On the $1 + 3$ Formalism in General Relativity

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Abstract We present in this paper the formalism for the splitting of a four-dimensional Lorentzian manifold by a set of time-like integral curves. Introducing the geometrical tensors characterizing the local spatial frames induced by the congruence (namely, the spatial metric tensor, the extrinsic curvature tensor and the Riemann curvature tensor), we derive the Gauss, Codazzi and Ricci equations, along with the evolution equation for the spatial metric. In the present framework, the spatial frames do not form any hypersurfaces as we allow the congruence to exhibit vorticity. The splitting procedure is then applied to the Einstein field equation and it results in an equivalent set of constraint and evolution equations. We discuss the resulting systems and compare them with the ones obtained from the 3+1 formalism, where the manifold is foliated by means of a family of three-dimensional space-like surfaces.

Keywords general relativity · differential geometry · splitting of space–time · threading of space–time · Gauss equation · Codazzi equation · Ricci equation

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1 Introduction

1.1 3 + 1 and 1 + 3 splittings

The Einstein field equation,

$$^{4}R - \frac{1}{2}^{4}Rg + \Lambda g = \frac{8\pi G}{c^4}T,$$

expresses from a four-dimensional point of view the dynamical coupling between the geometry and the content of the Universe. Recasting this relation into a set
of constraint and evolution equations in a three-dimensional framework allows for a more familiar and intuitive examination of the physical system at stake. It can bring for instance better understanding about the kinematical and geometrical aspects of particular relativistic solutions for which the field equations can be directly derived. It can also be of use, for instance, in the post-Newtonian treatment of weak gravitational fields and in the study of nonlinear structure formation in cosmology.

This reformulation is realized upon splitting the space–time into space and time, and then upon applying the prescription to Einstein’s general relativity. The splitting procedure itself can be carried out under two different perspectives: either by using the so-called 3+1 formalism, or slicing of space–time [1–10], or by using the so-called 1+3 formalism, or threading of space–time [11–31].

The first of these techniques is based upon the introduction of a preferred family of three-dimensional space-like surfaces, through the level sets of some scalar field. It provides a global space-like association of points. No other relation is assumed, and the identification of points located on different slices can be performed arbitrarily.

The second class of splitting is built upon a set of time-like integral curves, and it affords a global time-like relation between points. No other condition is supposed; however, in the case where the congruence exhibits vorticity, the local spatial frames orthogonal to the integral curves do not form any family of hypersurfaces.

When both the space-like and time-like conditions hold, the two splittings can be applied on the manifold. In the generic configuration, the manifold is covered by a family of space-like hypersurfaces and an independent set of time-like curves. The simplest case is made of a vorticity-free time-like congruence, the orthogonal frames of which globally form spatial hypersurfaces.

In the framework of general relativity, an alternative to the above procedures consists in splitting directly the Einstein field equation with the aid of the kinematics of the fluid filling the space–time. This prescription disregards the geometry of the spatial frames and places emphasis instead on the kinematical quantities of the fluid congruence (namely, on its expansion rate, shear and vorticity) [35–42]. The drawback of this approach lies in the impossibility of defining geometrically the extrinsic and Riemann curvatures of the spatial frames, and this change of perspective (from the kinematical to the geometrical point of view) can be requested as it constitutes an important aspect of Einstein’s theory. In the threading picture, on the contrary, this equivalence (from the geometrical to the kinematical point of view) is straightforward.

The present article offers a detailed description of the threading of a four-dimensional manifold. We shall provide the Gauss, Codazzi and Ricci equations associated with the congruence of curves, along with the evolution equation for the metric of the local spatial frames. We shall then apply the splitting to the Einstein field equation. As we shall see, the resulting sets of 1+3 equations closely resemble their 3+1 counterparts, with additional terms stemming from the non hypersurface-forming character of the spatial frames.

1 The reader is referred to the comprehensive reviews [32] and [33] for an introduction (and a historical presentation) of the 3+1 and 1+3 approaches, respectively.

2 This approach is also sometimes entitled ‘ADM formalism’ (for Arnowitt, Deser and Misner [34]). This denomination should be reserved, however, for the Hamiltonian formulation of general relativity only (see [32] for further comments).
Compared with previous works making use of the threading perspective, we therefore establish a formal correspondence with the 3+1 approach (i) by supplying the mentioned sets of equations, and (ii) by writing them with respect to an arbitrary four-dimensional vector basis and its dual. In preceding analyses, either the former or the latter item was not realized.

We do not intend, however, to provide an exhaustive investigation of the subject, and we shall restrict ourselves to the derivation of the mentioned equations along with the associated material. A unified and thorough description of the 1+3 and 3+1 techniques was supplied by Jantzen and collaborators in [25, 26, 33]. Notably, the 1+3 decomposition of the four-Riemann tensor (hence the Gauss, Codazzi and Ricci equations) can be found in [33], but it is valid in a specific basis only (the one adapted to the congruence of curves). As mentioned earlier, we provide in the present paper the general formulation for this decomposition (and hence the general formulation for the threaded Einstein equations). In addition to allow for a transparent comparison with the 3+1 approach, such an extension is needed in order to work along a specific congruence (that involved in the threading of space–time) while permitting the space-like basis to move along any other congruence.4,5

We proceed as follows. In Section 2, we recall the definitions of the covariant derivative and of the Lie derivative. This is followed, in Section 3, by the introduction of the two fundamental forms of the local spatial frames. Section 4 is devoted to the presentation of the spatial covariant derivative, while Section 5 focuses on the definition of the spatial curvature tensors. In Section 6, we provide in a first part the different projections of the four-Riemann curvature, which result in the Gauss, Codazzi and Ricci equations. The evolution equation for the spatial metric is also given. In a second part, we provide the different projections of the four-Ricci tensor. (All these relations are purely geometric and are valid independently of the gravitational theory.) We apply the splitting procedure to the Einstein field equation in Section 7. Finally, in Section 8, we summarize and discuss our results.

This presentation is complemented by two appendices. Appendix A provides the reformulation of the 1+3 Einstein equations in terms of the kinematical quantities of the fluid, and Appendix B is dedicated to the construction of bases and coordinates adapted to the congruence.

1.2 Notation and conventions

We consider in what follows a smooth four-dimensional manifold $\mathcal{M}$ endowed with the metric tensor $g$ of signature $(−, +, +, +)$, and a set of time-like integral curves $C$ in $\mathcal{M}$ described by the unit tangent vector field $u$. Our analysis will be conducted locally at a generic point $p$ of $\mathcal{M}$.

3 Our derivation and most of the notation we use are inspired from Gourgoulhon’s book on the slicing formalism [32].
4 This feature will be used in forthcoming works on tilted cosmologies and cosmological deviation theory.
5 For completeness, let us mention that the threaded approach to the thermodynamics of a perfect fluid is given in [43].
6 The main results of the paper are gathered in Propositions S1 to S4 and Corollaries S1 to S3 for the splitting of the manifold (‘S’ for splitting), and in Propositions E1 to E3 for the 1+3 form of Einstein’s equation (‘E’ for Einstein).
1.2.1 Tangent, cotangent spaces and canonical isomorphism

We denote by $T_p(M)$ and $T^*_p(M)$, respectively, the four-dimensional spaces of vectors and 1-forms on $M$ at $p$. These spaces are mapped onto each other by means of the canonical isomorphism induced by the metric. We denote by a flat the isomorphism $T_p(M) \to T^*_p(M)$ and by a sharp the reverse isomorphism $T^*_p(M) \to T_p(M)$. Hence

- for any vector $v$ in $T_p(M)$, $v^\flat$ stands for the unique linear form in $T^*_p(M)$ such that
  \[ \forall w \in T_p(M) \quad \langle v^\flat, w \rangle := v \cdot w, \] (1)

- for any 1-form $\omega$ in $T^*_p(M)$, $\omega^\sharp$ stands for the unique vector in $T_p(M)$ such that
  \[ \forall v \in T_p(M) \quad \omega^\sharp \cdot v := \langle \omega, v \rangle. \] (2)

This mapping is extended to multilinear forms as follows. For any tensor field $T$ of type $(0,2)$ and any two vectors $v$ and $w$ on $M$, we denote

- by $T^\sharp$ the tensor field of type $(1,1)$ such that
  \[ T^\sharp(v^\flat, w) := T(v, w), \] (3)

- by $T^{\sharp\sharp}$ the tensor field of type $(2,0)$ such that
  \[ T^{\sharp\sharp}(v^\flat, w^\flat) := T(v, w). \] (5)

The mapping of forms of higher types is defined following the same prescription.

Here and in the sequel, we employ a dot to indicate the scalar product of two vector fields taken with $g$,

\[ \forall (v, w) \in T_p(M) \times T_p(M) \quad v \cdot w := g(v, w), \]

and angle brackets to represent the action of linear forms on vector fields,

\[ \forall (\omega, v) \in T^*_p(M) \times T_p(M) \quad \langle \omega, v \rangle := \omega(v). \]

1.2.2 Spatial and temporal spaces

The local spatial frames induced by the congruence of curves (orthogonal to $u$) are collectively referred to as $\mathcal{F}_u$. They do not form any family of hypersurfaces in the present framework as we allow the congruence to manifest vorticity.

We denote by $\mathcal{T}_p(\mathcal{F}_u)$ the three-dimensional space of spatial vectors at $p$, such that

\[ \forall v \in \mathcal{T}_p(\mathcal{F}_u) \quad v \cdot u = 0, \]

and by $\mathcal{T}^*_p(\mathcal{F}_u)$ the three-dimensional space of spatial 1-forms at $p$, such that

\[ \forall \omega \in \mathcal{T}^*_p(\mathcal{F}_u) \quad \langle \omega, u \rangle = 0. \]
The one-dimensional space of \textit{temporal} vectors is identified by $\text{Vect}_p(u)$, and its elements are such that
\[
\forall v \in \text{Vect}_p(u) \quad \exists \lambda \in \mathbb{R} \quad v = \lambda u,
\]
and the one-dimensional space of \textit{temporal} 1-forms is identified by $\text{Vect}_p(u^\flat)$, with
\[
\forall \omega \in \text{Vect}_p(u^\flat) \quad \exists \lambda \in \mathbb{R} \quad \omega = \lambda u^\flat.
\]
A tensor field on $\mathcal{M}$ will be called spatial (resp. temporal) if it vanishes whenever one of its arguments is temporal (resp. spatial).

From the above definitions we write the orthogonal decomposition of the tangent space $T_p(\mathcal{M})$ as
\[
T_p(\mathcal{M}) = \mathcal{T}_p(F_u) \oplus \text{Vect}_p(u),
\]
and that of the cotangent space $T^*_p(\mathcal{M})$ according to
\[
T^*_p(\mathcal{M}) = \mathcal{T}^*_p(F_u) \oplus \text{Vect}_p(u^\flat).
\]

1.2.3 Bases and components

We denote by $\{e_\alpha\}$ an arbitrary basis of $T_p(\mathcal{M})$ and by $\{e^\alpha\}$ the dual basis in $T^*_p(\mathcal{M})$ satisfying by definition
\[
\langle e^\alpha, e^\beta \rangle := \delta^\alpha_\beta.
\]
If not explicitly specified, and unless otherwise stated, the components of any tensor are written with respect to these bases. We have for a vector $v$ and a 1-form $\omega$
\[
v = \langle e^\alpha, v \rangle e_\alpha =: v^\alpha e_\alpha, \quad \omega = \langle \omega, e_\alpha \rangle e^\alpha =: \omega_\alpha e^\alpha,
\]
and for a tensor $T$ of type $(k, l)$
\[
T =: T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \otimes e^{\beta_1} \otimes \ldots \otimes e^{\beta_l},
\]
with
\[
T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l} := T(e^{\alpha_1}, \ldots, e^{\alpha_k}, e_{\beta_1}, \ldots, e_{\beta_l}).
\]
In particular, we write the components of the metric as $g_{\alpha\beta}$ and those of its inverse as $g^{\alpha\beta}$, with
\[
g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta.
\]
The components of the linear form $v^\flat$ associated with $v$ (cf. Eq. (1)) and those of the vector $\omega^\flat$ associated with $\omega$ (cf. Eq. (2)) are expressed with respect to the components of $v$ and $\omega$ respectively as
\[
(v^\flat)_\alpha =: v_\alpha = g_{\alpha\gamma} v^\gamma, \quad (\omega^\flat)^\alpha =: \omega_\alpha = g^{\alpha\gamma} \omega_\gamma,
\]
Latin letters refer to space-like counters and indices, running in $\{1, 2, 3\}$. The objects $\{e_i\}$ and $\{e^i\}$ are space-like but not necessarily spatial. (For instance, for the $\{e_i\}$, we have $e_i \cdot e_i > 0$ but not necessarily $e_i \cdot u = 0$.) We adopt Einstein’s summation convention over repeated letters.
and the components of the tensors $T^5$, $T^\cdot1$ and $T^{5\cdot}$ associated with $T$ (cf. Eqs. (3), (4) and (5)) are written in terms of those of $T$ according to

$$
(T^5)_{\alpha\beta} =: T^\alpha_{\beta} = g^{\alpha\gamma} T^\gamma_{\beta}, \quad (T^\cdot5)_{\alpha}^{\beta} =: T^\alpha_{\beta} = g^{\beta\gamma} T^\alpha_{\gamma}, \quad (T^{5\cdot})^{\alpha\beta} =: T^\alpha_{\beta} = g^{\alpha\gamma} g^{\beta\delta} T^\gamma_{\delta}.
$$

(We drop the musical symbols in the component notation for the sake of clarity. No confusion should arise.)

2 Derivative operators

2.1 Covariant derivative

We consider in this work the torsionless and metric-compatible connection $\nabla$ on $\mathcal{M}$.

2.1.1 Definition

The covariant derivative constructs from a tensor field $T$ of type $(k, l)$ a new tensor field $\nabla T$ of type $(k, l + 1)$, the components of which are written

$$(\nabla T)^{\alpha_1...\alpha_k}_{\beta_1...\beta_l\gamma}.$$

The coefficients of the connection with respect to the bases $\{e_\alpha\}$ and $\{e^\alpha\}$ are defined by

$$\nabla e_\alpha =: \Gamma^\gamma_{\delta\alpha} e_\gamma \otimes e^\delta \Leftrightarrow \nabla e^\alpha =: -\Gamma^{\alpha}_{\delta\gamma} e^\gamma \otimes e^\delta.$$  \hspace{1cm} (6)

Remark 2.1. We follow Hawking and Ellis’ convention [44] for the order of the covariant indices of the connection coefficients. Note, however, that this choice does not affect the form of our main results.\(^8\)

The covariant derivative of $T$ along a vector $v$ defines a tensor field of the same type as $T$; it is written

$$\nabla_v T := \nabla T(\cdot\cdot\cdot, v),$$

and in component form

$$v^\gamma (\nabla T)^{\alpha_1...\alpha_k}_{\beta_1...\beta_l\gamma}. \hspace{1cm} (7)$$

\(^8\) See Propositions S1 to S4, Corollaries S1 to S3 and Propositions E1 to E3.
2.1.2 Covariant derivative of a tensor

The components of the covariant derivative of a tensor $T$ of type $(k, l)$ are given by

$$T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l ; \gamma} = e_\gamma \left( T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l} \right) + \sum_{i=1}^{k} T^{\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_k}_{\beta_1 \ldots \beta_l} R^{\alpha_i}_{\gamma \delta} + \sum_{i=1}^{l} T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \hat{\beta}_i \ldots \beta_l} R^{\delta}_{\gamma \beta_i}$$

where we use henceforth the short-hand

$$T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l ; \gamma} := (\nabla T)^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l ; \gamma}.$$  

For a function $f$ on $\mathcal{M}$, they reduce to

$$f_{,\alpha} := (\nabla f)_{\alpha} = e_\alpha (f).$$

Remark 2.2. We use a semicolon for the component form of the covariant derivative of a tensor and a comma when it comes to a function.

2.1.3 Torsion tensor

The torsion of the connection is defined by

$$T(v, w) := \nabla_v w - \nabla_w v - [v, w],$$

or, alternatively, by

$$T(v, w)(f) := \nabla_v \nabla f(v, w) - \nabla_w \nabla f(v, w),$$

with $f$ being a function and $v$ and $w$ two vectors on $\mathcal{M}$. It vanishes identically since we consider the connection to be Levi-Civita.

2.1.4 Curvature tensors

The Riemann curvature tensor of $\mathcal{M}$ is given by

$$\nabla^4_{\text{Riem}}(\omega, w, v_1, v_2) := \langle \omega, \nabla_{v_1} \nabla_{v_2} w - \nabla_{v_2} \nabla_{v_1} w - \nabla_{[v_1, v_2]} w \rangle,$$

with $\omega$ in $\mathcal{T}^*_p(\mathcal{M})$ and $w, v_1$ and $v_2$ in $\mathcal{T}_p(\mathcal{M})$, and the Ricci curvature tensor is defined as

$$\nabla^4_{\text{R))(w, v) := \nabla^4_{\text{Riem}}(e^\alpha, v, e_\alpha, w).$$

For convenience we denote the components of $\nabla^4_{\text{Riem}}$ by $\nabla^4_{\text{Riem}}^{\alpha \beta \mu \nu}$ rather than $\nabla^4_{\text{Riem}}^{\alpha \beta \mu \nu}$. In the case of a torsion-free connection, we have

$$v^\mu_{,\beta ; \alpha} = v^\mu_{,\alpha ; \beta} = \nabla^4_{\text{Riem}}^{\alpha \beta \mu \nu} v^\nu,$$

for any vector $v$ on $\mathcal{M}$. 
$2.2 \text{ Lie derivative}$

$2.2.1 \text{ Definition}$

The Lie derivative of a tensor $T$ of type $(k, l)$ along a vector $v$ defines a new tensor $L_v T$ of type $(k, l)$, the components of which are written

$L_v T^\alpha_1...\alpha_k \beta_1...\beta_l := (L_v T)^\alpha_1...\alpha_k \beta_1...\beta_l$.

$2.2.2 \text{ Lie derivative of a tensor}$

For a torsion-free connection, the components of the Lie derivative of $T$ along a vector $v$ are given by

$L_v T^\alpha_1...\alpha_k \beta_1...\beta_l = v^\gamma T^\alpha_1...\alpha_k \beta_1...\beta_l;\gamma - \sum_{i=1}^k T^\alpha_1...\alpha_k \beta_1...\beta_i;\gamma v^\alpha_i;\gamma + \sum_{i=1}^l T^\alpha_1...\alpha_k \beta_1...\gamma...\beta_i;\gamma v^\gamma;\beta_i$.

For a function $f$ on $\mathcal{M}$, they read

$L_v f = v^\alpha f,\alpha$.

$2.2.3 \text{ Lie bracket and structure coefficients}$

The Lie bracket of two vectors $v$ and $w$ is defined by

$[v, w](f) := v(w(f)) - w(v(f))$.

The structure coefficients of the vector basis $\{e_\alpha\}$ are defined from the Lie brackets of its elements:

$[e_\alpha, e_\beta](f) := \Gamma^\gamma_{\alpha\beta} e_\gamma(f)$,

and they are anti-symmetric in their two last indices. For a torsionless connection we have

$\Gamma^\gamma_{\alpha\beta} = 2 \Gamma^\gamma_{[\alpha\beta]}$,

where the brackets indicate anti-symmetrization over the indices enclosed.

$3 \text{ Fundamental forms}$

We introduce in this section the two fundamental forms of the local spatial frames of the congruence.
3.1 First fundamental form

3.1.1 Orthogonal projector

The orthogonal projector onto the spatial frames of \( \mathcal{C} \) is defined by

\[
\gamma^\sharp : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{F}_u) \\
v \mapsto v + (\langle u^\mu, v \rangle) u^\mu \tag{17}
\]

for vectors, and by

\[
\gamma^\sharp : \mathcal{T}^*_p(\mathcal{M}) \to \mathcal{T}^*_p(\mathcal{F}_u) \\
\omega \mapsto \omega + (\langle \omega, u \rangle) u^\beta \tag{18}
\]

for 1-forms. Its components are given by

\[
(\gamma^\sharp)^{\alpha \beta} := \gamma^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta . \tag{19}
\]

We construct the spatial projection of tensors of higher types by requiring the fulfillment of the relation

\[
(\gamma^\sharp) (T \otimes S) = (\gamma^\sharp T) \otimes (\gamma^\sharp S), \tag{20}
\]

for any two tensors \( T \) and \( S \). The components of the spatial projection \( \gamma^\sharp T \) of a tensor \( T \) of type \((k, l)\) are then given by

\[
(\gamma^\sharp T)^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l} = \gamma^{\alpha_1}_{\mu_1} \ldots \gamma^{\alpha_k}_{\mu_k} \gamma^{\nu_1}_{\beta_1} \ldots \gamma^{\nu_l}_{\beta_l} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} .
\]

**Proof.** We decompose the tensor \( T \) into

\[
T = T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_l} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \otimes e_{\beta_1} \otimes \ldots \otimes e_{\beta_l} .
\]

Applying recursively (20) on the projected expression, and noticing with the help of (17) and (19) that

\[
\gamma^\sharp (e_{\alpha}) = \gamma^\beta_{\alpha} e_{\beta} ,
\]

we obtain the sought-after expression.

3.1.2 Metric on the spatial frames

The orthogonal projector \( \gamma^\sharp \) introduced by Eqs. (17), (18) and (19) defines the first fundamental form of the spatial frames

\[
\gamma := g + u^\beta \otimes u^\beta ,
\]

and in component form

\[
\gamma_{\alpha \beta} = g_{\alpha \beta} + u_\alpha u_\beta . \tag{21}
\]

When its domain of definition is restricted to \( \mathcal{T}_p(\mathcal{F}_u) \times \mathcal{T}_p(\mathcal{F}_u) \), the symmetric, non-degenerate, bilinear form \( \gamma \) plays the role of the (Riemannian) spatial metric on the local spatial frames.
3.2 Second fundamental form

3.2.1 Weingarten map

The Weingarten map (or shape operator) of the spatial frames associates to a spatial vector the covariant derivative of the flow vector \( u \) along that vector:

\[
\chi: \mathcal{T}_p(F_u) \to \mathcal{T}_p(F_u) \\
v \mapsto \nabla_v u.
\]

The image of \( \mathcal{T}_p(F_u) \) under \( \chi \) is indeed in \( \mathcal{T}_p(F_u) \) as \( u \) is unitary.

Remark 3.1. From the torsion-free character of the connection together with (10), we find

\[
v \cdot \chi(w) = w \cdot \chi(v) + u \cdot [v, w],
\]

for any spatial vectors \( v \) and \( w \). Because the collection of spatial frames does not form any hypersurfaces, the Lie bracket \([v, w]\) is not spatial (cf. Frobenius’ theorem in, e.g., [45]). The term \( u \cdot [v, w] \) therefore does not cancel and the shape operator is not self-adjoint.

3.2.2 Extrinsic curvature

The second fundamental form of the spatial frames is defined for two spatial vectors by

\[
k: \mathcal{T}_p(F_u) \times \mathcal{T}_p(F_u) \to \mathbb{R} \\
(v, w) \mapsto -v \cdot \chi(w).
\]

We extend this definition to arbitrary vectors on \( \mathcal{M} \) upon writing

\[
k: \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R} \\
(v, w) \mapsto -\gamma^*(v) \cdot \chi(\gamma^*(w)).
\]

With the definition of the shape operator we can formulate the extrinsic curvature (or deformation tensor) as

\[
k(v, w) = -\gamma^*(v) \cdot \nabla_{\gamma^*(w)} u.
\]  

(23)

Remark 3.2. Because of the non self-adjointness of the Weingarten map, the extrinsic curvature fails to be symmetric. We have indeed, from Eqs. (22) and (23),

\[
k(v, w) - k(w, v) = -u \cdot [v, w],
\]

for any spatial vectors \( v \) and \( w \).
3.2.3 Relation between $k$ and $\nabla u^k$

Making use of Eq. (23) along with Eq. (17) we deduce, for two arbitrary vectors $v$ and $w$,

$$k(v, w) = -\nabla u^k(v, w) - \langle a^k, v \rangle \langle u^k, w \rangle,$$

where $a := \nabla_u u$ stands for the curvature vector of the congruence. (Note that $a$ is spatial.) This expression can be equivalently written

$$\nabla u^k = -k - a^k \otimes u^k,$$

and in component form

$$u_{\alpha \beta} = -k_{\alpha \beta} - a_{\alpha} u_{\beta}.$$

**Remark 3.3.** In Appendix A, we decompose $\nabla u^k$ into the kinematical quantities of the fluid filling the space-time, the world lines of which are identified with the integral curves of $\mathcal{U}$. Expressing the extrinsic curvature in terms of the fluid expansion rate, shear and vorticity, we there provide the kinematical formulation of the 1+3 Einstein equations.

4 Spatial connection

This section is devoted to the presentation of the spatial connection associated with the connection $\nabla$ on $\mathscr{M}$.

4.1 Introduction

4.1.1 Definition

We define the connection $D$ on the local spatial frames of the congruence by

$$DT := \gamma^i(\nabla T),$$

for any spatial tensor $T$.\(^9\) The spatial covariant derivative constructs from a spatial tensor of type $(k, l)$ a new spatial tensor of type $(k, l+1)$, the components of which satisfy the relation

$$(DT)_{\alpha_1...\alpha_k}^{\beta_1...\beta_l} = \gamma^{\alpha_1}_{\mu_1} ... \gamma^{\alpha_k}_{\mu_k} \gamma^{\nu_1}_{\beta_1} ... \gamma^{\nu_l}_{\beta_l} \gamma^{\delta}_{\gamma} (\nabla T)_{\mu_1...\mu_k \nu_1...\nu_l}^{\gamma \delta}.$$ (28)

**Remark 4.1.** Although the spatial connection only applies to spatial tensors, the object $DT$ can accept as arguments vectors and 1-forms not necessarily spatial ($DT$ is a tensor defined on $\mathscr{M}$). Accordingly we can write four-dimensional components for this term.

The spatial covariant derivative of $T$ along an arbitrary vector $v$ defines a spatial tensor field of the same type as $T$. It is written

$$D_v T := DT(\ldots, v),$$

and in component form

$$v^\gamma (DT)_{\alpha_1...\alpha_k}^{\beta_1...\beta_l}.$$ (29)

\(^9\) We do not extend this definition to non-spatial tensors, as for such objects the operator $D$ would lose its character of derivative.
4.1.2 Spatial covariant derivative of a tensor

The components of the spatial covariant derivative of a spatial tensor $T$ of type $(k, l)$ are given by

$$T_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l || \gamma} = \gamma^{\alpha_1} \mu_1 \ldots \gamma^{\alpha_k} \mu_k \gamma^\delta \varepsilon_\delta \left( T_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_l} \right)$$

$$+ \sum_{i=1}^k T_{\alpha_1 \ldots \hat{i} \ldots \alpha_k \beta_1 \ldots \beta_l} \Gamma[D]_{\gamma \alpha_i}^\delta - \sum_{i=1}^l T_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \hat{i} \ldots \beta_l} \Gamma[D]_{\gamma \beta_i}^\delta ,$$

where we have used the short-hand

$$T_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l || \gamma} := (DT)_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \gamma}$$

and defined the coefficients of the spatial connection as

$$\Gamma[D]_{\gamma \alpha \beta} := \gamma_{\delta \gamma \alpha} \gamma_{\beta}^\rho \Gamma^\delta_{\rho \sigma} .$$

For a function $f$ on $\mathcal{M}$ they read

$$f_{|| \alpha} = \gamma^{\beta} \alpha f, \beta = \gamma^{\beta} \alpha e_\beta (f).$$

**Proof.** Equations (30) and (32) are respectively obtained from (8) and (9) and upon using (28).

Remark 4.2. We use two strokes for the component form of the spatial covariant derivative of a tensor and only one when it comes to a function.

Remark 4.3. The coefficients of the spatial connection are spatial in the sense that any contraction with $u^\alpha$ or $u_\alpha$ vanishes.

4.2 Properties

4.2.1 Relations between $D$ and $\nabla$

For a spatial tensor $T$ and an arbitrary vector $v$ on $\mathcal{M}$, we have

$$D_v T = D_{\gamma^\delta (v)} T = \gamma^\delta (\nabla_{\gamma^\delta (v)} T) .$$

**Proof.** Using the spatial character of $DT$, we reformulate the components of $D_v T$ given by Eq. (29) into

$$v^{\gamma} \gamma^\delta (DT)_{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \delta} .$$

Noticing with (17) and (19) that the terms $v^{\gamma} \gamma^\delta$ denote the components of $\gamma^\delta (v)$, we deduce the first equality. The second equality is obtained by applying (28) and (7) to the above expression.

For two spatial vectors $v$ and $w$, we have the relation

$$D_v w = \nabla_v w + k(w, v) u .$$
Proof. The components of $\nabla_v w$ are decomposed orthogonally according to

$$
v^\beta w^\alpha;_\beta = v^\beta \delta^\alpha_\gamma \delta^\gamma_\delta w^\delta;_\beta = v^\beta w^\alpha;_\beta - v^\delta u^\alpha w^\gamma;_\gamma;_\delta = v^\beta w^\alpha;_\beta + v^\delta u^\alpha w^\gamma;_\gamma;_\delta,
$$

where we have used Eq. (19) and the spatial character of $v$ for the second equality, and Eq. (28) and the spatial character of $w$ for the third equality. Inserting Eq. (26) into the last line we get

$$
v^\beta w^\alpha;_\beta = v^\beta w^\alpha;_\beta - u^\alpha w^\gamma;_\gamma;_\delta,
$$

which, written in tensor form, concludes the proof. □

Similarly, for a spatial vector $v$ and a spatial 1-form $\omega$ we can deduce

$$D_v \omega = \nabla_v \omega + k^\gamma(\omega, v) u^\gamma.$$

4.2.2 Compatibility with the spatial metric

The components of $D_\gamma$ are written by means of (28) and (21) as

$$\gamma^\alpha_\beta \parallel_\gamma = \gamma^\alpha_\beta \gamma^\gamma_\delta g_{\rho \sigma} \delta.$$

From the compatibility of $\nabla$ with respect to $g$ we obtain

$$D_\gamma = 0.$$

The spatial connection is therefore compatible with the spatial metric.

4.2.3 Torsion tensor

Following Eq. (10), we define the torsion tensor of the spatial connection as

$$T[D](v, w) := D_v w - D_w v - [v, w], \quad (35)$$

for any spatial vectors $v$ and $w$. We extend this definition to arbitrary vectors on $\mathcal{M}$ upon writing

$$T[D](v, w) := D_{\gamma(v)} w - D_{\gamma(w)} v - \left[\gamma(v), \gamma(w)\right]. \quad (36)$$

Developing the right-hand side with the help of (34), and using the torsion-free character of the manifold connection along with (10), we obtain

$$T[D](v, w) = \left(k(w, v) - k(v, w)\right) u, \quad (37)$$

for any vectors $v$ and $w$. The torsion of $D$ is generated by the anti-symmetric part of the extrinsic curvature tensor or, equivalently, by the temporal part of the Lie bracket of two spatial vectors (cf. Eq. (24)). It is a tensor of type (1, 2), with a temporal contravariant part and a spatial covariant part.

Remark 4.4. In the 1+3 kinematical formulation, the torsion tensor of the spatial connection is induced by the fluid vorticity (see Appendix A).
Considering the analogue of (11) for the definition of the torsion,

\[ T[D](v, w)(f) := DD f(v, w) - DD f(w, v), \]

with \( f \) being a function and \( v \) and \( w \) two vectors on \( M \), also yields (37) as demonstrated in the following proof.

**Proof.** Definition (38) reads in component form

\[ v^\alpha w^\beta T[D]^\gamma_{\alpha\beta}(f) = v^\alpha w^\beta (f|_{\alpha||\beta} - f|_{\beta||\alpha}) . \]

From (28) we have

\[ f|_{\alpha||\beta} = \gamma^\gamma_{\alpha\beta} f(\gamma, f, \lambda)_{\gamma}. \]

Expanding the right-hand side and making use of (19), we write

\[ f|_{\alpha||\beta} = \gamma^\gamma_{\alpha\beta} f, \delta + \gamma^\gamma_{\alpha\beta} u^\lambda f, \lambda; \gamma, \delta. \]

With the help of (26) we then deduce

\[ f|_{\alpha||\beta} - f|_{\beta||\alpha} = \gamma^\gamma_{\alpha\beta} (f, \delta - f, \delta; \gamma) - \gamma^\gamma_{\alpha\beta} u^\lambda f, \lambda; f, \gamma. \]

Invoking the torsion-free character of the connection \( \nabla \) to cancel the first term of the sum and the spatial character of the extrinsic curvature to reformulate the second, we get

\[ f|_{\alpha||\beta} - f|_{\beta||\alpha} = (k_{\beta\alpha} - k_{\alpha\beta}) u^\lambda f, \lambda. \]

Inserting this expression back into (39) and using (9), we conclude the proof.

**Remark 4.5.** The torsion tensor is defined in [27-29] and [33] according to

\[ T[D](v, w) := D_v w - D_w v - \gamma^\gamma([v, w]). \]

(40)

(It vanishes for a Levi-Civita manifold connection.) This expression involves only the spatial part of \([v, w]\) in comparison with (35). Although it does not enter (40), the temporal part of \([v, w]\) plays the same role in these works as in ours. It only manifest itself in a different way, and more specifically through the so-called deficiency term in [27-29] and in an explicit form in [33]. As a consequence, choosing one or the other definition for the torsion does not affect the relations deduced from the threading of the manifold.

Note, however, that only our approach is consistent with the alternate definition (38). This implies that by picking definition (40), the commutation of two spatial covariant derivatives successively applied to a function induces a term that is not source of torsion.

### 5 Spatial curvature tensors

We introduce in this section the Riemann tensor, Ricci tensor and Ricci scalar of the spatial frames.
5.1 Spatial Riemann curvature

Following the definition of the Riemann curvature tensor of $\mathcal{M}$ (cf. Eq. (12)), one may want to define the spatial Riemann tensor according to

$$Riem(\omega, w, v_1, v_2) := \langle \omega, D_{v_1} D_{v_2} w - D_{v_2} D_{v_1} w - D_{[v_1, v_2]} w \rangle,$$  \hfill (41)

with $\omega$ in $T^*_p(F_u)$ and $v_1$, $v_2$, and $w$ in $T_p(F_u)$.

However, as it was noticed for instance in [27], the object hence defined is not a tensor because of its lack of linearity. We have indeed

$$Riem(\omega, f w, v_1, v_2) = f Riem(\omega, w, v_1, v_2) - u \cdot [v_1, v_2] u(f) \langle \omega, w \rangle,$$  \hfill (42)

for any function $f$ on $\mathcal{M}$.

**Proof.** From Eq. (41) we have

$$Riem(\omega, f w, v_1, v_2) = f Riem(\omega, w, v_1, v_2) - (D_{v_1} D_{v_2} f - D_{v_2} D_{v_1} f) \langle \omega, w \rangle.$$  \hfill (42)

Let us start by working on the first two terms between parentheses. For any spatial vector $v$ we write, by means of Eqs. (32) and (9),

$$D_v f = v^\alpha f_\alpha = v^\alpha \gamma^\beta_\alpha f, \beta = v^\alpha f_\alpha = v(f),$$  \hfill (43)

and thus we infer

$$D_{v_1} D_{v_2} f - D_{v_2} D_{v_1} f = [v_1, v_2](f).$$

The last term between parentheses, on the other hand, can be cast into

$$D_{[v_1, v_2]} f = [v_1, v_2]^\alpha \gamma^\beta_\alpha f, \beta$$

$$= [v_1, v_2]^\alpha f_\alpha + u_\alpha [v_1, v_2]^\alpha u^\beta f, \beta$$

$$= [v_1, v_2](f) + u \cdot [v_1, v_2] u(f).$$

Subtracting this expression from the one above and plugging the result into (42), we conclude the proof. \hfill \Box

The non-linearity of the object (41) stems from the fact that the temporal part of the Lie bracket $[v_1, v_2]$ does not vanish or, equivalently, from the fact that the extrinsic curvature of the spatial frames is not symmetric (cf. Eq. (24)). We present in what follows a definition that circumvents the issue.

**Remark 5.1.** In the 1+3 kinematical formulation, the non-linearity of (41) is caused by the vorticity of the fluid.

\footnote{Note that we have, from Eq. (33), $D_{[v_1, v_2]} w = D_{\gamma f([v_1, v_2])} w = \gamma f(\nabla_{\gamma([v_1, v_2])} w)$.}
5.1.1 Definition

The Riemann curvature tensor of the local spatial frames can be defined as

\[
\text{Riem}: \mathcal{F}_u^\ast \times \mathcal{F}_u^\ast \times \mathcal{F}_u^\ast \times \mathcal{F}_u^\ast \to \mathbb{R}
\]

\[
(\omega, w, v_1, v_2) \mapsto \left\langle \omega, D_{v_1} D_{v_2} w - D_{v_2} D_{v_1} w - \gamma^2 (\nabla_{[v_1,v_2]} w) \right\rangle.
\]  

(44)

Remark 5.2. This definition was introduced originally by Massa [22–24] and studied later by Jantzen and collaborators [26,33] and by Boersma and Dray [27]. In their works, Jantzen et al. derive the expressions of the Gauss, Codazzi and Ricci relations in a particular basis only (the one adapted to the congruence). On the other hand, Boersma and Dray derive, in tensorial form, the Gauss relation only, and they provide the component form of (44) in a specific basis only. In the present article we provide, as we shall see below, (i) the components of the spatial Riemann tensor in an arbitrary four-dimensional basis and its dual, and (ii) the different projections of the Riemann tensor of \(\mathcal{M}\) onto the congruence and the spatial frames.11

Remark 5.3. In the terminology of Jantzen et al. [26], Eq. (44) stands for the ‘Fermi–Walker spatial curvature’. Another definition for the spatial Riemann tensor (historically the first) was given by Zel’manov in [11,12] (see also [20,21,24,25]). For an analysis of the latter definition (‘Lie spatial curvature’) along with the proposal for yet another definition (‘co-rotating Fermi–Walker curvature’), we refer the reader to [25,33].12

We extend the definition of the spatial Riemann curvature (44) to arbitrary tensors on \(\mathcal{M}\) upon writing

\[
\text{Riem}(\omega, w, v_1, v_2) := \left\langle \gamma^2(\omega), D_{\gamma^2(v_1)} D_{\gamma^2(v_2)} \gamma^2(w) - D_{\gamma^2(v_2)} D_{\gamma^2(v_1)} \gamma^2(w)
- \gamma^2(\nabla_{[\gamma^2(v_1),\gamma^2(v_2)]}\gamma^2(w)) \right\rangle.
\]

(45)

Its components (with respect to an arbitrary vector basis and its dual) are given by13

\[
R^\gamma_{\beta \mu \nu} = \gamma^\alpha \gamma^\delta \gamma^\rho \gamma^\sigma \left\{ e_\rho \left( \Gamma[D]_{\gamma \sigma \delta} - \Gamma[D]_{\gamma \rho \delta} + \Gamma[D]_{\gamma \rho \sigma} \Gamma[D]_{\gamma \lambda} \right)
- \Gamma[D]_{\gamma \sigma \delta} \Gamma[D]_{\gamma \rho \lambda}
- \Gamma[D]_{\gamma \rho \sigma} \Gamma[D]_{\gamma \lambda \delta}
+ C[D]_{\gamma \rho \sigma} \Gamma[D]_{\gamma \lambda \delta}
+ e_\rho (u_\delta) e_\sigma (u_\gamma) - e_\rho (u_\gamma) e_\sigma (u_\delta) \right\},
\]

(46)

where we have defined the spatial structure coefficients \(C[D]\) according to

\[
C[D]_{\gamma \alpha \beta} := \gamma^\gamma \gamma^\rho \gamma^\sigma \gamma^\delta C_{\rho \sigma \delta} = 2 \Gamma[D]_{\gamma \alpha \beta}.
\]

(47)

11 These projections result in the Gauss, Codazzi and Ricci equations (cf. respectively Propositions S1, S2 and S4). Note that we supplement these relations with the equation for the variation of the spatial metric along the integral curves in Proposition S3.

12 In the present work we focus solely on (44) as it is the expression that most closely resembles (41) (see also footnote 10).

13 For simplicity, we denote the components of \(\text{Riem}\) by \(R^\gamma_{\beta \mu \nu}\) rather than \(Riem^\gamma_{\alpha \beta \mu \nu}\).
Proof. Let us first define the set of spatial starry vectors \( \{ e^\star_\alpha \} \) by
\[
e^\star_\alpha := \gamma^\delta (e_\alpha) = e_\alpha + u_\alpha u,
\] (48)
and the set of spatial starry 1-forms \( \{ e^\star_\alpha \} \) as
\[
e^\star_\alpha := \gamma^\delta (e^\alpha) = e^\alpha + u^\alpha u^\delta.\] (49)

For the purpose of the proof we list some of the properties fulfilled by these objects. We have for the vectors
\[
e^\star_\alpha = \gamma^\beta e_\beta, \quad e^\star_\alpha = \gamma^\beta e^\beta, \quad u^\alpha e^\star_\alpha = 0,
\] (50)
and, similarly, for the 1-forms
\[
e^\star_\alpha = \gamma^\beta e_\beta, \quad e^\star_\alpha = \gamma^\alpha e^\beta, \quad u_\alpha e^\star_\alpha = 0.
\] (51)
In addition, the starry sets satisfy the relation
\[
\langle e^\star_\alpha, e^\star_\beta \rangle = \gamma^\delta.\] (52)

With the help of (48) and (49) we can write the components of the spatial Riemann tensor (45) in the form
\[
R^\alpha_{\beta\mu\nu} = \langle e^\star_\alpha, D_{e^\beta} e^\star_\mu - D_{e^\mu} e^\star_\nu - \gamma^\delta (\nabla [e^\sigma_\mu, e^\sigma_\nu]) \rangle.\] (53)

For the sake of clarity, we divide the rest of the proof into the following four parts.

Part I. Let us begin with the decomposition of the term \( D_{e^\beta} e^\star_\alpha \). By means of the property
\[
\nabla(fT) = f\nabla T + T \otimes \nabla f,
\] with \( f \) being a function and \( T \) a tensor on \( \mathcal{M} \), together with Eqs. (27) and (48), we have
\[
D_{e^\beta} e^\star_\alpha = \gamma^\delta (\nabla e_\alpha + u_\alpha \nabla u + u \otimes \nabla u_\alpha).
\]
Using Eq. (6) for the first term between parentheses, Eq. (8) applied to \( u \) for the second term and Eqs. (20), (48) and (49) on the resulting expression, we find
\[
D_{e^\beta} e^\star_\alpha = \left( \Gamma^\gamma_{\delta\alpha} + u_\alpha \nabla^\alpha u^\gamma + u_\alpha \Gamma^\gamma_{\delta\lambda} u^\lambda \right) e^\star_\beta \otimes e^\delta.
\]
From Eqs. (19), (50), (51) and (31), this expression becomes
\[
D_{e^\beta} e^\star_\alpha = \left( \Gamma [D]^\gamma_{\delta\alpha} + u_\alpha \nabla^\alpha u^\gamma \right) e^\star_\beta \otimes e^\delta.
\]
At last, making use of Eq. (52), we obtain
\[
D_{e^\beta} e^\star_\alpha = \left( \Gamma [D]^\gamma_{\beta\alpha} + u_\alpha \nabla^\alpha u^\gamma \right) e^\star_\gamma.
\] (54)

The components of \( D_{e^\beta} e^\star_\alpha \) with respect to an arbitrary vector basis follow from Eq. (48). They are identical to those given in the above decomposition. We shall
however consider the form (54) in what follows as we will apply another spatial covariant derivative to $D_{e^\lambda} e^\mu e^\gamma$ (recall that this operator is only defined for spatial tensors).

**Part II.** We now turn to the decomposition of the first term of the right-hand side of (53). From (54) we have

$$D_{e^\gamma} D_{e^\beta} e^\mu = \left( \Gamma^\gamma_{\nu\beta} + u_\beta \gamma^\gamma_{\delta} e^\nu_{\delta}(u^\delta) \right) D_{e^\gamma} e^\mu + D_{e^\mu} \left( \Gamma^\gamma_{\nu\beta} + u_\beta \gamma^\gamma_{\delta} e^\nu_{\delta}(u^\delta) \right) e^\gamma.$$  

The first part of the sum is developed by using again Eq. (54) and the second part is expanded with the help of Eqs. (43) and (19). We reformulate the outcome by considering the spatial character of $\Gamma$ and the unitarity of $u$, and we obtain

$$D_{e^\mu} D_{e^\gamma} e^\mu e^\beta = \left\{ e^\mu_{\mu} \left( \Gamma^\gamma_{\nu\beta} \right) - u_\beta u^\delta e^\mu_{\delta} \left( \Gamma^\gamma_{\nu\beta} \right) + \Gamma^\gamma_{\mu\nu \delta} \Gamma^\gamma_{\mu\lambda} \right\} e^\gamma.$$  

From this expression and by means of Eqs. (15), (19) and (50), we write the first two terms of the right-hand side of (53) in the form

$$D_{e^\gamma} D_{e^\beta} e^\mu e^\beta = \left\{ \gamma^\gamma_{\beta} e^\mu_{\mu} \left( \Gamma^\gamma_{\nu\beta} \right) - \gamma^\gamma_{\beta} e^\nu_{\delta} \left( \Gamma^\gamma_{\nu\delta} \right) \right\} e^\gamma + \Gamma^\gamma_{\nu\beta} \Gamma^\gamma_{\mu\lambda} e^\mu_{\nu\delta} e^\mu_{\mu\lambda} - \Gamma^\gamma_{\nu\beta} \Gamma^\gamma_{\mu\lambda} e^\mu_{\nu\delta} \left( \Gamma^\gamma_{\mu\lambda} \right).  

**Interlude.** In order to proceed with the third part of the proof, we provide the expression of the Lie bracket of two starry vectors. From Eqs. (36) and (54) we have

$$[e^\mu, e^\beta] = \left( C^\gamma_{\alpha\beta} + u_\beta \gamma^\gamma_{\delta} e^\nu_{\delta}(u^\delta) - \Gamma^\gamma_{\delta\nu} e^\nu_{\mu}(u^\mu) \right) e^\gamma - T^\gamma_{\alpha\beta} e^\gamma,$$  

where the coefficients $C^\gamma$ are defined by (47).

**Part III.** We now decompose the last term of the right-hand side of (53). Using (56) and (48) we write

$$\nabla_{[e^\gamma, e^\beta]} e^\mu_{\mu} = \left( C^\delta_{\mu\nu} + u_\nu \gamma^\delta_{\lambda} e^\mu_{\lambda}(u^\lambda) - \Gamma^\delta_{\nu\lambda} e^\nu_{\mu}(u^\mu) \right) \nabla_{e^\gamma} e^\beta_{\mu} - \Gamma^\delta_{\nu\lambda} \left( \nabla_{e^\gamma} e^\beta_{\mu} + u_\mu \nabla_{e^\gamma} e^\beta_{\nu} + e^\lambda_{\nu}(u_{\beta}) u^\mu \right).$$  

The second expression in parentheses is developed with the help of Eq. (6), Eq. (8) written for $u$ and Eq. (19). We get

$$\nabla_{[e^\gamma, e^\beta]} e^\mu_{\mu} = \left( C^\delta_{\mu\nu} + u_\nu \gamma^\delta_{\lambda} e^\mu_{\lambda}(u^\lambda) - \Gamma^\delta_{\nu\lambda} e^\nu_{\mu}(u^\mu) \right) \nabla_{e^\gamma} e^\beta_{\mu} - \Gamma^\delta_{\nu\lambda} \left( \gamma^\delta_{\beta} \Gamma^\gamma_{\nu\lambda} e^\gamma_{\gamma} + u_\mu e^\lambda_{\lambda}(u^\nu) e^\gamma_{\gamma} + e^\lambda_{\nu}(u_{\beta}) u^\mu \right).$$
We project this equality onto the spatial frames and we apply Eqs. (33) and (54) on the first line and Eq. (48) on the second. Expanding the resulting expression and making use of the spatial character of $\Gamma^{D}$, we end up with

$$\nabla^{\gamma} \left( \nabla_{\epsilon^{\gamma} \epsilon^{\delta}} e^{*}_{\beta} \right) = \left\{ \begin{array}{l} -\gamma^{\delta}_{\beta} u^{\sigma} u_{\nu} e^{*}_{\mu} (\Gamma^{D}[\gamma]_{\sigma \delta}) + \gamma^{\delta}_{\beta} u_{\mu} e_{\nu} (\Gamma^{D}[\gamma]_{\rho \delta}) \\
+ C^{D}[\gamma]_{\mu \nu} \Gamma^{D}[\gamma]_{\delta \beta} + u_{\beta} \gamma^{\lambda}_{\alpha} C^{D}[\gamma]_{\mu \nu} e^{*}_{\delta}(u^{\lambda}) \\
+ u_{\beta} u_{\nu} \gamma^{\lambda}_{\alpha} e^{*}_{\mu}(u^{\delta}) e^{*}_{\delta}(u^{\lambda}) - u_{\beta} u_{\mu} \gamma^{\lambda}_{\alpha} e^{*}_{\nu}(u^{\delta}) e^{*}_{\delta}(u^{\lambda}) \\
- T^{D}[\gamma]_{\lambda \mu \nu} (\gamma^{\delta}_{\beta} \Gamma^{\gamma}_{\lambda \delta} + u_{\beta} e_{\lambda}(u^{\gamma})) \end{array} \right\} e^{*}_{\gamma}. \quad (57)$$

Part IV. Subtracting (57) from (55) and making use of (50), we obtain

$$D e^{\gamma}_{\epsilon^{\mu} \epsilon^{\nu}} e^{\delta}_{\epsilon^{\beta}} - D e^{\gamma}_{\epsilon^{\nu} \epsilon^{\mu}} e^{\delta}_{\epsilon^{\beta}} = \gamma^{\gamma}_{\beta} \gamma^{\rho}_{\mu} \gamma^{\sigma}_{\nu} \left\{ \epsilon^{\rho} \left( \Gamma^{D}[\gamma]_{\rho \delta} \right) - \epsilon_{\sigma} \left( \Gamma^{D}[\gamma]_{\rho \delta} \right) + \Gamma^{D}[\gamma]_{\sigma \delta} \Gamma^{D}[\gamma]_{\rho \lambda} \\
- \Gamma^{D}[\gamma]_{\rho \delta} \Gamma^{D}[\gamma]_{\sigma \lambda} - C^{D}[\gamma]_{\rho \sigma} \Gamma^{D}[\gamma]_{\lambda \delta} + T^{D}[\gamma]_{\rho \sigma} \Gamma^{\gamma}_{\lambda \delta} \\
+ \epsilon^{\rho}(u^{\delta}) \epsilon_{\sigma}(u^{\gamma}) - \epsilon_{\rho}(u^{\gamma}) \epsilon^{\sigma}(u^{\delta}) \right\} e^{*}_{\gamma}. \quad (58)$$

Finally, inserting this expression into (53) and using (52), we conclude the proof.

5.1.2 Properties

The spatial Riemann curvature satisfies the relations

$$R_{\alpha \beta \mu \nu} = -R_{\alpha \beta \nu \mu}, \quad R_{\alpha \beta \mu \nu} = -R_{\beta \alpha \mu \nu}, \quad R_{\alpha [\beta \mu \nu]} = 2k_{\alpha [\beta} k_{\mu \nu]} \quad (59)$$

Proof. From the symmetries of the four-Riemann tensor,

$$4R_{\alpha \beta \mu \nu} = -4R_{\alpha \beta \nu \mu}, \quad 4R_{\alpha \beta \mu \nu} = -4R_{\beta \alpha \mu \nu}, \quad 4R_{\alpha [\beta \mu \nu]} = 0,$$

together with Proposition S1 (given hereafter), we obtain the above expressions.

From the properties (58), we infer

$$R_{\alpha \beta \mu \nu} - R_{\mu \nu \alpha \beta} = k_{\mu \alpha} k_{\nu \beta} - k_{\mu \beta} k_{\nu \alpha} - k_{\mu \beta} k_{\nu \alpha} + k_{\beta \mu} k_{\alpha \nu}. \quad (59)$$

Remark 5.4. When the extrinsic curvature is symmetric, we recover the usual first Bianchi identity for the spatial Riemann tensor (cf. Eq. (58)) and relation (59) vanishes.
5.2 Spatial Ricci tensor

5.2.1 Definition

We define the Ricci curvature tensor of the spatial frames from the trace of the spatial Riemann tensor \( \mathcal{R} \) taken on the first and third arguments:

\[
\mathbf{R} : \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}
\]

\[
(v, w) \mapsto \mathbf{Riem}(e^\alpha, v, e_\alpha, w).
\] (60)

Remark 5.5. The other traces of the Riemann curvature of \( \mathcal{C} \) either vanish or are equal (possibly up to a sign) to the one here defined. (This can be shown from (58).)

The spatial Ricci scalar is defined from the trace of the spatial Ricci tensor. It is written

\[
R := R^\alpha_\beta(e^\alpha, e_\alpha).
\]

5.2.2 Property

The spatial Ricci tensor satisfies the relation

\[
R_{\alpha\beta} - R_{\beta\alpha} = -2k_{[\alpha\beta]} - 2k^\gamma(\alpha) k_{\beta\gamma},
\] (61)

where \( k \) is the trace of the extrinsic curvature tensor.

Proof. This expression is obtained by taking the trace of Eq. (59) on the first and third indices and by using definition (60).

Remark 5.6. Because the extrinsic curvature is not symmetric, the spatial Ricci tensor is not symmetric either.

Remark 5.7. In the 1+3 kinematical formulation, the spatial Ricci tensor fails to be symmetric on account of the fluid vorticity.

6 1 + 3 form of the curvature tensors

In this section, we provide the 1+3 formulation of the curvature tensors of the manifold in terms of the spatial curvature tensors introduced above.

The projections of the Riemann tensor of \( \mathcal{M} \), in terms of the Riemann tensor associated with \( \mathcal{C} \), are given in the first subsection. They yield the 1+3 version of the Gauss, Codazzi and Ricci equations. From these expressions, we supply in the ensuing part the different projections of the four-Ricci tensor in terms of the spatial Ricci tensor.

Our results are written in the form of propositions and corollaries. We here only add a few remarks; a thorough discussion follow in Section 8.
6.1 1 + 3 form of the Riemann tensor

6.1.1 Gauss equation

**Proposition S1.** In the 1 + 3 formalism the Gauss relation is given by

\[ \gamma_{\rho \gamma} \gamma_{\alpha \gamma} \gamma_{\beta} \gamma_{\delta} R^\rho_{\sigma \gamma \delta} = R^\mu_{\nu \alpha \beta} + k^\mu_{\alpha} k_{\nu \beta} - k^\mu_{\beta} k_{\nu \alpha} \]  

(62)

**Proof.** From Eq. (30) we write the components of \( DDv \), with \( v \) being a spatial vector, as

\[ v^\mu || \beta || \alpha = \gamma^\mu_{\rho \gamma} \gamma_{\alpha \gamma} \gamma_{\beta} \left( e^*_{\sigma} \left( R^\rho_{\sigma \gamma \delta} \right) \right) v^\delta + \Gamma^\lambda_{\sigma \delta} \Gamma^\gamma_{\rho \lambda} v^\delta \]

\[ - \Gamma^\lambda_{\rho \sigma} \Gamma^\gamma_{\lambda \delta} v^\delta - \Gamma^\lambda_{\rho \sigma} e^*_\gamma (v^\gamma) + \Gamma^\gamma_{\sigma \lambda} e^*_\rho (v^\lambda) \]

\[ + \Gamma^\gamma_{\rho \lambda} e^*_\sigma (v^\lambda) + e^*_\rho (e^*_\sigma (v^\gamma)) + u^\delta e^*_\rho (u^\gamma) e^*_\delta (v^\delta) \}

From Eqs. (47) and (56) and by means of the spatial character of \( v \), we then obtain

\[ v^\mu || \beta || \alpha - v^\mu || \alpha || \beta = \gamma^\mu_{\rho \gamma} \gamma_{\alpha \gamma} \gamma_{\beta} \left( \Gamma^\delta_{\rho \gamma} \left( e^*_\sigma \left( \Gamma^\rho_{\sigma \gamma \delta} \right) \right) - e^*_\sigma \left( \Gamma^\rho_{\sigma \gamma \delta} \right) \right) \]

\[ + \Gamma^\lambda_{\rho \sigma} \Gamma^\gamma_{\lambda \delta} \Gamma^\gamma_{\rho \lambda} - \Gamma^\lambda_{\rho \sigma} \Gamma^\gamma_{\lambda \delta} \Gamma^\gamma_{\rho \lambda} \]

\[ - C^\lambda_{\rho \sigma} \Gamma^\gamma_{\lambda \delta} \Gamma^\gamma_{\rho \lambda} + e^*_\rho (u^\gamma) e^*_\delta (u^\delta) \}

\[ v^\mu - T^\gamma_{\rho \gamma} v^\gamma - T^\rho_{\nu \alpha \beta} \Gamma^\gamma_{\sigma \gamma} v^\gamma \}

Using Eq. (50) to remove the stars, Eq. (46) to introduce the spatial Riemann curvature and Eq. (8) to insert the covariant derivative of \( v \), we find

\[ v^\mu || \beta || \alpha - v^\mu || \alpha || \beta = R^\mu_{\nu \alpha \beta} v^\nu - \gamma^\mu_{\rho \gamma} \Gamma^\gamma_{\sigma \gamma} v^\gamma \]

(63)

This relation concludes the first part of the proof.

Equation (63) strongly resembles the usual expression relating a spatial connection with torsion to its curvature tensor. The fact that the torsion of the spatial connection has a non-spatial contravariant part prevents the use of the spatial connection on the right-hand side of the expression.

We now express the left-hand side of (63) in terms of the covariant derivative of \( v \). From Eqs. (28) and (26) we have

\[ v^\mu || \beta || \alpha = \gamma^\mu_{\rho \gamma} \gamma_{\alpha \gamma} \gamma_{\beta} v^\rho ; \gamma - \gamma^\mu_{\rho \gamma} k_{\beta \alpha} u^\gamma v^\rho ; \lambda - k^\mu_{\alpha} k_{\lambda \beta} v^\lambda \]

With Eq. (13) and the spatial character of \( v \) we write

\[ v^\mu || \beta || \alpha - v^\mu || \alpha || \beta = \gamma^\mu_{\rho \gamma} \gamma_{\alpha \gamma} \gamma_{\beta} \gamma_{\delta} R^\rho_{\sigma \gamma \delta} v^\nu + \gamma^\mu_{\rho} (k_{\alpha \beta} - k_{\beta \alpha}) u^\lambda v^\rho ; \lambda \]

\[ - (k^\mu_{\alpha} k_{\lambda \beta} - k^\mu_{\beta} k_{\lambda \alpha}) v^\lambda \]

Inserting this expression into (63) and making use of (37) in component form we deduce

\[ \gamma^\mu_{\rho \gamma} \gamma_{\alpha \gamma} \gamma_{\beta} \gamma_{\delta} R^\rho_{\sigma \gamma \delta} v^\nu = (R^\mu_{\nu \alpha \beta} + k^\mu_{\alpha} k_{\nu \beta} - k^\mu_{\beta} k_{\nu \alpha}) v^\nu \]
Noticing at last that this relation can be written not only for spatial vectors but also for any vectors on \( \mathcal{M} \) (thanks to the presence of the orthogonal projector and the fact that both \( \text{Riem} \) and \( \mathbf{k} \) are spatial), we conclude the proof.

### 6.1.2 Codazzi equation

**Proposition S2.** In the 1+3 formalism the Codazzi relation is given by

\[
\gamma^\mu_{\rho\alpha} \gamma^\lambda_{\gamma\beta} R_a^\sigma_{\gamma\delta} = k^\mu_\alpha || \beta - k^\mu_\beta || \alpha - a^\mu (k_\alpha \beta - k_\beta \alpha) .
\] (64)

**Proof.** This equation is obtained by fully projecting (13) written for \( \mathbf{u} \) onto the spatial frames, and by using (26) together with (28).

**Remark 6.1.** The Codazzi relation (64) is anti-symmetric on the indices \( \alpha \) and \( \beta \).

### 6.1.3 Variation of the spatial metric

We now search for the evolution equation of the spatial metric along the congruence. (This will be of use in the sequel for the derivation of the Ricci equation and its reformulation.) We introduce to this aim the temporal evolution vector \( \mathbf{m} \), such that

\[
\mathbf{m} := M \mathbf{u} .
\]

\( M \) stands for the (positive) threading lapse function; it allows for the choice of the pace of evolution along the integral curves of \( \mathcal{C} \). (Further information about \( M \) can be found in Appendix B.)

**Remark 6.2.** Even in the case where we regard spatial tensor fields we work within a four-dimensional manifold. Hence there is no ‘evolution’ of spatial quantities per se, but rather a variation along a preferred time-like direction.

**Proposition S3.** The evolution of the spatial metric along the congruence of curves is given by

\[
\mathcal{L}_m \gamma_{\alpha\beta} = -M (k_{\alpha\beta} + k_{\beta\alpha})
\] (65)

for its covariant components,

\[
\mathcal{L}_m \gamma^{\alpha\beta} = M \left( k^{\alpha\beta} + k^{\beta\alpha} \right) + M u^{\alpha} \left( a^{\beta} - \frac{D^{\beta} M}{M} \right) + M u^{\beta} \left( a^{\alpha} - \frac{D^{\alpha} M}{M} \right)
\] (66)

for its contravariant components, and

\[
\mathcal{L}_m \gamma^{\alpha}_{\beta} = M u^{\alpha} \left( a_{\beta} - \frac{D_{\beta} M}{M} \right)
\] (67)

for its mixed components.

**Proof.** Each of these expressions is obtained by means of Eqs. (14), (21) and (26).

**Remark 6.3.** No particular expression can be written for the curvature vector when working in arbitrary bases, on the contrary to the slicing approach (see, e.g., [32]). In Appendix B, we provide an expression in a coordinate basis adapted to the congruence.
6.1.4 Ricci equation

**Proposition S4.** In the 1+3 formalism the Ricci equation is given by

\[ \gamma_{\alpha\rho} u^\gamma \delta^4 R_{\sigma\gamma\delta} = \frac{1}{M} \mathcal{L}_m k_{\alpha\beta} + k_{\gamma\alpha} k^{\gamma}_{\beta} + a_{\alpha||\beta} + a_{\alpha\beta} . \] (68)

**Proof.** Projecting Eq. (13) written for \( u \) twice onto the spatial frames and once onto the congruence, we obtain with the help of Eq. (26)

\[ \gamma_{\alpha\rho} u^\gamma \delta^4 R_{\sigma\gamma\delta} = \gamma_{\alpha\rho} \gamma^\gamma_\delta u^\delta k^{\rho}_{\gamma\delta} - k_{\alpha\gamma} k^{\gamma}_{\beta} + a_{\alpha||\beta} + a_{\alpha\beta} . \] (69)

On the other hand, we decompose from Eq. (14) the orthogonal projection of the Lie derivative of \( k \) along the evolution vector into

\[ \gamma_{\alpha\gamma\delta} \mathcal{L}_m k_{\gamma\delta} = M \gamma_{\alpha\gamma\delta} u^\lambda k_{\gamma\delta;\lambda} - M k^{\gamma}_{\beta} (k_{\alpha\gamma} + k_{\gamma\alpha}) . \]

The left-hand side of this relation is reformulated by means of (67) and the spatial character of \( k \) as

\[ \gamma_{\alpha\gamma\delta} \mathcal{L}_m k_{\gamma\delta} = \mathcal{L}_m k_{\alpha\beta} . \]

We use the resulting expression to substitute the first term of the right-hand side of (69), and we conclude the proof.

6.2 1 + 3 form of the Ricci tensor

6.2.1 Contracted Gauss equation and scalar Gauss equation

**Corollary S1.** In the 1+3 formalism the contracted Gauss relation is given by

\[ \gamma_{\alpha\gamma\delta} 4 R_{\gamma\delta} + \gamma_{\alpha\beta} u^\gamma u^\delta R_{\alpha\beta} = R_{\alpha\beta} + k k_{\alpha\beta} - k^{\gamma}_{\beta} k_{\alpha\gamma} , \] (70)

and the scalar Gauss relation reads

\[ 4 R + 2 u^\alpha u^\beta 4 R_{\alpha\beta} = R + k^2 - k_{\alpha\beta} k^{\beta\alpha} . \] (71)

**Proof.** The first equation is obtained by contracting (62) on the first and third indices and by using the idempotence of the orthogonal projector together with (19). The second equality stems from the trace of (70).

6.2.2 Contracted Codazzi equation

**Corollary S2.** In the 1+3 formalism the contracted Codazzi relation is written

\[ \gamma_{\alpha} u^\gamma 4 R_{\gamma\delta} = k_{\alpha} - k_{\alpha||\gamma} - a^\gamma (k_{\gamma\alpha} - k_{\alpha\gamma}) . \] (72)

**Proof.** This expression is derived by contracting Eq. (64) on the indices \( \mu \) and \( \alpha \).

**Remark 6.4.** The contraction of (64) on the indices \( \mu \) and \( \beta \) yields the same expression as (72) (cf. Remark 6.1).
6.2.3 Reformulation of the Ricci equation

**Corollary S3.** In the 1+3 formalism the Ricci equation can be alternatively written as

\[
\gamma^\beta_\alpha \gamma^\nu_\beta R_{\mu\nu} = R_{\alpha\beta} + k k_{\alpha\beta} - k^\gamma_\beta (k_{\alpha\gamma} + k_{\gamma\alpha}) - \frac{1}{M} \mathcal{L}_m k_{\alpha\beta} - a_{\alpha||\beta} - a_{\alpha\alpha}\beta.
\]

(73)

**Proof.** This expression is obtained by combining Eq. (68) with Eq. (70).

**Remark 6.5.** Taking the trace of (68) and using (66) and (71) on the outcome, we get

\[
4R = R + k^2 + k_{\alpha\beta}k^{\beta\alpha} - \frac{2}{M} \mathcal{L}_m k - 2a^\alpha_{||\alpha} - 2a^\alpha a_{\alpha}.
\]

(This relation is useful for instance for the derivation of the 1+3 form of the Hilbert action.)

7 1 + 3 form of Einstein’s equation

We present in this section the 1+3 formulation of the Einstein field equation

\[
4R - \frac{1}{2} 4Rg + \Lambda g = 8\pi T.
\]

(74)

\(T\) denotes the stress–energy tensor of the fluid filling the space–time, and \(\Lambda\) is the cosmological constant that we carry along for the sake of generality. (We use geometric units, for which \(G = c = 1\).)

We identify in what follows the four-dimensional Lorentzian manifold to the physical space–time, the integral curves of the congruence to the world lines of the fluid and the local spatial frames to the (instantaneous) rest-frames of the fluid.

The 1+3 decomposition of the space–time Ricci tensor is given in the above section; to provide the mentioned reformulation, we are left with the derivation of the 1+3 form of the stress–energy tensor.

7.1 1 + 3 form of the stress–energy tensor

The stress–energy tensor of the fluid is decomposed in its rest-frames according to

\[
T = \epsilon u^\alpha \otimes u^\beta + q \otimes u^\beta + u^\beta \otimes q + p \gamma + \zeta.
\]

It reads in component form

\[
T_{\mu\nu} = \epsilon u_\alpha u_\beta + 2 q_{(\alpha} u_{\beta)} + p \gamma_{\alpha\beta} + \pi_{\alpha\beta},
\]

where we have defined

\[
\epsilon := u^\alpha u_\beta T_{\alpha\beta}, \quad q_\alpha := -\gamma^\mu_\alpha u^\nu T_{\mu\nu}, \quad p \gamma_{\alpha\beta} + \zeta_{\alpha\beta} := \gamma^\mu_\alpha \gamma^\nu_\beta T_{\mu\nu}.
\]

(75)

\(\epsilon\) stands for the energy density of the fluid, \(p\) for its isotropic pressure, \(q\) defines its (spatial) heat vector and \(\zeta\) its (spatial, symmetric and traceless) anisotropic stress tensor.

The trace of the stress–energy tensor is given by

\[
T = -\epsilon + 3p.
\]

(76)
7.2 1 + 3 form of Einstein’s equation

The trace of Einstein’s equation yields the relation

$$4R = -8\pi T + 4\Lambda,$$  \hfill (77)

which, inserted back into (74), drives

$$4R = 8\pi \left(T - \frac{1}{2} T g\right) + \Lambda g,$$

and in component form

$$4R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta}.$$  \hfill (78)

We project this last expression on the world lines and the rest-frames of the fluid. From Eqs. (75) and (76) we get

$$u^\alpha u^\beta 4R_{\alpha\beta} = 4\pi \left(\epsilon + 3p\right) - \Lambda,$$  \hfill (79)

$$\gamma^\mu_{\alpha} u^\nu 4R_{\mu\nu} = -8\pi q_{\alpha},$$

$$\gamma^\mu_{\alpha} \gamma^\nu_{\beta} 4R_{\mu\nu} = 4\pi \left((\epsilon - p) \gamma_{\alpha\beta} + 2\zeta_{\alpha\beta}\right) + \Lambda \gamma_{\alpha\beta}.$$  \hfill (80)

The left-hand sides are then reformulated by way of the above corollaries, and we end up with the following set of relations, equivalent to the Einstein field equation (see Section 8 for a discussion).

7.2.1 Einstein–Gauss equation

**Proposition E1.** In the 1+3 formalism the Einstein–Gauss relation is given by

$$R + k^2 - k_{\alpha\beta} k^{\beta\alpha} = 16\pi \epsilon + 2\Lambda.$$  \hfill (81)

**Proof.** This expression is derived from Eq. (78), the scalar Gauss relation (71) and Eq. (77).

**Remark 7.1.** Eq. (81) is a scalar relation. It is equivalent to the two-temporal projection (78).

7.2.2 Einstein–Codazzi equation

**Proposition E2.** In the 1+3 formalism the Einstein–Codazzi relation is written

$$k^\gamma_{\alpha||\gamma} - k_{\gamma\alpha} + a^\gamma (k_{\gamma\alpha} - k_{\alpha\gamma}) = 8\pi q_{\alpha}.$$  \hfill (82)

**Proof.** This expression is obtained from Eq. (79) and the contracted Codazzi relation (72).

**Remark 7.2.** Eq. (82) is an expression relating spatial 1-forms. It comprises three independent relations, equivalent to those coming from the one-temporal, one-spatial projection (79).
7.2.3 Einstein–Ricci equation

**Proposition E3.** In the 1+3 formalism the Einstein–Ricci relation is given by

\[
\frac{1}{M} \mathcal{L}_m k_{(\alpha\beta)} = -a_{(\alpha \parallel \beta)} - a_{(\alpha a\beta)} + R_{(\alpha\beta)} + kk_{(\alpha\beta)} - k^\gamma_{(\beta k\alpha)} - k_{\gamma(\alpha k^\gamma\beta)} - 4\pi \left( (\epsilon - p) \gamma_{\alpha\beta} + 2 \zeta_{\alpha\beta} \right) - \Lambda_{\gamma\alpha\beta}. \tag{83}
\]

**Proof.** The symmetric part of (73) is written

\[
\gamma^\mu_{(\alpha \gamma \beta)} 4 R_{\mu\nu} = R_{(\alpha\beta)} + kk_{(\alpha\beta)} - k^\gamma_{(\beta k\alpha)} - k_{\gamma(\alpha k^\gamma\beta)} - \frac{1}{M} \mathcal{L}_m k_{(\alpha\beta)} - a_{(\alpha \parallel \beta)} - a_{(\alpha a\beta)}, \tag{84}
\]

where the parentheses imply symmetrization over the indices enclosed. Noticing that

\[
\gamma^\mu_{(\alpha \gamma \beta)} 4 R_{\mu\nu} = \gamma^\mu_{\alpha \gamma \beta} 4 R_{\mu\nu},
\]

because of the symmetry of the four-Ricci curvature, we combine (84) with (80), and we conclude the proof. 

**Remark 7.3.** Eq. (83) is an expression written for symmetric rank-2 spatial terms. It has six independent components, equivalent to those given by the two-spatial projection (80).

**Remark 7.4.** With the help of Eq. (61) we can write the symmetric part of the spatial Ricci curvature as

\[
R_{(\alpha\beta)} = R_{\alpha\beta} + kk_{[\alpha\beta]} + k^\gamma_{(a k^\gamma\beta)}.
\]

By means of this expression and using the fact that both \(k_{\gamma\alpha} k^\gamma\beta\) and \(a_{\alpha a\beta}\) are symmetric terms, we can reformulate (83) into

\[
\frac{1}{M} \mathcal{L}_m k_{(\alpha\beta)} = -a_{(\alpha \parallel \beta)} - a_{(\alpha a\beta)} + R_{(\alpha\beta)} + kk_{(\alpha\beta)} - k^\gamma_{(\beta k\alpha)} - k_{\gamma(\alpha k^\gamma\beta)} - 4\pi \left( (\epsilon - p) \gamma_{\alpha\beta} + 2 \zeta_{\alpha\beta} \right) - \Lambda_{\gamma\alpha\beta}.
\]

**Remark 7.5.** Combining directly (80) with (73) results in the expression

\[
\frac{1}{M} \mathcal{L}_m k_{\alpha\beta} = -a_{(\alpha \parallel \beta)} - a_{(\alpha a\beta)} + R_{\alpha\beta} + kk_{\alpha\beta} - k^\gamma_{(\beta k\alpha)} - k_{\gamma(\alpha k^\gamma\beta)} - 4\pi \left( (\epsilon - p) \gamma_{\alpha\beta} + 2 \zeta_{\alpha\beta} \right) - \Lambda_{\gamma\alpha\beta}.
\]

This equation is made of nine independent components. Six of them are given by the symmetric part (83) and are equivalent to the projected Einstein expression (80) (cf. Remark 7.3). The other three are given by the anti-symmetric part

\[
\frac{1}{M} \mathcal{L}_m k_{[\alpha\beta]} = -a_{[\alpha \parallel \beta]},
\]

which is a purely geometric expression. It can be indeed directly obtained by taking the anti-symmetric part of (73). Hence it does not involve the Einstein field equation for its formulation.
8 Conclusion

8.1 Summary

We have presented in this paper the formalism for the splitting of a four-dimensional Lorentzian manifold by a set of time-like integral curves. From the introduction of the geometrical tensors characterizing the local spatial frames (namely, the two fundamental forms in Section 3, the spatial covariant derivative in Section 4 and the spatial curvature tensors in Section 5), we have derived in a general space–time basis and its dual the Gauss, Codazzi and Ricci equations, along with the evolution equation for the spatial metric (Section 6.1). These relations were given in Propositions S1 to S4 and were used subsequently to obtain the different projections of the four-Ricci tensor (Section 6.2), which resulted in Corollaries S1 to S3. From these last expressions, we have provided the 1+3 formulation of the Einstein field equation in Propositions E1 to E3 (Section 7), valid as well in any bases.

8.2 Discussion

8.2.1 General remarks

Let us classify for the sake of argument the differences between the 1+3 and 3+1 descriptions in two categories: their construction and their formalism.

The 1+3 procedure is built from a congruence of time-like integral curves, and it provides a global time-like relation between points. In this type of splitting, the direction of time is chosen. The 3+1 procedure is based on the introduction of a family of space-like hypersurfaces, and it supplies a global space-like association of points. Here, the three directions of space are chosen.

When both the orientations of time and space are selected, both splittings can apply. In the generic configuration, a set of time-like integral curves and an independent family of space-like surfaces cover the manifold. The simplest setting is made of one vorticity-free time-like congruence: the integral curves provide the orientation of time, and the space-like orthogonal frames, which globally form hypersurfaces, provide the orientation of space. On account of this particular configuration, the 3+1 description is sometimes regarded as being identical to the 1+3 description without vorticity, although they differ in their construction.

In terms of formalism, the key difference between the 1+3 and 3+1 descriptions comes from the properties of the extrinsic curvature tensor. It is symmetric in the latter description, while it contains an anti-symmetric part in the former. As it was established, this additional term brings about several (interrelated) effects that are absent in the 3+1 perspective: (i) it prevents the spatial frames to form hypersurfaces, (ii) it sources the temporal part of the Lie bracket of two spatial vectors, (iii) it induces a torsion for the spatial connection, and at last (iv) it calls for a redefinition of the spatial Riemann curvature.

As another difference, let us mention that in the 1+3 approach the acceleration of the flow vector cannot be written (solely) in terms of a gradient. In a general setting, no particular expression can be actually supplied. When choosing bases and coordinates adapted to the congruence, we find that the acceleration can be expressed in terms of a gradient plus another term. This additional term, because
of the non-zero anti-symmetric part of the extrinsic curvature, does not vanish (see Appendix B).

8.2.2 1 + 3 form of the curvature tensors

The 1+3 formulation of the four-Riemann curvature is supplied in Propositions S1, S2 and S4. All other projections either vanish or come down to those given here, owing to the symmetries of the tensor. We can compare these expressions with their 3+1 analogues in two ways, either by regarding the extrinsic curvature as such or by considering its symmetric and anti-symmetric parts.14

Adopting the first point of view, Propositions S1 and S4 are identical to their 3+1 counterparts, provided that the indices of the extrinsic curvature are well ordered. Proposition S2, on the other hand, differs from its 3+1 counterpart by an additional term.

In the second point of view, the comparison is realized by decomposing the extrinsic curvature into its symmetric and anti-symmetric parts. The additional contributions are given, after expansion, by all the non-symmetric terms involving \( k \).

8.2.3 Variation of the spatial metric

The variation of the spatial metric along the congruence is supplied in Proposition S3. The expression for the covariant components is given by Eq. (65), that of the contravariant components by Eq. (66), and that of the mixed components by Eq. (67).

Equation (65) is the only expression identical to its 3+1 counterpart in both points of view. This stems from the fact that the right-hand side only involves the symmetric part of the extrinsic curvature. Equations (66) and (67), on the other hand, differ from their 3+1 counterparts as the acceleration of the flow vector cannot be expressed (solely) in terms of a gradient.

8.2.4 1 + 3 form of the Einstein equation

The 1+3 formulation of Einstein’s equation is given by Propositions E1, E2 and E3. The ten component relations are equivalent to the projected Einstein equations (78), (79) and (80), and accordingly to the Einstein field equation (74).

Adopting the first point of view to conduct the comparison, Propositions E1 and E3 are equivalent to their 3+1 counterparts. Proposition E2 on the other hand contains one additional term. In the second point view, Proposition E3 is also equivalent to its 3+1 counterpart, as all terms involved in the relation are symmetric.

The 1+3 system of Einstein equations (81), (82) and (83) can be supplemented by an evolution equation for the anti-symmetric part of \( k \) (cf. Remark 7.5),

\[
\frac{1}{M} \mathcal{L}_m k_{[\alpha \beta]} = -a_{[\alpha || \beta]},
\]  
(85)

14 See, e.g., [8], [10] and [32] for the 3+1 expressions.
and by a constraint also for the anti-symmetric part of $k$,
\[ k_{[\alpha\beta\parallel\mu]} = a_{[\mu} k_{\alpha\beta]} \cdot \]  
(86)

(This last relation is obtained upon taking the full anti-symmetric part of Eq. (64).)

The system can be in addition complemented by the once-contracted Bianchi identities, which provide two evolution and two constraint equations for the electric and magnetic parts of the Weyl tensor, and by the twice-contracted Bianchi identities, which yield from Einstein’s equation the energy and momentum conservation laws for the fluid. The system of equations is then closed by giving an equation of state for the fluid.15

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A Kinematical formulation of the 1 + 3 Einstein equations

In this appendix we reformulate the 1+3 Einstein equations, given by Propositions E1, E2 and E3, in terms of the kinematical quantities of the fluid.

A.1 Kinematical quantities of the fluid

The covariant derivative of the fluid 1-form $u^b$ can be decomposed into
\[ \nabla u^b = -a^b \otimes u^b + \frac{1}{3} \Theta \gamma + \sigma - \omega, \]  
(A.1)

or in component form
\[ u_{\alpha;\beta} = -a_{\alpha} u_{\beta} + \frac{1}{3} \Theta_{\gamma_{\alpha\beta}} + \sigma_{\alpha\beta} - \omega_{\alpha\beta}, \]  
where we have defined
\[ \Theta := u_{\alpha ;\alpha}, \quad \sigma_{\alpha\beta} := \gamma_{\alpha \beta}^{\gamma_\gamma \delta} u_{\delta \gamma} - \frac{1}{3} \Theta \gamma_{\alpha\beta}, \quad \omega_{\alpha\beta} := \gamma_{\alpha \beta}^{\gamma_{\gamma \delta}} u_{\delta \gamma}. \]  
(A.2)

$\Theta$ defines the expansion rate of the fluid, $\sigma$ its (spatial, symmetric and traceless) shear tensor, and $\omega$ its (spatial, anti-symmetric and traceless) vorticity tensor.16

A.2 Extrinsic curvature

Inserting Eq. (A.1) into Eq. (25), we get
\[ k = -\frac{1}{3} \Theta \gamma - \sigma + \omega, \]  
(A.3)

and in component form
\[ k_{\alpha\beta} = -\frac{1}{3} \Theta_{\gamma_{\alpha\beta}} - \sigma_{\alpha\beta} + \omega_{\alpha\beta}. \]  
(A.4)

15 The interested reader can find the formulation of these two additional sets of equations in, e.g., [41, 42].

16 Henceforth the symbol $\omega$ is employed for the (2-form) fluid vorticity, and no more for a generic 1-form as in the main text.
On using the symmetries of the fields $\gamma$, $\sigma$ and $\omega$, we deduce from (A.3)
\[
k(v, w) - k(w, v) = 2 \omega(v, w),
\]
for any spatial vectors $v$ and $w$. Plugging this expression into (24) and (37) respectively yields
\[
u \cdot [v, w] = -2 \omega(v, w), \quad T[\mathcal{D}](v, w) = -2 \omega(v, w) u.
\]
In the kinematical formulation, the temporal part of the Lie bracket of two spatial vectors, or, equivalently, the torsion of the spatial connection, is induced by the fluid vorticity.

**A.3 1 + 3 kinematical form of the Einstein field equation**

With the help of (A.4), we reformulate the 1+3 Einstein equations (81), (82) and (83) in terms of the kinematical quantities of the fluid. Introducing the rate of shear and the rate of vorticity respectively as
\[
\sigma^2 := \frac{1}{2} \sqrt{\sigma_{\alpha\beta} \sigma^{\alpha\beta}} , \quad \omega^2 := \frac{1}{2} \sqrt{\omega_{\alpha\beta} \omega^{\alpha\beta}},
\]
and adopting the notation
\[
\mathcal{T}_{(\alpha\beta)} := \mathcal{T}_{(\alpha\beta)} - \frac{1}{3} \mathcal{T} \gamma_{\alpha\beta}
\]
to denote the symmetric trace-free part of a rank-2 spatial tensor $\mathcal{T}$, we write:
- the Einstein–Gauss relation (81) as
  \[
  R + \frac{2}{3} \Theta^2 - 2 \sigma^2 + 2 \omega^2 = 16 \pi \epsilon + 2 \Lambda,
  \]
- the Einstein–Codazzi relation (82) as
  \[
  \frac{2}{3} \Theta |\alpha - \sigma^\gamma |\gamma + \omega^\gamma |\gamma + 2 \epsilon \omega_{\gamma\alpha} = 8 \pi q_\alpha ,
  \]
- the Einstein–Ricci relation (83) as
  \[
  \frac{1}{M} \mathcal{L}_m \Theta = -4 \pi (\epsilon + 3 p) + \Lambda - \frac{1}{3} \Theta^2 - 2 \sigma^2 + 2 \omega^2 + a^\alpha |\alpha + a^\alpha a_\alpha ,
  \]
for the trace-part, and
\[
\frac{1}{M} \mathcal{L}_m \sigma_{\alpha\beta} = a_{(\alpha\beta)} + a_{(\alpha} a_{\beta)} - R_{(\alpha\beta)} - \frac{1}{3} \Theta \sigma_{\alpha\beta} + 2 \sigma^\gamma |\gamma \sigma_{\beta\gamma} + 2 \sigma^\gamma (a_{\gamma\beta})_\gamma + 8 \pi \zeta_{\alpha\beta} ,
\]
for the symmetric trace-free part.

These relations constitute the kinematical formulation of the system of 1+3 equations (81), (82) and (83). They are equivalent to (the ten components of) the projected Einstein expressions (78), (79) and (80). (See the discussion in Section 8 for further remarks.)

**Remark A.1.** The kinematical formulation of the additional relations (85) and (86) is given by
\[
\frac{1}{M} \mathcal{L}_m \omega_{\alpha\beta} = -a_{[\alpha} |\beta] , \quad \omega_{[\alpha} |\beta] = a_{[\mu} \omega_{\alpha\beta]}. 
\]

**B Adapted bases and coordinates**

In this appendix, we introduce a four-dimensional vector basis adapted to the congruence of curves along with the dual 1-form basis. The time-like components of spatial tensors and the decomposition of the four-metric are specified. We introduce subsequently a set of adapted coordinates and we briefly discuss the implications for the 1+3 formalism.

For additional information about the choice of bases in the threading point of view, as well as a presentation of adapted local coordinates, we refer the reader to, e.g., [25, 26, 31, 33].
B.1 Bases adapted to the congruence

B.1.1 Construction

The congruence of integral curves upon which the threading procedure is built is characterized by the tangent vector field $u$. This provides the following natural choice for the four-dimensional basis $\{e_\alpha\}$:

$$e_0 := Mu, \quad e_i \cdot e_i > 0.$$  \hfill (B.1)

$M$ is the threading lapse function, introduced earlier via the evolution vector $m$. In the kinematical formulation, it relates the flow of an arbitrary parameter $t$ of a congruence line to the flow of the proper time $\tau$ of the fluid element moving along that line,

$$\frac{d\tau}{dt} = M.$$  \hfill (B.2)

The basis vectors $\{e_i\}$ are only required to be space-like (specifically, we do not ask them to be spatial). The form basis dual to (B.1) is given by

$$e^0 := -M^{-1} u^\flat + M, \quad \langle e^i, u \rangle = 0.$$  \hfill (B.3)

$M$ defines the threading shift 1-form, which provides the freedom of choice for the spatial part of the time-like 1-form basis,

$$\gamma^i(e^0) = M.$$  

The spatial character of the forms $\{e^i\}$ is implied by the dual condition $\langle e^i, e^0 \rangle = \delta^i_0$.

In these bases, the components of the flow vector $u$ and those of its dual $u^\flat$ are respectively written

$$u^\alpha = M^{-1}(1, 0), \quad u_\alpha = M(-1, M_i).$$  \hfill (B.4)

Remark B.1. It is worth mentioning the similarities with the construction of adapted bases in the slicing approach (see, e.g., [32]). Given the unit time-like vector $n$ everywhere orthogonal to a family of space-like hypersurfaces, the bases are then defined by

$$e^0 := -N^{-1} n^\flat, \quad g^{-1}(e^i, e^i) > 0,$$  \hfill (B.5)

for the 1-form basis, and

$$e_0 := Nn + N,$$  \hfill (B.6)

for the dual vector basis. $N$ is the slicing lapse function; it relates the flow of an arbitrary parameter $t$ of an integral curve of tangent vector $n$ to the flow of the proper time of the (Eulerian) observers moving along that line. $N$ defines the slicing shift vector, and it offers the freedom of choice for the hypersurface-tangent part of the time-like vector basis. In these bases, the components of $n$ and $n^\flat$ are respectively written

$$n^\alpha = N^{-1}(1, -N^i), \quad n_\alpha = -N(1, 0).$$

B.1.2 Components of spatial tensors

In the bases (B.1) and (B.3), the time-like covariant components of a spatial tensor $T$ of type $(k, l)$ vanish,

$$T^\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l = 0,$$

and the time-like contravariant components are given by

$$T^\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l = M_i T^{\alpha_1 \ldots i \ldots \alpha_k} \beta_1 \ldots \beta_i.$$
B.1.3 Decomposition of the four-metric

The metric tensor $g$ is decomposed in the basis (B.3) according to

$$g = -M^2 e^0 \otimes e^0 + 2M^2 M_i e^0 \otimes e^i + (\gamma_{ij} - M^2 M_i M_j) e^i \otimes e^j,$$

and its inverse $g^{-1}$ is decomposed in the basis (B.1) as

$$g^{-1} = -(M^{-2} - M_i M^i) e^0 \otimes e^0 + 2M^i e^0 \otimes e^i + \gamma_{ij} e^i \otimes e^j.$$

**Remark B.2.** Let us mention again the similarities with the slicing approach. In the bases (B.5) and (B.6), the metric tensor is written

$$g = -(N^2 - N_i N^i) e^0 \otimes e^0 + 2N_i e^0 \otimes e^i + h_{ij} e^i \otimes e^j,$$

and its inverse is decomposed as

$$g^{-1} = -N^2 e^0 \otimes e^0 + 2N^{-2} N_i e^0 \otimes e_i + (h_{ij} - N^{-2} N^i N^j) e_i \otimes e_j,$$

where $h := g + n \otimes n$ denotes the metric of the space-like hypersurfaces.

B.2 Adapted coordinates

B.2.1 Construction

We introduce the set of coordinates $(t, X^i)$ adapted to the congruence of curves as follows:

- the spatial coordinates $X^i$ label the integral curves of $C_i$,
- the temporal coordinate $t$ labels a family of hypersurfaces $\{\Sigma_t\}$.

The second item demands that the manifold is globally hyperbolic, which we suppose henceforth. As the congruence can exhibit vorticity, the space-like hypersurfaces cannot be orthogonal to $u$. We shall denote by $n$ their unit normal vector.

The vector basis associated with the coordinates $(t, X^i)$ is written

$$\{\partial_t, \partial_i\}.$$ (B.7)

By definition, the time-like vector $\partial_t$ is tangent to the line of constant spatial coordinates, hence it is tangent to the integral curves. We choose it to be

$$\partial_t := M u.$$ (B.8)

By definition, the space-like vectors $\{\partial_i\}$ are tangent to the line $t = \text{const.}$, $\{X^j = \text{const.}\}_{j \neq i}$; therefore they are tangent to the hypersurfaces. We have

$$\langle n^i, \partial_i \rangle = 0.$$

The vectors $\{\partial_i\}$ are not spatial (in the present terminology) since we have $u \neq n$.

We write the 1-form basis dual to (B.1) as

$$\{dt, dX^i\}.$$ (B.9)

Because the basis $\{\partial_t, \partial_i\}$ is a particular case of (B.1), we can identify its dual to (B.3) and write accordingly

$$dt := -M^{-1} u^0 + M, \quad (dX^i, u) = 0.$$ (B.10)

This concludes the construction of coordinates adapted to the congruence and of their related bases.
B.2.2 Lie derivative

We now specify the form taken by the operator
\[ \frac{1}{M} \mathcal{L}_m, \quad \text{with} \quad m := Mu, \]
in the coordinate basis \((B.7)\). From Eq. \((B.8)\), the definition of the Lie derivative (see, e.g., [32]) and the fact that \(\partial_t\) is a basis vector, we have
\[ \frac{1}{M} \mathcal{L}_m = \frac{1}{M} \partial_t. \quad (B.11) \]
Noticing that the partial derivative with respect to \(t\) is evaluated along the curves of constant coordinates \(X^i\), we can identify \(\partial_t\) to the total (convective) derivative \(d/d\tau\). Accordingly we can write, by means of \((B.2)\),
\[ \frac{1}{M} \mathcal{L}_m = \frac{d}{d\tau}. \]
The choice of the coordinates \((t, X^i)\) and of their related bases is thus completely adapted to the congruence. In the 1+3 kinematical formulation, they supply the Lagrangian point of view for the description of the fluid dynamics.

B.2.3 Vorticity and acceleration

Let us finally turn to the component forms of the fluid vorticity and acceleration in the introduced bases. Special attention should be given to the former quantity, as it constitutes a central aspect in the 1+3 (kinematical) formalism (see the discussion in Section 8).

In the bases \((B.7)\) and \((B.9)\) the components of the vorticity are written
\[ \omega_{ij} = M (M_{[i} M_{j]} t - M_{[i,j]}) \quad (B.12) \]

Proof. From Eq. \((A.2)\) and Eq. \((8)\) written for \(u\), we have
\[ \omega_{\alpha\beta} = \gamma_{[\alpha} \gamma_{\beta]} \left( e_\gamma (u_\delta) - \Gamma^\lambda_{\gamma\delta} u_\lambda \right). \]
With the help of \((16)\) and using the fact that the structure coefficients of \(\{\partial_t, \partial_i\}\) vanish, we obtain
\[ \omega_{\alpha\beta} = \gamma_{[\alpha} \gamma_{\beta]} e_\gamma (u_\delta). \]
\(\omega\) being a spatial tensor, its time-like covariant components are zero in the basis \((B.9)\). Hence we consider
\[ \omega_{ij} = \gamma_{[i} \gamma_{j]} e_\gamma (u_\delta). \]
From Eqs. \((19)\) and \((B.4)\) we conclude the proof. \(\square\)

Remark B.3. Equation \((B.12)\) shows that a vanishing shift 1-form implies a vanishing vorticity (the converse is not true). This can be understood in terms of the choice of the bases as follows. A zero shift implies that the flow form \(u^\flat\) can be formulated as a gradient (cf. Eq. \((B.10)\)). Hence its rotational vanishes and the vorticity cancels.

At last, we write the components of the acceleration with respect to the bases \((B.7)\) and \((B.9)\) as
\[ a_i = \frac{D_i M}{M} + M_{i,t}. \]

Proof. From Eqs. \((14)\) and \((28)\) we have
\[ \gamma^\beta_{\alpha} \mathcal{L}_m u_\beta = Mu_\alpha - D_\alpha M. \quad (B.13) \]
The components \(u_\alpha\) are written by means of \((B.10)\) as
\[ u_\alpha = M (-dt)_{\alpha} + M_{\alpha}. \quad (B.14) \]
Using the definition of the Lie derivative (see, e.g., [32]) and the fact that \( m = \partial_t \) is a basis vector and \( dt \) a basis 1-form, we find

\[
\gamma^\beta_\alpha \mathcal{L}_m u_\beta = M\gamma^\beta_\alpha \mathcal{L}_m M_\beta .
\]

From (67) and the spatial character of \( M \) we reformulate the right-hand side and we insert the outcome into (B.13) to obtain

\[
a_\alpha = \frac{D_\alpha M}{M} + \mathcal{L}_m M_\alpha .
\]

With the help of (B.11) we conclude the proof.

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