Gravity in a stabilized brane world model in five-dimensional Brans-Dicke theory

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Abstract
Linearized equations of motion for gravitational and scalar fields are found and solved in a stabilized brane world model in five-dimensional Brans-Dicke theory. The physical degrees of freedom are isolated, the mass spectrum of Kaluza-Klein excitations is found and the coupling constants of these excitations to matter on the negative tension brane are calculated.

1 Introduction
Nowadays models with extra dimensions with the fundamental energy scale lying in the $TeV$ range are widely discussed in scientific literature. One of the most known and interesting from the phenomenological point of view is the Randall-Sundrum model \cite{1}. It describes two branes interacting with gravity in a five-dimensional space-time and provides an original solution to the hierarchy problem of gravitational interaction \cite{1,2,3}. Nevertheless, the Randall-Sundrum model possesses an essential flaw: the distance between the branes is not fixed by the parameters of the model. This leads to the existence of the massless scalar field – the radion – in the four-dimensional effective theory on the branes. The coupling constant of this field to matter on the negative tension brane, which is assumed to trap the Standard Model fields, appears to be very large, which contradicts experimental data even at the level of classical experiments \cite{2,4}.

This problem was solved by introducing an extra scalar field living in the bulk. The most consistent model was proposed in paper \cite{5}, where exact solutions to equations of motion for the background metric and scalar field were found. The size of the extra dimension is defined by the boundary conditions on the branes. Nevertheless, the background solution for the metric in this model is more complicated compared to the simple solution in the Randall-Sundrum model. It turned out that the corresponding linearized equations of motion in this model can be solved analytically only for a certain choice of the model parameters, i.e. when the background solution for the metric can be approximated by the solution of the unstabilized Randall-Sundrum model \cite{6}. A question arises: is there a stabilized solution in the system of two branes admitting the simple Randall-Sundrum solution for the metric? In paper \cite{7} it was found that one can obtain such a solution in the case of non-minimal coupling of stabilizing scalar field to gravity. But the background solution for the scalar field in this paper has a rather complicated form.

One of the standard forms of the non-minimal coupling of scalar field to gravity is the linear interaction with the scalar curvature used in Brans-Dicke theory. It has been found that it is possible to get a stabilized model with two branes in five-dimensional Brans-Dicke theory, admitting the simple Randall-Sundrum solution for the metric, whereas the solution for the scalar field also has the form of a simple exponential function \cite{8}.
In the present paper we study linearized gravity in the stabilized brane world model proposed in [8]. It turned out that due to the simplicity of the background solution, the linearized equations of motion can be solved analytically for any physically interesting choice of the model parameters, contrary to the case of stabilized Randall-Sundrum model [5]. We also calculate the coupling constants of physical degrees of freedom to matter on the negative tension brane, where the hierarchy problem of gravitational interaction is solved, and describe the mass spectrum of Kaluza-Klein modes.

2 Background solution

Let us consider gravity in a five-dimensional space-time \( E = M_4 \times S^1/Z_2 \), interacting with two branes and with the scalar field \( \phi \). Let us denote coordinates in \( E \) by \( \{ x^M \} = \{ x^\mu, y \} \), \( M = 0, 1, 2, 3, 4 \), where \( \{ x^\mu \} \), \( \mu = 0, 1, 2, 3 \) are four-dimensional coordinates and the coordinate \( y \equiv x^4 \), \( -L \leq y \leq L \), corresponds to the extra dimension. The extra dimension forms the orbifold \( S^1/Z_2 \), which is a circle of diameter \( 2L/\pi \) with the points \( y \) and \( -y \) identified. Correspondingly, the metric \( g_{MN} \) and the scalar field \( \phi \) satisfy the orbifold symmetry conditions

\[
g_{\mu\nu}(x,-y) = g_{\mu\nu}(x,y), \quad g_{4\mu}(x,-y) = -g_{4\mu}(x,y), \quad g_{44}(x,-y) = g_{44}(x,y), \quad \phi(x,-y) = \phi(x,y).
\]

The branes are located at the fixed points of the orbifold \( y = 0 \ y = L \).

The action of the model has the form

\[
S = \int d^4x \int_{-L}^{L} dy \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \right] - \int_{y=0} \sqrt{-g} \lambda_1(\phi) d^4x - \int_{y=L} \sqrt{-g} \lambda_2(\phi) d^4x.
\]

Here \( V(\phi) \) is the scalar field potential in five-dimensional space-time, \( \lambda_{1,2}(\phi) \) are scalar field potentials on the branes, \( \omega \) is the five-dimensional Brans-Dicke parameter, \( g_{\mu\nu} \) denotes an induced metric on the branes. The signature of the metric \( g_{MN} \) is chosen to be \((-;++;+;+)\). Subscripts 1 and 2 label the branes.

We consider the following standard form of the background metric

\[
ds^2 = \gamma_{MN} dx^M dx^N = e^{2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2
\]

with \( \eta_{\mu\nu} \) being the flat Minkowski metric, which preserves the Poincaré invariance in four-dimensional flat space-time, and the following form of the background solution for the scalar field

\[
\phi(x, y) = \phi(y).
\]

Functions \( \sigma(y), \phi(y) \) are defined by the equations of motion [8]

\[
\frac{\omega}{\phi} (\phi'' + 4\sigma' \phi') - 4\sigma'' - 10(\sigma')^2 - \frac{1}{2} \frac{dV}{d\phi} - \frac{1}{2} \frac{d\lambda_1}{d\phi} \delta(y) - \frac{1}{2} \frac{d\lambda_2}{d\phi} \delta(y - L) = 0,
\]

\[
6(\sigma')^2 \phi - \frac{1}{2} \left( \frac{\omega}{\phi} (\phi')^2 - V \right) + 4\sigma' \phi' = 0,
\]

\[
3\sigma'' \phi + \frac{\omega}{\phi} (\phi')^2 + \phi'' - \sigma' \phi' + \frac{1}{2} \lambda_1 \delta(y) + \frac{1}{2} \lambda_2 \delta(y - L) = 0.
\]
Here and below $'=\partial_4 = \partial/\partial y$.

Let us consider the scalar field potentials to be

$$V(\phi) = \Lambda \phi, \quad \lambda_{1,2} = \pm \lambda \phi.$$  \hspace{1cm} (8)

In this case functions $\sigma(y), \phi(y)$, which are solution to equations (5)-(7), take the form

$$\sigma = -k|y| + C, \quad \phi = C_1 e^{-u|y|},$$  \hspace{1cm} (9)

where

$$u = \sqrt{-\frac{\Lambda}{(3\omega+4)(4\omega+5)}},$$

$$k = (\omega + 1)u,$$

$$\lambda = 4\sqrt{-\Lambda} \frac{3\omega+4}{4\omega+5}.$$  \hspace{1cm} (10)

Background solution for the metric has the same form as that in the unstabilized Randall-Sundrum model [1]. We consider the case when the matter is located on the brane at $y = L$, like in the Randall-Sundrum model. To this end we take $C = kL$, which makes the four-dimensional coordinates on this brane Galilean (see [2, 3, 4]).

To fix the size of the extra dimension, let us add the following terms to the brane potentials:

$$\Delta \lambda_{1,2} = \frac{\beta_{1,2}^2}{2} (\phi - v_{1,2})^2.$$  \hspace{1cm} (11)

The equations of motion appear to be satisfied provided

$$\phi|_{y=0} = C_1 = v_1, \quad \phi|_{y=L} = v_2.$$  \hspace{1cm} (12)

Thus, the distance between the branes is defined by the boundary conditions for the field $\phi$ and can be expressed through the parameters of the potentials as

$$L = \frac{1}{u} \ln \left( \frac{v_1}{v_2} \right).$$  \hspace{1cm} (13)

We suppose that $v_1 \simeq v_2$ and $uL < 1$. We also suppose that $kL \approx 35$, which can be achieved if $\omega > 35$. Note that the mechanism of stabilization is based on the dependence of the scalar field background solution on the coordinate of the extra dimension. Parameters $v_{1,2}$ of the potentials, made dimensionless by the fundamental five-dimensional energy scale $\Lambda$, should be positive values of the order of $O(1)$, i.e. there should not be any hierarchical difference.

As it was noted above, the five-dimensional scalar field usually minimally couples to gravity in stabilized brane world models [5]. In the case of non-minimal coupling, there exists a conformal transformation transforming the action from the Jordan frame to the Einstein frame, in which scalar field minimally couples to gravity. For the action of form (2) these transformations are presented in [8] (with the coordinate transformations bringing the background metric to the standard form). The difference between the initial and the transformed actions is characterized by the interaction of matter on the branes with gravity, i.e. by the metric which we are supposed to perceive. It appears that if we live in the world, in which five-dimensional scalar field is non-minimally coupled to five-dimensional curvature, it is more convenient to examine linearized gravity using the initial, untransformed action. It allows one to simplify the derivation of the Kaluza-Klein mode mass spectrum and the coupling constants of these modes to matter on the branes. But the use of the conformal transformations allows one to simplify the choice of gauge conditions, which are necessary for isolating the physical degrees of freedom of the theory. We will discuss this point in the next section.
3 Linearized equations of motion and the choice of gauge conditions

To study linearized gravity one should derive the second variation Lagrangian of the model. To this end let us parameterize the metric and the scalar field as

\[ g_{MN}(x, y) = \gamma_{MN}(y) + h_{MN}(x, y), \quad \phi(x, y) = \phi(y) + f(x, y), \]  

where \( \phi(y) \) is the background solution of the scalar field, substitute it into action (2) and retain the terms of the second order in fluctuations (below we will use the short notation \( \phi \) for the background solution \( \phi(y) \)). The corresponding second variation Lagrangian appears to be extremely large and we do not present it here in the explicit form. We only present the linearized equations of motion for fluctuations of metric and stabilizing scalar field, which follow from this Lagrangian:

1. \( \mu\nu \)-component

\[
-\frac{1}{2} \left[ \phi (\partial_\sigma \partial_\rho h_{\mu\nu} - \partial_\mu \partial_\sigma h_{\rho\nu} - \partial_\nu \partial_\rho h_{\sigma\mu} + \partial_\sigma \partial_\nu h_{\rho\mu}) + \phi \partial_\mu \partial_\nu \tilde{h} + \\
+ \phi \partial_\nu \partial_\rho h_{\mu44} - \phi \partial_\mu \partial_\nu h_{\rho44} + \partial_\sigma \partial_\nu h_{\mu44} - 2\sigma' \phi(\partial_\mu h_{\rho44} + \partial_\nu h_{\rho44}) + \\
+ \phi \gamma_{\mu\nu} \left( -\partial_\sigma \partial_\rho \tilde{h} - \partial_\sigma \partial_\rho h_{\mu44} - \partial_\rho \partial_\sigma h_{\mu44} - 4\sigma' \partial_\rho \tilde{h} + 3\sigma' \partial_\sigma h_{\mu44} + \\
+ \partial_\sigma \partial_\rho \sigma_{\sigma\tau} + 2\sigma' \partial_\rho \sigma_{\sigma\tau} + 4\sigma' \partial_\sigma \sigma_{\rho\tau} \right) - 2h_{\mu\nu}(2(\sigma')^2 + \sigma'' \phi + \sigma' \phi') + \\
+ 3h_{\mu44}(4(\sigma')^2 + \sigma'') - \phi'(\partial_\mu h_{\nu44} + \partial_\nu h_{\mu44} - \partial_4 h_{\mu44}) + 2\partial_\mu \partial_\nu f + \\
+ \gamma_{\mu\nu} \phi' \left( \partial_4 h_{\mu44} + 2\sigma' h_{\mu44} - \partial_4 \tilde{h} + 7\sigma' h_{\mu44} - 2\frac{\phi'}{\phi} f' - 8\frac{\phi''}{\phi} f \right) + \\
+ \gamma_{\mu\nu} \left( h_{\mu44} \phi'' - 2f'' - 2\partial_4 \phi f + 2f \phi'' + \\
+ 8f(\sigma')^2 + 2\frac{\phi''}{\phi} f(\phi')^2 - 2\frac{\phi''}{\phi} f \phi'' - 6\sigma' f' \right) \right] = 0,
\]

2. \( \mu4 \)-component

\[
\frac{1}{2} \left[ \phi \partial_4 (\partial_\sigma \tilde{h} - \partial_\sigma h_{\mu44}) + \phi \partial_\rho (\partial_\nu h_{\mu44} - \partial_\mu h_{\nu44}) - 3\sigma' \phi \partial_\mu h_{\mu44} + \\
+ 2\partial_\mu \partial_4 f - 2\sigma' \partial_\mu f - \phi' \partial_\mu h_{\mu44} + 2\frac{\phi''}{\phi} \partial_\mu f \phi' \right] = 0,
\]

3. 44-component

\[
-\frac{1}{2} \left[ \phi \partial_\mu (\partial_\nu h_{\mu44} - \partial_\mu \tilde{h}) + 6\phi' \partial_\mu h_{\mu44} - 3\phi \partial_\mu h_{\mu44} - 12f(\sigma')^2 + \\
+ 12\phi h_{\mu44}(\sigma')^2 + 2\phi'(\partial_\mu h_{\mu44} + 4\sigma' h_{\mu44} - \frac{1}{2} \partial_4 h_{\mu44}) - 2\partial_\mu \partial_\mu f - \\
- 8\sigma' f' - \frac{\phi''}{\phi} (\phi')^2 f + 2\frac{\phi''}{\phi} f' - \frac{\phi''}{\phi} (\phi')^2 h_{\mu44} - \frac{\partial_\mu f}{\partial_\phi} \right] = 0,
\]

4. equation for the field \( f \)

\[
\partial_\mu (\partial_\nu h_{\mu44} - \partial_\mu \tilde{h} + 2\partial_4 h_{\mu44}) - \partial_4 \partial_\mu \tilde{h} + \\
+ \partial_\nu h_{\mu44}(10\sigma' - 2\frac{\phi''}{\phi} \phi') + \partial_4 h_{\mu44}(4\sigma' - \frac{\phi''}{\phi} \phi') + \partial_4 \tilde{h}(5\sigma' + \frac{\phi''}{\phi} \phi') -
\]
\[-h_{44}\left(\frac{dV}{dy} + \frac{1}{2} \frac{d\lambda_1}{dy} \delta(y) + \frac{1}{2} \frac{d\lambda_2}{dy} \delta(y - L)\right) + \]
\[+ f\left(2 \frac{d\omega_0}{\phi} (\phi')^2 - 8 \frac{d\omega_0}{\phi} \sigma' \phi' - 2 \frac{d\omega_0}{\phi} \phi'' - \frac{d^2 V}{d\phi^2} - \frac{d^2 \lambda_1}{d\phi^2} \delta(y) - \frac{d^2 \lambda_2}{d\phi^2} \delta(y - L)\right) + f' \left(8 \frac{d\omega_0}{\phi} \sigma' \phi'\right) + 2 \frac{\omega_0}{\phi} f'' + 2 \frac{\omega_0}{\phi} \partial_\mu \partial_\mu f = 0,\]

where \( h = \gamma_{MN} h^{MN}, \) \( \tilde{h} = \gamma_{\mu\nu} h^{\mu\nu}. \) Note that these equations can also be obtained by linearizing the Einstein equations and the equation for the scalar field, which follow from action (2), in the background defined by solution of form (3), (4).

Let us now discuss the gauge invariance of the linearized theory. It is not difficult to check that the quadratic action is invariant under gauge transformations of the form

\[
h^{(t)}_{MN} = h_{MN} - (\nabla_M \xi_N + \nabla_N \xi_M),
\]
\[
f^{(t)} = f - \phi^4 \xi_4,
\]

where \( \nabla_M \) denotes the covariant derivative with respect to the background metric \( \gamma_{MN} \), if functions \( \xi^M \) satisfy the orbifold symmetry conditions

\[
\xi^\mu(x, -y) = \xi^\mu(x, y), \quad \xi^4(x, -y) = -\xi^4(x, y)
\]

(here \( (t) \) denotes the transformed field). The existence of these gauge transformations is a consequence of the invariance of action (2) under the general coordinate transformations. Analogous gauge transformations were discussed in the case of Randall-Sundrum model without the stabilizing scalar field and in the case of stabilized Randall-Sundrum model [4, 6, 9, 10]. One can use them to isolate the physical degrees of freedom of the fields \( h_{MN} \) and \( f \). Though this problem can be simplified. Indeed, for the case of the minimal coupling of the stabilizing scalar field there were found gauge conditions, which allows one to isolate the physical degrees of freedom in the general case [6]. Using the conformal transformations, supplemented by the coordinate transformations [8], one can obtain from the gauge conditions, corresponding to the minimal coupling of the scalar field, the gauge conditions, corresponding to action (2) and background metric (3):

\[
\partial_4 \left[ \left( h_{44} + \frac{2}{\sigma} \right) e^{2\sigma} \phi^{2/3} \right] = \frac{4}{3} \left( \omega + \frac{4}{3} \right) e^{2\sigma} \phi^4 \phi^4 f,
\]
\[
h_{\mu 4} = 0.
\]

Analogously we can find the substitution, which allows us to diagonalize equations of motion (15)-(18), as well as the second variation Lagrangian:

\[
h_{\mu \nu} = b_{\mu \nu} - \frac{1}{2} \gamma_{\mu \nu} h_{44} - \gamma_{\mu \nu} \frac{f}{\phi},
\]

where \( b_{\mu \nu} \) is a traceless-transverse field.

In gauge (20) and with substitution (21) equation (16) fulfills automatically. Equation (15) takes the form

\[
\frac{1}{2} \left( \partial_\sigma \partial^\sigma b_{\mu \nu} + \frac{\partial^2}{\partial y^2} b_{\mu \nu} \right) - b_{\mu \nu} \left( 2 (\sigma')^2 + \sigma'' + \sigma' \phi' \right) + \frac{\phi'}{2 \phi} b_{\mu \nu} = 0.
\]
Equation for 44-component (17) simplifies considerably, if one rewrites it in the interval \((0, L)\) using a new function \(g = e^{2\sigma(y)}\phi^{2/3}(h_{44}(x, y) + \frac{2}{3}\phi)\) and taking into account the relation between the potential \(V\) and functions \(\sigma, \phi\) (6):

\[
g'' + g' \left( \frac{5}{3} \phi' - 2\sigma' - 2\frac{\phi''}{\phi'} \right) - 2 \left( \frac{\phi'}{\phi} \right)^2 \left( \omega + \frac{4}{3} \right) g + \partial_\mu \partial^\mu g = 0. \tag{23} \]

In terms of function \(g\), substitution (21) and gauge conditions (20) take the form:

\[
h_{\mu\nu} = b_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} e^{-2\sigma} \phi^{-2/3} \frac{f}{\phi}, \tag{24} \]
\[
h_{44} = e^{-2\sigma} \phi^{-2/3} g - \frac{2 f}{3 \phi}, \tag{25} \]
\[
g' = \frac{4}{3} \left( \omega + \frac{4}{3} \right) e^{2\sigma} \frac{\phi'}{\phi^{4/3}} f, \tag{26} \]
\[
h_{\mu4} = 0, \quad \tilde{b} = \gamma_{\mu\nu} b^{\mu\nu} = 0, \quad \partial^\nu b_{\mu\nu} = 0. \tag{27} \]

Note, that the field \(g\) is a superposition of \(h_{44}\) component of the metric fluctuations and the fluctuation \(f\) of the stabilizing scalar field.

Substitution of (24)-(27) into equation (18) results in the equation, which can be obtained by differentiating (23) by \(y\), and in the boundary conditions on the branes:

\[
\left( \frac{\phi''}{\phi'} - \frac{1}{4} \frac{\phi}{(\omega+4/3)} \frac{d^2 \lambda_2}{d\phi^2} + \frac{1}{(3\omega+4)} (\omega \phi'' - 4\sigma') \right) g' - \partial_\mu \partial^\mu g|_{y=+0} = 0, \tag{28} \]
\[
\left( \frac{\phi''}{\phi'} + \frac{1}{4} \frac{\phi}{(\omega+4/3)} \frac{d^2 \lambda_2}{d\phi^2} + \frac{1}{(3\omega+4)} (\omega \phi'' - 4\sigma') \right) g' - \partial_\mu \partial^\mu g|_{y=L-0} = 0. \]

For the case of background solution (9), (10) the boundary conditions simplify considerably and take the form:

\[
\frac{1}{4} \frac{\phi}{(\omega+4/3)} \beta_1^2 g' + \partial_\mu \partial^\mu g|_{y=+0} = 0, \tag{29} \]
\[
\frac{1}{4} \frac{\phi}{(\omega+4/3)} \beta_2^2 g' - \partial_\mu \partial^\mu g|_{y=L-0} = 0. \]

It should be noted that such a simplification of the boundary conditions takes place for a class of background solutions in five-dimensional Brans-Dicke theory, namely, if the equations of motion for the background configuration of the fields can be reduced to first order differential equations [8].

### 4 Mass spectrum of Kaluza-Klein modes and four-dimensional effective Lagrangian

First let us consider the tensor modes of the field \(b_{\mu\nu}\), which satisfies equation (22). To find the mass spectrum and wave functions in the extra dimension, let us represent the field \(b_{\mu\nu}\) as

\[
b_{\mu\nu}(x, y) = \sum_{n=0}^\infty b_{\mu\nu}^n(x) \psi_n(y), \quad \square b_{\mu\nu}^n(x) = m_n^2 b_{\mu\nu}^n(x), \tag{30} \]
where $\Box = \eta^{\mu\nu} \partial_{\mu} \partial_{\nu}$. Substituting this into (22), we obtain
\[
e^{-2\sigma} m_n^2 \psi_n + \psi_n'' - \left(4(\sigma')^2 + 2\sigma'' + 2\sigma' \frac{\phi'}{\phi}\right) \psi_n + \frac{\phi'}{\phi} \psi_n' = 0. \tag{31}\]

The boundary conditions following from this equation can be obtained by integrating (31) in an infinitely small vicinity of the points $y = 0$, $y = L$ and have the form
\[
\psi_n' - 2\sigma' \psi_n|_{y=0} = 0, \tag{32}
\]
\[
\psi_n' - 2\sigma' \psi_n|_{y=L-0} = 0. \tag{33}\]

It follows from the general theory [11] that all eigenvalues of the problem under consideration are real and positive. Thus, the tensor sector does not contain tachyons. The eigenfunctions corresponding to different eigenvalues are orthogonal with the weight function defined by equation (31). The eigenfunctions can be normalized as follows
\[
\int_{-L}^{L} \phi e^{-2\sigma} \psi_n \psi_k dy = \delta_{nk}. \tag{34}\]

The solution for the zero mode is
\[
\psi_0(y) = \sqrt{\frac{k + u/2}{v_1}} \frac{e^{-kL}}{\sqrt{1 - e^{-2kL-uL}}} e^{2kL-2k|y|}, \tag{35}\]
and if $u \ll k$ and $kL \approx 35$
\[
\psi_0(L) \approx \sqrt{\frac{k}{v_1}} e^{-kL}. \tag{36}\]

For the case of massive modes equation (31) in the interval $(0, L)$ can be solved in a standard way (see, for example, [4, 6]) by passing to variable $z = m_n k e^{ky-kL}$, which leads to
\[
\psi_n(z) = z^a \left( AJ_{\alpha}(z) + BJ_{-\alpha}(z) \right), \tag{37}\]
where $J_{\alpha}(z)$ is the Bessel function, $a = \frac{u}{2k}$, $\alpha = 2 + \frac{u}{2k}$. In the next section we will show that one should take $kL \approx 35$ to obtain the (weak) Newtonian gravity on the brane at $L$ retaining a strong five-dimensional gravity. In this case one can use the approximation $z|_{y=0} = \frac{m_n}{k} e^{-kL} \approx 0$ with a good accuracy, which allows one to drop the singular term $J_{-\alpha}(z)$ in $\psi_n(z)$, because $B/A \sim e^{-2\alpha kL}$ and the corresponding corrections are negligible. The boundary condition at $L$ gives us the mass spectrum of tensor Kaluza-Klein modes, which is defined by
\[
J_{\alpha-1} \left( \frac{m_n}{k} \right) = 0. \tag{38}\]

Note that in the limit $u = 0$ we reproduce the equation for the mass spectrum of tensor modes in the unstabilized Randall-Sundrum model $J_1 \left( \frac{m_n}{k} \right) = 0$ [4]. Normalization constant $A$ is defined by formula (34), and the normalized wave functions of massive tensor modes look as follows:
\[
\psi_n(y) = \sqrt{\frac{k}{v_1}} e^{u|y|/2} J_{\alpha} \left( \frac{m_n}{k} e^{k|y|-kL} \right) \frac{J_{\alpha-1} \left( \frac{m_n}{k} \right)}{J_{\alpha} \left( \frac{m_n}{k} \right)}. \tag{39}\]
At the point $y = L$ we get a simple formula

$$
\psi_n(L) = \sqrt{\frac{k}{v_2}}. \quad (40)
$$

Now let us turn to the scalar sector. To find the mass spectrum of the scalar modes defined by equation (22) we represent $g$ as

$$
g(x, y) = e^{ipx} g_n(y), \quad p^2 = -\mu_n^2. \quad (41)
$$

As a result equation (23) and boundary conditions (29) for $g_n(y)$ take the form

$$
g''_n + g'_n \left( \frac{5}{3} \frac{\phi'}{\phi} - 2\sigma' - \frac{2}{3} \frac{\phi''}{\phi'} \right) - \frac{2}{3} \left( \frac{\phi'}{\phi} \right)^2 \left( \omega + \frac{4}{3} \right) g_n + e^{-2\sigma} \mu_n^2 g_n = 0, \quad (42)
$$

$$
\frac{1}{4} \frac{\phi}{(\omega + 4/3) \beta^2_1} g'_n + e^{-2\sigma} \mu_n^2 g_n|_{y=0} = 0, \quad (43)
$$

$$
\frac{1}{4} \frac{\phi}{(\omega + 4/3) \beta^2_2} g'_n - e^{-2\sigma} \mu_n^2 g_n|_{y=L-0} = 0. \quad (44)
$$

Equation (41) is written in the interval $(0, L)$, but it can be combined with the boundary conditions (42), (43), which results in a single equation on the circle $S^1$:

$$
\left( g'_n \frac{\phi^{5/3} e^{-2\sigma}}{\phi'^2} \right)' - \frac{2}{9} (3\omega + 4) e^{-2\sigma} \frac{\phi^{5/3} g_n}{\phi'^1/3} + \frac{\phi^{5/3} e^{-4\sigma}}{\phi'^2} \left( \frac{8(3\omega + 4)}{3\beta_1^2 \phi} \delta(y) + \frac{8(3\omega + 4)}{3\beta_2^2 \phi} \delta(y - L) + 1 \right) \mu_n^2 g_n = 0. \quad (45)
$$

With the help of this equation one can show that

$$
\int_{-L}^{L} dy \left( \frac{\phi^{5/3} e^{-2\sigma}}{\phi'^2} g'_n + \frac{2}{9} (3\omega + 4) e^{-2\sigma} \frac{\phi^{5/3} g_n g_k}{\phi'^1/3} \right) = 0 \quad (46)
$$

for $n \neq k$ (we suppose that $\mu_n \neq \mu_k$),

$$
\int_{-L}^{L} dy \frac{\phi^{5/3} e^{-2\sigma}}{\phi'^2} g_n^2 = \frac{2}{9} (3\omega + 4) e^{-2\sigma} \frac{\phi^{5/3} g_n^2}{\phi'^1/3}. \quad (47)
$$

From (46) it follows that the scalar sector has no zero mode with $\mu_0 = 0$, because if $\mu_0 = 0$, then $g_0(y) \equiv 0$. If $\beta_{1.2}^2 > 0$, then $\mu_n^2 > 0$, i.e. the scalar sector does not contain tachyons. Using the formulas presented above, we will normalize the wave functions as

$$
\mu_n^2 \int_{-L}^{L} dy \frac{\phi^{5/3} e^{-4\sigma}}{\phi'^2} \left( \frac{8(3\omega + 4)}{3\beta_1^2 \phi} \delta(y) + \frac{8(3\omega + 4)}{3\beta_2^2 \phi} \delta(y - L) + 1 \right) g_n g_k = \frac{8(3\omega + 4)}{27} \delta_{nk}. \quad (48)
$$

Five-dimensional scalar field $g(x, y)$ can be represented as a series

$$
g(x, y) = \sum_{n=1}^{\infty} \varphi_n(x) g_n(y), \quad (49)
$$
where four-dimensional scalar fields \( \varphi_n(x) \) have the masses \( \mu_n \).

With (9) and (10) equation (41) in the interval \((0, L)\) takes the form

\[
g''_n + g'_n \left( 2k + \frac{u}{3} \right) - \frac{2}{3} u^2 \left( \omega + \frac{4}{3} \right) g_n + e^{2ky} - 2kL \mu_n^2 g_n = 0. \quad (49)
\]

It can be solved by passing to the variable \( z = \frac{\mu_n k e^{ky} - k L}{u} \). After some calculations (which are absolutely equivalent to those carried out in [6] for the scalar sector of stabilized Randall-Sundrum model), we obtain

\[
g_n(z) = z^q \left( A_n J_\gamma(z) + B_n J_{-\gamma}(z) \right), \quad (50)
\]

with

\[
q = -1 - \frac{u}{6k} = -\frac{6\omega + 7}{6\omega + 6}, \\
\gamma = \sqrt{\frac{2u^2}{3k^2} \left( \omega + \frac{4}{3} \right) + q^2} = \frac{2\omega + 3}{2\omega + 2}.
\]

Substituting (50) into the boundary condition at zero, we can drop the singular term \( \sim J_{-\gamma}(z) \) (as for the tensor sector), i.e. \( B_n = 0 \). The boundary condition at \( L \) defines the mass spectrum and takes the form

\[
J_\gamma \left( \frac{\mu_n}{k} \right) \left[ 1 + \frac{v_2 \beta_2^2 k}{\mu_n^2 (2\omega + 2)} \right] = J_{\gamma - 1} \left( \frac{\mu_n}{k} \right) \frac{3v_2 \beta_2^2}{4\mu_n (3\omega + 4)}. \quad (51)
\]

The wave functions \( g_n(y) \) take the form

\[
g_n(y) = A_n \left( \frac{\mu_n}{k} e^{k|y| - kL} \right)^q J_\gamma \left( \frac{\mu_n}{k} e^{k|y| - kL} \right), \quad (52)
\]

\[
A_n = \frac{u}{3\mu_n J_\gamma \left( \frac{\mu_n}{k} \right)} \left( \frac{\mu_n}{k} \right)^{-q} \times \quad (53)
\]

\[
\times \left[ \frac{3}{8v_2^{1/3} k (3\omega + 4)} \left( 1 - \frac{\gamma^2 k^2}{q^2} + \left( \frac{4(3\omega + 4) \mu_n}{3v_2 \beta_2^2} - \frac{q k}{\mu_n} \right)^2 \right) + \frac{1}{\beta_2^2 v_2^{4/3}} \right]^{-\frac{1}{2}},
\]

where the normalization coefficient \( A_n \) is derived from (47).

To get an effective four-dimensional Lagrangian of the theory (which also allows one to check the correctness of normalization conditions (34) and (47) by checking the coefficients in front of the four-dimensional kinetic terms of the fields), we need the second variation Lagrangian of the model. The necessary part of the second variation Lagrangian is

\[
L_g/\sqrt{-\gamma} = \frac{1}{2} \left( -h^{\mu\nu} \times [eq. (15)]_{\mu\nu} - h^{14} \times [eq. (17)] + f \times [eq. (18)] \right), \quad (54)
\]

where we have taken into account \( h_{\mu4} \equiv 0 \), and [eq. (15)]_{\mu\nu}, [eq. (17)], [eq. (18)] are the left hand sides of equations (15), (17), (18) respectively. This result is not surprising: in fact, formula (54) follows from the definition of the second variation Lagrangian. Thus, in principle, one can obtain the second variation Lagrangian using only linearized equations of motion for the fields.
Substituting (30) and (48) into second variation Lagrangian (54), taking into account (24) - (27), (34), (47) and integrating over the extra dimension, we get

\[ S_{\text{eff}} = -\frac{1}{4} \sum_{k=0}^{\infty} \int d^4x \left( \partial^n b^k,\mu
u \partial_k b^k,\mu
u + m_k^2 b^k,\mu
u b^k,\mu
u \right) - \frac{1}{2} \sum_{k=1}^{\infty} \int d^4x \left( \partial_\nu \varphi_k \partial^\nu \varphi_k + \mu_k^2 \varphi_k \varphi_k \right). \]  

(55)

Thus, we have obtained the effective Lagrangian of the theory, which is the sum of the standard four-dimensional Lagrangians for the scalar and tensor fields. We note that the kinetic terms in the effective Lagrangian have the proper sign, i.e., there are no phantom fields in the four-dimensional effective theory.

5 Interaction with matter

Interaction of four-dimensional fields \( b^a_{\mu
u}(x) \) and \( \varphi_n(x) \) with the Standard Model fields on the branes is described by the interaction of the fluctuations of five-dimensional gravitational field \( h_{\mu
u} \) with matter on the branes, which has the standard form

\[ \frac{1}{2} \int_{B_1} h_{\mu
u}(x,0)T^{\mu\nu}_1 \sqrt{-\text{det} \gamma_\rho(0)} d^4x + \frac{1}{2} \int_{B_2} h_{\mu
u}(x,L)T^{\mu\nu}_2 \sqrt{-\text{det} \gamma_\rho(L)} d^4x, \]  

(56)

\( T^{\mu\nu}_1 \) and \( T^{\mu\nu}_2 \) being energy-momentum tensors of matter on brane 1 and brane 2 respectively.

We restrict ourselves to matter on the brane at \( y = L \), which is supposed to be "our" brane. Substituting expansions (30) and (48) into (56), taking into account (24), (25), (26) and (43), we get the formula describing interaction of tensor and scalar modes with matter on the brane at \( y = L \) in Galilean coordinates on that brane

\[ \frac{1}{2} \int_{B_2} \left( \psi_0(L)b^0_{\mu
u}(x)T^{\mu\nu}_1 + \sum_{n=1}^{\infty} \psi_n(L)b^n_{\mu
u}(x)T^{\mu\nu}_2 - \frac{1}{2} v_2^{-2/3} \sum_{n=1}^{\infty} \left( 1 - \frac{4\mu_n^2}{uv_2\beta^2} \right) g_n(L)\varphi_n(x)T^{\mu}_\mu \right) d^4x. \]  

(57)

The coefficient in front of the zero tensor mode \( b^0_{\mu
u}(x) \) defines the four-dimensional Planck mass on the brane, and with the use of (36) we obtain

\[ M_{Pl} = \psi_0^{-1}(L) \approx \sqrt{\frac{v_1}{k}} e^{kL}. \]  

(58)

If \( \sqrt{\frac{k}{v_1}} \sim 1 \text{ TeV}^{-1} \), whereas \( kL \approx 35 \), then \( M_{Pl} \sim 10^{19} \text{ GeV} \). Thus, the hierarchy problem of gravitational interaction on the brane at \( y = L \) is solved analogously to that in the Randall-Sundrum model. The coupling constants to matter on the brane have the form

\[ \frac{\psi_n(L)}{2} \approx \sqrt{\frac{k}{4v_2}} \sim 1 \text{ TeV}^{-1}, \]  

(59)

where we have used formula (10). When \( \omega > 35 \), one gets \( \alpha \approx 2 \) and the mass spectrum of the tensor modes is approximately the same as that in the unstabilized Randall-Sundrum model [4]. In this case \( m_1 \approx 3.8k \) and can be of the order of \( 3 - 4 \text{ TeV} \).
Now let us turn to the scalar sector. We would like to mention an interesting feature of the coupling constants: in principle it is possible that for appropriate values of the parameters
\[ \mu_j = \sqrt{\frac{uv_2\beta_2^2}{4}} \]
for some \( j \). In this case \( j \)-th mode of the scalar field does not interact with matter on the brane.

Now let us estimate the mass and the coupling constant of the lightest scalar mode, – the radion. To simplify the analysis we use the ”stiff brane potential” limit – \( \beta_2^2 \rightarrow \infty \). Let us suppose that \( \mu_1 < k \). Expanding the Bessel functions in (51) into a series, retaining the terms up to the second order in \( \frac{\mu_1}{k} \) and solving the resulting quadratic equation, we get
\[ \mu_1 \approx \frac{2k}{\sqrt{\omega}}. \]

Since \( \omega > 35 \), then \( \mu_1 < k \), which confirms the validity of the expansions of the Bessel functions.

Note, that if \( \beta_2^2 \rightarrow \infty \), then \( g'_n(L) = 0 \), which follows from the boundary condition (43). Thereby, only the first one of the two terms in (57), describing the interaction of the scalar modes with matter on the brane, remains, and the coupling constants look like
\[ \epsilon_n = -\frac{1}{4}v_2^{-2/3}g_n(L). \]

Taking into account (60) and \( \omega > 35 \), for the lightest mode – the radion – this constant is simply
\[ \epsilon_1 \approx -\frac{1}{2}\sqrt{\frac{k}{15v_2}} \approx -\frac{1}{4}\sqrt{\frac{k}{4v_2}}, \]
which is \( \epsilon_1 \sim 1 \text{TeV}^{-1} \) for \( k^3 \sim v_2 \sim 1 \text{TeV}^3 \). The radion mass in this case can be of the order of hundreds of GeV.

6 Conclusion

In the present paper we discussed a stabilized brane world model in five-dimensional Brans-Dicke theory and found equations of motion for the fields describing excitations above the background solution. A convenient gauge and a substitution were found allowing one to diagonalize the equations of motion and to isolate the physical degrees of freedom of the model. Analogously to the case of the stabilized model with the minimal coupling of scalar field to gravity, the tensor sector decouples from the scalar one. For the background solution (9) we found the mass spectra of tensor and scalar modes and the coupling constants to matter on the brane at \( y = L \), where our world is assumed to be. It was shown that the effective four-dimensional Lagrangian does not contain tachyons and phantom fields. It turned out that, contrary to the stabilized Randall-Sundrum model, the linearized equations of motion for the case of background solution (9) can be solved analytically for all physically interesting values of the parameters of the model, i.e. this model is exactly solvable in the linear approximation. This fact can be useful for obtaining estimates of the influence of the extra dimension on processes on the brane for different values of parameters of the five-dimensional theory.

It was shown that for a certain choice of the model parameters the radion mass may be of the order of hundreds of GeV, the inverse size of the extra dimension and the masses of
tensor excitations being of the order of $TeV$. The coupling constants of massive tensor and scalar modes appear to be of the order of $TeV^{-1}$, whereas the coupling constant of the massless graviton appears to be $\sim M_{Pl}^{-1}$, i.e. the hierarchy problem of gravitational interaction is solved on the brane at $L$.

Finally we would like to note that the explicit form of the background solution for functions $\sigma(y) \phi(y)$ was used only for calculating the mass spectrum and coupling constants. All the results related to the gauge choice, the diagonalization of linearized equations of motion and the structure of tensor and scalar sectors are valid for any scalar field potential in a stabilized brane world model in five-dimensional Brans-Dicke theory with background solution of the form (3), (4).

Acknowledgements

The work was supported by grant of Russian Ministry of Education and Science NS-1456.2008.2. M.S. acknowledges support of grant for young scientists MK-5602.2008.2 of the President of Russian Federation and grant of the ”Dynasty” Foundation.

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