A CONNECTION BETWEEN THE GOOD PROPERTY OF AN ARTINIAN GORENSTEIN LOCAL RING AND THAT OF ITS QUOTIENT MODULO SOCLE

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Abstract. Following Roos, we say that a local ring $R$ is good if all finitely generated $R$-modules have rational Poincaré series over $R$, sharing a common denominator. Rings with the Backelin-Roos property and generalised Golod rings are good due to results of Levin and Avramov respectively. Let $R$ be an Artinian Gorenstein local ring. The ring $R$ is shown to have the Backelin-Roos property if $R/\text{soc}(R)$ is a Golod ring. Furthermore the ring $R$ is generalised Golod if and only if $R/\text{soc}(R)$ is so.

We explore when connected sums of Artinian Gorenstein local rings are good. We provide a uniform argument to show that stretched, almost stretched Gorenstein rings are good and show further that the Auslander-Reiten conjecture holds true for such rings. We prove that Gorenstein rings of multiplicity at most eleven are good. We recover a result of Rossi-Şega [40] on the good property of compressed Gorenstein local rings in a stronger form by a shorter argument.

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1. Introduction

Let $R$ be a commutative Noetherian local ring with maximal ideal $m$ and residue field $k = R/m$. Let $M$ be a finitely generated module over $R$. The Poincaré series $P^R_M(t)$ of $M$ over $R$ is a formal power series in $\mathbb{Z}[\lvert t \rvert]$ defined as follows:

$$P^R_M(t) = \sum_{i \geq 0} \beta^R_i(M)t^i \in \mathbb{Z}[\lvert t \rvert],$$

where $\beta^R_i(M) = \dim_k \text{Tor}^R_i(M, k)$ denotes the $i$-th Betti number of $M$. In the 1950’s Serre and Kaplansky asked whether the Poincaré series $P^R_k(t)$ is a rational function. This question ties in closely with an analogous question in Algebraic topology attributed to Serre on the rationality of the Poincaré series of loop spaces of finite simply connected CW complexes (see [38]). Several authors made an attempt to find an affirmative answer. However in 1982 Anick found a counterexample answering the question in negative [6]. Research on this topic intensified since the appearance of Anick’s example. Bøgvad reworked Anick’s example and produced an Artinian Gorenstein local ring whose residue field has an irrational Poincaré series [14]. Jacobsson constructed a local ring $R$ with residue field $k$ and a cyclic $R$-module $M$ such that $P^R_k(t)$ is rational, but $P^R_M(t)$ is irrational [28, Corollary C]. Roos defined a ring $R$ to be good if there exists a polynomial $d_R(t) \in \mathbb{Z}[t]$ such that $d_R(t)P^R_M(t) \in \mathbb{Z}[t]$ for every finitely generated $R$-module $M$ [39, Definition 2.1]. He constructed bad (i.e. not good) affine rings defined by quadratic monomial relations [39, Theorem 2.4].

Nevertheless there are an abundance of good rings, e.g. Golod rings [27, Theorem 1], complete intersections [26, Corollary 4.2], rings $R$ with $\dim(R) - \text{depth}(R) \leq 4$ [10, Theorem 6.4], etc. (see [13] for more examples). McCullough and
Peeva mentioned in [36, Example 6.4] that to date we do not have a good understanding of how likely it is that rings with residue fields having rational or irrational Poincaré series occur. One of the main objectives of the present article is to show that at low multiplicities Gorenstein local rings are always good.

A rational expression for a Poincaré series $P_M^R(t)$ has practical applications. If $P_M^R(t)$ is rational, we write the denominator as $d_R(t) = 1 - \sum_{j=1}^a h_j t^j \in \mathbb{Q}[t]$. Then from $d_R(t)P_M^R(t) \in \mathbb{Q}[t]$ it follows that the Betti sequence $\{\beta_n^R(M)\}$ eventually satisfies a linear recurrent relation:

$$\beta_n^R(M) = \sum_{j=1}^a h_j \beta_{n-j}^R(M), \quad h_j \in \mathbb{Q}, n \gg 0.$$  

Precise information on the coefficients $h_j$ allows for efficient estimates of the asymptotic behaviour of the Betti sequence $\{\beta_n^R(M)\}$ [11]. Information on the common denominator is also useful to prove results on vanishing of co-homologies (see [42]). We refer the reader to [12], for a detailed list of applications.

Stretched Cohen-Macaulay local rings were introduced by Sally in [41]. Later Elias and Valla introduced almost stretched Cohen-Macaulay local rings in [20].

**Definition.** An Artinian local ring $R$ with maximal ideal $m$ is called stretched if $m^2$ is a principal ideal and almost stretched if $m^2$ is minimally generated by two elements.

Let $R$ be a Cohen-Macaulay local ring of dimension $d$ with maximal ideal $m$. Then $R$ is called stretched (almost stretched) if there exists a minimal reduction $x = x_1, \ldots, x_d$ of $m$ such that $R/(x)$ is a stretched (almost stretched) Artinian ring. Here by minimal reduction we mean that $x$ satisfies $m^{r+1} = (x_1, \ldots, x_d)m^r$ for some non-negative integer $r$.

Sally proved that $P_k^R(t)$ is rational for a stretched Cohen-Macaulay local ring $R$ if the characteristic of its residue field $k$ is not two [41, Theorem 2]. Elias and Valla imitating the proof of Sally, proved that $P_k^R(t)$ is rational if $R$ is an almost stretched Gorenstein ring and the residue field $k$ of $R$ has characteristic zero [21, Theorem 1.1]. A common step in the proof of both the results is to show that $P_k^{R/\text{soc}(R)}(t)$ is rational after reducing to the case when $R$ is in addition Artinian and Gorenstein. Rationality of $P_k^R(t)$ then follows by a theorem of Avramov and Levin which implies that if $R$ is an Artinian, Gorenstein ring, then proving rationality of $P_k^R(t)$ is tantamount to proving rationality of $P_k^{R/\text{soc}(R)}(t)$ (see Theorem 2.1). We raise the following question.

**Question 1.** Let $R$ be an Artinian Gorenstein local ring. Is $R$ good if $R/\text{soc}(R)$ is so and vice versa?

Tate described an iterated process of killing cycles by adjunctions of sets of $\Gamma$-variables in different degrees starting from a local ring $R$ to construct a DG algebra resolution $\epsilon : R(X) \rightarrow k$, $X = \{X_i : i \geq 1\}$ of its residue field $k$ [43, §2]. The ring $R$ is called Golod if $R(X_i : \deg(X_i) \leq 1)$ which is same as the Koszul complex of $R$ admits a trivial Massey operation (§2.4). In [12] Avramov generalised this notion by defining $R$ to be a generalised Golod ring of level $l$ if the DG algebra $R(X_i : \deg(X_i) \leq l)$ admits a trivial Massey operation (see §2.5 for details).

A local ring $R$ is said to have the Backelin-Roos property if its completion $\hat{R}$ with respect to its maximal ideal is a homomorphic image of a complete intersection under a Golod homomorphism (see §2.5). A classical result of Levin states that if a ring $R$ has the Backelin-Roos property, then $R$ is good. Avramov observed that rings with the Backelin-Roos property are generalised Golod rings of level 2. In particular, any complete intersection is a generalised Golod ring of level 2. In this article we prove the following two results.
Theorem II. (Theorem 4.3) Let $R$ be an Artinian Gorenstein local ring of embedding dimension $n \geq 2$ such that $R/\text{soc}(R)$ is a Golod ring. Let $\eta : Q \twoheadrightarrow R$ be a minimal Cohen presentation and $I = \ker(\eta)$. Then the following hold.

1. For any $f \in I \setminus nI$, the induced map $Q/(f) \twoheadrightarrow R$ is a Golod homomorphism, i.e. $R$ has the Backelin-Roos property.

2. Let $d_R(t) = 1 - t(P^{Q}_{R}(t) - 1) + t^{n+1}(1 + t)$. Then for any $R$-module $M$ we have $d_R(t)P^R_M(t) \in \mathbb{Z}[t]$.

Theorem III. (Theorem 4.6) Let $R$ be an Artinian Gorenstein local ring. Let $l \geq 2$. Then $R$ is a generalised Golod ring of level $l$ if and only if $R/\text{soc}(R)$ is so.

Avramov proved that generalised Golod rings are good (see Theorem 2.3). It is not known if the converse holds true (see [13, Problem 10.3.5]). So our Theorems II, III provide a satisfactory answer to Question I.

We find several interesting consequences of our theorems. In a recent article [2, Corollary 5.6], it is proved that stretched Cohen-Macaulay local rings are good. In Theorem 3.7, we show that quotients of stretched or almost stretched Artinian Gorenstein rings by non zero powers of maximal ideal are Golod rings. So from Theorem II, it follows that such rings are good without any assumption on characteristic of residue fields. More generally we prove that stretched Cohen-Macaulay and almost stretched Gorenstein rings are good (see Theorem 6.1, Corollary 6.2, 6.3). Using a result of Šega, we prove that the Auslander-Reiten conjecture holds true for such rings. We state our result in the Artinian Gorenstein case below where $\mu(-)$ denotes the minimal number of generators.

Theorem IV. (Theorems 6.1, Corollary 6.5) Let $R$ be an Artinian Gorenstein local ring with maximal ideal $m$ such that $\mu(m) = n$ and $\mu(m^2) \leq 2$. Let $M$ be a finitely generated $R$-module. Then the following hold.

1. If $n = 1$, then $P^{R}_{1}(t) = 1/t - 1$ and $(1 - t)P^{R}_{1}(t) \in \mathbb{Z}[t],

2. If $n \geq 2$, then $P^{R}_{n}(t) = 1/(1 - t)^{nt} - 1$ and $(1 + t)^n(1 - nt + t^2)P^{R}_{n}(t) \in \mathbb{Z}[t],

3. If $\text{Ext}^i(M, M \oplus R) = 0$ for all $i \geq 1$, then $M$ is a free $R$-module.

In [40, Theorem 5.1], Rossi and Šega proved that compressed Artinian Gorenstein local rings with socle degrees not equal to three have the Backelin-Roos property. We recover this result in a stronger form in Corollary 6.7 (see the discussion preceding Corollary 6.7). To prove this result, we identify using Theorem II above a certain quotient $C$ of the Koszul algebra $K^R$ of an Artinian Gorenstein ring $R$ such that $R$ has the Backelin-Roos property whenever $C$ is a Golod DG algebra (§2.4). We show that for such rings this quotient is a Golod algebra. By a similar approach, we prove that a Gorenstein local ring of codimension at most three has the Backelin-Roos property (see Corollary 6.8). Our version is a stronger form of a result of Avramov, Kustin and Miller [10, Theorem 6.4] when $R$ is Gorenstein.

Motivated by the description of cohomology algebras of connected sums of compact connected oriented $n$-manifolds, authors in [3] introduced the connected sum $R\#S$ of Gorenstein local rings $R$, $S$ with a common residue field $k$ (Definition 2.4). They obtained a formula for the Poincaré series of $k$ over $R\#S$ in terms of the Poincaré series of $k$ over $R$ and $S$ [3, Corollary 7.4]. In particular, if both $P^{R}_{k}(t)$, $P^{S}_{k}(t)$ are rational functions, so is $P^{R\#S}_{k}(t)$. This naturally raises the question of whether connected sums of good Gorenstein local rings are good. We prove the following.

Theorem V. (Theorem 5.5) Let $R$, $S$ be Artinian Gorenstein local rings with a common residue field $k$, $l \geq 2$ and $T = R\#S$ denote the connected sum of $R$ and $S$. Then $T$ is a generalised Golod ring of level $l$ if and only if both $R$, $S$ are so.
To prove the above theorem we consider the corresponding problem for fibre products of local rings. In [31, Theorem 4.1], Lescot proved that a fibre product (see Definition 2.4) of local rings is Golod if and only if the constituent rings are so. We prove a counterpart of Lescot’s theorem more generally for generalised Golod rings (see Theorem 5.4). The above theorem then follows from Theorem III.

Rationality of Poincaré series of residue fields of Gorenstein local rings of low multiplicities have been an active topic of research for a long time. The Poincaré series $P_R^k(t)$ is known to be rational when $R$ is a Gorenstein local ring of multiplicity at most $n$ with residue field $k$ in the cases: $n = 5$, char$(k) \neq 2$ ([41]); $n = 7$, char$(k) = 0$ ([21, Corollary 2.1]); $n = 10$, $k = \bar{k}$, char$(k) = 0$ ([15]). Our motivation on this topic comes from a different perspective. In [19] Eisenbud conjectured that modules with bounded Betti numbers are eventually periodic of period two. Several authors proved the conjecture for modules over various good local rings using concrete information on the common denominator (see [11, Theorem 1.6], [29, Corollary 5.2]). However Gasharov and Peeva in [24] found an example of a graded Artinian Gorenstein ring of multiplicity twelve over which the conjecture fails. Using a completely different technique they were able to prove the conjecture for Gorenstein local rings of multiplicity at most eleven. This observation motivates us to investigate whether such rings are good. Statement (1) of the theorem below is a generalisation of [16, Corollary 2.2].

**Theorem VI.** (Theorem 6.9) Let $R$ be a Gorenstein local ring. Then $R$ is good, i.e. the Betti sequence of modules over $R$ eventually satisfies a linear recurrent relation in the following cases.

1. The square $m^2$ of maximal ideal $m$ of $R$ is minimally generated by at most four elements and $m^4 = 0$.
2. The multiplicity of $R$ is at most 11.

To prove the above result, it is enough to assume further that $R$ is Artinian. We decompose $R$ as a connected sum of Gorenstein Artinian rings either of embedding dimension at most four or with the cube of maximal ideal being zero unless $R$ is already one of these types. The result then follows from Theorem V as rings of such types are good.

**Structure of the paper:** In §2 we review some background material that we will require. Readers familiar with [13], [25] may skip this section and start reading from §3. In §3 we prove two important results viz. Theorems 3.4, 3.5 both of which give criteria for connected sum decompositions. We use Theorem 3.4 to prove Theorem 3.7 whereas Theorem 3.5 is used to decompose Gorenstein rings concerned in the Theorem VI above. The next two sections §4, §6 consist of the main technical part of the article. The underlying idea is very simple. Suppose $S = R/I$ is a quotient of a local ring $R$. A priori there is no known method to compare the acyclic closures of $R$ and $S$. The key new idea of the present article is to show that when $R$ is an artinian Gorenstein ring of embedding dimension at least 2 and $I = soc(R)$, then the acyclic closure of $S$ is a semi-free extension of the tensor product of $S$ and the acyclic closure of $R$ (see Proposition 4.5). Proposition 5.3 concerns construction of the acyclic closure of fibre products and is similar in spirit. The reader may read both the propositions in one go. The proofs of our main results in these two sections extensively use a characterisation theorem for Golod algebras (see Theorem 2.2) and derivations on acyclic closures (see Lemma 4.1) whose construction dates back to the work of Gulliksen. The final section §6 contains applications of our Main results, viz. Theorems IV and VI.

All rings in this article are Noetherian local rings with $1 \neq 0$. All modules are nonzero and finitely generated. Throughout this article, the expression “local ring $(R, m, k)$” refers to a commutative
Noetherian local ring \( R \) with maximal ideal \( m \) and residue field \( k = R/\mathfrak{m} \). When information on the residue field is not necessary, we denote a local ring \( R \) with maximal ideal \( m \) simply by \( (R, m) \).

2. Preliminaries

In this section we set notations and discuss some background. Let \( (R, m) \) be a local ring. The embedding dimension of \( R \) is denoted by \( e. \dim(R) \). By a minimal Cohen presentation, we mean a surjection \( \eta : Q \to \hat{R} \) from a regular local ring \( (Q, \mathfrak{n}) \) onto the \( m \)-adic completion \( \hat{R} \) of \( R \) such that \( e. \dim(Q) = e. \dim(R) \) equivalently \( \ker(\eta) \subset \mathfrak{n}^2 \). Existence of such a map is guaranteed by Cohen’s structure theorem. For an \( R \)-module \( M \), the length of \( M \) is denoted by \( l(M) \) and the minimal number of generators of \( M \) is denoted by \( \mu(M) \). Moreover if \( R \) is an Artinian ring, the socle of \( R \) is defined as \( \text{soc}(R) = (0 :_R m) \) and Loewy length of \( R \) as \( \mathbb{L}(R) = \max\{n : m^n \neq 0\} \). If \( I \) is an ideal of \( R \), then the residue class \( \overline{(x+i)} \) of an element \( x \in R \) in the quotient ring \( R/I \) is denoted by \( \bar{x} \). By the phrase “a natural quotient map \( q : R \to R/I \)” we mean the obvious surjective map defined by \( q(x) = \overline{(x+i)} \).

Throughout the article unless otherwise stated an over-line is used to denote either a homology class or a residue class modulo an ideal, which should be understood from the context. We briefly recall some definitions and state results needed in latter sections. For all other unexplained notations and terminology we refer the reader to [13], [25].

2.1. DG algebras, DG\(\Gamma \) algebras. Let \( (R, m, k) \) be a local ring. A DG algebra \( (A, \partial) \) over the ring \( R \) consists of a non-negatively graded strictly skew-commutative \( R \)-algebra \( A = \bigoplus_{i \geq 0} A_i \) such that \( A_0 = R/I \) for some ideal \( I \) of \( R \) and an \( R \)-linear differential map \( \partial \) of degree \(-1\) satisfying Leibniz rule (see [25, Chapter 1, §1]). The DG algebra \( A \) is called minimal if \( \partial(A) \subset mA \).

In this article we say a DG algebra \( (A, \partial) \) augmented if it is equipped with a surjective algebra homomorphism \( \epsilon : A \to k \) such that \( \epsilon(A_{\geq 1}) = 0 \) and \( \epsilon \circ \partial = 0 \). The map \( \epsilon \) is called an augmentation map. The augmentation map \( \epsilon \) induces a map on homology \( \tilde{\epsilon} : H(A) \to k \). We set \( I(A) = \ker \epsilon \) called an augmentation ideal, \( IH(A) = \ker \tilde{\epsilon} \) and \( IZ(A) = IA \cap Z(A) \). If the DG algebra \( (A, \partial) \) over \( R \) satisfies \( \partial(A_1) \subset mA_0 \), then \( A \) is augmented naturally with the surjective map \( \epsilon = q \circ pr \) where \( pr : A \to A_0 \) is the projection and \( q : A_0 \to k \) is the natural quotient map.

A skew commutative non-negatively graded algebra is called a \( \Gamma \)-algebra if to every element \( x \) of even positive degree there is associated a sequence of elements \( x^{(k)} \), \( k \geq 0 \) called divided powers of \( x \) satisfying usual axioms (see [25, Chapter, §7]). A DG\(\Gamma \) algebra is a DG algebra which is a \( \Gamma \)-algebra in its own right and admits divided powers compatible with its differential, i.e. the differential \( \partial \) satisfies \( \partial(x^{(k)}) = \partial(x)(x^{(k-1)}) \), \( \text{deg}(x) \) being even. We treat the local ring \( R \) as an augmented DG\(\Gamma \) algebra concentrated at degree zero with the natural quotient map \( \epsilon : R \to k \) as an augmentation map.

Let \( f : A \to B \) be a homomorphism of DG\(\Gamma \) algebras. An \( A \)-linear derivation of degree \( n \) on the DG\(\Gamma \) algebra \( B \) is a \( R \)-linear map \( \eta : B \to B[n] \) such that \( \eta(A) = 0 \) (\( A \)-linearity); \( \eta \) satisfies Leibniz rule, i.e. \( \eta(xy) = \eta(x)y + (-1)^{\text{deg}(x)}x\eta(y) \) for \( x, y \in A \), \( \eta(x^{(n)}) = \eta(x)(x^{(n-1)}) \) for \( x \in A \), \( \text{deg}(x) \) being even and \( \eta \) commutes with the differential \( \partial \) of \( B \) in the graded sense, i.e. \( \eta \circ \partial = (-1)^{\text{deg}(\partial)} \partial \circ \eta \).

Unless otherwise stated, we shall always assume that DG algebras of the form \( (A, \partial) \) are piece-wise Noetherian, i.e. \( A_0 \) is Noetherian and all \( A_i \) are finitely generated \( A_0 \) modules.

2.2. Acyclic closures, distinguished DG\(\Gamma \) algebras, Tate resolutions. Assume that \( (R, m, k) \) is a local ring and \( (A, \partial) \) is an augmented DG\(\Gamma \) algebra over \( R \) with augmentation map \( \epsilon_A : A \to k \). Tate constructed an augmented semi-free extension \( i : A \hookrightarrow A^* = A(X), X = \{X_i\} \) by a recurrent process of adjoining sets of variables (called \( \Gamma \)-variables) to kill cycles such that the augmentation
map $\epsilon_A: A^* \to k$ satisfies $\epsilon_A \circ i = \epsilon_A$ and $A^*$ is acyclic, i.e. $IH(A^*) = 0$ (see [13, §6.3.1] and [25, Theorem 1.2.3]). We call $A^*$ the acyclic closure of $A$. The set $X$ of variables adjoined is countable since by our assumption $A$ is piecewise Noetherian. We order the set $X = \{X_n: n \in \mathbb{N}\}$ of variables to assume that $\deg X_i \leq \deg X_j$ for $i \leq j$. Henceforth we assume that the set of adjoined variables in acyclic closures of DG algebras are ordered in this manner.

The acyclic closure satisfies $B(A^*) \subset I(A)^* A^*$ [25, Theorem 1.6.2]. Following [12, §1.3], the augmented DG algebra $A(X_i: \deg(X_i) \leq d)$, $d \geq 1$ obtained in intermediate steps of adjunction of variables to construct the acyclic closure $A^* = A(X)$ are called distinguished DG algebras over $A$. Thus a distinguished DG algebra $B$ over $A$ is an augmented semi-free extension $B = A(X_1, \ldots, X_n)$, $\deg X_i \leq \deg X_j$ for $i \leq j$, $\deg(X_n) = d$ such that $\partial(X_i)$, $\deg(X_j) = i + 1$ minimally generate $IH_i(A(X_i: \deg(X_i) \leq i))$ for $0 \leq i \leq d - 1$. Since any cycle of $B$ is in the boundary of the acyclic closure $A^*$ extending $B$ we have $Z(B) \subset (IA)^* B$.

If $R(X)$ is the acyclic closure of the local ring $R$, then its augmentation map $\tilde{\epsilon}: R(X) \to k$ forms a minimal DG algebra resolution of $k$. In the literature it is called the Tate resolution. Since the augmentation ideal $I(R) = m$, the Tate resolution and distinguished DG algebras over $R$ are minimal. We denote the Koszul algebra of $R$ on a minimal set of generators of $m$ by $KR$. Note that $K^R$ can be identified with the distinguished DG algebra $R(X_i: \deg(X_i) \leq 1)$.

2.3. Homotopy Lie algebras. Let $(R, m, k)$ be a local ring and $(A, \partial)$ be an augmented DG algebra with augmentation map $\epsilon: A \to k$. We define $Tor^A(k, k) = H(\epsilon \otimes_k A^*)$, $A^*$ being the acyclic closure of $A$ and $k \otimes_k A^* = \frac{k \otimes_k A^*}{(\text{da:deker}\epsilon)}$. We have a Hopf $\Gamma$-algebra structure on $Tor^A(k, k)$ and the graded $k$-vector space dual $Tor^A(k, k)^\vee$ is the universal enveloping algebra of a uniquely defined graded Lie algebra over $k$ (see [8, Theorem 1.1, 1.2]). This Lie algebra is called the homotopy Lie algebra of $A$ and denote by $\pi(A)$.

Set $I Tor^A(k, k) = \ker(\tilde{\epsilon}: Tor^A(k, k) \to k)$. Let $I Tor^A(k, k)$ be the ideal of $Tor^A(k, k)$ generated by products $ab$ for $a, b \in I Tor^A(k, k)$ and divided powers $x^{(2)}$ of elements $x \in I Tor^A(k, k)$ of even positive degrees. Then $\pi(A) = \text{Hom}(\frac{1}{I Tor^A(k, k)}, k)$ (see [9, §3.1]). Thus $\pi(A)$ is the graded vector space dual of the space of $\Gamma$-indecomposable elements in $I Tor^A(k, k)$.

2.4. Golod algebras. If $W = \oplus_{i \geq 0} W_i$ is a graded $k$-vector space, we define the Hilbert series $H_W(t) = \sum_{i \geq 0} \dim_k W_i t^i \in \mathbb{Z}[t]$. Let $(R, m, k)$ be a local ring. Let $A$ be a minimal DG algebra over $R$ such that $H_0(A) = k$ and each $A_i$ is a free $R$-module of finite rank. Then there is a term-wise inequality of power series $P_k^R(t) < \frac{H_{\text{alg}}(t)}{1 - \sum_{i=1}^{\infty} b_i^R}$. Following Levin [34, Chapter 1, §4], we define $A$ to be a Golod algebra when equality holds.

In general an augmented DG algebra $A$ over $R$ with augmentation map $\epsilon: A \to k$ is called a Golod algebra if $A$ admits a trivial Massey operation, i.e. there is a graded $k$-basis $b_R = \{b_i\}_{i \in A}$ of $IH(A)$, a function $\mu: \bigcup_{i=1}^{\infty} b_i^R \to A$ such that $\mu(b_i) \in IZ(A)$ with $cl\mu \mu(h_{i}) = h_{i}$ and setting $\tilde{a} = (-1)^{i+1}a$ for $a \in A_i$ one has

$$\partial \mu(h_{i_1}, \ldots, h_{i_p}) = \sum_{j=1}^{p-1} \mu(h_{i_1}, \ldots, h_{i_j}) \mu(h_{i_{j+1}}, \ldots, h_{i_p}).$$

2.5. Golod homomorphisms, Backelin-Roos property and generalised Golod rings. Let $f: (R, m) \to (S, n)$ be a surjective local homomorphism. Let $R(X)$ be the acyclic closure of $R$. Then $f$ is called a Golod homomorphism if $S \otimes_k R(X)$ is a Golod algebra. If $f$ is Golod homomorphism, we have $P_k^S(t) = \frac{P_k^R(t)}{1 - (P_k^S(t) - 1)}$. Let $\eta: Q \to R$ be a minimal Cohen presentation of $R$. Then $R$ is called
a Golod ring if $\eta$ is a Golod homomorphism, i.e. the Koszul complex $R^k$ is a Golod algebra. The ring $R$ is said to have the Backelin-Roos property if there is a $Q$-sequence $a_1, \ldots, a_m$ in ker($\eta$) such that the induced map $Q/(a_1, \ldots, a_m) \to \hat{R}$ is a Golod homomorphism.

The ring $R$ is called a generalised Golod ring of level $n$ if the distinguished DG algebra $R(X_i : deg(X_i) \leq n)$ is a Golod algebra. A generalised Golod ring of level one is a Golod ring. If $R$ has the Backelin-Roos property, then $R$ is a generalised Golod ring of level 2 [12, §1.8]. For more examples we refer the reader to [12, §1.8].

The following theorem is proved in [33, Theorem 2].

**Theorem 2.1.** Let $(R, m, k)$ be an Artinian Gorenstein local ring such that $\text{e. dim}(R) \geq 2$. Then the quotient map $R \to R^{\text{soc}(R)}$ is a Golod homomorphism and $P_k^{R^{\text{soc}(R)}}(t) = \frac{P_k^R(t)}{1 - \epsilon^2 P_k^R(t)}$.

We recall the definition of graded Lie algebras and other necessary terminologies from [8, §1]. The following result follows from a blend of [34, Theorem 1.3], [9, Theorem 2.3].

**Theorem 2.2.** Let $(R, m, k)$ be a local ring and $A$ be a minimal DG$^*$ algebra over $R$ such that $H_0(A) = k$ and each $A_i$ is a free $R$-module of finite rank. Let $A^*$ be the acyclic closure of $A$. Consider $A$ augmented naturally. Then the following are equivalent.

1. The DG algebra $A$ is a Golod algebra.
2. We have $Z_{S+1}(A) \subset mA$ and the inclusion $i : mA \hookrightarrow mA^*$ induces an injective map $H_i(mA) \hookrightarrow H_i(mA^*) = \text{Tor}^R(k, mA)$.
3. The map $H(A \otimes_R k) \to H(A^* \otimes_R k) = \text{Tor}^R(k, k)$ is injective and $A$ admits a trivial Massey operation.
4. The map $H(A \otimes_R k) \to \text{Tor}^R(k, k)$ is injective and the homotopy Lie algebra $\pi(A)$ is a free Lie algebra.

The condition that the map $H(A \otimes_R k) \to \text{Tor}^R(k, k)$ is injective, is freely satisfied when $A$ is a distinguished DG$^*$ algebra. The following theorem was proved by Avramov in [12, Theorem A].

**Theorem 2.3.** A local ring $(R, m, k)$ is a generalised Golod ring of level $n$ if and only if $\pi^a(R)$ is a free Lie algebra. If $R$ is a generalised Golod ring of level $n$, then there exists a polynomial $d_R(t) \in \mathbb{Z}[t]$ such that for any finitely generated $R$-module $M$, the product $d_R(t)P^R_k(t) \in \mathbb{Z}[t]$. Furthermore, in this case $d_R(t)P^R_k(t)$ divides $\prod_{i \geq 2i} \#(1 + 2^{i+1})^e_{2i+1}$, where $e_{2i+1}$ denotes the number of exterior variables of degree $2i + 1$ in the distinguished DG$^*$ algebra $R(X_i : deg(X_i) \leq n)$.

**Definition 2.4.** Let $(R, m_R, k)$, $(S, m_S, k)$ be local rings with a common residue field $k$. Let $\pi_R : R \to k$, $\pi_S : S \to k$ be natural quotient maps from $R$, $S$ onto $k$ respectively. The fibre product of $R$ and $S$ is defined as the ring $R \times_k S = \{(r, s) \in R \times S : \pi_R(r) = \pi_S(s)\}$. The ring $R \times_k S$ is local with maximal ideal $m_R \oplus m_S$. The rings $R$, $S$ are modules over $R \times_k S$ by left actions defined by projection maps $(r, s) \mapsto r$, $(r, s) \mapsto s$ respectively.

Now assume that both $R$, $S$ are Artinian Gorenstein local rings with one dimensional socles $\text{soc}(R) = \langle \delta_R \rangle$, $\text{soc}(S) = \langle \delta_S \rangle$. Then the connected sum of $R$ and $S$ is defined as $R$#$S = \frac{R \times_k S}{\langle \delta_R - \delta_S \rangle}$. We say a Gorenstein local ring $Q$ is decomposable as a connected sum if there are rings $R$, $S$ such that $Q = R$#$S$, $l(R) < l(Q)$ and $l(S) < l(Q)$.

We recall the following without specific reference.

**Facts :**

1. The fibre product $T = R \times_k S$ is a local ring and the connected sum $Q = R$#$S$ is an Artinian Gorenstein local ring when both $R$, $S$ are so.
2. Let \( m_T, m_Q \) be the unique maximal ideals of \( T, Q \) respectively. Then \( m_T^i = m_T^i \oplus m_Q^i \) for \( i \geq 1 \). The maps \( m_T \rightarrow m_Q, m \mapsto (\hat{m}, 0) \) and \( m_S \rightarrow m_Q, n \mapsto (0, \hat{n}) \) are injective. We have \( \text{ll}(Q) = \max\{\text{ll}(R), \text{ll}(S)\} \) and \( Q/m^n \cong R/m^nS \cong (m^nR \cap S)/m^n \text{ll}(S) \) for \( 2 \leq n \leq \text{ll}(Q) \).

3. \( l(R \times_k S) = l(R) + l(S) - 1 \) and \( l(R\#S) = l(R) + l(S) - 2 \).

**Theorem 2.5.** [18, Satz 2] Let \( (S, m_S, k), (T, m_T, k) \) be two local rings and \( R = S \times_k T \). Then \( \frac{1}{p^m_Q(\text{ll}(S))} = \frac{1}{p^m_S(\text{ll}(S))} + \frac{1}{p^m_T(\text{ll}(S))} - 1 \). If \( M \) is an \( S \)-module, then \( \frac{1}{p^m_M(\text{ll}(S))} = \frac{1}{p^m_S(\text{ll}(S))} \left( \frac{1}{p^m_T(\text{ll}(S))} + \frac{1}{p^m_T(\text{ll}(S))} - 1 \right) \).

### 3. Criteria for Connected Sum Decompositions

In this section, we develop criteria to decompose an Artinian Gorenstein local ring as a connected sum. The following result follows from [4, Proposition 4.1] and Fact (2) of Definition 2.4.

**Theorem 3.1.** Let \( (Q, n, k) \) be a regular local ring and \( I \subset n^2 \) be an ideal such that \( R = Q/I \) is an Artinian Gorenstein local ring. Let \( n \) be minimally generated by \( x_1, \ldots, x_m, y_1, \ldots, y_n \) such that \( (x_1, \ldots, x_m)(y_1, \ldots, y_n) \subset I \). Let \( \max\{i : (x_1, \ldots, x_m)^i \not\subset I\} = s \), \( \max\{i : (y_1, \ldots, y_n)^i \not\subset I\} = t \). Then there are ideals \( I_1, I_2 \) in \( Q \) containing \( (x_1, \ldots, x_m), (y_1, \ldots, y_n) \) respectively such that the following hold:

1. The rings \( S = Q/I_1, T = Q/I_2 \) are Gorenstein rings. \( e. \text{dim}(S) = n, e. \text{dim}(T) = m, \text{ll}(S) = t, \text{ll}(T) = s \).
2. \( R = S \#_k T \).

**Lemma 3.2.** Let \( (R, m, k) \) be a local ring and \( I \) be an ideal of \( R \) such that \( \mu(xI) \leq 1 \) for all \( x \in I \setminus mI \). Then \( I^2 = yI \) for some \( y \in I \setminus mI \). In particular \( \mu(I^2) \leq 1 \).

**Proof.** Let \( I \) be minimally generated by \( x_1, x_2, \ldots, x_n \). We prove the lemma by induction on \( n = \mu(I) \). When \( n = 1 \), \( I^2 = yI = (y^2) \) for any \( y \in I \setminus mI \).

If we can show that \( I^2 = yI + mI^2 \) for some \( y \in I \setminus mI \), then by Nakayama’s lemma we have \( I^2 = yI \). So it is enough to prove that \( I^2 = yI + mI^2 \) for some \( y \in I \setminus mI \). We have \( \mu\left( \frac{xI + mI^2}{mI^2} \right) \leq 1 \) for all \( x \in I \setminus mI \). Therefore, there is no loss of generality in assuming that \( mI^2 = 0 \), i.e. \( I^2 \) is a \( k \)-vector space.

We consider the following two cases. We assume that \( I^2 \neq 0 \) to exclude the trivial case.

**Case - 1:** \( x_i^2 = 0 \) for \( 1 \leq i \leq n \).

After reordering the generators if necessary we may assume that \( x_1x_2 \neq 0 \). Then \( x_1I = (x_1, x_2) = x_2J \) since \( \dim_k(x_1I) = \dim_k(x_2I) = 1 \). We have \( x_1x_1 = \lambda_1x_1x_2 \) and \( x_2x_1 = \lambda_1'x_1x_2 \) for \( 3 \leq i \leq n \) and \( \lambda_1, \lambda_1' \in R \). Note that \( x_1(x_i - \lambda_1x_2 - \lambda_1'x_1) = x_2(x_i - \lambda_1x_2 - \lambda_1'x_1) = 0 \). Let \( J \) be the ideal generated by \( x_i - \lambda_1x_2 - \lambda_1'x_1, 3 \leq i \leq n \). Then \( I = (x_1, x_2) + J \) and \( (x_1, x_2)J = 0 \). For any \( x \in J \setminus mJ \) we have \( x \in I \setminus mI, xJ = xI \) and \( \mu(xJ) = \mu(xI) \leq 1 \). Since \( \mu(J) = \mu(I) - 2 \), by induction hypothesis we find \( y' \in J \setminus mJ \) such that \( y'J = J^2 \). Let \( y = x_1 + y' \). Then \( y \in I \setminus mI \) and \( I^2 = (x_1, x_2)J = (x_1 + y)'J = yI \).

**Case - 2:** Not all \( x_i^2 = 0 \) for \( 1 \leq i \leq n \).

Without loss of generality we assume that \( x_1^2 \neq 0 \) and \( x_1I = (x_1^2) \). Let \( x_1x_1 = \lambda_2x_1x_2 \), \( \lambda_2 \in R \) and \( 2 \leq i \leq n \). Then \( x_1(x_1 - \lambda_1x_2) = 0 \). Let \( J \) be the ideal generated by \( x_i - \lambda_2x_1, 2 \leq i \leq n \). Then \( I = (x_1) + J, x_1J = 0 \). For any \( x \in I \setminus mI \) we have \( x \in I \setminus mI, xJ = xI \) and \( \mu(xJ) = \mu(xI) \leq 1 \). Since \( \mu(J) = \mu(I) - 1 \), by induction hypothesis we find \( y' \in J \setminus mJ \) such that \( y'J = J^2 \). Let \( y = x_1 + y' \). Then \( y \in I \setminus mI \) and \( I^2 = (x_1^2) + J^2 = (x_1 + y')J = yI \). \( \square \)

**Lemma 3.3.** Let \( (R, m, k) \) be a local ring such that \( \mu(m^2) \leq 2 \). Then there exists an \( x \in m \setminus m^2 \) such that \( m^2 = xm \). If \( \text{ll}(R) \geq 3 \), then \( x^2 \not\in m^3 \).
Proof. If \( m^2 = xn + m^3 \), \( x \in m \setminus m^2 \), then by Nakayama’s lemma we have \( m^2 = xn \). So it is enough to assume that \( m^3 = 0 \), i.e. \( m^2 \) is a \( k \)-vector space. If \( m^2 = 0 \), then \( m^2 = xn = 0 \) for all \( x \in m \setminus m^2 \). If \( \mu(m^2) = 1 \), then for any \( x \in m \setminus m^2 \) such that \( xn \neq 0 \), we have \( m^2 = xn \). If \( \mu(m^2) = 2 \), by Lemma 3.2 we have \( x \in m \setminus m^2 \) such that \( \mu(xm) > 1 \). Then \( 2 \leq \dim_k xn \leq \dim_k m^2 = 2 \). Therefore, \( m^2 = xn \). This proves the first part of the lemma.

If \( x^2 \in m^3 \), then \( m^3 = x^2m \subset m^4 \). So by Nakayama’s lemma \( m^3 = 0 \) which is a contradiction. Therefore, \( x^2 \notin m^3 \) if \( l(R) \geq 3 \). \( \square \)

Theorem 3.4. Let \((R, m, k)\) be a local Artinian Gorenstein ring. Let \( e, \dim(R) = n, lI(R) \geq 3 \) and \( \dim_k m^2/m^3 = m < n \). Assume that \( m \) admits a generator \( x_1 \) such that \( m^2 = x_1m \). Then there exists a minimal generating set \( \{x_1, x_2, \ldots, x_n\} \) of \( m \) extending \( x_1 \) such that the following hold.

1. \( m^2 = (x_1^2, x_1x_2, \ldots, x_1x_m) \).
2. \( (x_1, \ldots, x_m)(x_{m+1}, \ldots, x_n) = 0 \).
3. \( (x_{m+1}, \ldots, x_n)^2 = \text{soc}(R) \).

The ring \( R \) decomposes as connected sum \( R = S \# T \) such that \( e, \dim(S) = m, e, \dim(T) = n - m, lI(S) = lI(R) \) and \( lI(T) = 2 \).

Proof. Since \( lI(R) \geq 3 \), \( x^2 \notin m^3 \). So we can choose a minimal generating set \( \{x_1, x_2, \ldots, x_n\} \) of \( m \) such that \( m^2 = (x_1^2, x_1x_2, \ldots, x_1x_m) \).

We have \( x_1x_j = \alpha_{1j}x_1^2 + \alpha_{2j}x_1x_2 + \ldots + \alpha_{mj}x_1x_m \), \( \alpha_{ij} \in R \) for \( 1 \leq i \leq m \) and \( m + 1 \leq j \leq n \). This gives \( x_1(x_j - \alpha_{1j}x_1 - \alpha_{2j}x_2 - \ldots - \alpha_{mj}x_m) = 0 \). So replacing \( x_j - \alpha_{1j}x_1 - \alpha_{2j}x_2 - \ldots - \alpha_{mj}x_m \) by \( x_j \), we assume that \( x_1x_j = 0 \) for \( m + 1 \leq j \leq n \).

If \( m = 1 \), the property (2) is satisfied. So we assume that \( \mu(m^2) = 2 \). Since \( m^2 \) is minimally generated by \( \{x_1^2, x_1x_2, \ldots, x_1x_m\} \) and \( x_1(x_{m+1}, \ldots, x_n) = 0 \), we have \( (0 : x_1) \subset m(x_1, \ldots, x_m) + (x_{m+1}, \ldots, x_n) \). This implies that residue classes of elements \( x_{m+1}, \ldots, x_n \) form a \( k \)-basis of \( (0 : x_1)m^2/m^3 \).

Therefore, \( \dim_k (0 : x_1)m^2/m^3 = n - m \).

Claim 1: \( m[(0 : x_1) \cap m^2] = \text{soc}(R) \).

Note that \( m^2(0 : x_1) = x_1m(0 : x_1) = 0 \). So \( m[(0 : x_1) \cap m^2] \subset \text{soc}(R) \). Therefore, it is enough to prove that \( m[(0 : x_1) \cap m^2] \neq 0 \). We have \( m^{i+1} = x_1m^{i} \) for \( i \geq 1 \). Thus \( \mu(m^{i+1}) \leq \mu(m^i), i \geq 1 \). Let \( t = \max\{i : \mu(m^i) = m\} \). Since \( m \geq 2 \), we have \( lI(R) \geq t + 1 \). The map \( m^i/m^{i+1} \to m^{i+1}/m^{i+2} \) is not injective since \( \dim_k(m^i/m^{i+1}) > \dim_k(m^{i+1}/m^{i+2}) \). So we find \( y \in m^i \setminus m^{i+1} \) such that \( x_1y \in m^{i+2} \). Note that \( m^{i+2} = x_1m^{i+1} \). So \( yx_1 = x_1m \) for some \( m \in m^{i+1} \). This implies that \( x_1(y - m) = 0 \).

Therefore, \( y - m \in [(0 : x_1) \cap m^2] \) and \( y - m \notin m^{i+1} \). Since \( \text{soc}(R) \subset m^{i+1} \), we have \( y - m \notin \text{soc}(R) \). So \( [(0 : x_1) \cap m^2] \notin \text{soc}(R) \). Consequently \( m[(0 : x_1) \cap m^2] \neq 0 \) and the claim is proved.

Claim 2: \( \dim_k \frac{m[(0 : x_1)m^2]}{m[(0 : x_1)m^3]} = m - 1 \).

We know \( (0 : x_1) = \text{Hom}_R(R/(x_1), R) \). We have \( m^2 = x_1m \subset (x_1) \). So by the Matlis duality \( l(0 : x_1) = l(R/x_1R) = l(R/m^2) - l(R/m^3^+) = 1 + n - 1 = n \). Now we have

\[
l((0 : x_1) \cap m^2) = l((0 : x_1) + m(m^2) - l((0 : x_1) + m^2) = l((0 : x_1) - l\left(\frac{(0 : x_1)m^2}{m^3}\right) = n - (n - m) = m.
\]

Therefore, \( \dim_k \frac{m[(0 : x_1)m^2]}{m[(0 : x_1)m^3]} = l((0 : x_1)m^2/m^3) = l((0 : x_1) \cap m^2)] - 1 = m - 1 \).

Claim 3: \( \) The pairing \( \left(\frac{m(x_2 + \cdots + x_m)}{m(x_3 + \cdots + x_m)}\right) \times \left(\frac{m(x_2 + \cdots + x_m)}{m(x_3 + \cdots + x_m)}\right) \to \text{soc}(R) \) given by \( (\bar{x}, \bar{y}) \to xy \) is well defined and non-degenerate.

We have \( m(x_2, \ldots, x_m)(0 : x_1) = 0 \) since \( m^2 = x_1m \). So \( (x_2, \ldots, x_m)(0 : x_1) \subset \text{soc}(R) \). Therefore, the above pairing exists. Note that \( m^2(x_{m+1}, \ldots, x_n) = 0 \). So if \( y \in [(0 : x_1) \cap m^2] \)
and \( y(x_2, \ldots, x_m) = 0 \), we have \( y \in \text{soc}(R) = m[(0 :_R x_1) \cap m^2] \). We also have \( \dim_k (x_2, \ldots, x_m) = \dim_k \frac{(x_2, \ldots, x_m)}{m(x_2, \ldots, x_m)} = m - 1 \). Therefore, the pairing is non-degenerate and the claim follows.

Note that \( \frac{m(x_2, x_3, \ldots, x_n)}{m(x_2, x_3, \ldots, x_n)_m} \) is a Gorenstein ring for some \( s \). We know that \( (0 :_R x_1) \cap m \) is a minimal generating set of \( R/I(m) \). By an easy inductive argument \( m^n = (0 : I)^n \) for all \( n \geq 2 \). Therefore, \( \max \{i : (x_1, \ldots, x_m)^i \neq 0\} \) is a principal ideal for some \( x \in R \). Now \( m^{-j+1} \subset (0 :_R m^j) \).

A careful study of [5] yields the following result.

**Theorem 3.5.** Let \((R, m, k)\) be a Gorenstein local ring with \( \text{ll}(R) \geq 3 \). Then the following are equivalent.

- (1) There are Gorenstein local rings \((S, p), (T, q)\) with \( \text{ll}(S) = \text{ll}(R) \), \( \text{ll}(T) = 2 \) such that \( R = S \# T \).
- (2) \( (0 : m^2) \not\subset m^2 \).

**Proof.** First we assume that (1) holds. Since \( \text{ll}(S), \text{ll}(T) \geq 2 \), we have \( m/m^2 = p/p^2 \oplus q/q^2 \). We choose \( y \in p \setminus q^2 \). Since \( q^3 = 0 \), we have \( (0, y)m^2 = 0 \). But \( (0, y) \in m \setminus m^2 \). So (2) follows.

Conversely we assume that (2) holds. We choose \( y_1, \ldots, y_n \in (0 : m^2) \) such that there images in \( \frac{(0 : m^2)}{m^2/0(m^2)} \) form a basis of k-vector space. We have \( (y_1, \ldots, y_n) \cap m^2 \subset \text{soc}(R) \). Let \( I = (y_1, \ldots, y_n) \subset (0 : m^2) \). If \( z \in I \cap (0 : I) \), then \( z \) kills both \( I \) and \( m^2 \). So \( z \) kills \( I + [m^2 \cap (0 : m^2)] = (0 : m^2) \).

This implies that \( z \in (0 : (0 : m^2)) = m^2 \) since \( R \) is a Gorenstein ring. So \( z \in m^2 \cap I \subset \text{soc}(R) \).

Thus we have \( I \cap (0 : I) \subset \text{soc}(R) \). Note that \( I \cap (0 : I) \neq 0 \) because otherwise \( I(I + (0 : I)) = I + (0 : I) = I(R) \), a contradiction since \( I + (0 : I) \) is a proper ideal. So \( I \cap (0 : I) = \text{soc}(R) \). Now \( I(I + (0 : I)) = I(I) = I(R) - 1 = I(m) \). Also \( I + (0 : I) \subset m \).

Therefore, \( I + (0 : I) = m \) and \( m/\text{soc}(R) \oplus (0 : I) = (0 : I) \).

Note that \( m^2 I = 0 \). So \( m \subset \text{soc}(R) \). If \( mI = 0 \), then \( I \subset \text{soc}(R) \subset m^2 \). This implies that \( (0 : m^2) = I + m^2 \cap (0 : m^2) \subset m^2 \), a contradiction. So \( m = \text{soc}(R) \). If \( m(0 : I) = 0 \), then \( m^2 = m[I + (0 : I)] = m^2 = \text{soc}(R) \) which is again a contradiction since \( \text{ll}(R) \geq 3 \). So \( \text{soc}(R) \subset (0 : I) \).

Therefore, we have \( m = m/\text{soc}(R) \oplus (0 : I) \). The ideal \( I \) is minimally generated by \( y_1, \ldots, y_n \). We choose a minimal generating set \( x_1, \ldots, x_m \) of \( (0 : I) \) such that \( x_1, \ldots, x_m, y_1, \ldots, y_n \) is a minimal generating set of \( m \).

Note that \( (x_1, \ldots, x_m)(y_1, \ldots, y_n) = 0 \). We have \( I^3 = 0 \). If \( I^2 = 0 \), then \( mI = 0 \), a contradiction. So \( \max \{i : (y_1, \ldots, y_n)^i \neq 0\} = 2 \). We have \( mI = \text{soc}(R) \subset (0 : I) \).

So \( m^2 = m(0 : I) = (0 : I)^2 \). By an easy inductive argument \( m^n = (0 : I)^n \) for all \( n \geq 2 \). Therefore, \( \max \{i : (x_1, \ldots, x_m)^i \neq 0\} \) is a principal ideal for some \( x \in R \).

Now the result follows from Theorem 3.1. \( \square \)

**Lemma 3.6.** Let \((R, m)\) be an Artinian Gorenstein local ring such that \( \dim(R) \geq 2 \) and \( \text{ll}(R) = s \). Then the quotient ring \( R/m^i \) can never be a Gorenstein ring for \( 2 \leq i \leq s \).

**Proof.** If possible assume that \( R/m^i \) is a Gorenstein ring for some \( i \) satisfying \( 2 \leq i \leq s \). Then the injective hull of \( k \) over \( R/m^i \), viz. \( E_{R/m^i}(k) \cong R/m^i \). We know that \( E_{R/m^i}(k) \cong \text{Hom}_R(R/m^i, R) = (0 :_R m^i) \). So \( (0 :_R m^i) = (x) \), a principal ideal for some \( x \in R \). Now \( m^{i-1} \subset (0 :_R m^i) \).
and \( m^{s-i+1} \nsubseteq m(0 :_RM') \) for otherwise \( m^{s-i+1} \subset m(0 :_RM') \subset (0 :_RM'') \) which implies that \( m^i = m^{s-i+1}m'^{-1} = 0 \), a contradiction. Since \( (0 :_RM') \) is principal we have \( (0 :_RM') = m^{s-i+1} = (x) \). This implies that \( m^i \) is a principal ideal for \( s-i+1 \leq j \leq s \). So \( l(0 :_RM') = l(m^{s-i+1}) = i \). By Matlis duality, \( l(R/m^i) = l(\text{Hom}(R/m^i, R) = (0 :_RM') = i \). We have \( \sum_{j=0}^{i-1}[l(m^{j}/m^{j+1}) - 1] = l(R/m^i) - i = 0 \) and each summand is non-negative. This shows that \( \text{e. dim}(R) = l(m/m^2) = 1 \), a contradiction.

**Theorem 3.7.** Let \((R, m, k)\) be an Artinian Gorenstein local ring such that \( \mu(m) = n \), \( \text{II}(R) \geq 2 \) and \( \mu(m^2) \leq 2 \). Let \( i \) be an integer satisfying \( 2 \leq i \leq \text{II}(R) \). Then \( R/m^i \) is a Golod ring and \( p_k^{R/m^i}(t) = \frac{1}{1-nt} \).

**Proof.** Fix \( i \) satisfying \( 2 \leq i \leq \text{II}(R) \) and set \( \tilde{R} = \frac{R}{m^i} \). The result is clear when \( n = 1 \) or \( \text{II}(R) = 2 \). We assume that \( n = 2 \) and \( \text{II}(R) > 2 \). A result of Scheja [23] states that a co-depth 2 local ring is either a Gorenstein or a Golod ring. The ring \( \tilde{R} \) cannot be a Gorenstein ring by Lemma 3.6. So \( \tilde{R} \) is a Golod ring. Since \( R \) is a complete intersection, the defining ideal of \( \tilde{R} \) is minimally generated by 3 elements. So, \( k^R(t) = \sum_{i \geq 0} \text{dim}_R(H_i(K^R)^{re}) = 1 + 3t + 2t^2 \) and \( (1-t(k^R(t) - 1)) = 1-t(1+3t+2t^2-1) = (1+t^2)(1-2t) \). We have \( p_k^R(t) = \frac{(1+t^2)}{1-t(2t^2)} = \frac{1}{1-2t} \).

Now we assume that \( n > 2 \) and \( \text{II}(R) > 2 \). By Lemma 3.3, Theorem 3.4, it follows that \( R = S \# T \) where \( (S, p), (T, q) \) are Gorenstein local rings, \( \text{e. dim}(S) = \mu(m^2) \leq 2 \), \( \text{e. dim}(T) + \text{e. dim}(S) = n \), \( \text{II}(S) = \text{II}(R) \) and \( \text{II}(T) = 2 \). So \( \tilde{R} = \tilde{S} \times_k \tilde{T} \) where \( \tilde{S} = S/p^t \) and \( \tilde{T} = T/q^t \) (see Facts, Definition 2.4). Both \( \tilde{S}, \tilde{T} \) are Golod rings. We have \( p_k^S(t) = \frac{1}{1-\text{e. dim}(S)\text{e}t}, p_k^T(t) = \frac{1}{1-\text{e. dim}(T)\text{e}t} \). The result follows from Theorem 2.5 and the fact that fibre products of Golod rings are Golod [31, Theorem 4.1].

4. Main results

Our goal in this section is to prove our main Theorems II, III as stated in the introduction. Suppose \((A, \partial)\) is an augmented DGI algebra and \( A^* = A(W), W = \{W_i\}_{i \geq 1} \) is the acyclic closure of \( A \). We recall the notion of (normal) \( \Gamma \)-monomials from [13, Remark 6.2.1]. Pick two \( \Gamma \)-monomials \( M, M' \) in \( A^* \). Then there exists an integer \( n \) large enough such that \( M = W_1^{c_1} \ldots W_n^{c_n} \) and \( M' = W_1^{c_1'} \ldots W_n^{c_n'} \) where \( c_j, c_j' \geq 0 \). By our assumption \( \text{deg}(W_i) \leq \ldots \leq \text{deg}(W_n) \). We define a total order on the set of \( \Gamma \)-monomials as \( M < M' \) if the last nonzero entry of \( (c_1' - c_1, \ldots, c_n' - c_n), \text{deg}(M') - \text{deg}(M) \) is positive. If \( \beta \) is an element of \( A^* \), then we can express \( \beta \) as \( \beta = \sum f_M M, f_M \in A \) where the sum is taken over finitely many \( \Gamma \)-monomials \( M \). We define the initial \( \Gamma \)-monomial of \( \beta \) as \( \text{in}(\beta) = \max \{M : f_M \neq 0\} \). The coefficient of \( \text{in}(\beta) \) in \( \beta \) is called the leading coefficient of \( \beta \). The following lemma is implicit in [25, Theorem 1.6.2].

**Lemma 4.1.** Let \((A, \partial)\) be an augmented DGI algebra over a local ring \((R, m, k)\) with augmentation map \( \epsilon : A \to k \) and \( A^* = A(W), W = \{W_i\}_{i \geq 1} \) be the acyclic closure of \( A \). Let \( F^n(A^*) = A(W_i : \text{deg}(W_i) \leq n) \) denote distinguished DGI algebras. Then the following hold.

1. There are \( A \)-linear derivations \( \partial_j, 1 \leq j \leq n \) such that \( \partial_j(W_j) = 1 \) and \( \partial_j(W_i) = 0 \) for \( i < j \).
2. \( \partial_j(F^k(A^*)) \subset F^j(A^*) \) for \( j \leq k \).
3. For a \( \Gamma \)-monomial \( M = W_1^{c_1} \ldots W_n^{c_n} \), define \( v_M = v_1^{c_1} \ldots v_n^{c_n} \). Then \( v_M(M') = \pm 1 \) and \( v_M(M') = 0 \) for \( M' < M \).

If \((R, m)\) is a Gorenstein local ring of embedding dimension \( n \geq 2 \), the Koszul homology algebra \( H(K^R) \) is a Poincaré algebra [7]. The multiplication map \( H_1(K^R) \times H_{n-1}(K^R) \to H_n(K^R) \) is a non-degenerate pairing. So for any \( z \in H_1(K^R) \), there is a \( z' \in H_{n-1}(K^R) \) such that \( zz' \) is a generator in \( H_n(K^R) \). The following lemma lays the groundwork for main results of this article.
Lemma 4.2. Let \((R, m)\) be an Artinian Gorenstein local ring of embedding dimension \(n \geq 2\). Let \(R(X)\) be the acyclic closure of \(R\) and \(\mathfrak{X} = R(X_j : 1 \leq j \leq n + i), i \geq 1\) be a distinguished DGΓ algebra over \(R\). Let \(soc(R) = (i) \leadsto \rfloor \) and \(\mathfrak{X} = \rfloor \otimes_R \mathfrak{X}\). Let \(z \in Z_1(R^\mathfrak{X}), z' \in Z_{n-1}(R^\mathfrak{X})\) be cycles such that \(\partial X_{n+1} = z\) and \(z z' = tX_1 \ldots X_n\). Define \(\mathfrak{Y} = \{x \in \mathfrak{X} : z' \partial(x) \in soc(R)X_1 \ldots X_n\mathfrak{X}\} and \(\mathfrak{Y} = \rfloor \otimes_R \mathfrak{Y}\). Then the following hold.

1. \(soc(R)K^R \subset (0 : m^2)B_{i+1}(R^\mathfrak{X}) \subset mB_{i+1}(R^\mathfrak{X})\) for \(i = 1, \ldots, n - 1\).
2. \(soc(R)X_1 \ldots X_n\mathfrak{X} = z' B(\mathfrak{Y})\).
3. \(soc(R)\mathfrak{X} = (0 : m^2)B(\mathfrak{X}) + z' B(\mathfrak{Y}) \subset mB(\mathfrak{X})\).
4. \(Z(\rfloor) = (0 : m^2)\rfloor + z' B(\rfloor)\).
5. The quotient map \(\mathfrak{X} \to \rfloor\mathfrak{X}\) induces injective maps \(H(\mathfrak{X}) \to H(\rfloor\mathfrak{X})\) and \(H(m\mathfrak{X}) \to H(m\rfloor\mathfrak{X})\) on homology.

6. For any two cycles \(v \in \partial Z(\mathfrak{X}), w \in Z(z' \mathfrak{X})\), there is an element \(v' \in z' \mathfrak{X}\) such that \(vw = \partial(v')\).

7. For any cycle \(\bar{v} \in \partial Z(\rfloor\mathfrak{X}), \bar{w} \in Z(z' \rfloor\mathfrak{X})\), there is an element \(\bar{v}' \in z' \rfloor\mathfrak{X} + (0 : m^2)\rfloor\mathfrak{X}\) such that \(\bar{v}w = \partial(\bar{v}')\).

Proof. Let \(\partial(X_i) = x_i, 1 \leq i \leq n\). Then the maximal ideal \(m\) is minimally generated by \(\{x_1, \ldots, x_n\}\). The statement (1) is proved in [33, Lemma 1.2]. It is proved in the line no 10, page no 80 in the proof of [33, Theorem 2] that \(tX_1 \ldots X_nR(\mathfrak{X}) = z' \partial(\mathfrak{X}_{n+1})R(\mathfrak{X}) \subset z' B(R(\mathfrak{X}))\). By a very similar argument, we have \(tX_1 \ldots X_n\mathfrak{X} = z' \partial(X_{n+1})\mathfrak{X} \subset z' B(\mathfrak{X})\). By definition of \(\mathfrak{Y}\), it follows that \(tX_1 \ldots X_n\mathfrak{X} = z' B(\mathfrak{Y})\). (2) is proved.

It is proved in the equation 2.3, page no 79 in the proof of [33, Theorem 2] that \(tR(\mathfrak{X}) \subset (0 : m^2)B(R(\mathfrak{X})) + tX_1 \ldots X_n\mathfrak{X}\). The same reasoning applies here and we have \(t\rfloor \subset (0 : m^2)B(\mathfrak{X} + tX_1 \ldots X_n\mathfrak{X}\). So from (2), we have \(t\rfloor \subset (0 : m^2)B(\mathfrak{X} + z' B(\mathfrak{Y}))\). The reverse inclusion is clear and (3) follows.

Both (4) and (5) follow from (3). The argument is implicit in [33, Theorem 2]. We skip the details.

We prove (6). We may assume that \(deg(v) \geq 1\) as the result is obvious when \(v \in m\). Let \(v = \sum_{i=1}^n x_iv_i, v_i \in \mathfrak{X}\). Then we have

\[
vw = \sum_{i=1}^n x_iv_iw = \sum_{i=1}^n [\partial(X_i)v_i]w + X_i \partial(v_i)w = \partial(v_i w) + \sum_{i=1}^n X_i \partial(v_i)w \text{ for } v_i = \sum_{i=1}^n X_i v_i.
\]

Note that \(X_i z' \subset X_1 \ldots X_n\mathfrak{X}, 1 \leq i \leq n\) and \(w \in Z(z' \mathfrak{X})\). So \(\sum_{i=1}^n X_i \partial(v_i)w = X_1 \ldots X_n u\) for \(u \in R(X_{n+1}, \ldots, X_{n+i})\). Now \(vw = \partial(v_i w) + X_1 \ldots X_n u\) and \(v, w\) are cycles. So the element \(X_1 \ldots X_n u\) is also a cycle. Expanding \(\partial(X_i \ldots X_n u)\) and comparing coefficients of both sides of \(\partial(X_i \ldots X_n u) = 0\), we conclude that coefficients of \(u\) are in \(soc(R)\). Thus \(u \in soc(R)R(X_{n+1}, \ldots, X_{n+i})\) and \(X_1 \ldots X_n u \in soc(R)X_1 \ldots X_n\mathfrak{X} \subset z' B(\mathfrak{X})\) by (2). Let \(X_1 \ldots X_n u = \partial(v'z')z', v' \in \mathfrak{X}\). Then we have \(vw = \partial(v' w) + \partial(v'z')z' = \partial(v')z'\) for \(v' = v'w + v'z'\). Clearly \(v' \in z' \mathfrak{X}\). So (6) follows.

Finally we prove (7). We assume that \(deg(v) \geq 1\) to exclude the obvious case. We observe from (4) that \(Z(\mathfrak{X})w = \rfloor \otimes_R Z(\mathfrak{X})w\). So we choose \(v_0 \in Z_{\geq 1}(\mathfrak{X})\) such that \(\bar{v}w = v_0 w\). Now \(w \in Z(z' \mathfrak{X})\). Again by (4), we may choose the lift

\[
w = z' w' = w_1 + z' w_2 + w_3 \text{ for } w_1 = \sum_{i=1}^n (0 : m^2)\mathfrak{X}, w_2, w_3 \in Z(\mathfrak{X}). \quad (1)
\]

Note that \(v_0 \in Z_{\geq 1}(\mathfrak{X}) \subset m\mathfrak{X}\). So \(v_0 w_1 \in soc(R)\mathfrak{X}\). By (6), \(v_0 z' = \partial(v_1 z')\) for some \(v_1 \in \mathfrak{X}\). So \(v_0 z' w_2 = \partial(v_1 z') w_2 = \partial(v_1 z' w_2) + (-1)^{\partial(v_1 z')}v_1 z' \partial(w_2)\). Now \(z' \partial(w_2) \in soc(R)X_1 \ldots X_n\mathfrak{X}\) since \(w_2 \in \mathfrak{Y}\). This implies that \(v_0 z' w_2 = \partial(v_1 z' w_2) + w_4\) for \(w_4 = (-1)^{\partial(v_1 z')}v_1 z' \partial(w_2) \in soc(R)\mathfrak{X}\).

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We write \( v_0 = \sum_{i=1}^{n} x_i a_i \) for \( a_i \in \mathfrak{x} \). So we have

\[
v_0 w_3 = \sum_{i=1}^{n} x_i a_i w_3 = \sum_{i=1}^{n} [\partial(X_i a_i) w_3 + X_i \partial(a_i) w_3] = \partial(v_2) w_3 + \sum_{i=1}^{n} X_i \partial(a_i) w_3 \text{ where } v_2 = \sum_{i=1}^{n} X_i a_i.
\]

Now from equation 1 it follows that \( w_3 = z'(w' - w_2) - w_1 \). So \( X_i w_3 \in X_1 \ldots X_n \mathfrak{x} + (0 : m^2) \mathfrak{x} \).

We have \( \partial(a_i) \in \mathfrak{m} \mathfrak{x} \). Therefore, \( \sum_{i=1}^{n} X_i \partial(a_i) w_3 = X_1 \ldots X_n u + w_5 \) for some \( u \in R(X_{n+1}, \ldots, X_{n+i}) \) and \( w_5 \in \text{soc}(R) \mathfrak{x} \). Now \( v_0 w_3 = \partial(v_2) w_3 + X_1 \ldots X_n u + w_5 \) and \( v_0, w_3, w_5 \) are cycles. So \( u \in \text{soc}(R) R(X_{n+1}, \ldots, X_{n+i}) \) and \( X_1 \ldots X_n u = \partial(v_3) z' \) for \( v_3 \in \mathfrak{y} \) by (2). Therefore, we have \( v_0 w_3 = \partial(v_3 w_3 + v_3 z') + w_5 \).

Finally we write \( v_0 w = v_0 (w_1 + z' w_2 + w_3) = (v_0 w_1 + w_4 + w_5) + \partial(v_1 z' w_2 + v_2 w_3 + v_3 z') \). We have \( v_0 w_1 + w_4 + w_5 \in \text{soc}(R) \mathfrak{x} \).

Define \( v' = v_1 z' w_2 + v_2 w_3 + v_3 z' = v_1 z' w_2 + v_2[z'(w' - w_2) - w_1] + v_3 z' = [v_1 z' w_2 + v_2 z'(w' - w_2)] + v_3 z' - v_2 w_1 \).

Since \( w_1 \in (0 : m^2) \mathfrak{x} \), we have \( v' \in \mathfrak{x} z' + (0 : m^2) \mathfrak{x} \). Therefore, going modulo socle, we have \( \overline{vw} = \overline{v_0 w} = \partial(v') \) and \( v' \in \mathfrak{x} \bar{z}' + (0 : m^2) \bar{x} \). Thus (7) follows. \( \square \)

**Theorem 4.3.** Let \( (R, m, k) \) be an Artinian Gorenstein local ring of embedding dimension \( n \geq 2 \) such that \( R/ \text{soc}(R) \) is a Golod ring. Let \( K^R = R(X_1 : 1 \leq X_i \leq n, \partial(X_i) = x_i) \) be the Koszul complex of \( R \) on a minimal set of generators \( x_1, \ldots, x_n \) of \( m \). Let \( \eta : (Q, n) \to (R, m) \) be a minimal Cohen presentation. Then the following hold.

1. Let \( z \in Z_1(K^R) \setminus B_1(K^R) \) and \( K^R(T : \partial(T) = z) \) be the extension of \( K^R \) by adjoining divided powers variable \( T \) of degree two to kill the cycle \( z \). Then \( K^R(T : \partial(T) = z) \) is a Golod algebra.

2. Let \( I = \ker(\eta) \). Then for any \( f \in I \setminus \text{nil} \), the induced map \( Q/f \to R \) is a Golod homomorphism, i.e. \( R \) has the Backelin-Roos property.

3. Let \( d_R(t) = 1 - t(P^Q_R(t) - 1) + t^n + (1 + t) \). Then for any \( R \)-module \( M \) we have \( d_R(t)P^R_M(t) \in \mathbb{Z}[t] \).

**Proof.** The DG\( \Gamma \) algebra \( K^R(T) \) can be extended to the acyclic closure \( \mathfrak{x} \) of \( R \). We have a derivation \( \nu : \mathfrak{x} \to \mathfrak{x} \) of degree \( -2 \) such that \( \nu(T) = 1 \) by Lemma 4.1. Set \( \bar{R} = R/ \text{soc}(R) \). Then \( K^R = R \otimes_R K^R \) is the Koszul complex of \( \bar{R} \) and \( K^R \) can be extended to the acyclic closure \( \mathfrak{y} \) of \( \bar{R} \). Now the augmentation \( \epsilon : \mathfrak{y} \to k \) is an algebra homomorphism over \( K^R \). The acyclic closure \( \mathfrak{x} \) is semi-free over \( K^R \). So the augmentation \( \epsilon : \mathfrak{x} \to k \) lifts to a DG\( \Gamma \) algebra homomorphism \( \beta : \mathfrak{x} \to \mathfrak{y} \) over \( K^R \) [13, Proposition 2.1.9]. Let \( \alpha : K^R \to K^R \) denote the quotient map. We have the following commutative diagram.

\[
\begin{array}{ccc}
mK^R & \xrightarrow{i} & m\mathfrak{x} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
mK^R & \xrightarrow{j} & m\mathfrak{y}
\end{array}
\]

To show that \( K^R(T) \) is a Golod algebra, we need to prove that the inclusion \( i : mK^R(T) \to m\mathfrak{x} \) induces an injective map \( i_* : H(mK^R(T)) \to H(m\mathfrak{x}) \) on homology (see Theorem 2.2). A cycle \( y \) in \( mK^R(T) \) can be written as \( y = \sum_{k=0}^{m} a_k T^{(m-k)} \), \( a_i \in mK^R \). Suppose \( i(y) \) is in the boundary of \( m\mathfrak{x} \). We prove by induction on \( m \) that \( y \) is in the boundary of \( mK^R(T) \).

First assume that \( m = 0 \). Then \( y \in mK^R \). Since \( i(y) \) is in the boundary of \( m\mathfrak{x} \), \( j \circ \alpha(y) \) is in the boundary of \( m\mathfrak{y} \) by the commutative diagram. Now \( \bar{R} \) is a Golod ring. So \( j \) induces an injective map \( j_* : H(mK^R) \to H(m\mathfrak{y}) \). Therefore, \( \alpha(y) \) is in the boundary of \( mK^R \). So we can
write \( y = s y_1 + \partial(y_2) \) for \( y_1 \in K^R \), \( y_2 \in mK^R \) and \( \soc(R) = (s) \). If \( \deg(y) = \deg(y_1) < n \), then \( sy_1 \in (0 : m^2)B(K^R) \) by (1) of Lemma 4.2. So \( y \in mB(K^R) \). If \( \deg(y) = n \), then \( y_2 = 0 \) and \( y = sy_1 = asX_1 \ldots X_n \), \( a \in R \). Since \( H(K^R) \) is a Poincaré algebra [7], there is a \( z' \in Z_{m-1}(K^R) \) such that \( zu = sX_1 \ldots X_n \). So \( y = az' = \partial(auz') \in mB(K^R(T)) \). Therefore, the induction step for \( m = 0 \) follows.

Now we assume that \( m > 0 \). Note that \( a_0 = v^m(i(y)) \). Since \( i(y) \in mB(\mathfrak{x}) \) and the derivation \( v \) commutes with differentials, we have \( a_0 \in mK^R \cap B(m\mathfrak{x}) \). Suppose \( \deg(a_0) = n \). Then \( a_0 = asX_1 \ldots X_n \), \( a \in R \). So \( a_0T^{(m)} = azT^{(m)} = \partial(auzT^{(m+1)}) \in mB(K^R(T)) \). Now \( a_0(\sum_{k=0}^{m-1} a_k T^{(m-k)}) = y - a_0T^{(m)} \) is in the boundary of \( m\mathfrak{x} \) and therefore is in the boundary of \( mK^R(T) \) by the induction hypothesis. So \( y \) is in the boundary of \( mK^R(T) \).

Suppose \( \deg(a_0) < n \). Then by the argument in the induction step \( m = 0 \), \( a_0 = \partial(y_1) \) for \( y_1 \in mK^R \). We can write \( y = \partial(y_1)T^{(m)} + \sum_{k=0}^{m-1} a_k T^{(m-k)} = \partial(y_1)T^{(m)} + [(-1)^{\deg(a_0)}y_1zT^{(m-1)} + \sum_{k=0}^{m-1} a_k T^{(m-k)}] \). The first summand is in \( mB(K^R(T)) \). So the second summand is in the boundary of \( m\mathfrak{x} \) and therefore is in the boundary of \( mK^R(T) \) by induction hypothesis. So \( y \) is in the boundary of \( mK^R(T) \). This completes the induction step and the proof of (1) follows.

It is well known that generators of \( I \in I \setminus nI \) are in one-one correspondence with the generators of \( H_1(K^R) \). Let the maximal ideal \( n \) of \( Q \) be minimally generated by \( y_1, \ldots, y_n \). The Koszul complex of \( Q \) is \( K^Q = (X_1 : \partial(X_1) = y_1, 1 \leq i \leq n) \). Let \( f = \sum_{i=1}^{n} a_i y_i \) and \( z = \sum_{i=1}^{n} a_i X_i \in K^Q \). Let \( P = Q(f) \). Then \( K^P = K^Q \otimes_Q P \), \( K^R = K^Q \otimes_Q R \) denote Koszul complexes of \( P \), \( R \) respectively. Since \( f \in I \setminus nI \), we have \( z \in Z_j(K^R) \setminus B_j(K^R) \). By (1), \( U = K^R(T) : \partial(T) = z \) is a Golod algebra. Now \( V = K^P(T) : \partial(T) = z \) is the acyclic closure of \( P \) and \( U = V \otimes_R R \). Therefore, the map \( P \to R \) is a Golod homomorphism and (2) follows.

The ring \( \bar{R} \) is Golod. In [40, Proposition 6.2], the Poincaré series of \( R \) is computed as \( P^R_k = \frac{(1+t)^p}{1-k\cdot(K^R_k)_{t-1}+p+1(1+t)} \). So the result follows from Theorem 2.3.

The following result is proved in [33, Theorem 1].

**Theorem 4.4.** Let \( (R, m) \) be an Artinian Gorenstein local ring of embedding dimension \( n \) and \( K^R \) be the Koszul complex on a minimal set of generators of the maximal ideal \( m \). Set \( \soc(R) = (s) \), \( \bar{R} = R/sR \) and \( K^R = \bar{R} \otimes_{R} K^R \), the Koszul complex of \( \bar{R} \). Define a DG algebra structure on \( K^R \otimes \frac{K^R}{mK^R}[-1] \) with multiplication : \( (k, \bar{l})(k', \bar{l}') = (kk', \bar{l}l') + (-1)^{\deg(k)\deg(k')} \bar{l}l' \) and differential : \( \partial(k, \bar{l}) = (\partial(k) + sl, 0) \). Then the chain map \( K^R \otimes \frac{K^R}{mK^R}[-1] \to K^R \), \( (k, \bar{l}) \mapsto \bar{k} \) is a quasi isomorphism. If \( \bar{H} = \frac{H(K^R)}{H(mK^R)} \), \( \bar{K} = \frac{K^R \otimes_{R} K^R}{K^R \otimes_{R} K^R}[-1] \), then \( H(K^R) = \bar{H} \otimes \bar{K} \).

A typical element \( (\alpha, \beta) \) of degree \( i \) in \( \bar{H} \otimes \bar{K} \) is given by a pair \( (\alpha, \beta) \) such that \( \alpha \in Z_i(K^R) \), \( \beta \in K^R_{i-1} \), \( \alpha \) is a lift of \( \bar{a} \) and \( \beta \) is a lift of \( \bar{b} \). By (1) of Lemma 4.2 choose \( \gamma \in K^R_i \) such that \( \partial(\gamma) = s\beta \) where \( s \) is a generator of \( \soc(R) \). Then the residue class of \( \alpha - \gamma \) in \( K^R \) is a cycle and its homology class corresponds to \( (\alpha, \beta) \) by the above isomorphism.

**Proposition 4.5.** Let \( (R, m, k) \) be an Artinian Gorenstein local ring with \( e.\dim(R) = m \geq 2 \) and \( R(X) \) be the acyclic closure of \( R \). Let \( \bar{R} = R/\soc(R) \). Then \( \bar{R} \) has an acyclic closure of the form \( \bar{R}(X, Y, Z) \) with the following properties.

1. \( Y = \{ Y_i : i \geq 1 \}, Z = \{ Z_j : j \geq 1 \}, \partial(Y_i) \in (0 : m^2)\bar{R}(X, Y, Z) \) and \( \partial(Z_j) \in \bar{z'}(\bar{R}(X, Y, Z) \) for some \( \bar{z'} \in Z_{m-1}(K^R) \setminus B_{m-1}(K^R) \).
The R-linear map $R(X) \to \check{R}(X, Y, Z)$ sending $X_i$ to $X_i$ induces injective maps on homology viz.
$H(R(X_i : \deg(X_i) \leq n)) \hookrightarrow H(\check{R}(X_i, Y_j, Z_k : \deg(X_i), \deg(Y_j), \deg(Z_k) \leq n))$ and
$H(mR(X_i : \deg(X_i) \leq n)) \hookrightarrow H(m\check{R}(X_i, Y_j, Z_k : \deg(X_i), \deg(Y_j), \deg(Z_k) \leq n))$ for all $n \geq 2$.

Proof. Set $\check{R} = R/ \soc(R)$, $U^n = R(X_i : \deg(X_i) \leq n)$ and $\check{U}^n = \check{R} \otimes_R U^n$ for $n \geq 1$. The DGI algebras $U^n$, $\check{U}^n$ admit natural augmentation maps. By construction, we have $IH_1(U^n) = 0$ for $0 \leq i \leq n - 1$ and the homology classes of the cycles $\partial(X_i)$, $\deg(X_i) = n + 1$ minimally generate $IH_n(U^n)$. Now $\deg(X_i) = 1$ for $1 \leq i \leq m = e, \dim(R)$. Let $\partial(X_i) = x_i$. Then $m$ is minimally generated by $\{x_i : 1 \leq i \leq m\}$. Clearly $U^1$ is the Koszul complex $K^R$ on $x_1, \ldots, x_m$. Let $z \in Z_1(K^R) \setminus B_1(K^R)$ be such that $\partial(x_m) = \check{z}$. We choose $z' \in Z_{m-1}(K^R)$ such that $zz' = sX_1 \ldots X_m$ (see [7]).

We construct inductively a sequence $V^n$ of distinguished DGI algebras over $\check{R}$ with the following properties.

1. $V^1 = K^R$, the Koszul complex of $\check{R}$.
2. $IH_1(V^n) = 0$ for $0 \leq i \leq n - 1$.
3. $V^n = \check{U}^n(y_{p_0-1}, \ldots, y_p, z_{q_0-1}, \ldots, z_{q_n}), n \geq 2$ for some nondecreasing sequences $\{p_i\}$, $\{q_i\}$ of integers. Homology classes of $\partial(X_i)$ for $\deg(X_i) = n$ and $\partial(Y_j), \partial(Z_k)$ for $p_{n-1} + 1 \leq j \leq p_n$ and $q_{n-1} + 1 \leq k \leq q_n$ minimally generate $H_{n-1}(V^{n-1})$, $n \geq 2$.
4. $\partial(y_j) \in (0 : m^2)V^{n-1}$ and $\partial(z_k) \in z'V^{n-1}$ for $p_{n-1} + 1 \leq j \leq p_n$ and $q_{n-1} + 1 \leq k \leq q_n$.
5. The composition $U^n \to \check{U}^n \hookrightarrow V^n$ induces injective maps $H(U^n) \hookrightarrow H(V^n)$ and $H(mU^n) \hookrightarrow H(mV^n)$ for $n \geq 2$.

Note that $\check{U}^1$ is a Koszul complex of $\check{R}$. We set $V^1 = \check{U}^1$ with natural augmentation. By Theorem 4.4, $H_1(V^1) = H_1(U^1) \otimes k$. The pairing $\frac{m}{m^2} \times \frac{(0:m^2)}{(0:m)} \to \soc(R), (\check{x}, \check{y}) \mapsto xy$ is non-degenerate. We choose $y_i \in (0 : m^2)$, $1 \leq i \leq m$ such that $xy_{ij} = 0$ for $i \neq j$ and $xy_{ij} = s$ for $i = j$ where $s$ is a generator of $\soc(R)$. Then homology classes of $\overline{\partial(X_i)}, \deg(X_i) = 2$ and $\check{y}_1X_1$ minimally generate $H_1(V^1)$. Set $V^2 = \check{U}^2(Y_1 : \partial(Y_1) = \check{y}_1X_1)$ with natural augmentation. Here $p_0 = p_1 = 0, p_2 = 1$, $q_0 = q_1 = q_2 = 0$.

Clearly $H_1(U^2) = H_1(V^2) = 0$. We prove that the map $H_1(U^2) \to H_1(V^2), i \geq 2$ is an injective map. Note that $Z_1(U^2) = B_1(U^2)$. We choose $\alpha \in Z_1(U^2), i \geq 2$ such that the image $\bar{\alpha}$ is in the boundary of $V^2$. Then $\bar{\alpha} = \partial(\beta), \beta \in \check{U}^2(Y_1)$. Let $\beta = a_0 + a_1Y_1 + \ldots + a_mY_1^{(m)}, a_i \in \check{U}^2$. By Lemma 4.1 we have a $\check{U}^2$-linear derivation $\nu$ on $V^2$ such that $\nu(Y_1) = 1$. We have $\nu(\beta) = a_m$.

Since $\bar{\alpha}$ is free from $Y^{(i)}$ terms, $\nu(\alpha) = 0$. So $\partial(a_m) = \nu(\nu(\beta)) = \nu(\nu(\partial(\beta))) = \nu(\partial(\nu(\beta))) = 0$.

This shows that $a_m \in Z(U^2)$. If $\deg(a_m) = 0$, then $m \geq 2$ as $\deg(\beta) \geq 3$. We have $\partial(a_{m-1} + a_mY) = \partial(\nu^{m-1}(\beta)) = \nu^{m-1}(\partial(\beta)) = 0$. So $a_{m-1} + a_mY \in Z_2(V^2)$. Since $V^2$ is a distinguished DGI algebra over $\check{R}, Z_{\geq 1}(V^2) \subseteq mV^2$. So $a_m \in m$ if $\deg(a_m) = 0$. Therefore, $a_m \in m$ if $\deg(a_m) = 0$. This shows that $a_m \in Z(U^2) \subseteq mU^2$ by (4) of Lemma 4.2.

This implies that $\partial(a_mY_1^{(m)}) = 0$ since $y_1m \subseteq \soc(R)$. So $\bar{\alpha} = \partial(\beta_1), \beta_1 = \beta - a_mY^{(m)}$. Repeating this argument $m$ times we have $\bar{\alpha} = \partial(\alpha_0)$. So $\alpha_0$ is in the boundary of $U^2$. By (5) of Lemma 4.2, $\alpha_0$ is in the boundary of $U^2$. This shows that the map $H(U^2) \hookrightarrow H(V^2)$ is injective. The proof of that the map $H(mU^2) \hookrightarrow H(mV^2)$ is injective follows similarly (in fact more easily as $a_i$ are already in $mU^2$).

Now we assume that $V^n, n \geq 2$ is constructed with aforesaid properties and construct $V^{n+1}$. Since $V^n$ is a distinguished DGI algebra over $\check{R}$, we have $IZ(V^n) \subseteq mV^n$.

Claim - 1: $Z_n(V^n) = B_n(V^n) + \check{R} \otimes_R Z_n(U^n) + (0 : m^2)V_n^n + z'V^n \cap Z_n(V^n), n \geq 2$. 


Set $S = B(V^n) + \bar{R} \otimes_R Z_n(U^n) + (0 : m^2)V^n + \alpha'V^n \cap Z_n(V^n)$. Clearly $S \subset Z_n(V^n)$. To show the reverse inclusion, choose $\alpha \in Z_n(V^n)$. Then $\alpha$ is a finite linear combination of $\Gamma$-monomials in $Y_j$ and $Z_k$ with coefficients in $\bar{U}^n$. Let the total degree of $\alpha$ in $Y_j$ and $Z_k$ be $d$. We prove $\alpha \in S$ by induction on $d$. If $d = 0$, then $\alpha$ has no terms involving $Y_j$ and $Z_k$. So $\alpha \in Z_n(\bar{U}^n)$. By (4) of Lemma 4.2, $\alpha \in \bar{R} \otimes_R Z_n(U^n) + (0 : m^2)\bar{U}^n + \alpha'\bar{U}^n \cap Z_n(\bar{U}^n)$. Since $\bar{U}^n \subset V^n$, we have $\alpha \in S$.

Now we assume that $d > 0$. We choose a $\Gamma$-monomial term in $Y_j, Z_k$ of the form $fM$ for $f \in \bar{U}^n$, $M = Y_{i_1}^{(c_1)} \ldots Y_{i_m}^{(c_m)} Z_{m_1}^{(d_1)} \ldots Z_{m_l}^{(d_l)}$ in the expression of $\alpha$ such that $\deg(M) = d$, $\deg(f) = e$ and $n = d + e$. Since $\alpha$ is a cycle and $M$ is a $\Gamma$-monomial of highest total degree in $Y_j, Z_k$, $f$ is a cycle in $\bar{U}^n$. So by (4) of Lemma 4.2, we have $f \in \bar{R} \otimes_R Z_n(U^n) + (0 : m^2)\bar{U}^n + Z_n(\bar{U}^n) \cap \alpha'\bar{U}^n$.

If $1 \leq e < n$, we have $Z_n(\bar{U}^n) = B(V^n)$. If $e = 0$, then $f \in \bar{R}$ since $\alpha \in Z_n(V^n) \subset mV^n$, $n \geq 2$. We write $f = \partial(g) + f' + f''$ for $g \in \bar{U}^{n+1}$, $f' \in (0 : m^2)\bar{U}^n$ and $f'' \in Z_n(\bar{U}^n) \cap \alpha'\bar{U}^n$. So $fM = \partial(gM) + f'M + f''M + (-1)^e g \partial(M)$. Note that $f'M \in (0 : m^2)V^n$ and $f''M \in z'V^n \cap Z_n(V^n)$ since $z'\partial(M) = 0$. Also $g \partial(M)$ has total degree in $Y_j$ and $Z_k$ strictly less than $d$. If we write each $\Gamma$-monomial term of total degree $d$ in $Y_j, Z_k$ appearing in the expression of $\alpha$ in this manner, we have $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in S$, $\alpha_2 \in Z_n(V^n)$ and total degree of $\alpha_2$ in $Y_j, Z_k$ is strictly less than $d$. By the induction hypothesis, $\alpha_2 \in S$. So $\alpha \in S$ and our claim follows.

The homology classes of cycles $\partial(X_i)$, $\deg(X_i) = n + 1$ minimally generate $H_i(U^n)$. Because of Claim - 1 and the fact that $H(U^n) \twoheadrightarrow H(V^n)$ is injective, we can adjoin $\Gamma$-variables $X_i$ for $\deg(X_i) = n + 1$ and $Y_j, Z_k$ for $p_n + 1 \leq j \leq p_{n+1}, q_n + 1 \leq k \leq q_{n+1}$ to $V^n$ such that homology classes of $\partial(X_i)$, $\partial(Y_j)$, $\partial(Z_k)$ minimally generate $H_i(V^n)$ and $\partial(Y_j) \in (0 : m^2)V^n$, $\partial(Z_k) \in z'V^n$. We define

$$V^{n+1} = V^n \langle X_i, Y_j, Z_k : \deg(X_i) = n + 1, p_n + 1 \leq j \leq p_{n+1}, q_n + 1 \leq k \leq q_{n+1} \rangle$$

$$\overline{U}^{n+1} \langle Y_j, Z_k : 1 \leq j \leq p_{n+1}, 1 \leq k \leq q_{n+1} \rangle$$

with the natural augmentation. Clearly $IH_i(V^{n+1}) = 0$ for $0 \leq i \leq n$.

Claim - 2: If $f \in IZ(\overline{U}^{n+1})$ and $M = Y_{i_1}^{(c_1)} \ldots Y_{i_m}^{(c_m)} Z_{m_1}^{(d_1)} \ldots Z_{m_l}^{(d_l)}$ is a $\Gamma$-monomial of total degree $d$, then $\partial(fM) = \partial(\beta)$ for some $\beta \in (0 : m^2)V^n + \alpha'V^n$ and the total degree of $\beta$ in $Y_j, Z_k$ is strictly less than $d$.

Let $\partial(M) = \sum_{i=1}^m g_i M_i$ where $g_i \in (0 : m^2)\overline{U}^{n+1} + \alpha'\overline{U}^{n+1}$ and $M_i$ are $\Gamma$-monomials in $Y_j, Z_k$ of total degree strictly less than $d$. One observes that $g_i \partial(Y_j)$, $g_i \partial(Z_k)$ are zero. So $\partial^2(M) = \sum_{i=1}^m \partial(g_i)M_i = 0$. This shows that $g_i \in Z(\overline{U}^{n+1}) \cap [0 : m^2]\overline{U}^{n+1} + \alpha'\overline{U}^{n+1}]$. So we can write $g_i = g_i' + g_i''$ for $g_i' \in (0 : m^2)\overline{U}^{n+1}$ and $g_i'' \in Z(\overline{U}^{n+1}) \cap z'\overline{U}^{n+1}$.

Since $f \in m\overline{U}^{n+1}$, we have $fg_i' = 0$. By (7) of Lemma 4.2, we have $fg_i = fg_i'' = \partial(h_i)$ for some $h_i \in (0 : m^2)\overline{U}^{n+1} + \alpha'\overline{U}^{n+1}$. Therefore, $\partial(fM) = (-1)^{\deg(f)} \sum_{i=1}^m fg_i M_i = (-1)^{\deg(f)} \sum_{i=1}^m \partial(h_i)M_i = \partial(\beta)$ where $\beta = (-1)^{\deg(f)} \sum_{i=1}^m h_i M_i \in (0 : m^2)V^{n+1} + \alpha'V^{n+1}$. So the claim follows.

Claim - 3: The composition map $U^{n+1} \twoheadrightarrow \overline{U}^{n+1} \hookrightarrow V^{n+1}$ induces injective maps $H(U^{n+1}) \hookrightarrow H(V^{n+1})$ and $H(mU^{n+1}) \hookrightarrow H(mV^{n+1})$.

We only show that $H(U^{n+1}) \hookrightarrow H(V^{n+1})$ is injective. The injective property of the second map will follow similarly. Note that $Z_i(\overline{U}^{n+1}) = B_i(\overline{U}^{n+1})$ for $1 \leq i \leq n$. Let $\alpha \in Z_i(\overline{U}^{n+1})$ for $i \geq n + 1$ be such that $\overline{\alpha} \in \overline{U}^{n+1}$ is in the boundary of $V^{n+1}$. Let $\overline{\alpha} = \partial(\beta)$, $\beta \in V^{n+1}$. We rename $Y_j, Z_k$ to $W_l, 1 \leq l \leq p_{n+1} + q_{n+1}$ such that $\deg(W_l) \leq \deg(W_l')$ for $l <' l'$. This is possible because for any positive integer $d$, there are only finitely many $Y_j, Z_k$ such that $\deg(Y_j) = \deg(Z_k) = d$.

Now $V^{n+1} = \overline{U}^{n+1} \langle W_l : 1 \leq l \leq p_{n+1} + q_{n+1} \rangle$ is a distinguished DG$\Gamma$ algebra over $\overline{U}^{n+1}$ since by the construction the homology classes of $\partial(W_l)$, $\deg(W_l) = d$ minimally generate $IH_{d-1}(\overline{U}^{n+1} \langle W_l : 1 \leq l \leq p_{n+1} + q_{n+1} \rangle)$.
deg(W_i) ≤ d − 1)) = 1H_{d−1}(V^{d−1})/\langle \text{cls}(\partial(X_i)) \rangle : deg(X_i) = d) for 1 ≤ d ≤ n + 1. So for each W_i, we have a $\bar{U}^{n+1}$-linear derivation on $V^{n+1}$ such that $v_i(W_j) = 1$ and $v_j(W_i) = 0$ for $i < j$ by Lemma 4.1.

Let $\text{in}(\beta) = M$ where $M = W^{(c_1)}_1 \ldots W^{(c_r)}_r$, $c_i ≥ 0, c_r ≥ 1$. Let the leading coefficient of $\beta$ be $f$. Then by Lemma 4.1, we have $v_M(β) = ±f$. Now $±\partial(f) = \partial(v_M(β)) = v_M(\partial(β)) = v_M(\bar{α}) = 0$ since $\bar{α}$ is free from $W_i$. So, we have $f ∈ Z(\bar{U}^{n+1})$. Then by Lemma 4.1, we have $\text{in}(\beta) = M$ and $\text{deg}(\beta) = \text{deg}(M) ≥ n + 2$ and $\text{deg}(W_i) ≤ n + 1$. We have $v_i(β) ∈ Z_\geq(\bar{U}^{n+1}) ⊂ mV^{n+1}$, $\text{in}(v_i(β)) = v_i(M)$ and the leading coefficient of $v_i(β)$ is $±f$. So $f ∈ m$ if $\text{deg}(f) = 0$. Therefore, we conclude that $f$ is in $IZ(\bar{U}^{n+1})$.

By Claim - 2, $\partial(f M)$ can be written as the boundary of an element in $V^{n+1}$ of total degree in $W_i$ (equivalently in $Y_j, Z_k$) strictly less than that of $\text{deg}(M)$. So we can write $\bar{α} = \partial(β_i)$, $\text{in}(β_i) < \text{in}(β)$. If we repeat this argument, we have after finitely many steps a $β_L ∈ V^{n+1}_{i+1}$ free from $W_i$ (equivalently from $Y_j, Z_k$) such that $\bar{α} = \partial(β_L)$. This means that $\bar{α}$ is in the boundary of $\bar{U}^{n+1}$. By (5) of Lemma 4.2, $α$ is in the boundary of $U^{n+1}$.

Therefore, our induction step of construction $V^{n+1}$ is complete. We define the acyclic closure of $R$ as $\bar{R}(X, Y, Z) = \text{proj lim} V^n$. The DGΓ algebras $\{V^n\}$ define a filtration of $\bar{R}(X, Y, Z)$. The properties (1) and (2) stated in the theorem, are satisfied by $\bar{R}(X, Y, Z)$ by the construction.

**Theorem 4.6.** Let $(R, m, k)$ be an Artinian Gorenstein local ring. Let $l ≥ 2$. Then $R$ is a generalised Golod ring of level $l$ if and only if $R/\text{soc}(R)$ is so.

**Proof.** Any Artinian local ring of embedding dimension one is a Golod ring. Therefore, we may assume that e. $\text{dim}(R) = n ≥ 2$. Let $R(X)$ be the acyclic closure of $R$ and $\bar{R}(X, Y, Z)$ be the acyclic closure of $R = R/\text{soc}(R)$ as constructed in Proposition 4.5. We use notations as set in the proof of Proposition 4.5. We have the following commutative diagram. The vertical maps are injective by (2) of Proposition 4.5.

$$
\begin{array}{ccc}
H(mU^l) & \longrightarrow & H(mR(X)) \\
\downarrow & & \downarrow \\
H(mV^l) & \longrightarrow & H(m\bar{R}(X, Y, Z))
\end{array}
$$

First we assume that $\bar{R}$ is a generalised Golod ring of level $l$. Then $V^l$ is a Golod algebra and the inclusion $mV^l \hookrightarrow mR(X, Y, Z)$ induces an injective map $H(mV^l) \hookrightarrow H(mR(X, Y, Z))$ by Theorem 2.2. So, the top horizontal map is injective. Therefore, $U^l$ is a Golod algebra and the ring $R$ is a generalised Golod ring of level $l$ by Theorem 2.2.

Now we assume that $R$ is a generalised Golod ring of level $l$. So $U^l$ is a Golod algebra and the inclusion $mU^l \hookrightarrow mR(X)$ induces an injective map $H(mU^l) \hookrightarrow H(mR(X))$ by Theorem 2.2. Let $α ∈ Z(mV^l)$ be a boundary in $mR(X, Y, Z)$. Then $α = \partial(β), β ∈ m\bar{R}(X, Y, Z)$. Since $V^n, n ≥ 1$ defines a filtration of $\bar{R}(X, Y, Z)$, we have $β ∈ mV^N$ for large $N$. We show that $α$ is in the boundary of $mV^l$ in two cases.

**Case - 1:** $α$ is free from $Y_j, Z_k$, i.e. $α ∈ mU^l$.

By the argument as in Claim - 3, Proposition 4.5, we may assume that $β$ is free from $Y_j$ and $Z_k$, i.e. $β ∈ mU^N$. Let $\bar{α}, \bar{β}$ be lifts of $α, β$ in $mU^l, mU^N$ respectively. Then $\bar{α} = \partial(\bar{β}) + w, w ∈ \text{soc}(R)R(X)$. This shows that $\bar{α} ∈ Z(mU^l)$. We see that $α$ is in the boundary of $mU^N$. So $\bar{α}$ is in the boundary of $mU^N$ by (5) of Lemma 4.2. The top horizontal map in the diagram is injective. So $\bar{α}$ is in the boundary of $mU^l$. This shows that $α$ is in the boundary of $mU^l$ in particular in $mV^l$. 

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Lemma 5.2. With the notations as above, assume further that $H_0(S(\langle X \rangle)) = H_0(T(\langle Y \rangle)) = k$ and $Z_{\geq 1}(\langle X \rangle) \subset pS(\langle X \rangle)$, $Z_{\geq 1}(T(\langle Y \rangle)) \subset qT(\langle Y \rangle)$. Set $R(\langle X \sqcup Y \rangle) = R(\langle X \rangle) \otimes_R R(\langle Y \rangle)$. Then the DG algebra $R(\langle X \sqcup Y \rangle)$ is Golod if and only if both $S(\langle X \rangle)$, $T(\langle Y \rangle)$ are Golod algebras.

Proof. We have the following exact sequence of $R$-modules.

$$0 \rightarrow R \xrightarrow{\alpha} S \times T \xrightarrow{\beta} k \rightarrow 0.$$ 

Here $\alpha(s, t) = (s, t)$ and $\beta(s, t) = s - t$. Taking tensor product with the algebra $R(\langle X \sqcup Y \rangle)$, we have the following short exact sequence of complexes of $R$-modules.

$$0 \rightarrow R(\langle X \sqcup Y \rangle) \xrightarrow{\alpha \otimes \text{id}} S \otimes_R R(\langle X \sqcup Y \rangle) \oplus T \otimes_R R(\langle X \sqcup Y \rangle) \xrightarrow{\beta \otimes \text{id}} k \otimes_R R(\langle X \sqcup Y \rangle) \rightarrow 0.$$
Here id denotes the identity map on the complex $R(X \sqcup Y)$. Note that $S \otimes_R R(X \sqcup Y) = S \langle X \rangle \otimes_S S \langle Y \rangle$. The differential $S \otimes_R \partial$ is zero on $S \langle Y \rangle$. Therefore, $H(S \otimes_R R(X \sqcup Y)) = H(S \langle X \rangle) \otimes_R k \langle Y \rangle$. Similarly we have $H(T \otimes_R R(X \sqcup Y)) = k \langle X \rangle \otimes_R H(T \langle Y \rangle)$. The differential on $k \otimes_R R(X \sqcup Y)$ is zero. So the long exact sequence of homology has the form

$$k(X \sqcup Y)[1 + 1] \xrightarrow{\partial} H(R(X \sqcup Y)) \xrightarrow{H(\alpha \otimes id)} H(S \langle X \rangle) \otimes_R k \langle Y \rangle \oplus k \langle X \rangle \otimes_R H(T \langle Y \rangle) \xrightarrow{H(\beta \otimes id)} k(X \sqcup Y).$$

Let $\ker(H(\beta \otimes id)) = K$ and $\coker(H(\beta \otimes id)) = C$. Then $H_{\beta \otimes id}(t) = \frac{1}{t} H_C(t) + H_K(t)$ where $H_C(t) = \sum_{i \geq 0} \dim_k H_i(C)$ and $H_K(t) = \sum_{i \geq 0} \dim_k H_i(K)$.

The quotient maps $\pi_R : S \to k$ and $\pi_S : T \to k$ induce maps $\pi_{S \langle X \rangle} : S \langle X \rangle \to k \langle X \rangle$ and $\pi_{T \langle Y \rangle} : T \langle Y \rangle \to k \langle Y \rangle$ respectively. Choose cycles $z \in Z(S \langle X \rangle)$, $z' \in Z(T \langle Y \rangle)$ and elements $u \in k \langle X \rangle$ and $v \in k \langle Y \rangle$. Let $[z]$, $[z']$ denote homology classes of $z$, $z'$ respectively. Note that $H(\beta \otimes id)([z] \otimes v, u \otimes [z']) = \pi_{S \langle X \rangle}(z)v - u \pi_{T \langle Y \rangle}(z')$.

By the given hypothesis $Z_{\geq 1}(S \langle X \rangle) \subset \pi S \langle X \rangle$ and $Z_{\geq 1}(T \langle Y \rangle) \subset \pi T \langle Y \rangle$. So $H(\beta \otimes id)$ is zero on $H_{\geq 1}(S \langle X \rangle) \otimes_R k \langle Y \rangle \oplus k \langle X \rangle \otimes_R H_{\geq 1}(T \langle Y \rangle)$. The restriction of $H(\beta \otimes id)$ on $H_0(S \langle X \rangle) \otimes_R k \langle Y \rangle \oplus k \langle X \rangle \otimes_R H_0(T \langle Y \rangle) = k \langle Y \rangle \oplus k \langle X \rangle$ has one dimensional kernel generated by $(1, 1)$.

So we have $K = H_{\geq 1}(S \langle X \rangle) \otimes_R k \langle Y \rangle \oplus k \langle X \rangle \otimes_R H_{\geq 1}(T \langle Y \rangle) \oplus k(1, 1)$ and $C = \frac{k \langle X \langle Y \rangle \rangle}{k \langle X \rangle \otimes_R k \langle Y \rangle \oplus k \langle X \rangle \otimes_R k \langle Y \rangle \oplus k(1, 1)}$. Note that $k \langle X \sqcup Y \rangle = k \langle X \rangle \otimes_R k \langle Y \rangle$. Therefore, we have $H_{k \langle X \langle Y \rangle \rangle}(t) = H_{k \langle X \rangle}(t) H_{k \langle Y \rangle}(t)$ and

$$H_{k \langle X \langle Y \rangle \rangle}(t) = \frac{1}{t} H_{k \langle X \rangle}(t) + H_{k \langle Y \rangle}(t)$$

$$= \frac{1}{t}[H_{k \langle X \rangle}(t) H_{k \langle Y \rangle}(t) - (H_{k \langle X \rangle}(t) - 1) - (H_{k \langle Y \rangle}(t) - 1) - 1]
+ (H_{k \langle S \langle X \rangle \rangle}(t) - 1) H_{k \langle Y \rangle}(t) + H_{k \langle X \rangle}(t) H_{k \langle T \langle Y \rangle \rangle}(t) - 1 + 1
= 1 + H_{k \langle Y \rangle}(t) H_{k \langle S \langle X \rangle \rangle}(t) - 1 + H_{k \langle X \rangle}(t) H_{k \langle T \langle Y \rangle \rangle}(t) - 1 + 1 - (H_{k \langle X \rangle}(t) - 1) (H_{k \langle Y \rangle}(t) - 1).$$

The above equality gives

$$1 - t(H_{k \langle X \langle Y \rangle \rangle}(t) - 1) = H_{k \langle X \rangle}(t) [1 - t(H_{k \langle T \langle Y \rangle \rangle}(t) - 1)] + H_{k \langle Y \rangle}(t) [1 - t(H_{k \langle S \langle X \rangle \rangle}(t) - 1)] - H_{k \langle X \rangle}(t) H_{k \langle Y \rangle}(t).$$

We have term wise inequality of power series

$$P_k^S(t) \leq \frac{H_{k \langle X \rangle}(t)}{1 - t(H_{k \langle S \langle X \rangle \rangle}(t) - 1)} \text{ and } P_k^T(t) \leq \frac{H_{k \langle Y \rangle}(t)}{1 - t(H_{k \langle T \langle Y \rangle \rangle}(t) - 1)}.$$ (2)

We have

$$\frac{1 - t(H_{k \langle X \langle Y \rangle \rangle}(t) - 1)}{H_{k \langle X \langle Y \rangle \rangle}(t)} = \frac{H_{k \langle X \rangle}(t)[1 - t(H_{k \langle T \langle Y \rangle \rangle}(t) - 1)] + H_{k \langle Y \rangle}(t)[1 - t(H_{k \langle S \langle X \rangle \rangle}(t) - 1)] - H_{k \langle X \rangle}(t) H_{k \langle Y \rangle}(t)}{H_{k \langle X \rangle}(t) H_{k \langle Y \rangle}(t)}$$

$$= \frac{1 - t(H_{k \langle T \langle Y \rangle \rangle}(t) - 1)}{H_{k \langle Y \rangle}(t)} + \frac{1 - t(H_{k \langle S \langle X \rangle \rangle}(t) - 1)}{H_{k \langle X \rangle}(t)} - 1$$

$$< \frac{1}{P_k^T(t)} + \frac{1}{P_k^S(t)} - 1 \text{ from (2)}$$

$$= \frac{1}{P_k^T(t)}.$$

The DGI algebra $R(X \sqcup Y)$ is Golod if and only if the above inequality is an equality. This is equivalent to that both inequalities in (2) are equalities, i.e. both $S \langle X \rangle$, $T \langle Y \rangle$ are Golod algebras. \(\square\)
Proposition 5.3. Let $(S, v, k), (T, o, k)$ be local rings. Let $(S(X, \partial_1), (T(Y), \partial_2)$ be acyclic closures of $S, T$ respectively. Set $R = S \times_k T$. Let $(R(X), \tilde{\partial}_1), (R(Y), \tilde{\partial}_2)$ be semi-free DGΓ extensions of $R$ as described in Lemma 5.1. Let $p : R(X) \to S \otimes_R R(X) = S(X), q : R(Y) \to T \otimes_R R(Y) = T(Y)$ be natural quotient maps. Then $R$ admits an acyclic closure of the form $(R(X, Y, Z), \partial)$ which satisfies the following properties.

1) The inclusions $\alpha : R(X) \hookrightarrow R(X, Y, Z), \beta : R(Y) \hookrightarrow R(X, Y, Z)$ are chain maps of complexes.
2) There are surjective homorphisms of DGΓ algebras $\phi : R(X, Y, Z) \twoheadrightarrow S(X)$ and $\psi : R(X, Y, Z) \twoheadrightarrow T(Y)$ such that $\phi \circ \alpha = 0, \psi \circ \alpha = 0$ and $\phi \circ \alpha = p, \psi \circ \beta = q$.

Proof. Set $U^n = S(X_i : \deg(X_i) \leq n)$ and $V^n = T(Y_i : \deg(Y_i) \leq n), n \geq 1$. Then $U^n, V^n, n \geq 1$ define filtrations of the acyclic closures $S(X), T(Y)$ respectively. Augmentation maps of $S(X), T(Y)$ restrict to augmentation maps of $U^n, V^n$ respectively. From the construction of acyclic closures, it follows that $I H_i(U^n) = I H_i(V^n) = 0$ for $0 \leq i \leq n - 1$, the homology classes of cycles $\partial_1(X_i)$ for $\deg(X_i) = n + 1$ minimally generate $I H_i(U^n)$ and the homology classes of cycles $\partial_2(Y_i)$ for $\deg(Y_i) = n + 1$ minimally generate $I H_i(V^n)$.

Set $\hat{U}^n = R(X_i : \deg(X_i) \leq n)$ and $\hat{V}^n = R(Y_i : \deg(Y_i) \leq n), n \geq 1$. Let $p^n : \hat{U}^n \twoheadrightarrow \hat{U}^n \otimes_R S = U^n, q^n : \hat{V}^n \twoheadrightarrow \hat{V}^n \otimes_R S = V^n$ be natural quotient maps.

We construct inductively a sequence of augmented DGΓ algebras $(F^n R, \partial^n)$ with natural augmentation $\varepsilon^n : F^n R \to k$ satisfying the following properties.

1) $F^1 R = K^R$, the Koszul complex of $R$.
2) $I H_i(F^n R) = 0$ for $0 \leq i \leq n - 1$.
3) There are inclusions $\alpha^n : \hat{U}^n \hookrightarrow F^n R$ and $\beta^n : \hat{V}^n \hookrightarrow F^n R$ of complexes.
4) There are surjective DGΓ algebra homomorphisms $\phi^n : F^n R \twoheadrightarrow U^n$ and $\psi^n : F^n R \twoheadrightarrow V^n$ such that the compositions $\phi^n \circ \alpha^n = p^n, \psi^n \circ \beta^n = q^n$ and $\phi^n \circ \alpha^n = 0, \psi^n \circ \beta^n = 0$.

When $n = 1$, we define $F^1 R = \hat{U}^1 \otimes_R \hat{V}^1 = R(X, Y) : \deg(X_i) = \deg(Y_j) = 1)$, the Koszul complex on a minimal set of generators of $p \oplus q$ with the natural augmentation $\varepsilon^1$. Define $\alpha^1 : \hat{U}^1 \hookrightarrow F^1 R, \beta^1 : \hat{V}^1 \hookrightarrow F^1 R$ to be natural inclusions and $\phi^1 : F^1 R \twoheadrightarrow U^1, \psi^1 : F^1 R \twoheadrightarrow V^1$ such that $\phi^1 \circ \alpha^1 = p^1, \psi^1 \circ \beta^1 = q^1$ and $\phi^1 \circ \alpha^1 = 0, \psi^1 \circ \beta^1 = 0$.

Now we assume that $F^k R, n \geq 2$ is constructed with the aforesaid properties. Let $U^{n+1} = U^n(X_{c+1}, \ldots, X_{c+n})$ and $V^{n+1} = V^n(Y_{d+1}, \ldots, Y_{d+n}), \partial_1(X_{c+i}) = s_i$ for $1 \leq i \leq u$ and $\partial_2(Y_{d+j}) = t_j$ for $1 \leq j \leq v$. Then homology classes of cycles $s_i$ form a basis of $H_n(U^n)$ and homology classes of cycles $t_j$ form a basis of $H_n(V^n)$. Note that cycles $s_i \in pU^n$ and $t_j \in qV^n$. So we can view cycles $s_i, t_j$ as elements in $\hat{U}^n$ and $\hat{V}^n$ respectively. We have $\psi^n \circ \alpha^n(s_i) = s_i, \psi^n \circ \alpha^n(s_i) = 0, \psi^n \circ \beta^n(t_j) = t_j$ and $\phi^n \circ \beta^n(t_j) = 0$. It is now easy to see that homology classes of cycles $\alpha^n(s_i), \beta^n(t_j)$, $1 \leq i \leq u, 1 \leq j \leq v$ are linearly independent in $H_n(F^n R)$.

We choose cycles $z_k, 1 \leq k \leq w \in Z_n(F^n R)$ such that homology classes of cycles $\alpha^n(s_i), \beta^n(t_j)$, $z_k, 1 \leq i \leq u, 1 \leq j \leq v, 1 \leq k \leq w$ form a basis of $H_n(F^n R)$. We subtract a finite $R$-linear sum of $\alpha^n(s_i), \beta^n(t_j)$ from $z_k$ if necessary to assume that $\alpha^n(s_k)$ is in the boundary of $U^n$ and $\alpha^n(s_k)$ is in the boundary of $V^n$. We choose $f_k \in U^n, g_k \in V^n$ of degree $n + 1$ such that $\partial_1(f_k) = \phi^n(z_k)$ and $\partial_2(g_k) = \psi^n(z_k)$. Since $p^n, q^n$ are surjective, we have $f_k \in \hat{U}^n, g_k \in \hat{V}^n$ such that $p^n(f_k) = f_k$ and $q^n(g_k) = g_k$.
By constructions the inclusions $\alpha^n, \beta^n$ extend to inclusions $\alpha^{n+1} : U^{n+1} \hookrightarrow F^n R$ and $\beta^{n+1} : V^{n+1} \hookrightarrow F^n R$ of chain complexes respectively. We only construct the map $\phi^{n+1}$. The construction of $\psi^{n+1}$ will follow similarly. Let $I^n = \ker \phi^n$. Then

$$
\phi^n \beta^{n+1}(Z_k - \alpha^n(\tilde{f}_k)) = \phi^n \phi^{n+1} (Z_k) - \phi^n \alpha^n(\tilde{f}_k)
$$

$$
= \phi^n (z_k) - \partial_1 \phi^n \alpha^n(\tilde{f}_k) = \phi^n (z_k) - \partial_1 p^n(\tilde{f}_k) = \phi^n (z_k) - \partial_1 (f_k) = 0.
$$

So $\beta^{n+1}(Z_k - \alpha^n(\tilde{f}_k)) \in I^n$. We also have $\partial^{n+1}(Y_{d+j}; 1 \leq j \leq v) + (Z_k - \alpha^n(\tilde{f}_k); 1 \leq k \leq w)$ is a DG ideal of $F^{n+1} R$. The augmentations $\epsilon^n$, the inclusions $\alpha^n, \beta^n$ and surjections $\phi^n, \psi^n$ patch together to form augmentation map $\epsilon : R(X, Y, Z) \twoheadrightarrow k$, the inclusions $\alpha : R(X) \hookrightarrow R(X, Y, Z), \beta : R(Y) \hookrightarrow R(X, Y, Z)$ of chain maps, surjections $\phi : R(X, Y, Z) \twoheadrightarrow S(X), \psi : R(X, Y, Z) \twoheadrightarrow T(Y)$ respectively with required properties. □

A surjective homomorphism $f : (R, m, k) \twoheadrightarrow (S, n, k)$ of local rings is called large if it induces an injective map $f^* : \text{Ext}_S(k, k) \rightarrow \text{Ext}_R(k, k)$ of Hopf algebras ([35]). The homotopy Lie algebra of a local ring is the set of primitive elements of its ext algebra. A homomorphism of Hopf algebras is injective if and only if it induces an injection on the sets of primitive elements [25, Proposition 3.9]. So $f$ is large if and only if the induced map $f^* : \pi(S) \rightarrow \pi(R)$ is an injective Lie algebra homomorphism.

A graded sub Lie algebra of a free graded Lie algebra on a positively graded vector space is free [30, Proposition A.1.10]. A proof in the case of characteristic zero can also be found in the corollary following [22, Proposition 21.4]. We use this fact in the next theorem which generalises a result [31, Theorem 4.1] of Lescot.

**Theorem 5.4.** Let $(S, \triangledown, k), (T, \triangledown, k)$ be local rings and $R = S \times_k T$. Then $R$ is a generalised Golod ring of level $n$ if and only if both $S, T$ are so.

**Proof.** Assume that $R$ is a generalised Golod ring of level $n$, i.e. $\pi^{\triangledown}(R)$ is a free Lie algebra. The projection maps $pr_1 : R \rightarrow S, pr_2 : R \rightarrow T$ are large homomorphisms [37, Theorem 3.4]. This means that $pr_1, pr_2$ induce injective Lie algebra homomorphisms $pr_1^* : \pi(S) \hookrightarrow \pi(R)$ and $pr_2^* : \pi(T) \hookrightarrow \pi(R)$. So both $\pi^{\triangledown}(S), \pi^{\triangledown}(T)$ are free as they are subalgebras of the free Lie algebra $\pi^{\triangledown}(R)$. Therefore, $S, T$ are generalised Golod rings of level $n$.

Now assume that both $S, T$ are generalised Golod rings of level $n$. Let $S(X), T(Y)$ be acyclic closures of $S, T$ respectively. Then both $S(X_i : \deg(X_i) \leq n), T(Y_j : \deg(Y_j) \leq n)$ are Golod algebras. So by Lemma 5.2, $U = R(X_i, Y_j : \deg(X_i), \deg(Y_j) \leq n)$ is a Golod DG algebra, i.e. $\pi(U)$ is a free Lie algebra. The acyclic closure of $R$ is of the form $R(X, Y, Z)$ by Proposition 5.3. Set $V = R(X_i, Y_j, Z_k : \deg(X_i), \deg(Y_j), \deg(Z_k) \leq n)$.

Consider the inclusion of DG algebras $i : U \hookrightarrow V$. We have an induced map $i_* : \text{Tor}^U(k, k) \rightarrow \text{Tor}^V(k, k)$. Since $R(X, Y, Z)$ is a semi-free extension of both $U, V$ and is a minimal resolution of $k$, we have $\text{Tor}^U(k, k) = H(k \otimes_U R(X, Y, Z)) = k \otimes V R(X, Y, Z)$ and $\text{Tor}^V(k, k) = H(k \otimes_V R(X, Y, Z)) = k \otimes_V R(X, Y, Z)$. Therefore, $i_*$ is surjective. So $i_*$ is also a surjective map from the space of $G$-indecomposable elements of $\text{Tor}^U(k, k)$ onto that of $\text{Tor}^V(k, k)$. Taking $k$-vector space dual, we
observe that $i$ induces an injective Lie algebra homomorphism $\hat{\pi} : \pi(V) \hookrightarrow \pi(U)$. Since $\pi(U)$ is a free Lie algebra, the Lie algebra $\pi(V)$ is also free. So $V$ is a Golod algebra. This implies that $R$ is a generalised Golod ring of level $n$.

**Theorem 5.5.** Let $(R, m_R, k), (S, m_S, k)$ be two Artinian Gorenstein local rings, $T = R\#S$ and $l \geq 2$. Then $T$ is a generalised Golod ring of level $l$ if and only if both $R, S$ are so.

**Proof.** We have $\frac{T}{\text{soc}(T)} = \frac{R}{\text{soc}(R)} \times_k \frac{S}{\text{soc}(S)}$. So the result follows from Theorems 4.6, 5.4.

The following result generalises [17, Theorem 3.8].

**Theorem 5.6.** Let $A, B$ be two local algebras over a field $k$. If both $A, B$ have the Backelin-Roos property, then so does the fibre product $A \times_k B$.

**Proof.** We may assume that both $A, B$ are complete with respect to maximal ideals. Let $A = \frac{k[[x_1, \ldots, x_m]]}{I_A}$ and $B = \frac{k[[y_1, \ldots, y_n]]}{I_B}$. Then $A \times_k B = \frac{k[[x_1, \ldots, x_m, y_1, \ldots, y_n]]}{(I_A+I_B) \langle x_1 y_1, \ldots, x_m y_n \rangle}$ where $x_i y_j, 1 \leq i \leq m, 1 \leq j \leq n$. Since $A, B$ have the Backelin-Roos property, we have a regular sequence $f_1, \ldots, f_t$ in $I_A$ and a regular sequence $g_1, \ldots, g_s$ in $I_B$ such that $f : P \to A$ and $g : Q \to B$ are Golod homomorphisms from complete intersections $P = \frac{k[[x_1, \ldots, x_m]]}{I_A}$ and $Q = \frac{k[[y_1, \ldots, y_n]]}{I_B}$ respectively. We claim that the surjective homomorphism $H : P \otimes_k Q \twoheadrightarrow A \times_k B$ is a Golod homomorphism.

Let $P(X), Q(Y)$ be acyclic closures of $P, Q$ respectively. By the hypothesis, $A(X) = A \otimes_P P(X), B(Y) = B \otimes_Q Q(Y)$ are Golod algebras. Now $P(X) \otimes_k Q(Y)$ is the acyclic closure of $P \otimes_k Q$ and by Lemma 5.2, $[P(X) \otimes_k Q(Y)] \otimes_{P \otimes_k Q} A \times_k B = (A \times_k B)(X \sqcup Y)$ is also a Golod algebra. So $H$ is a Golod homomorphism and the result follows.

6. Applications

**Theorem 6.1.** Let $(R, m, k)$ be an Artinian Gorenstein local ring such that $\dim(R) = n$ and $\mu(m^2) \leq 2$. Let $M$ be a finitely generated $R$-module. Let $[\eta : (Q, \mathfrak{m}) \to (R, \mathfrak{m})]$ be a minimal Cohen presentation and $I = \ker \eta \subset \mathfrak{m}^2$. Then the following hold.

1. If $\dim(R) = 1$, then $P^R_k(t) = \frac{1}{1 - t}$ and $(1 - t)P^R_M(t) \in \mathbb{Z}[t]$. 
2. If $\dim(R) \geq 2$, then for any $f \in I \setminus \mathfrak{m}^2$, the induced map $Q(f) \to R$ is a Golod homomorphism. $P^R_k(t) = \frac{1}{1 - nt + t^2}$. Moreover, $(1 + t)^n(1 - nt + t^2)P^R_M(t) \in \mathbb{Z}[t]$

**Proof.** If $n = 1$, $R$ is a Golod ring and $P^R_k(t) = \frac{1}{1 - t}$. The ring $R$ is a quotient of a DVR. Using the structure theorem of modules over PID, the module $M$ is a direct summand of quotients of $R$. So $P^R_M(t) = \mu(M) + \mu(Syz^R_1(M)) \frac{1}{1 - t}$. This implies that $(1 - t)P^R_M(t) \in \mathbb{Z}[t]$. So (1) follows.

The first part of (2) follows from Theorems 3.7, 4.3. The Poincaré series of $k$ is computed using Theorems 2.1 and 3.7. The last assertion on Poincaré series of $M$ follows from the result of Levin [10, Proposition 5.18].

Stretched Cohen-Macaulay rings are shown to be good in a recent article [2, Corollary 5.6]. We provide a different way to see this. Let $(R, m, k)$ be a stretched Artinian ring such that $\dim(R) = n$, $m = (x_1, \ldots, x_n), \text{Il}(R) = s$ and $\text{dim}_k(0 : m) = r$. If $\text{soc}(R) \subset m^2$, then $\text{soc}(R) = m^t$ and $R$ is Gorenstein. Otherwise assume that $x_1 \in \text{soc}(R) \setminus m^2$. We see $R = R/(x_1) \times_k R/(x_2, \ldots, x_n)$. Note that $R/(x_1)$ is also a stretched Artinian ring. Therefore, after a finite number of steps, we may write $R = S \times_k T$ where $(S, m_S)$ is a stretched Artinian Gorenstein ring and $(T, m_T)$ is a local ring with $m_T^2 = 0$. Clearly $r = 1 + \dim(T)$. 

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Proposition of DG algebras. By Theorem (2), (3) of Theorem 6.6.

Let $R$ be either a stretched Cohen-Macaulay ring or an almost stretched Gorenstein algebra with natural augmentation. Then $R$ has the following hold.

1. If $r = n - d$, then $R$ is a Golod ring, $P^R_k(t) = \frac{(1+d)^d}{1-(n-d)t}$ and $(1 - (n - d)t)P^R_M(t) \in \mathbb{Z}[t]$.

2. If $r \neq n - d$, then $P^R_k(t) = \frac{(1+d)^d}{1-(n-d)t} + (1 + t)^{n-d}(1 - (n - d)t + t^2)P^R_M(t) \in \mathbb{Z}[t]$.

**Corollary 6.2.** Let $(R, m, k)$ be a $d$-dimensional stretched Cohen-Macaulay local ring and $M$ be an $R$-module. Let $n = \dim_k m/m^2$ and $r = \dim_k \Ext^d_R(k, R)$ denote the type of $R$. Then the following hold.

1. If $r = n - d$, then $R$ is a Golod ring, $P^R_k(t) = \frac{(1+d)^d}{1-(n-d)t}$ and $(1 - (n - d)t)P^R_M(t) \in \mathbb{Z}[t]$.

2. If $r \neq n - d$, then $P^R_k(t) = \frac{(1+d)^d}{1-(n-d)t} + (1 + t)^{n-d}(1 - (n - d)t + t^2)P^R_M(t) \in \mathbb{Z}[t]$.

Corollary 6.3. Let $(R, m, k)$ be a $d$-dimensional almost stretched Gorenstein local ring and $M$ be an $R$-module. Let $n = \dim_k m/m^2$. Then $P^R_k(t) = \frac{(1+d)^d}{1-(n-d)t} + (1 + t)^{n-d}(1 - (n - d)t + t^2)P^R_M(t) \in \mathbb{Z}[t]$.

The following is due to Sega [42, Proposition 1.5].

**Proposition 6.4.** Let $R$ be a local ring such that there is a $d_R(t) \in \mathbb{Z}[t]$ satisfying $d_R(t)P^R_M(t) \in \mathbb{Z}[t]$ for each finitely generated $R$-module $M$. Let $d_R(t) = p(t)q(t)r(t)$ where $p(t)$ is 1 or irreducible; $q(t)$ has non-negative coefficients; $r(t)$ is 1 or irreducible and has no positive real root among its complex roots of minimal absolute value. Then the following hold for each pair of $R$-modules $M$, $N$.

1. If $\Tor^R_i(M, N) = 0$ for $i \gg 0$, then either $M$ or $N$ has finite projective dimension.

2. If $\Ext^R_i(M, N) = 0$ for $i \gg 0$, then either $M$ has a finite projective dimension or $N$ has a finite injective dimension.

Araya and Yoshino [1, 4.2] proved that if $M$ has finite CI-dimension, then the projective dimension of $M$ is same as $\sup(i : \Ext^i(M, M) \neq 0)$. Therefore, the following corollary follows from Proposition 6.4.

**Corollary 6.5.** Let $R$ be either a stretched Cohen-Macaulay ring or an almost stretched Gorenstein ring. Then $R$ satisfies Auslander-Reiten conjecture, i.e. $\Ext^{\geq 1}(M, M \oplus R) = 0$ for a finitely generated $R$ module $M$ implies that $R$ is free.

**Theorem 6.6.** Let $(R, m)$ be an Artinian Gorenstein local ring of embedding dimension $n \geq 2$. Let $K^R$ denote the Koszul complex on a minimal set of generators of $m$ and $C$ denote the quotient of $K^R$ defined by $C_i = K^R$ for $0 \leq i \leq n - 2$, $C_{n-1} = K^R_{B_{n-1}(K^R)}$ and $C_n = 0$. Assume that $C$ is a Golod DG algebra with natural augmentation. Then $R/\soc(R)$ is a Golod ring and $R$ satisfies assertions (1), (2), (3) of Theorem 4.3.

**Proof.** Let $\hat{q} : K^R \to C$ be the quotient map and $\soc(R) = (s)$. Let $G : \frac{k^R}{mK^R} \to K^R$, $H : \frac{C}{mC} \to C$ denote chain maps induced by multiplications by $s$ on $K^R$, $C$ respectively. We have $q \circ G = H \circ \hat{q}$ where $\hat{q} : \frac{k^R}{mK^R} \to \frac{C}{mC}$ is the map induced by $q$. Let $\alpha(G) = K^R \oplus \frac{k^R}{mK^R}[-1]$, $\alpha(H) = C \oplus \frac{C}{mC}[-1]$ be the cones of $G$, $H$ respectively. Define $\alpha : \alpha(G) \to \alpha(H)$ by $\alpha(k, l) = (q(k), \hat{q}(l))$. Both $\alpha(G)$, $\alpha(H)$ have DG algebra structure and $\alpha$ is a surjective DG algebra homomorphism. The kernel of $\alpha$ is the complex $0 \to \frac{k^R}{mK^R} \to K^R_n \to B_{n-1}(K^R) \to 0$ which is exact. Therefore, $\alpha$ is a quasi-isomorphism of DG algebras. By Theorem 4.4, $\alpha(G)$ is quasi-isomorphic to $K^R/\soc(R)$. Therefore, to show that $R/\soc(R)$ is a Golod ring it is enough to prove that $\alpha(H)$ is a Golod algebra.

Since $C$ is a Golod algebra, there is a $k$-basis $b_C = \{h_\lambda\}_{\lambda \in \Lambda}$ of $H_{\geq 1}(C)$ and a function (trivial Massey operation) $\mu : \sqcup_{i=1}^{\infty} b_C^i \to C$ such that $\mu(h_\lambda) \in Z_{\geq 1}(C)$ with $cls(\mu(h_\lambda)) = h_\lambda$ and
\[ \partial \mu(h_\lambda, \ldots, h_{\lambda_j}) = \sum_{j=1}^{p-1} \mu(h_\lambda, \ldots, h_{\lambda_j}) \mu(h_{\lambda_{j+1}}, \ldots, h_{\lambda_p}) \] 

By (2) of [13, Lemma 4.1.6], \( x \in C : \partial(x) \in m^2C \subset mC \). Since \( \mu(h_\lambda) \in mC \), by induction on \( p \) we conclude that \( \mu(h_\lambda, \ldots, h_{\lambda_p}) \in mC \).

By (1) of Lemma 4.2, \( sC \subset (0 : m^2)B(C) \). Let \( z \in Z(C) \) be such that \( (z, 0) = \partial(y_1, y_2) \) for \( (y_1, y_2) \in c(H) \). Then \( z = \partial(y_1) + sy_2 \in B(C) \). So the inclusion \( C \hookrightarrow c(H) \) induces an injective map \( H(C) \hookrightarrow c(H) \). By abuse of notation we write \( (c, 0) = c \) as \( c \). So \( b_C = \{ h_\lambda \}_{\lambda \in \Lambda} \) is a linearly independent set in \( H_{\geq 1}(c(H)) \) and \( \mu(h_\lambda, \ldots, h_{\lambda_p}) \in m(c(H)) \) satisfy properties above. We extend \( b_C \) to a basis \( b_{(\Lambda)} = \{ h_\lambda \}_{\lambda \in \Lambda \cup \{ R \}} \) of \( H_{\geq 1}(c(H)) \). Let \( h_{\lambda'}, \lambda' \in \Lambda' \) be the homology class of \( (e_{\lambda'}, d_{\lambda'}) \in Z_{\geq 1}(c(H)) \). Now \( \partial(e_{\lambda'}, d_{\lambda'}) = 0 \) implies that \( \partial(e_{\lambda'}) + sd_{\lambda'} = 0 \). We have \( sd_{\lambda'} = \partial(e_{\lambda'}) \) for some \( e_{\lambda'} \in (0 : m^2)C \). We write \( (e_{\lambda'}, d_{\lambda'}) = (e_{\lambda'} + e_{\lambda'}, 0) + (-e_{\lambda'}, d_{\lambda'}) \). Note that \( c_{\lambda'} + e_{\lambda'} \in Z(C) \). Therefore, after subtracting suitable \( R \)-linear combinations of \( h_\lambda, \lambda \in \Lambda \) from each \( h_{\lambda'}, \lambda' \in \Lambda' \) if necessary, we may assume that \( h_{\lambda'} \) is a homology class of some cycle in \( (0 : m^2)C \).

We define \( \mu(h_{\lambda'}) \), \( \lambda' \in \Lambda' \) to be an element in \( (0 : m^2)C \oplus \frac{C}{mC} \) whose homology class is \( h_{\lambda'} \). We extend \( \mu \) from \( b_C^i \) to \( b_{(\Lambda)}^i \), \( i > 1 \) such that \( \mu : b_{(\Lambda)}^i \setminus b_C^i \rightarrow (0 : m^2)C \), \( i > 1 \) by induction on \( i \). Note that \( \mu(h_{\lambda'}) \) satisfies desired relations are constructed for all \( i \leq p \). We choose \( (h_{\lambda_1}, \ldots, h_{\lambda_p}) \in b_{(\Lambda)}^{p+1} \) satisfying \( \mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) \) is an element in \( sC \). So we can choose \( \mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) \in (0 : m^2)C \) such that \( \partial(\mu(h_{\lambda_1}, \ldots, h_{\lambda_p})) = \sum_{j=1}^{p} \mu(h_{\lambda_1}, \ldots, h_{\lambda_j}) \mu(h_{\lambda_{j+1}}, \ldots, h_{\lambda_p}) \). Thus by induction \( \mu \) extends to a trivial Massey operation on \( b_{(\Lambda)} \). Therefore, \( c(H) \) is a Golod algebra and the result follows.

Poincaré series of Modules over compressed Gorenstein local rings were studied by Rossi and Şega in [40]. Under the hypothesis of Corollary 6.7 below they proved that if \( \eta : Q \twoheadrightarrow R \) is a minimal Cohen presentation, \( I = \ker \eta, f \in I \setminus n^{i+1}, t = \max\{i : I \subset n^i\} \), then \( Q/(f) \twoheadrightarrow R \) is a Golod homomorphism [40, Theorem 5.1]. They also showed that \( R/\text{soc}(R) \) is a Golod ring [40, Proposition 6.3]. We use the ideas developed by them and our Theorem 6.6 to produce a simple and short proof of their main result [40, Theorem 5.1]. Note that our result is stronger. Our theorem below shows that any choice of \( f \in I \setminus n^t \) is enough.

**Corollary 6.7.** Let \((R, m, k)\) be a compressed Gorenstein local ring such that \( e. \dim(R) = n \geq 2 \) and \( \text{dim}(R) = s, s \geq 2, s \neq 3 \). Then \( R/\text{soc}(R) \) is a Golod ring and \( R \) satisfies assertions (1), (2), (3) of Theorem 4.3.

**Proof.** We follow notations as set in Theorems 4.3 and 6.6. Let \( t = \max\{i : I \subset n^i\} \). From the structure theorem of compressed Gorenstein local rings, it follows that \( t = \left\lfloor \frac{n^t}{2} \right\rfloor \); the least integer no less than \( \frac{n^t}{2} \) [40, Proposition 4.2]. By [40, Lemma 1.4], the map \( H_{\geq 1}(R/m^t \otimes Q K^Q) \rightarrow H_{\geq 1}(R/m^t \otimes Q K^Q) \) induced by the surjection \( R/m^t \rightarrow R/m^t-1 \) is a zero map. This implies that \( Z_{\geq 1}(K^R) \subset B_{\geq 1}(K^R) + m^t-1 K^R \) and therefore \( Z_{\geq 1}(C) \subset B_{\geq 1}(C) + m^t-1 C \).

It is proved in [40, Lemma 4.4] that the map \( \psi : H_{\geq 0}(m^{i+1} K^R) \rightarrow H_{\geq 0}(m^t K^R) \) induced by the inclusion \( m^{i+1} \hookrightarrow m^t \) is zero for \( r = s+1-t \). Since, \( s \geq 2, s \neq 3 \), we have \( t-1 \leq r+1 \leq 4 \). This implies that the map \( H_{\geq 0}(m^t \otimes Q K^Q) \rightarrow H_{\geq 0}(m^t \otimes Q K^Q) \) is also zero since it factors through \( \psi \). Therefore, we have \( Z_{\leq 0}(m^2 K^R) \subset B(m^t-1 K^R) \) which implies \( Z(m^t-1 C) \subset B(m^t-1 C) \). It is worth pointing out that both the lemmas mentioned are independent of all other results in [40].

Thus we find a basis of \( H_{\geq 1}(C) \) represented by cycles in \( m^t-1 C \) such that the product of any two representatives is in the boundary of \( m^t-1 C \). So we can define inductively a trivial Massey operation on \( C \). Therefore, \( C \) is a Golod DG algebra and the result follows from Theorem 6.6. \( \square \)
Poincaré series of modules over a Gorenstein ring $R$ with $e \dim(R) - \dim(R) \leq 3$ are known to be rational (see [44, Satz 9], [10, Theorem 6.4]). We prove the following.

**Corollary 6.8.** Let $(R, m)$ be a Gorenstein local ring but not a complete intersection. Assume that $e \dim(R) - \dim(R) \leq 3$. Let $\eta : (Q, m) \to (\hat{R}, \hat{m})$ be a minimal Cohen presentation and $I = \ker \eta$. Then for any $f \in I \setminus m$, the induced map $Q/(f) \to \hat{R}$ is a Golod homomorphism. If $R$ is Artinian, then $R/\soc(R)$ is a Golod ring and assertions of Theorem 4.3 hold. Modules over a Gorenstein ring $R$ with $e \dim(R) - \dim(R) \leq 3$, have rational Poincaré series.

**Proof.** Let $m$ be minimally generated by $x_1, \ldots, x_d$ such that $x_1, \ldots, x_d$, $d = \dim(R)$ form an $R$-sequence. Now $S = R/(x_1, \ldots, x_d)$ is an Artinian Gorenstein local ring and $e \dim(S) \leq 3$. The Koszul complexes $K^R$ and $K^S$ are quasi-isomorphic [13, Lemma 4.1.6]. So the adjunction of a degree two variable to kill a degree one cycle in $K^R$ yields a Golod algebra if and only if the same holds true for $K^S$. Therefore, we may assume that $R$ is an Artinian Gorenstein local ring.

We construct $C$ as in Theorem 6.6. By [44, Satz 7], $H_1(K^R)^2 = 0$. So $H_1(C)^2 = 0$. Now $C$ is a DG algebra of length 2. So any basis of $H_{\geq 1}(C)$ admits a trivial Massey operation and the result follows by Theorem 6.6. The last assertion follows since modules over complete intersections have rational Poincaré series [26, Corollary 4.2].

To prove the following Theorem, we use the fact that Gorenstein rings of co-depth at most 4 have the Backelin-Roos property and therefore are generalised Golod rings of level 2 [10, Theorem 6.4].

**Theorem 6.9.** Let $(R, m)$ be a Gorenstein local ring. Then $R$ is a generalised Golod ring of level 2, and therefore good in the following cases.

1. $m^4 = 0$, $\mu(m^3) \leq 4$.
2. The multiplicity of $R$ is at most 11.

**Proof.** First we prove (1). We assume that $\ll(R) \geq 3$ because otherwise the result follows from Theorem 6.1. It is enough to assume that the ring is indecomposable as a connected sum because of Theorem 5.5. If $\mu(m) \geq 5$, then $l(0 : m^3) = l(R/m^3) = 1 + l(m/m^3) \geq 6$ and $l(m^2) = l(m^2/m^3) + l(m^3) \leq 5$. So $(0 : m^3) \subseteq m^3$, a contradiction due to Theorem 3.5. Therefore, $\mu(m) \leq 4$ and the result follows from [10, Theorem 6.4].

Now we prove (2). If we go modulo a nonzero divisor in $m \setminus m^2$, then the acyclic closure of $R$ does not change up to quasi-isomorphism. So we may assume by a standard argument that $R$ is Artinian. If we can show that either $e \dim(R) \leq 4$ or $\ll(R) \leq 2$ or else $R = S \# T$ for Gorenstein local rings $(S, \nu), (T, \eta)$ such that $e \dim(S) \leq 4$ and $\ll(T) = 2$, then we are done in view of [10, Theorem 6.4], Theorems 5.5, 6.1.

If possible assume that there is an Artinian Gorenstein local ring $R$ of multiplicity at most 11 such that $R$ is not of any of the types mentioned above. We choose $R$ of the least possible multiplicity. Then in view of Theorem 3.5 we must assume that $(0 : m^2) \subseteq m^2$. This implies that $l(R/m^2) = l(0 : m^2) \leq l(m^2) = l(R) - l(R/m^2)$. So $l(R/m^2) \leq \frac{1}{2}l(R) = \frac{11}{2}$ which gives that $e \dim(R) \leq l(R/m^2) - 1 \leq 4$, a contradiction. So the result follows.

We conclude this section with the following question.

**Question 6.10.** What is the least possible integer $n$ such that there is a Gorenstein bad local ring of multiplicity $n$?

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