Equilateral dimension of the Heisenberg group

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Abstract
Let $\mathcal{H}$ be the first Heisenberg group equipped with the Korányi metric $d$. We prove that the equilateral dimension of $\mathcal{H}$ is 4.

Keywords Heisenberg group · Korányi metric · Equilateral sets · Equilateral dimension.

Mathematics Subject Classification 22F30 · 54E35 · 57M50

1 Introduction

Let $(X, d)$ be a metric space. The equilateral dimension $\text{dim}_E(X)$ of a $X$ is the maximum number of points that are all at equal distances from each other. The equilateral dimension of the $n$-dimensional Euclidean space $(\mathbb{R}^n, e)$, where $e$ is the standard Euclidean metric, is known to be $n + 1$; it is achieved by a regular simplex. In the case of the $n$-dimensional vector space equipped with the $L^\infty$ norm the equilateral dimension is $2^n$ and it is achieved by a hypercube, see [5] as well as [10] and [11]. However, the equilateral dimension of an $n$-dimensional vector space equipped with the $L^1$ norm is not known; Kusner’s conjecture states (see [13]) that it is exactly $2^n$, achieved by a cross-polytope.

There has been an extensive study on the equilateral dimension of $L^p$-spaces, $1 < p < \infty$; in particular, for $p = 2$, $\text{dim}_E(X) = n + 1$ and for $2 < p < \infty$, $\text{dim}_E(X) \geq n + 1$ (see for instance [14]). Various results also exist for arbitrary normed spaces, illustrative references are [3], [12]. Also, for $n$-dimensional Riemannian manifolds $M$, we have $\text{dim}_E(M) = n + 1$ (for bounds of the equilateral dimension of manifolds with certain Ricci and sectional curvature see for instance [9]), and for Minkowski spaces we refer to [10].

Generally speaking, to track down the equilateral dimension of an arbitrary metric space is not an obvious task; to have a clear understanding of the properties of its similarity group is certainly helpful, as equilateral sets remain equilateral under the action of elements of

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the similarity group. As an elementary example we will present a short proof of the fact that \( \dim_E(\mathbb{R}^3, e) = 4 \). Since dilations are within the similarity group of \((\mathbb{R}^3, e)\), we may normalise any equilateral set \( S = \{p_0, p_1, \ldots\} \) so that \( e(p_i, p_j) = 1 \). We may further normalise so that \( p_0 = 0 = (0, 0, 0) \) and \( p_1 = 1 = (1, 0, 0) \) so that all points of \( S \) but \( p_0 \) lie on the Euclidean unit sphere \( S^2 \). If now \( p \in S\setminus\{0, 1\} \), \( p = (x, y, z) \), then the conditions 
\[
\theta = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \quad y^2 + z^2 = \frac{3}{4},
\]
that is, all points of \( S\setminus\{0, 1\} \) lie on the above circle which is the intersection of the unit sphere and the sphere centred at \( 1 \) and of radius 1. This circle is centred in \((1/2, 0, 0)\) and its radius is \( \sqrt{3}/2 \). From an elementary argument we then find that there can be three points on a circle which are at distance 1 from one another if and only if its radius is \( r = \sqrt{3}/3 \), otherwise there are fewer. Indeed, we may consider a planar circle of radius \( r > 0 \) centred at the origin and we may normalise so that the points
\[
p_0 = r, \quad p_1 = re^{i\theta_1}, \quad p_2 = re^{i\theta_2}
\]
are at distance 1 from one another. From the equations \( e(p_0, p_i) = 1 \), \( i = 1, 2 \) we find that \( \theta_1 = -\theta_2 \). Let \( \theta = \theta_1 \); from the equation \( e(p_1, p_2) = 1 \) we obtain \( \sin \theta = 1/(2r) \). But the relation \( e(p_0, p_i) = 1 \) also implies \( \cos \theta = 1 - 1/(2r^2) \). Since \( \cos^2 \theta + \sin^2 \theta = 1 \) we obtain
\[
r = \sqrt{3}/3.
\]

Therefore there can be at most two points on this circle having distance equal to 1, hence \( \dim_E(\mathbb{R}^3) = 4 \).

We will follow the rationale of this elementary approach in the case of the Heisenberg group \( \mathfrak{H} \) which is the object of our study in this article (for the definition, see Sect. 2.1). Contrary to the Euclidean case and to all cases mentioned before, \( \mathfrak{H} \) is equipped with the Korányi metric, a metric which is not induced by a norm. Thus we are not able to use general arguments from the study of equilateral dimension of finite normed spaces; rather, we concentrate on the use of the properties of the similarity group of \( \mathfrak{H} \). In this article we prove:

**Theorem 1.1** The equilateral dimension \( \dim_E(\mathfrak{H}) \) of \( \mathfrak{H} \) is 4.

Therefore \( \dim_E(\mathfrak{H}) \) is equal to its Hausdorff dimension (see [2]). This theorem answers in the negative to a question posed in Ref. [4] on whether there exist five equilateral points in \( \mathfrak{H} \). We also mention that in that article there exist examples of equilateral subsets comprising three points and they are characterized up to similarities. Furthermore, the existence of equilateral quadruples is also proved there. The novelty here is to prove that there cannot be 5-tuples of equidistant points.

This paper is organised as follows. In Sect. 2 we state well known facts about the Heisenberg group and the Korányi metric. The equilateral dimension of \( \mathbb{C} \)-circles is also calculated explicitly in 2.2 (for the definition of \( \mathbb{C} \)-circles, see Sect. 2.2). Section 3 is the section where Theorem 1.1 is proved. For the proof, several steps are taken; first, we study the case of equilateral sets where three of their points lie in the same \( \mathbb{C} \)-circle (Theorem 3.4). Then, we proceed to the case of equilateral sets such that only two of their points lie in the same \( \mathbb{C} \)-circle. Since a \( \mathbb{C} \)-circle can be either infinite or finite, we distinguish cases again: we study the infinite \( \mathbb{C} \)-circle case in Sect. 3.2 (Theorem 3.5) and the finite \( \mathbb{C} \)-circle case in Sect. 3.3 (Theorem 3.6). The proof of Theorem 1.1 follows from combining all the aforementioned theorems. Although elementary, the proof of Theorem 3.6 in particular, involves a series of long but mostly straightforward calculations. Some of them are contained into the Appendix.
2 Preliminaries

In Sect. 2.1 we define the metric space $(\mathcal{H}, d)$, where $\mathcal{H}$ is the Heisenberg group and $d$ is the Korányi metric, and we describe the similarity group $G$ of $(\mathcal{H}, d)$. We refer the reader to the standard book [7] for an extensive study of $\mathcal{H}$. In Sect. 2.1.1 we give a distance formula for points on the unit Korányi sphere. Finally, in Sect. 2.2 we calculate the equilateral dimension of $\mathbb{C}$-circles (Proposition 2.2).

2.1 Heisenberg group

By $\mathcal{H}$ we shall denote the Heisenberg group; recall that $\mathcal{H}$ is $\mathbb{C} \times \mathbb{R}$ with group law:

$$(z, t) \star (w, s) = (z + w, t + s + 2 \Im(z\overline{w})).$$

The Heisenberg norm (Korányi gauge) is given by

$$|(z, t)|_{\mathcal{H}} = |z|^2 - it|^{1/2},$$

Despite its name, $|\cdot|_{\mathcal{H}}$ is not a norm in the usual sense: subadditivity here is

$$|(z, t) \star (w, s)|_{\mathcal{H}} \leq |(z, t)|_{\mathcal{H}} + |(w, s)|_{\mathcal{H}}.$$  

(For certain norms in $\mathcal{H}$, see for instance [6]). But, from this we obtain a distance $d$ is given by

$$d((z, t), (w, s)) = |(w, s)^{-1} \star (z, t)|_{\mathcal{H}} = \left(|z - w|^4 + (t - s + 2 \Im(z\overline{w}))^2\right)^{1/4}. \quad (2.1)$$

This is the Korányi distance and it is invariant under

(a) Left-translations (or, Heisenberg translations) $L_{(w, s)}$ of $\mathcal{H}$,

$$(z, t) \mapsto (w, s) \star (z, t);$$

In particular, vertical translations are left-translations of the form $L_{(0, s)}$.

(b) Rotations $R_\phi$ around the $t$-axis,

$$(z, t) \mapsto (ze^{i\phi}, t) \quad \phi \in \mathbb{R};$$

(c) Conjugation $j$,

$$j(z, t) = (\overline{z}, -t).$$

Conjugation is an involution in $\mathcal{H}$: $j^2 = id$.

The distance $d$ is also scaled up to multiplicative constants by the action of Heisenberg dilations $D_r : (z, t) \mapsto (rz, r^2 t)$, $r \in \mathbb{R}_\times$. The similarity group $G = \text{Sim}(\mathcal{H}, d) \simeq \mathcal{H} \times \mathbb{R} \times \mathbb{R}_{>0}$, comprises all the above transformations. Inversion $I : \mathcal{H}\{o\} \to \mathcal{H}\{o\}$, $o = (0, 0)$, is defined by the formula

$$I(z, t) = \left(\frac{z}{-|z|^2 + it}, \frac{-t}{|z|^4 + t^2}\right).$$

Inversion is an involution in $\mathcal{H}\{o\}$: $I^2 = id$. Moreover, for any two $p, q \in \mathcal{H}\{o\}$ we have

$$d(I(p), o) = \frac{1}{d(p, o)}, \quad d(I(p), I(q)) = \frac{d(p, q)}{d(I(p), o) \cdot d(I(q), o)}.$$
All the above transformations can be extended into \( \mathbb{H} \cup \{ \infty \} \), that is, the boundary of \( \mathbb{H}^2_{\mathbb{C}} \). The group \( \text{SU}(2, 1) \), which is a triple cover of the group of holomorphic isometries of \( \mathbb{H}^2_{\mathbb{C}} \), comprises compositions of the extensions of the above transformations.

We note that \( d \) is not a path metric. Moreover, there is no Riemannian structure in \( \mathbb{H} \) with underlying metric \( d \). The Carnot-Carathéodory metric in \( \mathbb{H} \) which corresponds to its sub-Riemannian structure will not be of our concern here.

### 2.1.1 The unit Korányi sphere and a distance formula

A general Korányi sphere \( S_r^{1/2}(p_0) \) centred at \( p_0 \) with radius \( r \) is the locus comprising of points \( p \in \mathbb{H} \) such that \( d(p, p_0) = r \). The Korányi unit sphere

\[
S_{\mathbb{H}}^1 = \{ p \in \mathbb{H} | d(p, o) = 1 \},
\]

where \( o = (0, 0) \), is the surface described by the equation

\[
|z|^4 + t^2 = 1.
\]

When \( p_0 = (z_0, t_0) \) and \( p = (z, t) \) are points in \( S_{\mathbb{H}}^1 \), then (2.1) takes the form

\[
d^4(p, p_0) = 2 + 6|z|^2|z_0|^2 - 2tt_0 - 4(|z|^2 + |z_0|^2)\Re(z\overline{z_0}) + 4(t - t_0)\Im(z\overline{z_0}).
\]  

(2.2)

The above equation may be simplified even further by using Korányi-Reimann coordinates (see [8]):

\[
z = \sqrt{\cos \theta e^{i\phi}}, \quad t = \sin \theta,
\]

where \((\theta, \phi) \in [-\pi/2, \pi/2] \times [-\pi, \pi)\). In fact, Equation (2.2) then reads as

\[
d^4(p, p_0) = 2 + 6 \cos \theta \cos \theta_0 - 2 \sin \theta \sin \theta_0
- 8\sqrt{\cos \theta \cos \theta_0} \cos((\theta + \theta_0)/2) \cos((\phi + \phi_0)/2) - \phi_0 + \phi_0/2)) .
\]

(2.3)

For fixed \( p_0 \in S_{\mathbb{H}}^1 \) and \( p \in S_{\mathbb{H}}^1 \), the locus \( d(p, p_0) = 1 \) is the intersection of \( S_{\mathbb{H}}^1 \) with the Korányi sphere centred at \( p_0 \) with radius 1. Using (2.3) we find that this locus is a curve on \( S_{\mathbb{H}}^1 \) given parametrically by

\[
\cos((\phi + \theta/2) - (\phi_0 + \theta_0/2)) = \frac{1 + 6 \cos \theta \cos \theta_0 - 2 \sin \theta \sin \theta_0}{8\sqrt{\cos \theta \cos \theta_0} \cos((\theta + \theta_0)/2)},
\]

if \( p_0 \neq (0, \pm 1) \). If \( p_0 = (0, \pm 1) \), this curve is the Euclidean circle \( \theta = \pm \pi/6 \), respectively.

### 2.1.2 Convention

Let \( S \subset \mathbb{H} \) be a \( d_0 \) equilateral set of points: \( d(p, q) = d_0 \) for each \( p, q \in S \). By applying the dilation \( D_{1/d_0} \) we may always suppose that \( S \) is a 1-equilateral set, or simply equilateral set from now on. Throughout this paper we will further suppose that we deal with such sets.
2.2 C-circles and their equilateral dimension

Recall that the one point compactification $\mathcal{H} \cup \{\infty\}$ of the Heisenberg group $\mathcal{H}$ can be identified with the boundary of complex hyperbolic plane $\mathbb{H}_C^2$: this is the Siegel domain

$$\mathbb{H}_C^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid 2 \Re(z_1) + |z|^2 < 0\}.$$ 

A $\mathbb{C}$-circle in $\mathcal{H}$ is a topological circle which is the boundary of a complex geodesic of the Siegel domain. Such a circle comes in two flavours: (a) infinite $\mathbb{C}$-circles when one of the endpoints of the complex geodesic is $\infty$ and (b) finite $\mathbb{C}$-circles when none of the endpoints of the complex geodesic is $\infty$. Infinite $\mathbb{C}$-circles are lines vertical to the plane $t = 0$; they are all $G$-images of the $\mathbb{C}$-circle

$$C^{\infty} = \{p = (z, t) \in \mathcal{H} \mid z = 0\}. \tag{2.5}$$

On the other hand, finite $\mathbb{C}$-circles are ellipses which are all $G$ images of the Euclidean circle

$$C_1 = \{p = (z, t) \in \mathcal{H} \mid |z| = 1, \ t = 0\}. \tag{2.6}$$

We note here that (a) there is no element of $G$ that can map an infinite $\mathbb{C}$-circle to a finite $\mathbb{C}$-circle or vice versa and (b) any two points of $\mathcal{H}$ lie in a unique $\mathbb{C}$-circle: see Theorem 4.3.5 in [7]. For details on $\mathbb{H}_C^2$ and in particular $\mathbb{C}$-circles we refer to [7].

**Lemma 2.1** There can be at most two equilateral points lying in the same infinite $\mathbb{C}$-circle and at most three equilateral points lying in the same finite $\mathbb{C}$-circle.

**Proof** For the first statement, we may normalize so that the infinite $\mathbb{C}$-circle is the vertical axis which comprises points $(0, t)$. If $S = \{p_1, p_2, \ldots\}$ is an equilateral set of points on that $\mathbb{C}$-circle, by applying a suitable dilation and a vertical translation if necessary we may assume that $p_1 = (0, -1/2)$ and $p_2 = (0, 1/2)$ so that $d(p_1, p_2) = 1$. Now, if $(0, t)$ is equidistant from $p_1$ and $p_2$, short calculations lead $t = 0$. But this cannot be the case since $d((0, 0), p_1) = d((0, 0), p_2) = (1/4)^{1/4} \neq 1$

For the second statement, we may normalize so that the finite $\mathbb{C}$-circle is the planar circle $|z| = r$ for a certain $r$. If $S = \{p_1, p_2, \ldots\}$ is an equilateral set of points on that $\mathbb{C}$-circle, by performing a rotation we may suppose that $p_1 = (r, 0)$. Points $p = (re^{i\theta}, 0)$ in the circle such that $d(p, p_1) = 1$ must then satisfy the relation

$$8r^4(1 - \cos \theta) = 16r^4 \sin^2(\theta/2) = 1.$$

Since $\sin^2(\theta/2) < 1$, we obtain $r \geq 1/2$. In that case we obtain points $p = (re^{i\theta}, 0)$, with

$$\theta = \pm \arccos \left(1 - \frac{1}{8r^4}\right) = \pm \theta_0.$$

Now it is straightforward to show that $d\left((re^{i\theta_0}, 0), (re^{-i\theta_0}, 0)\right) = 1$ if and only if $r = 12^{-1/4}$ (and $\theta_0 = 2\pi/3$). In all other cases $d\left((re^{i\theta_0}, 0), (re^{-i\theta_0}, 0)\right) > 1$ which proves that there can be no equilateral triple of points in this case. \qed

From the proof of Lemma 2.1 we immediately have

**Proposition 2.2** Let $C$ be a $\mathbb{C}$-circle in $\mathcal{H}$. Then:

1. The maximum number of points in $C$ that are at mutual distances 1 is two if $C$ is infinite;
2. The maximum number of points in $C$ that are at mutual distances 1 is three if $C$ is the $G$-image of a Euclidean circle centred at the origin and with radius $r = 12^{-1/4}$. If $C$ is any other finite $\mathbb{C}$-circle then it is two.
3 Equilateral dimension of $\mathfrak{F}$

In this chapter, we prove our main theorem (Theorem 1.1). As the starting point of the proof, we use the fact that given any two points there is a $C$-circle through them. In Sect. 3.1 we study equilateral sets of points such that three of them lie in the same $C$-circle. By Lemma 2.1, this $C$-circle has to be finite. We prove in Theorem 3.4 that there can be at most four equidistant points such that three of them lie in the same $C$-circle. In Sect. 3.2 we prove in Theorem 3.5 that the maximum number of equilateral points in $\mathfrak{F}$ such that two of them lie in an infinite $C$-circle is 4. Finally in Sect. 3.3 we prove our result for the case where two points lie in a finite $C$-circle in Theorem 3.6.

3.1 Three points in the same finite $C$-circle

Given an equilateral set $S = \{p_1, p_2, p_3\}$ of points lying in the same finite $C$-circle $C$, let $P$ denote the Euclidean plane such that $S \subset P$. By applying a Heisenberg translation we may suppose that $P = \{(z, t) \in \mathfrak{F} | t = 0\}$ and that $C$ is a Euclidean circle centred at the origin. By Proposition 2.2 the radius of this circle is necessarily $r_0 = 12^{-1/4}$ and by applying a suitable dilation and a rotation if necessary, we may also normalise so that

$$p_1 = (r_0, 0), \quad p_2 = r_0 e^{i\theta_0}, \quad p_3 = r_0 e^{-i\theta_0},$$

where $\theta_0 = 2\pi/3$.

**Definition 3.1** The canonical 3-equilateral set is the set

$$S_3^{can} = \{(r_0, 0), (r_0 e^{i\theta_0}, 0), (r_0 e^{-i\theta_0}, 0)\}.$$

**Lemma 3.2** The only points in $\mathfrak{F}$ equidistant from $S_3^{can}$ are the points $(0, \pm t_0)$, where $t_0 = \sqrt{11/12}$.

**Proof** The proof is by elementary calculations. If $(z, t) \in \mathfrak{F}$ such that $d((z, t), (r_0, 0)) = 1$, then

$$|z - r_0|^4 + (t + 2r_0 \overline{z}(z))^2 = 1. \quad (3.1)$$

Also, $d((z, t), (r_0 e^{\pm i\theta_0}, 0)) = 1$ gives

$$|z - r_0 e^{\pm i\theta_0}|^4 + (t + 2r_0 \overline{z}(z e^{\mp i\theta_0}))^2 = 1. \quad (3.2)$$

After expanding, we subtract (3.2) from (3.1) to obtain

$$\Re \left( (|z|^2 + 12^{-1/2} + it) \cdot z(3 \pm i\sqrt{3}) \right) = 0.$$

In other words, we have the system of equations

$$(3x - \sqrt{3}y)(|z|^2 + 12^{-1/2}) - (3y + \sqrt{3}x)t = 0,$$

$$(3x + \sqrt{3}y)(|z|^2 + 12^{-1/2}) - (3y - \sqrt{3}x)t = 0.$$

Viewing this system as a homogeneous linear system in variables $|z|^2 + 12^{-1/2}$ and $t$, we find that if $z \neq 0$ then we must have $t = |z|^2 + 12^{-1/2} = 0$, a contradiction. Thus $z = 0$ and we obtain from both Eqs. (3.1) and (3.2) that $t = \pm t_0 = \sqrt{11/12}$. \hfill $\square$

**Definition 3.3** The canonical 4-equilateral set $S_4^{can}$ is the set $S_3^{can} \cup \{(0, t_0)\}$.
Any set of four equidistant points in \( \mathfrak{S} \) such that three of them lie in the same finite \( C \)-circle is \( G \)-equivalent to the canonical set \( S^\text{can}_4 \). If \( S = \{p_1, p_2, p_3, p_4\} \) and \( p_i, i = 1, 2, 3 \) lie in the same finite \( C \)-circle, then the subset \( S_0 = \{p_1, p_2, p_3\} \) is \( G \)-equivalent to the set \( S^\text{can}_3 \). From Lemma 3.2 we then have that the point \( p_4 \) is mapped either to \((0, t_0)\) or \((0, -t_0)\). By applying a conjugation if necessary, we may normalise so that the wished point is \((0, t_0)\).

**Theorem 3.4** The maximal number of equidistant points in \( \mathfrak{S} \) such that three of them lie in the same finite \( C \)-circle is 4.

**Proof** Suppose on the contrary that there exists an equilateral set \( S = \{p_1, p_2, p_3, p_4, p_5\} \) such that \( p_i, i = 1, 2, 3 \), lie in the same finite \( C \)-circle. Then the subset \( S_0 = \{p_1, p_2, p_3, p_4\} \) is \( G \)-equivalent to the set \( S^\text{can}_4 \). Since there exist no point equidistant to \( S^\text{can}_4 \), we obtain a contradiction \( d((0, t_0), (0, -t_0)) \neq 1) \). □

### 3.2 Two points in the same infinite \( C \)-circle

In this section we shall prove

**Theorem 3.5** The maximum number of equidistant points in \( \mathfrak{S} \) such that two of them lie in an infinite \( C \)-circle is 4.

**Proof** Let \( S = \{p_1, p_2, \ldots\} \) be an equilateral set such that \( p_1 \) and \( p_2 \) lie in the same infinite \( C \)-circle. By applying a left translation and a vertical translation if necessary, we may suppose that \( p_1 = (0, -1/2) \) and \( p_2 = (0, 1/2) \). All points equidistant to \( p_1 \) and \( p_2 \) then lie in the finite \( C \)-circle

\[
C = \{(z, 0) \in \mathfrak{S} \mid |z| = (3/4)^{1/4}\}.
\]

Since the radius of \( C \) is greater than \( 12^{-1/4} \), our result follows from Proposition 2.2. □

### 3.3 Two points in the same finite \( C \)-circle

In this section we shall prove

**Theorem 3.6** The maximum number of equidistant points in \( \mathfrak{S} \) such that at most two of them lie in the same finite \( C \)-circle is 4.
Parametric representation of $C_{3}^{\text{fin}}$ in $(-\pi/2, \pi/2)^2$.

We consider an equilateral set $S = \{p_1, p_2, \ldots\}$ such that there are no three points lying in the same finite $\mathbb{C}$-circle. By letting an element of $G$ acting on $S$ we may normalize so that

$$p_1 = (0, 0), \quad p_2 = (1, 0),$$

that is, they lie on the $\mathbb{C}$-circle which is the image of the Euclidean circle $|z| = 1/2$ lying on the plane $t = 0$ under the left-translation $L = L(1/2, 0)$ and in particular, $p_1 = L(-1/2, 0)$, $p_2 = L(1/2, 0)$.

Since all other points in $S$ must lie in the unit sphere $S^1_{\mathbb{H}} = \{(z, t) \in \mathbb{H} \mid |z|^4 + t^2 = 1\}$, we shall use Korányi–Reimann spherical coordinates

$$z = \sqrt{\cos \theta} e^{i\phi}, \quad t = \sin \theta, \quad (\theta, \phi) \in \Pi = (-\pi/2, \pi/2)^2,$$

to describe the points of $S$ from now on. Note that we may suppose that $\phi \in (-\pi/2, \pi/2)$ instead of $\phi \in [-\pi, 0]$ since we are only interested for the part of $S^1_{\mathbb{H}}$ which intersects the unit sphere $S^1_{\mathbb{H}}(1, 0)$, see below. Together with the notation $p = (z, t)$ we shall also use the notation $p = (\theta, \phi)$. Hence if $p = (\theta, \phi) \in S$ other than $p_1, p_2$, then by Eq. 2.4 the condition $d(p, p_2) = 1$ reads as

$$1 + 6 \cos \theta - 8 \sqrt{\cos \theta} \cos(\theta/2) \cos(\phi + \theta/2) = 0.$$  \hspace{1cm} (3.3)

Equation 3.3 represents a spherical curve which we shall denote by $C_{3}^{\text{fin}}$. Points in $C_{3}^{\text{fin}}$ are all of distance 1 from $p_1$ and $p_2$, that is, $C_{3}^{\text{fin}}$ is the intersection of $S^1_{\mathbb{H}}$ and $S^1_{\mathbb{H}}(1, 0)$. In the appendix (Section 4.1) we describe explicitly a parametric representation of $C_{3}^{\text{fin}}$: the curve $C_{3}^{\text{fin}}$ is the union of the graphs of the functions $\phi^\pm : I \to \mathbb{R}$ where

$$\phi^\pm(\theta) = -\frac{\theta}{2} \pm \arccos(f(\theta))$$  \hspace{1cm} (3.4)
and

\[ f(\theta) = \frac{1 + 6 \cos \theta}{8 \sqrt{\cos \theta \cos(\theta/2)}}, \quad \theta \in I. \tag{3.5} \]

Here, \( I = [\theta_0^*, \theta_0^*] \), \( \theta_0^* = \arccos(5/2 - \sqrt{6}) \). We wish to note at this point that there are some rather obvious symmetries of \( \mathcal{C}^3_{\text{fin}} \). Namely, we have the following three involutions:

- **The antipodal involution** \( j_1 \):
  \[ p = (\theta, \phi) \mapsto j_1(p) = (-\theta, -\phi). \]

- **The symmetric involution** \( j_2 \):
  \[ p = (\theta, \phi) \mapsto j_2(p) = (-\theta, \phi + \theta). \]

- **The vertical involution** \( j_3 \):
  \[ p = (\theta, \phi) \mapsto j_3(p) = (\theta, -\phi - \theta). \]

These involutions satisfy the relation

\[ j_1 \circ j_2 = j_2 \circ j_1 = j_3 \]

and they are studied in detail in the appendix (Section 4). Note that

- The antipodal involution is the restriction of conjugation \( j \) in \( \mathcal{C}^3_{\text{fin}} \);
- The symmetric involution is the restriction of inversion \( I \) in \( \mathcal{C}^3_{\text{fin}} \);
- The vertical involution is the restriction of conjugation \( j \) followed by the restriction of inversion \( I \) in \( \mathcal{C}^3_{\text{fin}} \).

We also mention here that we have the following result concerning antipodal points: for all \( p \in \mathcal{C}^3_{\text{fin}} \),

\[ 1.378 \approx (2\sqrt{6} - 3)^{1/2} \leq d(p, j_1(p)) \leq (15/4)^{1/4} \approx 1.391. \tag{3.6} \]

In both cases of symmetric and vertical points there exist certain pairs of points with distance 1 (see the appendix). From this we establish the existence of equilateral quadruples of points which are such that two of them lie in the same finite \( \mathbb{C} \)-circle. The next lemma shows that something much stronger holds.

**Lemma 3.7** For each fixed point \( p_0 \in \mathcal{C}^3_{\text{fin}} \) there exist at least two points \( p \in \mathcal{C}^3_{\text{fin}} \) such that \( d(p_0, p) = 1 \).

**Proof** Fix a point \( p_0 \in \mathcal{C}^3_{\text{fin}} \) and consider the function \( G_{p_0} : \mathcal{C}^3_{\text{fin}} \to \mathbb{R}_+ \) given by

\[ G_{p_0}(p) = d(p, p_0). \]

This function is continuous and bounded on both branches of \( \mathcal{C}^3_{\text{fin}} \) joining \( p_0 \) and \( j_1(p_0) \). Since from Eq. 3.6 we have \( d(p_0, j_1(p_0)) > 1 \), by the Intermediate Value Theorem there exists a point \( p^* \in \mathcal{C}^3_{\text{fin}} \) (at least one in each branch) such that \( G_{p_0}(p^*) = d(p^*, p_0) = 1 \).

However, in order to complete the proof of Theorem 3.6 we will show that for any fixed point \( p_0 \in \mathcal{C}^3_{\text{fin}} \) there are exactly two points \( p_1, p_2 \in \mathcal{C}^3_{\text{fin}} \) such that

\[ d(p_0, p_1) = d(p_0, p_2) = 1 \text{ and } d(p_1, p_2) \neq 1. \]

To do so, we will mostly use straightforward calculations.
3.3.1 Proof of Theorem 3.6

For a given \((\theta_0, \phi_0)\) such that
\[
\cos(\phi_0 + \theta_0/2) = \frac{1 + 6 \cos \theta_0}{8\sqrt{\cos \theta_0 \cos(\theta_0/2)}} = f(\theta_0) = f_0 = \frac{a_0}{b_0},
\]
we want to find \((\theta, \phi)\) such that
\[
\cos ((\phi + \theta/2) - (\phi_0 + \theta_0/2)) = \frac{1 + 6 \cos \theta \cos \theta_0 - 2 \sin \theta \sin \theta_0}{8\sqrt{\cos \theta \cos(\theta + \theta_0)/2}} = g(\theta, \theta_0) = g = \frac{A}{B}
\]
and
\[
\cos(\phi + \theta/2) = \frac{1 + 6 \cos \theta \cos \theta_0}{8\sqrt{\cos \theta \cos(\theta/2)}} = f(\theta) = f = \frac{a}{b}.\]

In the first place,
\[
g = ff_0 + \sin(\phi + \theta/2) \sin(\phi_0 + \theta_0/2);
\]
thus we have
\[
(g - ff_0)^2 = (1 - f^2)(1 - f_0^2)
\]
which we write again as
\[
(Abb_0 - Baa_0)^2 = B^2(b^2 - a^2)(b_0^2 - a_0^2). \tag{3.7}
\]
We calculate
\[
b^2 - a^2 = -4 \cos^2 \theta + 20 \cos \theta - 1,
\]
\[
b_0^2 - a_0^2 = -4 \cos^2 \theta_0 + 20 \cos \theta_0 - 1,
\]
\[
B^2 = 32 \cos \theta \cos \theta_0(1 + \cos(\theta + \theta_0)),
\]
and
\[
Abb_0 - Baa_0 = 8\sqrt{\cos \theta \cos \theta_0} (C \cos(\theta/2) + D \sin(\theta/2)),
\]
where
\[
C = \cos(\theta_0/2) (c_1 \cos \theta + c_2),
\]
\[
D = \sin(\theta_0/2) (d_1 \cos \theta + d_2),
\]
with
\[
c_1 = 6(2 \cos \theta_0 - 1), \quad c_2 = 7 - 6 \cos \theta_0,
\]
\[
d_1 = 10(2 \cos \theta_0 - 1), \quad d_2 = -5(2 \cos \theta_0 + 3).
\]
Thus Eq. 3.7 reads now as
\[
2(C \cos(\theta/2) + D \sin(\theta/2))^2 = (b^2 - a^2)(b_0^2 - a_0^2)(\cos \theta_0 \cos \theta - \sin \theta_0 \sin \theta + 1). \tag{3.8}
\]
We set \(t = \tan(\theta/2)\) and \(t_0 = \tan(\theta_0/2)\). Then we obtain the 6th degree polynomial equation
\[
P_{t_0}(t) = (5t_0(7 - 5t_0^2) \cdot t^3 - (31t_0^2 - 5) \cdot t^2 + 5t_0(7t_0^2 + 3) \cdot t - 7 + 5t_0^2)^2
\]
\[-(-25t_0^4 + 6t_0^2 + 15)(-25t^4 + 6t^2 + 15)(t_0t - 1)^2 = 0. \tag{3.9}
\]
As a function of variables $t_0$ and $t$, $P$ is a symmetric polynomial: $P_{t_0}(t) = P_t(t_0)$. Moreover, it represents the closed curve which is depicted in the following figure.

The curve $P = 0$.

The right term of (3.9) vanishes when $t_0 = \pm t^*, t^* = \tan(\theta^*/2), \theta^* = \arccos(5/2 - \sqrt{6})$. The equation is subsequently reduced to a cubic equation which has only one real root. This root correspond to the upper (resp. lower) point of the above curve. In all other cases we see that the equation has exactly two real roots; one of them is negative and the other is positive.

We will first prove a simple case before moving on to the general one. In the case of $t_0 = 0$ ($\theta_0 = 0$), (3.9) is reduced to

$$(5t^2 - 7)^2 = 15(-25t^4 + 6t^2 + 15).$$

This gives

$$t = \pm \sqrt{\frac{1 + 2\sqrt{3}}{5}} \implies \theta = \pm 2 \arctan \left( \sqrt{\frac{1 + 2\sqrt{3}}{5}} \right) = \pm \theta_0.$$

Now, $\theta_0 = 0$ corresponds to two points on the curve $C^\text{fin}_3$, namely the points $p_0 = (0, \arccos(7/8))$ and $j_1(p_0)$. From our solution we also have four points, namely

$q_0 = (\theta_0, \phi^+(\theta_0)), \quad j_i(q_0), \quad i = 1, 2, 3.$

We have

$$d(p_0, q_0) = d(p_0, j_2(q_0)) = 1,$$

$$d(j_1(p_0), j_1(q_0)) = d(j_3(p_0), j_3(q_0)) = 1.$$

It is straightforwardly verified that all other distances between these points are different than 1.
In the general case, let with no loss of generality $t_0 > 0$ corresponding to a fixed point $p_0 = (\theta_0, \phi_0) \in C^\text{fin}_3$. Let also $t_1 < 0$ and $t_2 > 0$ the two roots of $P_{t_0}$:

$$P_{t_0}(t_1) = P_{t_0}(t_2) = 0,$$

and these two roots correspond to points $p_1 = (\theta_1, \phi_1)$ and $p_2 = (\theta_2, \phi_2)$ in $C^\text{fin}_3$, respectively. Supposing that $d(p_1, p_2) = 1$ we would have $P_{t_1}(t_2) = 0$. But from symmetry of $P$ we would also have $P_{t_1}(t_0) = 0$, hence $P_{t_1}$ has two positive roots, a contradiction. This completes the proof.

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4. Appendix

4.1 Parametric representation of $C^\text{fin}_3$ 

The curve $C^\text{fin}_3$ intersects the $\theta$-axis at points $(\pm \arccos(1/4), 0)$ and the $\phi$-axis at points $(0, \pm \arccos(7/8))$. To find the range of $\theta$ we put $\phi + \theta/2 = 0$ and from

$$\frac{1 + 6 \cos \theta}{8 \sqrt{\cos \theta \cos(\theta/2)}} = \pm 1$$

we find $\theta \in I = [-\theta^*, \theta^*]$ where $\theta^* = \arccos(5/2 - \sqrt{6})$. We set

$$f(\theta) = \frac{1 + 6 \cos \theta}{8 \sqrt{\cos \theta \cos(\theta/2)}} , \quad \theta \in I,$$

as in 3.5. Its derivative in Int($I$) is given by

$$f'(\theta) = -\sin \theta \cdot \frac{4 \cos \theta - 1}{32 \cos^{3/2} \theta \cos^3(\theta/2)}.$$ 

The function $f$: 

- Attains its global maximum value 1 at the points $\pm \arccos(5/2 - \sqrt{6})$;
- Attains its global minimum value $\sqrt{5/8}$ at the points $\pm \arccos(1/4)$;
- Attains a locally maximum value $7/8$ at 0.

Moreover, $f$ is:

- Strictly monotone decreasing at the intervals $[ - \arccos(5/2 - \sqrt{6}), - \arccos(1/4) ]$, $[0, \arccos(1/4)]$;
- Strictly monotone increasing at the intervals $[ - \arccos(1/4), 0 ]$, $[ \arccos(1/4), \arccos(5/2 - \sqrt{6}) ]$. 

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The function $f$.

We note that

$$\sqrt{5}/8 \leq \cos(\phi + \theta/2) \leq 1,$$

which also reads as

$$-\arccos(\sqrt{5}/8) \leq \phi + \theta/2 \leq \arccos(\sqrt{5}/8).$$

We conclude that $C^\text{fin}_3$ is the union of the graphs of the functions $\phi^\pm : I \to \mathbb{R}$ as in 3.4.

4.2 Symmetries of $C^\text{fin}_3$

The involution $j : (z,t) \mapsto (\overline{z},-t)$ of the Heisenberg group is restricted to an involution $j_1$ defined in $C^\text{fin}_3$ which maps each $p = (\theta, \phi) \in C^\text{fin}_3$ to the point $j_1(p) = (-\theta, -\phi) \in C^\text{fin}_3$. That is to say, for each $\theta \in I$,

$$(\theta, \phi^+(\theta)) \mapsto (-\theta, \phi^-(\theta))$$

$$(\theta, \phi^-(\theta)) \mapsto (-\theta, \phi^+(\theta)).$$

We call $j_1$ the antipodal involution of $C^\text{fin}_3$; this defines antipodal pairs of points $(p, j_1(p))$, $p \in C^\text{fin}_3$.

Let $j_1$ be the antipodal involution and let the function $h : C^\text{fin}_3 \to \mathbb{R}^+$ given by

$$h(p) = d^4(p, j_1(p)).$$

Then $h$ as a function of $\theta \in I$ is given by:

$$h(\theta) = \frac{1}{2} \cdot \frac{8 \cos^3 \theta - 12 \cos^2 \theta + 12 \cos \theta + 7}{1 + \cos \theta}, \quad \theta \in I.$$

The following hold:
(a) $h$ attains its global maximum value $15/4$ at the three pairs of antipodal points

$$
q_1^\pm = \pm (0, \arccos(7/8)),
q_2^\pm = \pm (\arccos(1/4), 0),
q_3^\pm = \pm (\arccos(1/4), -\arccos(1/4)).
$$

Also,

(b) $h$ attains its global minimum value $(2\sqrt{6} - 3)^2$ at the three pairs of antipodal points

$$
r_1^\pm = \pm (\arccos((5 - 2\sqrt{6})/2), -(1/2) \arccos((5 - 2\sqrt{6})/2)),
r_2^\pm = \pm (\arccos((-1 + \sqrt{6})/2), -(1/2) \arccos((-1 + \sqrt{6})/2) + \arccos(\sqrt{5/8} \cdot (-2 + 3\sqrt{6})/5)),
r_3^\pm = \pm (\arccos((-1 + \sqrt{6})/2), -(1/2) \arccos((-1 + \sqrt{6})/2) - \arccos(\sqrt{5/8} \cdot (-2 + 3\sqrt{6})/5)).
$$

Finally,

(c) (Eq. 3.6) for all $p \in C_3^{fin}$,

$$
1.378 \approx (2\sqrt{6} - 3)^{1/2} \leq d(p, j_1(p)) \leq (15/4)^{1/4} \approx 1.391.
$$
\[
d^4(q_1^+, q_3^+) = d^4(q_2^+, q_1^-) = d^4(q_3^+, q_2^-) = d^4(q_1^-, q_3^+) = d^4(q_2^-, q_1^+),
\]
\[
d^4(q_1^+, q_3^-) = d^4(q_2^+, q_3^-) = d^4(q_3^+, q_2^+) = 9/4,
\]
\[
d^4(q_1^-, q_3^-) = d^4(q_2^-, q_3^+) = d^4(q_3^-, q_2^-) = 15/4.
\]

**Proof** For the first statement, let \( \omega = dt + 2\Im(\overline{z}dz) \) be the contact form of \( \mathcal{S} \). Then it is written in terms of Korányi-Reimann coordinates as
\[
\omega = \cos \theta (d\theta + 2d\phi).
\]

For \( \phi \) as in (3.4) we then have
\[
d\theta + 2d\phi = \pm \frac{2f'(\theta)}{\sqrt{1 - f^2(\theta)}} d\theta
\]
and the proof follows from the properties of \( f \) and \( h \). The proof of the second statement follows after straightforward calculations. \( \square \)

The symmetric involution \( j_2 \) maps each \( p = (\theta, \phi) \in C_3^{\text{fin}} \) to the point \( j_2(p) = (-\theta, \phi + \theta) \). It defines symmetric pairs of points \( p, j_2(p), p \in C_3^{\text{fin}} \). Let
\[
s(p) = d^4(p, j_2(p)), \quad p \in C_3^{\text{fin}}.
\]

As a function of \( \theta \in I \),
\[
s(\theta) = 2 + 4\cos^2 \theta + 2 - 8\cos \theta
\]
\[
= 4(1 - \cos \theta)^2 = 16\sin^4(\theta/2).
\]

The symmetric function \( s \).

The function \( s \):

1. Attains its global maximum value \( 4(\sqrt{6} - 3/2)^2 \) at points \( \pm \theta^* \).
2. Determines the following pairs of symmetric points with distance 1:
\[
(\pi/3, -\pi/6 + \arccos(\sqrt{6}/3)), \quad (-\pi/3, \pi/6 + \arccos(\sqrt{6}/3)),
\]
Finally, the vertical involution $j_3$ of $C_3^{fin}$ maps each $p = (\theta, \phi) \in C_3^{fin}$ to the point $j_3(p) = (\theta, -\phi - \theta)$. Note that $j_3$ is just the map $(\theta, \phi^+(\theta)) \mapsto (\theta, \phi^-(\theta))$; hence vertical points $p$ and $j_3(p)$ lie in the same spherical $C$-circle ($j_3(p)$ is the image of $p$ by some rotation of $\mathbb{H}$). Let $v(p) = d^3(p, j_3(p)), p \in C_3^{fin}$. As a function of $\theta \in I$,

$$v(\theta) = \frac{\cos \theta}{2(1 + \cos \theta)}(-4 \cos^2 \theta + 20 \cos \theta - 1).$$

The vertical function $v$.

The function $v(\theta)$:

1. attains its maximum value $15/4$ at 0;
2. determines the following pairs of vertical points with distance 1:

$$(\theta_0, \phi^+(\theta_0)), \quad (\theta_0, \phi^-(\theta_0)), \quad (-\theta_0, \phi^+(-\theta_0)), \quad (-\theta_0, \phi^-(\theta_0)),$$

where $\theta_0 = \arccos(0.42\ldots)$ is the absolute value of the two solutions of the equation

$$4\cos^3 \theta - 2\cos^2 \theta + 3\cos \theta + 2 = 0.$$ 

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