A Phase Front Instability in Periodically Forced Oscillatory Systems

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(March 31, 2022)

Multiplicity of phase states within frequency locked bands in periodically forced oscillatory systems may give rise to front structures separating states with different phases. A new front instability is found within bands where \( \omega_{\text{forcing}} / \omega_{\text{system}} = 2n \ (n > 1) \). Stationary fronts shifting the oscillation phase by \( \pi \) lose stability below a critical forcing strength and decompose into \( n \) traveling fronts each shifting the phase by \( \pi/n \). The instability designates a transition from stationary two-phase patterns to traveling \( n \)-phase patterns.

Periodic forcing of a single oscillator can lead to rich dynamics including quasiperiodic oscillations, frequency-locked bands ordered through the Farey construction, and low-dimensional chaos \([1–4]\). A typical feature of a locked bands ordered through the Farey construction, dynamics including quasiperiodic oscillations, frequency-locked bands phase fronts that shift the phase by different angles may coexist, for example, \( \pi \)-fronts connecting pairs of these states divide into two groups: \( \pi \)-fronts connecting states with a phase shift of \( \pi \)

\[
A_{-1 \rightarrow +1} = A_{+1} \tanh x, \quad A_{-i \rightarrow +i} = A_{+i} \tanh x,
\]

and \( \pi/2 \)-fronts connecting states with a phase shift of \( \pi/2 \) (see Fig. 3a). When \( \gamma_4 = 1/3 \) the \( \pi/2 \)-fronts are given by

\[
A_{+1 \rightarrow +i} = \frac{1}{2} \sqrt{\frac{3}{2}} [1 + i - (1 - i) \tanh x], \quad A_{-i \rightarrow +1} = \frac{1}{2} \sqrt{\frac{3}{2}} [1 - i + (1 + i) \tanh x],
\]

\[
A_{-1 \rightarrow -i} = -A_{+1 \rightarrow +i} \quad \text{and} \quad A_{+i \rightarrow -1} = -A_{-i \rightarrow +1}.
\]

Additional front solutions follow from the invariance of
Eqn. (2) under reflection, \( x \rightarrow -x \). For example, the symmetric counterparts of \( A_{+i \rightarrow +i}(x) \) and \( A_{+1 \rightarrow -1}(x) \) are \( A_{+1 \rightarrow +1}(x) = A_{+1 \rightarrow +1}(-x) \) and \( A_{-i \rightarrow -1}(x) = A_{+1 \rightarrow -1}(-x) \).

Stability analysis of the \( \pi \)-fronts (3) and (4) shows that they are stable for \( \gamma_4 > 1/3 \). To study the instability at \( \gamma_4 = 1/3 \) we rewrite Eqn. (2) in terms of \( U = \Re(A) + \Im(A) \) and \( V = \Re(A) - \Im(A) \):

\[
\begin{align*}
U_t &= U + \frac{1}{2} U_{xx} - \frac{2}{3} U^3 - \frac{d}{2} (U^2 - 3V^2)U, \\
V_t &= V + \frac{1}{2} V_{xx} - \frac{2}{3} V^3 - \frac{d}{2} (V^2 - 3U^2)V ,
\end{align*}
\]

where

\( d = \gamma_4 - 1/3 \).

At \( d = 0 \) the two equations decouple and assume the solutions \( U = \sigma_1 A_0(x - x_1) \) and \( V = \sigma_2 A_0(x - x_2) \), where \( A_0 = \sqrt{2} \tanh x, \sigma_{1,2} = \pm 1 \) and \( x_1 \) and \( x_2 \) are arbitrary constants. Consider now \( d \neq 0 \) but small. The coupling between \( U \) and \( V \) makes \( x_1 \) and \( x_2 \) slow dynamical variables and \( U \) and \( V \) can be written as

\[
\begin{align*}
U &= \sigma_1 A_0(x - x_1(t)) + u , \\
V &= \sigma_2 A_0(x - x_2(t)) + v ,
\end{align*}
\]

where \( u \) and \( v \) are corrections of order \( d \). Inserting these forms in Eqns. (3) we obtain

\[
\mathcal{H}_1 u = \sigma_1 \dot{x}_1 A_0'(x - x_1) \\
- \frac{1}{2} d \sigma_1 [A_0'^2(x - x_1) - 3A_0^2(x - x_2)] A_0(x - x_1) ,
\]

where \( \mathcal{H}_1 = -1 - \frac{1}{2} \frac{d^2}{dx^2} + 2A_0^2(x - x_1) \). A similar equation is obtained for \( v \) with the indices 1 and 2 interchanged. Solvability conditions lead to the equation

\[
\dot{x}_1 = -\frac{27}{16} d \int_{-\infty}^{\infty} dx \times \tanh(x - x_1) \sech^2(x - x_1) \tanh^2(x - x_2) ,
\]

and to a similar equation for \( x_2 \) with the indices 1 and 2 interchanged. Defining a translational degree of freedom \( \zeta = \frac{1}{2} (x_1 + x_2) \) and an order parameter \( \chi = \frac{1}{2} (x_2 - x_1) \), we obtain from (8)

\[
\dot{\chi} = 0 , \quad \dot{\zeta} = -\frac{27}{16} d J(\chi) ,
\]

where

\[
J(\chi) = I(a) = 6(a^{-1} - a^{-3}) + (1 - 3a^{-2})G(a) , \quad G(a) = (1 - a^{-2}) \ln \left( \frac{1+a}{1-a} \right) ,
\]

and \( a = \tanh 2\chi \). Note that Eqns. (9) are valid to all orders in \( \chi \) and to linear order around \( \gamma_4 = 1/3 \).

Figure 3 shows the potential \( V(\chi) = -\frac{27}{16} d \int I(z) dz \) associated with Eqn. (1) for \( d > 0 \) and \( \gamma_4 > 1/3 \) and \( d < 0 \). There is only one extremum point, \( \chi = 0 \), of \( V \). For \( d > 0 \) it is a minimum and \( \chi \) converges to zero. For \( d < 0 \) it is a maximum and \( \chi \) diverges to \( \pm \infty \). To see how the \( \chi \) dynamics affect the front solutions we rewrite the solution form (9) in terms of the amplitude \( A \) and bring it to the form

\[
A(x,t) = A_{-i \rightarrow -1}(x - x_1) + A_{+1 \rightarrow +1}(x - x_2) - \lambda + R ,
\]

where \( A_{-i \rightarrow -1} \) and \( A_{+1 \rightarrow +1} \) are given in Eqns. (3), \( R \) is a correction term of order \( d \) (related to \( u \) and \( v \) in Eqns. (3)) and for concreteness we chose \( \sigma_1 = -\sigma_2 = 1 \). Equation (10) describes two interacting \( \pi/2 \)-fronts centered at \( x_1 \) and \( x_2 \). When \( d > 0 \) \( \vert \chi \vert \) decreases in time and the two \( \pi/2 \)-fronts attract. As \( \chi \rightarrow 0 \) (or \( x_2 \rightarrow x_1 \) the two \( \pi/2 \)-fronts collapse into a single \( \pi \)-front, \( A(x,t) = A_{-i \rightarrow -1}(x - x_1) + R \), given by Eqn. (1). When \( d < 0 \) \( \vert \chi \vert \) increases in time and the two \( \pi/2 \)-fronts repel. Perturbing the unstable \( \chi = 0 \) solution, the single \( \pi \)-front decomposes into a pair of \( \pi/2 \)-fronts. Fig. 3b shows phase portraits of the \( \pi \) and \( \pi/2 \)-fronts (dashed lines) and the time evolution of an unstable \( \pi \)-front for \( d < 0 \) (solid lines) obtained by numerical integration of Eqn. (1). The approach of the phase portrait to the fixed point \( A_{+1} \) on the \( \Im(A) \) axis describes the decomposition into a pair of \( \pi/2 \)-fronts. This behavior persists arbitrarily close to \( d = 0 \) and is related to the absence of minima in the potential \( V(\chi) \) for \( d < 0 \) (see Fig. 3b).

At \( d = 0 \) there exists a continuous family of stationary solutions describing frozen (non-interacting) pairs of \( \pi/2 \)-fronts with arbitrary separations \( x_2 - x_1 \). This solution family spans the whole phase space inside the dashed triangle in Figure 3a. Because of the parity breaking symmetry \( \chi \rightarrow -\chi \) each \( \pi \)-front may decompose into one of two pairs of \( \pi/2 \)-fronts with phase portraits approaching the fixed points \( A_{+1} \) and \( A_{-1} \).

We extended the derivation of Eqns. (3) to the non-gradient equation (1), treating the constants \( \nu, \alpha, \beta \) as small parameters. The \( \chi \) equation remains unchanged. The \( \zeta \) equation takes the form

\[
\frac{\sigma_2}{\sigma_1} \dot{\zeta} = \nu F_\nu(\chi) + \alpha F_\alpha(\chi) + \beta F_\beta(\chi) ,
\]

where

\[
\begin{align*}
F_\nu &= -\frac{3}{4} G(a) - \frac{3}{2} a^{-1} , \\
F_\alpha &= \frac{3}{4} I(a) , \\
F_\beta &= 3a^{-1}(1 - \frac{3}{2} a^{-2}) - \frac{9}{4} a^{-2} G(a) .
\end{align*}
\]

Notice that \( F_\nu, F_\alpha, \) and \( F_\beta \) are odd functions of \( \chi \) and do not vanish when \( d = 0 \). When \( \vert \chi \vert \rightarrow \infty \) the right hand
side of (11) converges to \( \frac{3}{2}(\nu + \beta) \), the speed of a \( \pi/2 \)-front solution of Eqn. (1). The \( \chi = 0 \) solution (representing a \( \pi \)-front) remains stationary (\( \dot{\zeta} = 0 \)) in the nongradient case as well. At \( \gamma_4 = 1/3 \) (\( d = 0 \)) it loses stability and decomposes into a pair of \( \pi/2 \)-fronts which approach the asymptotic speed \( \frac{3}{2}(\nu + \beta) \). Depending on the initial sign of \( \chi \) the pair may propagate to the left or to the right.

This behavior is different from that near the Nonequilibrium Ising-Bloch front bifurcation within the 2:1 band. In that case, a stationary Ising front loses stability to a pair of counter-propagating Bloch fronts in a pitchfork bifurcation. Associated with the bifurcation is a transition from a single-well potential (Ising front) to a double-well potential (pair of Bloch fronts). A comparison with the potentials in Fig. 1 shows the essential difference between the two front instabilities. In the 2:1 band the Bloch fronts approach the Ising front and coincide with it as the distance to the bifurcation point diminishes to zero. In the 4:1 band, on the other hand, the asymptotic solutions just below \( \gamma_4 = 1/3 \) (the \( \pi/2 \)-front pairs as \( |\chi| \to \infty \)) are not smooth continuations of the stationary \( \pi \)-front (the \( \chi = 0 \) solution) at \( \gamma_4 = 1/3 \). In particular their speed remains finite (\( \frac{3}{2}(\nu + \beta) \)) as \( \gamma_4 \) approaches 1/3 from below. At \( \gamma_4 = 1/3 \), a whole family of propagating solutions appears with speeds ranging continuously from \( \frac{3}{2}(\nu + \beta) \) to zero (pertaining to \( \pi/2 \)-front pairs separated by distances ranging from infinity to zero).

The instability of \( \pi \)-fronts at \( \gamma_4 = 1/3 \) (\( d = 0 \)) determines the structure of stable periodic patterns below
FIG. 3. Collapse of a rotating four-phase spiral-wave into a stationary two-phase pattern. The left column is $|A|$ and the right column $\arg(A)$. (a) The initial four-phase spiral wave (computed with $\gamma_4 < 1/3$). (b) The spiral core, a 4-point vertex, splits into two 3-point vertices connected by a $\pi$-front. (c) A two-phase pattern develops as the 3-point vertices further separate. (d) The final stationary two-phase pattern. Parameters: $\gamma_4 = 0.6, \nu = 0.1, x = [0, 64], y = [0, 64]$.

and above the instability. In the range $\gamma_4 > 1/3$ two-phase patterns, involving domains separated by $\pi$-fronts, prevail. Below $\gamma_4 = 1/3$ four-phase patterns dominate. Four-phase patterns are not stable for $\gamma_4 > 1/3$ despite the stability of the $\pi/2$-fronts because of the attractive interactions among these fronts. In the gradient case [Eqn. (1)] all solutions are stationary. In the nongradient case [Eqn. (2)] the two-phase patterns are stationary while the four-phase patterns propagate. Fig. 3 shows a grey-scale map of a rotating four-phase spiral wave for $\gamma_4 < 1/3$. Figs. 3b, c, d show the collapse of this spiral wave into a stationary two-phase pattern as $\gamma_4$ is increased past 1/3. The collapse begins at the spiral core where the $\pi/2$-front interactions are the strongest. As pairs of $\pi/2$-fronts attract and collapse into $\pi$-fronts, the core splits into two vertices that propagate away from each other leaving behind a two-phase pattern.

To test whether the instability of $\pi$-fronts exists at higher resonances we integrated numerically the higher order equation

\[
A_t = \frac{1}{2} A_{xx} + A + \mu_4 |A|^2 A + \mu_6 |A|^4 A + \mu_8 |A|^6 A + \gamma_4 A^4 + \gamma_6 A^5 + \gamma_8 A^7.
\]

Indeed $\pi$-fronts within the 6:1 and 8:1 bands become unstable as $\gamma_6$ and $\gamma_8$, respectively, are decreased below a critical value. Fig. 2b shows the decomposition of a $\pi$-front into three $\pi/3$-fronts within the 6:1 band, and Fig. 2c shows the decomposition into four $\pi/4$-fronts within the 8:1 band. Our conjecture is that the instability is general, occurring within any $2n:1$ band ($n > 1$).

The phase front instability and the associated transition from stationary two-phase patterns to traveling four-phase patterns within the 4:1 band may be tested in experiments on the ruthenium-catalyzed Belousov-Zhabotinsky reaction subjected to periodic (in time) uniform illuminations [13].

We thank V. Petrov, H. L. Swinney, M. Clerc, R. Rojas and E. Tirapegui for helpful discussions. This research was supported in part by grant No. 95-00112 from the US-Israel Binational Science Foundation (BSF). C. E. acknowledges the support of the Catedra Presidencial en Ciencias.

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[16] No attempt has been made to include in Eqn. (12) all resonant terms. We have included only terms that appeared necessary for demonstrating the phase front instability in the higher order 6:1 and 8:1 bands.