THE QUIVER OF THE SEMIGROUP ALGEBRA OF A 
LEFT REGULAR BAND

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ABSTRACT. Recently it has been noticed that many interesting 
combinatorial objects belong to a class of semigroups called left 
regular bands, and that random walks on these semigroups encode 
several well-known random walks. For example, the set of faces of a 
hyperplane arrangement is endowed with a left regular band struc-
ture. This paper studies the module structure of the semigroup 
algebra of an arbitrary left regular band, extending results for the 
semigroup algebra of the faces of a hyperplane arrangement. In 
particular, a description of the quiver of the semigroup algebra is 
given and the Cartan invariants are computed. These are used to 
compute the quiver of the face semigroup algebra of a hyperplane 
arrangement and to show that the semigroup algebra of the free 
left regular band is isomorphic to the path algebra of its quiver.

CONTENTS

1. Introduction 2
2. Left Regular Bands 3
3. Representations of the Semigroup Algebra 5
4. Primitive Idempotents of the Semigroup Algebra 6
5. Projective Indecomposable Modules of the Semigroup Algebra 9
6. The Quiver of the Semigroup Algebra 10
7. An Inductive Construction of the Quiver 11
8. Example: The Free Left Regular Band 13
9. Example: The Face Semigroup of a Hyperplane Arrangement 14
10. Idempotents in the subalgebras $k\alpha S$ and $kS_{\geq X}$ 15
11. Cartan Invariants of the Semigroup Algebra 16
12. Example: The Face Semigroup of a Hyperplane Arrangement 18
13. Example: The Free Left Regular Band 18
1. INTRODUCTION

A left regular band is a semigroup $S$ satisfying $x^2 = x$ and $xyx = xy$ for all $x, y \in S$. Recent interest in left regular bands and their semigroup algebras arose due to the work of K. S. Brown [Brown, 2000], in which the representation theory of the semigroup algebra is used to study random walks on the semigroup. There are several interesting examples of such random walks, including the random walk on the chambers of a hyperplane arrangement. Several detailed examples are included in [Brown, 2000].

The starting point of this paper is the fact that the irreducible representations of the semigroup algebra of a left regular band are all 1-dimensional. This implies that there is a canonical quiver (a directed graph) associated to the left regular band, and that the semigroup algebra is a quotient of the path algebra of the quiver. This paper determines a combinatorial description of this quiver and the Cartan invariants of the semigroup algebras and illustrates the theory through detailed examples.

The paper is structured as follows. Section 2 recalls the definition and collects some properties of left regular bands, and introduces the examples that will be used throughout the paper. Section 3 describes the irreducible representations of the semigroup algebra of a left regular band. In Section 4 a complete system of primitive orthogonal idempotents for the semigroup algebra is explicitly constructed. Section 5 describes the projective indecomposable modules of the semigroup algebra. Sections 6 through 9 deal with computing the quiver of the semigroup algebra. Sections 10 through 13 compute the Cartan invariants of the semigroup algebras. Finally, Section 14 discusses future directions for this project.
2. Left Regular Bands

See [Brown, 2000, Appendix B] for foundations of left regular bands and for proofs of the statements presented in this section.

A left regular band is a semigroup $S$ satisfying the following two properties.

(LRB1) $x^2 = x$ for all $x \in S$.
(LRB2) $xyx = xy$ for all $x, y \in S$.

Define a relation on the elements of $S$ by $y \leq x$ iff $yx = x$. This relation is a partial order (reflexive, transitive and antisymmetric), so $S$ is a poset.

Define another relation on the elements of $S$ by $y \preceq x$ iff $xy = x$. This relation is reflexive and transitive, but not necessarily antisymmetric. Therefore we get a poset $L$ by identifying $x$ and $y$ if $x \preceq y$ and $y \preceq x$.

Let $\text{supp} : S \to L$ denote the quotient map. $L$ is called the support semilattice of $S$ and $\text{supp} : S \to L$ is called the support map.

**Proposition 2.1.** If $S$ is a left regular band, then there is a semilattice $L$ and a surjection $\text{supp} : S \to L$ satisfying the following properties for all $x, y \in S$.

1. If $y \leq x$, then $\text{supp}(y) \leq \text{supp}(x)$.
2. $\text{supp}(xy) = \text{supp}(x) \lor \text{supp}(y)$.
3. $xy = x$ iff $\text{supp}(y) \leq \text{supp}(x)$.
4. If $S'$ is a subsemigroup of $S$, then the image of $S'$ in $L$ is the support semilattice of $S'$.

Statement (1) says that $\text{supp}$ is an order-preserving poset map. (2) says that $\text{supp}$ is a semigroup map where we view $L$ as a semigroup with product $\lor$. (3) follows from the construction of $L$, and (4) follows from the fact that (3) characterizes $L$ up to isomorphism. If $S$ has an identity element then $L$ has a minimal element $\hat{0}$. If, in addition, $L$ is finite, then $L$ has a maximal element $\hat{1}$, and is therefore a lattice.
Example 2.2 (The Free Left Regular Band). The free left regular band $F(A)$ with identity on a finite set $A$ is the set of all (ordered) sequences of distinct elements from $A$ with multiplication defined by

$$(a_1, \ldots, a_l) \cdot (b_1, \ldots, b_m) = (a_1, \ldots, a_l, b_1, \ldots, b_m)$$

where $\Implies$ means “delete any element that has occurred earlier”. Equivalently, $F(A)$ is the set of all words on the alphabet $A$ that do not contain any repeated letters.

The empty sequence is an element of $F(A)$, therefore $F(A)$ contains an identity element. The support lattice of $F(A)$ is the lattice $L$ of subsets of $A$ and the support map $\text{supp} : F(A) \to A$ sends a sequence $(a_1, \ldots, a_l)$ to the set of elements in the sequence $\{a_1, \ldots, a_l\}$. Figure 1 shows the Hasse diagrams of the poset $(F(A), \leq)$ and the support lattice of $F(A)$, where $A = \{a, b, c\}$.

Example 2.3 (Hyperplane Arrangements). A (central) hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes containing the origin in some real vector space $V = \mathbb{R}^d$, for some $d \in \mathbb{N}$. For each hyperplane $H \in \mathcal{A}$, let $H^+$ and $H^-$ denote the two open half spaces of $V$ determined by $H$. The choice of labels $H^+$ and $H^-$ on the two open half spaces is arbitrary, but fixed throughout. For convenience, let $H^0$...
denote $H$. A face of the arrangement $A$ is a non-empty intersection of the form $\bigcap_{H \in A} H^{\varepsilon_H}$, where $\varepsilon_H \in \{0, +, -\}$. Let $F$ denote the set of all faces of $A$. Define a relation on $F$ by $x \leq y$ iff $x \subseteq \overline{y}$, where $\overline{y}$ denotes the closure of the set $y$. The relation is a partial order.

If $x = \bigcap_{H \in A} H^{\varepsilon_H}$ is a face, then let $\sigma_H(x) = \varepsilon_H$ and let $\sigma(x) = (\sigma_H(x))_{H \in A}$. The sequence $\sigma(x)$ is called the sign sequence of $x$. Define the product of two faces $x, y \in F$ to be the face $xy$ with sign sequence

$$
\sigma_H(xy) = \begin{cases} 
\sigma_H(x), & \text{if } \sigma_H(x) \neq 0, \\
\sigma_H(y), & \text{if } \sigma_H(x) = 0.
\end{cases}
$$

This product has a geometric interpretation: the product $xy$ of two faces $x, y$ is the face entered by moving a small positive distance along a straight line from any point in $x$ to a point in $y$. It is straightforward to verify that this product gives $F$ the structure of an associative left regular band. Since all the hyperplanes in the arrangement contain the origin, $F$ contains an identity element: $\bigcap_{H \in A} H$. The left regular band $F$ is called the face semigroup of $A$, and the semigroup algebra $kF$ of $F$ is called the face semigroup algebra of $A$.

Let $L$ denote the set of subspaces of $V$ that can be obtained as the intersection of some hyperplanes in $A$. Then $L$ is a finite lattice, called the intersection lattice of $A$, where the subspaces are ordered by inclusion and the meet operation is intersection. (Note that some authors order $L$ by reverse inclusion rather than inclusion.) $L$ is the support lattice of $F$ and the support map $\text{supp} : F \to L$ maps a face $x \in F$ to the intersection of all the hyperplanes of the arrangement that contain the face: $\text{supp}(x) = \bigcap_{\{H \in A : x \subseteq H\}} H$.

### 3. Representations of the Semigroup Algebra

Let $k$ denote a field and $S$ a left regular band. The semigroup algebra of $S$ is denoted by $kS$ and consists of all formal linear combinations $\sum_{s \in S} \lambda_s s$, with $\lambda_s \in k$ and multiplication induced by $\lambda_s s \cdot \lambda_t t = \lambda_s \lambda_t st$,
where \( st \) is the product of \( s \) and \( t \) in the semigroup \( S \). The following summarizes Section 7.2 of \[Brown, 2000\].

Since \( S \) and \( L \) are semigroups and \( \text{supp} : S \to L \) is a semigroup morphism, the support map extends linearly to a surjection of semigroup algebras \( \text{supp} : kS \to kL \). The kernel of this map is nilpotent and the semigroup algebra \( kL \) is isomorphic to a product of copies of the field \( k \), one copy for each element of \( L \). Standard ring theory implies that \( \ker(\text{supp}) \) is the Jacobson radical of \( kS \) and that the irreducible representations of \( kS \) are given by the components of the composition \( kS \xrightarrow{\text{supp}} kL \xrightarrow{\cong} \prod_{X \in L} k \). This last map sends \( X \in L \) to the vector with 1 in the \( Y \)-component if \( Y \geq X \) and 0 otherwise. The \( X \)-component of this surjection is the map \( \chi_X : kS \to k \) defined on the elements \( y \in S \) by

\[
\chi_X(y) = \begin{cases} 
1, & \text{if } \text{supp}(y) \leq X, \\
0, & \text{otherwise}. 
\end{cases}
\]

The elements

\[
E_X = \sum_{Y \geq X} \mu(X, Y)Y
\]

in \( kL \), one for each \( X \in L \), correspond to the standard basis vectors of \( \prod_{X \in L} k \) under the isomorphism \( kL \cong \prod_{X \in L} k \). In the above \( \mu \) denotes the M"obius function of the lattice \( L \) \[Stanley, 1997, \S3.7\]. The elements \( \{E_X\}_{X \in L} \) form a basis of \( kL \) and a complete system of primitive orthogonal idempotents for \( kL \) (see the next section for the definition).

4. Primitive Idempotents of the Semigroup Algebra

Let \( A \) be a \( k \)-algebra. An element \( e \in A \) is idempotent if \( e^2 = e \). It is a primitive idempotent if \( e \) is idempotent and we cannot write \( e = e_1 + e_2 \) where \( e_1 \) and \( e_2 \) nonzero idempotents in \( A \) with \( e_1e_2 = 0 = e_2e_1 \). Equivalently, \( e \) is primitive iff \( Ae \) is an indecomposable \( A \)-module. A set of elements \( \{e_i\}_{i \in I} \subset A \) is a complete system of primitive orthogonal idempotents for \( A \) if \( e_i \) is a primitive idempotent for every \( i \), if \( e_ie_j = 0 \)
for $i \neq j$ and if $\sum_i e_i = 1$. If $\{e_i\}_{i \in I}$ is a complete system of primitive orthogonal idempotents for $A$, then $A \cong \bigoplus_{i \in I} Ae_i$ as left $A$-modules and $A \cong \bigoplus_{i,j \in I} e_i Ae_j$ as $k$-vector spaces.

Let $S$ denote a left regular band with identity. For each $X \in L$, fix an $x \in S$ with supp$(x) = X$ and define elements in $kS$ recursively by the formula,

$$e_X = x - \sum_{Y > X} xe_Y.$$  

(4.1)

**Lemma 4.1.** Let $w \in S$ and $X \in L$. If supp$(w) \not\leq X$, then $we_X = 0$.

**Proof.** We proceed by induction on $X$. This is vacuously true if $X = \hat{1}$. Suppose the result holds for all $Y \in L$ with $Y > X$. Suppose $w \in S$ and $W = \text{supp}(w) \not\leq X$. Using the definition of $e_X$ and the identity $wxw = wx$ (LRB2),

$$we_X = wx - \sum_{Y > X} wxe_Y = wx - \sum_{Y > X} wx(we_Y).$$

By induction, $we_Y = 0$ if $W \not\leq Y$. Therefore, the summation runs over $Y$ with $W \leq Y$. But $Y > X$ and $Y \geq W$ iff $Y \geq W \vee X$, so the summation runs over $Y$ with $Y \geq W \vee X$.

$$we_X = wx - \sum_{Y > X} wx(we_Y) = wx - \sum_{Y \geq X \vee W} wxe_Y.$$  

Now let $z$ be the element of support $X \vee W$ chosen in defining $e_{X \vee W}$. So $e_{X \vee W} = z - \sum_{Y > X \vee W} ze_Y$. Note that $ze_{X \vee W} = e_{X \vee W}$ since $z = z^2$. Therefore, $z = \sum_{Y \geq X \vee W} ze_Y$. Since supp$(wx) = W \vee X = \text{supp}(z)$, it follows from Proposition 2.1 (3) that $wx = wxz$. Combining the last two statements,

$$we_X = wx - \sum_{Y \geq X \vee W} wxe_Y = wx \left(z - \sum_{Y \geq X \vee W} ze_Y\right) = 0. \quad \square$$

**Theorem 4.2.** Let $S$ denote a finite left regular band with identity and $L$ its support lattice. Let $k$ denote an arbitrary field. The elements $\{e_X\}_{X \in L}$ form a complete system of primitive orthogonal idempotents in the semigroup algebra $kS$.  

Proof. Complete. 1 is the only element of support \( \hat{0} \). Hence, \( e_0 = 1 - \sum_{Y > 0} e_Y \). Equivalently, \( \sum_X e_X = 1 \).

Idempotent. Since \( e_Y \) is a linear combination of elements of support at least \( Y \), \( e_Y z = e_Y \) for any \( z \) with \( \text{supp}(z) \leq Y \) (Proposition 2.1 3). Using the definition of \( e_X \), the facts \( e_X = xe_X \) and \( e_Y = e_Y y \), and Lemma 4.1

\[
e^2_X = \left( x - \sum_{Y > X} xe_Y \right) e_X = xe_X - \sum_{Y > X} xe_Y (ye_X) = xe_X = e_X.
\]

Orthogonal. We show that for every \( X \in L \), \( e_X e_Y = 0 \) for \( Y \neq X \). If \( X = \hat{1} \), then \( e_X e_Y = e_X xe_Y = 0 \) for every \( Y \neq X \) by Lemma 4.1 since \( X = \hat{1} \) implies \( X \not\leq Y \). Now suppose the result holds for \( Z > X \). That is, \( e_Z e_Y = 0 \) for all \( Y \neq Z \). If \( X \not\leq Y \), then \( e_X e_Y = 0 \) by Lemma 4.1. If \( X < Y \), then \( e_X e_Y = xe_Y - \sum_{Z > X} x(e_Z e_Y) = xe_Y - xe_Y^2 = 0 \).

Primitive. We’ll show that \( e_X \) lifts \( E_X = \sum_{Y \geq X} \mu(X, Y)Y \) (see Equation (3.1)) for all \( X \in L \), a primitive idempotent in \( kL \). (Then since \( e_X \) lifts a primitive idempotent, it is itself a primitive idempotent.) If \( X = \hat{1} \), then \( \text{supp}(e_1) = \hat{1} = E_1 \). Suppose the result holds for \( Y > X \). Then \( \text{supp}(e_X) = \text{supp}(x - \sum_{Y > X} xe_Y) = X - \sum_{Y > X} (X \lor E_Y) \). Since \( E_Y \) is a linear combination of elements \( Z \geq Y \), it follows that \( X \lor E_Y = E_Y \) if \( Y > X \). Therefore, \( \text{supp}(e_X) = X - \sum_{Y > X} E_Y \). The Möbius inversion formula [Stanley, 1997 §3] applied to \( E_X = \sum_{Y \geq X} \mu(X, Y)Y \) gives \( X = \sum_{Y \geq X} E_X \). Hence, \( \text{supp}(e_X) = X - \sum_{Y > X} E_Y = E_X \). \( \square \)

Remark 4.3. We can replace \( x \in S \) in Equation (4.1) with any linear combination \( \tilde{x} = \sum_{\text{supp}(x) = X} \lambda_x x \) of elements of support \( X \) whose coefficients \( \lambda_x \) sum to 1. The proofs still hold since the element \( \tilde{x} \) is idempotent and satisfies \( \text{supp}(\tilde{x}) = X \) and \( \tilde{x} y = \tilde{x} \) for all \( y \) with \( \text{supp}(y) \leq X \). Unless explicitly stated we will use the idempotents constructed above.

Corollary 4.4. The set \( \{ xe_{\text{supp}(x)} \mid x \in S \} \) is a basis of \( kS \) of primitive idempotents (not necessarily orthogonal idempotents).
Proof. Let \( y \in S \). Then by Theorem 4.2 and Lemma 4.1,

\[
y = y1 = y \sum_{Z} e_{Z} = \sum_{Z \geq \text{supp}(y)} ye_{Z} = \sum_{Z \geq \text{supp}(y)} (yz)e_{Z},
\]

where \( z \in S \) was the element used to define \( e_{Z} \). Since \( \text{supp}(yz) = \text{supp}(y) \lor \text{supp}(z) = Z \), every element \( y \in S \) is a linear combination of elements of the form \( xe_{\text{supp}(x)} \). So the elements \( xe_{\text{supp}(x)} \), one for each \( x \) in \( S \), span \( kS \). Since the number of these elements is the cardinality of \( S \), which is the dimension of \( kS \), the set forms a basis of \( kS \). The elements are idempotent since \( (xe_{X})^2 = (xe_{X})(xe_{X}) = xe_{X}^2 = xe_{X} \) (since \( xyx = xy \) for all \( x, y \in S \)). Since \( xe_{X} \) lifts the primitive idempotent \( E_{X} = \sum_{Y \geq X} \mu(X,Y)Y \in kL \), it is also a primitive idempotent (see the end of the proof of Theorem 4.2). \( \square \)

5. Projective Indecomposable Modules of the Semigroup Algebra

For \( X \in L \), let \( S_{X} \subset S \) denote the set of elements of \( S \) of support \( X \). For \( y \in S \) and \( x \in S_{X} \) define

\[
y \cdot x = \begin{cases} 
    yx, & \text{supp}(y) \leq \text{supp}(x), \\
    0, & \text{supp}(y) \not\leq \text{supp}(x).
\end{cases}
\]

Then \( \cdot \) defines an action of \( kS \) on the \( k \)-vector space \( kS_{X} \) spanned by \( S_{X} \).

Lemma 5.1. Let \( X \in L \). Then \( \{xe_{X} \mid \text{supp}(x) = X\} \) is a basis for \( (kS)e_{X} \).

Proof. Suppose \( \sum_{w \in S} \lambda_{w}we_{X} \in kSe_{X} \). If \( \text{supp}(w) \not\leq X \), then \( we_{X} = 0 \). So suppose \( \text{supp}(w) \leq X \). Then \( \text{supp}(wx) = \text{supp}(w) \lor X = X \). Therefore,

\[
\sum_{w \in S} \lambda_{w}we_{X} = \sum_{w \in S} \lambda_{w}(wx)e_{X} \in \text{span}_{k}\{ye_{X} \mid \text{supp}(y) = X\},
\]

where \( x \) is the element chosen in the construction of \( e_{X} \) (recall that \( e_{X} = xe_{X} \) since \( x^2 = x \)). So the elements span \( kSe_{X} \). These elements
are linearly independent being a subset of a basis of $kS$ (Corollary 4.4).

**Proposition 5.2.** There is a $kS$-module isomorphism $kS_X \cong kSe_X$ given by right multiplication by $e_X$. Therefore, the $kS$-modules $kS_X$ are all the projective indecomposable $kS$-modules. The radical of $kS_X$ is $\text{span}_k \{ y - y' \mid y, y' \in S_X \}$.

**Proof.** Define a map $\phi : kS_X \to kSe_X$ by $w \mapsto \phi(w) = we_X$. Then $\phi$ is surjective since $\phi(y) = ye_X$ for $y \in S_X$ and since $\{ ye_X \mid \text{supp}(y) = X \}$ is basis for $kSe_X$ (Lemma 5.1). Since $\dim kS_X = \# S_X = \dim kSe_X$, the map $\phi$ is an isomorphism of $k$-vector spaces.

$\phi$ is a $kS$-module map. Let $y \in S$ and let $x \in S_X$. If $\text{supp}(y) \leq X$, then $\phi(y \cdot x) = \phi(ye_X) = yxe_X = yx \phi(x)$. If $\text{supp}(y) \nleq X$, then $y \cdot x = 0$. Hence, $\phi(x \cdot y) = 0$. Also, since $\text{supp}(y) \nleq X$, it follows from Lemma 4.1 that $ye_X = 0$. Therefore, $y \phi(x) = yxe_X = yx(ye_X) = yx0 = 0$. So $\phi(y \cdot x) = y \phi(x)$. Hence $\phi$ is an isomorphism of $kS$-modules.

Since all the projective indecomposable $kS$-modules (upto isomorphism) are of the form $kSe_X$ for a complete system of primitive orthogonal idempotents $\{ e_X \}$, the $kS$-modules $kS_X$ are all the indecomposable projective $kS$-modules. \qed

6. **The Quiver of the Semigroup Algebra**

Let $A$ be a finite dimensional $k$-algebra whose simple modules are all 1-dimensional. The Ext-quiver or quiver of $A$ is the directed graph $Q$ with one vertex for each isomorphism class of simple modules and $\dim_k(\text{Ext}_A^1(M_X, M_Y))$ arrows from $X$ to $Y$, where $M_X$ and $M_Y$ are simple modules of the isomorphism classes corresponding to the vertices $X$ and $Y$, respectively. The path algebra $kQ$ of $Q$ is the $k$-algebra spanned by paths of $Q$ with multiplication induced by path composition: if two paths in $Q$ compose to form another path, then that is the product; if the paths do not compose, then the product is 0. If $Q$ is the quiver of $A$, then there exists a $k$-algebra surjection from $kQ$ onto $A$. Although the quiver $Q$ is canonical, this surjection is not.
Let $S$ be a left regular band with identity and let $L$ denote the support lattice of $S$. Let $X,Y \in L$ with $Y \leq X$ and fix $y \in S$ with $\text{supp}(y) = Y$. Define a relation on the elements of $S_X$ by $x \sim x'$ if there exists an element $w \in S$ satisfying $y < w$, $w < yx$ and $w < yx'$. (Equivalently, $yw = w$, $wx = yx$, $wx' = yx'$ and $\text{supp}(w) < X$.) Note that $x \sim x'$ iff $x \sim yx'$. Also note that for $X = \hat{1}$ and $Y = \hat{0}$, the relation becomes $x \sim x'$ iff there exists $w \neq 1$ such that $x > w$ and $x' > w$.

The relation $\sim$ is symmetric and reflexive, but not necessarily transitive. Let $\sim$ denote the transitive closure of $\sim$. Let $a_{XY} = \#(S_X/ \sim) - 1$, the number of equivalence classes of $\sim$ minus one. If $Y \not< X$, define $a_{XY} = 0$. In order to avoid confusion, we denote by $a^S_{XY}$ the number $a_{XY}$ computed in $S$. Since $u < v$ implies $yu < yv$ for all $u,v,y \in S$ (follows from (LRB2)), it follows that the relations $\sim$ and $\sim$ do not depend on the choice of $y$ with $\text{supp}(y) = Y$.

**Lemma 6.1.** Let $S$ be a finite left regular band with identity and $L$ its support lattice. Let $M_X$ and $M_Y$ denote the simple modules with irreducible characters $\chi_X$ and $\chi_Y$, respectively. Then

$$\dim(\text{Ext}_A^1(M_X, M_Y)) = a_{XY}.$$  

**Proof.** The proof is rather lengthy so we postpone it until page 15.

**Theorem 6.2.** Let $S$ be a left regular band with identity and $L$ the support lattice of $S$. Let $k$ denote a field. The quiver of the semigroup algebra $kS$ has $L$ as the vertex set and $a_{XY}$ arrows from the vertex $X$ to the vertex $Y$.

**7. An Inductive Construction of the Quiver**

In this section we describe how knowledge about the numbers $a^S_{XY}$ for certain subsemigroups $S'$ of $S$ determine all the numbers $a^S_{XY}$. This allows for an inductive construction of the quiver of a left regular band.

Suppose $S$ is a left regular band with identity. Let $X,Y \in L$ with $Y \leq X$ and let $y \in S$ be an element with $\text{supp}(y) = Y$. Then $yS =$
$\{yw : w \in S\}$ and $S_{\leq X} = \{w \in S : \text{supp}(w) \leq X\}$ are subsemigroups of $S$.

**Proposition 7.1.** Let $S$ be a left regular band with identity, and let $L$ denote the support lattice of $S$. Suppose $y \in S$ and $X \in L$. The quiver of the semigroup algebra $k(yS_{\leq X})$ of the left regular band $yS_{\leq X}$ is the full subquiver of the quiver of the semigroup algebra $kS$ on the vertices in the interval $[\text{supp}(y), X] \subset L$.

The Proposition follows from the following Lemma that shows the number of arrows from $X$ to $Y$ in the quiver of $kS$ is the number of arrows from $\hat{1}$ to $\hat{0}$ in the quiver of $k(yS_{\leq X})$, where $y \in S$ is any element of support $Y$. Recall that $a_{i_1}^{yS_{\leq X}}$ denotes the number $a_{i_0}$ computed in the left regular band $yS_{\leq X}$.

**Lemma 7.2.** Let $S$ be a left regular band with identity. Then $a_{xy}^S = a_{i_1}^{yS_{\leq X}}$. That is, the number $a_{xy}$ computed in $S$ is the number $a_{i_0}$ computed in $yS_{\leq X}$.

**Proof.** If $\text{supp}(y) \not\leq X$, then $yS_{\leq X}$ is empty. So $a_{xy}^S = 0 = a_{i_1}^{yS_{\leq X}}$. So suppose $\text{supp}(y) \leq X$.

Since $x \sim x'$ iff $x \sim yx'$ for any elements $x, x'$ of support $X$, every equivalence class of $\sim$ (on $S_X$) contains an element of $yS_X$. Therefore, $a_{xy} + 1$ is the number of equivalence classes of $\sim$ restricted to $yS_X$.

Since $yS_{\leq X}$ is a subsemigroup of $S$, the support lattice of $yS_{\leq X}$ is the image of $yS_{\leq X}$ in $L$. Therefore, the support lattice of $yS_{\leq X}$ is the interval $[Y, X]$ in $L$. Since the top and bottom elements of $[Y, X]$ are $X$ and $Y$ respectively, the number $a_{i_1}^{yS_{\leq X}} + 1$ is the number of equivalence classes of $\sim$ restricted to $yS_X$.  

Therefore, if the numbers $a_{i_1}^{yS_{\leq X}}$ are known for all the subsemigroups of $S$ of the form $yS_{\leq X}$, then the quiver of $kS$ is known. We illustrate this technique with two examples in the next two sections.
8. Example: The Free Left Regular Band

Let \( S = F(A) \) denote the free left regular band on a finite set \( A \) (defined in Example 2.2). Recall that the support lattice \( L \) of \( S \) is the set of subsets of \( A \).

Let \( y \in S \) and let \( Y \subset A \) denote the set of elements occurring in the sequence \( y \). Then \( yS \) is the set of all sequences of elements of \( A \) (without repetition) that begin with the sequence \( y \). Therefore, \( yS \) is isomorphic to the free left regular band on \( A \setminus Y \). If \( X \subset A \) (so \( X \in L \) ), then \( S_{\leq X} \) is the set of all sequences containing only elements from \( X \) (without repetition). Therefore, \( S_{\leq X} \) is also a free left regular band. It follows that \( yS_{\leq X} \) is a free left regular band for any \( y \in S \) and \( X \subset A \). Therefore, the quiver of \( S \) is determined once the numbers \( a_{\emptyset}^0 = a_{A\emptyset} \) are known for any free left regular band.

If two sequences \( x, y \in S \) begin with the same element \( a \in A \), then \( ax = x \) and \( ay = y \). Therefore, \( x \sim y \). Conversely, if \( x \sim y \), then there is a nonempty sequence \( w \) such that \( wx = x \) and \( wy = y \). Then \( x \) and \( y \) both begin with the first element of \( w \). Therefore, \( x \sim y \) iff \( x \) and \( y \) are sequences beginning with the same element. So the equivalence classes of \( \sim \) are determined by the first elements of the sequences in \( S \). Hence, \( a_{i0} = \#(A) - 1 \). This argument applies to any free left regular band with identity, so \( a_{XY} = \#(X \setminus Y) - 1 \) since \( yS_{\leq X} \) is isomorphic to the free left regular band on the elements \( X \setminus Y \).

**Theorem 8.1.** (K. S. Brown, private communication.) Let \( S = F(A) \) be the free left regular band on a finite set \( A \) and let \( k \) denote a field. Then the quiver of the semigroup algebra \( kS \) has one vertex \( X \) for each subset \( X \) of \( A \) and \( \#(X \setminus Y) - 1 \) arrows from \( X \) to \( Y \) if \( Y \subset X \) (and no other arrows or vertices).
9. Example: The Face Semigroup of a Hyperplane Arrangement

Let $F$ be the face semigroup of a central hyperplane arrangement $A$ and let $\mathcal{L}$ be the intersection lattice of $A$ (see Example 2.3.) Let $X, Y \in \mathcal{L}$ and $y$ a face of support $Y$. Then the subsemigroup $yF_{\leq X}$ is the semigroup of faces of a hyperplane arrangement with intersection lattice $[Y, X] \subset \mathcal{L}$. (Specifically, this hyperplane arrangement is given by $\{X \cap H : H \in A, Y \subset H, X \not\subset H\}$.) Therefore, we know all the numbers $a_{XY}$ for $F$ if we know the number $a_{\hat{1}\hat{0}}$ for the face semigroup of an arbitrary arrangement.

If $\mathcal{L}$ contains only one element, then $\hat{0} = \hat{1}$ and $a_{\hat{1}\hat{0}} = 0$. Suppose that $\mathcal{L}$ contains at least two elements. It is well-known that for any two distinct chambers $c$ and $d$, there exists a sequence of chambers $c_0 = c, c_1, \ldots, c_i = d$ such that $c_{j-1}$ and $c_j$ share a common codimension one face $w_j$ for each $1 \leq j \leq i$ [Brown, 1989 §I.4E Proposition 3]. Therefore, $c_{j-1} \sim c_j$ unless $w_j$ is of support $\hat{0}$, in which case $\mathcal{L}$ has two elements. Equivalently, $c \sim d$ iff the arrangement is of rank greater than 2. So if $\mathcal{L}$ has exactly two elements, then $a_{\hat{1}\hat{0}} = 1$ and if $\mathcal{L}$ has more than two elements then $a_{\hat{1}\hat{0}} = 1$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The support lattice and the quiver of the semigroup algebra of the free left regular band on three generators. See Figure 1.}
\end{figure}
Theorem 9.1 (Saliola, 2006, Corollary 8.4). The quiver $Q$ of the semigroup algebra $kF$ coincides with the Hasse diagram of $L$. That is, there is exactly one arrow $X \rightarrow Y$ iff $Y \preceq X$.

In [Saliola, 2006] the relations of the quiver are also determined. Let $I$ be the ideal generated by the following elements, one for each interval $[Z, X]$ of length two in $L$,

$$\sum_{Y: Y \preceq Z \preceq X} X \rightarrow Y \rightarrow Z.$$ 

Then $kF \cong kQ/I$ as $k$-algebras, where $kQ$ is the path algebra of $Q$.

10. Idempotents in the subalgebras $k(yS)$ and $kS_{\geq Y}$

This section describes the subalgebras of $kS$ generated by the subsemigroups $yS$ and $S_{\leq Y}$ of $S$.

Let $S$ be a left regular band. Recall that for $y \in S$, the set $yS = \{yw : w \in S\} = \{w \in S : w > y\}$ is a subsemigroup of $S$ (and hence a left regular band). Note that if $\text{supp}(y') = \text{supp}(y)$ then the left regular bands $yS$ and $y'S$ are isomorphic with isomorphism given by multiplication by $y$ (the inverse is multiplication by $y'$). Since $yS$ is a subsemigroup of $S$, the support lattice of $yS$ is the image of $yS$ in $L$ by Proposition 2.1, which is the interval $[Y, \hat{1}]$.

Proposition 10.1. Let $S$ be a left regular band, let $y \in S$ and let $Y = \text{supp}(y)$. There exists a complete system of primitive orthogonal idempotents $\{e_X : X \in L\}$ in $kS$ such that $\{e_X : X \geq Y\}$ is a complete system of primitive orthogonal idempotents in the semigroup algebra $k(yS)$. Moreover, $k(yS) = (\sum_{X \geq Y} e_X)kS$.

Proof. For each $X \in L$, fix $x \in S$ with $\text{supp}(x) = X$. If $X \geq Y$, then replace $x$ with $yx$. Note that $\text{supp}(yx) = \text{supp}(x)$ since $X \geq Y$. Therefore, $x > y$ if $X \geq Y$. The formula $e_X = x - \sum_{W > X} xe_W$ for $X \in L$ defines a complete system of primitive orthogonal idempotents for $kS$ (Theorem 4.2). And since the support lattice of $yS$ is $[Y, \hat{1}] \subset L$, the elements $e_X = x - \sum_{W > X} xe_W$ for $X \geq Y$ define a complete
system of primitive orthogonal idempotents in \( k(yS) \). Since \( y \) is the identity of \( yS \), we have \( y = \sum_{x \geq y} e_x \). Therefore, \( k(yS) = y(kS) = (\sum_{x \geq y} e_x)kS \).

If \( Y \in L \), then \( S_{\leq Y} = \{ w \in S : \text{supp}(w) \leq Y \} \) is a subsemigroup of \( S \). The support lattice of \( S_{\leq Y} \) is the interval \([0, Y]\) of \( L \). Let \( \text{proj}_{kS_{\leq Y}} : kS \to kS_{\leq X} \) denote the projection onto the subspace \( kS_{\leq X} \) of \( kS \).

**Proposition 10.2.** Let \( S \) be a left regular band and \( Y \in L \). Let \( \{ e_X : X \in L \} \) denote a complete system of primitive orthogonal idempotents of \( kS \). Then \( \{ \text{proj}_{kS_{\leq Y}}(e_X) : X \leq Y \} \) is a complete system of primitive orthogonal idempotents of \( kS_{\leq Y} \). Moreover, the semigroup algebra \( k(S_{\leq Y}) \) is isomorphic to \( kS(\sum_{X \leq Y} e_X) \).

**Proof.** The map \( \text{proj}_{kS_{\leq Y}} \) is an algebra morphism \( kS \to kS_{\leq Y} \). This follows from the fact that \( \text{supp}(wx) = \text{supp}(w) \vee \text{supp}(x) \) for any \( x, w \in S \). So if \( X \leq Y \), then \( \text{proj}_{kS_{\leq Y}}(e_X) = x - \sum_{W > X} x \text{proj}_{kS_{\leq Y}}(e_W) \) since \( e_X = x - \sum_{W > X} xe_W \). Therefore, the elements \( \text{proj}_{kS_{\leq Y}}(e_X) \) for \( X \leq Y \) form a complete system of primitive orthogonal idempotents for the semigroup algebra of the left regular band \( S_{\leq Y} \) (Theorem 4.2). Since \( \text{proj}_{kS_{\leq Y}} \) is an algebra morphism, it restricts to a surjective morphism of algebras \( \text{proj}_{kS_{\leq Y}} : kS(\sum_{X \leq Y} e_X) \to k(S_{\leq Y}) \). Since \( kS_X \cong (kS)e_X \) for all \( X \in L \) as \( kS \)-modules (Proposition 5.2), \( \dim(kS_{\leq Y}) = \dim(\sum_{X \leq Y}(kS)e_X) \). So \( \text{proj}_{kS_{\leq Y}} \) is an isomorphism. Its inverse is right multiplication by \( \sum_{X \leq Y} e_X \). \( \square \)

11. Cartan Invariants of the Semigroup Algebra

The Cartan invariants of a finite dimensional \( k \)-algebra \( A \) are the numbers \( \dim_k(\text{Hom}_A(Ae_X, Ae_Y)) \), where \( \{ e_X \}_{X \in I} \) is a complete system of primitive orthogonal idempotents for \( A \). They are independent of the choice of \( \{ e_X \}_{X \in I} \).

Let \( S \) be a left regular band with identity and let \( L \) denote the support lattice of \( S \). For \( X, Y \in L \), define numbers \( m(Y, X) \) follows. If \( Y \nleq X \), then \( m(Y, X) = 0 \). If \( Y \leq X \), then define \( m(Y, X) \) by the
formulas

\begin{equation}
\sum_{W \leq Y \leq X} m(Y, X) = \#(wS_X),
\end{equation}

one for each \( W \in L \), where \( w \) is an element of support \( W \). (Recall that the number \( \#(wS_X) \) does not depend on the choice of \( w \) with \( \text{supp}(w) = W \).) Equivalently,

\[ m(Y, X) = \sum_{Y \leq W \leq X} \mu(Y, W) \#(wS_X), \]

where \( \mu \) is the Möbius function of \( L \) [Stanley, 1997, §3.7].

**Proposition 11.1.** Let \( S \) be a left regular band with identity. Let \( \{e_X\}_{X \in L} \) denote a complete system of primitive orthogonal idempotents for \( kS \). Then for any \( X, Y \),

\[ \dim(e_YkSe_X) = \dim \text{Hom}_{kS}(kSe_Y, kSe_X) = m(Y, X). \]

Therefore, the numbers \( m(Y, X) \) are the Cartan invariants of \( kS \).

**Proof.** The first equality follows from the identity \( \text{Hom}_A(Ae, Af) \cong eAf \) for idempotents \( e, f \) of a \( k \)-algebra \( A \). If \( Y \nless X \), then it follows from (LRB2) and Lemma 4.1 that \( e_YkSe_X = 0 \). Suppose that \( Y \leq X \). From the previous section, \( k(yS) = \sum_{W \geq Y} e_W kS \) for some complete system of primitive orthogonal idempotents. Combined with the isomorphism \( kS_X \cong kSe_X \) we get \( k(yS_X) \cong \bigoplus_{Y \leq W \leq X} e_W kSe_X \).

Therefore,

\[ \sum_{Y \leq W \leq X} m(W, X) = \dim(k(yS_X)) = \sum_{Y \leq W \leq X} \dim(e_WkSe_X). \]

The result now follows by induction. If \( X = Y \), then \( \dim e_XkSe_X = m(X, X) \). Suppose the result holds for all \( W \) with \( Y < W \leq X \). Then

\[ \dim e_YkSe_X = \sum_{Y \leq W \leq X} m(W, X) - \sum_{Y < W \leq X} \dim e_WkSe_X \]

\[ = \sum_{Y \leq W \leq X} m(W, X) - \sum_{Y < W \leq X} m(W, X) \]

\[ = m(Y, X). \]

\( \square \)
12. Example: The Face Semigroup of a Hyperplane Arrangement

Let $\mathcal{F}$ denote the semigroup of faces of a hyperplane arrangement $\mathcal{A}$. Then $\#(w\mathcal{F}_X)$ is the number of faces of support $X$ containing $w$ as a face. Zaslavsky’s Theorem [Zaslavsky, 1975] gives that this is $\sum_{W \subseteq Y \subseteq X} |\mu(Y, X)|$, where $\mu$ is the Möbius function of the intersection lattice of $\mathcal{A}$. Comparing this with Equation (11.1) we conclude that the Cartan invariants of $k\mathcal{F}$ are $m(Y, X) = |\mu(Y, X)|$. These were also computed in [Saliola, 2006, Proposition 6.4].

13. Example: The Free Left Regular Band

Let $S$ be a free left regular band on a finite set $A$. The support lattice of $S$ is the lattice of subsets of $A$. Therefore, $\mu(Y, W) = (-1)^{\#(W\setminus Y)}$ [Stanley, 1997, Example 3.8.3] for any $Y, W \in L$. And $\#(wS_X) = \#(X\setminus W)!$ since the number of elements of maximal support in the free left regular band on $A$ is precisely $\#A!$. If $n = \# X$ and $j = \# Y$, and $Y \subseteq X$, then

$$m(Y, X) = \sum_{Y \subseteq W \subseteq X} \mu(Y, W) \#(wS_X)$$

$$= \sum_{Y \subseteq W \subseteq X} (-1)^{\#W-j} (n - \#W)!$$

$$= \sum_{i=j}^{n} \sum_{\substack{Y \subseteq W \subseteq X \\#W=i}} (-1)^{i-j} (n - i)!$$

$$= \sum_{i=j}^{n} (-1)^{i-j} (n - i)! \binom{n-j}{i-j}$$

$$= (n - j)! \sum_{i=j}^{n} \frac{(-1)^{i-j}}{(i-j)!}$$

$$= (n - j)! \sum_{i=0}^{n-j} \frac{(-1)^{i}}{i!}.$$
Therefore, the number $m(Y, X)$ depends only on the cardinality of $X \setminus Y$ and we denote it by $m_i$ where $i = \#(X \setminus Y)$.

We will now prove that these numbers count paths in the quiver of $kS$. For a set $A$ of cardinality $n$, let $Q_n$ be the directed graph with one vertex for each subset of $A$ and $\#(X \setminus Y) - 1$ arrows from $X$ to $Y$ if $Y \subset X$. Let $p_n$ denote the number of paths in $Q_n$ beginning at $A$ and ending at $\emptyset$. Note that if $Y \subset X \subset A$, then the number of paths beginning at $X$ and ending at $Y$ in $Q_n$ is $p_m$ where $m = \#(X \setminus Y)$.

For each $0 \leq i \leq n - 1$ there are $n - i - 1$ arrows from $A$ to sets of size $i$, and there are $\binom{n}{i}$ such sets, so

$$p_n = \sum_{0 \leq i \leq n-1} \binom{n}{i} (n - i - 1)p_i$$

for $n \geq 1$. Equivalently,

$$\sum_{0 \leq i \leq n} \binom{n}{i} p_i = \sum_{0 \leq i \leq n-1} \binom{n}{i} (n - i)p_i.$$

If the $m_i$ satisfy the above recurrence, then $m_i = p_i$ for all $i$ since $m_0 = 1 = p_0$. Well,

$$\sum_{0 \leq i \leq n-1} \binom{n}{i} (n - i)m_i$$

$$= \sum_{0 \leq i \leq n-1} \frac{n!}{(n - i - 1)! \cdot i!} m_i$$

$$= \sum_{0 \leq i \leq n-1} \frac{n!}{(n - i - 1)! \cdot i!} \left( i! \sum_{0 \leq j \leq i} \frac{(-1)^j}{j!} \right)$$

$$= \sum_{0 \leq i \leq n-1} \frac{n!}{(n - i - 1)! \cdot i!} \left( \sum_{0 \leq j \leq i} \frac{(-1)^j}{j!} \right)$$

$$= \sum_{1 \leq k \leq n} \frac{n!}{(n - k)!} \left( \sum_{0 \leq j \leq k-1} \frac{(-1)^j}{j!} \right)$$

$$= \sum_{1 \leq k \leq n} \frac{n!}{(n - k)!} \left( \sum_{0 \leq j \leq k} \frac{(-1)^j}{j!} - \frac{(-1)^k}{k!} \right)$$

$$= \sum_{1 \leq k \leq n} \frac{n!}{(n - k)!} \left( \sum_{0 \leq j \leq k} \frac{(-1)^j}{j!} \right) - \sum_{1 \leq k \leq n} \frac{n!}{(n - k)!} \left( \frac{(-1)^k}{k!} \right)$$
\begin{align*}
&= \sum_{1 \leq k \leq n} \binom{n}{k} \left( k! \sum_{0 \leq j \leq k} \frac{(-1)^j}{j!} \right) - \sum_{1 \leq k \leq n} \binom{n}{k} (-1)^k \\
&= \sum_{1 \leq k \leq n} \binom{n}{k} m_k + 1 \\
&= \sum_{1 \leq k \leq n} \binom{n}{k} m_k + \binom{n}{0} m_0 \\
&= \sum_{0 \leq k \leq n} \binom{n}{k} m_k.
\end{align*}

**Theorem 13.1.** (K. S. Brown, private communication.) Let $S = F(A)$ be the free left regular band on a finite set $A$. Then $kS \cong kQ$, where $kQ$ is the path algebra of the quiver $Q$ of $kS$.

**Proof.** Since $Q$ is the quiver of $kS$, there is an algebra surjection $kQ \to kS$, where $kQ$ is the path algebra of $Q$. The canonical basis for $kQ$ is the set of paths in $Q$, so using the fact that $m(Y, X) = \dim (e_Y kSe_X)$ counts the number of paths in $Q$ from $X$ to $Y$ (see the preceding two paragraphs), we have $\dim (kQ) = \sum_{Y,X} m(Y, X) = \sum_{Y,X} \dim (e_Y kSe_X) = \dim (kS)$. \qed

14. **Future Directions**

We conclude this paper by providing a few problems for future exploration.

Although this paper successfully determines the quiver of the semigroup algebra of a left regular band, it says nothing about the quiver relations. **Describe the quiver relations of the semigroup algebra of a left regular band with identity.**

The face semigroup algebra of a hyperplane arrangement is a Koszul algebra [Saliola, 2006, Proposition 9.4] and its Koszul dual is the incidence algebra of the opposite lattice of the support lattice of the semigroup. Since this algebra is the semigroup algebra of a left regular band, it is natural to ask this question for all left regular bands.
Determine which class of left regular bands give Koszul semigroup algebras and identify their Koszul duals. One source of examples of left regular bands giving Koszul algebras comes from interval greedoids \cite{Björner and Ziegler, 1992}. This will be explored in an upcoming paper.

Another nice property of the face semigroup algebra of a hyperplane arrangement is that the quiver of the semigroup algebra coincides with the support lattice of the semigroup. In fact, the support lattice completely determines the semigroup algebra. Determine the left regular bands \( S \) for which the quiver of \( kS \) coincides with the support lattice of \( L \). (From our description of the quiver of \( kS \), we have a description of these left regular bands in terms of the equivalence classes of \( \sim \).) Determine those \( S \) for which the support lattice \( L \) completely determines \( kS \).

A band is a semigroup \( B \) satisfying \( b^2 = b \) for all \( b \in B \). Since left regular bands are bands it is natural to try to generalize these results to arbitrary bands. Describe the quiver of the semigroup algebra \( kB \) of a band \( B \) with identity. Construct a complete system of primitive orthogonal idempotents for \( kB \). Determine the bands \( B \) for which \( kB \) is a Koszul algebra.

15. Appendix: Proof of Lemma 6.1

**Lemma 6.1.** Let \( S \) be a finite left regular band with identity and \( L \) its support lattice. Let \( M_X \) and \( M_Y \) denote the simple modules with irreducible characters \( \chi_X \) and \( \chi_Y \), respectively. Then

\[
\dim(\text{Ext}^1_A(M_X, M_Y)) = a_{XY}.
\]

**Proof.** As a vector space \( M_X = k \) and the action of \( kS \) on \( M_X \) is given by \( \chi_X \): if \( y \in S \) and \( \lambda \in k \), then \( y \cdot \lambda = \chi_X(y)\lambda \).

Since the following is a short exact sequence of \( kS \)-modules with \( kS_X \) projective,

\[
0 \longrightarrow \ker(\chi_X|_{kS}) \longrightarrow kS_X \xrightarrow{\chi_X} M_X \longrightarrow 0
\]

then

\[
\dim(\text{Ext}^1_A(M_X, M_Y)) = \dim(kS_X) = a_{XY}.
\]
Proposition 7.2 in Chapter V of [Cartan and Eilenberg, 1999] gives the exact sequence

\[ \text{Hom}_{kS}(kS_X, M_Y) \to \text{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y) \to \text{Ext}^1_{kS}(M_X, M_Y) \to 0. \]

Let \( K \) denote the kernel of \( \chi_X|_{kS_X} \). Then \( K \) is spanned by the differences of elements of support \( X \). If \( f \in \text{Hom}_{kS}(K, M_Y) \) and \( x, x' \) are elements of support \( X \), then \( f(x - x') = 1f(x - x') = \chi_Y(y)f(x - x') = y \cdot f(x - x') = f(y \cdot (x - x')) \), for any element \( y \) of support \( Y \). So if \( Y \not\subset X \) or if \( Y = X \), then \( f = 0 \). Therefore, \( \text{Hom}_{kS}(K, M_Y) = 0 \) if \( Y \not\subset X \). It follows that

\[ \text{Ext}^1_{kS}(M_X, M_Y) = 0 = a_{XY} \text{ for } Y \not\subset X. \]

Suppose \( Y < X \). If \( f \in \text{Hom}_{kS}(kS_X, M_Y) \), then for all \( x \in S_X \), \( f(x) = f(x^2) = f(x \cdot x) = x \cdot f(x) = \chi_Y(x)f(x) = 0f(x) = 0 \) for all \( x \in S \) with \( \text{supp}(x) = X \). Therefore, \( \text{Hom}_{kS}(kS_X, M_Y) = 0 \). Hence,

\[ \text{Ext}^1_{kS}(M_X, M_Y) \cong \text{Hom}_{kS}(K, M_Y) \text{ for } Y < X. \]

Suppose \( x \sim x' \). Then there exists a \( w \in S \) with \( y < w \), \( \text{supp}(w) < X \), \( wx = yx \) and \( wx = yx' \). Then \( x - x' \in K \), and for any \( f \in \text{Hom}_{kS}(K, M_Y) \) we have \( f(x - x') = \chi_Y(y)f(x - x') = f(yx - yx') = f(wx - wx') = f(w \cdot (x - x')) = w \cdot f(x - x') = \chi_Y(w)f(x - x') = 0f(x - x') = 0 \). Therefore, \( f(x - x') = 0 \) if \( x \sim x' \). If \( x \sim x' \), then there exist \( x_0 = x, x_1, \ldots, x_i = x' \) such that \( x_{j-1} \sim x_j \) for \( 1 \leq j \leq i \), and \( f(x - x') = f(x_0 - x_1) + f(x_1 - x_2) + \cdots + f(x_{i-1} + x_i) = 0 \). Therefore, \( f(x - x') = 0 \) if \( x \sim x' \). So \( f \) can only be nonzero on differences of elements in different equivalence classes of \( \sim \). Moreover, the equivalence classes determine \( f \): if \( u \sim x \) and \( u' \sim x' \), then \( f(u - u') = f(u - x) + f(x - x') + f(x' - u') = f(x - x') \). Therefore,

\[ \dim(\text{Ext}^1_{kS}(M_X, M_Y)) = \dim(\text{Hom}_{kS}(K, M_Y)) \leq a_{XY}. \]

Fix \( y \) with \( \text{supp}(y) = Y \) and let \( x, x' \in S_X \) with \( x \not\sim x' \). Since \( \{u - x : u \neq x, \text{supp}(u) = X\} \) is a basis for \( K \), we get a well-defined
linear function $f : K \to k$ by defining

$$f(u - x) = \begin{cases} 1, & \text{if } u \sim x', \\ 0, & \text{otherwise.} \end{cases}$$

We now show that $f : K \to M_Y$ is a $kS$-module map. That is, $f(w \cdot (u - x)) = \chi_Y(w) \cdot f(u - x)$ for all $w \in S$ and for all $u \in S_x$.

Suppose supp$(w) \not\subseteq Y$. Then $w \cdot f(u - x) = 0$ since $w$ acts trivially on $M_Y$. If supp$(w) \not\subseteq X$, then $w$ acts trivially on $K$ and so $w \cdot f(u - x) = 0 = f(w \cdot (u - x))$. So suppose supp$(w) < X$. Then $f(w \cdot (u - x)) = f(wu - wx) = f(wu - x) - f(wx - x)$. Since $v \sim x'$ iff $yw \sim x'$ for any $v \in S_X$, it follows that $f(wu - x) = f(ywu - x)$ and $f(wx - x) = f(ywx - x)$.

Suppose supp$(w) \subseteq Y$. Then $w$ acts as the identity on $M_Y$. Hence, $w \cdot f(u - x) = f(u - x)$. Since supp$(w) \subseteq Y$ and $Y \subseteq X$, we have that supp$(w) \subseteq X$. Therefore, $f(w \cdot (u - x)) = f(wu - wx) = f(wu - x) - f(wx - x)$. Since $v \sim x'$ iff $yw \sim x'$, we have $f(wu - x) = f(ywu_x - x) = f(yu - x) = f(u - x)$ since supp$(w) \subseteq Y$. Similarly, $f(wx - x) = f(x - x) = 0$. Therefore, $f(w \cdot (u - x)) = f(u - x)$.

This establishes that $f : K \to M_Y$ is a $kS$-module map. And since $f$ is nonzero only on differences of the form $u - u'$ with $u \sim x$ and $u' \sim x'$, there are exactly $a_{XY}$ such $kS$-module maps. These maps are linearly independent, therefore

$$\dim(\text{Ext}^1_{kS}(M_X, M_Y)) = \dim(\text{Hom}_{kS}(K, M_Y)) \geq a_{XY}. \square$$

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