FREE AND INTERACTING 2-D MAXWELL FIELD THEORY
ON CONFORMALLY FLAT SPACE-TIMES

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Abstract

The free Maxwell field theory is quantized in the Lorentz gauge on a two dimensional manifold $M$ with conformally flat background metric. It is shown that in this gauge the theory is equivalent, at least at the classical level, to a biharmonic version of the bosonic string theory. This equivalence is exploited in order to construct in details the propagator of the Maxwell field theory on $M$. The expectation values of the Wilson loops are computed. A trivial result is obtained confirming in the Lorentz gauge previous calculations. Finally the interacting case is briefly discussed taking the Schwinger model as an example. The two and three point functions of the Schwinger model are explicitly derived at the lowest order on a Riemann surface.

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1. INTRODUCTION

In recent times the two dimensional gauge invariant field theories have been a source of many interesting developments [1]. This paper concerns the quantization of the Maxwell Field Theory (MFT) on a two dimensional Riemannian manifold admitting conformally flat background metrics. Classically the MFT, like the more complicated Yang-Mills theories defined on Riemann surfaces, is well understood [2]. The quantum case however is not so well known. Earlier results about the Yang-Mills theories on the sphere and on the cylinder are given in [3]. On a general Riemann surface the MFT has been quantized for example in [4,5], where however the equations of motion were used in order to compute the vacuum expectation values (VEV's) of the gauge invariant quantities like the Wilson loops (WL).

Here we perform a full quantization of the MFT on a general two dimensional complex manifold. As examples the complex plane, the disk and the closed and orientable Riemann surfaces of any genus are considered. We have found very convenient to quantize the theory in the Lorentz gauge. In this way, in fact, the transversal and longitudinal components of the fields are decoupled (see e.g. [6]) and the transversal part can be expressed in terms of a purely imaginary scalar field.

The material contained in this paper is organized as follows:
In Section 2 the MFT on a two dimensional manifold is introduced using a set of complex coordinates. After choosing a conformally flat background metric, the Lorentz gauge is imposed so that just the trasverse field can propagate. Exploiting the Hodge decomposition of a general one form in exact, coexact and harmonic forms, it is shown that the MFT in the Lorentz gauge is equivalent to a theory of scalar fields $\varphi(z, \bar{z})$ with higher order derivatives. The residual gauge invariance, characteristic of the Lorentz gauge fixing, is analyzed and the proper boundary conditions in order to get rid of it are assigned.

In Section 3 the flat case is investigated. The biharmonic equations of motion for the scalar fields $\varphi$ are solved on a disk and on the complex plane and the explicit form of the propagators is derived. The difficulties of defining the propagators on noncompact manifolds are pointed out [7].
In Section 4 the results of Section 3 are extended to the case of the Riemann surfaces. The VEV of a WL is computed giving the trivial result of [4,5].

In Section 5 the MFT interacting with a theory of massless fermions is studied in the path integral formalism. The explicit calculations of the physical amplitudes are simplified by the fact that the longitudinal gauge fields can be integrated away. The result of this integration is a δ-functional expressing the physical requirement of the conservation of the fermionic number in the amplitudes. The computation of the two and three point functions of the Schwinger model on a Riemann surface is performed at the lowest order of the perturbation theory.

Finally Section 6 is a discussion about the possibility of coupling the MFT to the \( b - c \) systems and to the massless scalar fields of string theory. A model with higher order derivatives and with nonAbelian gauge group of symmetry is introduced on the complex sphere.

2. MAXWELL FIELD THEORY ON A COMPLEX BACKGROUND

On a general two-dimensional complex manifold \( M \) we consider the following functional:

\[
S[J_\mu, A_\mu] = \frac{1}{4} \int_M d^2x \sqrt{-g} \left( F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu \right)
\]

(2.1)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \mu, \nu = 1, 2, \) is the usual field strength and \( g_{\mu\nu} \) is a background metric with Minkowski signature. Finally \( J_\mu(x) \) represents an external current. As an explicit example of two-dimensional manifold \( M \) we consider the closed and orientable Riemann surface of genus \( g, \Sigma_g \). On \( \Sigma_g \) we define a canonical set of independent homology cycles \( A_i, B_j, i, j = 1, \ldots, g \). The classical equations of motion of the fields \( A_\mu \) are:

\[
(\sqrt{-g})^{-1} \partial_\nu \left( \sqrt{-g} F_{\mu\nu}(x) \right) = J^\mu(x)
\]

(2.2)

If we put \( J^\mu(x) = 0 \) and consider the antisymmetric character of the field strength \( F^{\mu\nu}(x) \), eq. (2.2) implies (see e.g. [5]):

\[
F = \frac{1}{2} [\epsilon]_{\mu\nu} F^{\mu\nu} = \alpha
\]

(2.3)
where \( \alpha \) is an arbitrary constant and \([\epsilon]_{\mu\nu}\) is the Levi-Civita tensor. Due to the fact that \( F^{\mu\nu} \) has only one nonvanishing component in two dimensions, namely \( F^{12} \), eq. (2.3) implies that the MFT described by eq. (2.1) has just discrete degrees of freedom depending on the topology of the background. In [4] eq. (2.3) was imposed also in the quantized version of the MFT inserting by hand the constraint \( F - \alpha = 0 \) in the path integral. Also in [5] eq. (2.3) was used in the computation of the VEV of the Wilson loops. In this case the classical constraint (2.3) was exploited in computing the commutator of the anti-BRST operator with the ghost fields. However, eq. (2.3) does not hold in general when the fields are quantum operators even when the two dimensional space-time has the topology of \( R^2 \).

In order to keep our results as general as possible we will forget in the following eq. (2.3). Before to quantize the MFT, we perform a Wick rotation in the \( x^2 \) axis: \( x^2 \to ix^2 \) and we choose a system of local complex coordinates on \( M \):

\[
\begin{aligned}
    z &= x^1 + ix^2 \\
    \bar{z} &= x^1 - ix^2
\end{aligned}
\]

\[\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \quad (2.4)\]

In this paper we will express the fields and their functionals using a local system of coordinates like that of eq. (4). The final results, as the propagators and the solutions of the equations of motion, will be however globally defined.

The nonvanishing component of the field strength \( F_{\mu\nu} \) is:

\[ F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z \quad (2.5) \]

Clearly the complex field strength \( \frac{1}{2}F_{z\bar{z}}dz\wedge d\bar{z} \) does not need the introduction of the covariant derivatives \( \nabla_z \) and \( \nabla_{\bar{z}} \) in order to transform covariantly under general diffeomorphisms. In complex coordinates the action (2.1) becomes:

\[ S[A_z, A_{\bar{z}}; J_z, J_{\bar{z}}] = \frac{i}{4} \int_M \sqrt{g} \left( F_{z\bar{z}}^2 + J_z A^z + J_{\bar{z}} A_{\bar{z}} \right) \]

In this equation the factor \( i \) coming from the Wick rotation has been neglected. We will restore it below as the path integral quantization will be introduced. One can simplify the above equation as follows:

\[ S[A_z, A_{\bar{z}}, J_z, J_{\bar{z}}] = \frac{i}{4} \int_M \left[ \frac{1}{\sqrt{g}} F_{z\bar{z}}^2 + \sqrt{g} J_z A^z + \sqrt{g} J_{\bar{z}} A_{\bar{z}} \right] \quad (2.6)\]
At this point we choose a conformally flat metric:

\[ g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad g_{z\bar{z}} = \frac{1}{2} e^{2\sigma(z, \bar{z})} \]

(2.7)

where \( \sigma(z, \bar{z}) \) is a real function of \( z \) and \( \bar{z} \). The only effect in eq. (2.6) is that now \( \sqrt{g} = g_{z\bar{z}} \) on a local patch. Substituting eqs. (2.5) and (2.7) in (2.6) we get:

\[ S[A, J] = \frac{1}{2} \int_M d^2z g^{z\bar{z}} \left( \partial_z A_{\bar{z}} \partial_{\bar{z}} A_z - \partial_{\bar{z}} A_z \partial_z A_{\bar{z}} + g^{z\bar{z}} J_z A_{\bar{z}} + \text{c.c.} \right) \]

(2.8)

Note that the external current \( J_{\mu} \) has been rescaled by a constant factor \( 1/2 \). From eq. (2.8) the equations of motion \( \frac{\delta S}{\delta A_z} = \frac{\delta S}{\delta A_{\bar{z}}} = 0 \) for the fields \( A_z \) and \( A_{\bar{z}} \) become:

\[
\begin{align*}
\partial_z \left( -g^{z\bar{z}} \partial_{\bar{z}} A_z + g^{z\bar{z}} \partial_z A_{\bar{z}} \right) &= J_z \\
\partial_{\bar{z}} \left( -g^{z\bar{z}} \partial_{\bar{z}} A_z + g^{z\bar{z}} \partial_z A_{\bar{z}} \right) &= J_{\bar{z}}
\end{align*}
\]

(2.9)

Now it is possible to quantize the MFT using the path integral formalism:

\[ Z[J] = \int D A_z D A_{\bar{z}} e^{-S[A, J]} \]

(2.10)

where \( S[A, J] \) is the action of eq. (2.8).

The above partition function is not well defined due to the \( U(1) \) gauge group of symmetry:

\[
\begin{align*}
A_z &\to A_z + \partial_z \lambda(z, \bar{z}) \\
A_{\bar{z}} &\to A_{\bar{z}} + \partial_{\bar{z}} \lambda(z, \bar{z})
\end{align*}
\]

(2.11)

where \( \lambda(z, \bar{z}) \) is a real function of \( z \) and \( \bar{z} \). In order to get rid of the spurious integration over the gauge degrees of freedom we impose the Lorentz gauge. To this purpose we express the vector fields \( A_z \) and \( A_{\bar{z}} \) in terms of a complex scalar field \( \chi(z, \bar{z}) \) as follows:

\[
\begin{align*}
A_z(z, \bar{z}) &= \frac{1}{2} \partial_z \chi(z, \bar{z}) + A_z^{\text{har}} \\
A_{\bar{z}}(z, \bar{z}) &= \frac{1}{2} \partial_{\bar{z}} \chi(z, \bar{z}) + A_{\bar{z}}^{\text{har}}
\end{align*}
\]

(2.12)

In eq. (2.12) \( A_{\mu}^{\text{har}}, \mu = z, \bar{z} \), denotes the zero mode content of a general vector field \( A_{\mu} \). On a Riemann surface we have [8]:

\[
A_z^{\text{har}} = \sum_{i,k=1}^{g} 2\pi i u_k (\Omega - \bar{\Omega})_{kj}^{-1} (\omega_z)_j(z)
\]

where \( \Omega \) is the period matrix, \( u_k \) is a vector with \( g \) real components and \( (\omega_z)_i(z), i = 1, \ldots, g, \) is a basis of zero modes normalized as follows:

\[
\oint_{A_i} \omega_j dz = 0 \quad \Omega_{ij} = \oint_{B_i} \omega_j(z) dz
\]
Let us denote with $\Re[T]$ and $\Im[T]$ the real and imaginary parts of a tensor $T$ respectively. If we put

$$\rho = \Re[\chi] \quad \varphi = i\Im[\chi]$$

so that $\chi = \varphi + \rho$, eq. (2.12) turns out to be the Hodge decomposition of a general vector field in the Euclidean space:

$$A_z = \frac{1}{2} \partial_z \varphi + \frac{1}{2} \partial_z \rho + A_z^{\text{har}}$$

$$A_{\bar{z}} = -\frac{1}{2} \partial_{\bar{z}} \varphi + \frac{1}{2} \partial_{\bar{z}} \rho + A_{\bar{z}}^{\text{har}}$$

The advantage of the decomposition (2.14) in our case is that the vectors $A_z^T = \partial_z \varphi$ and $A_z^L = \partial_z \rho$ describe, with their complex conjugate partners, the transverse and longitudinal components of $A_\mu$ respectively. This can be easily seen in the Minkowski space, where eq. (2.14) becomes:

$$A_\mu = \frac{1}{2} \epsilon_{\mu\nu} \partial^\nu \varphi + \frac{1}{2} \partial_\mu \rho + A_\mu^{\text{har}}$$

In the Euclidean space it is possible to check that the free part of the action (2.8), i.e. with $J_\mu$ set to zero, does not depend on the longitudinal vector fields $A_z^L$ and $A_{\bar{z}}^L$. One of the important properties of the decomposition (2.14) is that, in the scalar product $(A_\mu, A_\nu) \equiv \int d^2 z A_z A_{\bar{z}}$, the three different components of $A_z$, namely $A_z^T$, $A_z^L$ and $A_z^{\text{har}}$ are mutually orthogonal. This orthogonality property will be extensively used in the following.

Now we are ready to exploit the substitution (2.14) in the path integral (2.10). As in the case of flat space [6], it turns out that the integration over the unphysical longitudinal fields $A_z^L$, $A_{\bar{z}}^L$ can be easily performed leaving us only with the transverse fields $A_z^T$, $A_{\bar{z}}^T$. We notice that this way of fixing the gauge is equivalent of having chosen the Lorentz gauge fixing. As a consequence it remains a residual gauge invariance proper of the Lorentz gauge that will be discussed later.

In order to proceed, we have to decompose also the external currents as we did for the vector fields in eq. (2.14):

$$\begin{cases} J_z^T = \partial_z J \\ J_{\bar{z}}^T = -\partial_{\bar{z}} J \end{cases} \quad \begin{cases} J_z^L = \partial_z \tilde{J} \\ J_{\bar{z}}^L = \partial_{\bar{z}} \tilde{J} \end{cases}$$

where $J$ is a purely imaginary scalar field and $\tilde{J}$ is purely real. At this point we substitute
eqs. (2.14)-(2.15) in eq. (2.10). The functional measure in the path integral becomes:

\[ DA_z DA_{\bar{z}} = D\varphi D\rho \prod \delta A_z \delta A_{\bar{z}} = D\varphi D\rho \det(2\partial_z \partial_{\bar{z}}) \quad (2.16) \]

After a simple calculation eq. (2.10) yields:

\[ Z[J, \tilde{J}] = \int D\varphi D\rho \det(\partial_z \partial_{\bar{z}}) e^{\int_M d^2z \left[ g^{\bar{z}z} \partial_{\bar{z}} \varphi \partial_z \varphi + \partial_z J_{\bar{z}} \varphi + \partial_{\bar{z}} J_z \partial_z \varphi + A_z^{\text{har}} A_{\bar{z}}^{\text{har}} + \text{c.c.} \right]} \]

Notice that there is no dependence on the harmonic sector since we are free to set \( J_z^{\text{har}} = J_{\bar{z}}^{\text{har}} = 0 \). The integrations over \( D\rho \) and over the discrete degrees of freedom coming from \( A^{\text{har}} \) is trivial and the result can be factored out together with \( \det(\partial_z \partial_{\bar{z}}) \) which, in this context, may be considered as a constant depending on the moduli. At the end we get:

\[ Z[J] = \int D\varphi e^{S[\varphi, J]} \quad (2.17) \]

where

\[ S[\varphi, J, \tilde{J}] = \int_M d^2z \left[ \frac{g^{\bar{z}z}}{4} \Delta_z \varphi \Delta_{\bar{z}} \varphi + \partial_z J_{\bar{z}} \varphi + \text{c.c.} \right] \quad (2.18) \]

In the following we will use the convenient notation \( \Delta_z = \partial_{\bar{z}} \partial_z \). The positive sign with which the action in eq. (2.17) is exponentiated is correct if we remember that the field \( \varphi \) is purely imaginary as it follows from its definition in eq. (2.13). Eq. (2.18) can be also expressed in terms of the transverse fields \( A_z^T(z, \bar{z}) \). After some integrations by parts in the sector with the external currents we have:

\[ Z[J_z^T] = \int DA_z^T \det(\partial_z) e^{\int_M d^2z \left[ \frac{g^{\bar{z}z}}{2} \Delta_z A_z^T \Delta_{\bar{z}} A_z^T + g_{\bar{z}z} J_z^T A_z^T + \text{c.c.} \right]} \quad (2.19) \]

Remembering that \( \partial_{\bar{z}} A_z^T = -\partial_z A_{\bar{z}}^T \) due to eq. (2.14), we can cast the action appearing in eq. (2.19) in the more usual form:

\[ S[A_z^T, J_z^T] = \int_M d^2z \left[ -\frac{g^{\bar{z}z}}{2} \partial_z A_z^T \partial_{\bar{z}} A_z^T + J_z^T A_z^T + \text{c.c.} \right] \]

Apparently, there is just one degree of freedom in eq. (2.19) due to the fact that \( A_z^T = A_{\bar{z}}^T \). However, as it happens in the case of Chern-Simons field theory [9], one can show that also the remaining degree of freedom disappears in the canonical quantization. The equations of motion of the fields \( A_z^T \) and \( \varphi(z, \bar{z}) \) are

\[ \partial_z g^{\bar{z}z} \partial_{\bar{z}} A_z^T = J_z^T \quad (2.20) \]
\[ \triangle_z g^{z\bar{z}} \triangle_z \varphi = \triangle_z J \quad (2.21) \]

Apart from the presence of the metric tensor sandwiched between the two D'Alembertians, (2.21) is a biharmonic equation. The presence of the metric is however unavoidable when one has to define the biharmonic equation on a general manifold. A manifold is in fact covered by many charts and without the metric the operator \( \triangle_z^2 \) is no longer invariant after a transformation of local coordinates.

In the next section we will need the propagator of the scalar fields \( G(z, w) = \langle \varphi(z) \varphi(w) \rangle \). \( G(z, w) \) satisfies the following equations:

\[
\begin{cases}
\triangle_z g^{z\bar{z}} \triangle_z G(z, w) = \delta_{z\bar{z}}^{(2)}(z, w) \\
\triangle_w g^{w\bar{w}} \triangle_w G(z, w) = \delta_{w\bar{w}}^{(2)}(z, w)
\end{cases} \quad (2.22)
\]

In eq. (2.22) we have not taken into account the possible zero modes of the operator \( \triangle_z g^{z\bar{z}} \triangle_z \) which will be discussed later. Since the field \( \varphi(z, \bar{z}) \) is purely imaginary, we should in principle put a minus sign in front of the \( \delta \)-functions of eq. (2.22). However our choice of sign is motivated by the fact that eventually we are interested only in the propagator of the physical fields \( A^T_z, A^T_{\bar{z}} \). Once we have the propagator of the scalar fields we can construct in fact also the propagator of the vector fields \( A^T_z \):

\[ < A^T_z (z, \bar{z}) A^T_w (w, \bar{w}) > = \partial_z \partial_w G(z, w) \quad (2.23) \]

It is easy to show that

\[ \partial_z g^{z\bar{z}} \partial_{\bar{z}} < A^T_z (z, \bar{z}) A^T_w (w, \bar{w}) > = -\delta_{z\bar{z}}^{(2)}(z, w) \]

noting that \( A^T_w (w, \bar{w}) = \partial_w \varphi(w, \bar{w}) \) and that the derivative \( \partial_w \) commutes with the metric \( g^{z\bar{z}} \). The desired minus sign in the above equation is produced remembering that locally one has:

\[ \partial_z \partial_{\bar{z}} G(z, w) = g_{z\bar{z}} \log |z - w|^2 \]

and that \( \partial_w \partial_{\bar{z}} \log |z - w|^2 = -\delta_{z\bar{z}}^{(2)}(z, w) \). The form of the propagator given in eq. (2.23) is useful in computing the amplitudes of the Wilson loops. However, the formula (2.23) is just formal, since we need still to specify also the boundary conditions of \( G(z, w) \). We remember in fact that the Lorentz gauge fixing, which in complex coordinates has the form
\( \partial_z A_z + \partial_{\bar{z}} A_{\bar{z}} = 0 \), allows a residual gauge invariance given by the transformations of the kind:

\[
\begin{align*}
A_T^z &\rightarrow A_T^z + \partial_z \rho \\
A_T^{\bar{z}} &\rightarrow A_T^{\bar{z}} + \partial_{\bar{z}} \rho
\end{align*}
\]  \hspace{1cm} (2.24)

where \( \partial_z \partial_{\bar{z}} \rho = 0 \). On a compact Riemann surface the only possible solution to this equation is \( \rho = \text{const.} \) The residual gauge invariance amounts in this case to the shift \( \varphi \rightarrow \varphi + \text{const.} \) As in the usual bosonic string theory, this is not a serious problem and can be solved choosing a normalization of the propagator \( G(z, w) \) at some point. Alternatively one can compute just the physical amplitudes which should be invariant under translations of the scalar fields. One example is provided by the Wilson lines:

\[
\exp \left[ ic \int_a^b A_\mu dx^\mu \right] = \exp \left[ ic (\varphi(b) - \varphi(a)) \right] \times \text{c.c.}
\]

In order to get rid of the residual gauge invariance on a noncompact manifold like the complex plane, one has to require boundary conditions like \( A_z = A_{\bar{z}} = 0 \) when \( z \to \infty \). This means that

\[
\partial_z G(z, w) = \partial_{\bar{z}} G(z, w) = 0
\]

at \( z = \infty \).

If the manifold has a boundary \( \Omega \), the boundary conditions should be chosen in such a way that no harmonic functions satisfying \( \Delta_z \rho = 0 \) are possible. On a disk, for example, this purpose is achieved requiring eq. (2.25) to be valid on the boundary. Finally, additional problems can arise in constructing the biharmonic correlation functions \( G(z, w) \) on a noncompact manifold \( M \) [7]. In certain cases the direct construction of the propagator \( < A_T^z A_T^w > \) in the following way is preferable:

\[
< A_T^z A_T^w > = \int_M d^2tg_{tt} (R_z(z, t)R_w(w, t))
\]

where \( \partial_z R_z(z, t) = \delta_{zz}^{(2)}(z, t) \).
3. THE MAXWELL FIELD THEORY ON A FLAT SPACE-TIME

We begin to study the theory of eq. (2.17) of complex scalar fields with higher derivatives. This theory is equivalent to the MFT after the Lorentz gauge fixing has been chosen. First of all we treat the simple case in which the topology of the 2D space-time is just that of the complex plane $\mathbb{C}$. In order to solve the free equations of motion (2.21), i.e. with $J_\mu = 0$, we consider the auxiliary scalar field

$$\tilde{\varphi}(z, \bar{z}) = g^{zz} \Delta_z \varphi(z, \bar{z})$$ (3.1)

$\tilde{\varphi}(z, \bar{z})$ satisfies an harmonic equation:

$$\Delta_z \tilde{\varphi}(z, \bar{z}) = 0$$

whose solutions are:

$$\tilde{\varphi}(z, \bar{z}) = i (u(z) + \bar{u}(\bar{z}))$$

$u(z) = \sum_{n=0}^{\infty} a_n z^n$ being an harmonic function. The factor $i$ in the above equation is necessary since $\tilde{\varphi}(z, \bar{z})$ is defined as a purely imaginary field from eq. (2.13). At this point we can invert eq. (3.1) and obtain the final solution of eq. (2.21):

$$\varphi(z, \bar{z}) = \int_p^z \int_{\bar{p}}^{\bar{z}} dt \bar{t} g_{\bar{t} t} \ i (u(t) + \bar{u}(\bar{t})) + i \xi(z, \bar{z})$$ (3.2)

where $\xi(z, \bar{z})$ is a real harmonic function and $p$ represents an arbitrary basepoint. If $g_{t\bar{t}} = 1$, then eq. (3.2) gives for $p = 0$:

$$\varphi(z, \bar{z}) = iz\bar{z}(h(z) + \bar{h}(\bar{z})) + i\xi(z, \bar{z})$$

This is the most general solution of the biharmonic equation in the flat case. Eq. (3.2) represents all the zero modes of the operator $\Delta_z g^{zz} \Delta_z$ on the complex plane with an arbitrary metric. Let us notice that, due to the presence of the metric, the equations of motion (2.22) are invariant under conformal transformations in the following sense. After a conformal transformation $z = z(w)$, we have $\Delta_z g^{zz} \Delta_z = \left| \frac{dw}{dz} \right|^2 \Delta_w g^{w\bar{w}} \Delta_w$ which is the same operator as before. However the new metric is now $g^{w\bar{w}}(w, \bar{w}) = |dw/dz|^2 g^{zz}(z, \bar{z})$. 
Therefore eqs. (2.17)-(2.18) do not describe a conformal field theory. This is in agreement with the fact that MFT is conformal only in four dimensions. One of the consequences is that the energy momentum tensor is not traceless. To show this, we make the substitutions (2.14) in the general expression of the energy momentum tensor of MFT on a curved background:

\[ T_{\mu\nu} = -F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^\rho{}^\sigma \quad (3.3) \]

The result is:

\[ T_{zz} = T_{\bar{z}\bar{z}} = 0 \quad T_{z\bar{z}} = g_{z\bar{z}} \Delta z \varphi \Delta z \varphi \]

Therefore the energy momentum tensor has a nonvanishing trace. Moreover it consists in a pure trace term.

Let us now compute the propagator \( G(z, w) \) from eq. (2.22). We remember that, on the complex plane, the \( \delta \) function has the following expression:

\[ \delta^{(2)}_{z\bar{z}}(z, w) = \Delta z \log |z - w|^2 \quad (3.4) \]

Substituting eq. (3.4) in eqs. (2.22) we get:

\[
\begin{align*}
\Delta z G(z, w) &= g_{z\bar{z}} \log |z - w|^2 \\
\Delta w G(z, w) &= g_{\bar{z}w} \log |z - w|^2
\end{align*}
\quad (3.5)
\]

First of all we derive the Green function \( G(z, w) \) on a disk \( B_\rho \) of radius \( \rho \). The strategy is to obtain the propagator on the complex plane in the limit \( \rho \to \infty \). At the boundary, \( |z| = \rho \), there are many possible boundary conditions for \( G(z, w) \) [7], for example \( \varphi = \Delta z \varphi = 0 \) or \( \varphi = \partial_n \varphi = 0 \), where \( \partial_n \) denotes the inner normal derivative. However, in our case, we have to demand that the gauge fixing condition is true also at the boundary. This implies

\[ \Delta z G(z, w) = \Delta w G(z, w) = 0 \]

at the boundary. Moreover it is easy to show that the residual gauge invariance (2.24) requires the stronger boundary conditions (2.25). The propagator of the scalar fields with only a singularity in \( z = w \) and fulfilling the boundary conditions (2.25) is the following:

\[ G(z, w) = |z - w|^2 \log \left| \frac{\rho(z - w)}{\rho^2 - \bar{w}z} \right|^2 + \frac{1}{2\rho} (|z|^2 - \rho^2) (|w|^2 - \rho^2) \quad (3.6) \]
Therefore, using eq. (2.23), it is possible to obtain the correct propagator of the transverse vector fields of the MFT in the Lorentz gauge. Unfortunately eq. (3.6) is valid only in the case in which the metric \( g_{z\bar{z}} = 1 \).

Let us now consider the complex plane. Naively one would perform the limit \( \rho \to \infty \) in eq. (3.6), but it is easy to check that this limit does not exist. The difficulties we encounter are part of a more general problem in defining the propagator \( G(z, w) \) with given boundary conditions on a noncompact manifold [7]. Here we try to find the propagator of the vector fields. From eq. (2.20) we need to solve the equation:

\[
\partial_z g^{z\bar{z}} \partial_{\bar{z}} < A_z(z, \bar{z}) A_w(w, \bar{w}) >= -\delta_{z\bar{z}}^{(2)}(z, w)
\] (3.7)

and simultaneously an analogous equation in \( w \). Due to the fact that on the complex plane the \( \delta \) function is given by \( \delta_{z\bar{z}}^{(2)}(z, w) = \Delta z \log |z - w|^2 \), the solution to eq. (3.7) is:

\[
<A_z(z, \bar{z}) A_w(w, \bar{w})> = \int_C d^2t \partial_w \log |t - w|^2 \partial_{z} \log |t - z|^2
\] (3.8)

We notice that the propagator (3.8) satisfies the correct boundary conditions since it goes to zero in the limit in which \( z \) or \( w \) go to infinity. On the contrary, the naive solution \( \partial_z \partial_w G(z, w) = \log|z - w|^2 \), with \( G(z, w) = |z - w|^2 \log|z - w|^2 - |z - w|^2 \), is not correct since it has the wrong boundary conditions in the above limit. However, since \( \partial_w \log |t - w|^2 \sim 1/(t - w) \), the integral in eq. (3.8) is not well defined. In fact

\[
\partial_w \log |t - w|^2 \partial_{z} \log |t - z|^2 \sim \frac{1}{t^2}
\]

when \( t \to \infty \). To solve this problem we add to eq. (3.8) an infinite constant which improves the convergence of the integral. The following Green function:

\[
<A_z(z, \bar{z}) A_w(w, \bar{w})> = \int_C d^2t \partial_w \log |t - w|^2 \partial_{z} \log |t - z|^2 - 2 \int_a^{+\infty} \frac{dx_2}{x_2} - 2 \int_a^{+\infty} \frac{dx_1}{x_1}
\]

with \( a > 0 \) being a real constant and where we have used the notation of eq. (2.4), leads to a well defined propagator.

Now we return to the MFT in order to compute the VEV’s of the gauge invariant quantities, i.e. the Wilson loops \( W(L) \), where \( L \) is a closed path on \( \mathbb{C} \). Taking into
account the fact that in the Lorentz gauge it remains just one degree of freedom $A_z^T$, we have:

$$W(L) = \exp \left[ -e^2 \oint_L \oint_L dzdw \langle A_z^T(z, \bar{z}) A_w^T(w, \bar{w}) \rangle \right]$$  \hspace{0.5cm} (3.9)

Due to the form of the propagator of the transverse fields given in eq. (2.23), the VEV of the Wilson loop on the disk is trivial, i.e. $W(L) = 1$. In the case of the complex plane the result is surely trivial if the radial metric in eq. (3.8) is defined with $\alpha < -2$. However, when $0 > \alpha \geq -2$, the integration in $d^2t$ does not commute with the integration over the Wilson loops so that it is not possible to perform the calculation of the VEV explicitly.

4. THE MAXWELL FIELD THEORY ON A RIEMANN SURFACE

As a first step we consider the MFT on a complex sphere $\mathbb{C}P_1$ of genus $g = 0$. With respect to the complex plane, the sphere $\mathbb{C}P_1 = \mathbb{C} \cup \{\infty\}$ includes also the point at infinity. The sphere $\mathbb{C}P_1$ is covered here by two open sets $U, U'$ containing the points 0 and $\infty$ respectively. Local coordinates on $U$ and $U'$ are $z$ and $z'$. When the two open sets overlap, i.e. $U \cap U' \neq \emptyset$, the two systems of coordinates are related as follows: $z = 1/z'$. Choosing the metric $g_{zz}$ on $U$ we have

$$g_{zz} = g_{zz'} |z\bar{z}|^{-2} = g_{zz'} |z'|^2$$  \hspace{0.5cm} (4.1)

In the sense explained in Section 2, all the classical equations of motion and the action (2.18) are invariant under a change of coordinates like that of eq. (4.1).

Let us solve the equations of motion (2.21). We proceed as in the case of the complex plane. On a sphere the only possible solution of the equation $\triangle_z \tilde{\varphi}(z, \bar{z}) = 0$ is $\tilde{\varphi}(z, \bar{z}) = i \tilde{\varphi}_0$, where $\tilde{\varphi}_0$ is a real constant. Therefore, inverting eq. (3.1) we get:

$$\varphi(z, \bar{z}) = i \int_{\bar{p}} \int_{\bar{p}} d\tau \tilde{\varphi}_0$$

If we set $g_{tt} = 1$ on $U$, $\varphi(z, \bar{z})$ is a linear combination of the following independent zero modes, $z\bar{z}$, $z + \bar{z}$ and 1. At $z = \infty$ and supposing that $g_{tt} = 1$, the metric becomes $g_{tt'} = (t'^2 - p'^2)^{-2}$ and the above zero modes take the form $|z'|^{-2}$, $1/|z'|$, $1/|z'|$, 1. Due to the
presence of the metric in the operator $\triangle z g^{z\bar{z}} \triangle z$ these functions are zero modes despite of their singularities at $z = \infty$. It is easy to check that no other zero modes are possible. Using the same procedure as we did in the cases of the disk and of the complex plane, it is also possible to derive an expression of the Green function $G(z, w)$ on the complex sphere. The only residual gauge invariance is now given by the translation of the scalar field $\varphi(z)$ by a constant since the equation $\triangle z \rho$ has just the trivial solution. Moreover the $\delta$-function is expressed in terms of the prime form

$$E(z, w) = \log \left| \frac{z - w}{\sqrt{dz\bar{dw}}} \right|^2$$

in the following way:

$$\delta_{z\bar{z}}^{(2)}(z, w) = \triangle z \log |E(z, w)|^2$$

The prime form of eq. (4.2) does not have logarithmic singularities when $z, w \to \infty$ separately. The only possible singularity occurs at $z = w$. From eq. (4.3) we get:

$$G(z, w) = \int_{\mathbb{CP}_1} d^2tg_{ti} \log |E(t, w)|^2 \log |E(t, z)|^2$$

A metric on $\mathbb{CP}_1$ leading to a well defined integral is for example

$$g_{z\bar{z}}dzd\bar{z} = \frac{dzd\bar{z}}{(1 + z\bar{z})^2}$$

Again the computation of the Wilson loop in eq. (3.8) leads to the trivial result $W(L) = 1$. Finally we treat the MFT on a general Riemann surface $\Sigma_g$ of genus $g$. As in the case of the sphere, the only possible zero modes of eq. (2.21) are of the form:

$$\varphi(z, \bar{z}) = i \int_{\bar{p}} \int_{p} dtd\bar{t}g_{ti} \tilde{\varphi}_0$$

In computing the propagator $G(z, w)$ we use the notation of [10]. Let us consider the prime form:

$$E(z, w) = \frac{\theta \left[ \frac{\vec{a}_0}{\vec{b}_0} \right] (\int_{\bar{w}}^{\bar{z}} \omega \bar{\omega})}{h(z)h(w)}$$

where $\vec{a}_0$, $\vec{b}_0$ describe the period of an odd spin structure, $\omega \bar{\omega}$ is the vector $(\omega_1, \ldots, \omega_g)$ having as components the holomorphic differentials $\omega_i(z)$ and finally:

$$h(z) = \sum_{i=1}^{g} \omega_i(z) \partial_{e_i} \theta \left[ \frac{\vec{a}_0}{\vec{b}_0} \right] (e_i)|_{e_i=0}$$
A representation of the $\delta$-function on a Riemann surface is given by:

$$\delta^{(2)}_{z\bar{z}}(z, w) = \Delta_z \left[ \log |E(z, w)|^2 + \frac{\pi}{2} R(z, w) \right]$$

(4.7)

where

$$R(z, w) = \sum_{i,j=1}^{g} \left( \omega(z) - \omega(\bar{z}) \right)_i \mathfrak{Im} [\Omega]^{-1}_{ij} \left( \omega(w) - \omega(\bar{w}) \right)_j$$

(4.8)

and $\Omega$ is the period matrix defined in Section 2. Let us notice that the function

$$K(z, w) = \log |E(z, w)|^2 + \frac{\pi}{2} R(z, w)$$

(4.9)

is invariant under modular transformations and it is singlevalued around the nontrivial homology cycles. It is easy to see that the propagator of the scalar fields is provided by:

$$G(z, w) = \int_{\Sigma_g} d^2t g_{\bar{t}t} K(z, t) K(w, t)$$

(4.10)

First of all, for a sufficiently regular metric $g_{\bar{t}t}$, the integral of eq. (4.12) is well defined and does not diverge. In fact $\Sigma_g$ is compact and can be represented as a polygon whose sides are the homology cycles. Inside this polygon the integrand of eq. (4.10) is a square integrable function apart from the singularities at the points $t = z, t = w$. Hence it fulfills the existence conditions given in [7]. Moreover, remembering that

$$\Delta_z K(z, t) = \delta^{(2)}_{z\bar{z}}(z, t) - \frac{g_{\bar{z}z}}{N}$$

where

$$N = \int_{\Sigma_g} d^2t g_{\bar{t}t}$$

we have:

$$\Delta_z G(z, w) = g_{\bar{z}\bar{z}} K(z, w) - \int_{\Sigma_g} d^2t g_{\bar{t}t} K(w, t) g_{\bar{z}\bar{z}}$$

Applying the operator $\Delta_z g_{\bar{z}\bar{z}}$ on the above tensor we have the desired result:

$$\Delta_z g_{\bar{z}\bar{z}} \Delta_z G(z, w) = \delta^{(2)}_{z\bar{z}}(z, w) - \frac{g_{\bar{z}\bar{z}}}{N}$$

The term $g_{\bar{z}\bar{z}}/N$ appears due to the residual gauge invariance $\varphi \rightarrow \varphi + \text{const.}$ as discussed in section 2. Computing the VEV of a WL on a Riemann surface one finds that the contributions of the transverse gauge fields is trivial due to eq. (2.23). It remains the integration over the harmonic pieces, which yields a vanishing result as explained in [4].
5. QUANTUM ELECTRODYNAMICS IN TWO DIMENSIONS

In the following we consider the two dimensional MFT discussed above coupled to massless fermions (Schwinger model [11]):

\[ S_{\text{QED}^2}[A, \bar{\psi}, \psi, J, \xi] = \int_M d^2 z \left[ \frac{g_{z \bar{z}}}{2} F_{z \bar{z}} F^{z \bar{z}} + \bar{\psi}_\theta (\partial_z + A_z) \psi_\theta + \bar{\psi}_\bar{\theta} (\partial_{\bar{z}} + A_{\bar{z}}) \psi_{\bar{\theta}} + J_z A_z + J_{\bar{z}} A_{\bar{z}} + g_{\theta \bar{\theta}} (\xi_\theta \psi_\bar{\theta} + \xi_\bar{\theta} \psi_\theta) + g_{\theta \bar{\theta}} (\bar{\xi}_\theta \psi_{\bar{\theta}} + \bar{\xi}_{\bar{\theta}} \psi_{\theta}) \right] \]  

(5.1)

where \( \xi, \bar{\xi} \) are the external currents related to the fields \( \psi, \bar{\psi} \) respectively and \( g_{\theta \bar{\theta}} = \sqrt{g_{z \bar{z}}} \). The spinor indices are denoted with \( \theta, \bar{\theta} \). We assume that the “physical” boundary conditions for \( \psi \) and \( \bar{\psi} \) when transported along the homology cycles in the case in which \( M = \Sigma_g \) are given by the even spin structure \( m = \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \). \( \vec{a}_0 \) and \( \vec{b}_0 \) are two vectors of dimension \( g \) whose elements are half integers such that \( 4 \vec{a}_0 \cdot \vec{b}_0 = 0 \mod 2 \) [12].

Despite of the fact that QED is, at least in the flat case, solvable in two dimensions [13], we will use here a perturbative approach. The applications of the perturbation theory to the Schwinger model were investigate for example in [14]. Our aim is the computation of the Green functions of QED\( _2 \) at least at the lowest order. Exploiting the Hodge decomposition of eq. (2.14), eq. (5.1) becomes:

\[ S[\varphi, \rho, \bar{\psi}, \psi] = S_0 + S_1 + S_{z.m.} + S_c \]  

(5.2)

where \( S_0 \) is the free field action:

\[ S_0[\varphi, \bar{\psi}, \psi] = \int_M d^2 z \left[ g^{z \bar{z}} \varphi \bar{\varphi} + \bar{\psi}_\theta \partial_z \psi_\theta + \bar{\psi}_\bar{\theta} \partial_{\bar{z}} \psi_{\bar{\theta}} \right] \]  

(5.3a)

and \( S_1 \) contains the interaction part:

\[ S_1[\varphi, \rho, \bar{\psi}, \psi] = -\int_M d^2 z \varphi \left[ \partial_z (\bar{\psi}_\theta \psi_{\bar{\theta}}) - \partial_{\bar{z}} (\bar{\psi}_{\bar{\theta}} \psi_\theta) \right] - \int_M d^2 z \rho \left[ \partial_z (\bar{\psi}_\theta \psi_{\bar{\theta}}) + \partial_{\bar{z}} (\bar{\psi}_{\bar{\theta}} \psi_\theta) \right] \]  

(5.3b)

Moreover

\[ S_{z.m.} = \int_M d^2 z A_z^{\text{har}} \bar{\psi}_\theta \psi_{\bar{\theta}} + \int_M d^2 z A_{\bar{z}}^{\text{har}} \bar{\psi}_{\bar{\theta}} \psi_\theta \]  

(5.3c)

\( j_z = \bar{\psi}_\theta \psi_{\bar{\theta}} \) and \( j_{\bar{z}} = \bar{\psi}_{\bar{\theta}} \psi_\theta \) are the currents associated with the conservation of the number of fermions. We can use the Hodge decomposition (2.15) also for \( j_z, j_{\bar{z}} \) and, exploiting the
orthogonality property of this decomposition, it is easy to see that $S_{z.m.} = \int_M A^\text{har}_z j^\text{har}_z + \text{c.c.}$

Therefore $S_{z.m.}$ does not contribute to the computation of the Green functions, but becomes relevant in the computation of the partition function [15]. Finally $S_c$ contains the external currents. Again, due to the properties of the Hodge decompositions (2.14)-(2.15), the zero modes do not yield a relevant contribution in this sector so that we can write:

\[
S_c[J, \xi, \bar{\xi}] = \int_M d^2z \left[ (A^T_z J^T_z + A^L_z J^L_z + \text{c.c.}) + g_{\theta\bar{\theta}} (\xi_\theta \psi_{\bar{\theta}} + \bar{\xi}_{\bar{\theta}} \bar{\psi}_\theta + \text{c.c.}) \right]
\]

(5.3d)

The generating functional of the QED$_2$ Green functions becomes:

\[
Z[J, \bar{\xi}, \xi] = \int D\varphi D\rho D\bar{\psi} D\psi e^{-\left( S^0 + S_i + S_c \right)(\varphi, \bar{\psi}, \psi, J, \bar{\psi}, \psi)}
\]

with the action provided by eq. (5.2). Looking at equations (5.3b) and (5.3d) we see that the integration over $D\rho$ is not difficult and yields the following result in the limit in which the external currents $J^L_z, J^L_{\bar{z}}$ are zero:

\[
Z[J^T, \bar{\xi}, \xi] = \int D\varphi D\bar{\psi} D\psi e^{-\left( S^0 + S_i + S_c \right)(\varphi, \bar{\psi}, \psi, J, \bar{\psi}, \psi)} \delta \left( \Re \left( \partial_z (\bar{\psi}_\theta \psi_{\bar{\theta}}) \right) \right)
\]

(5.5)

The $\delta$-function in eq. (5.5) express the physical requirement that the total number of fermions should be conserved in the amplitudes. As a matter of fact the equation:

\[
\Re[\partial_z (\bar{\psi}_\theta \psi_{\bar{\theta}})] = \partial_z j_{\bar{z}} + \partial_{\bar{z}} j_z = 0
\]

(5.6)

means that the current $j_\mu$, $\mu = z, \bar{z}$, is conserved. Remembering that in the proper regularization scheme we have

\[
< 0|\partial_\mu j^\mu f(\bar{\psi}, \psi, A)|0 > = 0
\]

where $f(\bar{\psi}, \psi, A)$ is a polynomial in the fields $\bar{\psi}, \psi, A_\mu$, it is clear that we can factor out the $\delta$-function from eq. (5.5). The factor is an infinite constant $\delta(0)$ which corresponds to the infinite contribution of the longitudinal degrees of freedom. Therefore this way of quantizing the Schwinger model is unusual but consistent. The alternative is to treat also the scalar fields $\rho$ perturbatively. In the next section we will present other models in which this is not necessary. For example in the action of eq. (6.2), expressing the MFT interacting with massless scalar fields, the exact forms $\partial_z \rho$ and $\partial_{\bar{z}} \rho$ coming from the Hodge decomposition do not even appear.
At this point we are ready to compute the amplitudes of \( \text{QED}_2 \) perturbatively. If \( M = B_\rho, \mathbf{CP}_1, \Sigma_g \), the propagators of the scalar fields are given by eqs. (3.6), (4.4) and (4.10) respectively. The case of the disk needs some care due to the presence of the boundary. If \( M = \mathbf{C} \), eq. (3.8) provides the propagator of the transverse vector fields. Moreover, on \( \Sigma_g \), the propagator of the free fermionic fields with even spin structures is the well known Szegö kernel:

\[
S_{\theta\theta'}(z, w) \equiv < \bar{\psi}_\theta(z)\psi_{\theta'}(w) > = \frac{\theta \left[ \frac{a_0}{b_0} \right] (z - w)}{\theta \left[ \frac{a_0}{b_0} \right] (0) E(z, w)}
\]

(5.8)

The condition given by eq. (5.6) is trivially satisfied at the lowest order noting that with the usual regularization [16]:

\[
< j_z(z) > = \lim_{z \to w} \left( < \bar{\psi}(z)\psi(w) > - \frac{1}{z - w} \right)
\]

\(< j_z(z) > \) is independent of \( \bar{z} \) and has no poles in \( z \). We are now ready to compute the three point function for the coupling \( \varphi \bar{\psi}\psi \). Here we limit ourselves to the vertex \( V_{z_1\theta_2\theta_3}(z_1, z_2, z_3) \), which is defined as follows:

\[
V_{z_1\theta_2\theta_3}(z_1, z_2, z_3) = \langle \varphi(z_1, \bar{z}_1)\bar{\psi}_{\theta_2}(z_2, \bar{z}_2)\psi_{\theta_3}(z_3, \bar{z}_3) >
\]

(5.9)

The complex conjugate vertex can be computed in an analogous way. As in the flat case the three point function of eq. (5.9) is defined by:

\[
V_{z_1\theta_2\theta_3}(z_1, z_2, z_3) = -2 \frac{\delta}{\delta J(z_1, \bar{z}_1)} \frac{\delta}{\delta \theta_2(z_2, \bar{z}_2)} \frac{\delta}{\delta \theta_3(z_3, \bar{z}_3)} \int_M d^2z' \varphi(z', \bar{z}')\bar{\psi}_{\theta'}(z', \bar{z}')\psi_{\theta'}(z', \bar{z}') \exp \left[ -S_0 + S_c \right] \bigg|_{J=0, \xi, \zeta=0}
\]

(5.10)

where \( J \) is given by eq. (2.15). From eq. (5.10) we get the final expression of the vertex \( V_{z_1\theta_2\theta_3}(z_1, z_2, z_3) \) which is:

\[
V_{z_1\theta_2\theta_3}(z_1, z_2, z_3) = \int_M d^2z' \varphi(z', \bar{z}')G(z', z_1)S_{\theta\theta'}(z', z_3)S_{\theta_2\theta'}(z_2, z')
\]

(5.11)

As an upshot the vertex containing the vector field \( A_z^T \) becomes:

\[
< A_z^T(z_1)\bar{\psi}_{\theta_2}(z_2)\psi_{\theta_3}(z_3) > = \int_M d^2z' \varphi(z', \bar{z}')G(z, z_1)S_{\theta\theta'}(z', z_3)S_{\theta_2\theta'}(z_2, z')
\]

(5.12)

Substituting the propagators (4.10) and (5.8) in eqs. (5.11)-(5.12), we get an expression of the three point function \( V_{z_1\theta_2\theta_3}(z_1, z_2, z_3) \) on a Riemann surface. On the sphere one can use the propagator (4.4) for the scalar fields \( \varphi \) and the propagator \( S_{\theta\theta'}(z, w) = 1/(z - w) \) for the fermions. In an analogous way one can treat the case of the complex plane.
6. CONCLUSIONS

In this paper we have treated the quantization of the two dimensional Maxwell field theory in the Lorentz gauge and in the presence of a nontrivial background. The gauge fixed MFT becomes a theory of scalar fields with higher order derivatives at least at the tree level. This equivalence is useful since in this way we can exploit previous mathematical knowledge [7] in order to construct the propagators of the MFT on two dimensional manifolds. The propagator of these scalar fields satisfies in fact a biharmonic equation which has been studied long time ago in connection with a clamped thin plate subjected to a point load. On a general two dimensional manifold the problem loses its physical significance but it is still relevant for the biharmonic classification theory of Riemannian manifolds [7]. To this purpose we notice that the choice of a conformally flat metric in eq. (2.7) is very useful in treating the MFT in the interacting case. However, the derivation of the propagator $G(z,w)$ can be easily obtained also in the case of a general metric. From eq. (2.6) it is in fact clear that eqs. (3.8), (4.4) and (4.10) are still valid substituting $g_{\tilde{t}\tilde{t}}$ with $1/\sqrt{g}$. Particular attention has been devoted here to the boundary conditions that the biharmonic Green functions should satisfy in such a way that the original MFT remains invariant under the residual gauge transformations.

Moreover in the Lorentz gauge the functional integration over the longitudinal fields is trivial and, due to eq. (2.23), the transverse fields do not give any contribution to the VEV of the Wilson loops. Therefore the results of [4,5] are confirmed without using the equations of motion (2.3) explicitly.

Another advantage of having fixed the Lorentz gauge with the method explained in Section 2 is that one can easily derive the $n$-point functions of the MFT coupled with other field theories. In Section 5 we have briefly shown how this is possible in the case of the Schwinger model. Unfortunately interesting topics about the Schwinger model like the treatment of the anomalies [1], the derivation of the partition function [15] and the integrability on a Riemann surface [13], have been ignored being outside of the purposes of this paper.

In principle one can compute the $n$-point functions of the interacting MFT pertur-
batively also for theories which are different from the Schwinger model. In particular it is interesting the possibility that the gauge fields $A_z$ and $A_{\bar{z}}$ can interact with the fields appearing in string theory. The aim is to construct new string theories and to cancel, at least partially, the Lorentz and Weyl anomalies of two dimensional (chiral) conformal field theories as discussed in refs. [17].

The simplicity of the Schwinger model in the Lorentz gauge consists in the fact that the longitudinal fields can be integrated away without requiring a perturbative treatment. Other theories in which this is possible as well are the MFT coupled with the $b-c$ systems of string theory:

$$S[A,b,c] = \int d^2z \left( \frac{1}{4}g_{zz}F_{zz}F_{z\bar{z}} + b_{zz}(\partial_z + A_z)c_z + \bar{b}_{z\bar{z}}(\partial_{\bar{z}} + A_{\bar{z}})c_{\bar{z}} \right)$$  

and the MFT coupled to the scalar fields $X$ of string theory in the following way:

$$S[A,X] = \int d^2z \left( \frac{1}{4}g_{zz}F_{z\bar{z}}F_{\bar{z}z} + XF_{z\bar{z}} + \frac{1}{2}\partial_z X\partial_{\bar{z}} X \right)$$  

In eq. (6.1) the conservation of the ghost number is anomalous but the anomaly is a total derivative. Therefore, making the shift $J'_z \to J_z + \frac{3}{8}\partial_z \log(g_{zz})$ in the external currents we get the condition that the ghost number is conserved in analogy with eq. (5.6). Not so easy is the situation in which the gauge fields are minimally coupled to the scalar fields as in [18]. In this case the integration over the longitudinal gauge fields becomes complicated and one has to treat also the longitudinal degrees of freedom perturbatively. Finally we already noticed that on a manifold $M$ the usual biharmonic equation

$$\Delta^2_{z} \varphi^\mu(z, \bar{z}) = 0$$  

is not invariant under general transformations of coordinates and one has to introduce the operator $\Delta_z g^{z\bar{z}} \Delta_z$. In eq. (6.3) we have slightly generalized the previous discussion allowing the fields $\varphi$ to carry an index $\mu = 1, \ldots, N$. An exception, which merits a little digression, is provided by the complex sphere because it can be covered by two open sets with transition functions $z = 1/z'$. As a matter of fact eq. (6.3) is invariant under the transformations of coordinates

$$z \to \frac{az' + b}{cz' + d}, \quad ad - bc = 1$$  

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generating the group $\text{SL}(2, \mathbb{C})$ if $\varphi$ transforms as follows:

$$\varphi(z, \bar{z}) = J \varphi'(z', \bar{z}')$$ \hspace{1cm} (6.5)

where $J = | \frac{dz}{d\bar{z}} |$. Eqs. (6.4)-(6.5) form the socalled Mitchell transformations [19]. As an upshot the following action, dependent on a dimensional parameter $\gamma$:

$$S = \int d^2 z \left[ \varphi^\mu (\partial_z \partial_{\bar{z}})^2 \varphi_\mu + \frac{\gamma}{\varphi^\mu \varphi_\mu} \right]$$

is an example of a gauge invariant field theory with higher order derivatives [20]. The gauge invariance is preserved also if $\gamma$ is set to zero. In that limit eq. (6.6) provides one of the simplest gauge theories with nonabelian group of symmetry.

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