Deformation of the ABJM theory

Mir Faizal

Mathematical Institute, University of Oxford - Oxford OX1 3LB, UK, EU

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Abstract - In this paper we analyse the ABJM theory on a deformed spacetime. We show that this theory reduces to deformed super-Yang-Mills theory when one of the scalar superfields is given a non-vanishing vacuum expectation value. Our analysis is done in the $N=1$ superspace formalism.

Introduction. - The low-energy action for multiple $M2$-branes is thought to be given by the ABJM theory [1,2]. This theory is a three-dimensional Chern-Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$ at levels $k$ and $-k$ on the world-volume of $N$ $M2$-membranes placed at the fixed point of $R^8/Z_k$. This theory explicitly realizes the $N=6$ supersymmetry and it is expected to be enhanced to the full $N=8$ supersymmetry for $k=1,2$ [3]. The ABJM theory coincides with the Bagger-Lambert-Gustavsson (BLG) action [4–8] based on the Lie 3-algebra. The BLG model has been analysed in the $N=1$ superfield formalism [9]. Higgs mechanism for both the ABJM theory and BLG model has also been analysed in the $N=1$ superspace formalism [10,11].

The spacetime non-commutativity arises in the string theory because of the coupling of the theory to a $NS$ background [12–19]. Other background fields can similarly cause other deformations of the theory. A non-anticommutative deformation of the theory is caused by a $RR$ background [20,21]. Furthermore, the commutator of the spacetime coordinates with the fermionic coordinates does not vanish in the presence of a gravitino background [22]. Field theories with these kind of deformations have been thoroughly studied [21–27]. In fact, it is well known that the non-anticommutative deformation of a theory breaks half of the supersymmetry of that theory explicitly. In this paper we will study these deformations for $M$-theory.

As there is a duality between $M$-theory and $II$ string theory, we expect that any deformation on the $M$-theory side will also correspond to some deformation on the string theory side. In fact, $M2$-branes in $M$-theory are analogous objects to strings in string theory. This is because just like strings can end on $D$-branes in string theory, $M2$-branes can end on $M5$-branes in $M$-theory. Furthermore, as a three-form field strength occurs naturally in $M$-theory, we expect that coupling the ABJM theory to a background three-form field could lead to a deformation of the ABJM theory, just like coupling of $D$-branes to a background two-form field strength leads to a non-commutative deformation of the theory.

Coupling of the ABJM theory to background fields can have interesting uses in understanding the theory of multiple $M5$-branes. It may be noted that even though the action for a single $M5$-brane is known, the action for multiple $M5$-branes is not known [28–32]. Furthermore, in $M$-theory the action of a single $M5$-brane can be obtained by analyzing the the $\kappa$-symmetry of the open membrane ending on it [33]. It might be possible to perform a similar analysis using the ABJM theory coupled to a background three-form field strength and gain insight into the dynamics of multiple $M5$-branes.

ABJM theory in $N=1$ superspace. - Now we review the classical Lagrangian density for the ABJM theory in the $N=1$ superspace formalism with the gauge group $U(N) \times U(N)$,

$$\mathcal{L}_c = \mathcal{L}_M + \mathcal{L}_{CS} - \tilde{\mathcal{L}}_{CS},$$

(1)

where $\mathcal{L}_{CS}$ and $\tilde{\mathcal{L}}_{CS}$ are deformed Chern-Simons theories with gauge group’s $U(N)$ from $U(N) \times U(N)$, respectively. They can thus be expressed as

$$\mathcal{L}_{CS} = \frac{k}{2\pi} \int d^2\theta \text{Tr} \left[ \Gamma^a \Omega_a \right],$$

$$\tilde{\mathcal{L}}_{CS} = \frac{k}{2\pi} \int d^2\theta \text{Tr} \left[ \tilde{\Gamma}^a \tilde{\Omega}_a \right],$$

(2)
where $k$ is an integer and
\begin{align}
\Omega_a &= \omega_a - \frac{1}{6} [\Gamma^b, \Gamma_{ab}], \\
\omega_a &= \frac{1}{2} D^b D_a \Gamma_b - \frac{i}{2} [\Gamma^b, D_b \Gamma_a] - \frac{1}{6} [\Gamma^b, \{\Gamma_b, \Gamma_a\}], \\
\Gamma_{ab} &= -\frac{i}{2} [D_a \Gamma_b - i\{\Gamma_a, \Gamma_b\}], \\
\delta \Gamma_a &= \nabla_a \Lambda, \quad \delta \bar{\Gamma}_a = \nabla_a \bar{\Lambda}, \\
\delta X^I &= i(\Lambda X^I - X^I \Lambda), \quad \delta X^{I\dagger} = i(\bar{\Lambda} X^{I\dagger} - X^{I\dagger} \Lambda).
\end{align}

Thus, now the infinitesimal gauge transformations for these fields are given by
\begin{align}
\delta \Gamma_a &= \nabla_a \Lambda, \quad \delta \bar{\Gamma}_a = \nabla_a \bar{\Lambda}, \\
\delta X^I &= i(\Lambda X^I - X^I \Lambda), \quad \delta X^{I\dagger} = i(\bar{\Lambda} X^{I\dagger} - X^{I\dagger} \Lambda).
\end{align}

The Lagrangian for the ABJM theory is invariant under these gauge transformations:
\begin{align}
\delta \mathcal{L}_{ABJM} &= \delta \mathcal{L}_{kcs}(\Gamma) - \delta \bar{\mathcal{L}}_{kcs}(\bar{\Gamma}) + \delta \mathcal{L}_M \\
&= 0.
\end{align}

**Deformed ABJM theory.** In this section we shall construct a three-dimensional Chern-Simons theory on a deformed superspace. In order to analyse the deformation of the superspace both the Grassman coordinates and the spacetime coordinates are promoted to operators and a deformation of their superalgebra is imposed. In four dimensions the $N = 1$ supersymmetry is generated by the supersymmetric generator $Q_A$ which can be split into $Q_a$ and $Q_\alpha$. Furthermore, it is possible to break the supersymmetry corresponding to $Q_a$ and leave the supersymmetry corresponding to $Q_\alpha$ intact or vice versa [21]. Thus, it is possible to construct theories with $N = 1/2$ supersymmetry in four dimensions. However, in three dimensions both $Q_a$ and $Q_\alpha$ act as independent supercharges. So, we can view a theory with $N = 1$ supersymmetry in four dimensions as a theory with $N = 2$ supersymmetry in three dimensions. Thus, $N = 1/2$ supersymmetry in four dimensions corresponds to $N = 1$ supersymmetry in three dimensions [34]. It is not possible to obtain a $N = 1/2$ theory in three dimensions as there are not enough degrees of freedom in three dimensions to do that. So, if we deform a theory with $N = 1$ supersymmetry in three dimensions, we can either retain all the supersymmetry or break all of it. However, we cannot partially break $N = 1$ supersymmetry in three dimensions. As supersymmetry is very important in the analysis of the ABJM theory we will deform the superspace algebra in such a way that we do not break any supersymmetry. To do so we promote $\theta^a$ and $\bar{\theta}^\alpha$ to operators $\theta^a$ and $\bar{\theta}^\alpha$ which satisfy the following superspace algebra:
\begin{align}
[\theta^a, \bar{\theta}^\alpha] &= B^{ab}, \quad [\bar{\theta}^\alpha, \theta^a] = A^{\alpha a}.
\end{align}

We then use Weyl ordering and express the Fourier transformation of this superfield as
\begin{align}
\Gamma_a(\bar{y}, \hat{\theta}) &= \int d^4 k \int d^2 \pi e^{-iky - \pi \hat{\theta}} \Gamma_a(k, \pi).
\end{align}

Thus, we obtain a one-to-one map between a function of $\hat{\theta}, \bar{\theta}$ to a function of ordinary superspace coordinates $\theta, y$ via
\begin{align}
\Gamma_a(y, \theta) &= \int d^4 k \int d^2 \pi e^{-iky - \pi \theta} \Gamma_a(k, \pi).
\end{align}
We can express the product of two fields $\hat{\Gamma}^a(\hat{y}, \hat{\theta})\hat{\Gamma}_a(\hat{y}, \hat{\theta})$ on this deformed superspace as

$$
\hat{\Gamma}^a(\hat{y}, \hat{\theta})\hat{\Gamma}_a(\hat{y}, \hat{\theta}) = \int \frac{d^4k_1d^4k_2}{2\pi^2} \int d^2\pi_1d^2\pi_2 \exp -i((k_1 + k_2) \hat{y} + (\pi_1 + \pi_2) \hat{\theta}) \times \exp(i\Delta)\hat{\Gamma}^a(k_1, \pi_1)\hat{\Gamma}_a(k_2, \pi_2),
$$

where

$$
\exp(i\Delta) = \exp -i \left( B^{\mu\nu}k_+^\mu k_+^\nu + A^{\alpha\beta}(\pi_+^2k_+^\alpha - k_+^2\pi_+^\alpha) \right),
$$

This motivates the definition of the star product between ordinary functions, which is now defined as

$$
\Gamma^a(y, \theta)\star \Gamma_a(y, \theta) = \exp i \left( B^{\mu\nu}\partial_\mu \partial_\nu + A^{\alpha\beta}(D_\alpha^2 - D_\alpha D_\beta) \right) \times \Gamma^a(y_1, \theta_1)\Gamma_a(y_2, \theta_2) |_{y_1 = y_2 = y, \theta_1 = \theta_2 = \theta}.
$$

Here we have defined the star product between ordinary functions using super-derivative $D_\alpha$ rather than $\partial_\alpha$ because they commute with the generators of the supersymmetry $Q_a$ [34],

$$
\{Q_a, D_\beta\} = 0.
$$

Thus, to write the ABJM in this deformed superspace, we now use

$$
\hat{\Omega}_a = \hat{\omega}_a - \frac{i}{6} [\hat{\Gamma}^b, \hat{\Gamma}_{ab}],
$$

$$
\hat{\omega}_a = \frac{i}{2} D^\alpha D_\alpha \hat{\Gamma}_a - \frac{i}{2} [\hat{\Gamma}^b, D_b \hat{\Gamma}_a] - \frac{1}{6} [\hat{\Gamma}^b, \{\hat{\Gamma}_b, \hat{\Gamma}_a\}],
$$

$$
\hat{\Gamma}_{ab} = -\frac{i}{2} [D_a \hat{\Gamma}_b - i\{\hat{\Gamma}_b, \hat{\Gamma}_a\}],
$$

$$
\hat{\omega}_a = \frac{i}{2} D^\alpha D_\alpha \hat{\Gamma}_a - \frac{i}{2} [\hat{\Gamma}^b, D_b \hat{\Gamma}_a] - \frac{1}{6} [\hat{\Gamma}^b, \{\hat{\Gamma}_b, \hat{\Gamma}_a\}],
$$

$$
\hat{\Gamma}_{ab} = -\frac{i}{2} [D_a \hat{\Gamma}_b - i\{\hat{\Gamma}_b, \hat{\Gamma}_a\}].
$$

In order to analyse the gauge transformations of this deformed ABJM theory it will be useful to define

$$
\begin{align*}
\hat{u} &= [\exp(i\hat{\Lambda}^A T_a)] = [\exp(i\Lambda^A T_a)]_*, \\
\hat{v} &= [\exp(i\hat{\Lambda}^A T_a)] = [\exp(i\Lambda^A T_a)]_*.
\end{align*}
$$

The star product reduces to the usual Moyal star product for the bosonic non-commutativity in the limit $C^{ab} = A^{ab} = 0$ and for $A^{\mu\nu} = C^{\mu\nu} = 0$ it reduces to the standard fermionic star product.

Now we construct the classical Lagrangian density with the gauge group $U(N) \times U(N)$, on this deformed superspace,

$$
\mathcal{L}_c = \mathcal{L}_M + \mathcal{L}_{CS} - \hat{\mathcal{L}}_{CS},
$$

where $\mathcal{L}_{CS}$ and $\hat{\mathcal{L}}_{CS}$ are deformed Chern-Simons theories with gauge groups $U(N)$ from $U(N) \times U(N)$, respectively. They can thus be expressed as

$$
\mathcal{L}_{CS} = \frac{k}{2\pi} \int d^2\theta Tr [\hat{\Gamma}^a \hat{\Omega}_a],
$$

$$
\hat{\mathcal{L}}_{CS} = \frac{k}{2\pi} \int d^2\theta Tr [\hat{\Gamma}^a \hat{\Omega}_a].
$$

The Lagrangian for the matter fields is given by

$$
\mathcal{L}_M = \frac{1}{4} \int d^2\theta Tr [\nabla^a \hat{X}^I \nabla_a \hat{X}_I + \hat{\mathcal{V}}],
$$

where

$$
\nabla_a \hat{X}^I = \hat{D}_a \hat{X}^I + i\hat{\Gamma}_a \hat{X}^I - i\hat{X}^I \hat{\Gamma}_a,
$$

$$
\nabla_a \hat{X}^I = \hat{D}_a \hat{X}^I - i\hat{X}^I \hat{\Gamma}_a + i\hat{\Gamma}_a \hat{X}^I,
$$

and $\hat{\mathcal{V}}$ is the potential term given by

$$
\hat{\mathcal{V}} = \frac{16\pi}{k} \epsilon^{IJ} \epsilon_{KL} [\hat{X}_I \hat{X}^K \hat{X}_J \hat{X}^L].
$$

Now we can also express the Lagrangian for the matter fields as

$$
\mathcal{L}_M = \frac{1}{4} \int d^2\theta Tr [\nabla^a \hat{X}^I \nabla_a \hat{X}_I + \mathcal{V}_*],
$$

where

$$
\nabla_a \hat{X}^I = \hat{D}_a \hat{X}^I + i\Gamma_a \hat{X}^I - i\hat{X}^I \hat{\Gamma}_a,
$$

$$
\nabla_a \hat{X}^I = \hat{D}_a \hat{X}^I - i\hat{X}^I \hat{\Gamma}_a + i\hat{\Gamma}_a \hat{X}^I,
$$

and $\mathcal{V}_*$ is the potential term given by

$$
\mathcal{V}_* = \frac{16\pi}{k} \epsilon^{IJ} \epsilon_{KL} [X_I X^K X_J X^L].
$$

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Now if the full finite gauge transformation under which this ABJM theory invariant are given by
\[ \Gamma_a \rightarrow \Gamma_a + \lambda, \quad  \tilde{\Gamma}_a \rightarrow \tilde{\Gamma}_a + \tilde{\lambda}, \]
\[ X^I \rightarrow u^I X^I + u^I \tilde{\Gamma} + \ldots, \quad X^{I \dagger} \rightarrow u^{I \dagger} X^{I \dagger} + u^{I \dagger} \tilde{\Gamma} + \ldots. \]  
(40)

Thus now the infinitesimal gauge transformations for these fields is given by
\[ \delta \Gamma_a = \nabla_a \lambda, \quad \delta \tilde{\Gamma}_a = \nabla_a \tilde{\lambda}, \]
\[ \delta X^I = i(A \times X^I - X^I \cdot \Lambda), \quad \delta X^{I \dagger} = i(\tilde{\Lambda} \times X^{I \dagger} - X^{I \dagger} \cdot \Lambda). \]  
(41)

The Lagrangian for the ABJM theory is invariant under these gauge transformations:
\[ \delta L_{ABJM} = \delta L_{kes}(\Gamma) - \delta \tilde{L}_{kes}(\tilde{\Gamma}) + \delta L_M = 0. \]  
(42)

Higgs mechanism. – Now we take the vacuum expectation value of one of the scalar superfields say \( X \), to be a non-zero value:
\[ \langle X \rangle = \nu \neq 0. \]  
(43)

This spontaneously breaks the symmetry from \( U(N) \times U(N) \) to its diagonal subgroup, \( U(N) \). Now let \( A_a \) be the superfield associated with the broken gauge and \( B_a \) be associated with the unbroken gauge group. Then we have
\[ A_a = \frac{1}{2} (\Gamma_a - \tilde{\Gamma}_a), \]
\[ B_a = \frac{1}{2} (\Gamma_a + \tilde{\Gamma}_a). \]  
(44)

Now we can write the Chern-Simons part of the Lagrangian as
\[ L_{CS} = \frac{k}{2\pi} \int \theta \text{Tr} \left[ A^a \star \left( W_a + \frac{1}{6} [A^b, A_b] \star \right) \right], \]  
(45)

where
\[ W_a = \frac{1}{2} D^b D_a B_b - \frac{i}{2} [B^b, D_a B_a], \quad \frac{1}{6} [B^b, [B_b, B_a] \star], \]
\[ A_{ab} = -\frac{i}{2} [D_{(a} A_{b)} - i(A_a, A_b) \star]. \]  
(46)

Now as the gauge group is broken down to its diagonal subgroup, we can integrate out the field \( A \) by using its equations of motion. Using this value of \( A_a \) thus obtained we get the a theory whose first term corresponds to the non-commutative super-Yang-Mills theory. Thus the kinetic part of this theory has the terms
\[ L_{YM} = \frac{k^2}{16\pi^2 \nu^2} \int \theta \text{Tr} [W^a \star W_a + [O] \star]. \]  
(47)

There are higher derivative terms in this action corresponding to different values whose origin is not the non-commutative nature of the theory we have analysed. Similarly in the kinetic part of the matter field also contains higher derivatives. However, the first term is a usual gauge theory term with the covariant derivatives corresponding to the unbroken gauge field \( B_a \).
\[ L_{KM} = \frac{k^2}{16\pi^2 \nu^2} \int \theta [\nabla^a X^I \times \nabla_a X^I + [O] \star], \]  
(48)

where
\[ \nabla_a X^I = D_a X^I - i B_a X^I, \]
\[ \nabla_a X^{I \dagger} = D_a X^{I \dagger} + i B_a X^{I \dagger}. \]  
(49)

The potential term can now be rewritten as \( V \), using the equation of motion of \( A_a \). We identify the Yang-Mills coupling with
\[ g_Y M = \frac{2 \pi \nu}{k}. \]  
(50)

It is possible to keep \( g_Y M \) fixed in the limit \( v \rightarrow \infty \). However, it might be possible to rewrite the full Lagrangian as
\[ L_T = \frac{1}{g_Y M} \int \theta [W^a \star W_a + \nabla^a X^I \times \nabla_a X^I + V], \]  
(51)

Now the full finite gauge transformations under which this theory is invariant are given by
\[ B_a \rightarrow i \nu \nabla_a \star v^{-1}, \quad X^I \rightarrow v X^I, \]
\[ X^{I \dagger} \rightarrow v^{I \dagger} X^{I \dagger} v^{-1}. \]  
(52)

where
\[ v = u + \tilde{u}. \]  
(53)

We can write now \( v \) as follows:
\[ v = [\exp(i \lambda T_A)]_v. \]  
(54)

It may also be noted that
\[ \delta W_a = v \star W_a \star v^{-1}. \]  
(55)

Now the infinitesimal gauge transformations for these fields can be written as
\[ \delta B_a = \nabla_a \lambda, \quad \delta X^I = i \lambda X^I, \]
\[ \delta X^{I \dagger} = -i X^{I \dagger} \star \lambda. \]  
(56)

The Lagrangian for the ABJM theory is invariant under these gauge transformations:
\[ \delta L_T = 0. \]  
(57)

Thus this theory is just a super-Yang-Mills theory deformed by the background fields. This deformed super-Yang-Mills theory arises as a result of coupling \( D \)-branes to background fields. This motivates the question whether it is possible to write a non-linear Born-Infeld-type extension to the ABJM theory. As the gauge part is purely topological we expect no change in that part. However, it might be possible to write the kinetic part of the matter fields as a Born-Infeld-type theory.
Conclusion. – In this paper we have analysed the ABJM theory in the $N=1$ superspace on a non-commutative spacetime. It was shown that this theory is invariant under non-commutative gauge transformations, which in the commutative limit reduce to regular gauge transformations for the ABJM theory. Furthermore, it was demonstrated that this theory reduces to the non-commutative super-Yang-Mills theory when one of the scalar fields is given a vacuum expectation value.

Recently the supersymmetric Chern-Simons theory has also been used to study the fractional quantum Hall effect via holography [35] and analyse the $AdS_4/CFT_3$ correspondence [36–40]. It will be interesting to analyse similar effects for this deformed ABJM theory. It will also be interesting to analyse the BRST and the anti-BRST symmetries of this model. These symmetries for the non-commutative deformation of the ABJM theory in various values have been already analysed [41]. It will be interesting to analyse the BRST and the anti-BRST symmetries of the ABJM theory with non-anticommutative deformation.

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