**Entanglement and extreme planar spin squeezing**

G. Vitagliano,1,2,* G. Colangelo,3,† F. Martin Curiana,3 M. W. Mitchell,3,4 R. J. Sewell,3 and G. Tóth2,5,6

1Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria
2Department of Theoretical Physics, University of the Basque Country UPV/EHU, P.O. Box 644, E-48080 Bilbao, Spain
3ICFO-Institut de Ciencies Fotòniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain
4ICREA-Institució Catalana de Recerca i Estudis Avançats, 08010 Barcelona, Spain
5IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Spain
6Wigner Research Centre for Physics, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest, Hungary

(Received 27 June 2017; published 15 February 2018)

We introduce an entanglement-depth criterion optimized for planar quantum-squeezed (PQS) states. It is connected with the sensitivity of such states for estimating a phase generated by rotations about an axis orthogonal to its polarization. We compare numerically our criterion with the well-known extreme spin-squeezing condition of Sørensen and Mølmer [Phys. Rev. Lett. 86, 4431 (2001)] and show that our condition detects a higher depth of entanglement when both planar spin variances are squeezed below the standard quantum limit. We employ our theory to monitor the entanglement dynamics in a PQS state produced via quantum nondemolition measurements using data from a recent experiment [Phys. Rev. Lett. 118, 233603 (2017)].

DOI: 10.1103/PhysRevA.97.020301

**Introduction.** Detecting entanglement in large quantum systems is a major goal in quantum information science and underpins the development of quantum technologies [1,2]. Attention has now shifted toward the practical use of entanglement as a resource: In particular, entanglement-enhanced sensing using ensembles of $10^3–10^{12}$ atomic spins has emerged as a major application [3,4]. In this context, spin-squeezing inequalities can be used to quantify entanglement-enhanced sensitivity. Standard treatment studies spin-squeezed states (SSSs), characterized by a large spin polarization in the $z$ direction and a small variance in the $z$ direction via the parameter $\xi_2 := \frac{N^2}{\langle J_z^2 \rangle^2}$, where $J_v = \sum_{n=1}^{N} j_{v}^{(n)}$, where $v = x, y, z$ are the collective spin components, $j_{v}^{(n)}$ are single-particle spin operators, and $N$ is the total number of atoms. Then, states with $\xi_2 < 1$ provide quantum-enhanced sensitivity for estimating phases $\phi \approx 0$ due to small rotations around $J_z$ [5,6]. Such states have been produced using various platforms, including cold atoms [7–18], trapped ions [19], magnetic systems [20], and photons [21].

Their metrological sensitivity is strongly connected to entanglement: $\xi_2 < 1$ also implies entanglement for atoms with spin $j = 1/2$ [22]. More quantitatively, the amount of spin squeezing is also connected with the so-called depth of entanglement, i.e., the number of particles in the largest separable subset [23]. Several other highly entangled states have recently been found useful for quantum metrology. For example, Dicke states, which are unpolarized and have a large value of $\langle J_x^2 + J_y^2 \rangle$ and a small variance in the $z$ direction. Spin-squeezing inequalities have been developed to characterize entanglement in such states [24,25], which have been produced in experiments with photons [26,27] and Bose-Einstein condensates [24,28–31].

Here, we focus on so-called planar quantum-squeezed (PQS) states, studied theoretically in Refs. [32–34], and produced in a recent experiment [35,36]. They have reduced spin variances in two directions, i.e., $\langle \Delta J_x \rangle^2 := (\Delta J_x)^2 + (\Delta J_z)^2$ is small, and a large in-plane polarization, i.e., $\langle J_x \rangle \approx N j$. They provide quantum-enhanced sensitivity in estimating phases generated by rotations about the $\hat{x}$ axis (see Fig. 1 for an illustration) and are useful for tracking a changing phase shift or simultaneous estimation of phase and amplitude beyond classical limits [35,36]. The planar squeezing parameter, $\xi_2 := \frac{\langle \Delta J_x \rangle^2}{\langle J_x \rangle^2}$, where $\langle J_x \rangle := \sqrt{\langle J_x \rangle^2 + \langle J_z \rangle^2}$ is the in-plane polarization, was introduced by He and co-workers [32,33] to quantify such enhanced sensitivity and detect entanglement. The relation between their metrological usefulness and their degree of multiparticle entanglement is explored here.

In this Rapid Communication, we introduce a method to detect the depth of entanglement based on the planar-squeezing parameter $\xi_{2} J$. We present the condition, $\xi_{2} J \geq \xi_{2} J$, where $\xi_{2} J$ is the minimum value of the planar-squeezing parameter over single-particle states of spin $J$. We prove that, for all spin-$j$ ensembles that contain groups of at most $k$-entangled particles, called $k$ producible [23,37], Eq. (2) holds with $J = kj$. Thus, $\xi_{2} J < \xi_{2} J$ implies a depth of entanglement of at least $k + 1 = J/j + 1$. We can even estimate at least how many particles must be in fully entangled $(k + 1)$-particle groups. We stress that our criterion is very simple to use. We need to calculate $\xi_{2} J$ only once for the relevant range of $J$, then...
We consider a protocol in which a collective spin state arising from Poissonian fluctuations in the preparation; (blue vertical disk) a SSS produced by squeezing the $J_z$ variance; (red vertical spheroid) a PQS state produced by squeezing both $J_x$ and $J_z$; (green horizontal disk) a Dicke state. (b) Sensitivity advantage of a PQS state compared to a SSS in detecting an unknown phase $\phi$. The dashed black circles indicate the shot-noise limit $\Delta \phi = 1/\sqrt{N}$. The SSS provides enhanced sensitivity for detecting phases around $\phi \approx 0$ but reduced sensitivity for phases around $\phi \approx \pm \pi/2$. In contrast, although the sensitivity of the PQS state is slightly worse around $\phi \approx 0$, it provides enhanced sensitivity for all phases $0 \leq \phi \leq 2\pi$.

Eq. (2) can be applied for entanglement detection without any additional numerical optimization.

Finally, we examine the usefulness of our criterion. We compare it to the well-known criterion introduced by Sørensen and Mølmer in Ref. [23] and find that ours detects a higher entanglement depth even for nonideal PQS. We also test our theory using data from a recent experiment in which a PQS state was generated via a semicontinuous quantum nondemolition (QND) measurement [35,36].

**Link between our parameter and metrological sensitivity.** We consider a protocol in which a collective spin state is rotated about $J_z$ and accumulates a phase $\phi$ such that $J_z^{\text{out}} = J_z^0 \cos \phi - J_z^\alpha \sin \phi$. Afterwards, the phase is inferred from repeated measurements of $J_z^{\text{out}}$ with a sensitivity given by the error-propagation formula $\langle \Delta \phi \rangle^2 = \langle \Delta J_z^{\text{out}} \rangle^2 / |\delta \phi (J_z^{\text{out}})|^2$. We consider as reference an input state with uncertainties at $\phi = 0$.

By averaging over $J_z^{\text{out}}$ we find

$$\int_0^{2\pi} \frac{d\phi}{2\pi} (\Delta \phi)^2 / (\Delta \phi)_{\text{SQL}} = \frac{1}{2} s_{\text{SQL}}^2.$$  \hspace{1cm} (3)

Thus, the parameter appearing on the left-hand side of Eq. (2) also quantifies the average sensitivity enhancement over the interval $0 \leq \phi \leq 2\pi$ compared to the SQL.

**Entanglement criterion for planar squeezing.** Following an approach similar to past works [23,25], we derive a tight criterion to detect the depth of entanglement by computing the function,

$$G_k^{(j)}(X) := \frac{1}{k} \min_{\rho \in C_{\text{SSS}}}(\Delta L_{\rho}^x)^2 + (\Delta L_{\rho}^z)^2, \hspace{1cm} (4)$$

where $d = 2j + 1$, $j$ is the single-particle spin quantum number, and $L_{\rho}$'s are collective $k$-particle spin operators, i.e., $L_{\rho} = \sum_{\nu=1}^k j^{(\nu)}_{\nu}$, where $j^{(\nu)}_{\nu}$'s are single-particle spin-$j$ components. First, we find a tight lower bound on the planar spin variance valid for all states with a depth of entanglement smaller than $k$.

**Observation 1.** Every $k$-producible state of a spin-$j$ particle system with an average number of particles $\langle N \rangle$ must satisfy the tight inequality,

$$\langle (\Delta J_z)^2 \rangle \geq \langle N \rangle G_k^{(j)} \left( \frac{(|\langle J_z \rangle|)}{\sqrt{N}} \right). \hspace{1cm} (5)$$

where $G_k^{(j)}$ is defined as the convex hull of (4). Thus, every state that violates Eq. (5) must have a depth of entanglement of at least $k + 1$.

**Proof.** For pure $k$-producible states of (constant) $N$ particles we have $\langle (\Delta J_z^i)^2 \rangle \geq \sum_{n_1} n_1 j^{(i)}_{n_1} (\phi^{(n_1)/k_1})$, where $L^{(n_1)}_{\nu}$'s are collective operators of $0 \leq k_1 \leq k$ particles. The second inequality follows directly from the definition of $G_k^{(j)}$. Now, we use that $G_k^{(j)}$'s are as follows: (i) convex and (ii) decreasing for increasing the index, i.e., $G_k^{(j)} \leq G_r^{(j)}$ for $r \geq s$ and $k \geq k_1$ and $\sum_n n_1 = N$. Then, $\sum_n n_1 j^{(i)}_{n_1} (\phi^{(n_1)/k_1}) \geq \sum_n n_1 j^{(i)}_{n_1} (\phi^{(n_1)/k_1}) \geq N_j G_k^{(j)}$. The first inequality comes from property (ii) and the second comes from property (i) and Jensen’s inequality. Clearly, if $N$ is divisible by $k$, then the inequality (5) is tight by construction. Let us consider now a state with a nonzero particle number variance $\varrho = \sum N QN_N$, where $Q$ are states with fixed $N$'s and $Q_N$’s are probabilities. From the properties above it follows that $\langle (\Delta J_z)^2 \rangle \geq \sum N Q_N \langle (\Delta J_z)^2 \rangle \geq \sum N Q_N j^{(i)}_{Nj} (\phi^{(Nj)/k_j}) \geq (N) G_k^{(j)}$ holds, $\langle N \rangle = \sum N Q_N N$ being the average particle number.

**Numerical computation of $G_k^{(j)}$.** In order to detect the depth of entanglement with our criterion we need to carry out the optimization in Eq. (4) and then construct the convex hull $G_k^{(j)}$ that has properties (i) and (ii) mentioned in the proof of Observation 1. For $k = 1$ and $j = 1$, straightforward algebra yields $G_1^{(1)}(X) = \frac{1}{2} - X^2 - \frac{1}{2}\sqrt{1 - X^2}$. Analytical expressions are very hard to obtain even for the next simplest cases.

Numerically, the problem of finding the convex hull $G_k^{(j)}$ can be approached exploiting the Legendre transform in this framework defined as [38,39]

$$\mathcal{L}(\Delta L_{\phi}^z/k_j)(T) := \inf_{\phi} \left[ \frac{1}{k_j} (\Delta L_{\rho}^z)^2 - \langle T \rangle_{\phi} \right], \hspace{1cm} (6)$$

for the normalized planar variance $(\Delta L_{\rho}^z)^2/k_j$ as a function of $T = L_{\rho}/k_j$. Then, the lower bound $(\Delta L_{\rho}^z)^2 \geq G_k^{(j)}$ is obtained by means of another Legendre transform,

$$G_k^{(j)}(X) := \sup_{\lambda} \{ \lambda X - \mathcal{L}(\lambda (\Delta L_{\phi}^z/k_j))(\lambda L_{\rho}/k_j) \} \hspace{1cm} (7)$$

where $X$ is a real number. The function (7) is precisely the convex hull that we are looking for. Furthermore, (6) can be written as an eigenvalue problem (see also Refs. [40–42].
The straight line provides a lower bound on $G$ shown. (The dashed curve) The function $G$ computed on the symmetric subspace. The convex hull of $G^{(j)}(X)$, denoted by $G_k^{(j)}(X)$, is a linear function for $X \leq X_{\text{max}} = \text{argmin}|G_k^{(j)}(X)/X|$, whereas for $X > X_{\text{max}}$ it coincides with $G_k^{(j)}(X)$. $M$ denotes the point of the curve for which $X = X_{\text{min}}$. The straight line provides a lower bound on $G^{(j)}(X)$. (The inset) The parameter $\xi_j^2$ as a function of $J$.

addressing similar problems),

$$L[(\Delta L)^2]/kj = \frac{1}{k} \min_{\psi, \omega} \text{min}(H_{J, x, \lambda})$$

where the Hamiltonian $H_{J, x, \lambda} = (L_x - s_x)^2 + (L_z - s_z)^2 - \lambda L_y$ is a collective operator acting on a $k$-partite space of spin-$j$ particles. Moreover, by writing a general pure state (here we consider integer values of $k$) as $|\phi\rangle = \sum_j a_j |\psi_j\rangle$, the expectation value in Eq. (8) can be written as

$$\langle H_{J, x, \lambda} \rangle = \sum_{J=0}^{kJ} a_j^2 \langle (L_y^{(j)} - s_y)^2 + (L_z - s_z)^2 - \lambda L_y \rangle_{\psi_j}$$

where $L_m^{(j)}$'s are single spin-$J$ operators. In particular, for $k > 1$ and the $kj$ integer we can easily prove that $G_k^{(j)}(0) = 0$ where the value on the right-hand side is reached for $|\phi\rangle$ being the singlet. More in general, substituting Eqs. (8) and (9) into Eq. (7) one can see that the function $G_k^{(j)}(X)$ can be obtained with minimizations in spin-$J$ subspaces with $0 \leq J \leq kj$ (1/2 $\leq J \leq kj$ for the $kj$ half-integer). Thus, by increasing $k$ one has to minimize over a larger number of subspaces and consider a higher number of parameters $a_k$, which makes the resulting function decreasing with $k$, which is just property (ii) needed in the proof of Observation 1. When the minimization problem (4) is restricted to the symmetric subspace $J = kj$ we then call the resulting function $G_{kj}^{(j)}(X)$. Its convex roof can be obtained based on Eq. (9) if we set $a_k = 1$. In Fig. 2, we present a concrete example. The function $G_k^{(j)}(X)$ is plotted together with its convex hulls $G_k^{(j)}(X)$ and $G_k^{(j)}(X)$. We see in the figure that a simple linear function can be used as a lower bound to $G_k^{(j)}(X)$. This lower bound works in general, as we show in what follows.

FIG. 2. Lower bounds to $(\Delta L)^2/kj$ as functions of $(\Delta L)/kj$ for a system of $k$ spin-$j$ particles. The cases of $k = 4$ and $j = 1$ are shown. (The dashed curve) The function $G_k^{(j)}(X)$. (The solid line) The function $G_k^{(j)}(X)$ computed on the symmetric subspace. The convex hull of $G_k^{(j)}(X)$, denoted by $G_k^{(j)}(X)$ is a linear function for $X \leq X_{\text{max}} = \text{argmin}|G_k^{(j)}(X)/X|$, whereas for $X > X_{\text{max}}$ it coincides with $G_k^{(j)}(X)$. $M$ denotes the point of the curve for which $X = X_{\text{min}}$. The straight line provides a lower bound on $G_k^{(j)}(X)$. (The inset) The parameter $\xi_j^2$ as a function of $J$.

Table I. Values of $\xi_j^2$ for $0 \leq J \leq 27$.

| $J$ | $\xi_j^2$ | $J$ | $\xi_j^2$ | $J$ | $\xi_j^2$ |
|-----|----------|-----|----------|-----|----------|
| 1   | 0.45     | 10  | 0.2607   | 19  | 0.21111  |
| 2   | 0.44906  | 11  | 0.25262  | 20  | 0.20758  |
| 3   | 0.38945  | 12  | 0.2455   | 21  | 0.20428  |
| 4   | 0.35321  | 13  | 0.23913  | 22  | 0.2012   |
| 5   | 0.32779  | 14  | 0.23338  | 23  | 0.19826  |
| 6   | 0.30852  | 15  | 0.22815  | 24  | 0.19551  |
| 7   | 0.29318  | 16  | 0.22336  | 25  | 0.1929   |
| 8   | 0.28054  | 17  | 0.21896  | 26  | 0.19043  |
| 9   | 0.26986  | 18  | 0.21489  | 27  | 0.18809  |

Linear lower bound. As outlined above, the computation of $G_k^{(j)}(X)$ still requires some numerics, which can be hard for high $k$ and $J$. Here, we simplify further this task by finding a suitable lower bound that requires only the numerical computation of $G_k^{(j)}(X)$ with $J = kj$ and is thus easier than computing the full $G_k^{(j)}(X)$.

Observation 2. A convex lower bound to the curve $G_k^{(j)}(X)$ defined as in Eq. (4) is given by

$$G_k^{(j)}(X) \geq X \xi_j^2,$$

where $\xi_j^2 := \min_{\phi_{kj}} (\Delta L)^2/\langle \psi_j \rangle$. Thus, as a simple algorithm one can: (i) Find the ground states $|\phi_{kj}\rangle$ of $H_{kj}$ restricted to the symmetric subspace; (ii) compute $(\Delta L)^2/\langle \psi_j \rangle$; and finally take $\xi_j^2 = \min_{\phi_{kj}} (\Delta L)^2/\langle \psi_j \rangle$, which is feasible until very large $J$, up to the thousands. As an example the values of $\xi_j^2$ up to $J = 27$ are given in Table I, whereas the qualitative behavior can be observed in the inset of Fig. 2. Note that Eq. (10) is a tight approximation only for $k = 2$, independent of $j$. For $k = 1$ the original criterion given in Eq. (5) has to be used instead.

From Observations 1 and 2, we immediately obtain Eq. (2), which connects the metrological performance of PQS states to their entanglement depth. Next, we show that, apart from proving that the entanglement depth is $k + 1$, we also obtain information about how many particles are in fully entangled groups of $(k + 1)$. This provides a simple interpretation of the degree of the violation of Eq. (2).

Observation 3. Let us assume that the total polarization is equally distributed over all particles. Then, there is at least a fraction $f_{k+1} = (1 - \xi_j^2/\xi_k^2)$ of particles in fully entangled groups of $(k + 1)$ or more with $k$ given by $J/j$. The proof is given in the Appendix where the case of varying particle numbers is included in the model. We discuss that, without the assumption of equally split polarization, the above statement still holds for almost totally polarized states, i.e., with $(J_j) \approx N_j$ and that similar ideas work also for the Sørensen-Mølmer criterion.

Practical use of the criterion. Thus, our criterion can be employed to detect the depth of entanglement whenever two

020301-3
collective spin variances are known as well as the total in-plane polarization. With the same input information, it would be possible to use also the Sørensen-Mølmer extreme spin-squeezing condition [43]. Then, we can numerically compare the two criteria and study in which cases our criterion is more suitable to detect entanglement. To do this we parametrize the states with the ratio \( \alpha = (\Delta J_z)^2/(\Delta J_y)^2 \) between the two spin variances and the total in-plane polarization \( \beta = \langle J_y \rangle / N \).

We plot the lower bound on \( (\Delta J_z)^2 \) for \( k = 5 \) and \( j = 1 \) as a function of the ratio between the two planar spin variances and the in-plane polarization. Our criterion detects a depth of entanglement higher than that of Sørensen-Mølmer for the parameter values for which the red plot is above the blue one.

Next, we employ our criterion (2) to analyze entanglement in a PQS state produced in a recent experiment with an ensemble of \( N = 1.75 \times 10^6 \) cold \(^{87}\)Rb atoms via semicontinuous QND measurements [36]. In Fig. 4 we plot the observed planar-squeezing parameter \( \xi_1^2 \) as a function of the measurement strength, parametrized by the number of photons \( N_L \) used in the QND measurement. As \( N_L \) increases, the input spin-coherent state evolves into a planar-squeezed state with squeezing observed between \( N_L \simeq 2 \times 10^8 \) and \( N_L \simeq 3 \times 10^8 \) photons after which the spin variances increase due to noise and decoherence introduced by off-resonant scattering of probe photons.

For comparison, using the criterion developed by He and co-workers [32,33], one would detect a fraction 0.39 of atoms in fully entangled groups of \((k+1)\) or more, detected using our criterion. We observe the corresponding increase in entanglement depth with \( N_L \) up to the optimum of \( N_L = 2.47 \times 10^8 \) photons after which entanglement is gradually lost. At the optimum \( N_L \) we observe a spin-coherence \( \langle J_y \rangle \) of 0.83\(N\) and a planar-squeezing parameter \( \xi_1^2 = 0.32 \pm 0.02 \). For comparison, using the criterion developed by He and co-workers [32,33], one would detect a fraction 0.39 of atoms in entangled states without any information about the depth of entanglement. The details of the experiment are given in the Supplemental Material [44] (see also Ref. [45]).

**Conclusions.** We have introduced a criterion suitable to detect the depth of entanglement in planar-squeezed states and to distinguish them from traditional spin-squeezed states, detectable with the criterion of Sørensen-Mølmer [23]. Our criterion is simple to evaluate and directly connected with the sensitivity of the PQS states for phase estimations that do not require any prior knowledge of the phase. By numerical comparison, we have also shown that our criterion represents an important alternative to that of Sørensen-Mølmer suitable to detect entanglement in PQS states. Finally, we tested our criterion with data from a recent experiment in which a PQS state was generated via semicontinuous QND measurement [36].
At this point, we assume that $\langle L_y \rangle$ is distributed among the $N$ groups and the rest of the particles in proportion to the number of particles in these two groups, i.e., $\sum_{n=1}^N (L_y)_{\phi_n}/M_N j = (J_y)_{Nj}/N$. Hence, we arrive at $\langle \Delta J_y \rangle_N^2 \geq M_N \langle \mathcal{G}_{k}^{(j)}(\langle J_y \rangle_{Nj}/N) \rangle$. Due to the concavity of the variance and the convexity of $\mathcal{G}_{k}^{(j)}(X)$ this inequality also holds for mixtures of states of the type $|\Phi_N\rangle$ with a fixed particle number $N$, denoted by $\phi_N$. Hence, we obtain $\langle \Delta J_y \rangle_{\phi_N} \geq \langle \Delta J_y \rangle_N^2 \geq \langle \Delta \mathcal{G}_{k}^{(j)}(\langle J_y \rangle_{Nj}/N) \rangle$. Now we consider states $\phi = \sum_N r_N \phi_N$, where $r_N$'s are probabilities associated with different numbers of particles $N$ and groupings and define $Q = (M_N)/\langle (N) \rangle_{\phi}$, where $\langle (N) \rangle = \sum_N r_N N$ is the average particle number and $\langle \langle J_y \rangle_{Nj}/N \rangle_{\phi} = \sum_N r_N \langle (J_y)_{Nj}/N \rangle_{\phi} = \sum_N r_N (M_N)_{\phi} \mathcal{G}_{k}^{(j)}(\langle J_y \rangle_{Nj}/N)$. and by using the Jensen inequality we arrive at $\langle \Delta J_y \rangle_{\phi} \geq \langle \langle \Delta J_y \rangle_{Nj}/N \rangle_{\phi} \geq \langle \langle \Delta \mathcal{G}_{k}^{(j)}(\langle J_y \rangle_{Nj}/N) \rangle_{\phi} \rangle_{j}$. Using Eqs. (1) and (10), Observation 3 follows.

An argument similar to the above can be applied also to the criterion of Sørensen-Mølmer, which states that

$$\langle \Delta J_y \rangle_{\phi} \geq Nf_j \left( \frac{\langle J_y \rangle}{N} \right)$$

(A1)

holds in a system of spin-$j$ particles for states with an entanglement depth of at most $J_j$. Here, $f_j(X)$ is a convex function analogous to $\mathcal{G}_{j}^{(y)}(X)$ [23].

So far, in the derivations we made the assumption that the total polarization splits equally for the different subensembles of atoms. Without such an assumption, first for pure states, we analyze the worst-case scenario in which for a state, such as $|\Phi_N\rangle$, the polarization splits unequally and state $|\Psi_{\text{rest}}\rangle$ is polarized as much as possible. Hence, we assume $\langle J_y \rangle_{\text{rest}} = (N - M_N)j$, and it follows that $\sum_{n=1}^N (L_y)_{\phi_n} = (J_y)_{N} - (N - M_N)j$ and consequently $\langle \Delta J_y \rangle_{\phi} \geq (N - M_N)j$. Using Eq. (10), we obtain $\langle \Delta J_y \rangle_{\phi} \geq Nf_j \left( \frac{\langle J_y \rangle}{N} \right)$, which is clearly valid for mixed states with a varying particle number as $\langle \Delta J_y \rangle^2 \geq \mathcal{G}_{k}^{(j)}(\langle J_y \rangle_{Nj}/N - (N - M_N)j)).$ This can be further refined with states having a varying particle number $(\Delta J_y)^2 \geq \langle \mathcal{G}_{k}^{(j)}(\langle J_y \rangle_{Nj}/N - (N - M_N)j)\rangle_{\phi}$. For a state that is almost fully polarized, i.e., $\langle J_y \rangle_{Nj}/N \approx (N - M_N)j$, we recover the statement of Observation 3.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. 81, 865 (2009).

[2] O. Gühne and G. Tóth, Entanglement detection, Phys. Rep. 474, 1 (2009).

[3] G. Tóth and I. Apellaniz, Quantum metrology from a quantum information science perspective, J. Phys. A: Math. Theor. 47, 424006 (2014).

[4] L. Pezzé, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Non-classical states of atomic ensembles: Fundamentals and applications in quantum metrology, arXiv:1609.01609.

[5] M. Kitagawa and M. Ueda, Squeezed spin states, Phys. Rev. A 47, 5138 (1993).

[6] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, Phys. Rev. A 50, 67 (1994).

[7] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Spin Squeezed Atoms: A Macroscopic Entangled Ensemble Created by Light, Phys. Rev. Lett. 83, 1319 (1999).

[8] T. Fernholz, H. Krauter, K. Jensen, J. F. Sherson, A. S. Sørensen, and E. S. Polzik, Spin Squeezing of Atomic Ensembles via Nuclear-Electronic Spin Entanglement, Phys. Rev. Lett. 101, 073601 (2008).

[9] C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda, and M. A. Kasevich, Squeezed states in a bose-einstein condensate, Science 291, 2386 (2001).
A. Sørensen, L.-M. Duan, J. Cirac, and P. Zoller, Many-particle... Classical limit, Science 334, 773 (2011).

[29] C. Hamley, C. Gervung, T. Hoang, E. Bookjans, and M. Chapman, Spin-nematic squeezed vacuum in a quantum gas, Nat. Phys. 8, 305 (2012).

[30] T. M. Hoang, M. Anquez, M. J. Boguslawski, H. M. Bharath, B. A. Robbins, and M. S. Chapman, Adiabatic quenches and characterization of amplitude excitations in a continuous quantum phase transition, Proc. Natl. Acad. Sci. USA 113, 9475 (2016).

[31] X.-Y. Luo, Y.-Q. Zou, L.-N. Wu, Q. Liu, M.-F. Han, M. K. Tey, and L. You, Deterministic entanglement generation from driving through quantum phase transitions, Science 355, 620 (2017).

[32] Q. Y. He, S.-G. Peng, P. D. Drummond, and M. D. Reid, Planar quantum squeezing and atom interferometry, Phys. Rev. A 84, 022107 (2011).

[33] Q. Y. He, T. G. Vaughan, P. D. Drummond, and M. D. Reid, Entanglement, number fluctuations and optimized interferometric phase measurement, New J. Phys. 14, 093012 (2012).

[34] G. Puentes, G. Colangelo, R. J. Sewell, and M. W. Mitchell, Planar squeezing by quantum non-demolition measurement in cold atomic ensembles, New J. Phys. 15, 103031 (2013).

[35] G. Colangelo, F. M. Ciurana, L. C. Bianchet, R. J. Sewell, and M. W. Mitchell, Simultaneous tracking of spin angle and amplitude beyond classical limits, Nature (London) 543, 525 (2017).

[36] G. Colangelo, F. Martin Ciurana, G. Puentes, M. W. Mitchell, and R. J. Sewell, Entanglement-Enhanced Phase Estimation Without Prior Phase Information, Phys. Rev. Lett. 118, 233603 (2017).

[37] O. Gühne, G. Tóth, and H. J. Briegel, Multipartite entanglement in spin chains, New J. Phys. 7, 229 (2005).

[38] O. Gühne, M. Reimpell, and R. F. Werner, Estimating Entanglement Measures in Experiments, Phys. Rev. Lett. 98, 110502 (2007).

[39] J. Eisert, F. G. S. L. Brandão, and K. M. R. Audenaert, Quantitative entanglement witnesses, New J. Phys. 9, 46 (2007).

[40] L. Dammeier, R. Schwonnek, and R. F. Werner, Uncertainty relations for angular momentum, New J. Phys. 17, 093046 (2015).

[41] I. Apellaniz, M. Kleinmann, O. Gühne, and G. Tóth, Optimal witnessing of the quantum fisher information with few measurements, Phys. Rev. A 95, 032330 (2017).

[42] O. Marty, M. Cramer, G. Vitagliano, G. Tóth, and M. B. Plenio, Multipartite entanglement criteria for nonsymmetric collective variables, arXiv:1708.06986.

[43] Note that the Sörensen-Mølmer condition requires as input just one variance, that must be the one orthogonal to the polarization.

[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.97.020301 for details of the experiment.

[45] G. Colangelo, R. J. Sewell, N. Behbood, F. M. Ciurana, G. Triginer, and M. W. Mitchell, Quantum atom–light interfaces in the gaussian description for spin-1 systems, New J. Phys. 15, 103007 (2013).