Counting rooted nearly cubic planar maps

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Abstract. In this paper it is provided that a functional equation for enumerating rooted nearly cubic planar maps with the valency of root-vertex, the size and the valency of root-face of the maps as three parameters. In particular, the number of rooted nearly cubic planar maps with the valency of root-vertex and the size as two parameters, as an answer to open problem in 1989, can be also derived.

1. Introduction
The enumeration of rooted planar maps came from Tutte W T in 1962 [1]. Since then much work has been done by many scholars such as Bender [2], Brown [3], Cai [4], Gao [5], Liu [6-8], Long [9,10] and Mullin [11] and so on.

A planar map is a figure obtained by embedding a finite connected graph, having at least one vertex, in the Euclidean plane. In general, rooting a map means distinguishing one edge on the boundary of the outer face as the root-edge, and one end of that edge as the root-vertex. In diagrams we usually represent the root-edge as an edge with an arrow on the outer face, the arrow being drawn from the root-vertex to the other end. So the outer face is also called the root-face. A planar map with a rooting is said to be a rooted planar map. We say that two rooted planar maps are combinatorially equivalent or up to root-preserving isomorphism if they are related by 1-1 correspondence of their elements, which maps vertices onto vertices, edges onto edges and faces onto faces, which preserves incidence relations and which preserves the root-vertex, root-edge and root-face. Otherwise, combinatorially inequivalent or non-isomorphic here.

A map with all of its vertices being 3-valent is said to be a cubic map. A nearly cubic map is such a map that all of vertices but one are 3-valent. The exceptional vertex is always marked as the root-vertex of the map.

On problems of enumerating rooted nearly cubic planar maps with the valency of root-vertex and the size as two parameters, Liu [6] put forward an open problem in 1989 (See the equivalent problem 3.1, problem 3.2 and 4.1 in [6]). Although this problem had been studied at that time, because of the result too complex and for staggered summation form, it was ignored (See (4.5) in [6]). The problem was reintroduced in Liu's monograph in 1999 (See (5.5.1), (5.5.2) and (5.5.3) in [8]). A revised edition of this book in 2008, in which the question once again have been proposed (See (5.6.1), (5.6.2) and (5.6.3) in [8]). The main task of this paper is to use parameterized method to solve the problem that induces a positive solution are obtained.

Of course, due to the complexity of the issue its own attributes, the counting result also can have certain complexity.

Let $\mathcal{M}_{nc}$ be the set of all rooted nearly cubic planar maps and write the following formal series as
\[ f = f_{CM_c}(x, y, z) = \sum_{M \in CM_c} x^{n(M)} y^{m(M)} z^{l(M)}, \]

where \( m(M), n(M), l(M) \) denote the root-vertex valency, the size and the root-face valency of \( M \in CM_{nc} \), respectively. In addition, we have also \( h_{CM_c}(x, y, z) = f_{CM_c}(x, y, 1), \ F_{CM_c}(y, z) = f_{CM_c}(1, y, z), \) and \( H_{CM_c}(y) = f_{CM_c}(1, y, 1) = h_{CM_c}(1, y) = F_{CM_c}(1, y, 1). \)

Further, if we write that \( \mathcal{M}_{nc}^{(m)} = \{ M \in \mathcal{M}_{nc} : m(M) = m \} \) for \( m \geq 0 \), then we have \( \mathcal{M}_{nc} = \bigcup_{m \geq 0} \mathcal{M}_{nc}^{(m)} \), and write that 
\[ F_{M_{nc}^{(m)}}(y, z) = \partial_{y}^m f_{M_{nc}}(x, y, z), \quad H_{M_{nc}^{(m)}}(y) = \partial_{y}^m h_{M_{nc}}(x, y), \]

where \( \partial_{y}^m g(x) \) is the coefficient of \( x^m \) in \( g(x) \) as a power series of \( x \).

Terminologies and notations without definition here refer to [6].

2. Functional equations

For a map \( M \), let \( e_1, e_2, \ldots, e_{m(M)} \) be the rotation at the root-vertex of \( M \) such that \( R = e_i(M) \) is the root-edge of \( M \) and \( e_{m(M)} \) is in the boundary of the root-face. The angle \( \langle e_i, e_{i+1} \rangle \) on the rotation is said to be the \( i \)-th angle of \( M \), \( i = 1, 2, \ldots, m(M) \), where \( e_{m(M)+1} = e_1(M) \). For two maps \( M \) and \( M' \), we define a new map \( M + iM' \) as \( M \cup M' \) provided \( M \cap M' = \{ v_i \} \), the common root-vertex of \( M \) and \( M' \), and \( M' \) is inside the inner domain of the face to which the \( i \)-th angle of \( M \) is incident such that \( e_i(M + iM') = e_i(M) \), \( 1 \leq i \leq m(M) \).

For the sets \( \mathcal{M} \) and \( \mathcal{M}' \) of maps, we then write that
\[ \mathcal{M} \oplus_{i(M)} \mathcal{M}' = \{ M + iM' : M \in \mathcal{M}, \ M' \in \mathcal{M}' \}. \]

Particularly, we may define \( \mathcal{M} \oplus_{i(M)} \mathcal{M}' = \mathcal{M} \oplus \mathcal{M}' \). For a loop map \( O \), we define \( \{ O \} \oplus \mathcal{M} = \mathcal{M}^{(m)} \) called the set of all inner maps of \( \mathcal{M} \).

Further, for a map \( M \in \mathcal{M}_{nc} \), let \( M \ast R \) be such a map obtained from \( M \) by contracting the root-edge \( R = R(M) \) of \( M \) into a vertex as the new root-vertex of the resulting map. Let \( M \ast R \) be such a map obtained from \( M \) by deleting the root-edge \( R = R(M) \) and ignoring the vertex of degree 2 at the non-root end of the root-edge \( R = R(M) \).

For convenience, let the vertex map \( v \) be included in \( \mathcal{M}_{nc} \). For the set \( \mathcal{M}_{nc} \) we divide it into four subsets as
\[ \mathcal{M}_{nc} = \{ v \} + \mathcal{M}_{nc}^{(L)} + \mathcal{M}_{nc}^{(II)} + \mathcal{M}_{nc}^{(III)}. \] (1)

such that
\[ \mathcal{M}_{nc}^{(L)} = \{ M \in \mathcal{M}_{nc} : e_i(M) \text{ is a loop} \} ; \]
\[ \mathcal{M}_{nc}^{(II)} = \{ M \in \mathcal{M}_{nc} : e_i(M) \text{ is an isthmus} \}, \] (2)

and the set \( \mathcal{M}_{nc}^{(III)} \) denotes the other maps in \( \mathcal{M}_{nc} \). From the classification of \( \mathcal{M}_{nc} \) in (1) and the definitions of (2), it can be easily seen that the following Lemma 2.1 and Lemma 2.2 holds.

**Lemma 2.1** For the subset \( \mathcal{M}_{nc}^{(L)} \), we have
\[ \mathcal{M}_{nc}^{(L)} = \mathcal{M}_{nc}^{(m)} \oplus \mathcal{M}_{nc}. \] (3)

**Lemma 2.2** For the subset \( \mathcal{M}_{nc}^{(II)} \), we have
\[ \mathcal{M}_{nc}^{(II)} = \mathcal{M}_{nc}^{(1)} \oplus \mathcal{M}_{nc}. \] (4)

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Lemma 2.3 For the subset $\mathcal{M}_{nc}^{III}(R) = \{ M \star R : M \in \mathcal{M}_{nc}^{III} \}$, we have

$$\mathcal{M}_{nc}^{III} = \mathcal{M}_{nc} - \{p\} - \mathcal{M}_{nc}^{(1)} - \mathcal{M}_{nc}^{(2)} \oplus \mathcal{M}_{nc}. \quad (5)$$

Proof: For any $M \in \mathcal{M}_{nc}^{III}(R)$, there must be a map $M' \in \mathcal{M}_{nc}^{III}$ such that $M = M' \star R'$ where $R' = R(M')$ is the root-edge of $M'$. Since $R'$ is neither a loop nor an isthmus by (1) and (2), $M \in \mathcal{M}_{nc}$ and $m(M) \geq 3$ by the cubicness of $M'$. In other words, $M \in \mathcal{M}_{nc} - \{p\} - \mathcal{M}_{nc}^{(1)} - \mathcal{M}_{nc}^{(2)} \oplus \mathcal{M}_{nc}$, which is just the right hand side of (5).

Conversely, for any $M$ in the set on the right hand side of (9), it is easy to see that $m(M) \geq 3$. So we may construct a map $M$ by splitting the root-vertex of $M$ into two vertices $o_1$ and $o_2$ and adding $R' = (o_1, o_2)$ as the root-edge of $M'$ such that the non-rooted end $o_2$ is of valency 3. It is clear that $M' \in \mathcal{M}_{nc}^{III}(R)$. Therefore, from $M = M' \star R'$, $R' = (o_1, o_2)$ we have that $M$ is a member of the set on the left hand side of (5).

Lemma 2.4 For the set $\mathcal{M}_{nc}^{III}(R) = \{ M \star R : M \in \mathcal{M}_{nc}^{III} \}$, we have

$$\mathcal{M}_{nc}^{III}(R) = \mathcal{M}_{nc} - \{p\}. \quad (6)$$

Proof: For any $M \in \mathcal{M}_{nc}^{III}(R)$, it is clear $M \in \mathcal{M}_{nc}^{III} - \{p\}$.

Conversely, for any $M \in \mathcal{M}_{nc} - \{p\}$, let $v_1, v_2, \ldots, v_{l(M)}$ where $l(M) \geq 2$ be the vertices in order on the root-face boundary of $M$. Now taking a point inner $i = v_i v_i e^+$ as a new vertex $i$ and adding a new edge $R' = v_i u_i$ on the root face boundary of $M$, we can get a new map $M'$ with the root $R' = R(M')$ for $i = 1, 2, \ldots, l(M)$. Thus $M = M' \star R' \in \mathcal{M}_{nc}^{III}(R)$ since $M' \in \mathcal{M}_{nc}^{III}$.

According to (3) (4) (5), we have the expressions for the generating functions $f_{\mathcal{M}_{nc}^{III}}(x, y, z)$, $J = I, II, III$ as follows

$$f_{\mathcal{M}_{nc}^{III}}(x, y, z) = x^2 y^2 h f, \quad f_{\mathcal{M}_{nc}^{III}}(x, y, z) = x F_1 f,$$

$$f_{\mathcal{M}_{nc}^{III}}(x, y, z) = x^{-1} y^2 (f - 1 - x F_1 - x^2 F_2 f), \quad (7)$$

where $f = f_{\mathcal{M}_{nc}^{III}}(x, y, z)$, $h = h_{\mathcal{M}_{nc}^{III}}(x, y)$, $F_m = F_{\mathcal{M}_{nc}^{III}}(y, z)$, $H_m = H_{\mathcal{M}_{nc}^{III}}(y)$, and

$$F_1 = y z^2 F_2, \quad F_2 = y z (1 + F_3), \quad H_1 = H_2, \quad H_2 = y (1 + H_3). \quad (8)$$

Theorem 2.1 The generating function $f = f_{\mathcal{M}_{nc}^{III}}(x, y, z)$ for rooted nearly cubic planar maps with the valency of root-vertex, the size and the valency of root-face as three parameters satisfies the equation as follows

$$[x z - x^3 y z^2 h - y z^2 + x^3 (1 - z) F_1] f = x z - y z^2 - x y z^2 F_1,$$

where $h = h_{\mathcal{M}_{nc}^{III}}(x, y)$, $F_1 = F_{\mathcal{M}_{nc}^{III}}(y, z)$.

Proof: From (4) and (13) we have $f = 1 + x^2 y z h f + x F_1 f + x^{-1} y z (f - 1 - x F_1 - x^2 F_2 f)$.

By substituting $F_2 = y z^2 F_1$ of (8) into above equation and grouping the terms, the theorem follows immediately.
Corollary 2.1 The generating function $h = h_{\mathcal{C}_{6c}}(x, y)$ for rooted nearly cubic planar maps with the valency of root-vertex and the size as two parameters satisfies the equation as follows
\[ x^3 y h^2 - (x - y) h + x - y - xy H_1 = 0, \] (10)
where $H_1 = H_{\mathcal{C}_{6c}}(y)$.

Proof: By taking $z = 1$ from (9) the theorem holds.

Corollary 2.2 The generating function $H = H_{\mathcal{C}_{6c}}(y)$ for rooted nearly cubic planar maps with size as a parameter satisfies the equation as follows
\[ y H^2 - (1 - y) H + 1 - y - y H_1 = 0, \] (11)
where $H_1 = H_{\mathcal{C}_{6c}}(y)$.

Proof: By taking $x = 1$ from (10) the theorem holds.

Corollary 2.3 The generating function $F = F_{\mathcal{C}_{6c}}(y, z)$ for rooted nearly cubic planar maps with the size and the valency of root-face as two parameters satisfies the equation as follows
\[ [z - y z^2 (1 + H) + (1 - z) F_i] F = z - y z^2 (1 + F_i), \] (12)
where $H = H_{\mathcal{C}_{6c}}(y), F_i = F_{\mathcal{C}_{6c}}(y, z)$.

Proof: By taking $x = 1$ from (9) the theorem holds.

Theorem 2.2 The enumerating function $F_i = F_{\mathcal{C}_{6c}}(y, z)$ for rooted cubic planar maps with the size and the valency of root-face as two parameters satisfies the equation as follows
\[ (1 - z)^2 y^2 z^2 F_i^2 - (1 - z)(1 - z - z^2) y F_i^2 - (1 - z + y^2 z^4) F_i \]
\[ + y^3 z^5 (1 - z) + y^3 z^4 H_1 = 0, \] (13)
where $H_1 = H_{\mathcal{C}_{6c}}(y)$.

Proof: By (9) we have the equation system as follows
\[ 1 - x^2 y z h - x^{-1} y z + x z^{-1} (1 - z) F_i = 0, \]
\[ 1 - x^{-1} y z + y z F_i = 0. \] (14)
From (14) we find that
\[ x = y z (1 - y z F_i)^{-1}, \]
\[ h = y^{-2} z^{-2} (1 - y z F_i) (1 - y z^2 F_i) F_i. \] (15)
By substituting (15) into (10) and grouping the terms, (13) holds.

3. Main results
In this section, we shall find out the parametric expressions for the generating functions $H_3 = H_{\mathcal{C}_{6c}}(y)$ and $h = h_{\mathcal{C}_{6c}}(x, y)$ firstly, then find out their explicit formulae by using Lagrange inversion formula.

Lemma 3.1 The enumerating function $H_3 = H_{\mathcal{C}_{6c}}(y)$ for rooted cubic planar maps with the size as a parameter has the parametric expressions as follows
\[ y^3 = \eta (1 - 2 \eta) (1 - 2 \eta), \]
\[ y^3 (1 + H_3) = \frac{\eta (1 - 6 \eta)}{1 - 4 \eta}. \] (16)

Proof: Let the discriminant of (10) in $h = h_{\mathcal{C}_{6c}}(x, y)$ be $\delta(x, y)$, we then have $\delta(x, y) = (x - y)^2 - 4 x^3 y (x - y - xy H_1)$. Let $x = ty$, then $\delta(x, y)$ becomes
\[ \delta(t) = \delta(ty, y) = y^3 [1 - 2 t + t^2 + 4 y^3 t^2 - 4 y^3 (1 - y H_1) t^4]. \] (17)
Suppose that the discriminant $\delta(t)$ in (17) has the form:

$$y^2\delta(t)=(1-at)^2(1-2bt+ct^2).$$

(18)

where $a$, $b$ and $c$ all are functions of $y$. By comparing (17) with (18), we have the following equation system:

$$a + b = 1, \quad a^2 + 4ab + c = 1, \quad a^2b + ac = -2y^3, \quad a^2c = -4y^3(1-yH_i).$$

(19)

Let $b = 2\eta$ then $a = 1 - 2\eta$, and $c = 1 - a^2 - 4ab = -4\eta(1-3\eta)$. Thus, from (19) we have

$$a = 1 - 2\eta, \quad b = 2\eta, \quad c = -4\eta(1-3\eta).$$

(20)

By (19) and (20) we can get easily the parametric solution of $H_i = y^2(1+H_i)$.

**Lemma 3.2** The enumerating function $h = h_{\text{nc}}(x, y)$ for rooted nearly cubic planar maps with the valency of rood-vertex and the size as two parameters has the parametric expressions as follows

$$x = \frac{\xi y}{1 + 2\xi\eta + (1-2\eta)\xi^2}, \quad y = \eta(1-2\eta)(1-4\eta), \quad xh = \eta y\left(1 - \frac{-2\eta}{1-4\eta}\right)$$

(21)

**Proof:** According to (18) we can suppose that

$$1 - 2bt + ct^2 = \frac{u^2}{v^2}, \quad \xi = tv,$$

(22)

where $\xi$ is a new parameter introduced for $x$, and $u = 1 + u_1\xi + u_2\xi^2$, $v = 1 + v_1\xi + v_2\xi^2$. After evaluating and simplifying from (22) we have $v^2 - u^2 - 2btv^2 + ct^2v^2 = v^2 - u^2 - 2buv^2 + c\xi^2 = 0$.

Now, by comparing the formulae as above we have

$$[(v_1 - u_1)\xi + (v_2 - u_2)\xi^2] + [v_1^2 + (v_2 + u_2)\xi^2 - 2b(1 + v_1\xi + v_2\xi^2) + c\xi^2 = 0].$$

By regrouping it we have

$$(v_2^2 - u_2^2)\xi^2 + [(v_1 - u_1)(v_2 + u_2) + (v_1 + u_1)(v_2 - u_2) - 2bv_2]\xi^2$$

$$+ [v_1^2 - u_1^2 + 2(v_2 - u_2) - 2bv_1 + c]\xi + 2(v_1 - u_1 - b) = 0.$$ 

Since the coefficients of $\xi^i$ must be zero for $i = 0, 1, 2, 3$, we have $u_1 = 0$, $v_1 = b = 2\eta$, $u_2 = -v_2 = \frac{1}{4}(c - b^2) = -\eta(1-2\eta)$. Thus, we have

$$u = 1 - \eta(1-2\eta)\xi^2, \quad v = 1 + 2\eta\xi + \eta(1-2\eta)\xi^2,$$

(23)

and from (17) and (22) we have

$$t = xy^{-1} = \frac{\xi y}{1 + 2\eta\xi + \eta(1-2\eta)\xi^2}.$$ 

(24)

Now, by applying binomial theorem from (10) (17) (18) (19) (20) and (24) we have

$$2x^3yh = x - y + y(1-at) - \frac{u}{v} = \frac{1}{v}[xy - y(v-u) - axu] = \eta y\left(1 - \frac{-2\eta}{1-4\eta}\right) - \frac{(1-2\eta)u}{v}$$

$$= \frac{(1-2\eta)\xi^2}{v^2}[1 - 4\eta - (1-2\eta)\xi^2].$$

Equation (21) holds by grouping the terms.

**Theorem 3.1** The enumerating function $h = h_{\text{nc}}(x, y)$ for rooted nearly cubic planar maps with the valency of rood-vertex and the size as two parameters has the explicit expression as follows

$$h_{\text{nc}}(x, y) = 1 + xy^2 + x^2y + \sum_{m/2}^{m/2} \sum_{p=m+1}^{m+1} \sum_{i=0}^{m-p} \sum_{j=0}^{m-p} \theta_{ij}x^my^{p-m},$$

(25)
where \( \varphi_{ij} = \frac{2^{p-i+j-2}m!(2p-m-j-1)!(p+j-2)!\mu_{ij}}{p!i!(i+2)!(m-2i)!(p-i)!(j+1)!(p-m+i-j)!} \), and

\[
\mu_{ij} = (2p-m-j)(i+2)^j(j+1)[2p(p-j-1) - j(j-1)] 
- (m-2i)(p-i)(p+j-1)[2p(p-j-2) - j(j+1)].
\]

This is an answer to open problem 3.1 and problem 3.2 in 1989 [7].

**Proof:** From (21) we can find the enumerating factor of \( h = h_{\mathcal{C}_{nc}}(x, y, z) \) as follows

\[
\Delta_{(\xi, \eta)} = \frac{[1-(1-2\eta)\xi^2\eta][1-12\eta + 24\eta^2]}{(1-2\eta)(1-4\eta)[1+2\xi\eta+(1-2\eta)\xi^2\eta]}.
\]  

(26)

Now, by using Lagrangian inversion formula with two variables from (21) and (26) we can obtain

\[
xh = \sum_{m, r \geq 0} \partial_{(\xi, \eta)}^{(m, r)} \frac{[1+2\xi\eta+(1-2\eta)\xi^2\eta]^m}{(1-2\eta)^{r+1}(1-4\eta)^r} [1-(1-2\eta)\xi^2\eta][1-12\eta + 24\eta^2] [1-\xi^2\eta] x^m y^{3p-m+1}
\]

where \( \partial_{(\xi, \eta)}^{(i, j)} g(x, y) \) is the coefficient of \( x^i y^j \) in \( g(x, y) \) as a power series of \( x \) and \( y \). By grouping it we have

\[
h = \sum_{p \geq 2} \sum_{m \geq 0} \partial_{(\xi, \eta)}^{(m, p)} \left[ \frac{[1+2\xi\eta+(1-2\eta)\xi^2\eta]^m}{(1-2\eta)^{p+1}(1-4\eta)^p} [1-(1-2\eta)\xi^2\eta][1-12\eta + 24\eta^2] [1-\xi^2\eta] x^m y^{3p-m+1} \right]
\]

(27)

Thus

\[
h = 1 + xy^2 + x^2y + h_{(p \geq 2)}, \quad \text{where } h_{(p \geq 2)} = \sum_{p \geq 2} \sum_{m \geq 0} \partial_{(\xi, \eta)}^{(m, p)} \left[ \frac{[1+2\xi\eta+(1-2\eta)\xi^2\eta]^m}{(1-2\eta)^{p+1}(1-4\eta)^p} [1-(1-2\eta)\xi^2\eta][1-12\eta + 24\eta^2] [1-\xi^2\eta] x^m y^{3p-m+1} \right].
\]

It is due to \( h_{(p \geq 2)} \big|_{\eta=0} = 0 \) as there is not at all the map \( M \) in \( \mathcal{C}_{nc} \) with \( p \geq 2 \) and \( m(M) = 0 \).

It is equivalent to the theorem.

**Theorem 3.2** The enumerating function \( H_3 = H_{\mathcal{C}_{nc}^3}(y) \) for rooted cubic planar maps with the size as a parameter has the explicit expressions as follows

\[
H_{\mathcal{C}_{nc}^3}(y) = \sum_{n \geq 1} \sum_{i=0}^{n} \frac{2^{2n-i}(n+i)!}{i!(n+1)!(n+2)!}y^{3n},
\]

(27)

where \( \alpha_i(n) = 4+3i-i^2 + (3+4i)n - n^2 \).

**Proof:** By using Lagrangian inversion formula with one parameter from (16) we find that

\[
y^3(1 + H_3) = \sum_{n \geq 1} \frac{1}{n} \partial_{(\xi, \eta)}^{n-1} \left[ \frac{1-12\eta + 24\eta^2}{(1+2\eta)^n(1-4\eta)^{n+1}} \right] y^{3n}, \quad \text{That is}
\]

\[
H_3 = \sum_{n \geq 1} \partial_{(\xi, \eta)}^{n} \left[ \frac{1-12\eta + 24\eta^2}{(n+1)(1-2\eta)^{n+1}(1-4\eta)^{n+1}} \right] y^{3n} = \sum_{n \geq 1} \sum_{i=0}^{n} \frac{2^i}{i!} \partial_{(\xi, \eta)}^{n-i} \left[ \frac{1-12\eta + 24\eta^2}{(1-4\eta)^{n+1}} \right] y^{3n} = \sum_{n \geq 1} \sum_{i=0}^{n} \frac{2^{2n-i}(n+i)!}{i!(n+1)!} \left[ \frac{2}{n+2} \left( \frac{2n-i+2}{n+2} \right) - 6 \left( \frac{2n-i+1}{n+2} \right) + 3 \left( \frac{2n-i}{n+2} \right) \right] y^{3n}.
\]

By grouping the terms (27) holds.

Due to the difficulty of calculation, this paper only solved the calculation problem of counting functions \( h = h_{\mathcal{C}_{nc}^3}(x, y, z) \) and \( H_3 = H_{\mathcal{C}_{nc}^3}(y') \), and the remaining functions of calculation still
needs to be further studied. However, in the process of in-depth study, the method of this paper can be used for reference.

**Problem** Try to determine the explicit formulae of the generating functions $f = f_{\mathcal{M}_c}(x, y, z)$, $F = F_{\mathcal{M}_c^0}(y, z)$, $H = H_{\mathcal{M}_c^0}(y)$ and $F_3 = F_{\mathcal{M}_c^0}(y, z)$.

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