Double Categories in Mathematical Physics

S.S. Moskaliuk, A.T. Vlassov

Bogolyubov Institute for Theoretical Physics
14b, Metrolohichna str., Kiev, UA-252143, Ukraine
E-mail: mss@gluk.apc.org

June 2, 2008

Abstract

Expansion of the categorical point of view on many areas of the mathematics and mathematical physics will cause to deeper understanding of genuine features of these problems. New applications of categorical methods are connected with new additional structures on categories. One of such structures, the double category, is considered in this article. The double category structure is defined as generalization of the bicategory structure. It is shown that double categories exist in the topological and ordinary quantum field theories, and for dynamical systems with inputs and outputs. Morphisms of all these double categories are not maps of sets.

1 Double Category

Definition 1.1 A double category $D$ consists of the following:

1) A category $D_0$ of objects $\text{Obj}(D_0)$ and morphisms $\text{Mor}(D_0)$ of 0-level.

2) A category $D_1$ of objects $\text{Obj}(D_1)$ of 1-level and morphisms $\text{Mor}(D_1)$ of 2-level.

3) Two functors $d, r : D_1 \rightarrow D_0$.

4) A composition functor

\[
*: D_1 \times_{D_0} D_1 \rightarrow D_1
\]
where the bundle product is defined by commutative diagram

\[
\begin{array}{ccc}
D_1 \times_{D_0} D_1 & \xrightarrow{\pi_2} & D_1 \\
\pi_1 \downarrow & & \downarrow d \\
D_1 & \xrightarrow{r} & D_0
\end{array}
\]

(5) A unit functor \( ID : D_0 \rightarrow D_1 \), which is a section of \( d, r \).
The above data is subject to Associativity Axiom and Unit Axiom. If both of them are fulfilled only up to equivalence then the double category is called a weak double category, if they are fulfilled strictly then it is a strong double category.

Here we see that for two objects \( A, B \in \text{Obj}(D_0) \) there are 0-level morphisms \( D_0(A, B) \) which we note by ordinary arrows \( f : A \rightarrow B \), and 1-level morphisms \( D(1)(A, B) \) which we note by the arrows \( \xi : A \Rightarrow B \), for \( A = d(\xi) \) and \( B = r(\xi) \). So with a 2-level morphism \( \alpha : \xi \Rightarrow \xi' \), where \( \xi : A \Rightarrow B \) and \( \xi' : A' \Rightarrow B' \) we can associate the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\xi} & B \\
d(\alpha) \downarrow & & \downarrow r(\alpha) \\
A' & \xrightarrow{\xi'} & B'
\end{array}
\]

and arrow \( \alpha : d(\alpha) \Rightarrow r(\alpha) \)

The composition on 2-level associated with the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\xi} & B \\
d(\alpha) \downarrow & & \downarrow r(\alpha) \\
A' & \xrightarrow{\xi'} & B'
\end{array}
\]

\[
\begin{array}{ccc}
A'' & \xrightarrow{\xi''} & B'' \\
d(\alpha') \downarrow & & \downarrow r(\alpha') \\
A'' & \xrightarrow{\xi'''} & B''
\end{array}
\]

Now we can define for double categories double (category) functors and their morphisms, double subcategories, the category \( DCat \) of double categories, equivalence of double categories, dual double categories (changed direction off 1-level morphisms, i.e. \( d, r \) are transposed), and so on.
2 Examples of Double Categories

Examples considered below show that double categories are sufficiently natural for mathematics.

**Example 1:** Bicategories ([2]) is the partial case of double category $D$ when the category $D_0$ is trivial, i.e. has only identical morphisms and composition of 1-level and 2-level morphisms are associative.

**Example 2:** For each category $C$ we have the canonical double category $\text{Morph}(C)$ of morphisms. Let $C$ be a category, $T$ be the diagram $\bullet \to \bullet$, $TC$ be the category of diagrams in $C$ of type $T$, let $D_0 = C$ and $D_1 = TC$. The functor $d$ maps the diagram $f : A \to B$ into the object $A$, the functor $r$ maps this diagram into the object $B$, and so on. It is easy to see that we get a double category $D$ which is noted by $\text{Morph}(C)$. Here $\text{Obj}(D_1) = \text{Mor}(D_0)$, a 2-level morphism $f \Rightarrow g$ is a pair $(u, v)$ of morphisms $u, v \in \text{Mor}(C)$ from the commutative diagram

$$
\begin{array}{ccc}
A & \overset{u}{\to} & A' \\
\downarrow^f & & \downarrow^{f'} \\
B & \overset{v}{\to} & B'
\end{array}
$$

with usual composition.

**Example 3:** Let $C$ be a category with bundle products, i.e. for all morphisms $u, v$ to $Y$ the universal square

$$
\begin{array}{ccc}
X \times_Z Y & \longrightarrow & Y \\
\downarrow & & \downarrow^v \\
X & \overset{u}{\longrightarrow} & Z
\end{array}
$$

exists. And let $T$ be the following diagram

$$
\bullet \leftarrow \bullet \to \bullet,
$$

$TC$ be the category of diagrams in $C$ of type $T$. Now we define the double category $D$ with $D_0 = D$ and $D_1 = TC$. Two functors

$$
d, r : TC \to C,
$$

where the functor $d$ maps the diagram $A \leftarrow M \to B$ into the object $A$, the functor $r$ maps this diagram into the object $B$. The composition: for two
1-level morphisms $\xi = (A \xrightarrow{\pi} M \xrightarrow{f} B) : A \Rightarrow B$ and $\xi' = (B \xrightarrow{\pi'} M' \xrightarrow{f'} C) : B \Rightarrow C$ we define their composition $\xi' \circ \xi = (A \xrightarrow{\pi \pi_1} M \times_B M' \xrightarrow{f \circ f_1} C)$ where the bundle product is defined by the universal diagram

$$
\begin{array}{c}
M' \\
\pi_1 \\
\downarrow \\
M \\
\xrightarrow{f} \\
B
\end{array}
\quad \begin{array}{c}
\xrightarrow{\pi_2} \\
\downarrow \\
M
\end{array}
\quad \begin{array}{c}
\xrightarrow{\pi'} \\
\downarrow \\
B
\end{array}

A 2-level morphism is a triple $\alpha = (u, v, w) : \xi \Rightarrow \xi'$ from the following commutative diagram

$$
\begin{array}{c}
M \\
\xrightarrow{f} \\
\downarrow \pi \\
A \\
\xleftarrow{u} \\
\downarrow \\
A'
\end{array}
\quad \begin{array}{c}
B \\
\xleftarrow{v} \\
\downarrow w \\
M' \\
\xrightarrow{f'} \\
B'
\end{array}

$$

with the evident composition.

**Example 4**: Let $k$ be a ring, $\text{Alg}_k$ be the category of unital $k$-algebras. We define the double category with $D_0 = \text{Alg}_k$, $\text{Obj}(D_1)$ as a set of bimodules. So that for a left $A$- and right $B$-module (= $A \otimes_k B^\circ$-module) $M$ we suppose $d(M) = A$, $r(M) = B$. Composition of two morphisms $A \xrightarrow{M} B \xrightarrow{N} C$ then defined as $N \ast M = M \otimes_B N$.

For $A \xrightarrow{M} B$ and $C \xrightarrow{N} D$ the set $D_1(M, N)$ is the set of triples $(u, v, w)$, where $u : A \rightarrow C$, $v : B \rightarrow D$, $w : M \rightarrow N$ such that $w(amb) = u(a) \cdot w(m) \cdot w(b)$. Unit for $A \in \text{Obj}(\text{Alg}_k)$ is $A \otimes_k A^\circ$-module $A$. Subcategories, see [4]. There is the natural inclusion of double categories

$$\text{Morph}(\text{Alg}_k) \hookrightarrow \text{ALG}_k.$$

Here 1-level morphism $N : A \Rightarrow B$ we can consider as an algebra homomorphism $A \otimes_k B^\circ \rightarrow \text{End}_k(N)$ (what is a selected element $n_0$?). Analogue formulas appear in the next more general situation.

**Example 5**: For a multiplicative (tensor) category $(C, \otimes, U, u)$ (see [3]). Then we have the double category with $D_1 = C$, and $D_0 = (*, *)$, trivial category with one object and one morphism. The composition is

$$D_1 \times_{D_0} D_1 = C \times C \overset{\otimes}{\Rightarrow} C.$$
Other double category is $D$ with $D_0 = C$ and $D_1$ such that
$$\text{Obj}(D_1) = \{(X, x) | A, B, X \in \text{Obj}(C), \ x : X \otimes A \to B\}.$$ 
So, we write $\xi = (X, x) : A \Rightarrow B$ and for $\xi \in \text{Obj}(D_1)$ we denote $\xi = (X_\xi, x_\xi)$,
$$d(\xi) = A_\xi, \ r(\xi) = B_\xi.$$ 
2-level morphisms
$$D_1(\xi, \xi') = \{(f_1, f_2, f_3) | \text{commutative diagram}\}$$
$$f_3 \otimes f_1 \downarrow \downarrow f_2' \quad \quad X' \otimes A' \quad \quad X' \otimes B'$$
and $d(f_1, f_2, f_3) = f_1, \ r(f_1, f_2, f_3) = f_2$.
Composition $D_1 \times_{D_0} D_1 \to D_1$ is defined so
$$A \overset{\xi}{\Rightarrow} B \overset{\xi'}{\Rightarrow} B' \quad \xi \circ \xi' = (A, B', X'X, x''),$$
where $x''$ is the following composition
$$(X' \otimes X) \otimes A \overset{\varphi^{-1}}{\Rightarrow} X' \otimes (X \otimes A) \overset{id_{X' \otimes A}}{\Rightarrow} X' \otimes B \overset{x'}{\Rightarrow} B'.$$

### 3 Action of a double category

Double categories are categorical variants of usual monoids (and groups), and thus we have the corresponding variant for their actions. Below the definition of action of a double category $d, r : D_1 \to D_0$ on categories over $D_0$ is given. Thus we get an analog of group-theoretical methods in categorical frames.

**Definition 3.1** *(Left) action of a double category $d, r : D_1 \to D_0$ on a category $p : M \to D_0$ over $D_0$ is a functor $\varphi$ such that
(1)The diagram is commutative

\[
\begin{array}{ccc}
D_1 \times_{D_0} M & \xrightarrow{\varphi} & M \\
\downarrow r \circ \pi_1 & & \downarrow p \\
D_0 & & D_0
\end{array}
\]

where the bundle product $D_1 \times_{D_0} M$ is defined by the diagram

\[
\begin{array}{ccc}
D_1 \times_{D_0} M & \xrightarrow{\pi_2} & M \\
\downarrow \pi_1 & & \downarrow p \\
D_1 & \xrightarrow{r} & D_0
\end{array}
\]
(2) The diagram is commutative with precise to an isomorphism

\[
(D_1 \times_{D_0} D_1) \times_{D_0} M \cong D_1 \times_{D_0} (D_1 \times_{D_0} M) \xrightarrow{id_{D_1 \times_{D_0} D_0} \varphi} D_1 \times_{D_0} M \to M
\]

and there exists a functor isomorphism \( \varphi \) such that

\[
\forall \; \xi, \xi' \in \text{Obj}(D_1), \; m \in \text{Obj}(M) \quad \varphi_{\xi, \xi', m} : (\xi * \xi') * m \to \xi * (\xi' * m)
\]

(3) For the unit functor we have a functor isomorphism \( \chi : \varphi \circ (ID \times id_M) \cong id_M \) or for objects

\[
\forall \; A \in \text{Obj}(D_0), \; m \in \text{Obj}(M) \quad \chi_{A, m} : ID_A * m \cong m
\]

So we have the map of pair of objects \( \xi \in \text{Obj}(D_1), \; m \in \text{Obj}(M) \)

\[
(\xi \xrightarrow{\xi} \xi, \xi' \xrightarrow{\xi'} p(m), m) \mapsto \varphi(\xi, m) \text{ such that } p(\varphi(\xi, m)) = A, \text{ and of morphisms } \alpha \in D_1(\xi, \xi'), u \in M(m, m')
\]

\[
\begin{array}{c}
\xi \\
\alpha \\
\xi'
\end{array}
\begin{array}{c}
A \\
d(\alpha) \\
A'
\end{array}
\begin{array}{c}
p(m) \\
n(u) \\
p(m')
\end{array}
\begin{array}{c}
\varphi(\xi, m) \\
\varphi(\alpha, u) \\
\varphi(\xi', m')
\end{array}
\]

and here \( p(\varphi(\alpha, u)) = f \).

The definition of a **right action** is evident.

Invariant subcategories, ”orbits”, invariant properties.

An orbit of a subcategory is a subcategory.

**EXAMPLE 6**: Each double category \( D \) acts on itself one from left and from right by the composition \( \ast \).

**EXAMPLE 7**: Let \( SC \) be a subcategory of morphisms of the category \( C \) such that for all \( (\pi : M \to B) \in \text{Obj}(SC) \) and all \( (f : B' \to B) \in \text{Mor}(C) \) there exists the universal square

\[
f^*M = B' \times_B M \xrightarrow{\pi} M \xrightarrow{\pi} B
\]
and \( \Pi : SC \to C \) is the projection on base, i.e. \( \Pi : (\pi : M \to B) \mapsto B \).
Then we have the natural action of \( \text{Morph}(C) \) on \( SC \)
\[
\text{Morph}(C) \times_C SC \longrightarrow SC
\]
which maps the pair \(( (f : B' \to B), (\pi : M \to B) ) \) to \(( \pi_1 : f^*M \to B' ) \).

**Example 8:** Let \( \text{mod}_k \) be the category of left modules over \( k \)-algebras and \( \Pi : \text{mod}_k \to \text{Alg}_k \) be the natural projection, which an object \(( A, M ) \) maps to \( A \).
There is natural action of \( \text{ALG}_k \) on \( \text{mod}_k \)
\[
( \text{ALG}_k)_1 \times (\text{ALG}_k)_0 \text{mod}_k \to \text{mod}_k
\]
such that for \( \xi = N : A \Rightarrow B \) and \( B \)-module \( M \)
\[
\xi^*(B, M) = (A, N \otimes_B M).
\]

**Example 9:** Characteristic classes. Let a double category \( G \) act on \( p : M \to G_0 \), and there is a contravariant functor \( H : G^\circ \to \text{Morph}(M) \).
A characteristic class of \( m \in \text{Obj}(M) \) is \( c(m) \in H_0(p(m)) \), such that for all \( \xi : p(m') \Rightarrow p(m) \) we have
\[
c(\xi^*m) = H_1(\xi)(c(m)).
\]

**Example 10:** Equivariant functors. Let \( M \) be a category of manifolds (topological or smooth), \( L \) be a category of locally trivial bundles over objects of \( M \). Then \( \text{Morph}(M) \) acts on \( L \). Let \( G \) be a topological group, \( M_G \) be the category of \( G \)-manifolds, \( P \) is the category of principal bundles with structure group \( G \) over objects of \( M \). The functor
\[
P \times M_G \to L
\]
which maps \(( \eta, F )\) to the fiber bundle \( \eta[F] \) with fiber \( F \). This functor is equivariant relatively of action \( \text{Morph}(M) \) on \( P \) and \( L \).

**Example 11:** Let \( \text{ISO}(\mathcal{C}) \) be a sub double category of \( \text{Morph}(\mathcal{C}) \) such that \( (\text{ISO}(\mathcal{C})){_0} = \mathcal{C} \) and \( (\text{ISO}(\mathcal{C})){_1} \) is the full subcategory of \( (\text{Morph}(\mathcal{C})){_1} \)
with
\[
\text{Obj}(((\text{ISO}(\mathcal{C})){_1}) = \{ f \in \text{Mor}(\mathcal{C}) \mid f \text{ is an isomorphism } \}.
\]
For any forgetful functor $F : \mathcal{C}' \to \mathcal{C}$ (see the next section) the double category $ISO(\mathcal{C})$ acts on $\mathcal{C}'$ from left

$$ISO(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}' \to \mathcal{C}' : (u : B \to F(C), C) \mapsto u^*C$$

and $v^*(u^*C) \cong (u \circ v)^*C$.

4 Cobordism and Double Categories

Let $M_d$ be the category of oriented compact $d$-dimensional smooth manifolds (with boundary) and piecewise smooth maps (the sense of the condition we do not define more exactly here; this may be such continuous maps $f : M \to Y$ that are smooth on a dense open subset $U_f \subset M$), let $CM_d$ be its subcategory of closed (with empty boundary) manifolds and smooth maps, $CM_d \subset M_d$.

There are the following functors:

1. Disjoint union

$$\cup : M_d \times M_d \to M_d : (X, Y) \mapsto X \cup Y.$$ 

2. Changing of the orientation of manifolds on opposite

$$(-) : M_d \to M_d : X \mapsto -X.$$ 

3. Boundary operator

$$\partial : M_{d+1} \to CM_d : X \mapsto \partial X.$$ 

4. Multiplication on the unit segment $I = [0, 1]$

$$I \times \cdot : CM_d \to M_{d+1} : X \mapsto I \times X.$$ 

Now we define a double category $\mathcal{C}(d)$ with

1. $\mathcal{C}(d)_0 = CM_d$.

2. 1-level morphisms $\mathcal{C}(d)_{(1)}(X, X')$ is a set of pairs $(Y, f)$ where $Z$ is oriented compact $(d+1)$-dimensional smooth manifold with the boundary $\partial Y$ and $f$ is an diffeomorphism

$$f : (-X) \cup X' \to \partial Y,$$

where $\cup$ notes the disjoint union of $-X$ and $X'$. Thus we write $(Y, f) : X \Rightarrow X'$. 

8
(3) The composition of \((Y, f) : X \Rightarrow X'\) and \((Y', f') : X' \Rightarrow X''\) is the morphism
\[
(Y \cup_{X'} Y', (f|_X) \cup (f'|_{X'})) : X \Rightarrow X'',
\]
where \((Y \cup_{X'} Y')\) denotes the union \((Y \cup Y')\) after identification of each point \(f(y) \in f(Y)\) with the point \(f'(y) \in f'(Y)\) for all \(y \in Y\) and smoothing this topological manifold.

(4) The 1-level identical morphism \(I D_X\) is \((X \times [0; 1], id_{(-X) \cup X})\), because \(\partial(X \times [0; 1]) = (-X) \cup X\).

(5) 2-level morphisms of \(C(d)_{\xi, \xi'}\) from \(\xi = (Y, f : X' \cup (-X) \rightarrow \partial Y) : X \Rightarrow X'\) to \(\xi' = (Y', f' : X'' \cup (-X') \rightarrow \partial Y') : X' \Rightarrow X''\) are such triples of smooth maps \((f_1, f_2, f_3)\) that the following diagram is commutative
\[
\begin{array}{ccc}
(-X) \cup X' & \xrightarrow{f} & \partial Y \\
\downarrow f_1 \cup f_2 & & \downarrow f_3 \\
(-X') \cup X'' & \xrightarrow{f'} & \partial Y' \subset Y'
\end{array}
\]

It easy to see that functors \(\cup\) and \((-\)) may be expanded to double category functors
\[
\begin{align*}
\cup : C(d) & \rightarrow C(d), \\
(-) : C(d) & \rightarrow C(d)^\circ
\end{align*}
\]
and \((-\)) is an equivalence of the double categories.

**Remark.** It is interesting the appearance the following two formulas for 1-level morphisms in algebras and cobordisms
\[
f : A \otimes_k B^\circ \rightarrow End_k(N) \quad f : (-X) \cup Y \rightarrow \partial Z,
\]
where we have correspondence between the functors
\[
\begin{align*}
(-)^\circ & \leftrightarrow -(\_), \\
\otimes_k & \leftrightarrow \cup, \\
End_k & \leftrightarrow \partial.
\end{align*}
\]
5 Topological Quantum Field Theory

Topological quantum field theory is a functor $Z$ from the category $CM(d)$ of $d$-dimensional manifolds to the category of $H$ of (usually Hermitian) finite dimensional vector spaces and some axioms are satisfied ([1]). Really, the functor $Z$ is a functor between double categories.

Thus, topological quantum field theory in dimension $d$ is a functor

$$Z : C(d) \to \text{Morph}(H),$$

between double categories such that:

1. the disjoint union in $C(d)$ go to the tensor product

$$\bigcup \mapsto \otimes,$$

where $(\cdot)^* : H \to H^\circ$ is dualization of vector spaces.

2. changing of orientation in $C(d)_0$ go to dualization

$$(-) \mapsto (\cdot)^*$$

Thus, as consequence of double categorical functorial properties, we get

1. for each compact closed oriented smooth $d$-dimensional manifold $X \in Obj(C(d)_0)$ the value of the functor $Z(X)$ is a finite dimensional vector space over the field $\mathbb{C}$ of the complex numbers (usually with Hermitian metric),

2. for each $(Y, f) : X \Rightarrow X'$ from $Obj(C(d)_1)$ the value of the functor $Z(Y, f)$ is a homomorphism $Z(X) \to Z(X')$ of (Hermitian) vector spaces,

and the following well known axioms of topological quantum field theory are satisfied:

A(1) (involutivity) $Z(-X) = Z(X)^*$, where $-X$ denotes the manifold with opposite orientation, and $^*$ denotes the dual vector space.

A(2) (multiplicativity) $Z(X \cup X') = Z(X) \otimes Z(X')$, where $\cup$ denotes disconnected union of manifolds.
A(3) (associativity) For the composition \((Y'', f'') = (Y, f) \ast (Y', f')\) of cobordisms must be
\[
Z(Y'', f'') = Z(Y', f') \circ Z(Y, f) \in Hom_C(Z(X), Z(X'')).
\]
(Usually the identifications
\[
Z(X' - X) \cong Z(X)^* \otimes Z(X') \cong Hom_C(Z(X), Z(X'))
\]
allow us to identify \(Z(Y, f)\) with the element \(Z(Y, f) \in Z(\partial Y)\).

A(4) For the initial object \(\emptyset \in Obj(C(d)_0)\)
\[
Z(\emptyset) = C.
\]

A(5) (trivial homotopy condition) \(Z(X \times [0, 1]) = id_{Z(X)}\).

5.1 Field Theory

Here is sketch of categorical construction with double categories for the ordinary field theory, where we deal with fiber bundle and equations for their sections. Let us denote corresponding double category by \(F(d)\) where \((d + 1)\) is the dimension of the space-time, and it is similar to the double category \(C(d)\).

Objects of category \(F(d)_0\) are \(A = (\pi : V \to X, s \in \Gamma(V/X))\) where \(X\) is an oriented compact closed \(d\)-dimensional manifold, \(\pi : V \to X\) is a fiber bundle over \(X\) of some defined type, \(s\) is a section of \(\pi\) with some special properties. The definition of morphism is evident but there are variants. There are functors \(\cup\) and \(-\) as in \(C(d)_0\).

Objects of \(F(d)_1\) are defined the following way. Let \(\pi : E \to Y\) be a fiber bundle over an oriented compact \((d + 1)\)-dimensional manifold with a boundary, \(\mathcal{D}\) is a system of equations for sections of \(\pi\), \(s\) is a solution of \(\pi\) with some special properties. Here the construction of \(\partial E\) depends of type of the system \(\mathcal{D}\). An object of \(F(d)_1\) is \(\xi = (A, A', (E/Y, \mathcal{D}, f), \{\})\) where \(f\) is isomorphism
\[
f : (-A) \cup A' \to \partial(E/Y, \mathcal{D}, f).
\]
There exit the composition \(\xi \ast \xi'\). For the action integral \(S[\xi]\) we have \(S[\xi \ast \xi'] = S[\xi] + S[\xi']\). The system \(\mathcal{D}\) is an Euler-Lagrange equation for action \(S\), and boundary bundles and sections define unique gluing.

Thus it maybe we get categorical field theory.
References

[1] M.F. Atiyah, Topological quantum field, Publ. Math. Inst. Hautes Etudes Sci. Paris, 68, 1989, 175-186.

[2] P. Gabriel, M. Zisman. Calculus of fractions and homotopy theory. Berlin, Heidelberg, New York: Springer-Verlag, 1967.

[3] P. Deligne, J.S. Milne. Tannakian Categories. - In: P. Deligne, J.S. Milne, A. Ogus, Shih Kuang-yen. Hodge Cycles, Motives, and Shimura Varieties. - Lecture Notes in Math., No. 900, Berlin, Heidelberg, New York: Springer-Verlag, 1982, pp.101-228.

[4] J.-L. Loday. Cyclic Homology. Springer-Verlag, Berlin, Heidelberg, 1992.

[5] S. Maclane. Categories for Working Mathematician, Graduate Texts in Mathematics 5, Springer-Verlag, 1971.

[6] S.S. Moskaliuk, A.T. Vlassov. On some Categorical Constructions in Mathematical Physics. In: Proceed. of the Firth Wigner Workshop, August 25-30, 1997; Vienna (to appear).

[7] A.T. Vlassov. The Category of Dynamical Systems with Inputs and Outputs. In: Proceed. of the Firth Work-Shop on Mind, Brain and Neurocomputers. Byelorussian State University Press, Minsk, 1996 (to appear).