Subregion Action and Complexity

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Abstract

We evaluate finite part of the on-shell action for black brane solutions of Einstein gravity on different subregions of spacetime enclosed by null boundaries. These subregions include the intersection of WDW patch with past/future interior and left/right exterior for a two sided black brane. Identifying the on-shell action on the exterior regions with subregion complexity one finds that it obeys subadditivity condition. This gives an insight to define a new quantity named mutual complexity. We will also consider certain subregion that is a part of spacetime which could be causally connected to an operator localized behind/outside the horizon. Taking into account all terms needed to have a diffeomorphism invariant action with a well-defined variational principle, one observes that the main contribution that results to a nontrivial behavior of the on-shell action comes from joint points where two lightlike boundaries (including horizon) intersect. A spacelike boundary gives rise to a linear time growth, while we have a classical contribution due to a timelike boundary that is given by the free energy.
1 Introduction

Based on earlier works of [1, 2] it was conjectured that computational complexity associated with a boundary state may be identified with the on-shell action evaluated on a certain subregion of the bulk spacetime [3, 4]. The corresponding subregion is Wheeler-DeWitt (WDW) patch of the spacetime that is the domain of dependence of any Cauchy surface in the bulk whose intersection with the asymptotic boundary is the time slice on which the state is defined.

This proposal, known as “complexity equals action” (CA), has been used to explore several properties of computational complexity for those field theories that have gravitational dual 1. In particular the growth rate of complexity has been studied for an eternal black hole in [16]. It was shown that although in the late time the growth rate approaches a constant value that is twice of the mass of the black hole, the constant is approached from above, violating the Lloyd’s bound [17]. Of course this is not the case for a state followed by a global quench [18]. It is worth to mention that recently there has been some progress for studying the computational complexity of a state in field theory [19–28].

So far the main concern in the literature was the growth rate of complexity and therefore the on-shell action was computed up to time independent terms [29–31]. Moreover it was also shown that the time dependent effects are controlled by the regions behind the horizon. We note, however, that in order to understand holographic complexity better it is crucial to have the full expression of it. It is also important to evaluate the contribution of different parts (inside and outside of the

1We would like to stress that on the gravity side there is another proposal for computing the computational complexity, known as “complexity equals volume” (CV) [1, 2]. The generalization of CV proposal to subsystems has been done in [5] (see also [6–11]). Yet another approach to complexity based on Euclidean path-integral has been introduced in [12–14]. For a recent development and its possible relation with CA approach see [15].
horizon) of the WDW patch, specialty. It is also illustrative to compute on-shell action on a given subregion of spacetime enclosed by null boundaries, that is not necessarily the WDW patch. Indeed this is one of the aim of the present work to carry out these computations explicitly.

Since we are interested in the on-shell action, it is crucial to make clear what one means by “on-shell action”. In general an action could have several terms that might be important due to certain physical reason. In particular in order to have a well-defined variational principle with Dirichlet boundary condition one needs to add certain Gibbons-Hawking-York boundary terms at spacelike and timelike boundaries [32,33]. Moreover to accommodate null boundaries it is also crucial to add the corresponding boundary terms on the null boundaries as well as certain joint action at points where a null boundary intersects to other boundaries [34,35].

Restricted to Einstein gravity and assuming to have a well-defined variational principle one arrives at the following action [35]

\[
I^{(0)} = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g} (R - 2\Lambda) + \frac{1}{8\pi G_N} \int_{\Sigma^d} K_t \, d\Sigma_t \\
\pm \frac{1}{8\pi G_N} \int_{\Sigma^d} K_s \, d\Sigma_s \pm \frac{1}{8\pi G_N} \int_{\Sigma^d} K_n \, dS_d\lambda \pm \frac{1}{8\pi G_N} \int_{J^d} a \, dS. \tag{1.1}
\]

Here the timelike, spacelike, and null boundaries and also joint points are denoted by \(\Sigma^d_t, \Sigma^d_s, \Sigma^d_n\) and \(J^d\), respectively. The extrinsic curvature of the corresponding boundaries are given by \(K_t, K_s\) and \(K_n\). The function \(a\) at the intersection of the boundaries is given by the logarithm of the inner product of the corresponding normal vectors and \(\lambda\) is the null coordinate defined on the null segments. The sign of different terms depends on the relative position of the boundaries and the bulk region of interest (see [35] for more details).

As far as the variational principle is concerned the above action defines a consistent theory. Nonetheless one still has possibilities to add certain boundary terms that do not alter the boundary condition, but could have nontrivial contribution to the on-shell action. Therefore it is important to fix these terms using certain physical principles before computing the on-shell action.

In particular one can see that the above action is not invariant under a reparametrization of the null generators. Therefore one may conclude that the above action does not really define a consistent theory. Actually to maintain the invariance under a reparametrization of the null generators one needs to add an extra term to the action as follows [35]

\[
I^{\text{amb}} = \frac{1}{8\pi G_N} \int_{\Sigma^d} d^dxd\lambda \sqrt{\gamma} \Theta \log \frac{|\tilde{L}\Theta|}{d}, \tag{1.2}
\]

where \(\tilde{L}\) is an undetermined length scale and \(\gamma\) is the determinant of the induced metric on the joint point where two null segments intersect, and

\[
\Theta = \frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial \lambda}. \tag{1.3}
\]
Although even with this extra term the length scale $\tilde{L}$ remains undetermined, adding this term to action (1.1) would define a consistent theory. Therefore in what follows by evaluating “on-shell action” we mean to consider $I = I^{(0)} + I^{\text{amb}}$. We note, however, that the resultant on-shell action may or may not be UV finite. Thus, one may want to get finite on-shell action (as we do for gravitational free energy) that requires to add certain counterterms. Actually these terms are also required from holographic renormalization (see e.g. [36]). Of course in this paper we will not consider such counterterms and those needed due to null boundaries [37].

The aim of this article is to compute on-shell action on certain subregions behind and outside the horizon enclosed by null boundaries. We will consider an eternal black brane that provides a gravitational dual for a thermofield double state. Those subregions that are behind the horizon are UV finite and time dependent, though those outside the horizon are typically UV divergent and time independent.

To proceed we will consider a $(d+2)$-dimensional black brane solution in Einstein gravity whose metric is

$$ds^2 = \frac{L^2}{r^2} \left( -f(r) dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d} dx_i^2 \right), \quad f(r) = 1 - \left( \frac{r}{r_h} \right)^{d+1},$$

where $r_h$ is the radius of horizon and $L$ denotes the AdS radius. In terms of these parameters the entropy, mass and Hawking temperature of the corresponding black brane are

$$S_{th} = \frac{V_d L^d}{4G_N r_h^d}, \quad M = \frac{V_d L^d}{16\pi G_N} \frac{d}{r_h^{d+1}}, \quad T = \frac{d+1}{4\pi r_h},$$

with $V_d$ being the volume of $d$-dimensional internal space of the metric parametrized by $x_i$, $i = 1, \cdots d$. It is also useful to note that

$$\sqrt{-g} (R - 2\Lambda) = -2(d+1) \frac{L^d}{r_h^{d+2}}.$$  

The organization of the paper is as follows. In the next section we will consider on-shell action on the WDW patch that using CA proposal may be related to the holographic complexity of the dual state. Our main concern is to present a closed form for the on-shell action. We will also compute on-shell action for past patch that is obtained by continuing the past null boundaries all the way to the past singularity. We will also compute on-shell action on the intersection of WDW patch with past and future interiors. We study the time evolution of holographic uncomplexity too. In section three we will consider different patches that are outside the horizon. This includes the intersection of WDW patch with entanglement wedge that could thought of as CA subregion complexity. The last section is devoted to discussion and conclusion where we present the interpretation of our results.

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2Due to flat boundary of the black brane solution, we will be able to present our results in simple compact forms.
Figure 1: Penrose diagram of the WDW patch of an eternal AdS black hole assuming $t_R = t_L$. Left: WDW patch on which the on-shell action is computed to find the complexity. Right: Past patch corresponding to the WDW patch. The past patch may be identified as a part that is casually connected to an operator localized at $r = r_m$ behind the horizon.

2 Complexity and Subregions Behind the Horizon

2.1 CA Proposal

In this section using CA proposal we would like to evaluate complexity for the eternal two sided black brane which is dual to the thermofield double state in the boundary theory. Holographically one should compute on-shell action on WDW patch as depicted in the left panel of figure 1. Using the symmetry of the Penrose diagram of the eternal black hole, we shall consider a symmetric configuration with times $t_R = t_L = \frac{\tau}{2}$. Actually this question has been already addressed [16] where the full time dependence of complexity has been obtained where it was shown that the holographic complexity violates the Lloyd’s bound in this case\(^3\). Of course our main interest in the present paper is to study the finite part of the on-shell action. In this subsection we will present the results and computations rather in details. Due to the similarity of computations, in the rest of the paper the computations will be a little bit brief.

To proceed we note that the null boundaries of the corresponding WDW patch are (see left panel of figure 1)

\begin{align}
B_1 : & \quad t = t_R - r^*(\epsilon) + r^*(r), & B_2 : & \quad t = -t_L + r^*(\epsilon) - r^*(r), \\
B_3 : & \quad t = t_R + r^*(\epsilon) - r^*(r), & B_4 : & \quad t = -t_L - r^*(\epsilon) + r^*(r),
\end{align}

\(^3\)The one sided black hole was also discussed in [18,38–40] where it confirmed that in this case the Lloyd’s bound is respected.
by which the position of the joint point $m$ is given by\(^4\)

$$\tau \equiv t_L + t_R = 2(r^*(\epsilon) - r^*(r_m)). \quad (2.2)$$

Let us now compute the on-shell action over the corresponding WDW patch. As we already mentioned the action consists of several parts that include bulk, boundaries and joint actions. Using equation (1.6) the bulk action is\(^{[16]}\)

$$I_{\text{bulk WDW}} = -\frac{V_d L^d}{4\pi G_N} (d+1) \left( 2 \int_{\epsilon}^{r_{\text{Max}}} \frac{dr}{r^{d+2}} (r^*(\epsilon) - r^*(r)) + \int_{r_m}^{r_{\text{Max}}} \frac{dr}{r^{d+2}} \left( \frac{\tau}{2} - r^*(\epsilon) + r^*(r) \right) \right). \quad (2.3)$$

By making use of an integration by parts the above bulk action reads

$$I_{\text{bulk WDW}} = -\frac{V_d L^d}{4\pi G_N} \left( 2 \int_{\epsilon}^{r_{\text{Max}}} \frac{dr}{r^{d+1}f(r)} - \int_{r_m}^{r_{\text{Max}}} \frac{dr}{r^{d+1}f(r)} \right) \quad (2.4)$$

where $\tau_c = 2(r^*(\epsilon) - r^*(r_{\text{Max}}))$ is the critical time below which the time derivative of complexity vanishes. More explicitly one has

$$\tau_c = \frac{1}{2T} \frac{1}{\sin \frac{\pi}{d+1}}. \quad (2.5)$$

To find the boundary contributions we note that using the affine parametrization for the null directions, the corresponding boundary terms vanish\(^5\) and we are left with just a spacelike boundary at future singularity whose contribution is given by

$$I_{\text{surf WDW}} = -\frac{1}{8\pi G_N} \int d^d x \int_{-t_L}^{t_R} \left. \int_{-r^*(\epsilon)}^{r^*(r)} dt \sqrt{h} K_s \right|_{r=r_{\text{Max}}}, \quad (2.6)$$

where $K_s$ is the the trace of extrinsic curvature of the boundary at $r = r_{\text{Max}}$ and $h$ is the determinant of the induced metric. To compute this term it is useful to note that for a constant $r$ surface using the metric (1.4) one has

$$\sqrt{h} K = -\sqrt{g}^{rr} \partial_r \sqrt{h} = -\frac{1}{2} \frac{L^d}{r^d} \left( \partial_r f(r) - \frac{2(d+1)}{r} f(r) \right). \quad (2.7)$$

Plugging the above expression into (2.6) and evaluating it at $r = r_{\text{Max}}$ one finds

$$I_{\text{surf WDW}} = \frac{V_d L^d}{8\pi G_N} (d+1) \frac{\tau + \tau_c}{2 r^d_{\text{Max}}} . \quad (2.8)$$

\(^4\)Note that in our notation one has $r^*(r) \leq 0$.

\(^5\)For affine parametrization of the null direction, the extrinsic curvature of the null boundary will be zero and therefore there is no contribution from null boundaries. In this paper we always use this parametrization and therefore we do not need to consider the boundary terms for null boundaries.
There are also several joint points which may contribute to the on-shell action. Two of them are located at the future singularity that have zero contributions, while the contributions of the three remaining points at $r = \epsilon$ and $r = r_m$ are given by

$$I_{\text{joint}}^{WDW} = 2 \times \frac{-1}{8\pi G_N} \int_\epsilon^d x \sqrt{\gamma} \log \frac{|k_1 \cdot k_2|}{2} + \frac{1}{8\pi G_N} \int_{r_m}^d x \sqrt{\gamma} \log \frac{|k_1 \cdot k_2|}{2},$$  \hspace{1cm} (2.9)

where the factor of 2 is due to the two joint points at $r = \epsilon$ for left and right boundaries. Here $k_1$ and $k_2$ are the null vectors associated with the null boundaries

$$k_1 = \alpha \left( -dt + \frac{dr}{f(r)} \right), \quad k_2 = \beta \left( dt + \frac{dr}{f(r)} \right).$$  \hspace{1cm} (2.10)

Here $\alpha$ and $\beta$ are two constants appearing due to the ambiguity of the normalization of normal vectors of null segments. Therefore one gets

$$I_{\text{joint}}^{WDW} = -\frac{V_d L^d}{4\pi G_N} \frac{\log \alpha \beta}{L^2} \frac{L^2}{r_m^d} + \frac{V_d L^d}{8\pi G_N} \left( \frac{\log \alpha \beta}{r_m^d} \frac{L^2}{L^2} - \frac{\log |f(r_m)|}{L^2} \right).$$  \hspace{1cm} (2.11)

It is clear from the above expression that the result suffers from an ambiguity associated with the normalization of null vectors. This ambiguity may be fixed either by fixing the constants $\alpha$ and $\beta$ by hand or adding a proper term to the action. Actually as we have already mentioned in order to maintain the diffeomorphism invariance of the action we will have to add another term given by equation (1.2). Note that even with this term we are still left with an undetermined free parameter.

In the present case taking into account all four null boundaries one gets

$$I_{\text{amb}}^{WDW} = -\frac{V_d L^d}{8\pi G_N} \left( \frac{\log \alpha \beta L^2 r_m^2}{r_m^d} + \frac{2}{d r_m^d} \right) + \frac{V_d L^d}{4\pi G_N} \left( \frac{\log \alpha \beta L^2 \epsilon^2}{L^2} + \frac{2}{d \epsilon^d} \right).$$  \hspace{1cm} (2.12)

Now we have all terms in the action evaluated on the WDW patch. Therefore one arrives at

$$I_{WDW} = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{joint}} + I_{\text{amb}}$$

$$= \frac{V_d L^d}{8\pi G_N} \left[ \frac{2}{d \epsilon^d} \log \frac{\tilde{L}^2}{L^2} + \frac{d - 1}{2r_m^{d+1}} (r + \tau_e) - \frac{\log \frac{\tilde{L}^2 |f(r_m)|}{L^2}}{r_m^d} \right].$$  \hspace{1cm} (2.13)

It is important to note that in order to have a meaningful result the divergent term should be positive that is the case for $\tilde{L} \geq L$. On the other hand setting $\tilde{L} = L$ the divergent term will drop and one gets a finite result consisting of two contributions\(^{6}\): one from the future spacelike

\(^{6}\) Note that for boundary associated with $k_1$ one has $\frac{d}{d\lambda} = \alpha \frac{L^2}{T^2}$ and $\Theta = 2d\alpha \frac{T}{T^2}$.

\(^{7}\) Actually in the context of holographic renormalization one would add certain counterterms to make on-shell action

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7 Actually in the context of holographic renormalization one would add certain counterterms to make on-shell action
singularity and a contribution from the joint point at \( r = r_m \) given as follows

\[
I_{\text{WDW}} = \frac{V_d L^d}{8\pi G_N} \left[ \frac{d - 1}{2r_m^{d+1}} \left( r + \tau_c \right) - \frac{\log|f(r_m)|}{r_m^d} \right].
\]

(2.14)

It is also interesting to note that for \( r_m \to r_{\text{Max}} \) where \( \tau \to \tau_c \) one gets

\[
I_{\text{WDW}} = \frac{V_d L^d}{8\pi G_N} \frac{d - 1}{r_{\text{h}}^{d+1}} \tau_c = \frac{d - 1}{d + 1} \frac{S_{\text{th}}}{\sin \frac{\pi}{d-1}},
\]

(2.15)

which is identically zero for \( d = 1 \). This might be thought of as complexity of formation of the black brane. On the other hand, using the fact that \( \log|f(r_m)| \approx -(d+1)\tau \) for \( r_m \to r_{\text{h}} \) (see next section), one gets linear growth at late times

\[
I_{\text{WDW}} \approx \frac{V_d L^d}{8\pi G_N} \frac{d}{r_{\text{h}}^{d+1}} \tau = 2M\tau,
\]

(2.16)

as expected.

### 2.2 Past Patch

In this subsection we would like to compute on-shell action on the past patch defined by the colored triangle shown in the right panel of figure 1. Clearly the rate of change of on-shell action on the past patch is the same as that of the WDW patch. Another way to think of the past patch is to consider an operator localized at \( r = r_m \) behind the horizon. The part of spacetime that can be causally connected to the operator is the triangle depicted in figure 1. Following CA proposal one may think of the on-shell action evaluated on the past patch as the complexity associated with the operator.

Let us compute the on-shell action for the past patch. To proceed we note that using the notation of the previous subsection the contribution of the bulk term to the on-shell action is

\[
r_{\text{past}}^{\text{bulk}} = -\frac{V_d L^d}{8\pi G_N} (d + 1) \int_{r_m}^{r_{\text{Max}}} \frac{dr}{r^{d+2}} \int_{-t_{L}+r^*(0)-r^*(r)}^{t_{R}+r^*(0)+r^*(r)} dt
\]

\[
= \frac{V_d L^d}{4\pi G_N} \int_{r_m}^{r_{\text{Max}}} \frac{dr}{r^{d+1}} \frac{\frac{1}{2r_m^{d+1}} + 1}{d \frac{r^d m}{d+1}}.
\]

(2.17)

Here to get the second line we have performed an integration by parts. On the other hand the finite. In the present case to remove the divergent term one may add a counterterm in the following form

\[
I^c = \frac{1}{8\pi G_N} \int d\lambda d^d x \sqrt{\gamma} \Theta \log \frac{L^2}{L^2},
\]

which is essentially equivalent to set \( \tilde{L} = L \) and then we are left with finite on-shell action. Of course in this paper we keep the length scale \( \tilde{L} \) undetermined.
contribution of the spacelike boundary at past singularity is found to be

\[ I_{\text{surf}}^{\text{past}} = \frac{1}{8\pi G_N} \int d^d x \int_{t_L + r^*(0) - r^*(r)}^{t_R - r^*(0) + r^*(r)} dt \sqrt{h} K \bigg|_{r=r_{\text{Max}}} = \frac{V_d L^d}{8\pi G_N} (d + 1) \frac{\tau - \tau_c}{2r_h^{d+1}}. \tag{2.18} \]

There are also three joint points two of which at \( r = r_{\text{Max}} \) and one at \( r = r_m \). The corresponding contributions to the on-shell action for those at \( r_{\text{Max}} \) vanish for large \( r_{\text{Max}} \), while the contribution of that at \( r = r_m \) is given by

\[ I_{\text{joint}}^{\text{past}} = \frac{1}{8\pi G_N} \int d^{d-1} x \sqrt{\gamma} \log \frac{|k_1 \cdot k_2|}{2} = \frac{V_d L^d}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta r^2_m}{L^2}}{r^d_m} - \frac{\log |f(r_m)|}{r^d_m} \right). \tag{2.19} \]

Finally the contribution of term needed to remove the ambiguity is

\[ I_{\text{amb}}^{\text{past}} = -\frac{V_d L^d}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta \tilde{L}^2 + m}{L^2}}{r^d_m} + \frac{2}{d} \right). \tag{2.20} \]

Therefore altogether one arrives at

\[ I^{\text{past}} = \frac{V_d L^d}{8\pi G_N} \left( d - 1 \right) \left( \frac{\tau}{2r_h^{d+1}} - \frac{\log \frac{\tilde{L}^2 |f(r_m)|}{L^2}}{r^d_m} \right), \tag{2.21} \]

which is UV finite even with arbitrary finite length scale \( \tilde{L} \). Note that for \( r_m \to r_{\text{Max}} \) where \( \tau \to \tau_c \) the on-shell action for past patch vanishes identically. On the other hand, in the late times where \( r_m \to r_h \) one finds linear growth as expected.

### 2.3 Intersection of WDW Patch with Past and Future Interiors

Even for a static geometry, such as eternal black hole, the interior of black hole grows with time indicating that there could be a quantity in the dual field theory that grows with time far after the system reaches the thermal equilibrium. Indeed this was the original motivation for holographic computational complexity to be identified with the volume of the black hole interior.

In the previous subsection we have computed the on-shell action over whole WDW patch. The aim of this subsection is to compute on-shell action in the intersection of WDW patch with black brane interior. This consists of past and future interiors as shown in figure 2. Actually these subregions are the main parts that contribute to the time dependence of complexity of the dual state. It is, however, instructive to study these parts separately.\(^8\)

\(^8\)On-shell action for subregion in the black hole interior has been also studied in [42] where it was argued that complexity may be used as a probe to study the nature of different singularities.  

8
2.3.1 Past Interior

To begin with we first consider the intersection of WDW patch with past interior as shown in the left panel of figure 2. Actually one may use the results of the previous subsection to write different terms contributing to the on-shell action. To start with we note that for the bulk term one has

$$I_{\text{bulk}}^{\Pi_{1}} = \frac{V_{d}L_{d}}{4\pi G_{N}}(d+1) \int_{r_{h}}^{r_{m}} \frac{dr}{r^{d+2}} \left( \frac{r^{*}(r)}{2} + r^{*}(r) \right) = \frac{V_{d}L_{d}}{4\pi G_{N}} \left( \frac{1}{d} \frac{dr_{m}}{dr} - \frac{1}{d} \frac{dr_{h}}{dr} \right).$$  (2.22)

There are four joint points, one at \(r = r_{m}\) and three at \(r = r_{h}\) that contribute to the on-shell action. It is, however, important to note that those points at the horizon are not at the same point. In other words the radial coordinate \(r\) is not suitable to make a distinction between these points. Indeed to distinguish between these points, following [43], it is convenient to use the following coordinate system for the past interior

$$u = -e^{-\frac{1}{2}f(r_{h})(r^{*}(r)-t)}, \quad v = -e^{-\frac{1}{2}f(r_{h})(r^{*}(r)+t)}. \quad (2.23)$$

In this coordinate system the horizon is located at \(uv = 0\) (i.e. \(r^{*}(r_{h}) = -\infty\)). This equation has three nontrivial solutions given by \((u = 0, v \neq 0)\), \((u \neq 0, v = 0)\) and \((u = 0, v = 0)\) that correspond to three joint points at the horizon shown in figure 2. Since both \(r^{*}(r)\) and \(\log f(r)\) are singular at \(r = r_{h}\), one may regularize the contribution of these three points by setting the horizon at \(v = \epsilon_{v}\) and \(u = \epsilon_{u}\). In this notation the joint points are given by \((\epsilon_{u}, v_{m})\), \((u_{m}, \epsilon_{v})\) and \((\epsilon_{u}, \epsilon_{v})\) as depicted in figure 2. In what follows the radial coordinate associated with these three points are denoted by...
the extra term \((1.2)\) to the action. The resulting expression is then

\[
\log h = \log |f(r_m)| - \log |f(r_v)| - \log |f(r_e)| + \log \left( \frac{\alpha \beta r^2}{L^2} \right) - \log \left( \frac{\alpha \beta r^2}{L^2} \right) = \log \left( \frac{\alpha \beta r^2}{L^2} \right) - \log \left( \frac{\alpha \beta r^2}{L^2} \right).
\]

Here we have used the fact that \(\{r_m, r_v, r_e\} \approx r_h\). On the other hand by making use of the fact that \([43]\)

\[
\log |f(r_{u,v})| = \log |uv| + c_0 + O(uv) \quad \text{for} \quad uv \to 0,
\]

one gets

\[
\begin{align*}
\log |f(r_m)| &= \log |u_m \epsilon_v| + c_0 + O(\epsilon_v), \\
\log |f(r_v)| &= \log |\epsilon_u v_m| + c_0 + O(\epsilon_u), \\
\log |f(r_e)| &= \log |\epsilon_u \epsilon_v| + c_0 + O(\epsilon_u \epsilon_v),
\end{align*}
\]

which can be used to simplify \((2.24)\) as follows

\[
I_{\text{joint}}^{\text{PI}} = V_d L^d \left( \frac{\log |f(r_m)|}{r_m^d} - \frac{\log (u_m v_m)}{r_m^d} + \frac{\log |f(r_v)|}{r_v^d} - \frac{\log |f(r_e)|}{r_e^d} \right). \tag{2.27}
\]

Here \(c_0 = \psi(0)(1) - \psi(0)\left(\frac{1}{d+1}\right)\) is a positive number and \(\psi(0)(x) = \Gamma'(x) / \Gamma(x)\) is the digamma function.

Finally one has to remove the ambiguity due to the normalization of the null vectors by adding the extra term \((1.2)\) to the action. The resulting expression is then

\[
I_{\text{amb}}^{\text{PI}} = -V_d L^d \left( \frac{\log \frac{\alpha \beta L^2}{L^2}}{r_m^d} + \frac{2}{d} r_m^d \right) + \frac{V_d L^d}{8 \pi G_N} \left( \frac{\log \frac{\alpha \beta L^2}{L^2}}{r_h^d} + \frac{2}{d} r_h^d \right). \tag{2.28}
\]

Therefore altogether for the subregion given by the intersection of the WDW patch with the past interior shown in the left panel of figure 2 one gets

\[
I_{\text{PI}} = \frac{V_d L^d}{8 \pi G_N} \left( \frac{1}{r_h^d} \log \frac{\tilde{L}^2}{L^2} + c_0 - \frac{(d+1) \tau}{2 r_h^d + 1} \log \frac{\tilde{L}^2 |f(r_m)|}{r_m^d} \right), \tag{2.29}
\]

which depends on time through its \(r_m\) dependence, as expected. Here we have used the fact that

\[
\log (u_m v_m) = -f'(r_h) r^* (r_m) = -\frac{(d+1) \tau}{2 r_h}.
\]

Note that for \(r_m \to r_{\text{Max}}\) the time dependence of the
on-shell action drops out resulting to

\[ I_{\text{PI}} = \left( c_0 - \frac{(d+1)\tau_c}{2r_h} + \log \frac{\tilde{L}^2}{L^2} \right) S_{\text{th}} \frac{2}{2\pi}. \]  

(2.30)

Note also that at late times where \( r_m \to r_h \), using equation (2.25), the total on-shell action in the past interior vanishes.

### 2.3.2 Future Interior

Let us now compute on-shell action for the intersection of the WDW patch with the future interior shown in the right panel of figure 2. In this case, using our previous results, the bulk term of the action is

\[ I_{\text{FI}}^\text{bulk} = -\frac{V_dL^d}{4\pi G_N} (d+1) \int_{r_h}^{r_{\text{Max}}} \frac{dr}{r^{d+2}} \left( \frac{\tau}{2} + r^*(\epsilon) - r^*(r) \right) = -\frac{V_dL^d}{4\pi G_N} \left( \frac{1}{d r_h^d} + \frac{\tau + \tau_c}{2r_h^{d+1}} \right). \]  

(2.31)

There are five joint points two of which have zero contributions for large \( r_{\text{Max}} \), while the contributions of other three points are given by

\[ I_{\text{FI}}^\text{joint} = -\frac{V_dL^d}{8\pi G_N} \left( \frac{\log \frac{\alpha\beta r^2}{L^2 |f(r_\epsilon)|}}{r_\epsilon^d} - \frac{\log \frac{\alpha\beta r^2}{L^2 |f(r_{u_{m'}})|}}{r_{u_{m'}}^d} - \frac{\log \frac{\alpha\beta r^2}{L^2 |f(r_{v_{m'}})|}}{r_{v_{m'}}^d} \right) \]  

(2.32)

\[ = -\frac{V_dL^d}{8\pi G_N} \left( \frac{\log |f(r_\epsilon)|}{r_\epsilon^d} - \frac{\log |f(r_{u_{m'}})|}{r_{u_{m'}}^d} - \frac{\log |f(r_{v_{m'}})|}{r_{v_{m'}}^d} + \frac{\log \frac{\alpha\beta r^2}{L^2}}{r_h^d} \right) \]  

\[ = \frac{V_dL^d}{8\pi G_N} \left( \frac{\log |u_{m'}v_{m'}|}{r_h^d} + c_0 - \frac{\log \frac{\alpha\beta r^2}{L^2}}{r_h^d} \right). \]

The boundary terms associated with the null boundaries vanish using affine parametrization for the null directions and the only term we need to compute is the surface term at future singularity. This is indeed the term we have already computed in (2.8)

\[ I_{\text{FI}}^\text{surf} = \frac{V_dL^d}{8\pi G_N} (d+1) \frac{\tau + \tau_c}{2r_h^{d+1}}. \]  

(2.33)

The only remaining contribution to be computed is the term needed to remove the ambiguity

\[ I_{\text{FI}}^\text{amb} = \frac{V_dL^d}{8\pi G_N} \left( \frac{\log \frac{\alpha\beta L^2 r_h^2}{L^4}}{r_h^d} + \frac{2}{d r_h^d} \right). \]  

(2.34)
Taking all terms into account we have

\[ I_{FI} = \frac{V_d L^d}{8\pi G_N} \left( \frac{d\tau}{r_h^{d+1}} + \frac{(d-1)\tau_c}{2r_h^{d+1}} + \frac{c_0}{r_h^d} + \frac{1}{r_h^d} \log \frac{\tilde{L}^2}{L^2} \right). \] (2.35)

Here to get the final result we have used the fact that \( \log |u_m'v_m'| = \frac{(d+1)\tau}{2r_h} \).

It is also interesting to sum the contributions of both regions shown in figure 2 and compare the resultant expression with the on-shell action evaluated on the whole WDW patch

\[ I_{Ext} = I_{WDW} - (I_{PI} + I_{FI}) = 2 \times \frac{V_d L^d}{8\pi G_N} \left[ -\frac{c_0}{r_h^d} \log \left( \frac{\tilde{L}^2}{L^2} \right) \right]. \] (2.36)

that is time independent, as expected. In fact this is the contribution of the part of the WDW patch that is outside of the black hole horizon. The factor of two is a symmetric factor between left and right sides of the corresponding WDW patch. It is also interesting to note that the finite term is negative! We will consider the above result in the next section where we will study subregion complexity.

### 2.4 Late Time Behavior

In this section we will study the time derivative of the on-shell actions we have found in the previous subsections. To proceed we note that from definitions of \( r^* \) and \( r_m \) one has

\[ \frac{dr^*(r_m)}{d\tau} = -\frac{1}{2}, \quad \frac{dr_m}{d\tau} = \frac{1}{2} \tilde{f}(r_m), \] (2.37)

which can be used to show

\[ \frac{dI_{WDW}}{d\tau} = \frac{dI_{past}}{d\tau} = 2M \left( 1 + \frac{1}{2} \tilde{f}(r_m) \log \frac{\tilde{L}^2|f(r_m)|}{L^2} \right), \quad \tilde{f} = \frac{r_h^{d+1}}{r_m^{d+1}} - 1. \] (2.38)

It is also interesting to compute the time derivative of the on-shell action for the individual subregions we have considered before. Actually it is straightforward to see

\[ \frac{dI_{PI}}{d\tau} = M \tilde{f}(r_m) \log \frac{\tilde{L}^2|f(r_m)|}{L^2}, \quad \frac{dI_{FI}}{d\tau} = 2M, \quad \frac{dI_{Ext}}{d\tau} = 0. \] (2.39)

It is evident that summing up these contributions one gets (2.38), as expected. Note that at late times where \( r_m \to r_h \), the past interior has no contribution to the rate of complexity growth.

Of course it is known that the complexity obtained from WDW patch violates the Lloyd’s bound, though at late time it approaches \( 2M \). From the above results it is evident that the contribution to the late time behavior comes from the future interior of the black brane. It is also worth noting that the violation of the Lloyd’s bound is due to the contribution of the joint point located at the past interior. This, in turn, suggests that if one defines the complexity as on-shell action on the intersection of WDW patch and future interior the resultant complexity fulfills the bound and has
linear growth all the time! Of course if one wants appropriate UV divergences before regularizing
the complexity we should also add the contributions of the exterior region too. More explicitly one
has
\[ \tilde{I}_{\text{WDW}} = I_{FI} + I_{\text{Ext}} = 2M \left( \tau + \frac{(d-1)c_0 \tau}{d} + \left( \frac{2r_h^{d+1}}{d e^d} - \frac{r_h}{d} \right) \log \frac{\tilde{L}^2}{L^2} \right). \] (2.40)

It is also instructive to note that at late time where \( r_m \to r_h \), setting \( r_m - r_h = \xi \), from equation (2.2) one finds
\[ \tau = -\frac{2r_h}{d+1} \left( \log \left( \frac{d+1}{r_h} \right) \right) \sim \frac{\beta}{2\pi} \log \frac{r_h}{(d+1)\xi}, \quad \text{with } \beta = \frac{1}{T}. \] (2.41)

In particular when one is away from the horizon about a power of Planck scale \( \xi \sim \ell_d \ell_p r_d^{-1} h \) the above late time behavior reads
\[ \tau \sim \frac{\beta}{2\pi} \log S_{th}, \] (2.42)
in which the on-shell action reads \( I \sim S_{th} \log S_{th} \), that is the scrambling complexity. One may
also consider the case where the time is about \( \tau \sim \frac{\beta}{2\pi} e^{S_{th}} \) that could be the time where one gets
maximum complexity. At that time the on-shell action is
\[ I \sim S_{th} e^{S_{th}}, \] (2.43)
which could be thought of maximum complexity of the system.

2.5 Holographic Uncomplexity

Given a time slice and the associated WDW patch one may want to compute on-shell action on
a region that should be included in the WDW patch as time goes. The corresponding region is
shown in figure 3. Actually following [44] one may identify the on-shell action on this region with
“holographic uncomplexity” that is the gap between the complexity and the maximum possible
complexity (see also [45,46]). In other words the uncomplexity is a room for complexity to increase.
Alternatively one could thought of the holographic uncomplexity as the spacetime resource available
to an observer who intends to enter the horizon [44].

Clearly the on-shell action on the region depicted in figure 3 is given by a difference of on-shell
action evaluated on the future interior
\[ I_{\text{UC}} = I_{FI2} - I_{FI1} = 2M (\tau_2 - \tau_1). \] (2.44)

where \( \tau \) is the actual boundary time. It is also important to note that \( \tau_2 \) should be thought of a
time cut off and eventually we are interested in \( \tau_2 \to \infty \) limit for some fixed \( \tau_1 \). Indeed the time
cut off could be set to \( \tau_2 \sim \frac{\beta}{2\pi} e^{S_{th}} \).

As we mentioned the holographic uncomplexity is defined as a difference between maximum
complexity and the complexity of the state at a given time, it is then evident from (2.44) that this equation can not capture this difference. The crucial point is that the complexity, as we already mentioned, has two components: one from the boundary term and one from the joint point. The resultant uncomplexity given in equation (2.44) does not fully contain the contribution of joint point. To be precise using equation (2.13) one has

$$\Delta I_{WDW} = I_{WDW}^2 - I_{WDW}^1$$

$$= \frac{V_d L^d}{8 \pi G_N} \left[ \frac{d-1}{2 r_h^{d+1}}(\tau_2 - \tau_1) - \log \left| \frac{f(r_m^2)}{r_m^d} \right| + \log \left| \frac{f(r_m^1)}{r_m^d} \right| \right].$$

(2.45)

One observes that there is a joint contribution that the subregion shown in figure 3 can not see it and thus it is not equal to $I_{UC}$. Of course it approaches $I_{UC}$ when both $r_m^1$ and $r_m^2$ approach the horizon. Actually using the fact that $\tau_2$ should be thought of a cut off and therefore it is large (i.e. $r_m^2 \rightarrow r_h$) the above expression reads

$$\Delta I_{WDW} \approx 2M(\tau_2 - \tau_1) - \frac{V_d L^d}{8 \pi G_N} \left[ \frac{c_0}{r_h^d} - \frac{(d+1)\tau_1}{2 r_h^{d+1}} - \log \left| \frac{f(r_m^1)}{r_m^d} \right| \right].$$

(2.46)

Note that the second part is just the on-shell action evaluated on the past interior.

3 Subregion Complexity and Outside the Horizon

In the previous section we have computed on-shell action on the WDW patch and certain subregions located behind the horizon. Due to the contribution of the black hole interior the resultant on-shell actions are time dependent. Since the background is static, having found a time dependent quantity has to do with complexity [1]. Of course it needs further investigation to make this connection more precise.

In this section we shall consider on-shell action for the cases where the corresponding subregions
are located outside the horizon. In this case one expects that, typically, the resultant on-shell actions would be time independent, though even in the case we could find time dependent results.

3.1 Complexity of Layered Stretched Horizon

Let us consider a subregion in the black hole exterior in the shape of a triangle shown in the left panel of figure 4. The three faces of the corresponding triangle are given by two null and a timelike boundaries

\[ t = t_1 + r^*(\epsilon) - r^*(r), \quad t = t_2 - r^*(\epsilon) + r^*(r), \quad r = \epsilon. \]  

(3.1)

The null boundaries intersect at the point \( r = r_p \) that is given by

\[ \tilde{\tau} \equiv t_{R2} - t_{R1} = 2 (r^*(\epsilon) - r^*(r_p)). \]  

(3.2)

where \( \tilde{\tau} \) is the time interval. This should not be confused with the actual field theory time coordinate \( \tau \) we have used in the previous section.

Actually following [1] where the author has considered layered stretched horizon, this might be thought of as a bulk operator \( P \) localized at point \( r_p \). Indeed the corresponding triangle shows a region of the boundary involved in the construction of the operator \( P \). Now the aim is to compute the on-shell action in this subregion. Following [1] the result might be thought of as complexity of the operator localized on the corresponding layer.

To proceed let us start with the bulk contribution. From the notation depicted in figure 4 it is straightforward to see

\[ I_{\text{bulk}}^{T\text{H}} = - \frac{V_d L^d}{8 \pi G_N} (d + 1) \int_{\epsilon}^{r_p} \frac{dr}{r^{d+2}} (\tilde{\tau} - 2(r^*(\epsilon) - r^*(r))) \]  

15
\[\begin{align*}
&= -\frac{V_d L^d}{8\pi G_N} \left( \frac{1}{e^{d+1}} - \frac{1}{r_h^{d+1}} \right) \tilde{\tau} + \frac{V_d L^d}{4\pi G_N d} \left( \frac{1}{e^d} - \frac{1}{r_p^d} \right). \quad (3.3)
\end{align*}\]

As for the boundary terms we only need to consider the Gibbons-Hawking-York term at the timelike boundary \( r = \epsilon \)

\[I_{\text{surface}}^{\text{Tri}} = \frac{1}{8\pi G_N} \int \int_{t^2 \to \epsilon^2 r^*} dt \sqrt{h K_t} \bigg|_{r=\epsilon} = \frac{(d+1) V_d L^d}{8\pi G_N} \left( \frac{1}{e^{d+1}} - \frac{1}{2r_h^{d+1}} \right) \tilde{\tau}. \quad (3.4)\]

The normal vectors associated with the boundaries of the triangle given by (3.1) are

\[n_t = \frac{L d r}{\epsilon \sqrt{f(\epsilon)}}, \quad k_1 = \alpha \left( -dt + \frac{dr}{f(r)} \right), \quad k_2 = \beta \left( dt + \frac{dr}{f(r)} \right), \quad (3.5)\]

which can be used to compute the contribution of the joint points as follows

\[I_{\text{joint}}^{\text{Tri}} = \frac{1}{8\pi G_N} \int_{c_1} d^d x \sqrt{\gamma} \log |k_2 \cdot n_t| + \frac{1}{8\pi G_N} \int_{c_2} d^d x \sqrt{\gamma} \log |k_1 \cdot n_t| \]

\[-\frac{1}{8\pi G_N} \int_{r_p} d^d x \sqrt{\gamma} \log \left| \frac{k_1 \cdot k_2}{2} \right| \]

\[= \frac{V_d L^d}{8\pi G_N} \frac{\log \frac{\alpha \beta \ell^2}{L^2}}{e^d} - \frac{V_d L^d}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta \ell^2}{L^2}}{r_p^d} - \frac{\log f(r_p)}{r_p^d} \right). \quad (3.6)\]

The contribution of the term needed to remove the ambiguity is

\[I_{\text{amb}}^{\text{Tri}} = \frac{1}{8\pi G_N} \int_{\text{null}} d\lambda d^d x \sqrt{\gamma} \log \left| \frac{\tilde{L}}{d} \right| \]

\[= \frac{V_d L^d}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta \ell^2}{L^2}}{r_p^d} + \frac{2}{d r_p^d} \right) - \frac{V_d L^d}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta \ell^2}{L^2}}{e^d} + \frac{2}{d e^d} \right). \quad (3.7)\]

Therefore taking all contributions into account one arrives at

\[I_{\text{Tri}} = \frac{V_d L^d}{8\pi G_N} \left[ \left( \frac{d}{e^{d+1}} - \frac{d-1}{2 r_h^{d+1}} \right) \tilde{\tau} + \left( \frac{1}{r_p^d} - \frac{1}{e^d} \right) \log \frac{\tilde{L}^2}{L^2} + \frac{\log f(r_p)}{r_p^d} \right]. \quad (3.8)\]

Note that in the context of holographic renormalization it is known that when one has a timelike flat boundary (as in the present case at \( r = \epsilon \)) there is an extra counterterm as follows

\[I_{\text{ct}}^{\text{Tri}} = -\frac{1}{8\pi G_N} \int_{r=\epsilon} d^{d+1} x \sqrt{\gamma} \frac{d}{L} = \frac{V_d L^d}{8\pi G_N} \left( -\frac{d}{e^{d+1}} + \frac{d}{2 r_h^{d+1}} \right) \tilde{\tau}. \quad (3.9)\]
Combining the above result with equation (3.8), the total on-shell action reads

\[ I_{\text{Tri}} = \frac{V_d L^d}{8 \pi G N} \left[ -\frac{1}{e^d} \log \frac{\tilde{L}^2}{L^2} + \frac{\tilde{\tau}}{2r_{h}^{d+1}} + \frac{\log \frac{L^2 f(r_p)}{L^2}}{r_p^d} \right]. \]  

(3.10)

Note that since we have already assumed \( \tilde{L} \geq L \), from the above expression one finds that the most divergent term as well as the finite term are negative. This is of course in contrast with what one would expect from complexity. Actually the result is reminiscent of free energy of the black hole. Indeed denoting the contribution of joint point by \( J \) one has (dropping the divergent term)

\[ I_{\text{Tri}} = -F \tilde{\tau} + J_p \quad \text{with} \quad J_p = \frac{\log f(r_p)}{r_p^d}, \]  

(3.11)

where \( F = -\frac{V_d L^d}{16 \pi G N} \frac{1}{r_{h}^{d+1}} \) is the free energy of the corresponding black brane. To summarize we note that the on-shell action in this case consists of two parts: the first part that might be thought of as the classical contribution is the contribution of the timelike boundary that is the free energy of corresponding black brane, while the second one that comes from joint point should be treated as the new contribution associated with the complexity of the operator. Clearly when a given subregion does not contain a timelike boundary the free energy drops and the whole contributions come from joint points (see next subsection).

For the case where the point \( r_p \) is in the vicinity of the horizon, \( i.e., r_p = r_h - \xi \) for \( \xi \ll r_h \), from equation (3.10) one finds

\[ I_{\text{Tri}} \approx \frac{V_d L^d}{8 \pi G N} \left( \frac{1}{r_p^d} - \frac{1}{e^d} \right) \log \frac{\tilde{L}^2}{L^2} - \frac{V_d L^d}{16 \pi G N} \frac{1}{r_{h}^{d+1}} (d \tilde{\tau}). \]  

(3.12)

that shows the layer (operator) becomes more complex as one approach the horizon. In particular when one is away from the horizon about the Planck length one gets

\[ I_{\text{Tri}} \approx \frac{V_d L^d}{8 \pi G N} \left( \frac{1}{r_p^d} - \frac{1}{e^d} \right) \log \frac{\tilde{L}^2}{L^2} - \frac{1}{2 \pi (d + 1)} S_{\text{th}} \log S_{\text{th}} \]  

(3.13)

### 3.2 CA Proposal and Subregion Complexity

An immediate application of the result we have obtained in the previous section is to find on-shell action for a square subregion shown in orange in the right panel of figure 4. Indeed the desired result can be found by algebraic summation of three triangles identified by \( r_1, r_2 \) and \( r_p \). Actually using equation (3.8) one gets\(^9\)

\[ I_{\text{Sq}} = I_{r_p} - I_{r_1} - I_{r_2} \]  

(3.14)

\(^9\) One could have directly computed the on-shell action for the square region taking into account all terms in the action. Of course the result is the same as what we have found by an algebraic summation of three triangles.
\[
\left(\epsilon_v, \epsilon_u\right) \rightarrow \left(\epsilon_v, \epsilon_u\right)
\]

Figure 5: **Left:** Intersection of WDW patch and entanglement wedge for large entangling region at time slice \(t_R = 0\) for half of an eternal black hole. **Right:** Two subregions denoted by \(\ell_1\) and \(\ell_2\).

\[
I_{Sq} = \frac{V_d L^d}{8\pi G_N} \left( \frac{1}{\epsilon_d^d} - \frac{1}{\epsilon_h^d} \right) \log \frac{\tilde{L}^2}{L^2} - \frac{c_0}{2\pi} S_{th}.
\]  

It is important to note that although the most divergent term is positive for \(\tilde{L} > L\), the finite term is negative. We note that on-shell action for the subregion shown in the left panel of figure 5 has been recently studied [43] where the authors have not considered the term needed to remove the ambiguity and have fixed the ambiguity by hand. As a result the finite term they have found was positive. We note, however, that it is crucial to take into account the corresponding term to maintain the reparameterization invariance of the action. Note that for values of \(r_p\) it can be seen that the finite part of equation (3.14) is always negative\(^{10}\).

It is also interesting to compare on-shell action evaluated on different subregions and union of the subregions. To proceed we will consider two subregions denoted by \(\ell_1\) and \(\ell_2\) in the right panel.

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\(^{10}\)We would like to thank B. Swingle for discussions on this point.
of figure 5. Using the notation shown in the figure and setting \( \tilde{L} = L \), one has

\[
I_{\ell_1} = \frac{V_d L^d}{8\pi G_N} \left( \frac{\log |f(r_p)|}{r_p^d} - \frac{\log |f(r_1)|}{r_1^d} - \frac{\log |f(r_{wp})|}{r_{wp}^d} - \frac{\log |f(r_{wp})|}{r_{wp}^d} \right),
\]

\[
I_{\ell_2} = \frac{V_d L^d}{8\pi G_N} \left( \frac{\log |f(r_1)|}{r_1^d} - \frac{\log |f(r_2)|}{r_2^d} \right).
\]

Here in order to simplify \( I_{\ell_1} \) we have used (2.25). On the other hand the on-shell action evaluated on \( \ell_1 \cup \ell_2 \) is (3.15)

\[
I_{\ell_1 \cup \ell_2} = -\frac{V_d L^d}{8\pi G_N} \frac{c_0}{r_h^d}.
\]

Therefore one gets

\[
A = I_{\ell_1} + I_{\ell_2} - I_{\ell_1 \cup \ell_2} = \frac{V_d L^d}{8\pi G_N} \left( 2 \frac{\log |f(r_p)|}{r_p^d} - \frac{(d+1)r^*(r_p)}{r_h^{d+1}} - \frac{\log |f(r_1)|}{r_1^d} - \frac{\log |f(r_2)|}{r_2^d} \right).
\]

It is then important to determine the sign of \( A \). To do so, one first observes that \( A \) vanishes at both \( \{r_p, r_1, r_2\} \rightarrow r_h \) and \( \{r_p, r_1, r_2\} \rightarrow 0 \) limits. On the other hand one can show that at \( \{r_p, r_1, r_2\} \approx 0 \) the function \( A \) approaches zero from above leading to the fact that \( A \geq 0 \). This behavior may also be shown numerically. As a result we conclude that the on-shell action we have evaluated for subregions in the exterior of the black brane obeys subadditivity condition

\[
I_{\ell_1} + I_{\ell_2} \geq I_{\ell_1 \cup \ell_2}.
\]

4 Discussions and Conclusions

In this paper motivated by “complexity equals action” proposal we have evaluated on-shell action on certain spacetime subregions enclosed by null boundaries that of course includes the WDW patch itself too. Our main concern was to compute finite term of the on-shell action. It is contrary to the most studies in the literature where the main concern was to compute the growth rate of the complexity (see for example [48–61]).

Although we have computed on-shell action on a given subregion, taking into account all terms needed to have reparametrization invariance and well-defined variational principle, we have observed that the final result is given by contributions of joint points and timelike or spacelike boundaries. Removing the most divergent term by setting \( \tilde{L} = L \), the corresponding joint contribution, \( J \), and
timelike and spacelike surface contributions, $S_t$, and $S_s$ are given by

$$J = \frac{V_d L^d}{8\pi G_N} \log |f(r)| r^d, \quad S_t = \frac{V_d L^d}{8\pi G_N} \frac{\bar{\tau}}{2r^d_h+1}, \quad S_s = \frac{V_d L^d}{8\pi G_N} \frac{(d-1)\tau}{2r^d_h+1}. \quad (4.1)$$

Note that when the joint point occurs at the horizon one needs to take $r \to r_h$ limit from the above joint contribution $J$ that typically results to an expression proportional to $\log |uv| + c_0$.

Clearly when the joint point is located at the horizon one needs to regularize the joint contribution using equation (2.25). The sign of the joint contribution depends on the position of the corresponding joint point. If the joint point locates on the left or right of the given subregion, the sign is positive and for those that located up or down of the subregion, it is negative. It is also interesting to compute time derivative of the above expressions

$$\dot{J} = \frac{V_d L^d}{8\pi G_N} \left( \frac{d+1}{2r^d_h+1} + \frac{d}{2r^d} f(r) \log |f(r)| \right), \quad \dot{S}_t = 0, \quad \dot{S}_s = \frac{V_d L^d}{8\pi G_N} \frac{(d-1)\tau}{2r^d_h+1}, \quad (4.2)$$

showing that in the late time the joint point has nontrivial contribution.

Another observation we have made is that whenever the subregion contains a part of black hole interior the finite part of the action is positive and time dependent, while for the cases that the desired subregion is entirely in the exterior part of the black hole the corresponding finite term is time independent and negative. It is also important to note that for all cases, except one, the most divergent term exhibits volume law scaling with positive sign. These points should be taken into account when it comes to interpret the results from field theory point of view.

Throughout this paper we have been careful enough to clarify what we mean by on-shell action. Indeed there are several terms one may add to action that could alter the results once we compute the on-shell action. It is then important to fix them. Our physical constraints were to have reparametrization invariance and a well-defined variational principle. These assumptions enforce us to have certain boundary and joint actions. In particular it was crucial to consider the log term given by equation (1.2) that is needed to remove the ambiguity associated with the null vectors. Actually in our computations this term has played an essential role.

We note, however, that even with this term the resultant on-shell action still contains an arbitrary length scale. We have chosen the length scale so that the most divergent term of the on-shell action is positive. This is indeed required if one wants to identify the on-shell action with complexity, at least when evaluated inside the WDW patch. Of course following the general idea of the holographic renormalization one may add certain counterterms to remove all divergent terms including that associated with the undetermined length scale [37]. This is, actually, what we have done in this paper when we were only interested in the finite part of the on-shell action.

In fact if one wants to identify the on-shell action with the complexity we may not be surprised to have an arbitrary length scale. This might be related to choosing an arbitrary length scale in the definition of complexity in quantum field theory (see e.g. [20, 21]). Of course eventually we would like to find a way to fix the length scale or at least to make a constraint on it so that it could
naturally lead to a clear interpretation in terms of complexity.

The main question remains to be addressed is how to interpret our results from field theory point of view. It is well accepted that the on-shell action evaluated in the WDW patch is associated with the computational complexity that is the minimum number of gates one needs to reach the desired target state from reference state (usually the vacuum state). Of course this is the complexity defined for a pure state. It is then natural to look for a definition of complexity for mixed state. This has been partially addressed in [43,47].

When we are restricted to a subregion, even the whole system is in a pure state, we would have a mixed state density matrix and therefore a definition of mixed state complexity is required. Of course the resultant subregion complexity could as well depend on the state of whole system whether or not it is pure.

Actually different possible definitions of subregion complexity have been explored in [43]. Based on results we have found it seems that the on-shell action evaluated on a subregion in the exterior is better match with purification complexity that is the pure state complexity of a purified state minimizing over all possible purifications. The main observation supporting this proposal is the subadditivity condition satisfied by the corresponding on-shell actions\(^\text{11}\). Note that this is not the case for complexity obtained by CV proposal.

Based on this observation and motivated by the expression of mutual information in terms of entanglement entropy one may define a new quantity associated with two subregions \(A\) and \(B\) as follows
\[
C(A|B) = C(A) + C(B) - C(A \cup B),
\]
that could be thought of as mutual complexity which is always non-negative and symmetric under the exchange of \(A\) and \(B\) by definition. Here \(C\) stands for complexity evaluated using CA proposal. Note that according to our holographic results this quantity is finite (see eq. (3.18)). It would be interesting to explore properties of this quantity that might be thought of as a quantum measure that measures the correlation between the corresponding two subsystems.

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