Tunnelling through black rings

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Abstract
Hawking radiation of black ring solutions to 5-dimensional Einstein-Maxwell-dilaton gravity theory is analyzed by use of the Parikh-Wilczek tunnelling method. To get the correct tunnelling amplitude and emission rate, we adopted and developed the Angheben-Nadalini-Vanzo-Zerbini covariant approach to cover the effects of rotation and electronic discharge all at once, and the effect of back reaction is also taken into account. This constitute a unified approach to the tunnelling problem. Provided the first law of thermodynamics for black rings holds, the emission rate is proportional to the exponential of the change of Bekenstein-Hawking entropy. Explicit calculation for black ring temperatures agree exactly with the results obtained via the classical surface gravity method and the quasilocal formalism.

1 Introduction

The study of black hole radiation has long been an attractive field of study in gravitational physics as well as in modern theoretical frameworks such as string/brane dynamics. The reason for the problem of black hole radiation keeps attracting attentions of theoretical physicists is partly due to the fact that it indicates black holes are highly excited quantum states and therefore can only be expected to be fully understood in terms of quantum gravity; on the other hand, the understanding of black hole radiation is the key to make the second law of thermodynamics in spacetimes involving black holes consistent. Traditional approaches for the black hole radiation involve studying of QFT in a (fixed) curved spacetime in which a gravitational collapse occurs which lead to a purely thermo spectrum [1]. A more recent recurrence of interests on the black hole radiation problem has appeared, with emphasis on the inclusion of back reaction or gravitational self-interaction of the emitted particles on the black hole geometry [2, 3, 4], which result in a non-thermo radiation spectrum. In particular, in [5], Keski-Vakkuri and Kraus obtained a remarkable relationship between the emission rate and the entropy loss of the black hole, which is interpreted as the effect of the particle’s tunnelling across the horizon. Such a relationship, and together with the corresponding non-thermo spectrum, are believed to be helpful in understanding the information loss paradox and the underlying unitarity problem. The tunnelling method is then purified and popularized by the work of Parikh and Wilczek [6, 7], which is now known in the literature as Parikh-Wilczek tunnelling method. A lot of works has already been carried out for further development of this approach [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], most of them being focused on the study of Hawking radiation process from various black hole spacetimes and a few exceptional examples are [21, 22] which discussed the consistency of the tunnelling approach with the classical laws of thermodynamics, [23], which considered
the Plank scale corrections to the radiation spectrum via tunnelling method, and which studied the corrections to the Cardy-Verlinder entropy due to the tunnelling effect.

In this work, we shall concentrate ourselves to the problem of Hawking radiation of a certain class of black ring spacetimes. Unlike the usual black hole spacetimes in 4-dimensions which are found almost right after the birth of Einstein’s general theory of relativity, black ring solutions are explicitly constructed very recently. One reason that such spacetimes were not found until recently is that they are purely higher dimensional objects which have no 4-dimensional analogues. The topology of their event horizons is also quite exceptional ($S^2 \times S^1$, as their name indicates), and perhaps the most striking feature that black ring solutions brought to us is their infinite non-uniqueness, i.e. there exists infinitely many different black ring solutions carrying the same mass, angular momentum and/or electronic charge. In short words, black rings are the first explicit example for the non-uniqueness of black hole spacetimes in higher dimensions. Moreover, it was found in that black ring spacetimes are actually some very special intersecting brane configurations in string theory frameworks, and as such they provide us with a new class of theoretical laboratories for testing the string theory methods in counting the microstates of black holes/rings. To this end we feel that is also interesting to study various properties of black ring spacetimes – including the thermodynamic properties and radiation properties – using the more traditional methods, with the hope to make cross tests with results from string theory and/or other approaches.

The paper is organized as follows. In section 2 we give a brief description of the black ring solutions we shall study and extract some of their common properties in order to make the forthcoming analysis in a more unified form. In section 3, we first describe the tunnelling method adapted for the black ring solutions, which is basically the method of Angheben, Nadalini, Vanzo and Zerbini (henceforth referred to as the ANVZ approach) for rotating black holes with modifications to include the effect of electronic discharge, and the effect of back reaction is also included according to . The explicit results for the black ring spacetimes outlined in section 2 is then given after the presentation of the general framework, and some discussion on the Kaluza-Klein cases are also presented. Section 4 contains some discussions and concluding remarks.

2 Black ring solutions of Einstein-Maxwell-dilaton gravity

A significant amount of black ring solutions have been found since the pioneering works. The black ring spacetimes we shall study are only a special subset of them, i.e. those which are solutions of the Einstein-Maxwell-Dilaton gravity model (EMD) in 5-dimensions, the action of which being

$$I = \frac{1}{16\pi G} \int d^5x \sqrt{g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{-\alpha \phi} F^2 \right).$$

Except that these solutions have no supersymmetry, they carry every typical properties of all the black ring solutions known to this date, e.g. they all have horizon(s) of $S^2 \times S^1$ topology, they all have 3 Killing coordinates which determine their local symmetries, they must either be rotating along the $S^1$ or carry electronic charge or local dipole charge to prevent them from collapsing into black holes with horizons of $S^3$ topology, etc. For historical reasons the black ring solutions are all formulated in terms of C-metric-like coordinates, i.e. the 5 spacetime coordinates are chosen such that

$$x^\mu = (t, y, \psi, x, \varphi),$$

where $t$ is the coordinate time, $-\infty < y \leq -1$ represent a radial coordinate, $-1 \leq x \leq 1$ and $\psi, \varphi$ are 3 different angular coordinates, and to avoid conical singularities, the angular
coordinates $\psi$ and $\varphi$ respectively must be given certain unusual periods. The 3 Killing coordinates are respectively $t, \psi$ and $\varphi$, however the vector $\partial_t$ does not necessarily be always timelike (when it doesn’t, it implies the existence of an ergo-region).

For later use, let us now outline each black ring solutions of the EMD theory explicitly. They are

- Neutral black ring (first obtained R. Emparan in [26]):

$$ ds^2 = -\frac{F(y)}{F(x)} \left( dt + C(\nu, \lambda) R \frac{1 + y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} \frac{F(x)}{F(y)} \left[ -\frac{G(y)}{G(x)} \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 \right], $$

where

$$ F(\xi) = 1 + \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 + \nu \xi), $$

$$ C(\nu, \lambda) = \sqrt{\lambda(\lambda - \nu) \frac{1 + \lambda}{1 - \lambda}}. $$

In order to avoid conical singularities at $x = -1$ and $y = -1$, the angular coordinates $\varphi$ and $\psi$ are chosen to have the periodicity

$$ \Delta \varphi = \Delta \psi = 2\pi \sqrt{\frac{1 - \lambda}{1 - \nu}}. $$

The parameters $\lambda, \nu$ are dimensionless and take values in the range $0 < \nu \leq \lambda < 1$, and to remove the conical singularity also at $x = 1$, $\lambda$ and $\nu$ must be related to each other via

$$ \lambda = \frac{2\nu}{1 + \nu^2}. $$

The metric has a killing horizon at $y = y_H = -1/\nu$, where the coefficient of $dy^2$ diverges.

- Dipole black ring (also found in [26]):

$$ ds^2 = -\frac{F(y)}{F(x)} \left( \frac{H(x)}{H(y)} \right)^{\frac{N}{3}} \left( dt + C(\nu, \lambda) R \frac{1 + y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} \frac{F(x)}{F(y)} \left( \frac{H(x)}{H(y)} \right)^{\frac{N}{3}} \left[ -\frac{G(y)}{G(x)} \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} \frac{H(x)}{H(y)} \frac{dx^2}{G(x)} \frac{d\varphi^2}{N} \right], $$

where

$$ H(\xi) = 1 - \mu \xi, \quad 0 \leq \mu < 1, $$

$$ \alpha^2 = 4 - \frac{4}{N}, $$

This metric accompanied by the following electromagnetic one form potential and dilaton field:

$$ A = \left( C(\nu, -\mu) \sqrt{N} R \frac{1 + y}{H(x)} + k \right) d\varphi, $$

$$ e^{-\phi} = \left( \frac{H(x)}{H(y)} \right)^{\frac{N\alpha}{2}}. $$
The case $N = 1$, or $\alpha = \sqrt{8/3}$, corresponds to Kaluza-Klein coupling.

The condition for removing conical singularities at $x = -1$ and $y = -1$ gives rise to
periods for the coordinates $\varphi$ and $\psi$:

$$\Delta \varphi = \Delta \psi = 4\pi \frac{F(-1)H(-1)^N}{G'(-1)} = 2\pi \sqrt{(1 - \lambda)(1 + \mu)^N},$$

while the elimination of conical singularity at $x = +1$ requires

$$\Delta \varphi = 4\pi \frac{F(+1)H(+1)^N}{G'(1)} = 2\pi \sqrt{(1 + \lambda)(1 - \mu)^N}.$$

The horizon is also located at $y = y_H = -1/\nu$. At $\mu = 0$, this solution degenerates into the neutral black ring just mentioned.

- Electronically charged black ring: (found in [30])

\[
\begin{align*}
    ds^2 &= -V_k(x,y)^{-2N/3} \frac{F(x)}{F(y)} dt^2 + V_k(x,y)^{N/3} \frac{R^2}{(x-y)^2} \\
    &\times \left[-F(x) \left((1-y^2)d\psi^2 + \frac{F(y)}{1-y^2} dy^2 \right) + F(y)^2 \left(\frac{dx^2}{1-x^2} + \frac{1-x^2}{F(x)}d\varphi^2 \right) \right], \\
    A &= V_k(x,y)^{-1} k \frac{1 - F(x)}{1 - k^2} dt, \\
    e^{-\phi} &= V_k(x,y)^{\alpha/2},
\end{align*}
\]

where $0 < \lambda < 1$ and

\[
\begin{align*}
    F(\xi) &= 1 - \lambda \xi, \\
    V_k(x,y) &= \frac{1 - k^2 \frac{F(x)}{F(y)}}{1 - k^2}.
\end{align*}
\]

The horizon is located at $y = y_H = -\infty$. To avoid conical singularities at $x = -1$ and $y = -1$, the coordinates $\varphi$ and $\psi$ must be chosen to have the periods

$$\Delta \varphi = \Delta \psi = 2\pi \sqrt{1 + \lambda}.$$

At $x = +1$, then, there will be inevitably a conical singularity. Alternatively we may demand regularity at $x = +1$ by requiring the period for $\varphi$ to be

$$\Delta \varphi = 2\pi \sqrt{1 - \lambda},$$

but then at $x = -1$ a conical singularity arises. So, conical singularities always exist in this spacetime.

- Dyonic black ring (only known at KK coupling [30]):

\[
\begin{align*}
    ds^2 &= -\frac{F(x)}{F(y)} \left(\frac{dt + R\sqrt{\lambda\nu}(1+y) d\psi}{V_k(x,y)^{2/3}} \right)^2 \\
    &\quad+ \frac{R^2}{(x-y)^2} V_k(x,y)^{1/3} \left[-F(x) \left(G(y)d\psi^2 + \frac{F(y)}{G(y)} dy^2 \right) + F(y)^2 \left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 \right) \right], \\
    A &= V_k(x,y)^{-1} k \left[\frac{1 - F(x)}{1 - k^2} dt - R\sqrt{\lambda\nu}(1+y) \frac{F(x)}{F(y)} d\psi \right], \\
    e^{-\phi} &= V_k(x,y)^{\sqrt{2/3}},
\end{align*}
\]
where \( G(\xi) = (1 - \xi^2)(1 - \nu(1 - k^2)\xi) \). At \( \nu = 0 \), this solution degenerates into the static electronically charged black ring solution.

The condition to remove conical singularities at \( x = -1 \) and \( y = -1 \) reads

\[
\Delta \varphi = \Delta \psi = 2\pi \sqrt{\frac{1 + \lambda}{1 + \nu(1 - k^2)}}.
\]

Similar requirement at \( x = +1 \) yields

\[
\Delta \varphi = 2\pi \sqrt{\frac{1 - \lambda}{1 - \nu(1 - k^2)}},
\]

so one is led to the condition

\[
\lambda = \nu(1 - k^2).
\]

One sees that this is impossible if \( \nu = 0 \) but \( \lambda \neq 0 \). Moreover, the horizon located at

\[
y = y_H = \frac{1}{\nu(1 - k^2)}
\]

does not degenerate to the one for the electronically charged black ring in the limit \( \nu = 0 \).

In the next section, we shall study the Parikh-Wilczek tunnelling from the above black ring spacetimes. For this purpose, the original form of the metrics are not the most pertinent ones. Actually, it is the ADM form of the metrics which is most appropriate for our purpose. From the explicit form of the metrics list above, it can be seen that they all can be casted into the following form,

\[
ds^2 = -A dt^2 + B^{-1} dy^2 + g_{\psi\psi}(d\psi + N^\psi dt)^2 + g_{xx} dx^2 + g_{\varphi\varphi} d\varphi^2,
\]

(1)

where of course the coefficient functions \( A, B, g_{xx}, g_{\psi\psi}, g_{\varphi\varphi} \) and \( N^\psi \) are all functions of \( x, y \) only and for each of the above metrics they have their own different values. Moreover, the coefficients in the ADM metrics obey the following unique property

\[
A(x, y_H) = B(x, y_H) = 0, \quad N^\psi(x, y_H) = \Omega_H,
\]

where \( y_H \) represents the value of \( y \) at the event horizon and \( \Omega_H \) is just the angular velocity of the black ring at the event horizon. The electromagnetic one form potentials associated with the above black ring solutions also have some common properties. For instance, the components \( A_y \) and \( A_x \) always vanish,

\[
A_{\mu} = (A_t, 0, A_{\psi}, 0, A_{\varphi}),
\]

(2)

where \( A_t = -\Phi \) is the scalar potential of the electro-magnetic field. These facts will be very useful when we study the tunnelling from the black rings.

\(^1\)Notice that the angular velocity at the horizon given in [26] differs from the \( \Omega_H \) here by a factor of \( \frac{\Delta \psi}{2\pi} \). This is because the angular coordinate \( \psi \) has the unusual period \( \Delta \psi \) in order to avoid conical singularities, but the angular velocity in [26] is measured by a rescaled angle \( \psi' \) which has normal period \( 2\pi \). Here we prefer to use the original coordinate \( \psi \) to define angular velocity, because it is the time derivative of this variable which appears in the expression for the null Killing vector \( \xi = k + \Omega_H m = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \psi} \).
3 The tunnelling method and its application to the black ring spacetimes

3.1 The tunnelling method

The tunnelling approach is based on the following simple idea. Assume a (neutral) relativistic particle is emitted (via a virtual process) from inside the horizon (of a static, non-rotating black hole) to infinity. In this process the particle is travelling backward in time, so the action of the particle must involve an imaginary part which represents the tunnelling amplitude, i.e.

\[ \text{Amplitude} \propto e^{iI}. \]

Accordingly, the emission rate obeys the relation

\[ \Gamma \propto |\text{Amplitude}|^2 \propto e^{-2\text{Im}I}. \]

This is further identified with the Boltzmann weight for the radiation

\[ \Gamma \propto e^{-\beta E}, \]

with \( E \) being the energy of the emitted particle. In this consideration the back reaction of the emitted particle on the black hole metric is ignored and hence the radiation spectrum is thermo. The Hawking temperature of the black hole is easy to be read off as \( T = \frac{1}{\beta} \). A more careful analysis must involve the back reaction of the emitted particle on the black hole metric, which modifies the dependence of the exponent of \( \Gamma \) on the particle’s energy \( E \), which in general becomes nonlinear (and hence non-thermo spectrum). So the key to the tunnelling approach is the evaluation of the imaginary part of the action of the virtual particle.

The simplest example of tunnelling process is the case of Schwarzschild black hole studied in Parikh and Wilczek’s work \[5\]. The metric in a non-singular coordinate (which was first obtained by Painlevé \[32\] and brought back to the modern audience by Kraus and Wilczek in \[33\]) reads

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dt \, dr + dr^2 + r^2 d\Omega^2. \]  

(3)

The radial null geodesics obey the following equation:

\[ \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}. \]

Emission of a particle of energy \( \omega \) results in a back reaction which effectively sees the metric as one with black hole mass \( M - \omega \). So the imaginary part of the action is

\[ \text{Im}I = \text{Im} \int_{r_{in}}^{r_{out}} p_r \, dr = \text{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r \, dr \]

\[ = \text{Im} \int_{M-\omega}^{M} \int_{r_{in}}^{r_{out}} \frac{dr}{r} dH = -\text{Im} \int_0^{\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega)}{r}}} d\omega \]

\[ = 4\pi \omega \left(M - \frac{\omega}{2}\right). \]

Therefore, the emission rate is given by

\[ \Gamma \propto e^{-2\text{Im}I} = e^{\Delta S_{BH}}, \]

where \( \Delta S_{BH} \) is the difference of the Bekenstein-Hawking entropy of the hole after and before the tunnelling process. This relation is exactly the Keski-Vakkuri-Kraus relation \[4\] mentioned in the introduction, which is repeatedly found to hold in all black hole spacetimes.
From this example one sees that the direct application of Parikh-Wilczek’s approach to black ring metrics has some difficulties, e.g. it is cumbersome to find a non-singular coordinate as in (3); it is difficult to determine the pure radial geodesics because of non-spherical symmetries; it is difficult to determine the effect of back reaction, which is needed to make the integration over \( \omega \) and it is difficult to consider the lose of angular momentum and/or electronic discharge, etc. Some of these difficulties may be overcome after some elaborate works. However it is more advantageous to address all the difficulties at once by introducing a unified covariant framework for studying the tunnelling process. Fortunately, such a covariant framework already exist for the cases of rotating black holes, i.e. the ANVZ approach \[14\]. In the next subsection we will slightly adapt the ANVZ approach to the charged rotating cases which we then apply to the black ring solutions listed in section 2. The effect of back reactions will also be taken into account following \[15\].

### 3.2 ANVZ approach for charged rotating black rings

The most efficient way to find the imaginary part of the emitted particle is to solve the particle’s relativistic Hamilton-Jacobi equation

\[
g^{\mu\nu}(\partial_\mu I - QA_\mu)(\partial_\nu I - QA_\nu) + m^2 = 0, \tag{5}
\]

where \( Q \) is the electronic charge of the particle and \( m \) is its stationary mass. Before solving this equation, let us stress again that what is relevant here is only the imaginary part of the action, which comes from the contribution of the particle moving from somewhere very near but inside the horizon (\( y = y_{in} = y_H - 0^+ \)) to somewhere very near but outside the horizon (\( y = y_{out} = y_H + 0^+ \)). Therefore, we need only to consider the local behavior of the metrics near the horizon. Under such assumptions, we may make a local change of coordinate

\[
\psi \rightarrow \chi = \psi + \Omega_H t
\]

and keep the other coordinates unchanged, so that after the transformation, the ADM form of the metric \( (1) \) behaves locally like a diagonal metric,

\[
ds^2 = -Adt^2 + B^{-1}dy^2 + g_{\psi\psi}d\chi^2 + g_{xx}dx^2 + g_{\phi\phi}d\phi^2.
\]

Note that the above metric is exact only at the horizon. However, we shall see that this is the only point we shall be dealing with, so the above approximation actually gives rise to exact results for the tunnelling amplitude and the Hawking temperature.

Now let us make the following ansatz for the solution \( I, \)

\[
I = -\mathcal{E}t + \mathcal{J}\chi + \mathcal{L}\varphi + W(x, y), \tag{6}
\]

where \( \mathcal{E}, \mathcal{J} \) and \( \mathcal{L} \) are all real constants which represents the energy and angular momenta with respect to the angles \( \chi \) and \( \varphi \) respectively. It is clear that changing back into the coordinate \( \psi \) would only effectively change \( \mathcal{E} \) into \( \mathcal{E} - \Omega_H J \), i.e.

\[
I = -(\mathcal{E} - \Omega_H J)t + \mathcal{J}\psi + \mathcal{L}\varphi + W(x, y).
\]

Inserting this into \( (5) \), we get

\[
\frac{\partial}{\partial y} W(x, y) \simeq \frac{1}{\sqrt{AB}} \sqrt{(\mathcal{E} - \Omega_H J + QA_\eta)^2 - A(m^2 + \Psi(x, y))},
\]

\[
\Psi(x, y) \equiv (g_{xx})^{-1} W, (x, y)^2 + (g_{\phi\phi})^{-1} (\mathcal{L} - QA_\varphi)^2 + (g_{\psi\psi})^{-1} (J - QA_\psi)^2,
\]

where \( \simeq \) stands for “equals near the horizon”. Now limiting to the s-wave contribution, i.e. suppressing the angular contribution \( \Psi(x, y) \), the imaginary part of the action can be
written as
\[
\text{Im} I = \text{Im} W(x, y) \simeq \text{Im} \left[ \int_{y_{\text{in}}}^{y_{\text{out}}} dy \frac{\partial}{\partial y} W(x, y) \right]_{\Psi(x, y) \to 0} \\
= \text{Im} \left[ \int_{y_{\text{in}}}^{y_{\text{out}}} dy \frac{1}{\sqrt{AB}} (E - \Omega_H \mathcal{J} + Q A_t)^2 - A m^2 \right].
\]

The imaginary part of \( W(x, y) \) arises from the pole contribution of the factor \( \frac{1}{\sqrt{AB}} \) at \( y = y_H \). However, direct evaluation of the pole contribution is highly coordinate dependent because of lack of covariance. The correct way to make the integration is to make use of the proper spacial distance (this is where ANVZ make their important observation)
\[
d\sigma^2 = B^{-1}(x, y) dy^2 + g_{\psi\psi}(d\psi + \Omega_H dt)^2 + g_{xx} dx^2 + g_{\varphi\varphi} d\varphi^2.
\]

Under the \( s \)-wave approximation, we have
\[
\sigma = \int \frac{dy}{\sqrt{B(x, y)}}.
\]
In the near horizon limit, we get
\[
\sigma = \frac{2}{\sqrt{B_{,y}(x, y_H)}} \sqrt{y - y_H} + ... \\
\frac{1}{\sqrt{A(x, y)}} = \frac{2}{\sqrt{A_{,y}(x, y_H) B_{,y}(x, y_H)}} \frac{1}{\sigma} + ...
\]
Therefore,
\[
\text{Im} I = \frac{2}{\sqrt{A_{,y}(x, y_H) B_{,y}(x, y_H)}} \text{Im} \int d\sigma \frac{1}{\sigma} (E - \Omega_H \mathcal{J} + Q A_t).
\]

The integration on the RHS must be evaluated along a virtual path starting from a small negative value of \( \sigma \) to some positive value of \( \sigma \). Thus the pole contribution from the integrand \( \frac{1}{\sigma} \) is imaginary and can be obtained by slightly deforming the integration contour from the real \( \sigma \)-axis to the lower complex \( \sigma \)-plane which avoids the pole \( \sigma = 0 \) counterclockwise. In the end,
\[
\text{Im} I = \frac{2}{\sqrt{A_{,y}(x, y_H) B_{,y}(x, y_H)}} \times \frac{1}{2} \text{res}_{\sigma = 0} \left( \frac{E - \Omega_H \mathcal{J} + Q A_t}{\sigma} \right) \\
= \frac{2 \pi (E - \Omega_H \mathcal{J} - \Phi_H dQ)}{\sqrt{A_{,y}(x, y_H) B_{,y}(x, y_H)}}.
\]

Note that the above is only the contribution from a single emission particle and the back reaction of the particle on the black ring metric is not taken into account. For accumulated emission of energy \( E \) and angular momentum \( \mathcal{J} \), one changes the last formula into
\[
\text{Im} I = \frac{2 \pi}{\sqrt{A_{,y}(x, y_H) B_{,y}(x, y_H)}} \int (dE - \Omega_H d\mathcal{J} - \Phi_H dQ).
\]
The emission rate is then
\[
\Gamma \propto \exp \left( -2\text{Im} I \right) = \exp \left( -\beta \int (dE - \Omega_H d\mathcal{J} - \Phi_H dQ) \right).
\]
From the last formula one reads out the expression for the inverse temperature:
\[
\beta = \frac{4 \pi}{\sqrt{A_{,y}(x, y_H) B_{,y}(x, y_H)}} = \frac{1}{T}.
\]
Note that there are diverse proofs and/or verifications for the correctness of the first law of thermodynamics for black rings \[26, 34, 35\], i.e.

\[dE = T dS_{BH} + \Omega_H dJ + \Phi_H dQ \Rightarrow dS_{BH} = \frac{dE - \Omega_H dJ - \Phi_H dQ}{T},\]

where \(dE = -d\mathcal{E}, dJ = -d\mathcal{J}, dQ = -d\mathcal{Q}\), which are required by the conservation of mass, angular momentum and electronic charge. So, the emission rate can also be expressed as

\[\Gamma \propto e^{\int dS_{BH}} = e^{\Delta S_{BH}},\]

which is exactly the Keski-Vakkuri-Kraus relation first obtained in \[4\]. The present result add some more evidence on the universality of this formula. Note that the effect of electronic discharge as well as the back reaction of the emitted particles are both included in the above considerations.

### 3.3 Explicit results for black rings

The construction made in the last subsection has led to a very simple formula (8) for calculating the Hawking temperature of black rings. Now we apply this formula to the black ring metrics listed in section 2 to get the explicit results.

- Neutral black ring:

| \[A(x, y) = \frac{F(y)}{F(x)} \left(1 - \frac{1}{(x-y)^2} F(x)^2 G(y) + C(\nu, \lambda)^2 (1 + y)^2 \right)\] |
| \[B(x, y) = -\left(\frac{R^2}{(x-y)^2} F(y)\right)^{-1}\] |

\[\Rightarrow T = \frac{1}{4\pi R} \frac{1 + \nu}{\nu^{1/2}} \sqrt{\frac{1 - \lambda}{\lambda(1 + \lambda)}}\]

- Dipole black ring:

| \[A(x, y) = \frac{F(y)}{F(x)} \left(\frac{H(x)}{H(y)}\right)^{N/3} \left(1 - \frac{1}{(x-y)^2} F(x)^2 G(y) + C(\nu, \lambda)^2 (1 + y)^2 \right)\] |
| \[B(x, y) = -\left(\frac{R^2}{(x-y)^2} (H(x)H(y)^2)^{N/3} F(x) G(y)\right)^{-1}\] |

\[\Rightarrow T = \frac{1}{4\pi R} \frac{\nu^{(N-1)/2}(1 + \nu)}{(\mu + \nu)^{N/2}} \sqrt{\frac{1 - \lambda}{\lambda(1 + \lambda)}}\]

- Electronically charged:

\[T = \frac{1}{4\pi R \lambda} (1 - k^2)^{1/2},\]

- Dyonic:

\[T = \frac{1 - \nu(1 - k^2)}{4\pi R} \sqrt{1 - k^2 \frac{\lambda^2}{\lambda(1 - \nu(1 - k^2))}}.\]

All these results agree exactly with the surface gravity results presented in the original papers \[28, 30\]. For the dipole black ring the same Hawking temperature has also been obtained in \[34\] using the so-called quasilocal formalism.
3.4 Hawking radiation at the Kaluza-Klein coupling

Before ending the present section, let us make some further remarks on the black ring radiation at the Kaluza-Klein coupling $\alpha = \sqrt{\frac{8}{3}}$. In such cases, the tunnelling process can also be studied in the framework of 6-dimensional pure Einstein gravity theory. The Hamilton-Jacobi equation for the emitted particle can be written as

$$G_{MN} \partial_M I \partial_N I + m^2 = 0,$$

(9)

where $G_{MN}$ is given by the 6-dimensional Kaluza-Klein metric

$$ds^2_6 = e^{\sqrt{\frac{1}{6}\phi}} ds^2_5 + e^{-\sqrt{\frac{3}{2}\phi}} (dz + A)^2,$$

(10)

where $ds^2_5$ corresponds to the 5-dimensional black ring metric, $A = A_\mu dx^\mu$ is the electromagnetic one form potential and $z$ is the 6-th spacetime coordinate.

Now the ansatz (6) for the solution is (locally) changed into ($\zeta \equiv z - \Phi_H t$)

$$I = -\tilde{E} t + \mathcal{J} \chi + \mathcal{L} \varphi - \mathcal{Q} \zeta + W(x, y)$$

$$= -(\mathcal{E} - \Omega_H \mathcal{J} - \Phi_H \mathcal{Q}) t + \mathcal{J} \psi + \mathcal{L} \varphi - \mathcal{Q} z + W(x, y).$$

(11)

It is crucial to point out that in the 6-d metric (10), if $ds^2_5$ is written in the ADM form, then $ds^2_6$ is automatically an ADM metric, and the coefficient functions $A(x, y)$ and $B(x, y)$ undergo the following transformation:

$$A(x, y) \rightarrow e^{\sqrt{\frac{1}{6}\phi}(x, y)} A(x, y), \quad B(x, y) \rightarrow e^{-\sqrt{\frac{1}{6}\phi}(x, y)} B(x, y).$$

Therefore, inserting the ansatz (11) into the Hamilton-Jacobi equation (9) gives rise to the same solution (7) as in the 5-dimensional cases because the product $A(x, y)B(x, y)$ is kept unchanged. Notice that $\mathcal{Q}$ here is to be understood as the angular momentum which generates rotations along the $z$ direction. That the imaginary part of the emitted particle (and hence the Hawking temperature) is the same from both the 5-dimensional and 6-dimensional perspectives is not difficult to understand. Actually, since the Kaluza-Klein metric (10) contains only the zero KK mode, the tunnelling particles only escape from the black rings and run into the 5-dimensional sub-spacetime described by $ds^2_5$. If there were nonzero KK modes involved, the Hawking temperatures of from the 5-d and 6-d perspectives would have been different.

4 Discussions and concluding remarks

In this paper, we studied the problem of Hawking radiation from black ring solutions of the 5-dimensional EMD gravity theory. It is shown that provided the first law of thermodynamics for black rings is correct, the emission rate is related to the loss of Bekenstein-Hawking entropy of the black rings during the radiation. This result is in agreement with earlier studies of Hawking radiation in terms of tunnelling for other types of black holes and it seems to be a universal property for the tunnelling process. We also obtained a simple formula for calculating black ring temperatures, which leads to exactly the same result as compared to the classical surface gravity method and the quasilocal formalism. The method we used can be applied straightforwardly to the cases of other black rings/holes.

Since black rings are a new class of black spacetime solutions, it seems interesting to make further analysis on their various properties, especially the result of microstate counting using string theory methods is still to be compared with (an analysis on microstate counting for the dipole black ring to leading order is already made in [26]). Also, the stability analysis, both from dynamical and thermodynamical perspectives, for the black rings is also an interesting field of further study.
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