Differential-difference method to solve problems of hydrodynamics

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Abstract. Mathematical modelling of stationary processes of heat transfer, the problem of the current function in fluid mechanics and a number of other objects leads to the Poisson equation. The solution of multidimensional Poisson equations using various approximation schemes is usually brought to sweep methods or to problems of establishment, for which fictitious time is introduced. When solving two-dimensional problems of hydromechanics, the main part of the required machine time is spent on solving the equation of the current function, which has the form of the Poisson equation. When solving this problem, you can use the differential-difference method and with this eliminate the stage of matching the results of the sweep along different coordinate axes. In this paper, an algorithm based on this method, is proposed. The core of the differential-difference method is the diagonalization of the matrix equation using eigenvalues and vectors of the main three-diagonal transition matrix from differential to finite-difference operators. Twice applying the method of lines for a two-dimensional problem, solutions of the problem for new variables are obtained, which are linear combinations of the original grid unknowns. A formula for the reverse transition to the original unknowns is presented. The efficiency and accuracy of the algorithm for solving elliptic equations with stationary and non-stationary right-hand sides are proved using a number of test problems as examples.

1. Introduction
To solve multidimensional problems of mathematical physics, various splitting methods, a predictor-corrector, an alternating direction method, Lax-Wendroff and others [1-6] have been developed that require the use of method of establishing or successive approximation due to the use of the previous calculations for neighboring layers. The calculation time increases many times, and inconsistencies are accumulated, associated with machine rounding. These features are clearly manifested when solving the equations of hydrodynamics.

When solving the two-dimensional Navier-Stokes equations, one usually passes to separate equations for the current function, vorticity, and pressure. At the same time, for the first and third unknowns, Poisson’s equations (of elliptic type) are compiled, and for vorticity, the Burgers (parabolic type) equations with unknown coefficients before the first derivatives of the unknown are compiled. The first two equations are interrelated, and the pressure equation can be solved after the first and third equations are solved together with certain accuracy.

The process of coordinating the solutions of the current function and vorticity is due to two reasons: the nonlinearity of the equations to be solved and the use of the sweep method in different directions. For the second reason, as already noted, a fictitious time is introduced to solve the equation of the current
function and the method of iteration or establishing what is actually spent on the main part of the estimated time is organized.

Coordination of the results of the sweep process in different directions can be eliminated by using the provisions of the method of lines, one of the varieties of the differential-difference method. In works [7, 8], a way of applying the method of lines is presented, for which the elements of the fundamental and diagonal matrices for the Dirichlet problem are defined. In papers [9–12], methods are given for determining the elements of these matrices for problems with boundary conditions of the first and second kinds, as well as the second and first kinds.

The method of lines can be addressed when solving equations for the current function \( \psi \) and the pressure \( p \). For small values of the Reynolds criterion, we can also try to apply this method to solve the vorticity equation \( \zeta \). This is facilitated by the constancy of the coefficients of the second derivatives and the first type of boundary conditions for the current function and pressure. Below we demonstrate the features of this method for solving the current function equation.

2. Mathematical modeling of processes

Let's present the equation for the current function \( \psi \) in the form [1, 2, 6]:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\zeta.
\]

Here \( x, y \) - Cartesian coordinates corresponding to the directions of the components of the velocity vector \( u, v \); \( \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) - vorticity of flow, given function; the components of the velocity vector and the current function are interconnected by relations \( \frac{\partial \psi}{\partial y} = u, \frac{\partial \psi}{\partial x} = -v \) that uniquely satisfy the continuity equation.

The tangent to the graph of the current function indicates the direction of flow at a given point. The change in the current function between two points in the flow region physically means the flow rate of the fluid flowing through the segment, which combines these two points. The vorticity, in our problem, seems to be the only component of the vector of the same name, means the speed vector rotor.

A rectangular computational area is considered and, on the boundary of which, the value of the current function is given, and on the interior points there are known values of the right side of the equation, i.e. vorticity.

3. Differential-difference method for the current function.

We introduce a uniform grid [8]:

\[
\bar{x}_i = \left\{ x_i = ih_i, \; i = 0, 1, \ldots, N_x, N_x + 1; \; h_i = \frac{1}{N_x + 1} \right\},
\]

\[
\bar{y}_j = \left\{ y_j = jh_j, \; j = 0, 1, \ldots, N_y, N_y + 1; \; h_j = \frac{j}{N_y + 1} \right\}
\]

and grid functions \( u^n_{i,j}, v^n_{i,j}, \psi^n_{i,j}, \zeta^n_{i,j} \). The superscript refers to the time layer for solving the problem of vorticity, and the time step can be taken as variable \( \tau_n = t_{n+1} - t_n \).

According to the values of \( \zeta^n_{i,j} \), we find \( \psi^n_{i,j} \) in the internal nodes of the grid, and at the boundaries it has the specified values [1, 6]. In the general case, at the border of the settlement area we accept that \( \psi^n_{i,j} = \mu^n_{i,j} \). Using this we facilitate the use of the method for other tasks and boundary conditions.

For fixed values in the internal nodes \( j = 1..N_y \) of the grid \( i = 1..N_x \), the equation is approximated in the form:
\[ \psi^{n+1}_{i+1,j} - 2\psi^{n+1}_{i,j} + \psi^{n+1}_{i-1,j} + \frac{\partial^2 \psi^{n+1}_{i,j}}{\partial y^2} = -\zeta^n_{i,j}. \]

In the border nodes, the assumed boundary conditions are realized:
\[ \psi^{n+1}_{i,j} - 2\psi^{n+1}_{i,j} + \frac{\partial^2 \psi^{n+1}_{i,j}}{\partial y^2} = -\zeta^n_{i,j}, \]
\[ \mu^{n+1}_{i+1,j} - 2\psi^{n+1}_{i,j} + \frac{\partial^2 \psi^{n+1}_{i,j}}{\partial y^2} = -\zeta^n_{i,j}. \]

From the presented differential-difference equations we compose a matrix equation of the form [8]:

\[ \frac{1}{h_x^2} A_j^{(x)} U_j + \frac{\partial^2 U_j}{\partial y^2} = F_j, \tag{1} \]

here \( U_j = (\psi^{n+1}_{1,j}, \psi^{n+1}_{2,j}, \ldots, \psi^{n+1}_{N_x-1,j}, \psi^{n+1}_{N_x,j})^T \)
\[ F_j = \left( -\zeta^n_{1,j} - \mu^{n+1}_{1,j} h_x^2, \ldots, -\zeta^n_{N_x-1,j} - \mu^{n+1}_{N_x-1,j} h_x^2, -\zeta^n_{N_x,j} - \mu^{n+1}_{N_x,j} h_x^2 \right)^T = (f_{1,j}, f_{2,j}, \ldots, f_{N_x-1,j}, f_{N_x,j})^T, \]
\[ A_j^{(x)} = \left[ a_{ij}^{(x)} \right]_{N_x} = \begin{bmatrix}
-2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 
\end{bmatrix}. \]

The top sign "T" means the operation of transposing the matrix. The indices of the sought and matrix elements vary from 1 to \( N_x \).

The matrix \( A_j^{(x)} \) has a diagonal predominance, belongs to the Jacobi type and therefore it can be represented as [7, 8]:
\[ A_j^{(x)} = B_x \Lambda_x B_x^{-1}, \]
where \( B_x \) is the fundamental matrix composed of \( A_j^{(x)} \) matrix eigenvectors; \( B_x^{-1} \) - inverse matrix of \( B_x \);
\( \Lambda_x \) - a diagonal matrix whose nonzero entries represent the eigenvalues of the matrix \( A_j^{(x)} \).

In [7] the values of the elements of the diagonal matrix are defined:
\[ \lambda_x^{(x)} = -2 \left( 1 + \cos \frac{\pi s}{N_x + 1} \right), \]

as well as the values of the elements of the fundamental \( B_x \) and inverse \( B_x^{-1} \) matrices:
\[ b_{i,p}^{(x)} = b_{i,p}^{(x)-1} = (-1)^{i+p} \sqrt{\frac{2}{N_x + 1}} \sin \frac{\pi s p}{N_x + 1}. \]

Let us multiply both sides of equation (1) on the left to \( B_x^{-1} \) and get:
\[ \frac{1}{h_x^2} B_x^{-1} A_j^{(x)} U_j + \frac{\partial^2 B_x^{-1} U_j}{\partial y^2} = B_x^{-1} F_j. \]

Here, the commutative property of the operations of differentiation and multiplication of matrices was taken into account.

Since \( A_j^{(x)} = B_x \Lambda_x B_x^{-1} \) [7], then:
\[ B_x^{-1} A_j^{(x)} U_j = B_x^{-1} B_x \Lambda_x B_x^{-1} U_j = B_x^{-1} B_x \Lambda_x \left( B_x^{-1} U_j \right) = \Lambda_x \tilde{U}_j = \]

\[ \Lambda_x \]
\[ \psi_{i,j} = \left( \psi_{i-1,j}, \psi_{i,j}, \psi_{i+1,j}, \psi_{i,j-1}, \psi_{i,j+1} \right)^T. \]

It follows that if we introduce a new column vector:

\[ \bar{U}_j = \left( \psi_{i,j}, \psi_{i+1,j}, \psi_{i,j-1}, \psi_{i,j+1} \right)^T = B_j \psi = B_j \psi = \left( \sum_{p=1}^{N_1} b_{i,p}^{(1)} \psi_{i,p}^{(1)}, \psi_{i-1,j}, \psi_{i+1,j}, \psi_{i,j-1}, \psi_{i,j+1} \right)^T, \]

then the equation takes the form:

\[ \frac{1}{h_x} \Lambda \bar{U}_j + \frac{\partial^2 \bar{U}_j}{\partial y^2} = \bar{F}_j, \tag{2} \]

where \( \bar{F}_j = \left( \tilde{f}_{i,j}, \tilde{f}_{i,j}, \tilde{f}_{i,j}, \tilde{f}_{i,j} \right)^T = B_j \bar{F} = B_j \bar{F} = \left( \sum_{p=1}^{N_1} b_{i,p}^{(1)} \psi_{i,p}^{(1)}, \psi_{i-1,j}, \psi_{i+1,j}, \psi_{i,j-1}, \psi_{i,j+1} \right)^T. \]

From the last matrix equation, we can distinguish a separate ordinary equation with respect to \( \psi_{i,j} \) when \( i=1..N_x \) and \( j=1..N_y \):

\[ \frac{1}{h_x^2} \lambda_i^{(s)} \psi_{i,j} + \frac{\partial^2 \psi_{i,j}}{\partial y^2} = \bar{f}_{i,j}. \tag{3} \]

According to the introduced change of variables \( \psi_{i,j} = \sum_{p=1}^{N} b_{i,p}^{(s)} \psi_{i,p}^{(s+1)} \), we form the boundary conditions for the function \( \psi_{i,j} \). Then at \( j = 0 \) and \( j = N_y + 1 \) we have:

\[ \bar{\mu}_{i,0} = \psi_{i,0} = \sum_{p=1}^{N_1} b_{i,p}^{(1)} \mu_{p,0}, \quad \bar{\mu}_{i,N_y+1} = \psi_{i,N_y+1} = \sum_{p=1}^{N_1} b_{i,p}^{(1)} \mu_{p,N_y+1}. \]

Now we approximate equation (3) when \( i=1..N_x \) for internal nodes \( j=1..N_y \):

\[ \frac{1}{h_x^2} \lambda_i^{(s)} \psi_{i,j} + \frac{\partial^2 \psi_{i,j}}{\partial y^2} = \bar{f}_{i,j}. \]

When \( j=1 \) and \( j=N_y \) are implemented the boundary conditions:

\[ \frac{1}{h_x^2} \lambda_i^{(s)} \psi_{i,1} + \frac{\partial^2 \psi_{i,1}}{\partial y^2} = \tilde{f}_{i,1}, \quad \frac{1}{h_x^2} \lambda_i^{(s)} \psi_{i,N_y} + \frac{\partial^2 \psi_{i,N_y}}{\partial y^2} = \tilde{f}_{i,N_y}. \]

Then we create a new matrix equation:

\[ \frac{1}{h_x^2} \bar{\Lambda} \bar{U}_j + \frac{1}{h_y^2} \Lambda^{(s)} \bar{U}_j = \bar{F}_j. \tag{4} \]

Here \( \bar{U}_j = \left( \psi_{i,1}, \psi_{i,2}, ..., \psi_{i,N_y}, \psi_{i,N_y} \right)^T \) (i.e. \( \psi_{i,j} = \psi_{i,0} \)),

\[ \bar{F}_j = \left( \tilde{f}_{i,1}, \tilde{f}_{i,2}, ..., \tilde{f}_{i,N_y-1}, \tilde{f}_{i,N_y} \right)^T, \]

\[ \bar{\Lambda} \bar{U}_j = \left( \lambda_i^{(s)} \psi_{i,1}, \lambda_i^{(s)} \psi_{i,2}, ..., \lambda_i^{(s)} \psi_{i,N_y-1}, \lambda_i^{(s)} \psi_{i,N_y} \right)^T. \]
Now for $A_{i}^{(s)}$ we compose a fundamental matrix $B_{y}$ with elements 
\[ b_{s,p}^{(y)} = b_{i,p}^{(y)} = (-1)^{i+p} \frac{2}{N_{y} + 1} \sin \frac{\pi sp}{N_{y} + 1} \] 
and a diagonal matrix $\Lambda_{y}$ with nonzero elements 
\[ \lambda_{y}^{(s)} = -2 \left( 1 + \cos \frac{\pi s}{N_{y} + 1} \right). \] 

The equality $B_{y}^{-1} = B_{y}$ is appropriate for the inverse matrix $B_{y}$. For $N_{x} = N_{y}$ they coincide with like matrices for the case of diagonalization of the transition matrix by $x$.

Multiply both sides of equation (4) $B_{y}^{-1}$ on the left and get:
\[ \frac{1}{h_{y}^{2}} B_{y}^{-1} \Lambda U_{j} + \frac{1}{h_{y}^{2}} B_{y}^{-1} A_{i}^{(s)} U_{i} = B_{y}^{-1} F_{i}. \]

We introduce a vector column:
\[ B_{y}^{-1} \hat{U}_{i} = B_{y} U_{i} = \left( \sum_{q=1}^{N_{i}} b_{1,q}^{(y)} \psi_{i,q}, \sum_{q=1}^{N_{i}} b_{2,q}^{(y)} \psi_{i,q}, \ldots, \sum_{q=1}^{N_{i}} b_{N_{i},q}^{(y)} \psi_{i,q} \right)^{T} = \left( \psi_{i,1}, \psi_{i,2}, \ldots, \psi_{i,N_{y}} \right) = \hat{U}_{j}. \]

The remaining parts of the equation are converted as follows:
\[ B_{y}^{-1} A_{i}^{(s)} \hat{U}_{i} = B_{y}^{-1} B_{y} \Lambda_{y} B_{y}^{-1} \hat{U}_{i} = \Lambda_{y} \hat{U}_{i} = \left( \lambda_{y}^{(1)} \psi_{i,1}, \lambda_{y}^{(2)} \psi_{i,2}, \ldots, \lambda_{y}^{(N_{y})} \psi_{i,N_{y}} \right)^{T}, \]
\[ B_{y}^{-1} \Lambda \hat{U}_{i} = B_{y} \left( \lambda_{y}^{(1)} \psi_{i,1}, \lambda_{y}^{(2)} \psi_{i,2}, \ldots, \lambda_{y}^{(N_{y})} \psi_{i,N_{y}} \right)^{T} = \lambda_{y}^{(1)} \left( \sum_{q=1}^{N_{i}} b_{1,q}^{(y)} \lambda_{y}^{(1)} \psi_{i,q}, \sum_{q=1}^{N_{i}} b_{2,q}^{(y)} \lambda_{y}^{(1)} \psi_{i,q}, \ldots, \sum_{q=1}^{N_{i}} b_{N_{i},q}^{(y)} \lambda_{y}^{(1)} \psi_{i,q} \right)^{T} = \lambda_{y}^{(1)} \left( \sum_{q=1}^{N_{i}} b_{1,q}^{(y)} \lambda_{y}^{(1)} \psi_{i,q}, \sum_{q=1}^{N_{i}} b_{2,q}^{(y)} \lambda_{y}^{(1)} \psi_{i,q}, \ldots, \sum_{q=1}^{N_{i}} b_{N_{i},q}^{(y)} \lambda_{y}^{(1)} \psi_{i,q} \right)^{T} = \lambda_{y}^{(1)} \left( \psi_{i,1}, \psi_{i,2}, \ldots, \psi_{i,N_{y}} \right)^{T} = \lambda_{y}^{(1)} \hat{U}_{i}, \]
\[ B_{y}^{-1} F_{i} = B_{y} \tilde{F}_{i} = \left( \sum_{q=1}^{N_{i}} b_{1,q}^{(y)} \tilde{f}_{i,q}, \sum_{q=1}^{N_{i}} b_{2,q}^{(y)} \tilde{f}_{i,q}, \ldots, \sum_{q=1}^{N_{i}} b_{N_{i},q}^{(y)} \tilde{f}_{i,q} \right)^{T} = \left( \tilde{f}_{i,1}, \tilde{f}_{i,2}, \ldots, \tilde{f}_{i,N_{y}} \right)^{T} = \tilde{F}_{i}. \]

As a result of these changes, the matrix equation takes the form:
\[ \frac{1}{h_{y}^{2}} \Lambda \hat{U}_{i} + \frac{1}{h_{y}^{2}} \Lambda \hat{U}_{j} = \tilde{F}_{i}. \]

From here you can select a separate equation:
\[
\frac{1}{h_i^2} \lambda_i^{(1)} \psi_{i,j} + \frac{1}{h_j^2} \lambda_j^{(2)} \psi_{i,j} = \tilde{f}_{i,j}.
\]

From this linear equation we find the solution of \( \psi_{i,j} \) for the internal nodes:

\[
\psi_{i,j} = \tilde{f}_{i,j} \left( \frac{\lambda_i^{(1)}}{h_i^2} + \frac{\lambda_j^{(2)}}{h_j^2} \right). 
\]

The inverse transition to the desired function, according to the introduced substitutions and the self-conjugation of the matrices \( B_x \) and \( B_y \), can be done by the formula \[8\]:

\[
\bar{\psi}_{i,j} = \sum_{p=1}^{N_x} b_{i,p} \sum_{q=1}^{N_y} b_{j,q} \psi_{p,q},
\]

where the introduction of a superscript \( n \) took into account the possibility of a non-stationary statement of the problem, when the boundary conditions depend on time or the method of establishment together with vorticity is applied.

4. The discussion of the results

As can be seen, using the differential-difference method, we obtain the solution of the current function with an accuracy that is inherent in the used approximation scheme in two coordinates. We made a new system of linear equations and each of them was solved according to the provisions of the method of lines. In this case, neither a sequential approximation nor the introduction of fictitious time was required to coordinate the results of the sweep along different coordinates.

In order to verify the viability of the proposed variant of the differential-difference method for solving an elliptic equation, we developed a program and conducted a computational experiment.

The first test calculation concerned the function \( \psi(x, y) = 2 - x^2 - y^2 \). The corresponding boundary conditions were \( \psi(x,0) = 2 - x^2 \), \( \psi(x,1) = 1 - x^2 \), \( \psi(0, y) = 2 - y^2 \) and \( \psi(1, y) = 1 - y^2 \). The result of the solution will be true if the vorticity is \( -\zeta(x, y) = -4 \).

Calculations were performed for integration steps \( h_x = h_y = 0.1 \); 0.05 and 0.02. The values \( N_x \) and \( N_y \) were equal to 9, 19 and 49.

Numerical results showed that with the smallest integration step \( h_x = h_y = 0.1 \) used, the results were in the first four digits after the decimal point match. But in order to get a more detailed visual picture, the calculations were carried out for \( h_x = h_y = 0.02 \).

As can be seen from fig. 1, the isolines of the current function represent concentric circles in the first quadrant of a Cartesian coordinate system.

The second test calculation was performed for the boundary conditions of \( \psi(x,0) = 3x^3 \), \( \psi(x,1) = 3x^3 - 2x^2 - 1 \), \( \psi(0, y) = -y^2 \) and \( \psi(1, y) = 3 - 2y - y^2 \) (Fig. 2). The vorticity was taken in the form of \( -\zeta(x, y) = 18x - 4y - 2 \). The corresponding analytical solution of the problem is \( \psi(x, y) = 3x^3 - 2x^2 y - y^2 \).

Here we obtained the result of comparing the exact and numerical solution of the problem in terms of accuracy that we had for the first test problem.
The third test calculation was related to the nonstationary case, and the desired function had the form of \( \psi(x, y) = \sin \pi (x + y) t \).

The boundary conditions for the problem were \( \psi(x, 0, t) = \sin \pi x t \), \( \psi(x, 1, t) = \sin \pi (x + 1) t \), \( \psi(0, y, t) = \sin \pi y t \), \( \psi(1, y, t) = \sin \pi (y + 1) t \). The solution is true if \( -\zeta(x, y) = -2\pi^2 t^2 \sin \pi (x + y) t \) is taken.

We can get a picture of isolines by setting a specific time value or by looping over time values. In fig. 3 the results obtained at \( t=2 \) sec are shown. Isolines correspond to families of curves \( x + y = \text{const} \). Diagonally, starting from the lower left corner to the upper right corner, the value of the current function passes through two periods of the sine. There were the bands covering the largest and smallest values of the sine. Here we achieved the coincidence of the exact and numerically found solutions with accuracy of \( 10^{-5} \) at the step of \( h_x = h_y = 0.05 \).
5. Conclusion

To solve the Poisson equation for the stream function in two-dimensional problems of hydromechanics, a modification of the differential-difference method is proposed. The equation and conditions are approximated with the second order of accuracy for a uniform grid. The method eliminates the steps of reconciling the results of the sweep in two directions and ensures the accuracy of solving finite-difference equations within the framework of the accuracy of machine rounding, as practice has shown, contributes to a multiple reduction in the calculation time.

When developing numerical methods for solving various one-dimensional and multi-dimensional problems, the following requirements are set [1]:

- computational stability;
- the accuracy of the calculation of the main characteristics of the object, ensuring the adequacy within the requirements of the application of the results;
- efficiency; minimum amount of used memory; ease of implementation.

The method presented in the framework of this work meets all these requirements, and the algorithm developed on its basis allows implementation when solving one- and multidimensional parabolic and hyperbolic equations in Cartesian coordinates, i.e. taking into account the derivative of the desired function with respect to time in the equations of conservation of substances does not create fundamental difficulties. The method can be used to solve the equation of the current function in cylindrical coordinates. The use of the method of straight lines along the longitudinal coordinate and the method of the usual sweep along the radial coordinate allows one to exclude the introduction of fictitious time, respectively; the calculation time is greatly reduced.

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