On Some Properties of $K$- type Block Matrices in the context of Complementarity Problem

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Abstract

In this article we introduce $K$-type block matrices which include two new classes of block matrices namely block triangular $K$-matrices and hidden block triangular $K$-matrices. We show that the solution of linear complementarity problem with $K$-type block matrices can be obtained by solving a linear programming problem. We show that block triangular $K$-matrices satisfy least element property. We prove that hidden block triangular $K$-matrices are $Q_0$ and processable by Lemke's algorithm. The purpose of this article is to study properties of $K$-type block matrices in the context of the solution of linear complementarity problem.

Keywords: Z-matrix, Hidden Z-matrix, linear programming problem, linear complementary problem, semi-sublattice, P-matrix, $Q_0$-matrix.

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1 Introduction

The linear complementarity problem is a combination of linear and nonlinear system of inequalities and equations. The problem may be stated as follows: Given $M \in R^{n \times n}$ and a vector $q \in R^n$, the linear complementarity problem, LCP($M, q$) is the problem of finding a solution $w \in R^n$ and $z \in R^n$ to the following system of linear equations and inequalities:

\begin{align*}
    w - Mz &= q, \quad w \geq 0, \quad z \geq 0 \quad (1.1) \\
    w^Tz &= 0. \quad (1.2)
\end{align*}

In complementarity theory several matrix classes are considered due to the study of theoretical properties, applications and its solution methods. For details see [17], [20], [15], [19], [3]. The complementarity problem plays an important role in the formulation of structured stochastic game problems. For details see [12], [18], [22], [13], [14]. The complementarity problem establishes an important connections with
multiobjective programming problem for KKT point and the solution point \[11\]. The complementarity problems are considered with respect to principal pivot transforms and pivotal method to its solution point of view. For details see \[21\], \[2\], \[16\]. It is well known that the linear complementarity problem can be solved by a linear program if \(M\) or its inverse is a \(Z\)-matrix, i.e. a real square matrix with non-positive off diagonal elements. A number of authors have considered the special case of the linear complementarity problem under the restriction that \(M\) is a \(Z\)-matrix. Chandrasekharan \[24\] considered \(Z\)-matrix solving a sequence of linear inequalities. Lemke’s algorithm is a well known technique for solving linear complementarity problem \[1\]. Mangasarian \[9\] showed that the following linear program

\[
\begin{align*}
\text{minimize} & \quad p^T u \\
\text{subject to} & \quad q + Mu \geq 0, \\
& \quad u \geq 0
\end{align*}
\]

for an easily determined \(p \in \mathbb{R}^n\) solves the linear complementarity problem for a number of special cases specially when \(M\) is a \(Z\)-matrix. Mangasarian \[9\] proved that least element of the polyhedral set \(\{u : q + Mu \geq 0, u \geq 0\}\) in the sense of Cottle-Veinott can be obtained by a single linear program. It is well known that the quadratic programming problem

\[
\begin{align*}
\text{minimize} & \quad q^T u + \frac{1}{2} u^T Mu \\
\text{subject to} & \quad u \geq 0
\end{align*}
\]

can be formulated as a linear complementarity problem when \(M\) is symmetric positive semidefinite. Mangasarian showed that this problem can be solved using single linear program if \(M\) is a \(Z\)-matrix. Hidden \(Z\)-matrices are the extension of \(Z\)-matrices. A matrix \(M\) is said to be a hidden \(Z\)-matrix if \(\exists\) two \(Z\)-matrices \(X\) and \(Y\) such that

(i) \(MX = Y\)

(ii) \(r^TX + s^TY > 0\), for some \(r, s \geq 0\).

For details, see \[6\], \[7\]. In this paper we introduce block triangular \(K\)-matrix and hidden block triangular \(K\)-matrix. We call these two classes collectively as \(K\)-type block matrix. We discuss the class of \(K\)-type block matrices in solution aspects for linear complementarity problem.

The paper is organized as follows. Section 2 presents some basic notations, definitions and results. In section 3, we establish some results of these two matrix classes. We show that a linear complementarity problem with block triangular \(K\)-matrix and hidden block triangular \(K\)-matrix can be solved using linear programming problem.

## 2 Preliminaries

We denote the \(n\) dimensional real space by \(\mathbb{R}^n\). \(\mathbb{R}_+^n\) denotes the nonnegative orthant of \(\mathbb{R}^n\). We consider vectors and matrices with real entries. Any vector \(x \in \mathbb{R}^n\) is a
column vector and $x^T$ denotes the row transpose of $x$. $e$ denotes the vector of all 1. A matrix is said to be nonnegative or $M \geq 0$ if $m_{ij} \geq 0 \ \forall \ i,j$. A matrix is said to be positive if $m_{ij} > 0 \ \forall \ i,j$. Let $M$ and $N$ be two matrices with $M \geq N$, then $M - N \geq 0$. If $M$ is a matrix of order $n$, $\alpha \subseteq \{1,2,\cdots,n\}$ and $\bar{\alpha} \subseteq \{1,2,\cdots,n\} \setminus \alpha$ then $M_{\alpha\bar{\alpha}}$ denotes the submatrix of $M$ consisting of only the rows and columns of $M$ whose indices are in $\alpha$ and $\bar{\alpha}$ respectively. $M_{\alpha\alpha}$ is called a principal submatrix of $M$ and $\det(M_{\alpha\alpha})$ is called a principal minor of $M$. Given a matrix $M \in R^{n \times n}$ and a vector $q \in R^n$, we define the feasible set $\text{FEA}(M,q) = \{z \in R^n : z \geq 0, q + Mz \geq 0\}$ and the solution set of $\text{LCP}(M,q)$ by $\text{SOL}(M,q) = \{z \in \text{FEA}(M,q) : z^T(q + Mz) = 0\}$.

We state the results of two person matrix games in linear system with complementary conditions due to von Neumann [25] and Kaplansky [8]. The results say that there exist $\bar{x} \in R^m, \bar{y} \in R^m$ and $v \in R$ such that

$$
\sum_{i=1}^m \bar{x}_i a_{ij} \leq v, \ \forall \ j = 1, 2, \cdots, n,
$$

$$
\sum_{j=1}^n \bar{y}_j a_{ij} \geq v, \ \forall \ i = 1, 2, \cdots, m.
$$

The strategies $(\bar{x}, \bar{y})$ are said to be optimal strategies for player I and player II and $v$ is said to be minimax value of game. We write $v(A)$ to denote the value of the game corresponding to the payoff matrix $A$. The value of the game, $v(A)$ is positive(nonnegative) if there exists a $0 \neq x \geq 0$ such that $Ax > 0 (Ax \geq 0)$. Similarly, $v(A)$ is negative(nonpositive) if there exists a $0 \neq y \geq 0$ such that $y^T A < 0 (y^T A \leq 0)$.

A matrix $M \in R^{n \times n}$ is said to be

- PSD-matrix if $x^T M x \geq 0 \ \forall \ 0 \neq x \in R^n$.
- $P(P_0)$-matrix if all its principal minors are positive (nonnegative).
- $S$-matrix [23] if there exists a vector $x > 0$ such that $Mx > 0$ and $\bar{S}$-matrix if all its principal submatrices are $S$-matrix.
- $Z$-matrix if off-diagonal elements are all non-positive and $K(K_0)$-matrix if it is a $Z$-matrix as well as $P(P_0)$-matrix.
- $Q$-matrix if for every $q$, $\text{LCP}(M,q)$ has at least one solution.
- $Q_0$-matrix if for $\text{FEA}(q,A) \neq \emptyset \implies \text{SOL}(q,A) \neq \emptyset$.

Now we give some definitions, lemmas, theorems which will be required for discussion in the next section.

**Lemma 2.1.** [1] If $A$ is a $P$-matrix, then $A^T$ is also $P$-matrix.

**Lemma 2.2.** Let $A$ be a $P$-matrix. Then $v(A) > 0$.

**Definition 2.1.** [1] A subset $S$ of $R^n$ is called a meet semi-sublattice(under the componentwise ordering of $R^n$) if for any two vectors $x$ and $y$ in $S$, their meet, the vector $z = \min(x,y)$ belongs to $S$.

**Definition 2.2.** [5] The spectral radius $\sigma(M)$ of $M$ is defined as the maximum of the moduli $|\lambda|$ of all proper values $\lambda$ of $M$.

**Lemma 2.3.** [5] Let $M$ be a nonnegative matrix. Then there exists a proper value $p(M)$ of $M$, the Perron root of $M$, such that $p(M) \geq 0 \ \text{and} \ |\lambda| \leq p(M)$ for every proper value $\lambda$ of $M$. If $0 \leq M \leq N$ then $p(M) \leq p(N)$. Moreover, if $M$ is
irreducible, the Perron-Frobenius root \( p(M) \) is positive, simple and the corresponding proper value may be chosen positive. According to the Perron-Frobenius theorem, we have \( \sigma(M) = p(M) \) for nonnegative matrices.

**Definition 2.3.** A matrix \( W \) is said to have dominant principal diagonal if \( |w_{ii}| > \sum_{k \neq i} |w_{ik}| \) for each \( i \).

**Lemma 2.4.** [5] If \( W \) is a matrix with dominant principal diagonal, then \( \sigma(I - H^{-1}W) < 1 \), where \( H \) is the diagonal of \( W \).

**Theorem 2.1.** [5] The following four properties of a matrix are equivalent:
(i) All principal minors of \( M \) are positive.
(ii) To every vector \( x \neq 0 \) there exists an index \( k \) such that \( x_k y_k > 0 \) where \( y = Mx \).
(iii) To every vector \( x \neq 0 \) there exists a diagonal matrix \( D_x \) with positive diagonal elements such that the inner product \( (Mx, D_x x) > 0 \).
(iv) To every vector \( x \neq 0 \) there exists a diagonal matrix \( H_x \geq 0 \) such that the inner product \( (Mx, H_x x) > 0 \).
(v) Every real proper value of \( M \) as well as of each principal minor of \( M \) is positive.

**Lemma 2.5.** [1] If \( F \) is a nonempty meet semi-sublattice that is closed and bounded below, then \( F \) has a least element.

**Lemma 2.6.** [10] If \( z \) solves the linear program \( \min p^T z \) subject to \( Mz + q \geq 0, z \geq 0 \) and if the corresponding optimal dual variable \( y \) satisfies \( (I - M^T)y + p > 0 \), then \( z \) solves the linear complementarity problem LCP\((M, q)\).

## 3 Main Results

In this paper we introduce block triangular \( K \)-matrix and hidden block triangular \( K \)-matrix, which are defined as follows: A matrix \( M \in \mathbb{R}^{mn \times mn} \) is said to be a block triangular \( K \)-matrix if it is formed with block of \( K \)-matrices \( M_{ij} \in \mathbb{R}^{m \times m} \), either in upper triangular forms or in lower triangular forms. Here \( i \) and \( j \) vary from 1 to \( n \).

For block upper triangular form of \( M \), the blocks \( M_{ij} = 0 \) for \( i < j \) and for block lower triangular form of \( M \), the blocks \( M_{ij} = 0 \) for \( i > j \).

Consider

\[
M = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1.5 & 2 & 0 & 0 & 0 & 0 \\
3 & -1 & 1 & -1 & 0 & 0 \\
-1 & 4 & -0.75 & 1 & 0 & 0 \\
1 & -1 & 1 & -0.5 & 5 & -1 \\
-0.5 & 1 & -0.5 & 1 & -10 & 6
\end{bmatrix},
\]

which is a block triangular \( K \)-matrix.

The matrix \( N \in \mathbb{R}^{mn \times mn} \) is said to be hidden block triangular \( K \)-matrix if there
exist two block triangular $K$-matrices $X$ and $Y$ such that $NX = Y$. $N$ is formed with block matrices either in upper triangular forms or in lower triangular forms. For block upper triangular form of $N$, the blocks $N_{ij} = 0$ for $i < j$ and $X, Y$ are formed with $K$ matrices in upper triangular form. Similarly for block lower triangular form of $N$, the blocks $N_{ij} = 0$ for $i > j$ and $X, Y$ are formed with $K$ matrices in lower triangular form.

Consider $N = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 5 & 4 & 0 & 0 \\ -4.5 & -3 & 1 & 0.5 \\ 4 & 3.875 & -0.25 & 0.3125 \end{bmatrix}$, 

$$X = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 3 & 0 & 4 & -1 \\ -2 & 1 & 0 & 4 \end{bmatrix}$$ and $Y = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$, 

such that $NX = Y$. Then $N$ is a hidden block triangular $K$-matrix.

**Theorem 3.1.** Let $M$ be a block triangular $K$-matrix. Then $LCP(M, q)$ is processable by Lemke’s algorithm.

**Proof.** Let $M$ be a block triangular $K$-matrix. Then $\exists z \in R^n$ such that $z_i(Mz)_i \leq 0 \quad \forall i \iff (z_1)_i(M_{11}z_1)_i \leq 0 \quad \forall i \iff z_1 = 0$, as $M_{11} \in K$; $(z_2)_i(M_{21}z_1 + M_{22}z_2)_i \leq 0 \quad \forall i \iff (z_2)_i(M_{22}z_2)_i \leq 0 \quad \forall i \iff z_2 = 0$, as $M_{22} \in K$. In similar way $(z_n)_i(M_{n1}z_1 + M_{n2}z_2 + \cdots + M_{nn}z_n)_i \leq 0 \quad \forall i \iff (z_n)_i(M_{nn}z_n)_i \leq 0 \quad \forall i \iff z_n = 0$, as $M_{nn} \in K$ and $z_1 = z_2 = \cdots = z_{n-1} = 0$. Hence $z = 0$. So $M$ is a $P$-matrix. Therefore $LCP(M, q)$ is processable by Lemke’s algorithm.

**Remark 3.1.** [4] Let $M$ be a block triangular $K$-matrix. Then $LCP(M, q)$ is solvable by criss-cross method.

**Theorem 3.2.** If $M$ is a block triangular $K$-matrix and $q$ is an arbitrary vector, then the feasible region of $LCP(M, q)$ is a meet semi-sublattice.

**Proof.** Let $F = \text{FEA}(M, q)$. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \in F$ are two feasible vectors. So $x \geq 0, y \geq 0, Mx + q \geq 0, My + q \geq 0$. 

5
Let $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = \min(x, y)$. Then

$$Mx + q = \begin{bmatrix} M_{11}x_1 + q_1 \\ M_{21}x_1 + M_{22}x_2 + q_2 \\ M_{31}x_1 + M_{32}x_2 + M_{33}x_3 + q_3 \\ \vdots \\ M_{n1}x_1 + M_{n2}x_2 + M_{n3}x_3 + \cdots + M_{nn}x_n + q_n \end{bmatrix} \geq 0.$$}

$$\implies x_1 \in \text{FEA}(M_{11}, q_1), x_2 \in \text{FEA}(M_{22}, M_{21}x_1 + q_1), \ldots, x_n \in \text{FEA}(M_{nn}, M_{n1}x_1 + M_{n2}x_2 + \cdots + M_{n(n-1)}x_{n-1} + q_n).$$

In similar way $y_1 \in \text{FEA}(M_{11}, q_1), y_2 \in \text{FEA}(M_{22}, M_{21}y_1 + q_1), \ldots, y_n \in \text{FEA}(M_{nn}, M_{n1}y_1 + M_{n2}y_2 + \cdots + M_{n(n-1)}y_{n-1} + q_n).$ Suppose $z = \min(x, y) \implies z_1 = \min(x_1, y_1), z_2 = \min(x_2, y_2), \ldots, z_n = \min(x_n, y_n).$ $M_{ij} \in K \implies z_1 \in \text{FEA}(M_{11}, q_1) \implies M_{11}z_1 + q_1 \geq 0, z_2 \in \text{FEA}(M_{22}, M_{21}z_1 + q_2) \implies M_{22}z_2 + M_{21}z_1 + q_2 \geq 0, \ldots, z_n \in \text{FEA}(M_{nn}, M_{n1}z_1 + M_{n2}z_2 + \cdots + M_{n(n-1)}z_{n-1} + q_n) \implies M_{n1}z_1 + M_{n2}z_2 + \cdots + M_{n(n-1)}z_{n-1} + M_{nn}z_n + q_n \geq 0.$ So $z = \min(x, y) \in \text{FEA}(M, q).$ Hence the feasible region is a meet semi-sublattice.

Cottle et al.\cite{1} showed that if $F$ is a nonempty meet semi-sublattice, that is closed and bounded below, then $F$ has a least element by lemma 2.5. Now we show that if the LCP($M, q$) is feasible, where $M$ is a block triangular $K$-matrix, then FEA($M, q$) contains a least element $u.$

**Theorem 3.3.** Let $M$ be a block triangular $K$-matrix and $q$ be an arbitrary vector. If the LCP($M, q$) is feasible, then FEA($M, q$) contains a least element $u,$ where $u$ solves the LCP($M, q$).

**Proof.** Let $F = \text{FEA}(M, q).$ By theorem 3.2, $F$ is a meet semi-sublattice. Let LCP($M, q$) be feasible. Then the set $F$ is obviously nonempty and bounded below by zero. Then the existence of the least element $l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{bmatrix}$ follows from lemma
\[ 2.5 \] That is \( l = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} \preceq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x \ \forall \ x \in F \text{ and } l \in F. \]

Let \( F_i = \text{FEA}(M_{ii}, M_{i(i-1)}z_{i-1} + M_{i(i-2)}z_{i-2} + \cdots + M_{i2}z_2 + M_{i1}z_1 + q_i). \) Now it is clear that \( y_1 \in F_1, y_2 \in F_2, \cdots, y_n \in F_n, \) where \( y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in F. \) As \( M_{ii} \) are \( Z- \) matrices, \( l_i \) is the least element of \( F_i \ \forall \ i \in \{1, 2, \cdots, n\} \) and \( l_i \) solves \( \text{LCP}(M_{ii}, M_{i(i-1)}z_{i-1} + M_{i(i-2)}z_{i-2} + \cdots + M_{i2}z_2 + M_{i1}z_1 + q_i). \) So \( l = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} \) solves \( \text{LCP}(M, q). \)

Mangasarian \[10\] showed that if \( z \) solves the linear program, \( \min p^Tz \) subject to \( Mz + q \geq 0, \ z \geq 0 \) and if the corresponding optimal dual variable \( y \) satisfies \( (I - M^T)y + p > 0, \) then \( z \) solves the linear complementarity problem \( \text{LCP}(M, q) \) by lemma \[2.6\]. Here we show that if \( \text{LCP}(M, q) \) with \( M, \) a block triangular \( K \)-matrix, has a solution which can be obtained by solving the linear program \( \min p^T x \) subject to \( Mx + q \geq 0, \ x \geq 0. \)

**Theorem 3.4.** The linear complementarity problem \( \text{LCP}(M, q), \) where \( M \) is a block triangular \( K \)-matrix, has a solution which can be obtained by solving the linear program \( \min p^T x \) subject to \( Mx + q \geq 0, \ x \geq 0, \) where \( p = r \geq 0 \) and \( Z_1 \) is a block triangular \( K \)-matrix with \( r^TZ_1 > 0. \)

**Proof.** Let \( M \) be a block triangular \( K \)-matrix. The linear program, \( \min p^T x \) subject to \( Mx + q \geq 0, \ x \geq 0 \) and the dual linear program, \( \max -q^Ty \) subject to \( -M^Ty + p \geq 0, \ y \geq 0 \) have solutions \( x \) and \( y \) respectively. \( M \) can be written as \( D - U, \) where

\[ D = \begin{bmatrix} D_{11} & 0 & 0 & \cdots & 0 \\ D_{21} & D_{22} & 0 & \cdots & 0 \\ D_{31} & D_{32} & D_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{n1} & D_{n2} & D_{n3} & \cdots & D_{nn} \end{bmatrix}, \]

\( D_{ij} \)'s are diagonal matrices with positive entries and
Thus the series \( I + Q + Q^2 + \cdots \) converges to the matrix \((I - Q)^{-1} = (tM)^{-1} \geq 0\), since \( Q^k \geq 0 \) for \( k = 1, 2, \ldots \). Therefore \( M^{-1} \) exists and \( M^{-1} \geq 0 \).

**Proof.** Assume that \( Q = I - tM \geq 0, t > 0 \). Let \( p(Q) \) be the Perron-root of \( Q \). Then we have \( \det[(1 - p(Q))I - tM] = \det[Q - p(Q)I] = 0 \). By theorem 2.1, \( 0 < p(Q) < 1 \). Thus the series \( I + Q + Q^2 + \cdots \) converges to the matrix \((I - Q)^{-1} = (tM)^{-1} \geq 0\), since \( Q^k \geq 0 \) for \( k = 1, 2, \ldots \). Therefore \( M^{-1} \) exists and \( M^{-1} \geq 0 \).
Theorem 3.6. Let $N$ be a block triangular $K$-matrix and $M$ be a $Z$-matrix such that $M \leq N$. Then both $M^{-1}$ and $N^{-1}$ exist and $M^{-1} \geq N^{-1} \geq 0$.

**Proof.** Let $N$ be a block triangular $K$-matrix and $M$ be a $Z$-matrix such that $M \leq N$. Assume that $R = I - \alpha N \geq 0, \alpha > 0$. Let $p(R)$ be a Perron root of $R$. Then we have $\det[(1 - p(R))I - \alpha N] = \det[R - p(R)I] = 0$. By theorem 2.1, $0 < p(R) < 1$. Thus the series $I + R + R^2 + \cdots$ converges to the matrix $(I - R)^{-1} = (\alpha N)^{-1}$. Since $S^k \geq R^k \geq 0$, for $k = 1, 2, \cdots$, the series $I + S^2 + \cdots$ converges to the matrix $(I - S)^{-1} = (\alpha M)^{-1}$. Therefore $M^{-1}$ and $N^{-1}$ exist and $M^{-1} \geq N^{-1} \geq 0$.

**Corollary 3.2.** Assume that $M, N$ are block triangular $K$-matrices such that $M \leq N$. Then both $M^{-1}$ and $N^{-1}$ exist and $M^{-1} \geq N^{-1} \geq 0$.

**Theorem 3.7.** Let $N$ be a hidden block triangular $K$-matrix. Then every diagonal block of $N$ is a hidden $Z$-matrix.

**Proof.** Let $N$ be a hidden block triangular $K$-matrix with $NX = Y$, where $X$ and $Y$ are block triangular $K$-matrices. Let

$$
N = 
\begin{bmatrix}
N_{11} & 0 & 0 & \cdots & 0 \\
N_{21} & N_{22} & 0 & \cdots & 0 \\
N_{31} & N_{32} & N_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N_{n1} & N_{n2} & N_{n3} & \cdots & N_{nn}
\end{bmatrix},
$$

$$
X = 
\begin{bmatrix}
X_{11} & 0 & 0 & \cdots & 0 \\
X_{21} & X_{22} & 0 & \cdots & 0 \\
X_{31} & X_{32} & X_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nn}
\end{bmatrix}
$$

and

$$
Y = 
\begin{bmatrix}
Y_{11} & 0 & 0 & \cdots & 0 \\
Y_{21} & Y_{22} & 0 & \cdots & 0 \\
Y_{31} & Y_{32} & Y_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Y_{n1} & Y_{n2} & Y_{n3} & \cdots & Y_{nn}
\end{bmatrix}.
$$
The block diagonal of \( NX \) are \( N_{ii}X_{ii} \) for \( i \in \{1, 2, \ldots, n\} \). So \( N_{ii}X_{ii} = Y_{ii} \) for \( i \in \{1, 2, \ldots, n\} \). \( X_{ii}, Y_{ii} \) are \( K \)-matrices. Then \( X_{ii}^T, Y_{ii}^T \) are also \( K \)-matrices. So \( v(X_{ii}^T) > 0, v(Y_{ii}^T) > 0 \). Let \( r_i, s_i \in R^{m_+} \) such that \( X_{ii}^Tr_i + Y_{ii}^Ts_i > 0 \implies r_i^TX_{ii} + s_i^TY_{ii} > 0 \). Hence the block diagonals of \( N \) are hidden \( Z \)-matrices.

**Theorem 3.8.** Let \( N \) be a hidden block triangular \( K \)-matrix. Then every determinant of block diagonal matrices of \( N \) are positive.

**Proof.** Let \( N \) be a hidden block triangular \( K \)-matrix with \( NX = Y \), where \( X \) and \( Y \) are block triangular \( K \)-matrices. Let

\[
N = \begin{bmatrix}
N_{11} & 0 & 0 & \cdots & 0 \\
N_{21} & N_{22} & 0 & \cdots & 0 \\
N_{31} & N_{32} & N_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N_{n1} & N_{n2} & N_{n3} & \cdots & N_{nn}
\end{bmatrix},
\]

\[
X = \begin{bmatrix}
X_{11} & 0 & 0 & \cdots & 0 \\
X_{21} & X_{22} & 0 & \cdots & 0 \\
X_{31} & X_{32} & X_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nn}
\end{bmatrix}
\text{ and } Y = \begin{bmatrix}
Y_{11} & 0 & 0 & \cdots & 0 \\
Y_{21} & Y_{22} & 0 & \cdots & 0 \\
Y_{31} & Y_{32} & Y_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Y_{n1} & Y_{n2} & Y_{n3} & \cdots & Y_{nn}
\end{bmatrix}.
\]

The block diagonal of \( NX \) are \( N_{ii}X_{ii} \) for \( i \in \{1, 2, \ldots, n\} \). So \( N_{ii}X_{ii} = Y_{ii} \) for \( i \in \{1, 2, \ldots, n\} \). \( X_{ii}, Y_{ii} \) are \( K \)-matrices. Then \( \det(X_{ii}), \det(Y_{ii}) > 0 \forall i \). Hence \( \det(N_{ii}) > 0 \forall i \).

**Corollary 3.3.** Every block triangular \( K \)-matrix is a hidden block triangular \( K \)-matrix.

**Proof.** Let \( M \) be a block triangular \( K \)-matrix. Taking \( X = I \), the identity matrix, it is clear that \( M \) is a hidden block triangular \( K \)-matrices.

**Theorem 3.9.** The linear complementarity problem \( LCP(N, q) \), where \( N \) is a hidden block triangular \( K \)-matrix with \( NX = Y \), \( X \) and \( Y \) are block triangular \( K \)-matrices, has a solution which can be obtained by solving the linear program \( \min \ p^T x \) subject to \( Nx + q \geq 0, x \geq 0 \), where \( p = r + NTs \geq 0 \) and \( r, s \geq 0 \) such that \( X^Tr > 0 \) and \( Y^Ts > 0 \).

**Proof.** Let \( N \) be a hidden block triangular \( K \)-matrix with \( NX = Y \), where \( X \) and \( Y \) are block triangular \( K \)-matrices. The linear program, \( \min \ p^T x \) subject to \( Nx + q \geq 0, x \geq 0 \) and the dual linear program, \( \max -q^T y \) subject to \( -N^Ty + p \geq 0, y \geq 0 \) have solutions \( x \) and \( y \) respectively. \( X \) can be written as \( D - U \), where
Let \( D \) be a hidden block triangular \( K \)-matrix. Consider the \( \text{LCP}(\mathcal{N}, \bar{q}) \), where

\[
\mathcal{N} = \begin{bmatrix}
0 & -N^T \\
N & 0
\end{bmatrix}, \quad \bar{q} = \begin{bmatrix}
r + N^T s \\
q
\end{bmatrix}
\]

and \( r, s \) as mentioned in theorem 3.3. If

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} \in \text{FEA}(\mathcal{N}, \bar{q}),
\]

then \((I - N^T)y + p > 0\), where \( p = r + N^T s \).
Proof. Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{FEA}(\mathcal{N}, \bar{q})$. Since $N$ is a hidden block triangular $K$-matrix, there exist two block triangular $K$-matrices $X$ and $Y$ such that $NX = Y$ and $r, s \geq 0$, $r^T X + s^T Y > 0$. Let $X = D - U$ and $Y = D - V$, where $U$ and $V$ are two square matrices with all nonnegative entries and $D$ is a block triangular diagonal matrix with positive entries as mentioned in theorem 3.9. Then $0 < r^T X + s^T Y = r^T X + s^T N X = p^T (D - U) = p^T (D - U) + y^T (Y - N X) = p^T (D - U) + y^T (D - V - N (D - U)) = (-y^T N + p^T) (D - U) + y^T (D - V) \leq (y^T (I - N) + p^T) D$ since $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{FEA}(\mathcal{N}, \bar{q})$, $U \geq 0, V \geq 0$. Since $D$ is a positive block triangular diagonal matrix, $(I - N^T) y + p > 0$.

Theorem 3.10. LCP($\mathcal{N}, \bar{q}$) has a solution iff LCP($N, q$) has a solution.

Proof. Suppose LCP($\mathcal{N}, \bar{q}$) has a solution. Let $\bar{z} = \begin{bmatrix} x \\ y \end{bmatrix} \in \text{SOL}(\mathcal{N}, \bar{q})$. From the complementarity condition it follows that $x^T (p - N^T y) + y^T (N x + q) = 0$. Since $p - N^T y, N x + q, x, y \geq 0$, and $x^T (p - N^T y) = 0, y^T (N x + q) = 0$. By lemma 3.1, it follows that $y + (p - N^T y) > 0$. This implies for all $i$ either $(p - N^T y)_i > 0$ or $y_i > 0$. Now if $(p - N^T y)_i > 0$, then $x_i = 0$. If $y_i > 0$ then $(q + N x)_i = 0$. This implies $x_i (q + N x)_i = 0 \forall i$. Therefore $x$ solves LCP($N, q$).

Conversely, $x$ solves LCP($N, q$). Let $y = s$, where $s$ as mentioned in theorem 3.9. Here $p - N^T y = r + N^T s - N^T y = r + N^T s - N^T s = r \geq 0$. So $\bar{z} = \begin{bmatrix} x \\ s \end{bmatrix} \in \text{FEA}(\mathcal{N}, \bar{q})$. Further $\mathcal{N}$ is PSD-matrix, which implies that $\mathcal{N} \in Q_0$. Therefore $\bar{z}$ solves the LCP($\mathcal{N}, \bar{q}$).

Theorem 3.11. All hidden block triangular $K$-matrices are $Q_0$.

Proof. Let $N$ be a hidden block triangular $K$-matrix. It is clear that feasibility of LCP($N, q$) implies the feasibility of LCP($\mathcal{N}, \bar{q}$). Note that $\mathcal{N} \in Q_0$. This implies that the feasible point of LCP($\mathcal{N}, \bar{q}$) is also a solution of LCP($\mathcal{N}, \bar{q}$). Hence by theorem 3.10 feasibility of LCP($N, q$) ensures the solvability of LCP($N, q$). Therefore $N$ is a $Q_0$-matrix.

Remark 3.2. Let $M = \begin{bmatrix} M_{11} & 0 & 0 & \cdots & 0 \\
M_{21} & M_{22} & 0 & \cdots & 0 \\
M_{31} & M_{32} & M_{33} & \cdots & 0 \\
& \vdots & \vdots & \vdots & \vdots \\
M_{n1} & M_{n2} & M_{n3} & \cdots & M_{nn} \end{bmatrix}$, where $M_{ij} \in \mathbb{R}^{m \times m}$ are $K$-matrices.
Let $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix}$ and $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix}$, where $z_i, q_i \in R^m$.

Then $Mz + q = \begin{bmatrix} M_{11}z_1 + q_1 \\ M_{21}z_1 + M_{22}z_2 + q_2 \\ M_{31}z_1 + M_{32}z_2 + M_{33}z_3 + q_3 \\ \vdots \\ M_{n1}z_1 + M_{n2}z_2 + M_{n3}z_3 + \cdots + M_{nn}z_n + q_n \end{bmatrix}$.

First we solve $LCP(M_{11}, q_1)$ and get the solution $w_1 = M_{11}z_1 + q_1, w_1^Tz_1 = 0$. Then we solve $LCP(M_{22}, M_{21}z_1 + q_2)$ and get the solution $w_2 = M_{22}z_2 + M_{21}z_1 + q_2, w_2^Tz_2 = 0$. Finally we solve $LCP(M_{nn}, M_{nn}z_n + \cdots + M_{n(n-1)}z_{n-1} + q_n)$ and get the solution $w_n = M_{nn}z_n + M_{n1}z_1 + M_{n2}z_2 + M_{n3}z_3 + \cdots + M_{n(n-1)}z_{n-1} + q_n, w_n^Tz_n = 0$.

So $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix}$ and $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix}$ solve $LCP(M, q)$.

4 Conclusion

In this article, we introduce the class of block triangular $K$-matrix and the class of hidden block triangular $K$-matrix in the context of solution of linear complementarity problem. We call these two classes jointly as $K$-type block matrices. We show that the linear complementarity problem with $K$-type block matrix is solvable by linear program. The linear complementarity problem with block triangular $K$-matrix is also processable by Lemke’s algorithm as well as criss-cross method. We show that the hidden block triangular $K$-matrix is a $Q_0$-matrix.

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References

[1] RW Cottle, JS Pang, and RE Stone. The linear complementarity problem. 1992. *AP, New York*.

[2] AK Das. Properties of some matrix classes based on principal pivot transform. *Annals of Operations Research*, 243(1):375–382, 2016.

[3] AK Das and R Jana. Finiteness of criss-cross method in complementarity problem. In *International Conference on Mathematics and Computing*, pages 170–180. Springer, 2017.

[4] AK Das, R Jana, and Deepmala. Finiteness of criss-cross method in complementarity problem. In *Mathematics and Computing*, eds: Debasis Giri, Ram N. Mohapatra, Heinrich Begehr, Mohammad S. Obaidat, pages 170–180. Springer, 2017.

[5] Miroslav Fiedler and Vlastimil Pták. On matrices with non-positive off-diagonal elements and positive principal minors. *Czechoslovak Mathematical Journal*, 12(3):382–400, 1962.

[6] R Jana, AK Das, and A Dutta. On hidden Z-matrix and interior point algorithm. *Opsearch*, 56(4):1108–1116, 2019.

[7] R Jana, A Dutta, and AK Das. More on hidden Z-matrices and linear complementarity problem. *Linear and Multilinear Algebra*, 69(6):1151–1160, 2021.

[8] Irving Kaplansky. A contribution to von neumann’s theory of games. *Annals of Mathematics*, pages 474–479, 1945.

[9] Olvi L Mangasarian. Linear complementarity problems solvable by a single linear program. *Mathematical Programming*, 10(1):263–270, 1976.

[10] Olvi L Mangasarian. Linear complementarity problems solvable by a single linear program. *Mathematical Programming*, 10(1):263–270, 1976.

[11] SR Mohan, SK Neogy, and AK Das. A note on linear complementarity problems and multiple objective programming. *Mathematical programming*, 100(2):339–344, 2004.

[12] Prasenjit Mondal, S Sinha, SK Neogy, and AK Das. On discounted ar–at semi-markov games and its complementarity formulations. *International Journal of Game Theory*, 45(3):567–583, 2016.

[13] SK Neogy, R B Bapat, AK Das, and Biswabrata Pradhan. Optimization models with economic and game theoretic applications. *Annals of Operations Research*, 243(1):1–3, 2016.
[14] SK Neogy and AK Das. Linear complementarity and two classes of structured stochastic games. *Operations Research with Economic and Industrial Applications: Emerging Trends, eds: SR Mohan and SK Neogy, Anamaya Publishers, New Delhi, India*, pages 156–180, 2005.

[15] SK Neogy and AK Das. On almost type classes of matrices with $Q$-property. *Linear and Multilinear Algebra*, 53(4):243–257, 2005.

[16] SK Neogy and AK Das. Principal pivot transforms of some classes of matrices. *Linear algebra and its applications*, 400:243–252, 2005.

[17] SK Neogy and AK Das. Some properties of generalized positive subdefinite matrices. *SIAM journal on matrix analysis and applications*, 27(4):988–995, 2006.

[18] SK Neogy and AK Das. *Mathematical programming and game theory for decision making*, volume 1. World Scientific, 2008.

[19] SK Neogy and AK Das. On singular $N_0$-matrices and the class $Q$. *Linear algebra and its applications*, 434(3):813–819, 2011.

[20] SK Neogy and AK Das. On weak generalized positive subdefinite matrices and the linear complementarity problem. *Linear and Multilinear Algebra*, 61(7):945–953, 2013.

[21] SK Neogy, AK Das, and Abhijit Gupta. Generalized principal pivot transforms, complementarity theory and their applications in stochastic games. *Optimization Letters*, 6(2):339–356, 2012.

[22] SK Neogy, AK Das, S Sinha, and A Gupta. On a mixture class of stochastic game with ordered field property. In *Mathematical programming and game theory for decision making*, pages 451–477. World Scientific, 2008.

[23] Jong-Shi Pang. Hidden $Z$-matrices with positive principal minors. Technical report, WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER, 1977.

[24] Jong-Shi Pang and Ramaswamy Chandrasekaran. Linear complementarity problems solvable by a polynomially bounded pivoting algorithm. In *Mathematical Programming Essays in Honor of George B. Dantzig Part II*, pages 13–27. Springer, 1985.

[25] John Von Neumann. A certain zero-sum two-person game equivalent to the optimal assignment problem. *Contributions to the Theory of Games*, 2:5–12, 1953.