Tsallis information dimension of complex networks

Qi Zhang\textsuperscript{a}, Meizhu Li\textsuperscript{a}, Yong Deng\textsuperscript{a,b,c,*}, Sankaran Mahadevan\textsuperscript{c}

\textsuperscript{a}School of Computer and Information Science, Southwest University, Chongqing, 400715, China
\textsuperscript{b}School of Automation, Northwestern Polytechnical University, Xian, Shaanxi 710072, China
\textsuperscript{c}School of Engineering, Vanderbilt University, Nashville, TN, 37235, USA

Abstract

The fractal and self-similarity properties are revealed in many complex networks. In order to show the influence of different part in the complex networks to the information dimension, we have proposed a new information dimension based on Tsallis entropy namely Tsallis information dimension. The Tsallis information dimension can show the fractal property from different perspective by set different value of $q$.

Keywords: Complex networks, Information dimension, Tsallis entropy, Tsallis information dimension

1. Introduction

The complex networks have been applied in many disciplines \cite{1, 2, 3, 4, 5, 6, 7}. Researchers have revealed several properties of the complex networks, such as small-world phenomena \cite{8}, scale-free degree \cite{9}, fractal, self-
similarity and community structure. The fractal and self-similarity properties have shown the structure characteristic of the complex networks, many researchers have been attracted to explore it. In order to describe the fractal properties, Song et al. proposed the dimension of the complex networks.

Recently, an information dimension of the complex networks has been proposed by Wei et al. in. In the information dimension, the boxes which contain more nodes have a maximum effect to the information dimension. However, sometimes those boxes contain few nodes may play an important role in the fractal property. In order to show the influence of the boxes which have different mounts of nodes to the information dimension. A new information dimension based on Tsallis entropy is proposed in this paper. In the proposed method, setting different values of $q$ means chose different part as the main effect of the information dimension.

The rest of this paper is organised as follows. Section 2 introduces some preliminaries of this work. In section 3, a new information dimension of complex networks based on the Tsallis entropy is proposed. The application of the proposed method is illustrated in section 4. Conclusion is given in Section 5.

2. Preliminaries

2.1. Box-covering algorithm of complex networks

Song et al. have proposed a new box-covering algorithm for complex networks. It contains a new definition for the box size $l_B$ which is
based on the distances between the nodes in the complex networks.

Figure 1: The classical box-covering algorithm for complex networks, where \( l = 3 \). The network \( G_1 \) is original network with 6 nodes and 6 edges. The network \( G_2 \) is obtained by only connecting to nodes which distance between them not less than 3 in network \( G_3 \). The network \( G_3 \) is obtained when the greedy algorithm is used for node coloring on \( G_2 \).

For a given network \( G_1 \) and box size \( l_B \), a box is a set of nodes where all distances \( l_{ij} \) between any two nodes \( i \) and \( j \) in the box are smaller than \( l_B \). The minimum number of boxes required to cover the entire network is denoted by \( N_B \).

For \( l_B = 1 \), \( N_B \) is obviously equals to the size of the network \( N \), while \( N_B = 1 \) for \( l_B \geq l_B^{\text{max}} \), where \( l_B^{\text{max}} \) is the diameter of the network plus one, the diameter of the network equals to the maximum distance in the network [20].

If the distance between two nodes in \( G_1 \) is greater than \( l_B \), these two neighbors cannot belong in the same box. According to the construction of
$G_2$, these two nodes will be connected in $G_2$ and thus they will not belong in the same box in $G_1$. On the contrary, if the distance between two nodes in $G_1$ is less than $l_B$, it is possible that these nodes belong in the same box. In $G_2$ these two nodes will not be connected and it is allowed for these nodes to carry the same color, it will belong to the same box in $G_1$ \[20\]. More details are shown in Fig. \[1\].

The box-covering algorithm is used to calculate the minimum numbers of box $N_l$ by Song et al. Then the fractal dimension $d_B$ of the complex networks can be described by the relationship between $N_l$ and $l_B$. The details are shown in Eq. \[1\].

$$d_B = -\lim_{l_B \to 0} \frac{\ln N_l}{\ln l_B}$$

(1)

2.2. Tsallis entropy

The entropy is defined by Clausius for thermodynamics \[23\], connects the macroscopic and microscopic worlds. For a finite discrete set of probabilities the definition of the Boltzmann-Gibbs \[21\] entropy is given as follows:

$$S_{BG} = -k \sum_{i=1}^{N} p_i \ln p_i$$

(2)

Where $BG$ stands for Boltzmann – Gibbs, the $S_{BG}$ represents the Boltzmann-Gibbs entropy. The conventional constant $k$ is the Boltzmann universal constant for thermostatistical systems, the value of $k$ will being taken to be unity in information theory \[24, 25\].

In 1988, a more general form for entropy have been proposed by Tsallis \[22\]. It is shown as follows:
\[ S_q = -k \sum_{i=1}^{N} p_i \ln_q p_i \]  \hspace{1cm} (3)

The \( q - \text{logarithmic} \) function in the Eq. (3) is presented as follows [24]:

\[
\ln_q p_i = \frac{p_i^{1-q} - 1}{1 - q} (p_i > 0; q \in \Re; \ln_1 p_i = \ln p_i) \]  \hspace{1cm} (4)

Based on the Eq. (4), the Eq. (3) can be rewritten as follows:

\[
S_q = k \frac{1 - \sum_{i=1}^{N} p_i^q}{q - 1} \]  \hspace{1cm} (5)

Where \( N \) is the number of the subsystems.

2.3. Information dimension

Based on the information entropy and the box-covering algorithm, an information dimension has been proposed by Wei et al. in [21].

The information of the complex networks is shown as follows:

\[ I = - \sum_{i=1}^{N_b} p_i \ln(p_i) \]  \hspace{1cm} (6)

The \( p_i \) in the Eq. (6) represents the probability of the nodes in the \( i \)th box. It is shown in Eq. (7).

\[ p_i = \frac{n_i}{n} \]  \hspace{1cm} (7)

Where \( n_i \) is the node number in the \( i \)th box, \( n \) is the total number of the nodes in the complex networks [21].
Depends on the relationship between information of the complex networks and the box size. The information dimension of the complex networks is shown in Eq. (8) [21].

\[
d_b = -\lim_{l \to 0} \frac{I}{\ln l} = \lim_{l \to 0} \frac{\sum_{i=1}^{N_b} p_i \ln(p_i)}{\ln l} \quad (8)
\]

Where \( d_b \) is the information dimension of the complex network. Based on Eq. (6), the Eq. (8) can be rewritten as follows:

\[
d_b = \lim_{l \to 0} \frac{\sum_{i=1}^{N_b} n_i(l) \ln\left(\frac{n_i(l)}{n}\right)}{\ln l} \quad (9)
\]

3. Tsallis information dimension

In this section, a Tsallis information dimension of the complex networks, \( d_T \), is proposed as follows:

\[
d_T = \frac{1 - \sum_{i=1}^{N} p_i(l)^q}{q - 1} \ln l \quad (q \in \mathbb{R}) \quad (10)
\]

Where \( l \) is the box size in the box-covering algorithm. The numerator is the Tsallis entropy which is defined in Eq. [5]. It can be easily seen that when \( q = 1 \) the Tsallis information dimension is degenerated to the information dimension of complex networks in [21].

Similar to Shannon’s information volume, we use the Tsallis entropy to define the information volume of complex networks as follows,
\[ I_v = \frac{1 - \sum_{i=1}^{N_v} p_i^q}{q - 1} \] (11)

We discuss the relationship between the parameter \( q \) and the information dimension of the complex networks.

**CASE 1**, **when** \( q \to -\infty \), the boxes with the minimum probability have the maximum effect on the information dimension of the complex networks.

**CASE 2**, **when** \( q \to 0 \), the boxes with different probability have the same effect on the information dimension of the complex networks.

**CASE 3**, **when** \( q \to 1 \), the Tsallis information dimension is degenerated to the information dimension in [21].

**CASE 4**, **when** \( q \to \infty \), the boxes with the maximum probability have the maximum effect on the information dimension of the complex networks. The information dimension of the complex networks is closed to 0.

It can be easily found that, with the increase of \( q \), the information dimension of the complex networks is decreased.

4. Application

In this section, we use the proposed method to calculate the information dimension of four real networks, namely, the US-airlines networks [26], Email networks [26] and the Germany highway networks [27]. The results are given in Table [1].

The comparison between the information dimension in [21] and the proposed information dimension is shown in Table [2].
Table 1: Tsallis information dimension of real networks

|                      | Germany highway [27] | Us-airline [26] | Email [26] |
|----------------------|-----------------------|-----------------|------------|
| Nodes                | 1168                  | 500             | 1133       |
| edges                | 2486                  | 5962            | 10902      |
| \(d_T(q=0.1)\)       | 61.88                 | 62.89           | 175.20     |
| \(d_T(q=0.5)\)       | 10.646                | 15.780          | 23.778     |
| \(d_T(q=1.0)\)       | 1.9384                | 2.9682          | 3.5132     |
| \(d_T(q=1.5)\)       | 0.66732               | 1.0585          | 1.1131     |
| \(d_T(q=2.0)\)       | 0.35145               | 0.5758          | 0.5817     |
| \(d_T(q=10)\)        | 0.0268                | 0.0564          | 0.0562     |
| \(d_T(q=100)\)       | 0.0009                | 0.0037          | 0.0039     |
| \(d_T(q=1000)\)      | 0.0003                | 0.0002          | 0.00029    |

Table 2: The results with different methods

| Networks             | Nodes | edges | \(d_b\) | \(d_T(q=0.1)\) | \(d_T(q=1)\) | \(d_T(q=1000)\) |
|----------------------|-------|-------|---------|----------------|--------------|-----------------|
| Germany highway [27] | 1168  | 2486  | 1.9384  | 61.8816        | 1.9384       | 0.00030         |
| Us-airline [26]      | 500   | 5962  | 2.9682  | 62.8919        | 2.9682       | 0.00020         |
| Email [26]           | 1133  | 10902 | 3.5132  | 175.21         | 3.5132       | 0.00029         |

In the Table 1 and Table 2, the \(d_b\) represents the information dimension of the complex networks which is calculated by the method in [21]. The \(d_T(q=x)\) represents the information dimension of complex networks which is calculated by the proposed method.

The slope of the straight lines in the Figure (2-4) represents the information dimension of complex network. The results have shown that the
Figure 2: The Germany highway network

Figure 3: The Email network
information dimension is in inverse proportion to the value of $q$.

5. Conclusion

The information dimension is widely used to illuminate the fractal and self-similarity properties of the complex networks. In this article, a general method to calculate the information dimension of complex networks has been proposed based on the Tsallis entropy. It can be used to describe the influence of different parts in the complex networks to the fractal property. The proposed Tsallis information dimension is a generalization of the existing information dimension to the complex networks.
Acknowledgments

The work is partially supported by National Natural Science Foundation of China (Grant No. 61174022), Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20131102130002), R&D Program of China (2012BAH07B01), National High Technology Research and Development Program of China (863 Program) (Grant No. 2013AA013801), the open funding project of State Key Laboratory of Virtual Reality Technology and Systems, Beihang University (Grant No.BUAA-VR-14KF-02).

References

[1] M. Newman, The structure and function of complex networks, SIAM Review (2003) 167–256.

[2] W. Yu, G. Chen, J. Lü, On pinning synchronization of complex dynamical networks, Automatica 45 (2) (2009) 429–435.

[3] Q. Song, J. Cao, F. Liu, Synchronization of complex dynamical networks with nonidentical nodes, Physics Letters A 374 (4) (2010) 544–551.

[4] C. Hu, J. Yu, H. Jiang, Z. Teng, Synchronization of complex community networks with nonidentical nodes and adaptive coupling strength, Physics Letters A 375 (5) (2011) 873–879.

[5] M. Vidal, M. Cusick, A. Barabási, Interactome networks and human disease, Cell 144 (6) (2011) 986–998.
[6] D. Wei, X. Deng, Y. Deng, S. Mahadevan, Identifying influential nodes in weighted networks based on evidence theory, Physica A: Statistical Mechanics and its Applications 392 (2013) 2564–2575.

[7] Q. Zhang, J. Chen, L. Wan, Impulsive generalized function synchronization of complex dynamical networks, Physics Letters A 377 (39) (2013) 2754–2760.

[8] D. Watts, S. Strogatz, Collective dynamics of small-world networks, Nature 393 (6684) (1998) 440–442.

[9] A. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (5439) (1999) 509–512.

[10] S. Fortunato, Community detection in graphs, Physics Reports 486 (3-5) (2010) 75–174.

[11] B. Mandelbrot, D. Passoja, A. Paullay, Fractal character of fracture surfaces of metals, Nature 308 (1984) 721–722.

[12] M. Locci, G. Concas, R. Tonelli, I. Turnu, Three algorithms for analyzing fractal software networks, WSEAS Transactions on Information Science and Applications 7 (3) (2010) 371–380.

[13] M. A. Moret, L. Q. Antonio, H. B. Pereira, Classical and fractal analysis of vehicle demand on the ferry-boat system, Physica A: Statistical Mechanics and its Applications 391 (4) (2012) 1657–1661.
[14] I. I. Smolyaninov, Metamaterial model of fractal time, Physics Letters A 376 (16) (2012) 1315–1317.

[15] F. O. Redelico, A. N. Proto, Empirical fractal geometry analysis of some speculative financial bubbles, Physica A: Statistical Mechanics and its Applications 391 (21) (2012) 5132–5138.

[16] V. Rajagopalan, Z. Liu, D. Allexandre, L. Zhang, X. Wang, E. P. Pioro, G. H. Yue, Brain white matter shape changes in amyotrophic lateral sclerosis (als): a fractal dimension study, PloS one 8 (9) (2013) e73614.

[17] P. Chelminiak, Emergence of fractal scale-free networks from stochastic evolution on the cayley tree, Physics Letters A 377 (40) (2013) 2846–2850.

[18] C. Song, S. Havlin, H. Makse, Self-similarity of complex networks, Nature 433 (7024) (2005) 392–395.

[19] C. Song, S. Havlin, H. Makse, Origins of fractality in the growth of complex networks, Nature Physics 2 (4) (2006) 275–281.

[20] C. Song, L. Gallos, S. Havlin, H. Makse, How to calculate the fractal dimension of a complex network: the box covering algorithm, Journal of Statistical Mechanics: Theory and Experiment 2007 (2007) P03006.

[21] W. Daijun, W. Bo, H. Yong, Z. Haixin, Y. Deng, A new information dimension of complex networks, Physics Letters A (2014) 1091–1094.
[22] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, Journal of Statistical Physics 52 (1-2) (1988) 479–487.

[23] R. Clausius, The mechanical theory of heat: with its applications to the steam-engine and to the physical properties of bodies, J. van Voorst, 1867.

[24] C. Tsallis, U. Tirnakli, Nonadditive entropy and nonextensive statistical mechanics–some central concepts and recent applications, in: Journal of Physics: Conference Series, Vol. 201, 2010, p. 012001.

[25] C. E. Shannon, A mathematical theory of communication, ACM SIGMOBILE Mobile Computing and Communications Review 5 (1) (2001) 3–55.

[26] Pajek datasets, http://vlado.fmf.uni-lj.si/pub/networks/data/ (2014).

[27] Tore Opsahl, http://toreopsahl.com/datasets/ (2014).