THE TETRAD FRAME CONSTRAINT ALGEBRA

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ABSTRACT. It is shown via the principle of path independence that the (time gauge) constraint algebra derived in [3] for vielbein General Relativity is a generic feature of any covariant theory formulated in a vielbein frame. In the process of doing so, the relationship between the coordinate and orthonormal frame algebra is made explicit.

INTRODUCTION

It is by now well-known that the canonical constraint algebra for any covariantly constructed field theory is of a fixed form regardless of the theory in question [21]. This algebra reflects the embeddability of a spatial surface into the spacetime manifold, that is, whether the theory is consistent with the foliation of spacetime into arbitrary spacelike hypersurfaces; an assumption that has been used in a derivation of Geometrodynamics [10]. These results were derived using coordinate frames on the spatial hypersurface, and the original translation of the algebra to an orthonormal frame in tetrad gravity [20] was not complete. The correct algebra [4, 5] was derived within a specific model and therefore not obviously a generic feature of tetrad theories. Here it will be shown that the (time gauge) constraint algebra given in [3] may be derived using similar arguments to those in [21], and is therefore generic. Since the result must hold on the entire phase space (not just on the constraint surface), it is important for any quantisation that considers states that do not satisfy the constraints [8].

To prove this result we will proceed in three stages, each with an associated section. First the standard geometry of embeddings is reviewed and extended to the case where one is considering an arbitrary (nondegenerate) linear frame on the spatial hypersurface. We also give the description of the hypersurface geometry that we will be using throughout. Next we review and generalise the principle of path independence that relates the diffeomorphism constraint algebra to the geometry of the hypersurface. The bulk of the paper is devoted to the third section, in which we consider the generators of hypersurface deformation as a means of determining the structure functions that appear in the constraint algebra. In doing so,
we consider two situations explicitly: the first where the frame is chosen to be independent of the embedding, yielding results that may be easily related to the original coordinate frame approach (which is a special case), and the second where the form of the metric is left unaltered by the action of the deformation generators, from which a limit to an orthonormal frame is straightforward.

1. the Embeddings and Hypersurface Geometry

The basic object throughout this work will be the embedding \( e \in \text{Emb}_g(\Sigma, \mathbf{M}) : \Sigma \to \mathbf{M} \), which maps an \( n \)-dimensional, Riemannian manifold \( \Sigma \) into an \( n + 1 \)-dimensional, pseudo–Riemannian (Lorentzian) manifold \( \mathbf{M} \). In coordinate charts represented by the coordinates \( x^i \) on \( \Sigma \) and \( x^\mu \) on \( \mathbf{M} \) (lowercase Greek and Roman indices will represent spacetime and spatial coordinate indices throughout), the embedding takes the form \( e : x^i \to e^a(x^i) \). Although we will not discuss the geometry of hyperspace explicitly herein, much of the formalism is directly adapted from Kuchař [15, 16] with slight changes in notation.

We introduce an arbitrary linear frame (and dual coframe) on \( \Sigma \) related through the vielbein \( E^a \) to a coordinate frame and coframe as \[ E^a = E^a_i(x) \partial_{x^i} \] and \[ \theta^a = dx^i E^a_i(x) \], respectively, where the vielbeins satisfy the frame duality conditions \( E^b_i E^i_a = \delta^a_b \) and \( E^a_j E^j_b = \delta^a_b \), and \( \partial_{x^i} \) indicates the partial derivative with respect to \( x^i \) on \( \Sigma \). (Note that by construction we are singling-out a normal frame vector, and so the results derived will correspond to the time gauge of [3].) We will use \( a, b, c \ldots \in \{1, 2, 3, \ldots n\} \) to indicate the components of a tensor in this surface frame (all results herein will apply to hypersurfaces of arbitrary dimension \( n \)), and \( A, B, C \ldots \in \{0, 1, 2, 3, \ldots n\} \) for a frame above \( \mathbf{M} \). This linear frame and its dual are used as a basis in the tangent bundle \( T\Sigma \) and tensor bundles associated to it. This is by now a standard procedure [13, 14] which has been applied to general relativity [14, 15].

The transformations that (locally) make a change of frame—or frame rotation—on \( \Sigma \) are elements of \( \text{GL}(n, \mathbb{R}) \), acting on the frame and coframe (and similarly on the components of tensors) as: \( e_a \to |M|^{-1} e_b^a e_b \) and \( \theta^a \to M^a_b \theta^b \), where \( M \in \text{GL}(n, \mathbb{R}) \). Considering the infinitesimal form of these transformations \( M_a^b \to \delta_a^b + \Omega^a_b \), one finds the the Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \) is generated by an operator \( \Delta^a_{bx} \), defined to act on vectors and covectors as (extendible to arbitrary tensors in the usual manner)

\[
\Delta^a_{bx}[V_c(y)] = -\delta^a_c V_b(y)\delta(y, x), \quad \Delta^a_{bx}[V^c(y)] = \delta^a_b V^c(y)\delta(y, x),
\]

and thus \( \Omega^a_b(x) \Delta^a_{bx} \) is the infinitesimal form of the frame rotation considered above. The Lie algebra of these frame rotation generators is straightforward to compute, yielding the standard
result

\[ [\Delta^a_{bx}, \Delta^c_{dy}] = \delta^a_b \Delta^d_{cx} \delta(x, y) - \delta^a_d \Delta^c_{by} \delta(y, x). \]  

Note that throughout we will be considering the full set of \( \mathfrak{gl}(n, \mathbb{R}) \) generators; reducing the frame bundle to \( \text{SO}(n) \) and considering therefore the generators of \( n \)-dimensional rotations is equivalent to considering only the antisymmetric generators.

Since the embedding clearly maps curves in \( \Sigma \) to those in \( M \), it induces the pullback map \( e_* : T_{e(x)}M \to T_x \Sigma \) which has the local form

\[ e^0_a(x) = E_{ax}[e^0(x)] = E^i_a(x) \partial_x [e^0(x)], \]

where the presence of the vielbein is required in order to map the components of vectors in \( T_{e(x)}M \) to vectors in \( T \Sigma \) that are expanded in the chosen linear frame. The set of vectors \( e_a := e^0_a \partial_x \) defines an \( n \)-dimensional subspace of \( TM \), the remaining dimension spanned by the normal vector \([11]\), defined to be a unit vector in \( T_{e(x)}M \) that is orthogonal to \( e_a \):

\[ g(n, e_a) = n_\alpha e^\alpha_a = 0, \quad g(n, n) = n^\alpha n_\alpha = 1. \]

These conditions combined with the fact that the spacetime metric \( g_{\alpha\beta} \) has been chosen to have \((+, - , - , - , \ldots)\) signature, reflect the fact that \( \Sigma \) is a spacelike embedded surface. The chosen normal vector combined with the pullback map (viewed as \( n \) vector fields on \( M \)) \( \{ e_A \} := \{ n, e_a \} \) define a frame on which all spacetime tensors may be decomposed, separating them into normal and tangential components to \( \Sigma \) respectively. For example, a vector field is decomposed as \( V^\alpha = V_n n^\alpha + e^\alpha_a V^a \), where \( V_n := V^\alpha n_\alpha \) is the scalar component perpendicular to \( \Sigma \) and \( V^a := V^\alpha e^\alpha_a \) are the vector components that are tangential to \( \Sigma \). The (negative-definite) metric over \( \Sigma \) is defined as the pullback of the spacetime metric \( g_{\alpha\beta} := [e_* g]_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \). The spacetime metric takes on the projected form \( g_{\alpha\beta} = n_\alpha n_\beta + g_{ab} e^\alpha_a e^\beta_b \), or in terms of the dual basis \( \{ n, \theta^a \} \) (where \( \theta^a[e_b] = \delta^a_b \)) is \( g = n \otimes n + g_{ab} \theta^a \otimes \theta^b \). Throughout Greek and Roman indices are ‘raised’ and ‘lowered’ using \( g_{\alpha\beta} \) and \( g_{ab} \) respectively. This projection is extendible to higher-order tensors in a straightforward manner \([15]\).

The foliation of \( M \) by non-overlapping spacelike hypersurfaces is accomplished by introducing a family of embeddings \( e(t) \) that cover \( M \) indexed by a coordinate \( t \), effectively realizing the diffeomorphism \( e(t, x) : \mathbb{R} \times \Sigma \to M \). The resulting family of pullback maps then give \( e_*(t, x) : T_{e(t, x)}M \to T_t \mathbb{R} \times T_x \Sigma \), where a tangent to \( \mathbb{R} \) is related to the lapse function \( N \) and shift vector \( N^a \) by

\[ N := n_\alpha \partial_t [e^\alpha], \quad N^a := e^\alpha_a \partial_t [e^\alpha]. \]
This decomposition makes clear that the lapse function and shift vector represent the normal and tangential (to \( \Sigma \)) respectively of the time vector \( \partial_t \). It is also clear that the local ‘flow of time’ (or choice of family of embeddings) near a hypersurface is parameterised by a choice of \( N \) and \( N^a \). A sans-serif font will be used in order to indicate nontrivial dependence on the embedding (i.e., not just via the surface projection); notable exceptions to this will be the extrinsic curvature \( k_{ab} \) and in trinsic connection coefficients \( \Gamma^a_{bc} \).

The coordinate frame inverse of the pull-back map: \( \frac{\partial t}{\partial x^\mu} = \frac{\partial t}{\partial y^\mu} \) and \( \frac{\partial x^i}{\partial y^\mu} = e^i_\alpha - \frac{N^i}{N} n^\alpha \), may be used to make the mapping between \( \{ \partial_t, E_a \} \) and the spacetime, surface-compatible frame \( \{ n, e_a \} \):

\[
n = \partial_n := n^\alpha \partial_\alpha = N^{-1}(\partial_t - N^a E_a), \quad e^a_\alpha \partial_\alpha = E_a = E_a^i \partial_i,
\]

may be derived. This allows us to do a projection of the spacetime geometry onto \( \Sigma \) and in general do away with the spacetime coordinates \( x^\alpha \) altogether, so that, for example, the argument of \( F(x) \) indicates dependence on a point in the spatial hypersurface.

In the frame \( \{ n, e_a \} \) we may compute the Levi-Civita connection coefficients from the metric compatibility conditions and the condition of vanishing torsion, to find: \( \Gamma^a_{nn} = \Gamma^a_{an} = 0 \), \( \Gamma^a_{na} := a_a = -g_{ab} \Gamma^b_{nn}, \Gamma^a_{ab} = \Gamma^a_{ba} := k_{ab} = -g_{ac} \Gamma^c_{bn} \), and \( \Gamma^a_{bn} = \Gamma^a_{nb} + C_{bn}^a \). In these, we have used (6) to derive the nonvanishing structure constants of the pulled-back basis vectors: \( [e_a, e_b] = C_{ab}^c e_c \) and \( [n, e_a] = a_a n + C_{na}^b e_b \), where \( C_{ab}^c = \frac{E_i}{E_a^c(E_a[E_b] - E_b[E_a]}) \) are the structure constants for the basis defined on \( \Sigma \), and

\[
a_a := C_{na}^b = -\Gamma^\nu_{\mu\beta} n^\mu e^\nu_{\beta/a} = n^\mu \partial_\nu [e^\mu_{\beta/a}] - n^\mu E_a [n^\mu] = E_a [\ln N],
\]

\[
C_{na}^b = e^b_\mu \partial_\nu [e^\mu_{\alpha/n}] - e^b_\mu E_a [n^\mu] = N^{-1}(E_a N^b) + N^c C_{ac}^\nu E^\nu_{\mu} + \partial_\nu [E_a i_E^i] E^b_i.
\]

All of these quantities have been written in terms of the hypersurface geometry and the local parameterisation of the family of embeddings via \( N \) and \( N^a \).

In [7], we have introduced the coordinate components of the Levi-Civita connection on \( \mathcal{M} \), which may be related to that on \( \Sigma \) by

\[
\Gamma^a_{bc} = e^a_\mu E_{b} [e^\mu_{\beta/c}] + e^\nu_{\beta/c} e^\mu_\alpha \Gamma^a_{\mu\nu},
\]

which satisfies the intrinsic metric-compatibility conditions \( \nabla_{\Sigma} [g]_{ab} := E_c [g_{bc}] - \Gamma^d_{ca} g_{db} - \Gamma^d_{db} g_{ad} = 0 \). The presence of the spacetime metric implies that such a connection is uniquely defined, and so may be introduced regardless of whether or not it plays a fundamental role in the theory in question. Other projections of the spacetime compatibility conditions result in
n^\mu \partial_\alpha [n_\mu] = n_\mu n^\nu \Gamma^\mu_{\alpha \nu}, \text{ and define the extrinsic curvature } k_{ab} \text{ of } \Sigma

\partial_n [g_{ab}] - 2g_{c(b}C_{na)}^{\ c} + 2k_{ab} = 0, \quad (9a)

k_{ab} := n_\alpha E_a[e^\alpha_b] + n_\gamma e^\alpha_b \Gamma^\gamma_{\alpha \beta} = -e^\beta_b E_a[n^\beta] - n^\gamma e_{ab} e^\gamma_{\alpha \mu}, \quad (9b)

which may be shown to be a hypersurface tensor under frame rotations. Note that this structure is identical to what appears when introducing a surface adapted basis \cite{11}, and as in that reference, we will use the minimal number of defined quantities \(a_\alpha, k_{ab}, \Gamma^a_{bc},\) and \(C_{\alpha a}^b\) throughout.

We will also require some of the projected components of the spacetime Riemann tensor defined in general by \(R^A_{BCD} = E_C[G^A_{DB}] - E_D[G^A_{CB}] + \Gamma^E_{DB} \Gamma^A_{CE} - \Gamma^E_{CB} \Gamma^A_{DE} - C_{CDE} \Gamma^A_{EB}.\) The components with one normal projection are related by \(R^b_{can} = -g^{bd}R^n_{adc},\) where \(R^n_{abc} := e^\alpha_m e^\beta_n e^\gamma_{\alpha \mu} n^\mu \Gamma^\alpha_{\alpha \beta} \text{ and } R^n_{abc} := e^\gamma_a e^\alpha_c n_\mu R^\alpha_{\gamma \alpha \beta} = \nabla_b[k]_c - \nabla_c[k]_b,\) in particular implying that

\begin{align*}
R^a_{bcdn} &= -\nabla_c[k]^a_b - k^a_b a_c - k^a_c a_b + k_{cb} a^a + E_c[C_{nb}^a] + C_{nb}^a a_c \\
&- \partial_n[\Gamma^a_{cb}] + C_{nb}^b \Gamma^a_{ce} - C_{ne}^a \Gamma^e_{cb} + C_{nc}^b \Gamma^a_{eb} \\
&= \nabla_c[k]^b_a - g^{bd} \nabla_d[k]_a.
\end{align*}

The intrinsic Riemannian curvature of \(\Sigma\) defined in terms of the intrinsic connection components \(\Gamma^a_{bc}\) will be denoted \(\Xi^a_{bcd},\) and should not be confused with the spatially projected components of the spacetime Riemann tensor \(R^a_{bcd} := e^\alpha_m e^\beta_n e^\gamma_{\alpha \mu} R^\alpha_{\beta \mu \nu}.\)

Note that the results of this work do not require that one consider a metric compatible theory nor even a metric theory. The reason is that for any such theory, one is still required to define a normal vector in order to pass to the Hamiltonian formalism. This definition requires a symmetric tensor that plays the role of a metric in \(\Xi,\) and therefore also defines the Riemannian geometry described in this section; it is this metric that will therefore appear in the constraint algebra. Note also that this construction will not tell one anything about the consistency of the Cauchy problem (as may be seen by the fact that the original investigations of the algebra from this point of view consider both Lorentzian and Riemannian signatures for the spacetime metric), merely whether the dynamics are consistent with the geometry.

2. The Principle of Path Independence

The ‘principle of path independence’ expresses the conviction that a physical system set up on an initial Cauchy surface \(\Sigma_i\) should evolve to a unique state on a later hypersurface \(\Sigma_f\) regardless of how one looks at the evolution in-between \cite{21, 17}. In order to examine the consequences of this principle, note that the lapse function and shift vector \(N^A := (N, N^f)\) (as defined through the family of embeddings by \cite{3} or as metric components in the usual
GR definition [2]) are geometrical objects that encode the direction in which the surface is evolving in spacetime, or, equivalently, how the surface coordinate labels are evolving with time. If we consider infinitesimal evolution from $\Sigma_i$ along $N^A_1$ followed by evolution along $N^A_2$, the resulting hypersurface $\Sigma_f = \Sigma_{12}$ is in general a different hypersurface than $\Sigma_{21}$ which results from evolution along $N^A_2$ followed by $N^A_1$. The deformation vector $N^A$ that evolves $\Sigma_{21}$ to $\Sigma_f$ will be related to $N^A_1$ and $N^A_2$ in general through the structure functions $\kappa^{A}_{BC}$ defined by [21]

$$
N^A(x) = \int_{\Sigma} dy_1 \int_{\Sigma} dy_2 \kappa^{A}_{BC}(x;y_1,y_2)N^B_1(y_1)N^C_2(y_2).
$$

(11)

Here we assume for simplicity that $\Sigma_{21}$ is completely to the past of $\Sigma_f$ so that we may consider further evolution from $\Sigma_{21}$ to $\Sigma_f$. As we shall see, the structure functions $\kappa^{A}_{BC}$ are determined completely from the hypersurface geometry alone (i.e., they are intrinsic) and are the same structure functions that appear in the constraint algebra (as we will discuss in the following paragraphs). It is the computation of these structure functions which makes up the bulk of the original coordinate frame work [21] as well as the present manuscript.

Translating the principle of path independence to a Hamiltonian system is a fairly straightforward procedure; we merely consider the evolution of initial data on $\Sigma_i$ to data on $\Sigma_f$ along the same two paths considered above. The real content of the argument rests in recognising the fact that the general form of the Hamiltonian for a diffeomorphism invariant system may be put into a very specific form. We will begin by reviewing the coordinate frame argument of [21, 10], and afterwards see that the generalisation to an arbitrary choice of linear frame is fairly straightforward.

It is by now well-known that the lapse function and shift vector will appear undifferentiated in the action for any covariantly constructed Lagrangian. That this is so follows from the fact that a covariant action allows the use of any family of embeddings that cover $M$, and therefore the local parameterisation of the family (the lapse and shift) should be freely specifiable. Therefore considering the case of gravitational theories for which the fields that describe the intrinsic geometry appear in phase space as canonical coordinates, treating $N$ and $N^i$ as Lagrange multipliers the Hamiltonain may be written in the form $H = \int_{\Sigma} dx(N\mathcal{H} + N^i\mathcal{H}_i)$ (modulo surface terms which will be systematically ignored throughout this work), immediately resulting in the Diffeomorphism constraints $\mathcal{H} \approx \mathcal{H}_i \approx 0$. Although we will consider this case exclusively, it is not difficult to extend the results to parameterised field theories propagationg on a fixed spacetime background where where $\mathcal{H}$ and $\mathcal{H}_i$ are the energy and momentum respectively of the field, as well as to unparameterised fields where there is a slight change in the ensuing constraint algebra [10].
The evolution of a scalar functional $F$ on phase space as it evolves from an initial hypersurface to a final one related to it via the infinitesimal deformation given by the parameters $N^A := (N, N^f)$ is determined by

$$\delta F = \dot{F} = \{F, H\} = \int_{\Sigma} dx \{F, N^A \mathcal{H}_A\},$$

(12)

where the standard Poisson brackets \[9\] have been assumed. Using the path independence argument, the change in $F$ as the system evolves from $\Sigma_{21}$ to $\Sigma_f$ is given by

$$F_{\Sigma_{12}} - F_{\Sigma_{21}} = \int_{\Sigma} dx \{F, N^A \mathcal{H}_A\}$$

$$=- \int_{\Sigma} dy_1 \int_{\Sigma} dy_2 \left\{F, \{N^A_1(y_1) \mathcal{H}_A(y_1), N^A_2(y_2) \mathcal{H}_A(y_2)\}\right\},$$

(13)

the first of which is the direct evolution using $N^A$ as defined by (11), and the latter is the difference as given by the evolution along the alternating paths (the Jacobi identity has been used to write it in this form).

Inserting the structure functions (11) and noting that the lapse and shift may be pulled through the Poisson brackets (we are assuming that the diffeomorphism constraints are satisfied), we find the condition

$$\{F, \{\mathcal{H}_B(y_1), \mathcal{H}_B(y_2)\} + \int_{\Sigma} dx \kappa^A_{BC}(x; y_1, y_2) \mathcal{H}_A(x)\} = 0.$$ 

(14)

Removing the functional $F$ from this condition is straightforward \[21\]. If the argument of the overall Poisson bracket ($\{F, \cdot\}$) depends in a nontrivial way on canonical variables, then $F$ may be chosen as a functional that is conjugate to it, and (14) would not be satisfied. Therefore the argument must be a constant on phase space, and since the diffeomorphism constraints are satisfied, this constant factor must be identically zero, resulting in the condition

$$\{\mathcal{H}_B(y_1), \mathcal{H}_B(y_2)\} = - \int_{\Sigma} dx \kappa^A_{BC}(x; y_1, y_2) \mathcal{H}_A(x),$$

(15)

which is the relation between dynamics and geometry that we have been seeking.

For the more general case at hand, this argument is actually extended in a very straightforward manner. Since we have introduced an arbitrary frame on $\Sigma$, the Lagrangian will possess a $\text{GL}(n, \mathbb{R})$ symmetry in addition to diffeomorphism invariance. Using the argument in \[6\] we know that variations with respect to the spatial vielbeins $E_i^a$ are not independent of those with respect to the surface metric $g_{ab}$, and therefore when passing to the Hamiltonian formalism there will be $n^2$ constraints that must be imposed via Lagrange multipliers $N^a_b$ (which may be chosen to be the time component of the Ricci coefficient in tetrad gravity \[3\]). Since the lapse
and shift play exactly the same role as they did previously, the Hamiltonian takes the form

\[ H = \int \Sigma \, dx (N^a H_a + N^b \mathcal{J}_a^b), \]  

(16)

and since by construction we have that \( \mathcal{J}_a^a \approx 0 \), the argument therefore proceeds exactly as before, with now \( N^a := (N, N^a, N^b) \).

The decomposition (16) of the constraints is far from unique, however there is at least a natural identification of \( \mathcal{J}_a^b \) with the generators of \( \mathfrak{gl}(n, \mathbb{R}) \). Here we will assume throughout that \( \mathcal{J}_a^b \) are the phase space representation of the Lie algebra of frame transformations in \( \Delta_a^b \) as described in (1). In fact, just as the coordinate frame surface diffeomorphism generators \( \mathcal{H}_i \) may be deduced purely from the fields in question [11], it is also always possible to construct the generators \( \mathcal{J}_a^b \) purely from the chosen parameterisation of phase space. The argument is a straightforward extension of the following example.

Since we are dealing with a general linear frame on \( \Sigma \) there are two densities to take into account: \( \sqrt{-g} \) as well as \( E := \det(E^i_a) \). If we have chosen the lapse function and shift vector to be of weight zero (i.e., an ordinary scalar and vector respectively) then all of the constraints must be of weight one in each, in particular \( \mathcal{J}_a^b := E \sqrt{-g} J_a^b \) for \( J_a^b \) an ordinary tensor. Consider the simple case of a hypersurface vector field \( Q^a := \sqrt{-g} Q^a \) and its conjugate \( P_a := EP_a \) that appear as coordinates in phase space. It is straightforward to see that \( \mathcal{J}_a^b = Q^a P_b + \delta^a_b Q^c P_c \) properly generates frame rotations on the \( (Q^a, P_a) \) sector of phase space (note that effect of a frame rotation on the densities is nontrivial) without affecting any other canonical variable. (In fact, the action of \( \mathcal{J}_a^b \) is only identical to that of \( \Delta_a^b \) up to a sign, since \( \Delta_a^b \) acts from the left and \( \mathcal{J}_a^b \) acts from the right via the Poisson bracket; see the comments following (21).) This is easily generalised to vectors of arbitrary weight by adjusting the coefficient of trace term, and higher-order tensors by considering the possible \((1,1)\) tensors built from contractions of the coordinate with the momenta.

That one may do this is important since it guarantees that the chosen form of \( \mathcal{J}_a^b \) will satisfy the Lie algebra (2) of \( \mathfrak{gl}(n, \mathbb{R}) \) strongly (i.e., on all of phase space), and \( \mathcal{J}_a^b \approx 0 \) are therefore first class constraints, closing separately from the diffeomorphism constraints. In contrast, we will not assume any particular form or action the diffeomorphism generators (which may clearly mix linearly with the frame rotation generators as, for example \( \tilde{\mathcal{H}} = \mathcal{H} + \kappa \mathcal{J}_a^a \) and \( \tilde{\mathcal{H}}_a = \mathcal{H}_a + \lambda \nabla_a (\mathcal{J}_a^b) \)). In particular, we will consider the following two cases: The first where \( \mathcal{H}_a \) acts on the components of tensors as a Lie derivative without affecting the vielbeins at all (as in the standard coordinate frame approach), and the second where \( \mathcal{H}_a \) acts on the components of tensors to give a covariant derivative (and therefore the action on the surface metric vanishes using metric compatibility) while at the same time producing a rotation of the
spatial vielbeins. (In this case there is also a mixing of $H$ with $J_{abc}$ in order to guarantee that the form of the spatial metric is not affected by the action of $H$.)

3. The Hypersurface Deformation Algebras

As mentioned in the previous section, we will be considering the deformation generators in order to compute the structure functions $\kappa^A_{BC}$. These generators are defined by

$$\delta \alpha x := \delta / \delta e^\alpha (x), \quad \delta \eta x := n^\alpha (x) \delta_{\alpha x}, \quad \delta a x := e^\alpha_a (x) \delta_{\alpha x},$$

and generate deformations of the image of $\Sigma$ in $M$. This may be seen by considering a deformation of $\Sigma$ that is represented by an infinitesimal change in the embedding as $e \to e + \delta e$.

A functional of the embedding will then change by

$$F[ e ] \to F[ e + \delta e ] = F[ e ] + V^\alpha \delta_{\alpha x} [ F ]|_e,$$

where $V^\alpha$ represents the vector along which the surface is deformed. This is, of course, the basis for the geometry of hyperspace considered extensively by Kuchar, and to quote [12]: “A reader perverse enough to ask for more details is referred to [15, 16].” In particular, if we deform $\Sigma$ along a chosen family of embeddings represented locally by $N$ and $N^a$, then the (coordinate frame) deformation operator $\int_{\Sigma} dx (N(x) \delta_{\alpha x} + N^a(x) \delta_{a x})$ will be identical to the time derivative operator $\partial_t$ on all tensors, consistent with the fact that $N$ and $N^a$ are the normal and tangential projections respectively of $\partial_t \in T_t \mathbb{R}$.

Here we will actually have to deal with generalisations of the time derivative operator since the components of any tensor are defined with respect to a spatial frame, and there is a difference between the partial derivative of the components of a tensor and the partial derivative of a tensor expanded in the local frame. Hence we will identify two such operators, on of which is the partial derivative operator that acts on tensor components as $\partial_t : T^a_{mn\cdots} \to \partial_t [ T^a_{mn\cdots} ]$, and the other is the total derivative operator that acts as $d_t : T^a_{mn\cdots} \to \theta^a \otimes \theta^b \cdots E_m \otimes E_n \cdots \partial_t [ T^a_{mn\cdots} E_{\alpha'} \otimes E_{b'} \cdots \theta^{\alpha'} \otimes \theta^{b'} \cdots ]$ (i.e., that takes into account the evolution of the frame as well). Clearly these operators are identical when operating on scalars and the tensors themselves (not just the components) and are related, for example on a covector field by

$$d_t [ V_a ] = [ \partial_t [ V_b \theta^b ] ] [ E_a ] = \partial_t [ V_a ] - \partial_t [ E_a ] E_t^b V_b.$$ (18)

The linear frame generalization of the variation of $e_\alpha$ and $n^\alpha$ given in [21, 11] may be easily computed from (3), yielding

$$\delta_{\alpha x} [ e^\alpha_b (y) ] = \delta_{\alpha x}^b E_{by} [ \delta (y, x) ] + e^\alpha_a (y) \delta_{\alpha x} [ E_b^i (y) ] E_i^a (y).$$ (19)
for the embedding, and from the variation of (1)

\[ \delta_n^\mu(y) = \partial_n n^\mu(y) \delta(y, x) - e^{ab}(y) (E_{by}[\delta(y, x)] - a_b(y) \delta(y, x)), \]  
\[ \delta_a^\mu(y) = E_{ay}[n^\mu(y)] \delta(y, x), \]  
(20a)

where \( \delta(x, y) \) is defined as a scalar at \( x \) and a density at \( y \). These tell us how the pullback and normal change as the surface is deformed and depend in general on the chosen family of embeddings via the presence of \( N \) in (20a).

We begin by choosing the frame on \( \Sigma \) such that it is completely decoupled from the embedding and geometric structure of \( M \) (i.e., \( \delta_a^x y = 0 \)). This does not mean that the vierbein is in any way trivial, just that it is chosen without reference to the embedding; we will come back to this later on in this section. Using (19) and (20), and the fact that the coordinate frame deformation vectors commute \( [\delta_n^x y, \delta_a^b y] = 0 \), we calculate the generalized commutator algebra

\[ [\delta_n^x y, \delta_n^x y] = g^{ab}(x) E_{ax}[\delta(x, y)] \delta_{bx} - g^{ab}(y) E_{ay}[\delta(y, x)] \delta_{by}, \]  
\[ [\delta_a^x y, \delta_n^x y] = -E_{ax}[\delta(x, y)] \delta_{nx}, \]  
\[ [\delta_a^x y, \delta_b^y] = E_{by}[\delta(y, x)] \delta_{ay} - E_{ax}[\delta(x, y)] \delta_{bx} - C_{ab}^c(y) \delta(y, x) \delta_{cy}. \]  
(21a)

Due to the scalar and vector quality of \( \delta_n^x y \) and \( \delta_a^x y \) one finds

\[ [\Delta_{by}^a, \delta_n^x y] = 0, \quad [\Delta_{bx}^a, \delta_{cy}] = -\delta_{cx}^a \delta(y, x) \delta_{by}, \]  
(21b)

and (21) combined with (2) complete the commutation algebra of the set \( (\delta_n^x, \delta_a^x, \Delta_a^b) \) of surface deformations and frame rotations. (In deriving these, the general rule

\[ f(x) \partial_{y^i} \partial_{y^j} \cdots [\delta(y, x)] = \partial_{y^i} \partial_{y^j} \cdots [f(y) \delta(y, x)], \]  
(22)

is useful, as is the variation of the spacetime point with respect to the embedding, as in the case of a scalar \( \delta_a^x y [f \varepsilon(y)] = \partial_a \varepsilon(y) \delta(y, x). \) )

There are two ways to see that the structure functions in (21) are identical to those appearing in (15). The most straightforward is to note that one would expect the generators of hypersurface deformations to have an equivalent action as the generators of dynamical evolution (and frame rotation) as given by the Hamiltonian system. This is in fact the case, and was used in (19) in an identical manner to derive the constraint algebra in a coordinate frame. Therefore we may relate the operators \( \delta[\cdot] \) directly to \( \{\cdot, \mathcal{H}\} \), and merely replace the operators \( (\delta_n^x, \delta_a^x, \Delta_a^b) \) in (2) and (21) with \( (\mathcal{H}, \mathcal{H_a}, \mathcal{J}_a^b) \) (up to a sign since the former act from the left and the latter act from the right (15).) The result of this is that the structure constants \( \kappa_{BC}^A \).
may be read off of (21) directly, and in the coordinate frame limit \((E_a^i(x) = \delta^i_a)\) agrees with previous results \([21, 15]\). Alternatively one may construct (11) directly, which results from smearing (21) with \(N^1_1(x)\) and \(N^2_2(y)\). That \(N^1_1(x)\) should occur outside the commutators is due to the fact that by assumption they represent families of embeddings with respect to any hypersurface, that is, the components \(N^A_{1,2}(x)\) are taken to be the same on any hypersurface.

That the structure functions should be determined by geometry alone is merely a reflection of the fact that the setting itself is purely geometrical; we have not committed ourselves to a particular dynamical model, we have stated that it should be true regardless of the model in question, and in fact the method that we will employ here in order to compute these structure functions will reflect this. The original derivation of Teitelboim \([21]\) consisted of a direct computation of \(N^A\) from \(N^A_{1,2}\) via Taylor expansion, however instead we will follow the more geometric and systematic procedure of \([10]\) whereby we consider the action of the generators of surface deformations directly. Thus the construction is far more powerful since the structure functions may be related to any tensor, and we end up with generators that in fact represent the geometrical content of any (covariant) dynamical system represented by a Hamiltonian.

When transferred to the constraints this algebra may be related to the Bianchi identities; if the evolution equations are all satisfied, what remains are the evolution equations for the constraints \([1]\), and \(\mathcal{H}\) and \(\mathcal{H}_a\) are just combinations of (21). This is made more explicit in \([1]\), where the algebra appears as evolution equations for the constraints on the space of gravitational degrees of freedom that satisfy the evolution equations but not (necessarily) the constraints. This is sensible since both are consequences of diffeomorphism invariance, and thus the constraint algebra may be considered to be the Hamiltonian form of the Bianchi identities.

The commutator algebra (21) and (2) may also be determined explicitly from the action of the generators on various objects. By assumption, the variations do not affect the frames themselves \((\delta_{ax}[E_a^i] = 0)\), and their action on tensors above \(\mathcal{M}\) that have been pulled-back to tensors above \(\Sigma\) may be determined from the explicit form of the pull-back \((i.e., from V_a = e_a^\alpha V_\alpha)\). The action of the perpendicular generator is perhaps slightly more complicated than would be expected, yielding for example on a covector

\[
\delta_{nz}[V_a(y)] = \partial_{ny}[V_a(y)]\delta(y, x) - C_{na}{}^b(y)V_b(y)\delta(y, x) + V_n(y)(E_{ay}[\delta(y, x)] - a_n(y)\delta(y, x)),
\]

which mixes the spatial and perpendicular projections of tensors. However, if one smears this with respect to the lapse function \(N\), one finds the familiar result \(\int_S dx N(x)\delta_{nz}[V_a(y)] = N(y)(\partial_{ny}[V_a(y)] - C_{na}{}^b(y)V_b(y))\), which is the surface-covariant normal derivative operator \([1]\),

\[
\int_S dx N(x)\delta_{nz}[V_a(y)] = N(y)(\partial_{ny}[V_a(y)] - C_{na}{}^b(y)V_b(y)),
\]
One can explicitly show that the operator \( \delta_{nx} \) does not change the \( \Sigma \) tensor character of objects (i.e., it is a scalar operator, for example if \( V_a \to |M^{-1}|_a^b V_b \), then \( \delta_{nx} [V_a] \to |M^{-1}|_a^b \delta_{nx} [V_b] \)) which results in \( (21b) \). The tangential generators \( \delta_{ax} \) act as, for example

\[
\delta_{ax} [V_b(y)] = E_{ay} [V_b(y)] \delta(y, x) + E_{by} [\delta(y, x)] V_a(y) - C_{ab}^c (y) V_c(y) \delta(y, x),
\]

which, when contracted with a vector field and integrated over \( \Sigma \), yields the Lie derivative defined on \( \Sigma \) (e.g., \( \int_\Sigma dx M^a(x) \delta_{ax} [V_b(y)] = L_M [V_b(y)] \)). Therefore \( \delta_{ax} \) is said to generate infinitesimal diffeomorphisms, and the algebra \( (21c) \) is that of \( LDiff \Sigma \). The form of the commutators \( (21b) \) and \( (21c) \) may be derived by taking into account the fact that \( \delta_{nx} \) and \( \delta_{ax} \) are vector and scalar density operators of weight one respectively. With this choice of generators, we naturally find the partial time derivative operator \( \partial_t = \int_\Sigma dx (N(x) \delta_{nx} + N^a(x) \delta_{ax}) \).

The set of generators considered thus far \( (\delta_{nx}, \delta_{ax} \text{ and } \Delta_{ax}^b) \) is convenient for considering frames that have been fixed independently of the foliation, however not for considering the opposite case, namely, where the spatial metric has a fixed form. Explicitly, the action of the generators on the components of the spatial metric is

\[
\delta_{nx} [g_{ab}(y)] = \partial_{ny} [g_{ab}(y)] \delta(y, x) - C_{na}^c(y) g_{cb}(y) \delta(y, x) - C_{nb}^c(y) g_{ac}(y) \delta(y, x),
\]

\[
\delta_{ax} [g_{bc}(y)] = E_{ay} [g_{bc}(y)] \delta(y, x) + E_{by} [\delta(y, x)] g_{ac}(y) + E_{cy} [\delta(y, x)] g_{ba}(y)
- C_{ab}^d(y) g_{dc}(y) \delta(y, x) - C_{ac}^d(y) g_{bd}(y) \delta(y, x),
\]

in neither case preserving its form. This means that if we wanted to specialise to an orthonormal frame on \( \Sigma \) by taking \( g_{ab} = -\delta_{ab} \), the action of the deformation generators (and therefore also the related constraints) would not respect this.

In order to deal with this case (which is what is considered in \( [4, 3] \)) we will define the following generators which mix the hypersurface deformation generators with the generators of frame rotations:

\[
\delta'_{nx} := \delta_{nx} - \Delta_x, \quad \delta'_{ax} := \delta_{ax} - \Delta_{ax}^+, \quad \delta'_{ax} := \delta_{ax} - \Delta_{ax}^-. \quad (26a)
\]

where

\[
\Delta_x := k^a_b(x) \Delta_{ax}^b, \quad (26b)
\]

and the action of the \( \Delta_{ax} \) and \( \Delta_{ax}^+ \) is defined to be

\[
\int_\Sigma dx f^a(x) \Delta_{ax} = - \int_\Sigma dx \nabla_b [f]^a(x) \Delta_{ax}^b, \quad (26c)
\]

\[
\int_\Sigma dx f^a(x) \Delta_{ax}^+ = - \int_\Sigma dx g^{ac}(x) \nabla_b [f]_{c}(x) \Delta_{ax}^b. \quad (26d)
\]
(Note that $\Delta_{ax}^+$ is the contribution to $\Delta_{ax}$ from the symmetric generators $\Delta_{xy}^{(ab)}$ where $\Delta_{xy}^{ab} := g^{ac}(x)\Delta_{by}^b$. Symmetrization and antisymmetrization on a pair of indices is indicated by ( ) and [ ] respectively e.g., $T_{[ab]} := \frac{1}{2}(T_{ab} - T_{ba})$.)

In order to compute the algebra based on the set of generators $(\delta_n^a, \delta_a^n, \Delta_{by}^b)$ or $(\delta_n^a, \delta_a^n, \Delta_{by}^b)$, we will need to generate some intermediate results. In particular, $k_{ab}$ and $\Gamma^a_{bc}$ appear in (26) although as it turns out, we will only need to compute their normal variations. That this computation is nontrivial follows from the fact that, unlike most of the tensors over $\Sigma$ that we have been considering, $k_{ab}$ and $\Gamma^a_{bc}$ are not merely surface projections of spacetime tensors. Nevertheless, the dependence on the embedding is given explicitly by the spacetime definitions given in (1) and (8) respectively. The required results are:

$$\delta_{nx}[k_{ab}(y)] = \partial_n[k_{ab}(y)]\delta(y, x) - k_{ac}(y)C_{nb}^c(y)\delta(y, x) - k_{bc}(y)C_{na}^c(y)\delta(y, x)$$

(27a)

$$- \nabla_a[a]_b(y)\delta(y, x) - a_a(y)a_b(y)\delta(y, x) + E_{by}[E_{by}[\delta(y, x)]] - \Gamma^c_{ab}(y)E_{cy}[\delta(y, x)],$$

$$\delta_{nx}[\Gamma^a_{bc}(y)] = - \Sigma R^a_{bmn}(y)\delta(y, x) - \nabla_b[k^c_n(y)\delta(y, x)]$$

$$- k^o_n(y)E_{cy}[\delta(y, x)] - k^o_n(y)E_{by}[\delta(y, x)] + k_{bc}(y)g^{ad}(y)E_{dy}[\delta(y, x)].$$

(27b)

From the scalar and vector nature of $\Delta_x$ and $\Delta_{ax}$ respectively, it is easy to compute

$$[\Delta_x, \Delta_y] = 0, \quad [\Delta_{ax}^b, \Delta_y] = 0, \quad [\Delta_{by}^a, \Delta_{cy}] = - \delta^n_c \delta(y, x)\Delta_{by},$$

(28a)

and using (27), the remaining commutators that are necessary to compute the algebra are given by

$$[\Delta_{ax}, \Delta_y] = k^b_n(y)\delta(y, x)\Delta_{by}$$

$$+ \left(\delta^d_aE_{cy}[\delta(y, x)] + \Gamma^{d}_c(y)\delta(y, x)\right)k^b_d(y)\Delta_{by}$$

$$- \left(\delta^d_aE_{dy}[\delta(y, x)] + \Gamma^{d}_a(y)\delta(y, x)\right)k^c_d(y)\Delta_{by},$$

(28b)

$$[\delta_{nx}, \Delta_y] = (\partial_n[k^b_n(y)]\delta(y, x) + C_{na}^c(y)k^d_b(y)\delta(y, x) - C_{nb}^d(y)k^c_n(y)\delta(y, x))\Delta_{by}$$

$$- g^{ac}(y)(\nabla_b[a]_c(y) + a_b(y)a_c(y))\delta(y, x)\Delta_{ay}$$

$$+ g^{ac}(y)(E_{by}[E_{by}[\delta(y, x)]] - \Gamma_{bc}(y)E_{dy}[\delta(y, x)])\Delta_{cy},$$

(28c)

$$[\delta_{nx}, \Delta_{ay}] = (R^b_{can}(y)\delta(y, x) + \nabla_c[k^b_n(y)\delta(y, x)])\Delta_{by}$$

$$+ \left(k^c_b(y)E_{ay}[\delta(y, x)] + k^b_n(y)E_{cy}[\delta(y, x)] - k_{ac}(y)g^{bd}(y)E_{dy}[\delta(y, x)]\right)\Delta_{cy},$$

(28d)

$$[\delta_{ax}, \Delta_y] = k^b_{n}(y)\delta(y, x)\delta_{by} + \nabla_a[k^b_n(y)\delta(y, x)\Delta_{by}$$

$$+ \left(\delta^d_aE_{cy}[\delta(y, x)] + \Gamma^{d}_c(y)\delta(y, x)\right)k^b_d(y)\Delta_{cy}$$

$$- \left(\delta^d_aE_{dy}[\delta(y, x)] + \Gamma^{d}_a(y)\delta(y, x)\right)k^c_d(y)\Delta_{by},$$

(28e)
Using these results, we find the commutator algebra of the set \( \delta'_{nx}, \delta'_{ax}, \Delta'_{bc} \) to be given by (29) and

\[
\begin{align*}
[\delta'_{nx}, \delta'_{ny}] &= g^{ab}(x)E_{ax}[\delta(x, y)]\delta^a_{by} - g^{ab}(y)E_{ay}[\delta(y, x)]\delta^a_{by}, \\
[\delta'_{ax}, \delta'_{ny}] &= -E_{ax}[\delta(x, y)]\delta'_{nx} - k^b_a(y)\delta(y, x)\delta'_{by} \\
&\quad + R^b_{can}(y)\delta(y, x)\Delta'_{by} + 2k_a[b(x)E_{cl}][\delta(x, y)]\Delta'_{bc}, \\
[\delta'_{ax}, \delta'_{by}] &= \Sigma R^c_{dab}(y)\delta(y, x)\Delta'_{cy}, \\
[\Delta'_{by}, \delta'_{nx}] &= 0, \quad [\Delta'_{by}, \delta'_{cy}] = -\delta^a_c\delta(y, x)\delta'_{by},
\end{align*}
\]

where these new generators act on tensors to give, for example

\[
\begin{align*}
\delta'_{nx}[V_a(y)] &= \partial_{ny}[V_a(y)]\delta(y, x) - C_{na}^b(y)V_b(y)\delta(y, x) + k^b_a(y)V_b(y)\delta(y, x) \\
&\quad + V_n(y)(E_{ay}[\delta(y, x)] - a_a(y)\delta(y, x)), \\
\delta'_{ax}[V_b(y)] &= \nabla_a[V_b(y)]\delta(y, x),
\end{align*}
\]

and rotates the vielbeins through

\[
\begin{align*}
\delta'_{nx}[E_b^i(y)] &= k^a_b(y)E_a^i(y)\delta(y, x), \\
\delta'_{ax}[E_b^i(y)] &= -\left(\delta^c_a E_{by}[\delta(y, x)] + \Gamma^c_{ba}(y)\delta(y, x)\right)E_c^i(y).
\end{align*}
\]

The commutator (29c) was derived by noting that \( \delta'_a \) acts on tensors as a covariant derivative, the commutator of which results in the curvature operator; this is why only the normal variations of \( k_{ab} \) and \( \Gamma^a_{bc} \) were required.

In contrast to the unprimed generators, the action of this new set of generators on the components of the spatial metric is

\[
\begin{align*}
\delta'_{nx}[g_{ab}(y)] &= \partial_{ny}[g_{ab}(y)]\delta(y, x) - C_{na}^c(y)g_{cb}(y)\delta(y, x) - C_{nb}^c(y)g_{ac}(y)\delta(y, x) \\
&\quad + k^c_a(y)g_{cb}(y)\delta(y, x) + k^c_b(y)g_{ac}(y)\delta(y, x), \\
\delta'_{ax}[g_{bc}(y)] &= \nabla_a[g_{bc}(y)]\delta(y, x),
\end{align*}
\]

both of which vanish due to (4) and the spatial metric compatibility conditions respectively.

It is this set of deformation generators that represent the total time derivative operator

\[
d_t = \int_\Sigma dx \left( N(x)\delta'_{nx} + N^a(x)\delta'_{ax} \right)
\]

and correspond to the mixing of the original diffeomorphism constraints by

\[
\mathcal{H}' := \mathcal{H} - k_b^a J_a^b, \quad \mathcal{H}'_a := \mathcal{H} - \nabla_b J_a^b.
\]

(33)
Due to (32), these constraints will not alter the form of the spatial metric, and the reduction to orthonormal frames on \( \Sigma \) is a simple matter of choosing \( g_{ab} = -\delta_{ab} \) and restricting \( \Delta^a_b \) (or \( J^a_b \)) to correspond to generators of \( \mathfrak{so}(n) \).

Furthermore, defining the coordinate components \( \delta'_{ix} := E_i^a(x)\delta'_{ax} \) and using (31), the constraint algebra related to the coordinate generators \( (\delta'_a, \delta'_i, \Delta^a_b) \) is identically the time gauge result given in \( \text{[3]} \):

\[
\begin{align*}
[\delta'_{nx}, \delta'_{ny}] &= g^{ij}(x)\partial_{x_i}[\delta(x,y)]\delta^+_j - g^{ij}(y)\partial_{y_i}[\delta(y,x)]\delta^+_j, \\
[\delta'_{lx}, \delta'_{ly}] &= -\partial_{x_i}[\delta(x,y)]\delta'_{nx} + R^b_{cim}(y)\delta(y,x)\Delta^c_{by} + 2E_{i}^{a}(x)k_{a[b}(x)E_{c]x}[\delta(x,y)]\Delta^c_{by}, \\
[\delta'_{ix}, \delta'_{iy}] &= \partial_{y_i}[\delta(y,x)]\delta'_{iy} - \partial_{x_i}[\delta(x,y)]\delta'_{jx} + \Sigma R^c_{diy}(y)\delta(y,x)\Delta^d_{cy}, \\
[\Delta^b_{by}, \delta'_{nx}] &= 0, \quad [\Delta^b_{by}, \delta'_{iy}] = 0.
\end{align*}
\]

If one were to consider instead the set \( (\delta'_{nx}, \delta'_{ax}, \Delta^a_{bx}) \), then (29b) and (29c) would be replaced by the following:

\[
\begin{align*}
\int_{\Sigma} dx \, dy \ A^a(x)B(y)[\delta_{ax}^+, \delta_{ay}^+] &= \int_{\Sigma} dx \left(-A^a\nabla_a[B]\delta_{nx}^+ - B A^a_k b^{b}_{ax} \right. \\
& \left. + 2A^a k_{ab}\nabla_c[B]\Delta^b_{cx} + 2B k^b_{a} \nabla_c[A]_b \Delta^a_{bx} \right), \\
\int_{\Sigma} dx \, dy \ A^a(x)B^b(y)[\delta_{ax}^+, \delta_{by}^+] &= \int_{\Sigma} dx g^{ab}(B^c \nabla_c[A]_b) + A^c \nabla_c[B]_b\delta_{ax}^+ \\
& - 2\int_{\Sigma} dx g^{cd} \nabla_{(a}[A]_c)\nabla_{(b[B]_d)\Delta^a_{bx}},
\end{align*}
\]

which have been smeared over an appropriate choice of tensor field for ease of display. The coordinate components related to \( \text{[3]} \) correspond to the those initially derived in \( \text{[3]} \), Equation (2.6), before the additional frame rotation has been performed.

Instead of requiring that the frame be completely independent of the embedding, it may be constrained so that the action of the deformation generators on the components of the spatial metric is trivial: \( \delta_{ax}[g_{ab}(y)] = 0 \), yielding the condition

\[
\delta_{ax}[E_{(a}^{i}(y)]E_{ib)}(y) = -\frac{1}{2}\delta_{ax}[g_{\mu
u}(y)]e^{a}_{\mu}(y)e^{b}_{\nu}(y) - e_{\alpha(a}(y)E_{b)}y[\delta(y,x)].
\]

Choosing the vielbein to satisfy

\[
\begin{align*}
\delta_{nx}[E_{a}^{i}(y)]E_{i}^{b}(y) &= k_{b}^{y}(y)\delta(y,x), \\
\delta_{ax}[E_{b}^{i}(y)]E_{i}^{c}(y) &= -\left(\delta_{a}^{c}E_{by}[\delta(y,x)] + \Gamma_{ba}^{c}(y)\delta(y,x)\right),
\end{align*}
\]

we find that the derived generators and algebra are equivalent to that of \( (\delta'_{nx}, \delta'_{ax}, \Delta^a_{bx}) \) considered above, and removing the antisymmetric part of the right hand side of (37b), one recovers
the generator $\delta_{ax}^+$. Thus we have two ultimately identical ways of approaching the problem; the first by looking for an equivalent form of the generators that preserve the form of the spatial metric, and the second is to constrain the vierbein so that the same condition holds.

Clearly there are other representations of the constraint algebra that correspond to different ways of mixing the diffeomorphism constraints with the generators of frame rotations, however the two cases that we have dealt with here have the most direct physical interpretation. Note that it is also possible to determine how the algebra is altered by a (partial) fixing of the remaining SO($n$) invariance. This would show up in an additional condition on the frame that would have to be maintained under the action of the generators, leading to further alterations of the constraint algebra.

**Conclusions**

What we have developed herein is, in fact, a general formalism for determining how a choice of frame affects the diffeomorphism constraint algebra of a theory. The two important limits, namely, that of coordinate frame diffeomorphism generators and that of orthonormal frame generators, are reached through a mixing of the diffeomorphism constraints with the generators of the generalised frame rotations. In both of these limits, the standard results (of [21] and [3] respectively) are recovered, however since the analysis has not been restricted to a particular model, represents the generalisation of the latter results to the algebra of any covariant model written in a tetrad frame.

The conclusion drawn from these results is that the diffeomorphism constraint algebra is purely a geometrical relation *in any frame*. Once one has chosen the coordinate system and frames of reference in which a particular system is to be described, the constraint algebra of the generators of the diffeomorphism algebra is fixed independently of the model. In any consistent quantisation of the model, the operators that play the role of these generators must faithfully represent the algebra on the Hilbert space in question. Since classical (and presumably quantum) General Relativity should relate observations made in different frames of reference, one would like any potential quantisation (of General Relativity or of quantum fields on a curved background) to reproduce this algebra not just in a particular choice of frame, but for any choice of frame.

It is also interesting to note that any covariant combination of the generators has vanishing commutator with itself providing there is no dependence on the embedding other than that due to the spacetime point. In particular, taking $\delta_V := V^\alpha \delta_\alpha$ (for some future pointing vector field $V^\alpha$, guaranteeing that $\delta_V$ generates deformations of the embedded surface forwards with respect to the foliation), one finds that $[\delta_{Vx},\delta_{Vy}] = 0$ and the rest of the algebra in (21) remains
the same. The resulting algebra does not depend on canonical coordinates and is therefore a true Lie algebra. Another special case of this which is more relevant to the case of vacuum general relativity is the combination
\[ \delta g := g^{\alpha \beta} \delta \alpha \delta \beta = \delta_n^2 + g^{ab} \delta a \delta b \]
recently discussed in [17]. It would be interesting to determine whether the more general combinations discussed in [18] could also be considered as covariant combinations of the coordinate frame constraints, and indeed, what the action of the resulting generators would be.

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