Balanced varieties

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Abstract

After the work of Bloch and Srinivas on correspondences and algebraic cycles we begin the study of a birational class of algebraic varieties determined by the property that a multiple of the diagonal is rationally equivalent to a cycle supported on proper subschemes.

Introduction

One of the main results due to Mumford is the observation that a complex projective non singular surface having a non zero global holomorphic 2-form has a non zero Albanese kernel; the result was obtained after Severi’s work on algebraic cycles and holomorphic forms. Because of Mumford’s result the functor $A_0$ given by 0-cycles of degree zero modulo rational equivalence is not representable, in contrast with the codimension 1 case treated by Grothendieck and Murre. These facts lead to the study of those varieties for which the Albanese kernel is zero and to “weak” representability.

Bloch’s proof of the mentioned result by Mumford reveals the motivic (read algebraic) nature of the problem: he observed that the vanishing of the Albanese kernel of a surface $X$ at a sufficiently large base field extension yields the existence of 1-dimensional subschemes $Z_1$ and $Z_2$, 2-dimensional cycles $\Gamma_1$ and $\Gamma_2$ on $X \times X$, supported on $Z_1 \times X$ and $X \times Z_2$ respectively, such that some non-zero multiple of the diagonal is rationally equivalent to $\Gamma_1 + \Gamma_2$ over the base field. Then by investigating the action of the correspondences $\Gamma_1, \Gamma_2$ on the transcendental (read not algebraic) part of $\ell$-adic étale cohomology one immediately obtains its vanishing.

The subsequent paper by Bloch and Srinivas generalizes Bloch’s argument to varieties of dimension > 2 showing its influence on algebraic, Hodge and arithmetic cycles.

After all of that one has the temptation of understanding the geometry of those varieties $X$ of pure dimension $n$, for which we assume given proper closed subschemes $Z_1$ and $Z_2$, $n$-dimensional cycles $\Gamma_1$ and $\Gamma_2$ on $X \times X$, supported on $Z_1 \times X$ and $X \times Z_2$ respectively, and a positive integer $N$ such that the following equation

$$N\Delta_X = \Gamma_1 + \Gamma_2$$

This Note is the germ of a work in progress, I wrote it down for seminal use; the first lecture I gave on the subject was on February 16, 1996 at the Universitat de Barcelona, Departament d’Algebra i Geometria, which I thank for the invitation.

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holds in the Chow group $CH_n(X \times X)$. I like to call such a variety balanced because it appears rather natural to regard $X$ “balanced” by $\Gamma_1$ and $\Gamma_2$ on $Z_1$ and $Z_2$ in a motivic way as explained above or as will become more clear in the following.

The purpose of this Note is to begin the study (in Section 1) of “balanced varieties and balanced morphisms” by showing some basic geometric properties e.g. balancing is a birational property, stable under products; I also explain the methods by Bloch and Srinivas (in Section 2) obtaining applications in the “language” of balanced varieties; I’ll sketch some examples (in Section 3) and, finally by the way, I would draw a picture of some expected properties of balanced varieties.

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1 Basic geometry of balancing

A variety will be an equidimensional reduced separated scheme of finite type over a fixed base field $k$. A non singular variety will be a regular scheme which is a variety.

**Definition 1.1:** A balanced variety is a variety $X$ of dimension $n$ over $k$ such that there exist (i) proper closed subschemes $Z_1$ and $Z_2$ in $X$ which are of finite type over $k$, (ii) $n$-dimensional cycles $\Gamma_1$ and $\Gamma_2$ on $X \times_k X$ which are supported on $Z_1 \times_k X$ and $X \times_k Z_2$ respectively, and (iii) a positive integer $N$, such that the following equation

$$N\Delta_X = \Gamma_1 + \Gamma_2$$

holds in the Chow group $CH_n(X \times X)$, where $\Delta_X$ is the canonical cycle associated with the diagonal imbedding.

In this case we will shortly say that we have a balancing of a variety. We then will say that $X$ is balanced by $\Gamma_1$ and $\Gamma_2$ on $Z_1$ and $Z_2$. We will say that $Z_1$ and $Z_2$ are the balance pans. The weight of $X$ w.r.t. $Z_1$ and $Z_2$ is the positive integer $\min\{\dim Z_1, \dim Z_2\}$, which we denote by $w(X)$.

The basic nice properties of balanced varieties are the following.

**Proposition 1.2** Let $X$ be a variety of dimension $n$. Let $U$ be a Zariski open dense subset of $X$ e.g. $X$ integral and $U \subset X$ any non-empty Zariski open. Then $X$ is balanced if and only if $U$ is balanced.

**Proof** Since $U$ contains all generic points we have that $\dim Z \xrightarrow{\text{def}} X - U < n$ whence the diagonal cycle $\Delta_X$ restricts to the diagonal cycle $\Delta_U$: so we may restrict the balancing as well. Conversely, if $U$ is balanced we have $N\Delta_U = \Gamma_1^U + \Gamma_2^U$ where $\Gamma_1^U$ is a cycle supported on $(Z_1 \cap U) \times U$ and $\Gamma_2^U$ is a cycle supported on $U \times (Z_2 \cap U)$ for some $Z_1$ and $Z_2$ closed subschemes of $X$. Now we can lift $\Gamma_1^U$ to a cycle $\Gamma_1 + \zeta_1$ on $X \times X$ where $\Gamma_1$ is supported on $Z_1 \times X$ and $\zeta_1$ is supported on $(Z_1 \cap Z) \times Z$; in the same way $\Gamma_2^U$ yields a cycle $\Gamma_2 + \zeta_2$ with $\Gamma_2$ supported on $X \times Z_2$ and $\zeta_2$ supported on $Z \times (Z_2 \cap Z)$. Since the restriction of $\Gamma_1 + \Gamma_2 + \zeta_1 + \zeta_2$ to $U \times U$ is $\Gamma_1^U + \Gamma_2^U$ and the restriction of $N\Delta_X$ is $N\Delta_U$ we have that $\Gamma_1 + \Gamma_2 + \zeta_1 + \zeta_2 - N\Delta_X$ is a cycle supported on $Z \times Z$. We then obtain an equation

$$N\Delta_X = \Gamma_1 + \Gamma_2 + \zeta$$
for some cycle ζ supported on \(Z \times Z\), which is a balancing of \(X\).

**Proposition 1.3** Let \(X\) be a balanced variety. Let \(f : X \to X'\) be a proper surjective morphism to a variety \(X'\) with \(\dim X = \dim X'\). Then \(X'\) is balanced.

**Proof** Since \(f\) is generically finite we have that the fundamental cycle \([X]\) has proper push-forward \(f_*[X] = \deg(f)[X']\). Let denote by \(\delta_X\) and \(\delta_{X'}\) the diagonal imbeddings of \(X\) and \(X'\). We clearly have that \(\delta_{X'} \circ f_* = (f \times f)_* \circ \delta_X\) whence we have

\[
(f \times f)_*(\Delta_X) = \deg(f)\Delta_{X'}
\]

by the definition of \(\Delta\). Applying \((f \times f)_*\) to (1) we have

\[
N(f \times f)_*(\Delta_X) = (f \times f)_*(\Gamma_1 + \Gamma_2)
\]

so that we obtain

\[
N\deg(f)\Delta_{X'} = (f \times f)_*(\Gamma_1) + (f \times f)_*(\Gamma_2)
\]

Then \(X'\) is balanced by \((f \times f)_*(\Gamma_1)\) and \((f \times f)_*(\Gamma_2)\) on \(f(Z_1)\) and \(f(Z_2)\).

We then have:

**Corollary 1.4** Let \(f : X' \to X\) be a proper dominant rational map with \(\dim X = \dim X'\): if \(X'\) is balanced then \(X\) is balanced. Balancing is a birational property of arbitrary varieties over arbitrary fields.

**Proof** We may indeed restrict \(f\) to open dense subsets \(U'\) of \(X'\) and \(U\) of \(X\) such that \(U' \to U\) is a proper morphism and make use of the Propositions 1.2 and 1.3. If \(X\) is birational to \(X'\) then they have isomorphic open dense subsets and we just use Proposition 1.2.

**Proposition 1.5** Let \(X\) be a balanced variety. Then the product of \(X\) with any variety is balanced.

**Proof** Let \(X'\) be any variety over \(k\). We need the following simple Lemma.

**Lemma 1.6** Let \(X\) and \(X'\) be varieties over \(k\). Let

\[
\sigma : X \times X \times X' \times X' \to X \times X' \times X \times X'
\]

be the isomorphism given by \(\sigma(x_1, x_2, x'_1, x'_2) = (x_1, x'_1, x_2, x'_2)\). Then

\[
\sigma_*(\Delta_X \times \Delta_{X'}) = \Delta_{X \times X'}
\]

**Proof** Because of the functoriality of the exterior product of cycles we have

\[
(\delta_X \times \delta_{X'})_*(- \times \dagger) = \delta_X(-) \times \delta_{X'}(\dagger)
\]

Moreover we clearly have

\[
\sigma_*(\delta_X \times \delta_{X'}) = \delta_{X \times X'}
\]

So then
\[ \sigma_*(\Delta_X \times \Delta_{X'}) = \]
\[ = \sigma_*(\delta_{X}[X] \times \delta_{X'}[X']) = \text{by (3)} \]
\[ = \sigma_*(\delta_X \times \delta_{X'})_*(X \times X') = \text{by (4)} \]
\[ = (\delta_{X \times X'})_*[X \times X'] = \]
\[ = \Delta_{X \times X'} \]

as claimed.

Let assume \(X\) balanced by \(\Gamma_1\) and \(\Gamma_2\) on \(Z_1\) and \(Z_2\) and let (1) holds. We then have:

\[ N\Delta_{X \times X'} = \]
\[ = N\sigma_*(\Delta_X \times \Delta_{X'}) = \text{by (2)} \]
\[ = \sigma_*((\Gamma_1 + \Gamma_2) \times \Delta_{X'}) = \]
\[ = \sigma_*(\Gamma_1 \times \Delta_{X'}) + \sigma_*(\Gamma_2 \times \Delta_{X'}) \]

where the support of \(\sigma_*(\Gamma_1 \times \Delta_{X'})\) is contained in \(Z_1 \times X \times X'\) and the support of \(\sigma_*(\Gamma_2 \times \Delta_{X'})\) is in \(X \times X' \times Z_2 \times X\). Then the product \(X \times_k X'\) is balanced on \(Z_1 \times X'\) and \(Z_2 \times X'\). By symmetry also the product \(X' \times_k X\) is balanced.

We are now going to give another description of the variation of a balancing under birational maps between non-singular varieties. We recall that for any regular algebraic scheme \(V\) of pure dimension \(n\) and \(S \subset V\) a closed subscheme we have a duality isomorphism

\[ H^p_S(V,K_p) \cong CH_{n-p}(S) \] (5)

where \(K_p\) is the Zariski sheaf on \(V\) associated with Quillen \(K\)-theory (cf. [3]). If we have a cartesian square

\[
\begin{array}{ccc}
S' & \rightarrow & V' \\
\downarrow & & \downarrow f \\
S & \rightarrow & V
\end{array}
\]

we then have a commutative diagram

\[
\begin{array}{ccc}
H^p_S(V,K_p) & \rightarrow & H^p(V,K_p) \\
\downarrow & & \downarrow & \\
H^p_S(V',K_p) & \rightarrow & H^p(V',K_p)
\end{array}
\]

whence by (3) the following commutative diagram of Chow groups

\[
\begin{array}{ccc}
CH_{n-p}(S) & \rightarrow & CH_{n-p}(V) \\
\downarrow & & \downarrow f^* \\
CH_{n'-p}(S') & \rightarrow & CH_{n'-p}(V')
\end{array}
\] (6)

where \(i\) and \(i'\) are the imbedding of \(S\) in \(V\) and \(S'\) in \(V'\) respectively.

Let \(f : X' \rightarrow X\) be a proper birational morphism between non-singular varieties. For
$f : X' \to X$ as above we let consider the cartesian square

$$
\begin{array}{c}
E_f \\
\downarrow e \\
X' \times X' \\
\downarrow f \times f \\
X \times X
\end{array}
\xrightarrow{\pi} \xrightarrow{\delta} \xrightarrow{\pi'} \xrightarrow{\delta} X
$$

where $\delta$ is the diagonal imbedding. Because of (6) we obtain $(n = \dim X = \dim X')$

$$
CH_n(X \times X) \xrightarrow{(f \times f)^*} CH_n(X' \times X') \\
\uparrow \delta_* \uparrow \pi_* \\
CH_n(X) \xrightarrow{\pi^*} CH_n(E_f)
$$

and moreover:

(i) there is a closed imbedding $i : X' \hookrightarrow E_f$ such that $eoi = \delta_{X'}$ and $\pi oi = f$;

(ii) if $Z' = f^{-1}(Z)$ is a closed in $X'$ such that $f : X' - Z' \sim X - Z$ then $e(E_f - X') \subset Z' \times Z' - \Delta_{Z'}$;

(iii) $E_f$ can also be regarded as the fiber product of $f$ with itself i.e.

$$
\begin{array}{c}
E_f \\
\downarrow \pi' \\
X' \\
\downarrow f \\
X
\end{array}
\xrightarrow{\pi} \xrightarrow{f} X
$$

is a cartesian square, where $f \circ \pi' = \pi$ and $\pi' : E_f \to X'$ has a section given by $i : X' \hookrightarrow E_f$; we then have a splitting exact sequence

$$
0 \to CH_n(X') \xrightarrow{(f \times f)^*} CH_n(E_f) \to CH_n(E_f - X') \to 0
$$

because of $\pi^*oi_* = 1$.

**Lemma 1.7** Notations and hypothesis as above. Then there is a cycle $\zeta_f$ on $X' \times X'$ supported on $Z' \times Z'$ such that the following equation

$$
(f \times f)^* \Delta_X = \Delta_{X'} + \zeta_f
$$

holds in $CH_n(X' \times X')$.

**Proof** Because of (6) we have that

$$
(f \times f)^* \Delta_X = (f \times f)^*(\delta_{X'}[X]) = e_*\pi^*[X]
$$

So we are left to show the equation $e_*\pi^*[X] = \Delta_{X'} + \zeta_f$. The projection $\pi'_*\pi^*[X] \in CH_n(E_f)$ is $[X']$ since the composite of

$$
H^n_X(X \times X, \mathcal{K}_n) \to H^n_{E_f}(X' \times X', \mathcal{K}_n) \to H^n_X(X \times X, \mathcal{K}_n)
$$
is the identity, by [3, Section 7] because $f \times f$ is birational, and the composite of

$$CH_n(E_f) \xrightarrow{\pi_*} CH_n(X') \xrightarrow{f_*} CH_n(X)$$

is $\pi_*$ by (iii) above, where $f_*$ is the isomorphism sending $[X']$ to $[X]$ because $f$ is mapping birationally each component of $X$ to a distinct component of $X'$.

Moreover by (ii) we have a commutative diagram

$$
\begin{array}{ccc}
CH_n(E_f - X') & \xrightarrow{e_*} & CH_n(Z' \times Z' - \Delta_{Z'}) \\
\downarrow & & \downarrow \\
CH_n(X' \times X' - \Delta_{X'}) & \xleftarrow{f_*} & CH_n(X' \times X')
\end{array}
$$

since $Z'$ has dimension $\leq n - 1$ whence $CH_n(Z') = 0$. Thus we are able to give a description of $e_*\pi^*[X]$ w.r.t. the splitting (5): from the latter diagram we see that the restriction of $\pi^*[X]$ to $CH_n(E_f - X')$ (i.e. a component of $\pi^*[X]$) has image (i.e. $e_*$ of it in $CH_n(X' \times X')$) is a cycle $\zeta_f$ supported on $Z' \times Z'$; the component of $\pi^*[X]$ in $CH_n(X')$ (i.e. $i_*[X']$) has image $\Delta_X'$ because of (i) i.e. $e_*\otimes_i = \delta_{X'}$. The claimed formula (6) is obtained.

Assuming $X$ balanced we have:

$$(f \times f)^*(\Gamma_1 + \Gamma_2) = \text{ by (1)}$$

$$= N(f \times f)^*\Delta_X = \text{ by (8)}$$

yielding the equation

$$N\Delta_{X'} = (f \times f)^*(\Gamma_1) + (f \times f)^*(\Gamma_2) - N\zeta_f$$

Now, because of (9) and (8), we have that the cycle $(f \times f)^*(\Gamma_1)$ is supported on $f^{-1}(Z_1) \times X'$ and $(f \times f)^*(\Gamma_2) - N\zeta_f$ on $X' \times f^{-1}(Z_2) \cup Z' \times Z'$. Then $X'$ is balanced on $f^{-1}(Z_1)$ and $f^{-1}(Z_2) \cup Z'$.

In the following any flat morphism will have a relative dimension.

**Definition 1.8:** Let $p : X \to S$ be a morphism of varieties over $k$ where $S$ is irreducible of dimension $i$. We say that $p : X \to S$ is balanced or that $X$ is balanced over $S$ if $p : X \to S$ is flat of relative dimension $n$ and the equation $\Delta_{X/S} = \Gamma_1 + \Gamma_2$ holds in $CH_{n+i}(X \times_S X)$ where $\Gamma_1$ is supported on $Z_1 \times_S X$ and $\Gamma_2$ is supported on $X \times_S Z_2$ for some $Z_1$ and $Z_2$ proper closed subschemes of $X$ which are flat over $S$.

**Proposition 1.9** We have the following basic properties.

**Base change** Let $p : X \to S$ be a balanced morphism, $\varphi : S' \to S$ be any flat morphism (but $S'$ irreducible) and $X' = X \times_S S'$. Then $p' : X' \to S'$, obtained by base extension, is balanced.
Composition If $p : X \to S$ is balanced and $\varphi : S \to T$ is flat ($T$ irreducible) then $\varphi \circ p : X \to T$ is balanced.

Product If $X$ is balanced over $S$ and $Y$ is flat over $S$ then $X \times_S Y$ is balanced over $S$.

Proof A balancing for the base extension $p' : X' \to S'$ can be obtained by making use of external products of flat families of cycles because of the equation $\Delta_{X'/S'} = \Delta_{X/S} \times_S [S']$ (cf. the proof of Proposition 1.5). The composition $\varphi \circ p : X \to T$ is known to be flat and the imbedding $j : X \times_S X \hookrightarrow X \times_T X$ is known to be closed, yielding the equation $j_* \Delta_{X/S} = \Delta_{X/T}$. Finally: $X \times_S Y \to S$ is the composition of the flat morphism $Y \to S$ with the base extension.

By the above, if $X$ is balanced over $S$ then $X$ is balanced over the base field. Moreover $X$ can be view as an algebraic family and, for example, we have that $X_s$, the fibre at the generic point $s \in S$, is balanced over $K(S)$ (the function field of $S$). Indeed, let $V$ be any Zariski open neighborhood of the generic point $s$ in $S = \{s\}$: by flat base change we have that $X \times_S V \to V$ is balanced. The claim above is obtained by taking the limit over $V$ because we have

$$CH_n(X_s) \cong \lim_{\to V} CH_n(X \times_S V)$$

and $\Delta_{X_s/K(S)} = \lim_{\to V} \Delta_{X \times_S V/V}$.

Definition 1.10: Let $X$ be a variety over a field $k$. We say that $X$ is $K$-balanced if $X_K$ is balanced over the field extension $k \subseteq K$. If $\Omega$ is a universal domain in the sense of Weil and $X$ is $\Omega$-balanced we say that $X$ is universally balanced.

We will see in the next Section, that universally balanced varieties are also balanced over the base field.

2 Actions of a balancing

Let $(H^*, H_*)$ be an appropriate duality theory in the sense of [3, Sections 6-7] e.g. De Rham theory or $\ell$-adic étale theory. The twisted cohomology functor $H^*(-, \cdot)$ is then equipped with a functorial ‘cycle class’ map $c_\ell : H^p(X, \mathcal{H}^p(p)) \to H^{2p}(X, p)$ for $X$ smooth over $k$. An algebraic ‘$\mathcal{H}$-correspondence’ from $X$ to $Y$ is an $\mathcal{H}$-cohomology class $\alpha \in H^{n+r}(X \times Y, \mathcal{H}^{n+r}(n+r))$ ($X$ has pure dimension $n$) and it will be denoted by $\alpha : X \sim Y$. As usual $\alpha$ acts on the cohomology groups of non-singular projective varieties

$$\alpha^*_\ell : H^i(X, j) \to H^{i+2r}(Y, j+r)$$

where $\alpha^*_\ell(-) \overset{\text{def}}{=} p_{Y,*}(\overline{c_\ell(\alpha) \cdot p_X^*(-)})$ and $p_Y, p_X$ are the projections of $X \times Y$ on $Y$ and $X$ respectively. Because of our assumptions there is a canonical ring homomorphism from the Chow ring to the $\mathcal{H}$-cohomology ring (cf. [3, Sections 5.5 and 6.3]). The action of any algebraic cycle on a fixed cohomology theory is given via its $\mathcal{H}$-cohomology incarnation e.g. any cycle
algebraically equivalent to zero acts as zero on the singular cohomology of the associated analytic space. In particular any \( \alpha : X \sim Y \) as above acts on the \( \mathcal{H} \)-cohomology groups
\[
\alpha_2 : H^p(X, \mathcal{H}^q(j)) \rightarrow H^{p+r}(Y, \mathcal{H}^{q+r}(j+r))
\]
in a compatible way w.r.t. the coniveau spectral sequences.
An easy application of the projection formula (for \( \mathcal{H} \)-cohomologies we need \cite[Section 5.4]{4}) yields the following useful Lemma.

**Lemma 2.1** Let \( f : X' \rightarrow X \) and \( g : Y' \rightarrow Y \) be morphisms of non-singular projective varieties. For \( \alpha : X' \sim Y \) we have
\[
(f \times 1_Y)_*(\alpha)_2 = \alpha_2 \circ f^*
\]
For \( \beta : X \sim Y' \) we have
\[
(1_X \times g)_*(\beta)_2 = g_\ast \circ \beta_2
\]

**Proof** The proof is left as an exercise for the reader.

Let now consider a balanced variety \( X \) which is non-singular and projective; we are going to make use of \cite{1} w.r.t. the various cohomology theories. We need to assume resolution of singularities and we let assume that \( Z_1 \) and \( Z_2 \) are equidimensional. We then may consider a resolution \( Z_1' \rightarrow Z_1 \) and we have that \( \Gamma_1 = (f \times 1_X)_*(\Gamma_1') \) where \( \Gamma_1' \) is a cycle on \( Z_1' \times X \) of codimension equal to the dimension of \( Z_1 \) and \( f : Z_1' \rightarrow X \), so that by (10) we obtain
\[
(\Gamma_1)_2 = (f \times 1_X)_*(\Gamma_1'_2) = (\Gamma_1'_2) \circ f^*
\]
since \( \Gamma_1' : Z_1' \sim X \); i.e. we have a commutative triangle
\[
\begin{array}{ccc}
H^p(X, \mathcal{H}^q(\cdot)) & \xrightarrow{\Gamma_1'_2} & H^p(X, \mathcal{H}^q(\cdot)) \\
\downarrow & & \uparrow \\
H^p(Z_1', \mathcal{H}^q(\cdot)) & & \\
\end{array}
\]
We consider as well a resolution of singularities \( Z_2' \rightarrow Z_2 \) and we have \( \Gamma_2 = (1_X \times g)_*(\Gamma_2') \) where \( \Gamma_2' \) is a cycle on \( X \times Z_2' \) whence the degree of the correspondence \( \Gamma_2' : X \sim Z_2' \) is \( -\text{codim}_X(Z_2) \); by (10) we then have
\[
(\Gamma_2)_2 = (1_X \times g)_*(\Gamma_2'_2) = g_\ast \circ (\Gamma_2'_2)_2
\]
i.e. the following triangle
\[
\begin{array}{ccc}
H^p(X, \mathcal{H}^q(\cdot)) & \xrightarrow{\Gamma_2'_2} & H^p(X, \mathcal{H}^q(\cdot)) \\
\downarrow & & \uparrow \\
H^{p-c}(Z_2', \mathcal{H}^{q-c}(\cdot-c)) & & \\
\end{array}
\]
commutes, where \( c = \text{codim}_X(Z_2) \). If \( Z_2 \) (or \( Z_1 \)) is not equidimensional we may consider each smooth component of its resolution \( Z_2' \) acting as above.
Proposition 2.2 Let consider any cohomology theory $H^*$ as above. Let $cd(k)$ be the “cohomological dimension” of the field $k$ i.e. we assume that $H^i(U) = 0$ if $i > \dim U + cd(k)$ and $U$ is affine. If $X$ is balanced of weight $w$, and either the pans are smooth or can be resolved, then $H^0(X, \mathcal{H}^q(\cdot))$ is $N$-torsion for $q > w + cd(k)$ and some positive integer $N$.

Proof Is essentially the same of [3, Th. 1, ii–iii]. By interchanging the pans we may assume that $w = \dim Z_1$. The action of the cycle $N\Delta_X$ is the multiplication by $N$ which is given by $\Gamma_{k^2} + \Gamma_{k^2}$. Because of the assumptions and (13) we have $\Gamma_{k^2} = 0$ since $\mathcal{H}^q = 0$ on $Z_1'$ if $q > w + cd(k)$. Because of (15) $\Gamma_{k^2} = 0$ since $\text{codim}_k(Z_2) > 0$.

Whenever the sheaves $\mathcal{H}^q(\cdot)$ are torsion free we have that $H^0(X, \mathcal{H}^q(\cdot)) = 0$ under the assumptions in the Proposition above. Because of (2) (cf. [1], [4]) we then have:

Corollary 2.3 Let $X$ be balanced over $\mathcal{C}$ of weight $w$ and let assume that Kato’s conjecture hold true for function fields. If $q > w$ we then have

$$H^0(X, \mathcal{H}^q(Z)) = 0$$

and

$$H^0(X, \mathcal{H}^{q+1}(\mathcal{Z}(q)_{\mathcal{D}})) = 0$$

where $\mathcal{H}^q(Z)$ and $\mathcal{H}^{q+1}(\mathcal{Z}(q)_{\mathcal{D}})$ are the Zariski sheaves associated with singular cohomology and Deligne–Beilinson cohomology respectively.

As in [3, Th. 1] by the above one obtains the following: for $X$ balanced over $\mathcal{C}$ of weight $\leq 2$ algebraic and homological equivalence coincide for codimension 2 cycles and for $X$ of weight $\leq 1$ also homological and Abel-Jacobi equivalence coincide in codimension 2. Indeed the Griffiths group is a quotient of $H^0(X, \mathcal{H}^3(Z))$ and the Abel-Jacobi kernel is a quotient of the torsion free group $H^0(X, \mathcal{H}^3(Z(2)_{\mathcal{D}}))$ (cf. [3] for surfaces).

Moreover:

Proposition 2.4 Let $X$ (projective, non-singular) be balanced over $\mathcal{C}$ of weight $w$. If $q > w$ then

$$H^0(X, \Omega_X^q) = 0$$

Proof The previous arguments applies to the twisted cohomology theory given by $F \cdot H^*$ i.e. the De Rham filtration, and $F^qH^q(X) = H^0(X, \Omega_X^q)$. Now $\Gamma_{k^2} = 0$ because $\Omega^2_{Z_1} = 0$ if $q > w = \dim Z_1$ and (13). Moreover $\Gamma_{k^2} = 0$ by $F^{q-c}H^{q-2c}(Z_2') = 0$ since $c > 0$ i.e. the codimension of any component of $Z_2'$ cannot be zero. Since $F \cdot H^*$ are real vector spaces we are done.

Proposition 2.5 If $X$ (projective, non-singular) is balanced of weight $w$ over a field $k$, and either the pans are smooth or can be resolved, then there exists a closed subscheme $Y$ of $X$ such that $\text{CH}^i(X - Y) = 0$ for all $i > w$. For 0-cycles $Y$ exists of dimension $w$. If, moreover, $k$ is algebraically closed then $A^i(X - Y) = 0$ for all $i > w$ and $A_0(X - Y) = 0$ as above; finally, in this case, the Albanese kernel of $X$ is contained in the Albanese kernel of the resolved pan of dimension $w$. 
Proof We may assume that $Z_1$ is irreducible and non-singular of dimension $w$: therefore we obtain that $\Gamma_{1} = 0$ on $CH^i(X)$ if $i > w$, by (13) and $CH^i(Z_1) = 0$. Then $\Gamma_{2} \otimes \mathbb{Q}$ is the isomorphism induced by the multiplication by $N$ on $CH^i(X)_{\mathbb{Q}}$ if $i > w$; then, by (13), the Chow group of $Z_2$ surjects onto $CH^i(X)_{\mathbb{Q}}$ if $i > w$. By taking $Z_2$ as $Y$ we have that $CH^i(X - Y)_{\mathbb{Q}} = 0$ as claimed. If $i = \dim X$ i.e. for 0-cycles, we then can take $Z_2$ of dimension $w$ and we conclude as above. If $k$ is closed it is well known that $A_0(X)$ is divisible and the Albanese kernel is uniquely divisible; by the same argument we obtain the claimed results.

• Following Bloch, Srinivas and Jannsen (cf. [5], [6], [7, Remark 1.7]) we have:

Proposition 2.6 Let $X$ be a non-singular projective variety over a field $k$. Then $X$ is universally balanced of weight $\leq 1$ if and only if the Chow group of 0-cycles of degree zero is representable.

Proof By the definition, $X$ is universally balanced if $X_{\Omega}$ is balanced over a universal domain $\Omega$. By the Proposition 2.3 the Albanese kernel $X_{\Omega}$ is contained in the Albanese kernel of (at most) a smooth curve over $\Omega$, which is zero. Conversely, if the Chow group of 0-cycles is representable then by [6, Prop. 1.6] there is (at most) a smooth projective curve over $\Omega$ and a morphism $g : C \to X_{\Omega}$ such that $A_0(X_{\Omega} - g(C)) = 0$ hence by [6, Prop. 1] $X_{\Omega}$ is balanced of weight $\leq 1$.

Proposition 2.7 Let $X$ be a non-singular projective variety over a field $k$. If $X$ is universally balanced then it is balanced.

Proof The proof is similar to [6, Proof of Th. 3.5.(a)-(b)] which is quite the same of the proof of Proposition 1 of [6].

Following Jannsen [7, § 3] we have:

Corollary 2.8 If $X$ is universally balanced of weight $\leq w$ then $H^i(X,j)$ (= any twisted cohomology theory in the sense of [7, § 3]) is of coniveau 1 for $i > w$.

Finally, by Proposition 2.4 and Proposition 2.6 we can easily obtain the following.

Corollary 2.9 (Mumford-Roitman Theorem) If $H^0(X,\Omega^q_X) \neq 0$ for some $q > 1$ then the Chow group of 0-cycles of degree zero is not representable.

See also [7, Cor. 3.7].

3 Paradigma

We now give some examples.

Example 3.1: The projective space $P^n_k$ over any field $k$ is balanced of weight 0 i.e. the pans are given by any couple of closed points $x_0$ and $x_1$. In fact we have that

$$CH_n(P^n_k \times P^n_k) \cong (P^n_k \times P^n_k) \mathbb{Z} \bigoplus (P^n_k \times x_1) \mathbb{Z}$$
Example 3.2: Because of Corollary [1.4] and the above, unirational varieties are balanced over any field. More generally, because of Proposition [1.5], we have that uniruled varieties are balanced, since are dominated by a product with $\mathbb{P}^1_k$.

Example 3.3: If $X$ is a smooth closed subvariety of an abelian variety then $X$ is not balanced over $\mathbb{C}$, since $H^0(X, \Omega^n_X) \neq 0$ for $n = \dim X$ and Proposition [2.4].

Example 3.4: The Kummer 3-fold over $\mathbb{C}$ is balanced of weight 2 by Bloch and Srinivas [6] since $A_0(X)$ is generated by cycles supported on a finite number of surfaces. This is an example of balanced variety which is quite far from ruled varieties.

Some remarks and questions

1. Of course one might like to define balanced schemes starting from an equidimensional separated scheme over an arbitrary base scheme but then does one have anything to say about it? Nevertheless one case that sounds interesting is the case of arithmetic schemes i.e. regular schemes which are projective and flat over the integers, by mean of arithmetic cycles and correspondences in the sense of H. Gillet and C. Soulé. So I expect a parallel theory of “arithmetically balanced varieties” by analysing the action of arithmetic correspondences.

2. In the language of balanced varieties, Bloch’s conjecture give us a numerical criterion for balanced surfaces i.e. $p_g = 0$ for weight 1 and $p_g = q = 0$ for weight zero. We may expect that the following numerical criterion

$$X \text{ balanced of weight } w \iff H^0(X, \Omega^q_X) = 0 \text{ for } q > w$$

holds true for any projective complex manifold $X$.

More in general, for $X$ defined over any field $k$, $X$ will possibly be universally balanced of weight $\leq w$ if and only if any suitable ‘cohomology theory’ will be of coniveau 1 in degrees $> w$.

3. Since the global forms are invariants under a smooth deformation, the previous speculation is suggesting us that the balancing should be a deformation property. We remark that uniruled varieties do have this property, thanks to A. Fujiki and M. Levine (deformation invariance of rationally connected varieties is due to J. Kollar, Y. Miyaoka and S. Mori).

4. Let say that a flat family of varieties, parametrized by a nice variety, is balanced if its general member is a balanced variety. Is there a section?
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