Exact Anti-Self-Dual four-manifolds with a Killing symmetry by similarity transformations

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Abstract

We study the group properties and the similarity solutions for the constraint conditions of anti-self-dual null Kähler four-dimensional manifolds with at least a Killing symmetry vector. Specifically we apply the theory of Lie symmetries to determine all the infinitesimal generators of the one-parameter point transformations which leave the system invariant. We use these transformations to define invariant similarity transformations which are used to simplify the differential equations and find the exact form of the spacetime. We show that the constraint equations admit an infinite number of symmetries which can be used to construct an infinite number of similarity transformations.

Keywords: Lie symmetries; invariants; null-Kähler metrics; Similarity transformations

1 Introduction

By definition, an Anti-Self-Dual Riemannian space has the property its self dual Weyl curvature to be zero, Locally conformally flat spacetimes are known examples of ASD manifolds. The property a manifold is ASD is usually related with the context of integrability for specific field equations of Riemannian manifolds. The pioneer work of Penrose relates four-dimensional ASD manifolds with three-dimensional complex twistor spaces with certain algebraic properties [1]. Furthermore, Ward [2] generalized the result of Penrose by introducing the cosmological constant in the field equations. There are various studies in the literature on the investigation of ASD four-manifolds. For instance, in [3], Gibbons and Hawking found a new family of class of hyper–Kähler metrics characterized by the existence of an isometry. LeBrun [4] generalized that result; he proved that self-dual four-manifolds with $G_1$ symmetry can be constructed from the solutions of a linear differential equation

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on a three-dimensional hyperbolic space. Such approach was applied in [5] to determine self-dual manifolds of higher dimension with a $G_n$ symmetry group.

In this study we focus on ASD null Kähler metrics of signature $(++--)$ which admit a covariantly constant real spinor and an isometry vector field. Hence, we assume the line element

$$ds^2 = W_x (dy^2 - 4dxdt + 4H_x dt^2) - W_x^{-1} (dz - W_x dy - 2W_y dt)^2$$

which we shall say that it is ASD null Kähler geometry if and only if functions $H(t, x, y)$ and $W(t, x, y)$ satisfy the following system [6]

$$H_{yy} - H_{tx} + H_x H_{xx} = 0,$$

$$W_{yy} - W_{tx} + (H_x W_x)_{,x} = 0,$$

where the admitted Killing symmetry is the vector field $K = \partial_z$, that is, $\mathcal{L}_K g_{\mu\nu} = 0$, where $\mathcal{L}_K$ is the Lie derivative with respect to the vector field $K$ on the metric tensor $g_{\mu\nu}$ with line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ as defined by expression (1) and $x^\mu = (t, x, y, z)$. We observe that for $W = \frac{H_{xx}}{2} + f(t)$ the line element describes a pseudo hyper-Kähler metric, while when $W_x \neq \frac{H_{xx}}{2}$ the line element describes a non Ricci-flat space. For $H(t, x, y) = H_0$, the space reduces to the Gibbons-Hawking solution [6].

We are interested on the algebraic properties of system (2), (3) and in the derivation of exact solutions. Specifically, we consider the theory of the symmetries of differential equations and we determine all the Lie point symmetries for the gravitational system (2), (3). Lie symmetry analysis is a robust method for the study of nonlinear differential equations. Nowadays it is the standard approach for the computation of solutions and the description of the algebra for nonlinear differential equations [7–10], for applications see [11–25] and references therein. In this work we take some interest in the application of Lie symmetries in the classification of exact solutions. This specific approach has been widely studied in various gravitational systems from General Relativity to alternative theories of gravity with very interesting results [26–30]. For a recent review on the application of Lie symmetries in cosmological studies we refer the reader to [31]. Indeed, as we shall see in the following, with the application of the Lie point symmetries we find new families of solutions for the system (2), (3). We recover previous well-known solutions, thus we find new solutions which have not been presented before in the literature. Indeed for the plethora of these solutions the resulting spacetime admits additional Killing vectors. However, there are exact solutions where no additional Killing vectors exists for the line element (1).

As far as the Kadomtsev–Petviashvili equation is concerned, the group properties have been investigated in various studies in the literature [32–34]. Kadomtsev–Petviashvili equation is related with the Einstein-Weyl equations in three-dimensional manifolds. Indeed, if a three-dimensional Einstein-Weyl structure admits an isometry then its structure is defined by the Kadomtsev–Petviashvili equation [35]. The plan of the paper is as follows.

In Section 2 we present the basic properties and definitions for Lie symmetries and the main definition of the one-dimensional optimal system. We continue our analysis by investigating two separate cases; the spacetime (1) being static or non static. The Lie symmetries of the field equations (2), (3) for the static spacetime are investigated in Section 3. We find that the system admits a nine-dimensional Lie algebra. We calculate the commutators and the adjoint representation, the latter results are used to define the one-dimensional optimal system which for simplicity of the presentation we give it terms of various families. The Lie point symmetries
are applied to define similarity transformations which are used to reduce the gravitational field equations into a system of ordinary differential equations. For all the reductions of our analysis the reduced system can be solved in terms of quadratures. However, in this work we investigate the possibility of the exact closed-form solutions. For the exact spacetimes we study the admitted isometries which can be used for the physical description of the solutions. The nonstatic case is investigated in Section 4. In contrast with the static case, we find that the gravitational field equations admit infinite Lie point symmetries, which are summarized in nine families of Lie vector fields. These families provide eight coefficient functions which depend on the time parameter. We demonstrate the application of these infinite Lie symmetries in the derivation of nonstatic exact closed-form solutions. Finally in Section 5, we summarize our results and we draw our conclusions.

2 Lie symmetries and one-dimensional optimal system

For the convenience of the reader in the following lines we present the basic properties and definitions for the theory of Lie symmetries.

Let us consider the infinitesimal one-parameter point transformation

\[ t' = t + \varepsilon \xi^t(t,x,y,H,W), \]  
\[ x' = x + \varepsilon \xi^x(t,x,y,H,W), \]  
\[ y' = y + \varepsilon \xi^y(t,x,y,H,W), \]  
\[ H' = H + \varepsilon \eta^H(t,x,y,H,W), \]  
\[ W' = W + \varepsilon \eta^W(t,x,y,H,W) \]

with infinitesimal generator

\[ X = \xi^\mu \partial_\mu + \eta^A \partial_A, \xi^\mu = (\xi^t, \xi^x, \xi^y), \eta^A = (\eta^H, \eta^W) \]

and second extension/prolongation

\[ X^{[2]} = \xi^\mu \partial_\mu + \eta^A \partial_A + \eta^{A[1]} \partial_{\mu A} + \xi^\mu \partial_\mu + \eta^{A[2]} \partial_{\mu \nu A}. \]

in which the coefficients \( \eta^{A[1]}, \eta^{A[2]} \) are defined by the following formula

\[ \eta^{[n]} = D_\mu \eta^{[n-1]} - u_{\mu_1 \mu_2 \ldots \mu_{n-1}} D_\mu (\xi^\nu). \]

Therefore, under the action of the latter point transformation the gravitational system \( t, x, y, H, W \) remain invariant if and only if

\[ \lim_{\varepsilon \to 0} \frac{\mathbf{F}(t,x,y,H,W;\varepsilon) - \mathbf{F}(t,x,y,H,W;\varepsilon)}{\varepsilon} = 0, \]

where \( \mathbf{F} = \left( H_{yy} - H_{tx} + H_x H_{xx}, W_{yy} - W_{tx} + (H_x W_x)_{,x} \right) \). The symmetry condition \( \text{(11)} \) can be written in the equivalent form

\[ \mathcal{L}_{X^{[2]}}(\mathbf{F}) = \lambda \mathbf{F}, \mod \mathbf{F} = 0 \]

where \( \lambda \) is a function that should be determined and \( \mathcal{L}_{X^{[2]}} \) denotes the Lie derivative with respect to the vector field \( X^{[2]} \).

The main application of the Lie symmetries is summarized in the derivation of similarity transformations which are used to simplify the differential equation by reduce the number of the independent variables or by
reducing the order of the differential equation. The final purpose for the application of the Lie symmetries is to reduce the given differential equation into the form of another well-known equation or write the differential equation into a simple form where an exact solution can be found.

The solutions which follow from the application of the Lie invariants are mainly known as similarity solutions. A basic property for the admitted Lie symmetries of a differential equation is that they form a Lie group. Therefore, in order to perform a complete derivation of all the possible similarity solutions we should find the admitted one-dimensional optimal system.

Consider the \( n \)-dimensional Lie algebra \( G_n \) with elements \( X_1, X_2, \ldots, X_n \) admitted by the system \( H^A \). Then the vector fields

\[
Z = \sum_{i=1}^{n} a_i X_i, \quad W = \sum_{i=1}^{n} b_i X_i, \quad a_i, b_i \text{ are constants.} \tag{13}
\]

are equivalent if and only if

\[
W = Ad(\exp(\varepsilon X_i)) Z \tag{14}
\]
or

\[
W = cZ, \quad c = \text{const.} \tag{15}
\]

where the operator \( Ad(\exp(\varepsilon X_i)) \) \( X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] + \ldots \) is called the adjoint representation. It is clear that the adjoint representation of the admitted Lie symmetries should be calculated in order to find all the possible independent similarity transformations. The latter set of one-dimensional Lie algebras which do not connect thought the adjoint representation form the so-called one-dimensional optimal system for the given differential equation.

We proceed our analysis by considering first static spacetime with \( H = H(x,y) \), \( W = W(x,y) \) and then the general case. We follow that analysis for the convenience of the reader and the presentation of the results.

### 3 Static spacetime

In the case where the spacetime is static \( (1) \), that is, it admits the second isometry vector \( \partial_t \), the gravitational system \( (2), (3) \) is simplified as

\[
H_{yy} + H_x H_{xx} = 0, \tag{16}
\]

\[
W_{yy} + (H_x W_x)_x = 0. \tag{17}
\]

From the symmetry condition \( (12) \) for the latter system we find the following symmetry vectors

\[
X^1 = \partial_H, \quad X^2 = \partial_W, \quad X^3 = \partial_x, \quad X^4 = \partial_y, \quad X^5 = W \partial_W,
\]

\[
X^6 = y \partial_H, \quad X^7 = y \partial_W, \quad X^8 = H \partial_H - \frac{y}{2} \partial_y, \quad X^9 = x \partial_x + \frac{3}{2} y \partial_y.
\]

The corresponding commutators and the adjoint representation for the admitted Lie symmetries are presented in Tables 1 and 2 respectively. With the results of the tables we can determine the one-dimensional optimal
system, consisting of the following one-dimensional Lie algebras

\[
\begin{align*}
\{X^1\}, \{X^2\}, \{X^3\}, \{X^4\}, \{X^5\}, \{X^6\}, \{X^7\}, \{X^8\}, \{X^9\}, \\
\{a_2X_2 + a_8X_8 + a_9X_9\}, \{a_3X_3 + a_5X_5 + a_8X_8\}, \{a_1X_1 + a_5X_5 + a_9X_9\}, \\
\{a_1X_1 + a_2X_2 + a_9X_9\}, \{a_2X_2 + a_3X_3 + a_8X_8\}, \{a_3X_3 + a_4X_4 + a_5X_5\}, \\
\{a_3X_3 + a_4X_4 + a_6X_6 + a_7X_7\}, \{a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6\}, \\
\{a_2X_2 + a_3X_3 + a_4X_4 + a_6X_6\}, \{a_1X_1 + a_2X_2 + a_3X_3 + a_6X_6\}, \\
\{a_1X_1 + a_3X_3 + a_4X_4 + a_5X_5\}, \{a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4\}
\end{align*}
\]

in which \(a_I\) are real numbers. We have used the coefficients \(a_I\) such as to simplify the presentation of the one-dimensional optimal system.

We proceed with the application of the Lie point symmetries for the derivation of the similarity transformations which will be used to reduce the number of the independent variables of the gravitational system (15), (17). The system (15), (17) is reduced into a system of ordinary differential equation which as we shall see in all cases can be solved by quadratures.

### 3.1 Similarity transformations

Among the elements of the one-dimensional Lie algebra not all the vector fields reduce the number of the independent variables. For instance, the symmetry vector \(X^1 = \partial_H\) cannot be used for the reduction of the system of partial differential equations. However, these symmetries are essential when the system reduces into a system of ordinary differential equations.

#### 3.1.1 \(\{X^3\}\)

Application of the Lie symmetry vector \(X^3\) leads to the reduced system \(H(x, y) = h(y)\), \(W(x, y) = w(y)\) in which \(h_{yy} = 0\), \(w_{yy} = 0\). However such solution is not physically accepted.
while the nonzero component of the Einstein Tensor 

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

is the Gibbons-Hawking solution while the second solution is that of the maximal symmetric spacetime of zero curvature, that is, it describes the flat space.

3.1.3 \{X^8\}

Reduction with respect the Lie symmetry vector \(X^8\) provides \(H(x,y) = h(x) \ y^{-2}\), \(W(x,y) = w(x)\) where \(h_{xx}h_{x} + 6h = 0\), \((h_{x}w_{x})_{x} = 0\) with solutions \(\{h(x) = h_{0}\}, \\{w(x)\}\) and \(\{h(x) = h_{0}(x - x_{1})\}, \\{w(x) = w_{0}(x - x_{2})\}\). The latter systems can be solved in terms of quadratures. However, a special solution is \(h(x) = \frac{1}{3}x^3\) with \(w(x) = w_{1}x^{-3} + w_{0}\). For the latter exact solution the background space admits a four dimensional Lie algebra consisting of the generic vector field \((c_{1}\ z + c_{2}) \partial_{x} + (c_{4} - 4v_{1}c_{1}t + c_{2}w_{x}) \partial_{y} + c_{3}\partial_{z}\).

Furthermore, for this exact solution the Ricciscalar of the spacetime is found to be zero, that is, \(R = 0\), while the nonzero component of the Einstein Tensor \(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\) is the \(G_{zz} = \frac{3}{(w_{2})^3} x^4\).

3.1.4 \{X^9\}

From the Lie symmetry vector \(X^9\) it follows the similarity transformation \(H(x,y) = h(\sigma)\), \(W(x,y) = w(\sigma)\) in which \(\sigma = yx^{-\frac{2}{3}}\) and \(h(\sigma)\), \(w(\sigma)\) satisfy the system \(8h_{\sigma\sigma} - 27\sigma^3h_{\sigma} + 45\sigma^2 (h_{\sigma})^2 = 0\), \(8w_{\sigma\sigma} - 27\sigma^3w_{\sigma} + 90\sigma^2w_{\sigma\sigma} = 0\). The latter system can be solved in terms of quadratures. However, a special solution of the system is \(h(\sigma) = -\frac{1}{3}y^{-2}\), \(w(\sigma) = w_{0} + w_{1}\sigma^\frac{2}{3}\), that is, \(H(x,y) = -\frac{1}{3}y^{-2}x^3\), \(W(x,y) = w_{0} + w_{1}y^\frac{2}{3}x^{-\frac{2}{3}}\). For that particular exact solution the Ricciscalar of the spacetime is calculated \(R = \frac{4}{3w_{2}}x^{\frac{2}{3}}y^{-\frac{12}{5}}\) while the spacetime admits a three dimensional Killing algebra with generator \((c_{1}t + c_{2}) \partial_{t} + 3c_{1}\partial_{x} + 2c_{1}\partial_{y} - (2c_{1}z - c_{3})\partial_{z}\). Finally the nonzero components of the Einstein tensor are derived

\[
G_{tt} = \frac{244}{45}x^{2}y^{-4}, \quad G_{ty} = -\frac{16}{15}xy^{-3}, \quad G_{tz} = \frac{20}{9}x^{\frac{13}{2}}y^{-\frac{17}{2}}
\]
The reduced system is written as
\[ G_{yz} = -\frac{2}{3w_1}x^\frac{2}{3}y^{-\frac{1}{3}}, \quad G_{zz} = \frac{29}{9}x^\frac{16}{9}y^{-\frac{2}{3}}. \]

### 3.1.5 \{a_2X_2 + a_8X_8 + a_9X_9\}

The generic symmetry vector \(a_2X_2 + a_8X_8 + a_9X_9\) provides the similarity transformation \(H(x, y) = h(\xi) x^{\frac{a_8}{a_9}} \), \(W(x, y) = \frac{a_2}{a_9} \ln x + w(\xi)\), \(\xi = yx^{\frac{3a_8}{a_9}}\). Hence, after the application of the similarity transformation the gravitational field equations are simplified in the following system

\[
0 = h_{\xi\xi} \left( (a_8 - 3a_9)^3 h_{\xi} + 8(a_9)^3 + 2a_8 (a_8 - 3a_9)^2 \xi^2 h \right) + (a_8 - a_9) (4a_8 h + (a_8 - 3a_9) h_{\xi} + (a_8 - 3a_9) h_{\xi\xi}), \tag{18}
\]

\[
0 = w_{\xi\xi} \left( (a_8 - 3a_9)^3 h_{\xi} + 8(a_9)^3 + 2a_8 (a_8 - 3a_9)^2 \xi^2 h \right) + (a_8 - 3a_9)^2 \xi^2 h_{\xi\xi} (2a_2 + \xi (a_8 - 3a_9) h_{\xi}) + 2(a_8 - 3a_9) h_{\xi} ((3a_8 - 5a_9) (a_8 - 3a_9) w_{\xi} + a_2 (5a_8 - 7a_9)) + 2a_8 h (2a_8 - 7a_9) w_{\xi} + 4a_2 (a_8 - 2a_9). \tag{19}
\]

The latter system can be solved by quadratures. Thus a closed-form solution is

\[
h(\xi) = -\frac{1}{3} \xi^{-2}, \quad w(\xi) = -w_1 \xi^{-\frac{2a_9}{a_8 - a_9}} - 2 \frac{a_2}{(a_8 - a_9)} \ln \xi + w_0, \quad a_8 \neq a_9. \tag{20}
\]

On the other hand in the special case where \(a_8 = a_9\) the closed-form solution is

\[
h(\xi) = h_1 + h_0, \quad w(\xi) = \frac{1}{2} \frac{a_2}{a_9} \ln (1 + h_0 \xi^2) + \frac{w_1}{\sqrt{h_0}} \arctan \left( \sqrt{h_0} \xi \right) + w_0. \tag{21}
\]

For these specific solutions the spacetime is found to admit only two isometries. In Figs. 1 and 2 we present the qualitative evolution for the similarity solutions 20, 21.

### 3.1.6 \{a_3X_3 + a_5X_5 + a_8X_8\}

From the vector field \(a_3X_3 + a_5X_5 + a_8X_8\) it follows \(H(x, y) = e^{\frac{a_8}{a_9}} h(\xi)\), \(W(x, y) = e^{\frac{a_8}{a_9}} w(\xi)\), \(\xi = ye^{\frac{a_8}{a_9}}\). The reduced system is written as

\[
0 = h_{\xi\xi} \left( 8(a_3)^3 + (a_8)^3 \xi^2 (\xi h + 2h) \right) + (a_8)^3 (\xi h + 2h) (5\xi h + 4h), \tag{22}
\]

\[
0 = w_{\xi\xi} \left( 8(a_3)^3 + (a_8)^3 \xi^2 (\xi h + 2h) \right) + (a_8)^3 \xi^2 h_{\xi\xi} (a_8 \xi w + 2a_5 w) + a_8^2 \xi w (\xi (3a_8 + a_5) h_{\xi} + (3a_8 + 4a_5) h) + a_8 a_8 w (\xi (2a_5 + 5a_8) h_{\xi} + 4(a_5 + a_8) h), \tag{23}
\]

which can be solved in terms of quadratures.

In the special case where \(a_3 = 0\), the similarity transformation is calculated \(H(x, y) = h(x) y^{-\frac{2a_8}{a_9}} \), \(W(x, y) = w(x) y^{-\frac{2a_8}{a_9}}\) with reduced system \(h_{xx} h_x + 6h = 0, 2a_5 (2a_5 + a_8) w + a_8 (h_x w_x)_x = 0\). A closed-form solution of the latter system is \(h(x) = -\frac{1}{4} x^2, \quad w(x) = w_1 x^2 J_\frac{1}{2} (\bar{a} x^2) + w_2 x^2 I_\frac{1}{2} (\bar{a} x^2)\), in which \(\bar{a} = 2\sqrt{3a_5 (2a_5 + a_8)}\) and \(J(x), I(x)\) are the Bessel functions. For the spacetime we find that it admits a two dimensional Killing algebra. The qualitative evolution for the latter similarity solution is given in Fig. 3.
Figure 1: Qualitative evolution for the similarity solution (20) of functions $H(x,y)$, $W(x,y)$ provided by the application of the symmetry vector $a_2 X_2 + a_8 X_8 + a_9 X_9$. The plots are for $(w_1, a_2, a_8, a_9) = (1, 1, 2, 1)$.

Figure 2: Qualitative evolution for the similarity solution (21) of functions $H(x,y)$, $W(x,y)$ provided by the application of the symmetry vector $a_2 X_2 + a_8 X_8 + a_9 X_9$. The plots are for $(w_1, a_2, a_8, a_9) = (1, 1, 2, 1)$. 
Figure 3: Qualitative evolution for the similarity solution of functions $H(x, y)$, $W(x, y)$ provided by the application of the symmetry vector $a_5X_5 + a_8X_8$. The plots are for $a_5 = \frac{1}{2}$, $a_8 = 2$, $w_1 = 1$ and $w_2 = 0$.

3.1.7 $\{a_1X_1 + a_5X_5 + a_9X_9\}$

We proceed with the application of the Lie symmetry vector $a_1X_1 + a_5X_5 + a_9X_9$. The corresponding similarity transformation is $H(x, y) = a_1a_9\ln x + h(\sigma)$, $W(x, y) = w(\sigma)x^{\frac{a_1}{a_9}}$, where $\sigma = yx^{-\frac{3}{2}}$. The reduced system is

$$0 = a_9h_{\sigma\sigma} (8a_9 + 18a_1\sigma^2 - 27a_9\sigma^3h_\sigma) - (15a_9\sigma h_\sigma - 4a_1)(3a_9\sigma h_\sigma - 2a_1), \quad (24)$$

$$0 = a_9^2w_{\sigma\sigma} (8a_9 - 18a_1\sigma^2 + 27a_9\sigma^3h_\sigma + 9a_9^2\sigma^2h_{\sigma\sigma}(2a_5h - 3a_9\sigma w_\sigma) + 6a_9\sigma w_\sigma(a_1(7a_9 - 4a_5) + 3a_9(2a_5 - 5a_9)\sigma h_\sigma) + -6a_5a_9(2a_5 - 7a_9)wh_\sigma + 8a_1a_5(a_5 - 2a_9)w. \quad (25)$$

in which the solution can be expressed in terms of quadratures.

On the other hand, for $a_1 = 0$ we find the closed-form solution $h(\sigma) = -\frac{1}{3}\sigma^{-2}$, $w(\sigma) = w_1\sigma^{\frac{2(a_5+a_9)}{a_9}} + w_2\sigma^{\frac{2a_5}{a_9}}$. For the latter exact solution the spacetime admits two-dimensional Killing algebra, however when $w_1 = 0$, or $a_5 + a_9 = 0$, the spacetime admits the scaling symmetry $-2a_5 + 2a_9 + a_5t\partial_t - \frac{2(a_5+a_9)}{2a_5+a_9}y\partial_y + z\partial_z$ as an additional isometry vector. The qualitative evolution for the latter similarity solution is presented in Fig. 4.

3.1.8 $\{a_1X_1 + a_2X_2 + a_9X_9\}$

From the Lie symmetry vector $a_1X_1 + a_2X_2 + a_9X_9$ it follows $H(x, y) = \frac{a_1}{a_9}\ln x + h(\sigma)$, $W(x, y) = \frac{a_2}{a_9}\ln x + w(\sigma)$, $\sigma = yx^{-\frac{3}{2}}$. For the latter similarity transformation the gravitational system is reduced to the following system
of ordinary-differential equations

\[ 0 = a_9 h_\sigma \left( 18 a_1 \sigma^2 + 8 a_9 - 27 a_9 \sigma^3 h_\sigma \right) - 2 (3 a_9 \sigma h_{gs} - 2 a_1) (15 a_9 \sigma h_\sigma - 4 a_1), \quad (26) \]

\[ 0 = -a_9 w_\sigma \left( 18 a_1 \sigma^2 + 8 a_9 - 27 a_9 \sigma^3 h_\sigma \right) + a_8 \sigma^2 h_\sigma \left( 3 a_9 \sigma h_\sigma - 2 a_2 \right) + \\
+ 42 a_9 \sigma \left( a_1 w_\sigma + a_2 h_\sigma \right) - 90 a_9^2 \sigma^2 h_\sigma - 16 a_1 a_2. \quad (27) \]

The later system can be solved by quadratures. However for \( a_1 = 0 \), we are able to determine the exact solution \( h(\sigma) = -\frac{1}{3} \sigma^{-2} \), \( w(\sigma) = w_1 \sigma^2 + \frac{2 a_9}{a_8} \ln \sigma + w_0 \). It easily follows that when \( w_1 a_2 = 0 \), a scaling symmetry similar to that found in the previous case exists as an isometry for the gravitational background space. In Fig. 5 we present the evolution of the functions \( H(x,y) \), \( W(x,y) \) for the similarity solution given by the symmetry vector \( \{ a_2 X_2 + a_9 X_9 \} \).

### 3.1.9 \( \{ a_2 X_2 + a_3 X_3 + a_8 X_8 \} \)

In a similar way, from the Lie symmetry vector \( a_2 X_2 + a_3 X_3 + a_8 X_8 \) it follows the similarity transformation \( H(x,y) = e^{a_2 x} h(\zeta) \), \( W(x,y) = \frac{a_2}{a_3} x + w(\zeta) \) with reduced system

\[ 0 = h_{\zeta \zeta} \left( 8 (a_3)^3 + (a_8)^3 \zeta^2 (2h + \zeta h_\zeta) \right) + (a_8)^3 (4h + 5\zeta h_\zeta)^{2}(2h + \zeta h_\zeta), \quad (28) \]
solution the spacetime admits an additional isometry which is a similarity solution. From the generator the reduced system is found to be 

\[ \text{Figure 5: Qualitative evolution for the similarity solution of functions } H(x,y), W(x,y) \text{ provided by the application of the symmetry vector } a_2X_2 + a_9X_9. \text{ The plots are for } a_2 = 1, a_9 = 1. \]

\[ 0 = w_{\zeta\zeta} \left( 8(a_3)^3 + (a_8)^3 \zeta^2 (2h + \zeta g_{\zeta}) \right) + (a_8)^2 \zeta^2 h_{\zeta\zeta} (2a_2 + a_8 \zeta h_\zeta) + 2(a_8)^2 (3a_8 \zeta w_\zeta (\zeta h_\zeta + h) + a_2 (5\zeta h_\zeta + 4h)). \]

Again the solution can be written in terms of quadratures. However a closed-form solution exists when \( a_3 = 0 \).

Indeed for \( a_3 = 0 \), the similarity transformation is \( H(x,y) = y^{-2}h(x), \ W(x,y) = -\frac{a_8}{a_3} \ln y^2 + w(x) \) with reduced system \( h_{xx}h_x + 6h_x = 0, 2a_2 + a_8(h_xw_x)_x = 0 \). The closed-form solution is \( h(x) = -\frac{1}{3}x^3, \ w(x) = w_1x^{-1} + \frac{a_8}{a_3} \ln x^2 + w_0 \). The qualitative evolution for this closed-form solution is presented in Fig. 6. Finally, for this exact solution the spacetime admits always a two-dimensional Killing algebra.

### 3.1.10 \( \{ a_3X_3 + a_4X_4 + a_5X_5 \} \)

From the generator \( a_3X_3 + a_4X_4 + a_5X_5 \), it follows \( H(x,y) = h(\mu), W(x,y) = e^{\frac{a_8}{a_3}x}w(\mu) \) in which \( \mu = y - \frac{a_4}{a_3}x \).

The reduced system is found to be \( h_{\mu\mu} \left( (a_4)^3 h_\mu - (a_3)^3 \right) = 0, w_{\mu\mu} \left( (a_4)^3 h_\mu - (a_3)^3 \right) + (a_4)^2 h_{\mu\mu} (a_4 h_\mu - a_5 h) - a_4 a_5 h_\zeta (2w_\zeta a_4 - a_5 w) = 0. \)

The analytic solution of the reduced system is given by the closed form solution \( h(\mu) = \left( \frac{a_4}{a_3} \right)^3 h_{0}, w(\mu) = w_1 e^{\frac{a_8}{a_3}x} \) or \( h(\mu) = h_1 \mu + h_0, w(\mu) = w_1 e^{M+\mu} + w_2 e^{M-\mu} \) in which \( M_{\pm} = a_5 \left( \frac{h_1 a_4^2 \pm a_3 \sqrt{h_1 a_4^2 - a_3}}{h_1 a_4^2 - a_3} \right) \). For this exact solution the spacetime admits an additional isometry which is \( \frac{a_8}{a_3} \partial_x + \partial_y \).

For \( a_3 = 0 \), the similarity transformation is \( H(x,y) = h(x), W(x,y) = e^{\frac{a_8}{a_3}y}w(x) \), while the corresponding similarity solution is \( h(x) = h_1 x + h_0, w(x) = w_1 \sin \left( \frac{a_8}{a_4 \sqrt{a_3}}x \right) + w_2 \cos \left( \frac{a_8}{a_4 \sqrt{a_3}}x \right) \). In this case, the additional
The vector field \( a \) spacetime admits the additional isometry

\[
\text{the closed-form solution}
\]

cition of the symmetry vector

\[
\text{for both the unknown functions.}
\]

terms of quadratures. However for specific values of the coefficient constants we can find closed-form solutions

\[
3.1.11
\]

\[
\{ a_3 X_3 + a_4 X_4 + a_6 X_6 + a_7 X_7 \} \]

The vector field \( a_3 X_3 + a_4 X_4 + a_6 X_6 + a_7 X_7 \) provides the similarity transformation \( H(x, y) = -\frac{a_6 a_4}{2a_3^2} x^2 + \frac{a_6}{a_3} x y + h(\mu) \), \( W(x, y) = -\frac{a_7 a_4}{2a_3^2} x^2 + \frac{a_7}{a_3} x y + w(\mu) \), \( \mu = y - \frac{a_4}{a_3} x \). The reduced system is

\[
0 = h_{\mu \mu} \left( (a_4)^3 h_\mu - (a_3)^3 - (a_4)^2 a_6 \mu \right) + a_4 a_6 \left( a_4 h_\mu - a_6 \mu \right),
\]

(30)

\[
0 = w_{\mu \mu} \left( (a_4)^3 h_\mu - (a_3)^3 - (a_4)^2 a_6 \mu \right) + (a_4)^2 h_{\mu \mu} \left( a_4 w_\mu - a_7 \mu \right) + a_4 \left( a_4 a_6 w_\mu + a_4 a_7 h_\mu - 2a_6 a_7 \mu \right).
\]

(31)

For the latter system we find \( h(\mu) = \left( \frac{a_4}{a_3} \right)^3 \mu + \frac{a_6 a_4}{2a_3^2} \mu^2 + \frac{(a_3)^{\frac{2}{3}} (a_3)^{\frac{1}{3}} + 2a_6 a_7 h_0}{3(a_3)^{\frac{2}{3}} a_6} \), while \( w(\mu) \) is expressed in terms of quadratures. However for specific values of the coefficient constants we can find closed-form solutions for both the unknown functions.

When \( a_4 = 0 \), the similarity transformation reads \( H(x, y) = \frac{a_6}{a_3} x y + h(y) \), \( W(x, y) = \frac{a_7}{a_3} x y + w(y) \) with the closed-form solution \( h(y) = h_1 + h_0 \), \( w(y) = w_1 y + w_0 \). For this exact solution the spacetime admits an additional isometry, the vector fields \( \frac{a_1}{2a_3} \partial_x + t \partial_z \).

For \( a_3 = 0 \), the similarity transformation reads \( H(x, y) = \frac{a_6}{a_4} y^2 + h(x) \), \( W(x, y) = \frac{a_7}{a_4} y^2 + w(x) \) with the exact solution \( h(x) = \frac{2\sqrt{2}}{3} \sqrt{-\frac{a_6}{a_4} x^2} \), \( w(x) = w_1 \sqrt{x} - a_7 \sqrt{-\frac{1}{a_6 a_4} x^2} + w_0 \). Hence, for this exact solution the spacetime admits the additional isometry \( \frac{a_1}{2a_3} \partial_y + t \partial_z \).

Figure 6: Qualitative evolution for the similarity solution of functions \( H(x, y) \), \( W(x, y) \) provided by the application of the symmetry vector \( a_2 X_2 + a_8 X_8 \). The plots are for \( a_2 = 1 \), \( a_8 = -1 \).
Figure 7: Qualitative evolution for the similarity solution of functions $H(x,y)$, $W(x,y)$ provided by the application of the symmetry vector $a_3 X_3 + a_5 X_5 + a_6 X_6$. The plots are for $a_3 = 1$, $a_5 = 2$, $a_6 = 1$, $h_1 = 1$.

3.1.12 $\{a_3 X_3 + a_4 X_4 + a_5 X_5 + a_6 X_6\}$

From the Lie symmetry vector $a_3 X_3 + a_4 X_4 + a_5 X_5 + a_6 X_6$ we find the similarity transformation $H(x,y) = -\frac{a_6 a_4}{2 a_5} x^2 + \frac{a_6}{a_3} x y + h(\mu) , W(x,y) = w(\mu) e^{\frac{a_5}{a_3} x} , \mu = y - a_4 x$ while the reduced system

$$0 = h_{\mu\mu} \left( (a_4)^3 h_{\mu} - (a_3)^3 - (a_4)^2 a_6 h_{\mu} \right) + a_4 a_6 (a_4 h_{\mu} - a_6 h_{\mu}) ,$$

(32)

$$0 = w_{\mu\mu} \left( (a_3)^3 + (a_4)^2 a_6 h_{\mu} - (a_4)^3 h_{\mu} \right) - (a_4)^2 h_{\mu \mu} (a_4 w_{\mu} - a_5 h) + a_4 \left( w_{\mu} (a_4 a_6 - 2a_5 a_6 h_{\mu}) + (a_5)^2 w h_{\mu} \right) - a_5 a_6 (a_4 - a_5 h w).$$

(33)

which can be solved by quadratures.

For $a_4 = 0$, the closed form solution is expressed in terms of the Airy functions, that is, $H(x,y) = \frac{a_6}{a_3} x y + h(y) , W(x,y) = w(y) e^{\frac{a_5}{a_3} x}$, where $h(y) = h_1 y + h_0$, $w(y) = w_1 Ai(\bar{a} y) + w_2 Bi(\bar{a} y)$ in which $\bar{a} = -\left( a_6 (a_5)^2 (a_3)^{-3} \right)^{\frac{1}{3}}$. The resulting spacetime admits a two-dimensional Killing algebra. In Fig. 7 the qualitative evolution of the functions $H(x,y)$ and $W(x,y)$ is given.

On the other hand for $a_3 = 0$, we end with the closed form solution $H(x,y) = \frac{a_6}{2a_4} y^2 + h(x) , W(x,y) = \frac{a_6}{2a_4} y^2 + w(x)$ where $h(x) = \frac{2\sqrt{2}}{3} \sqrt{-a_6 a_4 x^2}$ and $w(x) = w_1 F_{2/3} \left( a' x^{\frac{3}{2}} \right) + w_2 \sqrt{2} F_{4/3} \left( a' x^{\frac{3}{2}} \right)$ where $F$ is the hypergeometric function and $a' = \frac{2\sqrt{2} (a_5)^2}{9 \sqrt{-a_6 (a_4)^2}}$. Again we find that the spacetime admits a two-dimensional Killing algebra.
Moreover, from the vector field $a_3 X_3 + a_4 X_4 + a_6 X_6$ it follows $H(x, y) = -\frac{a_6 a_4}{a_3} x^2 + \frac{a_6}{a_3} x y + h(\mu), W(x, y) = \frac{a_6}{a_3} x + w(\mu)$, $\mu = y - \frac{a_4}{a_3} x$, with reduced system

$$0 = h_{\mu\mu} \left( (a_4)^3 - a_6 (a_4)^2 \mu - (a_4)^3 h_{\mu} \right) + a_4 a_6 (a_4 h_{\mu} - \mu a_6),$$

$$0 = w_{\mu\mu} \left( (a_4)^3 + (a_4)^2 a_6 \mu - (a_4)^3 h_{\mu} \right) - (a_4 w_{\mu} - a_2) \left( (a_4)^2 h_{\mu\mu} - a_4 a_6 \right),$$

Therefore, the closed-form solution of the latter system is $h(\mu) = \left( \frac{a_4}{a_3} \right)^{\frac{1}{3}} \mu + \frac{a_6}{2 a_3} \mu^2 + \frac{(a_4)^2}{3(a_4)^3 a_6} + \frac{(a_4)^3}{2 a_4 (a_4)^3} + \frac{1}{2} (a_3)^2 + 2 a_4 w_{\mu} + h_{0}, w(\mu) = \frac{a_6}{a_3} y + w(x)$ in which $h(x) = \frac{1}{2} \sqrt{-a_4 a_6} x^3 + h_{0}, w(x) = w_1 \sqrt{x} + w_{0}$. In this case, the resulting spacetime admits an additional Killing symmetry the vector field $\partial_y$.

From the vector field $a_1 X_1 + a_2 X_2 + a_3 X_3 + a_6 X_6$ it follows $H(x, y) = \frac{a_2}{a_3} x + a_4 y h (\mu), W(x, y) = \frac{a_6}{a_3} y + w(\mu)$ in which $y_{yy} = 0, w_{yy} = 0$, that is $h(y) = h_{1} y + h_{0}$ and $w(y) = w_{1} y + w_{0}$. In this case the spacetime is maximally symmetric with zero Ricciscalar, that is, it becomes the flat space.

Moreover, from the vector field $a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5$ we find the similarity transformation $H(x, y) = \frac{a_4}{a_3} y + h(x), W(x, y) = w(x) e^{\frac{a_4}{a_3} y}$, where the reduced system is $h_{xx} h_{xx} = 0, a_5 w + (h_{2} w_{x}) x = 0$, with solution $h(x) = h_{1} x + h_{0}, w(x) = w_{1} e^{i \frac{a_4}{a_3} y_{1} x} + w_{2} e^{-i \frac{a_4}{a_3} y_{1} x}$. For $w_{1} w_{2} = 0$, the spacetime admits a three dimensional Killing algebra. The third Killing symmetry is the vector field $\sqrt{-h_{1}} \partial_{x} - \partial_{y}$ for $w_{2} = 0$, or $\sqrt{-h_{1}} \partial_{x} + \partial_{y}$ for $w_{1} = 0$.

In the case where we apply the symmetry vector $a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4$ we end with the similarity solution $H(x, y) = \frac{a_1}{a_3} x + h(\nu), W(x, y) = w(\nu), \nu = y - \frac{2 a_4}{a_3} x$; where $h(\nu) = h_{1} \nu + h_{0}, w(\nu) = w_{1} \nu + w_{0}$ or $h(\nu) = \frac{a_1^3 + 4 a_4 a_6}{8 (a_1)^3} \nu + h_{0}$ with $w(\nu)$ arbitrary. In the first solution the spacetime becomes maximally symmetric and specifically the flat space, while in the case with arbitrary $w(\nu)$ the background geometry admits as additional symmetry vector a translation symmetry as in the previous case.

In this Section we investigate exact static ASD four-manifolds with at least a $G_2$ Lie algebra. In Tables 3 and 4 we summarize all the cases where the unknown functions are solved with the use of closed-form expressions from the application of similarity transformations.
### Table 3: Similarity transformations and exact solutions of static ASD four-manifolds with at least a $G_2$ Lie algebra (1/2)

| Symmetry vector | Similarity transformation | Exact Solution |
|-----------------|---------------------------|----------------|
| $X^4$           | $H = h(x), W = w(x)$      | $h(x) = h_0, w(x)$; $h(x) = h_0(x - x_1), w(x) = w_0(x - x_2)$ |
| $X^8$           | $H = h(x)y^{-2}, W = w(x)$| $h(x) = -\frac{1}{3}x^3, w(x) = w_1x^{-1}$ |
| $X^9$           | $H = h(\sigma), W = w(\sigma), \sigma = yx^{-\frac{3}{2}}$ | $h(\sigma) = -\frac{1}{3}\sigma^{-2}, w(\sigma) = w_0 + w_1\sigma^2$ |
| $a_2X_2 + a_8X_8 + a_9X_9$ | $H = h(\xi)\frac{a_2}{a_9}x\frac{a_8}{a_9}w(x), \xi = \frac{a_2}{a_9}x\frac{a_8}{a_9}$ | $h(\xi) = -\frac{1}{3}\xi^{-2}, w(\xi) = -w_1\xi^{-\frac{2a_8}{a_9}a_9} - 2\frac{a_2}{(a_8 - a_9)}\ln(\xi)$ |
| $a_2X_2 + a_9X_8 + a_9X_9$ | $H = h(\xi)x\frac{a_2}{a_9}lnx + w(\xi), \xi = yx^{-1}$ | $h(\xi) = h_1\xi + h_0, w(\xi) = \frac{a_2}{a_9}\ln(1 + h_0\xi^2) + 2\frac{a_2}{\sqrt{h_0}}\arctan\left(\sqrt{h_0}\xi\right)$ |
| $a_2x_3 + a_8X_8 + a_9X_8$ | $H = h(\sigma)\frac{a_2}{a_9}lnx + w(\sigma), \sigma = yx^{-\frac{3}{2}}$ | $h(\sigma) = -\frac{1}{3}\sigma^{-2}, w(\sigma) = w_1\sigma^{-\frac{2a_8}{a_9}a_9} + 2\frac{a_2}{a_9}\ln(\sigma)$ |
| $a_3X_3 + a_4X_4 + a_5X_5$ | $H = h(\mu)\frac{a_2}{a_9}x\frac{a_5}{a_3}w(\mu), \mu = y - \frac{a_4}{a_3}x$ | $h(\mu) = -\frac{1}{3}\mu^3, w(x) = w_1x^{-1} + \frac{a_2}{a_9}\ln(\xi)$ |
| $a_4X_4 + a_5X_5$ | $H = h(x), W = e^{\frac{a_2}{a_9}y}w(x)$ | $h(x) = \left(\frac{a_2}{a_9}\right)^{\frac{3}{4}}h_0, w = w_1e^{\frac{a_2}{a_9}y}; h(x) = h_1x + h_0,w = w_1e^{M\xi} + w_2e^{M_{-}\xi}$ |

Note: The table continues on the next page with similar transformations and solutions.
| Symmetry vector | Similarity transformation | Exact Solution |
|----------------|--------------------------|----------------|
| $a_3X_3 + a_6X_6 + a_7X_7$ | $H = \frac{a_3}{a_3} xy + h (y)$, $W = \frac{a_3}{a_3} xy + w (y)$ | $h (y) = h_1y + h_0$, $w (y) = w_1y + w_0$. |
| $X_4 + a_6X_6 + a_7X_7$ | $H = \frac{a_6}{a_3} y^2 + h (x)$, $W = \frac{a_7}{a_3} y^2 + w (x)$ | $h (x) = \frac{2\sqrt{2}}{3} \sqrt{-\frac{a_6}{a_4} x^3}$, $w (x) = w_1\sqrt{x} - a_7\sqrt{-\frac{1}{a_6a_4} x^3}$ |
| $a_3X_3 + a_5X_5 + a_6X_6$ | $H = \frac{a_3}{a_3} x y + h (y)$, $W = w (y) e^{\frac{a_3}{a_3} x}$ | $h (y) = h_1y + h_0$, $w (y) = w_1Ai (\bar{a}y) + w_2Bi (\bar{a}y)$ |
| $a_4X_4 + a_5X_5 + a_6X_6$ | $H = \frac{a_6}{a_3} y^2 + h (x)$, $W = \frac{a_7}{a_3} y^2 + w (x)$ | $h (\mu) = \left(\frac{a_4}{a_4}\right)^3 \mu + \frac{a_6}{a_4} \mu^2 + \frac{(a_3)^2}{3(a_4)^{3/2}a_0} \left(\frac{a_3^2 + 2a_6a_7\mu + h_0}{a_0}\right)^{3/2}$, $w (\mu) = w_1\sqrt{2a_6\mu + (a_3)^2} - h_0 + \frac{a_7}{a_4} \mu$ |
| $a_2X_2 + a_3X_3 + a_4X_4 + a_6X_6$ | $h (\mu) = \left(\frac{a_4}{a_4}\right)^3 \mu + \frac{a_6}{a_4} \mu^2 + \frac{(a_3)^2}{3(a_4)^{3/2}a_0} \left(\frac{a_3^2 + 2a_6a_7\mu + h_0}{a_0}\right)^{3/2}$, $w (\mu) = w_1\sqrt{2a_6\mu + (a_3)^2} - h_0 + \frac{a_7}{a_4} \mu$ |
| $a_2X_2 + a_4X_4 + a_6X_6$ | $H = \frac{a_6}{a_3} y^2 + h (x)$, $W = \frac{a_6}{a_3} y + w (x)$ | $h (x) = \frac{2\sqrt{3}}{3} \sqrt{-\frac{a_6}{a_4} x^2}$, $w (x) = w_1\sqrt{x} + w_0$. |
| $a_1X_1 + a_2X_2 + a_3X_3 + a_6X_6$ | $H (x, y) = \frac{a_1}{a_4} x + h (y)$, $W (x, y) = \frac{a_1}{a_3} x + w (y)$ | $h (y) = h_1y + h_0$, $w (y) = w_1y + w_0$. |
| $a_1X_1 + a_3X_3 + a_4X_4 + a_5X_5$ | $H (x, y) = \frac{a_1}{a_4} y + h (x)$, $W (x, y) = w (x) e^{\frac{a_1}{a_4} y}$ | $h (x) = h_1x + h_0$, $w (x) = w_1e^{-i\frac{a_6}{a_3} x} + w_2e^{-i\frac{a_7}{a_3} h_1 x}$ |
| $a_1X + a_2X_2 + a_3X_3 + a_4X_4$ | $H (x, y) = \frac{a_1}{a_3} x + h (\nu)$, $W (x, y) = w (\nu)$, $\nu = -2\frac{a_4}{a_3} x$ | $h = h_1\nu + h_0$, $w = w_1\nu; \ h = \frac{a_3^2 + 4a_7^2a_3^2}{8(a_4)^2} \nu + h_0$, $w$ arbitrary |
4 Nonstatic spacetime

In the case where the spacetime is nonstatic, the Lie point symmetries of the system (2), (3) are

\[ Y^1 = \beta_1 (t) \partial_H , \quad Y^2 = \beta_2 (t) \partial_W , \quad Y^3 = \left( \frac{1}{2} \beta_3 (t) x + \frac{1}{2} (\beta_3)_{tt} \right) + \beta_3 (t) \partial_x , \]
\[ Y^4 = \left( \frac{1}{2} (\beta_4 (t))_{tt} xy - \frac{1}{2} (\beta_4 (t))_{ttt} y^3 \right) + \frac{1}{2} (\beta_4 (t))_{t} \partial_x + \beta_4 (t) \partial_y , \]
\[ Y^5 = W \partial_W , \quad Y^6 = \beta_6 (t) y \partial_H , \quad Y^7 = \beta_7 (t) y \partial_W , \]
\[ Y^8 = \left( -\frac{1}{3} H (\beta_8)_{t} - \frac{1}{6} (\beta_8)_{tt} x^2 - (\beta_8)_{ttt} xy^4 - \frac{1}{12} (\beta_4)_{tttt} y^4 \right) \partial_H - \frac{W}{3} (\beta_8)_{t} \partial_W \]
\[ + \left( \frac{1}{3} x (\beta_8)_t + \frac{1}{6} y^2 (\beta_8)_{tt} \right) + \frac{2}{3} y (\beta_8)_t \partial_y + \beta_8 (t) \partial_t \]
\[ Y^9 = \frac{2}{3} H \partial_H - \frac{1}{3} W \partial_W + \frac{1}{3} x \partial_x + \frac{1}{6} y \partial_y . \]

We observe that the gravitational system (2), (3) admits an infinite Lie algebra. The latter infinite Lie algebra reduces to the finite Lie algebra of the static case if we assume that all the coefficient function \( \beta_8 (t) \) is constant and then apply the similarity transformation which follows from \( Y^8 \), the reduced system is that which was studied in the previous section. The infinite Lie symmetries can be categorized in terms of nine families, where the eight families describe infinite symmetries with arbitrary parameters the functions \( \beta (t) \).

The nonzero commutators of the admitted Lie algebra are

\[ [Y^1, Y^8] = -\frac{1}{3} (\beta_1)_{t} \partial_H - 3 \beta_6 \beta_8 (t) \partial_H = -\frac{1}{3} Y^1 , \quad [Y^1, Y^9] = \frac{2}{3} Y^1 , \]
\[ [Y^2, Y^5] = Y^2 , \quad [Y^2, Y^8] = -\frac{1}{3} (\beta_2)_{t} + 3 \beta_6 \beta_8 (t) \partial_W = Y^2 , \quad [Y^2, Y^9] = -\frac{1}{3} Y^2 , \]
\[ [Y^3, Y^4] = \frac{1}{2} (2 \beta_4 (\beta_3)_t - \beta_3 (\beta_4)_t) y \partial_H = Y^6 , \quad [Y^3, Y^8] = -\frac{1}{3} Y^3 (3 \beta_8 (\beta_3)_t - \beta_3 (\beta_8)_t) , \]
\[ [Y^3, Y^9] = \frac{1}{3} Y^3 , \quad [Y^4, Y^6] = \beta_4 Y^6 = Y^6 , \quad [Y^4, Y^7] = \beta_4 Y^7 = Y^7 , \]
\[ [Y^4, Y^8] = -\frac{1}{3} Y^8 (3 \beta_8 (\beta_3)_t - 2 \beta_4 (\beta_3)_t) , \quad [Y^4, Y^9] = \frac{1}{6} Y^4 , \quad [Y^5, Y^7] = -Y^7 , \]
\[ [Y^6, Y^8] = Y^6 ((\beta_8)_t , \quad [Y^6, Y^9] = \frac{1}{2} Y^6 , \quad [Y^7, Y^8] = Y^7 ((\beta_7)_t) , \quad [Y^7, Y^9] = -\frac{1}{2} Y^7 , \]

and

\[ [Y^8 (\beta_8) , Y^8 (\beta_8)] = Y^8 \left( \beta_8 (\beta_8')_t - \beta_8' (\beta_8)_t \right) . \]

From the above result it follows that the gravitational system admits infinite number of similarity transformations which can be used to determine an infinite number of similarity transformations. Let us demonstrate the application of the Lie symmetries to determine nonstatic ASD exact solutions.

Consider now the Lie symmetry vector \( Y^8 \) with \( \beta_8 (t) = t \), that is

\[ Y^8_{\beta_8 \rightarrow t} = t \partial_t + \frac{1}{3} (x \partial_x + 2 y \partial_y - H \partial_H - W \partial_W) . \]
Application of the latter symmetry in the system (2), (3) provides the similarity transformation $H(t, x, y) = t^{-\frac{1}{3}} h(\chi, \psi)$, $W(t, x, y) = w(\chi, \psi)$ in which $\chi = xt^{-\frac{1}{2}}$ and $\psi = yt^{-\frac{1}{2}}$. The reduced equations for a system of two partial differential equations are

$$3h_{\psi\psi} + \chi h_{\chi x} + 2\psi h_{\psi x} + 2h_{\chi x} + 3h_{x} h_{\chi x} = 0,$$  \hspace{1cm} (36)

$$3w_{\psi\psi} + \chi w_{\chi x} + 2\psi w_{\psi x} + 2w_{\chi x} + 3(w_{x} h_{\chi x}) = 0.$$  \hspace{1cm} (37)

The latter system consists a nine dimensional Lie algebra with elements

$$Y_{1'} = \partial H, \quad Y_{2'} = \partial W, \quad Y_{3'} = -\frac{1}{3} \left( \chi + \frac{2}{3} \psi^{2} \right) + \partial \chi,$$

$$Y_{4'} = \left( \frac{1}{5} \chi \psi - \frac{2}{81} \psi^{3} \right) \partial H + \frac{1}{3} \partial x + \partial y,$$

$$Y_{5} = W \partial W, \quad Y_{6'} = Y \partial H, \quad Y_{7'} = Y \partial W,$$

$$Y_{8'} = \left( - \frac{2}{9} (\chi^{2} - \chi \psi^{2} + \frac{1}{6} \psi^{4}) \right) \partial H + \left( \frac{2}{9} \chi + \frac{2}{9} \psi^{2} \right) \partial \chi + \psi \partial \psi,$$

$$Y_{9'} = H \partial H + \frac{1}{2} x \partial x + \frac{1}{4} \partial y.$$

These vector fields form a finite Lie algebra which can be used for further reduction as we did in the previous section. All these symmetry vectors are reduced symmetries which follow from the infinite symmetries of the time-dependent system. We demonstrate the application of the Lie symmetries by present some exact solutions which follows from the application of the latter similarity transformations.

From $Y_{9'}$ there follows the similarity solution $h(\chi, \psi) = -\frac{1}{3} \chi \psi^{2}$, $w(\chi, \psi) = \left( \frac{\psi^{2}}{\chi} \right)^{\frac{1}{3}} F\left( \frac{1}{2}; \frac{2}{3}; \frac{4}{3}; \left( \frac{\psi^{2}}{\chi} \right) \right)$ in which $F$ is the hypergeometric function. For the latter solution the spacetime is nonstatic. Moreover, from the symmetry vector $Y_{3'} + Y_{2'}$ we find the similarity solution $h(\chi, \psi) = -\frac{\chi^{2}}{6} + \frac{\chi \psi}{9} - \frac{5}{324} \psi^{4} + h_{1} \psi + h_{0}$, $w(\chi, \psi) = \chi - \frac{1}{3} \psi^{2} + w_{1} \psi + w_{0}$ which also provides a nonstatic spacetime. In a similar way from the vector field $Y_{4'}$ there follows the similarity transformation $h(\chi, \psi) = -\frac{1}{6} \chi^{2} + \frac{1}{3} \chi \psi^{2} - \frac{5}{324} \psi^{4} + h_{0}$, $w(\chi, \psi) = w(\psi^{2} - 6\chi)$ where $w$ is an arbitrary function, or $h(\chi, \psi) = -\frac{1}{6} \chi^{2} + \frac{1}{3} \chi \psi^{2} - \frac{5}{324} \psi^{4} + h_{1} (\psi^{2} - 6\chi) + h_{0}$, $w(\chi, \psi) = w_{1} \left( \psi^{2} - 6\chi \right) + w_{0}$. Again for this solution the resulting spacetime admits only one isometry vector, the field $\partial_{z}$.

5 Conclusions

In this work we investigated exact ASD four-dimensional null Kähler manifolds by using the theory of Lie symmetries. In particular, we studied the existence of one-parameter point transformations where the system (2), (3) remains invariant. In the general case we found that there is an infinite number of Lie point symmetries, which belong to eight families of infinite Lie symmetries which depend on eight time-dependent coefficient variables. The infinite number of symmetries can be related with the property that an ASD null Kähler manifold under a conformal transformation remains an ASD manifold.

Furthermore, we studied the algebraic properties of the admitted Lie point symmetries and we demonstrated the application of them on the determination of exact solutions. We focused on the special case where the spacetime admits a two dimensional Killing algebra and specifically when the spacetime is static. In such
specific case the admitted Lie point symmetries of the reduced system are finite; they are nine. All these symmetries are reduced symmetries from the infinite symmetries of the general case.

For the static case and for the finite Lie algebra we calculated the commutators and the adjoint representation. The latter applied in order to construct the one-dimensional optimal system, an essential tool for the complete classification of the independent similarity transformations. Finally, we use the invariants of the Lie point symmetries to reduce the original system (2), (3) into a system of ordinary differential equations which can be solved in all cases by quadratures. Thus, we were able to determine exact closed-form solutions. For this exact spacetimes we investigate the admitted isometries. For simplicity of the presentation the results were presented in tables.

In a future work we plan to investigate the more general scenario of ASD spacetimes without any isometry.

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