Golden-Thompson’s inequality
for
deformed exponentials

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Abstract

Deformed logarithms and their inverse functions, the deformed exponentials, are important tools in the theory of non-additive entropies and non-extensive statistical mechanics. We formulate and prove counterparts of Golden-Thompson’s trace inequality for $q$-exponentials with parameter $q$ in the interval $[1,3]$.

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1 Introduction and main result

Tsallis [7] generalised in 1988 the standard Boltzmann-Gibbs entropy to a non-extensive quantity $S_q$ depending on a parameter $q$. In the quantum version it is given by

$$S_q(\rho) = \frac{1 - \text{Tr} \rho^q}{q - 1} \quad q \neq 1,$$

where $\rho$ is a density matrix. It has the property that $S_q(\rho) \to S(\rho)$ for $q \to 1$, where $S(\rho) = -\text{Tr} \rho \log \rho$ is the von Neumann entropy. The Tsallis entropy may be written on a similar form

$$S_q(\rho) = -\text{Tr} \rho \log_q(\rho),$$

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where the deformed logarithm \( \log_q \) is given by

\[
\log_q x = \int_1^x t^{q-2} \, dt = \begin{cases} 
x^{q-1} - 1 & q > 1 \\
q - 1 & q = 1 \\
\log x & q = 1
\end{cases}
\]

for \( x > 0 \). The deformed logarithm is also denoted the \( q \)-logarithm. The inverse function \( \exp_q \) is called the \( q \)-exponential and is given by

\[
\exp_q(x) = (x(q - 1) + 1)^{1/(q-1)} \quad \text{for} \quad x > \frac{-1}{q - 1}.
\]

The \( q \)-logarithm and the \( q \)-exponential functions converge, respectively, to the logarithmic and the exponential functions for \( q \to 1 \).

The aim of this article is to generalise Golden-Thompson’s trace inequality \([2, 6]\) to deformed exponentials. The main result is the following:

**Theorem 1.1.** Let \( A \) and \( B \) be positive definite matrices.

(i) If \( 1 \leq q < 2 \) then

\[
\text{Tr} \, \exp_q(A + B) \leq \text{Tr} \, \exp_q(A)^{2-q}(A(q - 1) + \exp_q B).
\]

(ii) If \( 2 \leq q \leq 3 \) then

\[
\text{Tr} \, \exp_q(A + B) \geq \text{Tr} \, \exp_q(A)^{2-q}(A(q - 1) + \exp_q B).
\]

Notice that we for \( q = 1 \) recovers Golden-Thompson’s trace inequality

\[
\text{Tr} \, \exp(A + B) \leq \text{Tr} \, \exp(A) \exp(B).
\]

This inequality is valid for arbitrary self-adjoint matrices \( A \) and \( B \). However, it is sufficient to know the inequality for positive definite matrices, since the general form follows by multiplication with positive numbers.

2 Preliminaries

We collect a few well-known results that we are going to use in the proof of the main theorem.
The $q$-logarithm is a bijection of the positive half-line onto the open interval $(-(q - 1)^{-1}, \infty)$, and the $q$-exponential is consequently a bijection of the interval $(-(q - 1)^{-1}, \infty)$ onto the positive half-line. For $q > 1$ we may thus safely apply both the $q$-logarithm and the $q$-exponential to positive definite operators. We also notice that
\[
\frac{d}{dx} \log_q(x) = x^{q-2} \quad \text{and} \quad \frac{d}{dx} \exp_q(x) = \exp_q(x)^{2-q}.
\]
The proof of the following lemma is rather easy and may be found in [4, Lemma 5].

**Lemma 2.1.** Let $\varphi : D \to A_{sa}$ be a map defined in a convex cone $D$ in a Banach space $X$ with values in the self-adjoint part of a $C^*$-algebra $A$. If $\varphi$ is Fréchet differentiable, convex and positively homogeneous then
\[
d\varphi(x)h \leq \varphi(h).
\]
for $x, h \in D$.

Let $H$ be any $n \times n$ matrix. The map
\[
A \to \text{Tr}(H^* A^p H)^{1/p},
\]
defined in positive definite $n \times n$ matrices, is concave for $0 < p \leq 1$ and convex for $1 \leq p \leq 2$, cf. [11, Theorem 1.1]. By a slight modification of the construction given in Remark 3.2 in the same reference, cf. also [3], we obtain that the mapping
\[
(A_1, \ldots, A_k) \to \text{Tr}(H_1^* A_1^p A_1 + \cdots + H_k^* A_k H_k)^{1/p},
\]
defined in $k$-tuples of positive definite $n \times n$ matrices, is concave for $0 < p \leq 1$ and convex for $1 \leq p \leq 2$; for arbitrary $n \times n$ matrices $H_1, \ldots, H_k$.

### 3 Deformed trace functions

**Theorem 3.1.** Let $H_1, \ldots, H_k$ be matrices with $H_1^* H_1 + \cdots + H_k^* H_k = 1$ and define the function
\[
\varphi(A_1, \ldots, A_k) = \text{Tr} \exp_q \left( \sum_{i=1}^{k} H_i^* \log_q(A_i) H_i \right)
\]
in $k$-tuples of positive definite matrices. Then $\varphi$ is positively homogeneous of degree one. It is concave for $1 \leq q \leq 2$ and convex for $2 \leq q \leq 3$. 
Proof. For \( q > 1 \) we obtain

\[
\varphi(A_1, \ldots, A_k) = \text{Tr} \exp_q \left( \sum_{i=1}^{k} H_i^* \log_q (A_i) H_i \right)
\]

\[
= \text{Tr} \left( (q - 1) \left( \sum_{i=1}^{k} H_i^* \log_q (A_i) H_i \right) + 1 \right)^{1/(q-1)}
\]

\[
= \text{Tr} \left( (q - 1) \left( \sum_{i=1}^{k} H_i^* A_i^{q-1} - \frac{1}{q-1} H_i \right) + 1 \right)^{1/(q-1)}
\]

\[
= \text{Tr} \left( \sum_{i=1}^{k} H_i^* (A_i^{q-1} - 1) H_i + 1 \right)^{1/(q-1)}
\]

\[
= \text{Tr} \left( H_1^* A_1^{q-1} H_1 + \cdots + H_k^* A_k^{q-1} H_k \right)^{1/(q-1)}.
\]

From this identity it follows that \( \varphi \) is positively homogeneous of degree one. The concavity for \( 1 < q \leq 2 \) and the convexity for \( 2 \leq q \leq 3 \) now follows from (2). The statement for \( q = 1 \) follows by letting \( q \) tend to one. QED

**Corollary 3.2.** Let \( L \) be positive definite, and let \( H_1, \ldots, H_k \) be matrices such that \( H_1^* H_1 + \cdots + H_k^* H_k \leq 1 \). Then the function

\[
\varphi(A_1, \ldots, A_k) = \text{Tr} \exp_q \left( L + H_1^* \log_q (A_1) H_1 + \cdots + H_k^* \log_q (A_k) H_k \right),
\]

defined in \( k \)-tuples of positive definite matrices, is concave for \( 1 \leq q \leq 2 \) and convex for \( 2 \leq q \leq 3 \).

Proof. We may without loss of generality assume \( H_1^* H_1 + \cdots + H_k^* H_k < 1 \) and put \( H_{k+1} = (1 - (H_1^* H_1 + \cdots + H_k^* H_k))^{1/2} \). We then have

\[
H_1^* H_1 + \cdots + H_k^* H_k + H_{k+1}^* H_{k+1} = 1
\]

and may use the preceding theorem to conclude that the function

\[
(A_1, \ldots, A_{k+1}) \rightarrow \text{Tr} \exp_q \left( H_1^* \log_q (A_1) H_1 + \cdots + H_{k+1}^* \log_q (A_{k+1}) H_{k+1} \right)
\]

of \( k + 1 \) variables is concave for \( 1 \leq q \leq 2 \) and convex for \( 2 \leq q \leq 3 \). Since \( H_{k+1} \) is invertible we may choose

\[
A_{k+1} = \exp_q \left( H_{k+1}^{-1} L H_{k+1}^{-1} \right)
\]

which makes sense since \( H_{k+1}^{-1} L H_{k+1}^{-1} \) is positive definite. Concavity for \( 1 \leq q \leq 2 \) and convexity for \( 2 \leq q \leq 3 \) in the first \( k \) variables of the above function then yields the result. QED
Setting \( q = 1 \) we recover in particular [3, Theorem 3].

**Corollary 3.3.** Let \( H_1, \ldots, H_k \) be matrices with \( H_1^* H_1 + \cdots + H_k^* H_k \leq 1 \), and let \( L \) be self-adjoint. The trace function

\[
(A_1, \ldots, A_k) \rightarrow \text{Tr} \exp \left( L + H_1^* \log(A_1) H_1 + \cdots + H_k^* \log(A_k) H_k \right)
\]

is concave in positive definite matrices.

**Corollary 3.4.** The trace function \( \varphi \) defined in [3] satisfies

\[
\varphi(B_1, \ldots, B_k) \leq \text{Tr} \exp \left( \sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) \sum_{j=1}^k H_j^* (d \log_q(A_j) B_j) H_j
\]

for \( 1 \leq q \leq 2 \) and

\[
\varphi(B_1, \ldots, B_k) \geq \text{Tr} \exp \left( \sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) \sum_{j=1}^k H_j^* (d \log_q(A_j) B_j) H_j
\]

for \( 2 \leq q \leq 3 \), where \( A_1, \ldots, A_k \) and \( B_1, \ldots, B_k \) are positive definite matrices.

**Proof.** For \( 1 \leq q \leq 2 \) we obtain

\[
d\varphi(A_1, \ldots, A_k)(B_1, \ldots, B_k) \geq \varphi(B_1, \ldots, B_k)
\]

by Lemma [2.1]. By the chain rule for Fréchet differentiable mappings between Banach spaces we therefore obtain

\[
\varphi(B_1, \ldots, B_k) \leq \sum_{j=1}^k d_j \varphi(A_1, \ldots, A_k) B_j
\]

\[
= \sum_{j=1}^k \text{Tr} \exp \left( \sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) H_j^* (d \log_q(A_j) B_j) H_j
\]

\[
= \sum_{j=1}^k \text{Tr} \exp \left( \sum_{i=1}^k H_i^* \log_q(A_i) H_i \right) 2^{-q} H_j^* (d \log_q(A_j) B_j) H_j
\]

where we used the identity \( \text{Tr} \, df(A) B = \text{Tr} \, f'(A) B \) valid for differentiable functions. This proves the first assertion. The result for \( 2 \leq q \leq 3 \) follows similarly. \textbf{QED}
4 Proof of the main theorem

In order to prove Theorem 1.1 (i) we set $k = 2$ in Corollary 3.4 and obtain
\[
\varphi(B_1, B_2) \leq \text{Tr} \exp_q(X)^{2-q}(H_1^*(d\log_q(A_1)B_1)H_1 + H_2^*(d\log_q(A_2)B_2)H_2)
\]
for $1 \leq q \leq 2$ and positive definite matrices $A_1, A_2$ and $B_1, B_2$ where
\[
X = H_1^* \log_q(A_1)H_1 + H_2^* \log_q(A_2)H_2.
\]
If we set $A_1 = B_1$ and $A_2 = 1$ the inequality reduces to
\[
\varphi(B_1, B_2) \leq \text{Tr} \exp_q(H_1^* \log_q(B_1)H_1)^{2-q}(H_1^* B_1^{q-1}H_1 + H_2^* B_2 H_2).
\]
We now set $H_1 = \varepsilon^{1/2}$ for $0 < \varepsilon < 1$, and to fixed positive definite matrices $L_1$ and $L_2$ we choose $B_1$ and $B_2$ such that
\[
L_1 = H_1^* \log_q(B_1)H_1 = \varepsilon \log_q(B_1)
\]
\[
L_2 = H_2^* \log_q(B_2)H_2 = (1 - \varepsilon) \log_q(B_2).
\]
It follows that
\[
B_1 = \exp_q(\varepsilon^{-1}L_1) \quad \text{and} \quad B_2 = \exp_q((1 - \varepsilon)^{-1}L_2).
\]
Inserting in the inequality we now obtain
\[
\text{Tr} \exp_q(L_1 + L_2)
\]
\[
\leq \text{Tr} \exp_q(L_1)^{2-q}(\varepsilon \exp_q(\varepsilon^{-1}L_1)^{q-1} + (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2))
\]
\[
= \text{Tr} \exp_q(L_1)^{2-q}(L_1(q - 1) + \varepsilon + (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2)).
\]
This expression decouple $L_1$ and $L_2$ and reduces the minimisation problem over $\varepsilon$ to the commutative case. We furthermore realise that minimum is obtained by letting $\varepsilon$ tend to zero and that
\[
\lim_{\varepsilon \to 0}(1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2) = \exp_q(L_2).
\]
We finally replace $L_1$ and $L_2$ with $A$ and $B$. This proves the first statement in Theorem 1.1.

The proof of the second statement is virtually identical to the proof of the
first. Since now \( 2 \leq q \leq 3 \) the second inequality in Corollary 3.4 applies. Setting \( k = 2 \) and applying the same substitutions as in the proof of the first statement we arrive at the inequality

\[
\text{Tr } \exp_q(L_1 + L_2) \\
\geq \text{Tr } \exp_q(L_1)^{2-q}(L_1(q - 1) + \varepsilon + (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2)).
\]

Since \( 2 \leq q \leq 3 \) the function

\[
\varepsilon \to \varepsilon + (1 - \varepsilon) \exp_q((1 - \varepsilon)^{-1}L_2)
\]

is now decreasing, and we thus maximise the right hand side in the above inequality by letting \( \varepsilon \) tend to zero. This proves the second statement in Theorem 1.1.

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