THE RIGHT ORTHOGONAL CLASS $\mathcal{GP}(R)^\perp$ VIA $\text{Ext}$

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Abstract. In this paper, we study the pair $(\mathcal{GP}(R), \mathcal{GP}(R)^\perp)$ where $\mathcal{GP}(R)$ is the class of all Gorenstein projective modules. We prove that it is complete hereditary cotorsion theory provided $l.Ggldim(R) < \infty$. We discuss also, when every Gorenstein projective module is Gorenstein flat.

1. Introduction

Throughout this paper, $R$ denotes a non-trivial associative ring and all modules-if not specified otherwise- are left and unitary. The definitions and notations employed in this paper are based on those introduced by Holm in [12]. Let $R$ be a ring, and let $M$ be an $R$-module. As usual we use $\text{pd}_R(M)$, $\text{id}_R(M)$ and $\text{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of $M$. We use also $\text{gldim}(R)$ and $\text{wdim}(R)$ to denote, respectively, the classical global and weak dimension of $R$.

For a two-sided Noetherian ring $R$, Auslander and Bridger [1] introduced the G-dimension, $\text{Gdim}_R(M)$, for every finitely generated $R$-module $M$. They showed that there is an inequality $\text{Gdim}_R(M) \leq \text{pd}_R(M)$ for all finite $R$-modules $M$, and equality holds if $\text{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [5, 6] defined the notion of Gorenstein projective dimension ($G$-projective dimension for short), as an extension of $G$-dimension to modules which are not necessarily finitely generated, and the Gorenstein injective dimension ($G$-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [8] introduced the Gorenstein flat dimension. Some references are [4, 5, 6, 8, 12].

Recall that an $R$-module $M$ is called Gorenstein projective if, there exists an exact sequence of projective $R$-modules:

$\mathbf{P} : \ldots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots$

such that $M \cong \text{Im}(P_0 \to P^0)$ and such that the operator $\text{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever $Q$ is a projective. The resolution $\mathbf{P}$ is called a complete projective resolution.

The Gorenstein injective $R$-modules are defined dually.

And an $R$-module $M$ is called Gorenstein flat if, there exists an exact sequence of
flat $R$-modules:

$$F : \ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

such that $M \cong \text{Im}(P_0 \to P^0)$ and such that the operator $I \otimes_R -$ leaves $F$ exact whenever $I$ is a right injective $R$-module. The resolution $F$ is called complete flat resolution.

The Gorenstein projective, injective and flat dimensions are defined in terms of resolution and denoted by $\text{Gpd}(-)$, $\text{Gid}(-)$ and $\text{Gfd}(-)$ respectively (see [3,7,12]).

**Notation.** By $\mathcal{P}(R)$ and $\mathcal{I}(R)$ we denote the classes of all projective and injective $R$-modules respectively and by $\overline{\mathcal{P}}(R)$ and $\overline{\mathcal{I}}(R)$ we denote the classes of all modules with finite projective dimension and injective dimension respectively. Furthermore, we let $\mathcal{GP}(R)$ and $\mathcal{GI}(R)$ denote the classes of all Gorenstein projective and injective $R$-modules respectively. The character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$. 

In [2], the authors prove the equality:

$$\sup\{\text{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup\{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\}$$

They called the common value of the above quantities the left Gorenstein global dimension of $R$ and denoted it by $l.\text{Ggldim}(R)$. Similarly, they set

$$l.\text{wGgldim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}$$

which they called the left weak Gorenstein global dimension of $R$.

Given a class $\mathcal{X}$ of $R$-modules we set:

$$\mathcal{X}^\perp = \ker \text{Ext}^1_R(\mathcal{X}, -) = \{M \mid \text{Ext}^1_R(\mathcal{X}, M) = 0 \text{ for all } X \in \mathcal{X}\},$$

$$\perp \mathcal{X} = \ker \text{Ext}^1_R(-, \mathcal{X}) = \{M \mid \text{Ext}^1_R(M, \mathcal{X}) = 0 \text{ for all } X \in \mathcal{X}\}$$

**Definition 1.1 (Precovers and Preenvelopes).** Let $\mathcal{X}$ be any class of $R$-modules, and let $M$ be an $R$-module.

- An $\mathcal{X}$-precover of $M$ is an $R$-homomorphism $\varphi : X \to M$, where $X \in \mathcal{X}$, and such that the sequence,

$$\text{Hom}_R(X', X) \xrightarrow{\text{Hom}_R(X', \varphi)} \text{Hom}_R(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. An $\mathcal{X}$-precover is called special if $\varphi$ is surjective and $\ker(\varphi) \in \mathcal{X}^\perp$.

- An $\mathcal{X}$-preenvelope of $M$ is an $R$-homomorphism $\varphi : M \to X$, where $X \in \mathcal{X}$, and such that the sequence,

$$\text{Hom}_R(X, X') \xrightarrow{\text{Hom}_R(\varphi, X')} \text{Hom}_R(M, X') \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. An $\mathcal{X}$-preenvelope is called special if $\varphi$ is injective and $\text{coker}(\varphi) \in \perp \mathcal{X}$.

For more details about precovers (and preenvelopes), the reader may consult [7] Chapters 5 and 6.

**Definition 1.2 (Resolving classes 1.1 , [12]).** For any class $\mathcal{X}$ of $R$-modules.
• We call \( \mathfrak{X} \) projectively resolving if \( \mathcal{P}(R) \subseteq \mathfrak{X} \), and for every short exact sequence \( 0 \to X' \to X \to X'' \to 0 \) with \( X'' \in \mathfrak{X} \) the conditions \( X' \in \mathfrak{X} \) and \( X \in \mathfrak{X} \) are equivalent.

• We call \( \mathfrak{X} \) injectively resolving if \( \mathfrak{I}(R) \subseteq \mathfrak{X} \), and for every short exact sequence \( 0 \to X' \to X \to X'' \to 0 \) with \( X' \in \mathfrak{X} \) the conditions \( X'' \in \mathfrak{X} \) and \( X \in \mathfrak{X} \) are equivalent.

A pair \((\mathfrak{X}, \mathfrak{Y})\) of classes of \( R \)-modules is called a cotorsion theory \((\mathfrak{I}, \mathfrak{P})\) if \( \mathfrak{X} = \mathfrak{Y} \) and \( \mathfrak{Y} = \mathfrak{X} \). In this case we call \( \mathfrak{X} \cap \mathfrak{Y} \) the kernel of \((\mathfrak{X}, \mathfrak{Y})\). Note that each element \( K \) of the kernel is a splitter in the sense of [11], i.e., \( \operatorname{Ext}^1_R(K, K) = 0 \). If \( \mathfrak{C} \) is any class of modules, then \((\mathfrak{C}^\perp, (\mathfrak{C}^\perp)^\perp)\) is easy seen to be a cotorsion theory, called a cotorsion theory generated by \( \mathfrak{C} \) (see please [13, Definition 1.10]). A cotorsion theory \((\mathfrak{X}, \mathfrak{Y})\) is called complete \((\mathfrak{I}, \mathfrak{P})\) if every \( R \)-module has a special \( \mathfrak{Y} \)-preenvelope and a special \( \mathfrak{X} \)-precover. A cotorsion theory \((\mathfrak{X}, \mathfrak{Y})\) is said to be hereditary \((\mathfrak{I}, \mathfrak{P})\) if whenever \( 0 \to L' \to L \to L'' \to 0 \) is exact with \( L, L'' \in \mathfrak{X} \) then \( L' \) is also in \( \mathfrak{X} \), or equivalently, if \( 0 \to M' \to M \to M'' \to 0 \) is exact \( M', M \in \mathfrak{Y} \) then \( M'' \) is also in \( \mathfrak{Y} \).

**Note:** Above we have only proved the results concerning Gorenstein projective modules. The proof of the Gorenstein injective ones is dual and we can find a dual of results using in the proofs in [12].

### 2. Main Results

The aim of this section is the study of the pair \((\mathcal{GP}(R), \mathcal{GP}(R)^\perp)\). The class \( \mathcal{GP}(R) \) verified the following properties.

**Theorem 2.1.** For any ring \( R \) the following holds:

1. \( \operatorname{Ext}^i_R(G, M) = 0 \) for all \( i > 0 \), all \( G \in \mathcal{GP}(R) \) and all \( M \in \mathcal{GP}(R)^\perp \).
2. \( \operatorname{Ext}^i_R(M, G') = 0 \) for all \( i > 0 \), all \( G' \in \mathcal{GP}(R) \) and all \( M \in \mathcal{GP}(R)^\perp \).
3. \( \mathcal{GP}(R)^\perp \) and \( \mathcal{GP}(R) \) are projectively resolving.
4. \( \mathcal{GP}(R)^\perp \) and \( \mathcal{GP}(R) \) are injectively resolving.

**Proof.** (1). Consider \( M \in \mathcal{GP}(R)^\perp \). For any Gorenstein projective module \( G \) and any \( n > 1 \) pick an exact sequence \( 0 \to G' \to P_1 \to \cdots \to P_n \to G \to 0 \) where all \( P_i \) are projective. Clearly, \( G' \) is also Gorenstein projective ([12, Theorem 2.5]). So, we have, \( \operatorname{Ext}^i_R(G, M) = \operatorname{Ext}^i_R(G', M) = 0 \), as desired.

(2). Dual to (1).

(3). We claim that \( \mathcal{GP}(R)^\perp \) is projectively resolving. Using the long exact sequence in homology, we conclude that \( \mathcal{GP}(R)^\perp \) is closed by extension, i.e., if \( 0 \to M \to M' \to M'' \to 0 \) where \( M \) and \( M'' \) are in \( \mathcal{GP}(R)^\perp \) then so is \( M' \). Clearly \( \mathcal{P}(R) \subseteq \mathcal{GP}(R)^\perp \) ([12, Proposition 2.3]). Now, consider a short exact sequence \( 0 \to M \to M' \to M'' \to 0 \) where \( M' \) and \( M'' \) are in \( \mathcal{GP}(R)^\perp \). For an arbitrary Gorenstein projective \( R \)-module \( G \) consider a short exact sequence \( 0 \to G \to P \to G' \to 0 \) where \( P \) is projective and \( G' \) is Gorenstein projective (such sequence exists by definition of Gorenstein projective modules). From the long exact sequence of homology, we have

\[
\ldots \to \operatorname{Ext}^1_R(G', M') \to \operatorname{Ext}^2_R(G', M) \to \operatorname{Ext}^2_R(G', M') \to \ldots
\]

Then, \( \operatorname{Ext}^2_R(G', M) = 0 \) since \( \operatorname{Ext}^2_R(G', M'') = \operatorname{Ext}^2_R(G', M') = 0 \) (from (1)). Thus, \( \operatorname{Ext}^2_R(G, M) = \operatorname{Ext}^2_R(G', M) = 0 \), as desired.

(4) We claim that \( \mathcal{GP}(R)^\perp \) is injectively resolving. Clearly, \( \mathcal{I}(R) \subseteq \mathcal{GP}(R)^\perp \) and
Thus, from (1), Ext\_1^R(G,M) → Ext\_1^R(G,M'') → Ext\_2^R(G,M) →...

Thus, from (1), Ext\_1^R(G,M'') = 0 for all Gorenstein projective module as desired.

Hence, we conclude the following two Corollarys. The second once was proved by Holm in [12].

**Corollary 2.2.** For any ring R,

1. \( \mathcal{P}(R) = \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp \)
2. \( \mathcal{I}(R) = \mathcal{GI}(R) \cap \mathcal{GI}(R)^\perp \)

**Proof.** (1). Consider \( M ∈ \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp \) and for such module M consider a short exact sequence \( 0 → P → M' → M → 0 \) where P is projective. Since \( \mathcal{GP}(R)^\perp \) is projectively resolving (from Theorem 2.1), \( M' ∈ \mathcal{GP}(R)^\perp \). Then, Ext\_R(M,M') = 0 and so this short exact sequence splits. Therefore, M is a direct summand of P and so projective, as desired.

(2). Dual proof.

(1) [12] Proposition 2.27] Every Gorenstein projective (resp., injective) module with finite projective (resp., injective) dimension is projective (resp., injective).

(2) Every Gorenstein projective (resp., injective) module with finite injective (resp., projective) dimension is projective (resp., injective).

**Proof.** (1). If M is a Gorenstein projective module with finite projective dimension, then \( M ∈ \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp \) (from [12] Proposition 2.3) and then projective (by Corollary 2.2). The injective case is dual.

(2). Note that every module I with finite injective dimension is an element of \( \mathcal{GP}(R)^\perp \). Indeed, by definition, for every Gorenstein projective module G we can find an exact sequence \( 0 → G → P\_n → ... → P\_0 → G' → 0 \) where all \( P\_i \) are projective and \( G' \) is Gorenstein projective with \( n = id\_R(I) \). Thus, we have Ext\_R^1(G,I) = Ext\_R^1(G',I) = 0, as desired.

Now, if M is Gorenstein projective with finite injective dimension then \( M ∈ \mathcal{GP}(R) \cap \mathcal{GP}(R)^\perp \) and then projective (by Corollary 2.2). Dually, we can prove that every module with finite projective dimension is an element of \( \mathcal{GI}(R) \). And then every Gorenstein injective module with finite projective dimension is injective (by Corollary 2.2).

The main result of this paper is the following Theorem:

**Theorem 2.4.** If Ggldim(R) < ∞, then (\( \mathcal{GP}(R) \), \( \mathcal{GP}(R)^\perp \)) and (\( \mathcal{GI}(R), \mathcal{GI}(R)^\perp \)) are complete hereditary cotorsion theory.

**Proof.** (1). To prove that (\( \mathcal{GP}(R), \mathcal{GP}(R)^\perp \)) is cotorsion theory, we have to prove that \( \mathcal{GP}(R)^\perp \) is \( \mathcal{GP}(R) \). Let M be in \( \mathcal{GP}(R)^\perp \). Thus, Ext\_R^1(M,N) = 0 for all \( N ∈ \mathcal{GP}(R)^\perp \). Since Gpd \( _R(M) < ∞ \) and from [12] Theorem 2.10, M admits a surjective \( \mathcal{GP}(R) \)-precover \( φ : G → M \), where \( K = ker(φ) \) satisfies pd \( _R(K) < ∞ \). Since \( K ∈ \mathcal{GP}(R)^\perp \), G is a special \( \mathcal{GP}(R) \)-precover and this short exact sequence splits since Ext\_R^1(M,K) = 0. Thus, M is a direct summand of
Corollary 2.7. If \( \text{l.Ggldim}(R) < \infty \), then
\[
\mathcal{GP}(R) = \mathcal{P}(R) = \mathcal{I}(R) = \mathcal{G}(R).
\]

Proof. Clearly \( \mathcal{P}(R) \subseteq \mathcal{GP}(R) \) (from [12, Proposition 2.3]). Now, let \( M \in \mathcal{GP}(R) \) and \( N \) an arbitrary \( R \)-module and set \( n = \text{l.Ggldim}(R) \). Then, \( \text{Gpd}_R(N) \leq n \). So, from [12, Theorem 2.20], we can find an exact sequence
\[
0 \to G \to P_n \to \ldots \to P_1 \to N \to 0
\]
where all \( P_i \) are projective and \( G \) is Gorenstein projective. Thus, \( \text{Ext}_R^j(G, M) = 0 \) for all \( j > 0 \) (by Theorem 2.1). Then, \( \text{id}_R(M) \leq n \). Using [2, Corollary 2.7], \( \mathcal{P}(R) = \mathcal{I}(R) \) since \( \text{l.Ggldim}(R) < \infty \). Then, \( M \in \mathcal{P}(R) \). Similarly, we have \( \mathcal{GP}(R) = \mathcal{I}(R) \). This complete the proof.

Recall that the finitistic projective dimension of \( R \) is the global dimension defined as:
\[
\text{FPD}(R) = \text{sup}\{\text{pd}_R(M) \mid M \text{ is an } R \text{-module with } \text{pd}_R(M) < \infty\}
\]

Proposition 2.5. If \( \text{l.Ggldim}(R) < \infty \), then,
\[
\mathcal{GP}(R) = \mathcal{P}(R) \subseteq \mathcal{I}(R) = \mathcal{G}(R).
\]

Proof. Clearly \( \mathcal{P}(R) \subseteq \mathcal{GP}(R) \) (from [12, Proposition 2.3]). Now, let \( M \in \mathcal{GP}(R) \) and \( N \) an arbitrary \( R \)-module and set \( n = \text{l.Ggldim}(R) \). Then, \( \text{Gpd}_R(N) \leq n \). So, from [12, Theorem 2.20], we can find an exact sequence
\[
0 \to G \to P_n \to \ldots \to P_1 \to N \to 0
\]
where all \( P_i \) are projective and \( G \) is Gorenstein projective. Thus, \( \text{Ext}_R^j(G, M) = 0 \) for all \( j > 0 \) (by Theorem 2.1). Then, \( \text{id}_R(M) \leq n \). Using [2, Corollary 2.7], \( \mathcal{P}(R) = \mathcal{I}(R) \) since \( \text{l.Ggldim}(R) < \infty \). Then, \( M \in \mathcal{P}(R) \). Similarly, we have \( \mathcal{GP}(R) = \mathcal{I}(R) \). This complete the proof.

Recall that the finitistic projective dimension of \( R \) is the global dimension defined as:
\[
\text{FPD}(R) = \text{sup}\{\text{pd}_R(M) \mid M \text{ is an } R \text{-module with } \text{pd}_R(M) < \infty\}
\]

Proposition 2.6. If \( \mathcal{GP}(R) = \mathcal{P}(R) \) and \( \mathcal{GP}(R) = \mathcal{I}(R) \), then \( \text{FPD}(R) = \text{l.Ggldim}(R) \).

Proof. From [13, Theorem 2.2], every \( R \)-module admits a special \( \mathcal{GP}(R) \)-preenvelope.

Corollary 2.7. If \( \text{FPD}(R) < \infty \), then the following are equivalent:

1. \( \text{l.Ggldim}(R) < \infty \).
2. \( \mathcal{GP}(R) = \mathcal{P}(R) \) and \( \mathcal{GP}(R) = \mathcal{I}(R) \).
Proof. (1 $\Rightarrow$ 2). The first equality follows from [12, Theorem 2.20] and the second from Proposition 2.5. (2 $\Rightarrow$ 1). Follows from Proposition 2.6. □

Now, we discuss the rings over which "every Gorenstein projective module is Gorenstein flat".

**Proposition 2.8.** For any ring $R$, the following are equivalents:

1. Every Gorenstein projective module is Gorenstein flat.
2. $I^+ \in \mathcal{GP}(R)^+$ for every right injective module $I$.
3. $F^{++} \in \mathcal{GP}(R)^+$ for every flat module $F$.

Proof. (1 $\Rightarrow$ 2). Let $I$ be a right injective $R$-module. Since every Gorenstein projective $R$-module is Gorenstein flat and by definition of Gorenstein flat module, we have $\text{Tor}^1_R(I, G) = 0$ for all $G \in \mathcal{GP}(R)$. By adjointness, we have $\text{Ext}^1_R(G, I^+) = (\text{Tor}_R^1(I, G))^+ = 0$. Hence, $I^+ \in \mathcal{GP}(R)^+$, as desired.

(2 $\Rightarrow$ 1). Consider a complete projective resolution

$$P = \cdots \rightarrow P_{-1} \xrightarrow{f_{-1}} P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} \cdots$$

We decompose it into short exact sequences $0 \rightarrow G_i \rightarrow P_i \rightarrow G'_i \rightarrow 0$ where $G_i = \ker(f_i)$ and $G'_i = \text{Im}(f_i)$. From [12, Observation 2.2], $G_i$ and $G'_i$ are Gorenstein projective. Now let $I$ be an injective right $R$-module. We have $\text{Tor}^1_R(I, G') = 0$ since $(\text{Tor}_R^1(I, G'))^+ = \text{Ext}^1_R(G', I^+) = 0$. Thus,

$$0 \rightarrow I \otimes_R G' \rightarrow I \otimes_R P_i \rightarrow I \otimes_R G' \rightarrow 0$$

is exact. So, $I \otimes_R \_ -$ keep the exactness of $P$. Then, it is a complete flat resolution. Consequently, every Gorenstein projective module is Gorenstein flat, as desired.

(2 $\Rightarrow$ 3). Let $F$ be a flat $R$-module. Then, $F^+$ is right injective. So, $F^{++} \in \mathcal{GP}(R)^+$.

(3 $\Rightarrow$ 2). Let $I$ be a right injective $R$-module. There exist a flat $R$-module $F$ such that $F \rightarrow I^+ \rightarrow 0$ is exact. Then, $0 \rightarrow I^{++} \rightarrow F^+$ is exact. But $0 \rightarrow I \rightarrow I^+$ is exact (by [9, Proposition 3.32]). Then, $0 \rightarrow I \rightarrow F^+$ is exact and then $I$ is a direct summand of $F^+$. Hence, $I^+$ is a direct summand of $F^{++}$. On the other hand, it is easy to see that $\mathcal{GP}(R)^+$ is closed under direct summands. Thus, $I^+ \in \mathcal{GP}(R)^+$, as desired. □

**Proposition 2.9.** For any ring $R$, $\sup\{\text{Gfd}_R(M) \mid M \text{ is Gorenstein projective}\} = 0$ or $\infty$.

Proof. Recall that if $\text{Gfd}_R(M) \leq n$, we have $\text{Tor}^i_R(I, M) = 0$ for all $i > n$. Indeed, the case $n = 0$ is from the definition of Gorenstein flat modules and the case $n > 0$ is deduced from the first case by the $n$-step projective resolution of $M$.

Suppose that $\sup\{\text{Gfd}_R(M) \mid M \text{ is Gorenstein projective}\} = n < \infty$. Then, $\text{Ext}_R^{n+1}(G, I^+) = (\text{Tor}_R^{n+1}(I, G))^+ = 0$ for all injective right module $I$ and all Gorenstein projective module $G$. But for every Gorenstein projective module $G$ we can find an exact sequence $0 \rightarrow G \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow G' \rightarrow 0$ where all $P_i$ are projective and $G'$ is Gorenstein projective. Thus, $\text{Ext}_R^1(G, I^+) = \text{Ext}_R^{n+1}(G', I^+) = 0$. So, $I^+ \in \mathcal{GP}(R)^+$ for every injective right module $I$. Then, from Proposition 2.8, every Gorenstein projective is Gorenstein flat. Consequently, $\sup\{\text{Gfd}_R(M) \mid M \text{ is Gorenstein projective}\} = 0$, as desired. □

A direct consequence of the above Proposition is the following Corollary:
Corollary 2.10. If $\mathrm{l.wGgldim}(R) < \infty$, then every Gorenstein projective $R$-module is Gorenstein flat.

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