Longitudinal and transverse structure functions in high Reynolds-number magneto-hydrodynamic turbulence

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Abstract

We investigate the scaling behavior of longitudinal and transverse structure functions in homogeneous and isotropic magneto-hydrodynamic (MHD) turbulence by means of an exact hierarchy of structure function equations as well as by direct numerical simulations of two- and three-dimensional MHD turbulence. In particular, rescaling relations between longitudinal and transverse structure functions are derived and utilized in order to compare different scaling behavior in the inertial range. It is found that there are no substantial differences between longitudinal and transverse structure functions in MHD turbulence. This finding stands in contrast to the case of hydrodynamic turbulence which shows persistent differences even at high Reynolds numbers. We propose a physical picture that is based on an effective reduction of pressure contributions due to local regions of same magnitude and alignment of velocity and magnetic field fluctuations. Finally, our findings underline the importance of the pressure term for the actually observed scaling differences in hydrodynamic turbulence.

1. Introduction

The question whether longitudinal and transverse structure functions possess different scaling behavior in highly turbulent flows is still an open and unsolved problem. Symmetry considerations [1] of the underlying Navier–Stokes equation suggest no difference in scaling behavior. However, experimental data [2–4] and high resolution numerical simulations [5–9] show a consistent difference between longitudinal and transverse structure functions. So far it is absolutely not clear what the cause of this observation is and possible explanations include a remaining small-scale anisotropy of the flow [10, 11] or a finite Reynolds number effect [12]. Thus a natural question and approach is to ask whether these differences between longitudinal and transverse structure functions exist or vanish in other turbulent systems in order to better understand and identify a possible cause.

For instance, it was shown in the context of two-dimensional electron magneto-hydrodynamics [13] that the differences between longitudinal and transverse structure functions in the direct cascade range decreased with increasing Reynolds number.

In this paper we investigate the behavior of longitudinal and transverse structure functions in the context of magneto-hydrodynamic (MHD) turbulence. The motivation for looking at MHD turbulence arose from the known fact that current and vortex sheets play a key role for the understanding of intermittency in this particular flow. Since sheets represent quite anisotropic structures compared to vortex filaments as their hydrodynamic counterparts, the hope was to attribute the different scaling of longitudinal and transverse structure functions to the nature of dissipative structures. It turned out that this statement is correct but in a way that was not anticipated and which is counterintuitive at first glance. The result of this paper is that there is no substantial difference between longitudinal and transverse structure functions in the inertial range of high Reynolds
number MHD flows. Moreover, this observation can be attributed to regions of preferential alignment of the magnetic and the velocity field which result in an effective depletion of pressure contributions. In reverse, these results also open up the way for a better understanding of the problem in hydrodynamic turbulence. The outline of the paper is as follows: first-of-all, we derive a hierarchy of structure function relations from the basic MHD equations, a rather technical part in section 2 which will be accompanied by appendix A to G. These relations are then used in section 3, in order to derive rescaling relations between transverse and longitudinal structure functions along the lines of Grauer, Homann and Pinton [9]. The scaling behavior of longitudinal and transverse structure functions in direct numerical simulations of MHD turbulence will then be investigated with the help of these rescaling relations in section 4. The paper concludes with a simple examination of local regions of same magnitude and alignment of velocity and magnetic field contributions and their depleting effect on the total pressure.

2. Hierarchy of structure functions in MHD turbulence

The aim of this section is to establish relations between longitudinal and transverse structure functions in MHD turbulence similar to the ones that have been derived by Hill [12], Hill and Boratav [14] and Yakhot [15] for the case of hydrodynamic turbulence. To this end, we make use of the calculus of isotropic tensors in MHD turbulence introduced by Chandrasekhar [16]. We thus consider the MHD equations in the following form

\[
\frac{\partial}{\partial t}u_i + u_n \frac{\partial}{\partial x_n} u_i - h_n \frac{\partial}{\partial x_n} h_i = - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left( \rho + \frac{1}{2} \rho |h|^2 \right) + \nu \nabla^2 u_i, \tag{1}
\]

\[
\frac{\partial}{\partial t} h_i + u_n \frac{\partial}{\partial x_n} h_i - h_n \frac{\partial}{\partial x_n} u_i = \lambda \nabla^2 h_i, \tag{2}
\]

where summation over equal indices is implied. Here, \( p(x, t) \) denotes the hydrodynamic pressure, \( \rho \) the density, \( \nu \) the kinematic viscosity and \( \lambda \) the magnetic diffusivity of the fluid. In the following, the density of the fluid is set to one. Furthermore, it should be noted that in this convenient form of the MHD equations, the magnetic field \( h(x, t) \) has the dimensions of a velocity [16]. From these equations (1) and (2), Chandrasekhar derived the MHD analogon of the Friedmann–Keller correlation function hierarchy [17] of hydrodynamic turbulence and made use of the calculus of isotropic tensors [18, 19] in order to derive, among other things, the corresponding von Kármán–Howarth equation of MHD turbulence.

In our approach, which is more concerned with the local isotropy of MHD turbulence, we introduce velocity and magnetic field increments \( v_i(x, x', t) = u_i(x, t) - u_i(x', t) = u_i - u'_i \) and \( h_i(x, x', t) = h_i(x, t) - h(x', t) = h_i - h'_i \). The evolution equations for these increments can be derived in following a procedure devised by Hill [12] (see appendix A for further details) and take the form

\[
\frac{\partial}{\partial t} v_i + v_n \frac{\partial}{\partial x_n} v_i + U_n \frac{\partial}{\partial x_n} v_i - h_n \frac{\partial}{\partial x_n} h_i - H_n \frac{\partial}{\partial x_n} h_i = -p_i + \nu \left( \nabla^2 v_i + \nabla^2 h_i \right) v_i, \tag{3}
\]

\[
\frac{\partial}{\partial t} h_i + v_n \frac{\partial}{\partial x_n} h_i + U_n \frac{\partial}{\partial x_n} h_i - h_n \frac{\partial}{\partial x_n} v_i - H_n \frac{\partial}{\partial x_n} v_i = \lambda \left( \nabla^2 v_i + \nabla^2 h_i \right) h_i, \tag{4}
\]

where we have introduced the mean fields \( U_i = \frac{u_i + u'_i}{2}, H_i = \frac{h_i + h'_i}{2} \), and where we have switched to relative and center coordinates \( r = x - x' \) and \( X = \frac{x + x'}{2} \). Furthermore, the total pressure gradient increment has been introduced according to

\[
P_i = \frac{\partial}{\partial X_i} \left[ p - p' + \frac{1}{2} (|h|^2 - |h'|^2) \right]. \tag{5}
\]

Equations (3) and (4) are the point of departure for the derivation of the structure function hierarchy, the main objective in this section. The usual procedure consists in multiplying equations (3) and (4) by certain increment components and subsequently taking the ensemble average in order to make use of the assumption of homogeneity and isotropy.

Before we address this issue, however, we want to address certain implications of the additional magnetic field on the structure function procedure along the lines of Hill [12]. First-of-all, it is important to take notice of the influence of the mean magnetic field \( H \) in the increment evolution equations (3) and (4): in contrast to the mean velocity field \( U \), which can be removed in a system comoving with the mean velocity, the mean magnetic field will be present in all moment equations derived from the increment equations. The influence of these terms can only be removed by certain combinations of the moments in addition to the assumption of homogeneity. Furthermore, as far as the tensorial character of the moments derived from equations (3) and (4) is concerned, we have to deal with certain statistical quantities that are not invariant under the full rotation group [18, 19]. This difference arises due to \( h \) being an axial vector which is unchanged under a reflexion, contrary to the true polar
vector \( \mathbf{u} \) which changes signs. Therefore, tensorial quantities that involve an odd number of magnetic field components exhibit a lack of mirror symmetry and are thus skew tensors. By contrast, an even number of magnetic field components leads to the usual tensorial forms encountered in hydrodynamic turbulence (see appendix B for further discussion). Concerning the defining scalars of these tensors, an important restriction emerges from the incompressibility condition of the velocity and the magnetic field, i.e., \( \frac{\partial}{\partial x_i} u_i(\mathbf{x}, t) = 0 \) and \( \frac{\partial}{\partial x_i} B_i(\mathbf{x}, t) = 0 \). The incompressibility conditions give rise to a first relation between the transverse \( S_{yy}^r(r, t) \) and longitudinal \( S_{yy}^l(r, t) \) velocity (magnetic) field structure function of second order, namely

\[
S_{yy}^r(r, t) = \frac{1}{2r} \frac{\partial}{\partial r} (r^2 S_{yy}^l(r, t)) \quad \text{and} \quad S_{yy}^l(r, t) = \frac{1}{2r} \frac{\partial}{\partial r} (r^2 S_{yy}^r(r, t)).
\]  

(6)

These relations are the well-known von Kármán–Howarth relations and are a direct consequence of the incompressibility, homogeneity and isotropy of the MHD flow discussed in appendix B.1. The existence of such direct relations between higher-order longitudinal and transverse structure functions, however, is far less obvious. Therefore, we have to rely on structure function relations that are directly derived from the evolution equations of the increments, i.e., equations (3) and (4).

A first evolution equation for the symmetric tensor of second order \( \langle v_i v_j + b_i b_j \rangle \) is derived in appendix E according to

\[
\frac{\partial}{\partial t} \langle v_i v_j + b_i b_j \rangle + \frac{\partial}{\partial x_n} \langle v_n (v_i v_j + b_i b_j) \rangle - \frac{\partial}{\partial x_n} \langle b_n (v_i v_j + v_j v_i) \rangle + \frac{\partial}{\partial x_n} (U_n (v_i v_j + b_i b_j)) - \frac{\partial}{\partial x_n} (H_n (v_i v_j + v_j v_i)) = \langle b_i P_j + v_i P_j \rangle
\]

(7)

\[
= 2 \nu \left( \nabla_i^2 + \frac{1}{4} \nabla_X^2 \right) (v_i v_j) - 2 \langle \epsilon_{u j}^{uu} \rangle + 2 \lambda \left( \nabla_i^2 + \frac{1}{4} \nabla_X^2 \right) (b_i b_j) - 2 \langle \epsilon_{h j}^{hh} \rangle.
\]

Here, we have introduced the tensors of the local energy dissipation rates

\[
\epsilon_{ij}^{uu} = \nu \sum_n \left[ \frac{\partial u_i}{\partial x_n} \frac{\partial u_j}{\partial x_n} + \frac{\partial u_i}{\partial x_n} \right],
\]

(8)

\[
\epsilon_{ij}^{hh} = \lambda \sum_n \left[ \frac{\partial h_i}{\partial x_n} \frac{\partial h_j}{\partial x_n} + \frac{\partial h_i}{\partial x_n} \right].
\]

(9)

Equation (7) is the first equation in a chain of transport equations and couples to tensors of third order via the nonlinear terms. A further simplification of equation (7) arises from the assumption of homogeneity which enables us to neglect terms that involve a center derivative acting on the ensemble average, i.e., \( \frac{\partial}{\partial x_n} \langle v_i v_j \rangle = 0 \). At this stage of the hierarchy, pressure contributions also vanish on the basis of homogeneity \[20\].

The averaged equation of energy balance in MHD turbulence can be obtained from equation (7) in summing over equal \( i = j \), which is performed in appendix E. The latter equation can be used to derive the MHD analagon of Kolmogorov’s 4/5-law of hydrodynamic turbulence. In addition to the longitudinal velocity structure function of third order, the 4/5-law in MHD turbulence involves the mixed correlation function \( C_{ij,3}^{uu} \) that is symmetric in \( i, j \) and the antisymmetric correlation function

\[
C_{ij,3}^{uh} = \langle (h_i u_j - u_i h_j) h_3 \rangle = C_{ij,3}^{uh}(r, t) \left( \frac{r}{r} \delta_{im} - \frac{r}{r} \delta_{jm} \right).
\]

(10)

In the following, tensorial forms of the antisymmetric type (10) will be indicated by a semicolon between the antisymmetric indices. The notation of the correlation functions follows a similar approach than the notation used by Chandrasekhar in his seminal discussion of correlation functions in MHD turbulence \[16\]. Antisymmetric tensors such as (10) will be encountered throughout the entire structure function hierarchy in MHD turbulence and lead to modified scaling relations in comparison to the ordinary symmetric tensors encountered in hydrodynamics. This can be seen from the 4/5-law in MHD turbulence (see appendix E.1) in the inertial range

\[
S_{yy}^{rvv} = - 12 C_{ij,3}^{uh}(r) - \frac{24}{r^4} \int_0^r dr' r'^3 C_{ij,3}^{uh}(r') = - \frac{4}{5} \langle \epsilon_{uu} + \epsilon_{hh} \rangle r.
\]

(11)

The corresponding averaged local energy dissipation rates \( \langle \epsilon_{uu} \rangle \) and \( \langle \epsilon_{hh} \rangle \) can be recovered from equations (8) and (9), as it is further discussed in appendix E. In the absence of the antisymmetric tensor, for instance in the case of a vanishing electromotive force in equation (10), we recover the relation established by Politián and Pouquet \[21\].

The next order equation relates structure functions of third and fourth order. It is also the first order which provides a relation between the longitudinal and transverse structure functions based on the MHD equations. By
contrast, the von Kármán–Howarth relations \( (6) \) are a pure statement of the corresponding incompressibility conditions. Furthermore, in the next order of the hierarchy we have to deal for the first time with statistical quantities that contain the pressure gradient increment \( \nu_P \). The evolution equation for the symmetric tensor of third order is derived in appendix \( T \) and reads

\[
\frac{\partial}{\partial t} \left\langle \nu_i \nu_j \nu_k + \nu_i b_j b_k + b_i \nu_j b_k + b_i b_j \nu_k \right\rangle + \frac{\partial}{\partial n} \left\langle \nu_i \nu_j \nu_k n - b_i b_j b_k n \right\rangle \\
+ \frac{\partial}{\partial n} \left( \left( \nu_i b_j b_k + b_i \nu_j b_k + b_i b_j \nu_k \right) - \frac{\partial}{\partial n} \left( b_i (\nu_j b_k + b_j \nu_k + v_j b_k) \right) \right) \\
= - \left\langle (\nu_i \nu_j + b_i b_j) P_k + (\nu_k \nu_j + b_i b_k) P_j + (\nu_k \nu_j + b_i b_k) P_j \right\rangle,
\]

where dissipative terms have been neglected in the inertial range \( [14] \). Obviously, this can only be a crude approximation since these terms contain the joint statistics of the the velocity (magnetic) field increment and the local energy dissipation rates \( (8) \) and \( (9) \), which are known to contribute even in the vicinity of \( \nu \). Nonetheless, for now we proceed in introducing the notations

\[
S_{ij \ k \ n}^{\nu \nu \nu} (r, t) = \left\langle \nu_i \nu_j \nu_k \nu_n \right\rangle,
\]

\[
S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r, t) = \left\langle b_i b_j b_k b_n \right\rangle,
\]

\[
S_{ij \ k \ n}^{\nu \b b b} (r, t) = \left\langle \nu_i b_j b_k \nu_k \right\rangle,
\]

\[
T_{ij} (r, t) = \left\langle (\nu_i b_j + b_i b_j) P_k + (\nu_k b_j + b_i b_k) P_j + (\nu_k b_j + b_i b_k) P_j \right\rangle,
\]

which enable us to rewrite equation \( (12) \) in shorter form as

\[
\frac{\partial}{\partial n} \left( S_{ij \ k \ n}^{\nu \nu \nu} (r, t) - S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r, t) \right) \\
- \frac{\partial}{\partial n} \left( S_{ij \ k \ n}^{\nu \b b b} (r, t) + S_{ij \ k \ n}^{\nu \b b b} (r, t) + S_{ij \ k \ n}^{\nu \b b b} (r, t) \right) = T_{ij} (r, t).
\]

Here, the tensors in the first line are ordinary symmetric tensors of fourth order, described in appendix \( D \), whereas the mixed tensor possesses another tensorial form

\[
S_{ij \ k \ n}^{\nu \b b b} (r, t) = S_{ij \ k \ n}^{\nu \b b b} (r, t) \left( \frac{\partial r}{\partial r} \delta_k n - \frac{\partial r}{\partial r} \delta_k n \right),
\]

since it is antisymmetric in exchanging \( i \) against \( kn \). It can therefore be considered as the next-order equivalent tensor of equation \( (10) \). Inserting the corresponding tensors (appendix \( D \)) yields

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \left( S_{ij \ k \ n}^{\nu \nu \nu} (r) - S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r) \right) \right] - \frac{6}{r} \left( S_{ij \ k \ n}^{\nu \nu \nu} (r) - S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r) - S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r) \right) = - T_{rr} (r)
\]

for the longitudinal structure functions and

\[
\frac{1}{r^4} \frac{\partial}{\partial r} \left[ r^4 \left( S_{ij \ k \ n}^{\nu \nu \nu} (r) - S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r) \right) \right] - \frac{4}{3r} \left( S_{ij \ k \ n}^{\nu \nu \nu} (r) - S_{ij \ k \ n}^{\nu \nu \nu 
\nu} (r) \right) + \frac{\partial}{\partial r} S_{ij \ k \ n}^{\nu \n \nu} (r) = - T_{rn} (r),
\]

for the mixed structure functions.

These equations represent the generalization of the next-order structure function relations in hydrodynamic turbulence derived in \([14, 15]\) in the presence of a magnetic field. As in the hydrodynamic case the longitudinal and transverse structure functions of fourth order are coupled via the mixed terms \( S_{i k n} (r) \) in equations \( (19) \) and \( (20) \). Furthermore, the presence of the pressure contributions \( T_{rr} (r) \) and \( T_{rn} (r) \) in the inertial range are considered to be responsible for the persistent different scaling behavior between the longitudinal and transverse structure functions in hydrodynamic turbulence \([9, 14, 22]\). Nevertheless, equations \( (19) \) and \( (20) \) distinguish themselves from their hydrodynamic counterparts in several points: first-of-all, the presence of the antisymmetric tensor \( S_{ij \ k \ n}^{\nu \b b b} (r) \) leads to novel terms, especially in the mixed velocity-magnetic structure function equation \( (20) \) that involves \( S_{ij \ k \ n}^{\nu \b b b} (r) \) in a new differential relation compared to the other mixed structure functions. Furthermore, only differences between pure velocity and magnetic structure functions enter the relation, which can lead to cancellation effects, e.g., in case of alignment solutions. In addition, the pressure contributions on the right-hand side of \( (19) \) and \( (20) \) include contributions from the magnetic pressure next to the hydrodynamic pressure which opens up the way to discuss the role of pressure in MHD turbulence via a local Bernoulli law along the lines of Gotoh and Nakano \([22]\) for the case of hydrodynamic turbulence.
3. Rescaling relations between longitudinal and transverse structure functions

In this section, we briefly want to review rescaling relations between longitudinal and transverse structure functions of all orders that were derived recently [9]. We shall then proceed to motivate such rescaling relations for MHD turbulence, namely by means of equations (19) and (20). The point of departure for hydrodynamic turbulence is the observation by Siefert and Peinke [23] that $S_{yy}^m(r)$ is a smooth function of $r$ for which we can interpret the right-hand side of the von Kármán–Howarth relation (6) as the first two terms of a Taylor expansion around $r$

$$S_{yy}^m(r + \frac{r}{2}) \approx S_{yy}^m(r) + \frac{r}{2} \frac{\partial}{\partial r} S_{yy}^m(r).$$

(21)

Here, we neglect terms of higher order under the assumption that $r/2$ is much smaller than the integral length scale $L$. In this approximation, the transverse structure function is simply the longitudinal structure function rescaled by the factor 3/2 according to

$$S_{yt}^m(r) \approx S_{yy}^m(\frac{3}{2}r).$$

(22)

This procedure can be generalized for structure functions of order $n$ in hydrodynamic turbulence [9], which is discussed at the example of the fourth order equations (19) and (20) in the hydrodynamic limit

$$\frac{1}{r^2} \frac{\partial}{\partial r} [r^2 S_{yy}^m(r)] - \frac{6}{r} S_{y1y1}^m(r) = - T_{yy}(r),$$

(23)

$$\frac{1}{r^2} \frac{\partial}{\partial r} [r^4 S_{yt}^m(r)] - \frac{4}{3} S_{y1t1}^m(r) = - T_{yt}(r).$$

(24)

These two equations are solely related by the mixed term $S_{y1t1}^m(r)$, in contrast to the MHD equations, where the antisymmetric mixed terms $S_{y1bb}^m(r)$ appear. The rescaling properties can be repeated under the neglect of the pressure terms, so that we obtain

$$3 S_{y1t1}^m(r) \approx S_{yy}^m(r) + \frac{r}{2} \frac{\partial}{\partial r} S_{yy}^m(r) \approx S_{yy}^m(\frac{3}{2}r),$$

(25)

$$\frac{1}{3} S_{y1t1}^m(r) \approx S_{yt}^m(r) + \frac{r}{4} \frac{\partial}{\partial r} S_{yt}^m(r) \approx S_{yt}^m(\frac{5}{4}r),$$

(26)

which yields the transverse structure function of fourth order in terms of the rescaled longitudinal structure function of fourth order according to

$$S_{yt}^m(r) \approx S_{yy}^m(\frac{5}{4}r).$$

(27)

It is clear that neglecting the pressure terms in equations (23) and (24) in combination with the corresponding Taylor expansions has to be treated with caution. Nevertheless, it has been shown [9] that the rescaling relations are an accurate method to map scales without altering the corresponding scaling behavior. It is therefore an essential tool for the investigation of possible different intermittency behavior of the longitudinal and transverse structure functions. In considering higher order structure function equations [12], a general relation between even order transverse structure function of order $n$ can be derived according to

$$S_{yn}^m(r) \approx S_{yn}^m(\frac{5}{4}r) = S_{yn}^m(\frac{n + 1}{n}r) = S_{yn}^m(\frac{n + 1}{n}!!r).$$

(28)

Turning to the case of MHD turbulence, the von Kármán–Howarth relation (6) suggests that equation (21) is also valid for the magnetic field structure functions of second order. The next order equations (19) and (20), however, are more complicated due to the additional appearance of the anti-symmetric tensor $S_{y1t1}^b(r)$. Nevertheless, assuming that the contribution of the latter are rather small, we can—in a first approximation—establish the same relation (27) between $S_{y1t1}^b(r) - S_{y1t1}^m(r)$ and $S_{yt}^b(r) - S_{yt}^m(r)$. Moreover, if velocity and magnetic structure functions possess a unique power law in the inertial range, it is appropriate to assume that the general rescaling relation (28) holds independently for velocity and magnetic structure functions of all orders, given that higher order relations similar to equations (19) and (20) exist.

4. Direct numerical simulations of 3D MHD turbulence and comparison of longitudinal and transverse structure functions

In the previous section, we argued that rescaling relations of the form (28) apply for all even order velocity and magnetic structure functions. We have examined these relations in direct numerical simulations of 3D MHD
Table 1. Characteristic parameters of the direct numerical simulations of 3D MHD and hydrodynamic turbulence: Taylor–Reynolds number \( Re_h = \frac{12 \text{num} h}{\nu} \), root mean square velocity \( u_{\text{rms}} \), root mean square magnetic field \( b_{\text{rms}} = \sqrt{\langle b^2 \rangle} \), averaged kinetic energy dissipation rate \( \langle \varepsilon \rangle \), kinematic viscosity \( \nu \) and magnetic diffusivity \( \eta \), dissipation length \( \lambda = \frac{\eta}{\varepsilon} \), dissipation time \( \tau_\eta = \frac{1}{\langle \varepsilon \rangle} \), integral length scale \( L = \frac{12 \nu}{\langle \nabla u \cdot \nabla b \rangle} \), large-eddy turn-over time \( T_L = \frac{1}{\langle \nabla \cdot (\theta + b) \rangle} \) and resolution \( N \).

| Run  | \( Re_h \) | \( u_{\text{rms}} \) | \( b_{\text{rms}} \) | \( \langle \varepsilon \rangle \) | \( \eta \) | \( \lambda \) | \( \tau_\eta \) | \( L \) | \( T_L \) | \( N^3 \) |
|------|-------------|-----------------|-----------------|-----------------|--------|---------|---------|------|------|------|
| 3D NS | 460         | 0.19            | 0               | \( 3.6 \times 10^{-3} \)  | 2.5 \times 10^{-3} | 1.45 \times 10^{-3} | 0.083 | 1.85 | 9.9 | 2048^3 |
| 3D MHD | 430         | 0.23            | 0.36            | \( 1.2 \times 10^{-2} \) | 7 \times 10^{-3} | 2.3 \times 10^{-3} | 0.078 | 2.75 | 6.5 | 2048^3 |

Figure 1. (a) Rescaled longitudinal and transverse structure functions in 3D MHD (green: magnetic field, blue: velocity field, Elsässer field \( z^m \) : red) and hydrodynamic turbulence (black). The scales are mapped accurately via the rescaling relation (29). The logarithmic derivative plot in figure 1 suggests that the rescaled transverse structure function possesses a slightly more intermittent character than the longitudinal structure function (dashed black line) whereas the rescaled MHD structure functions do not show considerable differences. (b) Logarithmic derivatives of the structure functions from (a). Power law scaling should manifest itself as a flat curve.

turbulence. Table 1 summarizes the corresponding characteristic parameters of the simulations. As an example, figure 1 (a) shows the rescaling between longitudinal and transverse structure functions of order 10. The scales of the velocity field structure functions (blue) and the magnetic field structure functions (green) are mapped accurately onto each other. This is also true for the structure functions of the Elsässer field \( z^m(x, t) = u(x, t) + h(x, t) \). Furthermore, figure 1 shows no substantial difference between rescaled structure functions of order 10 in MHD turbulence. In opposition to the finding from hydrodynamic turbulence (see the black line in figure 1) which suggests that the rescaled transverse structure function possesses a slightly more intermittent character than the longitudinal one, we thus arrive at a somewhat contradictory conclusion: although that structure functions in MHD turbulence lie above the ones in hydrodynamic turbulence. The latter fact is commonly attributed to dissipative structures in MHD turbulence that consist mainly of vortex and current sheets and are believed to possess a more singular character than the vortex tubes in hydrodynamic turbulence [24]. Nevertheless, figure 1 shows that these geometric considerations are not an adequate explanation for different scaling behavior between longitudinal and transverse structure functions in general.

The negative finding from MHD turbulence thus suggests that a more subtle mechanism is at the heart of the problem of longitudinal and transverse structure function differences. Before we discuss such a mechanism on the basis of the pressure contributions that enter in equations (19) and (20), it is in order to briefly discuss the notion of scaling in MHD turbulence in general. The logarithmic derivative plot in figure 1 (b) shows that a clear power law scaling of the structure functions is somehow hard to anticipate. In opposition, the structure functions from hydrodynamic turbulence (black) manifest themselves by rather flat curves (power law behavior) in the logarithmic derivative plot. Especially the magnetic longitudinal structure functions show a pronounced bump on larger scales \( r \) and are thus missing a clear power law behavior as it can be seen from figure 2 (b).

Moreover, the bump is increasing with increasing order of the structure functions and seems to hint at the existence of two different power law behaviors in the inertial range. Since at this point, we are only interested in...
relative differences between longitudinal and transverse structure functions, we only take note of this problem of power laws in MHD turbulence and leave its evaluation to further work.

5. Alignment of dissipative structures and depletion of pressure

Apparently, the identical scaling behavior of the longitudinal and transverse structure functions in MHD turbulence discussed in the previous section is not directly related to the singular structures of the MHD flow. Therefore, we want to address a different mechanism that considers the influence of the pressure gradient on the longitudinal and transverse structure functions. In hydrodynamic turbulence, relations (23) and (24) suggest that the scaling behavior of the longitudinal and transverse structure functions is altered solely by pressure contributions represented by $T_{rr}$ and $T_{tt}$.

Consequently, pressure contributions in MHD turbulence that enter on the rhs of equations (19) and (20) have to be slightly milder than the ones in hydrodynamic turbulence. In order to quantitatively discuss this behavior it is convenient to consider the total pressure $p_{\text{tot}}(x, t)$ that can be solved with the usual method of Green’s functions according to

$$p_{\text{tot}}(x, t) = \frac{1}{4\pi} \int \frac{dx'}{|x - x'|} \frac{\partial^2 (u_i(x', t)u_i(x', t) - h_i(x', t)h_i(x', t))}{\partial x'_i \partial x'_i}. \quad (29)$$

As in hydrodynamic turbulence, the total pressure is thus determined nonlocally from the nonlinear terms in equation (1). Therefore, if the MHD flow possesses local regions of alignment and equal magnitude, i.e.

$$u(x, t) = \pm h(x, t), \quad (30)$$

which results in an effective depletion of pressure contributions that are mainly located in these regions due to the $1/|x - x'|$ dependence in equation (29) (we refer the reader to the monograph [25] for a further discussion of these so-called equipartition solutions in MHD turbulence). As a consequence, the pressure gradient increment (5) that enters in the pressure contributions of equations (19) and (20) is also reduced. The described mechanism can thus be considered as a first explanation of the observations made in section 4. We emphasize that our alignment mechanism bears similarities with the phenomenological theory of scale-dependent alignment proposed by Boldyrev [26, 27] which is in accordance with experimentally obtained energy spectra of MHD turbulence. Boldyrev argued that small-scale turbulent eddies spontaneously develop alignment of magnetic and velocity field polarizations with respect to a large-scale magnetic field. The latter does not necessarily have to be an external magnetic field, but can also be apprehended as an effective magnetic field formed by the large-scale vortical motions. Similar to the case of decaying MHD turbulence which ultimately reaches the equilibrium state in equation (30), the alignment causes a depletion of the nonlinearities in the MHD equations. In externally forced MHD turbulence, however, the tendency for dynamic alignment is reduced due to the energy flux across scales that preserves the nonlinear interactions. Therefore, at each scale $r$ of the energy cascade the alignment should reach a maximum that is consistent with the constant energy flux through this scale. In imposing the latter condition, Boldyrev derived the field-perpendicular energy spectrum according to

$$E(k_\perp) \sim k_\perp^{-3/2}. \quad (29)$$

In order to get a first idea and for reasons of graphical representation, we examine the impact of local regions of alignment on the pressure field by direct numerical simulations of forced 2D MHD turbulence. The characteristic parameters can be found in table 2. Here, we made use of the concept of hyperviscosity by

![Figure 2. (a) Logarithmic derivative of the longitudinal velocity field structure functions of order 2–10. (b) Logarithmic derivative of the longitudinal magnetic field structure functions of order 2–10.](image)
replacing \( \nu_t \rightarrow (\nu_t)^2 \) in equations (1) and (2) in order to attain higher Reynolds numbers and to extend the inertial range. As a first impression, we have depicted a snapshot of the vorticity field \( \omega(x, t) = \nabla \times u(x, t) \) and the current density \( j(x, t) = \nabla \times h(x, t) \) in figure 3. The singular structures in the direct cascading range mainly consist of current and vortex sheets. The alignment angle \( \varphi(x, t) \) between \( u(x, t) \) and \( h(x, t) \) is depicted in figure 4(a). It can already be seen that regions of preferential alignment and anti-alignment do in fact exist.

Replacing this snapshot no. 1 they occur in a nearly dipolar manner, for instance in the dipole on the left bottom. The clustering of like-signed alignment angles seems therefore to be in agreement with the arguments established by Boldyrev \[26, 27\]. The total pressure \( p_{\text{tot}}(x, t) \) of snapshot no. 1 is depicted in figure 5, and is mainly concentrated in dipolar boundary regions of the alignment angle, where flips out of the alignment (30) occur. This seems to support our hypothesis that pressure contributions are depleted in regions of alignment.

Table 2: Characteristic parameters of the numerical simulations: Taylor–Reynolds number \( \text{Re}_t = u_{\text{rms}} \left( \frac{L_{\text{rms}}^2}{\eta} + \frac{h_{\text{rms}}^2}{\eta} \right)^{1/4} \), root mean square velocity \( u_{\text{rms}} = \sqrt{\langle u^2 \rangle} \), root mean square magnetic field \( h_{\text{rms}} = \sqrt{\langle h^2 \rangle} \), dissipation length \( \eta = \left( \frac{\nu_t}{\nu} \right)^{1/4} \), dissipation time \( \tau_{\eta} = \left( \frac{h_{\text{rms}}^4}{\nu_{\text{rms}}^2} \right)^{1/4} \), integral length scale \( L = \left( \frac{L_{\text{rms}}^3 + h_{\text{rms}}^3}{\nu_{\text{rms}}^2} \right)^{1/4} \), and large-eddy turn-over time \( T_L = \frac{h_{\text{rms}}}{\nu_{\text{rms}}} \).

| Run    | \( \text{Re}_t \) | \( u_{\text{rms}} \) | \( h_{\text{rms}} \) | \( \nu^{(2)} = \lambda^{(2)} \) | \( dx \) | \( \eta \) | \( \tau_{\eta} \) | \( L \) | \( T_L \) | \( N^2 \) |
|--------|------------------|----------------|----------------|-----------------|------|------|------------|-----|------|-------|
| 2D MHD | 110–170          | 0.831          | 0.759          | \( 2 \times 10^{-10} \) | \( 2.05 \times 10^{-3} \) | \( 1.64 \times 10^{-3} \) | 0.037      | 4.741 | 5.705 | \( (3072)^2 \) |

Figure 3. (a) Snapshot no. 1 of the vorticity field \( \omega(x, t) \) from direct numerical simulations of 2D MHD turbulence. (b) Same snapshot no. 1 of the current density \( j(x, t) \).

Figure 4. (a) Snapshot no. 1 of the alignment angle \( \varphi(x, t) \) between velocity field and magnetic field. (b) Snapshot no. 1 of the filter function \( \chi(x, t) \) introduced in equation (31). The covered area is 35.33% of the total area.
and comparable magnitude of velocity and magnetic field. In order to further underline our hypothesis, we introduce a filter function

\[
\chi(x, t) = \begin{cases} 
1 & \text{where } |\cos(\varphi(x, t))| > 0.88 \quad \text{and} \\
|u(x, t)^2 - h(x, t)^2| < 0.1 \times \max|u(x, t)^2 - h(x, t)^2|, \\
0 & \text{else}.
\end{cases}
\]  

(31)

The filter function can roughly be interpreted as a measure of deviation of ten per cent of the alignment relation (30). Applying this simple filter function to the total pressure in figure 5 shows that it is indeed depleted in regions of approximate alignment and equal magnitude. Since regions of maximal pressure contributions in figure 5(b) are filtered, they can be assigned to regions where equation (30) is not fulfilled, i.e., \( u \) and \( h \) are not aligned. The same argument holds for the norm of the pressure gradient, whose maximal values are filtered out by the filter function, which becomes apparent from the comparison of figure 6(a) to figure 6(b). A second snapshot no. 2 is presented in figures 7–10. The snapshot no. 2 reveals an increased preference of aligned regions in comparison to snapshot no. 1. The area covered by the filter function is 44.57\% (snapshot no. 2) in comparison to 35.33\% (snapshot no. 1). The alignment angle of snapshot no. 2 in figure 8 shows a pronounced dipole structure of alignment (anti-alignment). Pressure contributions in figures 9–10 are also filtered to a great extent. In order to quantify and support our statement that the alignment of velocity and magnetic field causes the absence of differences of longitudinal and transverse structure function scaling we constructed a two-dimensional histogram, where the joint probability density function of pressure and alignment angle is shown. Figure 11(a) demonstrates that with high probability the pressure is nearly zero if the alignment angle is either 0
Figure 7. (a) Snapshot no. 2 of the vorticity field $\omega(x, t)$ from direct numerical simulations of 2D MHD turbulence. (b) Same snapshot no. 2 of the current density $j(x, t)$.

Figure 8. (a) Snapshot no. 2 of the alignment angle $\varphi(x, t)$ between velocity field and magnetic field. (b) Snapshot no. 2 of filter function $\chi(x, t)$ in equation (31). The covered area is 44.57% of the total area.

Figure 9. (a) Snapshot no. 2 of the total pressure field $p_{\text{tot}}(x, t)$. (b) Snapshot no. 2 of the filtered pressure field via the filter function $\chi(x, t)$ from equation (31). Again, the pressure field is effectively reduced.
Moreover, figure 11(b) shows the same histogram but now conditioned on the current density above a threshold (20% of the maximum). Here, the vanishing of the pressure at angles 0 and π is even more pronounced and supports the above statement even stronger.

In the final part of this section, we extend our investigations to 3D MHD turbulence which was the basis of our findings in section 4. To this end, we have depicted an isosurface plot of the vorticity in figure 12(a). The singular structures can be perceived as vortex sheets, in contrast to the hydrodynamic case that is dominated by tube-like vortical structures. Here, the plot consists of a 512^3-segment of the full 2048^3-box from a snapshot of the simulation in table 1. Moreover, the volumes have been colored with the magnitude of the total pressure gradient. The total pressure gradient is seen to be small on strong vortex sheets. Figure 12(b) shows an isosurface plot of the filter function from equation (31). The isosurface has additionally been colored with the magnitude of the total pressure gradient. Again, the regions of preferential alignment and same magnitude of velocity and magnetic field directly correspond to regions of depleted total pressure gradient. A slice of the filter function at constant $z = \pi/4$ is depicted in figure 13(b). The covered volume of the filter function is 9.14% of the entire 512^3-box, which is far less than in the 2D case. Figure 14(a) shows a slice for constant $z = \pi/4$ of the total pressure field from figure 12(b). The filtered total pressure in figure 14(b) and the filtered total pressure gradient figure 15(b) supports our hypothesis that alignment regions effectively reduce pressure contributions.
6. Conclusion and outlook

We derived a hierarchy of structure functions from the basic MHD equations. The deployed procedure yielded exact equations between longitudinal and transverse structure functions similar to the next order equations by Hill and Boratav [14] for the purely hydrodynamic limit case. Furthermore, neglecting the pressure contributions in these equations allowed us to establish rescaling relations between longitudinal and transverse structure functions in MHD turbulence. The latter were used to directly compare the scaling behavior of longitudinal and transverse structure functions in the inertial range of direct numerical simulations of 3D MHD turbulence. In contrast to hydrodynamic turbulence, no clear scaling differences could be observed. This rather unexpected finding was explained by an effective reduction of the total pressure gradient due to regions of preferential alignment and same magnitude of velocity field and magnetic field fluctuations. In a first attempt, this potential mechanism was tested with the help of direct numerical simulations of 2D and 3D MHD turbulence. It could be shown that pressure contributions are indeed depleted in such regions of preferential alignment. Further work will be dedicated to the identification of a similar mechanism in hydrodynamic turbulence.
Figure 13. (a) Slice of the alignment angle $\varphi(x, t)$ between velocity field and magnetic field at constant $z = \pi/4$ for 3D MHD turbulence. (b) Slice of filter function $\chi(x, t)$ in equation (31). The covered volume of the $512^3$-segment is 9.14%.

Figure 14. (a) Slice of the pressure field at constant $z = \pi/4$ from figure 12 (b). (b) Slice of the filtered pressure field via the filter function $\chi(x, t)$ from equation (31).

Figure 15. (a) Slice of the norm of the pressure gradient field $|\nabla P_{\text{tot}}(x, t)|$ at constant $z = \pi/4$ from figure 12 (b). (b) Slice of the filtered norm of the pressure gradient field via the filter function $\chi(x, t)$ from equation (31).
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**Appendix A. Derivation of the evolution equations for velocity and magnetic increments**

The derivation of the evolution equations for velocity and magnetic field increments from the MHD equations (1) and (2) follows Hill’s procedure [12] for the case of hydrodynamic turbulence. We introduce the velocity and magnetic field increments

\[ v_i(x, x', t) = u_i(x, t) - u_i(x', t) \]  
\[ b_i(x, x', t) = h_i(x, t) - h_i(x', t) \]  

For brevity, let us denote \( u_i(x', t) = u'_i \) and \( h_i(x', t) = h'_i \). Furthermore, it is required that \( x \) and \( x' \) have no relative motion so that terms like \( \frac{\partial}{\partial x} u'_i \) and \( \frac{\partial}{\partial x} u_i \) vanish.

Subtracting the evolution equation of the velocity field (1) at point \( x' \) from the same equation at point \( x \) and performing the same procedure for the induction equation (2) yields

\[ \frac{\partial}{\partial t} v_i + u_n \frac{\partial}{\partial x_n} v_i + u'_n \frac{\partial}{\partial x_n} v_i - h_n \frac{\partial}{\partial x_n} b_i - h'_n \frac{\partial}{\partial x_n} b_i = -P_i + \nu (\nabla^2_x + \nabla^2_{x'}) v_i \]  
\[ \frac{\partial}{\partial t} b_i + u_n \frac{\partial}{\partial x_n} b_i + u'_n \frac{\partial}{\partial x_n} b_i - h_n \frac{\partial}{\partial x_n} v_i - h'_n \frac{\partial}{\partial x_n} v_i = \lambda (\nabla^2_x + \nabla^2_{x'}) b_i \]

where

\[ P_i(x, x', t) = \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x'_i} \right) \left[ p(x, t) - p(x', t) + \frac{1}{2} (|h(x, t)|^2 - |h(x', t)|^2) \right] \]

is the total pressure increment.

In the following, we introduce the mean velocity and magnetic fields

\[ U_i(x, x', t) = \frac{u_i(x, t) + u_i(x', t)}{2} \]  
\[ H_i(x, x', t) = \frac{h_i(x, t) + h_i(x', t)}{2} \]

and perform a coordinate transform to relative and center coordinates according to

\[ r = x - x' \quad \text{and} \quad X = \frac{x + x'}{2} \]

The coordinate transform to relative and center coordinate implies the following relations for the derivatives

\[ \frac{\partial}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial X_i} - \frac{\partial}{\partial r_i} \quad \text{and} \quad \frac{\partial}{\partial x'_i} = \frac{1}{2} \frac{\partial}{\partial X_i} + \frac{\partial}{\partial r_i} \]

In making use of these relations, we obtain the equations of motion for the velocity and the magnetic field increment

\[ \frac{\partial}{\partial t} v_i + v_n \frac{\partial}{\partial r_n} v_i + U_n \frac{\partial}{\partial X_n} v_i - h_n \frac{\partial}{\partial X_n} b_i - H_n \frac{\partial}{\partial X_n} b_i = -\frac{\partial}{\partial X_i} P + \nu (\nabla^2_x + \nabla^2_{x'}) v_i \]
\[ \frac{\partial}{\partial t} b_i + v_n \frac{\partial}{\partial r_n} b_i + U_n \frac{\partial}{\partial X_n} b_i - h_n \frac{\partial}{\partial X_n} v_i - H_n \frac{\partial}{\partial X_n} v_i = \lambda (\nabla^2_x + \nabla^2_{x'}) b_i \]

By means of these conversions, the implication of homogeneity for the statistical interpretation of the increment equations (A.10) and (A.11) can be seen directly: if \( \frac{\partial}{\partial X} \) acts on a statistical quantity, then \( \frac{\partial}{\partial X} (...) = 0 \), since this quantity should be independent of the rate of change with respect to the place where the measurement is performed [12].
Appendix B. Structure functions of second order

As it has been pointed out by Chandrasekhar [16], the MHD equations provide the evolution equations for three different correlation functions of second order, namely \( \langle u_i(x, t)u_j(x', t) \rangle, \langle h_i(x, t)h_j(x', t) \rangle \) and the cross helicity correlation function \( \langle u_i(x, t)h_j(x', t) \rangle \). Under the assumption of homogeneity, the first two correlation functions possess the ordinary form of a tensor of order two, i.e., the tensorial form that is going to be introduced in equation (B.9). The third tensor, however, is a quantity that is not invariant under the full rotation group. This difference arises due to \( h \) being an axial vector which is unchanged under a reflection whereas the true polar vector \( u \) changes signs. Due to this lack of mirror symmetry, the cross helicity correlation function possesses the following tensorial form

\[
C_{ij}^{\text{uh}}(r, t) = \langle u_i(x, t)h_j(x', t) \rangle = C_{ij}^{\text{uh}}(r, t) \epsilon_{ijs} \frac{r_s}{r}.
\]  

(B.1)

In the following, our aim is to discuss a similar treatment for the structure functions of second order in MHD turbulence. To this end, let us consider the magnetic structure function of order two (the same treatment applies to the velocity structure function of order two):

\[
S_{ij}^{bb}(x, x', t) = \langle h_i(x, t) - h_i(x', t) \rangle \langle h_j(x, t) - h_j(x', t) \rangle
= \langle h_i(x, t)h_j(x', t) \rangle + \langle h_i(x')h_j(x, t) \rangle - \langle h_i(x, t)h_j(x', t) \rangle - \langle h_i(x', t)h_j(x, t) \rangle
= 2 \langle h_i(x, t)h_j(x', t) \rangle - 2 \langle h_i(x, t)h_j(x', t) \rangle,
\]  

(B.2)

where homogeneity and isotropy were used in the last step, so that

\[
\langle h_i(x', t)h_j(x', t) \rangle = \langle h_i(x, t)h_j(x, t) \rangle = C_{ij}^{bb}(0, t)
\]

\[
\langle h_i(x, t)h_j(x', t) \rangle = \langle h_i(x', t)h_j(x, t) \rangle = C_{ij}^{bb}(r, t).
\]  

(B.3)

Therefore, the structure function \( S_{ij}^{bb}(r, t) \) can be written in terms of the correlation function \( C_{ij}^{bb}(r, t) \) according to

\[
S_{ij}^{bb}(r, t) = 2 (C_{ij}^{bb}(0, t) - C_{ij}^{bb}(r, t)),
\]  

(B.4)

whereas the cross helicity structure function behaves in another way due to the lack of mirror symmetry

\[
S_{ij}^{hb}(x, x', t) = \langle u_i(x, t) - u_i(x', t) \rangle \langle h_j(x, t) - h_j(x', t) \rangle
= \langle u_i(x, t)h_j(x, t) \rangle + \langle u_i(x', t)h_j(x', t) \rangle - \langle u_i(x, t)h_j(x', t) \rangle - \langle u_i(x', t)h_j(x, t) \rangle
= 2 \langle u_i(x, t)h_j(x, t) \rangle.
\]  

(B.5)

In the last step we made use of the assumption of homogeneity

\[
\langle u_i(x, t)h_j(x, t) \rangle = \langle u_i(x', t)h_j(x', t) \rangle = C_{ij}^{uh}(0, t)
\]  

(B.6)

and

\[
\langle u_i(x, t)h_j(x', t) \rangle = - \langle u_i(x', t)h_j(x, t) \rangle = C_{ij}^{uh}(r, t).
\]  

(B.7)

This last relation is responsible for the different decomposition behavior of the second order tensor of the cross helicity structure function, which can be expressed in terms of the cross helicity correlation function according to

\[
S_{ij}^{hb}(r, t) = 2C_{ij}^{uh}(0, t).
\]  

(B.8)

The cross helicity structure function thus becomes a purely local quantity.

B.1. The von Kármán–Howarth relation

In general we are interested in relations between longitudinal and transverse structure functions. Under the assumption of isotropy and homogeneity, the tensorial form of the second order magnetic structure function reads

\[
S_{ij}^{bb}(r, t) = (S_{ij}^{bb}(r, t) - S_{ij}^{bb}(r, t)) \frac{r_i r_j}{r^2} + S_{ij}^{bb}(r, t) \delta_{ij},
\]  

(B.9)

where the subscript \( rr \) denotes the longitudinal and \( tt \) the transverse structure function. Equation (B.4) implies that

\[
\frac{\partial}{\partial r_i} S_{ij}^{bb}(r, t) = 0,
\]  

(B.10)
due to the incompressibility condition for the magnetic field. Inserting the tensorial form (B.9) yields

\[
\frac{\partial}{\partial t} \mathcal{S}_{ij}^{bb}(r, t) = \frac{\partial}{\partial r} \left( S_{ij}^{bb}(r, t) - \mathcal{S}_{ij}^{bb}(r, t) \right) \frac{r_{j}}{r} + 2 \frac{2}{r} \left( S_{ij}^{bb}(r, t) - S_{ij}^{bb}(r, t) \right) \frac{r_{j}}{r} + \frac{\partial}{\partial r} S_{ij}^{bb}(r, t) \frac{r_{j}}{r} = 0, \tag{B.11}
\]

where we made use of \( \frac{\partial}{\partial t} = \frac{\partial}{\partial r} \). We finally obtain a first relation between the longitudinal and the transverse structure function of second order

\[
S_{ij}^{bb}(r, t) = \frac{1}{2r} \frac{\partial}{\partial r} (r^{2} S_{ij}^{bb}(r, t)), \tag{B.12}
\]

which is known as the von Kármán–Howarth relation. The same relation holds also for the velocity structure function of second order whereas the cross helicity structure function bears no such relation.

Summing over equal \( i \) and \( j \) in (B.9) and making use of relation (B.12) yields

\[
\left\langle v^{2}(r, t) \right\rangle = S_{ij}^{vv}(r, t) + 2S_{ij}^{bb}(r, t) = \frac{1}{r^{2}} \frac{\partial}{\partial r} (r^{2} S_{ij}^{vv}(r, t)) \tag{B.13}
\]

and similarly

\[
\left\langle b^{2}(r, t) \right\rangle = S_{ij}^{bb}(r, t) + 2S_{ij}^{bb}(r, t) = \frac{1}{r^{2}} \frac{\partial}{\partial r} (r^{2} S_{ij}^{bb}(r, t)), \tag{B.14}
\]

which is needed for the averaged equation of energy balance in MHD turbulence (E.14).

**Appendix C. Structure functions of third order**

The velocity structure function of third order reads

\[
S_{ij}^{vv}(x, x', t) = \langle (u_{i}(x, t) - u_{i}(x', t))(u_{j}(x, t) - u_{j}(x', t))(u_{a}(x, t) - u_{a}(x', t)) \rangle. \tag{C.1}
\]

Since \( \langle u_{i}(x, t) u_{i}(x, t) u_{a}(x, t) \rangle = 0 \), the structure function decomposes under the assumption of homogeneity and isotropy according to

\[
S_{ij}^{vv}(r, t) = -2C_{ij}^{uuu}(r, t) + C_{j}^{uuu}(r, t) + 2C_{n}^{uuu}(r, t), \tag{C.2}
\]

where \( C_{ij}^{uuu}(r, t) \) is a third order tensor that is symmetric in \( ij \) and solenoidal in \( n \), as it is described in [19]. It can thus solely be expressed by its longitudinal part

\[
C_{ij}^{uuu}(r, t) = -\frac{r^{2}}{2} \frac{\partial}{\partial r} \left( \frac{C_{ij}^{uuu}(r, t)}{r} \right) \frac{r_{j} r_{n}}{r^{3}} + \frac{1}{4r} \frac{\partial}{\partial r} (r^{2} C_{ij}^{uuu}(r, t)) \left( \frac{r_{j}}{r} \delta_{in} + \delta_{jn} \right) - \frac{C_{ij}^{uuu}(r, t)}{2} \frac{r_{n}}{r} \delta_{jn}. \tag{C.3}
\]

Inserting this tensorial form in equation (C.2) yields

\[
S_{ij}^{vv}(r, t) = 3r^{2} \frac{\partial}{\partial r} \left( \frac{C_{ij}^{uuu}(r, t)}{r} \right) \frac{r_{j} r_{n}}{r^{3}} - \frac{\partial}{\partial r} (r^{2} C_{ij}^{uuu}(r, t)) \left( \frac{r_{j}}{r} \delta_{in} + \delta_{jn} \right) \tag{C.4}
\]

Now, we contract this tensor to a tensor of second order and obtain

\[
S_{ij}^{vv}(r, t) = \frac{r_{j}}{r} S_{ij}^{vv}(r, t)
= \left( r \frac{\partial}{\partial r} C_{ij}^{uuu}(r, t) - 5C_{ij}^{uuu}(r, t) \right) \frac{r_{j} r_{n}}{r^{2}} - \frac{\partial}{\partial r} (r^{2} C_{ij}^{uuu}(r, t)) \delta_{jn}. \tag{C.4}
\]

Comparing this relation to the general form of a tensor of second order, for instance (B.9) gives

\[
S_{ij}^{vv}(r, t) = -\frac{\partial}{\partial r} (r C_{ij}^{uuu}(r, t)), \tag{C.5}
\]

\[
S_{ij}^{vv}(r, t) = \left( \frac{\partial}{\partial r} C_{ij}^{uuu}(r, t) - 5C_{ij}^{uuu}(r, t) \right) + S_{ij}^{vv}(r, t) = -6C_{ij}^{uuu}(r, t). \tag{C.6}
\]

From these equations, a relation between the mixed and the longitudinal velocity structure function of third order can be derived as

\[
S_{ij}^{vv}(r, t) = \frac{1}{6} \frac{\partial}{\partial r} (r S_{ij}^{vv}(r, t)). \tag{C.7}
\]
Summing (C.4) over equal indices \( j = n \) gives

\[
S^{\nu\nu}(r, t) = \langle \nu_i(r, t) \nu(r, t) \rangle = S_{r^i r^j}(r, t) + 2S_{r^j r^i}(r, t),
\]

which can be rewritten with relation (C.7) as

\[
S^{\nu\nu}(r, t) = \frac{1}{3r^2} \frac{\partial}{\partial r} (r^2 S_{\nu\nu})(r, t).
\]

This is the average which is needed for the averaged equation of energy balance in MHD turbulence (E.14). The other average stems from the mixed third order tensor

\[
S_{\nu j j}(r, t) = S^{\nu\nu}(r, t) = U_{ij n}(r, t) + H_{ijn}(r, t),
\]

from equation (E.11) and can be divided into

\[
U_{ijn}(r, t) = -2 \left( \langle h_j h_n u_i \rangle - \langle h_i h_n u_j \rangle + \langle h_i h_j u_n \rangle \right)
\]

and

\[
H_{ijn}(r, t) = -2 \left( \langle u_n h_j - u_j h_n \rangle h_i \rangle + \langle u_n h_i - u_i h_n \rangle h_j \rangle - \langle u_i h_i + u_j h_j \rangle h_n \rangle \right).
\]

Turning first to \( U_{ijn}(r, t) \), we have to evaluate correlation functions like \( (\nu_i \nu_j r_{ij}) \). Since this tensor is again symmetric in \( ij \) and solenoidal in \( n \), it has the same tensorial form as \( C^{\nu \nu}(r, t) \) in equation (C.4) and can solely be expressed in terms of \( C^{\nu \nu}(r, t) \)

\[
U_{ijn}(r, t) = \left( C^{\nu \nu}(r, t) - \frac{\partial}{\partial r} C^{\nu \nu}(r, t) \right) \frac{f_1 f_2 f_n}{r^3} - C^{\nu \nu}(r, t) \left( \frac{r_1}{r} \delta_{jm} + \frac{r_2}{r} \delta_{jm} \right) + \left( 3 C^{\nu \nu}(r, t) + \frac{\partial}{\partial r} C^{\nu \nu}(r, t) \right) \frac{f_1}{r} \delta_{ij}.
\]

This yields the following relations between the structure function \( U_{ijn}(r, t) \) and the correlation function \( C^{\nu \nu}(r, t) \)

\[
U_{irr}(r, t) = 2C^{\nu \nu}(r, t),
\]

\[
U_{iir}(r, t) = -C^{\nu \nu}(r, t),
\]

\[
U_{iir}(r, t) = 3C^{\nu \nu}(r, t) + \frac{\partial}{\partial r} C^{\nu \nu}(r, t) (r, t).
\]

For \( H_{ijn}(r, t) \) we need the antisymmetric tensor

\[
C_{i j n}^{\nu \nu}(r, t) = \langle (h_j u_i - u_i h_j) h_n \rangle = C_{i j n}^{\nu \nu}(r, t) \left( \frac{r_1}{r} \delta_{jm} - \frac{r_2}{r} \delta_{jm} \right)
\]

and its symmetric counterpart

\[
C_{i j n}^{\nu \nu}(r, t) = \langle (h_j u_i + u_i h_j) h_n \rangle,
\]

which again fulfills the same equation as \( C^{\nu \nu}(r, t) \), namely (C.4). We obtain

\[
H_{ij n}(r, t) = \left( C^{\nu \nu}(r, t) - \frac{\partial}{\partial r} C^{\nu \nu}(r, t) \right) \frac{f_1 f_2 f_n}{r^3} + \left( \frac{1}{2r} \frac{\partial}{\partial r} (r^2 C^{\nu \nu}(r, t)) + 2C^{\nu \nu}(r, t) \left( \frac{r_1}{r} \delta_{jm} + \frac{r_2}{r} \delta_{jm} \right) \right) + (-C^{\nu \nu}(r, t) - 4C^{\nu \nu}(r, t) \frac{f_1}{r} \delta_{ij}.
\]

This yields the following relations

\[
H_{iir}(r, t) = 2C^{\nu \nu}(r, t),
\]

\[
H_{iir}(r, t) = \frac{1}{2r} \frac{\partial}{\partial r} (r^2 C^{\nu \nu}(r, t)) + 2C^{\nu \nu}(r, t),
\]

\[
H_{iir}(r, t) = -C^{\nu \nu}(r, t) - 4C^{\nu \nu}(r, t).
\]

We need the function \( S^{\nu \nu}(r, t) - S^{\nu \nu}(r, t) \) for the averaged equation of energy balance in spherical coordinates. By making use of

\[
C^{\nu \nu}(r, t) = -2C^{\nu \nu}(r, t),
\]
we get
\[ \mathbf{S}^{hhv}(r, t) - S^{vhh}(r, t) = \langle v_i(r, t) b_j(r, t) \rangle - 2 \langle b_i(r, t) v_j(r, t) \rangle = U_{mr}(r, t) + 2 \delta_{mr}(r, t) + H_{rr}(r, t) + 2 H_{rr}(r, t) \]
\[ = -\frac{4}{r^2} \frac{\partial}{\partial r} \left( r^4 \mathbf{S}^{hhv}_{rr}(r, t) \right) - 8 C^{vhh}_{rr}(r, t). \] (C.24)

Interestingly, the contributions from the symmetric correlation tensor \( \langle (u_i h_i + u_i h_j) h^i_0 \rangle \) vanish from this expression.

**Appendix D. Structure functions of fourth order**

The tensor of fourth order, symmetric in all four indices is given by Monin and Yaglom [28] in vol II by formula (13.82). It has the form
\[ S_{ijkn}(r, t) = (S_{mr}(r, t) - 6 S_{mrt}(r, t) + S_{mnt}(r, t)) \frac{r^3 r_j r_k r_n}{r^4} \]
\[ + (S_{mrt}(r, t) - \frac{2}{3} S_{mnt}(r, t)) \left[ \frac{r_j r_k}{r^2} \delta_{ln} + \frac{r_j r_n}{r^2} \delta_{lk} + \frac{r_k r_n}{r^2} \delta_{lj} + \frac{r_j r_k}{r^2} \delta_{ln} \right], \] (D.1)
If we calculate its divergence we get a tensor of third order, whose tensorial form is given by Monin and Yaglom [28] in vol II by formula (13.80), therefore we obtain
\[ \frac{\partial}{\partial \tau_n} S_{ijkn}(r, t) = \left( \frac{\partial}{\partial \tau_n} S_{mr}(r, t) - \frac{2}{3} \frac{\partial}{\partial \tau_n} S_{mrt}(r, t) \right) \frac{r^3 r_n r_j r_k}{r^4} \]
\[ + \frac{\partial}{\partial \tau_n} S_{mnt}(r, t) \left[ \frac{r_j r_k}{r^2} \delta_{ln} + \frac{r_n r_k}{r^2} \delta_{lj} + \frac{r_n r_l}{r^2} \delta_{jk} \right]. \] (D.2)
This tensorial form can be compared with the original calculations. We get
\[ \frac{\partial}{\partial \tau_n} S_{mr}(r, t) - \frac{3}{2} \frac{\partial}{\partial \tau_n} S_{mrt}(r, t) = \left( \frac{\partial}{\partial \tau_n} + \frac{2}{r} \right) \left( S_{mr}(r, t) - 6 S_{mrt}(r, t) + S_{mnt}(r, t) \right) \]
\[ + \left[ \frac{3}{2} \frac{\partial}{\partial \tau_n} \right] \left( S_{mrt}(r, t) - \frac{1}{3} S_{mnt}(r, t) \right), \] (D.3)
and
\[ \frac{\partial}{\partial \tau_n} S_{mnt}(r, t) = \left( \frac{\partial}{\partial \tau_n} + \frac{4}{r} \right) S_{mnt}(r, t) - \frac{4}{3r} S_{mnt}(r, t). \] (D.4)
The last equation has to be inserted into (D.3) in order to get the equation for the longitudinal structure function \( \frac{\partial}{\partial \tau_n} S_{mnt}(r, t) \).

For the antisymmetric tensor
\[ S_{ijkn}^{vhh}(r, t) = \langle v_i b_j b_k - b_i b_j v_k \rangle = S_{ijl}^{vhh}(r, t) \left( \frac{r^3 r_j r_k}{r^4} \delta_{ln} - \frac{r_k r_n}{r^2} \delta_{lj} \right), \] (D.5)
we calculate the divergence as
\[ \frac{\partial}{\partial \tau_n} S_{ijkn}^{vhh}(r, t) = \frac{\partial}{\partial r} S_{ijl}^{vhh}(r, t) \left[ \frac{r^3 r_j r_k}{r^4} \delta_{ln} - \frac{r_k r_n}{r^2} \delta_{lj} \right] \]
\[ + \frac{S_{ijl}^{vhh}(r, t)}{r} \left[ \frac{r_j r_k}{r^2} \delta_{ln} + \frac{r_n r_k}{r^2} \delta_{lj} - 2 \frac{r_j r_k}{r^2} \delta_{ln} - 2 \frac{r_k r_n}{r^2} \delta_{lj} \right]. \] (D.6)
Let us define \( A_{ijkn}(r, t) \) as
\[ A_{ijkn}(r, t) = S_{ijkn}^{vhh}(r, t) + S_{ijkn}^{vhh}(r, t) + S_{ijkn}^{vhh}(r, t). \] (D.7)
We obtain
\[
\frac{\partial}{\partial t} A_{ijk,n}(r, t) = \left(3 \frac{\partial}{\partial r} S_{r_{ij},l}^{vbh}(r, t) - 6 \frac{\partial}{\partial r} r_{ji,l}^{vbh}(r, t) \right) \frac{v_j r_f r_k}{r^3} \\
- \frac{\partial}{\partial r} S_{r_{ij},l}^{vbh}(r, t) \left[ \frac{f_l}{r} \delta_{jk} + \frac{f_j}{r} \delta_{ik} + \frac{f_k}{r} \delta_{lj} \right].
\] (D.8)

Therefore, the coefficients can be read of analogous to (D.3) and (D.4) as
\[
\frac{\partial}{\partial t} A_{mn}(r, t) - 3 \frac{\partial}{\partial r} A_{mn}(r, t) = \left(3 \frac{\partial}{\partial r} - 6 \frac{1}{r} \right) S_{r_{ij},l}^{vbh}(r, t) \] (D.9)

and
\[
\frac{\partial}{\partial t} A_{mn}(r, t) = - \frac{\partial}{\partial r} S_{r_{ij},l}^{vbh}(r, t). \] (D.10)

**Appendix E. The averaged equation of energy balance for velocity and magnetic field increments in MHD turbulence**

In this section, we derive an evolution equation for the symmetric tensor of second order \((v_i v_j + b_i b_j)\). To this end, let us multiply (A.10) by \(v_i\) and then do the same procedure for interchanged indices. Adding the corresponding equations together yields
\[
\frac{\partial}{\partial t} v_i v_j + \frac{\partial}{\partial r} v_i v_i v_j + \frac{\partial}{\partial X_n} U_n v_i v_j - b_n v_j \frac{\partial}{\partial r} b_i - b_n v_i \frac{\partial}{\partial r} b_j - H_n v_i \frac{\partial}{\partial X_n} b_i - H_n v_i \frac{\partial}{\partial X_n} b_j \\
= - v_i P_j - v_j P_i + \nu v_i (\nabla_x^2 + \nabla_h^2) v_j + \nu v_j (\nabla_x^2 + \nabla_h^2) v_i,
\] (E.1)

where we made use of the incompressibility condition for \(v_i\) and \(U_n\) in order to pull the divergences in front of the expressions. This is especially suitable for the direct use of homogeneity after the averaging of equation (E.1), since terms like \(\frac{\partial}{\partial X_n} \langle \ldots \rangle = 0\). Unfortunately, this treatment cannot be applied for terms in equation (E.1) that involve advection by either \(b\) or \(H\). However, we are able to derive the complementary terms from the evolution equation of the magnetic increment (A.11). To this end, we multiply equation (A.11) by \(b_j\) and again interchange the indices, which leads to
\[
\frac{\partial}{\partial t} b_i b_j + \frac{\partial}{\partial r} v_i v_i b_j - \nu v_i b_j - \nu v_j b_i - \nu v_i b_j - \nu v_j b_i - H_n v_i \frac{\partial}{\partial X_n} b_i - H_n v_i \frac{\partial}{\partial X_n} b_j \\
= \nu b_i (\nabla_x^2 + \nabla_h^2) b_j + \nu b_j (\nabla_x^2 + \nabla_h^2) b_i.
\] (E.2)

It can readily be seen that equations (E.1) and (E.2) are linked together by terms in (A.10) and (A.11) that are advected by either \(b\) or \(H\), since we are not able to write the balance equation for the kinetic and magnetic increments in a closed form separately. This can be seen as a significant feature of locally isotropic MHD turbulence.

Adding equation (E.1) to (E.2) and taking the averages, gives an evolution equation for the symmetric tensor \((v_i v_j + b_i b_j)\) in MHD turbulence
\[
\frac{\partial}{\partial t} \langle v_i v_j + b_i b_j \rangle + \frac{\partial}{\partial r} \langle v_i v_i v_j + b_i v_j \rangle - \frac{\partial}{\partial r} \langle b_i (v_i b_j + v_j b_i) \rangle \\
+ \frac{\partial}{\partial X_n} \langle U_n (v_i v_j + b_i b_j) \rangle - \frac{\partial}{\partial X_n} \langle H_n (v_i b_j + v_j b_i) \rangle = \langle v_i P_j + v_j P_i \rangle
\] (E.3)

\[
= 2 \nu \left(\nabla_x^2 + \frac{1}{4} \nabla_h^2\right) \langle v_i v_j \rangle + 2 \langle \epsilon_{ij}^{uu} \rangle + 2 \lambda \left(\nabla_x^2 + \frac{1}{4} \nabla_h^2\right) \langle b_i b_j \rangle - 2 \langle \epsilon_{ij}^{bb} \rangle,
\]

where we have introduced the tensors of the local energy dissipation rate \((8)\) and \(9\). The treatment of the viscous terms is described in appendix G. Under the assumption of homogeneity, terms that stand behind the center derivative \(\frac{\partial}{\partial X_n}\) can be neglected. Furthermore, the pressure term vanishes on the basis of homogeneity, which has been discussed by Hill [20].

Summing over equal indices \(i = j\) in equation (E.3) leads to the averaged equation of energy balance of MHD turbulence in a briefer form.
\[
\frac{\partial}{\partial t} \left( \frac{v^2}{2} + b^2(t, r, t) \right) \\
+ \nabla \cdot \{ v(r, t) \left( \frac{v^2}{2} + b^2(t, r, t) \right) \} - \nabla \cdot \{ b(r, t) \cdot v(r, t) \cdot b(r, t) \} \\
= v \nabla ^2 \left( v^2(r, t) \right) + \lambda \nabla ^2 \{ b^2(r, t) \} - 2 \langle \varepsilon ^{uu} (x, t) + \varepsilon ^{hh} (x, t) \rangle + Q(r, t),
\]

where

\[
\varepsilon ^{uu} (x, t) = \sum _{i,j} \varepsilon _{ii} ^{uu} = \frac{\nu}{2} \sum _{i,j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) ^2,
\]

\[
\varepsilon ^{hh} (x, t) = \sum _{i,j} \varepsilon _{ii} ^{hh} = \frac{\lambda}{2} \sum _{i,j} \left( \frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i} \right) ^2,
\]

denote the corresponding local energy dissipation rates and \( Q(r, t) = \langle v(r, t) \cdot F(r, t) \rangle + \langle b(r, t) \cdot G(r, t) \rangle \) takes into account a suitable forcing procedure.

**E.1. The 4/5 law in MHD turbulence**

In the following we introduce the tensors

\[
S_{ij} ^{vv} (r, t) = \langle v_i v_j \rangle,
\]

\[
S_{ij} ^{bb} (r, t) = \langle h_i h_j \rangle,
\]

\[
S_{ij} ^{vh} (r, t) = \langle v_i h_j \rangle,
\]

\[
S_{ij} ^{bh} (r, t) = \langle b_i h_j \rangle,
\]

where the subscripts denote the corresponding increments. The last tensor (E.10) decomposes in terms of the corresponding correlation functions according to

\[
S_{ij} ^{bh} (r, t) - S_{ij} ^{gb} (r, t) = \langle h_i b_j \rangle - \langle h_i b_j \rangle = \langle (v_i h_j + v_j h_i) b_0 \rangle - \langle (v_i h_j + v_j h_i) b_0 \rangle,
\]

where terms like \( \langle u_i h_j h_0 \rangle - \langle u_i h_j h_0 \rangle \) vanish under the assumption of homogeneity. It can readily be seen, that the structure function does not decompose into the corresponding correlation functions like the structure function in the hydrodynamic case (E.9) that we considered in the appendix C. This is crucial for the appearance of the antisymmetric tensor

\[
C_{ij} ^{abh} (r, t) = \langle h_i b_j \rangle - \langle h_j b_i \rangle = C_{ij} ^{abh} (r, t) \left( \frac{r_j}{r} S_{in} ^{bh} - \frac{r_i}{r} S_{jn} ^{bh} \right),
\]

whose defining scalar \( C_{ij} ^{abh} (r, t) \) is not touched by the incompressibility condition. The second correlation function that enters in equation (E.11) follows the usual tensorial form (C.4) and reads

\[
C_{ij} ^{abh} (r, t) = \langle h_i b_j \rangle.
\]

The averaged equation of energy balance in spherical coordinates can now be written in the form

\[
\frac{1}{2} \frac{\partial}{\partial t} (S^{vv} (r, t) + S^{bb} (r, t)) + \frac{1}{2r^2} \frac{\partial}{\partial r} \{r^2 S^{vv} (r, t) + r^2 (S^{bb} (r, t) - S^{vb} (r, t)) \}
= \nu \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} S^{vv} (r, t) \right) + \lambda \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} S^{bb} (r, t) \right) - 2 \langle \varepsilon ^{uu} + \varepsilon ^{hh} \rangle + Q(r, t),
\]

where we have inserted the following functions, which can be expressed in terms of longitudinal structure functions and correlation functions as derived in the appendix B.1 and appendix C

\[
S^{vv} (r, t) = \langle v^2 (r, t) \rangle = \frac{1}{r^2} \frac{\partial}{\partial r} \{r^2 S_{rr} ^{vv} (r, t) \},
\]

\[
\frac{\partial}{\partial r} S^{vv} (r, t) = \frac{1}{r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial}{\partial r} S_{rr} ^{vv} (r, t) \right),
\]

\[
S^{bb} (r, t) = \langle b^2 (r, t) \rangle = \frac{1}{r^2} \frac{\partial}{\partial r} \{r^2 S_{rr} ^{bb} (r, t) \},
\]

\[
\frac{\partial}{\partial r} S^{bb} (r, t) = \frac{1}{r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial}{\partial r} S_{rr} ^{bb} (r, t) \right).
\]
For stationary turbulence the fields are driven by a time independent source term \( Q(r, t) = Q(r) \), and we obtain
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{6} S_{rr}^{vv}(r) - 2C_{r,t}^{hh}(r) - \nu \frac{\partial}{\partial r} S_{r}^{yy}(r) - \lambda \frac{\partial}{\partial r} S_{r}^{bb}(r) \right) \right) - \frac{4}{r^2} \frac{\partial}{\partial r} r^2 C_{r,t}^{uhh}(r) = -2(e_{uu} + e_{hh}) + Q(r).
\]

(E.21)

Two integrations with respect to \( r \) yield
\[
S_{r}^{vv}(r) - 12C_{r,t}^{hh}(r) - \frac{24}{r^4} \int_0^r dr' r'^3 C_{r,t}^{hh}(r') = -\frac{4}{5}(e_{uu} + e_{hh}) r + 6\nu \frac{\partial}{\partial r} S_{r}^{yy}(r) + 6\lambda \frac{\partial}{\partial r} S_{r}^{bb}(r) + q(r),
\]
where the source \( q(r) \) is given by
\[
q(r) = \frac{6}{r^4} \int_0^r dr' r'^3 \int_0^{r'} dr'' r''^2 Q(r'').
\]

(E.23)

Equation (E.23) is the generalization of the 4/5-law from hydrodynamic turbulence in the presence of a magnetic field. The 4/5-law in hydrodynamic turbulence is an exact relation between the second and third order longitudinal velocity structure function and the energy dissipation rate \( \langle \varepsilon_{uu} \rangle \). However, in the case of MHD turbulence this relation is not closed, since the source term from \( C_{r,t}^{hh}(r) \) does not vanish in the inertial range.

In the following, we want to discuss the implications of equation (E.23).

**E.1.1. Dissipation range.** The third order longitudinal velocity structure function \( S_{r}^{vv}(r) \) scales as \( \sim r^3 \) for small \( r \), whereas the mixed correlation functions only scale as \( r \), which first has been established by Chandrasekhar [16]. Furthermore, he was able to show that the mixed correlation functions for small \( r \) are related by
\[
\begin{align*}
C_{r,t}^{hh}(r) &= -2C_{0}r, \\
C_{r,t}^{hh}(r) &= 5C_{0}r,
\end{align*}
\]

where \( C_{0} \) can be seen as the contribution to the magnetic energy from the stretching of the lines of force by the velocity field. The source term can thus be expressed in terms of the defining scalar \( C_{r,t}^{hh}(r) \), namely
\[
C_{r,t}^{hh}(r) = -\frac{5}{2}C_{r,t}^{hh}(r).
\]

(E.24)

(E.25)

In the dissipation range \( (r < \min(q_{uu}, \eta_{hh})) \) the velocity structure function of third order \( S_{r}^{vv}(r) \) can thus be neglected. Furthermore, by inserting (E.25) into (E.23), one can readily see that the two mixed correlation terms exactly cancel each other, which implies that energy is only lost due to dissipative effects. In the dissipation range equation (E.23) thus reads
\[
6\nu \frac{\partial}{\partial r} S_{r}^{yy}(r) + 6\lambda \frac{\partial}{\partial r} S_{r}^{bb}(r) = \frac{4}{5}(e_{uu} + e_{hh}) r,
\]

(E.26)

where the forcing is assumed to take place on larger scales, which implies \( q(r) = 0 \). An integration with respect to \( r \) yields
\[
S_{r}^{yy}(r) + \frac{1}{Pm} S_{r}^{bb}(r) = \frac{\langle \varepsilon_{uu} + \varepsilon_{hh} \rangle}{15} r^2,
\]

(E.27)

where \( Pm = \frac{\nu}{\lambda} \) is the magnetic Prandtl number.

**E.1.2. Inertial range.** In the inertial range the viscous and forcing terms can be neglected and the 4/5-law (E.23) reads
\[
S_{r}^{vv}(r) - 12C_{r,t}^{hh}(r) - \frac{24}{r^4} \int_0^r dr' r'^3 C_{r,t}^{hh}(r') = -\frac{4}{5}(e_{uu} + e_{hh}) r.
\]

(E.28)
Appendix F. Next-order structure function equation in MHD turbulence

In this appendix we derive an evolution equation that contains the fourth-order structure functions in MHD turbulence in analogy to the next-order equations given by Hill and Boratav [14]. In multiplying (3) and (4) by the corresponding increments we obtain a first evolution equation for the triple velocity structure function \( \langle v_i v_j v_k \rangle \), which also involves the dynamics of three terms similar to \( \langle v_i b_j b_k \rangle \) (indices interchanged). These three terms have to be added in order to arrive at closed expressions in much the same way as it has been done for the symmetric tensor of second order \( \langle v_i v_j v_k \rangle \) in Appendix E. The tensor \( \langle v_i v_j v_k + v_i b_j b_k + v_i v_j b_k + v_i b_j v_k \rangle \) can therefore be considered as the symmetric tensor of third order in MHD turbulence. We obtain

\[
\frac{\partial}{\partial t} \langle v_i v_j v_k + v_i b_j b_k + v_i v_j b_k + v_i b_j v_k \rangle \\
+ \frac{\partial}{\partial n} \langle v_i (v_j v_k + v_i b_j b_k + v_i v_j b_k + v_i b_j v_k) \rangle \\
+ \frac{\partial}{\partial X_n} \left[ U_n (v_i v_j v_k + v_i b_j b_k + v_i v_j b_k + v_i b_j v_k) \right] \\
- \langle (v_i v_j v_k + v_i b_j b_k + v_i v_j b_k + v_i b_j v_k) \rangle \\
+ \nu \langle v_i v_j (\nabla_i^2 + \nabla_j^2) v_k + v_i v_k (\nabla_i^2 + \nabla_k^2) v_j + v_i v_j (\nabla_i^2 + \nabla_j^2) v_k \rangle \\
+ \nu \langle b_i b_j (\nabla_i^2 + \nabla_j^2) b_k + b_i b_k (\nabla_i^2 + \nabla_k^2) b_j + b_i b_j (\nabla_i^2 + \nabla_j^2) b_k \rangle \\
+ \lambda \langle v_i b_j (\nabla_i^2 + \nabla_j^2) b_k + b_i (\nabla_i^2 + \nabla_k^2) b_j v_k + b_i (\nabla_i^2 + \nabla_j^2) b_j \rangle \\
+ \lambda \langle v_i b_j (\nabla_i^2 + \nabla_j^2) b_k + v_i b_k (\nabla_i^2 + \nabla_k^2) b_j + v_i b_j (\nabla_i^2 + \nabla_j^2) b_j \rangle.
\]  

We introduce the following tensors

\[
S_{i j k}^{v v v} (r, t) = \langle v_i v_j v_k \rangle,
\]

\[
S_{i j k}^{v b b} (r, t) = \langle v_i b_j b_k + b_i v_j b_k + b_i b_j v_k \rangle,
\]

\[
S_{i j k}^{v v v v} (r, t) = \langle v_i v_j v_k v_l \rangle,
\]

\[
S_{i j k}^{v b b v} (r, t) = \langle v_i b_j b_k v_l \rangle,
\]

\[
S_{i j k}^{v v v b} (r, t) = \langle v_i v_j b_k v_l \rangle,
\]

\[
S_{i j k}^{v b b b} (r, t) = \langle v_i b_j b_k b_l \rangle,
\]

\[
T_{i j k} (r, t) = \langle (v_i v_j + b_i b_j) P_k + (v_i v_k + b_i v_k) P_j + (v_i b_k + b_i v_k) P_j \rangle.
\]

For the viscous terms, we restrict ourselves to the case where the magnetic Prandtl number \( P_m = \frac{\nu}{\chi} \) is unity, which simplifies the treatment. This restriction seems somehow arbitrary, but for later times we are only interested in the inertial range behavior of the fourth order structure functions, where it was shown for the hydrodynamic case [14] that the viscous terms should have no contribution. We require homogeneity in order to neglect terms in \( \frac{\partial}{\partial x_n} \) acts on an ensemble average and arrive at the following equation

\[
\frac{\partial}{\partial t} \left( S_{i j k}^{v v v} (r, t) + S_{i j k}^{v b b} (r, t) \right) + \frac{\partial}{\partial n} \left( S_{i j k}^{v v v v} (r, t) - S_{i j k}^{v b b v} (r, t) \right)
\]

\[
- \frac{\partial}{\partial n} \left( S_{i j k}^{v v v} (r, t) + S_{i j k}^{v b b} (r, t) + S_{i j k}^{v v v b} (r, t) \right)
\]

\[
= - T_{i j k} (r, t) + 2\nu \left[ \nabla_i^2 \left( S_{i j k}^{v v v} (r, t) + S_{i j k}^{v b b} (r, t) \right) \right] - Z_{i j k}^{v v v} (r, t) - Z_{i j k}^{v b b} (r, t),
\]

where we have introduced

\[
Z_{i j k}^{v v v} (r, t) = \langle v_i c_{j k}^{u u} + v_j c_{i k}^{u u} + v_k c_{i j}^{u u} \rangle
\]

\[
Z_{i j k}^{v b b} (r, t) = \langle v_i c_{j k}^{u b} + v_j c_{i k}^{u b} + v_k c_{i j}^{u b} + b_i (c_{j k}^{u b} + c_{k j}^{u b}) + b_j (c_{k i}^{u b} + c_{i k}^{u b}) + b_k (c_{i j}^{u b} + c_{j i}^{u b}) \rangle
\]

and

\[
c_{i j}^{u u} = \left( \frac{\partial u_i}{\partial x_j} \right) \left( \frac{\partial u_i}{\partial x_j} \right) + \left( \frac{\partial u_i}{\partial x_i} \right) \left( \frac{\partial u_j}{\partial x_j} \right)
\]
\[
\varepsilon_{ij} = \left( \frac{\partial u_i}{\partial x_j} \right) \left( \frac{\partial u_j}{\partial x_i} \right) + \left( \frac{\partial u_i'}{\partial x_j} \right) \left( \frac{\partial u_j'}{\partial x_i'} \right).
\]  

(F.10)

The treatment of the viscous terms follows the same procedure as described in appendix G.

In the inertial range (F.8) the viscous terms can be neglected and we require statistical stationarity, which yields

\[
\frac{\partial}{\partial r_n} \left( S_{ij}^{vvv} (\mathbf{r}, t) - S_{ij}^{bb} (\mathbf{r}, t) \right) - \frac{\partial}{\partial r_n} \left( c_{ij}^{vv} (\mathbf{r}, t) + c_{ij}^{bb} (\mathbf{r}, t) + S_{ij}^{vv} (\mathbf{r}, t) \right) = T_{ij} (\mathbf{r}, t).
\]  

(F.11)

Appendix G. The viscous term

As an example of the treatment of the viscous terms in (E.3) we focus on the viscous velocity contributions. The calculation is the same for the diffusive magnetic field contributions. The viscous terms in (E.3) read

\[
v_j (\nabla_x^2 + \nabla_x^2) v_i + v_i (\nabla_x^2 + \nabla_x^2) v_j.
\]

(G.1)

We rewrite the Laplacian in \(x\) - and \(x'\) -space according to

\[
\nabla_x^2 = \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} \text{ and } \nabla_x^2 = \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n'}
\]

(G.2)

and make use of the identity

\[
\frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} (\mathbf{g}) = f \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} g + 2 \left( \frac{\partial f}{\partial x_n} \frac{\partial g}{\partial x_n} \right) + g \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} f.
\]

(G.3)

Therefore, we can rewrite (G.1) as

\[
v_j \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} + \frac{\partial}{\partial x_n'} \frac{\partial}{\partial x_n'} \right) v_i + v_i \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} + \frac{\partial}{\partial x_n'} \frac{\partial}{\partial x_n'} \right) v_j
\]

\[= \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} + \frac{\partial}{\partial x_n'} \frac{\partial}{\partial x_n'} \right) v_i v_j - 2 \left( \frac{\partial v_i}{\partial x_n} \frac{\partial v_j}{\partial x_n} + \frac{\partial v_i}{\partial x_n'} \frac{\partial v_j}{\partial x_n'} \right).
\]

(G.4)

Note that \( \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} \right) = \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} \right) \) and \( \left( \frac{\partial}{\partial x_n'} \frac{\partial}{\partial x_n'} \right) = \left( \frac{\partial}{\partial x_n'} \frac{\partial}{\partial x_n'} \right) \) and that

\[
\left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} + \frac{\partial}{\partial x_n'} \frac{\partial}{\partial x_n'} \right) = 2 \left( \frac{\partial}{\partial r_n} \frac{\partial}{\partial r_n} + \frac{1}{4} \frac{\partial}{\partial X_n} \frac{\partial}{\partial X_n} \right),
\]

(G.5)

where we made use of (A.9). If we rewrite the equations again with the Laplacian in \(r\) - and \(X\) -space, we get

\[
v_j (\nabla_x^2 + \nabla_x^2) v_i + v_i (\nabla_x^2 + \nabla_x^2) v_j = 2 \left( \frac{\nabla_x^2}{4} + \nabla_x^2 \right) v_i v_j - \varepsilon_{ij}^{uu} \]

(G.6)

with

\[
\varepsilon_{ij}^{uu} = \left( \frac{\partial u_i}{\partial x_n} \right) \left( \frac{\partial u_j}{\partial x_n} \right) + \left( \frac{\partial u_i'}{\partial x_n'} \right) \left( \frac{\partial u_j'}{\partial x_n'} \right).
\]

(G.7)

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