Spectral Parameter Power Series Analysis of Supersymmetric Schrödinger Equations with Balanced Gain and Loss Potentials

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Abstract. In this work the Zakharov-Shabat system is addressed to obtain a pair of supersymmetric Schrödinger equations. The scattering and resonance states of these equations are investigated. Explicit solutions for the equations are obtained in the form of power series of the spectral parameter. In the case of the scattering states, we obtain expressions for the transmission and reflection coefficients. In the case of the resonance states we obtain the characteristic equation that defines their complex energies. We show that finding approximate complex energies of the resonance states reduces to calculating polynomial roots from certain characteristic polynomial. Some cases of interest are numerically implemented.

1. Introduction

The Zakharov-Shabat system arises in the inverse scattering transform when investigating the time evolution of the nonlinear Schrödinger equation [1,2]. This system also arises in the linear applications of quantum mechanics [3–6]. By performing a decoupling process, the Zakharov-Shabat system leads to a pair of Darboux-associated Schrödinger equations [7,8]. The Zakharov-Shabat system has also wide applications in the scattering of waves, for describing the interaction of solitons in stable media, in one-dimensional problems involving self-modulation of dispersive waves, among others (see, e.g., [1,2,6,9–11]).

In the present work we analyze the scattering and resonance states of a pair of supersymmetric Schrödinger equations resulting from the Zakharov-Shabat system. Scattering states are analyzed from the general solutions of supersymmetric equations. The analysis leads to explicit expressions for the transmission and reflection coefficients. From the transmission coefficients we determine the transparency energies of the potentials. On the other hand, those solutions of the Schrödinger equations associated to complex energies \( \epsilon = E_0 - i\Gamma/2 \) (\( \Gamma > 0 \)) and satisfying the outgoing wave conditions describe resonance states, also called Gamow-Siegert states [12–14]. Resonance states of quantum systems having \( \mathcal{P}\mathcal{T} \)-symmetry have also been considered in the literature [15–20].
In this work we use the spectral parameter power series (SPPS) method [21] for obtaining exact solutions of the supersymmetric equations. These solutions are given by uniformly convergent power series of the spectral parameter. From these solutions we obtain a SPPS representation of the characteristic equation of the resonance states, which takes the form of a polynomial equation in its numerical implementation. Hence, calculating approximate complex energies of resonance states reduces to determining complex roots of a polynomial equation.

The outline of this work is the following. In Section 2 we present the Zakharov-Shabat system and its relation with supersymmetric partners. In Section 3 we construct a general solution of Schrödinger equation in a SPPS form. In Sections 4 and 5 we analyse the scattering and resonance states resulting of considering real- and complex-valued potentials. Finally, some concluding remarks are presented in Section 6.

2. The Zakharov-Shabat system

Supersymmetry in quantum mechanics leads to finding new solvable potentials from the factorization of Hamiltonians. Given a potential its supersymmetric partner can be obtained from the solution of a Ricatti equation that follows from the original potential. In this paper, however, we obtain a pair of supersymmetric partners not by invoking Ricatti equations, but the Zakharov-Shabat system. The obtained superpotentials have the property of possessing balanced gains and losses [15, 20, 22, 23]. The so-called Zakharov-Shabat system is a pair of first-order differential equations of the form [24,26]

\[ n'_1(x) - \kappa n_2(x) = U(x) n_1(x), \]
\[ n'_2(x) + \kappa n_1(x) = -U(x) n_2(x), \]

where the vector-valued function \((n_1, n_2)^\top\) denotes its solution, \(\kappa\) is a complex parameter, and \(U: \mathbb{R} \to \mathbb{C}\) is a function (also called potential) satisfying \(U \in L^1(\mathbb{R})\). The link between the Zakharov-Shabat system and a pair of supersymmetric Schrödinger equations is deduced by introducing the following functions

\[ u := n_1 + in_2, \quad v := n_1 - in_2, \quad \eta := iU. \]

From these functions system (1) can be written in the form

\[ (\partial + \eta) u = \kappa v, \quad (2a) \]
\[ (\partial - \eta) v = \kappa u, \quad (2b) \]

where \((u, v)^\top\) is the solution in the new variables, and \(\partial\) represents the derivative with respect to the real variable \(x \in \mathbb{R}\). The application to the left of operator \((\partial - \eta)\) on equation (2a) and the operator \((\partial + \eta)\) on equation (2b) give a pair of decoupled Schrödinger equations

\[ S_V u := (-\partial^2 + V) u = \kappa^2 u, \]
\[ S_W v := (-\partial^2 + W) v = \kappa^2 v, \]

being \(V := \eta^2 - \partial \eta\) and \(W := \eta^2 + \partial \eta\) the corresponding complex-valued superpotentials [25], which exhibit balanced gain and loss properties [27,28]. These potentials are analogous to those complex refractive indexes found in optics [29,30]. Schrödinger equations (3) are also known as Darboux associated partners [7,31].

Let \(\lambda := \kappa^2\) be the spectral parameter of equations (3), which in general is a complex number having the polar representation \(\lambda = |\lambda| e^{i\phi}\) for \(\lambda \neq 0\), where \(0 \leq \phi < 2\pi\). A branch of \(\sqrt{\lambda} =: \kappa\) is defined as

\[ \kappa = |\lambda|^{1/2} e^{i\theta}, \quad 0 \leq \theta < \pi. \]

Hence, the analysis of the solutions of equations (3) will be carried out in the upper half of the complex \(\kappa\)-plane thus defined.
3. SPPS analysis of Schrödinger equations with complex potentials

Let us focus on Schrödinger equation (3a) involving the potential \( V = \eta^2 - \partial \eta \), where \( \eta = iU \), and \( \lambda = \kappa^2 \) takes the role of an energy parameter. Let us assume that \( V \) vanishes as \( |x| \to \infty \). Hence, \( V \) can be suitably approximated by the function

\[
\tilde{V}(x) = \begin{cases} 
V_0(x), & a < x < b, \\
0, & \text{elsewhere,}
\end{cases}
\]

where \( V_0 = V|_{(a,b)} \) is the restriction of \( V \) in \((a,b)\). The numbers \( a < b \) can be chosen so that \( |V(x)| < \delta \), where \( \delta > 0 \) is an arbitrary given small number. The definition of this short-range potential leads to the boundary conditions

\[
[u(x)]_a = 0, \quad [u'(x)]_a = 0; \quad [u(x)]_b = 0, \quad [u'(x)]_b = 0, \quad \text{(5)}
\]

which define the continuity of the function \( u \) and its derivative \( u' \) at the points of discontinuity \( x = \{a,b\} \). In conditions (5) the notation \([f(x)]_{x_0} := f(x_0 + 0) - f(x_0 - 0)\) specifies the magnitude of the (finite) jump of function \( f \) at the point \( x = x_0 \) provided that one-sided limits \( f(x_0 \pm 0) \) exist.

A general solution of the equation \( \mathcal{S}_{\tilde{V}} u = \kappa^2 u \) can be sought in the form

\[
\psi(x;\kappa) = \begin{cases} 
C_1 e^{i\kappa(x-a)} + C_2 e^{-i\kappa(x-a)}, & -\infty < x < a, \\
\psi(x;\kappa), & a < x < b, \\
C_3 e^{i\kappa(x-b)} + C_4 e^{-i\kappa(x-b)}, & b < x < \infty,
\end{cases}
\]

where \( \kappa = \sqrt{\lambda} \) is the branch defined in (4), \( C_m \) \((m = 1, \ldots, 4)\) are arbitrary coefficients, and \( \psi \) is a solution of the Schrödinger equation

\[
(-\partial^2 + V_0) \psi = \kappa^2 \psi, \quad a < x < b \quad \text{(7)}
\]

corresponding to the interaction region \((a,b)\). This solution is defined by

\[
\psi(x) = B_1 \psi_1(x) + B_2 \psi_2(x),
\]

where \( \psi_1, \psi_2 \) are two linearly independent solutions, and \( B_1, B_2 \) are arbitrary coefficients. Continuity conditions (5) give the following expressions

\[
B_1 = \frac{1}{w[\psi_1,\psi_2](b)} (C_3 \mu_{22} + C_4 \mu_{12}), \quad B_2 = \frac{-1}{w[\psi_1,\psi_2](b)} (C_3 \mu_{21} + C_4 \mu_{11}). \quad \text{(8)}
\]

\[
C_1 = \frac{1}{2i\kappa} (B_1 \gamma_{11} + B_2 \gamma_{12}), \quad C_2 = \frac{-1}{2i\kappa} (B_1 \gamma_{21} + B_2 \gamma_{22}),
\]

where \( w[\psi_1,\psi_2](x) \) is the Wronskian of solutions \( \psi_1 \) and \( \psi_2 \)

\[
w[\psi_1,\psi_2](x) = \begin{vmatrix} 
\psi_1(x) & \psi_2(x) \\
\psi_1'(x) & \psi_2'(x)
\end{vmatrix},
\]

and \( \gamma_{pq} \) and \( \mu_{pq} \) \((p, q = 1, 2)\) are coefficients defined by

\[
\gamma_{pq}(\kappa) := i\kappa \psi_q(a;\kappa) + (-1)^{p+1} \psi_p(a;\kappa), \quad \mu_{pq}(\kappa) := (-1)^{p+q+1} i\kappa \psi_q(b;\kappa) + (-1)^q \psi_p(b;\kappa).
\]
3.1. On the construction of the particular solutions $\psi_1$ and $\psi_2$

Here we employ the SPPS method [21] for determining the linearly independent solutions $\psi_1$ and $\psi_2$ of Schrödinger equation (7) as power series of the spectral parameter $\kappa^2$. Let us assume that the homogeneous equation

$$(-\partial^2 + V_0) \psi_0 = 0, \quad a < x < b,$$

(9)

possesses a particular solution $\psi_0$ satisfying the conditions $\psi_0^2, \psi_0^{-2} \in C([a, b])$. Then the series

$$\psi_1 (x) = \psi_0 (x) \sum_{k=0}^{\infty} \kappa^{2k} \tilde{X}^{(2k)} (x), \quad \psi_2 (x) = \psi_0 (x) \sum_{k=0}^{\infty} \kappa^{2k} X^{(2k+1)} (x)$$

with the functions $\tilde{X}^{(n)}$, $X^{(n)}$ defined by

$$\tilde{X}^{(0)} (x) = 1, \quad \tilde{X}^{(n)} := \begin{cases} \int_{0}^{x} \tilde{X}^{(n-1)} (s) \psi_0^2 (s) \, ds, & n \text{ odd}, \\ -\int_{x}^{0} \tilde{X}^{(n-1)} (s) \psi_0^{-2} (s) \, ds, & n \text{ even}, \end{cases}$$

$$X^{(0)} (x) = 1, \quad X^{(n)} := \begin{cases} -\int_{x}^{0} X^{(n-1)} (s) \psi_0^{-2} (s) \, ds, & n \text{ odd}, \\ \int_{0}^{x} X^{(n-1)} (s) \psi_0^2 (s) \, ds, & n \text{ even}. \end{cases}$$

are linearly independent solutions of equation (7), being $x_0$ an arbitrary point of $[a, b]$. Series $\psi_1, \psi_2$ with the functions $\tilde{X}^{(n)}$, $X^{(n)}$ thus defined converge uniformly on $[a, b]$. Furthermore these solutions satisfy the Cauchy conditions

$$\psi_1 (x_0) = \psi_0 (x_0), \quad \psi_1' (x_0) = \psi_0' (x_0), \quad \psi_2 (x_0) = 0, \quad \psi_2' (x_0) = -\frac{1}{\psi_0 (x_0)}.$$

Liouville-Ostrogradskii identity [32] establishes that

$$w [\psi_1, \psi_2] (a) = w [\psi_1, \psi_2] (x), \quad \forall \quad x \in [a, b],$$

hence by setting $x_0 = a$ we have that $w (a) = 1$, thereby $w [\psi_1, \psi_2] (x) \equiv 1$ for each $x \in [a, b]$. Note that the particular solution $\psi_0$ of equation (9) carries the information of the potential $V_0$ in the interaction region. This solution, for instance, can be determined by means of numerical methods, though the SPPS method can also be used for its construction [21,33].

Coefficients $\gamma_{pq}$ ($p, q = 1, 2$) can be written in a SPPS form as follows

$$\gamma_{11} (\kappa) = (i\kappa \psi_0 (a) + \psi_0' (a)) \sum_{k=0}^{\infty} \kappa^{2k} \rho_k (a) + \frac{1}{\psi_0 (a)} \sum_{k=1}^{\infty} \kappa^{2k} \varrho_k (a),$$

$$\gamma_{12} (\kappa) = (i\kappa \psi_0 (a) + \psi_0' (a)) \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k (a) + \frac{1}{\psi_0 (a)} \sum_{k=0}^{\infty} \kappa^{2k} \tau_k (a),$$

$$\gamma_{21} (\kappa) = (i\kappa \psi_0 (a) - \psi_0' (a)) \sum_{k=0}^{\infty} \kappa^{2k} \rho_k (a) - \frac{1}{\psi_0 (a)} \sum_{k=0}^{\infty} \kappa^{2k} \varrho_k (a),$$

$$\gamma_{22} (\kappa) = (i\kappa \psi_0 (a) - \psi_0' (a)) \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k (a) - \frac{1}{\psi_0 (a)} \sum_{k=0}^{\infty} \kappa^{2k} \tau_k (a),$$
also the coefficients $\mu_{pq}$ ($p, q = 1, 2$) admit a SPPS representation

$$
\mu_{11}(\kappa) = - (i\kappa \psi_0(b) + \psi'_0(b)) \sum_{k=0}^{\infty} \kappa^{2k} \rho_k(b) - \frac{1}{\psi_0(b)} \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k(b),
$$

$$
\mu_{12}(\kappa) = (i\kappa \psi_0(b) + \psi'_0(b)) \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k(b) + \frac{1}{\psi_0(b)} \sum_{k=0}^{\infty} \kappa^{2k} \tau_k(b),
$$

$$
\mu_{21}(\kappa) = (i\kappa \psi_0(b) - \psi'_0(b)) \sum_{k=0}^{\infty} \kappa^{2k} \rho_k(b) - \frac{1}{\psi_0(b)} \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k(b),
$$

$$
\mu_{22}(\kappa) = (-i\kappa \psi_0(b) + \psi'_0(b)) \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k(b) + \frac{1}{\psi_0(b)} \sum_{k=0}^{\infty} \kappa^{2k} \tau_k(b),
$$

where $\rho_k(x) := \tilde{X}^{(2k)}(x)$, $\sigma_k(x) := \tilde{X}^{(2k-1)}(x)$, $\tau_k(x) := X^{(2k+1)}(x)$, and $\kappa$ are the amplitudes of the reflected and transmitted waves, respectively, which are denoted by $R_t$ and $T_t$. For an incident wave with unit amplitude this process is described by

$$
u_l(x; \kappa) = \begin{cases} e^{i\kappa(x-a)} + R_l e^{-i\kappa(x-a)}, & -\infty < x < a, \\
B_1 \psi_1(x; \kappa) + B_2 \psi_2(x; \kappa), & a < x < b, \\
T_l e^{i\kappa(x-b)}, & b < x < \infty, \end{cases}
$$

where

$$
R_l = \frac{\mu_{22} \gamma_{21} + \mu_{21} \gamma_{22}}{\mu_{22} \gamma_{11} + \mu_{21} \gamma_{12}}, \quad T_l = \frac{2i\kappa}{\mu_{22} \gamma_{11} + \mu_{21} \gamma_{12}},
$$

are functions of $\kappa$. Coefficients $B_{1,l}$ and $B_{2,l}$ are calculated from expressions (8). We identify $|T_l(\kappa)|^2$ and $|R_l(\kappa)|^2$ as the transmission and reflection coefficients, respectively. If $V_0$ were real-valued these coefficients would satisfy the conservation of probability $|R_l(\kappa)|^2 + |T_l(\kappa)|^2 = 1$, [34, 35]. However, in the present case $V_0 = (-U^2 - i\partial U)\big|_{(a,b)}$ is a complex-valued function, hence the conservation is observed globally [15].

Similarly, in the propagation from right to left, the waves generated by a source of particles located at $x \rightarrow \infty$ are described by the undulatory process

$$
u_r(x; \kappa) = \begin{cases} T_r e^{-i\kappa(x-a)}, & -\infty < x < a, \\
B_1 \psi_1(x; \kappa) + B_2 \psi_2(x; \kappa), & a < x < b, \\
R_r e^{i\kappa(x-b)} + e^{-i\kappa(x-b)}, & b < x < \infty, \end{cases}
$$

where coefficients $B_{1,r}$ and $B_{2,r}$ are calculated from expressions (8). By $R_r$ and $T_r$ we denote the amplitudes of the reflected and transmitted waves, respectively. These amplitudes are given as functions of the parameter $\kappa$ by

$$
R_r = \frac{\gamma_{12} \mu_{11} + \gamma_{11} \mu_{12}}{\gamma_{11} \mu_{22} + \gamma_{12} \mu_{21}}, \quad T_r = \mu_{11} \mu_{22} - \mu_{12} \mu_{21},
$$

3.2. A SPPS analysis of the scattering states

Scattering states are described by solutions of Schrödinger equation $S_{\psi}u = \kappa^2 u$ associated with the spectral parameter $\lambda = \kappa^2 > 0$, that is $\kappa > 0$. Since it is a linear differential equation, the scattering of quantum waves propagating from left to right, and those from right to left can be treated as independent processes.

In the propagation from left to right a source of particles is assumed to be located at $x \rightarrow -\infty$. This implies that $C_4 \equiv 0$. In this sense $C_1$ represents the amplitude of an incident wave, and $C_2$, $C_3$ are the amplitudes of the reflected and transmitted waves, respectively, which are denoted by $R_t$ and $T_t$. For an incident wave with unit amplitude this process is described by
and \(|T_\kappa (\kappa)|^2\), \(|R_\kappa (\kappa)|^2\) correspond to the transmission and reflection coefficients from the right, respectively.

Those values \(\kappa_T\) for which the reflection coefficient \(|R(\kappa_T)|^2\) vanishes define transparency energies \(E_T = \kappa_T^2\). At such energies the short-range potential \(\tilde{V}\) is transparent to the incident waves. A similar analysis for Schrödinger equation (3b) involving the potential \(W\) can be carried out by defining an approximate short range potential \(\tilde{W}\).

4. SPPS study of resonance states

Resonance in quantum mechanics is a physical phenomenon in which a particle is temporally captured by the interaction potential and then it is scattered. During the capture, the particle stays in a quasi-bound state [13,36]. Gamow [14] and Siegert [37] introduced complex energies and the purely outgoing wave conditions for the study of certain nuclear processes modeled as resonance states. In general, solutions of the Schrödinger equation associated to complex energies and satisfying the outgoing wave conditions describe resonance states, also known as Gamow-Siegert functions [13,14].

Let us consider the Schrödinger equation

\[
S\tilde{V} u = \left( -\partial_x^2 + \tilde{V} \right) u = \lambda u, \quad x \in \mathbb{R},
\]

where the complex energy \(\lambda\) is a parameter of the form \(\epsilon = E - i\Gamma/2\), where \(\Gamma > 0\), \(E \in \mathbb{R}\). The parameter \(\Gamma\) is related to the inverse of the lifetime of the resonance state. The solution \(u\) representing a resonance state must fulfill the outgoing wave conditions

\[
\lim_{x \to \pm \infty} \left( u' \pm i \kappa u \right) = 0. \quad (10)
\]

Let us write the complex parameter \(\kappa \neq 0\) in the rectangular form \(\kappa = \kappa' + i\kappa''\). It follows that

\[
\epsilon = \kappa'^2 = \kappa'^2 - \kappa''^2 + i2\kappa' \kappa'' = E - i\Gamma/2,
\]

where \(E = \kappa'^2 - \kappa''^2\) and \(\Gamma = -4\kappa' \kappa''\). This last equality implies that \(\kappa'\) and \(\kappa''\) must have opposite signs. In order for \(\kappa\) to lie on the upper half of the complex \(\kappa\)-plane it is necessary that \(\kappa' < 0\) and \(\kappa'' > 0\). These conditions restrict \(\kappa\) to live in the region \(\frac{\pi}{2} < \arg \kappa < \pi\) of the complex \(\kappa\)-plane for the description of resonance states.

For obtaining the Gamow-Siegert functions of operator \(S\tilde{V}\) we begin from general solution (6). Outgoing wave conditions (10) imply that \(C_2 = C_3 = 0\). Continuity at \(x = a\) and \(x = b\) leads to \(C_1 = \psi (a)\) and \(C_4 = \psi (b)\). We can choose \(\psi (a) = 1\) so that \(\psi' (a) = i\kappa\). The condition \(C_3 = 0\) gives the characteristic equation of resonance states

\[
\psi (b; \kappa) + \frac{1}{i\kappa} \psi' (b; \kappa) = 0. \quad (11)
\]

If \(\kappa_j = \kappa''_j + i\kappa''_j\) is a zero of equation (11) such that \(\kappa''_j < 0\) and \(\kappa''_j > 0\), then \(\epsilon_j = \kappa_j^2 = E_j - i\Gamma_j/2\) is the complex energy of the \(j\)-th resonance state, which is described by the piecewise continuous function

\[
u (x; \kappa_j) = \begin{cases} 
  e^{i\kappa_j (x-a)}, & -\infty < x < a, \\
  \psi (x; \kappa_j), & a < x < b, \\
  \psi (b; \kappa_j) e^{-i\kappa_j (x-b)}, & b < x < \infty,
\end{cases}
\]

where the solution \(\psi\) is explicitly given by the convergent series

\[
\psi (x; \kappa) = \frac{\psi_0 (x)}{\psi_0 (a)} \sum_{k=0}^{\infty} \kappa^{2k} X^{(2k)} (x) + \psi_0 (x) \left( \psi'_0 (a) - i\kappa \psi_0 (a) \right) \sum_{k=0}^{\infty} \kappa^{2k} X^{(2k+1)} (x).
\]
From previous expression we obtain a SPPS representation of the characteristic equation for the resonance states

\[
\left(\psi_0'(b) + 2i\kappa \psi_0(b)\right) \sum_{k=0}^{\infty} \kappa^{2k} \rho_k(b) - \frac{1}{\psi_0(b)} \sum_{k=1}^{\infty} \kappa^{2k} \varrho_k(b) +
\psi_0(a) \left(\psi_0'(a) - i\kappa \psi_0(a)\right) \left(\psi_0'(b) + 2i\kappa \psi_0(b)\right) \sum_{k=0}^{\infty} \kappa^{2k} \sigma_k(b) - \frac{1}{\psi_0(b)} \sum_{k=0}^{\infty} \kappa^{2k} \tau_k(b) = 0. \tag{12}
\]

5. Numerical implementation

The functions \(X^{(n)}\) and \(\tilde{X}^{(n)}\), also called formal powers, are defined by integrals that can be evaluated numerically. This is achieved by dividing the interval \((a, b)\) into \(M\) subintervals on which the integrand is approximated by interpolating functions, say of the spline type. The accuracy of integrals evaluated numerically is a standard question of numerical analysis. On the other hand, the infinite power series must be truncated for its numerical implementation. Owing to the uniform convergence of the series, the bounded error due to truncation is mainly determined by the number of terms in which the series are truncated. In addition, the numerical precision of the results should be considered since the formal powers may take very small values as their order increase so that machine-precision is not enough for correctly storing those numbers in the memory of a computer.

Next we show some numerical examples involving potentials that possess balanced properties of gain and loss. The real-valued case is also considered. We use Wolfram Mathematica 12 with the following settings:

(i) The interval \([a, b]\) is divided into \(M = 1000\) subintervals.

(ii) The formal powers are truncated in \(n = 100\) terms.

(iii) The numerical precision is set to 60 decimal places.

5.1. Scattering problems

Next, we consider real-valued bounded potentials \(q\) and calculate the dispersion parameters of the operator \(S_q\) by using the SPPS theory here developed.

**Example 1.** Consider a cosine potential barrier defined by

\[
q(x) = \begin{cases} 
10 \cos x, & 0 < x < 0.5, \\
0, & \text{otherwise}, 
\end{cases}
\]

see Figure 1.a. The maxima \(\kappa_T\) of the transmission coefficient \(T_l(\kappa)\) in the segment \(\kappa \in [0, 60]\) are shown in Table 1. Figure 1.b shows the corresponding plots of the reflection and transmission coefficients.

**Example 2.** Consider a truncated symmetric Pöschl-Teller potential barrier defined by the function

\[
q(x) = \begin{cases} 
80 \text{sech}^2 x, & -1 < x < 1, \\
0, & \text{otherwise}, 
\end{cases}
\]

see Figure 2.a. The maxima \(\kappa_T\) of transmission coefficient \(T_l(\kappa)\) in the segment \(\kappa \in [0, 150]\) are shown in Table 2. Figure 2.b shows the plots of the corresponding reflection and transmission coefficients.
Table 1. Some transparency energies of the potential $q(x)$ from Example 1.

| $j$ | Maxima $\kappa_T$ | Transparency energies $E_T = \kappa_T^2$ |
|-----|--------------------|----------------------------------|
| 1   | 12.77819933975655  | 163.28237821881586 |
| 2   | 19.1112357679424   | 365.26562280522353  |
| 3   | 31.44014441310595  | 988.4745080698187   |
| 4   | 48.58916496747884  | 2360.902219345374   |
| 5   | 63.52528224711569  | 4035.461484016157   |

Figure 1. (a) Cosine potential barrier from Example 1. (b) Plots of the transmission and reflection coefficients.

Table 2. Some transparency energies of the potential $q(x)$ from Example 2.

| $j$ | Maxima $\kappa_T$ | Transparency energies $E_T = \kappa_T^2$ |
|-----|--------------------|----------------------------------|
| 1   | 89.04603686608982  | 7929.196681557028   |
| 2   | 101.99817774951632 | 10403.628264221925  |
| 3   | 124.19176398456759 | 15423.59421598556   |
| 4   | 151.39297165346864 | 22919.8318666068026 |

Figure 2. (a) Truncated Pöschl-Teller potential barrier from Example 2. (b) Plots of the transmission and reflection coefficients.
Table 3. Some resonance energies corresponding to superpotentials (13) from Example 3.

| $j$ | Zeros $\kappa_j$ of (12) | $\mathcal{S}_{\tilde{V}}u = \epsilon u$ | Resonance energies $\epsilon_j = \kappa_j^2$ |
|-----|--------------------------|---------------------------------|---------------------------------|
| 1   | $-53.050497799965 + i 3.758361280790$ | $2800.230037307015 - i 398.76587371604705$ |
| 2   | $-55.510683355754 + i 10.331565191028$ | $2974.694727326322 - i 1147.0244877769708$ |
| 3   | $-68.068951170690 + i 8.260114318194$ | $4565.152624928145 - i 1124.5146363789295$ |

Solutions $\tilde{W}$ and $\tilde{V}$ are shown in Figure 3. Resonance energies $\epsilon_j$ of supersymmetric partners $S_{\tilde{V}}u = \epsilon u$, $S_{\tilde{W}}v = \epsilon v$ are shown in Table 3. It is expected that supersymmetric partners would have the same resonance energies. The differences between the results shown in the table are mainly due to the truncation of the series of the characteristic equation corresponding to each partner. In Figure 4 we can see the resonant states $u_j$ and $v_j$ ($j = 1, 2, 3$) associated with the complex energies $\kappa_j$ of the Table 3.

Figure 3. $\mathcal{P}\mathcal{T}$-symmetric potentials from Example 3.

5.2. Resonance problems

Now, we consider some examples with balanced gain-loss potentials, and determine their resonance energies with the SPPS approach here developed.

Example 3. Let us consider the following $\mathcal{P}\mathcal{T}$-symmetric potentials

$$\tilde{V}(x) = \begin{cases} -A^2 \text{sech}^2 x + i A \text{sech} x \tanh x, & -1 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{W}(x) = \begin{cases} -A^2 \text{sech}^2 x - i A \text{sech} x \tanh x, & -1 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $A = 2\sqrt{15}$, which are shown in Figure 3. Resonance energies $\epsilon_j$ of supersymmetric partners $S_{\tilde{V}}u = \epsilon u$, $S_{\tilde{W}}v = \epsilon v$ are shown in Table 3. It is expected that supersymmetric partners would have the same resonance energies. The differences between the results shown in the table are mainly due to the truncation of the series of the characteristic equation corresponding to each partner. In Figure 4 we can see the resonant states $u_j$ and $v_j$ ($j = 1, 2, 3$) associated with the complex energies $\kappa_j$ of the Table 3.

Example 4. Let us consider the $\mathcal{P}\mathcal{T}$-symmetric potentials defined in (13) but with $A = 5$. These potentials are shown in Figure 5. From the SPPS approach we obtain the complex energies $\epsilon_j = \kappa_j^2$ of supersymmetric partners $S_{\tilde{V}}u = \epsilon u$ and $S_{\tilde{W}}v = \epsilon v$, from the zeros $\kappa_j$ of characteristic
Figure 4. Plots of the real parts of the resonance states $u_i$ and $v_i$ ($i = 1, 2, 3$) corresponding to the complex energies $\epsilon_i = \kappa_i^2$ (see Table 3) from Example 3.

equation (12). The results are shown in Table 4. In Figure 6 we can see the first resonance states $u_1$ and $v_1$ associated to the complex energy $\epsilon_1 = \kappa_1^2$ of Table 4.

6. Concluding remarks

We have studied the Zakharov-Shabat system as a model that leads to supersymmetric Schrödinger equations. We identify the Darboux transformation embedded in the process of obtaining the supersymmetric partners. By means of the SPPS method we obtained an explicit representation for the characteristic equation that defines the complex energies of resonance states. Also, explicit expressions for the amplitudes of the reflected and transmitted waves in the scattering processes were obtained. The obtained results are general enough to be applied on (almost) arbitrary regular potentials.

In order for the potentials $V = -U^2 - i\partial U$ and $W = -U^2 + i\partial U$ to be $\mathcal{PT}$-symmetric
Table 4. Some complex energies from Example 4.

| j | Zeros $\kappa_j$ of (12) | $\mathcal{S}_\psi u = \epsilon u$ | Resonance energies $\epsilon_j = \kappa_j^2$ |
|---|--------------------------|----------------------------------|------------------------------------------|
| 1 | $-24.256128876630 + i 2.391102340277$ | $582.6424176780056 - i 115.99777304594103$ | |
| 2 | $-28.1684556764972 + i 7.492125284893$ | $737.3299538202732 - i 422.0831979955481$ | |
| 3 | $-38.867354093256 + i 11.624320410283$ | $1375.5463892096218 - i 903.15149598643$ | |
| 4 | $-54.87892795246 + i 18.153179961069$ | $2694.1304113625097 - i 1996.4202271841402$ | |
| 5 | $-76.074233786694 + i 25.55528884796$ | $5134.216258138287 - i 3888.1980365874533$ | |

\[ \mathcal{S}_\psi v = \epsilon v \]

| j | Zeros $\kappa_j$ of (12) | $\mathcal{S}_\phi v = \epsilon v$ |
|---|--------------------------|----------------------------------|
| 1 | $-24.256128873032 + i 2.391102340969$ | $582.6424175001493 - i 115.99777306230514$ |
| 2 | $-28.168455676095 + i 7.492125280326$ | $737.3299536900288 - i 422.0831977522754$ |
| 3 | $-38.867354090552 + i 11.624320404896$ | $1375.5463891246677 - i 903.151447824323$ |
| 4 | $-54.878927938378 + i 18.153179940208$ | $2694.130411915452 - i 1996.410248224415$ |
| 5 | $-76.074233792578 + i 25.55528889162$ | $5134.216262030347 - i 3888.1980279671147$ |

Figure 5. $\mathcal{P}\mathcal{T}$-symmetric potentials from Example 4 with support on $[-1,1]$.

Figure 6. Plots of the real parts of the first resonance states corresponding to the complex energy $\epsilon_1 = \kappa_1^2$ (see Table 4) from Example 4.
it is necessary that potential $U$ to be a real, even function. Indeed $\mathcal{PT}$-symmetry implies that $V(x) = V^*(-x)$, where $V^*$ denotes the complex conjugate of $V$. This equality leads to $U^2(-x) = U^2(x)$ and $-\partial U(-x) = \partial U(x)$, which are satisfied if $U$ is even. The same conclusion holds for the equality $W(x) = W^*(-x)$. Note that $\mathcal{PT}$-symmetry is not a sufficient condition for obtaining a real spectrum (see [15,38]). For instance, the potentials of Examples 3 and 4 are $\mathcal{PT}$-symmetric but they have complex eigenvalues. The conditions for which a complex-valued potential may possess a real spectrum have been considered in the literature, see, e.g., [39,40].

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