On a conjecture of Hefetz and Keevash on Lagrangians of intersecting hypergraphs and Turán numbers

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Abstract

Let \( S'(n) \) be the \( r \)-graph on \( n \) vertices with parts \( A \) and \( B \), where the edges consist of all \( r \)-tuples with 1 vertex in \( A \) and \( r - 1 \) vertices in \( B \), and the sizes of \( A \) and \( B \) are chosen to maximise the number of edges. Let \( M_r^t \) be the \( r \)-graph with \( t \) pairwise disjoint edges. Given an \( r \)-graph \( F \) and a positive integer \( p \geq |V(F)| \), we define the extension of \( F \), denoted by \( H_p^F \) as follows: Label the vertices of \( F \) as \( v_1, \ldots, v_{|V(F)|} \). Add new vertices \( v_{|V(F)|+1}, \ldots, v_p \). For each pair of vertices \( v_i, v_j, 1 \leq i < j \leq p \) not contained in an edge of \( F \), we add a set \( B_{ij} \) of \( r - 2 \) new vertices and the edge \( \{v_i, v_j\} \cup B_{ij} \), where the \( B_{ij} \)'s are pairwise disjoint over all such pairs \( \{i, j\} \). Hefetz and Keevash conjectured that the Turán number of the extension of \( M_r^t \) is \( \frac{1}{r}n \cdot \left( \frac{|V(F)|}{r-1} \right) \) for \( r \geq 4 \) and sufficiently large \( n \). Moreover, if \( n \) is sufficiently large and \( G \) is an \( H_p^F \)-free \( r \)-graph with \( n \) vertices and \( \frac{1}{r}n \cdot \left( \frac{|V(F)|}{r-1} \right) \) edges, then \( G \) is isomorphic to \( S'(n) \). In this paper, we confirm the above conjecture for \( r = 4 \).

Key Words: Turán number, Hypergraph Lagrangian, Intersecting family

1 Introduction

For a set \( V \) and a positive integer \( r \) we denote by \( V^{(r)} \) the family of all \( r \)-subsets of \( V \). An \( r \)-uniform graph or \( r \)-graph \( G \) consists of a vertex set \( V(G) \) and an edge set \( E(G) \subseteq V(G)^{(r)} \). We sometimes write the edge set of \( G \) as \( E(G) \). Let \( |G| \) denote the number of edges of \( G \). An edge \( e = \{a_1, a_2, \ldots, a_r\} \) will be simply denoted by \( a_1a_2\ldots a_r \). An \( r \)-graph \( H \) is a subgraph of an \( r \)-graph \( G \), denoted by \( H \subseteq G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). A subgraph \( G \) induced by \( V' \subseteq V \), denoted as \( G[V'] \), is the \( r \)-graph with vertex set \( V' \) and edge set \( E' = \{e \in E(G) : e \subseteq V'\} \). Let \( K_r^t \) denote the complete \( r \)-graph on \( t \) vertices.

Given an \( r \)-uniform hypergraph \( F \), an \( r \)-uniform hypergraph \( G \) is called \( F \)-free if it does not contain a copy of \( F \) as a subgraph. The Turán number of \( F \), denoted by \( ex(n, F) \), is the maximum number of edges in an \( F \)-free \( r \)-uniform hypergraph on \( n \) vertices. An averaging argument of Katona, Nemetz and

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Simonovits [8] implies the sequence $ex(n, F)/\binom{n}{r}$ decreases. So $\lim_{n \to \infty} ex(n, F)/\binom{n}{r}$ exists. The Turán density of $F$ is defined as

$$\pi(F) = \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{r}}.$$ 

For 2-graphs, Erdős-Stone-Simonovits determined the Turán densities of all graphs except bipartite graphs. Very few results are known for hypergraphs and a survey on this topic can be found in Keevash’s survey paper [9]. Lagrangian has been a useful tool in estimating the Turán density of a hypergraph.

**Definition 1.1** For an $r$-graph $G$ with the vertex set $[n]$, edge set $E(G)$ and a weighting $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the Lagrangian function of $G$ as

$$\lambda(G, \bar{x}) = \sum_{e \in E(G)} \prod_{i \in e} x_i.$$ 

The Lagrangian of $G$, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \bar{x}) : \bar{x} \in \Delta\},$$

where

$$\Delta = \{\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for every } i \in [n]\}.$$ 

The value $x_i$ is called the weight of the vertex $i$ and a weighting $\bar{x} \in \Delta$ is called a feasible weighting. A weighting $\bar{y} \in \Delta$ is called an optimum weighting for $G$ if $\lambda(G, \bar{y}) = \lambda(G)$.

**Fact 1.2** If $G' \subseteq G$ then $\lambda(G') \leq \lambda(G)$.

Given an $r$-graph $F$, the Lagrangian density $\pi_\lambda(F)$ of $F$ is defined as

$$\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ is an } F\text{-free } r\text{-graph}\}.$$ 

Usually, the difficulty part of obtaining Turán density is to get a good upper bound. The following remark says that the Turán density of an $r$-graph is no more than its Lagrangian density.

**Remark 1.3** (see Remark 1.2 in [7]) $\pi(F) \leq \pi_\lambda(F)$.

The Lagrangian method for hypergraph Turán problems were developed independently by Sidorenko [19] and Frankl-Füredi [4], generalizing work of Motzkin-Straus [10] and Zykov [22]. More recent developments of the method were obtained by Pikhurko [18] and Norin and Yepremyan [13]. More recent results based on these developments will be introduced later.

Let $r \geq 3$, $F$ be an $r$-graph and $p \geq |V(F)|$. Let $K_p^F$ denote the family of $r$-graphs $H$ that contains a set $C$ of $p$ vertices, called the core, such that the subgraph of $H$ induced by $C$ contains a copy of $F$ and such that every pair of vertices in $C$ is covered in $H$ (A pair of vertices $i, j$ of $H$ is covered if there exists an edge of $H$ containing both $i, j$). Let $H_p^F$ be a member of $K_p^F$ obtained as follows. Label the vertices of $F$ as $v_1, \ldots, v_{|V(F)|}$. Add new vertices $v_{|V(F)|+1}, \ldots, v_p$. Let $C = \{v_1, \ldots, v_p\}$. For each pair of vertices $v_i, v_j \in C$ not covered in $F$, we add a set $B_{ij}$ of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the $B_{ij}$’s are pairwise disjoint over all such pairs $\{i, j\}$. We call $H_p^F$ the extension of $F$.

Frankl and Füredi [4] conjectured that for all $r \geq 4$, if $n \geq n_0(r)$ is sufficiently large then $ex(n, K_{r+1}^L) = ex(n, H_{r+1}^L)$, where $L$ is the graph on $r+1$ vertices consisting of two edges sharing $r-1$ vertices.
Let $T_r(n,l)$ denote the balanced complete $l$-partite $r$-graph on $n$ vertices. Pikhurko [17] proved the conjecture for $r = 4$, showing that $ex(n, K_{5}^{6}) = ex(n, H_{6}^{5}) = e(T_{4}(n, 4))$, with the $T_{4}(n, 4)$ being the unique extremal graph. Recently, Norin and Yepremyan [13] proved the conjecture for $r = 5$ and $r = 6$, moreover, extremal graphs are blowups of the unique $(11, 5, 4)$ and $(12, 6, 5)$ Steiner systems for $r = 5$ and $r = 6$, respectively. For all $n, p, r$, Mubayi [11] and Pikhurko [12] showed that $ex(n, K_{p}^{r}) = ex(n, H_{p}^{r}) = e(T_{r}(n, p - 1))$ with the unique extremal graph $T_{r}(n, p - 1)$, where $F$ is the $r$-uniform empty graph. Mubayi and Pikhurko [12] showed that for all $r \geq 3$ and all sufficiently large $n$, $ex(n, H_{r+1}^{F}) = e(T_{r}(n, r))$, where $f$ is a single $r$-set. Moreover, $T_{r}(n, r)$ is the unique extremal graph. Brandt-Irwin-Jiang [23] and independently Norin and Yepremyan [14] showed that for a large family of $r$-graphs $F$ and sufficiently large $n$, $ex(n, H_{r}^{F}) = e(T_{r}(n, p - 1))$ with the unique extremal graph being $T_{r}(n, p - 1)$.

Let $M_{r}^{t}$ be the $r$-graph with $t$ pairwise disjoint edges, called $r$-uniform $t$-matching. Hefetz and Keevash in [6] determined the Lagrangian density of $M_{3}^{2}$, and showed $ex(n, H_{6}^{M_{3}^{2}}) = e(T_{3}(n, 5))$ for large $n$ and $T_{3}(n, 5)$ is the unique extremal graph. More generally, Jiang-Peng-Wu in [7] determined the Lagrangian density of $M_{r}^{3}$ and showed that $ex(n, H_{3t}^{M_{r}^{3}}) = e(T_{3}(n, 3t - 1))$ for large $n$ and $T_{3}(n, 3t - 1)$ is the unique extremal graph.

Let $S^{t}(n)$ be the $r$-graph on $[n]$ with parts $A$ and $B$, where the edges consist of all $r$-tuples with $1$ vertex in $A$ and $r - 1$ vertices in $B$, and the sizes of $A$ and $B$ are chosen to maximise the number of edges (so $|A| = n/r$). Write $s^{t}(n) = |S^{t}(n)|$. When $r = 4$, then $|A| = \lfloor \frac{4n}{5} \rfloor$ for $n \equiv 0$ or $1$ (mod $4$) and $|A| = \lfloor \frac{4n}{3} \rfloor$ or $\lceil \frac{4n}{3} \rceil$ for $n \equiv 3$ (mod $4$). In [6], Hefetz and Keevash proposed the following conjecture.

Conjecture 1.4 (D. Hefetz and P. Keevash, [6]) $ex(n, H_{2r}^{M_{r}^{2}}) = \frac{1}{r} n \cdot \left(\frac{4n}{r-1}\right)$ for $r \geq 4$ and sufficiently large $n$. Moreover, if $n$ is sufficiently large and $G$ is an $H_{2r}^{M_{r}^{2}}$-free $r$-graph on $[n]$ with $\frac{1}{r} n \cdot \left(\frac{4n}{r-1}\right)$ edges, then $G \cong S^{t}(n)$.

In this paper, we confirm the above conjecture for $r = 4$.

Theorem 1.5 $ex(n, H_{8}^{M_{4}^{3}}) = \left(\frac{1}{4} n\right) \cdot \left(\frac{4n}{3}\right)$ for sufficiently large $n$. Moreover, if $n$ is sufficiently large and $G$ is an $H_{8}^{M_{4}^{3}}$-free $4$-graph on $[n]$ with $\left(\frac{1}{4} n\right) \cdot \left(\frac{4n}{3}\right)$ edges, then $G \cong S^{4}(n)$.

Let $S_{n}$ be the $4$-graph on vertex set $[n]$ with all edges containing a fixed vertex, we call it a star. $S_{n}$ is denoted by $S$ while $n$ goes to infinity. We apply the Lagrangian method in the proof and show the following result that the maximum Lagrangian among all $M_{4}^{3}$-free $4$-graphs is uniquely achieved by $S$.

Theorem 1.6 Let $F$ be an $M_{4}^{3}$-free $4$-graph on $[n]$. If $F \not\subseteq S_{n}$ then $\lambda(F) < 0.0169 < \frac{9}{512}$. Otherwise, $\lambda(F) \leq \frac{9(n-2)(n-3)}{512(n-1)^{2}}$.

Corollary 1.7 $\pi_{4}(M_{4}^{3}) = 4! \lambda(S) = \frac{27}{64}$.
We sometimes drop the subscript $G$. If $L(i \setminus j) = L(j \setminus i)$ and \{i, j\} is not contained in any edge of $G$, then we say that $i$ and $j$ are equivalent and write $i \sim j$. We say $G$ on vertex set $[n]$ is left-compressed if for every $i, j$, $1 \leq i < j \leq n$, $L_G(j \setminus i) = \emptyset$. Given $i, j \in V(G)$, define

$$\pi_{ij}(G) = (E(G) \setminus \{(j) \cup F : F \in L_G(j \setminus i)\}) \bigcup \{(i) \cup F : F \in L_G(j \setminus i)\}.$$ 

By the definition of $\pi_{ij}(G)$, it’s easy to see the following fact.

**Fact 2.1** Let $G$ be an $r$-graph on vertex set $[n]$. Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weighing of $G$. If $x_i \geq x_j$, then $\lambda(\pi_{ij}(G), \bar{x}) \geq \lambda(G, \bar{x})$.

The following lemma plays an important role in the proof.

**Lemma 2.2** (see e.g. [3]) Let $G$ be an $M_r^*$-free $r$-graph on vertex set $[n]$. Then for every pair $i, j$ with $1 \leq i \neq j \leq n$, $\pi_{ij}(G)$ is $M_r^*$-free.

An $r$-graph $G$ is dense if and only if every proper subgraph $G'$ of $G$ satisfies $\lambda(G') < \lambda(G)$. This is equivalent to that all optimum weightings of $G$ are in the interior of $\Delta$, which means no coordinate in an optimum weighting is zero.

**Algorithm 2.3** (Dense and compressed [7])

**Input:** An $M_r^*$-free $r$-graph $G$ on $[n]$.

**Output:** A dense and left-compressed $M_r^*$-free $r$-graph with non-decreasing Lagrangian and an optimum weighting $\bar{x}$ satisfying $x_i \geq x_j$ if $i \leq j$.

**Step 1.** If $G$ is not dense, then replace $G$ by a dense subgraph with the same Lagrangian. Otherwise, go to Step 2.

**Step 2.** Let $\bar{x}$ be an optimum weighting of $G$. If $G$ is left-compressed, then terminate. Otherwise, relabel the vertices of $G$ such that $x_i \geq x_j$ for all $i < j$ if necessary. So there exist vertices $i, j$ such that $x_i \geq x_j$ and $L_G(j \setminus i) \neq \emptyset$, then replace $G$ by $\pi_{ij}(G)$ and go to step 1.

Note that the algorithm terminates after finite many steps since Step 1 reduces the number of vertices by at least 1 each time and Step 2 reduces the parameter $s(G) = \sum_{e \in G} \sum_{i \in e} i$ by at least 1 each time. Applying Fact 2.1, Lemma 2.2 and the fact that Algorithms 2.3 terminate after finite many steps, we get the following lemma.

**Lemma 2.4** Let $G$ be an $M_r^*$-free $r$-graph. Then there exists an $M_r^*$-free dense and left-compressed $r$-graph $G'$ with $|V(G')| \leq |V(G)|$ such that $\lambda(G') \geq \lambda(G)$.

We need the following result to estimate the Lagrangians of some hypergraphs.

**Theorem 2.5** ([16]) Let $m$ and $l$ be positive integers satisfying \((\binom{l-1}{3}) \leq m \leq \binom{l-1}{3} + \binom{l-2}{2}\). Let $G$ be a $3$-graph with $m$ edges and $G$ contains a clique of order $l - 1$. Then $\lambda(G) = \lambda([l - 1]^{(3)})$.

Let $G$ be an $r$-graph on $[n]$ and $\bar{x} = (x_1, x_2, \ldots, x_n)$ be a weighing of $G$. Fix $i \in [n]$, denote

$$L_G(x_i) = \frac{\partial \lambda(G, \bar{x})}{\partial x_i} = \sum_{e \in E(G) \setminus e \ni \{i\}} \prod_{j \in e \setminus \{i\}} x_j.$$
Fact 2.6 (5) Let $G$ be an $r$-graph on $[n]$. Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weighing for $G$ with
$k \leq n$ nonzero weights $x_1, x_2, \ldots, x_k$. Then, for every $\{i, j\} \in [k]^{(2)}$,
(1) $L_G(x_i) = r \lambda(G)$ ;
(2) The pair $i$ and $j$ is covered.

Fact 2.7 (5) Let $G = (V, E)$ be a dense $r$-graph. Then every pair of vertices $i, j \in V$ is covered.

Lemma 2.8 (6) Let $G$ be an $r$-graph on vertex set $[n]$. If the pair $\{i, j\} \subseteq [n]$ is not covered, then
$\lambda(G) = \max \{\lambda(G \setminus \{i\}), \lambda(G \setminus \{j\})\}$. Further more, if $L_G(i) \subseteq L_G(j)$ then $\lambda(G) = \lambda(G \setminus \{i\})$.

Given disjoint sets of vertices $V_1, \ldots, V_s$, denote $\Pi_{i=1}^s V_i = V_1 \times V_2 \times \ldots \times V_s = \{(x_1, x_2, \ldots, x_s) : \forall i = 1, \ldots, s, x_i \in V_i\}$. We will also use $\Pi_{i=1}^s V_i$ to denote the set of the corresponding unordered $s$-sets. Let $F$ be a hypergraph on $[m]$, a blowup of $F$ is a hypergraph $G$ whose vertex set can be partitioned into $V_1, \ldots, V_m$ such that $E(G) = \bigcup_{e \in F} \prod_{i \in e} V_i$.

Corollary 2.9 (5) Given an $r$-graph $F$. Let $G$ be a blowup of $F$, then $\lambda(G) = \lambda(F)$.

Lemma 2.10 (6) Let $G$ be an $r$-graph on vertex set $[n]$. If $L(i \setminus j) = L(j \setminus i)$, then there is an optimum weighting $\bar{x} = (x_1, x_2, \ldots, x_n)$ such that $x_i = x_j$.

Lemma 2.11 Let $G$ be a 3-graph obtained by removing two edges intersecting at two vertices from $K_5^3$. Then $\lambda(G) \leq 0.0673 < \frac{9}{128}$.

Proof: Without loss of generality, suppose that $G = K_5^3 \setminus \{245, 345\}$. Let $\bar{x} = (x_1, \ldots, x_5)$ be an optimum weighting of $G$. By Lemma 2.10 we can assume that $x_1 = a$, $x_2 = x_3 = \frac{b}{2}$ and $x_4 = x_5 = \frac{c}{2}$. So $a + b + c = 1$. Then

$$\lambda(G, \bar{x}) = \frac{ab^2}{4} + abc + \frac{ac^2}{4} + \frac{b^2c}{4}$$

$$= \frac{a}{4} (b + c)^2 + \frac{1}{2} abc + \frac{b^2c}{4}$$

$$= \frac{a}{4} (1 - a)^2 + \frac{bc}{4} (2a + b)$$

$$= \frac{a}{4} (1 - a)^2 + \frac{2.2b \times 3.2c \times (2a + b)}{4 \times 2.2 \times 3.2}$$

$$\leq \frac{a}{4} (1 - a)^2 + \frac{1}{4 \times 2.2 \times 3.2} \left( \frac{2a + 3.2b + 3.2c}{3} \right)^3$$

$$= \frac{a}{4} (1 - a)^2 + \frac{1}{4 \times 2.2 \times 3.2 \times 27} (3.2 - 1.2a)^3$$

$$= \frac{2943a^3 - 5724a^2 + 2394a + 512}{4 \times 27 \times 110}.$$ 

Let $g(a) = 2943a^3 - 5724a^2 + 2394a + 512$. Then $g'(a) = 0$ implies $a = \frac{212 - \sqrt{3268}}{327}$. So $\lambda(G, \bar{x}) \leq g\left(\frac{212 - \sqrt{3268}}{327}\right) < 0.0673.$
3 Lagrangians of intersecting 4-graphs

We first calculate $\lambda(S)$.

**Lemma 3.1** $\lambda(S_n) = \frac{9(n-2)(n-3)}{312(n-1)}$ for $n \geq 4$ and $\lambda(S) = \frac{9}{312}$.

**Proof:** Note that $S_n = \{1ijk : 2 \leq i < j < k \leq n\}$. Let $\bar{x} = (x_1, \ldots, x_n)$ be a feasible weighting of $S_n$, then

$$\lambda(S_n, \bar{x}) = x_1 \sum_{2 \leq i < j < k \leq n} x_i x_j x_k \leq x_1 \left(\frac{n-1}{3}\right) \left(\frac{1-x_1}{n-1}\right)^3 \leq \frac{9(n-2)(n-3)}{512(n-1)^2},$$

equality holds if and only if $x_1 = \frac{1}{2}$ and $x_2 = \cdots = x_n = \frac{3}{4(n-1)}$. So $\lambda(S) = \lim_{n \to \infty} \lambda(S_n, \bar{x}) = \frac{9}{312}$. ■

**Proof of Theorem 1.6** Let $\mathcal{F}$ be an $M_2^4$-free 4-graph on $[n]$. By Lemma 2.4, we may assume that $\mathcal{F}$ is left-compressed and dense. If $n \leq 7$, then $\mathcal{F} \subseteq K^4_7$. Therefore $\lambda(\mathcal{F}) \leq \lambda(K^4_7) = \frac{5}{16} < 0.0169$. Now assume that $n \geq 8$. Let $\bar{x} = (x_1, \ldots, x_n)$ be an optimum weighting of $\mathcal{F}$, satisfying $x_1 \geq x_2 \geq \cdots \geq x_n > 0$. The proof classifies such 4-graphs into several cases and verifies the required bound in each case.

Since $\mathcal{F}$ is left-compressed and dense, then every pair $i, j$ with $3 \leq i < j \leq n$ satisfies $12ij \in \mathcal{F}$. We claim that $\{1, 2\}$ is a vertex cover of $\mathcal{F}$ (i.e., every edge of $\mathcal{F}$ contains 1 or 2), otherwise $3456 \in \mathcal{F}$, then $\{3456, 1278\}$ forms a copy of $M_2^4$ in $\mathcal{F}$, a contradiction. Furthermore, we have $2468 \notin \mathcal{F}$, otherwise $\{2468, 1357\}$ forms a copy of $M_2^4$ in $\mathcal{F}$, a contradiction.

**Case 1.** $2567 \in \mathcal{F}$.

Since $\mathcal{F}$ is $M_2^4$-free, we have $1348 \notin \mathcal{F}$. Then

$$\mathcal{F} \subseteq \mathcal{F}_1 = \{12ij : 3 \leq i < j \leq n\} \cup \{ijkl : i \in [2], 3 \leq j < k < l \leq 7\}.$$

**Lemma 3.2** $\lambda(\mathcal{F}_1) \leq \frac{1}{108} < 0.0169$.

**Proof:** Let $\bar{x} = (x_1, \ldots, x_n)$ be an optimum weighting of $\mathcal{F}_1$. Note that $L_{\mathcal{F}_1}(8) = \{12i : 3 \leq i \leq n, i \neq 8\}$. By Fact 2.6 (1), we have $\lambda(\mathcal{F}_1) = \frac{1}{4}\lambda(L_{\mathcal{F}_1}(8), \bar{x}) \leq \frac{1}{4}\lambda(L_{\mathcal{F}_1}(8)) = \frac{1}{108}$. ■

**Case 2.** $2567 \notin \mathcal{F}$ and $2467 \in \mathcal{F}$.

In this case, we have $1358 \notin \mathcal{F}$.

$$\mathcal{F} \subseteq \mathcal{F}_2 = \{12ij, 134k, 13lm, 14lm, 1567, 234k, 23lm, 24lm : 3 \leq i < j \leq n, 5 \leq k \leq n, 5 \leq l < m \leq 7\}.$$

**Lemma 3.3** $\lambda(\mathcal{F}_2) \leq \frac{1}{64} < 0.0169$.

**Proof:** Let $\bar{x} = (x_1, \ldots, x_n)$ be an optimum weighting of $\mathcal{F}_2$. Note that $L_{\mathcal{F}_2}(8) = \{12i, 134, 234 : 3 \leq i \leq n, i \neq 8\}$. Since the pair of vertices $\{4, 5\}$ is not covered in $L_{\mathcal{F}_2}(8)$, we can assume that $L_{\mathcal{F}_2}(8)$ is on $[4]$ by Lemma 2.3. Moreover, $L_{\mathcal{F}_2}(8)[\{4\}] = \{123, 124, 134, 234\}$. By Fact 2.6 (1), we have $\lambda(\mathcal{F}_2) = \frac{1}{4}\lambda(L_{\mathcal{F}_2}(8), \bar{x}) \leq \frac{1}{4}\lambda(L_{\mathcal{F}_2}(8)) = \frac{1}{4}\lambda(K_4^2) = \frac{1}{64}$. ■

**Case 3.** $2467 \notin \mathcal{F}$ and $2368 \in \mathcal{F}$.

In this case, we have $1457 \notin \mathcal{F}$.

$$\mathcal{F} \subseteq \mathcal{F}_3 = \{12ij1, 13i2j2, 1456, 23i3j3, 2456 : 3 \leq i_1 < j_1 \leq n, 4 \leq i_2 < j_2 \leq n, 4 \leq i_3 < j_3 \leq n\}.$$
Lemma 3.4 \( \lambda(F_3) \leq \frac{1}{64} < 0.0169 \).

**Proof:** Let \( \vec{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( F_3 \). Note that
\[
L_{F_3}(8) = \{ 12i, 13j, 23k : 3 \leq i \leq n, 4 \leq j, k \leq n, i, j, k \neq 8 \}.
\]
Since the pair of vertices \( \{4, 5\} \) is not covered in \( L_{F_3}(8) \). By Fact 2.6 (1) and Lemma 2.8, we have
\[
\lambda(F_3) = \frac{1}{4} \lambda(L_{F_3}(8), \vec{x}) \leq \frac{1}{4} \lambda(L_{F_3}(8)) = \frac{1}{4} \lambda(K^3_4) = \frac{1}{64}.
\]

Case 4. \( 2467, 2368 \notin F \) and \( 2458 \in F \).
In this case, we have \( 1367 \notin F \). Then
\[
F \subseteq F_4 = \{ 12ij, 134k, 145l, 234k, 235l, 245l : 3 \leq i < j \leq n, 5 \leq k \leq n, 6 \leq l \leq n \}.
\]

Lemma 3.5 \( \lambda(F_4) \leq \frac{1}{243} < 0.0169 \).

**Proof:** Let \( \vec{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( F_4 \). By Lemma 2.10, we can assume that \( x_1 = x_2 = a, x_3 = x_4 = x_5 = b \) and \( x_6 + \cdots + x_n = c \). So \( 2a + 3b + c = 1 \). Note that
\[
L_{F_4}(1) = \{ 2ij, 34k, 35l, 45l : 3 \leq i < j \leq n, 5 \leq k \leq n, 6 \leq l \leq n \}.
\]
Then
\[
\lambda(L_{F_4}(1), \vec{x}) \leq a \frac{(1 - 2a)^2}{2} + b^3 + 3bc
\]
\[
= a \frac{(1 - 2a)^2}{2} + b^2(b + 3c)
\]
\[
= a \frac{(1 - 2a)^2}{2} + \frac{b^2}{16} \times 4b \times 4b \times (b + 3c)
\]
\[
\leq a \frac{(1 - 2a)^2}{2} + \frac{1}{16} (3b + c)^3.
\]
Since \( 3b + c = 1 - 2a \), we have
\[
\lambda(L_{F_4}(1), \vec{x}) \leq a \frac{(1 - 2a)^2}{2} + \frac{(1 - 2a)^3}{16}
\]
\[
= \frac{(1 - 2a)^2}{16} (1 + 6a)
\]
\[
= \frac{3}{2} \times \frac{(1 - 2a)^2}{16} \left( \frac{2}{3} + 4a \right)
\]
\[
\leq \frac{3}{2} \times \frac{1}{16} \times \left( \frac{8}{9} \right)^3
\]
\[
= \frac{16}{243}.
\]
Hence \( \lambda(F_4) = \frac{1}{4} \lambda(L_{F_4}(1), \vec{x}) \leq \frac{1}{243} \).

Case 5. \( 2467, 2368, 2458 \notin F \), \( 2367 \in F \).
In this case, we have \( 1458 \notin F \). Then
\[
F \subseteq F_5 = \{ 12ij, 136kl, 1456, 1457, 1567, 2345, 234m, 235m, 2367, 2456, 2457 : 3 \leq i < j \leq n, 4 \leq k < l \leq n, 6 \leq m \leq n \}.
\]
Lemma 3.6 \( \lambda(F_5) \leq \frac{1}{64} < 0.0169 \).

Proof: Let \( \vec{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( F_5 \). Note that

\[
L_{F_5}(8) = \{12i, 13j, 234, 235 : 3 \leq i \leq n, 4 \leq j \leq n, i, j \neq 8 \}.
\]

By Fact 2.6 (1) and Lemma 2.8 we have

\[
\lambda(F_5) = \frac{1}{4} \lambda(L_{F_5}(8), \vec{x}) \leq \frac{1}{4} \lambda(L_{F_5}(8)) = \frac{1}{4} \lambda(K_5^{(3)} \setminus \{245, 345\}) = \frac{1}{64}.
\]

Case 6. 2458, 2367 \( \notin F \) and 2457 \( \in F \).

In this case, we have 1368 \( \notin F \). Then

\[
F \subseteq F_6 = \{12ij, 134k, 135l, 1367, 145l, 1467, 1567, 234k, 235l, 2456, 2457 : 3 \leq i < j \leq n, 5 \leq k \leq n, 6 \leq l \leq n \}.
\]

Lemma 3.7 \( \lambda(F_6) < 0.0169 \).

Proof: Let \( \vec{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( F_6 \). Note that

\[
L_{F_6}(8) = \{12i, 134, 135, 145, 234, 235 : 3 \leq i \leq n, i \neq 8 \}.
\]

Since the pair of vertices \( \{5, 6\} \) is not covered in \( L_{F_6}(8) \), we have

\[
\lambda(F_6) = \frac{1}{4} \lambda(L_{F_6}(8), \vec{x}) \leq \frac{1}{4} \lambda(L_{F_6}(8)) = \frac{1}{4} \lambda(K_5^{(3)} \setminus \{245, 345\}) < 0.0169,
\]

according to Fact 2.6 (1), Lemma 2.8 and Lemma 2.11.

Case 7. 2457, 2367 \( \notin F \) and 2358 \( \in F \).

In this case, we have 1468 \( \notin F \). Then

\[
F \subseteq F_7 = \{12ij, 13kl, 145m, 2345, 234m, 235m, 2456 : 3 \leq i < j \leq n, 4 \leq k < l \leq n, 6 \leq m \leq n \}.
\]

Lemma 3.8 \( \lambda(F_7) < 0.0169 \).

Proof: Let \( \vec{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( F_7 \). Note that

\[
L_{F_7}(8) = \{12i, 13j, 145, 234, 235 : 3 \leq i \leq n, 4 \leq j \leq n, i, j \neq 8 \}.
\]

Since the pair of vertices \( \{5, 6\} \) is not covered in \( L_{F_7}(8) \), we have

\[
\lambda(F_7) = \frac{1}{4} \lambda(L_{F_7}(8), \vec{x}) \leq \frac{1}{4} \lambda(L_{F_7}(8)) = \frac{1}{4} \lambda(K_5^{(3)} \setminus \{245, 345\}) < 0.0169,
\]

according to Fact 2.6 (1), Lemma 2.8 and Lemma 2.11.

Case 8. 2457, 2367, 2358 \( \notin F \), and 2357 \( \in F \).

In this case, we have 1468 \( \notin F \). Then

\[
F \subseteq F_8 = \{12ij, 13kl, 145m, 1467, 234m, 2345, 2356, 2357, 2456 : 3 \leq i < j \leq n, 4 \leq k < l \leq n, 6 \leq m \leq n \}.
\]
Lemma 3.9 \( \lambda(F) \leq \frac{1}{16} < 0.0169 \).

Proof: Note that \( L_{F_8}(8) = \{12i, 13j, 14k, 234 : 3 \leq i \leq n, 4 \leq j \leq n, i, j \neq 8 \} \). Since the pair of vertices \{5, 6\} is not covered by any edge in \( L_{F_8}(8) \), we can assume that \( L_{F_8}(8) \) is on \([5]\) by Lemma 2.8. Moreover, \( L_{F_8}([5]) = \{123, 124, 125, 134, 135, 145, 234\} \), hence we have \( \lambda(L_{F_8}(8)) = \lambda(K_4^4) = \frac{1}{16} \) by Theorem 2.15. So \( \lambda(F) \leq \frac{1}{16} \lambda(L_{F_8}(8)) = \frac{1}{16} \).

Case 9. \( 2357 \notin F \) and \( 2348 \in F \).

In this case, we have \( 1567 \notin F \). Then

\[
F \subseteq F_9 = \{12i, 13kl, 14st, 234p, 2356, 2456 : 3 \leq i < j \leq n, 4 \leq k < l \leq n, 5 \leq s < t \leq n, 5 \leq p \leq n \}.
\]

Lemma 3.10 \( \lambda(F_9) \leq \frac{3}{16} < 0.0169 \).

Proof: Note that \( L_{F_9}(8) = \{12i, 13j, 14k, 234 : 3 \leq i \leq n, 4 \leq j \leq n, 5 \leq k \leq n, i, j, k \neq 8 \} \). Since the pair of vertices \{5, 6\} is not covered in \( L_{F_9}(8) \), we can assume that \( L_{F_9}(8) \) is on \([5]\) by Lemma 2.8. Moreover, \( L_{F_9}([5]) = \{123, 124, 125, 134, 135, 145, 234\} \), hence we have \( \lambda(L_{F_9}(8)) = \lambda(K_4^4) = \frac{1}{16} \) by Theorem 2.15. So \( \lambda(F_9) \leq \frac{1}{16} \lambda(L_{F_9}(8)) = \frac{3}{16} \).

Case 10. \( 2357, 2348 \notin F \) and \( 2456 \in F \).

In this case, we have \( 1378 \notin F \). Then

\[
F \subseteq F_{10} = \{12i, 134k, 135l, 136m, 145l, 146m, 156m, 2345, 2346, 2347, 2356, 2456 : 3 \leq i < j \leq n, 5 \leq k \leq n, 6 \leq l \leq n, 7 \leq m \leq n \}.
\]

Lemma 3.11 \( \lambda(F_{10}) \leq \frac{2}{11} < 0.0169 \).

Proof: Let \( \bar{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( F_{10} \). Note that

\[
L_{F_{10}}(8) = \{12i, 134, 135, 136, 145, 146, 156 : 3 \leq i \leq n, i \neq 8 \}.
\]

Since the pair of vertices \{6, 7\} is not covered in \( L_{F_{10}}(8) \), then by Lemma 2.8 \( \lambda(L_{F_{10}}(8)) = \lambda(\{1ij : 2 \leq i \leq j \leq 6\}) \). Let \( G = \{1ij : 2 \leq i \leq j \leq 6\} \) and \( \bar{x} = (x_1, \ldots, x_6) \) be an optimum weighting of \( G \). By Lemma 2.10 we can assume that \( x_1 = x, x_2 = \cdots = x_6 = \frac{1}{2} \). So

\[
\lambda(L_{F_{10}}(8)) = \lambda(G) = x \left( \frac{5}{2} \right) \left( \frac{1 - x}{5} \right)^2 = \frac{1}{5} \cdot 2x \cdot (1 - x)^2 \leq \frac{1}{5} \left( \frac{2}{3} \right)^3 = \frac{8}{135}.
\]

So \( \lambda(F_{10}) \leq \frac{1}{16} \lambda(L_{F_{10}}(8)) \leq \frac{2}{11} \).

Case 11. \( 2357, 2348, 2456 \notin F \) and \( 2346 \in F \).

In this case, we have \( 1478 \notin F \). Then

\[
F \subseteq F_{11} = \{12i_1, j_1, 13i_2j_2, 145k, 146l, 156m, 2345, 2346, 2347, 2356 : 3 \leq i_1 < j_1 \leq n, 4 \leq i_2 < j_2 \leq n, 6 \leq k \leq n, 7 \leq l \leq n \}.
\]
Lemma 3.12 \( \lambda(\mathcal{F}_{11}) \leq \frac{2}{135} < 0.0169 \).

**Proof:** Note that \( L_{\mathcal{F}_{11}}(8) = \{12i, 13j, 14k, 15l : 3 \leq i \leq n, 4 \leq j \leq n, 5 \leq k \leq n, i, j, k \neq 8 \} \). Since the pair of vertices \( \{6, 7\} \) is not covered in \( L_{\mathcal{F}_{11}}(8) \), then by Lemma 2.8, \( \lambda(L_{\mathcal{F}_{11}}(8)) = \lambda(\{1ij : 2 \leq i < j \leq 6\}) \). So

So \( \lambda(\mathcal{F}_{11}) \leq \frac{1}{4} \lambda(L_{\mathcal{F}_{11}}(8)) \leq \frac{2}{135} \) from (1).

Case 12. \( 2348, 2356 \notin \mathcal{F} \) and \( 2347 \in \mathcal{F} \).

In this case, we have \( 1568 \notin \mathcal{F} \). Then

\[
\mathcal{F} \subseteq \mathcal{F}_{12} = \{12i_1j_1, 13i_2j_2, 14i_3j_3, 1567, 2345, 2346, 2347 : 3 \leq i_1 < j_1 \leq n, 4 \leq i_2 < j_2 \leq n, 5 \leq i_3 < j_3 \leq n \}.
\]

Lemma 3.13 \( \lambda(\mathcal{F}_{12}) \leq \frac{1}{72} < 0.0169 \).

**Proof:** Let \( \bar{x} = (x_1, \ldots, x_n) \) be an optimum weighting of \( \mathcal{F}_{12} \). Note that \( L_{\mathcal{F}_{12}}(8) = \{12i, 13j, 14k, 15l : 3 \leq i \leq n, 4 \leq j \leq n, 5 \leq k \leq n, i, j, k \neq 8 \} \). Since the pair of vertices \( \{5, 6\} \) is not covered in \( L_{\mathcal{F}_{12}}(8) \), then by Lemma 2.8, \( \lambda(L_{\mathcal{F}_{12}}(8)) = \lambda(\{1ij : 2 \leq i < j \leq 5\}) \). Similar to (1), \( \lambda(L_{\mathcal{F}_{12}}(8)) \leq \frac{1}{135} \). Then we have \( \lambda(\mathcal{F}_{12}) \leq \frac{1}{72} \).

Case 13. \( 2356, 2347 \notin \mathcal{F} \) and \( 2346 \in \mathcal{F} \).

In this case, we have \( 1578 \notin \mathcal{F} \). Then

\[
\mathcal{F} \subseteq \mathcal{F}_{13} = \{12i_1j_1, 13i_2j_2, 14i_3j_3, 156k, 2345, 2346 : 3 \leq i_1 < j_1 \leq n, 4 \leq i_2 < j_2 \leq n, 5 \leq i_3 < j_3 \leq n, 7 \leq k \leq n \}.
\]

Lemma 3.14 \( \lambda(\mathcal{F}_{13}) \leq \frac{2}{135} < 0.0169 \).

**Proof:** Note that \( L_{\mathcal{F}_{13}}(8) = \{12i, 13j, 14k, 15l : 3 \leq i \leq n, 4 \leq j \leq n, 5 \leq k \leq n, i, j, k \neq 8 \} \). Since the pair of vertices \( \{6, 7\} \) is not covered in \( L_{\mathcal{F}_{13}}(8) \), then by Lemma 2.8, \( \lambda(L_{\mathcal{F}_{13}}(8)) = \lambda(\{1ij : 2 \leq i < j \leq 6\}) \). So \( \lambda(\mathcal{F}_{13}) \leq \frac{1}{4} \lambda(L_{\mathcal{F}_{13}}(8)) \leq \frac{2}{135} \) from (1).

Case 14. \( 2346 \notin \mathcal{F} \) and \( 2345 \in \mathcal{F} \).

In this case, we have \( 1678 \notin \mathcal{F} \). Then

\[
\mathcal{F} \subseteq \mathcal{F}_{14} = \{12i_1j_1, 13i_2j_2, 14i_3j_3, 15i_4j_4, 2345 : 3 \leq i_1 < j_1 \leq n, 4 \leq i_2 < j_2 \leq n, 5 \leq i_3 < j_3 \leq n, 6 \leq i_4 < j_4 \leq n \}.
\]

Lemma 3.15 \( \lambda(\mathcal{F}_{14}) \leq \frac{2}{135} < 0.0169 \).

**Proof:** Note that \( L_{\mathcal{F}_{14}}(8) = \{12i, 13j, 14k, 15l : 3 \leq i \leq n, 4 \leq j \leq n, 5 \leq k \leq n, 6 \leq l \leq n, i, j, k, l \neq 8 \} \). Since the pair of vertices \( \{6, 7\} \) is not covered in \( L_{\mathcal{F}_{14}}(8) \), then by Lemma 2.8, \( \lambda(L_{\mathcal{F}_{14}}(8)) = \lambda(\{1ij : 2 \leq i < j \leq 6\}) \). So \( \lambda(\mathcal{F}_{14}) \leq \frac{1}{4} \lambda(L_{\mathcal{F}_{14}}(8)) \leq \frac{2}{135} \) from (1).

Case 15. \( 2345 \notin \mathcal{F} \).

In this case, \( \mathcal{F} \subseteq \mathcal{F}_{15} = \{1ijk : 2 \leq i < j < k \leq n \} = \mathcal{S}_n \). By Lemma 3.1, we have \( \lambda(\mathcal{F}_{15}) = \frac{9}{512} \cdot \frac{(n-2)(n-3)}{2} \leq \frac{9}{512} \).

If \( \mathcal{F} \nsubseteq \mathcal{S}_n \), then by the argument in Cases 1-14, \( \mathcal{F} \) is a subgraph of one of the \( K_7^2 \) and \( \mathcal{F}_i, i \in [14] \). So \( \lambda(\mathcal{F}) < 0.0169 \). Hence we complete the proof. ■
4 An application to a hypergraph Turán problem

Given \( r \)-graphs \( F \) and \( G \), a function \( f : V(F) \rightarrow V(G) \) is a homomorphism if it preserves edges, i.e., \( f(i_1)f(i_2) \ldots f(i_r) \in G \) if \( i_1, i_2, \ldots, i_r \in F \). We say \( G \) is \( F \)-hom-free if there is no homomorphism from \( F \) to \( G \). We need the following connection between Turán densities and Lagrangians due to Sidorenko.

**Lemma 4.1** (see e.g. [9], Section 3) Given an \( r \)-graph \( F \), \( \pi(F) \) is the supremum of \( r!\lambda(G) \) over all dense \( F \)-hom-free \( r \)-graphs \( G \).

Write \( H_8^{M_2} \) as \( K_{4,4}^4 \). From Theorem 1.6 and Lemma 4.1 we get that

**Theorem 4.2** \( \pi(K_{4,4}^4) = \frac{27}{512} \).

**Proof:** Since \( \mathcal{S} \) is \( K_{4,4}^4 \)-free, we get the lower bound. For the upper bound, by Lemma 4.1 it suffices to show that \( \lambda(G) \leq \frac{9}{512} \) for any dense \( K_{4,4}^4 \)-free 4-graph \( G \). For every dense \( K_{4,4}^4 \)-hom-free 4-graph \( G \), we claim that \( G \) is \( M_2^4 \)-free. Otherwise suppose there are two disjoint edges \( e, f \in G \). Since \( G \) cover pairs, then there is an edge \( e_{ab} \in G \) with \( \{a, b\} \subseteq e_{ab} \) for every \( a \in e \) and \( b \in f \). Thus \( \{e, f\} \cup \{e_{ab} : a \in e, b \in f\} \) forms a copy of \( K_{4,4}^4 \) which contradicts \( G \) being \( K_{4,4}^4 \)-hom-free. Then \( \lambda(G) \leq \frac{9}{512} \) by Theorem 1.6. \( \blacksquare \)

**Corollary 4.3** \( ex(n, K_{4,4}^4) = \frac{9}{512}n^4 \) for sufficiently large \( n \).

4.1 Stability

In order to prove Theorem 1.6, we will first prove the following stability result.

**Theorem 4.4** For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) and an integer \( n_0 \) such that if \( F \) is a \( K_{4,4}^4 \)-free 4-graph with \( n \geq n_0 \) vertices and at least \( \left( \frac{9}{512} - \delta \right)n^4 \) edges, then there exists a partition \( \mathcal{V}(F) = A \cup B \) of the vertex set of \( F \) such that \( |\{e \in F : |e \cap A| \geq 2\}| + |\{e \in F : e \subseteq B\}| < \varepsilon n^4 \).

We first show that it suffices to prove the result under the assumption that \( F \) is \( K_{4,4}^4 \)-hom-free.

**Remark 4.5** Given an \( r \)-graph \( F \) and \( p \geq |V(F)| \), \( G \) is \( H_p^F \)-hom-free if and only if \( G \) is \( K_{p}^F \)-free.

**Proof:** We first prove that if \( G \) is \( H_p^F \)-hom-free then \( G \) is \( K_{p}^F \)-free. Otherwise suppose that \( G \) contains a member of \( K_{p}^F \), say \( K \). By the definition of \( K_{p}^F \), \( K \) contains a core \( C \) of size \( p \) such that \( K[C] \) contains \( F \) as a subgraph. We now construct a map \( f \) from \( V(H_p^F) \) to \( V(K) \). Map the core of \( H_p^F \) to \( C \), that is, the core of \( K \), such that \( F \) in \( H_p^F \) is mapped to a copy of \( F \) in \( K \). For other vertices in \( V(H_p^F) \), e.g., \( B_{ij} \) such that \( \{i, j\} \cup B_{ij} \in H_p^F \), where \( i, j \in \mathcal{C} \), \( B_{ij} \) maps \( B_{ij} \) to \( B'_{ij} \), where \( \{f(i), f(j)\} \cup B'_{ij} \in K \) (since each pair in \( C \) is covered by an edge in \( K \), \( \{f(i), f(j)\} \) is contained in some edge of \( K \)). So \( f \) is a homomorphism from \( H_p^F \) to \( K \), a contradiction.

Now we prove that if \( G \) is \( K_{p}^F \)-free then \( G \) is \( H_p^F \)-hom-free. Otherwise suppose that \( G \) is not \( H_p^F \)-hom-free, that is, there is a homomorphism \( g \) from \( H_p^F \) to \( G \). Denote the core of \( H_p^F \) as \( C' \) and \( \mathcal{C}' = \{g(v) : v \in C'\} \). We first show that \( |\mathcal{C}'| = |C| = p \). Otherwise suppose that \( |\mathcal{C}'| < p \), then \( \exists u, v \in C \) such that \( g(u) = g(v) \), which contradicts that \( g \) is a homomorphism, since \( u, v \) is contained in some edges of \( H_p^F \). This implies that \( F \subseteq G(C') \). For every pair \( x, y \in C \) that is not covered by \( F \), fix one \((r - 2)\)-set as \( B_{xy} \) such that \( \{x, y\} \cup B_{xy} \in H_p^F \). Then \( g(w) : w \in \{x, y\} \cup B_{xy} \) is an edge of \( G \) for
all pairs \( x, y \in C \) that is not covered by \( F \), since \( g \) is a homomorphism. Hence \( \{ (g(v) : v \in e) : e \in H_p^F \} \) is a member of \( K_p^F \), a contradiction.

In Section 7 of \( [2] \), Brandt-Irwin-Jiang proved that every \( H_p^F \)-free graph on \( [n] \) can be made \( K_p^F \)-free by removing \( \Theta(n^{r-1}) \) edges. Let \( F = M_2^p \) and \( p = 8 \), then it suffices to prove Theorem 4.4 under the assumption that \( F \) is \( K_4^4 \)-hom-free.

Part of our proof follows the approach in [17, 6] by Pikhurko and Hefetz-Keevash. We gradually adjust \( F \) by iterating a process which is called \textit{Symmetrization}. This process consists of two parts: \textit{Cleaning} is to delete vertices with ‘small’ degree, and \textit{Merging} is to replace the link of a vertex \( v \) by the link of a vertex \( u \) if \( d(v) \leq d(u) \) and the pair \( u, v \) is not covered by an edge. It terminates if we can no longer clean any vertex. We show the terminating-4-graph is isomorphic to \( S^4(n') \) with \( n' = (1 - o(1))n \). Then we trace back and show that the symmetrization process is ‘stable’ (does not change the 4-graph much).

Now let us give the proof precisely. Clearly, we can also assume that \( \varepsilon \) is sufficiently small and \( \delta \ll \varepsilon \). Let \( \alpha, \beta, \gamma \) and \( \delta \) be real numbers satisfying

\[
1 \gg \gamma \gg \beta \gg \alpha \gg \varepsilon \gg \delta \gg n_0^{-1}.
\]

We operate the symmetrization process for a pointed 4-graph: by this we mean a triple \((G, P, U)\), where \( G = (V, E) \) is a 4-graph, \( P = \{ \mathcal{P}_u : u \in U \} \) is a partition of \( V \), i.e., for any \( v \in V \) there exists some \( u \in U \) such that \( v \in \mathcal{P}_u \) and we give an order for vertices in \( \mathcal{P}_u \) for every \( u \in U \), and \( U \subseteq V \) is a transversal of \( P \) and every \( u \in U \) is the representative of \( \mathcal{P}_u \). Let us describe the process precisely.

### Cleaning:

**Input:** A pointed 4-graph \((G, P, U)\) on \( n \) vertices.

**Output:** A pointed 4-graph \((G', P', U')\) on \( n' \leq n \) vertices.

**Process:** If \( \delta(G) \geq (9/128 - \alpha)n^3 \) or \( V(G) = \emptyset \) then stop and return \((G', P', U') = (G, P, U)\), where \( n' = |V(G')| \). Otherwise, let \( u \in U \) be an arbitrary vertex such that \( d_G(u) < (9/128 - \alpha)n^3 \). If \( \mathcal{P}_u = \{ u \} \) then apply cleaning to \((G - \{ u \}, P - \{ \mathcal{P}_u \}, U - \{ u \})\). Otherwise let \( v \in \mathcal{P}_u \) be the vertex with the maximum order and apply cleaning to \((G - \{ v \}, P - \{ \mathcal{P}_u \}) \cup \{ \mathcal{P}_u - \{ v \} \}, U\), where \( n' = |V(G')| < n \).

Note that this algorithm will always terminate and \( \delta(G') \geq (9/128 - \alpha)n^3 \) or \( G' \) is empty. A vertex set \( U \) is covered by \( G \) if for every pair \( u, v \in U \), \( \{ u, v \} \) is contained in an edge of \( G \).

### Merging:

**Input:** A pointed 4-graph \((G, P, U)\) on \( n \) vertices.

**Output:** A pointed 4-graph \((G', P', U')\) on the same vertex set of \( G \).

**Process:** If \( U \) is covered by \( G \) then stop and return \((G', P', U') = (G, P, U)\). Otherwise, let \( u, v \in U \) be arbitrary vertices such that \( \{ u, v \} \) is not covered. Assume that \( d_G(v) \leq d_G(u) \). Merge \( P_v \) into \( \mathcal{P}_u \) (clone), that is, let \( \mathcal{P}_u' = \mathcal{P}_u \cup P_v \) and \( P_w' = \mathcal{P}_w \) for every \( w \in U \setminus \{ u, v \} \). Moreover, let \( U' = U \setminus \{ v \} \) and let \( G' \) be a blowup of \( G'[U'] \) with partition sets \( \{ \mathcal{P}_w' : w \in U' \} \), i.e., \( E(G') = \bigcup_{e \in E(G'[U'])} \prod_{w \in e} \mathcal{P}_w' \). Suppose the order of \( P_u \) is \( u_1 \prec u_2 \prec \cdots \prec u_k \) and the order of \( P_v \) is \( v_1 \prec v_2 \prec \cdots \prec v_t \), then we give \( P_w' \) an order as \( u_1 \prec \cdots \prec u_k \prec v_1 \prec \cdots \prec v_t \). Let \( P' = \{ P_w' : w \in U' \} \) and return \((G', P', U')\).

Clearly, \( P' = \{ P_u' : u \in U' \} \) is a partition of \( V(G') \) satisfying for any \( v \in V(G') \) there exists some \( u \in U' \) such that \( v \in \mathcal{P}_u' \), and \( U' \subseteq V(G') \) is a transversal of \( P' \). Note that the merging processes at most one step and for every \( u \in U \), \( \forall v, w \in P_u \), we have \( v \sim w \). Now we are ready to describe the symmetrization process.
Symmetrization:

Input: A $K^4_{4,4}$-free 4-graph $F = (V, E)$.
Output: A pointed 4-graph $(F^*, P, U)$.

Initiation: Let $H_0 = F = (V, E)$, $P_0 = \{P_{0,v}: v \in U_0\}$, $U_0 = V$, where $P_{0,v} = \{v\}$ for every $v \in V$. And the order of every part $P_{0,v}$ which has only one single vertex is trivial. Set $i = 0$.

Iteration: Apply Cleaning to $(H_i, P_i, U_i)$ and let $(H_{i+1}', P_{i+1}', U_{i+1}')$ be the output, where $H_{i+1}' = (V_{i+1}', E_{i+1}')$, and $P_{i+1}' = \{P_{i+1,v}' : w \in U_{i+1}'\}$. Apply Merging to $(H_{i+1}' = (V_{i+1}', E_{i+1}'), P_{i+1}' = \{P_{i+1,v}' : w \in U_{i+1}'\}$, $U_{i+1}'$) and let $(H_{i+1} = (V_{i+1}, E_{i+1}), P_{i+1} = \{P_{i+1,v} : w \in U_{i+1}\}$) be the output. If $(H_{i+1}, P_{i+1}, U_{i+1}) = (H_{i+1}', P_{i+1}', U_{i+1}')$, then stop and return $(F^*, P, U) = (H_i, P_i, U_i)$. Otherwise, increase $i$ by one and repeat Cleaning and Merging.

Let $(F^*, P, U)$ be the output of applying Symmetrization to $F$. Let

$$H_0 = F, H_1, H_2, \ldots, H_t, H_t = F^*$$

be the sequence of 4-graphs produced during this process, where $H_i' = (V_i', E_i')$ and $H_i = (V_i, E_i)$ for every $1 \leq i \leq t$.

We split the proof into two stages. In the first stage we show that $F^*$ is contained in a large blowup of $S_k$, where $k \approx 3n/4$. In the second stage we show that $F[V_t]$ is a subgraph of a blowup of $S_k$ (Just clone the vertex that all edges of $S_k$ intersect for about $n/4$ copies). Then Theorem 4.4 will follow easily from this.

We start with the first stage, which we prove by a series of lemmas.

Lemma 4.6 The following properties hold for every $0 \leq i \leq t$.

1. For every $0 \leq j \leq i$ the set $U_j \cap V_i$ is a transversal for the partition $\{P_{j,v} \cap V_i : v \in U_j \cap V_i\}$.
2. $H_i[|U_i|] = F[U_i]$.
3. $|e \cap P_{j,v}| \leq 1$ for every $e \in E_i$ and every $v \in V_i$.
4. For every $u \in U_i$, $\forall v, w \in P_{j,u}$, $v \sim w$.
5. For every $i \geq 1$, $U_i \subseteq U_{i-1}$ and $V_i \subseteq V_{i-1}$.

Proof: By the process of symmetrization algorithm, properties (2)-(5) hold obviously. Now we prove property (1). It is sufficient to show that $\{P_{j,v} \cap V_i : v \in U_j \cap V_i\}$ is a partition of $V_i$. (i) For any distinct vertices $u, v \in U_j \cap V_i$, $(P_{j,u} \cap V_i) \cap (P_{j,v} \cap V_i) \subseteq P_{j,u} \cap P_{j,v} = \emptyset$. (ii) Since $V_i \subseteq V_{j}$ by property (5) and $V_j = \bigcup_{u \in U_j} P_{j,u}$ (a partition), then $\forall v \in V_i$, $\exists u \in U_j$ such that $v \in P_{j,u} \cap V_i$. On the other hand, $\bigcup_{u \in U_j \cap V_i} (P_{j,u} \cap V_i) = (\bigcup_{u \in U_j \cap V_i} P_{j,u}) \cap V_i \subseteq V_j \cap V_i = V_i$. So $\bigcup_{u \in U_j \cap V_i} (P_{j,u} \cap V_i) = V_i$.

Lemma 4.7 Let $1 \leq i \leq t$ and suppose that $P'_{i,u}$ was merged into $P'_{i,v}$ during the $i$th merging step. Then $P'_{i,u} \cap V(F^*) = \emptyset$ implies $P'_{i,v} \cap V(F^*) = \emptyset$.

Proof: Suppose $P'_{i,u} \cap V(F^*) = \emptyset$, this implies that $u$ has been cleaned in some $j$th merging step, where $i < j \leq t$. Since $u$ is the last vertex among $P'_{i,v} \cup P'_{i,u}$ that be cleaned, so all vertices in $P'_{i,v}$ have been cleaned. Therefore, $P'_{i,v} \cap V(F^*) = \emptyset$.

Since merging preserves the property of $K^4_{4,4}$-hom-freeness and deleting vertices certainly also does, we get the following lemma.

Lemma 4.8 $H_i$ is $K^4_{4,4}$-hom-free for every $0 \leq i \leq t$. 

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The next Lemma asserts that merging does not decrease the number of edges.

**Lemma 4.9** \( e(H_i) \geq e(H'_i) \) holds for every \( 1 \leq i \leq t \).

**Lemma 4.10** \( \mathcal{F}[U_i] \) is \( M_2^4 \)-free.

**Proof:** Assume for the sake of contradiction that there exist two disjoint edges \( e, f \) in \( \mathcal{F}[U_i] \). Note that \( \mathcal{F}[U_i] = \mathcal{F}^*[U_i] \) and \( \mathcal{F}[U_i] \) cover pairs, then \( \forall a \in e \) and \( \forall b \in f \), \( \exists e_{ab} \in \mathcal{F}[U_i] \) such that \( a, b \in e_{ab} \).

Hence \( \{e, f\} \cup \{e_{ab} : a \in e, b \in f\} \subseteq \mathcal{F}[U_i] \subseteq \mathcal{F} \), which contradicts that \( \mathcal{F} \) is \( K_{4,4}^4 \)-hom-free.

The following proposition follows immediately from the definition and is implicit in many papers (see [9] for instance). We include a short proof of it for completeness.

**Proposition 4.11** Let \( F \) be an \( r \)-graph. Let \( L \) be an \( \mathcal{F} \)-free \( r \)-graph. Let \( H \) be a blow-up of \( L \) with \( n \) vertices. Then \( |H| \leq \frac{\pi_4(F)}{r!} n^r \). In particular, \( |\mathcal{F}^*| \leq \frac{9}{512} n^4 \).

**Proof:** Let \( \tilde{x} = (x_1, x_2, \ldots, x_n) \) be a weighting of \( H \), where \( x_i = \frac{1}{n} \) for all \( i \in [n] \). Then

\[
\frac{|H|}{n^r} = \lambda(H, \tilde{x}) \leq \lambda(H) = \lambda(L) \leq \frac{\pi_4(F)}{r!},
\]

where \( \lambda(H) = \lambda(L) \) is from Corollary [2,3]. Therefore \( |H| \leq \frac{\pi_4(F)}{r!} n^r \). Since \( \mathcal{F}^* \) is a blowup of \( \mathcal{F}^*[U_i] \) and \( \mathcal{F}^*[U_i] \) is an \( M_2^4 \)-free 4-graph by Lemma [1,10], then \( |\mathcal{F}^*| \leq \frac{\pi_4(M_2^4)}{4!} n^4 = \frac{9}{512} n^4 \).

Let \( C_i \) be the vertex set deleted by the \( i \)-th cleaning, where \( 0 \leq i \leq t - 1 \), i.e. \( C_i = V_i \setminus V_{i+1} \). Let \( C = \bigcup_{i=0}^{t-1} C_i \), so \( C = V \setminus V_t \) is the set of vertices removed by the symmetrization algorithm. The next lemma asserts that the symmetrization process does not delete too many vertices.

**Lemma 4.12** \( |V(\mathcal{F}^*)| \geq (1 - \alpha)n \).

**Proof:** Write \( s = |C| \). By Lemma [4,9] and the definition of cleaning we have

\[
|\mathcal{F}^*| \geq |\mathcal{F}| - \sum_{i=0}^{s-1} (9/128 - \alpha) (n - i)^3
\]

\[
\geq \left(9/512 - \delta\right) n^4 - \left((9/512 - \alpha/4) (n^4 - (n - s)^4) + \delta n^4\right)
\]

since

\[
\sum_{i=0}^{s-1} (n - i)^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n-s} i^3 = \frac{1}{4} n^2 (n + 1)^2 - \frac{1}{4} (n - s)^2 (n - s + 1)^2 \geq \frac{1}{4} n^4 - \frac{1}{4} (n - s)^4 - \delta n^4.
\]

On the other hand, since \( \mathcal{F}^* \) is \( K_{4,4}^4 \)-hom-free, then by Proposition [4,11] we have

\[
|\mathcal{F}^*| \leq 9/512 (n - s)^4.
\]

This yields

\[
\frac{\alpha}{4} (n - s)^4 \geq \left(\frac{\alpha}{4} - 2\delta\right) n^4.
\]

Hence

\[
\left(\frac{n - s}{n}\right)^4 \geq \frac{\alpha/4 - 2\delta}{\alpha/4} > 1 - \alpha
\]

for \( \delta \ll \alpha \). Hence

\[
\frac{n - s}{n} > (1 - \alpha)^{1/4} \geq 1 - \alpha.
\]

So \( s \leq \alpha n \) and \( |V(\mathcal{F}^*)| \geq (1 - \alpha)n \).
Lemma 4.13 \( F^*[U_i] \subseteq S_{[U_i]} \). Let 1 be the vertex in \( U_i \) that intersects all edges of \( F^*[U_i] \), then \( |P_{i,1}| = (\frac{\gamma}{4} + \beta)|V(F^*)| \).

Proof: Suppose \( U_i = [m] \). Let \( \bar{x} = \{x_1, \ldots, x_m\} \) be a weighting of \( F^*[U_i] \) such that \( x_i = |P_{i,1}| / |V_i| \) for every \( i \in [m] \). So \( x_i \geq 0 \) and \( \sum_{i=1}^{m} x_i = 1 \). Then

\[
\lambda(F^*[U_i], \bar{x}) = \sum_{e \in F^*[U_i]} \prod_{i \in e} |P_{i,1}| / |V_i| = \frac{1}{|V_i|^4} e(F^*).
\]

Since \( d_{F^*}(x) \geq (\frac{9}{128} - \alpha)|V_i|^3 \) for every \( x \in V(F^*) \), so \( e(F^*) \geq (\frac{9}{128} - \frac{9}{4})|V_i|^4 \). Then \( \lambda(F^*[U_i], \bar{x}) = \frac{1}{|V_i|^4} e(F^*) \geq \frac{9}{512} - \frac{9}{4} \). Since \( \alpha \) is small enough, then by Theorem 1.4 we have \( F^*[U_i] \subseteq S_{[U_i]} \). This proves the first part.

For the second part, suppose that \( |P_{i,1}| \neq (\frac{9}{128} + \beta)|V(F^*)| \). Denote \( A = P_{i,1} \) and \( B = V(F^*) \setminus A \).

Case 1. \( |A| > (\frac{9}{128} + \beta)|V(F^*)| \). Then \( |B| < (\frac{9}{128} - \beta)|V(F^*)| \). Then for any \( v \in A \), \( d_{F^*}(v) \leq ((\frac{9}{128} - \beta)(V(F^*))^3) \leq (\frac{9}{128} - \frac{9}{4})|V(F^*)|^3 \), which contradicts that \( \delta(F^*) \geq (\frac{9}{128} - \delta)|V(F^*)|^3 \).

Case 2. \( |A| < (\frac{9}{128} - \beta)|V(F^*)| \). Suppose that \( |A| = (\frac{9}{128} - \mu)|V(F^*)| \), so \( |B| = (\frac{9}{128} + \mu)|V(F^*)| \), where \( \mu > \beta \). Then for any \( v \in B \), \( d_{F^*}(v) \leq ((\frac{9}{128} + \mu)(V(F^*))^3) \leq (\frac{9}{128} - \frac{9}{4})|V(F^*)|^3 \), a contradiction.

This completes the first stage of the proof. The second stage is to show that \( F[V_i] \) is a subgraph of a blowup of \( S_{[U_i]} \), that is, \( F[V_i] \subseteq A' \times (B')^3 \) for some \( \{A', B\} \) which is a partition of \( V_i \) satisfying \( |A'| \approx |B'|/3 \). To do so, we will trace back the Merging steps performed during symmetrization.

Recall that 1 is the vertex in \( U_i \) that intersects all edges of \( F^*[U_i] \). Let \( A \) be the set \( P_{i,1} \) and \( B = V(F_i) \setminus A \). By Lemma 4.13 we have \( V_i = A \cup B \) with \( |A| \approx \frac{1}{3}|B| \). For every \( 0 \leq i \leq t \) we will find a partition \( Q_i = \{Q_{i,j} : j \in [2]\} \) of \( V_i \) which satisfies the following properties:

(P1) \( 1 \in Q_{i,1} \);
(P2) For every \( v \in U_i \), \( P_{i,v} \cap V_i \subseteq Q_{i,1} \) or \( P_{i,v} \cap V_i \subseteq Q_{i,2} \);
(P3) For every \( e \in \mathcal{H}_i[V_i] \) we have \( |e \cap Q_{i,1}| = 1 \) (so \( |e \cap Q_{i,2}| = 3 \)), that is, \( \mathcal{H}_i[V_i] \subseteq Q_{i,1} \times (Q_{i,2})^3 \).

Set \( Q_{i,1} = A \) and \( Q_{i,2} = B \). It follows by Lemma 4.13 that \( Q_i = \{Q_{i,1}, Q_{i,2}\} \) satisfies (P1)-(P3).

Assume that we have already found a partition \( Q_i \) which satisfies (P1)-(P3) for some \( i \in [t] \), now we will find a partition \( Q_{i-1} \) with the desired properties.

Write

\[
m = |V_i|, \ \mathcal{G}_i = \mathcal{H}_i[V_i], \ \mathcal{G}_{i-1} = \mathcal{H}_{i-1}[V_i] \text{ and } \mathcal{B}_i = Q_{i,1} \times (Q_{i,2})^3.
\]

We first prove the following Lemma which is vital for finding \( Q_{i-1} \) with the desired properties.

Lemma 4.14 Let \( 0 \leq i \leq t \).

(i) \( \delta(G_i) \geq (\frac{9}{128} - 2\alpha)m^3 \), so \( e(G_i) \geq (\frac{9}{128} - \frac{9}{4})m^4 \).

If a partition \( Q_i = \{Q_{i,1}, Q_{i,2}\} \) of \( V_i \) satisfies (P1)-(P3), then

(ii) \( |Q_{i,1}| = (\frac{9}{128} + \beta)m \), so \( |Q_{i,2}| = (\frac{9}{128} - \beta)m \)

and

(iii) \( d_{\mathcal{H}_i[V_i]}(x) \leq \gamma m^3 \) for all \( x \in V_i \).

Proof: By definition of cleaning, \( d_{\mathcal{H}_i}(x) \geq (\frac{9}{128} - \alpha)|V_i \setminus V_i|^3 \) holds for every \( x \in V_i \). It follows by Lemma 4.12 that

\[
\delta(G_i) \geq (\frac{9}{128} - \alpha)m^3 - |V_i \setminus V_i|^3 \geq (\frac{9}{128} - \alpha)m^3 - \frac{1}{2} \alpha \left( \frac{m}{1 - \alpha} \right)^3 \geq (\frac{9}{128} - 2\alpha)m^3.
\]
This proves (i).

Next, assume for the sake of contradiction that $|Q_{i,1}| \neq (\frac{1}{4} \pm \beta) m$. First suppose that $|Q_{i,1}| > (\frac{1}{4} + \beta) m$. Then $|Q_{i,2}| < (\frac{1}{4} - \beta) m$. For any $v \in Q_{i,1}$, $d_{F^*}(v) \leq \left(\frac{4}{3} - \beta\right)m \leq \left(\frac{9}{128} - \frac{9}{83} \beta\right)m^3$, which contradicts to $\delta(F^*) \geq \left(\frac{128}{125} - \delta\right)m^4$. Now assume that $|Q_{i,1}| < (\frac{1}{4} - \beta) m$. Suppose that $|Q_{i,1}| = (\frac{1}{4} - \mu)m$, where $\beta \leq \mu \leq 1/4$ is a real number. So $|Q_{i,2}| = (\frac{3}{4} + \mu)m$. For any $v \in Q_{i,2}$, $d_{F^*}(v) \leq (\frac{1}{4} - \mu)|V(F^*)| \cdot \left(\frac{4}{3} + \mu\right)m = \left(\frac{1}{16} - \frac{3}{10} \mu - \frac{7}{4} \mu^2 - \mu^3\right)m^3 \leq \left(\frac{9}{128} - \frac{9}{83} \beta\right)m^3$, a contradiction. This proves (ii).

Finally, let $x \in Q_{i,1}$ and $y \in Q_{i,2}$ be two arbitrary vertices. By (P3) and (ii) we have

$$d_{B_i}(x) \leq \left(\frac{3}{4} + \beta\right)m \leq \left(\frac{9}{128} + \frac{9}{128} \beta\right)m^3$$

and

$$d_{B_i}(y) \leq \left(\frac{1}{4} + \beta\right)m \left(\frac{4}{3} + \beta\right)m \leq \left(\frac{9}{128} + \frac{9}{128} \beta\right)m^3.$$ 

Since $G_i \subseteq B_i$ by (P3) for $Q_i$, this implies (iii), using (i) and $\alpha \ll \gamma$.

**Lemma 4.15** Let $c$ be a real number satisfying $\gamma \ll c \leq 10^{-2}$. Let $G$ be a $K^\alpha_{4,4}$-hom-free 4-graph with vertex set $Q_{i,1} \cup Q_{i,2}$, where $Q_{i,1} \cap Q_{i,2} = \emptyset$ and $B = Q_{i,1} \times (Q_{i,2})$. Let $m = |Q_{i,1} \cup Q_{i,2}|$. If there are vertex sets $A \subseteq Q_{i,1}$ and $B \subseteq Q_{i,2}$ satisfying that

1. $|A| \geq cm$ and $|B| \geq cm$,
2. $d_G(x) \geq (9/128 - 2\alpha)m^3$ for every $x \in Q_{i,1} \cup Q_{i,2}$ and
3. $d_{B[A \cup B] \setminus G[A \cup B]}(x) \leq \gamma m^3$ for every $x \in A$,

then there is no edge $e$ of $G$ such that $|e \cap A| \geq 2$.

**Proof:** Assume for the sake of contradiction that there exists one edge $e \in G$ such that $|e \cap A| \geq 2$. Let $a, b \in e$ satisfy $a, b \in A$. For $x = a$ or $b$ let

$$B(x) := \{ x' \in B \setminus e : \exists e' \in G \text{ such that } \{ x, x' \} \subseteq e' \}.$$ 

By condition (3) we have $|B(x)| \geq |B| - cm/10$. Let $Q' = B(a) \cap B(b)$. It is clear that $|Q'| \geq 3cm/4$. Let $Q_1', Q_2', Q_3'$ be an arbitrary partition of $Q'$ satisfying $|Q_j'| \geq cm/4$ for every $j \in [3]$. Fix $D \subseteq A \setminus e$ satisfying $|D| \geq cm/2$. Denote the maximum set of the disjoint triples in $Q_1'$ belonging to $L_G(a)$ as $M_1$. We claim that $|M_1| \geq cm/20$. Otherwise suppose that $|M_1| < cm/20$. Since for every triple $g \in Q_1' \setminus (\cup_{f \in M_1} f)$, $g \notin L_G(a)$. Then there are at least $|Q_1'| \left(\frac{1}{5} \left| (\cup_{f \in M_1} f) \right| \right) > \left(\frac{cm}{100}\right)$ triples in $Q_1'$ not belonging to $L_G(a)$, which contradicts to (3). Similarly, there are at least $cm/20$ pairwise disjoint triples in $Q_2'$, denoted as $M_2$, belonging to $L_G(b)$. Since $G$ is $K^\alpha_{4,4}$-hom-free, then $\forall f_1 \in M_1, \forall f_2 \in M_2$, $\exists v_1 \in f_1, v_2 \in f_2, v_1v_2u \notin G$ for all $u \in D$ and $v_2 \in Q_2'$. Let $v$ be an arbitrary vertex in $D$. Then there are at least $|M_1| \cdot |M_2| \cdot |Q_3'|$ triples not belonging to $L_{B[A \cup B] \setminus G[A \cup B]}(v)$, which contradicts to (3).

Let $u, v \in U_i'$ be such that in the $i$th Merging step $P_{i,u}'$ was merged into $P_{i,w}'$. Note that $u \in U_i$ and $v \notin U_i$. Denote

$$A_u = P_{i,u}' \cap V_i$$

and

$$A_v = P_{i,v}' \cap V_i.$$ 

We can view $G_i$ as being obtained from $G_{i-1}$ by Merging $A_u$ to $A_v$. Since $G_i$ satisfies (P2), then $A_u \cup A_v \subseteq Q_{i,1}$ or $A_v \cup A_u \subseteq Q_{i,2}$. In both cases, let

$$W_1 = Q_{i,1} \setminus A_v \text{ and } W_2 = Q_{i,2} \setminus A_v.$$
Suppose the partition \( \{W'_1, W'_2\} \) of \( V_i \) is obtained by adding \( A_v \) to \( W_1 \) or \( W_2 \) such that
\[
\Sigma := |\{ e \in E_i : |e \cap W'_1| = 2 \text{ or } e \subseteq W'_2 \}| + 2|\{ e \in E_i : |e \cap W'_1| = 3 \}| + 3|\{ e \in E_i : e \subseteq W'_1 \}| \]
is the smaller one.

Let \( Q_{i-1,1} = W'_1 \) and \( Q_{i-1,2} = W'_2 \) and let \( Q_{i-1} = \{Q_{i-1,1}, Q_{i-1,2}\} \). We call an edge \( e \) of \( G_{i-1} \) bad if \( |e \cap W'_1| = 2 \) or \( e \subseteq W'_2 \), very bad if \( |e \cap W'_1| = 3 \), worst if \( |e \cap W'_1| = 4 \) or good otherwise. We will prove that \( Q_{i-1} \) satisfies (P1)-(P3). This is immediate for (P2), and (P1) follows since \( W_1 \neq \emptyset \) and \( 1 \notin A_v \) by the definition of Merging. It remains to prove (P3), i.e. all edges are good. Equivalently, we need to show that \( \Sigma = 0 \), as every bad edge is counted exactly once in \( \Sigma \), every very bad edge is counted exactly twice in \( \Sigma \) and every worst edge is counted exactly three times in \( \Sigma \), whereas good edges are not counted at all.

Note that any \( e \in G_{i-1} \) that is not good satisfies \( |e \cap A_v| = 1 \). Since \( |e \cap A_v| \leq 1 \) by the definition of Merging and \( |e \cap A_v| \geq 1 \) by (P3) for \( Q_{i-1} \).

We say that a vertex of \( A_v \) is bad if it is contained in at least \( 10^{-3} m^3 \) edges which are not good. Before proving that all edges are good, we will prove that a vertex of \( A_v \) cannot be contained in too many edges which are not good.

**Lemma 4.16** There are no bad vertices in \( A_v \).

**Proof:** Assume for the sake of contradiction that \( x \in A_v \) is a bad vertex. Then every vertex in \( A_v \) is bad. We divide it into two cases according to \( A_v \cup A_u \subseteq Q_{i,1} \) or \( A_v \cup A_u \subseteq Q_{i,2} \).

**Case 1.** \( A_v \cup A_u \subseteq Q_{i,1} \). Note that \( W_1 = Q_{i,1} \setminus A_v \) and \( W_2 = Q_{i,2} \) in this case.

Subcase 1. Adding \( A_v \) to \( W_1 \) minimises \( \Sigma \). Since every bad edge (if there exist bad edges) intersects \( A_v \), so every bad edge intersect \( W_1 \). Fix an edge \( xaa \in G_{i-1} \) such that \( a \in W_1 \). Note that \( a \notin A_v \).

Claim 1.1.1: There are at least \( \frac{1}{30} m^3 \) good edges containing \( x \). Otherwise we consider the partition \( \{W''_1, W''_2\} \) of \( V_i \) obtained by adding \( A_v \) to \( W_2 \) rather than \( W_1 \). This new partition implies that every good edge containing a vertex of \( A_v \) turns bad, this contributes to \( \Sigma \) at most \( \frac{1}{30} |A_v| m^3 \). Every bad edge turns good, every very bad edges turns bad and every worst edge turns very bad, so this reduces \( \Sigma \) by at least \( (9/128 - 2a - 1/30)|A_v| m^3 \geq \frac{1}{30} |A_v| m^3 \), which contradicts the minimality of \( \Sigma \).

Claim 1.1.2: There are at least \( 10^{-3} m \) pairwise disjoint triples in \( W_2 \) belonging to \( L_{G_{i-1}}(x) \). Otherwise, since every such triple intersects with less than \( 3\binom{m}{2} \) triples in \( W_2 \), then there are at most \( 3\binom{m}{2} \cdot 10^{-3} m < m^3/30 \) good edges of \( F \) containing \( x \), contradicting to Claim 1.1.1.

Fix a set of such triples of size \( 10^{-3} m \) as \( M(x) \). Let
\[
B(x) := \{ x' \in W_2 : \exists e \in G_{i-1} \text{ such that } \{x, x'\} \subseteq e \}.
\]

Claim 1.1.3: \( |B(x) \setminus \bigcup_{e \in M(x)} e| \geq m/10 \). Since all good edges of \( F \) containing \( x \) are contained in \( \{x\} \times \binom{B(x)}{3} \), then \( \binom{B(x)}{3} \leq \frac{1}{30} m^3 \) and Claim 1.1.3 follows.

Denote the maximum set of the disjoint triples in \( B(x) \setminus \bigcup_{e \in M(x)} e \) belonging to \( L_{G_{i-1}}(a) \) (in this sense, they belong to \( L_{G_{i-1}}(a) \) as well) as \( M(a) \).

Claim 1.1.4: \( |M(a)| \geq m/60 \). Otherwise suppose that \( |M_a| < m/60 \). Since for every triple \( g \) in \( B(x) \setminus \bigcup_{e \in M(x) \cup M(a)} f \), \( g \notin L_{G_{i-1}}(a) \) (or \( L_{G_{i-1}}(a) \)). Then there are at least \( \frac{m}{3^3} \) triples in \( W_2 \) not belonging to \( L_{G_{i-1}}(a) \), which contradicts Lemma 4.13 (iii).
Thus we get a contradiction with Lemma 4.14 (iii).

Subcase 2. Adding $A_v$ to $W_2$ minimizes $\Sigma$.

We first prove that all edges which is not good are contained in $W_2 \cup A_v$.

Claim 1.2.1: $|A_v| < |Q_{i,1}| - m/20$ (note that $W_1 = Q_{i,1} \setminus A_v$). Suppose that $|A_v| \geq |Q_{i,1}| - m/20$. Then $|W_1| \leq m/20$. Let $w \in A_v$. We bound the number of good edges containing $w$, denoted by $d_{\text{good}}(w)$, and the number of edges containing $w$ which are not good, denoted by $d_{\text{bad}}(w)$ in $G_{i-1}$. Then

$$d_{\text{good}}(w) \leq |W_1| \left( \frac{|W_2|}{2} \right) \leq \frac{m}{20} \left( \frac{19m/20}{2} \right) \leq 0.023 m^3.$$

So

$$d_{\text{bad}}(w) \geq (9/128 - 2\alpha) m^3 - 0.023 m^3 \geq 0.047 m^3.$$

Now we consider the partition $\{W'_1, W'_2\}$ of $V_t$ obtained by adding $A_v$ to $W_1$ rather than $W_2$. The number increasing $\Sigma$ is at most

$$|W_1| \left( \frac{|W_2|}{2} \right) \cdot |A_v| + \left| \frac{|W_1|}{3} \right| \cdot |A_v| < 0.02 |A_v| m^3,$$

whereas the number decreasing $\Sigma$ is at least

$$\sum_{w \in A_v} d_{\text{bad}}(w) - \left( \frac{|W_1|}{2} \right) |W_2| = \frac{|W_1|}{3} \geq 0.03 |A_v| m^3.$$

This contradicts the minimality of $\Sigma$.

Let $A = W_1$ and $B = W_2$. By Lemma 4.14 and the relationship between $G_i$ and $G_{i-1}$, then $G_{i-1}$ with $Q_{i-1,1} \cup Q_{i-1,2}$ and $A, B$ satisfy the conditions of Lemma 4.15. Applying Lemma 4.15 we get that all bad edges are contained in $W_2 \cup A_v$. Fix an edge $xyzw \in G_{i-1}$ satisfying $y, z, w \in W_2$. For $j = 1, 2$, let

$$B_j(x) := \{ x' \in W_j \setminus xyzw : \exists e \in G_{i-1} \text{ such that } \{x, x'\} \subseteq e \}.$$

Claim 1.2.2: $|B_1(x)| \geq 10^{-3} m$ and $|B_2(x)| \geq 10^{-1} m$. We first prove that $|B_1(x)| \geq 10^{-3} m$. Otherwise suppose that $|B_1(x)| < 10^{-3} m$. Now we consider the partition $\{W'_1, W'_2\}$ of $V_t$ obtained by adding $A_v$ to $W_1$ rather than $W_2$. Since there are no very bad edges and worst edges in the partition of $\{W'_1, W'_2\}$, so every bad edge turns good. There are at least $|A_v|10^{-3} m^3$ such edges (Recall that every vertex in $A_v$ is bad). On the other side, every good edge containing a vertex of $A_v$ becomes bad. There are less than $|A_v| \cdot |B_1(x)| m^2 < |A_v|10^{-3} m^3$ such edges. However, this contradicts the minimality of $\Sigma$. Now we prove that $|B_2(x)| \geq 10^{-1} m$. Since there are at least $10^{-3} m^3$ edges containing $x$ in $W_2 \cup A_v$, then $|B_2(x)| \geq 10^{-3} m^3$.

Claim 1.2.3: Let $M_x$ be the maximum matching of such edges satisfying one vertex in $B_1(x)$ and another three vertices in $B_2(x)$, then $|M_x| \geq 10^{-4} m$. Otherwise suppose that $|M_x| < 10^{-4} m$. Denote $V_x = \cup_{f \in M_x} f$. Since for every vertex $x_1 \in B_1(x) \setminus V_x$ and every triple $\{x_2, x_3, x_4\} \in B_2(x) \setminus V_x$,
\[ \{x_1, x_2, x_3, x_4\} \in B_i \setminus G_{i-1} \text{ (in } B_i \setminus G_i \text{ as well). Then there are at least }|B_i(x) \setminus V_x| \left| \frac{(B_2(x) \setminus V_x)}{3} \right| \geq \varepsilon n^4 \text{ edges of } B_i \setminus G_i, \text{ which contradicts (iii) of Lemma 4.14.} \]

Since \( G_{i-1} \) is \( K_4^{4}\)-hom-free, then \( \forall f_1 \in M_z, \exists a \in f_1 \) and \( q \in \{y, z, w\} \) such that \( aqq'' \notin G_{i-1} \) for all pairs \( q', q'' \in V_i \).

Consider the subcase that there are at least \( 10^{-5}m \) vertices in \( B_1(x) \cap (\cup_{f \in M_z} f) \) such that for every such vertex \( b, \exists q \in \{y, z, w\} \) such that \( bqq'q'' \notin G_{i-1} \) for all pairs \( q', q'' \in V_i \). By pigeonhole principle, there is \( q \in \{y, z, w\} \) such that there are at least \( 1/3 \cdot 10^{-5}m \cdot (m/2) > 10^{-7}m^3 \) edges of \( B_i \setminus G_i \) containing \( q \), which contradicts (iii) of Lemma 4.14.

In the remaining subcase that there are at least \( 10^{-5}m \) vertices in \( B_2(x) \cap (\cup_{f \in M_z} f) \) such that for every such vertex \( b', \exists p \in \{y, z, w\} \) such that \( bpq'q'' \notin G_{i-1} \) for all pairs \( q', q'' \in V_i \). By pigeonhole principle, there is \( p' \in \{y, z, w\} \) such that there are at least \( \frac{1}{3} \cdot 10^{-5}m \cdot \frac{m}{2} > 10^{-7}m^3 \) edges of \( B_i \setminus G_i \) containing \( p' \), which contradicts (iii) of Lemma 4.14.

Case 2. \( A_v \cup A_v \subseteq Q_{1, 2} \).

Subcase 1. Adding \( A_v \) to \( W_1 \) minimizing \( \Sigma \).

Claim 2.1.1: \( |A_v| < m/4 \). Suppose that \( |A_v| \geq m/4 \). In this case, there are no bad edges contained in \( W_2 \). Consider the vertex \( u \in A_v \). So all edges containing \( u \) are good. We now count the degree of \( u \) in \( G_{i-1} \) (in \( G_i \) as well),

\[
d_{G_{i-1}}(u) \leq |W_1| \left( |W_2| \frac{|W_2|}{2} \right) \leq (m/4 + \gamma) \cdot \left( \frac{3m/4 - m/4 + \gamma}{2} \right) < (9/128 - 2\alpha)m^4,
\]

which contradicts Lemma 4.14 (i).

Let \( A = W_1 \) and \( B = W_2 \). By Lemma 4.14 and the relationship between \( G_i \) and \( G_{i-1} \), then \( G_{i-1} \) with \( Q_{i-1, 1} \cup Q_{i-1, 2} \) and \( A, B \) satisfy the conditions of Lemma 4.15. Applying Lemma 4.15 we are done.

Subcase 2. Adding \( A_v \) to \( W_2 \) minimizing \( \Sigma \).

Denote \( W = W_1 \cup W_2 = V_i \setminus A_v \). We first show that there is no edge of \( G_{i-1} \) intersecting \( W_1 \) with two or three vertices. Let \( u' \in A_v \), note that every edge of \( G_{i-1} \) containing \( u' \) is good. Then \( (9/128 - 2\alpha)m^3 \leq d_{G_{i-1}}(u') \leq |W_1| \cdot \left( |W_2| \frac{|W_2|}{2} \right) \), note that \( |W_1| = |Q_{i, 1}| = (1/4 + \gamma)m, \) so \( |W_2| \geq m/2 \). Let \( A = W_1 \) and \( B = W_2 \). By Lemma 4.14 and the relationship between \( G_i \) and \( G_{i-1} \), then \( G_{i-1} \) with \( Q_{i-1, 1} \cup Q_{i-1, 2} \) and \( A, B \) satisfy the conditions of Lemma 4.15. Applying Lemma 4.15 we get that there is no edge of \( G_{i-1} \) that contains two or three vertices of \( W_1 \). Hence there are at least \( 10^{-3}m^3 \) edges containing \( x \) contained in \( W_2 \cup A_v \). Fix one such edge \( xyzw \in G_{i-1} \), where \( y, z, w \in W_2 \). For \( j = 1, 2 \), let

\[
B_j(x) := \{x' \in W_j \setminus \{y, z, w\} : \exists e \in G_{i-1} \text{ such that } \{x, x'\} \subseteq e\}.
\]

We claim that \( |B_1(x)| \geq 10^{-4}m \) and \( |B_2(x)| \geq 10^{-2}m \). We first prove that \( |B_1(x)| \geq 10^{-4}m \). Otherwise suppose that \( |B_1(x)| < 10^{-4}m \). Now we consider the partition \( \{W_1', W_2'\} \) of \( V_i \) obtained by adding \( A_v \) to \( W_1 \) rather than \( W_2 \). Every bad edge contained in \( W_2 \) turns good. There are at least \( |A_v|10^{-3}m^3 \) such edges. On the other hand, every good edge containing a vertex of \( A_v \) becomes bad, every bad edge that intersects \( W_1 \) with two vertices turns very bad and every very bad edge turns worst. There are less than \( |A_v| \cdot |B_1(x)|m^2 < |A_v|10^{-3}m^3 \) such edges. However, this contradicts the minimality of \( \Sigma \). Now we prove that \( |B_2(x)| \geq 10^{-2}m \). Since there are at least \( 10^{-3}m^3 \) edges containing \( x \) in \( W_2 \) and \( (|B_2(x)|) \geq 10^{-3}m^3, \) then \( |B_2(x)| > 10^{-2}m \).
Let $M'(x)$ be the maximum matching of such edges satisfying one vertex in $B'_1(x)$ and another three vertices in $B'_2(x)$. Similar to Claim 1.2.3, we have $|M'(x)| \geq 10^{-5}m$. Since $G_{i-1}$ is $K^4_{4,4}$-free, then $\forall f_1 \in M'(x)$, $\exists q \in f_1$ and $q \in \{y, z, w\}$ such that $aqq''q'' \notin G_{i-1}$ for all pairs $q', q'' \in V_i$.

Consider the subcase that there are at least $10^{-6}m$ vertices in $D \subseteq B'_1(x) \cap (\cup_{f \in M(x)})$ such that $\forall b \in D$, $\exists q \in \{y, z, w\}$ such that $bqq''q'' \notin G_{i-1}$ for all pairs $q', q'' \in V_i$. By pigeonhole principle, there is $q \in \{y, z, w\}$ such that there are at least $1/3 \cdot 10^{-6}m \cdot (m/3) > 10^{-8}m^3$ edges of $B_i \setminus G_i$ containing $q$, which contradicts (iii) of Lemma 4.14.

In the remaining subcase that there are at least $9 \cdot 10^{-6}m$ edges $e \in M'(x)$ such that $\exists d \in e \cap W_2$ and $d' \in \{y, z, w\}$ such that $dd'q'q'' \notin G_{i-1}$ for all pairs $q', q'' \in V_i$. By pigeonhole principle, there is $p' \in \{y, z, w\}$ such that there are at least $1/3 \cdot 9 \cdot 10^{-6}m \cdot m/3 > 10^{-8}m^3$ edges of $B_i \setminus G_i$ containing $p'$, which contradicts (iii) of Lemma 4.14.

In our next lemma we will conclude the second stage of the proof by showing that every edge of $G_{i-1}$ is good. First we show that $|W'_1| \geq m/5$ and $|W'_2| \geq 3m/5$ (so $m/5 \leq |W'_1| \leq 2m/5$ and $3m/5 \leq |W'_2| \leq 4m/5$). Note that $W'_1$ and $W'_2$ are obtained by adding $A_v$ to $W_1$ or $W_2$. If $W'_1 = Q_{i,1}$ then this holds by Lemma 4.13 (ii). Now we consider the remaining two cases. In both cases, let $x \in A_v$.

Case 1. Adding $A_v$ from $Q_{i,1}$ to $W_2$. For this case $Q_{i,2} \subseteq W'_2$, so it suffices to prove $|W'_1| \geq m/5$. Suppose that $|W'_1| < m/5$. By Lemma 4.13 (ii) and Lemma 4.16, we have

\[
d_{G_{i-1}}(x) \leq 10^{-3}m^3 + |W'_1| \left( \frac{|W'_2|}{2} \right) \leq 10^{-3}m^3 + \frac{m}{5} \cdot \frac{8m^2}{25} < \left( \frac{9}{128} - \alpha \right)m^3,
\]

which contradicts Lemma 4.14 (i).

Case 2. Adding $A_v$ from $Q_{i,2}$ to $W_1$. For this case $Q_{i,1} \subseteq W'_1$, so it suffices to prove $|W'_2| \geq 3m/5$. Otherwise suppose $|W'_2| < 3m/5$. By Lemma 4.14 (ii) and Lemma 4.16, we have

\[
d_{G_{i-1}}(x) \leq 10^{-3}m^3 + \left( \frac{|W'_2|}{3} \right) < \left( \frac{9}{128} - \alpha \right)m^3,
\]

which contradicts Lemma 4.14 (i). We can now state our next lemma.

**Lemma 4.17** Every edge of $G_{i-1}$ is good.

**Proof:** Assume for the sake of contradiction that $e \in G_{i-1}$ is not a good edge.

**Case 1.** $|e \cap W'_1| \geq 2$.

Let $a, b \in e$ satisfy $a, b \in W'_1$. For $x = a$ or $b$ let

\[
B(x) := \{x' \in W'_2 \setminus e : \exists e' \in G_{i-1} \text{ such that } \{x, x'\} \subseteq e'\}.
\]

We claim that $|B(a) \cap B(b)| \geq m/2$. Otherwise $|B(a)| \leq |W'_2|/2 + m/4$ or $|B(b)| \leq |W'_2|/2 + m/4$. Without loss of generality we can assume that $|B(a)| \leq |W'_2|/2 + m/4$, then $|B(a)| \leq 2m/5 + m/4 = 13m/20$. Hence $d_{G_{i-1}}(a) \leq (\frac{13m}{20}) + 10^{-3}m^3 < (\frac{9}{128} - \alpha)m^3$, which contradicts Lemma 4.14 (i).
We claim that there are at least $m/100$ disjoint triples in $B(a) \cap B(b)$ belonging to $L_{G_{i-1}}(a)$. Otherwise $d_{G_{i-1}}(a) \leq \left( \frac{4m}{3} \right) - \left( \frac{m}{2} - \frac{3m}{100} \right) + 10^{-3}m^3 < \left( \frac{9}{125} - \alpha \right)m^3$, which contradicts Lemma 4.14(i). Fix such a set of $m/100$ triples as $M_a$.

We also claim that there are at least $m/100$ disjoint triples in $(B(a) \cap B(b)) \setminus \left( \cup_{f \in M_a} f \right)$ belonging to $L_{G_{i-1}}(b)$. Otherwise $d_{G_{i-1}}(b) \leq \left( \frac{4m}{3} \right) - \left( \frac{m}{2} - \frac{6m}{100} \right) + 10^{-3}m^3 < \left( \frac{9}{125} - \alpha \right)m^3$, which contradicts Lemma 4.14(i). Fix such a set of $m/100$ triples as $M_b$.

Since $G_{i-1}$ is $K^4_{4,4}$-hom-free, then $\forall f_1 \in M_a$ and $f_2 \in M_b$, $\exists w_1 \in f_1$ and $w_2 \in f_2$ such that $\{w_1, w_2\}$ is not contained in any edge of $G_{i-1}$. Let $w_3 \in W_2 \setminus (\cup_{f \in M_a \cup M_b} f)$ (there exists such a vertex), then $d_{B \setminus G_i}(w_3) \geq |M(a)| \cdot |M(b)| \cdot \left( |W_1| - m/50 \right) > \gamma m^3$, which contradicts Lemma 4.14(iii).

Case 2. Let $e = xyzw \subseteq W_2$.

For $j = 1, 2$ and $a \in \{x, y, z, w\}$ let

$$B_j(a) := \{ a' \in W_j : \exists e' \in G_{i-1} \text{ such that } \{a, a'\} \subseteq e' \}.$$ 

We claim that $|B_1(a)| \geq m/6$ and $|B_2(a)| \geq m/2$ for every $a \in \{x, y, z, w\}$. We first prove that $|B_1(a)| \geq m/6$. Otherwise suppose that $|B_1(a)| < m/6$. Then $d_{G_{i-1}}(a) < 10^{-3}m^3 + |B_1(a)| \cdot |W_2^j| \leq 10^{-3}m^3 + m \cdot \left( \frac{4m}{3} \right) < \left( \frac{9}{125} - \alpha \right)m^3$, which contradicts Lemma 4.14(i). Now we prove that $|B_2(a)| \geq m/2$. Otherwise, in a similar way as above, $d_{G_{i-1}}(a) < 10^{-3}m^3 + |W_1| \cdot \left( \frac{m}{2} \right) \leq 10^{-3}m^3 + 2m/5 \cdot \left( \frac{m}{2} \right) < \left( \frac{9}{125} - \alpha \right)m^3$, which contradicts Lemma 4.14(i).

Now we show that almost all vertices in $B_1(a) \cup B_2(a)$ are adjacent to $y, z, w$. Otherwise without loss of generality assume there are $10\gamma m$ vertices of $B_1(a)$ which are not adjacent to $y$. Then

$$d_{B_1 \setminus G_i}(y) > 10\gamma m \left( \frac{3m/5}{2} \right) > \gamma m^3,$$

contradicting to Lemma 4.14(iii). Let $B_1 \subseteq W_1^j$ and $B_2 \subseteq W_2^j$ be the common neighbourhood of $x, y, z, w$. Since almost all vertices in $V_i$ are the neighbours of every vertex in $W_2^j$, so we can assume that $|B_1| \geq m/7$ and $|B_2| \geq m/3$. It is clearly that there is an edge $e' \in G_{i-1}$ with one vertex in $B_1$ and another three vertices in $B_2$. Then $\forall a \in xyzw$ and $b \in e'$, there exists one edge $e_{ab}$ of $G_{i-1}$ such that $\{a, b\} \subseteq e_{ab}$. Hence $e_{ab} : a \in \{x, y, z, w\}, b \in e' \cup \{xyzw, e'\}$ forms a configuration contradicting that $G_{i-1}$ is $K^4_{4,4}$-hom-free. 

This shows that $Q_{i-1}$ satisfies (P3), so splitting has the required properties. It terminates with $Q_0$ such that $F[V_i] = H_0[V_i] \subseteq Q_{0,1} \times \left( Q_{0,2} \right)$. Let $A = Q_{0,1} \cup (V(F) \setminus V_i)$ and $B = Q_{0,2}$. Then by Lemma 4.12, we have $|\{ e \in F : |e \cap A| \geq 2 \}| + |\{ e \in F : e \subseteq B \}| < \varepsilon n^4$. This concludes the proof of Theorem 1.3.

### 4.2 Proof of Theorem 1.5

Let $F = (V, E)$ be a maximum $K^4_{4,4}$-free 4-graph on $n$ vertices, where $n$ is sufficiently large. Since $S^4(n)$ is $K^4_{4,4}$-free, $|F| \geq |S^4(n)|$. Similar to [10] (in the first paragraph of Section 4.2), it suffices to prove Theorem 1.5 under the assumption that the minimum degree of $F$ is at least $\delta(S^4(n))$. Indeed, assume we have proved Theorem 1.5 for every maximum $K^4_{4,4}$-free 4-graph $F$ on $n \geq n_0$ vertices and minimum degree at least $\delta(S^4(n))$. Let $H_n$ be a maximum $K^4_{4,4}$-free 4-graph on $n \geq n_0^2$ vertices, delete the vertex of $H_n$ with degree less than $\delta(S^4(n))$ until $\delta(H_m) \geq \delta(S^4(m))$ or all vertices have been deleted. By some
Claim 2. Let $c_1, c_2, c_3$ and $\varepsilon > 0$ be real numbers satisfying
$$
\varepsilon \ll c_3 \ll c_2 \ll c_1 \ll 1.
$$
Let $V = W_1 \cup W_2$ be a partition of the vertex set of $\mathcal{F}$ which minimizes
$$
\Sigma' := \{\{e \in E : |e \cap W_1| = 2 \text{ or } e \subseteq W_2\} + 2\{e \in E : |e \cap W_1| = 3\} + 3\{e \in E : e \subseteq W_1\}\}.
$$
By Theorem 4.4 we can assume that $\Sigma' < \varepsilon n^4$. Similar to before, we have
$$
|W_1| = (\frac{1}{4} \pm c_3)n \text{ and } |W_2| = (\frac{3}{4} \pm c_3)n.
$$
Similar to the proof of Theorem 4.4 we call an edge $e \in E$ bad if $|e \cap W_1| = 2$ or $e \subseteq W_2$, very bad if $|e \cap W_1| \geq 3$, good otherwise. Equivalently, we need to show that $\Sigma' = 0$. We say that a vertex $v \in V$ is bad if it is contained in at least $30c_1 n^3$ edges which are not good. Before proving that all edges are good, we will prove that any vertex of $V$ cannot be contained in too many edges which are not good.

**Lemma 4.18** There are no bad vertices.

**Proof:** Assume for the sake of contradiction that $x \in V$ is a bad vertex. First suppose that $x \in W_1$. Then all edges containing $x$ which are not good intersect $W_1$ with at least two vertices. Denote $F_x = \{e \in E : x \in e, |e \cap W_1| \geq 2\}$ and $I_x = \{f \in \binom{V}{3} : f \cup \{x\} \subseteq F_x\}$. Then $|F_x| = |I_x| \geq 30c_1 n^3$.

Claim 1. There is a set $K \subseteq I_x$ with $|K| = 10c_1 n$ such that $\forall f_1, f_2 \in K$, $f_1 \cap f_2 = \emptyset$. This is because that $\forall f \in I_x$, $f$ intersects less than $3\binom{n}{2}$ elements of $I_x$. Denote
$$
B(x) := \{v \in W_2 \setminus (\cup f \in I_x, f) : \{v, x\} \text{ is contained in at least } 40n \text{ edges of } \mathcal{F}\}.
$$
Claim 2. $|B(x)| \geq n/20$. By the minimality of $\Sigma'$, the number of good edges of $\mathcal{F}$ containing $x$ is no less than the number of edges in $\mathcal{F}$ which are not good containing $x$. Then $|W_2 \setminus B(x)| + |B(x)| \binom{|W_2|}{2} > d(x)/2 \geq (9/256 - c_1/2)n^3$, so $|B(x)| \geq n/20$.

Claim 3. There are at least $|K|/2 = 5c_1 n$ elements $f = \{a, b, c\} \in K$ with $a \in W_1$ such that there is $B_{f_a} \subseteq B(x)$ of size $c_1 n$ satisfying that $\forall v, a$ is contained in at least $40n$ edges of $\mathcal{F}$ for every $v \in B_{f_a}$. Otherwise there are at least $|K|/2$ elements $f = \{a, b, c\} \in K$ with $a \in W_1$ such that for all but at most $c_1 n$ vertices $v$ in $B(x)$, $\{v, a\}$ is contained in at most $40n$ edges of $\mathcal{F}$. Then there are at least $|K|/2 \text{ elements } f = \{a, b, c\} \in K$ with $a \in W_1$ such that for all but at most $c_1 n$ vertices $v$ in $B(x)$, $\{v, a\}$ is contained in at least $40n$ edges of $\mathcal{F}$. Then there are at least $\frac{|K|}{2}\left(\binom{|B(x)\setminus B_{f_a}|}{3} - 40n^2\right) > \varepsilon n^4$ edges in $\left(W_1 \setminus \binom{W_2}{3}\right) \setminus \mathcal{F}$, contradicting that $|E| \geq |S^4(n)|$. Denote the set of elements $f \in K$ satisfying the property above as $K'$.

Claim 4. There is $f \in K'$ with $a \in W_1 \cap f$ such that there is a set $M_{f_a} \subseteq \left(\binom{B_{f_a}}{3}\right)$ of size $c_2 n$ satisfying $e_1 \cup \{a\} \in \mathcal{F}$ and $e_1 \cup \{x\} \in \mathcal{F}$ and $e_1 \cap e_2 = \emptyset$ for all different $e_1, e_2 \in M_{f_a}$. Otherwise for each $f \in K'$ with $a \in W_1 \cap f$, fix a maximum set $M_{f_a} \subseteq \left(\binom{B_{f_a}}{3}\right)$ satisfying $e_1 \cup \{a\}$, $e_1 \cup \{x\} \in \mathcal{F}$ and $e_1 \cap e_2 = \emptyset$ for all different $e_1, e_2 \in M_{f_a}$. Then $|M_{f_a}| < c_2 n$ and for every triple $g \in \binom{B_{f_a}\setminus (\cup a \in M_{f_a})}{3}$, $g \cup \{a\} \notin \mathcal{F}$ or $g \cup \{x\} \notin \mathcal{F}$ (or both). So there are at least $\frac{|K|}{2}\left(\binom{|B_{f_a}|}{3} - 3c_2 n\right) > \varepsilon n^4$ edges in $\left(W_1 \setminus \binom{W_2}{3}\right) \setminus \mathcal{F}$, contradicting that $|E| \geq |S^4(n)|$.

Since $\mathcal{F}$ is $K^4_{4,4}$-free, for every pair $e_1, e_2 \in M_{f_a}$, $\exists u \in e_1$ and $v \in e_2$ such that there are at most $40n$ edges of $\mathcal{F}$ containing $\{u, v\}$. Otherwise we can greedily choose vertices to extend $e_1 \cup \{x\}$ and
Consider the family of triples $L \cup \{x\}$ containing $x$ which are not good contained in $W_2$. The proof of both cases is similar to the above. We first prove the case that there are at least $15c_1n^3$ edges containing $x$ which are not good contained in $W_2$. Fix a set of those edges removing $x$ as $L'(x)$. Consider a maximum matching $M(x)$ in $L'(x)$. Then $|M(x)| \geq c_1n$. Fix $M' \subseteq M(x)$ with $|M'| = c_2n$. Note that $\bigcup_{f \in M'} f \subseteq W_2$. For $j = 1, 2$, let $B_j(x) := \{x' \in W_j : \text{there are at least } 40n \text{ edges of } F \text{ containing } \{x, x'\}\}$.

Claim 5. $|B_1'(x)| \geq 3c_2n$ and $|B_2'(x)| \geq 5c_2n$. Since there are at least $15c_1n^3$ edges containing $x$ which are not good such that each is contained in $W_2$, it is clearly that $|B_2'(x)| \geq 5c_2n$. If $|B_1'(x)| < 3c_2n$, we get a new partition by moving $x$ from $W_2$ to $W_1$ with smaller $\sum$, which contradicts the minimality of $\Sigma'$. Fix $B_1'(x) \subseteq B_1'(x)$ and $B_2'(x) \subseteq B_2'(x) \setminus \bigcup_{f \in M', f} |W_1| = 2c_2n$ and $|B_2'(x)| = 4c_2n$. Consider a maximum matching of $F$, denoted as $M''$, with one vertex in $B_1'(x)$ and another three vertices in $B_2'(x)$.

Claim 6. $|M''| \geq c_2n$. Otherwise there are at least $c_2n(\frac{n}{2}) > \varepsilon n^4$ edges in $(W_1 \times (W_2^3)) \setminus F$, a contradiction. Since $F$ is $K_{4,4}$-free, for every $f \in M'$ and $e \in M''$ there are $a \in f$ and $b \in e$ such that there are at most $40n$ edges of $F$ containing $\{a, b\}$; otherwise we can greedily choose vertices to extend $f \cup \{x\}$ and $e$ to a copy of $K_4$. Hence we can find more than $|M'| \cdot |M''| \cdot \frac{n}{2} \cdot \frac{n}{2} - 40n^3 > \varepsilon n^4$ elements in $(W_1 \times (W_2^3)) \setminus F$ and this contradicts that $|E| \geq |S^4(n)|$.

The last case is that there are at least $15c_1n^3$ edges containing $x$ which are not good such that each intersects $W_1$ with at least two vertices. Denote $F_x' = \{e \in F : x \in e, |e \cap W_1| \geq 2\}$ and $I_x' = \{f : f \cup \{x\} \subseteq F_x\}$. Since $\forall f \in I_x'$, $f$ intersects less than $3\binom{n}{2}$ elements of $I_x'$, there is a $P \subseteq I_x'$ with size $5c_1n$ such that $\forall f_1$, $f_2 \in P$, $f_1 \cap f_2 = \emptyset$.

Claim 7. There are at least $c_1n$ elements $f = \{y, z, w\} \in P$, denoted as $P'$, where $y, z \in W_1$ such that there is a set $V_{yz} \subseteq W_2$ with size at least $n/10$ such that $\{y, v\}$ and $\{z, w\}$ are both contained in at least $40n$ edges of $F$ for every $v \in V_{yz}$; otherwise there are at least $c_1n \cdot \left(\binom{W_2 \cdot n/10}{3} - 40n^2\right) > \varepsilon n^4$ edges in $(W_1 \times (W_2^3)) \setminus F$, a contradiction.

For every $f = \{y, z, w\} \in P'$, where $y, z \in W_1$ consider the set $J_{yz} \subseteq (V_{yz} \setminus \{y\}) \cap L(y) \cap L(z)$ satisfying all elements of $J_{yz}$ are pairwise disjoint.

Claim 8. There is at least an element $f' = \{y', z', w'\}$, where $y', z' \in W_1$ such that $|J_{y'z'}| \geq c_1n$; otherwise there are at least $|P'| \cdot \left(\binom{V_{y'z'}^3\setminus J_{y'z'}}{3} - 3c_1n\right) > \varepsilon n^4$ edges in $(W_1 \times (W_2^3)) \setminus F$, a contradiction.

Since $F$ is $K_{4,4}$-free, for every pair $f_1, f_2 \in J_{y'z'}$, there are $a \in f_1$ and $b \in f_2$ such that there are at most $40n$ edges of $F$ containing $\{a, b\}$; otherwise we can greedily choose vertices to extend $f_1 \cup \{y'\}$ and $f_2 \cup \{z'\}$ to a copy of $K_{4,4}$. Hence we can find at least $\left(\binom{V_{y'z'}^3}{3}\right) \left(|W_1| \cdot |W_2 \setminus \bigcup_{f \in J_{y'z'}, f} f| - 40n\right) > \varepsilon n^4$ edges in $(W_1 \times (W_2^3)) \setminus F$, a contradiction.

Finally, we show that every edge of $F$ is good. Suppose that $e = xyzw \in F$ is an edge which is not good. First we assume that $|e \cap W_1| \geq 2$. Without loss of generality assume that $x, y \in W_1$. Denote $B(x) := \{v \in W_2 : v \in \{x, y\}\}$ is contained in at least $40n$ edges of $F$.

We claim that $|B(x)| \geq 7n/10$. Otherwise $d_F(x) \leq (7n/10) + 40n^2 + 30c_1n^3 < \delta(S^4(n))$, a contradiction. Consider the family of triples $L_x \subseteq (B(x))$ satisfying $f \cap g = \emptyset$ for every pair $f, g \in L_x$. Similar
as before, we claim that \(|L_x| \geq n/5\). Otherwise suppose that \(|L_x| < n/5\), then \(d_F(x) \leq \binom{|W_2|}{3} \cdot 40n^2 + 30c_1n^3 < \delta(S^4(n))\), a contradiction. Fix \(L'_x \subseteq L_x\) with \(|L'_x| = n/10\). Let

\[B(y) := \{v \in B(x) \mid (\cup_{e \in L'_x} e) : \{v, y\}\ is\ contained\ in\ at\ least\ 40n\ edges\ of\ F\}\]

and \(L_y \subseteq \binom{B(y)}{3}\) satisfying \(f \cap g = \emptyset\) for every pair \(f, g \in L_y\). Similarly, we have \(|L_y| \geq n/10\).

Since \(F\) is \(K_{i,4}\)-free, then for every \(f_1 \in L'_x\) and \(f_2 \in L_y\), there are \(a \in f_1\) and \(b \in f_2\) such that there are at most 40 edges of \(F\) containing \(\{a, b\}\); otherwise we can greedily choose vertices to extend \(f_1 \cup \{x\}\) and \(f_2 \cup \{y\}\) to form a copy of \(K_{i,4}\). Hence we can find at least \(|L'_x| \cdot |L_y| \cdot \frac{2}{5} \cdot \frac{3n}{5} - 40n^3 > \varepsilon n^4\) edges in \((W_1 \times \binom{|W_2|}{3})\) \(F\), a contradiction.

Now assume that \(e = xyzw \subseteq W_2\). For \(j = 1, 2\), let

\[B_j(e) := \{v \in W_j : \text{there are at least 40n edges of } F \text{ containing } \{u, v\} \text{ for every } u \in e\}\]

We claim that \(|B_1(e)| \geq n/5\) and \(|B_2(e)| \geq 3n/5\). Otherwise, first suppose that \(|B_1(e)| < n/5\). Since for every \(v \in W_1 \setminus B_1(e)\), there is at least one \(u \in e\) such that \(\{u, v\}\) is contained in less than 40n edges of \(F\). So there exists at least one \(u \in e\) and at least \(\frac{1}{4}|W_1 \setminus B_1(e)|\) vertices \(v \in W_1 \setminus B_1(e)\) such that \(\{u, v\}\) is contained in less than 40n edges of \(F\). Then \(d_F(u) \leq \frac{1}{4} \cdot 40n|W_1 \setminus B_1(e)| + (|B_1(e)| + \frac{1}{4}|W_1 \setminus B_1(e)|) \binom{|W_2|}{3} + 30c_1n^3 \leq 10n^2 + \left(\frac{1}{2} + \frac{3}{4} + \frac{5}{2}\right)n + \frac{3}{4}n^3 \leq \delta(S^4(n))\), a contradiction. Now suppose that \(|B_2(e)| < 3n/5\). Similarly, there exists \(u \in e\) such that \(d_F(u) \leq \frac{1}{2} \cdot 40n|W_2 \setminus B_2(e)| + |W_1|((|B_2(u)| + \frac{1}{4}|W_2 \setminus B_2(e)|) + 30c_1n^3 < \delta(S^4(n))\), a contradiction.

Let \(M\) be a maximum matching of edges with one vertex in \(B_1(e)\) and another three vertices in \(B_2(e)\). We claim that \(|M| \geq n/6\). Otherwise there is a vertex \(u \in B_1(e)\) such that there are at least \(\frac{n^3}{2}\) triples in \(B_2(e)\) not belonging to \(L_F(u)\). Then \(d_F(u) \leq (\frac{1}{2} + \frac{3}{4} + \frac{1}{2})n + 30c_1n^3 < \delta(S^4(n))\), a contradiction.

In fact, as long as there is one edge (good edge) \(e'\) in \(M\) we would get a contradiction, since we can find a copy of \(K_{4,4}\) as follows: for each vertex \(u \in e'\) and each vertex \(v \in xyzw\), we can find an edge \(e_{uv}\) of \(F\) greedily such that the other two vertices of all these \(e_{uv}\) are all different. This contradicts that \(F\) is \(K_{4,4}\)-free. Then \(F\) is isomorphic to \(W_1 \times \binom{|W_2|}{3}\). By the maximality of the number of edges of \(F\), \(|W_1| = \left\lceil \frac{4}{3}n \right\rceil\), or \(|W_1| = \left\lfloor \frac{4}{3}n \right\rfloor\) for \(n \equiv 3\pmod{4}\). \(\blacksquare\)

**Remarks.** We learned that Norin, Watts, and Yepremyan\(^{15}\) had a proof of Conjecture\(^{14}\) for all \(r\). Our work here is independent of theirs. This manuscript uses some ideas from \(^7\) such as reducing to left compressed and dense hypergraphs and the method may apply to \(r\)-graphs without a matching of any fixed size as in \(^7\) for \(r = 3\). The method can also be extended to prove Conjecture\(^{14}\) for \(r = 5\).

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