GLOBAL MODES FOR THE COMPLEX GINZBURG-LANDAU EQUATION

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Abstract

Linear global modes, which are time-harmonic solutions with vanishing boundary conditions, are analysed in the context of the complex Ginzburg-Landau equation with slowly varying coefficients in doubly infinite domains. The most unstable modes are shown to be characterized by the geometry of their Stokes line network: they are found to generically correspond to a configuration with two turning points issued from opposite sides of the real axis which are either merged or connected by a common Stokes line. A region of local absolute instability is also demonstrated to be a necessary condition for the existence of unstable global modes.

1 Introduction

Wakes, jets, counterflow mixing layers are well-known to exhibit under certain flow conditions self-sustained oscillations. It is well-established that these vortical structures result from a Hopf bifurcation and are due to a global destabilization of the underlying non-parallel basic flow (Mathis et al. 1984, Monkewitz et al. 1990, Strykowski & Niccum 1991). Furthermore, they seem to be connected to the appearance of a sufficiently large region of local absolute instability (Monkewitz 1990). Most theoretical investigations have so far used a WKBJ approach which assumes the basic flow to be weakly non-parallel in the streamwise direction. As explained in Huerre & Monkewitz (1990), it permits to define local stability properties and to link the global spatio-temporal behavior of the perturbation in the streamwise direction to the local dispersion relation.

The complex Ginzburg-Landau equation with slowly varying coefficients appears to be the simplest model providing a scenario of transition consistent with experiments (Huerre & Monkewitz 1990). Global modes, which are time-harmonic solutions satisfying vanishing boundary conditions have been analysed, either with linear or quadratic

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coefficients (Chomaz et al. 1988), or under specific assumptions on the number of turning points involved in the frequency selection criterion (Chomaz et al. 1991, Le Dizès et al. 1994).

The main objective of the present article is to obtain a characterization of the most unstable global modes without any assumption on the coefficients. An upper bound for the growth rate $\Im m[\omega]$ of the most unstable global mode is obtained in section 2. In section 3, the evolution with respect to the frequency of separation curves -or Stokes lines- is analysed in the complex plane in order to identify most unstable global modes configurations. Characteristic properties of global modes and expressions satisfied by their frequency are given in section 4. The last section recalls the main results and briefly discusses their extension to the weakly nonlinear régime.

2 Necessary conditions for the existence of a global mode

In the present framework, a global mode of global frequency $\omega$ is a time-harmonic function of the form

$$\Psi(x, t; \omega, \varepsilon) = \psi(X; \omega, \varepsilon)e^{-i\omega t},$$

where $\psi$ satisfies the Ginzburg-Landau equation

$$\left[\omega + \frac{1}{2}\omega_{kk}(X)\frac{\partial^2}{\partial x^2} - i\omega_{kk}(X)k_0(X)\frac{\partial}{\partial x} - \left(\frac{1}{2}\omega_{kk}(X)k_0^2(X) + \omega_0(X)\right)\right] \psi(X; \omega, \varepsilon) = 0,$$

(2)

and vanishing boundary conditions at $+\infty$ and $-\infty$. The coefficients $\omega_0(X)$, $\omega_{kk}(X)$ and $k_0(X)$ are assumed to be complex analytic functions of the slow variable $X = \varepsilon x$ where $\varepsilon$ is a small positive parameter, and $\omega_{kk}(X)$ is nonzero in the entire complex plane.

The local dispersion relation associated to (2) has the following form:

$$\omega = \omega_0(X) + \frac{\omega_{kk}(X)}{2}(k - k_0(X))^2.$$  

(3)

Causality implies that there exists a local maximum growth rate for the perturbation of the basic flow, which means that the quantity $\omega_{i,\text{max}}(X) = \max_{k \in R}(\omega_i(k, X))$ is everywhere defined as well as its maximum over all $X$ real denoted $\omega_{i,\text{max}}^\text{max}$. This condition guarantees also the existence of $\omega_{0,i,\text{max}}^\text{max}$ and $\omega_{i,\text{max}}(\infty)$ respectively defined as being the maximum of $\omega_{0,i}(X)$ over all $X$ real and the larger of the two limiting values taken
by \( \omega_{i,\text{max}}(X) \) at \( X = +\infty \) and \( X = -\infty \). Note that the wavenumber \( k = k_0(X) \) has a vanishing group velocity \( \frac{\partial \omega}{\partial k}(k_0) = 0 \). According to Bers (1983), \( k_0(X) \) and \( \omega_0(X) = \omega(k_0, X) \) are indeed the local absolute wavenumber and the local absolute frequency, respectively. The sign of \( \omega_{0,i}(X) \) characterises the absolute/convective nature of the local instability. If \( \omega_{0,i}(X) > 0 \), the location \( X \) is said to be (locally) absolutely unstable. It is convectively unstable or stable, otherwise.

It is well-known that there exist two formal solutions of (2) which admit the following WKBJ approximations as \( \varepsilon \to 0 \) (Bender & Orszag 1978, Wasow 1985)

\[
\psi_{\pm} \sim A_{\pm}(X; \omega) e^{\pm \int X k_{\pm}(S, \omega) dS},
\]

where the two local wavenumbers \( k^+ \) and \( k^- \) are defined as

\[
k_{\pm}(X; \omega) = k_0(X) \pm \sqrt{2 \left( \frac{\omega - \omega_0(X)}{\omega_{kk}(X)} \right)}. \quad (5)
\]

The square root can be arbitrarily defined, but for convenience, we shall use in this section the value of the square root of positive imaginary part with a branch cut on the positive real axis.

The solutions \( \psi^+ \) and \( \psi^- \) are not asymptotically valid in a full complex neighborhood of turning points defined by

\[
k^+(X; \omega) = k^-(X; \omega). \quad (6)
\]

However, they are known to be uniformly valid approximations of solutions of (2) in any (bounded) domain where any two points can be connected by a path along which \( \Im \left[ \int X (k^+(S; \omega) - k^-(S; \omega)) dS \right] \) is a differentiable and nondecreasing function (Wasow 1985, Fedoriuk 1987).

In the present context, WKBJ approximations are useful only if boundary conditions can be applied to them, and therefore only if they are uniformly valid from arbitrarily large negative \( X \) to arbitrarily large positive \( X \). Their use then requires a precise analysis in the complex \( X \) plane of the separation curves \( \Im \left[ \int X (k^+(S; \omega) - k^-(S; \omega)) dS \right] = \text{Cst} \) in order to identify the region of uniform validity, as well as a study of the signs of \( \Im[k^+] \) and \( \Im[k^-] \) near infinity along the real axis to determine the behavior of each WKBJ approximation for large \( X \).

According to the above definitions, the signs of the imaginary part of both branches \( k^+ \) and \( k^- \) are easily determined. For all \( \omega \) such that \( \omega_i > \omega_{i,\text{max}} \), and all real \( X \) there is

\footnote{Note that these points are the branch points of the square root and satisfy also \( \omega_0(X) = \omega \).}
no real $k$ solution of the local dispersion relation (3), and when $\omega_i \to +\infty$, the branches $k^+$ and $k^-$ are opposite. It follows that for all $\omega_i > \omega_{i,\text{max}}^{\text{max}}$ and all real $X$, $\Im[m[k^+]]$ and $\Im[m[k^-]]$ are differentiable, of constant and opposite sign as illustrated on Figure 1(a). When $\omega_i$ becomes inferior to $\omega_{i,\text{max}}^{\text{max}}$, $\Im[m[k^+]]$ and $\Im[m[k^-]]$ eventually change sign. But, by definition, as long as $\omega_i > \omega_{i,\text{max}}^{\text{max}}$, these changes do not occur near $\infty$ [Figure 1(b)]. Behaviors for large $X$ of both WKBJ approximations are then invariant for all $\omega_i > \omega_{i,\text{max}}^{\text{max}}$: $\psi^+$ is exponentially small for large positive real $X$ and exponentially large for large negative real $X$ and the opposite holds for $\psi^-$. When $\omega_i > \omega_{0,i}^{\text{max}}$, the spatial branches $k^+$ and $k^-$ satisfy an additional property: according to the definition of the square root prescribed above, both branches are analytic complex functions for all real $X$ and satisfy $\Im[m[k^+(X;\omega) - k^-(X;\omega)]] > 0$ [Figure 1(c)]. The function $\Im[m[\int^X (k^+(S;\omega) - k^-(S;\omega))dS]]$ is then a differentiable and nondecreasing function on the real $X$ axis, so that $\psi^+$ and $\psi^-$ are uniformly valid asymptotic solutions in any interval of the real axis.

The behaviors of these two independent solutions for large $X$ finally imply that no solution vanishing on both sides at $\infty$ can be constructed. In other words, all global modes have a frequency satisfying

$$\omega_i \leq \max[\omega_{i,\text{max}}(\infty), \omega_{0,i}^{\text{max}}] \quad (7)$$

In physical terms, this in particular means that a medium which is stable at infinity

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3The uniform validity near infinity of WKBJ approximations is not essential for the proof, although it can be achieved under certain conditions (Fedoriuk 1987).
$(\omega_{i,max}(\infty) < 0)$ is globally stable if there is no region of local absolute instability, i.e. no location where $\omega_{0,i}(X) > 0$.

Note that this result is stated in Chomaz et al. (1991) with an incomplete proof and proved in Le Dizès et al. (1994) by another method in the case of two-turning-point configurations.

3 Stokes line network deformations to a global mode configuration

In this section, we assume that $\omega_{i,max}(\infty) < \omega_{0,i}^{max}$ and show that the analysis of the separation curves $\Im \int X(k^+(S;\omega) - k^-(S;\omega))dS = Cst$ in the complex plane permits to identify the most unstable global mode configurations of frequency $\omega$ such that $\omega_i > \omega_{i,max}(\infty)$.

Figure 2: Example of separation curve network for a frequency $\omega$ such that $\omega_i > \omega_{0,i}^{max}$.

If $\omega_i > \omega_{0,i}^{max}$, the geometry close to the real axis of the separation curves is particularly simple: with the definition of the square root prescribed above, the real $X$ axis crosses each line once and only once, and branch cuts and turning points are far from the real axis. The domain of uniform validity of WKBJ approximations can be extended in the complex $X$ plane by following the simple procedure that consists in deforming its frontier without cutting any branch cut, nor twice separation curves of the same label.
The largest domain that can be obtained is limited by the Stokes lines\(^4\) that do not cross the real axis\(^5\) as illustrated on Figure 2.

As long as \(\omega_i > \omega_{i,max}(\infty)\), none of the WKBJ approximations satisfies simultaneously both boundary conditions, as obtained above (see Figure 1(b)). Furthermore, for these values, the geometry of the separation curves remains unchanged near infinity because no turning points goes to infinity and none of the separation curves become asymptotic to the real axis near \(\infty\). However, when \(\omega_i\) is smaller than \(\omega_{i,max}^0\), some turning points may have crossed the real axis and some separation curves become twisted but the domain of uniform validity of WKBJ approximations may still connect \(+\infty\) and \(-\infty\) on the real axis, which guarantees the non-existence of global modes. This fails as soon as \textbf{two turning points or two Stokes lines which were on opposite sides of the real axis for large \(\omega_i\) collide.}

\[\int_{X_2}^{X_1}(k^+(S; \omega) - k^-(S; \omega))dS\] becomes stationary at the double turning point \(X_s\) in case (a), and on the Stokes line connecting the single turning points \(X_1\) and \(X_2\) in case (b). The uniform validity of the WKBJ approximation from large negative \(X\) to large positive \(X\) is then no longer guaranteed and the existence of a global mode cannot be excluded by the former argument.

\[^4\text{i.e. separation curves issued from a turning point}\]
\[^5\text{A redefinition of the square root may be needed to maintain branch cuts in the grey sector limited by Stokes lines of Figure 2}\]
One can indeed demonstrate that the frequency that corresponds to one of these Stokes line networks is actually the leading-order approximation to a discrete number of global frequencies. The next section gives a outline of the proof.

4 Most unstable global modes characteristics

The simplest proof is based on the local analysis of turning point regions. Since WKBJ approximations are still uniformly valid on each side of the region where pinching has occurred, the existence of a global mode is equivalent to the correct matching of the WKBJ approximations prescribed by the boundary conditions at infinity across either the double turning point $X_s$ [Figure 3(a)], or the two single turning points $X_1$ and $X_2$ connected by a Stokes line [Figure 3(b)].

This kind of problem is treated in several textbooks for real configurations (see for instance Bender & Orszag, 1981, chap. 10). The same analysis can easily be applied in the complex plane and leads to the following results.

![Figure 4: Global modes approximations in the complex X plane; (a): Double-turning-point global mode, (b): Two-single-turning-points global mode.](image)

For the double-turning-point configuration, matching is possible if and only if the local approximation near the double-turning-point region is a Hermite function [see Figure 3(a)]. This gives a discrete number of global modes whose leading-order frequency

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6i.e. both WKBJ approximations subdominant at $+\infty$ and $-\infty$, respectively.
\( \omega^{(0)} \) is the local absolute frequency taken at the double turning point \( X_s \):

\[
\omega^{(0)} = \omega_0(X_s). \tag{8}
\]

For the two-single-turning-point configuration, one has to proceed to a double matching because there are three large separated regions [see Figure 3(b)]. Along the Stokes line connecting both turning points, any solution admits a uniform decomposition into both families of WKBJ waves. But, that approximation breaks down near each turning point where a local approximation is built in terms of Airy functions. These local approximations can themselves be matched to each subdominant WKBJ approximation if and only if the following integral relation is satisfied by the leading order global frequencies \( \omega^{(0)}_n \)

\[
\frac{1}{\pi \varepsilon} \int_{X_1}^{X_2} \sqrt{\frac{2(\omega^{(0)}_n - \omega_0(X))}{\omega_{kk}(X)}} \, dX \sim n + \frac{1}{2}. \tag{9}
\]

This relation, where \( n \) is an integer of order \( 1/\varepsilon \), constitutes the leading-order expression satisfied by a discrete number of global modes. When \( n \) is of order unity, both turning points are distant of \( \varepsilon \) and one recovers the double-turning-point case. The same results can also be obtained using a method based on uniform approximations as shown by Le Dizès et al. (1994).

## 5 Discussion

We have shown that a global mode for the Ginzburg-Landau equation has always a growth rate \( \omega_i \) smaller than \( \max[\omega_{i,\text{max}}(\infty), \omega_{0,i}^{\text{max}}] \). Furthermore, the most unstable global modes of growth rate larger than \( \omega_{i,\text{max}}(\infty) \) have been demonstrated to be described by specific Stokes line configurations. They are generically\(^7\) associated to the first pinching, when \( \omega_i \) is decreased, of two Stokes lines or turning points which were on opposite sides of the real axis for large \( \omega_i \).

The global frequencies are discretized at order \( \varepsilon \) and satisfy the interesting property of being only (explicitly) dependent on the local dispersion relation (3) in the region where the pinching has occurred. Approximations for these modes can easily be obtained in the “pinched region” and in the domains of uniform validity of the WKBJ approximations (white domains of Figures 3), but one must emphasize that approximations are not necessarily known or easily accessible everywhere on the real axis.

\(^7\)One can point out that nothing can in general be said concerning more stable global modes. Furthermore, non-generic cases corresponding to a first pinching of more than two Stokes lines or turning points may also give rise to a global mode which is not described by the present analysis.
The extension in the weakly nonlinear régime has been studied in Chomaz et al. (1990), Le Dizès et al. (1993) and Le Dizès (1994) by adding a cubic nonlinearity to the right-hand side of equation (2). In the simplest case where the single destabilized linear global mode and its adjoint mode admit uniform WKBJ approximations on the real axis, the weakly nonlinear analysis “à la Stuart-Landau” has given the surprising conclusion that a physically acceptable solution is only obtained under very restrictive conditions (Le Dizès 1994). It is then very likely that there only exists a small class of linear global modes which can effectively saturate through a weakly nonlinear process when destabilized. This class remains to be determined in order to fully justify the application to a physical problem of the linear frequency selection criterion obtained in the present article.

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