A Classification of the Veldkamp Lines of the Near Hexagon $L_3 \times GQ(2, 2)$

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Abstract
Using a standard technique sometimes (inaccurately) known as Burnside’s Lemma, it is shown that the Veldkamp space of the near hexagon $L_3 \times GQ(2, 2)$ features 158 different types of lines. We also give an explicit description of each type of a line by listing the types of the three geometric hyperplanes it consists of and describing the properties of its core set, that is the subset of points of $L_3 \times GQ(2, 2)$ shared by the three geometric hyperplanes in question.

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1 Introduction

Brouwer \textit{et al.} \cite{Brouwer} proved that there are eleven isomorphism types of slim dense near hexagons. Of these eleven, the near hexagons of sizes 27, 45 and 81 are the most promising for physical applications. This paper is devoted to a study of the second of these three examples and its Veldkamp space. The first of the three examples was described in our paper \cite{GreenSaniga1}, and we plan to study the third case in a future work. The 45 point space we study here is the product $L_3 \times GQ(2, 2)$, where $L_3$ is the line containing three points and GQ(2, 2) is the generalized quadrangle of order two.

2 Near polygons, quads, geometric hyperplanes and Veldkamp spaces

In this section we gather all the basic notions and well-established theoretical results that will be needed in the sequel.

A \textit{near polygon} (see, e.g., \cite{Brouwer} and references therein) is a connected partial linear space $S = (P, L, I)$, $I \subseteq P \times L$, with the property that given a point $x$ and a line $L$, there always exists a unique point on $L$ nearest to $x$. (Here distances are measured in the point graph, or collinearity graph of the geometry.) If the maximal distance between two points of $S$ is equal to $d$, then the near polygon is called a near $2d$-gon. A near 0-gon is a point and a near 2-gon is a line; the class of near quadrangles coincides with the class of generalized quadrangles.

A nonempty set $X$ of points in a near polygon $S = (P, L, I)$ is called a subspace if every line meeting $X$ in at least two points is completely contained in $X$. A subspace $X$ is called
geodetically closed if every point on a shortest path between two points of \(X\) is contained in \(X\). Given a subspace \(X\), one can define a sub-geometry \(S_X\) of \(S\) by considering only those points and lines of \(S\) that are completely contained in \(X\). If \(X\) is geodetically closed, then \(S_X\) clearly is a sub-near-polygon of \(S\). If a geodetically closed sub-near-polygon \(S_X\) is a non-degenerate generalized quadrangle, then \(X\) (and often also \(S_X\)) is called a quad.

A near polygon is said to have order \((s, t)\) if every line is incident with precisely \(s + 1\) points and if every point is on precisely \(t + 1\) lines. If \(s = t\), then the near polygon is said to have order \(s\). A near polygon is called dense if every line is incident with at least three points and if every two points at distance two have at least two common neighbours. A near polygon is called slim if every line is incident with precisely three points. It is well known (see, e.g., \[4\]) that there are, up to isomorphism, three slim non-degenerate generalized quadrangles. The \((3 \times 3)\)-grid is the unique generalized quadrangle of order \((2, 1)\), \(GQ(2, 1)\). The unique generalized quadrangle of order 2, \(GQ(2, 2)\), is the generalized quadrangle of the points and lines of \(PG(3, 2)\) that are totally isotropic with respect to a given symplectic polarity. The points and lines lying on a given nonsingular elliptic quadric of \(PG(5, 2)\) define the unique generalized quadrangle of order \((2, 4)\), \(GQ(2, 4)\). Any slim dense near polygon contains quads, which are necessarily isomorphic to either \(GQ(2, 1)\), \(GQ(2, 2)\) or \(GQ(2, 4)\).

Next, a geometric hyperplane of a partial linear space is a proper subspace meeting each line (necessarily in a unique point or the whole line). The set of points at non-maximal distance from a given point \(x\) of a dense near polygon \(S\) is a hyperplane of \(S\), usually called the singular hyperplane (or perp-set) with deepest point \(x\). Given a hyperplane \(H\) (or any subset of points \(C\)) of \(S\), one defines the order of any of its points as the number of lines through the point that are fully contained in \(H\) (\(C\)); a point of \(H\) (\(C\)) is called deep if all the lines passing through it are fully contained in \(H\) (\(C\)). If \(H\) is a hyperplane of a dense near polygon \(S\) and if \(Q\) is a quad of \(S\), then precisely one of the following possibilities occurs: (1) \(Q \subseteq H\); (2) \(Q \cap H = x^+ \cap Q\) for some point \(x\) of \(Q\); (3) \(Q \cap H\) is a sub-quadrangle of \(Q\); and (4) \(Q \cap H\) is an ovoid of \(Q\). If case (1), case (2), case (3), or case (4) occurs, then \(Q\) is called, respectively, deep, singular, sub-quadrangular, or ovoidal with respect to \(H\). If \(S\) is slim and \(H_1\) and \(H_2\) are its two distinct hyperplanes, then the complement of symplectic difference of \(H_1\) and \(H_2\), \(H_1 \Delta H_2\), is again a hyperplane; this means that the totality of hyperplanes of a slim near polygon form a vector space over the Galois field with two elements, \(\mathbb{F}_2\). In what follows, we shall put \(\overline{H_1 \Delta H_2} \equiv H_1 \oplus H_2\) and call it the (Veldkamp) sum of the two hyperplanes.

Finally, we shall introduce the notion of the Veldkamp space of a point-line incidence geometry \(\Gamma(P, L)\), \(\mathcal{V}(\Gamma)\) \[5\]. Here, \(\mathcal{V}(\Gamma)\) is the space in which (i) a point is a geometric hyperplane of \(\Gamma\) and (ii) a line is the collection \(H' H''\) of all geometric hyperplanes \(H\) of \(\Gamma\) such that \(H' \cap H'' = H' \cap H = H'' \cap H\) or \(H = H', H''\), where \(H'\) and \(H''\) are distinct points of \(\mathcal{V}(\Gamma)\). Following \[6, 7\], we adopt also here the definition of Veldkamp space given by Buekenhout and Cohen \[5\] instead of that of Shult \[8\], as the latter is much too restrictive by requiring any three distinct hyperplanes \(H', H''\) and \(H'''\) of \(\Gamma\) to satisfy the following two conditions: i) \(H'\) is not properly contained in \(H''\) and ii) \(H' \cap H'' \subseteq H'''\) implies \(H' \subset H'''\) or \(H' \cap H''' = H' \cap H''\). The two definitions differ in the crucial fact that whereas the Veldkamp space in the sense of Shult is always a linear space, that of Buekenhout and Cohen needs not be so; in other words, Shult’s Veldkamp lines are always of the form \(\{H \in \mathcal{V}(\Gamma) \mid H \supseteq H' \cap H''\}\) for certain geometric hyperplanes \(H'\) and \(H''\).

### 3 The near hexagon \(L_3 \times GQ(2, 2)\)

The near hexagon \(L_3 \times GQ(2, 2)\) has recently \[9\] caught an attention of theoretical physicists due to the fact that its main constituent, the generalized quadrangle \(GQ(2, 2)\), reproduces the commutation relations of the 15 elements of the two-qubit Pauli group (see, e.g., \[10\]), with each of its ten embedded copies of \(GQ(2, 1)\) playing, remarkably, the role of the so-called
The other ovoids, described earlier, we find that an example of an ovoid is the set

\[ \text{in some order, then the sum} \]

\[ e_i + e_j \] (for \( 1 \leq i < j \leq 6 \)) is the perp-set of the point \( \{i, j\} \). If we have

\[ \{1, 2, 3, 4, 5, 6\} = \{i, j, k, l, m, n\} \]

in some order, then the sum \( e_i + e_j + e_k \) is the grid whose elements are the nine points

\[ \{\{a, b\} : a \in \{i, j, k\} \text{ and } b \in \{l, m, n\}\} \].

It follows that the six ovoids are a spanning set for \( V_{GQ(2, 2)} \). Since each point of \( GQ(2, 2) \) lies in precisely two ovoids, it follows that we have the relation

\[ e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = 0, \]

where 0 denotes the subset of \( GQ(2, 2) \) consisting of all 15 points. Since we have an isomorphism \( V_{GQ(2, 2)} \cong \text{PG}(4, 2) \), it follows by a counting argument that this is the only nontrivial dependence relation between the \( e_i \), and thus that the ovoids \( e_1, \ldots, e_5 \) form a basis for \( V_{GQ(2, 2)} \).

The points of the near hexagon \( L_3 \times GQ(2, 2) \) are simply the 45 ordered pairs \( (p, q) \) where \( p \) is a point of \( L_3 \) and \( q \) is a point of \( GQ(2, 2) \). Two points \( (p_1, q_1) \) and \( (p_2, q_2) \) of \( L_3 \times GQ(2, 2) \) are collinear if either

(i) \( p_1 = p_2 \) and \( q_1 \) is collinear to \( q_2 \), or

(ii) \( p_1 \) is collinear to \( p_2 \) and \( q_1 = q_2 \).

The lines of \( L_3 \times GQ(2, 2) \) are of two types. The type-one lines are the 15 lines of the form \( \{p, q : p \in L_3\} \) for a fixed point \( q \in GQ(2, 2) \). The type-two lines are the 45 lines of the form \( \{p, q : q \in L\} \) for a fixed \( p \in L_3 \) and some line \( L \) of \( GQ(2, 2) \).

The near hexagon \( L_3 \times GQ(2, 2) \) has a number of obvious automorphisms. One type of automorphism involves permuting the three \( GQ(2, 2) \)-quads, but making no other changes. The subgroup of all such automorphisms is isomorphic to \( S_3 \). Another type of automorphism

**Mermin magic square** [11] — the smallest configuration of two-qubit observables furnishing a very important proof of contextuality of quantum mechanics. A well-known construction of \( GQ(2, 2) \) identifies the points with two-element subsets of \( \{1, 2, 3, 4, 5, 6\} \), with two points being collinear if and only if they are equal or disjoint. The natural action of \( S_6 \) on this set of size 6 induces automorphisms of \( GQ(2, 2) \). In fact, when considered in this way, \( S_6 \) turns out to be the full automorphism group.

It is known that every geometric hyperplane of a slim dense near polygon arises from its universal embedding. It can be shown from this that, equipped with the operation of Veldkamp sum, the Veldkamp space \( V_{GQ(2, 2)} \) is isomorphic to \( \text{PG}(4, 2) \), the projective space obtained from a 5-dimensional space over \( \mathbb{F}_2 \) (see also [6]). It follows that \( GQ(2, 2) \) has 2⁵ − 1 = 31 geometric hyperplanes, which turn out to be of three types:

(i) 15 perp-sets, with 7 points each;

(ii) 10 grids (copies of \( GQ(2, 1) \)), with 9 points each;

(iii) 6 ovoids, with 5 points each.

In other words, there are three orbits of geometric hyperplanes under the action of \( S_6 \).

Identifying the points of \( GQ(2, 2) \) with two-element subsets of the set \( \{1, 2, 3, 4, 5, 6\} \) as described earlier, we find that an example of an ovoid is the set

\[ e_1 := \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}. \]

The other ovoids, \( e_2, e_3, \ldots, e_6 \) are obtained from \( e_1 \) by acting by the transposition \( (1, i) \) for \( i = 2, 3, \ldots, 6 \) respectively.

The Veldkamp sum \( e_i + e_j \) (for \( 1 \leq i < j \leq 6 \)) is the perp-set of the point \( \{i, j\} \). If we have

\[ \{1, 2, 3, 4, 5, 6\} = \{i, j, k, l, m, n\} \]

in some order, then the sum \( e_i + e_j + e_k \) is the grid whose elements are the nine points

\[ \{\{a, b\} : a \in \{i, j, k\} \text{ and } b \in \{l, m, n\}\}. \]

It follows that the six ovoids are a spanning set for \( V_{GQ(2, 2)} \). Since each point of \( GQ(2, 2) \) lies in precisely two ovoids, it follows that we have the relation

\[ e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = 0, \]
involves acting diagonally on the three $GQ(2,2)$-quads by $S_6$, the automorphism group of $GQ(2,2)$. This action commutes with the action of $S_3$ just mentioned, and produces a group of automorphisms isomorphic to $S_6 \times S_3$. It turns out that this is the full automorphism group, as shown by Brouwer et al. [11].

From now on, let us denote the Veldkamp space of $L_3 \times GQ(2,2)$ by $V$. Some features of $V$ are close to obvious, which stems from Sec. 2. One of these is that the intersection of one of the three $GQ(2,2)$-quads with a point of $V$ (regarded as a subset of the 45 points) can take one of two forms. Either the $GQ(2,2)$-quad is completely filled in (i.e., it is deep), or takes the form of one of the geometric hyperplanes of $GQ(2,2)$ (i.e., it is singular, sub-quadrangular or ovoidal). Furthermore, the Veldkamp sum of any two of the layers (regarded as subsets of $GQ(2,2)$ under some obvious identification) must be equal to the third layer. It follows from this that $V$ contains $2^{10} - 1 = 1023$ points.

The above discussion shows that, as an $S_6 \times S_3$-module over $F_2$, $V$ is isomorphic to $M \otimes N$, where $M$ is the 5-dimensional module for $S_6$ described earlier, and $N$ is the $S_3$-module obtained by quotienting the 3-dimensional permutation module $\{f_1, f_2, f_3\}$ for $S_3$ by the submodule spanned by $f_1 + f_2 + f_3$. The set $\{f_1, f_2\}$ then form a basis for $N$, and the set

$$\{e_i \otimes f_j : 1 \leq i \leq 5, 1 \leq j \leq 2\}$$

forms a basis for $V$. We will write this basis for short as $\{e_1, \ldots, e_{10}\}$, where for $1 \leq i \leq 5$, $e_i$ denotes $e_i \otimes f_1$, and for $6 \leq i \leq 10$, $e_i$ denotes $e_{i-5} \otimes f_2$.

4 The classification of hyperplanes

The geometric hyperplanes of $L_3 \times GQ(2,2)$ were classified in [9]. Up to automorphisms, there are eight types of them, denoted by $H_1$ to $H_8$ and described in detail in [9, Table 2]. We now explain how these eight types can be reconstructed using the results in the previous section.

The description of the hyperplanes of $GQ(2,2)$ above can be used to identify each hyperplane with one of the 31 nontrivial set partitions of a 6-element into two pieces. If $S$ and $T$ are disjoint nonempty sets for which

$$S \cup T = \{1, 2, 3, 4, 5, 6\},$$

then we identify the pair $\{S, T\}$ with the hyperplane

$$\sum_{i \in S} e_i = \sum_{j \in T} e_j.$$

Under these identifications, the partitions of 6 given by $(5,1)$, $(4,2)$ and $(3,3)$ correspond, via set partitions, to ovoids, perp sets and grids, respectively.

The Veldkamp sum operation on $V_{GQ(2,2)}$ described in the previous section may now be defined purely in terms of sets: the Veldkamp sum of the two set partitions $\{A|B\}$ and $\{C|D\}$ is given by

$$\{(A \cap C) \cup (B \cap D)|(A \cap D) \cup (B \cap C)\}.$$ 

This identification extends to a set-theoretic description of the hyperplanes of $L_3 \times GQ(2,2)$. The hyperplanes of this larger space may be put into bijection with ordered quadruples of pairwise disjoint sets $\{A, B, C, D\}$ such that (a) no three of the sets are empty and (b) the union of the four sets is $\{1, 2, 3, 4, 5, 6\}$. Such a quadruple corresponds to the hyperplane given by the ordered triple of partitions

$$\{(\{A \cup B|C \cup D\}, \{A \cup C|B \cup D\}, \{A \cup D|B \cup C\}).$$

Here, the leftmost component of the ordered triple describes the hyperplane of $GQ(2,2)$ appearing in the uppermost $GQ(2,2)$-quad of $L_3 \times GQ(2,2)$, and so on. For example, if the sets
Table 1: A classification of geometric hyperplanes of $L_3 \times GQ(2, 2)$.

| Name | Partition | Orbit size | Stabilizer | Order |
|------|-----------|------------|------------|-------|
| $H_1$ | $(3, 3)$ | 30 | $S_3 \times S_3 \times S_2 \times S_2$ | 144 |
| $H_2$ | $(4, 2)$ | 45 | $S_4 \times S_2 \times S_2$ | 96 |
| $H_3$ | $(5, 1)$ | 18 | $S_5 \times S_2$ | 240 |
| $H_4$ | $(2, 2, 1, 1)$ | 270 | $S_2 \times S_2 \times S_2 \times S_2$ | 16 |
| $H_5$ | $(2, 2, 2)$ | 90 | $S_2 \times S_2 \times S_2 \times S_3$ | 48 |
| $H_6$ | $(3, 1, 1, 1)$ | 120 | $S_3 \times S_3$ | 36 |
| $H_7$ | $(3, 2, 1)$ | 360 | $S_3 \times S_3$ | 12 |
| $H_8$ | $(4, 1, 1)$ | 90 | $S_4 \times S_2$ | 48 |

$C$ and $D$ are empty, the top $GQ(2, 2)$-quad will be deep and the other two will be identical to each other, being either singular, sub-quadrangular or ovoidal.

The correspondence between the ordered quadruples and the hyperplanes is four-to-one, because the quadruples $(A, B, C, D)$, $(B, A, D, C)$, $(C, D, A, B)$ and $(D, C, B, A)$ all index the same hyperplane. It follows that acting by an element of the Klein four-group $V_4$ on an ordered quadruple leaves the corresponding hyperplane invariant. The group $S_6 \times S_4$ acts on the quadruples, where $S_6$ acts diagonally on each of the set partitions $A$, $B$, $C$ and $D$, and $S_4$ acts by place permutation. This induces an action of $S_6 \times S_4$ on the hyperplanes of $L_3 \times GQ(2, 2)$, and since the action of $V_4 \leq S_4$ is trivial, this in turn induces an action of $S_6 \times (S_4/V_4) \cong S_6 \times S_3$ on the hyperplanes, thus recovering the full automorphism group of $L_3 \times GQ(2, 2)$ in which $S_3$ acts by permuting the $GQ(2, 2)$-quads.

This approach yields another way to deduce that the number of hyperplanes of $L_3 \times GQ(2, 2)$ is $2^{10} - 1$, as follows. There are $4^6$ possible quadruples of pairwise disjoint sets $(A, B, C, D)$ whose union is $\{1, 2, 3, 4, 5, 6\}$, and four of these quadruples have three empty components. Since the correspondence between quadruples and hyperplanes is four-to-one, the number of hyperplanes is $(4^6 - 4)/4$.

The correspondence described above induces a natural correspondence between $S_6 \times S_4$-orbits (or $S_6 \times S_3$-orbits) of hyperplanes on the one hand, and partitions of 6 into two, three or four parts on the other. There are eight such partitions; they are shown in Table 1, together with their orbit sizes, stabilizers isomorphism types, stabilizer orders, and their name in the $H_1 - H_8$ notation of [9, Table 2].

5 Counting and classifying different types of Veldkamp lines

The orbits of lines in the Veldkamp space $V$ may be enumerated using a standard technique sometimes (inaccurately) known as Burnside’s Lemma, which proves the following.

Let $G$ be a finite group acting on a finite set $X$ with $t$ orbits, and for each $g \in G$, let $X^g$ denote the number of elements of $X$ fixed by $g$. Then we have $t = \frac{1}{|G|} \sum_{g \in G} |X^g|$. Furthermore, if $C$ is a set of conjugacy class representatives of $G$, then we have

$$t = \frac{1}{|G|} \sum_{g \in C} |C||X^g|.$$ 

Using this technique, we can recover known results about orbits of lines under the action of the automorphism group $S_6$ of $GQ(2, 2)$: there are 3 orbits of hyperplanes (Veldkamp points)
and 5 orbits of Veldkamp lines. We can also recover the result the Veldkamp space $V$ has 8 orbits of hyperplanes under the automorphism group $S_6 \times S_3$.

The same idea can be adapted to count the orbits of Veldkamp lines of $V$. The counting argument is more complicated than for the case of Veldkamp points, because it is possible for a line to be fixed by a group element $g$ without the three individual points being fixed. There are three possibilities to consider, which we denote by (1), (2) and (3) in Table 2.

1. Every point of the Veldkamp line is fixed by $g$. Such lines lie entirely within the fixed point space of $g$.

2. One point of the Veldkamp line is fixed by $g$, and the other two are exchanged. To enumerate such lines, we take one point $x$ outside the fixed point space of $g$. The other two points are the point $g(x)$, and the point collinear with both of them (which is fixed by $g$). We then divide by 2 to correct for the overcount.

3. The element $g$ rotates the three points of the Veldkamp line in a 3-cycle. These are much harder to enumerate, but fortunately this happens relatively rarely.

We identify the group $S_6 \times S_3$ in the obvious way with the subgroup of $S_9$ fixing setwise each of the subsets $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9\}$. Since there are 11 partitions of 6 and 3 partitions of 3, it follows that $S_6 \times S_3$ has 33 conjugacy classes, and it is straightforward to find conjugacy class representatives. Table 2 shows the calculation for the Veldkamp lines of $L_3 \times GQ(2, 2)$. The grand total of

\[ 682560 = |S_6 \times S_3| \times 158 = 720 \times 6 \times 158 \]

proves that there are 158 orbits of Veldkamp lines of the near hexagon. (The highlighted entry of 85 in the table corresponds to some remarkable exceptional behaviour.)

All 158 types are then listed in Table 3. Here, each type is characterized by its composition (columns 9 to 16) and the properties of the core $C$ of the line, that is the set of points that are common to all the three geometric hyperplanes of a line of the given type. In particular, for each type (column 1) we list the number of points (column 2) and lines (column 3) of the core as well as the distribution of the orders of its points. The last three columns show the intersection of $C$ with each of the three $GQ(2, 2)$-quads. Here, ‘g-perp’ stands for a perp-set in a certain $GQ(2, 1)$ located in the particular $GQ(2, 2)$, and ‘unitr/tritr’ abbreviates a unicentric/tricentric triad. If two or more types happen to possess the same string of parameters, the distinction between them is given by an explanatory remark/footnote.
Table 2: Orbits of Veldkamp lines of $L_3 \times GQ(2,2)$.

| Conjugacy class | Fix(1) | Fix(2) | Fix(3) | Size of class | Product |
|-----------------|--------|--------|--------|---------------|---------|
| id              | 174251 | 0      | 0      | 1             | 174251  |
| (12)            | 10795  | 384    | 0      | 15            | 167685  |
| (12)(34)        | 651    | 480    | 0      | 45            | 50895   |
| (12)(34)(56)    | 651    | 480    | 0      | 15            | 16965   |
| (123)           | 651    | 0      | 5      | 40            | 26240   |
| (123)(456)      | 1      | 0      | 85     | 40            | 3440    |
| (1234)          | 35     | 24     | 0      | 90            | 5310    |
| (1234)(56)      | 35     | 24     | 0      | 90            | 5310    |
| (123)(45)       | 35     | 24     | 5      | 120           | 7680    |
| (12345)         | 1      | 0      | 0      | 144           | 144     |
| (123456)        | 1      | 0      | 5      | 120           | 720     |
| (78)            | 155    | 496    | 0      | 3             | 1953    |
| (12)(78)        | 155    | 496    | 0      | 45            | 29295   |
| (12)(34)(78)    | 155    | 496    | 0      | 135           | 87885   |
| (12)(34)(56)(78)| 155    | 496    | 0      | 45            | 29295   |
| (123)(78)       | 7      | 28     | 1      | 120           | 4320    |
| (123)(45)(78)   | 0      | 1      | 5      | 120           | 720     |
| (1234)(78)      | 7      | 28     | 0      | 270           | 9450    |
| (1234)(56)(78)  | 7      | 28     | 0      | 270           | 9450    |
| (123)(45)(78)   | 7      | 28     | 1      | 360           | 12960   |
| (12345)(78)     | 0      | 1      | 0      | 432           | 432     |
| (123456)(78)    | 0      | 1      | 1      | 360           | 720     |
| (789)           | 0      | 0      | 1023   | 2             | 2046    |
| (12)(789)       | 0      | 0      | 255    | 30            | 7650    |
| (12)(34)(789)   | 0      | 0      | 63     | 90            | 5670    |
| (12)(34)(56)(789)| 0    | 0      | 63     | 30            | 1890    |
| (123)(789)      | 1      | 0      | 64     | 80            | 5200    |
| (123)(45)(789)  | 35     | 0      | 8      | 80            | 3440    |
| (1234)(789)     | 0      | 0      | 15     | 180           | 2700    |
| (1234)(56)(789) | 0      | 0      | 15     | 180           | 2700    |
| (123)(45)(789)  | 1      | 0      | 16     | 240           | 4080    |
| (12345)(789)    | 0      | 0      | 3      | 288           | 864     |
| (123456)(789)   | 1      | 0      | 4      | 240           | 1200    |

Total: 682560
Table 3: The types of Veldkamp lines of \( L_3 \times GQ(2,2) \).

| \( T_p \) | \( P_t \) | \( L_n \) | \# of Points of Order \( 0 \) \( 1 \) \( 2 \) \( 3 \) \( 4 \) | Composition | \( H_1 \) | \( H_2 \) | \( H_3 \) | \( H_4 \) | \( H_5 \) | \( H_6 \) | \( H_7 \) | \( H_8 \) | 1st | 2nd | 3rd |
|-----------|---------|---------|----------------|--------------|------|------|------|------|------|------|------|------|------|------|------|------|
| 1         | 27      | 27      | 0 0 0 27 0     | 3 - - - - - -| grid | grid | grid |
| 2         | 25      | 24      | 0 0 10 10 5   | 2 1 - - - - -| full | g-perp| g-perp |
| 3         | 23      | 19      | 0 0 12 11 0   | 2 - - 1 - - -| grid | g-perp| grid |
| 4         | 21      | 20      | 0 0 6 12 3    | 3 - - - - - -| full | line | line |
| 5         | 21      | 18      | 0 0 6 12 3    | 1 1 - - - - -| full | unitr| unitr|
| 6         | 21      | 18      | 0 0 6 12 3    | 3 - - - - - -| full | tritr| tritr|
| 7         | 21      | 16      | 0 2 12 6 1    | 1 1 - - - - -| perp | grid | g-perp |
| 8         | 21      | 16      | 0 0 18 0 3    | 3 - - - - - -| perp | perp | perp |
| 9         | 19      | 15      | 0 0 12 7 0    | 1 - - - 2 - -| grid | g-perp| g-perp |
| 10        | 19      | 13      | 0 4 10 5 0    | 1 - - - 2 - -| grid | g-perp| g-perp |
| 11        | 19      | 12      | 0 6 9 4 0     | 1 1 - - - - -| perp | grid | unitr|
| 12        | 17      | 16      | 0 2 0 14 1    | - 1 - 2 - - -| full | point| point |
| 13        | 17      | 12      | 0 2 12 2 1    | - 1 - 2 - - -| perp | g-perp| g-perp |
| 14        | 17      | 12      | 0 2 11 4 0    | - 1 - 2 - - -| grid | line | g-perp |
| 15        | 17      | 10      | 0 8 6 2 1 1   | - 1 - 1 1 - -| g-perp| g-perp| g-perp |
| 16        | 17      | 10      | 1 4 10 2 0    | 1 - - - 1 - -| grid | unitr| g-perp |
| 17        | 17      | 10      | 0 8 7 0 2    | - 2 - - - 1 -| perp | line | perp |
| 18        | 17      | 10      | 1 4 10 2 0    | - 1 - 2 - - -| grid | tritr| g-perp |
| 19        | 17      | 10      | 0 8 6 2 1    | - 1 - 2 - - -| perp | g-perp| g-perp |
| 20        | 17      | 9       | 2 6 6 3 0 1  | - 1 - 1 - - -| perp | grid | g-perp |
| 21        | 17      | 9       | 0 8 8 1 0 1   | - 1 - 1 - - -| perp | tritr| g-perp |
| 22        | 17      | 9       | 0 9 6 2 0    | - 2 - - - - 1| perp | tritr| g-perp |
| 23        | 15      | 11      | 0 0 12 3 0    | - - - 3 - - -| g-perp| g-perp| g-perp |
| 24        | 15      | 9       | 0 6 6 3 0 1   | - - - - - - 2| unitr| grid | unitr|
| 25        | 15      | 9       | 0 6 6 3 0    | - - - - - 3 -| g-perp| g-perp| g-perp |
| 26        | 15      | 9       | 0 6 6 3 0    | - - - - - 3 -| g-perp| g-perp| g-perp |
| 27        | 15      | 8       | 2 4 7 2 0    | - 1 - 1 - - -| grid | tritr| unitr|
| 28        | 15      | 8       | 2 3 9 1 0    | - 1 - 1 - - -| line | grid | unitr|
| 29        | 15      | 8       | 2 4 7 2 0    | - 1 - 1 - - -| grid | unitr| unitr|
| 30        | 15      | 8       | 0 6 9 0 0    | - - - 3 - - -| g-perp| g-perp| g-perp |
| 31        | 15      | 7       | 1 8 5 1 0 1  | - - - 1 1 - -| perp | g-perp| unitr|
| 32        | 15      | 7       | 4 2 8 1 0 1  | - - - 1 - - -| perp | unitr| grid |
| 33        | 15      | 7       | 1 8 5 1 0    | - 1 - 1 - - -| perp | unitr| g-perp |
| 34        | 15      | 7       | 0 9 6 0 0    | - - - 3 - - -| g-perp| g-perp| g-perp |
| 35        | 15      | 6       | 2 10 1 2 0 1 | - - - 1 - - -| perp | unitr| g-perp |
| 36        | 15      | 6       | 3 6 6 0 0 1  | - - - 2 - - -| g-perp| g-perp| g-perp |
| 37        | 15      | 6       | 2 9 3 1 0    | - 1 1 - - - -| g-perp| g-perp| g-perp |
| 38        | 15      | 5       | 0 15 0 0 0    | - - - 2 - - -| g-perp| g-perp| g-perp |
| 39        | 13      | 8       | 0 4 8 0 1    | - 1 - - - 2 -| perp | line | line |
| 40        | 13      | 8       | 0 3 9 1 0    | - 1 - - - 2 -| line | grid | point |
| Tp | Pt | Ln | # of Points of Order | Composition | 1st       | 2nd       | 3rd       |
|----|----|----|----------------------|-------------|-----------|-----------|-----------|
| 41 | 13 | 8  | 0 4 7 2 0            | line g-perp | g-perp    |           |           |
| 42 | 13 | 7  | 2 2 8 1 0            | grid unitr  | point     |           |           |
| 43 | 13 | 6  | 0 9 3 1 0            | perp tritr  | tritr     |           |           |
| 44 | 13 | 6  | 0 9 3 1 0            | perp line   | line      |           |           |
| 45 | 13 | 6  | 0 9 3 1 0            | perp point  | point     |           |           |
| 46 | 13 | 6  | 0 10 2 1 0           | perp unitr  | unitr     |           |           |
| 47 | 13 | 6  | 0 9 3 1 0            | g-perp g-perp|           |           |           |
| 48 | 13 | 6  | 1 6 6 0 0            | tritr g-perp| g-perp    |           |           |
| 49 | 13 | 6  | 0 8 5 0 0            | g-perp g-perp|           |           |           |
| 50 | 13 | 6  | 1 6 6 0 0            | g-perp unitr|           |           |           |
| 51 | 13 | 5  | 2 8 2 1 0            | perp line   | tritr     |           |           |
| 52 | 13 | 5  | 2 8 2 1 0            | perp unitr  | unitr     |           |           |
| 53 | 13 | 5  | 2 8 2 1 0            | tritr g-perp| g-perp    |           |           |
| 54 | 13 | 5  | 0 11 2 0 0           | line g-perp | g-perp    |           |           |
| 55 | 13 | 5  | 2 7 4 0 0            | tritr g-perp| g-perp    |           |           |
| 56 | 13 | 5  | 2 8 2 1 0            | perp unitr  | unitr     |           |           |
| 57 | 13 | 5  | 2 7 4 0 0            | g-perp unitr|           |           |           |
| 58 | 13 | 4  | 4 8 0 0 1            | perp unitr  | unitr     |           |           |
| 59 | 13 | 4  | 4 8 0 0 1            | perp ovoid  | point     |           |           |
| 60 | 13 | 4  | 4 8 0 0 1            | perp unitr  | unitr     |           |           |
| 61 | 13 | 4  | 4 8 0 0 1            | perp tritr  | tritr     |           |           |
| 62 | 13 | 4  | 4 7 1 1 0            | perp tritr  | tritr     |           |           |
| 63 | 13 | 4  | 4 7 1 1 0            | line g-perp | ovoid     |           |           |
| 64 | 13 | 4  | 4 7 1 1 0            | perp unitr  | unitr     |           |           |
| 65 | 13 | 4  | 4 6 3 0 0            | g-perp ovoid|           |           |           |
| 66 | 13 | 4  | 4 8 0 0 1            | perp unitr  | unitr     |           |           |
| 67 | 13 | 3  | 6 6 0 1 0            | perp unitr  | unitr     |           |           |
| 68 | 13 | 3  | 6 6 0 1 0            | ovoid g-perp|           |           |           |
| 69 | 11 | 6  | 2 0 9 0 0            | grid point  | point     |           |           |
| 70 | 11 | 5  | 0 7 4 0 0            | g-perp point| point     |           |           |
| 71 | 11 | 4  | 2 7 1 1 0            | perp unitr  | unitr     |           |           |
| 72 | 11 | 4  | 2 7 1 1 0            | line g-perp | unitr     |           |           |
| 73 | 11 | 4  | 2 6 3 0 0            | line unitr  | unitr     |           |           |
| 74 | 11 | 4  | 2 6 3 0 0            | g-perp tritr| g-perp    |           |           |
| 75 | 11 | 4  | 2 6 3 0 0            | g-perp unitr|           |           |           |
| 76 | 11 | 4  | 2 6 3 0 0            | perp point  | point     |           |           |
| 77 | 11 | 4  | 2 6 3 0 0            | g-perp unitr|           |           |           |
| 78 | 11 | 4  | 1 8 2 0 0            | point g-perp| g-perp    |           |           |
| 79 | 11 | 3  | 4 6 0 1 0            | perp unitr  | unitr     |           |           |
| 80 | 11 | 3  | 4 6 0 1 0            | perp unitr  | unitr     |           |           |
Table 3: (Continued.)

| Tp | Pt | Ln | # of Points of Order | Composition | 1st   | 2nd   | 3rd   |
|----|----|----|----------------------|-------------|-------|-------|-------|
|    |    |    | 0 | 1 | 2 | 3 | 4 | H₁ | H₂ | H₃ | H₄ | H₅ | H₆ | H₇ | H₈ | 1st   | 2nd   | 3rd   |
| 81 | 11 | 3  | 2 | 9 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | unitr | unitr | ovoid |
| 82 | 11 | 3  | 4 | 6 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | unitr | g-perp | unitr |
| 83 | 11 | 3  | 4 | 6 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | unitr | g-perp |
| 84 | 11 | 3  | 4 | 5 | 2 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 85 | 11 | 3  | 3 | 7 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 86 | 11 | 3  | 4 | 6 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 87 | 11 | 3  | 4 | 6 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 88 | 11 | 3  | 4 | 5 | 2 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 89 | 11 | 3  | 4 | 5 | 2 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 90 | 11 | 2  | 6 | 4 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | line  | unitr | ovoid |
| 91 | 11 | 2  | 6 | 4 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | unitr | g-perp | unitr |
| 92 | 11 | 2  | 6 | 4 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 93 | 11 | 2  | 6 | 4 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 94 | 11 | 2  | 6 | 4 | 0 | 1 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | g-perp | unitr |
| 95 | 11 | 1  | 8 | 3 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | ovoid  | unitr | ovoid |
| 96 | 11 | 1  | 8 | 3 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | unitr | unitr | ovoid |
| 97 | 11 | 0  | 11| 0 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 2     | unitr | ovoid |
| 98 | 11 | 0  | 11| 0 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | tritr | ovoid | ovoid |
| 99 | 9  | 6  | 0 | 0 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | line  | line  | line  |
| 100| 9  | 4  | 0 | 1 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 2     | point | point |
| 101| 9  | 3  | 2 | 6 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | point |
| 102| 9  | 3  | 2 | 6 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | unitr |
| 103| 9  | 3  | 0 | 9 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | line  | line  | line  |
| 104| 9  | 3  | 2 | 5 | 2 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 2     | point | g-perp |
| 105| 9  | 3  | 0 | 9 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | g-perp |
| 106| 9  | 3  | 2 | 5 | 2 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 2     | point | g-perp |
| 107| 9  | 3  | 1 | 7 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | g-perp |
| 108| 9  | 3  | 0 | 9 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 3     | point | g-perp |
| 109| 9  | 3  | 1 | 7 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 2     | point | g-perp |
| 110| 9  | 3  | 0 | 9 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 3     | point | unitr |
| 111| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | line  | unitr | unitr |
| 112| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | line  | unitr | unitr |
| 113| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | unitr |
| 114| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | g-perp |
| 115| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 2     | point | g-perp |
| 116| 9  | 2  | 3 | 6 | 0 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 3     | point | g-perp |
| 117| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | g-perp |
| 118| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 1     | point | g-perp |
| 119| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 3     | point | g-perp |
| 120| 9  | 2  | 4 | 4 | 1 | 0 | 0 | 0 |    |    |    |    |    |    |    |    | 3     | point | g-perp |
| Tp | Pt | Ln | # of Points of Order | Composition | 1st | 2nd | 3rd |
|----|----|----|---------------------|-------------|-----|-----|-----|
|    |    |    |                     |             |     |     |     |
| 121| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - 1 1 - - - - - - 1 | unitr | line | unitr |
| 122| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - 3 - - - - - - - - 2 | tritr | tritr | line |
| 123| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - - - - - - - - - - - 3 | - - | line | tritr |
| 124| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - 1 - - - - - - - - 2 | tritr | tritr | tritr |
| 125| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - - - - - - - - - - - 3 | - - | line | unitr |
| 126| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - - - - - - - - - - - 3 | - - | line | unitr |
| 127| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - - - - - - - - - - - 3 | - - | tritr | unitr |
| 128| 9  | 1  | 6 3 0 0 0 0 0       | - - - - - - - - - - - - - - - - 3 | - - | tritr | unitr |
| 129| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - 1 - - - - - - - - 2 | tritr | point | ovoid |
| 130| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 131| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 132| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 133| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 134| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 135| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 136| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 137| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
| 138| 9  | 0  | 9 0 0 0 0 0 0       | - - - - - - - - - - - - - - - - 2 | tritr | point | ovoid |
|    |    |    |                     |             |     |     |     |
| 139| 7  | 2  | 2 4 1 0 0 0         | - - - - - - 1 - - - - - - - - 1 | point | unitr | line |
| 140| 7  | 2  | 2 4 1 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | g-perp | point |
| 141| 7  | 1  | 4 3 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 142| 7  | 1  | 4 3 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 143| 7  | 1  | 4 3 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 144| 7  | 1  | 4 3 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 145| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 146| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 147| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 148| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 149| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 150| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 151| 7  | 0  | 7 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 152| 5  | 1  | 2 3 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 153| 5  | 0  | 5 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 154| 5  | 0  | 5 0 0 0 0 0         | - - - - - - - - - - - - - - - - 2 | point | no point | point |
| 155| 3  | 1  | 0 3 0 0 0 0         | - - - - - - - - - - - - - - - - 3 | point | no point | point |
| 156| 3  | 0  | 3 0 0 0 0 0         | - - - - - - - - - - - - - - - - 3 | point | no point | point |

Table 3: (Continued.)
Explanatory remarks:

1 Two (25) or no two (26) of the g-perps are such that their centers are joined by a type-one line.
2 The center of the g-perp does (77) or does not (76) lie on the type-one line passing through the center of one of the two unicentric triads.
3 The centers of the two unicentric triads are (86) or are not (87) joined by a type-one line.
4 One line (88) or no line (89) of the g-perp is incident with the type-one line passing through the center of one of the two unicentric triads.
5 The five type-one lines through the points of the two triads do (114) or do not (113) cut a doily-quad in an ovoid.
6 One line (120) or no line (119) of type-two through the point is incident with the type-one line through the center of the g-perp.
7 One (133) or none (134) of the unicentric triads is such that the type-one lines through two of its points pass through the centers of the other two triads.
8 The centers of the two unicentric triads are (143) or are not (144) joined by a type-one line.
9 The point does (147) or does not (148) lie on the type-one line passing through a center of the tricentric triad.
10 The point does (149) or does not (150 and 151) lie on the type-one line passing through the center of one of the two unicentric triads.
11 The centers of the two unicentric triads do (150) or do not (151) belong to the same grid-quad.

⋆ This type splits into two distinct orbits according as the type-one line through the center of the triad passes through the center of the one or the other g-perp.
† This type splits into two distinct orbits according as the point is incident with the type-one line through the center of the one or the other unicentric triad.

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References

[1] A.E. Brouwer, A.M. Cohen, J.I. Hall, H.A. Wilbrink, Near Polygons and Fischer spaces, Geom. Dedicata 49 (1994) 349–368.
[2] R.M. Green, M. Saniga, The Veldkamp space of the smallest slim dense near hexagon, Int. J. Geom. Methods Mod. Physics 10 (2013) 1250082 (arXiv:0908.0989).
[3] B. De Bruyn, Near Polygons, Birkhäuser, Basel, 2006.
[4] S.E. Payne, J.A. Thas, Finite Generalized Quadrangles, Pitman, Boston – London – Melbourne, 1984; see also K. Thas, Symmetry in Finite Generalized Quadrangles, Birkhäuser, Basel, 2004.
[5] F. Buekenhout, A.M. Cohen, Diagram Geometry (Chapter 10.2), Springer, Berlin – Heidelberg – New York – London – Paris – Tokyo – Hong Kong – Barcelona – Budapest, 2009; preprints of separate chapters can be also found at http://www.win.tue.nl/~amc/buek/.
[6] M. Saniga, M. Planat, P. Pracna, H. Havlicek, The Veldkamp space of two-qubits, SIGMA 3 (2007) 075 (arXiv:0704.0495).
[7] M. Saniga, R.M. Green, P. Lévy, P. Pracna, P. Vrana, The Veldkamp space of GQ(2, 4), Int. J. Geom. Methods Mod. Phys. 7 (2010) 1133–1145 (arXiv:0903.0715).
[8] E.E. Shult, On Veldkamp lines, Bull. Belg. Math. Soc. 4 (1997) 299–316.
[9] M. Saniga, P. Lévy, M. Planat, P. Pracna, Geometric hyperplanes of the near hexagon $L_3 \times GQ(2, 2)$, Lett. Math. Physics 91 (2010) 93–104 (arXiv:0908.3363).
[10] M. Planat, M. Saniga, On the Pauli graphs of N-qudits, Quantum Inf. Comput. 8 (2008) 127–146 (arXiv:quant-ph/0701211).
[11] N.D. Mermin, Hidden variables and two theorems of John Bell, Rev. Mod. Phys. 65 (1993) 803–15.