Constrained N-body problems

Wojciech Szumiński and Maria Przybylska

Institute of Physics,
University of Zielona Góra, Licealna 9,
PL-65-407, Zielona Góra, Poland

June 13, 2016

Abstract

We consider a problem of mass points interacting gravitationally whose motion is subjected to certain holonomic constraints. The motion of points is restricted to certain curves and surfaces. We illustrate the complicated behaviour of trajectories of these systems using Poincaré cross sections. For some models we prove the non-integrability analysing properties of the differential Galois group of variational equations along certain particular solutions of considered systems. Also some integrable cases are identified.

Key words: n-body problem; non-integrability; Morales–Ramis theory; differential Galois theory; Poincaré sections; chaotic Hamiltonian systems.

1 Introduction

Let us consider several point masses interacting mutually according to a certain law. This is just the n-body problem. For the classical gravitational, or electrostatic interactions such problem with \( n > 2 \) is not integrable. Let us restrict the motion of points to certain surfaces or curves. These holonomic constraints modify interactions of points. In some cases these modifications lead to the non-integrability, and in others to the integrability. The described constrained classical \( n \)-body problems can be considered as a source of toy models for testing various methods and tools for study dynamics of classical systems. In fact this paper arose from such investigations. Several simple examples show that, in fact, one can meet interesting and difficult problems investigating this kind of systems and moreover, such, let us say, academic investigations, give unexpected results.

To describe them let us recall the anisotropic Kepler problem which appears in quantum mechanics of solid. It was thoroughly investigated by Guz twiller [5]. The rescaled Hamiltonian of the problem is given by

\[
H = \frac{1}{2} \left( p_1^2 + p_2^2 + p_3^2 \right) - \frac{1}{\sqrt{x^2 + \mu(y^2 + z^2)}}, \tag{1.1}
\]
where \( \mu \) is a positive constant. For the two-degrees of freedom version of this problem the Hamiltonian reads

\[
H = \frac{1}{2} \left( p_1^2 + p_2^2 \right) - \frac{1}{\sqrt{x^2 + \mu y^2}}. \tag{1.2}
\]

Unexpectedly, these systems can be considered as gravitational two body problems with constraints. To see this, let us consider two masses, one mass moving along a line, and the second mass moving along a perpendicular line, see Fig. 1(a). The Hamiltonian of the system is following

\[
H_1 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Gm_1m_2}{\sqrt{x^2 + y^2}}. \tag{1.3}
\]

So, by a simple rescaling we obtain Hamiltonian (1.2). Similarly, let one mass moves along a line, and the other moves in a plane perpendicular to this line, see Fig. 1(b). The Hamiltonian has the form

\[
H_2 = \frac{p_1^2}{2m_1} + \frac{1}{2m_2} (p_2^2 + p_3^2) - \frac{Gm_1m_2}{\sqrt{x^2 + y^2 + z^2}} \tag{1.4}
\]

and again its simple rescaling gives (1.1).

Figure 1: Motion of two masses on: (a) perpendicular axes, (b) plane and perpendicular axis.

As we can see the Hamiltonians (1.2) and (1.1) differ from the Hamiltonians of standard planar and spatial Kepler problem only in the parameter \( \mu \). For \( \mu \neq 0 \), contrary to standard Kepler problem, the force is not radial. The dynamics of anisotropic Kepler problem is dramatically different from that of the standard Kepler problem.

The chaotic behaviour of the anisotropic Kepler problems was investigated in numerous papers, see e.g. [2, 3, 5] and the non-integrability of planar problem was proved in [4] and for planar and spatial problem in [1]. The non-integrability proof in [1] uses the differential Galois approach and authors state that for \( \mu \notin \{0, 1\} \) there is no meromorphic integrals besides the Hamiltonian itself. But there is no written about meromorphic functions of what variables authors say. If one consider meromorphic functions of coordinates and momenta, then already Hamiltonian is not meromorphic function, thus system trivially is not meromorphically integrable for all values of \( \mu \). Thus below we formulate these theorems in a more precise way.
Theorem 1.1. Hamiltonian system defined by (1.2) is integrable in the Liouville sense with first integrals which are meromorphic in \((x, y, p_1, p_2, r)\) where \(r = \sqrt{x^2 + \mu y^2}\), if and only if \(\mu \in \{0, 1\}\).

In the case when \(\mu = 1\) this system has two additional functionally independent additional first integrals

\[
I_1 = p_2x - p_1y, \quad I_2 = p_2(p_1y - p_2x) + \frac{x}{\sqrt{x^2 + y^2}},
\]

thus it is super-integrable.

Spatial anisotropic Kepler problem defined by (1.1) has an invariant subspace defined by \(z = p_3 = 0\). In this subspace it coincides with the previous system. Thus, the necessary conditions of the integrability are the same as for the previous system.

Theorem 1.2. Hamiltonian system defined by (1.1) is integrable in the Liouville sense with first integrals which are meromorphic in \((x, y, z, p_1, p_2, p_3, r)\), where 

\[r = \sqrt{x^2 + \mu(y^2 + z^2)},\]

if and only if, \(\mu \in \{0, 1\}\).

In the case \(\mu = 1\) it coincides with three dimensional standard Kepler problem, and it has the following first integrals

\[
c = r \times p, \quad e = p \times c - \frac{r}{r},
\]

where \(r = (x, y, z)\), \(p = (p_1, p_2, p_3)\), and \(r = \sqrt{x^2 + y^2 + z^2}\). Among them one can find three functionally independent and pairwise commuting.

Hamiltonian (1.2) (and also (1.1)) because of presence of square root \(r\) is not single-valued and meromorphic in coordinates and momenta. Thus, formally, in order to apply the differential Galois theory approach to such a Hamiltonian system we have to extend it to the corresponding Poisson system introducing \(r\) as additional variable. However, in calculations one can work with the original Hamiltonian system, and the only trace of this extension is the fact that we study the integrability in the class of meromorphic functions of not only coordinates and momenta but also of \(r\). This extension procedure as well as its application to a certain three-body problem was given in [7]. The similar trick is applied to all remaining Hamiltonian systems with algebraic potentials considered in this paper.

The above examples show that it is reasonable to examine similar classes of constrained \(n\)-body systems. In the next section we will give several examples of such systems with a few degrees of freedoms. In a case when the considered system reduces to a system with two degrees of freedom the Poincaré cross sections give us quickly insight into the dynamics of the systems. However, a challenging problem is to prove that they are non-integrable and to find values of parameters for that they become integrable. For some presented problems we prove their non-integrability using the so-called Morales-Ramis theory [8]. It is based on analysis of differential Galois group of variational equations obtained by linearisation of equations of motion along a particular solution. The main theorem of this theory states that if the considered system is meromorphically integrable in the Liouville sense, then the identity component of the differential Galois group of the variational equations is Abelian. For a precise definition of the differential Galois group and differential Galois theory, see, e.g., [9].
2 Integrability analysis of several restricted n-body problems

Model 3: Two masses on two inclined straight lines

The direct generalisation of the model 1 from Fig.1(a) is following. Assume that mass \( m_1 \) moves along horizontal line \( q_2 = 0 \) and it has coordinates \((q_1, 0)\), and mass \( m_2 \) with coordinates \( q_2(\cos \phi, \sin \phi) \) moves along a straight line inclined to the horizontal line, see Fig.2. The Hamiltonian function is given by

\[
H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{G m_1 m_2}{\sqrt{q_1^2 + q_2^2 - 2 \cos \phi q_1 q_2}}. \tag{2.1}
\]

In Appendix we will prove the following theorem.

**Theorem 2.1.** The system governed by Hamiltonian (2.1) is integrable in the class of functions meromorphic in \((q_1, q_2, p_1, p_2, r)\) where \( r = \sqrt{q_1^2 + q_2^2 - 2 q_1 q_2 \cos \phi} \), iff

- either \( \phi \in \{0, \pi\} \) and \( m_1, m_2 \in \mathbb{R} \), or

- \( \phi \in \{\pi/2, 3\pi/2\} \) and \( m_2 = m_1 \).

Model 4: Masses moving on the parallel lines

Let us consider a problem of \( n \) masses moving in parallel lines, see Fig.3. As a generalised coordinates we use the relative displacements \( q_i = x_i - x_{i-1} \) along axis \( x \), for \( i = 2, \ldots, n \) and \( q_1 = x_1 \).

The Lagrange and the Hamiltonian functions do not depend on variable \( q_1 \), which is a cyclic variable and its corresponding momentum \( p_1 = c \) becomes a parameter. Thus, we obtain the reduced system with \( n - 1 \) degrees of freedom. Model of \( n = 2 \) masses is integrable. The reduced system with \( n = 3 \) masses has two degrees of freedom and it is described by the following Hamiltonian

\[
H = \frac{1}{2} \left( \frac{(c-p_2)^2}{m_1} + \frac{(p_2-p_3)^2}{m_2} + \frac{p_3^2}{m_3} - \frac{2Gm_2 m_3}{\sqrt{(a-b)^2+q_3^2}} + m_1 \left( -\frac{2Gm_2}{\sqrt{a^2+q_3^2}} - \frac{2Gm_3}{\sqrt{b^2+(q_2+q_3)^2}} \right) \right). \tag{2.2}
\]

We assumed that masses \( m_2 \) and \( m_3 \) move along horizontal curves \( y = a \) and \( y = b \), respectively. Fig.4 shows the Poincaré cross sections related to (2.2). Clearly, the system is generally non-integrable. However, a proof of this fact is an open question.
Figure 4: Poincaré sections for model 4. Parameters: \( m_1 = 1, \ m_2 = 2, \ m_3 = 1, \ G = 1, \ a = 3, \ b = 1, \ c = 0, \) cross-plain \( q_2, \ p_2 > 0. \)

**Model 5: Two masses moving on an ellipse and a straight line parallel to the main axis of the ellipse**

In Fig. 5 the geometry of the system is shown. Now we assume that the mass \( m_1 \) moves on the ellipse with coordinates \((\rho \cos \phi, \rho \sin \phi)\), where \( \rho = c/(1 + e \cos \phi) \), and mass \( m_2 \) moves along a straight line parallel to the main axis of ellipse with coordinates \((x, a)\). The Hamiltonian function is given by

\[
H = \frac{1}{2} \left( \frac{p_x^2}{m_2} + \frac{p_\phi^2 (1 + e \cos \phi)^4}{c^2 m_1 (1 + e^2 + 2x \cos \phi)} - \frac{2G m_1 m_2}{\sqrt{\left( \frac{c \cos \phi}{1 + e \cos \phi} - x \right)^2 + \left( \frac{c \sin \phi}{1 + e \cos \phi} - a \right)^2}} \right).
\]

Fig. 6 shows the Poincaré cross sections. They present that for certain fixed values of parameters, the system is not integrable. In fact we can prove the following theorem.

**Theorem 2.2.** If \( a = 0, \) and \( m_1 \neq m_2, \ m_1 m_2 \neq 0, \) then the system governed by Hamiltonian (2.3) is not completely integrable with first integrals which are meromorphic in \((x, \phi, p_1, p_2, r)\), where

\[
r = \sqrt{\left( \frac{c \cos \phi}{1 + e \cos \phi} - x \right)^2 + \left( \frac{c \sin \phi}{1 + e \cos \phi} \right)^2}.
\]

This theorem is in particular true for the circle when \( e = 0 \) and \( c = \rho. \)
Figure 6: Poincaré sections for model 5. Parameters: \( m_1 = 1, m_2 = 2, G = 1, a = 3, c = 2, e = 0.5, \) cross-plain \( x, p_x > 0. \)

\[
\begin{align*}
E &= -1, \\
E &= -0.5.
\end{align*}
\]

Figure 7: Motion of two masses on: (a) two confocal ellipses, and (b) two concentric ellipses with parallel main axes.

**Model 6: Two mass points moving in two conics**

In Fig. 7(a) the geometry of the system is presented. In this case, masses \( m_1 \) and \( m_2 \) move along two confocal ellipses with coordinates \((\rho_1 \cos \phi_1, \rho_1 \sin \phi_1)\) and \((\rho_2 \cos \phi_2, \rho_2 \sin \phi_2)\), where

\[
\rho_1 = \frac{c_1}{1 + e_1 \cos \phi_1}, \quad \rho_2 = \frac{c_2}{1 + e_2 \cos \phi_2},
\]

and interact gravitationally. Hamiltonian function takes the form

\[
H = \frac{(1 + e_1 \cos \phi_1)^4 \rho_1^2}{2c_1^2 m_1 (2e_1 \cos \phi_1 + e_1^2 + 1)} + \frac{(1 + e_2 \cos \phi_2)^4 \rho_2^2}{2c_2^2 m_2 (2e_2 \cos \phi_2 + e_2^2 + 1)} - \frac{G m_1 m_2}{B},
\]

\[
B = \sqrt{\left( \frac{c_1 \cos \phi_1}{1 + e_1 \cos \phi_1} - \frac{c_2 \cos \phi_2}{1 + e_2 \cos \phi_2} \right)^2 + \left( \frac{c_1 \sin \phi_1}{1 + e_1 \cos \phi_1} - \frac{c_2 \sin \phi_2}{1 + e_2 \cos \phi_2} \right)^2}.
\] (2.4)

To present the dynamics of considered system we make several Poincaré cross sections, see Figs. 8-9.
Figure 8: Poincaré sections for model 6. Parameters: $m_1 = 2$, $m_2 = 2$, $G = 1$, $c_1 = 1$, $c_2 = 2$, $e_1 = \frac{1}{2}$, $e_2 = \frac{3}{2}$, cross-plain $\phi_1$, $p_1 > 0$.

Figure 9: Poincaré sections for model 6. Parameters: $m_1 = 2$, $m_2 = 1$, $G = 1$, $c_1 = 3$, $c_2 = 1$, $e_1 = \frac{3}{5}$, $e_2 = 0$, cross-plain $\phi_1$, $p_1 > 0$. 
Model 7: Two masses moving in concentric ellipses with parallel main axes

The geometry of the system is shown in Fig. 7(b). In this case, masses \( m_1 \) and \( m_2 \) move in two ellipses which have common centres and parallel main axes. Using the standard trigonometric parametrizations of points on ellipses \((a_i \cos \phi_i, b_i \sin \phi_i)\) for \( i = 1, 2 \), we can derive the Hamiltonian

\[
H = \frac{1}{2} \left( \frac{p_1^2}{a_1^2 m_1 \cos \phi_1^2 + a_2^2 m_1 \sin \phi_1^2} + \frac{p_2^2}{b_1^2 m_2 \cos \phi_2^2 + b_2^2 m_2 \sin \phi_2^2} \right) - \frac{2G m_1 m_2}{\sqrt{(a_2 \cos \phi_1 - b_2 \cos \phi_2)^2 + (a_1 \sin \phi_1 - b_1 \sin \phi_2)^2}},
\]

where \( a_1, a_2 \) and \( b_1, b_2 \) are major and minor semi-axes. The Poincaré cross sections are shown in Fig. 10.

Figure 10: Poincaré sections related to model 7. Parameters: \( m_1 = 1, G = 1, a_1 = 0.8, a_2 = 1.1, b_1 = 1, b_2 = \frac{a_2 b_1}{a_1} = 1.4, m_2 = \frac{m_1 \theta_1}{a_2} = 0.73 \), cross-plain \( \phi_1, p_1 > 0 \).

Model 8: N-masses moving in the circles

Let us consider the motion of \( n \)-masses moving on the concentric circles which interact gravitationally. As a generalized coordinates we use the relative angles \( \theta_i \), see Fig. 11. Similarly to the fourth model the Hamiltonian function has one cyclic variable \( \theta_1 \) and its corresponding momentum \( p_1 = c \) is a first integral of the system. Thus, we get the reduced system with \( n - 1 \) degrees of freedom. Case of two masses is of course
integrable, but the model of \( n = 3 \) has much more complex dynamics. To present this complexity we make several Poincaré sections, see Fig. [12]. Hamiltonian of this reduced system has the form

\[
H = \frac{1}{2} \left( \frac{(c - p_2)^2}{\rho_1^2 m_1} + \frac{(p_2 - p_3)^2}{\rho_2^2 m_2} + \frac{p_3^2}{\rho_3^2 m_3} - \frac{2G m_1 m_2}{\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \cos \theta_2}} \right) - \frac{2G m_2 m_3}{\sqrt{\rho_2^2 + \rho_3^2 - 2\rho_2 \rho_3 \cos \theta_3}} - \frac{2G m_1 m_3}{\sqrt{\rho_1^2 + \rho_3^2 - 2\rho_1 \rho_3 \cos (\theta_2 + \theta_3)}}.
\]  

(2.6)

Figure 12: Poincaré sections for model 8. Parameters: \( m_1 = 1 \), \( m_2 = 2 \), \( m_3 = 1 \), \( G = 1 \), \( \rho_1 = 2 \), \( \rho_2 = 3 \), \( \rho_3 = 1 \), \( c = 0 \), cross-plain \( \theta_2 \), \( p_2 > 0 \).

### Acknowledgement

The authors are very grateful to Andrzej J. Maciejewski for many helpful comments and suggestions. This research has been supported by grant No. DEC-2011/02/A/ST1/00208 of National Science Centre of Poland.

### Appendix: Proof of Theorem 2.1

Hamilton equations for Hamiltonian (2.1) have the form

\[
\dot{q}_1 = \frac{p_1}{m_1}, \quad \dot{q}_2 = \frac{p_2}{m_2}, \quad \dot{p}_1 = \frac{G m_1 m_2 (aq_2 - q_1)}{(q_1^2 + q_2^2 - 2a q_1 q_2)^{3/2}}, \quad \dot{p}_2 = \frac{G m_1 m_2 (aq_1 - q_2)}{(q_1^2 + q_2^2 - 2a q_1 q_2)^{3/2}}.
\]  

(2.7)
where \( \alpha := \cos \phi \). In order to simplify calculations, we make the following non-canonical transformation

\[
\begin{bmatrix}
q_1 \\
q_2 \\
p_1 \\
p_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{\beta^2 - \mu_2^2}} & \frac{\sqrt{\beta^2 - \mu_1^2}(\mu_1 + \mu_2)}{(\mu_2 - \sqrt{\beta^2}) \sqrt{\mu_1^2 - \mu_2^2}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{\mu_1 - \mu_2}{2 \sqrt{\beta^2 - 2\mu_2}} & \frac{\sqrt{\beta^2 - \mu_2^2} \sqrt{\mu_1^2 - \mu_2^2}}{(\mu_2 - \sqrt{\beta^2})(\mu_1 + \mu_2)} \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{bmatrix},
\]

\( \mu_1 = m_1 + m_2, \mu_2 = m_2 - m_1, \beta = \sqrt{(m_1 - m_2)^2 + 4m_1 m_2 \alpha^2} \).

System (2.7) after this transformation takes the form

\[
\begin{align*}
\dot{x}_1 &= y_1, \\
\dot{y}_1 &= \frac{Gx_1 (\mu_1 - \beta) (\mu_2 - \beta)^3}{2 \left(-2x_2x_1 (\beta + \mu_1) \sqrt{\frac{\beta^2 - \mu_1^2}{\mu_1^2 - \mu_2^2}} + \frac{2 \beta^2 x_2^2 (\beta + \mu_1)(\beta - \mu_2)}{\mu_1 - \mu_2} + x_1^2 \right)^{3/2}}, \\
\dot{x}_2 &= \frac{2y_2}{\mu_1 + \mu_2}, \\
\dot{y}_2 &= -\frac{G (\mu_1 + \mu_2) [(\mu_1 - \mu_2) (\beta - \mu_2)]^{3/2}}{8 \sqrt{2} x_2^2 \sqrt{\beta^3 (\beta + \mu_1)}}.
\end{align*}
\]

It has invariant manifold \( N = \{(x_1, x_2, y_1, y_2) \in \mathbb{C}^4 \mid x_1 = y_1 = 0\} \) and its restriction to \( N \) is

\[
\begin{align*}
\dot{x}_2 &= \frac{2y_2}{\mu_1 + \mu_2}, \\
\dot{y}_2 &= -\frac{G (\mu_1 + \mu_2) [(\mu_1 - \mu_2) (\beta - \mu_2)]^{3/2}}{8 \sqrt{2} x_2^2 \sqrt{\beta^3 (\beta + \mu_1)}}.
\end{align*}
\]

Let be the particular solution of (2.9) defined by (2.10), and \( Z = [X_1, X_2, Y_1, Y_2]^T \) denotes the variations of \([x_1, x_2, y_1, y_2]^T\). Then, the variational equations along this particular solution have the form \( \dot{Z} = AZ \), where

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{2}{\mu_1 + \mu_2} \\
\frac{G(\mu_1 - \beta)(\mu_1 - \mu_2)^{3/2}(\mu_2 - \beta)^3}{4 \sqrt{2} \beta^{3/2} x_2^3 (\beta + \mu_1)^{3/2}} & 0 & 0 & 0 \\
\frac{G(\beta + \mu_1)(\mu_1 - \mu_2)^{3/2}(\beta - \mu_2)^{3/2}(\mu_1 + \mu_2)}{16 \sqrt{2} \beta^{3/2} x_2^3 (\beta + \mu_1)^{3/2}} & 0 & 0 & 0
\end{bmatrix}.
\]

Equations for \( X_1 \) and \( Y_1 \) form a subsystem of normal variational equations and can be rewritten as a one second-order differential equation for variable \( X \equiv X_1 \)

\[
\dot{X} + \left( -\frac{G (\mu_1 - \mu_2)^{3/2} (\beta - \mu_1)(\beta - \mu_2)^{3/2}}{4 \sqrt{2} \beta^{3/2} x_2^3 (\beta + \mu_1)^{3/2}} \right) X = 0.
\]

We transform this equation using the following change of independent variable

\[
t \rightarrow z = -\frac{4 \sqrt{2} E \sqrt{\frac{\mu_1 + \mu_2}{(\mu_1 - \mu_2)(\beta - \mu_2)}} x_2(t),
\]

\( (2.12) \).
where $E$ is a level of Hamiltonian transformed by means of (2.8) and restricted to $N$. Then normal variational equation (2.11) takes the form

$$X'' + pX' + qX = 0, \quad p = -\frac{1}{2z} + \frac{1}{2(z-1)}, \quad q = \frac{-\beta + \mu_1}{2(\beta + \mu_1)z^2} + \frac{\beta - \mu_1}{2(\beta + \mu_1)(z-1)z'},$$

where $'$ $\equiv \frac{d}{dz}$. We recognize that this equation is a Riemann $P$ equation, see e.g., [6, 8].

$$\frac{d^2X}{dz^2} + \left(1 - a - a' + \frac{1 - c - c'}{z - 1}\right) \frac{dX}{dz} + \left(aa' + \frac{cc'}{z^2} + \frac{bb' - aa' - cc'}{z(z-1)}\right) X = 0,$$

with exponents

$$a = \frac{1}{4} \left(3 + \sqrt{1 + \frac{16\beta}{\nu_1 + \nu}}\right), \quad a' = \frac{1}{4} \left(3 - \sqrt{1 + \frac{16\beta}{\nu_1 + \nu}}\right), \quad b = c' = 0, \quad b' = -1, \quad c = \frac{1}{2}.$$

The differences of exponents are given by

$$\lambda = a - a' = \frac{1}{2} \sqrt{\frac{17\gamma + 1}{\gamma + 1}}, \quad \sigma = b - b' = 1, \quad v = c - c' = \frac{1}{2},$$

where $\gamma = \frac{\beta}{\mu_1}$. Riemann $P$ equation is solvable iff one of the four numbers $\lambda + \sigma + v, -\lambda + \sigma + v, \lambda - \sigma + v, \lambda + \sigma - v$ is an odd integer or $\lambda$ or $-\lambda$ and $\sigma$ or $-\sigma$ and $v$ or $-v$ belong (in an arbitrary order) to the so-called Schwarz’s table [6, 8]. Conditions $\pm \lambda + \sigma + v = 2p + 1$, where $p \in \mathbb{Z}$, give the following expression for $\gamma$

$$\gamma = \frac{-3 + 5p - 2p^2}{1 - 5p + 2p^2},$$

that takes only two non-negative values 0 and 1. Similarly, conditions $\lambda - \sigma + v = 2p + 1$, and $\lambda + \sigma - v = 2p - 1$, where $p \in \mathbb{Z}$, give

$$\gamma = \frac{p - 2p^2}{-2 - p + 2p^2}, \quad \gamma = \frac{-1 + 3p - 2p^2}{-1 - 3p + 2p^2},$$

respectively, that only take two non-negative values 0 and 1.

Since two differences of exponents are equal to 1/2 and 1, only the first case in the Schwarz’s table is admissible that leads to the condition $\lambda = 1/2 + p$, where $p \in \mathbb{Z}$. It gives

$$\gamma = -\frac{p + p^2}{-4 + p + p^2},$$

and this expression takes only two non-negative values 0 and 1. Value $\gamma = 1$ gives $\alpha = \cos \phi = \pm 1$, and that implies $\phi \in \{0, \pi\}$. Parameter $\gamma$ vanishes only when $\beta = 0$, that gives $m_2 = m_1$, and simultaneously $\alpha = \cos \phi = 0$. These are the only cases when the identity component of differential Galois group of Riemann $P$ equation (2.14) with exponents (2.15) is solvable that is necessary for Abelianity and the integrability of the system.
References

[1] Arribas, M., Elipe, A., Riaguas, A.: Non-integrability of anisotropic quasi-homogeneous Hamiltonian systems. Mech. Res. Comm. 30(3): 209–216 (2003)

[2] Casasayas J., Llibre J.: Qualitative Analysis of the Anisotropic Kepler Problem. Memoirs of Amer. Math. Soc. 52, no. 312(1984)

[3] Devaney R. L.: Blowing up Singularities in Classical Mechanical Systems. Amer. Math. Monthly. 89:535–552 (1982)

[4] Gutzwiller M.C.: Bernoulli sequences and trajectories in the anisotropic Kepler problem. J. Math. Phys. 18(4):806–823 (1977)

[5] Gutzwiller M.C.: Chaos in classical and quantum mechanics. Springer-Verlag, New York (1990)

[6] Kimura T.: On Riemann’s equations which are solvable by quadratures. Funkcial. Ekvac. 12:269–281 (1970)

[7] Maciejewski A.J., Przybylska M., Simpson L., Szumiński W.: Non-integrability of the dumbbell and point mass problem. Celestial Mech. Dynam. Astronom. (2013), to appear. doi: 10.1007/s10569-013-9514-7

[8] Morales Ruiz J.J.: Differential Galois Theory and Non-Integrability of Hamiltonian systems. Birkhäuser, Basel (1999)

[9] Van der Put M, Singer M.F.: Galois theory of linear differential equations. Springer-Verlag, Berlin (2003)