Effective behaviour of critical-contrast PDEs: micro-resonances, frequency conversion, and time dispersive properties. II.

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Abstract

We construct an order-sharp theory for a double-porosity model in the full linear elasticity setup. Crucially, we uncover time and frequency dispersive properties of highly oscillatory elastic composites.

Keywords: Elasticity theory · Highly oscillatory media · Resolvent asymptotics · Time and frequency dispersion

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1 Introduction

Quantitative asymptotic methods in the analysis of parameter-dependent families of PDEs, see e.g. [63, 4, 32, 65, 35, 25], serve as a natural replacement of the classical ad hoc asymptotic approach, which is known to lead to errors, as pointed out in, e.g., [11, 12, 21, 14]. Their key feature is the pursuit of an estimate, in a uniform operator topology, on the difference between the “exact” (usually inaccessible) solution and its asymptotic approximation. This problem formulation has brought forth the physically relevant possibility to account for degenerate problems (such as the “double porosity”, “flipped double porosity” [16], and thin network [17] setups). The principal goal of the quantitative analysis is to develop a rigorous mathematical framework for metamaterials [62, 10]. It can be argued, see e.g. the discussion in [19], that generically metamaterial-like behaviour is accounted for by the “corrector” terms in the asymptotic expansion [16]. Indeed, if one assumed that the family admitted a “limiting” operator in a strong enough topology, then the latter must inherit the positive-definiteness of the original formulation, which would not permit the “negative” effects expected of a metamaterial. This calls for quantitatively tight asymptotic expansions capturing the key features of the medium at hand.

In connection with this goal, the operator theory has emerged as a source of powerful tools, a subset of which is based on the analysis of resolvents. Arguably, the norm-resolvent topology is indispensable if one seeks to control both the convergence of spectra and that of (generalised) eigenvectors. Furthermore, the specific choice of topological models is governed by the established consensus that non-trivial, and in particular metamaterial-like, properties arise by equipping the medium with an infinite array of small resonators. In light of the recent advances in the operator-theoretic treatment of boundary value problems, spectral theory thus assumes a new prominent rôle. Indeed, for problems involving resonators as well as heterogeneous thin structures, (rods, plates, shells, and their combinations – see [23, 9, 24, 17]), the relevant operator-theoretic setup is given by a parametrised family of “transmission” or “boundary value” problems for PDEs.

In [15, 16, 18] we proposed to utilise the link (facilitated by the classical Krein formula) between the resolvent and the Dirichlet-to-Neumann (DtN) operator on the interface between the medium and the resonators, to obtain sharp operator-norm convergence estimates. This has been done in a scalar version of the model commonly known as double porosity [3, 2, 64]. We point out that the idea to use DtN maps can be viewed as natural in this area, the first example of its application being traceable to [31]. However, prior to our work [15, 16, 18, 22, 23, 24] no attempts were made at employing this machinery to establish norm-resolvent convergence. In the moderate contrast setting, a theory covering a wide class of problems has been known since the beginning of this century, due to seminal work [4, 5]. Up to the publication of the first part of the present work [16], nothing of the kind has been available for degenerate problems, of which the double-porosity setup is arguably the most well-studied, as the degenerating coefficients make the problem considerably more challenging. The results we obtain can be viewed as running in parallel with those of [4], see Section 2.3. Also, while we have treated the whole-space setting, bounded regions can be dealt with in a standard way, as in [49].

Apart from addressing the specific problem of double porosity, we point out several generalisations. First, although in view of clarity we only treat the prototypical model, a wide range of similar problems is amenable to the same approach, see e.g. [31]. Second, essentially the same technique with minor modifications, see [17], is directly applicable to problems with a “geometric” contrast, e.g., elastic networks thinning to metric graphs.
2 Setup and main results

2.1 Notation

For a vector $a \in \mathbb{C}^k$, we denote by $a_j$, $j = 1, \ldots, k$ its components. Similarly, the entries of a matrix $A \in \mathbb{C}^{k \times k}$ are referred to as $A_{ij}$, $i, j = 1, \ldots, k$. $\text{sym} A$ denotes the symmetric part of $A$. The vectors of the standard orthonormal basis in $\mathbb{C}^k$ are denoted by $e_i$, $i = 1, \ldots, k$. Furthermore, for $a, b \in \mathbb{C}^k$, we denote by $a \otimes b \in \mathbb{C}^{k \times k}$ the matrix with entries $a_i b_j$, and set $a \otimes b := \text{sym}(a \otimes b)$. The Frobenius inner product of matrices $A, B$ is denoted by $A : B := \text{Tr}(B^T A)$ where $B^*$ stands for the adjoint of $B$, and we set $|A| := (A : A)^{1/2}$.

For an operator $\mathcal{A}$ (or a sesquilinear form $a$) the domain of $\mathcal{A}$ (respectively $a$) is denoted by $\mathcal{D}(\mathcal{A})$ (respectively $\mathcal{D}(a)$). We use the notation $\overline{\mathcal{A}}$ for the closure of a closable operator $\mathcal{A}$, and denote by $\sigma(\mathcal{A})$ (respectively, $\rho(\mathcal{A})$) the spectrum (respectively, the resolvent set) of an operator $\mathcal{A}$. For normed vector spaces $X, Y$, we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators from $X$ to $Y$. Furthermore, when indicating a function space $X$ in the notation for a norm $\| \cdot \|_X$, we omit the physical domain on which functions in $X$ are defined whenever it is clear from the context. For example, we often write $\| \cdot \|_{L^2}$, $\| \cdot \|_p$ instead of $\| \cdot \|_{L^2(\Omega^3)}$, $\| \cdot \|_{L^p(\Omega^3)}$, $k \in \mathbb{N}$.

For $A, B \in \mathbb{C}^k$, by $\text{dist}(A, B)$ we denote the distance between the sets $A$ and $B$. For $f \in L^1(A)$, we set $\langle f \rangle := \int_A f$ and $\mathbb{I}_A$ denotes the indicator function of $A$. Finally, $\delta_{ij}$ denotes the Kronecker delta, and $C$ generically stands for a positive constant whose value is of no importance.

2.2 Operator of linear elasticity

Consider the “reference cell” $Y := [0, 1)^3 \subset \mathbb{R}^3$ (which is without loss of generality for what is to follow), and let $Y_{\text{soft}} \subset Y$ be a connected open set with $C^{1,1}$ boundary $\Gamma$ such that the closure of $Y_{\text{soft}}$ is a subset of the interior of $Y$ and $Y_{\text{soft}} = Y \backslash Y_{\text{soft}}$. For a fixed period $\varepsilon > 0$ of material oscillations, we are interested in the behaviour of a composite elastic medium with components whose properties are in contrast to one another. We refer to the component materials as “soft” and “stiff” accordingly. With this goal in mind, we view $\mathbb{R}^3$ as being composed of two complementary subsets, the stiff part $\Omega_{\text{stiff}}$ (“matrix”) and the soft complement $\Omega_{\text{soft}}$ (“inclusions”), see Fig. 1:

$$\Omega_{\text{stiff}} := \mathbb{R}^3 \backslash \Omega_{\text{soft}}, \quad \Omega_{\text{soft}} := \bigcup_{\varepsilon \in \mathbb{D}} \{ \varepsilon Y_{\text{soft}} + z \}.$$

We are interested in the approximation properties, when $\varepsilon \rightarrow 0$, of the operator family $(\mathcal{A}_\varepsilon)_{\varepsilon > 0}$, where, for every $\varepsilon > 0$, the operators $\mathcal{A}_\varepsilon$ are defined as self-adjoint unbounded operators on $L^2(\mathbb{R}^3; \mathbb{C}^3)$ corresponding to the differential expressions $-\text{div}(\mathbb{A}_\varepsilon(x/\varepsilon) \text{sym } \nabla v)$ with domains $\mathcal{D}(\mathcal{A}_\varepsilon) \subset H^1(\mathbb{R}^3, \mathbb{C}^3)$. These operators are defined by the sesquilinear forms

$$a_\varepsilon(u, v) := \int_{\mathbb{R}^3} \mathbb{A}_\varepsilon(x/\varepsilon) \text{sym } \nabla u : \text{sym } \nabla v, \quad u, v \in H^1(\mathbb{R}^3; \mathbb{C}^3),$$

where the tensor-valued function $\mathbb{A}_\varepsilon$ represents the spatially varying elastic moduli of the medium. The properties of the stiff and soft components, modelled by tensor-valued functions $\mathbb{A}_{\text{stiff}}(\mathbf{y})$, $\mathbb{A}_{\text{soft}}(\mathbf{y})$, are assumed to be in “critical” contrast [64] to each other, so the ratio between the stiff and soft moduli is of order $\varepsilon^{-2}$:

$$\mathbb{A}_\varepsilon(\mathbf{y}) = \begin{cases} \mathbb{A}_{\text{stiff}}(\mathbf{y}), & \mathbf{y} \in Y_{\text{stiff}}, \\ \varepsilon^2 \mathbb{A}_{\text{soft}}(\mathbf{y}), & \mathbf{y} \in Y_{\text{soft}}, \end{cases}$$

The function $\mathbb{A}_\varepsilon$ is defined on the unit cell $Y$ and extended to $\mathbb{R}^3$ by periodicity. We make the following assumptions about $\mathbb{A}_{\text{stiff}}(\mathbf{y})$.

**Assumption 2.1.**

- Uniform positive-definiteness and uniform boundedness on symmetric matrices: there exists $\nu > 0$ such that $\nu \| \xi \|^2 \leq \mathbb{A}_{\text{stiff}}(\mathbf{y}) \xi : \xi \leq \nu^{-1} \| \xi \|^2 \quad \forall \xi \in \mathbb{R}^{3 \times 3}, \quad \xi^T = \xi \quad \forall \mathbf{y} \in Y$.

- Material symmetries $[\mathbb{A}_{\text{stiff}}(\mathbf{y})]_{ijkl} = [\mathbb{A}_{\text{stiff}}(\mathbf{y})]_{jikl} = [\mathbb{A}_{\text{stiff}}(\mathbf{y})]_{ijkl} = \mathbb{A}_{\text{stiff}}(\mathbf{y})_{ij}, \quad i, j, k, l \in \{1, 2, 3\}$.

- Lipschitz continuity: $[\mathbb{A}_{\text{stiff}}(\mathbf{y})]_{ijkl} \in C^{0,1}(Y_{\text{soft}})$, $i, j, k, l \in \{1, 2, 3\}$.
Proof. A definition 2.2 (Macroscopic operator) defines the following operators, which are key components of the leading-order term of the resolvent asymptotics.

Furthermore, we denote by $P_{\text{soft}}$ the orthogonal projection from $L^2(\mathbb{R}^3;C^1)$ onto $L^2_{\text{soft}}$. To state the main result, we define the following operators, which are key components of the leading-order term of the resolvent asymptotics.

**Definition 2.2** (Macroscopic operator). Consider the tensor $A_{\text{macro}} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ defined by

$$A_{\text{macro}} \xi : \eta = \int_{Y_{\text{soft}}} A_{\text{soft}} \left( \text{sym} \nabla u_{\xi} + \xi \right) : \left( \text{sym} \nabla u_{\eta} + \eta \right), \quad \xi, \eta \in \mathbb{R}^{3 \times 3}, \quad \xi^T = \xi, \quad \eta^T = \eta. \quad (2)$$

where $u_{\xi} \in H^1_{\text{stiff}}(Y_{\text{soft}}; \mathbb{R}^3)$ is the unique solution (guaranteed by the Lax-Milgram lemma) to the problem

$$\int_{Y_{\text{soft}}} A_{\text{soft}} \left( \text{sym} \nabla u_{\xi} + \xi \right) : \text{sym} \nabla \nu = 0 \quad \forall \nu \in H^1_{\text{stiff}}(Y_{\text{soft}}; \mathbb{R}^3), \quad \int_{Y_{\text{soft}}} u_{\xi} = 0.$$ 

Henceforth $H^1_{\text{stiff}}(Y_{\text{soft}}; \mathbb{R}^3)$ is the closure of $Y$-periodic smooth vector functions on $Y_{\text{soft}}$ in the $H^1(Y_{\text{soft}}, \mathbb{R}^3)$ norm.

We define the macroscopic operator (the operator of perforated domain) $A_{\text{macro}}$ as the self-adjoint unbounded operator on $L^2(\mathbb{R}^3;C^1)$ corresponding to the differential expression $-\text{div}(A_{\text{macro}} \text{sym} \nabla)$, with domain $\mathcal{D}(A_{\text{macro}}) \subset H^1(\mathbb{R}^3;C^1)$, defined by the sesquilinear form (cf. [59])

$$a_{\text{macro}}(u, v) := \int_{\mathbb{R}^3} A_{\text{macro}} \text{sym} \nabla u : \text{sym} \nabla v, \quad u, v \in H^1(\mathbb{R}^3;C^1). \quad (3)$$

We will require the following lemma, whose proof is standard.

**Lemma 2.3.** There tensor $A_{\text{macro}}$ is symmetric, in the sense that $[A_{\text{macro}}]_{ijkl} = [A_{\text{macro}}]_{jkil}$, and positive definite: there exists a constant $\eta > 0$ such that $A_{\text{macro}} \xi : \xi \geq \eta |\xi|^2$ for all $\xi \in \mathbb{R}^{3 \times 3}, \xi^T = \xi$.

**Proof.** The proof is based on a standard extension argument, see e.g. [9, Proposition 3.4].

In addition to the properties highlighted in Lemma 2.3, the leading-order term in the resolvent asymptotics retains information on the microstructure, via the spectrum of the “Bloch operator” $\mathcal{A}_{\text{Bloch}}$ associated with the bilinear form

$$a_{\text{Bloch}}(u, v) := \int_{Y_{\text{soft}}} A_{\text{soft}} \text{sym} \nabla u : \text{sym} \nabla v, \quad u, v \in H^1_{\text{stiff}}(Y_{\text{soft}};C^1),$$
as a non-negative self-adjoint operator on $L^2(Y_{\text{soft}}; \mathbb{C}^3)$. Furthermore, we define the matrix-valued “Zhikov function” $\mathcal{B}$ by

$$\mathcal{B}(z)_{ij} := z\delta_{ij} + z^2 \sum_{k=1}^{\infty} \frac{\langle \varphi_k \rangle \langle \varphi_k \rangle_j}{\eta_k - z}, \quad i, j \in 1, 2, 3,$$

where $\eta_1 \leq \cdots \leq \eta_k \leq \cdots \to \infty$ are the eigenvalues (indexed taking multiplicities into account) and $\varphi_k$, $k \in \mathbb{N}$, are the corresponding (orthonormal) eigenfunctions of the associated Bloch operator $\mathcal{A}_\text{Bloch}$ on $L^2(Y_{\text{soft}}; \mathbb{C}^3)$, and $z \neq \eta_k$ for all $k \in \mathbb{N}$. The function $\mathcal{B}$ has appeared in the context of “qualitative” analysis of high contrast (see, e.g., [66, 13]).

From this perspective, the results of the present paper can be viewed as demonstrating how it enters quantitative estimates, with sharp error control, in the context of elasticity.

2.3 Main results

We will now state the main results of the paper, which provide $O(\varepsilon)$ and $O(\varepsilon^2)$ approximations of the resolvent of the operator $\mathcal{A}_\varepsilon$. When restricted to the stiff component, the $O(\varepsilon^3)$ approximation involves a pseudodifferential operator, which leads to a second-order differential operator at the cost of an $O(\varepsilon)$ correction. The operator estimates we prove (Theorems 2.4, 2.5) involve certain “approximating” and “effective” operators $\Theta_\varepsilon^{\text{app}}$ and $\mathcal{A}_\varepsilon^{\text{eff}}$, which are described explicitly in Section 5.3 and Section 6, respectively. The proof of Theorem 2.4 is given at the end of Section 5.3, the proof of claim (a) of Theorem 2.5 is contained Section 6, while its claim (b) is discussed in Section 7.

Theorem 2.4. There exists $C > 0$ that depends only on $\sigma$ and $\text{diam}(K_\varepsilon)$ such that for all $z \in K_\varepsilon$ one has

$$\left\| \left( \mathcal{A}_\varepsilon - zI \right)^{-1} - \Theta_\varepsilon^{\text{app}} \left( \mathcal{A}_\varepsilon^{\text{app}} - zI \right)^{-1} \Theta_\varepsilon^{\text{app}} \right\|_{L^2(\mathbb{R}^3; \mathbb{C}^3)} \leq C\varepsilon^2.$$  \hspace{1cm} (5)

The operator $\Theta_\varepsilon^{\text{app}}$ is an orthogonal projection (see Section 5.3), and $\mathcal{A}_\varepsilon^{\text{app}}$ is defined uniquely on $\Theta_\varepsilon^{\text{app}} L^2(\mathbb{R}^3; \mathbb{C}^3)$ and then extended somehow (e.g., by the zero operator) to its orthogonal complement.

Theorem 2.5. There exists $C > 0$ that depends only on $\sigma$ and $\text{diam}(K_\varepsilon)$ such that, for all $z \in K_\varepsilon$:

(a) The “whole-space” homogenisation estimate

$$\left\| \left( \mathcal{A}_\varepsilon - zI \right)^{-1} - \Theta_\varepsilon^{\text{eff}} \left( \mathcal{A}_\varepsilon^{\text{eff}} - zI \right)^{-1} \Theta_\varepsilon^{\text{eff}} \right\|_{L^2(\mathbb{R}^3; \mathbb{C}^3)} \leq C\varepsilon$$  \hspace{1cm} (6)

holds. Here $\Theta_\varepsilon^{\text{eff}}$ is an orthogonal projection, and $\mathcal{A}_\varepsilon^{\text{eff}}$ is a self-adjoint operator initially defined uniquely on $\Theta_\varepsilon^{\text{eff}} L^2(\mathbb{R}^3; \mathbb{C}^3)$ and then extended somehow (e.g., by the zero operator) to its orthogonal complement.

(b) The estimate on the “stiff” component

$$\left\| P_\varepsilon^{\text{eff}} (\mathcal{A}_\varepsilon - zI)^{-1} P_\varepsilon^{\text{eff}} - P_\varepsilon^{\text{eff}} \left( \mathcal{A}_\text{macro} - \mathcal{B}(z) \right)^{-1} P_\varepsilon^{\text{eff}} \right\|_{L^2(\mathbb{R}^3; \mathbb{C}^3)} \leq C\varepsilon$$

holds. Here $\mathcal{A}_\text{macro}$ is the differential operator of linear elasticity with constant coefficients defined by the form (3) and $P_\varepsilon^{\text{eff}}$ is the orthogonal projection onto the subspace of $L^2(\mathbb{R}^3; \mathbb{C}^3)$ consisting of functions vanishing on the soft component $\Omega_\varepsilon^{\text{soft}}$ of the medium.

For additional discussions about the form of approximating operators in high-contrast homogenisation and the comparison of the above results to the ones of [4] in the moderate-contrast setting, see Remarks 6.14, 7.6 below.

One can also extend the claims of Theorems 2.4 and 2.5 beyond $K_\varepsilon$, provided the spectral parameter $z$ is confined to a bounded region away from the spectrum. Under this extension, Corollary 2.6 reveals an explicit dependence of the constant $C$ on the distance of $z$ from the spectrum of the operator $\mathcal{A}_\varepsilon$ and the modulus of $z$, as follows.

Corollary 2.6. The claim of Theorem 2.4 can be extended to all $z \notin \sigma(\mathcal{A}_\varepsilon) \cup \sigma(\mathcal{A}_\varepsilon^{\text{app}})$ and the constant $C = C(z)$ depends on $z$ as follows:

$$C(z) = C \left( 1 + (|z| + 1)/\text{dist}(z, \sigma(\mathcal{A}_\varepsilon)) \right) \left( 1 + (|z| + 1)/\text{dist}(z, \sigma(\mathcal{A}_\varepsilon^{\text{app}})) \right),$$

where $C$ is independent of $z$. A similar statement holds for the claim of Theorem 2.5 (a).

Proof. For the unbounded operators $\mathcal{A}$ and $\mathcal{B}$ on the Banach space $X$, the orthogonal projection $P$ that commutes with the operator $\mathcal{B}$, and $z_1, z_2 \notin \sigma(\mathcal{A}) \cup \sigma(\mathcal{B})$, it is easy to check the identity

$$\left( \mathcal{A} - z_1I \right)^{-1} - P(\mathcal{B} - z_1I)^{-1}P = P(\mathcal{B} - z_2I)^{-1}P \left( \left( \mathcal{A} - z_1I \right)^{-1} - P(\mathcal{B} - z_1I)^{-1}P \right) (\mathcal{A} - z_1I) (\mathcal{A} - z_2I)^{-1} \hspace{1cm} (7)$$

$$+ \left( I - P \right)(\mathcal{A} - z_1I)^{-1}(\mathcal{A} - z_1I)(\mathcal{A} - z_2I)^{-1}. $$
Notice also that by the functional calculus for a self-adjoint $\mathcal{A}$ we have
\[
\|(\mathcal{A} - z_1 I)(\mathcal{A} - z_2 I)^{-1}\|_{X \to X} \leq 1 + \frac{|z_2 - z_1|}{\text{dist}(z_2, \sigma(\mathcal{A}))}.
\]
We set $z_1 = i$, $\mathcal{A} = \mathcal{A}_e$, $\mathcal{B} = \mathcal{A}_{\text{eff}}^{\text{pp}}$, and $\mathcal{P} = \Theta_e^{\text{pp}}$ in (7) and combine it with Theorem 2.4, where we set $z = i$, bearing in mind that, also as a consequence of Theorem 2.4, one has
\[
\|(I - \Theta_e^{\text{pp}}) (\mathcal{A}_e - i I)^{-1}\|_{L^2(\mathbb{R}^3; \mathbb{C})} \leq C \epsilon^2.
\]
The claim now follows immediately. □

We next discuss implications of the main results for the asymptotic behaviour of the spectra of the operators $\mathcal{A}_e$. It is well known that these have a band-gap structure (see [64]). Theorem 2.4 and Theorem 2.5 enable us to estimate the gaps in the spectrum of $\mathcal{A}_e$ on any compact interval by the gaps in the spectra of $\mathcal{A}_{\text{eff}}^{\text{pp}}$ and $\mathcal{A}_{\text{eff}}^{\text{eff}}$, respectively.

**Corollary 2.7.** For every $M > 0$, one has
\[
\text{dist}\left(\sigma(\mathcal{A}_e) \cap [-M, M], \sigma(\mathcal{A}_{\text{eff}}^{\text{pp}}) \cap [-M, M]\right) \leq C(M + 1)^2 \epsilon^2,
\]
\[
\text{dist}\left(\sigma(\mathcal{A}_e) \cap [-M, M], \sigma(\mathcal{A}_{\text{eff}}^{\text{eff}}) \cap [-M, M]\right) \leq C(M + 1)^2 \epsilon.
\]
where $C > 0$ is independent of $M$.

**Proof.** The proof is obtained by setting $z = i$ in Theorem 2.4 and Theorem 2.5 (a). It is well known that for self-adjoint bounded linear operators $\mathcal{A}, \mathcal{B}$ on a Hilbert space $X$, one has $\text{dist}(\sigma(\mathcal{A}), \sigma(\mathcal{B})) \leq \|\mathcal{A} - \mathcal{B}\|_{X \to X}$, see e.g. [34]. Furthermore, noting that
\[
\sigma((\mathcal{A}_e + I)^{-1}) = \{(\lambda + 1)^{-1} : \lambda \in \sigma(\mathcal{A}_e)\},
\]
\[
\sigma(\Theta_e^{\text{pp}}(\mathcal{A}_{\text{eff}}^{\text{pp}} + I)^{-1}) = \{(\lambda + 1)^{-1} : \lambda \in \sigma(\mathcal{A}_{\text{eff}}^{\text{pp}})\},
\]
\[
\sigma(\Theta_e^{\text{eff}}(\mathcal{A}_{\text{eff}}^{\text{eff}} + I)^{-1}) = \{(\lambda + 1)^{-1} : \lambda \in \sigma(\mathcal{A}_{\text{eff}}^{\text{eff}})\},
\]
the claim follows from the fact that for arbitrary $\lambda, \mu \in \mathbb{R}$ one has
\[
|\lambda - \mu| = |\lambda + 1| |\mu + 1| \left| (\lambda + 1)^{-1} - (\mu + 1)^{-1} \right| \leq (|\lambda + 1| + |\mu + 1|) \left| (\lambda + 1)^{-1} - (\mu + 1)^{-1} \right|.
\]
Remark 2.8. By combining Corollary 2.7 and 2.6 one can make the constant $C = C(\epsilon)$ in Theorem 2.4 and 2.5 dependent only on $|z|$ and $\text{dist}(z, \sigma(\mathcal{A}_e))$, i.e. only on $|z|$ and $\text{dist}(z, \sigma(\mathcal{A}_{\text{eff}}^{\text{pp}}))$. Also, an explicit numerical value for $C > 0$ in Corollary 2.7 and 2.6 that corresponds to $z = i$ in Theorem 2.4 and 2.5 can be provided.

### 2.4 Gelfand transform

The purpose of this chapter is to decompose the original differential operator into a family of differential operators with compact resolvents that act on functions defined on the unit cell $Y$. This is carried out in a standard way by the Gelfand transform.

The Gelfand transform $\mathcal{G}$ is defined on $L^2(\mathbb{R}^3; \mathbb{C})$ by the formula
\[
(\mathcal{G}u)(y, \chi) := (2\pi)^{-3/2} \sum_{n \in \mathbb{Z}^3} e^{-iy \cdot (y + n)} u(y + n), \quad y \in Y, \quad \chi \in Y',
\]
where $Y' = [-\pi, \pi]^3$. (As noted above, without loss of generality we assume that $Y = [0, 1]^3$. Lattices with other periods can be treated by the same analysis with minor modifications — the associated “Brillouin zone” $Y'$ is then adjusted appropriately.) In the case of The Gelfand transform is a unitary operator:
\[
\mathcal{G} : L^2(\mathbb{R}^3; \mathbb{C}) \to L^2(Y'; L^2(Y; \mathbb{C}^3)) = \int_{Y'} L^2(Y; \mathbb{C}^3, \chi) d\chi,
\]
in the sense that $\langle u, v \rangle_{L^2(\mathbb{R}^3; \mathbb{C})} = \langle \mathcal{G} u, \mathcal{G} v \rangle_{L^2(Y'; L^2(Y; \mathbb{C}^3))}$ for all $u, v \in L^2(\mathbb{R}^3; \mathbb{C})$. A function can be reconstructed from its Gelfand transform as follows:
\[
u(x) = (2\pi)^{-3/2} \int_{Y'} e^{ix \cdot \chi} (\mathcal{G} u)(x, \chi) d\chi, \quad x \in \mathbb{R}^3.
\]
For an overview of the properties of Gelfand transform in relation to homogenisation problems, we refer to [4].

In order to deal with the setting of highly oscillating material coefficients, we consider the following scaled version of Gelfand transform. For a fixed $\varepsilon > 0$ and all $u \in L^2(\mathbb{R}^3; \mathbb{C}^3)$, we set

$$(G_{\varepsilon} u)(y, \chi) := \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \sum_{n \in \mathbb{Z}^3} e^{-i(y+n)\chi} u(y+n), \quad y \in Y, \ \chi \in \chi'.'$$

Note that $G_{\varepsilon}$ is a composition of $G$ and the unitary scaling operator $S_\varepsilon : L^2(\mathbb{R}^3; \mathbb{C}^3) \to L^2(\mathbb{R}^3; \mathbb{C}^3)$ defined by

$$(S_\varepsilon u)(x) := \varepsilon^{3/2} u(\varepsilon x), \quad u \in L^2(\mathbb{R}^3; \mathbb{C}^3).$$

It follows that $G_{\varepsilon}$ is also unitary, i.e.

$$\langle u, v \rangle_{L^2(\mathbb{R}^3; \mathbb{C}^3)} = \langle G_{\varepsilon} u, G_{\varepsilon} v \rangle_{L^2(\chi'')} \quad \forall u, v \in L^2(\mathbb{R}^3; \mathbb{C}^3).$$

The original function is recovered from its Gelfand transform by the formula

$$u(x) = (2\pi \varepsilon)^{-3/2} \int_{\chi'} e^{ikx/\varepsilon}(G_{\varepsilon} u)(x,k,\chi)dk.$$

(8)

Also, by noting that for the scaled Gelfand transform of a derivative of $u \in H^1(\mathbb{R}^3; \mathbb{C}^3)$ one has

$$G_{\varepsilon}(\partial_y u) = \varepsilon^{-1}(\partial_y (G_{\varepsilon} u) + i\chi (G_{\varepsilon} u)), \quad \chi = 1, 2, 3,$$

we infer that

$$G_{\varepsilon}(\text{sym} \nabla u)(y, \chi) = \varepsilon^{-1}(\text{sym} \nabla_y (G_{\varepsilon} u) + i\chi (G_{\varepsilon} u)), \quad \chi = 1, 2, 3.$$

(9)

where the for each $\chi \in \chi'$ the operator $X_\chi$ acting on $L^2(Y; \mathbb{C}^3)$ is defined by

$$X_\chi u = \text{sym} \ (u \otimes \chi) = \begin{bmatrix} \chi_1 u_1 & \frac{1}{2}(\chi_1 u_2 + \chi_2 u_1) & \frac{1}{2}(\chi_1 u_3 + \chi_3 u_1) \\ \frac{1}{2}(\chi_1 u_2 + \chi_2 u_1) & \chi_2 u_2 & \frac{1}{2}(\chi_3 u_2 + \chi_2 u_3) \\ \frac{1}{2}(\chi_3 u_1 + \chi_1 u_3) & \frac{1}{2}(\chi_3 u_2 + \chi_2 u_3) & \chi_3 u_3 \end{bmatrix}, \quad u \in L^2(Y; \mathbb{C}^3).$$

Remark 2.9. Note that in the setting of 2D elasticity the operator $X_\chi$ takes the form

$$X_\chi u = \begin{bmatrix} \chi_1 u_1 & \frac{1}{2}(\chi_1 u_2 + \chi_2 u_1) \\ \frac{1}{2}(\chi_1 u_2 + \chi_2 u_1) & \chi_2 u_2 \end{bmatrix}, \quad u \in L^2(Y; \mathbb{C}^2).$$

It is straightforward to show the existence of $C_1, C_2 > 0$ such that that

$$C_1 ||u||_{L^2(Y; \mathbb{C}^3)} \leq ||X_\chi u||_{L^2(Y; \mathbb{C}^3)} \leq C_2 ||u||_{L^2(Y; \mathbb{C}^3)} \quad \forall u \in L^2(Y; \mathbb{C}^3).$$

(10)

Denote by $H^1_\chi(Y; \mathbb{C}^3)$, $H^2_\chi(Y; \mathbb{C}^3)$ the spaces of $Y$-periodic functions in $H^1(Y; \mathbb{C}^3)$, $H^2(Y; \mathbb{C}^3)$, respectively. We use similar notation when $Y$ is replaced by $Y_{\text{ndf}}$. For Gelfand transform $G_{\varepsilon}$, one can show [37] that

$$a_{\varepsilon}(u, v) = \int_{\chi'} \varepsilon^{-2} a_{\varepsilon}(G_{\varepsilon} u, G_{\varepsilon} v) d\chi,$$

where

$$a_{\varepsilon}(u, v) := \int_{\chi'} A_{\varepsilon}(\text{sym} \nabla + iX_\chi) u : (\text{sym} \nabla + iX_\chi) v, \quad u, v \in H^1_\chi(Y; \mathbb{C}^3).$$

(11)

For each $\chi \in \chi'$ and $\varepsilon > 0$, we introduce the self-adjoint operator

$$A_{\varepsilon}(\chi) := (\text{sym} \nabla + iX_\chi)' \cdot (\text{sym} \nabla + iX_\chi) : D(A_{\varepsilon}(\chi)) \subset H^1_\chi(Y; \mathbb{C}^3) \to L^2(Y; \mathbb{C}^3)$$

associated with the positive definite form $a_{\varepsilon}(\cdot, \cdot)$. Here we use the notation $(\cdot)'$ for the formal adjoint of the operator. Applying the scaled Gelfand transform to the resolvent yields

$$\left(A_{\varepsilon} - \zeta I\right)^{-1} = \left(G_{\varepsilon}^{-1} \left(\int_{\chi'} \left(\frac{1}{\varepsilon^2} A_{\varepsilon}(\chi) - \zeta I\right)^{-1} d\chi\right) \right) G_{\varepsilon}, \quad \zeta \in \rho(A_{\varepsilon}).$$

(12)
which is an example of the classical von Neumann direct integral formula. Due to the compactness of the embedding $H^1_0(Y; C^3) \hookrightarrow L^2(Y; C^3)$, the resolvents $(\epsilon^{-2}A_{\chi,\epsilon} - zI)^{-1}$ are compact. We interpret (12) as follows: by applying the Gelfand transform to the problem, we have decomposed the resolvent operator $(A_{\chi} - zI)^{-1}$ into a continuum family of resolvent operators $(\epsilon^{-2}A_{\chi,\epsilon} - zI)^{-1}$ indexed by $\chi \in Y$. In contrast to the original resolvent operator, this family consists of compact operators, which have discrete spectra.

For each $\epsilon > 0$, the $\chi$-fibre ($\chi \in Y$) resolvent problem for $A_{\chi}$ consists in finding, for a fixed $z \in \rho(A_{\chi,\epsilon})$ and every $f \in L^2(Y; C^3)$, the solution $u \in D(A_{\chi,\epsilon})$ to the equation $(\epsilon^{-2}A_{\chi,\epsilon} - zI)u = f$.

**Remark 2.10** (Transmission boundary value problem). The equation $(\epsilon^{-2}A_{\chi,\epsilon} - zI)u = f$ can be formally recast as follows: find $u_{\text{soft}} \in H^2_0(Y_{\text{soft}}, C^3)$, $u_{\text{soft}} \in H^2(Y_{\text{soft}}, C^3)$ such that

\[
\epsilon^{-2}(\text{sym} \nabla + iX_\epsilon)^* A_{\text{soft}} \text{sym} \nabla u_{\text{soft}} = f \quad \text{on} \quad Y_{\text{soft}},
\]

\[
(\text{sym} \nabla + iX_\epsilon)^* A_{\text{soft}} \text{sym} \nabla u_{\text{soft}} - u_{\text{soft}} = f \quad \text{on} \quad Y_{\text{soft}},
\]

\[
u_{\text{soft}} = u_{\text{soft}} \quad \text{on} \quad \Gamma,
\]

where $n_{\text{soft}}$, $n_{\text{soft}}$, denote the outward-pointing unit normals to $\Gamma$ from $Y_{\text{soft}}$, $Y_{\text{soft}}$. The question of making this reformulation rigorous is that of regularity of functions in the domain of $A_{\chi,\epsilon}$.

The above regularity question can alternatively be framed in terms of the solution $u \in H^2_0(Y; C^3)$ to the weak formulation of (13):

\[
a_{\chi,\epsilon}(u, v) + \int \nabla u \cdot \nabla v = \int f \cdot \nabla v \quad \forall v \in H^1_0(Y; C^3).
\]

It is then tempting to interpret the problem (13) via understanding its first two equations in the sense of distributions, the penultimate equation in the sense of equality in $H^{1/2}(\Gamma; C^3)$, and the last equation in the sense of equality in $H^{-1/2}(\Gamma; C^3)$. However, even this interpretation calls for some caution, in view of the moderate regularity assumptions made in Section 2.2 about the coefficient tensors $\alpha_{\chi}^{\text{soft}}$ and interface $\Gamma$.

We do not address the $H^2$ regularity in the present work, as it has no bearing on our results. As the formulation (13) is often preferred in the engineering community, at present its link to the operators $A_{\chi,\epsilon}$ is an open question, to which some of the machinery of [45] may be relevant.

We note however that, in the infinitely smooth smooth (i.e., $C^\infty$ coefficients and interface), the above regularity question was settled positively in [54]. Under the assumptions we made in the present paper (see Section 2.2), the results we present below can be shown to yield at least $H^{1/2}$ regularity. In other words, the weak formulation (14) is shown to be equivalent to looking for $u_{\text{soft}} \in H^{3/2}_0(Y_{\text{soft}}, C^3)$, $u_{\text{soft}} \in H^{3/2}(Y_{\text{soft}}, C^3)$ such that (13) holds.

### 3 Operator theoretic approach: Ryzhov triples

The purpose of this section is to introduce an abstract framework for the transmission problem (13). In Section 3.1 we recall a general construction due to Ryzhov [52], while in Sections 3.2, 3.3 we show how the problem (13) can be seen as part of this construction and prove key properties of the operators emerging in the process.

We start by introducing some basic objects required to work with Ryzhov triples, an operator framework convenient for the analysis of boundary value problems for PDEs.

#### 3.1 Abstract notion of a Ryzhov triple

The concept of a Ryzhov triple was introduced in [51, 52]. The main results of this section are Theorems 3.13, 3.14, which provide an operator-theoretic formula for the solution to an abstract spectral boundary value problem.

**Definition 3.1.** Let $\mathcal{H}$ be a separable Hilbert space and $E$ an auxiliary Hilbert space. Suppose that:

- $A_0$ is a self-adjoint operator on $\mathcal{H}$ with $0 \in \rho(A_0)$,
- $\Pi: E \rightarrow \mathcal{H}$ is a bounded operator such that $D(A_0) \cap \mathcal{R}(\Pi) = \{0\}$, $\ker(\Pi) = \{0\}$,
- $\Lambda$ is a self-adjoint operator on the domain $D(\Lambda) \subset E$.

We refer to the triple $(A_0, \Pi, \Lambda)$ as a Ryzhov triple on $(\mathcal{H}, E)$.

Following [52], we introduce the following objects.
Proposition 3.5. The next proposition lists key properties of the and therefore on $D$.

Remark 3.7. In the case when the operators $\partial$ are normal derivatives, the formula

\[ \partial u = \Pi \partial \partial u + \Lambda g, \quad f \in \mathcal{H}, g \in \mathcal{E}. \]

Remark 3.6. In the case when the operators $\partial$ are normal derivatives, the formula

\[ \partial u = \Pi \partial \partial u + \Lambda g, \quad f \in \mathcal{H}, g \in \mathcal{E}. \]

Definition 3.2. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{E}$ an auxiliary Hilbert space, and $(\mathcal{A}, \Pi, \Lambda)$ a Ryzhov triple on $(\mathcal{H}, \mathcal{E})$. Define the operators $\mathcal{A}$ as follows:

\[
\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0) + \Pi(\mathcal{E}), \quad \mathcal{A} : \mathcal{A}_0^{-1} f + \Pi g \rightarrow f, \quad f \in \mathcal{H}, g \in \mathcal{E},
\]

\[
\mathcal{D}(\Pi_0) = \mathcal{D}(\mathcal{A}_0) + \Pi(\mathcal{E}), \quad \Pi_0 : \mathcal{A}_0^{-1} f + \Pi g \rightarrow g, \quad f \in \mathcal{H}, g \in \mathcal{E},
\]

\[
\mathcal{D}(\Pi_1) = \mathcal{D}(\mathcal{A}_0) + \Pi(\mathcal{E}), \quad \Pi_1 : \mathcal{A}_0^{-1} f + \Pi g \rightarrow \Pi^* f + \Lambda g, \quad f \in \mathcal{H}, g \in \mathcal{D}(\mathcal{A}).
\]

We say that $(\mathcal{A}, \Pi_0, \Pi_1)$ is the boundary triple associated with the Ryzhov triple $(\mathcal{A}_0, \Pi, \Lambda)$.

Remark 3.5. In the case when $\mathcal{A}_0$ is an (unbounded) differential operator, and the operators $\Pi_0$ and $\Pi_1$ assume the roles of the trace of a function and of its co-normal derivative on the boundary, respectively. The next result is then well expected and can be found in [52].

Theorem 3.3 (Green’s formula). Let $(\mathcal{A}_0, \Pi, \Lambda)$ be a Ryzhov triple. Then for the associated boundary triple $(\mathcal{A}, \Pi_0, \Pi_1)$ the following identity holds:

\[ \langle \mathcal{A}u, v \rangle_{\mathcal{H}} - \langle u, \mathcal{A}v \rangle_{\mathcal{H}} = \langle \Pi_0 u, \Pi_0 v \rangle_{\mathcal{E}} - \langle \Pi_1 u, \Pi_1 v \rangle_{\mathcal{E}} \quad \forall u, v \in \mathcal{D}(\mathcal{A}_0) + \Pi(\mathcal{D}(\mathcal{A})). \]

Definition 3.4. Suppose $z \in \rho(\mathcal{A}_0)$. Define the operator $S(z)$ mapping $g \in \mathcal{E}$ to the solution $u \in \mathcal{D}(\mathcal{A}) = D(\Pi_0)$ of the spectral boundary value problem

\[ \mathcal{A}u = z u, \quad \Pi_0 u = g. \]

The operator-valued function $M(z)$ defined on $\mathcal{D}(\mathcal{A})$ by $M(z) := \Pi S(z)$, is called the Weyl $M$-function of the Ryzhov triple $(\mathcal{A}_0, \Pi, \Lambda)$.

In [52] the following formulae for the operators $S(z)$ and $M(z)$ were proven ($z \in \rho(\mathcal{A}_0)$):

\[ S(z) = (I - z \mathcal{A}_0^{-1})^{-1} \Pi, \quad M(z) = \Pi_1 (I - z \mathcal{A}_0^{-1})^{-1} \Pi. \]

Note also that

\[ (I - z \mathcal{A}_0^{-1})^{-1} = I + z \mathcal{A}_0^{-1} (I - z \mathcal{A}_0^{-1})^{-1} = I + z(\mathcal{A}_0 - zI)^{-1}, \]

and therefore

\[ S(z) = \Pi + z(\mathcal{A}_0 - zI)^{-1} \Pi. \]

The next proposition lists key properties of the $M$-function.

Proposition 3.5. (Properties of the Weyl $M$-function)

- The following representation holds:

\[ M(z) = \Lambda + z \Pi^* (I - z \mathcal{A}_0^{-1})^{-1} \Pi, \quad z \in \rho(\mathcal{A}_0). \]

- $M(z)$ is an operator-valued function with values in the set of closed operators on $\mathcal{E}$ with (z-independent) domain $\mathcal{D}(\Lambda)$ such that $M(z) - \Lambda$ is analytic.

- For $z, \xi \in \rho(\mathcal{A}_0)$ the operator $M(z) - M(\xi)$ is bounded, and $M(z) - M(\xi) = (z - \xi) (S(\xi))^* S(\xi)$.

- For $u \in \ker(\mathcal{A} - zI) \cap \{ \mathcal{D}(\mathcal{A}_0) + \Pi \mathcal{D}(\mathcal{A}) \}$, the following formula holds:

\[ M(z) \Gamma_1 u = \Gamma_1 u. \]

Remark 3.6. In the case when the operators $\Gamma_0$ and $\Gamma_1$ represent the trace of a function and the trace of its co-normal derivative, the formula (22) clearly reveals the $M$-function $M(z)$ to be the DtN map associated with the resolvent problem.

Remark 3.7. Due to the fact that $\mathcal{D}(M(z)) = \mathcal{D}(M(z)^*) = \mathcal{D}(\Lambda)$ independently on $z \in \mathbb{C}$, one can define the operators

\[ \mathcal{R}M(z) = 2^{-1}(M(z) + M(z)^*), \quad \mathcal{I}M(z) = (2i)^{-1}(M(z) - M(z)^*) \]

on $\mathcal{D}(\Lambda)$. Note that by (21) and the fact that $\Lambda$ is self-adjoint, one has $M(z)^* = M(\overline{z})$, and therefore

\[ \mathcal{R}M(z) = (2i)^{-1}(M(z) - M(z)^*) = \mathcal{I}z (S(\overline{z}))^* S(\overline{z}), \quad \mathcal{I}M(z) = \mathcal{S}(M(z) - \Lambda) = S(M(z) - M(0)). \]
Remark 3.8. It is clear from (21) and (19) that
\[ M(z) = \Lambda + z\Pi^*\Pi + z^2\Pi^*(A_0 - zI)^{-1}\Pi. \]
This formula will prove to be one of the key elements in deriving the asymptotics, as \( \varepsilon \to 0 \), of the resolvents \((A^\varepsilon - zI)^{-1}\) of the operators \( A^\varepsilon \) introduced in Section 2.4.

For a given \( A_0 \), we define \( A_{00} \) to be the restriction of \( A_0 \) to the set \( D(A_{00}) := \ker(\Gamma_1) \cap \ker(\Gamma_0) \).

Remark 3.9. It was shown in [52] that \( D(A_{00}) \) does not actually depend on the choice of the operator \( \Gamma_1 \) (or \( \Lambda \)) and can be characterised as the subspace of \( D(A_0) \) consisting of those elements \( u \) for which \( A_0 u \) is orthogonal to the range of \( \Pi \).

One can characterise a wide class of densely defined closed extensions of \( A_0 \), contained in \( \mathcal{A} \), to be the operators \( \mathcal{A}_{\beta_0,\beta_1} \) associated with the spectral boundary value problem \( Au - zu = f \) subject to an abstract Robin-type condition
\[ (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = 0, \]
by varying over the choice of the operators \( \beta_0, \beta_1 \) on \( E \). The rigorous definition of the extension operators \( \mathcal{A}_{\beta_0,\beta_1} \) is postponed to Theorem 3.14. Note that \( A_0 \) is then the self-adjoint extension of \( A_{00} \) corresponding to the choice \( \beta_0 = I, \beta_1 = 0 \). However, in order to clarify the meaning of (25), it is necessary to make additional assumptions on the operators \( \beta_0, \beta_1 \), e.g., as follows.

Assumption 3.10. The operators \( \beta_0, \beta_1 \) are linear in \( E \) and such that \( D(\beta_0) \supset D(\Lambda) \) and \( \beta_1 \) is bounded on \( E \). The operator \( \beta_0 + \beta_1\Lambda \), defined on \( D(\Lambda) \), is closable in \( E \). We denote its closure by \( \mathcal{B} \).

Under the above assumption, the operator \( \beta_0 + \beta_1\mathcal{M}(z) \) is also closable. The condition (25) is shown to be well posed on a certain Hilbert space associated with the closure of \( \beta_0 + \beta_1\Lambda \) in \( E \).

Definition 3.11. Consider a separable Hilbert space \( \mathcal{H} \), an auxiliary Hilbert space \( E \), and suppose that \( (\mathcal{A}_0, \Pi, \Lambda) \) is a Ryzhov triple on \( (\mathcal{H}, E) \). Suppose also that \( \beta_0, \beta_1 \) are linear operators on \( E \) satisfying Assumption 3.10. Consider the space
\[ \mathcal{H}_{\beta_0,\beta_1} := D(\mathcal{A}_0) + \Pi(D(\mathcal{A}_0)) \subset \{ \mathcal{A}_0^{-1}f + \Pi g : f \in \mathcal{H}, g \in E \}, \]
equipped with the norm
\[ \|\mathcal{A}_0^{-1}f + \Pi g\|_{\mathcal{H}_{\beta_0,\beta_1}} := (\|f\|_{\mathcal{H}}^2 + \|g\|_{E}^2 + \|\Pi g\|_{E}^2)^{1/2}. \]

The following lemma is proved in [52].

Lemma 3.12. The space \( (\mathcal{H}_{\beta_0,\beta_1}, \|\cdot\|_{\mathcal{H}_{\beta_0,\beta_1}}) \) is a Hilbert space. The operator \( \beta_0\Gamma_0 + \beta_1\Gamma_1 : \mathcal{H}_{\beta_0,\beta_1} \rightarrow \mathcal{E} \) is bounded.

We are now in a position to assign a meaning to the abstract spectral boundary value problem and establish its well-posedness.

Theorem 3.13. Suppose that \( z \in \rho(\mathcal{A}_0) \) is such that the operator \( \beta_0 + \beta_1\mathcal{M}(z) \) is boundedly invertible in \( E \). Then, for given \( f \in \mathcal{H}, g \in E \), the unique solution \( u \in \mathcal{H}_{\beta_0,\beta_1} \) to the spectral boundary value problem
\[ Au - zu = f, \quad (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = g, \]
is provided by the formula
\[ u = (\mathcal{A}_0 - zI)^{-1}f + (I - z\mathcal{A}_0^{-1})^{-1}\Pi(\beta_0 + \beta_1\mathcal{M}(z))^{-1}(g - \beta_1\Pi^* (I - z\mathcal{A}_0^{-1})^{-1}f). \]

By setting \( g = 0 \), the formula (26) defines the resolvent of a closed, densely defined operator in \( \mathcal{H} \) that is an extension of \( \mathcal{A}_0 \).

Theorem 3.14. Suppose that \( z \in \rho(\mathcal{A}_0) \) be such that the operator \( \overline{\beta_0 + \beta_1\mathcal{M}(z)} \) defined on \( D(\mathcal{B}) \) is boundedly invertible in \( E \). Then the operator \( \mathcal{R}_{\beta_0,\beta_1}(z) \) defined by
\[ \mathcal{R}_{\beta_0,\beta_1}(z) := (\mathcal{A}_0 - zI)^{-1} - (I - z\mathcal{A}_0^{-1})^{-1}\Pi(\beta_0 + \beta_1\mathcal{M}(z))^{-1}\beta_1(\Pi^* (I - z\mathcal{A}_0^{-1})^{-1}) \]
\[ = (\mathcal{A}_0 - zI)^{-1} - S(z)(\beta_0 + \beta_1\mathcal{M}(z))^{-1}\beta_1S^*(z) \]
is the resolvent \( (\mathcal{A}_{\beta_0,\beta_1} - zI)^{-1} \) of a closed densely defined operator \( \mathcal{A}_{\beta_0,\beta_1} \) in \( \mathcal{H} \) such that
\[ \mathcal{A}_{00} \subset \mathcal{A}_{\beta_0,\beta_1} \subset \mathcal{A}, \quad D(\mathcal{A}_{\beta_0,\beta_1}) \subset \ker(\beta_0\Gamma_0 + \beta_1\Gamma_1). \]

Remark 3.15. Notice that the pointwise nature of Theorem 3.13 with respect to the parameter \( z \) yields the same formula (26) under an even more general assumption that the operator \( \beta_1 \) depends on \( z \), see also [26].

Remark 3.16. Corollary 5.9 in [52] states that if \( \Lambda \) is boundedly invertible then the map \( \mathcal{M}(z) \) is also boundedly invertible for all \( z \) in the complement of the set \( \sigma(\mathcal{A}_0) \cup \sigma(\mathcal{A}_1) \). In the case that we will analyse below, the operator \( \Lambda \) will be boundedly invertible for all non-zero values of quasimomentum \( \chi \).
3.2 Operators associated with boundary value problems

In this section we introduce some operators required to reformulate the transmission value problem (13) in the context of the abstract theory of Ryzhov triples. First, we define the spaces
\[
\mathcal{H} := L^2(Y; \mathbb{C}^3), \quad E := L^2(\Gamma; \mathbb{C}^3), \quad \mathcal{H}^{\text{diff}} := L^2(Y^{\text{diff}}; \mathbb{C}^3), \quad \mathcal{H}^{\text{soft}} := L^2(Y^{\text{soft}}; \mathbb{C}^3).
\]
Note that we can identify \(\mathcal{H}^{\text{diff(soft)}}\) with the following spaces:
\[
\mathcal{H}^{\text{diff}} = \{ u \in L^2(Y; \mathbb{C}^3), \quad u = 0 \text{ on } Y^{\text{soft}} \}, \quad \mathcal{H}^{\text{soft}} = \{ u \in L^2(Y; \mathbb{C}^3), \quad u = 0 \text{ on } Y^{\text{diff}} \}.
\]
We also define the associated orthogonal projections \(P^{\text{soft}} : \mathcal{H}^{\text{soft}} \mapsto \mathcal{H}^{\text{soft}}, \quad P^{\text{diff}} : \mathcal{H}^{\text{diff}} \mapsto \mathcal{H}^{\text{diff}}\), so the following orthogonal decomposition holds:
\[
\mathcal{H} = P^{\text{soft}} \mathcal{H} \oplus P^{\text{diff}} \mathcal{H} = \mathcal{H}^{\text{soft}} \oplus \mathcal{H}^{\text{diff}}.
\] (28)

3.2.1 Differential operators of linear elasticity

Relative to the decomposition (28), we define self-adjoint operators \(\mathcal{A}^{\text{diff}}_{0,\chi} : \mathcal{A}^{\text{soft}}_{0,\chi}\) on the spaces \(\mathcal{H}^{\text{diff}}, \mathcal{H}^{\text{soft}}\), respectively, by the sesquilinear forms
\[
a^{\text{diff}}_{0,\chi}(u, v) := \int_{Y^{\text{diff}}} A^{\text{diff}}(\text{sym} \nabla + iX_\chi)u \cdot (\text{sym} \nabla + iX_\chi)v, \quad u, v \in D(a^{\text{diff}}_{0,\chi}),
\]
\[
a^{\text{soft}}_{0,\chi}(u, v) := \int_{Y^{\text{soft}}} A^{\text{soft}}(\text{sym} \nabla + iX_\chi)u \cdot (\text{sym} \nabla + iX_\chi)v, \quad u, v \in D(a^{\text{soft}}_{0,\chi}),
\] (29)
\[
D(a^{\text{diff}}_{0,\chi}) := \{ u \in H^1(Y^{\text{diff}}; \mathbb{C}^3), \quad u|_\Gamma = 0 \}, \quad D(a^{\text{soft}}_{0,\chi}) := \{ u \in H^1(Y^{\text{soft}}; \mathbb{C}^3), \quad u|_\Gamma = 0 \}.
\]
The following basic property is easy to prove.

**Proposition 3.17.** The forms \(a^{\text{diff (soft)}}_{0,\chi}\) are uniformly coercive, symmetric and closed.

**Proof.** Due to Assumption 2.1, there exist \(\chi\)-independent constants \(C_1, C_2 > 0\) such that
\[
C_1 \left\| (\text{sym} \nabla + iX_\chi)u \right\|_{L^2(Y^{\text{diff}}; \mathbb{C}^3 \times \mathbb{C}^3)} \leq a^{\text{diff(soft)}}_{0,\chi}(u, u) \leq C_2 \left\| (\text{sym} \nabla + iX_\chi)u \right\|_{L^2(Y^{\text{diff}}; \mathbb{C}^3 \times \mathbb{C}^3)} \quad \forall u \in D(a^{\text{diff(soft)}}_{0,\chi}).
\]
Furthermore, by Proposition A.8 (see the Appendix), there exists a \(\chi\)-independent \(C > 0\) such that
\[
a^{\text{diff(soft)}}_{0,\chi}(u, u) \geq C \left\| u \right\|^2_{H^1(Y^{\text{diff(soft)}}; \mathbb{C}^3)}, \quad \forall u \in D(a^{\text{diff(soft)}}_{0,\chi}),
\]
so both forms are uniformly coercive. \(\square\)

Clearly, the operators \(\mathcal{A}^{\text{diff(soft)}}_{0,\chi}\) correspond to the differential expressions
\[
(\text{sym} \nabla + iX_\chi)^* \mathcal{A}^{\text{diff(soft)}}_{0,\chi} (\text{sym} \nabla + iX_\chi) \quad \text{on } \mathcal{H}^{\text{diff(soft)}} \quad \text{on } \mathcal{H}^{\text{diff(soft)}}.
\] (30)
subject to the zero boundary condition on \(\Gamma\).

3.2.2 Lift operators

Here we introduce the lift operators required for the analysis of boundary value problems via the Krein formula. First, we introduce classical lift operators \(\Pi^{\text{diff(soft)}}_{\chi}\) as the operators mapping \(g \in H^{1/2}(\Gamma; \mathbb{C}^3)\) to the weak solutions \(u \in H^1_{\chi}(Y^{\text{diff(soft)}}; \mathbb{C}^3)\) of the boundary value problems
\[
\begin{cases}
(\text{sym} \nabla + iX_\chi)^* \mathcal{A}^{\text{diff(soft)}}_{\chi} (\text{sym} \nabla + iX_\chi)u = 0 & \text{on } Y^{\text{diff(soft)}}, \\
u = g & \text{on } \Gamma, \quad u \text{ is } Y\text{-periodic (in the case of } Y^{\text{diff}}). 
\end{cases}
\] (31)
As a consequence of Proposition A.8, the following statement holds.
Proposition 3.18. For every $\chi \in Y$ the problem (31) has a unique weak solution. The associated operator $\tilde{\Pi}_\chi$ is bounded and satisfies the following bounds for all $g \in H^{1/2}(\Gamma; C^3)$:
\[
\|\tilde{\Pi}_\chi \|_{H^{1/2}(\Gamma; C^3)} \leq C \|g\|_{H^{1/2}(\Gamma; C^3)},
\]
where $C > 0$ is independent of $\chi$.

Proof. The proof uses the classical approach that involves rewriting the problem (31) with zero boundary condition and non-zero right-hand side and using Proposition A.3 and Proposition A.8.

We next introduce specific realisations $\Pi_{\chi}^{\text{soft}}$ of the abstract lift operators $\Pi$ (see Section 3.1), natural for the problem at hand, by their adjoints $\Xi_{\chi}^{\text{soft}}$ with respect to the pair of inner products $\langle \cdot , \cdot \rangle_{Y^{\text{soft}}(\Omega)}$ and $\langle \cdot , \cdot \rangle_{E}$, namely
\[
\Pi_{\chi}^{\text{soft}} : E \rightarrow \mathcal{H}_{\chi}^{\text{soft}}, \quad \langle \Pi_{\chi}^{\text{soft}}(g) , f \rangle_{\mathcal{H}_{\chi}^{\text{soft}}} = \langle g , \Xi_{\chi}^{\text{soft}}(f) \rangle_E.
\]
We use the following definition:
\[
-c_{\chi}^{\text{soft}}(\mathcal{A}_{\Omega,\chi}^{\text{soft}}) (\mathcal{A}_{\Omega,\chi}^{\text{soft}})^{-1} = \Xi_{\chi}^{\text{soft}} : \mathcal{H}_{\chi}^{\text{soft}} \rightarrow E.
\]
Here $c_{\chi}^{\text{soft}}(\mathcal{A}_{\Omega,\chi}^{\text{soft}})$ is the trace of the co-normal derivative $c_{\chi}^{\text{soft}}(u) = (A_{\chi}^{\text{soft}}(\text{sym} \nabla + iX_\chi)u \cdot n_{\chi}^{\text{soft}})_{|\Gamma}$. The adjoints of $\Xi_{\chi}^{\text{soft}}$ coincide with the closures of $\Pi_{\chi}^{\text{soft}}$ in the space $E = L^2(\Gamma; C^3)$. Indeed, we have the following theorem.

Theorem 3.19. The operators $\Xi_{\chi}^{\text{soft}}$ defined by (33) are compact. Their adjoints $\Pi_{\chi}^{\text{soft}}$ are (compact) closures in $E$ of the classical lift operators $\tilde{\Pi}_{\chi}$, and (cf. Definition 3.1)
\[
\ker(\Pi_{\chi}^{\text{soft}}) = \{0\}, \quad \mathcal{D}(\mathcal{A}_{\Omega,\chi}^{\text{soft}}) \cap \mathcal{R}(\Pi_{\chi}^{\text{soft}}) = \{0\}.
\]

Proof. Due to results on elliptic regularity, under Assumption 2.1 the operators $(\mathcal{A}_{\Omega,\chi}^{\text{soft}})^{-1}$ are bounded from $\mathcal{H}_{\chi}^{\text{soft}}$ to $H^2(\Omega_{\chi}^{\text{soft}}; C^3)$ by Lemma A.16. Thus, using the trace theorem, we infer that $\Xi_{\chi}^{\text{soft}}$ is bounded as an operator to $H^{1/2}(\Gamma; C^3)$. Due to the compactness of the embedding $H^{1/2}(\Gamma; C^3) \hookrightarrow E$, it follows that $\Xi_{\chi}^{\text{soft}}$ is compact. Next, one can easily verify by integration by parts that for all $g \in H^{1/2}(\Gamma; C^3)$, $f \in \mathcal{H}_{\chi}^{\text{soft}}$ one has
\[
\langle \Pi_{\chi}^{\text{soft}}(g) , f \rangle_{\mathcal{H}_{\chi}^{\text{soft}}} = \langle g , -c_{\chi}^{\text{soft}}(\mathcal{A}_{\Omega,\chi}^{\text{soft}}) (\mathcal{A}_{\Omega,\chi}^{\text{soft}})^{-1} f \rangle_E,
\]
and hence
\[
\Pi_{\chi}^{\text{soft}}|_{H^{1/2}(\Gamma; C^3)} = \tilde{\Pi}_{\chi}^{\text{soft}}.
\]
Proceeding to the proof of (34), note that all $g \in E$ satisfy
\[
(\text{sym} \nabla + iX_\chi)^* A_{\chi}^{\text{soft}} (\text{sym} \nabla + iX_\chi) \Pi_{\chi}^{\text{soft}}(g) = 0
\]
in the sense of distributions. Provided $\Pi_{\chi}^{\text{soft}}(g) \in \mathcal{D}(\mathcal{A}_{\Omega,\chi}^{\text{soft}})$, one then has, for all $v \in C_c^\infty(Y_{\chi}^{\text{soft}}; C^3)$,
\[
\int_{Y_{\chi}^{\text{soft}}} \mathcal{A}_{\Omega,\chi}^{\text{soft}} \Pi_{\chi}^{\text{soft}}(g) : v = \int_{Y_{\chi}^{\text{soft}}} A_{\chi}^{\text{soft}} (\text{sym} \nabla + iX_\chi) \Pi_{\chi}^{\text{soft}}(g) : (\text{sym} \nabla + iX_\chi) v = 0,
\]
from which it follows immediately that $\Pi_{\chi}^{\text{soft}}(g) = 0$. This proves the second property in (34).

To prove the first property in (34), choose an arbitrary $g \in H^1(\Gamma; C^3)$ and let $u \in H^2(Y_{\chi}^{\text{soft}}; C^3)$ be such that
\[
-c_{\chi}^{\text{soft}}(u)|_\Gamma = f, \quad u|_\Gamma = 0, \quad u \text{ is } Y \text{-periodic}.
\]
The existence of such $u$ is guaranteed by the Lemma A.18. Now, denoting
\[
f := (\text{sym} \nabla + iX_\chi)^* A_{\chi}^{\text{soft}} (\text{sym} \nabla + iX_\chi) u \in \mathcal{H}_{\chi}^{\text{soft}},
\]
it is clear that $g = \Xi_{\chi}^{\text{soft}} f$, and, due to the density of $H^1(\Gamma; C^3)$ in $E$, we conclude that
\[
\ker(\Pi_{\chi}^{\text{soft}}) = \mathcal{R}(\Xi_{\chi}^{\text{soft}}) = \{0\}.
\]

\[\Box\]

\[\Box\]

\[\Box\]

\[\Box\]

\[\Box\]
Henceforth, we will be using the notation \( \Pi_{k}^{\text{soft}} \) also for the operator \( \tilde{\Lambda}_{k}^{\text{soft}} \), as it will always be clear from the context which one we are referring to.

A more precise statement than that of Theorem 3.19 is available, although we do not use it in what follows – as it is of an independent interest, we include it next.

**Theorem 3.20.** The operators \( \Sigma_{k}^{\text{soft}} \) are bounded from \( \mathcal{H}^{\text{soft}} \) to \( H^{1/2}(\Gamma) \), and the their adjoints \( \Pi_{k}^{\text{soft}} \) are bounded from \( L^{2}(\Gamma) \) to \( H^{1/2}(Y_{\text{soft}}) \).

**Proof.** The claim pertaining to the operators \( \Sigma_{k}^{\text{soft}} \), which are the adjoints of \( \Pi_{k}^{\text{soft}} \), is verified in the proof of Theorem 3.19 above.

Passing over to the operators \( \Pi_{k}^{\text{soft}} \), first notice that by Sobolev duality \( \Sigma_{k}^{\text{soft}} \) admit bounded extensions to operators acting from \( H^{-1/2}(\Gamma) \) to \( \mathcal{H}^{\text{soft}} \). The claim now follows by linear interpolation [61] between this fact and the boundedness of \( \Pi_{k}^{\text{soft}} \) as operators from \( H^{1/2}(\Gamma) \) to \( H^{1}(Y_{\text{soft}}) \), established in the estimate (32).

### 3.2.3 Dirichlet-to-Neumann maps

Here we introduce specific realisations \( \Lambda_{k}^{\text{soft}} \) of the operator \( \Lambda \) (see Section 3.1) and prove their self-adjointness. The main result of this section is Theorem 3.21.

We begin by noting that, due to the elliptic regularity, under Assumption 2.1 one can define the operators \[ \Lambda_{k}^{\text{soft}} : H^{1/2}(\Gamma; \mathbb{C}) \to H^{1/2}(\Gamma; \mathbb{C}), \quad \Lambda_{k}^{\text{soft}} \mathbf{g} = -\mathbf{c}_{v}^{\text{soft}} \mathbf{u}, \]
where \( \mathbf{g} \in H^{1/2}(\Gamma; \mathbb{C}) \) and \( \mathbf{u} \) is the unique solution to (31). Notice that as a consequence of Lemma A.16 we have

\[ \| \Lambda_{k}^{\text{soft}} \mathbf{g} \|_{H^{1/2}(\Gamma; \mathbb{C})} \leq C \| \mathbf{g} \|_{H^{1/2}(\Gamma; \mathbb{C})}, \tag{35} \]

where the constant \( C \) is independent of \( \chi \).

**Theorem 3.21.** The operators \( \Lambda_{k}^{\text{soft}} \) can be uniquely extended to unbounded non-positive self-adjoint operators \( \Lambda_{k}^{\text{soft}} \) on \( \mathcal{E} \) with domains \( \mathcal{D}(\Lambda_{k}^{\text{soft}}) = H^{1}(\Gamma; \mathbb{C}) \). Furthermore, \( \Lambda_{k}^{\text{soft}} \) have compact resolvents.

**Proof.** The first step consists in obtaining a larger extension of the operator \( \Lambda_{k}^{\text{soft}} \) for which the desired operator \( \Lambda_{k}^{\text{soft}} \) is simply a restriction onto \( H^{1}(\Gamma; \mathbb{C}) \). The second step is to show that this restriction is, in fact, self-adjoint. For this, it suffices to show that the restriction is symmetric, and that the resolvent set of the restriction contains at least one real number, for example the unity.

We begin by taking \( \mathbf{g} \in H^{3/2}(\Gamma; \mathbb{C}), \mathbf{h} \in H^{1/2}(\Gamma; \mathbb{C}) \) and considering the solutions \( \mathbf{u} \in H^{2}(Y_{\text{soft}}; \mathbb{C}), \mathbf{v} \in H^{1}(Y_{\text{soft}}; \mathbb{C}) \) to the problem (31) such that \( \mathbf{u} = \Pi_{k}^{\text{soft}} \mathbf{g}, \mathbf{v} = \Pi_{k}^{\text{soft}} \mathbf{h} \). Using Green’s formula, we obtain

\[
\int_{\Gamma} \Lambda_{k}^{\text{soft}} \mathbf{g} : \mathbf{h} = -\int_{Y_{\text{soft}}} \Lambda_{k}^{\text{soft}} (\text{sym } \nabla + iX_{\lambda}) \mathbf{u} : (\text{sym } \nabla + iX_{\lambda}) \mathbf{v} = -\int_{Y_{\text{soft}}} \Lambda_{k}^{\text{soft}} (\text{sym } \nabla + iX_{\lambda}) \Pi_{k}^{\text{soft}} \mathbf{g} : (\text{sym } \nabla + iX_{\lambda}) \Pi_{k}^{\text{soft}} \mathbf{h}. \tag{36}
\]

It follows that \( \Lambda_{k}^{\text{soft}} \mathbf{g} \) defines an element of \( H^{-1/2}(\Gamma; \mathbb{C}) \). Due to (32), the bound

\[ \| \Lambda_{k}^{\text{soft}} \mathbf{g} \|_{H^{-1/2}(\Gamma; \mathbb{C})} \leq C \| \mathbf{g} \|_{H^{1/2}(\Gamma; \mathbb{C})}, \tag{37} \]

holds with \( C \) independent of \( \chi \). In particular, we can define a unique bounded extension

\[ \Lambda_{k}^{\text{soft}} : H^{1/2}(\Gamma; \mathbb{C}) \to H^{-1/2}(\Gamma; \mathbb{C}), \quad \Lambda_{k}^{\text{soft}} \big|_{H^{1/2}(\Gamma; \mathbb{C})} = \Lambda_{k}^{\text{soft}}. \]

Since \( (H^{1}(\Gamma; \mathbb{C}), L^{2}(\Gamma; \mathbb{C})) \) is an interpolation pair [61, 36] with respect to the pairs \( (H^{3/2}(\Gamma; \mathbb{C}), H^{1/2}(\Gamma; \mathbb{C})) \) and \( (H^{1/2}(\Gamma; \mathbb{C}), H^{-1/2}(\Gamma; \mathbb{C})) \), we conclude that \( \Lambda_{k}^{\text{soft}} \) is bounded as an operator from \( H^{1}(\Gamma; \mathbb{C}) \) to \( L^{2}(\Gamma; \mathbb{C}) \) and denote by \( \Lambda_{k}^{\text{soft}} \) the restriction of \( \Lambda_{k}^{\text{soft}} \) to \( H^{1}(\Gamma; \mathbb{C}) \).

Proceeding to the second step, we prove that \( \Lambda_{k}^{\text{soft}} \) is self-adjoint as an unbounded operator on \( L^{2}(\Gamma; \mathbb{C}) \) with domain \( \mathcal{D}(\Lambda_{k}^{\text{soft}}) = H^{1}(\Gamma; \mathbb{C}) \). Notice that \( \Lambda_{k}^{\text{soft}} \) is non-positive and symmetric. Indeed, as a consequence of
(36) holding for $g \in H^{1/2}(\Gamma; C^3)$, $h \in H^{1/2}(\Gamma; C^3)$, one has
\[
\int_{\Gamma} \Lambda_{\text{st}(\text{soft})}^\prime g \cdot h = -\int_{\partial \Omega} \Lambda_{\text{st}(\text{soft})}^\prime (\text{sym} \nabla + iX_\nu) \Pi_{\text{st}(\text{soft})}^\prime g : \left(\text{sym} \nabla + iX_\nu\right) \Pi_{\text{st}(\text{soft})}^\prime h
\]
\[\forall g, h \in H^1(\Gamma; C^3).\]

In order to prove self-adjointness of $\Lambda_{\text{st}(\text{soft})}^\prime$, it suffices to show that $\rho(\Lambda_{\text{st}(\text{soft})}^\prime) \cap \mathbb{R} \neq \emptyset$. We claim that
\[(-\Lambda_{\text{st}(\text{soft})}^\prime + I)^{-1} : L^2(\Gamma; C^3) \to H^1(\Gamma; C^3)\]
is bounded. To see this, first assume that $h \in H^{1/2}(\Gamma; C^3)$ and seek the solution $g \in H^{1/2}(\Gamma; C^3)$ to the problem $-\Lambda_{\text{st}(\text{soft})}^\prime g + g = h$. This is equivalent to seeking $u$ such that
\[
\begin{cases}
(\text{sym} \nabla + iX_\nu)^\prime \Lambda_{\text{st}(\text{soft})}^\prime (\text{sym} \nabla + iX_\nu) u = 0 & \text{on } Y_{\text{st}(\text{soft})}, \\
\Lambda_{\text{st}(\text{soft})}^\prime u + u = h & \text{on } \Gamma
\end{cases}
\]
in the weak sense, and then setting $g := u|_\Gamma$. Invoking Lemma A.15 for the existence and uniqueness of such a solution, we thus obtain an operator $(-\Lambda_{\text{st}(\text{soft})}^\prime + I)^{-1} : H^{1/2}(\Gamma; C^3) \to H^{1/2}(\Gamma; C^3)$.

Second, consider the form $a_{\text{st}(\text{soft})}^\prime$ defined by the same expression as $a_{\text{st}(\text{soft})}$, see (29), but on the domain $D(a_{\text{st}(\text{soft})}^\prime) := H^1(Y_{\text{st}(\text{soft})}; C^3)$. Applying Korn’s inequality (see Proposition A.8) to the weak form of (38), namely
\[
a_{\text{st}(\text{soft})}^\prime(u, v) + \int_{\Gamma} u \cdot \nu = \int_{\Gamma} h \cdot \nu & \forall v \in H^1(Y_{\text{st}(\text{soft})}; C^3),
\]
where we set $v = u$, we obtain the apriori estimate $\|u\|_{L^2(Y_{\text{st}(\text{soft})}; C^3)} \leq C \|h\|_{H^{1/2}(\Gamma; C^3)}$. Therefore, the resolvent
\[(-\Lambda_{\text{st}(\text{soft})}^\prime + I)^{-1}\]
can be extended to a bounded operator from $H^{-1/2}(\Gamma; C^3)$ to $H^{1/2}(\Gamma; C^3)$. Using an interpolation argument once again, we conclude that $(-\Lambda_{\text{st}(\text{soft})}^\prime + I)^{-1}$ is bounded from $L^2(\Gamma; C^3)$ to $H^1(\Gamma; C^3)$ and is therefore compact as an operator on $L^2(\Gamma; C^3)$. Thus, unity is in the resolvent set of $\Lambda_{\text{st}(\text{soft})}^\prime$.

**Remark 3.22.** Notice that as a consequence of the linear interpolation theory (see, e.g., [61, 36]) as well as the bounds (35), (37), we have
\[
\|\Lambda_{\text{st}(\text{soft})}^\prime g\|_{L^2(\Gamma; C^3)} \leq C\|g\|_{H^1(\Gamma; C^3)} \quad \forall g \in L^2(\Gamma; C^3),
\]
where the constant $C$ does not depend on $\chi$.

Applying the Ryzhov triple framework (15) to the lift and DtN operators provided by Theorems 3.19, 3.21, we define the operators $\Lambda_{\text{st}(\text{soft})}^\prime, \Gamma_{\text{st}(\text{soft})}, \Omega_{\text{st}(\text{soft})}$. In particular, one has
\[
\Gamma_{\text{st}(\text{soft})} u = u|_\Gamma \quad \forall u \in H^1(Y_{\text{st}(\text{soft})}; C^3) + H^1(\Gamma; C^3),
\]
\[
\Omega_{\text{st}(\text{soft})} u = -c_{\text{st}(\text{soft})} u|_\Gamma \quad \forall u \in H^1(Y_{\text{st}(\text{soft})}; C^3),
\]
\[
\Lambda_{\text{st}(\text{soft})} u = (\text{sym} \nabla + iX_\nu) a_{\text{st}(\text{soft})} (\text{sym} \nabla + iX_\nu) u \quad \forall u \in H^1(Y_{\text{st}(\text{soft})}; C^3).
\]
(For clarity, we note that the domains of the operators $\Gamma_{\text{st}(\text{soft})}$ and $\Lambda_{\text{st}(\text{soft})}$ are wider than the indicated spaces.)

An alternative approach to defining the operators $-\Lambda_{\text{st}(\text{soft})}^\prime$ (and thus $\Lambda_{\text{st}(\text{soft})}^\prime$) can be developed under weaker assumptions on the regularity of the domain and the operator coefficients by considering sesquilinear forms $\Lambda_{\text{st}(\text{soft})}^\prime : H^{1/2}(\Gamma; C^3) \times H^{1/2}(\Gamma; C^3) \to \mathbb{R}$, as follows:
\[
\Lambda_{\text{st}(\text{soft})}^\prime (g, h) := a_{\text{st}(\text{soft})} (\Pi_{\text{st}(\text{soft})}^\prime g, \Pi_{\text{st}(\text{soft})}^\prime h), \quad g, h \in H^{1/2}(\Gamma; C^3).
\]
The next proposition provides a coercivity estimate for the form $\Lambda_{\text{st}(\text{soft})}^\prime$, which implies an estimate for the lowest eigenvalue of the operator $\Lambda_{\text{st}(\text{soft})}^\prime$, see Section 4.2.

**Proposition 3.23.** Let the boundary $\Gamma$ be of class $C^4$, and suppose the entries of the elasticity tensor $\Lambda_{\text{st}(\text{soft})}$ are in $L^\infty(Y_{\text{st}(\text{soft})})$. The forms $\Lambda_{\text{st}(\text{soft})}^\prime$ are symmetric, non-negative, closed, and densely defined in $\mathcal{E}$. Furthermore, there exists a $\chi$-independent constant $C > 0$ such that
\[
\|\Lambda_{\text{st}(\text{soft})}^\prime g\|^2_{H^{1/2}(\Gamma; C^3)} \leq C\Lambda_{\text{st}(\text{soft})}^\prime (g, g) \quad \forall g \in H^{1/2}(\Gamma; C^3).
\]
Proof. For \( g \in H^{1/2}(\Gamma; \mathbb{C}^3) \), due to the pointwise coercivity of \( A_{\text{soft}} \), one has
\[
\|(\text{sym} \nabla + iX_k) \Pi_{\text{soft}} g\|_{L^2(Y_{\text{ref}}; \mathbb{C}^3)} \leq C_k A_{\text{soft}}(g, g).
\]
Furthermore, due to the estimate (A.8) and the trace theorem, we obtain
\[
|\chi|^2 \|g\|_{H^{1/2}(\Gamma; \mathbb{C}^3)} \leq |\chi|^2 \|\Pi_{\text{soft}} g\|_{L^2(Y_{\text{ref}}; \mathbb{C}^3)} \leq C k A_{\text{soft}}(g, g). \tag*{\hfill \square}
\]

A representation theorem by Kato [34] yields the existence of (non-positive) self-adjoint operators \( A_{\text{soft}} \) : \( \mathcal{E} \cong \mathcal{D}(A_{\text{soft}}) \to \mathcal{E} \) such that
\[
(\mathcal{E} \ni \langle A_{\text{soft}} \rangle g, v)_{H^{1/2}(\Gamma; \mathbb{C}^3)} = -A_{\text{soft}}(g, v) \quad \forall g, v \in H^{1/2}(\Gamma; \mathbb{C}^3).
\]

Here, \( (\cdot, \cdot)_{H^{1/2}} \) denotes the duality pairing between the spaces \( H^{-1/2} \) and \( H^{1/2} \). This approach to defining DtN maps allows us to relax the regularity assumptions (on the boundary and the operator coefficients). However, as a result, we lose the information on the domains of these maps. In particular, we no longer have the equality \( \mathcal{D}(A_{\text{soft}}) = \mathcal{D}(A_{\text{soft}}^\dagger) = H^1(\Gamma; \mathbb{C}^3) \), as in the Theorem 3.21. We will not pursue this approach, but we will use Proposition 3.23 for estimating the smallest eigenvalues of the operator \( A_{\text{soft}} \). The eigenvalues and eigenfunctions of DtN maps are usually referred to as Steklov eigenvalues and eigenfunctions, respectively.

### 3.3 Transmission problem: reformulation in terms of Ryzhov triple

The decoupled operator \( \mathcal{A}_{0,\chi} \) is defined by
\[
\mathcal{A}_{0,\chi} := \mathcal{A}_{0,\chi} \oplus \mathcal{A}_{0,\chi} \oplus \mathcal{R}_{\mathcal{A}_{0,\chi}}, \quad \mathcal{D}(\mathcal{A}_{0,\chi}) = \mathcal{D}(\mathcal{A}_{0,\chi}) \oplus \mathcal{D}(\mathcal{A}_{0,\chi}),
\]
relative to the decomposition (28). Equivalently, its form is given by
\[
a_{0,\chi}(u, v) := \int_Y A^2(\text{sym} \nabla + iX_k) u : (\text{sym} \nabla + iX_k) v, \quad u, v \in \mathcal{D}(a_{0,\chi}) = \{ u \in H^1(Y; \mathbb{C}^3), u|_{\Gamma} = 0 \}.
\]

We also define the coupled lift operator by
\[
\Pi_{\chi} : \mathcal{E} \to \mathcal{H}_{\text{soft}} \oplus \mathcal{H}_{\text{soft}}, \quad \Pi_{\chi} g := \Pi_{\text{soft}} \Pi_{\chi} g, \quad g \in \mathcal{E}.
\]

From Theorem 3.19 we have \( \ker(\Pi_{\chi}) = \{ 0 \} \), \( \mathcal{D}(\mathcal{A}_{0,\chi}) \cap \mathcal{R}(\Pi_{\chi}) = \{ 0 \} \). In order to describe the condition of continuity of normal derivatives of (15), we introduce the operator \( A_{\chi,\chi} := \mathcal{K}^2 - A_{\text{soft}} + A_{\text{soft}} \).

Both \( A_{\text{soft}} \) and \( A_{\text{soft}} \) are self-adjoint on the common domain \( H^1(\Gamma; \mathbb{C}^3) \) and non-negative. Therefore, the operator \( A_{\chi,\chi} \) is self-adjoint on \( \mathcal{D}(A_{\chi,\chi}) := H^1(\Gamma; \mathbb{C}^3) \). This fact very probably belongs to the domain of folklore, still we include the proof of it as kindly shared with the authors by Dr. V. Sloushch, to whom we express our deep gratitude.

**Theorem 3.24.** Assuming there are two self-adjoint operators \( \mathcal{A} \geq 0 \) and \( \mathcal{B} \geq 0 \) with the same domain \( \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{B}) \) in the Hilbert space \( \mathcal{H} \), the sum \( \mathcal{A} + \mathcal{B} \) is self-adjoint on the same domain.

**Proof.** Without loss of generality, assume that \( \mathcal{A} \) and \( \mathcal{B} \) are positive definite. By the closed graph theorem, \( \mathcal{A}^{-1} \) and \( \mathcal{B}^{-1} \) are bounded in \( \mathcal{H} \). Pick a \( \kappa \) such that \( |\kappa| \mathcal{B}^{-1} \leq 1 \). One clearly has \( \mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\kappa \mathcal{B}) \) and \( \| \kappa \mathcal{B} u \| \leq \| \mathcal{A} u \| \) for \( u \in \mathcal{D}(\mathcal{A}) \).

By the Heinz inequality (see, e.g., [6, Chapter 10, Section 4.2, Theorem 3]),
\[
\mathcal{D}(\mathcal{A})^{1/2} \subseteq \mathcal{D}(\kappa \mathcal{B})^{1/2}, \quad \| \kappa^{1/2} \mathcal{B}^{1/2} u \| \leq \| \mathcal{A}^{1/2} u \|, \quad u \in \mathcal{D}(\mathcal{A})^{1/2},
\]
hence \( \mathcal{D}(\mathcal{A})^{1/2} \subseteq \mathcal{D}(\mathcal{B})^{1/2} \) and \( B^{1/2} A^{-1/2} \) is bounded. Swapping the rôles of \( \mathcal{A} \) and \( \mathcal{B} \), \( \mathcal{B}(\mathcal{B})^{1/2} \subset \mathcal{D}(\mathcal{A})^{1/2} \) and \( A^{-1/2} B^{1/2} \) is bounded.

It follows that \( B^{-1/2} A^{-1/2} \) is bounded and non-negative. Since obviously
\[
(\mathcal{A} + \mathcal{B}) u = B^{1/2} (\mathcal{B}^{-1/2} A^{-1/2} + I) \mathcal{B}^{1/2} u \quad \forall u \in \mathcal{D}(\mathcal{A}),
\]
and \( B^{-1/2} A^{-1/2} + I \) is bounded invertible, \( (\mathcal{A} + \mathcal{B})^{-1} \) is bounded and defined on the whole \( \mathcal{H} \), and therefore closed. Therefore, \( \mathcal{A} + \mathcal{B} \) is self-adjoint on \( \mathcal{D}(\mathcal{A}) \), as required. \( \square \)
Obviously, one has to deal with the auxiliary DtN maps, and consequently, the asymptotics of their spectral projections.

It turns out to be sufficient for our purposes to obtain the asymptotics of the resolvents of DtN maps, and consequently, the asymptotics of their spectral projections. We conclude by providing simple approximations for the boundary operators on the soft component (Section 4.4).

### Transmission problem: Ryzhov triple asymptotics

The goal of this section is to provide operator asymptotics with respect to the quasimomentum $\chi \in \chi'$ for the operators of the Ryzhov triple associated with the stiff component that were introduced in the previous section. As we will see, the approximation on the stiff component pays a key role in the overall approximation, cf. Remark 7.6 below.

We also show that the eigenspace of the DtN map $\Lambda_{\chi,\omega}$ corresponding, for small $\chi$, to Steklov eigenvalues of order $|\chi|^2$ is finite-dimensional. This fact is one of the key ingredients for providing resolvent asymptotics for the transmission problem (13).

As we are about to see, the vector nature of the problem does not allow one to infer an asymptotic expansion for Steklov eigenvalues. However, it turns out to be sufficient for our purposes to obtain the asymptotics of the resolvents of DtN maps, and consequently, the asymptotics of their spectral projections.

We conclude by providing simple approximations for the boundary operators on the soft component (Section 4.4).
4.1 Lift operators: asymptotic properties

We are interested in the asymptotics for the lift operator \( \Pi^\text{diff}_\chi^\ast : H^{1/2}(\Gamma; C^3) \to H^1(\chi; \mathbb{C}^3) \) defined by (31). (Recall that for each \( \chi \) the operator \( \Pi^\text{diff}_\chi^\ast \) is nothing but the closure of \( \Pi^\text{diff}_\chi \), which is defined by (31).) Naturally, the leading-order term is the operator \( \Pi^\text{diff}_0^\ast : H^{1/2}(\Gamma; C^3) \to H^1(\chi; \mathbb{C}^3) \), which does not depend on \( \chi \). Note that for \( g \in H^{1/2}(\Gamma; C^3) \) the function \( \Pi^\text{diff}_0^\ast \ g \in H^1(\chi; \mathbb{C}^3) \) satisfies the identity

\[
\int_{\chi^\ast} A^\text{diff} \left[ \text{sym} \nabla \Pi^\text{diff}_0^\ast \ g \right] : \text{sym} \nabla \varphi = 0 \quad \forall \varphi \in H^1(\chi; \mathbb{C}^3), \ \varphi|_{\Gamma} = 0. \tag{46}
\]

The operator \( \Pi^\text{diff}_0^\ast \) satisfies the estimate provided by the following lemma.

**Lemma 4.1.** There exist constants \( C_1, C_2 > 0 \) such that for all \( g \in H^{1/2}(\Gamma; C^3) \) one has

\[
C_1 \left\| \text{sym} \nabla \Pi^\text{diff}_0^\ast \ g \right\|_{L^2(\chi; \mathbb{C}^3)} \leq \left\| g - \frac{1}{|\Gamma|} \int_{\Gamma} g \right\|_{H^{1/2}(\Gamma; C^3)} \leq C_2 \left\| \text{sym} \nabla \Pi^\text{diff}_0^\ast \ g \right\|_{L^2(\chi; \mathbb{C}^3)},
\]

**Proof.** The right-hand inequality is proved in Proposition A.12 (see the Appendix). To prove the left-hand one, we observe that

\[
\left\| \text{sym} \nabla \Pi^\text{diff}_0^\ast \ g \right\|_{L^2(\chi; \mathbb{C}^3)} \leq \left\| \text{sym} \nabla \Pi^\text{diff}_0^\ast \left( g - \frac{1}{|\Gamma|} \int_{\Gamma} g \right) \right\|_{L^2(\chi; \mathbb{C}^3)} \leq \left\| \Pi^\text{diff}_0^\ast \left( g - \frac{1}{|\Gamma|} \int_{\Gamma} g \right) \right\|_{H^1(\chi; \mathbb{C}^3)},
\]

which concludes the proof. \( \square \)

**Remark 4.2.** Notice here that the operator \( \Pi^\text{diff}_0^\ast \) lifts constant functions on the boundary \( g \equiv C \in \mathbb{C}^3 \gets \mathcal{E} \) to constant functions defined by the same constant: \( \Pi^\text{diff}_0^\ast \ G = G, \ G \equiv C \in \mathbb{C}^3 \rightsquigarrow \mathcal{H}^\text{diff} \).

The following theorem is crucial for understanding the asymptotics of the DtN map. As its proof follows a standard asymptotic argument \cite{53,31}, we provide it in the Appendix.

**Theorem 4.3.** For each \( n \in \mathbb{N} \), the operator \( \Pi^\text{diff}_\chi \) admits the asymptotic expansion

\[
\Pi^\text{diff}_\chi = \Pi^\text{diff}_0 + \tilde{\Pi}^\text{err}_1 + \tilde{\Pi}^\text{err}_2 + \cdots + \tilde{\Pi}^\text{err}_n + \tilde{\Pi}^\text{err},
\]

where the operators \( \tilde{\Pi}^\text{err}_k, \tilde{\Pi}^\text{err} : H^{1/2}(\Gamma; C^3) \to H^1(\chi; \mathbb{C}^3) \), \( k = 1, \ldots, n \) are bounded and satisfy

\[
\left\| \tilde{\Pi}^\text{err}_k \right\|_{H^{1/2}(\Gamma; C^3) \to H^1(\chi; \mathbb{C}^3)} \leq C |\chi|^k, \ k = 1, \ldots, n, \quad \left\| \tilde{\Pi}^\text{err} \right\|_{H^{1/2}(\Gamma; C^3) \to H^1(\chi; \mathbb{C}^3)} \leq C |\chi|^{n+1}, \tag{47}
\]

and the constant \( C > 0 \) does not depend on \( \chi \in \mathcal{Y}_\chi \). Furthermore,

\[
\tilde{\Pi}^\text{err}_k = \Pi_k : \chi^\otimes k, \quad [\Pi_k]_{(1, \ldots, k)} \in \mathfrak{P}\left( H^{1/2}(\Gamma; C^3), H^1(\chi; \mathbb{C}^3) \right),
\]

and the operator-valued tensors \( \Pi_k \) are symmetric, i.e., \( [\Pi_k]_{(\sigma(1, \ldots, k))} = [\Pi_k]_{(1, \ldots, k)} \) for all \( \sigma \in S_n \), where \( S_n \) is the permutation group of order \( n \).

**Remark 4.4.** Here we make an observation that proves to be crucial in the understanding of the homogenisation properties of the effective operator (see Lemma 4.8). By following the proof of Theorem 4.3 and taking into account the equation (A.22), which actually serves as the definition of the operator \( \tilde{\Pi}^\text{err}_1 \), one concludes that for \( g \equiv C \in \mathbb{C}^3 \gets \mathcal{E} \) one has

\[
\int_{\chi^\ast} A^\text{diff} \left( \text{sym} \nabla \tilde{\Pi}^\text{err}_1 \ g + i \chi^\ast \ u \right) : \text{sym} \nabla \varphi = 0 \quad \forall \varphi \in H^1(\chi; \mathbb{C}^3), \ \varphi|_{\Gamma} = 0. \tag{48}
\]

The following lemma yields estimates on the stiff component that are useful in the spectral analysis to follow. We identify the space of constant functions on \( \chi^\ast \) and \( \Gamma \) with the space \( C^3 \).

**Proposition 4.5.** There exists a constant \( C > 0 \) such that for all \( \chi \in \mathcal{Y}_\chi \setminus \{0\} \):

- For every \( g \in H^{1/2}(\Gamma; C^3) \), one has

\[
\left\| g \right\|_{H^{1/2}(\Gamma; C^3)} \leq C |\chi|^{-1} \left\| \left( \text{sym} \nabla + i \chi^\ast \right) \ Pi^\text{diff}_\chi \ g \right\|_{L^2(\chi^\ast; \mathbb{C}^3)}; \tag{49}
\]
• For every $g \in H^{1/2}(\Gamma; C^3)$, $g \perp C^3$ (in $L^2(\Gamma; C^3)$ inner product), one has

$$\|g\|_{H^{1/2}(\Gamma; C^3)} \leq C \left\| \text{sym} \nabla + iX_\chi \right\|_{L^2(\Gamma; \mathbb{C}^{3 \times 1})}.$$

(50)

Proof. The estimate (49) is a straightforward consequence of Proposition A.9 and the trace theorem. In order to prove (50), it suffices to show that for $g \in H^{1/2}(\Gamma; C^3)$, $g \perp C^3$ one has

$$\|g\|_{L^2(\Gamma; C^3)} \leq C \left\| \text{sym} \nabla + iX_\chi \right\|_{L^2(\Gamma; \mathbb{C}^{3 \times 1})}.$$  

(51)

Indeed, suppose (51) holds. Next, employing Proposition A.8 and the trace theorem, we obtain

$$\|g\|_{H^{1/2}(\Gamma; C^3)} \leq C \left\| \Pi_{\text{sym}} g \right\|_{H^1(\Gamma; \mathbb{C}^3)} \leq C \left\| (\text{sym} \nabla + iX_\chi) \Pi_{\text{sym}} g \right\|_{L^2(\Gamma; \mathbb{C}^{3 \times 1})} + C \|g\|_{L^2(\Gamma; C^3)} \leq C \left\| (\text{sym} \nabla + iX_\chi) \Pi_{\text{sym}} g \right\|_{L^2(\Gamma; \mathbb{C}^{3 \times 1})}.$$ 

Finally, (51) is is obtained by plugging $u = \Pi_{\text{sym}} g$ into the second estimate of Proposition A.12. □

4.2 Smallest Steklov eigenvalues

The operator $\Lambda_{\chi}^{\text{diff}}$ on $L^2(\Gamma; C^3)$ has discrete spectrum, due to the compactness of its resolvent established above. The eigenvalues $\lambda^k$ of $\Lambda_{\chi}^{\text{diff}}$ are equivalently characterised as solutions to either of the following two problems:

$$\Lambda_{\chi}^{\text{diff}} g = \lambda^k g, \quad g \in \mathcal{D}(\Lambda_{\chi}^{\text{diff}}) \setminus \{0\},$$

subject to

$$\left\{ \begin{array}{l}
\mathcal{R}_{\chi} g = 0, \\
\Gamma_{\chi} g = \lambda^k \Gamma_{\chi} g.
\end{array} \right.$$ 

Next, we define the Rayleigh quotient associated with $\Lambda_{\chi}^{\text{diff}}$, namely

$$\mathcal{R}_{\chi}(g) := \frac{\lambda^k(g, g)}{\|g\|_{L^2(\Gamma; C^3)}^2}, \quad g \in H^{1/2}(\Gamma; C^3),$$

where $\lambda^k_{\chi} (\cdot)$ is defined by (43). The sequence $(\nu_n^k)_{n \in \mathbb{N}}$ is characterised by the min-max principle

$$-\nu_n^k = \min_{g \in H^{1/2}(\Gamma; C^3)} \max_{\dim G = n} \mathcal{R}_{\chi}(g), \quad n \in \mathbb{N}.$$ 

(52)

The following lemma quantifies the behaviour of the smallest eigenvalues.

Lemma 4.6. There exist constants $C_1 > C_2 > 0$ such that

- $\mathcal{R}_{\chi}(g) \geq C_2 |x|^2 \quad \forall g \in H^{1/2}(\Gamma; C^3),$
- $\mathcal{R}_{\chi}(g) \leq C_1 |x|^2 \quad \forall g \in C^3,$
- $\mathcal{R}_{\chi}(g) \geq C_2 \quad \forall g \in (C^3)^\perp.$

Proof. The proof of the first and third points is a direct consequence of Proposition 4.5. The second point is verified by a direct computation. □

The asymptotics of eigenvalues with respect to $|x|$ is given in the following lemma.

Lemma 4.7. The three smallest eigenvalues of $\Lambda_{\chi}^{\text{diff}}$ are of order $O(|x|^2)$, and the remaining eigenvalues uniformly separated from zero. Namely, there exist constants $c_1, c_2 > 0$ that do not depend on $\chi$ such that

$$c_1 |x|^2 \leq -\nu_n^k \leq c_2 |x|^2, \quad n = 1, 2, 3, \quad c_1 \leq -\nu_n^k, \quad n \geq 4.$$ 

(53)

Proof. The proof is a direct consequence of (52) and Lemma 4.6. □

In what follows, we refer to $\nu_n^k, n = 1, 2, 3$, as $O(|x|^2)$ eigenvalues and to $\nu_n^k, n \geq 4$, as $O(1)$ eigenvalues. Consider the decomposition

$$\mathcal{E} := \mathcal{E}_{\chi} \oplus \mathcal{E}_{\chi} = \mathcal{P}_{\chi} \mathcal{E} \oplus \mathcal{P}_{\chi} \mathcal{E},$$

(53)
where \( \hat{P}_\chi \) is the orthogonal projection onto the three-dimensional space \( \hat{E}_\chi < E \) spanned by the eigenfunctions associated with order \( O(|\chi|^2) \) eigenvalues of \( \Lambda^{\text{sym}}_\chi \), and \( \tilde{P}_\chi \) is the orthogonal projection onto \( \tilde{E}_\chi < E \), the infinite-dimensional space spanned by the eigenfunctions associated with order \( O(1) \) eigenvalues of \( \Lambda^{\text{sym}}_\chi \), so that \( \hat{P}_\chi = I - \tilde{P}_\chi \). Since the decomposition (53) is spectral for \( \Lambda^{\text{sym}}_\chi \), we have

\[
\Lambda^{\text{sym}}_\chi = \begin{bmatrix}
\tilde{\Lambda}^{\text{sym}}_\chi & 0 \\
0 & \hat{\Lambda}^{\text{sym}}_\chi
\end{bmatrix},
\]

where

\[
\tilde{\Lambda}^{\text{sym}}_\chi := \hat{P}_\chi \Lambda^{\text{sym}}_\chi |_{\tilde{E}_\chi}, \quad \hat{\Lambda}^{\text{sym}}_\chi := \tilde{P}_\chi \Lambda^{\text{sym}}_\chi |_{\hat{E}_\chi}.
\]

Both \( \tilde{\Lambda}^{\text{sym}}_\chi \) and \( \hat{\Lambda}^{\text{sym}}_\chi \) are self-adjoint operators on \( \tilde{E}_\chi \) and \( \hat{E}_\chi \), respectively, since \( \hat{P} \) is a spectral projection for \( \Lambda^{\text{sym}}_\chi \). The operator \( \hat{\Lambda}^{\text{sym}}_\chi \) is clearly bounded since it is finite-rank. Note also that the domain of the second operator is a subset of \( H^1(\Gamma; \mathbb{C}^3) \), see Section 3.2.3. Furthermore, due to Lemma 4.7 we have the uniform bound \( \| \tilde{\Lambda}^{\text{sym}}_\chi \|_{\ell E \to \ell E} \leq C |\chi|^2 \). On the other hand, the same lemma implies that the operator \( \hat{\Lambda}^{\text{sym}}_\chi \), while unbounded, is uniformly bounded from below, where the estimate does not depend on \(|\chi|\), namely \( \| (\tilde{\Lambda}^{\text{sym}}_\chi)^{-1} \|_{\ell E \to \ell E} \leq C \). Moreover, \( (\tilde{\Lambda}^{\text{sym}}_\chi)^{-1} \) is compact and

\[
\| (\tilde{\Lambda}^{\text{sym}}_\chi)^{-1} g \|_{H^1(\Gamma; \mathbb{C}^3)} \leq C \| g \|_{L^2(\Gamma; \mathbb{C}^3)}, \quad \forall g \in L^2(\Gamma; \mathbb{C}^3),
\]

where the constant \( C \) is independent of \( \chi \).

### 4.3 Asymptotics of \( (|\chi|^{-2} \Lambda^{\text{sym}}_\chi - I)^{-1} \)

While the results of this section are not necessary for the proof of Theorem 2.4, they are essential in the proof of Theorem 2.5, see Sections 6, 7. The main tool for proving the latter theorem is finding an approximating homogenised operator by developing an asymptotics of the DtN map using its variational definition (see (56)). Note that for that we cannot follow the approach of [16], since for PDE systems one cannot expand eigenfunctions or eigenvalues with respect to the quasimomentum \( \chi \), see, e.g., [34, Example 5.12]. Instead, we construct an asymptotics for the resolvent of the DtN map; in Sections 6, 7 this suffices to prove Theorem 2.5. Note also that the variational definition of the approximating operator (56) implies its characterisation in terms of \( \Lambda^{\text{macro}}_\chi \), see Lemma 4.8.

We calculate the estimates on the distance between the resolvents of \( -|\chi|^{-2} \Lambda^{\text{sym}}_\chi \) and \( -|\chi|^{-2} \Lambda^{\text{hom}}_\chi \), where the latter plays the rôle of the effective DtN map and is introduced below. In Corollary 4.12 we use this to obtain the approximation error with respect to the resolvents of \( \varepsilon^2 \)-scaled operators. A similar approach (in a different context) was used in [23, 24].

Recall that the lift operator \( \Pi^{\text{lift}}_{\chi} : H^{1/2}(\Gamma; \mathbb{C}^3) \to H^1(\chi; \mathbb{C}^3) \) admits the decomposition

\[
\Pi^{\text{lift}}_{\chi} = \Pi^{\text{lift}}_0 + \Pi^{\text{lift}}_{\chi,1} + \Pi^{\text{error}}_{\chi,1},
\]

in the sense of Theorem 4.3, where the dependence of the operator \( \Pi^{\text{lift}}_{\chi,1} \) on the parameter \( \chi \) is linear. The error term \( \Pi^{\text{error}}_{\chi,1} \) satisfies the bound (see (47)) \( \| \Pi^{\text{error}}_{\chi,1} \|_{H^{1/2}(\Gamma; \mathbb{C}^3) \to H^1(\chi; \mathbb{C}^3)} \leq C |\chi|^2 \). For each \( \chi \in \chi' \), consider the expression \( \Psi_{\chi} := \im \Pi^{\text{lift}}_0 + \text{sym} \nabla \Pi^{\text{lift}}_{\chi,1} \). We define the homogenised operator to be a constant matrix \( \Lambda^{\text{hom}}_\chi \in \mathbb{C}^{3 \times 3} \) such that

\[
\langle -\Lambda^{\text{hom}}_\chi e, d \rangle_{\mathbb{C}^3} := \frac{1}{|\chi|} \int_{\chi'} \Lambda^{\text{lift}} |\text{sym} \nabla \Pi^{\text{lift}}_0 e + \Psi_{\chi} d| : \overline{\text{sym} \nabla \Pi^{\text{lift}}_0 d} \quad \forall e, d \in \mathbb{C}^3,
\]

where \( e, d \in H^1(\chi; \mathbb{C}^3) \) is the unique solution with \( \Pi^{\text{error}}_{\chi,1} e = 0 \) to the cell problem

\[
\int_{\chi'} \Lambda^{\text{lift}} |\text{sym} \nabla \Pi^{\text{lift}}_0 e + \Psi_{\chi} e| : \text{sym} \nabla \Pi^{\text{lift}}_0 \varphi = 0 \quad \forall \varphi \in \text{sym} \nabla H^{1/2}(\Gamma; \mathbb{C}^3).
\]

In next lemma we provide important properties of the matrix \( \Lambda^{\text{hom}}_\chi \).

**Lemma 4.8.** The matrix \( \Lambda^{\text{hom}}_\chi \) is quadratic in \( \chi \), in particular \( \Lambda^{\text{hom}}_\chi = -|\Gamma|^{-1} (\im \chi)^* \Lambda^{\text{macro}}_\gamma \im \chi \), where \( \Lambda^{\text{macro}}_\gamma \) is the constant symmetric tensor defined by (2). As a consequence, \( |\chi|^{-2} \Lambda^{\text{hom}}_\chi \) is positive and bounded uniformly in \( \chi \).

**Proof.** Note first that for \( e \in \mathbb{C}^3 \) the solution \( e_{\text{corr}} \) to (57) satisfies

\[
\int_{\chi'} \Lambda^{\text{lift}} |\text{sym} \nabla \Pi^{\text{lift}}_0 e_{\text{corr}} + \Psi_{\chi} e| : \text{sym} \nabla \varphi = 0 \quad \forall \varphi \in \text{sym} \nabla H^1(\chi; \mathbb{C}^3).
\]
This is seen by noting that for an arbitrary \( v \in H^s_0(Y_{\text{stiff}}; \mathbb{C}^3) \) one has the decomposition
\[
v = \Pi_{0,\text{macro}}^\text{stiff} h + \omega, \quad h \in H^{1/2}(\Gamma; \mathbb{C}^3), \quad \omega \in H^1_0(Y_{\text{stiff}}; \mathbb{C}^3), \quad \omega|_{\Gamma} = 0.
\]
The identity (58) when \( v = \Pi_{0,\text{macro}}^\text{stiff} h, \ h \in H^{1/2}(\Gamma; \mathbb{C}^3) \) is stated in (57), while for \( v = \omega \in H^1_0(Y_{\text{stiff}}; \mathbb{C}^3), \ \omega|_{\Gamma} = 0 \) is covered by (46) and (48).

For arbitrary \( e_1, e_2 \in \mathbb{C}^3 \), define \( e_{\text{corr}}^j, \ j = 1, 2 \), as in (57) and introduce the notation \( u_{\text{corr}}^j := \Pi_{0,\text{macro}}^\text{stiff} e_{\text{corr}}^j + \Gamma_{k,l} e^j \), \( j = 1, 2 \). Invoking (58), we obtain
\[
\int_{Y_{\text{stiff}}} A_{\text{stiff}} \left[ \text{sym} \nabla u_{\text{corr}}^j + i X^j \Pi_{0,\text{macro}}^\text{stiff} e^j \right] : \text{sym} \nabla v = 0 \quad \forall v \in H^1_0(Y_{\text{stiff}}; \mathbb{C}^3), \quad j = 1, 2,
\]
as well as
\[
\left< -\Lambda^\text{hom}_x e^1, e^2 \right>_{\mathbb{C}^3} := \int_{Y_{\text{stiff}}} A_{\text{stiff}} \left[ \text{sym} \nabla u_{\text{corr}}^1 + i X^1 \Pi_{0,\text{macro}}^\text{stiff} e^1 \right] : \left[ \text{sym} \nabla u_{\text{corr}}^2 + i X^2 \Pi_{0,\text{macro}}^\text{stiff} e^2 \right] = \frac{1}{|\Gamma|} \int_{Y_{\text{stiff}}} A_{\text{stiff}} \left[ \text{sym} \nabla u_{\text{corr}}^1 + i X^1 e^1 \right] : \left[ \text{sym} \nabla u_{\text{corr}}^2 + i X^2 e^2 \right] = \frac{1}{|\Gamma|} \Lambda_{\text{macro}} X^1 e^1 : i X^2 e^2,
\]
where we have employed the definition of the macroscopic tensor (2). Using Lemma 2.3 completes the proof. \( \square \)

Next, we state a theorem on the norm-resolvent estimates of DtN maps.

**Theorem 4.9.** There exists a constant \( C > 0 \), which does not depend on \( \chi \), such that the following norm-resolvent estimate holds:
\[
\left\| \left( |\chi|^{-2} \Lambda_{x,\text{stiff}} - I \right)^{-1} - \left( |\chi|^{-2} \Lambda_{x,\text{hom}} - I \right)^{-1} S \right\|_{\mathcal{L}(L^2(\Gamma; \mathbb{C}^3) \to H^{1/2}(\Gamma; \mathbb{C}^3))} \leq C|\chi|.
\]

Here \( S : h \mapsto |\Gamma|^{-1} \int_{\Gamma} h \) is the averaging operator on \( \Gamma \), and \( \Lambda_{x,\text{hom}} \) is the effective operator defined by (56).

**Proof.** The proof is by construction, following a version of the standard asymptotic expansion approach. We start with the weak formulation of the resolvent problem for the operator \( |\chi|^{-2} \Lambda_{x,\text{stiff}} \). For \( h \in L^2(\Gamma; \mathbb{C}^3) \), our goal is to expand the solution \( g \) of the problem
\[
\int_{Y_{\text{stiff}}} A_{\text{stiff}} \left( \text{sym} \nabla + i X^1 \right) \Pi_{0,\text{macro}}^\text{stiff} g : \left( \text{sym} \nabla + i X^2 \right) \Pi_{0,\text{macro}}^\text{stiff} v = \int_{\Gamma} g \cdot v = \int_{\Gamma} h \cdot \bar{v} \quad \forall v \in H^{1/2}(\Gamma; \mathbb{C}^3)
\]
in the form \( g = g_0 + g_1 + g_2 + g_{\text{corr}} \), where the terms satisfy the bounds
\[
g_0 = O(1), \quad g_1 = O(|\chi|), \quad g_2 = O(|\chi|^2), \quad g_{\text{corr}} = O(|\chi|^3),
\]
with respect to the \( H^{1/2}(\Gamma; \mathbb{C}^3) \) norm, and are calculated from a sequence of boundary value problems obtained from (59), the first few of which are introduced below, see (61). By equating the terms in (59) which are of order \( O(1) \), we obtain the following identity for the leading-order term \( g_0 \):
\[
\int_{Y_{\text{stiff}}} A_{\text{stiff}} \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} g_0 : \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} v = 0 \quad \forall v \in H^{1/2}(\Gamma; \mathbb{C}^3),
\]
hence \( g_0 \in \mathbb{C}^3 \). Furthermore, by combining the terms of order \( O(|\chi|) \), we obtain the identity
\[
\int_{Y_{\text{stiff}}} A_{\text{stiff}} \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} g_1 : \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} v = - \int_{Y_{\text{stiff}}} A_{\text{stiff}} \Psi^1 g_0 : \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} v \quad \forall v \in H^{1/2}(\Gamma; \mathbb{C}^3),
\]
which has a unique solution satisfying \( \int_{\Gamma} g_1 = 0 \). Next, comparing the terms of order \( O(|\chi|^2) \), we define \( g_2 \) as the solution to the identity
\[
\int_{Y_{\text{stiff}}} A_{\text{stiff}} \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} g_2 : \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} v =
\]
\[
- \int_{Y_{\text{stiff}}} A_{\text{stiff}} \left( \Psi^1 g_1 + i X^1 \Gamma_{k,l} g_0 + \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} g_0 \right) : \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} v
\]
\[
- \int_{Y_{\text{stiff}}} A_{\text{stiff}} \left( \text{sym} \nabla \Pi_{0,\text{macro}}^\text{stiff} g_1 + \Psi^1 g_0 \right) : \text{sym} \nabla v - |\chi|^2 \int_{\Gamma} g_0 \cdot \bar{v} - |\chi|^2 \int_{\Gamma} h \cdot \bar{v} \quad \forall v \in H^{1/2}(\Gamma; \mathbb{C}^3).
\]
The existence and uniqueness of solution to (61) can be established under two additional constraints. We require \( g_2 \) to have zero mean, \( \int g_2 = 0 \), and the right-hand side of (61) to vanish when tested against constants.\(^2\) The second part of this requirement is satisfied by choosing an appropriate \( g_0 \in \mathbb{C}^3 \). Namely, setting \( v = v_0 \in \mathbb{C}^3 \) in (61) yields
\[
0 = -\int_{\mathcal{Y}_{\text{int}}} A_{\text{eff}} (\text{sym} \nabla \Pi_0^{\text{eff}} g_1 + \Psi \lambda g_0) : \Psi \lambda v_0 - |\chi|^2 \int_{\mathcal{Y}} g_0 \cdot \overline{v_0} + |\chi|^2 \int_{\mathcal{Y}} h \cdot \overline{v_0} \quad \forall v_0 \in \mathbb{C}^3.
\]
By virtue of (56) and (60), it follows that
\[
-\frac{1}{|\chi|^2} \langle \Lambda_0 \text{hom} g_0, v_0 \rangle_{\mathbb{C}^3} + \frac{1}{|\chi|^2} \int_{\mathcal{Y}} g_0 \cdot \overline{v_0} = \frac{1}{|\chi|^2} \int_{\mathcal{Y}} h \cdot \overline{v_0} \quad \forall v_0 \in \mathbb{C}^3,
\]
or, equivalently,
\[
-\frac{1}{|\chi|^2} \Lambda_0 \text{hom} g_0 + g_0 = \frac{1}{|\chi|^2} \int_{\mathcal{Y}} h.
\]
(62)
Thus, by defining the leading-order term as the solution to the above resolvent problem, we have ensured the solvability of (61).

Next, we prove the estimates for the correctors and the final error estimate. It is clear from (62) and the coercivity of \( \Lambda_0 \text{hom} \) that \( \|g_0\|_{H^2(Y; \mathbb{C}^3)} \leq C \|h\|_{L^2(Y; \mathbb{C}^3)} \). Furthermore, it follows from (60) that
\[
\|\text{sym} \nabla \Pi_0^{\text{eff}} g_1\|_{L^2(Y; \mathbb{C}^3)} \leq \|iX_\lambda \nabla \Pi_0^{\text{eff}} g_0\|_{L^2(Y; \mathbb{C}^3)} + \|\text{sym} \nabla X_\lambda g_0\|_{L^2(Y; \mathbb{C}^3)}.
\]
By virtue of Theorem 4.3 and Proposition A.12, we obtain
\[
\|g_1\|_{H^2(Y; \mathbb{C}^3)} \leq C |\chi| \|h\|_{L^2(Y; \mathbb{C}^3)}.
\]
(63)
In a similar manner, it can be seen from (61) that
\[
\|g_2\|_{H^2(Y; \mathbb{C}^3)} \leq C |\chi| \|h\|_{L^2(Y; \mathbb{C}^3)}.
\]
(64)
Next, we formulate the equation for the error term \( g_{\text{err}} := g - g_0 - g_1 - g_2 \), where \( g \) is the solution to the full problem (59). Using the above identities for \( g_0, g_1, g_2 \), it is straightforward to infer that
\[
\frac{1}{|\chi|^2} \int_{\mathcal{Y}_{\text{int}}} A_{\text{eff}} (\text{sym} \nabla + iX_\lambda) \Pi_0^{\text{eff}} g_{\text{err}} : (\text{sym} \nabla + iX_\lambda) \Pi_0^{\text{eff}} v = \frac{1}{|\chi|^2} \mathcal{R}_{\text{err}}(v) \quad \forall v \in H^{1/2}(\Gamma; \mathbb{C}^3),
\]
(65)
where the functional \( \mathcal{R}_{\text{err}} \) satisfies the bound \( |\mathcal{R}_{\text{err}}(v)| \leq C |\chi|^2 \|v\|_{H^{1/2}(\Gamma; \mathbb{C}^3)} \|h\|_{L^2(Y; \mathbb{C}^3)} \). Thus, by testing (65) with \( g_{\text{err}} \) and invoking (49), we obtain
\[
\|g_{\text{err}}\|_{H^{1/2}(\Gamma; \mathbb{C}^3)} \leq C |\chi|^{-2} \|\text{sym} \nabla + iX_\lambda\|_{H^{1/2}(Y; \mathbb{C}^3)} \|g_{\text{err}}\|_{L^2(Y; \mathbb{C}^3)} \leq C |\chi| \|g_{\text{err}}\|_{H^2(Y; \mathbb{C}^3)} \|h\|_{L^2(Y; \mathbb{C}^3)}.
\]
Therefore, one has \( \|g - g_0 - g_1 - g_2\|_{H^{1/2}(\Gamma; \mathbb{C}^3)} \leq C |\chi| \|h\|_{L^2(Y; \mathbb{C}^3)} \). It now follows from (63) and (64) that \( \|g - g_0\|_{H^2(Y; \mathbb{C}^3)} \leq C |\chi| \|h\|_{L^2(Y; \mathbb{C}^3)} \), as required.

\[\square\]

Remark 4.10. The averaging operator \( S : E \to E \) coincides with the projection operator \( \hat{P}_0 = \hat{P}_I |_{\chi=0} \).

Remark 4.11. The norm-resolvent estimate provided in Theorem 4.9 also yields
\[
\left\| (|\chi|-2\Lambda_0 \text{stiff} - zI)^{-1} - (|\chi|-2\Lambda_0 \text{hom} - zI)^{-1} S \right\|_{L^2(Y; \mathbb{C}^3) \to H^{1/2}(\Gamma; \mathbb{C}^3)} \leq C(z) |\chi|,
\]
where the constant \( C(z) \) depends on the distance of \( z \) to the spectrum of \( |\chi|-2\Lambda_0 \text{stiff(hom)} \). (It is bounded uniformly in \( z \) if \( z \) belongs to a set for which both \( |z| \) and \( \{ \text{dist}(z, \sigma(|\chi|-2\Lambda_0 \text{stiff(hom)})) \}^{-1} \) are bounded.) This can be seen by using resolvent identities or by revisiting the proof of Theorem 4.9.

Next, we discuss the resolvent asymptotics with respect to \( \varepsilon \) of the DN maps on the stiff component.

Corollary 4.12. There exists a constant \( C > 0 \), independent of \( \chi \in Y' \), \( \varepsilon > 0 \), such that the following norm-resolvent estimate holds:
\[
\left\| (|\varepsilon|^2\Lambda_0 \text{stiff} - I)^{-1} - (|\varepsilon|^2\Lambda_0 \text{hom} - I)^{-1} S \right\|_{L^2(Y; \mathbb{C}^3) \to H^{1/2}(\Gamma; \mathbb{C}^3)} \leq C \varepsilon,
\]
where the homogenised operator \( \Lambda_0 \text{hom} \) is defined by the formula (56).

\[\text{This is in fact the Fredholm alternative.}\]
Before proceeding to the proof, we provide some auxiliary results. There are several important points to make that allow us to rewrite norm-resolvent estimates in terms of the parameter $\varepsilon$. Both operators $|\chi|^{-2}A^\text{stiff}_\chi$ and $|\chi|^{-2}A^\text{hom}_\chi$ (where the latter is in fact a multiplication by a constant matrix depending on $\chi$) have exactly $3$ eigenvalues of order $O(1)$, and the set of these eigenvalues can be enclosed with a fixed contour $\gamma \subset \mathbb{C}$ uniformly in $|\chi|$ (for small enough $|\chi|$). These properties are summarised in the following lemma.

**Lemma 4.13.** There exist $\xi, \eta > 0$ and a contour $\gamma \subset \{z \in \mathbb{C} : \Re(z) < -\eta\}$ such that, for all $0 < |\varepsilon| \leq \xi$:

- All eigenvalues of the operators $|\chi|^{-2}A^\text{stiff}_\chi$ and $|\chi|^{-2}A^\text{hom}_\chi$ that admit an $O(1)$ estimate in $|\varepsilon|$ are enclosed by $\gamma$;
- No other eigenvalue of $|\chi|^{-2}A^\text{stiff}_\chi$ and $|\chi|^{-2}A^\text{hom}_\chi$ is enclosed by $\gamma$;
- There exists $\rho_0 > 0$ such that for all eigenvalues $\varepsilon$ of $|\chi|^{-2}A^\text{stiff}_\chi$ and $|\chi|^{-2}A^\text{hom}_\chi$ one has $\inf_{\mathbb{R} \gamma} |z - \varepsilon| \geq \rho_0$.

**Proof.** Note that for small enough $|\varepsilon|$ the spectrum of order $O(1)$ is uniformly separated from the remaining spectrum – this is guaranteed by the estimates in Lemma 4.7. The same estimates also show that this part of the spectrum lies in a fixed $\chi$-independent interval that does not contain zero. □

For every fixed $\varepsilon > 0$, $\chi \neq 0$ we consider the function $g_{\varepsilon,\chi}(z) := \left(\varepsilon^{-2}|\chi|^2z - 1\right)^{-1}$, $\Re(z) < 0$, which satisfies the following lemma.

**Lemma 4.14.** For every fixed $\eta > 0$, the function $g_{\varepsilon,\chi}$ is bounded in the half-plane $\{z \in \mathbb{C}, \Re(z) < -\eta\}$:

$$|g_{\varepsilon,\chi}(z)| \leq C(\eta) \left(\max \left\{\varepsilon^{-2}|\chi|^2, 1\right\}\right)^{-1}.$$  

**Proof.** The required bound follows from the estimate

$$|g_{\varepsilon,\chi}(z)|^{-1} = |\varepsilon^{-2}|\chi|^2z - 1| \geq |\varepsilon^{-2}|\chi|^2\eta + 1 \geq C(\eta) \max \left\{\varepsilon^{-2}|\chi|^2, 1\right\}.$$ □

**Proof of Corollary 4.12.** Applying the integral Cauchy formula, we obtain

$$\hat{P}_\chi \left(\frac{1}{\varepsilon^2A^\text{stiff}_\chi} - I\right)^{-1} \hat{P}_\chi = \frac{1}{2\pi i} \int_{\gamma} g_{\varepsilon,\chi}(z) \left(1 - \frac{1}{|\chi|^2 A^\text{stiff}_\chi}\right)^{-1} dz,$$

$$\left(\frac{1}{\varepsilon^2A^\text{hom}_\chi} - I\right)^{-1} S = \frac{1}{2\pi i} \int_{\gamma} g_{\varepsilon,\chi}(z) \left(1 - \frac{1}{|\chi|^2 A^\text{hom}_\chi}\right)^{-1} S dz,$$

where $\hat{P}_\chi$ is the operator of projection onto the 3-dimensional span of the eigenfunctions of $A^\text{stiff}_\chi$ associated with eigenvalues of order $|\chi|^2$. Furthermore, since $\hat{P}_\chi$ is a spectral projection for $A^\text{stiff}_\chi$, using the estimates from Lemma 4.7 yields

$$\left\|\left(\varepsilon^{-2}A^\text{stiff}_\chi - I\right)^{-1} \hat{P}_\chi \left(\varepsilon^{-2}A^\text{stiff}_\chi - I\right)^{-1} \hat{P}_\chi\right\|_{L^2(\Gamma,\mathbb{C}^3) \rightarrow H^{1/2}(\Gamma,\mathbb{C}^3)} \leq C \varepsilon^2, \tag{66}$$

On the other hand,

$$\left\|\hat{P}_\chi \left(\varepsilon^{-2}A^\text{stiff}_\chi - I\right)^{-1} \hat{P}_\chi - \left(\varepsilon^{-2}A^\text{stiff}_\chi - I\right)^{-1} S\right\|_{L^2(\Gamma,\mathbb{C}^3) \rightarrow H^{1/2}(\Gamma,\mathbb{C}^3)} \leq \frac{1}{2\pi i} \int_{\gamma} \left\|g_{\varepsilon,\chi}(z)\right\| \left\|\left(1 - \frac{1}{|\chi|^2 A^\text{stiff}_\chi}\right)^{-1}\left(1 - \frac{1}{|\chi|^2 A^\text{hom}_\chi}\right)^{-1} S\right\| dz \tag{67}$$

$$\leq C \varepsilon \left(\max \left\{\varepsilon^{-2}|\chi|^2, 1\right\}\right)^{-1} \leq C \varepsilon,$$

where we have used Remark 4.11. The proof is concluded by combining (66) and (67). □

Theorem 4.9 has another direct consequence, namely the asymptotics of the spectral projections $\hat{P}_\chi$ and the truncated lift operators $\Pi^{\text{stiff}}_\chi \hat{P}_\chi$ with respect to the quasimomentum $\chi$.  

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where the operator $A$ functions. Then the corresponding sequence of eigenfunctions of $O$

Moreover, the eigenvalues of the operator $O$

For the boundary operators on the soft component, one has

$4.4$ Soft component asymptotics

where the assertion of Theorem $4.3$ this yields

the same bound with respect to the same norm, by virtue of (71) and the fact that

Now note that $P$ estimate in the

is the projection onto the eigenspace corresponding to

as a consequence of Remark $4.11$. This proves the estimate

as (68)

The operators $P$ and $\Pi^{\text{ stiff}}$ satisfy the asymptotics

$$\|P - P\|_{L^2(\Gamma; C^3)} \leq C|\gamma|,$$

$$\|\Pi^{\text{ stiff}} - \Pi^{\text{ stiff}}\|_{L^2(\Gamma; C^3)} \leq C|\gamma|,$$

$$\|P (\Pi^{\text{ stiff}})^* - P_0 (\Pi^{\text{ stiff}})^*\|_{L^2(Y_{\text{ soft}}; C^3)} \leq C|\gamma|.$$

where the constant $C > 0$ does not depend on $\chi \in Y'$.

Proof. By choosing a contour $\gamma$ as above, and applying the Cauchy formula to the constant function $g(z) = 1$, we obtain the asymptotics of projections $P$ defined by (53):

$$P = \frac{1}{2\pi i} \int_{\gamma} \left( zI - \frac{1}{|x|^2} \Lambda^{\text{ stiff}} \right)^{-1} dz = \frac{1}{2\pi i} \int_{\gamma} \left( zI - \frac{1}{|x|^2} \Lambda^{\text{ hom}} \right)^{-1} dz + 1, \int_{\gamma} \tilde{R}_k^{\text{ corr}}(z) dz = P_0 + \int_{\gamma} \tilde{R}_k^{\text{ corr}}(z) dz,$$

as a consequence of Remark $4.11$. This proves the estimate

$$\|P - P_0\|_{L^2(\Gamma; C^3)} \leq C|\gamma|.$$ (71)

To upgrade it to an $H^1$ estimate, we write

$$\|P - P_0\|_{L^2(\Gamma; C^3)} = \|P - P_0\|_{L^2(\Gamma; C^3)} \leq C|\gamma|.$$ (72)

Now note that $P : L^2(\Gamma; C^3) \rightarrow H^1(\Gamma; C^3)$ is bounded, owing to the facts that $\mathcal{D}(\Lambda^{\text{ stiff}}) = H^1(\Gamma; C^3)$ and that $P$ is the projection onto the eigenspace corresponding to $O(|\gamma|^2)$ eigenvalues. The first term in (72) admits an $O(|\gamma|)$ estimate in the $L^2(\Gamma; C^3) \rightarrow H^1(\Gamma; C^3)$ norm, due to (71) and the boundedness of $P$. The second term in (72) admits the same bound with respect to the same norm, by virtue of (71) and the fact that $P_0$ is finite-rank. Together with the assertion of Theorem $4.3$ this yields

$$\Pi^{\text{ stiff}} P = (\Pi^{\text{ stiff}} + \Pi^{\text{ error}}) (P_0 + \tilde{R}_k) = \Pi^{\text{ stiff}} P_0 + O(|\gamma|),$$

where $O(|\gamma|)$ is understood in the sense of the $L^2(\Gamma; C^3) \rightarrow H^1(\Gamma; C^3)$ norm. Similarly,

$$\Pi^{\text{ stiff}} P = (\Pi^{\text{ stiff}})^* = (P_0 (\Pi^{\text{ stiff}})^*)^* + O(|\gamma|),$$

where $O(|\gamma|)$ is understood in the sense of the $L^2(\Gamma; C^3) \rightarrow L^2(\gamma; C^3)$ norm.

$4.4$ Soft component asymptotics

Next, we state some simpler asymptotic properties of the boundary operators for the soft component. These results will be used in Section $6$ for proving Theorem $2.5$ (a).

Lemma $4.16$. For the boundary operators on the soft component, one has

$$\Pi^{\text{ soft}} = e^{-i\gamma \chi} \Pi^{\text{ soft}} e^{i\gamma \chi}, \quad \Lambda^{\text{ soft}} = e^{-i\gamma \chi} \Lambda^{\text{ soft}} e^{i\gamma \chi}, \quad \Gamma^{\text{ soft}}_{\chi, \chi'} = e^{-i\gamma \chi} \Gamma_{\chi, \chi'} e^{i\gamma \chi}. $$

Moreover, the eigenvalues of the operator $\mathcal{A}^{\text{ soft}}_{\chi, \chi'}$ are independent of $\gamma$, and

$$e^{-i\gamma \chi} (\mathcal{A}^{\text{ soft}}_{\chi, \chi'} - zI)^{-1} e^{-i\gamma \chi} = (\mathcal{A}^{\text{ soft}}_{\chi, \chi'} - zI)^{-1},$$

where the operator $\mathcal{A}^{\text{ soft}}_{\chi, \chi'}$ is defined by (30) with $\chi = 0$ (and so coincides with $\mathcal{A}^{\text{ Bloch}}$ introduced in Section $2.2$.) Furthermore, let $\{\eta_i\}$ be a sequence of eigenvalues of the operator $\mathcal{A}^{\text{ soft}}_{0, \chi}$ and $\{\varphi_i\}$ the associated sequence of eigenfunctions. Then the corresponding sequence of eigenfunctions of $\mathcal{A}^{\text{ soft}}_{0, \chi}$ is given by $\{\varphi_i \chi\} = \{e^{-i\gamma \chi} \varphi_i\}$. 

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Proof. By the definition of $\Pi^\text{soft}_0$, one has
\[
\Pi^\text{soft}_0 g = u \iff \begin{cases} 
- \text{div} \left( A_{\text{soft}} \text{sym} \nabla u \right) = 0 & \text{on } Y_{\text{soft}}, \\
u = g & \text{on } \Gamma.
\end{cases}
\]

Writing $u = e^{i\chi} w \in H^1(Y_{\text{soft}}; C^1)$, we obtain
\[
\Pi^\text{soft}_0 g = u \iff \begin{cases} 
- \text{div} \left( A_{\text{soft}} \text{sym} \nabla \left( e^{i\chi} w \right) \right) = 0 & \text{on } Y_{\text{soft}}, \\
e^{i\chi} w = g & \text{on } \Gamma
\end{cases}
\iff \begin{cases} 
- \text{div} \left( e^{i\chi} A_{\text{soft}} \left( \text{sym} \nabla + iX_\chi \right) w \right) = 0 & \text{on } Y_{\text{soft}}, \\
w = e^{-i\chi} g & \text{on } \Gamma
\end{cases}
\iff w = \Pi^\text{soft}_\chi e^{-i\chi} g \iff u = e^{i\chi} \Pi^\text{soft}_\chi e^{-i\chi} g.
\]

Similarly, one has
\[
\Lambda^\text{soft}_0 g = h \iff \begin{cases} 
- \text{div} \left( A_{\text{soft}} \text{sym} \nabla u \right) = 0 & \text{on } Y_{\text{soft}}, \\
u = g & \text{on } \Gamma.
\end{cases}
\iff \begin{cases} 
- \text{div} \left( e^{i\chi} A_{\text{soft}} \left( \text{sym} \nabla + iX_\chi \right) w \right) = 0 & \text{on } Y_{\text{soft}}, \\
w = e^{-i\chi} g & \text{on } \Gamma
\end{cases}
\iff e^{-i\chi} h = \Lambda^\text{soft}_\chi e^{-i\chi} g \iff h = e^{i\chi} \Lambda^\text{soft}_\chi e^{-i\chi} g.
\]

\[\blacktriangleleft\]

Corollary 4.17. For the $M$-function $M^\text{soft}_\chi(z)$ of the soft component, one has
\[
M^\text{soft}_\chi(z) = e^{-i\chi} M^\text{soft}_0(z) e^{i\chi}, \quad M^\text{soft}_0(z) = \Lambda^\text{soft}_0 + z \left( \Pi^\text{soft}_0 \right)^* \Pi^\text{soft}_0 + z^2 \left( \Pi^\text{soft}_0 \right)^* \left( \mathcal{A}^\text{soft}_{0,0} - zI \right)^{-1} \Pi^\text{soft}_0.
\]

Furthermore, there exists a $\chi$-independent constant $C = C(z) > 0$ such that
\[
\| \hat{P}_\chi M^\text{soft}_\chi(z) \hat{P}_\chi - \hat{P}_0 M^\text{soft}_0(z) \hat{P}_0 \|_{L^2(\Gamma; C^1)} \leq C |\chi|.
\]

Proof. Recalling the identity (24), we have the representation formula
\[
M^\text{soft}_\chi(z) = \Lambda^\text{soft}_\chi + z \left( \Pi^\text{soft}_0 \right)^* \Pi^\text{soft}_0 + z^2 \left( \Pi^\text{soft}_0 \right)^* \left( \mathcal{A}^\text{soft}_{0,0} - zI \right)^{-1} \Pi^\text{soft}_0.
\]

Employing the identities (73), (74), Remark 3.22, and the estimate (68) yields the claim. \[\blacktriangleleft\]

Remark 4.18. Notice that, due to the fact that $\Lambda^\text{soft}_0|_{\bar{E}_0} = 0$, one also has
\[
M^\text{soft}_0(z)|_{\bar{E}_0} = z \left( \Pi^\text{soft}_0 \right)^* \Pi^\text{soft}_0|_{\bar{E}_0} + z^2 \left( \Pi^\text{soft}_0 \right)^* \left( \mathcal{A}^\text{soft}_{0,0} - zI \right)^{-1} \Pi^\text{soft}_0|_{\bar{E}_0} \tag{75}
\]

Combining the above lemma with (68) yields the claim. \[\blacktriangleleft\]

Corollary 4.19. The operator $\Pi^\text{soft}_\chi \hat{P}_\chi$ satisfies the estimates
\[
\| \Pi^\text{soft}_\chi \hat{P}_\chi - \Pi^\text{soft}_0 \hat{P}_0 \|_{L^2(\Gamma; C^1)} \leq C |\chi|, \tag{76}
\]
\[
\| \hat{P}_\chi \left( \Pi^\text{soft}_0 \right)^* - \hat{P}_0 \left( \Pi^\text{soft}_0 \right)^* \|_{L^2(Y_{\text{soft}}; C^1)} \leq C |\chi|.
\]

where the constant $C > 0$ does not depend on $\chi \in Y'$.
5 Transmission problem: $O(\varepsilon^2)$ resolvent asymptotics

In this section we aim at proving Theorem 2.4. The starting point is the Krein formula (45). The approximation is carried out in two steps. The first step (see Section 5.1, Theorem 5.2) is to use the Schur-Frobenius inversion formula by restricting traces to the space $E_{\delta}$ and by imposing the equality of projections onto the same space of the traces of co-normal derivatives. The second step (Section 5.2, Theorem 5.6) is to approximate the $M$-function on the stiff component. Recalling (19), we write

$$S^\text{soft}(\varepsilon^2) = \left( I + \varepsilon^2 z \left( \mathcal{A}^\text{soft}_{0\varepsilon} - \varepsilon^2 z I \right)^{-1} \right) \Pi^\text{soft} = \Pi^\text{soft} + O(\varepsilon^2),$$

(77)

where $O(\varepsilon^2)$ is understood in the sense of the $\mathcal{E} \mapsto \mathcal{H}^\text{soft}$ norm. The formula (24) yields

$$M^\text{soft}(\varepsilon^2) = \lambda^\text{soft} + \varepsilon^2 z \left( \Pi^\text{soft} \right)^* \left( I - z \left( \mathcal{A}^\text{soft}_{0\varepsilon} \right)^{-1} \right)^{-1} \Pi^\text{soft}$$

$$= \lambda^\text{soft} + \varepsilon^2 z \left( \Pi^\text{soft} \right)^* \Pi^\text{soft} + \varepsilon^2 z \left( \Pi^\text{soft} \right)^* \left( \mathcal{A}^\text{soft}_{0\varepsilon} - \varepsilon^2 z I \right)^{-1} \Pi^\text{soft}.$$  

(78)

5.1 Steklov truncation

Similarly to (54), we define the following truncated DtN maps:

$$\tilde{\lambda}^\text{soft}_{\delta} := \tilde{\mathcal{P}} \lambda^\text{soft}_{\delta} \chi^\text{soft}_{\delta}, \quad \tilde{\lambda}^\text{soft} := \tilde{\mathcal{P}} \lambda^\text{soft} \chi^\text{soft}, \quad \tilde{\lambda}_{\delta,\varepsilon} := \tilde{\mathcal{P}} \lambda_{\delta,\varepsilon} \chi^\text{soft}_{\delta},$$

so one obviously has

$$\tilde{\lambda}_{\delta,\varepsilon} = \varepsilon^{-2} \tilde{\lambda}^\text{soft} + \tilde{\lambda}^\text{soft}, \quad \tilde{\lambda}_{\delta,\varepsilon} = \varepsilon^{-2} \tilde{\lambda}^\text{soft} + \tilde{\lambda}^\text{soft}.$$  

(79)

The first operator sum in (79) is self-adjoint due to the fact that its terms are finite-rank self-adjoint (Hermitian) operators. The second sum in (79) also defines a self-adjoint operator (noting that $\lambda^\text{soft}$ is also self-adjoint, which can be checked by considering the associated sesquilinear form), by an argument similar to that of Theorem 3.24, where the operator domains are given by

$$\mathcal{D}(\tilde{\lambda}_{\delta,\varepsilon}) = \mathcal{D}(\tilde{\lambda}^\text{soft}) = \mathcal{D}(\tilde{\lambda}^\text{soft}) = \tilde{\mathcal{P}} \mathcal{D}(\lambda_{\delta,\varepsilon}).$$

Additionally, we denote the truncated lift operators by

$$\tilde{\Pi}^\text{soft}(\delta) := \tilde{\mathcal{P}} \Pi^\text{soft}(\delta) \chi^\text{soft}(\delta), \quad \tilde{\Pi}^\text{soft}(\delta) := \tilde{\mathcal{P}} \Pi^\text{soft}(\delta) \chi^\text{soft}(\delta), \quad \tilde{\Pi}_{\delta} := \tilde{\mathcal{P}} \Pi^\text{soft}(\delta) \chi^\text{soft}(\delta), \quad \tilde{\Pi}_{\delta} := \tilde{\mathcal{P}} \Pi^\text{soft}(\delta) \chi^\text{soft}(\delta).$$

Thus, we have defined the following “Steklov-truncated” Ryzhov triples

$$\left( \mathcal{A}^\text{soft}_{0\varepsilon}(\delta), \tilde{\Pi}^\text{soft}(\delta), \tilde{\lambda}^\text{soft}(\delta) \right) \text{ on } \left( \mathcal{H}^\text{soft}(\delta), \mathcal{E}_{\delta} \right), \quad \left( \mathcal{A}^\text{soft}_{0\varepsilon}(\delta), \tilde{\Pi}^\text{soft}(\delta), \tilde{\lambda}^\text{soft}(\delta) \right) \text{ on } \left( \mathcal{H}^\text{soft}(\delta), \mathcal{E}_{\delta} \right),$$

as well as the coupled Steklov-truncated triples $\left( \mathcal{A}_{0\varepsilon,\delta}, \tilde{\Pi}_{\delta}, \mathcal{E}_{\delta} \right), \left( \mathcal{A}_{0\varepsilon,\delta}, \tilde{\Pi}_{\delta}, \mathcal{E}_{\delta} \right)$ on $\left( \mathcal{H}, \mathcal{E}_{\delta} \right), \left( \mathcal{H}, \mathcal{E}_{\delta} \right)$, respectively. The triple properties as stated in the Definition 3.1 are easily checked. In particular, one has

$$\mathcal{D}(\mathcal{A}_{0\varepsilon,\delta}) \cap \mathcal{R}(\tilde{\Pi}_{\delta}) = \mathcal{D}(\mathcal{A}_{0\varepsilon,\delta}) \cap \Pi_{\delta}(\tilde{\mathcal{P}}(\mathcal{E}_{\delta})) \subset \mathcal{D}(\mathcal{A}_{0\varepsilon,\delta}) \cap \Pi_{\delta}(\mathcal{E}) = \{0\},$$

$$\ker(\tilde{\Pi}_{\delta}) = \ker(\Pi_{\delta}) \subset \ker(\Pi_{\delta}) = \{0\}.$$

One can define the boundary triples and $M$-functions associated with the Ryzhov triples introduced above, denoted in the same fashion. Notice that, for example, the domain of the operator $\mathcal{A}^\text{soft}_{0\varepsilon}$ coincides with $\mathcal{D}(\mathcal{A}^\text{soft}_{0\varepsilon}) + \tilde{\mathcal{P}}(\mathcal{E}_{\delta})$, so the trace operator $\mathcal{G}_{0\varepsilon}$ takes values in $\mathcal{E}_{\delta}$. By recalling that representation formula (21) decomposes the $M$-function into the sum of a self-adjoint operator (DtN map) and a bounded operator, we know that its domain actually coincides with the domain of the associated “$\Lambda$-operator”, and thus one has

$$\tilde{M}_{\delta}(z) = \tilde{\lambda}^\text{soft}(z) + z \left( \tilde{\Pi}^\text{soft}(z) \left( I - z \left( \mathcal{A}^\text{soft}_{0\varepsilon} \right)^{-1} \right)^{-1} \right)^{-1} \tilde{\Pi}^\text{soft}(z)$$

$$= \tilde{\mathcal{P}} \left( \left( \tilde{\lambda}^\text{soft}(z) + z \left( \tilde{\Pi}^\text{soft}(z) \left( I - z \left( \mathcal{A}^\text{soft}_{0\varepsilon} \right)^{-1} \right)^{-1} \right)^{-1} \tilde{\Pi}^\text{soft}(z) \right) \chi^\text{soft}(\delta) \right).$$

and similar claims hold for $\tilde{M}_{\delta}(z), \tilde{M}_{\delta,\varepsilon}(z)$, and $\tilde{M}_{\delta,\varepsilon}(z)$.

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We introduce the notation $\tilde{\mathcal{H}}_{\chi}^{\text{stiff}(\text{soft})} := \Pi_k^{\text{stiff}(\text{soft})} \tilde{\mathcal{H}}_k$, together with the notation $P_{\tilde{\mathcal{H}}_{\chi}^{\text{stiff}(\text{soft})}}$ for the respective orthogonal projections (with respect to the $\mathcal{H}$ inner product). We also define $\Theta_{\chi} : \mathcal{H} \rightarrow \mathcal{H}$ as the orthogonal projection
\[
\Theta_{\chi}(\mathbf{u}_{\text{soft}} \oplus \mathbf{u}_{\text{stiff}}) = \mathbf{u}_{\text{soft}} \oplus P_{\tilde{\mathcal{H}}_{\chi}^{\text{stiff}}} \mathbf{u}_{\text{stiff}}.
\]
(80)

Before stating our approximation result, we need one helpful lemma, whose proof is found in the Appendix. It establishes the equivalence of the $H^1$ and $L^2$ norms on $\tilde{\mathcal{H}}_{\chi}^{\text{stiff}(\text{soft})}$ uniformly in the quasimomentum $\chi$. The proof of this lemma can be found in the Appendix.

**Lemma 5.1.** There exists a $\chi$-independent constant $C > 0$ such that
\[
\|f\|_{L^2(Y_{\text{stiff}}(\chi); \mathbb{C})} \leq C \|f\|_{H^1(Y_{\text{stiff}}(\chi); \mathbb{C})} \quad \forall f \in \tilde{\mathcal{H}}_{\chi}^{\text{stiff}(\text{soft})}.
\]

The next theorem provides the basis for Theorem 2.4.

**Theorem 5.2.** There exists $C > 0$ such that for the resolvent of the transmission problem (13) one has
\[
\left\| \left( (\mathcal{A}_{\chi,\varepsilon})_0 - zI \right)^{-1} - \left( (\mathcal{A}_{\chi,\varepsilon})_p - zI \right)^{-1} \right\|_{E \rightarrow E} \leq C \varepsilon^2 \quad \forall \chi \in \chi', \varepsilon \in K_{\varepsilon}.
\]

**Proof.** Notice that, as a consequence of (24) and the second equality in (44), we can write
\[
M_{\chi,\varepsilon}(z) = \Lambda_{\chi,\varepsilon} + B_{\chi,\varepsilon}(z) = \varepsilon^{-2} \Lambda_{\chi}^{\text{soft}} + \Lambda_{\chi}^{\text{stiff}} + B_{\chi,\varepsilon}(z),
\]
(81)
where the operator $B_{\chi,\varepsilon}$ is bounded uniformly in $\chi, \varepsilon$. So, the question of boundedness of a certain truncation of $M_{\chi,\varepsilon}$ actually comes down to the boundedness of associated truncation of DtN map. But, since $\tilde{\mathcal{E}}_{\chi} \subset \mathcal{D}((\Lambda_{\chi})_{\text{soft}}) = \mathcal{D}(\Lambda_{\chi}^{\text{soft}})$ and $\tilde{\mathcal{E}}_{\chi}$ is finite-dimensional, one has
\[
\left\| \tilde{P}_{\chi} \Lambda_{\chi,\varepsilon} \tilde{P}_{\chi} \right\|_{E \rightarrow E} \leq \left\| \Lambda_{\chi,\varepsilon} \right\|_{E \rightarrow E}, \quad \left\| \tilde{P}_{\chi} \Lambda_{\chi,\varepsilon} \tilde{P}_{\chi} \right\|_{E \rightarrow E} \leq \left\| \Lambda_{\chi,\varepsilon} \right\|_{E \rightarrow E},
\]
and thus the operators $\tilde{P}_{\chi} M_{\chi,\varepsilon}(z) \tilde{P}_{\chi}, \tilde{P}_{\chi} M_{\chi,\varepsilon}(z) \tilde{P}_{\chi}$, are bounded as well, uniformly in $\chi, \varepsilon$. We next show that the operator $\tilde{P}_{\chi} M_{\chi,\varepsilon}(z) \tilde{P}_{\chi}$ is boundedly invertible with a bound depending on $\varepsilon$. This is the point where we stress the importance of Steklov truncations and the bound (55).

To prove the mentioned boundedness, our first observation is that the operator $\tilde{\Lambda}_{\chi}^{\text{soft}} \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1}$ is boundedly independent of $\chi$ as a consequence of (39) and (55). Using the formula (81), we have
\[
\tilde{M}_{\chi,\varepsilon}(z) = \left( 1 + \varepsilon^2 \tilde{\Lambda}_{\chi}^{\text{soft}} \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1} + \varepsilon^2 \tilde{P}_{\chi} B_{\chi,\varepsilon}(z) \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1} \right) \varepsilon^{-2} \tilde{\Lambda}_{\chi}^{\text{stiff}}.
\]
(82)
Since the operators $\tilde{\Lambda}_{\chi}^{\text{stiff}} \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1}$ and $B_{\chi,\varepsilon} \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1}$ are bounded uniformly in $\chi$ and $\varepsilon$, we can choose $\varepsilon$ small enough so that (82) is invertible and
\[
\left\| \left( 1 + \varepsilon^2 \tilde{\Lambda}_{\chi}^{\text{soft}} \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1} + \varepsilon^2 \tilde{P}_{\chi} B_{\chi,\varepsilon}(z) \left( \tilde{\Lambda}_{\chi}^{\text{stiff}} \right)^{-1} \right)^{-1} \right\|_{E \rightarrow E} \leq \frac{1}{2}.
\]
(83)
Combining (82) and (83) yields $\|\tilde{M}_{\chi,\varepsilon}(z)\|_{E \rightarrow E} \leq C \varepsilon^2$. We write $M_{\chi,\varepsilon}$ as a block operator matrix relative to the decomposition (53):
\[
M_{\chi,\varepsilon} = \begin{bmatrix} A & B \\ E & F \end{bmatrix},
\]
where
\[
A := \tilde{M}_{\chi,\varepsilon}(z), \quad B := \tilde{P}_{\chi} M_{\chi,\varepsilon}(z) \tilde{P}_{\chi}, \quad E := \tilde{P}_{\chi} M_{\chi,\varepsilon}(z) \tilde{P}_{\chi}, \quad F := \tilde{M}_{\chi,\varepsilon}(z).
\]
We have shown that $A$, $B$, $E$ are bounded (where the bound of $A$ depends on $\varepsilon$), and $F$ is boundedly invertible: $\|F^{-1}\|_{E \rightarrow E} \leq C \varepsilon^2$, where $C$ does not depend on $\varepsilon$. Our next objective is to show that $A$ is boundedly invertible with a $\chi$-independent bound. To this end, notice that (78) implies
\[
\varepsilon^{-2} \tilde{M}_{\chi}^{\text{stiff}}(\varepsilon^2 z) = \varepsilon^{-2} \tilde{\Lambda}_{\chi}^{\text{stiff}} + z(\tilde{\Pi}_{\chi}^{\text{stiff}} \ast \tilde{\Pi}_{\chi}^{\text{stiff}} + O(\varepsilon^2),
\]
(84)
where $O(\varepsilon^2)$ is understood in the sense of the $L^2 \rightarrow L^2$ operator norm, uniformly in $\chi$. Using (23), (84), Lemma 5.1, the trace inequality, and the fact that $z \in K_\varepsilon$, we infer the existence of a $\chi$-independent constant $C > 0$ such that, for all $f \in L^2(\Gamma; \mathbb{C})$ and $\varepsilon$ small enough, one has
\[
\left| \left\langle 3 \tilde{M}_{\chi}(z) f, f \right\rangle_{E} \right| = \left| \left\langle \varepsilon^{-2} \tilde{M}_{\chi}^{\text{stiff}}(\varepsilon^2 z) f, f \right\rangle_{E} \right| \geq \left| \left\langle \varepsilon^{-2} \tilde{M}_{\chi}^{\text{stiff}}(\varepsilon^2 z) f, f \right\rangle_{E} \right| \geq C \|z\|_{L^2(\Gamma; \mathbb{C})} \|f\|_{L^2(\Gamma; \mathbb{C})} + O(\varepsilon^2) \|f\|_{L^2(\Gamma; \mathbb{C})} \geq C \|\tilde{\Pi}_{\chi}^{\text{stiff}} f\|_{L^2(Y_{\text{stiff}}(\chi); \mathbb{C})} + O(\varepsilon^2) \|f\|_{L^2(Y_{\text{stiff}}(\chi); \mathbb{C})} \geq C \|f\|_{L^2(Y_{\text{stiff}}(\chi); \mathbb{C})}.
\]
By virtue of Corollary A.2, it now follows that
\[ \|A^{-1}\|_{E \to E} \leq C, \]  
where \( C > 0 \) does not depend on \( \chi \). Using the Schur-Frobenius inversion formula, see [60], we have
\[ M_{x,e}^{-1}(z) = \left[ \frac{\beta}{\alpha} \frac{\beta}{\alpha} \right]^{-1} = \left[ \begin{array}{cc} \alpha^{-1} & 0 \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{cc} A^{-1}BB^{-1} & EA^{-1} \\ S^{-1}E & 0 \end{array} \right], \]
where \( S := F - EA^{-1}B \). Furthermore, since \( \|S^{-1}\|_{E \to E} \leq \|(I - F^{-1}EA^{-1}B)^{-1}F^{-1}\|_{E \to E} \leq C\varepsilon^2 \), we write
\[ M_{x,e}^{-1}(z) = \left[ \begin{array}{cc} (\tilde{P}_xM_{x,e}(z)\tilde{P}_x)^{-1} & 0 \\ 0 & 0 \end{array} \right] + O(\varepsilon^2) = \tilde{P}_x(M_{x,e}(z)\tilde{P}_x)^{-1}\tilde{P}_x + O(\varepsilon^2). \]

On the other hand, the Schur-Frobenius formula implies
\[ (\tilde{P}_x + \tilde{P}_xM_{x,e}(z))^{-1} = \left[ \begin{array}{cc} \tilde{P}_xM_{x,e}(z)\tilde{P}_x & 0 \\ 0 & I \end{array} \right]^{-1} = \left[ \begin{array}{cc} (\tilde{P}_xM_{x,e}(z)\tilde{P}_x)^{-1} & 0 \\ 0 & I \end{array} \right] \tilde{P}_x. \]
Now, clearly \( M_{x,e}^{-1}(z) = (\tilde{P}_x + \tilde{P}_xM_{x,e}(z))^{-1}\tilde{P}_x + O(\varepsilon^2) \). Thus, by putting \( \beta_0 = \tilde{P}_x, \beta_1 = \tilde{P}_x \) and recalling the Theorem 3.14 we recognise that the operator-valued function
\[ z \mapsto (\mathcal{A}_{0,x,e} - zI)^{-1} - S_{x,e}(z)(\tilde{P}_x + \tilde{P}_xM_{x,e}(z))^{-1}\tilde{P}_x S_{x,e}(z)^* \]
is the resolvent of the closed extension of \( \mathcal{A}_{0,x,e} \) associated, in the sense of Theorem 3.14, with the boundary condition \((\tilde{P}_x\Gamma_{0,x} + \tilde{P}_x\Gamma_{1,x})u = 0\), and the result follows. \( \square \)

**Remark 5.3.** The resolvent \( ((\mathcal{A}_{x,e})\tilde{P}_x,\tilde{P}_x - zI)^{-1} \) is the solution operator of the boundary value problem \((f \in \mathcal{H})\)
\[ \mathcal{A}_{x,e}u - zu = f, \quad (\tilde{P}_x\Gamma_{0,x} + \tilde{P}_x\Gamma_{1,x})u = 0. \]
The solution \( u \in \mathcal{H} \) satisfies the following constraints:
- The 3-dimensional projections of the traces on \( \Gamma \) of conormal derivatives of \( u \) (from inside \( \text{Y}_\text{soft} \), \( \text{Y}_\text{stiff} \)) onto the space \( \tilde{E}_\chi \) coincide.
- The traces of \( u \) from both \( \text{Y}_\text{soft} \) and \( \text{Y}_\text{stiff} \) belong to the 3-dimensional space \( \tilde{E}_\chi \) (they clearly coincide, as per Remark 2.25).

Thus, by approximating the resolvent \((((\mathcal{A}_{x,e})_{0,1} - zI)^{-1} \) associated with the transmission problem (13) by the resolvent \((((\mathcal{A}_{x,e})\tilde{P}_x,\tilde{P}_x - zI)^{-1} \), one relaxes the condition on the continuity of co-normal derivatives and tightens the constraint on the traces, which leads to an order of \( \varepsilon^2 \).

Intuitively, the homogenisation procedure should replace the solution on the stiff component with a 3-dimensional constant vector. However, at this point only the trace of the solution is finite-dimensional.

### 5.2 Approximation refinement: truncation of \( \hat{M}^\text{stiff}_\chi \)
We introduce the following notation for the truncated solution operator \( \hat{S}_{x,e}(z) := S_{x,e}(z)|_{\hat{E}_e} \). Using (20), we have
\[ \hat{S}_{x,e}(z) := S_{x,e}(z)|_{\hat{E}_e} = (\Pi_x + z(\mathcal{A}_{0,x,e} - zI)^{-1}\Pi_x)|_{\hat{E}_e} = \hat{\Pi}_x + z(\mathcal{A}_{0,x,e} - zI)^{-1}\hat{\Pi}_x, \]
which is precisely the solution operator associated with the triple \((\mathcal{A}_{0,x,e},\hat{\Pi}_x,\hat{\chi}_{x,e})\) in the sense of Definition 3.4. Similar representation formulae are obtained for the operators \( \hat{S}^\text{soft}(z) \), which are defined in an obvious way.

**Remark 5.4.** Notice that
\[ \hat{S}^\text{stiff}(z) = \hat{P}_x(S^\text{stiff}(z))^*, \quad (\hat{\Pi}^\text{stiff}(z))^* = \hat{P}_x(\Pi^\text{stiff}(z))^*. \]

Also, one has
\[ P^\text{stiff}(z) = \hat{P}_x^\text{stiff}(z), \quad (\hat{\Pi}^\text{stiff}(z))^* = \hat{P}_x^\text{stiff}(z)^*, \]
which follows directly from the definition of the operators involved.
The operator $\hat{\Gamma}_{0,\chi}$ is the left inverse of the operator $\hat{\Pi}_\chi$ in the sense of Definition 3.2, so by virtue of (86) it is the left inverse of $\hat{S}_{\chi,\epsilon}(z)$ as well. A similar claim applies to the operators $\hat{S}_{\chi,\epsilon}^{\text{soft}}(z)$. In particular, we have

$$\hat{\Gamma}_{0,\chi} \hat{S}_{\chi,\epsilon}(z) = \hat{\Gamma}_{0,\chi} \hat{S}_{\chi,\epsilon}^{\text{soft}}(z) = I |_{\hat{E}_\chi}.$$ 

In Theorem 5.2 we have obtained an approximation of the original resolvent in terms of the resolvent of another operator, where the relative simplification is not immediately evident. However, by doing simple additional approximations the result becomes much more transparent. We carry out these additional approximations by analyzing the block components of the resolvent (see (87)) separately.

$$((\mathcal{A}_{\chi,\epsilon}) \hat{P}_\chi \hat{P}_\chi - z I)^{-1} = (\mathcal{A}_{0,\chi,\epsilon} - z I)^{-1} - \hat{S}_{\chi,\epsilon}(z) \hat{P}_\chi (\hat{P}_\chi M_{\chi,\epsilon}(z) \hat{P}_\chi)^{-1} \hat{P}_\chi \hat{S}_{\chi,\epsilon}(z)^*$$

$$= (\mathcal{A}_{0,\chi,\epsilon} - z I)^{-1} - \hat{S}_{\chi,\epsilon}(z) \hat{M}_{\chi,\epsilon}(z)^{-1} \hat{S}_{\chi,\epsilon}(z)^*,$$

relative to the decomposition (28).

It follows from (84) and (85) that

$$\hat{M}_{\chi,\epsilon}(z)^{-1} = (\epsilon^{-2} \hat{X}^{\text{soft}}(z) + \hat{M}^{\text{soft}}(z))^{-1} = \left(\epsilon^{-2} \hat{X}^{\text{soft}} + z(\hat{\Pi}^{\text{soft}})^* \hat{\Pi}^{\text{soft}} + \hat{M}^{\text{soft}}(z)\right)^{-1} + O(\epsilon^2),$$

which we use in the proof of Theorem 5.6 below. We introduce the following operator-valued function featuring prominently in our homogenisation results.

**Definition 5.5.** We refer to the operator-valued function $\hat{Q}^{\text{app}}_{\chi,\epsilon}(z) : \hat{E}_\chi \to \hat{E}_\chi$ given by

$$\hat{Q}^{\text{app}}_{\chi,\epsilon}(z) := \epsilon^{-2} \hat{X}^{\text{soft}} + z(\hat{\Pi}^{\text{soft}})^* \hat{\Pi}^{\text{soft}} + \hat{M}^{\text{soft}}(z)$$

(90)

as the transmission function.

We next prove a result on resolvent asymptotics that simplifies the solution on the stiff component.

**Theorem 5.6.** There exists $C > 0$, which depends only on $\sigma$ and $\text{diam}(K_\sigma)$, such that for the resolvent of the transmission problem (13) one has

$$\|((\mathcal{A}_{\chi,\epsilon})_{|_{\mathcal{H}}} - z I)^{-1} - \mathcal{R}^{\text{app}}_{\chi,\epsilon}(z)\|_{\mathcal{H} \to \mathcal{H}} \leq C\epsilon^2$$

for all $\chi \in \chi'$.

where the operator-valued function $\mathcal{R}^{\text{app}}_{\chi,\epsilon}(z)$ is defined by

$$\mathcal{R}^{\text{app}}_{\chi,\epsilon}(z) := \left[\begin{array}{cc}
(\mathcal{A}_{0,\chi} - z I)^{-1} - \hat{S}^{\text{soft}}(z) \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{soft}}(z)^* & -\hat{S}^{\text{soft}}(z) \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} (\hat{\Pi}^{\text{soft}})^* \\
-\hat{\Pi}^{\text{soft}} \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{soft}}(z)^* & -\hat{\Pi}^{\text{soft}} \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} (\hat{\Pi}^{\text{soft}})^*,
\end{array}\right],$$

(91)

and the block-operator matrix is understood relative to the decomposition $\mathcal{H}^{\text{soft}} \oplus \mathcal{H}^{\text{stiff}}$.

**Proof.** The proof consists in applying the formula (89) to the individual blocks of the resolvent $((\mathcal{A}_{\chi,\epsilon})_{|_{\mathcal{H}}} - z I)^{-1}$. We have

$$P^{\text{soft}} ((\mathcal{A}_{\chi,\epsilon})_{|_{\mathcal{H}}} - z I)^{-1} P^{\text{soft}} = (\mathcal{A}^{\text{soft}}_{0,\chi} - z I)^{-1} - \hat{S}^{\text{soft}}(z) \hat{M}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{soft}}(z)^*$$

$$= (\mathcal{A}^{\text{soft}}_{0,\chi} - z I)^{-1} - \hat{S}^{\text{soft}}(z) \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{soft}}(z)^* + O(\epsilon^2).$$

Next, we use the asymptotic formula (77) for $S^{\text{stiff}}$.

$$P^{\text{stiff}} ((\mathcal{A}_{\chi,\epsilon})_{|_{\mathcal{H}}} - z I)^{-1} P^{\text{stiff}} = -\hat{S}^{\text{stiff}}(z) \hat{M}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{stiff}}(z)^* = -\hat{\Pi}^{\text{stiff}} \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{stiff}}(z)^* + O(\epsilon^2),$$

$$P^{\text{soft}} ((\mathcal{A}_{\chi,\epsilon})_{|_{\mathcal{H}}} - z I)^{-1} P^{\text{stiff}} = -\hat{S}^{\text{soft}}(z) \hat{M}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{stiff}}(z)^* = -\hat{S}^{\text{soft}}(z) \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} (\hat{\Pi}^{\text{stiff}})^* + O(\epsilon^2).$$

For calculating the remaining block, we use the fact that

$$P^{\text{stiff}} (\mathcal{A}_{0,\chi} - z I)^{-1} P^{\text{stiff}} = (\epsilon^{-2} \mathcal{A}^{\text{stiff}}_{0,\chi} - z I)^{-1} = O(\epsilon^2)$$

in the $\mathcal{H} \to \mathcal{H}$ operator norm. Finally, we have

$$P^{\text{stiff}} ((\mathcal{A}_{\chi,\epsilon})_{|_{\mathcal{H}}} - z I)^{-1} P^{\text{stiff}} = (\epsilon^{-2} \mathcal{A}^{\text{stiff}}_{0,\chi} - z I)^{-1} - \hat{S}^{\text{stiff}}(z) \hat{M}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{stiff}}(z)^*$$

$$= -\hat{\Pi}^{\text{stiff}} \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} (\hat{\Pi}^{\text{stiff}})^* + O(\epsilon^2),$$

which, combined with Theorem 5.2, completes the proof.

**Remark 5.7.** Note that one can rewrite (91) as follows:

$$\mathcal{R}^{\text{app}}_{\chi,\epsilon}(z) := (\mathcal{A}^{\text{soft}}_{0,\chi} - z I)^{-1} P^{\text{Hoa}} - \left[\hat{S}^{\text{soft}}(z) \hat{\Pi}^{\text{stiff}} \hat{Q}^{\text{app}}_{\chi,\epsilon}(z)^{-1} \hat{S}^{\text{soft}}(z) \hat{\Pi}^{\text{stiff}}\right]^*.$$  

(92)
5.3 Fiberwise approximating operator

It remains to provide an explicit description of the selfadjoint operator whose resolvent is given by (91). To this end, we consider the Hilbert space \( \mathcal{H}^{soft} \oplus \mathcal{H}^{soft}_x \) and define the operator \( \mathcal{A}_{app, x} \) as follows:

\[
\mathcal{D}(\mathcal{A}_{app, x}^{pp}) := \{(u, \hat{u})^\top \in \mathcal{H}^{soft} \oplus \mathcal{H}^{soft}_x, \ u \in \mathcal{D}(\mathcal{A}^{soft}_x), \ \hat{u} = \tilde{\Gamma}^{soft}_0 x u\},
\]

\[
\mathcal{A}_{app, x}^{pp}[u, \hat{u}] := \begin{bmatrix}
\mathcal{A}^{soft}_x[u, \hat{u}] & 0 \\
-(\mathcal{P}^{soft}_x)^{-1} \tilde{\Gamma}^{soft}_0 x - e^{-2}(\mathcal{P}^{soft}_x)^{-1} \tilde{\Gamma}^{soft}_1 x & 0
\end{bmatrix}.
\]  

(93)

The following theorem links \( \mathcal{A}_{app, x}^{pp} \), see (91), to the resolvent of \( \mathcal{A}_{app, x}^{pp} \).

**Theorem 5.8.** For every \( \chi \in \mathcal{Y} \), the operator \( \mathcal{A}_{app, x}^{pp} \) is self-adjoint and its resolvent for all \( z \in \rho(\mathcal{A}_{app, x}^{pp}) \) is given by the formula (91), relative to the decomposition \( \mathcal{H}^{soft} \oplus \mathcal{H}^{soft}_x \).

**Proof.** First we show that the operator \( \mathcal{A}_{app, x}^{pp} \) is symmetric. For \( (u, \hat{u})^\top, (v, \hat{v})^\top \in \mathcal{D}(\mathcal{A}_{app, x}^{pp}) \) one has

\[
\langle \mathcal{A}_{app, x}^{pp}, (u, \hat{u})^\top, (v, \hat{v})^\top \rangle = \langle \mathcal{A}_{app, x}^{pp}, (v, \hat{v})^\top, (u, \hat{u})^\top \rangle.
\]

By Green’s formula (see (17)) and the self-adjointness of \( \mathcal{A}^{soft}_x \), one has

\[
\langle \mathcal{A}_{app, x}^{pp}, (u, \hat{u})^\top, (v, \hat{v})^\top \rangle = \langle \mathcal{A}_{app, x}^{pp}, (v, \hat{v})^\top, (u, \hat{u})^\top \rangle.
\]

Next, we fix \( f \in \mathcal{H}^{soft}, \hat{f} \in \mathcal{H}^{soft}_x \). For every \( z \in \rho(\mathcal{A}_{app, x}^{pp}) \) we consider the problem

\[
(\mathcal{A}_{app, x}^{pp} - zI)[u, \hat{u}] = [f, \hat{f}] , \quad [u, \hat{u}] \in \mathcal{D}(\mathcal{A}_{app, x}^{pp}).
\]

Component-wise, we have

\[
\begin{cases}
\tilde{\Gamma}^{soft}_x u - zu = f, \\
-\tilde{\Gamma}^{soft}_1 x u - e^{-2}\tilde{\Gamma}^{soft}_x u - z(\mathcal{P}^{soft}_x)^{-1} \tilde{\Gamma}^{soft}_1 x \hat{u} = \hat{f},
\end{cases}
\]

(94)

which is equivalent to

\[
\begin{cases}
\tilde{\Gamma}^{soft}_x u - zu = f, \\
-\tilde{\Gamma}^{soft}_1 x u - e^{-2}\tilde{\Gamma}^{soft}_x u - z(\mathcal{P}^{soft}_x)^{-1} \tilde{\Gamma}^{soft}_1 x \hat{u} = (\mathcal{P}^{soft}_x)^{-1} \hat{f},
\end{cases}
\]

(95)

Due to the fact that \( \hat{u} = \tilde{\Gamma}^{soft}_0 x u \), the problem (95) is equivalent to finding a vector \( u \in \mathcal{D}(\mathcal{A}^{soft}_x) \) such that

\[
\begin{cases}
\tilde{\Gamma}^{soft}_x u - zu = f, \\
-\tilde{\Gamma}^{soft}_1 x u - e^{-2}\tilde{\Gamma}^{soft}_x u - z(\mathcal{P}^{soft}_x)^{-1} \tilde{\Gamma}^{soft}_1 x \tilde{\Gamma}^{soft}_0 x u = (\mathcal{P}^{soft}_x)^{-1} \hat{f},
\end{cases}
\]

and then setting \( \hat{u} = \tilde{\Gamma}^{soft}_0 x u \). By recalling (16), one has \( \tilde{\Gamma}^{soft}_0 x \tilde{\Gamma}^{soft}_x = \tilde{\Lambda}^{soft} \), and therefore the above is equivalent to finding \( u \in \mathcal{D}(\mathcal{A}^{soft}_x) \) such that

\[
\tilde{\Gamma}^{soft}_x u - zu = f, \quad \tilde{\Gamma}^{soft}_1 x u + (e^{-2}\tilde{\Lambda}^{soft} + z(\mathcal{P}^{soft}_x)^{-1} \tilde{\Gamma}^{soft}_1 x) \tilde{\Gamma}^{soft}_0 x u = -(\mathcal{P}^{soft}_x)^{-1} \hat{f}.
\]

\[\text{Since the operators } \tilde{\Gamma}^{soft}_0 x \text{ are acting from 3-dimensional space to a 3-dimensional space they themselves and their inverses can be represented by the appropriate matrices.}\]
We next define the operators \( \beta_{0,1}(z) := e^{-2\hat{A}_x^{\text{soft}}} + z(\hat{\Pi}_x^{\text{soft}})\hat{\Pi}_x^{\text{soft}} \), \( \beta_1 = I \), so the transmission function (90) can be written as

\[
\hat{Q}_{\xi,x}^{\text{app}}(z) = \beta_{0,1}(z) + \beta_1 \hat{Q}_{\xi,x}^{\text{soft}}(z).
\]

The operator \( \hat{Q}_{\xi,x}^{\text{app}}(z) \) is boundedly invertible (as can be seen by considering its imaginary part and using Corollary A.2) and satisfies the assumptions of Theorem 3.13. The solution \( u \) is then given by (26) with \( g = -\hat{\Pi}_x^{\text{soft}} \cdot \hat{f} \):

\[
u = (\mathcal{A}_{0,1}^{\text{soft}} - z I)^{-1} f - S_x^{\text{soft}}(z) \hat{Q}_{\xi,x}^{\text{app}}(z)^{-1} (S_x^{\text{soft}}(z))^* f + (\hat{\Pi}_x^{\text{diff}} \cdot \hat{f})
\]

\[
= \left[ (\mathcal{A}_{0,1}^{\text{soft}} - z I)^{-1} - S_x^{\text{soft}}(z) \hat{Q}_{\xi,x}^{\text{app}}(z)^{-1} (S_x^{\text{soft}}(z))^* \right] f + (\hat{\Pi}_x^{\text{diff}} \cdot \hat{f})
\]

Now, one has

\[
\hat{u} = \hat{\Pi}_x^{\text{diff}} G_{0,1}^{\text{soft}} u = -\hat{\Pi}_x^{\text{diff}} \hat{\Pi}_x^{\text{soft}} S_x^{\text{soft}}(z) \hat{Q}_{\xi,x}^{\text{app}}(z)^{-1} (S_x^{\text{soft}}(z))^* f + (\hat{\Pi}_x^{\text{diff}} \cdot \hat{f})
\]

\[
= \left[ -\hat{\Pi}_x^{\text{diff}} \hat{Q}_{\xi,x}^{\text{app}}(z)^{-1} (S_x^{\text{soft}}(z))^* \right] f + (\hat{\Pi}_x^{\text{diff}} \cdot \hat{f})
\]

□

Another insight into the operator \( \mathcal{A}_{\xi,x}^{\text{app}} \) is obtained by considering its sesquilinear form.

**Lemma 5.9.** The sesquilinear form \( a_{\xi,x}^{\text{app}}(\mathcal{H} \times \mathcal{H}) \) on \( \mathcal{H} \) with \( \mathcal{A}_{\xi,x}^{\text{app}} \) is given by

\[
\mathcal{D} (a_{\xi,x}^{\text{app}}) := \{ (u, \hat{u}, \hat{v}) : u \in \mathcal{A}_{\xi,x}^{\text{soft}} \mathcal{H}, \hat{u} = \hat{\Pi}_x^{\text{soft}} u \},
\]

\[
a_{\xi,x}^{\text{app}}(u, \hat{u}, \hat{v}) := \int_{Y_{\xi,x}} \mathcal{A}_{\xi,x}^{\text{soft}} (\text{sym} \nabla + iX_{\xi,x}) u : (\text{sym} \nabla + iX_{\xi,x}) \hat{v} + \frac{1}{\varepsilon} \int_{Y_{\xi,x}} \mathcal{A}_{\xi,x}^{\text{soft}} (\text{sym} \nabla + iX_{\xi,x}) \hat{u} : (\text{sym} \nabla + iX_{\xi,x}) \hat{v} \quad \forall (u, \hat{u}, \hat{v}) \in \mathcal{D}(a_{\xi,x}^{\text{app}}).
\]

The proof is obtained by a direct computation.

**Remark 5.10.** For \( (u, \hat{u}, \hat{v}) \in \mathcal{D}(a_{\xi,x}^{\text{app}}) \), one has

\[
u = \hat{u} + \hat{\Pi}_x^{\text{diff}} g_u, \quad \hat{v} = \hat{\Pi}_x^{\text{diff}} g_v, \quad \hat{u} = \hat{\Pi}_x^{\text{diff}} g_u, \quad \hat{v} = \hat{\Pi}_x^{\text{diff}} g_v,
\]

where \( \hat{u}, \hat{v} \in \mathcal{D}(a_{\xi,x}^{\text{soft}}) \) and \( g_u, g_v \in \mathcal{E}_x \). With this notation at hand, the form \( a_{\xi,x}^{\text{app}} \) can be written as

\[
a_{\xi,x}^{\text{app}}(u, \hat{u}, \hat{v}) = \int_{Y_{\xi,x}} \mathcal{A}_{\xi,x}^{\text{soft}} (\text{sym} \nabla + iX_{\xi,x}) u : (\text{sym} \nabla + iX_{\xi,x}) \hat{v} + \frac{1}{\varepsilon} \int_{Y_{\xi,x}} \mathcal{A}_{\xi,x}^{\text{soft}} (\text{sym} \nabla + iX_{\xi,x}) \hat{u} : (\text{sym} \nabla + iX_{\xi,x}) \hat{v} + \frac{1}{\varepsilon} \int_{Y_{\xi,x}} \mathcal{A}_{\xi,x}^{\text{soft}} (\text{sym} \nabla + iX_{\xi,x}) \hat{u} : (\text{sym} \nabla + iX_{\xi,x}) \hat{v} + \frac{1}{\varepsilon} \int_{Y_{\xi,x}} \mathcal{A}_{\xi,x}^{\text{soft}} (\text{sym} \nabla + iX_{\xi,x}) \hat{u} : (\text{sym} \nabla + iX_{\xi,x}) \hat{v} \quad \forall (u, \hat{u}, \hat{v}) \in \mathcal{D}(a_{\xi,x}^{\text{app}}).
\]

The following theorem contains the main result of this section.

**Theorem 5.11.** There exists \( C > 0 \), which depends only on \( \sigma \) and \( \text{diam}(K_x) \), such that for the resolvent of the transmission problem (13) one has

\[
\left\| \left( (\mathcal{A}_{\xi,x}^{\text{app}} - z I)^{-1} - \Theta_x (\mathcal{A}_{\xi,x}^{\text{app}} - z I)^{-1} \Theta_x \right) \right\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq C \varepsilon^2 \quad \forall \chi \in Y',
\]

where the operator \( \mathcal{A}_{\xi,x}^{\text{app}} \) is defined by (93), and \( \Theta_x : \mathcal{H} \rightarrow \mathcal{H} \) is an orthogonal projection defined by \( \Theta_x (u_{\text{soft}} \oplus u_{\text{diff}}) := u_{\text{soft}} \oplus P_{\mathcal{H}_{\xi,x}} u_{\text{diff}} \) with respect to the \( L^2(Y; 
abla^3) \) inner product.

**Proof.** The proof consists in combining (88) and (92) with Theorem 5.8, to infer that \( \mathcal{R}_{\xi,x}^{\text{app}}(z) = \Theta_x (\mathcal{A}_{\xi,x}^{\text{app}} - z I)^{-1} \Theta_x \). The orthogonality of \( \Theta_x \) is obvious. □

**Proof of Theorem 2.4.** The proof is a direct consequence of Theorem 5.11, using the fact that the (scaled) Gelfand transform is an isometry and defining \( \Theta_x^{\text{app}} := \mathcal{G}_x^{-1} \Theta_x \mathcal{G}_x, \mathcal{A}_{\xi,x}^{\text{app}} := \mathcal{G}_x^{-1} \mathcal{A}_{\xi,x}^{\text{app}} \mathcal{G}_x \). □
5.4 General outlook on the approach

An alternative way to rewrite (91) is as follows:

\[ R_{\chi,\varepsilon}^{\text{app,soft}}(z) := \left[ \begin{array}{c} \mathcal{R}_{\chi,\varepsilon}^{\text{app,soft}}(z) \\ \hat{\Pi}_{\chi}^{\text{soft}} \mathcal{G}_{\chi,\varepsilon}^{\text{app,soft}}(z) (\chi_{0,\varepsilon} - z I)^{-1} \end{array} \right] \left[ \begin{array}{c} \left( \mathcal{R}_{\chi,\varepsilon}^{\text{app,soft}}(z) - (\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} \right) (\mathcal{G}_{\chi,\varepsilon}^{\text{soft}} \mathcal{G}_{\chi,\varepsilon}^{\text{soft}})^* \\ \hat{\Pi}_{\chi}^{\text{soft}} \mathcal{G}_{\chi,\varepsilon}^{\text{app,soft}}(z) - (\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} \end{array} \right], \]

relative to the decomposition \( \mathcal{H}^{\text{soft}} \oplus \hat{\mathcal{H}}_{\chi}^{\text{soft}} \), where

\[
\mathcal{R}_{\chi,\varepsilon}^{\text{app,soft}}(z) := (\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} - \hat{S}_{\chi,\varepsilon}^{\text{soft}}(z) \hat{Q}_{\chi,\varepsilon}^{\text{app,soft}}(z)^{-1} \hat{S}_{\chi,\varepsilon}^{\text{soft}}(z)^* = (\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} - \hat{S}_{\chi,\varepsilon}^{\text{soft}}(z) (\hat{M}_{\chi,\varepsilon}^{\text{soft}}(z) + \varepsilon^{-2} \hat{M}_{\chi,\varepsilon}^{\text{diff}}(z^2) + z (\hat{\Pi}_{\chi}^{\text{soft}})^* \hat{\Pi}_{\chi}^{\text{soft}})^{-1} \hat{S}_{\chi,\varepsilon}^{\text{soft}}(z)^*. \]

The operator-valued function \( \mathcal{R}_{\chi,\varepsilon}^{\text{app,soft}}(z) \) is the solution operator (in the sense of Theorem 3.14) to the problem of finding \( u \in \mathcal{D}(\mathcal{A}_{\chi,\varepsilon}^{\text{soft}}) \) such that

\[
\hat{\mathcal{R}}_{\chi,\varepsilon}^{\text{soft}} u - z u = f, \quad \beta_{0,\varepsilon}(z) \hat{\mathcal{R}}_{\chi,\varepsilon}^{\text{soft}} u + \beta_{\chi,\varepsilon}(z) \hat{\mathcal{R}}_{\chi,\varepsilon}^{\text{soft}} u = 0, \]

where transmission operators are given by \( \beta_{0,\varepsilon}(z) = \varepsilon^{-2} \hat{M}_{\chi,\varepsilon}^{\text{diff}}(z^2) + z (\hat{\Pi}_{\chi}^{\text{soft}})^* \hat{\Pi}_{\chi}^{\text{soft}}, \beta_{\chi,\varepsilon} = I \). Problems like this, namely those where the boundary conditions depend on the spectral parameter, are often referred to as impedance boundary value problems. One should note that the “impedance” here is linear in the spectral parameter.

On the other hand, one can consider the “sandwiched resolvent”

\[
\mathcal{R}_{\chi,\varepsilon}^{\text{soft}}(z) := P_{\text{soft}} ((\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} - 1) P_{\text{soft}},
\]

and use (45) to infer that

\[
\mathcal{R}_{\chi,\varepsilon}^{\text{soft}}(z) = P_{\text{soft}} ((\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} - S_{\chi,\varepsilon}(z) M_{\chi,\varepsilon}(z) - S_{\chi,\varepsilon}(z)^*) P_{\text{soft}} = (\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} - S_{\chi,\varepsilon}(z) (M_{\chi,\varepsilon}(z) + \varepsilon^{-2} M_{\chi,\varepsilon}^{\text{diff}}(z^2))^{-1} S_{\chi,\varepsilon}(z)^*.
\]

Comparing this to (27) yields the following proposition.

Proposition 5.12. The generalised resolvent \( \mathcal{R}_{\chi,\varepsilon}^{\text{soft}}(z) \) is the solution operator of the “impedance” boundary value problem on \( \mathcal{H}^{\text{soft}} \) that consists in finding \( u \in \mathcal{D}(\mathcal{A}_{\chi,\varepsilon}^{\text{soft}}) \) such that

\[
\mathcal{R}_{\chi,\varepsilon}^{\text{soft}} u - z u = f, \quad \beta_{0,\varepsilon}(z) \mathcal{R}_{\chi,\varepsilon}^{\text{soft}} u + \beta_{\chi,\varepsilon}(z) \mathcal{R}_{\chi,\varepsilon}^{\text{soft}} u = 0,
\]

where the transmission operators are given by \( \beta_{0,\varepsilon}(z) := \varepsilon^{-2} M_{\chi,\varepsilon}(z^2), \beta_{\chi,\varepsilon} = I \).

The “impedance” of the boundary value problem (97) is highly nonlinear, due to the structure of the \( M \)-function \( M_{\chi,\varepsilon}(z) \). On the abstract level, both solution operators \( \mathcal{R}_{\chi,\varepsilon}^{\text{app,soft}} \) and \( \mathcal{R}_{\chi,\varepsilon}^{\text{soft}} \) are generalised resolvents [46, 47, 56, 57, 58].

A generalised resolvent can be equivalently characterised as either an operator of the form \( P(\mathcal{A} - z I)^{-1} P \) for a self-adjoint \( \mathcal{A} \) in a Hilbert space \( \mathcal{H} \) and an orthogonal projection \( P \), or a solution operator of an abstract spectral boundary value problem

\[
\mathcal{A} u = z u, \quad \Gamma u = \mathcal{B}(z) \Gamma u.
\]

where \( \mathcal{A} \) is the densely defined linear operator on \( P \mathcal{H} \) and \( (\mathcal{E}, \Gamma_0, \Gamma_1) \) is an abstract boundary triple of \( \mathcal{A} \), while \( -\mathcal{B}(z) \) is an analytic in the upper half-plane operator-valued function with positive imaginary part (i.e., an operator R-function) on \( \mathcal{E} \), extended into the region \( \Re z < 0 \) by the identity \( \mathcal{B}(z) = \mathcal{B}^*(z) \).

The system (98) can be thus re-cast in the form of the operator equation \( \mathcal{A}_{\varepsilon} u = z u \), where \( \mathcal{A}_{\varepsilon} \) is a closed densely defined linear operator on \( P \mathcal{H} \) with domain \( \mathcal{D}(\mathcal{A}_{\varepsilon}) = \{ u \in \mathcal{D}(\mathcal{A}) \subset P \mathcal{H} : \Gamma_{\varepsilon} u = \mathcal{B}(z) \Gamma_{\varepsilon} u \} \). The operator \( \mathcal{A}_{\varepsilon} \) is shown to be maximal dissipative for \( z \in C_- \) and maximal antidiissipative for \( z \in C_+ \).

From the point of view of generalised resolvents, one can therefore view the homogenisation procedure we have performed above as obtaining the main order term in the asymptotic expansion of the generalised resolvent \( \mathcal{R}_{\chi,\varepsilon}^{\text{soft}} \) for every fixed \( \chi \in Y \) as \( \varepsilon \to 0 \).

Moreover, we point out that in order to determine the main order term of \( (\mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I)^{-1} \) as \( \varepsilon \to 0 \), or in other words to recover the operator describing the homogenised medium, it is in fact necessary and sufficient to construct an asymptotic expansion of the generalised resolvent described above. This follows from the fact that under a natural
and non-restrictive “minimality” condition the operator \( A \) giving rise to the generalised resolvent \( P(\mathcal{A} - zI)^{-1}\big|_{\mathcal{P}^H} \) is in fact uniquely determined based on the latter up to a unitary gauge transform \( \Phi \) such that \( \Phi|_{\mathcal{P}^H} = I|_{\mathcal{P}^H} \).

This can be viewed in the homogenisation problem at hand as taking the “down, right, up” detour in the commutative diagram

\[
\begin{array}{c}
(\mathcal{A}_{\chi}\chi,\varepsilon - zI)^{-1} \quad (\mathcal{A}_{\chi}\chi,\varepsilon)^{app} - zI)^{-1} \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
\mathcal{R}_{\chi,\varepsilon}^{soft} \quad \mathcal{R}_{\chi,\varepsilon}^{app}
\end{array}
\]

\[
(99)
\]

where the double solid line represents the unitary gauge.

As far as the asymptotic analysis of the generalised resolvent \( \mathcal{R}_{\chi,\varepsilon}^{soft} \) is concerned, the required analysis is essentially reduced to the derivation of the asymptotics of the operator \( \mathcal{B}(\chi,\varepsilon) \) which governs its impedance boundary conditions. This, due to (96), in turn reduces to a well-understood problem of perturbation theory for the DtN map pertaining to the stiff component of the medium and thus presents no complications.

Having said that, we point out that however appealing this argument appears, it meets two significant difficulties. Firstly, at present we don’t have an explicit way to construct the operator \( \mathcal{A}_{\chi,\varepsilon}^{app} \) in (99) for arbitrary impedance boundary conditions parameterised by a generic \( (-R) \)-operator function \( B(z) \) in (98). In the problem at hand, this presents no challenge as the main order term (94) of the boundary operator is in fact linear in \( z \). Generalised resolvents of this form have already appeared in problems of dimension reduction, most notably in the works concerned with the convergence of PDEs defined on “thin” networks to ODEs on limiting metric graphs, see, e.g., [50, 38, 39, 27], and in particular our recent paper [17] where an approach akin to the one utilised in the present work is extended to the context of thin networks. In the area of linear elasticity in particular this analysis is thought to be applicable to the analysis of pentamodes [43], which will be further discussed elsewhere.

Secondly and crucially, once the asymptotics of the family of generalised resolvents is obtained in some desired strong topology, the same type of convergence for the family of resolvents \( ((\mathcal{A}_{\chi},\varepsilon)_{0,-} - zI)^{-1} \) cannot be inferred from the general operator theory. In fact, one can argue that norm-resolvent convergence of \( \mathcal{R}_{\chi,\varepsilon}^{soft} \) only yields strong convergence of \( ((\mathcal{A}_{\chi},\varepsilon)_{0,1} - zI)^{-1} \). It is here that the specifics of the problem at hand must play a crucial rôle in the analysis, leading to a result of the type formulated in Theorem 5.6 above.

Despite the deficiencies of the general operator-theoretic outlook based on generalised resolvents explained above, we point out that this way of considering the dimension reduction problem at hand is very natural in that it presents one with a physically motivated understanding of the problem.

As the argument of [29, 30], see also references therein, demonstrates, generalised resolvents appear naturally in physical setups where one forcefuly removes certain degrees of freedom from consideration in an otherwise conservative setting in view of simplifying the latter. Conversely, the procedure of reconstructing the self-adjoint generator of conservative dynamics must be viewed as adding those “hidden”, or concealed, degrees of freedom back in a proper way. In doing so, one frequently faces a situation (and in particular, in the setup of linear elasticity discussed in the present paper) where the resulting model is drastically simplified owing to only a certain limited number of concealed degrees of freedom appearing in it in a handily transparent way. The procedure of the diagram (99) can be therefore seen as a non-trivial generalisation of the seminal idea of Lax and Phillips [40], with a dissipative generator expressing the scattering properties of the system being replaced by a more general one, corresponding to an \( R \)-function which non-trivially depends on the spectral parameter \( z \).

At the same time, as explained in [19] (see also references therein), the concept of dilating (in the sense of (98)) a generalised resolvent to a resolvent of a self-adjoint generator gives rise to the understanding of homogenisation limits in the setup of double porosity models as essentially operators on soft component of the media with singular surface potentials possessing internal structure, see also [21] where a similar argument applied to high-contrast ODEs has led to a Kronig-Penney-type model. In the problem considered in the present paper, the mentioned singular surface can be shown to be the periodic lattice of the original composite.

Moreover, the argument of [20] can be immediately invoked for the homogenised family (93) to obtain its functional model in an explicit form in certain explicitly constructed Hilbert space of complex-analytic functions, giving rise to a Clark–Alexandrov measure serving as the spectral measure of the family. This latter program will be pursued elsewhere, together with the study of effective scattering problems of the high-contrast composite which can be considered naturally on this basis.
6 Transmission problem: $O(\varepsilon)$ resolvent asymptotics

The goal of this section is to further approximate the resolvent related to the transmission problem and prove Theorem 2.5, (a). In doing so, we will worsen the order in $\varepsilon$ of the estimate but will obtain more familiar objects in the asymptotics.

Our aim is to provide a further approximation to the operator $\left[\hat{S}_x^{\text{soft}}(z) \hat{\Pi}^{\text{stiff}}_x \hat{Q}_{x,\varepsilon}^{\text{app}}(z) \right]^{-1} \left[\hat{S}_x^{\text{soft}}(z) \hat{\Pi}^{\text{stiff}}_x \right]^{*}$ entering the resolvent (92) so the associated error is not worse than $O(\varepsilon)$. For this, the following estimate on the inverse of the transmission function is crucial.

**Lemma 6.1.** There exists $C > 0$ which does not depend on $\varepsilon > 0$, $z \in K_\sigma$, $\chi' \in Y'$, such that

$$\|\hat{Q}_{x,\varepsilon}^{\text{app}}(z) \|^2_{E \rightarrow E} \leq C \min \{ |\chi'|^{-2} \varepsilon^2, 1 \}. \quad (100)$$

**Proof.** First, note that

$$\mathcal{G} \left( \hat{Q}_{x,\varepsilon}^{\text{app}}(z) \right) = \mathcal{G} \left( \hat{S}_x^{\text{soft}}(z) \right)^* \hat{S}_x^{\text{soft}}(z) + \mathcal{G} \left( \hat{\Pi}^{\text{stiff}}_x \right)^* \hat{\Pi}^{\text{stiff}}_x, \quad \mathcal{R} \left( \hat{Q}_{x,\varepsilon}^{\text{app}}(z) \right) = \varepsilon^{-2} \hat{\Pi}^{\text{stiff}}_x + \mathcal{R} \hat{\Pi}^{\text{stiff}}_x + \mathcal{R} \hat{\Pi}^{\text{stiff}}_x + \mathcal{R} \hat{\Pi}^{\text{stiff}}_x.$$  

For $u \in \hat{E}_x$ using Lemma 5.1 and the trace inequality, we write

$$\langle \mathcal{G} \left( \hat{Q}_{x,\varepsilon}^{\text{app}}(z) \right) u, u \rangle_{\mathcal{G}} = |\mathcal{G} \left( \langle \hat{\Pi}^{\text{stiff}}_x u, \hat{\Pi}^{\text{stiff}}_x u \rangle_{\mathcal{G}} + \langle \hat{S}_x^{\text{soft}}(z) u, \hat{S}_x^{\text{soft}}(z) u \rangle_{\mathcal{G}} \right) |$$

$$\geq |\mathcal{G} | \| \hat{\Pi}^{\text{stiff}}_x u \|^2_{\mathcal{G}} \geq C |\mathcal{G} | \| \hat{\Pi}^{\text{stiff}}_x u \|^2_{\mathcal{G}} \geq C |\mathcal{G} | \| u \|^2_{\mathcal{G}},$$

where $C > 0$ depends only on $K_\sigma$. Thus, due to Corollary A.2, one has

$$\|\hat{Q}_{x,\varepsilon}^{\text{app}}(z) \|^2_{E \rightarrow E} \leq C, \quad (101)$$

where $C > 0$ is independent of $\chi'$ and $z$. Furthermore, by Corollary 4.17 and Remark 4.18, we infer the existence of $\hat{C} > 0$, which depends on $|z|$ and $\sigma$, such that

$$\|\hat{\Pi}^{\text{stiff}}_x(z)\|_{L^2(\Gamma,\mathbb{C}^n) \rightarrow L^2(\Gamma,\mathbb{C}^n)} \leq \hat{C}.$$  

Using Lemma 4.7 and the fact that $\hat{\Pi}^{\text{stiff}}_x$ is uniformly bounded, we infer the existence of constants $D, C_2$, independent of $\chi'$, $\varepsilon$, such that for $|\chi'| \geq D \varepsilon$, one has

$$\left| \langle \mathcal{R} \left( \hat{Q}_{x,\varepsilon}^{\text{app}}(z) \right) u, u \rangle_{\mathcal{G}} \right| \geq C_2 \varepsilon^{-2} |\chi'|^2 \| u \|^2_{\mathcal{G}} \quad \forall u \in \hat{E}_x.$$  

For such $|\chi'|$, by applying Corollary A.2, we obtain

$$\|\hat{Q}_{x,\varepsilon}^{\text{app}}(z) \|^2_{E \rightarrow E} \leq C_2 |\chi'|^{-2} \varepsilon^2,$$

which, combined with (101), concludes the proof. \hfill \Box

Next we introduce the version of the transmission function that will appear in the final homogenisation result. The above two lemmata allow us to replace the subscript $\chi'$ by zero everywhere except $\hat{\Lambda}^{\text{hom}}$, which leads to a more transparent result involving a differential, rather than a pseudodifferential, form of the operator asymptotics.

**Definition 6.2.** We refer to the operator valued function $\hat{Q}_{\chi,\varepsilon}^{\text{eff}}(z) : \hat{E}_0 \rightarrow \hat{E}_0$ given by

$$\hat{Q}_{\chi,\varepsilon}^{\text{eff}}(z) := \varepsilon^{-2} \hat{\Pi}^{\text{hom}}_0 + z \left( \hat{\Pi}^{\text{stiff}}_0 \right)^* \hat{\Pi}^{\text{stiff}}_0 + \hat{\Pi}^{\text{soft}}_0(z) \quad (102)$$

as the effective transmission function. We introduce the following associated operator-valued function on $\mathcal{H}$ :

$$\mathcal{R}_{\chi,\varepsilon}^{\text{eff}}(z) := \left( \mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I \right)^{-1} P_{\mathcal{H}_{\text{rig}}} - \left[ \hat{S}_x^{\text{soft},\text{eff}}(z) \right] \hat{\Pi}^{\text{stiff}}_0 \hat{Q}_{\chi,\varepsilon}^{\text{app}}(z) \hat{Q}_{\chi,\varepsilon}^{\text{eff}}(z)^{-1} \left[ \hat{S}_x^{\text{soft},\text{eff}}(z) \right]^* \hat{\Pi}^{\text{stiff}}_0,$$

where the effective solution operator $\hat{S}_x^{\text{soft},\text{eff}}(z) : \hat{E}_0 \rightarrow \mathcal{H}_{\text{soft}}$ is defined by

$$\hat{S}_x^{\text{soft},\text{eff}}(z) := \hat{\Pi}^{\text{soft}}_0 + z \left( \mathcal{A}_{\chi,\varepsilon}^{\text{soft}} - z I \right)^{-1} \hat{\Pi}^{\text{soft}}_0.$$
Remark 6.3. Due to the estimate (76) and the boundedness of the resolvent \((\mathcal{A}_{0,\chi}^\text{eff} - zI)^{-1}\), we have
\[
\|\tilde{\mathcal{A}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{soft,eff}(z)\tilde{\mathcal{P}}_0\|_{L^2(\Gamma;C^3)\rightarrow L^2(\Gamma_{\text{eff}};C^3)} \leq C|\chi|,
\]
where the constant \(C > 0\) is independent of \(\chi\) and \(z\). Using the estimate (70) and the identity (88) yields
\[
\|\left(\tilde{\Pi}_\chi^\text{eff}\right)^* - \left(\tilde{\Pi}_0^\text{eff}\right)^*\|_{L^2(\Gamma_{\text{eff}};C^3)\rightarrow L^2(\Gamma;C^3)} \leq C|\chi|.
\]
For the inverse of the effective transmission function we have an estimate similar to (100).

**Lemma 6.4.** There exists \(C > 0\) which does not depend on \(\varepsilon > 0\), \(z \in K_\varepsilon\), \(\chi \in Y'\), such that
\[
\|\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0\|_{E \rightarrow E} \leq C \min \{|\chi|^2, 1\}.
\]

**Proof.** The proof follows the steps of the proof of Lemma 6.1 while utilizing Lemma 4.8. \(\square\)

The following lemma provides an estimate on the distance between the two transmission functions.

**Lemma 6.5.** There exists a constant \(C > 0\), independent of \(\varepsilon > 0\), \(z \in K_\varepsilon\), \(\chi \in Y'\), such that
\[
\|\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0\|_{E \rightarrow E} \leq C \max \{|\chi|^2, |\chi|\}.
\]

**Proof.** The case of \(\chi = 0\) is trivial. It is clear that for all \(\chi \in Y' \setminus \{0\}\) we have
\[
\frac{1}{|\chi|^2} \tilde{\mathcal{A}}_{\chi}^\text{eff} \tilde{\mathcal{P}}_\chi - \frac{1}{|\chi|^2} \tilde{\mathcal{A}}_{\chi}^\text{hom} \tilde{\mathcal{P}}_0 = \frac{1}{2\pi i} \int_\gamma z \left( \left( zI - \frac{1}{|\chi|^2} \tilde{\mathcal{A}}_{\chi}^\text{eff} \right)^{-1} - \left( zI - \frac{1}{|\chi|^2} \tilde{\mathcal{A}}_{\chi}^\text{hom} \right)^{-1} \right) dz,
\]
where \(\gamma\) is the contour provided by Lemma 4.13. Therefore, by applying the Theorem 4.9 (cf. Remark 4.11), we obtain
\[
\|\varepsilon^{-1} \tilde{\mathcal{A}}_{\chi}^\text{eff} \tilde{\mathcal{P}}_\chi - \varepsilon^{-1} \tilde{\mathcal{A}}_{\chi}^\text{hom} \tilde{\mathcal{P}}_0\|_{E \rightarrow E} \leq C\varepsilon^{-2} |\chi|^3.
\]
The claim now follows from (90) and (102), by invoking Corollary 4.17 and Corollary 4.15. \(\square\)

The following lemma is crucial for obtaining \(\varepsilon\)-order asymptotics of the resolvent \((\mathcal{A}_{\chi,e}^\text{eff} - zI)^{-1}\) and relating it to an object that incorporates the effective transmission function.

**Lemma 6.6.** There exists a constant \(C > 0\) which does not depend on \(\varepsilon > 0\), \(z \in K_\varepsilon\), \(\chi \in Y'\), such that
\[
\|\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0\|_{E \rightarrow E} \leq C\varepsilon.
\]

**Proof.** By a direct calculation, we see that
\[
\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0 = I + II + III,
\]
where
\[
I := \left(\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0\right)\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0,
II := \left(\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0\right)\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0,
III := \left(\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0\right)\hat{\mathcal{A}}_{\chi,e}^\text{eff}(z)\tilde{\mathcal{P}}_\chi - \tilde{\mathcal{S}}_{\chi}^\text{eff}(z)\tilde{\mathcal{P}}_0.
\]
Next, using Lemma 6.1, Lemma 6.4 and Lemma 6.5, we obtain
\[
\|I\|_{E \rightarrow E} \leq C \min \left\{ \frac{\varepsilon^2}{|\chi|^2}, 1 \right\} \max \left\{ |\chi|, \frac{|\chi|^3}{\varepsilon^2} \right\} \min \left\{ \frac{\varepsilon^2}{|\chi|^2}, 1 \right\} \leq \left\{ \begin{array}{ll}
1 \cdot \varepsilon \cdot 1 & \text{if } |\chi| \leq \varepsilon, \\
\frac{\varepsilon^2}{|\chi|^2} \cdot \frac{|\chi|^3}{\varepsilon^2} = \varepsilon^2 & \text{if } |\chi| \geq \varepsilon.
\end{array} \right.
\]
Furthermore, by employing Corollary 4.15 and Lemma 6.1, one easily estimates
\[
\|II\|_{E \rightarrow E} \leq C |\chi| \min \left\{ \frac{\varepsilon^2}{|\chi|^2}, 1 \right\} \leq \left\{ \begin{array}{ll}
|\chi| \cdot \varepsilon \cdot 1 & \text{if } |\chi| \leq \varepsilon, \\
\frac{\varepsilon^2}{|\chi|^2} \cdot \frac{\varepsilon^2}{|\chi|^2} = \varepsilon^2 & \text{if } |\chi| \geq \varepsilon.
\end{array} \right.\leq \left(103\right)
\]
Similarly, using again Corollary 4.15 and Lemma 6.4 we estimate
\[
\|III\|_{E \rightarrow E} \leq C |\chi| \min \left\{ \frac{|\chi|^2 \varepsilon^2}{1}, 1 \right\} \leq C\varepsilon,
\]
which concludes the proof. \(\square\)
Finally, we can summarise these results as the following theorem.

**Theorem 6.7.** There exists a constant $C > 0$ which does not depend on $\epsilon > 0$, $z \in K_\sigma$, $\chi \in Y'$, such that
\[
\|R_{\chi,x}^{\text{soft}}(z) - R_{\chi,x}^{\text{eff}}(z)\|_{Y \to Y} \leq Ce,
\]
where
\[
R_{\chi,x}^{\text{eff}}(z) := \left[ \left( \mathcal{A}_{\chi,x}^{\text{soft}} - zI \right)^{-1} - \hat{S}_{\chi,x}^{\text{soft,eff}}(z) \right] \left( \hat{S}_{\chi,x}^{\text{soft,eff}}(z) \right)^{-1} \left( \hat{S}_{\chi,x}^{\text{soft,eff}}(z) \right)^* - \hat{S}_{\chi,x}^{\text{soft,eff}}(z) \hat{\Pi}_{\chi,x}^{\text{eff}}(z) \left( \hat{\Pi}_{\chi,x}^{\text{eff}}(0) \right)^* \right],
\]
and the block decomposition is relative to the decomposition $H^\text{soft} \oplus H^\text{eff}$.

**Proof.** The proof of this fact consists of estimating the four blocks of the matrix
\[
\begin{pmatrix}
\hat{S}_{\chi,x}^{\text{soft}}(z) & \hat{\Pi}_{\chi,x}^{\text{eff}}(z)
\end{pmatrix}^{-1}
\begin{pmatrix}
\hat{S}_{\chi,x}^{\text{soft}}(z) & \hat{\Pi}_{\chi,x}^{\text{eff}}(z)
\end{pmatrix}^{-1}
\begin{pmatrix}
\hat{S}_{\chi,x}^{\text{soft}}(z) & \hat{\Pi}_{\chi,x}^{\text{eff}}(z)
\end{pmatrix}^{-1}
\begin{pmatrix}
\hat{S}_{\chi,x}^{\text{soft}}(z) & \hat{\Pi}_{\chi,x}^{\text{eff}}(z)
\end{pmatrix}^{-1}
\]
for which all the arguments go analogously. Thus, we prove the estimate only for one of the blocks. Combining the triangle inequality, Corollary 4.19, identity (87), Lemma 6.1, and Remark 6.3, we obtain, similarly to (103):
\[
\hat{\Pi}_{\chi,x}^{\text{eff}}(z) \left( \hat{\Pi}_{\chi,x}^{\text{eff}}(0) \right)^* - \hat{\Pi}_{\chi,x}^{\text{eff}}(z) \left( \hat{\Pi}_{\chi,x}^{\text{eff}}(0) \right)^* = \hat{\Pi}_{\chi,x}^{\text{eff}} \left( \left( \hat{S}_{\chi,x}^{\text{soft}}(z) \right)^{-1} \hat{\Pi}_{\chi,x}^{\text{eff}} \left( \hat{S}_{\chi,x}^{\text{soft}}(z) \right)^{-1} \hat{\Pi}_{\chi,x}^{\text{eff}}(0) \right)^* + O(\epsilon),
\]
where $O(\epsilon)$ is of order $\epsilon$ with respect to the $L^2 \to L^2$ norm. Using Lemma 6.6 concludes the estimate for this block.

**Definition 6.8.** We define the effective operator $\mathcal{A}_{\chi,x}^{\text{eff}}$, as follows:
\[
\mathcal{D}(\mathcal{A}_{\chi,x}^{\text{eff}}) := \{(u, \mathcal{U}) \in H^\text{soft} \oplus H^\text{eff} : u \in \mathcal{D}(\mathcal{A}_{\chi,x}^{\text{soft}}) + \hat{\Pi}_{\chi,x}^{\text{soft}} \hat{E}_0 \}, \quad \mathcal{A}_{\chi,x}^{\text{eff}} u := -\left( \begin{matrix}
-\left( \hat{\Pi}_{\chi,x}^{\text{eff}}(0) \right)^* & -\epsilon\frac{\hat{S}_{\chi,x}^{\text{soft,eff}}(0)}{\epsilon_0} & \hat{\Pi}_{\chi,x}^{\text{eff}}(0)
\end{matrix} \right) \left( \begin{matrix}
u \\
\mathcal{U}
\end{matrix} \right),
\]
where
\[
\mathcal{D}(\mathcal{A}_{\chi,x}^{\text{soft}}) := \mathcal{D}(\mathcal{A}_{\chi,x}^{\text{soft}}) + \hat{\Pi}_{\chi,x}^{\text{soft}} \hat{E}_0, \quad \mathcal{A}_{\chi,x}^{\text{soft}} : (\mathcal{A}_{\chi,x}^{\text{soft}})^{-1} f + \hat{\Pi}_{\chi,x}^{\text{soft}} g \to f, \quad f \in \mathcal{H}, g \in \hat{E}_0.
\]
\[
\mathcal{D}(\hat{\Pi}_{\chi,x}^{\text{eff}}) := \mathcal{D}(\mathcal{A}_{\chi,x}^{\text{eff}}) + \hat{\Pi}_{\chi,x}^{\text{eff}} \hat{E}_0, \quad \hat{\Pi}_{\chi,x}^{\text{eff}} : (\mathcal{A}_{\chi,x}^{\text{soft}})^{-1} f + \hat{\Pi}_{\chi,x}^{\text{soft}} g \to \left( \hat{\Pi}_{\chi,x}^{\text{eff}} \right)^* f + \hat{\Pi}_{\chi,x}^{\text{soft}} g, \quad f \in \mathcal{H}, g \in \hat{E}_0.
\]

**Remark 6.9.** It is straightforward to check that $\left( \hat{\Pi}_{\chi,x}^{\text{soft}}(0) \right)^* = |Y_{\text{soft}}(0)|^{-1} \hat{\Pi}_{\chi,x}^{\text{eff}} P_{\chi,x}^{\text{soft}}(0)$. Recall that $\hat{\Lambda}_{\chi,x}^{\text{soft}} g = 0$ for every $g \in \hat{E}_0$, as $\hat{E}_0 \subset H^\text{soft}$ consists of constant functions. Similarly to Theorem 5.8, one can establish the following statement, whose proof we omit.

**Theorem 6.10.** For every $\chi \in Y'$, the operator $\mathcal{A}_{\chi,x}^{\text{eff}}$ is self-adjoint and its resolvent is given, for all $z \in \rho(\mathcal{A}_{\chi,x}^{\text{eff}})$, by the formula (104) to the decomposition $H^\text{soft} \oplus H^\text{eff}$.

**Remark 6.11.** The sesquilinear form $a_{\chi,x}^{\text{eff}}$ on $\mathcal{H} \times \mathcal{H}$ associated with the operator (105) is given by
\[
\mathcal{D}(a_{\chi,x}^{\text{eff}}) := \{(u, \mathcal{U}) \in H^\text{soft} \oplus H^\text{eff} : u \in \mathcal{D}(a_{\chi,x}^{\text{soft}}) + \hat{\Pi}_{\chi,x}^{\text{soft}} \hat{E}_0 \}, \quad a_{\chi,x}^{\text{eff}} \left( \begin{matrix}
u \\
\mathcal{U}
\end{matrix} \right) := \int_{\chi \in X} \hat{\Lambda}_{\chi,x} \left( \text{sym} \mathcal{V} + iX_\chi \right) u : \left( \text{sym} \mathcal{V} + iX_\chi \right) \nu + \frac{1}{\epsilon} \hat{\Lambda}_{\chi,x}^{-1} \hat{\Pi}_{\chi,x}^{\text{eff}} \hat{\Pi}_{\chi,x}^{\text{eff}} \nu, \quad \left( \begin{matrix}
u \\
\mathcal{U}
\end{matrix} \right) \in \mathcal{D}(a_{\chi,x}^{\text{eff}}).
\]
Recalling Lemma 4.8, one can see that a similar form was obtained in [12] as an $O(\epsilon)$-approximation in the case of a scalar equation by using a different technique.

By a slight abuse of notation, Remark 6.11 allows us to identify the operator $\mathcal{A}_{\chi,x}^{\text{eff}}$ with an operator acting in a subspace of $H^\text{soft} \oplus H^\text{eff} = \mathcal{H}$. We then extend it by zero to the whole $\mathcal{H}$, while still keeping the same notation for the extension, hoping that it does not lead to any confusion.

The following theorem provides norm-resolvent asymptotics of order $\epsilon$ in the form of a “sandwiched” resolvent of the effective operator $\mathcal{A}_{\chi,x}^{\text{eff}}$. It is a direct consequence of Theorem 6.7, cf. the proof of Theorem 5.11.
Theorem 6.12. There exists $C > 0$, independent of $\varepsilon \in K_\varepsilon$ and $e$, such that for the resolvent of the transmission problem (13) one has
\[
\|\left((\mathcal{A}_{\varepsilon,e} z - zI)^{-1} - \Theta_0 \left(\mathcal{A}_{\varepsilon,e} z - zI\right)^{-1}\Theta_0 \right)_{H^1 \rightarrow H^1}\| \leq C\varepsilon, \quad \forall \chi \in Y',
\]
where the operator $\mathcal{A}_{\varepsilon,e}$ is defined by (105), and $\Theta_0$ is the orthogonal projection (80).

Proof of Theorem 2.5 (a). This is a direct consequence of Theorem 6.12, based on the fact that the (scaled) Gelfand transform is an isometry, by setting $\Theta_{\text{eff}} := y^{1/2} \Theta_0 y^{1/2}, \mathcal{A}_{\text{eff}} := y^{1/2} \mathcal{A}_{\varepsilon,e} y^{1/2}$.

Remark 6.13. In [16], a version of Theorem 6.12 is proved by expanding the least eigenvalue and the corresponding eigenfunction of the operator $\Lambda_{\text{eff}}^\text{stiff}$ with respect to the quasimomentum $\chi$. As explained in the introduction to Section 4.3, this is not possible when dealing with systems. Thus we expand the resolvent of an appropriately scaled operator $\Lambda_{\text{eff}}^\text{stiff}$, which as we have shown, suffices to prove Theorem 6.12. We also improve the error estimate $O(\varepsilon^{2/3})$ obtained in [16] to $O(\varepsilon)$.

Remark 6.14. For every $\varepsilon > 0$ we define the space $S_{\varepsilon,\text{eff}}^\text{stiff} \subset H^1(\mathbb{R}^3; \mathbb{R}^3)$ as the space of functions whose scaled Gelfand transform is constant for every $\chi \in Y'$. It is easy to see that this space consists of functions $u$ such that $\mathcal{F}(u)(\xi) = 0$ when $|\xi|_{\mathbb{R}^3} > (2\varepsilon)^{-1}$, where $|\xi|_{\mathbb{R}^3} = \max\{|\xi_1|, |\xi_2|, |\xi_3|\}$ (cf. (8)). Here $\mathcal{F}(\cdot)$ stands for the Fourier transform and $\xi \in C^3$ is the Fourier variable. We also introduce the space $S_{\varepsilon}^\text{stiff} := H^1(\mathbb{R}^3; \mathbb{R}^3) \cap L_{\varepsilon}^{\text{soft}}$. Define a bilinear form $\mathcal{a}_{\varepsilon}^\text{eff}$ by
\[
\mathcal{a}_{\varepsilon}^\text{eff}(u + \hat{u}, v + \hat{v}) := \varepsilon^2 \int_{\Omega_{\text{soft}}} \mathcal{A}_{\text{soft}} \text{sym} \nabla (u + \hat{u}) : \text{sym} \nabla (v + \hat{v}) + \int_{\Omega_{\text{macro}}} \mathcal{A}_{\text{macro}} \text{sym} \hat{\nabla} \hat{u} : \text{sym} \hat{\nabla} \hat{v},
\]
for $(u + \hat{u}, (v + \hat{v}) \in \mathcal{D}(\mathcal{a}_{\varepsilon}^\text{eff})$.

It is easily seen that the scaled Gelfand transform of the form $\mathcal{a}_{\varepsilon}^\text{eff}$ equals $\mathcal{a}_{\varepsilon,x}^\text{eff}$. By an appropriate modification of the definition of $S_{\varepsilon,x}^\text{stiff}$ we can also treat the form from Remark 5.10 that defines the operator $\mathcal{R}_{\varepsilon,x}^{\text{app}}$.

7 Stiff component analysis

In this section we study implications of the estimates of the previous section. Our goal here is to prove Theorem 2.5 (b). We are interested in the properties of the effective operator (105) when restricted to the stiff component. A representation formula for this operator will be obtained that will bring to focus some known features of high-contrast homogenisation. To this end, we define the following operators that unitarily identify the spaces $E_0$ and $\tilde{H}_0^\text{stiff}$ (spanned by constant functions) with $C^3$:
\[
\iota_{\text{eff}} : E_0 \rightarrow C^3, \quad \iota_{\text{eff}} c = |\Gamma|^{-1/2} c, \quad c \in E_0, \quad \iota_{\text{eff}} : \tilde{H}_0^\text{stiff} \rightarrow C^3, \quad \iota_{\text{eff}} c = |\Gamma|^{-1/2} \tilde{c}, \quad c \in \tilde{H}_0^\text{stiff}.
\]
Notice that (cf. Remark 6.9)
\[
\tilde{\Pi}_{\text{eff}} = |\Gamma|^{-1/2} |\Gamma|^{-1/2} \tilde{c}^{\ast} \iota_{\text{eff}} \iota_{\text{eff}}^{\ast}.
\]
With these operators at hand, we obtain the following representation formula (recall Lemma 4.8 and (75)):
\[
P_{\gamma_{\text{soft}}} \left(\mathcal{R}_{\varepsilon,x}^\text{eff} z - zI\right)^{-1}|_{\gamma_{\text{soft}}} = -\tilde{\Pi}_{\text{eff}} \tilde{Q}_{\gamma_{\text{soft}}} (z)^{-1} \tilde{\Pi}_{\text{eff}}^{\ast} = -\tilde{\Pi}_{\text{eff}} \left(e^{-2 \tilde{\Lambda}_{\varepsilon}^{\text{hom}} + z|\Gamma|^{-1} \tilde{\Pi}_{\text{eff}}^{\ast} + \tilde{\Pi}_{\text{eff}} (z)}\right)^{-1} \left(\tilde{\Pi}_{\text{eff}}^{\ast} \right)^{\ast}
\]
\[
= -|\Gamma|^{-1/2} \tilde{\Pi}_{\text{eff}}^{\ast} \left(e^{-2 \tilde{\Lambda}_{\varepsilon}^{\text{hom}} + z|\Gamma|^{-1} \tilde{\Pi}_{\text{eff}}^{\ast} + \tilde{\Pi}_{\text{eff}} (z)}\right)^{-1} \tilde{\Pi}_{\text{eff}}^{\ast}
\]
\[
= -|\Gamma|^{-1/2} \tilde{\Pi}_{\text{eff}}^{\ast} \left(e^{-2 (i\Gamma)^{\ast} \tilde{\Lambda}_{\varepsilon}^{\text{hom}} + z|\Gamma|^{-1} \tilde{\Pi}_{\text{eff}}^{\ast} + \tilde{\Pi}_{\text{eff}} (z)}\right)^{-1} \tilde{\Pi}_{\text{eff}}^{\ast}
\]
where the matrix-valued function $\mathcal{B}(z)$ is defined by
\[
\mathcal{B}(z) := z|\Gamma|^{-1/2} \tilde{\Pi}_{\text{eff}}^{\ast} (z)\tilde{\Pi}_{\text{eff}}^{\ast}.
\]
In what follows, we show (see (118), (119)) that, owing to (75), a natural matrix representation of $\mathcal{B}$ is given by (4).
Remark 7.1. Regarding the estimates from above and below for the operator $B(z)$, note that for $c \in C^3$ one has
\begin{equation}
|\langle \delta (B(z)) e, c \rangle | = |\delta (z) | | Y_{\text{stiff}}(e, c) \rangle |_C^1 + | \langle \delta_{0, \text{stiff}}(z) \xi, \delta_{0, \text{stiff}}(z) \xi e \rangle \rangle_{\gamma \text{eff}} | \geq C |e|^2, \quad (107)
\end{equation}
and thus, by Corollary, A.2 one has $|B(z)^{-1}|_{C^0} \leq C_1$, where $C_1 > 0$ depends on the set $K_z$. Also, clearly
\begin{equation}
|B(z)|_{C^0} \leq C_2, \quad (108)
\end{equation}
where $C_2 > 0$ depends only on the $\max_{z \in K_z} |z|$.

The above is an explicit proof of the property for a Herglotz matrix function to be bounded together with its inverse away from the real line.

Remark 7.2. We will keep the same notation $B(z)$ for the operator of (pointwise) multiplication by the said matrix.

In order to pass to the real domain, what remains is to apply the inverse Gelfand transform. Before doing so, we introduce a smoothing operator: $\Xi : L^2(\mathbb{R}; C^3) \to L^2(\mathbb{R}; C^3)$ defined by
\begin{equation}
\Xi u := \mathcal{G}^{-1}_z \int_y (\mathcal{G}_u)(y, \cdot) dy.
\end{equation}
Next, we note that the projection operator $P_{\text{stiff}}$ is simply a multiplication with an indicator function associated with $Y_{\text{stiff}}$, namely $P_{\text{stiff}} u = \|Y_{\text{stiff}}(y) u, u \in \mathcal{H}$. Similarly, for the operator $P_{\text{eff}}$ i.e. the orthogonal projector from $L^2(\mathbb{R}; C^3)$ onto $L^2_{\text{eff}}$, which is defined by (1), we have
\begin{equation}
P_{\text{eff}} u = \|Y_{\text{eff}}(y) u, u \in L^2(\mathbb{R}; C^3).
\end{equation}
Also, for $u \in \mathcal{H}_{\text{stiff}}$, we have
\begin{equation}
P_{\text{eff}} u = \|Y_{\text{eff}}(y) u, u \in L^2(\mathbb{R}; C^3).
\end{equation}
Note that
\begin{equation}
\Xi P_{\text{eff}}(y) u = \|Y_{\text{eff}}(y) \Xi u, \Xi u \in \mathcal{H}.
\end{equation}
We have the following lemma.

Lemma 7.3. The following formula holds:
\begin{equation}
\mathcal{G}^{-1}_z \left( \int_y \left( \frac{1}{\xi^2} (IX)^\ast A_{\text{macro}} IX - B(z) \right)^{-1} t_{\text{stiff}} P_{\text{eff}} d\xi \right) \mathcal{G}_z = \frac{1}{\sqrt{|Y_{\text{eff}}|}} \left( \mathcal{A}_{\text{macro}} - B(z) \right)^{-1} \Xi P_{\text{eff}}, \quad (109)
\end{equation}
where the operator $A_{\text{macro}}$ is defined by the form (3).

Proof. To see this, we consider the operator $A_{\text{macro}}$ on $\mathcal{H}$ with the sesquilinear form
\begin{equation}
u \langle A_{\text{macro}}(u, v) = \int_y A_{\text{macro}} (\text{sym \nabla} + iX_y) u : (\text{sym \nabla} + iX_y) v, \quad u, v \in H^1_y(\mathbb{R}; C^3).
\end{equation}
By invoking the properties of the Gelfand transform (9), it is clear that
\begin{equation}
\mathcal{G}^{-1}_z \left( \int_y \left( \frac{1}{\xi^2} A_{\text{macro}} \right) d\xi \right) \mathcal{G}_z = A_{\text{macro}}.
\end{equation}
By virtue of (109), it remains to show that
\begin{equation}
\left( e^{-2} A_{\text{macro}} - B(z) \right)^{-1} t_{\text{stiff}} P_{\text{eff}} = \left( e^{-2} (IX_y)^\ast A_{\text{macro}} IX_y - B(z) \right)^{-1} t_{\text{stiff}} P_{\text{eff}}.
\end{equation}
First conclusion is that for $z \in K_z, u \in \mathcal{H}$, we have
\begin{equation}
\mathcal{S} \left( e^{-2} (IX_y)^\ast A_{\text{macro}} IX_y - B(z) \right) = - \mathcal{S} \left( B(z) \right).
\end{equation}

\textsuperscript{4}The operator $\mathcal{G}^{-1}_z$ is here applied to a function depending only on $\gamma i.e. for fixed $\gamma$ constant in $y$.

\textsuperscript{5}Again, the inverse Gelfand transform is applied to a function depending only on $\gamma i.e. for fixed $\gamma$ constant in $y$. 

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so the operators are invertible by taking into account Corollary A.2 and the estimates (107). In order to show (111), we take \( f \in \mathcal{H} \) and consider the unique solution \( u \in C^2 \) to the resolvent problem

\[
e^{-2} (iX_\delta)^* A_{\text{macro}} iX_\delta u - \mathcal{B}(z) u = t_{\text{stiff}} P_{\text{out}} f.
\]

Multiplying the above equation with arbitrary \( v \in C^3 \), and integrating over \( Y \) one obtains

\[
\frac{1}{e^2} \int_Y A_{\text{macro}} iX_\delta u : iX_\delta v - \int_Y \mathcal{B}(z) u \cdot \bar{v} = \int_Y t_{\text{stiff}} P_{\text{out}} f \cdot \bar{v}.
\]

Furthermore, it is checked that, as an element of \( H_0^1(Y; C^3) \), the constant function \( u \) solves the problem

\[
\frac{1}{e^2} \int_Y A_{\text{macro}} (\text{sym } v + iX_\delta) u : (\text{sym } v + iX_\delta) v - \int_Y \mathcal{B}(z) u \cdot \bar{v} = \int_Y t_{\text{stiff}} P_{\text{out}} f \cdot \bar{v} \quad \forall v \in H_0^1(Y; C^3),
\]

which is unique. The formula (110) now follows from (111).

\[\square\]

The following lemma allows us to drop the smoothing operator \( \Xi_\delta \) from the resolvent asymptotics while not making the error of higher order than \( e^2 \).

**Lemma 7.4.** Let \( z \in K_\delta \). There exists a constant \( C > 0 \) such that

\[
\left\| \left( (A_{\text{macro}} - \mathcal{B}(z))^{-1} (I - \Xi_\delta) \right) \right\|_{L^2(\mathbb{R}^3; C^1) \rightarrow L^2(\mathbb{R}^3; C^1)} \leq C e^2,
\]

where \( A_{\text{macro}} \) is a differential operator of linear elasticity with constant coefficients defined by the form (3).

**Proof.** We start with the identity

\[
\mathcal{F}(\Xi_\delta f)(\xi) = \mathbb{P}_{-1/(2\varepsilon)} \mathcal{F}(f)(\xi) \quad \forall f \in L^2(\mathbb{R}^3; C^3),
\]

where \( \mathcal{F} \) denotes, as before, the Fourier transform, and \( \xi \in C^3 \) is the Fourier variable (see, e.g. [24, Section 2.5.3]). The estimate (112) follows from the fact that for \( f \in L^2(\mathbb{R}^3; C^3) \) one has

\[
\mathcal{F} \left( (A_{\text{macro}} - \mathcal{B}(z))^{-1} f \right)(\xi) = \mathbb{P}_{-1/(2\varepsilon)} \mathcal{F}(f)(\xi),
\]

Namely, introducing \( u(\xi) := (iX_\delta)^* A_{\text{macro}} iX_\delta - \mathcal{B}(z))^{-1} \mathcal{F}(f)(\xi) \), one has

\[
|u(\xi)| \leq (1 - e^2 C_2)^{-1} e^2 |\mathcal{F}(f)(\xi)| \leq C e^2 |\mathcal{F}(f)(\xi)|, \quad |\xi| > (2e)^{-1}, \quad e \in (0, C_2^{-1/2}),
\]

where \( C_2 \) is given by (108).

\[\square\]

Finally, we proceed to the proof of Theorem 2.5 (b).

**Proof of Theorem 2.5 (b).** The asymptotic estimate (106) immediately yields

\[
\left\| p_{\text{stiff}} \left( (A_{\text{app}} \chi, \varepsilon)_{0_I} - zI \right)^{-1} P_{\text{stiff}} - \left| Y_{\text{stiff}} \right|^{1/2} P_{\text{stiff}} \left( e^{-2} (iX_\delta)^* A_{\text{macro}} iX_\delta - \mathcal{B}(z) \right)^{-1} t_{\text{stiff}} P_{\text{out}} \right\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq C e
\]

Invoking (110), we obtain

\[
\mathcal{G}_\delta^{-1} \int_{Y'} \left( p_{\text{stiff}} \left( (A_{\text{app}} \chi, \varepsilon)_{0_I} - zI \right)^{-1} P_{\text{stiff}} - \left| Y_{\text{stiff}} \right|^{1/2} P_{\text{stiff}} \left( e^{-2} (iX_\delta)^* A_{\text{macro}} iX_\delta - \mathcal{B}(z) \right)^{-1} t_{\text{stiff}} P_{\text{out}} \right) dY \mathcal{G}_\delta
\]

\[
= p_{\text{stiff}}(\mathcal{A}_\delta - zI)^{-1} p_{\text{stiff}} - p_{\text{stiff}}(\mathcal{A}_{\text{macro}} - \mathcal{B}(z))^{-1} \Xi_\delta p_{\text{stiff}}.\]

Combining this with (113) and using the fact that Gelfand transform is a unitary operator, we obtain

\[
\left\| p_{\text{stiff}}(\mathcal{A}_\delta - zI)^{-1} P_{\text{stiff}} - P_{\text{stiff}}(\mathcal{A}_{\text{macro}} - \mathcal{B}(z))^{-1} \Xi_\delta p_{\text{stiff}} \right\|_{L^2(\mathbb{R}^3; C^3) \rightarrow L^2(\mathbb{R}^3; C^3)} \leq C e.
\]

The last step is to drop the smoothing operator for which we use Lemma 7.4.

\[\square\]

**Remark 7.5.** The operator \( A_{\text{macro}} - \mathcal{B}(z) \), which plays the rôle of the leading-order term in the resolvent asymptotics of Theorem 2.5 (b), is clearly a second-order differential operator with constant coefficients.

**Remark 7.6.** The operator \( \mathcal{A}_{\text{app}} \) has a more concise form than the operator \( \mathcal{A}_{\text{app}} \), and admits no further simplification.

On the other hand, in what applies to the operator \( \mathcal{A}_{\text{app}} \), one can still obtain a simpler approximation, by going further in the expansion of DnN map in Section 4.3. The error bound thus obtained can be seen as \( O(e^2) \), i.e., the same as for the \( \mathcal{A}_{\text{app}} \). It can be further seen the thus obtained operator will non-local in the spatial variable, i.e. non-differential. For brevity, we refrain from discussing this in detail.
7.1 Dispersion relation

The assertion of Theorem 6.12, as well as a more precise (at the cost of being more involved) statement of Theorem 5.11, pertains to the asymptotic behaviour of the resolvent in the whole space. In applications however, and in particular in applications to periodic problems, it is often desirable to relate the spectrum of the problem to the so-called wave vector, or equivalently quasimomentum $\chi$, of the problem at hand. Indeed, the dispersion relation, which expresses the mentioned relationship, becomes of a paramount importance when the question of which monochromatic waves are supported by the medium at hand, as well as when the group velocity of wave packets spreading in the medium is brought to the forefront of investigation. The latter question is to arise naturally in the context of metamaterials, to which the model considered in the present paper is thought to be intimately related (the precise mathematical formulation of this relationship will however be discussed elsewhere; see also [19] and references therein for a related discussion), as in these both the phase and group velocity need to negate, [62].

Definition 7.7. We refer to the operator valued function defined by

$$K_{\chi,x}^{\text{app}}(z) := -(\hat{\Pi}^{\text{stiff}}_{\chi})^{-1} \hat{Q}_{\chi,x}^{\text{app}}(z) \hat{\Pi}^{\text{stiff}}_{\chi}^{-1} + zI = -((\hat{\Pi}^{\text{stiff}}_{\chi})^{*})^{-1} (e^{-2\hat{\Lambda}^{\text{stiff}}_{\chi}} + \hat{M}^{\text{soft}}_{\chi}(z)) \hat{\Pi}^{\text{stiff}}_{\chi}^{-1}$$

as the dispersion function associated with the operator $\mathcal{H}_{\chi,x}^{\text{app}}$. Similarly, we define the effective dispersion function associated with the operator $\mathcal{H}_{\chi,x}^{\text{eff}}$:

$$K_{\chi,x}^{\text{eff}}(z) := -(\hat{\Pi}^{\text{stiff}}_{\chi})^{-1} \hat{Q}_{\chi,x}^{\text{app}}(z) \hat{\Pi}^{\text{stiff}}_{\chi}^{-1} + zI = -((\hat{\Pi}^{\text{stiff}}_{\chi})^{*})^{-1} (e^{-2\hat{\Lambda}^{\text{hom}}_{\chi}} + \hat{M}^{\text{soft}}_{\chi}(z)) \hat{\Pi}^{\text{stiff}}_{\chi}^{-1}.$$

Remark 7.8. Notice that

$$\langle K_{\chi,x}^{\text{app}}(z) - zI \rangle^{-1} = -\hat{\Pi}^{\text{stiff}}_{\chi} \hat{Q}_{\chi,x}^{\text{app}}(z) \hat{\Pi}^{\text{stiff}}_{\chi}^{*}, \quad \langle K_{\chi,x}^{\text{eff}}(z) - zI \rangle^{-1} = -\hat{\Pi}^{\text{stiff}}_{\chi} \hat{Q}_{\chi,x}^{\text{app}}(z) \hat{\Pi}^{\text{stiff}}_{\chi}^{*}.$$

By comparing this with resolvent formulas: (91) and (104), one can see that these operators are, in fact, resolvents of the appropriate operators sandwiched with projections onto the stiff component:

$$P_{q^{\text{eff}}} \langle \mathcal{H}_{\chi,x}^{\text{app}}(z) - zI \rangle^{-1} P_{q^{\text{eff}}} = \langle K_{\chi,x}^{\text{eff}}(z) - zI \rangle^{-1}.$$

Denote by $\{\psi_i\}_{i=1}^{3} \subset \hat{E}_{\chi}$ an orthonormal basis of eigenfunctions of $\hat{\Lambda}^{\text{stiff}}_{\chi}$ and by $\{v_i^\chi\}_{i=1}^{3}$ the associated set of eigenvalues. Note that the $\{\hat{\Pi}^{\text{stiff}}_{\chi} \psi_i^\chi\}_{i=1}^{3}$ is a basis in $\hat{H}^{\text{eff}}_{\chi}$. The function $\tilde{u} \in \hat{H}^{\text{eff}}_{\chi}$ is represented in this basis with a vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ as

$$\tilde{u} = \sum_{i=1}^{3} \alpha_i \hat{\Pi}^{\text{eff}}_{\chi} \psi_i^\chi.$$  

Furthermore, we denote by $\{\Psi_i\}_{i=1}^{3} \subset \hat{H}^{\text{eff}}_{\chi}$ the contravariant dual for the basis $\{\hat{\Pi}^{\text{stiff}}_{\chi} \psi_i^\chi\}_{i=1}^{3}$, namely, the set of functions such that

$$\langle \hat{\Pi}^{\text{stiff}}_{\chi} \psi_i^\chi, \Psi_j^\chi \rangle_{q^{\text{eff}}} = \delta_{ij}, \quad i, j = 1, 2, 3.$$

One can easily check that $\Psi_j^\chi = ((\hat{\Pi}^{\text{stiff}}_{\chi})^{*})^{-1} \psi_j^\chi$, $j = 1, 2, 3$. Thus we find the following expressions for the coefficients in (116):

$$\alpha_j = \langle \hat{\Gamma}_{0,\chi} \tilde{u}, \psi_j^\chi \rangle_{q^{\text{eff}}}, \quad j = 1, 2, 3.$$

Next, we calculate the matrix $K_{\chi,x}^{\text{app}}(z)$ of the operator $K_{\chi,x}^{\text{app}}(z)$ in the basis $\{\hat{\Pi}^{\text{stiff}}_{\chi} \psi_i^\chi\}_{i=1}^{3}$:

$$K_{\chi,x}^{\text{app}}(z)_{ij} = \langle K_{\chi,x}^{\text{app}}(z) \hat{\Pi}^{\text{stiff}}_{\chi} \psi_j^\chi, ((\hat{\Pi}^{\text{stiff}}_{\chi})^{*})^{-1} \psi_i^\chi \rangle_{q^{\text{eff}}}.$$

By invoking the formulas (19), (18) and (114), the dispersion function can be expressed as

$$K_{\chi,x}^{\text{app}}(z) := -((\hat{\Pi}^{\text{stiff}}_{\chi})^{*})^{-1} (e^{-2\hat{\Lambda}^{\text{stiff}}_{\chi}} + \hat{M}^{\text{soft}}_{\chi}(z) \hat{\Pi}^{\text{stiff}}_{\chi})^{-1} \hat{\Pi}^{\text{stiff}}_{\chi}^{*}.$$

Denote now by $(\eta_k)_{k \in \mathbb{N}} \subset \mathbb{R}^+$, $(\varphi_k^\chi)_{k \in \mathbb{N}} \subset \mathcal{H}^{\text{soft}}_{\chi}$ the eigenvalues and associated eigenfunctions of the operator $\mathcal{H}^{\text{soft}}_{\chi}$. The resolvent of $\mathcal{H}^{\text{soft}}_{\chi}$ admits the expansion

$$(\mathcal{H}^{\text{soft}}_{\chi} - zI)^{-1} = \sum_{k=1}^{\infty} (\eta_k - z)^{-1} \varphi_k^\chi, \quad \varphi_k^\chi := \langle \cdot, \varphi_k^\chi \rangle_{q^{\text{eff}}} \varphi_k^\chi.$$
Upon a straightforward computation, we obtain

$$
\mathbb{K}_{\chi,x}^{\text{app}}(z)_{ij} = -e^{-2} v_0^2 \langle \psi_j, \mathbb{H}_x \psi_i \rangle_{\mathbb{E}^7} - \langle \Lambda^{\text{soft}}_x \psi_j, \mathbb{H}_x \psi_i \rangle_{\mathbb{E}^7} - z \langle \Pi^{\text{soft}}_x \psi_j, \Pi^{\text{soft}}_x \psi_i \rangle_{\mathbb{E}^7} = -z^2 \left( \frac{z^2}{\eta_k - z} \right) \langle \Pi^{\text{soft}}_x \psi_j, \mathbb{Q}_k \psi_i \rangle_{\mathbb{E}^7}^{\mathbb{H}_x \psi_i} - z^2 \langle \Pi^{\text{soft}}_x \psi_j, \mathbb{Q}_k \psi_i \rangle_{\mathbb{E}^7}^{\mathbb{H}_x \psi_i}.
$$

where $\mathbb{H}_x = \left( (\Pi^{\text{diff}})^* \mathbb{F}_x \right)^{-1} : \mathbb{E}_x \to \mathbb{E}_x$.

Similarly, by using (115), one establishes the matrix representation $\mathbb{K}_x^{\text{eff}}(z)$ of the effective dispersion function $\mathbb{K}_x^{\text{eff}}(z)$ in the canonical basis $\{e_j\}_{j=1}^N$ of $\mathbb{C}^3$:

$$
\mathbb{K}_x^{\text{eff}}(z)_{ij} = -e^{-2} \langle \Lambda^{\text{hom}}_x e_j, e_i \rangle_{\mathbb{E}_0} - \langle \hat{M}_0^{\text{soft}}(z) e_j, e_i \rangle_{\mathbb{E}_0}.
$$

Recalling (75), we obtain

$$
\langle \hat{M}_0^{\text{soft}}(z) e_j, e_i \rangle_{\mathbb{E}_0} = z \langle \Pi^{\text{soft}}_0 e_j, \Pi^{\text{soft}}_0 e_i \rangle_{\mathbb{E}_0} + z^2 \left( \mathcal{A}^{\text{soft}}_{0,0} - zI \right)^{-1} \langle \Pi^{\text{soft}}_0 e_j, \Pi^{\text{soft}}_0 e_i \rangle_{\mathbb{E}_0}
$$

where $z \langle \Pi^{\text{soft}}_0 e_j, \Pi^{\text{soft}}_0 e_i \rangle_{\mathbb{E}_0}$ is given by $z \langle \Lambda^{\text{hom}}_0 e_j, e_i \rangle_{\mathbb{E}_0} - z \langle \hat{M}_0^{\text{soft}}(z) e_j, e_i \rangle_{\mathbb{E}_0}$.

The calculations above prove the following theorem.

**Theorem 7.9.** The dispersion relation for the operator $\mathcal{A}_{\chi,x}^{\text{app}}(z)$ is given by

$$
\det \left( \mathcal{A}_{\chi,x}^{\text{app}}(z) - z \mathbb{Y} \right) = 0,
$$

It links the parameters $z$ and $\chi$, where the matrix $\mathbb{K}_{\chi,x}(z) \in \mathbb{C}^{N \times N}$ is given by (117). The effective dispersion relation (see also [55, 13]) is given by

$$
\det \left( \mathcal{A}_{\chi,x}^{\text{eff}} - \mathbb{B}(z) \right) = 0,
$$

where (cf. Lemma 4.8)

$$
\mathcal{A}_{\chi,x}^{\text{eff}} := e^{-2} X^\theta \mathcal{A}_{\text{macro}} X^\theta, \quad \mathbb{B}(z)_{ij} = z \delta_{ij} + \sum_{k=1}^N \frac{z^2}{\eta_k - z} \langle \varphi_k \rangle_{\Lambda_{\text{macro}}} \langle \varphi_k \rangle_{\Lambda_{\text{macro}}}.
$$

Denoting $\theta := \frac{1}{|\chi|} |\chi|^{-1} \chi$, one has $\mathcal{A}_{\chi,x}^{\text{eff}} = |\chi|^2 \mathcal{A}_{\chi,x}^{\text{eff}} = e^{-2} |\chi|^2 X^\theta \mathcal{A}_{\text{macro}} X^\theta$. The matrix $X^\theta \mathcal{A}_{\text{macro}} X^\theta$ is positive definite with the lower bound uniform with respect to $\theta$, with $|\eta, C_1 > 0$:

$$
\langle X^\theta \mathcal{A}_{\text{macro}} X^\theta \xi, \xi \rangle \geq |\eta| \xi \cdot \xi \geq |\eta| C_1 |\xi|^2 = |\eta| C_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^2,
$$

where we use the coercivity estimate of Lemma 2.3 and the lower bound in (10). Denote by $\mathcal{A}_0^{\text{app}}$ the positive definite (symmetric) square root of $X^\theta \mathcal{A}_{\text{macro}} X^\theta$. Clearly, the dispersion relation (120) is equivalent to

$$
\det \left( |\chi|^2 I - e^{-2} \mathcal{A}_0^{\text{app}} \mathbb{B}(z) \mathcal{A}_0^{\text{app}} \right) = 0,
$$

Note that for fixed $z > 0$ and $\theta$, the matrices $\mathbb{B}(z)$ and $\mathcal{A}_0^{\text{app}} \mathbb{B}(z) \mathcal{A}_0^{\text{app}}$ have the same number of non-negative eigenvalues. The eigenvalues $\beta(z)$ of $\mathbb{B}(z)$ are shown to be strictly increasing in $z$ in every interval of analyticity of $\mathbb{B}$ on the real line. This follows from the Herglotz property of $\mathbb{B}(z)$, since the latter implies the positive-definiteness of the derivative $\mathbb{B}'(z)$ on the real line. It follows that the same holds for the eigenvalues of $\mathcal{A}_0^{\text{app}} \mathbb{B}(z) \mathcal{A}_0^{\text{app}}$ and therefore all the branches of the multivalued function $z \mapsto |\chi(z)|$ defined implicitly by (121) are strictly increasing. This, in turn, implies that the gradient of $z$ is parallel to $\chi$:

$$
\nabla_x z = \left( |\chi| \right) \frac{X}{|\chi|} = \left( \frac{d|\chi(z)|}{dz} \right)^{-1} \frac{X}{|\chi|}.
$$

We have thus proved the following statement.

**Theorem 7.10.** For every direction $\theta$, the number of solutions $|\chi|$ to (121) is equal to the number of non-negative eigenvalues of $\mathbb{B}(z)$. Furthermore, the corresponding group velocities $[7, 44]$ are positive $[10]$ on every interval of analyticity of $\mathbb{B}$. 

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the perforated domain \( Y \) where for \( t \) the elements of an orthonormal basis \( A \)
This results in an explicit construction of the matrix \( A \):

\[
A_{ijkl}(y) = \begin{cases} 
\lambda \delta_{ij} \delta_{kl} + 2\mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}) =: A_{ijkl}^{\text{soft}}, & y \in Y_{\text{soft}}, \\
\varepsilon^2 (A \delta_{ij} \delta_{kl} + 2\mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj})) =: A_{ijkl}^{\text{eff}}, & y \in Y_{\text{eff}}.
\end{cases}
\]

The macroscopic tensor \( A_{\text{macro}} \), which constitutes the fiberwise effective operator \( A_{\text{eff},\chi} \), is represented by its action on the elements of an orthonormal basis \( \{E_i\}_{i=1}^3 \) of \( \mathbb{R}^{2 \times 2}_{\text{sym}} \):

\[
A_{\text{macro}} E_j : E_i := \int_{Y_{\text{eff}}} A_{\text{eff}} (\text{sym } \nabla E_i + E_i) : E_j, \quad i, j = 1, 2, 3.
\]

where for \( E \in \mathbb{R}^{2 \times 2}_{\text{sym}} \) the displacement \( u_E \in H^1_0(Y_{\text{eff}}; \mathbb{R}^2) \), \( \int_{Y_{\text{eff}}} u_E = 0 \) is calculated by solving the cell problem for the perforated domain \( Y_{\text{eff}} \):

\[
\int_{Y_{\text{eff}}} A_{\text{eff}} (\text{sym } \nabla u_E + E) : \text{sym } \nabla v = 0 \quad \forall v \in H^1_0(Y_{\text{eff}}; \mathbb{R}^2), \quad \int_{Y_{\text{eff}}} v = 0.
\]

This results in an explicit construction of the matrix \( A_{\text{eff},\chi} \). The function \( B(z) \) is approximated by a finite sum

\[
B^\alpha(z)_{i,j} = z \delta_{ij} + \sum_{k=1}^n \frac{z^2}{\eta_k - z} \langle \varphi_k \rangle \langle \varphi_k \rangle,
\]

where \( \eta_k, \quad k = 1, \ldots, n \), are the \( n \) smallest eigenvalues of \( \mathcal{A}^{\text{eff}}_{\chi,0} \), see Fig. 2. The graphs of the real eigenvalues \( \beta_1(z) \), \( \beta_2(z) \) (ordered by the relation \( \beta_1(z) \leq \beta_2(z) \)) of the symmetric matrix-valued function \( B^\alpha(z), z > 0 \), are shown in Fig. 3. The dispersion surfaces for the effective problem, which are determined by the relation (120) are shown in Fig. 4.

\[\eta = 41.4271\]
\[\eta = 41.555\]
\[\eta = 52.2137\]
\[\eta = 64.7445\]
Appendix

A.1 Operator theory

Lemma A.1. Let $\mathcal{A}$ be a closed, densely defined linear operator on a complex Hilbert space $\mathcal{H}$, such that $|\langle \mathcal{A} x, x \rangle| \geq C\|x\|^2$, $\forall x \in D(\mathcal{A})$ and $\ker(\mathcal{A}^*) \subset D(\mathcal{A})$. Then the inverse $\mathcal{A}^{-1}$ exists and $\|\mathcal{A}^{-1}\| \leq C^{-1}$.

Proof. First, we show that $\mathcal{R}(\mathcal{A}) = \mathcal{H} = \mathcal{R}(\mathcal{A})$. To this end, take $x \in \ker(\mathcal{A}^*)$. Since $x \in D(\mathcal{A})$, one has

$$0 = |\langle x, \mathcal{A}^* x \rangle| = |\langle \mathcal{A} x, x \rangle| \geq C\|x\|^2,$$

and therefore $x = 0$. Therefore, $\mathcal{R}(\mathcal{A}) = (\ker(\mathcal{A}^*))^\perp = \{0\}^\perp = \mathcal{H}$. Clearly, $\mathcal{A}$ is an injection. Thus $\mathcal{A}^{-1}$ exists. For $y \in \mathcal{R}(\mathcal{A})$ we put $x = \mathcal{A}^{-1}y$. Thus, one has

$$\|x\|^2 = \|\mathcal{A}^{-1}y\|^2 \leq C^{-1}\|y, \mathcal{A}^{-1}y\| \leq C^{-1}\|y\|\|\mathcal{A}^{-1}y\|,$$

and the claim follows. □

Corollary A.2. Let $\mathcal{A}$ be a closed, densely defined linear operator on a complex Hilbert space $\mathcal{H}$ such that $D(\mathcal{A}) = D(\mathcal{A}^*)$. Denote by $\Re \mathcal{A}$ and $\Im \mathcal{A}$ the real and imaginary part of $\mathcal{A}$. Assume that max $\{|\langle \Im \mathcal{A} x, x \rangle|, |\langle \Re \mathcal{A} x, x \rangle|\} \geq C\|x\|^2$ for all $x \in D(\mathcal{A})$ for some $C > 0$. Then the inverse $\mathcal{A}^{-1}$ exists and $\|\mathcal{A}^{-1}\| \leq C^{-1}$.

Proof. The claim follows immediately from Lemma A.1, since

$$|\langle \mathcal{A} x, x \rangle| = \sqrt{|\langle \Re \mathcal{A} x, x \rangle|^2 + |\langle \Im \mathcal{A} x, x \rangle|^2} \geq \max \{|\langle \Im \mathcal{A} x, x \rangle|, |\langle \Re \mathcal{A} x, x \rangle|\} \quad \forall x \in D(\mathcal{A}).$$

□

A.2 Auxiliary estimates

Here we state some of the results which we use in the main text. In this part we state various versions of Korn’s and trace inequalities. First, we state a special case of the classical trace theorem.

Proposition A.3. There exists $C > 0$ such that for every $g \in H^{1/2}(\Gamma; C^1)$ there is an extension $G \in H^1_0(\text{Y}^\text{soft}, C^1)$ satisfying $\|G\|_{H^1_0(\text{Y}^\text{soft}, C^1)} \leq C \|g\|_{H^{1/2}(\Gamma; C^1)}$.

Next, we recall a well-known Korn’s inequality (see, e.g., [48]).

Proposition A.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary. There exists a constant $C > 0$ such that for every $u \in H^1(\Omega; C^1)$, we have

$$\|u\|_{H^1(\Omega; C^1)} \leq C \left( \|\text{sym} \nabla u\|_{L^2(\Omega; C^{1,1})} + \|u\|_{L^2(\Omega; C^1)} \right),$$

where the constant $C$ depends only on the domain $\Omega$.

Figure 3: Plots of the eigenvalues $\beta_1(z), \beta_2(z)$ of the “truncation” $\mathcal{B}^n(z)$, see (122), for $n = 11$. The $x$-axis represents the frequency $z$, the blue curve is the function $\beta_1(z)$ and the red curve is $\beta_2(z)$. The regions of $z$ for which both $\beta_1(z)$ and $\beta_2(z)$ are negative yield no solutions to (120).
Figure 4: Solutions \((\chi, z)\) to the dispersion relation (120): \(\chi\) is horizontal, \(z\) is vertical. Left panel: side view in the direction of \(\chi^2\); right panel: 3D view. The number of dispersion surfaces at every \(z\) is the number of non-negative eigenvalues of \(\mathcal{B}(z)\). As \(\varepsilon \to 0\), the gaps between the surfaces converge to the regions in which \(\mathcal{B}(z)\) is negative-definite.

We use the following version of Korn’s inequality, proved by contradiction from Proposition A.4.

**Proposition A.5.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded open connected set with Lipschitz boundary and \(B \subset \partial \Omega, |B| > 0\). Then there exists a constant \(C > 0\) such that for every \(u \in H^1(\Omega; \mathbb{C}^3)\) we have

\[
\|u\|_{H^1(\Omega; \mathbb{C}^3)} \leq C \left( \|\text{sym} \nabla u\|_{L^2(\Omega; \mathbb{C}^{3 \times 3})} + \|u\|_{L^2(B; \mathbb{C}^3)} \right),
\]

where the constant \(C\) depends only on \(\Omega\) and \(B\).

**Proof.** By the inequality (A.1) we see that it suffices to show that

\[
\|u\|_{L^2(\Omega; \mathbb{C}^3)} \leq C \left( \|\text{sym} \nabla u\|_{L^2(\Omega; \mathbb{C}^{3 \times 3})} + \|u\|_{L^2(B; \mathbb{C}^3)} \right).
\]

Suppose the contrary, namely that there is a sequence \((u_k)_{k \in \mathbb{N}} \subset H^1(\Omega; \mathbb{C}^3), \|u_k\|_{L^2(\Omega; \mathbb{C}^3)} = 1\) such that

\[
\left( \|\text{sym} \nabla u_k\|_{L^2(\Omega; \mathbb{C}^{3 \times 3})} + \|u_k\|_{L^2(B; \mathbb{C}^3)} \right) < 1/k \quad \forall k \in \mathbb{N}.
\]

By (A.1), the sequence \((u_k)_{k \in \mathbb{N}}\) is bounded in \(H^1(\Omega; \mathbb{C}^3)\) and therefore converges (up to a subsequence) weakly in \(H^1(\Omega; \mathbb{C}^3)\) and strongly in \(L^2(\Omega; \mathbb{C}^3)\) to some \(u \in H^1(\Omega; \mathbb{C}^3), \|u\|_{L^2(\Omega; \mathbb{C}^3)} = 1\). By lower semicontinuity of the norm, we have

\[
\|\text{sym} \nabla u\|_{L^2(\Omega; \mathbb{C}^{3 \times 3})} \leq \lim_{k \to \infty} \|\text{sym} \nabla u_k\|_{L^2(\Omega; \mathbb{C}^{3 \times 3})} = 0.
\]
Also, since \( u_k - u \) in \( H^1(\Omega; \mathbb{C}^3) \), by trace compactness we infer that \( u|_B = 0 \). By (A.3) one has \( u|_B = Ax + c \) for some skew-symmetric \( A \in \mathbb{C}^{3 \times 3} \) and \( c \in \mathbb{C}^3 \) (see, e.g., [48]). Since \( |B| > 0 \), this implies that \( A = 0, \ c = 0 \). \( \square \)

Another useful version of Korn’s inequality is also well known and is a direct consequence of [48, Theorem 2.5].

**Proposition A.6.** Let \( \Omega \) be a bounded open set with Lipschitz boundary. There exists \( C > 0 \) dependent only on \( \Omega \) such that for every \( u \in H^1(\Omega; \mathbb{C}^3) \) the estimate \( \| u - w \|_{H^1(\Omega; \mathbb{C}^3)} \leq C \| \text{sym} \nabla u \|_{L^p(\Omega; \mathbb{C}^3 \times \mathbb{C}^3)} \) holds with \( w = Ax + c \), where

\[
A = \begin{bmatrix}
0 & d & a \\
-d & 0 & b \\
a & -b & 0
\end{bmatrix}, \quad c = \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}, \quad c_j = \int_{\Omega} (u_j), \quad j = 1, 2, 3.
\]

\[a = \int_{\Omega} (\partial_1 u_1 - \partial_2 u_3), \quad b = \int_{\Omega} (\partial_3 u_1 - \partial_1 u_3), \quad d = \int_{\Omega} (\partial_1 u_3 - \partial_3 u_1),\]

We proceed with the following simple assertion.

**Lemma A.7.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded open set. There exist constants \( C_0, C_1 > 0 \), independent of \( \chi \in Y' \), such that

\[
C_0 \| u \|_{H^1(\Omega; \mathbb{C}^3)} \leq \| e^{i\chi} u \|_{H^1(\Omega; \mathbb{C}^3)} \leq C_1 \| u \|_{H^1(\Omega; \mathbb{C}^3)} \quad \forall \chi \in Y', \ \ u \in H^1(\Omega; \mathbb{C}^3).
\]

**Proof.** We clearly have (due to \( |e^{i\chi}| = 1 \))

\[
\| e^{i\chi} u \|_{L^2(\Omega; \mathbb{C}^3)} = \| u \|_{L^2(\Omega; \mathbb{C}^3)}, \quad \| \nabla (e^{i\chi} u) \|_{L^2(\Omega; \mathbb{C}^3)} = \| \nabla u + u \otimes i \chi \|_{L^2(\Omega; \mathbb{C}^3)}.
\]

Now, we calculate

\[
\| e^{i\chi} u \|_{H^1(\Omega; \mathbb{C}^3)}^2 = \| u \|_{L^2(\Omega; \mathbb{C}^3)}^2 + \| \nabla u + u \otimes i \chi \|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq (1 + |\chi|^2) \| u \|_{L^2(\Omega; \mathbb{C}^3)}^2 + \| \nabla u \|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C \| u \|_{H^1(\Omega; \mathbb{C}^3)}^2.
\]

Conversely:

\[
\| u \|_{H^1(\Omega; \mathbb{C}^3)}^2 = \| u \|_{L^2(\Omega; \mathbb{C}^3)}^2 + \| \nabla u \|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq \| u \|_{L^2(\Omega; \mathbb{C}^3)}^2 + \| \nabla u + u \otimes i \chi \|_{L^2(\Omega; \mathbb{C}^3)}^2 + |\chi|^2 \| u \|_{L^2(\Omega; \mathbb{C}^3)}^2
\]

\[= (1 + |\chi|^2) \| u \|_{L^2(\Omega; \mathbb{C}^3)}^2 + \| \nabla u + u \otimes i \chi \|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C \| e^{i\chi} u \|_{H^1(\Omega; \mathbb{C}^3)}.
\]

\( \square \)

Using the above lemma as a starting point, we prove five propositions.

**Proposition A.8.** Let \( \Omega \) be a bounded open set with Lipschitz boundary and \( B \subset \partial \Omega \), \( |B| > 0 \). There exists a constant \( C \), depending only on the domain \( \Omega \), such that for every \( u \in H^1(\Omega; \mathbb{C}^3) \) and \( |\chi| \in Y' \) one has

\[
\| u \|_{H^1(\Omega; \mathbb{C}^3)} \leq C \left( \| \text{sym} \nabla \chi_1 u \|_{L^2(\Omega; \mathbb{C}^3 \times \mathbb{C}^3)} + \| u \|_{L^2(\Omega; \mathbb{C}^3)} \right).
\]

**Proof.** Plugging \( w = e^{i\chi} u, \ \ u \in H^1(\Omega; \mathbb{C}^3) \) into the inequality (A.2), we obtain, by virtue of Lemma A.7,

\[
\| u \|_{H^1(\Omega; \mathbb{C}^3)} \leq \| w \|_{H^1(\Omega; \mathbb{C}^3)} \leq C \left( \| \text{sym} \nabla w \|_{L^2(\Omega; \mathbb{C}^3 \times \mathbb{C}^3)} + \| w \|_{L^2(\Omega; \mathbb{C}^3)} \right) = C \left( \| \text{sym} \nabla \chi_1 u \|_{L^2(\Omega; \mathbb{C}^3 \times \mathbb{C}^3)} + \| u \|_{L^2(\Omega; \mathbb{C}^3)} \right),
\]

where we have also used (A.4). \( \square \)

**Proposition A.9.** There exists a constant \( C > 0 \) such that for all \( u \in H^1_{y_0}(Y; \mathbb{C}^3), \ \chi \in Y' \setminus \{0\} \) one has

\[
\| u \|_{H^1_{y_0}(Y; \mathbb{C}^3)} \leq \frac{C}{|\chi|} \| \text{sym} \nabla \chi_1 u \|_{L^2(Y; \mathbb{C}^3 \times \mathbb{C}^3)}, \quad (A.5)
\]

\[
\| \nabla u \|_{L^2(Y; \mathbb{C}^3 \times \mathbb{C}^3)} \leq C \| \text{sym} \nabla \chi_1 u \|_{L^2(Y; \mathbb{C}^3 \times \mathbb{C}^3)},
\]

\[
\| u - \int_Y u \|_{H^1_{y_0}(Y; \mathbb{C}^3)} \leq C \| \text{sym} \nabla \chi_1 u \|_{L^2(Y; \mathbb{C}^3 \times \mathbb{C}^3)} . \quad (A.6)
\]
Proof. For a function $u \in H^1_b(Y; \mathbb{C}^3)$ we have the Fourier series decomposition
\[
u u = \sum_{k \in \mathbb{Z}^3} e^{2\pi ik \cdot y} a_k \otimes (2\pi k),
\]
from which, by Parseval’s identity,
\[
\|u\|^2_{L^2(Y; \mathbb{C}^3)} = \sum_{k \in \mathbb{Z}^3} |a_k|^2, \quad \|\nabla u\|^2_{L^2(Y; \mathbb{C}^3)} = \sum_{k \in \mathbb{Z}^3} |2\pi|^2 |a_k \otimes k|^2.
\]
It follows that
\[
\|\nabla u + u \otimes i\chi\|^2_{L^2(Y; \mathbb{C}^3)} = \sum_{k \in \mathbb{Z}^3} |a_k \otimes (2\pi k + i\chi)|^2,
\]
and therefore
\[
\|(\text{sym } \nabla + iX_1)u\|^2_{L^2(Y; \mathbb{C}^3)} = \sum_{k \in \mathbb{Z}^3} |a_k \otimes (2\pi k + i\chi)|^2.
\]
Combining this with the inequality $|a \otimes b| \geq \frac{|a| |b|}{\sqrt{3}}$, we infer (A.5)–(A.6). \qed

**Proposition A.10.** There is an extension operator mapping $u \in H^1_b(Y_{\text{stiff}}; \mathbb{C}^3)$ to an element of $H^1_b(Y; \mathbb{C}^3)$ (for which we keep the same notation) such that
\[
\|u\|_{H^1(Y_{\text{stiff}}; \mathbb{C}^3)} \leq C \|u\|_{H^1(Y; \mathbb{C}^3)}, \quad \|(\text{sym } \nabla + iX_1)u\|_{L^2(Y_{\text{stiff}}; \mathbb{C}^3)} \leq C \|(\text{sym } \nabla + iX_1)u\|_{L^2(Y; \mathbb{C}^3)},
\]
for all $\chi \in Y$, where the constant $C$ depends only on $Y_{\text{stiff}}$.

**Proof.** This is a straightforward consequence of an analogous result for the extension operator applied to quasiperiodic functions. For the related construction, see [48]. \qed

**Proposition A.11.** There exists a constant $C > 0$ such that for every $u \in H^1_b(Y_{\text{stiff}}; \mathbb{C}^3)$, $\chi \in Y \setminus \{0\}$ we have the following estimate:
\[
\|u\|_{H^1(Y_{\text{stiff}}; \mathbb{C}^3)} \leq C |\chi|^{-1} \|(\text{sym } \nabla + iX_1)u\|_{L^2(Y_{\text{stiff}}; \mathbb{C}^3)},
\]
for all $\chi \in Y$, where the constant $C$ depends only on $Y_{\text{stiff}}$.

**Proof.** By Proposition A.10, the function $u$ is extended to $u \in H^1_b(Y; \mathbb{C}^3)$. Combining (A.5) and (A.7) yields
\[
\|u\|_{H^1(Y_{\text{stiff}}; \mathbb{C}^3)} \leq \|u\|_{H^1(Y; \mathbb{C}^3)} \leq C |\chi|^{-1} \|(\text{sym } \nabla + iX_1)u\|_{L^2(Y; \mathbb{C}^3)}.
\]
\qed

**Proposition A.12.** There exists a constant $C > 0$ such that for all $u \in H^1_b(Y_{\text{stiff}}; \mathbb{C}^3)$, $\chi \in Y'$ one has
\[
\left\|u - \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{H^1(Y_{\text{stiff}}; \mathbb{C}^3)} \leq C \left\| (\text{sym } \nabla + iX_1)u \right\|_{L^2(Y_{\text{stiff}}; \mathbb{C}^3)},
\]
and
\[
\left\|u - \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{H^1(\Gamma; \mathbb{C}^3)} \leq C \left\| (\text{sym } \nabla + iX_1)u \right\|_{L^2(\Gamma; \mathbb{C}^3)}.
\]

**Proof.** The bound (A.9) is deduced from (A.6) and Proposition A.10 similar to how Proposition A.11 was established. To prove (A.10), note first that by (A.9) and the continuity of traces, one has
\[
\left\|u - \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{L^2(Y; \mathbb{C}^3)} \leq C \left\| (\text{sym } \nabla + iX_1)u \right\|_{L^2(Y_{\text{stiff}}; \mathbb{C}^3)}.
\]
Second, from Proposition A.8 and the trace inequality, one has
\[
\left\|u - \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{H^1(\Gamma; \mathbb{C}^3)} \leq C \left( \left\| (\text{sym } \nabla + iX_1)u \right\|_{L^2(Y_{\text{stiff}}; \mathbb{C}^3)} + \left\| u - \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{L^2(Y; \mathbb{C}^3)} \right),
\]
and
\[
\left\|u - \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{H^1(\Gamma; \mathbb{C}^3)} \leq C \left( \left\| (\text{sym } \nabla + iX_1)u \right\|_{L^2(Y_{\text{stiff}}; \mathbb{C}^3)} + |\chi| \left\| \frac{1}{|\Gamma|} \int_{\Gamma} u \right\|_{L^2(Y; \mathbb{C}^3)} \right).
\]

The bound (A.10) now follows from (A.12) by using (A.11), (A.8), and the continuity of traces. \qed
A.3 Well-posedness and regularity of elasticity boundary value problems

Here we state some results regarding the properties of the weak solution to the boundary value problem

\[- \text{div}(A \, \text{sym} \nabla u) = f \quad \text{on } \Omega, \quad (A \, \text{sym} \nabla u)n_{|\partial \Omega} + u = g \quad \text{on } \partial \Omega, \tag{A.13}\]

where \(\Omega \subset \mathbb{R}^3\) is a bounded Lipschitz domain with the exterior normal \(n, f \in L^2(\Omega; \mathbb{C}^3), g \in H^{-1/2}(\partial \Omega; \mathbb{C}^3)\), and the tensor of material properties \(A \in L^2(\Omega; \mathbb{R}^{3 \times 3 \times 3 \times 3})\) satisfies the following assumptions (cf. Assumption 2.1).

Assumption A.13. • Uniform positive-definiteness and uniform boundedness on symmetric matrices: there exists \(\nu > 0\) such that \(\nu|\xi|^2 \leq A(x)|\xi|^2 \leq \nu^{-1}|\xi|^2\) for all \(\xi \in \mathbb{R}^{3 \times 3}, \xi^T = \xi, x \in \Omega\).  
• Material symmetries: \([A]_{ijkl} = [A]_{jikl} = [A]_{iljk}, i, j, k, l \in \{1, 2, 3\} \).

The weak form of this problem is stated in the following definition.

Definition A.14 (Robin boundary problem for elasticity). For given functions \(f \in H^{-1}(\Omega; \mathbb{C}^3), g \in H^{-1/2}(\partial \Omega; \mathbb{C}^3)\), find \(u \in H^1(\Omega; \mathbb{C}^3)\) such that

\[
\int_{\Omega} A \, \text{sym} \nabla u : \text{sym} \nabla v + \int_{\Omega} u \cdot v = \int_{\Omega} f \cdot v + \int_{\Omega} g \cdot \nu \quad \forall v \in H^1(\Omega; \mathbb{C}^3). 
\]

Notice that the map \(v \mapsto \int_{\partial \Omega} g \cdot \nu \) is a bounded linear functional on \(H^1(\Omega; \mathbb{C}^3)\). Also, the form

\[(u, v) \mapsto \int_{\Omega} A \, \text{sym} \nabla u : \text{sym} \nabla v + \int_{\partial \Omega} u \cdot \nu,\]

is coercive on \(H^1(\Omega; \mathbb{C}^3)\) by Proposition A.5. Thus by the Lax-Milgram lemma, there exists a unique weak solution \(u \in H^1(\Omega; \mathbb{C}^3)\) of (A.13) in the sense of Definition A.14, and

\[\|u\|_{H^1(\Omega; \mathbb{C}^3)} \leq C(\|f\|_{H^{-1}(\Omega; \mathbb{C}^3)} + \|g\|_{H^{-1/2}(\partial \Omega; \mathbb{C}^3)}).\]

The following two lemmata can be found in, e.g., [42].

Lemma A.15 (Regularity of the solution of Robin problem). \(Let \Omega \subset \mathbb{R}^3\) be a bounded domain with boundary \(\partial \Omega\) of class \(C^{1,1}\). Let \(f \in L^2(\Omega; \mathbb{C}^3), g \in H^{1/2}(\partial \Omega; \mathbb{C}^3)\) and, in addition to Assumption A.13, \(A \in C^{0,1}(\Omega; \mathbb{R}^{3 \times 3 \times 3 \times 3})\). Then the unique weak solution to the problem (A.13) belongs to \(H^2(\Omega; \mathbb{C}^3)\), and

\[\|u\|_{H^2(\Omega; \mathbb{C}^3)} \leq C(\|f\|_{L^2(\Omega; \mathbb{C}^3)} + \|g\|_{H^{1/2}(\partial \Omega; \mathbb{C}^3)}).\]

We are also interested in the solution to the Dirichlet boundary value problem

\[
\begin{cases}
- \text{div}(A \, \text{sym} \nabla u) = f & \text{on } \Omega, \\
u \quad u = g & \text{on } \partial \Omega,
\end{cases}
\tag{A.14}
\]

where \(f \in H^{-1}(\Omega; \mathbb{C}^3), g \in H^{1/2}(\partial \Omega; \mathbb{C}^3)\). Applying the Lax-Milgram lemma, we infer that it has a unique weak solution \(u \in H^1(\Omega; \mathbb{C}^3)\).

Lemma A.16 (Regularity of the solution of Dirichlet problem). Suppose that \(\Omega \subset \mathbb{R}^3\) be a bounded domain with \(C^{1,1}\) boundary \(\partial \Omega\), \(f \in L^2(\Omega; \mathbb{C}^3), g \in H^{1/2}(\partial \Omega; \mathbb{C}^3)\), and \(A \in C^{0,1}(\Omega; \mathbb{R}^{3 \times 3 \times 3 \times 3})\). Then the solution to (A.14) belongs to \(H^2(\Omega; \mathbb{C}^3)\), and

\[\|u\|_{H^2(\Omega; \mathbb{C}^3)} \leq C(\|f\|_{L^2(\Omega; \mathbb{C}^3)} + \|g\|_{H^{1/2}(\partial \Omega; \mathbb{C}^3)}).\]

where the constant \(C\) depends only on \(\Omega\) and the \(C^{0,1}\)-norm of \(A\).

A.3.1 Trace extension lemma

A version of the following theorem can be found in [8].

Theorem A.17. Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain with \(C^{0,1}\) boundary \(\partial \Omega\). Let \(g_0 \in H^1(\partial \Omega; \mathbb{C}^3), g_1 \in L^2(\partial \Omega; \mathbb{C}^3)\). Then there exists \(u \in H^2(\Omega; \mathbb{C}^3)\) such that \(\partial_n u = g_1\) and \(u = g_0\) on \(\partial \Omega\) if and only if

\[\nabla\partial_\Omega g_0 + g_1 \otimes n \in H^{1/2}(\partial \Omega; \mathbb{C}^3),\]

where \(\nabla\partial_\Omega\) is the tangential gradient.
This leads us to the trace extension lemma.

**Lemma A.18.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial \Omega$ of class $C^{1,1}$ and $A \in C^{0,1}(\overline{\Omega} ; \mathbb{R}^{3 \times 3 \times 3})$. Let $g \in H^{1/2}(\partial \Omega; \mathbb{C}^3)$. Then there exists $u \in H^2(\Omega; \mathbb{C}^3)$ such that

$$
(A.\text{sym} \nabla u)n = g \quad \text{on } \partial \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
$$

(A.16)

**Proof.** The first step is to note that for $u \in H^2(\Omega; \mathbb{C}^3)$ such that $u|_\Gamma = 0$ one has

$$
\nabla u|_{\partial \Omega} = \partial_{n} u|_{\partial \Omega} \otimes n.
$$

(A.17)

Indeed, for an arbitrary point on $\partial \Omega$ consider an arbitrary vector $v = a_1 \tau_1 + a_2 \tau_2 + a_3 n$, where $\tau_1, \tau_2, n$ is an orthonormal basis of vectors $\tau_1, \tau_2$ tangential on $\partial \Omega$ and the vector $n$ is normal to $\partial \Omega$. Then

$$
\nabla u|_{\partial \Omega} \cdot v = a_1 \partial_{\tau_1} u|_{\partial \Omega} + a_2 \partial_{\tau_2} u|_{\partial \Omega} + a_3 \partial_n u|_{\partial \Omega} = a_3 \partial_n u|_{\partial \Omega},
$$

due to the fact that $\partial_{\tau_1} u|_{\partial \Omega} = \partial_{\tau_2} u|_{\partial \Omega} = 0|_{\partial \Omega}$. But, since $a = v \cdot n$, one has

$$
\nabla u|_{\partial \Omega} v = (v \cdot n) \partial_{n} u|_{\partial \Omega} = \left( (\partial_{n} u|_{\partial \Omega} \otimes n) \right) v,
$$

and (A.17) follows. With this in hand, the first equation in (A.16) becomes $(A \partial_n u|_{\partial \Omega} \otimes n)n = g$. We proceed by introducing the new variable $\omega := \partial_{n} u|_{\partial \Omega}$ and consider the problem of finding $\omega$ such that

$$
(A.\omega \otimes n)n = g.
$$

(A.18)

The second step of the proof consists in showing that there exists a unique solution $\omega \in H^{1/2}(\partial \Omega; \mathbb{C}^3)$ to (A.18). Note that this is an algebraic equation for every $x \in \partial \Omega$. For a fixed point $x \in \partial \Omega$, we introduce an operator $L_x : \mathbb{C}^3 \to \mathbb{C}^3$ as follows: $L_x \omega := (A(\omega \otimes n(x)))n(x)$. It is symmetric and positive definite:

$$
\langle L_x \omega, w \rangle_{\mathbb{C}^3} = \langle (A(\omega \otimes n(x)))n(x), w \rangle_{\mathbb{C}^3} = A(\omega \otimes n(x)) : (w \otimes n(x)) = \langle L_x w, \omega \rangle_{\mathbb{C}^3},
$$

$$
\langle L_x \omega, \omega \rangle_{\mathbb{C}^3} = A(\omega \otimes n(x)) : (\omega \otimes n(x)) \geq \frac{1}{2} |\omega|^2 |n(x)|^2 = \frac{1}{2} |\omega|^2,
$$

(A.19)

where we have used Assumption A.13. Thus, the problem $L_x \omega(x) = g(x)$ has a unique solution for a.e. $x \in \partial \Omega$. It follows from (A.19) that det $(L_x) \geq 0$, uniformly in $x \in \partial \Omega$. By $C^{0,1}$ regularity of both the material coefficients and the normal $n$, we know that both det $(L_x)$ and $(\text{det}(L_x))^{-1}$ are of class $C^0$. Then, by virtue of Cramer’s rule, the function $\omega(x) := (L_x)^{-1} g(x)$ belongs to $H^{1/2}(\partial \Omega; \mathbb{C}^3)$. Thus, we have reduced the problem (A.16) to finding $u \in H^2(\Omega; \mathbb{C}^3)$ such that $\partial_{n} u = \omega$ on $\partial \Omega$, $u = 0$ on $\partial \Omega$, where $\omega$ is the solution of (A.18). To complete the proof, it remains to check the validity of the condition (A.15) and apply Theorem A.17, which is possible as $\omega \otimes n \in H^{1/2}(\partial \Omega; \mathbb{C}^3)$ due to the $C^{0,1}(\partial \Omega; \mathbb{R}^3)$ regularity of $n$. 

**A.4 Auxiliary proofs**

**Proof of Theorem 4.3.** The definition of $\Pi^\text{diff}_x$ can be restated as follows: $\Pi^\text{diff}_x g := G + u$, where $u \in H^1_y(Y^\text{diff}; \mathbb{C}^3)$, $u|\Gamma = 0$ is such that

$$
a_u^{\text{diff}_y}(u, v) = -a_u^{\text{diff}_y}(G, v) \quad \forall v \in H^1_y(Y^\text{diff}; \mathbb{C}^3), \quad v|\Gamma = 0,
$$

and $G \in H^1_y(Y^\text{diff}; \mathbb{C}^3)$ is an extension of $g \in H^{1/2}(\Gamma; \mathbb{C}^3)$ satisfying the bound $||G||_{H^{1/2}(\Gamma; \mathbb{C}^3)} \leq C ||g||_{H^{1/2}(\Gamma; \mathbb{C}^3)}$, see Proposition A.3. Recall that $a_u^{\text{diff}} : H^1_y(Y^\text{diff}; \mathbb{C}^3) \times H^1_y(Y^\text{diff}; \mathbb{C}^3) \to \mathbb{R}$ is the sesquilinear form associated with the operator $\mathcal{J}^{\text{diff}_x}$. We aim to find an expansion of the solution $u \in H^2_y(Y^\text{diff}; \mathbb{C}^3), u|\Gamma = 0$, to

$$
\int_{Y^\text{diff}} A^{\text{diff}} \left( \text{sym} \nabla + iX_y \right) u(y) : (\text{sym} \nabla + iX_y) v(y) dy
$$

$$
\quad = \int_{Y^\text{diff}} A^{\text{diff}} \left( \text{sym} \nabla + iX_y \right) G(y) : (\text{sym} \nabla + iX_y) v(y) dy
$$

$$
\quad \forall v \in H^2_y(Y^\text{diff}; \mathbb{C}^3), \quad v|\Gamma = 0.
$$

(A.20)

in the form

$$
u = u_0 + \sum_{n=1}^\infty u_n.
$$

(A.21)
where $u_n|_{\Gamma} = 0$ for all $n$, the $H^1$ norm of $u_n$ is of order $|\chi|^n$, while the error of approximation

$$u_{\text{error, n}} := u - u_0 - \sum_{k=1}^{n} u_k$$

is of order $|\chi|^{n+1}$. The leading-order term $u_0$ corresponds to the harmonic lift in the case $\chi = 0$. With this apriori intention, we plug the expansion (A.21) into (A.20) and equate the terms which would be of order $O(1)$ to obtain

$$\int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla u_0(y) : \text{sym} \nabla v(y) dy = -\int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla G(y) : \text{sym} \nabla v(y) dy \quad \forall v \in H^1(\Gamma_{\text{stiff}}; \mathbb{C}^3), \quad v|_{\Gamma} = 0.$$

We recognise here the definition of the order-zero lift operator $\Pi_0^{\text{stiff}}$, namely $\Pi_0^{\text{stiff}} g := u_0 + G =: \tilde{u}$. It is clear that

$$\|u_0\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} \leq C \|G\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} \leq C \|g\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{C}^3)}.$$

We proceed to define $u_1$ as the solutions to

$$\int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla u_1(y) : \text{sym} \nabla v(y) dy = -\int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 \tilde{u}(y) : \text{sym} \nabla v(y) dy = \int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla \tilde{u}(y) : iX_1 v(y) dy \quad \forall v \in H^1(\Gamma_{\text{stiff}}; \mathbb{C}^3), \quad v|_{\Gamma} = 0. \quad (A.22)$$

Clearly, one has $\|u_1\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} \leq C |\chi| \|g\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{C}^3)}$. The next-order term of the expansion is defined by

$$\int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla u_2(y) : \text{sym} \nabla v(y) dy = -\int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 \tilde{u}(y) : iX_1 v(y) dy - \int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 u_1(y) : \text{sym} \nabla v(y) dy$$

$$= -\int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla u_1(y) : iX_1 v(y) dy \quad \forall v \in H^1(\Gamma_{\text{stiff}}; \mathbb{C}^3), \quad v|_{\Gamma} = 0.$$

We see that $\|u_2\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} \leq C |\chi|^2 \|g\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{C}^3)}$. Continuing this process by induction for $n \geq 3$, we define $u_n$ by the recurrence relation

$$\int_{Y_{\text{ext}}} A_{\text{stiff}} \text{sym} \nabla u_n(y) : \text{sym} \nabla v(y) dy = -\int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 u_{n-2}(y) : iX_1 v(y) dy$$

$$-\int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 u_{n-1}(y) : \text{sym} \nabla v(y) dy \quad \forall v \in H^1(\Gamma_{\text{stiff}}; \mathbb{C}^3), \quad v|_{\Gamma} = 0,$$

so that $\|u_n\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} \leq C |\chi|^n \|g\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{C}^3)}$ for all $n \in \mathbb{N}$. The error term

$$u_{\text{error, n}} := u - \sum_{k=0}^{n} u_k, \quad n \in \mathbb{N},$$

satisfies

$$\int_{Y_{\text{ext}}} A_{\text{stiff}} (\text{sym} \nabla + iX_1) u_{\text{error, n}}(y) : (\text{sym} \nabla + iX_1) v(y) dy$$

$$= -\int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 u_n(y) : iX_1 v(y) dy - \int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 u_{n-1}(y) : iX_1 v(y) dy$$

$$-\int_{Y_{\text{ext}}} A_{\text{stiff}} iX_1 u_1(y) : \text{sym} \nabla v(y) dy \quad \forall v \in H^1(\Gamma_{\text{stiff}}; \mathbb{C}^3), \quad v|_{\Gamma} = 0.$$

It follows that $\|u_{\text{error, n}}\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} \leq C |\chi|^{n+1} \|g\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{C}^3)}$. \hfill \Box

**Proof of Lemma 5.1.** The proof is by contradiction. Assume that for all $n \in \mathbb{N}$ there exist $\chi_n \in Y'$ and $f_n \in H_{\text{stiff}}^{\text{soft}}(Y_{\text{ext}})$ such that $\|f_n\|_{H^1(Y_{\text{ext}}; \mathbb{C}^3)} = 1$ and $\lim_{n \to \infty} \|f_n\|_{L^2(Y_{\text{ext}}; \mathbb{C}^3)} = 0$. From this, it is clear that

$$f_n \xrightarrow{H^1(Y_{\text{ext}}; \mathbb{C}^3)} 0. \quad (A.23)$$
We proceed by defining $g_n := f_n|_{E_n}$, i.e., $f_n = \Pi_{E_n}^{\text{stiff}} g_n$. Since $\|f_n\|_{H^1(Y_{\text{surf}(\Omega)}; \mathbb{C}^3)} = 1$, due to the continuity of the trace operator, we conclude that the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $H^{1/2}(\Gamma; \mathbb{C}^3)$ and there exists $g \in \mathcal{E}$ such that (on a subsequence) $g_n \xrightarrow{H^{1/2}(\Gamma; \mathbb{C}^3)} g$. Furthermore, due to Proposition 3.18, the sequence $(\|g_n\|_{H^{1/2}(\Gamma; \mathbb{C}^3)})_{n \in \mathbb{N}}$ is bounded below by a $\chi$-independent $C > 0$:

$$\|g_n\|_{H^{1/2}(\Gamma; \mathbb{C}^3)} \geq C.$$  \hspace{1cm} (A.24)

However, due to the compactness of the trace operator acting from $H^1(Y_{\text{stiff}}; \mathbb{C}^3)$ to $L^2(\Gamma; \mathbb{C}^3)$, from (A.23) we infer that $g_n \xrightarrow{L^2(\Gamma; \mathbb{C}^3)} 0$, hence $g = 0$. Next, we note that, on the one hand,

$$\langle \Lambda_{\chi, a}^{\text{eff}} g_n, g_n \rangle_{L^2(\Gamma; \mathbb{C}^3)} + \langle g_n^*, g_n \rangle_{L^2(\Gamma; \mathbb{C}^3)} \leq (C|\chi|^2 + 1) \|g_n\|^2_{L^2(\Gamma; \mathbb{C}^3)},$$

where we use the fact that $g_n \in \hat{E}_{\chi, a}$. On the other hand, one has

$$\langle \Lambda_{\chi, a}^{\text{eff}} g_n, g_n \rangle_{L^2(\Gamma; \mathbb{C}^3)} + \langle g_n^*, g_n \rangle_{L^2(\Gamma; \mathbb{C}^3)} = \Lambda_{\chi, a}^{\text{eff}}(g_n, g_n) + \|g_n\|^2_{L^2(\Gamma; \mathbb{C}^3)} \geq C \left( \|\text{sym } \nabla + i\chi \nabla\Pi_{E_n}^{\text{stiff}} g_n\|^2_{L^2(Y_{\text{surf}}; \mathbb{C}^3)} + \|g_n\|^2_{L^2(\Gamma; \mathbb{C}^3)} \right) \geq C \left( \|\Pi_{E_n}^{\text{stiff}} g_n\|^2_{L^2(Y_{\text{surf}}; \mathbb{C}^3)} \right) \geq C \|g_n\|^2_{H^{1/2}(\Gamma; \mathbb{C}^3)},$$

where we use the coercivity of the sesquilinear form (43), Proposition (A.8), and the continuity of the trace operator. It follows that

$$\|g_n\|_{H^{1/2}(\Gamma; \mathbb{C}^3)} \leq C \|g_n\|_{L^2(\Gamma; \mathbb{C}^3)} \rightarrow 0,$$

which contradicts (A.24).

\hspace{1cm} \square

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References

[1] M. S. Agranovich, 2006. On a mixed Poincaré–Steklov type spectral problem in a Lipschitz domain. Russian J. Math. Phys. 13(3), 239-244.

[2] G. Allaire, 1992. Homogenization and two-scale convergence, SIAM J. Math. Anal. 23, 1482–1518.

[3] T. Arbogast, J. Douglas, U. Hornung, 1990. Derivation of the double porosity model of single phase flow via homogenisation theory. SIAM J. Math. Anal. 21(4), 823–836.

[4] M. Sh. Birman, T. A. Suslina, 2004. Second order periodic differential operators. Threshold properties and homogenisation. St. Petersburg Math. J. 15(5), 639–714.

[5] M. Sh. Birman, T. A. Suslina, 2006. Averaging of periodic elliptic differential operators taking a corrector into account. St. Petersburg Math. J. 17(6), 897–973.

[6] M. Sh. Birman, M. Z. Solomjak. Spectral Theory of Selfadjoint Operators in Hilbert Space. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[7] L. Brillouin, 1953. Wave Propagation in Periodic Structures. Dover Publications, New York.

[8] A. Buffa, G. Geymonat, 2001. On traces of functions in $W^{2,p}(\Omega)$ for Lipschitz domains in $\mathbb{R}^3$. C. R. Acad. Sci. Paris Sér. I Math. 332(8), 699–704.
[9] M. Bužančić, K Cherednichenko, I. Velčić, J. Žubrinić, 2022. Spectral and evolution analysis of composite elastic plates with high contrast. *J. Elast.* 152, 79–177.

[10] F. Capolino, 2009. *Theory and Phenomena of Metamaterials.* Taylor & Francis.

[11] K. Cherednichenko, S. Cooper, 2015. Homogenisation of the system of high-contrast Maxwell equations. *Mathematika* 61(2), 475–500.

[12] K. Cherednichenko, S. Cooper, 2016. Resolvent estimates for high-contrast homogenisation problems. *Arch. Rational Mech. Anal.* 219(3), 1061–1086.

[13] K. Cherednichenko, S. Cooper, 2018. Asymptotic behaviour of the spectra of systems of Maxwell equations in periodic composite media with high contrast. *Mathematika* 64(2), 583–605.

[14] K. Cherednichenko, S. D’Onofrio, 2023. Order-sharp norm-resolvent homogenisation estimates for Maxwell equations on periodic singular structures: the case of non-zero current and the general system. *arXiv: 2007.04836*, 25 pp.

[15] K. D. Cherednichenko, Yu. Yu. Ershova, A. V. Kiselev, 2019. Time-dispersive behaviour as a feature of critical contrast media, *SIAM J. Appl. Math.* 79(2), 690–715.

[16] K. D. Cherednichenko, Yu. Yu. Ershova, A. V. Kiselev, 2020. Effective behaviour of critical-contrast PDEs: micro-resonances, frequency conversion, and time dispersive properties. I. *Comm. Math. Phys.* 375, 1833–1884.

[17] K. D. Cherednichenko, Yu. Yu. Ershova, A. V. Kiselev, 2023. Norm-resolvent convergence for Neumann Laplacians on manifolds thinning to graphs. *arXiv: 2205.04397*, 24 pp.

[18] K. D. Cherednichenko, Yu. Yu. Ershova, A. V. Kiselev, S. N. Naboko, 2021. Functional model for generalised resolvents and its application to time-dispersive media. *Comm. Math. Phys.* 349(2), 441–480.

[19] K. D. Cherednichenko, Yu. Yu. Ershova, S. N. Naboko, 2022. Operator-norm resolvent asymptotic analysis of continuous media with low-index inclusions. *Math. Notes* 111(3–4), 373–387.

[20] K. Cherednichenko, I. Velčić, 2022. Sharp operator-norm asymptotics for linearised elastic plates with rapidly oscillating periodic properties. *J. London Math. Soc.* 105(3), 1634–1680.

[21] K. Cherednichenko, I. Velčić, J. Žubrinić, 2023. Operator-norm resolvent estimates for thin elastically periodically heterogeneous rods in moderate contrast. *Calc. Var. Partial Differential Equations* 62(5), Paper No. 147, 72 pp.

[22] S. Cooper, I. Kamotski, V. P. Smyshlyaev, 2023. Uniform asymptotics for a family of degenerating variational problems and multiscale approximations with error estimates. *arXiv: 2307.13151*, 69 pp.

[23] V. Derkach, 2015. Boundary triples, Weyl functions, and the Kre˘ın formula, in *Operator Theory: Living Reference Work* (Springer, Basel). https://doi.org/10.1007/978-3-0348-0692-3_32-1

[24] A. Figotin, J. H. Schenker, 2005. Spectral analysis of time dispersive and dissipative systems, *J. Stat. Phys.* 118(1–2), 199–263.

[25] A. Figotin, J. H. Schenker, 2007. Hamiltonian structure for dispersive and dissipative dynamical systems. *J. Stat. Phys.* 128(4), 969–1056.
[31] L. Friedlander, 2002. On the density of states of periodic media in the large coupling limit. *Comm. Partial Differential Equations* 27(1–2), 355–380.

[32] G. Griso, 2004. Error estimate and unfolding for periodic homogenization. *Asymptot. Anal.* 40(3–4), 269–286.

[33] G. Grubb, 1984. Singular Green operators and their spectral asymptotics. *Duke Math. J.* 51(3), 477–528.

[34] T. Kato, 1995. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin.

[35] C. E. Kenig, F. Lin, Z. Shen, 2012. Convergence rates in $L^2$ for elliptic homogenization problems. *Arch. Rational Mech. Anal.* 203(3), 1009–1036.

[36] S. G. Krein, Ju. I. Petunin, E. M. Semenov, 1982. *Interpolation of Linear Operators*. Transl. Math. Monogr. 54, American Mathematical Society, Providence, RI.

[37] P. Kuchment, 1993. *Floquet Theory for Partial Differential Equations*. Birkhäuser.

[38] P. Kuchment, H. Zeng, 2001. Convergence of spectra of mesoscopic systems collapsing onto a graph. *J. Math. Anal. Appl.* 258(2), 671–700.

[39] P. Kuchment, H. Zeng, 2004. Asymptotics of spectra of Neumann Laplacians in thin domains. *Contemporary Mathematics* 327, Amer. Math. Soc., Providence, RI, 199–213.

[40] P. D. Lax, R. S. Phillips, 1967. *Scattering Theory*. Pure and Applied Mathematics, Vol. 26. Academic Press, New York-London.

[41] Y.-S. Lim, 2023. A high-contrast composite with annular inclusions: Norm-resolvent asymptotics. *arXiv: 2212.10206*, 54 pp.

[42] W. McLean, 2000. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge.

[43] G. W. Milton, A. V. Cherkaev, 1995. Which elasticity tensors are realizable? *J. Eng. Mater. Technol.* 117(4), 483–493.

[44] P. W. Milonni, 2005. *Fast Light, Slow Light and Left-Handed Light*. Taylor & Francis, New York.

[45] D. Mitrea, 2018. *Distributions, Partial Differential Equations, and Harmonic Analysis*. Springer, New York.

[46] M. Neumark, 1940. Spectral functions of a symmetric operator. (Russian) *Bull. Acad. Sci. URSS. Ser. Math.* 4, 277–318.

[47] M. Neumark, 1943 Positive definite operator functions on a commutative group. (Russian) *Bull. Acad. Sci. URSS Ser. Math.* 7, 237–244.

[48] O. A. Olešnik, A. S. Shamaev, G. A. Yosifian, 1992. *Mathematical Problems in Elasticity and Homogenization*. North-Holland Publishing Co., Amsterdam.

[49] M. A. Pakhnin, T. A. Suslina, 2013. Operator error estimates for the homogenization of the elliptic Dirichlet problem in a bounded domain. *St. Petersburg Math. J.* 24(6), 949–976.

[50] O. Post, 2012. *Spectral Analysis on Graph-Like Spaces*, Lecture Notes in Mathematics 2039, Springer.

[51] V. Ryzhov, 2007. Functional model of a class of non-selfadjoint extensions of symmetric operators. In: Operator theory, analysis and mathematical physics. *Oper. Theory Adv. Appl.* 174, 117–158.

[52] V. Ryzhov, 2020. Linear operators and operator functions associated with spectral boundary value problems. In: Analysis as a Tool in Mathematical Physics. *Oper. Theory: Adv. Appl.* 276, 576–626.

[53] J. Selden, 2005. Periodic operators in high-contrast media and the integrated density of states function. *Comm. Partial Differential Equations* 30(7–9), 1021–1037.

[54] M. Schechter, 1960. A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa* 14(3), 207–36.

[55] V. P. Smyshlyaev, 2009. Propagation and localization of elastic waves in highly anisotropic periodic composites via two-scale homogenization. *Mech. Mater.* 41(4), 434–447.
[56] A. V. Štraus, 1954. Generalised resolvents of symmetric operators. (Russian) Izv. Akad. Nauk SSSR, Ser. Mat. 18, 51–86.

[57] A. V. Štrauss, 1970. Extensions and generalized resolvents of a non-densely defined symmetric operator. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 34, 175–202.

[58] A. V. Štraus, 1999. Functional models and generalized spectral functions of symmetric operators. St. Petersburg Math. J. 10(5), 733–784.

[59] T. A. Suslina, 2018. Spectral approach to homogenization of elliptic operators in a perforated space. Rev. Math. Phys. 30(8), 1840016, 57 pp.

[60] C. Tretter, 2008. Spectral Theory of Block Operator Matrices and Applications. World Scientific.

[61] H. Triebel, 1978. Interpolation Theory, Function Spaces, Differential Operators. North-Holland.

[62] V. G. Veselago, 1968. The electrodynamics of substances with simultaneously negative values of \( \varepsilon \) and \( \mu \). Soviet Phys. Uspekhi 10, 509–514.

[63] V. V. Zhikov, 1989. Spectral approach to asymptotic diffusion problems. Differential equations 25(1), 33-39.

[64] V. V. Zhikov, 2000. On an extension of the method of two-scale convergence and its applications, Sb. Math. 191(7), 973–1014.

[65] V. Zhikov, S. Pastukhova, 2005. On operator estimates for some problems in homogenization theory. Russian J. Math. Phys. 12(4), 515–524.

[66] V. V. Zhikov, S. E. Pastukhova, 2013. On gaps in the spectrum of the operator of elasticity theory on a high contrast periodic structure. J. Math. Sci. 188(3), 227–240.