Quantum theory of ultracold atom-ion collisions

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We study atom-ion scattering in the ultracold regime. To this aim, an analytical model based on the multichannel quantum defect formalism is developed and compared to close-coupled numerical calculations. We investigate the occurrence of magnetic Feshbach resonances focusing on the specific $^{40}\text{Ca}^{+}+\text{Na}$ system. The presence of several resonances at experimentally accessible magnetic fields should allow the atom-ion interaction to be precisely tuned. A fully quantum-mechanical study of charge exchange processes shows that charge-exchange rates should remain small even in the presence of resonance effects. Most of our results can be cast in a system-independent form and are important for the realization of the charge-neutral ultracold systems.

Advances in trapping, cooling, manipulation and read-out of single atoms and ions have led over recent years to a range of fundamental as well as applied investigations on the quantum properties of such systems. Nowadays, an increasing number of experimental groups worldwide are starting experiments with combined charged-neutral systems in various configurations [1]. While the theory of atom-ion collisions is well established for high collision energies [2, 3], a theoretical description in the ultracold domain is still largely missing.

This letter presents the first study of magnetic Feshbach resonances and the first fully quantum study of the radiative charge exchange process for ultracold atom-ion systems that includes effects of Feshbach and shape resonances. Here we consider only two-body collisions in free space, a necessary prelude to further studies incorporating effects of ion micromotion or trap confinement. We develop a reliable yet manageable effective model of atom-ion collisions by applying multichannel quantum defect theory (MQDT) [4, 5, 6] based on the long range ion-induced-dipole potential that varies as $r^{-4}$ at large ion-atom distance $r$. This powerful tool has proven effective as a few-parameter approach for describing scattering and bound states in electron-ion core [4], electron-atom [4] and neutral atom systems [10]. Although the literature on the subject is rich, here we discuss some details of MQDT illustrating how it works in the ultracold domain, so we can reveal the new and interesting ultracold ion-atom physics. We adapt MQDT to the atom-ion realm, utilizing the analytical solutions for the $r^{-4}$ asymptotic potential [4] and applying the frame transformation [10, 12] at short distances to reduce the number of quantum defect parameters in the model. We verify the model predictions by comparing to our own numerical close-coupled calculations, taking $^{40}\text{Ca}^{+}-^{23}\text{Na}$ as a reference system.

We describe the $S$-state atom and $S$-state ion collisions with the close-coupled radial Schrödinger equation

\begin{equation}
\frac{\partial^2 \mathbf{F}}{\partial r^2} + \frac{2\mu}{\hbar^2} [E - \mathbf{W}(r)] \mathbf{F}(r) = 0. \tag{1}
\end{equation}

Here, $\mu = m_i m_a / (m_i + m_a)$ denotes the reduced mass, $\mathbf{W}(r)$ is the interaction matrix, and $\mathbf{F}(r)$ is the matrix of radial solutions. The wave function for $N$ scattering channels reads $\Psi_i(r) = \sum_{j=1}^{N} A_j Y_j(r) F_{ij}(r)/r$ where $Y_j(r)$ denotes the angular part of the solution for the scattering channel $j$, and the constant vector $\mathbf{A}$ is determined by the boundary conditions at $r \to \infty$. The asymptotic channel states can be characterized by the hyperfine quantum numbers $f_1, m_{f_1}$ and $f_2, m_{f_2}$ for ion and atom respectively, and by the angular-momentum quantum numbers $l$ and $m_l$ of the relative motion of the atom and ion centers of mass. We denote the asymptotic channel states by $|\Psi_i(B)\rangle$, where $\alpha = \{f_1 f_2 m_{f_1} m_{f_2} l m_l\}$. In the presence of a magnetic field $B$, the field-dressed channel states $|\Psi_i(B)\rangle$ are linear combinations of the bare $(B = 0)$ channel states $|\Psi_i(B)\rangle = \sum_{\alpha'} Z_{\alpha\alpha'}(B)|\Psi_i\rangle$. In the asymptotic channel basis the interaction matrix is diagonal at large distances

\begin{equation}
W_{ij}(r) \to \infty \left[ E_{i}^{\infty} + \frac{\hbar^2 (l+1)}{2\mu r^2} - \frac{C_4}{r^4} \right] \delta_{ij} + O(r^{-6}), \tag{2}
\end{equation}

where $E_{i}^{\infty}$ are the threshold energies for the channel $i$ including the hyperfine energies and Zeeman shifts, and $C_4 = \alpha e^2 / 2$ with $\alpha$ denoting the static dipolar polarizability of the atom and $e$ the ion charge. At typical distances $R_0$ where the short-range exchange interaction takes place, the interaction matrix becomes diagonal in the $IS$ representation, characterized by total electron spin $S = s_1 + s_2$, the total nuclear spin $I = i_1 + i_2$, the total hyperfine angular momentum $F = f_1 + f_2 = I + S$, and its projection $M_F$ on the axis of quantization, where $s_1, s_2$ are electron spin of ion and atom respectively, and $i_1, i_2$ denote their nuclear spins, respectively. We label those channels by $\beta = \{ISFM_F l m_l\}$.

With the long-range atom-ion one can associate the length scale $R^* = \sqrt{2C_4 \mu / \hbar^2}$ and energy scale $E^* = \hbar^2 / [2 \mu (R^*)^2]$. The length $R^*$ can be also related to the position of the last node of the zero-energy s-wave radial wave function for the scattering length $a = 0$: $L = 2/\pi R^*$. The maxima due to the centrifugal bar-
rrier occur at \( r_{\text{max}} = \sqrt{2/\sqrt{l(l+1)}} R^* \) and have heights of \( E_{\text{max}} = \frac{1}{4} l^2 (l+1)^2 E^* \). Hence \( E^* \) determines the contribution of higher partial waves in the atom-ion scattering.

**MQDT.** The basic idea of MQDT \cite{4,5,6} is the separation between the short-range and the long-range properties of the scattering wave function. The short-range wave function is insensitive to the total energy \( E \), and in the case of atom-ion collisions also to the relative orbital angular momentum \( \ell \). On the contrary, at large distances the wave function vary rapidly with \( E \), as well as other scattering quantities defined at \( r \to \infty \). By introducing a set of short-range quantum-defect parameters, MQDT allows one to determine all scattering and bound-state properties in a wide range of collision energies.

In MQDT one starts by choosing a set of reference potentials \( \{ V_j (r) \} \), that reproduce the asymptotic behavior of the interaction matrix at large distances: \( V_j (r) \to W_{jj} (r) \) , but otherwise can be arbitrary. With the reference potentials \( V_i (r) \) one can associate a pair of linearly independent solutions \( \{ \tilde{f}_i (r) \} \) and \( \{ \tilde{g}_i (r) \} \)

\[
\tilde{f}_i (r) = \alpha_i (r) \sin \beta_i (r), \quad \tilde{g}_i (r) = \alpha_i (r) \cos \beta_i (r). \]

The amplitude \( \alpha_i (r) \) fulfills the inhomogeneous Milne equation: \( [d^2/dr^2 + k_i (r)^2] \alpha (r) = \alpha^{-1} (r) Y_j (E) \| A \), with the local wavevector \( k_i (r) = \sqrt{2 \mu (E - V_i (r))/\hbar} \), while the phase \( d \beta_i /dr = 1/\alpha_i^2 \). We impose WKB-like boundary conditions at small distances: \( \alpha_i (r) \approx 1/\sqrt{k_i (r)} \), \( \alpha_i' (r) \approx 0 \), which makes the functions \( \tilde{f}_i (r) \) and \( \tilde{g}_i (r) \) weakly dependent on energy. The exact solution to Eq. (1) at large distances (where \( V_j (r) \to W_{jj} (r) \)) can be expressed in terms of \( \{ \tilde{f}_i (r) \} \) and \( \{ \tilde{g}_i (r) \} \) . $F (r) \to \diag [\tilde{f}_i (r)], \tilde{g}_i (r) = \diag [\tilde{g}_i (r)]$, and $Y (E)$ is the energy insensitive (\( Y (E) \approx \delta_y \)) quantum-defect matrix that plays a central role in the quantum-defect analysis.

**Analytical solutions.** The Schrödinger equation with the polarization potential \( C_A / r^4 \) can be solved analytically, in terms of Mathieu functions of imaginary argument \( \nu = \frac{\pi}{2} \ell \). For our choice of MQDT reference potentials: \( V_i (r) = C_A / r^4 + \ell (l+1)/r^2 + E_i^\infty \), the solutions:

\[
\tilde{f} (r) \approx r \sin (- R^* r + \varphi), \quad \tilde{g} (r) \approx r \cos (- R^* r + \varphi), \quad (r \to 0)
\]

Here, \( \varphi \) denotes some short-range phase, that can be related to the \( s \)-wave scattering length \( a = - R^* \cot \varphi \) of the \( C_A / r^4 \) potential \( \| A \). In view of the arbitrariness of the reference potential, we set \( \varphi = 0 \), which does not affect the physical scattering matrices, that are not sensitive to the short-range behavior of the reference potentials.

**Frame transformation.** The frame transformation (FT) is a unitary transformation between channels \( \alpha \) and \( \beta \): \( U_{\alpha \beta} = \{ f_{j\beta} | m_{f\beta} | f_{j\alpha} \} \| ISFMF \), that can be expressed in terms of the usual Clebsch-Gordan and Wigner 9j symbols (see e.g. \cite{10}). The application of FT requires \( R_0 \ll R^* \), which is typically very well fulfilled in the atom-ion collisions. At distances \( r \gg R_0 \) the exchange interaction is no longer present, and for \( r \ll R^* \) we can safely neglect both the centrifugal potential and hyperfine splittings, and also ignore the higher order dispersion terms: \( C_6 / r^6, C_8 / r^8 \) that lead only to small corrections (see below). Then for \( R_0 \ll R^* \), \( W_{jj} (r) \approx \delta_j C_A / r^4 \), and the wave function in each of the channel is simply given by the linear combination of the two short-range solutions \( \| A \) and \( \| B \), both in the asymptotic and (\( IS \)) representations.

In view of the assumed form of the reference potentials, and the choice \( \varphi = 0 \), the quantum-defect matrix in \( IS \) representation takes the form \( Y_{\beta \beta'}^{(IS)} = \delta_{\beta \beta'} | a_S (\beta) |^{-1} \), where \( S (\beta) \) is the total electron spin in the channel \( \beta \), that takes values \( S (\beta) = 0, 1 \) corresponding to the singlet \( a_s \) and triplet \( a_t \) scattering lengths, respectively. Then, $Y$ in the basis of the asymptotic channel states reads $Y = Z (B) Y^{(IS)} U^{(IS)} Z' (B)$ where $Z (B)$ is the transformation from bare to the magnetic-field-dressed states. Finally, the \( S \) and \( K \) matrices describing all the scattering properties including the cross-sections, are calculated directly from $Y$ with the help of the quantum-defect functions \( \| A \), \( \| B \), that are determined by the asymptotic behavior of the short-range normalized solutions \( \tilde{f}, \tilde{g} \).

**Results.** We now apply our model to a system of $^{40}$Ca$^+$ and $^{23}$Na for which the long-range parameter \( C_A = 81.35 \) a.u. and the short-range \( ab-initio \) potential curves are ap-

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**FIG. 1:** (Color online) Elastic collision rate in the singlet channel $A^1 \Sigma^+ \to X^1 \Sigma^+$ charge transfer (lower panel) versus collision energy. Different theoretical approaches are compared (see text). Both the elastic and the total inelastic rate are decomposed on selected partial waves.
proximately known \[13\]. The close-coupled Schrödinger equation \([11]\) is solved both within our MQDT approach and exactly using standard numerical methods.

Unfortunately, potentials calculated by ab-initio methods are usually not sufficiently accurate to predict the value of the associated scattering lengths. Therefore, for our calculation we will either assume typical magnitudes of \(a_s\) and \(a_l\) on the order of \(R^*\) or we vary them. It is interesting to note that for a \(r^{-4}\) potential the probabilities to find a positive or a negative scattering length are equal in contrast to van der Waals \(r^{-6}\) interaction where their ratio is 3:4 \[18\].

The \(^{40}\)Ca ion has vanishing nuclear spin \((i_1 = 0)\), whereas the \(^{23}\)Na atom has \(i_2 = 3/2\), resulting in a total hyperfine angular momentum \(I = 2\). Neglecting small anisotropic spin interactions, rotational invariance implies that \(l, m_l, \) and \(M_F\) are conserved quantities. In our calculations we will consider collisions in the \(M_F = 1/2\) block, and use the FT to obtain \(v_l\) in the asymptotic channel representation. At \(B = 0\) the off-diagonal elements of \(v_l\) and \(\sigma_l\) are proportional to \(1/a_s = 1/a_s - 1/a_l\), where \(a_s\) is an effective scattering length characterizing the strength of channel coupling.

We begin our analysis by considering a single \(A^1\Sigma^+\) potential as entrance channel. The upper panel of Fig. 1 shows the rate \(v\sigma_l\) of elastic \(Ca^+ + Na\) collisions as a function of \(r\) for different partial waves, with \(v\) the relative atomic velocity and \(\sigma_l\) the partial wave cross section. The calculation is performed using a potential parameterized by a scattering length of \(a_s = -0.961R^* = -2000a_0\). One can observe several dips of the collisional rates, corresponding to Ramsauer minima. We note that because of the relatively low centrifugal barriers, partial waves with \(l > 0\) give a significant contribution already in the \(\mu K\) regime. At small energies the rates for \(l > 0\) behave asymptotically as \(\sigma_l v \sim E^{3/2}\), in agreement with the threshold behavior typical of \(r^{-4}\) potentials \[19\]. For \(l = 4\) and at higher energies some discrepancies between the numerical and MQDT approach can be observed, resulting from breakdown of the assumption of angular-momentum insensitivity of the quantum-defect matrix in the analytical model. On the basis of the MQDT model, one can argue that the elastic collisional rates in the triplet channels, would be of the similar order, with the particular structure of maximum/Ramsauer minima determined by the value of \(a_l\).

The lower panel of Fig. 1 shows the total rate \(v\sigma_l\) of inelastic \(A^1\Sigma^+ \rightarrow X^1\Sigma^+\) collisions associated with radiative charge transfer (see \[13\] for details). The numerical results (first three entries in the legend) have been obtained in the distorted-wave Born approximation by summing contributions from all free-free and free-bound transitions \[20\]. Note that molecular ions are formed with very high probability (\(\sim 96\%\)) in the electron exchange process. The figure also shows the total charge-exchange rate obtained by a computationally much simpler single channel calculation with a position-dependent Einstein coefficient \(A(r)\) (see e.g. \[21\]). In addition we show the result of our analytical model where the charge-transfer probability has been described in the semiclassical theory, while MQDT provides a proper rescaling of the wave function between reaction and asymptotic zones. One can observe that the three numerical and analytical approaches give virtually the same result for the total (free-free plus free-bound) charge-exchange rate at low energy. It is only at higher energies, in particular near the peaks associated to shape resonances, that the MQDT model becomes less accurate. This can again result from the modification of the quantum-defect matrix \(Y\) for higher \(l\).

In contrast to the purely semiclassical result of Ref. \[13\], both elastic and inelastic quantum cross sections appear to be strongly affected by the presence of shape resonances up to \(mK\) regime; see Fig. 1. We have verified that this is a general behavior not related to the particular choice of the singlet scattering length assumed for the calculation. The influence of shape resonances can also be observed in the thermally averaged rate which only begins to saturate to the asymptotic value in the \(mK\) regime.

Next, we take into account multichannel hyperfine effects and the presence of an external magnetic field. Fig. 2 shows the variation of the effective scattering length \(a\) versus magnetic field for sample values of the sin-
The range of magnetic fields in which the first Feshbach resonances should occur can be estimated from the energy location \( E^\infty \) of the associated bound states for \( a = \infty \) and \( B = 0 \). For a pure \( r^{-4} \) potential such energies are \( E^\infty/E^\ast = 105.8, 1180, 5208, \ldots \) Letting \( \Delta \mu \) be the difference of magnetic moments between the Feshbach molecule and free atoms and assuming only linear Zeeman shift for small values of \( B \), we expect the first resonances to be found at \( B < B_{\text{max}} = E^\infty/\Delta \mu \), with \( B_{\text{max}}[\text{mT}] = 0.086, 0.962, 4.26 \) for Na-Ca\(^{+}\) system.

In order to investigate the dependence of the position and strength of Feshbach resonances on the interatomic potentials, we show in Fig. 3 the variation of the quantity \( 2\arctan(a/R)/\pi \) with the singlet and triplet short-range phases \( \phi_{s,t} = -\arctan 1/a_t \) and magnetic field \( B \). Please note that typically several resonances should be observable below \( B \sim 20 \text{ mT} \). One can remark that the resonances are usually relatively broad and only become very narrow when \( a_s \) and \( a_t \) are comparable, and the interchannel coupling \( 1/a_t \) is small. Apart from the vicinity of resonances the field dependent scattering length is mainly determined by the value of \( a_t \), that again is in turn related to the structure of the \( Y \) matrix.

In conclusion, a relatively simple quantum-defect model accurately describes atom-ion collisions in the ultracold domain, as verified by comparison with numerical coupled-channel calculations for the reference system \( ^{40}\text{Ca}^{+} \) and \( ^{23}\text{Na} \). Our calculations predict that several magnetic Feshbach resonances should be available at relatively small values of magnetic fields to control the atom-ion interaction. Radiative charge exchange rates remain relatively small even in the presence of Feshbach and shape resonances so that elastic collisions can be controlled without introducing unwanted losses.

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[1] A. T. Grier, M. Cetina, F. Oručević, V. Vuletić, arXiv:0808.3620
[2] B. H. Bransden and M. R. C. McDowell, Charge Exchange and the Theory of Ion-Atom Collisions (Oxford University Press, Oxford, 1992).
[3] J. B. Delos, Rev. Mod. Phys. 53, 287(1981).
[4] M.J. Seaton, Rep. Prog. Phys. 46, 167 (1983).
[5] Ch.H. Greene, A.R.P. Rau and U. Fano, Phys. Rev. A 26, 2441 (1982).
[6] F.H. Mies, J. Chem. Phys. 80, 2514 (1984).
[7] M.J. Seaton, L. Steenman-Clark, J. Phys. B 10, 2639 (1977); R.J. Drachman, ibid 12, L699 (1979).
[8] Apart from neutral-charged systems, the long-range \( r^{-4} \) potential describes effective interactions of polar molecules in a strong electric field: A. V. Avdeenkov and J. L. Bohn, Phys. Rev. A 66, 052718 (2002).
[9] S. Watanabe and Ch. H. Greene, Phys. Rev. A 22, 1190 (1954).
[10] B. Gao, E. Tiesinga, C.J. Williams, and P.S. Julienne, Phys. Rev. A 72, 042719 (2005).
[11] E. Vogt, and G.H. Wannier, Phys Rev. 95, 1190 (1954).
[12] J. P. Burke, Jr., Ch.H. Greene, and J.L. Bohn, Phys. Rev. Lett. 81, 3355 (1998).
[13] O.P. Makarov, R. Côté, H. Michels, and W.W. Smith,
Phys. Rev. A\textbf{67}, 042705 (2003).

[14] B. Gao, Phys. Rev. A \textbf{64}, 010701(R) (2001).

[15] W.E. Milne, Phys. Rev. \textbf{35}, 863 (1930).

[16] R.M. Spector, J. Math. Phys. \textbf{5}, 1185 (1964).

[17] Z. Idziaszek, T. Calarco, and P. Zoller, Phys. Rev. A\textbf{76}, 033409 (2007).

[18] G.F. Gribakin and V.V. Flambaum, Phys. Rev. A \textbf{48}, 546 (1993).

[19] In the numerical calculation of elastic and charge-exchange rates we neglect higher order dispersion terms $C_6$ and $C_8$ that give rise to negligible corrections.

[20] P.S. Julienne and F.H. Mies, Phys. Rev. A \textbf{34}, 3792 (1986).

[21] P.S. Julienne, J. Chem. Phys \textbf{68}, 32 (1978).