Non Commutativity, Fluctuations and Unification

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Abstract
We point out that in the non commutativity and breakdown of conventional spacetime at micro scales lies the seed to the unification of gravitation and electromagnetism.

1 Introduction

Feynman had speculated that gravitation may be a result of some type of a fluctuation, motivated by the fact that the attractive Van der Waal force can be attributed to the dipole moments induced by the fluctuation in molecular distribution [1, 2]. On the other hand the author had pointed out that in a fluctuational scheme, not only do we get a key to electromagnetism, but also we are able to deduce a consistent cosmology in which supposedly accidental large number relations including the so called Weinberg relation between the Hubble constant and the pion mass, arise naturally, as a consequence of the theory [3, 4, 5, 6]. The question is, can gravitation also be included in such a scheme, and if so can this point to the long sought after unification of gravitation and electromagnetism?

We will now argue that indeed this is so - it is an underlying non commutativity of the spacetime that has kept apart gravitation which relies on a smooth spacetime manifold on the one hand, and Quantum Theory on the other hand, though the latter also uses a smooth manifold as an approximation.

Relatively recent work by scholars like Ord, Nottale, El Naschie, the author and others, particularly several papers in Chaos, Solitons and Fractals
have highlighted the non smooth and even fractal nature of spacetime\[^7, 8, 9, 10, 11\]. This again can be the key to Einstein’s much sought after unification\[^12, 13, 14, 15\].

An understanding of the underlying substructure of spacetime is required for such a unification program. Indeed Einstein himself had noted in the Journal of the Franklin Institute,\[^16\] “...it has ben pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature that is to the elimination of continuous functions from physics. Then however, we must also give up, by principle the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At present however, such a program looks like an attempt to breathe in empty space”. Wheeler\[^17\] on the other hand notes, ”the most evident shortcoming of the geometrodynamic model as it stands is this, that it fails to supply any completely natural place for spin $\frac{1}{2}$ in general and for the neutrino in particular”, while ”it is impossible to accept any description of elementary particles that does not have a place for spin half.” Indeed he describes the four dimensional spacetime as a classical approximation.

2 Non Commutativity and Fluctuations

We are beginning to realize now, and as we will also see in the sequel, that the problem is that of reconciling the usual classical spacetime with the spacetime of Quantum Theory, or as Witten puts it\[^18\] Bosonic spacetime with Fermionic spacetime.

Our starting point is the effect of an infinitesimal parallel displacement on a vector\[^19\]:

$$\delta a^\sigma = -\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu \quad (1)$$

Equation (1) represents the extra effect in displacements, due to the curvature of space. In terms of partial derivatives with respect to the $\mu^{th}$ coordinate, (1) leads to,

$$\frac{\partial a^\sigma}{\partial x^\mu} \rightarrow \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu \quad (2)$$
where the \( \Gamma \)'s are the Christoffel symbols. We consider the second term on the right side of (2) as:

\[
-\Gamma^\lambda_{\mu\nu} g^\nu_{\lambda a} \sigma = -\Gamma^\nu_{\mu\nu} a^\sigma
\]

where we have utilized the linearity property that in the above formulation

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},
\]

\( \eta_{\mu\nu} \) being the Minkowski metric and \( h_{\mu\nu} \) a small correction whose square is neglected.

So from (2) we get,

\[
\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu}
\]

From (3) we can deduce that

\[
\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \rightarrow \frac{\partial}{\partial x^\lambda} \Gamma^\nu_{\mu\nu} - \frac{\partial}{\partial x^\mu} \Gamma^\nu_{\lambda\nu}
\]

If we now impose the condition that the right hand side in (4) does not vanish, then we have a non commutativity of the momentum components in Quantum Theory. Indeed the left side of (4) can be written as

\[
\frac{1}{\hbar^2} \left[ p_\lambda, p_\mu \right] \approx \frac{0(1)}{l^2}
\]

where \( l \) is the Compton wavelength and \( \hbar \) the reduced Planck length wherein we have utilised the fact that at the extreme scale of the Compton wavelength, the Planck scale being a special case, the momentum is \( mc \).

We can, in view of (3) identify the right side of (4) as the electromagnetic Field Tensor

\[
\frac{e}{c\hbar} \Gamma^{\mu\nu}
\]

What we have shown here is that once we consider the non commutativity of the right or left side of (4), then it is meaningful to identify

\[
A^\mu = \hbar \Gamma^{\mu\nu}_v
\]

with the electromagnetic four potential, thus leading to a unification of electromagnetism with gravitation theory. This unification is not possible in the usual commutative spacetime, that is when the right or left side of (4) vanish.
Equation (6), as we shall see can be deduced alternatively. Equation (5) is the manifestation of non commutative geometry: This has been discussed in detail[15, 20]. The non commutative geometry is given by, apart from (5), by relations like:

\[
[x, y] = 0(l^2)
\]

\[
[x, p_x] = i\hbar[1 + (a/h)^2 p_x^2];
\]

\[
[t, p_t] = i\hbar[1 - (a/\hbar c)^2 p_t^2];
\]

\[
[x, p_y] = [y, p_x] = i\hbar(a/\hbar)^2 p_x p_y;
\]

\[
[x, p_t] = c^2[p_x, t] = i\hbar(a/\hbar)^2 p_x p_t; \text{etc.}
\]

Interestingly not only can the Dirac equation itself be deduced therefrom, but as also the existence of the magnetic monopole at extreme Compton scales can be deduced. Indeed this follows from (5) and the identification of the electromagnetic Field Tensor: We have infact

\[
B l^2 \approx \frac{\hbar c}{e}
\]

which is the celebrated monopole equation. It must be mentioned that such strong magnetic fields can be shown to be associated with a non commutative geometry from an alternative point of view[21].

The relations (5) and (7) are a manifestation of the non point like, spinorial structure of spacetime. In Quantum SuperString Theory, Witten has called it Fermionic spacetime, as against the usual commutative geometry of Classical or Bosonic spacetime[18, 22]. This non commutativity is at the root of the emergence of electromagnetism and spin as can be seen from (5) or (6) and subsequent remarks.

Another way of looking at the non commutative relations (5) or (7) is that, as pointed out by Witten, they provide a correction to the Heisenberg Uncertainty Principle[23] viz.,

\[
\Delta x \sim \alpha \frac{\Delta p}{\hbar}
\]

where, \(\alpha \equiv l^2\), this as discussed in the references cited referring to the well known duality, one encounters in Quantum SuperString Theory. In this case \(\Delta x\) is of the order of the radius of the universe and is given by

\[
\Delta x \sim \sqrt{N} l
\]
which is the well known Eddington Large Number formula, \( N \sim 10^{80} \) being the number of particles in the universe.

We now use the fact that the number of fluctuationally created particles is \( \sim \sqrt{N} \) \cite{3,4}, so that the Uncertainty in the energy momentum is given by

\[
\Delta p \sim \sqrt{N}mc, \Delta E \sim \sqrt{N}mc^2
\]

(10)

Using equations (8), (9) and (10) along with equation (6) we get, remembering that \( h_{00} \) represents the gravitational potential except for a factor \( c^2 \),

\[
A_0 \sim \hbar \frac{\partial}{\partial x^0} h_{00} \sim \sqrt{N}mc^2 \left( \frac{Gm}{c^2} \right)
\]

whence we get the celebrated large number "coincidence"

\[
e^2/Gm^2 \sim \sqrt{N}
\]

(11)

Equation (11) provides the link between electromagnetism and gravitation and arises from equation (6) and preceding considerations with the fluctuation input, very much in the spirit of Feynman. Infact the fluctuational underpinning for interactions has been commented on earlier\cite{24}.

3 Remarks

We observe that the identification of (6) with the electromagnetic four potential, leads to a unification of electromagnetism with gravitation theory. This unification is not possible in the usual commutative spacetime, that is when the right or left side of (4) vanishes.

We now make a number of remarks which corroborate the above deductions. The identification of (6) with the electromagnetic vector potential was deduced and discussed at length though from a completely different and infact Quantum Mechanical point of view\cite{14,3,25}. There the spinorial or pseudo vector property of the Dirac four spinor was used, in a purely Quantum Mechanical derivation.

This has been discussed at length in the references cited. But briefly, if the Dirac bispinor is written as \( \begin{pmatrix} \Theta \\ \phi \end{pmatrix} \), then at the Compton scale, it is the spinor \( \phi \) which predominates and moreover, under reflection,

\[ \phi \rightarrow -\phi \]
This was shown to immediately lead to (3) or (6).
It was pointed out there that interestingly (6) was mathematically similar to Weyl’s original formulation except that Weyl had introduced it ad hoc, in fact as an external element, without any internal derivation. This was why Weyl’s formulation was rejected\cite{26, 19}. However as can be seen from the above, (6) is a consequence of the pseudo spinorial behaviour at the Compton scale, which again is related to the non commutativity of equation (4) and manifested in the non commutative geometry contained in equations (5) and (7).

As string Theorist Greene puts it\cite{27}, ”...wild electromagnetic field oscillation, weak and strong free field fluctuations - quantum mechanical uncertainty tells us the universe is a teeming, chaotic, frenzied arena on microscopic scales...this frenzy is the obstacle to merging general relativity and quantum mechanics.” Only when we take into account the breakdown of the smooth spacetime manifold do we begin to make progress.

**APPENDIX**

In a recent paper El Naschie has introduced the concept of a fluctuation \cite{11}, a result of geometric fluctuation which could lead towards a unification of fundamental forces. It is pointed out here that recent work by the author does indeed emphasize the underpinning of fluctuations for fundamental interactions.

In this recent work\cite{25, 3, 28, 15, 29, 30}, it was pointed out firstly that the fluctuation of the electromagnetic field (or the Zero Point Field) leads to\cite{17},

\[
\Delta B \sim \sqrt{\hbar c/L^2},
\]

where $L$ is the spatial extent. It was pointed out that if $L \sim$ Compton wavelength of a typical elementary particle then from (12) we recover the mass and energy of this particle. In other words at the Compton wavelength the elementary particle ”condenses” out of the background Zero Point Field. Similarly a fluctuation in the metric leads to (Cf.refs.\cite{16, 25}),

\[
\Delta \Gamma \sim \frac{\Delta g}{L} \sim l_P/L^2
\]

where $l_P \sim 10^{-33}cms \sim$ Planck scale. Unlike in equation (12), if $L$ in (13) is taken to be $\sim l_P$ then from (13) we get the gravitational interaction.
That fluctuations tie up equations (12) and (13) can be seen explicitly as follows. As is known, given \( N \sim 10^{80} \) elementary particles in the universe, the fluctuation in the particle number is \( \sim \sqrt{N} \) which leads to a fluctuational electromagnetic energy which in the above scheme is the energy of the typical elementary particle, so that we have (Cf. also [31])

\[
e^2 \sqrt{N} = mc^2
\]  

(14)

Using in (14) the fact that [3, 4],

\[
R = \frac{GNm}{e^2}
\]

we get the well known relation

\[
e^2 \sim Gm^2 \cdot \sqrt{N} = Gm^2 \cdot 10^{40}
\]  

(15)

Equation (15) is usually interpreted as an adhoc or empirical relation comparing the strengths of gravitational and electromagnetic forces. But once the fluctuational underpinning has been taken into account, we have deduced (15) and can now see the connection between electromagnetic and gravitational interactions. Indeed from (15) one can deduce that [32] at the Planck scale the electromagnetic and gravitational forces become equal, or alternatively the Planck scale of mass \( \sim 10^{-5} \text{gms} \) is a Schwarszchild black hole.

Indeed in the model referred to earlier, elementary particles like electrons are Kerr-Newman type black holes giving at once both the electromagnetic and gravitational fields including the Quantum Mechanical anomalous gyromagnetic ratio \( g = 2 \). [14]

From this point, it was shown that the strong interactions follow at the Compton wavelength scale itself, where the dimensionality is low (Cf.ref. [29, 28, 15]). Infact within the same scheme, it was shown that the very puzzling characteristics of quarks namely their fractional charge, handedness and confinement besides the order of their masses can be deduced.

It is by the same argument of the fluctuation of the number of particles that it was shown that the weak interactions can also be explained [30, 33]. Indeed similar arguments in a different context were put forward years ago by Hayakawa [31].
Briefly if the weak force is mediated by a particle of mass $M$ and Compton wavelength $L$ we get from the fluctuation of particle number, this time

$$g^2\sqrt{NL^2} \approx Mc^2 \sim 10^{-14},$$

whence the weak interaction can be characterised.

The conclusion is that the spirit of fluctuations is vindicated (Cf.also ref.[34]).

References

[1] R.P. Feynman, Lectures on Gravitation, Ed. B. Hatfield, Addison-Wesley, New York, 1995.

[2] M.S. El Naschie, Chaos, Solitons & Fractals, Vol.8, No.5, 1997, p.753-759.

[3] B.G. Sidharth, Int.J.Mod.Phys.A, 13 (15), 1998, p.2599ff.

[4] B.G. Sidharth, Int.J.Th.Phys., 37 (4), 1998, p.1307ff.

[5] B.G. Sidharth, Astronomy and Geophysics, Royal Astronomical Society, London, April 1999, p.2.8.

[6] B.G. Sidharth, Nuovo Cimento, 115B (12,2), 2000, pp.151ff.

[7] B.G. Sidharth, Chaos, Solitons & Fractals, 12(2001), p.173-178.

[8] B.G. Sidharth, "The Chaotic Universe", Nova Science Publishers, New York, 2001 (in press).

[9] L. Nottale, Chaos, Solitons & Fractals, (1994) 4, 3, p.361-388 and references therein.

[10] G.N. Ord, 04817 Elsevier Science CHAOS Ms 1036, MFC September 1998 (Chaos, Solitons and Fractals).

[11] M.S. El Naschie, Chaos, Solitons & Fractals, 1999, 10(11), p.1947-1954.

[12] El Naschie, Chaos, Soliton & Fractals, 12(2001), p.1361-1368.

[13] El Naschie, Chaos, Soliton & Fractals, 12(2001), p.969-988.
[14] B.G. Sidharth, Gravitation and Cosmology, 4 (2) (14), 1998, p.158ff.
[15] B.G. Sidharth, Chaos, Solitons & Fractals, 11(2000), p.1269-1278.
[16] M.S. El Naschie, Chaos, Solitons & Fractals, 10(2/3), 1999, p.163.
[17] C.W. Misner, K.S. Thorne and J.A. Wheeler, ”Gravitation”, W.H. Freeman, San Francisco, 1973, pp.819ff.
[18] W. Witten, Physics Today, April 1996, pp.24-30.
[19] P.G., Bergmann, ”Introduction to the Theory of Relativity”, Prentice-Hall, New Delhi, 1969, p248ff.
[20] Y. Ne’eman, in Proceedings of the First International Symposium, ”Frontiers of Fundamental Physics”, Eds. B.G. Sidharth and A. Burinskii, Universities Press, Hyderabad, 1999, pp.83ff.
[21] T. Saito, Gravitation and Cosmology, 6(2000), No.22, pp.130-136.
[22] B.G. Sidharth, ”Quantum Super Strings and Quantized Fractal Spacetime”, to appear in Chaos, Solitons & Fractals.
[23] B.G. Sidharth, Proceedings of Fourth International Symposium on ”Frontiers of Fundamental Physics” Kluwer Academy, New York (in press).
[24] B.G. Sidharth, Chaos, Solitons & Fractals, 11(2000), p.2155-2156.
[25] B.G. Sidharth, Ind.J.Pure and Appl.Phys., Vol.35, July 1997, pp.456-471.
[26] A. Einstein, ”The Meaning of Relativity”, Oxford & IBH, New Delhi, 1965, pp.93-94.
[27] B. Greene, ”The Elegant Universe”, Vintage, London, 2000, p.120.
[28] B.G. Sidharth, Mod.Phys.Lett.A., Vol.14, No.5, 1999, p.387-389.
[29] B.G. Sidharth, in Instantaneous Action at a Distance in Modern Physics: ”Pro and Contra”, Eds., A.E. Chubykalo et. al., Nova Science Publishing, New York, 1999.
[30] B.G. Sidharth, Chaos, Solitons & Fractals, 12(2001), p.1101-1109.

[31] S. Hayakawa, Suppl of PTP Commemorative Issue, 1965, 532-541.

[32] B.G. Sidharth, Chaos, Solitons & Fractals, 12(2001), p.795-799.

[33] B.G. Sidharth, Chaos, Solitons & Fractals, 12(2001), p.1449-1457.

[34] M.S. El Naschie, ”Towards the Unification of Fundamental Interactions....”, to appear in Chaos Solitons and Fractals.