Exact phase shifts for atom interferometry

Ch. Antoine and Ch. J. Bordé

Abstract

In the case of an external Hamiltonian at most quadratic in position and momentum operators, we use the ABCDξ formulation of atom optics to establish an exact analytical phase shift expression for atom interferometers with arbitrary spatial or temporal beam splitter configurations. This result is expressed in terms of coordinates and momenta of the wave packet centers at the interaction vertices only.

1 Introduction

Recently atom interferometers have been described by the ABCDξ formalism of Gaussian atom optics which yields an exact formulation of phase shifts taking into account the wave packet structure of atom waves.

For the theory of atom interferometers two basic stages are required:

1. a proper description of the propagation of wave packets between the beam splitters.
2. an adequate modelization of the beam splitters themselves.

The first stage is achieved through the ABCDξ theorem whose main results are briefly recalled in section 2. The second problem is addressed by the ttt theorem which provides a simple model for the phase introduced by the splitting process.

In this paper we give a compact way to express the atom interferometer phase shifts in terms of the coordinates and momenta of the wave packet centers only. For this purpose we derive two new theorems (the four end-points theorem and the phase shift theorem) valid for a Hamiltonian at-most-quadratic in position and momentum operators.

1 E-mail: antoinec@ccr.jussieu.fr
2 E-mail: chbo@ccr.jussieu.fr
2 The ABCD\(\xi\) theorem

In this framework we consider a Hamiltonian which is the sum of an internal Hamiltonian \(H_0\) (with eigenvalues written with rest masses \(m_i\)) and an external Hamiltonian \(H_{\text{ext}}\):

\[
H_{\text{ext}} = \frac{1}{2m} \dot{p}_{\text{op}} \cdot g (t) \cdot \dot{p}_{\text{op}} - \frac{m}{2} \dot{q}_{\text{op}} \cdot \dot{g} (t) \cdot \dot{q}_{\text{op}} - \dot{\Omega} (t) \cdot (\dot{q}_{\text{op}} \times \dot{p}_{\text{op}}) - m \ddot{g} (t) \cdot \ddot{q}_{\text{op}}
\]

where one recognizes several usual gravito-inertial effects: rotation through \(\dot{\Omega} (t)\), gravity through \(\dot{g} (t)\), gradient of gravity through \(\ddot{g} (t)\,... and where \(\dot{g} (t)\) is usually equal to the unity tensor (for simplicity we omit the transposition sign \(\sim\) on vectors).

For a wave packet \(\psi (q, t_1) = \wp (q, t_1, q_1, p_1, X_1, Y_1)\), where \(q_1\) is the initial mean position of the wave-packet, \(p_1\) its initial mean momentum, and \((X_1, Y_1)\) its initial complex width parameters in phase space, one obtains the ABCD\(\xi\) theorem [2]:

\[
\psi (q, t_2) = \int d^3q'.K (q, t_2, q', t_1) \cdot \wp (q', t_1, q_1, p_1, X_1, Y_1)
= e^{i \frac{\hbar}{\bar{\hbar}} S_{\text{cl}} (t_2, t_1, q_1, p_1)} \cdot \wp (q, t_2, q_2, p_2, X_2, Y_2)
\]

where \(K\) and \(S_{\text{cl}}\) are the quantum propagator and classical action respectively, and where \(q_2, p_2, X_2, Y_2\) obey the ABCD law (\(G\) and \(U\) are the representative matrices of \(\dot{g} (t)\) and the rotation \(\dot{\Omega} (t)\), and we write \(A_{21}\) instead of \(A (t_2, t_1)\) for simplicity):

\[
\begin{pmatrix}
q_2 \\
p_2/m
\end{pmatrix} =
\begin{pmatrix}
U_{21} & \xi_{21} \\
G_{21}^{-1} & U_{21} \xi_{21}
\end{pmatrix}
+ \begin{pmatrix}
A_{21} & B_{21} \\
C_{21} & D_{21}
\end{pmatrix}
\cdot
\begin{pmatrix}
q_1 \\
p_1/m
\end{pmatrix}
\]

\[
\begin{pmatrix}
X_2 \\
Y_2
\end{pmatrix} =
\begin{pmatrix}
A_{21} & B_{21} \\
C_{21} & D_{21}
\end{pmatrix}
\cdot
\begin{pmatrix}
X_1 \\
Y_1
\end{pmatrix}
\]

For example the phase of a gaussian wave packet is:

\[
S_{\text{cl}} (t_2, t_1, q_1, p_1) / \hbar + p_2 \cdot (q - q_2) / \hbar + \frac{m}{2\hbar} (q - q_2) \cdot \Re (Y_2 X_2^{-1}) \cdot (q - q_2)
\]

and in this case the main phase shift recorded between \(t_1\) and \(t_2\) is equal to:

\[
S_{\text{cl}} (t_2, t_1, q_1, p_1) / \hbar + p_1 q_1 / \hbar - p_2 q_2 / \hbar
\]
3 The ttt theorem

When the dispersive nature of a laser beam splitter is neglected (wave packets structure preserved), its effect may be summarized by the introduction of both a phase and an amplitude factor (see [13] and [4] for a detailed proof):

\[ M_{ba} e^{-i(\omega t^* - k^* q^* + \varphi^*)} \quad (7) \]

where \( t^* \) and \( q^* \) depend on \( t_A \) and \( q_A \), the mean time and position of the laser wave.

For a temporal beam splitter:

\[
\begin{align*}
  t^* &\equiv t_A \\
  q^* &\equiv q_{cl}(t_A) \\
  k^* &\equiv k \\
  \varphi^* &\equiv \varphi \text{ (laser phase)}
\end{align*}
\quad (8)
\]

For a spatial beam splitter:

\[
\begin{align*}
  q^* &\equiv q_A \\
  t^* \text{ such that } q_{cl}(t^*) &\equiv q_A \\
  k^* &\equiv k + \delta k \\
  \varphi^* &\equiv \varphi + \delta \varphi
\end{align*}
\quad (9)
\]

where \( \delta k \) is the additional momentum transferred to the excited atoms out of resonance, and where \( \delta \varphi \) is a laser phase: \( \delta \varphi \equiv -\delta k \cdot q_A \) (see [4]).

Let us emphasize that these calculations do not rely on the assumption that the splitter is infinitely thin or that the atom trajectories are classical.

4 The four end-points theorem for a Hamiltonian at most quadratic in position and momentum operators

We shall cut any interferometer into as many slices as there are interactions on either arm and thus obtain several path pieces (see section 5). From now on we shall consider systematically pairs of these homologous paths (see Fig. 1) in the case of a Hamiltonian at most quadratic.
These two classical trajectories are labelled by their corresponding mass \((m_\alpha\) and \(m_\beta\)), their initial position and momentum \((q_{\alpha 1}, p_{\alpha 1}, q_{\beta 1}\) and \(p_{\beta 1}\)) and their common drift time \(T = t_2 - t_1\).

Before establishing the first new theorem let us consider the expression of the classical action for the \(\alpha\) path (see [2]):

\[
S_{cl}(t_2, t_1, q_{\alpha 1}, p_{\alpha 1}) = \frac{\dot{q}_{\alpha 2}}{m_\alpha} G^{-1} (A q_{\alpha 1} + B p_{\alpha 1}/m_\alpha) + \frac{L}{m_\alpha} dt + \frac{1}{2} \frac{\dot{\xi}}{G^{-1}} q_{\alpha 2} - \frac{1}{2} \frac{\xi}{G^{-1}} \frac{p_{\alpha 2}}{m_\alpha}
\]

which can be rewritten as:

\[
S_{cl}(t_2, t_1, q_{\alpha 1}, p_{\alpha 1}) = \frac{p_{\alpha 2}}{2 m_\alpha} q_{\alpha 2} - \frac{p_{\alpha 1}}{2 m_\alpha} q_{\alpha 1} - \frac{1}{2} \frac{\dot{\xi}}{G^{-1} U} q_{\alpha 2} - \frac{1}{2} \frac{\xi}{G^{-1}} \frac{p_{\alpha 2}}{m_\alpha}
\]

with the help of the definition of \(q_{\alpha 2}\) and \(p_{\alpha 2}/m_\alpha\) (see [3]). Then we can use the \(\beta\) path to replace \(\dot{\xi} G^{-1}\) with:

\[
\dot{\xi} G^{-1} = \frac{p_{\beta 2}}{m_\beta} - C q_{\beta 1} - D \frac{p_{\beta 1}}{m_\beta}
\]

Consequently we get:

\[
S_{cl}(t_2, t_1, q_{\alpha 1}, p_{\alpha 1}) = \frac{1}{2} \frac{p_{\alpha 2}}{m_\alpha} \frac{p_{\alpha 1}}{m_\beta} - \frac{1}{2} \frac{p_{\beta 2}}{m_\alpha} \frac{p_{\beta 1}}{m_\beta} + h(t_2, t_1) + f(\alpha, \beta)
\]

where \(h(t_2, t_1)\) is independent of positions and momenta and where \(f(\alpha, \beta) = f(\beta, \alpha)\). The same goes for the expression of \(S_{cl}(t_2, t_1, q_{\beta 1}, p_{\beta 1})/m_\beta\) which is obtained by exchanging \(\alpha\) and \(\beta\). Finally we arrive at the first new theorem (a more general demonstration starting with Hamilton principal functions is given in appendix 1):
Theorem 1

\[
\frac{S_{\text{cl}}(t_2, t_1, q_{\alpha 1}, p_{\alpha 1})}{m_{\alpha}} = \frac{1}{2} \left( \frac{p_{\alpha 2}}{m_{\alpha}} + \frac{p_{\beta 2}}{m_{\beta}} \right) q_{\alpha 2} + \frac{1}{2} \left( \frac{p_{\alpha 1}}{m_{\alpha}} + \frac{p_{\beta 1}}{m_{\beta}} \right) q_{\alpha 1}
\]

or equivalently:

\[
\frac{S_{\text{cl}}(t_2, t_1, q_{\alpha 1}, p_{\alpha 1})}{m_{\alpha}} = \frac{1}{2} \left( \frac{p_{\alpha 2}}{m_{\alpha}} + \frac{p_{\beta 2}}{m_{\beta}} \right) q_{\beta 2} + \frac{1}{2} \left( \frac{p_{\alpha 1}}{m_{\alpha}} + \frac{p_{\beta 1}}{m_{\beta}} \right) q_{\beta 1}
\]

which gives the main part of the phase shift expressed with the half sums of the coordinates and the momenta of the four end-points only.

In the case of identical masses \((m_{\alpha} = m_{\beta})\) this expression simplifies to:

\[
\frac{S_{\text{cl}}(t_2, t_1, q_{\alpha 1}, p_{\alpha 1})}{m_{\alpha}} - \frac{p_{\alpha 2}}{m_{\alpha}} q_{\alpha 2} + \frac{p_{\alpha 1}}{m_{\alpha}} q_{\alpha 1}
\]

\[
- \left[ \frac{S_{\text{cl}}(t_2, t_1, q_{\beta 1}, p_{\beta 1})}{m_{\beta}} - \frac{p_{\beta 2}}{m_{\beta}} q_{\beta 2} + \frac{p_{\beta 1}}{m_{\beta}} q_{\beta 1} \right]
\]

\[
= \left( \frac{p_{\beta 2} - p_{\alpha 2}}{m_{\beta}} \right) \left( \frac{q_{\alpha 2} + q_{\beta 2}}{2} \right) - \left( \frac{p_{\beta 1} - p_{\alpha 1}}{m_{\beta}} \right) \left( \frac{q_{\alpha 1} + q_{\beta 1}}{2} \right)
\]

(15)

5 The phase shift theorem for a Hamiltonian at most quadratic in position and momentum operators

In this section we draw on the results of previous sections to establish the interferometer phase shift expression for an arbitrary beam splitters configuration.

For a sequence of pairs of homologous paths (an interferometer geometry) (see Fig. 2)
one can infer the general sum for the main coordinate dependant part of the global phase shift:

\[
\frac{p_{\beta D} - p_{\alpha D}}{\hbar} \left( q - \frac{q_{\alpha D} + q_{\beta D}}{2} \right) - \frac{p_{\alpha 1} + p_{\beta 1}}{2\hbar} \left( q_{\beta 1} - q_{\alpha 1} \right) + \sum_{i=1}^{N} \left( k_{\beta i} - k_{\alpha i} \right) \frac{q_{\alpha i} + q_{\beta i}}{2} \]

(17)

If now we take into account the other terms of the phase shift we finally obtain the following result (given here for a Gaussian wave packet):

**Theorem 2**

\[
\Delta \phi(q, t_{N+1} = t_D) = (p_{\beta D} - p_{\alpha D}) \cdot \left( q - \frac{q_{\alpha D} + q_{\beta D}}{2} \right) / \hbar - \frac{p_{\alpha 1} + p_{\beta 1}}{2\hbar} \left( q_{\beta 1} - q_{\alpha 1} \right) + \sum_{i=1}^{N} \left( k_{\beta i} - k_{\alpha i} \right) \frac{q_{\alpha i} + q_{\beta i}}{2} - \left( \omega_{\beta i} - \omega_{\alpha i} \right) t_i + \varphi_{\beta i} - \varphi_{\alpha i} \right]

+ \sum_{i=1}^{N} \left( \frac{m_{\beta i}}{m_{\alpha i}} - 1 \right) \cdot \left[ \frac{S_{\alpha i}}{\hbar} + \frac{p_{\alpha i+1}}{2\hbar} \cdot (q_{\beta,i+1} - q_{\alpha,i+1}) - \frac{p_{\alpha i} + \hbar k_{\alpha i}}{2\hbar} \cdot (q_{\beta i} - q_{\alpha i}) \right]

+ \frac{m_{\beta,N}}{2\hbar} (q - q_{\beta D}) \cdot \text{Re} \left( Y_D \cdot X_D^{-1} \right) . (q - q_{\beta D})

- \frac{m_{\alpha,N}}{2\hbar} (q - q_{\alpha D}) \cdot \text{Re} \left( Y_D \cdot X_D^{-1} \right) . (q - q_{\alpha D})

(18)

where \( S_{\alpha i} \equiv S_d(t_{i+1}, t_i, q_{\alpha i}, p_{\alpha i} + \hbar k_{\alpha i}, m_{\alpha i}) \).
This fundamental result is valid for a time-dependent Hamiltonian and takes into account all the mass differences which may occur. It allows to calculate exactly the phase shift for all the interferometer geometries which can be sliced as above: symmetrical Ramsey-Bordé (Mach-Zehnder), atomic fountain clocks,... All these particular cases will be detailed in a forthcoming paper (see [6]).

Let us point out that the nature (temporal or spatial) of beam splitters leads to different slicing of the paths. In the spatial case, indeed, the number of different $t^*_i$ may be twice as great as in the temporal case (see the definition of $t^*_i$ in these two different cases in section 3).

6 Phase shift after spatial integration

In an actual interferometer one has to integrate spatially the output wave packet over the detection region. With Gaussian wave packets this integration leads to a mid-point theorem [3]: "The first term of $\Delta \phi (q, t_D)$ disappears when the spatial integration is performed".

Furthermore the terms which depend on the wave packets structure ($Y$ and $X$) vanish when $m_{\beta,N} = m_{\alpha,N}$ (which is always the case). One obtains finally:

$$\Delta \phi (t_D) = -\frac{p_{\alpha 1} + p_{\beta 1}}{2\hbar} (q_{\beta 1} - q_{\alpha 1})$$

$$+ \sum_{i=1}^{N} \left[ (k_{\beta i} - k_{\alpha i}) \frac{q_{\alpha i} + q_{\beta i}}{2} - (\omega_{\beta i} - \omega_{\alpha i}) t_i + \varphi_{\beta i} - \varphi_{\alpha i} \right]$$

$$+ \sum_{i=1}^{N} \left( \frac{m_{\beta i} - m_{\alpha i}}{2\hbar} \right) \right.$$ 

$$\left\{ \left( \frac{S_{\alpha i}}{m_{\alpha i}} + \frac{p_{\alpha, i+1}}{2m_{\alpha i}} \right) (q_{\beta, i+1} - q_{\alpha, i+1}) - \frac{p_{\alpha i} + \hbar k_{\alpha i}}{2m_{\alpha i}} (q_{\beta i} - q_{\alpha i}) \right\}$$

$$+ \left( \frac{S_{\beta i}}{m_{\beta i}} + \frac{p_{\beta, i+1}}{2m_{\beta i}} \right) (q_{\alpha, i+1} - q_{\beta, i+1}) - \frac{p_{\beta i} + \hbar k_{\beta i}}{2m_{\beta i}} (q_{\alpha i} - q_{\beta i}) \right\}$$

7 Identical masses and symmetrical case

The case of identical masses is an important approximation which is commonly used for the modelization of many devices like gravimeters and gyrometers [8].
If $m_{\alpha i} = m_{\beta i} = m$, $\forall i$, this general phase shift becomes:

$$\Delta \phi (t_D) = -\frac{p_{\alpha 1} + p_{\beta 1}}{2h} (q_{\beta 1} - q_{\alpha 1}) + \sum_{i=1}^{N} (k_{\beta i} - k_{\alpha i}) \frac{q_{\alpha i} + q_{\beta i}}{2}$$

$$+ \sum_{i=1}^{N} \left[ \varphi_{\beta i} - \varphi_{\alpha i} - (\omega_{\beta i} - \omega_{\alpha i}) t_i \right]$$

(20)

We can also specify the form of this phase shift when the interferometer geometry is symmetrical (see Fig.3).

**Fig. 3:** A typical symmetrical interferometer

This symmetry is expressed as: $k_{\beta i} + k_{\alpha i} = 0$, $\forall i \in [2, N - 1]$, i.e. it is a symmetry with respect to the direction of the particular vector: $\vec{p}_{\text{initial}} + \frac{\hbar k_{\text{initial}}}{2}$.

Consequently:

$$\Delta \phi (t_N) = k_1 q_1 + 2 \sum_{i=2}^{N-1} k_i \frac{q_{\alpha i} + q_{\beta i}}{2} + k_N \frac{q_{\alpha N} + q_{\beta N}}{2} - \sum_{i=1}^{N-1} (\varphi_{\beta i} - \varphi_{\alpha i})$$

(21)

But $\forall i \in [2, N - 1]$:

$$\frac{q_{\alpha i+1} + q_{\beta i+1}}{2} = \xi_{i+1,i} + A_{i+1,i} \cdot \frac{q_{\alpha i} + q_{\beta i}}{2} + \frac{B_{i+1,i}}{m} \cdot \frac{p_{\alpha i} + p_{\beta i}}{2}$$

$$= \xi_{i+1,i} + A_{i+1,i} \cdot q_1 + \frac{B_{i+1,i}}{m} \cdot (p_1 + \frac{\hbar k_1}{2})$$

$$\equiv Q (t_{i+1})$$

(22)

which can be calculated with the ABCD$\xi$ law.

It depends only on $q_1$ (initial position) and $p_1 + \frac{\hbar k_1}{2}$ (“Bragg initial momentum”).
Therefore:

\[ \Delta \phi (t_N) = \sum_{i=1}^{N} (k_{\beta i} - k_{\alpha i})Q(t_i) - \sum_{i=1}^{N-1} (\varphi_{\beta i} - \varphi_{\alpha i}) \] (23)

which has a very simple form when the origin of coordinates is chosen such that \( q_1 = 0 \), and when the Bragg condition \( p_1 + \frac{\hbar k_1}{2} = 0 \) is satisfied.

8 Conclusion

In this paper we have used the ABCDξ formulation of atom optics and the \( \text{ttt} \) theorem to establish two theorems valid for a time-dependent Hamiltonian at most quadratic in position and momentum operators. The first one gives a compact expression of the action difference between two homologous paths. The second one gives an analytical expression of the global phase shift for atom interferometers in the case of such a Hamiltonian.

Consequently this analytical expression provides a simple way to calculate exactly the phase shift in this case, and then to calculate perturbatively for example the effect of a third-order term in the external Hamiltonian (necessary for space missions like HYPER [10]). For example, one can calculate exactly the global phase shift due to gravity plus a gradient of gravity plus a rotation, and then calculate perturbatively the effect of a gradient of gradient of gravity. These calculations and the application to specific cases (gravimeters, gyroimeters, atomic clocks...) will be detailed in a forthcoming article [6] where we recover well-known perturbative results ([5], [9], [11], [12]) from exact expressions.

9 Appendix 1

In a case of a Hamiltonian at most quadratic in position and momentum operators, the Hamilton principal functions concerning two pairs of homologous points are also at most quadratic in positions (owing to the Hamilton-Jacobi equation, see [2]):

\[ S_\alpha (q_{\alpha 1}, q_{\alpha 2}) / m_\alpha = a + b.q_{\alpha 1} + c.q_{\alpha 2} + q_{\alpha 1}.d.q_{\alpha 1} + q_{\alpha 1}.e.q_{\alpha 2} + q_{\alpha 2}.f.q_{\alpha 2} \] (24)

\[ S_\beta (q_{\beta 1}, q_{\beta 2}) / m_\beta = a + b.q_{\beta 1} + c.q_{\beta 2} + q_{\beta 1}.d.q_{\beta 1} + q_{\beta 1}.e.q_{\beta 2} + q_{\beta 2}.f.q_{\beta 2} \] (25)

where \( a \) is a scalar, \( b \) and \( c \) are vectors, and \( d, e \) and \( f \) are matrices (see [3]).
We can define $p_{\alpha 1}$, $p_{\alpha 2}$, $p_{\beta 1}$, $p_{\beta 2}$ such that:

$$p_{\alpha 1} \equiv -\nabla q_{\alpha 1} \left( \frac{S_{\alpha}}{m_{\alpha}} \right) = -b - 2d.q_{\alpha 1} - e.q_{\alpha 2} \quad (26)$$

$$p_{\alpha 2} \equiv \nabla q_{\alpha 2} \left( \frac{S_{\alpha}}{m_{\alpha}} \right) = c + 2f.q_{\alpha 2} + \tilde{e}.q_{\alpha 1} \quad (27)$$

$$p_{\beta 1} \equiv -\nabla q_{\beta 1} \left( \frac{S_{\beta}}{m_{\beta}} \right) = -b - 2d.q_{\beta 1} - e.q_{\beta 2} \quad (28)$$

$$p_{\beta 2} \equiv \nabla q_{\beta 2} \left( \frac{S_{\beta}}{m_{\beta}} \right) = c + 2f.q_{\beta 2} + \tilde{e}.q_{\beta 1} \quad (29)$$

and obtain the following expression:

$$\frac{S_{\alpha}}{m_{\alpha}} - \frac{S_{\beta}}{m_{\beta}} = \frac{1}{2} (p_{\alpha 2} + p_{\beta 2}) \cdot (q_{\alpha 2} - q_{\beta 2}) - \frac{1}{2} (p_{\alpha 1} + p_{\beta 1}) \cdot (q_{\alpha 1} - q_{\beta 1}) \quad (30)$$

The same relation holds for the classical action concerning two actual paths with a common drift time (homologous paths). This yields an other demonstration of the first theorem expressed in section 4.

References

[1] Atom interferometry, ed. P. Berman, Academic Press (1997)
[2] Ch. J. Bordé, Theoretical Tools for atom optics and interferometry, C.R. Acad. Sci. Paris, t.2, Série IV, p509 (2001)
[3] Atomic clocks and inertial sensors, Metrologia, 39, in press (2002)
[4] Ch. J. Bordé, An elementary quantum theory of atom-wave beam splitters: the ttt theorem, Lecture notes for a mini-course, Institut für Quantenoptik, Universität Hannover (2002) and to be published
[5] J. Audretsch and K.-P. Marzlin, Atom interferometry with arbitrary laser configurations : exact phase shift for potentials including inertia and gravitation, J. Phys. II (France) 4 (1994) 2073
[6] Ch. Antoine and Ch. J. Bordé, in Journal of Optics B, in preparation (2003)
[7] A. Peters, K.Y. Chung and S. Chu, High-precision gravity measurements using atom interferometry, Metrologia 38 (2001) 25
[8] M.J. Snadden, J.M. McGuirk, P. Bouyer, K.G. Haritos and M.A. Kasevich, Measurement of the Earth’s gravity gradient with an atom interferometer-based gravity gradiometer, Phys. Rev. Lett. 81 (1998) 971
[9] P. Wolf and Ph. Tourrenc, *Gravimetry using atom interferometers: Some systematic effects*, Phys. Lett. A 251 (1999) 241

[10] R. Bingham et al., HYPER, *Hyper-Precision Cold Atom Interferometry in Space*, Assessment Study Report, ESA-SCI (2000)

[11] Ch.J. Bordé, *Atomic interferometry with internal state labelling*, Phys. Lett. A 140 (1989)

[12] Ch.J. Bordé, *Atomic interferometry and laser spectroscopy*, in: Laser Spectroscopy X, World Scientific (1991) 239-245

[13] J. Ishikawa, F. Riehle, J. Helmcke and Ch.J. Bordé, *Strong-field effects in coherent saturation spectroscopy of atomic beams*, Phys. Rev. A 49 (1994) 4794-4825