The extremal number of longer subdivisions

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Abstract

For a multigraph $F$, the $k$-subdivision of $F$ is the graph obtained by replacing the edges of $F$ with pairwise internally vertex-disjoint paths of length $k + 1$. Conlon and Lee conjectured that if $k$ is even, then the $(k - 1)$-subdivision of any multigraph has extremal number $O(n^{1 + \frac{1}{k}})$, and moreover, that for any simple graph $F$ there exists $\varepsilon > 0$ such that the $(k - 1)$-subdivision of $F$ has extremal number $O(n^{1 + \frac{1}{k} - \varepsilon})$. In this paper, we prove both conjectures.

1 Introduction

For a multigraph $F$, a subdivision of $F$ is a graph obtained by replacing the edges of $F$ with pairwise internally vertex-disjoint paths of arbitrary lengths. The $k$-subdivision of $F$ is the graph obtained by replacing the edges of $F$ with pairwise internally vertex-disjoint paths of length $k + 1$, and is denoted by $F^k$.

Many researchers have studied the problem of estimating the number of edges needed in a graph $G$ on $n$ vertices to guarantee that it contains as a subgraph a subdivided copy of a fixed graph. The first result in this direction is due to Mader [12] who proved that for any graph $F$ there exists a constant $c_F = c$ such that if an $n$-vertex graph $G$ contains at least $cn$ edges, then $G$ contains a subdivision of $F$ as a subgraph. In this result the size of the subdivided graph can grow with $n$, which is necessary since an $n$-vertex graph with $cn$ edges need not contain a cycle of bounded length.

Answering a question of Erdős about planar subgraphs [5], Kostochka and Pyber [11] proved that any $n$-vertex graph with at least $4t^2 n^{1 + \varepsilon}$ edges contains a subdivided $K_t$ with at most $\frac{7t^2 \log t}{\varepsilon}$ vertices. This is the first result that guarantees a subdivided $K_t$ of bounded size.

For a family $\mathcal{F}$ of graphs, we let $\text{ex}(n, \mathcal{F})$ be the maximum number of edges in an $n$-vertex graph not containing any $F \in \mathcal{F}$ as a subgraph. When $\mathcal{F} = \{F\}$, we write $\text{ex}(n, F)$ for the same function.

Let $\mathcal{F}_{t,k}$ be the family of graphs that can be obtained by replacing the edges of $K_t$ with pairwise internally vertex-disjoint paths of length at most $k$. Jiang [9] proved that for any $t \in \mathbb{N}$ and any $0 < \varepsilon < 1/2$, we have $\text{ex}(n, \mathcal{F}_{t,\lceil 10/\varepsilon \rceil}) = O(n^{1+\varepsilon})$. Here the asymptotic notation means that $n \to \infty$ and other parameters are constant. We follow the same convention throughout the paper.

Note that Jiang’s result improves that of Kostochka and Pyber in two ways. Firstly, any $F \in \mathcal{F}_{t,\lceil 10/\varepsilon \rceil}$ has at most $\frac{2t^2}{\varepsilon}$ vertices, so a log factor is saved. Secondly, the edges in Jiang’s theorem are replaced by uniformly short paths not depending on $t$. However, they can still have different lengths. The next result of Jiang and Seiver guarantees a subdivided $K_t$ with prescribed path lengths.

**Theorem 1.1** (Jiang–Seiver [10]). For any $t \in \mathbb{N}$ and any even $k \in \mathbb{N}$,

$$\text{ex}(n, K_t^{k-1}) = O(n^{1+\frac{16}{k}}).$$

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Note that if \( k \) is odd, then \( K_t^{k-1} \) is not a bipartite graph, so \( \text{ex}(n, K_t^{k-1}) = \Theta(n^2) \).

Conlon and Lee conjectured that the following two strengthenings hold.

**Conjecture 1.2** (Conlon–Lee [4]). Let \( F \) be a multigraph and let \( k \geq 2 \) be even. Then
\[
\text{ex}(n, F^{k-1}) = O(n^{1+\frac{1}{k}}).
\]

**Conjecture 1.3** (Conlon–Lee [4]). Let \( F \) be a simple graph and let \( k \geq 2 \) be even. Then there exists some \( \varepsilon > 0 \) such that
\[
\text{ex}(n, F^{k-1}) = O(n^{1+\frac{1}{k}-\varepsilon}).
\]

In the case \( k = 2 \), Conjecture 1.2 follows from the \( r = 2 \) case of a result of Füredi [7] and Alon, Krivelevich and Sudakov [1], which states that any bipartite graph with maximum degree at most \( r \) on one side has extremal number \( O(n^{2-1/r}) \). The \( k = 2 \) case of Conjecture 1.3 was proved by Conlon and Lee [4], and improved bounds were given by the author [8].

Very recently, Conlon, Janzer and Lee proved Conjecture 1.3 for every bipartite graph \( F \).

**Theorem 1.4** (Conlon–Janzer–Lee [3]). Let \( F \) be a simple bipartite graph and let \( k \geq 1 \). Then there exists some \( \varepsilon > 0 \) such that
\[
\text{ex}(n, F^{k-1}) = O(n^{1+\frac{1}{k}-\varepsilon}).
\]

As a simple corollary, they significantly improved the bound in Theorem 1.1.

**Theorem 1.5** (Conlon–Janzer–Lee [3]). Let \( F \) be a simple graph and let \( k \geq 2 \) be even. Then there exists some \( \varepsilon > 0 \) such that
\[
\text{ex}(n, F^{k-1}) = O(n^{1+\frac{2}{k}-\varepsilon}).
\]

In this paper, we prove both Conjecture 1.2 and Conjecture 1.3.

**Theorem 1.6**. Let \( F \) be a multigraph and let \( k \geq 2 \) be even. Then
\[
\text{ex}(n, F^{k-1}) = O(n^{1+\frac{1}{k}}).
\]

**Theorem 1.7**. Let \( F \) be a simple graph and let \( k \geq 2 \) be even. Then there exists some \( \varepsilon > 0 \) such that
\[
\text{ex}(n, F^{k-1}) = O(n^{1+\frac{1}{k}-\varepsilon}).
\]

Note that these results are tight. Indeed, by a result of Conlon [2], the Theta graph \( \theta_{k,\ell} \) has extremal number \( \Theta(n^{1+1/k}) \) for all \( \ell \geq \ell_0(k) \), showing that Theorem 1.6 is tight. Moreover, Erdős-Rényi random graphs show that \( \text{ex}(n, K_t^{k-1}) = \Omega(n^{1+1/k-c_{k,\ell}}) \) where \( c_{k,\ell} \to 0 \) as \( t \to \infty \), so Theorem 1.7 is also tight.

The rest of the paper is organised as follows. In Section 2 we introduce some of the key definitions and give the high-level structure of the proof, with the key technical lemmas deferred to Sections 3 and 4.
2 The high-level structure of the proof

A graph $G$ is called $K$-almost-regular if $\max_{v \in V(G)} d(v) \leq K \min_{v \in V(G)} d(v)$, where $d(v)$ is the degree of vertex $v$. The following lemma, which is a small modification of a result proved by Erdős and Simonovits [8], allows us to restrict our attention to almost regular host graphs.

Lemma 2.1 (Jiang–Seiver [10]). Let $\varepsilon, c$ be positive reals, where $\varepsilon < 1$ and $c \geq 1$. Let $n$ be a positive integer that is sufficiently large as a function of $\varepsilon$. Let $G$ be a graph on $n$ vertices with $e(G) \geq cn^{1+\varepsilon}$. Then $G$ contains a $K$-almost-regular subgraph $G'$ on $m \geq n^{\frac{1}{1+\varepsilon}}$ vertices such that $e(G') \geq \frac{2c}{\varepsilon}m^{1+\varepsilon}$ and $K = 20 \cdot 2^{\frac{1}{2\varepsilon} + 1}$.

Using this lemma, Theorem 1.6 and Theorem 1.7 reduce to the following two statements, respectively. For notational convenience, we have dropped the assumption that $k$ is even, and replaced $k$ by $2k$.

Theorem 2.2. Let $F$ be a multigraph and let $k \geq 1$. Suppose that $G$ is a $K$-almost-regular graph on $n$ vertices with minimum degree $\delta = \omega(n^{\frac{1}{2k}-1})$. Then, for $n$ sufficiently large, $G$ contains a copy of $F^{2k-1}$.

Theorem 2.3. Let $F$ be a simple graph and let $k \geq 1$. Then there exists $\varepsilon > 0$ with the following property. Suppose that $G$ is a $K$-almost-regular graph on $n$ vertices with minimum degree $\delta = \omega(n^{\frac{1}{2k}-i})$. Then, for $n$ sufficiently large, $G$ contains a copy of $F^{2k-1}$.

From now on we let $F$ be an arbitrary fixed multigraph and write $H = F^{2k-1}$. Moreover, throughout the paper we tacitly assume that $n$ is sufficiently large.

The next definition was introduced in [3], and was used to prove Theorem 1.4.

Definition 2.4. Let $L$ be a positive real and let $f(\ell, L) = L^{5\ell}$ for $1 \leq \ell \leq 2k$. We recursively define the notions of $L$-admissible and $L$-good paths of length $\ell$ in a graph. Any path of length 1 is both $L$-admissible and $L$-good. For $2 \leq \ell \leq 2k$, we say a path $P = v_0v_1\ldots v_\ell$ is $L$-admissible if every proper subpath of $P$ is $L$-good, i.e., $v_i v_{i+1}\ldots v_j$ is $L$-good for every $(i, j) \neq (0, \ell)$. The path $P$ is $L$-good if it is $L$-admissible and the number of $L$-admissible paths of length $\ell$ between $v_0$ and $v_\ell$ is at most $f(\ell, L)$.

The next lemma will be used several times later.

Lemma 2.5. Let $\ell \geq 2$ and let $L > \ell$. If a path $P = v_0\ldots v_\ell$ is $L$-admissible, but not $L$-good, then there exist at least $L$ pairwise internally vertex-disjoint paths of length $\ell$ from $v_0$ to $v_\ell$.

Proof. Take a maximal set of pairwise internally vertex-disjoint paths of length $\ell$ from $v_0$ to $v_\ell$ and assume that it consists of fewer than $L$ paths. These paths contain at most $L(\ell - 1)$ internal vertices in total and any path of length $\ell$ between $v_0$ and $v_\ell$ intersects at least one of these vertices. Since there are at least $L^{5\ell}$ $L$-admissible paths of length $\ell$ between $v_0$ and $v_\ell$, it follows by pigeon hole that there exist some $1 \leq i \leq \ell - 1$ and some $x \in V(G)$ such that there are at least $\frac{L^{5\ell}}{(\ell - 1)L(\ell - 1)}$ $L$-admissible paths of the form $u_0u_1\ldots u_\ell$ with $u_0 = v_0, u_i = x, u_{\ell} = v_\ell$. Observe that $\frac{L^{5\ell}}{(\ell - 1)L(\ell - 1)} > L^{5\ell}L^{5\ell-i}$, so either there are more than $L^{5\ell} L$-good paths of length $i$ between $v_0$ and $x$ or there are more than $L^{5\ell-i}$ $L$-good paths of length $\ell - i$ between $x$ and $v_\ell$. In either case, we contradict the definition of an $L$-good path. \qed

Our strategy will be to prove that, roughly speaking, in any almost regular $H$-free graph there are many good paths of length $2k$. As we will see in Section 3 the techniques in [3] can be easily applied to prove this for paths of length $k$. The novelty of this paper is the machinery that allows us to extend this to longer paths, using very different techniques. This is given in Section 4 where we prove the following lemma.
Lemma 2.6. Let $G$ be an $H$-free $K$-almost-regular graph on $n$ vertices with minimum degree $\delta \geq L^{100^k|V(H)|}$, and let $S \subset V(G)$. Then, provided that $L$ is sufficiently large compared to $|V(H)|$ and $K$, $|S| = \omega(\frac{n}{\delta^{2k}})$ and $|S| = \omega(\frac{n}{\delta^{2k}})$, the number of $L$-good paths of length $2k$ with both endpoints in $S$ is $\Omega\left(\frac{|S|^2 \delta^{2k}}{n}\right)$.

Note that in this result and everywhere else in the paper, the asymptotic notation $\Omega$ allows the implied constant to depend on $k$, $|V(H)|$ and $K$, which are thought of as constants, while $\delta$ and $L$ are functions of $n$.

With Lemma 2.6 in hand, the proof of Theorem 2.2 is immediate.

Proof of Theorem 2.2. Suppose that $G$ does not contain $H = F^{2k-1}$ as a subgraph. Since $\delta = \omega(n^{1/k})$, we may choose $L$ with $L = \omega(1)$, $L^{100^k|V(H)|} \leq \delta$ and $n^2 f(2k, L) = o(n \delta^{2k})$. Then we may apply Lemma 2.6 with $S = V(G)$ to get that the number of $L$-good paths of length $2k$ in $G$ is $\Omega(n \delta^{2k})$, which is $\omega(n^2 f(2k, L))$. However, by the definition of $L$-goodness, between any two vertices there can be at most $f(2k, L)$ such paths, which is a contradiction. \hfill $\Box$

The proof of Theorem 2.3 is slightly more complicated, and it uses ideas from [8].

Proof of Theorem 2.3. Firstly note that $F$ is a subgraph of $K_t$ for some $t$, so it suffices to prove the result for $F = K_t$. Let $\varepsilon > 0$ be sufficiently small, to be specified, and let $G$ be a $K$-almost-regular graph on $n$ vertices with minimum degree $\delta = \omega(n^{1/k} - \varepsilon)$. Assume that $G$ does not contain a copy of $H = F^{2k-1}$.

For vertices $u, v \in V(G)$, let us write $u \sim v$ if there is a path of length $2k$ between $u$ and $v$. Also, let us say that $u$ and $v$ are distant if for every $1 \leq i \leq 4k - 2$, the number of walks of length $i$ between $u$ and $v$ is at most $\delta^{i-2k+1/2}$. Observe that for any $u \in V(G)$ the number of walks of length $i$ starting from $u$ is at most $(K \delta)^i$, so the number of vertices $v \in V(G)$ for which there are at least $\delta^{i-2k+1/2}$ walks of length $i$ from $u$ to $v$ is at most $\frac{(K \delta)^i}{\delta^{i-2k+1/2}} = K^i \delta^{2k-1/2}$. Thus, the number of $v \in V(G)$ for which $u$ and $v$ are not distant is $O(\delta^{2k-1/2})$.

Define $c_0 = \varepsilon$ and $c_{\ell + 1} = (3 \cdot 5^{2k} + 1)c_\ell + 2k \varepsilon$ for $0 \leq \ell \leq t - 1$. Assume that $\varepsilon$ is small enough so that

$$3 \cdot 100^k |V(H)| \cdot c_\ell \leq \frac{1}{2k} - \varepsilon$$

for all $0 \leq \ell \leq t$. Then in particular $c_\ell \leq \frac{1}{\delta^{2k} - \varepsilon/2}$ holds for all $0 \leq \ell \leq t$. For future reference, note that then

$$n^{c_\ell} \leq n^{1/k - \varepsilon/2} = o(\delta^{1/2}).$$

Claim. For any $0 \leq \ell \leq t$, there exist distinct vertices $x_1, \ldots, x_\ell \in V(G)$ and a set $S_\ell \subset V(G)$ such that

(i) there is a copy of $K_t^{2k-1}$ in $G$ with the vertices of the subdivided $K_\ell$ being $x_1, \ldots, x_\ell$

(ii) $x_i \sim y$ for every $1 \leq i \leq \ell$ and every $y \in S_\ell$

(iii) $|S_\ell| = \Omega(n^{1-c_\ell})$ and

(iv) $x_i$ and $x_j$ are distant for every $1 \leq i < j \leq \ell$.

Note that in particular for $\ell = t$, condition (i) guarantees the existence of a subgraph $K_t^{2k-1}$, so it suffices to prove the claim.

Proof of Claim. We proceed by induction on $\ell$. For $\ell = 0$, we may take $S_0 = V(G)$. Assume now that we have verified the claim for $\ell$.
Suppose that for some $y \in S_\ell$ there exist $1 \leq i < j \leq \ell$ and two paths of length $2k$, one (called $P_i$) from $x_i$ to $y$ and one (called $P_j$) from $x_j$ to $y$, which share a vertex other than $y$. Let them intersect at some vertex $z \neq y$. Now let the subpath of $P_i$ between $x_i$ and $z$ have length $\alpha$ and let the subpath of $P_j$ between $x_j$ and $z$ have length $\beta$. Then there is a walk of length $\alpha + \beta$ from $x_i$ to $x_j$ through $z$. Moreover, there is a path of length $2k - \alpha$ from $z$ to $y$. Observe that $2k - \alpha \leq 4k - (\alpha + \beta) - 1$.

Let $Y$ be the set of $y \in S_\ell$ for which there exist some $1 \leq i < j \leq \ell$ and a walk $W$ of length $\gamma \leq 4k - 2$ between $x_i$ and $x_j$ such that for some vertex $w$ on $W$ the distance of $y$ from $w$ is at most $4k - \gamma - 1$. By condition (iv), there are at most $\frac{\delta^2k - \delta^{k + 1}}{1}$ walks of length $\gamma$ between any $x_i$ and $x_j$ so there are $O(\frac{\delta^{2k - 2}}{1})$ vertices appearing in at least one of these walks. Therefore the number of vertices at distance at most $4k - \gamma - 1$ from at least one of these vertices is $O(\frac{\delta^{2k - 2} \cdot \delta^{k - \gamma - 1}}{1}) = O(\frac{\delta^{2k - 2}}{1})$. That is, $|Y| = O(\frac{\delta^{2k - 2}}{1})$.

Notice that by the discussion above, for any $y \in S_\ell \setminus Y$ and any $i \neq j$, a path of length $2k$ from $x_i$ to $y$, and a path of length $2k$ from $x_j$ to $y$ have no common vertex other than $y$. Thus, by condition (iii) there exist $\ell$ paths of length $2k$, one from each $x_i$ to $y$ which are pairwise vertex-disjoint apart from at $y$. Moreover, these paths are also vertex-disjoint from the paths forming the $K_{\ell - 1}$ guaranteed by condition (i), apart from the trivial intersections at $x_1, \ldots, x_\ell$ (else, there is a path of length at most $2k - 1$ from $y$ to a point on a path of length $2k$ between some $x_i$ and $x_j$, which contradicts the fact that $y \notin Y$). Thus, for any $y \in S_\ell \setminus Y$ there is a copy of $K_{\ell - 1}$ in $G$ with the vertices of the subdivided $K_{\ell + 1}$ being $x_1, \ldots, x_\ell, y$.

Let $Z$ be the set of $z \in S_\ell$ which are not distant to $x_i$ for at least one $1 \leq i \leq \ell$. By the second paragraph in this proof, $|Z| = O(\frac{\delta^{2k - 1}}{1})$.

Let $S'_\ell = S_\ell \setminus (Y \cup Z)$. Recall that $|Y| = O(\frac{\delta^{2k - 1}}{1})$. Note that if $\delta = \omega(n^{\frac{1}{ck^2}})$, then, by Theorem 2.2, $G$ contains $H$ as a subgraph, so we may assume that $\delta = o(n^{\frac{1}{ck^2}})$. Then $\frac{\delta^{2k - 1}}{1} = O(n^{\frac{1}{ck^2}})$, which is $o(n^{1 - \epsilon_L})$ by equation (2). Thus, $|Y \cup Z| = o(n^{1 - \epsilon_L})$ and so $|S'_\ell| = \Omega(n^{1 - \epsilon_L})$.

Let $L = n^{\epsilon_L}$. Then, by equation (1), we have $L^{100k|V(H)|} \leq n^{\frac{1}{ck^2} - \epsilon} = o(\delta)$. Moreover, by equation (2), we have $n^{1 - \epsilon_L} = \omega(n^{\frac{1}{ck^2}})$, and by the definition of $L$, we have $n^{1 - \epsilon_L} = \omega(n^{\frac{1}{ck^2}})$. Hence, by Lemma 2.6, the number of $L$-good paths of length $2k$ with both endpoints in $S'_\ell$ is $\Omega(\frac{|S'_\ell|^2 \delta^{2k}}{n^2})$. Between any two vertices in $S'_\ell$ there are at most $f(2k, L)$-good paths of length $2k$, so the number of pairs $(z, y) \in S'_\ell \times S'_\ell$ with $z \sim y$ is $\Omega(\frac{|S'_\ell|^2 \delta^{2k}}{n^2 f(2k, L)^2})$. Thus, there exists some $x_{\ell+1} \in S'_\ell$ such that the number of $y \in S'_\ell$ with $x_{\ell+1} \sim y$ is $\Omega(\frac{|S'_\ell|^2 \delta^{2k}}{n^2 f(2k, L)^2}) \geq \Omega(n^{1 - \epsilon_L - 2k + 3 \epsilon_L \delta^{2k}}) = \Omega(n^{1 - \epsilon_L - 2k - 3 \epsilon_L \delta^{2k}}) = \Omega(n^{1 - \epsilon_L - 2k + 3 \epsilon_L \delta^{2k}})$. Set $S'_{\ell+1}$ to be the set of these $y \in S'_\ell$, and note that properties (i)-(iv) are satisfied for $\ell + 1$.

## 3 Short paths

Our aim in this section is to prove the following lemma.

**Lemma 3.1.** Let $G$ be an $H$-free $K$-almost-regular graph on $n$ vertices with minimum degree $\delta \geq L^{100k|V(H)|}$. Then, provided that $L$ is sufficiently large compared to $|V(H)|$ and $K$, the number of paths of length $k$ that are not good is $O(\frac{4k^2}{n})$.

The proof of this is almost identical to that of Lemma 6.4 in [3], nevertheless we include it here for completeness and since some minor details need to be modified.

The next definition is for notational convenience.

**Definition 3.2.** A pair of distinct vertices $(x, y)$ in $G$ is said to be $(\ell, L)$-bad for some $2 \leq \ell \leq 2k$ and some $L$ if there is an $L$-admissible, but not $L$-good, path of length $\ell$ from $x$ to $y$. 

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In what follows, for \( v \in V(G) \), we shall write \( \Gamma_i(v) \) for the set of vertices \( u \in V(G) \) for which there exists a path of length \( i \) from \( v \) to \( u \) and write \( N(v) = \Gamma_1(v) \). The next lemma is a slight variant of Lemma 6.7 from [3].

**Lemma 3.3.** Let \( 2 \leq \ell \leq k \) and \( 1 \leq i \leq \ell \). Let \( G \) be a \( K \)-almost-regular graph on \( n \) vertices with minimum degree \( \delta > 0 \). Let \( X, Y, Z \subseteq V(G) \) be such that \( |X| \leq L^{1/10} \), \( |Y| \leq (K\delta)^{\ell-1} \) and, for any \( x \in X \), the number of \( y \in Y \) such that \( (x, y) \) is \( (\ell, L) \)-bad is at least \( \frac{(K\delta)^{\ell-1}}{f(\ell-1, L)^2} \).

Then, provided that \( L \) is sufficiently large compared to \( k \) and \( K \), there exist a path of length \( 2i \) in \( G \), disjoint from \( Z \), whose endpoints form a set \( R \subseteq Y \), and a subset \( X' \subseteq X \) such that \( |X'| \geq |X \setminus Z|/(16f(\ell-1, L)^2) \) and \( (x', r) \) is \( (\ell, L) \)-bad for every \( x' \in X' \) and \( r \in R \).

**Proof.** After replacing \( X \) by \( X \setminus Z \), we may assume \( X \cap Z = \emptyset \). Let \( Y' \) be the set of those \( y \in Y \) for which the number of \( x \in X \) such that \((x, y)\) is \((\ell, L)\)-bad is at least \( \frac{|X|}{2f(\ell-1, L)^2} \). Then the number of \((x, y) \in X \times (Y \setminus Y')\) which are \((\ell, L)\)-bad is at most \( \frac{|X||Y|}{2f(\ell-1, L)^2} \leq \frac{|X|(|K\delta|^{\ell-1})}{2f(\ell-1, L)^2} \), so the number of \((x, y) \in X \times Y'\) which are \((\ell, L)\)-bad is at least \( \frac{|X|(|K\delta|^{\ell-1})}{2f(\ell-1, L)^2} \). Now there exists some \( x' \in X \) such that there are at least \( \frac{(|K\delta|^{\ell-1})}{2f(\ell-1, L)^2} \) choices \( y \in Y' \) for which \((x', y)\) is \((\ell, L)\)-bad. If a pair \((x', y)\) is \((\ell, L)\)-bad, then there are at least \( f(\ell, L) \) paths of length \( \ell \) from \( x' \) to \( y \). Hence, there are at least \( \frac{(|K\delta|^{\ell-1})}{2f(\ell-1, L)^2} \cdot f(\ell, L) = \Omega(f(\ell-1, L)^3\delta^{\ell-1}) \) paths of length \( \ell \) starting at \( x' \) and ending in \( Y' \).

The number of such paths intersecting \( Z \) is at most \(|Z|(|K\delta|^{\ell-1}) \). Indeed, there are at most \(|Z|\) choices for the element of \( Z \) in the path, at most \( \ell \) choices for its position in the path and, given a fixed choice for these, at most \((K\delta)^{\ell-1}\) choices for the other \( \ell-1 \) vertices in the path. (Note that as \( X \cap Z = \emptyset \), the vertex in \( Z \) is not \( x' \).) But \(|Z|(|K\delta|^{\ell-1}) \leq L^{1/10}K\ell^{-1} \delta^{-1} \), so, for \( L \) sufficiently large there are \( \Omega(f(\ell-1, L)^3\delta^{\ell-1}) \) paths of length \( \ell \) starting at \( x' \) and ending in \( Y' \) avoiding \( Z \). Moreover, since \(|\Gamma_{\ell-i}(x')| \leq (K\delta)^{\ell-i} \), it follows that there exists some \( u \in \Gamma_{\ell-i}(x') \) such that there are \( \Omega(f(\ell-1, L)^3\delta^{\ell-1}) \) paths of length \( i \) from \( u \) to \( Y' \), all avoiding \( Z \).

Take now a maximal set of such paths which are pairwise vertex-disjoint apart from at \( u \). We claim that there are \( \Omega(f(\ell-1, L)^3) \) such paths. Suppose otherwise. Then all the \( \Omega(f(\ell-1, L)^3\delta^{\ell-1}) \) paths of length \( i \) from \( u \) to \( Y' \) intersect a certain set of size \( o(f(\ell-1, L)^3) \) not containing \( u \). But there are \( o(f(\ell-1, L)^3\delta^{\ell-1}) \) such paths, which is a contradiction.

So we have \( r = \Omega(f(\ell-1, L)^3) \) paths \( P_1, \ldots, P_r \) of length \( i \) from \( u \) to \( Y' \) which are pairwise vertex-disjoint except at \( u \) and avoid \( Z \). Let the endpoints of these paths be \( y_1, \ldots, y_r \). Since \( y_j \in Y' \) for all \( j \), the number of pairs \((x, y_j)\) with \( x \in X \) which are \((\ell, L)\)-bad is at least \( \frac{|X|}{2f(\ell-1, L)^2} \). Therefore, by Jensen's inequality, for an average \( x \in X \) there are at least \( (r/(2f(\ell-1, L)^2))^2 \) choices \( 1 \leq j_1 < j_2 \leq r \) such that both \((x, y_{j_1})\) and \((x, y_{j_2})\) are \((\ell, L)\)-bad. Since \( (r/(2f(\ell-1, L)^2))^2 \geq (\frac{1}{4f(\ell-1, L)^2})^2 \), there exist \( 1 \leq j_1 < j_2 \leq r \) such that the set

\[ X' = \{ x \in X : (x, y_{j_1}) \text{ and } (x, y_{j_2}) \text{ are } (\ell, L)\text{-bad} \} \]

has size at least \( |X|/(4f(\ell-1, L)^2)^2 \). We can now take \( R = \{ y_{j_1}, y_{j_2} \} \), and the union of the paths \( P_{j_1} \) and \( P_{j_2} \) is a suitable path of length \( 2i \).

The following lemma is a small modification of Lemma 6.8 from [3].

**Lemma 3.4.** Let \( G \) be an \( H \)-free \( K \)-almost-regular graph on \( n \) vertices with minimum degree \( \delta \geq L^{100^3|V(H)|} \). Let \( 2 \leq \ell \leq k \) and any \( v \in V(G) \). Then, provided that \( L \) is sufficiently large compared to \( |V(H)| \) and \( K \), the number of \( L \)-admissible, but not \( L \)-good, paths of the form \( v_0v_1v_2v_3 \ldots v_{2i} \) is at most \( \frac{2(K\delta)^{\ell}}{f(\ell-1, L)^2} \).
Proof. Suppose otherwise. Let $Y = \Gamma_{\ell-1}(v)$ and note that $|Y| \leq (K\delta)^{\ell-1}$. For any $x \in N(v)$ and any $y \in Y$, the number of $L$-admissible paths of the form $xuv_2 \ldots v_{\ell-1}y$ is at most $f(\ell-1, L)$. Indeed, in any such path, the subpath $vv_2v_3 \ldots v_{\ell-1}y$ is $L$-good, and for any fixed $y \in Y$ there are at most $f(\ell-1, L)$ such $L$-good paths. Hence, by assumption, the number of pairs $(x, y) \in N(v) \times Y$ such that there is an $L$-admissible, but not $L$-good, path of the form $xuv_1 \ldots v_{\ell-1}y$ is at least $\frac{2(K\delta)^{\ell}}{f(\ell-1, L)^2} \geq \frac{2|N(v)|(K\delta)^{\ell-1}}{f(\ell-1, L)^2}$. By definition, any such pair $(x, y)$ is $(\ell, L)$-bad. Let $X$ consist of those $x \in N(v)$ for which there are at least $\frac{|N(v)|(K\delta)^{\ell-1}}{f(\ell-1, L)^2}$ choices of $y \in Y$ such that $(x, y)$ is $(\ell, L)$-bad. Then the number of pairs $(x, y) \in X \times Y$ which are $(\ell, L)$-bad is at least $\frac{|N(v)|(K\delta)^{\ell-1}}{f(\ell-1, L)^2}$, and so $|X| \geq \frac{|N(v)|}{f(\ell-1, L)^2} \geq \frac{\delta}{\ell}$. Our aim now is to find a copy of $H$ in $G$, which will yield a contradiction. Write $k = j \ell + i$ with $1 \leq i \leq \ell$.

Note that if $L$ is sufficiently large, then

$$|X| \geq \frac{\delta}{f(\ell-1, L)^2} \geq \frac{L^{100\theta}|V(H)|}{f(\ell-1, L)^2} \geq \frac{f(\ell-1, L)^2|V(H)|}{f(\ell-1, L)^2} \geq 2L(16f(\ell-1, L)^2)^{|V(H)|},$$

so we may apply Lemma 8.3 repeatedly $|E(F)| + |V(H)| \leq 2|V(H)|$ times and still get a set $X'$ of size at least $L$. Thus, we find disjoint paths $P_e$ of length $2i$ for every $e \in E(F)$ whose endpoint sets are $R_e \subset Y$, and sets $X_{\text{final}} \subset X$ and $U \subset Y$ with $|X_{\text{final}}| = |U| = |V(H)|$ such that $V(P_e), X_{\text{final}}$ and $U$ are pairwise disjoint and any pair $(x, y)$ with $x \in X_{\text{final}}$ and $y \in U \cup \bigcup_{e \in E(F)} R_e$ is $(\ell, L)$-bad. For $e \in E(F)$, let $y_{e-k}y_{e-k+1} \ldots y_e$ be the path of length $2k$ replacing the edge $e$.

A copy of $H$ in $G$ can now be constructed as follows. For each $e \in E(F)$, map the path $y_{e-k}y_{e-k+1} \ldots y_e$ to $P_e$. Then map, for each $e \in E(F)$, the vertices $y_{e-k}, y_{e-k+1}, \ldots, y_e$ in an arbitrary manner. Also, map each $y_{e_{i+1}}, y_{e_{i+2}}, \ldots, y_{e_{i+\ell}}$ to $U$ in an arbitrary injective manner. More generally, map the vertices $y_{e_{i+1}}, y_{e_{i+2}}, \ldots, y_{e_{i+\ell}}$ with $a \geq 1$ odd to $X_{\text{final}}$ in an arbitrary injective manner and map the vertices $y_{e_{i+1}}, y_{e_{i+2}}, \ldots, y_{e_{i+\ell}}$ with $a \geq 2$ even to $U$ in an arbitrary injective manner. We then just need to find paths of length $\ell$ connecting $y_{e_{i+1}}$ and $y_{e_{i+1+1}}$, respectively, which are disjoint from each other and from the images of the already mapped vertices. Since $(x, y)$ is $(\ell, L)$-bad for every $x \in X_{\text{final}}$ and $y \in U \cup \bigcup_{e \in E(F)} R_e$, such paths exist by Lemma 2.5 provided that $L$ is sufficiently large.

**Corollary 3.5.** Let $G$ be an $H$-free $K$-almost-regular graph on $n$ vertices with minimum degree $\delta \geq L^{100\theta}|V(H)|$. Then, provided that $L$ is sufficiently large compared to $|V(H)|$ and $K$, for any $2 \leq \ell \leq k$, the number of $L$-admissible, but not $L$-good, paths of length $\ell$ is at most $n^{2(K\delta)^{\ell}}$. 

Now we are in a position to prove Lemma 3.1.

**Proof of Lemma 3.1** Suppose that the path $u_0u_1 \ldots u_k$ is not $L$-good. Take $0 \leq i < j \leq k$ with $j - i$ minimal such that $u_iu_{i+1} \ldots u_j$ is not $L$-good. Then $u_i \ldots u_j$ is $L$-admissible. For any fixed $i, j$, by Corollary 8.3, the number of such paths is at most $n^{2(K\delta)^{j-i}} \cdot 2(K\delta)^{j-i} = 4K^k n^{2(K\delta)^{\ell}} \leq 4K^k n^{2(K\delta)^{\ell}}$. Using that $i$ and $j$ can take at most $k+1$ values each, it follows that the number of not $L$-good paths of length $k$ is at most $(k+1)^2 4K^k n^{2(K\delta)^{\ell}}$.

**4 Long paths**

In what follows, for a vertex $x \in V(G)$ and a nonnegative integer $i$, we write $P_i(x)$ for the set of directed paths of length $i$ starting at $x$. For an element $P \in P_i(x)$, we let $v(P)$ be the endpoint of the path $P$.
Definition 4.1. Let \( i, j \) be nonnegative integers with \( i + j < 2k \). Call a pair \((x, y)\) of distinct vertices \((i, j)\)-rich if the number of pairs \((P, Q) \in \mathcal{P}_i(x) \times \mathcal{P}_j(y)\) such that there are at least \(|V(H)| + 2)(2k + 1) + 1\) pairwise internally vertex-disjoint paths of length \(2k - i - j\) between \(v(P)\) and \(v(Q)\) is more than \((2(i + j)|V(H)|)(2k + 1) + 2(i + 1)j(K\delta)^{i+j-1}\). Otherwise (including when \(x = y\)) call it \((i, j)\)-poor.

Lemma 4.2. Let \( G \) be a graph with maximum degree at most \(K\delta\). Let \( x, y \in V(G)\) and let \( i, j \) be nonnegative integers with \( i + j < 2k\). If \((x, y)\) is \((i, j)\)-rich, then there exist \(|V(H)|\) pairwise internally vertex-disjoint paths of length \(2k\) between \(x\) and \(y\).

Proof. Choose a maximal set of pairwise internally vertex-disjoint paths \(R_1, \ldots, R_\alpha\) between \(x\) and \(y\) and assume that \(\alpha < |V(H)|\). Let \(T\) be the set of the vertices appearing in at least one of these paths. Note that \(|T| < |V(H)|)(2k + 1)\).

Claim. If there is a pair \((P, Q) \in \mathcal{P}_i(x) \times \mathcal{P}_j(y)\) such that

(i) \(P\) is disjoint from \(T \setminus \{x\}\)

(ii) \(Q\) is disjoint from \(T \setminus \{y\}\)

(iii) \(P\) and \(Q\) are vertex-disjoint and

(iv) there are at least \(|V(H)| + 2)(2k + 1) + 1\) pairwise internally vertex-disjoint paths of length \(2k - i - j\) between \(v(P)\) and \(v(Q)\),

then there is a path of length \(2k\) between \(x\) and \(y\) which is internally vertex-disjoint from all of \(R_1, \ldots, R_\alpha\).

Proof of Claim. Clearly, it suffices to find a path of length \(2k - i - j\) between \(v(P)\) and \(v(Q)\) which is disjoint from the vertices of \(R_1, \ldots, R_\alpha, P, Q\), except for \(v(P)\) and \(v(Q)\). But such a path exists since there are at most \((\alpha + 1) \cdot (2k + 1)\) vertices in one of \(R_1, \ldots, R_\alpha, P, Q\) and there are at least \(|V(H)| + 2)(2k + 1) + 1\) pairwise internally vertex-disjoint paths of length \(2k - i - j\) between \(v(P)\) and \(v(Q)\).

A path provided by the claim would contradict the maximality of \(R_1, \ldots, R_\alpha\), so it suffices to prove that there are \(P, Q\) satisfying (i)-(iv) above.

Since the maximum degree of \(G\) is at most \(K\delta\), the number of paths of length \(i - 1\) in \(G\) intersecting \(T\) is at most \(i|T|(K\delta)^{i-1}\), so the number of \(P \in \mathcal{P}_i(x)\) which have a vertex in \(T \setminus \{x\}\) is at most \(2i|T|(K\delta)^{i-1}\). Since \(|\mathcal{P}_j(y)| \leq (K\delta)^j\), the number of pairs \((P, Q) \in \mathcal{P}_i(x) \times \mathcal{P}_j(y)\) failing condition (i) above is at most \(2i|T|(K\delta)^{i-1}(K\delta)^j\). Similarly, the number of pairs failing (ii) is at most \(2j|T|(K\delta)^{j-1}(K\delta)^i\). Finally, for every \(P \in \mathcal{P}_i(x)\), the number of paths of length \(j - 1\) which intersect \(P\) is at most \((i + 1)j(K\delta)^{j-1}\), so the number of pairs \((P, Q) \in \mathcal{P}_i(x) \times \mathcal{P}_j(y)\) for which \(P\) and \(Q\) share a vertex other than \(y\) is at most \((K\delta)^{i+1}(K\delta)^{j-1}\). So the number of pairs which fail at least one of (i),(ii),(iii) is at most \((2(i + j)|T| + 2(i + 1)j)(K\delta)^{i+j-1} \leq (2(i + j)|V(H)|(2k + 1) + 2(i + 1)j)(K\delta)^{i+j-1}\). By the definition of \((i, j)\)-richness of \((x, y)\) it follows that there is a pair \((P, Q)\) satisfying (i)-(iv).

Definition 4.3. For a vertex \(v \in V(G)\) and some \(1 \leq \ell \leq k\), define an auxiliary graph \(G_\ell(v)\) as follows. The vertices of \(G_\ell(v)\) are the \((k + 1)\)-tuples \((u_0, u_1, \ldots, u_k) \in V(G)^{k+1}\) with \(u_0 = v\) such that \(u_i u_{i+1} \in E(G)\) for all \(i\). Vertices \((u_0, \ldots, u_k)\) and \((u_0', \ldots, u_k')\) are joined by an edge if \(v, u_1, u_2, \ldots, u_k, u_1', u_2', \ldots, u_k'\) are distinct and there exist \(0 \leq i, j \leq k - 1\) such that the pair \((u_i, u_j')\) is \((i, j)\)-rich. Since the vertex set of \(G_\ell(v)\) does not depend on \(\ell\), we may define \(G(v)\) to be the union \(\bigcup_{1 \leq \ell \leq k} G_\ell(v)\).
Lemma 4.4. Let $G$ be a graph with maximum degree at most $K\delta$ which does not contain $H$ as a subgraph. Let $t = |V(F)|$. Then for any $v \in V(G)$ and any $1 \leq \ell \leq k$, the graph $G_\ell(v)$ is $K_t$-free.

Moreover, let $r = R_k(t)$ be the $k$-colour Ramsey number. Then $G(v)$ is $K_r$-free.

Proof. Suppose that $G_\ell(v)$ contains $K_t$ as a subgraph. Let the corresponding vertices be the vectors $u_1', \ldots, u_t'$. Let their respective $(\ell + 1)$th coordinate be $u_1^{\ell+1}', \ldots, u_t^{\ell+1}'$. For every $a \neq b$, since $u_a^{\ell+1}u_b^{\ell+1}$ is an edge in $G_\ell(v)$, it follows that $u_a^\ell$ and $u_b^\ell$ are distinct, and, by Lemma 4.2, there exist $|V(H)|$ pairwise internally vertex-disjoint paths of length $2k$ between them. It is not hard to see that this implies that there is a copy of $H$ in $G$ in which the vertices of $F$ are mapped to $u_1', \ldots, u_t'$. This is a contradiction, so $G_\ell(v)$ is indeed $K_t$-free.

Suppose there is a copy of $K_r$ in $G(v)$. Then each edge in this $K_r$ can be coloured with one of the colours $1, 2, \ldots, k$ such that if an edge gets colour $i$, then it lies in $G_\ell(v)$. By the definition of $r$, there exists a monochromatic $K_t$ in this $k$-edge-coloured $K_r$, which gives a $K_t$ in some $G_\ell(v)$, contradicting the first paragraph. \hfill \Box

The next lemma provides us a large set of walks of length $2k$ with both endpoints in $S$. Later, we will argue that most of them are $L$-good paths.

Lemma 4.5. Let $r = R_k(t)$ denote the $k$-colour Ramsey number where $t = |V(F)|$. Let $G$ be an $H$-free $K$-almost-regular graph on $n$ vertices with minimum degree $\delta$ and let $S \subset V(G)$ such that $|S| \geq 2nr/\delta^k$. Then there are at least $|S|^{2\delta^k}/4r^k$ vectors $(u_{-k}, \ldots, u_k) \in V(G)^{2k+1}$ with the following properties

(i) $u_{-k} \in S$, $u_k \in S$

(ii) $u_\ell u_{\ell+1} \in E(G)$ for every $-k \leq \ell \leq k - 1$

(iii) $(u_{-\ell}, u_k)$ is $(i, j)$-poor for every $1 \leq \ell \leq k$ and every $0 \leq i, j \leq k - 1$.

Proof. Since the minimum degree of $G$ is $\delta$, the number of $(k + 1)$-tuples $(v_0, v_1, \ldots, v_k) \in V(G)^{k+1}$ with $v_0 \in S$ and $v_i v_{i+1} \in E(G)$ for every $0 \leq i \leq k - 1$ is at least $|S|^{\delta^k}$. Writing $T(v_0)$ for the set of such vectors for a fixed $v_0$ and letting $g(v_0) = |T(v_0)|$, we get that $\sum_{v_0 \in V(G)} g(v_0) \geq |S|^{\delta^k}$. Note that $\sum_{v_0 \in V(G); g(v_0) < r} g(v_0) \leq nr \leq \frac{|S|^{2\delta^k}}{2}$, so

\[ \sum_{v_0 \in V(G): g(v_0) \geq r} g(v_0) \geq \frac{|S|^{2\delta^k}}{2}. \] (3)

Note that $T(v_0) \subset V(G)$. By Lemma 4.4, the graph $G(v_0)[T(v_0)]$ is $K_r$-free. This graph has $g(v_0)$ vertices, so if $g(v_0) \geq r$, then the number of non-edges in $G(v_0)[T(v_0)]$ is at least $\frac{1}{(2k+1)} \left( \frac{g(v_0)^2}{r^2} \right) \geq \frac{g(v_0)^2}{r^2}$. But if $v = (v_0, v_1, \ldots, v_k) \in T(v_0)$ and $v' = (v_0', v_1', \ldots, v_k') \in T(v_0)$ are such that $vv'$ is not an edge in $G$, then $(u_{-k}, \ldots, u_k) = (v_k', v_{k-1}', \ldots, v_0)'$ satisfies all three properties in the statement of the lemma. Therefore the number of such $(2k + 1)$-tuples with $u_0 = v_0$ is at least $\frac{g(v_0)^2}{r^2}$ provided that $g(v_0) \geq r$. By (3) and Jensen’s inequality, we get $\sum_{v_0 \in V(G); g(v_0) \geq r} \frac{g(v_0)^2}{r^2} \geq \frac{|S|^{2\delta^k}}{4r^k}$, and the proof is complete. \hfill \Box

The following simple lemma shows that most walks of length $2k$ are paths.

Lemma 4.6. Let $G$ be a graph on $n$ vertices with maximum degree at most $K\delta$. Then the number of $(2k + 1)$-tuples $(u_{-k}, \ldots, u_k) \in V(G)^{2k+1}$ such that $u_i u_{i+1} \in E(G)$ for every $i$ and $u_i = u_j$ for some $i \neq j$ is at most $\left( \frac{2k+1}{2k-1} \right) K^{2k-1} n^{2k-1}$. 

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Proof. There are \( \binom{2k+1}{2} \) ways to choose the pair \( \{i,j\} \) and there are \( n \) ways to choose \( u_i = u_j \). Given any such choices, there are at most \( (K\delta)^{2k-1} \) ways to choose the vertices \( u_b \) for \( b \not\in \{i,j\} \) since any vertex in \( G \) has degree at most \( K\delta \).

Our strategy now is to take all the paths guaranteed by Lemmas 4.5 and 4.6 and discard those which contain a subpath of length \( k \) which is not \( L \)-good. The next result shows that doing this we discard only a small proportion of the paths.

Lemma 4.7. Let \( G \) be an \( H \)-free \( K \)-almost-regular graph on \( n \) vertices with minimum degree \( \delta \geq \frac{1}{100} |V(H)| \). Then, provided that \( L \) is sufficiently large compared to \( |V(H)| \) and \( K \), the number of paths \( u_{-k}u_{-k+1}\ldots u_k \) of length \( 2k \) in \( G \) with the property that there is some \(-k \leq j < 0\) for which the path \( u_ju_{j+1}\ldots u_{j+k} \) is not \( L \)-good is \( O\left(\frac{n^{2k}}{L}\right) \).

Proof. By Lemma 4.3, there are \( O\left(\frac{n^{2k}}{L}\right) \) paths \( u_ju_{j+1}\ldots u_{j+k} \) which are not \( L \)-good, and since the maximum degree of \( G \) is at most \( K\delta \), there are at most \( 2(K\delta)^k \) ways to extend such a path to a path \( u_{-k}u_{-k+1}\ldots u_k \) of length \( 2k \). The result follows after summing these terms for all \(-k \leq j < 0\).

The next lemma is the first step to relate the notion of \( L \)-goodness with the notion of \((i,j)\)-richness.

Lemma 4.8. Suppose that \( u_{-k}u_{-k+1}\ldots u_k \) is a path in \( G \) which is not \( L \)-good but each of its subpaths of length \( k \) is \( L \)-good. Then, provided that \( L \) is sufficiently large compared to \( |V(H)| \), there exist \( 1 \leq \alpha, \beta \leq k \) with \( \alpha + \beta > k \) such that there exist \( (|V(H)|+2)(2k+1)+1 \) pairwise internally vertex-disjoint paths of length \( \alpha + \beta \) between \( u_{-\alpha} \) and \( u_\beta \).

Proof. Choose \(-k \leq i < j \leq k \) with \( j - i \) minimal such that \( u_iu_{i+1}\ldots u_j \) is not \( L \)-good. By the minimality of \( j - i \), every proper subpath of \( u_iu_{i+1}\ldots u_j \) is \( L \)-admissible. By Lemma 2.5, there exist \( (|V(H)|+2)(2k+1)+1 \) pairwise internally vertex-disjoint paths of length \( j - i \) between \( u_i \) and \( u_j \).

By the assumption that every subpath of \( u_{-k}u_{-k+1}\ldots u_k \) of length \( k \) is good, we have \( j - i > k \), so \( i < 0 \) and \( j > 0 \). Thus, the choices \( \alpha = -i \) and \( \beta = j \) satisfy the conditions described in the lemma.

The next result is the final ingredient to the proof of Lemma 2.6.

Lemma 4.9. Let \( G \) be a graph on \( n \) vertices with maximum degree at most \( K\delta \). Then there are \( O(n\delta^{2k-1}) \) paths \( u_{-k}u_{-k+1}\ldots u_k \) in \( G \) with the following two properties

\( (i) \) \( (u_{-\ell}, u_\ell) \) is \((i,j)\)-poor for every \( 1 \leq \ell \leq k \) and every \( 0 \leq i, j \leq k-1 \) and

\( (ii) \) there exist \( 1 \leq \alpha, \beta \leq k \) with \( \alpha + \beta > k \) such that there exist \( (|V(H)|+2)(2k+1)+1 \) pairwise internally vertex-disjoint paths of length \( \alpha + \beta \) between \( u_{-\alpha} \) and \( u_\beta \).

Proof. Fix a pair \((\alpha, \beta)\) with \( 1 \leq \alpha, \beta \leq k \) and \( \alpha + \beta > k \). It suffices to prove that the number of paths satisfying \( (i) \) and \( (ii) \) for this pair \((\alpha, \beta)\) is \( O(n\delta^{2k-1}) \).

Let \( \ell = \alpha + \beta - k \). Note that \( 1 \leq \ell \leq k \). Also, let \( i = \alpha - \ell = k - \beta \) and \( j = \beta - \ell = k - \alpha \). Observe that \( 0 \leq i, j \leq k - \ell \) - 1.

Suppose that \( u_{-\ell}u_{-\ell+1}\ldots u_\ell \) is a path such that \( (u_{-\ell}, u_\ell) \) is \((i,j)\)-poor. By the definition of \((i,j)\)-poorness, the number of pairs of paths \( (u_{-\ell}u_{-\ell+1}\ldots u_{-\alpha}, u_\ell u_{\ell+1}\ldots u_\beta) \) such that there exist \( (|V(H)|+2)(2k+1)+1 \) pairwise internally vertex-disjoint paths of length \( \alpha + \beta = 2k - i - j \) between \( u_{-\alpha} \) and \( u_\beta \) is \( O(\delta^{i+j-1}) \). Thus, the number of ways to extend \( u_{-\ell}u_{-\ell+1}\ldots u_\ell \) to a path \( u_{-k}u_{-k+1}\ldots u_k \) possessing property \( (ii) \) with our fixed choice of \( \alpha \) and \( \beta \) is \( O(\delta^{i+j-1} \cdot (K\delta)^{k-\alpha+k-\beta}) = O(\delta^{2k-2\ell-1}) \), where the first factor bounds the number
of possible ways to extend to \( u^{-\alpha}u^{-\alpha+1}\ldots u_{\beta} \), and the second factor bounds the number of possible ways to extend that to \( u^{-k}u_{-k+1}\ldots u_{k} \). The number of possible choices for \( u^{-\ell}u^{-\ell+1}\ldots u_{\ell} \) is \( O(n^{2k}) \), so the result follows.

We are now in a position to complete the proof of Lemma 2.6.

**Proof of Lemma 2.6** The condition \(|S| = \omega(n^{\frac{L}{2}})\) implies that \( n^{2k-1} = o\left(\frac{|S|\cdot 2^{2k}}{n}\right) \), so by Lemmas 1.3 and 1.6 there are \( \Omega\left(\frac{|S|\cdot 2^{2k}}{n}\right) \) paths \( u^{-k}u_{-k+1}\ldots u_{k} \) with both endpoints in \( S \) such that \((u_{-\ell}, u_{\ell})\) is \((i, j)-\text{poor}\) for every \( 1 \leq \ell \leq k \) and every \( 0 \leq i, j \leq k - 1 \). Discard all those paths among these in which there is a subpath of length \( k \) which is not \( L\text{-good} \). By Lemma 4.7, we discarded \( O\left(n^{2k}\right) \) paths, which is \( o\left(\frac{|S|\cdot 2^{2k}}{n}\right) \), by the condition \(|S| = \omega(n^{\frac{L}{2}})\). Of the remaining paths, discard all those for which there exist \( 1 \leq \alpha, \beta \leq k \) with \( \alpha + \beta > k \) such that there exist \((|V(H)| + 2)(2k + 1) + 1\) pairwise internally vertex-disjoint paths of length \( \alpha + \beta \) between \( u^{-\alpha} \) and \( u_{\beta} \). By Lemma 1.9 there are \( O(n^{2k-1}) \) such paths, which is again \( o\left(\frac{|S|\cdot 2^{2k}}{n}\right) \). Hence, we are left with \( \Omega\left(\frac{|S|\cdot 2^{2k}}{n}\right) \) paths.

We claim that each such path is \( L\text{-good} \). Suppose otherwise, and take a path \( u^{-k}u_{-k+1}\ldots u_{k} \) which is not \( L\text{-good} \). Since each of its subpaths of length \( k \) is \( L\text{-good} \), by Lemma 1.8 there exist \( 1 \leq \alpha, \beta \leq k \) with \( \alpha + \beta > k \) such that there exist \((|V(H)| + 2)(2k + 1) + 1\) pairwise internally vertex-disjoint paths of length \( \alpha + \beta \) between \( u^{-\alpha} \) and \( u_{\beta} \). But we discarded these paths, which is a contradiction, and the proof is complete.

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