Canonical Quantization
of Spherically Symmetric Gravity
in Ashtekar’s Self-Dual Representation

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Abstract

We show that the quantization of spherically symmetric pure gravity can be carried out completely in the framework of Ashtekar’s self-dual representation. Consistent operator orderings can be given for the constraint functionals yielding two kinds of solutions for the constraint equations, corresponding classically to globally nondegenerate or degenerate metrics. The physical state functionals can be determined by quadratures and the reduced Hamiltonian system possesses 2 degrees of freedom, one of them corresponding to the classical Schwarzschild mass squared and the canonically conjugate one representing a measure for the deviation of the nonstatic field configurations from the static Schwarzschild one. There is a natural choice for the scalar product making the 2 fundamental observables self-adjoint. Finally, a unitary transformation is performed in order to calculate the triad-representation of the physical state functionals and to provide for a solution of the appropriately regularized Wheeler-DeWitt equation.

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1 Introduction

The introduction of new canonical variables for the Hamiltonian framework of general relativity by Ashtekar [1] has considerably enhanced the chances of finding a consistent quantum theory for gravity because the new constraint functionals depend only polynomially on the field variables, in contrast to the much more complicated ADM-approach. Up to now the effort to find solutions of the quantum constraint equations has been concentrated mainly on the loop representation introduced by Rovelli and Smolin. We refer to several recent review articles [2]-[5] for the discussion of the underlying ideas and of the status of solved and unsolved problems.

In the following we shall show that Ashtekar’s nonperturbative quantization programme can be carried out completely for spherically symmetric field configurations by using the self-dual representation originally suggested by Ashtekar.

The basic canonical variables in Ashtekar’s approach to quantum gravity are the connection coefficients $A_{ia}(x)$ as configuration variables and the densitized triads $\tilde{E}_a^i(x)$, $a = 1, 2, 3, i = 1, 2, 3$ as momentum variables characterized as follows:

Let $q_{ab}, a, b = 1, 2, 3$ be the spatial metric on the 3-dimensional Cauchy hypersurface $\Sigma$, then the cotriad coefficients $E_a^i(x)$ obey the relations

$$q_{ab}(x) = E_a^i(x) E_b^j(x) \delta_{ij}, \quad E_a^i(x) E_a^j(x) = \delta^i_j. \quad (1.1)$$

The indices $a, b$ are space indices and the indices $i, j$ are internal indices with respect to the Lie-algebra $\mathcal{L}(SO(3)) : E_a(x) = E_a^i(x) \lambda_i, \lambda_i \in \mathcal{L}(SO(3)), tr(\lambda_i \lambda_j) = \delta_{ij}$. The densitized triad coefficients $\tilde{E}_a^i(x)$ are defined by

$$\tilde{E}_a^i(x) = \sqrt{q(x)} E_a^i(x), \quad q(x) = \det(q_{ab}(x)). \quad (1.2)$$

The complex coefficients $A_{ia}(x)$ of the Ashtekar connection $A_a = A_{ia}^i \lambda_i$ are the pull-backs to the spatial surface $\Sigma$ of the self-dual part of the spacetime spin-connection. They can also be defined by the linear combination

$$A_{ia}(x) \lambda_j = \Gamma_{ia}^j(x) \lambda_j + iK_{ia}^j(x) \lambda_j, \quad K_{ia}^j = K_{ab} E_b^k \delta^{ji}. \quad (1.3)$$

where $\Gamma_{ia}^j(x)$ are the coefficients of the spatial spin connection expressed by the triads and $K_{ab}$ are the coefficients of the extrinsic curvature of $\Sigma$ which
play the role of canonical momenta in the ADM formalism. The last equation may be viewed as a complex canonical transformation.

The action for gravity formulated in terms of Ashtekar’s variables takes the form

\[ S = \frac{1}{\kappa} \int_{R^3} \{ \Theta_L - (i\Lambda^i \circ G_i - iN^a \circ H_a + \frac{1}{2}N \circ H)dt \}, \quad (1.4) \]

\[ \Theta_L = -i\tilde{E}_i^a \circ dA^i_a, \quad (1.5) \]

\[ G_i = D_a\tilde{E}_i^a, \quad (1.6) \]

\[ H_a = \epsilon_{abc}B_i^a\tilde{E}_i^b, \quad (1.7) \]

\[ H = \epsilon_{abc}\epsilon^{ijk}B_i^a\tilde{E}_j^b\tilde{E}_k^c. \quad (1.8) \]

The notation is the usual one: \( \Lambda^i, N^a \) and \( N \sim \) are the Lagrange-multipliers corresponding to the Gauss, vector and scalar constraints \( G_i, H_a \) and \( H \) respectively, that is \( \Lambda^i = A^i_0 \), whereas \( N^a \) and \( N = \det(q)^{1/2}N \) are the usual shift and lapse functions. The covariant differential with respect to \( A^i \) is denoted by \( D \), \( \Theta_L \) is the Liouville-form and the ’magnetic fields’ with respect to the curvature 2-form \( F^i \) of the connection \( A^i \) are given by

\[ B_i^a := \frac{1}{2}\epsilon^{abc}F_i^{bc}. \quad (1.9) \]

The convention

\[ f \circ g := \int_{\Sigma} d^3x f(x)g(x) \quad (1.10) \]

has been used, where \( f(x)g(x) \) is a scalar density of weight 1.

The paper is organized as follows: Section 2 contains the basic formulas for the theory concerning spherically symmetric field configurations only. Due to their \( SO(3) \) transformation properties the connection variables and the triads can be expressed by 6 functions \( A_I(t, x), \) and \( E^I(t, x), I = 1, 2, 3. \) Here \( t \) is a ’time’ variable and \( x \) is a (local) spatial variable which becomes the usual Euclidean radial variable \( r \) at spatial infinity.

The metric \( (q_{ab}) \) now takes the form

\[ (q_{ab}) = \text{diag}(\frac{E}{2E^1}, E^1, E^1 \sin^2 \theta), \quad \det(q_{ab}) = \frac{1}{2}E^1 E \sin^2 \theta, \quad E = (E^2)^2+(E^3)^2, \quad (1.11) \]

which shows that the variables \( E^1 \) and \( E \) determine the sign and the degeneracies of the spatial metric.
The 4-dimensional line element is given by
\[ ds^2 = -(N(x, t) \, dt)^2 + q_{xx}(x, t)(dx + N^x(x, t) \, dt)^2 + q_{\theta\theta}(t, x)(d\theta^2 + \sin^2 \theta d\phi^2) . \] (1.12)
Section 3 discusses properties of the classical phase space, especially the fall-off properties of different quantities for asymptotically flat spaces motivating the choice of function spaces on which the quantum states have support.
Section 4 is the central part of the paper giving the solutions of the constraint equations in the self dual connection (Schrödinger) representation, where
\[ \hat{A}_I(x)\Psi[A_I] = A_I(x)\Psi[A_I], \quad \hat{E}^I\Psi[A_I] = \frac{\delta}{\delta A_I}\Psi[A_I], \] (1.13)
\(\Psi[A_I]\) being a holomorphic functional of the classical field variables \(A_I\) on which the multiplication operators \(\hat{A}_I(x)\) and the functional derivative operators \(\hat{E}^I\) act in the way just described. The crucial point as to the solution of the constraint equations is that they can be transformed into equivalent ones containing the functional derivatives with respect to \(A_I\) at most linearly.
Two types of solutions emerge:
The first one,
\[ \Psi_I[A_I] = \Psi(Q), \quad Q = -i \int_{\Sigma} \frac{A_2B^2 + A_3B^3}{A}(B^1)^{-2} . \] (1.14)
corresponds to configurations which classically have a singular metric at most at isolated points. Here
\[ B^1 = (A - 2)/2, \quad A = (A_2)^2 + (A_3)^2, \] (1.15)
\[ B^2 = A'_3 + A_1A_2, \quad f' = \frac{df(x)}{dx}, \] (1.16)
\[ B^3 = -A'_2 + A_1A_3, \] (1.17)
are the ”magnetic fields” associated with the spherically reduced connection represented by the coefficients \(A_I\) and \(\Psi(Q)\) is any smooth function of \(Q\).
The integrand in the quantity \(Q\) bears a strong resemblance to the integrand in the Chern-Simons functional
\[ \int_{\Sigma} \frac{A_2B^2 + A_3B^3}{A}B^1 . \] (1.18)
of the theory. 
The second type of solutions,
\[ \Psi_{II}[A_J] = \Psi\left( \int_{\Sigma} dx A_1(x) \right), \quad (1.19) \]
belongs to configurations which have globally singular metrics, namely \( E=0 \).
We shall mainly be concerned with the first type of solutions:
The gauge and diffeomorphism invariant quantity \( Q \) is weakly real and emerges
as one of the 2 basic observables of the system.
The second, canonically conjugate observable \( P \) is
\[ P = \int_{\Sigma} dx T(x)(B^1(x))^2 E^1(x), \int_{\Sigma} dx T(x) = 1, \quad (1.20) \]
- \( T(x) \) is a test function - which acts as a derivative operator \( \dot{P} = -id/dQ \)
on the state functionals \( \Psi(Q) \) on which \( \dot{Q} \) acts as multiplication operator,
that is we have
\[ [\dot{Q}, \dot{P}] = i. \quad (1.21) \]
The Hilbert space of physical states consists of all square-integrable functionals \( \Psi(Q) \) with the scalar product
\[ < \Psi_1, \Psi_2 > = \int_{R} dQ \bar{\Psi}_1(Q)\Psi_2(Q). \quad (1.22) \]
This form of the scalar product is almost obvious but can be justified in
detail by BRST-type arguments. = const. configuration.
Section 5 discusses the problem why there are 2 observables, \( P \) and \( Q \) be-
cause, according to Birkhoff’s theorem concerning the uniqueness of the
Schwarzschild solutions, one expects just 1 physical degree of freedom for
spherically symmetric gravity, namely the Schwarzschild mass \( m \). The rea-
son for the occurrence of a second observable, \( Q \), is related to that discussed
in reference [6]:
For stationary classical spacetimes the quantity \( Q \) can be expressed as
\[ Q = \frac{1}{4c} \int_{\Sigma} dx \frac{N^x}{N} \frac{(E^1)'}{E^1(1 + \sqrt{E^1/c})}. \quad (1.23) \]
The real parameter \( c \) is a constant of integration which takes the value
\( 16m^2 \), \( m \) : Schwarzschild mass, for the Schwarzschild solution. For the stan-
dard Schwarzschild solution associated with a static foliation (slicing of space
and time) one gets $Q = 0$ because $N^x = 0$ here and $P = 4m^2$. If $Q \neq 0$ then we must have $N^x \neq 0$, i.e. we cannot have purely static field configurations. Here the following difference between the notions of gauge in the Lagrangean and the Hamiltonian formulation respectively comes into play [6]:

While one uses any spacetime-diffeomorphism to relate gauge-equivalent metrics in the Lagrangean formulation, only those diffeomorphisms that are generated by the constraint functionals define gauge transformations in the Hamiltonian formulation. Thus, according to Birkhoff one can make $N^x$ vanish by an appropriate spacetime gauge, but this is in general not possible in the Hamiltonian framework.

Section 6 presents the solution of the constraints in the triad representation where the operators $A_I$ act as functional derivative operators and the operators $\hat{E}^I$ as multiplication operators on the functionals $\tilde{\Psi}[E^I]$. The physical functionals $\tilde{\Psi}^c[E^I]$ can be obtained explicitly either by transforming the constraints in the triad representation into a linearized form (now with respect to $A_I$) or by performing a (unitary) Laplace transformation of the physical connection functionals $\Psi^c(Q)$ into the triad representation. Both methods yield the same $\tilde{\Psi}^c[E^I]$ which turns out to be a spherically symmetric solution of the Wheeler-DeWitt equation if the operator regularization of the latter is appropriately chosen.

\section{Definition of the model}

The reduction of the various classical geometrical quantities of the Ashtekar formalism to the spherically symmetric case has been done in refs. [7]. We here give mainly the basic ingredients necessary for our purposes:

Let the 3 rotational Killing-fields be denoted by $L_i$ and the generators of $O(3)$ by $T_i$. Then the reduction is obtained by imposing

\begin{equation}
\mathcal{L}_{L_j}E_i = -[T_j, E_i],
\end{equation}

where on the rhs the bracket means the commutator in the Lie-algebra of $SO(3)$. $\mathcal{L}_{L_j}$ is the Lie derivative with respect to $L_j$. Hence it is required that the rotation of the triads in the tangent bundle is compensated by an internal rotation in the $SO(3)$-bundle. Evaluation of eqn. (2.1) yields the general form of the reduced triads. It is not difficult to compute the reduced form of $LSO(3)$-valued vector densities of weight one and of covectors

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as, for example, the densitized triads $\tilde{E}_i^a$ and the Ashtekar-connection $A^i_a$ respectively, which make up the basic variables of the theory:

\[
(\tilde{E}^x, \tilde{E}^\theta, \tilde{E}^\phi) = (E^1 n_x \sin(\theta), \sqrt{\frac{1}{2} (E^2 n_\theta + E^3 n_\phi)} \sin(\theta), \sqrt{\frac{1}{2} (E^2 n_\phi - E^3 n_\theta)}),
\]

(2.2)
\[ (A_x, A_\theta, A_\phi) = (A_1 n_x, \sqrt{\frac{1}{2}}(A_2 n_\theta + (A_3 - \sqrt{2}) n_\phi), \sqrt{\frac{1}{2}}(A_2 n_\phi - (A_3 - \sqrt{2}) n_\theta) \sin(\theta)) \).  

Here we have simultaneously introduced our notation:

We denote by \( \theta, \phi \) the (global) Killing-parameters, \( t \) and \( x \) are the local time and space coordinates. The up to now arbitrary complex functions \( E^I = E^I(t, x), A_I = A_I(t, x); I = 1, 2, 3 \) depend on \( t \) and \( x \) only. The vectors \( n_x, n_\theta \) and \( n_\phi \) are the usual unit vectors on the sphere but are to be understood here as the generators of \( SO(3) \) (thereby making use of the fact that the adjoint- and defining representation of \( SO(3) \) are equivalent), i.e. the basis of \( LSO(3) \) here is angle-dependent.

One now simply inserts this into Ashtekar’s action of full gravity (1.4), integrates out the angles (in particular the factor \( \sin \theta \) contained in \( N \) drops out) and obtains the basic quantities of the model summarized in the following list (\( \Lambda = \Lambda^i(n_x)^i; G = G_i(n_x)^i; \kappa/8\pi \) is Newton constant):

**Action:**

\[ S = \frac{4\pi}{\kappa} \int_R \{ \Theta_L - [b + \Phi] dt \}, \]  

**Liouville-form:**

\[ \Theta_L = -i E^I \circ dA_I, \]  

**Constraint functional:**

\[ \Phi = i \Lambda \circ G - i N_x \circ H_x + N \circ H, \]  

where

Gauss constraint function (i.e. the integrand of the functional):

\[ G = D_x E^1 = (E^1)' + A_2 E^3 - A_3 E^2, \]  

**Vector constraint function:**

\[ H_x = B^2 E^3 - B^3 E^2, \]  

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Diffeomorphism constraint function:

\[ \xi = H_x - A_1 \mathcal{G} = -A_1(E^1)' + (A_2)'E^2 + (A_3)'E^3, \]  

(2.9)

Scalar constraint function:

\[ H = \frac{1}{2}(E^2(2B^2E^1 + B^1E^2) + E^3(2B^3E^1 + B^1E^3)), \]  

(2.10)

Boundary term:

\[ b = \int_{\partial \Sigma} (q + p + e), \]  

(2.11)

O(2)-charge density:

\[ q = -i\Lambda E^1, \]  

(2.12)

ADM-momentum density:

\[ p = iN^x(A_2E^2 + (A_3 - \sqrt{2})E^3), \]  

(2.13)

ADM-energy density:

\[ e_{ADM} = \sim N(A_2E^3 - (A_3 - \sqrt{2})E^2)E^1, \]  

(2.14)

Poisson-brackets:

\[ \{A_I(x), E^J(y)\} = i\delta_I^J\delta(x, y), \quad \{A_I(x), A_J(y)\} = \{E^I(x), E^J(y)\} = 0 \]  

(2.15)

Eqs. of (gauge transformations and) motion:

\[ \frac{d}{d\tau} A_I = \{A_I, \Phi + b\}, \quad \frac{d}{d\tau} E^I = \{E^I, \Phi + b\}, \]  

(2.16)

explicitly:

\[ \frac{d}{d\tau} A_1 = i[(-i\Lambda' + \sim N(B^2E^2 + B^3E^3)], \]  

(2.17)

\[ \frac{d}{d\tau} A_2 = i[(-i\Lambda A_3 + iN^x B^3 + \sim N(B^2E^1 + B^1E^2)], \]  

\[ \frac{d}{d\tau} A_3 = i[(+i\Lambda A_2 - iN^x B^2 + \sim N(B^3E^1 + B^1E^3)], \]  

\[ \frac{d}{d\tau} E^1 = -i[-iN^x(A_2E^3 - A_3E^2) + \sim N(A_2E^2 + A_3E^3)E^1], \]  


\[
\frac{d}{d\tau} E^2 = -i[+i(\Lambda - N^x A_1)E^3 + i(N^x E^2)'] + N(A_1 E^1 E^2 + \frac{1}{2} E A_2) \\
+ (N E^1 E^3)',
\]
\[
\frac{d}{d\tau} E^3 = -i[-i(\Lambda - N^x A_1)E^2 + i(N^x E^3)'] + N(A_1 E^1 E^3 + \frac{1}{2} E A_3) \\
- (N E^1 E^2)'].
\]

The classical canonical constraint algebra is:
\[
\{ \Lambda_1 \circ \mathcal{G}, \Lambda_2 \circ \mathcal{G} \} = 0, \tag{2.18}
\]
\[
\{ N \ast \xi, \Lambda \circ \mathcal{G} \} = -iNN' \circ \mathcal{G},
\]
\[
\{ \mathcal{N} \circ H, \Lambda \circ \mathcal{G} \} = 0,
\]
\[
\{ M \circ \xi, N \circ \xi \} = i(MN' - M'N) \circ \xi,
\]
\[
\{ M \circ \xi, \mathcal{N} \circ H \} = i(M \mathcal{N}' - M' \mathcal{N}) \circ H,
\]
\[
\{ M \circ H, \mathcal{N} \circ H \} = i(M \mathcal{N}' - M' \mathcal{N}) \circ (E^1)^2 H_x.
\]

For the model the ”magnetic fields” take the form
\[
(B^x, B^\theta, B^\phi) = (B^1 n_x \sin(\theta), \sqrt{\frac{1}{2} (B^2 n_\theta + B^3 n_\phi) \sin(\theta)}, \sqrt{\frac{1}{2} (B^2 n_\phi - B^3 n_\theta)}),
\]
\[
(B^1, B^2, B^3) = \left( \frac{1}{2}((A_2)^2 + (A_3)^2 - 2), A'_3 + A_1 A_2, -A'_2 + A_1 A_3 \right), \tag{2.19}
\]

A prime denotes derivation with respect to x, and \( \circ \) - compare eq. (1.10) - denotes now an integral over \( \Sigma^1 \) with local variable \( x \) only, i.e. the integral over the unit sphere has been carried out.

In order to make contact with ref. [1], one has to exchange the labels I=2 and I=3 and to replace \( A_3 + \sqrt{2} \) there by \( A_3 \) in order to get \( A_3 \) here. This shift of \( A_3 \) by \( \sqrt{2} \) on the other hand is essential to read off the model’s kinematical part of the gauge group from the table as \( O(2) \times Diff(\Sigma^1) \), which seems to have been overlooked in ref. [1].

It is then obvious from the above list that there is a strong similarity between our model and full gravity, so that one can hope to learn something from the model, valid for full gravity: the diffeomorphisms have been frozen to the x-direction, the internal rotations to the \( n_x \)-direction. \( A_1 \) plays
the role of an $O(2)$ gauge potential, $E^1$ is $O(2)$-invariant while the vectors $(E^2, E^3), (A_2, A_3)$ transform according to the defining representation of $O(2)$. $A_1, E^2, E^3$ are densities of weight one, while $E^1, A_2, A_3$ are scalars. This can be seen from the transformation formulas (2.17). Moreover, the constraint algebra is again first class and, as one can show, of rank 1 in BRST-terminology, too (see ref. [8]).

Remark: Although the shift by $\sqrt{2}$ drops out from derivatives of $A_3$, the fall-off conditions discussed in ref. [7] are now to be imposed on $A_3 - \sqrt{2}$ (see next section).

Frequent use will be made of the following abbreviations:

\[
\begin{align*}
\alpha & := \arctan\left(\frac{A_3}{A_2}\right), & \eta & := \arctan\left(\frac{E^3}{E^2}\right), & \beta & := \arctan\left(\frac{B^3}{B^2}\right), \\
A & := (A_2)^2 + (A_3)^2, & E & := (E^2)^2 + (E^3)^2, & B & := (B^2)^2 + (B^3)^2,
\end{align*}
\]

(2.20)

The metric is given by

\[
(q_{ab}) = \text{diag}(\frac{E^2}{2E^1}, E^1, E^1 \sin^2(\theta)).
\]

(2.21)

For the discussion of the reality conditions the reduction to spherical symmetry of the spin connection is needed:

\[
\begin{align*}
(\Gamma_x, \Gamma_\theta, \Gamma_\phi) & = (\Gamma_1 n_x, \sqrt{\frac{1}{2}}(\Gamma_2 n_\theta + (\Gamma_3 - \sqrt{2}) n_\phi), \\
& \sqrt{\frac{1}{2}}(\Gamma_2 n_\phi - (\Gamma_3 - \sqrt{2}) n_\theta) \sin(\theta)), \\
(\Gamma_1, \Gamma_2, \Gamma_3) & = (-\eta', -(E^1)'\frac{E^3}{E}, (E^1)'\frac{E^2}{E}), & \eta' & = \frac{E^2 E^3 - E^3 E^2'}{E},
\end{align*}
\]

(2.22)

(2.23)

and for later use in section 5 we record the spherically symmetric reduction of the extrinsic curvature (which is a Lie-algebra-valued covector like $A^i_a$):

\[
\begin{align*}
(K_1, K_2, K_3) & = \frac{1}{N E E^1}(E^1(\dot{q}_{xx} - (q_{xx})'N^x - 2q_{xx}(N^x)'), \\
& E^2(\dot{E}^1 - N^x(E^1)'), E^3(\dot{E}^1 - N^x(E^1)').
\end{align*}
\]

(2.24)

where the dot means the derivative with respect to the variable $t$. 

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3 Definition of the phase space

The following analysis is a simplification of that given in reference [7]. The classical Poisson-brackets can be read off from the Liouville form (we work with geometrical units and set \( \hbar = 1 \); the factor \( 4\pi \) from angle-integration is also dropped because it is a common pre-factor of all spherically symmetric terms):

\[
\{ A^I(x), A^J(y) \} = \{ E^I(x), E^J(y) \} = 0,
\]
\[
\{ A_I(x), E^J(y) \} = i\delta^I_J \delta(x, y) \quad \text{for all } x, y \text{ in } \Sigma.
\] (3.1)

We next wish to determine the function spaces to which our basic variables \( A_I \) and \( E^I \) belong for the case of asymptotically flat field configurations of the metric. We follow here the definition of asymptotic flatness given, for example, in ref. [9], that is, in simplified form:

**Definition 3.1**

i) A Riemannian manifold \((\Sigma^3, q)\) is called asymptotically Euclidean iff there exists a compact subset \( K^3 \) of \( \Sigma^3 \) so that \( \Sigma^3 - K^3 = \bigcup \Sigma^3_A \), where any 'end' \( \Sigma^3_A \) is diffeomorphic to the complement of a ball \( B \) in \( \mathbb{R}^3 \) and \( q \) tends to the Euclidean metric at infinity of \( \Sigma_A \) in a way to be specified.

ii) A spacetime \((M = \mathbb{R} \times \Sigma, g)\) is called asymptotically flat, if i) holds and moreover lapse \( N \) and shift \( \vec{N} \) tend to their Minkowskian values in a way to be specified.

Note that \( \mathbb{R}^3 \) is homeomorphic to \( \mathbb{R}^+ \times S^2 \) and \( \mathbb{R}^3 - B \) to \( \mathbb{R}^{> \rho} \times S^2, \rho > 0 \), where \( \mathbb{R}^+ = \{ x \in \mathbb{R}; x \geq 0 \} \), \( \mathbb{R}^{> \rho} = \{ x \in \mathbb{R}; \rho < x < \infty \} \), so that one encounters the following difficulty after integrating the action over \( S^2 \):

According to our definition of asymptotic flatness we have in the spherically symmetric case

\[
\Sigma^3 - K^3 = (\Sigma^1 - K^1) \times S^2 \cong \bigcup \Sigma_A (\mathbb{R}^3 - B_A) = \bigcup (\mathbb{R}^{> \rho_A} A) \times S^2.
\] (3.2)

Thus \( \Sigma^1 - K^1 \cong \bigcup \mathbb{R}^{> \rho_A} \), (3.3)

where \( K^1 \) is again a compact subset of \( \Sigma^1 \), which may be empty. Thus, if one chooses a spatial topology with only one end (asymptotic region) one has at a Cauchy hypersurface possessing a boundary that does not correspond to infinity. In that case one would also have to impose boundary conditions at the finite boundary. We assume that for this situation a consistent choice
of the boundary conditions exists. This problem is, of course, an artefact of
the reduction process and is of no physical relevance from a 3-dimensional
viewpoint. A possible choice, for example, is that the fields have support
outside the origin or that one restricts oneself to (connected) manifolds \( \Sigma^1 \)
such that \( \Sigma^1 - K^1 \) is diffeomorphic to the union of at least 2 copies of \( R^\infty \).
Thinking of Kruskal's extension of the Schwarzschild solution, the latter op-
tion is physically reasonable.
In the following, we will drop the superscript on \( \Sigma := \Sigma^1 \) and need only treat
the fall-off at infinity in more detail.

The usually imposed fall-off conditions in an asymptotically cartesian frame
are as follows:

\[
(q_{ab}) \to (\delta_{ab} + \frac{1}{r} f_{ab}(\vec{n}, t) + \frac{1}{r^{1+\epsilon}} g_{ab}(\vec{n}, t)), \quad (3.4)
\]

where here and in what follows the asymptotically radial coordinate which
belongs to the given end has been called \( r \); \( f \) and \( g \) are smooth tensors of the
angles and the time coordinate \( t \).
Translating this into the model yields (\( \epsilon > 0 \))

\[
(E^1, E) = (r^2 (1 + f^1(t)/r + O(r^{-1-\epsilon})), 2r^2 (1 + f(t)/r + O(r^{-1-\epsilon}))) \quad (3.5)
\]

which implies

\[
(E^2, E^3) = r \sqrt{2} (e^2 + f^2(t)/r + O(r^{-1-\epsilon}), e^3 + f^3(t)/r + O(r^{-1-\epsilon})), \quad (3.6)
\]

where \( f^I, f \) are the obvious analogues of the tensors given in (3.4) for the
spherically symmetric case and \( e^2, e^3 \) are at this stage complex numbers, sub-
ject to the constraint \( (e^2)^2 + (e^3)^2 = 1 \). The angle-dependence has completely
dropped out while integrating the action over \( S^2 \).

The fall-off-conditions for the conjugate variables \( A_I \) are prescribed by re-
quiring that the Liouville-form is well defined:

\[
(A_1, A_2, A_3 - \sqrt{2}) = \left( \frac{1}{r^2} (a_1 + c_1 \frac{1}{r^\epsilon}) + \frac{b_1(t)}{r^\epsilon}, \frac{1}{r} (a_2 + c_2 \frac{1}{r^2+\epsilon}) + \frac{b_2(t)}{r^{2+\epsilon}}, \frac{1}{r} (a_3 + c_3 \frac{1}{r^\epsilon}) + \frac{b_3(t)}{r^{2+\epsilon}} \right), \quad (3.7)
\]

where \( \delta a_I, \delta c_I = 0 \) is a possible choice for the variations of the leading order
coefficients of the connections (the variations are infinitesimal dynamical or
gauge transformations). Note that it makes a difference whether one requires $E^I \circ \delta A_I$ or $A_I \circ \delta E^I$ to be a finite 1-form on the phase space (this is, of course, also true for fullgravity).

The Lagrange-multipliers are test functions that regularize the constraint distributions. The induced fall-off properties of the latter are as follows:

If one insists on a non-vanishing ADM-energy for improper gauge transformations (the terminology is explained, for instance, in [7]; roughly speaking, proper gauge transformations act as identity on the constraint surface of the phase space which implies that in this case the boundary term $b$ in eqn. (2.11) vanishes) inspecting the boundary term $b$ shows that one has to choose the following fall-off properties of the test functions that appear in the action:

Proper gauge transformations:

\begin{align*}
(\Lambda, N^x, N^\sim) &= (O(r^{-2+\epsilon}), O(r^{-\epsilon}), O(r^{-(2+\epsilon)}), \quad (3.9)
\end{align*}

Improper gauge transformations:

\begin{align*}
(\Lambda, N^x, N^\sim) &= (O(r^{-2}) \text{ or } O(r^{-(2+\epsilon)}), O(1), O(r^{-2}). \quad (3.10)
\end{align*}

This choice simultaneously makes the action functional finite provided one imposes the following additional conditions:

\begin{align*}
a_1 \sqrt{2} + a_2 &= 0, \quad a_3(1 - e^2) = 0, \quad (3.11)
\end{align*}

where the stronger fall-off of $\Lambda$ corresponds to the choice $a_3 = 0$. Note that one usually (see ref. [3]) does not allow the $O(3)$-charge in full gravity to be non-vanishing, but it turns out that for the Schwarzschild solution only the weaker fall-off is possible because $a_3$ is essentially the Schwarzschild mass, so that its analogue, the $O(2)$-charge, does not vanish, while $a_1, a_2$ turn out to vanish seperately (see ref. [4]). Formula (3.11) then suggests to restrict further:

\begin{align*}
a_1 = a_2 = 0, \quad \sqrt{2}c_1 + c_2 = 0, \quad (3.12)
\end{align*}

which will prove essential for the existence of physical states.
4 Realization of the Quantization Programme

4.1 Step 1: Definition of the $\star$–algebra $\mathcal{A}$

We will now carry out all the steps of the quantization programme as proposed by Ashtekar and discussed at length in refs. [2]-[5].

In the first step one is asked to state the equal time canonical commutation relations (CCR) as well as the reality conditions ($\star$-relations) which defines an abstract $\star$-algebra. The algebra $\mathcal{A}$ we choose is the natural one suggested by the Liouville-form and is defined by

\[
[\hat{A}_I(x), \hat{E}^J(y)] = -\delta^J_I \delta(x, y), \quad (4.1)
\]

\[
[\hat{A}_I(x), \hat{A}_J(y)] = [\hat{E}^I(x), \hat{E}^J(y)] = 0, \quad (4.2)
\]

\[
(\hat{E}^I(x))^\dagger - \hat{E}^I(x) = 0, \quad (4.3)
\]

\[
(A_I(x) - \Gamma_I(x))^\dagger + (A_I(x) - \Gamma_I(x)) = 0. \quad (4.4)
\]

However, we will not use the just defined $\star$-relations (4.3) and (4.4) in order to find the scalar-product (or induced ones on other operators, even not in their polynomial version, see, for instance, ref. [10]) for the following reason:

Normally ([2]-[5]) one argues that the inner product is to be determined by imposing the above $\star$-relations to become adjointness-conditions with respect to the scalar product. The integration measure of this scalar product will then in general be highly non-trivial in order to account for the complicated adjointness conditions (4.3) and (4.4). However, even if one succeeds to construct such a measure this will in general not imply that the induced adjointness-relations on quantum-observables reflect the reality-conditions of their classical counterparts, i.e. even if an observable is classically real and therefore satisfies a necessary condition to be measurable it is far from granted that the associated quantum operator becomes self-adjoint (example: 1-dimensional quantum mechanics; $O = qp$ is classically real but in quantum theory it fails to be self-adjoint). Furthermore, there will in general exist non-polynomial observables (as for example the quantity $Q$ in this model). So, imposing the adjointness-conditions (4.3) and (4.4) will then result in mathematically horrible objects. It appears therefore much more natural to impose that the classically induced reality conditions on basic observables should become adjointness-conditions with respect to the scalar
product (roughly speaking one uses the observables as the basic coordinates of the theory without caring about the 'substructure' in terms of the gauge-variant variables $A^i_a$ and $E^a_i$ i.e., if $O = \bar{O}$ is a classical observable, then we want to fix the scalar product by $O^\dagger = O$ whatever the expression of $O$ in terms of $A^i_a$ and $E^a_i$ might be). This requirement is reasonable due to the following argument:

The scalar product should integrate only over those coordinates on which physical states depend. But then the observables are anyway the only operators for which the scalar product is defined because the observables are precisely those operators, which leave $H_{phys}$ invariant. So there is then in general no necessity and no possibility any more to implement the conditions (4.3) and (4.4).

This then has the consequence that one has to get rid of the 'gauge group volume' of the naively defined scalar product, just as in the usual Yang-Mills field theories. Since the gauge 'group' of general relativity is no Lie group, the Faddeev-Popov (FP) procedure will have to be substituted by its extension, the Batalin-Vilkovisky-Fradkin (BFV) method (see ref. [8]).

We will apply this idea with success to the present model:

4.2 Step 2 : Choice of a representation of $\mathcal{A}$ on a linear space

We choose the self-dual (Schrödinger-) representation:

$$\hat{A}_I(x)\Psi = A_I(x)\Psi,$$

$$\hat{E}^I(x)\Psi = \frac{\delta}{\delta A_I(x)}\Psi,$$

where the up to now arbitrary holomorphic functionals $\Psi = \Psi[A_I]$ of the configuration variables $A_I$ form a linear space $\mathcal{H}$ (in the language of geometric quantization (see ref. [11]) the self-dual representation corresponds to the polarization $\mathcal{P} = \text{span}(\delta/\delta E^I)$).

The triad representation on the other hand is given by

$$\hat{A}_I(x)\tilde{\Psi} = -\frac{\delta}{\delta E^I(x)}\tilde{\Psi},$$

$$\hat{E}^I(x)\tilde{\Psi} = E^I(x)\tilde{\Psi},$$

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where $\tilde{\Psi} = \tilde{\Psi}[E^I]$ is an up to now arbitrary holomorphic functional of the momentum variables $E^I$.

In what follows, we will drop the hat again, which indicated an operator in this subsection.

### 4.3 Step 3 : Determination of the physical quantities

#### 4.3.1 The physical subspace $\mathcal{H}_{\text{phys}}$

Following Dirac’s prescription ([12]) a state functional $\Psi$ belongs to $\mathcal{H}_{\text{phys}}$ iff it is annihilated by the (smeared) operator-valued distributions that correspond to the classical constraint functionals. Now this is quite an ambiguous definition because an infinite number of inequivalent operators have the same classical limit, which is defined to be the functional that one obtains when turning back the operators into commuting functions (this is the so-called operator-ordering problem). However, there exists a selection principle:

**Definition 4.1**

i) An ordering of a set of first class constraint operators is called consistent iff its algebra is (weakly) closed. ii) An operator is called a quantum observable iff it (weakly) commutes with all constraints.

(Note that this definition does not imply that the classical counterpart of an observable is real!)

If no consistent ordering exists, then the theory simply cannot be quantized in the given representation. One can show that the set of quantum observables is identical with the set of operators that leave $\mathcal{H}_{\text{phys}}$ invariant.

These and further results are proven, for example, in ref. [12].

The constraint operators consist of polynomials of basic operators which are located at the same point $x$ in $\Sigma^1$, so they are a priori ill-defined and must first be regularized. However, regularizations via point-splitting suggested for the self-dual representation of quantum gravity in the literature poses serious problems (see refs. [13], [14]). Its most naive form (that is simply separating the points $x$ occurring in the operator product by an amount of $\epsilon$ which one sets equal to zero after all commutators and actions on states are carried out) turns all operators in principle into commuting objects, so that one is effectively dealing with a semi-classical theory. This is one of the reasons, why the self-dual representation is hard to make consistent (in contrast to the loop representation).
We will now show that
1. for our model consistent operator-orderings exist that need no regularization,
2. the choice of ordering has physical significance.
In order to do that, we begin with a classical analysis of the structure of the constraint functionals. This provides us with a reformulation of the constraints which are at most linear in the momenta, a property which allows the quantum constraints of our model to be solved:

**Classical analysis of the constraints**
There is a qualitative difference between the cases $E \neq 0$ and $E = 0$ (see eqs. (2.20), (2.21)) the first one is associated with non-degenerate the second one with degenerate solutions.

**Sector I: $E \neq 0$, nondegenerate metrics**
We first dispose of the trivial case $B^1 = 0$:
As $E \neq 0$ it follows from the vector and scalar constraints (2.8) and (2.9) that $B^2 = B^3 = 0$ if $E^1 \neq 0$. This implies $\gamma := A_1 + \alpha' = (A_2 B^2 + A_3 B^3)/A = 0$.
It is not difficult to show that the canonical coordinates $\alpha$ (see eq. (2.20)) $\gamma$ and $B^1$ have the canonical momenta $\mathcal{G}$, $E^1$ and $\sqrt{E/2} \cos(\alpha - \beta)$ respectively.
Now we have 3 (gauge) conditions ($\gamma = 0, B^1 = 0, \mathcal{G} = 0$) for the above 6 canonical momenta. This implies that the physical phase space consists of just one point, as can be seen as follows:
If $q(x), p(x)$ form a conjugate pair and $\lambda(x)$ is the Lagrangean multiplier for the gauge condition $p(x) = 0$ then it follows from $\{q(x), \int_\Sigma dy \lambda(y)p(y)\} = \lambda(x)$ that $p(x)$ generates just one gauge orbit with respect to $q$, a representative of which is $q = 0$. As $p = 0$ we see the validity of the assertion.
The argument can be generalized immediately to our case by choosing an appropriate polarization of the phase space in which $\gamma, B^1$ and $\mathcal{G}$ are the canonical momenta.
The operator constraint method on the other hand shows that $\mathcal{H}_{phys}$ is the linear span of the single functional $\delta[A - 2] \delta[\gamma]$ where $\delta[\gamma] := \prod_{x \in \Sigma} \delta(\gamma(x))$ and is therefore isomorphic to $\mathbb{C}$ (complex numbers). The quantum theory is thus also trivial: observables are only the constant functionals, the natural scalar product is given by $<\Psi_1|\Psi_2> = \bar{\Psi}_1 \Psi_2$.
If $B^1 = 0$ and $E^1 = 0$ we no longer can conclude that $B^2 = B^3 = 0$. However, we have the same situation as above, the only difference being the
interchange of the roles of $\gamma$ and $E^1$. Note that the metric (2.21) becomes highly singular if $E^1 = 0$.
We do not consider this trivial sectors any further.

We now turn to the main case ($E \neq 0, B^1 \neq 0$) which we call ”sector I” and which corresponds to metrics that are degenerate at most at isolated points.

Taking suitable linear combinations of the constraint functions yields

\[ E^2 H - E^3 E^1 H_x = E(B^2 E^1 + \frac{1}{2} B^1 E^2), \quad (4.9) \]
\[ E^3 H + E^2 E^1 H_x = E(B^3 E^1 + \frac{1}{2} B^1 E^3). \quad (4.10) \]

So, for sector I the constraint functions $H$ and $H_x$ are classically equivalent to the constraint functions

\[ \phi_2 := E^2 + \frac{2B^2}{B^1} E^1, \quad (4.11) \]
\[ \phi_3 := E^3 + \frac{2B^3}{B^1} E^1, \quad (4.12) \]

because conversely

\[ \frac{B^1}{2}(E^2 \phi_2 + E^3 \phi_3) = H, \quad (4.13) \]
\[ -B^3 \phi_2 + B^2 \phi_3 = H_x. \quad (4.14) \]

It is obvious that the assumption $B^1 \neq 0$ is needed for the eqs. (4.11)-(4.14) to make sense.

Finally we conclude from the Gauss constraint that

\[ (B^1)^2 G = (B^1)^2 (E^1)' + (B^1)^2 A_2 (\phi_2 - 2B^2 E^1) - B^1 A_3 (\phi_3 - 2B^3 E^1) \]
\[ = ((B^1)^2 (E^1)' + ((B^1)^2)' E^1) + (B^1)^2 (A_2 \phi_2 - A_3 \phi_3) \]
\[ = ((B^1)^2 E^1)' + (B^1)^2 (A_2 \phi_2 - A_3 \phi_3). \quad (4.15) \]

The relations (4.13), (4.14) and (4.16) imply that the constraint functions $H, H_x$ and $G$ are then equivalent to the set $\phi_I, I = 1, 2, 3$, where

\[ \phi_1 = B^1 (E^1)' + 2(B^1)' E^1. \quad (4.16) \]
On the constraint surface the eqs. \( \phi_I = 0 \) can be integrated to give

\[
E^1 = \frac{c}{(B^1)^2},
\]
(4.17)

\[
E^2 = -\frac{2c}{(B^1)^3} B^2,
\]
(4.18)

\[
E^3 = -\frac{2c}{(B^1)^3} B^3,
\]
(4.19)

where \( c \) is up to now an arbitrary complex function of time.

Note:
The above equations (4.17)-(4.19) can also be obtained by the methods of Capovilla et al.[16]. They are equivalent to the following (inverse) 'CDJ-matrix':

\[
(\Psi_{ij})^{-1} = \frac{c}{(B^1)^3} \text{diag}(1, -2, -2).
\]
(4.20)

Compared to the CDJ-approach we are able to solve the Gauss constraint of our model, too, because it is rather simple.

**Sector II:** \( E = 0 \), **degenerate metrics**

For sector II on the other hand it follows from the reality conditions (4.3) and (4.4) and Gauss' law (2.7) that we have on the constraint surface: \( (E^1)' = E^2 = E^3 = 0 \Rightarrow E^1 = c \), \( c \) some real (due to the reality conditions) function of \( t \), while the \( A_I \) remain undetermined. This corresponds to a globally degenerate metric (see formula (1.11)). Hence the constraint functions \( H, H_x \) and \( G \) are here equivalent to the set

\[
\chi_1 = (E^1)', \chi_2 = E^2, \chi_3 = E^3.
\]
(4.21)

This is a remarkable result: The constraint surface splits into precisely 2 disjoint nontrivial pieces (which is associated with the fact that 2 Riemannian manifolds, one of which possesses a globally degenerate metric while the other one is degenerate at most at isolated points, cannot be diffeomorphic if the vector constraint functional does generate diffeomorphisms because one has to observe that the associated test functions (see eqs. (3.9), (3.10)) cannot be chosen arbitrarily but in particular have to be smooth).

Furthermore, the constraints that restrict the phase space to these sectors can be chosen \textit{at most linear in the momenta} \( E^I \). On each sector the original set of constraint functions \( H, H_x, G \), henceforth called 'the canonical set', is
equivalent to the new set \( \phi_I(\chi_I) \), henceforth called the BRST-set (this name will be justified in section (4.4)), where we define 2 sets of constraint functionals to be equivalent iff they restrict the phase space to the same constraint surface.

Quantum analysis of the constraints
Let us turn now to the quantum theory and compute the physical subspace with the help of operator constraints. The basic idea is to use the constraint functions \( \phi_I \) and \( \chi_I \) instead of the original ones. This is justified by showing that there is a regular (operator-valued) transformation between the 2 sets on the operator level. We refer to the appendix for the proof.

For the moment, consider the set of equations (4.17)-(4.19)). Translating these into quantum theory, one obtains simple functional differential equations of the type

\[
\frac{\delta F}{\delta A_I(x)} = f_I(A_J)(x),
\]

which means heuristically that the \((I,x)\)-th partial derivative of the functional \( F \) is prescribed. The existence of a solution (a ‘potential’ for the ‘vector field’ \( (f_I(A_J)(x))_{I=1.3,x\in\Sigma} \)) of this problem is guaranteed at least locally, iff the integrability conditions are satisfied. But for this model one can apply the following theorem:

**Theorem 4.1** Provided that one is given a field theory such that
1) the constraint algebra is of 1st class,
2) the number of configuration space variables coincides with the number of constraints (3 per spatial point \( x \) in this model), then the integrability conditions are automatically satisfied.

Proof:
Let \( E^I(x) := f^I(A_J,x) \) (the notation means that \( f^I \) may be a functional of \( A_J \) and a function of \( x \)) an arbitrary solution of the classical constraints \( \phi_K = \phi_K[A_J,E^I] = 0 \), where one needs only to know that their Poisson-algebra weakly closes. Let \( \eta := M^I \circ \phi_I, \zeta := N^I \circ \phi_I \) \((M^I,N^I\text{test functions of, say, compact support})\). From \( \eta[A_K,E^L = f^L] \equiv 0 \) we have

\[
0 = \frac{\delta}{\delta A_I(x)} \eta[A_K,E^L = f^L]
\]
\[
\frac{\delta}{\delta A_I(x)} [A_K, E^L] \big|_{E^P = f^P} + \int dy \left( \frac{\delta}{\delta E^J(y)} [A_K, E^L] \right) \big|_{E^P = f^P} \frac{\delta f^J(y)}{\delta A_I(x)}
\]

Therefore
\[
\left( \frac{\delta \eta}{\delta A_I(x)} \right) \big|_{E^P = f^P} = - \left( \frac{\delta \eta}{\delta E^J(x)} \right) \big|_{E^P = f^P} \circ \frac{\delta f^J}{\delta A_I(x)}
\]
and
\[
\left( \frac{\delta \eta}{\delta A_I(x)} \right) \big|_{E^P = f^P} \circ \left( \frac{\delta \zeta}{\delta E^J(x)} \right) \big|_{E^P = f^P}
\]
\[
\begin{align*}
&= - \int \Sigma dx \int \Sigma dy \int \Sigma dz \left[ \frac{\delta \zeta}{\delta E^J(x)} \big|_{E^P = f^P} \frac{\delta \eta}{\delta E^J(y)} \big|_{E^P = f^P} \frac{\delta f^J(y)}{\delta A_I(x)} \right. \\
&\quad \left. - \frac{\delta f^J(x)}{\delta A_I(x)} \right]
\end{align*}
\]

Exchanging \( \eta \) and \( \zeta \) as well as the summation-indices, \((x, I) \leftrightarrow (y, J)\), and finally using the definition of the Poisson-bracket yields
\[
\left\{ \eta, \zeta \right\} \big|_{E^P = f^P} = -i \int \Sigma dx \int \Sigma dy \int \Sigma dz U(x, y; z)_I^J K \phi_K
\]

Hence
\[
\left\{ \eta, \zeta \right\} \big|_{E^P = f^P} = 0
\]

Finally, exploiting the 2nd hypothesis, one realizes that the functions
\[
\tilde{M}^I := \left( \frac{\delta \eta}{\delta E^I(x)} \right) \big|_{E^J = f^J}, \quad \tilde{N}^I := \left( \frac{\delta \zeta}{\delta E^I(x)} \right) \big|_{E^J = f^J}
\]
due to the arbitrariness of the test functions \( M^I, N^I \), can be varied independently of the given field distribution of the \( A_I \), as long as they do not vanish identically (which will not be the case, for example, if the constraints are at most linear and homogenous in momenta). Thus they can be interpreted as new test functions and the validity of the integrability conditions
\[
\int dx \int dy \tilde{M}^I(x) \tilde{N}^J(y) \left[ \frac{\delta f^J(y)}{\delta A_I(x)} - \frac{\delta f^I(x)}{\delta A_J(y)} \right] = 0 \forall \tilde{M}^I, \tilde{N}^J, A_K
\]
is shown. □

The ‘potential’ can now be computed by generalizing the potential-formula for an integrable vector field on finite dimensional manifolds to infinite dimensional ones (written down for a field theory based on the configuration variable $\phi$):

$$F[\varphi] - F[\varphi_0] = \int dx (\varphi(x) - \varphi_0(x)) \int_0^1 dt \frac{\delta F}{\delta \varphi(x)}|_{\varphi_t}, \quad (4.30)$$

where $\varphi_t := \varphi_0 + t(\varphi - \varphi_0)$ and $\varphi_0$ is some fixed field configuration.

In order to apply this theorem to our model one only has to check that the new set of constraints $\phi_I$ again has a closed Poisson algebra. This is done in the appendix. In our model the general solution of the quantum constraints $\lambda^i \phi_I \Psi = 0$, $\lambda^I$ suitable test functions, can therefore be written down at once. For simplicity we choose $\varphi_0 = 0$, which is allowed because the zero-field distribution is an element of the phase space; the constant $\Psi[\varphi_0]$ may be neglected, because the constant state functional always solves the constraints in Ashtekar’s formulation of general relativity.

**Sector I:**

Let $F := \ln(\Psi_I)$, then

$$F[A_I] = \int_\Sigma dx A_I(x) \int_0^1 dt \frac{\delta F}{\delta A_I(x)}|_{tA_I}$$

$$= c \int_\Sigma dx \int_0^1 dt [A^1 f(t^2 A_I) + 2A^2 (B^2 \dot{f})|_{tA_I} + 2A^3 (B^3 \dot{f})|_{tA_I}] (x)$$

$$= c \int_\Sigma dx \int_0^1 dt [A^1 f(t^2 A_I) + t[2t A f(t^2 A)]A_I + [2t A f(t^2 A)]\alpha'] (x)$$

$$= c \int_\Sigma dx [(A^1 + \alpha') f(t^2 A_I)]_0^1 (x) = c \int_\Sigma dx (\gamma f)(x) - \frac{c}{2} \int_\Sigma dx \alpha'(x)$$

$$\iff \Psi_I[A_I] = \exp(c \int_\Sigma dx (\gamma f)(x)), \quad (4.31)$$

where we used the abbreviations $\gamma = A_1 + \alpha', f := 1/(B^1)^2$ and the integral over the total differential was neglected because it is a constant: 23
\[ \alpha \rightarrow \lim_{r \to \infty} \arctan(r^{1+\epsilon}\sqrt{2}/c_2). \]

We must show now that the integrand of the functional \( F \) is not divergent at infinity:

We have \( A_1 + \alpha' = (A_2B^2 + A_3B^3)/A \). The fall-off of the magnetic fields is given by (see section 3)

\[
B^1 = \frac{1}{2}((A_2)^2 + (A_3)^2 - 2) \rightarrow \frac{(a_3)^2}{r^2} + \frac{a_3\sqrt{2}}{r} + O(r^{-2-\epsilon}) = O(r^{-1}), \tag{4.32}
\]

\[
B^2 = A'_3 + A_1A_2 \rightarrow -\frac{a_3}{r^2} + O(r^{-3-\epsilon}) = O(r^{-2}),
\]

\[
B^3 = -A'_2 + A_1A_3 \rightarrow \frac{c_2 + c_1\sqrt{2}}{r^{2+\epsilon}} + O(r^{-3-\epsilon}) = O(r^{-3-\epsilon}),
\]

because we imposed \( \sqrt{2}c_1 + c_2 = 0 \), compare (3.12), whence

\[
\frac{\gamma}{(B^1)^2} = O(r^{-1-\epsilon}). \tag{4.33}
\]

Finally one has to show that no surface term survives when acting with the (smeared) constraint-operators on \( F \) (note that the fall-off of the associated test-functions corresponds to a proper gauge transformation because the Dirac condition only says that a physical state shall be gauge invariant. It needs not be invariant under a symmetry transformation, which corresponds to non-vanishing boundary terms of the action functional). This is equivalent to the proof that with our choice of function space to which the connections belong the functional \( F \) is indeed (functionally) differentiable. By definition, the functional derivative of a functional \( f = f[\varphi] \) (if it exists) is given by

\[
\int_\Sigma dxT(x)\frac{\delta f}{\delta \varphi(x)} := \lim_{t \to 0} \frac{1}{t} (f[\varphi + tT] - f[\varphi]), \tag{4.34}
\]

where the test function \( T = : \delta \varphi \), the variation of \( \varphi \), must fit into the function space of field distributions \( \varphi \). Moreover, the so defined function \( \delta f/\delta \varphi(x) \) must belong to the allowed set functions of \( \varphi \) (in our case, this set consists of smooth functions on the phase space).

Since \( (A_1)' \) does not appear in \( F \) we need only inspect the surface terms that arise when varying with respect to \( A_2, A_3 \). We have

\[
\lim_{t \to 0} \frac{1}{t} (F[A_I + t\delta A_2] - F[A_I])_{\text{surface term}} = - \int_{\partial \Sigma} \frac{A_3}{A(B^1)^2} \delta A_2, \tag{4.35}
\]

\[
\lim_{t \to 0} \frac{1}{t} (F[A_I + t\delta A_3] - F[A_I])_{\text{surface term}} = + \int_{\partial \Sigma} \frac{A_2}{A(B^1)^2} \delta A_3.
\]
Inspecting formula (3.7) one can easily see that the fall-off defined there suffices to make the surface terms vanish.

**Sector II :**
Let $G := \ln(\Psi_{II})$

$$G[A_I] = \int_{\Sigma} dx A_I(x) \int_0^1 dt \left( \frac{\delta F}{\delta A_I(x)} \right) |_{tA_I} = c \int_{\Sigma} dx A_1(x) \int_0^1 dt = c \int_{\Sigma} dx A_1(x)$$

$$\iff \Psi_{II} = \exp(c \int_{\Sigma} dx A_1(x)).$$

The functional $G$ is well defined because, according to the choice of function space (=phase space) to which the connections belong, the integrand falls off at infinity as $O(r^{-2-\epsilon})$.

Thus, we have found the general solution of the quantum constraints: $\mathcal{H}_{phys}$ consists of 2 sectors. On sector I physical states are yet arbitrary functions of the functional $F$, $\Psi[A_I] = \Psi_I(F)$, on sector II physical states are yet arbitrary functions of the functional $G$, $\Psi[A_I] = \Psi_{II}(G)$.

It is now of considerable interest to see whether the states just computed are in fact also annihilated by the constraint operators of the canonical set. This is however obvious, since the constraint operators of the canonical set can be ordered in such a way that they are just linear combinations with operator-valued coefficients of the BRST-set (D is any function of the connections and its spatial derivatives): 

$$G = \frac{1}{B^1} \phi_1 + A_2 \phi_3 - A_3 \phi_2,$$

$$H_x = B^2 \phi_3 - B^3 \phi_2,$$

$$H = \frac{1}{2D} (E^2 B^1 D \phi_2 + E^3 B^1 D \phi_3),$$

where the constraints of the BRST-set appear always on the rhs. An interesting choice is given by

$$D = (B^1 B)^{-1/2}, B = (B^2)^2 + (B^3)^2,$$

because then the scalar constraint functional takes the form of a d’Alembert-Beltrami functional differential operator:

$$H = \frac{1}{2} \sqrt{-\det(G)} E^I G_{IJ} \sqrt{-\det(G)} E^J,$$
with respect to the 'magnetic metric'

\[
(G_{IJ}) = \begin{pmatrix}
0 & B^2 & B^3 \\
B^2 & B^1 & 0 \\
B^3 & 0 & B^1
\end{pmatrix},
\]

(4.40)

with determinant \(-\det(G) = -\det((G^{IJ})) = (B^1B)^{-1}\).

We now show that the choice of ordering indeed has physical significance:

Namely, the physical states that correspond to sector II of the constraint surface \((G = \int_\Sigma dx A_1)\) are in fact not annihilated by the scalar constraint operator ordered as above for a general choice of D. For D=1, for example, we get:

\[
\sim N \circ H \Psi_{II}(G) = \int_\Sigma dx \sim N(E^2B^2 + E^3B^3)\dot{\Psi}_{II} \equiv \dot{\Psi}_{II} \delta(0) \int_\Sigma dx \sim A_1 = \infty \neq 0.
\]

(4.41)

Although there is at least one choice for D such that the scalar constraint annihilates the states of both sectors, namely \(D = 1/B\) (which may be checked by calculations similar to those above), we were not able to find observables with respect to a choice of D that are well-defined on both sectors simultaneously (a proof that \(D = 1/B\) fails to fulfil this requirement will be given in the next subsection. In case that such a choice of D would exist we would have a nice selection principle for D at our disposal, but we suspect that no such D can be found and conjecture that the following 'superselection rule' holds:

It is impossible to superpose states belonging to different sectors, because they either fail to be both annihilated by the same ordering of the scalar constraint or no observables common to both sectors do exist.

This corresponds to the classical situation that the constraint surface is unconnected. Thus, the ordering has physical relevance and it is an important structural element of quantum theory. Furthermore, one can argue that at least a regularization of the scalar constraint by the naive version of pointsplitting mentioned in section (4.3.1), would have given a wrong result, because it would not differentiate between the 2 sectors.

It should be clear by now that the ordering has great influence on the resulting quantum theory. In particular if the above mentioned function D would exist the number of degrees of freedom would double (see next subsection),
because physical states would depend on F and G, which were then both observables.

At the end of this subsection we display, for completeness sake, an operator ordering of the canonical constraints that is appropriate for sector II:

\[ H = \frac{1}{2}((2B^2E^1 + B^1E^2)E^2 + (2B^3E^1 + B^1E^2)E^3). \] (4.42)

The ordering of the kinematical constraints remains the same.

### 4.3.2 Observables and reality conditions

From now on we deal with sector I only.

As already stated at the beginning of this section, (quantum) observables are exactly those operators that leave the physical subspace invariant. As the latter here consists of yet arbitrary, complex functions of F, an observable O is an operator which maps a function \( \Psi = \Psi(F) \) of F into a new function \( \Phi = \Phi(F) \). We define the dense subspace \( \mathcal{P} \) of \( \mathcal{H}_{phys} \) with respect to, say, the supremum-topology (or any finer one) with F restricted to a bounded domain in C, to be the polynomials of F of finite degree. Consider then the class \( \mathcal{P}_n \) of monomials of a given degree n. An observable maps these classes into each other or finite sums thereof. All observables are known if one knows any one observable that changes the degree n by +1,-1,0. These are precisely the multiplication operator F, the derivation operator \( \partial/\partial F \) and the unit operator 1, which are the basic observables. Any other lies in the algebra generated by these 3 operators. They are densely defined (on \( \mathcal{P} \)).

Let us represent these operators in terms of Ashtekar’s variables. For F this is already done:

\[ \hat{Q} := -i \int_{\Sigma} dx(A_1 + \alpha')(B^1)^{-2} = -iF, \] (4.43)

where the motivation for the factor i will become clear in a moment. For the derivative operator a convenient choice is

\[ \hat{P} := \int_{\Sigma} dxT(x)(B^1(x))^2E^1(x). \] (4.44)

Here T is any smooth test function normalized to 1,

\[ \int_{\Sigma} dxT(x) = 1, \] (4.45)
introduced in order that the basic operators \( \hat{Q} \) and \( \hat{P} \) become conjugate:

\[
[\hat{Q}, \hat{P}] = i .
\] (4.46)

Then for any function \( \Psi \) of \( Q = -iF \):

\[
\hat{Q}\Psi(Q) = Q\Psi(Q), \quad \hat{P}\Psi(Q) = -i \frac{\partial}{\partial Q}\Psi(Q) .
\] (4.47)

Although the theorem mentioned after definition (4.1) ensures that the observables \( \hat{Q} \) and \( \hat{P} \) in fact weakly commute with the constraint operators, an explicit proof will be given in the appendix.

Now it is easy to show that these observables are in fact no observables on both sectors for the choice \( D = 1/B \) mentioned in the last subsection, because the product functional (taking \( \Psi = \int_{\Sigma} dx A_1 =: G \) as a physical state)

\[
i\hat{Q}G = FG = \int_{\Sigma} dx (\frac{A_1 + \alpha'}{(B^1)^2})(x) \int_{\Sigma} dy A_1(y),
\] (4.48)

which should be a physical state again, is not annihilated by the scalar constraint. In fact, a straightforward but rather involved computation reveals that

\[
N \circ \hat{H}FG = -2 \int_{\Sigma} dx N \frac{B}{(B^1)^3} \neq 0 .
\] (4.49)

Since the 'multiplication-operator' \( \hat{Q} \) is a basic ingredient of the quantum theory it must not be neglected and thus we are forced to select a sector, in order that \( \hat{Q} \) is well-defined. This is the proof promised in the last subsection.

Let us now determine the classical reality conditions of the observables on the constraint surface. Since observables are invariant along the gauge orbit, these reality conditions are then also valid on the reduced phase space. An essential theorem is the following (which holds for full gravity, too):

**Theorem 4.2** If \( O \) is an observable, then Re\((O)\) and Im\((O)\) are also separately observables.

**Proof:**

One has to show that \( \{\hat{O}, \Phi\} \approx 0 \ (\approx 0 \text{ means } =0 \text{ on the constraint surface}) \) for all (smeared) linear combinations of the canonical constraint functions \( \Phi \).
We have (indices are suppressed) $\bar{O}(A, E) := O(\bar{A}, E) = O(A, E)$. Viewing $E$ and the anti-self-dual connection $\bar{A}$ as independent variables (where $\bar{A} = -A + 2\Gamma(E) \Rightarrow \delta/\delta \bar{A} = -\delta/\delta A$), it follows from the definition of the Poisson bracket that

$$0 \approx \{O, \Phi\} = -i(\frac{\delta O}{\delta A} \circ \frac{\delta \Phi}{\delta E} - \frac{\delta O}{\delta E} \circ \frac{\delta \Phi}{\delta A}) = i(\frac{\delta \bar{O}}{\delta A} \circ \frac{\delta \bar{\Phi}}{\delta E} - \frac{\delta \bar{O}}{\delta E} \circ \frac{\delta \bar{\Phi}}{\delta A})$$

$$= -i(\frac{\delta \bar{O}}{\delta A} \circ \frac{\delta \bar{\Phi}}{\delta E} - \frac{\delta \bar{O}}{\delta E} \circ \frac{\delta \bar{\Phi}}{\delta A}) = \{\bar{O}, \bar{\Phi}\}.$$  \hfill (4.50)

But $\bar{\Phi} \approx 0$, just as in full gravity, as can easily be shown.

\[\square\]

Hence, without loss of generality one can always choose a set of basic real observables (note that 2 observables are identified, if they agree on the constraint surface). The next theorem, unfortunately, does not carry over to full gravity.

\textbf{Theorem 4.3} The classical magnetic fields are (weakly) real.

\textbf{Proof :}

Using eqn. (2.23) we obtain

$$\bar{A} = (A_2 - 2\Gamma_2)^2 + (A_3 - 2\Gamma_3)^2 = A - 4(A_2\Gamma_2 + A_3\Gamma_3) + 4((\Gamma_2)^2 + (\Gamma_3)^2)$$

$$= A + 4\left(\frac{E^1}{E}\right)'(A_2E^3 - A_3E^2) + \frac{4((E^1)'^2}{E} = A + \frac{4(E^1)' E}{E} \approx A,$$  \hfill (4.51)

hence $B^1$ is already real. Next we solve the constraints for $B^2, B^3$:

$$B^2 = \frac{E^3}{E} H_x + \frac{E^2}{E^1 E} H - \frac{E^2}{2E^1} B^1$$  \hfill (4.52)

$$B^3 = -\frac{E^2}{E} H_x + \frac{E^3}{E^1 E} H - \frac{E^3}{2E^1} B^1,$$  \hfill (4.53)

to conclude by means of the preceding theorem and the (weak) reality of $B^1$, that $\bar{B}^2 \approx B^2, \bar{B}^3 \approx B^3$.

\[\square\]

With these results it is an easy task to show that the observables $Q$ and $P$ are indeed (weakly) real, thereby justifying the factor $i$ contained in $Q$. It is
already obvious that $P$ is real while for $Q$

\[
\gamma := A_1 + \alpha' = \frac{1}{A}(A_2B^2 + A_3B^3) \Rightarrow \\
\bar{\gamma} \approx \frac{1}{A}((-A_2 + 2\Gamma_2)B^2 + (-A_3 + 2\Gamma_3)B^3) \\
= -\gamma + \frac{2(E_1')}{AE}(B^2(-E^3) + B^3E^2) \\
= -\gamma - \frac{2(E_1')}{AE}H_x \approx -\gamma ,
\]

so that finally

\[
\bar{Q} = -i \int_{x} dx \gamma(B_1)^{-2} \approx Q .
\]

These are the reality conditions for the *classical* basic observables. Since all gauge invariant quantities can be constructed from them, there are only 2 basic variables left on the reduced phase space which thus turns out to be 2-dimensional.

As proposed in section (4.3.1), the scalar product should simultaneously turn the (real) observables into self-adjoint operators on the physical subspace, which then acquires the structure of a Hilbert space.

Physical states depend only on the real observable $Q$, so the scalar product should only integrate over $Q$. One can thus already intuitively guess that the desired Hilbert-space is given by $\mathcal{H}_{phys} = L_2(R, dQ)$. In the next subsection we will sketch the main ideas to justify this more systematically. The details are given elsewhere (see ref. [17]).

### 4.4 Step 4 : Construction of the scalar product

The motivation for the following construction is the expression for the pre-scalar product in the framework of geometric quantization (see, e.g., ref. [4]) for the self-dual representation:

\[
< \Psi | \Phi > = \int_{P} [idA_i \wedge dE_i] \exp(k) \bar{\Psi}[A] \Phi[A] ,
\]

where $k$ is some strictly real functional of $\bar{E}_i$ and $A_i$, which guarantees convergence of the functional integral over the whole phase space $P$. Geometric
quantization, unfortunately, does not show how this inner product has to be modified when the integration extends only over the Lagrangean subspace (see reference [11]) determined by the choice of polarization, in order that it becomes a scalar product. Furthermore, no notice is taken of possible divergencies that appear typically in gauge theories when integrating along the fibres.

The latter observation suggests to look at the partition function of gauge theories. Here one cures the problem by the Faddeev-Popov (FP) procedure, or more generally if the gauge group is not a proper Lie group, by the Batalin-Fradkin-Vilkovisky (BFV) method ([8]):

\[ Z[j] = \int_P d\mu_L \exp(-I[j] + \int_{t_1}^{t_2} dt \{G, \Omega\}) , \]

(4.57)

where \( I[j] \) is the Euclidean action, \( j \) the external current to generate the n-point-Schwinger-functions, \( \mu_L \) is the Liouville-measure on the ghost-extended phase space, \( \Omega \) is the BRST-charge and \( G \) some gauge-fixing functional. The important point is that one encounters precisely the same situation in the case of the pre-scalar product: \( I[0] \) and \( \bar{\Psi}\Phi \) are both gauge invariant, because physical states depend only on observables, \( \{G, \Omega\} \) and \( k \) are both gauge fixing exponents. But while the BFV-theorem guarantees that the partition function does not depend on \( G \), this is not obvious for the pre-scalar product.

The basic idea is now simply to replace the above exponent \( k \) by an analogous expression, thereby introducing ghosts and to extend the BFV theorem to our case. If it is possible to integrate out the ghosts and the momenta then this will provide us with an elegant method to obtain a gauge invariant inner product that integrates over the configuration coordinates of the reduced phase space only. As proved in the last subsection these can always chosen to be real because they are observables and therefore it should be possible to turn their quantum version into self-adjoint operators on \( \mathcal{H}_{phys} \) simply by integrating over the real line. Thus, it will be not necessary to introduce a complicated measure on the full phase space in order to raise the reality conditions on \( A^a_i \) and \( \tilde{E}^a_i \) to the operator level which, as already argued in section (4.1.1), will in general not even guarantee the observables to become self-adjoint operators.

We will apply the above idea in the following, which is, of course, only practicable because we know all the observables.
The proposal for the scalar product in full pure quantum gravity is then

\[ < \Psi | \Phi > = \frac{1}{N} \int_P \left[ i d A^i_a \wedge d \tilde{E}^a_i \wedge dc^i_a \wedge d \rho_a^i \right] \exp(\{G, \Omega\}) \bar{\Psi}[\tilde{A}] \Phi[\tilde{A}] , \]

(4.58)

where \( N \) is some yet unknown normalization constant, \( c^i_a \) and \( \rho_a^i \) are the ghosts and their conjugate momenta respectively and the accent circumflex appearing in the argument of the wave functional is to indicate that the observables on which the latter depend have to be BRST-extended in general, in order that they strongly commute with the BRST charge \( \Omega \) (which, according to \cite{8}, is always possible for any choice of a set of constraints). The proof that this scalar product is in fact independent of the gauge fixing functional \( G \) follows along the chain of arguments of the BFV-theorem (compare chapter 9 of reference \cite{8}) and will be omitted here.

Let us now apply this construction to our model:

The method is, of course, only of practical interest if it is possible to carry out all ghost and momentum integrations. There are 2 means to achieve this:

1) the choice of the set of constraints,
2) the choice of the gauge fixing functional.

One can show (compare \cite{8}) that the rank of the algebra as well as the 'length' of the BRST-extension of observables depends on the choice of the constraints. One should try to get these as small as possible.

As it is easy to integrate Gaussian bosonic integrals, one should keep the momenta outside the structure functions of the constraint algebra, since otherwise they would appear inside the ghost determinant after having carried out the ghost integrations. This latter condition turns out to be in accordance with the requirement to have a short BRST-extension of the observables because then the extension (Poisson-) commutes with that part of \( \Omega \) which is of higher order in the ghosts and anti-ghosts. Finally, the gauge fixing functional should also contain no momenta if possible, for otherwise a quartic ghost term would in general survive in \( \{G, \Omega\} \) because we argued that the structure functions should depend on the connections only.

Fortunately, the BRST-set of constraints in our model obeys all these requirements, thereby giving reason for its name. Namely, according to the appendix

1) the rank of the BRST-algebra is 1,
2) the structure functions do not involve momenta,
3) as the BRST-set is at most linear in the momenta the annihilation of $Q$
by the (smeared) constraints $\phi_I$ is equivalent to the (strong) commutativity
of $Q$ with $\phi_I$. As the structure functions do not depend on the momenta the
observable $Q$ is already BRST-closed, thus no BRST-extension occurs.

It then turns out that one can reduce the scalar product

$$<\Psi|\Phi> = \frac{1}{N}\int_P [iA_I \wedge dE^I \wedge dc^I \wedge d\rho_I] \exp(\{G, \Omega\}) \bar{\Psi}(Q)\Phi(Q)$$  (4.59)

simply by integrating out the undesired coordinates $c^I, \rho_I, E^I$ and the con-
tinuous degrees of freedom contained in $A_I$, after imposing a suitably cho-

sen gauge-fixing functional $G$ compatible with the definition of phase space,
which is independent of the $E^I$ (a possible choice for $G$ can be obtained by
requiring that the connection coefficients $A_I$ fall off with a specific power at
infinity, which is compatible with the requirements of section 3, see ref. [17]).

For the sake of completeness consider the complications we would have en-
countered had we insisted on implementing the adjointness-conditions (4.3)
and (4.4). The reader may convince himself that one could not give any sense
to $Q^\dagger$ in that case because the integrand of $Q$ depends non-analytically on
the connections, not to speak of operator-ordering difficulties arising in the
definition of an infinite series of functional derivatives.

5 Why two degrees of freedom?

The classical Birkhoff theorem states that any spherically symmetric solution
of the Einstein equations in vacuum is \textit{gauge equivalent to a Schwarzschild}

\textit{solution} expressed in its standard static foliation (slicing of space and time),
where 2 solutions $g, \bar{g}$ are defined to be gauge-related if there exists a diffeo-

morphism $(M, g) \rightarrow (M, \bar{g})$. The Schwarzschild solutions are parametrized
by one real parameter, the Schwarzschild mass $m$. Moreover, there exists a
1-1-correspondence between the numbers of gauge-inequivalent classical solu-
tions and gauge-inequivalent Cauchy data for (gauge) field theories such as
general relativity that admit a Hamiltonian formulation (see ref. [18]). The latter, in turn, can be identified with the number of degrees of freedom, i.e. the dimension of the reduced phase space. Thus we seem to have a serious problem: How can our analysis be reconciled with Birkhoff’s theorem? The answer is similar to that in ref. [6]: the Lagrangean formulation of a field theory on which the proof of Birkhoff’s theorem is based and its corresponding Hamiltonian formulation use different notions of gauge: Roughly speaking, from the spacetime (Lagrangean) point of view one uses any 4-diffeomorphism to relate metrics, where, for instance, it is not important whether the diffeomorphism diverges at infinity or not (see [6]). On the other hand, from the Hamiltonian point of view only those ‘diffeomorphisms’ are allowed that 1) fit into the definition of phase space and 2) are generated by the constraints. Now in our model, the observables $P$ and $Q$ (Poisson-) commute weakly with all the constraints which in particular means that they are gauge independent, and thus cannot be gauged away. We will now show that for any solution of the classical equations of motion $P$ is essentially just the Schwarzschild mass squared, while the observable $Q$ can be gauged to zero from the spacetime point of view. The latter can be expected from the following expression for $Q$ (obtained by using the formulas (2.23),(2.24) and the relation $A^I = \Gamma^I + iK^I$):

$$Q = \frac{1}{4c} \int_{\Sigma} dx \frac{N^x}{\tilde{N}} \frac{(E^1)'}{E^1(1 + \sqrt{\frac{E^1}{c}})},$$

valid for those special field configurations for which the functions $E^1$, $E$, $N^x$ and $\tilde{N}$ are stationary. According to Birkhoff’s theorem the shift-vector $N^x$ can be made to zero by a diffeomorphism. One immediately sees that one is dealing with non-static foliations ($N^x \neq 0$) if $Q \neq 0$ in this special case. The situation is thus as follows: We know that $Q$ is gauge invariant from the Hamiltonian point of view but can be gauged to zero from the Lagrangean point of view. There are 2 possible resolutions of this apparent contradiction:

Either we have $Q = 0$ identically for any physical field configuration or the Hamiltonian and Lagrangean notions of gauge are here different, too. The subsequent discussion is intended to rule out the first option by explicit calculations. It shows in particular that the classical physical spectrum of
Q is continuous (and not discrete as one might expect) and that there is therefore a well-defined 2-dimensional reduced phase space. We do this - for the sake of simplicity - by considering stationary solutions of the equations of motion for sector I. The calculations reveal that $Q=0$ for all Schwarzschild solutions in the standard static slicing, but that in general $Q \neq 0$, e.g. for stationary slicings.

Inserting the solution (formulas (4.17)-(4.19)) of the constraint equations into the evolution equations (2.17) one can check that those for $E^I$ are identically satisfied as they have to be, so that one ends up with

$$\dot{A}_1 = \Lambda' - 2icN \frac{B}{(B^1)^3} , \quad (5.2)$$

$$\dot{A}_2 = \Lambda A_3 - N^x B^3 - icN \frac{B^2}{(B^1)^2},$$

$$\dot{A}_3 = -\Lambda A_2 + N^x B^2 - icN \frac{B^3}{(B^1)^2} .$$

On the constraint surface the independent metric components read

$$q_{\theta\theta} = \frac{c}{(B^1)^2}, \quad q_{xx} = \frac{2cB}{(B^1)^4} . \quad (5.3)$$

We will impose coordinate (gauge) conditions on $B^1, B$ later.

Assuming the system to be stationary ($\dot{A}_I = 0$) but not necessarily static the evolution equations can be uniquely solved for the multipliers:

$$\Lambda = ik(B^1)^2 , \quad k = \text{const.} , \quad (5.4)$$

$$N_x = \frac{\Lambda}{B} \sqrt{2B(1 + B^1) - ((B^1)')^2} ,$$

$$N = \frac{(B^1)^3}{2icB} \Lambda' = 2k \frac{(B^1)'}{q_{xx}} ,$$

while the reality conditions $Re(A_I) = \Gamma_I$ turn out to be equivalent to imposing the reality of the magnetic fields because, remarkably enough, the 'electric' spin connections on the constraint surface become precisely the 'magnetic' ones,

$$(\Gamma_1, \Gamma_2, \Gamma_3) = (-\beta', -(B^1)' \frac{B^3}{B}, (B^1)' \frac{B^2}{B}) \quad (5.5)$$

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(compare formula (2.23)). They have the general solution (the equations (5.2) are not needed to derive the following relations):

\[
(\text{Im}(A_2), \text{Im}(A_3)) = \rho (\text{Re}(A_3), -\text{Re}(A_2)),
\]

\[
(\text{Re}(A_1), \text{Im}(A_1)) = (\arctan(\frac{\text{Re}(A_3)}{\text{Re}(A_2)}))', 
\rho' + \frac{\rho (A_R')}{2A_R},
\]

\[
\rho := \frac{1}{(B_1')^\prime} \sqrt{2B(1 + B^1) - ((B^1)'^2},
\]

\[
A_R := (\text{Re}(A_2))^2 + (\text{Re}(A_3))^2 = 2 \frac{1 + B^1}{1 - \rho^2},
\]

which implies that all quantities may be expressed in terms of \(B^1, B\) and \(\text{Re}(A_1)\).

On the constraint surface the observables are classically given by

\[
P = c, Q = i \int_{\Sigma} dx \frac{1}{2(1 + B^1)(B^1)^2} \sqrt{2B(1 + B^1) - ((B^1)'^2} := \int_{\Sigma} dx h,
\]

and here one even does not have to know \(\text{Re}(A_1)\).

We are now able to impose coordinate conditions compatible with the definition of phase space. In view of formula (5.3) and according to section 3 we make the following ansatz

\[
q_{\theta\theta} = x^2 \Rightarrow B^1 = \pm \frac{\sqrt{c}}{x}, q_{xx} = (1 + \sigma(x))^{-1},
\]

where \(\sigma\) is assumed to possess a Laurent-expansion in \(x\). It is, however, not arbitrary because the rest of the conditions derived in section 3 are not satisfied yet. They ensured in particular that the integral over \(h\) in equation (5.7) converges. Computing \(h\) for our ansatz yields

\[
h = i \frac{x^2}{2c} (1 \pm \sqrt{c} \frac{x}{x})^{-1} \sqrt{\frac{c}{x^4} 1 + \sigma(x)} - \frac{c}{x^3} = \pm i \frac{1}{2\sqrt{c}}(1 \pm \sqrt{c})^{-1} \sqrt{\frac{\pm \sqrt{c} - \sigma(x)}{1 + \sigma(x)}}
\]

\[
= O(r^{-1-\epsilon}) \text{ (} x \to r \to \infty \).
\]

Thus, in order to ensure convergence of the integral over \(h\) we must require \(\sigma(x) = \pm \frac{\sqrt{c}}{x} + z(x), z(r) = O(r^{-3})\). This shows that \(\pm \sqrt{c} = -2m\), where
m is the Schwarzschild mass. Since \( h \) is real we have to require that \( z \) is a strictly positive function. Furthermore, recalling the fall-off of \( B^1 \), formula (4.32) and comparing with formula (5.8) we deduce that \( \sqrt{2}a_3 = -2m \).

So far we have imposed coordinate conditions in the asymptotic region only and are still free to join smoothly other conditions in other charts. But, in order to discuss a concrete example, let us assume that \( \Sigma \) is topologically \( R^1 \), i.e. has 2 ends, and that it is possible to choose a global chart. Then a choice that makes \( h \) integrable in this chart and everywhere well-defined is, for instance,

\[
z(x) := \frac{a}{x^4}(1 \pm \sqrt{c})^3 \Rightarrow (5.10)
\]

\[
h(x) = \pm \frac{1}{2\sqrt{c}\sqrt{\frac{x^4}{a} + (1 \pm \sqrt{c})^2}}, \quad (5.11)
\]

\( a \geq 0 \) being a constant and for \( a > 0 \) the observable \( Q \) does not vanish, while for \( a = 0 \) we arrive at the usual Schwarzschild configuration. One may easily check that the metric encounters the usual Schwarzschild-type coordinate singularities and that there is a singularity at \( x=0 \) of a scalar curvature-polynomial. We do not care about that here because the integral over the densitized curvature polynomial is not an observable of our theory. \( Q \) also cannot be gauged away by use of the constraint-generators due to its very definition as an observable. Indeed, the metric that corresponds to this solution of the equations of motion is not static but only stationary, which is indicated by the non-vanishing of the shift-vector \( N^x \):

\[
N^x = \pm 2k\sqrt{c}(1 + \sigma(x)) \frac{1 \pm \sqrt{c}}{\sqrt{\frac{x^4}{a} + (1 \pm \sqrt{c})^2}}. \quad (5.12)
\]

Note that the fall-off behaviour of \( N^x \) fits into the definition of phase space of section 3 and that \( k \) has to be real. Obviously \( N^x \) vanishes iff \( Q \) vanishes. How does this come about?

Usually, when deriving the Schwarzschild metric and Birkhoff’s theorem, (see [18]) one argues that \( N^x \) can be made to vanish because ‘one may choose the coordinates \( t, x \) arbitrarily in the 2-surface \( \theta, \phi = \text{const.} \).’ More precisely this means the following:

In the spacetime picture, where \( g_{tt} = N^2 - (N^x)^2q_{xx}, g_{xt} = q_{xx}N^x, g_{xx} = q_{xx} \)
(recall formula (1.12)), we can choose a new time function \( t = t(x, \tau) \) such that for the metric \( \hat{g} \) diffeomorphic to \( g \) the relation \( \hat{g}_{xx} = 0 \) holds. This implies

\[
\delta g_{xx} := \hat{g}_{xx} - g_{xx} = g_{tx}^2 / g_{tt}, \tag{5.13}
\]

where we use the symbol \( \delta \) to denote the difference between 2 field configurations which are equivalent by an infinitesimal gauge transformation, whereas we denote the (infinitesimal) difference between 2 solutions \( \hat{g} \) and \( g \) of the equations of motion by \( \Delta g = \hat{g} - g \).

Since (up to 4-diffeomorphisms) solutions of the Hamiltonian equations of motion also solve the Euler-Lagrange equations we may use the above solutions \( \hat{g}_{xx} := g_{xx}(a \neq 0) \) and \( g_{xx} := g_{xx}(a = 0) \) in the Lagrangian framework and find \( \Delta g_{xx} = \delta g_{xx} \). This is of course expected, since Einstein’s equations are 4-diffeomorphism covariant.

We will now show that \( \hat{g} \) and \( g \), however, are not gauge-equivalent from the Hamiltonian point of view:

Consider first any constrained field theory with canonical coordinates \( \phi \), Lagrange multipliers \( \lambda \) and constraints \( C = 0 \). In order to arrive at a representative of an initial data set a specific gauge one has to impose a coordinate condition, \( \chi = 0 \), which we also did in (5.8). The consistency conditions that this gauge is preserved under time evolution (\( \dot{\chi} = 0 \)) lead to a unique initial data set \((\phi; \lambda)\).

In the Hamiltonian picture one regards the configurations \((\hat{\phi}; \hat{\lambda})\) and \((\phi; \lambda)\) as infinitesimally gauge-related iff the infinitesimal gauge transformation effected by \( \delta \lambda = \hat{\lambda} - \lambda \) solves \( \delta \varphi := \{ \phi, \delta \lambda \circ C \} \), i.e. iff the transformation is generated by the constraints. Hence the necessary and sufficient condition for 2 initial data set \((\hat{\phi}; \hat{\lambda})\) and \((\phi; \lambda)\) to be gauge equivalent, \( \Delta \varphi = \delta \varphi \), in the Hamiltonian picture takes the form \( \Delta \varphi = \delta \varphi := \{ \phi, \Delta \lambda \circ C \} \).

In our case, we have \((\phi; \lambda) \to (g; N^x, N)\) and are dealing with the above solutions of the equations of motion that correspond to vanishing or non-vanishing parameter \( a \) respectively. According to formula (5.4) we have

\[
\Delta N^x = N^x(a) - N^x(a = 0) = N^x(a) = O(r^{-2}), \quad (\delta N^x)' = O(r^{-3}), \\
\Delta \tilde{N} = \tilde{N}(a) - \tilde{N}(a = 0) = O(r^{-6}), \tag{5.14}
\]

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so that the induced change of the metric components is given by

$$\delta q_{xx} = \frac{1}{E_1}(\frac{1}{2}(\Delta N^x)E' + (\Delta N^x)'E - (\Delta N^x)q_{xx}(E^1)' - i[(\Delta N^x E^1)(A_1 - \Gamma_1)])$$

(5.15)

where the 1st bracket is $O(r^{-3})$, the 2nd $O(r^{-6})$, whereas $\Delta q_{xx} = O(r^{-4})$ (see eqn. (5.10)).

This is the resolution of the apparent contradiction: the Hamiltonian and the Lagrangean picture simply have different understandings of 'gauge', a conclusion first drawn in ref. [6].

This is not unexpected: from the above analysis it is obvious that in the Lagrangean and Hamiltonian picture respectively, the change of $q_{xx}$ involves different powers of $\Delta N^x$, namely $\delta q_{xx} \propto (\Delta N^x)^2$ and $\delta q_{xx} \propto \Delta N^x$ respectively (see eqs. (5.13) and (5.15)).

It is worthwhile to compare the functional $Q$ with the Chern-Simons functional $C_2$ for the spherically symmetric case: Up to a boundary term which vanishes with our choice of phase space and a numerical factor that depends on the definition of the Chern-Simons term one obtains

$$C_2 = \int_\Sigma (A_1 + \alpha')B^1 = 2\int_\Sigma (A_2 B^2 + A_3 B^3)(B^1 + 1)^{-1}B^1,$$

(5.16)

the integrand of which coincides up to a power of $B^1$ with the integrand of the observable $Q$. This is, however, no coincidence but related to the fact that the Chern-Simons term is the generating functional of the magnetic fields which play an important role in the Ashtekar framework (see [16]).

6 Unitary transformation to the triad representation

Now that we have the physical Hilbert space it is interesting to see what it looks like in the triad representation because this representation lies nearest to the old metric formulation of quantum gravity and thus is more suitable to an interpretation. The most straightforward way to find the physical states in this representation turns out to solve simply the BRST-set of constraints for the connections and apply the theorem of section (4.3.1).
Inspecting formulas (4.17)-(4.19) and recalling that on the constraint surface the magnetic fields are real we find that the constant $c$ is real. In the following, let us absorb the sign ambiguity in taking the square root of $c$ into $\sqrt{c}$. Then, using 'cylinder coordinates' $(A_1, \sqrt{A}, \alpha)$ and $(E^1, \sqrt{E}, \eta)$, the 1-parameter-set of constraint functionals

$$\zeta_1 = E^1 - \frac{c}{(B^1)^2},$$

$$\zeta_2 = E^2 + \frac{2cB^2}{(B^1)^3},$$

$$\zeta_3 = E^3 + \frac{2cB^3}{(B^1)^3}$$

can be replaced by the constraint functionals at most linear in the connection coefficients $A_I$ (4c has been replaced by c in this section)

$$\tilde{\zeta}_1 = A_1 + \eta' + \left( \arcsin \left( \frac{(E^1)'}{\sqrt{E(2 + \sqrt{E^1})}} \right) \right)' + \frac{1}{4c(2 + \sqrt{E^1})} \left( \sqrt{\frac{c}{E^1}} \right)^3 \sqrt{E(2 + \sqrt{\frac{c}{E^1}}) - ((E^1)')^2},$$

$$\tilde{\zeta}_2 = A_2 + \frac{E^3(E^1)'}{E} - \frac{E^2}{E} \sqrt{E(2 + \sqrt{\frac{c}{E^1}}) - ((E^1)')^2},$$

$$\tilde{\zeta}_3 = A_3 - \frac{E^2(E^1)'}{E} - \frac{E^3}{E} \sqrt{E(2 + \sqrt{\frac{c}{E^1}}) - ((E^1)')^2}.$$
It is precisely the reduction to spherical symmetry of that for full gravity (compare \[2\]), which can easily be checked by inserting the expressions given in section 2.

Recalling that in the triad representation \( A^I = -\delta / \delta E^I \) and defining \( \Phi := \ln(\Psi) - F \) we arrive at the following ‘gradient components’:

\[
\frac{\delta \Phi}{\delta E^1} = \left( \arcsin\left( \frac{(E^1)' \sqrt{E(2 + \sqrt{\frac{c}{tE^1}})}}{\sqrt{2 + \sqrt{\frac{c}{E^1}}}} \right) \right)' + \frac{\sqrt{\frac{c}{(E^1)')))}{4(2 + \sqrt{\frac{c}{E^1}})} \sqrt{E(2 + \sqrt{\frac{c}{E^1}})} - ((E^1)')^2, \tag{6.6}
\]

\[
\frac{\delta \Phi}{\delta E^2} = -\frac{E^2}{E} \sqrt{E(2 + \sqrt{\frac{c}{E^1}})} - ((E^1)')^2, \tag{6.7}
\]

\[
\frac{\delta \Phi}{\delta E^3} = -\frac{E^3}{E} \sqrt{E(2 + \sqrt{\frac{c}{E^1}})} - ((E^1)')^2, \tag{6.8}
\]

which can, by the method presented in section (4.3.1), be integrated to give the ‘potential’

\[
\Phi[E^I] = \int_\Sigma dx E^I(x) \int_0^1 dt \frac{\delta \Phi[F]}{\delta F^I}|_{F^I=tE^I} = -\int_\Sigma dx \int_0^1 dt ((E^1)' \arcsin\left( \frac{(E^1)'}{\sqrt{E(2 + \sqrt{\frac{c}{tE^1}})}} \right)
- \frac{1}{4(2 + \sqrt{\frac{c}{tE^1}})} \sqrt{E(2 + \sqrt{\frac{c}{tE^1}})} - ((E^1)')^2 + \sqrt{E(2 + \sqrt{\frac{c}{tE^1}})} - ((E^1)')^2. \tag{6.9}
\]

Introducing the new variables

\[
R := \sqrt{E}, \ Y := \frac{(E^1)'}{R}, \ \rho^2 := 2 + \sqrt{\frac{c}{tE^1}}, \tag{6.10}
\]

\( \Phi \) takes the form

\[
\Phi[E^I] = -\int_\Sigma R dx \int_0^1 dt [Y \arcsin\left( \frac{Y}{\rho} \right) + \sqrt{E(2 + \sqrt{\frac{c}{tE^1}})} - Y^2 - \frac{1}{4 \sqrt{t \rho}} \sqrt{\frac{c}{E^1}} \sqrt{\rho^2 - Y^2}], \tag{6.11}
\]

in which the two first terms are easily recognized as the integral of \( \arcsin(Y/\rho) \) with respect to \( Y \). On the other hand (note that \( \rho = \rho(t) \)) we have

\[
\frac{d}{dt} \arcsin\left( \frac{Y}{\rho} \right) = \frac{Y}{t \sqrt{\rho^2 - Y^2}} \frac{1}{4 \sqrt{t \rho}} \sqrt{\frac{c}{E^1}} = \frac{d}{dY} \left( -\frac{1}{4 \sqrt{t \rho}} \sqrt{\frac{c}{E^1}} \sqrt{\rho^2 - Y^2} \right), \tag{6.12}
\]
whence

\[ \Phi[E] = -\int_\Sigma R dx \int \int_0^Y ds [\arcsin(\frac{s}{\rho}) \frac{d}{ds} (-\frac{1}{4\sqrt{t\rho}} \sqrt{c E^1 \sqrt{\rho^2 - s^2}})] = -\int_\Sigma R dx \int \int_0^Y ds \frac{d}{dt} \arcsin(\frac{s}{\rho(t)}) = -\int_\Sigma R dx \int_0^Y ds \arcsin(\frac{s}{\rho(1)}) \]

\[ = -\int dx [((E^1)' \arcsin(\frac{(E^1)'}{\sqrt{E(2 + \sqrt{c/E^1})}}) + \sqrt{E(2 + \sqrt{c/E^1})} - ((E^1)')^2)] \tag{6.13} \]

where the lower boundary \( Y_0 \) was chosen appropriately.

Hence \( \mathcal{H}_{\text{phys}} \) in the triad representation is spanned by the following states:

\[ \tilde{\Psi}_c[E] = \exp(-\int_\Sigma dx [-E^1 \Gamma_1 + ((E^1)') \arcsin(\frac{(E^1)'}{\sqrt{E(2 + \sqrt{c/E^1})}}) + (2 + \sqrt{c/E^1}) - ((E^1)')^2]). \tag{6.14} \]

Thus, up to \( \Gamma_1 \), we are now able to write the theory in metric language, because \( q_{\theta\theta} = E^1, E = 2E^1 q_{xx} = 2q_{\theta\theta}q_{xx} \). One may easily check that the fall-off conditions derived in section 3 need further specification in order to make the integrand of \( \Psi_c[E] \) converge. We refrain from giving it here because, at the moment, we are only interested in formal manipulations for the triad representation.

Note that in the triad representation the constraint functionals cannot be chosen at most linear with respect to the momenta \( A_I \) because the constraint functionals of the canonical set are inhomogenous in \( A_I \). Nevertheless, the physical states can be computed if one uses the classical solutions of the constraints ((6.2)-(6.4)) as the appropriate set of constraint functionals.

It is in this sense that one can argue that a 1-parameter family of exact solutions of the Wheeler-Dewitt-equation has been found for the spherically symmetric case. It can be verified that these states are indeed annihilated by the Wheeler-DeWitt constraint operator in its original form if one pointsplits the second functional derivatives and omits the factor \( \exp(E^1 \circ \Gamma_1) \).

The conclusion of all this might be that the usual quantization procedure should be modified in such a way that one first chooses a set of constraint functionals that contain the momenta at most linearly and then applies the Dirac-procedure thereby eliminating the operator-ordering difficulties right from the beginning.
We will now sketch the proof that the above states can be interpreted as formal Laplace-transforms of \( \exp(i c Q) \), i.e.

\[
\tilde{\Psi}_c[E^I] = \int [dA_1 \wedge dA_2 \wedge dA_3] \exp(A_I \circ E^I) \exp(i c Q).
\] (6.15)

That this is only a formal transformation corresponds to the fact that one has to define non-analytical functions of operators by their spectral-resolution. Solving \( \zeta_I \) for \( A_I \) to obtain \( \tilde{\zeta}_I \) means that there exists an (operator-valued) matrix \( M^J_I \) such that

\[
M^J_I \tilde{\zeta}_J = \zeta_I.
\]

Let us define \( \Psi_c[A_I] \) formally by

\[
\tilde{\Psi}_c[E^I] = \int [dA_1 \wedge dA_2 \wedge dA_3] \exp(A_I \circ E^I) \Psi_c[A_I].
\] (6.16)

Then by the very definition of \( \tilde{\Psi}_c \) and a functional integration by parts (\( \lambda \) is a suitable test function) we get

\[
0 = \lambda M^J_I \circ \tilde{\zeta}_J \tilde{\Psi}_c[E^I] = \lambda \circ \zeta_I(\hat{A}_J = \frac{\delta}{\delta E^J}, \hat{E}^J = E^J) \tilde{\Psi}_c[E^I]
\]

\[
= \int [dA_1 \wedge dA_2 \wedge dA_3] \exp(A_I \circ E^I) \lambda \circ \zeta_I(\hat{A}_J = A_J, E^J = \frac{\delta}{\delta A_J}) \Psi_c[A_I].
\] (6.17)

Inverting the Laplace-transform shows that

\[
\lambda \circ \zeta_I(\hat{A}_J = A_J, E^J = \frac{\delta}{\delta A^J}) \Psi_c[A_I] = 0
\] (6.18)

for any (suitable) test function \( \lambda \). But in section (4.3.1) we showed that these 3 equations have the (up to a constant factor) solution \( \Psi_c(A_I) = \exp(-i c Q) \), which completes the proof. Unfortunately we were not able to give a direct proof by doing the functional integral (6.15) explicitly. However, one can show that a saddle-point approximation gives the correct result.

The scalar product for the triad representation can be taken over from the self-dual representation simply by defining a measure that makes the Laplace-transformation unitary. This measure \( \mu \) turns out to be non-local as was to be expected recalling the general results of ref. [19] because the Ashtekar-transformation is a complex symplectomorphism (see section (4.4) for notation):

\[
\langle \Psi_1 | \Psi_2 \rangle := \int [d(E_1)^I][d(E_2)^J][\mu[[((E_1)^I, (E_2)^J)]\tilde{\Psi}_1[(E_1)^I]\tilde{\Psi}_2[(E_2)^J]], \quad (6.19)
\]

\[
\mu[[((E_1)^I, (E_2)^J)] := \int [dE^J dA_J dc^J d\rho_J] \exp\{G, \Omega\} + A_J \circ (E_2)^J + \overline{A_J} \circ (E_1)^J).
\]
We have the usual interpretation (due to positive definiteness of the scalar product) that for any \( f \in L^2(R, dc) \) and physical state
\[
\Psi = \int_R dc f(c) \Psi_c
\] (6.20)
\( \tilde{\Psi}[E^I] \) is the probability amplitude for the pure spherically symmetric gravitating system to adopt the field configuration \( E^I \) in the state \( \Psi \). The restriction on \( f \) ensures that \( \Psi \) is normalizable:
\[
< \Psi | \Psi > = \int_R dc_1 \int_R dc_2 < \Psi_{c_1} | \Psi_{c_2} > \tilde{f}(c_1) f(c_2) = 2\pi \int_R dc |f(c)|^2
\] (6.21)
because
\[
< \Psi_{c_1} | \Psi_{c_2} > = \int_R dQ \exp(i(c_1 - c_2)Q) = 2\pi \delta(c_1 - c_2).
\]
As an application let us compute the extrema of \( |\tilde{\Psi}[E^I]|^2 \) for a given state \( \Psi \). For a general \( f \) this is not easy. However, we are interested mainly in the eigenstates of observables. Now the observable \( P \) has the simple eigenstates \( \Psi_c(Q) = \exp(icQ) \) and the spectrum is the real line: \( c \in R \), because \( P \) is self-adjoint. Since \( \Psi_c \) is not normalizable we construct the state \( \hat{\Psi}_c := \int_R dc' f_c(c')\Psi_{c'} \), where \( f_c \) may be, for instance, a Gaussian function as sharply peaked around \( c \) as we like. We then compute the extrema of \( |\hat{\Psi}_c[E^I]|^2 \), which approximate the extrema of the probability density of \( \hat{\Psi}_c \) as well as we like. Before doing this one has to take into account the following fact: The pre-factor \( \exp(\int_{\Sigma} dx E^I \Gamma_1) \) common to all physical states must not be varied when determining the extrema of the absolute square of the states because it is the result of imposing the reality condition \( A^+_I + A_I = 2\Gamma_1 \) on the states and thus eventually belongs to the measure of the triad scalar product which could be defined as follows
\[
< \Psi | \Phi > := \int_R [dE^1 \wedge dE^2 \wedge dE^3] \tilde{\Psi}[E^I] \exp(2 \int_{\Sigma} dx \Gamma_1 E^1) \Phi[E^J],
\] (6.22)
and implies that \( (A^I)_I + A_I = 2\Gamma_I \). The latter result is again in nice agreement with ref. [19].
The solution of our variational problem is then
\[
\sqrt{2 + \sqrt{\frac{c}{E^1}}} = \pm \frac{(E^1)'}{\sqrt{E}}.
\] (6.23)
The classical Schwarzschild solution is contained in this class of field configurations: choose $E^1 = r^2, E = 2E^1(1 - \frac{2m}{r})^{-1}, c = 16m^2$ and the negative sign for $\sqrt{c}$.

Note that the integrand in the exponential of eqn. (6.14) reduces to $-E^1 \Gamma_1$ if eqn. (6.23) holds.

7 Conclusions

Ashtekar’s quantization programme for gravity could be carried out completely in the self-dual representation with restriction to spherical symmetry. Although the model has many aspects common with full gravity, there is one property that is not shared by full gravity and which simplified important steps of the quantization procedure used by us, namely that the constraint functionals could be replaced by constraint functionals at most linear in the momenta. It should also be mentioned that in such a situation the reduced phase space method and the operator constraint method give equivalent results.

There may be other elements of our analysis which could be applied to more general systems, e.g. the treatment of observables or the construction of the scalar product.

It also turned out that the operator ordering is of physical significance and that care must be taken when regularizing full quantum gravity in order not to destroy physically important properties of the theory by point-splitting. Even more interesting, the operator ordering implied that the classical structure functions do not resemble their quantum counterparts. This latter result should be valid for the quantization of full gravity in the self-dual representation, too.

Finally, it is remarkable that not the Ashtekar-constraints, but the BRST-constraints seem to be the natural ones. This might indicate that a rearrangement of Ashtekar’s constraints is useful in other models, too. Moreover, as explained in section 6, it could well be that the quantization in a general representation via the Dirac-procedure is possible only, if one first computes the constraint surface explicitly, i.e. casts the constraints into a form in which the momenta appear at most linearly if one does not want to make use of point-splitting.
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A Appendix

In the present paper, 2 sets of constraint functionals are used. They are given here only for the more physical sector I. Also they are written down directly in the correct ordering for quantum theory. In order to obtain the corresponding Poisson algebras one simply has to turn off the ordering and to multiply the rhs by a factor of (-i). For completeness, we also give the fall-off behaviour of the corresponding Lagrange-multipliers.

All appearing integrals have $\Sigma$ as domain of integration. The various surface terms all vanish, as classical functions as well as operators on $H_{\text{phys}}$, on which they are null-operators, due to the fall-off behaviour of the test-functions defined in section 3. This can be seen as follows:

In the following calculations the first and last line contain operators $O_1$ and $O_2$ that annihilate $H_{\text{phys}}$. Suppose that during the calculation we neglected a surface operator $S$, i.e. $O_1 = O_2 + S$. Then $S H_{\text{phys}} = 0$. Since $S$ is a surface term, it cannot be proportional to the constraint functionals, so it must vanish identically on $H_{\text{phys}}$ and thus may be neglected. The technical reason is that the fields (together with the multipliers) that appear in the surface term after applying $S$ to physical states fall off strongly enough at infinity, so that the surface term vanishes.

i) Canonical set

\begin{align*}
\mathcal{G} &= (E^1)' + A_2 E^3 - A_3 E^2, \\
H_x &= -A_1 (E^1)' + A_2' E^2 + A_3' E^3, \\
H &= \frac{1}{2D} (E^2 D[2B^2 E^1 + B^1 E^2] + E^3 D[2B^3 E^1 + B^1 E^3]),
\end{align*}

(A.1)
\[(A, N^x, \mathcal{N}) = (O(r^{-2-\epsilon}), O(r^{-\epsilon}), O(r^{-2-\epsilon}))\, , \]

where D was defined in connection with eqs. (4.37).

ii) BRST-set

\[
\begin{align*}
\phi_1 & = 2((B^1)' E^1 + B^1 (E^1)'), \\
\phi_2 & = E_2 + \frac{B_2}{B_1} E_1, \\
\phi_3 & = E_3 + \frac{B_3}{B_1} E_1,
\end{align*}
\]

\[(\lambda^1, \lambda^2, \lambda^3) = (O(r^{-1-\epsilon}), O(r^{-2-\epsilon}), O(r^{-2-\epsilon}))\, , \]

where the multipliers \(\lambda^I\) are defined to be the coefficients of the constraint functionals of the BRST-set when replacing the canonical set by the \(\phi_I\) in the action. The algebra is again of rank 1 and the BRST-charge computed by standard methods (see [8]) is given by \((c^I, \rho_I\) are ghosts and anti-ghosts)

\[
\Omega = c^A \circ \phi_A + \frac{1}{2} U [c^A, c^B]_{AB} c^C \circ \rho_C \\
= c^A \circ \phi_A - \frac{i}{2(B_1)^2} (2B_1^1 c^1 (A_2 c^2 + A_3 c^3) + 2c^2 c^3) \circ \rho_1 .
\]

The algebra of the BRST-set is as follows:

\[
[M \circ \phi_1, N \circ \phi_1] = 0 ,
\]

since only the momentum \(E^1\) but no \(A_1\) is contained in \(\phi_1\). Furthermore

\[
[M \circ \phi_2, N \circ \phi_2]
= \int dx \int dy (M(x)N(y) - M(y)N(x)) ([E^2(x), \frac{2B^2}{B_1^1}(y)] E^1(y) \\
+ \frac{2B^2}{B_1^1}(x)[E^1(x), \frac{2B^2}{B_1^1}(y)] E^1(y)) = 0
\]

since no spatial derivative survives in either of the commutators. Similarly we have

\[
[M \circ \phi_3, N \circ \phi_3] = 0 .
\]
The calculation of the rest of the commutators is slightly more difficult:

\[
[M \circ \phi_1, N \circ \phi_2] = \int dx \int dy M(x)N(y)(2[A'(x), E^2(y)]E^1(x) + [(A - 2)(x), E^2(y)][E^1]'(x)
\]
\[
+ (2A'(x)[E^3(y), B^2(y)] + (A - 2)(x)[(E^1)'(x), B^2(y)]) \frac{4}{A - 2}(y)E^1(y))
\]
\[
= \int dx(4NA_2(ME^1)' - 2MNA_2(E^1)' + (M2A' - N(M(A - 2))A_2)\frac{4}{A - 2}E^1
\]
\[
= MN \frac{A_2}{B^1} \circ \phi_1.
\]

Similarly, or by O(2)-symmetry:

\[
[M \circ \phi_1, N \circ \phi_3] = MN \frac{A_3}{B^1} \circ \phi_1.
\]

Finally

\[
[M \circ \phi_2, N \circ \phi_3] = [M \circ \phi_1, N \circ \phi_3] + [M \circ \phi_2, N \circ \phi_1] = [M \circ \phi_1, N \circ \phi_2] + [M \circ \phi_3, N \circ \phi_1] = MN \frac{A_2}{B^1} \circ \phi_1.
\]

As to the algebra of the canonical set it is impossible to choose the above mentioned function D in such a way that the structure 'functions' for the canonical algebra become the classical ones given in eqs. (2.18) although, of course, these are recovered in the classical limit. Moreover, the structure
functions become \textit{nonanalytical in the momenta}. This is, of course, quite undesirable. However, it is the price to pay in order that the constraint operators always appear to the right, i.e. that the algebra strictly closes. In this way one never has to make sense of a functional derivative in the denominator. Nevertheless it is obvious that the canonical set is not the natural one for the model.

The commutators of the generators of the kinematical subgroup $O(2) \times Diff \Sigma$ yield the same structure functions as their classical counterparts, because they are at most linear in the momenta. The remaining commutators, however, are rather involved for a general D and we refrain from displaying them here, but rather wish to give a rigorous general argument showing the closure of the algebra in the operator ordering defined.

The following list of equations gives the 'transformation matrix' for sector I between the 2 equivalent sets of constraints, where the sequence of the operators is important:

\begin{align*}
\chi_1 & := \mathcal{G} = \frac{1}{B^1} \phi_1 + A_2 \phi_3 - A_3 \phi_2, \quad (A.10) \\
\chi_2 & := H_x = B^2 \phi_3 - B^3 \phi_2, \\
\chi_3 & := H = \frac{1}{2D}(E^2 B^1 D \phi_2 + E^3 B^1 D \phi_3) \\
\Rightarrow & \\
\phi_1 & = B^3 \mathcal{G} - A_2 \phi_3 + A_3 \phi_2, \quad \text{where} \quad (A.11) \\
\phi_2 & = \frac{1}{E^2 B^2 + B^3 E^3} \frac{B^2}{B^1 D} [2DH - E^3 B^1 D H_x], \\
\phi_3 & = \frac{1}{E^2 B^2 + B^3 E^3} \frac{B^3}{B^1 D} [2DH + E^2 B^1 D H_x].
\end{align*}

The idea is now as follows: let the operator-valued matrices $c^I_J$ and $d^I_J$ be defined by $\chi_I =: c^I_J \phi_J$ and $\phi_I =: d^I_J \chi_J$ and let the structure 'functions' of the BRST set as derived above be denoted by $U$. Then

\begin{align*}
&M \circ \chi_I, N \circ \chi_J \\
= & \ [M \circ c^K_I \phi_K, N \circ c^L_J \phi_L] \\
= & \ M c^K_I \circ [\phi_K, N \circ c^L_J \phi_L] + [Mc^K_I, N \circ \chi_J] \circ \phi_K \\
= & \ M c^K_I \circ [\phi_K, N c^L_J] \circ \phi_L + Mc^K_I \circ N c^L_J \circ [\phi_K, \circ \phi_L] + [Mc^K_I, N \circ \chi_J] \circ \phi_K \\
= & \ (Mc^K_I \circ [\phi_K, N c^L_J] + U[Mc^K_I, N c^L_J]_{KL} P + [Mc^P_I, N \circ \chi_J]) \circ \phi_P
\end{align*}
\[(Mc^K_I \circ [\phi_K, Nc^P_J] + U[Mc^K_I, Nc^L_J]_K + [Mc^P_I, N \circ \chi_J])d^Q_P \circ \chi_\delta(A.12)\]

Finally, in order to see what problems arise with the canonical set we choose D=1 and compute the commutator between 2 scalar constraints. The result is
\[\[M \circ H, N \circ H\] = \frac{1}{2}(MN' - M'N) \circ E^1(E^3B^1 \phi_2 - E^2B^1 \phi_3), \quad (A.13)\]

where the two constraints of the BRST-set on the rhs must be expressed, via the above transformation, in terms of the canonical set. Clearly, the structure 'functions' are then not polynomial, not even analytic, in the momenta although the constraints stand always on the right. The classical limit of the operator on the rhs of (A.13) is the expected one: \((E^1)^2H_x\) (see eqs. (2.18)).

We finally show the weak commutativity of P and Q with the constraint functionals of the canonical set.

As the BRST-set of constraints is at most linear in the momenta, the annihilation of Q by the (smeared) BRST-constraints therefore implies that it strongly commutes with these constraints. So far for Q. For P we get
\[\[P, M \circ \phi_1\] = 0 \quad (A.14)\]
trivially, since no \(A_1\) but only \(E^1\) is contained in both operators. Furthermore
\[\[P, M \circ \phi_2\] = \int_\Sigma dx \int_\Sigma dy M(y)(B^1(x)^2[E^1(x), B^2(y)] \frac{2}{B^1(y)}E^1(y) + [B^1(x)^2, E^2(y)]E^1(x)) = \int_\Sigma dx M(2B^1A_2E^1 - 2A_2B^1E^1) = 0, \quad (A.15)\]
and similarly
\[\[P, M \circ \phi_3\] = 0. \quad (A.16)\]
So the BRST-constraint functionals even strongly commute with both observables. Proving weak commutativity with all the canonical constraint functionals directly, only using the CCR, is hard work. We will give a more elegant indirect proof, based on the following argument which exploits the existence of the transformation (A.10) and (A.11) between the two sets of
constraint functionals. Adopting the same notation as above we have for any observable $O$ with respect to the BRST-set

$$
[O, N \circ \chi_I] = N c_I^j \circ [O, \phi_J] + [O, N c_J^K] \circ \phi_K
$$

(A.17)

where the structure functions of $O$ are defined by

$$
[O, M \circ \phi_I] := V[I]^I_J M \circ \phi_J := \int_\Sigma dx \int_\Sigma dy V[I]^I_J(x,y)M(x)\phi_J(y),
$$

(A.18)

and vanish for our observables $P$ and $Q$. This completes the proof.

For completeness sake we will display the Poisson-bracket of the observable $Q$ with the scalar constraint:

$$
\{M \circ \mathcal{H}, Q\} = \int dx M \left[ \frac{2}{A} f((B^2 A_3' - B^3 A_2')G - (B^2 A_2 + B^3 A_3)\xi)) + \frac{2}{(A')} (A'E^1 + (A - 2)(E^1)'((B^2 A_3' - B^3 A_2')G - (B^2 A_2 + B^3 A_3)A)) \right]
$$

(A.19)

which shows that one never would have found $Q$ by using the original definition of an observable, that is $\{Q, M \circ \chi_I\} \approx 0$. Furthermore it is obvious that the canonical set is completely inappropriate for the construction of the scalar product as presented in section (4.4), because with these complicated first order structure functions its BRST-extension is probably 'long'.

**References**

[1] A. Ashtekar, Phys. Rev. D36 (1987)1587

[2] A. Ashtekar, New Perspectives in Canonical Gravity (Monographs and Textbooks in Physical Science, Bibliopolis, Napoli, 1988)

[3] A. Ashtekar, Lectures on Non-Perturbative Canonical Gravity (World Scientific, Singapore, 1991)
[4] C. Rovelli, Class. Quantum Grav. 8(1991)1613

[5] C.J. Isham, in: Recent Aspects of Quantum Fields (Proceedings Schladming, Austria 1991) ed. by H. Mitter and H. Gausterer (Springer-Verlag, Berlin, 1991) p. 123

[6] A. Ashtekar and J. Samuel, Class. Quantum Grav. 8 (1991)2191.

[7] I. Bengtsson, Class. Quantum Grav. 7(1990)27;
R. Benguria, P. Cordero and C. Teitelboim, Nucl. Phys. B122(1977)61

[8] M. Henneaux, Phys. Rep. 126(1985)1

[9] Y. Choquet-Bruhat, in: Relativity, Groups and Topology II, ed. by B.S. DeWitt and R. Stora, (North Holland, Amsterdam, 1984)p. 739

[10] A. Ashtekar et al., Phys.Rev. D40 (1989)2572.

[11] N. Woodhouse, Geometric Quantization (Clarendon Press, Oxford, 1980)

[12] P.A.M. Dirac, Lectures on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, New York, 1964)

[13] J.L. Friedman and I. Jack, Phys. Rev. D37(1988)3495

[14] T. Jacobson and L. Smolin, Nucl. Phys. B299(1988)295

[15] I. Bengtsson, Class. Quantum Grav. 8(1991)1847
M. Varadarajan, Class. Quantum Grav. 8(1991)L235

[16] R. Capovilla, T. Jacobson and J. Dell, Phys. Rev. Lett. 63(1989)2325;
R. Capovilla et al., Class. Quantum Grav. 8(1991)41;
R. Capovilla et al., Class. Quantum Grav. 8(1991)57

[17] T.Thiemann, diplom thesis, RWTH Aachen, January 1992, and to be published.

[18] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time ( Cambridge University Press, Cambridge, 1973)

[19] T. Fukuyama and K. Kamimura, Phys. Rev. D41(1989)1105