On superluminal propagation and information velocity

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This paper examines some of the recent experiments on superluminal propagation. It is well known that Sommerfeld and Brillouin analyzed a rectangular sinusoidal signal propagating through a dispersive medium and derived expressions to describe the precursors and the main signal. In this paper, the impulse response of this dispersive medium is derived as exact expression using Taylor series expansion and output signal for any causal input signal is shown to be zero for time less than vacuum transit time and implications for superluminal information velocity is analyzed.

I. INTRODUCTION

Let us start with the well known formulation used by Sommerfeld and Brillouin\(^3\) for a rectangular sinusoidal signal propagating through a dispersive medium and using the modern notations\(^1\) and ignoring reflections at the interface, we can write as follows.

\[
u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{i k(\omega) x - i\omega t} d\omega
\]

\[
k(\omega) = \frac{\omega n(\omega)}{c}
\]

\[
A(\omega) = \int_{-\infty}^{\infty} u(0, t) e^{i\omega t} dt
\]

\[
u(0, t) = \text{rect}( \frac{t - T}{T} ) \sin(\omega_0 t)
\]

\[
A(\omega) = \frac{1}{2\pi} [A_0(\omega + \omega_0) - A_0(\omega - \omega_0)]
\]

\[
A_0(\omega) = \frac{2}{\omega} \sin(\omega \frac{T}{2}) e^{i\omega T}
\]

(1)

where \(n(\omega)\) is the refractive index of this dispersive medium, \(k(\omega)\) is the wave number and \(c\) is the speed of light in vacuum. Let \(L\) be the length of this dispersive medium. At \(x = 0\), we have the input signal is given by \(x(t) = u(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{-i\omega t} d\omega = \text{rect}(\frac{t - T}{T}) \sin(\omega_0 t)\). At \(x = L\), we get the output signal given by \(y(t) = u(L, t) = \frac{L}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{i k(\omega) L - i\omega t} d\omega = \frac{L}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{i\omega T} e^{-i\omega t} d\omega\). Considering the dispersive medium as a Linear Time Invariant system(LTI), we can write the output signal as the convolution of the input signal and impulse response given by \(y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau\) where \(h(t)\) is the impulse response of the medium. Applying Fourier Transform to this equation we get \(Y(\omega) = X(\omega) H(\omega)\) where \(H(\omega)\) is the frequency response of the medium given by \(H(\omega) = \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt = \frac{Y(\omega)}{X(\omega)} = e^{i\omega n(\omega) L}\). Let us denote \(t_0 = \frac{T}{c}\) as the vacuum transit time. We have \(H(\omega) = e^{i\omega n(\omega) t_0}\).

It is well known that the refractive index of the dispersive absorbing medium is given by \(n(\omega) = [1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}]^{\frac{1}{2}}\) and \(\omega_p, \omega_0, \gamma\) refer to the plasma frequency, resonance frequency and damping constant respectively. Given the recent experiments\(^4,5\) involving subluminal and superluminal light velocity in absorbing media and gain media, it may be of interest to derive an exact expression for the signal propagating through such media. In this paper, the impulse response of this absorbing medium is derived as exact expressions using Taylor series expansion for \(\frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\) < 1. For Gain media, \(\omega_p^2\) is merely replaced by \(-\omega_p^2\).
II. SECTION 2

Let the refractive index be given by \( n(\omega) = [1 + \chi(\omega)]^{1/2} \) where \( \chi(\omega) = \frac{\omega^2_p}{\omega^2 - \omega^2 - i\gamma\omega} \) denotes electric susceptibility. We can expand \( n(\omega) \) as follows for the case \( |\chi(\omega)| < 1 \) in the range \(-\infty \leq \omega \leq \infty\). \( |\chi(\omega)| < 1 \) is possible when \( \omega^2_p < \gamma\omega_0 \).

\[
n(\omega) = \sum_{r=0}^{\infty} \left( \frac{1}{r!} \right) \left[ \frac{\omega^2_p}{\omega^2 - \omega^2 - i\gamma\omega} \right]^r
\]

Hence we can expand the medium frequency response \( H(\omega) \) in Taylor’s series as follows:

\[
H(\omega) = e^{i\omega n(\omega) t_0} = \sum_{k=0}^{\infty} \frac{1}{k!} [i\omega n(\omega) t_0]^k = 1 + i\omega n(\omega) t_0 + \frac{1}{12} (i\omega n(\omega) t_0)^2 + \ldots
\]

Rearranging the even terms and odd terms of this Taylor series expansion and writing \( \frac{\omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{A}{(\omega - \omega_1)(\omega - \omega_2)} \)

where \( A = -\omega_p^2, \omega_1 = -i\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} + \frac{\omega_0^2}{4}}, \omega_2 = -i\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} + \frac{\omega_0^2}{4}} \), \( \gamma = n^2(\omega) = [1 + \frac{A}{(\omega - \omega_1)(\omega - \omega_2)}] \), we get

\[
H(\omega) = [1 + \frac{1}{12} (i\omega t_0)^2] + \frac{1}{14} (i\omega t_0)^4 y^2 + \ldots + \frac{1}{16} (i\omega t_0)^6 y^2 + \ldots
\]

Using the fact that \( y^{1/2} = n(\omega) = \sum_{r=0}^{\infty} \left( \frac{1}{r!} \right) \left[ \frac{A_r}{(\omega - \omega_1)(\omega - \omega_2)} \right] \) and the fact that \( y^k = \sum_{n=0}^{k} \frac{k}{n} \frac{A^n}{(\omega - \omega_1)^n(\omega - \omega_2)^n} \)

we get

\[
H(\omega) = \sum_{k=0}^{\infty} \left( \frac{-\omega^2 t_0^2}{(2k)!} \right) \sum_{n=0}^{k} \left( \frac{k}{n} \frac{A^n}{(\omega - \omega_1)^n(\omega - \omega_2)^n} \right) + (\omega t_0)^{\infty} \sum_{n=0}^{\infty} \left( \frac{-\omega^2 t_0^2}{(2k+1)!} \right) \sum_{r=0}^{\infty} \frac{1}{(k+1)} \sum_{n=0}^{\infty} \left( \frac{k}{n} \frac{A^{n+r}}{(\omega - \omega_1)^{n+r}(\omega - \omega_2)^{n+r}} \right)
\]

For \( t < t_0 \), it is easy to show that \( h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega = 0 \). Using \( n(\omega) = 1 + \sum_{r=1}^{\infty} \left( \frac{1}{r!} \right) \left[ \frac{A_r}{(\omega - \omega_1)(\omega - \omega_2)} \right] = 1 + n_1(\omega) \), we have \( h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega n_1(\omega) t_0 e^{-i\omega(t-t_0)}} d\omega \). Given that \( e^{i\omega n_1(\omega) t_0 e^{-i\omega(t-t_0)}} \) is analytic in the upper-half complex-plane with no singularities for \( t < t_0 \), it is easy to show that \( h(t) = 0 \) for \( t < t_0 \).

We wish to find \( h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega \) by using contour integration and Cauchy’s theorem of residues for \( t = t_0 \). Given that \( H(\omega) e^{-i\omega t} \) is analytic in the lower-half complex-plane except at singularities at \( \omega = \omega_1 \) and \( \omega = \omega_2 \) (from Eq.5), using Cauchy’s Residue theorem, we can compute the residues at these singularities as follows. Let us examine factors of the form \( \frac{\omega^n}{(\omega - b)^n} \) which figure in the above equation. We can write its derivatives 0, 1, 2, ... \( n \) with respect to \( \omega \) as follows.

\[
f_0(\omega, b, n, a) = \frac{\omega^n}{(\omega - b)^n}
\]

\[
f_1(\omega, b, n, a) = \frac{-n\omega^{n-1}}{(\omega - b)^{n+1}} + \frac{a\omega^n}{(\omega - b)^n}
\]

\[
f_2(\omega, b, n, a) = \frac{n(n+1)\omega^n}{(\omega - b)^{n+2}} + \frac{-2n\omega^{n-1}}{(\omega - b)^{n+1}} + \frac{a(n-1)\omega^{n-2}}{(\omega - b)^n} + \frac{a(a-1)\omega^{n-3}}{(\omega - b)^{n-1}}
\]

\[
f_3(\omega, b, n, a) = \frac{-n(n+1)(n+2)\omega^n}{(\omega - b)^{n+3}} + \frac{3n(n+1)a\omega^{n-1}}{(\omega - b)^{n+2}} + \frac{-3na(n-1)\omega^{n-2}}{(\omega - b)^{n+1}} + \frac{a(a-1)(a-2)\omega^{n-3}}{(\omega - b)^{n}}
\]
\[ f_{n-1}(\omega, b, n, a) = \sum_{r=0}^{n-1} (-1)^{n-r-1} \frac{(n-1)!}{r!(n-r-1)!} \frac{\omega^{a-r} \prod_{l=0}^{r-2} (n+l) \prod_{l=0}^{r-1} (a-l)}{(\omega - b)^{2n-r-1}} \]  

(6)

Using this, we can write the impulse response of the absorbing medium for \( t > t_0 \) as follows, using the theory of residues. Using \( h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega \) and expanding \( e^{-i\omega t} \) in Taylor series, we have

\[ h(t) = \frac{1}{2\pi} \sum_{m=0}^{\infty} \frac{(-it)^m}{m!} \sum_{k=0}^{\infty} \frac{(-t_0^2)^m}{(2k)!} \sum_{n=0}^{k} \frac{k!}{n!} G(2k + m, n) + (it_0) \sum_{k=0}^{\infty} \frac{(-t_0^2)^m}{(2k + 1)!} \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{n=0}^{k} \frac{k!}{n!} G(2k + m + 1, n + r) \]

where \( G(2k + m, n), G(2k + m + 1, n + r) \) are the residues.

\[ G(a, n') = 2\pi i \frac{A'}{(n' - 1)!} [f_{n'-1}(\omega_1, \omega_2, n', a) + f_{n'-1}(\omega_2, \omega_1, n', a)] \]

\[ f_{n-1}(\omega, b, n', a) = \sum_{r=0}^{n'-1} (-1)^{n'-r-1} \frac{(n'-1)!}{r!(n'-r-1)!} \frac{\omega^{a-r} \prod_{l=0}^{r-2} (n'+l) \prod_{l=0}^{r-1} (a-l)}{(\omega - b)^{2n'-r-1}} \]

(7)

Let us use the expression derived for \( h(t) \) in the above section to develop expressions for the output signal \( y(t) = u(L, t) \). We know that \( x(t) = u(0, t) = rect(\frac{t-\frac{L}{2}}{\frac{L}{c}}) \sin(\omega_0 t) \) and \( y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \) and \( h(t) = 0 \) for \( t < t_0 \). Hence we can see that \( y(t) = 0 \) for \( t < t_0 \). Hence superluminal propagation and superluminal information velocity seems not possible in absorbing medium, for \( |\chi(\omega)| < 1 \). For Gain media, \( \omega_p^2 \) is merely replaced by \( -\omega_p^2 \) and superluminal propagation and superluminal information velocity seems not possible.

### III. SECTION 3

We can expand the refractive index \( n(\omega) = [1 + \chi(\omega)]^A \) where \( \chi(\omega) = \frac{\omega^2}{\omega_0^2 - \omega^2 - \gamma c} \) as follows for the case \( |\chi(\omega)| > 1 \).

Let \( z = \chi(\omega) \) and we can express \( (1 + z)^{\frac{1}{2}} \) for \( |z| > 1 \) as follows\(^6\).

It is well known that the Taylor series expansion of \((1 + z)^{\frac{1}{2}}\) is given by

\[ (1 + z)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{A}{n} z^n \]  

(8)

for \(|z| < 1\) where \( \binom{A}{n} \) is the binomial choose function. It is well known that the series expansion does not converge for \(|z| > 1\) where \( A \) is a real number which is not equal to zero or a positive integer.

We could obtain a limited series expansion for \(|z| > 1\) by writing the above expression as follows

\[ (1 + z)^{\frac{1}{2}} = (1 + \frac{z}{2} + \frac{z^2}{2})^{\frac{1}{2}} = (1 + \frac{z}{2})^A(1 + \frac{\frac{z}{2}}{1 + \frac{z}{2}})^A = (1 + \frac{z}{2})^A(1 + \frac{z}{z + 2})^A \]

(9)
The second term in the above equation has a convergent series representation, given that \(|\frac{z}{z+2}| < 1\). If \(|\frac{z}{2}| > 1\), we can write

\[
(1 + \frac{z}{2})^A = (1 + \frac{z}{4})^A (1 + \frac{z}{z+4})^A
\]  

(10)

Repeating this procedure iteratively, if \(m_0\) is the minimum value for which \(|\frac{z}{2m_0}| < 1\), we can write

\[
(1 + z)^A = (1 + \frac{z}{2m_0})^A \prod_{r=1}^{m_0} (1 + \frac{z}{z+2r})^A
\]  

(11)

Each of the terms in the above product of terms has a convergent series representation. Given that we can write the convergent series expansion for each of the terms above as

\[
(1 + \frac{z}{2m_0})^A = \sum_{n=0}^{\infty} \binom{A}{n} (\frac{z}{2m_0})^n
\]

and

\[
(1 + \frac{z}{z+2r})^A = \sum_{m=0}^{\infty} \binom{A}{m} (\frac{z}{z+2r})^m
\]

where \(\binom{A}{n}\) represents the Choose function[2], we have the **series expansion for** \((1 + z)^A\) expressed as a **product of convergent series, which converges for** \(|z| > 1\) as follows:

\[
(1 + z)^A = \left[ \sum_{n=0}^{\infty} \binom{A}{n} (\frac{z}{2m_0})^n \right] \prod_{r=1}^{m_0} \sum_{m=0}^{\infty} \binom{A}{m} (\frac{z}{z+2r})^m
\]  

(12)

Now we can substitute \(z = \chi(\omega)\) and \(A = \frac{1}{2}\) in the above expression and obtain the series expansion of \(n(\omega)\) and substitute it for \(y_1\) in Eq.4 and use the procedure outlined in Section 2 to derive similar expression for the impulse response \(h(t)\) and \(y(t)\).

**IV. CONCLUSIONS**

Sommerfeld and Brillouin analyzed a rectangular sinusoidal signal propagating through a dispersive medium and derived expressions to describe the precursors and the main signal. In this paper, the impulse response of this dispersive medium is derived as exact expression using Taylor series expansion and output signal for any causal input signal is shown to be zero for time \(t\) less than vacuum transit time and hence superluminal propagation and superluminal information velocity seems not possible in absorbing and gain media.

**V. REFERENCES**

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