Chang’s lemma via Pinsker’s inequality

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Abstract

Extending the idea in [9] we give a short information theoretic proof for Chang’s lemma that is based on Pinsker’s inequality.

1 Introduction and the proof

In recent years, there is a growing interest in applying information theoretic arguments to combinatorics and theoretical computer science. For example, Fox’s new proof [5] of the graph removal lemma, Tao’s solution [12, 13] of the Erdős discrepancy problem, and the application of information theory to communication complexity by Braverman et al. [3]. For more discussion, see the surveys [11, 8, 14, 2]. The purpose of this note is to give a short information theoretic proof of Chang’s lemma, an important result and tool in both additive combinatorics and theoretical computer science.

For every function $f : \{-1, 1\}^n \to \mathbb{R}$, and every $S \subseteq [n]$, define $\hat{f}(S) = \mathbb{E}_{x \sim \{-1, 1\}^n} f(x) \chi_S(x)$, where $\chi_S(x) = \prod_{i \in S} x_i$. The numbers $\hat{f}(S)$ are called Fourier coefficients of $f$, and $f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(x)$ is called the Fourier expansion of $f$. Given these definitions, Chang’s lemma states the following.

Theorem 1 (Chang’s lemma, [4]). Let $A \subseteq \{-1, 1\}^n$ have density $\alpha = \frac{|A|}{2^n}$. Let $f = 1_A$ denote the characteristic function of $A$, that is $f(x) = 1$ if $x \in A$, and $f(x) = 0$ if $x \notin A$. Then,

$$\sum_{i=1}^n \hat{f}(\{i\})^2 \leq 2\alpha^2 \ln \frac{1}{\alpha}.$$ 

The original Chang’s lemma [4] is stated for more general groups than $\{-1, 1\}^n$, and its proof relies on Rudin’s inequality. Our proof is inspired by [9], where a proof for $\{-1, 1\}^n$ is given using entropy and Taylor expansion. However, our proof is shorter and more direct by replacing the Taylor expansion with Pinsker’s inequality.

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Let $p$ and $q$ be two probability distributions on a finite space $\Omega$. The Shannon entropy of $p$ is defined as $H(p) = -\sum_{x \in \Omega} p(x) \ln p(x)$. The Kullback-Leibler divergence from $q$ to $p$ is defined as $D(p||q) = \sum_{x \in \Omega} p(x) \ln \frac{p(x)}{q(x)}$, assuming $p(x) = 0$ whenever $q(x) = 0$. Observe that $D(p||q) = H(q) - H(p)$ if $q$ is the uniform distribution. Let $\|\cdot\|_1$ denote the $L_1$ norm: $\|g\|_1 = \sum_x |g(x)|$.

Pinsker’s inequality states $D(p||q) \geq \frac{1}{2} \|p - q\|_1^2$.

**Proof of Theorem** [7] Let $p$ be the uniform distribution on the set $A$, and $q$ be the uniform distribution on $\{-1, 1\}^n$. For every $i \in [n]$, denote the corresponding marginal distribution $p_i$ of $p$ as the pair $p_i = (\alpha_i, 1 - \alpha_i)$ where $\alpha_i = \Pr[x_i = 1| x \in A]$. The marginal distributions of $q$ are $q_i = (1/2, 1/2)$, i.e., they are uniform distributions on $\{-1, 1\}$. As the marginals of $q$ are independent, we have $H(q) = \sum_{i=1}^n H(q_i)$. Observe

$$\hat{f}(\{i\})^2 = (\mathbb{E}_x f(x)x_i)^2 = \alpha^2 (\alpha_i - (1 - \alpha_i))^2$$

$$= \alpha^2 \left(\left|\alpha_i - \frac{1}{2}\right| + \left|1 - \alpha_i - \frac{1}{2}\right|\right)^2 = \alpha^2 \|p_i - q_i\|_1^2. \quad (1)$$

By the subadditivity of Shannon entropy and Pinsker’s inequality,

$$\ln \frac{1}{\alpha} = D(p||q) = H(q) - H(p)$$

$$\geq \sum_{i=1}^n \left( H(q_i) - H(p_i) \right) = \sum_{i=1}^n D(p_i||q_i) \geq \frac{1}{2} \sum_{i=1}^n \|p_i - q_i\|_1^2. \quad (2)$$

Combining (1) and (2) gives the desired bound. $\square$

## 2 Concluding remarks

Firstly, let $W^k = \sum_{|S|=k} \hat{f}(S)^2$. In the analysis of boolean functions, Chang’s Lemma is also called as the level-1 inequality (see [10]), since it gives an upper bound for $W^1$. There is a generalization of Chang’s lemma that states $\sum_{|S| \leq k} \hat{f}(S)^2 \leq (\frac{2k}{k} \ln (1/\alpha))^k \alpha^2$ whenever $k \leq 2 \ln (1/\alpha)$. This is called the level-$k$ inequality in [10]. Can our argument be generalized to give a simple proof of the level-$k$ inequality? On the one hand, the level-$k$ inequality [10] can be derived from hypercontractivity which adopts some entropic proofs (see [7, 11, 6]). This indicates some hope. On the other hand, the level-$k$ inequality only holds for sets $A$ with small density depending on $k$ for every $k \geq 2$. However, it is unclear how this constraint on the density can appear in an informational argument.

For example, Pinsker’s inequality does not have any constraint.

Secondly, we show that the inequality $D(p||q) \geq \sum_{i=1}^n D(p_i||q_i)$ that appears in (2) can be generalized to the case whenever $q$ is a product distribution. Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ be a finite product space. Let $p$ and $q$ be two probability distributions on $\Omega$, and let $p_i$ and $q_i$ denote the marginal distribution of $p$ and $q$ on $\Omega_i$, respectively.
Lemma 1 (supadditivity). If \( q \) is a product distribution, then

\[
D(p||q) \geq \sum_{i=1}^{n} D(p_i||q_i).
\]

Proof. By induction, it suffices to prove it for \( n = 2 \). Now suppose \( n = 2 \). For notational clarity, denote \( p \) by \( p(X,Y) \) where \( (X,Y) \in \Omega_1 \times \Omega_2 \), and similarly for \( q \). By the chain rule of divergence

\[
D(p(X,Y)||q(X,Y)) = D(p(X)||q(X)) + D(p(Y|X)||q(Y|X)).
\]

Hence, it suffices to show that \( D(p(Y|X)||q(Y|X)) \geq D(p(Y)||q(Y)) \). By the definition of divergence, one has

\[
D(p(Y|X)||q(Y|X)) - D(p(Y)||q(Y)) \geq \sum_{(x,y)\in \Omega_1 \times \Omega_2} p(x,y) \ln \frac{p(y|x)}{q(y|x)} - \sum_{y\in \Omega_2} p(y) \ln \frac{p(y)}{q(y)}.
\]

One can apply Lemma 1 directly in (2) without using Shannon entropy. We point out that the supadditivity of the Kullback-Leibler divergence in Lemma 1 is not necessarily true if \( q \) is not a product distribution. Let \( p(X,Y), q(X,Y) \) be two distributions given by

\[
p = \begin{pmatrix}
1/4 & 1/4 \\
1/4 & 1/4
\end{pmatrix}, \quad q = \begin{pmatrix}
1/4 - 3\epsilon & 1/4 + \epsilon \\
1/4 + \epsilon & 1/4 + \epsilon
\end{pmatrix},
\]

where \(-1/4 < \epsilon < 1/12\). In particular, \( p \) is a product distribution. We will choose \( \epsilon \) such that \( q \) is not a product distribution. The marginal distributions are: \( p(X) = (1/2,1/2) \), \( p(Y) = (1/2,1/2) \), and \( q(X) = (1/2 - 2\epsilon,1/2 + 2\epsilon) \), \( q(Y) = (1/2 - 2\epsilon,1/2 + 2\epsilon) \). Let \( \epsilon = 0.01 \), using Wolfram Mathematica,

\[
D(p(X,Y)||q(X,Y)) \approx 0.0025 > D(p(X)||q(X)) + D(p(Y)||q(Y)) \approx 0.0016.
\]

Let \( \epsilon = -0.2 \), using Wolfram Mathematica,

\[
D(p(X,Y)||q(X,Y)) \approx 0.90 < D(p(X)||q(X)) + D(p(Y)||q(Y)) \approx 1.02.
\]

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References

[1] Eric Blais and Li-Yang Tan. Hypercontractivity via the entropy method. *Theory of Computing*, 9(29):889–896, 2013.

[2] Mark Braverman. Interactive information and coding theory. In *Proceedings of the International Congress of Mathematicians*, pages 535–559. Citeseer, 2014.

[3] Mark Braverman, Ankit Garg, Denis Pankratov, and Omri Weinstein. From information to exact communication. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 151–160. ACM, 2013.

[4] Mei-Chu Chang. A polynomial bound in freiman’s theorem. *Duke mathematical journal*, 113(3):399–419, 2002.

[5] Jacob Fox. A new proof of the graph removal lemma. *Annals of Mathematics*, pages 561–579, 2011.

[6] Ehud Friedgut. An information-theoretic proof of a hypercontractive inequality. *arXiv preprint arXiv:1504.01506*, 2015.

[7] Ehud Friedgut and Vojtech Rödl. Proof of a hypercontractive estimate via entropy. *Israel Journal of Mathematics*, 125(1):369–380, 2001.

[8] David Galvin. Three tutorial lectures on entropy and counting. *arXiv preprint arXiv:1406.7872*, 2014.

[9] Russell Impagliazzo, Cristopher Moore, and Alexander Russell. An entropic proof of chang’s inequality. *SIAM Journal on Discrete Mathematics*, 28(1):173–176, 2014.

[10] Ryan O’Donnell. *Analysis of boolean functions*. Cambridge University Press, 2014.

[11] Jaikumar Radhakrishnan. Entropy and counting. *Computational mathematics, modelling and algorithms*, 146, 2003.

[12] Terence Tao. The erdős discrepancy problem. *Discrete Analysis*, 5202016(1):609, 2016.

[13] Terence Tao. The logarithmically averaged chowla and elliott conjectures for two-point correlations. In *Forum of Mathematics, Pi*, volume 4. Cambridge University Press, 2016.

[14] Julia Wolf. Some applications of relative entropy in additive combinatorics. *Surveys in Combinatorics 2017*, 440:409, 2017.