Generalized Euler–Poisson–Darboux equation and singular Klein–Gordon equation

Elina L. Shishkina
Department of Applied Mathematics, Informatics and Mechanics, Voronezh State University, 394006, Universitetskaya pl., 1, Voronezh, Russia
E-mail: ilina_dico@mail.ru

Abstract. Solutions of the Cauchy problem for generalized Euler–Poisson–Darboux equation were obtained by Hankel transform method.

1. Introduction
The Euler–Poisson–Darboux equation is the one of the most extensively studied singular linear hyperbolic equation. This equation is closely related to the spherical mean transform, which integrates functions over spheres of arbitrary radius and centers located on a sphere and to the Riesz potential theory (see [1], [2]). In our article we study generalized Euler–Poisson–Darboux equation with both timelike and spacelike singularities and with a function multiplied by a constant on the right-hand side. In the particular case, such an equation is the singular Klein–Gordon equation and must be interesting for relativists ([3], [4]). We treat the problem with sufficient generality to develop generalized and regular solutions.

We apply Hankel transform method to solve the initial value problem

\[ \sum_{i=1}^{n} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i \partial x_i} \right) - \left( \frac{\partial^2}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t} \right) u = c^2 u, \]

\[ u(x,0;k) = f(x), \quad u_t(x,0;k) = 0, \quad u(x,t), \quad \gamma_i > 0, \quad x_i > 0, \quad i = 1, ..., n, \quad t > 0. \]  \[
\tag{1}
\tag{2}
\]

We will call (1) the generalized Euler–Poisson–Darboux equation. We obtain the distributional solution of (1)–(2) in convenient space. Besides, we give formulas for regular solution of (1)–(2) in particular case of \( k \) and of Cauchy the the singular Klein–Gordon equation.

Equation

\[ \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial^2 u}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t} \right), \quad u = u(x,t) \]

appears in Leonard Euler’s work (see [5] p. 227) and later was studied by Siméon Denis Poisson in [6], by Gaston Darboux in [7] and by Bernhard Riemann in [8].

Alexander Weinstein (see [9]) and then his students from “University of Maryland College Park” made significant progress in studies of (1) when \( \gamma_i = 0, \quad i = 1, ..., n \) and \( c = 0 \) in the classical sense. David Fox (one of the students of Weinstein) obtained the solution of (1)–(2) when \( c = 0 \) and \( k \neq -1, -3, -5, ... \) in terms of the Lauricella function (see [10]). Carrol
and Showalter (see [11]) applied method of transmutation operators based on the ideas of Delsarte to study Euler–Poisson–Darboux type equations. The benefits of the transmutation operators method see [12]. The solutions of the Euler–Poisson–Darboux type equations in the distributional sense were obtained by D. W. Bresters (see [13], [14]). Abstract Euler–Poisson–Darboux equation were studied in [15], [16], [17].

Let expand briefly on each of article sections. First, in the section 1, we give necessary definitions. Namely, we define space of weighted functionals, some weighted functionals connected with quadratic form, generalized convolution operator and Hankel transform. Next, in the section 2, we find explicit representation for Hankel transform of weighted functionals connected with quadratic form. In the section 3 we obtain an explicit form for the solution for arbitrary $k$ (1)–(2). In the same section we show that for $c = 0$ one obtains again the results of [18] and write the solution of a Cauchy for the singular Klein–Gordon equation. Finally, in the section 3 we show that for $k > n + |\gamma|$ distributional solution of (1)–(2) is regular and give some examples.

2. Definitions and propositions

We deal with the subset of the Euclidean space

$$R^n_+ = \{x = (x_1, \ldots, x_n) \in R^n, \ x_1 > 0, \ldots, x_n > 0\}.$$  

Let denote $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and $\Omega$ be finite or infinite open set in $R^n$ symmetric with respect to each hyperplane $x_i = 0, i = 1, \ldots, n,$ $\Omega_+ = \Omega \cap R^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{R^n}_+$ where

$$\overline{R^n}_+ = \{x = (x_1, \ldots, x_n) \in R^n, \ x_1 \geq 0, \ldots, x_n \geq 0\}.$$  

The function $f$ is said to be of class $C^\infty(\overline{\Omega}_+)$ if it has derivatives of all orders on $\overline{\Omega}_+$. Function $f \in C^\infty(\overline{\Omega}_+)$ we will call even with respect to $x_i, i = 1, \ldots, n$ if $\frac{\partial^{k+1}f}{\partial x_i^{k+1}x_j^n|_{x=0}} = 0$ for all nonnegative integer $k$ (see [19], p. 21). Class $C^\infty_{ev}(\overline{\Omega}_+)$ consists of functions from $C^\infty(\overline{\Omega}_+)$ even with respect to each variable $x_i, i = 1, \ldots, n$. Let $C^\infty_{ev}(\overline{\Omega}_+)$ be the space of all functions $f \in C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support. Elements of $C^\infty_{ev}(\overline{\Omega}_+)$ we will call test functions and use the notation $C^\infty_{ev}(\overline{\Omega}_+) = D_+(\overline{\Omega}_+)$. We will use notation $D_+ = D_+(\overline{R^n}_+)$.

As the space of basic functions we will use the subspace of the space of rapidly decreasing functions:

$$S_{ev}(R^n_+) = \left\{ f \in C^\infty_{ev} \ : \ \sup_{x \in R^n_+} |x^\alpha D^{\beta} f(x)| < \infty \ \forall \alpha, \beta \in Z^n_+ \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n),$ $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are integer nonnegative numbers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$ $D^{\beta} = D_{x_1}^{\beta_1} \cdots D_{x_n}^{\beta_n},$ $D_{x_j} = \frac{\partial}{\partial x_j}$.

We deal with multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$ consists of positive fixed reals $\gamma_i > 0, i = 1, \ldots, n,$ $|\gamma| = \gamma_1 + \ldots + \gamma_n$. Let $L_p^s(\Omega_+), 1 \leq p < \infty,$ be the space of all measurable in $\Omega_+$ functions even with respect to each variable $x_i, i = 1, \ldots, n$ such that

$$\int_{\Omega_+} |f(x)|^p x^\gamma dx < \infty,$$

where and further

$$x^\gamma = \prod_{i=1}^{n} x_i^{\gamma_i}.$$
For a real number \( p \geq 1 \), the \( L^p_\gamma(\Omega_+) \)-norm of \( f \) is defined by
\[
||f||_{L^p_\gamma(\Omega_+)} = \left( \int_{\Omega_+} |f(x)|^p x^\gamma \, dx \right)^{1/p}.
\]

Weighted measure of \( \Omega_+ \) is denoted by \( \text{mes}_\gamma(\Omega) \) and is defined by formula
\[
\text{mes}_\gamma(\Omega) = \int_{\Omega} x^\gamma \, dx.
\]

For every measurable function \( f(x) \) defined on \( \mathbb{R}^n_+ \) we consider
\[
\mu_\gamma(f,t) = \text{mes}_\gamma \{ x \in \mathbb{R}^n_+ : |f(x)| > t \} = \int_{\{x \in \mathbb{R}^n_+: |f(x)| > t\}} x^\gamma \, dx.
\]

We will call the function \( \mu_\gamma = \mu_\gamma(f,t) \) a *weighted distribution function* \( |f(x)| \).

A space \( L^\infty_\gamma(\Omega_+) \) is defined as a set of measurable on \( \Omega_+ \) and even with respect to each variable functions \( f(x) \) such as
\[
||f||_{L^\infty_\gamma(\Omega_+)} = \inf_{a \in \Omega_+} \{ \mu_\gamma(f,a) = 0 \} < \infty.
\]

For \( 1 \leq p \leq \infty \) the \( L^p_{\gamma,\text{loc}}(\Omega_+) \) is the set of functions \( u(x) \) defined almost everywhere in \( \Omega_+ \) such that \( uf \in L^p_\gamma(\Omega_+) \) for any \( f \in C_c^\infty(\Omega_+) \). Each function \( u(x) \in L^p_{\gamma,\text{loc}}(\Omega_+) \) will be identified with the functional \( u \in \mathcal{D}_\gamma'(\Omega_+) \) acting according to the formula
\[
(u,f)_\gamma = \int_{\mathbb{R}^n_+} u(x) f(x) x^\gamma \, dx, \quad f \in C_c^\infty(\mathbb{R}^n_+).
\]

Functionals \( u \in \mathcal{D}_\gamma'(\Omega_+) \) acting by the formula 3 will be called *regular weighted functionals*. All other linear functionals \( u \in \mathcal{D}_\gamma'(\Omega_+) \) will be called *singular weighted functionals*.

### 3. Singular Bessel differential operator, Laplace–Bessel operator, Bessel functions and Hankel transform

We will deal with the **singular Bessel differential operator** \( B_\nu \) (see, for example, [19], p. 5):
\[
(B_\nu)f = \frac{\partial^2}{\partial t^2} + \frac{\nu}{t} \frac{\partial}{\partial t} = \frac{1}{t^{\nu}} \frac{\partial}{\partial t} t^{\nu} \frac{\partial}{\partial t}, \quad t > 0,
\]
and the elliptical singular operator or the Laplace–Bessel operator \( \Delta_\gamma \):
\[
\Delta_\gamma = (\Delta_\gamma)_x = \sum_{i=1}^n (B_{\gamma_i})_x = \sum_{i=1}^n \left( \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i} \right) = \sum_{i=1}^n \frac{1}{x_i^{\nu}} \frac{\partial}{\partial x_i} x_i^{\gamma_i} \frac{\partial}{\partial x_i}.
\]

The operator 4 belongs to the class of B–elliptic operators by I. A. Kipriyanovs’ classification (see [19]).

The symbol \( j_\nu \) is used for the normalized Bessel function:
\[
j_\nu(t) = \frac{2^{\nu} \Gamma(\nu + 1)}{t^\nu} J_\nu(t),
\]
where $J_\nu(t)$ is the Bessel function of the first kind of order $\nu$ (see [20]):

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$ 

Using formulas 9.1.27 from [21] we obtain that the function $j_\nu(t)$ is an eigenfunction of a linear operator $B_\nu$:

$$(B_\nu)j_{\nu-1}(\tau t) = -\tau^2 j_{\nu-1}(\tau t). \tag{6}$$

We also will need some other Bessel functions (see [20]). Bessel functions of the second kind $Y_\alpha$ for non–integer $\alpha$ is related to $J_\alpha$ by:

$$Y_\alpha(x) = J_\alpha(x) \cos(\alpha\pi) - J_{-\alpha}(x) \sin(\alpha\pi). \tag{7}$$

In the case of integer order $n$, the function $Y_n$ is defined by taking the limit as a non–integer $\alpha$ tends to $n$,

$$Y_n(x) = \lim_{\alpha \to n} Y_\alpha(x).$$

Hankel functions of the first and second kind $H^{(1)}_\alpha(x)$ and $H^{(2)}_\alpha(x)$, defined by:

$$H^{(1)}_\alpha(x) = J_\alpha(x) + i Y_\alpha(x) \tag{7}$$

and

$$H^{(2)}_\alpha(x) = J_\alpha(x) - i Y_\alpha(x). \tag{8}$$

Modified Bessel functions of the first and second kind $I_\alpha(x)$ and $K_\alpha(x)$ are defined by:

$$I_\alpha(x) = i^{-\alpha}J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \tag{9}$$

$$K_\alpha(x) = \frac{\pi I_{-\alpha}(x) - I_\alpha(x)}{2 \sin(\alpha\pi)}, \tag{10}$$

when $\alpha$ is not an integer and when $\alpha$ is an integer, then the limit is used.

The **multidimensional Hankel (Fourier–Bessel) transform** of a function $f(x)$ is given by (see [22]):

$$F_\gamma[f](\xi) = (F_\gamma)_x[f(x)](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n_+} f(x) j_\gamma(x; \xi) x^\gamma dx,$$

where

$$j_\gamma(x; \xi) = \prod_{i=1}^{n} j_{\gamma_{i}-1}(x_i \xi_i), \quad \gamma_1 > 0, ..., \gamma_n > 0.$$

For $f \in S_{ev}$ inverse multidimensional Hankel transform is defined by

$$F^{-1}_\gamma[\hat{f}(\xi)](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod_{j=1}^{n} \Gamma^{2} \left(\frac{\gamma_{j}+1}{2}\right)} \int_{\mathbb{R}^n_+} j_\gamma(x, \xi) \hat{f}(\xi) \xi^\gamma d\xi.$$

We will use the generalized convolution operator defined by the formula

$$(f \ast g)_\gamma = \int_{\mathbb{R}^n_+} f(y) (y^\gamma T_y g)(x) y^\gamma dy,$$
where $\gamma T^y$ is multidimensional generalized translation

$$
\gamma T^y = \gamma_1 T^y_1 \ldots \gamma_n T^y_n,
$$

each one-dimensional operator $\gamma_i T^y_i$, $i = 1, \ldots, n$ acts according to (see [23])

$$
\gamma_i T^y_i f(x) = \left( \frac{2i+1}{2} \right) \Gamma \left( \frac{1}{2} \right) \times \int_0^\pi f(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos \alpha_i, x_{i+1}, \ldots, x_n}) \sin^{n-1} \alpha_i d\alpha_i.
$$

Based on the multidimensional generalized translation $\gamma T^y$ the weighted spherical mean $M^\gamma_0 [f(x)]$ of a suitable function is defined by the formula

$$
M^\gamma_0 [f(x)] = \frac{1}{|S^+_1(n)|} \int_{S^+_1(n)} \gamma T^y f(x) \theta^\gamma dS,
$$

where $\theta^\gamma = \prod_{i=1}^n \theta_i^\gamma$, $S^+_1(n) = \{ \theta : |\theta| = 1, \theta \in \mathbb{R}^n \}$ and $|S^+_1(n)|_\gamma = \prod_{i=1}^n \frac{\Gamma \left( \frac{2i+1}{2} \right)}{2^{n-1} \Gamma \left( \frac{n+2i}{2} \right)}$. It is easy to see that

$$
M^\gamma_0 [f(x)] = f(x), \quad \frac{\partial}{\partial t} M^\gamma_t [f(x)] \bigg|_{t=0} = 0.
$$

**Lemma 1.** Let $u \in S_{ev}$ then

$$
\textbf{F}_\gamma[\Delta_\gamma f](\xi) = -|\xi|^2 \textbf{F}_\gamma[f](\xi).
$$

**Proof.** We have

$$
\textbf{F}_\gamma[\Delta_\gamma f](\xi) = \int_{\mathbb{R}^n_+} [\Delta_\gamma f(x)] \jmath_\gamma(x; \xi) x^\gamma dx =
$$

$$
= \sum_{i=1}^n \int_{\mathbb{R}^n_+} \frac{1}{x_i^n} \frac{\partial}{\partial x_i} x_i^n \frac{\partial}{\partial x_i} f(x) \jmath_\gamma(x; \xi) x^\gamma dx.
$$

Integrating by parts by variable $x_i$ and using formula (6), we obtain

$$
\textbf{F}_\gamma[\Delta_\gamma f](\xi) = \sum_{i=1}^n \int_{\mathbb{R}^n_+} f(x) \left[ \frac{1}{x_i^n} \frac{\partial}{\partial x_i} x_i^n \frac{\partial}{\partial x_i} \jmath_\gamma(x; \xi) \right] x^\gamma dx =
$$

$$
= \sum_{i=1}^n (-\xi_i^2) \int_{\mathbb{R}^n_+} f(x) \jmath_\gamma(x; \xi) x^\gamma dx = -|\xi|^2 \int_{\mathbb{R}^n_+} f(x) \jmath_\gamma(x; \xi) x^\gamma dx = -|\xi|^2 \textbf{F}_\gamma[f](\xi).
$$

This completes the proof.

**Lemma 2.** The integral $\int_{S^+_1(n)} \jmath_\gamma(r^\theta, \xi) \theta^\gamma dS$ is calculated by the formula

$$
\int_{S^+_1(n)} \jmath_\gamma(r^\theta, \xi) \theta^\gamma dS = \frac{\prod_{i=1}^n \Gamma \left( \frac{2i+1}{2} \right)}{2^{n-1} \Gamma \left( \frac{n+|\xi|}{2} \right)} \frac{j^{n+|\xi|-1}(r|\xi|)}.
$$

(14)
Proof. It is well known that the function \( j_\gamma(x, \xi) \) is related to \( e^{-i(x, \xi)} \) by the formula (see [24], p. 32):

\[
  j_\gamma(r \theta, \xi) = P_\xi^\gamma \left( e^{-ir(\theta, \xi)} \right)
\]

therefore, one can use formula

\[
  \int_{S_1^+(n)} P_\xi^\gamma f((\sigma, \xi)) \sigma^\gamma dS_\sigma = \frac{\prod_{i=1}^n \Gamma \left( \frac{\gamma+1}{2} \right)}{\sqrt{\pi} 2^{n-1} \Gamma \left( \frac{|\gamma|+n-1}{2} \right)} \int_{-1}^1 f(|\xi|p) \left( 1 - p^2 \right)^{\frac{n+|\gamma|-3}{2}} dp.
\]

which is proved in [25]. We obtain

\[
  \int_{S_1^+(n)} j_\gamma(r \theta, \xi) \theta^\gamma dS = \frac{\prod_{i=1}^n \Gamma \left( \frac{\gamma+1}{2} \right)}{\sqrt{\pi} 2^{n-1} \Gamma \left( \frac{|\gamma|+n-1}{2} \right)} \int_{-1}^1 e^{-irp|\xi|} \left( 1 - p^2 \right)^{\frac{n+|\gamma|-3}{2}} dp.
\]

Replacing \( p \) by \(-p\), we get

\[
  \int_{-1}^1 e^{-irp|\xi|} \left( 1 - p^2 \right)^{\frac{n+|\gamma|-3}{2}} dp = \int_{-1}^1 e^{irp|\xi|} \left( 1 - p^2 \right)^{\frac{n+|\gamma|-3}{2}} dp.
\]

It is well known that the above-obtained integral satisfies the next formula (e.g., see [26], formula 2.3.5.3)

\[
  \int_{-a}^a e^{itp} (a^2 - p^2)^{\beta-1} dp = \sqrt{\pi} (2a)^{\beta-\frac{1}{2}} \Gamma(\beta) J_{\beta - \frac{1}{2}}(at).
\]

Therefore

\[
  \int_{S_1^+(n)} j_\gamma(r \theta, \xi) \theta^\gamma dS = \frac{\prod_{i=1}^n \Gamma \left( \frac{\gamma+1}{2} \right)}{\sqrt{\pi} 2^{n-1} \Gamma \left( \frac{|\gamma|+n-1}{2} \right)} \frac{\sqrt{\pi} 2^{\frac{n+|\gamma|-1}{2}} \Gamma \left( \frac{n+|\gamma|-1}{2} \right)}{(r|\xi|)^\frac{n+|\gamma|-1}{2}} J_{\frac{n+|\gamma|-1}{2}}(r|\xi|) =
\]

\[
  = \frac{\prod_{i=1}^n \Gamma \left( \frac{\gamma+1}{2} \right)}{2^{n-1} \Gamma \left( \frac{n+|\gamma|-1}{2} \right)} J_{\frac{n+|\gamma|-1}{2}}(r|\xi|).
\]

That gives (14).

4. Weighted functionals connected with quadratic form and their Hankel transforms

We will use regular weighted functional \( (w^2 - |x|^2)^\lambda_{+,-} \) defined by the formula

\[
  ((w^2 - |x|^2)^\lambda_{+,-}, \varphi)_\gamma = \int_{\{x \in \mathbb{R}^n_+: |x| < w\}} (w^2 - |x|^2)^\lambda \varphi(x) x^\gamma dx, \quad \lambda \in C.
\]

Here and further let \( \varphi \in \mathcal{D}_+ \).

Let \( \lambda \in C, p, q \in \mathbb{N}, n = p + q \) and

\[
  P = |x|^2 - |x'|^2 = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2.
\]
where \( x = (x_1, \ldots, x_n) = (x', x'') \in \mathbb{R}^n_+ \), \( x' = (x_1, \ldots, x_p) \), \( x'' = (x_{p+1}, \ldots, x_{p+q}) \). We define the weighted generalized functions \( P_{\gamma, \pm}^\lambda \) by

\[
(P_{\gamma, +}^\lambda, \varphi)_\gamma = \int_{\{P(x) > 0\}^+} P^\lambda(x) \varphi(x) x^\gamma dx
\]  

and

\[
(P_{\gamma, -}^\lambda, \varphi)_\gamma = \int_{\{P(x) < 0\}^+} (-P^\lambda(x)) \varphi(x) x^\gamma dx,
\]

where \( \{\pm P(x) > 0\}^+ = \{x \in \mathbb{R}^n_+ : \pm P(x) > 0\} \), \( \lambda \in \mathbb{C} \).

If quadratic form \( P = P_1 + iP_2 \) has positive definite imaginary part \( P_2 \) then we can define a single–valued analytic function of \( \lambda \)

\[
P^\lambda = e^{\lambda(\ln|P|+i\arg P)}, \quad 0 < \arg P < \pi.
\]

To the function \( P^\lambda \) we compare the generalized function \( P_{\gamma}^\lambda = (P_1 + iP_2)_\gamma^\lambda \) by formula

\[
(P_{\gamma}^\lambda, \varphi)_\gamma = \int_{\mathbb{R}^n_+} P^\lambda(x) \varphi(x) x^\gamma dx.
\]

Let us consider the non–degenerate quadratic form with real coefficients

\[
P = \sum_{k=1}^{n} a_k x_k^2, \quad a_k \in \mathbb{R}.
\]

Suppose that the quadratic form \( P \) has \( p \) positive summands and \( q \) negative, \( p + q = n \). Let

\[
P = P + iP',
\]

where \( P' \) is positive definite quadratic form with real coefficients. Without loss of generality, we assume

\[
P' = \varepsilon (x_1^2 + \ldots + x_n^2), \quad \varepsilon > 0.
\]

If \( P_{\gamma}^\lambda = (P + iP')_{\gamma}^\lambda \) then \( (P + i0)_{\gamma}^\lambda \) and \( (P - i0)_{\gamma}^\lambda \) when \( \Re \lambda > 0 \) are defined by formulas

\[
(P + i0)_{\gamma}^\lambda = \lim_{\varepsilon \to 0} (P + iP')_{\gamma}^\lambda,
\]

\[
(P - i0)_{\gamma}^\lambda = \lim_{\varepsilon \to 0} (P - iP')_{\gamma}^\lambda,
\]

in which we pass to the limit under the integral sign \( \int_{\mathbb{R}^n_+} P^\lambda \varphi(x^\gamma) dx \). When \( \Re \lambda < 0 \), \( \lambda \neq -k \), \( \lambda \neq -\frac{n+|\gamma|}{2} - k + 1 \), \( k \in \mathbb{N} \) for determining \( (P + i0)_{\gamma}^\lambda \) and \( (P - i0)_{\gamma}^\lambda \) we first apply

\[
B_{\gamma, a}^k(\partial) P_{\gamma}^{\lambda+k} = 4^k (\lambda + 1) \ldots (\lambda + k) \left( \lambda + \frac{n+|\gamma|}{2} \right) \ldots \left( \lambda + \frac{n+|\gamma|}{2} + k - 1 \right) P_{\gamma}^\lambda,
\]

where

\[
B_{\gamma, a}^k = \sum_{k=1}^{n} \frac{1}{a_k} \left( \frac{\partial^2}{\partial x_k^2} + \frac{\gamma_k}{x_k} \frac{\partial}{\partial x_k} \right)
\]

7
and then we pass to the limit as \( \varepsilon \to 0 \) (see [27]).

In addition, we consider similarly defined functions \((c^2 + P + i0)^r\gamma\) and \((c^2 + P - i0)^r\gamma\), where \(P\) is an arbitrary quadratic form and \(c \in R\) does not depend on \(x\).

**Lemma 3.** We have the following formula

\[
(F_\gamma)_x (w^2 - |x|^2)_+^{\lambda, \gamma} = \int_{B^+_n (n)} j_\gamma (x, \xi) (w^2 - |x|^2)^{\lambda, \gamma} \, dx = \{x = r\theta, r = |x|\} = \int_0^w (w^2 - r^2)^{\lambda^*_{n+|\gamma|-1}} \, dr = \int_{S^+_n (n)} j_\gamma (r\theta, x) \theta^\gamma \, dS = \frac{\prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right)}{2^{n-1} \Gamma \left( \frac{n + |\gamma|}{2} \right)} \int_0^w (w^2 - r^2)^{\lambda^*_{n+|\gamma|-1}} (r|x|)^{n+|\gamma|-1} \, dr = |x|^{\gamma - n+|\gamma| - \frac{1}{2}} \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right) \int_0^w (w^2 - r^2)^{\lambda^*_{n+|\gamma|-1}} (r|x|)^{n+|\gamma| - \frac{1}{2}} \, dr.
\]

Using formula 2.12.4.6 from [28] which has the form

\[
\int_0^w (w^2 - r^2)^{\beta-1} J_{\beta}(\mu r) \, dr = \frac{2^{\beta-1} w^\beta \Gamma(\beta)}{\mu^\beta} J_{\beta}(\mu w), \quad w > 0, \, \text{Re} \, \beta > 0, \, \text{Re} \, \nu > -1
\]

we get

\[
\int_0^w (w^2 - r^2)^{\lambda} J_{n+|\gamma|-1}(r|x|) r^\frac{n+|\gamma|}{2} \, dr = \frac{2^{\lambda} w^{\frac{n+|\gamma|}{2} + \lambda} \Gamma(\lambda + 1)}{|x|^\lambda+1} J_{n+|\gamma|+\lambda}(|x|\omega)
\]

for Re \( \lambda > -1 \) and

\[
(F_\gamma)_x (w^2 - |x|^2)_+^{\lambda, \gamma} = \frac{w^{n+|\gamma|+2\lambda} \Gamma(\lambda + 1) \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right)}{2^n \Gamma \left( \frac{n + |\gamma|}{2} + \lambda + 1 \right)} J_{n+|\gamma|+\lambda} (w|x|),
\]

which coincides with (20). So we get (20) for Re \( \lambda > -1 \). For other values of \( \lambda \) such that \( \lambda \neq -1, -2, -3, \ldots \) it can be analytically continued (see [29]).

The residue of \( \frac{(w^2 - |x|^2)^{\lambda}}{\Gamma(\lambda+1)} \) at \( \lambda = -m, m \in N \) is (see [27], [31])

\[
\lim_{\lambda \to -m} \frac{(w^2 - |x|^2)^{\lambda}}{\Gamma(\lambda + 1)} = \delta^{(m-1)}(w^2 - |x|^2)
\]
and for $\lambda = -m$
\[
\left( F_\gamma \right)_x \left[ \frac{(w^2 - |x|^2)^\lambda}{\Gamma(\lambda + 1)} \right](\xi) = \int_{R^m_+} j_\gamma(x, \xi) \delta^{(m-1)}(w^2 - |x|^2) x^\gamma \, dx = \]
\[
= \frac{w^{n+|\gamma|-2m} \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right)}{2^n \Gamma \left( \frac{n+|\gamma|}{2} - m + 1 \right)} j_{2^{|\gamma|-m}}(w|x|).
\]

This completes the proof.

**Lemma 4.** We have the following formulas
\[
F_\gamma(w^2 + P + i0)_{\lambda} = \]
\[
= 2^{\frac{|\gamma|-n}{2} + \lambda + 1} e^{-\frac{1}{2} \pi i} w^{\frac{n+|\gamma|}{2} + \lambda} \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right) \left[ \frac{K_{n+|\gamma|+\lambda} \left( wQ_{\gamma+,+}^2 \right)}{Q_{\gamma+,+}^2 \left( \frac{\gamma_i + 1}{2} \right)} + \frac{i \pi}{2} H^{(1)}_{-\frac{n+|\gamma|}{2} - \lambda} \left( wQ_{\gamma+,-}^2 \right) \right] \] 
(22)

and
\[
F_\gamma(w^2 + P - i0)_{\lambda} = \]
\[
= 2^{\frac{|\gamma|-n}{2} + \lambda + 1} e^{\frac{1}{2} \pi i} w^{\frac{n+|\gamma|}{2} + \lambda} \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right) \left[ \frac{K_{n+|\gamma|+\lambda} \left( wQ_{\gamma+,+}^2 \right)}{Q_{\gamma+,+}^2 \left( \frac{\gamma_i + 1}{2} \right)} - \frac{i \pi}{2} H^{(2)}_{\frac{n+|\gamma|}{2} + \lambda} \left( wQ_{\gamma+,-}^2 \right) \right], \] 
(23)

where $Q = \sum_{i=1}^n \frac{1}{a_i} \xi_i^2$ is the quadratic form dual to $P = \sum_{i=1}^n a_i x_i^2$, $\Delta$ is the determinant of the coefficients of $P$, $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ are the Hankel functions of the first and second kind and $K_\alpha$ is modified Bessel function.

**Proof.** First we consider the Hankel transform of the generalized function $(e^2 + P)_{\lambda}$ for a positive definite quadratic form $P = |x|^2$ and $\text{Re} \, \lambda < \frac{n+|\gamma|}{2}$. Applying (14) we obtain
\[
F_\gamma[(w^2 + P)_{\lambda}^2](\xi) = \int_{R^m_+} j_\gamma(x, \xi) (e^2 + |x|^2)^\lambda x^\gamma \, dx = \{x = r\theta, r = |x|\} = \]
\[
= \int_0^{\infty} (w^2 + r^2)^\lambda r^{n+|\gamma|-1} \, dr \int_{S^1_n} j_\gamma(r\theta, x) \theta^\gamma \, dS = \]
\[
= \prod_{i=1}^n \Gamma \left( \frac{n+1}{2} \right) \int_0^{\infty} (w^2 + r^2)^\lambda j_{n+|\gamma|-2} (r|\xi|) r^{n+|\gamma|-1} \, dr = \]
\[
= \lambda^{1-\frac{n+|\gamma|}{2} - \frac{|\gamma|-n}{2}} \prod_{i=1}^n \Gamma \left( \frac{\gamma_i + 1}{2} \right) \int_0^{\infty} (w^2 + r^2)^\lambda J_{n+|\gamma|-2} (r|\xi|) r^{n+|\gamma|} \, dr.
\]

Using formula 2.12.4.28 from [28]
\[
\int_0^{\infty} x^{\nu+1} (x^2 + z^2)^{-\rho} J_\nu(c x) \, dx = \frac{c^{\rho-1} z^{\nu+1}}{2^{\nu-1} \Gamma(\rho)} \lambda_{-\rho} \nu^{\rho+1} (cz)
\]

9
we obtain

\[ \mathbf{F}_\gamma((w^2 + P)^\lambda_\gamma)(\xi) = \frac{2^{\frac{|\lambda - n|}{2}} w^{\frac{n + |\lambda|}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\xi^{\frac{n + |\lambda|}{2} + \lambda} \Gamma(-\lambda)} K_{\frac{n + |\lambda|}{2} + \lambda}(w|\xi|), \]  

(24)

where \( \lambda < \frac{1 - |\lambda|}{4} \). For other values of \( \lambda \) the Hankel transform \( \mathbf{F}_\gamma(w^2 + P)^\lambda_\gamma \) remains valid by analytic continuation in \( \lambda \).

Now let \( P \) be any real quadratic form. We wish to consider the generalized functions \((w^2 + P + i\varepsilon)^\lambda_\gamma\) and \((w^2 + P - i\varepsilon)^\lambda_\gamma\) defined by

\[ (w^2 + P \pm i\varepsilon)^\lambda_\gamma = \lim_{\varepsilon \to 0} (w^2 + P \pm i\varepsilon P_1)^\lambda_\gamma, \]

where \( \varepsilon > 0 \) and \( P_1 \) is a positive definite quadratic form. According to the uniqueness of analytic continuation, (24) implies that

\[ \mathbf{F}_\gamma((w^2 + P \pm i\varepsilon)^\lambda_\gamma)(\xi) = \frac{2^{\frac{|\lambda - n|}{2}} w^{\frac{n + |\lambda|}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i + 1}{2}\right)}{(Q \mp i\varepsilon)^\frac{1}{2}(\frac{n + |\lambda|}{2} + \lambda) \Gamma(-\lambda)\sqrt{\Delta}} K_{\frac{n + |\lambda|}{2} + \lambda}(w(Q \mp i\varepsilon)^\frac{1}{2}), \]

(25)

where \( Q = \sum_{i=1}^{n} \frac{1}{a_i} \xi_i^2 \) is the quadratic form dual to \( P = \sum_{i=1}^{n} a_i x_i^2 \). Considering the definitions of modified Bessel functions of the first and second kind (9) and (10) we get

\[ \frac{K_{\frac{n + |\lambda|}{2} + \lambda}(w(Q \mp i\varepsilon)^\frac{1}{2})}{(Q \mp i\varepsilon)^\frac{1}{2}(\frac{n + |\lambda|}{2} + \lambda)} = \frac{\pi}{2} \frac{I_{\frac{n + |\lambda|}{2} - \lambda}(w(Q \mp i\varepsilon)^\frac{1}{2}) - I_{\frac{n + |\lambda|}{2} + \lambda}(w(Q \mp i\varepsilon)^\frac{1}{2})}{\sin \left(\left(\frac{n + |\lambda|}{2} + \lambda\right)\pi\right)(Q \mp i\varepsilon)^\frac{1}{2}(\frac{n + |\lambda|}{2} + \lambda)} = \]

\[ = \frac{\pi}{2 \sin \left(\left(\frac{n + |\lambda|}{2} + \lambda\right)\pi\right)} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{w^{2m - \frac{n + |\lambda|}{2} - \lambda}}{2^{2m - \frac{n + |\lambda|}{2} - \lambda} \Gamma \left( m - \frac{n + |\lambda|}{2} - \lambda + 1 \right) (Q \mp i\varepsilon)^m_m - \right) \]

\[ \frac{w^{2m + \frac{n + |\lambda|}{2} + \lambda}}{2^{2m + \frac{n + |\lambda|}{2} + \lambda} \Gamma \left( m + \frac{n + |\lambda|}{2} + \lambda + 1 \right) (Q \mp i\varepsilon)^m_m}. \]

Weighted generalized functions \((Q + i\varepsilon)^\lambda_\gamma\) and \((Q - i\varepsilon)^\lambda_\gamma\) can be expressed through weighted generalized functions \(Q^\lambda_\gamma\) and \(Q^\lambda_{\gamma_\gamma}\) defined by (see [27]):

\[ (Q \mp i\varepsilon)^\mu_\gamma = Q^\mu_{\gamma_\gamma} + e^{\mp \pi i \mu} Q^\mu_{\gamma_\gamma}. \]

(26)

Then we get

\[ \frac{K_{\frac{n + |\lambda|}{2} + \lambda}(w(Q \mp i\varepsilon)^\frac{1}{2})}{(Q \mp i\varepsilon)^\frac{1}{2}(\frac{n + |\lambda|}{2} + \lambda)} = \]

\[ = \frac{\pi}{2 \sin \left(\left(\frac{n + |\lambda|}{2} + \lambda\right)\pi\right)} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{w^{2m - \frac{n + |\lambda|}{2} - \lambda}}{2^{2m - \frac{n + |\lambda|}{2} - \lambda} \Gamma \left( m - \frac{n + |\lambda|}{2} - \lambda + 1 \right) \times \right) \]

\[ \times \left( Q^\mu_{\gamma_\gamma} - \right) - e^{\mp \pi i \left( m - \frac{n + |\lambda|}{2} - \lambda \right)} Q^\mu_{\gamma_\gamma} \left( m - \frac{n + |\lambda|}{2} - \lambda \right) - \]
Noticing that we have Corollary. If $P$ is positive definite then we have

\[ F_{\gamma}(w^2 + P + io)_{\gamma} = \frac{2^{\left|\frac{n}{2} - n\right| + \lambda + 1} e^{-\frac{1}{2} q\pi i w_{\frac{n+1}{2} + \lambda}} \prod_{i=1}^{n} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(-\lambda) \sqrt{\Delta Q_{\gamma,+}^{\left(\frac{n+1}{2} + \lambda\right)}}} F_{\gamma}(w_{\frac{n+1}{2} + \lambda} Q_{\gamma,+}) \]  

(27)

and

\[ F_{\gamma}(w^2 + P - io)_{\gamma} = \frac{2^{\left|\frac{n}{2} - n\right| + \lambda + 1} e^{\frac{1}{2} q\pi i w_{\frac{n+1}{2} + \lambda}} \prod_{i=1}^{n} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(-\lambda) \sqrt{\Delta Q_{\gamma,+}^{\left(\frac{n+1}{2} + \lambda\right)}}} F_{\gamma}(w_{\frac{n+1}{2} + \lambda} Q_{\gamma,+}) \]  

(28)

while if $P$ is negative definite

\[ F_{\gamma}(w^2 + P + io)_{\gamma} = \frac{i\pi 2^{\left|\frac{n}{2} - n\right| + \lambda} e^{-\frac{1}{2} q\pi i w_{\frac{n+1}{2} + \lambda}} \prod_{i=1}^{n} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(-\lambda) \sqrt{\Delta Q_{\gamma,+}^{\left(\frac{n+1}{2} + \lambda\right)}}} H_{\gamma}^{(1)}(w_{\frac{n+1}{2} + \lambda} Q_{\gamma,+}) \]  

(29)

and

\[ F_{\gamma}(w^2 + P - io)_{\gamma} = \frac{i\pi 2^{\left|\frac{n}{2} - n\right| + \lambda} e^{\frac{1}{2} q\pi i w_{\frac{n+1}{2} + \lambda}} \prod_{i=1}^{n} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(-\lambda) \sqrt{\Delta Q_{\gamma,+}^{\left(\frac{n+1}{2} + \lambda\right)}}} H_{\gamma}^{(2)}(w_{\frac{n+1}{2} + \lambda} Q_{\gamma,+}) \]  

(30)

where $Q = \sum_{i=1}^{n} \frac{1}{n} x_{i}^{2}$ is the quadratic form dual to $P = \sum_{i=1}^{n} a_{i} x_{i}^{2}$ and $H_{\gamma}^{(1)}$ and $H_{\gamma}^{(2)}$ are the Hankel functions of the first and second kind and $K_{\alpha}$ is modified Bessel function.
5. Solution of the Cauchy problem using the Hankel transform

The main tool used here consist of using the Hankel transform for solving Cauchy problem for generalized Euler–Poisson–Darboux equation. The greatest advantage of this method is that it gives solutions for all values of \( k \). The case when \( \gamma_i = 0, i = 1, \ldots, n \) was studied in [13].

We will be concerned with the solutions of the following initial value hyperbolic problem

\[
[(\Delta_{\gamma})_x - (B_k)_t] u = c^2 u, \quad k \in R, \quad c > 0, \quad u(x, t; k), \quad x \in R^n_+, \quad t > 0. \tag{31}
\]

\[
u(x, 0; k) = f(x), \quad u_t(x, 0; k) = 0. \tag{32}
\]

We will call (31) the generalized Euler–Poisson–Darboux equation. We are looking for the solution \( u \in S'_{ev}(R^n_+) \times C^2(0, \infty) \) of (31)–(32). Notation \( u \in S'_{ev}(R^n_+) \times C^2(0, \infty) \) means that \( u(x, t; k) \) belongs to \( S'_{ev}(R^n_+) \) by variable \( x \) and belongs to \( C^2(0, \infty) \) by variable \( t \).

**Theorem 1.** The solution \( u \in S'_{ev}(R^n_+) \times C^2(0, \infty) \) of the (31)–(32) for \( k \neq -1, -3, -5, \ldots \) is unique and defined by the formula

\[
u(x, t; k) = C(n, \gamma, k) \left(t^{1-k} (t^2 - |x|^2)_+^{k-n-|\gamma|-1} \right) \frac{j_{k-n-|\gamma|-1}(t^2 - |x|^2)_+^2 \cdot c * f(x)}{\gamma}, \tag{33}\]

where

\[
C(n, \gamma, k) = \frac{2^n \Gamma \left(\frac{k+1}{2}\right)}{\Gamma \left(\frac{k-n-|\gamma|+1}{2}\right) \prod_{i=1}^{\gamma} \Gamma \left(\frac{\gamma_i+1}{2}\right)}.\]

In the case when \( k < 0 \) of the (31)–(32) is not unique. When \( k < 0 \) and \( k \neq -1, -3, -5, \ldots \) the difference between two arbitrary solutions is always of the form

\[
At^{1-k}u(t, x; 2-k), \quad A = \text{const}, \tag{34}\]

where \( u(t, x; 2-k) \) is solution of the Cauchy problem

\[
[(\Delta_{\gamma})_x - (B_{2-k})_t] u = c^2 u, \quad u(x, 0; 2-k) = \psi(x), \quad u_t(x, 0; 2-k) = 0,
\]

\( \psi(x) \) is an arbitrary function or distribution belonging to \( S'_{ev} \). When \( k = -1, -3, -5, \ldots \) a nonunique solution of the Cauchy problem (31)–(32) will contain a terms (51) and

\[
\frac{\frac{e^{\pm \frac{i\pi n}{2} \Pi \Gamma \left(\frac{n+|\gamma|-k+1}{2}\right)}}{2^n \Gamma \left(\frac{1-k}{2}\right) \prod_{i=1}^{n} \Gamma \left(\frac{\gamma_i+1}{2}\right)} t^{1-k} \left(t^2 - |x|^2 \mp i0\right)_+^{k-n-|\gamma|-1} * f(x) }{\gamma}. \tag{35}\]

**Proof.** Applying multidimensional Hankel transform to (31) with respect to the variables \( x_1, \ldots, x_n \) only and using (13) we obtain

\[
\left(\xi^2 + c^2 + \frac{\partial^2}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t}\right) \hat{\nu}(\xi, t; k) = 0, \tag{36}\]

\[
\hat{\nu}(\xi, 0; k) = \hat{f}(\xi), \quad \hat{\nu}_t(\xi, 0; k) = 0, \tag{37}\]

where \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n_+ \) corresponds to \( x = (x_1, \ldots, x_n) \in R^n_+ \), \( |\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 \),

\[
\hat{\nu}(\xi, t; k) = (F_{\gamma})(x; u(x, t; k) \xi^2) = \int_{R^n_+} u(x, t; k \xi \xi^2) d\xi
\]
Besides, we can see that the difference between two different solutions (38) is always of the form

\[ \xi \text{ of problem (31)--(32)}. \]

The inverse transform \( \hat{G}^k(\xi, t) \) was obtained in [13]. We have different solutions for nonnegative and negative values of \( k \), specifically

(i) for \( k \geq 0 \)

\[ \hat{G}^k(\xi, t) = j_{\frac{k+1}{2}}(\sqrt{\xi^2 + c^2} t), \quad (37) \]

(ii) for \( k < 0, k \neq -1, -3, -5, ... \)

\[ \hat{G}^k(\xi, t) = At^{1-k}j_{\frac{k+1}{2}}(\sqrt{\xi^2 + c^2} t) + j_{\frac{k-1}{2}}(\sqrt{\xi^2 + c^2} t), \quad (38) \]

where \( A \) is arbitrary complex number which depend on \( \xi \) and \( c \),

(iii) for \( k = -1, -3, -5, ... \)

\[ \hat{G}^k(\xi, t) = Bt^{1-k}j_{\frac{k+1}{2}}(\sqrt{\xi^2 + c^2} t) - \frac{\pi^\frac{k+1}{2}2^k}{\Gamma\left(1 - \frac{k}{2}\right)} \left[ \sqrt{\xi^2 + c^2} t \right]^\frac{k+1}{2} Y_{\frac{k+1}{2}}(\sqrt{\xi^2 + c^2} t), \quad (39) \]

where \( B \) denotes an arbitrary complex number which depend on \( \xi \) and \( c \).

From (37)--(39) we conclude that problem (31)--(32) has a unique solution for \( k \geq 0 \) only. Besides, we can see that the difference between two different solutions (38) is always of the form

\[ At^{1-k}j_{\frac{k+1}{2}}(\sqrt{\xi^2 + c^2} t). \quad (40) \]

Now let find \( G^k(x, t) = \left( (\mathbf{F}^{-1}_\gamma)_{\xi} \hat{G}^k(\xi, t) \right)(x) \). We call \( G^k(x, t) \) the fundamental solution of problem (31)--(32). The inverse transform \( \left( (\mathbf{F}^{-1}_\gamma)_{\xi} \hat{G}^k(\xi, t) \right)(x) \) is most easily found by considering \( c \) as an additional independent variable. Setting \( \xi' = (\xi_1, ..., \xi_n, c) \) we can write \( \hat{G}^k(\xi, t) = j_{\frac{k-1}{2}}(|\xi'| t) \) for \( k \geq 0 \) and find an inverse Hankel transform of \( j_{\frac{k-1}{2}}(|\xi'| t) \) by variable \( \xi' \) using (20). We obtain

\[ \left( (\mathbf{F}^{-1}_\gamma)_{\xi'} j_{\frac{k-1}{2}}(|\xi'| t) \right)(x') = \frac{2^{n+1} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma'|}{2}\right) \prod_{i=1}^{n+1} \Gamma\left(\frac{\gamma_i+1}{2}\right)} t^{1-k} (t^2 - |x'|^2)^{\frac{k-n-|\gamma'|}{2}}, \]

where \( \gamma' = (\gamma_1, ..., \gamma_n, \gamma_{n+1}) \) and \( \gamma_{n+1} \) is an arbitrary positive number, \( x' = (x, \sigma) \), \( \sigma \in R_+ \) is dual to \( c \) variable. Now in order to find \( G^k(x, t) \) we need to apply a direct Hankel transform only on a one-dimensional variable \( \sigma \). We have

\[ G^k(x, t) = \frac{2^{n+1} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma'|}{2}\right) \prod_{i=1}^{n+1} \Gamma\left(\frac{\gamma_i+1}{2}\right)} t^{1-k} \left( (\mathbf{F}_n)_{\sigma} (t^2 - x^2 - \sigma^2)^{\frac{k-n-|\gamma'|}{2}} \right)(c) = \]

\[ \text{(specific solution)}. \]
\[ 0 \text{ has the form} \]
\[ k< \]
\[ \text{fundamental solution is no longer concentrated within the part of the sphere} \]
\[ \text{convolutions in exist for arbitrary} \]
\[ \text{Therefore, the solution of (31)-(32) for} \]
\[ \text{will also be a fundamental solution of our problem. Then using (29) and (30) we obtain} \]
\[ \text{It is clear that} \]
\[ \text{where} \]
\[ \text{Consequently, the difference between two arbitrary solutions for} \]
\[ \text{Finally we consider the case} \]
\[ \text{where} \]
\[ \text{and} \]
\[ \text{is solution of the Cauchy problem} \]
\[ \text{and} \]
\[ \text{Finally we consider the case} \]
\[ \text{In this case the solution will be of a different character than the solutions for other values of} \]
\[ \text{It is clear that} \]
\[ \text{and} \]
\[ \text{will also be a fundamental solution of our problem. Then using (29) and (30) we obtain} \]
\[ \text{Since} \]
\[ \text{has its support in the interior of the part of the sphere} \]
\[ \text{when} \]
\[ \text{we may conclude that in the case} \]
\[ \text{the generalized convolutions in exist for arbitrary} \]
\[ \text{However, in the case} \]
\[ \text{the fundamental solution is no longer concentrated within the part of the sphere} \]
Corollary 1.

The solution \( u \in S'_{ev}(R^n_+) \times C^2(0, \infty) \) of the following initial value problem for the singular Klein–Gordon equation

\[
\left[ (\Delta_\gamma) x - \frac{\partial^2}{\partial t^2} \right] v = c^2 v, \quad c > 0, \quad v = v(x, t), \quad x \in R^n, \quad t > 0. \tag{44}
\]

\( v(x, 0) = f(x), \quad v_t(x, 0) = 0, \quad f(x) \in S'_{ev} \tag{45} \)

is

\[
v(x, t; k) = \frac{2^n \sqrt{\pi}}{\Gamma \left( \frac{1-n-|\gamma|}{2} \right) \prod_{i=1}^{n} \Gamma \left( \frac{\gamma_i+1}{2} \right)} t \left( (t^2 - |x|^2)^{\frac{n-|\gamma|}{2} - 1} \right) f(x). \tag{46}
\]

This solution was obtained by letting \( k \) tend to 0 in (50).

The Klein–Gordon equation

\[
\left[ \Delta_z - \frac{\partial^2}{\partial t^2} \right] v = c^2 v, \quad v = v(z, t), \quad z \in R^N \tag{47}
\]

is the most frequently used wave equations for the description of particle dynamics in relativistic quantum mechanics. When we have that function \( v \) is radially symmetric by some groups of variables \( z_1, ..., z_N \) in (47) we obtain (44) with a smaller number of spatial variables. In this case numbers \( \gamma_i, i = 1, .., n \) in (44) will integer.

Corollary 2. \( u \in S'_{ev}(R^n_+) \times C^2(0, \infty) \)

\[
((\Delta_\gamma)_{x} - (B_k)_{t}) u = 0, \quad k \in R, \quad u = u(x, t; k), \quad x \in R^n, \quad t > 0. \tag{48}
\]

\( u(x, 0; k) = f(x), \quad u_t(x, 0; k) = 0. \tag{49} \)

for \( k \neq -1, -3, -5, ... \) is defined by the formula

\[
u(x, t; k) = \frac{2^n \Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{k-n-|\gamma|+1}{2} \right) \prod_{i=1}^{n} \Gamma \left( \frac{\gamma_i+1}{2} \right)} t^{1-k} \left( (t^2 - |x|^2)^{\frac{k-n-|\gamma|}{2} - 1} \right) f(x). \tag{50}
\]

Corollary 3. In the case \( k > n + |\gamma| - 1 \) the integral in (50) integral converges in the usual sense and we obtain unique classical solution of (31)–(32)

\[
u(x, t; k) = A(n, \gamma, k) t^{1-k} \int_0^t \left( t^2 - r^2 \right)^{\frac{k-n-|\gamma|}{2} - 1} \left( c \sqrt{t^2 - r^2} \right)^{n+|\gamma|-1} M_r^n[f(x)] dr, \tag{51}
\]

\[
A(n, \gamma, k) = \frac{2\Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{n+|\gamma|}{2} \right) \Gamma \left( \frac{k-n-|\gamma|+1}{2} \right)}. \tag{52}
\]

Proof. For \( k > n + |\gamma| - 1 \) passing to spherical coordinates we obtain

\[
u(x, t; k) = C(n, \gamma, k) t^{1-k} \int_{B^r_+ (n)} \left( t^2 - |y|^2 \right)^{\frac{k-n-|\gamma|}{2} - 1} \left( (t^2 - |y|^2)^{\frac{1}{2} - c} \right)^{\gamma} T^y f(x) y^\gamma dy = \]
\[ \begin{align*}
&= C(n, \gamma, k) \int_{B_+^n(n)} \left(1 - |y|^2 \right)^{\frac{k-n-|\gamma|}{2}} j_{\frac{k-n-|\gamma|}{2}} \left((1 - |y|^2)^{\frac{1}{2}} \cdot tc\right) \gamma T^\gamma f(x) y^\gamma dy = \\
&= C(n, \gamma, k) \int_{0}^{1} \left(1 - r^2 \right)^{\frac{k-n-|\gamma|}{2}} j_{\frac{k-n-|\gamma|}{2}} \left((1 - r^2)^{\frac{1}{2}} \cdot tc\right) r^{n+|\gamma|-1} dr \int_{S^+_{1}(n)} \gamma T^\theta f(x) \theta^\gamma dS = \\
&= \frac{2\Gamma \left(\frac{k+1}{2}\right)}{\Gamma \left(\frac{n+|\gamma|}{2}\right)} \Gamma \left(\frac{k-n-|\gamma|-1}{2}\right) \int_{0}^{1} \left(1 - r^2 \right)^{\frac{k-n-|\gamma|}{2}} j_{\frac{k-n-|\gamma|-1}{2}} \left((1 - r^2)^{\frac{1}{2}} \cdot tc\right) r^{n+|\gamma|-1} M^\gamma_{r}\{f(x)\}dr = \\
&= \frac{2\Gamma \left(\frac{k+1}{2}\right)}{\Gamma \left(\frac{n+|\gamma|}{2}\right)} \Gamma \left(\frac{k-n-|\gamma|-1}{2}\right) \int_{0}^{t} \left(t^2 - r^2 \right)^{\frac{k-n-|\gamma|-1}{2}} j_{\frac{k-n-|\gamma|-1}{2}} \left((t^2 - r^2)^{\frac{1}{2}} \cdot c\right) r^{n+|\gamma|-1} M^\gamma_{r}\{f(x)\}dr.
\end{align*} \]

**Corollary 4.** In the case \(k > n + |\gamma| - 1\) the solution of

\[ [(\triangle_\gamma) x - (B_k) x] u = 0, \quad k \in R, \quad u = u(x, t; k), \quad x \in R^n_+, \quad t > 0. \quad (52) \]

\[ u(x, 0; k) = f(x), \quad u_t(x, 0; k) = 0 \quad (53) \]

is unique and gives by the formula

\[ u(x, t; k) = A(n, \gamma, k) t^{1-k} \int_{0}^{t} \left(t^2 - r^2 \right)^{\frac{k-n-|\gamma|}{2}} r^{n+|\gamma|-1} M^\gamma_{r}\{f(x)\}dr, \quad (54) \]

\[ A(n, \gamma, k) = \frac{2\Gamma \left(\frac{k+1}{2}\right)}{\Gamma \left(\frac{n+|\gamma|}{2}\right)} \Gamma \left(\frac{k-n-|\gamma|-1}{2}\right). \]

**Remark.** Solution of the problem close to (52)–(53) was obtain in [10] (see also [11] p. 243) when \(k \neq -1, -3, -5, \ldots\) in terms of the Lauricella function ([30], p. 33)

\[ F^{(n)}_{\gamma}(a_1, \ldots, a_n, b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} z_1^{m_1} \cdots z_n^{m_n}}{(c)_{m_1 + \cdots + m_n} m_1! \cdots m_n!}, \]

\[ \max\{|z_1|, \ldots, |z_n|\} < 1. \]

More precisely, in [10] the solution of the problem

\[ \frac{\partial^2 v}{\partial t^2} + k \frac{\partial v}{\partial t} - \sum_{i=1}^{n} \left(\frac{\partial^2 v}{\partial x_i^2} + \frac{\lambda_i}{x_i^2} v\right) = 0, \quad v = v(t, x), \quad (55) \]

\[ v(0, x) = T(x), \quad \frac{\partial v}{\partial t} \bigg|_{t=0} = 0 \quad (56) \]

has the form

\[ v(t, x) = \frac{\Gamma \left(\frac{k+1}{2}\right)}{\pi^{\frac{n}{2}} \Gamma \left(\frac{k-n+1}{2}\right)} \int_{|x - \xi| = |t|} |t|^{1-k} (t^2 - |x - \xi|^2)^{\frac{k-n-1}{2}} T(\xi) \times \]

\[ \times F^{(n)}_{\gamma}(a_1, \ldots, a_n, b_1, \ldots, b_n; \frac{k-n+1}{2}; z_1, \ldots, z_n) dS_\xi, \quad (57) \]
In this case the solution is unique and gives by the formula (51):

\[
[1 + \sqrt{1 - 4\lambda_1} \ldots, a_n = 1 + \sqrt{1 - 4\lambda_n}], \quad b_1 = 1 - \sqrt{1 - 4\lambda_1} \ldots, b_n = 1 - \sqrt{1 - 4\lambda_n},
\]

\[
z_1 = \frac{t^2 - |x - \xi|^2}{2x_1\xi_1}, \ldots, z_n = \frac{t^2 - |x - \xi|^2}{2x_n\xi_n}.
\]

If \( \lambda_k = \frac{2\pi}{k} (1 - \frac{2\pi}{k} i), \ i = 1, \ldots, n \) and \( u = x^2 v = x_1^2 \ldots x_n^2 v \) then we get our problem (52)–(53). As we see, the expression (54) gives a much more convenient formula for solving the problem (52)–(53). Also note that, in [18] in two different ways including \( k = -1, -3, -5, \ldots \).

### 6. Examples

Let consider the Cauchy problem for \( k > n + |\gamma| - 1 \)

\[
[(\Delta_{\gamma})_x - (B_k)u] = c^2 u, \quad u(x, 0; k) = j_\gamma(x; \xi) \quad \text{and} \quad u_t(x, 0; k) = 0.
\]

In this case the solution is unique and gives by the formula (51):

\[
u(x, t; k) = A(n, \gamma, k) t^{1-k} \int_0^t (t^2 - r^2)^{\frac{k-n-|\gamma|-1}{2}} j_{k-n-|\gamma|-1} \left( c\sqrt{t^2 - r^2} \right) r^{n+|\gamma|-1} M_n^\gamma j_\gamma(x; \xi) dr,
\]

\[
A(n, \gamma, k) = \frac{2\Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{n+|\gamma|}{2} \right) \Gamma \left( \frac{k-n-|\gamma|+1}{2} \right)}.
\]

For \( M_n^\gamma j_\gamma(x; \xi) \) we have the next formula (see [32])

\[
M_n^\gamma j_\gamma(x; \xi) = j_\gamma(x; \xi) j_{\frac{n+|\gamma| - 1}{2}}(r|\xi|).
\]

Using (60) and (5) we get

\[
u(x, t; k) = \frac{2\Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{n+|\gamma|}{2} \right) \Gamma \left( \frac{k-n-|\gamma|+1}{2} \right)} t^{1-k} j_\gamma(x; \xi) \times
\]

\[
\times \int_0^t (t^2 - r^2)^{\frac{k-n-|\gamma|-1}{2}} r^{n+|\gamma|-1} j_{k-n-|\gamma|-1} \left( c\sqrt{t^2 - r^2} \right) j_{\frac{n+|\gamma| - 1}{2}}(r|\xi|) dr =
\]

\[
= \frac{2^{\frac{k-1}{2}} \Gamma \left( \frac{k+1}{2} \right)}{c^{\frac{k-n-|\gamma|-1}{2}} |\xi|^{-\frac{n+|\gamma|}{2}-1}} t^{1-k} j_\gamma(x; \xi) \times
\]

\[
\times \int_0^t (t^2 - r^2)^{\frac{k-n-|\gamma|-1}{2}} r^{n+|\gamma|} J_{k-n-|\gamma|-1} \left( c\sqrt{t^2 - r^2} \right) J_{\frac{n+|\gamma| - 1}{2}}(r|\xi|) dr.
\]

Applying formula 2.12.35.2 from [28] of the form

\[
\int_0^t (t^2 - x^2)^{m+\frac{1}{2}} x^{\nu+1+2l} M_\mu (c\sqrt{t^2 - x^2}) J_\nu(hx) dx =
\]
\[ = t^{\mu + \nu - m - l + 1} c^{\mu} h^{\nu} \left( \frac{\partial}{\partial c \partial c} \right)^{m} \left( \frac{\partial}{\partial h \partial h} \right)^{l} \left[ (c^{2} + h^{2})^{\frac{\mu + \nu + m + l + 1}{2}} J_{\mu + \nu + m + l + 1} \left( t \sqrt{c^{2} + h^{2}} \right) \right], \]

\[ t > 0, \quad \text{Re} \nu > -l - 1, \quad \text{Re} \mu > -m - 1. \]

We have \( k = m = 0, \nu = \frac{n + |\gamma|}{2} - 1, \mu = \frac{k - n - |\gamma|}{2} - 1, h = |\xi| \) and

\[
\int_{0}^{t} (t^{2} - r^{2})^{\frac{k - n - |\gamma| - 1}{2}} r^{\frac{n + |\gamma|}{2}} J_{\frac{k - n - |\gamma|}{2} - 1} \left( c \sqrt{t^{2} - r^{2}} \right) J_{\frac{n + |\gamma|}{2} - 1} (r |\xi|) dr =
\]

\[ = t^{\frac{k - 1}{2} c^{\frac{k - n - |\gamma|}{2} - 1}} |\xi|^{\frac{n + |\gamma|}{2} - 1} \frac{\Gamma \left( \frac{k + 1}{2} \right)}{\sqrt{	ext{c}^{2} + |\xi|^{2}}} J_{\frac{k - 1}{2}} \left( t \sqrt{c^{2} + |\xi|^{2}} \right). \]

Therefore

\[ u(x, t; k) = \frac{2^{\frac{k - 1}{2}} \Gamma \left( \frac{k + 1}{2} \right)}{c^{\frac{k - n - |\gamma|}{2} - 1}} t^{\frac{k - 1}{2} c^{\frac{k - n - |\gamma|}{2} - 1}} |\xi|^{\frac{n + |\gamma|}{2} - 1} \frac{\Gamma \left( \frac{k + 1}{2} \right)}{\sqrt{	ext{c}^{2} + |\xi|^{2}}} J_{\frac{k - 1}{2}} \left( t \sqrt{c^{2} + |\xi|^{2}} \right) = \]

\[ = j_{\gamma}(x; \xi) j_{\frac{k - 1}{2}} \left( t \sqrt{c^{2} + |\xi|^{2}} \right). \]

The following three figures show the graphs of the solutions of the last problem for \( n = 1, k = \frac{1}{3}, \gamma = \frac{1}{3} \) and for \( c = 0, c = 1, c = 2, c = 4 \), accordingly.

Figure 1. \( u(x, t; k) = j_{\frac{1}{3}}(x) j_{\frac{1}{3}} \left( t \sqrt{c^{2} + 1} \right), \quad c = 0. \)

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Figure 2. \( u(x, t; k) = j_{-\frac{c}{2}}(x) j_{-\frac{3}{4}} \left( t\sqrt{c^2 + 1} \right), \quad c = 1. \)

Figure 3. \( u(x, t; k) = j_{-\frac{c}{2}}(x) j_{-\frac{3}{4}} \left( t\sqrt{c^2 + 1} \right), \quad c = 2. \)

Figure 4. \( u(x, t; k) = j_{-\frac{c}{2}}(x) j_{-\frac{3}{4}} \left( t\sqrt{c^2 + 1} \right), \quad c = 4. \)

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