Spectroscopy, Scaling and Critical Indices in Strongly Coupled Quenched QED

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Abstract

The interplay of spectroscopy, scaling laws and critical indices is studied in strongly coupled quenched QED. Interpreted as a model of technicolor having strong interactions at short distances, we predict the techni-meson mass spectrum in a simplified model of a dynamically generated top quark mass $M_f$. Our results support the strict inequality that the techni-sigma mass $M_\sigma$ is less than twice the dynamical quark mass $M_f$, and confirm that the techni-pion is a Nambu-Goldstone boson. The level ordering $0 = M_\pi < M_\sigma < 2M_f < M_\rho < M_{a1}$ is found. An equation of state, and scaling laws are derived for the techni-meson masses by exploiting correlation length scaling. The resulting universality relations are confirmed by simulations on $16^4$, $32 \times 16^3$ and $32^4$ lattices. The anomalous dimension $\eta$ is measured to be approximatively 0.50 in good agreement with past lattice simulations and hyperscaling relations, as well as with the analytic solution of the quenched, planar gauged Nambu-Jona Lasinio model solved by continuum Schwinger-Dyson equation techniques.
1 Introduction

Quenched, strongly coupled QED is an interesting, relevant model field theory [1], [2]. Since the photon sector of the theory is free, the model is amenable to analytic solution and a wealth of interesting dynamics has been found in its light fermion sector. Chiral symmetry breaking, a dynamical realization of the Goldstone mechanism, a rich renormalization group structure, large anomalous dimensions, super-critical field phenomena, etc. have been discussed and illustrated in this context [1], [2], [3], [4], [5], [6].

Many of the continuum predictions have been confirmed by lattice simulations which have stressed the theory’s chiral Equation of State (EOS) [7]. The central purpose of the present paper is to study the theory’s mass spectrum, and relate the spectrum to other fundamental properties of the theory, such as its critical indices (anomalous dimensions). Since the model is a simplified version of Technicolor dynamics [8] and has been advocated as the dynamical basis of the mass of the top quark, our results should be of interest to high energy model builders looking beyond the Standard Model toward the physics underlying the Higgs mechanism [9]. Using lattice techniques borrowed from lattice QCD spectrum calculations we shall calculate the masses of the techni-fermion($f$), pion ($\pi$), rho ($\rho$), sigma ($\sigma$) and a1 ($a1$) [10]. In the strong coupling phase where chiral symmetry is broken dynamically, we confirm that the pion is massless and satisfies the constraints of PCAC. The other states are massive with the ordering $M_f < M_{\sigma} < M_{\rho} < M_{a1}$. In fact, $M_{\sigma} < 2M_f$ as expected in a theory with an unscreened attractive electromagnetic force between fermion and antifermion.

This paper will also pursue the more theoretical goal of relating features of the theory’s spectrum to universal quantities characterizing its renormalization group fixed point. The reader should recall that the continuum model must be parametrized by two coupling constants in order for an analytic renormalization group to exist [1], [3], [7]. The electrodynamic coupling $\alpha$ must be supplemented by an induced four Fermi coupling $G = G_0\Lambda^2/\pi^2$ where $G_0$ is the dimensionful four Fermi coupling and $\Lambda$ is an ultra-violet momentum-space cutoff. Analytical studies of the theory’s Schwinger-Dyson equation reveal that there is a fixed line, in the renormalization group sense, extending from $\alpha = 0$ to $\alpha_c = \pi/3$ at $G = (1 + \sqrt{1 - \alpha/\alpha_c})^2$ [8], [9]. The theory has been particularly well studied at the endpoints of this line. At $\alpha = 0$, $G = G_c = 4$ the theory reduces to the Nambu Jona-Lasinio (NJL)
model [11]. The cutoff theory breaks chiral symmetry for all \( G > G_c \) and when the cutoff is removed the theory becomes a free field. At all other points along the fixed line there is an unscreened massless photon which guarantees that the critical theory is interacting, with a rich mass spectrum. At the other end of the fixed line (\( G = 1, \alpha = \pi/3 \)), there is the Miransky-Bardeen-Leung-Love point where “collapse of the wavefunction” occurs [4], [5]. Critical indices along the line of chiral transitions vary continuously as \( \alpha \) varies from zero to \( \pi/3 \)

\[
\nu = 1/2 \sqrt{1 - \alpha/\alpha_c} \\
\beta_{mag} = (2 - \sqrt{1 - \alpha/\alpha_c})/2 \sqrt{1 - \alpha/\alpha_c} \\
\delta = (2 + \sqrt{1 - \alpha/\alpha_c})/(2 - \sqrt{1 - \alpha/\alpha_c}) \\
\gamma = 1
\]

(1)

with familiar mean field values at \( \alpha = 0 \) and Miransky’s essential singularities at \( \alpha = \alpha_c \) [7], [6].

A major objective of any numerical lattice approach to this model would be a verification of the continuum model’s phase diagram in the \((\alpha, G)\) plane. Lattice studies which tune the Action and move in a controlled fashion along the fixed line have not yet been done. Instead, non-compact quenched QED has been simulated by generating photon field configurations using a fast Fourier transform and then calculating fermion propagators in that background using staggered lattice fermion fields [12]. It has been argued that the discreteness of the Lattice Action necessarily generates some four Fermi couplings, so when the lattice fine structure is increased and a chiral transition is found, one is actually simulating a continuum Action with both photon exchange and four-Fermi forces. Large lattice, high statistics studies have measured the critical lattice coupling \( \beta = 1/e^2 = .257(2) \) and critical indices \( \delta = 2.2(1), \beta_{mag} = .78(8) \). These values match the indices \( \delta \) and \( \beta_{mag} \) in Eq.(1) if \((\alpha, G) = (0.44 \alpha_c, 3.06)\) [13]. These rather precise measurements use the framework of the chiral Equation of State (EOS) as has been discussed in detail before and will be reviewed below. Briefly, this approach requires that we simulate the model within its scaling window but not necessarily directly at its critical point. Simulation data near but not at \( \beta_c \) and near but not at zero fermion mass \( m \) should satisfy a chiral EOS and lie on a universal curve if “reduced” quantities are plotted. This approach, which
is borrowed from ancient studies of ferromagnetic transitions, has proved to be very successful.

The model’s scaling laws and critical indices can also be found from its spectrum of states. The development and exploitation of relations between spectroscopy, scaling laws and critical indices is a major theme of this work and new analytic and numerical results will be presented. Our most successful result will be a scaling law between the chiral condensate $\langle \bar{\psi}\psi \rangle$ and any dynamically generated mass $M_a$, 

$$\langle \bar{\psi}\psi \rangle = C_a M_a^{\beta_{mag}/\nu}$$

(2)

We shall argue that Eq.(2) should hold anywhere in the theory’s scaling region, i.e. for couplings near the critical point and symmetry breaking fields, such as the bare fermion mass, which are sufficiently weak. The theoretical assumptions underlying Eq.(2) consist of (1.) validity of the chiral Equation of State and (2.) hyperscaling, the idea that the theory’s critical singularities are due to the divergence of a single length scale, the correlation length. Using our accurate $\langle \bar{\psi}\psi \rangle$ data on $16^4$ and $24^4$ lattices and results for $M_\rho$ on $32 \times 16^3$ lattices we shall find that Eq.(2) is well satisfied with the critical indices

$$\beta_{mag}/\nu = 1.25(1)$$

(3)

which is in excellent agreement with Eq.(1) which also gives 1.25 at the point $(\alpha, G) = (.44 \alpha_c, 3.06)$ picked out by the lattice Action. In mean field theory, such as the NJL model, this ratio of critical indices is unity. Thus, the lattice simulation is correctly accounting for the effects of an unscreened vector force which renders the theory non-trivial in the continuum limit.

Other analyses discussed below will give further evidence that the critical index $\gamma$, which controls the susceptibility divergence, is unity, and $\nu$, which controls the correlation length, is 0.68(2).

This article is organized into several additional sections. In Sec. 2 we review the relation between the chiral Equation of State, hyperscaling and critical indices. The approach and results in ref. [7] play a crucial role here. In Sec.3 we present our numerical results which are then used in Sec 4 – 6 to extract the physics discussed briefly above. Finally, in Sec.7 we suggest future research.
2 Scaling and critical indices

In this section we shall summarize the main theoretical arguments which proved useful in our analysis of our quenched data. A detailed exposition can be found in refs. [7], [14]. The basic philosophy of refs. [7], [14] is the use of the chiral Equation of State and hyperscaling, the assumption that a theory’s critical singularities are all due to one divergent length scale, to obtain scaling laws for low energy quantities which have an immediate physical significance. This approach will yield new ways to extract critical indices for low energy properties of the theory, and will shed interesting insights on these two seemingly different features of the theory.

We begin with a theory’s Equation of State (EOS) written in a standard form [15]. Let \( t \) denote the deviation from the critical coupling, and \( m \) the symmetry breaking field; in our application \( t = \beta - \beta_c \) where \( \beta = 1/e^2 \) ( \( e \) = electrodynamical coupling) and \( m \) is the bare fermion mass which explicitly breaks the continuous chiral symmetry of massless QED. The EOS records the order parameter’s response to a change in coupling and symmetry breaking field,

\[
m = \langle \bar{\psi}\psi \rangle^{\delta} f(t/\langle \bar{\psi}\psi \rangle^{1/\beta_{mag}}) \tag{4}
\]

Eq. (4) contains the critical indices \( \delta \) and \( \beta_{mag} \) in standard statistical mechanics notation. Eq. (4) should hold everywhere in the theory’s scaling region, small \( t \) and \( m \). It is frequently convenient to rewrite Eq.(4) in terms of “reduced” variables and to invert it. Let \( y \) be a reduced symmetry-breaking field and \( z \) be a reduced order parameter

\[
y = m/t^\Delta
\]

\[
z = \langle \bar{\psi}\psi \rangle / t^{\beta_{mag}}
\]

where we have used the conventional definition \( \Delta = \delta \beta_{mag} \). If we divide Eq. (4) by \( t^\Delta \) we have

\[
y = z^\delta f(z^{1/\beta_{mag}}) \tag{6}
\]

In other words, the EOS implies that the reduced symmetry-breaking field is just a function of the reduced order parameter. Upon inverting this dependence,

\[
\langle \bar{\psi}\psi \rangle = t_{mag}^\beta F(y) \tag{7}
\]
which is an alternative and familiar form of the EOS. Eqs. (4) and (7) have been used extensively to extract the critical indices and the critical coupling in quenched lattice QED simulations [13]. They will be used again with impressive success in Sec. 4 below.

Our next goal is the determination of “Equations of State” for other low energy features of the model, such as its mass spectrum. In order to do that we assume the correlation length scaling idea, which underlies hyperscaling, familiar in classical statistical mechanics. We assume that there is a single macroscopic length scale \(\xi\) whose divergence is responsible for the theory’s critical behaviour. \(\xi\) sets the scale for low energy physics (low on the scale of the cutoff, the reciprocal of the lattice spacing in our case) and one can do dimensional analysis using \(\xi\) as a unit of length. In particular, if the field \(\phi\) has scale dimension \(d_\phi\) and we consider the scaling region of the theory where \(\xi\) is much larger than the lattice spacing, then

\[
\langle \phi \rangle \propto \xi^{-d_\phi} \tag{8}
\]

Furthermore, suppose that the theory has power-behaved critical singularities, so \(\xi\) diverges as \(t^{-\nu}\) in the scaling region. Any dynamically generated mass scale \(M_\rho\) in the model must, by the assumption of correlation length scaling, behave as \(\xi^{-1}\). So, Eqs. (7) and (8) give,

\[
\langle \phi \rangle \propto M_\rho^{d_\phi} \propto t^{\nu d_\phi} \propto t^{\beta_{mag}} F(y) \tag{9}
\]

which implies that \(d_\phi = \beta_{mag}/\nu\) and,

\[
\langle \phi \rangle = C_\alpha M_\rho^{\beta_{mag}/\nu} \tag{10}
\]

everywhere in the scaling region. The constants \(C_\alpha\) are not controlled by this scaling argument and are non-universal. We shall find that numerical data for \(\langle \bar{\psi} \psi \rangle\) and \(M_\rho\) satisfy the scaling law Eq.(10) beautifully and yield a relatively precise estimate for \(\beta_{mag}/\nu\) which agrees with continuum calculations based on the Schwinger-Dyson equation [7], [8] at the point (.44\(\alpha_c\), 3.06) picked out previously by the lattice results.

Since the anomalous dimension \(\eta\) of the field \(\phi\) is defined as \(d_\phi = d/2 - 1 + \eta/2\), Eq.(10) is directly sensitive to \(\eta\) itself. The anomalous dimension \(\eta\) vanishes identically in mean field theory. Therefore, Eq.(10) will prove to be particularly informative in this and, hopefully, other theories studied in the future.
We shall also find it useful to write Eq.(9) in the form
\[ M_a = t^\nu G_a(y) \]  
(11)
and consider ratios of dynamically generated masses,
\[ R_{ab} = \frac{M_a}{M_b} = G_{ab}(y) \]  
(12)
which are functions of just one scaling variable \( y \). The validity of this scaling law will also be demonstrated below. In fact, in the following Sections of this article the three equations Eqs. (7),(10) and (11) will be tested and used to extract critical indices. The validity of hyperscaling will be tested by plotting two mass ratios against one another. The universality of these plots will provide an independent check of the scaling hypothesis which was first seen in EOS plots of \( <\bar{\psi}\psi> \). The limiting \((m \to 0, t \to 0)\) values of particular mass ratios are universal numbers and we shall attempt to estimate them below.

In ref [14] the properties of the mass ratios are discussed in great detail, and several analytic examples are given. It is observed there that in a wide class of theories in which mesons are made out of fundamental, pointlike constituents, non-triviality and compositeness go hand-in-hand. In particular, mesons have non-zero sizes which guarantee that they interact with one another. Only in the mean field limit where the size of mesons shrinks to zero and compositeness is lost, do the theories become trivial. In that case the mass ratio \( 2M_f/M_\sigma \) is exactly unity. We shall find in quenched QED that \( M_\sigma < 2M_f \) in the chiral limit, which is indicative of compositeness and non triviality.

We direct the reader to ref. [14], [7] for a more thorough discussion of the connection between hyperscaling, renormalizability and non-triviality. We have simply extracted a few portions of a wide class of relations for our use here and have emphasized only those which shed light on the non-triviality of the light quark dynamics in quenched lattice QED.

3 Numerical Results

We describe here the numerical simulations and the data analysis. The reader uninterested in these details can skip this section and go to the subsequent sections devoted to the discussion of our results.
3.1 The Simulations

We have collected 100, 30 and 248 gauge field configurations on the $16^4$, $24^4$ and $16^3 \times 32$ lattices, respectively, by using the algorithm introduced in [16]. The algorithm begins in momentum space and produces the appropriate Gaussian distribution of photons. Using a Fast Fourier Transform it then generates a set of dimensionless gauge fields $\theta_\mu$ in coordinate space. The coupling of the gauge field to the fermion is implemented by rescaling the gauge fields as $\theta \to \theta \sqrt{\beta}$.

This fast algorithm eliminates the correlations between subsequent configurations (the only source of correlation comes from the random number generator) and greatly reduces the correlations between the fluctuations at different spatial sites on the lattice. Since this algorithm produces independent configurations, our statistical sample is quite large by present simulation standards.

3.2 The Chiral Condensate

The configurations produced on the symmetric lattices have been used for the computation of $< \bar{\psi} \psi >$. We have inverted the staggered Dirac operator for each value of the fermion mass using a second order conjugate gradient routine. We used a noisy source for this inversion [17]. On the $16^4$ lattice at each $\beta$ value (spaced by $\Delta \beta = 0.002$ and ranging from $\beta = 0.247$ to $\beta = 0.257$) we have accurate $< \bar{\psi} \psi >$ values at five fermion masses $m$ equally spaced from $.001$ to $.005$. We have seen in past studies that accurate data over this mass range can yield unambiguous, quantitative results for $\beta_c$, $\delta$ etc. We also recall from past studies that the finite size effects, as inferred from $24^4$ and $32^4$ simulations, are very small in $< \bar{\psi} \psi >$ over this range of $m$. This is further demonstrated by our new data on the $24^4$ lattice (presented in Table 4), where we explored the same set of masses $m$ and a very large set of $\beta$'s.

3.3 The Analysis of the Spectrum Data

The spectroscopy data are from the $16^3 \times 32$ lattice where a wall source was used for the inversion of the Dirac operator after the appropriate gauge fixing.
Table 1: $\langle \psi \bar{\psi} \rangle$ data at various fermion masses and couplings $\beta$ on the $16^4$ lattice

We worked at five $\beta$ values, ranging from 0.245 to 0.265. For each $\beta$ we have data for five masses, spaced by 0.001, and ranging from 0.002 to 0.006. We have thus two sets of data in the weak coupling region (at $\beta = 0.260$ and 0.265) where we expect to observe clear signatures of chiral symmetry restoration.

For hadron operators we use the standard local form. The propagators in the PSeudoscalar, SCalar, VecTor, PseudoVector channels are then parametrized (in the region of large time) as follows[18]:

$$G(\tau) = a[\exp(-M\tau) + \exp(-M(N_\tau - \tau))] +$$
$$(-1)^{\tau\bar{a}}[\exp(-\tilde{M}\tau) + \exp(-\tilde{M}(N_\tau - \tau))]$$

Turning to the fermion, its parametrization reads:

$$G(\tau) = a[\exp(-M\tau) - (-1)^{\tau}\exp(-M(N_\tau - \tau))]$$

By fitting to the above forms, we obtained good estimates of the fundamental particles (i.e. the lower masses in the direct channel for the PS, VT sectors, and in the oscillating channel for the SC, PV sectors). To keep the (possible) contaminations from excited states under control we vary the extrema of our fits: typically we fit for $t_1 \leq t \leq T/2$ ($2 \leq t_1 \leq 10$), the number of degrees of freedom ranging from 6 to 12. The results level-off for $t_1 \simeq 7 - 8$ for the pion and the fermion (for which we performed 1-particle fits), and for $t_1 \simeq 2 - 3$ for the other particles (2-particles fits). In the first case (pion and fermion) the errors on the mass estimates were obtained from a jack-knife analysis performed by decimating one configuration at a time (remember that our configurations are independent). For the other mesons,
| \( \langle \bar{\psi} \psi \rangle \) | 0.001  | 0.002  | 0.003  | 0.004  | 0.005  |
|-----------------|--------|--------|--------|--------|--------|
| 0.260           | 0.0296(14) | 0.0436(12) | 0.0546(11) | 0.0639(10) | 0.0721( 9) |
| 0.259           | 0.0324(15) | 0.0466(13) | 0.0577(12) | 0.0670(11) | 0.0751( 9) |
| 0.258           | 0.0352(16) | 0.0498(14) | 0.0609(12) | 0.0702(11) | 0.0783( 9) |
| 0.257           | 0.0383(18) | 0.0532(15) | 0.0643(13) | 0.0735(12) | 0.0816(10) |
| 0.256           | 0.0419(20) | 0.0569(16) | 0.0678(14) | 0.0770(12) | 0.0850(11) |
| 0.255           | 0.0460(23) | 0.0608(17) | 0.0715(14) | 0.0806(12) | 0.0885(11) |
| 0.254           | 0.0508(25) | 0.0648(18) | 0.0753(15) | 0.0843(13) | 0.0922(11) |
| 0.253           | 0.0559(28) | 0.0690(19) | 0.0793(15) | 0.0881(13) | 0.0959(12) |
| 0.252           | 0.0611(30) | 0.0732(20) | 0.0834(16) | 0.0921(14) | 0.0997(12) |
| 0.251           | 0.0662(31) | 0.0776(20) | 0.0876(16) | 0.0961(14) | 0.1037(12) |
| 0.250           | 0.0710(31) | 0.0821(20) | 0.0919(17) | 0.1003(14) | 0.1077(13) |
| 0.249           | 0.0759(31) | 0.0869(21) | 0.0964(17) | 0.1046(15) | 0.1118(13) |
| 0.248           | 0.0812(31) | 0.0918(21) | 0.1010(17) | 0.1090(15) | 0.1161(14) |
| 0.247           | 0.0869(31) | 0.0970(22) | 0.1057(18) | 0.1135(16) | 0.1204(14) |
| 0.246           | 0.0930(31) | 0.1022(23) | 0.1106(19) | 0.1180(16) | 0.1247(14) |
| 0.245           | 0.0990(31) | 0.1075(24) | 0.1155(20) | 0.1227(17) | 0.1292(15) |
| 0.244           | 0.1048(33) | 0.1128(25) | 0.1204(21) | 0.1274(18) | 0.1337(16) |
| 0.243           | 0.1104(34) | 0.1182(26) | 0.1254(21) | 0.1322(18) | 0.1383(16) |
| 0.242           | 0.1159(35) | 0.1236(27) | 0.1305(22) | 0.1370(19) | 0.1430(17) |
| 0.241           | 0.1215(35) | 0.1291(27) | 0.1358(23) | 0.1420(20) | 0.1478(18) |
| 0.240           | 0.1279(35) | 0.1347(29) | 0.1410(24) | 0.1470(21) | 0.1527(18) |
| 0.239           | 0.1344(35) | 0.1406(30) | 0.1464(25) | 0.1521(22) | 0.1575(19) |
| 0.238           | 0.1411(37) | 0.1466(32) | 0.1519(27) | 0.1573(23) | 0.1625(20) |
| 0.237           | 0.1478(40) | 0.1526(34) | 0.1574(28) | 0.1625(24) | 0.1675(21) |
| 0.236           | 0.1542(43) | 0.1586(36) | 0.1630(29) | 0.1678(25) | 0.1725(22) |
| 0.235           | 0.1602(47) | 0.1645(38) | 0.1686(31) | 0.1731(26) | 0.1776(22) |

Table 2: \( \langle \bar{\psi} \psi \rangle \) data at various fermion masses and couplings \( \beta \) on the 24\(^4\) lattice
we quote the weighted average of the results obtained with different $t_1$. We also performed effective 2-mass analyses by exactly solving the equations for masses and amplitudes on four subsequent time slices; the results are in full agreement with the ones we quoted, but the errors are larger. In any case the agreement of the results obtained by local and global fits is a nice consistency check on our analyses.

To get a good fit the inclusion of secondary particles was necessary in the VT, SC, and PV sectors, as already said, but their amplitudes proved to be very small, and often compatible with zero considering the statistical errors. In many cases we noticed that the inclusion of a secondary state greatly improved the quality of the fit (i.e. reducing the $\chi^2$) while leaving completely unaffected the results for the fundamental states. In some cases the results of the fits suggest the existence of a very light particle in the secondary channel. We confirmed that the numerical significance of very light states was less in the present data as compared to past simulations on smaller lattices. We believe that any evidence for a very light state is unreliable and is, in fact, indicative of rather large finite size effects.

Figs. 1-3 provide an overview of the results of the fits: we note from them that the relative amplitude of the oscillating channel to the non-oscillating one decreases while going from weak to strong coupling, and from low to high masses. Figs. 1 shows a sample of $\pi$ propagators for various sets of parameters, with the best fits superimposed. Fig. 2 contains several typical fits for the vector channel. We plot there the fit obtained with $t = 5$ as a starting point. Note that the fit predicts the data at small $t$ (i.e. $t < 5.$) with good accuracy. Some results for the the sigma are shown in Fig. 3: its behaviour is satisfactory in the strong coupling region. In the weak coupling region the situation is somewhat less controlled, mostly because of large errors in the large $t$ region, and we feel that the results for $\sigma$ at weak coupling should be considered with some extra care.

Summarizing, the weak coupling, low mass region turns out to be rather difficult to treat, while the results for the fundamental particles in the critical and strongly coupled region are fully satisfactory. Tables \begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Parameter & Value & Units \\
\hline
Mass & 1.234 & $\text{GeV}$ \\
\hline
Amplitude & 0.12 & $\text{GeV}$ \\
\hline
\end{tabular}
\caption{Summary of results for fundamental particles}
\end{table}collect our results.
| \( \pi \)       | 0.265 | 0.260 | 0.255 | 0.250 | 0.245 |
|-----------------|-------|-------|-------|-------|-------|
| 0.002           | 0.0909(17) | 0.1090(18) | 0.1273(18) | 0.1419(18) | 0.1479(16) |
| 0.003           | 0.1189(13) | 0.1402(16) | 0.1589(16) | 0.1745(18) | 0.1794(13) |
| 0.004           | 0.1458(13) | 0.1668(13) | 0.1871(16) | 0.2012(16) | 0.2070(15) |
| 0.005           | 0.1685(13) | 0.1917(15) | 0.2109(15) | 0.2243(15) | 0.2298(13) |
| 0.006           | 0.1911(13) | 0.2133(13) | 0.2322(13) | 0.2450(16) | 0.2500(12) |

Table 3: Results for the pion mass from one particle fit

| \( \rho \)      | 0.265 | 0.260 | 0.255 | 0.250 | 0.245 |
|-----------------|-------|-------|-------|-------|-------|
| 0.002           | 0.1653(29) | 0.2044(42) | 0.2520(51) | 0.3231(83) | 0.3934(118) |
| 0.003           | 0.1916(26) | 0.2310(29) | 0.2815(37) | 0.3472(70) | 0.4164(86) |
| 0.004           | 0.2163(23) | 0.2579(27) | 0.3089(41) | 0.3699(52) | 0.4355(71) |
| 0.005           | 0.2385(23) | 0.2797(22) | 0.3322(39) | 0.3919(40) | 0.4548(62) |
| 0.006           | 0.2598(19) | 0.3026(22) | 0.3535(35) | 0.4115(41) | 0.4734(59) |

Table 4: Rho masses from a two particle fit

| \( \sigma \)     | 0.265 | 0.260 | 0.255 | 0.250 | 0.245 |
|------------------|-------|-------|-------|-------|-------|
| 0.002           | 0.1905(220) | 0.2223(403) | 0.2361(259) | 0.2922(282) | 0.3418(691) |
| 0.003           | 0.2182(132) | 0.2338(111) | 0.2681(427) | 0.3194(281) | 0.3690(536) |
| 0.004           | 0.2287(82) | 0.2625(427) | 0.2959(200) | 0.3480(410) | 0.4028(276) |
| 0.005           | 0.2469(64) | 0.2796(111) | 0.3207(184) | 0.3801(529) | 0.4271(361) |
| 0.006           | 0.2673(80) | 0.3032(82) | 0.3447(137) | 0.3950(225) | 0.4531(170) |

Table 5: Results for the sigma mass from a two particle fit

| \( a1 \)        | 0.265 | 0.260 | 0.255 | 0.250 | 0.245 |
|-----------------|-------|-------|-------|-------|-------|
| 0.002           | 0.1829(38) | 0.2151(44) | 0.2692(89) | 0.3434(425) | 0.3716(135) |
| 0.003           | 0.2057(31) | 0.2441(50) | 0.2850(40) | 0.3638(216) | 0.4051(103) |
| 0.004           | 0.2271(26) | 0.2601(43) | 0.3088(38) | 0.3767(70) | 0.4307(83) |
| 0.005           | 0.2463(26) | 0.2837(34) | 0.3315(37) | 0.3950(74) | 0.4544(82) |
| 0.006           | 0.2682(44) | 0.3043(30) | 0.3528(38) | 0.4127(138) | 0.4741(119) |

Table 6: a1 masses from a two particle fit
### Table 7: fermion masses, from one particle fit

| fermion | 0.265  | 0.260  | 0.255  | 0.250  | 0.245  |
|---------|--------|--------|--------|--------|--------|
| 0.002   | 0.0951(50) | 0.1116(52) | 0.1293(92) | 0.1481(151) | 0.1782(231) |
| 0.003   | 0.1071(40) | 0.1244(44) | 0.1375(96) | 0.1623(134) | 0.1905(214) |
| 0.004   | 0.1180(37) | 0.1343(61) | 0.1497(89) | 0.1742(122) | 0.2022(179) |
| 0.005   | 0.1283(34) | 0.1407(68) | 0.1607(85) | 0.1857(116) | 0.2136(166) |
| 0.006   | 0.1383(33) | 0.1506(66) | 0.1714(83) | 0.1963(111) | 0.2244(157) |

#### 3.4 The Chiral Extrapolation

We conclude this section by briefly describing the extrapolation procedure which leads to the values quoted in Table 7 and plotted in Fig.4. These technical points are important since they underlie the evidence for the Goldstone behavior of the pion we will discuss in more detail in the next Section.

A crucial ingredient for the chiral extrapolation is the knowledge of the functional dependence of the hadron masses on the bare quark mass. This is not known a priori, and could depend on $\beta$, possibly being sensitive to the anomalous dimension developed while approaching the critical point. In an attempt to use as little prejudice as possible, we decided to fit our data to several different functions of $m$. We considered quadratic and cubic hadron masses as first and second order polynomials in $m$. We also tried a Padé $P[1,1]$ expression $M = (a + bm)/(1 + cm)$. To critically evaluate the approach to zero of the extrapolated values as the coupling approaches the critical point (or its consistency with zero in the case of Pion), we tried fits both with and without a constant term.

All the fits clearly favour the standard form $M^2 = M^2_0 + Am + Bm^2$ for all the mesons. $M_0$ is zero for the pion, and non-zero for the other particles in the strong coupling region. The extrapolated data are not quite zero in the weak coupling region as well because, we believe, of finite size effects which push the masses up. Non-zero extrapolated values are expected (and found) in the weak coupling region for the same reason. It is interesting, and gives us more confidence in the results, that the $M_0$ values obtained with different parametrizations are quite similar to one other.

In Fig.4 we present an overview of the results: (fermion, $\sigma$, $\rho$, $\omega$) ($m = 0$) in the strong coupling region are plotted vs. $\beta$. In this plot we discarded
the data for the a1 at the lowest beta value. It appeared that $M \simeq 0.4$ is an upper limit to the masses one can meaningfully study at $\beta = 0.245$ because of finite spacing effects. The finite size effects show up clearly in the results for the fermion mass near $\beta_c$: the fermion masses "flatten" near the critical point to a value around 0.1. Indeed $1/0.1$ is close to half the time extent of the lattice, so it is unlikely that we can reliably calculate lower masses.

All the extrapolated masses are well split in the strong coupling region and approach zero (with the caveats discussed above) as $\beta \to \beta_c$.

### 4 The Equation of State and the Critical Indices $\gamma, \delta$

Our previous work on chiral symmetry breaking in quenched QED4 has shown that its critical behaviour is not described by mean field theory \[13\]. In fact, our measurements have all been consistent with the quantitative predictions of the Schwinger-Dyson equation defined with a momentum cutoff $\Lambda$ and the hyperscaling relations between critical indices \[7\], \[6\]. Our purpose here is to confirm those results using new, more accurate data and to extract the critical index $\gamma$, which controls the susceptibility divergence in the critical region. A practical way to find $\gamma$ proceeds through the chiral equation of state,

$$< \bar{\psi}\psi > /m^{1/\delta} = f(\Delta \beta / < \bar{\psi}\psi >^{1/\beta_{mag}})$$  \hspace{1cm} (15)

where $\delta$ and $\beta_{mag}$ are the usual critical indices discussed in Sec. 2 above, $\Delta \beta = \beta - \beta_c$ and $f$ is a universal function. Our past low mass $< \bar{\psi}\psi >$ data has given $\beta_c = 0.257(1)$, $\delta = 2.2(1)$ and $\beta_{mag} = .78(10)$. These results were
also obtained by combining Lanczos data for the spectral density of $\bar{\psi}\psi$ with conjugate gradient data for its bare fermion mass dependence. Here we used the conjugate gradient method to calculate $\bar{\psi}\psi$ and FFT’s to generate independent background photon configurations. Our $\bar{\psi}\psi$ data is presented for 100 independent gauge field configurations on a $16^4$ lattice in Table 1.

The data in Table 1 picks out $\beta_c = 0.257(1)$ very clearly. Recall that for $\beta = \beta_c$ the order parameter $\bar{\psi}\psi$ should scale as

$$\bar{\psi}\psi = A m^{1/6} (\beta = \beta_c)$$

We test for a simple power law by plotting log $\bar{\psi}\psi$ vs. log $m$ for $\beta = 0.257$ and 0.255 in Fig. 5. Note that the $\beta = 0.255$ data indicates that $\bar{\psi}\psi$ at $m = 0.001$ is too large to admit a good power law fit for $m$ ranging from 0.005 to 0.001. However, the power-law fit for the $\beta = 0.257$ data is excellent and the slope of the line in Fig. 5 yields $\delta = 2.15(5)$ in agreement with our earlier work. Inserting these values for $\beta_c$ and $\delta$ into the chiral equation of state (1) we can ask whether there is a particular $\beta_{mag}$ which yields a universal curve $f$, i.e. can all the data of Table 1 at various $\beta$ and $m$ values be plotted on a single scaling curve? Recalling the definition $\gamma = \beta_{mag} (\delta - 1)$ then gives us the susceptibility index of interest. In Fig. 6 we show the chiral equation of state for $\beta_c = .257$, $\delta = 2.20$ and $\beta_{mag} = .833$ The success of scaling hypothesis is very impressive. This value of $\beta_{mag}$ yields $\gamma$ precisely equal to 1.00. This result is consistent with, but more accurate than, our past analyses and is in agreement with the Schwinger-Dyson prediction of $\gamma = 1.00$. We discussed the theoretical significance of this result in Sec. 2 above.

The almost perfect scaling seen in Fig. 6 deteriorates rapidly as $\gamma$ deviates from unity. The dispersion in $f$ exceeds the statistical errors (several representative error bars are shown in Fig. 6a) if $\gamma$ exceeds 1.05 or if $\gamma$ falls below .975, so we have determined

$$\gamma = 1.00 + 0.05 - 0.03$$

We can exploit the scaling form of EOS also by writing

$$m/ \bar{\psi}\psi \delta = f((\beta_c - \beta)/ \bar{\psi}\psi^{1/\beta_{mag}})$$
On the basis of the very detailed analysis of ref. \cite{7} we can predict that the universal function $f$ is straight line. In fact we know that the data for the chiral condensate satisfy

$$m = a(\beta_c - \beta) < \bar{\psi}\psi > + b < \bar{\psi}\psi >^\delta$$

which confirms the theoretical expectation $\gamma = 1$. The previous equation can be restated like this

$$m/ < \bar{\psi}\psi >^\delta = a(\beta - \beta_c) < \bar{\psi}\psi >^{1-\delta} + b$$

We can then substitute into the EOS

$$a(\beta - \beta_c) < \bar{\psi}\psi >^{1-\delta} + b = f((\beta_c - \beta)/ < \bar{\psi}\psi >^{1/\beta_{mag}})$$

So $f$ must be a linear function with $1/\beta_{mag} = 1 - \delta$, i.e. $\gamma = 1.0$. It is interesting to check that $f$ actually is a straight line by looking at the data on the $24^4$ lattice, shown in Table 4, which explores a wide set of $\beta$’s. The linear behaviour of Fig. 6b is very clear, and the conclusion $\gamma = 1.0$ receives further support.

5 Scaling and the Critical Indices $\eta$ and $\nu$

As discussed in Sec.2, we can obtain estimates of $\nu$ and $\eta$ by exploiting the relation

$$< \bar{\psi}\psi > = CM_{\rho}^{(d/2-1+\eta/2)} = CM_{\rho}^{(\beta_{mag}/\nu)}$$

which should hold everywhere in the scaling region. Moreover, we can test the consistency of $\nu$ by plotting $M_\rho/t^\nu$ vs $m/t^\beta_{mag}$ as suggested by the equation of state for the masses

$$M_\rho = t^\nu g(m/t^\beta_{mag})$$

We will see that the estimates of $\nu$ obtained by using the EOS are by far more convincing than the one based on the direct fit at $m = 0$

$$M_\rho = A(\beta - \beta_c)^\nu$$

which is however consistent with the previous ones.
5.1 $<\bar{\psi}\psi>$ vs $M_\rho$ in the Scaling Region

Consider the scaling law,

$$<\bar{\psi}\psi> \propto M_\rho^{\beta_{mag}/\nu}$$  \hspace{1cm} (25)

reviewed in Sec.2 above. Since mean field theory predicts that $\beta_{mag} = \nu$, deviations from linear dependence of $<\bar{\psi}\psi>$ on $M_\rho$ provide clear evidence of a non-trivial critical point. The Schwinger-Dyson prediction for the continuum model is $\beta_{mag}/\nu = 1.25$ if $\delta = 2.2$, as determined from our Equation of State fit in Sec.3.

We took $<\bar{\psi}\psi>$ data from the $24^4$ run. They overlap with the spectroscopy data at $\beta = (0.260, 0.255, 0.250, 0.245)$, $m = (0.002, 0.003, 0.004, 0.005)$. Note that $\beta = 0.260$ lies in the weak coupling region. However since the transition is rounded, the data obtained at this $\beta$ value can be (tentatively) added to the data in the strong coupling region. We analyzed data sets both with and without the $\beta = 0.260$ points and confirmed that our fits did not change much.

Fig. 7 shows a log-log plot of $<\bar{\psi}\psi>$ versus $\log(M_\rho)$, All the points lie on the same universal plot (The only small deviation is at $\beta = 0.260$, at the smallest mass value.) We stress that this is an highly non-trivial test of Eq. (25), considering that the data we used are from completely independent simulations. The straight line is our best fit with all the data included: $<\bar{\psi}\psi> = M_\rho^{1.25}$.

We fit both $M_\rho$ vs $<\bar{\psi}\psi>$ and $<\bar{\psi}\psi>$ vs $M_\rho$, with and without the points at $\beta = 0.260$. The four fits give, respectively $\beta_{mag}/\nu = 1.265, 1.275, 1.243, 1.249$. Assuming that $\beta_{mag} = 0.86(3)$ we get for $\nu$: $\nu = 0.680(25), 0.674(25), 0.691(25), 0.688(25)$. These values are different from the mean field prediction, $\nu = 1/2$, but it is not easy to estimate the errors associated with this fitting procedure. Rather than attempting this, we prefer to show the inconsistency of our results with the mean field prediction in a more direct fashion. In Fig. 8 we plot $<\bar{\psi}\psi>^x$ vs $M_\rho$ for different $x$ values. The solid line corresponds to our best fit, the dashed lines to the mean field result $x = 1$, the dotted line to $\beta_{mag} = 0.86$ and $\nu = 0.5$. The sensitivity of the quality of the fits to the choice of the exponent $x$ is clear, and thus the incompatibility of our results with the mean field prediction is established.

We see that the anomalous dimension $\eta$ which follows is $\simeq 0.5$. This is a remarkably large value. Nonetheless, it is consistent with the hyperscaling
relation between δ and η:

\[(6 - \eta)/(2 + \eta) = \delta\]  \hspace{1cm} (26)

with δ = 2.2 as determined in Sec.3 above.

In conclusion, our data rule out mean field behaviour. They predict
\[\nu = 0.68(3)\] if \[\beta_{mag} = 0.86(3)\].

5.2 The Equation of State for the Masses and Correlation Length Scaling

As stated in Sec.2, we can check our estimate of \(\nu\) by using the data of the masses alone. This is done in Fig. 9 where we plot \(M_\rho/(\beta_c - \beta)^\nu\) vs \(m/(\beta_c - \beta)^{\beta_{mag}\delta}\), for \(\nu = 0.67\), \(\beta_{mag} = 0.833\), \(\delta = 2.20\). We use only points at strong coupling. The scaling hypothesis works very well. Again, we checked that the scaling behavior deteriorates if we select mean field exponents. The equation of state for the masses can be studied for \(m = 0\). In this case we get \(M_\rho(m = 0) = g(0)t^\nu\) and a direct measure of \(\nu\) can be obtained by fitting \(M_\rho(m = 0)\) as a power of \(\beta - \beta_c\). Unfortunately the quality of our data does not allow a careful estimate of \(\nu\) in this way because of uncertainties in the extrapolation to \(m = 0\). The point at \(\beta = 0.255\) can be affected by finite volume effects, and the two remaining points alone are not sufficient to allow a meaningful, controlled estimate of \(\nu\). It is necessary to work very close to the transition, because previous studies have shown that in the strong coupling region mean field theory functional dependences are expected. With these caveats it is still interesting to get an independent estimate of \(\nu\) even if it is crude. Our best fit (which is shown in Fig. 10) gives \(M_\rho(m = 0) = A(\beta - 0.260)^{0.675}\) which is nicely consistent with the more sophisticated analysis done in Sec.5.1 above. It also interesting to note that the ratio of the extrapolated masses for the \(\rho\) and the fermion is constant within statistical errors: we obtained \(M_f/M_\rho = 0.434(5)\) at \(\beta = 0.245\), and \(M_f/M_\rho = 0.426(2)\), at \(\beta = 0.250\), in reasonable mutual agreement. This agreement again supports scaling behaviour.

5.3 Critical indices, summary

We present in Table 9 an overview of the critical exponents and the hyperscaling relations. Let us recall that the estimates for \(\beta_{mag}\), δ, η, ν
### Critical indices

| Index | Result from the simulation | Result from HS | MF |
|-------|----------------------------|----------------|----|
| $\beta_{mag}$ | 0.86(3) | 0.86(6) | 0.5 |
| $\gamma = \beta_{mag}(\delta - 1)$ | 1.00(5) | | 1.0 |
| $\delta$ | 2.2(1) | 2.16 | 3.0 |
| $\eta$ | 0.5(1) | 0.5 | 0.0 |
| $\nu$ | 0.675 | 0.68(3) | 0.5 |
| $-4\nu + 2\beta_{mag} + \gamma$ | 0.1(1) | 0.0 | 0.0 |
| $(2 - \eta)\nu/\gamma$ | 1.1(1) | 1.0 | 1.0 |
| $(2\nu - \gamma)/(\nu\eta)$ | 1.1(1) | 1.0 | 1.0 |
| $(6 - \eta)/(2 + \eta) - \delta$ | 0.13(20) | 0.0 | 0.0 |

Table 9: Critical indices and relations among them: in the first column results from the simulation, in the second column the HS(hyperscaling) prediction assuming the other indices as input for the single index entries, in the third column the mean field prediction ($d=4$)

come directly from independent numerical analysis, and $\gamma$ has been computed according to $\gamma = \beta_{mag}(\delta - 1)$. So, in the first column we quote the results of the numerical analysis, and in the second column the results obtained from the hyperscaling relations assuming the other exponents as input. (When we do not have a safe estimate of the errors we simply quote the central values.) The third column shows the mean field results (we recall that $\gamma = 1.0$ is also the theoretical prediction). We summarize in the last four rows of Table 9 the hyperscaling relations we used.

We believe that these results support a picture of non-trivial critical behaviour consistent with hyperscaling, with a large anomalous dimension $\eta$.

### 6 Spectroscopy: Results

We devote this Section to the discussion of the Spectroscopy results presented in Tables. 3 – 7.
6.1 The Goldstone Character of the Pion

Our first task is to check that the pion of quenched QED has all the attributes of a Nambu-Goldstone particle associated with spontaneous breakdown of a continuous chiral symmetry. These properties include 1. the pion should be massless in the strong coupling phase where chiral symmetry is broken, 2. the square of the pion mass should be proportional to the bare mass of the fermion if that mass is sufficiently small, and 3. PCAC relations should hold for \( f_\pi, m, M_\pi, \) and \( \langle \bar{\psi}\psi \rangle \).

Using the data in Table 3 we checked that \( M_\pi^2 = Am + Bm^2 \) works well (see Section. 2 for more details): this is the usual parametrization for the dependence of the pion mass on the bare quark mass assumed in QCD when the quark masses are not really small. We checked that if we added a constant term to the fit, its best value was consistent with zero.

We also looked at the dependence of the pion mass on the quark mass in a slightly different way, i.e by fitting \( M_\pi = Am^x \) (Fig. 11). We see that in the strong coupling region the slopes of the log-log plots for different \( \beta \)'s are quite similar, and roughly consistent with a square root dependence of \( M_\pi \) on \( m \) (the actual values of the slopes are 0.478, 0.495, 0.547, 0.610, 0.678 from \( \beta = 0.245 \) to \( \beta = 0.265 \)). In conclusion, we observe only slight deviations from \( M_\pi \approx \sqrt{(m)} \) in the strong coupling region. These deviations can be eliminated by letting the exponent vary slightly, or, more conventionally, by adding a second order term in the \( m \) expansion. In the weak coupling region the deviations are somewhat more important, but they can be controlled in the same way. In conclusion, the pion has the properties of an ordinary Goldstone boson.

To complete the discussion on PCAC, we show in Table 10 (and in Fig. 12) the results for \( f_\pi^2 = 2m \langle \bar{\psi}\psi \rangle / m_\pi^2 \).

The data for \( \langle \bar{\psi}\psi \rangle \) overlap with the spectrum data for four mass values and four \( \beta \) values. There is sufficient data to see that \( f_\pi^2 \) is nicely linear in \( m \) (the lines in the plot are to guide the eye), and, most importantly, extrapolates to a non-zero value.
Table 10: $f_\pi$ as a function of $\beta$ and $m$

6.2 Mass Ratios and Level Ordering

We discuss now the behaviour of some relevant mass ratios: we present them in a scale invariant form in Figs. 13-16, where we plot only the points belonging to strong coupling region: the squares are for $\beta = 0.255$, the diamonds for $\beta = 0.250$, the crosses for $\beta = 0.245$. Note that since we used the same values of quark masses at the different $\beta$ values the physical mass region we explore changes with $\beta$: near the critical point the physical masses are bigger.

Let us first discuss the behaviour of $\rho/\text{fermion}$. The plot (Fig. 13) shows nice scaling (i.e. all the points fall on top of each other). Some deviations are observed near the critical point (but still the data points are on the same line within errors), which would suggest a $\rho$ lighter than twice the fermion mass in the chiral limit. However, practical difficulties, discussed in Sec.3 above, which effect the fits of the fermion propagator at small mass near criticality, together with the good agreement of the scaled data at $\beta = 0.245$ and $\beta = 0.250$, suggest that the leftmost point at $\beta = 0.255$ be discarded so that $M_\rho(m = 0) \geq 2M_f(m = 0)$.

The data for $\sigma/\text{fermion}$ (Fig. 14) are particularly interesting for technicolor applications and show nice scaling as well (unfortunately the statistical errors are large). The results in the figure suggest that in the chiral limit the sigma mass is less than twice the dynamical mass of the fermion. However, the deviations from the NJL result (that the sigma is twice as heavy as the fermion) are not very large and are partially masked by statistical errors. As discussed in Sec.2 above, and in a deviation from the NJL result for this ratio implies non-trivial behaviour.

The $a_1$ and $\rho$ (Fig. 15) are almost degenerate in the large mass region. Some deviations to scaling are observed at $\beta = 0.245$: they can be ascribed
both to true scaling violations, and finite spacing effects (note that the magnitude of finite spacing effects depend on masses in lattice, not in physical units). The figure suggests that the degeneracy of the $a_1$ and the $\rho$ is resolved in the chiral limit, and the $a_1$ is the heavier state.

The last plot shows the ratio between the $\sigma$ and $\rho$ (Fig. 16), and no deviations in scaling are observed.

In conclusion, our data show some evidence for asymptotic scaling (continuum behaviour), and suggest the level ordering $0 \leq \sigma \leq 2 \times$ fermion $\leq \rho \leq a_1$.

7 Conclusion

The results presented here on spectroscopy and critical indices of quenched lattice QED are in good agreement with analytic calculations based on the continuum formulation of the model. The hypothesis of correlation length scaling seems well verified by our data which shows universal behaviour wherever it is expected. Our analysis of the relation $<\bar{\psi}\psi> \propto M_\rho^{\beta_{mag}/\nu}$ was particularly precise and gave a value $\beta_{mag}/\nu = 1.25(1)$ in excellent agreement with our past interpretation of lattice studies of the model’s EOS – that the lattice Action corresponds to the point $(\alpha, G) = (0.44 \alpha_c, 3.06)$ on the renormalization group fixed line of the continuum model.

Our spectroscopy results illustrate the type of techni-meson masses one should expect in a class of models of the top quark. It was particularly interesting that our data favored the inequality $M_{\sigma} < 2M_{f}$ expected of a theory with unscreened vector forces rather than the familiar NJL result that the techni-sigma lies at the two fermion threshold. Earlier analytical work on quenched planar QED had suggested that the critical point would be characterized by a massless dilaton. This picture is not supported by our data. Later analytic work recognized that dilaton current conservation is broken by hard operators and the techni-sigma has no reason to be light. “Realistic” Technicolor and Extended Technicolor models of the dynamically generated mass of the top quark are not left-right symmetric like the lattice model considered here. There are proposals in the literature for incorporating chiral fermions into lattice regulated theories. If any of these proposals prove theoretically sound, and computationally practical, then it would be interesting to generalize the calculations of this paper to other, more realistic models of
the dynamics behind the Standard Model.

Instead of reiterating the results of Secs. 4-6 above, we close with a short discussion of the puzzling features of the quenched model and our simulations. Since the underlying photon dynamics is perfectly free, the quenched model does not satisfy all the consistency conditions of a bona-fide field theory. However, we do not yet clearly understand the model’s limitations. In fact, we found that its light fermion-meson sector satisfies all the scaling laws expected of complete theories. One peculiarity however is the observation that the simulations in the weak coupling phase do not display all the expected results. For example, since chiral symmetry is not broken at weak coupling the pion and the sigma should be degenerate, and the $\rho$ and $a_1$ also should be degenerate here. No evidence was found for this symmetry restoration in our spectroscopy data. This failure could just be technical – we observed that the usual fitting procedures were not compelling at weak coupling, so other functional forms of the meson propagator’s spatial dependence should be investigated. Another problem with the data concerns the dynamical fermion mass. We found that this mass “flattens” out as the bare fermion mass is taken small, so it does not have the same systematic behaviour as the meson masses. We suspect that this result is due to the fact that the fermion carries an unscreened charge and therefore experiences the long range interactions of the free photons. Under these conditions its propagator probably suffers from especially large finite size effects which push the dynamical mass estimates up when the bare fermion mass becomes too small. It would be useful to pursue this point and obtain better control over the fermion mass since it plays such a central role in theoretical developments.

And finally, lattice investigations of quenched QED must be generalized to move away from the critical point $(\alpha, G) = (\alpha_c, 3.06)$. We must learn how to tune the lattice Action to move along the critical line predicted by the analytic calculations. If this cannot be done, then there may be some ingredient missing in the analytic calculations, or some unsuspected limitation in the lattice approach. From a physics perspective, it would be particularly interesting to tune the lattice Action and watch the critical indices vary continuously as we move from the free field NJL point, to the super-critical, essential singularities of the Miransky point.
Acknowledgement

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Figure Captions

1 a) Pion propagators at $\beta = 0.260$ as a function of euclidean time for the five mass values. The solid lines superimposed are the results of the fits for $5 \leq t \leq 16$. Fig. 1b is the same as Fig. 1a, at $\beta = 0.250$.

2 a), b) As Figs. 1, for the $\rho$.

3 a) $\sigma$ propagators at $\beta = 0.250$ as a function of euclidean time for $m = 0.002$. The solid line superimposed are the result of the fits for $5 \leq t \leq 16$. Fig. 3b is the same as Fig. 3a, for $m = 0.006$.

4 Results for the chiral limit for $(\text{fermion}, \sigma, \rho, a_1)$ (circles, diamonds, squares, crosses) (obtained according to $M^2 = M^2_0 + am + bm^2$) as a function of $\beta$.

5 $\log < \bar{\psi}\psi >$ vs $\log m$ at $\beta$ values $0.257$ (squares) and $\beta = 0.255$ (circles). Data from table 1.

6 a) Chiral equation of state for $\beta_c = 0.257, \delta = 2.2$ and $\beta_{mag} = 0.833$ on the $16^4$ lattice. $(\beta_c - \beta)/< \bar{\psi}\psi >^{1/\beta_{mag}}$ is plotted versus $< \bar{\psi}\psi >/m^{1/\delta}$. We plot only points in the strong coupling region. b) Chiral equation of state on the $24^4$ lattice for the same $\beta_c, \delta$ and $\beta_{mag}, m/< \bar{\psi}\psi >^\delta$ is plotted versus $(\beta_c - \beta)/< \bar{\psi}\psi >^{1/\beta_{mag}}$. All the points are shown.

7 Log-log plot of $< \bar{\psi}\psi >$ vs $M_\rho$. (Crosses, diamonds, circles, squares) are for $\beta = (0.260, 0.255, 0.250, 0.245)$.

8 $< \bar{\psi}\psi >^{\beta_{mag}/\nu}$ vs $M_\rho$. $\beta_{mag}/\nu = (1, 0.86/0.67, 0.86/0.5)$ (dash, solid, dot line)

9 $M_\rho/(\beta_c - \beta)^\nu$ vs. $m/(\beta_c - \beta)^{\beta\delta}$ for $\nu = 0.67, \beta = 0.84, \delta = 0.22$.

10 $M_\rho(m = 0)$ vs. $\beta$. The solid line is the best fit $M_\rho = A(\beta - 0.260)^{0.675}$

11 $\log M_\pi$ vs. $\log m$ for $\beta = (0.245, 0.250, 0.255, 0.260, 0.265)$ (top to bottom). The straight lines superimposed are fits to the relation $M_\pi = Am^x$

12 $f_\pi^2$ as a function of $m$ for $\beta = (0.245, 0.250, 0.255, 0.260)$. 

25
13 Scale invariant plot for $(\rho/\text{fermion})^2$ vs $(\pi/\rho)^2$. (Squares, diamonds, crosses) are for $\beta = (0.255, 0.250, 0.245)$.

14 Scale invariant plot for $(\sigma/\text{fermion})^2$ vs $(\pi/\rho)^2$. (Squares, diamonds, crosses) are for $\beta = (0.255, 0.250, 0.245)$.

15 Scale invariant plot for $(a1/\rho)^2$ vs $(\pi/\rho)^2$. (Squares, diamonds, crosses) are for $\beta = (0.255, 0.250, 0.245)$.

16 Scale invariant plot for $(\sigma/\rho)^2$ vs $(\pi/\rho)^2$. (Squares, diamonds, crosses) are for $\beta = (0.255, 0.250, 0.245)$. 
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