On some new global existence results for 3D magnetohydrodynamic equations

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Abstract

This paper is devoted to the incompressible magnetohydrodynamic equations in $\mathbb{R}^3$. We prove that if the difference between the magnetic field and the velocity is small initially then it will remain forever, thus resulting in a global strong solution without the smallness restriction on the size of initial velocity or magnetic field. In other words, magnetic field can indeed regularize Navier–Stokes equations, due to cancellation.

Keywords: magnetohydrodynamics equations, global strong solution, cancellation

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1. Introduction

This paper is devoted to the study of the incompressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^3$,\footnote{951-7715/14/020343+10$33.00 © 2014 IOP Publishing Ltd & London Mathematical Society Printed in the UK}

$$\begin{align*}
\frac{\partial u}{\partial t} - \frac{1}{Re} \Delta u + (u \cdot \nabla)u - \nu (B \cdot \nabla)B + \nabla \left( P + \frac{S}{2} |B|^2 \right) &= 0, \\
\frac{\partial B}{\partial t} - \frac{1}{Rm} \Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \\
\text{div } u &= 0, \\
\text{div } B &= 0.
\end{align*}$$

(1.1)
with the following initial conditions:

\[
\begin{cases}
    u(x, 0) = u_0(x), \\
    B(x, 0) = B_0(x).
\end{cases}
\] (1.2)

Here \(u, P, B\) are non-dimensional quantities corresponding to the velocity of the fluid, its pressure and the magnetic field, respectively. The non-dimensional number \(Re > 0\) is the Reynolds number, \(Rm > 0\) is the magnetic Reynolds number and \(S = M^2/(ReRm)\) with \(M\) being the Hartman number. For simplicity of writing, we can assume that \(S = 1\), otherwise, let \(\tilde{B}(x, t) = \sqrt{S} B(x, t)\). And let \(P\) denote the total pressure \(P + S|B|^2/2\).

The MHD system (1.1) was studied by Duvaut and Lions [3]. They established the local existence and uniqueness of a solution in the classical Sobolev spaces \(H^s(\mathbb{R}^N), s \geq N\). But whether this unique local solution can exist globally for large initial data is a challenging open problem in mathematical fluid mechanics. Later, Sermange and Temam [13] showed the regularity for weak solutions in the case of three dimensions under the assumption that \((u, B)\) belongs to \(L^\infty(0, T; H^1(\mathbb{R}^3))\). Some other kinds of regularity criteria are established in [2, 16–18] and references therein. On the other hand, Duvant–Lions [3] also proved the global existence of the strong solution for small initial data. For some extension, refer to [11, 14]. It should be noted that all these global existence results of smooth solutions require that both the velocity field and magnetic field be sufficiently small. This is mainly because MHD equations share the same nonlinear convection structure as incompressible Navier–Stokes equations. However, the relation between velocity field and magnetic field in existence theory is not clear.

Recently, some efforts have been made to characterize the different roles played by the velocity field and the magnetic field in the regularity of weak solutions. Partial developments are achieved and partial results are obtained in this direction. Namely, it was shown in [5] that a weak solution \((u, B)\) is smooth provided that the velocity field satisfies any one of the following assumptions: (1) \(u \in L^p(0, T; L^q(\mathbb{R}^3))\) with \(1/p + 3/2q \leq 1/2\) for \(q > 3\); (2) \(u \in C([0, T]; L^3(\mathbb{R}^3))\); (3) \(\nabla u \in L^\alpha(0, T; L^\beta(\mathbb{R}^3))\) with \(1/\alpha + 3/2\beta \leq 1\) for \(3 \leq \beta < \infty\); (4) Let \(\omega(x, t) = \text{curl } u(x, t)\). There exist some positive constants \(K\), \(M\) and \(\rho\), such that

\[
|\omega(x + y, t) - \omega(x, t)| \leq K|\omega(x + y, t)||y|^{1/2}
\]

holds for any \(t \in [0, T]\), if both \(|y| \leq \rho\) and \(|\omega(x + y, t)| \geq M\). This result was then generalized, for more references see [6, 8, 10, 15, 19]. These regularity criteria imply that the velocity field of the fluid seems to play a dominant role in the theory of regularity of weak solutions in some sense.

On the other hand, there is some evidence indicating that the magnetic field should have some dissipation, due to the numerical simulations of Politano et al in [12] and the observations of space and laboratory plasmas alike in [4]. Then the solutions to the incompressible MHD equations should exhibit a greater degree of regularity than the ordinary incompressible Navier–Stokes equations, in some sense. In this direction, Bardos–Sulem–Sulem [1] first established the global strong classical solution for the inviscid MHD equations with strong magnetic field. Inspired by the results in [4, 12], we study the cancellation between the velocity field and the magnetic field.

First, reformulate equation (1.1) using Elsasser’s variables \(W^*, W^-\) as follows:

\[
W^* = u + B, \quad W^- = u - B, \quad W^*_0 = u_0 + B_0, \quad W^-_0 = u_0 - B_0.
\]
Then MHD equations (1.1)–(1.2) can be rewritten as

\[
\begin{aligned}
\frac{\partial W^+}{\partial t} - \kappa \Delta W^+ - \lambda \Delta W^- + (W^- \cdot \nabla) W^+ + \nabla P &= 0, \\
\frac{\partial W^-}{\partial t} - \kappa \Delta W^- - \lambda \Delta W^+ + (W^+ \cdot \nabla) W^- + \nabla P &= 0,
\end{aligned}
\]

(1.3)

with the following initial conditions

\[
\begin{aligned}
W^+(x, 0) &= W_{0+}^+(x), \\
W^-(x, 0) &= W_{0-}^-(x).
\end{aligned}
\]

(1.4)

Here

\[
\kappa = \frac{1}{2Re} + \frac{1}{2Rm}, \quad \lambda = \frac{1}{2Re} - \frac{1}{2Rm}.
\]

Note that

\[
\kappa > |\lambda|.
\]

In this paper, we will show that magnetic field can regularize the Navier–Stokes equations. More precisely, there exists a unique global strong solution to 3D MHD equations with large initial velocity as long as \(|\lambda|/\kappa \ll 1\) and the magnetic field is comparable to the velocity initially.

2. Main result

Before stating our main results, we introduce some function spaces. Let \(C_{0,\alpha}^\infty(\mathbb{R}^3)\) denote the set of all \(C^\infty\) vector-valued functions \(\phi\) with compact support in \(\mathbb{R}^3\), such that \(\text{div}\phi = 0\). \(L^\beta(\mathbb{R}^3), W^{k,\beta}(\mathbb{R}^3)\) \((1 \leq \beta \leq \infty)\), \(H^s(\mathbb{R}^3)\) \((s > 0)\) are the standard Sobolev spaces. The fractional-order homogeneous Sobolev space \(\dot{H}^s(\mathbb{R}^3)\) \((s > 0)\) is defined as the space of tempered distributions \(u\) over \(\mathbb{R}^3\) for which the Fourier transform \(\mathcal{F}u\) belongs to \(L^1_{\text{loc}}(\mathbb{R}^3)\) and which satisfy

\[
\|u\|_{\dot{H}^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi < \infty.
\]

Let \(L^\beta_{\alpha}(\mathbb{R}^3)\) and \(\dot{H}^s_{\alpha}(\mathbb{R}^3)\) be the closure of \(C_{0,\alpha}^\infty(\mathbb{R}^3)\) with the respect to the \(L^\beta\)-norm and \(\dot{H}^s\)-norm.

This paper is devoted to the existence of global strong solutions to the problem (1.1)–(1.2). We provide a new smallness condition on the initial data \(W_{0+}^- (x)\) and \(|\lambda|/\kappa\) rather than the initial velocity and magnetic field \((u_0, B_0)\).

To illustrate our main idea in a clear way, we study a special case first, for which \(Re = Rm\), and prove the following theorem.

**Theorem 2.1.** Let \((W_{0+}^+, W_{0-}^-) \in L^3_{\alpha}(\mathbb{R}^3)\) and \(\lambda = 0\). Then there exist two generic constants \(\epsilon_0\) and \(C_0\), independent of initial data and \(\kappa\), such that if

\[
\kappa^{-3} \|W_{0+}^-\|_{L^3_{\alpha}}^3 \cdot \exp \left\{ C_0 \kappa^{-3} \|W_{0+}^+\|_{L^3_{\alpha}}^3 \right\} < \epsilon_0,
\]

(2.1)

or

\[
\kappa^{-3} \|W_{0+}^-\|_{L^3_{\alpha}}^3 \cdot \exp \left\{ C_0 \kappa^{-3} \|W_{0-}^-\|_{L^3_{\alpha}}^3 \right\} < \epsilon_0,
\]

(2.2)

then system (1.3)–(1.4) admits a global strong solution

\((W^+, W^-) \in C([0, \infty), L^3_{\alpha}(\mathbb{R}^3)) \cap C((0, \infty), W^{2,3}(\mathbb{R}^3))\).
Remark 2.1. Condition (2.1) holds for
\[ W_0^+(x) = 0, \]  
while condition (2.2) holds for
\[ W_0^-(x) = 0. \]  

We say a few words on this special case. Taking (2.3) for example, one can easily deduce that
\[ W_0^-(x, t) = \nabla P = 0, \]  
and \( W_0^+(x, t) \) is a solution to the following heat equation:
\[ \partial_t W_0^+ - \kappa \Delta W_0^+ = 0. \]  
There is no doubt that (2.6) has a global solution, which is smooth when \( t > 0 \).

Remark 2.2. Very recently, a similar version of theorem 2.1 was indicated in [9].

Next, we generalize the same idea to a general case, where \( \text{Re} \neq \text{Rm} \).

**Theorem 2.2.** Let \( (W_0^+, W_0^-) \in \dot{H}^\frac{1}{2} (\mathbb{R}^3) \). Then there exist two generic positive constants \( \epsilon_0 < \frac{1}{2} \) and \( C_0 \), independent of initial data, \( \kappa \) and \( \lambda \), such that if
\[
\left( \kappa^{-2} \| W_0^- \|^2_{\dot{H}^\frac{1}{2}} + \frac{\lambda^2}{\kappa^2} \left( \kappa^{-2} \| W_0^- \|^2_{H^\frac{1}{2}} + \frac{\lambda^2}{\kappa^2} \right) \right) \exp \left\{ C_0 \left( \kappa^{-4} \| W_0^+ \|^4_{\dot{H}^{\frac{1}{2}}} + \frac{\lambda^4}{\kappa^4} \right) \right\} < \epsilon_0 \]  
or
\[
\left( \kappa^{-2} \| W_0^+ \|^2_{H^\frac{1}{2}} + \frac{\lambda^2}{\kappa^2} \left( \kappa^{-2} \| W_0^- \|^2_{H^\frac{1}{2}} + \frac{\lambda^2}{\kappa^2} \right) \right) \exp \left\{ C_0 \left( \kappa^{-4} \| W_0^- \|^4_{H^\frac{1}{2}} + \frac{\lambda^4}{\kappa^4} \right) \right\} < \epsilon_0, \]  
then system (1.3)–(1.4) admits a global strong solution
\[ (W^+, W^-) \in C([0, \infty), \dot{H}^\frac{1}{2} (\mathbb{R}^3)) \cap C((0, \infty), \dot{H}^2 (\mathbb{R}^3)). \]

**Remark 2.3.** Either condition (2.7) or (2.8) automatically implies
\[ \frac{\| \lambda \|}{\kappa} \leq \epsilon_0 \ll 1, \]  
which corresponds to \((|\text{Re} - \text{Rm}|/(\text{Re} + \text{Rm})) < 1\) in original MHD system. Indeed, in astrophysical magnetic phenomena, both the Reynolds number and the magnetic Reynolds number are huge and the difference between \( \text{Re} \) and \( \text{Rm} \) is not critical. In this aspect our assumption is reasonable.

**Remark 2.4.** From mathematical point of view, it is important to study the regularizing effect of magnetic field to the Navier–Stokes equations. As shown in theorems 2.1–2.2, one type of regularizing effects is due to cancellation. Consequently, synchronous diffusion speed is a good candidate for maintaining the effect of cancellation all the time which turns out to ask \((|\text{Re} - \text{Rm}|/(\text{Re} + \text{Rm})) < 1\).

**Remark 2.5.** There is no smallness condition imposed on the initial velocity. It indicates that one can generate solutions with large initial velocity, which are smooth for all the time \( t > 0 \), as long as magnetic field and velocity are comparable initially and \((|\lambda|/\kappa) \ll 1\). In other words, magnetic field can indeed regularize the Navier–Stoke equations.

**Remark 2.6.** In fact, theorem 2.2 cannot cover theorem 2.1 completely. We require the initial data belongs to \( \dot{H}^\frac{1}{2} (\mathbb{R}^3) \), instead of \( L^3 (\mathbb{R}^3) \). Hence it is left open whether a real generalized version of theorem 2.1 can be derived.

**Remark 2.7.** Due to the symmetric structure of system (1.3), we need only to prove theorems 2.1 and 2.2 under condition (2.1) and (2.7), respectively.
3. Proof of theorem 2.1

Since \((W^*_0, W^-_0) \in L^4(\mathbb{R}^3)\) is equivalent to \((u_0, B_0) \in L^3(\mathbb{R}^3)\), it is well known that there are \(T_0 > 0\) and a unique strong solution \((u, B)\) to the MHD equations in \((0, T_0]\) and the solution is classical when \(t > 0\). Hence, in the following, we assume that the solution \((W^*, W^-)\) is sufficiently smooth on \([0, T]\) and deduce the uniform \(a \ priori\) strong estimates under the assumptions of theorem 2.1, which guarantee the extension of the local strong solution.

Here and thereafter, \(C, C_1\) will denote a generic constant which is independent of \(\lambda, \kappa\), the initial data \((W^*_0, W^-_0)\) and time \(T\).

3.1. \(a \ priori\) estimates

Given a strong solution \((W^*, W^-)\) on \(\mathbb{R}^3 \times [0, T]\) for \(T < T_0\), define

\[
A^- (T) = \kappa^{-3} \sup_{0 \leq t \leq T} \|W^-(t)\|^3_{L^3}.
\]

(3.10)

Actually, we have the following proposition.

**Proposition 3.1.** Let \((W^*, W^-)\) be a strong solution to (1.3)–(1.4) in \(\mathbb{R}^3 \times [0, T]\), then there exist two positive constants \(\epsilon_0\) and \(C_0\), such that if

\[
A^- (T) \leq 2\epsilon_0,
\]

then it in fact holds that

\[
A^- (T) \leq \epsilon_0 \quad \text{and} \quad \|W^*\|_{L^3(0, T; L^9)} \leq C\kappa^{-\frac{1}{2}} \|W^*_0\|_{L^3},
\]

provided

\[
\kappa^{-3} \|W^-\|^3_{L^3} \cdot \exp [C_0\kappa^{-3} \|W^*_0\|^3_{L^3}] \leq \epsilon_0.
\]

(3.12)

The remainder of this subsection consists in proving this key result.

**Lemma 3.1.** Let \((W^*, W^-)\) be a strong solution to (1.3)–(1.4) in \(\mathbb{R}^3 \times [0, T]\). There exist two positive constants \(\epsilon_0\) and \(C_0\), such that if

\[
A^- (T) \leq 2\epsilon_0,
\]

(3.13)

then it holds that

\[
\sup_{0 \leq t \leq T} \|W^+(t)\|^3_{L^3} + \kappa \int_0^T \left( \|W^+\|^2_{L^2} \right) dt \leq \|W^*_0\|^3_{L^3},
\]

(3.14)

and

\[
\sup_{0 \leq t \leq T} \|W^-(t)\|^3_{L^3} \leq \|W^*_0\|^3_{L^3} e^{C\kappa^{-3} \|W^*_0\|^3_{L^3}}.
\]

(3.15)

**Proof.**

*Step 1.* Multiplying the first equation of (1.3) by \(3|W^+| W^+\) and integrating over \(\mathbb{R}^3\), we obtain

\[
\frac{d}{dt} \|W^+\|_{L^3} + \kappa \int \|W^+\|^2_{L^2} dx + \frac{k}{3} \int \left| \nabla \left( |W^+|^\frac{3}{2} \right) \right|^2 dx \leq 3 \int |P \cdot \text{div} (W^+ | W^+) | dx
\]

\[
\leq C \|P\|_{L^\infty} \|W^+\|^\frac{3}{2} \|W^+|^\frac{1}{2} \nabla W^+ \|_{L^3}.
\]

(3.16)
Note that the equation for $W^+$ can be written as
\[ (\partial_t W^+ - \kappa \Delta W^+) + \nabla P = -\text{div}(W^- \otimes W^+), \]  
where the left-hand side is viewed as the Helmholtz–Weyl decomposition of the right-hand one. From this equation, one has
\[ \|P\|_{L^2} \leq C \|W^- \otimes W^+\|_{L^2} \leq C \|W^-\|_{L^2} \|W^+\|_{L^2}. \]  

Insert estimate (3.18) into (3.16), then
\[ \frac{d}{dt} \|W^+\|_{L^2}^2 + 3\kappa \int |W^+| |\nabla W^+|^2 \, dx + \frac{4}{3}\kappa \int |\nabla (|W^+|^2)|^2 \, dx \leq C \|W^+\|_{L^2} \|W^+\|_{L^6} \|W^+\|_{L^2} \|\nabla W^+\|_{L^2} \]  
where the Sobolev embedding inequality and Young’s inequality were used. Now if we choose $\epsilon_0$ small enough such that
\[ A^- (T) \leq (2\epsilon_0)^\frac{1}{2} \leq \kappa/(3C_1), \]  
then (3.14) follows immediately.

Step 2. Multiplying the second equation of (1.3) by $3|W^-|W^-$, with the help of integration by parts, we have
\[ \frac{d}{dt} \|W^-\|_{L^2}^2 + 3\kappa \int |W^-| |\nabla W^-|^2 \, dx + \frac{4}{3}\kappa \int |\nabla (|W^-|^2)|^2 \, dx \leq C \|P\|_{L^2} \|W^+\|_{L^2} \|W^-\|_{L^2} \|\nabla W^-\|_{L^2}. \]  
Substituting estimate (3.18) into (3.21) and employing the Sobolev embedding inequality, we have
\[ \frac{d}{dt} \|W^-\|_{L^2}^2 + 3\kappa \int |W^-| |\nabla W^-|^2 \, dx + \frac{4}{3}\kappa \int |\nabla (|W^-|^2)|^2 \, dx \leq C \|W^-\|_{L^2} \|W^+\|_{L^2} \|W^-\|_{L^2} \|\nabla W^-\|_{L^2}. \]  
One can easily deduce from (3.22) after using Young’s inequality that
\[ \frac{d}{dt} \|W^-\|_{L^2}^2 \leq C\kappa^2 \|W^-\|_{L^2}^3 \|\nabla (|W^+|^2)|_{L^2}^2. \]  
Hence, in view of (3.14), one has
\[ \sup_{0 \leq t \leq T} \|W^- (t)\|_{L^2}^3 \leq \|W^-_0\|_{L^2}^3 e^{C\kappa^-3 \|W^+_0\|_{L^3}^3}. \]  
This completes the proof of lemma 3.1.

It follows from lemma 3.1 that
\[ A^- (T) \leq \kappa^{-3} \|W^-_0\|_{L^2}^3 e^{C\kappa-3 \|W^+_0\|_{L^3}} \leq \epsilon_0. \]
if (3.12) holds. And also the estimate
\[ \| W^+ \|_{L^1(0,T;L^3)} \leq C \kappa^{-\frac{1}{2}} \| W^0 \|_{L^3} \] (3.26)
is implied in lemma 3.1, more precisely, (3.14). Hence we complete the proof of proposition 3.1.

3.2. Proof of theorem 2.1

With the \textit{a priori} estimates in the previous subsection in hand, we are prepared for the proof of theorem 2.1.

\textbf{Proof.} In view of classical results, there exists $T_0 > 0$ such that the Cauchy problem (1.3)–(1.4) has a unique local strong solution $(W^+, W^-)$ in $\mathbb{R}^3 \times (0, T_0)$. We will show that this local solution can be extended to a global one provided condition (2.1) holds.

Since the local strong solution is continuous in $L^3$, there exists a $T_1 \in (0, T_0)$ such that (3.11) holds for $T = T_1$. So we set
\[ \bar{T} = \sup\{T \mid (W^+, W^-) \text{ is a strong solution in } \mathbb{R}^3 \times (0, T) \text{ and } A^- (T) \leq 2\varepsilon_0\} \]
and
\[ T^* = \sup\{T \mid (W^+, W^-) \text{ is a strong solution in } \mathbb{R}^3 \times (0, T)\}. \]
Obviously, $0 < T_1 \leq \bar{T} \leq T^*$. However, in fact, it follows from proposition 3.1 that
\[ \bar{T} = T^*, \] (3.27)
if condition (2.1) holds. We claim that $T^* = \infty$, for which we will argue by contradiction. Suppose $T^* < \infty$, as proved in proposition 3.1,
\[ \sup_{0 \leq T \leq T^*} \| W^+ \|_{L^1(0,T;L^3)} \leq C \kappa^{-\frac{1}{2}} \| W^0 \|_{L^3}^3, \] (3.28)
which guarantees that the local solution will not blow up at $T^*$, according to the blowup criterion in [7]. Hence it contradicts the definition of $T^*$.
\[ \square \]

4. Proof of theorem 2.2

The main idea of the proof for theorem 2.2 is the same as that for theorem 2.1. The computations and techniques here are a bit more complicated. Since $(W^0_+, W^0_-) \in \dot{H}^{\frac{1}{2}} (\mathbb{R}^3)$ is equivalent to $(u_0, B_0) \in \dot{H}^{\frac{1}{2}} (\mathbb{R}^3)$, there are $T_0 > 0$ and a unique strong solution $(u, B)$ to the MHD equations in $(0, T_0]$. In fact, the solution is classical when $t > 0$. Hence, in the following we assume that the solution $(W^+, W^-)$ is sufficiently smooth on $[0, T]$ and deduce the uniform \textit{a priori} estimates under the assumptions of theorem 2.2, which guarantee the extension of the local strong solution.

For simplicity of writing, we first scale $W^+, W^-, P$ to $V^+, V^-, \bar{P}$ as follows:
\[ \begin{cases} V^+(x, t) = \kappa^{-1} W^+(x, \kappa^{-1} t), \\ V^-(x, t) = \kappa^{-1} W^-(x, \kappa^{-1} t), \\ \bar{P}(x, t) = \kappa^{-2} P(x, \kappa^{-1} t). \end{cases} \] (4.29)
Then the system for \((V^+, V^-, \tilde{P})\) becomes
\[
\begin{align*}
\frac{\partial V^+}{\partial t} &= -\Delta V^+ - \frac{\lambda}{\kappa} V^+ + (V^- \cdot \nabla) V^+ + \nabla \tilde{P} = 0, \\
\frac{\partial V^-}{\partial t} &= -\Delta V^- - \frac{\lambda}{\kappa} V^- + (V^+ \cdot \nabla) V^- + \nabla \tilde{P} = 0, \\
\text{div } V^+ &= \text{div } V^- = 0,
\end{align*}
\] (4.30)
with the following initial conditions
\[
\begin{align*}
V^+(x, 0) &= \kappa^{-1} W_0^+(x), \\
V^-(x, 0) &= \kappa^{-1} W_0^-(x).
\end{align*}
\] (4.31)

Given a strong solution \((V^+, V^-, \tilde{P})\) in \(\mathbb{R}^3 \times [0, T]\) for \(T < T_0\), define
\[
A^-(T) = \sup_{0 \leq t < T} \| V^-(\cdot, t) \|_{H^\frac{1}{2}}^2 + \int_0^T \| V^-(\cdot, t) \|_{H^\frac{1}{2}}^2 \, dt.
\] (4.32)
Similarly, we have the following proposition.

**Proposition 4.1.** Let \((V^+, V^-, \tilde{P})\) be a strong solution to (4.30)–(4.31) in \(\mathbb{R}^3 \times [0, T]\), then there exist two positive constants \(\epsilon_0 < \frac{1}{2}\) and \(C_0\), such that if
\[
A^-(T) \leq 2\epsilon_0,
\] (4.33)
then it in fact holds that
\[
A^-(T) \leq \epsilon_0,
\]
provided
\[
\left( \kappa^{-2} \| W_0^+ \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \left( \kappa^{-2} \| W_0^- \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \right) \right) \cdot \exp \left\{ C_0 \left( \kappa^{-4} \| W_0^+ \|_{H^\frac{1}{2}}^4 + \frac{\lambda^4}{\kappa^4} \right) \right\} \leq \epsilon_0.
\] (4.34)

**Lemma 4.1.** Let \((V^+, V^-)\) be a strong solution to (4.30)–(4.31) in \(\mathbb{R}^3 \times [0, T]\). There exist two positive constants \(\epsilon_0 < \frac{1}{2}\) and \(C_0\), such that if
\[
A^-(T) \leq 2\epsilon_0,
\] (4.35)
then it holds that
\[
\sup_{0 \leq t \leq T} \| V^+(t) \|_{H^\frac{1}{2}}^2 + \int_0^T \| V^+ \|_{H^\frac{1}{2}}^2 \, dt \leq \kappa^{-2} \| W_0^+ \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2},
\] (4.36)
and
\[
\sup_{0 \leq t \leq T} \| V^-(t) \|_{H^\frac{1}{2}}^2 + \int_0^T \| V^- \|_{H^\frac{1}{2}}^2 \, dt \leq \left( \kappa^{-2} \| W_0^- \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \left( \kappa^{-2} \| W_0^+ \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \right) \right) \exp \left\{ C_0 \left( \kappa^{-4} \| W_0^+ \|_{H^\frac{1}{2}}^4 + \frac{\lambda^4}{\kappa^4} \right) \right\}.
\] (4.37)
Proof.

Step 1. Multiplying the first equation of (4.30) by \((-\Delta)^{\frac{1}{2}} V^+\) and integrating over \(\mathbb{R}^3\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| V^+ \|_{H^\frac{1}{2}}^2 + \| V'^+ \|_{H^\frac{1}{2}}^2 = \frac{\lambda}{\kappa} \int \Delta V^- \cdot (-\Delta)^{\frac{1}{2}} V^+ dx - \int (V^- \cdot \nabla) V'^+ \cdot (-\Delta)^{\frac{1}{2}} V^+ dx
\]

\[
\leq \frac{|\lambda|}{\kappa} \| V^- \|_{H^\frac{1}{2}} \| V'^+ \|_{H^\frac{1}{2}} + C \| V^- \|_{L^6} \| \nabla V'^+ \|_{L^6}^2
\]

\[
\leq \frac{1}{4} \| V'^+ \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \| V^- \|_{H^\frac{1}{2}}^2 + C \| V^- \|_{H^\frac{1}{2}} \| V'^+ \|_{H^\frac{1}{2}}^2,
\]

(4.38)

where we used the Sobolev embedding inequality and Calderón–Zygmund operator theory.

If we choose \(\epsilon_0\) small enough such that

\[
\epsilon_0^2 \leq \min \left\{ \frac{1}{8C_1}, \frac{1}{\sqrt{2}} \right\},
\]

then it holds that

\[
\sup_{0 \leq t \leq T} \| V'^+ \|_{H^\frac{1}{2}}^2 + \int_0^T \| V'^+ \|_{H^\frac{1}{2}}^2 \, dt \leq \| V^+_0 \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} = \kappa^{-2} \| W_0^+ \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2},
\]

(4.40)

Step 2. Multiplying the second equation of (4.30) by \((-\Delta)^{\frac{1}{2}} V^-\) and integrating over \(\mathbb{R}^3\), we obtain after the interpolation inequality and Sobolev embedding inequality that

\[
\frac{1}{2} \frac{d}{dt} \| V^- \|_{H^\frac{1}{2}}^2 + \| V'^- \|_{H^\frac{1}{2}}^2 = \frac{\lambda}{\kappa} \int \Delta V^+ \cdot (-\Delta)^{\frac{1}{2}} V^- dx - \int (V^+ \cdot \nabla) V'^- \cdot (-\Delta)^{\frac{1}{2}} V^- dx
\]

\[
\leq \frac{|\lambda|}{\kappa} \| V^+ \|_{H^\frac{1}{2}} \| V'^- \|_{H^\frac{1}{2}} + \| V^+ \|_{L^6} \| \nabla V'^- \|_{L^6} \| \nabla V^- \|_{L^6}
\]

\[
\leq \frac{1}{4} \| V'^- \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \| V'^+ \|_{H^\frac{1}{2}}^2 + C \| V'^+ \|_{L^6} \| V^+ \|_{L^6} \| \nabla V'^- \|_{L^6} \| \nabla V^- \|_{L^6}
\]

\[
\leq \frac{1}{2} \| V'- \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \| V'^+ \|_{H^\frac{1}{2}}^2 + C \| V'^+ \|_{L^6} \| V^+ \|_{L^6} \| \nabla V'^- \|_{L^6}
\]

(4.41)

where the interpolation inequality and Sobolev embedding inequality were employed.

It follows from the Gronwall’s inequality that

\[
\sup_{0 \leq t \leq T} \| V^-(t) \|_{H^\frac{1}{2}}^2 + \int_0^T \| V^- \|_{H^\frac{1}{2}}^2 \, dt \leq e^{C \| V^+ \|_{L^6}(t)_{t \leq T}} \left( \kappa^{-2} \| W_0^- \|_{H^\frac{1}{2}}^2 + \frac{2\lambda^2}{\kappa^2} \| V'^- \|_{L^6(0,T;H^\frac{1}{2})}^2 \right).
\]

(4.42)

Using the interpolation theory, one has

\[
\int_0^T \| \nabla V'^- \|_{L^6}^2 \, dt \leq C \| V'^+ \|_{L^6(0,T;H^\frac{1}{2})} \| V^+ \|_{L^6(0,T;H^\frac{1}{2})}.
\]

(4.43)

Substitute this inequality into (4.42), and one can easily deduce after (4.40) that

\[
\sup_{0 \leq t \leq T} \| V^-(t) \|_{H^\frac{1}{2}}^2 + \int_0^T \| V^- \|_{H^\frac{1}{2}}^2 \, dt \leq \left( \kappa^{-2} \| W_0^- \|_{H^\frac{1}{2}}^2 + \frac{\lambda^2}{\kappa^2} \right) \exp \left\{ C \left( \kappa^{-4} \| W_0^+ \|_{H^\frac{3}{2}}^4 + \frac{\lambda^4}{\kappa^4} \right) \right\},
\]

(4.44)

which is our desired estimate (4.37).
It follows from lemma 4.1 that \( A^{-}(T) \leq \epsilon_0 \), if (4.34) holds. Hence we complete the proof of proposition 4.1. The remaining part of the proof for theorem 2.2 follows the same lines as section 3, so we omit the details.

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