) defined by this equation we obtain

\[
\frac{1}{\alpha_0} = \frac{1}{\alpha'} - \frac{1}{\pi} \ln \frac{\Lambda}{\mu} - \frac{\alpha'}{\pi^2} \ln \frac{\Lambda}{\mu} - \frac{(\alpha')^2}{\pi^3} \left( \frac{1}{2} \ln^2 \frac{\Lambda}{\mu} - \ln \frac{\Lambda}{\mu} \sum_{I=1}^n c_I \ln \frac{M_I}{\Lambda} - \frac{3}{2} \ln \frac{\Lambda}{\mu} \right) + O((\alpha')^3). \tag{116}
\]

This choice is a particular case of Eq. (63) with \( b_1 = b_2 = b_3 = 0 \). Therefore, the function \( d^{-1}(\alpha_0(\alpha', \Lambda/\mu), \Lambda/p) \) does not depend on \( \Lambda \) in the limit \( \Lambda \to \infty \). Moreover, it is evident that

\[
\alpha_0(\alpha', x = 0) = \alpha' + O((\alpha')^5). \tag{117}
\]

Similarly, one can construct the function \( Z'(\alpha', \Lambda/\mu) \) in the two-loop approximation. As a starting point we consider the renormalization constant (57). Then we find

\[
g(\alpha) = Z(\alpha, x = 0) = 1 + \frac{\alpha}{\pi} g_1 + \frac{\alpha^2}{\pi^2} g_2 + O(\alpha^3). \tag{118}
\]

Writing this function in terms of \( \alpha' \) defined by Eq. (115) we obtain

\[
g(\alpha(\alpha')) = 1 + \frac{\alpha'}{\pi} g_1 + \frac{\alpha'^2}{\pi^2}(g_2 - g_1 b_1) + O((\alpha')^3). \tag{119}
\]

As a consequence,

\[
Z'(\alpha', \Lambda/\mu) = g(\alpha(\alpha'))^{-1} Z(\alpha(\alpha'), \Lambda/\mu)
= 1 + \frac{\alpha'}{\pi} \ln \frac{\Lambda}{\mu} + \frac{(\alpha')^2}{\pi^2} \ln^2 \frac{\Lambda}{\mu} - \frac{(\alpha')^2}{\pi^2} \left( \sum_{I=1}^n c_I \ln \frac{M_I}{\Lambda} + \frac{3}{2} \right) + O((\alpha')^3). \tag{120}
\]

This function can be obtained from Eq. (57) if one sets \( g_1 = g_2 = 0 \). (Earlier we have already set \( b_1 = 0 \).) Therefore, the finiteness of the renormalized two-point Green function of the matter superfields is guaranteed. Moreover, the function \( Z' \) evidently satisfies the required condition

\[
Z'(\alpha', x = 0) = 1 + O((\alpha')^3). \tag{121}
\]

Thus, the NSVZ scheme is constructed in the three-loop approximation. Certainly, this procedure can be generalized to higher orders. For example, considering the next order of the perturbation theory we find a scheme in which Eq. (117) is valid also for terms proportional to \( \alpha_0^5 \), and Eq. (121) is valid for terms proportional to \( \alpha_0^3 \).

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