ON VECTOR-VALUED AUTOMORPHIC FORMS ON BOUNDED SYMMETRIC DOMAINS

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Abstract. We prove a spanning result for vector-valued Poincaré series on a bounded symmetric domain. We associate a sequence of holomorphic automorphic forms to a submanifold of the domain. When the domain is the unit ball in \( \mathbb{C}^n \), we provide estimates for the norms of these automorphic forms and we find asymptotics of the norms (as the weight goes to infinity) for a class of totally real submanifolds. We give an example of a CR submanifold of the ball, for which the norms of the associated automorphic forms have a different asymptotic behaviour.

MSC 2010: 32N15, 53C99

Keywords: holomorphic automorphic forms, Poincaré series, spanning set, domain, canonical bundle, Bergman kernel, complex hyperbolic space, submanifold, asymptotics.

1. Introduction

Let \( D \) be a bounded symmetric domain (in \( \mathbb{C}^n \), for \( n \geq 1 \)). Suppose \( \Gamma \) is a cocompact discrete subgroup of \( Aut(D) \). Let \( k \) be a positive integer. A holomorphic function \( f : D \rightarrow \mathbb{C} \) is called a holomorphic automorphic form of weight \( k \) for \( \Gamma \) if \( f(\gamma z)J(\gamma, z)^k = f(z) \) for all \( \gamma \in \Gamma \), \( z \in D \). Here \( J(\gamma, z) \) denotes the determinant of the Jacobi matrix of \( \gamma \) at \( z \). Let \( m \) be a positive integer and let \( \rho : \Gamma \rightarrow GL(m, \mathbb{C}) \) be a unitary representation of \( \Gamma \). A holomorphic function \( F : D \rightarrow \mathbb{C}^m \) is called a holomorphic \( \mathbb{C}^m \)-valued automorphic form of weight \( k \) (for the pair \((\Gamma, \rho)) \) if \( F(\gamma z) = \rho(\gamma)F(z)J(\gamma, z)^{-k} \) for all \( \gamma \in \Gamma \), \( z \in D \). Holomorphic automorphic forms correspond to holomorphic sections of \( L^{\otimes k} \), where \( L \) is the canonical bundle on \( M = \Gamma\backslash D \), and \( \mathbb{C}^m \)-valued holomorphic automorphic forms correspond to holomorphic sections of \( E_{\rho} \otimes L^{\otimes k} \), where \( E_{\rho} \rightarrow M \) is the flat vector bundle defined by \( \rho \).

The theory of automorphic forms is a vast subject that has strong interaction with many areas of mathematics, including representation theory, number theory, semiclassical analysis and quantization. One connection between automorphic forms and quantization is as follows: \( M \) is a Kähler manifold, \( L \) is a quantum line bundle, \( \frac{1}{\hbar} \) is interpreted as \( \hbar \) (the Planck constant), and the space of holomorphic automorphic forms, with the Petersson inner product, is isomorphic to the Hilbert space \( H^0(M, L^{\otimes k}) \) used in quantization\[4\]. Berezin-Toeplitz quantization or Kähler quantization is usually studied for \( \mathbb{C} \)-valued observables. Extending the theory to \( \mathbb{C}^m \)-valued functions on \( D \) is a non-trivial task which is interesting from the mathematical point of view and physically meaningful (see e.g. work by S.T. Ali and M. Englis [1] on domains in \( \mathbb{C}^n \)).

There are many different kinds of automorphic/modular forms, and generalizations. Vector-valued automorphic forms are ubiquitous and go back to classical works of Borel,
Selberg and others (see, for example [8]). Applications include work of R. Borcherds on singular Howe correspondence, work of S. Kudla on arithmetic cycle $s$, physics-related work by T. Gannon, G. Mason and others. The space of Jacobi forms is isomorphic to a space of vector-valued modular forms. Various kinds of vector-valued forms for $G = SU(n, 1)$ (i.e. when $D$ is the $n$-dimensional complex hyperbolic space, or, equivalently, the open unit ball in $\mathbb{C}^n$ with the complex hyperbolic metric: $D = \mathbb{B}^n \simeq SU(n, 1)/SU(n) \times U(1))$ have been studied, in particular, in recent papers by E. Freitag, G. van der Geer and others [11], [18], in work by Kato including [25], and in work by Kojima including [29]. It is well known that modular forms appear in generating functions for arithmetic or algebraic objects in Calabi-Yau varieties. It would be interesting to see if Picard modular forms could play a similar role.

Poincaré series is a standard and powerful tool that is used in automorphic forms, spectral theory, complex analysis, Teichmüller theory, algebraic geometry, and other areas. In [14, 15] T. Barron (Foth) studied automorphic forms for compact smooth $M = \Gamma \backslash D$ (in [15] $D = \mathbb{B}^n$), constructing explicitly the automorphic form $f_p$ ($p \in D$) with the property $(g, f_p) = g(p)$ for any other holomorphic automorphic form $g$. Here $(.,.)$ denotes the Petersson inner product. Such $f_p$ is constructed via Poincaré series and is related to the (weighted) Bergman kernel and to the concept of a coherent state. Choosing a $q$-form on a $q$-dimensional submanifold, and integrating $f_p$, one can get automorphic forms associated to submanifolds of $D$. In this paper we extend this framework to vector-valued holomorphic automorphic forms. In Section 3 we prove that sufficiently many of these vector-valued Poincaré series span the space of vector-valued holomorphic automorphic forms - Theorem 3.3.

There are somewhat different, but closely related results in literature: it is known that $\mathbb{C}$-valued Poincaré series of polynomials in $z_1,...,z_n$ span the space of holomorphic automorphic forms on a bounded symmetric domain (for sufficiently large weights) [6, 14, 42]. In David Bell’s thesis [6] it is stated that similar results also hold for vector-valued automorphic forms on classical domains and it is explained how the proofs for $\mathbb{C}$-valued case can be extended to the vector-valued case.

To give general context to Section 4 we observe that associating an automorphic form or, more generally, a section of a vector bundle, to a submanifold of a Kähler manifold is an idea that is used in many contexts. In particular, relative Poincaré series can be associated to closed geodesics on a hyperbolic Riemann surface [26, 27]. S. Katok and T. Foth (Barron) generalized this construction from compact Riemann surfaces of genus $g \geq 1$ to ball quotients in [16, 17] (where they addressed the spanning problem), and more recently T.B. addressed the non-vanishing question in [5]. In [31, 41] the submanifold is a closed geodesic or, more generally, a totally geodesic submanifold. To mention a somewhat different kind of such technique, there is a way to associate a section of a line bundle to a Bohr-Sommerfeld Lagrangian submanifold, which is used in semiclassical analysis and symplectic geometry (see, in particular, [9, 10, 13, 20, 24, 35]). In [21, 23] sections of vector bundles are associated to isotropic submanifolds.

Here, in Section 4 we take advantage of the fact that the Kähler manifold is $M = \Gamma \backslash D$, the holomorphic sections of the vector bundle on $M$ correspond to holomorphic vector-valued functions on $D$, and we associate an automorphic form to a submanifold of a fundamental domain of $\Gamma$ in $D$, and not to a submanifold of $M$ (see Remark 4.10). In the case when the domain is the unit ball in $\mathbb{C}^n$, we provide asymptotic (as the weight goes to infinity) statements about the inner products - Theorems 4.6, 4.9. In particular, in Theorem 4.6(ii)
we show that if two submanifolds are at a positive distance from each other, then the inner product of the associated automorphic forms decreases rapidly as the weight goes to infinity.

In Theorem 4.9(ii) we show, in particular, that for a class of totally real submanifolds, as $k \to \infty$, the square of the norm of $\Theta^{(j,k)}_X$ grows as a positive constant times $k^{n-\frac{q}{2}}$, where $X$ is such a submanifold, $\Theta^{(j,k)}_X$ is one of the $m$ $\mathbb{C}^m$-valued Poincaré series associated to $X$ ($j \in \{1,2,\ldots,m\}$), and $q$ is the real dimension of $X$. We work out several examples. In Example 4.16 we estimate the asymptotics of the norms for a 3-dimensional submanifold which is CR and not totally real, and we show that the leading term in the asymptotics the square of the norm of the associated Poincaré series is not $\text{const} \cdot k^{n-\frac{q}{2}}$.

This paper contains results from the Ph.D. thesis of N.A. [2] written under the supervision of T.B.

Acknowledgments. We are thankful to A. Dhillon, Y. Karshon, M. Pinsonnault, E. Schippers, A. Uribe and N. Yui for related discussions. We acknowledge the referee’s efforts.

2. Preliminaries

Let $D = G/K \subset \mathbb{C}^n$, for $n \geq 1$, be a bounded symmetric domain ($G = \text{Aut}(D)$ is a real semisimple Lie group that acts transitively on $D$, $K$ a maximal compact subgroup of $G$). Denote by $z_1,\ldots,z_n$ the complex coordinates. Also denote $z_j = x_j + iy_j$ ($x_j,y_j \in \mathbb{R}$, $1 \leq j \leq n$) and denote the Euclidean volume form by

$$dV_e = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n.$$ 

Let $\mathcal{K}(.,.)$ be the Bergman kernel for $D$. It has the reproducing property:

$$f(z) = \int_D f(w)\mathcal{K}(z,w)dV_e(w),$$ 

$z \in D$, for all functions $f$ that are holomorphic on $D$ and such that $\int_D |f(z)|^2dV_e(z) < \infty$.

Also $\mathcal{K}(z,w) = \overline{\mathcal{K}(w,z)}$ for $z,w \in D$, and

$$(1) \quad J(\gamma,z)J(\gamma,w)\mathcal{K}(\gamma z,\gamma w) = \mathcal{K}(z,w)$$

for $z,w \in D$, $\gamma \in G$, where $J(\gamma,z)$ is the complex Jacobian of the transformation $D \to D$ at $z$ defined by $\gamma$. The $(1,1)$-form $\omega = i\partial \bar{\partial} \log K(z,z)$ is a $G$-invariant Kähler form on $D$.

Let $k \in \mathbb{N}$ be a positive integer. It will be usually assumed that $k$ is sufficiently large. The volume form $dV(z) = \mathcal{K}(z,z)dv_e(z)$ is $G$-invariant. The reproducing kernel for the Hilbert space of holomorphic functions on $D$ satisfying $\int_D |f(z)|^2\mathcal{K}(z,z)^{-k}dV(z) < \infty$ is $c(D,k)\mathcal{K}(z,w)^k$, where $c(D,k)$ is a constant (note: the proof of this fact in [39] uses the assumption that $\text{Aut}(D)$ acts transitively on $D$). The reproducing property is: for any such function $f$

$$(2) \quad f(z) = c(D,k)\int_D f(w)\mathcal{K}(z,w)^k\mathcal{K}(w,w)^{-k}dV(w),$$

$z \in D$. The value of the constant is determined by (1.6)[39]:

$$(3) \quad c(D,k)\int_D \frac{\mathcal{K}(z,w)^k\mathcal{K}(w,z)^k}{\mathcal{K}(w,w)^k}dV(w) = \mathcal{K}(z,z)^k$$
for any $z \in D$.

Let $\Gamma$ be a discrete subgroup of $G$ such that the quotient $M = \Gamma \backslash D = \Gamma \backslash G/K$ is smooth and compact. Let $m$ be a positive integer. Let $\rho : \Gamma \to GL(m, \mathbb{C})$ be a unitary representation of $\Gamma$.

**Definition 2.1.** [3] A function $f : D \to \mathbb{C}$ is called a (holomorphic) $\Gamma$-automorphic form of weight $k$ if $f$ is holomorphic and

$$f(\gamma z) J(\gamma, z)^k = f(z) \quad \forall \gamma \in \Gamma, z \in D. \quad (4)$$

**Definition 2.2.** [39] A vector-valued automorphic form of weight $k$ for $(\rho, \Gamma)$ is $F = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}$, where $F_j : D \to \mathbb{C}$, $j = 1, \ldots, m$, are holomorphic functions, and

$$J(\gamma, z)^k F(\gamma z) = \rho(\gamma) F(z) \quad \forall \gamma \in \Gamma, z \in D. \quad (5)$$

Denote the space of holomorphic $\Gamma$-automorphic forms of weight $k$ on $D$ by $\mathcal{A}(\Gamma, k)$. Denote the space of $\mathbb{C}^m$-valued holomorphic $(\rho, \Gamma)$-automorphic forms of weight $k$ on $D$ by $\mathcal{A}(\Gamma, m, k, \rho)$.

**Remark 2.3.** In a more general case when $M$ is of finite volume and not compact the definitions should include an appropriate condition at the cusps. The condition ”$M$ is smooth” can be relaxed to allow $\Gamma$ such as, for example, $SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R}) \simeq SU(1, 1)$ or $SU(2, 1) \cap SL(3, \mathbb{Z}[i])$.

Define the inner product on the space $\mathcal{A}(\Gamma, m, k, \rho)$ as follows:

$$(F, G) = \int_{\Gamma \backslash D} F(z)^T \overline{G(z)} \mathcal{K}(z, z)^{-k} dV$$

for $F, G \in \mathcal{A}(\Gamma, m, k, \rho)$. This is well-defined because the function $F(z)^T \overline{G(z)} \mathcal{K}(z, z)^{-k}$ is $\Gamma$-invariant (note: for that it is essential that $\rho$ is unitary).

Define the inner product on the space $\mathcal{A}(\Gamma, k)$ by

$$(f, g) = \int_{\Gamma \backslash D} f(z) \overline{g(z)} \mathcal{K}(z, z)^{-k} dV$$

for $f, g \in \mathcal{A}(\Gamma, k)$.

Denote by $K_M$ the canonical bundle on $M$ and by $K_D$ the canonical bundle on $D$.

**Remark 2.4.** We have isomorphisms of Hilbert spaces: $\mathcal{A}(\Gamma, k) \cong H^0(M, K_M^\otimes k)$, $\mathcal{A}(\Gamma, m, k, \rho) \cong H^0(M, E_\rho \otimes K_M^\otimes k)$. In particular, a holomorphic function $f$ on $D$ satisfies (4) if and only if $f(z)(dz_1 \wedge \ldots \wedge dz_n)^\otimes k$ is a $\Gamma$-invariant holomorphic section of $K_D^\otimes k$ (and thus descends to a holomorphic section of $K_M^\otimes k$).

**Remark 2.5.** There are irreducible unitary representations of the fundamental group of a compact Riemann surface of genus $\geq 2$ for each $m \in \mathbb{N}$ (Proposition 2.1 [34]). The proof in [34] provides explicit examples of such representations.
3. Poincaré series and a spanning result

Let \( D \) be a bounded symmetric domain, and let \( \Gamma \) be a discrete subgroup of \( Aut(D) \) such that the quotient \( M = \Gamma \backslash D \) is smooth and compact. Let \( k \) and \( m \) be positive integers, and let \( \rho \) be an \( m \)-dimensional unitary representation of \( \Gamma \). This is the setting for this section.

For an integrable holomorphic function \( F : D \to \mathbb{C}^m \) we define, formally, the Poincaré series of weight \( k \)

\[
\Theta_F(z) = \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) F(\gamma z) J(\gamma, z)^k
\]

(here we omit \( k \) from notation and write simply \( \Theta_F \)). If the series converges uniformly on compact sets in \( D \), then \( \Theta_F \in \mathcal{A}(\Gamma, m, k, \rho) \). Indeed, since the convergence is uniform on compact sets, it follows that \( \Theta_F \) is holomorphic. To verify (5), we observe: for \( g \in \Gamma, z \in D \)

\[
\Theta_F(gz) = \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) F(\gamma g z) J(\gamma, g z)^k = \sum_{\gamma \in \Gamma} \rho(g(\gamma g)^{-1}) F(\gamma g z) J(\gamma g, z)^k F(g(\gamma g)^{-1}) J(g, z)^k = \rho(g) J(g, z)^{-k} \Theta_F(z).
\]

Choose \( p \in D \). In [14] the \( \mathbb{C} \)-valued Poincaré series

\[
\theta_p(z) = \sum_{\gamma \in \Gamma} \left( \mathcal{K}(\gamma z, p) J(\gamma, z) \right)^k \in \mathcal{A}(\Gamma, k)
\]

(convergent absolutely and uniformly on compact sets for sufficiently large \( k \), and having the property \((f, \theta_p) = \text{const}(D, k) f(p) \) for any \( f \in \mathcal{A}(\Gamma, k) \)) were considered, and it was shown that such Poincaré series for an appropriate number of points in general position form a basis in \( \mathcal{A}(\Gamma, k) \). Note that the property \((f, \theta_p) = \text{const}(D, k) f(p) \) reflects the fact that the Bergman kernel for \( K_D^{\otimes k} \) is the Poincaré series of the Bergman kernel for \( K_D^{\otimes k} \) (Theorem 2 [33] or Theorem 1 [32]).

Let us now generalize the construction from [14] by associating to a point \( p \in D \) \( m \) vector-valued Poincaré series

\[
\Theta_p^{(j,k)}(z) = c(D, k) \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) T_p(\gamma z) J(\gamma, z)^k, \quad j = 1, \ldots, m
\]

where \( T_p(z) = \begin{pmatrix} (T_p)_1(z) \\ \vdots \\ (T_p)_m(z) \end{pmatrix} \), \((T_p)_j(z) = \mathcal{K}(z, p)^k \) and \((T_p)_j(z) = 0 \) for \( l \neq j \) (i.e. \( T_p(\gamma z) \) is the vector-function whose components, except for the \( j \)-th one, are zero, and \((T_p)_j(\gamma z) = \mathcal{K}(\gamma z, p)^k \).

We shall also use the notation \( \hat{\Theta}^{(j,k)}(z, p) \) for the function

\[
\hat{\Theta}^{(j,k)} : D \times D \to \mathbb{C}^m
\]

\[(z, p) \to \Theta_p^{(j,k)}(z) \].

**Lemma 3.1.** Let \( p \in D \). For \( k \geq 2 \) the series \( \sum_{\gamma \in \Gamma} (\mathcal{K}(\gamma z, p) J(\gamma, z))^k \) converges absolutely and uniformly on compact sets of \( D \).

The proof is in the Appendix.
Proposition 3.2. Let \( j \in \{1, \ldots, m\} \) and let \( p \in D \). Suppose \( k \) is sufficiently large.

(i) The series \( (7) \) converges absolutely and uniformly on compact sets.

(ii) For each \( H \in \mathcal{A}(\Gamma, m, k, \rho) \)

\[
(H, \Theta^{(j,k)}_p) = H_j(p).
\]

Theorem 3.3. For sufficiently large \( k \), for sufficiently many points \( p_1, \ldots, p_d \) in general position, the \( \mathbb{C} \)-linear span of \( \{\Theta^{(j,k)}_{p_i}|1 \leq l \leq d; 1 \leq j \leq m\} \) is \( \mathcal{A}(\Gamma, m, k, \rho) \).

Proof of Proposition 3.2. Proof of (i). For \( 1 \leq l \leq m \)

\[
\left| \left( \rho(\gamma^{-1}) T_p(\gamma z) J(\gamma, z)^k \right) \right| \leq \sqrt{\left( \rho(\gamma^{-1}) T_p(\gamma z) J(\gamma, z)^k \right)^T \rho(\gamma^{-1}) T_p(\gamma z) J(\gamma, z)^k} = |K(\gamma z, p) J(\gamma, z)|^k.
\]

The statement now follows from Lemma 3.1

Proof of (ii). Let \( \mathcal{F} \) be a Dirichlet fundamental domain for \( \Gamma \) (or a canonical fundamental domain \( [40] \)). Denote \( w = \gamma z \) for \( \gamma \in \Gamma, z \in \mathcal{F} \). By \( (5) \)

\[
H(z)^T = H(w)^T (\rho(\gamma)^{-1})^T J(\gamma, z)^k.
\]

Using \( (1), (2), (6), (7) \), we get:

\[
(H, \Theta^{(j,k)}_p) = c(D, k) \int_{\mathcal{F}} \left( H_1(z) \ldots H_m(z) \right) \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) T_p(\gamma z) J(\gamma, z)^k K(z, z)^{-k} dV(z) = c(D, k) \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} H(w)^T T_p(w) K(w, w)^{-k} dV(w) = c(D, k) \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} H_j(w) K(w, p)^k K(w, w)^{-k} dV(w) = c(D, k) \int_{\mathcal{F}} H_j(w) K(p, w)^k K(w, w)^{-k} dV(w) = H_j(p).
\]

Proof of Theorem 3.3 Let \( k \in \mathbb{N} \). The holomorphic vector bundle \( W = E_\rho \otimes K^*_M \) is positive. Using the notations similar to those in Chapters 2, 3 \([28]\), denote by \( P(W) \) the fibre bundle over \( M \) whose fibre at \( x \) is \( \mathbb{P}(W_x) \) (i.e. \( P(W) = (W - \{\text{zero section}\})/\mathbb{C}^* \)), denote by \( \pi : P(W) \rightarrow M \) the projection, and by \( L(W) \) the tautological line bundle over \( P(W) \) (i.e. the subbundle of \( \pi^*W \) with the fiber \( L(W)_x \) at \( \xi \in P(W) \) being the complex line in \( W_x(\xi) \) represented by \( \xi \)). Also denote by \( L(W^*) \) the tautological line bundle over \( P(W^*) = (W^* - \{\text{zero section}\})/\mathbb{C}^* \) and by \( \hat{\pi} : P(W^*) \rightarrow M \) the projection. We note that a section \( s \) of \( W \) produces a section \( \hat{s} \) of \( (L(W^*))^* \rightarrow P(W^*) \). Specifically, \( \hat{s} = h \circ s \circ \hat{\pi} \), where \( h \) is the holomorphic surjection \( \hat{\pi}^*W \rightarrow (L(W^*))^* \simeq L(W) \) given, fiberwise, by the quotient map \( W_x \rightarrow W_x/\ker f \) over \( (x, [f]) \in P(W^*) \), where \( x \in M, f \in W_x^*, f \neq 0 \).

Suppose \( k \) is large enough, so that the line bundle \( (L(W^*))^* \rightarrow P(W^*) \) is very ample. Let \( d = \dim H^0(P(W^*), (L(W^*))^*) \) and let \( \tilde{p}_1, \ldots, \tilde{p}_d \) be points in \( P(W^*) \) in general position (i.e. such that their images under the projective embedding given by \( (L(W^*))^* \) are not on the same hyperplane in \( \mathbb{P}(H^0(P(W^*), (L(W^*))^*))^* \)). Such \( d \) points exist because the linear system is base point free. Select a Dirichlet fundamental domain \( \mathcal{F} \) for \( \Gamma \) and for each \( j \in \{1, \ldots, d\} \) let \( p_j \) be the point in \( \mathcal{F} \) that corresponds to \( \hat{\pi}(\tilde{p}_j) \).
Now, to prove the statement of the theorem, suppose \( H \in \mathcal{A}(\Gamma, m, k, \rho) \) is not in the linear span of \( \{ \Theta_{p_i}^{(j;k)} \}_{1 \leq i \leq d} \), for all \( l \in \{1, \ldots, d\} \) and all \( j \in \{1, \ldots, m\} \). By Proposition 3.2(ii) \( H(p_1) = \ldots = H(p_d) = 0 \). Let \( s \) be the section of \( W \) corresponding to \( H \). This section vanishes at \( p_1, \ldots, p_d \). Therefore \( \tilde{s}(\tilde{p}_1) = \ldots = \tilde{s}(\tilde{p}_d) = 0 \). Since \( \tilde{p}_1, \ldots, \tilde{p}_d \) are in general position, we conclude that \( \tilde{s} \equiv 0 \). It follows that \( s = 0 \). Hence \( H = 0 \). \( \square \)

4. Automorphic forms and submanifolds

Let \( D \) be a bounded symmetric domain, and let \( \Gamma \) be a discrete subgroup of \( Aut(D) \) such that the quotient \( M = \Gamma \backslash D \) is smooth and compact. Let \( k \) and \( m \) be positive integers, and let \( \rho \) be an \( m \)-dimensional unitary representation of \( \Gamma \).

Let \( \Lambda \) be a \( q \)-dimensional submanifold of \( D \) (\( q \geq 1 \)) such that \( \Lambda \subset \bar{B}(z_0, r_0) \subset D \), where \( \bar{B}(z_0, r_0) \) is the closed ball centered at \( z_0 \) of radius \( r_0 \) with respect to the Euclidean metric, for some \( z_0 \in D, r_0 > 0 \). Let \( \nu \) be a nonzero volume form (a real \( q \)-form) on \( \Lambda \) such that \( \int_{\Lambda} \nu > 0 \). Set

\[
\Theta_{(j;k)}^{(\Lambda)}(z) = \int_{\Lambda} \Theta^{(j;k)}(z, p)K(p, p)^{-\frac{1}{2}}\nu(p)
\]

for \( j \in \{1, \ldots, m\} \). By a standard differentiation under the integral sign argument \( \Theta_{(j;k)}^{(\Lambda)} \) is holomorphic. Moreover, \( \Theta_{(j;k)}^{(\Lambda)} \in \mathcal{A}(\Gamma, m, k, \rho) \) and

\[
(H, \Theta_{(j;k)}^{(\Lambda)}) = \int_{\Lambda} H(z)K(z, z)^{-\frac{1}{2}}\nu(z)
\]

for any \( H \in \mathcal{A}(\Gamma, m, k, \rho) \).

Remark 4.1. The statement analogous to Proposition 3.2(ii), but written for the corresponding sections of \( E_\rho \otimes K_M^{\otimes k} \), would mean that the section of \( E_\rho \otimes K_M^{\otimes k} \), corresponding to \( \Theta_{p_i}^{(j;k)} \), is the \( j \)-th row of the Bergman kernel for this vector bundle, where the Bergman kernel is written as an \( m \times m \) matrix. The general idea of “integrating the Bergman kernel over a submanifold \( \Lambda \)” was used in [9] (to obtain sections of powers of a line bundle, with \( \Lambda \) being a Bohr-Sommerfeld Lagrangian submanifold of a compact Kähler manifold) and it is used in a recent preprint [23] (to obtain sections of certain vector bundles, with \( \Lambda \) being an isotropic Bohr-Sommerfeld submanifold of a compact symplectic manifold).

In this section, the domain \( D \) will be the unit ball \( \mathbb{B}^n \subset \mathbb{C}^n \) (\( n \geq 1 \)), with its Bergman metric. Recall that \( SU(n, 1) = \{ A \in SL(n + 1, \mathbb{C}) \mid A^T \sigma A = \sigma \} \), where \( \sigma = \begin{pmatrix} 1_{n \times n} & 0 \\ 0 & -1 \end{pmatrix} \).

The ball is a bounded realization of the Hermitian symmetric space \( SU(n, 1)/SU(1, 1) \) (note that for \( n = 1 \) \( D \) is the unit disc: \( D \cong SU(1, 1)/U(1) \cong SL(2, \mathbb{R})/SO(2) \)). The group \( SU(n, 1) \) acts on \( \mathbb{B}^n \) by fractional-linear transformations: for \( \gamma = (a_{jk}) \in SU(n, 1) \) the corresponding automorphism \( \mathbb{B}^n \rightarrow \mathbb{B}^n \) is

\[
z = (z_1, \ldots, z_n) \mapsto \left( \frac{a_{11}z_1 + \ldots + a_{1n}z_n + a_{1,n+1}}{a_{n+1,1}z_1 + \ldots + a_{n+1,n}z_n + a_{n+1,n+1}}, \ldots, \frac{a_{n1}z_1 + \ldots + a_{nn}z_n + a_{n,n+1}}{a_{n+1,1}z_1 + \ldots + a_{n+1,n}z_n + a_{n+1,n+1}} \right).
\]
The complex Jacobian is $J(\gamma, z) = 1/(a_{n+1,1}z_1 + \ldots + a_{n+1,n}z_n + a_{n+1,n+1})^{n+1}$.

**Remark 4.2.** Each element of the center of $SU(n, 1)$ acts as the identity map on $\mathbb{B}^n$, and $\text{Aut}(\mathbb{B}^n)$ is isomorphic to $PU(n, 1)$. We will represent automorphisms of the ball by matrices from $SU(n, 1)$, and we will use the same letter to denote the matrix and the corresponding automorphism.

We will denote by $0$ the point $(0, \ldots, 0) \in \mathbb{B}^n$. Also, for $z, w \in \mathbb{B}^n$ denote

$$
\langle z, w \rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n - 1.
$$

The $SU(n, 1)$-invariant Kähler form on $\mathbb{B}^n$ is, up to a positive constant factor,

$$
i\partial \bar{\partial} \log(-\langle z, z \rangle) = \frac{i}{2} \left( \sum_{j=1}^n \bar{z}_j dz_j \right) \wedge \left( \sum_{l=1}^n z_l d\bar{z}_l \right) - \langle z, z \rangle \sum_{r=1}^n dz_r \wedge d\bar{z}_r \right].
$$

Denote by $\tau(z, w)$ the distance between $z$ and $w$ with respect to the complex hyperbolic metric. Note that

$$
\cosh^2 \frac{\tau(z, w)}{2} = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}
$$

(see e.g. [19] 3.1.7). It is a standard fact (see e.g. [38] or [37]) that for the ball

$$
K(z, w) = \frac{n!}{\pi^n} (-\langle z, w \rangle)^{(n+1)}.
$$

**Lemma 4.3.** For $D = \mathbb{B}^n$ the constant $c(D, k)$ from Section 2 is $c(\mathbb{B}^n, k) = \left(\frac{(n+1)(k-1)+n}{n}\right)$.

This follows from Theorem 2.2 [44] with $\alpha = (n+1)(k-1)$ (the constant $c(D, k)$ comes out to be $c_k$ given by (2.2) [44]). This also can be verified in another way, by a direct calculation (see the Appendix).

**Remark 4.4.** Applying the Stirling formula $N! \sim (\frac{N}{e})^N \sqrt{2\pi N} \left(1 + O\left(\frac{1}{N}\right)\right)$ as $N \to \infty$ [12], we get: $c(\mathbb{B}^n, k) \sim \frac{(n+1)^n}{n!} k^n \left(1 + O\left(\frac{1}{k}\right)\right)$ as $k \to \infty$.

Let $\Gamma$ be a discrete subgroup of $SU(n, 1)$ such that the quotient $M = \Gamma \backslash \mathbb{B}^n$ is smooth and compact. Let $k$, $m$ be positive integers, and let $\rho$ be an $m$-dimensional unitary representation of $\Gamma$. Denote by $\pi : \mathbb{B}^n \to M$ the covering map. Let $\mathcal{F}$ be a Dirichlet fundamental domain for $\Gamma$ [30]. Suppose $X$ and $Y$ are submanifolds of $\mathbb{B}^n$ of dimensions $q_X > 0$ and $q_Y > 0$ respectively, such that $X = \pi^{-1}(X') \cap \mathcal{F}$, $X \cong X'$, and $Y = \pi^{-1}(Y') \cap \mathcal{F}$, $Y \cong Y'$, where $X'$ and $Y'$ are submanifolds of $M$, and $\cong$ stands for diffeomorphism. Let $\nu_X$ be a nonzero volume form on $X$ (a real $q_X$-form) such that $\int_X \nu_X > 0$ and let $\nu_Y$ be a nonzero volume form on $Y$ (a real $q_Y$-form) such that $\int_Y \nu_Y > 0$. Denote $\tilde{X} = \Gamma X$, $\tilde{Y} = \Gamma Y$. Define the $q_X$-form $\nu_{\tilde{X}}$ on $\tilde{X}$ by $\nu_{\tilde{X}}|_{\gamma^{-1}(X)} = \gamma^* \nu_X$ for each $\gamma \in \Gamma$. Define $\nu_{\tilde{Y}}$ the same way. Note that $\nu_{\tilde{X}}$, $\nu_{\tilde{Y}}$ are $\Gamma$-invariant. Assume $\int_{\tilde{X}} |K(z, w)|^2 \frac{\nu_{\tilde{X}}(w)}{\kappa(w, w)} < \infty$ for all $z \in \mathcal{F}$, $\int_{\tilde{Y}} |K(z, w)|^2 \frac{\nu_{\tilde{Y}}(w)}{\kappa(w, w)} < \infty$ for all $z \in \mathcal{F}$ (the last condition is satisfied, for example, when $Y$ is a small ball and $\nu_{\tilde{Y}} = dV|_{\tilde{Y}}$, because $K(., w)$ is square-integrable on $\mathbb{B}^n$).

For two subsets $A$, $B$ of $\mathbb{B}^n$ we will denote

$$
dist(A, B) = \inf\{\tau(z, w) \mid z \in A, w \in B\}.$$
Since $\tau$ is $\Gamma$-invariant, the same notation can be used for subsets $A$, $B$ of $M$.

**Remark 4.5.** Recall that if $(a_k)$, $(b_k)$ are two sequences of complex numbers, then notation $a_k \sim b_k$ as $k \to \infty$ means $\lim_{k \to \infty} \frac{a_k}{b_k} = 1$.

**Theorem 4.6.** Let $r, j \in \{1, \ldots, m\}$.

(i) Suppose $\text{dist}(\tilde{X} - X, Y) > 0$ and $r \neq j$. Then for any $l \in \mathbb{N}$ there is a constant $C = C(l; n, X, Y, \Gamma, \nu_X, \nu_Y)$ such that, as $k \to \infty$

$$|(\Theta_X^{(r;k)}, \Theta_Y^{(j;k)})| \leq \frac{C}{k^l}.$$

(ii) Suppose $\text{dist}(\tilde{X}, Y) > 0$. Then for any $l \in \mathbb{N}$ there is a constant $C = C(l; n, X, Y, \Gamma, \nu_X, \nu_Y)$ such that, as $k \to \infty$

$$|(\Theta_X^{(r;k)}, \Theta_Y^{(j;k)})| \leq \frac{C}{k^l}.$$

**Remark 4.7.** If $\text{dist}(X, \partial\mathcal{F}) > 0$ or $\text{dist}(Y, \partial\mathcal{F}) > 0$, then $\text{dist}(\tilde{X} - X, Y) > 0$.

**Remark 4.8.** If $\text{dist}(\tilde{X}, Y) > 0$, then $\text{dist}(X', Y') > 0$.

**Theorem 4.9.** Suppose $Y \subset X$, $\text{dist}(X, \partial\mathcal{F}) > 0$, and $j \in \{1, \ldots, m\}$.

(i) If $q_X \leq n$, then

$$(\Theta_X^{(jk)}, \Theta_Y^{(jk)}) \leq \text{const}(n, X, Y, \nu_X, \nu_Y) k^{n-q_X}$$

as $k \to \infty$.

(ii) If $X \subset \{z \in \mathbb{B}^n | y_1 = \ldots = y_n = 0\}$, then

$$(\Theta_X^{(jk)}, \Theta_Y^{(jk)}) \sim C(n, X, Y, \nu_X, \nu_Y) k^{n-q_X}$$

as $k \to \infty$, where $C(n, X, Y, \nu_X, \nu_Y)$ is a positive constant.

**Remark 4.10.** Because of the assumptions in Theorem 4.9(ii), in this part of the theorem the submanifolds $X$ and $Y$ are isotropic submanifolds of $\mathbb{B}^n$. In Theorem 4.9(i) and in Theorem 4.6 the submanifolds are not necessarily isotropic. Note that $X$ and $Y$ are submanifolds of $\mathbb{B}^n$, the universal cover of $M$, and not of $M$. We do not require $X$ and $Y$ to satisfy a Bohr-Sommerfeld condition. In the usual procedure of associating a section of a line bundle, $L$, to a Lagrangian or isotropic submanifold, $\Lambda$, of a Kähler manifold, the Bohr-Sommerfeld condition ensures the existence of a covariant constant nonvanishing section, $\varphi$, of $L^*|_{\Lambda}$.

Having such $\varphi$, from a holomorphic section $s$ of $L^{\otimes k}$, one obtains a function on $\Lambda$, $\varphi^{\otimes k}(s)$, which then can be integrated over $\Lambda$. This provides a linear functional on the space of holomorphic sections of $L^{\otimes k}$. We do not need such $\varphi$, since in (9) we are already integrating a function.

**Proof of Theorem 4.6.** Using (7), (8), (9), we get:

$$|(\Theta_X^{(r;k)}, \Theta_Y^{(j;k)})| = | \int_X (\Theta_X^{(r;k)}(z)) j \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) | =$$

$$| \int_X \int_Y (\hat{\Theta}^{(r;k)}(z, \zeta)) j \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) | =$$
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X \sum_{\gamma \in \Gamma} \rho(\gamma^{-1})_{jr} \mathcal{K}(\gamma z, \zeta) J(\gamma, z)^k \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_{X}(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| \leq \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X \sum_{\gamma \in \Gamma} |\rho(\gamma^{-1})_{jr}||\mathcal{K}(\gamma z, \zeta) J(\gamma, z)^k \mathcal{K}(\zeta, \zeta)^{-\frac{1}{2}} \nu_{X}(\zeta) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right|. \]

Setting \( \zeta = \gamma w \), we get, using (1):
\[ |(\Theta_X^{r;k}, \Theta_Y^{j;k})| \leq c(\mathbb{B}^n, k) \left| \int_Y \sum_{\gamma \in \Gamma} \int_X |\rho(\gamma^{-1})_{jr}||\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right|. \]

Since \( \rho(\gamma^{-1}) \) is a unitary matrix, we have: \( |\rho(\gamma^{-1})_{jr}| \leq 1 \). Using (10), (11), we get:
\[ |(\Theta_X^{r;k}, \Theta_Y^{j;k})| \leq c(\mathbb{B}^n, k) \left| \int_Y \sum_{\gamma \in \Gamma} \int_X |\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| = \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X |\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| = \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X |\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| = \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X \left( \frac{\tau(z, w)}{2} \right)^{(n+1)(\frac{k}{2}-1)} |K(z, w)|^2 \frac{\nu_X(w) \nu_Y(z)}{K(w, w) K(z, z)} \right| \leq \]
\[ c(\mathbb{B}^n, k) \left( \frac{1}{\cosh^{[\frac{1}{2} \text{dist}(X, Y)]}} \right)^{(n+1)(k-2)} \left| \int_Y \int_X |K(z, w)|^2 \frac{\nu_X(w) \nu_Y(z)}{K(w, w) K(z, z)} \right|. \]

For \( r \neq j \), since \( \rho(\text{id})_{jr} = 0 \), the argument above can be modified:
\[ |(\Theta_X^{r;k}, \Theta_Y^{j;k})| \leq c(\mathbb{B}^n, k) \left| \int_Y \sum_{\gamma \in \Gamma, \gamma \neq \text{id}_Y} \int_X |\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| = \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X |\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| = \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X |\mathcal{K}(z, w) J^k \mathcal{K}(w, w)^{-\frac{1}{2}} \nu_{X}(w) \mathcal{K}(z, z)^{-\frac{1}{2}} \nu_Y(z) \right| = \]
\[ c(\mathbb{B}^n, k) \left| \int_Y \int_X \left( \frac{\tau(z, w)}{2} \right)^{(n+1)(\frac{k}{2}-1)} |K(z, w)|^2 \frac{\nu_X(w) \nu_Y(z)}{K(w, w) K(z, z)} \right| \leq \]
\[ c(\mathbb{B}^n, k) \left( \frac{1}{\cosh^{[\frac{1}{2} \text{dist}(X, Y)]}} \right)^{(n+1)(k-2)} \left| \int_Y \int_X |K(z, w)|^2 \frac{\nu_X(w) \nu_Y(z)}{K(w, w) K(z, z)} \right|. \]
Since \( \cosh \varepsilon > 1 \) for \( \varepsilon > 0 \), and with Remark 4.4 the statements follow. \( \Box \)

**Proof of Theorem 4.9.** Using (7), (8), (9), we get:

\[
(\Theta_X^{(j;k)}, \Theta_Y^{(j;k)}) = \int_Y (\Theta_X^{(j;k)}(z))_j \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) =
\]

\[
\int \int (\Theta^{(j;k)}(z, \zeta))_{1, k} \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) =
\]

\[
c(\mathbb{B}^n, k) \int \int \sum_{\gamma \in \Gamma} \rho(\gamma^{-1})_{jj}(\mathcal{K}(\gamma z, \zeta) J(\gamma, z)^{j} \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) = I_1 + I_2,
\]

where \( I_1 \) is the term with \( \gamma = \text{id} \) and \( I_2 \) is the rest. Thus,

\[
I_1 = c(\mathbb{B}^n, k) \int \int \mathcal{K}(z, \zeta)^{k} \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z)
\]

and

\[
I_2 = c(\mathbb{B}^n, k) \int \int \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} \rho(\gamma^{-1})_{jj}(\mathcal{K}(\gamma z, \zeta) J(\gamma, z)^{j} \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z).
\]

Since \( \rho(\gamma^{-1}) \) is a unitary matrix, \( |\rho(\gamma^{-1})_{jj}| \leq 1 \). Setting \( \zeta = \gamma w \) and using (1), (10), (11), we get:

\[
|I_2| \leq c(\mathbb{B}^n, k) \int \int \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} |\mathcal{K}(\gamma z, \zeta) J(\gamma, z)^{j} \mathcal{K}(\zeta, \zeta)^{-\frac{k}{2}} \nu_X(\zeta) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) =
\]

\[
c(\mathbb{B}^n, k) \int \sum_{\gamma \in \Gamma, \gamma \neq \text{id}} \int \mathcal{K}(z, w)^{k} \mathcal{K}(w, w)^{-\frac{k}{2}} \nu_X(w) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) =
\]

\[
c(\mathbb{B}^n, k) \int \int \mathcal{K}(z, w)^{k} \mathcal{K}(w, w)^{-\frac{k}{2}} \nu_X(w) \mathcal{K}(z, z)^{-\frac{k}{2}} \nu_Y(z) =
\]

\[
c(\mathbb{B}^n, k) \int \int \left( \frac{\langle z, z \rangle \langle w, w \rangle}{\langle z, w \rangle \langle w, z \rangle} \right)^{(n+1)(\frac{k}{2} - 1)} |\mathcal{K}(z, w)|^{2} \frac{\nu_X(w)}{\mathcal{K}(w, w) \mathcal{K}(z, z)}
\]
Because of Remark 4.7, \( \cosh\left(\frac{1}{2}\right) \) is over the part of \( Y \times X \) where \( \tau(z, \zeta) > \delta \) and since \( \cosh \frac{1}{2} \delta > 1 \), it follows that \( I_2 \) has the property: for any \( l \in \mathbb{N} \) there is a constant \( C = C(l; n, X, Y, \delta, \nu_X, \nu_Y) \) such that
\[
|I_2| \leq \frac{C}{k^l}
\]
as \( k \to \infty \).

Now we consider
\[
I_1 = c(\mathbb{B}^n, k) \int_Y \int_X \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k/2}}{\langle -z, \zeta \rangle^{(n+1)k}} \nu_X(\zeta) \nu_Y(z).
\]
We use Fubini's theorem to switch to the integral over \( Y \times X \) with respect to the product measure, then choose and fix a sufficiently small \( \delta > 0 \), and split \( I_1 \) into two parts: \( I_1^{(1)} \), where the integration is over the part of \( Y \times X \) where \( \tau(z, \zeta) \leq \delta \) and \( I_1^{(2)} \), where the integration is over the part of \( Y \times X \) where \( \tau(z, \zeta) > \delta \). Using (10), we get:

\[
I_1^{(2)} = c(\mathbb{B}^n, k) \int_{Y \times X} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k/2}}{\langle -z, \zeta \rangle^{(n+1)k}} \nu_Y(z) \nu_X(\zeta),
\]

\[
|I_1^{(2)}| \leq c(\mathbb{B}^n, k) \int_{Y \times X} \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle^{(n+1)k/2}}{\langle -z, \zeta \rangle^{(n+1)k}} \nu_Y(z) \nu_X(\zeta) =
\]

\[
c(\mathbb{B}^n, k) \int_{Y \times X} \frac{1}{\cosh(\frac{\delta}{2})^{(n+1)k}} \nu_Y(z) \nu_X(\zeta),
\]

therefore by Remark 4.4 and since \( \cosh \frac{\delta}{2} > 1 \), it follows that \( I_1^{(2)} \) has the property: for any \( l \in \mathbb{N} \) there is a constant \( C = C(l; n, X, Y, \delta, \nu_X, \nu_Y) \) such that
\[
|I_1^{(2)}| \leq \frac{C}{k^l}
\]
as \( k \to \infty \).
It remains to investigate the term

$$I_1^{(1)} = c(\mathbb{B}^n, k) \int_{Y \times X} \left( \frac{(z, z)(\zeta, \zeta)}{(\zeta, \zeta)(z, z)} \right)^{\frac{(n+1)k}{2}} \nu_Y(z)\nu_X(\zeta).$$

To proceed with the proof of (i), we observe:

$$|I_1^{(1)}| \leq c(\mathbb{B}^n, k) \int_{Y \times X} \left( \frac{(z, z)(\zeta, \zeta)}{(\zeta, \zeta)(z, z)} \right)^{\frac{(n+1)k}{2}} \nu_Y(z)\nu_X(\zeta).$$

For the proof of (ii): if $z \in Y$ and $\zeta \in X$, then $(z, \zeta) = (\zeta, z)$ and

$$I_1^{(1)} = c(\mathbb{B}^n, k) \int_{Y \times X} \left( \frac{(z, z)(\zeta, \zeta)}{(\zeta, \zeta)(z, z)} \right)^{\frac{(n+1)k}{2}} \nu_Y(z)\nu_X(\zeta).$$

Thus, to finish the proof of the theorem we need to treat the integral

$$c(\mathbb{B}^n, k) \int_{Y \times X} \left( \frac{(z, z)(\zeta, \zeta)}{(\zeta, \zeta)(z, z)} \right)^{\frac{(n+1)k}{2}} \nu_Y(z)\nu_X(\zeta).$$

Using (10), we get:

$$\int_{Y \times X} \left( \frac{(z, z)(\zeta, \zeta)}{(\zeta, \zeta)(z, z)} \right)^{\frac{(n+1)k}{2}} \nu_Y(z)\nu_X(\zeta) = \int_{Y \times X} \left( \cosh \frac{\tau(z, \zeta)}{2} \right)^{-(n+1)k} \nu_Y(z)\nu_X(\zeta) =$$

$$\int_{Y \times X} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_Y(z)\nu_X(\zeta) = \int_{Y \{z \in X \mid \tau(z, \zeta) \leq \delta\}} \int_{\{z \in X \mid \tau(z, \zeta) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(\zeta)\nu_Y(z).$$

Let $A_z \in SU(n, 1)$, $z \in Y$, be a smooth family of automorphisms $\mathbb{B}^n \to \mathbb{B}^n$ such that $A_z 0 = 0$. Denote $\hat{X} = \cup_{z \in Y} A_z(X)$. Let $\{U_j\}$ be a finite cover of $\hat{X}$ by open subsets of $\mathbb{B}^n$ with smooth boundary, let $t_{1}^{(j)}, ..., t_{q_X}^{(j)}$ be local coordinates on $U_j \cap \hat{X}$, and let $\{\psi^{(j)}\}$ be a partition of unity subordinate to the cover $\{U_j\}$.

For a fixed $z \in Y$ consider the integral

$$\int_{\{z \in X \mid \tau(z, \zeta) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(z, \zeta)}{2}} \nu_X(\zeta) = \int_{\{w \in A_z(X) \mid \tau(0, w) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{\tau(w, 0)}{2}} [(A_z^{-1})^*\nu_X](w),$$

where $w = A_z \zeta$. Note: $\tau(0, w) = \tau(A_z z, A_z \zeta) = \tau(z, \zeta)$. We have: $(A_z^{-1})^*\nu_X|_{U_j} = f^{(j)}(t)dt_1^{(j)} \wedge ... \wedge dt_{q_X}^{(j)}$, and the integral becomes

$$\sum_j \int_{\{w \in A_z(X) \mid \tau(w, 0) \leq \delta\} \cap U_j} e^{-(n+1)k \ln \cosh \frac{\tau(w, 0)}{2}} \psi^{(j)}(t)f^{(j)}(t)dt_1^{(j)} \wedge ... \wedge dt_{q_X}^{(j)}.$$
Now we will work with the integral
\[
\int_{\{w \in A(X) | r(w, 0) \leq \delta \} \cap U_j} e^{-\frac{(n+1)k}{2} \ln \cosh^2 \frac{r(w, 0)}{2}} \psi^{(j)}(t)(t) dt^{(j)}_{1} ... dt^{(j)}_{q_X}
\]
Apply the multivariable Laplace method. If the point \( w = 0 \) is in \( U_j \) or on the boundary of \( U_j \), then the appropriate statement is, respectively, Theorem 3 p. 495 or (5.15) p. 498 in [13]. If the point \( w = 0 \) is not in \( U_j \), then it follows that the contribution from the \( j \)-th integral is rapidly decreasing as \( k \to \infty \), by an argument similar to the one that has already been used earlier.

In order to use the Laplace method, we need to show that the Hessian matrix \( H_\zeta \) of the function \( \ln \cosh^2 \frac{r(w, 0)}{2} = -\ln(-\langle w, w \rangle) \) at \( w = 0 \) is positive definite. We have: for \( l \in \{1, ..., q_X\} \), \( p \in \{1, ..., q_X\} \)
\[
\frac{\partial}{\partial t_p} (-\ln(-\langle w, w \rangle)) = \frac{1}{-\langle w, w \rangle} \sum_{r=1}^{n} \left( w_r \frac{\partial w_r}{\partial t_p} + \bar{w}_r \frac{\partial \bar{w}_r}{\partial t_p} \right)
\]
\[
\frac{\partial^2}{\partial t_l \partial t_p} (-\ln(-\langle w, w \rangle)) \bigg|_{w=0} = \sum_{r=1}^{n} \left( \frac{\partial w_r}{\partial t_l} \frac{\partial \bar{w}_r}{\partial t_p} + \frac{\partial \bar{w}_r}{\partial t_l} \frac{\partial w_r}{\partial t_p} \right).
\]
Therefore \( H_\zeta = B_\zeta \bar{B}_\zeta^T + \bar{B}_\zeta B_\zeta^T \), where \( B_\zeta \) is the \( q_X \times n \) matrix \((\frac{\partial w_1}{\partial t_1}, ..., \frac{\partial w_n}{\partial t_1}, \frac{\partial w_1}{\partial t_2}, ..., \frac{\partial w_n}{\partial t_2}, ..., \frac{\partial w_1}{\partial t_{q_X}}, ..., \frac{\partial w_n}{\partial t_{q_X}})\). The matrix \( H_\zeta \) is symmetric. The matrices \( H_\zeta, B_\zeta \bar{B}_\zeta^T, \bar{B}_\zeta B_\zeta^T \) are positive semidefinite, because for a vector \( v \in \mathbb{C}^{q_X} \) \((B_\zeta \bar{B}_\zeta^T v)^T \bar{v} = (\bar{B}_\zeta B_\zeta^T v)^T \bar{v} = (B_\zeta v)^T \bar{B}_\zeta \bar{v} = (\bar{B}_\zeta B_\zeta^T v)^T \bar{v} \). It remains to show: if \( H_\zeta v = 0 \), then \( v = 0 \). If \( H_\zeta v = 0 \), then \( \bar{v}^T H_\zeta v = 0 \), and it follows that \( B_\zeta \bar{B}_\zeta^T v = 0 \). But \( \text{rk}(B_\zeta^T) = q_X \), hence \( \text{dim ker } B_\zeta^T = 0 \), therefore \( v = 0 \). Thus \( H_\zeta \) is positive definite.

If the point \( w = 0 \) is in \( U_j \), then the integral (12) is asymptotic, as \( k \to \infty \), to
\[
\left( \frac{4\pi}{(n+1)k} \right)^{\frac{q_X}{2}} \psi^{(j)}(t) \bigg|_{w=0}^{} (\det H_\zeta)^{-\frac{1}{2}}, \text{ and if } 0 \text{ is on the boundary of } U_j, \text{ then the integral (12) is asymptotic to } \frac{1}{2} \left( \frac{4\pi}{(n+1)k} \right)^{\frac{q_X}{2}} \psi^{(j)}(t) \bigg|_{w=0}^{} (\det H_\zeta)^{-\frac{1}{2}}. \text{ We conclude:}
\]
\[
c(\mathbb{B}^n, k) \int_{\mathcal{Y}} \int_{\{\zeta \in X | r(z, \zeta) \leq \delta\}} e^{-(n+1)k \ln \cosh \frac{r(z, \zeta)}{2}} \nu_X(z) \nu_Y(z) =
\]
\[
c(\mathbb{B}^n, k) \sum_{j} \int_{\{w \in A(X) | r(w, 0) \leq \delta\} \cap U_j} e^{-\frac{(n+1)k}{2} \ln \cosh^2 \frac{r(w, 0)}{2}} \psi^{(j)}(t) f^{(j)}(t) dt^{(j)}_{1} ... dt^{(j)}_{q_X} \nu_Y(z) \sim
\]
\[
c(\mathbb{B}^n, k) Ck^{-\frac{q_X}{2}}
\]
and the statements (i), (ii) now follow from Remark 4.4. For the constant \( C \) we have: \( C > 0 \), because the number \( f^{(j)} \bigg|_{w=0}^{} (\det H_\zeta)^{-\frac{1}{2}} \) is positive for each \( j \), the value of the function \( \psi^{(j)} \) at the point \( w = 0 \) is nonnegative for each \( j \), and there is \( j_0 \) such that the point \( w = 0 \) is in \( U_{j_0} \) and \( \psi^{(j_0)} \bigg|_{w=0}^{} > 0 \). □
Remark 4.11. The remainder in Theorem 4.9(ii) is determined by \( I_2, I_1^{(2)} \), the error term in the Laplace approximation and the error in the Stirling formula.

In the examples below, for specific \( X \) and \( Y \), we will work out the integral

\[
I_1 = c(\mathbb{B}^n, k) \int \int_X \left( \frac{(1 - t^2)(1 - T^2)}{1 - tT} \right)^{\frac{(n+1)k}{2}} dt dT.
\]

This term appeared in the proof of Theorem 4.9 as the term that determines the behaviour of \((\Theta_X^{(j,k)}, \Theta_Y^{(j,k)})\) as \( k \to \infty \).

Example 4.12. Let \( Y = X \subset \mathbb{B}^n \) be a (1-dimensional) line segment defined by \( z_j = t \cos \varphi, \) 
\(-\alpha < t < \alpha \), where \( \alpha \in (0, 1) \) and \( \varphi \in [0, \frac{\pi}{2}] \) are fixed, \( z_j = 0 \) for \( j > 1 \), and let \( \nu_X = dt \). If \( n = 1 \), then \( X \) is a Lagrangian submanifold of \( \mathbb{B}^1 \). For arbitrary \( n \) such \( X \) is totally real. We have:

\[
I_1 = c(\mathbb{B}^n, k) \int \int_X \left( \frac{(1 - t^2)(1 - T^2)}{1 - tT} \right)^{\frac{(n+1)k}{2}} dt dT.
\]

Here \( \zeta = T e^{i\varphi} \). For a fixed \( T \) denote \( f(t) = \left( \frac{(1-t^2)(1-T^2)}{1-tT} \right)^{1/2} \). We have:

\[
f(T) = 1, \quad \frac{df}{dt} \bigg|_{t=T} = 0, \quad \frac{d^2f}{dt^2} \bigg|_{t=T} = -\frac{1}{(1 - T^2)^2} < 0.
\]

The function \( f \) has a maximum at \( t = T \). Applying the 1-dimensional Laplace approximation ((1.5) p. 60 [13] or (5.1.21) [7]) we get: as \( k \to \infty \)

\[
\int_{-\alpha}^{\alpha} f(t)^{(n+1)k} dt \sim \left( \frac{-2\pi}{(n+1)k f''(T)} \right)^{\frac{1}{2}} = \sqrt{\frac{2\pi}{(n+1)k}} (1 - T^2),
\]

hence

\[
I_1 \sim c(\mathbb{B}^n, k) \sqrt{\frac{2\pi}{(n+1)k}} \int_{-\alpha}^{\alpha} (1 - T^2) dT = c(\mathbb{B}^n, k) \sqrt{\frac{2\pi}{(n+1)k}} 2(\alpha - \frac{\alpha^3}{3}) \sim c(n) (\alpha - \frac{\alpha^3}{3}) k^{-\frac{3}{2}},
\]

where \( c(n) = \frac{(n+1)^{-\frac{3}{2}}}{2\sqrt{2\pi}} \).

Example 4.13. Let \( Y = X \subset \mathbb{B}^n \) \((n \geq 2)\) be the circle of radius \( 0 < \alpha < 1 \) in the \( x_1 x_2 \)-plane centered at \((x_1, x_2) = (0, 0)\): \( z_1 = x_1 = \alpha \cos \Theta, \) \( z_2 = x_2 = \alpha \sin \Theta, \) \( 0 \leq \Theta < 2\pi, \) \( y_1 = y_2 = 0, \) \( z_j = 0 \) for \( j > 2 \). Let \( \nu_X = d\Theta \). For arbitrary \( n \geq 2 \) such \( X \) is totally real. We have:

\[
I_1 = c(\mathbb{B}^n, k) \int \int_X \frac{[(1 - x_1^2 - x_2^2)(1 - (\text{Re}(\zeta_1))^2 - (\text{Re}(\zeta_2))^2)]^{\frac{(n+1)k}{2}}}{(1 - x_1 \text{Re}(\zeta_1) - x_2 \text{Re}(\zeta_2))^{(n+1)k}} d\Theta d\varphi =
\]

\[
c(\mathbb{B}^n, k) 2\pi \int_{0}^{\alpha} \int_{0}^{\frac{1}{1 - \alpha^2 \cos(\Theta - \varphi)}} \frac{1 - \alpha^2}{1 - \alpha^2 \cos(\Theta - \varphi)}^{\frac{(n+1)k}{2}} d\Theta d\varphi.
\]

Here \( \zeta_1 = \alpha \cos \varphi, \) \( 0 \leq \varphi < 2\pi, \) \( \zeta_2 = \alpha \sin \varphi, \) \( \text{Im}(\zeta_1) = \text{Im}(\zeta_2) = 0, \) \( \zeta_j = 0 \) for \( j > 2 \). For a fixed \( \varphi \) denote \( \phi(\Theta) = \frac{1 - \alpha^2}{1 - \alpha^2 \cos(\Theta - \varphi)}. \) We have:

\[
f(\phi) = 1, \quad \frac{df}{d\Theta} \bigg|_{\Theta=\varphi} = 0, \quad \frac{d^2f}{d\Theta^2} \bigg|_{\Theta=\varphi} = -\frac{\alpha^2}{1 - \alpha^2} < 0,
\]
$f$ has a local maximum at $\Theta = \varphi$. Applying the 1-dimensional Laplace approximation (\cite{[43]} or (5.1.21) \cite{[7]}) we get:

$$\int_0^{2\pi} f(\Theta)^{(n+1)k} d\Theta \sim \left(\frac{-2\pi}{(n+1)kf''(\varphi)}\right)^{\frac{1}{2}} \sqrt{\frac{2\pi}{(n+1)k} \frac{\sqrt{1-\alpha^2}}{\alpha}},$$

hence

$$I_1 \sim c(\mathbb{B}^n, k) \sqrt{\frac{2\pi}{(n+1)k}} \frac{\sqrt{1-\alpha^2}}{\alpha} \int_0^{2\pi} d\varphi = c(\mathbb{B}^n, k) \sqrt{\frac{2\pi}{(n+1)k}} \frac{\sqrt{1-\alpha^2}}{\alpha} 2\pi \sim k^{n-\frac{1}{2}} c(n) \frac{\sqrt{1-\alpha^2}}{\alpha},$$

where $c(n) = \frac{(n+1)^{n-\frac{1}{2}}}{\pi^n} 2\pi^{\frac{n}{2}}$.

**Example 4.14.** Let $Y = X \subset \mathbb{B}^n (n \geq 2)$ be the disc of radius $\alpha \in (0, 1)$ in the $x_1x_2$-plane centered at $(x_1, x_2) = (0, 0)$. Thus, $X$ is defined by $x_1^2 + x_2^2 < \alpha^2$, $y_1 = y_2 = 0$, $z_j = 0$ for $j > 2$. Let $\nu_X = dx_1 \wedge dx_2$. For arbitrary $n \geq 2$ such $X$ is totally real. If $n = 2$, then $X$ is a Lagrangian submanifold of $\mathbb{B}^2$.

$$I_1 = c(\mathbb{B}^n, k) \int_X \int_X f\left(\frac{1-x_1^2-x_2^2}{1-x_1u_1-x_2u_2}\right)^{(n+1)k}dx_1dx_2du_1du_2 =$$

$$c(\mathbb{B}^n, k) \int_X \int_X e^{\frac{(n+1)k}{2} \ln \frac{(1-x_1^2-x_2^2)(1-u_1^2-u_2^2)}{(1-x_1u_1-x_2u_2)^2}} dx_1dx_2du_1du_2,$$

where $u_1 = \text{Re}(\zeta_1)$, $u_2 = \text{Re}(\zeta_2)$. For fixed $u_1$, $u_2$ let $f(x_1, x_2) = -\ln \frac{(1-x_1^2-x_2^2)(1-u_1^2-u_2^2)}{(1-x_1u_1-x_2u_2)^2}$. We have: $f(u_1, u_2) = 0$,

$$\frac{\partial f}{\partial x_j} = 2\left(\frac{x_j}{1-x_1^2-x_2^2} - \frac{u_j}{1-x_1u_1-x_2u_2}\right), \quad j = 1, 2$$

$$\left. \frac{\partial f}{\partial x_1}\right|_{(u_1, u_2)} = \left. \frac{\partial f}{\partial x_2}\right|_{(u_1, u_2)} = 0$$

$$\frac{\partial^2 f}{\partial x_j^2} = 2\left(\frac{1-x_1^2-x_2^2 + 2x_j^2}{(1-x_1^2-x_2^2)^2} - \frac{u_j^2}{(1-x_1u_1-x_2u_2)^2}\right), \quad j = 1, 2$$

$$\left. \frac{\partial^2 f}{\partial x_1\partial x_2}\right|_{(u_1, u_2)} = 2\left(\frac{2x_1x_2}{(1-x_1^2-x_2^2)^2} - \frac{u_1u_2}{(1-x_1u_1-x_2u_2)^2}\right)$$

$$\left. \frac{\partial^2 f}{\partial x_1^2}\right|_{(u_1, u_2)} = 2\left(\frac{1-u_1^2}{(1-u_1^2-u_2^2)^2}\right) > 0$$

$$H(u_1, u_2) = \left. \left(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1\partial x_2}\right)^2\right)\right|_{(u_1, u_2)} = \frac{4}{(1-u_1^2-u_2^2)^3} > 0$$

Using Laplace approximation in $\mathbb{R}^2$ (\cite{[43]} p. 495 or Theorem 2 \cite{[22]}) we get: for a fixed $\zeta$

$$\int_X e^{-\frac{(n+1)k}{2} f(x_1, x_2)} dx_1dx_2 \sim \frac{4\pi}{(n+1)k\sqrt{H(u_1, u_2)}} = \frac{2\pi}{(n+1)k} (1-u_1^2-u_2^2)^{\frac{3}{2}}$$
and
\[
I_1 \sim c(\mathbb{B}^n, k) \frac{2\pi}{(n+1)k} \int_X (1 - u_1^2 - u_2^2)^{\frac{3}{2}} du_1 du_2 = c(\mathbb{B}^n, k) \frac{4\pi^2}{5} \frac{1}{(n+1)k} (1 - (1 - \alpha^2)^{\frac{3}{2}}) \sim \\
\frac{k^{n+1}4\pi^2(n+1)^{n-1}}{n!} (1 - (1 - \alpha^2)^{\frac{3}{2}})
\]

**Example 4.15.** Let \( \beta \in (0, 1) \) and \( \alpha \in (\beta, 1) \) be fixed. Let \( Y \) be the line segment in \( \mathbb{B}^n \) \((n \geq 2)\) defined by \(-\beta < x_1 < \beta, y_1 = 0, z_j = 0 \) for \( j > 1 \), and let \( X \) be the disc defined by \( x_1^2 + x_2^2 < \alpha^2, y_1 = y_2 = 0, z_j = 0 \) for \( j > 2 \). Let \( \nu_X = dx_1 \wedge dx_2 \) and \( \nu_Y = dx_1 \).

\[
I_1 = c(\mathbb{B}^n, k) \int_Y \int_X \left( \frac{(1 - x_1^2 - x_2^2)(1 - u_1^2)}{(1 - x_1 u_1)^2} \right)^{(n+1)k} dx_1 dx_2 du_1 = \\
c(\mathbb{B}^n, k) \int_Y \int_X e^{(n+1)k} ln \left( \frac{1 - x_1^2 - x_2^2(1 - u_1^2)}{(1 - x_1 u_1)^2} \right) dx_1 dx_2 du_1,
\]

where \( u_1 = Re(\zeta_1) \). For a fixed \( u_1 \) let \( f(x_1, x_2) = -\ln \left( \frac{1 - x_1^2 - x_2^2(1 - u_1^2)}{(1 - x_1 u_1)^2} \right) \). We have: \( f(u_1, 0) = 0 \),

\[
\frac{\partial f}{\partial x_1} = 2\left( \frac{x_1}{1 - x_1^2 - x_2^2} - \frac{u_1}{1 - x_1 u_1} \right), \quad \frac{\partial f}{\partial x_2} = \frac{2x_2}{1 - x_1^2 - x_2^2}
\]

\[
\frac{\partial^2 f}{\partial x_1^2} = 2\left( \frac{1 + x_1^2 - x_2^2(1 - u_1^2)^2}{(1 - x_1^2 - x_2^2)^2} - \frac{u_1^2}{(1 - x_1 u_1)^2} \right), \quad \frac{\partial^2 f}{\partial x_2^2} = \frac{2}{1 - x_1^2 - x_2^2}
\]

\[
H(u_1, 0) = \left( \frac{\partial^2 f}{\partial x_1^2} \frac{\partial^2 f}{\partial x_2^2} - \left( \frac{\partial^2 f}{\partial x_1 \partial x_2} \right)^2 \right) \bigg|_{(u_1, 0)} = \frac{4}{(1 - u_1^2)^3} > 0
\]

Using Laplace approximation in \( \mathbb{R}^2 \) ([43] p. 495 or Theorem 2 [22]) we get: for a fixed \( \zeta \)

\[
\int_X e^{-\frac{(n+1)k}{2} f(x_1, x_2)} dx_1 dx_2 \sim \frac{4\pi}{(n+1)k \sqrt{H(u_1, 0)}} = \frac{2\pi}{(n+1)k} (1 - u_1^2)^{\frac{3}{2}}
\]

and as \( k \to \infty \)

\[
I_1 \sim c(\mathbb{B}^n, k) \frac{2\pi}{(n+1)k} \int_{-\beta}^{\beta} (1 - u_1^2)^{\frac{3}{2}} du_1 \sim \text{const}(n, \beta) k^{n-1}.
\]

**Example 4.16.** Let \( \alpha \in (0, 1) \) be fixed. Define a submanifold of \( \mathbb{B}^n, n \geq 2 \), by

\[
X = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{B}^n | x_1^2 + y_1^2 + x_2^2 < \alpha^2, x_2 > 0, y_2 = 0, x_j = y_j = 0 \text{ for } j > 2 \}
\]

and set \( \nu_X = dx_1 \wedge dy_1 \wedge dx_2 \).

As a remark, we point out that \( X \) is a CR submanifold which is not totally real and not complex (see subsection A.3 of the Appendix).
We will now estimate $I_1$, with $X = Y$:

$$I_1 = c(\mathbb{R}^n, k) \int_X \int_X \left( \frac{\langle z, z \rangle \langle \zeta, \zeta \rangle}{\langle -z, \zeta \rangle} \right)^{(n+1)k/2} \nu_X(z) \nu_X(\zeta).$$

We set all coordinates, except for $x_1$, $y_1$ and $x_2$, to be zero. We will use the spherical coordinates:

$$x_1 = \rho \sin \Phi \cos \Theta,$$
$$y_1 = \rho \sin \Phi \sin \Theta,$$
$$x_2 = \rho \cos \Phi,$$
$$0 < \rho < \alpha, \ 0 \leq \Theta < 2\pi, \ 0 \leq \Phi < \frac{\pi}{2},$$
$$u_1 = r \sin \psi \cos \beta,$$  
$$v_1 = r \sin \psi \sin \beta,$$  
$$u_2 = r \cos \psi,$$
$$0 < r < \alpha, \ 0 \leq \beta < 2\pi, \ 0 \leq \psi < \frac{\pi}{2},$$

where $\zeta_1 = u_1 + iv_1$ $(u_1, v_1 \in \mathbb{R})$, $Re(\zeta_2) = u_2$. The integral becomes

$$I_1 = c(\mathbb{R}^n, k) \int_X \int_X \left[ \frac{(1 - x_1^2 - y_1^2 - x_2^2)(1 - u_1^2 - v_1^2 - u_2^2)}{(1 - (x_1 + iy_1)(u_1 - iv_1) - x_2u_2)^{(n+1)k/2}} \right] dx_1dx_2du_1dv_1du_2.$$

For fixed $u_1, v_1, u_2$, the integral

$$\int_X \frac{(1 - x_1^2 - y_1^2 - x_2^2)^{(n+1)k/2}}{(1 - (x_1 + iy_1)(u_1 - iv_1) - x_2u_2)^{(n+1)k}} \rho \sin \Phi \ d\rho \ d\Phi \ d\Theta =$$

$$\int_0^{\frac{\pi}{2}} \sin \Phi \int_0^\alpha (1 - \rho^2)^{(n+1)k/2} \rho^2 d\rho \int_0^{2\pi} \frac{1}{(1 - \rho \sin \Phi e^{i\Theta}(u_1 - iv_1) - u_2 \rho \cos \Phi)^{(n+1)k}} d\Theta.$$  

Apply the Laplace method ([13] Theorem 3 p. 495) to the integral

$$\int_0^\alpha (1 - \rho^2)^{(n+1)k/2} \rho^2 \frac{1}{(1 - u_2 \rho \cos \Phi)^{(n+1)k}} d\rho = \int_0^\alpha e^{-(n+1)k f(\rho)} \rho^2 d\rho,$$

where

$$f(\rho) = -\ln \left( \frac{\sqrt{1 - \rho^2}}{1 - u_2 \rho \cos \Phi} \right).$$

We have:

$$\frac{df}{d\rho} = -\frac{\rho - u_2 \cos \Phi}{(1 - \rho^2)(1 - u_2 \rho \cos \Phi)},$$

$$\frac{d^2f}{d\rho^2} \bigg|_{\rho = u_2 \cos \Phi} = \frac{1}{(1 - (u_2 \cos \Phi)^2)^2} > 0,$$
and as $k \to \infty$
\[
\int_0^\alpha e^{-(n+1)kf(\rho)} \rho^2 d\rho \sim e^{-(n+1)kf(u_2 \cos \Phi)} \left[ \sqrt{\frac{2\pi}{(n+1)kf''(u_2 \cos \Phi)}} (u_2 \cos \Phi)^2 + c_1 ((n+1)k)^{-\frac{5}{2}} + c_2 ((n+1)k)^{-\frac{7}{2}} + \ldots \right]
\]
where the constants $c_1, c_2, \ldots$ depend on $\Phi$ and $u_2$. So,
\[
\int_0^\alpha e^{-(n+1)kf(\rho)} \rho^2 d\rho \sim (1 - (u_2 \cos \Phi)^2)^{-(n+1)k} \left[ \sqrt{\frac{2\pi}{(n+1)k}} (u_2 \cos \Phi)^2 (1 - (u_2 \cos \Phi)^2) + c_1 ((n+1)k)^{-\frac{5}{2}} + c_2 ((n+1)k)^{-\frac{7}{2}} + \ldots \right].
\]
Therefore
\[
I_1 \sim c(\mathbb{B}^n, k) 2\pi \int_X \int_0^{\frac{\pi}{2}} (1 - (u_2 \cos \Phi)^2)^{-(n+1)k} \left[ \sqrt{\frac{2\pi}{(n+1)k}} (u_2 \cos \Phi)^2 (1 - (u_2 \cos \Phi)^2) + c_1 ((n+1)k)^{-\frac{5}{2}} + c_2 ((n+1)k)^{-\frac{7}{2}} + \ldots \right] \sin \Phi \ d\Phi \ du_1 du_2 =
\]
\[
c(\mathbb{B}^n, k) 2\pi \int_0^{\frac{\pi}{2}} d\beta \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} \sin \Phi \ r^2 \sin \psi \ d\Phi \ dr \ d\psi =
\]
\[
c(\mathbb{B}^n, k) 4\pi^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} \sin \Phi \ r^2 \sin \psi \ dr \ d\Phi \ d\psi =
\]
\[
c(\mathbb{B}^n, k) 4\pi^2 \sqrt{\frac{2\pi}{(n+1)k}} \int_0^{\frac{\pi}{2}} (\cos \psi)^2 \sin \psi \int_0^{\frac{\pi}{2}} (\cos \Phi)^2 \sin \Phi \int_0^{\alpha} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} \sin \Phi \ r^2 \sin \psi \ dr d\Phi d\psi +
\]
\[
c(\mathbb{B}^n, k) 4\pi^2 \int_0^{\frac{\pi}{2}} \sin \psi \int_0^{\frac{\pi}{2}} \sin \Phi \int_0^{\alpha} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} \sin \Phi \ r^2 \sin \psi \ dr d\Phi d\psi +
\]
\[
\left[ c_1 ((n+1)k)^{-\frac{5}{2}} + c_2 ((n+1)k)^{-\frac{7}{2}} + \ldots \right] (1 - r^2)^{-(n+1)k} r^2 \ dr \ d\Phi \ d\psi.
\]
For fixed values of $\Phi, \Psi$, apply the Laplace method (Theorem 3 p. 495) to the integral
\[
\int_0^\alpha e^{-(n+1)kf(r)} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} r^4 dr = \frac{1}{2} \int_{-\alpha}^\alpha e^{-(n+1)kf(r)} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} r^4 dr
\]
where
\[
f(r) = -\frac{1}{2} \ln \left( \frac{1 - r^2}{1 - (r \cos \psi \cos \Phi)^2} \right).
\]
We get:
\[
\frac{df}{dr} = \frac{r - r(\cos \Phi \cos \psi)^2}{(1 - r^2)(1 - (r \cos \Phi \cos \psi)^2)}
\]
and as \( k \to \infty \)
\[
\int_{-\alpha}^{\alpha} e^{-(n+1)kf(r)}(1 - (r \cos \psi \cos \Phi)^2)r^4dr \sim \]
\[
e^{-\frac{2\pi}{(n+1)kf''(0)}(1 - (r \cos \psi \cos \Phi)^2)r^4}\left|_{r=0} + O(k^{-\frac{3}{2}}) \right].
\]

We also have, for fixed values of \( \Phi, \psi \):
\[
\left| \int_{0}^{\alpha} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} \frac{1}{2} \left[ c_1((n+1)k)^{-\frac{3}{2}} + c_2((n+1)k)^{-\frac{3}{2}} + \ldots \right] (1 - r^2)^{\frac{(n+1)k}{2}}r^2dr \right|
\]
\[\leq \text{const } k^{-3},\]

since by the Laplace method, as above,
\[
\int_{0}^{\alpha} (1 - (r \cos \psi \cos \Phi)^2)^{-(n+1)k} \frac{1}{2} (1 - r^2)^{\frac{(n+1)k}{2}}r^2dr = \frac{1}{2} \int_{-\alpha}^{\alpha} e^{-(n+1)kf} \ln \frac{1 - (r \cos \psi \cos \Phi)^2}{1 - r^2} r^2dr \sim \]
\[O(k^{-\frac{3}{2}}).\]

We recall: \( c(B^n,k) \sim \frac{(n+1)^n}{n!}k^n(1 + O(\frac{1}{k})) \) (Remark 4.4). Combining all this together, we conclude:
\[|I_1| \leq \text{const } k^{n-2}.\]

It follows that
\[(\Theta^{(j,k)}_X, \Theta^{(j,k)}_X) \leq \text{const } k^{n-2}.\]

We have: \( \dim_{\mathbb{R}} X = 3 \), for \( n > 2 \) we are in the setting of part (i) of Theorem 4.9 the asymptotic inequality holds and it is a strict inequality, and the asymptotic behaviour we observe here is different from the asymptotics for totally real submanifolds in Theorem 4.9(ii), for which \( (\Theta^{(j,k)}_X, \Theta^{(j,k)}_X) \) would have been asymptotic to \( Ck^{n-\frac{4}{2}} \) with \( C > 0 \).

Remark 4.17. In [9] it is shown that the square of the norm of the sections of powers of the line bundle associated to a Bohr-Sommerfeld Lagrangian submanifold of a compact \( n \)-dimensional Kähler manifold grows as a constant times \( k^{\frac{7}{2}} \). In [23] an analogous result for a \( q \)-dimensional isotropic Bohr-Sommerfeld submanifold of a \( 2n \)-dimensional symplectic manifold and associated sections of vector bundles gives the leading term of \( \text{const } k^{n-\frac{4}{2}} \).

Our Theorem 4.9(ii) gives the same leading term for norms of vector-valued automorphic forms associated to isotropic submanifolds of ball quotients. In the Example 4.16 we have a submanifold \( X \) which is not isotropic and \( (\Theta^{(j,k)}_X, \Theta^{(j,k)}_X) \) does not have the same kind of asymptotics. All this raises a general question how the geometric properties of submanifolds are reflected in asymptotics of the associated sections of vector bundles.
A.1. Proof of Lemma 3.1. First, we observe that this statement is contained in the general framework of Chapters 5 and 7 of [30]. Now we will present an actual proof. This is a modification of the proof of Prop. 1, p. 44 [3], which will use that for a fixed framework of Chapters 5 and 7 of [30]. Now we will present an actual proof. This is a Proof of Lemma 3.1.

For $a \in B$ where $w = \gamma z$. We get:

$$\sum_{\gamma \in \Gamma} |K(\gamma a, p)^2 J(\gamma, a)^2| \leq \frac{1}{\delta} \sum_{\gamma \in \Gamma, \gamma P_a} |K(w, p)|^2 dV_e(w) \leq \frac{m_0}{\delta} \int_D |K(w, p)|^2 dV_e(w).$$

The last inequality is justified by observing that if $\gamma P_a \cap \gamma' P_a \neq \emptyset$ for $\gamma, \gamma' \in \Gamma$, then $\gamma^{-1} \gamma' \in \{g \in \Gamma | gB \cap B \neq \emptyset\}$, so each $w \in D$ is in at most $m_0$ of the sets $\gamma P_a$, $\gamma \in \Gamma$. This settles the case $k = 2$. Therefore for $a \in A$ $|K(\gamma a, p)J(\gamma, a)| < 1$ for all but at most finitely many $\gamma \in \Gamma$. When $|K(\gamma a, p)J(\gamma, a)| < 1$, $|K(\gamma a, p)J(\gamma, a)|^k$ is a decreasing function of $k \geq 2$. The desired statement follows. \hfill $\Box$

A.2. Proof of Lemma 4.3. From [3] for $D = \mathbb{B}^n$ with $z = 0$ we get:

$$c(\mathbb{B}^n, k) \frac{n!}{\pi^n} \int_{\mathbb{B}^n} (-\langle w, w \rangle)^{(n+1)(k-1)} \left(\frac{i}{2}\right)^n dw_1 \wedge dw_1 \wedge ... \wedge dw_n \wedge d\bar{w}_n = 1.$$ 

The integral in the left hand side is equal to $\pi^n \frac{((n+1)(k-1))!}{(n+1)(k-1)+n!}$, and the statement readily follows. To calculate this integral apply a change of variables $(w_1, \bar{w}_1) \rightarrow (R_1, \varphi_1)$, where $0 \leq R_1 \leq 1$, $0 \leq \varphi_1 < 2\pi$, $w_1 = R_1 e^{i\varphi_1} \sqrt{1 - |w_2|^2 - ... - |w_n|^2}$. We get:

$$\int_{\mathbb{B}^n} (1 - |w_1|^2 - ... - |w_n|^2)^{(n+1)(k-1)} \left(\frac{i}{2}\right)^n dw_1 \wedge dw_1 \wedge ... \wedge dw_n \wedge d\bar{w}_n =$$

$$\left(\frac{i}{2}\right)^{n-1} \int_{\mathbb{B}^n} (1 - |w_2|^2 - ... - |w_n|^2)^{(n+1)(k-1)+1} (1 - R_1^2)^{(n+1)(k-1)} R_1 dR_1 \wedge d\varphi_1 \wedge dw_2 \wedge d\bar{w}_2 \wedge ... \wedge dw_n \wedge d\bar{w}_n,$$ 

then apply the change of variables $(w_2, \bar{w}_2) \rightarrow (R_2, \varphi_2)$, where $0 \leq R_2 \leq 1$, $0 \leq \varphi_2 < 2\pi$, $w_2 = R_2 e^{i\varphi_2} \sqrt{1 - |w_3|^2 - ... - |w_n|^2}$ to transform the integral into

$$\left(\frac{i}{2}\right)^{n-2} \int_{\mathbb{B}^n} (1 - |w_3|^2 - ... - |w_n|^2)^{(n+1)(k-1)+2} (1 - R_2^2)^{(n+1)(k-1)} R_1 (1 - R_2^2)^{(n+1)(k-1)+1} R_2 dR_1 \wedge d\varphi_1 \wedge$$

$$dR_2 \wedge d\varphi_2 \wedge dw_3 \wedge d\bar{w}_3 \wedge ... \wedge dw_n \wedge d\bar{w}_n,$$
and so on. At the end we get:

\[ (2\pi)^{n-1} \int_0^1 (1 - R_1^2)^{(n+1)(k-1)} R_1 dR_1 \int_0^1 (1 - R_2^2)^{(n+1)(k-1)+1} R_2 dR_2 \ldots \]

\[ \int_0^1 (1 - R_{n-1}^2)^{(n+1)(k-1)+n-2} R_{n-1} dR_{n-1} \int_{|w_n| \leq 1} (1 - |w_n|^2)^{(n+1)(k-1)+(n-1)} \frac{i}{2} dw_n \wedge d\bar{w}_n, \]

and with \( w_n = R_n e^{i\varphi_n}, 0 \leq R_n \leq 1, 0 \leq \varphi_n < 2\pi \), the last integral is

\[ 2\pi \int_0^1 (1 - R_n^2)^{(n+1)(k-1)+n-1} R_n dR_n. \]

An elementary calculation now yields the answer. □

A.3. Note for Example 4.16. In this subsection we explain why the submanifold

\[ X = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{B}^n \mid x_1^2 + y_1^2 + x_2^2 < \alpha^2, x_2 > 0, y_2 = 0, x_j = y_j = 0 \text{ for } j > 2 \}. \]

of \( \mathbb{B}^n \), where \( n \geq 2 \) and \( \alpha \in (0, 1) \), is a CR submanifold, which is not totally real and not complex. The relevant part of the complex hyperbolic metric (3.3)\textsuperscript{[19]} is, up to a positive factor,

\[ \frac{1}{(1 - x_1^2 - y_1^2 - x_2^2 - y_2^2)^2} [(1 - x_1^2 - y_1^2)(dx_1^2 + dy_1^2) + (1 - x_1^2 - y_1^2)(dx_2^2 + dy_2^2) + 2(x_1 y_2 - y_1 x_2)(dx_1 dx_2 + dy_1 dy_2) + 2(-y_1 x_2 + x_1 y_2)(dx_1 dy_2 - dy_1 dx_2)]. \]

The complex structure \( J : T\mathbb{B}^n \to T\mathbb{B}^n \) acts as follows:

\[ \frac{\partial}{\partial x_j} \mapsto -\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial y_j} \mapsto \frac{\partial}{\partial x_j} \]

for \( j = 1, 2 \).

The distribution \( D = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\} \) is a holomorphic distribution on \( X \). The complementary orthogonal distribution \( D^\perp = \text{span}\{\frac{\partial}{\partial x_2} - \frac{x_1 y_2}{1-x_2^2} \frac{\partial}{\partial x_1} - \frac{x_2 y_1}{1-x_2^2} \frac{\partial}{\partial y_1}\} \) is a totally real distribution.

Indeed, \( J(\frac{\partial}{\partial x_2} - \frac{x_1 y_2}{1-x_2^2} \frac{\partial}{\partial x_1} - \frac{x_2 y_1}{1-x_2^2} \frac{\partial}{\partial y_1}) \) is orthogonal to \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} \), and is, therefore, in the normal bundle of \( X \).

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