Reciprocal Multi-Agent Systems with Triangulated Laman Graphs

Part I: The Morse-Bott index Formula

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Abstract—In this paper, we will consider a special class of reciprocal multi-agent (RMA) systems. The interaction pattern of each RMA system in this class is defined by a particular type of Laman graphs, as we call the triangulated Laman graphs. We develop, among other things, a basic formula for computing the Morse-Bott index of a critical orbit in a RMA system. This formula relates the Morse-Bott index to a geometric partition of a critical configuration. This partition decomposes a critical configuration into union of critical line sub-configurations, and the formula then computes the Morse-Bott index of the associated critical orbit by summing over the Morse-Bott indices of critical orbits of these decomposed sub-configurations. This Morse-Bott index formula has a potential impact on design and control of RMA systems because it enables us to locate or place critical orbits with various Morse-Bott indices over the entire configuration space.

I. INTRODUCTION

The class of reciprocal multi-agent systems (or in short, RMA systems) has been one of the most-studied models in multi-agent systems. Each RMA system is defined by an undirected graph, together with a family of interaction laws. To be more specific, we let $G := (V,E)$ be an undirected graph with $V := \{1, \cdots, N\}$ the set of vertices, and $E$ the set of edges. Let $V_i$ be the set of vertices adjacent to vertex $i$, two vertices are said to be adjacent if there is an edge in between. The equations of motions for a set of $N$ agents $\vec{x}_1, \cdots, \vec{x}_N$ in $\mathbb{R}^2$ are then described by

$$\dot{\vec{x}}_i = \sum_{j \in V_i} f_{ij}(d_{ij}) \cdot (\vec{x}_j - \vec{x}_i), \quad \forall i = 1, \cdots, N \tag{1}$$

Each $f_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous differentiable function defined over the set of positive real numbers, modeling the interaction between $\vec{x}_i$ and $\vec{x}_j$. As usual, we assume in this paper that the interaction between $\vec{x}_i$ and $\vec{x}_j$ depends only on their relative distance $d_{ij}$, as seen from equation (1). Also, we assume that $f_{ij}$ is identical with $f_{ji}$ for all $(i,j) \in E$, i.e., interactions among agents are reciprocal.

An important property about the class of RMA systems is that agents evolve as a gradient flow, with the associated potential function defined by

$$\Phi(\vec{x}_1, \cdots, \vec{x}_N) := \sum_{(i,j) \in E} \int_1^{d_{ij}} x f_{ij}(x) dx \tag{2}$$

We note that the potential function $\Phi$ depends only on relative distances between agents, thus it is invariant if we translate and/or rotate the entire configuration. In mathematics, it simply says that $\Phi$ is an equivariant function with respect to the group action of rigid motion, as we will describe in detail in the next section.

As steepest descent equations provide the direct demonstration of the existence of a local minima and provide an easily implemented algorithm. So RMA systems are utilized as models for decentralized formation control. Interaction laws are designed so that the target formation becomes one of the local minima of the potential function. Questions concerning the level of interaction laws for organizing such systems, questions about system convergence, and questions about local stability, robustness and performance have all been treated to some degree. We here refer readers to [1]–[11] for some of the related works.

It has observed that the potential function associated with a RMA system often has saddle points, and multiple local minima. Thus, the resulting stable equilibria divide the configuration space into disjoint regions of attraction. Knowledges about the total number of stable equilibria, knowledges about the locations of these stable equilibria, and knowledges about how regions of attractions of stable equilibria divide the underlying space will be valuable. But, in general, they are all hard problems, even counting critical line configurations are challenging [3].

In this paper, we will focus on a special class of RMA systems, equipped with a particular type Laman graphs, as we call the triangulated Laman graphs. We establish, among other things, two relevant facts about this class of RMA systems. We first show that for each equilibrium configuration, there is a geometric decomposition of the configuration into union of line sub-configurations each of which is also an equilibrium. We then follow this decomposition to develop a formula for computing the Morse-Bott index of a critical orbit (i.e., the orbit of an equilibrium with respect to the group action of rigid motion, a precise definition will be given in the next section). The formula says that the Morse-Bott index of a critical orbit can be computed as the sum of the Morse-Bott indices of critical orbits of these decomposed line sub-configurations.

This formula, together with the geometric decomposition, is a useful tool for counting and locating critical orbits of various Morse-Bott indices in the configuration space. For example, we have applied this formula in [10] to find all the stable critical orbits. In addition, the Morse-Bott index formula will also be used to establish a fundamental property of this class RMA systems. We show in [11] that the potential

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A. About system convergence

This paper.

tions which are necessary for stating the main theorems of RMA systems, and meanwhile introduce some key defini-

be the

organization of the proof will be given after the statements devoted to the proof of these two theorems, and a detailed organization of the proof will be given after the statements of the main theorems.

II. Preliminary results

In this section, we will describe some known facts about RMA systems, and meanwhile introduce some key defini-
tions which are necessary for stating the main theorems of this paper.

A. About system convergence

Let \( G = (V, E) \) be an undirected graph of \( N \) vertices. Let \( P \) be the configuration space defined by

\[
P := \{ (\bar{x}_i, \cdots, \bar{x}_N) \in \mathbb{R}^{2 \times N} | \bar{x}_i \neq \bar{x}_j, \forall (i, j) \in E \}
\]

It is the underlying space of system \([1]\). Since we will assume infinite repulsion at zero separation, we exclude configurations with collisions of adjacent agents so that equation \([1]\) is well-defined over the entire set \( P \).

Now suppose each interaction law \( f_{ij} \) is continuous differentiable, satisfying the next two conditions

1. Strong repulsion.

\[
\begin{align*}
\lim_{d \to 0} df_{ij}(d) & \to -\infty \\
\lim_{d \to 0} \int_1^d x f_{ij}(x)dx & \to \infty
\end{align*}
\]

2. Long-range attraction. \( \lim_{d \to \infty} f_{ij}(d) > 0 \)

Then each solution of system \([1]\), with an initial condition contained in \( P \), exists for all time, and converge to the set of equilibria of system \([1]\). In other words, there is neither collision nor escape of agents along the evolution. In fact, the same results will still be valid if we relax the long-

range attraction condition by assuming fading attraction, i.e, \( f(d) > 0 \) for sufficiently large \( d \) but \( \lim_{d \to \infty} df_{ij}(d) = 0 \). A complete proof can be found in [9].

In this paper, we assume that each \( f_{ij} \) is continuous differentiable, and satisfies the two conditions above. Thus, we will assume the result about convergence of system \([1]\).

B. About equivariance of the potential function

We first define the group action of rigid motion on the configuration space. Let \( SE(2) \) be the special Euclidean group for \( \mathbb{R}^2 \), each element \( \gamma \) in \( SE(2) \) can be represented by a pair \((\theta, \bar{v})\) with \( \theta \) in the special orthogonal group \( SO(2) \) and \( \bar{v} \) a vector in \( \mathbb{R}^2 \). In this representation, the group multiplication of two elements \( \gamma_1 = (\theta_1, \bar{v}_1) \) and \( \gamma_2 = (\theta_2, \bar{v}_2) \) is given by

\[
\gamma_2 \cdot \gamma_1 = (\theta_2 \theta_1, \theta_2 \bar{v}_1 + \bar{v}_2)
\]

We now define a \( SE(2) \)-action on \( P \) by sending \( \gamma \) in \( SE(2) \) and \( p = (\bar{x}_1, \cdots, \bar{x}_N) \) in \( P \) to

\[
\gamma \cdot p := (\theta \bar{x}_1 + \bar{v}, \cdots, \theta \bar{x}_N + \bar{v})
\]

This group action is often referred as the group action of rigid motion because it preserves the shape of a configuration. In this paper, we let

\[
\mathcal{O}_p := SE(2) \cdot p
\]

be the orbit of \( p \) with respect to the \( SE(2) \)-action. We here note the fact that each orbit \( \mathcal{O}_p \) is a smooth submanifold of \( P \), diffeomorphic to \( S^1 \times \mathbb{R}^2 \), with \( \dim \mathcal{O}_p = 3 \).

We here note that the group action of rigid motion also applies to any sub-configuration. Let \( \mathcal{G}_i = (V_i, E_i) \) be a sub-graph of \( \mathcal{G} \) with \( V_i = \{i_1, \cdots, i_k\} \). Let \( P_i \) be the collection of sub-configurations induced by \( \mathcal{G}_i \), i.e,

\[
P_i := \{ (\bar{x}_{i_1}, \cdots, \bar{x}_{i_k}) | \bar{x}_{ij} \neq \bar{x}_{ik}, \forall (ij, ik) \in E_i \}
\]

The group action of rigid motion on \( P_i \) is then simply defined by sending \( \gamma = (\theta, \bar{v}) \) and \( p_i = (\bar{x}_{i_1}, \cdots, \bar{x}_{i_k}) \) to

\[
\gamma \cdot p_i = (\theta \bar{x}_{i_1} + \bar{v}, \cdots, \theta \bar{x}_{i_k} + \bar{v})
\]

and we denote by \( \mathcal{O}_{p_i} \) the orbit of \( p_i \) in \( P_i \).

An important observation of the the potential function \( \Phi \) is that \( \Phi \) is an equivariant function with respect to the \( SE(2) \)-action, i.e,

\[
\Phi(p) = \Phi(\gamma \cdot p)
\]

for any \( p \in P \) and any \( \gamma \in SE(2) \). Similarly, this property of equivariance applies to any sub-system. Let \( \mathcal{G}_i = (V_i, E_i) \) be a sub-graph of \( \mathcal{G} \) with \( V_i = \{i_1, \cdots, i_k\} \). Let \( \Phi_i \) be the potential function associated with the sub-system induced by \( \mathcal{G}_i \), i.e,

\[
\Phi_i(p_i) = \sum_{(j,k) \in E_i} \int_1^{d_{jk}} x f_{jk}(x)dx
\]

then we have \( \Phi_i(\gamma \cdot p_i) = \Phi_i(p_i) \).

A consequence of having an equivariant gradient system is that we have continuum equilibria. Suppose \( p \) is an equilibrium of system \([1]\), then so is \( p' \in \mathcal{O}_p \). In this case, we say \( \mathcal{O}_p \) is a critical orbit.
III. TRIANGULATED LAMAN GRAPHS

In this section, we will introduce the notion of triangulated Laman graph, as one of the key notions of this paper.

First we say an undirected graph $G$ of $N$ vertices is a Laman graph if each $k$-vertex subgraph of $G$, $1 \leq k \leq N$, has at most $(2k-3)$ edges, while the whole graph has exactly $(2N-3)$ edges. It is well known that each Laman graph can be constructed via a Henneberg construction [12].

Each Laman graph $G$ is rigid in $\mathbb{R}^2$ [13], i.e., if we place the vertices of $G$ on a plane in a general position (i.e., coordinates of vertices are algebraically independent over $\mathbb{R}$), then only rotations and translations will preserve the lengths of all graph edges. A Laman graph $G$ is also minimally rigid, i.e., if we remove an edge from the graph, then the resulting graph won’t be rigid anymore.

Because of this property of minimal rigidity, network topologies in multi-agent systems are often designed to be Laman graphs. In many cases, they are most effective and parsimonious way for agents to communicate with each other to maintain rigid formation in the plane. Also in problems concerning operations on network topology such as merging and splitting, minimal rigid graphs are more convenient to deal with.

In this paper, we will consider a special class of Laman graphs, as we call the triangulated Laman graphs (TLGs). A Laman graph is a TLG if it can be constructed via a special Henneberg construction. We start with one single edge, and then join a new vertex, at each step, to two adjacent existing vertices via two new edges. An example of a TLG is illustrated in figure 1.

![An example of a TLG. Start with edge (1,2), then subsequently join vertices 3, 4 and 5 to two existing adjacent vertices.](image)

We here note that the author in [14] considers a similar, but directed formation control framework, known as the leader/first-follower formation. This formation framework is constructed by starting from a single directed edge, and then subsequently joining a new vertex via two directed edges pointing to two existing vertices. Though the construction is very similar to the way we define a TLG, yet this leader/first-follower formation framework relies on the existence of a well-established hierarchy, especially the existence of a leader and its first follower. But such a hierarchy is unnecessary in our case. In fact, as we will see in section VII a TLG is base-edge irrelevant in the sense that for any edge of a TLG, there is a Henneberg construction with this edge as its base edge.

In the rest of this paper, we will assume that a RMA system is equipped with a TLG, and we explore relevant properties associated with this special class of RMA systems. Two main results, summarized in theorem 1 and theorem 2, will be stated in the next section.

IV. THE MAIN THEOREMS

In this section, we will state the two main theorems of this paper.

Theorem 1 (The canonical partition): Let $G = (V,E)$ be a TLG, and let $p$ be a configuration in $P$. There is a unique partition of $E$ into disjoint nonempty subsets $E_1, \cdots, E_m$ satisfying the next four conditions

a. let $G_i = (V_i, E_i)$ be the subgraph induced by $E_i$, then each $G_i$ is a TLG;
b. let $p_i$ be the sub-configuration of $p$ induced by $G_i$, then each $p_i$ is a line configuration;
c. if, in addition, $p$ is an equilibrium, then each $p_i$ is an equilibrium of the sub-system induced by $G_i$;
d. if there is another partition of $E$ into $E'_1, \cdots, E'_m$ which satisfies both conditions a) and b), then each $E'_j$ is a subset of $E_j$ for some $j = 1, \cdots, m$.

In this paper, we refer to this unique partition as the canonical partition of $E$ associated with $p$ (or simply the canonical partition if there is no confusion).

The second theorem is about a formula for computing the Morse-Bott index formula. So before stating the theorem, we need to first define what is the Morse-Bott index of a critical orbit. Let $\mathcal{O}_p$ be a critical orbit of system (1), and let $H_p$ be the Hessian matrix of $\Phi$ at $p$, i.e.,

$$H_p := \frac{\partial^2 \Phi(p)}{\partial p^2}$$

Let $n_-(H_p)$, $n_0(H_p)$, and $n_+(H_p)$ be the numbers of positive, zero, and negative eigenvalues of $H_p$, respectively. The Morse-Bott index and co-index of $\mathcal{O}_p$ are then defined to be $n_-(H_p)$ and $n_+(H_p)$ respectively. (We note that the set of eigenvalues of $H_p$ is invariant as $p'$ varies over $\mathcal{O}_p$, so the definition above is independent of the choice of $p'$.)

Theorem 2 (The Morse-Bott index formula): Let $G$ be a TLG, and let $\mathcal{O}_p$ be a critical orbit of system (1) in $P$. Let $\{G_i\}_{i=1}^m$ and $\{p_i\}_{i=1}^m$ be sub-graphs of $G$ and sub-configurations of $p$ respectively, associated with the canonical partition. Let $\Phi_i$ be the potential function associated with the sub-system induced by $G_i$, and let $H_{p_i}$ be the Hessian matrix of $\Phi_i$ at $p_i$. Then we have

$$\begin{cases} n_-(H_p) = \sum_{i=1}^m n_-(H_{p_i}) \\ n_+(H_p) = \sum_{i=1}^m n_+(H_{p_i}) \end{cases}$$
and in this paper, we refer to this set of expressions as the Morse-Bott index formula.

As advertised earlier, the rest of this paper is devoted to the proof of the two theorems above. Section V through section VII are for theorem 1, while section VIII through section X are for theorem 2. In section V, we will define the canonical partition by following a Henneberg construction, and we will verify both conditions a) and b) along the construction. In section VI and section VII, we will verify condition c) and condition d) respectively. To prove theorem 2, we will first investigate a geometric question in section VIII. We ask whether we can perturb one sub-configuration while preserving shapes of all the others? The answer is confirmative and we will prove this fact in the same section. In section IX, we will explicitly compute the Hessian matrix $H_p$ of $\Phi$ at an equilibrium $p$, as well as the Hessian matrix $\mathbb{H}_p$ of $\Phi$ at any sub-configuration $p_i$. A key expression which relates $H_p$ to the set $\{H_p(1) \}_{i=1}^m$ will be given there. In the last section, we will prove theorem 2, and an important implication of the Morse-Bott index formula will also be given there.

V. THE CANONICAL PARTITION

In this section, we will explicitly construct the canonical partition. In particular, we construct it in a way that it automatically satisfies conditions a) and b) in the statement of theorem 1. The rest two conditions will be verified in the next two sections.

Let $\mathcal{G} = (V,E)$ be a TLG, and let $p$ be a configuration in $P$. We will now introduce the canonical partition of $E$ associated with $p$. Choose a Henneberg construction of $\mathcal{G}$, and we label the vertices with respect to the order of the construction. The partition is then defined inductively by following the construction.

**Base case.** Start with the subgraph $G' = (V',E')$ of $G$ consisting of vertices $V' = \{1,2\}$. Since there is only one edge $(1,2)$ in $E'$, the partition of $E'$ is trivial.

**Inductive step.** Suppose now $G' = (V',E')$ is a subgraph of $G$ consisting of vertices $V' = \{1,\cdots,n-1\}$, and we have partitioned $E'$ into disjoint subsets as

$$E' = E'_1 \cup \cdots \cup E'_m$$

(13)

We assume that the vertex $n$ joins to vertices $i$ and $j$ via edges $(i,n)$ and $(j,n)$, and we will describe the rule of updating the partition by taking into account $(i,n)$ and $(j,n)$.

Without loss of generality, we assume that the edge $(i,j)$ is in $E'_1$, then there are two cases:

**Case I.** If $\bar{x}_i$, $\bar{x}_j$ and $\bar{x}_n$ are aligned, then we update the partition by adding $(i,n)$ and $(j,n)$ into $E'_1$.

**Case II.** If $\bar{x}_i$, $\bar{x}_j$ and $\bar{x}_n$ are not aligned, then we update the partition by joining two singletons, i.e,

$$E'_1 \cup \cdots \cup E'_m \cup \{(i,n)\} \cup \{(j,n)\}$$

(14)

By following the Henneberg construction, we then derive the canonical partition of $E$ associated with $p$. An example of the canonical partition is illustrated in figure 2.

We note that the definition of the canonical partition apparently depends on the choice of a Henneberg construction of $G$, yet the next lemma shows that it does not.

**Lemma 3:** The canonical partition is independent of the choice of the Henneberg construction.

**Proof:** The proof is done by induction on the number agents.

**Base case.** The lemma is trivially true in the case $N = 2$ because there is only one Henneberg construction.

**Inductive step.** Suppose the lemma hold for $N < n$, and we prove for the case $N = n$. First we choose a Henneberg construction, call it $\tau_1$. Label the vertices of the graph with respect to the order of $\tau_1$. Let $i$ and $j$ be the two vertices that the last vertex $n$ joins to in $\tau_1$.

Now choose another Henneberg construction, call it $\tau_2$. We may as well assume that neither $(i,n)$ nor $(j,n)$ is the base edge because otherwise, say $(i,n)$ is the base edge in $\tau_2$, then $j$ has to be the next vertex joining to the graph so then the 3-cycle $(i,j,n)$ is formed. But this is the same as we start with the base edge $(i,j)$ and then joins $n$ to the graph.

On the other hand, if neither $(i,n)$ nor $(j,n)$ is the base edge, then vertex $n$ has to join to vertices $i$ and $j$ via edges $(i,n)$ and $(j,n)$, and there is no vertex joining to $n$ in $\tau_2$. So it does not matter when we join $n$ to the graph as long as vertices $i$ and $j$ are present along $\tau_2$, and in the rest of this proof we will assume that vertex $n$ is the last vertex joining to the graph along $\tau_2$.

Let $G' = (V',E')$ be the subgraph of $G$ defined by restricting $G$ to vertices $V' = \{1,\cdots,n-1\}$, then $G'$ is a TLG. As the vertex $n$ is the last vertex joining to $G$ along both $\tau_1$ and $\tau_2$, so there are two well-defined Henneberg constructions $\tau_1'$ and $\tau_2'$ of $G'$ by restricting $\tau_1$ and $\tau_2$ on $G'$.

By induction, the two Henneberg constructions $\tau_1'$ and $\tau_2'$ of $G'$ give rise to the same canonical partition, say $E' = \cup_{i=1}^m E'_i$. Suppose $E'_1$ contains the edge $(i,j)$, then either $\tau_1'$
or \( \tau_2 \) will add the two edges \((i, n)\) and \((j, n)\) into \(E'_1\). In other words, the canonical partition derived via \( \tau_1 \) coincides with the canonical partition via \( \tau_2 \).

Let \( G \) be a TLG, and let \( p \) be a configuration. Let \( \{G_i\}_{i=1}^m \) and \( \{p_i\}_{i=1}^m \) be subgraphs of \( G \) and sub-configurations of \( p \) respectively, associated with the canonical partition. Then it is straightforward by construction that each \( G_i \) is a TLG, and each \( p_i \) is a line configuration by following the Henneberg construction. Thus, we have shown that the canonical partition satisfies both conditions a) and b).

We here also note a fact that the canonical partition, by its way of construction, is invariant with respect to the \( SE(2) \)-action. Let \( p' \) be any configuration in the orbit \( \mathcal{O}_p \), then the canonical partition of \( E \) associated with \( p' \) agrees with the canonical partition of \( E \) associated with \( p \). So there will be no ambiguity by saying the canonical partition of \( E \) associated with an orbit \( \mathcal{O}_p \).

VI. ON THE EQUILIBRIUM CONDITION ASSOCIATED WITH THE CANONICAL PARTITION

In this section, we assume that \( p \) is an equilibrium of system \( \Gamma \), and we verify condition c) in the statement of theorem \( \Gamma \).

Lemma 4: Let \( G \) be a TLG, and let \( p \) be an equilibrium of system \( \Gamma \). Let \( \{G_i\}_{i=1}^m \) and \( \{p_i\}_{i=1}^m \) be subgraphs and sub-configurations respectively, associated with the canonical partition. Then each \( p_i \) is an equilibrium of the sub-system induced by \( G_i \).

Proof: The proof goes along with a Henneberg construction and will be done by induction on the number of agents.

Base case. Suppose \( G \) consists only of two vertices, then the statement is trivially true.

Inductive step. Suppose the lemma holds for any \( N \) with \( N < n \), and we prove for the case \( N = n \). We again assume that the vertices of \( G \) are labeled with respect to the order of a Henneberg construction, and we assume that the vertex \( n \) joins to the graph via edges \((i, n)\) and \((j, n)\) to \( i \) and \( j \). There are two cases depending on whether or not the three agents \( \bar{x}_i, \bar{x}_j \) and \( \bar{x}_n \) are aligned or not.

Case I. We assume that \( \bar{x}_i, \bar{x}_j \) and \( \bar{x}_n \) are not aligned. Then \( f_{mn}(d_{in}) \) and \( f_{jn}(d_{jn}) \) have to vanish because the two vectors \((\bar{x}_i-\bar{x}_n)\) and \((\bar{x}_j-\bar{x}_n)\) are linearly independent, and

\[
\bar{x}_n = f_{mn}(d_{in})(\bar{x}_i-\bar{x}_n) + f_{jn}(d_{jn})(\bar{x}_j-\bar{x}_n) = 0
\]  

Let \( G' := (V', E') \) be a subgraph of \( G \) defined by restricting \( G \) to the set of vertices \( V' := \{1, \ldots, n-1\} \), and let \( p' \) be the sub-configuration formed by agents \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1} \). Since \( f_{mn}(d_{in}) = f_{mn}(d_{jn}) = 0 \), the agent \( \bar{x}_n \) does not interact with any other agent at \( p \). So \( p' \) is an equilibrium of the sub-system induced by \( G' \).

Let \( E'_1, \ldots, E'_{m'} \) be the subsets of \( E' \) derived by applying the canonical partition of \( E' \) associated with \( p' \). Then by induction, each sub-configuration \( p'_i \) is an equilibrium of the sub-system induced by \( \mathcal{G}' \). On the other hand, the canonical partition of \( E \) associated with \( p \) is given by

\[
E = E'_1 \cup \cdots \cup E'_{m'} \cup \{(i, n) \cup (j, n)\}
\]

and the two sub-configurations \( p_{m'+1} = (\bar{x}_i, \bar{x}_n) \) and \( p_{m'+2} = (\bar{x}_j, \bar{x}_n) \) are both equilibria since \( f_{mn}(d_{in}) = f_{jn}(d_{jn}) = 0 \).

Case II. We assume that \( \bar{x}_i, \bar{x}_j \) and \( \bar{x}_n \) are aligned. Since agent \( \bar{x}_n \) is balanced, we then have

\[
f_{mn}(d_{in}) \cdot (\bar{x}_i - \bar{x}_n) + f_{jn}(d_{jn}) \cdot (\bar{x}_j - \bar{x}_n) = 0
\]

We now choose a function \( g_{ij} \) of compact support in \( C^1(\mathbb{R}_+, \mathbb{R}) \), with the value \( g_{ij}(d_{ij}) \) satisfying the condition

\[
g_{ij}(d_{ij}) \cdot (\bar{x}_j - \bar{x}_i) = f_{mn}(d_{in}) \cdot (\bar{x}_i - \bar{x}_n) + f_{jn}(d_{jn}) \cdot (\bar{x}_j - \bar{x}_n)
\]

We then construct a new system of \((n-1)\) agents by ruling out \( \bar{x}_n \), and meanwhile replacing \( f_{ij} \) with

\[
f_{ij} := f_{ij} + g_{ij}
\]

This function \( f_{ij} \) satisfies both the condition of strong repulsion and the condition of long-range attraction. Furthermore, the sub-configuration \( p' \) of \( p \), formed by agents \( \bar{x}_1, \ldots, \bar{x}_{n-1} \), is an equilibrium of this new system.

Let \( G' \) be the sub-graph of \( G \) by restricting \( G \) to vertices \( 1, \ldots, n-1 \). Let \( p'_1, \ldots, p'_{m'} \) be the sub-configurations of \( p' \) associated with the canonical partition, then by induction each \( p'_i \) is an equilibrium. Now suppose agents \( \bar{x}_i \) and \( \bar{x}_j \) are in \( p'_1 \), it then suffices to show that if we restore the system to the original one, i.e,

1. replace \( f_{ij} \) with \( f_{ij} \)
2. put \( \bar{x}_n \) on its original position, and retrieve \( f_{mn} \) and \( f_{jn} \)

then the line configuration \( p_1 \), formed as a union of \( p'_i \) and \( \bar{x}_n \), is still an equilibrium. But this holds by our construction of the function \( g_{ij} \).

So far, we have showed that the canonical partition satisfies conditions a), b) and c).

VII. ON THE LATTICE OF PARTITIONS OF THE EDGE SET

In this section, we will verify that the canonical partition satisfies condition d) in the statement of theorem \( \Gamma \). In particular, we will introduce a lattice, with a partial order, of all partitions satisfying conditions a) and b).

Let \( G = (V, E) \) be a TLG, and let \( p \) be a configuration. Let \( \sigma = \{E_1, \ldots, E_m\} \) be a partition of \( E \) (not necessary the canonical), and we assume that \( \sigma \) satisfies both conditions a) and b) in the statement of theorem \( \Gamma \). Now let \( \Sigma \) be the collection of all such partitions of \( E \), and we will impose a partial order on \( \Sigma \). Let \( \sigma = \{E_1, \ldots, E_m\} \) and \( \sigma' = \{E'_1, \ldots, E'_{m'}\} \) be two distinct partitions in \( \Sigma \), we denote by \( \sigma \gtrsim \sigma' \) if each \( E'_i \) is a subset of some \( E_j \). Our goal in this section is then prove the next theorem.

Theorem 5: Let \( G \) be a TLG, and let \( p \) be a configuration. Let \( \Sigma \) be the lattice defined above, then the canonical partition
is the unique maximal element in $\Sigma$ with respect to the partial order.

We proof theorem \[5\] by first observing a relevant property of a TLG.

**Lemma 6:** Let $G$ be a TLG, and let $G^*$ be a subgraph of $G$ which is also a TLG. There exists a Henneberg construction of $G$ with $G^*$ its top priority, i.e., the subgraph $G^*$ is built-up prior to any other vertices and edges in $G$ during the construction.

**Proof:** The proof is done by induction on the number of agents.

**Base case.** In the case $N = 2$, the lemma is trivial true.

**Inductive step.** Suppose the lemma holds for $N < n$, and we prove for the case $N = n$. Choose a Henneberg construction, and label the vertices with respect to the order of this construction. Let $i$ and $j$ be the two vertices that the last vertex $n$ joins to. We let $G'$ be the subgraph of $G$ defined by restricting $G$ to vertices $1, \ldots, n - 1$.

There are two cases depending whether or not the subgraph $G^*$ contains the vertex $n$.

**Case I.** We assume that $G^*$ does not contain the vertex $n$, then $G^*$ is a subgraph of $G$. By induction we can choose a Henneberg construction of $G'$ with $G^*$ its top priority. We then build up $G$ by joining the vertex $n$ to $G'$ via edges $(i,n)$ and $(j,n)$.

**Case II.** We assume that $G^*$ contains the vertex $n$, and we may as well assume that $G^*$ consists of more than two vertices. This, in particular, implies that the subgraph $G^*$ also contains vertices $i$ and $j$.

Suppose we remove the vertex $n$, together with the two edges $(i,n)$ and $(j,n)$ from $G^*$, then the remaining graph $G''$ is still a TLG, and is a subgraph of $G'$. So by induction, there is a Henneberg construction of $G'$ with $G''$ its top priority. We now modify this Henneberg construction to build up $G$. The modification goes as follows: right after the subgraph $G''$ is formed, we pause the construction and join the vertex $n$ to the two vertices $i$ and $j$ via edges $(i,n)$ and $(j,n)$, and then resume the construction. Then the modified Henneberg construct sets up $G$ with $G^*$ its top priority.

We will now be ready to prove theorem \[5\].

**Proof of theorem \[5\]** Let $\sigma = (E_1, \ldots, E_m)$ be the canonical partition of $E$, then $\sigma$ is in $\Sigma$. Let $\sigma' = (E'_1, \ldots, E'_m)$ be another element in $\Sigma$, and we show $\sigma \succ \sigma'$. Let $G_i = (V_i, E_i)$ and $G'_i = (V'_i, E'_i)$ be the subgraphs of $G$ induced by $E_i$ and $E'_i$, respectively. For each fixed $i$, we show that $G'_i$ is a subgraph of $G_j$ for some $j$. Since $G'_i$ is a TLG, by lemma \[6\] there is a Henneberg construction of $G$ with $G'_i$ its top priority. Now suppose $G'_i$ consists of $k$ vertices, with $V'_i = \{i_1, \ldots, i_k\}$, and the base edge $(i_1,i_2)$ of $G'_i$ is contained in $G_j$ for some $j$. Then by just following the chosen Henneberg construction, we know that $G'_i$ is a subgraph of $G_j$.

So now we have verified all the four conditions in the statement of theorem \[1\].

**VIII. Perturbation of One Sub-Configuration While Preserving Shapes of the Others**

In this section, we will investigate a geometric question. Let $p$ be a configuration, and let $p_1, \ldots, p_m$ be sub-configurations of $p$ associated with the canonical partition. We ask whether we can perturb one sub-configuration while preserving shapes of the others? The answer is confirmative, as summarized in the next theorem.

**Theorem 7:** Let $G$ be a TLG, and let $p$ be a configuration. Let $\{G_i = (V_i, E_i)\}_{i=1}^m$ and $\{p_i\}_{i=1}^m$ be subgraphs of $G$ and sub-configurations of $p$ respectively, associated with the canonical partition. For a fixed $i$, there is an open neighborhood $W_i$ of $p_i$ in $\mathbb{R}^{2 \times |V_i|}$ such that if $p_i$ is perturbed within $W_i$, then there is a unique displacement $\delta \bar{x}_j \in \mathbb{R}^2$ for each agent $\bar{x}_j$, with $j \in V - V_i$, such that the next two conditions are satisfied:

1. **Shape preserving.** If we update the position of agent $\bar{x}_j$ to $\bar{x}_j + \delta \bar{x}_j$ for each $j \in V - V_i$, then the distance $d_{ij}$ will remain the same for all $(k,l) \in E - E_i$. (Thus, the shape of each $p_i$ is preserved after perturbation)

2. **Smoothness.** Each $\delta \bar{x}_j$, with $j \notin V_i$, is a smooth function over $W_i$ with $\delta \bar{x}_j(p_i) = 0$.

We start by investigating a simple case where we have only three agents, and they form a nondegenerate triangle.

**Lemma 8:** Suppose the three agents $\bar{x}_1, \bar{x}_2$ and $\bar{x}_3$ form a nondegenerate triangle, i.e, $\bar{x}_3 - \bar{x}_1$ and $\bar{x}_3 - \bar{x}_2$ are linearly independent. There are open neighborhoods $U_1$ and $U_2$ of $\bar{x}_1$ and $\bar{x}_2$ in $\mathbb{R}^2$ such that as long as $\bar{x}_1 + \delta \bar{x}_1$ lies in $U_1$ and $\bar{x}_2 + \delta \bar{x}_2$ lies in $U_2$, then there is a unique smooth function $\delta \bar{x}_3$ defined on $U_1 \times U_2$ such that the two distances $d_{13}$ and $d_{23}$ are preserved, and $\delta \bar{x}_3 = 0$ if $\delta \bar{x}_1 = \delta \bar{x}_2 = 0$.

**Proof:** Choose an open neighborhood $U_i$ of $\bar{x}_i$ in $\mathbb{R}^2$ for each $i = 1,2,3$ such that the triangle $p' = (\bar{x}_1', \bar{x}_2', \bar{x}_3')$ is nondegenerate if $p'$ is in $\Pi_3^1 U_i$. We then define a smooth function $\phi : \Pi_3^1 U_i \to \mathbb{R}^2$ by

\[ \phi : (\bar{x}_1', \bar{x}_2', \bar{x}_3') \mapsto \left( \begin{array}{c} |\bar{x}_1' - \bar{x}_2'|^2 - |\bar{x}_3 - \bar{x}_1'|^2 \\ |\bar{x}_3' - \bar{x}_2'|^2 - |\bar{x}_3 - \bar{x}_2'|^2 \end{array} \right) \]  

and we compute the partial derivative

\[ \frac{\partial \phi(\bar{x}_1, \bar{x}_2, \bar{x}_3)}{\partial \bar{x}_3} = \left( \begin{array}{c} \bar{x}_1' - \bar{x}_2' \\ \bar{x}_3' - \bar{x}_2' \end{array} \right) \]  

It is nonsingular because two vectors $\bar{x}_3 - \bar{x}_1$ and $\bar{x}_3 - \bar{x}_2$ are linearly independent. By the inverse function theorem we know that by shrinking $U_1$ and $U_2$, if necessary, there is a unique smooth function $\rho : U_1 \times U_2 \to U_3$ such that

\[ \phi(\bar{x}_1', \bar{x}_2', \rho(\bar{x}_1', \bar{x}_2')) = 0 \]  

with $\rho(\bar{x}_1, \bar{x}_2) = \bar{x}_3$. We then let $\delta \bar{x}_3 := \rho - \bar{x}_3$ which satisfies all the conditions we need.

We will now be able to prove theorem \[7\].

**Proof of theorem \[7\]** We prove for the case where $p_1$ is perturbed. By lemma \[6\] we can choose a Henneberg...
construction of $G$ with $G_1 = (V_1,E_1)$ its order of vertices, we then label the vertices of $G$ with respect to the order of the construction.

Suppose $V_1$ consists of $k$ vertices. Choose an open neighborhood $U_i$ of $\bar{x}_i$ in $\mathbb{R}^2$ for each $i$ in $V_1$, and let $W_i := \Pi_{j=1}^k U_j$. We may shrink $U_i$, and hence $W_i$ along the proof if necessary. Let $\delta \bar{x}_i$ be a perturbation of $\bar{x}_i$ within $U_i$, and we will find $\delta \bar{x}_{i+1}, \ldots, \delta \bar{x}_{n}$ subsequently. This is done by induction.

Suppose we have found smooth displacements $\delta \bar{x}_{i+1}, \ldots, \delta \bar{x}_{n-1}$, and we show there exists a unique displacement $\delta \bar{x}_n$ satisfies the two conditions in theorem [7].

Let $i$ and $j$ the two vertices that $n$ joins to. There are two situations.

**Case I.** Both vertices $i$, $j$ are in $V_1$. The triangle formed by $\bar{x}_i$, $\bar{x}_j$, and $\bar{x}_n$ must then be nondegenerate. By lemma [9] there is a unique displacement $\delta \bar{x}_n$, as a smooth function of $\bar{x}_i$, and $\delta \bar{x}_j$, that preserves the two distances $d_{ij}$ and $d_{jn}$ as long as the two open neighborhoods $U_i$ and $U_j$ are small enough. Furthermore, we know that $\delta \bar{x}_n = 0$ if $\delta \bar{x}_i = \delta \bar{x}_j = 0$. So given an open neighborhood $U$ of $\bar{x}_n$, we may shrunk $U_i$ and $U_j$, if necessary, so that $\bar{x}_n + \delta \bar{x}_n$ is in $U_n$ as long as $(\bar{x}_i + \delta \bar{x}_i, \bar{x}_j + \delta \bar{x}_j)$ is in $U_i \times U_j$.

**Case II.** There is at least a vertex, say $i$, not contained in $V_1$. Consequently the edge $(i,j)$ is not in $E_1$. So by induction, we have already found $\delta \bar{x}_i$ which preserves the distance $d_{ij}$. We may as well assume that the three agents $\bar{x}_i$, $\bar{x}_j$, and $\bar{x}_n$ are aligned because otherwise, the analysis will be the same as we did in the previous case. But we then define

$$\delta \bar{x}_n := \delta \bar{x}_i + \frac{d_{in}}{d_{ij}} (\delta \bar{x}_j - \delta \bar{x}_i) \quad (23)$$

This displacement $\delta \bar{x}_n$ of $\bar{x}_n$ will keep the three agents aligned after perturbation, and it is the unique solution that preserves both $d_{in}$ and $d_{jn}$. It is clear that $\delta \bar{x}_n$ is a linear (hence smooth) function in $\delta \bar{x}_i$ and $\delta \bar{x}_j$, and $\delta \bar{x}_n = 0$ if $\delta \bar{x}_i = \delta \bar{x}_j = 0$.

**Remark.** Let $\delta p_j$, with $j \neq i$, be the displacement of the subconfiguration $p_j$. Then by the condition of shape preserving and by the fact that the subgraph $G_j$ is a TLG, we know that $p_j + \delta p_j$ will still be a line configuration, and it is contained in $\mathcal{C}_{p_j}$.

We will now follow theorem [7] to consider infinitesimal displacements of agents. We assume that $V_j$ consists of vertices $i_1, \ldots, i_k$. Let $W_j$ be the open neighborhood of $p_j$ in $\mathbb{R}^{2 \times k}$, defined in the statement of theorem [7]. We now define a smooth function $\Delta_j : W_j \rightarrow \mathbb{R}^{2 \times N}$ by

$$\Delta_j : p_j + (\delta \bar{x}_{i_1}, \ldots, \delta \bar{x}_{i_k}) \mapsto p_j + (\delta \bar{x}_{i_1}, \ldots, \delta \bar{x}_{i_k}) \quad (24)$$

each $\delta \bar{x}_j$, with $j \in V_j$, is a perturbation of $\bar{x}_j$, while each $\delta \bar{x}_k$, with $k \not\in V_j$, is the associated displacement of $\bar{x}_k$. Let $d\Delta_j : \mathbb{R}^{2 \times k} \rightarrow \mathbb{R}^{2 \times N}$ be the derivative of $\Delta_j$ at $p_j$. Let $\bar{v}$ be any vector in $\mathbb{R}^{2 \times k}$, describing the infinitesimal motion of $p_j$. Let $d\Delta_j(\bar{v}) = (u_1, \ldots, u_N)$ with each $u_j$ a vector in $\mathbb{R}^2$, and let

$$\bar{u}_{p_j} := (u_{j_1}, \ldots, u_{j_k}) \quad (26)$$

Then $\bar{u}_{p_j}$ describes the associated infinitesimal motion of $p_j$. By the definition of $\Delta_j$, we know

$$\bar{u}_{p_j} = \bar{v} \quad (27)$$

By the previous remark, we know

$$\bar{u}_{p_j} \in T_{p_j} \mathcal{C}_{p_j}, \quad \forall j \neq i \quad (28)$$

where $T_{p_j} \mathcal{C}_{p_j}$ is the tangent space of $\mathcal{C}_{p_j}$ at $p_j$. This fact will be useful later proving theorem [2].

**IX. THE HESSIAN MATRIX AT AN EQUILIBRIUM**

In this section, we will focus on the computation of the Hessian matrix of $\Phi$ at an fixed equilibrium $p$. Let $\{G_i = (V_i,E_i)\}_{i=1}^m$ and $\{p_i\}_{i=1}^m$ be subgraphs of $G$ and subconfigurations of $p$ associated with the canonical partition. Let $\Phi_p$ be the potential function of the sub-system induced by $G_i$, and we let $H_{p_i}$ be the Hessian matrix of $\Phi_i$ at $p_i$. In this section we will relate $H_p$ to the set $\{H_{p_i}\}_{i=1}^m$.

Before stepping further, we will first rearrange entries of $p$. In the rest of this paper, we assume that the $a$-axis and the $b$-axis are the two axes of $\mathbb{R}^2$, and we let $a_i$ and $b_i$ be the two coordinates of $\bar{x}_i$. Let

$$\begin{align*}
\bar{a} &:= (a_1, \ldots, a_N) \\
\bar{b} &:= (b_1, \ldots, b_N)
\end{align*} \quad (29)$$

and we rearrange entries of $p$ so that

$$p = (\bar{a}, \bar{b}) \quad (30)$$

We again assume that each $V_i$ consist of vertices $i_1, \ldots, i_k$, with $i_1 < \cdots < i_k$. Then similarly, we arrange entries of $p_i$ such that

$$\begin{align*}
p_i &:= (\bar{a}_{p_i}, \bar{b}_{p_i}) \\
\bar{a}_{p_i} &:= (a_{i_1}, \ldots, a_{i_k}) \\
\bar{b}_{p_i} &:= (b_{i_1}, \ldots, b_{i_k})
\end{align*} \quad (31)$$

The Hessian matrix $H_p$, as well as the Hessian matrix $H_{p_i}$, will then be computed with respect to the order above.

To compute $H_p$ and $H_{p_i}$, we will first introduce three $N$-by-$N$ matrices $A_{p_i}$, $B_{p_i}$ and $C_{p_i}$. All these three matrices are symmetric, of zero-row/column-sum. So we define these three matrices by specifying their off-diagonal matrices. For each pair $(j,k)$ with $j \neq k$, we define the off-diagonal entries of these three matrices by

$$A_{p_{i,j,k}} = \begin{cases}
f_{jk}(d_{jk}) + \frac{f_{a_{jk}}(d_{jk})}{d_{jk}} \cdot (a_{j} - a_{k})^2 & \text{if } (j,k) \in E_i \\
0 & \text{otherwise}
\end{cases} \quad (32)$$

$$B_{p_{i,j,k}} = \begin{cases}
f_{jk}(d_{jk}) + \frac{f_{b_{jk}}(d_{jk})}{d_{jk}} \cdot (b_{j} - b_{k})^2 & \text{if } (j,k) \in E_i \\
0 & \text{otherwise}
\end{cases} \quad (33)$$

$$C_{p_{i,j,k}} = \begin{cases}
f_{a_{jk}}(d_{jk}) \cdot (a_{j} - a_{k})(b_{j} - b_{k}) & \text{if } (j,k) \in E_i \\
0 & \text{otherwise}
\end{cases} \quad (34)$$
We then define a matrix $H^*_p$ by

$$H^*_p = \begin{pmatrix} A_p & C_p \\ C_p & B_p \end{pmatrix} \quad (33)$$

This set of auxiliary matrices $\{H^*_p\}$ then helps connect $H_p$ and the set $\{H^*_p\}$.

First, the Hessian matrix $H_p$ can be derived by removing the $j$-th row/column out of $H^*_p$ as long as $j$ is not contained in $V_i$. We note that every deleted row/column is a zero-row/column. The Hessian matrix $H_p$ is given by

$$H_p = \sum_{i=1}^{m} H^*_p \quad (34)$$

This expression plays a key role in establishing the Morse-Bott index formula as we will do in the next section.

At the end of this section, we will discuss about the null spaces of the Hessian matrices. The null space of $H_p$ at least contains $T_p\partial_p$, i.e., the tangent space of $\partial_p$ at $p$. Let $\vec{e}$ be a vector all ones in $\mathbb{R}^N$, and we define three vectors in $\mathbb{R}^{2	imes k_i}$ by

$$\begin{cases} \vec{a}_i := (\vec{e}, 0) \\ \vec{b}_i := (0, \vec{e}) \\ \vec{r}_i := (-\vec{b}, \vec{a}) \end{cases} \quad (35)$$

The first two vectors $\vec{a}_i, \vec{b}_i$ represent the infinitesimal motions of translation of $p$ along $a$-axis and $b$-axis respectively. The last vector $\vec{r}_i$ represents the infinitesimal motion of clockwise rotation of $p$. Thus, these three vectors form a basis of $T_p\partial_p$, and is contained in the null space of $H_p$. Similarly, the null space of each Hessian matrix $H_{p_i}$ at least contains $T_{p_i}\partial_{p_i}$, which is spanned by

$$\begin{cases} \vec{a}_{p_i} := (\vec{e}_{p_i}, 0) \\ \vec{b}_{p_i} := (0, \vec{e}_{p_i}) \\ \vec{r}_{p_i} := (-\vec{b}_{p_i}, \vec{a}_{p_i}) \end{cases} \quad (36)$$

where we denote by $\vec{e}_{p_i}$ a vector of all ones in $\mathbb{R}^{k_i}$.

**X. The Morse-Bott Index Formula**

In this section, we will establish the Morse-Bott index formula.

We will still assume the arrangement of entries of $p$, as well as $p_i$, introduced at the beginning of section [X]. Thus, the map $\Lambda_i$ we defined at the end of section [VIII] needs to be adapted to this arrangement. If we write $\delta x_i = (\delta a_i, \delta b_i)$, then the map $\Lambda_i$ is now given by

$$\Lambda_i : p_i + (\delta a_{p_i}, \delta b_{p_i}) \mapsto p + (\delta a, \delta b) \quad (37)$$

and its derivative $d\Lambda_i$ at $p_i$ is modified correspondingly.

We again assume that each $V_i$ consists of vertices $i_1, \ldots, i_{k_i}$. As each $G_i = (V_i, E_i)$ is a TLG, thus we have

$$l_i := |E_i| = 2|V_i| - 3 = 2k_i - 3 \quad (38)$$

Now for each fixed $i$, we let $\{\vec{v}_{i_1}, \ldots, \vec{v}_{i_{k_i}}\}$ be a set of orthonormal eigenvectors of $H_{p_i}$ with respect to eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_{k_i}}$. We assume that these $l_i$ vectors are chosen so that they are perpendicular to $T_{p_i}\partial_{p_i}$, which are spanned by $\vec{a}_{p_i}, \vec{b}_{p_i}$, and $\vec{r}_{p_i}$. We then let

$$\vec{u}_{i_j} := d\Lambda_i(\vec{v}_{i_j}) \quad (39)$$

for all $j = 1, \ldots, l_i$. Let

$$\begin{cases} U := U_1 \cup \cdots \cup U_m \cup \{\vec{u}_{i_1}, \vec{u}_b, \vec{r}_p\} \\ U_i := \{\vec{u}_{i_1}, \ldots, \vec{u}_{i_k}\}, \quad \forall i = 1, \ldots, m \end{cases} \quad (40)$$

and our goal in the rest of this section is to prove the next two lemmas.

**Lemma 9:** Let $U$ be defined above, then there are $2N$ vectors in $U$, and they are all linearly independent.

**Lemma 10:** Let $\Lambda := U^T H_p U$, then $\Lambda$ is a diagonal matrix given by $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m, 0, 0, 0)$ where each $\lambda_i$ is given by $\lambda_i := \text{diag}(\lambda_{i_1}, \ldots, \lambda_{i_{k_i}})$.

Then we define a matrix $H_p$ and let $\Lambda$ be defined above, then there are $2N$ vectors in $U$, and they are all linearly independent.

There is a relevant implication of lemma [10] as we will describe below. We say a critical orbit $\partial_p$ is nondegenerate if and only if the null space of $H_p$ coincides with $T_p\partial_p$, which, in our case, is equivalent to say $n_0(H_p) = 3$. The next corollary show an direct application of the Morse-Bott index formula to the problem of determining nondegenerate critical orbits.

**Corollary 11:** Let $G$ be a TLG, and let $\partial_p$ be a critical orbit. Then $\partial_p$ is nondegenerate if and only if each $\lambda_j$ is nonzero. In other words, a critical orbit $\partial_p$ is nondegenerate if and only if each $\partial_{p_i}$ is nondegenerate.

The rest of this section is this divided into two parts which are proofs of lemma [9] and lemma [10] respectively.

**A. Proof of lemma [9]**

We first show that the total number of vectors in $U$ is $2N$. This holds because

$$|U| = \sum_{i=1}^{m} |U_i| + 3 = |E| + 3 = 2N \quad (41)$$

Again, we have used the fact that $G$ is a TLG, and hence $|E| = 2|V| - 3 = 2N - 3$.

Before stepping further, we introduce some useful notations. Let $\vec{u} = (\vec{u}', \vec{u}'')$ be a vector in $\mathbb{R}^{2\times N}$, with both $\vec{u}'$ and $\vec{u}''$ in $\mathbb{R}^N$. Let $u'_i$ be the $i$-th entry of $\vec{u}'$, and $u''_i$ the $i$-th entry of $\vec{u}''$. For each $p_i$, we let

$$\vec{u}_{p_i} := (u'_{i_1}, \ldots, u'_{i_k}, u''_{i_1}, \ldots, u''_{i_k}) \quad (42)$$

be the restriction of $\vec{u}$ to entries associated with $p_i$.

We will now prove that the $2N$ vectors in $U$ are linearly independent. Suppose there is a set of real coefficients $\{\alpha_{ij} \mid 1 \leq j \leq k_i, 1 \leq i \leq m\}$, and $\{\beta_1, \beta_2, \beta_3\}$ such that

$$\sum_{i=1}^{m} \sum_{j=1}^{k_i} \alpha_{ij} \cdot \vec{u}_{ij} + \beta_1 \cdot \vec{a}_b + \beta_2 \cdot \vec{b}_p + \beta_3 \cdot \vec{r}_p = 0 \quad (43)$$
so then for any $i = 1, \cdots, m$, we have
\[ \sum_{i=1}^{m} \sum_{j=1}^{k_i} \alpha_{ij} \cdot \bar{u}_{ij} |_{p_i} + \beta_1 \cdot \bar{u}_{i} |_{p_i} + \beta_2 \cdot \bar{v}_{i} |_{p_i} + \beta_3 \cdot \bar{r}_{p} |_{p_i} = 0 \]  
(44)

We have showed at the end of section VIII that
\[
\begin{align*}
\bar{u}_{ij} |_{p_i} &= \bar{v}_{ij} & \text{if } i' = i \\
\bar{u}_{ij} |_{p_i} &\in T_{p_i} \mathcal{G}_{p_i} & \text{if } i' \neq i
\end{align*}
\]
(45)

Also by verifying the definitions, we have
\[
\begin{align*}
\bar{t}_{pi} &= \bar{t}_{pi} \\
\bar{r}_{pi} &= \bar{r}_{pi}
\end{align*}
\]
(46)

for all $i = 1, \cdots, m$. So then equation (44) implies
\[ \sum_{i=1}^{k_i} \alpha_{ij} \cdot \bar{v}_{ij} + \bar{v} = 0 \]  
(47)

where $\bar{v}$ is a vector in $T_{p_i} \mathcal{G}_{p_i}$. Thus $\alpha_{ij} = 0$ for all $j = 1, \cdots, k_i$. As this holds for all $i = 1, \cdots, m$, so then equation (45) implies
\[ \beta_1 \cdot \bar{t}_{a} + \beta_2 \bar{v}_{a} + \beta_3 \bar{r}_{p} = 0 \]  
(48)

and hence, each $\beta_i$ is also zero. This completes the proof.

B. Proof of lemma 10

We again start the proof by introducing some useful notations. We fix an $i = 1, \cdots, m$, and let $w$ be a vector in $\mathbb{R}^{2 \times k_i}$. We now extend the vector $\bar{w}$ to a vector $\text{ext}_{p_i}(\bar{w})$ in $\mathbb{R}^{2 \times N}$ by filling with zeros, and we add zeros in a way such that
\[ \text{ext}_{p_i}(\bar{w}) |_{p_i} = \bar{w} \]  
(49)

for all $\bar{w}$ in $\mathbb{R}^{2 \times k_i}$.

Recall the set of matrices $\{ H_{p_i} \}_{i=1}^{m}$ defined in section IX and the equality
\[ H_p = \sum_{i=1}^{m} H_{p_i}^* \]  
(50)

So then
\[ H_p \cdot \bar{u}_{ij} = \sum_{i=1}^{m} H_{p_i}^* \cdot \bar{u}_{ij} \]  
(51)

As the $k$-th row/column of $H_{p_i}^*$ is zero if $k \notin V_i$, so then
\[ H_{p_i}^* \cdot \bar{u}_{ij} = \text{ext}_{p_i} \left( H_{p_i}^* \cdot \bar{u}_{ij} |_{p_i} \right) = \begin{cases} \lambda_{ij} \cdot \text{ext}_{p_i}(\bar{v}_{ij}) & \text{if } i' = i \\ 0 & \text{otherwise} \end{cases} \]  
(52)

the last equality holds by expression (45). So then
\[ H_p \cdot \bar{u}_{ij} = \lambda_{ij} \cdot \text{ext}_{p_i}(\bar{v}_{ij}) \]  
(53)

It is then clear that by our choice of $\bar{v}_{ij}$, we have
\[ \langle \bar{u}_{ij}, H_p \cdot \bar{u}_{ij} \rangle = \begin{cases} \lambda_{ij} \langle \bar{u}_{ij} |_{p_i}, \bar{v}_{ij} \rangle & \text{if } i = i' = j \\ 0 & \text{otherwise} \end{cases} \]  
(54)

On the other hand, we know that
\[ H_p \cdot \bar{t}_{a} = H_p \cdot \bar{v}_{b} = H_p \cdot \bar{r}_{p} = 0 \]  
(55)

All combined then establish lemma 10.

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