Symplectic aspects of the first eigenvalue

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There are two themes in the present paper. The first one is spelled out in the title, and is inspired by an attempt to find an analogue of Hersch-Yang-Yau estimate for $\lambda_1$ of surfaces in symplectic category. In particular we prove that every split symplectic manifold $T^4 \times M$ admits a compatible Riemannian metric whose first eigenvalue is arbitrary large. On the other hand for Kähler metrics compatible with a given integral symplectic form an upper bound for $\lambda_1$ does exist. The second theme is the study of Hamiltonian symplectic fibrations over $S^2$. We construct a numerical invariant called the size of a fibration which arises as the solution of certain variational problems closely related to Hofer’s geometry, K-area and coupling. In some examples it can be computed with the use of Gromov-Witten invariants. The link between these two themes is given by an observation that the first eigenvalue of a Riemannian metric compatible with a symplectic fibration admits a universal upper bound in terms of the size.

1. Introduction and main results

1.1 Motivation and an overview

Let $M$ be a smooth closed manifold endowed with some ”geometric” structure. In many cases this structure determines in a canonical way a class $\mathcal{R}$ of Riemannian metrics on $M$. Traditionally, one asks the following

Question. Does there exist a universal upper bound $\lambda_1(M, g) \leq C$ for all $g \in \mathcal{R}$?

Here and below we write $\lambda_1(M, g)$ for the first positive eigenvalue of the Laplacian of $(M, g)$ acting on functions. Let us give several examples.

1.1.A Topology (no additional structure on $M$). Here $\mathcal{R}$ is the class of all metrics on $M$ whose sectional curvatures at each point belong to $[-1; 1]$. In this

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case existence of such a universal bound leads to some non-trivial topological restrictions on \( M \) (Wu [W], 1995).

1.1.B Surfaces endowed with an area form. Let \( \mathcal{R} \) be the class of all metrics on an orientable closed surface \( M \) with a given area form. Then for all \( g \in \mathcal{R} \) holds
\[
\lambda_1(M, g) \leq \frac{8\pi (\text{genus}(M) + 1)}{\text{Area}(M)}.
\]
This was proved by Hersch [He], 1970 for \( M = S^2 \) and by Yang and Yau [Y-Y], 1980 for surfaces of higher genus.

The beauty of this result inspired many attempts to find its multi-dimensional analogue (see Gromov [G1], 1993 for a stimulating discussion). The difficulty here is to recognize "to which category" belongs this estimate. Indeed, in dimension 2 one cannot distinguish between a volume form and a symplectic structure, and between Riemannian and Kähler geometry. Some important developments in this direction are reflected in the next two examples.

1.1.C Higher-dimensional manifolds endowed with a volume form. Let \( \mathcal{R} \) be the class of all Riemannian metrics on \( M \) with a given volume form. In 1993 Colbois and Dodziuk [C-D] proved that \( \sup_{g \in \mathcal{R}} \lambda_1(M, g) = \infty \) for every \( M \) of dimension \( \geq 3 \).

1.1.D Complex manifolds with a distinguished cohomology class. Let \( \mathcal{R} \) be the class of all Kähler metrics on a complex manifold \( M \) whose Kähler form represents a given cohomology class. In various interesting cases a universal upper bound for \( \lambda_1(M, g), g \in \mathcal{R} \) exists and is related to algebraic-geometric properties of \( M \). This was discovered by Li and Yau [L-Y] in 1982, and by Bourguignon, Li and Yau [B-L-Y] in 1994 (see 1.2, 1.5 and 4.1 below for more details).

In the present paper we study non-existence/existence of universal upper bounds for \( \lambda_1 \) in the symplectic context, which is a natural generalization of an area form on an oriented surface. We fix a closed symplectic manifold \((M, \Omega)\) and consider certain classes \( \mathcal{R} \) of compatible Riemannian metrics, that is metrics \( g \) of the form
\[
g(\xi, \eta) = \Omega(\xi, J\eta), \quad \xi, \eta \in TM,
\]
where \( J \) is an almost complex structure on \( M \).

At the beginning we focus on two cases: \( \mathcal{R} \) consists of all compatible metrics, and \( \mathcal{R} \) consists of Kähler compatible metrics (that is metrics associated to integrable \( J \)'s). In a number of examples we establish "flexibility" (that is \( \sup \lambda_1 = \infty \)) in the first case (see 1.2.A below) vs. "rigidity" (\( \sup \lambda_1 < \infty \)) in the second one (see 1.2.B below). Let us mention that our point of view on the Kähler case is different from one in 1.1.D: we fix \( \Omega \) and vary \( J \), while in 1.1.D the complex structure \( J \) is fixed and the compatible symplectic form \( \Omega \) varies in a given cohomology class.
However, this is not the end of the story. It turns out that there exists an intermediate (with respect to "degree of integrability" of $J$’s) category where in many situations an upper bound for the first eigenvalue does exist, namely symplectic fibrations over $S^2$ endowed with a class of Riemannian metrics which satisfy a natural compatibility condition (see 1.4 and 1.5 below). Our upper bound for $\lambda_1$ of a symplectic fibration turns out to be related to such objects of Symplectic Topology as Hofer’s metric, K-area and coupling (see 1.9 below). In this aspect the present research continues the study of "hard" numerical invariants of symplectic fibrations initiated in [P1-P3].

1.2 $\lambda_1$ for a symplectic manifold

Let $(M, \Omega)$ be a symplectic manifold. Set $\lambda_1(M, \Omega) = \sup_g \lambda_1(M, g)$, where $g$ runs over all compatible metrics. As we have seen in 1.1.B the quantity $\lambda_1(M, \Omega)$ is finite when $M$ is a closed surface. Our first result states that at least after suitable stabilization this property disappears and hence the situation becomes flexible as in 1.1.C.

More precisely, consider the 4-torus $T^4 = \mathbb{R}^4 / \mathbb{Z}^4$ endowed with the standard symplectic form $\sigma = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, where $(p_1, q_1, p_2, q_2)$ are coordinates on $\mathbb{R}^4$.

**Theorem 1.2.A.** Let $(M, \Omega)$ be a closed symplectic manifold. Then

$$\lambda_1(T^4 \times M, \sigma \oplus \Omega) = \infty.$$  

Notice that this result remains true when $M$ is a point. The proof which is given in section 2 below is based on the construction of an auxiliary hypoelliptic operator in the spirit of [B-B]. It would be interesting to understand to which extent the stabilization is important.

The situation changes drastically when one considers a smaller class of Kähler metrics whose Kähler class equals to $\Omega$.

**Theorem 1.2.B.** Let $(M, \Omega)$ be a closed symplectic manifold of real dimension $2n$ such that the cohomology class of $\Omega$ is integral: $[\Omega] \in H^2(M, \mathbb{Z})$. Then for every compatible Kähler metric $g$ on $M$ holds

$$\lambda_1(M, g) \leq \text{constant}(n + 2 - \frac{(c_1(TM) \cup [\Omega]^{n-1}, [M])}{([\Omega]^n, [M])}),$$

where the constant depends only on $n$.

The proof which is a simple combination of two known results goes as follows. Given a Kähler projective manifold, one can estimate $\lambda_1$ in terms of its degree [B-L-Y]. On the other hand, there exists an upper bound for the projective degree of an integral Kähler manifold in purely cohomological terms. This bound which can be considered as an effective version of the classical Kodaira embedding...
Theorem was established by Demailly [De]. The details of this argument can be found in section 4.

Remark 1.2.C Clearly, 1.2.B automatically implies an upper bound for symplectic manifolds whose symplectic form represents a rational cohomology class, that is \([\Omega] \in \frac{1}{N} H^2(M, \mathbb{Z})\), where \(N\) is an integer. However this upper bound goes to infinity with \(N\). Therefore the approximation argument does not work, and the situation in the case when \([\Omega]\) is irrational remains completely unclear even for simplest manifolds (for instance for the 4-dimensional torus).

1.3 Symplectic fibrations over \(S^2\)

Let \((M, \Omega)\) be a closed symplectic manifold. A fibration \(p : P \to S^2\) with fiber \(M\) over an oriented 2-sphere is called symplectic if each fiber \(p^{-1}(x)\) is endowed with a symplectic form \(\Omega_x\) which depends smoothly on the base point \(x\), and is symplectomorphic to \(\Omega\). Such fibrations which were introduced in [G-L-S] appear in various problems of Symplectic Topology, in particular in the study of loops on the group of symplectomorphisms. We are going now to explain this link in more details.

Let \(\{f_t\}, t \in S^1, f_0 = f_1 = 1\) be a loop of symplectic diffeomorphisms of \((M, \Omega)\) which represents an element \(\gamma \in \pi_1(\text{Symp}(M, \Omega), 1)\). Let \(D_+\) and \(D_-\) be two copies of the disc \(D^2\) bounded by \(S^1\). Consider a map \(\Psi : M \times S^1 \to M \times S^1\) given by \((z, t) \to (f_t z, t)\). Define now a new manifold \(P(\gamma) = (M \times D_-) \cup_{\Psi} (M \times D_+)\).

It is clear that \(P\) has the canonical structure of a symplectic fibration over \(S^2\) which depends only on the element \(\gamma\). In what follows we assume that the base \(S^2\) is oriented, and the orientation comes from \(D_+\). Note that the choice of an orientation on \(S^2\) is equivalent to the choice of an orientation on \(P\).

Clearly, this construction can be reversed. Namely, given a symplectic fibration \(p : P \to S^2\) over an oriented 2-sphere with the fiber \((M, \Omega)\), one can reconstruct the corresponding element \(\gamma \in \pi_1(\text{Symp}(M, \Omega))\) up to conjugation. For that purpose, fix a point, say \(x\), on \(S^2\) and consider a loop \(\alpha(s), s \in S^1\) in the space \(L S^2\) of all loops on \(S^2\) based at \(x\). Assume that \(\alpha(0)\) is the constant loop, and that \(\alpha\) represents a positive generator in \(\pi_1(L S^2) = \pi_2(S^2)\) (here we use the choice of an orientation on \(S^2\)). Choose a symplectic connection on \(P\) (we refer the reader to [G-L-S], [McD-S] for basic notions of the theory of symplectic connections). Then the holonomies of this connections over loops \(\alpha(s)\) form a loop in the group of symplectomorphisms of the fiber \(P_x\) over \(x\), which defines the needed element \(\gamma\).

Throughout this paper, we restrict our attention to the subgroup \(\text{Ham}(M, \Omega) \subset \text{Symp}(M, \Omega)\) of all Hamiltonian diffeomorphisms of \((M, \Omega)\). In particular we consider only Hamiltonian loops and corresponding symplectic fibrations, which we call Hamiltonian symplectic fibrations.
fact that there exists a cohomology class on the total space whose restriction to fibers coincides with the class of the symplectic form (see [Se]). Notice that the difference between "symplectic" and "Hamiltonian" disappears when the manifold $M$ is simply connected.

1.4 $\lambda_1$ of a symplectic fibration over $S^2$

Let $p : P \to S^2$ be a Hamiltonian symplectic fibration. A pair $(\omega, j)$ consisting of a symplectic form $\omega$ on $P$ and an almost complex structure $j$ on $P$ is called a compatible quasi-Kähler structure if the following holds:

(i) The form $\omega(\xi, j\eta), \xi, \eta \in TP$ is a Riemannian metric on $P$;

(ii) The restriction of $\omega$ to fibers coincides with the symplectic structure on fibers;

(iii) The projection $p$ is $(j, i)$-holomorphic for some complex structure $i$ on $S^2$ which respects the orientation.

We say that a Riemannian metric $g$ on $P$ is compatible if it comes from a compatible quasi-Kähler structure.

A natural way to construct compatible quasi-Kähler structures on $P$ is as follows. Fix a field $J_x, x \in S^2$ of almost complex structures on the fibers of $P$ which are compatible with symplectic forms on the fibers. Pick up a complex structure $i$ on $S^2$. Let $\omega$ be a symplectic form on $P$ which extends symplectic structures on the fibers and defines the positive orientation on $P$. The field of $\omega$-orthogonal subspaces to the fibers defines a connection on $P$, that is a splitting $T(x)P = T_zP_x \oplus T_xS^2$ where $x \in S^2, P_x$ is a fiber over $x$ and $z \in P_x$. Consider an almost complex structure $j = J_x \oplus i$ on $P$. Clearly, $(\omega, j)$ satisfies (i) - (iii) above.

Now we are ready to define a spectral invariant of a symplectic fibration. Namely, set

$$\lambda_1(P) = \sup_g \lambda_1(P, g),$$

where the supremum is taken over all compatible Riemannian metrics on $P$.

It turns out that in some interesting examples this quantity is finite. Consider for instance the case $(M, \Omega) = (\mathbb{C}P^{n-1}, \Omega_{st})$, where $\Omega_{st}$ is the Fubini-Study form normalized in such a way that the integral over a projective line equals to 1. Let $\gamma_{q,n} \in \pi_1(\text{Ham}(\mathbb{C}P^{n-1}, \Omega_{st}))$ be the element represented by the $S^1$-action

$$f_{q,n} : (z_1 : \ldots : z_n) \to (e^{2\pi i t} z_1 : \ldots : e^{2\pi i t} z_q : z_{q+1} : \ldots : z_n),$$

where $q \in \{1, \ldots, n-1\}$. Let $P_{q,n}$ be the corresponding symplectic fibration.

**Theorem 1.4.A.** The following inequality holds:

$$\lambda_1(P_{q,n}) \leq \frac{8\pi n}{n - q}.$$
Moreover, this estimate is stable in the following sense. Let \((T^4, \sigma)\) be the standard symplectic 4-torus as in 1.2. Take the product \(\tilde{P}_{q,n} = T^4 \times P_{q,n}\) which we consider as a symplectic fibration over \(S^2\) with the fiber

\[(T^4 \times \mathbb{CP}^{n-1}, \sigma \oplus \Omega_{st}).\]

**Theorem 1.4.A’.** The following inequality holds:

\[\lambda_1(\tilde{P}_{q,n}) \leq \frac{8\pi n}{n - q}.\]

The proof of both estimates is given in 1.9 below. It seems to be a difficult problem to compute this invariant even in simple examples. Consider for instance the simplest case of the trivial symplectic fibration with the fiber \((M, \Omega)\), that is \(P_{\text{trivial}} = M \times S^2\). Clearly, in this situation we can consider split compatible metrics, and making the area of the base \(S^2\) arbitrarily small we get that

\[\lambda_1(P_{\text{trivial}}) \geq \lambda_1(M, \Omega).\]

Applying results of 1.2 we see that \(\lambda_1\) is infinite for trivial fibrations with a stabilized fiber of the form \(T^4 \times M\). Comparing this with 1.4.A’ above we come to the following conclusion:

**Theorem 1.4.A’.** The following inequality holds:

\[\lambda_1(P_{q,n}) \leq \frac{8\pi n}{n - q} < 4\pi n \leq \lambda_1(P_{\text{trivial}}),\]

and hence our spectral invariant distinguishes \(P_{q,n}\) from the trivial fibration.

**Remark 1.4.B.** It is instructive to compare 1.4.A,A’ above with the results of 1.2. As we have seen in 1.2.B the integrability of a compatible almost complex structure imposes a global restriction on the geometry, namely it prevents the first eigenvalue from being arbitrary large. The almost complex structures considered in the present section are ”integrable in one direction”, namely by definition they admit a holomorphic mapping to \(\mathbb{CP}^1\). The message of theorems 1.4.A,A’ above (and of the more general statement 1.5.B below) is that this small amount of integrability is still sufficient for the global geometric restriction of the same nature. However theorem 1.2.A shows that it disappears if one considers general non-integrable almost complex structures.
1.5 The Li-Yau estimate revisited.

The first step towards getting an upper bound for $\lambda_1(P)$ is the following minor generalization of a theorem by Li and Yau [L-Y]. Let ($\omega, j$) be a compatible quasi-Kähler structure on a Hamiltonian symplectic fibration $P$ with a fiber $(M, \Omega)$, and let $g$ be the corresponding Riemannian metric. Set

$$ s(\omega) = \frac{\text{Volume}(M, \Omega)}{\text{Volume}(P, \omega)}. $$

Then $\lambda_1(P, g) \leq 8\pi s(\omega)$. The proof of this inequality goes exactly as in [L-Y, Theorem 3] (let us mention that our normalization conventions are slightly different from the ones in [L-Y]). Note that in our situation $j$ is not assumed to be integrable on the fibers, however this does not play any role.

**Definition 1.5.A** Let $P \to S^2$ be a Hamiltonian symplectic fibration. Define its size as follows:

$$ \text{size}(P) = \sup_\omega s(\omega), $$

where $\omega$ runs over all symplectic forms on $P$ which extend the symplectic structures on the fibers and define the positive orientation.

With this language our previous discussion leads to the next statement.

**Theorem 1.5.B.** The following inequality holds:

$$ \lambda_1(P) \leq 8\pi \text{size}(P). $$

Now we face the following purely symplectic question. Given a symplectic fibration, is its size finite? It turns out that $\text{size}(P)$ admits three other equivalent definitions in more familiar terms. Namely, one can describe it using Hofer’s geometry, K-area and coupling (see 1.9.A below). Having in mind this description, we get the finitness of the size (and in fact compute it explicitly) in a number of examples using results of [P1].

1.6 Hofer’s geometry

Given a path $\{f_t\}$ in $\text{Ham}(M, \Omega)$ ($t \in [0; 1]$), define its length as follows. Let $F : M \times S^1 \to \mathbb{R}$ be the associated Hamiltonian function normalized so that the mean value of $F_t$ over $M$ vanishes for all $t \in S^1$. Then $\text{length}\{f_t\} = \max|F|$. With this language the distance between two Hamiltonian diffeomorphisms is just the minimal possible length of a path between them (see [H-Z], [McD-S] for an introduction to Hofer’s geometry). One can show that this distance, which was called coarse Hofer’s distance in [P1], in fact coincides with the one associated to the ”maximum of the absolute value” norm on the Lie algebra of all normalized Hamiltonian functions.

In the present paper, we deal mainly not with the Hofer’s length itself, but with its positive and negative parts (see [E-P, 4.4]). Namely set $\text{length}_+\{f_t\} = \max F$ and $\text{length}_-\{f_t\} = -\min F$. 7
For an element $\gamma \in \pi_1(\text{Ham})$ define its norm as

$$||\gamma|| = \inf \text{length}\{f_t\},$$

where the infimum is taken over all loops $\{f_t\}$ with $f_0 = f_1 = \mathbb{1}$ representing $\gamma$. Define also its positive and negative parts as

$$||\gamma||_+ = \inf \text{length}_+\{f_t\},$$

and

$$||\gamma||_- = \inf \text{length}_-\{f_t\}.$$ 

Clearly, $||\gamma|| \geq \max(||\gamma||_+, ||\gamma||_-)$. Interestingly enough, in all known examples in fact the equality take place, however we cannot prove this in general situation.

Note also that $||\gamma||_- = ||\gamma^{-1}||_+$ (see 1.7.A below for further discussion about this duality).

### 1.7 K-area of symplectic fibrations

Let $p : P \to S^2$ be a Hamiltonian symplectic fibration with fiber $(M, \Omega)$. A connection $\nu$ on $P$ is called symplectic if the parallel transport preserves symplectic forms on fibers. The curvature $\rho^\nu$ of $\nu$ is a 2-form on the base which take values in the Lie algebra of the group of symplectic diffeomorphisms of a fiber, that is in the space of locally Hamiltonian vector fields. A symplectic connection is called Hamiltonian if its curvature takes values in Hamiltonian vector fields. In this case we make a natural identification and consider $\rho^\nu(\xi, \eta)$ as a Hamiltonian function on the fiber with zero mean value. We call such Hamiltonian functions normalized. It is not hard to show that a symplectic fibration is Hamiltonian in the sense of 1.3 if and only if it admits a Hamiltonian symplectic connection.

Given an area form, say $\tau$ on $S^2$, one can write $\rho^\nu = L^\nu \tau$, where $L^\nu$ is a function on $P$.

Fix an area form $\tau$ on $S^2$ of the total area 1 which respects the orientation (recall, that $S^2$ is always assumed to be oriented). Define the K-area of $P$ as follows ([P1], cf. [G3]):

$$\chi(P) = \sup_{\nu} \frac{1}{\max_P |L^\nu|},$$

where the supremum is taken over all Hamiltonian symplectic connections $\nu$ on $P$. Notice that the definition in [P1] contained an additional multiple $\frac{1}{2\pi}$.

In what follows we shall consider also positive and negative parts of the K-area:

$$\chi_+(P) = \sup_{\nu} \frac{1}{\max_P L^\nu}$$
and

$$\chi_-(P) = \sup_{\nu} \frac{1}{-\min_P L^\nu}.$$ 

Note that $\chi(P) \leq \min(\chi_-(P), \chi_+(P))$, and in all known examples in fact the equality holds. Note also that all three quantities do not depend on the specific choice of the area form $\tau$.

**Remark 1.7.A (Duality).** Let $P \to S^2$ be a Hamiltonian symplectic fibration over an oriented 2-sphere associated to an element $\gamma \in \pi_1(\text{Ham}(M, \Omega))$. Change the orientation of the base $S^2$ and denote the new object by $P^*$. Clearly $P^*$ as a Hamiltonian symplectic fibration over the oriented 2-sphere is associated with the element $\gamma^{-1}$. Note that the following identity holds: $\chi_-(P) = \chi_+(P^*)$.

1.8 Coupling

Let $p : P \to S^2$ be a symplectic fibration associated to an element $\gamma \in \pi_1(\text{Ham}(M, \Omega))$. Denote by $a$ the positive generator of $H^2(S^2, \mathbb{Z})$. Let $c \in H^2(P, \mathbb{R})$ be the unique class whose restriction on fibers coincides with $[\Omega]$ and whose top power vanishes. Following [G-L-S] we call $c$ the coupling class of $P$.

The weak coupling construction [G-L-S] prescribes that for a sufficiently small $\varepsilon > 0$ there exists a smooth family of closed 2-forms $\{\omega_t\}$, $t \in [0; \varepsilon)$ on $P$ with the following properties:

(i) $\omega_0$ is the lift of an area form on $S^2$ (the total area equals to 1);

(ii) $[\omega_t] = tc + [p^*a]$;

(iii) the restriction of $\omega_t$ to each fiber of $P$ coincides with a multiple of the symplectic form on the fiber;

(iv) $\omega_t$ is symplectic for $t > 0$.

Following [P1], define $\epsilon(P)$ as the supremum of all such $\epsilon$ (this invariant measures how strong the weak coupling can be).

1.9 On the size of a symplectic fibration

Now we are ready to state our main result on the size of a symplectic fibration.

**Theorem 1.9.A.** Let $P \to S^2$ be a symplectic fibration associated to an element $\gamma \in \pi_1(\text{Ham}(M, \Omega))$. The following identity holds:

$$\text{size}(P) = \epsilon(P) = \chi_+(P) = \frac{1}{||\gamma||_+}.$$ 

The proof is given in section 3 below.

**Remark 1.9.B** We will prove also that $\chi(P) = \frac{1}{||\gamma||_+}$ (see 3.1.B; note that the inequalities $\chi(P) \geq \frac{1}{||\gamma||_+}$ and $\chi(P) \leq \epsilon(P)$ were proved in [P1]). This identity in a sense reflects a principle of the Yang-Mills theory that the critical values of the Yang-Mills functional on a given $G$-bundle over $S^2$ correspond to the
lengths of closed geodesics on the Lie group $G$, see [A-B], [Gra]. Our proof in 3.3 is somewhat similar to the argument in [Gra].

Clearly, the size of the trivial fibration is infinite. The general phylosophy of Hofer’s geometry suggests that $||\gamma||_+$ is strictly positive for non-trivial elements $\gamma$, and hence one can hope that non-trivial symplectic fibrations has finite size. In a number of situations this was confirmed in [P1], [P2] with the use of Gromov-Witten invariants. In particular, one can use these results in order to prove 1.4.A and 1.4.A’.

**Proof of 1.4.A and 1.4.A**: We use notations of 1.4. It was shown in [P1] that $\epsilon(P_{q,n}) = \frac{n}{n-q}$. Moreover, the argument of [P1] remains valid under stabilization, thus $\epsilon(\tilde{P}_{q,n}) = \frac{n}{n-q}$. Thus 1.9.A implies that

$$\text{size}(P_{q,n}) = \text{size}(\tilde{P}_{q,n}) = \frac{n}{n-q}.$$  

The desired statements follow immediately from 1.5.B.

The rest of the paper is organized as follows. In section 2 we prove Theorem 1.2.A, in section 3 we prove Theorem 1.9.A, and in section 4 we prove Theorem 1.2.B.

2. Collapsing symplectic manifolds

In this section we prove theorem 1.2.A (see 2.2 below).

2.1 A deformation of an almost complex structure.

Recall that a finite set of vector fields on a manifold $M$ satisfies Hörmander condition if these fields together with their iterated brackets generate $TM$ at every point. A distribution (that is a field of tangent subspaces) on $M$ satisfies Hörmander condition if locally it can be generated by such vector fields.

**Theorem 2.1.A.** Let $(M, \Omega)$ be a closed symplectic manifold which admits an isotropic distribution satisfying Hörmander condition. Then $\lambda_1(M, \Omega) = \infty$.

**Proof:** Let $J$ be an arbitrary compatible almost complex structure on $M$. Denote by $g$ the corresponding Riemannian metric. Let $L$ be an isotropic distribution which satisfies Hörmander condition. Consider the following $g$-orthogonal decomposition:

$$TM = L \oplus JL \oplus V.$$  

Define a family of compatible almost complex structures $J_t$ on $M$ as follows: $J_t = t^{-1}J$ on $L$, $J_t = tJ$ on $JL$, and $J_t = J$ on $V$. Notice that this deformation somewhat reminds the Teichmüller deformation on surfaces.

Let $g_t = t^{-1}g \oplus tg \oplus g$ be the Riemannian metric associated with $J_t$. It suffices to show that $\lambda_1(M, g_t)$ goes to infinity when $t \to \infty$.  

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In order to prove this, choose vector fields $X_1, ..., X_N$ which are tangent to $L$ and satisfy Hörmander condition. Denote by $D_j$ the Lie derivative along $X_j$, and by $D_j^*$ its $L^2$-adjoint. (The $L^2$-scalar product on functions is associated to the canonical volume form on $M$). Set $D = \sum_{j=1}^N D_j^* D_j$. The Hörmander theory [H] implies that $D$ has a discrete spectrum. Moreover, all eigenvalues are non-negative, and the zero eigenspace consists of constant functions only.

Let $H$ be the space of smooth functions on $M$ with the zero mean. The previous discussion implies that $(Df, f) \geq C(f, f)$ for some positive constant $C$ and all $f \in H$. Denote by $\| \cdot \|_t$ and $\nabla_t$ the length on vectors and the gradient in the sense of the metric $g_t$ respectively. Choose a constant $K > 0$ such that $g(X_j, X_j) \leq K$ at every point for all $j = 1, ..., N$. Then

$$\begin{align*}
(Df, f) &= \sum_{j=1}^N \int_M (D_j f)^2 d\text{Vol} \\
&\leq \Sigma_{j=1}^N \int_M [\nabla_t f]^2_t [X_j]^2_t d\text{Vol} \\
&\leq NK^{-1} \int_M [\nabla_t f]^2_t d\text{Vol}.
\end{align*}$$

Thus for some positive constant $C_1$ and for every $f \in H$ holds

$$\int_M [\nabla_t f]^2_t d\text{Vol} \geq C_1 t \int_M f^2 d\text{Vol}.$$ 

Therefore $\lambda_1(M, g_t) \to \infty$ when $t \to \infty$. This completes the proof.

**Remark 2.1.B.** The idea of a collapse to a non-integrable distribution was extensively discussed in the literature on the first eigenvalue, see e.g. [B-B], [Fu], [Ge].

**2.2 How to construct "generic" isotropic distributions?**

In general, existence of an isotropic distribution imposes topological restrictions on $M$. However this difficulty disappears after stabilization.

**Proposition 2.2.A.** Let $(M, \Omega)$ be a closed symplectic manifold. Then the stabilized manifold $P = (T^4 \times M, \sigma \oplus \Omega)$ admits a field of isotropic 2-planes which satisfies Hörmander condition.

**Proof of 2.2.A:** The theorem follows immediately from 2.1.A and 2.2.A.

**Proof of 2.2.A:** Clearly $P$ admits a 2-dimensional isotropic foliation. A discussion in [G2] suggests that its generic perturbation satisfies Hörmander condition. We will use an explicit argument.

1) Let $k$ be a positive integer. Consider $2k$ functions $\phi_1(s), ..., \phi_{2k}(s)$ defined as follows: $\phi_{2j-1}(s) = \sin js$ and $\phi_{2j}(s) = \cos js$. Denote by $\phi^{(i)}$ the $i$-th derivative of a function $\phi$. We claim that at every point $s$ the $2k \times 2k$ matrix
$(\phi_j^{(i)})$ is invertible. This is so since our functions form a basis of solutions of a $2k$-th order differential equation

$$\Pi_j^{k}(\frac{\partial^2}{\partial s^2} + j^2)\phi = 0.$$  

2) Let $(p_1, q_1, p_2, q_2)$ be coordinates on $T^4$ such that the standard symplectic form is written as $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Define an auxiliary set of vector fields $Z_j$ on $P$ as follows. Set $Z_1 = \frac{\partial}{\partial p_1}$ and $Z_2 = \frac{\partial}{\partial p_2}$. Let $Z_3, ..., Z_{2k}$ be vector fields on $M$ which span $TM$ at every point (here $k$ is as large as needed). Finally, set

$$X_1 = \frac{\partial}{\partial q_1} + \sin q_1 \frac{\partial}{\partial p_2},$$

and

$$X_2 = \frac{\partial}{\partial q_2} + \sum_{j=1}^{2k} \phi_j(q_1)Z_j.$$

Clearly $X_1$ and $X_2$ generate an isotropic distribution on $P$. Let us verify that it satisfies Hörmander condition. For that purpose denote $D = [X_1, \cdot]$. Then

$$D^m X_2 = \sum_{j=1}^{2k} \phi_j^{(m)}(q_1)Z_j.$$

It follows from the step 1 of the proof that the span of vector fields

$$DX_2, ..., D^{2k}X_2$$

at each point contains vector fields $Z_1, ..., Z_{2k}$. Therefore together with $X_1$ and $X_2$ these fields generate the whole $TP$. This completes the proof. \qed

3. K-area, coupling and Hofer’s norm

3.1 Two propositions

Let $p : P \to S^2$ be a Hamiltonian symplectic fibration with the fiber $(M, \Omega)$. Theorem 1.9.A splits to the following two statements.

**Proposition 3.1.A.** The following identity holds: $\chi_+(P) = \text{size}(P) = \epsilon(P)$.  

**Proposition 3.1.B.** Suppose that the fibration $P$ is associated to an element $\gamma \in \pi_1(\text{Ham}(M, \Omega))$. Then the following identities hold: $\chi_+(P) = \frac{1}{||\gamma||}$, and $\chi(P) = \frac{1}{||\gamma||}$.  

The proofs are given in 3.2 and 3.3 respectively.
3.2 The coupling form

Let $n$ be the complex dimension of $P$. Recall [G-L-S],[McD-S] that the coupling form $\delta^\nu$ of a Hamiltonian symplectic connection $\nu$ is the unique closed 2-form on $P$ whose restriction to each fiber coincides with the symplectic form, which defines the connection $\nu$ and such that the image of its top power under the fiber integration map vanishes. Here the fiber integration is a map, say $FI$ which maps 2$n$-forms on $P$ to 2-forms on $S^2$ as follows:

$$FI(\beta)(\xi, \eta) = \int_{P_x} i(\tilde{\xi} \wedge \tilde{\eta})\beta,$$

where $\xi, \eta$ are tangent vectors to $S^2$ at a point $x$, and $\tilde{\xi}, \tilde{\eta}$ denote their (horizontal) lifts to the points of the fiber $P_x$ over $x$.

It is proved in [G-L-S],[McD-S] that for all $z \in P_x$ the coupling form can be written as $\delta^\nu = \Omega -\rho^\nu(z)$, where $TP = TP_x \oplus TS^2$ is the splitting associated to the connection $\nu$, and $\rho^\nu(z)$ means the evaluation of the curvature form at the point $z$ (remember that $\rho^\nu$ is a function-valued form). Let us mention also, that this formula is cited in [P1, proof of 1.6.B] with the opposite sign at $\rho^\nu$.

The sign convention. The Hamiltonian vector field $X$ generated by a Hamiltonian function on $M$ is given by $dH = -i_X\Omega$.

Denote by $a$ the positive generator of $H^2(S^2, \mathbb{Z})$. The proof of 3.1.A,B is based on the following simple fact.

**Lemma 3.2.A.** Let $\omega$ be a symplectic form on $P$ whose restriction to fibers coincides with a a multiple of the symplectic form and whose cohomology class equals to $p^*a + uc$ for some $u > 0$. Let $\nu$ be the symplectic connection defined by $\omega$-orthogonal complements to the fibers. Then $\omega - u\delta^\nu = p^*\tau$, where $\tau$ is an area form on $S^2$ with the total area 1.

**Proof:** Set $r = nu^{n-1}\int_M \Omega^{n-1}$, and define a 2-form $\tau$ on $S^2$ by

$$\tau = \frac{1}{r}FI(\omega^n).$$

Notice that the image of every volume form, say $\beta$ on $P$ under the fiber integration is always an area form on $S^2$. Indeed for each positive frame $(\xi, \eta)$ on $S^2$ the form $i(\xi \wedge \eta)\beta$ does not vanishes on the tangent space to the corresponding fiber. Thus $\tau$ is an area form.

It easily follows from the definition of $FI$ above that $FI((\omega - p^*\tau)^n) = 0$. In particular $(p^*a + uc - p^*[\tau])^n = 0$, thus the total area of $\tau$ equals to 1. Moreover, the form $u^{-1}(\omega - p^*\tau)$ satisfies the definition of the coupling form of $\nu$. This completes the proof.

$\square$
Proof of 3.1.A:

1) Let $\omega$ be an arbitrary symplectic form on $P$ which extends symplectic structures on the fibers and defines the positive orientation. Then $[\omega] = u^* a + c$ for some $u > 0$. Write $u^{-1}\omega = p^* \tau + u^{-1} \delta'$, where the area form $\tau$ and the connection $\nu$ are defined in 3.2.A. Decompose $\delta'$ as $\Omega \oplus -L^\nu \tau$. Then $u^{-1}\omega = \Omega \oplus (1 - u^{-1}L^\nu) \tau$. Since this form is symplectic, we get that $\frac{1}{\max L^\nu} > u^{-1}$.

Note now that $s(\omega) = u^{-1}$. Therefore $\chi_+(P) \geq \text{size}(P)$.

2) Take an arbitrary deformation of symplectic forms satisfying (i) - (iv) of 1.8, and take one of these forms $\omega_u$. Set $\omega = u^{-1}\omega_u$. Then $\omega$ belongs to the class of symplectic forms which is used for the definition of the size. Since $[\omega] = c + u^{-1} p^* a$ we compute that $\text{Vol}(P, \omega) = u^{-1} \text{Vol}(M, \Omega)$, thus $s(\omega) = u$. Taking $u$ arbitrarily close to $\epsilon(P)$ we conclude that $\text{size}(P) \geq \epsilon(P)$.

3) Let $\nu$ be a Hamiltonian symplectic connection on $P$. Assume that for some area form $\tau$ of the total area $1$ holds $\frac{1}{\max L^\nu} = r$. Writing $\delta' = \Omega \oplus -L^\nu \tau$ we see that the deformation $u\delta' + \tau$ satisfies the conditions (i)-(iv) of 1.8 provided $u < r$. Hence $\epsilon(P) \geq r$, and taking $r$ arbitrarily close to $\chi_+(P)$ we get the estimate $\epsilon(P) \geq \chi_+(P)$. Together with the results of the previous steps this completes the proof.

3.3 Computing holonomy of a symplectic connection

Here we prove 3.1.B. The inequality $\chi(P) \geq \frac{1}{||\gamma||}$ was proved in [P1]. Exactly the same argument gives that $\chi_+(P) \geq \frac{1}{||\gamma_+||}$. (One should be careful with the sign conventions at this point!) So it remains to prove the opposite inequalities. In what follows we will represent the 2-sphere $S^2$ as the unite square $K = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$ whose all boundary points are identified. We consider Hamiltonian symplectic fibrations over $K$ trivialized over the boundary, and Hamiltonian symplectic connections with trivial parallel transport along every segment of the boundary. Let $P$ be such a fibration, and $\nu$ be such a connection on $P$. We write $M$ for the fiber over the boundary.

Fix an area form $\tau = dx \wedge dy$ on $K$. Assume that the curvature of $\nu$ is given by $L(x, y, z) \tau$, where $(x, y) \in K$ and $z$ belongs to the fiber over $(x, y)$.

Consider a family of paths on $K$ given in $(x, y)$-coordinates by $\alpha(s) = \{(s, t)|t \in [0; 1]\}$, where $s \in [0; 1]$. Note that $\{\alpha(s)\}$ represents a positive generator of $\pi_1(LS^2)$ (here one takes the boundary of $K$ as the base point on $S^2$). Denote by $f_s : M \to M$ the holonomy of $\nu$ along $\alpha(s)$. Clearly, $\{f_s\}$ is a loop of Hamiltonian diffeomorphisms of $M$. The needed inequalities follow immediately from the next statement.
Lemma 3.3.A. The following inequalities hold:

\[ \text{length}_{+}\{ f_s \} \leq \max L \]

and

\[ \text{length}\{ f_s \} \leq \max |L|. \]

Proof: Let \( X, Y \) be the horizontal lifts of the vector fields \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) respectively. Let \( X^s, Y^t \) be the flows of \( X \) and \( Y \) respectively. Set \( h_{t,s} = Y^t X^s \) (warning: such a map is defined on a domain which depends on \( t \) and \( s \)).

Note that

\[(s, 1, f^s(z)) = Y^1 X^s(0, 0, z) = h_{1,s}(0, 0, z).\]

Our goal is to calculate the vector field

\[ v = \frac{\partial h_{1,s}}{\partial s}(0, 0, z). \]

Note that

\[ \frac{\partial h_{t,s}}{\partial t} = Y, \quad \frac{\partial h_{t,s}}{\partial s} = Y^t X, \]

and therefore

\[
\frac{\partial}{\partial t} \frac{\partial h_{t,s}}{\partial s} = \frac{\partial}{\partial s} \frac{\partial h_{t,s}}{\partial t} - \left[ \frac{\partial h_{t,s}}{\partial t}, \frac{\partial h_{t,s}}{\partial s} \right] = -[Y, Y^t X] = -Y^t [Y, X].
\]

Notice that \([X, Y]\) is a vertical vector field which equals to the Hamiltonian field \( \text{sgrad}L \), thus

\[ \frac{\partial}{\partial t} \frac{\partial h_{t,s}}{\partial s} = \text{sgrad}(L \circ Y^{-t}). \]

Taking into account that

\[ \frac{\partial h_{0,s}}{\partial s}(0, 0, z) = \frac{\partial}{\partial x}, \]

we get that

\[ v = \frac{\partial}{\partial x} + \text{sgrad} \int_0^1 L \circ Y^{-t} dt, \]

and therefore the Hamiltonian function generating the path \( \{ f_s \} \) is given by

\[ F(s, z) = \int_0^1 L(s, 1 - t, Y^{-t}(s, 1, z)) dt. \]

Clearly, \( \max F \leq \max L \) and \( \max |F| \leq \max |L| \). This completes the proof of the lemma, and hence of the proposition 3.1.B. \( \Box \)
4. Estimating $\lambda_1$ on Kähler manifolds

In this section we prove Theorem 1.2.B. Below $a$ denotes the positive generator of $H^2(\mathbb{CP}^N, \mathbb{Z})$. We start with two auxiliary results.

4.1 Bourguignon-Li-Yau estimate [B-L-Y]
Let $(M, \Omega, J, g)$ be a closed Kähler manifold of complex dimension $n$, and let $\phi : M \to \mathbb{CP}^N$ be a holomorphic embedding. Then
\[
\lambda_1(M, g) \leq 8\pi n \frac{\phi^*a \cup [\Omega]^{n-1}, [M]}{([\Omega]^n, [M])}.
\]

4.2 Demailly’s theorem [De, 8.5, p.58]
Let $(M, J)$ be a complex manifold of complex dimension $n$ with the canonical bundle $K$. There exists a constant $\alpha(n)$ which depends only on $n$ such that for every ample line bundle $L \to M$, the bundle
\[
\alpha(n)(K + (n+2)L)
\]
is very ample. In particular $M$ admits a holomorphic embedding $\phi$ to a projective space with
\[
\phi^*a = \alpha(n)(-c_1(TM) + (n+2)c_1(L)).
\]

4.3 Proof of Theorem 1.2.B:
Let $J$ be a compatible complex structure on $(M, \Omega)$. Since $[\Omega]$ is an integral class, there exists a Hermitian holomorphic line bundle $L \to M$ whose Chern form equals to $\Omega$. The Kodaira embedding theorem [Gr-Ha] implies that $L$ is ample. Using 4.2 we get a holomorphic projective embedding $\phi$ of $M$ with
\[
\phi^*a = \alpha(n)(-c_1(TM) + (n+2)[\Omega]).
\]
The required inequality 1.2.B follows now from 4.1.

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References

[A-B] M. Atyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond. A* 308 (1982), 523-615.

[B-B] L. Berard Bergery and J.-P. Bourguignon, Laplacians and Riemannian submersions with totally geodesic fibers, *Illinois J. of Math.* 26 (1982), 181-200.

[B-L-Y] J.-P. Bourguignon, P. Li and S.T. Yau, Upper bound for the first eigenvalue of algebraic submanifolds, *Comm. Math. Helv.* 69 (1994), 199-207.

[C-D] B. Colbois and J. Dodziuk, Riemannian metrics with large $\lambda_1$, *Proc. AMS* 122 (1994), 905-906.

[De] J.-P. Demailly, $L^2$-vanishing theorems for positive line bundles and adjunction theory, in *Transcendental methods in Algebraic Geometry*, F. Catanese and C. Ciliberto eds., Lect. Notes in Math. 1646, Springer, 1996, pp. 1-97.

[E-P] Y. Eliashberg and L. Polterovich, Biinvariant metrics on the group of Hamiltonian diffeomorphisms, *International J. of Math.* 5 (1993), 727-738.

[Fu] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of Laplacian operators, *Invent. Math.* 87 (1987), 517-547.

[G1] M. Gromov, Metric invariants of Kähler manifolds, in Proc. of the Workshop on Differential Geometry and Topology, Algero, Italy, Caddeo, Tricerri eds. World Sci., 1993, pp. 90-117.

[G2] M. Gromov, Carnot-Carathéodory spaces seen from within, Preprint IHES M/94/06, 1994.

[G3] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, in *Functional Analysis on the eve of the 21st century*, S. Gindikin, J. Lepowsky, R. Wilson eds., Birkhäuser, 1996.

[G-L-S] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, 1996.

[Ge] Z. Ge, Collapsing Riemannian metrics to Carnot-Carathéodory metrics and Laplacians to sub-Laplacians, *Can. J. Math.* 45 (1993), 537-553.

[Gr-Ha] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & sons, 1978.

[Gra] J. Gravesen, Loop groups and Yang-Mills theory in dimension two, *Comm. Math. Phys* 127 (1990), 597-605.
[H] L. Hörmander, Hypoelliptic second order differential equations, *Acta Math.* **199** (1967), 147-171.

[H-Z] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, 1994.

[He] J. Hersch, Quatre propriétés isoperimétriques des membranes sphériques homogènes, *C. R. Acad. Sc. Paris* **270** (1970), 1645-1648.

[L-Y] P. Li and S.T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces, *Invent. Math.* **69** (1982), 269-291.

[McD-S] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clarendon Press, Oxford, 1995.

[P1] L. Polterovich, Gromov’s K-area and symplectic rigidity, *Geom. and Funct. Analysis* **6** (1996), 726-739.

[P2] L. Polterovich, Hamiltonian loops and Arnold’s principle, Preprint, 1996.

[P3] L. Polterovich, Precise measurements in Symplectic Topology, Proceedings of the ECM2, Budapest, to appear.

[Se] P. Seidel, $\pi_1$ of symplectic automorphisms groups and invertibles in quantum (co)homology rings, Preprint, 1996.

[W] J.-Y. Wu, The topological spectrum of a smooth closed manifold, *Indiana University Math. J.* **44** (1995), 511-534.

[Y-Y] P. Yang and S.T. Yau, Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds, *Annali della Sc. Sup. di Pisa*, **7** (1980), 55-63.