L∞-algebra of an unobstructed deformation functor

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Abstract

This is a comment on the Kuranishi method of constructing analytic deformation spaces. It is based on a simple observation that the Kuranishi map can always be inverted in the category of L∞-algebras. The L∞-structure obtained by this inversion is used to define an "unobstructed" deformation functor which is always representable by a smooth pointed moduli space.

The singular nature of the original Kuranishi deformation space emerges in this setting merely as a result of the truncation of this "naive" L∞-algebra controlling the deformations to a usual differential Lie algebra.

1 Introduction

1.0. Typical moduli problems in geometry are plagued with obstructions and associated singularities. It is noticed in this paper that, in the situations when the Kuranishi method of constructing an analytic deformation space applies, there is always a "naive" way of bypassing (rather than overcoming) the obstructions by suitably extending the deformation problem from the categories of associative or Lie algebras to the associated strong homotopy versions. The resulting moduli spaces are always smooth.

The singular nature of the original Kuranishi deformation space emerges in this setting merely as a result of the truncation of the "non-obstructed" L∞-algebra controlling the deformations to a usual differential Lie algebra.

The paper is based on a simple observation that, roughly speaking, the Kuranishi map can always be inverted in the category of L∞-algebras.

To make the above statements more clear, we shall remind in the next subsection a few facts about the Deformation Theory, and then formulate the main results again.

1.1. The deformation functor. The standard approach to constructing the analytic deformation space of a given mathematical structure A consists of two key steps [4, 11, 16, 17, 18]:

1) Associate to A a "controlling" differential graded Lie algebra (g = Ωk∈Z g^k, d, [, ]) over a field k (which is usually R or C) and define the deformation functor

Def^0_B : \begin{cases} 
\text{the category of Artin } \\
\text{k-local algebras } B \\
\text{with maximal ideals } m_B 
\end{cases} \rightarrow \{ \text{the category of sets} \}

more precisely, quasi-isomorphic to a differential Abelian Lie algebra
as follows
\[ \text{Def}^0_\mathfrak{g}(\mathcal{B}) = \left\{ \Gamma \in C(\mathfrak{g} \otimes m_B)^1 \mid d\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0 \right\}, \]
where \(C(\mathfrak{g} \otimes m_B)^1\) is a complement to the 1-coboundaries, \(d(\mathfrak{g} \otimes m_B)^0\), in \((\mathfrak{g} \otimes m_B)^1\) and \(\mathcal{B}\) is viewed as a \(\mathbb{Z}\)-graded algebra concentrated in degree zero (so that \((\mathfrak{g} \otimes m_B)^i = \mathfrak{g}^i \otimes m_B)\).

2) Obtain a Hodge theory on \(\mathfrak{g}\) and apply the Kuranishi construction to represent the deformation functor by a germ, \(O_p\), of the structure sheaf on a pointed analytic space \((\mathcal{K}_\mathfrak{g}^0, p \in \mathcal{K}_\mathfrak{g}^0)\).

The basic examples are the differential graded algebras \((TM \otimes \Omega^{0,d} M, \partial, \text{Schouten bracket})\) and \((E \otimes E^* \otimes \Omega^{0,d} M, \bar{\partial}, \text{standard bracket})\). The first controls deformations of a given complex structure on a smooth manifold \(M\), while the second controls deformations of holomorphic structures on a given complex vector bundle \(E \rightarrow M\).

The tangent space, \(\text{Def}^0_\mathfrak{g}(k[\varepsilon])\), to the functor \(\text{Def}^0_\mathfrak{g}\) is isomorphic to the first cohomology group \(H^1(\mathfrak{g})\) of the complex \((\mathfrak{g}, d)\). If one extends in the obvious way the above deformation functor to the category of arbitrary \(\mathbb{Z}\)-graded \(k\)-local Artin algebras (which may not be concentrated in degree 0), one gets the functor \(\text{Def}^Z_\mathfrak{g}\) with the tangent space isomorphic to the full cohomology group \(H^*(\mathfrak{g})\). Moreover, if \((\mathfrak{g}, d, [\ , \ ]\) happens to be formal, one can construct an associated Kuranishi moduli space \(\mathcal{K}_\mathfrak{g}^Z\) representing \(\text{Def}^Z_\mathfrak{g}\) [8].

This extended deformation functor \(\text{Def}^Z_\mathfrak{g}\) has been used recently, in the mirror symmetry context, by Barannikov and Kontsevich [4, 3] in constructing the smooth extended moduli space of complex structures on a Calabi-Yau manifold. According to Kontsevich’s homological mirror symmetry conjecture [11], the “extended holomorphic deformations” of a complex vector bundle on a Calabi-Yau manifold may also play an important role in mirror symmetry (cf. [22]). The extended moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold was constructed in [18].

1.2. Obstructions. If the Lie algebra \((\mathfrak{g}, d, [\ , \ ])\) is such that \(H^2(\mathfrak{g}) \neq 0\), then the associated deformation functor \(\text{Def}^0_\mathfrak{g}\) is usually obstructed in the sense that the local moduli space \(\mathcal{K}_\mathfrak{g}^0\) has singularities.

On the other hand, if the Lie algebra \((\mathfrak{g}, d, [\ , \ ])\) is formal but the induced Lie bracket on its cohomology,
\[ [\ , \] : H^*(\mathfrak{g}) \times H^*(\mathfrak{g}) \rightarrow H^*(\mathfrak{g}), \tag{1} \]
is non-vanishing, the same problem plagues the extended deformation functor \(\text{Def}^Z_\mathfrak{g}\) — the associated Kuranishi moduli space \(\mathcal{K}_\mathfrak{g}^Z\) is singular.

1.3. Bypassing the obstructions. It is shown in Sect. 3 of this paper that the Hodge theory on \((\mathfrak{g}, d, [\ , \ ])\) gives rise canonically to the structure of a \(L_\infty\)-algebra on the vector space \(\mathfrak{g}\), i.e. to a set of linear maps \(\mu_n : \Lambda^n\mathfrak{g} \rightarrow \mathfrak{g}[2-n]\) satisfying the higher Jacobi identities. We use this structure to define a modified deformation functor,
\[ \text{MDef}^0_\mathfrak{g}(\mathcal{B}) = \left\{ \Gamma \in C(\mathfrak{g} \otimes m_B)^1 \mid \sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2}}{k!} \mu_k(\Gamma, \ldots, \Gamma) = 0 \right\}, \]
which associates to a concentrated in degree zero $\mathbb{Z}$-graded Artin $\mathbb{F}$ algebra $B$ with the maximal ideal $m_B$ a set of solutions to the Maurer-Cartan equations in the $L_\infty$ algebra $(g \otimes m_B, \mu_\ast)$. Here $C(g \otimes m_B)^1$ is a complement to $d((g \otimes m_B)^0)$ in $(g \otimes m_B)^1$ (alternatively, we may replace above the subspace $C(g \otimes m_B)^1$ by $(g \otimes m_B)^1$ and take the quotient by a natural gauge equivalence; this change is possible because $(g, \mu_\ast)$ is quasi-isomorphic to a differential Abelian Lie algebra).

Analogously, one defines the extended deformation functor $M\text{Def}_g^Z$ (by simply omitting the words concentrated in degree zero in the sentence above).

The main results of the paper are:

(i) The functor $M\text{Def}_g^0$ is always representable by an analytic germ, $O_p$, of a smooth pointed moduli space $(\mathcal{M}_g^0, p \in \mathcal{M}_g^0)$; the tangent space to $\mathcal{M}_g^0$ at $p$ is canonically isomorphic to $H^1(g)$.

(ii) The original Kuranishi moduli space $\mathcal{K}_g^0$ space arises in this setting as an analytic subspace in $\mathcal{M}_g^0$ where both functors, the classical one, $\text{Def}_g^0$, and the modified one, $M\text{Def}_g^0$, agree.

If the obstructions vanish, $H^2(g) = 0$, then $M\text{Def}_g^0 \simeq \text{Def}_g^0$ and $\mathcal{K}_g^0 \simeq \mathcal{M}_g^0$.

(iii) The functor $M\text{Def}_g^Z$ is always representable by an analytic germ, $O_p$, of a pointed smooth moduli superspace $(\mathcal{M}_g^Z, \mu \in \mathcal{M}_g^Z)$; the tangent space to $\mathcal{M}_g^Z$ at $p$ is canonically isomorphic to $H^1(g)$.

(iv) If the differential Lie superalgebra $(g, d, [\cdot, \cdot])$ is formal and the obstruction map $[\frac{d}{d\mu}]$ vanishes, then $M\text{Def}_g^Z \simeq \text{Def}_g^Z$ and $\mathcal{M}_g^Z \simeq \mathcal{K}_g^Z$.

The paper is organized as follows. In Sect. 2 we give a brief introduction into the theory of $L_\infty$-algebras and list a few necessary facts about Maurer-Cartan equations. In Sect. 3 we construct a particular $L_\infty$-algebra used in the definition of the functor and prove the main results (i)-(iv) stated above.

A few words about notations. The category $\mathbb{Z}$-graded vector spaces over a field $k$ contains an object $[1] = \bigoplus_{i \in \mathbb{Z}} [1]^i$ defined by

$$[1]^i = \begin{cases} k & \text{if } i = -1, \\ 0 & \text{if } i \neq -1. \end{cases}$$

Its tensor powers are denoted by $[n]$, and the tensor product of a graded vector space $g = \bigoplus_{i \in \mathbb{Z}} g^i$ with $[n]$ is denoted by $g[n]$. A homogeneous vector $v \in g$ viewed as an element of $g[n]$ is denoted by $v[n]$.

For a homogeneous element $v \in g^i \subset g$ we write $\tilde{v} := i \mod 2\mathbb{Z} \in \mathbb{Z}_2$. Analogously, for an integer $n \in \mathbb{Z}$ we write $\tilde{n} = n \mod 2\mathbb{Z}$.  

\footnote{Our apologies for this awkward series of adjectives, but its truncation leads to a very different object, see below.}

\footnote{With $M\text{Def}_g^0$ one can canonically associate a truncated deformation functor, $m\text{Def}_g^0(B) := \{ \Gamma \in C(g \otimes m_B)^1 \mid d\Gamma + \frac{1}{2} \mu_2(\Gamma, \Gamma) = 0, \mu_{n \geq 3}(\Gamma, \ldots, \Gamma) = 0 \}$, which is representable by a pointed analytic subspace $(M_{g}^0, p) \subset (\mathcal{M}_g^0, p)$; the word agree in the text means that $m\text{Def}_g^0 \simeq \text{Def}_g^0$ and $M_{g}^0 \simeq K_g^0$.}
If one is interested in the extended deformations functors $\text{Def}_g^Z$ or $\text{MDef}_g^Z$ only, the $\mathbb{Z}$-grading above and below can be safely replaced by the associated $\mathbb{Z}_2$-grading — no essential information will be lost.

2 Strong homotopy algebras and Maurer-Cartan equations

2.1. $L_\infty$-algebras. A strong homotopy Lie algebra, or shortly $L_\infty$-algebra, is by definition a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ equipped with linear maps,

$$\mu_k : \Lambda^k \mathfrak{g} \rightarrow \mathfrak{g}[2 - k]$$

$$v_1 \otimes \ldots \otimes v_k \mapsto \mu_k(v_1, \ldots, v_k), \quad k \geq 1,$$

satisfying, for any $n \geq 1$ and any $v_1, \ldots, v_n \in V$, the following higher order Jacobi identities,

$$\sum_{k+l=n+1} \sum_{\sigma \in \text{Sh}(k,n)} (-1)^{\bar{\sigma} + k + l - 1} e(\sigma; v_1, \ldots, v_n) \mu_l \left( \mu_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(n)} \right) = 0,$$

where $\text{Sh}(k,n)$ is the set of all permutations $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ which satisfy $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k + 1) < \ldots < \sigma(n)$. The symbol $e(\sigma; v_1, \ldots, v_n)$ (which we abbreviate from now on to $e(\sigma)$) stands for the Koszul sign defined by the equality

$$v_{\sigma(1)} \wedge \ldots v_{\sigma(n)} = (-1)^{\bar{\sigma}} e(\sigma) v_1 \wedge \ldots \wedge v_n,$$

$\bar{\sigma}$ being the parity of the permutation $\sigma$.

This notion and the associated notion of $A_\infty$-algebra (reminded below) are due to Stasheff \cite{19, 20, 21}.

The first three higher order Jacobi identities have the form

$n = 1$: \quad $d^2 = 0$,

$n = 2$: \quad $d[v_1, v_2] = [dv_1, v_2] + (-1)^{\bar{\nu}_1} [v_1, dv_2]$,

$n = 3$: \quad $[[v_1, v_2], v_3] + (-1)^{\bar{\nu}_1 + \bar{\nu}_2 + \bar{\nu}_3} [v_3, [v_1, v_2]] + (-1)^{\bar{\nu}_1 + \bar{\nu}_2 + \bar{\nu}_3} [v_2, v_3], v_1] = -d\mu_3(v_1, v_2, v_3) - \mu_3(dv_1, v_2, v_3) - (-1)^{\bar{\nu}_1 + \bar{\nu}_2} \mu_3(v_1, dv_2, v_3) - (-1)^{\bar{\nu}_1 + \bar{\nu}_2} \mu_3(v_1, v_2, dv_3),$

where we denoted $dv_1 := \mu_1(v_1)$ and $[v_1, v_2] := \mu_2(v_1, v_2)$.

Therefore $L_\infty$-algebras with $\mu_k = 0$ for $k \geq 3$ are nothing but the usual graded differential Lie algebras with the differential $\mu_1$ and the Lie bracket given by $\mu_2$. If, furthermore, $\mu_1 = 0$, one gets the class of usual graded Lie algebras.
2.2. Another (conceptually better) definition of \( L_\infty \)-algebra. Let \( W \) be a \( \mathbb{Z} \)-graded vector space and let \( \circ^*W = \bigoplus_{n=1}^{\infty} \circ^nW \) be the associated symmetric tensor algebra with its induced \( \mathbb{Z} \)-grading. We equip \( \circ^*W \) with the structure of cosymmetric coalgebra by setting

\[
\Delta(w_1 \circ \ldots \circ w_n) = \sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n)} e(\sigma) \left( w_{\sigma(1)} \circ \ldots \circ w_{\sigma(i)} \right) \otimes \left( w_{\sigma(i+1)} \circ \ldots \circ w_{\sigma(n)} \right).
\]

2.2.1. Fact [14]. A \( L_\infty \)-algebra structure on a graded vector space \( \mathfrak{g} \) is equivalent to a codifferential on the coalgebra \((\circ^*(\mathfrak{g}[1]), \Delta)\), i.e. to a linear map

\[
Q : \circ^*(\mathfrak{g}[1]) \longrightarrow (\circ^*(\mathfrak{g}[1]))[1]
\]

satisfying the conditions

(i) \( \Delta \circ Q = (Q \otimes \text{Id} + \text{Id} \otimes Q) \circ \Delta \),

(ii) \( Q^2 = 0 \).

The first condition simply says that \( Q \) is a coderivation of the coalgebra \((\circ^*(\mathfrak{g}[1]), \Delta)\) and hence is completely determined by the compositions (“values in cogenerators”),

\[
\hat{\mu}_n : \circ^n(\mathfrak{g}[1]) \longrightarrow \circ^*(\mathfrak{g}[1]) \longrightarrow (\circ^*(\mathfrak{g}[1]))[1] \longrightarrow \mathfrak{g}[2].
\]

for all \( n \geq 1 \). Then the second conditions imposes an (infinite, in general) set of quadratic equations for these tensor maps \( \hat{\mu}_n \).

The natural isomorphism

\[
\circ^n(\mathfrak{g}[1]) \simeq (\Lambda^* \mathfrak{g})[n]
\]

identifies the maps \( \hat{\mu}_n : \circ^n(\mathfrak{g}[1]) \rightarrow \mathfrak{g}[2] \) with the maps \( \mu_n : \Lambda^n \mathfrak{g} \rightarrow \mathfrak{g}[2-n] \),

\[
\hat{\mu}(v_1[1], \ldots, v_n[1]) = (-1)^{\sum_{i=1}^{n}(n-i)e_i+n} \mu_n(v_1, \ldots, v_n)[n].
\]

The homological condition \( Q^2 = 0 \) translates then precisely into the higher Jacobi identities (3) (see [14] for the proof).

2.3. \( L_\infty \)-morphisms. Given two \( L_\infty \)-algebras, \((\mathfrak{g}, \mu_*)\) and \((\mathfrak{g}', \mu'_*)\). A \( L_\infty \)-morphism \( F \) from the first one to the second is, by definition, a differential coalgebra homomorphism

\[
F : (\circ^*(\mathfrak{g}[1]), \Delta, Q) \longrightarrow (\circ^*(\mathfrak{g}'[1]), \Delta, Q').
\]

It is completely determined by a set of linear maps \( \hat{F}_n : \circ^n(\mathfrak{g}[1]) \rightarrow \mathfrak{g}'[1] \) (or, equivalently, by \( F_n : \Lambda^n \mathfrak{g} \rightarrow \mathfrak{g}'[1-n] \)) satisfying an (infinite, in general) set of equations. In the case when \((\mathfrak{g}', \mu'_n)\) is a differential Abelian Lie algebra (i.e. \( \mu'_n = 0 \) for \( n \geq 2 \)) these equations take the form (3)

\[
dF_n(v_1, \ldots, v_n) = \sum_{k+l=n+1} \sum_{\sigma \in Sh(k,n)} (-1)^{\sigma + k(l-1)} e(\sigma) F_l \left( \mu_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+l)}, \ldots, v_{\sigma(n)} \right),
\]

(3)
where we denoted \( d = \mu' \).

A \( L_\infty \)-morphism \( F : (\mathfrak{g}, \mu_\ast) \rightarrow (\mathfrak{g}', \mu'_\ast) \) is called a quasi-isomorphism if its first component \( F_1 : \mathfrak{g} \rightarrow \mathfrak{g}' \) induces an isomorphism between cohomology groups of complexes \((\mathfrak{g}, \mu_1)\) and \((\mathfrak{g}', \mu'_1)\). It is called a \( L_\infty \)-isomorphism, if \( F_1 : \mathfrak{g} \rightarrow \mathfrak{h} \) is an isomorphism of graded vector spaces.

A differential Lie algebra \((\mathfrak{g}, d, [, , ])\) is called formal if it is quasi-isomorphic to its cohomology \((H^*(\mathfrak{g}), 0, [, , ])\).

2.3.1. Fact \([12]\). If the \( L_\infty \)-morphism \( F : (\mathfrak{g}, \mu_\ast) \rightarrow (\mathfrak{g}', \mu'_\ast) \) is a quasi-isomorphism, then there exists a \( L_\infty \)-morphism \( F' : (\mathfrak{g}', \mu'_\ast) \rightarrow (\mathfrak{g}, \mu_\ast) \) which induces the inverse isomorphism between cohomology groups of complexes \((\mathfrak{g}, \mu_1)\) and \((\mathfrak{g}', \mu'_1)\).

A similar statement holds true for \( L_\infty \)-isomorphisms.

2.4. Maurer-Cartan equations. Let \( (\mathfrak{g}, \mu_\ast) \) be a \( L_\infty \)-algebra. A subset \( \mathcal{MC}(\mathfrak{g}) \subset \mathfrak{g} \) of solutions to Maurer-Cartan equations is defined formally as follows:\(^4\)

\[
\mathcal{MC}(\mathfrak{g}) = \left\{ \Gamma \in \mathfrak{g}^1 \mid \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)/2}}{k!} \mu_k(\Gamma, \ldots, \Gamma) = 0 \right\}
\]

\[
= \left\{ \Gamma \in \mathfrak{g}^1 \mid d\Gamma + \frac{1}{2!} \mu_2(\Gamma, \Gamma) - \frac{1}{3!} \mu_3(\Gamma, \Gamma, \Gamma) - \frac{1}{4!} \mu_4(\Gamma, \Gamma, \Gamma, \Gamma) + \ldots = 0 \right\}.
\]

If \( \mu_n = 0 \) for \( n \geq 3 \), the equation on the r.h.s. reduces to the standard Maurer-Cartan equation in a differential Lie algebra.

2.4.1. Fact \([8, 12]\). Let \( F : (\mathfrak{g}, \mu_\ast) \rightarrow (\mathfrak{g}', \mu'_\ast) \) be a \( L_\infty \)-morphism between two \( L_\infty \)-algebras. If \( F \) is such that \( F_1 \) provides an isomorphism of complexes \((\mathfrak{g}, \mu_1)\) and \((\mathfrak{g}', \mu'_1)\), then \( \mathcal{MC}(\mathfrak{g}) = \mathcal{MC}(\mathfrak{g}') \). In particular, the deformation functors \( \text{MDef}_{\mathfrak{g}} \) and \( \text{MDef}_{\mathfrak{g}'} \) are equivalent. Moreover, the last statement remains true under a weaker assumption that \( F \) is a quasi-isomorphism.

2.5. A geometric interpretation of a \( L_\infty \)-algebra. The dual of the free cocommutative coalgebra \( \odot^*(\mathfrak{g}[1]) \) can be identified with the algebra of formal power series on the vector superspace \( \mathfrak{g}[1] \) viewed as a formal \( \mathbb{Z} \)-graded supermanifold (to emphasize this change of thought we denote the supermanifold structure on \( \mathfrak{g}[1] \) by \( M_{\mathfrak{g}[1]} \)). With this identification the codifferential \( Q \) on \( \odot^*(\mathfrak{g}[1]) \) goes into a degree +1 vector field \( Q \) on \( M_{\mathfrak{g}[1]} \) satisfying the following two conditions \([1, 12]\)

a) \( Q^2 = 0 \);

b) \( Q|_0 = 0 \).

The set \( \mathcal{MC}(\mathfrak{g}) \) is then precisely the subset in \( M_{\mathfrak{g}[1]} \) where \( Q \) vanishes \([12]\). A \( L_\infty \)-morphism \( F \) between two \( L_\infty \)-algebras \((\mathfrak{g}, Q)\) and \((\mathfrak{g}', Q')\) is nothing but a \( Q \)-equivariant map between pointed formal graded supermanifolds \((M_{\mathfrak{g}[1]}, 0)\) and \((M_{\mathfrak{g}'[1]}, 0)\). In this setting

\(^4\)In our context \( \mathcal{MC} \) is applied only to \( L_\infty \)-algebras of the form \( \mathfrak{g} \otimes m \), where \( m \) is the maximal ideal of an Artin algebra (or its completion); hence no convergence problem arises.
the Fact 2.4.1 becomes very transparent: the morphism $F$ evidently maps zeros of $Q$ into zeros of $Q'$.

Since $Q^2 = 0$, for any vector field $\alpha$ on $M_{g[1]}$ the associated vector field $[Q, \alpha]_{MC(g)}$ is tangent to $\mathcal{MC}(g)$. Moreover, since

$$[[Q, \alpha], [Q, \beta]] = [Q, [\alpha, [Q, \beta]]],$$

such vector fields form an integrable distribution on $\mathcal{MC}(g)$ \cite{12}. The leaves of this distribution define a natural gauge equivalence on $\mathcal{MC}(g)$.

We refer to the nice exposition of Kontsevich \cite{12} for more details on this geometrical model.

2.6. **Γ-deformed $L_\infty$-structure.** Let $(g, \mu_\ast)$, or equivalently $(g, Q)$, be a $L_\infty$-structure on a graded vector space $g$. There is an odd (more precisely, degree +1) linear morphism

$$\Psi : g \longrightarrow \Gamma(M_{g[1]}, TM_{g[1]})$$

identifying elements of $g$ with constant vector fields on the supermanifold $M_{g[1]}$. In particular, an (odd) element $\Gamma$ of $g^1$ give rise to an (even) constant vector field $\Psi(\Gamma)$ on $M_{g[1]}$ which in turn defines a local diffeomorphism (the shift by $-\Gamma[1]$) and hence an associated morphism of Lie algebras of formal vector fields,

$$e^{ad\Psi(\Gamma)} : \Gamma(M_{g[1]}, TM_{g[1]}) \longrightarrow \Gamma(M_{g[1]}, TM_{g[1]})$$

$$V \longrightarrow e^{ad\Psi(\Gamma)}(V) = V + [\Psi(\Gamma), V] + \frac{1}{2!} [\Psi(\Gamma), [\Psi(\Gamma), V]] + \ldots.$$

2.6.1. **Theorem.** Let $(g, Q)$ be a $L_\infty$-algebra. For any $\Gamma \in MC(g)$, the associated data

$$(g, Q_\Gamma := e^{ad\Psi(\Gamma)}Q)$$

is again a $L_\infty$-algebra.

**Proof.** The $\Gamma$-deformed vector field $Q_\Gamma$ is clearly homological,

$$[Q_\Gamma, Q_\Gamma] = [e^{ad\Psi(\Gamma)}Q, e^{ad\Psi(\Gamma)}Q] = e^{ad\Psi(\Gamma)}[Q, Q] = 0.$$

The main point is that the condition

$$Q_\Gamma|_0 = 0$$

holds precisely when $\Gamma$ is a solution of the Maurer-Cartan equations with respect to the original $L_\infty$-structure $Q$. \qed

The differential of the $\Gamma$-deformed $L_\infty$-structure $(g, Q_\Gamma)$ is given explicitly by

$$d_\Gamma v = - \sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2}}{(k-1)!} \mu_k(\Gamma, \ldots, \Gamma, v)$$

$$= d\Gamma + \mu_2(\Gamma, v) - \frac{1}{2!} \mu_3(\Gamma, \Gamma, v) - \frac{1}{3!} \mu_4(\Gamma, \Gamma, \Gamma, v) + \ldots,$$

where $v \in g$. 

\[7\]
2.7. $A_\infty$-algebras. A strong homotopy algebra, or shortly $A_\infty$-algebra, is by definition a $\mathbb{Z}$-graded vector space $A$ equipped with linear maps,

$$m_k : \otimes^k A \rightarrow A[2 - k]$$

$$a_1 \otimes \ldots \otimes a_k \mapsto m_k(a_1, \ldots, a_k), \quad k \geq 1,$$

which satisfy, for any $n \geq 1$ and any $a_1, \ldots, a_n \in A$, the following higher order associativity conditions,

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r m_k(a_1, \ldots, a_j, m_l(a_{j+1}, \ldots, a_{j+l}), a_{j+l+1}, \ldots, a_n) = 0,$$

(5)

where $r = \bar{l}(\bar{a}_1 + \ldots + \bar{a}_j) + \bar{j}(\bar{l} - 1) + (k - 1)\bar{l}$.

It is easy to see from (5) that $A_\infty$-algebras with $m_k = 0$ for $k \geq 3$ are nothing but the usual graded differential associative algebras $(A, d, \cdot)$ with the differential $d = \mu_1$ and the associative multiplication given by $a_1 \cdot a_2 = m_2(a_1, a_2)$.

There is a natural functor

$$\{ \text{the category of } A_\infty\text{-structures on a graded vector space } V \} \rightarrow \{ \text{the category of } L_\infty\text{-structures on a graded vector space } V \}$$

given by [13]

$$\Phi(m_n)(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} e(\sigma) m_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}),$$

where $S_n$ is the permutation group on $n$ elements. This is a generalization of the usual construction of the Lie algebra out of an associative algebra.

3 An unobstructed deformation functor

3.1. Theorem. Let $(\mathfrak{g}, d, [,])$ be a differential graded Lie algebra, and let $\eta : \mathfrak{g} \rightarrow \mathfrak{g}[-1]$ be any linear map. Then the formulae

$$\mu_1(v_1) := dv_1$$

$$\mu_2(v_1, v_2) := (d\eta + \eta d)(v_1, v_2)$$

$$\mu_3(v_1, v_2, v_3) := -\eta [\mu_2(v_1, v_2), v_3] + (-1)^{\bar{v}_2\bar{v}_3} \eta [\mu_2(v_1, v_3), v_2] - (-1)^{\bar{v}_1(\bar{v}_2 + \bar{v}_3)} \eta [\mu_2(v_2, v_3), v_1]$$

$$\ldots$$

$$\mu_n(v_1, \ldots, v_n) := (-1)^n \sum_{\sigma \in Sh(n-1,n)} (-1)^{\sigma} e(\sigma) \eta [\mu_{n-1}(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}), v_{\sigma(n)}]$$

(6)

define inductively a structure of $L_\infty$-algebra on the graded vector space $\mathfrak{g}$.

Proof (an outline). We have to show that the tensors

$$\Phi_n(v_1, \ldots, v_n) := \sum_{k+l=n+1} \sum_{\sigma \in Sh(k,n)} (-1)^{\bar{\sigma} + k(l-1)} e(\sigma) \eta (\mu_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+l)}, \ldots, v_{\sigma(n)})$$
\[ d\mu_n(v_1, \ldots, v_n) = d\mu_n(v_1, \ldots, v_n) + \sum_{i=1}^{n} (-1)^{n-1+i+\tilde{v}_1+\ldots+\tilde{v}_{i-1}} \mu_n(v_1, \ldots, v_i, dv_i, v_{i+1}, \ldots, v_n) + (-1)^{n-1} \sum_{\sigma \in Sh(n-1, n)} (-1)^{\tilde{\sigma}} e(\sigma) \mu_2(\mu_{n-1}(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}), v_{\sigma(n)}) + \sum_{k+l=n+1, k+l \geq 1} \sum_{\sigma \in Sh(k, n)} (-1)^{\tilde{\sigma}+k(l-1)} e(\sigma) \mu_l(\mu_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+l)}, \ldots, v_{\sigma(n)}) \]

can be constructed out of the maps \( \mu_n \) defined above vanish for all \( n \geq 2 \) and all \( v_1, \ldots, v_n \in g \).

For \( n \geq 3 \),
\[
\begin{align*}
d\mu_n(v_1, \ldots, v_n) &= (-1)^n \sum_{\sigma \in Sh(n-1, n)} (-1)^{\tilde{\sigma}} e(\sigma) \eta \mu_n(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}, v_{\sigma(n)}) \\
&= (-1)^n \sum_{\sigma \in Sh(n-1, n)} (-1)^{\tilde{\sigma}} e(\sigma) \mu_2(\mu_{n-1}(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}), v_{\sigma(n)}) \\
&\quad + (-1)^{n+1} \sum_{\sigma \in Sh(n-1, n)} (-1)^{\tilde{\sigma}} e(\sigma) \eta \mu_l(\mu_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+l)}, \ldots, v_{\sigma(n)}) .
\end{align*}
\]
Substituting this expression into \( \Phi_n(v_1, \ldots, v_n) \), one gets, after tedious but straightforward algebraic manipulations, a recursion formula
\[
\Phi_n(v_1, \ldots, v_n) = (-1)^{n+1} \sum_{\sigma \in Sh(n-1, n)} (-1)^{\tilde{\sigma}} e(\sigma) Q(\Phi_{n-1}(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}), v_{\sigma(n)}) ; \quad n \geq 3 .
\]

Since \( \Phi_1(v_1) = d^2 v_1 = 0 \) and
\[
\Phi_2(v_1, v_2) = d\mu(v_1, v_2) - \mu_2(dv_1, v_2) - (-1)^{\tilde{v}_1} \mu_2(v_1, dv_2) = d\eta[dv_1, v_2] - \eta[\mu_2(dv_1, v_2)] = 0,
\]
the required statement follows. \( \square \)

**3.1.1. Remark.** Let \( (g, d, [\cdot, \cdot], \eta : g \to g) \) be the same data as in Theorem 3.1. Setting formally \( \lambda_1 := -\eta^{-1} \) we define a series of linear maps,
\[
\lambda_n : \Lambda^n g \longrightarrow g[2-n], \quad n \geq 2,
\]
by a recursive formula (cf. [17])
\[
\lambda_n(v_1, \ldots, v_n) := \sum_{k+l=n} \sum_{\sigma \in Sh(k, n)} (-1)^{\tilde{\sigma}+r} e(\sigma) \left[ \eta \lambda_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \eta \lambda_l(v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) \right],
\]
where \( r = k + 1 + (l-1)(\tilde{v}_{\sigma(1)} + \ldots + \tilde{v}_{\sigma(k)}) \). Then the data
\[
m_1 := d, \quad m_n := (1 - [d, \eta]) \lambda_n, \quad \text{for } n \geq 2,
\]
define a structure of \( L_\infty \)-algebra on \( g \) which is, in a sense, complementary to the one given in Theorem 3.1. We do not use this structure in the paper and hence omit the proof.
defines a graded associative algebra and let $\eta : A \to A[-1]$ be any linear operator. Then the formulae

\[
\begin{align*}
m_1(a_1) & := dv_1 \\
m_2(a_1,a_2) & := (d\eta + \eta d)(a_1 \cdot a_2) \\
m_3(a_1,a_2,a_3) & := \eta (-m_2(a_1,a_2) \cdot a_3 + a_1 \cdot m_2(a_2,a_3)) \\
& \quad \quad \quad \quad \quad \quad \quad \ldots \\
m_n(a_1, \ldots, a_n) & := \eta \left( (-1)^n m_{n-1}(a_1, \ldots, a_{n-1}) \cdot a_n + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} \mu_k (g_{k+1}, \ldots, g_{k+1}) \cdot m_{n-k}(a_1, \ldots, a_{n-k}) \right) \\
& \quad \quad \quad \quad \quad \quad \quad \ldots
\end{align*}
\]

define a structure of $A_\infty$-algebra on the vector superspace $A$. We omit the proof.

3.2. Theorem. Let $(\mathfrak{g}, d, [ , ])$ be a differential graded Lie algebra and $\eta : \mathfrak{g} \to \mathfrak{g}[-1]$ any linear map. Then the Kuranishi map $K : \Lambda^n \mathfrak{g} \to \mathfrak{g}[1 - *]$ given by its homogeneous components as follows

\[
K_1(v_1) := v_1, \\
K_2(v_1,v_2) := \eta [v_1, v_2], \\
K_n(v_1, \ldots, v_n) := 0 \quad \text{for } n \geq 3,
\]

defines a $L_\infty$-isomorphism between the $L_\infty$-structure $(\mathfrak{g}, \mu_\ast)$ induced on $\mathfrak{g}$ by Theorem 3.1 and the differential Abelian Lie algebra $(\mathfrak{g}, d, 0)$.

Proof. The equations (3) defining $L_\infty$-morphisms into an a differential Abelian Lie algebra take, in our case, the form

\[
\begin{align*}
n = 1 : & \quad dv_1 - dv_1 = 0, \\
n = 2 : & \quad dQ[v_1,v_2] - \mu_2(v_1,v_2) + Q[dv_1,v_2] + (-1)^{\tilde{\sigma}} Q[v_1, dv_2] = 0, \\
n \geq 3 : & \quad \mu_n(v_1, \ldots, v_n) - \sum_{\sigma \in S(n-1,n)} (-1)^{\tilde{\sigma}} e(\sigma) \eta \left[ \mu_{n-1}(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}), v_{\sigma(n)} \right] = 0.
\end{align*}
\]

They are all obviously satisfied. $\square$

Therefore, the “naive” $L_\infty$-structure (3) on $\mathfrak{g}$ is precisely the one obtained from the differential Abelian Lie algebra $(\mathfrak{g}, d, 0)$ by inverting (in the category of $L_\infty$-algebras) the Kuranishi map $K$. This is a key observation of the paper from which everything else follows.

3.3. Corollary. It follows from Theorem 3.2 and Fact 2.4.1 that the deformation functor

\[
\text{MDef}_{\mathfrak{g}} : \left\{ \begin{array}{c}
\text{the category of graded Artin} \\
\text{k-local algebras}
\end{array} \right\} \rightarrow \left\{ \text{the category of sets} \right\}
\]

\[
\mathcal{B} \rightarrow \{ \Gamma \in (\mathfrak{g} \otimes m_\mathcal{B})^1 | \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mu_k (\Gamma, \ldots, \Gamma) = 0 \}
\]

is equivalent to the deformation functor

\[
\text{Def}_{\mathfrak{g}} : \mathcal{B} \rightarrow \left\{ \Gamma \in (\mathfrak{g} \otimes m_\mathcal{B})^1 | \frac{d\Gamma = 0}{\text{Im } d} \right\},
\]
which is pro-representable by the graded algebra \( k[[t_{H^*}]] \) of formal power series on the \([1]-\)shifted cohomology space,

\[
H^* \equiv H^*(g, d)[1] := \frac{\text{Ker } d}{\text{Im } d}[1],
\]

viewed as a graded linear supermanifold. Hence \( \text{MDef}^Z_g \) is always pro-representable by a germ of a pointed formal supermanifold \( (\mathcal{M}^Z_g, p) \) which is isomorphic to the completion of the analytic germ \( \mathcal{O}_0 \) of the pointed supermanifold \( (H^*, 0) \).

Similarly, the deformation functor \( \text{MDef}^0_g \) (defined in the Subsect. 1.3) is always pro-representable by the algebra of formal power series on the vector superspace \( H^1 \equiv \Pi H^1(g, d), \Pi \) being the parity change functor. The associated formal pointed manifold we denote by \( (\mathcal{M}^0_g, p) \).

### 3.4. A versal solution to the Maurer-Cartan equations.

Let us fix a basis \( \{\gamma_\alpha\} \) of \( H^*(g, d) \). It defines an associated basis \( \gamma_\alpha[1] \) in \( H^* \) and hence a set of linear coordinates \( \{t^\alpha\} \). Let \( \gamma_\alpha \in \text{Ker } d \) be any representatives of the cohomology classes \( \{\gamma_\alpha\} \) in \( g \). Then

\[
\tilde{\Gamma}(t) := \sum \gamma_\alpha t^\alpha
\]

is a versal solution to the Maurer-Cartan equation in the Abelian Lie algebra \( (g \otimes k[[t_{H^*}]], d, 0) \).

By Theorem 3.3 and Fact 2.4.1 any formal power series

\[
\Gamma(t) = \sum_{\alpha} t^\alpha \Gamma_\alpha + \sum_{\alpha_1, \alpha_2} \Gamma_{\alpha_1, \alpha_2} t^{\alpha_1} t^{\alpha_2} + \ldots \in (g \otimes k[[t_{H^*}]])^1
\]

satisfying the equation

\[
\tilde{\Gamma}(t) = \sum_{n=1}^{\infty} \frac{1}{n!} K_n(\Gamma(t), \ldots, \Gamma(t)) = \Gamma(t) + \frac{1}{2} \eta[\Gamma, \Gamma]
\]

(7)

gives a versal solution to the Maurer-Cartan equation,

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2}}{k!} \mu_k(\Gamma(t), \ldots, \Gamma(t)) = 0,
\]

(8)

in the \( L_\infty \)-algebra \( (g \otimes k[[t_{H^*}]], \mu_s) \). The equation (7) is easily solved by

\[
\Gamma(t) = \Gamma_1(t) + \Gamma_2(t) + \ldots + \Gamma_n(t) + \ldots
\]

where

\[
\begin{align*}
\Gamma_1 &= \sum \gamma_\alpha t^\alpha \\
\Gamma_2 &= -\frac{1}{2} \eta[\Gamma_1(t), \Gamma_1(t)] \\
\Gamma_3 &= -\frac{1}{2} \eta ([\Gamma_1(t), \Gamma_2(t)] + [\Gamma_2(t), \Gamma_1(t)]) \\
&\vdots \\
\Gamma_n &= -\frac{1}{2} \eta \left( \sum_{k=1}^{n-1} \mu_{n-k}(\Gamma_k(t), \Gamma_{n-k}(t)) \right)
\end{align*}
\]

(9)
This is a well known power series \[10, 13, 14, 9\] playing a key role in the deformation theory. Therefore, it has a very simple algebraic interpretation within the category of \(L_\infty\)-algebras.

### 3.4.1. Remark

The same formulae \([\mathfrak{g}]\) describe a versal solution of the Maurer-Cartan equations in the \(L_\infty\)-algebra \((\mathfrak{g} \otimes k[[t]], \mu_*)\).

### 3.5. Hodge structures

Assume that the graded differential Lie algebra \((\mathfrak{g}, d, [\ , \ ])\) is equipped with a norm \(||\ ||\) on each \(\mathfrak{g}_i \subset \mathfrak{g}\) making \(\mathfrak{g}_i\) into a normed vector space such that both the differential \(d: \mathfrak{g}_i \rightarrow \mathfrak{g}_{i+1}\) and the Lie bracket \([\ , \]: \mathfrak{g}_i \times \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}\) are continuous. Assume also that there exists a projection \(P_H: \mathfrak{g} \rightarrow H\) and a linear operator \(\eta: \mathfrak{g} \rightarrow \mathfrak{g}[-1]\) such that the Hodge decomposition holds

\[
\text{Id} = P_H + d\eta + \eta d,
\]

and similarly for the completion of \(\mathfrak{g}\) with respect to the norm. In the usual Hodge theory \(\eta = d^*G\), where \(d^*\) is the adjoint to \(d\) and \(G\) is the Green function.

#### 3.5.1. Smoothness

Using the implicit function theorem in a Banach space as in \([8]\), one easily shows that the pointed formal supermanifold \((M_{\mathfrak{g}}^Z, p)\) representing the functor \(\text{MDef}_{\mathfrak{g}}^Z\) has a natural smooth analytic structure which makes \((M_{\mathfrak{g}}^Z, p)\) analytically diffeomorphic (with respect to the Kuranishi map) to a neighbourhood of zero in the vector space \(H^*\) (it is also easy to show that, for sufficiently small \(t\), the power series \([\mathfrak{g}]\) is convergent \([10, 9]\)). This proves the Claim (iii) in the Introduction.

In a similar way one provides the pointed moduli space \((M_0\mathfrak{g}, p)\) representing the functor \(\text{Def}_{\mathfrak{g}}^0\) with a smooth analytic structure in such a way that the Kuranishi map becomes an analytic equivalence between \((M_0\mathfrak{g}, p)\) and \((H^1, 0)\). Hence the Claim (i) follows.

#### 3.5.2. Vanishing obstructions

Assume that in a formal differential Lie algebra \((\mathfrak{g}, d, [\ , \ ])\) the induced map

\[
[,\ ]: H^*(\mathfrak{g}) \times H^*(\mathfrak{g}) \rightarrow H^*(\mathfrak{g}),
\]

is zero. Put another way, for any \(v_1\) and \(v_2\) in \(\mathfrak{g}\),

\[
P_H[P_H(v_1), P_H(v_2)] = 0.
\]

Then there is a chain of quasi-isomorphisms

\[
(\mathfrak{g}, \mu_*) \xrightarrow{\text{Kuranishi map}} (\mathfrak{g}, d, 0) \xrightarrow{P_H} (H, 0, 0) \leftarrow (\mathfrak{g}, d, [\ , \]),
\]

connecting the \(L_\infty\)-structure \([\mathfrak{g}]\) with the original differentiable Lie algebra structure on \(\mathfrak{g}\). By Fact 2.4.1, the deformation functors \(\text{MDef}_{\mathfrak{g}}^Z\) and \(\text{Def}_{\mathfrak{g}}^Z\) are equivalent. This in turn implies that the completion of the analytic germ of the pointed moduli space \((M_0\mathfrak{g}, p)\) (constructed in Subsect. 3.5.1) is isomorphic to the completion of the analytic germ of the classical Kuranishi pointed moduli space \((K_0\mathfrak{g}, p)\) (constructed in \([13, 14, 8]\)). By \([\mathfrak{g}]\), the analytic equivalence of \((M_0\mathfrak{g}, p)\) and \((K_0\mathfrak{g}, p)\) follows. This completes the proof of the Claim (iv) in the Introduction.
3.5.3. Non-vanishing obstructions. The classical Kuranishi moduli space $K_0^0$ is an analytic subspace in $\mathcal{M}_0^0$ given by the equations \[ P_H[\Gamma(t), \Gamma(t)] = 0, \] where $\Gamma(t) \in \mathfrak{g}^1 \otimes k[[t_{H^1}]]$ is the versal solution (3) (see Remark 3.4.1).

For any $t \in K_0^0$ one has
\[
\mu_2(\Gamma(t), \Gamma(t)) = (d\eta + \eta d)[\Gamma(t), \Gamma(t)] = (1 - P_H)[\Gamma(t), \Gamma(t)] = [\Gamma(t), \Gamma(t)],
\]
and hence
\[
\mu_3(\Gamma(t), \Gamma(t), \Gamma(t)) = -3\eta[[\Gamma(t), \Gamma(t)], \Gamma(t)] = 0,
\]
and hence
\[
\mu_n(\Gamma(t), \ldots, \Gamma(t)) = (-1)^n n\eta [\mu_{n-1}(\Gamma(t), \ldots, \Gamma(t)), \Gamma(t)] = 0
\]
for all $n \geq 3$. Therefore, the classical Kuranishi space $K_0^0$ is precisely the subspace of $\mathcal{M}_0^0$ where the Maurer-Cartan equation (8) of the $L_\infty$-algebra (4) degenerates into the following one,
\[
d\Gamma(t) + \frac{1}{2}[\Gamma(t), \Gamma(t)] = 0.
\]
This is exactly the Maurer-Cartan equation of the differential Lie algebra $(\mathfrak{g}, d, [\cdot, \cdot])$. This explains the Claim (ii) in the Subsect. 1.3.

3.6. Open questions and speculations. If $H^2(\mathfrak{g}, d) = 0$, then, evidently, $\mathcal{M}_0^0 \simeq K_0^0$ and the deformed differential (4) degenerates into a usual linear connection
\[
d_\Gamma = d + ad\Gamma.
\]
If obstructions do not vanish, then $d_\Gamma$ has terms higher order in $\Gamma$. Though it is still “flat”, $d_\Gamma^2 = 0$, this object is no more a linear connection. It seems that, for a geometric interpretation of the smooth moduli space $\mathcal{M}_0^Z$ in the case $(\mathfrak{g}, d, [\cdot, \cdot]) = (E \otimes E^* \otimes \Omega^{0\cdot}M, \bar{\partial}, \text{standard bracket})$, $E$ being a holomorphic vector bundle over a complex manifold $M$, one should switch from the category of projective modules to the category of strong homotopy modules over the differential algebra $(\Omega^{0\cdot}M, \bar{\partial})$ (cf. [3]) or even over its $A_\infty$-versions.

If $M$ is a Calabi-Yau manifold, then the moduli space $\mathcal{M}_0^Z$ associated with the differential graded algebras $(\Lambda^{\bullet}TM \otimes \Omega^{0\cdot}M, \bar{\partial}, \text{Schouten bracket})$ is precisely the Baramnikov-Kontsevich [4] extended moduli space of complex structures. To understand $\mathcal{M}_0^Z$ for a general compact complex manifold, one should probably think about a strong homotopy generalization of the notion of odd contact structure on a complex supermanifold.
References

[1] M. Alexandrov, M. Kontsevich, A. Schwarz and O. Zaboronsky, *The geometry of master equations and topological quantum field theory*, Intern. J. Mod. Phys. **12** (1997), 1405-1429.

[2] M. Artin, *On solutions to analytic equations*, Invent. Math. **5** (1968), 277-291.

[3] S. Barannikov, *Generalized periods and mirror symmetry in dimensions n > 3*, preprint.

[4] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices, no.4 (1998), 201-215.

[5] I. Ciocan-Fontanine and M. Kapranov, *Derived Quot schemes*, preprint math.AG/9905174.

[6] P. Deligne, unpublished.

[7] W.M. Goldman and J.J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Publ. Math. I.H.E.S. **67** (1988), 43-96.

[8] W.M. Goldman and J.J. Millson, *The homotopy invariance of the Kuranishi space*, Illinois J. Math. **34** (1990), 337-367.

[9] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, Princeton, 1987.

[10] K. Kodaira, *Complex manifolds and deformations of complex structures*, Springer 1986.

[11] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol.1 (Zürich, 1994) (Birkhäuser, Basel), 1995, pp. 120-139.

[12] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, math/9709040.

[13] M. Kuranishi, *On the locally complete families of complex analytic structures*, Ann. Math. **75** (1962), 536-577.

[14] M. Kuranishi, *Deformations of complex manifolds*, Les Presses de l’Université de Montréal, 1971.

[15] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. Algebra **23** (1995), 2147-2161.

[16] T. Lada and J. Stasheff, *Introduction to sh Lie algebras for physicists*, Intern. J. Theor. Phys. **32** (1993), 1087-1103.

[17] S.A. Merkulov, *Strong homotopy algebras of a Kähler manifold*, Intern. Math. Research Notices **3** (1998), 153-164.

[18] S.A. Merkulov, *The extended moduli space of special Lagrangian submanifolds*, Commun. Math. Phys. (1999), to appear.

[19] J.D. Stasheff, *On the homotopy associativity of H-spaces, II*, Trans. Amer. Math. Soc. **108** (1963), 293-312.

[20] J.D. Stasheff, *Higher homotopy algebras: string field theory and Drinfeld’s quasi-Hopf algebras*, in Proceedings of the XXth International Conference on Differential Geometric methods in Theoretical Physics (New York, 1991)(River Edge, NJ), vol.1, 1992, pp.408-425.
[21] J.D. Stasheff, *Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli space*, Perspectives on Mathematics and Physics (R.C. Penner and S.Y. Yau, eds.), International Press, 1994, hep-th/930461, pp. 265-288.

[22] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, hep-th/9804131.

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