Supersymmetric Fluid Mechanics

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Abstract

When anticommuting Grassmann variables are introduced into a fluid dynamical model with irrotational velocity and no vorticity, the velocity acquires a nonvanishing curl and the resultant vorticity is described by Gaussian potentials formed from the Grassmann variables. Upon adding a further specific interaction with the Grassmann degrees of freedom, the model becomes supersymmetric.

I. INTRODUCTION

An isentropic fluid is described by a matter density field $\rho$ and a velocity field $\mathbf{v}$. These satisfy the continuity equation, which involves the current $\mathbf{j} = \mathbf{v}\rho$,

$$ \dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1.1) $$

and the force equation

$$ \dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P \quad (1.2) $$

where $P$ is the pressure. (Over-dot denotes differentiation with respect to time.) We show that it is possible to supplement the $(\rho, \mathbf{v})$ bosonic/commuting variables with Grassmann/anticommuting variables $\psi$ such that the entire system exhibits a centrally extended supersymmetry. Moreover, when the bosonic system is irrotational, so that its vorticity vanishes, $\omega_{ij} \equiv \partial_i v^j - \partial_j v^i = 0$, and the velocity is the gradient of a velocity potential $\mathbf{v} = \nabla \theta$.

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the Grassmann variables give rise to nonvanishing vorticity and provide the Gaussian potentials in a Clebsch representation for the total velocity (see below).

The specific system that we analyze devolves from the dynamics for a membrane (a 2-dimensional extended object), which propagates in (3 + 1) dimensional space-time. The emergent fluid propagates in two spatial dimensions. When the membrane involves just bosonic variables, the fluid is irrotational \[1,2\]. Our supersymmetric fluid is derived by a similar construction, starting from a supermembrane \[3\].

In the remainder of this section, we review the action/Hamiltonian formulation for the system \((1.1)–(1.2)\). In the next section, we directly present the supersymmetric model. Section III is devoted to a derivation of this supersymmetric fluid from a supermembrane, while concluding remarks comprise the last Section IV.

For isentropic fluids, the pressure \(P\) is a function only of the density, and the right side of \((1.2)\) may also be written as \(-\nabla V'(\rho)\), where \(V'\) is the enthalpy, \(V''(\rho) = \frac{1}{\rho}P'(\rho)\), and \(\sqrt{P'}\) is the sound speed (dash denotes differentiation with respect to argument). Moreover, equations \((1.1)\) and \((1.2)\) can be obtained by (Poisson) bracketing with the Hamiltonian

\[
H = \int dr \left( \frac{1}{2} \rho v^2 + V(\rho) \right)
\]

\[
\dot{\rho} = \{H, \rho\}
\]

\[
\dot{v} = \{H, v\}
\]

provided the nonvanishing brackets of the fundamental variables \((\rho, v)\) are taken to be

\[
\{v^i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i \delta(\mathbf{r} - \mathbf{r}')
\]

\[
\{v^i(\mathbf{r}), v^j(\mathbf{r}')\} = -\frac{\omega_{ij}(\mathbf{r})}{\rho(\mathbf{r})} \delta(\mathbf{r} - \mathbf{r}') .
\]

(The fields in the brackets are at equal times, hence the time argument is suppressed.) An equivalent, more transparent version of the algebra \((1.5)\) is satisfied by the field momentum density, which also coincides with the current \(j\).

\[
\mathcal{P} = \rho v
\]

As a consequence of \((1.3)\) we have

\[
\{\mathcal{P}^i(\mathbf{r}), \rho(\mathbf{r}')\} = \rho(\mathbf{r}) \partial_i \delta(\mathbf{r} - \mathbf{r}')
\]

\[
\{\mathcal{P}^i(\mathbf{r}), \mathcal{P}^j(\mathbf{r}')\} = \mathcal{P}^j(\mathbf{r}) \partial_i \delta(\mathbf{r} - \mathbf{r}') + \mathcal{P}^i(\mathbf{r}') \partial_j \delta(\mathbf{r} - \mathbf{r}') .
\]

This is the familiar algebra of momentum densities. The Jacobi identity is satisfied by \((1.3)\) and \((1.7)\). The above holds in any dimension \[4\].

One naturally asks whether there is a canonical 1-form that leads to the symplectic structure \((1.5)\), \((1.7)\); that is, one seeks a Lagrangian whose canonical variables can be used to derive \((1.3)\) and \((1.7)\) from canonical brackets. When the velocity is irrotational, the vorticity vanishes, \(v\) can be written as \(\nabla \theta\), and \((1.5)\) is satisfied by postulating that
\[ \{ \theta(r), \rho(r') \} = \delta(r - r') \] (1.8)

that is, the velocity potential is conjugate to the density, so that the Lagrangian can be taken as

\[ L \big|_{\text{irrotational}} = \int dr \, \dot{\theta} \dot{\rho} - H \] (1.9)

where \( H \) is given by (1.3) with \( \mathbf{v} = \nabla \theta \).

With nonvanishing vorticity, the canonical formulation is more indirect. One writes the velocity in a Clebsch decomposition, which in two and three spatial dimensions reads

\[ \mathbf{v} = \nabla \theta + \alpha \nabla \beta . \] (1.10)

Then

\[ L = - \int dr \, \rho (\dot{\theta} + \alpha \dot{\beta}) - H . \] (1.11)

Here \( \alpha \) and \( \beta \) are the “Gauss potentials”, and from (1.11) is seen that \( \{ \theta, \rho \} \) as well as \( \{ \beta, \rho \alpha \} \) are canonically conjugate. It then follows that \( \mathbf{v} \), given by (1.10), and \( \rho \) satisfy (1.5).

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Some more observations on the Clebsch decomposition of the vector field \( \mathbf{v} \): In three dimensions, Eq. (1.10) involves the same number of functions on the left and right sides of the equality: three. Nevertheless the Gauss potentials are not uniquely determined by \( \mathbf{v} \). The following is the reason why a canonical formulation of (1.5) requires using the Clebsch decomposition (1.10). Although the algebra (1.5) is consistent in that the Jacobi identity is satisfied, it is degenerate in that the kinematic helicity

\[ h = \int d^3r \, \mathbf{v} \cdot (\nabla \times \mathbf{v}) = \int d^3r \, \mathbf{v} \cdot \mathbf{\omega} \]

(\( \omega^i = \frac{1}{2} \epsilon^{ijk} \omega_{jk} \)) has vanishing bracket with \( \rho \) and \( \mathbf{v} \). (Note that \( h \) is just the Abelian Chern-Simons term of \( \mathbf{v} \).) Consequently, a canonical formulation requires eliminating the algebra, that is, neutralizing \( h \). This is achieved by the Clebsch decomposition: \( \mathbf{v} = \nabla \theta + \alpha \nabla \beta \), \( \mathbf{\omega} = \nabla \alpha \times \nabla \beta \), \( \mathbf{v} \cdot \mathbf{\omega} = \nabla \theta \cdot (\nabla \alpha \times \nabla \beta) = \nabla \cdot (\theta \nabla \alpha \times \nabla \beta) \). Thus in the Clebsch parameterization the helicity is given by a surface integral \( h = \frac{1}{2} \int dS \cdot \mathbf{\omega} \) — it possesses no bulk contribution, and the obstruction to a canonical realization of (1.5) is removed.

In two spatial dimensions, the Clebsch parameterization is redundant, involving three functions to express the two velocity components. Moreover, the kernel of (1.5) in two dimensions comprises an infinite number of quantities

\[ k_n = \int d^2r \, \rho (\frac{\mathbf{\omega}}{\rho})^n \]

for which the Clebsch parameterization offers no simplification. (Here \( \mathbf{\omega} \) is the two-dimensional vorticity \( \omega_{ij} = \epsilon_{ij} \omega \).) Nevertheless, a canonical formulation in two dimensions also uses Clebsch variables to obtain an even-dimensional phase space.
II. SUPERSYMMETRIC FLUID MECHANICS

A. The Model

The bosonic fluid model in two spatial dimensions, which descends from a bosonic Nambu-Goto action, is supplemented by Grassmann variables $\psi_a$ that are Majorana spinors (real, two-component: $\psi_a^* = \psi_a$, $a = 1, 2$). The Lagrange density reads

$$\mathcal{L} = -\rho(\dot{\theta} - \frac{1}{2}\psi\dot{\psi}) - \frac{1}{2}\rho(\nabla\theta - \frac{1}{2}\psi\nabla\psi)^2 - \frac{\lambda}{\rho} - \frac{\sqrt{2}}{2}\psi\alpha \cdot \nabla\psi .$$  \hspace{1cm} (2.1)

Here $\alpha_i$ are two ($i = 1, 2$), $2 \times 2$, real symmetric Dirac “alpha” matrices; in terms of Pauli matrices we can take $\alpha_1 = \sigma_1$, $\alpha_2 = \sigma_3$. Note that the matrices satisfy the following relations, which are needed to verify subsequent formulas

$$\epsilon_{ab}^i \alpha_{bc}^j = \epsilon^{ij} \alpha_{ac}$$  
$$\alpha_{ab}^i \alpha_{bc}^j = \delta^{ij} \delta_{ac} - \epsilon^{ij} \epsilon_{ac}$$

$$\alpha_{ab} \alpha_{cd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} .$$  \hspace{1cm} (2.2)

$\epsilon_{ab}$ is the $2 \times 2$ antisymmetric matrix $\epsilon \equiv i\sigma^2$. In (2.1) $\lambda$ is a coupling strength, taken positive. The density-dependent potential $V(\rho) = \lambda/\rho$ corresponds to a negative pressure $P = -2\lambda/\rho$ and to sound velocity $\sqrt{2\lambda/\rho}$. These describe the “Chaplygin gas”. The Grassmann term enters with coupling $\sqrt{2}\lambda$, so chosen to ensure supersymmetry (see below). It is evident that the velocity should be defined as

$$\mathbf{v} = \nabla\theta - \frac{1}{2}\psi\nabla\psi .$$  \hspace{1cm} (2.3)

The Grassmann variables directly give rise to a Clebsch formula for $\mathbf{v}$, and provide the Gauss potentials. The two-dimensional vorticity reads $\omega = \epsilon^{ij} \partial_i \mathbf{v}^j = -\frac{1}{2}\epsilon^{ij} \partial_i \psi \partial_j \psi = -\frac{1}{2}\nabla\psi \times \nabla\psi$. The variables $\{\theta, \rho\}$ remain a canonical pair, while the canonical 1-form in (2.1) indicates that the canonically independent Grassmann variables are $\sqrt{\rho}\psi$ so that the antibracket of the $\psi$’s is

$$\{\psi_a(r), \psi_b(r')\} = -\frac{\delta_{ab}}{\rho(r)} \delta(r - r') .$$  \hspace{1cm} (2.4)

One verifies that the algebra (1.3) or (1.6) is satisfied, and further, one has

$$\{\theta(r), \psi(r)\} = -\frac{1}{2\rho(r)} \psi(r) \delta(r - r')$$  \hspace{1cm} (2.5a)

$$\{\mathbf{v}(r), \psi(r')\} = -\frac{\nabla\psi(r)}{\rho(r)} \delta(r - r')$$  \hspace{1cm} (2.5b)

$$\{P(r), \psi(r')\} = -\nabla\psi(r) \delta(r - r') .$$  \hspace{1cm} (2.5c)
The equations of motion read

\[ \dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0 \] (2.6a)

\[ \dot{\theta} + \mathbf{v} \cdot \nabla \theta = \frac{1}{2} \mathbf{v}^2 + \frac{\lambda}{\rho^2} + \frac{\sqrt{2\lambda}}{2\rho} \psi \mathbf{\alpha} \cdot \nabla \psi \] (2.6b)

\[ \dot{\psi} + \mathbf{v} \cdot \nabla \psi = \frac{\sqrt{2\lambda}}{\rho} \mathbf{\alpha} \cdot \nabla \psi \] (2.6c)

and together with (2.3) they imply

\[ \dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \frac{\lambda}{\rho^2} + \frac{\sqrt{2\lambda}}{\rho} \nabla \psi \mathbf{\alpha} \cdot \nabla \psi \] (2.6d)

All these equations may be obtained by bracketing with the Hamiltonian

\[ H = \int d^2r \left( \frac{1}{2} \rho \mathbf{v}^2 + \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \mathbf{\alpha} \cdot \nabla \psi \right) \] (2.7)

when (1.5), (1.8) and (2.5) are used.

We record the components of the energy-momentum tensor, and the continuity equations they satisfy. The energy density \( E = T^{oo} \), given by

\[ T^{oo} = \frac{1}{2} \rho \mathbf{v}^2 + \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \mathbf{\alpha} \cdot \nabla \psi \] (2.8)

satisfies a continuity equation with the energy flux \( T^{oj} \).

\[ \dot{T}^{oo} + \partial_j T^{oj} = 0 \] (2.9a)

\[ T^{oj} = \frac{1}{2} \rho \mathbf{v}_j \mathbf{v}^j - \frac{\lambda \mathbf{v}^j}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \mathbf{\alpha} \cdot \nabla \psi - \frac{\lambda}{\rho} \psi \partial_j \psi + \frac{\lambda}{\rho} \epsilon^{jkl} \psi \partial_k \psi \] (2.9b)

This ensures that the total energy, that is, the Hamiltonian, is time-independent. Conservation of the total momentum

\[ \mathbf{P} = \int d^2r \mathbf{P} \] (2.10)

follows from the continuity equation satisfied by the momentum density \( \mathbf{P}^i = T^{io} \)

\[ \dot{T}^{io} + \partial_j T^{ij} = 0 \] (2.11a)

\[ T^{ij} = \rho \mathbf{v}^i \mathbf{v}^j - \delta^{ij} \left( \frac{2\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2} \psi \mathbf{\alpha} \cdot \nabla \psi \right) + \frac{\sqrt{2\lambda}}{2} \psi \mathbf{\alpha} \cdot \nabla \psi \] (2.11b)

but the momentum flux \( T^{ij} \), that is, the stress tensor, is not symmetric in its spatial indices, owing to the presence of spin in the problem. However, rotational symmetry makes it possible to effect an “improvement”, which modifies the momentum density by a total derivative term,
leaving the integrated total momentum unchanged (provided surface terms can be ignored) and rendering the stress tensor symmetric. The improved quantities are

\[ P_i^i = T_i^i = \rho v^i + \frac{1}{8} \epsilon^{ij} \partial_j (\rho \psi \psi) \]  

(2.12)

\[ \dot{T}^{io} + \partial_j T_i^{kj} = 0 \]  

(2.13a)

\[ T_i^{ij} = \rho v^i v^j - \delta^{ij}\left(\frac{2\lambda}{\rho} + \frac{\sqrt{2}\lambda}{2} \psi \alpha \cdot \nabla \psi\right) + \frac{\sqrt{2}\lambda}{4} \left(\psi \alpha^i \partial_j \psi + \psi \alpha^j \partial_i \psi\right) \]

\[ - \frac{1}{8} \partial_k \left(\epsilon^{ki} v^j + \epsilon^{kj} v^i\right) \rho \psi \psi \]  

(2.13b)

It immediately follows that the angular momentum

\[ M = \int d^2r \epsilon^{ij} r^i P_j^i = \int d^2r \rho \epsilon^{ij} r^i v^j + \frac{1}{4} \int d^2r \rho \psi \psi \]  

(2.14)

is conserved. The first term is clearly the orbital part (which still receives a Grassmann contribution through \( v \)), whereas the second, coming from the improvement, is the spin part. Indeed, since \( \epsilon^{ij} = \frac{1}{2} \sigma^2 \equiv \Sigma \), we recognize this as the spin matrix in (2+1) dimensions. The extra term in the improved momentum density \( \frac{1}{8} \epsilon^{ij} \partial_j (\rho \psi \psi) \) can then be readily interpreted as an additional localized momentum density, generated by the nonhomogeneity of the spin density. This is analogous to the magnetostatics formula giving the localized current density \( j_m \) in a magnet in terms of its magnetization \( m \): \( j_m = \nabla \times m \). All in all, we are describing a fluid with spin.

Also the total number

\[ N = \int d^2r \rho \]  

(2.15)

is conserved by virtue of the continuity equation (2.6a) satisfied by \( \rho \). Finally, the theory is Galileo invariant, as is seen from the conservation of the Galileo boost,

\[ B = tP - \int d^2r \rho \]  

(2.16)

which follows from (2.6a) and (2.10). The generators \( H, P, M, B \) and \( N \) close on the (extended) Galileo group. [The theory is not Lorentz invariant in (2+1)-dimensional space-time, hence the energy flux \( T^{\mu\nu} \) does not coincide with the momentum density, improved or not.]

We observe that \( \rho \) can be eliminated from (2.1) so that \( \mathcal{L} \) involves only \( \theta \) and \( \psi \). From (2.6b) and (2.6d) it follows that

\[ \rho = \left(\dot{\theta} - \frac{1}{2} \dot{\psi} \psi + \frac{1}{2} v^2\right)^{-\frac{1}{2}}. \]  

(2.17)

Substituting into (2.1) leaves

\[ \mathcal{L} = -\sqrt{2}\lambda \left\{ \sqrt{2}\theta - \psi \psi + \left(\nabla \theta - \frac{1}{2} \psi \nabla \psi\right)^2 + \frac{1}{2} \psi \alpha \cdot \nabla \psi \right\}. \]  

(2.18)

Note that the coupling strength has disappeared from the dynamical equations, remaining only as a normalization factor for the Lagrangian. Consequently the above elimination of \( \rho \) cannot be carried out in the free case, \( \lambda = 0 \). 
B. Supersymmetry

The theory also possesses supersymmetry. This can be established, first of all, by verifying that the following two-component dynamics-dependent supercharges are time-independent Grassmann quantities.

$$Q_a = \int d^2r \left[ \rho \mathbf{v} \cdot (\alpha_{ab} \psi_b) + \sqrt{2\lambda} \psi_a \right]. \quad (2.19)$$

Taking a time derivative and using the evolution equations (2.6) establishes that $\dot{Q}_a = 0$.

Next, the transformation rule for the dynamical variables is found by considering the Grassmann charge contracted with a constant Grassmann parameter $\eta^a$, giving a bosonic symmetry generator $Q = \eta^a Q_a$. Using the canonical brackets one verifies the field transformation rules

\begin{align*}
\delta \rho &= \{Q, \rho\} = -\nabla \cdot \rho (\eta \alpha \psi) \quad (2.20a) \\
\delta \theta &= \{Q, \theta\} = -\frac{1}{2} (\eta \alpha \psi) \cdot \nabla \theta - \frac{1}{4} (\eta \alpha \psi) \cdot \psi \nabla \psi + \frac{\sqrt{2\lambda}}{2 \rho} \eta \psi \quad (2.20b) \\
\delta \psi &= \{Q, \psi\} = -(\eta \alpha \psi) \cdot \nabla \psi - \mathbf{v} \cdot \alpha \eta - \frac{\sqrt{2\lambda}}{\rho} \eta \\
\delta \mathbf{v} &= \{Q, \mathbf{v}\} = -(\eta \alpha \psi) \cdot \nabla \mathbf{v} + \frac{\sqrt{2\lambda}}{\rho} \eta \nabla \psi. \quad (2.20d)
\end{align*}

Supersymmetry is reestablished by determining the variation of the action $\int dt L$, consequent to the above field variations: the action is invariant. One then reconstructs the supercharges (2.19) by Noether’s theorem. Finally, upon computing the bracket of two supercharges, one finds

$$\{\eta_1^a Q_a, \eta_2^b Q_b\} = 2(\eta_1 \eta_2) H \quad (2.21)$$

which again confirms that the charges are time-independent:

$$\{H, Q_a\} = 0. \quad (2.22)$$

Additionally a further, kinematical, supersymmetry can be identified. According to the equations of motion the following two supercharges are also time-independent:

$$\tilde{Q}_a = \int d^2r \rho \psi_a. \quad (2.23)$$

$\tilde{Q} = \tilde{\eta}^a \tilde{Q}_a$ effects a shift of the Grassmann field:

\begin{align*}
\tilde{\delta} \rho &= \{\tilde{Q}, \rho\} = 0 \quad (2.24a) \\
\tilde{\delta} \theta &= \{\tilde{Q}, \theta\} = -\frac{1}{2} (\tilde{\eta} \psi) \quad (2.24b) \\
\tilde{\delta} \psi &= \{\tilde{Q}, \psi\} = -\tilde{\eta} \quad (2.24c) \\
\tilde{\delta} \mathbf{v} &= \{\tilde{Q}, \mathbf{v}\} = 0. \quad (2.24d)
\end{align*}
This transformation leaves the Lagrangian invariant, and Noether’s theorem reproduces (2.23). The algebra of these charges closes on the total number $N$.

$$\{\tilde{\eta}_a^i \tilde{Q}_a, \tilde{\eta}_b^j \tilde{Q}_b\} = (\tilde{\eta}_1 \tilde{\eta}_2) N$$

while the algebra with the generators (2.19), closes on the total momentum, together with a central extension, proportional to volume of space $\Omega = \int d^2 r$.

$$\{\tilde{\eta}_a^i \tilde{Q}_a, \eta^j Q_b\} = (\tilde{\eta} \alpha \eta) \cdot P + \sqrt{2} \beta \epsilon \eta \Omega.$$ (2.26)

The supercharges $Q_a, \tilde{Q}_a$, together with the Galileo generators ($H, P, M, \text{and } B$), with $N$ form a superextended Galileo algebra. The additional, nonvanishing brackets are

$$\{M, Q_a\} = \frac{1}{2} \epsilon^{ab} Q_b$$

$$\{M, \tilde{Q}_a\} = \frac{1}{2} \epsilon^{ab} \tilde{Q}_b$$

$$\{B, Q_a\} = \alpha_{ab} \tilde{Q}_b.$$ (2.29)

### III. MEMBRANE CONNECTION

The equations for a supersymmetric Chaplygin fluid devolve from the supermembrane Lagrangian, $L_M$. We shall give two different derivations of this result, which make use of two different parameterizations for the parameterization-invariant membrane action and give rise, respectively, to (2.1) and (2.18).

We work in a light-cone gauge-fixed theory: The membrane in 4-dimensional space-time is described by coordinates $x^\mu$ ($\mu = 0, 1, 2, 3$), which are decomposed into light-cone components $x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^3)$ and transverse components $x^i$ ($i = 1, 2$). These depend on an evolution parameter $\tau$ and two space-like parameters $\phi^r$ ($r = 1, 2$). Additionally there are two-component, real Grassmann spinors $\psi$, which also depend on $\tau$ and $\phi^r$. In the light-cone gauge, $x^+$ is identified with $\tau$, $x^-$ is renamed $\theta$, and the supermembrane Lagrangian is [3]

$$L_M = \int d^2 \phi \sqrt{G} - \frac{1}{2} \epsilon^{rs} \partial_r \psi \partial_s \psi \cdot x$$

where $G = \det G_{\alpha \beta}$; $G_{\alpha \beta} =

$$\begin{pmatrix}
G_{\alpha \beta}
\end{pmatrix} =
\begin{pmatrix}
G_0 & G_o & G_{os} \\
G_{ro} & -g_{rs}
\end{pmatrix}
$$

$$\begin{pmatrix}
2 \partial_r \theta - \partial_r x \cdot \partial_r x - \psi \partial_r \psi & u_a \\
u_r & -g_{rs}
\end{pmatrix}$$

$$G = g \Gamma$$

$$\Gamma \equiv 2 \partial_r \theta - \partial_r x \cdot \partial_r x - \psi \partial_r \psi + g^{rs} u_r u_s$$

$$g_{rs} \equiv \partial_r x \cdot \partial_s x, \quad g = \det g_{rs}$$

$$u_r \equiv \partial_r \theta - \frac{1}{2} \psi \partial_r \psi - \partial_r x \cdot \partial_r x.$$ (3.3)
Here $\partial_\tau$ signifies differentiation with respect to the evolution parameter $\tau$, while $\partial_r, \partial_s$ differentiate with respect to the space-like parameters ($\phi^r, \phi^s$), and $g^{rs}$, the inverse of $g_{rs}$, is used to move the $(r, s)$ indices. Note that the dimensionality of the transverse coordinates $x^i$ is the same as of the parameters $\phi^r$, namely two.

A. First Derivation

To give our first derivation, we rewrite the Lagrangian in canonical, first-order form, with the help of canonical momenta defined by

$$\frac{\partial L}{\partial \partial_\tau x} = p = -\Pi \partial_r x - \Pi u^r \partial_r x$$

$$\frac{\partial L}{\partial \partial_\tau \theta} = \Pi = \sqrt{g/\Gamma}$$

$$L_M = p \cdot \partial_\tau x + \Pi \partial_\theta \theta - \frac{1}{2} \Pi \psi \partial_r \psi + \frac{1}{2 \Pi} (p^2 + g) + \frac{1}{2} \epsilon^{rs} \partial_r \psi \alpha \partial_s \psi \cdot x$$

$$+ u^r \left( \partial_r x \cdot p + \Pi \partial_\tau \theta - \frac{1}{2} \Pi \psi \partial_r \psi \right).$$

In (3.5) $u^r$ serves as a Lagrange multiplier enforcing a subsidiary condition on the canonical variables. The equations that follow from (3.5) coincide with the Euler-Lagrange equations for (3.1)–(3.3). The theory still possesses an invariance against redefining the spatial parameters with a $\tau$-dependent function of the parameters. This freedom may be used to set $u_r$ to zero and fix $\Pi$ at $-1$. Next we introduce the hodographic transformation [5], whereby independent-dependent variables are interchanged, namely we view the $\phi^r$ to be functions of $x^i$. It then follows that the constraint on (3.5), which with $\Pi = -1$ reads

$$\partial_r x \cdot p - \partial_\tau \theta + \frac{1}{2} \psi \partial_r \psi = 0$$

becomes

$$\partial_r x \cdot \left( p - \nabla \theta + \frac{1}{2} \psi \nabla \psi \right) = 0.$$  

Here $p$, $\theta$ and $\psi$ are viewed as functions of $x$, renamed $r$, with respect to which acts the gradient $\nabla$. Also we rename $p$ as $v$, which according to (3.6b) is

$$v = \nabla \theta - \frac{1}{2} \psi \nabla \psi.$$  

From the chain rule, it follows that

$$\partial_r = \partial_t + \partial_r x \cdot \nabla$$

and according to (3.4a) (at $\Pi = -1$, $u^r = 0$) $\partial_r x = p = v$. Finally, the measure transforms according to $d^2 \phi \rightarrow d^2 r \frac{1}{\sqrt{g}}$. Thus the Lagrangian for (3.3) becomes, after setting $u^r$ to zero and $\Pi$ to $-1$,
But \( \epsilon^{rs} \partial_s x^i \partial_r x^j = \epsilon^{ij} \text{det} \partial_r x^i = \epsilon^{ij} \sqrt{g} \). After \( \sqrt{g} \) is renamed \( \sqrt{2} \lambda/\rho \), (3.8) finally reads

\[
L_M = \left( \frac{1}{\sqrt{2} \lambda} \right) \int d^2 r \left( -\rho (\dot{\theta} - \frac{1}{2} \psi \dot{\psi}) - \frac{1}{\rho} (\nabla \theta - \frac{1}{2} \psi \nabla \psi)^2 - \frac{\lambda}{2} \frac{\sqrt{2} \lambda}{2} \psi \alpha \times \nabla \psi \right). \tag{3.9}
\]

Upon replacing \( \psi \) by \( \frac{1}{\sqrt{2}} (1 - \epsilon) \psi \), this is seen to reproduce the Lagrange density (2.1), apart from an overall factor.

**B. Second Derivation**

For our second derivation, we return to (3.1)–(3.3) and use the remaining reparameterization freedom to equate the two \( x^i \) variables with the two \( \phi^r \) variables, renaming both as \( r^i \) [6]. Also \( \tau \) is renamed as \( t \). In (3.1)–(3.3) \( g_{rs} = \delta_{rs} \), and \( \partial_{\tau} x = 0 \), so that (3.3) becomes simply

\[
G = \Gamma = 2 \dot{\theta} - \psi \dot{\psi} + u^2 \tag{3.10}
\]

\[
u = \nabla \theta - \frac{1}{2} \psi \nabla \psi. \tag{3.11}
\]

Therefore the Nambu-Goto action (3.1) reads

\[
L_M = - \int d^2 r \left\{ \sqrt{2} \dot{\theta} - \psi \dot{\psi} + (\nabla \theta - \frac{1}{2} \psi \nabla \psi)^2 + \frac{1}{2} \psi \alpha \times \nabla \psi \right\}. \tag{3.12}
\]

Again a replacement of \( \psi \) by \( \frac{1}{\sqrt{2}} (1 - \epsilon) \psi \) demonstrates that the integrand coincides with the Lagrange density in (2.18) (apart from a normalization factor).

**C. Further Consequences of the Supermembrane Connection**

The supermembrane dynamics is Poincaré invariant in (3+1)-dimensional space-time. This invariance is hidden by the choice of light-cone parameterization: only the light-cone subgroup of the Poincaré group is left as a manifest invariance. This is just the \((2 + 1)\) Galileo group generated by \( H, \ P, \ M, \ B, \) and \( N \). (The light-cone subgroup of the Poincaré group is isomorphic to the Galileo group in one lower dimension.) The Poincaré generators not included in the above list correspond to Lorentz transformations in the “–” direction.

We expect therefore that these generators are “dynamical”, that is, hidden and unexpected conserved quantities of our supersymmetric Chaplygin gas, similar to the situation with the purely bosonic model [4].

One verifies that the following quantities

\[
D = t H - \int d^2 r \rho \dot{\theta} \tag{3.13}
\]

\[
G = \int d^2 r \left( r \mathcal{E} - \theta \mathcal{P}_I - \frac{1}{8} \psi \alpha \alpha \cdot \mathcal{P}_I \psi \right) \tag{3.14}
\]

\[
= \int d^2 r \left( r \mathcal{E} - \theta \mathcal{P} - \frac{1}{4} \psi \alpha \alpha \cdot \mathcal{P} \psi \right) \tag{3.15}
\]
are time-independent by virtue of the equations of motion (2.6), and they supplement the Galileo generators to form the full (3+1) Poincaré algebra, which becomes the super-Poincaré algebra once the supersymmetry is taken into account.

**IV. CONCLUSION**

We have shown how fluid dynamics can be extended to include Grassmann variables, which also enter in a supersymmetry-preserving interaction. Since our construction is based on a supermembrane in (3+1)-dimensional space-time, the fluid model is necessarily a planar Chaplygin gas. It remains to be shown how this construction could be extended to arbitrary dimensions and to different interactions. Note that Grassmann Gauss potentials can be used even in the absence of supersymmetry. For example, our theory (2.7), with the last term omitted, possesses a conventional, bosonic Hamiltonian without supersymmetry, while the Grassmann variables are hidden in $v$ and occur only in the canonical 1-form. In a related investigation, conventional fluid mechanics is generalized, so that it possesses a non-Abelian gauge symmetry \[8\].

*Note Added:* J. Hoppe has informed us that some of the above results were obtained by him in unpublished research: Karlsruhe preprints KA-THEP-6-93 and KA-THEP-9-93 (hep-th/9311059).
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