A Master Equation with Generalized Lindblad Form and a Unitary Transformation by the Squeezing Operator

Kazuyuki FUJII *

†Department of Mathematical Sciences
Yokohama City University
Yokohama, 236–0027
Japan

Abstract

In the preceding paper arXiv : 0802.3252 [quant-ph] we treated a model given by a master equation with generalized Lindblad form, and examined the algebraic structure related to some Lie algebras and constructed an approximate solution.

In this paper we apply a unitary transformation by the squeezing operator to the master equation. Then the generalized Lindblad form is transformed to the usual Lindblad one, while the (original) Hamiltonian is transformed to somewhat complicated one.

As a result we have two different representations based on the Lie algebra $su(1,1)$.

We examine new algebraic structure and construct some approximate solution.

Quantum Computation (Computer) is one of main subjects in Quantum Physics. To realize it we must overcome severe problems arising from Decoherence, so we need to study Quantum Open System to control decoherence (if possible).

*E-mail address : fujii@yokohama-cu.ac.jp
This paper is a series of [1], [2] and [3] and we continue to study dynamics of a quantum open system. First we explain our purpose in a short manner. See [4] as a general introduction to this subject.

We consider a quantum open system \( S \) coupled to the environment \( E \). Then the total system \( S + E \) is described by the Hamiltonian

\[
H_{S+E} = H_S \otimes 1_E + 1_S \otimes H_E + H_I
\]

where \( H_S, H_E \) are respectively the Hamiltonians of the system and environment, and \( H_I \) is the Hamiltonian of the interaction.

Then under several assumptions (see [4]) the reduced dynamics of the system (which is not unitary!) is given by the Master Equation

\[
\frac{\partial}{\partial t} \rho = -i[H_S, \rho] - \mathcal{D}(\rho) 
\]

with the dissipator being the usual Lindblad form

\[
\mathcal{D}(\rho) = \frac{1}{2} \sum_{\{j\}} \left( A_j^\dagger A_j \rho + \rho A_j^\dagger A_j - 2A_j \rho A_j^\dagger \right) .
\]

Here \( \rho \equiv \rho(t) \) is the density operator (or matrix) of the system.

Similarly, the master equation of quantum damped harmonic oscillator (see [4], Section 3.4.6 or also [5]) is given by

\[
\frac{\partial}{\partial t} \rho = -i[\omega a^\dagger a, \rho] - \frac{\mu}{2} \left( a^\dagger a \rho + \rho a a^\dagger - 2a \rho a^\dagger \right) - \frac{\nu}{2} \left( a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a \right),
\]

where \( a \) and \( a^\dagger \) are the annihilation and creation operators of the system (for example, an electro–magnetic field mode in a cavity), and \( \mu, \nu (\mu > \nu \geq 0) \) are some real constants depending on the system (for example, a damping rate of the cavity mode).

In [2] the general solution was given in the **operator algebra level**. This is a very important step. Moreover, in [1] we treated the master equation with generalized Lindblad (or Kossakowski–Lindblad [6]) form given by

\[
\frac{\partial}{\partial t} \rho = -i[\omega a^\dagger a, \rho] - \frac{\mu}{2} \left( a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger \right) - \frac{\nu}{2} \left( a a^\dagger \rho + \rho a a^\dagger - 2a^\dagger \rho a \right) - \frac{\kappa}{2} \left( a^2 \rho + \rho a^2 - 2a \rho a \right) - \frac{\tilde{\kappa}}{2} \left( (a^\dagger)^2 \rho + \rho (a^\dagger)^2 - 2a^\dagger \rho a^\dagger \right)
\]
where $\kappa$ is a complex constant satisfying the condition $\mu \nu \geq |\kappa|^2$ which ensures the positivity. Then we examined the algebraic structure related to the Lie algebras $su(1,1)$ and $su(2)$, and constructed some approximate solutions by use of it.

In this paper we apply a unitary transformation by the squeezing operator to (4), which was developed by An et al. [7], [8]. Then the generalized Lindblad form is transformed to the usual Lindblad one, while the (original) Hamiltonian is transformed to somewhat complicated one. We examine new algebraic structure in detail.

The squeezing operator $S(\epsilon)$ is given by

$$S(\epsilon) = \exp \left( \frac{1}{2} (\epsilon (a^\dagger)^2 - \bar{\epsilon} a^2) \right), \quad \epsilon = |\epsilon| e^{i\phi} \in \mathbb{C}$$

(|$\epsilon$| and $\phi$ are chosen later on) and it has well–known remarkable property

$$S(\epsilon)aS(\epsilon)^\dagger = c(|\epsilon|) a - e^{i\phi}s(|\epsilon|) a^\dagger, \quad S(\epsilon)a^\dagger S(\epsilon)^\dagger = c(|\epsilon|) a^\dagger - e^{-i\phi}s(|\epsilon|) a$$

where $c(|\epsilon|) \equiv \cosh(|\epsilon|)$ and $s(|\epsilon|) \equiv \sinh(|\epsilon|)$ for simplicity, see for example [9].

We act the squeezing operator to the master equation (4) as adjoint action

$$\frac{\partial}{\partial t} \rho_S = -i[\omega a^\dagger a, \rho_S]$$

$$-\frac{\mu}{2} (a^\dagger a \rho_S + \rho_S a^\dagger a - 2a \rho_S a^\dagger) - \frac{\nu}{2} (a^\dagger a \rho_S + \rho_S a^\dagger a - 2a^\dagger \rho_S a)$$

$$-\frac{\kappa}{2} (a^2 \rho_S + \rho_S a^2 - 2a \rho_S a) - \frac{\bar{\kappa}}{2} ((a^\dagger)^2 \rho_S + \rho_S (a^\dagger)^2 - 2a^\dagger \rho_S a^\dagger)$$

(7)

where $\rho_S = S \rho S^\dagger$ and $a_S = S a S^\dagger$. Now let us calculate the right hand side.

It is not difficult to check

$$A_S \equiv a^\dagger a S \rho_S + \rho_S a^\dagger a - 2a \rho_S a^\dagger$$

$$= c^2 \{ a^\dagger a \rho_S + \rho_S a^\dagger a - 2a \rho_S a^\dagger \} + s^2 \{ a a^\dagger \rho_S + \rho_S a^\dagger a - 2a^\dagger \rho_S a \}$$

$$- e^{-i\phi} c s \{ a^2 \rho_S + \rho_S a^2 - 2a \rho_S a \} - e^{i\phi} c s \{ (a^\dagger)^2 \rho_S + \rho_S (a^\dagger)^2 - 2a^\dagger \rho_S a^\dagger \}$$

(8)

\footnote{1 In the papers the positivity condition $\mu \nu \geq |\kappa|^2$ has been neglected}
and
\[ B_S = a_s a_s^\dagger \rho_S + \rho_s a_s a_s^\dagger - 2a_s^\dagger \rho_s a_s \]
\[ = s^2 \{ a^\dagger a \rho_S + \rho_s a^\dagger a - 2a \rho_s a^\dagger \} + c^2 \{ a a^\dagger \rho_S + \rho_s a a^\dagger - 2a^\dagger \rho_s a \}
- e^{-i\phi} a^2 \rho_S + \rho_s a^2 - 2a \rho_s a \} - e^{i\phi} c \{(a^\dagger)^2 \rho_S + \rho_s (a^\dagger)^2 - 2a^\dagger \rho_s a \} \]\[
(9)\]
and
\[ C_S = a_s^2 \rho_S + \rho_s a_s^2 - 2a \rho_s a_s \]
\[ = -e^{i\phi} a^\dagger a \rho_S + \rho_s a^\dagger a - 2a \rho_s a^\dagger \} - e^{i\phi} c \{(a^\dagger)^2 \rho_S + \rho_s (a^\dagger)^2 - 2a^\dagger \rho_s a \}
+ c^2 \{ a^2 \rho_S + \rho_s a^2 - 2a \rho_s a \} + e^{2i\phi} s^2 \{(a^\dagger)^2 \rho_S + \rho_s (a^\dagger)^2 - 2a^\dagger \rho_s a \} \]\[
(10)\]
and
\[ D_S = (a_s^\dagger)^2 \rho_S + \rho_s (a_s^\dagger)^2 - 2a_s^\dagger \rho_s a_s \]
\[ = -e^{-i\phi} a^\dagger a \rho_S + \rho_s a^\dagger a - 2a \rho_s a^\dagger \} - e^{-i\phi} c \{(a^\dagger)^2 \rho_S + \rho_s (a^\dagger)^2 - 2a^\dagger \rho_s a \}
+ e^{-2i\phi} s^2 \{ a^2 \rho_S + \rho_s a^2 - 2a \rho_s a \} + c^2 \{(a^\dagger)^2 \rho_S + \rho_s (a^\dagger)^2 - 2a^\dagger \rho_s a \} \]\[
(11)\]
where \( c = c(|\epsilon|) \) and \( s = s(|\epsilon|) \) for simplicity.

Therefore by \((5) \sim (11)\) the generalized Lindblad form in \((7)\) becomes
\[
-\frac{1}{2} \left( \mu c^2 + \nu s^2 - \kappa c e^{i\phi} s - \kappa e^{-i\phi} cs \right) A - \frac{1}{2} \left( \mu s^2 + \nu c^2 - \kappa e^{i\phi} cs - \kappa e^{-i\phi} cs \right) B \]
\[
-\frac{1}{2} \left( -\mu e^{-i\phi} cs - \nu e^{-i\phi} cs + \kappa c^2 + e^{-2i\phi} \kappa s^2 \right) C - \frac{1}{2} \left( -\mu e^{i\phi} cs - \nu e^{i\phi} cs + \kappa c^2 + e^{2i\phi} \kappa s^2 \right) D \]
\[
(12)\]
\[
(13)\]
where \( A = A_{S=1} \) et al.

Now we can remove the terms \((13)\) if we choose \( \epsilon \) suitably. In fact, we set
\[
\kappa = |\kappa| e^{-i\phi} = k e^{-i\phi} \]
then we have a quadratic equation
\[
-(\mu + \nu) c s + k(c^2 + s^2) = 0 \iff k t^2 - (\mu + \nu) t + k = 0 \]
\[
(14)\]
for \( t \equiv \frac{2}{\epsilon} = \tanh(|\epsilon|) \). The discriminant \( D \) is

\[
D = (\mu + \nu)^2 - 4k^2 = (\mu - \nu)^2 + 4(\mu \nu - k^2) \geq 0
\]

by the positivity condition \( \mu \nu \geq k^2 \), so there is a solution on \( t \) \( \Rightarrow \) on \( |\epsilon| \).

**A comment is in order.** We cannot remove the second term in (12) against the result in [8], [7] by the positivity condition. We leave it to readers.

Therefore, the generalized Lindblad form becomes

\[
- \frac{\mu c^2 + \nu s^2 - 2kcs}{2} (a^\dagger \rho_S + \rho_S a^\dagger a - 2a \rho_S a^\dagger) - \frac{\mu s^2 + \nu c^2 - 2kcs}{2} (aa^\dagger \rho_S + \rho_S aa^\dagger - 2a^\dagger \rho_S a) .
\]

(15)

Next let us calculate the Hamiltonian in (17), which is easy to see

\[
- i [\omega a^\dagger a_S, \rho_S] = - i [\omega \{ (c^2 + s^2) a^\dagger a - e^{i \phi} cs(a^\dagger)^2 - e^{-i \phi} cs a^2 \} , \rho_S].
\]

(16)

As a result the master equation (7) becomes

\[
\frac{\partial}{\partial t} \rho_S = - i [\omega \{ (c^2 + s^2) a^\dagger a - e^{i \phi} cs(a^\dagger)^2 - e^{-i \phi} cs a^2 \} , \rho_S]
- \frac{\mu c^2 + \nu s^2 - 2kcs}{2} (a^\dagger \rho_S + \rho_S a^\dagger a - 2a \rho_S a^\dagger)
- \frac{\mu s^2 + \nu c^2 - 2kcs}{2} (aa^\dagger \rho_S + \rho_S aa^\dagger - 2a^\dagger \rho_S a)
\]

(17)

under the squeezing operator \( S(\epsilon) \) with some \( \epsilon = |\epsilon| e^{i \phi} \).

Next we apply the method in [1], [2] to the equation (17) to make the algebraic structure clearer. Before that let us make some mathematical preparations.

A matrix representation of \( a \) and \( a^\dagger \) on the usual Fock space

\[
\mathcal{F} = \text{Vect}_C \{|0\}, |1\}, |2\}, |3\}, \cdots \}; \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle
\]

is given by
where $e^{i\theta}$ is some phase. Note that $aa^\dagger = a^\dagger a + 1 = N + 1$.

For a matrix $X = (x_{ij}) \in M(\mathcal{F})$

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we correspond to the vector $\hat{X} \in \mathcal{F}^{\dim \mathcal{F}}$ as

$$X = (x_{ij}) \rightarrow \hat{X} = (x_{11}, x_{12}, x_{13}, \cdots ; x_{21}, x_{22}, x_{23}, \cdots ; x_{31}, x_{32}, x_{33}, \cdots ; \cdots)^T$$

where $T$ means the transpose. The following formula

$$\widehat{AXB} = (A \otimes B^T)\hat{X}$$

holds for $A, B, X \in M(\mathcal{F})$.

If we set

$$K_3 = \frac{1}{2}(N \otimes 1 - 1 \otimes N), \quad K_+ = \frac{1}{2}\{(a^\dagger)^2 \otimes 1 - 1 \otimes ((a^\dagger)^2)^T\}, \quad K_- = \frac{1}{2}\{a^2 \otimes 1 - 1 \otimes (a^2)^T\}$$

(22)
where \( N^T = N \), then we have
\[
[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.
\] (23)

If we also set
\[
\tilde{K}_3 = \frac{1}{2}(N \otimes 1 + 1 \otimes N + 1 \otimes 1), \quad \tilde{K}_+ = a^\dagger \otimes a^T, \quad \tilde{K}_- = a \otimes (a^\dagger)^T,
\] (24)
we have
\[
[\tilde{K}_3, \tilde{K}_+] = \tilde{K}_+, \quad [\tilde{K}_3, \tilde{K}_-] = -\tilde{K}_-, \quad [\tilde{K}_+, \tilde{K}_-] = -2\tilde{K}_3.
\] (25)
Namely, \( \{K_3, K_+, K_-\} \) and \( \{\tilde{K}_3, \tilde{K}_+, \tilde{K}_-\} \) are a set of generators of \( su(1, 1) \) algebra.

Now the equation (17) can be written as
\[
\frac{\partial}{\partial t} \hat{\rho}_S = \left\{ -2i\omega(c^2 + s^2)K_3 + 2i\omega e^{i\phi}csK_+ + 2i\omega e^{-i\phi}csK_- + \frac{\mu - \nu}{2} 1 \otimes 1 - ((\mu + \nu)(c^2 + s^2) - 4kcs) \tilde{K}_3 + (\mu s^2 + \nu c^2 - 2kcs) \tilde{K}_+ + (\mu c^2 + \nu s^2 - 2kcs) \tilde{K}_- \right\} \hat{\rho}_S
\equiv \left( \frac{\mu - \nu}{2} 1 \otimes 1 + A + \tilde{A} \right) \hat{\rho}_S ;
\] (26)
\[
A \equiv -2i\omega(c^2 + s^2)K_3 + 2i\omega e^{i\phi}csK_+ + 2i\omega e^{-i\phi}csK_-,
\] (27)
\[
\tilde{A} \equiv -((\mu + \nu)(c^2 + s^2) - 4kcs) \tilde{K}_3 + (\mu s^2 + \nu c^2 - 2kcs) \tilde{K}_+ + (\mu c^2 + \nu s^2 - 2kcs) \tilde{K}_- \] (28)
by use of (21) and generators (22), (24). The solution is formally given by
\[
\hat{\rho}_S(t) = e^{\frac{\mu - \nu}{2} t} e^{t(A + \tilde{A})} \hat{\rho}_S(0).
\] (29)

Now let us calculate the commutators of \( \{K_3, K_+, K_-\} \) and \( \{\tilde{K}_3, \tilde{K}_+, \tilde{K}_-\} \) :
\[
[K_3, \tilde{K}_+] = [K_3, \tilde{K}_-] = [K_3, \tilde{K}_3] = 0
\]
and
\[
[K_+, \tilde{K}_+] = -a^\dagger \otimes (a^\dagger)^T, \quad [K_+, \tilde{K}_-] = -a^\dagger \otimes (a^\dagger)^T, \quad [K_+, \tilde{K}_3] = -\frac{1}{2} \{(a^\dagger)^2 \otimes 1 + 1 \otimes ((a^\dagger)^2)^T\},
\]
\[
[K_-, \tilde{K}_+] = a \otimes a^T, \quad [K_-, \tilde{K}_-] = a \otimes a^T, \quad [K_-, \tilde{K}_3] = \frac{1}{2} \{a^2 \otimes 1 + 1 \otimes (a^2)^T\}.
\]
Since \( \{K_3, K_+, K_-\} \) and \( \{\tilde{K}_3, \tilde{K}_+, \tilde{K}_-\} \) don’t commute it is reasonable to assume
\[
\hat{\rho}_S(t) \approx e^{\frac{\mu - \nu}{2} t} e^{tA} e^{t\tilde{A}} \hat{\rho}_S(0)
\] (30)
as the first approximation.

The disentangling formula (which is well–known) for the system \( \{L_3, L_+, L_-\} \) based on the Lie algebra \( su(1,1) \) is given by

\[
e^{t(2aL_3+bL_++cL_-)} = e^{G(t)L_+} e^{-2\log(F(t))L_3} e^{E(t)L_-}
\]  

(31)

with

\[
G(t) = \frac{b}{\sqrt{a^2 - bc}} \sinh \left(t \sqrt{a^2 - bc}\right) \\
F(t) = \cosh \left(t \sqrt{a^2 - bc}\right) - \frac{a}{\sqrt{a^2 - bc}} \sinh \left(t \sqrt{a^2 - bc}\right),
\]

\[
E(t) = \frac{c}{\sqrt{a^2 - bc}} \sinh \left(t \sqrt{a^2 - bc}\right) \\
\]

(32)

(33)

See for example [9], [10].

From this formula we can calculate \( e^{tA} \) and \( e^{\tilde{t}A} \) in (30) as follows.

\[
e^{tA} = e^{G(t)K_+} e^{-2\log(F(t))K_3} e^{E(t)K_-}
\]

with

\[
G(t) = \frac{b}{\sqrt{a^2 - bc}} \sinh \left(t \sqrt{a^2 - bc}\right) \\
F(t) = \cosh \left(t \sqrt{a^2 - bc}\right) - \frac{a}{\sqrt{a^2 - bc}} \sinh \left(t \sqrt{a^2 - bc}\right),
\]

\[
E(t) = \frac{c}{\sqrt{a^2 - bc}} \sinh \left(t \sqrt{a^2 - bc}\right)
\]

(34)

and

\[
a = -i\omega(c^2 + s^2), \quad b = 2i\omega e^{i\phi cs}, \quad c = 2i\omega e^{-i\phi cs}.
\]

(35)

Then it is easy to see \( \sqrt{a^2 - bc} = i\omega \) and

\[
G(t) = \frac{2ie^{i\phi cs} \sin(t\omega)}{\cos(t\omega) + i(c^2 + s^2) \sin(t\omega)}, \quad F(t) = \cos(t\omega) + i(c^2 + s^2) \sin(t\omega),
\]

\[
E(t) = \frac{2ie^{-i\phi cs} \sin(t\omega)}{\cos(t\omega) + i(c^2 + s^2) \sin(t\omega)}
\]

(36)

Next,

\[
e^{\tilde{t}A} = e^{\tilde{G}(t)\tilde{K}_+} e^{-2\log(\tilde{F}(t))\tilde{K}_3} e^{\tilde{E}(t)\tilde{K}_-}
\]

(37)
with

$$\tilde{G}(t) = \frac{b}{\sqrt{a^2 - bc}} \sinh \left( t \sqrt{a^2 - bc} \right) - \frac{a}{\sqrt{a^2 - bc}} \sinh \left( t \sqrt{a^2 - bc} \right),$$

$$\tilde{F}(t) = \cosh \left( t \sqrt{a^2 - bc} \right) - \frac{a}{\sqrt{a^2 - bc}} \sinh \left( t \sqrt{a^2 - bc} \right),$$

$$\tilde{E}(t) = \frac{c}{\sqrt{a^2 - bc}} \sinh \left( t \sqrt{a^2 - bc} \right) - \frac{a}{\sqrt{a^2 - bc}} \sinh \left( t \sqrt{a^2 - bc} \right)$$

and

$$a = -\frac{(\mu + \nu)(c^2 + s^2) - 4kcs}{2}, \quad b = \mu s^2 + \nu c^2 - 2kcs, \quad c = \mu c^2 + \nu s^2 - 2kcs. \quad (39)$$

Then it is not difficult to see

$$\sqrt{a^2 - bc} = \frac{\mu - \nu}{2}.$$

However, by use of it we cannot simplify (38) like (36).

Therefore our approximate solution (30) becomes

$$\hat{\rho}_S(t) \approx e^{\frac{\mu + \nu}{2} t} e^{G(t)K_+ e^{-2\log(F(t))K_+ e^{G(t)} K_- e^{G(t)} K_+ e^{-2\log(F(t))K_+ e^{G(t)} K_-}} \hat{\rho}_S(0) \quad (40)$$

and we restore this form to the usual one by use of (21). The result is

$$\rho_S(t) \approx e^{\frac{\mu + \nu}{2} t} \exp \left( \frac{G(t)}{2} (a^\dagger)^2 \right) \{ \exp ( - \log(F(t)) N ) \times$$

$$\left\{ \exp \left( \frac{E(t)}{2} a^2 \right) \phi(t) \exp \left( - \frac{E(t)}{2} a^2 \right) \right\} \exp ( \log(F(t)) N ) \} \exp \left( - \frac{G(t)}{2} (a^\dagger)^2 \right) \quad (41)$$

and

$$\phi(t) = \frac{1}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \left\{ \exp \left( - \log(F(t)) N \right) \times$$

$$\left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho_S(0) (a^\dagger)^m \right\} \exp \left( - \log(F(t)) N \right) \} a^n. \quad (42)$$

This is indeed complicated. Compare this with the corresponding one in [1].

In this paper we revisited the quantum damped harmonic oscillator with generalized Lindblad form and applied the unitary transformation by the squeezing operator to the master
equation, and examined the new algebraic structure and next constructed some approximate solution in the operator algebra level.

The model is very important to understand several phenomena related to quantum open systems, so the general solution is indeed required. To obtain it (like in \[2\]) is almost impossible at the present time.

Lastly, we conclude the paper by stating our motivation. We are studying a model of quantum computation (computer) based on Cavity QED (see \[11\] and \[12\]), so in order to construct a more realistic model of (robust) quantum computer we have to study severe problems coming from decoherence.

For example, we have to study the quantum damped Jaynes–Cummings model (in our terminology) whose phenomenological master equation for the density operator is given by

\[
\frac{\partial}{\partial t} \rho = -i[H_{JC}, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (aa^\dagger \rho + \rho aa^\dagger - 2a^\dagger \rho a),
\]  

(43)

where \(H_{JC}\) is the well–known Jaynes-Cummings Hamiltonian given by

\[
H_{JC} = \frac{\omega_0}{2} \sigma_3 \otimes 1 + \omega_0 1_2 \otimes a^\dagger a + \Omega (\sigma_+ \otimes a + \sigma_- \otimes a^\dagger)
\]  

\[
= \begin{pmatrix}
\frac{\omega_0}{2} + \omega_0 N & \Omega a \\
\Omega a^\dagger & -\frac{\omega_0}{2} + \omega_0 N
\end{pmatrix}
\]

(44)

with

\[
\sigma_+ = \begin{pmatrix}0 & 1 \\0 & 0\end{pmatrix}, \quad \sigma_- = \begin{pmatrix}0 & 0 \\0 & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0 \\0 & -1\end{pmatrix}, \quad 1_2 = \begin{pmatrix}1 & 0 \\0 & 1\end{pmatrix}.
\]

Note that \(\rho \in M(2; \mathbb{C}) \otimes M(\mathcal{F}) = M(2; M(\mathcal{F}))\), where \(M(\mathcal{F})\) is the set of all operators on the Fock space \(\mathcal{F}\). See for example \[13\], \[14\].

Furthermore, it may be possible to treat the generalized master equation given by

\[
\frac{\partial}{\partial t} \rho = -i[H_{JC}, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger) - \frac{\nu}{2} (aa^\dagger \rho + \rho aa^\dagger - 2a^\dagger \rho a)
\]

\[-\frac{\kappa}{2} (a^2 \rho + \rho a^2 - 2a \rho a) - \frac{\overline{\kappa}}{2} ((a^\dagger)^2 \rho + \rho (a^\dagger)^2 - 2a^\dagger \rho a^\dagger),
\]

(45)

with the condition \(\mu \nu \geq |\kappa|^2\) similarly in this paper.

These equations (\[13\], \[15\]) are very hard to solve in the operator algebra level, so even constructing approximate solutions is not easy. This is our future task.
References

[1] K. Fujii : Algebraic Structure of a Master Equation with Generalized Lindblad Form, arXiv : 0802.3252 [quant-ph].

[2] R. Endo, K. Fujii and T. Suzuki : General Solution of the Quantum Damped Harmonic Oscillator, to appear in Int. J. Geom. Meth. Mod. Phys, arXiv : 0710.2724 [quant-ph].

[3] K. Fujii : An Approximate Solution of the Master Equation with the Dissipator being a Set of Projectors, to appear in Yokohama Math. J, arXiv : 0708.4047 [quant-ph].

[4] H. -P. Breuer and F. Petruccione : The theory of open quantum systems, Oxford University Press, New York, 2002.

[5] W. P. Schleich : Quantum Optics in Phase Space, WILEY–VCH, Berlin, 2001.

[6] R. Alicki, F. Benatti and R. Floreanini : Charge Oscillations in Superconducting Nanodevices Coupled to External Environments, arXiv : 0711.0812 [quant-ph].

[7] J-H An, S-J Wang and H-G Luo : Entanglement production and decoherence-free subspace of two single-mode cavities embedded in a common environment, J. Phys. A: Math. Gen 38 (2005), 3579, quant-ph/0505087.

[8] J-H An, S-J Wang, H-G Luo and C-L Jia : Production of squeezed state of single mode cavity field by the coupling of squeezed vacuum field reservoir in nonautonomous case, Chin. Phys. Lett. 21 (2004), 1, quant-ph/0505078.

[9] K. Fujii : Introduction to Coherent States and Quantum Information Theory, quant-ph/0112090.

[10] K. Fujii : Matrix Elements of Generalized Coherent Operators, Yokohama Math. J, 53 (2007), 101, quant-ph/0202081.
[11] K. Fujii, K. Higashida, R. Kato and Y. Wada: Cavity QED and Quantum Computation in the Weak Coupling Regime, J. Opt. B: Quantum and Semiclass. Opt, 6 (2004), 502, quant-ph/0407014.

[12] K. Fujii, K. Higashida, R. Kato and Y. Wada: Cavity QED and Quantum Computation in the Weak Coupling Regime II: Complete Construction of the Controlled–Controlled NOT Gate, Trends in Quantum Computing Research, Susan Shannon (Ed.), Chapter 8, Nova Science Publishers, 2006 and Computer Science and Quantum Computing, James E. Stones (Ed.), Chapter 1, Nova Science Publishers, 2007, quant-ph/0501046.

[13] M. Scala, B. Militello, A. Messina, S. Maniscalco, J. Piilo and K.-A. Suominen: Cavity losses for the dissipative Jaynes-Cummings Hamiltonian beyond Rotating Wave Approximation, J. Phys. A: Math. Theor. 40 (2007), 14527, arXiv:0709.1614 [quant-ph].

[14] M. Scala, B. Militello, A. Messina, S. Maniscalco, J. Piilo and K.-A. Suominen: Population trapping due to cavity losses, arXiv:0710.3701 [quant-ph].