The Hamiltonian dynamics of two-component spherically symmetric null dust is studied with regard to the quantum theory of gravitational collapse. The components—the ingoing and outgoing dusts—are assumed to interact only through gravitation. Different kinds of singularities, naked or “clothed”, that can form during collapse processes are described. The general canonical formulation of the one-component null-dust dynamics by Bičák and Kuchař is restricted to the spherically symmetric case and used to construct an action for the two components. The transformation from a metric variable to the quasilocal mass is shown to simplify the mathematics. The action is reduced by a choice of gauge and the corresponding true Hamiltonian is written down. Asymptotic coordinates and energy densities of dust shells are shown to form a complete set of Dirac observables. The action of the asymptotic time translation on the observables is defined but it has been calculated explicitly only in the case of one-component dust (Vaidya metric).

I. INTRODUCTION

The theory of gravitational collapse and black holes has acquired many facets, extending from detailed astrophysical models of activity in galactic nuclei to the sophisticated calculations of the black-hole entropy. Still, some long-standing central questions persist: Is the outcome of collapse always a black hole or can in some cases naked singularities be formed? Even if the cosmic censorship conjecture will sometime be formulated as a theorem, the final singularities inside black holes will require new physics. Hopes in curing singularities by a suitable form of quantum gravity are older than the concept of a black hole itself. Despite new advances towards a general formulation of quantum gravity, simplified models play still important role, not least because within such models one may address oneself more easily to a concrete physical question.

The spherical gravitational collapse of null dust belongs to this type of problems. Since Vaidya’s discovery of an exact solution of Einstein’s equations describing a “radiating spherical star” [1], null dust – also referred to in literature as “pure radiation field” – has been widely used as a simple matter source. Its energy-momentum tensor

\[ T_{\alpha\beta} = \rho k_\alpha k_\beta, \]

where \( k_\alpha \) is null,

\[ k_\alpha k^\alpha = 0, \] (1.2)

may be interpreted as an incoherent superposition of waves with random phases and polarizations moving in a single direction. Null dust exhibits all features of the geometrical optics limit of Maxwell’s theory except for the polarization properties [2]. In this limit the mass distribution \( \rho \) can be identified with the square of the scalar wave amplitude. Alternatively, the energy-momentum tensor (1.1) can be considered as representing photons that move along null rays with the flux vector determined by the amplitude and the null vector \( k^\alpha \) [2]. The light-like particles may be massless scalar particles or neutrinos. The same energy-momentum tensor describes high-frequency gravitational waves. One can also reinterpret some exact solutions of Einstein’s equations with null dust as spacetimes produced by zero-rest-mass fields. All these roles of null dust indicate that it is more closely related to fundamental matter fields than ordinary dust from phenomenological massive particles. For more details and references on the null dust and geometrical optics, see Appendix A in [3]; for the list of various exact solutions with null dust, see [4], and Appendix B in [3], where some applications are also mentioned.

Vaidya metric is an interesting model in respect of studying the singularities and their possible removal by quantum phenomena. Indeed, classical solutions with null dust contain both naked and “clothed” singularities. In the next Section 2 we shall briefly review how spherical null dust solutions can be used to study the formation of naked singularities, or to describe mass inflation inside black holes.

The main purpose of the present work is to construct the canonical theory of spherically symmetric spacetimes with null dusts. The general Lagrangian and Hamiltonian framework which includes (single-component) null dust as a source into canonical gravity was developed in
In Sec. 3, we generalize the Hamiltonian action given in Ref. [8] to include both radially ingoing and outgoing null dusts. We assume that the null dust components interact only gravitationally, i.e., even in the region where the two streams interpenetrate, the energy-momentum tensors of both components are conserved separately. In the same section, the action is reduced by spherical symmetry. Since in this paper we are interested in the interaction of outgoing and ingoing null dusts, let us add that recently the similarity solutions representing interacting null dusts (not necessarily with the same type of interaction between the components as we have chosen) were studied [9]; and the exact static solution for two colliding spherically symmetric null dust streams in equilibrium was presented [10].

Although the present paper does not address any quantum problem directly, our main motivation for this generalization originates in some problems of the quantum theory of gravitational collapse. In Refs. [7] and [8], quantum gravitational collapse of a spherically symmetric null-dust thin shell has been studied (this is a kind of limit of the Vaidya metric). The quantum mechanics constructed there is unitary and describes a bounce and re-expansion of the shell in spite of the fact that the corresponding classical solution contains a horizon and a singularity inside it. The problem mentioned above is: what is the nature of the quantum spacetime geometry around the quantum shell? The metric itself is not a gauge invariant quantity in the sense that different gauges lead to unitarily inequivalent metric operators (see, e.g., [3]). We have proposed, therefore, to study the quantum theory of a system consisting of two shells so that the second shell can probe the geometry around the first one and vice versa, carrying some information about it to the infinity where it may be deciphered by the asymptotic observers. The investigations of such a system [11] has revealed a rich set of states in which the shells cross. Such crossing can be described by a finite-step canonical transformation in spite of the crossing process being instantaneous. The reason, of course, is that the shells are infinitesimally thin but carry finite energy and momenta. The canonical transformation describing a crossing is rather involved and there is no obvious way of how it can be “quantized.” (Usually, we quantize an infinitesimal canonical transformation by making the function that generates it to an operator.) As yet, we have not managed in a direct way to find a suitable quantum version (i.e., a unitary operator) of the canonical transformation for the crossing. One hope, however, is that it could be found indirectly, if we study some limit of the system of two null-dust thick shells crossing each other: there must be a well-defined Hamiltonian for this process. Such a study is initiated by the present work.

The gravitational part of the constraints can be simplified by Kuchař’s canonical transformations introduced originally in the vacuum Schwarzschild geometrodynamics [11] and used subsequently for a null-dust thin shell model in [12]. In Kuchař’s procedure the quasilocal mass plays an important role. In case of colliding spherical layers of null dust the quasilocal mass is well identified in Bondi-type coordinates used by Bardeen [13] to study the effects of back reaction of the Hawking radiation. These coordinates will be used extensively throughout the paper. The Kuchař transformation is carried out in Sec. 4, based on a careful analysis of the resulting boundary terms.

In Sec. 5, we reduce the action by a gauge choice so that the Hamiltonian for cross-streaming null dusts can be found. The gauge is shown to be regular, and the Hamiltonian is calculated. The result, Eq. (5.12), is non-local—one must be careful in calculating its variation. Then the dynamical equations implied by Hamiltonian (5.12) are obtained and, in Sec. 6, compared with the Einstein equations in the form that has been given by Bardeen [13].

In Sec. 7, we analyze the dynamical equations looking for integrals of motion. Two explicit integrals of motion per point are found. From them a complete set of Dirac observables is constructed. The transformation between the original phase space variables and the Dirac observables so constructed is, however, not known in an explicit form. Similarly, the expression of the Hamiltonian in terms of the Dirac observables is not known explicitly. Nevertheless, the transformation properties, as well as the physical meaning of the Dirac observables are available. And the lack of the explicit algebra and dynamics of our Dirac observables does not obstruct our project of finding the quantum version of the shell crossing. We can work with the canonical coordinates of Sec. 5 and with the Hamiltonian (5.12). This will be done in the subsequent work.

It turns out that the explicit transformation between the Dirac observables and the variables of the reduced theory, as well as the expression of the Hamiltonian in terms of the Dirac observables, can be found in the special case of one-component dust. This is shown in Sec. 8. To achieve this aim, we start from the extended phase space and constraints (before the reduction by a gauge choice of Sec. 5). We use the Vaidya coordinates as our covariant gauge condition and introduce the corresponding embedding variables. The coordinates in the extended phase space are transformed to the embedding variables and the Dirac observables. Then the reduction to the physical phase space spanned by the Dirac observables is carried out and all desired transformations are written down explicitly. In this way the action is transformed into the Kuchař form [15]—cf. Eq. (5.13). This section can also illustrate how a nontrivial dynamics of Dirac observables based on the asymptotic symmetries can be constructed. These ideas are explained thoroughly in Ref. [14].

The canonical theory of one-component spherical null dust, starting from the general formalism of Ref. [8] and Kuchař’s transformations, was recently developed by Vaz et al [15]. The authors also discussed the quantization procedure. However, their final form of the action is not
in the Kuchař form and their theory is not gauge invariant.

II. NULL DUST AND FORMATION OF NAKED SINGULARITIES

As we have already stated in the Introduction, the null dust models have been used to clarify, in classical context, the formation of naked singularities during a spherical gravitational collapse. It has been known for about 30 years that during spherical collapse of ordinary dust, the shells of dust may cross under suitable condition to generate the density singularity outside the event horizon (see e.g., [16] for a review). Since it is possible to determine the motion of dust through the singularity, this “shell-crossing singularity” is considered as relatively innocent. More serious is the “shell-focusing singularity” which may form at the center of a collapsing spherically symmetric inhomogeneous dust cloud described by a Tolman-Bondi metric. When collapse to the center proceeds sufficiently slowly, a “past-null” singularity which is visible from infinity may arise at the center [15]. Such singularity is a “strong curvature singularity”, as defined e.g. in [18].

That a shell focusing null singularity may form in case of imploding null dust, was first noticed by Hiscock et al [19], and analyzed in detail by Kuroda [20] and Papapetrou [21]. Hollier [22] proved that this singularity is a “strong curvature singularity”, as defined in the Kuchař form and their theory is not gauge invariant. By taking \( \lambda k_\alpha = \lambda l_\alpha \), \( \lambda > 0 \), we can simultaneously rescale the “density” \( \rho \), \( \rho = \lambda^{-2} \rho \), so that the form (1.1) of the energy-momentum tensor is preserved. By taking \( \lambda = \rho^{1/2} \) we can eliminate the scalar \( \rho \) and give the energy-momentum tensor just in terms of a single null vector,

\[
T^\alpha = \rho^{1/2} k_\alpha \tag{2.1}
\]

so that the form (1.1) of the energy-momentum tensor is preserved. By taking \( \lambda = \rho^{1/2} \) we can eliminate the scalar \( \rho \) and give the energy-momentum tensor just in terms of a single null vector,

\[
T^\alpha = \rho^{1/2} k_\alpha \tag{2.3}
\]

in the form

\[
T^{\alpha \beta} = l^\alpha l^\beta \tag{2.4}
\]

This choice maximally simplifies the canonical description of null dust. It will be employed in the following.

The simplest form of the Vaidya metric describing spherical implosion of null dust is achieved by using the incoming null coordinate \( V \), together with the Schwarzschild curvature coordinate \( R \):

\[
ds^2 = -\left(1 - \frac{2M(V)}{R}\right) dV^2 + 2dVdR + R^2 d\Omega^2 \tag{2.5}
\]

Here \( M(V) \) – the “advanced mass”, measured at past null infinity \( \mathcal{I}^- \) – may be an arbitrary, non-decreasing function. In the time-reversed case of outgoing spherical cloud of null dust, the simplest description is provided in terms of the outgoing null coordinate \( U \):

\[
ds^2 = -\left(1 - \frac{2M(U)}{R}\right) dU^2 - 2dUdR + R^2 d\Omega^2 \tag{2.6}
\]

where the “retarded mass” \( M(U) \), measured at future null infinity \( \mathcal{I}^+ \), may be an arbitrary, non-increasing function. For \( M = \) constant, the metrics (2.5) and (2.6) go over into the Schwarzschild metric in the ingoing, respectively outgoing Eddington-Finkelstein coordinates. The metrics remain simple if, instead of null coordinate \( V \), respectively \( U \), Eddington time-coordinate \( T^E = V - R \), respectively \( T^E = U + R \), is introduced. However, Vaidya’s metric becomes more complicated in the Schwarzschild-type coordinates [1].

In case of ingoing null dust the metric (2.5) together with the Einstein equations imply

\[
T^-_{\alpha \beta} = \frac{M_V}{4\pi R^2} k^-_{\alpha} k^-_{\beta} = l^-_{\alpha} l^-_{\beta} \tag{2.7}
\]

where \( M_V = \partial M/\partial V \geq 0 \),

\[
k^-_{\alpha} = -V_{\alpha}, \quad l^-_{\alpha} = -\left(\frac{M_V}{4\pi R^2}\right)^{1/2} V_{\alpha} \tag{2.8}
\]

Analogously, for outgoing null dust we get

\[
T^+_{\alpha \beta} = \frac{M_U}{4\pi R^2} k^+_{\alpha} k^+_{\beta} = l^+_{\alpha} l^+_{\beta} \tag{2.9}
\]

where \( M_U = \partial M/\partial U \leq 0 \),

\[
k^+_{\alpha} = U_{\alpha}, \quad l^+_{\alpha} = \left(-\frac{M_U}{4\pi R^2}\right)^{1/2} U_{\alpha} \tag{2.10}
\]

Consider a spherical layer of null dust imploding at \( V = 0 \) from \( \mathcal{I}^- \) (see Figs. 1 and 2). In region \( V < 0 \) spacetime is flat. For \( V \geq 0 \) the gravitational field is given by Vaidya’s metric (2.5), the energy-momentum tensor by Eq. (2.8). The mass function \( M(V) = 0 \) for \( V \leq 0 \), \( M(V) > 0 \) is non-decreasing for \( V > 0 \). The scalar invariant quadratic in the Riemann tensor is equal to \( 48M^2(V)/R^6 \) – a naked singularity may arise at \( R = 0 \), \( V \to 0_+ \). If there exists an outgoing radial null geodesic which starts at \( R = 0 \), \( V = 0_+ \) and ends at \( \mathcal{I}^+ \), the singularity is globally naked. The analysis of such geodesics is not easy – Kruskal-type (“double-null”) coordinates which would enable one a suitable description of both incoming and outgoing null geodesics can be introduced explicitly only when \( M(V) \) is a linear or exponential function [24]. In the linear case, \( M(V) = \alpha V, \alpha = \text{constant}, \)
lies below the outer horizon. We refer a reader in particular to Fig. 3 in [20] in which – as in our Fig. 3 in the following – there are four regions with different values of the mass parameter outside cross-flowing streams. Poisson and Israel show that the mass parameter inflates to classically arbitrarily large values on the Cauchy horizon inside the black hole. The timelike Reissner-Nordström singularity, which is locally naked, is thus probably ‘converted’ into a null or spacelike singularity. The mass inflation and the instability of the Cauchy horizon inside black holes have been also studied by other authors [27].

The mass inflation represents another classical effect which, in a quantum context, requires a canonical theory for cross-streaming null dust flows. We now turn to its formulation.

III. CANONICAL FORMALISM FOR SPHERICALLY SYMMETRIC SPACETIMES WITH TWO-COMPONENT NULL DUST

We shall start from the geometrodynamical approach to a spherically symmetric geometry as described in [11]. Up to very few exceptions (noted in the following) we adopt the same notation and natural units $G = c = 1$.

In geometrodynamics, a spherically symmetric spacetime is generated by evolving a spherically symmetric three-dimensional Riemannian space $\Sigma$ with the line element described by two functions, $\Lambda(r)$ and $R(r)$, of a radial variable $r$:

$$ \text{d}s^2 = \Lambda^2(r)\text{d}r^2 + R^2(r) \text{d}\Omega^2. \quad (3.1) $$

Here $\text{d}\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, $R(r)$ is the curvature radius of the 2-sphere $r = \text{constant}$; under transformations of $r$, $R(r)$ behaves as a scalar, $\Lambda(r)$ as a scalar density. The range of the radial variable $r$ depends on the problem we wish to discuss. If null dust falls into a primordial Schwarzschild black hole (which thus existed before an "irradiation" by dust), variable $r \in (-\infty, +\infty)$ covers both ("right" and "left") infinities, whereas if an infalling "ball" of null dust creates a Schwarzschild black hole, the point $r = 0$ is the center of the ball, and $r \in < 0, +\infty)$. As discussed in the Introduction, our main interest will be in treating the thick shell of null dust falling in or out with Minkowski space inside. Then again $r \in < 0, +\infty)$. Clearly, appropriate boundary conditions must be formulated and boundary terms analyzed depending on the physical situation considered.

If a spherically symmetric spacetime is foliated by spherically symmetric leaves $\Sigma$, labelled by a time parameter $t \in \mathbb{R}$, the metric functions $\Lambda$ and $R$ depend also on $t$. The leaves are related by the lapse function $N(t,r)$, and by the only non-vanishing component of the shift vector $N^\nu(r,t)$. Modulo boundary terms, the vacuum dynamics of the spherically symmetric gravitational field follows from the canonical form of the ADM action, integrated over $\theta$ and $\varphi$. It reads (cf. [11], Eqs. (31)-
where the gravitational super-Hamiltonian
\[ H^G = R^{-1} P_R P_A + \frac{1}{2} \left( R^2 \Lambda^2 + \Lambda^{-1} R R' \right) - \Lambda^{-2} R R' \Lambda' + \frac{1}{2} \Lambda^{-1} R^2 - \frac{1}{2} \Lambda, \] (3.3)
and supermomentum
\[ H^G = P_R R' - \Lambda P_A', \] (3.4)

For \( W_i \), we have \( \bar{W}_i = (\partial Z_i / \partial \bar{Z}^i) W_j \), so it transforms as a covector under the transformation (3.12). The Lagrangian is manifestly invariant. Observe that the transformation group is \( \text{Diff} \Sigma \), an infinite-dimensional group, but it cannot be considered as a gauge group. The transformations (3.12) are “rigid” or “global” in the sense that they are independent of spacetime points.

In Ref. 3, the transition to the Hamiltonian action for the null dust is performed with the result
\[ S^{ND} = \int dt \int d^3x (P_k \dot{Z}^k - NH^{ND} - N^a H_a^{ND}), \] (3.13)
where
\[ H^{ND} = \sqrt{g} h_a^{ND} h_b^{ND}, \] (3.14)
and
\[ H_a^{ND} = P_k Z^k_a; \] (3.15)

the variable \( P_k \) is the canonically conjugate momentum to \( Z^k \) and it transforms as a covector with respect to (3.12); \( g^{ab} \) is the inverse of the metric \( g_{ab} \) induced on the leaves \( \Sigma \) by the spacetime metric.

During the transition to Eqs. (3.13)–(3.16), important relations have been obtained and used. Some of them cannot be derived from the reduced action (3.13) any more. Thus, it has to be supplied by the formula for the vector field \( l^a \) which follows from the Hamiltonian equations (see Ref. 3, Eqs. (6.29)–(6.31); notice that on the right-hand side of Eq. (6.30) the sign should be “+”):
\[ l^a = \sqrt{W} h^a - g^{ab} P_k Z^k_b \sqrt{|g|} Y^a, \]

where \( n^a \) is the unit future-oriented normal vector to the foliation and \( Y^a_\alpha \) is determined by the description \( Y^a = Y^a(x^1, x^2, x^3) \) of the foliation in the coordinates \( Y^a_\alpha \) of the spacetime and \( x^k \) of the foliation surfaces. The abbreviations
\[ W = \sqrt{g^{ij} P_i P_j / |g|}, \quad g^{ij} = g^{ab} Z^a_i Z^b_j, \quad |g| = \det g_{ab}, \]
are used; \( g^{ij} \) is the induced metric on \( \Sigma \) expressed in the basis \( Z^a_i \).

To reduce action (3.13) by spherical symmetry, we choose
\[ (Z^1, Z^2, Z^3) \equiv (\Phi, \theta, \varphi), \]
where \( \Phi \) labels the comoving spherical surfaces of the dust while \( \theta \) and \( \varphi \) mark the radial lines of the dust. From the symmetry, we have \( \theta = \varphi = \theta' = \varphi' = 0 \); we assume that the spacetime coordinates \( \theta \) and \( \varphi \) are identical with the dust coordinates. The conjugate momenta are (cf. Eq. (6.18) of 3)
\[ (P_1, P_2, P_3) \equiv (\Pi, 0, 0). \]
For the metrics, we have
\[ ds^2 = -\left[N^2 - \Lambda^2(N^r)^2\right]dt^2 + 2\Lambda^2N^r dt dr + \Lambda^2 dr^2 + R^2 d\Omega^2 \]
and
\[ (N_1, N_2, N_3) = (N^r, 0, 0) . \]
Then,
\[ g_{ij} = \text{diag}(\Lambda^2 \Phi_r^2, R^2, R^2 \sin^2 \theta) , \]
\[ W = \frac{|\Pi \Phi|}{\Lambda \sqrt{|g|}} , \]
and
\[ l^n = \frac{\sqrt{|\Pi \Phi|}}{\Lambda R} \left(n^n - \frac{\epsilon}{\Lambda} Y^n\right) , \]
where \( \epsilon = \text{sgn}(\Pi \Phi) \) describes the two possibilities of radially out- and ingoing dust. Thus, we obtain after integration over the angles:
\[ S^{ND} = \int dt dr \left[ \Pi(\Phi - \frac{\epsilon N}{\Lambda} + N^r \Phi) \right] . \]
As \( g_{00} < 0 \), we must have \( N/\Lambda > N^r \). If \( \Phi \) is chosen to be positive, we must have \( \epsilon \Phi > 0 \). For ingoing dust, \( \Phi > 0 \), while for the outgoing one, \( \Phi < 0 \). Hence, \( \epsilon = +1 \) for the in- and \( \epsilon = -1 \) for the outgoing dust.

The generalization from one to several dust components that do not directly interact with each other is simple: several terms in the action similar to the one-component action must be added. This, for two components, leads to the null-dust action of the form:
\[ S_{2}^{ND} = \int dt dr \left[ \Pi_+ \Phi_+ + \Pi_- \Phi_- \right. \]
\[ \left. - \frac{N}{\Lambda} (|\Pi_+ \Phi'_+| + |\Pi_- \Phi'_-|) - N^r (|\Pi_+ \Phi'_+| + |\Pi_- \Phi'_-|) \right] . \]
The total action is
\[ S_2 = \int dt dr (P_\Lambda \dot{\Lambda} + P_R \dot{R} + \Pi_+ \dot{\Phi}_+ + \Pi_- \dot{\Phi}_- - N \dot{H} - N^r \dot{H}_r) , \]
where
\[ H = \frac{P_R P_\Lambda}{R} + \frac{\Lambda P_\Phi^2}{2 R^2} + \frac{RR'' - RR'\Lambda}{\Lambda^2} + \frac{R^2}{2\Lambda} - \frac{\Lambda}{2} \]
\[ + \frac{|\Pi_+ \Phi'_+| + |\Pi_- \Phi'_-|}{\Lambda} , \]
\[ H_r = P_R R' - \Lambda P_\Phi' + \Pi_+ \Phi'_+ + \Pi_- \Phi'_- . \]
Eq. (3.19) must be written for each dust component defining in our case the vector fields \( \ell^\alpha_+ \) and \( \ell^\alpha_- \), so that
\[ T^{\alpha \beta} = \frac{1}{4\pi} \left( \ell^\alpha_+ \ell^\beta_+ + \ell^\alpha_- \ell^\beta_- \right) , \]
the total stress-energy tensor is the sum of the component terms because the total null-dust action is the sum of two terms of the form (3.10).

IV. KUCHAŘ TRANSFORMATION

The gravitational part of constraint (3.20) can be simplified by a canonical transformation that replaces \( \Lambda \) by the quasilocal mass \( M \) as it has been first shown by [11]. One way to define the mass is to use the coordinates introduced by Bardeen [12]. The metric reads
\[ ds^2 = -Fe^{2\psi} dV^2 + 2e^{\psi} dV dR + R^2 d\Omega^2 \]
in the advanced Bardeen coordinates (ABC), and
\[ ds^2 = -Fe^{2\psi} dU^2 - 2e^{\psi} dU dR + R^2 d\Omega^2 \]
in the retarded Bardeen coordinates (RBC). Here, we use the abbreviation
\[ F = 1 - 2M/R , \]
and the pairs of functions \( M(V, R), \psi(V, R) \) or \( M(U, R), \phi(U, R) \) specify the geometry. The function \( M \) is a scalar field (it is the quasilocal mass of the spherically symmetric spacetimes) while \( \psi \) and \( \phi \) are potentials determined up to the transformation
\[ \psi \mapsto \psi + \ln v^\prime , \quad \phi \mapsto \phi + \ln u^\prime , \]
because the null coordinates \( U \) and \( V \) are determined up to
\[ V \mapsto v(V) , \quad U \mapsto u(U) , \]
\( u \) and \( v \) being two arbitrary functions of one variable (the prime denoting the ordinary derivative of a function of one variable). We shall work with both ABC and RBC but derive our formulae explicitly only for ABC.

Since we foliate the spacetime by spherically symmetric spacelike leaves \( \Sigma \), labelling the leaves by a time parameter \( t \in \mathbb{R} \), and use a radial label \( r \) on the leaves, we can substitute the foliation
\[ V = V(t, r) , \quad R = R(t, r) \]
into the line element (4.1) and find
\[ ds^2 = -(Fe^{2\psi} \dot{V}^2 - 2e^{\psi} \dot{V} \dot{R})dt^2 \]
\[ + 2(-Fe^{2\psi} \dot{V} \dot{V}' + e^{\psi} \dot{V} R' + V' \dot{R})dt dr \]
\[ + (-Fe^{2\psi} \dot{V}'^2 + 2e^{\psi} V' \dot{R})dr^2 + R^2 d\Omega^2 , \]
where the dots and commas again denote \( \partial/\partial t \) and \( \partial/\partial r \). Comparing (4.5) with the general ADM form of the spherically symmetric line element (3.10), we can express function \( \Lambda \), and the lapse and shift functions in the form:
\[ \Lambda^2 = -Fe^{2\psi} V^2 + 2e^{\psi} V' R' , \]
\[ N = \frac{e^{\psi} (\dot{V} R' - V' \dot{R})}{\Lambda} , \]
Returning now back to the expression (4.6), for the momentum \( P_\Lambda \), we may substitute for \( \Lambda, N, \) and \( N^r \) to obtain

\[
P_\Lambda = -\frac{R(Fe^{\psi}V' - R')}{\Lambda}.
\]  

(4.9)

From the last equation we can express \( V' \) regarding (4.6) in the form

\[
-e^{\psi}V' = -\frac{\Lambda P_\Lambda}{FR} + \frac{R'}{R}.
\]  

(4.10)

From Eq. (4.9), we can calculate function \( F \), Eq. (4.3), in terms of the canonical data:

\[
F = \left( \frac{R'}{N} \right)^2 - \left( \frac{P_\Lambda}{R} \right)^2.
\]  

(4.11)

Substituting for \( F \) into Eq. (4.10), we obtain \( e^\psi V' \) also entirely in terms of the canonical data. Finally, combining Eqs. (4.3) and (4.11) we can express the mass \( M \) as a function of the canonical data,

\[
M = \frac{1}{2} R^{-1} P_\Lambda^2 - \frac{1}{2} \Lambda^{-2} RR'^2 + \frac{1}{2} \Lambda R.
\]  

(4.12)

Notice that the equations (4.11) and (4.12) are of exactly the same form as in the vacuum Schwarzschild geometry, cf. [11]. They can also be interpreted in a similar way.

Clearly, the corresponding expressions in RBC are obtained by replacing

\[ e^\psi dV \rightarrow -e^\psi dU \]

everywhere, and by changing the sign of \( N \) (keeping \( N^r \) positive).

In vacuum spherically symmetric spacetimes one can find a canonical transformation to a new set of variables in terms of which the constraints remarkably simplify [11]. The transformation replaces the canonical variables \((\Lambda, P_\Lambda, R, P_R)\) by new canonical variables \((M, P_M, R, P_R)\) where \( M \) is given by Eq. (4.12),

\[
P_M = R^{-1} F^{-1} \Lambda P_\Lambda,
\]  

(4.13)

\[
R = R,
\]  

(4.14)

\[
P_R = P_R - \frac{1}{2} R^{-1} \Lambda P_\Lambda - \frac{1}{2} F^{-1} \Lambda P_\Lambda
\]

\[
- R^{-1} \Lambda^{-2} F^{-1} \left[ (\Lambda P_\Lambda)'(RR') - (\Lambda P_\Lambda)(RR')' \right],
\]  

(4.15)

where \( F \) is given in terms of “old” canonical variables by (4.11). This transformation was shown [11] to be canonical without employing constraints or dynamical equations. We can make easily sure that it remains a canonical transformation also on the extended phase space which includes the null dust canonical variables \( \Phi_+, \Pi_+ \) and \( \Phi_-, \Pi_- \).

The only point we have to check is that the boundary terms vanish as in the case of primordial vacuum Schwarzschild black holes analyzed in [11]. We consider the following arrangement: An outgoing thick shell of dust starts at the past singularity and runs through the spacetime to reach \( \mathcal{I}^+ \); another thick shell starts at \( \mathcal{I}^- \), crosses the first one and reaches the future singularity (Fig. 3). Under both shells, the spacetime is flat (Schwarzschild mass parameter \( M = 0 \)). In the future of both shells, we have Schwarzschild spacetime with mass \( M = M_3 \), in their common past \( M = M_1 \), and above both shells, \( M = M_2 \). The hypersurface \( \Sigma \) is assumed to run from the regular origin \( R = 0 \) in Minkowski spacetime to the \( i^0 \) of the Schwarzschild spacetime above the thick shells, as indicated in Fig. 3.

FIG. 3: The Penrose diagram of two crossing thick spherical layers (shaded) of null dust. One layer is collapsing from \( \mathcal{I}^- \) into a future Schwarzschild singularity of mass \( M_3 \). The other is expanding from the past singularity of mass \( M_1 \) into \( \mathcal{I}^+ \). Below the layers the spacetime is flat \( (M = 0) \), above the layers the Schwarzschild mass is \( M_2 \). A general Cauchy surface \( \Sigma \) going from the origin \( R = 0 \) through the crossing region to \( i^0 \) is shown. The past and future event horizons are indicated by dashed lines.

The ABC near \( R = 0 \) of \( \Sigma \) must be a transform of the advanced Minkowski coordinates \( V_M, R, \theta, \varphi \):

\[
V_M = v(V),
\]

where the function \( v \) could in principle be determined by boundary conditions at infinity and integration of Einstein’s equations through the thick shells. For \( V_M, R, \theta \) and \( \varphi \), the Bardeen potential \( \psi \) is zero, hence

\[
\psi(V) = \ln v'(V).
\]

At \( \mathcal{I}^- \), we assume \( V \) to be the advanced time. Thus, \( V \) is the Eddington-Finkelstein coordinate above and in
the past of the thick shells. Within the ingoing shell and above the outgoing one, $V$ is the advanced Vaidya coordinate. We can, therefore, set $\psi = 0$ in a neighbourhood of $I^-$, and in the part of $I^+$ lying above the outgoing shell.

We have thus to pay attention to the behaviour of the canonical variables $\Lambda$, $R$, $P_\Lambda$ and $P_R$ at the origin and at spatial infinity. Since we assume that thick shells of radiation do not extend to spatial infinities, there our fall-off conditions are identical to those considered in [11]. Therefore, we assume the canonical variables at infinities to behave as follows:

$$
\Lambda(t, r) = 1 + M_2(t)|r|^{-1} + O(|r|^{-(1+\varepsilon)}),
$$

$$
R(t, r) = |r| + O(|r|^{-\varepsilon}),
$$

$$
P_\Lambda(t, r) = O(|r|^{-\varepsilon}),
$$

$$
P_R(t, r) = O(|r|^{-(1+\varepsilon)}),
$$

(4.16)

where $O(r^{-\varepsilon})$ means that this term falls off as $r^{-\varepsilon}$, whereas its $j$-th spatial derivatives as $r^{-(\varepsilon+j)}$; the absolute values appear since in the case of primordial black holes $r \in \mathbb{R}$. The mass term can depend on $t$. The Schwarzschild time $T$ outside the thick shells along a slice $\Sigma$ is assumed to behave as

$$
T(r) = T_\infty(t) + O(r^{-1}),
$$

(4.17)

whereas the Lagrange multipliers in the action, the lapse and shift, satisfy conditions

$$
N = N_\pm(t) + O(|r|^{-\varepsilon}), \quad N^r = O(|r|^{-\varepsilon}).
$$

(4.18)

Since advanced time $V$ is given in terms of canonical coordinates by Eqs. (4.10), (4.11) with $\psi = 0$, the fall-off conditions (4.16) imply

$$
V = |r| + 2M_\infty \ln \left( \frac{r}{2M_\infty} \right) + T_\infty(t) + O(r^{-\varepsilon}),
$$

(4.19)

so that

$$
V' = 1 + 2M_\infty \left[ \frac{1}{|r|} + O(|r|^{-1-\varepsilon}) \right].
$$

(4.20)

At the origin of Minkowski space the boundary conditions are the same as in [11]. The 4-metric is flat, the 3-metric must be smooth. The hypersurfaces $\Sigma$ must meet the origin parallelly to $T = \text{constant}$ hypersurfaces to avoid conical singularities. Therefore, we require at $r \to 0$

$$
T'(r) = 0, \quad R = r + O(r^2),
$$

(4.21)

$$
V_M = T + R = T_0 + r, \quad V' = 1.
$$

(4.22)

The boundary terms, given by the expression (95) in [11],

$$
B = \frac{1}{2} R \delta R \ln \left| \frac{R'R' + \Lambda P_\Lambda}{R'R' - \Lambda P_\Lambda} \right|, \quad (4.23)
$$

evaluated at the boundaries, can also be written in terms of the advanced time variable $V$ and function $F$ as

$$
B = \frac{1}{2} R \delta R \ln \left| \frac{V'e^V FR - 2RR'}{e^V FR} \right|, \quad (4.24)
$$

where $V'e^V = V_M'$ at $R = 0$ and $V'e^V = V'$ at $R \to \infty$. Regarding the boundary conditions (4.10), (4.11), which imply $\delta R \sim O(|r|^{-\varepsilon})$ at infinity(ies) and $\delta R \sim O(r^2)$ at the origin, we easily see that the boundary terms vanish in all the situations we are dealing with.

The constraints (4.20) and (4.21) acquire now much simpler form:

$$
\Lambda H = -F^{-1}M'R' - FP_M P_R + \Pi_+ \Phi'_+ + \Pi_- \Phi'_-, \quad (4.25)
$$

$$
H_r = \mathcal{P}_R R' + P_M M' + \Pi_+ \Phi_+ + \Pi_- \Phi_-, \quad (4.26)
$$

where $F$ is again given by (4.14), but $M(r)$ and $R(r)$ are now understood as new canonical variables. (Notice that in [12] we scaled $H$ by $\Lambda$.) From Eq. (4.10) we find the advanced time coordinate in terms of new canonical variables also in a simplified form:

$$
-e^V V' = P_M - F^{-1} R'. \quad (4.27)
$$

As we have seen above, we may assume that $\Pi_- \Phi'_- > 0$ for the ingoing and $\Pi_+ \Phi'_+ < 0$ for the outgoing dust. We then combine Eqs. (4.24) and (4.26) to obtain:

$$
H_r \pm \Lambda H = (R' \mp FP_M)(\mathcal{P}_R \mp F^{-1} M') + \left\{ \begin{array}{ll}
2\Pi_- \Phi'_- \\
2\Pi_+ \Phi'_+
\end{array} \right\}, \quad (4.28)
$$

The total action (3.19)–(3.21) can now be written in the form

$$
S = \int dt dr \left[ P_M \dot{M} + \mathcal{P}_R \dot{R} + \Pi_+ \dot{\Phi}_+ + \Pi_- \dot{\Phi}_- - N_+ H_+ - N_- H_- - N(\infty) M(\infty) \right], \quad (4.29)
$$

where

$$
H_+ = (R' + FP_M)(\mathcal{P}_R + F^{-1} M') + 2\Pi_+ \Phi'_+,
$$

$$
H_- = (R' - FP_M)(\mathcal{P}_R - F^{-1} M') + 2\Pi_- \Phi'_-.
$$

(4.29)

and

$$
N_+ = \frac{1}{2} \left( N^r - \frac{N}{\Lambda} \right), \quad (4.30)
$$

$$
N_- = \frac{1}{2} \left( N^r + \frac{N}{\Lambda} \right). \quad (4.31)
$$

The action has also been supplied with the ADM boundary term.
V. REDUCTION BY A GAUGE CHOICE

In this section, we shall reduce the action (4.29) by choosing a gauge and by solving the constraints. The equations will simplify strongly, and this will help us to perform further transformations.

Let us try the gauge defined by the conditions
\[ r - R = 0 , \quad P_M = 0 \]  
(5.1)

We have to check that this is a regular gauge, that is, the gauge functions defined by the left-hand sides of Eqs. (5.1) must have vanishing Poisson brackets with the generators of gauge transformations. From the action (4.29), we obtain
\[ \dot{R} = (N_+ - N_-) F P_M + (N_+ + N_-) R' , \]  
(5.2)

\[ \dot{P}_M = [(N_+ + N_-) P_M]' + \frac{R'}{F} (N_+ - N_-)' \]
\[ + \frac{2}{R} (N_+ - N_-) \left( P_M P_R - \frac{R' M'}{F^2} \right) . \]  
(5.3)

For the Poisson brackets it holds, of course, that \( \{ R, N_+ + N_- H_+ \} = \hat{R} \) and \( \{ P_M, N_+ H_+ + N_- H_- \} = \hat{P}_M \), where \( N_+ H_+ + N_- H_- \) is a generator of gauge transformations for suitable values of \( N_+ \) and \( N_- \). Regarding Eqs. (5.2) and (5.3), we see that these will be zero everywhere at the gauge surface only if
\[ N_+ + N_- = 0 , \]
and
\[ \left( \frac{N_+ - N_-}{F} \right)' - \frac{2 M'}{RF^2} (N_+ - N_-) = 0 . \]
These equations are equivalent to
\[ N_+ = -K F \exp \left( 2 \int dR \frac{M'}{RF} \right) , \]  
(5.4)
\[ N_- = K F \exp \left( 2 \int dR \frac{M'}{RF} \right) , \]  
(5.5)

where \( K \) is a constant.

The constraints can be solved for \( \mathcal{P}_R \) and \( M' \):
\[ \mathcal{P}_R = -\Pi_+ \Phi'_+ - \Pi_- \Phi'_- , \]  
(5.6)
\[ M' = F (\Pi_+ \Phi'_+ + \Pi_- \Phi'_-) . \]  
(5.7)

Eq. (5.7) shows that \( M' \geq 0 \) for \( F > 0 \) while \( M' \leq 0 \) for \( F < 0 \). Moreover, the integrands in Eqs. (5.4) and (5.5) are always regular and non negative and they converge for our arrangement: at \( R = 0 \) and near \( R = \infty \), there is no dust. Hence, both \( N_+ \) and \( N_- \) will be non zero at \( R = \infty \), and so the generators for which the Poisson brackets of the gauge functions vanish are not gauge generators. Hence, the gauge is regular. Eqs. (5.4) and (5.5) determine also the values of \( N_+ \) and \( N_- \) for our gauge. Eqs. (4.30) and (4.31) then imply
\[ N' = 0 , \quad N = 2K \Lambda F \exp \left( 2 \int_0^R dx \frac{h(x)}{x} \right) , \]
where
\[ h(R) = -\Pi_+ \Phi'_+ + \Pi_- \Phi'_- . \]  
(5.8)

It follows that the gauge (5.1) is \( R \)-orthogonal one: the \( t \)-lines are the curves \( R = \) constant and the foliation is orthogonal to them. The foliation is determined by the second equation of (5.1), but the time function that is to be constant along the folios is not. We can choose the function, which will be called \( T \), by imposing the condition on \( N \): \( N(\infty) = 1 \). This determines the constant \( K \) so that the definitive \( N \) is
\[ N(R) = \Lambda F \exp \left( -2 \int_R^\infty dx \frac{h(x)}{x} \right) . \]  
(5.9)

Eq. (5.6) together with the Hamiltonian equation for \( M \) and the above relations for the lapse and shift yield:
\[ M = F N \Lambda (\Pi_+ \Phi'_+ + \Pi_- \Phi'_-) . \]  
(5.10)

Eq. (5.7) can be considered as a differential equation for \( M \) if the function \( F \) is written out according to Eq. (4.29). This equation can be easily integrated:
\[ M(R) = \int_0^R dx h(x) \exp \left( -2 \int_x^\infty dy \frac{h(y)}{y} \right) . \]  
(5.11)

The Hamiltonian of the reduced theory is the value of the boundary term in the action (4.29) that can be now calculated from the above equation with the result:
\[ H_{\text{red}} = \int_0^\infty dR h(R) \exp \left( -2 \int_R^\infty dx \frac{h(x)}{x} \right) . \]  
(5.12)

The reduced action is, therefore, given by
\[ S_{\text{red}} = \int_0^\infty dR \left( \Pi_+ \Phi'_+ + \Pi_- \Phi'_- + H_{\text{red}} \right) . \]  
(5.13)

The Hamiltonian (5.12) is non-local. We can obtain the canonical equations for the Hamiltonian \( H_{\text{red}} \) as follows. First, we vary \( H_{\text{red}} \) with respect to \( h(R) \):
\[ \delta H_{\text{red}} = \int_0^R dR \delta h(R) \exp \left( -2 \int_R^\infty dx \frac{h(x)}{x} \right) - 2 \int_0^\infty dR u'(R) v(R) , \]
where we have introduced the abbreviations
\[ u(R) = \int_0^R dx h(x) \exp \left( -2 \int_x^\infty dy \frac{h(y)}{y} \right) , \]
and

\[ v(R) = \int_R^\infty dx \frac{\delta h(x)}{x} . \]

It is easy to express \( u(R) \) in terms of \( N(R) \) and \( M(R) \) using relations (5.11) and (5.9):

\[ u(R) = \frac{N(R)M(R)}{\Lambda F} . \]

We can see immediately that \( u(0) = 0 \) and \( v(\infty) = 0 \) so that the last integral in \( \delta H_{\text{red}} \) is

\[ -2 \int_0^\infty dR u'(R)v(R) = -\int_0^\infty dR \frac{2N(R)M(R) \delta h(R)}{\Lambda F} . \]

Collecting all terms, we obtain

\[ \delta H_{\text{red}} = \int_0^\infty dR \delta h(R) \frac{N(R)}{\Lambda} . \quad (5.14) \]

The calculation of the canonical equations is now simple if we substitute for \( h(R) \) Eq. (5.10):

\[ \dot{\phi}_- - \frac{N}{\Lambda} \phi'_- = 0 \quad (5.15) \]

\[ \dot{\phi}_+ + \frac{N}{\Lambda} \phi'_+ = 0 \quad (5.16) \]

\[ \dot{\Pi}_- - \left( \frac{N}{\Lambda} \Pi_- \right)' = 0 \quad (5.17) \]

\[ \dot{\Pi}_+ + \left( \frac{N}{\Lambda} \Pi_+ \right)' = 0 \quad (5.18) \]

The canonical equations (5.15)–(5.18) together with Eqs. (5.9) and (5.11) determine the dynamics of the dust and the metric. For example, Eq. (5.10) can be derived from Eq. (5.11) by expressing \( M(R) \) first as a linear functional of \( h \), similarly to the calculation of \( \delta M \) in terms of \( \delta h \), and then by transforming \( h \) with the help of Eqs. (5.9)–(5.11).

The spacetime metric in the coordinates \( T, R, \vartheta \) and \( \varphi \) has the form, due to our choice of gauge,

\[ ds^2 = -N^2 dT^2 + \Lambda^2 dR^2 + R^2 d\Omega^2 . \quad (5.19) \]

The function \( \Lambda \) can be obtained from Eq. (4.10). Indeed, the gauge condition \( R = r \) and Eq. (4.8) with \( N^r = 0 \) imply

\[ -F e^\vartheta V' + 1 = 0 ; \]

this substituted into Eq. (4.6) leads to

\[ \Lambda = \frac{1}{\sqrt{F}} , \quad (5.20) \]

where \( M(R) \) is given by Eq. (5.11). Eqs. (5.9) and (5.20) determine the function \( N \).

\section{VI. ReDerivation of Bardeen’s Equations}

The variation principle (6.10) can be checked by deriving Einstein’s equations in Bardeen’s coordinates from it. The equations to be derived have the form (12):

\[ \frac{\partial \psi}{\partial \tau} = 4\pi RT_{RR} , \quad (6.1) \]

\[ \frac{\partial M}{\partial \tau} = -4\pi R^2 T_V^V , \quad (6.2) \]

\[ \frac{\partial M}{\partial V} = 4\pi R^2 T_V^R . \quad (6.3) \]

They are written in the ABC; analogous formulae hold in the RBC.

To derive Eqs. (6.1)–(6.3), we have first to express the stress–energy tensor using Eqs. (3.22) and (3.17). Transforming the metrics (4.1) and (4.2) into the coordinates \( T, R \) by the embedding relations \( V = V(T, R) \) and \( U = U(T, R) \), and using the \( R \)-orthogonality of our gauge, we obtain

\[ \dot{V} = \frac{N}{e^\vartheta \sqrt{F}} , \quad \dot{U} = \frac{N}{e^\vartheta \sqrt{F}} , \quad (6.4) \]

and

\[ V' = \frac{1}{e^\vartheta F} , \quad U' = -\frac{1}{e^\vartheta F} , \quad (6.5) \]

where the formulae for the RBC are also written out for later use; \( F \) is to be calculated from Eqs. (4.3) and (5.11). The \( R \)-orthogonality of the gauge implies, in the ABC,

\[ Y^a_{\varphi R} = (V', 1) , \quad n^a = \nu(1, 0) , \quad (6.6) \]

where \( \nu \) is a normalization factor. Eqs. (4.1) and (6.5) yield

\[ Y^a_{\varphi R} = \left( \frac{1}{e^\vartheta F}, 1 \right) , \quad n^a = \left( \frac{1}{e^\vartheta \sqrt{F}}, 0 \right) . \quad (6.6) \]

Substituting this into Eqs. (3.22) and (3.17), we obtain the right-hand sides of (6.1)–(6.3):

\[ T_{RR} = \frac{-1}{\pi R^2} \Pi_+ \Phi'_+ , \quad (6.7) \]

\[ T_V^V = \frac{F}{2\pi R^2} \Pi_+ \Phi'_+ , \quad (6.8) \]

\[ T_V^R = \frac{e^\vartheta F^2}{4\pi R^2} (\Pi_+ \Phi'_+ + \Pi_- \Phi'_- ) . \quad (6.9) \]

On the other hand, we also find the useful formula

\[ T_{\alpha\beta} = \frac{\Pi_+ \Phi'_+}{4\pi R^2} F^2 e^{2\vartheta} U_\alpha U_\beta , \]

and, in an analogous way,

\[ T_{\alpha\beta} = \frac{\Pi_- \Phi'_-}{4\pi R^2} F^2 e^{2\vartheta} U_\alpha U_\beta . \]
Next, the derivatives with respect to ABC that occur on the left-hand sides of Eqs. (5.20), (5.3) are to be expressed in terms of the primes and dots. Clearly, we have for any function $X(V, R)$:

$$
\dot{X} = \left( \frac{\partial X}{\partial V} \right)_R \dot{V}, \quad X' = \left( \frac{\partial X}{\partial V} \right)_R V + \left( \frac{\partial X}{\partial R} \right)_V \dot{R} ,
$$

so that Eqs. (6.4) and (6.5) imply

$$
\left( \frac{\partial X}{\partial V} \right)_R = \frac{e^{\psi} \sqrt{F}}{N} \dot{X}, \quad \left( \frac{\partial X}{\partial R} \right)_V = X' - \frac{1}{N \sqrt{F}} \dot{X} .
$$

(6.10)

Analogous formulae for the RBC and $X(U, R)$ read

$$
\left( \frac{\partial X}{\partial U} \right)_R = \frac{e^{\psi} \sqrt{R}}{N} \dot{X}, \quad \left( \frac{\partial X}{\partial U} \right)_U = X' + \frac{1}{N \sqrt{R}} \dot{X} .
$$

(6.11)

Now, Eqs. (5.7), (5.10), (6.8), (6.9) and (6.10) yield Eqs. (6.12)

To derive Eq. (6.14), we start from Eqs. (5.7), (5.10), (6.10) and obtain

$$
\dot{V} = \exp \left( -2 \int_R^\infty dx \frac{h(x)}{x} \right) .
$$

(6.12)

Taking primes of both sides and using Eq. (6.12) again gives

$$
e^{\psi} \dot{V}' + e^{\psi} \dot{V} = e^{\psi} \dot{V} \frac{2h(R)}{R} .
$$

To calculate $\dot{V}'$, we substitute for $V'$ from Eq. (6.3):

$$
\dot{V}' = \left( \frac{1}{e^{\psi} F} \right)' = - \dot{V} \frac{2 \dot{\psi} - \frac{\partial F}{\partial V}}{e^{\psi} F^2 + \frac{\partial F}{\partial V}} .
$$

Of course, from Eq. (4.133) we have

$$
\frac{\partial F}{\partial V} = -2 \frac{\partial M}{R \partial V} ,
$$

so that Eq. (6.12) together with the last three equations, entails

$$
\frac{\partial \psi}{\partial R} = \frac{2}{R} \left( \frac{h - 1}{e^{\psi} F^2} \frac{\partial M}{\partial V} \right) .
$$

Then, Eqs. (6.10), (6.10), (6.2) and (6.3) yield Eq. (6.14).

### VII. DIRAC OBSERVABLES

The equations of motion imply conservation laws, two for each dust component. We can use these conserved quantities as Dirac observables.

Consider Eq. (5.14); with the help of Eqs. (6.20) and (6.4), it can be rewritten in the form

$$
\left( \frac{\partial}{\partial T} - e^{\psi} F \frac{\partial}{\partial R} \right) \Phi = 0 .
$$

However, Eqs. (5.20), (6.1) and (6.3) also imply that

$$
\left( \frac{\partial}{\partial T} - e^{\psi} F \frac{\partial}{\partial R} \right) V = 0 .
$$

Hence, $\Phi_-$ is conserved along $V$–curves:

$$
\left( \frac{\partial \Phi}{\partial R} \right)_V = 0 .
$$

(7.1)

The same result, of course, follows from applying Eq. (6.10).

The second integral of motion is connected with the conserved current of Eq. (4.10) of $\Psi$. It is interesting that it remains conserved even under the conditions of cross-streaming dusts. It has the form

$$
\left( \frac{\partial(e^{\psi} F \Pi_-)}{\partial R} \right)_V = 0 .
$$

(7.2)

To prove the equation, we express the left-hand side as follows:

$$
\left( \frac{\partial(e^{\psi} F \Pi_-)}{\partial R} \right)_V = \left( \frac{\partial(e^{\psi} F)}{\partial R} \right)_V \Pi_+ + e^{\psi} F \left( \frac{\partial \Pi_-}{\partial R} \right)_V ,
$$

and calculate the second term by means of Eqs. (6.10), (6.20), (6.1) and (6.17):

$$
\left( \frac{\partial \Pi_-}{\partial R} \right)_V = \Pi_+ - 1 \left( e^{\psi} F \right) \frac{\partial \Pi_-}{\partial R} ,
$$

(7.4)

We also obtain from Eqs. (6.10) and (6.1) that

$$
\left( e^{\psi} F \right)' = \dot{V} \left( \frac{\partial(e^{\psi} F)}{\partial R} \right)_V .
$$

(7.5)

Then, Eq. (7.2) follows immediately.

Analogous conservation laws, this time along $U$–lines, hold for the outgoing dust component:

$$
\left( \frac{\partial \Phi_+}{\partial R} \right)_U = 0 , \quad \left( \frac{\partial(e^{\psi} F \Pi_+)}{\partial R} \right)_U = 0 .
$$

(7.3)

These conservation laws can be utilized for the construction of a nice set of Dirac observables for the two-component dust. Let us define two functions $\zeta_-(V)$ and $\mu_-(V)$ by

$$
\zeta_-(V) = \Phi_- , \quad \mu_-(V) = e^{\psi} F \Pi_- .
$$

as well as two functions $\zeta_+(U)$ and $\mu_+(U)$ by

$$
\zeta_+(U) = \Phi_+ , \quad \mu_+(U) = e^{\psi} F \Pi_+ .
$$

The original null-dust canonical variables can then be expressed by means of these functions as follows

$$
\Phi_- (R) = \zeta_-(V(R)) , \quad \Pi_- (R) = \mu_-(V(R)) V'(R) , \quad \Phi_+ (R) = \zeta_+(U(R)) , \quad \Pi_+ (R) = -\mu_+(U(R)) U'(R) .
$$

(7.4)

(7.5)
where we have substituted $V'$ or $-U'$ for $e^\psi F$ and $e^{\Phi} F$ with the help of Eq. (6.8).

Let us finally define the first pair $v(\zeta_-)$ and $u(\zeta_+)$ of the desired integrals of motion to be the functions inverse to $\zeta_-(V)$ and $\zeta_+(U)$:

$$
\zeta_-(v(x)) = x, \quad \zeta_+(u(x)) = x. \tag{7.6}
$$

The other pair $\mu_-(\zeta_-)$ and $\mu_+(\zeta_+)$ is then defined by

$$
\mu_-(\zeta_-) = \mu_-(v(\zeta_-)), \quad \mu_+(\zeta_+) = \mu_+(u(\zeta_+)). \tag{7.7}
$$

The two pairs form a complete set of Dirac observables. We assume that the material space $\mathcal{Z}$ with coordinates $\zeta_-$ and $\zeta_+$ consists of two components, each with topology $\mathbb{R}^1$:

$$
\zeta_- \in \mathbb{R}^1, \quad \zeta_+ \in \mathbb{R}^1.
$$

We also assume that the functions $v(\zeta_-)$ and $u(\zeta_+)$ are non-decreasing.

Relations (7.4) and (7.5) determine the transformation between $(\Pi_-, \Phi_-)$ and $(\mu_-, v)$ only implicitly, and so it is for the “$-$” variables. The function $V(T, R)$ that occurs in them is, in fact, a complicated functional of the canonical variables. The reason is that the transformation between the coordinates $(V, R)$ and $(T, R)$ is solution dependent, and it can, therefore, be written as

$$
V = V[\Pi_-(T, R), \Phi_-(T, R), \Pi_+(T, R), \Phi_+(T, R); T, R],
$$
or

$$
V = V[\mu_-(\zeta_-), v(\zeta_-), \mu_+(\zeta_+), u(\zeta_+); T, R],
$$

where the functions $\Pi_-(T, R), \Phi_-(T, R), \Pi_+(T, R)$ and $\Phi_+(T, R)$ in the first formula are considered as the initial datum for a solution (including the metric) at the Cauchy surface corresponding to a time value $T$.

Formulas (7.4) and (7.5) can, nevertheless, serve to find the transformation properties and physical meaning of the Dirac observables. Indeed, $\Pi_\pm$ is a density in the material space as well as at the Cauchy surface $T = \text{constant}$. Divided by $V'$ or $U'$, it becomes a scalar at the Cauchy surface and a density in the space with coordinates $V$ or $U$. Thus

$$
\mu_+(\zeta_+) u(\zeta_+) d\zeta_+, \quad \mu_-(\zeta_-) \psi(\zeta_-) d\zeta_-
$$

are scalars with respect to diffeomorphisms in $\mathcal{Z}_\pm$ and with respect to general transformations of the coordinates $U$ and $V$.

Let us express the total energy $M$ in terms of the Dirac observables. Eqs. (5.3), (6.3), (7.4) and (7.5) yield

$$
M = \int_0^\infty dR F \left( -\Pi_+ \Phi'_+ + \Pi_- \Phi'_- \right)
= \int_0^\infty dR \frac{e^{-\phi} \Pi_+ \Phi'_+}{U'} + \int_0^\infty dR \frac{e^{-\psi} \Pi_- \Phi'_-}{V'}
= \int_0^\infty dR e^{-\phi} \mu_+ d\zeta_+ + \int_0^\infty dR e^{-\psi} \mu_- d\zeta_-, \quad
$$

and we obtain:

$$
M = \int_{z_+} d\zeta_+ e^{\phi} \mu_+ + \int_{z_-} d\zeta_- e^{-\psi} \mu_- + \int_{z_0} d\zeta_0. \tag{7.8}
$$

With the correcting factors $e^{-\phi}$ or $e^{-\psi}$, the integrands are just densities in the material space and the energy is a scalar. However, the explicit dependence of $\phi$ or $\psi$ on the Dirac observables is difficult to calculate.

The physical meaning of the quantities $\mu_\pm(\zeta_\pm)$ can be made clear if we consider the arrangement of two crossing thick shells, Fig. 3. In a neighbourhood of $\mathcal{I}^+$ that is intersected only by the outgoing thick shell, the solution can be written in the form of Vaidya metric that is given by Eq. (1.2) with $\phi = 0$ and $M = M(U)$. Then, Eq. (7.4) implies

$$
\frac{dM}{dU} U = -F \Pi_+ \Phi'_+.
$$

Using Eqs. (6.5) and (7.5) with $\phi = 0$, we have

$$
\mu_+ = \frac{dM}{dU} \frac{d\zeta_+}{d\zeta}, \quad \mu_- = \frac{dM}{d\zeta_-}
$$

and, similarly,

$$
\mu_+ = \frac{dM}{d\zeta_+}
$$

near $\mathcal{I}^-$ for the ingoing thick shell. Hence, $\mu_\pm(\zeta_\pm)$ are asymptotic energy densities in the material space. The corresponding variables $u(\zeta_+)$ and $v(\zeta_-)$ must then be defined by the retarded or advanced times $U$ at $\mathcal{I}^+$ and $V$ at $\mathcal{I}^-$; $u(\zeta_+)$ is the retarded time at which the material sphere $\zeta_+$ arrives at $\mathcal{I}^+$ and similarly $v(\zeta_-)$ is the advanced time at which the material sphere $\zeta_-$ starts at $\mathcal{I}^-$.

These simple results suggest that we may obtain more explicit formulae for the special case of one-component dust. This will be shown in the next section.

**VIII. VAIKYA SOLUTION**

Let us set $\Pi_+ = 0$ so that there is only the ingoing component of null dust; from now on, we shall leave out the index “$-$”. The case with only the outgoing component can be treated in an analogous way.

Eqs. (6.1) and (6.7) then imply that $\psi = \psi(V)$ and so $\psi$ can be transformed away by a suitable choice of coordinate $V$. Moreover, Eqs. (6.2) and (6.8) show that $M = M(V)$. Thus, we end up with the Vaidya metric.

The canonical theory of the ingoing dust can start with the action (4.20) written out for $\Pi_+ = 0$:

$$
S = \int dt dr \left[ P_M M + P_R \dot{R} + \Pi \dot{\Phi}
- N_+ H_+ - N_- H_- - N(\infty)M(\infty) \right], \tag{8.1}
$$
where

\[ H_+ = (R' + FP_M)(P_R + F^{-1}M') \]  \hspace{1cm} (8.2)

\[ H_- = (R' - FP_M)(P_R - F^{-1}M') + 2\Pi \Phi' \]  \hspace{1cm} (8.3)

The variables in the action (8.1) can be transformed to embeddings and Dirac observables \((\text{Kuchař decomposition})\) as follows.

Since Eqs. (4.10) and (4.27) together with \(\psi = 0\) imply

\[ FP_M + R' = \frac{\Lambda^2}{V} \]  \hspace{1cm} (8.4)

the first factor on the right-hand side of Eq. (8.2) is generally non-zero. Hence the factor can be included in the Lagrange multiplier which we shall call \(M\): (a primordial Schwarzschild black hole.)

The constraint equation \(H_+ = 0\) implies \(P_R + F^{-1}M' = 0\); substituting this back into the constraint \(H_- = 0\), we get

\[ \left( \frac{P_R - R'}{F} \right) M' + \Pi \Phi' = 0 \]  \hspace{1cm} (8.6)

An equivalent action can thus be written as

\[ S = \int dt \int_0^\infty dr \left[ P_M \dot{M} + P_R \dot{R} + \Pi \dot{\Phi} \right. \]

\[ - 2N_- \left( (P_M - R'F^{-1})M' + \Pi \Phi' \right) \]

\[ - N^R (P_R + F^{-1}M') \]  \hspace{1cm} (8.7)

\[ - \int dt M_\infty \dot{T}_\infty \]

where \(F\) is determined in terms of canonical variables by Eq. (8.8). The Regge-Teitelboim boundary term in the action (8.8) acquires the form \(M_\infty \dot{T}_\infty\) after the parametrization at infinity, as it is analyzed in detail in \([11]\). (There would be another term of this type, corresponding to the left infinity, if null dust is collapsing into a primordial Schwarzschild black hole.)

The action (8.7) can be simplified by introducing new canonical momenta conjugate to \(M\) and \(R\) by

\[ \Pi_M = P_M - F^{-1}R' \]  \hspace{1cm} (8.8)

and

\[ \Pi_R = P_R + F^{-1}M' \]  \hspace{1cm} (8.9)

Substituting this into the Liouville form in (8.8), we find

\[ \int_0^\infty dr \left[ P_M \delta M + P_R \delta R + \Pi \delta \dot{\Phi} \right] = \int_0^\infty dr \left[ \Pi_M \delta M + \Pi_R \delta R + \Pi \delta \dot{\Phi} \right] \]

\[ + \int_0^\infty dr \left[ F^{-1}(R' \delta M - M' \delta R) \right] \]  \hspace{1cm} (8.10)

One can check by a short calculation that

\[ F^{-1}(R' \delta M - M' \delta R) = \]

\[ = \left( R\delta R \ln F \right)' - \delta \left( \frac{R' \ln F}{2} \right) \]  \hspace{1cm} (8.11)

so that the transformation defined by Eqs. (8.8) and (8.9) is indeed canonical, since with our boundary conditions the boundary terms resulting from the first term on the right-hand side of Eq. (8.11) drop out.

Action (8.7) then takes the form

\[ S = \int dt \int_0^\infty \left[ \Pi_M \dot{M} + \Pi_R \dot{R} + \Pi \dot{\Phi} \right. \]

\[ - 2N_- \left( \Pi_M M' + \Pi \Phi' \right) \]

\[ - N^R \Pi_R \]  \hspace{1cm} (8.12)

\[ - \int dt M_\infty \dot{T}_\infty \]

The next step is to transform the variables \(M\) and \(\Pi_M\) to \(V\) and \(\Pi_V\). Eqs. (4.27) with \(\psi = 0\) and (8.8) imply

\[ \Pi_M = -V' \]  \hspace{1cm} (8.13)

Let us define \(\Pi_V\) by

\[ M(r) = - \int_0^r d\tilde{r} \tilde{\Pi}_V(\tilde{r}) \]  \hspace{1cm} (8.14)

so that we have

\[ \int_0^\infty dr \Pi_M \delta M - M_\infty \delta T_\infty = \]

\[ = \int_0^\infty dr V' \int_0^r d\tilde{r} \delta \tilde{\Pi}_V(\tilde{r}) - M_\infty \delta T_\infty \]

\[ = \lim_{r \to \infty} \left[ V(r) \int_0^r d\tilde{r} \delta \tilde{\Pi}_V(\tilde{r}) \right] \]

\[ - \int_0^\infty dr V(r) \delta \tilde{\Pi}_V(r) - M_\infty \delta T_\infty \].

Taking into account the boundary condition (4.19) and Eq. (8.11), we have

\[ \int_0^\infty dr \Pi_M \delta M - M_\infty \delta T_\infty = \]

\[ = \int_0^\infty dr \tilde{\Pi}_V(r) \delta V(r) - \delta \int_0^\infty dr V(r) \tilde{\Pi}_V(r) \]

\[ - \lim_{r \to \infty} \left[ \left( r + 2M_\infty \ln \frac{r}{2M_\infty} \right) \delta M(r) \right] \]

\[ - T_\infty \delta M_\infty - M_\infty \delta T_\infty \].

Since sufficiently large \(r\) always lie outside the matter, we can set \(M(r) = M_\infty\) within the limit so that the limit can be written as follows:

\[ \lim_{r \to \infty} \delta \left( rM_\infty + 2 \int_0^{M_\infty} dM M \ln \frac{r}{2M} \right). \]
Discarding all such total differentials in the Liouville form, we obtain finally
\[
\int_0^\infty dr \Pi_M \delta M - M_\infty \delta T_\infty = \int_0^\infty dr \Pi_V(r) \delta V(r) .
\]
Hence, the transformation (8.13) and (8.14) leads to the action
\[
S = \int dt \int_0^\infty dr \left[ \Pi_V V + \Pi_R \dot{R} + \Pi \dot{\Phi} \right] - N^V (\Pi_V + \Pi \dot{\Phi}/V') - N R \Pi_R ,
\]
where \( N^V = 2V' N_\infty \). The action has been transformed so that the original ADM coordinates \( \Lambda(r) \) and \( R(r) \) have been replaced by the embeddings \( V(r) \) and \( R(r) \). We observe that the ADM boundary term does not occur explicitly any more in the action (8.15); this seems to be a general property of embedding variables (another example is given in Ref. [6]). It seems that the role of the asymptotic time can be taken over by the embedding variables. The constraints in the action (8.15) are not identical with the conjugate momenta \( \Pi_V(r) \) and \( \Pi_R(r) \) as it should be for an action after a Kuchař decomposition. The reason is, of course, that the remaining variables \( \Phi(r) \) and \( \Pi(r) \), which describe the physical degrees of freedom, are not Dirac observables and cannot have zero Poisson brackets with the constraints. In order to make the Kuchař decomposition complete, we have to pass to the Dirac observables \( \mu(\xi) \) and \( v(\xi) \).

Eq. (7.4) that relates \( \Pi, \Phi \) and \( V \) to \( \mu \) and \( v \) is valid for an arbitrary embedding \( V(r) \) and \( R(r) \):
\[
\Phi(r) = \zeta(V(r)) , \quad \Pi(r) = \mu(V(r)) V'(r) ,
\]
but the variable \( V(r) \) plays now a different role in the formalism: it is itself a canonical coordinate in the action (8.15). The problem that the function \( V(r) \) in Eq. (7.4) has not been explicitly known as a functional of canonical variables (of the reduced theory) has disappeared.

Variations of the functions \( \Phi(r) \), \( \zeta(v) \) and \( V(r) \) are related by Eq. (8.16):
\[
\delta \Phi = \delta \zeta + \frac{d \zeta}{dv} \delta V .
\]
Hence,
\[
\Pi \dot{\Phi} dr = \mu \dot{\zeta} V' dr + \Pi \frac{d \zeta}{dv} \dot{V} dr = \mu \dot{\zeta} + \frac{\Pi \dot{\Phi}}{V} \dot{V} dr .
\]
The variations of the functions \( \zeta(v) \) and \( v(\zeta) \) are related by Eq. (7.6),
\[
\delta \zeta + \frac{d \zeta}{dv} \delta v = 0 ,
\]
so that
\[
\Pi \dot{\Phi} dr + \Pi \dot{V} \dot{V} = -\mu \dot{v} \delta \zeta + \left( \Pi \dot{V} + \frac{\Pi \dot{\Phi}}{V} \right) \dot{V} dr .
\]
If we introduce the momentum \( \Pi_V \) by
\[
\Pi_V = \dot{V} + \frac{\Pi \dot{\Phi}}{V} ,
\]
then the transformation defined by Eqs. (8.13) and (8.17) is canonical. Substituting this into the action (8.15), we obtain finally
\[
S = \int dt \int_0^\infty \left[ - \int_{-\infty}^\infty d\zeta \mu(\zeta) \dot{\zeta} + \frac{\Pi \dot{\Phi}}{V} \dot{V} - N R \Pi_R - N^V \Pi_V \right] .
\]
This action is in the Kuchař form [3].

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