On the asymptotic stability of solutions of stochastic differential delay equations of second order

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ABSTRACT
In this paper, we consider a non-linear stochastic differential delay equation (SDDE) of second order. We derive new sufficient conditions which guarantee stochastically stability and stochastically asymptotically stability of the zero solution of that SDDE. Here, the technique of the proof is based on the definition of a suitable Lyapunov-Krasovskiifunctional, which gives meaningful results for the problem under consideration. The derived results extend and improve some result of in the relevant literature, which are related to the qualitative properties of solutions of a SDDE of second order. The results of this paper are new and have novelty, and they do a contribution to the topic and relevant literature. As an application, an example is given to show the effectiveness and applicability of the obtained results. Finally, by the results of this paper, we extend and improve some recent results that can be found in the relevant literature.

1. Introduction
Since then 1950s the qualitative properties of solutions, such as stability analysis, convergence analysis, asymptotic analysis, chaotic behaviour, oscillation, globally existence of solutions, existence of periodic solutions and so on, for linear and non-linear delay differential equations (DDEs) of second order have been extensively investigated since such delay differential equations and in addition ordinary differential equations have been successfully applied in many fields such as physical, biological, control theory, engineering, medical, social sciences, economics, finance and so on. As a very particular and limited information, see [1–58], and the references therein.

In fact, when DDEs are subject to environmental disturbances, they can be characterized by SDDEs, see [21]. The basic theory of SDDEs has been systematically established in [18] and there are many interesting results on the qualitative properties of solutions of SDDEs in the literature, see, for example, [11–25]. We would not like to present here more details about the extensive literature on the subject.

One of the important issues in the study of qualitative properties of solutions of SDDEs is placed on the stability analysis of solutions. Over past few decades, many excellent and interesting works on the stochastic stability of solutions of SDDEs have been developed in [11–25,55,57,58] and the references therein. In addition, in the relevant literature, some fundamental methods or theorems such as the fixed point theorems, the Razumikhin-type theorem, LaSalle-type theorem, the variation of parameters formula, the perturbation theory, the comparison principle, the direct method of Lyapunov, the Lyapunov-Krasovskii functional approach and so on are used to analyse the stability of DDEs and SDDEs, see [1–55] and the references therein.

Despite the existence of a lot of papers on the qualitative analysis of solutions of DDEs second order and SDDEs of first order, to the best of our information, we observe only a few papers from the literature on the qualitative analysis of SDDEs of second order, see [1–4,23,27]. In these works, based on the suitable Lyapunov-Krasovskii functionals, the qualitative analysis of solutions have been proceeded in that papers.

The proper reason for so less the existence of a few papers may be the difficulty of finding suitable Lyapunov- functionals and the difficulty the topic.

In 2017, Abou-El-Ela et al. [2] considered the following SDDE of second order with a constant time lag, $r$:

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)f(x(t - r)) + g(t,x)\dot{\omega}(t) = 0. \quad (1)$$

Abou-El-Ela et al. established sufficient conditions for the stochastically asymptotically stability of the zero
solution of the SDDE (1). In [2], the authors defined a suitable auxiliary functional, Lyapunov-Krasovskii functional, to prove the stability result, and they also introduced two examples to show the effectiveness and applicability of the obtained result.

In this paper, motivated by the work of Abou-El-Ela et al. [2] and SDDE (1), we deal with the following SDDE of the second order with multiple constant retardations \( \tau_j \) (\( j = 1, 2, \ldots, m \)):

\[
\frac{d^2x}{dt^2} + a(t)f(x, \frac{dx}{dt}) + b_0(t)g_0(x) + \sum_{j=1}^{m} b_j(t)g_j(x(t - \tau_j)) + g(t, x)\dot{\omega}(t) = 0
\]  

(2)

where \( \tau_j \) (\( j = 0, 1, 2, \ldots, m \)), are fixed time lags, \( t \in \mathbb{R}^+ \), \( \mathbb{R}^+ = [0, \infty) \); the functions \( a, b_j, f, g_j \) (\( j = 0, 1, 2, \ldots, m \)), and \( g \) are continuous on \( \mathbb{R}^+ \times \mathbb{R}^+ \), \( \mathbb{R}^2 \), and \( \mathbb{R}^m \times \mathbb{R}^m \), respectively, with \( g(t, 0) = 0 \) (0, 0), and it is assumed that \( \omega(t) \in \mathbb{R}^m \) is a standard Weiner process. Since the functions \( a, b_j, f, g_j \) and \( g \) are continuous, then the existence of the solutions of the SDDE (2) is guaranteed by the continuity of these functions. In addition, we suppose that \( f, g_j \) (\( j = 0, 1, 2, \ldots, m \)), and \( g \) satisfy a Lipschitz condition in \( x, x, x(t - \tau_1), \ldots, x(t - \tau_m) \). Hence, we can guarantee the uniqueness of solutions of SDDE (2). Further, through this paper, it is supposed the existence and continuity of the derivatives \( b'_0(t) = \frac{db_0(t)}{dt}, b'_j(t) = \frac{db_j(t)}{dt}, g'_j(x) (j = 1, 2, \ldots, m), \) and \( \frac{dx_j}{dx} = f_j(x, x) \). Finally, for brevity in notation, if a function is written without its argument, we mean that the argument is always \( t \). For example, \( x \) represents \( x(t) \) and \( x' \) represents \( x'(t) \).

We can write SDDE (2) as the following equivalent stochastic delay differential system (SDDS):

\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -a(t)f(x, y)y - \sum_{j=0}^{m} b_j(t)g_j(x) + \sum_{j=1}^{m} b_j(t) \int_{t-\tau_j}^{t} g'_j(x(s)y(s))ds - g(t, x)\dot{\omega}(t)
\]  

(3)

The motivation for considering SDDE (2) and studying the qualitative properties of solutions of this equation come from the paper of Abou-El-Ela et al. [2] and the works that can be found in the references of this paper, see [1–58]. If we compare SDDE (2) with SDDE (1), then it can be easily seen that SDDE (2) includes and improve SDDE (1). Here, we define a new Lyapunov-Krasovskii functional and benefit from some elementary inequalities, then we obtain some new criteria for the stochastically stable and stochastically asymptotically stability of solutions of SDDE (2). We establish a new theorem and corollary on the stochastically stable and stochastically asymptotically stability of zero solution of SDDE (2). The theorem and the corollary extend, improve and complement the result of Abou-El-Ela et al. [2] and that can be found in [1,4,23,27]. In addition, we give an example, which satisfies our assumptions and shows the applicability of them. We also would like to point out that the results of this paper are new, and they different from that can be found in the related literature.

Let \( (\Omega, F, \{F_t\}_{t \geq 0}, P) \) be a complete probability space with a filtration \( \{F_t\}_{t \geq 0} \) satisfying the usual conditions. That is, \( \Omega \) is a set called the sample space, \( F \) is a \( \sigma \)-field of subsets of \( \Omega \) and \( P \) is a probability measure on \( (\Omega, F) \). \( (\Omega, F, \{F_t\}_{t \geq 0}, P) \) is filtered by a non-decreasing right-continuous family \( \{F_t\}_{t \geq 0} \) of \( \sigma \)-fields of \( F \).

Let

\[
B(t) = (B_1(t), B_2(t), \ldots, B_m(t))
\]

be \( m \)- dimensional Brownian motion defined on the probability space.

Let us consider \( n \)- dimensional non-linear SDE as the following:

\[
dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \quad t \geq 0
\]  

(4)

with the initial value \( x(0) = x_0, x_0 \in \mathbb{R}^n \). We assume that the functions \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \) are continuous with \( f(t, 0) = g(t, 0) = 0 \) satisfy the local Lipschitz condition and the growth condition (see Wu and Mao [50]). Then, the SDDE (4) has a unique and continuous solution on \( t \geq 0 \).

**Definition 1:** The zero solution of SDDE (4) is said to be stochastically stable or stable in the probability if for every pair \( \varepsilon \in (0, 1) \) and \( r > 0 \), there exists \( \delta = \delta(\varepsilon, r) > 0 \) such that

\[
P(|x(t, x_0)| < r \text{ for all } t \geq 0) \geq 1 - \varepsilon,
\]

whenever implies that \( |x_0| < \delta \). Otherwise, the zero solution of SDDE (4) is said to be stochastically unstable.

**Definition 2:** The zero solution of SDDE (4) is said to be stochastically asymptotically stable if it is stochastically stable, and moreover for every pair \( \varepsilon \in (0, 1) \) and \( r > 0 \), there exists \( \delta_0 = \delta_0(\varepsilon, r) > 0 \) such that

\[
P(\lim_{t \to \infty} x(t, x_0) = 0) \geq 1 - \varepsilon,
\]

Whenever it implies that \( |x_0| < \delta \).

We define an operator \( L \) acting on \( C^{1,2}([\Omega \times \mathbb{R}^n, \mathbb{R}^+]) \) functions by

\[
LV(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}\text{trace}(g^T(t, x)V_{xx}(t, x)g(t, x)),
\]

where \( V_t = (V_{t1}, V_{t2}, \ldots, V_{tn}) \) and \( V_{xx} = (V_{xk,l})_{n \times n}, \) \( (i, j = 1, 2, \ldots, n) \).

Further, let \( K \) denote the family of all continuous and non-decreasing functions \( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( \rho(0) = 0 \) and \( \rho(r) > 0 \), if \( r > 0 \).
**Theorem 1 ([18]):** Suppose that there exist a functional \( C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+) \) and \( r \in K \) such that
\[
V(t,0) = 0, \; r(|x|) \leq V(t,x)
\]
and
\[
LV(t,x) \leq 0 \text{ for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.
\]
Then the zero solution of the SDDE (4) is stochastically stable.

**Theorem 2 ([18]):** Suppose that there exist \( V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+) \) and \( r_1, r_2 \) and \( r_3 \in K \) such that \( r_1(|x|) \leq V(t,x) \leq r_2(|x|) \) and
\[
LV(t,x) \leq -r_3(|x|) \text{ for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.
\]
Then the zero solution of SDDE (4) is stochastically asymptotically stable.

**2. Stochastic stability**

**Assumptions**

The following assumptions are needed for the results of this paper.

We suppose that there are positive constants \( \delta_j, \beta_j, \mu_j, \alpha_0, \alpha_i, \alpha, f_0, \alpha, \rho_j \) and \( C \) such that the following assumptions hold.

\[ (A1) \quad \frac{1}{2} \leq \alpha_0 \leq a(t) \leq \alpha_1, \quad a'(t) \leq \alpha, \]
\[ 0 < \delta_j \leq b_j(t) \leq \beta_j, \quad (j = 0, 1, 2, \ldots, m), \]
\[ b'_j(t) \leq 0, \quad b'_j(t) \leq \mu_j, \forall t \in \mathbb{R}^+ \cup \{0\}, (j = 1, 2, \ldots, m). \]

\[ (A2) \quad 1 \leq f(x,y) \leq f_0, \quad f_j(x,y) \geq 0, \forall t \in \mathbb{R}^+ \cup \{0\}, \forall x, y \in \mathbb{R}. \]

\[ (A3) \quad g_j(0) = 0, \quad \frac{g_j(x)}{x} \geq \alpha_j, \quad (x \neq 0), \forall x \in \mathbb{R}, \quad (j = 0, 1, 2, \ldots, m), \]
\[ 0 < g'_j(x) \leq \rho_j, \quad (j = 1, 2, \ldots, m), \forall t \in \mathbb{R}^+ \cup \{0\}, \forall x, y \in \mathbb{R}. \]

**Theorem 3:** Let assumptions (A1) – (A3) be hold. If
\[
\tau < \min \left[ \frac{A_0 - 2 \alpha f_0 - C^2 - B_0}{C_0}, \frac{2 \alpha_0 - 1}{C_0} \right],
\]
with
\[
C_0 > 0, \quad 2 \alpha_0 - 1 > 0, \quad A_0 - 2 \alpha f_0 - C^2 - B_0 > 0,
\]
\[
\tau = \max_{1 \leq j \leq m} \left( \frac{\delta_j}{\beta_j} + \alpha_j + \alpha_0 \right), \quad A_0 = \sum_{j=0}^{m} (\delta_j \alpha_j),
\]
\[
B_0 = \frac{1}{2} \sum_{j=1}^{m} (\beta_j \mu_j), \quad C_0 = \frac{1}{2} \sum_{j=1}^{m} (\alpha_j \beta_j),
\]
then the zero solution of SDDE (2) is stochastically asymptotically stable.

**Proof:** Define a Lyapunov-Krasovskii functional by
\[
U(t,x_t,y_t) = 2 \sum_{j=0}^{m} b_j(t) \int_{0}^{x} g_j(s) ds + y^2
\]
\[
+ \alpha(t) \int_{0}^{x} f(s,0) ds + xy
\]
\[
+ \sum_{j=1}^{m} \lambda_j \int_{t-\tau_j}^{t} r^2 \int_{t-\tau_j}^{t} (\theta) d\theta d\theta,
\]
where \( \lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_m > 0 \) and \( \lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{R}, \ldots, \lambda_m \in \mathbb{R} \), and we choose these constants later.

By using assumptions (A2), (A3) and the Lyapunov-Krasovskii functional \( U(.) \), that is, (5), we observe
\[
2 \sum_{j=0}^{m} b_j \int_{0}^{x} g_j(s) ds = 2 \sum_{j=0}^{m} b_j \int_{0}^{x} \frac{g_j(s)}{s} ds
\]
\[
\geq \sum_{j=0}^{m} (\delta_j \alpha_j) x^2, \quad a(t) \int_{0}^{x} f(s,0) ds
\]
\[
\geq \frac{1}{2} a_0 x^2.
\]

Hence, we have
\[
U(t,x_t,y_t) \geq \sum_{j=0}^{m} (\delta_j \alpha_j) x^2 + \frac{1}{2} a_0 x^2 + y^2 + xy
\]
\[
= \left( \sum_{j=0}^{m} (\delta_j \alpha_j) + a_0 - 1 \right) x^2 + \left[ x + \frac{1}{2} y^2 \right] + \frac{3}{4} y^2.
\]

Since \( \sum_{j=0}^{m} (\delta_j \alpha_j) + a_0 > 1 \), then we can derive
\[
U(t,x_t,y_t) \geq D_1 (x^2 + y^2), \quad D_1 > 0, D_1 \in \mathbb{R}. \quad (6)
\]

By considering assumptions (A1) – (A3), the Lyapunov-Krasovskii functional \( U(.) \) in (5) and the inequality \( 2|ab| \leq a^2 + b^2 \), we have
\[
2 \sum_{j=0}^{m} b_j \int_{0}^{x} g_j(s) ds
\]
\[
= 2 \sum_{j=0}^{m} b_j \int_{0}^{x} \frac{g_j(s)}{s} ds \leq \sum_{j=0}^{m} (\beta_j \rho_j) x^2,
\]
\[
x y \leq \frac{1}{2} x^2 + \frac{1}{2} y^2,
\]
\[
a(t) \int_{0}^{x} f(s,0) ds \leq \frac{1}{2} (a_1 f_0) x^2,
\]
\[
\lambda_j \int_{t-\tau_j}^{t} \int_{t-\tau_j}^{t} y^2 (\theta) d\theta d\theta
\]
\[
= \lambda_j \int_{t-\tau_j}^{t} (\theta - t - \tau_j) y^2 (\theta) d\theta d\theta
\]
\[
\leq \lambda_j |y|^2 \int_{t-\tau_j}^{t} (\theta - t - \tau_j) d\theta d\theta = \frac{1}{2} \lambda_j \tau_j |y|^2.
\]
Hence, we observe
\[
\sum_{j=1}^{m} \lambda_j \int_{t-	au_j}^{t} y^2(\theta) d\theta ds = \frac{1}{2} ||y||^2 \sum_{j=1}^{m} (\lambda_j \tau_j).
\]
Then, it follows that
\[
U(t, x_t, y_t) \leq \sum_{j=0}^{m} (\beta_j \eta_j) x^2 + y^2 + \frac{1}{2} (a_1 f_0) x^2 \\
+ \frac{1}{2} x^2 + 2 y^2 + \frac{1}{2} ||y||^2 \sum_{j=1}^{m} (\lambda_j \tau_j).
\]
Let \( D = \frac{1}{2} \sum_{j=1}^{m} (\lambda_j \tau_j) \). By this equality, the last inequality and (6), we can obtain
\[
U(t, x_t, y_t) \leq D_2(x^2 + y^2) + D ||y||^2,
\]
\[
D_2 > 0, D > 0, D_2, D \in \mathbb{R}.
\]
By combining the inequalities (6) and (7), it follows that
\[
D_1(x^2 + y^2) \leq U(t, x_t, y_t) \leq D_2(x^2 + y^2) + D ||y||^2.
\]
By an easy calculation, the time derivative of the Lyapunov-Krasovskii functional \( U(t, x_t, y_t) \) along any solution \((x, y)\) of stochastic delay differential system (3) gives
\[
\frac{d}{dt} U(t, x_t, y_t) = -2a(t)f(x, y)xy + a'(t) \int_{0}^{x} f(s, 0) ds + a(t)f(x, 0)xy - \sum_{j=0}^{m} b_j(t)g_j(x) + g^2(t, x) \\
+ x \sum_{j=1}^{m} b_j(t) \int_{t-	au_j}^{t} g_j'(x(s)) y(s) ds \\
+ 2y \sum_{j=1}^{m} b_j(t) \int_{t-	au_j}^{t} g_j'(x(s)) y(s) ds \\
+ 2y \\sum_{j=0}^{m} b'_j(t) \int_{0}^{\tau_j} g_j(s) ds \\
+ \sum_{j=1}^{m} (\lambda_j \tau_j) y^2 - \sum_{j=1}^{m} \lambda_j \\int_{t-	au_j}^{t} y^2(s) ds.
\]
In view of assumptions (A1) – (A3) and the inequality \(2|ab| \leq a^2 + b^2\), we can observe the following inequalities for the below terms, which are included in the derivative of \( U(t, x_t, y_t) \):
\[
2a(t)f(x, y)xy \geq 2a_0 y^2, \\
da'(t) \int_{0}^{x} f(s, 0) ds \leq \frac{1}{2}(a_0 f_0) x^2,
\]
\[
\frac{d}{dt} U(t, x_t, y_t) \leq - \sum_{j=0}^{m} (\delta_j \alpha_j) x^2 + \frac{1}{2} (a_0 f_0) x^2 + C^2 x^2 - a(t)xf(x, y)y^2 \\
+ \frac{1}{2} \sum_{j=0}^{m} (\rho_j \mu_j) x^2 + \frac{1}{2} \sum_{j=1}^{m} (\gamma_j \beta_j) x^2 \\
- 2a_0 y^2 + y^2 + \sum_{j=1}^{m} (\rho_j \beta_j) y^2 \\
+ \sum_{j=1}^{m} (\lambda_j \tau_j) y^2 + \frac{3}{2} \sum_{j=1}^{m} (\gamma_j \beta_j) \int_{t-	au_j}^{t} y^2(s) ds.
\]
The next result of this paper is the following the corollary.

**Corollary 1:** If hypotheses (A1) – (A3) hold, then the zero solution of SDDE (2) is stochastically stable.

The proof of Corollary 1 can be easily completed by considering Theorem 1 and Theorem 3.

### 3. Illustrative example

In this section, in a particular case, a numerical example is given to demonstrate the accuracy and applicability of the obtained results.

**Example 1:** In particular case of SDDE (2), let us consider the following SDDE of second order with two constant delays, \( \tau_1 \) and \( \tau_2 \):

\[
\frac{d^2 x}{dt^2} + (4 + \exp(-2^{-1}t)) \left( 1 + \frac{1}{1 + x^2} \right) \frac{dx}{dt} + \left( 1 + \frac{1}{t^2 + 1} \right) \left( \frac{x}{t^2 + 1} + x \right) + \left( 1 + \frac{1}{t + 1} \right) x \left( t - \frac{1}{10} \right) + \left( \frac{tx}{t^2 + 1} \right) \omega(t) = 0, \tag{9}
\]

which is included by SDDE (2), and where \( \tau_1 = \frac{1}{10} \) and \( \tau_2 = \frac{1}{100} \).

We can write SDDE (9) as the following equivalent SDDS

\[
\frac{dy}{dt} = - (4 + \exp(-2^{-1}t)) \left( 1 + \frac{1}{1 + x^2} \right) y
- \left( 1 + \frac{1}{t^2 + 1} \right) \left( \frac{x}{t^2 + 1} + x \right)
- \left( 1 + \frac{1}{t + 1} \right) x
- 2 \left( 1 + \frac{1}{t^2 + 1} \right) \int_{t - \frac{1}{10}}^{t} \frac{1 - x^2(s)}{(x^2(s) + 1)^2} \, ds
+ (4 + \exp(-2^{-1}t)) \int_{t - \frac{1}{10}}^{t} y(s) \, ds - \frac{tx}{t^2 + 1} \omega(t).
\]

If we compare SDDS (10) with SDDS (3), then we observe

\[
a(t) = 4 + \exp(-2^{-1}t), \quad a_0 = 4 \leq a(t) = 4 + \exp(-2^{-1}t) \leq 5 = a_1,
\]
\( a'(t) = -\frac{1}{2} \exp(-2^{-1} t) \leq 0, \quad t \geq 0. \)

We can choose
\[
\alpha = \frac{1}{1000000},
\]

\[
b_0(t) = 1 + \frac{1}{t^4 + 1},
\]

\[
\delta_0 = 1 \leq b_0(t) = 1 + \frac{1}{t^4 + 1} \leq 2 = \beta_0,
\]

\[
b_0'(t) = -\frac{2 t^3}{(t^4 + 1)} \leq 0, \quad t \geq 0,
\]

\[
b_1(t) = 1 + \frac{1}{t^2 + 1},
\]

\[
\delta_1 = 1 \leq b_1(t) = 1 + \frac{1}{t^2 + 1} \leq 2 = \beta_1,
\]

\[
b_1'(t) = -\frac{2 t}{(t^2 + 1)^{3/2}} \leq 0, \quad t \geq 0,
\]

\[
b_2(t) = 1 + \frac{1}{t + 1},
\]

\[
\delta_2 = 1 \leq b_2(t) = 1 + \frac{1}{t + 1} \leq 2 = \beta_2,
\]

\[
b_2'(t) = -\frac{1}{(t + 1)^2} \leq 0, \quad t \geq 0,
\]

and we can choose \( \mu_0 = \frac{1}{1000}, \quad \mu_1 = \frac{1}{1000}, \quad \mu_2 = \frac{1}{1000}. \)

\[
g_0(x) = \frac{x}{x^4 + 1} + x, \quad g_0(0) = 0,
\]

\[
g_0(x) = \frac{1}{x^4 + 1} + 1,
\]

\[
g_0(x) \geq 1 = \alpha_0,
\]

\[
g_1(x) = \frac{x}{x^2 + 1} + x, \quad g_1(0) = 0,
\]

\[
g_1(x) = \frac{1}{x^2 + 1} + 1,
\]

\[
g_1(x) \geq 1 = \alpha_1,
\]

\[
g_1'(x) = \frac{1 - x^2}{(x^2 + 1)^2} + 1 \leq 2 = \rho_1,
\]

\[
g_2(x) = 2x, \quad g_2(0) = 0,
\]

\[
g_2(x) = 2 = \alpha_2,
\]

\[
g_2'(x) = 2 = \rho_2,
\]

\[
f(x, y) = 1 + \frac{1}{1 + x^2},
\]

\[
1 \leq f(x, y) = 1 + \frac{1}{1 + x^2} \leq 2 = f_0,
\]

\[
f_j(x, y) = 0,
\]

\[
g(t, x) = \frac{tx}{t^2 + 1},
\]

\[
g^2(t, x) = \frac{x^2 t^2}{(t^2 + 1)^2},
\]

\[
g^2(t, x) \leq \frac{1}{4} x^2, \quad C = \frac{1}{2}, \quad t \in \mathbb{R}^+.
\]

Thus, SDDE (9) satisfies all assumptions of Theorem 3, (A1)–(A3). Therefore, if
\[
\tau < \min \left[ \frac{4 - 10^{-6} - \frac{1}{4} - \frac{25}{20}}{4}, \frac{749999}{1000000} \right] = \frac{7}{20},
\]

then the zero solution of SDDS (10) is stable and stochastically asymptotically stable. Hence, we can conclude the same for the zero solution of SDDE (10). This fact shows the applicability of Theorem 3 and Corollary 1.

4. Conclusion

In this paper, the second functional method of Lyapunov-Krasovskii and some elementary inequalities are used to obtain the stochastically stability and stochastically asymptotically stability of the zero solution of a non-linear SDDE. Here, some new sufficient conditions are obtained by defining a suitable Lyapunov-Krasovskii functional and using some elementary inequalities. It is thought that the obtained results may be useful for researchers working in the various fields of sciences and engineering, for instance, in biology, mechanics, economy, control theory, population dynamics, medicine, engineering and so on. To the best of our knowledge, there are no more results on the stability of the solutions of SDDEs of higher order for that kind of functional differential equations of higher order in the literature. The possible reason could be the difficulty of the topic and construction of suitable Lyapunov-Krasovskii functionals for proper problems under study. By this paper, our purpose to do a contribution to the related literature and topic. The idea of this paper can be also applied to the different types of SDDEs of higher order, that is, neutral, advanced and delay SDDEs of higher order. These facts can be considered as open new problems.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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