Six-loop Konishi anomalous dimension from the Y-system

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We compute the Konishi anomalous dimension perturbatively up to six loop using the finite set of functional equations (FiNLIE) derived recently in [1]. The recursive procedure can be in principle extended to higher loops, the only obstacle being the complexity of the computation.

I. INTRODUCTION

Using integrability in conjunction with the AdS/CFT correspondence led to some of the main achievements in theoretical high energy physics of the last decade. One of them is determining the spectrum of the anomalous dimensions of the planar $\mathcal{N} = 4$ SUSY gauge theory, or equivalently finding the spectrum of free string theory in $\text{AdS}_5 \times S^5$ background. For a recent review on the subject see [2]. For operators with large charges, the anomalous dimension is determined by a set of Bethe Ansatz equations, [2], while for operators with small charges, the information on the spectrum is encapsulated into an infinite set of functional equations derivable from the thermodynamic Bethe Ansatz (TBA) equations [3] and known under the name of $Y$-system. For the AdS/CFT integrable system, the $Y$-system was conjectured in [4] and derived from the TBA equations in [5]. The anomalous dimension of the Konishi operator, the shortest operator not protected by supersymmetry, is used as a testing ground for these equations, both for analytical and numerical computations. In particular, in perturbation four [4] and five loop [6, 7] corrections to the Bethe Ansatz were computed using the $Y$-system, and they were found to coincide both with the corresponding Lüscher corrections [8, 9] and with the four- [10] and five-loop [11] perturbative gauge theory computations. At strong coupling, the result of the extrapolation to short, Konishi-like operators [12] agrees both with the numerical results [13, 14] and with the string predictions [15].

Recently [1, 16], the $Y$-system was reformulated in terms of a finite closed set of functional equations. In the present work, we set up a recursive procedure to solve in perturbation the equations of [1] and we perform the explicit computation up to six loops. The procedure can be in principle continued to higher loops, in particular to the double-wrapping order at eight loops. Double wrapping was already attained for the ground state energy in the twisted AdS/CFT [17], however our case is significantly more complicated because of necessity to account the displacement of the Bethe roots.

II. THE SET OF FUNCTIONAL EQUATIONS

The wrapping corrections are encoded in a finite set of functional equations [1] which were derived from the AdS/CFT Y-system by solving the Hirota equation in the semi-infinite bands of the $T$-hook defined in [4]. The different solutions are then glued together using analyticity constraints which reflect the physical properties the $Y$-system has to satisfy. We give below a brief summary of this set of functional equations.

a. Input data. The functional equations depend on a specific operator through the position of the Bethe roots $u_j$, via the objects $B_{(\pm)} = \prod_{j=1}^M \sqrt{x - \hat{x}^j}$, $\mathcal{R}_{(\pm)} = \prod_{j=1}^M \sqrt{x - \hat{x}^j}$ and the Baxter polynomial $Q(x) = (-1)^M B_{(\pm)} \mathcal{R}_{(\pm)}$. Here we use the conventions $j_{(a)} = f(u+ia/2)$, $\mathcal{F}(x) = f(\pm [x])$, and $x(u) = u/2g+i\sqrt{1-u^2/4g^2}$ is the Zhukovsky variable in the so-called mirror regime, with a cut on $Z \equiv (-\infty, -2g] \cup [2g, \infty)$. For the Zhukovsky variable in the physical regime, with a branch cut on the interval $Z \equiv [-2g, 2g]$, we use the notation $\hat{x}(u) = u/2g + \sqrt{u/2g - 1}/\sqrt{u/2g + 1}$. By convention, we denote with a hat the quantities which depend on $\hat{x}(u)$, if we want to emphasize the position of the branch cut.

For the Konishi operator there are two magnons, $M = 2$, with $u_1 = -u_2 = 1/\sqrt{2} + O(g^2)$ and $Q(u) = u^2 - u^2_1$.

b. Parameterization of the $T$- and $q$-functions.

When considering a state from the $sl(2)$ sector, as it is the case for Konishi, the $Y$-system can be determined from only three functions: two real-valued densities with a finite support on $\hat{x}$, $\rho$ and $\rho_2$, and a complex-valued function $U$ analytic in the upper half-plane. A relatively simple formulation of the set of functional equations can be obtained using $T$-functions in two different gauges denoted $T_{a,s}$ and $\tilde{T}_{a,s}$. The first gauge gives a simple solution of the Hirota equation in the right band $s \geq a$

$$T_{0,s} = 1, \quad T_{1,s} = s + \mathcal{K} \ast \rho, \quad T_{2,s} = \mathcal{T}_{1,s}^{1+s} \mathcal{T}_{1,s}^{-s},$$

with $(\mathcal{K} \ast f)(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dv \frac{f(v)}{(v-u)}$ and $\mathcal{K} \ast \mathcal{K} = \mathcal{K}[\mathcal{K}] - \mathcal{K}[\mathcal{K}]$. The second gauge gives a solution of the
Hirota equation in the upper band of the $T$-hook, $a \geq |s|$, 
\[ \mathcal{F}_{a,2} = q_{[a]} q_{[-a]}, \quad \mathcal{F}_{a,-1} = (U_{a}|a| U_{-a})^2 \mathcal{F}_{a,1}, \]  
(2) 
\[ \mathcal{F}_{0,1} = q_{12} q_{12} + q_{12} q_{12} + q_{12} q_{12} + q_{12} q_{12}, \]  
\[ \mathcal{F}_{0,0} = q_{12} q_{12} + q_{12} q_{12} - q_{12} q_{12} - q_{12} q_{12}, \]  
We will also use the combinations $\mathcal{F}_{c}^{\alpha}$, defined by (2) for $a < |s|$. The q-functions related among themselves by the Plucker relations [1] are determined by $\rho_2$ and $U$ as follows

\[
q_1 = 1, \quad q_2 = -iu + K \ast \rho_2 - K \ast \bar{W}_{pv}, \tag{3a}
\]
\[
q_{12} = (u - u_1 - \alpha) (u + u_1 + \bar{\alpha}) \equiv Q + \delta q_{12}, \tag{3b}
\]
\[
\{q_{13}, q_{23} \equiv q_{14}, q_{24} \} = \sum_{k=0}^{2k+1} \left( U^{2}_{(1,q_2^2)} ; q_{12} |^{2k+1} \right), \tag{3c}
\]
\[
q_{34} q_{12} = q_{13} q_{24} - q_{12} q_{23}, \quad q_{24} q_{12} = q_{22} - q_{22}^{2}, \tag{3d}
\]
\[
W_{\alpha} = q_{3} + q_{3}^{2}, \quad W_{\alpha} = W_{0}. \tag{3f}
\]
Above, $\bar{W}_{pv} = \frac{1}{2}(\bar{W}^{[+]} + \bar{W}^{[-]})$. The definition of $\rho_2(u)$ differs slightly from the one in [1], so that here $\rho_2(u)$ is of the square root type, in the sense that $\rho_2(u) \sqrt{4g^2 - u^2}$ is analytic in the vicinity of the real axis.

c. Auxiliary integer equations. In the intermediate steps of the computations one needs to compute 3 quantities, $Y_{11}, Y_{22}, h$. $Y_{11}, Y_{22}$ are determined from the following relations, considered at $\text{Im}(u) > 0$:

\[
\ln \frac{Y_{11}}{Y_{22}^{2}} \mathcal{F}_{0,0} \left[ \frac{\mathcal{F}_{2,1}}{\mathcal{T}_{2,1}} \right]^{2} = 2iK \ast \text{Im} \left[ \frac{\mathcal{F}_{0,0}^{2}}{\mathcal{T}_{0,1}^{2}} \right], \tag{4a}
\]
\[
\frac{1}{\bar{x} - \bar{x}^{-1} \ln} \frac{1}{Y_{11} Y_{22}} \mathcal{F}_{0,0} = 2K \ast \text{Re} \left[ \frac{1}{\bar{x} - \bar{x}^{-1} \ln} \mathcal{F}_{0,0} \right]. \tag{4b}
\]

The function $\bar{h}$ is found from equations (5.38) and (6.14) in [1]. Only its large-volume asymptotic solution (11) is needed for the six-loop computation.

d. Equations for $\rho_2, U$. After $Y_{11}, Y_{22}, h$, are found, the set of functional equations can be closed by finding $\rho_2$ from

\[
1 + Y_{22}^{2} = \mathcal{F}_{c}^{\alpha} \mathcal{F}_{0,1}^{2}, \quad 1 + Y_{22}^{2} = \mathcal{F}_{c}^{\alpha} \mathcal{F}_{0,1}^{2}, \tag{5}
\]
e. Supplementary constraints. Let us now emphasize the role of the Bethe roots. Bethe roots appear as the zeroes of the following functions:

\[
\mathcal{F}_{1,0}^{u_j} = 0, \text{ fixes the value of } \alpha, \tag{7a}
\]
\[
\mathcal{F}_{1,0}^{u_j} = 0, \text{ fixes the ambiguity in (5).} \tag{7b}
\]
Using them and the Hirota equation $\mathcal{F}_{1,0}^{u_j} = \mathcal{F}_{0,0}^{u_j} + (U^t \bar{U}) \mathcal{F}_{0,1}^{u_j}$, we conclude that $\mathcal{F}_{0,0}^{u_j}$ should have a double zero at Bethe root which appears to be a double zero of $\mathcal{F}_{0,0}^{u_j}$. The overall normalization of $U$ is not fixed by (6) and should be defined from

\[
U \bar{U} = \frac{1}{\mathcal{F}_{0,0}^{u_j}} \left( 1 + Y_{11} Y_{22} \right) \mathcal{F}_{0,1}^{u_j} \quad u \in \bar{Z}. \tag{8}
\]

f. Exact Bethe equations. The set of equations above has a solution for a range of values of $u_1$. To get the correct answer for the energy, one has to insert the value of $u_1$ which is fixed by the exact Bethe equation, which can be written [1] in the form

\[
\frac{\bar{h}^{-2}}{h^2} \left( Y_{22}^{2} \mathcal{T}_{1,2}^{0,2} \right)^{2} = 1 \quad \text{at } u = u_j. \tag{9}
\]

Asymptotic solution. The equations listed above depend on the parameter $L$ which sets the large $u$ behavior of $\bar{h}$. In the large volume limit $L \rightarrow \infty$, the $Y$-system, and hence the functional equations above, can be solved explicitly [4]. All the functions $q_{12}$, with the exception of $q_{12}$, and $q_3$ and $q_4$ are suppressed at least by a factor $\bar{x} \bar{L}$ with respect to $q_1$ and $q_2$, so they are zero in the asymptotic expressions of $\mathcal{F}_{a,s}$ and $q_{12}$ are in this limit given by $(q_{12})_{as} = -iu + K \ast \rho_2, \quad (q_{12})_{as} = Q$. The asymptotic values of $\rho_2$ are

\[
\rho_{as} = 4 \sqrt{4g^2 - u^2}, \quad \rho_{as} = -4 \sqrt{4g^2 - u^2} \tag{10}
\]

\[
\text{where } E_{as} \text{ is defined in (21). One can check that both the equations (4) and (5) are satisfied by } (Y_{11} Y_{22})_{as} = \left( \frac{B(-\bar{L})}{B(\bar{L})} \right) \text{ and } \left( \frac{Y_{22}}{Y_{22}^{2}} \right)_{as} = \left( \frac{B_{as}^{2} + 2}{\bar{B}_{as}^{2} - 2} \right) \frac{Q + B_{as}^{2} x^{-2}}{Q - B_{as}^{2} x^{-2}}. \tag{11}
\]

The asymptotic values of $\bar{h}$ and $\bar{U}$ are given by

\[
\frac{\bar{h} \bar{x}^{+1} \bar{h}_{as}}{\Lambda_{h}} = \frac{\bar{h} \bar{x}^{+1} \bar{h}_{as}}{\Lambda_{h}} \frac{B_{as}^{2} + 2}{\bar{B}_{as}^{2} - 2} \text{ and } \frac{\bar{h} \bar{x}^{+1} \bar{h}_{as}}{\Lambda_{h}} \frac{Q + B_{as}^{2} x^{-2}}{Q - B_{as}^{2} x^{-2}}. \tag{12}
\]

where for $\chi(x,y)$ one can use the BES perturbative expansion [18]. The shift operator $D$ is defined such that $DF = f^D = f^{+}$. The overall normalization $\Lambda_\gamma$ is irrelevant, cf. (9) and (6), whereas $\Lambda_U = \frac{3}{2} \sqrt{E_{as} (E_{as} - 2)} \times \exp(6 g^2 (-1 + 4g^2 - 28g^4 + 186g^4 - 24g^6 (3c_3 + 5c_5)))$ is fixed by (8). The constant $\gamma$ depends on the regularization scheme for diverging sum. We use the prescription $\frac{1}{1 - \frac{1}{u} u} = -i \psi (-iu)$ for which $\gamma$ is the Euler constant.

Although for Konishi-like operators $L$ is not large, the asymptotic solution is still valid up to at least $L$ loops.
(four loops for Konishi), so the weak coupling is effectively a large volume limit. We use as a constraint that the full solution should reduce to the asymptotic one at weak coupling. In the following, we will continue to call ‘asymptotic’ the above quantities evaluated at the exact position of the Bethe roots $u_j$, determined by equations (9). These quantities will therefore incorporate part of the wrapping corrections via the corrections to the Bethe roots.

### III. WEAK COUPLING EXPANSION

In order to solve perturbatively the functional equations, our strategy is to subtract from the exact equations the asymptotic ones and to use the fact that the deviations of the $T$ functions from the asymptotic values are small. The resulting equations depend on the ratio of the exact and the asymptotic values, so we denote $(T)_r = T/(T)_{as}$. The following quantities enter the functional equations

$$H = \ln\left(\mathcal{F}_{1,0}/\mathcal{F}_{0,0}\right), \quad r = \ln\left(\mathcal{F}_{1,1}/\mathcal{F}_{1,1}\right), \quad r_u = \ln\left(\hat{q}_2^- - \hat{q}_2^+ \right). \quad (12)$$

All these quantities are small in perturbation, and this will allow performing the expansion of the functional equations. For example, up to seven loops, one has

$$H \approx \frac{\delta q_{12}^-}{Q^2} - \frac{\delta q_{12}^+}{Q^2} + \frac{(q_{24}^+ + 2q_{14}q_{24} + q_{43}q_2)}{(Q^2)2Q^2} \frac{U^2}{(Q^2)^2}. \quad (13)$$

The quantities $U$ and $q_2$ are determined asymptotically from (3a) and (11), and at the leading order they are equal to

$$U^2 = -\frac{2g^4}{u^2} + \ldots, \quad q_2 = -i u - \frac{i}{3} \frac{1}{u} + \ldots. \quad (14)$$

As for the $q$-functions $q_{13}, q_{23} = q_{14}, q_{24}$, they are given by the sums in (3c) and in perturbation they have an array of equidistant poles which will be responsible for the appearance of the zeta functions in the final answer. For example, at the leading order, we have

$$q_{13} = 18 g^4 (-i u + Q \psi^{(1)}(-i u + 1/2)), \quad (15)$$

where $\psi^{(n)}$ denotes the $n$-th derivative of the digamma function. The functions $q_3$ and $q_4$ have a similar structure. $q_{34}$ is given by a double sum and it contributes only at eight loops and higher. A priori, the infinite sum in (3c) creates also infinitely many poles at positions of the shifted Bethe roots. However, because of the equality $(U^+/U^-)^2 = -Q^{[+2]}/Q^{[-2]}$ which holds at the Bethe root at first four nontrivial orders and which is just the asymptotic Bethe equation, the poles in $q_{13}$ cancel out pairwise (except for the first one, which is cancelled by an overall factor $Q$). The cancellation mechanism still holds for $q_{14}$ and $q_{24}$ because $(\hat{q}_2^+ + \hat{q}_2^-)_{as} = (\mathcal{F}_{1,1})_{as}$ and $\mathcal{F}_{1,1}(u_1) = 0$. At least up to seven loops, the $q$-functions are given by linear combination of the multiple Hurwitz zeta functions, which we define by

$$\eta_{a_1,a_2,\ldots,a_n}(u) = \sum_{k=0}^{\infty} \frac{1}{u + ik} a_1^{a_2^{\ldots} a_n(u + i(k + 1))},$$

$$\eta_a(u) = \frac{i^a}{(a - 1)!} \psi^{(a-1)}(-i u), \quad a \geq 1, \quad (16)$$

with coefficients which are rational functions of $u$.

The algorithm for computing the perturbative computation is summarized in figure 1. The interior loop determines the densities $\rho$ and $\rho_2$, or rather their variations with respect to the asymptotic values, $\delta \rho$ and $\delta \rho_2$, from equations (4), analytically continued to $u \in \mathbb{Z}$, and from (5). In order to preserve the square root structure of the two densities, we need to make a rescaling $z = u/2g$. When performing various integrations, we encounter two different situations. In the first, the integration is (parallel to) the real axis and we deform the contour to pass just below it and to avoid the possible singularities on the real axis. This allows us to perform uniform expansion in $g$. In the second situation the Cauchy kernel $K$ acts on a function with a finite support, like $\delta \rho$ and $\delta \rho_2$, then it can be expanded in terms of moments:

$$K[\rho] \ast \rho = \sum_{n \geq 0} \frac{i(2g)^{2n+1}}{(u + is/2)^{2n+1}} \int_0^1 \frac{dz}{2\pi} z^{2n} \rho(z), \quad (17)$$

except for $s = 0$, where $K[\rho_{\pm 2}] \ast \rho = \pm \rho/2 + \mathcal{K} \ast \rho$, the slash meaning principal value. The equations (4) and (5) finally reduce to the linear system

$$\begin{pmatrix} 2 & 12 & 13 \\ 1 & 3 & v_1 \\
\end{pmatrix} \begin{pmatrix} v_1 \\ g^2 v_2 \end{pmatrix} = g^9 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (18)$$

where $C_1$ and $C_2$ are Taylor series in $g^2$ and $z$ is known from the previous orders of perturbative expansion (at the leading order in $g$ they are constants) and

$$v_1 = \frac{\delta \rho}{\sqrt{1 - z^2}}, \quad v_2 = \frac{(1 - 2z^2)\delta \rho_2}{\sqrt{1 - z^2}} - 4iz \mathcal{K} \ast \delta \rho_2. \quad (19)$$

By definition of $\rho$ (and $\rho_2$), we know that $\rho(2gz)/\sqrt{1 - z^2}$ is analytic in the vicinity of the real axis and hence can be Taylor-expanded. At a given order in $g$, this means that $\delta \rho$ and $\delta \rho_2$ are polynomial in $z$ times $\sqrt{1 - z^2}$. Using this information, the equation (18) gives, at leading order,

$$\delta \rho_2 = g^7 (A + B z^2) \sqrt{1 - z^2}, \quad \delta \rho = g^9 C \sqrt{1 - z^2}, \quad (20)$$

FIG. 1. Structure of the perturbative computation.
with \(6A = C_1 - 2C_2, \quad 2C = -C_1 + 4C_2\). The value of \(B\)
is unconstrained by (18), and it is fixed by (7b).

The result for the energy. The energy can be computed [1] from the behavior at large \(u\) of \(\ln Y_{1,2,2}\), \(\ln Y_{1,1}\). Isolating the asymptotic and the wrapping part in the above expression, \(E = E_{\text{as}} + E_{\text{wrap}}\), one obtains

\[
E_{\text{as}} = 2 - 8g \Im \left( \frac{1}{w_1} \right), \quad E_{\text{wrap}} = \int_{R-\infty}^{R-10} \frac{-H(u)du}{\pi \sqrt{1 - 4(\frac{u}{4})^2}}.
\]

The Asymptotic Bethe Ansatz [18] predicts up to 6 loops:

\[
E_{\text{BAE}} = E_{\text{as}}(u_1 \to u_{1,BAE}) = 2 + 12g^2 - 48g^4 + 336g^6 - (2820 + 288\zeta_5)g^8 + (26508 + 4320\zeta_3 + 2880\zeta_5)g^{10} - (269148 + 55296\zeta_3 + 44064\zeta_4 + 30240\zeta_7)g^{12}.
\]

At finite volume \((L = 2)\) for the Konishi operator, the energy receives corrections both from \(E_{\text{as}}\) through the correction of the position of Bethe roots \(u_1\), and from \(E_{\text{wrap}}\). The corrections to the Bethe equations leading to displacement of the Bethe roots start at 5 loops, where they are due to correction of \(Y_{2,2}\) only. At 6 loops one should take into account the first corrections to \(\rho_2\), whereas effects from correction to \(\rho\) and \(\hat{h}\) are delayed at least up to 7 loops. \(E_{\text{wrap}}\) is non-zero starting from 4 loops. The single-wrapping corrections, up to seven loop, follow from computations within interior loop in figure 1, whereas exterior loop has to be run only once, to find the explicit analytic expression (11). For \(H\) one can use the approximation (13). The double wrapping effects, in particular correction to \((U)_{\text{as}}\), are important starting from 8 loops. We have performed the explicit perturbative expansions discussed in previous sections and computed \(\delta E = E - E_{\text{BAE}}\) up to six loops, i.e. up to \(g^{12}\) term. Intermediate expressions are too bulky to be presented here. We summarized them in the Mathematica notebook file [19]. They contain \(\eta\)-functions (16) and their residues at the Bethe root. However, the final expression is significantly simpler and is given in terms of zeta-functions:

\[
\delta E_{4k5 \, \text{loop}} = (324 + 864\zeta_3 - 1440\zeta_5)g^8 + (-11340 + 2592\zeta_3 - 11520\zeta_5 - 5184\zeta_7^2 + 30240\zeta_7)g^{10},
\]

\[
\delta E_{6 \, \text{loop}} = (261468 - 207360\zeta_3 - 20736\zeta_5 + 156384\zeta_5 + 155520\zeta_3\zeta_5 + 105840\zeta_7 - 849888\zeta_9)g^{12}.
\]

At four and five loops we reproduced the already known answers [8, 9]. Our final expression for the Energy of the Konishi operator is

\[
E = 2 + 12g^2 - 48g^4 + 336g^6 + (-2496 + 576\zeta_3)(-1440\zeta_5)g^8 + (15168 + 6912\zeta_3 - 5184\zeta_7^2 - 8640\zeta_5 + 30240\zeta_7^2)(-7680 - 26256\zeta_3 - 20736\zeta_7^2 + 112320\zeta_5 + 155520\zeta_3\zeta_5 + 75600\zeta_7 - 489888\zeta_9)g^{12}.
\]

We were informed that Z. Bajnok and R. Janik have obtained [20] six and seven-loop corrections using Lüscher’s method; our result coincides with their six loop result. Also, fitting the known numerical results [13, 14, 21] with a diagonal Padé approximant, we were able to fix the 6-loop energy with 5% confidence. Our analytic result is compatible with this numerical estimation.

IV. Conclusion

We have computed the wrapping corrections of the Konishi operator up to 6-loop order using the functional equations proposed in [1]. We have adjusted the structure of the functional equations for a systematic perturbative expansion and we hope to be able to apply our methods to reach double-wrapping orders for the Konishi states. An interesting question to explore is whether the cancellation of poles at Bethe roots observed when computing \(q_{ij}\) holds at any order and if it can be used as a regularity condition implying the exact Bethe equation. Another question is what type of functions appear in the final answer. So far the expression for the energy is reducing to Euler-Zagier sums and we believe that it will be always so.

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