Classical Solutions of Path-dependent PDEs and Functional Forward-Backward Stochastic Systems*

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Abstract. In this paper we study the relationship between functional forward-backward stochastic systems and path-dependent PDEs. In the framework of functional Itô calculus, we introduce a path-dependent PDE and prove that its solution is uniquely determined by a functional forward-backward stochastic system.

Keywords: Functional Itô calculus, Functional forward-backward systems, Path-dependent PDEs, Classical solutions.

1 Introduction

It is well known that quasilinear parabolic partial differential equations are related to Markovian forward-backward stochastic differential equations (see [12], [10] and [9]), which generalizes the classical Feynman-Kac formula. Recently in the framework of functional Itô calculus, a path-dependent PDE was introduced by Dupire [5] and the so-called functional Feynman-Kac formula was also obtained. For a recent account and development of this theory we refer the reader to [1], [2], [3], [14], [13], [6] and [4].

In this paper, we study a functional forward-backward system and its relation to a quasilinear parabolic path-dependent PDE. In more details, the functional forward-backward system is described by the following forward-backward SDE:

\begin{align}
X^{\gamma_{t},x}(s) &= \gamma_{t}(t) + \int_{t}^{s} b(X^{\gamma_{t}}_{r})dr + \int_{t}^{s} \sigma(X^{\gamma_{t}}_{r})dW(r), \\
Y^{\gamma_{t}}(s) &= g(X^{\gamma_{T}}_{T}) - \int_{s}^{T} h(X^{\gamma_{r}}, Y^{\gamma_{r}}(r), Z^{\gamma_{r}}(r))dr - \int_{s}^{T} Z^{\gamma_{r}}(r)dW(r), \quad s \in [t,T].
\end{align}

After establishing some estimates and regularity results for the solution with respect to paths, we prove that the solution of (1.2) is the unique classical solution of the following path-dependent PDE

\begin{align}
D_{t}u(\gamma_{t}) + \mathcal{L}u(\gamma_{t}) &= h(\gamma_{t}, u(\gamma_{t}), D_{x}u(\gamma_{t})\sigma(\gamma_{t})), \\
u(\gamma_{T}) &= g(\gamma_{T}), \quad \gamma_{T} \in \Lambda.
\end{align}

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where

\[ \mathcal{L}u = \frac{1}{2} tr[\sigma \sigma^T D_{xx}u] + \langle b, D_xu \rangle. \]

The paper is organized as follows: in section 2, we give the notations and results on functional Itô calculus and functional SDEs. Some estimates and regularity results for the solution of FBSDEs are established in section 3. Finally, we prove the relationship between functional FBSDEs and path-dependent PDEs in section 4.

## 2 Preliminaries

### 2.1 Functional Itô calculus

The following notations and tools are mainly from Dupire [5]. Let \( T > 0 \) be fixed. For each \( t \in [0, T] \), we denote by \( \Lambda_t \) the set of càdlàg \( \mathbb{R}^d \)-valued functions on \([0, t] \). For each \( \gamma \in \Lambda_T \) the value of \( \gamma \) at time \( s \in [0, T] \) is denoted by \( \gamma(s) \). Thus \( \gamma = \gamma(s)_{0 \leq s \leq T} \) is a càdlàg process on \([0, T] \) and its value at time \( s \) is \( \gamma(s) \). The path of \( \gamma \) up to time \( t \) is denoted by \( \gamma_t \), i.e., \( \gamma_t = \gamma(s)_{0 \leq s \leq t} \in \Lambda_t \). We denote \( \Lambda = \bigcup_{t \in [0, T]} \Lambda_t \). For each \( \gamma_t \in \Lambda \) and \( x \in \mathbb{R}^d \) we denote by \( \gamma_t(s) \) the value of \( \gamma_t \) at \( s \in [0, t] \) and \( \gamma^x_t := (\gamma_t(s)_{0 \leq s < t}, \gamma_t(t) + x) \) which is also an element in \( \Lambda_t \).

Let \((\cdot, \cdot)\) and \(| \cdot |\) denote the inner product and norm in \( \mathbb{R}^n \). We now define a distance on \( \Lambda \). For each \( 0 \leq t, \bar{t} \leq T \) and \( \gamma_t, \bar{\gamma}_t \in \Lambda \), we denote

\[
\| \gamma_t \| := \sup_{s \in [0, t]} |\gamma_t(s)|,
\]

\[
\| \gamma_t - \bar{\gamma}_t \| := \sup_{s \in [0, t \wedge \bar{t}]} |\gamma_t(s \wedge t) - \bar{\gamma}_t(s \wedge \bar{t})|,
\]

\[
d_{\infty}(\gamma_t, \bar{\gamma}_t) := \sup_{0 \leq s \leq t \vee \bar{t}} |\gamma_t(s \wedge t) - \bar{\gamma}_t(s \wedge \bar{t})| + |t - \bar{t}|.
\]

It is obvious that \( \Lambda_t \) is a Banach space with respect to \( \| \cdot \| \) and \( d_{\infty} \) is not a norm.

**Definition 2.1.** A function \( u : \Lambda \mapsto \mathbb{R} \) is said to be \( \Lambda \)-continuous at \( \gamma_t \in \Lambda \), if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for each \( \bar{\gamma}_t \in \Lambda \) with \( d_{\infty}(\gamma_t, \bar{\gamma}_t) < \delta \), we have \( |u(\gamma_t) - u(\bar{\gamma}_t)| < \varepsilon \). \( u \) is said to be \( \Lambda \)-continuous if it is \( \Lambda \)-continuous at each \( \gamma_t \in \Lambda \).

**Definition 2.2.** Let \( u : \Lambda \mapsto \mathbb{R} \) and \( \gamma_t \in \Lambda \) be given. If there exists \( p \in \mathbb{R}^d \), such that

\[ u(\gamma^x_t) = u(\gamma_t) + \langle p, x \rangle + o(|x|) \text{ as } x \to 0, \quad x \in \mathbb{R}^d. \]

Then we say that \( u \) is (vertically) differentiable at \( \gamma_t \) and denote the gradient of \( D_x u(\gamma_t) = p \). \( u \) is said to be vertically differentiable in \( \Lambda \) if \( D_x u(\gamma_t) \) exists for each \( \gamma_t \in \Lambda \). We can similarly define the Hessian \( D_{xx} u(\gamma_t) \). It is an \( \mathbb{S}(d) \)-valued function defined on \( \Lambda \), where \( \mathbb{S}(d) \) is the space of all \( d \times d \) symmetric matrices.

For each \( \gamma_t \in \Lambda \) we denote

\[ \gamma_{t,s}(r) = \gamma_t(r) \mathbf{1}_{[0,t]}(r) + \gamma_s(r) \mathbf{1}_{[t,s]}(r), \quad r \in [0, s]. \]

It is clear that \( \gamma_{t,s} \in \Lambda_s \).
Definition 2.3. For a given $\gamma_t \in \Lambda$ if we have
\[
u(\gamma_{t,s}) = \nu(\gamma_t) + \alpha(s - t) + o(|s - t|) \quad \text{as} \quad s \to t, \quad s \geq t,
\]
then we say that $\nu(\gamma_t)$ is (horizontally) differentiable in $t$ at $\gamma_t$ and denote $D_t \nu(\gamma_t) = \alpha$. $\nu$ is said to be horizontally differentiable in $\Lambda$ if $D_t \nu(\gamma_t)$ exists for each $\gamma_t \in \Lambda$.

Definition 2.4. Define $C^{j,k}(\Lambda)$ as the set of function $u := (\nu(\gamma_t))_{\gamma_t \in \Lambda}$ defined on $\Lambda$ which are $j$ times horizontally and $k$ times vertically differentiable in $\Lambda$ such that all these derivatives are $\Lambda$-continuous.

The following Itô formula was firstly obtained by Dupire [5] and then generalized by Cont and Fournié [1, 2] and [3].

Theorem 2.1 (Functional Itô’s formula). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a probability space, if $X$ is a continuous semi-martingale and $u$ is in $C^{1,2}(\Lambda)$, then for any $t \in [0,T)$,
\[
u(X_t) - \mathcal{F}_t = \int_0^t D_s \nu(X_s) \, ds + \int_0^t D_s \nu(X_s) \, dX(s) + \frac{1}{2} \int_0^t D_{ss} \nu(X_s) \, d\langle X \rangle(s) \quad P \text{-a.s.}
\]

2.2 Functional FBSDEs

Let $\Omega = C([0,T]; \mathbb{R}^d)$ and $P$ the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$. We denote by $W = (W(t)_{t \in [0,T]})$ the canonical Wiener process, with $W(t, \omega) = \omega(t), \ t \in [0,T], \ \omega \in \Omega$. For any $t \in [0,T]$ we denote by $\mathcal{F}_t$ the $P$-completion of $\sigma(W(s), s \in [0,t])$.

For any $t \in [0,T]$, we denote by $L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$ the set of all square integrable $\mathcal{F}_t$-measurable random variables, $M^2(0,T; \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $x(\cdot)$ such that
\[
E \int_0^T |x(s)|^2 \, ds < +\infty.
\]

Let $t \in [0,T]$ and $\gamma_t \in \Lambda$. For every $s \in [t,T]$, we consider the following functional forward-backward SDEs:

\[
X_{\gamma}^\tau(s) = \gamma_t(t) + \int_t^s b(X_{\gamma}^\tau) \, dr + \int_t^s \sigma(X_{\gamma}^\tau) \, dW(r), \quad (2.1)
\]
\[
Y_{\gamma}^\tau(s) = g(X_{\gamma}^\tau) - \int_s^t h(X_{\gamma}^\tau, Y_{\gamma}^\tau(r), Z_{\gamma}^\tau(r)) \, dr - \int_s^T Z_{\gamma}^\tau(r) \, dW(r), \quad (2.2)
\]

where
\[
X_{\gamma}^\tau(s) := \gamma(s), \ s \in [0,t].
\]

The processes $X, Y, Z$ take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n \times d}$; $b, h, \sigma$ and $g$ take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n \times d}$ and $\mathbb{R}^n$. (2.1) and (2.2) can be rewritten as:

\[
\begin{align*}
&dX_{\gamma}^\tau(s) = b(X_{\gamma}^\tau) \, ds + \sigma(X_{\gamma}^\tau) \, dW(s), \\
&dY_{\gamma}^\tau(s) = h(X_{\gamma}^\tau, Y_{\gamma}^\tau(s), Z_{\gamma}^\tau(s)) \, ds + Z_{\gamma}^\tau(s) \, dW(s), \\
&X_{\gamma}^\tau(t) = \gamma_t(t), \quad Y_{\gamma}^\tau(T) = g(X_{\gamma}^\tau_T).
\end{align*}
\]
For \( z \in \mathbb{R}^{n \times d} \), we define \(|z| = \{tr(zz^T)\}^{1/2}\). For \( z^1 \in \mathbb{R}^{n \times d}, z^2 \in \mathbb{R}^{n \times d}, \)

\[
((z^1, z^2)) = tr(z^1(z^2)^T),
\]

and for \( u^1 = (y^1, z^1) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}, u^2 = (y^2, z^2) \in \mathbb{R}^n \times \mathbb{R}^{n \times d} \)

\[
[u^1, u^2] = (y^1, y^2) + ((z^1, z^2)).
\]

We give the following assumption:

**Assumption 2.1.** \( \forall x^1_1, x^2_1 \in \Lambda \) and \( t \in [0, T] \), there exists a constant \( c_1 > 0 \), such that

\[
|b(x^1_1) - b(x^2_1)| + |\sigma(x^1_1) - \sigma(x^2_1)| \leq c_1 \|x^1_1 - x^2_1\|, \text{ a.e.}
\]

and \( \forall x_t \in \Lambda, \)

\[
|b(x_t)| + |\sigma(x_t)| \leq c_1 (1+ \|x_t\|), \text{ a.e.}
\]

**Definition 2.5.** \( X : [0, T] \times \Omega \rightarrow \mathbb{R}^n \) is called an adapted solution of the Eqs. (2.1), if \( X \in M^2(0, T; \mathbb{R}^n) \), and it satisfies (2.1) \( P - a.s. \)

Then we have the following theorem (see [5]):

**Theorem 2.2.** Let Assumptions 2.1 hold, then there exists a unique adapted solution \( X \) for Eqs. (2.1).

### 3 Regularity

We first recall some notions in Pardoux and Peng [10]. \( C^n(\mathbb{R}^p; \mathbb{R}^q), C^n_p(\mathbb{R}^p; \mathbb{R}^q), C^n_p(\mathbb{R}^p; \mathbb{R}^q) \) will denote respectively the set of functions of class \( C^n \) from \( \mathbb{R}^p \) into \( \mathbb{R}^q \), the set of those functions of class \( C^n_0 \) whose partial derivatives of order less than or equal to \( n \) are bounded, and the set of those functions of class \( C^n_p \) which, together with all their partial derivatives of order less than or equal to \( n \), grow at most like a polynomial function of the variable \( x \) at infinity.

Now we give the definition of derivatives in our context. Under the above Assumption 2.1 we have

\[
dX^\gamma_t(s) = b(X^\gamma_s^s)ds + \sigma(X^\gamma_s^s)dW(s),
\]

\[
X^\gamma_t(t) = \gamma_t(t),
\]

has a uniqueness solution. For \( t \leq s \leq T \), set

\[
\hat{\Lambda}_{\gamma_t,s} := \{\hat{\gamma}_s : \hat{\gamma}(h) = X^\gamma_t(h, \omega), 0 \leq h \leq s, \omega \in \Omega\},
\]

\[
\hat{\Lambda}_{t,s} := \bigcup_{\gamma_t \in \hat{\Lambda}_t} \hat{\Lambda}_{\gamma_t,s} \text{ and } \hat{\Lambda}_t := \bigcup_{0 \leq s \leq T} \hat{\Lambda}_{t,s}.
\]

Then the following definition of derivatives will be used frequently in the sequel.

**Definition 3.1.** An \( \mathbb{R}^n \)-valued function \( g \) is said to be in \( C^2(\hat{\Lambda}_{\gamma_t,T}) \), if for \( \gamma_1 \in \hat{\Lambda}_{\gamma_t,T} \) and \( \gamma_2 \in \hat{\Lambda}_{\gamma_t,T} \), there exist \( p_1 \in \mathbb{R}^d \) and \( p_2 \in \mathbb{R}^d \times \mathbb{R}^d \) such that \( p_2 \) is symmetric,

\[
g(\gamma_2) - g(\gamma_1) = \langle p_1, y \rangle + \frac{1}{2}\langle p_2(y, y) + o(|y|^2), x \in \mathbb{R}^d \rangle.
\]
We denote $g'_{\gamma}(\gamma) := p_1$, and $g''_{\gamma}(\gamma) := p_2$. $g$ is said to be in $C_{l,\text{lip}}^2(\tilde{\Lambda}_t, \mathbb{T})$ if $g'_{\gamma}(\gamma)$ and $g''_{\gamma}(\gamma)$ exist for each $\gamma \in \Lambda_t$, and there exists some constants $C \geq 0$ and $k \geq 0$ depending only on $g$ such that for each $\gamma, \tilde{\gamma} \in \Lambda_T, t, s \in [0, T],$

$$| g(\gamma) - g(\tilde{\gamma}) | \leq C(\| \gamma \|_k + \| \tilde{\gamma} \|_k) \| \gamma - \tilde{\gamma} \|,$$

and for each $\gamma \in \tilde{\Lambda}_t, \tilde{\gamma} \in \tilde{\Lambda}_s, t, s \in [0, T],$

$$| \Phi_{\gamma}(\tau) - \Phi_{\gamma}(\tilde{\gamma}) | \leq C(\| \gamma \|_k + \| \tilde{\gamma} \|_k)(| t - s | + \| \gamma - \tilde{\gamma} \|)$$

with $\Phi = g'_{\gamma}(\gamma), g''_{\gamma}(\gamma)$. We can also define $C^2(\tilde{\Lambda}_t), C_{l,\text{lip}}(\tilde{\Lambda}_t), C_{l,\text{lip}}(\tilde{\Lambda}_t)$ and $C^2(\tilde{\Lambda}_t), C_{l,\text{lip}}(\tilde{\Lambda}_t), C_{l,\text{lip}}(\tilde{\Lambda}_t)$. Moreover $g \in C_{l,\text{lip}}^2(\tilde{\Lambda}_t, \mathbb{T})$ with the Lipschitz constants $C$ and $k$.

**Assumption 3.1.** Let $g$ is an $\mathbb{R}^n$-valued function on $\Lambda_T$. Moreover $g \in C_{l,\text{lip}}^2(\tilde{\Lambda}_t, \mathbb{T})$ with the Lipschitz constants $C$ and $k$.

**Assumption 3.2.** Let $h(\gamma, y, z) = \tilde{h}(t, \gamma(t), y, z)$, where $\tilde{h} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \mapsto \mathbb{R}^n$ is such that $(t, r, y, z) \mapsto \Psi(t, r, y, z)$ is of class $C^0([0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}; \mathbb{R}^n)$ and the first order partial derivatives in $r, y$ and $z$ are bounded, as well as their derivatives of up to order two with respect to $y, z$.

It is obvious under Assumption 2.1, 3.1, 3.2 the FBSDE (2.1) and (2.2) has a uniqueness solution (see [7, 11] and [8]).

### 3.1 Regularity of the solution of FBSDEs

We assume the Lipschitz constants with respect to $b, \sigma, h$ are $C$ and $k$. Then we have the following estimates for the solution of FBSDE (2.1) and (2.2).

**Lemma 3.1.** Under Assumption 2.1, 3.1, 3.2 there exists $C_2$ and $q$ depending only on $C, T, k, x$ such that

$$E[ \sup_{s \in [t, T]} | X^\gamma(s) |^2 ] \leq C_2(1 + \| \gamma \|_p^2),$$

$$E[ \sup_{s \in [t, T]} | Y^\gamma(s) |^2 ] \leq C_2(1 + \| \gamma \|_q^q),$$

$$E[ (\int_t^T | Z^\gamma(s) |^2 ds ) ] \leq C_2(1 + \| \gamma \|_q^q).$$

**Proof.** To simplify presentation, we only study the case $n = d = 1$. Applying Itô’s formula to $(Y_{\gamma}(x(s)))^2 e^{\beta_1 s}$ yields that

$$(Y_{\gamma}(s))^2 e^{\beta_1 s} + \int_s^T e^{\beta_1 r}[(Z_{\gamma}(r))^2 + \beta_1(Y_{\gamma}(r))^2]dr$$

$$= g^2(X_{\gamma}^r)e^{\beta_1 T} - \int_s^T 2e^{\beta_1 r}Y_{\gamma}(r)h(X_{\gamma}^r, Y_{\gamma}(r), Z_{\gamma}(r))dr - \int_s^T 2e^{\beta_1 r}Y_{\gamma}(r)Z_{\gamma}(r)dW(r).$$
So

\[(Y^{\gamma_t}(s))^2 + E[\int_s^T e^{\beta_t(s-r)} [(Z^{\gamma_t}(r))^2 + \beta_1(Y^{\gamma_t}(r))^2] dr \mid \mathcal{F}_s] \]

\[= E[g^2(X_T^\gamma)e^{\beta(T-s)} \mid \mathcal{F}_s] - E[\int_s^T 2e^{\beta_t(s-r)}Y^{\gamma_t}(r)h(X_T^\gamma, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr \mid \mathcal{F}_s]. \]

Then we have

\[E \sup_{t \leq s \leq T}(Y^{\gamma_t}(s))^2 + E[\int_t^T e^{\beta_t(s-r)}[(Z^{\gamma_t}(r))^2 + \beta_1(Y^{\gamma_t}(T))^2] dr] \]

\[\leq E[g^2(X_T^\gamma)e^{\beta(T-t)}] + E[\int_t^T e^{\beta_t(s-r)} \frac{2}{\bar{\gamma}_1} h^2(X_T^\gamma, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr] + E[\int_t^T e^{\beta_t(s-r)(\frac{2}{\bar{\gamma}_1})} (Y^{\gamma_t}(r))^2 d(r)]. \]

and

\[E \sup_{t \leq s \leq T}(Y^{\gamma_t}(s))^2 + E[\int_t^T e^{\beta_t(s-r)}[(Z^{\gamma_t}(r))^2 + \beta_1(Y^{\gamma_t}(T))^2] dr] \]

\[\leq E[g^2(X_T^\gamma)e^{\beta(T-t)}] + E[\int_t^T e^{\beta_t(s-r)} \frac{2}{\bar{\gamma}_1} h^2(X_T^\gamma, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr]. \]

(3.3)

Applying Itô’s formula to \((X^{\gamma_t}(s))^2\) yields that

\[(X^{\gamma_t}(s))^2 = \gamma_t(t)^2 + \int_t^s 2X^{\gamma_t}(r)h(X_T^\gamma) dr + \int_t^s 2X^{\gamma_t}(r)\sigma(X_T^\gamma) dW(r) + \int_t^s \sigma^2(X_T^\gamma) dr \]

By inequality \(2ab \leq a^2 + b^2\) and Burkholder-Davis-Gundy’s inequality, there is a \(C_0\) such that,

\[E \sup_{t \leq s \leq T}(X^{\gamma_t}(r))^2 \leq C_0 [\gamma_t(t)^2 + E \int_t^s b^2(X_T^\gamma) dr + E \int_t^s \sigma^2(X_T^\gamma) dr]. \]

By Assumption 2.1 and Gronwall’s inequality, we have (note that \(C_0\) will change line by line)

\[E \sup_{t \leq s \leq T}(X^{\gamma_t}(r))^2 \leq C_0(1 + \| \gamma_t \|^2). \]

By Assumptions 3.1 and 3.2 and taking \(\beta_1 = 4C^2 + 1\), we have

\[E[\sup_{t \leq s \leq T}(Y^{\gamma_t}(s))^2 + E[\int_t^T [(Z^{\gamma_t}(r))^2 + (Y^{\gamma_t}(r))^2] dr] \]

\[\leq C_0(1 + \| \gamma_t \|^q) \]

where \(q = 2(1 + k)\). This completes the proof. \(\square\)

Now we study the regularity properties of the solution of FBSDE (2.1), (2.2) with respect to the "parameter" \(\gamma_t\). For \(0 \leq s < t \leq T\), define \(Y^{\gamma_t}(s) = Y^{\gamma_t}(s \vee t)\) and \(Z^{\gamma_t}(s) = 0\).

**Theorem 3.1.** Under Assumptions 2.1, 3.2 and 3.3, there exist \(C_2\) and \(q\) depending only on \(C, c_2, x\) such that for any \(t, \bar{t} \in [0, T]\), \(\gamma_t, \bar{\gamma}_t\), and \(h, \bar{h} \in \mathbb{R} \setminus \{0\}\).

(i) \(E[\sup_{u \in [s, \bar{t}, T]} Y^{\gamma_t}(u) - Y^{\gamma_t}(u) \mid \mathcal{F}_s] \leq C_2(1 + \| \gamma_t \|^q + \| \bar{\gamma}_t \|^q)(\| \gamma_t - \bar{\gamma}_t \|^2 + | t - \bar{t} |)\),

(ii) \(E[\int_{t \vee \bar{t}}^T | Z^{\gamma_t}(u) - Z^{\gamma_t}(u) |^2 du] \leq C_2(1 + \| \gamma_t \|^q + \| \bar{\gamma}_t \|^q)(\| \gamma_t - \bar{\gamma}_t \|^2 + | t - \bar{t} |)\),

(iii) \(E[\sup_{u \in [s, \bar{t}, T]} | \Delta^{\gamma}_t Y^{\gamma_t}(u) - \Delta^{\gamma}_t Y^{\gamma_t}(u) |^2] \leq C_2(1 + \| \gamma_t \|^q + \| \bar{\gamma}_t \|^q + | h |^q + | \bar{h} |^q)(\| h - \bar{h} \|^2 + \| \gamma_t - \bar{\gamma}_t \|^2 + | t - \bar{t} |)\),
Consider where (with $U$)

Then the same calculus implies that

where

and $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

**Proof.** $(Y^{\gamma_i} - Y^{\gamma_i}, Z^{\gamma_i} - Z^{\gamma_i})$ can be formed as a linearized BSDE: for each $s \in [t \vee \bar{t}, T]$,

$$Y^{\gamma_i}(s) - Y^{\gamma_i}(s) = g(X^{\gamma_i}_s) - g(X^{\gamma_i}_T) + \int_t^T [h(X^{\gamma_i}_s, Y^{\gamma_i}(s), Z^{\gamma_i}(s)) - h(X^{\gamma_i}_s, Y^{\gamma_i}(r), Z^{\gamma_i}(r))]dr + \int_t^T (Z^{\gamma_i}(r) - Z^{\gamma_i}(r))dW(r)$$

where (with $U^{\gamma_i} = (Y^{\gamma_i}, Z^{\gamma_i})$)

$$\hat{\alpha}_{\gamma_i, \gamma_i}(r) = h(X^{\gamma_i}_s, Y^{\gamma_i}(s), Z^{\gamma_i}(s)) - h(X^{\gamma_i}_s, Y^{\gamma_i}(r), Z^{\gamma_i}(r)),
\hat{\beta}_{\gamma_i, \gamma_i}(r)(Y^{\gamma_i}(r) - Y^{\gamma_i}(r)) = \int_s^T \frac{\partial h}{\partial y}(X^{\gamma_i}_r, U^{\gamma_i}(r) + \theta(U^{\gamma_i}(r) - U^{\gamma_i}(r)))d\theta,
\hat{\delta}_{\gamma_i, \gamma_i}(r)(Z^{\gamma_i}(r) - Z^{\gamma_i}(r)) = \int_s^T \frac{\partial h}{\partial z}(X^{\gamma_i}_r, U^{\gamma_i}(r) + \theta(U^{\gamma_i}(r) - U^{\gamma_i}(r)))d\theta.$$

Under Assumptions 3.1, 3.2, using the same method as in Lemma 3.1, we get the first three inequalities.

For the next three inequalities, we write $(\Delta_h^{\gamma_i} Y^{\gamma_i}, \Delta_h^{\gamma_i} Z^{\gamma_i})$ as the solution of the following linearized BSDE:

$$\Delta_h^{\gamma_i} Y^{\gamma_i}(s)$$

$$= \frac{1}{h}(g(X^{\gamma_i}_{h}\bar{\gamma}_{\gamma_i}) - g(X^{\gamma_i}_T)) - \int_s^T \frac{1}{h} (\hat{\delta}_{\gamma_i, \gamma_i}(r)) \Delta_h^{\gamma_i} Y^{\gamma_i}(r) + \hat{\beta}_{\gamma_i, \gamma_i}(r) \Delta_h^{\gamma_i} Y^{\gamma_i}(r) + \hat{\delta}_{\gamma_i, \gamma_i}(r) \Delta_h^{\gamma_i} Z^{\gamma_i}(r)dr$$

$$= -\int_s^T \Delta_h^{\gamma_i} Z^{\gamma_i}(r) dW(r).$$

Then the same calculus implies that

$$E[\sup_{s \in [t, T]} |\Delta_h^{\gamma_i} Y^{\gamma_i}(s)|^2 + |\int_t^T |\Delta_h^{\gamma_i} Z^{\gamma_i}|^2 dr |] \leq C_2(1 + \|\gamma_i\|^q + \|h\|^q).$$

Consider

$$\Delta_h^{\gamma_i} Y^{\gamma_i}(s) - \Delta_h^{\gamma_i} Y^{\gamma_i}(s)$$

$$= \frac{1}{h}(g(X^{\gamma_i}_{h}\bar{\gamma}_{\gamma_i}) - g(X^{\gamma_i}_T)) - \frac{1}{h}(g(X^{\gamma_i}_T) - g(X^{\gamma_i}_{h}\bar{\gamma}_{\gamma_i})) - \int_s^T (\Delta_h^{\gamma_i} Z^{\gamma_i}(r) - \Delta_h^{\gamma_i} Z^{\gamma_i}(r))dW(r)$$

$$- \int_s^T \frac{1}{h} (\hat{\delta}_{\gamma_i, \gamma_i}(r)) - \hat{\delta}_{\gamma_i, \gamma_i}(r) \Delta_h^{\gamma_i} Y^{\gamma_i}(r) - \hat{\delta}_{\gamma_i, \gamma_i}(r) \Delta_h^{\gamma_i} Z^{\gamma_i}(r)dr$$

$$+ \hat{\delta}_{\gamma_i, \gamma_i}(r) \Delta_h^{\gamma_i} Z^{\gamma_i}(r) - \hat{\delta}_{\gamma_i, \gamma_i}(r) \Delta_h^{\gamma_i} Z^{\gamma_i}(r)dr.$$
We have the following results about second order derivative of $Y$

Then it solves the following BSDE

$$
\tilde{Y}(s) = \frac{1}{\gamma}(g(X_s^\gamma) - g(X_T^\gamma)) - \frac{1}{\gamma}(g(X_s^{\gamma \xi}) - g(X_T^{\gamma \xi}))
- \int_s^T \left[ \tilde{\beta}_{\gamma, r} \gamma h_r(u) \tilde{Y}(r) + \tilde{\delta}_{\gamma, r} \gamma h_r(u) \tilde{Z}(r) + \tilde{h}(r) \right] dr - \int_s^T \tilde{Z}(r)dW(r),
$$

where

$$
\tilde{h}(r) := \left[ \tilde{\beta}_{\gamma, r} \gamma h_r(u) - \tilde{\beta}_{\gamma, r} \gamma h_r(u) \right] \Delta_h Y^\gamma(r) + \left[ \tilde{\delta}_{\gamma, r} \gamma h_r(u) - \tilde{\delta}_{\gamma, r} \gamma h_r(u) \right] \Delta_h Z^\gamma(r) + \frac{1}{\gamma} \tilde{\alpha}_{\gamma, r} h_r(u) - \frac{1}{\gamma} \tilde{\alpha}_{\gamma, r} h_r(u).
$$

Thus, under Assumptions 3.1, 3.2, similarly as in Lemma 3.1, we can get the last three inequalities. □

**Theorem 3.2.** For each $\gamma \in \Lambda$, $\{Y^\gamma(s), s \in [t, T], z \in \mathbb{R}^n\}$ has a version which is a.e. of class $C^{0,2}([0, T] \times \mathbb{R}^n)$.

**Proof.** We only consider one dimensional case. Applying Lemma 3.1, for each $h, \tilde{h} \in \mathbb{R} \setminus \{0\}$ and $k, \tilde{k} \in \mathbb{R},$

$$
E[\sup_{u \in [t, T]} | Y^\gamma(u) - Y^\gamma(u) |^2] \leq C_p(1 + \| \gamma \|^9) \| k - \tilde{k} \|^2,
$$

$$
E[\int_t^T | Z^\gamma(u) - Z^\gamma(u) |^2 du \| \leq C_q(1 + \| \gamma \|^9) \| k - \tilde{k} \|^2,
$$

$$
E[\sup_{u \in [t, T]} | \Delta_h Y^\gamma(u) - \Delta_h Y^\gamma(u) |^2] \leq C_2(1 + \| \gamma \|^9 + \| k - \tilde{k} \|^2 + \| h - \tilde{h} \|^2),
$$

$$
E[\int_t^T | \Delta_h Z^\gamma(u) - \Delta_h Z^\gamma(u) |^2 du \| \leq C_2(1 + \| \gamma \|^9 + \| k - \tilde{k} \|^2 + \| h - \tilde{h} \|^2).
$$

By Kolmogorov’s criterion, there exists a continuous derivative of $Y^\gamma(s)$ with respect to $z$. There also exists a mean-square derivative of $Z^\gamma(s)$ with respect to $z$, which is mean square continuous in $z$. We denote them by

$$(D_x Y^\gamma, D_x Z^\gamma).$$

By Theorem 3.1 and definition 3.1, $(D_x Y^\gamma, D_x Z^\gamma)$ is the solution of the following BSDE:

$$
D_x Y^\gamma(s) = g'_{\gamma}(X_T^\gamma) - \int_s^T \left[ h'_{\gamma}(X_T^\gamma, Y^\gamma(r), Z^\gamma(r)) - \frac{1}{\gamma} \int_s^T D_x Z^\gamma(r)dW(r)
+ h'_{\gamma}(X_T^\gamma, Y^\gamma(r), Z^\gamma(r))D_x Y^\gamma(r) + h'_{\gamma}(X_T^\gamma, Y^\gamma(r), Z^\gamma(r))D_x Z^\gamma(r) \right] dr.
$$

It is easy to check that the above BSDE has a uniqueness solutions. Thus the existence of a continuous second order derivative of $Y^\gamma(z)$ with respect to $z$ is proved in a similar way. □

Define

$$u(\gamma) := Y^\gamma(t), \text{ for } \gamma \in \Lambda.$$

We have the following results about $u(\gamma)$.
Lemma 3.2. \( \forall t \leq s \leq T, \) we have \( u(X^\gamma_t) = Y^\gamma(s) \).

Proof. For given \( \gamma_t, t_1 < t \), set \( X(r) = xI_{0 \leq r \leq t_1} \). Consider the solution of FBSDE (2.1) and (2.2) on \([t, T]\):

\[
X^\gamma_t = \gamma_t(t) + \int_t^s b(X^\gamma_r)dr + \int_t^s \sigma(X^\gamma_r)dW(r),
\]

\[
Y^\gamma(s) = g(X^\gamma_T) - \int_s^T h(X^\gamma_r, Y^\gamma(r), Z^\gamma(r))dr - \int_s^T Z^\gamma(r)dW(r), \quad s \in [t, T].
\]

We need to prove \( u(X^\gamma_t) = Y^\gamma_t(t) \). Define

\[
X^{N, \gamma_t}_t := \sum_{i=1}^N I_{A_i} x^i_t,
\]

where \( \{A_i\}_{i=1}^N \) is a division of \( \mathcal{F}_t \), \( x^i_t \in A_i \cap A \), \( i = 1, 2, \ldots, N \). For any \( i \), \( (Y^{x^i_t}; a^i(s), Y^{x^i_t}; a^i(s)) \) is the solution of the following BSDE:

\[
Y^{x^i_t}(s) = g(X^{x^i_t}_T) - \int_s^T h(X^{x^i_t}_r, Y^{x^i_t}(r), Z^{x^i_t}(r))dr - \int_s^T Z^{x^i_t}(r)dW(r), \quad s \in [t, T].
\]

Multiplying by \( I_{A_i} \) and adding the corresponding terms, we obtain:

\[
\sum_{i=1}^N I_{A_i} Y^{x^i_t}(s) = g(\sum_{i=1}^N I_{A_i} X^{x^i_t}_T) - \int_s^T h(\sum_{i=1}^N I_{A_i} X^{x^i_t}_r, \sum_{i=1}^N I_{A_i} Y^{x^i_t}(r), \sum_{i=1}^N I_{A_i} Z^{x^i_t}(r))dr
\]

\[
- \int_s^T \sum_{i=1}^N I_{A_i} Z^{x^i_t}(r)dW(r), \quad s \in [t, T].
\]

By the uniqueness and existence theorem of BSDE, we get \( Y^\gamma_{s_i} = \sum_{i=1}^N I_{A_i} Y^{x^i_t}(s) \), \( Z^\gamma_{s_i} = \sum_{i=1}^N I_{A_i} Z^{x^i_t}(s) \) a.s. Then, by the definition of \( u \), we get

\[
Y^\gamma_t(t) = \sum_{i=1}^N I_{A_i} Y^{x^i_t}(t) = \sum_{i=1}^N I_{A_i} u(x^i_t) = u(X^{N, \gamma}_t).
\]

For the general case, following the method in Peng and Wang [14] (Lemma 4.3), we choose a simple adapted process \( \{\gamma^i_t\}_{i=1}^\infty \) such that \( E \| \gamma^i_t - X^{N, \gamma}_t \| \) convergence to 0 as \( i \to \infty \). We obtain

\[
E | Y^{\gamma^i_t}(t) - Y^\gamma_t(t) |^2 \leq CE \| \gamma^i_t - X^{N, \gamma}_t \|
\]

This completes the proof. \( \Box \)

By Theorem 3.1 and 3.2 and the definition of vertical derivative, we have the following corollary.

**Corollary 3.1.** \( u(\gamma_t) \) is \( \Lambda \)-continuous and \( D_x u(\gamma_t), D_{zz} u(\gamma_t) \) exist, moreover they are both \( \Lambda \)-continuous.

**Proof.** By Theorem 3.2 we know that \( D_x u(\gamma_t) \) and \( D_{zz} u(\gamma_t) \) exist. In the following, we only prove \( u(\gamma_t) \) is \( \Lambda \)-continuous. The proof for the continuous property of \( D_x u(\gamma_t) \) and \( D_{zz} u(\gamma_t) \) is similar. Taking expectation on both sides of equation (2.2),

\[
u(\gamma_t) = Eg(X^\gamma_T) - E \int_t^T h(X^\gamma_r, Y^\gamma(r), Z^\gamma(r))dr. \tag{5.9}
\]
For \( \gamma_t, \tilde{\gamma}_t \in \Lambda, t \geq t \), we have

\[
| u(\gamma_t) - u(\tilde{\gamma}_t) | \leq \mathbb{E}[g(X^\gamma_t) - g(X^{\tilde{\gamma}}_t)] + \mathbb{E}[f_t^T | h(X^\gamma_t, Y^\gamma_t(r), Z^\gamma_t(r)) | dr] \\
+ \mathbb{E}[f_t^T | h(X^{\tilde{\gamma}}_t, Y^{\tilde{\gamma}}_t(r), Z^{\tilde{\gamma}}_t(r)) - h(X^\gamma_t, Y^\gamma_t(r), Z^\gamma_t(r)) | dr] \\
\leq \mathbb{E}[C_1(1 + \| X^\gamma_t \|^k + \| X^{\tilde{\gamma}}_t \|^k) \| \gamma_t - \tilde{\gamma}_t \| \\
+ 3(t - s)^{2}(f_t^T | h(X^\gamma_t(r), 0, 0) |^2 + | CY^{\gamma}_t(r) |^2 \\
+ | CY^{\gamma}_t(r)|^2 dr) + C f_t^T (| Y^{\gamma}_t(r) - Y^{\tilde{\gamma}_t} | + | Z^{\gamma}_t(r) - Z^{\tilde{\gamma}}_t |) dr] .
\]

By Theorem 3.1, for some constant \( C_1 \) depending only in \( C, k \) and \( T \),

\[
| u(\gamma_t) - u(\tilde{\gamma}_t) | \leq C_1(1 + \| \gamma_t \|^k + \| \tilde{\gamma}_t \|^k)(\| \gamma_t - \tilde{\gamma}_t \| + | t - \tilde{t} |^2).
\]

This completes the proof. □

### 3.2 Path regularity of process \( Z \)

In Pardoux and Peng [10], BSDE is only state-dependent, i.e., \( h = h(t, \gamma(t), y, z) \) and \( g = g(\gamma(T)) \). Under appropriate assumptions, \( Y \) and \( Z \) are related in the following sense:

\[
Z^{\gamma}(s) = \nabla_x u(s, \gamma_t(t) + W(s) - W(t)), \quad P - a.s.
\]

Peng and Wang [14] extends this result to the path-dependent case. The corresponding BSDE is

\[
Y^{\gamma}(s) = g(W^{\gamma}_{T}) - \int_{s}^{T} h(W^{\gamma}_{r}, Y^{\gamma}_{r}(r), Z^{\gamma}(r)) dr - \int_{s}^{T} Z^{\gamma}(r)dW(r), \quad s \in [t, T].
\]

where \( W^{\gamma}_{T} = I_{s \leq t} \gamma_t(s) + I_{t < s \leq T}(\gamma_t(t) + W(s) - W(t)) \). Then under some assumptions, they obtained

\[
Z^{\gamma}(s) = D_x u(W^{\gamma}_{s}), \quad P - a.s.
\]

In our context, we have the following theorem:

**Theorem 3.3.** Under Assumption 2.1, 3.1 and 3.2, for each \( \gamma_t \in \Lambda \), the process \( (Z^{\gamma}(s))_{s \in [t, T]} \) has a continuous version with the form,

\[
Z^{\gamma}(s) = \sigma(X^{\gamma}_s)D_x u(X^{\gamma}_s), \quad for \quad s \in [t, T] \quad P - a.s.
\]

To prove the above Theorem, we need the following lemma essentially from Pardoux and Peng [10].

**Lemma 3.3.** Let \( \gamma_t \) and some \( \hat{t} \in [t, T] \) be given. Suppose that

\[
g(\gamma, z) = \varphi(\gamma(\hat{t}), \gamma(T) - \gamma(\hat{t}), z),
\]

where \( \varphi \) is in \( C^3_p(\mathbb{R}^{2d} \times \mathbb{R}^m; \mathbb{R}^m) \). For \( \phi = h, \sigma \) and \( h \), suppose that

\[
h(\gamma_t, y, z) = h_1(s, \gamma_s(s), y, z)I_{[0, \hat{t}]}(s) + h_2(s, \gamma_s(s) - \gamma_s(\hat{t}), y, z)I_{[\hat{t}, T]}(s),
\]

\[
\sigma(\gamma_t, y, z) = \sigma_1(s, \gamma_s(s), y, z)I_{[0, \hat{t}]}(s) + \sigma_2(s, \gamma_s(s) - \gamma_s(\hat{t}), y, z)I_{[\hat{t}, T]}(s),
\]

\[
\varphi(\gamma(\hat{t}), \gamma(T) - \gamma(\hat{t}), z) = \varphi_1(s, \gamma_s(s), \gamma(T) - \gamma(\hat{t}), z)I_{[0, \hat{t}]}(s) + \varphi_2(s, \gamma_s(s) - \gamma_s(\hat{t}), \gamma(T) - \gamma(\hat{t}), z)I_{[\hat{t}, T]}(s).
\]

\[
\nabla_x u(s, \gamma_t(t) + W(s) - W(t)) = \nabla_x u(s, \gamma_t(t) + W(s) - W(t)) + \nabla_x u(s, W(s) - W(t)).
\]

This completes the proof. □
Then we obtain
\[ Z^{\gamma_1}(s) = \sigma(X^{\gamma_1}_s)D_xu(X^{\gamma_1}_s), \quad \text{for} \quad s \in [t, T] \quad P - a.s. \]

**Proof.** We only consider the one dimensional case. For \( s \in [\bar{t}, T] \), the BSDE (2.2) can be rewritten as
\[
Y^{\gamma_1}(u) = \varphi(\gamma_1(\bar{t}), X^{\gamma_1}(T) - \gamma_1(\bar{t})) - \int_{\bar{t}}^{T} Z^{\gamma_1}(r)dW(r) \\
- \int_{\bar{t}}^{T} h_2(r, \gamma_1(\bar{t}), X^{\gamma_1}(r) - \gamma_1(\bar{t}), \gamma_1(t), Z^{\gamma_1}(r))dr, \quad u \in [s, T].
\]

For \( s \in [t, \bar{t}] \),
\[
Y^{\gamma_1}(u) = \varphi(X^{\gamma_1}(\bar{t}), X^{\gamma_1}(T) - X^{\gamma_1}(\bar{t})) - \int_{\bar{t}}^{T} Z^{\gamma_1}(r)dW(r) \\
- \int_{\bar{t}}^{T} h_2(r, X^{\gamma_1}(\bar{t}), X^{\gamma_1}(r) - X^{\gamma_1}(\bar{t}), \gamma_1(t), Z^{\gamma_1}(r))dr, \quad u \in [t, \bar{t}],
\]
\[
Y^{\gamma_1}(u) = Y^{\gamma_1}(\bar{t}) - \int_{\bar{t}}^{\bar{t}} h_1(r, X^{\gamma_1}(\bar{t}), Y^{\gamma_1}(r), Z^{\gamma_1}(r))dr - \int_{\bar{t}}^{T} Z^{\gamma_1}(r)dW(r), \quad u \in [s, \bar{t}].
\]

Now consider the following system of quasilinear parabolic differential equations, which is defined on \([\bar{t}, T] \times \mathbb{R}^2\) and parameterized by \( x \in \mathbb{R} \),
\[
\partial_t u_2(s, x, y) + \mathcal{L}u_2(s, x, y) = h_2(s, x, y, u_2(s, x, y), \partial_y u_2(s, x, y)\sigma(r)),
\]
\[
u_2(t, x, y) = \varphi(x, y).
\]

where \( \mathcal{L} = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x} \). The other one is defined on \([t, \bar{t}] \times \mathbb{R}^2\):
\[
\partial_t u_1(s, x) + \mathcal{L}u_1(s, x) = h_1(s, x, u_1(s, x), \partial_y u_1(s, x)\sigma(r)),
\]
\[
u_1(t, x) = u_2(t, x, 0).
\]

where \( \mathcal{L} = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x} \). By the above Corollary 3.1 and Theorem 3.1, 3.2 of Paroux-Peng [10], we have \( u_2 \in C^{1,2}([\bar{t}, T] \times \mathbb{R}^2; \mathbb{R}), u_1 \in C^{1,2}([t, \bar{t}] \times \mathbb{R}; \mathbb{R}) \) and
\[
u(u_{\gamma_1}) = u_1(s, \gamma_1(s))I_{[t, \bar{t}]}(s) + u_2(s, \gamma_1(s), \gamma_1(s) - \gamma_1(\bar{t}))I_{[\bar{t}, T]}(s).
\]

Then we obtain
\[
Y^{\gamma_1}(s) = u_1(s, X^{\gamma_1}(s)), \quad t \leq s < \bar{t},
\]
\[
Y^{\gamma_1}(s) = u_2(s, X^{\gamma_1}(\bar{t}), X^{\gamma_1}(s) - X^{\gamma_1}(\bar{t})), \quad \bar{t} \leq s \leq T,
\]
\[
Z^{\gamma_1}(s) = \partial_x u_1(s, X^{\gamma_1}(s))\sigma(X^{\gamma_1}_s), \quad t \leq s < \bar{t},
\]
\[
Z^{\gamma_1}(s) = \partial_x u_2(s, X^{\gamma_1}(\bar{t}), X^{\gamma_1}(s) - X^{\gamma_1}(\bar{t}))\sigma(X^{\gamma_1}_s), \quad \bar{t} \leq s \leq T.
\]

Finally, for each \( s \in [t, T] \),
\[
\sigma(X^{\gamma_1}_s)D_xu(X^{\gamma_1}_s) = Z^{\gamma_1}(s) \quad P - a.s.
\]

In particular,
\[
\sigma(\gamma_t)D_xu(\gamma_t) = Z^{\gamma_1}(t). \quad \gamma_t \in \Lambda.
\]
This completes the proof. □

Now we give the proof of Theorem 3.3.

Proof. For each fixed $t \in [0, T]$ and positive integer $n$, we introduce a mapping $\gamma^n(\tilde{\gamma}_{\alpha}): \Lambda_{\alpha} \mapsto \Lambda_{\alpha}$

$$
\gamma^n(\tilde{\gamma}_{\alpha})(\bar{r}) = \gamma_{\alpha}(r)I_{[0,t]} + \sum_{k=0}^{n-1} \gamma_{\alpha}(t_k^{n+1} \wedge s)I_{[t_k^{n+1},t]}(s) + \gamma_{\alpha}(s)I_{(s, \bar{r})}(r), \quad s \in [0, T],
$$

where $t_k^{n} = t + \frac{k(T-t)}{n}$, $k = 0, 1, \ldots, n$

$$
g^n(\tilde{\gamma}) := g(\gamma^n(\tilde{\gamma})), \quad h^n(\bar{\gamma}_{\alpha}, y, z) := h(\gamma^n(\tilde{\gamma}_{\alpha}), y, z).
$$

For each $n$, there exists some functions $\phi_n$ defined on $\Lambda_t \times \mathbb{R}^{n \times d}$ and $\psi_n$ defined on $[t, T] \times \Lambda_t \times \mathbb{R}^{n \times d} \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ such that

$$
g^n(\tilde{\gamma}) = \phi_n(\tilde{\gamma}_t, \bar{\gamma}(t_1^n) - \bar{\gamma}(t), \ldots, \bar{\gamma}(t_n^n) - \bar{\gamma}(t_{n-1}^n)),
$$

$$
h^n(\bar{\gamma}_t, y, z) = \psi_n(s, \bar{\gamma}_t, \bar{\gamma}(t_1^n \wedge s) - \bar{\gamma}(t), \ldots, \bar{\gamma}(t_n^n \wedge s) - \bar{\gamma}(t_{n-1}^n \wedge s), y, z).
$$

Indeed, if we set

$$
\tilde{\psi}_n(\tilde{\gamma}_t, x_1, \ldots, x_n) := g((\tilde{\gamma}_t(s)I_{[0,t]}(s) + \sum_{k=1}^{n} x_k I_{(t_k^{n+1},t)}(s) + x_n I_{(t, T]}(s))_{0 \leq s \leq T}),
$$

$$
\tilde{\phi}_n(\tilde{\gamma}_t, x_1, \ldots, x_n) := \tilde{\phi}_n(\tilde{\gamma}_t, \bar{\gamma}_t + x_1, \bar{\gamma}_t(t) + x_1 + x_2, \ldots, \bar{\gamma}_t(t) + \sum_{i=1}^{n} x_i),
$$

then by Assumptions 2.1, 3.1 and 3.2, we obtain that, for each fixed $\gamma_t$, $\phi_n(\gamma_t, x_1, \ldots, x_n)$ is a $C_p^3$-function of $x_1, \ldots, x_n$. In particular, for each $\gamma_t \in \Lambda$,

$$
\partial_{x_i} \phi_n(\tilde{\gamma}_t, \bar{\gamma}(t_1^n) - \bar{\gamma}(t), \ldots, \bar{\gamma}(t_n^n) - \bar{\gamma}(t_{n-1}^n)) = g'_{\gamma_{t_{i-1}+1}^n}(\gamma^n(\tilde{\gamma})).
$$

For any $\ell \geq t$, $\tilde{\gamma}_\ell \in \Lambda_\ell$, we consider the following BSDE:

$$
Y^{n, \gamma_{\ell}}(s) = g^n(X_T^{n, \gamma_{\ell}}) - \int_s^T h^n(X^n_{t, \gamma_{\ell}}, Y^{n, \gamma_{\ell}}(r), Z^{n, \gamma_{\ell}}(r))dr - \int_s^T Y^{n, \gamma_{\ell}}(r)dW(r).
$$

we denote

$$
u^n(\tilde{\gamma}_{\ell}) := Y^{n, \gamma_{\ell}}(t), \quad \gamma_{\ell} \in \Lambda.
$$

Following the argument as in Lemma 3.3, for each $s \in [t, T]$, we have

$$
\sigma^n(X_s^{n, \gamma_{\ell}})D_s u^n(\tilde{\gamma}_{\ell}) = Z^{n, \gamma_{\ell}}(s) \quad a.s.
$$

Let $C_0$ be a constant depending only on $C, T$ and $k$, which is allowed to change from line by line. Following the similar calculus as in Lemma 3.1 and Theorem 3.1, we get that

$$
u^n(\gamma_{\ell}) - u(\gamma_{\ell})
$$

= $\gamma^n(\gamma_{\ell})(\ell) - \gamma_{\ell}(\ell) + g^n(X_{\ell}^{n, \gamma_{\ell}}) - g(X_{\ell}^{n, \gamma_{\ell}})$

$$
+ \int_{\ell}^{T}[h^n(X_{t, \gamma_{\ell}}, Y^{n, \gamma_{\ell}}(r), Z^{n, \gamma_{\ell}}(r)) - h(X_{t, \gamma_{\ell}}, Y^{\gamma_{\ell}}(r), Z^{\gamma_{\ell}}(r))]dr + \int_{\ell}^{T}[Z^{n, \gamma_{\ell}}(r) - Z^{\gamma_{\ell}}(r)]dW(r)
$$

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For \( X_{s}^{n,\gamma_{t}} = \gamma^{n}(X_{T}^{'}) \), we have the next result,

\[
\lim_{n} X_{T}^{n,\gamma_{t}} = X_{T}^{'}, \ P - a.s.
\]

\[
\lim_{n}(Y^{n,\gamma_{t}}(s), Z^{n,\gamma_{t}}(s)) = (Y^{\gamma_{t}}(s), Z^{\gamma_{t}}(s)), \ a.e.s \in [t, T], \ P - a.s.
\]

We can get

\[
\lim_{n} u^{n}(\gamma_{t}) = u(\gamma_{t}), \ \lim_{n} D_xu^{n}(\gamma_{t}) = D_xu(\gamma_{t}), \ \lim_{n} D_{xx}u^{n}(\gamma_{t}) = D_{xx}u(\gamma_{t}),
\]

and

\[
\lim_{n}(u^{n}(X_{s}^{n,\gamma_{t}}), D_xu^{n}(X_{s}^{n,\gamma_{t}}), D_{xx}u^{n}(X_{s}^{n,\gamma_{t}})) = (u(X_{s}^{\gamma_{t}}), D_xu(X_{s}^{\gamma_{t}}), D_{xx}u(X_{s}^{\gamma_{t}})),
\]

\( \ a.e.s \in [t, T], \ P - a.s. \)

So that

\[
Z^{\gamma_{t}}(s) = \sigma(X_{s}^{\gamma_{t}})D_xu(X_{s}^{\gamma_{t}}), \ a.e.s \in [t, T], \ P - a.s.
\]

This completes the proof. □

4 The related path-dependent PDEs

In this section, we relate FBSDE (2.1), (2.2) to the following path-dependent partial differential equation:

\[
D_{t}u(\gamma_{t}) + Lu(\gamma_{t}) - h(\gamma_{t}, u(\gamma_{t}), \sigma(\gamma_{t})D_xu(\gamma_{t})) = 0,
\]

\[
u(\gamma_{T}) = g(\gamma_{T}), \ \gamma_{T} \in \Lambda.
\]

where

\[
L u = \frac{1}{2}tr[(\sigma\sigma^{T})D_{xx}u] + (b, D_xu).
\]

**Theorem 4.1.** Suppose Assumption 2.1, 3.1 and 3.2 hold, and if \( u \in C^{1,2}(\Lambda) \) and \( u \) is the solutions of equation (4.1), \( u \) is uniformly Lipschitz continuous, and bounded by \( C(1 + \| \gamma_{t} \|) \), then the solution is uniqueness, and for any \( \gamma_{t} \in \Lambda, u(\gamma_{t}) \) is determined by equation (2.1) and (2.2).

**Proof.** By the assumptions of this theorem, we know that \( b(\gamma_{t}) \) and \( \sigma(\gamma_{t}) \) is uniformly Lipschitz continuous and the following SDE has a uniqueness solution.

\[
dX^{\gamma_{t}}(s) = b(X_{s}^{\gamma_{t}})ds + \sigma(X_{s}^{\gamma_{t}})dW(s),
\]

\[
X_{t} = \gamma_{t}, \ s \in [t, T].
\]

Set \( Y(s) = u(X_{s}^{\gamma_{t}}), \ t \leq s \leq T \). Applying Itô’s formula to \( Y(s) = u(X_{s}^{\gamma_{t}}) \), we have

\[
dY(s) = -h(X_{s}^{\gamma_{t}}, Y(s), Z(s))dr - \sigma(X_{s}^{\gamma_{t}})D_xu(X_{s}^{\gamma_{t}})dW(s),
\]

\[
Y(T) = g(X_{T}^{\gamma_{t}}) \ s \in [t, T].
\]
Then by the uniqueness and existence theorem of the functional FBSDE, we obtain the result. □

Now we prove the converse to the above result.

**Theorem 4.2.** Under Assumption 2.1, 3.1 and 3.2. The function $u(\gamma_t) = Y^{\gamma_t}(t)$ is the unique $C^{1,2}(\Lambda)$-solution of the path-dependent PDE (4.1).

**Proof.** We only study the one dimensional case. $u \in C^{0,2}(\Lambda)$ follows from Corollary 3.1. Let $\delta > 0$ be such that $t + \delta \leq T$. By Lemma 3.2 we can get

$$u(X^{\gamma_{t+\delta}}_{t+\delta}) = Y^{\gamma_{t+\delta}}(t + \delta).$$

Hence

$$u(\gamma, t+\delta) - u(\gamma_{t+\delta}) = u(\gamma, t) - u(X^{\gamma_{t+\delta}}_{t+\delta}) + u(X^{\gamma_{t+\delta}}_{t+\delta}) - u(\gamma_{t+\delta}).$$

By the proof of Theorem 3.3, we obtain

$$u(\gamma, t+\delta) - u(\gamma_{t+\delta}) = \lim_{n \to \infty} [u^n(\gamma, t+\delta) - u^n(X^{\gamma_{t+\delta}}_{t+\delta})] + \int_t^{t+\delta} h(X^{\gamma_{t+\delta}}_{s+\delta}, Y^{\gamma_{t+\delta}}_{s+\delta}) ds + \int_t^{t+\delta} Z^{\gamma_{t+\delta}}(s) dW(s).$$

By Lemma 3.1 and Theorem 3.2 of Pardoux and Peng [10] and Theorem 4.4 of Peng and Wang [14], we deduce that

$$u^n(\gamma, t+\delta) - u^n(X^{\gamma_{t+\delta}}_{t+\delta}) = \int_t^{t+\delta} D_s u^n(\gamma, s) ds - \int_t^{t+\delta} D_s u^n(X^{\gamma_{t+\delta}}_{s+\delta}) ds - \int_t^{t+\delta} D_x u^n(X^{\gamma_{t+\delta}}_{s+\delta}) dX^{\gamma_{t+\delta}}(s) - \frac{1}{2} \int_t^{t+\delta} D_{xx} u^n(X^{\gamma_{t+\delta}}_{s+\delta}) d(X^{\gamma_{t+\delta}})(s).$$

Thus by the dominated convergence theorem, we have

$$u(\gamma, t+\delta) - u(\gamma_{t+\delta}) = -\int_t^{t+\delta} D_x u^n(X^{\gamma_{t+\delta}}_{s+\delta}) dX^{\gamma_{t+\delta}}(s) - \frac{1}{2} \int_t^{t+\delta} D_{xx} u^n(X^{\gamma_{t+\delta}}_{s+\delta}) d(X^{\gamma_{t+\delta}})(s) + \int_t^{t+\delta} h(X^{\gamma_{t+\delta}}_{s+\delta}, Y^{\gamma_{t+\delta}}_{s+\delta}) ds + \int_t^{t+\delta} Z^{\gamma_{t+\delta}}(s) dW(s) + \lim_{n \to \infty} C^n,$$

where

$$C^n = \int_t^{t+\delta} D_s u^n(\gamma, s) ds - \int_t^{t+\delta} D_s u^n(X^{\gamma_{t+\delta}}_{s+\delta}) ds.$$

Note that $u^n(\gamma_t) \in C^{0,2}_{t,T}(\Lambda)T$. By Lemma 3.1 and 3.3, we get

$$| D_s u^n(\gamma, s) - D_s u^n(X^{\gamma_{t+\delta}}_{s+\delta}) | \leq c \| \gamma_{t+\delta} - X^{\gamma_{t+\delta}} \|, \quad a.e.s \in [t, T] \quad P - a.s$$

for some constant $c$ depending on $C, T, \gamma_t$ and $k$. Hence

$$| C^n | \leq c \delta \sup_{s \in [t, t+\delta]} | X^{\gamma_{t+\delta}}(s) - \gamma_{t+\delta}(t) |. \quad P - a.s$$

Taking expectation on both sides of (4.2), we have

$$\lim_{\delta \to 0} \frac{u(\gamma, t+\delta) - u(\gamma_t)}{\delta} = -Lu(\gamma_t) + h(\gamma_t, u(\gamma_t), D_x u(\gamma_t) x(\gamma_t)).$$

Thus $u(\gamma_t)$ belongs to $C^{1,2}(\Lambda)$ and satisfies the equation (4.3). □
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