CATEGORICALLY CLOSED COUNTABLE SEMIGROUPS

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Abstract. In this paper we establish a connection between categorical closedness and non-topologizability of semigroups. In particular, for the class $T_1S$ of $T_1$ topological semigroups we prove that a countable semigroup $X$ with finite-to-one shifts is injectively $T_1S$-closed if and only if it is $T_1S$-discrete in the sense that every $T_1$ semigroup topology on $X$ is discrete. Moreover, a countable cancellative semigroup $X$ is absolutely $T_1S$-closed if and only if every homomorphic image of $X$ is $T_1S$-discrete. Also, we introduce and investigate a new notion of a polybounded semigroup. It is proved that a countable semigroup $X$ with finite-to-one shifts is polybounded if and only if it is $T_1S$-closed if and only if $X$ is $T_1S$-closed, where $T_1S$ is the class of Tychonoff zero-dimensional topological semigroups. We show that polybounded cancellative semigroups are groups, and polybounded $T_1$ paratopological groups are topological groups.

1. Introduction

In many cases, completeness properties of various objects of General Topology and Topological Algebra can be characterized externally as closedness in ambient objects. For example, a metric space $X$ is complete if and only if $X$ is closed in any metric space containing $X$ as a subspace. A uniform space $X$ is complete if and only if $X$ is closed in any uniform space containing $X$ as a uniform subspace. A topological group $G$ is Raïkov complete if and only if it is closed in any topological group containing $G$ as a subgroup.

This motivates the following general definition.

Definition 1.1. Let $C$ be a class of topological semigroups. A topological semigroup $X$ is defined to be

- $C$-closed if for any isomorphic topological embedding $e: X \to Y$ into a topological semigroup $Y \in C$ the image $e[X]$ is closed in $Y$;
- injectively $C$-closed if for any injective continuous homomorphism $i: X \to Y$ to a topological semigroup $Y \in C$ the image $i[X]$ is closed in $Y$;
- absolutely $C$-closed if for any continuous homomorphism $h: X \to Y$ to a topological semigroup $Y \in C$ the image $h[X]$ is closed in $Y$.

Clearly, for any topological semigroup the following implications hold:

$$\text{absolutely } C\text{-closed} \Rightarrow \text{injectively } C\text{-closed} \Rightarrow \text{ } C\text{-closed}.$$
were investigated in [21, 22, 36, 37, 39, 40, 52]. In particular, closure operators in different categories were studied in [17, 18, 19, 20, 24, 25, 41, 62, 65]. Categorically closed topological semilattices were investigated in [8, 11, 43, 44, 59]. For more information about complete topological semilattices and pospaces we refer to the recent survey of the authors [9].

In this paper we shall be mainly interested in the categorical closedness of discrete topological semigroups. Since each semigroup can be endowed with the discrete topology and thus identified with a discrete topological semigroup, it is natural to define (injectively or absolutely) $C$-closed semigroups as follows.

**Definition 1.2.** A semigroup $X$ is defined to be *(injectively or absolutely) C-closed* if so is the topological semigroup $X$ endowed with the discrete topology.

The results of this paper show that the categorical closedness of semigroups is related to their topologizability properties. Many such properties can be defined as follows.

**Definition 1.3.** Let $C$ be a class of topological semigroups. A semigroup $X$ is called
- $C$-discrete (or else $C$-nontopologizable) if for every injective homomorphism $i : X \to Y$ to a topological semigroup $Y \in C$, the image $i[X]$ is a discrete subspace of $Y$;
- $C$-topologizable if $X$ is not $C$-discrete;
- projectively $C$-discrete if for every homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$, the image $h[X]$ is a discrete subspace of $Y$.

If the class $C$ is closed under taking topological subsemigroups, then a semigroup is $C$-topologizable if and only if it is algebraically isomorphic to a non-discrete topological semigroup in the class $C$. Topologizable and nontopologizable semigroups are actively studied in Topological Algebra since the seminal paper of Markov [55], see [12, 32, 46, 50, 51, 54, 58, 61, 64, 67].

For any congruence $\approx$ on a semigroup $X$, the quotient set $X/\approx$ has a unique structure of a semigroup such that the quotient map $X \to X/\approx$ is a homomorphism. The semigroup $X/\approx$ is called a *quotient semigroup* of $X$. A subset $I$ of a semigroup $X$ is called an *ideal* if for any $x \in I$ and $y \in X$ the elements $xy$ and $yx$ belong to $I$. Every ideal $I$ determines the congruence $(I \times I) \cup \{(x, y) \in X \times X : x = y\}$ on $X$. The quotient semigroup of $X$ by this congruence is denoted by $X/I$. The semigroup $X$ can be identified with the quotient semigroup $X/\emptyset$.

A quotient semigroup of a $C$-closed semigroup is not necessarily $C$-closed (see Example 1.8 from [10]). This fact motivates the following definition.

**Definition 1.4.** A semigroup $X$ is defined to be
- *projectively C-closed* if for every congruence $\approx$ on $X$ the quotient semigroup $X/\approx$ is $C$-closed;
- *ideally C-closed* if for every ideal $I \subseteq X$ the quotient semigroup $X/I$ is $C$-closed.

For any semigroup we have the implications:

$$\text{absolutely } C\text{-closed} \iff \text{projectively } C\text{-closed} \iff \text{ideally } C\text{-closed} \iff \text{injectively } C\text{-closed} \iff C\text{-closed}.$$
A topology $\tau$ on a semigroup $X$ is called a semigroup topology if the binary operation of $X$ is continuous with respect to the topology $\tau$.

**Definition 1.5.** A semigroup $X$ is called zero-closed if $X$ is closed in $(X^0, \tau)$ for any $T_1$ semigroup topology $\tau$ on $X$.

For $i \in \{0, 1, 2, 3, 3_2\}$ we denote by $T_iS$ the class of topological semigroups satisfying the separation axiom $T_i$ (see [35, §1.5] for more details). The class of all zero-dimensional topological semigroups (i.e., $T_1$ topological semigroups possessing a base consisting of clopen sets) is denoted by $T_2S$. Note that any zero-dimensional space is Tychonoff and any $T_1$ space with a unique non-isolated point is zero-dimensional. This yields the following chain of implications holding for every semigroup:

$$T_1S\text{-closed} \Rightarrow T_2S\text{-closed} \Rightarrow T_3S\text{-closed} \Rightarrow T_{3_2}S\text{-closed} \Rightarrow T_2S\text{-closed} \Rightarrow \text{zero-closed}.$$ 

By a semigroup polynomial on a semigroup $X$ we understand a function $f : X \to X$ of the form $f(x) = a_0xa_1 \cdots xa_n$ for some elements $a_0, \ldots, a_n \in X^1$ where the number $n \geq 1$ is called a degree of $f$. Note that every semigroup polynomial $f$ on a topological semigroup $X$ is continuous.

**Definition 1.6.** A subset $A$ of a semigroup $X$ is called polybounded in $X$ if

$$A \subseteq \bigcup_{p \in P} \bigcup_{b \in B} \{x \in X : p(x) = b\}$$

for a finite set $B \subseteq X$ and a finite set $P$ of semigroup polynomials on $X$.

A semigroup $X$ is called polybounded if $X$ is polybounded in $X$.

Polybounded semigroups will be studied in Section 4. In Section 5 we shall study polybounded semigroups with finite-to-one shifts.

**Definition 1.7.** A semigroup $X$ is defined to have finite-to-one shifts if for any $a, b \in X$ the sets $\{x \in X : ax = b\}$ and $\{x \in X : xa = b\}$ are finite.

The class of semigroups with finite-to-one shifts includes all groups and, more generally, all cancellative semigroups. Let us recall that a semigroup $X$ is cancellative if for any $a, b \in X^1$ the two-sided shift $s_{a,b} : X \to X, s_{a,b} : x \mapsto axb$, is injective.

The following description of the structure of polybounded cancellative semigroups will be proved in Section 6.

**Theorem 1.8.** Every nonempty polybounded cancellative semigroup is a group.

One of the main results of this paper is the following characterization of $C$-closed countable semigroups with finite-to-one shifts.

**Theorem 1.9.** For every countable semigroup $X$ with finite-to-one shifts, the following conditions are equivalent:

1. $X$ is $T_1S$-closed;
2. $X$ is $T_2S$-closed;
3. $X$ is zero-closed;
4. $X$ is polybounded.
By Proposition 4.4, a homomorphic image of a polybounded semigroup is polybounded. Since any homomorphic image of a group is a group, Theorems 1.8 and 1.9 imply the following characterization.

Corollary 1.10. For a countable cancellative semigroup $X$ the following conditions are equivalent:

1. $X$ is polybounded;
2. $X$ is zero-closed;
3. $X$ is $T_1S$-closed;
4. $X$ is projectively $T_1S$-closed.

Corollary 1.10 is specific for countable cancellative semigroups and does not generalize to groups of arbitrary cardinality: by [6], for every infinite cardinal $\kappa$ with $\kappa^+ = 2^\kappa$, there exists a simple group of cardinality $\kappa^+$ which is absolutely $T_1S$-closed but not polybounded. A group $X$ is simple if every normal subgroup of $X$ is either trivial or equals $X$.

On the other hand, the equivalence (3) $\iff$ (4) in Corollary 1.10 does hold for arbitrary groups and more generally for arbitrary semigroups whose all homomorphic images have finite-to-one shifts.

Theorem 1.11. Let $X$ be a semigroup such that every homomorphic image of $X$ has finite-to-one shifts. Let $i \in \{1, 2, 3, 3_1, 2\}$. The semigroup $X$ is $T_1S$-closed if and only if $X$ is projectively $T_1S$-closed.

The following corollary of Theorem 1.11 answers the “group” part of Question 9.2 in [10].

Corollary 1.12. Let $i \in \{1, 2, 3, 3_1, 2\}$. A group $X$ is $T_1S$-closed if and only if $X$ is projectively $T_1S$-closed.

Next, we characterize the injective (and absolute) $T_1S$-closedness of (cancellative) countable semigroups with finite-to-one shifts in terms of their (projective) $T_1S$-discreteness.

Theorem 1.13. A countable semigroup $X$ with finite-to-one shifts is injectively $T_1S$-closed if and only if $X$ is $T_1S$-discrete.

Theorem 1.14. A countable cancellative semigroup $X$ is absolutely $T_1S$-closed if and only if $X$ is projectively $T_1S$-discrete.

In fact, the “only if” parts of the characterization Theorems 1.13 and 1.14 hold for an arbitrary semigroup. This motivates the following problem.

Problem 1.15. Is every $T_1S$-discrete group $T_1S$-closed?

The answer to Problem 1.15 is affirmative for commutative groups as follows from our next characterization.

Theorem 1.16. Let $C$ be a class of topological semigroups such that $T_2S \cap TG \subseteq C \subseteq T_1S$. For a commutative cancellative semigroup $X$ the following conditions are equivalent:

1. $X$ is injectively $C$-closed;
2. $X$ is absolutely $C$-closed;
3. $X$ is $C$-discrete;
4. $X$ is finite.
Theorem 1.16 is specific for commutative cancellative semigroups. For non-commutative groups the situation is totally different. In the following theorem by a bounded semigroup we understand a semigroup $X$ for which there exists $n \in \mathbb{N}$ such that the $n$-th power $x^n$ of any element $x \in X$ is an idempotent.

**Theorem 1.17.** Every countable bounded group without elements of order 2 is a subgroup of an absolutely $T_1S$-closed countable bounded simple group.

*Proof.* By Theorem 2.3 in [50], every countable bounded group $G$ without elements of order 2 is a subgroup of a countable simple bounded group $X$, which is $T_G$-discrete for the class $T_G$ of Hausdorff topological groups. Since each semigroup topology on a bounded group is a group topology, the $T_G$-discrete group $X$ is $T_1S$-discrete. By Theorem 1.13 the group $X$ is injectively $T_1S$-closed. To show that $X$ is absolutely $T_1S$-closed, take any homomorphism $h : X \to Y$ to a $T_1$ topological semigroup $Y$. Let $e$ be the unique idempotent of $X$. Since $X$ is a group, the image $h[X]$ is a group and $H = h^{-1}(h(e))$ is a normal subgroup of $X$. Since the group $X$ is simple, $H = X$ or $H = \{e\}$. If $H = X$, then the image $h[X]$ is a singleton and hence $h[X]$ is closed in $Y$. If $H = \{e\}$, then the homomorphism $h : X \to Y$ is injective and $h[X]$ is closed in $Y$ by the injective $T_1S$-closedness of $X$. □

The following counterpart of Theorem 1.17 for uncountable groups is proved in [9].

**Theorem 1.18.** Let $\kappa$ be a cardinal such that $\kappa^+ = 2^\kappa$. Every group $H$ of cardinality $|H| \leq \kappa$ is a subgroup of an absolutely $T_1S$-closed 36-Shelah simple group $G$ of cardinality $|G| = \kappa^+$.

A semigroup $X$ is called $n$-Shelah if $X = \{a_1a_2 \cdots a_n : a_1, \ldots, a_n \in A\}$ for any subset $A \subseteq X$ of cardinality $|A| = |X|$.

Theorem 1.17 is applied in the following example showing that Corollary 1.10 and Theorem 1.11 cannot be generalized to the class of all semigroups with finite-to-one shifts.

**Example 1.19.** There exists a countable semigroup $X$ such that

1. $X$ has finite-to-one shifts;
2. $X$ is $T_1S$-discrete but not projectively $T_2S$-discrete;
3. $X$ is injectively $T_1S$-closed but not absolutely $T_2S$-closed.

*Proof.* Take any countable infinite bounded commutative group $A$ without elements of order 2. By Theorem 1.17 $A$ is a subgroup of an absolutely $T_1S$-closed countable group $G$. Let 2 be the two-element semilattice $\{0, 1\}$ endowed with the operation of minimum. We claim that the subsemigroup $X = (\{0\} \times G) \cup (\{1\} \times A)$ of $2 \times G$ has the required properties. It is easy to see that the semigroup $X$ has finite-to-one shifts. To see that $X$ is $T_1S$-discrete, take any injective homomorphism $h : X \to Y$ to a $T_1$ topological semigroup $Y$. By Theorem 1.13 the injective $T_1S$-closedness of the group $G$ implies that $G$ is $T_1S$-discrete and hence the image $h[\{0\} \times G]$ is a discrete subspace of $Y$. To see that $h[X]$ is discrete, take any $x \in X$ and let $e$ be the unique idempotent of the subgroup $\{0\} \times G$ of $X$. Since $h[\{0\} \times G]$ is a discrete subspace of $Y$, there exists an open set $U \subseteq Y$ such that $U \cap h[\{0\} \times G] = \{h(ex)\}$. By the continuity of shifts in the topological semigroup $Y$, the set $V = \{y \in h[X] : ey \in U\} = \{y \in h[X] : ey = ex\}$ is open in $h[X]$. It is easy to see that this set contains exactly two elements, one of which is $x$. Since $Y$ is a $T_1$-space, the singleton $V \setminus \{x\}$ is closed in $Y$ and hence \{x\} = $V \setminus (V \setminus \{x\})$ is open in $h[X]$, witnessing that the space $h[X]$ is discrete and the semigroup $X$ is $T_1S$-discrete. By Theorem 1.13 the semigroup $X$ is injectively $T_1S$-closed.
To see that $X$ is not absolutely $T_2S$-closed, consider the homomorphism $h : X \to A^0$, defined by

$$h(i, x) = \begin{cases} x & \text{if } i = 1; \\ 0 & \text{if } i = 0; \end{cases}$$

for any $(i, x) \in X$. By Theorem 1.16 the infinite abelian group $A$ is not injectively $T_2S$-closed, which implies that its 0-extension $A^0$ is not injectively $T_2S$-closed and the semigroup $X$ is not absolutely $T_2S$-closed. Also Theorem 1.16 ensures that the group $A$ is not $T_2S$-discrete, which implies that its 0-extension $A^0$ is not $T_2S$-discrete and hence $X$ is not projectively $T_2S$-discrete. \qed

2. Auxiliary results

In this paper we denote by $\omega$ the set $\{0, 1, 2, \ldots\}$ of nonnegative integers and by $\mathbb{N}$ the set $\{1, 2, \ldots\}$ of positive integers. For a set $X$, we denote by $|X|$ the cardinality of $X$.

The following technical lemma yields a sufficient condition of non-zero-closedness and will be used in the proof of Theorem 6.4.

**Lemma 2.1.** Let $X$ be a semigroup and $K$ be a countable family of infinite subsets of $X$ such that

1. for any $K, L \in K$ there exists $M \in K$ such that $KL \subseteq M$;
2. for any $a, b \in X^1$ and $K \in K$ there exists $L \in K$ such that $aKb \subseteq L$;
3. for any $K \in K$ and $a, b, c \in X^1$ the set $\{x \in K : axb = c\}$ is finite;
4. for any $K, L \in K$ and $c \in X$ the set $\{(x, y) \in K \times L : xy = c\}$ is finite.

Then the topology $\tau^0 = \{V \subseteq X^0 : 0 \in V \Rightarrow \forall K \in K (|K \setminus V| < \omega)\}$ turns $X^0$ into a Hausdorff topological semigroup with a unique non-isolated point $0$.

**Proof.** Let $\{K_n\}_{n \in \omega}$ be an enumeration of the countable family $K$. For every $n \in \omega$ let $U_n = \bigcup_{i < n} K_i$. Note that $U_0 = \emptyset$ and $(U_n)_{n \in \mathbb{N}}$ is an increasing sequence of infinite subsets of $X$. The conditions (1)–(4) imply the conditions:

1. for any $n \in \omega$ there exists $m \in \omega$ such that $U_n \subseteq U_m$;
2. for any $a, b \in X^1$ and $n \in \omega$ there exists $m \in \omega$ such that $aU_nb \subseteq U_m$;
3. for any $n \in \omega$ and $a, b, c \in X^1$ the set $\{x \in U_n : axb = c\}$ is finite;
4. for any $n \in \omega$ and $c \in X$ the set $\{(x, y) \in U_n \times U_n : xy = c\}$ is finite.

Observe that the topology $\tau^0$ coincides with the Hausdorff topology

$$\{V \subseteq X^0 : 0 \in V \Rightarrow \forall n \in \omega (|U_n \setminus V| < \omega)\}.$$ 

The definition of the topology $\tau^0$ guarantees that $0$ is the unique non-isolated point of the topological space $(X^0, \tau^0)$. So, it remains to prove that $(X^0, \tau^0)$ is a topological semigroup.

First we show that for every $a, b \in X^1$ the shift $s_{a,b} : X^0 \to X^0$, $x \mapsto axb$, is continuous. Since $0$ is a unique non-isolated point of $(X^0, \tau^0)$, it suffices to check the continuity of the shift $s_{a,b}$ at $0$. Given any neighborhood $V \subseteq \tau^0$ of $0$, we need to show that the set $s_{a,b}^{-1}(V) = \{x \in X^0 : axb \in V\}$ belongs to the topology $\tau^0$. This will follow as soon as we check that for every $n \in \omega$ the set $U_n \setminus s_{a,b}^{-1}(V)$ is finite. Fix any $n \in \omega$. By condition (2'), there exists a number $m \in \omega$ such that $aU_nb \subseteq U_m$. Then

$$U_n \setminus s_{a,b}^{-1}(V) = \{x \in U_n : axb \notin V\} = \{x \in U_n : axb \in U_m \setminus V\}.$$
Since the set $V$ is open, the definition of the topology $\tau^0$ guarantees that the difference $U_m \setminus V$ is finite. Applying condition (3'), we conclude that the set $U_n \setminus s_{a,b}^{-1}(V)$ is finite. Therefore, the shift $s_{a,b}$ of the semigroup $X^0$ is continuous with respect to the topology $\tau^0$ and the semigroup operation is continuous on the subset $(X^0 \times X) \cup (X \times X^0)$.

So, it remains to check the continuity of the semigroup operation at $(0,0)$. Fix any neighborhood $V \in \tau^0$ of $0 = 00$. By condition (1'), for every $n \in \omega$ there exists $m_n \in \omega$ such that $U_n U_n \subseteq U_{m_n}$. The definition of the topology $\tau^0$ ensures that for every $n \in \omega$ the set $U_{m_n} \setminus V$ is finite. Applying condition (4'), we conclude that for every $n \in \omega$ the set

$$
\Pi_n = \{ (x, y) \in U_n \times U_n : xy \notin V \} = \{ (x, y) \in U_n \times U_n : xy \in U_{m_n} \setminus V \}
$$

is finite. Then we can find a finite set $P_n \subseteq U_n$ such that $\Pi_n \subseteq P_n \times P_n$.

Consider the set

$$
W = \{ 0 \} \cup \bigcup_{n \in \mathbb{N}} ((U_n \setminus U_{n-1}) \setminus P_n)
$$

and observe that for every $n \in \mathbb{N}$ the complement

$$
U_n \setminus W \subseteq \bigcup_{1 \leq k \leq n} P_k
$$

is finite. Then $W \in \tau^0$ by the definition of the topology $\tau^0$. We claim that $WW \subseteq V$. Assuming the opposite, find $x, y \in W$ such that $xy \notin V$. Let $n, k \in \mathbb{N}$ be unique numbers such that $x \in (U_n \setminus U_{n-1}) \setminus P_n$ and $y \in (U_k \setminus U_{k-1}) \setminus P_k$. If $n \leq k$, then $(x, y) \in \Pi_k$ and $x, y \in P_k$, which contradicts the choice of $y$. If $k \leq n$, then $(x, y) \in \Pi_n$ and $x, y \in P_n$, which contradicts the choice of $x$. In both cases we obtain a contradiction, which completes the proof of the continuity of the semigroup operation on $(X^0, \tau^0)$. \qed

A nonempty family $\mathcal{F}$ of nonempty subsets of a set $X$ is called a filter on $X$ if $\mathcal{F}$ is closed under intersections and taking supersets in $X$. A subfamily $\mathcal{B} \subseteq \mathcal{F}$ is called a base of a filter $\mathcal{F}$ if each set $F \in \mathcal{F}$ contains some set $B \in \mathcal{B}$. In this case we say that the filter $\mathcal{F}$ is generated by the base $\mathcal{B}$. A filter $\mathcal{F}$ on $X$ is

- **free** if $\bigcap \mathcal{F} = \emptyset$;
- **principal** if $\{ x \} \in \mathcal{F}$ for some $x \in X$;
- an **ultrafilter** if for any $F \subseteq X$ either $F \in \mathcal{F}$ or $X \setminus F \in \mathcal{F}$.

The set $\varphi(X)$ of filters on a semigroup $X$ has a natural structure of a semigroup: for any filters $\mathcal{E}, \mathcal{F}$ on $X$ their product $\mathcal{E} \mathcal{F}$ is the filter generated by the base $\{ EF : E \in \mathcal{E}, F \in \mathcal{F} \}$ where $EF = \{ xy : x \in E, y \in F \}$. Identifying each element $x \in X$ with the principal filter $\{ F \subseteq X : x \in F \}$, we identify the semigroup $X$ with a subsemigroup of the semigroup $\varphi(X)$.

**Proposition 2.2.** A semigroup $X$ is $T_1 S$-closed if for any free ultrafilter $\mathcal{F}$ on $X$ there are elements $a_0, \ldots, a_n \in X^1$ such that the filter $a_0 \mathcal{F} a_1 \mathcal{F} \cdots \mathcal{F} a_n$ is neither free nor principal.

**Proof.** To derive a contradiction, assume that a semigroup $X$ is not $T_1 S$-closed, but for any free ultrafilter $\mathcal{F}$ on $X$ there are elements $a_0, \ldots, a_n \in X^1$ such that the filter $a_0 \mathcal{F} a_1 \mathcal{F} \cdots \mathcal{F} a_n$ is neither free nor principal. By our assumption, $X$ is a non-closed subsemigroup of some $T_1$ topological semigroup $Y$ whose topology is denoted by $\tau_Y$. Take any element $y \in X \setminus X$ and consider the filter $\mathcal{H}$ on $X$ generated by the family $\{ X \cap U : y \in U \in \tau_Y \}$. Since the space $Y$ is $T_1$, the filter $\mathcal{H}$ is free. Let $\mathcal{F}$ be any ultrafilter on $X$ which contains $\mathcal{H}$. By our assumption, there exist elements $a_0, \ldots, a_n \in X^1 \subset Y^1$ such that the filter $a_0 \mathcal{F} \cdots \mathcal{F} a_n$ is neither free nor principal. Since this filter is not free, there exists an element $z \in \bigcap_{F \in \mathcal{F}} a_0 \mathcal{F} \cdots \mathcal{F} a_n$. We claim
that $a_0y_1 \cdots y_n = z$. In the opposite case, we can find a neighborhood $U \subseteq Y$ of $y$ such that $a_0Ua_1 \cdots Ua_n \subseteq Y \setminus \{z\}$. Then for the set $F = U \cap X \in \mathcal{H} \subseteq \mathcal{F}$ we obtain $z \notin a_0Fa_1 \cdots Fa_n$, which contradicts the choice of $z$. Hence $a_0y_1 \cdots y_n = z$. Since $X$ is a discrete subspace of $Y$, there exists an open set $V \subseteq Y$ such that $V \cap X = \{z\}$. By the continuity of the semigroup operation on $Y$, the point $y$ has a neighborhood $W \subseteq Y$ such that $a_0Wa_1 \cdots Wa_n \subseteq V$. Then the set $F = X \cap W \in \mathcal{F}$ has the property $a_0Fa_1 \cdots Fa_n \subseteq X \cap V = \{z\}$, implying that the filter $a_0Fa_1 \cdots \mathcal{F}a_n$ is principal. But this contradicts the choice of the points $a_0, \ldots, a_n$. □

**Proposition 2.3.** A semigroup $X$ is injectively $T_1\mathcal{S}$-closed if for any free ultrafilter $\mathcal{F}$ on $X$ there are elements $a_0, \ldots, a_n \in X^1$ and distinct elements $u, v \in X$ such that

$$\{u, v\} \subseteq \bigcap_{F \in \mathcal{F}} a_0Fa_1 \cdots Fa_n.$$

**Proof.** Assuming that $X$ is not injectively $T_1\mathcal{S}$-closed, we can find a continuous injective homomorphism $h : X \to Y$ into a topological semigroup $(Y, \tau_Y) \in T_1\mathcal{S}$ such that $h[X]$ is not closed in $Y$. Fix any element $y_0 \in h[X] \setminus h[X]$. Since the space $Y$ is $T_1$, the filter $\mathcal{H}$ on $X$ generated by the base $\{h^{-1}[U] : y_0 \in U \subseteq \tau_Y\}$ is free. Let $\mathcal{F}$ be any ultrafilter on $X$ which contains $\mathcal{H}$. By the assumption, there exist elements $a_0, \ldots, a_n \in X^1$ and distinct elements $u, v \in X$ such that $\{u, v\} \subseteq a_0Fa_1 \cdots Fa_n$ for any $F \in \mathcal{F}$. Since $(Y, \tau_Y)$ is a $T_1$-space, the point $h(a_0)y_0h(a_1) \cdots y_0h(a_n) \in Y$ has a neighborhood $W \subseteq \tau_Y$ such that $|W \cap \{h(u), h(v)\}| \leq 1$ and hence $|\{u, v\} \cap h^{-1}[W]| \leq 1$. By the continuity of semigroup operation on $Y$, there exists a neighborhood $U \subseteq \tau_Y$ of $y_0$ such that $h(a_0)Uh(a_1) \cdots Uh(a_n) \subseteq W$. Note that $h^{-1}[U] \in \mathcal{H} \subseteq \mathcal{F}$ and hence

$$\{u, v\} \subseteq a_0h^{-1}[U]a_1 \cdots h^{-1}[U]a_n = h^{-1}[h(a_0)Uh(a_1) \cdots Uh(a_n)] \subseteq h^{-1}[W],$$

which contradicts the choice of $W$. □

The following example constructed by Taimanov [60] shows that there exists an injectively $T_1\mathcal{S}$-closed semigroup which is not ideally $T_2\mathcal{S}$-closed, and hence for any class $\mathcal{C}$ with $T_2\mathcal{S} \subseteq \mathcal{C} \subseteq T_1\mathcal{S}$, the injective $\mathcal{C}$-closedness does not imply the ideal $\mathcal{C}$-closedness.

**Example 2.4.** Given an infinite cardinal $\kappa$, consider the semigroup $X = (\kappa, \ast)$ endowed with the binary operation

$$x \ast y = \begin{cases} 1, & \text{if } x = y > 1; \\ 0, & \text{otherwise}. \end{cases}$$

Observe that for any free filter $\mathcal{F}$ on $X$ we have $\{0, 1\} \subseteq \bigcap_{F \in \mathcal{F}} FF$. By Proposition 2.3 the Taimanov semigroup is injectively $T_1\mathcal{S}$-closed. We claim that for the ideal $J = \{0, 1\} \subseteq X$, the quotient semigroup $X/J$ is not $T_2\mathcal{S}$-closed. Observe that $X/J$ is a semigroup with trivial multiplication, i.e., $ab = J \in X/J$ for any $a, b \in X/J$. Take any Hausdorff zero-dimensional space $Y$ containing $X/J$ as a non-closed discrete subspace. Endow $Y$ with the continuous semigroup operation defined by $xy = J \in X/J \subseteq Y$ for all $x, y \in Y$. Since $X/J$ is a non-closed discrete subspace of zero-dimensional topological semigroup $Y$, the semigroup $X$ is not ideally $T_2\mathcal{S}$-closed.

The following trivial characterization of $T_0\mathcal{S}$-closed semigroups explains why we restrict our attention to topological semigroups satisfying the separation axioms $T_i$ for $i \geq 1$.

**Proposition 2.5.** For a topological semigroup $X \in T_0\mathcal{S}$ the following conditions are equivalent:

1. $X$ is $T_0\mathcal{S}$-closed.
2. For any free ultrafilter $\mathcal{F}$ on $X$ there are elements $a_0, \ldots, a_n \in X^1$ and distinct elements $u, v \in X$ such that $\{u, v\} \subseteq a_0Fa_1 \cdots Fa_n$.
3. For any free ultrafilter $\mathcal{F}$ on $X$ there are elements $a_0, \ldots, a_n \in X^1$ and distinct elements $u, v \in X$ such that $\{u, v\} \subseteq a_0Fa_1 \cdots Fa_n$. 

□
(1) $X$ is $T_0S$-closed;
(2) $X$ is absolutely $T_0S$-closed;
(3) $X = \emptyset$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. To see that (1) $\Rightarrow$ (3), assume that the semigroup $X$ is not empty. On the 0-extension $X^0$ of $X$ consider the topology $\tau^0 = \{X^0\} \cup \tau$ where $\tau$ is the topology of $X$. It is easy to see that $(X^0, \tau^0)$ is a $T_0$ topological semigroup containing $X$ as a non-closed subsemigroup. Consequently, $X$ is not $T_0S$-closed. □

3. $C$-closedness and $C$-discreteness

In this section we outline a connection between (injectively) $C$-closed and $C$-discrete semigroups.

Proposition 3.1. Every $T_1S$-closed topological semigroup $X \in T_1S$ is discrete.

Proof. To derive a contradiction, assume that the topological semigroup $X$ contains a non-isolated point $a$. Take any point $b \notin X$ and consider the set $Y = X \cup \{b\}$ endowed with the semigroup operation defined as follows:

- $X$ is a subsemigroup of $Y$;
- $bb = aa$;
- for every $x \in X$, $bx = ax$ and $xb = xa$.

By $\tau_X$ we denote the topology of $X$. Let $\tau_Y$ be the topology on $Y$ generated by the base $\tau_X \cup \{(U \setminus \{a\}) \cup \{b\} : a \in U \in \tau_X\}$. It is straightforward to check that $(Y, \tau_Y)$ is a $T_1$ topological semigroup containing $X$ as a non-closed subsemigroup, which contradicts the $T_1S$-closedness of $X$. □

Proposition 3.2. For a semigroup $X$ the following conditions are equivalent:

(1) $X$ is injectively $T_1S$-closed;
(2) $X$ is $T_1S$-closed and $T_1S$-discrete.

Proof. (1) $\Rightarrow$ (2): Assume that the semigroup $X$ is injectively $T_1S$-closed. Then $X$ is $T_1S$-closed. To see that $X$ is $T_1S$-discrete, consider any injective homomorphism $i : X \to Y$ to a $T_1$ topological semigroup $Y$. The injective $T_1S$-closedness of $X$ implies the (injective) $T_1S$-closedness of the topological semigroup $i[X]$. By Proposition 3.1, the topology of $i[X]$ is discrete, which means that the semigroup $X$ is $T_1S$-discrete.

(2) $\Rightarrow$ (1): Assume that a semigroup $X$ is $T_1S$-closed and $T_1S$-discrete. Consider any injective homomorphism $i : X \to Y$ into a $T_1$ topological semigroup $Y$. Since $X$ is $T_1S$-discrete, the topological space $i[X]$ is discrete and the map $i : X \to Y$ is a topological embedding of $X$ endowed with the discrete topology into $Y$. Since the discrete topological semigroup $X$ is $T_1S$-closed, the image $i[X]$ is closed in $Y$. □

Proposition 3.3. For a semigroup $X$ the following conditions are equivalent:

(1) $X$ is absolutely $T_1S$-closed;
(2) $X$ is projectively $T_1S$-closed and projectively $T_1S$-discrete.

Proof. Let $X$ be an absolutely $T_1S$-closed semigroup. Then $X$ is projectively $T_1S$-closed and every homomorphic image of $X$ is injectively $T_1S$-closed. Proposition 3.2 implies that $X$ is projectively $T_1S$-discrete.
Assume that a semigroup $X$ is projectively $T_1$-$S$-closed and projectively $T_1$-$S$-discrete. Fix any homomorphism $h : X \to Y$ to a $T_1$ topological semigroup $Y$. Since $X$ is projectively $T_1$-$S$-discrete, the subspace $h[X]$ of $Y$ is discrete. Since $X$ is projectively $T_1$-$S$-closed, the image $h[X]$ is closed in $Y$, witnessing that $X$ is absolutely $T_1$-$S$-closed. □

The above propositions motivate the problem of recognizing $C$-discrete (or else $C$-nontopologizable) semigroups. This problem has been considered by many authors, see [12, 32, 50, 51, 54, 55, 61, 63, 67]. The $T_1$-$S$-discreteness of countable semigroups can be characterized with the help of suitable Zariski topologies, see [27, 28, 29, 30, 31, 33, 36, 38].

For a semigroup $X$ its

- Zariski topology $Z_X$ is the topology on $X$ generated by the subbase consisting of the sets $\{x \in X : f(x) \neq b\}$ and $\{x \in X : f(x) \neq g(x)\}$ where $b \in X$ and $f, g$ are semigroup polynomials on $X$;
- Zariski $T_1$-topology $Z'_X$ is the topology on $X$ generated by the subbase consisting of the sets $\{x \in X : f(x) \neq b\}$ where $b \in X$ and $f$ is a semigroup polynomial on $X$.

The following theorem was proved independently by Podewski [56] and Taimanov [61].

**Theorem 3.4** (Podewski–Taimanov). A countable semigroup $X$ is $T_1$-$S$-discrete if and only if its Zariski $T_1$-topology $Z'_X$ is discrete.

The next theorem was announced by Taimanov [61] and proved by Kotov [51].

**Theorem 3.5** (Taimanov–Kotov). A countable semigroup $X$ is $T_2$-$S$-discrete if and only if $X$ is $T_2$-$S$-discrete if and only if the Zariski topology $Z_X$ is discrete.

We finish this section with the following example of a projectively $T_1$-$S$-closed and absolutely $T_2$-$S$-closed semilattice which is not injectively $T_1$-$S$-closed. We recall that a semilattice is a commutative semigroup of idempotents.

**Example 3.6.** Given an infinite cardinal $\kappa$, consider the semilattice $X_\kappa = (\kappa, \ast)$ endowed with the binary operation

$$x \ast y = \begin{cases} x, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that for every free filter $\mathcal{F}$ on $X_\kappa$ the filter $\mathcal{F}\mathcal{F}$ is neither principal nor free. Applying Proposition 2.2, we conclude that the semilattice $X_\kappa$ is $T_1$-$S$-closed. Since any homomorphic image of $X_\kappa$ is isomorphic to the semilattice $X_\lambda$ for some nonzero cardinal $\lambda \leq \kappa$, the semilattice $X_\kappa$ is projectively $T_1$-$S$-closed. By [3], the semilattice $X_\kappa$ is absolutely $T_2$-$S$-closed. Since $X_\kappa$ admits a non-discrete $T_1$ semigroup topology $\tau = \{U \subseteq \kappa : 0 \in U \implies (|\kappa \setminus U| < \omega)\}$, the semigroup $X_\kappa$ is $T_1$-$S$-topologizable. Then Proposition 3.2 implies that $X$ is not injectively $T_1$-$S$-closed.

4. Polybounded semigroups

In this section we establish some structural properties of polybounded semigroups. We start with the following three characterizations of polyboundedness.

**Proposition 4.1.** For a nonempty semigroup $X$ the following conditions are equivalent:

1. $X$ is polybounded;
2. for every $a, b \in X$ the semigroup $aXb$ is polybounded;
3. for some $a, b \in X$ the semigroup $aXb$ is polybounded.
Proof. To prove that (1) ⇒ (2), assume that the semigroup \( X \) is polybounded and hence \( X = \bigcup_{i=1}^{n} p_i^{-1}(c_i) \) for some elements \( c_1, \ldots, c_n \in X \) and semigroup polynomials \( p_1, \ldots, p_n \) on \( X \). Given any elements \( a, b \in X \), for every \( i \in \{1, \ldots, n\} \), consider the function \( f_i : aXb \rightarrow aXb \) defined by \( f_i(x) = ap_i(bbxaa)b \), and observe that \( f_i \) is a semigroup polynomial on \( aXb \). We claim that \( X = \bigcup_{i=1}^{n} f_i^{-1}(ac_i b) \). Indeed, for every \( x \in aXb \) there exists \( i \in \{1, \ldots, n\} \) such that \( p_i(bbxaa) = c_i \) and hence \( f_i(x) = ap_i(bbxaa)b = ac_ib \) and finally \( x \in f_i^{-1}(ac_i b) \), witnessing that the semigroup \( aXb \) is polybounded.

The implication \((2) \Rightarrow (3)\) is trivial.

To prove that \((3) \Rightarrow (1)\), assume that for some \( a, b \in X \) the semigroup \( aXb \) is polybounded. Then \( aXb = \bigcup_{i=1}^{n} p_i^{-1}(c_i) \) for some elements \( c_1, \ldots, c_n \in aXb \) and some semigroup polynomials \( p_1, \ldots, p_n \) on \( aXb \). For every \( i \in \{1, \ldots, n\} \), consider the semigroup polynomial \( f_i : X \rightarrow X \), \( f_i : x \mapsto p_i(axb) \), and observe that \( X = \bigcup_{i=1}^{n} f_i^{-1}(c_i) \), witnessing that the semigroup \( X \) is polybounded. \( \Box \)

Proposition 4.2. A nonempty semigroup \( X \) is polybounded if and only if \( X = \bigcup_{i=1}^{n} p_i^{-1}(b) \) for some \( b \in X \) and semigroup polynomials \( p_1, \ldots, p_n \) on \( X \).

Proof. The “if” part is trivial. To prove the “only if” part, assume that \( X \) is polybounded and find elements \( b_1, \ldots, b_n \in X \) and semigroup polynomials \( p_1, \ldots, p_n \) on \( X \) such that \( X = \bigcup_{i=1}^{n} p_i^{-1}(b_i) \). We lose no generality assuming that \( n > 1 \).

Let \( b = b_1 \cdots b_n \). For every \( i \in \{1, \ldots, n\} \), consider the semigroup polynomial \( f_i : X \rightarrow X \) defined by \( f_i(x) = \begin{cases} p_1(x)b_2 \cdots b_n & \text{if } i = 1 < n; \\ b_1 \cdots b_{n-1}p_n(x) & \text{if } 1 < i = n; \\ b_1 \cdots b_{i-1}p_i(x)b_{i+1} \cdots b_n & \text{if } 1 < i < n. \end{cases} \)

It is easy to see that \( p_i^{-1}(b_i) \subseteq f_i^{-1}(b) \) and hence \( X = \bigcup_{i=1}^{n} p_i^{-1}(b) \subseteq \bigcup_{i=1}^{n} f_i^{-1}(b) \subseteq X \). \( \Box \)

Proposition 4.3. A nonempty semigroup \( X \) is polybounded if and only if the semigroup \( K = \bigcap_{x \in X} X^1xX^1 \) is nonempty and polybounded.

Proof. To prove the “only if” part, assume that the semigroup \( X \) is polybounded. By Proposition 4.2, \( X = \bigcup_{i=1}^{n} p_i^{-1}(b) \) for some \( b \in X \) and some semigroup polynomials \( p_1, \ldots, p_n \). We claim that \( b \) belongs to the ideal \( K = \bigcap_{x \in X} X^1xX^1 \). Indeed, given any \( x \in X \), we can find \( i \in \{1, \ldots, n\} \) such that \( b = p_i(x) \in X^1xX^1 \). Therefore, the semigroup \( K \) contains the element \( b \) and hence \( K \) is not empty. In fact, \( K \) is the smallest nonempty ideal in \( X \). To see that the semigroup \( K \) is polybounded, for every \( i \in \{1, \ldots, n\} \) consider the function \( f_i : K \rightarrow K \), \( f_i(x) = p_i(bxb) \), and observe that \( f_i \) is a semigroup polynomial on \( K \). By the choice of the polynomials \( p_1, \ldots, p_n \), for every \( x \in K \) there exists \( i \in \{1, \ldots, n\} \) such that \( bxb \in p_i^{-1}(b) \). It follows that \( x \in f_i^{-1}(b) \), witnessing that \( K = \bigcup_{i=1}^{n} f_i^{-1}(b) \). Hence the semigroup \( K \) is polybounded.

To prove the “if” part, assume that the semigroup \( K = \bigcap_{x \in X} X^1xX^1 \) is nonempty and polybounded. By Proposition 4.2, \( K = \bigcup_{i=1}^{n} p_i^{-1}(b) \) for some \( b \in K \) and some semigroup polynomials \( p_1, \ldots, p_n \) on \( K \). Since \( K \) is an ideal in \( X \), for every \( x \in X \) the element \( bxb \) belongs to \( K \). Then for every \( i \in \{1, \ldots, n\} \), the function \( f_i : X \rightarrow X \), \( f_i : x \mapsto p_i(bxb) \), is a well-defined semigroup polynomial on \( X \). The equality \( K = \bigcup_{i=1}^{n} p_i^{-1}(b) \) implies the equality \( X = \bigcup_{i=1}^{n} f_i^{-1}(b) \), witnessing that the semigroup \( X \) is polybounded. \( \Box \)
Next, we show that the class of polybounded semigroups is closed under taking finite products and quotients.

**Proposition 4.4.** Quotients and finite products of polybounded semigroups are polybounded.

**Proof.** Assume that $X$ is a polybounded semigroup, and hence $X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$ for some finite set $B \subseteq X$ and some finite set $P$ of semigroup polynomials on $X$. For every polynomial $p \in P$ find a number $n_p \in \mathbb{N}$ and elements $a_{p,0}, a_{p,1}, \ldots, a_{p,n_p} \in X^1$ such that $p(x) = a_{p,0}xa_{p,1}x \cdots xa_{p,n_p}$ for all $x \in X$.

Let $Y = X/\sim$ be a quotient semigroup of $X$, and $q : X \to Y$ be the quotient homomorphism. For every $p \in P$ and $i \in \{0, \ldots, n_p\}$, let $\tilde{a}_{p,i} = q(a_{p,i})$. Let $\tilde{p}$ be the semigroup polynomial on $Y$ defined by $\tilde{p}(y) = \tilde{a}_{p,0}ya_{p,1}y \cdots ya_{p,n_p}$. Let $\tilde{P} = \{\tilde{p} : p \in P\}$ and $\tilde{B} = \{\tilde{b} : b \in B\}$, where $\tilde{b} = q(b)$ for $b \in B$. Since $q$ is a homomorphism, for every $p \in P$, $b \in B$ and $x \in p^{-1}(b)$ we have $\tilde{p}(q(x)) = q(p(x)) = q(b) = \tilde{b}$. Then $Y = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$ implies $Y = \bigcup_{\tilde{p} \in \tilde{P}} \bigcup_{\tilde{b} \in \tilde{B}} \tilde{p}^{-1}(\tilde{b})$, which means that the quotient semigroup $Y = X/\sim$ is polybounded.

To show that finite products of polybounded semigroups are polybounded, it suffices to prove that for any nonempty polybounded semigroups $X, Y$ their product $X \times Y$ is polybounded. By Proposition 4.4, $X = \bigcup_{f \in P_X} f^{-1}(b_X)$ for some $b_X \in X$ and some finite set $P_X$ of semigroup polynomials on $X$. By the same reason, $Y = \bigcup_{g \in P_Y} g^{-1}(b_Y)$ for some $b_Y \in Y$ and some finite set $P_Y$ of semigroup polynomials on $Y$.

For every polynomial $f \in P_X$, find $\deg(f) \in \mathbb{N}$ and elements $a_{f,0}, \ldots, a_{f,\deg(f)} \in X^1$ such that $f(x) = a_{f,0}xa_{f,1}x \cdots xa_{f,\deg(f)}$ for all $x \in X$. Also for every polynomial $g \in P_Y$ find $\deg(g) \in \mathbb{N}$ and elements $a_{g,0}, a_{g,1}, \ldots, a_{g,\deg(g)} \in Y^1$ such that $g(y) = a_{g,0}ya_{g,1}y \cdots ya_{g,\deg(g)}$ for all $y \in Y$.

For any semigroup polynomials $f \in P_X$ and $g \in P_Y$, consider the function

$$p_{f,g} : X \times Y \to X \times Y, \quad p_{f,g} : (x, y) \mapsto (f(x)^{\deg(g)}, g(y)^{\deg(f)}),$$

and observe that $p_{f,g}$ is a semigroup polynomial (of degree $\deg(f) \cdot \deg(g)$) on $X \times Y$. Since

$$X \times Y = \bigcup_{f \in P_X} \bigcup_{g \in P_Y} p_{f,g}^{-1}(b_X^{\deg(g)} \times b_Y^{\deg(f)}),$$

the semigroup $X \times Y$ is polybounded. \hfill \Box

The following example provides a simple method for constructing polybounded groups.

**Example 4.5.** For any commutative group $X$ with neutral element $0$, consider the group $X \rtimes \{-1, 1\}$ endowed with the group operation

$$\langle x, i \rangle \ast \langle y, j \rangle = \langle xy^j, ij \rangle.$$

The group $X \rtimes \{-1, 1\}$ is polybounded since

$$X \rtimes \{-1, 1\} = \{x \in X \rtimes \{-1, 1\} : ax^2ax^2 = e\},$$

where $e = (0, 1)$ and $a = (0, -1)$.

**Lemma 4.6.** A semigroup $X$ is polybounded, if some point of $X$ is isolated in the Zariski topology $\mathcal{Z}_X$. 


Proof. Let $a$ be an isolated point of the topological space $(X, \mathcal{O}_X)$. By the definition of the topology $\mathcal{O}_X$, there exist semigroup polynomials $f_1, \ldots, f_n$ on $X$ and elements $b_1, \ldots, b_n \in X$ such that

$$\{a\} = X \setminus \bigcup_{i=1}^n \{x \in X : f_i(x) = b_i\}.$$

Consider the semigroup polynomial $f_0 : X \to X$, $f_0 : x \mapsto x$, and let $b_0 = a$. Then $X = \bigcup_{i=0}^n \{x \in X : f_i(x) = b_i\}$, witnessing that $X$ is polybounded.

□

Remark 4.7. It is straightforward to check that the group $G = \mathbb{Z} \times \{-1, 1\}$ (see Example 4.5) is polybounded, but the space $(G, \mathcal{O}_G)$ is homeomorphic to the topological sum of two countable spaces endowed with the cofinite topology, implying that the space $(G, \mathcal{O}_G)$ has no isolated points. Hence the implication of Lemma 4.6 cannot be reversed even for countable groups.

Remark 4.8. By [6], for every infinite cardinal $\kappa$ with $\kappa^+ = 2^\kappa$ there exists a non-polybounded absolutely $T_1$-S-closed simple group $G$ of cardinality $|G| = \kappa^+$. By Proposition 3.3 the absolutely $T_1$-S-closed group $G$ is projectively $T_1$-discrete. Since $G$ is not polybounded, Lemma 4.6 ensures that the Zariski topology $\mathcal{O}_X$ has no isolated points. This example shows that Podewski–Taimanov Theorem 3.4 does not extend to uncountable (semi)groups.

5. POLYBOUNDED SEMIGROUPS WITH FINITE-TO-ONE SHIFTS

In this section we establish some specific properties of polybounded semigroups with finite-to-one shifts. The principal tool here is the notion of a pruned polynomial.

A semigroup polynomial $p : X \to X$, $p : x \mapsto a_0xa_1 \cdots xa_n$, on a semigroup $X$ is said to be pruned if $a_0 = 1 = a_n$. Obviously, for every semigroup polynomial $f$ on $X$ there exists a pruned semigroup polynomial $p$ on $X$ and elements $a, b \in X_1$ such that $f(x) = ap(x)b$ for every $x \in X$.

Lemma 5.1. For every polybounded semigroup $X$ with finite-to-one shifts there exist a finite set $B \subseteq X$ and a finite set $P$ of pruned semigroup polynomials on $X$ such that $X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$.

Proof. By the polyboundedness of $X$, there exist a finite set $A \subseteq X$ and a finite set $F$ of semigroup polynomials on $X$ such that $X = \bigcup_{f \in F} \bigcup_{a \in A} f^{-1}(a)$. For every semigroup polynomial $f \in F$, find a pruned semigroup polynomial $p_f : X \to X$ and elements $c_f, d_f \in X_1$ such that $f(x) = c_f p_f(x) d_f$ for all $x \in X$. Since the semigroup $X$ has finite-to-one shifts, the set $B = \bigcup_{f \in F} \bigcup_{a \in A} \{x \in X : c_f x d_f = a\}$ is finite. Let $P = \{p_f : f \in F\}$. It is easy to see that $X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$. □

An element $x$ of a semigroup $X$ is called regular if $x = xx^{-1}x$ for some element $x^{-1} \in X$. A semigroup $X$ is regular if every element of $X$ is regular.

Lemma 5.2. For every polybounded semigroup $X$ with finite-to-one shifts, there exist a finite set $P$ of pruned semigroup polynomials on $X$ and a finite set $B$ of regular elements of $X$ such that $X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$.

Proof. By Lemma 5.1, $X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$ for some finite set $B \subseteq X$ and some finite set $P$ of pruned semigroup polynomials on $X$. We can assume that the cardinality of the set $B$ is the smallest possible.
Consider the semigroup polynomial \( s : X \to X, s : x \mapsto xx \). Since \( BB \subseteq X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b) \), for every \( b \in B \) there exists \( \varphi_b \in P \) such that \( \varphi_b(b^2) \in B \). We claim that \( \varphi_b(b^2) = b \) for every \( b \in B \). Assuming that \( \varphi_b(b^2) \neq b \) for some \( b \in B \), we can consider the set

\[
P' = P \cup \{ \varphi_b \circ s \circ \varphi_b \}
\]

of pruned semigroup polynomials on \( X \). We claim that \( X = \bigcup_{p \in P'} \bigcup_{c \in B \setminus \{b\}} p^{-1}(c) \). Take any element \( x \in X \). If \( p(x) = c \) for some \( p \in P \) and \( c \in B \setminus \{b\} \), then \( x \in \bigcup_{p \in P'} \bigcup_{c \in B \setminus \{b\}} p^{-1}(c) \). Otherwise, \( p(x) = b \) for each \( p \in P \). In particular, \( \varphi_b(x) = b \) and, consequently, \( \varphi_b \circ s \circ \varphi_b(x) = \varphi_b(b^2) \in B \setminus \{b\} \). Therefore, \( X = \bigcup_{p \in P'} \bigcup_{c \in B \setminus \{b\}} p^{-1}(c) \) which contradicts the minimality of \( B \). This contradiction shows that \( \varphi_b(b^2) = b \) for all \( b \in B \). Since the polynomial \( \varphi_b \) is pruned, there exist elements \( a_1, \ldots, a_{n-1} \in X \) such that

\[
b = \varphi_b(b^2) = b^2 a_1 b^2 \cdots a_{n-1} b^2 = bb^{-1}b
\]

for the element \( b^{-1} = ba_1 b^2 \cdots a_{n-1} b \in X \), which means that the elements of the set \( B \) are regular. \( \square \)

For a semigroup \( X \) we denote by \( E(X) \) the set \( \{ x \in X : xx = x \} \) of idempotents of \( X \).

**Proposition 5.3.** For any nonempty polybounded semigroup \( X \) with finite-to-one shifts, the set \( E(X) \) is finite and nonempty.

*Proof.* By Lemma 5.2 \( X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b) \) for a finite nonempty set \( P \) of pruned semigroup polynomials on \( X \) and a finite nonempty set \( B \) of regular elements of \( X \). By the regularity, for every \( b \in B \) there exists \( b^{-1} \in X \) such that \( b = bb^{-1}b \) and hence \( bb^{-1} \) is an idempotent. Therefore, the set \( E(X) \supseteq \{ bb^{-1} : b \in B \} \) is not empty. Since the semigroup \( X \) has finite-to-one shifts, the set \( F = \bigcup_{b \in B} \{ x \in X : xbb^{-1} = bb^{-1} \} \) is finite. We claim that \( E(X) \subseteq F \). Indeed, for every \( e \in E(X) \) we can find a pruned polynomial \( p \in P \) and an element \( b \in B \) such that \( b = p(e) = ece \) for some \( c \in X^1 \). Then \( ebb^{-1} = e(cec)bb^{-1} = ecebb^{-1} = bb^{-1} \) and hence \( e \in F \). \( \square \)

**Question 5.4.** Does every nonempty polybounded semigroup contain an idempotent?

Now let us recall some standard definitions related to minimal (left or right) ideals.

A nonempty subset \( I \) of a semigroup \( X \) is called a left ideal (resp. right ideal) if \( IX \subseteq I \) (resp. \( IX \subseteq I \)). A left (resp. right) ideal \( I \) is called minimal if \( I = J \) for any left (resp. right) ideal \( J \subseteq X \) with \( J \subseteq I \).

An ideal \( I \) in a semigroup \( X \) is called the minimal ideal if \( I \neq \emptyset \) and \( I \subseteq J \) for every nonempty ideal \( J \subseteq X \). A semigroup \( X \) has the minimal ideal if and only if the semigroup \( K = \bigcap_{x \in X} X^1 xx^1 \) is not empty, in which case \( K \) is the minimal ideal of \( X \).

A semigroup \( X \) is called

- *simple* if each nonempty ideal in \( X \) coincides with \( X \);
- *completely simple* if \( X \) is simple and contains a minimal left ideal and a minimal right ideal;
- *completely regular* if \( X \) is a union of subgroups.

By Theorem 3.3.2 of [17], a nonempty simple semigroup is completely simple if and only if it is completely regular.

**Proposition 5.3** implies that for every polybounded semigroup \( X \) its minimal ideal \( K = \bigcap_{x \in X} X^1 xx^1 \) is a simple polybounded semigroup. Our next aim is to show that for a polybounded semigroup with finite-to-one shifts, its minimal ideal \( K \) is completely simple.
Lemma 5.5. Every nonempty polybounded semigroup $X$ with finite-to-one shifts contains a minimal left ideal and a minimal right ideal.

Proof. By Lemma 5.2, $X = \bigcup_{p \in P} \bigcup_{b \in B} p^{-1}(b)$ for some finite set $P$ of pruned polynomials and some finite set $B$ of regular elements.

To derive a contradiction, assume that the semigroup $X$ has no minimal right ideals. Then we can inductively construct sequences $(b_n)_{n \in \omega} \in B^\omega$ and $(x_n)_{n \in \omega} \in X^\omega$ such that for every $n \in \omega$ the following conditions are satisfied:

- $x_n \in b_n X^1$ and $x_n X^1 \neq b_n X^1$;
- $b_{n+1} \in x_n X^1$.

To start the inductive construction of the sequences $(b_n)_{n \in \omega}$ and $(x_n)_{n \in \omega}$, take any element $b_0 \in X$. Since $X$ has no minimal right ideals, the right ideal $b_0 X^1$ is not minimal and hence there exists an element $x_0 \in b_0 X^1$ such that $x_0 X^1 \neq b_0 X^1$. Assume that for some $n \in \omega$ we have constructed finite sequences $(b_k)_{k \leq n}$ and $(x_k)_{k \leq n}$. By the choice of the sets $P$ and $B$, there exist an element $b_{n+1} \in B$ and a pruned polynomial $p \in P$ such that $b_{n+1} = p(x_n) \in x_n X^1$. This completes the inductive step. Since the set $B$ is finite, there are two numbers $n < m$ such that $b_n = b_m$.

Then 
\[ b_m X^1 \subseteq x_{m-1} X^1 \subseteq b_{m-1} X^1 \subseteq x_{m-2} X^1 \subseteq \ldots \subseteq b_n X^1 \]

and hence $b_m X^1 \neq b_n X^1 = b_m X^1$, which is a contradiction showing that $X$ has a minimal right ideal.

By analogy we can prove that $X$ has a minimal left ideal. $\square$

Proposition 5.6. If a nonempty polybounded semigroup $X$ has finite-to-one shifts, then its minimal ideal $K = \bigcap_{x \in X} X^1 x X^1$ is a completely simple semigroup, which is the union $K = \bigcup_{e \in E(K)} e Ke$ of finitely many polybounded groups $e Ke$, $e \in E(X)$.

Proof. By Proposition 4.3, the semigroup $K = \bigcap_{x \in X} X^1 x X^1$ is nonempty and polybounded. It is clear that $K$ is simple and has finite-to-one shifts. By Lemma 5.5, the polybounded semigroup $K$ has a minimal left ideal and a minimal right ideal and hence the simple semigroup $K$ is completely simple. By Theorem 3.3.2 in [47], the completely simple semigroup $S$ is completely regular and hence $K = \bigcup_{e \in E(K)} e Ke$ is the union of subgroups. By Proposition 5.3, the set $E(K)$ is finite, and by Proposition 4.1 for every idempotent $e \in E(K)$ the group $e Ke$ is polybounded. $\square$

For a semigroup $X$ let 
\[ Z(X) = \{ z \in X : \forall x \in X \ (xz = zx) \} \]

be the center of $X$. We recall that a semigroup $X$ is called bounded if there exists $n \in \mathbb{N}$ such that the $n$th power $x^n$ of any element $x \in X$ is an idempotent. It is clear that a bounded semigroup with finitely many idempotents is polybounded. The converse implication holds for commutative semigroups with finite-to-one shifts.

Proposition 5.7. Let $X$ be a semigroup. If $X$ has finite-to-one shifts and $Z(X)$ is polybounded in $X$, then $Z(X)$ is bounded.

Proof. Assuming that $Z(X)$ is not bounded, we conclude that $Z(X)$ and $X$ are nonempty. By the polyboundedness of $Z(X)$ in $X$, for some $n \in \mathbb{N}$ there exist semigroup polynomials $f_1, \ldots, f_n$ on $X$ and elements $b_1, \ldots, b_n \in X$ such that $Z(X) \subseteq \bigcup_{i=1}^n f_i^{-1}(b_i)$. For every $i \leq n$
there exist a number $p_i \in \mathbb{N}$ and an element $a_i \in X^1$ such that $f_i(x) = a_i x^{p_i}$ for all $x \in Z(X)$. Let $p = \max_{i \leq n} p_i$. Since $X$ has finite-to-one shifts, the set $F = \bigcup_{i \leq n} \{ x \in Z(X) : a_i x = b_i \}$ is finite. Since $Z(X)$ is not bounded, there exists an element $z \in Z(X)$ such that for any distinct numbers $i, j \in \{ 1, \ldots, p(1 + n|F|) \}$ the powers $z^i, z^j$ are distinct. Since

$$Z(X) = \bigcup_{i=1}^{n} \{ x \in Z(X) : f_i(x) = b_i \} = \bigcup_{i=1}^{n} \{ x \in Z(X) : a_i x^{p_i} = b_i \},$$

for every $j \in \mathbb{N}$ there exists $i_j \in \{ 1, \ldots, n \}$ such that $a_{i_j} (z^j)^{p_{i_j}} = b_{i_j}$. By the Pigeonhole principle, for some $i \in \{ 1, \ldots, n \}$ the set $J_i = \{ j \in \{ 1, \ldots, 1 + n \cdot |F| \} : i = i_j \}$ has cardinality $|J_i| > |F|$. Then for every $j \in J_i$ we have $a_i z^{j_{p_i}} = b_i$ and hence $z^{j_{p_i}} \in F$. Since $|J_i| > |F|$, there are two numbers $j < j'$ in $J_i$ such that $z^{j_{p_i}} = z^{j'_{p_i}}$. Since $\max\{ j_{p_i}, j'_{p_i} \} \leq (1 + n|F|)p$, we obtain a contradiction with the choice of $z$. □

We apply Proposition 5.7 in the proofs of the following characterizations of polybounded commutative semigroups (with finite-to-one shifts).

**Proposition 5.8.** A commutative semigroup $X$ with finite-to-one shifts is polybounded if and only if $X$ is bounded and the set $E(X)$ is finite.

*Proof.* To prove the “if” part, assume that $X$ is bounded and has finitely many idempotents. By the boundedness of $X$, there exists $n \in \mathbb{N}$ such that $X = \bigcup_{e \in E(X)} \{ x \in X : x^n = e \}$, which means that $X$ is polybounded.

To prove the “only if” part, assume that $X$ is polybounded. Being commutative, the semigroup $X$ coincides with its center $Z(X)$. By Proposition 5.7, the semigroup $Z(X) = X$ is bounded and by Proposition 5.3 the set $E(X)$ is finite. □

**Proposition 5.9.** A nonempty commutative semigroup $X$ is polybounded if and only if the semigroup $K = \bigcap_{x \in X} X^1 x X^1$ is nonempty and bounded.

*Proof.* Assume that the commutative semigroup $X$ is commutative and polybounded. Proposition 5.3 implies that the semigroup $K = \bigcap_{x \in X} X^1 x X^1$ is nonempty and polybounded. Clearly, $K$ is the minimal ideal of $X$. Using the commutativity of $X$ it can be checked that $K = aK = Ka$ for any $a \in K$, which implies that $K$ is a group, see [23, p. 5]. By Proposition 5.8, the group $K$ is bounded.

Now assume that the semigroup $K = \bigcap_{x \in X^1} X^1 x X^1$ is bounded and nonempty. The above arguments imply that $K$ is a group. Proposition 5.8 ensures that group $K$ is polybounded. By Proposition 5.3 the semigroup $X$ is polybounded. □

The following simple example shows that bounded commutative semigroups are not necessarily polybounded.

**Example 5.10.** The commutative semigroup $\omega$ with the operation of maximum has finite-to-one shifts and is bounded but not polybounded.

**Remark 5.11.** By Proposition 5.8 the infinite product of finite cyclic groups $\prod_{n \in \mathbb{N}} (\mathbb{Z}/n\mathbb{Z})$ is not polybounded. Also, Example 4.5 shows that the polyboundedness is not inherited by taking subgroups, whereas the boundedness is a hereditary property.
6. $C$-CLOSEDNESS AND POLYBOUNDEDNESS

In this section we study a relationship between polybounded and $C$-closed semigroups.

**Lemma 6.1.** Let $X$ be a semigroup with finite-to-one shifts and $A$ be a polybounded subset in $X$. Then $A$ is closed in any $T_1$ topological semigroup $Y$ that contains $X$ as a discrete subsemigroup.

**Proof.** We lose no generality assuming that the semigroup $X$ is nonempty. Being polybounded in $X$, the set $A$ is contained in the union $\bigcup_{i=1}^{m} f_i^{-1}(b_i)$ for some semigroup polynomials $f_1, \ldots, f_m$ on $X$ and some elements $b_1, \ldots, b_m \in X$. Each semigroup polynomial $f_i$ is of the form $f_i(x) = a_{i,0}x \ldots x a_{i,n_i}$ for some $n_i \in \mathbb{N}$ and $a_{i,0}, \ldots, a_{i,n_i} \in X^1$. Let $\tilde{f}_i : Y \to Y$ be the semigroup polynomial on $Y$, defined by $\tilde{f}_i(y) = a_{i,0}y \ldots ya_{i,n_i}$ for $y \in Y$. Since the space $Y$ is $T_1$ and the polynomials $\tilde{f}_i$, $i \leq m$, are continuous, the subset $\bigcup_{i=1}^{m} \tilde{f}_i^{-1}(b_i)$ of $Y$ is closed and hence contains the closure of $A$ in $Y$. Assuming that the set $A$ is not closed in $Y$, take any element $y \in \overline{A} \setminus A$ and find $i \in \{1, \ldots, m\}$ such that $\tilde{f}_i(y) = b_i$. Taking into account that the subspace $X$ of $Y$ is discrete, we conclude that $y \in \overline{A} \setminus A \subseteq X \setminus Y$. By the discreteness of $X$, there exists an open set $V \subseteq Y$ such that $V \cap X = \{b_i\}$. By the continuity of the semigroup operation on $Y$, there exists an open neighborhood $U \subseteq Y$ of $y$ such that $a_{i,0}Ua_{i,1} \cdots Ua_{i,n_i} \subseteq V$. Since $y \in \overline{X} \setminus X$, the set $U \cap X$ is infinite. Fix any element $u \in U \cap X$ and observe that the set

$$\{x \in X : a_{i,0}x(a_{i,1}u \cdots ua_{i,n_i}) = b_i\} \supseteq U \cap X$$

is infinite. But this is impossible, because $X$ has finite-to-one shifts. \hfill \Box

Lemma 6.1 implies the following:

**Corollary 6.2.** Each polybounded semigroup with finite-to-one shifts is $T_1$-$\mathcal{S}$-closed.

**Remark 6.3.** For every semigroup $S$ its $0$-extension $S^0$ is polybounded since $S^0 = \{x \in S^0 : 0x = 0\}$. This trivial example shows that the finite-to-one shift property is essential in Lemma 6.1 and Corollary 6.2.

**Theorem 6.4.** Every zero-closed countable semigroup is polybounded.

**Proof.** Assume that a countable semigroup $X$ is not polybounded. Let $(b_n)_{n \in \mathbb{N}}$ be an enumeration of elements of $X$. We shall construct inductively a sequence $A = \{x_n\}_{n \in \omega} \subseteq X$ such that for each $n \in \omega$ the point $x_n$ satisfies the following condition:

$$*(n) \quad \text{for every } k \leq n \text{ and elements } a_0, \ldots, a_k, a_{k+1} \in \{1\} \cup \{b_i\}_{1 \leq i \leq n} \cup \{x_i\}_{i < n}^n \subseteq X^1 \text{ we have } f(x_n) \neq a_{k+1} \text{ where } f(x) = a_0xa_1 \cdots xa_k.$$ 

To start the inductive construction, choose any point $x_0 \in X$ and observe that the condition $*(0)$ is satisfied, as $1 \notin X$. Assume that for some $n \in \omega$ we have already chosen pairwise distinct elements $x_0, \ldots, x_n$ such that for each $i \leq n$ the condition $*(i)$ is satisfied. The finiteness of the set $F_{n+1} = \{(1) \cup \{b_k : 1 \leq k \leq n + 1\} \cup \{x_k : k \leq n\}\}^{n+1}$ implies that there exist only finitely many semigroup polynomials of the form $f(x) = a_0xa_1 \cdots xa_k$, where $k \leq n + 1$ and $\{a_0, \ldots, a_k\} \subseteq F_{n+1}$. Since the semigroup $X$ is not polybounded, there exists an element $x_{n+1} \in X \setminus \{x_i : i \leq n\}$ which satisfies the condition $*(n+1)$. After completing the inductive construction we obtain the desired set $A = \{x_n\}_{n \in \omega}$.

Consider the countable family

$$\mathcal{K} = \bigcup_{n=1}^{\infty} \{a_0Aa_1 \cdots Aa_n : a_0, \ldots, a_n \in X^1\}$$
of subsets of \( X \). Since
\[
(a_0 A a_1 \cdots A a_n) \cdot (b_0 A b_1 \cdots A b_m) = a_0 A a_1 \cdots A (a_n b_0) A b_1 \cdots A b_m
\]
and
\[
b(a_0 A a_1 \cdots A a_n) c = (b a_0) A a_1 \cdots A (a_n c),
\]
the family \( K \) satisfies conditions (1) and (2) of Lemma 2.1.

To show that \( K \) satisfies condition (3) of Lemma 2.1, fix any set \( K \in \mathcal{K} \) and elements \( a, b, c \in X^1 \). Find \( n \in \mathbb{N} \) and elements \( a_0, \ldots, a_n \in X^1 \) such that \( K = a_0 A a_1 \cdots A a_n \). Next, find \( m \geq 2n \) such that
\[
\{a_0, a_n b, c\} \cup \{a_0, a_1, \ldots, a_n\} \subseteq \{b_0, \ldots, b_m\}.
\]
We claim that \( \{z \in K : azb = c\} \subseteq \{a_0 x_{i_1} a_1 \cdots x_{i_n} a_n : i_1, \ldots, i_n < m\} \). In the opposite case there exists an element \( z \in K \) such that \( azb = c \) and \( z = a_0 x_{j_1} a_1 \cdots x_{j_n} a_n \) for some numbers \( j_1, \ldots, j_n \in \omega \) with \( j := \max\{j_1, \ldots, j_n\} \geq m \). Let \( P = \{p \leq n : j_p = j\} \) and write \( P \) as \( P = \{p(1), \ldots, p(t)\} \) for some numbers \( p(1) < \cdots < p(t) \). Let
\[
a_0' = a_0 a_0 x_{j_1} a_1 \cdots x_{j_{p(1) - 1}} a_{p(1) - 1}, \quad a_i' = a_{p(t)} x_{j_{p(t) + i}} a_{p(t) + i} \cdots x_{j_n} a_n b,
\]
and for every \( 0 < i < t \) put
\[
a_i' = a_{p(i)} x_{j_{p(i) + i}} a_{p(i) + i} \cdots x_{j_{p(i) + 1}} a_{p(i) + 1} \cdots x_{j_n} a_n b,
\]
for the semigroup polynomial \( f(x_i) = a_0' x a_1' \cdots x a_t' \). Observe that
\[
\{c, a_0', \ldots, a_t'\} \subset (\{1\} \cup \{aa_0, a_n b, c\} \cup \{a_i\}_{i \leq n} \cup \{x_i\}_{1 < j})^{2n-1} \subseteq (\{1\} \cup \{b_i\}_{1 \leq i \leq n} \cup \{x_i\}_{i < j})^j.
\]
Then \( c = azb = f(x_j) \) which is not possible, since \( x_j \) satisfies the condition \((*j)\). This contradiction shows that the set \( \{z \in K : azb = c\} \subseteq \{a_0 x_{i_1} a_1 \cdots x_{i_n} a_n : i_1, \ldots, i_n < m\} \) is finite and hence the family \( K \) satisfies condition (3) of Lemma 2.1.

To show that \( K \) satisfies the condition (4) of Lemma 2.1, fix any \( c \in X \) and sets \( K, L \in \mathcal{K} \). Find elements \( a_0', a_0'', \ldots, a_k' \), \( a_0', a_0'', \ldots, a_k'' \in X^1 \) such that \( K = a_0' A a_1' \cdots A a_k' \) and \( L = a_0'' A a_1'' \cdots A a_k'' \). Find a number \( m \geq 2(k + l) \) such that
\[
\{a_k', a_0, c\} \cup \{a_0', \ldots, a_k'\} \cup \{a_0'', \ldots, a_i''\} \subseteq \{b_0, \ldots, b_m\}.
\]
Repeating the argument of the proof of condition (3), we can show that the set \( \{(x, y) \in K \times L : xy = c\} \subseteq \{(a_0' x_{i_1} \cdots x_{i_k} a_k', a_0'' x_{j_1} \cdots x_{j_l} a_l'') : \max\{i_1, \ldots, i_k, j_1, \ldots, j_l\} < m\} \) is finite.

Applying Lemma 2.1, we conclude that the semigroup \( X \) is not zero-closed. \( \square \)

**Remark 6.5.** By [9], for every infinite cardinal \( \kappa \) with \( \kappa^+ = 2^\kappa \) there exists a non-polybounded absolutely \( T_\exists \) -closed group of cardinality \( \kappa^+ \). This shows that Theorem 6.4 does not generalize to semigroups of arbitrary cardinality. On the other hand, according to [7], the set-theoretic assumption \( \text{cov}(\mathcal{M}) = \kappa \) implies that a zero-closed semigroup \( X \) is polybounded if \( X \) admits a compact Hausdorff semigroup topology or \( X \) has a separable complete subinvariant metric. Here \( \text{cov}(\mathcal{M}) \) is the smallest cardinality of a cover of the real line by nowhere dense subsets. The Baire Theorem implies that \( \omega_1 \leq \text{cov}(\mathcal{M}) \leq \kappa \). By Theorem 7.13 in [15], the equality \( \text{cov}(\mathcal{M}) = \kappa \) is equivalent to Martin’s Axiom for countable posets.
7. Proofs of the main results

Proof of Theorem 1.15: Given a nonempty polybounded cancellative semigroup \( X \), we should prove that \( X \) is a group. By Lemma 5.2, \( X = \bigcup_{b \in B} X \) for some finite set \( B \) of pruned semigroup polynomials on \( X \) and some finite set \( B \) of regular elements of \( X \). Then for every \( b \in B \) we can find an element \( b^{-1} \in X \) such that \( bb^{-1}b = b \). It follows that the elements \( bb^{-1} \) and \( b^{-1}b \) are idempotents.

By the cancellativity, \( X \) contains a unique idempotent. Indeed, for any idempotents \( e, f \) in \( X \), the equalities \( e(fg) = (ef)g = e(fg) \) imply \( f = e = e \). Now we see that \( X \) contains a unique idempotent \( e \) and every element \( b \in B \) is invertible in the sense that \( bb^{-1} = e = b^{-1}b \) for some element \( b^{-1} \in X \). By the cancellativity, for every \( x \in X \) the equalities \( xe = x(ee) = (xe)e \) and \( ex = (ee)x = e(ex) \) imply \( xe = x = ex \), which means that \( e \) is the unit of the semigroup \( X \).

For every \( x \in X \) we can find \( p \in P \) and \( b \in B \) such that \( p(x) = b \). Since the semigroup polynomial \( p \) is pruned, \( p(x) = xy \) for some \( y \in X \). Then
\[
x(yb^{-1})x = (xy)b^{-1}x = p(x)b^{-1}x = bb^{-1}x = ex = x,
\]
which means that the semigroup \( X \) is regular. By \cite[Ex.11]{47}, the regular cancellative semigroup \( X \) is a group.

Proof of Theorem 1.16: Given a countable semigroup \( X \) with finite-to-one shifts, we should prove the equivalence of the following conditions:

1. \( X \) is \( T_1 \)-closed;
2. \( X \) is \( T_2 \)-closed;
3. \( X \) is zero-closed;
4. \( X \) is polybounded.

The implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are trivial. The implications (3) \( \Rightarrow \) (4) and (4) \( \Rightarrow \) (1) follows from Theorem 6.4 and Corollary 6.2, respectively.

Proof of Theorem 1.17: Let \( i \in \{1, 2, 3, 3\frac{1}{2}, 2\} \) and \( X \) be a semigroup whose all homomorphic images have finite-to-one shifts. Assuming that \( X \) is \( T_1 \)-closed, we will prove that \( X \) is projectively \( T_1 \)-closed. Let \( h : X \to Y \) be any homomorphism to a topological semigroup \( Y \in T_1 \) such that \( h[X] \) is a discrete subsemigroup of \( Y \). We should prove that \( h[X] \) is closed in \( Y \). Since \( h[X] \) is a topological semigroup in the class \( T_1 \), we lose no generality assuming that \( h[X] \) is dense in \( Y \).

By our assumption, the semigroup \( h[X] \) has finite-to-one shifts. We claim that \( I = Y \setminus h[X] \) is an ideal in \( Y = h[X] \). Assuming the opposite, we can find \( x \in I \) and \( y \in Y \) such that \( xy \notin I \) or \( yx \notin I \). First we consider the case \( xy \notin I \). It follows from \( xy \notin I \) that \( xy \notin h[X] \). Since \( h[X] \) is a discrete subspace of \( Y \), there exists an open neighborhood \( O_{x,y} \subseteq Y \) of \( xy \) in \( Y \) such that \( O_{xy} \cap h[X] = \{xy\} \). By the continuity of the semigroup operation on \( Y \), there exist neighborhoods \( O_x \) and \( O_y \) of the points \( x \) and \( y \) in \( Y \) such that \( O_x \cdot O_y \subseteq O_{xy} \). Choose any \( a \in O_x \cap h[X] \) and observe that \( a(O_y \cap h[X]) = O_{xy} \cap h[X] = \{xy\} \). Since the set \( O_y \cap h[X] \) is infinite, the semigroup \( h[X] \) fails to have finite-to-one shifts. This contradiction shows that \( xy \in I \). By analogy we can show that \( yx \in I \). So, \( I \) is an ideal in \( Y \).

Consider the semigroup \( U_h(X, Y) := X \cup (Y \setminus h[X]) \) that contains \( X \) and \( Y \setminus h[X] \) as subsemigroups and such that for any \( x \in X \) and \( y \in Y \setminus h[X] \), the products \( xy \) and \( yx \) in \( U_h(X, Y) \) are defined by \( xy = h(x)y \) and \( yx = yh(x) \) in \( Y \setminus h[X] \). The semigroup \( U_h(X, Y) \) is endowed with the topology \( \tau \) consisting of the sets \( W \subseteq U_h(X, Y) \) such that for any
such that $\{z \in Y \setminus h[X] : y \in Y \}$ there exists a neighborhood $V_y$ of $y$ in $Y$ such that $(V_y \setminus h[X]) \cup h^{-1}[V_y] \subseteq W$. By [3] Theorem 18, $U_h(X,Y)$ is a topological semigroup in the class $\mathcal{T}_S$ and $X$ is a discrete subsemigroup of $U_h(X,Y)$. Since the semigroup $X$ is $\mathcal{T}_S$-closed, the set $X$ is closed in $U_h(X,Y)$ and consequently, $h[X]$ is closed in $Y$, witnessing that the semigroup $h[X]$ is projectively $\mathcal{T}_S$-closed.

Proof of Theorem 1.13: We should prove that a countable semigroup $X$ with finite-to-one shifts is injectively $\mathcal{T}_S$-closed if and only if $X$ is $\mathcal{T}_S$-discrete. The “only if” part of this characterization follows from Proposition 3.2 (and requires no assumption on the semigroup $X$). To prove the “if” part, assume that the semigroup $X$ is $\mathcal{T}_S$-discrete. By Podewski–Taimanov Theorem 3.4, the space $(X, \mathcal{Z}_Y)$ is discrete, and by Lemma 4.6 the semigroup $X$ is polybounded. Corollary 6.2 implies that the polybounded semigroup $X$ is $\mathcal{T}_S$-closed and Proposition 3.2 implies that $X$ is injectively $\mathcal{T}_S$-closed.

Proof of Theorem 1.14: We should prove that a countable cancellative semigroup $X$ is absolutely $\mathcal{T}_S$-closed if and only if $X$ is projectively $\mathcal{T}_S$-discrete. The “only if” part of this characterization follows from Proposition 5.3 (and requires no assumptions on the semigroup $X$). To prove the “if” part, assume that the semigroup $X$ is projectively $\mathcal{T}_S$-discrete. We lose no generality assuming that the semigroup $X$ is nonempty. By Podewski–Taimanov Theorem 5.4 the space $(X, \mathcal{Z}_Y)$ is discrete, and by Lemma 4.6 the semigroup $X$ is polybounded. By Theorem 1.8 the nonempty polybounded cancellative semigroup $X$ is a group. To prove that the group $X$ is absolutely $\mathcal{T}_S$-closed, take any homomorphism $h : X \to Y$ into a $\mathcal{T}_1$ topological semigroup $Y$. The projective $\mathcal{T}_S$-discreteness of the group $X$ implies the $\mathcal{T}_S$-discreteness of the group $h[X] \subseteq Y$. By Podewski–Taimanov Theorem 3.4 the space $(h[X], \mathcal{Z}_Y)$ is discrete, and by Lemma 4.6 the group $h[X]$ is polybounded. Corollary 6.2 ensures that the group $h[X]$ is closed in $Y$.

Proof of Theorem 1.16: Given a commutative cancellative semigroup $X$, and a class $\mathcal{C}$ of topological semigroups with $\mathcal{T}_S \cap TG \subseteq \mathcal{C} \subseteq \mathcal{T}_S$, we should prove the equivalence of the following conditions:

1. $X$ is absolutely $\mathcal{C}$-closed;
2. $X$ is injectively $\mathcal{C}$-closed;
3. $X$ is $\mathcal{C}$-discrete;
4. $X$ is finite.

Since (4) trivially implies (1)–(3), and (1) $\Rightarrow$ (2), it suffices to prove that the negation of (4) implies the negations of the conditions (2) and (3). So, assume that $X$ is an infinite commutative cancellative semigroup. By [23] p.34, the cancellative commutative semigroup $X$ is a subsemigroup of a commutative group $G$ such that $G = X - X$. Separately we consider two cases.

First assume that the group $G$ is bounded. Then the bounded subsemigroup $X$ of $G$ is a subgroup of $G$ and hence $G = X - X = X$. By Baer–Prüfer Theorem 17.2 [35], the bounded group $G$ is isomorphic to the direct sum $\oplus_{\alpha \in \kappa} G_\alpha$ of finite cyclic groups $G_\alpha$. Since $X$ is infinite, the indexing cardinal $\kappa$ is infinite, too. Taking into account that $\oplus_{\alpha \in \kappa} G_\alpha$ is a non-closed non-discrete subgroup of the zero-dimensional compact topological group $\prod_{\alpha \in \kappa} G_\alpha \in \mathcal{T}_S \cap TG \subseteq \mathcal{C}$, we conclude that the semigroup $X = G$ is not injectively $\mathcal{C}$-closed and not $\mathcal{C}$-discrete.

Next, assume that the group $G$ is unbounded. Then the semigroup $X$ is unbounded as well. Choose any countable divisible subgroup $D$ of the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that $\{z \in \mathbb{C} : \exists n \in \mathbb{N} (z^n = 1)\} \subseteq D$ and $D$ contains an element of infinite order.
Denote by $H$ the set of all homomorphisms from the group $G$ to the group $D$. By Baer Theorem \[81\] 21.1, the diagonal homomorphism $\delta : G \to D^H$, $\delta : x \mapsto (\varphi(x))_{\varphi \in H}$, is injective. Endow the group $D^H$ with the subspace topology inherited from the compact topological group $T^H$.

We claim that the space $\delta[X]$ has no isolated points. In the opposite case, we could find a point $x \in X$ and an open set $U$ in $T^H$ such that $\delta[X] \cap U = \{\delta(x)\}$. Find an open neighborhood $V$ of the identity $e$ in the compact topological group $T^H$ such that $\delta(x)V V^{-1} \subseteq U$. By the compactness of the topological group $T^H$, there exists a finite set $F \subseteq T^H$ such that $T^H = VF$. Since the semigroup $X$ is unbounded, there exists an element $b \in X$ such that $e \neq \delta(b^i) \neq \delta(b^j)$ for any positive numbers $i < j \leq |F| + 1$. Since $\{\delta(b^i) : 1 \leq i \leq |F| + 1\} \subseteq VF$, by the Pigeonhole Principle, there exist $f \in F$ and positive numbers $i < j \leq |F| + 1$ such that $\{\delta(b^i), \delta(b^j)\} \subseteq V f$. Then $\delta(xb^{j-i}) = \delta(b^i)\delta(b^{j-i})^{-1} \in V f (V f)^{-1} = V V^{-1}$ and then $\delta(xb^{j-i}) \in \delta[X] \cap \delta(x)V V^{-1} \subseteq \delta[X] \cap U = \{\delta(x)\}$ and hence $\delta(b^{j-i}) = e$, which contradicts the choice of $b$. This contradiction shows that the subspace $\delta[X]$ of the zero-dimensional Hausdorff group $D^H$ has no isolated points and hence the semigroup $X$ is not $C$-discrete.

To show that $X$ is not injectively $C$-closed, we need the following claim.

**Claim 7.1.** There exists a homomorphism $h : X \to D$ whose image $h[X]$ is infinite.

**Proof.** If the semigroup $X$ has an element $x$ of infinite order, then we can apply Baer Theorem \[81\] 21.1 on extension of homomorphisms into divisible groups and find a homomorphism $h : G \to D$ such that $h(x)$ is the element of infinite order in $D$. In this case the set $h[X] \supseteq \{h(x^n) : n \in \mathbb{N}\}$ is infinite.

If every element of the semigroup $X$ has finite order, then $X = G$ is an unbounded periodic group. In this case a homomorphism $h : G \to D$ with infinite image $h[G] = h[X]$ exists by Claim 5.10 from \[34\].

Let $\delta[X]$ be the closure of the semigroup $\delta[X]$ in the compact topological group $T^H$. For every $h \in H$ let $pr_h : T^H \to \mathbb{T}$, $pr_h : y \mapsto y(h)$, denote the $h$-coordinate projection. By Claim 7.1 there exists a homomorphism $\varphi \in H$ whose image $\varphi[X]$ is infinite and hence $\varphi[X] \subseteq D$ is a dense subsemigroup of the compact topological group $T$. Observe that $\varphi = \varphi \circ \delta$ and hence $\varphi[X] = pr_{\varphi}[\delta[X]] \subseteq \varphi[\delta[X]] \subseteq \varphi[\delta[X]] = \varphi[X] = \mathbb{T}$. The compactness of the set $pr_{\varphi}[\delta[X]]$ and the density of $\varphi[X]$ in $T$ imply that $pr_{\varphi}[\delta[X]] = \mathbb{T}$. So, we can choose an element $y \in \delta[X] \subseteq T^H$ such that $pr_{\varphi}(y) \in \mathbb{T} \setminus D$ and hence $y \in \delta[X] \setminus \delta[X]$.

For every $h \in H$, let $D_h$ be the countable subgroup of $\mathbb{T}$, generated by the set $D \cup \{pr_h(y)\}$. Consider the group $\Pi = \prod_{h \in H} D_h$ endowed with the topology inherited from the compact topological group $T^H$. It is easy to see that $\Pi$ is a zero-dimensional topological group containing $\delta[X]$ as a non-closed subsemigroup and witnessing that the semigroup $X$ is not injectively $C$-closed.

8. **APPLICATION OF THE POLYBOUNDEDNESS TO PARATOPOLOGICAL GROUPS**

In this section we present some applications of polyboundedness to paratopological groups.

A *paratopological group* is a group endowed with a semigroup topology. A paratopological group $G$ is a *topological group* if the operation of taking the inverse $G \to G$, $x \mapsto x^{-1}$, is continuous. In this case the topology of $G$ is called a *group topology*.

The problem of automatic continuity of the inversion in paratopological groups goes back to the classical work of Ellis \[34\]. This problem was investigated by many authors (see...
surveys [13 and 63 for more information]. A typical result says that the continuity of the inversion in paratopological groups follows from a suitable property of compactness type. For example, a paratopological group \( G \) is topological if \( G \) possesses one of the following properties: locally compact, sequentially compact, totally countably compact, regular feebly compact or quasiregular 2-pseudocompact [63].

The next proposition shows that every polybounded \( T_1 \) paratopological group is topological.

**Proposition 8.1.** Let \( G \) be a \( T_1 \) paratopological group and \( H \subseteq G \) be a subgroup which is polybounded in \( G \). Then \( H \) is a topological group.

**Proof.** Being polybounded, the subgroup \( H \) is contained in the union \( \bigcup_{i=1}^{n} f_i^{-1}(b_i) \) for some semigroup polynomials \( f_1, \ldots, f_n : G \to G \) and some elements \( b_1, \ldots, b_n \in G \). Observe that if \( f_i(x) = a_0xa_1 \cdots xa_n \) for some \( a_0, \ldots, a_n \in G \), then \( f_i^{-1}(b_i) = f_i^{-1}(e) \) for the semigroup polynomial \( f_i(x) = xa_1 \cdots xa_nb_i^{-1}a_0 \). So, we can assume that for every \( i \leq n \) there exists a semigroup polynomial or a constant self-map \( g_i \) of \( X \) such that \( f_i(x) = xg_i(x) \) for all \( x \in G \), and \( b_i = e \). It follows from

\[
H \subseteq \bigcup_{i=1}^{n} f_i^{-1}(e) = \bigcup_{i=1}^{n}\{x \in G : xg_i(x) = e\}
\]

that \( x^{-1} \in \{g_i(x)\}_{i \leq n} \) for every \( x \in H \).

To prove that \( H \) is a topological group we should prove that for any open neighborhood \( U \subseteq G \) of \( e \) there exists a neighborhood \( V \subseteq G \) of \( e \) such that \( (H \cap V)^{-1} \subseteq U \). Fix any neighborhood \( U \) of \( e \) in \( G \). Since \( G \) satisfies the separation axiom \( T_1 \), we can replace \( U \) by a smaller neighborhood and assume that for every \( i \leq n \), \( g_i(e)^{-1} \notin U \), whenever \( g_i(e) \neq e \). By the continuity of the semigroup operation in \( G \), there exists a neighborhood \( W \subseteq U \) of \( e \) such that \( WW \subseteq U \). By the continuity of the functions \( g_1, \ldots, g_n \), there exists a neighborhood \( V \subseteq W \) of \( e \) such that for any \( i \in \{1, \ldots, n\} \) we have \( g_i[V] \subseteq Wg_i(e) \). We claim that \( (H \cap V)^{-1} \subseteq U \). Indeed, fix any \( x \in H \cap V \). By the choice of the functions \( g_1, \ldots, g_n \), there exists \( i \in \{1, \ldots, n\} \) such that \( xg_i(x) = e \). Then

\[
e = xg_i(x) \in Vg_i[V] \subseteq VWg_i(e) \subseteq WWg_i(e) \subseteq Ug_i(e).
\]

Recall that if \( g_i(e) \neq e \), then, by the choice of \( U \), \( g_i(e)^{-1} \notin U \). Consequently, \( e \notin Ug_i(e) \) which contradicts to the above inclusion. Hence \( g_i(e) = e \). Finally,

\[
x^{-1} = g_i(x) \in g_i[V] \subseteq Wg_i(e) = Wf = W \subseteq U.
\]

□

**Proposition 8.2.** If a Hausdorff topological semigroup \( Y \) contains a dense polybounded subgroup \( X \), then \( Y \) is a topological group.

**Proof.** Let \( e \) be the identity of the group \( X \). The density of \( X \) in \( Y \) and the Hausdorff property of \( Y \) implies that the (closed) set \( \{y \in Y : ye = y = ey \} \) coincides with \( Y = X \), which means that \( e \) is the identity of the semigroup \( Y \).

Since the group \( X \) is polybounded, there are elements \( b_1, \ldots, b_n \in X \) and semigroup polynomials \( f_1, \ldots, f_n \) on \( X \) such that

\[
X = \bigcup_{i=1}^{n} f_i^{-1}(b_i).
\]
CATEGORICALLY CLOSED COUNTABLE SEMIGROUPS

23

Using the same trick as in the proof of Proposition 8.1 we can assume that each polynomial $f_i(x)$ is of the form $xg_i(x)$ where $g_i$ is a semigroup polynomial or a constant self-map of $X$ and $b_i = e$ for every $i \leq n$. Then for every $x \in X$ there exists $i \in \{1, \ldots, n\}$ such that $xg_i(x) = e$, which means that $g_i(x) = x^{-1}$ and $xg_i(x) = e = g_i(x)x$.

The function $g_i : X \to X$ is of the form $g_i(x) = a_{i,0}x \cdots xa_{i,n_i}$ for some $n_i \in \omega$ and $a_{i,0}, \ldots, a_{i,n_i} \in X$. Let $\bar{g}_i : Y \to Y$ be the continuous function on $Y$ defined by $\bar{g}_i(y) = a_{i,0}y \cdots ya_{i,n_i}$ for $y \in Y$. It follows that the set $\bigcup_{i=1}^{\infty} \{ y \in Y : y\bar{g}_i(y) = e = \bar{g}_i(y)y \}$ coincides with $Y$, because it is closed and contains a dense subset $X$ in $Y$. Consequently, the semigroup $Y$ is polybounded and for every $y \in Y$ there exists an element $z \in \{ \bar{g}_i(y) : i \leq n \} \subset Y$ such that $yz = e = zy$, which means that $Y$ is a group. By Proposition 8.1, $Y$ is a topological group.

□

Conditions implying that a cancellative topological semigroup $S$ is a topological group is another widely studied topic in Topological Algebra (see, [16, 45, 57]). The following corollary of Theorem 1.8 and Proposition 8.1 contributes to this field.

Corollary 8.3. Every nonempty polybounded cancellative $T_1$ topological semigroup is a topological group.

Acknowledgements

The authors express their sincere thanks to the referee for many valuable suggestion which helped the authors to improve essentially the final version of the paper.

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