Constructing phase space distributions with internal symmetries

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We discuss an ab initio world-line approach to constructing phase space distributions in systems with internal symmetries. Starting from the Schwinger-Keldysh real time path integral in quantum field theory, we derive the most general extension of the Wigner phase space distribution to include color and spin degrees of freedom in terms of dynamical Grassmann variables. The corresponding Liouville distribution for colored particles, which obey Wong’s equation, has only singlet and octet components, while higher moments are fully constrained by the Grassmann algebra. The extension of phase space dynamics to spin is represented by a generalization of the Pauli-Lubanski vector; its time evolution via the Bargmann-Michel-Telegdi equation also follows from the phase space trajectories of the underlying Grassmann coordinates. Our results for the Liouville phase space distribution in systems with both spin and color are of interest in fields as diverse as chiral fluids, finite temperature field theory and polarized parton distribution functions. We also comment on the role of the chiral anomaly in the phase space dynamics of spinning particles.

I. INTRODUCTION

Classical phase space distributions involving internal degrees of freedom such as spin and color are useful in describing a wide range of physics across energy scales. The construction of classical phase space distributions with internal symmetries through the use of Grassmann variables was pioneered by Berezin and Marinov \cite{1}. They demonstrated straightforwardly how one recovers in this framework the Bargmann-Michel-Telegdi (BMT) equation \cite{2} describing the precession of spins in background fields. Several contemporaneous studies complemented the work of Berezin and Marinov and extended the Grassmann algebra description of internal symmetries to describing the dynamics of both spin and color degrees of freedom \cite{3–10}. The Wong equations \cite{11, 12} describing the precession of non-Abelian point particles in colored background fields are also recovered in this approach.

The world-line formalism \cite{13–22} provides an elegant method, from first principles in quantum field theory, to derive classical phase space distributions in systems with internal symmetries. In this approach, one-loop effective actions are expressed by quantum mechanical point-particle path integrals with internal symmetries incorporated through Grassmann degrees of freedom. In this work, we will develop the Schwinger-Keldysh (SK) generalization of the world-line formalism for applications to nonequilibrium physical processes\cite{4}. The classical color phase space limit is obtained explicitly from the saddle point of the world-line effective action and the phase space dynamics of the extended color and spin symplectic algebra is obtained from Grassmannian classical Poisson brackets as well as a Grassmann valued phase space measure.

For colored degrees of freedom, a canonical coordinate transformation from Grassmann variables to Grassmann bilinears relates our approach to that of Alekseev, Faddeev and Shatashvili (AFS) \cite{23}. The AFS approach described the symplectic orbits of compact Lie groups with functional integrals involving phase space Darboux variables. In particular, the AFS construction of symplectic structures for internal symmetries contains a classical action with a topological Berry monopole \cite{26}. In contrast, no such topological terms arise in our derivation of the classical limit by a saddle-point approximation to the SK world-line path integral. We also demonstrate that the Grassmann algebra allows one to express the Liouville density for colored particles uniquely in terms of two independent color structures.

One can similarly construct classical phase space distributions in the world-line approach for both massless and massive fermions recovering the results of Berezin and Marinov \cite{1}. Here too, a canonical transformation from Grassmann spin variables to their commuting bilinear phase space counterpart, and the corresponding simple rules for the spin phase space measure, generates a unique and elegant form for the classical phase space distribution function for spin in these variables. This derivation provides a many-body generalization of the relativistic Pauli-Lubanski spin pseudo-vector \cite{1}, whose time evolution reproduces the BMT equation.

One can also define, in this language of Grassmann bilinears, chiral vector and and axial vector currents as classical phase space averages weighted by the generalized Liouville density. We show that a naive generalization of our approach misses the effects of the chiral anomaly. Recovering the anomaly requires a careful treatment whereby one introduces auxiliary axial vector fields in the Schwinger Keldysh world-line path integral.

Our manuscript is organized as follows: In section \ref{sec:app} we provide a short derivation of classical phase space distributions with internal symmetries from a saddle point expansion of the world-line path integral expression for the one loop effective action in quantum field theory.
A more detailed computation will be presented in [24]. We derive in the Schwinger-Keldysh formalism, via a “truncated Wigner” (or classical-statistical) approximation, the Wigner distribution for the extended symplectic phase space involving Grassmann variables[3]. Our construction involves canonical phase space variables and we will show that the semi-classical limit of quantum phase space is incompressible [33–35].

In section III we discuss the structure of the Grassmann valued Wigner phase space for color degrees of freedom and demonstrate that the group algebra is exactly realized in the classical limit for any finite dimensional representation. An intuitive description of the extended phase space for color is obtained by expressing our results in terms of Grassmanian bilinear color charges whose dynamics is given by Wong’s equations [11]. This allows us to replace the Grassmann formulation of the extended Wigner phase space distribution (and the corresponding Grassmann valued measure) by a classical phase space distribution (and phase space measure), whose elements are color charges.

Section IV is devoted to the Grassmann description of relativistic spin and chirality. In analogy to the case of color, the spin Wigner function can be replaced by a classical phase space distribution including a many-body generalization of the Grassmanian bilinear Pauli-Lubanski spin pseudo-vector satisfying the BMT equation.

In section V we briefly describe from first principles how the chiral anomaly manifests itself in this phase space approach. We first show that the expectation value of the axial-vector current, constructed from a semi-classical phase space average, is naively conserved. We show that the proper treatment of its violation by the anomaly is obtained by introducing an auxiliary axial-vector gauge field in the path integral; the four-derivative of the corresponding axial vector current is robust when this axial vector field is subsequently put to zero and generates the well known expression for the axial anomaly.

Our derivation provides a clean and transparent derivation in quantum field theory of classical phase space distributions in terms of internal degrees of freedom that have a simple physical interpretation but are nevertheless fully constrained by the underlying Grassmann algebra. The framework developed here potentially has wide ranging applications in nonequilibrium processes; in our concluding remarks, we briefly outline our work in progress on some of these problems.

Our work is supplemented by appendices: In appendix A we discuss Liouville’s theorem and the (in-)compressibility of phase space in the semi-classical limit. We provide details of the classical color phase space construction in appendix B. In appendix C we outline how the incompressibility of phase space may be affected by

the presence of a Berry phase [26, 36, 12] and discuss possible ambiguities in its interpretation.

II. WORLD-LINE FORMULATION OF REAL-TIME QUANTUM FIELD THEORY

We begin by reviewing the world-line computation of one-loop effective actions [13–22, 43] including internal symmetries such as color and spin. For simplicity, we begin with a scalar (spin-less) massless particle coupled to a background non-Abelian gauge field, where the one-loop effective action is

\[
\Gamma[A] = - \text{Tr} \log(-iD^2[A]) .
\]

Here \(D_\mu = \partial_\mu - igA_\mu\) is the non-Abelian covariant derivative in arbitrary representation and \(\text{Tr}\) denotes an infinite dimensional functional trace over spatial coordinates as well as internal symmetries. Using the heat-kernel representation of the logarithm, Strassler [13] showed that Eq. (1) can be written as a quantum mechanical path integral

\[
\Gamma[A] = \int_0^\infty \frac{dT}{T} N^2 \text{tr} \mathcal{P} \exp \left[ \int_0^T d\tau \left( \frac{\dot{\phi}^2}{2\epsilon} + gA_\mu(x)\dot{A}^\mu \right) \right],
\]

where the functional trace over position is now written as a path integral of a pointlike particle satisfying a trajectory in proper time \(\tau\) with the position eigenvalue \(x_\mu(\tau)\). The einbein \(\epsilon\) is an arbitrary positive parameter that we will discuss further shortly. Eq. (2) contains the trace in color of the matrix valued Wilson line of the external gauge field along the world-line.

Remarkably, this proper-time ordered color trace can be expressed as a functional integral just as is the case for the spatial coordinates. In particular, D’Hoker and Gagne derived the following very elegant identity for an \(n \times n\) Hermitian matrix \(M(\tau)\)

\[
\text{tr} \mathcal{P} \exp \left[ \int_0^T d\tau M(\tau) \right] = \int D\phi \int D\lambda D\lambda^\dagger e^{i\phi(\lambda^\dagger \lambda + \frac{1}{2})} \times \exp \left[ \int_0^T d\tau (i\lambda^\dagger \frac{d\lambda}{d\tau} + \lambda^\dagger M \lambda) \right],
\]

where \(\text{tr} D\phi \equiv (\frac{1}{\sqrt{2\pi}})^n \sum_\phi e^{i\phi(\lambda^\dagger \lambda + \frac{1}{2})}\) is a constraint.

\[\text{\footnotesize 2} \text{ General properties of fermionic Wigner functionals are discussed in e.g. [23, 42].}\]

\[\text{\footnotesize 3} \text{ A similar representation was suggested previously in [10]. World-line representations for mixed symmetries are discussed in [14].}\]
In this expression, 

\[ \chi_A(x^+_i, x^-_i, \lambda^+_i, \lambda^-_i, \lambda^{i+}_i, \lambda^{i-}_i) \]

represents the density matrix at initial time \( \tau_0 \) with \( (x^+_i, x^-_i, \lambda^+_i, \lambda^-_i, \lambda^{i+}_i, \lambda^{i-}_i)(\tau_0) \), denotes support on the upper/lower Keldysh contour and \( C \) denotes integration along this contour with the massless SK action given by

\[ S_C[A] = \int d\tau \left\{ \frac{\dot{x}^2}{2} + g \dot{x}_\mu \lambda^\mu A^\nu \lambda + i\lambda^A \dot{\lambda} + \phi |\lambda^A \lambda + \frac{n}{2} - 1| \right\}. \]

\[ (5) \]

In going from Eq. (2) to Eq. (4), we replaced the \( dT/T \) integral by a more general expression, where the einbein is promoted to a dynamical variable and integrated over. The integration over the einbein can be understood as integration over a gauge redundancy, which results from the world line action in Eq. (2) being invariant under rescaling of the world-line parameter \( \tau \rightarrow \tau'/\tau' \). As shown in the Euclidean formulation of [21], one can “BRST-fix” this gauge freedom in Eq. (4) to arrive at Eq. (2), where the \( dT/T \) integral is a remnant of this construction.

The einbein representation in Eq. (4) is advantageous when taking the saddle point of the SK path integral. To make contact with the phase space formulation of the quantum mechanics, we can express the initial density matrix in terms of its Wigner transform \( W_A^\chi \)

\[ \chi_A(x^+_i, x^-_i, \lambda^+_i, \lambda^-_i, \lambda^{i+}_i, \lambda^{i-}_i) \equiv \int \frac{d^4p_\mu}{(2\pi)^4} W_A^\chi(\vec{x}_i, \vec{p}_i, \vec{\lambda}_i, \vec{\lambda}_i) e^{i\vec{p}_\mu \cdot \vec{x}_i + \frac{1}{2} \lambda^{i+}_i \lambda_i - \frac{1}{2} \lambda^{i-}_i \lambda_i} \].

\[ (8) \]

Substituting Eq. (7) and Eq. (8) into Eq. (4), the path integral can be performed,

\[ \Gamma_c \approx \int d^4x_i d^4p_\mu d\lambda_i d\bar{\lambda}_i W_A^\chi(\vec{x}_i, \vec{p}_i, \vec{\lambda}_i, \vec{\lambda}_i) \]

\[ \times \int D\epsilon D\phi D\lambda D\bar{\lambda} \exp \left\{ iS_0 + i \int d\tau \left( \left[ \frac{\dot{p}}{\bar{\phi}} \frac{\partial H}{\partial \bar{\phi}} \right] \bar{\phi} - \left[ \frac{\dot{\bar{\phi}}}{\phi} \frac{\partial H}{\partial \phi} \right] \phi - \left[ \frac{\dot{\phi}}{\bar{\phi}} \frac{\partial H}{\partial \bar{\phi}} \right] \bar{\phi} - \left[ \frac{\dot{\bar{\phi}}}{\phi} \frac{\partial H}{\partial \phi} \right] \phi \right) \right\} \]

\[ \times \int D\epsilon D\phi \prod_\tau \delta(\dot{p}(\tau) - \epsilon_0) \delta(\dot{\bar{\phi}}(\tau) - \epsilon_0) \delta(\dot{\phi}(\tau) - \lambda_0) \delta(\dot{\bar{\phi}}(\tau) - \lambda_0) \]

\[ \times \delta(\dot{\bar{\lambda}}(\tau) - \bar{\lambda}_0) \delta(\dot{\phi}(\tau) - \lambda_0) \].

\[ (9) \]

The variables \( \epsilon_0, \lambda_0, \bar{\lambda}_0, \lambda_0, \bar{\lambda}_0 \) satisfy the classical equations of motion which we will specify shortly in section III. The last two delta functions in Eq. (9) impose classical spectral constraints and are obtained by integration over \( \bar{\epsilon} \) and \( \bar{\phi} \).

Eq. (9) contains all the ingredients to construct classical phase space distributions with internal symmetries. Specifically, \( W_A^\chi(\vec{x}_i, \vec{p}_i, \vec{\lambda}_i, \vec{\lambda}_i) \) is a generalized Wigner distribution that coincides with the classical Liouville density in the \( \hbar = 0 \) limit. The Wigner distribution at a given world-line proper time can be obtained from its

\[ ^4 \text{This can be interpreted as the integral over equivalent configuration normalized by the volume of the gauge group (not explicitly written).} \]
initial distribution at $\tau_0$ from the relation
\[
W_A^\chi[\bar{x}(\tau), \bar{p}(\tau), \bar{\lambda}(\tau), \bar{\lambda}^\dagger(\tau)] = \int d^3\bar{x} d^3\bar{p} d\bar{\lambda} d\bar{\lambda}^\dagger W_A^\chi[\bar{x}, \bar{p}, \bar{\lambda}, \bar{\lambda}^\dagger]
\]
\[
\times \prod_{\tau' < \tau} \delta(\tilde{\dot{\bar{p}}}(\tau') - \tilde{\dot{\bar{p}}}_c(\tau')) \delta(\tilde{x}(\tau') - \tilde{x}_{c\tau}) \delta(\tilde{\lambda}(\tau') - \tilde{\lambda}_{c\tau})
\]
\[
\times \delta(\tilde{\lambda}^\dagger(\tau') - \tilde{\lambda}^\dagger_{c\tau}) \delta(P^2\tilde{\tau}(\tau'))\delta(\tilde{\lambda}_{a\tau}\tilde{\lambda}_a(\tau') + \frac{n}{2} - 1),
\]
where $D\tilde{z} \equiv \prod_{\tau' < \tau} d\tilde{z}(\tau')$. This relation can equivalently be expressed as the solution to Liouville’s equation,
\[
\frac{d}{d\tau} W_A^\chi = \left( \tilde{x}_\mu \frac{\partial}{\partial x_\mu} + \tilde{\dot{p}}_\mu \frac{\partial}{\partial \tilde{P}_\mu} + \tilde{\lambda}_a \frac{\partial}{\partial \lambda_a} + \tilde{\lambda}^\dagger_a \frac{\partial}{\partial \tilde{\lambda}^\dagger_a} \right) W_A^\chi.
\]

III. PHASE SPACE REPRESENTATIONS OF COLOR

In this section, we will discuss further the properties of the Wigner distribution $W_A^\chi(x, \vec{p}, \lambda, \lambda^\dagger)$ which obeys the Liouville equation defined in Eq.\(\text{(11)}\). We will show that this Wigner distribution can equivalently be reexpressed in terms of the color charges $Q^a$ that are constructed from Grassmann bilinears. As we will discuss, this formulation is elegant and potentially powerful.

In Eq.\(\text{(11)}\), the Euler-Lagrange equations of motion derived from the world-line effective action are
\[
\dot{x}^\mu = v^\mu, \\
\dot{\bar{p}}_\mu = g_{\lambda_b^A} F^{\mu\nu} \bar{e}_{ab} \lambda_b \nu, \\
\dot{\bar{\lambda}}_a = -ig \nu^\mu \bar{e}_{ab} A^\mu_a \lambda_b, \\
\dot{\lambda}_a = ig \nu^\mu \bar{e}_{ab} A^\mu_b \lambda_a,
\]
where $\nu^\mu \equiv e P^\mu = e[p_\mu - g A^\mu_a(x)Q^a]$. The corresponding Dirac brackets are defined as
\[
\{A, B\} = A \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial \tilde{P}_\mu} \right) B - i A \left( \frac{\partial}{\partial \lambda_a} + \frac{\partial}{\partial \tilde{\lambda}^\dagger_a} \right) B + i A \left( \frac{\partial}{\partial \tilde{\lambda}_a} - \frac{\partial}{\partial \lambda^\dagger_a} \right) B,
\]
and give
\[
\{x^\mu, P^\nu\} = g^{\mu\nu}, \\
\{P^\mu, P^\nu\} = g_{\lambda_b^A} F^{\mu\nu} \lambda_b, \\
\{\lambda_b^A, \lambda_a\} = -i \delta_{ab}, \\
\{P^\mu, F_{\alpha\beta}\} = -(D^\mu F_{\alpha\beta})
\]
The classical color phase space measure at fixed proper time $\tau$ follows directly from the saddle point limit of the SK path integral, Eq.\(\text{(10)}\):
\[
\int d\lambda(\tau) d\lambda^\dagger(\tau) \equiv \int d^n\lambda(\tau) d^n\lambda^\dagger(\tau) \delta(\lambda^\dagger_0(\tau)\lambda_a(\tau) + \frac{n}{2} - 1),
\]
where the integration is over the “classical” Keldysh coordinates which we denoted with bars in Eq.\(\text{[4]}\). The phase space measure represents the integration over the phase space variables for one specific slice in proper time; from now on, we will drop the explicit label $\tau$. Also, $n$ is the dimension of the representation of the color group; we will restrict ourselves to SU(3) in this work.

The Liouville density $W_A^\chi(x, P, \lambda, \lambda^\dagger)$ can be expanded as a power series in $\lambda, \lambda^\dagger$, and is Grassmann even and real. It can therefore be parametrized by the bilinears $\lambda_a^\dagger \Gamma_{ab} \lambda_b$.  

\[\text{[5]}\]
\[\text{[6]}\]
\[\text{[7]}\]
\[\text{[8]}\]

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6 We emphasize that the delta functions in Eq.\(\text{[10]}\) include the last $\tau$ slice, while the path integral excludes it.
7 The reverse is not necessarily true.
8 For simplicity, we will here, and henceforth, drop the bar symbol representing classical phase space coordinates.
where $\Gamma$ are Hermitian matrices. A natural choice for $\Gamma$ are the eight $SU(3)$ generators and the identity, which span a complete trace-orthonormal set,

$$ Q^a \equiv \lambda^a_{\ell c} \epsilon_{cd} \lambda_d, \quad (22) $$

where insertion of the identity $\lambda^a_\ell \delta_{ab} \lambda_b$ produces a phase space invariant and can be dropped. The classical Dirac brackets of the bilinears follow from Eq. (19),

$$ \{ Q^a, Q^b \} = \lambda^\dagger [t^\ell, t^b] \lambda = i f^{abc} Q^c, \quad (23) $$

realizing the $SU(N_c)$ group algebra. Eqs. (19) along with Eq. (22), reproduce Wong’s equations for the precession of color charges in a non-Abelian background field, thereby providing a first principles derivation for the same [4, 5], but also cleanly establishing its provenance and regime of applicability in QCD.

The phase space measure of the $Q^a$ charges was not discussed in these papers. However, by simply defining

$$ \int dQ \equiv \int d\lambda d\lambda^\dagger, \quad (24) $$

the following identities are obtained:

$$ \int dQ = 0, \quad (25) $$

$$ \int dQ Q^a = 0, \quad (26) $$

$$ \int dQ Q^a Q^b = \frac{1}{2} \delta^{ab}, \quad (27) $$

$$ \int dQ Q^a Q^b Q^c = \frac{A_R}{2} d^{abc}, \quad (28) $$

where $A_R$ is the so-called anomaly coefficient of the representation $[61]$. Integrals of higher polynomials of $Q$’s vanish by Grassmann nilpotency. (Eq. (24) and Eq. (28) are proven in appendix [13]).

The most general form of the phase space distribution is thus

$$ W^\lambda(x, P, \lambda, \lambda^\dagger) \to f_A(x, P, Q) = c_0 f(x, P) + c_1 f^a(x, P)Q^a + c_2 f^{ab}(x, P)Q^b Q^a + c_3 f^{abc}(x, P)Q^b Q^c Q^a, \quad (29) $$

with real coefficients $c_i$, which can be determined by taking moments,

$$ f(x, P) \equiv \int dQ f(x, P, Q), \quad (30) $$

$$ f^a(x, P) \equiv \int dQ Q^a f(x, P, Q), \quad (31) $$

$$ f^{ab}(x, P) \equiv \int dQ Q^a Q^b f(x, P, Q), \quad (32) $$

$$ f^{abc}(x, P) \equiv \int dQ Q^a Q^b Q^c f(x, P, Q). \quad (33) $$

Using the properties of the Grassmann algebra, one obtains the form of the phase space distribution to be

$$ f(x, P, Q) = f(x, P)[1 + \frac{2}{A_R d^2} d^{abc} Q^a Q^b Q^c] + 2 f^a(x, P)Q^a, \quad (34) $$

where $d^2 \equiv d^{abc} d^{abc} = N_c^2 - 4$.

Eq. (34) is a key result of our study and is a direct consequence of the Grassmann color algebra obtained from the saddle point limit of the SK world-line path integral. Details of this derivation can be found in appendix [13] where we also show that higher color moments of the phase space distribution are completely determined by the lower singlet and octet moments:

$$ f^{ab}(x, P) = \frac{1}{2} \delta^{ab} f(x, P) + A_R d^{abc} f^c(x, P), \quad (35) $$

$$ f^{abc}(x, P) = \frac{A_R}{2} d^{abc} f(x, P). \quad (36) $$

Similar phase space representations were discussed previously [52, 53, 62, 66]. However the underlying Grassmann algebra was not considered in these works; in its absence, one may conclude that an infinite tower of independent moments analogous to Eqs. (30-33) is feasible. In contrast, we have shown that the entire content of the phase space dynamics is captured by two moments giving the simple form in Eq. (34). Our construction is therefore potentially very useful in further development of the transport theory of color charges.

The bilinears $Q^a(\tau)$ parametrize orbits of the gauge group $SU(3)$ in the classical limit. They coincide with the commuting Darboux variables discussed by Alekseev, Faddeev and Shatashvili [23] (AFS) – albeit their origin in terms of Grassmann coherent states was not considered there. In that work, AFS used spherical coordinates to derive an action principle parametrizing orbits of various compact Lie groups. Specifically, for $SU(2)$, they specified the Darboux variables by the spherical coordinates $\phi, \theta$ and imposed the quantization constraint $\sqrt{Q^a Q^a} = m \sim h$. As a result, the authors arrived at the topological action

$$ S = m \int \cos \theta \, d\phi + \gamma \int d\phi, \quad (37) $$

containing a Berry monopole at $Q^a Q^a = m^2 = 0$. Here $\gamma = 1/2$ if $m$ is half-integer and $\gamma = 0$ if $m$ is integer. This construction is for fixed spin length $m \sim h$ and it is unclear what happens for $h \to 0$. In contrast, no topological term arises in our world-line construction. The classical limit $h \to 0$ is understood to be the saddle point approximation to the SK world-line path integral, where classical and quantum degrees of freedom are manifest in the Keldysh basis [13, 47].
IV. REPRESENTATIONS OF SPIN AND CHIRALITY

We shall now extend the construction developed in the previous section to include spin, with a focus on relativistic spin-1/2 fermions. The Grassmann representation of phase space was obtained by Berezin and Marinov over 40 years ago [1]. The Hamiltonian for a massive colored spin-1/2 fermion is

\[ H = \frac{\epsilon}{2} (P^2 + m^2) + ig \psi^\mu F^a_{\mu \nu} (x) \Gamma^{a}_{bc} \chi^c \psi^\nu \]

where \( g \) is the QCD coupling, \( \psi^\mu, \psi_5 \) are Grassmann spin variables and \( \chi \) is a Grassmann valued supersymmetric partner of the einbein parameter \( \epsilon \). Using this expression for the Hamiltonian and Eq.(16), the classical equations of motion are

\[ \dot{\psi}^\mu = \epsilon P^\mu, \]

\[ \dot{P}^\mu = ig F^a_{\alpha \beta} Q^a \psi^\mu, \]

\[ \dot{\psi}^\alpha = ig F^a_{\alpha \beta} Q^a \psi^\beta, \]

\[ \dot{\lambda}_a^\dagger = -ig \psi^\mu \epsilon_{ab} A^b \lambda^\dagger_a - \frac{ig}{2} \psi^\mu F^a_{\mu \nu} \lambda^\dagger_a \psi^\nu, \]

\[ \dot{\lambda}_a = ig \psi^\mu \epsilon_{ab} A^b \lambda_a + \frac{ig}{2} \psi^\mu F^a_{\mu \nu} \lambda_a \psi^\nu, \]

where \( \psi^\mu \equiv \epsilon P^\mu \) and \( Q^a \) as in Eq.(22). For a suitable choice of \( \chi, \psi_5 \) is not dynamical and can be dropped from the equations of motion. The Grassmann coordinates obey \( \{ \psi^\mu, \psi^\nu \} = -ig \delta^{\mu \nu} \).

The most general form of the Liouville distribution with spin written down by Berezin and Marinov [1] now extended to color – is

\[ W^A_\lambda (x, P, \lambda, \lambda^\dagger, \psi) = W^A_\lambda (x, P, \lambda, \lambda^\dagger) \left[ \sigma_\mu (x, P, \lambda, \lambda^\dagger) \times v_\lambda \right] \]

\[ \psi^\mu \psi^\lambda - \frac{i}{6} \epsilon^{\alpha \mu \beta} v_\lambda \psi^\nu \psi^\alpha \psi^\beta \psi^\lambda, \]

which is uniquely parametrized by a pseudo-vector \( \sigma_\mu (x, P, \lambda, \lambda^\dagger) \). As shown in [1], this form of the distribution function equivalently imposes the spectral constraint [10] given by the second term of Eq.(28). As in the case of color discussed in the previous section, we can organize the Grassmann spin variables into the bilinears

\[ S_{\mu \nu} = -i \psi^\mu \psi^\nu, \]

which can straightforwardly shown to satisfy the Poisson relation

\[ \{ S_{\mu \nu}, S_{\alpha \beta} \} = -g \epsilon_{\alpha \mu \beta} + 2g \sigma_\alpha F^a_{\mu \nu} Q^a_{\nu \beta} v_\mu \]

\[ -v_\nu (D^a F^{\alpha \beta}) Q^a_{\alpha \beta} \sigma_\mu v_\mu. \]

\[ \{ S_{\mu \nu}, S_{\alpha \beta} \} = -g \epsilon_{\alpha \mu \beta} v_\mu + 2g \sigma_\alpha F^a_{\mu \nu} Q^a_{\nu \beta} v_\mu \]

\[ + v_\nu (D^a F^{\alpha \beta}) Q^a_{\alpha \beta} \sigma_\mu v_\mu. \]

Using Eq.(13) and Eqs.(40-42)

\[ \dot{\sigma}_\mu = \{ \sigma_\mu, H \} = \frac{g}{F^0} F^0_{\mu \nu} Q^a \sigma^a + \frac{2g}{F^0} \sigma_\alpha F^a_{\alpha \beta} Q^a_{\nu \beta} v_\mu \]

\[ + \frac{g}{2(F^0)^2} \sigma_\nu (D^a F^{\alpha \beta}) Q^a_{\alpha \beta} \sigma_\mu v_\mu \]

\[ - v_\nu (D^a F^{\alpha \beta}) Q^a_{\alpha \beta} \sigma_\mu v_\mu. \]

where \( F^a_{\alpha \beta} = \epsilon^{\mu \nu \alpha \beta} F^a_{\mu \nu} /2 \). Note that this is the covariant generalization of the spin precession of the quantum mechanical spin three vector defined to be \( \sigma^i = -\frac{1}{2} \epsilon^{ijk} \psi^j \psi^k \).

Our derivation also allows us to identify \( \Sigma^{\mu} (x, P, Q) \) as the proper definition of the Pauli-Lubanski spin pseudo-
vector in quantum field theory. Relative to Eq. (55), the time evolution of $\Sigma_\mu$ has the simpler form

$$\dot{\Sigma}_\mu(x,P,Q) = \frac{g}{P_0} F^a_{\mu\nu} Q^a \Sigma^\nu(x,P,Q) + \frac{2g}{P_0} \Sigma_\alpha(x,P,Q) F^{\alpha\beta}_{\mu\nu} Q^\nu \psi_\beta \psi_\mu .$$  \hspace{1cm} (56)$$

Eq. (56) is the Bargmann-Michel-Telegdi (BMT) equation mentioned previously. It describes the precession of the spin pseudo-vector generalized to relevant inhomogeneous QCD color backgrounds. This is a physical quantity; the relativistic BMT equation (in QED) is routinely used in high energy accelerator physics to describe spin precession of relativistic spinning electrons and protons in external electromagnetic fields. Similarly, from Eqs. (12–13), we obtain Wong’s equation for the color precession of spin-1/2 fermions in background fields.\(^{13}\)

$$\dot{Q}^a = -igv^\mu f^{abc} A^B_\mu Q^c - \frac{gf}{2} f^{abc} \psi_\mu \psi^a \psi^c Q^c .$$  \hspace{1cm} (57)$$

To summarize our discussion, we have obtained a novel expression for the Liouville phase space distribution (given by $f_A(x,P,Q,S)$ in Eq. (15)) and the corresponding phase space measures (given by Eq. (25)–Eq. (28) and Eq. (50)–Eq. (52)) that describes the phase space dynamics of particles with both color and spin internal degrees of freedom. To do so, we identified from first principles in quantum field theory, the canonical Grassmann bilinear variables $Q^a$ and $S^{\mu\nu}$ that satisfy the Wong equation in Eq. (57) and the covariant BMT equation in Eq. (56). We note that besides providing a compact and transparent formulation of semi-classical transport phenomena in quantum field theory, our world-line construction allows one in principle to devise a systematic route beyond the truncated Wigner/classical-statistical semi-classical approximation employed here.

V. SEMI-CLASSICAL CURRENTS AND THE CHIRAL ANOMALY

A transport problem of great interest in a wide range of fields – from astrophysical phenomena to heavy-ion collisions to strongly correlated electronic materials – is chiral kinetic theory. The challenge is to find semi-classical phase space distributions for chiral fermions and to account for the chiral anomaly, when defining vector and axial-vector currents. This is also important in first principles constructions of anomalous hydrodynamics in a number of different physical contexts.\(^{35\text{–}40}\)

Let’s consider first in this chiral kinetic theory context the semi-classical vector and axial vector currents defined naively as phase space averages in our generalized phase space framework. To simplify our discussion, we shall consider only the Abelian case, where the currents are obtained from the phase space average over $\partial S/\partial A_\mu$.

$$\langle J^\mu_{L/R}(x) \rangle \equiv e \int d^4P \, dS \, \epsilon^\mu_{\mu\nu} \, \partial_\nu f(x,P,S) .$$  \hspace{1cm} (58)$$

Here, $\epsilon$ is the electromagnetic coupling. Using the Abelian equivalent of Eq. (58) and the fact that semi-classically $\Sigma^\mu_{L/R} = \pm P^\mu/2P^0$ for left and right handed particles in the chiral limit, one can explicitly perform the Grassmann integral over $S$ to obtain

$$\langle J^\mu_L(x) \rangle = \langle J^\mu_R(x) \rangle = e \int d^4P \, \epsilon^\mu_{\nu} \, f(x,P) ,$$  \hspace{1cm} (59)$$

$$\langle J^\mu_5(x) \rangle = \langle J^\mu_P(x) \rangle = 0 .$$  \hspace{1cm} (60)$$

Both currents are classically conserved, as $\partial_\mu \langle J^\mu_L(x) \rangle = 0$ follows directly from Liouville’s equation and $\partial_\mu \langle J^\mu_5(x) \rangle = 0$ from the antisymmetry of $\epsilon^{\mu
u\alpha\beta}$. This straightforward implementation clearly misses the effects of the anomaly.

The missing piece is obtained from a more careful derivation within the world-line formalism. We will sketch below a variational approach to the derivation of the semi-classical limit of the world-line representation for chiral fermions by carefully repeating the derivation of section II this time for chiral fermions. The corresponding Schwinger-Keldysh real-time many-body world-line path integral is

$$\Gamma[A,B] \equiv \text{tr} \, \int d^4x^+_1 d^4x^-_1 d^4\psi^+_1 d^4\psi^-_1 \, \zeta^{A,B}_\mu (x^+_1,x^-_1,\psi^+_1,\psi^-_1) \int \mathcal{D}x \mathcal{D}P \int \mathcal{D}\psi \int \mathcal{D}\chi \mathcal{D}\chi' \mathcal{D}\chi'' \mathcal{D}c \mathcal{D}c' e^{iS_c[A,B]} ,$$  \hspace{1cm} (61)$$

where the trace is over left and right handed chiral sectors and the initial density matrix $\zeta^{A,B}_\mu$ is in $2 \times 2$ matrix form, respectively. This path integral is embedded in a larger path integral including dynamical gauge fields $A_\mu$ and a novel nondonynamical variational axial-vector gauge field $B_\mu$ that will be put to zero at the end of the derivation. The exponential in Eq. (61) can be written as

$$e^{iS_c[A,B]} \equiv \begin{pmatrix} e^{iS_c[A+B]} & 0 \\ 0 & e^{iS_c[A-B]} \end{pmatrix} .$$  \hspace{1cm} (62)$$

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12 We remark that $\sigma_\mu$ defined in Eq. (57) is equivalent to the phase space coordinates suggested in Eq. (63). This identification however did not take into account the Grassmann origin of $S_{\mu\nu}$. More importantly, $\sigma_\mu$ is not a canonical variable; replacing $S_{\mu\nu}$ by $\sigma_\mu$ fails to preserve the Poisson bracket algebra satisfied by the former.

13 Solutions to Wong’s equations without spin are discussed in [77].
with the spinning particle action given by

$$S[A ± B] = \int d\tau C \left[ p_\mu x^\mu + i 2\psi_\mu \dot{\psi}^\mu - H[A ± B] \right] , \quad (63)$$

where $C$ denotes the 'in-in' (closed) Keldysh time contour. The generalization of the Hamiltonian in Eq.\(\text{(65)}\) is given by

$$H[A ± B] \equiv \frac{\epsilon}{2} (p^2 + i e \psi_\mu F_{\mu\nu}[A ± B] \psi^\nu)$$

$$+ i 4 \left( P_\mu \psi_\mu + \frac{i}{3} \epsilon_{\mu\nu\alpha\beta} P^\nu \psi^\alpha \psi^\beta \right) \chi , \quad (64)$$

with kinetic momentum $P_\mu \equiv p_\mu - e(A ± B)_\mu$. The anti-commuting Lagrange multiplier $\chi$ ensures that the Weyl spectral condition is independently satisfied for both left and right handed fermions [49, 107].

It is sufficient to expand the effective action to linear order in the variational parameter $B_\mu(y)$,

$$\Gamma[A, B] = \Gamma[A] + \int d^4y \frac{\delta \Gamma[A, B]}{\delta B_\mu(y)} |_{B=0} B_\mu(y) , \quad (65)$$

where the linear term $\delta \Gamma / i \delta B_\mu(y) \equiv \langle J_\mu(y) \rangle$ is the expectation value of the axial vector current. The worldline representation for the lowest order term $\Gamma[A]$ is

$$\Gamma[A] = \text{tr} \int d^4 x^+_i d^4 x^-_i d^4 \psi^+_i d^4 \psi^-_i \zeta^A(x^+_i, x^-_i, \psi^+_i, \psi^-_i)$$

$$\times \int dx dp \int D\psi D\chi e^{i S[A]} , \quad (66)$$

which is equivalent to Eq.\(\text{(4)}\) if we replace color by spin. The linear term in Eq.\(\text{(65)}\) can then be written as

$$\frac{\delta \Gamma[A, B]}{\delta B_\mu(y)} |_{B=0} = \text{tr} \int d^4 x^+_i d^4 x^-_i d^4 \psi^+_i d^4 \psi^-_i$$

$$\times \left[ \zeta^{A,B}(x^+_i, x^-_i, \psi^+_i, \psi^-_i) \int dx dp \int D\psi \int D\chi \delta S[A, B] e^{i S[A]} \right. \left. + \frac{i \delta S[A, B]}{\delta B_\mu(y)} e^{i S[A]} \right]$$

$$\times \int dx dp \int D\psi \int D\chi e^{i S[A, B]} \bigg] |_{B=0} . \quad (67)$$

The first term of Eq.\(\text{(67)}\) in the square brackets is the expectation value of the axial-vector current given previously in Eq.\(\text{(68)}\).

$$\langle J^\mu_\mu(y) \rangle = \text{tr} \int d^4 x^+_i d^4 x^-_i d^4 \psi^+_i d^4 \psi^-_i \zeta^A(x^+_i, x^-_i, \psi^+_i, \psi^-_i)$$

$$\times \int dx dp \int D\psi \int D\chi \left( J^\mu_\mu 0 \right) e^{i S[A]} , \quad (68)$$

where

$$J^\mu_{L/R} \equiv \frac{\delta S[A ± B]}{i \delta B_\mu(y)} |_{B=0}$$

$$= \pm e \int d\tau C \left( P^\mu - i \psi^\mu \psi^\nu \partial_\nu \right) \delta[x - y] . \quad (69)$$

To compute the second term of Eq.\(\text{(67)}\), we split the initial density matrix,

$$\zeta \equiv \zeta^{(0)} + \zeta^{(1)} ,$$

where $\zeta^{(0)}$ parametrizes arbitrary occupation numbers of left and right handed fermions at initial time and is independent of $B_\mu$:

$$\zeta^{(0)} \equiv \left( \begin{array}{cccc} \zeta_0^{A}[x^+_i, x^-_i, \psi^+_i, \psi^-_i] & 0 \\ 0 & \zeta_0^{A}[x^+_i, x^-_i, \psi^+_i, \psi^-_i] \end{array} \right) . \quad (70)$$

Only the second piece $\zeta^{(1)}$ is $B_\mu$-dependent and therefore contributes to the second term in Eq.\(\text{(67)}\). It was computed previously in [49] to be

$$\zeta^{(1)} = 2 \bar{\psi} x_{2 \times 2} \left[ \partial_\mu B_\mu(\bar{x}) - \left\{ \partial_\mu, B_\mu(\bar{x}) \right\} \bar{\psi}^\nu \bar{\psi}^\mu \right]$$

$$\times \delta(x^+_i - x^-_i) \delta(\psi^+_i - \psi^-_i) , \quad (71)$$

where $\bar{x} = (x^+_i + x^-_i)/2$ and $\bar{\psi} = (\psi^+_i + \psi^-_i)/2$. This term can be interpreted as a vacuum contribution to the initial density matrix and survives as an anomalous contribution to $\langle \partial_\mu J^\mu_\mu \rangle$ when we set $B_\mu \rightarrow 0$, thereby modifying the expectation value of the axial vector current in Eq.\(\text{(68)}\).

In [49], we showed by analytic continuation in Euclidean space that the anomalous non-conservation of the axial-vector current arises from this contribution and yields the well known expression

$$\langle \partial_\mu J^\mu_\mu(y) \rangle = -\frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}(y) , \quad (72)$$

where $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2$. Eq.\(\text{(68)}\) can be reexpressed, using the canonical transformation to Grassmannian bilinears, as the second equation in Eq.\(\text{(61)}\), while, as we have seen, the derivation of Eq.\(\text{(70)}\) is quite nontrivial.

In some of the condensed matter literature, the emergence of the chiral anomaly is identified with the compressibility of phase space in the presence of a Berry
monopole [26]. However, as we have shown, in our construction, phase space is incompressible and the derivation of the anomaly equation is distinct from these considerations. Indeed, we showed previously that in a Euclidean formulation of the world-line path integral, Berry’s phase is derived from the real part of the effective action; the anomaly, in contrast, arises from the imaginary phase that is made manifest by introducing the auxiliary gauge field $B_\mu$ [48]. Note that a similar observation concerning the connection between Berry’s phase and anomalies was discussed recently in [106–109]. In appendix C we will discuss further formulations of chiral kinetic theory including a Berry term and its interpretation in terms of the chiral anomaly.

VI. SUMMARY AND OUTLOOK

We presented in this manuscript a first principles construction of classical phase space with color and spin internal symmetries employing the Schwinger-Keldysh generalization of the world-line approach to many-body quantum field theory. In this path integral formalism, internal symmetries are expressed by elements of a Grassmann algebra. We obtained the classical phase space limit by taking the saddle-point limit of the quantum mechanical SK world line path integral in the truncated Wigner approximation, which is also equivalent to the classical statistical approximation in field theory. For $SU(N_c)$ color, we derived the quantum Wigner distribution whose dynamics is defined by classical Poisson brackets and a Grassmann valued phase space measure.

A canonical coordinate transformation connects our approach to that of Alekseev, Faddeev and Shatashvili [22] who used commuting Darboux variables to represent the color algebra. The underlying Grassmann algebra has a number of advantages. Without it, color phase space density and Liouville equation are decomposed into an infinite tower of equations, preventing any practical solution. In contrast, we obtained a unique form for the color phase space distribution function, containing only singlet and octet components. Secondly, the action generating the symplectic algebra for color using commuting Darboux variables in [22] contains a topological Berry monopole which is problematic in the classical limit. In contrast, no such topological term arises in our approach and the classical phase space limit is conceptually clean.

We discussed further the construction of classical phase space distributions and the corresponding phase space measure for spin-1/2 fermions. Starting from the SK world-line formulation, we recovered the results of Berezin and Marinov [3]. We explicitly demonstrated that one can write the most general form for the Wigner phase space distribution function in terms of Grassmann spin bilinears and discussed the properties of the corresponding phase space measure. The phase space distribution is parametrized by a pseudo-vector, which is the well-known Pauli-Lubanski vector satisfying the relativistic Bargmann-Michel-Telegdi equation.

We proceeded to construct the vector and axial-vector currents of massless spin-1/2 fermions as phase space averages over the associated Liouville phase space distribution. A naive representation of the latter shows that it is conserved. We showed that the nonconservation of this current due to the anomaly can be traced back to the proper gauge invariant regularization of the spectrum; this is shown to be encoded in the initial conditions for the quantum Wigner distribution in the Schwinger-Keldysh representation.

The results presented in this manuscript are part of a larger effort in constructing a consistent chiral kinetic theory. In related work [23] in preparation, we derive a kinetic theory including collision terms from a fluctuation analysis of the Liouville equation using the relations derived here. We have applied this formalism to the case of a nonequilibrium QCD plasma applying the consistent power counting pioneered by Bödeker [52–110], Arnold, Son and Yaffe [111], and Moore [112–113]. The physics goal of this specific application is to follow the spacetime development of the chiral magnetic current [114, 115] generated in the Glasma produced in ultrarelativistic heavy-ion collisions [116–117], from the initial generation of net topological charge by sphaleron transitions [118], to the transport of this charge in background electromagnetic fields all the way, and ultimately to quantitatively ascertain the physical consequences of the same in experimental observables. While anomalous hydrodynamics is a robust approach to describe the late time dynamics of the chiral current in heavy-ion collisions, the results are very sensitive to the initial conditions. The initial spacetime development of the chiral magnetic current vial sphaleron transitions can be followed using classical statistical methods [119, 120]; however the validity of this approach fails before hydrodynamic equilibrium is attained. It is at this step that chiral kinetic theory is crucial for the subsequent description of chiral transport in the topological QCD background. Our work develops the framework for the implementation of chiral kinetic theory in this context.

Phase space formulations for internal symmetries are also important for future experimental investigations of the spin structure of the proton. For the first time, the proposed Electron-Ion Collider (EIC) [121] will allow a unique three-dimensional tomography of the proton, thus extending previous one-dimensional parton distribution functions to fully five-dimensional quantum Wigner distributions [122–124]. Specifically, the EIC will allow to measure the decomposition of the proton spin’s spin into intrinsic and orbital angular momentum contributions of its parton constituents [125–128]. Recently, there has been considerable progress in studies of polarized parton distributions at small $x$ [129–130]. An outstanding question in this regard is the role of the chiral anomaly [131–132]. The world-line framework is ideally suited for this discussion. We have recently adapted the world-line framework to study unpolarized and polarized parton dis-
tributions at small $x$ \cite{134} and plan to address this issues in the near future \cite{135}.

This formalism may also be useful in extending the regime of validity of effective field theory descriptions \cite{58,60} of gluon saturation because this framework naturally includes a Wess-Zumino term \cite{136,137}.

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Appendix A: Derivation of Liouville’s theorem

In this appendix, we will derive Liouville’s theorem from quantum phase space in the semi-classical approximation. Firstly, we demonstrate how Eq.(11) follows from Eq.(10). We write Eq.(10) in the more compact form

$$ W^\chi_A[x(\tau), \bar{p}(\tau), \bar{\lambda}(\tau), \bar{\lambda}^I(\tau)] = \int d^4\bar{x} d^4\bar{p} d^4\bar{\lambda} d^4\bar{\lambda}^I W^\chi_A[\bar{x}, \bar{p}, \bar{\lambda}, \bar{\lambda}^I] $$

$$ \times \delta[x(\tau) - x(c)(\tau; x_i)] \delta[p(\tau) - p(c)(\tau; p_i)] $$

$$ \times \delta[\lambda(\tau) - \lambda(c)(\tau; \lambda_i)] \delta[\lambda^I(\tau) - \lambda^I(c)(\tau; \lambda^I_i)] , \quad (A1) $$

where $x(c)(\tau; x_i)$ denotes the solution to the classical equations of motion at time $\tau$ for given initial conditions $x_i$. Secondly, taking the derivative $d/d\tau$ yields Liouville’s equation, Eq.(11),

$$ \frac{d}{d\tau} W^\chi_A = \int d^4\bar{x} d^4\bar{p} d^4\bar{\lambda} d^4\bar{\lambda}^I W^\chi_A[\bar{x}, \bar{p}, \bar{\lambda}, \bar{\lambda}^I] $$

$$ \times \left( \frac{\partial}{\partial x^{\mu}} + \bar{p} \frac{\partial}{\partial p^{\mu}} + \lambda \frac{\partial}{\partial \lambda} + \lambda^I \frac{\partial}{\partial \lambda^I} \right) $$

$$ \times \left[ \delta[x(\tau) - x(c)(\tau; x_i)] \delta[p(\tau) - p(c)(\tau; p_i)] $$

$$ \times \delta[\lambda(\tau) - \lambda(c)(\tau; \lambda_i)] \delta[\lambda^I(\tau) - \lambda^I(c)(\tau; \lambda^I_i)] \right] $$

$$ = \left( \frac{\partial}{\partial x^{\mu}} + \bar{p} \frac{\partial}{\partial p^{\mu}} + \lambda \frac{\partial}{\partial \lambda} + \lambda^I \frac{\partial}{\partial \lambda^I} \right) W^\chi_A , \quad (A2) $$

where we used the identity $(d/d\tau) \delta[x(\tau) - x(c)(\tau; x_i)] = \delta[\lambda(\tau) - \lambda(c)(\tau; \lambda_i)] \delta[\lambda^I(\tau) - \lambda^I(c)(\tau; \lambda^I_i)]$ and similar identities for $p, \lambda, \lambda^I$. Alternatively, Eq.(A2) can be written as

$$ \frac{dW^\chi_A}{d\tau} = \{ W^\chi_A, H_W \} , \quad (A3) $$

where $H_W$ is the Weyl symbol of the Hamiltonian and $\{.,.\}$ denote Dirac brackets. Fixing a specific world line parameterization, $\tau = \tau(x^0)$, allows one to write the right-hand of Eq.(A2) in non-covariant form,

$$ (\partial_\tau - \bar{x} \frac{\partial}{\partial x} - \bar{p} \frac{\partial}{\partial p} - \lambda \frac{\partial}{\partial \lambda} + \lambda^I \frac{\partial}{\partial \lambda^I}) W^\chi_A = 0 , \quad (A4) $$

which demonstrates that indeed $dW^\chi_A/d\tau = 0$. The absence of an explicit $\tau$-dependence can alternatively be understood as a gauge symmetry related to reparametrization invariance on the worldline.

Appendix B: Color phase space: Some identities

In this appendix, we will fill in some details of classical phase space using Grassmanian variables for color. First, Eq.(24) is demonstrated through direct integration

$$ \int dQ^a Q^b = \int \lambda d\lambda^I (\chi^{a_e}_{\cdot c} \chi^{a_d}_{\cdot c})(\lambda^{b_c}_{\cdot e} \lambda^{b_f}_{\cdot f}) $$

$$ = -\epsilon_{eg} \epsilon_{df} t^a_{cd} t^b_{ef} = \text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab} , \quad (B1) $$

and similarly for Eq.(25)

$$ \int dQ^a Q^b Q^c = \int \lambda d\lambda^I (\chi^{a_e}_{\cdot c} \chi^{a_d}_{\cdot c})(\lambda^{b_c}_{\cdot e} \lambda^{b_f}_{\cdot f} \lambda^{c_h}_{\cdot h}) $$

$$ = -\epsilon_{eg} \epsilon_{df} (t^a_{de} t^b_{ef} t^c_{gh}) = \text{Tr}(t^a t^b t^c) = \frac{A_R}{2} \delta^{abc} . \quad (B2) $$

To compute the coefficients $c_i$ in Eq.(34), we take moments according to Eqs.(30-33). We obtain

$$ f^a(x, P) = \int dQ^a f(x, P, Q) $$

$$ = \frac{c_1}{2} f^a(x, P) + \frac{A_R}{2} c_2 f^{abc}(x, P) d^{abc} , \quad (B3) $$

$$ f^{ab}(x, P) = \int dQ^a Q^b f(x, P, Q) $$

$$ = \frac{c_0}{2} \delta^{ab} f(x, P) + \frac{A_R}{2} c_1 d^{abc} f^c(x, P) , \quad (B4) $$

$$ f^{abc}(x, P) = \int dQ^a Q^b Q^c f(x, P, Q) $$

$$ = \frac{A_R}{2} c_0 d^{abc} f(x, P) , \quad (B5) $$

$$ f(x, P) = \int dQ f(x, P, Q) = \frac{A_R}{2} c_3 d^{abc} f^{abc} , \quad (B6) $$

where we have used $\langle \lambda \rangle = 0$ and $\langle \lambda^I \rangle = 0$.
so that $c_1 = 2$, $c_2 = 0$, $c_3 = 4/(A_R^2 c_0 d^2)$ and $c_0 = 1$. In the context of Liouville’s equation, one encounters derivatives with respect to color variables and we will find the following identities to be useful:

\[ \int dQ dP f(x,P,Q) = \frac{3}{d^2} d^{abcd} f(x,P), \hspace{1cm} (B7) \]
\[ \int dQ dP f(x,P,Q) = \delta^{ab} f(x,P). \hspace{1cm} (B8) \]

**Appendix C: Compressibility of quantum phase space and the chiral anomaly**

It is well known [33] that the quantum mechanical formulation of phase space violates Liouville’s theorem by compressible corrections to Eq. [A3] at $O(h^2)$. These can be understood from Moyal’s equation

\[ \frac{dW}{d\tau} = -2H_W \sin \left( \frac{\Lambda}{2} \right) W = \{W, H_W\} + O(h^2), \hspace{1cm} (C1) \]

with the Poisson/Dirac brackets as defined in Eq. [I6] and the bilinear operator $\Lambda$ which satisfies

\[ A \Lambda B \equiv \{A, B\}. \hspace{1cm} (C2) \]

In the more general formulation of Eq. [C1], one can systematically derive quantum corrections to the semi-classical approximations presented in this manuscript.

One might also speculate about a connection between the quantum anomaly and (compressible) quantum corrections in Moyal’s equation. We wish to argue that there is no such connection. In fact, one can demonstrate that the generalized coordinates $(x^\mu, p^\mu, \psi^\mu)$, introduce Xkd in section IV are canonical variables and that phase space is incompressible in the semi-classical limit. Yet, in this limit the anomaly is manifest from the SK path integral, as discussed in section V.

In this context, let us consider semi-classical phase space formulations including a Berry term [26, 37, 38, 40], which involves non-canonical variables and compressible phase space in the semi-classical limit [15]. Son and Yamamoto proposed a semi-classical effective many-body theory for chiral fermions following the description of Xiao, Shi and Niu [37] for semi-classical Bloch electrons in weak electromagnetic fields. A similar discussion may also be found in Sundaram and Niu [36]. This theory is summarized in the effective classical equations,

\[ x = \frac{1}{\hbar} \epsilon_n(p) - k \times \Omega_n(p), \hspace{1cm} (C3) \]
\[ \hbar p = eE(x) - e\dot{r} \times B(x), \hspace{1cm} (C4) \]

where $\Omega$ is the Berry curvature [26] and $\epsilon_n$ the energy of the $n$-th energy band. Xiao et al. [37] demonstrated that phase space is compressible in this theory, with the volume element

\[ \Delta V = \frac{\Delta V_0}{1 + eB \cdot \Omega}. \hspace{1cm} (C5) \]

and $\Delta V_0 \equiv d^3xd^3p$ given in the absence of magnetic fields (in the asymptotic past). They further suggested that the particle number density at zero temperature and finite chemical potential should be defined to be

\[ n_c = \int \mu \frac{d^3p}{(2\pi)^3} \left[ 1 + \frac{eB \cdot \Omega}{\hbar} \right], \hspace{1cm} (C6) \]

where the integrand includes modes with energy below the chemical potential $\mu$.

Son and Yamamoto [38, 40] interpret Eq. [C6] as the chiral charge density and obtain their well known anomaly result by assuming constant $\mu$.

In contrast, Xiao et al. [37] give a different interpretation of Eq. [C6], where particle number is conserved. They note that the chemical potential is not constant in a magnetic field. Its time dependence causes a change in the fermi volume [37], precisely compensating the correction term to the electron density in Eq. [C6]. This interpretation is more in line with our understanding that the compressibility of phase space and the chiral anomaly have different origins. However it is indeed remarkable that the expression obtained by Son and Yamamoto from the compressibility of phase space in the presence of a Berry term recovers precisely the form of the anomaly. The puzzle this presents, and its definitive resolution, deserves further study.

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