Time and harmonic study of strongly controllable group systems, group shifts, and group codes

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September 6, 2018

ABSTRACT

In this paper we give a complementary view of some of the results on group systems by Forney and Trott. We find an encoder of a group system which has the form of a time convolution. We consider this to be a time domain encoder while the encoder of Forney and Trott is a spectral domain encoder. We study the outputs of time and spectral domain encoders when the inputs are the same, and also study outputs when the same input is used but time runs forward and backward. In an abelian group system, all four cases give the same output for the same input, but this may not be true for a nonabelian system. Moreover, time symmetry and harmonic symmetry are broken for the same reason. We use a canonic form, a set of tensors, to show how the outputs are related. These results show there is a time and harmonic theory of group systems.
1. INTRODUCTION

The idea of group shifts and group codes is important in several areas of mathematics and engineering such as symbolic dynamics, linear systems theory, and coding theory. Research in this area started with the work of Kitchens [1], Willems [2], Forney and Trott [3], and Loeliger and Mittelholzer [4].

Kitchens [1] introduced the idea of a group shift [13] and showed that a group shift has finite memory, i.e., it is a shift of finite type [13]. Using the work of Willems [2] on linear systems, Forney and Trott [3] describe the state group and state code of a set of sequences with a group property, which they term a group code $C$. A time invariant group code is essentially a group shift. They show that any group code that is complete (any global constraints can be determined locally, see [3]) can be wholely specified by a sequence of connected labeled group trellis sections (which may vary in time) which form a group trellis $C$. They explained the important idea of “shortest length code sequences” or generators. A generator is a code sequence which is not a combination of shorter sequences. In a strongly controllable group code, the nontrivial portions of all generators have a bounded length. They give an encoder whose inputs are generators and whose outputs are codewords in the group code. At each time $t$, a finite set of generators is used to give a symbol in the codeword.

Loeliger and Mittelholzer [4] obtain an analog of the derivation of Forney and Trott starting with a group trellis $C$ instead of the group of sequences $C$. To derive their encoder, they use an intersection of paths which split and merge to the identity path in the trellis, an analog of the quotient group of code sequences (granule) used in [3].

Forney and Trott also suggest the term group system in place of group code. Here we generally use the term group system rather than group code because some results have analogues in classical systems theory and harmonic analysis. We only consider time invariant group systems; therefore the results here also apply to group shifts. In addition, we only consider strongly controllable group systems, in which there is a fixed integer $\ell$ such that for any time $t$, for any sequence on $(-\infty, t]$ there exists a valid path of length $\ell$ to any sequence on $[t + \ell, \infty)$. Then the nontrivial lengths of the generators are at most $\ell$.

Forney and Trott have shown that any group system $C$ can be reduced to a group trellis $C$ whose vertices are the states of the group system. The states are defined using a group theoretic construction as quotient groups. Each component of the trellis is a trellis section, a collection of branches which forms a branch group $B^t$ at time $t$. We call group trellis $C$ the first canonic form of the group system. The group system can be implemented with an encoder. The encoder has a shift register structure and the outputs give a trellis which is graph isomorphic to $C$.

In this paper, we consider several problems that arise from their discussion. First, their encoder is implemented going forward in time. It is natural to ask what is the encoder if we go backwards in time, and if both forward and backward encoders are filled with the same sequence of generators, are their outputs the same. We answer this question here.

The Forney and Trott encoder does not have the form of a time convolution. Next, we find another encoder which has the form of a convolution.
For this reason, we call this encoder a time domain encoder, and the Forney and Trott encoder a spectral domain encoder. The time domain encoder can be implemented for forward and backward time, and the same question applies as for the spectral domain encoder. The time domain encoder uses the same input sequences of generators as the spectral domain encoder. So we may also compare the outputs of the time and spectral domain encoder if both use the same input.

In this paper, we show how the time and spectral domain encoders, and forward and backward time encoders, are related. In the abelian group system, we show that all four encoders give the same outputs if the same input is used. But in the nonabelian system, these symmetries can break, and we do not necessarily get the same output for the same input. Moreover, time symmetry and harmonic symmetry break for the same reason. It is interesting to observe how these symmetries break since a group system is possibly the most elementary nonlinear system in mathematics with a time and spectral domain interpretation.

When time symmetry or harmonic symmetry breaks, we show how the two different outputs are related. To do this we use a second canonic form of the group system. The second canonic form is a set of tensors $R$. Each tensor is a sequence of generators. At each time $t$, a component of the tensor is a matrix, called a static matrix. Each static matrix is formed by $\ell + 1$ shift matrices at times $t-j$, for $j = 0, 1, \ldots, \ell$. A row in a shift matrix is a generator vector, the nontrivial components of a generator.

The entries in a static matrix are components of different generator vectors. We show that these elements are the representatives of a coset decomposition chain of the branch group $B^t$ at time $t$. And so each tensor is a sequence of branches which is a path in the group trellis $C$. Moreover, the static matrix at time $t$ can be used to define group theoretic input and output states which are isomorphic to the quotient group states defined for $C$. This means a group trellis $C$ can be reduced to a set of tensors $R$.

We believe $R$ is more revealing of the structure of a group system than group trellis $C$. The group trellis $C$ emphasizes the branch group $B^t$ of a trellis section. But the set of tensors $R$ shows that $B^t$ is a secondary object which is a snapshot at time $t$ of $\ell + 1$ shift matrices formed by generator vectors. In addition, time reversal appears deceptively simple in $C$, but canonic form $R$ shows that it is not.

The canonic form $R$ has a natural shift structure which arises from quotient groups in the coset decomposition chain of $B^t$. Then $R$ can also be written as a trellis, which is graph isomorphic to $C$. The labels of the branches in the trellis are matrices.

The spectral domain encoder has a set theoretic description of its states which is graph isomorphic to the group theoretic states of $C$, but the isomorphism has not been described. There is also a set theoretic construction of the states of $R$ which matches the set theoretic construction of the spectral domain encoder. This explains the isomorphism between states of the Forney and Trott encoder and states of $C$. Therefore each tensor $r \in R$ can be used as an input to any of the four encoders.

The representatives in a tensor set $R$ can be replaced with integers. This gives a tensor set $U$. There is a 1-1 correspondence $u \leftrightarrow r$ between a tensor $u \in U$ and a tensor $r \in R$, and between shift vectors in $u$ and shift vectors in...
r. \( \mathcal{U} \) can also be realized as a trellis. If the tensors in \( \mathcal{R} \) are used as inputs to each of the four encoders, the outputs form \( C \). The outputs of one encoder are related to the outputs of another encoder by a graph automorphism of the trellis of \( \mathcal{U} \).

A selection of a set of generator vectors at each time \( t \) that is necessary and sufficient to generate \( C \) forms a basis \( B \). Each basis \( B \) gives a tensor set \( \mathcal{U} \). Two different bases give two different tensor sets; this is called a change of basis. The two different tensor sets can be used as inputs to the same encoder. The tensor set \( \mathcal{U} \) is independent of basis, and when there is a change of basis, the outputs of the same encoder are related by a graph automorphism of the trellis of \( \mathcal{U} \).

The set of all graph automorphisms of \( \mathcal{U} \) forms a permutation group under composition. This is termed the \textit{full symmetry system} in \cite{9}. We calculate the full symmetry system of \( \mathcal{U} \). Any symmetry is specified by a finite set of separating permutations at each time \( t \). Using the separating permutations, we give an algorithm to construct any symmetry.

We show that any symmetry in the full symmetry system takes each tensor \( u \in \mathcal{U} \) to another tensor \( \hat{u} \in \mathcal{U} \), and takes each shift vector in \( u \) to another shift vector in \( \hat{u} \) of the same length \( k \), for the same time \( t \). This induces a permutation of \( \mathcal{R} \) which takes each tensor \( r \in \mathcal{R} \) to another tensor \( \hat{r} \in \mathcal{R} \), and takes each generator vector in \( r \) to another generator vector in \( \hat{r} \) of the same length \( k \), for the same time \( t \). The permutation of a generator vector of length \( k \) at time \( t \) in \( r \) is only affected by generator vectors of length at least \( k \) at time \( t \) in \( r \). The permutation of all tensors in tensor set \( \mathcal{U} \) or \( \mathcal{R} \) can be performed iteratively, starting with a permutation of the sequence of longest generator vectors, and working down.

The product \( c \mathcal{C} \), where \( c \) is a path in \( \mathcal{C} \), permutes the paths of group trellis \( \mathcal{C} \), and therefore induces a symmetry of \( \mathcal{U} \). The set of symmetries induced by \( \{c \mathcal{C} : c \in \mathcal{C} \} \) forms a group which we call the natural symmetry system \( \mathcal{N} \). \( \mathcal{N} \) is a subgroup of the full symmetry system, and \( \mathcal{N} \) is isomorphic to \( \mathcal{C} \).

Since the product \( c \mathcal{C} \) induces a symmetry, we can study multiplication in \( \mathcal{C} \) using the natural symmetry system \( \mathcal{N} \). We show how two paths \( c_1 \) and \( c_2 \) multiply in terms of the two tensors \( r_1 \) and \( r_2 \) that encode to \( c_1 \) and \( c_2 \), respectively. We show that multiplication in \( \mathcal{C} \) implies that any group system has an underlying commutative property.

Since \( \mathcal{C} \) is time invariant, the natural symmetry system of \( \mathcal{C} \) is time invariant. Therefore the natural symmetry system \( \mathcal{N} \) of \( \mathcal{C} \) can be specified by a finite set of separating permutations which is constant for all time \( t \). This approach can be used to construct \( \mathcal{C} \).

This paper is organized as follows. We start with a group system \( \mathcal{C} \), as in \cite{3}. Any group system \( \mathcal{C} \) can be reduced to a group trellis \( \mathcal{C} \) with a group trellis section, or branch group \( B^t \) \cite{3}; this is reviewed in Section 2. We study an \( \ell \)-controllable group system and group trellis, in which each state can be reached from any other state in \( \ell \) branches \cite{3}.

In group trellis \( \mathcal{C} \), the sequence of branches that split from the identity path and merge to the identity path form two normal chains \cite{4}. The Schreier refinement theorem can be applied to these two normal chains to obtain another normal chain, a refinement of the two chains that we call a Schreier series. The Schreier series is a normal chain of the branch group at time \( t \).
of the group trellis. The Schreier series can be written in the form of a matrix, with rows and columns determined by branches of the splitting and merging trellis paths. When the group trellis is strongly controllable, the matrix reduces to a triangular form, called the static matrix. The static matrix is an echo of matrix ideas used in classical linear systems analysis.

The static matrix is defined over time interval $[t, t]$. Since the group system is assumed to be time invariant, we can replace the branches in column $j$ of the static matrix with the same branches at time $t + j$. The resulting matrix is defined over the time interval $[t, t + \ell]$, and is called the shift matrix; it is also a triangular form. We show the shift matrix has a natural shift property, and in fact the shift matrix forms a part of the group trellis, the truncation of the ray of paths splitting from the identity path at time $t$. This is discussed in Section 3.

We show that the rows of the shift matrix can be used to form quotient groups, and the generator sequences of Forney and Trott are a transversal of the quotient groups. The coset representatives of the generators in the transversal are also a triangular form, a shift matrix which we call a generator matrix. The rows of the generator matrix are the nontrivial portion of a generator sequence, called a generator vector. At time $t$, the components of the generators form a complete set of coset representatives for the Schreier series decomposition of branch group $B^t$. The same set of coset representatives can be used for the Schreier series decomposition of the branch group of the time reversed group trellis. This is discussed in Section 4.

In Section 5, based on the generator matrix, we give a causal minimal encoder structure for a group trellis and group system. We can think of the encoder as an estimator. As in [3, 4], the encoder uses shortest length generator sequences, but here the components of the generator sequences give a time domain convolution. Therefore this appears to be a natural time domain encoder for a group system, whereas the encoders in [3, 4] can be viewed as spectral domain encoders.

In Section 6, we show the first canonic form, group trellis $C$, can be reduced to the second canonic form $\mathcal{R}$. The tensor set $\mathcal{R}$ depends on basis $\mathbf{B}$. We find a tensor set and trellis $\mathcal{U}$ which corresponds to $\mathcal{R}$ but is independent of basis. We show that the four encoders are related by graph automorphisms of $\mathcal{U}$; the same holds for a change of basis. In Section 7, we find the structure of graphs automorphisms of $\mathcal{U}$, a permutation group called the full symmetry system. In Section 8, we study the natural symmetry system of $C$ and multiplication in $C$ and $\mathcal{R}$.
2. GROUP SYSTEMS

This section gives a very brief review of some fundamental concepts in [3], and introduces some definitions used here. We follow the notation of Forney and Trott as closely as possible. One significant difference is that subscript $k$ in [3] denotes time; we use $t$ (an integer) in place of $k$. In any notation, a superscript is used exclusively to indicate time; thus $t$ always appears as a superscript in any notation.

Forney and Trott study a collection of sequences with time axis defined on the set of integers $\mathbb{Z}$, whose components $a^t$ are taken from an alphabet $A^t$ at each time $t$, $t \in \mathbb{Z}$. The set of sequences is a group under componentwise addition in $A^t$. We call this a group system or group code $C$ [3]. In this paper, we assume the group system is time invariant, so for each $t$, $A^t$ is the same as a fixed common group $A$. A sequence $a$ in $C$ is given by

\[ a = \ldots, a^{t-1}, a^t, a^{t+1}, \ldots, \tag{1} \]

where $a^t \in A^t$ is the component at time $t$.

The group system $C$ is assumed to be complete [2, 3]; an important consequence is that local behavior is sufficient to describe global behavior. Completeness is the same as closure in symbolic dynamics [13]. Therefore a time invariant complete group system $C$ is the same thing as a group shift in symbolic dynamics. In this paper, we use the language associated with group systems [3] rather than group shifts [13].

Define $C^{t^+}$ to be the set of all codewords in $C$ for which $b^n = 1^n$ for $n < t$, where $1^n$ is the identity component at time $n$. Define $C^{t^-}$ to be the set of all codewords in $C$ for which $b^n = 1^n$ for $n \geq t$. The group system satisfies the axiom of state: whenever two sequences pass through the same state at a given time, the concatenation of the past of either with the future of the other is a valid sequence [3]. The canonic state space $\Sigma_t$ at time $t$ is defined to be

\[ \Sigma_t \overset{\text{def}}{=} \frac{C}{C^{t^-} \cup C^{t^+}}. \]

The canonic state space is unique. For a time invariant group system, for each time $t$, the state space $\Sigma_t$ is the same as a common fixed group $\Sigma$.

![Diagram](Figure 1: Definition of $C^{t^+}$ and $C^{t^-}$.)

The state $\sigma^t(a)$ of a system sequence $a$ at time $t$ is determined by the natural map

\[ \sigma^t : C \rightarrow C/(C^{t^-} \cup C^{t^+}) = \Sigma^t, \]

a homomorphism. There is therefore a well defined state sequence $\sigma(a) = \{\sigma^t(a) : t \in \mathbb{Z}\}$ associated with each $a \in C$, and a well defined state code $\sigma(C) = \{\sigma(a) : a \in C\}$ associated with $C$. The canonic realization $C$ of a
group system $C$ is the set of all pairs of sequences $(a, \sigma(a))$:  
\[
\{(a, \sigma(a)) : a \in C\},
\]  
where $\sigma(a)$ is the state sequence of $C$. The state spaces of the canonic realization are $\Sigma^t$. The canonic realization is a minimal realization of a group system.

An element of the canonic realization $C$ is denoted  
\[
b = \ldots, b_{t-1}, b_t, b_{t+1}, \ldots,
\]  
where component $b_t$ is given by $b_t = (s^t, a^t, s^{t+1})$, where $s^t \in \Sigma^t$ is the canonic state at time $t$, and $s^{t+1} \in \Sigma^{t+1}$ is the canonic state at time $t + 1$; we think of component $b_t$ stretching over the time interval $[t, t+1]$. We say $s^t$ is the left state of $b_t$, and use notation $(b_t)^- = s^t$. In addition, we say $s^{t+1}$ is the right state of $b_t$, and use notation $(b_t)^+ = s^{t+1}$. For any path $b$, as given in (2), it is clear that for $b' = (s^t, a^t, s^{t+1})$ and $b^{t+1} = (s^{t+1}, a^{t+1}, s^{t+2})$, we must have $s^{t+1} = s^{t+1}$ or equivalently $(b_t)^+ = (b_{t+1})^-$. Let $\sigma(C)$ be the state code of $C$, the sequences of states $\ldots, s^{t-1}, s^t, s^{t+1}, \ldots$ in each $b \in C$.

**Theorem 1** There is a group isomorphism from $C$ to $C$ given by the 1-1 correspondence $a \leftrightarrow b$, where $a \in C$ and $b \in C$. If  
\[
a = \ldots, a_{t-1}, a^t, a^{t+1}, \ldots,
\]  
and  
\[
b = \ldots, b_{t-1}, b^t, b_{t+1}, \ldots,
\]  
then for each time $t$, $a_t \mapsto b_t = (s^t, a^t, s^{t+1})$ is the assignment of the group isomorphism.

**Proof.** There is a well defined state sequence $\sigma(a)$ associated with each $a \in C$. This means each $a \in C$ is assigned to a well defined $b \in C$ by the assignment $a_t \mapsto b_t = (s^t, a^t, s^{t+1})$ for each time $t$. This map is a bijection since if $a \in C$ and $\bar{a} \in C$ are both assigned to the same $b \in C$, then we must have $a_t = \bar{a}_t$ for each time $t$, so $a$ and $\bar{a}$ are the same. 

We will be interested in canonic realization $C$ rather than group system $C$ in the remainder of the paper. There is no loss in generality in considering $C$ rather than $C$ because of the above 1-1 correspondence and isomorphism.

The canonic realization can be described with a graph [3]. Any other minimal realization is graph isomorphic to the canonic realization [4]. We think of component $b_t$ as a branch in a trellis section $T_t$ or an element in branch group $B^t$. Trellis section $T^t$ is a bipartite graph where the left vertices are states in $\Sigma^t$, the right vertices are states in $\Sigma^{t+1}$, and the label of a branch $(s^t, a^t, s^{t+1})$ between state $s^t$ and state $s^{t+1}$ is $a^t \in A^t$. $B^t$ is the group of branches $b_t$, which is a subdirect product, a subgroup of the direct product group $\Sigma^t \times A^t \times \Sigma^{t+1}$. Clearly there is a branch $(s^t, a^t, s^{t+1})$ in $T^t$, with label $a^t$ between two vertices $s^t$ and $s^{t+1}$, if and only if $(s^t, a^t, s^{t+1}) \in B^t$. Then $C$ can be described by a group trellis, a connected sequence of trellis sections, where $T^t$ and $T^{t+1}$ are joined together using the common states in
We refer to this as group trellis $C$. We regard group trellis $C$ as the first canonic form of group system $C$.

The states of $B^t$ are $\Sigma^t$ and $\Sigma^{t+1}$. We now describe state groups of $B^t$ isomorphic to $\Sigma^t$ and $\Sigma^{t+1}$. Consider the projection map $\pi_L : B^t \rightarrow \Sigma^t$ onto the left states of $B^t$, given by the assignment $(s^t, a^t, s^{t+1}) \mapsto s^t$. This is a homomorphism with kernel $X_0^t$, where $X_0^t$ is the subgroup of all elements of $B^t$ of the form $(1^t, a^t, s^{t+1})$, where $1^t$ is the identity of $\Sigma^t$. Then by the first homomorphism theorem $B^t/X_0^t \cong \Sigma^t$. Also consider the projection map $\pi_R : B^t \rightarrow \Sigma^{t+1}$ onto the right states of $B^t$, given by the assignment $(s^t, a^t, s^{t+1}) \mapsto s^{t+1}$. This is a homomorphism with kernel $Y_0^t$, where $Y_0^t$ is the subgroup of all elements of $B^t$ of the form $(s^t, a^t, 1^{t+1})$, where $1^{t+1}$ is the identity of $\Sigma^{t+1}$. Then by the first homomorphism theorem $B^t/Y_0^t \cong \Sigma^{t+1}$. Thus any branch $b^t \in B^t$ is of the form $b^t = (s^t, a^t, s^{t+1})$ where $s^t \in \Sigma^t \cong B^t/X_0^t$ and $s^{t+1} \in \Sigma^{t+1} \cong B^t/Y_0^t$. These results show there is a state group isomorphism $B^t/Y_0^t \cong \Sigma^{t+1} \cong B^{t+1}/X_0^{t+1}$ at each time $t+1$.

Since $C$ is time invariant, we can regard $C$ as the sofic shift [13] of a graph $T$ which is graph isomorphic to $T^t$ for all $t$. The branches of $T$ form a branch group $B$ which is isomorphic to $B^t$, and the states of $T$ form a state group $\Sigma$ which is isomorphic to $\Sigma^t$, for all $t$. We can regard $C$ as the edge shift of $T$ and $\sigma(C)$ as the vertex shift of $T$ [9].

Let $C$ be a group trellis, and let $b$ be a trellis path in $C$. Using [13], define the projection map at time $t$, $\chi^t : C \rightarrow B^t$, by the assignment $b \mapsto b^t$. Define the projection map $\chi^{[t_1, t_2]} : C \rightarrow B^{t_1} \times \cdots \times B^{t_2}$ by the assignment $b \mapsto (b^{t_1}, \ldots, b^{t_2})$. We say that $(b^{t_1}, \ldots, b^{t_2})$ is a trellis path segment of length $t_2 - t_1 + 1$. We say that codeword $b$ has span $t_2 - t_1 + 1$ if $b^{t_1} \neq 1$, $b^{t_2} \neq 1$, and $b^n = 1$ for $n < t_1$ and $n > t_2$.

For any integer $l > 0$, we say a group trellis $C$ is $l$-controllable if for any time epoch $t$, and any pair of states $s$ and $s'$, where $s \in \Sigma^t$ and $s' \in \Sigma^{t+l}$, there is a trellis path segment of length $l$ connecting the two states. A group trellis $C$ is strongly controllable if it is $l$-controllable for some integer $l$. The least integer $l$ for which a group trellis is strongly controllable is denoted as $\ell$. In this paper, we only study the case $l = \ell$. 
3. THE STATIC MATRIX AND SHIFT MATRIX

In this section, we write the coset decomposition chain of a group as a matrix, and call this a matrix chain. We study a matrix chain called a static matrix. There is another matrix of group elements called a shift matrix. We use both matrices to construct a tensor.

In like manner to C, define $C^{t+}$ to be the set of all codewords in $C$ for which $b^n = 1^n$ for $n < t$, where $1^n$ is the identity component of $B^n$ at time $n$. Define $C^{t-}$ to be the set of all codewords in $C$ for which $b^n = 1^n$ for $n ≥ t$. For all integers $j$, define

$$X_j^t \overset{\text{def}}{=} \{x^t(b) : b ∈ C^{(t-j)^+}\}. \quad (4)$$

Note that $X_0^t$ is consistent with the definition previously given in Section 2. We have $X_j^t = 1^t$ for $j < 0$. For all integers $i$, define

$$Y_i^t \overset{\text{def}}{=} \{x^t(b) : b ∈ C^{(t+i+1)^-}\}. \quad (5)$$

Note that $Y_0^t$ is consistent with the definition previously given in Section 2. We have $Y_0^t = 1^t$ for $i > 0$. It is clear that $X_j^t < B^t$, and $Y_i^t < B^t$ for any time $t$ and any integer $j$.

The groups $X_j^t$ and $Y_i^t$ were first introduced in [4]. The group intersections $X_j^t \cap Y_i^{t-j}$, for $0 ≤ j ≤ \ell$, are the groups used in [4] to give an abstract characterization of the branch group of an $\ell$-controllable group trellis.

For any set $H^t ∈ B^t$, define $(H^t)^+$ to be the set of right states of $H^t$, or $\{s^{t+1} : b' = (s^t, a^t, s^{t+1}) ∈ H^t\}$, and define $(H^t)^-$ to be the set of left states of $H^t$, or $\{s^t : b' = (s^t, a^t, s^{t+1}) ∈ H^t\}$.

For sets $H_1^t ⊂ B^t$ and $H_1^{t+1} ⊂ B^{t+1}$ such that $(H_1^t)^+ = (H_2^t)^-$, define the concatenation of $H_1^t$ and $H_2^{t+1}$, $H_1^t \wedge H_2^{t+1}$, to be all the (valid) trellis path segments of length two with first component in $H_1^t$ and second component in $H_2^{t+1}$.

Note that $(X_j^t)^+ = (X_j^{t+1})^-$ and $(Y_i^t)^+ = (Y_i^{t+1})^-$ for all integers $i, j$.

Then $X_j^t \wedge X_j^{t+1}$ and $Y_i^t \wedge Y_i^{t+1}$ are sets of trellis path segments of length two.

The next result follows directly from Proposition 7.2 of [4], using our notation.

**Proposition 2** The group trellis $C$ is $\ell$-controllable if and only if $X_0^t = B^t$, or equivalently, if and only if $Y_0^t = B^t$, for each time $t$.

The group $B^t$ has two normal series (and chief series)

$$1^t = X_{t-1}^t ⊂ X_0^t ⊂ X_1^t ⊂ ⋯ ⊂ X_t^t = B^t,$$

and

$$1^t = Y_{t-1}^t ⊂ Y_0^t ⊂ Y_1^t ⊂ ⋯ ⊂ Y_t^t = B^t.$$

We denote these normal series by $\{X_j^t\}$ and $\{Y_i^t\}$.

The Schreier refinement theorem used to prove the Jordan-Hölder theorem [11] shows how to obtain a refinement of $\{X_j^t\}$ by inserting $\{Y_i^t\}$; we call this the forward Schreier series of $\{X_j^t\}$ and $\{Y_i^t\}$. Since $\{X_j^t\}$ and $\{Y_i^t\}$ are chief series, the forward Schreier series of $\{X_j^t\}$ and $\{Y_i^t\}$ is a chief series.
In equation (8), we have written the forward Schreier series as a matrix of \( \ell + 1 \) columns and \( \ell + 2 \) rows. Note that the terms in the bottom row form the sequence \( X^t_1, X^t_0, X^t_{\ell-1}, \ldots, X^t_{\ell-2}, X^t_\ell \), and the terms in the top row form the sequence \( X^t_0, X^t_1, X^t_2, \ldots, X^t_{\ell-1}, X^t_\ell \). Thus (8) is indeed a refinement of the normal series \( \{X^t_j\} \). We call (8) the matrix chain of the forward Schreier series of \( \{X^t_j\} \) and \( \{Y^t_i\} \).

**Proposition 3** If the group trellis \( C \) is \( \ell \)-controllable, then

\[
X^t_{j-1}(X^t_j \cap Y^t_{\ell-j}) = X^t_j,
\]

for each \( t \), for \( j \geq 0 \).

**Proof.** If the group trellis \( C \) is \( \ell \)-controllable, then from Proposition 7.2 of [4], in our notation,

\[
(X^t_0 \cap Y^t_\ell)(X^t_1 \cap Y^t_{\ell-1}) \cdots (X^t_j \cap Y^t_{\ell-j}) = X^t_j
\]

for all \( j \geq 0 \). This means we can rewrite (7) as (6).

The diagonal terms of the matrix chain (8) are \( X^t_{j-1}(X^t_j \cap Y^t_{\ell-j}) \) for \( j = 0, \ldots, \ell \). Proposition 3 shows that the diagonal terms satisfy \( X^t_{j-1}(X^t_j \cap Y^t_{\ell-j}) = X^t_j \) for \( j = 0, \ldots, \ell \), if the group trellis is \( \ell \)-controllable. For \( j \in [1, \ell] \), this means all column terms above the diagonal term are the same as the diagonal term. Then we can reduce the matrix chain to a triangular form as shown in (9). A triangle can be formed in two ways, depending on whether the columns in (8) are shifted up or not; we have shifted the columns up since it is more useful here. We call (9) the \( X^t_{[t,t]} \) static matrix. To make this notation clearer, the bracketed term \([t, t]\) only appears in the paper as the superscript of a matrix defined over the time interval \([t, t]\) (except in this sentence). A typical entry in the matrix is \( X^t_{j-1}(X^t_j \cap Y^t_{k-j}) \).

**Theorem 4** The \( X^t_{[t,t]} \) static matrix is a description (normal chain and chief series) of the branch group \( B^t \) of an \( \ell \)-controllable group trellis.

**Proof.** Both \( \{X^t_j\} \) and \( \{Y^t_i\} \) are normal chains of the branch group \( B^t \). Then by the Schreier refinement theorem, the forward Schreier series is a normal chain of \( B^t \).

For each \( t \), we can replace \( B^t \) in the group trellis \( C \) with \( X^t_{[t,t]} \). We denote the resulting structure by \( x \); note that \( x \) is a tensor. Since \( X^t_{[t,t]} \) is a coset decomposition chain of \( B^t \), then \( x \) is a description of the coset structure of group trellis \( C \). Each path \( b \in C \) traverses some sequence of cosets in \( x \).

Note that the first column of (9) is a description of \( X^t_0 \), which we can think of as an input. The remaining columns are a description of \( B^t/X^t_0 \), which is isomorphic to the state \( \Sigma^t \). Thus columns of the static matrix contain information about the input and state. Therefore an isomorphic copy of the state code \( \sigma(C) \) is embedded in \( x \).
Since $C$ is time invariant, for any $t$, the elements in $X_{j-1}^t$ and $X_{j-1}^{t+j}$ are the same, the elements in $X_j^t$ and $X_j^{t+j}$ are the same, and the elements in $Y_{k-j}^t$ and $Y_{k-j}^{t+j}$ are the same. Therefore we replace the column containing $X_{j-1}^t(X_j^t \cap Y_{k-j}^t)$ in (9) with the column containing $X_{j-1}^{t+j}(X_j^{t+j} \cap Y_{k-j}^{t+j})$ in (10). Doing this for each column in (9) gives the matrix shown in (10). Since the time index is changed from one column to the next in (10), we no longer have the inclusion from one column to the next as in (9). However the coset decomposition within each column is preserved. We call (10) the $X^{[t,j]}$ shift matrix. Notice the shift matrix extends over the time interval $[t, t + \ell]$. A typical entry in the matrix is $X_{j-1}^{t+j}(X_j^{t+j} \cap Y_{k-j}^{t+j})$.

For $j = 0, \ldots, \ell$, the $j$-th column of static matrix $X^{[t,\ell]}$ is the $j$-th column of a shift matrix $X^{[t-j,t-j+\ell]}$ at time $t - j$. Thus the static matrix $X^{[t,\ell]}$ is a composite of columns of $\ell + 1$ shift matrices.

The forward Schreier series evolves forward in time. There is a dual of the forward Schreier series that evolves backward in time. The backward Schreier series of $\{X_j^t\}$ and $\{Y_i^t\}$ is a refinement of $\{Y_i^t\}$ obtained by inserting $\{X_j^t\}$. The static matrix of the backward Schreier series is $Y^{[t,\ell]}$, the dual of $X^{[t,\ell]}$, and the shift matrix is $Y^{[t - \ell,\ell]}$, the dual of $X^{[t,t+\ell]}$. As an example, the static matrix $Y^{[t,\ell]}$ is shown in (11). $Y^{[t,\ell]}$ is a reflection of $X^{[t,\ell]}$ about the vertical axis. In (9), index $j$ increases from left to right, while in (11), index $i$ increases from right to left. This reflects the symmetry in the definitions of $\{X_j^t\}$ and $\{Y_i^t\}$.

We now show that the $X^{[t,t+\ell]}$ shift matrix (10) has a kind of shift property, after some preliminary results. The discussion will show that the shift matrix has a physical interpretation as the quotient group of certain paths
that split from the identity path.

For any time \( t \), for each branch \( b^t \in B^t \), we define the following branch set \( \mathcal{F}(b^t) \) to be the set of branches that can follow \( b^t \) at the next time epoch \( t + 1 \) in valid trellis paths. In other words, branch \( b^{t+1} \in \mathcal{F}(b^t) \) if and only if \( (b^t)^+ = (b^{t+1})^- \). Then the following branch set \( \mathcal{F}(b^t) \) represents the contraction, correspondence, and expansion given by

\[
b^t \mapsto b^t Y^t_0 \leftarrow b^{t+1} X^{t+1}_0,\]

where \( \eta \) is the 1-1 correspondence \( B^t/Y^t_0 \stackrel{\eta}{\rightarrow} B^{t+1}/X^{t+1}_0 \) given by the state group isomorphism \( B^t/Y^t_0 \simeq B^{t+1}/X^{t+1}_0 \).

It is clear that \( b^t \in B^t \) and \( \mathcal{F}(b^t) \subset B^{t+1} \). However note that \( \mathcal{F} \) is not a function with domain \( B^t \) and range \( B^{t+1} \). But we can think of \( \mathcal{F} \) as a relation on \( B^t \times B^{t+1} \). In this relation, we can think of \( \mathcal{F} \) as an assignment of set \( \mathcal{F}(b^t) \) to branch \( b^t \), or \( \mathcal{F} : b^t \mapsto \mathcal{F}(b^t) \).

**Proposition 5** If \( (b^t)^+ = (b^{t+1})^- \), the following branch set \( \mathcal{F}(b^t) \) of a branch \( b^t \) in \( B^t \) is the coset \( b^{t+1} X^{t+1}_0 \) in \( B^{t+1} \), or the assignment \( \mathcal{F} : b^t \mapsto b^{t+1} X^{t+1}_0. \)

Define the following branch set \( \mathcal{F} : B^t \to B^{t+1} \) such that for any set \( H^t \subset B^t \), the set \( \mathcal{F}(H^t) \) is the union \( \cup_{b^t \in H^t} \mathcal{F}(b^t) \). The set \( \mathcal{F}(H^t) \) always consists of cosets of \( X^{t+1}_0 \). In particular, \( \mathcal{F}(X^t_j) = X^{t+1}_j \) for all integers \( j \geq -1 \).

For a set \( H^t \subset B^t \) and integer \( j > 0 \), define \( \mathcal{F}^j(H^t) \) to be the \( j \)-fold composition \( \mathcal{F}^j(H^t) = \mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F}(H^t) \). For \( j = 0 \), define \( \mathcal{F}^0(H^t) = \mathcal{F}(H^t) \) to be just \( H^t \). If \( H^t \) is a set of trellis branches at time epoch \( t \), then \( \mathcal{F}^j(H^t) \) is the set of trellis branches at time epoch \( t + j \), such that for each
$b^{t+j} \in \mathcal{F}^j(H^t)$ there is a $b' \in H^t$ and a path in the trellis from $b'$ to $b^{t+j}$. Note that $X_j^{t+j} = \mathcal{F}^j(X_0^t)$.

For a set $H^t \subset B^t$ and integer $k \geq 0$, define $\mathcal{F}^{[0,k]}(H^t)$ to be the set of all trellis path segments $(b', \ldots, b^{t+k})$ on time interval $[t, t + k]$ that start with a branch $b' \in H^t$.

**Proposition 6** For any subsets $G^t, H^t$ of $B^t$, we have $(G^t H^t)^+ = (G^t)^+(H^t)^+$, $(G^t H^t)^- = (G^t)^-(H^t)^-$, and $\mathcal{F}(G^t H^t) = \mathcal{F}(G^t) \mathcal{F}(H^t)$.

**Proof.** It is clear that $(G^t H^t)^+ = (G^t)^+(H^t)^+$. Then it follows that $\mathcal{F}(G^t H^t) = \mathcal{F}(G^t) \mathcal{F}(H^t)$.

**Proposition 7** For any subsets $G^t, H^t$ of $B^t$, we have $(G^t \cap H^t)^+ = (G^t)^+ \cap (H^t)^+$, $(G^t \cap H^t)^- = (G^t)^- \cap (H^t)^-$, and $\mathcal{F}(G^t \cap H^t) = \mathcal{F}(G^t) \cap \mathcal{F}(H^t)$.

We index the rows and columns of (10), and denote terms, in a definite way. We index the columns with $j$, for $0 \leq j \leq \ell$, and rows with $k$, for $0 \leq k \leq \ell$, starting with $(j, k) = (0,0)$ in the bottom left corner. In general, we indicate a term in the shift matrix by $X_{j-1}(X_j^t \cap Y_{k-1}^t)$, where the subscripts mean definite things. The subscript $\alpha$ of $X$ in the factor term $(X_\alpha \cap Y_\beta)$ always indicates the column, and the sum of the subscripts $\alpha + \beta$ of $X$ and $Y$ in the factor term always indicates the row. So the term $X_j^t(Y_{k+1}^t \cap X_{j+1}^t)$ is in column $j$ and row $k$. We do not include terms of the form $X_{j+1}^t(1^{t+j})$. For example, $X_{-1}^t(X_0^t \cap Y_0^t)$ is the bottom left corner term, in column $j = 0$ and row $k = 0$. As other examples, the factor term $(X_j^t \cap Y_{k+1}^t)$ is in column $j$ and row $k - 1$, and the factor term $(X_{j+1}^t \cap Y_{k+1}^t)$ is in column $j + 1$ and row $k - 1$. Note that row $k$ of the shift matrix has (length) $k+1$ terms, ignoring the last term $X_{k+1}^t(1^{t+k+1})$.

We now show the $X^{[t,t+\ell]}$ shift matrix preserves shifts, that is, it has a shift property.

**Proposition 8** Fix $k$, $0 \leq k \leq \ell$, and fix $j$, $0 \leq j \leq k$. The shift matrix has a shift property: the term $X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k+1}^t)$ in column $j+1$ and row $k$ is a shift of the term $X_j^t(Y_{k-1}^t \cap X_{j+1}^t)$ in column $j$ and row $k$, that is,

\[
\mathcal{F}(X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k+1}^t)) = X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k-1}^t).
\]

**Proof.** Fix $k$, $0 \leq k \leq \ell$, and fix $j$, $0 \leq j \leq k$. We have

\[
\begin{align*}
\mathcal{F}(X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k+1}^t)) &= \mathcal{F}(X_{j+1}^t) \mathcal{F}(X_{j+1}^t \cap Y_{k+1}^t) \\
&= X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k+1}^t) \\
&= X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k+1}^t) \\
&= X_{j+1}^t(1^{t+j+1}) (X_{j+1}^t \cap Y_{k-1}^t),
\end{align*}
\]

where the fourth equality follows from the Dedekind Law (if $H$, $K$, and $L$ are subgroups of group $G$ with $H \subset L$, then $HK \cap L = H(K \cap L)$).
The first column of the shift matrix \( [10] \) will be important to us so we define \( \Delta_k^t = X_0^t \cap Y_k^t \) for \(-1 \leq k \leq \ell \). We now show that row \( k \) of the shift matrix is just \( F_x^t \chi \) for \( k \) from the identity state at time epoch \( t \). We have \( \Delta_k^t \) to show it is true for \( j = n + 1 \). Then use \( [12] \) to show it is true for \( j = n + 1 \). Then \( [13] \) shows the \( k \)-th row of the shift matrix is just the terms in \( F_x^t(\Delta_k^t) \).

\[ \text{Theorem 9} \quad \text{Fix} \ k, \ 0 \leq k \leq \ell. \ \text{We have} \]
\[ F_x^t(\Delta_k^t) = X_j^{t+j} \cap Y_k^{t+j}, \]
for \( 0 \leq j \leq k \). And \( F_x^t(\Delta_k^t) = X_j^{t+j} \cap Y_k^{t+j} \) for \( j > \ell \). The \( k \)-th row of the shift matrix is just the terms in \( F_x^t(\Delta_k^t) \).

\[ \text{Proof.} \quad \text{We prove} \ [13] \ \text{by induction. Assume it is true for} \ j = n. \ \text{Then use} \ [12] \ \text{to show it is true for} \ j = n + 1. \ \text{Then} \ [13] \ \text{shows the} \ k \text{-th row of the shift matrix is just the terms in} \ F_x^t(\Delta_k^t). \]

Note that \( \chi^{t,t+\ell}(C^t) \) are the trellis path segments in a truncated ray, paths in the trellis which split from the identity state at time epoch \( t \). Further we have \( \chi^{t,t+\ell}(C^t) = F_x^t(X_0^t) \).

\[ \text{Theorem 10} \quad \text{The} \ X^{t,t+\ell} \ \text{shift matrix describes the coset structure of the truncated ray} \ \chi^{t,t+\ell}(C^t) = F_x^t(X_0^t) \ \text{of an} \ \ell \text{-controllable group trellis}. \]

Using Theorem 9 we can represent a quotient group of adjacent terms in the same column of shift matrix \( [10] \) in two equivalent ways:

\[ F_x^t(\Delta_k^t) = \frac{X_j^{t+j} \cap Y_k^{t+j}}{X_j^{t+j} \cap Y_k^{t+j}}, \]
for \( 0 \leq j \leq k \).

\[ \text{Proposition 11} \quad F_x^t(\Delta_k^t) \text{ and } F_x^t(\Delta_k^t) \text{ are groups.} \]

\[ \text{Proof.} \quad \Delta_k^t = X_0^t \cap Y_k^t \text{ is a group so the trellis path segments in } F_x^t(\Delta_k^t) \text{ are a group.} \]

\[ \text{Proposition 12} \quad F_x^t(\Delta_k^t) \mid F_x^t(\Delta_k^t) \text{ if and only if } \Delta_k^t \mid \Delta_k^t \text{ if and only if } \Delta_k^t \mid \Delta_k^t. \ \text{Then as a result } F_x^t(\Delta_k^t) \mid F_x^t(\Delta_k^t) \text{ if and only if } F_x^t(\Delta_k^t) \mid F_x^t(\Delta_k^t). \]

\[ \text{Theorem 13} \quad \text{We have} \]
\[ F_x^t(\Delta_k^t) \mid F_x^t(\Delta_k^t) \sim \Delta_k^t \]
\[ \text{Proof.} \quad \text{The projection } \chi^t : F_x^t(\Delta_k^t) \to \Delta_k^t \text{ is onto. It is a homomorphism with kernel } F_x^t(\Delta_k^t). \ \text{The projection } \chi^t : F_x^t(\Delta_k^t) \to \Delta_k^t \text{ is onto. It is a homomorphism with kernel } F_x^t(\Delta_k^t). \ \text{Therefore, by the first homomorphism theorem,} \]
\[ \frac{F_x^t(\Delta_k^t)}{F_x^t(\Delta_k^t)} \sim \Delta_k^t, \]
\[ \frac{F_x^t(\Delta_k^t)}{F_x^t(\Delta_k^t)} \sim \Delta_k^t. \]

Now use the correspondence theorem and third isomorphism theorem to complete the proof. \[ \bullet \]
Proposition 14 \( F^j(\Lambda_{t-1}) \) and \( F^j(\Delta^t_k) \) are groups.

Proof. See (13) or note that \( F^j(\Delta^t_k) \) is the projection of group \( F^{[0,k]}(\Delta^t_k) \) on the time interval \([t+j, t+j]\).

Proposition 15 \( F^j(\Delta^t_{k-1}) \triangleleft F^j(\Delta^t_k) \).

Proof. See (14).

Theorem 16 For \( 0 \leq j \leq k \), we have

\[
\frac{F^{[0,k]}(\Delta^t_k)}{F^{[0,k]}(\Delta^t_{k-1})} \cong \frac{F^j(\Delta^t_k)}{F^j(\Delta^t_{k-1})}.
\]

Proof. The projection \( \chi^{t+j} : F^{[0,k]}(\Delta^t_k) \to F^j(\Delta^t_k) \) is onto. It is a homomorphism with kernel \( K_k \), the path segments in \( F^{[0,k]}(\Delta^t_k) \) that are the identity at time \( t+j \). The projection \( \chi^{t+j} : F^{[0,k]}(\Delta^t_{k-1}) \to F^j(\Delta^t_{k-1}) \) is onto. It is a homomorphism with kernel \( K_{k-1} \), the path segments in \( F^{[0,k]}(\Delta^t_{k-1}) \) that are the identity at time \( t+j \). Therefore, by the first homomorphism theorem,

\[
\frac{F^{[0,k]}(\Delta^t_k)}{K_k} \cong F^j(\Delta^t_k),
\]

\[
\frac{F^{[0,k]}(\Delta^t_{k-1})}{K_{k-1}} \cong F^j(\Delta^t_{k-1}).
\]

We now show \( K_k = K_{k-1} \); we first show \( K_k \subseteq K_{k-1} \). Let \( (b', \ldots, b'^{t+j}, \ldots, b'^{t+k}) \) be a path segment in \( F^{[0,k]}(\Delta^t_k) \) that is the identity at time \( t+j \), \( 0 \leq j \leq k \). But then \( b' \) must be in \( (X^t_0 \cap Y^t_{j-1}) = \Delta^t_{j-1} \). Since \( j \leq k \), then \( \Delta^t_{j-1} \subseteq \Delta^t_{k-1} \) and \( b' \in \Delta^t_{k-1} \). Then \( (b', \ldots, b'^{t+j}, \ldots, b'^{t+k}) \in F^{[0,k]}(\Delta^t_{k-1}) \) and \( (b', \ldots, b'^{t+j}, \ldots, b'^{t+k}) \in K_{k-1} \). Therefore \( K_k \subseteq K_{k-1} \).

We now show \( K_{k-1} \subseteq K_k \). Let \( (b^t, \ldots, b^{t+j}, \ldots, b^{t+k}) \) be a path segment in \( F^{[0,k]}(\Delta^t_{k-1}) \) that is the identity at time \( t+j \), \( 0 \leq j \leq k \). But then \( b^j \) must be in \( (X^t_0 \cap Y^t_{j-1}) = \Delta^t_{j-1} \). Since \( j \leq k \), then \( \Delta^t_{j-1} \subseteq \Delta^t_{k} \) and \( b^j \in \Delta^t_{k} \). Then \( (b^t, \ldots, b^{t+j}, \ldots, b^{t+k}) \in F^{[0,k]}(\Delta^t_k) \) and \( (b^t, \ldots, b^{t+j}, \ldots, b^{t+k}) \in K_k \). Therefore \( K_{k-1} \subseteq K_k \).

We have just shown \( K_k = K_{k-1} \). Now use the correspondence theorem and third isomorphism theorem to complete the proof.

Note that the proof breaks down if we try to go further. In other words, we cannot show that for \( 0 \leq j \leq k \), we have

\[
\frac{F^{[0,k]}(\Delta^t_k)}{F^{[0,k]}(\Delta^t_{k-2})} \cong \frac{F^j(\Delta^t_k)}{F^j(\Delta^t_{k-2})}.
\]

Define

\[
\Lambda^{[t, t+k]} \overset{\text{def}}{=} \frac{F^{[0,k]}(\Delta^t_k)}{F^{[0,k]}(\Delta^t_{k-1})}.
\]

Corollary 17 For \( 0 \leq j \leq k \), the \( t+j \)-th components of a transversal of \( \Lambda^{[t, t+k]} \) are a transversal of

\[
\frac{F^j(\Delta^t_k)}{F^j(\Delta^t_{k-1})}.
\]

(15)
Theorem 16 shows that the projection \( \chi^{t+j}(\Lambda^{[t,t+k]}) \) gives a 1-1 correspondence between cosets of \( \Lambda^{[t,t+k]} \) and cosets of \( \chi^{t+j} \). Therefore the projection \( \chi^{t+j} \) of a transversal of \( \Lambda^{[t,t+k]} \) is a transversal of \( \chi^{t+j} \).

**Corollary 18** For \( 0 \leq k \leq \ell \), and \( 0 \leq j \leq k \), we have

\[
\frac{\Delta^t_k}{\Delta^t_{k-1}} \simeq \frac{\mathcal{F}^{[0,k]}(\Delta^t_k)}{\mathcal{F}^{[0,k]}(\Delta^t_{k-1})} \simeq \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})},
\]

Remark: This result can be regarded as a rectangle criterion for a shift matrix, with \( \Delta^t_k, \Delta^t_{k-1}, \mathcal{F}^t(\Delta^t_k), \) and \( \mathcal{F}^t(\Delta^t_{k-1}) \) as the corners of a rectangle in (10). It is similar in spirit to a quadrangle criterion for a Latin square [14] or a configuration theorem for a net [15]. In fact, the rectangle condition can be generalized further by starting with groups \( \Delta^t_k \) and \( \Delta^t_{k-m} \), for \( m > 1 \). These more general results are not needed.

We can use (14) and Corollary 18 to create a tensor. Fix \( j \) such that \( 0 \leq j \leq \ell \), and define \( X^{t+j}_j \parallel X^{t+j}_{j-1} \) to be the column vector of quotient groups

\[
X^{t+j}_j \parallel X^{t+j}_{j-1} \equiv \left( \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})} \cdots \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})} \right)^T
\]

for \( k \) such that \( j \leq k \leq \ell \). This is the vector of quotient groups formed from groups in the normal chain in the center column of (10). Then using (14),

\[
X^{t+j}_j \parallel X^{t+j}_{j-1} \equiv \left( X^t_0 \parallel X^t_{-1} \quad X^t_1 \parallel X^t_0 \quad \cdots \quad X^t_j \parallel X^t_{j-1} \quad \cdots \quad X^t_\ell \parallel X^t_{\ell-1} \right).
\]

This is a second example of a shift matrix. The \( k \)-th row of shift matrix \( X^{[t,t+\ell]}_j \), \( 0 \leq k \leq \ell \), is a shift vector

\[
\left( \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})} \cdots \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})} \right).
\]

The shift vector is just all the components of

\[
\Lambda^{[t,t+k]} = \frac{\mathcal{F}^{[0,k]}(\Delta^t_k)}{\mathcal{F}^{[0,k]}(\Delta^t_{k-1})}.
\]

Corollary 18 shows the shift matrix \( X^{[t,t+\ell]}_j \) preserves isomorphism of quotient groups, and each shift of a quotient group in a row gives the next quotient group in the row. Therefore we can regard a shift matrix \( X^{[t,t+\ell]}_j \) as the natural shift structure of a strongly controllable group system.

Fix \( j \) such that \( 0 \leq j \leq \ell \), and define \( X^t_j \parallel X^t_{j-1} \) to be the column vector of quotient groups

\[
X^{t+j}_j \parallel X^{t+j}_{j-1} \equiv \left( \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})} \cdots \frac{\mathcal{F}^t(\Delta^t_k)}{\mathcal{F}^t(\Delta^t_{k-1})} \right).
\]
for $k$ such that $j \leq k \leq \ell$. This is the vector of quotient groups formed from groups in the normal chain in the center column of (19). For $j = 0, \ldots, \ell$, we obtain the column vectors $X_j^{t} / X_j^{t-1}$, which can be used to form the static matrix $X_j^{[t,t]}$,

$$X_j^{[t,t]} \defeq \begin{pmatrix} X_0^{t} / X_0^{t-1} & X_1^{t} / X_0^{t-1} & \cdots & X_j^{t} / X_j^{t-1} & \cdots & X_{\ell}^{t} / X_{\ell-1}^{t-1} \end{pmatrix}.$$  \hspace{1cm} (19)

Note that the definition of $X_j^{t+j} / X_j^{t-1}$ and $X_j^{t} / X_j^{t-1}$ is consistent since $X_j^{t+j} / X_j^{t-1}$ is defined using (16) and $X_j^{t} / X_j^{t-1}$ is defined using (18), and (16) and (18) are consistent. Note that

$$X_j^{t} / X_j^{t-1} = X_j^{(t-j)+j} / X_j^{(t-j)+j},$$

and we can think of $X_j^{(t-j)+j} / X_j^{(t-j)+j}$ as the definition $X_j^{t'j} / X_j^{t'j}$ with time $t'$ defined by the parentheses term $(t-j)$. Then we can also think of static matrix (19) as

$$X_j^{[t,t]} \defeq \begin{pmatrix} X_0^{t} / X_0^{t-1} & X_1^{(t-1)+1} / X_0^{(t-1)+1} & \cdots & X_j^{(t-j)+j} / X_j^{(t-j)+j} & \cdots & X_{\ell}^{(t-\ell)+\ell} / X_{\ell-1}^{(t-\ell)+\ell} \end{pmatrix}.$$  \hspace{1cm} (20)

Now it is clear that each term in (20) is from one of $\ell + 1$ different shift matrices.

We can relate a static matrix $X_j^{[t,t]}$ to a shift matrix $X_j^{[t,t+\ell]}$ using the tensor description shown in (21). Time increases as we move up the page. The vectors in the shift matrix (17) are the vectors along the diagonal in (21), and the vectors in the static matrix (20) are the vectors in a row of (21). The superscript parentheses terms in (21), like $(t-j)$, indicate terms that all belong to the same shift matrix. For example, the diagonal terms

$$X_0^{(t-j)} / X_0^{(t-j)}, X_1^{(t-j)+1} / X_0^{(t-j)+1}, \ldots, X_j^{(t-j)+j} / X_j^{(t-j)+j}, \ldots, X_{\ell}^{(t-\ell)+\ell} / X_{\ell-1}^{(t-\ell)+\ell},$$

all belong to the shift matrix starting at time $t-j$, $X_j^{[(t-j),(t-j)+\ell]}$. The center row in (21) is (20), which reduces to (19), which is just the static matrix $X_j^{[t,t]}$.

$$\begin{pmatrix}
\vdots & & & & X_\ell^{(t-\ell)+\ell} / X_{\ell-1}^{(t-\ell)+\ell} \\
\vdots & \ddots & & & \vdots \\
\vdots & & \ddots & & \vdots \\
X_0^{(t-j)} / X_0^{(t-j)} & X_1^{(t-j)+1} / X_0^{(t-j)+1} & \cdots & X_j^{(t-j)+j} / X_j^{(t-j)+j} & \cdots & X_{\ell}^{(t-\ell)+\ell} / X_{\ell-1}^{(t-\ell)+\ell} \\
\vdots & \ddots & & & \vdots \\
\vdots & & \ddots & & \vdots \\
X_0^{(t-j)} / X_0^{(t-j)} & X_1^{(t-j)+1} / X_0^{(t-j)+1} & \cdots & \cdots & \vdots \\
\vdots & \ddots & & & \vdots \\
\vdots & & \ddots & & \vdots \\
\vdots & & & \ddots & \vdots \\
\vdots & & & & \ddots \\
\vdots & & & & & \ddots 
\end{pmatrix}$$  \hspace{1cm} (21)
We let $\mathbf{x}_g$ denote the tensor in (21), and say $\mathbf{x}_g$ is a chain tensor. For a given group trellis $C$, there is only one chain tensor $\mathbf{x}_g$. The tensor $\mathbf{x}_g$ is a description of the coset structure of group trellis $C$. The tensor $\mathbf{x}_g$ has a dual nature of having both shift matrices and static matrices. The most natural and important way to understand $\mathbf{x}_g$ is to look at (21) along the diagonals, in terms of shift matrices.

**Theorem 19** For each time $t$, the diagonals of (21) are a description of the quotient groups $\Lambda^{[t,t+k]}$ for $k$ such that $0 \leq k \leq \ell$.

In the next two sections, we will show how to recover paths $\mathbf{b} \in C$ from generators, which are representatives of the coset structure described by $\mathbf{x}_g$. 
4. GENERATORS AND THE GENERATOR MATRIX

We now show the Forney-Trott generators are a transversal of \( \Lambda^{[t,t+k]} \), and components of the generators are a transversal of \([15]\), for \( 0 \leq j \leq k \). Forney and Trott \([3]\) define a generator for a group code \( C \) using the quotient group

\[
\mathcal{T}^{[t,t+k]} \defeq \frac{C^{[t,t+k]}}{C^{[t,t+k]}C^{[t,t+k]}},
\]

for \( 0 \leq k \leq \ell \), where \( \mathcal{T}^{[t,t+k]} \) is called a granule. A coset representative of \( \mathcal{T}^{[t,t+k]} \) is called a generator. The coset representative of \( C^{[t,t+k]}C^{[t,t+k]} \) is always taken to be the identity sequence. In case \( \mathcal{T}^{[t,t+k]} \) is isomorphic to the identity group, the identity sequence is the only coset representative. A nonidentity generator is an element of \( C^{[t,t+k]} \) but not of \( C^{[t,t+k]} \) or of \( C^{[t,t+k]} \), so its span is exactly \( k+1 \). Thus every nonidentity generator is a codeword that cannot be expressed as a combination of shorter codewords \([3]\). A basis of \( C \) is a minimal set of shortest length generators that is sufficient to generate the group system \( C \) \([7]\). It is a set of coset representatives of \( \mathcal{T}^{[t,t+k]} \), for \( 0 \leq k \leq \ell \).

Since a group trellis is a group system, we can transcribe the generator approach of \([3]\) to the group trellis \( C \), used here, as

\[
\Gamma^{[t,t+k]} \defeq \frac{C^{[t,t+k]}}{C^{[t,t+k]}C^{[t,t+k]}}
\]

where quotient group \( \Gamma^{[t,t+k]} \) is a granule. If \( Q \) is any quotient group, let \([Q]\) denote a transversal of \( Q \). Let \([\Gamma^{[t,t+k]}]\) be a transversal of \( \Gamma^{[t,t+k]} \). A coset representative of \( \Gamma^{[t,t+k]} \), or an element of \( \Gamma^{[t,t+k]} \), is a generator \( g^{[t,t+k]} \), or a generator at time \( t \). Then transversal \([\Gamma^{[t,t+k]}]\) is a set of representatives \( g^{[t,t+k]} \) of \( \Gamma^{[t,t+k]} \) at time \( t \). For each time \( t \), let vector basis \( B^t \) be the set of generators \( \{g^{[t,t+k]} \in [\Gamma^{[t,t+k]}] : 0 \leq k \leq \ell \} \) in all transversals at time \( t \). We allow \( B^t \) to vary with time, e.g., \( B^{t+1} \) need not be just a time shift of \( B^t \). The sequence of vector bases, \( \ldots, B^t, B^{t+1}, \ldots \), gives a basis \( B = \{B^t : t \in Z\} \).

We also consider a constant basis \( B_c = \{\ldots, B, B, \ldots\} \) where \( B^t \) is the same vector basis \( B \) for all \( t \in Z \).

We now show that the projection \( \chi^{[t,t+k]} \) of generators in \( \Gamma^{[t,t+k]} \) is also a transversal of \( \Lambda^{[t,t+k]} \). Therefore a basis \( B \) of \( C \) can be found using representatives of either \( \Gamma^{[t,t+k]} \) or \( \Lambda^{[t,t+k]} \).

**Lemma 20** The set of paths formed by the concatenation of groups

\[
\ldots, l^{t-2}, l^{t-1} \wedge (X_{t}^{t} \cap Y_{k}^{t}) \wedge \cdots \wedge (X_{j}^{t+j} \cap Y_{k-j}^{t+j}) \wedge (X_{j}^{t+j+1} \cap Y_{k-j-1}^{t+j+1}) \wedge \cdots \wedge (X_{k}^{t+k} \cap Y_{0}^{t+k}) \wedge 1^{t+k+1}, 1^{t+k+2}, \ldots
\]

is \( C^{[t,t+k]} \).

**Proof.** From the proof of Proposition \([8]\) we have

\[
\mathcal{F}(X_{j}^{t+j} \cap Y_{k-j}^{t+j}) = X_{0}^{t+j+1}(X_{j}^{t+j+1} \cap Y_{k-j-1}^{t+j+1}).
\]

This means the set of paths formed by the concatenation of groups in \([22]\) is well defined: for any branch \( b^{t+j} \in X_{j}^{t+j} \cap Y_{k-j}^{t+j} \), there is a branch \( b^{t+j+1} \in X_{j}^{t+j+1} \cap Y_{k-j-1}^{t+j+1} \) such that \( b^{t+j+1} = (b^{t+j})^{-1} \), and \( b^{t+j}, b^{t+j+1} \) is a trellis path segment of length two. The paths in \([22]\) consist of sequences which
split from the identity state at time $t$ and merge to the identity state at time $t + k + 1$. Therefore, any path in (22) must be in $C^{[t,t+k]}$. Fix integer $k$ such that $0 \leq k \leq \ell$. Let $b$ be a sequence in $C^{[t,t+k]}$. We now show $b$ is in (22). If $b \in C^{[t,t+k]}$, then for each $j$, $0 \leq j \leq k$, $b^{t+j}$ must be in $X_j^{t+j}$, but cannot be in $X_m^{t+j}$, $m > j$. Similarly, $b^{t+j}$ must be in $Y_k^{t+j}$. Then $b^{t+j} \in X_j^{t+j} \cap Y_k^{t+j}$ for all $j \in [0,k]$. Since (22) contains all code sequences whose component $b^{t+j} \in X_j^{t+j} \cap Y_k^{t+j}$ for all $j \in [0,k]$, then $b$ is in (22).

**Lemma 21** For $j$, $0 \leq j \leq k$, we have $\chi^{t+j}(C^{[t,t+k]}) = X_j^{t+j} \cap Y_k^{t+j}$. For example, this means $X_0^{t} \cap Y_0^{t} = \chi^{t}(C^{[t,t+k]})$ and $Y_0^{t+k} \cap X_k^{t+k} = \chi^{t+k}(C^{[t,t+k]})$.

**Proof.** From (22), we know $\chi^{t+j}(C^{[t,t+k]}) \subseteq X_j^{t+j} \cap Y_k^{t+j}$.

We now show $X_j^{t+j} \cap Y_k^{t+j} \subseteq \chi^{t+j}(C^{[t,t+k]})$. The proof of Lemma 20 shows that for any branch $b^{t+j} \in X_j^t \cap Y_k^{t+j}$, there is a branch $b^{t+j+1} \in X_j^{t+j+1} \cap Y_k^{t+j+1}$ such that $(b^{t+j}, b^{t+j+1})$ is a trellis path segment of length two. We can continue this argument: for any branch $b^{t+j+1} \in X_j^{t+j+1} \cap Y_k^{t+j+1}$, there is a branch $b^{t+j+2} \in X_j^{t+j+2} \cap Y_k^{t+j+2}$ such that $(b^{t+j+1}, b^{t+j+2})$ is a trellis path segment of length two. Continuing the argument further shows that for any branch $b^{t+j} \in X_j^{t+j} \cap Y_k^{t+j}$, there is a trellis path segment of length $k-j+1$, $(b^{t+j}, b^{t+j+1}, \ldots, b^{t+k})$, which merges to the identity state at time $t+k+1$. This argument works in reverse time as well: for any branch $b^{t+j} \in X_j^{t+j} \cap Y_k^{t+j}$, there is a branch $b^{t+j+1} \in X_j^{t+j+1} \cap Y_k^{t+j+1}$ such that $(b^{t+j+1}, b^{t+j})$ is a trellis path segment of length two, and so on. Thus we see that for any $b^{t+j} \in X_j^{t+j} \cap Y_k^{t+j}$, there is a sequence $b \in C^{[t,t+k]}$ such that $\chi^{t+j}(b) = b^{t+j}$. Thus we have shown $X_j^{t+j} \cap Y_k^{t+j} \subseteq \chi^{t+j}(C^{[t,t+k]})$.

**Lemma 22** We have

\[
\chi^{[t,t+k]}(C^{[t,t+k]}) \subseteq \mathcal{F}^{[0,k]}(\Delta_k^{t}),
\]

and

\[
\chi^{[t,t+k]}(C^{[t,t+k]}C^{(t,t+k)}) \subseteq \mathcal{F}^{[0,k]}(\Delta_{k-1}^{t}).
\]

**Proof.** We have (23) holds if and only if $\chi^{t}(C^{[t,t+k]}) \subseteq \Delta_k$. But this follows from Lemma 21. We have (24) holds if and only if $\chi^{t}(C^{[t,t+k]}C^{(t,t+k)}) \subseteq \Delta_{k-1}$. But $\chi^{t}(C^{[t,t+k]}C^{(t,t+k)}) = \chi^{t}(C^{[t,t+k]}) \Delta_{k-1}$ from Lemma 21.

**Theorem 23** There is an isomorphism

\[
\Gamma^{[t,t+k]} \cong \Lambda^{[t,t+k]},
\]

where the 1-1 correspondence $\mu$ between cosets of $\Gamma^{[t,t+k]}$ and $\Lambda^{[t,t+k]}$ is given by

\[
\mu : C^{[t,t+k]}C^{(t,t+k)} \mapsto \mathcal{F}^{[0,k]}(\chi^{t}(C^{[t,t+k]}C^{(t,t+k)}b)).
\]

**Proof.** Using Lemma 22, we have

\[
\chi^{t}(C^{[t,t+k]}C^{(t,t+k)}b) = \chi^{t}(C^{[t,t+k]}b) = \chi^{t}(C^{[t,t+k]})\chi^{t}(b) = \Delta_{k-1}b^{t}.
\]
Since \( \Lambda^{[t,t+k]} = \mathcal{F}^{[0,k]}(\Delta^t_k)/\mathcal{F}^{[0,k]}(\Delta^t_{k-1}) \), this shows we can properly define the 1-1 correspondence \( \mu \) between cosets of \( \Gamma^{[t,t+k]} \) and \( \Lambda^{[t,t+k]} \) as given in (25).

Forney and Trott [3] define an input chain \( F^t_k \subset F^t_1 \subset \cdots \subset F^t_\ell \) by the projection \( F^t_k = \chi^t(C^{[t,t+k]}) \) for \( k = 0, 1, \ldots, \ell \). Using Lemma 21, this gives \( F^t_k = \Delta^t_k \). In their Input Granule Theorem [3], Forney and Trott show that \( \Gamma^{[t,t+k]} \simeq F^t_k/F^t_{k-1} \) for \( k \) such that \( 0 < k \leq \ell \). Then we have

\[
\Gamma^{[t,t+k]} \simeq F^t_k/F^t_{k-1} = \Delta^t_k/\Delta^t_{k-1}.
\]

Combining this with Theorem 13 gives

\[
\Gamma^{[t,t+k]} \simeq \Delta^t_k/\Delta^t_{k-1} \simeq \Lambda^{[t,t+k]}.
\]

Then following the correspondences given in the Input Granule Theorem of [3] and Theorem 13 shows that the isomorphism \( \Gamma^{[t,t+k]} \simeq \Lambda^{[t,t+k]} \) is given by \( \mu \).

**Corollary 24** Let \( [\Gamma^{[t,t+k]}] \) be a set of generators which is a transversal of \( \Gamma^{[t,t+k]} \). Then \( \{\chi^{[t,t+k]}(g^{[t,t+k]}_{j,k}) : g^{[t,t+k]}_{j,k} \in [\Gamma^{[t,t+k]}]\} \) is a transversal of \( \Lambda^{[t,t+k]} \).

The above corollary shows that any set of Forney-Trott generators can equally well be found from a transversal of \( \Lambda^{[t,t+k]} \).

If \( Q \) is any quotient group, there is another way we denote a transversal of \( Q \) besides \( [Q] \). If \( \{q\} \) is a set of coset representatives of \( Q \) which is a transversal of \( Q \), we let \( \{\{q\}\} \) denote a transversal of \( Q \).

Fix \( k \) such that \( 0 \leq k \leq \ell \). Let generator \( g^{[t,t+k]}_{j,k} \) be a representative in \( \Gamma^{[t,t+k]} \),

\[
g^{[t,t+k]} = \ldots, 1^{t-2}, 1^{t-1}, r^{t+1}_{0,k}, r^{t+1}_{1,k}, \ldots, r^{t+1}_{j,k}, \ldots, r^{t+1}_{k,k}, 1^{t+1}, 1^{t+k+1}, 1^{t+k+2}, \ldots.
\]

From (22) we know component \( r^{t+1}_{j,k} \) is an element of \( X^{t+1}_{j} \cap Y^{t+1}_{k-j} \), and from Corollaries 24 and 17 we know \( r^{t+1}_{j,k} \) is a representative of

\[
\mathcal{F}^{j}(\Delta^t_k) \quad \frac{X_{j}^{t+1} \cap Y_{k-j}^{t+1}}{X_{j-1}^{t+1} \cap Y_{k-j-1}^{t+1}},
\]

for \( j = 0, 1, \ldots, k \). If we pick a set of generators \( g^{[t,t+k]} \) which is a transversal of \( \Gamma^{[t,t+k]} \), \([\Gamma^{[t,t+k]}]\), then \([\Gamma^{[t,t+k]}]\) induces a transversal \([\{r^{t+1}_{j,k}\}]\) of (27), for \( j = 0, 1, \ldots, k \).

Pick a generator \( g^{[t,t+k]} \) in \( \Gamma^{[t,t+k]} \) for each \( k, 0 \leq k \leq \ell \). We can arrange the nontrivial components of these generators in a matrix as shown in (29), which is called a shift matrix, or also a generator matrix, at time \( t \), and denoted \( F^{[t,t+\ell]} \). The \( k \)-th row of matrix \( F^{[t,t+\ell]} \), \( 0 \leq k \leq \ell \), is a shift vector, also called a generator vector, denoted \( r^{[t,t+k]} \), where

\[
r^{[t,t+k]} \overset{\text{def}}{=} (r_{0,k}^{t}, r_{1,k}^{t+1}, \ldots, r_{j,k}^{t+j}, \ldots, r_{k,k}^{t+k}).
\]
A generator vector \( r^{[t, t+\ell]} \) is the nontrivial components of the generator \( g^{[t, t+k]} \).

\[
\begin{array}{cccccccc}
  r_{0,\ell}^t & r_{1,\ell}^{t+1} & \cdots & \cdots & r_{j,\ell}^{t+j} & \cdots & \cdots & r_{t-1,\ell}^{t+\ell} & r_{t,\ell}^t \\
  r_{0,\ell-1}^t & r_{1,\ell-1}^{t+1} & \cdots & \cdots & r_{j,\ell-1}^{t+j} & \cdots & \cdots & r_{t-1,\ell-1}^{t+\ell} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  r_{0,k}^t & r_{1,k}^{t+1} & \cdots & \cdots & r_{j,k}^{t+j} & \cdots & \cdots & r_{k,k}^{t+k} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  r_{0,2}^t & r_{1,2}^{t+1} & r_{2,2}^{t+2} \\
  r_{0,1}^t & r_{1,1}^{t+1} \\
  r_{0,0}^t
\end{array}
\tag{29}
\]

We define \( r_{j}^{t+j} \) to be a column vector in \(29\), for \( 0 \leq j \leq \ell \), where

\[
r_{j}^{t+j} \overset{\text{def}}{=} \begin{pmatrix} r_{j,\ell}^{t+j} & \cdots & r_{j,k}^{t+j} & \cdots & r_{j,j}^{t+j} \end{pmatrix}^T.
\]

Then we can rewrite \(29\) as

\[
R^{[t, t+\ell]} = (r_0^t, r_1^{t+1}, \ldots, r_j^{t+j}, \ldots, r_\ell^{t+\ell}).
\tag{30}
\]

There is another related form, shown in \(31\), called the **static matrix** \( R^{[t,\ell]} \), where component \( r_{j,k}^t \) is just an element in \( X_j^t \cap Y_k^t \). As can be seen, all components of the static matrix occur at time \( t \). For a generator matrix, the first column specifies the matrix completely. For a static matrix, the first column does not determine the static matrix uniquely.

\[
\begin{array}{cccccccc}
  r_{0,\ell}^t & r_{1,\ell}^t & \cdots & \cdots & r_{j,\ell}^t & \cdots & \cdots & r_{t-1,\ell}^t & r_{t,\ell}^t \\
  r_{0,\ell-1}^t & r_{1,\ell-1}^t & \cdots & \cdots & r_{j,\ell-1}^t & \cdots & \cdots & r_{t-1,\ell-1}^t \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  r_{0,k}^t & r_{1,k}^t & \cdots & \cdots & r_{j,k}^t & \cdots & \cdots & r_{k,k}^t \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  r_{0,2}^t & r_{1,2}^t & r_{2,2}^t \\
  r_{0,1}^t & r_{1,1}^t \\
  r_{0,0}^t
\end{array}
\tag{31}
\]

We can rewrite \(31\) as

\[
R^{[t,\ell]} = (r_0^t, r_1^t, \ldots, r_j^t, \ldots, r_\ell^t).
\tag{32}
\]

We can relate a static matrix \( R^{[t,\ell]} \) to a generator matrix \( R^{[t, t+\ell]} \) using the tensor description shown in \(35\). Time increases as we move up the page. The vectors in the generator matrix \(30\) are the vectors along the diagonal in \(35\), and the vectors in the static matrix \(32\) are the vectors in a row of \(35\). The superscript parentheses terms in \(35\), like \((t - j)\),
indicate terms that all belong to the same generator matrix. For example, the diagonal terms $r_0^{(t-j)}, r_1^{(t-j)+1}, \ldots, r_j^{(t-j)+j}, \ldots, r_{\ell}^{(t-j)+\ell}$ all belong to the generator matrix starting at time $t-j$, $R^{(t-j),(t-j)+\ell}$. The center row in (35) is

\[
(r_0^{(t)}, r_1^{(t-1)+1}, \ldots, r_j^{(t-j)+j}, \ldots, r_{\ell}^{(t-\ell)+\ell}),
\]

where each entry is itself a column; this reduces to

\[
(r_0^t, r_1^t, \ldots, r_j^t, \ldots, r_{\ell}^t),
\]

which is just the static matrix $R^{[t,\ell]}$. Notice that each term in (33) and (34) is from one of $\ell+1$ different shift matrices.

Theorem 25 Fix time $t$. A finite sequence of $\ell+1$ generator matrices $R^{[(t-j),(t-j)+\ell]}$ at times $t-j$, for $j = 0, \ldots, \ell$, uniquely determines a static matrix $R^{[t,\ell]}$, where column $j$ of generator matrix $R^{[(t-j),(t-j)+\ell]}$, denoted $r_j^{(t-j)+j}$, is column $j$ of static matrix $R^{[t,\ell]}$, denoted $r_j^t$.

Proof. The center row in (35) is (33), which reduces to (34), which is just static matrix $R^{[t,\ell]}$. But entry $r_j^{(t-j)+j}$ in (33) is just the $(j+1)$-th column of the generator matrix $R^{[(t-j),(t-j)+\ell]}$ at time $t-j$. \hfill \bullet

We let $r$ denote the tensor in (35), and say $r$ is a representative tensor. We can regard $r$ in two different ways, as a sequence of static matrices or as a sequence of shift matrices. In the first way we can write $r$ as

\[
r = \ldots, r^t, r^{t+1}, \ldots,
\]

where each $r^t$ is a static matrix $R^{[t,\ell]}$ in the set of all static matrices, denoted $R^t$. Therefore (36) is equivalent to

\[
r = \ldots, R^{[t,\ell]}, R^{[t+1,\ell+1]}, \ldots.
\]

We have just seen from Theorem 25 that each $r^t$ is determined by $\ell+1$ shift matrices. Then tensor $r$ in (36) is also determined by a sequence of shift matrices. We denote this interpretation of $r$ using notation

\[
r \sim \ldots, R^{[t,t+\ell]}, R^{[t+1,t+1+\ell]}, \ldots,
\]
where each shift matrix \( R^{[t,t+\ell]} \) is in the set of all possible shift matrices, denoted \( R^{[t,t+\ell]} \).

We define tensor set \( \mathcal{R} \) to be the set of representative tensors \( r \) determined by the Cartesian product of all possible shift matrices,

\[
\mathcal{R} \sim \prod_{t=-\infty}^{\infty} R^{[t,t+\ell]}.
\]

Note that \( \mathcal{R} \) depends on choice of basis \( B \). Because \( \mathcal{R} \) is the product of all possible shift matrices, we say \( \mathcal{R} \) is full.

For a given group trellis \( C \), there is only one coset tensor \( x_{\parallel} \), and at each time \( t \), there is only one shift matrix \( X_{\parallel}^{[t,t+\ell]} \) and one static matrix \( X_{\parallel}^{[t,t]} \). A group trellis \( C \) can have many bases \( B \). Each basis \( B \) is a selection of one coset representative (generator vector) from each of the cosets in each of the quotient groups \( \{ \Lambda^{[t,t+k]} : 0 \leq k \leq \ell \} \) in \( X_{\parallel}^{[t,t+\ell]} \), at each time \( t \). Now fix basis \( B \) and fix the corresponding tensor set \( \mathcal{R} \). Each tensor \( r \in \mathcal{R} \) is a selection of one coset representative from a single coset of each of the quotient groups \( \{ \Lambda^{[t,t+k]} : 0 \leq k \leq \ell \} \) in \( X_{\parallel}^{[t,t+\ell]} \), at each time \( t \). Thus for each basis \( B \), there are many possible \( r \in \mathcal{R} \).

Each tensor \( r \in \mathcal{R} \) gives one shift matrix \( R^{[t,t+\ell]} \) and one static matrix \( R^{[t,t]} \) at each time \( t \). A different tensor \( \tilde{r} \in \mathcal{R} \) may have a different shift matrix \( \tilde{R}^{[t,t+\ell]} \) and different static matrix \( \tilde{R}^{[t,t]} \) at each time \( t \). \( R^{[t,t+\ell]} \) is a selection of one coset representative from a single coset of each quotient group in \( X_{\parallel}^{[t,t+\ell]} \). Thus \( R^{[t,t+\ell]} \) has the same form and time indices as the \( X_{\parallel}^{[t,t+\ell]} \) shift matrix. Similarly \( R^{[t,t]} \) is a selection of one coset representative from a single coset of each quotient group in \( X_{\parallel}^{[t,t]} \). Thus \( R^{[t,t]} \) has the same form and time indices as the \( X_{\parallel}^{[t,t]} \) static matrix. This explains why tensor \( r \) in (35) has the same form as tensor \( x_{\parallel} \) in (21).

A given \( r \in \mathcal{R} \) produces a sequence of shift matrices \( R^{[t,t+\ell]} \) and a sequence of static matrices \( R^{[t,t]} \). Any sequence of shift matrices corresponds to some \( r \in \mathcal{R} \) and uniquely determines a sequence of static matrices. But an arbitrary sequence of static matrices may not correspond to a valid sequence of generator vectors and therefore an \( r \in \mathcal{R} \). In this paper we regard shift matrices and shift vectors as the primary objects; these have intrinsic meaning since they are related to generators. The static matrix is formed by an interleaving of columns of different shift matrices and is regarded as a secondary object.

**Lemma 26** Fix \( j \) such that \( 0 \leq j \leq \ell \). Fix \( k \) such that \( j \leq k \leq \ell \). Let \( \Gamma^{[t-j,t-j+k]} \) be a set of generators \( \{ g_{j-k,t-j+k} \} \) which is a transversal of \( \Gamma^{[t-j,t-j+k]} \). The \( (t-j)+j \)-th components of generators \( g_{j-k,t-j+k} \in \Gamma^{[t-j,t-j+k]} \) form a transversal

\[
[\{ X_{j-1}^{t} (X_{j}^{t} \cap Y_{k-j}^{t}) \}] = [\{ r_{j,k}^{(t-j)+j} \}] = [\{ r_{j,k}^{t} \}]
\]

(37)
of

\[
\frac{X_{j-1}^{t} (X_{j}^{t} \cap Y_{k-j}^{t})}{X_{j-1}^{t} (X_{j}^{t} \cap Y_{k-j-1}^{t})}
\]

(38)
**Proof.** Fix \( j \), where \( 0 \leq j \leq \ell \), and examine time \( t - j \). Fix \( k \) such that \( j \leq k \leq \ell \). Pick a set of generators \( g^{(t-j),(t-j)+k} \) which is a transversal of \( \Gamma^{(t-j),(t-j)+k} \), denoted \( \Gamma^{(t-j),(t-j)+k} \). Then \( \Gamma^{(t-j),(t-j)+k} \) induces a transversal \( \{ r_{j,k}^{(t-j)+m} \} \) of
\[
\frac{X_{j-1}^{(t-j)+m} (X_j^{(t-j)+m} \cap Y_{k-j}^{(t-j)+m})}{X_{j-1}^{(t-j)+m} (X_j^{(t-j)+m} \cap Y_{k-j-1}^{(t-j)+m})}, \tag{39}
\]
for \( m = 0, 1, \ldots, k \). Choose \( m = j \). Then \( \{ r_{j,k}^{(t-j)+j} \} \) is a transversal \( \{ r_{j,k}^{(t-j)+j} \} = \{ r_{j}^{t} \} \) of \( \{ 39 \} \) for \( m = j \), which is the same as \( \{ 38 \} \).

Note that the set of transversals \( \{ r_{j,k}^{(t-j)+j} \} \) for \( k \) such that \( j \leq k \leq \ell \) are the coset representatives of all cosets in quotient groups in column \( j \) of shift matrix \( X_j^{(t-j),(t-j)+\ell} \), which is column \( X_j^{(t-j)+j} / X_j^{(t-j)+j} \). And the set of transversals \( \{ r_{j,k}^{t} \} \) for \( k \) such that \( j \leq k \leq \ell \) are the coset representatives of all cosets in quotient groups in column \( j \) of static matrix \( X_j^{t} \), which is column \( X_j^{t} / X_{j-1}^{t} \). By selecting one coset representative from each quotient group of \( X_j^{t} \), we obtain a complete set of coset representatives for the normal chain of \( B^t \) given by the \( X_j^{t} \) static matrix. This gives the following result.

**Theorem 27** For \( 0 \leq j \leq \ell \), for \( k \) such that \( j \leq k \leq \ell \), let \( \Gamma^{[t-j,t-j+k]} \) be a set of generators \( \{ g^{[t-j,t-j+k]} \} \) which is a transversal of \( \Gamma^{[t-j,t-j+k]} \). The \( (t-j)+j \)-th components of generators \( g^{[t-j,t-j+k]} \) in \( \Gamma^{[t-j,t-j+k]} \) form a transversal \( \{ 37 \} \) of \( \{ 38 \} \) for \( 0 \leq j \leq \ell \), for \( j \leq k \leq \ell \). The set of transversals, \( \{ r_{j,k}^{t} \} \), for \( 0 \leq j \leq \ell \), for \( j \leq k \leq \ell \), forms a complete set of coset representatives for the normal chain of \( B^t \) given by the \( X_j^{t} \) static matrix.

Any branch \( b^t \in B^t \) can be written using elements of this complete set of coset representatives as
\[
b^t = \prod_{j=0}^{\ell} \left( \prod_{k=j}^{\ell} r_{j,k}^{t} \right). \tag{40}
\]

By the convention used here, equation \( \{ 40 \} \) is evaluated as
\[
b^t = r_{t,0}^{t} r_{t-1,0}^{t} r_{t-1,0}^{t} \ldots r_{j,0}^{t} \ldots r_{j,k}^{t} \ldots r_{j,k}^{t} \ldots r_{2,0}^{t} r_{1,0}^{t} r_{1,0}^{t} \ldots r_{0,0}^{t} r_{0,0}^{t} \cdot r_{0,0}^{t} \cdot r_{0,0}^{t} \cdot r_{0,0}^{t}. \tag{41}
\]
Note that \( b^t \) is the product of terms in some static matrix \( R_j^{[t]} \), where the inner product in parentheses in \( \{ 40 \} \) is just the product of terms in the \( j \)-th
column of $R^{[t,t]}$. Using (37), (40) can be written in equivalent forms as

\[ b' = \prod_{j=0}^{\ell} \left( \prod_{k=j}^{\ell} r_{j,k}^t \right) \]

(42)

\[ = \prod_{j=0}^{\ell} \left( \prod_{k=j}^{\ell} r_{j,k}^{(t-j)+j} \right) \]

(43)

\[ = \prod_{j=0}^{\ell} \left( \prod_{k=j}^{\ell} \chi^{t} (g^{t-j,t+j+k}) \right). \]

(44)

We have just shown that for any time $t$, we can find any branch $b' \in B'$ using a selected set of generators at times $t - j$, for $j = 0, \ldots, \ell$. However we have not shown we can construct any path in $C$ this way. We do this in the next section.

We now give a development dual to the forward Schreier series using the backward Schreier series. We show that components of the same generators form a complete set of coset representatives for two normal chains. Define $\Delta_{t,k}^{Y} = Y_{t} \cap X_{k}^{t}$, for $-1 \leq k \leq \ell$. Define the previous branch set $P(b)$ to be the time reversal of $F(b)$. The time reversal of quotient group $\Lambda^{[t,t+k]}$ is $\Lambda^{[t-k,t]}_{Y}$, where $P^{[-k,0]}$ is the time reversal of $F^{[0,k]}$. The time reversal of quotient group $\Gamma^{[t,t+k]}$ is $\Gamma^{[t-k,t]}_{Y}$,

\[ \Lambda^{[t-k,t]}_{Y} \]  

where $P^{[-k,0]}$ is the time reversal of $F^{[0,k]}$. The time reversal of quotient group $\Gamma^{[t,t+k]}$ is $\Gamma^{[t-k,t]}_{Y}$,

\[ \Gamma^{[t-k,t]}_{Y} \]

The representatives of quotient group $\Gamma^{[t-k,t]}_{Y}$ are generators $g^{[t-k,t]}$. Previously we defined a vector basis $B'$ using generators $g^{[t,t+k]}$ which begin at time $t$, for $0 \leq k \leq \ell$. Now we define a vector basis $B'_{Y}$ using generators $g^{[t,k,t]}$ which begin at time $t$, for $0 \leq k \leq \ell$. This defines a basis $B_{Y}$ and constant basis $B_{Y}$. The vector bases $B'$ and $B'_{Y}$ have an inherent asymmetry with respect to time. The asymmetry of $B'$ and $B'_{Y}$ is reflected in $B$ and $B_{Y}$ also.

Using these definitions, the arguments in Lemma 22 and Theorem 23 can be reversed in time. In place of the input chain $[3]$ in the proof of Theorem 23 the last output chain $[3]$ is used. This gives the following time reversed version of Theorem 23 and Corollary 24.

**Theorem 28** There is an isomorphism

\[ \Gamma^{[t-k,t]}_{Y} \approx B^{[t-k,t]} \]

where the 1-1 correspondence $\mu'$ between cosets of $\Gamma^{[t-k,t]}_{Y}$ and $\Lambda^{[t-k,t]}_{Y}$ is given by

\[ \mu' : C^{[t-k,t]} \cdot C^{[t-k,t]} \rightarrow P^{[-k,0]}(\chi^{t}(C^{[t-k,t]} \cdot C^{[t-k,t]})). \]

**Corollary 29** Let $[\Gamma^{[t-k,t]}_{Y}]$ be a set of generators which is a transversal of $\Gamma^{[t-k,t]}_{Y}$. Then $\{ \chi^{t-k,t}(g^{t-k,t}) : g^{t-k,t} \in [\Gamma^{[t-k,t]}_{Y}] \}$ is a transversal of $\Lambda^{[t-k,t]}_{Y}$. 

27
The generator matrix of the backward Schreier series is $R_{Y}^{[t-\ell,t]}$ and the static matrix is $R_{Y}^{[t,t]}$, shown in (45). To distinguish representatives in the forward and backward Schreier series, we have added an additional subscript $Y$ to representatives in the backward Schreier series.

$$
\begin{array}{cccccc}
  r_{Y,t,0}^{1} & r_{Y,t-1,1}^{1} & \cdots & \cdots & r_{Y,t,1}^{1} & r_{Y,t,0}^{1} \\
r_{Y,t-1,0}^{1} & r_{Y,t-2,1}^{1} & \cdots & \cdots & r_{Y,t-1,1}^{1} & r_{Y,t-1,0}^{1} \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
r_{Y,t,k,k}^{1} & r_{Y,t-1,k}^{1} & \cdots & \cdots & r_{Y,t-1,1}^{1} & r_{Y,t-1,0}^{1} \\
r_{Y,t,1,i}^{1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
r_{Y,2,2}^{1} & r_{Y,1,1}^{1} & \cdots & \cdots & r_{Y,1,0}^{1} & r_{Y,0,0}^{1} \\
\end{array}
$$

(45)

The generator matrix $R_{Y}^{[t-\ell,t]}$ consists of representatives $r_{Y,i,k}^{t-\ell-i}$ from generators $g^{[t-k,t]}$ for $0 \leq k \leq \ell$. From Theorem 25 and Corollary 29 we may use the same generators for the forward and backward Schreier series. Then in the forward Schreier series, $g^{[t-k,t]}$ is a generator which begins at time $t - k$ and ends at time $t$. In the backward Schreier series, we consider $g^{[t-k,t]}$ to be a generator which begins at time $t$ and ends at time $t - k$. In the forward Schreier series, the generator $g^{[t-k,t]}$ is written as

$$
g^{[t-k,t]} = \ldots, 1^{t-k-2}, 1^{t-k-1}, r_{0,k}^{t-k}, \ldots, r_{1,k}^{t-k+1}, \ldots, r_{t,k}^{t-k}, 1^{t+1}, 1^{t+2}, \ldots \tag{46}
$$

while in the backward Schreier series, the generator $g^{[t-k,t]}$ is written as

$$
g^{[t-k,t]} = \ldots, 1^{t-k-2}, 1^{t-k-1}, r_{Y,k}^{t-k}, r_{Y,k-1}^{t-k+1}, \ldots, r_{Y,1}^{t-k}, r_{Y,0}^{t-k}, 1^{t+1}, 1^{t+2}, \ldots \tag{47}
$$

Note that $r_{j,k}^{t-k-j} = r_{Y,i,k}^{t-i}$ when $j = k - i$.

The first column $r_{0}^{t}$ in $R_{Y}^{[t,t+\ell]}$ and $R_{Y}^{[t,t]}$ is composed of representatives from generators $g_{[t,t+k]}$ that begin at time $t$, for $0 \leq k \leq \ell$. The first column $r_{Y,0}^{t}$ in $R_{Y}^{[t-\ell,t]}$ and $R_{Y}^{[t,t]}$ is composed of representatives from generators $g^{[t-k,t]}$ that end at time $t$ going forward in time, or begin at time $t$ going backward in time, for $0 \leq k \leq \ell$. If the same generators are used for the forward Schreier series and backward Schreier series, the first column $r_{0}^{t}$ in $R_{Y}^{[t,t+\ell]}$ and $R_{Y}^{[t,t]}$ are the representatives in the diagonal terms $r_{Y,i}^{t}$ of $R_{Y}^{[t,t]}$ for $0 \leq i \leq \ell$. And the first column $r_{Y,0}^{t}$ in $R_{Y}^{[t-\ell,t]}$ and $R_{Y}^{[t,t]}$ are the representatives in the diagonal terms $r_{j,t}^{t}$ of $R_{Y}^{[t,t]}$ for $0 \leq j \leq \ell$.

A representative tensor $r_{Y} \in R_{Y}$ in the backward Schreier series is constructed using $R_{Y}^{[t-\ell,t]}$ and $R_{Y}^{[t,t]}$ in a dual manner to constructing $r \in \mathcal{R}$.

Assume that basis $B = \{B^{t} : t \in \mathbb{Z}\}$ is chosen. Then tensor set $\mathcal{R}$ can be found. Fix time $t$. For $k = 0, \ldots, \ell$, a generator $g^{[t-k,t]}$ in vector basis $B^{t-k}$ of basis $B$ ends at time $t$. We can use these generators to form a vector basis $B_{Y}^{t}$. The vector bases $B_{Y}^{t}$, for each $t \in \mathbb{Z}$, form a basis $B_{Y}$, and we say $B$ and $B_{Y}$ formed in this way have a natural correspondence, denoted
\( B \equiv B_Y \). We can use \( B_Y \) to find a basis set \( \mathcal{R}_Y \) and say \( \mathcal{R} \equiv \mathcal{R}_Y \). If \( B \equiv B_Y \) and \( \mathcal{R} \equiv \mathcal{R}_Y \), then there is a 1-1 correspondence \( \mathcal{R} \leftrightarrow \mathcal{R}_Y \) such that for each \( r \in \mathcal{R} \), there is an \( r_Y \in \mathcal{R}_Y \) which uses the same sequence of generators. In other words, \( r \) and \( r_Y \) are the same tensor, and we say there is a natural correspondence \( r \equiv r_Y \).

**Theorem 30** Fix basis \( B \) and tensor set \( \mathcal{R} \). We can find a basis \( B_Y \) and tensor set \( \mathcal{R}_Y \) such that there is a natural correspondence \( B \equiv B_Y \), \( \mathcal{R} \equiv \mathcal{R}_Y \), and \( r \equiv r_Y \) for each \( r \in \mathcal{R} \).

If \( r \equiv r_Y \) then at each time \( t \), the representatives in the static matrices \( R^{[t,t]} \) and \( R^{[t,t]}_Y \) are the same aside from a change in index as shown in (46)-(47). In other words, a representative \( r_{j,k}^t \) in (31) is the same as representative \( r_{j,k}^t \) when \( j = k - i \), for \( 0 \leq j \leq \ell \) and \( j \leq k \leq \ell \). We write this as \( R^{[t,t]} \equiv R^{[t,t]}_Y \), so if \( r \equiv r_Y \), then \( R^{[t,t]} \equiv R^{[t,t]}_Y \) at each time \( t \).

Using a development dual to Theorem 25 and Lemma 26 we obtain the following theorem dual to Theorem 27.

**Theorem 31** For \( 0 \leq i \leq \ell \), for \( k \) such that \( i \leq k \leq \ell \), let \( [\Gamma_Y^{[t+i-k+1,t+i]}] \) be a set of generators \( \{ g^{[t+i-k+1,t+i]} \} \) which is a transversal of \( \Gamma_Y^{[t+i-k+1,t+i]} \). The \((t+i) - i\)-th components of generators \( g^{[t+i-k+1,t+i]} \in [\Gamma_Y^{[t+i-k+1,t+i]}] \) form a transversal

\[
\{ \{ \chi^t(g^{[(t+i)-k,(t+i)]}) \} \} = \{ \{ r_{Y,i,k}^{(t+i)-i} \} \} = \{ \{ r_{Y,i,k}^t \} \} \tag{48}
\]

of

\[
\frac{Y_{i-1}^t(Y_i^t \cap X_{k-i}^t)}{Y_{i-1}^t(Y_i^t \cap X_{k-i-1}^t)} \tag{49}
\]

for \( 0 \leq i \leq \ell \), for \( i \leq k \leq \ell \). The set of transversals, \( \{ \{ r_{Y,i,k}^t \} \} \), for \( 0 \leq i \leq \ell \), for \( i \leq k \leq \ell \), forms a complete set of coset representatives for the normal chain of \( B^t \) given by the \( Y^{[t,t]} \) static matrix.

From (37) we have

\[
\chi^t(g^{[(t-j),(t-j)+k]}) = r_{j,k}^{(t-j)+j},
\]

and from (48) we have

\[
\chi^t(g^{[(t+i)-(k),(t+i)]}) = r_{Y,i,k}^{(t+i)-i}.
\]

The generators \( g^{[(t-j),(t-j)+k]} \) and \( g^{[(t+i)-(k),(t+i)]} \) have the same endpoints when \( j = k - i \). If \( r \equiv r_Y \), the generators are the same, and then \( r_{j,k}^t = r_{Y,i,k}^t \) for \( j = k - i \). Then \( R^{[t,t]} \equiv R^{[t,t]}_Y \). Fix \( i \) such that \( 0 \leq i \leq \ell \). Let \( j = k - i \). Then there is a 1-1 correspondence between the set of transversals \( \{ \{ r_{j,k}^t \} \} \) for \( j \leq k \leq \ell \), and the set of transversals \( \{ \{ r_{Y,i,k}^t \} \} \) for \( i \leq k \leq \ell \), such that transversals with the same index \( k \) are the same.

**Corollary 32** There is one set of transversals, either \( \{ \{ r_{j,k}^t \} \} \) for \( 0 \leq j \leq \ell \) and \( j \leq k \leq \ell \), or \( \{ \{ r_{Y,i,k}^t \} \} \) for \( 0 \leq i \leq \ell \) and \( i \leq k \leq \ell \), that forms a complete set of coset representatives for two normal chains, the normal chain of \( B^t \) given by the \( X^{[t,t]} \) static matrix and the normal chain of \( B^t \) given by the \( Y^{[t,t]} \) static matrix.
Note that for the forward Schreier series, a generator $g^{[t-j,t-j+k]}$ is selected at time $t-j$, while for the backward Schreier series, the same generator $g^{[t-j,t-j+k]} = g^{[t+i-k,t+i]}$ where $j = k - i$, is selected at time $t - j + k$. Thus in both cases there is a causal collection of generators at time $t$.

We previously calculated a branch $b^t \in B^t$ using representatives in $R^{[t,t]}$ in (40) and (41). We now calculate a branch $b_Y^t \in B^t$ using representatives in $R_Y^{[t,t]}$. Then

$$b_Y^t = \prod_{i=0}^{\ell} \left( \prod_{k=i}^{\ell} r_{Y,i,k}^t \right). \quad (50)$$

By the convention used here, equation (50) is evaluated as

$$b_Y^t = r_{Y,\ell,\ell}^t r_{Y,\ell-1,\ell}^t \cdots r_{Y,i,\ell}^t \cdots r_{Y,i,k}^t \cdots r_{Y,1,1}^t r_{Y,0,0}^t \cdots \cdots r_{Y,0,2}^t r_{Y,0,1}^t r_{Y,0,0}^t. \quad (51)$$

If $R^{[t,t]} = R_Y^{[t,t]}$, then $r_{j,k}^t$ in (31) is the same as $r_{Y,i,k}^t$ in (45) when $j = k - i$, and we can rewrite $b_Y^t$ in terms of representatives $r_{j,k}^t$ in the forward Schreier series as

$$b_Y^t = r_{0,0,\ell}^t r_{0,1,\ell}^t \cdots r_{0,-i,\ell}^t \cdots r_{k-1,k}^t \cdots r_{2,2}^t r_{1,1}^t r_{0,0}^t. \quad (52)$$

If $R^{[t,t]} = R_Y^{[t,t]}$, product (52) is a rearrangement of product (41). If $B^t$ is abelian, then rearrangements of the same terms give the same result, and then $b^t = b_Y^t$. If $B^t$ is not abelian, this may not be true.
5. THE TIME DOMAIN ENCODER

For $0 \leq j < \ell$, we know that in shift matrix $R^{[t,t+\ell]}$ there is a column vector
\[ r_j^{t+j} = \begin{pmatrix} r_{j,k}^{t+j} & \cdots & r_{j,k}^{t+j} \\ r_{j+1,k}^{t+j} & \cdots & r_{j+1,k}^{t+j} \end{pmatrix}^T, \]  
and a column vector
\[ r_{j+1}^{t+j+1} = \begin{pmatrix} r_{j+1,k}^{t+j+1} & \cdots & r_{j+1,k}^{t+j+1} \end{pmatrix}^T. \]

Note that column $r_{j+1}^{t+j+1}$ is completely determined by column $r_j^{t+j}$. Then we can think of $r_{j+1}^{t+j+1}$ as a shift of $r_j^{t+j}$. For $0 \leq j \leq \ell$, let $R_j^{t+j}$, $R_{j+1}^{t+j+1}$ be the set of all columns $r_j^{t+j}$, $r_{j+1}^{t+j+1}$ in all possible shift matrices $R^{[t,t+\ell]}$. For $0 \leq j < \ell$, define a column shift map $\sigma : R_j^{t+j} \rightarrow R_{j+1}^{t+j+1}$ by the assignment $\sigma : r_j^{t+j} \rightarrow r_{j+1}^{t+j+1}$, where this assignment is given by $\sigma : r_{j,k}^{t+j} \rightarrow r_{j+1,k}^{t+j+1}$ for $j < k \leq \ell$. Note that $\sigma_j^{t+j}$ is not defined since $r_j^{t+j}$ “shifts out”. We abbreviate $\sigma_j^{t+j}$ as $\sigma r_j^{t+j}$ and $\sigma r_{j,k}^{t+j}$ as $\sigma_{j,k}^{t+j}$. (The notation $\sigma r_{j,k}^{t+j}$ and $\sigma_{j,k}^{t+j}$ is slightly inconsistent, but any ambiguity in $\sigma$ or $\sigma$ is resolved by looking at its argument. In addition $\sigma$ and $\sigma$ should have a time index, but again this ambiguity is resolved by looking at its argument. Although somewhat inconsistent and incomplete, this notation is simple and helps to clarify the basic argument.)

Define
\[ \sigma r_t^t \overset{\text{def}}{=} (\sigma r_0^t, \sigma r_1^t, \ldots, \sigma r_j^t, \ldots, \sigma r_{\ell-1}^t, \sigma r_{\ell}^t). \]

**Theorem 33** Let $w = \ldots, r^t, r^{t+1}, \ldots$ be an arbitrary sequence, not necessarily a tensor in $\mathcal{R}$, where $r^t \in \mathcal{R}^t$ for each time $t \in \mathbb{Z}$. Then $w$ is a tensor in $\mathcal{R}$ if and only if for each time $t$, $r^{t+1} = (r_0^{t+1}, \sigma r^t)$ where input $r_0^{t+1}$ is any element of $\mathcal{R}_0^{t+1}$.

**Proof.** First assume $w \in \mathcal{R}$. Then we know $w$ is formed from a sequence of shift matrices. Consider $(r^t, r^{t+1})$ where $r^t \in \mathcal{R}^t$ and $r^{t+1} \in \mathcal{R}^{t+1}$. Fix $0 \leq j < \ell$. We know column $r_j^t$ of $r^t$ is a column $r_j^{(t-j)+j}$ in shift matrix $R^{[(t-j),(t-j)+\ell]}$. From the preceding discussion of shifts, we know
\[ \sigma r_j^t = \sigma r_j^{(t-j)+j} = r_{j+1}^{(t-j)+j+1} = r_{j+1}^{t+1}, \]
where $r_{j+1}^{(t-j)+j+1}$ is a column in shift matrix $R^{[(t-j),(t-j)+\ell]}$ and $r_{j+1}^{t+1}$ is a column in $r^{t+1}$. Then
\[ (r_1^{t+1}, r_2^{t+1}, \ldots, r_j^{t+1}, \ldots, r_{\ell}^{t+1}) = (\sigma r_0^t, \sigma r_1^t, \ldots, \sigma r_j^t, \ldots, \sigma r_{\ell-1}^t, \sigma r_{\ell}^t) = \sigma r^t \]
and $r^{t+1} = (r_0^{t+1}, \sigma r^t)$ where $r_0^{t+1} \in \mathcal{R}^{t+1}$.

Conversely, if $r^{t+1} = (r_0^{t+1}, \sigma r^t)$ for each $t \in \mathbb{Z}$, then it can be shown $w$ is a sequence of shift matrices, and therefore a tensor in $\mathcal{R}$.  

\[ \blacksquare \]
Theorem 34 shows the tensor set $\mathcal{R}$ has a natural shift structure. In the remainder of this section, we show that any path $b \in C$ is the encoding of some $r \in \mathcal{R}$. Then the group trellis $C$ can be considered to have a natural shift structure. The fact that a group code $C$ has an encoder with a shift structure was first proven by Forney and Trott [3] using a spectral domain encoder. We prove $C$ has a natural shift structure using a time domain approach.

An encoder of the group trellis is a finite state machine that, given a sequence of inputs, can produce any path (any sequence of states and branches) in the group trellis. An encoder can help to explain the structure of a group trellis. We give an encoder here which has a sliding block structure and uses the same generators as in [3], but the encoder is different. The encoder is given in (40) and (43)-(44). It is useful to think of (40) and (43)-(44) as equivalent forms of the same encoder; each version is useful in the following discussion.

Assume we have found a basis $B$. Then we have found generators $g_{t,t+k}^r \in [\Gamma_{t,t+k}]$ for each $t \in \mathbb{Z}$, for $0 \leq k \leq \ell$. Fix time $t$. The nontrivial components of the selected generators in encoder (44) form a generator matrix $R_{(t-j),(t-j)+\ell}$, for $j = 0, \ldots, \ell$. From Theorem 25, these generator matrices uniquely determine a static matrix $R_{t,\ell}$, where column $j$ of generator matrix $R_{(t-j),(t-j)+\ell}$, $r_j^{(t-j)+j}$, is column $j$ of static matrix $R_{t,\ell}$, $r_j$. Then we can see that (35) has the form of a sliding block encoder. At each time $t$, we select a new generator matrix $R_{(t),(t)+\ell}$ whose column vectors are shown along the diagonals in (35). The column vectors $r_j^{(t-j)+j}$ of the generator matrix at time $t-j$, $R_{(t-j),(t-j)+\ell} = (r_0^{(t-j)}, r_1^{(t-j)+1}, \ldots, r_j^{(t-j)+j}, \ldots, r_\ell^{(t-j)+\ell})$, and column vectors $r_j^{(t-j)+j}$ of the generator matrix at time $t$, $R_{(t),(t)+\ell} = (r_0^{(t)}, r_1^{(t)+1}, \ldots, r_j^{(t)+j}, \ldots, r_\ell^{(t)+\ell})$, are shown along the diagonals of (35). As time increases, we slide along the infinite matrix in (35) from left to right. At time $t$, the output branch $b'$ of the sliding block encoder is calculated from the static matrix

$$R_{t,\ell} = (r_0^{(t)}, r_1^{(t-1)+1}, \ldots, r_j^{(t-j)+j}, \ldots, r_\ell^{(t-\ell)+\ell}) = (r_0^t, r_1^t, \ldots, r_j^t, \ldots, r_\ell^t),$$

whose terms are shown in the center row in (35). The first term in the center row is the new input $r_0^t$, the first column vector of the new generator matrix $R_{(t),(t)+\ell}$ selected at time $t$, and the remaining terms $r_1^{(t-1)+1}, \ldots, r_j^{(t-j)+j}, \ldots, r_\ell^{(t-\ell)+\ell}$ are from previous generator matrices selected at times $t-1, \ldots, t-j, \ldots, t-\ell$, respectively. To calculate branch $b'$ at time $t$, the sliding block encoder uses time window $[t-\ell, t]$, and therefore the encoder is causal. We now show that we can use (35) to implement (40) as a sliding block encoder.

**Lemma 34** Fix $r^t \in R^t$. Consider all $(r^t, r^{t+1}) \in R^t \times R^{t+1}$ that appear in any tensor $r \in \mathcal{R}$. Then encoder (40) encodes $(r^t, r^{t+1})$ into a trellis path segment $(b^t, b^{t+1})$ of length 2 in group trellis $C$. In other words, $b^t \in B^t$, $b^{t+1} \in B^{t+1}$, and $b^{t+1} \in F(b^t)$.
Proof. Using (40), the encoding of \((r^t, r^{t+1})\) is

\[
\left( \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} r^t_{j,k}, \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} r^{t+1}_{j,k} \right) = \left( \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} r^t_{j,k}, \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} r^{t+1}_{0,k} \right) \left( \prod_{j=0}^{\ell-1} \prod_{k=j+1}^{\ell} r^{t+1}_{j+1,k} \right) \right). \tag{55}
\]

We know that \(r^{t+1}\) is of the form \((r^{t+1}_0, \sigma r^t)\). Then for \(0 \leq j \leq \ell - 1\), \(r^{t+1}_{j+1,k}\) is a shift of \(r^{t}_{j,k}\). Since

\[
r^{t}_{j,k} = \chi^t(g^{[t-j,t-j+k]}),
\]

then

\[
r^{t+1}_{j+1,k} = \chi^{t+1}(g^{[t-j,t-j+k]}).
\]

This means that we can rewrite (55) in terms of generators (see (41)) as

\[
\left( \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} \chi^t(g^{[t-j,t-j+k]}), \prod_{k=0}^{\ell} \chi^{t+1}(g^{[t+1,t+1+k]}) \right) \left( \prod_{j=0}^{\ell-1} \prod_{k=j+1}^{\ell} \chi^{t+1}(g^{[t-j,t-j+k]}) \right). \tag{56}
\]

Since \(\chi^{t+1}(g^{[t-j,t-j+k]}) = 1^{t+1}\) for \(0 \leq j \leq \ell\), we can change the limits of the last double product in (56) as

\[
\left( \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} \chi^t(g^{[t-j,t-j+k]}), \prod_{k=0}^{\ell} \chi^{t+1}(g^{[t+1,t+1+k]}) \right) \left( \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} \chi^{t+1}(g^{[t-j,t-j+k]}) \right) \right). \tag{57}
\]

Note that the term

\[
\prod_{k=0}^{\ell} \chi^{t+1}(g^{[t+1,t+1+k]})
\]

involves generators from vector basis \(B^{t+1}\), and the other terms involve generators from vector bases \(B^{t-j}\) for \(j = 0, \ldots, \ell\).

First consider the case where \(r^{t+1}\) is \(\hat{r}^{t+1} = (1^{t+1}_0, \sigma r^t)\). Let \(\hat{b}^{t+1}\) be the encoding of \(\hat{r}^{t+1}\). Since \(r^{t+1}_0 = 1^{t+1}_0\), then components \(r^{t+1}_0\) are the identity for \(0 \leq k \leq \ell\). Then we can rewrite (57) as

\[
(b^t, \hat{b}^{t+1}) = \left( \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} \chi^t(g^{[t-j,t-j+k]}), \prod_{j=0}^{\ell} \chi^{t+1}(g^{[t-j,t-j+k]}) \right). \tag{58}
\]

Note that (58) just involve generators from vector bases \(B^{t-j}\) for \(j = 0, \ldots, \ell\).

We can pair terms in (58) as

\[
(b^t, \hat{b}^{t+1}) = \prod_{j=0}^{\ell} \prod_{k=j}^{\ell} [\chi^t(g^{[t-j,t-j+k]}), \chi^{t+1}(g^{[t-j,t-j+k]})], \tag{59}
\]

where the product multiplication in the inner square bracket is component by component, i.e., \([a, b] *[c, d] = [a * c, b * d]\). But note that

\[
[\chi^t(g^{[t-j,t-j+k]}), \chi^{t+1}(g^{[t-j,t-j+k]})]
\]
is a valid trellis path segment of length 2, for $0 \leq j \leq \ell$, for $j \leq k \leq \ell$. Then (59) is a product of trellis path segments of length 2, and hence by properties of the group trellis, $(b^t, b^{t+1})$ is a trellis path segment of length 2. This means $b^{t+1} \in \mathcal{F}(b^t)$.

Now consider the case where $r^{t+1} = (r_0^{t+1}, \sigma \mathbf{r}^t)$. Let $b^{t+1}$ be the encoding of $r^{t+1}$. Then using (57) and (58), we have

$$(b^t, b^{t+1}) = \left( b^t, \left( \prod_{k=0}^\ell \chi^{t+1}(g^{[t+1, t+1+k]}) \right) \hat{b}^{t+1} \right). \quad (60)$$

But

$$\left( \prod_{k=0}^\ell \chi^{t+1}(g^{[t+1, t+1+k]}) \right) = \prod_{k=0}^\ell r_{0,k}^{t+1},$$

and this is some branch $\tilde{b}^{t+1} \in X_0^{t+1}$. Then

$$(b^t, b^{t+1}) = (b^t, \tilde{b}^{t+1})$$

where $b^{t+1} = \tilde{b}^{t+1} b^{t+1}$ and $b^{t+1} \in \mathcal{F}(b^t)$.

Notice that we can think of the encoder as an estimator. The encoding of $r^{t+1} = (1^{t+1}, \sigma \mathbf{r}^t)$ gives an initial estimate $\hat{b}^{t+1}$ where $\hat{b}^{t+1} \in \mathcal{F}(b^t)$. Then at time $t + 1$, we use new input $r_0^{(t+1)}$ to find $\tilde{b}^{t+1} \in X_0^{t+1}$ to correct the initial estimate $\hat{b}^{t+1}$ so that $b^{t+1} = \tilde{b}^{t+1} b^{t+1}$ and $b^{t+1} \in \mathcal{F}(b^t)$.

**Corollary 35** Fix $r^t \in \mathcal{R}^t$. Let $A^t_{\mathcal{R}}$ be the set of components $(r^t, r^{t+1})$ that appear in any $r \in \mathcal{R}$. Let $r^t$ encode to $b^t$ using (40). Let $A^t_C$, be the set of trellis path segments $(b^t, b^{t+1})$ of length 2 in $C$. Encoder (40) encodes $(r^t, r^{t+1}) \in A^t_{\mathcal{R}}$ into $(b^t, b^{t+1}) \in A^t_C$. This map is 1-1 and onto.

**Proof.** We know from Lemma 34 that $(r^t, r^{t+1})$ encodes to $(b^t, b^{t+1})$ using (40). Therefore (40) maps $A^t_{\mathcal{R}}$ into $A^t_C$. But $b^{t+1}$ is uniquely determined by $r^{t+1}$, and specifically $r_0^{t+1}$. There are $|X_0^{t+1}|$ possible $r_0^{t+1}$, and $|X_0^{t+1}|$ possible $b^{t+1} \in \mathcal{F}(b^t)$. Therefore the map from $A^t_{\mathcal{R}}$ into $A^t_C$ is 1-1 and onto.

**Corollary 36** Fix any $b \in C$. For any time $t$, consider a trellis path segment $(b^t, b^{t+1})$ of length 2 in group trellis $C$. In other words, $b^t \in B^t$, $b^{t+1} \in B^{t+1}$, and $b^{t+1} \in \mathcal{F}(b^t)$. Then there is some $(r^t, r^{t+1}) \in \mathcal{R}^t \times \mathcal{R}^{t+1}$ such that $b^t$ decodes to $r^t$ using (40), $b^{t+1}$ decodes to $r^{t+1}$ using (40), and $r^{t+1} = (r_0^{t+1}, \sigma \mathbf{r}^t)$.

**Proof.** Use Corollary 35.

**Theorem 37** Each tensor $r \in \mathcal{R}$ can be encoded into a path $b \in C$ using (40).

**Proof.** Lemma 34 shows that for each time $t$, if $(r^t, r^{t+1}) \in \mathcal{R}$, then $(b^t, b^{t+1})$ is a trellis path segment of length 2 in $C$. To show that we obtain a path $b \in C$ using (40), we have to show the trellis path segments of length 2 can be connected. Fix $r$ and fix time $t$. Then $(r^t, r^{t+1})$ gives a trellis path segment $(b^t, b^{t+1})$ of length 2. Now use $(r^{t+1}, r^{t+2})$ to obtain a trellis path
segment $(b^{t+1}, b^{t+2})$ of length 2. But the encoding of $r^{t+1}$ using (40) is unique so $b^{t+1} = \hat{b}^{t+1}$. Therefore we have obtained a trellis path segment $(b^t, b^{t+1}, b^{t+2})$ of length 3 in $C$.

Continuing forward in this way, we can find a trellis path segment $b^{[t, \infty)}$ on $[t, \infty)$ in $C$. Given $r$, the trellis path segment $b^{[t, \infty)}$ is unique since for each time $t$, (40) is a unique function of $r^t$. But since we know how to find a unique trellis path segment $b^{[t, \infty)}$ on $[t, \infty)$ in $C$, we can apply the same argument again starting with $r^{t-1}$ to find a unique trellis path segment $b^{[t-1, \infty)}$ on $[t-1, \infty)$. Given $r$, the trellis path segments $b^{[t, \infty)}$ and $\hat{b}^{[t-1, \infty)}$ must agree on $[t, \infty)$ since again (40) is a unique function of $r^t$. Then we have found a unique trellis path segment on $[t, \infty)$.

\begin{lemma}
If tensor $r \in \mathcal{R}$ is encoded into $b \in C$ using (40), then $r$ is the only tensor in $\mathcal{R}$ that encodes to $b$ using (40).
\end{lemma}

\begin{proof}
Fix time $t$. If $r^t$ encodes to $b^t$ using (40), $r^t$ is unique because $b^t$ is a unique function of the coset representatives in $r^t$ (see (41)). Since this holds for each $t$, $r$ must be unique.
\end{proof}

\begin{theorem}
Each path $b \in C$ can be decoded into a unique tensor $r \in \mathcal{R}$.
In other words, for each path $b \in C$, there is a unique $r \in \mathcal{R}$ that can be encoded to $b$ using (40).
\end{theorem}

\begin{proof}
The proof is analogous to the proof of Theorem 37 but with Corollary 36 in place of Lemma 34.
\end{proof}

\begin{corollary}
There is a 1-1 correspondence $\mathcal{R} \leftrightarrow C$ given by $r \leftrightarrow b$, where $b$ is an encoding of $r$ using (40).
\end{corollary}

\begin{proof}
Combine Theorem 37 and Theorem 39.
\end{proof}

Consider the triple $(\mathcal{R}, C; B)$. $\mathcal{R}$ is a tensor set that depends on choice of basis $B$. If $B$ is fixed, then $\mathcal{R}$ is fixed, and there is a 1-1 correspondence $\mathcal{R} \leftrightarrow C$. Each $r \in \mathcal{R}$ can be encoded into a $b \in C$, and each $b \in C$ can be decoded into an $r \in \mathcal{R}$. Note that the restriction that $C$ be $\ell$-controllable is transparent from the structure of $\mathcal{R}$.

We can reverse time in the argument just given for encoder $b^t$ in (40) and obtain analogous results for encoder $b^t_Y$ in (50). In particular the analog of Corollary 40 is the following.

\begin{corollary}
There is a 1-1 correspondence $\mathcal{R}_Y \leftrightarrow C$ given by $r_Y \leftrightarrow b$, where $b$ is an encoding of $r_Y$ using (50).
\end{corollary}

We now review the encoder construction in [3]. Forney and Trott [3] define the $k$-controllable subcode $C_k$ of a group code $C$. We can transcribe their approach to the group trellis $C$ used here. The $k$-controllable subcode $C_k$ of a group trellis $C$ is defined as the set of combinations of code sequences of span $k + 1$ or less:

$$C_k = \prod_t C^{[t, t+k]}.$$
They show 

\[ C_0 \subset C_1 \subset \ldots C_{k-1} \subset C_k \subset \ldots C_\ell = C \]

is a normal series. Then in their Code Granule Theorem, they show \( C_k/C_{k-1} \) is isomorphic to a direct product,

\[ C_k/C_{k-1} \cong \prod_t \Gamma^{[t,t+k]}, \]

where \( \Gamma^{[t,t+k]} \) is a granule. Let \( \Gamma^{[t,t+k]} = \{ g^{[t,t+k]} \} \) be a set of coset representatives for the granule \( \Gamma^{[t,t+k]} \). Then it follows (p. 1509) that the set \( \prod_t \Gamma^{[t,t+k]} \) is a set of coset representatives for the cosets of \( C_{k-1} \) in \( C_k \). This means (Generator Theorem) that every code sequence \( b \) can be uniquely expressed as a product

\[ b = \prod_{0 \leq k \leq \ell} \prod_t g^{[t,t+k]} \]  

(61)

of generators \( g^{[t,t+k]} \). Thus every code sequence \( b \) is a product of some sequence of generators, and conversely, every sequence of generators corresponds to some code sequence \( b \). It is clear that for any particular time \( t \), only the generators \( g^{[t-j,t-j+k]} \) are relevant in calculating an output, for \( k \) such that \( 0 \leq k \leq \ell \), for \( 0 \leq j \leq k \). Therefore the equation (61) can be realized as a minimal encoder with a shift register structure, as discussed and diagrammed in [3]. Using our notation, the output at time \( t \), denoted as branch \( b^t_s \) for the spectral domain encoder, is given by

\[ b^t_s = \prod_{k=0}^\ell \left( \prod_{j=k}^0 \chi^t (g^{[t-j,t-j+k]}) \right) \]  

(62)

\[ = \prod_{k=0}^\ell \left( \prod_{j=k}^0 r^{(t-j)+j}_{j,k} \right) \]  

(63)

\[ = \prod_{k=0}^\ell \left( \prod_{j=k}^0 r^t_{j,k} \right) \]  

(64)

The output at time \( t \) for the time reversed spectral domain encoder, denoted \( b^t_{s,Y} \), is given by

\[ b^t_{s,Y} = \prod_{k=0}^\ell \left( \prod_{i=k}^0 r^t_{Y,i,k} \right) \]  

(65)

We now compare the two forward time encoders (12) and (64). In the Forney and Trott encoder (64), for fixed \( k \) the inner product (the term in parentheses of (64)) is a product of terms in a single row of the generator matrix, and the outer product can be considered to be a column product. Thus we refer to the Forney and Trott encoder as a column-row encoder, and the product in (64) as a column-row product. In (12), for fixed \( j \) the inner product (the term in parentheses of (12)) is a product of terms in a single column of the generator matrix, and the outer product can be considered to be a row product. Then we refer to encoder (12) as a row-column encoder, and its product in (12) as a row-column product. This terminology points out a distinct difference between the two encoders. However note that it is
easy to transform (42) to (64) by merely interchanging the inner and outer product and then reverse ordering terms in each row.

We can observe an important feature of the encoder (43) or (44). The term in the parentheses of (43) or (44) is a column which is some function of time $t - j$, say $h_{t-j}$. Then $b^t = \prod_{j=0}^{\ell} h_{t-j}$. Thus the encoder has the form of a time convolution, reminiscent of a linear system. The Forney-Trott encoder [3] and Loeliger-Mittelholzer encoder [4] do not have the form of a convolution. The term in the parentheses of (63) or (62) is some function of time $t - j$ but this term is a row. Therefore the overall encoder, the column-row product, is not a time convolution. This is the reason we think of the encoder (43) or (44) given here as a *time domain encoder*, while the encoders in [3, 4] are thought of as *spectral domain encoders*.

We have discussed four different encoders, the forward time domain encoder giving $b^t$ in (40), the backward time domain encoder giving $b^t_Y$ in (50), the forward spectral domain encoder giving $b^t_s$ in (64), and the backward spectral domain encoder giving $b^t_{s,Y}$ in (65). Each encoder encodes an $r \in R$ or $r_Y \in R_Y$ into a path $b \in C$, and each encoder gives a 1-1 correspondence $R \leftrightarrow C$ by $r \leftrightarrow b$, or a 1-1 correspondence $R_Y \leftrightarrow C$ by $r_Y \leftrightarrow b$. We show how the four encoders are related in Subsection 6.3.
6. THE NATURAL SHIFT STRUCTURE AND CANONIC STRUCTURE

6.1 The tensor set \( R \)

We use the time domain encoder for forward time to show the group trellis \( C \) can be reduced to tensor set \( R \). We think of \( R \) as a second canonical form, the \textit{forward time canonical form} of a group system \( C \). We show the tensor set \( R \) has a natural shift structure and is a natural shift register graph \( D^\infty(R, B) \), which is graph isomorphic to \( C \). The paths in \( D^\infty(R, B) \) are tensors in \( R \). Then we give a dual result using the time domain encoder for backward time and define the \textit{backward time canonical form}.

Note that static matrix \( R[t,t] \) has the triangular form \((31)\). We now introduce a triangle notation to describe certain subsets of entries in \( r^t \). For \( r^t \in R^t \), we let \( \nabla_{j,k}(r^t) \) be the entries in \( r^t \) specified by the triangle with lower vertex \( r^t_{j,k} \) and upper vertices \( r^t_{j,k+\ell-k} \). These are the entries \( r^t_{m,n} \) where \( m, n \) satisfy \( k \leq n \leq \ell \) and \( j \leq m \leq (j + n - k) \). Let \( \nabla_{j,k}(R^t) \) be the set of all possible triangles \( \nabla_{j,k}(r^t) \), \( \nabla_{j,k}(R^t) = \{ \nabla_{j,k}(r^t) : r^t \in R \} \).

A path \( b \) in \( C \) is
\[
\ldots, b^{t-1}, b^t, b^{t+1}, \ldots, (66)
\]
where \( b^{t-1} = (s^{t-1}, a^{t-1}, s^t) \), \( b^t = (s^t, a^t, s^{t+1}) \), and \( b^{t+1} = (s^{t+1}, a^{t+1}, s^{t+2}) \). We know \( B^t/X^t_0 \simeq \Sigma^t \). We rewrite path \((66)\) in \( C \) as
\[
\ldots, (b^t X^t_0, b^t, b^{t+1} X^t_{0+1}), (b^{t+1} X^t_{0+1}, b^{t+1}, b^{t+2} X^t_{0+2}), \ldots. (67)
\]
We let the rewritten paths in \((67)\) give trellis \( C' \). Clearly \( C' \) is graph isomorphic to \( C \), written as \( C' \simeq C \).

Now replace \( b^t \) in \((67)\) with \( r^t \) that encodes to it using \((40)\). This gives path
\[
\ldots, (b^t X^t_0, r^t, b^{t+1} X^t_{0+1}), (b^{t+1} X^t_{0+1}, r^{t+1}, b^{t+2} X^t_{0+2}), \ldots.
\]
Call this trellis \( C'' \). Then \( C'' \simeq C' \simeq C \).

\textbf{Theorem 42} \textit{The labels } \ldots, r^t, r^{t+1}, \ldots \textit{ of paths in } C'' \textit{are the paths in } R.

\textbf{Proof.} By Corollary 40, there is a 1-1 correspondence \( R \leftrightarrow C \) given by \( r \leftrightarrow b \), where \( b \) is an encoding of \( r \) using \((40)\).

The set of transversals, \( \{ r^t_{j,k} \} \), for \( 0 \leq j \leq \ell \) and \( j \leq k \leq \ell \), forms a complete set of coset representatives for the normal chain of \( B^t \) given by the \( X^t_{[t,t]} \) static matrix. We can calculate any \( b^t \in B^t \) using these representatives as in \((40)-(41)\). In terms of these representatives note that \( b^t X^t_0 = g^t X^t_0 \) where
\[
g^t = r^t_{1,\ell} r^t_{1,\ell-1} \cdots r^t_{1,2} r^t_{1,1}. (68)
\]
Then all edges \( \hat{b}^t \) out of state \( b^t X^t_0 \) must have \( \nabla_{1,1}(\hat{r}^t) = \nabla_{1,1}(r^t) \). Then there is a 1-1 correspondence
\[
B^t/X^t_0 \leftrightarrow \nabla_{1,1}(R^t)
\]
given by
\[
g^t X^t_0 \leftrightarrow \nabla_{1,1}(r^t).
\]
So we can define $\nabla_{1,1}(r^t)$ to be the left state or left vertex of $r^t$, and $\nabla_{1,1}(r^{t+1})$ to be the right state or right vertex of $r^t$. As a result we can replace paths in $C''$ with paths
\[ \ldots , (\nabla_{1,1}(r^t), r^t, \nabla_{1,1}(r^{t+1})), (\nabla_{1,1}(r^{t+1}), r^{t+1}, \nabla_{1,1}(r^{t+2})), \ldots \] (69)
This gives trellis $C'''$. Then $C'''$ is graph isomorphic to $C$, since $C''' \simeq C'' \simeq C$. We rename trellis $C'''$ as $D^\infty(\mathcal{R}, \mathcal{B})$. Then we have shown $D^\infty(\mathcal{R}, \mathcal{B}) \simeq C$.

**Theorem 43** $D^\infty(\mathcal{R}, \mathcal{B})$ is a graph trellis of $\mathcal{R}$ and $D^\infty(\mathcal{R}, \mathcal{B})$ is graph isomorphic to group trellis $C$, $D^\infty(\mathcal{R}, \mathcal{B}) \simeq C$. The isomorphism maps vertices of $D^\infty(\mathcal{R}, \mathcal{B})$ to vertices of $C$.

Note that $B^t/X_0^t \simeq \Sigma^t$ is a group theoretic description of the states of $C$, and $\nabla_{1,1}(R^t)$ is a set theoretic description of the same states in $D^\infty(\mathcal{R}, \mathcal{B})$. The following result uses the set theoretic description of states to show that $D^\infty(\mathcal{R}, \mathcal{B})$ is a shift register trellis.

**Theorem 44** Let $r = \ldots , r^t, r^{t+1}, \ldots$ be a path in $\mathcal{R}$. In graph trellis $D^\infty(\mathcal{R}, \mathcal{B})$, edge $r^t = (r_0^t, r_1^t, \ldots, r_{t+1}^t)$ has left vertex $\nabla_{1,1}(r^t)$ in $\nabla_{1,1}(R^t)$ and right vertex $\nabla_{1,1}(r^{t+1})$ in $\nabla_{1,1}(R^{t+1})$. We have $r^{t+1} = (r_0^{t+1}, \sigma r^t)$, where $r_0^{t+1}$ is a new input at time $t + 1$, and columns $\sigma r^t = (r_1^{t+1}, \ldots, r_{0}^{t+2})$ of $r^{t+1}$ are a shift of columns $(r_0^{t}, \ldots, r_{t-1}^t)$ of $r^t$, i.e., $\sigma r_0^t = r_{j+1}^{t+1}$ for $0 \leq j \leq t-1$.

Note that $\nabla_{1,1}(r^{t+1}) = \sigma r^t$, a shift of $r^t$. Therefore the right vertex of $r^t$ is completely specified by $r^t$.

Theorem 44 shows that $D^\infty(\mathcal{R}, \mathcal{B})$ is a shift register trellis. We can think of graph trellis $D^\infty(\mathcal{R}, \mathcal{B})$ as composed of trellis sections $D(R^t, B^t)$. At each time $t$, $D^\infty(\mathcal{R}, \mathcal{B})$ is a bipartite graph $D(R^t, B^t)$ having edges $r^t \in R^t$, left vertices $\nabla_{1,1}(r^t)$ in vertex set $\nabla_{1,1}(R^t)$, and right vertices $\nabla_{1,1}(r^{t+1}) = \sigma r^t$ in vertex set $\nabla_{1,1}(R^{t+1})$.

**Theorem 45** $D(R^t, B^t)$ is graph isomorphic to $B^t$ given by trellis section $T^t$ in group trellis $C$.

At each time $t$, the graph isomorphism is given by mapping left vertex $\nabla_{1,1}(r^t)$ of $D(R^t, B^t)$ to state $s^t$ in $B^t$ corresponding to coset $g^t X_0^t \in B^t/X_0^t \simeq \Sigma^t$, where $g^t$ is given in (68), and mapping right vertex $\nabla_{1,1}(r^{t+1}) = \sigma r^t$ of $D(R^t, B^t)$ to state $s^{t+1}$ in $B^t$ corresponding to coset $g^{t+1} X_0^{t+1} \in B^{t+1}/X_0^{t+1} \simeq \Sigma^{t+1}$, where $g^{t+1}$ is analogous to $g^t$ and only depends on $\sigma r^t$. And finally mapping edge $r^t$ in $D(R^t, B^t)$ to edge $b^t$ in $B^t$, where $b^t$ is determined from $r^t$ using encoding (69).

We now describe two encoders of $\mathcal{R}$, or equivalently $D^\infty(\mathcal{R}, \mathcal{B})$, for forward time. $D^\infty(\mathcal{R}, \mathcal{B})$ consists of sequences of the form (69). We define a time domain encoder $E(D^\infty(\mathcal{R}, \mathcal{B}))$ of $D^\infty(\mathcal{R}, \mathcal{B})$ by replacing sequences of the form (69) with sequences of the form
\[ \ldots , (\nabla_{1,1}(r^t), b^l, \nabla_{1,1}(r^{t+1})), (\nabla_{1,1}(r^{t+1}), b^{l+1}, \nabla_{1,1}(r^{t+2})), \ldots \] (70)
where $b^l$ is an encoding of $r^t$ using time domain encoder (40). We define a spectral domain encoder $E_s(D^\infty(\mathcal{R}, \mathcal{B}))$ of $D^\infty(\mathcal{R}, \mathcal{B})$ by replacing sequences of the form (69) with sequences of the form
\[ \ldots , (\nabla_{1,1}(r^t), b^l, \nabla_{1,1}(r^{t+1})), (\nabla_{1,1}(r^{t+1}), b^{l+1}, \nabla_{1,1}(r^{t+2})), \ldots \] (71)
where \( b_s^t \) is an encoding of \( r^t \) using spectral domain encoder (54).

The Forney-Trott encoder in [3] is an encoding of the sequence \( \ldots, r^t, r^{t+1}, \ldots \) into \( \ldots, b_s^t, b_s^{t+1}, \ldots \), where \( r^{t+1} = (r_0^{t+1}, \sigma r^t) \) is composed of an input \( r_0^{t+1} \) and a shift \( \sigma r^t \) of \( r^t \). Their encoder is of the form state, input, shift to next state, next input, and so on. The states of their encoder are set theoretic constructions and appear to have no group theoretic interpretation in the spectral domain. The state of their encoder at time \( t \) can be regarded as \( \nabla_{1,1}(r^t) \) as in (71), and the state at time \( t+1 \) as \( \nabla_{1,1}(r^{t+1}) \) as in (71). Therefore the encoder \( E_s(D^\infty(\mathcal{R}, \mathcal{B})) \) is an exact replica of the Forney-Trott encoder. The Forney-Trott encoder is a minimal realization of \( \mathcal{C} \) and all minimal realizations are graph isomorphic to the canonical realization, or group trellis \( \mathcal{C} \) [3]. Therefore \( E_s(D^\infty(\mathcal{R}, \mathcal{B})) \) is graph isomorphic to \( \mathcal{C} \). This gives the following result.

**Theorem 46** The time domain encoder \( E(D^\infty(\mathcal{R}, \mathcal{B})) \) and spectral domain encoder \( E_s(D^\infty(\mathcal{R}, \mathcal{B})) \) are graph isomorphic to group trellis \( \mathcal{C} \). The isomorphism maps vertices of \( D^\infty(\mathcal{R}, \mathcal{B}) \) to vertices of \( \mathcal{C} \).

We now give the dual result for backward time. A path \( b \) in \( \mathcal{C} \) is given in (60). We know \( B^\ell/Y_0^t \simeq \Sigma^\ell+1 \). We rewrite path (60) in \( \mathcal{C} \) as

\[
\ldots, (b_Y^{t-1}Y_0^{t-1}, b_Y^{t-1}, b_Y^{t-1}Y_0^{t}), (b_Y^{t-1}Y_0^{t}, b_Y^{t}, b_Y^{t+1}Y_0^{t+1}), \ldots
\]

(72)

We let the rewritten paths in (72) give trellis \( \mathcal{C}_Y' \). Clearly \( \mathcal{C}_Y' \) is graph isomorphic to \( \mathcal{C} \), written as \( \mathcal{C}_Y' \simeq \mathcal{C} \).

Now replace \( b_Y^{t} \) in (72) with \( r_Y^{t} \) that encodes to it using (50). This gives path

\[
\ldots, (b_Y^{t-1}Y_0^{t-1}, r_Y^{t-1}, b_Y^{t-1}Y_0^{t}), (b_Y^{t-1}Y_0^{t}, r_Y^{t}, b_Y^{t+1}Y_0^{t+1}), \ldots
\]

Call this trellis \( \mathcal{C}_Y'' \). Then \( \mathcal{C}_Y'' \simeq \mathcal{C}_Y' \simeq \mathcal{C} \).

**Theorem 47** The labels \( \ldots, r_Y^{t-1}, r_Y^{t}, \ldots \) of paths in \( \mathcal{C}_Y'' \) are the paths in \( \mathcal{R}_Y \).

**Proof.** By Corollary 41 there is a 1-1 correspondence \( \mathcal{R}_Y \leftrightarrow \mathcal{C} \) given by \( r_Y \leftrightarrow b \), where \( b \) is an encoding of \( r_Y \) using (50).

The set of transversals, \( \{ (r_Y^{t+j})_j \} \), for \( 0 \leq j \leq \ell \) and \( j \leq k \leq \ell \), forms a complete set of coset representatives for the normal chain of \( B^\ell \) given by the \( Y[\ell, \ell] \) static matrix. We can calculate any \( b_Y^{t} \in B^\ell \) using these representatives as in (50)-(51). In terms of these representatives note that \( b_Y^{t}Y_0^{t} = hY_0^{t} \)

\[
h = r_Y^{t, t+1}Y_0^{t, t+1}r_Y^{t, t+1}Y_0^{t+1, t+1}r_Y^{t, t+1}Y_0^{t+1, t+1} \ldots
\]

(73)

Then all edges \( b_Y^{t} \) into state \( b_Y^{t}Y_0^{t} \) must have \( \nabla_{0,1}(b_Y^{t}) = \nabla_{0,1}(r_Y^{t}) \). Then there is a 1-1 correspondence

\[
B^\ell/Y_0^{t} \leftrightarrow \nabla_{0,1}(\mathcal{R}_Y)
\]

given by

\[
hY_0^{t} \leftrightarrow \nabla_{0,1}(r_Y^{t}).
\]
So we can define \( \triangledown_{0,1}(r_{1}^{t}) \) to be the right state or right vertex of \( r_{1}^{t} \), and \( \triangledown_{0,1}(r_{1}^{-1}) \) to be the left state or left vertex of \( r_{1}^{t} \). As a result we can replace paths in \( C_{Y}^{\mu} \) with paths
\[
\ldots, (\triangledown_{0,1}(r_{1}^{t-1}), r_{1}^{t-1}, \triangledown_{0,1}(r_{1}^{-1})), (\triangledown_{0,1}(r_{1}^{t-1}), r_{1}^{t}, \triangledown_{0,1}(r_{1}^{t})), \ldots \quad (74)
\]
This gives trellis \( C_{Y}^{\mu} \). Then \( C_{Y}^{\mu} \) is graph isomorphic to \( C_{Y} \), since \( C_{Y}^{\mu} \simeq C_{Y} \simeq C_{Y} \). We rename trellis \( C_{Y}^{\mu} \) as \( D^{\infty}(R_{Y}, B_{Y}) \). Then we have shown \( D^{\infty}(R_{Y}, B_{Y}) \simeq C \).

**Theorem 48** \( D^{\infty}(R_{Y}, B_{Y}) \) is a graph trellis of \( R_{Y} \) and \( D^{\infty}(R_{Y}, B_{Y}) \) is graph isomorphic to group trellis \( C \), \( D^{\infty}(R_{Y}, B_{Y}) \simeq C \). The isomorphism maps vertices of \( D^{\infty}(R_{Y}, B_{Y}) \) to vertices of \( C \).

There are analogies of Theorems 44 and 45 which show \( D^{\infty}(R_{Y}, B_{Y}) \) is a shift register trellis with trellis section \( D(R_{Y}, B_{Y}) \) graph isomorphic to \( B_{t} \) at each time \( t \).

We now describe two encoders of \( R_{Y} \), or equivalently \( D^{\infty}(R_{Y}, B_{Y}) \), for backward time. \( D^{\infty}(R_{Y}, B_{Y}) \) consists of sequences of the form \((74)\). We define a time domain encoder \( E_{Y}(D^{\infty}(R_{Y}, B_{Y})) \) of \( D^{\infty}(R_{Y}, B_{Y}) \) by replacing sequences of the form \((74)\) with sequences of the form
\[
\ldots, (\triangledown_{0,1}(r_{1}^{t-1}), b_{1}^{t-1}, \triangledown_{0,1}(r_{1}^{-1})), (\triangledown_{0,1}(r_{1}^{t-1}), b_{1}^{t}, \triangledown_{0,1}(r_{1}^{t})), \ldots \quad (75)
\]
where \( b_{1}^{t} \) is an encoding of \( r_{1}^{t} \) using time domain encoder \((50)\). We define a spectral domain encoder \( E_{s,Y}(D^{\infty}(R_{Y}, B_{Y})) \) of \( D^{\infty}(R_{Y}, B_{Y}) \) by replacing sequences of the form \((74)\) with sequences of the form
\[
\ldots, (\triangledown_{0,1}(r_{1}^{t-1}), b_{s,Y}^{t-1}, \triangledown_{0,1}(r_{1}^{-1})), (\triangledown_{0,1}(r_{1}^{t-1}), b_{s,Y}^{t}, \triangledown_{0,1}(r_{1}^{t})), \ldots \quad (76)
\]
where \( b_{s,Y}^{t} \) is an encoding of \( r_{1}^{t} \) using spectral domain encoder \((65)\).

**Theorem 49** The time domain encoder \( E_{Y}(D^{\infty}(R_{Y}, B_{Y})) \) and spectral domain encoder \( E_{s,Y}(D^{\infty}(R_{Y}, B_{Y})) \) are graph isomorphic to group trellis \( C \). The isomorphism maps vertices of \( D^{\infty}(R_{Y}, B_{Y}) \) to vertices of \( C \).

### 6.2 The tensor set \( \mathcal{U} \)

We now describe a tensor set \( \mathcal{U} \) that is closely related to \( R \). The advantage of \( \mathcal{U} \) is that it is independent of basis \( B \). There is a 1-1 correspondence \( \mathcal{U} \leftrightarrow R \) for any basis \( B \).

Previously we defined a vector basis \( B_{t} \) using representatives \( g_{[t,t+k]} \) of quotient group \( \Gamma_{[t,t+k]} \) for \( 0 \leq k \leq \ell \). We now number the cosets of \( \Gamma_{[t,t+k]} \) and assign an integer sequence to generator vector \( r_{[t,t+k]} \) of \( g_{[t,t+k]} \). Let integer \( Q_{k}^{t} \) be the number of cosets in \( \Gamma_{[t,t+k]} \). We number the cosets of \( \Gamma_{[t,t+k]} \) with integers \( q_{k}^{t} \) in the set \( \{0, 1, \ldots, |Q_{k}^{t}| - 1\} \). Define the map \( \gamma_{k}^{t} : \Gamma_{[t,t+k]} \to \{0, 1, \ldots, |Q_{k}^{t}| - 1\} \) such that if coset \( \gamma_{k}^{t} \in \Gamma_{[t,t+k]} \), then \( \gamma_{k}^{t} \) is assigned an integer \( q_{k}^{t} \) in the set \( \{0, 1, \ldots, |Q_{k}^{t}| - 1\} \); this gives assignment \( \gamma_{k}^{t} \mapsto q_{k}^{t} \). The numbering is arbitrary except we number the identity coset with integer 0.

Fix basis \( B \). Let \( g_{[t,t+k]} \) be the representative of a coset \( \gamma_{k}^{t} \) in \( \Gamma_{[t,t+k]} \) numbered with \( q_{k}^{t} \). We assign a constant integer sequence \( u_{[t,t+k]} \),

\[
u_{[t,t+k]} \overset{\text{def}}{=} (u_{0,k}, u_{1,k}, \ldots, u_{t+j,k}, \ldots, u_{t+k,k}), \quad (77)
\]
to generator vector \( \mathbf{r}^{[t,t+k]} \) of \( \mathbf{g}^{[t,t+k]} \), where \( u^t_{j,k} = q^t_k \) for \( 0 \leq j \leq k \). For 
\( 0 \leq k \leq \ell \) and \( 0 \leq j \leq k \), \( u^{t+j}_{j,k} \) is an integer in the set of integers \( U^{t+j}_{j,k} \equiv \{0,1,\ldots,|Q^t_k|\} \). Then we define the map 
\[
\lambda^t_{B,k} : [\Gamma^{[t,t+k]}] \to U^t_{0,k} \times U^{t+1}_{1,k} \times \cdots \times U^{t+j}_{j,k} \times \cdots \times U^{t+k}_{k,k}
\]
with assignment \( \lambda^t_{B,k} : \mathbf{r}^{[t,t+k]} \mapsto \mathbf{u}^{[t,t+k]} \). Then in place of generator matrix \( R^{[t,t+\ell]} \) in (29), we can define a shift matrix \( U^{[t,t+\ell]} \) shown in (78). The shift matrix \( U^{[t,t+\ell]} \) is the same as generator matrix \( R^{[t,t+\ell]} \) in (29) with \( r \) replaced by \( u \). The \( k \)-th row of matrix \( U^{[t,t+\ell]} \), \( 0 \leq k \leq \ell \) is a shift vector \( \mathbf{u}^{[t,t+k]} \) which is the constant integer sequence assigned to row \( \mathbf{r}^{[t,t+k]} \) of \( R^{[t,t+\ell]} \).

\[
\begin{array}{cccccccc}
  u^t_{0,0} & u^t_{0,1} & \cdots & u^t_{0,\ell-1} & u^{t+1}_{1,0} & \cdots & u^{t+1}_{1,\ell-1} & \cdots & u^{t+1}_{1,k-1} \\
  u^t_{0,1} & u^t_{0,2} & \cdots & \vdots & u^{t+1}_{1,0} & \cdots & u^{t+1}_{1,\ell-1} & \cdots & u^{t+1}_{1,k-1} \\
  & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  & & & & \vdots & \vdots & \ddots & \vdots & \vdots \\
  & & & & & \ddots & \vdots & \vdots & \vdots \\
  & & & & & & \ddots & \vdots & \vdots \\
  & & & & & & & \ddots & \vdots \\
  & & & & & & & & \ddots \\
  \end{array}
\]  

(78)

We define \( \mathbf{u}^{t+j}_j \) to be a column vector in (78), for \( 0 \leq j \leq \ell \), where 
\[
\mathbf{u}^{t+j}_j \overset{\text{def}}{=} \left( \begin{array}{c} u^{t+j}_{j,\ell} \cdots u^{t+j}_{j,k} \cdots u^{t+j}_{j,j} \end{array} \right)^T.
\]

Then we can rewrite (78) as 
\[
U^{[t,t+\ell]} = (\mathbf{u}^t_0, \mathbf{u}^{t+1}_1, \ldots, \mathbf{u}^{t+j}_j, \ldots, \mathbf{u}^{t+\ell}_\ell).
\]

(79)

Fix tensor \( \mathbf{r} \in \mathcal{R} \). We know \( \mathbf{r} \) is defined by the collection of generator vectors \( \{ \mathbf{r}^{[t,t+k]} : 0 \leq k \leq \ell, t \in \mathbb{Z} \} \). Using the 1-1 correspondence given by \( \lambda^t_{B,k} : \mathbf{r}^{[t,t+k]} \mapsto \mathbf{u}^{[t,t+k]} \), for \( 0 \leq k \leq \ell \), for each \( t \in \mathbb{Z} \), gives a collection of shift vectors \( \{ \mathbf{u}^{[t,t+k]} : 0 \leq k \leq \ell, t \in \mathbb{Z} \} \). This collection defines a coset tensor \( \mathbf{u} \), shown in (80), which corresponds to tensor \( \mathbf{r} \) in (35). Let map \( \lambda_B \) give the assignment \( \lambda_B : \mathbf{r} \mapsto \mathbf{u} \). Let \( \mathcal{U} \) be the tensor set of all tensors \( \mathbf{u} \).
that can be constructed from \( r \in \mathcal{R} \) in this way.

\[
\begin{pmatrix}
\vdots \\
p \in \mathcal{U} @ \in \mathcal{U} + \ell \\
\vdots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\ldots \quad u_j^{(t)+j} \\
\vdots \\
\ldots \quad u_j^{(t)+\ell} \\
\end{pmatrix}
\]

**Theorem 50** For a given basis \( \mathbf{B} \), there is a 1-1 correspondence \( \lambda_{\mathbf{B},k} \) between shift vectors \( \mathbf{u}^{[t:t+k]} \) in \( \mathbf{u} \in \mathcal{U} \) and generator vectors \( \mathbf{r}^{[t:t+k]} \) in \( \mathbf{r} \in \mathcal{R} \), and therefore between shift matrices \( \mathbf{U}^{[t:t+\ell]} \) in \( \mathbf{U} \in \mathcal{U} \) and generator matrices \( \mathbf{R}^{[t:t+\ell]} \) in \( \mathbf{r} \in \mathcal{R} \). This gives a 1-1 correspondence \( \lambda_{\mathbf{B}} \) between tensors \( \mathbf{u} \in \mathcal{U} \) and tensors \( \mathbf{r} \in \mathcal{R} \), \( \lambda_{\mathbf{B}} : \mathbf{r} \leftrightarrow \mathbf{u} \), and therefore between tensors \( \mathbf{u} \in \mathcal{U} \) and paths \( \mathbf{b} \in \mathcal{C} \).

Each basis \( \mathbf{B} \) determines a tensor set \( \mathcal{R} \) and a map \( \lambda_{\mathbf{B}} : \mathcal{R} \to \mathcal{U} \). As \( \mathbf{B} \) changes, \( \mathcal{R} \) changes and \( \lambda_{\mathbf{B}} \) changes, but \( \mathcal{U} \) does not change. Consider the 4-tuple \( (\mathcal{U}, \mathcal{R}, C; \mathbf{B}) \) that includes the triple \( (\mathcal{R}, C; \mathbf{B}) \) previously discussed in Section 5. \( \mathcal{R} \) depends on choice of basis \( \mathbf{B} \) but \( \mathcal{U} \) does not. For any basis \( \mathbf{B} \), the map \( \lambda_{\mathbf{B}} \) gives a 1-1 correspondence \( \mathcal{U} \leftrightarrow \mathcal{R} \). If \( \mathbf{B} \) is fixed, then \( \mathcal{R} \) is fixed, and there is a 1-1 correspondence \( \mathcal{U} \leftrightarrow \mathcal{R} \leftrightarrow \mathcal{C} \).

The superscript parentheses terms in (80), like \( (t - j) \), indicate terms that all belong to the same shift matrix. For example, the diagonal terms \( u_0^{(t-j)}, u_1^{(t-j)+1}, \ldots, u_j^{(t-j)+j}, \ldots, u_\ell^{(t-j)+\ell} \) all belong to the shift matrix starting at time \( t - j \), \( U^{[(t-j)\cdots(t-j+\ell)]} \). The center row in (80) is

\[
(u_0^{(t)}, u_1^{(t-1)+1}, \ldots, u_j^{(t-j)+j}, \ldots, u_\ell^{(t-\ell)+\ell}),
\]

where each entry is itself a column; this reduces to

\[
(u_0^t, u_1^t, \ldots, u_j^t, \ldots, u_\ell^t),
\]

which is just the static matrix \( U^{[t;\ell]} \). Notice that each term in (81) and (82) is from one of \( \ell + 1 \) different shift matrices.

**Theorem 51** Fix time \( t \). A finite sequence of \( \ell + 1 \) shift matrices \( U^{[(t-j)\cdots(t-j+\ell)]} \) at times \( t - j \), for \( j = 0, \ldots, \ell \), uniquely determines a static matrix \( U^{[t;\ell]} \), where column \( j \) of shift matrix \( U^{[(t-j)\cdots(t-j+\ell)]} \), denoted \( u_j^{(t-j)+j} \), is column \( j \) of static matrix \( U^{[t;\ell]} \), denoted \( u_j^t \).

The static matrix \( U^{[t;\ell]} \) is the same as static matrix \( R^{[t;\ell]} \) in (81) with \( r \) replaced by \( u \). We define \( u^t \) to be a static matrix \( U^{[t;\ell]} \), or \( u^t \overset{\text{def}}{=} U^{[t;\ell]} \). The
set of all static matrices $u^t = U^{[t,t]}$ is the set $U^t$ of all triangular matrices of $\ell + 1$ rows and $\ell + 1$ columns over the sets $U^t_{j,k}$, $0 \leq j \leq k$, $0 \leq k \leq \ell$. Let $U^t_j$ be the set of all $j$-th columns of $U^t$, $0 \leq j \leq \ell$. Then for $u^t \in U^t$, we have $u^t = (u^t_0, u^t_1, \ldots, u^t_j, \ldots, u^t_\ell)$, where $u^t_j$ is a column in $U^t_j$. We denote any column $u^t_j$ with all entries 0 by $0^t_j$.

**Theorem 52** For a given basis $B$, there is a 1-1 correspondence between static matrices $u^t = U^{[t,t]} \in U^t$ and static matrices $r^t = R^{[t,t]} \in R^t$, induced by the 1-1 correspondence between shift matrices $U^{[(t-j),(t-j)+\ell]}$ and generator matrices $R^{[(t-j),(t-j)+\ell]}$ at times $t - j$, for $j = 0, \ldots, \ell$.

We define a shift property of tensor $u$ that mimics the shift property of tensor $r$. For $0 \leq j \leq \ell$, let $U^t_j, U^t_{j+1}+1$ be the set of all columns $u^t_j, u^t_{j+1}+1$ in all possible shift matrices $U^{[t,t]}$. For $0 \leq j < \ell$, define a column shift map $\sigma : U^t_j \rightarrow U^t_{j+1}+1$ by the assignment $\sigma : u^t_j \rightarrow u^t_{j+1}+1$, where this assignment is given by $\sigma : u^t_{j+1} \rightarrow u^t_j$ for $j < k \leq \ell$. Note that $\sigma u^t_{j+1}$ is not defined since $u^t_j$ “shifts out”. We abbreviate $\sigma(u^t_j)$ as $\sigma u^t_j$. Define

$$\sigma u^t = (\sigma u^t_0, \sigma u^t_1, \ldots, \sigma u^t_j, \ldots, \sigma u^t_{\ell-1}, \sigma u^t_\ell).$$

We have used the same shift notation in $\sigma r^t$ and $\sigma u^t$, but the meaning is clear by context. We obtain the following result for $U$ in the same way as Theorem 52 is obtained for $R$.

**Theorem 53** Let $w = \ldots, u^t, u^t+1, \ldots$ be an arbitrary sequence, not necessarily a tensor in $U$, where $u^t \in U^t$ for each time $t \in Z$. Then $w$ is a tensor in $U$ if and only if for each time $t$, $u^t+1 = (u^t+1, \sigma u^t)$ where input $u^t+1$ is any element of $U^t$.

Theorem 53 shows that $U$ has a natural shift structure in the same way that $R$ does. In Subsection 6.1, we interpreted a tensor $r \in R$ as a path in graph trellis $D^\infty(R, B)$, given by (69). We define $\nabla_{j,k}(u^t)$ and $\nabla_{j,k}(U^t)$ in analogous way to $\nabla_{j,k}(r^t)$ and $\nabla_{j,k}(R^t)$. In the same way as for $R$, we can interpret a tensor $u \in U$ as a path in a graph trellis $D^\infty(U)$, given by

$$\ldots, (\nabla_{1,1}(u^t), u^t, \nabla_{1,1}(u^t+1)), (\nabla_{1,1}(u^t+1), u^t+1, \nabla_{1,1}(u^t+2)), \ldots.$$  (83)

We have the following analogy to Theorem 44.

**Theorem 54** Let $u = \ldots, u^t, u^t+1, \ldots$ be a path in $U$. In graph trellis $D^\infty(U)$, edge $u^t = (u^t_0, u^t_1, \ldots, u^t_\ell)$ has left vertex $\nabla_{1,1}(u^t)$ in $\nabla_{1,1}(U^t)$ and right vertex $\nabla_{1,1}(u^t+1)$ in $\nabla_{1,1}(U^t+1)$. We have $u^t+1 = (u^t+1, \sigma u^t)$, where $u^t+1$ is a new input at time $t + 1$, and columns $\sigma u^t = (u^t+1, \ldots, u^t_\ell)$ of $u^t+1$ are a shift of columns $(u^t_0, \ldots, u^t_\ell)$ of $u^t$, i.e., $\sigma u^t_j = u^t_{j+1}$ for $0 \leq j \leq \ell - 1$. Note that $\nabla_{1,1}(u^t+1) = \sigma u^t$, a shift of $u^t$. Therefore the right vertex of $u^t$ is completely specified by $u^t$.

Theorem 54 shows that $D^\infty(U)$ is a shift register trellis. We can think of graph trellis $D^\infty(U)$ as composed of trellis sections $D(U^t)$. At each time $t$, $D^\infty(U)$ is a bipartite graph $D(U^t)$ having edges $u^t \in U^t$, left vertices
\(\nabla_{1,1}(u^t)\) in vertex set \(\nabla_{1,1}(U^t)\), and right vertices \(\nabla_{1,1}(u^{t+1}) = \sigma u^t\) in vertex set \(\nabla_{1,1}(U^{t+1})\).

For each path \(69\) in \(D^\infty(R, B)\), there is a path \(83\) in \(D^\infty(U)\) induced by the 1-1 correspondence \(\lambda_B\).

**Theorem 55** \(D^\infty(U)\) is graph isomorphic to \(D^\infty(R, B)\). The graph isomorphism is given by the 1-1 correspondence \(u \leftrightarrow r\) induced by \(\lambda_B\),

\[
\ldots, (\nabla_{1,1}(u^t), u^t, \nabla_{1,1}(u^{t+1})), \ldots \leftrightarrow \ldots, (\nabla_{1,1}(r^t), r^t, \nabla_{1,1}(r^{t+1})), \ldots
\]

Then we write \(D^\infty(U) \simeq D^\infty(R, B)\). The graph isomorphism maps vertices of \(D^\infty(U)\) to vertices of \(D^\infty(R, B)\). For each time \(t\), the graph isomorphism \(D^\infty(U) \simeq D^\infty(R, B)\) is given by the graph isomorphism \(D(U^t) \simeq D(R^t, B^t)\), where the 1-1 correspondence of branches and states is induced by \(\lambda^t_B\), the time \(t\) component of \(\lambda_B\).

We can reverse time in the preceding results and obtain dual results for \(B_Y, R_Y, U_Y\), and \(\lambda_{B_Y}\). The dual of Theorem 55 is the following.

**Theorem 56** \(D^\infty(U_Y)\) is graph isomorphic to \(D^\infty(R_Y, B_Y)\). The graph isomorphism is given by the 1-1 correspondence \(u_Y \leftrightarrow r_Y\) induced by \(\lambda_{B_Y}\),

\[
\ldots, (\nabla_{0,1}(u_{Y}^{t-1}), u_{Y}^{t-1}, \nabla_{0,1}(u_{Y}^{t})), \ldots \leftrightarrow \ldots, (\nabla_{0,1}(r_{Y}^{t-1}), r_{Y}^{t-1}, \nabla_{0,1}(r_{Y}^{t})), \ldots
\]

Then we write \(D^\infty(U_Y) \simeq D^\infty(R_Y, B_Y)\). The graph isomorphism maps vertices of \(D^\infty(U_Y)\) to vertices of \(D^\infty(R_Y, B_Y)\). For each time \(t\), the graph isomorphism \(D^\infty(U_Y) \simeq D^\infty(R_Y, B_Y)\) is given by the graph isomorphism \(D(U_Y^t) \simeq D(R_Y^t, B_Y^t)\), where the 1-1 correspondence of branches and states is induced by \(\lambda^t_{B_Y}\), the time \(t\) component of \(\lambda_{B_Y}\).

### 6.3 Change of basis, time equivalence, and harmonic equivalence

Given basis \(B = \{B^t : t \in \mathbb{Z}\}\) and encoder \(E \stackrel{\text{def}}{=} E(D^\infty(R, B))\), there is a 1-1 correspondence \(U \leftrightarrow R \leftrightarrow C\) given by

\[
u \stackrel{\lambda_B}{\leftrightarrow} r \stackrel{E}{\leftrightarrow} b,
\]

where correspondence \(u \leftrightarrow r\) is induced by \(\lambda_B\), and correspondence \(r \leftrightarrow b\) is induced by encoder \(E\). Now consider two different bases \(B_1 = \{B^t_1 : t \in \mathbb{Z}\}\) and \(B_2 = \{B^t_2 : t \in \mathbb{Z}\}\), and two different encoders \(E_1 \stackrel{\text{def}}{=} E(D^\infty(R_1, B_1))\) and \(E_2 \stackrel{\text{def}}{=} E(D^\infty(R_2, B_2))\). We say there is a change of basis. For encoder \(E_1\), there is a 1-1 correspondence \(U \leftrightarrow R_1 \leftrightarrow C\) given by

\[
u_1 \stackrel{\lambda_{B_1}}{\leftrightarrow} r_1 \stackrel{E_1}{\leftrightarrow} b_1,
\]

and for encoder \(E_2\), there is a 1-1 correspondence \(U \leftrightarrow R_2 \leftrightarrow C\) given by

\[
u_2 \stackrel{\lambda_{B_2}}{\leftrightarrow} r_2 \stackrel{E_2}{\leftrightarrow} b_2.
\]

In general, if \(u_1 = u_2\), then \(b_1 \neq b_2\), and conversely, if \(b_1 = b_2\) then \(u_1 \neq u_2\).
Theorem 57 Consider two time domain encoders $E_1 = E(D^\infty(R_1, B_1))$ and $E_2 = E(D^\infty(R_2, B_2))$ for bases $B_1$ and $B_2$, respectively; both encoders go forward in time. There is a graph automorphism of $D^\infty(U)$ which makes $E_1$ and $E_2$ graph isomorphic.

**Proof.** From Theorem 46 we know encoder $E_1$ is graph isomorphic to group trellis $C$ and so is encoder $E_2$. For encoder $E_1$, the graph isomorphism is given by a mapping of vertices of $D^\infty(R_1, B_1)$ to states of $C$, and the same holds for $E_2$. Therefore there must be a mapping of vertices of $D^\infty(R_1, B_1)$ to vertices of $D^\infty(R_2, B_2)$ which makes $E_1$ and $E_2$ graph isomorphic. But from Theorem 55, there is a mapping of vertices of $D^\infty(R_1, B_1)$ to vertices of $D^\infty(U)$ which makes $D^\infty(R_1, B_1)$ and $D^\infty(U)$ graph isomorphic. The same holds for $D^\infty(R_2, B_2)$ and $D^\infty(U)$. Therefore there is a mapping of vertices of $D^\infty(U)$ to vertices of $D^\infty(U)$ which makes $E_1$ and $E_2$ graph isomorphic, or a graph automorphism of $D^\infty(U)$.

Since $C$ is time invariant, we know that we can replace any basis $B_1 = \{B'_1 : t \in \mathbb{Z}\}$ with a constant basis $B_{c,1} = \{\ldots, B_1, B_1, \ldots\}$. Similarly we can replace any basis $B_2 = \{B'_2 : t \in \mathbb{Z}\}$ with a constant basis $B_{c,2} = \{\ldots, B_2, B_2, \ldots\}$. In general we assume $B_1 \neq B_2$. Then $C$ can be constructed from a time domain encoder $E_{c,1} \overset{\text{def}}{=} E(D^\infty(R_{c,1}, B_{c,1}))$ and a time domain encoder $E_{c,2} \overset{\text{def}}{=} E(D^\infty(R_{c,2}, B_{c,2}))$ where $B_1$ and $B_2$ are constant vector bases. We say a graph automorphism of $D^\infty(U)$ is constant if the mapping of states and edges is constant for each time $t$.

Theorem 58 Consider two time domain encoders $E_{c,1} = E(D^\infty(R_{c,1}, B_{c,1}))$ and $E_{c,2} = E(D^\infty(R_{c,2}, B_{c,2}))$. There is a constant graph automorphism of $D^\infty(U)$ which makes $E_{c,1}$ and $E_{c,2}$ graph isomorphic.

**Proof.** From Theorem 46 we know encoder $E_{c,1}$ is graph isomorphic to group trellis $C$ and so is encoder $E_{c,2}$. For encoder $E_{c,1}$, the graph isomorphism is given by a mapping of vertices of $D^\infty(R_{c,1}, B_{c,1})$ to states of $C$. Since $C$ is time invariant, and basis $B_1$ is time invariant, the mapping of vertices of $D^\infty(R_{c,1}, B_{c,1})$ to states of $C$ must be time invariant. The same holds for $E_{c,2}$. Therefore there must be a time invariant mapping of vertices of $D^\infty(R_{c,1}, B_{c,1})$ to vertices of $D^\infty(R_{c,2}, B_{c,2})$ which makes $E_{c,1}$ and $E_{c,2}$ graph isomorphic. But from Theorem 55 there is a mapping of vertices of $D^\infty(R_{c,1}, B_{c,1})$ to vertices of $D^\infty(U)$ which makes $D^\infty(R_{c,1}, B_{c,1})$ and $D^\infty(U)$ graph isomorphic. This mapping is time invariant by construction of $U$. The same holds for $D^\infty(R_{c,2}, B_{c,2})$ and $D^\infty(U)$. Therefore there is a time invariant mapping of vertices of $D^\infty(U)$ to vertices of $D^\infty(U)$ which makes $E_{c,1}$ and $E_{c,2}$ graph isomorphic, or a constant graph automorphism of $D^\infty(U)$.

We now compare time domain encoders for forward time and backward time. We consider two different bases, a basis $B = \{B^t : t \in \mathbb{Z}\}$ in the forward time direction and a basis $B_{Y} = \{B^t_{Y} : t \in \mathbb{Z}\}$ in the backward time direction. At each time $t$, we select $B^t$ and $B^t_{Y}$ arbitrarily and independently of one another.
Theorem 59 Consider two time domain encoders, a forward time encoder \( E = E(\mathcal{D}^\infty(\mathcal{R}, \mathcal{B})) \) and a backward time encoder \( E_Y \overset{\text{def}}{=} E_Y(\mathcal{D}^\infty(\mathcal{R}_Y, \mathcal{B}_Y)) \). There is a graph isomorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) to \( \mathcal{D}^\infty(\mathcal{U}_Y) \) which makes \( E \) and \( E_Y \) graph isomorphic.

Proof. From Theorem 46 we know encoder \( E \) is graph isomorphic to group trellis \( C \), and from Theorem 49 we know encoder \( E_Y \) is graph isomorphic to group trellis \( C \). For encoder \( E \), the graph isomorphism is given by a mapping of vertices of \( \mathcal{D}^\infty(\mathcal{R}, \mathcal{B}) \) to states of \( C \), and for encoder \( E_Y \), the graph isomorphism is given by a mapping of vertices of \( \mathcal{D}^\infty(\mathcal{R}_Y, \mathcal{B}_Y) \) to states of \( C \). Therefore there must be a mapping of vertices of \( \mathcal{D}^\infty(\mathcal{R}, \mathcal{B}) \) to vertices of \( \mathcal{D}^\infty(\mathcal{R}_Y, \mathcal{B}_Y) \) which makes \( E \) and \( E_Y \) graph isomorphic. But from Theorem 55 there is a mapping of vertices of \( \mathcal{D}^\infty(\mathcal{R}, \mathcal{B}) \) to vertices of \( \mathcal{D}^\infty(\mathcal{U}) \) which makes \( \mathcal{D}^\infty(\mathcal{R}, \mathcal{B}) \) and \( \mathcal{D}^\infty(\mathcal{U}) \) graph isomorphic. And from Theorem 56 the same holds for \( \mathcal{D}^\infty(\mathcal{R}_Y, \mathcal{B}_Y) \) and \( \mathcal{D}^\infty(\mathcal{U}_Y) \). Therefore there is a mapping of vertices of \( \mathcal{D}^\infty(\mathcal{U}) \) to vertices of \( \mathcal{D}^\infty(\mathcal{U}_Y) \) which makes \( E \) and \( E_Y \) graph isomorphic.\[\square\]

There is a natural isomorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) to \( \mathcal{D}^\infty(\mathcal{U}_Y) \). We can look at a generator \( \mathbf{g}^{[t,t+k]} \) as beginning at time \( t \) or ending at time \( t + k \). In constructing tensor \( \mathbf{u} \in \mathcal{U} \), at each time \( t \), we have collected the generators \( \mathbf{g}^{[t,t+k]} \), \( 0 \leq k \leq \ell \), that begin at time \( t \) to form a shift matrix \( \mathbf{R}^{[t,t+\ell]} \). In constructing tensor \( \mathbf{u}_Y \in \mathcal{U}_Y \), at each time \( t \), we have collected the generators \( \mathbf{g}^{[t-k,t]} \), \( 0 \leq k \leq \ell \), that end at time \( t \) to form a shift matrix \( \mathbf{R}^{[t-\ell,t]} \). Thus for each tensor \( \mathbf{u} \in \mathcal{U} \), there is a natural correspondence \( \mathbf{u} \equiv \mathbf{u}_Y \) with a tensor \( \mathbf{u}_Y \in \mathcal{U}_Y \) that uses the same shift vectors. The state of \( \mathbf{u} \) at time \( t \) is \( \nabla_{1,1}(\mathbf{u}^t) \) and the state at time \( t + 1 \) is \( \nabla_{1,1}(\mathbf{u}^{t+1}) \). The state of \( \mathbf{u}_Y \) at time \( t + 1 \) is \( \nabla_{0,1}(\mathbf{u}_Y^{t+1}) = \nabla_{0,1}(\mathbf{u}^{t+1}) \) and the state at time \( t \) is \( \nabla_{0,1}(\mathbf{u}_Y^t) = \nabla_{0,1}(\mathbf{u}^t) \). Any graph isomorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) to \( \mathcal{D}^\infty(\mathcal{U}_Y) \) is a graph automorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) composed with the natural (graph) isomorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) to \( \mathcal{D}^\infty(\mathcal{U}_Y) \) given by the natural correspondence. This gives the following result.

Corollary 60 Consider two time domain encoders, a forward time encoder \( E = E(\mathcal{D}^\infty(\mathcal{R}, \mathcal{B})) \) and a backward time encoder \( E_Y = E_Y(\mathcal{D}^\infty(\mathcal{R}_Y, \mathcal{B}_Y)) \). There is a graph automorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) composed with the natural isomorphism to \( \mathcal{D}^\infty(\mathcal{U}_Y) \) which makes \( E \) and \( E_Y \) graph isomorphic.

We say a group system has time equivalence if, when time domain encoder \( E(\mathcal{D}^\infty(\mathcal{R}, \mathcal{B})) \) (forward time) is loaded with \( \mathbf{r} \in \mathcal{R} \), and time domain encoder \( E_Y(\mathcal{D}^\infty(\mathcal{R}_Y, \mathcal{B}_Y)) \) (backward time) is loaded with \( \mathbf{r}_Y \in \mathcal{R}_Y \), where \( \mathbf{B} \equiv \mathcal{B}_Y \), \( \mathcal{R} \equiv \mathcal{R}_Y \), and \( \mathbf{r} \equiv \mathbf{r}_Y \), the outputs of both encoders are the same. In other words, if both encoders are loaded with the same sequence of generators, then both encoders give the same output \( \mathbf{c} \in \mathcal{C} \).

Theorem 61 Any abelian group system has time equivalence, but this is not necessarily true for a nonabelian group system. For the abelian group system, there is a trivial graph automorphism of \( \mathcal{D}^\infty(\mathcal{U}) \) composed with the natural isomorphism to \( \mathcal{D}^\infty(\mathcal{U}_Y) \) which makes \( E \) and \( E_Y \) graph isomorphic.

Proof. We have seen in Section 4 that if \( \mathbf{r} \equiv \mathbf{r}_Y \), then (40) and (50) give the same result. But if \( \mathbf{r} \equiv \mathbf{r}_Y \) then \( \mathbf{u} \equiv \mathbf{u}_Y \).\[\square\]
The standardized V.32 code is shown to be nonabelian in [5]. It can be shown time equivalence does not hold for this code.

Since \( C \) is time invariant, we know that we can replace any basis \( B = \{ B^t : t \in \mathbb{Z} \} \) with a constant basis \( B_0 = \{ \ldots, B, B, \ldots \} \). Similarly we can replace any basis \( B_Y = \{ B_Y^t : t \in \mathbb{Z} \} \) with a constant basis \( B_{Y,c} = \{ \ldots, B_Y, B_Y, \ldots \} \). Then \( C \) can be constructed from a time domain encoder \( E_c \text{ def } = E(D^\infty(\mathcal{R}_c, B_c)) \) (forward time) and a time domain encoder \( E_{Y,c} \text{ def } = E_Y(D^\infty(\mathcal{R}_{Y,c}, B_{Y,c})) \) (backward time) where \( B_c \) and \( B_{Y,c} \) are constant vector bases. We say a graph isomorphism of \( D^\infty(U) \) to \( D^\infty(U_Y) \) is constant if the mapping of states and edges is constant for each time \( t \).

**Theorem 62** Consider two time domain encoders, a forward time encoder \( E_c = E(D^\infty(\mathcal{R}_c, B_c)) \) and a backward time encoder \( E_{Y,c} = E_Y(D^\infty(\mathcal{R}_{Y,c}, B_{Y,c})) \). There is a constant graph isomorphism of \( D^\infty(U) \) to \( D^\infty(U_Y) \) which makes \( E_c \) and \( E_{Y,c} \) graph isomorphic.

**Proof.** The proof is a mix of the proof of Theorem 58 combined with the proofs of Theorem 59 and Corollary 60.

We now compare spectral domain encoders for forward time and backward time. The following result and proof is an analog of Theorem 59 and proof for the spectral domain.

**Theorem 63** Consider two spectral domain encoders, a forward time encoder \( E_s \text{ def } = E_s(D^\infty(\mathcal{R}, B)) \) and a backward time encoder \( E_{s,Y} \text{ def } = E_{s,Y}(D^\infty(\mathcal{R}_Y, B_Y)) \). There is a graph isomorphism of \( D^\infty(U) \) to \( D^\infty(U_Y) \) which makes \( E_s \) and \( E_{s,Y} \) graph isomorphic.

An analog of Corollary 60 holds as well.

**Corollary 64** Consider two spectral domain encoders, a forward time encoder \( E_s = E_s(D^\infty(\mathcal{R}, B)) \) and a backward time encoder \( E_{s,Y} = E_{s,Y}(D^\infty(\mathcal{R}_Y, B_Y)) \). There is a graph automorphism of \( D^\infty(U) \) composed with the natural isomorphism to \( D^\infty(U_Y) \) which makes \( E_s \) and \( E_{s,Y} \) graph isomorphic.

There are also analogs of Theorems 61 and 62.

We now compare the time and spectral domain encoders for forward time. We consider the two different bases \( B_1 \) and \( B_2 \) used previously. The following result and proof is similar to Theorem 57.

**Theorem 65** Consider a time domain encoder \( E_1 = E(D^\infty(\mathcal{R}_1, B_1)) \) and a spectral domain encoder \( E_{s,2} \text{ def } = E_s(D^\infty(\mathcal{R}_2, B_2)) \); both encoders go forward in time. There is a graph automorphism of \( D^\infty(U) \) which makes \( E_1 \) and \( E_{s,2} \) graph isomorphic.

As before, we replace basis \( B_1 \) with a constant basis \( B_{c,1} \) and replace basis \( B_2 \) with a constant basis \( B_{c,2} \). Then we have the following analog of Theorem 58.
Theorem 66 Consider a time domain encoder $E_{c,1} = E(D^\infty(R_{c,1}, B_{c,1}))$ and a spectral domain encoder $E_{s,c,2} \overset{\text{def}}{=} E_s(D^\infty(R_{c,2}, B_{c,2}))$; both encoders go forward in time. There is a constant graph automorphism of $D^\infty(U)$ which makes $E_{c,1}$ and $E_{s,c,2}$ graph isomorphic.

We say a group system has harmonic equivalence if, when time domain encoder $E = E(D^\infty(R, B))$ (forward time) and spectral domain encoder $E_s = E_s(D^\infty(R, B))$ (forward time) are loaded with the same sequence of generators, i.e., the same $r \in R$, then both encoders give the same output $c \in C$. In other words, if the group system is harmonically equivalent, then any path $c$ has a decomposition in the time domain and spectral domain into the same $r \in R$.

Theorem 67 Any abelian group system has harmonic equivalence, but this is not necessarily true for a nonabelian group system. For the abelian group system, there is a trivial graph automorphism of $D^\infty(U)$ which makes $E$ and $E_s$ graph isomorphic.

Proof. For an abelian group system, we see that the rearrangement of $b_t$ in (64) gives $b_t$ in (10).

For each of the four comparisons of encoders, $E_1$ and $E_2$, $E$ and $E_Y$, $E_s$ and $E_s,Y$, and $E_1$ and $E_{s,2}$, we see there is a graph automorphism of $D^\infty(U)$ which makes the two encoders graph isomorphic, composed with the natural isomorphism to $D^\infty(U_Y)$ in the second and third comparisons. If the bases are constant, the graph automorphism of $D^\infty(U)$ is constant. In the next section, we analyze the structure of any graph automorphism of $D^\infty(U)$. 

49
7. THE FULL SYMMETRY SYSTEM OF THE COSET TENSOR SET $\mathcal{U}$

7.1 Analysis of a symmetry permutation

As defined in [9], the full symmetry system of $\mathcal{U}$ is the set of all permutations or bijections of $\mathcal{U}$. This is a group under composition operation. Note that $\mathcal{U}$ and $\mathcal{D}^\infty(\mathcal{U})$ are equivalent: the paths of $\mathcal{U}$ are the paths of $\mathcal{D}^\infty(\mathcal{U})$ and vice versa. Therefore the full symmetry system of $\mathcal{U}$ is the set of all graph automorphisms of $\mathcal{D}^\infty(\mathcal{U})$. A symmetry $\Phi$ of $\mathcal{D}^\infty(\mathcal{U})$ is a graph automorphism of $\mathcal{D}^\infty(\mathcal{U})$. If $u$ is a path in $\mathcal{D}^\infty(\mathcal{U})$, then $\Phi(u)$ is a path in $\mathcal{D}^\infty(\mathcal{U})$, and we say $\Phi$ preserves paths in $\mathcal{D}^\infty(\mathcal{U})$. So a symmetry $\Phi$ of $\mathcal{D}^\infty(\mathcal{U})$ is a 1-1 and onto map of the states and edges of $\mathcal{D}(U_t)$ at each time $t$ that preserves paths in $\mathcal{D}^\infty(\mathcal{U})$. In this subsection we analyze the structure of any symmetry $\Phi$, and then in Subsection 7.2 we show how to construct any symmetry. In Subsection 7.3 we study the full symmetry system.

Let a symmetry $\Phi$ of $\mathcal{D}^\infty(\mathcal{U})$ be denoted as $\Phi = \ldots, \varphi^t, \varphi^{t+1}, \ldots$, where $\varphi^t : U_t \mapsto U_t$. Define a component form of $\varphi^t$ by $\varphi^t = (\varphi^t_0, \ldots, \varphi^t_\ell)$, where function $\varphi^t_j : U_t \mapsto U^t_j$ gives the $j$-th component of $\varphi^t$, $j = 0, \ldots, \ell$. We say $\varphi^t_j$ is independent of component $u_m'\ell$ if

$$\varphi^t_j(u_0', \ldots, u_{m-1}', u_m', u_{m+1}', \ldots, u_\ell') = \varphi^t_j(u_0', \ldots, u_{m-1}', 0_m', u_{m+1}', \ldots, u_\ell')$$

for all $u' = (u_0', \ldots, u_{m-1}', u_m', u_{m+1}', \ldots, u_\ell') \in U_t$. We denote this property as $\varphi^t_j(u_0', \ldots, u_{m-1}', u_m', u_{m+1}', \ldots, u_\ell') \in U_t$. We denote this property as $\varphi^t_j(u_0', \ldots, u_{m-1}', u_m', u_{m+1}', \ldots, u_\ell')$, where the bullet “$\bullet$” means $\varphi^t_j$ is independent of that component. For $0 \leq j \leq \ell$, define function $\varphi^t_{[j, \ell]} : U_t \mapsto U^t_j \times \cdots \times U^t_\ell$ to be the components $\varphi^t_m$ of $\varphi^t$ for $m \in [j, \ell]$, i.e.,

$$\varphi^t_{[j, \ell]} \overset{\text{def}}{=} (\varphi^t_j, \varphi^t_{j+1}, \ldots, \varphi^t_\ell).$$

First we review this important result about paths in $\mathcal{D}^\infty(\mathcal{U})$, which is a corollary of Theorem 53 for tensors in $\mathcal{U}$.

**Corollary 68** Let $w = \ldots, u^t, u^{t+1}, \ldots$ be an arbitrary sequence, not necessarily a path in $\mathcal{D}^\infty(\mathcal{U})$, where $u^t \in U_t$ for each time $t \in \mathbb{Z}$. Then $w$ is a path in $\mathcal{D}^\infty(\mathcal{U})$ if and only if for each time $t$, $u^{t+1} = (u_0^{t+1}, \sigma u^t)$ where input $u_0^{t+1}$ is any element of $U_0^{t+1}$.

We know if $w$ is a path, then the symmetry $\Phi(w)$ is also a path, and Corollary 68 applies to both $w$ and $\Phi(w)$. Therefore the commutative diagram Figure 2 holds.

For each $t$, a component $\varphi^t$ of $\Phi$ must be a 1-1 and onto map $\varphi^t : U^t \mapsto U^t$. Therefore the maps $\varphi^t : U^t \mapsto U^t$ and $\varphi^{t+1} : U^{t+1} \mapsto U^{t+1}$ in Figure 2 must be 1-1 and onto. The map $\varphi^{t+1}$ must be 1-1 and onto, but we know as well that all branches $u^{t+1}$ and $u^{t+1}$ which split from states $\sigma u^t$ and $\sigma u^t$ must map to each other. Or, in other words, state $\sigma u^t$ must map to state $\sigma u^t$. This gives commutative diagram Figure 3 and Theorem 69.

**Theorem 69** $\Phi = \ldots, \varphi^t, \varphi^{t+1}, \ldots$ is a symmetry of $\mathcal{D}^\infty(\mathcal{U})$ if and only if the following two conditions hold for each $t \in \mathbb{Z}$:

(i) $\varphi^t : U^t \mapsto U^t$ is 1-1 and onto,

(ii) $\sigma \varphi^t(u^t) = \varphi^{t+1}_{[1, \ell]}(\sigma u^t), \quad \text{(84)}$

for each $u^t \in U^t$. 

50
We can write (84) in component form as
\[ \Phi = \ldots, \varphi^t, \varphi^{t+1}, \ldots \] is a symmetry of \( D^\infty(U) \) if and only if the following three conditions hold for each \( t \in \mathbb{Z} \):

1. \( \varphi^t : U^t \to U^t \) is 1-1 and onto,
2. for \( 1 \leq j \leq \ell \), \( \varphi^j \) is independent of \( u_0^t, \ldots, u_{t-1}^t \), e.g., \( \varphi^j(u_0^t, u_1^t, \ldots, u_{t-1}^t) \),
3. for \( 0 \leq j \leq \ell - 1 \), for each \( u^t \in U^t \),
\[ \sigma \varphi^j(u^t) = \varphi_j^{t+1}(u_0^{t+1}, \sigma u^t) \]

**Proof.** We can write (84) in component form as
\[ \sigma \varphi^j(u^t) = \varphi_j^{t+1}(u_0^{t+1}, \sigma u^t) \]
for \( j = 0, \ldots, \ell - 1 \), for each \( u^t \in U^t \).

We can use Corollary 70 to further characterize a symmetry \( \Phi \) of \( D^\infty(U) \) as follows. Using (ii), we can rewrite (iii) as
\[ \sigma \varphi^j_0(u_0^t, u_1^t, \ldots, u_{t-1}^t) = \varphi_j^{t+1}(u_0^{t+1}, \sigma u_0^t, \ldots, \sigma u_{t-1}^t) \]
for \( j = 0, \ldots, \ell - 1 \). We can reduce the set of equations (87) further. Start with \( j = 1 \),
\[ \sigma \varphi_1^{t+1}(u_0^{t+1}, u_1^{t+1}, \ldots, u_{t-1}^{t+1}) = \varphi_2^{t+2}(u_0^{t+2}, \sigma u_0^{t+1}, \ldots, \sigma u_{t-1}^{t+1}) \]
for \( j = 1, \ldots, \ell - 1 \). We can reduce the set of equations (87) further. Start with \( j = 1 \),
are fixed on the right hand side. Then to have equality, \( \varphi_2^{t+2} \) must be independent of \( \sigma u_0^{t+1} \), or
\[
\sigma \varphi_2^{t+1}(u_0^{t+1}, u_1^{t+1}, \ldots, u_{t+1}) = \varphi_2^{t+2}(u_0^{t+2}, u_1^{t+2}, \sigma u_1^{t+1}, \ldots, \sigma u_{t-1}^{t+1}).
\] (88)

Now look at the case \( j = 2 \),
\[
\sigma \varphi_2^{t+2}(u_0^{t+2}, u_1^{t+2}, \ldots, u_{t+2}) = \varphi_3^{t+3}(u_0^{t+3}, \sigma u_0^{t+2}, \ldots, \sigma u_{t-1}^{t+2}).
\]

Using the result (88), we obtain
\[
\sigma \varphi_2^{t+2}(u_0^{t+2}, u_1^{t+2}, \ldots, u_{t+2}) = \varphi_3^{t+3}(u_0^{t+3}, u_1^{t+3}, u_2^{t+3}, \sigma u_2^{t+2}, \ldots, \sigma u_{t-1}^{t+2}).
\]

Now fix \( u_2^{t+2}, \ldots, u_{t+2} \) on the left hand side. Then to have equality, \( \varphi_3^{t+3} \) must be independent of \( \sigma u_0^{t+2} \) and \( \sigma u_1^{t+2} \), so we have
\[
\sigma \varphi_2^{t+2}(u_0^{t+2}, u_1^{t+2}, u_2^{t+2}, \ldots, u_{t+2}) = \varphi_3^{t+3}(u_0^{t+3}, u_1^{t+3}, u_2^{t+3}, \sigma u_2^{t+2}, \ldots, \sigma u_{t-1}^{t+2}).
\]

Continuing this process in the same manner, we finally reduce the last equation, \( j = \ell \), to
\[
\sigma \varphi_{\ell-1}^{t+\ell-1}(u_0^{t+\ell-1}, u_1^{t+\ell-1}, \ldots, u_{\ell-2}^{t+\ell-1}, u_{\ell-1}^{t+\ell-1}, u_{\ell}^{t+\ell-1}) = \varphi_{\ell}^{t+\ell}(u_0^{t+\ell}, u_1^{t+\ell}, \ldots, u_{\ell-1}^{t+\ell}, \sigma u_{\ell-1}^{t+\ell-1}).
\]

Summarizing our results, we can rewrite (87) as
\[
\sigma \varphi_j^{t+j}(u_0^{t+j}, u_1^{t+j}, \ldots, u_{j-1}^{t+j}, u_j^{t+j}, \ldots, u_{\ell}^{t+j}) = \varphi_{j+1}^{t+j+1}(u_0^{t+j+1}, u_1^{t+j+1}, \ldots, u_{j+1}^{t+j+1}, \sigma u_j^{t+j}, \ldots, \sigma u_{\ell}^{t+j}),
\]
(89)

for \( j = 1, \ldots, \ell - 1 \). With the understanding that the left hand side of (89) is \( \sigma \varphi_0^{t}(u_0, u_1, \ldots, u_{\ell}) \) when \( j = 0 \), then (89) also includes (86), and we can assume (89) holds for \( j = 0, \ldots, \ell - 1 \). Note that (89) can be explained using a commutative diagram.

Equation (89) shows that \( \varphi_j^{t+j} \) is independent of components \( u_0^{t+j}, \ldots, u_{j-1}^{t+j} \), for \( j = 1, \ldots, \ell \). This means that
\[
\varphi_j^{t+j}(u_0^{t+j}, \ldots, u_{j-1}^{t+j}, u_j^{t+j}, \ldots, u_{\ell}^{t+j}) = \varphi_j^{t+j}(0^{t+j}, \ldots, 0_{j-1}^{t+j}, u_j^{t+j}, \ldots, u_{\ell}^{t+j})
\]
(90)

for all \( u^{t+j} \in U^{t+j} \). We refer to this property by saying \( \varphi_j^{t+j} \) is a function of the form
\[
\varphi_j^{t+j} : (u^{t+j}_0, \ldots, u^{t+j}_{j-1}) \times U^{t+j}_j \times \cdots \times U^{t+j}_{\ell} \to U^{t+j}_j,
\]
(91)

where \( (u^{t+j}_0, \ldots, u^{t+j}_{j-1}) \) means the function is independent of these components.

For \( j = 0, \ldots, \ell \), let \( r \varphi_j^{t+j} \) be the restriction of \( \varphi_j^{t+j} \) to \( U_j^{t+j} \times \cdots \times U^{t+j}_{\ell} \). Then \( r \varphi_j^{t+j} \) is a function
\[
r \varphi_j^{t+j} : U_j^{t+j} \times \cdots \times U^{t+j}_{\ell} \to U_j^{t+j}.
\] (92)

For \( j = 0, \varphi_j^{t+j} \) and \( r \varphi_j^{t+j} \) are the same. With \( r \varphi_j^{t+j} \) the restriction of \( \varphi_j^{t+j} \), we have that (89) holds for \( j = 0, \ldots, \ell - 1 \) if and only if
\[
\sigma r \varphi_j^{t+j}(u_j^{t+j}, \ldots, u_{\ell}^{t+j}) = r \varphi_{j+1}^{t+j+1}(\sigma u_j^{t+j}, \ldots, \sigma u_{\ell}^{t+j})
\]
(93)

holds for \( j = 0, \ldots, \ell - 1 \).

If \( \varphi_j^{t+j} \) is a function of the form (91), then \( r \varphi_j^{t+j} \) is uniquely defined. Conversely if \( r \varphi_j^{t+j} \) is a function defined as in (92), then there is a unique function \( \varphi_j^{t+j} \) of the form (91) whose restriction is \( r \varphi_j^{t+j} \).
Lemma 71. Fix time \( t \in \mathbb{Z} \). For \( j = 1, \ldots, \ell \), suppose that \( \varphi_j^t \) has the property in (71)-(72) that \( \varphi_j^t \) is independent of components \( u_0^t, u_1^t, \ldots, u_{j-1}^t \).

With this property of \( \varphi_j^t \), we have that \( \varphi^t = (\varphi_0^t, \ldots, \varphi_j^t, \ldots, \varphi_{\ell}^t) \) is 1-1 and onto. We show the function \( r\varphi_j^t : U_j^t \times (u_{j+1}^t, \ldots, u_{\ell}^t) \to U_j^t \) is 1-1 and onto for each fixed \( (u_{j+1}^t, \ldots, u_{\ell}^t) \in U_{j+1}^t \times \cdots \times U_{\ell}^t \), for \( j \) such that \( 0 \leq j \leq \ell \). For \( j = \ell \), this is understood to mean \( r\varphi_j^t : U_j^\ell \to U_j^t \) is 1-1 and onto.

Proof. Assume \( \varphi^t \) is 1-1 and onto. Fix \( (u_{j+1}^t, \ldots, u_{\ell}^t) \in U_{j+1}^t \times \cdots \times U_{\ell}^t \). It is clear \( \varphi^t \) cannot be onto unless \( r\varphi_j^t : U_j^t \times (u_{j+1}^t, \ldots, u_{\ell}^t) \to U_j^t \) is onto. But by hypothesis the restriction \( r\varphi_j^t : U_j^t \times (u_{j+1}^t, \ldots, u_{\ell}^t) \to U_j^t \) is 1-1 and onto. We use proof by induction. Consider the function \( (r\varphi_j^t, r\varphi_{j+1}^t, \ldots, r\varphi_{\ell}^t) : U_j^t \times \cdots \times U_{\ell}^t \to U_j^t \times \cdots \times U_{\ell}^t \). Assume this function is 1-1 and onto. We show the function \( (r\varphi_j^t, r\varphi_{j+1}^t, \ldots, r\varphi_{\ell}^t) : U_j^t \times \cdots \times U_{\ell}^t \to U_j^t \times \cdots \times U_{\ell}^t \) is 1-1 and onto. By hypothesis, the restriction \( r\varphi_j^t : U_j^t \times (u_{j+1}^t, \ldots, u_{\ell}^t) \to U_j^t \) is 1-1 and onto for each fixed \( (u_{j+1}^t, \ldots, u_{\ell}^t) \in U_{j+1}^t \times \cdots \times U_{\ell}^t \). Then it follows that \( (r\varphi_j^t, r\varphi_{j+1}^t, \ldots, r\varphi_{\ell}^t) \) is 1-1 and onto. But by hypothesis the restriction \( r\varphi_j^t : U_j^t \to U_j^t \) is 1-1 and onto. Then by induction the function \( (r\varphi_0^t, r\varphi_1^t, \ldots, r\varphi_{\ell}^t) : U_0^t \times \cdots \times U_{\ell}^t \to U_0^t \times \cdots \times U_{\ell}^t \) is 1-1 and onto. But the function \( (r\varphi_0^t, r\varphi_1^t, \ldots, r\varphi_{\ell}^t) \) has the same values as \( (\varphi_0^t, \varphi_1^t, \ldots, \varphi_{\ell}^t) \). This proves that \( \varphi^t : U^t \to U^t \) is 1-1 and onto.

We now formalize the properties of \( \varphi_j^t \) and \( r\varphi_j^t \).

Definition 72 (Definition of \( \omega_j^{t+j} : 0 \leq j \leq \ell \))

Fix \( j \) such that \( 0 \leq j \leq \ell \). We define a function \( \omega_j^{t+j} : U_j^{t+j} \to U_j^{t+j} \) with the following two properties:

(i) The function \( \omega_j^{t+j} \) is a function of the form

\[
\omega_j^{t+j} : (u_0^{t+j}, \ldots, u_{j-1}^{t+j}) \times U_j^{t+j} \times \cdots \times U_{\ell}^{t+j} \to U_j^{t+j}.
\]

(ii) The restriction of \( \omega_j^{t+j} \) to \( U_j^{t+j} \times \cdots \times U_{\ell}^{t+j} \) is a function \( \beta_j^{t+j} : U_j^{t+j} \times \cdots \times U_{\ell}^{t+j} \to U_j^{t+j} \) which is a 1-1 and onto function

\[
\beta_j^{t+j} : U_j^{t+j} \times (u_{j+1}^{t+j}, \ldots, u_{\ell}^{t+j}) \to U_j^{t+j}
\]

from \( U_j^{t+j} \) to \( U_j^{t+j} \) for each fixed \( (u_{j+1}^{t+j}, \ldots, u_{\ell}^{t+j}) \in U_{j+1}^{t+j} \times \cdots \times U_{\ell}^{t+j} \).

We call a function \( \omega_j^{t+j} \) with the properties in Definition 72 a separating function, and function \( \beta_j^{t+j} \) a restricted separating function.

Lemma 73. The function \( \varphi_j^{t+j} \) is a separating function, and \( r\varphi_j^t \) is a restricted separating function.

Using Lemma 71 and 73 we are able to characterize a symmetry \( \Phi \) of \( \mathcal{D}^\infty(\mathcal{U}) \) as follows.
Theorem 74 (Analysis) \( \Phi = \ldots, \varphi^t, \varphi^{t+1}, \ldots \) is a symmetry of \( D^\infty(U) \) if and only if, for each \( t \in \mathbb{Z} \), \( \varphi^t = (\varphi_0^t, \ldots, \varphi_j^t, \ldots, \varphi_\ell^t) \) where the following two equivalent conditions hold:

(i) \( \varphi_j^t \) is a separating function, for \( j \) such that \( 0 \leq j \leq \ell \), and (89) is satisfied for \( j = 0, \ldots, \ell - 1 \);

(ii) \( \varphi_j^t \) is a restricted separating function, for \( j \) such that \( 0 \leq j \leq \ell \), and (93) is satisfied for \( j = 0, \ldots, \ell - 1 \).

For each \( t, \ell \in \mathbb{Z} \), a sequence of functions

\[
\Psi^t \overset{\text{def}}{=} (\varphi_0^t, \varphi_1^t, \ldots, \varphi_{j+1}^t, \ldots, \varphi_\ell^t)
\]

such that for \( j = 0, \ldots, \ell - 1 \), each pair \((\varphi_{j+1}^t, \varphi_{j+1}^t)\) satisfies (89), and such that for \( j = 0, \ldots, \ell \), \( \varphi_j^t \) is a separating function, is called a \( t \)-tower \( \Psi^t \) or just tower. Then an essential conclusion of Theorem 74 is that any symmetry of \( D^\infty(U) \) gives rise to a sequence of \( t \)-towers \( \ldots, \Psi^t, \Psi^{t+1}, \ldots \) that is, a \( t \)-tower for each \( t \in \mathbb{Z} \). We utilize \( t \)-towers in the construction algorithm below.

\[
\begin{pmatrix}
\vdots \\
\varphi_0^{(t)} & \ldots & \varphi_j^{(t)+j} & \ldots & \varphi_\ell^{(t)+\ell} \\
\varphi_1^{(t)} & \ldots & \varphi_j^{(t)+j} & \ldots & \varphi_\ell^{(t)+\ell} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\varphi_{j-1}^{(t)} & \ldots & \varphi_j^{(t)+j} & \ldots & \varphi_\ell^{(t)+\ell} \\
\varphi_j^{(t)} & \ldots & \varphi_j^{(t)+j} & \ldots & \varphi_\ell^{(t)+\ell} \\
\varphi_{j+1}^{(t)} & \ldots & \varphi_j^{(t)+j} & \ldots & \varphi_\ell^{(t)+\ell} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\end{pmatrix}
\]

(95)

Equation (95) shows a \( t \)-tower \( \Psi^t \) and \((t - j)\)-tower \( \Psi^{t-j} \) as diagonals in an infinite matrix of towers. The theorem shows that the sequence of towers \( \ldots, \Psi^{t-j}, \ldots, \Psi^t, \ldots \) in (95) defines a symmetry and any such matrix (95) of towers defines a symmetry. A component \( \varphi^t \) of the symmetry is defined by going “across the row” in (95),

\[
\varphi^t = (\varphi_0^{(t)}, \varphi_1^{(t)+1}, \ldots, \varphi_{j+1}^{(t)+j}, \ldots, \varphi^{(t)+\ell}) = (\varphi_0^t, \varphi_1^t, \ldots, \varphi_j^t, \ldots, \varphi_\ell^t),
\]

(96)

and more explicitly, \( \varphi^t : U^t \rightarrow U^t \) is defined by

\[
\varphi^t(u^t) = (\varphi_0^t(u^t), \varphi_1^t(u^t), \ldots, \varphi_j^t(u^t), \ldots, \varphi_\ell^t(u^t)).
\]

(97)

Note that each component \( \varphi_j^t \) in (97) is selected from a different tower. For example, \( \varphi_j^t \) in (97) is component \( \varphi_j^{(t-j)+j} \) in \((t - j)\)-tower \( \Psi^{t-j} \):

\[
\varphi_j^t(u^t) = \varphi_j^{(t-j)+j}(u^{(t-j)+j}).
\]
7.2 Construction of a symmetry permutation

We now use Theorem 74 to construct any symmetry \( \omega = \ldots, \omega^t, \omega^{t+1}, \ldots \) of \( \mathcal{D}^\infty(\mathcal{U}) \). We can solve the set of equations (89) or (93) by starting with \( j = \ell \) and working backwards, for \( j = \ell, \ell-1, \ldots, 1, 0 \). At each step \( j \), we want to find a separating function \( \omega_j^{t+j} : U^{t+j}_\ell \to U^{t+j}_j \) that satisfies (89), or a restricted separating function \( \beta_j^{t+j} : U^{t+j}_j \times \ldots \times U^{t+j}_\ell \to U^{t+j}_j \) that satisfies (93).

Algorithm 75 (Construction) Any solution of the set of equations (89) or (93) which is a symmetry \( \omega = \ldots, \omega^t, \omega^{t+1}, \ldots \) of \( \mathcal{D}^\infty(\mathcal{U}) \) can be found as follows.

DO
1. Fix time \( t \).
2. Let \( \beta^{t+\ell}_\ell : U^{t+\ell}_\ell \to U^{t+\ell}_\ell \) be any restricted separating function. Define \( \omega^{t+\ell}_\ell \) to be the unique separating function whose restriction is \( \beta^{t+\ell}_\ell \).
3. FOR \( j = \ell-1, \ldots, 0 \) (counting down in order), find a restricted separating function \( \beta^{t+j}_j : U^{t+j}_j \times \ldots \times U^{t+j}_\ell \to U^{t+j}_j \) such that
   \[
   \sigma_{\beta^{t+j}_j} (u^{t+j}_j, \ldots, u^{t+j}_\ell) = \beta^t_{j+1} (\sigma u^{t+j}_j, \ldots, \sigma u^{t+j}_{\ell-1}).
   \] 
   Define \( \omega^{t+j}_j \) to be the unique separating function whose restriction is \( \beta^{t+j}_j \).
ENDFOR
ENDDO
4. For each time \( t \), steps 1-3 produce a \( t \)-tower \( Y^t \), where
   \[
   Y^t \overset{\text{def}}{=} (\omega^t_0, \omega^t_1, \ldots, \omega^t_j, \ldots, \omega^t_\ell).
   \] 
   A sequence of any \( t \)-towers \( \ldots, Y^t, Y^{t+1}, \ldots \) defines a symmetry \( \omega = \ldots, \omega^t, \omega^{t+1}, \ldots \) of \( \mathcal{D}^\infty(\mathcal{U}) \) in the following manner. For each \( t \), define a function \( \omega^t : U^t \to U^t \) by
   \[
   \omega^t = (\omega^t_0, \omega^t_1, \ldots, \omega^t_j, \ldots, \omega^t_\ell),
   \] 
   where \( \omega^t_j \) is component function \( \omega^{(t-j)+j}_j \) in \( (t-j) \)-tower \( Y^{t-j} \):
   \[
   \omega^t_j (u^t) = \omega^{(t-j)+j}_j (u^{(t-j)+j}).
   \]
Then \( \omega = \ldots, \omega^t, \omega^{t+1}, \ldots \) is a symmetry of \( \mathcal{D}^\infty(\mathcal{U}) \), and the set of all possible symmetries \( \omega \) obtained this way is the full symmetry system of \( \mathcal{D}^\infty(\mathcal{U}) \).

Proof. Note that if \( \beta_{j+1}^{t+j+1} \) on the right hand side of (98) is a restricted separating function, then
   \[
   \beta_{j+1}^{t+j+1} : U^{t+j+1}_{j+1} \times (\sigma u^{t+j}_{j+1}, \ldots, \sigma u^{t+j}_{\ell-1}) \to U^{t+j+1}_{j+1}
   \] 
is a 1-1 and onto function for each fixed \( (\sigma u^{t+j}_{j+1}, \ldots, \sigma u^{t+j}_{\ell-1}) \in U^{t+j+1}_{j+1} \times \ldots \times U^{t+j+1}_{\ell} \). This means we can find a function \( \beta^{t+j}_j \) on the left hand side of (98) such that
   \[
   \beta^{t+j}_j : U^{t+j}_j \times (u^{t+j}_{j+1}, \ldots, u^{t+j}_\ell) \to U^{t+j}_j
   \]
is a 1-1 and onto function for each fixed \((u_{j+1}^{t+j}, \ldots, u_{\ell}^{t+j}) \in U_{j+1}^{t+j} \times \cdots \times U_{\ell}^{t+j}\). Then \(\beta_{j}^{t+j}\) is a restricted separating function.

A sequence of any \(t\)-towers \(\ldots, \Upsilon^t, \Upsilon^{t+1}, \ldots\) defines a symmetry. Note that in general, \(\Upsilon^t\) can be different for each \(t\), i.e., we need not have \(\Upsilon^t = \Upsilon^{t+1}\).

Define \(\nabla_{j,k}(u^t)\) to be the same as \(\nabla_{j,k}(u^t)\) except missing entry \(u_{j,k}^t\), and likewise define \(\nabla_{j,k}(U^t)\) to be the same as \(\nabla_{j,k}(U^t)\) except missing entry \(u_{j,k}\). Define \(\Delta_{j,k}(u^t)\) to be all the entries \(u_{m,n}\) in \(u^t\) except those in \(\nabla_{j,k}(u^t)\), and define \(\Delta_{j,k}(U^t)\) to be \(\Delta_{j,k}(U^t) = (\Delta_{j,k}(u^t) : u^t \in U^t)\). If a function is independent of entries in \(\Delta_{j,k}(U^t)\), we denote this by \(\Delta_{j,k}(\bullet^t)\).

We now define functions \(\omega_{j,k}^{t+j}\) and \(\beta_{j,k}^{t+j}\) and then show these functions can be used to construct a separating function \(\omega_j^{t+j}\).

**Definition 76 (Definition of \(\omega_{j,k}^{t+j}\))**

Fix \(k\) such that \(0 \leq k \leq t\). Fix \(j\) such that \(0 \leq j \leq k\). We define a function \(\omega_{j,k}^{t+j} : U_{j,k}^{t+j} \rightarrow U_{j,k}^{t+j}\) with the following two properties:

(i) The function \(\omega_{j,k}^{t+j}\) is a function of the form

\[
\omega_{j,k}^{t+j} : \Delta_{j,k}(\bullet^t) \times U_{j,k}^{t+j} \xrightarrow{\nabla_{j,k}} U_{j,k}^{t+j}.
\]

(ii) The restriction of \(\omega_{j,k}^{t+j}\) to \(\nabla_{j,k}(U_{j,k}^{t+j})\) is a function \(\beta_{j,k}^{t+j} : U_{j,k}^{t+j} \times \nabla_{j,k}(U_{j,k}^{t+j}) \rightarrow U_{j,k}^{t+j}\) which is a 1-1 and onto function

\[
\beta_{j,k}^{t+j} : U_{j,k}^{t+j} \times \nabla_{j,k}(u_{j,k}^{t+j}) \rightarrow U_{j,k}^{t+j}
\]

from \(U_{j,k}^{t+j}\) to \(U_{j,k}^{t+j}\) for each fixed \(\nabla_{j,k}(u_{j,k}^{t+j}) \in \nabla_{j,k}(U_{j,k}^{t+j})\).

For \(j = k = \ell\), (i) is understood to mean \(\omega_{j,k}^{t+j}\) is a function of the form

\[
\omega_{\ell,\ell}^{t+\ell} : \Delta_{\ell,\ell}(\bullet^{t+\ell}) \times U_{\ell,\ell}^{t+\ell} \rightarrow U_{\ell,\ell}^{t+\ell}.
\]

For \(j = k = 0\), (i) is understood to mean \(\omega_{j,k}^{t+j}\) is a function of the form

\[
\omega_{0,0} : U_{0,0}^{t} \times \nabla_{0,0}(U^t) \rightarrow U_{0,0}^{t}.
\]

Again we call a function \(\omega_{j,k}^{t+j}\) with the properties in Definition 76 a separating function and \(\beta_{j,k}^{t+j}\) a restricted separating function.

We now use these definitions and results to simplify Algorithm 75 by solving (98) of Step 3 in Algorithm 75. Given a separating function \(\beta_{j+1}^{t+j}\), we want to find a separating function \(\beta_{j}^{t+j}\) that satisfies (98). We first find properties of any function \(\beta_{j}^{t+j}\) that satisfies (98) and then give a necessary and sufficient condition that it be a separating function.

It is sufficient to construct \(\beta_{j}^{t+j}\) for arbitrary fixed \((u_{j+1}^{t+j}, \ldots, u_{\ell}^{t+j}) \in U_{j+1}^{t+j} \times \cdots \times U_{\ell}^{t+j}\). For fixed \((u_{j+1}^{t+j}, \ldots, u_{\ell}^{t+j})\), \(\beta_{j}^{t+j}\) is a function with domain \(U_{j+1}^{t+j}\) and range \(U_{j}^{t+j}\):

\[
\beta_{j}^{t+j} : U_{j}^{t+j} \times (u_{j+1}^{t+j}, \ldots, u_{\ell}^{t+j}) \rightarrow U_{j}^{t+j}.
\]
We decompose \([101]\) into two functions by dividing range \(U_{j,j}^{t+j}\) into two pieces: \(U_{j,j}^{t+j}\) and \(U_{j,j}^{t+j}\), where \(U_{j,j}^{t+j}\) are all the vectors \(u_{j,j}^{t+j}\) except component \(u_{j,j}^{t+j}\) is deleted; denote these vectors by \(u_{j,j}^{t+j}\). The first function, defined to be \(\beta_{j,j}^{t+j}\), has domain \(U_{j,j}^{t+j}\) and range \(U_{j,j}^{t+j}\) for fixed \((u_{j,j}^{t+j}, \ldots, u_{\ell}^{t+j}) \in U_{j,j}^{t+j} \times \cdots \times U_{\ell}^{t+j}:
\[
\beta_{j,j}^{t+j} : U_{j,j}^{t+j} \times (u_{j,j}^{t+j}, \ldots, u_{\ell}^{t+j}) \rightarrow U_{j,j}^{t+j}.
\]
(102)

The second function, defined to be \(\beta_{j,j}^{t+j}\), has domain \(U_{j,j}^{t+j}\) and range \(U_{j,j}^{t+j}\) for fixed \((u_{j,j}^{t+j}, \ldots, u_{\ell}^{t+j}) \in U_{j,j}^{t+j} \times \cdots \times U_{\ell}^{t+j}:
\[
\beta_{j,j}^{t+j} : U_{j,j}^{t+j} \times (u_{j,j}^{t+j}, \ldots, u_{\ell}^{t+j}) \rightarrow U_{j,j}^{t+j}.
\]
(103)

(At this point we do not assume that \(\beta_{j,j}^{t+j}\) is a restricted separating function.)

Since any \(u_{j,j}^{t+j} \in U_{j,j}^{t+j}\) can be uniquely expressed as \(u_{j,j}^{t+j} = (u_{j,j}^{t+j}, u_{j,j}^{t+j})\), it is clear that given \(\beta_{j,j}^{t+j}\) in \([101]\), then \(\beta_{j,j}^{t+j}\) in \([102]\) and \(\beta_{j,j}^{t+j}\) in \([103]\) are completely specified, and the reverse is also true. Thus specifying \(\beta_{j,j}^{t+j}\) and \(\beta_{j,j}^{t+j}\) will completely specify \(\beta_{j,j}^{t+j}\). We will see that \(\beta_{j,j}^{t+j}\) is completely specified by \([58]\), but \([58]\) has nothing to say about \(\beta_{j,j}^{t+j}\).

We first determine \(\beta_{j,j}^{t+j}\). Since the range of \(\beta_{j,j}^{t+j}\) is \(U_{j,j}^{t+j}\), we can rewrite \([58]\) as
\[
\beta_{j,j}^{t+j}(u_{j,j}^{t+j}, \ldots, u_{\ell}^{t+j}) = \beta_{j,j}^{t+j+1}(\sigma u_{j,j}^{t+j}, \ldots, \sigma u_{\ell}^{t+j}.
\]
(104)

The function \(\beta_{j,j}^{t+j}\) on the left hand side of \([104]\) is a function of \(\nabla_{j,j}(u^{t+j})\), and \(\beta_{j,j}^{t+j+1}\) on the right hand side is a function of \(\nabla_{j,j+1}(\sigma u^{t+j})\). Clearly \(\nabla_{j,j+1}(\sigma u^{t+j})\) is a shift of \(\nabla_{j,j}(u^{t+j})\). We can divide \(\nabla_{j,j}(u^{t+j})\) into two pieces. One piece is \(\nabla_{j,j+1}(u^{t+j})\), and the other piece is the remaining diagonal terms in \(\nabla_{j,j}(u^{t+j})\); denote the later piece by \(\nabla_{j,j}^{t+j}(u^{t+j})\). Note that the first piece shifts to \(\nabla_{j,j+1}(\sigma u^{t+j})\), and the second piece, the diagonal terms, shifts out. In fact, aside from time index, the first piece is identical in integer values to \(\nabla_{j,j+1}(\sigma u^{t+j})\). We refer to this by saying \(\nabla_{j,j}(u^{t+j})\) is shift equivalent to \(\nabla_{j,j+1}(\sigma u^{t+j})\) on \(\nabla_{j,j+1}(u^{t+j})\), written as \(\nabla_{j,j}(u^{t+j}) \equiv \nabla_{j,j+1}(\sigma u^{t+j})\).

For fixed \(\nabla_{j,j+1}(\sigma u^{t+j})\), the elements \(\nabla_{j,j}(w^{t+j}) \in \nabla_{j,j}(U^{t+j})\) which shift to \(\nabla_{j,j+1}(\sigma u^{t+j})\) are all the elements in which \(\nabla_{j,j+1}(w^{t+j})\) is the same as \(\nabla_{j,j+1}(u^{t+j})\), but the remaining diagonal of terms \(\nabla_{j,j}^{t+j}(w^{t+j})\) can be anything. We refer to such elements by saying \(\nabla_{j,j}(w^{t+j})\) is shift equivalent to \(\nabla_{j,j+1}(\sigma u^{t+j})\) on \(\nabla_{j,j+1}(w^{t+j})\), written as \(\nabla_{j,j}(w^{t+j}) \equiv \nabla_{j,j+1}(\sigma u^{t+j})\). Therefore any element \(\nabla_{j,j}(w^{t+j})\) which is shift equivalent to \(\nabla_{j,j+1}(\sigma u^{t+j})\) on \(\nabla_{j,j+1}(w^{t+j})\) will shift to \(\nabla_{j,j+1}(\sigma u^{t+j})\).

Again fix \(\nabla_{j,j+1}(\sigma u^{t+j})\). Then the value \(u_{j,j}^{t+j+1}\) of \(\beta_{j,j}^{t+j+1}\) on the right hand side of \([101]\) is fixed. Now examine \(\beta_{j,j}^{t+j}\) on the left hand side. With the right hand side fixed, the value \(u_{j,j}^{t+j+1}\) of \(\beta_{j,j}^{t+j+1}(w^{t+j})\) must be the same for the set of all elements \(w^{t+j}\) such that \(\nabla_{j,j}(w^{t+j}) \equiv \nabla_{j,j+1}(\sigma u^{t+j})\). We can look at the function \(\beta_{j,j}^{t+j}(w^{t+j})\) of all these elements \(w^{t+j}\) in a slightly different way. The only components of \(w^{t+j}\) which remain fixed in this set are \(\nabla_{j,j+1}(w^{t+j})\). Therefore we can regard \(\beta_{j,j}^{t+j}\) as a map from these fixed
values $\nabla_{j,j+1}(u^{t+j})$ to $u^{t+j}_{j,j}$,

$$\beta^{t+j}_{j,j+1} : \nabla_{j,j+1}(u^{t+j}) \mapsto u^{t+j}_{j,j+1},$$

(105)

which ignores components in the diagonal of terms $\gamma_{j,j}$ ($w^{t+j}$). In other words, $\beta^{t+j}_{j,j+1}$ is a function of the form

$$\beta^{t+j}_{j,j+1} : \gamma_{j,j}(e^{t+j}) \times \nabla_{j,j+1}(U^{t+j}) \mapsto U_{j,j+1}^{t+j}.$$  

(106)

On the right hand side of (104) we know that $\beta^{t+j+1}_{j+1}$ is an assignment of $\nabla_{j+1,j+1}(\sigma u^{t+j})$ to $u^{t+j+1}_{j+1}$,

$$\beta^{t+j+1}_{j+1} : \nabla_{j+1,j+1}(\sigma u^{t+j}) \mapsto u^{t+j+1}_{j+1}.$$  

(107)

Therefore (104) reduces to

$$\beta^{t+j}_{j,j+1}(\gamma_{j,j}(e^{t+j}), \nabla_{j,j+1}(u^{t+j})) = \beta^{t+j+1}_{j+1}(\nabla_{j+1,j+1}(\sigma u^{t+j})).$$  

(108)

But we know that for $\beta^{t+j+1}_{j+1}$, the assignment of (107) is an assignment

$$\beta^{t+j+1}_{j+1} : \sigma u^{t+j}_{j} \times \nabla_{j+2,j+2}(\sigma u^{t+j}) \mapsto u^{t+j+1}_{j+1},$$

which is 1-1 and onto from $U^{t+j+1}_{j+1}$ to $U^{t+j+1}_{j,j+1}$ for each fixed $\nabla_{j+2,j+2}(\sigma u^{t+j})$. Moreover from (108), the assignment of (105) must be identical, aside from time index, to the assignment of (107). Therefore we must have that the assignment of (105) is

$$\beta^{t+j}_{j,j+1} : u^{t+j}_{j,j} \times \nabla_{j+1,j+2}(u^{t+j}) \mapsto u^{t+j}_{j,j+1},$$

which is 1-1 and onto from $U^{t+j}_{j,j}$ to $U^{t+j}_{j,j+1}$ for each fixed $\nabla_{j+1,j+2}(u^{t+j})$. Then we can rewrite (108) as

$$\beta^{t+j}_{j,j+1}(\gamma_{j,j}(e^{t+j}), u^{t+j}_{j,j}, \nabla_{j+1,j+2}(u^{t+j})) = \beta^{t+j+1}_{j+1}(\sigma u^{t+j}_{j,j}, \nabla_{j+2,j+2}(\sigma u^{t+j})).$$

(109)

Note that $\nabla_{j,j}(u^{t+j})$ is shift equivalent to $(\sigma u^{t+j}_{j,j}, \nabla_{j+2,j+2}(\sigma u^{t+j}))$ on $\nabla_{j,j+1}(u^{t+j}) = (u^{t+j}_{j,j}, \nabla_{j+1,j+2}(u^{t+j}))$. Therefore we refer to the property of $\beta^{t+j}_{j,j+1}$ given in (109) by saying $\beta^{t+j}_{j,j+1}$ is shift equivalent to $\beta^{t+j+1}_{j+1}$ on $\nabla_{j,j+1}(U^{t+j})$, and write this as $\beta^{t+j}_{j,j+1} \equiv \beta^{t+j+1}_{j+1}$. We know $\beta^{t+j}_{j,j+1}$ is a function of the form (106) which is 1-1 and onto from $U^{t+j}_{j,j}$ to $U^{t+j}_{j,j+1}$ for each fixed $\nabla_{j+1,j+2}(u^{t+j})$. It follows that $\beta^{t+j}_{j,j}$ is a function which is 1-1 and onto from $U^{t+j}_{j,j}$ to $U^{t+j}_{j,j+1}$ for each fixed $(u^{t+j}_{j,j}, \ldots, u^{t+j}_{\ell}) \in U^{t+j}_{j,j+1} \times \cdots \times U^{t+j}_{\ell}$. But $\beta^{t+j}_{j,j}$ in (108) must be a restricted separating function which is 1-1 and onto from $U^{t+j}_{j,j}$ to $U^{t+j}_{j,j+1}$ for each fixed $(u^{t+j}_{j,j}, \ldots, u^{t+j}_{\ell}) \in U^{t+j}_{j,j+1} \times \cdots \times U^{t+j}_{\ell}$. Therefore in order for $\beta^{t+j}_{j,j}$ to have this property, it is necessary and sufficient that $\beta^{t+j}_{j,j}$ be any function of the form

$$\beta^{t+j}_{j,j} : \nabla_{j,j}(U^{t+j}) \mapsto U^{t+j}_{j,j}.$$  

(110)

which is 1-1 and onto from $U^{t+j}_{j,j}$ to $U^{t+j}_{j,j}$ for each fixed $(u^{t+j}_{j,j}, u^{t+j}_{j,j+1}, \ldots, u^{t+j}_{\ell}) \in U^{t+j}_{j,j+1} \times U^{t+j}_{j,j+1} \times \cdots \times U^{t+j}_{\ell}$,

$$\beta^{t+j}_{j,j} : U^{t+j}_{j,j} \times (u^{t+j}_{j,j}, u^{t+j}_{j,j+1}, \ldots, u^{t+j}_{\ell}) \mapsto U^{t+j}_{j,j}.$$  

(111)
In other words, $\beta_{j,i}^{t+j}$ must be a restricted separating function. With $\beta_{j,i}^{t+j}$ specified as in (106) and (109), and $\beta_{j,j}^{t+j}$ specified as in (110) and (111), $\beta_{j,j}^{t+j}$ is completely determined, and $\beta_{j,j}^{t+j}$ is a restricted separating function with the desired properties.

We can summarize these results as follows.

**Theorem 77** The solution $\beta_{j,j}^{t+j}$ of (98) is a restricted separating function composed of a function $\beta_{j,i}^{t+j}$, given in (106) and (109), and a function $\beta_{j,j}^{t+j}$, given in (110) and (111). The function $\beta_{j,j}^{t+j}$ is shift equivalent to $\beta_{j,j}^{t+j+1}$ on $\nabla_{j,j+1}U_{t+j}$. Therefore, it is 1-1 and onto from $U_{j,j+1}^{t+j}$ to $U_{j,j}^{t+j}$ for each fixed $\nabla_{j+1,j+2}(u_{j,j})$. The function $\beta_{j,j}^{t+j}$ is a restricted separating function.

This gives the following algorithm.

**Algorithm 78** Any solution of the set of equations (89) or (93) which is a symmetry $\omega = \omega^{t}, \omega^{t+1}, \ldots$ of $D_{t}^{\infty}(U)$ can be found as follows.

**DO**

1. Fix time $t$.

2. Let $\beta_{t}^{t+j} : U_{t}^{t+j} \rightarrow U_{t}^{t+j}$ be any restricted separating function. Define $\omega_{t}^{t+j}$ to be the unique separating function whose restriction is $\beta_{t}^{t+j}$.

3. FOR $j = \ell - 1, \ldots, 0$ (counting down in order),

   (i) Find the unique function $\beta_{j,i}^{t+j} : U_{j}^{t+j} \times \cdots \times U_{t}^{t+j} \rightarrow U_{j,j}^{t+j}$ of the form (106) that satisfies (109), or in other words, $\beta_{j,j}^{t+j}$ is shift equivalent to $\beta_{j,j}^{t+j+1}$ on $\nabla_{j,j+1}U_{j}^{t+j}$.

   \[ \beta_{j,j}^{t+j} \equiv \beta_{j,j}^{t+j+1}. \]

   (ii) Define any restricted separating function $\beta_{j,j}^{t+j} : U_{j}^{t+j} \times \cdots \times U_{t}^{t+j} \rightarrow U_{j,j}^{t+j}$.

Now combine the $\beta_{j,j}^{t+j}$ and $\beta_{j,j}^{t+j}$ to form $\beta_{j,j}^{t+j}$.

FOR each $u_{j,j}^{t+j} \in U_{j}^{t+j} \times \cdots \times U_{t}^{t+j}$,

   (i) Define $u_{j,j}^{t+j} \in U_{j,j}^{t+j}$ by

   \[ u_{j,j}^{t+j} \equiv \beta_{j,j}^{t+j}(u_{j,j}^{t+j}, \ldots, u_{t}^{t+j}), \]

   and define $u_{j,j}^{t+j} \in U_{j,j}^{t+j}$ by

   \[ u_{j,j}^{t+j} \equiv \beta_{j,j}^{t+j}(u_{j,j}^{t+j}, \ldots, u_{t}^{t+j}). \]

   (ii) Define $\beta_{j,j}^{t+j} : U_{j}^{t+j} \times \cdots \times U_{t}^{t+j} \rightarrow U_{j,j}^{t+j}$ by $\beta_{j,j}^{t+j}(u_{j,j}^{t+j}, \ldots, u_{t}^{t+j}) \equiv u_{j,j}^{t+j}$, where

   \[ u_{j,j}^{t+j} = (u_{j,j}^{t+j}T, u_{j,j}^{t+j}T). \]

ENDFOR
Define $\omega_j^{t+j}$ to be the unique separating function whose restriction is $\beta_j^{t+j}$.

ENDFOR

ENDDO

4. Step 4 same as in Algorithm 75.

We now show that Theorem 77 and Algorithm 78 can be refined to use separating functions $\omega_j^{t+j}$ and restricted separating functions $\beta_j^{t+j}$. We first show how to construct a function $f_j^{t+j} : U_j^{t+j} \times \cdots \times U_{\ell}^{t+j} \to U_j^{t+j}$ (not necessarily a restricted separating function) using the set of restricted separating functions $\{\beta_j^{t+j} : j \leq k \leq \ell\}$.

Definition 79 (Construction of $f_j^{t+j}$)

Let $\{\beta_j^{t+j} : j \leq k \leq \ell\}$ be a set of restricted separating functions, as defined in Definition 76. Define a function $f_j^{t+j} : U_j^{t+j} \times \cdots \times U_{\ell}^{t+j} \to U_j^{t+j}$ as follows.

FOR each fixed $\nabla_{j,j}(u^{t+j}) \in \nabla_{j,j}(U^{t+j})$,

FOR each $k$ such that $j \leq k \leq \ell$,

define $v_{j,k}^{t+j} \in U_{j,k}^{t+j}$ by

$$v_{j,k}^{t+j} \overset{\text{def}}{=} \beta_{j,k}^{t+j}(\nabla_{j,k}(u^{t+j})).$$

ENDFOR

Define $f_j^{t+j}(\nabla_{j,j}(u^{t+j}))$ to be the vector $v_j^{t+j}$ in $U_j^{t+j}$ given by

$$v_j^{t+j} = \left( v_{j,\ell}^{t+j} \cdots v_{j,k}^{t+j} \cdots v_{j,j}^{t+j} \right)^T.$$ 

ENDFOR

Consistent with (112) and (113), we can represent $f_j^{t+j}$ by the vector of functions

$$f_j^{t+j} = \left( \beta_{j,\ell}^{t+j} \cdots \beta_{j,k}^{t+j} \cdots \beta_{j,j}^{t+j} \right)^T.$$ 

If Definition 79 holds, we say $f_j^{t+j}$ is constructed from the set of restricted separating functions $\{\beta_j^{t+j} : j \leq k \leq \ell\}$. Given a set of restricted separating functions $\{\beta_j^{t+j} : j \leq k \leq \ell\}$ as defined in Definition 76 the construction in Definition 79 gives a unique function $f_j^{t+j}$.

Theorem 80 (Induction hypothesis) Assume the function $\beta_{j+1,k+1}^{t+j+1}$ on the right hand side of (98) is a restricted separating function $\beta_{j+1,k+1}^{t+j+1} : U_{j+1}^{t+j+1} \times \cdots \times U_{\ell}^{t+j+1} \to U_{j+1}^{t+j+1}$ such that $\beta_{j+1,k+1}^{t+j+1}$ is constructed from a set of restricted separating functions $\{\beta_{j+1,k}^{t+j+1} \}$, where $\beta_{j+1,k}^{t+j+1}$ is shift equivalent to $\beta_{k,k}^{t+k}$ on $\nabla_{j+1,k}(U^{t+j+1})$,

$$\beta_{j+1,k}^{t+j+1} \equiv \beta_{k,k}^{t+k},$$

for $k$ such that $j+1 < k \leq \ell$, and where

$$\beta_{j+1,j+1}^{t+j+1}(u^{t+j+1})$$

(115)
is any restricted separating function, for \( k = j + 1 \).

Then there exists a solution \( \beta_{j}^{t+j} \) on the left hand side of (98) which is a restricted separating function \( \beta_{j}^{t+j} : U_{j}^{t+j} \times \cdots \times U_{j}^{t+j} \to U_{j}^{t+j} \) such that \( \beta_{j}^{t+j} \) is constructed from the set of restricted separating functions \( \{ \beta_{j,k}^{t+j} \}_{k=1}^{\ell} \), where \( \beta_{j,k}^{t+j} \) is shift equivalent to \( \beta_{j,k}^{t+j} \) on \( \nabla_{j,k}(U_{j}^{t+j}) \),

\[
\beta_{j,k}^{t+j} \equiv \beta_{j,k}^{t+j} \tag{116}
\]

for \( k \) such that \( j < k \leq \ell \), and where

\[
\beta_{j,j}^{t+j}(u_{j}^{t+j}) \tag{117}
\]

is any restricted separating function, for \( k = j \).

**Proof.** We use proof by induction. Assume we have found \( \beta_{j+1}^{t+j+1} \) on the right hand side of (98) and assume \( \beta_{j+1}^{t+j+1} \) can be constructed from the set of restricted separating functions \( \{ \beta_{j+1,k}^{t+j+1} : j + 1 \leq k \leq \ell \} \). We then show the solution \( \beta_{j}^{t+j} \) on the left hand side of (98) can be constructed from a set of restricted separating functions \( \{ \beta_{j,k}^{t+j} : j \leq k \leq \ell \} \), which are related to the set \( \{ \beta_{j,k}^{t+j+1} : j + 1 \leq k \leq \ell \} \).

From (118), we have that

\[
\beta_{j+1}^{t+j}(\nabla_{j,j}(u_{j}^{t+j})) = \beta_{j+1}^{t+j+1}(\sigma u_{j}^{t+j}), \tag{118}
\]

where

\[
u_{j}^{t+j}(u_{j}^{t+j}) = \begin{pmatrix} u_{j}^{t+j} & \cdots & u_{j}^{t+j} & u_{j+1}^{t+j} \end{pmatrix}^T,
\]

and

\[
\sigma u_{j}^{t+j} = \begin{pmatrix} \sigma u_{j}^{t+j} \cdots \sigma u_{j}^{t+j} \cdots \sigma u_{j+1}^{t+j} \end{pmatrix}^T.
\]

On the right hand side of (118), we know that \( \beta_{j+1}^{t+j+1} \) is constructed from the set of restricted separating functions \( \{ \beta_{j+1,k}^{t+j+1} : j + 1 \leq k \leq \ell \} \). If \( \beta_{j+1}^{t+j+1}(\sigma u_{j}^{t+j}) = v_{j+1}^{t+j+1} \in U_{j+1}^{t+j+1} \), where

\[
v_{j+1}^{t+j+1} = \begin{pmatrix} v_{j+1}^{t+j+1} & \cdots & v_{j+1}^{t+j+1} & v_{j+1}^{t+j+1} \end{pmatrix}^T,
\]

then \( \beta_{j+1}^{t+j+1} \) can be represented by the vector of functions

\[
\beta_{j+1}^{t+j+1} = \begin{pmatrix} \beta_{j+1,1}^{t+j+1} \cdots \beta_{j+1,j}^{t+j+1} \cdots \beta_{j+1,1}^{t+j+1} \end{pmatrix}^T,
\]

where the \( k \)-th coordinate \( \beta_{j+1,k}^{t+j+1} \) of \( \beta_{j+1}^{t+j+1} \) gives the \( k \)-th coordinate \( v_{j+1,k}^{t+j+1} \) of \( v_{j+1}^{t+j+1} \). Fix \( k \) such that \( j + 1 \leq k \leq \ell \). Then

\[
v_{j+1,k}^{t+j+1} = \beta_{j+1,k}^{t+j+1}(\sigma u_{j}^{t+j})
\]

\[
= \beta_{j+1,k}^{t+j+1}(\sigma v_{j,k}^{t+j}, \nabla_{j+1,k}(\sigma u_{j}^{t+j})).
\]

Let \( \beta_{j,j}^{t+j}(\nabla_{j,j}(u_{j}^{t+j})) = v_{j,j}^{t+j} \in U_{j,j}^{t+j} \), where

\[
v_{j,j}^{t+j} = \begin{pmatrix} v_{j}^{t+j} & \cdots & v_{j}^{t+j} & v_{j,j}^{t+j} \end{pmatrix}^T.
\]

(120)
For $k$ such that $j + 1 \leq k \leq \ell$, let $f_{j,k}^{t+j} : \nabla j_j(U^{t+j}) \to U_{j,k}^{t+j}$ be the function which gives the $k$-th coordinate $v_{j,k}^{t+j}$ of $v_{j,j}^{t+j}$. Then we can represent $\beta_{j,j}^{t+j}$ by the vector of functions

$$\beta_{j,j}^{t+j} = \left( f_{j,k}^{t+j} \ldots f_{j,k}^{t+j} \ldots f_{j,j+1}^{t+j} \right)^T.$$

Then from (118), we must have

$$f_{j,k}^{t+j} (\sigma_{j,k}^{t+j}, u_{j,k}^{t+j}, \nabla_{j,j+1}^{t+j}(u_{j,k}^{t+j})) = \beta_{j+1,j,k}^{t+j+1}(\sigma_{j,k}^{t+j}, \nabla_{j+1,k}^{t+j}(\sigma u_{j,k}^{t+j})). \quad (121)$$

Now use the same argument as given for finding $\beta_{j,j}^{t+j}$ given $\beta_{j+1,j}^{t+j+1}$. The right hand side of (121) is a function of $(\sigma_{j,k}^{t+j}, \nabla_{j+1,k}^{t+j}(\sigma u_{j,k}^{t+j}))$. Therefore the left hand side must be a function of the set $(u_{j,k}^{t+j}, \nabla_{j,k}^{t+j}(u_{j,k}^{t+j}))$ which is shift equivalent to $(\sigma_{j,k}^{t+j}, \nabla_{j+1,k}^{t+j}(\sigma u_{j,k}^{t+j}))$, and independent of other components. Therefore the left hand side is some function $h_{j,k}^{t+j} : \nabla_{j,k}^{t+j}(U^{t+j}) \to U_{j,k}^{t+j}$ such that

$$h_{j,k}^{t+j}(u_{j,k}^{t+j}, \nabla_{j,k}^{t+j}(u_{j,k}^{t+j})) = \beta_{j+1,j,k}^{t+j+1}(\sigma_{j,k}^{t+j}, \nabla_{j+1,k}^{t+j}(\sigma u_{j,k}^{t+j})). \quad (122)$$

And since $\beta_{j+1,j,k}^{t+j+1}$ is a function 1-1 and onto from $U_{j+1,k}^{t+j+1}$ to $U_{j+1,k}^{t+j+1}$ for each fixed $\nabla_{j+1,k}^{t+j}(\sigma u_{j,k}^{t+j})$, then $h_{j,k}^{t+j}$ must be a function 1-1 and onto from $U_{j,k}^{t+j}$ to $U_{j,k}^{t+j}$ for each fixed $\nabla_{j,k}^{t+j}(u_{j,k}^{t+j})$. In other words, $h_{j,k}^{t+j}$ is a restricted separating function $\beta_{j,k}^{t+j}$, and (122) gives $\beta_{j,k}^{t+j} \equiv \beta_{j+1,j,k}^{t+j+1}$ on $\nabla_{j,k}(U^{t+j})$. Since $\beta_{j+1,j,k}^{t+j+1} \equiv \beta_{j,k}^{t+j}$ on $\nabla_{j+1,k}(U^{t+j+1})$ from (114), then we see that $\beta_{j,k}^{t+j} \equiv \beta_{j,k}^{t+k}$ on $\nabla_{j,k}(U^{t+j})$.

Clearly the induction hypothesis holds for $j + 1 = \ell$ because $\beta_{\ell,j}^{t+k}$ is a restricted separating function $\beta_{\ell,j}^{t+k}$. This completes the proof by induction. Thus we have proven the following algorithm, using results (116) and (117) above.

**Algorithm 81** Any solution of the set of equations (89) which is a symmetry $\omega = \ldots, \omega^t, \omega^{t+1}, \ldots$ of $\mathcal{D}^\infty(U)$ can be found as follows.

**DO**

1. Fix time $t$.

2. **FOR** $k = \ell, \ldots, 0, \ldots$,
   define any separating function $\omega_{k,k}^{t+k} : U^{t+k} \to U_{k,k}^{t+k}$.

3. **FOR** $j$ satisfying $0 \leq j < k$,
   define a separating function $\omega_{j,k}^{t+j} : U^{t+j} \to U_{j,k}^{t+j}$ by
   $$\omega_{j,k}^{t+j} \equiv \omega_{k,k}^{t+k} \quad (123)$$
   on $\nabla_{j,k}(U^{t+j})$.

**ENDDO**

3. Now combine the $\omega_{j,k}$ directly to form $\omega^t$.
Theorem 82 Any symmetry $\omega$ of $D^\infty(U)$ is uniquely specified by the collection of separating functions $\omega_{k,k}^t(u^{k+k})$, for $k$ such that $0 \leq k \leq \ell$, for each $t \in Z$.

In this subsection we have given three algorithms to construct all the symmetries of $D^\infty(U)$. Algorithm 75 is the basic algorithm. It can be shown that Algorithm 78 is the best algorithm to construct any group system $C$. Algorithm 81 is a very simple algorithm and the best for finding all the symmetries of $D^\infty(U)$.

7.3 The full symmetry system

In the same way as (95), we can diagram Step 4 of Algorithms 75 and 78 as shown in (124).

\[
\begin{pmatrix}
\vdots \\
\omega_{\ell+1-t}^t \\
\vdots \\
\omega_{1}^{t+1} & \omega_{1}^{t+1} & \omega_{1}^{t+1} & \omega_{1}^{t+1} \\
\vdots \\
\omega_{0}^{t} & \omega_{0}^{t} & \omega_{0}^{t} & \omega_{0}^{t} \\
\vdots \\
\omega_{0}^{t} & \omega_{0}^{t} & \omega_{0}^{t} & \omega_{0}^{t} \\
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\omega_{0}^{t} & \omega_{0}^{t} & \omega_{0}^{t} & \omega_{0}^{t} \\
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\end{pmatrix}
\tag{124}
\]

Equation (124) shows a $t$-tower $T^t$ and $(t-j)$-tower $T^{t-j}$ as diagonals in an infinite matrix of towers. Algorithms 75 and 78 show that the sequence of towers $\ldots, T^{t-j}, \ldots, \gamma^{t}, \ldots$ in (124) defines a symmetry and any such matrix (124) of towers defines a symmetry. A component $\omega^t$ of the symmetry $\omega$ is defined by going “across the row” in (124),

$$\omega^t = (\omega_0^t, \omega_1^t, \omega_2^t, \omega_3^t, \ldots, \omega_{\ell-1}^t, \omega_{\ell}^t) \tag{125}$$

and more explicitly, $\omega^t : U^t \rightarrow U^t$ is defined by

$$\omega^t(u^t) = (\omega_0^t(u^t), \omega_1^t(u^t), \omega_2^t(u^t), \omega_3^t(u^t), \ldots, \omega_{\ell}^t(u^t)). \tag{126}$$

Note that each component $\omega_j^t$ in (126) is selected from a different tower.

Let $M$ be the full symmetry system of $U$ obtained using Algorithms 75, 78, or 81 to find each symmetry $\omega \in M$. As just discussed, we have the following.
Theorem 83 The full symmetry system $\mathcal{M}$ is a set of tensors.

And at the beginning of Section 7, we noted this.

Theorem 84 The full symmetry system $\mathcal{M}$ is a group system.

Consider the 5-tuple family $(\mathcal{M}, \mathcal{U}, \mathcal{R}, C; B)$, which includes the 4-tuple family $(\mathcal{U}, \mathcal{R}, C; B)$ already considered in Section 6. $\mathcal{M}$ is a tensor set like $\mathcal{U}$ and $\mathcal{R}$. And $\mathcal{M}$ is also a group system like $C$. $\mathcal{M}$ only depends on $\mathcal{U}$ and does not depend on basis $B$. $\mathcal{M}$ acts on $\mathcal{U}$ and acts on $C$ also. The induced action of $\mathcal{M}$ on $C$ means symmetry $\omega \in \mathcal{M}$ gives a permutation of the paths of $C$.

From the form of (124) and (125)-(126), we can regard $\omega_{0}^{(t)}$, or $\omega_{0}^{t}$, as the input at time $t$, and $(\omega_{1}^{(t-1)+1}, \ldots, \omega_{j}^{(t-j)+j}, \ldots, \omega_{\ell}^{(t-\ell)+\ell})$, or $(\omega_{1}^{t}, \ldots, \omega_{j}^{t}, \ldots, \omega_{\ell}^{t})$, as the state at time $t$. Note that the state at time $t$ is composed of shifts of previous inputs, i.e., $\omega_{j}^{(t-j)+j}$ is a shift of the input $\omega_{j}^{(t-j)}$ at time $t-j$.

Because the structure of a symmetry $\omega$ mirrors the structure of a tensor $u \in \mathcal{U}$, we see that component $\omega_{j}^{t}+j$ has the same form as component $u_{j}^{t}$ in tensor $u$, which is a static matrix $U[t, t]$. Therefore in the same way as (80), we regard the diagonals of (124) as columns $\omega_{j}^{t}+j$ in a shift matrix $\Omega[t, t]$, $\Omega[t, t] = (\omega_{0}^{t}, \omega_{1}^{t+1}, \ldots, \omega_{j}^{t+j}, \ldots, \omega_{\ell}^{t+\ell}).$

A diagonal of (124) is a $t$-tower. Therefore a $t$-tower $\Upsilon[t]$ is a shift matrix $\Omega[t, t]$ at time $t$. A shift vector $\omega^{[t, t+k]}$ is a row in $\Omega[t, t+k]$, for $0 \leq k \leq \ell$, where

$$\omega^{[t, t+k]} \equiv (\omega_{0,k}^{t}, \omega_{1,k}^{t+1}, \ldots, \omega_{j,k}^{t+j}, \ldots, \omega_{k,k}^{t+k}).$$

The shift vector is determined by shifts of the separating permutation $\omega_{k,k}^{t+k}$.

Step 4 of Algorithms 75 and 78 can be viewed as the construction of the static matrix $\omega^{t} = \Omega[t, t]$ using a sequence of shift matrices, exactly analogous to the procedure in Theorem 25 for generator matrices.

A path in $C$ is denoted $c$, where

$$c = \ldots, c^{t}, \ldots$$

Each component $c^{t}$ is a branch. Every group system has an identity sequence. The identity path of $C$ is the path where each component, or branch, $c^{t}$, is the identity $1^{t}$.

A path in $\mathcal{M}$ is denoted $\omega$, where

$$\omega = \ldots, \omega^{t}, \ldots$$
and \( \omega^t = (\omega^t_0, \omega^t_1, \ldots, \omega^t_j) \). We can think of component \( \omega^t \) as a branch in a bipartite or unipartite graph. The vertices (states) of the graph are given by \( (\omega^t_1, \ldots, \omega^t_j) \), and the input is \( \omega^t_0 \). The next state is \( \sigma \omega^t = \sigma(\omega^t_0, \omega^t_1, \ldots, \omega^t_j) \). This mimics the description of the graph \( D(U^t) \). The identity path of \( M \) is the path \( \omega \) where each component, or branch, \( \omega^t \), is given by \( (1^t_0, 1^t_1, \ldots, 1^t_j) \). The identity sequence is obtained using inputs \( \omega^t_0 \), where \( \omega^t_0 \) is the identity \( I^t_0 \) for each time \( t \).

The equation (80) was used in the analysis of a symmetry permutation. We can think of this equation in shorthand form as \( \sigma \varphi^t_{j} = \varphi^{t+j+1} \_j \) for \( j = 0, \ldots, \ell - 1 \), and \( \varphi^{t+j+1} \_j \) can be regarded as a “shift” of \( \varphi^t_{j} \). In the construction of a symmetry, we solved the same equation \( \sigma \omega^t_j = \omega^{t+j+1} \_j \) going backwards, from \( j = \ell - 1 \) to \( j = 0 \). However it is clear that we can also go forward, and once \( \omega^t_0 \) is found, we can find all \( \omega^t_j \), \( 1 \leq j \leq \ell \). Thus a shift matrix \( \Omega^{t,t+\ell} \),

\[
\Omega^{t,t+\ell} = (\omega^t_0, \omega^t_1, \ldots, \omega^t_j, \ldots, \omega^t_{\ell}),
\]

is completely determined by \( \omega^t_0 \). This situation is completely analogous to that for a shift matrix \( U^{t,t+\ell} \), where (124) is analogous to (80), shift matrix \( U^{t,t+\ell} \) is completely determined by \( u^t_0 \), and an analogous equation \( \sigma u^t_j = u^{t+j+1} \) holds. This gives the following result.

**Proposition 85** A symmetry \( \omega \) in \( M \) is completely determined by a sequence of inputs \( \omega^t_0 \), for \( t \in \mathbb{Z} \).

Using (124) and Proposition 85 it is easy to define a sliding block encoder of the full symmetry system. The encoder slides along the matrix in (124) from left to right as time increases. At each time \( t \), a new input \( \omega^t_0 \) is selected from a set of inputs. The encoder output, component \( \omega^t \) of symmetry \( \omega \), is defined by going “across the row” in (124), as given in (125)-(126). Note that this is equivalent to just forming the static matrix \( \Omega^{t,t+\ell} \).

**Theorem 86** The full symmetry system \( M \) of \( C \) is \( \ell \)-controllable, the same as \( C \).

**Proof.** \( M \) is completely determined by a sequence of inputs, which can be selected arbitrarily. Therefore we can go from any state of \( M \) to any other state in \( \ell \) steps, by a suitable choice of inputs. \( \blacksquare \)

\( M^t \deq \chi^t(M) \) are the time \( t \) components of the symmetries in \( M \),

\[
M^t \deq \{ \omega^t : \omega = \ldots, \omega^t, \ldots, \omega \in M \}.
\]

\( M^t \) is called a branch group.

An element \( u^t \in U^t \) is a static matrix \( U^{t,t+\ell} \). The static matrix \( U^{t,t+\ell} \) is permuted by component \( \omega^t \) in symmetry \( \omega \), and \( \omega^t \) is a static matrix \( \Omega^{t,t+\ell} \), with components \( \omega^t_{j,k} \) for \( 0 \leq j \leq k \leq \ell \). Component \( \omega^t_{j,k} \) in \( \Omega^{t,t+\ell} \) permutes component \( u^t_{j,k} \) in \( U^{t,t+\ell} \).

We now study the action of the full symmetry system on \( U \). Fix symmetry \( \omega \in M \). Fix tensor \( u \in U \). Fix time \( t \) and fix \( k \), \( 0 \leq k \leq \ell \). Let \( u \) have shift vector \( u^{t,t+k} \). The shift vector \( u^{t,t+k} \) is a finite sequence

\[
(u^t_{0,k}, u^t_{1,k}, \ldots, u^t_{j,k}, \ldots, u^{t+k}_{k,k}), \tag{127}
\]
where \( t_{j,k}^{t+j} \) is the same integer for \( 0 \leq j \leq k \). From the form of the solution of the full symmetry system, we know symmetry \( \omega \) acts on this finite sequence with the finite sequence of permutations
\[
(\omega_{0,k}^t, \omega_{1,k}^{t+1}, \ldots, \omega_{j,k}^{t+j}, \ldots, \omega_{k,k}^{t+k}),
\]
which is a shift vector \( \omega_t^{[t,t+k]} \) in \( \omega \). Then the action of (128) on (127) gives
\[
(\omega_{0,k}^t(u_{r,0}^t; \nabla_{0,k}(u^t)), \omega_{1,k}^{t+1}(u_{r,1}^{t+1}; \nabla_{1,k}(u^{t+1})), \ldots, \omega_{j,k}^{t+j}(u_{r,j}^{t+j}; \nabla_{j,k}(u^{t+j})), \ldots, \omega_{k,k}^{t+k}(u_{r,k}^{t+k}; \nabla_{k,k}(u^{t+k}))).
\]
But from (123), we have
\[
\omega_{j,k}^{t+j}(u_{j,k}^{t+j}; \nabla_{j,k}(u^{t+j})) = \omega_{k,k}^{t+k}(u_{j,k}^{t+j}; \nabla_{j,k}(u^{t+j}))
\]
for \( 0 \leq j \leq k \). But the contents of memory \( \nabla_{j,k}(u^{t+j}) \) is the same for \( 0 \leq j \leq k \), and integer \( u_{r,j}^{t+j} \) is the same integer for \( 0 \leq j \leq k \). Then the action of (128) on (127) gives
\[
(\tilde{u}_{r,0}^{t+1}, \tilde{u}_{r,1}^{t+1}, \ldots, \tilde{u}_{r,j}^{t+j}, \ldots, \tilde{u}_{r,k}^{t+k}),
\]
where \( \tilde{u}_{r,j}^{t+j} \) is the same integer for \( 0 \leq j \leq k \). But then (129) is a shift vector \( \tilde{u}^{[t,t+k]} \) in \( U \). Thus the shift vector (127) has been changed to shift vector (129).

It is clear that the action of \( \omega_t^{[t,t+k]} \) on \( u^{[t,t+k]} \) is completely determined by the first component \( \omega_{0,k}^t(u_{r,0}^t; \nabla_{0,k}(u^t)) \). The argument of \( \omega_{0,k}^t \) is a function of \( u_t \). The state of \( u \) at time \( t \) is \( \nabla_{0,1}(u^t) \), and the input at time \( t \) is \( u_{0}^t \).

Then for each time \( t \in \mathbb{Z} \) and each \( k, 0 \leq k \leq \ell \), symmetry \( \omega \) permutes shift vector \( u^{[t,t+k]} \) in \( U \) to another shift vector \( \tilde{u}^{[t,t+k]} \) in \( U \). The collection of shift vectors \( \{u^{[t,t+k]} : t \in \mathbb{Z}, 0 \leq k \leq \ell \} \) specifies a unique tensor \( \tilde{u} \in U \). Thus \( \omega \) permutes tensor \( u \) to tensor \( \tilde{u} \).

**Theorem 87** Fix symmetry \( \omega \in \mathcal{M} \). Fix tensor \( u \in U \). Fix time \( t \) and fix \( k, 0 \leq k \leq \ell \). Let \( u \) have shift vector \( u^{[t,t+k]} \). The symmetry \( \omega \) permutes shift vector \( u^{[t,t+k]} \) to another shift vector \( \tilde{u}^{[t,t+k]} \) in \( U \). The permutation is solely determined by component \( \omega_{0,k}^t \) of symmetry input \( \omega_t^t \) at time \( t \). The argument of \( \omega_{0,k}^t \) is a part of the state \( \nabla_{0,k}(u^t) \), and part of the input \( u_{0,m}^t \), for \( k \leq m \leq \ell \), which is part of the state \( \nabla_{0,1}(u^t) \) of \( u \), and part of the input \( u_{0}^t \) of \( u \), respectively, at time \( t \).

Then for each time \( t \in \mathbb{Z} \) and each \( k, 0 \leq k \leq \ell \), symmetry \( \omega \) permutes shift vector \( u^{[t,t+k]} \) in \( u \) to another shift vector \( \tilde{u}^{[t,t+k]} \) in \( U \). The collection of shift vectors \( \{u^{[t,t+k]} : t \in \mathbb{Z}, 0 \leq k \leq \ell \} \) specifies a unique tensor \( \tilde{u} \in U \). Thus \( \omega \) permutes tensor \( u \) to tensor \( \tilde{u} \).

**Theorem 88** Fix symmetry \( \omega \in \mathcal{M} \). Fix tensor \( u \in U \). Fix time \( t \) and fix \( k, 0 \leq k \leq \ell \). Let \( u \) have shift vector \( u^{[t,t+k]} \). The symmetry \( \omega \) permutes shift vector \( u^{[t,t+k]} \) to another shift vector \( \tilde{u}^{[t,t+k]} \) in \( U \). Fix basis \( B \). There is a 1-1 correspondence \( U \leftrightarrow R \). From this correspondence, let \( u \leftrightarrow r \) and \( u^{[t,t+k]} \leftrightarrow r^{[t,t+k]} \). Then through the 1-1 correspondence \( U \leftrightarrow R \), symmetry \( \omega \) induces an assignment that takes generator vector \( r^{[t,t+k]} \) to another generator vector \( \hat{r}^{[t,t+k]} \) in \( R \), where \( r^{[t,t+k]} \leftrightarrow \hat{r}^{[t,t+k]} \). The permutation is solely determined by component \( \omega_{0,k}^t \) of symmetry input \( \omega_t^t \) at time \( t \). Since \( u_t^t \leftrightarrow r_t^t \), the permutation effectively depends on \( \nabla_{0,k}(r_t^t) \), a part of the state \( \nabla_{0,1}(r_t^t) \) of \( r \) at time \( t \), and \( r_{0,m}^t \), for \( k \leq m \leq \ell \), a part of the input \( r_{0}^t \) of \( r \) at time \( t \).
Then for each time \( t \in \mathbb{Z} \) and each \( k, 0 \leq k \leq \ell \), symmetry \( \omega \) induces a permutation of generator vector \( r^{[t,t+k]} \) in \( r \) to another generator vector \( \hat{r}^{[t,t+k]} \) in \( \hat{r} \). The collection of generator vectors \( \{ \hat{r}^{[t,t+k]} ; t \in \mathbb{Z}, 0 \leq k \leq \ell \} \) specifies a unique tensor \( \hat{r} \in \hat{R} \). Thus \( \omega \) effectively permutes tensor \( r \) to tensor \( \hat{r} \).

In Subsection 6.3, for each of the four comparisons of encoders, \( E_1 \) and \( E_2 \), \( E \) and \( E_Y \), \( E_s \) and \( E_{s,Y} \), and \( E_1 \) and \( E_{s,2} \), we saw there was a graph automorphism of \( D^\infty(U) \) which made the two encoders graph isomorphic, composed with the natural isomorphism to \( D^\infty(U_Y) \) in the second and third comparisons. If the bases are constant, the graph automorphism of \( D^\infty(U) \) is constant. In this section, we analyzed the structure of any graph automorphism of \( D^\infty(U) \). Theorem 87 shows that any graph automorphism of \( D^\infty(U) \) is a symmetry \( \omega \) which permutes shift vector \( u^{[t,t+k]} \) in tensor \( u \in U \) to shift vector \( \hat{u}^{[t,t+k]} \) in tensor \( \hat{u} \in U \). Theorem 88 shows that symmetry \( \omega \) induces a permutation of \( R \) which permutes generator vector \( r^{[t,t+k]} \) in tensor \( r \in \hat{R} \) to generator vector \( \hat{r}^{[t,t+k]} \) in tensor \( \hat{r} \). For each time \( t \), this means generator vector \( r^{[t,t+k]} \) in vector basis \( B^t \) is taken to generator vector \( \hat{r}^{[t,t+k]} \) in \( B^t \). The permutation depends on \( \nabla_{0,k}(r^t) \), a part of the state of \( r \) at time \( t \), and \( r_{0,m}^t \), for \( k \leq m \leq \ell \), a part of the input of \( r \) at time \( t \). For a constant basis, the permutation is constant. In the case of the comparison \( E \) and \( E_s \), the permutation of generator vectors gives a transformation between the time domain and spectral domains.
8. THE NATURAL SYMMETRY SYSTEM

8.1 The natural symmetry system

Rotman \[\text{[11]}\] gives the Cayley theorem and proof for finite groups (Theorem 3.12). Let \( S_n \) be the symmetric group on integers \( \{1, \ldots, n\} \).

**Theorem 89 (Cayley theorem)** Let \( |G| = n \). Every group \( G \) can be imbedded as a subgroup of \( S_n \).

**Proof.** Note that a bijection is a permutation and a permutation is a bijection. Left translation \( L_g : G \to G \) defined by assignment \( h \mapsto gh \) is a bijection, so \( L_g \in S_n \). The map \( L : G \to S_n \) defined by the assignment \( g \mapsto L_g \) is an injection and homomorphism. Then \( G \simeq \text{im}(L) \). \( \bullet \)

We have just seen the set \( \{L_g : g \in G\} \) is a group \( \text{im}(L) \) and \( G \simeq \text{im}(L) \) under the 1-1 correspondence \( g \mapsto L_g \). The operation in \( \text{im}(L) \) is composition defined as follows. If \( L_g_1 \in \text{im}(L) \) and \( L_g_2 \in \text{im}(L) \), then \( L_{g_1} \circ L_{g_2} \in \text{im}(L) \), and in fact \( L_{g_1} \circ L_{g_2} = L_{g_1 g_2} \).

We now want to extend the Cayley theorem for finite groups to group system \( C \). The following result is just the Cayley theorem and proof restated for group system \( C \). Let \( S_C \) be the symmetric group on group system \( C \). This is the group of all permutations of paths in \( C \) with composition operation.

**Theorem 90** Every group system \( C \) can be imbedded as a subgroup of \( S_C \).

**Proof.** Note that a bijection is a permutation and a permutation is a bijection. Left translation \( L_b : C \to C \) defined by assignment \( c \mapsto bc \) is a bijection, so \( L_b \in S_C \). The map \( L : C \to S_C \) defined by the assignment \( b \mapsto L_b \) is an injection and homomorphism. Then \( C \simeq \text{im}(L) \). \( \bullet \)

Note that left translation \( L_b \) is essentially just \( bC \), and the map \( L : C \to S_C \) defined by the assignment \( b \mapsto L_b \) is essentially just \( b \mapsto bC \). Then we have just seen the set \( \{bC : b \in C\} \) is a group \( \text{im}(L) \) and \( C \simeq \text{im}(L) \) under the 1-1 correspondence \( b \mapsto bC \). The operation in \( \text{im}(L) \) is composition defined as follows. If \( b_1 C \in \text{im}(L) \) and \( b_2 C \in \text{im}(L) \), then \( b_1 C \circ b_2 C \in \text{im}(L) \), and in fact \( b_1 C \circ b_2 C = (b_1 b_2)C \). We now show \( bC \) is essentially a symmetry.

**Lemma 91** Left translation \( L_b \), a bijection on \( C \), induces a symmetry \( \omega_b \) of \( D^\infty(U) \), a bijection on \( U \).

**Proof.** The paths of \( C \) are described by sequences of the encoder \( E(D^\infty(\mathcal{R}, \mathcal{B})) \). Then multiplication by \( b \) in product \( bC \) permutes the sequences of \( E(D^\infty(\mathcal{R}, \mathcal{B})) \), and therefore the vertices of \( D^\infty(\mathcal{R}, \mathcal{B}) \) so the sequences are preserved. But \( D^\infty(\mathcal{R}, \mathcal{B}) \) is graph isomorphic to \( D^\infty(U) \). Therefore the product \( bC \) must induce a permutation of vertices of \( D^\infty(U) \) that preserves paths. \( \bullet \)

**Lemma 91** shows we can define an isomorphism from \( \text{im}(L) \) into \( \mathcal{M} \). Let \( bC \simeq \omega_b \). Then \( b_1 C \circ b_2 C \simeq \omega_{b_1} \circ \omega_{b_2} \). Therefore \( \text{im}(\alpha) \) is a subgroup of \( \mathcal{M} \) with composition operation. We let \( \text{im}(\alpha) \) be \( \mathcal{N} \), the natural symmetry system of \( C \). A symmetry in \( \mathcal{N} \) is denoted by \( \omega_b \), where \( \omega_b \) is the symmetry induced by \( bC \).
Theorem 92 There is an isomorphism $C \simeq \text{im}(L) \overset{\alpha}{\simeq} \mathcal{N}$, where $\mathcal{N}$ is the group of symmetries induced by the iterated mapping $b \mapsto bC \mapsto \omega_b$, where $b \in C$ and $bC \in \text{im}(L)$. Thus every group system $C$ can be imbedded as a subgroup $\mathcal{N}$ of $\mathcal{M}$.

The isomorphism $C \simeq \text{im}(L) \overset{\alpha}{\simeq} \mathcal{N}$ gives the assignments $b \mapsto bC \mapsto \omega_b$. If $b$ is a generator $g_{[t,t+k]}$ in $C$, then we have the assignments

$$g_{[t,t+k]} \mapsto g_{[t,t+k]}C \mapsto \omega_{g_{[t,t+k]}}.$$  

For a generator $g_{[t,t+k]}$ in $C$, $g_{[t,t+k]}C$ is a generator in $\text{im}(L)$ and the corresponding symmetry $\omega_{g_{[t,t+k]}}$ is a generator in $\mathcal{N}$. We see that a generator in $\mathcal{N}$ can be more complicated than a generator $g_{[t,t+k]}$ in $C$ because it involves multiplication $g_{[t,t+k]}C$.

Based on [3], (61) gives a decomposition of any path $b \in C$ as a product of generators $g_{[t,t+k]}$, $t \in \mathbb{Z}$, $0 \leq k \leq \ell$. We now consider the equality (61) in $\text{im}(L)$. $L$ gives the assignments

$$L : b \mapsto bC,$$

and

$$L : \prod_{0 \leq k \leq \ell} \prod_t g_{[t,t+k]} \mapsto \bigotimes_{0 \leq k \leq \ell} \bigotimes_t g_{[t,t+k]}C,$$

where $\bigotimes$ indicates an iterated series of compositions in $\text{im}(L)$. Then we can rewrite (61) in $\text{im}(L)$ as

$$bC = \bigotimes_{0 \leq k \leq \ell} \bigotimes_t g_{[t,t+k]}C.$$  

Using the isomorphism $\text{im}(L) \overset{\alpha}{\simeq} \mathcal{N}$, this gives

$$\omega_b = \bigotimes_{0 \leq k \leq \ell} \bigotimes_t \omega_{g_{[t,t+k]}}.$$  

Thus a symmetry $\omega_b$ in $\mathcal{N}$ is a composition of generators $\omega_{g_{[t,t+k]}}$ in $\mathcal{N}$. Then to study any symmetry $\omega_b$, it is sufficient to study the generator $\omega_{g_{[t,t+k]}}$. Alternatively we may study the product $g_{[t,t+k]}C$ or the product $g_{[t,t+k]}c$ for any $c \in C$.

Theorem 93 Let $b \in C$ be composed of generators $g_{[t,t+k]} \in C$. A symmetry $\omega_b$ in $\mathcal{N}$ is a composition of generators $\omega_{g_{[t,t+k]}}$ in $\mathcal{N}$, where $\omega_b$ and $\omega_{g_{[t,t+k]}}$ satisfy the 1-1 correspondence in the isomorphism $C \simeq \text{im}(L) \overset{\alpha}{\simeq} \mathcal{N}$.

Consider the 5-tuple family $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{C}; \mathcal{B})$. Like $\mathcal{M}$ in 5-tuple family $(\mathcal{M}, \mathcal{U}, \mathcal{R}, \mathcal{C}; \mathcal{B})$, $\mathcal{N}$ is a tensor set and group system. And like $\mathcal{M}$, $\mathcal{N}$ acts on $\mathcal{U}$ and through the 1-1 correspondence $\mathcal{U} \leftrightarrow \mathcal{R} \leftrightarrow C$, $\mathcal{N}$ implicitly acts on $\mathcal{R}$ and $C$ also. Unlike $\mathcal{M}$, $\mathcal{N} \simeq C$ and $\mathcal{N}$ depends on basis $\mathcal{B}$.

In Section 8 we use a second notation to denote shift vectors $r_{[t,t+k]}$, $u_{[t,t+k]}$, and $\omega_{[t,t+k]}$. If shift vector $r_{[t,t+k]}$ is in a tensor $r \in \mathcal{R}$, we let $r_{[t,t+k]}$ be denoted by

$$\nu_{[t,t+k]}(r) \overset{\text{def}}{=} r_{[t,t+k]}.$$
Similarly, if shift vector $u^{[t,t+k]}$ is in tensor $u \in U$, let
\[ v^{[t,t+k]}(u) \overset{\text{def}}{=} u^{[t,t+k]}, \]
and if shift vector $\omega^{[t,t+k]}$ is in tensor $\omega \in M$, let
\[ v^{[t,t+k]}(\omega) \overset{\text{def}}{=} \omega^{[t,t+k]}.
\]

Left translation $L_b : C \rightarrow C$ gives the assignment $L_b : c \mapsto bc$. Let $bc = \bar{c}$. Consider the 1-1 correspondences $u \leftrightarrow r \leftrightarrow c$ and $\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}$. $L_b$ gives the assignment $L_b : c \mapsto \bar{c}$. The symmetry $\omega_b$ corresponding to $L_b$ gives the corresponding assignment $\omega_b : u \mapsto \bar{u}$. The commutative diagram Figure 4 relates $L_b$ and $\omega_b$.

Through the 1-1 correspondence $U \leftrightarrow R$, a symmetry $\omega_b : U \rightarrow U$ induces a function $\bar{\omega}_b : R \rightarrow R$ such that if $\omega_b : u \mapsto \bar{u}$, then $\bar{\omega}_b : r \mapsto \bar{r}$, as shown by the commutative diagram Figure 4. Tensor $r$ is composed of shift vectors $v^{[t,t+k]}(r)$ for $t \in \mathbb{Z}$ and $0 \leq k \leq \ell$. If
\[ \omega_b : r \mapsto \bar{r}, \quad (130) \]
then shift vector $v^{[t,t+k]}(r)$ in $r$ is changed to shift vector $v^{[t,t+k]}(\bar{r})$ in $\bar{r}$ for $t \in \mathbb{Z}$ and $0 \leq k \leq \ell$. We abuse notation (130) slightly and indicate this as
\[ \omega_b : v^{[t,t+k]}(r) \mapsto v^{[t,t+k]}(\bar{r}). \quad (131) \]

Let $L_b$, $\omega_b^t$, and $\omega_b^t$ be the time $t$ components of $L_b$, $\omega_b$, and $\omega_b$ respectively.

\[ \begin{array}{cccc}
\text{c} & \xrightarrow{L_b(c)} & \bar{c} = bc \\
\uparrow & & & \uparrow \\
u & \xrightarrow{\omega_b(u)} & \bar{u} \\
\uparrow & & & \uparrow \\
r & \xrightarrow{\omega_b(r)} & \bar{r} \\
\end{array} \]

Figure 4: Commutative diagram relating $L_b$, $\omega_b$, and $\omega_b$.

From Theorems 87 and 88, we know that each symmetry in $M$ takes each shift vector in $u \in U$ to another shift vector in $\bar{u} \in U$ and each generator vector in $r \in R$ to another generator vector in $\bar{r} \in R$, where $u \leftrightarrow r$ and $\bar{u} \leftrightarrow \bar{r}$ are in 1-1 correspondence. We can now give a result on multiplication in $C$.

**Theorem 94** The multiplication by $b$ in $bC$ corresponds to changing each generator vector in $R$ to another, at each time $t$ and length $k$, $0 \leq k \leq \ell$.  

70
Proof. From Lemma 91 left translation induces a symmetry of $D^\infty(U)$. 

We consider the effect of multiplication in $C$ on tensor set $\mathcal{R}$ further in the next section.

**Proposition 95** The natural symmetry system $\mathcal{N}$ of $C$ is $\ell$-controllable, the same as $C$.

**Proof.** Fix any $b_1, b_2 \in C$. Consider any $b_1^{(-\infty,t]}$ and any $b_2^{[t+\ell,\infty)}$. Since $C$ is $\ell$-controllable, there is always a path $b$ such that $\chi^{(-\infty,t]}(b) = b_1^{(-\infty,t]}$ and $\chi^{[t+\ell,\infty)}(b) = b_2^{[t+\ell,\infty)}$. Now fix any $b_1^{(-\infty,t]} C$ and any $b_2^{[t+\ell,\infty)} C$. Since $C$ is $\ell$-controllable, there is always a path $b$ such that $\chi^{(-\infty,t]}(b C) = b_1^{(-\infty,t]} C$ and $\chi^{[t+\ell,\infty)}(b C) = b_2^{[t+\ell,\infty)} C$. Thus $\mathcal{N}$ is $\ell$-controllable. 

8.2 Multiplication in $\mathcal{R}$

We study multiplication in $\mathcal{R}$ and show this is related to the abelian and nonabelian structure of a group system and the structure of the natural symmetry system $\mathcal{N}$.

Multiplication in $C$ is easy. For $b, c \in C$, product $bc$ is given by $b'c'$ for each $t \in \mathbb{Z}$. We now want to consider multiplication in $\mathcal{R}$. Multiplication in $\mathcal{R}$ gives more insight into the structure of a group system than multiplication in $C$. In Theorem 93 we showed that products in $C$ can be decomposed into terms of the form $g^{[t+k]} C$ or $g^{[t+k]} c$ for any $c \in C$. In this subsection we study the term $g^{[t+k]} c$ using the time domain and natural symmetry system $\mathcal{N}$. First we give some useful definitions.

For any time $t \in \mathbb{Z}$, we have given an expansion, or coset representative chain, of branch $b'$ in terms of coset representatives in $[11]$:

$$b' = r_{t,0,0} \cdots r_{t,1,0} \cdots r_{t,\ell-1,0} r_{t-1,\ell-1} \cdots \cdots r_{t,j,k} \cdots r_{t-2,2,1} \cdots r_{t,j',k} \cdots r_{t,1,0} \cdots r_{t,0,2} r_{t,0,1} r_{t,0,0}.$$  

Fix representative $r_{t,j,k}$. The components $r_{t,m,n}$ to the left of $r_{t,j,k}$ in [11] are called ascendants of $r_{t,j,k}$. These are “above” $r_{t,j,k}$ in the coset representative chain. The components $r_{t,m,n}$ to the right of $r_{t,j,k}$ in [11] are called descendants of $r_{t,j,k}$. These are “below” $r_{t,j,k}$ in the coset representative chain.

We say two time intervals $[t, t + k]$ and $[t', t' + n]$ overlap if $[t, t + k] \cap [t', t' + n]$ is not empty. We say two generators $g^{[t+k]}$ and $g^{[t'+n]}$ overlap if the time intervals $[t, t + k]$ and $[t', t' + n]$ overlap. We now give conditions under which a component $r_{t,m,n}^{t'}$ of generator vector $g^{[t'+n]}$ is an ascendant and a descendant of component $r_{t,j,k}^{t+j}$ in generator vector $g^{[t+k]}$.

**Lemma 96** Fix $r$. Fix time $t + j$. Fix $r_{t,j,k}^{t+j} \in g^{[t+k]}$. Fix $r_{t',m,n}^{t'+m} \in g^{[t'+n]}$. Then $r_{t,m,n}^{t'+m}$ is an ascendant of $r_{t,j,k}^{t+j}$ if and only if these 3 conditions hold: $[t, t + k]$ and $[t', t' + n]$ overlap with $t' \leq t$, if $t = t'$ then $n > k$, and $t' + m = t + j$. And $r_{t,j,k}^{t+j}$ is a descendant of $r_{t',m,n}^{t'+m}$ if and only if these 3 conditions hold: $[t, t + k]$ and $[t', t' + n]$ overlap with $t' \geq t$, if $t = t'$ then $n < k$, and $t' + m = t + j$.

We say a shift vector $v^{[t', t'+n]}(\bar{r})$ in $\mathcal{R}$ is subordinate to $[t, t + k]$ if $[t', t' + n] \subset [t, t + k]$ and $n < k$. We say a shift vector $v^{[t', t'+n]}(\bar{r})$ in $\mathcal{R}$ is superordinate to $[t, t + k]$ if $[t, t + k] \subset [t', t' + n]$ and $n > k$.
We say representative \( r_{t,m}^{t+j} \) is a direct descendant of \( r_{j,k}^{t+j} \), where \( t' + m = t + j \), if it is a component of a shift vector \( v_{t',t'+n}^{t+j}(\tilde{r}) \) superordinate to \([t, t + k]\). We say representative \( r_{m,n}^{t+j} \) is a direct descendant of \( r_{j,k}^{t+j} \), where \( t' + m = t + j \), if it is a component of a shift vector \( v_{t',t'+n}^{t+j}(\tilde{r}) \) subordinate to \([t, t + k]\). By Lemma 96, it is easy to see both definitions are well defined. Ascendants that are not direct ascendants are called indirect ascendants. Descendants that are not direct descendants are called indirect descendants.

**Proposition 97** The direct ascendants of \( r_{j,k}^{t+j} \) are all the components in \( \nabla_{j,k}(r^{t+j}) \).

It can be seen the components that are direct descendants give a parallelogram shape in \( \nabla_{0,0}(r^{t+j}) \) with upper right corner \( r_{j,k}^{t+j} \).

**Lemma 98** Fix time \( t \). Consider components \( c^t \) and \( \tilde{c}^t \) from two paths \( c \) and \( \tilde{c} \). Let \( c \leftrightarrow r \) and \( \tilde{c} \leftrightarrow \tilde{r} \). We have \( c^t = \tilde{c}^t \) if and only if \( r_{j,k}^t = \tilde{r}_{j,k}^t \) for \( 0 \leq j \leq k \) and \( 0 \leq k \leq \ell \).

**Lemma 99** Fix time \( t \). Consider components \( c^t \) and \( \tilde{c}^t \) from two paths \( c \) and \( \tilde{c} \). Let \( c \leftrightarrow r \) and \( \tilde{c} \leftrightarrow \tilde{r} \). We have \( c^t = \tilde{c}^t \) if and only if shift vector \( v_{t',t'+\alpha}^{t+k}(\tilde{r}) \) in \( \tilde{r} \) is shift vector \( v_{t',t'+\alpha}^{t+k}(\tilde{r}) \) in \( \tilde{r} \) satisfy \( v_{t',t'+\alpha}^{t+k}(\tilde{r}) = v_{t',t'+\alpha}^{t+k}(\tilde{r}) \) for any \([t', t' + \alpha]\) such that \( t \in [t', t' + \alpha] \).

**Proof.** Shift vector \( v_{t',t'+\alpha}^{t+k}(\tilde{r}) \) is uniquely determined by any of its components \( \tilde{r}_{j,k}^t, t' \leq t \leq t' + k \), for fixed basis \( B \).

**Theorem 100** Consider the product \( g_{[t,t+k]}^c = c \) or \( L_{g_{[t,t+k]}^c}(c) \). Let \( c \leftrightarrow r \) and \( \tilde{c} \leftrightarrow \tilde{r} \). After multiplication, the decomposition \( r \) of \( c \) changes to that of \( \tilde{r} \). The only shift vectors in \( r \) which can change from \( r \) to \( \tilde{r} \) are those subordinate to \([t, t + k]\).

**Proof.** From Theorem 91, we know the product \( g_{[t,t+k]}^c \) changes shift vectors in \( r \) to shift vectors in \( \tilde{r} \), for each time \( t \) and length \( k \), \( 0 \leq k \leq \ell \). Since \( c^t = \tilde{c}^t \) for \( t \) outside time interval \([t, t + k]\), then we can apply Lemma 99. This means the only shift vectors in \( r \) which can change from \( r \) to \( \tilde{r} \) are those subordinate to \([t, t + k]\).

**Corollary 101** Consider the product \( g_{[t,t+k]}^c = c \) or \( L_{g_{[t,t+k]}^c}(c) \). Let \( c \leftrightarrow r \) and \( \tilde{c} \leftrightarrow \tilde{r} \). After multiplication, the decomposition \( r \) of \( c \) changes to that of \( \tilde{r} \). The only representatives in \( r \) which can change from \( r \) to \( \tilde{r} \) are \( r_{j,k}^{t+j} \) and direct descendants of \( r_{j,k}^{t+j} \), for \( 0 \leq j \leq k \).

**Proof.** The representative \( r_{j,k}^{t+j} \) and direct descendants of \( r_{j,k}^{t+j} \), for \( 0 \leq j \leq k \), are the representatives of shift vectors in \( r \) which are subordinate to \([t, t + k]\).

We can use these results to find the form of the symmetry \( \omega_{g_{[t,t+k]}^c} \) corresponding to the product \( g_{[t,t+k]}^c \). In particular, we want to find \( g_{[t,t+k]}^c \) or \( L_{g_{[t,t+k]}^c}(c) \), for each \( c \in C \). Let \( u \leftrightarrow r \leftrightarrow c \) and \( \tilde{u} \leftrightarrow \tilde{r} \leftrightarrow \tilde{c} \). As a
consequence of Theorem 100, we know the form of function $\omega_{g^{[t,t+k]}}(u)$ corresponding to $L_{g^{[t,t+k]}}(c)$. All functions $\omega_{m,n}^{t'+m}$ in $\omega_{g^{[t,t+k]}}$ are trivial except possibly those belonging to any shift vector $u$ if consequently this multiplication also induces a change from shift vector $v$ to shift vector $v^{[t',t'+n]}(u)$ in $u$,

$$v^{[t',t'+n]}(\omega_{g^{[t,t+k]}}) = \left(\omega_{0,n}^{t'}, \omega_{1,n}^{t'+1}, \ldots, \omega_{m,n}^{t'+m}, \ldots, \omega_{n,n}^{t'+n}\right),$$

(132)

where $[t', t' + n] \subseteq [t, t + k]$. Then the multiplication $g^{[t,t+k]}c = \overline{c}$ induces a change from shift vector

$$v^{[t',t'+n]}(u) = \left(v_{0,n}^{t'}, v_{1,n}^{t'+1}, \ldots, v_{m,n}^{t'+m}, \ldots, v_{n,n}^{t'+n}\right)$$

(133)

in $u$ to shift vector $v^{[t',t'+n]}(\overline{u})$ in $\overline{u}$,

$$\omega_{g^{[t,t+k]}} : v^{[t',t'+n]}(u) \mapsto v^{[t',t'+n]}(\overline{u}),$$

(134)

if $[t', t' + n] \subseteq [t, t + k]$, but all other shift vectors are unchanged. Consequently this multiplication also induces a change from shift vector $v^{[t',t'+n]}(r)$ in $r$ to shift vector $v^{[t',t'+n]}(\overline{r})$ in $\overline{r}$,

$$\omega_{g^{[t,t+k]}} : v^{[t',t'+n]}(r) \mapsto v^{[t',t'+n]}(\overline{r}),$$

(135)

if $[t', t' + n] \subseteq [t, t + k]$, but all other shift vectors are unchanged.

In any component function $\omega_{m,n}^{t'+m}(u_{m,n}^{t'+m}, \nabla_{m,n}(u^{t'+m}))$ of a shift vector (132), partial argument $\nabla_{m,n}(u^{t'+m})$ has a triangle shape. As $c$ varies among elements of $C$ in product $g^{[t,t+k]}c$, $u$ changes and therefore entries in $\nabla_{m,n}(u^{t'+m})$ change. We know that entries in $\nabla_{m,n}(u^{t'+m})$ correspond 1-1 with entries in $\nabla_{m,n}(r^{t'+m})$. From Proposition 97 an entry in the triangle $\nabla_{m,n}(r^{t'+m})$ corresponds to a shift vector $v^{[t'',t'']+1]}(r)$ in $r$ superordinate to $[t', t' + n]$, and each shift vector $v^{[t'',t'']+1]}(r)$ in $r$ superordinate to $[t', t' + n]$ corresponds to an entry in the triangle. This means that in the product $g^{[t,t+k]}c = \overline{c}$, the change from $v^{[t',t'+n]}(r)$ in $r$ to $v^{[t',t'+n]}(\overline{r})$ in $\overline{r}$ is only affected by shift vectors in $r$ which are superordinate to $[t', t' + n]$.

**Theorem 102** Let $[t', t' + n] \subseteq [t, t + k]$. In the product $g^{[t,t+k]}c = \overline{c}$, the change in (135) from $v^{[t',t'+n]}(r)$ in $r$ to $v^{[t',t'+n]}(\overline{r})$ in $\overline{r}$ is only affected by shift vectors in $r$ superordinate to $[t', t' + n]$.

**Corollary 103** Let $[t', t' + n] \subseteq [t, t + k]$. Fix $m$ such that $t' \leq t' + m \leq t' + n$. Let $r_{m,n}^{t'+m}$ be a component in shift vector $v^{[t',t'+n]}(r)$. In the product $g^{[t,t+k]}c = \overline{c}$, the change from representative $r_{m,n}^{t'+m}$ in $r$ to representative $r_{m,n}^{t'+m}$ in $\overline{r}$ is only affected by representatives in $r$ which are direct ascendants of $r_{m,n}^{t'+m}$.

**Proof.** Consider $r_{m,n}^{t'+m}(c^{t'+m})$. Since $g^{[t,t+k]}c = \overline{c}$ is only affected by shift vectors in $r$ superordinate to $[t', t' + n]$, then for $0 \leq m \leq n$, $r_{m,n}^{t'+m}(c^{t'+m})$ is only affected by representatives in $r$ which are direct ascendants of $r_{m,n}^{t'+m}$. 

In particular we now want to study the effect of multiplication $g^{[t,t+k]}c$ on shift vector $v^{[t',t'+n]}(u)$ in $u$ when $[t', t' + n] = [t, t + k]$. Then (132) - (135) become

$$v^{[t,t+k]}(\omega_{g^{[t,t+k]}}) = (\omega_{0,k}^{t+k}, \omega_{1,k}^{t+k+1}, \ldots, \omega_{j,k}^{t+j}, \ldots, \omega_{k,k}^{t+k}),$$

(136)
\[
\v^{[t,t+k]}(u) = (u_{0,k}^{t,k}, u_{1,k}^{t,k+1}, \ldots, u_{j,k}^{t+j}, \ldots, u_{k,k}^{t+k}),
\]
(137)

\[
\omega_{g^{[t,t+k]}} : \v^{[t,t+k]}(u) \mapsto \v^{[t,t+k]}(\bar{u}),
\]
(138)

\[
\varpi_{g^{[t,t+k]}} : \v^{[t,t+k]}(r) \mapsto \v^{[t,t+k]}(\bar{r}),
\]
(139)

where \(0 \leq j \leq k\). We have just shown the assignment in hence only depends on direct descendants of \(u_{j,k}^{t+j}\), for \(0 \leq j \leq k\), and the assignment in only depends on direct descendants of \(r_{j,k}^{t+j}\), for \(0 \leq j \leq k\).

We now consider two different choices for \(c\), \(\bar{c}\) and \(\bar{c}\). Let \(\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}\) and \(\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}\). We select \(\bar{c}\) and \(\bar{c}\) so that

\[
\v^{[t,t+k]}(\bar{u}) = \v^{[t,t+k]}(\bar{u}).
\]

In \(\bar{u}\), all the ascendants of \(u_{j,k}^{t+j}\), for \(0 \leq j \leq k\), are trivial. In \(\bar{u}\), all the direct ascendants of \(u_{j,k}^{t+j}\), for \(0 \leq j \leq k\), are trivial, but the indirect ascendants can be arbitrary. Let \(g^{[t,t+k]}\bar{c} = \bar{c}\) and \(g^{[t,t+k]}\bar{c} = \bar{c}\). Let \(\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}\) and \(\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}\). Then for multiplication \(g^{[t,t+k]}\bar{c} = \bar{c}\), we have

\[
\omega_{g^{[t,t+k]}} : \v^{[t,t+k]}(\bar{u}) \mapsto \v^{[t,t+k]}(\bar{u}),
\]
(140)

\[
\varpi_{g^{[t,t+k]}} : \v^{[t,t+k]}(\bar{r}) \mapsto \v^{[t,t+k]}(\bar{r}),
\]
(141)

and for multiplication \(g^{[t,t+k]}\bar{c} = \bar{c}\), we have

\[
\omega_{g^{[t,t+k]}} : \v^{[t,t+k]}(\bar{u}) \mapsto \v^{[t,t+k]}(\bar{u}),
\]
(142)

\[
\varpi_{g^{[t,t+k]}} : \v^{[t,t+k]}(\bar{r}) \mapsto \v^{[t,t+k]}(\bar{r}).
\]
(143)

But since \(\v^{[t,t+k]}(\bar{u})\) in \(\bar{u}\) is the same as \(\v^{[t,t+k]}(\bar{u})\) in \(\bar{u}\), and since the direct ascendants of \(u_{j,k}^{t+j}\) in \(\bar{u}\) are the same as the direct ascendants of \(u_{j,k}^{t+j}\) in \(\bar{u}\), for \(0 \leq j \leq k\), we must have \(\v^{[t,t+k]}(\bar{u}) = \v^{[t,t+k]}(\bar{u})\). This gives the commutative diagram Figure 5. But since \(\v^{[t,t+k]}(\bar{u}) = \v^{[t,t+k]}(\bar{u})\), we must have \(\v^{[t,t+k]}(\bar{r}) = \v^{[t,t+k]}(\bar{r})\), and consequently commutative diagram Figure 6 also holds. Thus we have shown the following.

**Lemma 104** Consider the products \(g^{[t,t+k]}\bar{c} = \bar{c}\) and \(g^{[t,t+k]}\bar{c} = \bar{c}\). Let \(\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}\) and \(\bar{u} \leftrightarrow \bar{r} \leftrightarrow \bar{c}\). Then \(\v^{[t,t+k]}(\bar{r}) = \v^{[t,t+k]}(\bar{r})\). In other words, we have \(\bar{r}_{j,k}^{t+j} = \bar{r}_{j,k}^{t+j}\) for \(0 \leq j \leq k\).

We know that \(r_{j,k}^{t+j}\) is the \((j,k)\) component of \(\bar{r}^{t+j}\) and \(r_{j,k}^{t+j}\) is the \((j,k)\) component of \(\bar{r}^{t+j}\). And using Figure 4 we know that \(\bar{r}^{t+j}\) is the decomposition of \(\bar{c}^{t+j}\), where

\[
\bar{c}^{t+j} = \chi^{t+j}(g^{[t,t+k]}\bar{c}^{t+j}) = r_{j,k}^{t+j}\bar{c}^{t+j},
\]

as shown in Figure 7. Similarly, \(\bar{r}^{t+j}\) is the decomposition of \(\bar{c}^{t+j}\), where

\[
\bar{c}^{t+j} = r_{j,k}^{t+j}\bar{c}^{t+j}.
\]

74
Consider the products $g^{t,t+k}|\hat{c} = \hat{c}$ and $g^{t,t+k}|\hat{c} = \hat{c}$. Let $\hat{u} \leftrightarrow \hat{r} \leftrightarrow \hat{c}$ and $\hat{u} \leftrightarrow \hat{r} \leftrightarrow \hat{c}$. Fix $j$ such that $0 \leq j \leq k$. The $(j,k)$ component $r^{t+j}_{j,k}$ of the decomposition $\hat{r}^{t+j}$ of product $r^{t+j}_{j,k}(\hat{c}^{t+j})$ is the same as the $(j,k)$ component $r^{t+j}_{j,k}$ of the decomposition $\hat{r}^{t+j}$ of product $r^{t+j}_{j,k}(\hat{c}^{t+j})$, that is, $r^{t+j}_{j,k} = r^{t+j}_{j,k}$.

We now evaluate $r^{t+j}_{j,k} \hat{c}^{t+j}$ and $r^{t+j}_{j,k} \hat{c}^{t+j}$ and use these results to show that Theorem 105 explains a commutative property of any group system $C$. We can calculate $r^{t+j}_{j,k} \hat{c}^{t+j}$ as

$$r^{t+j}_{j,k}(\hat{c}^{t+j}) = r^{t+j}_{j,k} \left( \prod_{m=0}^{\ell} \prod_{n=m}^{\ell} \hat{c}^{t,m,n} \right),$$

(144)

for $0 \leq j \leq k$. The representatives $\hat{c}^{t,m,n}$ are the identity except for $\hat{c}^{t,j,k}$. Then

$$r^{t+j}_{j,k}(\hat{c}^{t+j}) = r^{t+j}_{j,k}(\hat{c}^{t,j,k}).$$

(145)

We can calculate $r^{t+j}_{j,k} \hat{c}^{t+j}$ as

$$r^{t+j}_{j,k}(\hat{c}^{t+j}) = r^{t+j}_{j,k} \left( \prod_{m=0}^{\ell} \prod_{n=m}^{\ell} \hat{c}^{t,m,n} \right),$$

(146)

for $0 \leq j \leq k$.

Fix $j$ such that $0 \leq j \leq k$. Consider an $\hat{r}$ such that

$$\hat{c}^{t+j} = \hat{r}^{t+j}_{p,q} \hat{r}^{t+j}_{j,k} \hat{r}^{t,j} \cdots \hat{r}^{t,0},$$

where for some $p, q$, $\hat{r}^{t+j}_{p,q}$ is a nontrivial ascendant of $\hat{r}^{t,j}_{j,k}$ but not a direct ascendant. The remaining ascendants are trivial. Since $\hat{r}^{t+j}_{j,k} = \hat{r}^{t+j}_{j,k}$, we can
rewrite $\dot{\mathcal{c}}^{t+j}$ as $\dot{\mathcal{c}}^{t+j} = \mathcal{r}^{t+j}_p q \mathcal{r}^{t+j}_{j,k} \mathcal{r}^{t+j}_{j,k-1} \cdots \mathcal{r}^{t+j}_{0,0}$. Then
\[
\mathcal{r}^{t+j}_{j,k}(\dot{\mathcal{c}}^{t+j}) = \mathcal{r}^{t+j}_{j,k}(\mathcal{c}^{t+j}_p q \mathcal{c}^{t+j}_{j,k} \mathcal{c}^{t+j}_{j,k-1} \cdots \mathcal{c}^{t+j}_{0,0}).
\] (147)

From (27) we know $\mathcal{r}^{t+j}_{j,k}$ is a representative of quotient group
\[
\frac{\mathcal{F}^j(\Delta^t_k)}{\mathcal{F}^j(\Delta^t_{k-1})} = \frac{X^{t+j}_{j-1}(X^{t+j}_j \cap Y^{t+j}_{k-j})}{X^{t+j}_{j-1}(X^{t+j}_j \cap Y^{t+j}_{k-j-1})},
\] (148)
for $j = 0, 1, \ldots, k$. Consider the quotient group (149), determined by representative $\mathcal{r}^{t+j}_{j,k}$ and (148).
\[
\frac{B^{t+j}}{X^{t+j}_{j-1}(X^{t+j}_j \cap Y^{t+j}_{k-j-1})}.
\] (149)

This quotient group contains the cosets of (148). Consider representatives $\mathcal{r}^{t+j}_{m,n}$ for $(m,n)$ satisfying $m = j$, $n \geq k$, and $j < m \leq \ell$, $m \leq n \leq \ell$. Then $\mathcal{r}^{t+j}_{m,n}$ is a representative of some coset (149). If $\mathcal{L}(\mathcal{r}^{t+j}_{m,n})$ is a coset in (149) such that representative $\mathcal{r}^{t+j}_{m,n} \in \mathcal{L}(\mathcal{r}^{t+j}_{m,n})$, then $\mathcal{r}^{t+j}_{m,n}$ is a lifting of $\mathcal{L}(\mathcal{r}^{t+j}_{m,n})$.

Going the other way, given a representative $\mathcal{r}^{t+j}_{m,n}$ of a coset $\mathcal{L}(\mathcal{r}^{t+j}_{m,n})$ in (149), we say $\mathcal{L}(\mathcal{r}^{t+j}_{m,n})$ is a reverse lifting of $\mathcal{r}^{t+j}_{m,n}$. Let $\mathcal{L}(\mathcal{r}^{t+j}_{j,k})$ be the reverse lifting of $\mathcal{r}^{t+j}_{j,k}$ to the quotient group (149). Similarly let $\mathcal{L}(\mathcal{r}^{t+j}_{p,q})$ be the reverse lifting of $\mathcal{r}^{t+j}_{p,q}$ to the quotient group (149).

The $(j,k)$ component $\mathcal{r}^{t+j}_{j,k}$ of the decomposition $\mathcal{r}^{t+j}$ of product (143) is the same as the $(j,k)$ component $\mathcal{r}^{t+j}_{j,k}$ of the decomposition $\mathcal{r}^{t+j}$ of product (147). This means that $\mathcal{r}^{t+j}_{j,k}(\mathcal{r}^{t+j}_{j,k})$ must be in the same coset of (148) in (149) as $\mathcal{r}^{t+j}_{j,k}(\mathcal{r}^{t+j}_{p,q})$. Let $\mathcal{r}^{t+j}_{j,k}$ be the representative that satisfies
\[
\mathcal{r}^{t+j}_{j,k} \cdot \mathcal{r}^{t+j}_{p,q} = \mathcal{r}^{t+j}_{j,k} \cdot \mathcal{r}^{t+j}_{p,q}. \tag{150}
\]

Then we must have $\mathcal{r}^{t+j}_{j,k} \cdot \mathcal{r}^{t+j}_{p,q}$ is in the same coset of (148) in (149) as $\mathcal{r}^{t+j}_{j,k} \cdot \mathcal{r}^{t+j}_{j,k}$. This is true if and only if $\mathcal{r}^{t+j}_{j,k}$ is in the same coset of (148) in (149) as $\mathcal{r}^{t+j}_{j,k}$. This is true if and only if $\mathcal{L}(\mathcal{r}^{t+j}_{j,k})$, the reverse lifting of $\mathcal{r}^{t+j}_{j,k}$ to the quotient group (149), is the same coset of (148) in (149) as $\mathcal{L}(\mathcal{r}^{t+j}_{j,k})$. Then from (150), this is true if and only if coset $\mathcal{L}(\mathcal{r}^{t+j}_{j,k})$ commutes with coset $\mathcal{L}(\mathcal{r}^{t+j}_{p,q})$ in (149). This gives a commutative property that holds for any strongly controllable group system.
Theorem 106  Fix any representative $r_{j,k}^{t+j}$. Fix any $j,k$ such that $0 \leq j \leq \ell$, $j \leq k \leq \ell$. Let $\hat{r}$ be any tensor in $\mathcal{R}$ which has $r_{j,k}^{t+j}$ as a component. Let $r_{p,q}^{t+j}$ be any representative in $\hat{r} \in \mathcal{R}$ such that $r_{p,q}^{t+j}$ is an ascendant but not a direct ascendant of $r_{j,k}^{t+j}$. Then the coset $L(r_{j,k}^{t+j})$ in quotient group (149) determined by $r_{j,k}^{t+j}$ commutes with coset $L(r_{p,q}^{t+j})$ in (149).

There are 3 extreme cases of this result. Element $r_{\ell,\ell}^{t+j}$ has no ascendants so this result does not apply. However $r_{\ell,\ell}^{t+j}$ is an indirect ascendant of any representative in $\nabla_{0,1}(r^{t+j})$, and so there is a commutative property with all these representatives. Element $r_{0,\ell}^{t+j}$ has no direct ascendants so there is a commutative property with all representatives in $\nabla_{1,1}(r^{t+j})$. Element $r_{0,0}^{t+j}$ has no indirect ascendants so this result does not apply. In general $r_{p,q}^{t+j}$ is an indirect ascendant of any representative $r_{j,k}^{t+j}$ that is not a direct descendant of $r_{p,q}^{t+j}$, and so a commutative property holds.
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