New Relativistic Wave Equations for Two-Particle Systems

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We seek to introduce a mathematical method to derive the relativistic wave equations for two-particle system. According to this method, if we define stationary wave functions as special solutions like $\Psi(r_1, r_2, t) = \psi(r_1, r_2)e^{-iEt/\hbar}$, $\psi(r_1, r_2) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$, and properly define the relativistic reduced mass $\mu_0$, then some new relativistic two-body wave equations can be derived. On this basis, we obtain the two-body Sommerfeld fine-structure formula for relativistic atomic two-body systems such as the pionium and pionic hydrogen atoms bound states, using which, we discuss the pair production and annihilation of $\pi^+$ and $\pi^−$.

Keywords: Relativistic two-body wave equations, Two-body Sommerfeld fine-structure formula

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1. Introduction

The lack of an analytically solvable relativistic wave equation for two-body atomic systems has compelled physicists to use second-order perturbation theory in calculating energy levels to order $\alpha^6$ in systems such as positronium (See [1], [2]). In Reference [3] and [4] a two-particle Sommerfeld fine-structure formula was derived from the Bethe-Salpeter equation for two spin-1/2 constituent particles bound by a single-photon-exchange kernel in the Coulomb gauge. It is

$$E = \sqrt{m^2 + M^2 + \frac{2mM}{\sqrt{1 + \frac{Z^2\alpha^2}{(n+\epsilon+1)^2}}} c^2}. \quad (1)$$

Here $\alpha$ is the fine-structure constant, $n$ is the radial quantum number, $m$ is the mass of the electron and $M$ is the mass of the other particle. In this work we will not be able to derive an angular equation for $\epsilon$. Nevertheless, from the two-particle Sommerfeld formula (1) alone, without knowing $\epsilon$, we have found two predictions at order $\alpha^6$ which are verified in particular atomic systems by previous calculations which used second-order perturbation theory. These results suggest that it may soon be possible to find an analytically solvable

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relativistic atomic two-body wave equation which would eliminate the need for second-order perturbation calculations to obtain energies to order $\alpha^6$. Both special relativity and experiments indicate that, the mass of a many-particle system in a bound state is less than the sum of the rest mass of every particle forming the system, and the difference gives the mass defect of the system, while the product of the mass defect and the square of the speed of light gives the binding energy of the system. As the binding energy is quantized, the sum of it and the rest mass of every particle forming the system is the energy level of the system. For instance, the mass of an atomic nucleus is obviously less than the sum of the rest mass of every nucleon forming the atomic nucleus. Therefore, in order to express the mass defect explicitly, there is a necessity to introduce the concept of system mass, which differs from the sum of the rest mass of every particle forming the system. By introducing the concept of the system mass and applying proper mathematical skills, the relativistic wave equations for two-particle system is derived. On this basis, let us properly define the relativistic reduced mass to further derive the new relativistic two-body wave equations. The main results of this paper are expressed as

1. The relativistic two-body wave equations

$$E' \psi = - \frac{2(m_1 \mu + m_0 \mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_0 \mu_0} \nabla_1^2 \psi - \frac{2(m_0 \mu + m_0 \mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_0 \mu_0} \nabla_2^2 \psi$$

$$+ \frac{2 \mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi$$

$$- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi$$

$$+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) (\nabla_1^2 + \nabla_2^2) \psi$$

$$+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right]$$

$$- \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \psi.$$ 

2. In the center-of-momentum frame, it is simplified as

$$E' \psi = - \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)} \left( m - \frac{U}{c^2} \right)^2 \nabla^2 \psi + \frac{2 \mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi$$

$$+ \frac{2 \hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \psi$$

$$+ \frac{4 \hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \psi$$

$$- \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \psi.$$
Where $m_0 = m_{01} + m_{02}$, $E = mc^2$, and $m, \mu_0, \mu$ respectively denote $m = m_0 + \frac{1}{c^2}E'$, $\mu_0 = \frac{2m_0m_{02}}{m_0 + m}$, $\mu = \mu_0 + \frac{1}{c^2}E'$.

3. If $U = -Z\epsilon^2/r$, $\epsilon_s = e(4\pi\varepsilon_0)^{-1/2}$, $r = |\mathbf{r}_1 - \mathbf{r}_2|$ and $E' < 0$, then

$$E_n = \left[ m^2_{01} \pm 2m_0m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m^2_{02} \right]^{1/2}c^2.$$

$$\sigma_l = l + 1 + \frac{d_0}{2(n - \sigma_l)} - \sqrt{\left( l + \frac{1}{2} \right)^2 - Z^2\alpha^2 + \frac{3d_0}{2} - \frac{d_0}{n - \sigma_l}}, \quad \alpha = \frac{e^2}{hc}.$$

$$d_0 = 2Z^2\alpha^2D, \quad D = \frac{\mu(m_0 + m)}{2m^2}, \quad m = E_n/c^2, \quad \mu = \pm \mu_0 \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2}.$$

2. Relativistic Two-body Wave Equations

As we know, arbitrary wave function is equal to the linear superposition of the plane waves of free particles with all possible momentum, namely

Let $E$ be the total energy of the system, $p$ be the momentum of particle, then

$$\Psi(r, t) = \int \int \int c(p, t)\Psi_p(r, t) \, dp_x dp_y dp_z,$$

$$c(p, t) = \int \int \int \Psi(r, t)\Psi_p^*(r, t) \, dx \, dy \, dz.$$  \hfill (2)

Where $\Psi_p$ is

$$\Psi_p(r, t) = A \exp(-i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar).$$  \hfill (3)

For the one-particle system, $A = (2\pi\hbar)^{-3/2}$. Clearly, (2) are the Fourier transform and its inversion, which can be extended to many-particle systems. For two-particle systems, let $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ be position vectors of two particles in the laboratory reference frame respectively, and corresponding momentum vectors be $\mathbf{p}_1 = (p_{x_1}, p_{y_1}, p_{z_1})$ and $\mathbf{p}_2 = (p_{x_2}, p_{y_2}, p_{z_2})$. Thus related physical quantities in (2) and (3) are extended to $\mathbf{r} = (r_1, r_2)$, $\mathbf{p} = (p_1, p_2)$, $A = (2\pi\hbar)^{-3}$, $dp_x = dp_{x_1} dp_{x_2}$, $dp_y = dp_{y_1} dp_{y_2}$, $dp_z = dp_{z_1} dp_{z_2}$, $dx = dx_1 dx_2$, $dy = dy_1 dy_2$, $dz = dz_1 dz_2$.

Assuming that any particle with the rest mass $m_0$, no matter how high the speed is, no matter it is in a potential field or in free space, and no matter how it interacts with other particles, its kinetic energy is:

$$E_k = (c^2p^2 + m_0^2c^4)^{1/2} - m_0c^2.$$

\hfill (4)
On this basis, we can establish the relation between the system energy $E$ and the momentum $p$ using proper mathematical skills, thus obtain the relativistic Hamiltonian. Therefore, we introduce the mathematical method of Reference [5]-[8] to quantum mechanics. Similarly to Reference [5], we can introduce the relevant concepts in quantum mechanics:

**Definition 1** $\Psi_p(r,t)$ in the right-hand side of (2) is defined as the base function of quantum mechanics, where $E$ and $p$ are called the characters of base functions, while $E$ and $p$ are not only suitable for free particles, but also suitable for any system, and the relation between $E$ and $p$ is called the characteristic equation of wave equations. Different system has different characteristic equations.

According to differential laws, we have

$$i\hbar \frac{\partial}{\partial t} \Psi_p = E\Psi_p, \quad -i\hbar \nabla_j \Psi_p = p_j \Psi_p, \quad j = 1, 2, \ldots.$$  \hspace{1cm} (5)

**Definition 2** Let $m_0 = m_{01} + m_{02} + \cdots + m_{0N}$ be the total rest mass of an $N$-particle system, $E'$ be the sum of the kinetic energy and potential energy of all the $N$ particles, then the actual mass of the system, which is called the system mass, is defined as

$$m = m_0 + \frac{1}{c^2} E'.$$ \hspace{1cm} (6)

**Definition 3** If the system is in a bound state ($E' < 0$), then the absolute value of $E'$ is

$$|E'| = m_0c^2 - mc^2 = \Delta mc^2,$$

which is called the binding energy of the system, where $\Delta m = m_0 - m$ is the mass defect of the system.

**Definition 4** The total energy of the system $E$ is defined as the sum of the rest energy, kinetic energy and potential energy of all the particles forming the system, namely $E = m_0c^2 + E'$

According to Definition 2 and 4, the total energy of the system is equal to the product of the system mass and the square of the speed of light, namely $E = mc^2$, thus the system mass is uniquely determined by the energy levels of the system.

**Definition 5** In relativistic quantum mechanics, the stationary wave function for two-particle system is defined as the following special solution:

$$\Psi(r_1, r_2, t) = \psi(r_1, r_2) \exp(-iEt/\hbar), \quad \psi(r_1, r_2) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3).$$ \hspace{1cm} (7)

Where $E$ is the total energy of the two-particle system.

Applying (2) to Definition 5, we have
\[ \Psi(r_1, r_2, t) = \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(p_1, p_2) \exp \left( \frac{i}{\hbar} (p_1 \cdot r_1 + p_2 \cdot r_2 - Et) \right) \, dp_1 \, dp_2 \, dp_3, \]

\[ c(p_1, p_2) = \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(r_1, r_2) \exp \left( -\frac{i}{\hbar} (p_1 \cdot r_1 + p_2 \cdot r_2) \right) \, dx \, dy \, dz. \]  

(8)

Where \( r_1 = (x_1, y_1, z_1) \) and \( r_2 = (x_2, y_2, z_2) \) are position vectors of two particles in the laboratory reference frame respectively, and \( dx = dx_1 dx_2, dy = dy_1 dy_2, dz = dz_1 dz_2 \). Corresponding momentum vectors are respectively \( p_1 = (p_{x_1}, p_{y_1}, p_{z_1}) \) and \( p_2 = (p_{x_2}, p_{y_2}, p_{z_2}) \), and \( dp_x = dp_{x_1} dp_{x_2}, dp_y = dp_{y_1} dp_{y_2}, dp_z = dp_{z_1} dp_{z_2} \). Because of \( \psi(r_1, r_2) \in \mathcal{S} (\mathbb{R}^3 \times \mathbb{R}^3) \) it satisfies natural boundary conditions: \( \psi(r_1, r_2) \to 0, \ r \to \infty \).

In relativistic quantum mechanics, due to the relativistic effect that mass varies with speed, the center of mass system is no longer a proper description framework, instead, we use the center-of-momentum frame, which is a coordinate system that the total momentum equals zero. If \( v_1, v_2 \) respectively denote the speed of particles in the two-particle system, then their momentum respectively are

\[ p_1 = m_1 v_1 = \frac{m_0_1 v_1}{\sqrt{1 - (v_1/c)^2}}, \quad p_2 = m_2 v_2 = \frac{m_0_2 v_2}{\sqrt{1 - (v_2/c)^2}}, \]

and \( p_1 = -p_2 \). If \( v \) denotes the relative speed between two particles, then we can properly define the relativistic reduced mass \( \mu \) to make the relative momentum \( p = \mu v \) satisfy \( |p_1| = |p_2| = |p| \), namely

\[ p_{x_1}^2 + p_{y_1}^2 + p_{z_1}^2 = p_{x_2}^2 + p_{y_2}^2 + p_{z_2}^2 = p_x^2 + p_y^2 + p_z^2. \]  

(9)

In other words, the reduced mass \( \mu \) can be determined using (9) and the relativistic velocity addition formula. As it is related to speed, in order to distinguish it from another type of reduced mass, we call this one the speed-type reduced mass. For instance, if two particles of a two-particle system are restricted to movement along the same line, then its speed-type reduced mass is defined as

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \left( 1 + \frac{v_1 v_2}{c^2} \right). \]  

(10)

Where

\[ m_1 = \frac{m_0_1}{\sqrt{1 - (v_1/c)^2}}, \quad m_2 = \frac{m_0_2}{\sqrt{1 - (v_2/c)^2}}. \]

Therefore, using the center-of-momentum frame in (8), \( p_2 = -p_1, \ |p_1| = |p| \). Substituting them into (8), we have
\[
\psi(r_1, r_2) = \frac{1}{(2\pi\hbar)^3} \iint_{-\infty}^{\infty} c(p_1, p_2) \exp(i p \cdot (r_1 - r_2)/\hbar) \, dp_x dp_y dp_z,
\]

\[
c(p_1, p_2) = \frac{1}{(2\pi\hbar)^3} \iint_{-\infty}^{\infty} \psi(r_1, r_2) \exp(-i p \cdot (r_1 - r_2)/\hbar) \, dx \, dy \, dz.
\]

Where \(|p|\) is the relative momentum of the two-particle system. Relative coordinate is denoted by \(r = r_1 - r_2\), then the result can be expressed by a more symmetric form

\[
\Psi(r_1, r_2, t) = \frac{1}{(2\pi\hbar)^3} \iint_{-\infty}^{\infty} c(p_1, p_2) \exp(-i(Et - p \cdot r)/\hbar) \, dp_x dp_y dp_z,
\]

\[
c(p_1, p_2) = \frac{1}{(2\pi\hbar)^3} \iint_{-\infty}^{\infty} \Psi(r_1, r_2, t) \exp(i(Et - p \cdot r)/\hbar) \, dx \, dy \, dz.
\]

Where \(\Psi(r_1, r_2, t) = \psi(r_1, r_2) \exp(-iEt/\hbar)\) is the relativistic stationary wave functions for the two-particle system. Therefore, \(p = \mu \nu\), which is the relative momentum of the two particle system in the center-of-momentum frame, is definitely equivalent to the differential operator \(-i\hbar \nabla\) with respect to the relative coordinate \(r = r_1 - r_2\), \( Brent becomes

\[
(i\hbar \frac{\partial}{\partial t}) \Psi_p = EP_p, \quad -i\hbar \nabla \Psi_p = p \Psi_p.
\]

Considering an isolated two-particle system, if the interaction energy between two particles is denoted by \(U(r_1, r_2)\), then according to Definition 2 and (14), we have

\[
E' - U = (c^2 p_1^2 + m_{01}^2 c^4)^{1/2} - m_{01}c^2 + (c^2 p_2^2 + m_{02}^2 c^4)^{1/2} - m_{02}c^2.
\]

Where \(m_{01}, m_{02}\) are the rest mass of two particles respectively, and corresponding momentum are \(p_1 = |p_1|, p_2 = |p_2|\). This is the characteristic equation of relativistic wave equations for the two-particle system, thus we obtain

\[
\sqrt{c^2 p_1^2 + m_{01}^2 c^4} + \sqrt{c^2 p_2^2 + m_{02}^2 c^4} + U])\psi = E\psi.
\]

This is the spin-less Salpeter equation (See [9], [10]), it is an important relativistic two-body wave equation. In order to make it easier to solve the corresponding relativistic wave equation, the characteristic equation (14) should be transformed to remove the fractional power, then we have

\[
(E' - U + m_{01}c^2 + m_{02}c^2)^2 = c^2 p_1^2 + m_{01}^2 c^4 + c^2 p_2^2 + m_{02}^2 c^4
\]

\[
+ 2(c^2 p_1^2 + m_{01}^2 c^4)^{1/2}(c^2 p_2^2 + m_{02}^2 c^4)^{1/2}.
\]
Expanding the left-hand side of (16) and applying (6), we have
\[
(m_0 + m)E' - 2mU + U^2/c^2 = p_1^2 + p_2^2 + 2(p_1^2 + m_{01}^2c^2)^{1/2}(p_2^2 + m_{02}^2c^2)^{1/2} - 2m_{01}m_{02}c^2. \tag{17}
\]

Further, removing the radical sign, we have
\[
[(m_0 + m)E' + 2m_{01}m_{02}c^2 - p_1^2 - p_2^2 - 2mU + U^2/c^2]^2 = 4(p_1^2 + m_{01}^2c^2)(p_2^2 + m_{02}^2c^2). \tag{18}
\]

**Definition 6** In relativistic quantum mechanics, a type of relativistic reduced mass \(\mu_0\) of two-particle systems is defined as
\[
\mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}, \quad \mu = \mu_0 + \frac{1}{c^2}E'. \tag{19}
\]

Where \(m_0 = m_{01} + m_{02}\), \(m\) is the system mass of the two-particle system, \(E'\) is the sum of kinetic energy and potential energy of the two particles, \(\mu\) is called the system mass corresponding to \(\mu_0\). Unless otherwise stated, the reduced mass referred in our paper from now on is defined in this way, which should not be confused with the speed-type reduced mass mentioned previously.

According to (19), we have
\[
\frac{m_{01}\mu + m_{02}\mu_0}{m_{01}} + \frac{m_{02}\mu + m_{01}\mu_0}{m_{02}} = \frac{2m^2}{m_0 + m}. \tag{20}
\]
\[
(m_0 + m)E' + 2m_{01}m_{02}c^2 = (m_0 + m)\mu c^2. \tag{21}
\]
\[
((m_0 + m)E' + 2m_{01}m_{02}c^2)^2 = (m_0 + m)^2(\mu_0 + \mu)c^2E' + 4m_{01}^2m_{02}^2c^4. \tag{22}
\]

By (21) and (22), the characteristic equation (18) becomes
\[
4(p_1^2p_2^2 + m_{02}^2c^2p_1^2 + m_{01}^2c^2p_2^2) = (m_0 + m)^2(\mu_0 + \mu)c^2E' - 2(m_0 + m)\mu c^2(p_1^2 + p_2^2 + 2mU - U^2/c^2) + (p_1^2 + p_2^2 + 2mU - U^2/c^2)^2. \tag{23}
\]

According to (23), the relativistic Hamiltonian of two-particle systems can be ex-
pressed as

\[
H = E = E' + m_0c^2 = \frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_0} \nabla_1^2 \Psi_p + \frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_0} \nabla_2^2 \Psi_p
\]

\[
+ \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi_p
\]

\[
+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \nabla_1^2 + \nabla_2^2 \right) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi_p \right]
\]

\[
+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \nabla_1^2 \Psi_p
\]

\[
- \frac{1}{(\mu_0 + \mu)c^2} \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \Psi_p
\]

\[
- \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla_2^2 \Psi_p + m_0c^2 \Psi_p.
\]

Therefore, taking (24) as the characteristic equation, multiplying both sides of the equation by the base function \(\Psi_p(r, t)\), and by using (4), we have

\[
i\hbar \frac{\partial \Psi_p}{\partial t} = -\frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_0} \nabla_1^2 \Psi_p - \frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_0} \nabla_2^2 \Psi_p
\]

\[
+ \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi_p
\]

\[
+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \nabla_1^2 + \nabla_2^2 \right) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi_p \right]
\]

\[
+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \nabla_1^2 \Psi_p
\]

\[
- \frac{1}{(\mu_0 + \mu)c^2} \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \Psi_p
\]

\[
- \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla_2^2 \Psi_p + m_0c^2 \Psi_p.
\]

From (8) here \(\Psi_p\) is expressed as

\[
\Psi_p(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{1}{(2\pi\hbar)^3} \exp \left( \frac{i}{\hbar} (\mathbf{p}_1 \cdot \mathbf{r}_1 + \mathbf{p}_2 \cdot \mathbf{r}_2 - Et) \right).
\]

Where \(U(\mathbf{r}_1, \mathbf{r}_2)\) denotes the potential energy of the interaction between two particles, \(\nabla_1^2, \nabla_2^2\) are Laplace operators respectively corresponding to \(\mathbf{r}_1, \mathbf{r}_2\). According to (8), in the operator equation which is tenable for the base function \(\Psi_p(\mathbf{r}_1, \mathbf{r}_2, t)\), as long as each operator in the operator equation is a linear operator and each linear operator does not explicitly contain the characters \(E, \mathbf{p}_1\) and \(\mathbf{p}_2\) of \(\Psi_p(\mathbf{r}_1, \mathbf{r}_2, t)\), then this operator equation is also tenable for an arbitrary wave function \(\Psi(\mathbf{r}_1, \mathbf{r}_2, t)\). Whereas, considering that the
system mass $m$ is equivalent to the character $E$, this operator equation is not tenable for arbitrary wave functions, but tenable for an stationary wave function like (7). In other words, if $i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$, from (8) we have

$$H \Psi(r_1, r_2, t) = H \int \int \int c(p_1, p_2) \Psi dp_x dp_y dp_z = \int \int \int c(p_1, p_2) H \Psi dp_x dp_y dp_z$$

$$= \int \int \int c(p_1, p_2) i\hbar \frac{\partial}{\partial t} \Psi dp_x dp_y dp_z = i\hbar \frac{\partial}{\partial t} \int \int \int c(p_1, p_2) \Psi dp_x dp_y dp_z$$

$$= i\hbar \frac{\partial}{\partial t} \Psi(r_1, r_2, t), \ \forall \psi(r_1, r_2) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3).$$

So we get the following results:

An isolated two-particle system, the total spin angular momentum of which is zero, is described by the stationary wave function $\Psi(r_1, r_2, t)$ or $\psi(r_1, r_2)$, any stationary wave function

$$\Psi(r_1, r_2, t) = \psi(r_1, r_2) \exp(-iEt/\hbar), \ \psi(r_1, r_2) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$$

satisfies the following relativistic wave equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{2(m_0 \mu + m_{02} \mu_0)}{(m_0 + m)(\mu_0 + \mu)} m_{01} \nabla_1^2 \Psi - \frac{2(m_0 \mu + m_{01} \mu_0)}{(m_0 + m)(\mu_0 + \mu)} m_{02} \nabla_2^2 \Psi$$

$$+ \frac{2 \mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi$$

$$- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi$$

$$+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) (\nabla_1^2 + \nabla_2^2) \Psi$$

$$+ \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \right]$$

$$- \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \Psi + m_0 c^2 \Psi. \ \ (25)$$

Where $m_0 = m_{01} + m_{02}$, $E' = E - m_0 c^2$, $E = mc^2$. $m, \mu_0, \mu$ respectively denote

$$m = m_0 + \frac{1}{c^2} E', \ \mu_0 = \frac{2m_0 m_{02}}{m_0 + m}, \ \mu = \mu_0 + \frac{1}{c^2} E'.$$

Clearly, for non-relativistic limits, we have

$$\mu \rightarrow \mu_0 \rightarrow \frac{m_0 m_{02}}{m_0} = \frac{m_0 m_{02}}{m_0 + m_{02}}.$$
In other words, the relativistic wave function $\psi(r_1, r_2)$ for two-particle systems is determined by the following relativistic wave equation and natural boundary conditions:

$$E'\psi = \frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)m_{01}} \hbar^2 \nabla_1^2 \psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)m_{02}} \hbar^2 \nabla_2^2 \psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi$$

$$- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right]$$

$$- \frac{\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \psi. \quad (26)$$

For bound states, the total energy $E$ of the system is quantized, which is called the system energy level. The system mass $m = E/c^2$ is uniquely determined by the system energy level $E$. Clearly, for non-relativistic limits, this equation turns out to be the Schrödinger equation of two-particle systems. If the system is in the external field, then the system potential energy $U(r_1, r_2)$ includes both the potential energy of the system in the external field and the interaction energy between particles.

Using the center-of-momentum frame, then according to (9) we have $p_1^2 = p_2^2 = p^2$, where $p$ is the relative momentum. Considering (20) or

$$\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)m_{01}} + \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)m_{02}} = \frac{4m^2}{(\mu_0 + \mu)(m_0 + m)^2}, \quad (27)$$

then in the center-of-momentum frame, (24) becomes

$$H = E = E' + m_0c^2 = \left( \frac{2m}{m_0 + m} \right)^2 \frac{p^2}{\mu_0 + \mu} - \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)$$

$$- \frac{2}{(m_0 + m)c^2} \frac{p^2}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) + \frac{2}{(m_0 + m)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \frac{p^2}{\mu_0 + \mu}$$

$$- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 + m_0c^2. \quad (28)$$

Taking (28) as the characteristic equation, similarly we have: Considering an isolated two-particle system in the center-of-momentum frame, if the total spin angular momentum
of the system is zero, then the stationary wave function

\[ \Psi(r, t) = \psi(r) \exp(-iEt/\hbar) \]

is determined by the following relativistic wave equation and natural boundary conditions:

\[
\frac{i\hbar}{\partial t} \psi = -\left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\
+ \frac{2}{(m_0 + m)c^2} \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right] \\
+ \frac{2}{(m_0 + m)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi \\
- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi + m_0c^2 \psi. 
\tag{29}
\]

\[
E' \psi = -\left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\
+ \frac{2}{(m_0 + m)c^2} \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right] \\
+ \frac{2}{(m_0 + m)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi \\
- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi. 
\tag{30}
\]

These are also the expressions of relativistic wave equations \(25\) and \(26\) in the center-of-momentum frame respectively, where \(\nabla^2\) is the Laplace operator corresponding to the relative coordinate \(r = r_1 - r_2\).

Relativistic wave equations \(29\) and \(30\) can be further expressed as

\[
\frac{i\hbar}{\partial t} \psi = -\left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi \\
+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \psi \\
+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \nabla \psi \\
- \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \Psi + m_0c^2 \psi. 
\tag{31}
\]
\[ E'\psi = -\left(\frac{2m}{m_0 + m}\right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left(2mU - \frac{U^2}{c^2}\right) \psi \]

\[ \begin{align*}
  &+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left(2mU - \frac{U^2}{c^2}\right) \right] \psi \\
  &+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left(2mU - \frac{U^2}{c^2}\right) \cdot \nabla \psi \\
  &+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(2mU - \frac{U^2}{c^2}\right) \nabla^2 \psi \\
  &- \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(2mU - \frac{U^2}{c^2}\right)^2 \psi. \end{align*} \]

(32)

On the right-hand side of (32), if combining the first and the fifth terms, the stationary relativistic wave equation for two-particle systems in the center-of-momentum frame is expressed as

\[ E'\psi = -\frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(\frac{m - U}{c^2}\right)^2 \nabla^2 \psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left(2mU - \frac{U^2}{c^2}\right) \psi \]

\[ \begin{align*}
  &+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left(2mU - \frac{U^2}{c^2}\right) \right] \psi \\
  &+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left(2mU - \frac{U^2}{c^2}\right) \cdot \nabla \psi \\
  &+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(2mU - \frac{U^2}{c^2}\right) \nabla^2 \psi \\
  &- \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(2mU - \frac{U^2}{c^2}\right)^2 \psi. \end{align*} \]

(33)

As we know, the spin angular momentum not only has the general property of angular momentum \( S \times S = i\hbar S \), but also has its own particularity. For instance, for electrons and protons, the projection of the spin angular momentum \( S \) in any direction only takes two values \( \pm \hbar/2 \).

For the convenience of studying this type of angular momentum, a type of dimensionless vector \( \sigma \) is introduced, determined by

\[ \sigma \times \sigma = 2i\sigma, \quad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1. \]  

(34)

Using this type of vector \( \sigma \), the spin angular momentum is expressed as \( S = (\hbar/2)\sigma \). According to (34), if \( A, B \) are two arbitrary vectors commuted with \( \sigma \), we have

\[ (\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B). \]  

(35)
If the relativistic Hamiltonian of two-particle systems (24) is rewritten as

\[
E' = \frac{2(m_{01\mu} + m_{02\mu})}{(m_0 + m)(\mu + \mu)} \frac{p^2_1}{m_0} + \frac{2(m_{02\mu} + m_{01\mu})}{(m_0 + m)(\mu + \mu)} \frac{p^2_2}{m_0} + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)
\]

Thus the stationary relativistic wave equation for two-particle systems can be expressed as

\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = -\frac{2(m_{01\mu} + m_{02\mu})}{(m_0 + m)(\mu + \mu)} \frac{\hbar^2}{m_0} \nabla^2_1 \Psi - \frac{2(m_{02\mu} + m_{01\mu})}{(m_0 + m)(\mu + \mu)} \frac{\hbar^2}{m_0} \nabla^2_2 \Psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi
\]

Thus the stationary relativistic wave equation for two-particle systems can be expressed as

\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = -\frac{2(m_{01\mu} + m_{02\mu})}{(m_0 + m)(\mu + \mu)} \frac{\hbar^2}{m_0} \nabla^2_1 \Psi - \frac{2(m_{02\mu} + m_{01\mu})}{(m_0 + m)(\mu + \mu)} \frac{\hbar^2}{m_0} \nabla^2_2 \Psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi
\]

The two particles of the system are spin 1/2 particles, where \(U(\mathbf{r}_1, \mathbf{r}_2)\) denotes the potential energy of the system in the external field and the interaction energy between particles.
Let $S_1$ be the spin angular momentum of the first particle, and $S_2$ be that of the second one. In the central field, according to (35), (37) can be express as

\[
\frac{i\hbar}{\partial_t} \Psi = -\frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{h^2}{m_0} \nabla_1^2 \Psi - \frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{h^2}{m_0} \nabla_2^2 \Psi
\]

\[
+ \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi
\]

\[
+ \frac{4\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{dU}{dr_1} \frac{\partial \Psi}{\partial r_1} + \frac{dU}{dr_2} \frac{\partial \Psi}{\partial r_2} \right)
\]

\[
+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\nabla_1^2 + \nabla_2^2) \Psi
\]

\[
+ \frac{h^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (\nabla_1^2 + \nabla_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \Psi
\]

\[
- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi
\]

\[
- \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{1}{r_1 dr_1} S_1 \cdot L_1 \Psi + \frac{1}{r_2 dr_2} S_2 \cdot L_2 \Psi \right)
\]

\[
- \frac{h^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2) \Psi + m_0c^2 \Psi.
\]

(38)

\[
E' \psi = -\frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{h^2}{m_0} \nabla_1^2 \psi - \frac{2(m_0\mu + m_0\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{h^2}{m_0} \nabla_2^2 \psi
\]

\[
+ \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi
\]

\[
+ \frac{4\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{dU}{dr_1} \frac{\partial \psi}{\partial r_1} + \frac{dU}{dr_2} \frac{\partial \psi}{\partial r_2} \right)
\]

\[
+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\nabla_1^2 + \nabla_2^2) \psi
\]

\[
+ \frac{h^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (\nabla_1^2 + \nabla_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi
\]

\[
- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi
\]

\[
- \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{1}{r_1 dr_1} S_1 \cdot L_1 \psi + \frac{1}{r_2 dr_2} S_2 \cdot L_2 \psi \right)
\]

\[
- \frac{h^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2) \psi.
\]

(39)

Where $L_1$ is the orbital angular momentum of the first particle, and $L_2$ is that of the
second one. \( \Psi \) is the stationary relativistic wave function

\[
\Psi(r_1, r_2, s_{1z}, s_{2z}, t) = \psi(r_1, r_2, s_{1z}, s_{2z}) \exp(-iEt/\hbar).
\]

For an isolated two-particle system, in the center-of-momentum frame, \( p_2 = -p_1 \), \( |p_1| = |p| \), and \( |p| \) is the relative momentum of the two-particle system. Based on the corresponding relation between momentum operators and gradient operators, according to

\[
p = 2\frac{\hbar}{i} \partial \Psi/\partial r,
\]

we have

\[
\nabla = 2\frac{\hbar}{i} \partial \Psi/\partial r.
\]

Therefore, if using the center-of-momentum frame in the wave equation (38), supposing an isolated two-particle system, then Using \( \nabla_2 = -\nabla_1 = -\nabla \), where \( \nabla_1, \nabla_2 \) and \( \nabla \) are gradient operators corresponding to the coordinates \( r_1, r_2 \) and \( r = r_1 - r_2 \).

Considering (27), the wave equation (38) can be expressed as

\[
\begin{align*}
\text{i}\hbar \frac{\partial \Psi}{\partial t} &= -\frac{4m^2\hbar^2}{(\mu + \mu)(m_0 + m)^2} \nabla^2 \Psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \\
&+ \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \Psi}{\partial r} \\
&+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{2mU - U^2}{c^2} \right) \nabla^2 \Psi \\
&+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( \frac{2mU - U^2}{c^2} \right) \right] \Psi \\
&- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi \\
&- \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} (S_1 + S_2) \cdot (r \times \Psi) + m_0c^2 \Psi.
\end{align*}
\]

The total spin angular momentum of the system is \( S = S_1 + S_2 \), and the orbital angular momentum \( \mathbf{L} \) is

\[
\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = r_1 \times \mathbf{p}_1 + r_2 \times \mathbf{p}_2 = (r_1 - r_2) \times \mathbf{p}_1 = r \times \mathbf{p}.
\]
Therefore, the total orbital angular momentum $L$ of the two-particle system in the center-of-momentum frame is equal to the cross product of the relative coordinate $r$ and the relative momentum $p$. Combining the first and the fourth terms on the right-hand side of (40), we have

Let $S$, $L$ be the total spin angular momentum and total orbital angular momentum operators of the two-particle system respectively, $U(r)$ be the interaction energy between particles, then in the central field, the stationary wave function for the two-particle system in the center-of-momentum frame

$$\Psi(r, s_z, t) = \psi(r, s_z) \exp(-iEt/\hbar)$$

is determined by the following relativistic wave function and natural boundary conditions, namely

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{4\hbar^2}{(\mu_0 + \mu)(m_0 + m)^2} \left( m - \frac{U}{c^2} \right)^2 \nabla^2 \Psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \Psi + \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \Psi}{\partial r} + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \Psi - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \Psi - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} S \cdot L \Psi + m_0c^2\Psi. \quad (41)$$

$$E' \psi = -\frac{4\hbar^2}{(\mu_0 + \mu)(m_0 + m)^2} \left( m - \frac{U}{c^2} \right)^2 \nabla^2 \psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi + \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \psi}{\partial r} + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \psi - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} S \cdot L \psi. \quad (42)$$
is clearly correct according to (33). As for $S \neq 0$, the correctness of (42) still needs to be further verified.

3. Relativistic Energy Levels for Two-Particle Systems with Zero Total Spin Angular Momentum

A hydrogen-like atom that the total spin angular momentum is zero, and the pionium composed by $\pi^-$ and $\pi^+$, are both two-particle systems. This type of potential energy of interaction between particles is $U = -Ze^2_s/r$, $e_s = e(4\pi\varepsilon_0)^{-1/2}$, considering

$$\nabla^2 \frac{1}{r} = -4\pi\delta(r), \quad r = |r_1 - r_2|,$$

the wave equation (33) can be further expressed as

$$E'\psi = -\frac{4\hbar^2}{(m + m)^2(\mu_0 + \mu)} \left( m + \frac{Ze^2_s c^2}{r} \right)^2 \nabla^2 \psi$$

$$- \frac{2\mu}{(\mu_0 + \mu)(m + m)} \left( \frac{2mZe^2_s}{r} + \frac{Z^2e^4_s}{c^2r^2} \right) \psi$$

$$+ \frac{8\hbar^2}{(m + m)^2(\mu_0 + \mu)c^2} \left( m + \frac{Ze^2_s c^2}{r} \right) \frac{Ze^2_s}{r^2} \frac{\partial}{\partial r} \psi$$

$$- \frac{1}{(m + m)^2(\mu_0 + \mu)} \left( \frac{2mZe^2_s}{r} + \frac{Z^2e^4_s}{c^2r^2} \right)^2 \psi$$

$$- \frac{4\hbar^2}{(m + m)^2(\mu_0 + \mu)c^2} \frac{Z^2e^4_s}{c^2r^4} \psi$$

$$+ \frac{16\pi m\hbar^2 Ze^2_s}{(m + m)(\mu_0 + \mu)c^2} \delta(r)\psi. \quad (43)$$

Now let us solve the wave equation (43) under the condition of $r > 0$, when $\delta(r) = 0$. Using the spherical polar coordinates, the Laplace operator $\nabla^2$ is expressed as

$$\nabla^2 = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right].$$

Supposing $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$, substituting it into (43) and considering $\delta(r) = 0$,
we have
\[
\frac{(m_0 + m)^2(\mu_0 + \mu)E'^2}{4\hbar^2[m + Ze_s^2/(c^2r)]^2} + \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{2Ze_s^2}{[m + Ze_s^2/(c^2r)]^2} \frac{1}{dR} R dr + \frac{\mu(m_0 + m)r^2}{2\hbar^2[m + Ze_s^2/(c^2r)]^2} \left( \frac{2mZe_s^2}{r} + \frac{Z^2e_s^4}{c^2r^2} \right)
\]
\[
+ \frac{r^2}{4\hbar^2c^2[m + Ze_s^2/(c^2r)]^2} \left( \frac{2mZe_s^2}{r} + \frac{Z^2e_s^4}{c^2r^2} \right)^2
\]
\[
+ \frac{1}{[m + Ze_s^2/(c^2r)]^2} \frac{Z^2e_s^4}{c^4r^2}
\]
\[
= - \frac{1}{Y} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} \right] = \lambda.
\]

According to (45), denoting \( \lambda = l(l + 1), \ l = 0, 1, 2, \ldots \), obviously the solution of the equation is the spherical harmonics \( Y_{lm}(\theta, \varphi) \).

Now let us solve the radial equation (44), discussing the situation of the bound state (\( E' < 0 \)). Let
\[
\alpha' = \frac{m_0 + m}{\hbar} (\mu_0 + \mu)|E'|^{1/2}, \ \ \rho = \alpha' r,
\]
\[
\beta = \frac{\mu(m_0 + m)Ze_s^2}{\alpha' m \hbar^2} = \frac{Ze_s^2}{\hbar} \left[ \frac{\mu^2}{(\mu_0 + \mu)|E'|} \right]^{1/2},
\]
using the variable substitution \( \rho = \alpha' r \), then (44) can be expressed as:
\[
\left( 1 + \frac{d_0}{\beta \rho} \right)^2 \frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR}{d\rho} \right) - \left( 1 + \frac{d_0}{\beta \rho} \right) \frac{2d_0 dR}{\beta \rho^2 d\rho} + \frac{\beta}{\rho} \left( 1 + \frac{d_0}{2\beta \rho} \right) R
\]
\[
- \frac{1}{4} R + \left( 1 + \frac{d_0}{2\beta \rho} \right)^2 \frac{Z^2 \alpha'^2}{\rho^2} R + \frac{d_0^2}{\beta^2 \rho^4} R - \left( 1 + \frac{d_0}{\beta \rho} \right)^2 \frac{l(l + 1)}{\rho^2} R = 0.
\]

Where \( \alpha \) denotes the fine structure constant. \( d_0 \), which is a small parameter, denotes
\[
d_0 = 2Z^2 \alpha^2 D, \ \ D = \frac{\mu(m_0 + m)}{2m^2}, \ \ \alpha = \frac{e^2}{\hbar c}.
\]
Let $R(\rho) = u(\rho)/\rho$, considering

$$\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR}{d\rho} \right) = \frac{1}{\rho} \frac{d^2}{d\rho^2}(\rho R),$$

then (48) can be expressed as:

$$\left( 1 + \frac{d_0}{\beta \rho} \right)^2 \frac{d^2 u}{d\rho^2} - \left( 1 + \frac{d_0}{\beta \rho} \right) \frac{2d_0}{\beta \rho^2} \frac{du}{d\rho} + \frac{2d_0}{\beta \rho^3} u + \frac{\beta}{\rho} \left( 1 + \frac{d_0}{2\beta \rho} \right) u
- \frac{1}{4} u + \left( 1 + \frac{d_0}{2\beta \rho} \right)^2 \frac{Z^2 \alpha^2}{\rho^2} u + \frac{3d_0^2}{\beta^2 \rho^4} u - \left( 1 + \frac{d_0}{\beta \rho} \right)^2 \frac{l(l+1)}{\rho^2} u = 0. \quad (50)$$

Firstly, let us study the asymptotic behavior of this equation, when $\rho \to \infty$, the equation can be transformed into the following form:

$$\frac{d^2 u}{d\rho^2} - \frac{1}{4} u = 0, \quad u(\rho) = \exp(\pm \rho/2).$$

As $\exp(\rho/2)$ is in conflict with the finite conditions of wave functions, we substitute $u(\rho) = \exp(-\rho/2)f(\rho)$ into the equation, then we have the equation satisfied by $f(\rho)$:

$$\left( 1 + \frac{d_0}{\beta \rho} \right)^2 \frac{d^2 f}{d\rho^2} - \left( 1 + \frac{d_0}{\beta \rho} \right) \left( 1 + \frac{d_0}{\beta \rho} + \frac{2d_0}{\beta \rho^2} \right) \frac{df}{d\rho}
+ \left( \frac{\beta}{\rho} + \frac{d_0}{2\beta \rho} \right) \left( 1 + \frac{d_0}{2\beta \rho} \right) f
+ \left( 1 + \frac{2}{\rho} + \frac{d_0}{\beta \rho} + \frac{3d_0}{\beta \rho^2} \right) \frac{d_0}{\beta \rho^2} f
+ \left( 1 + \frac{d_0}{2\beta \rho} \right)^2 \frac{Z^2 \alpha^2}{\rho^2} f - \left( 1 + \frac{d_0}{\beta \rho} \right)^2 \frac{l(l+1)}{\rho^2} f = 0. \quad (51)$$

Thus solving for the radial wave function $R(\rho)$ comes down to solving for $f(\rho)$, namely

$$R(\rho) = \frac{1}{\rho} \exp \left( -\frac{1}{2} \rho \right) f(\rho), \quad \rho = \frac{2Z}{\beta a_0} r, \quad a_0 = \frac{2m}{m_0 + m \mu e_s^2}. \quad (52)$$

According to (47) we have

$$\beta^2 = \frac{Z^2 e_s^4}{h^2} \frac{\mu^2}{(\mu_0 + \mu)|E'|}.$$ 

Substituting $\mu = \mu_0 - |E'|/c^2$ into the equation above, we have

$$(Z^2 \alpha^2 + \beta^2)|E'|^2 - 2 \mu_0 c^2 (Z^2 \alpha^2 + \beta^2)|E'| + \mu_0 c^4 Z^2 \alpha^2 = 0. \quad (53)$$

Solving (53), we obtain two roots of $|E'|$:

$$|E'| = \mu_0 c^2 \pm \mu_0 c^2 \left( 1 + \frac{Z^2 \alpha^2}{\beta^2} \right)^{-1/2}.$$
Thus we obtain the system mass $\mu$ corresponding to the reduced mass $\mu_0$:

$$\mu = \pm \mu_0 \left(1 + \frac{Z^2 \alpha^2}{\beta^2}\right)^{-1/2}. \quad (54)$$

According to Definition 6, $\mu_0$, $\mu$ respectively denote

$$\mu_0 = \frac{2m_{01}m_{02}}{m_m}, \quad \mu = \mu_0 - \frac{1}{c^2} |E'|.$$

Considering $m = m_0 - |E'|/c^2$, $m_0 = m_{01} + m_{02}$, $|E'|$ is derived by (54). As the total energy of the system $E = m_0c^2 - |E'| = mc^2$, we can further obtain the total energy $E$ and the system mass $m$. According to (54), $\mu$ has positive and negative values. When $\mu$ takes on positive values, $E$ is expressed as

$$E = \pm \left[ m_{01}^2 + 2m_{01}m_{02} \left(1 + \frac{Z^2 \alpha^2}{\beta^2}\right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2. \quad (55)$$

When $\mu$ takes on negative values, $E$ is expressed as

$$E = \pm \left[ m_{01}^2 - 2m_{01}m_{02} \left(1 + \frac{Z^2 \alpha^2}{\beta^2}\right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2. \quad (56)$$

Therefore we have two positive and two negative energy solutions. Negative energy solutions are related to the ubiquity of antimatter, which will not be discussed in this paper. Taking on positive energy solutions, we have

$$E = \left[ m_{01}^2 + \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2/\beta^2}} + m_{02}^2 \right]^{1/2} c^2. \quad (57)$$

$$E = \left[ m_{01}^2 - \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2/\beta^2}} + m_{02}^2 \right]^{1/2} c^2. \quad (58)$$

Corresponding to the two positive energy solutions, the system mass $m$ has two expressions:

$$m = \left[ m_{01}^2 + \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2/\beta^2}} + m_{02}^2 \right]^{1/2}. \quad (57)$$

$$m = \left[ m_{01}^2 - \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2/\beta^2}} + m_{02}^2 \right]^{1/2}. \quad (58)$$

Clearly, when (54) takes on positive values, the system mass $m$ is expressed by (57). But when (54) takes on negative values, $m$ is expressed by (58).
In (55) and (56), the quantization of $E$ is mirrored by the fact that $\beta$ is related to both the principal quantum number $n$ and the angular quantum number $l$, i.e. $\beta = \beta(n,l)$. Solving the equation (51) we obtain the expression of $\beta(n,l)$. Therefore (55) and (56) are the general expressions of the relativistic energy levels for two-particle systems.

It seems difficult to accurately solve (51). Let us solve this equation for approximate solutions to obtain the approximate expression of $\beta(n,l)$. First, (51) is expressed by the standard form of second-order ordinary differential equations

$$f'' + p(\rho)f' + q(\rho)f = 0.$$ 

Where $p(\rho), q(\rho)$ respectively denote

$$p(\rho) = -\left(1 + \frac{d_0}{\beta \rho}\right)^{-1} \left(1 + \frac{d_0}{\beta \rho} + \frac{2d_0}{\beta^2 \rho^2}\right),$$

$$q(\rho) = \left(1 + \frac{d_0}{\beta \rho}\right)^{-2} \left[\left(\beta + \frac{d_0}{2 \beta}\right) \frac{1}{\rho} + \left(Z^2 \alpha^2 - l(l + 1) + \frac{d_0}{2} + \frac{d_0}{\beta} + \frac{d_0^2}{4 \beta^2}\right) \frac{1}{\rho^2}\right]$$

$$+ \left(1 + \frac{d_0}{\beta \rho}\right)^{-2} \left(Z^2 \alpha^2 - 2l(l + 1) + 2 \frac{d_0}{\beta} \right) \frac{d_0}{\beta} \frac{1}{\rho^3}$$

$$+ \left(1 + \frac{d_0}{\beta \rho}\right)^{-2} \left(Z^2 \alpha^2 - 4l(l + 1) + 12 \frac{d_0^2}{4 \beta^2} \frac{1}{\rho^4}\right).$$

Considering $d_0$ is very small, we have $p(\rho) \approx -1$, and $q(\rho)$ is approximately expressed as

$$q(\rho) \approx \left(\beta + \frac{d_0}{2 \beta}\right) \frac{1}{\rho} + \left(Z^2 \alpha^2 - l(l + 1) - \frac{3d_0}{2} + \frac{d_0}{\beta}\right) \frac{1}{\rho^2}.$$ 

(51) is approximately expressed as

$$\frac{d^2 f}{d\rho^2} - \frac{df}{d\rho} + \left[\left(\beta + \frac{d_0}{2 \beta}\right) \frac{1}{\rho} + \left(Z^2 \alpha^2 - l(l + 1) - \frac{3d_0}{2} + \frac{d_0}{\beta}\right) \frac{1}{\rho^2}\right] f = 0. \quad (59)$$

Thus $\rho = 0$ is a regular singular point of the equation (59). Suppose the series solution of this equation can be expressed as

$$f(\rho) = \sum_{\nu=0}^{\infty} b_{\nu} \rho^{s+\nu}, \quad b_0 \neq 0. \quad (60)$$

In order to guarantee the finiteness of $R = u/\rho$ at $\rho = 0$, $s$ should be no less than 1. By substituting (60) into (59), as the coefficient of $\rho^{s+\nu-1}$ is equal to zero, we have the relation satisfied by $b_{\nu}$:

$$b_{\nu+1} = \frac{s + \nu - \lceil\beta + d_0/(2 \beta)\rceil}{(s + \nu)(s + \nu + 1) - l(l + 1) + Z^2 \alpha^2 - 3d_0/2 + d_0/\beta} b_{\nu}. \quad (61)$$
If the series are infinite series, then when \( \nu \to \infty \) we have \( b_{\nu+1}/b_{\nu} \to 1/\nu \). Therefore, when \( \rho \to \infty \), the behaviour of the series is the same as that of \( e^\rho \), thus \( f(\rho) \) in (52) tends to infinity when \( \rho \to \infty \), which is in conflict with the finite conditions of wave functions. Therefore, the series should only have finite terms. Let \( b_n, \rho^{s+n_\nu} \) be the highest-order term, then \( b_{n\nu+1} = 0 \). By substituting \( \nu = n_\nu \) into (61) we have \( \beta + d_0/(2\beta) = n_\nu + s \). On the other hand, the series starts from \( \nu = 0 \), therefore, \( b_{-1} = 0 \). Substituting \( \nu = -1 \) into (61), considering \( b_0 \neq 0 \), we have \( s(s-1) = l(l+1) - Z^2\alpha^2 + 3d_0/2 - d_0/\beta \). Denoting \( n = n_\nu + l + 1 \), then the following set of equations can be solved for \( s \) and \( \beta \):

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad s(s-1) = l(l+1) - Z^2\alpha^2 + 3d_0/2 - d_0/\beta \\
\quad \beta + d_0/(2\beta) = n_\nu + s \\
\quad n = n_\nu + l + 1
\end{array} \right.
\end{align*}
\]

(62)

We derive \( s = 1/2 \pm \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta} \). Considering \( s \) should not be less than 1, \( s = 1/2 + \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta} \). Thus we obtain a specific expression of \( \beta(n,l) \):

\[
\beta = n - l - 1/2 - d_0/2\beta + \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta} = n - \sigma_l. \tag{63}
\]

Where \( \sigma_l = l + 1/2 + d_0/(2\beta) - \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta} \).

Therefore, the relativistic energy levels for two-particle systems (55)-(56), the system mass \( (57)-(58) \) and \( (54) \) can be respectively expressed as

\[
E_n = \left[ m_{01}^2 + 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2. \tag{64}
\]

\[
E_n = \left[ m_{01}^2 - 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2. \tag{65}
\]

\[
m = \left[ m_{01}^2 + 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2}. \tag{66}
\]

\[
m = \left[ m_{01}^2 - 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2}. \tag{67}
\]

\[
\mu = \pm \mu_0 \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2}, \quad \mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}. \tag{68}
\]

\[
\sigma_l = l + 1/2 + \frac{d_0}{2(n - \sigma_l)} - \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - \frac{d_0}{n - \sigma_l}}. \tag{69}
\]
\[ d_0 = 2Z^2 \alpha^2 D, \quad D = \frac{\mu (m_0 + m)}{2m^2}. \]  

(70)

Considering a two-particle system, which is a hydrogen-like atom composed by a spin-zero nucleus (like the deuteron) and a \( \pi^- \), called a pionic hydrogen atom, \( m_{01}, m_{02} \) are the rest mass of \( \pi^- \) and nucleus \( (m_{01} \ll m_{02}) \), then (64) can be expanded as the following fast convergent infinite series

\[
E_n = m_{02}c^2 + m_{01}c^2 \left( 1 + \frac{Z^2 \alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} \\
+ \frac{1}{2} m_{01}^2 c^2 \frac{m_{01}}{m_{02}} \frac{Z^2 \alpha^2}{(n - \sigma_l)^2} \left( 1 + \frac{Z^2 \alpha^2}{(n - \sigma_l)^2} \right)^{-1} \\
- \frac{1}{2} m_{01} c^2 \left( m_{01} \right)^2 \frac{Z^2 \alpha^2}{(n - \sigma_l)^2} \left( 1 + \frac{Z^2 \alpha^2}{(n - \sigma_l)^2} \right)^{-3/2} + \cdots .
\]

Clearly, we obtain the normal energy levels for two-particle systems. Using (64), we can calculate the energy spectrum of pionic hydrogen atoms more accurately. The energy levels expressed by (65) are called the abnormal energy levels. Unlike the normal energy levels, the abnormal energy levels decrease with increasing the principal quantum number \( n \).

According to (64)-(70), we need to use iterative methods for calculation. As \( d_0 \) is very small, taking \( d_0 = 0 \) in the expression of \( \sigma_l \), we can obtain the zeroth order approximation of \( \sigma_l \) to calculate that of \( m \) and \( \mu \), then substitute them into \( d_0 \) to obtain its zeroth order approximation. Substituting the zeroth-order approximation of \( d_0 \) and \( \sigma_l \) into \( \sigma_l \) to calculate its first-order approximation, repeating this process, using the first-order approximation of \( \sigma_l \) to calculate that of \( m \) and \( \mu \), substitute them into \( d_0 \) to obtain its first-order approximation. Substituting the first-order approximation of \( d_0 \) and \( \sigma_l \) into \( \sigma_l \) to calculate its second-order approximation, and repeating this process we can calculate the \( n \)th-order approximation of \( \sigma_l \), further, we can obtain the energy levels \( E_n \) and reach the required accuracy. This calculation process can also be realized by computer programming. Note that (68) means \( \mu \) has both positive and negative values. When calculating normal energy levels, \( \mu > 0 \), the system mass uses (66). When calculating abnormal energy levels, \( \mu < 0 \), \( m \) uses (67).

Then what is the physical meaning of abnormal energy levels? A bound state composed of a positive and a negative pion is called the pionium. The positronium, pionium, protonium, neutronium, etc., are generally called the particleium. For this type of system, we have \( Z = 1 \) and \( m_{01} = m_{02} \). Therefore, when the pionium is at abnormal energy levels, according to (65) we have

\[
\lim_{n \to \infty} E_n \to (m_{01}^2 - 2m_{01}m_{02} + m_{02}^2)^{1/2}c^2 = (m_{01} - m_{02})c^2 = 0.
\]
What is the physical meaning of this result? According to $E_n = mc^2$, we have $m = 0$, which means the disappearance of the particle system and the annihilation of a pair of positive and negative pions. In relativistic quantum mechanics, the meaning of the vacuum state should not be restricted to a state that the energy is zero. For any bound state composed of a particle-antiparticle pair, if it is at abnormal energy levels, it is in the vacuum state. Therefore, the annihilation of a pair of positive and negative pions has two phases. The first one composes the pionium, while the second one is its transition from normal energy level expressed by (64) to abnormal energy levels expressed by (65). If this process produces $\gamma$ photons, it means a pair of positive and negative pions annihilates into photons. The reverse process is the pair production of positive and negative pions. A reasonable extension of this concept is that after the annihilation of any particle-antiparticle pair, a small percentage of the energy is generally given to the abnormal energy levels of the particleium. Thus this percentage of energy is also quantized, and its energy spectrum is given by the abnormal energy levels of the particleium. For instance, let $m_\pi$ be the rest mass of $\pi^+$, then the abnormal energy levels of the pionium can be expressed as

$$E_n = \sqrt{2m_\pi c^2} \sqrt{1 - \left(1 + \frac{\alpha^2}{(n - \sigma)^2}\right)^{-1/2}}.$$ 

The non-relativistic approximation of this formula can be easily obtained:

$$E_n = \frac{\alpha m_\pi c^2}{n}, \quad n = 1, 2 \ldots$$

which is completely different from the energy spectrum of normal matter expressed by (64). The abnormal energy levels expressed by (65) clearly can not be applied to atoms, but may be applied to the production and annihilation of pionium.

4. Conclusion

In conclusion, by introducing Definition 1-6, using (2) and assuming the relativistic kinetic expression is tenable on a wider scale, the relativistic wave equations for two-particle systems is derived, and the new relativistic two-body wave equations are obtained. By applying this type of wave equations to pionium and pionic hydrogen atoms, the general expression and specific calculation formulas of relativistic energy levels for two-particle systems are derived. Besides, we further find the relativistic abnormal energy levels, thus the pair production and annihilation of particles and antiparticles boil down to the transition between normal and abnormal energy levels of two-particle systems.
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