CONTINUOUS AND DISCRETE NOETHER’S FRACTIONAL CONSERVED QUANTITIES FOR RESTRICTED CALCULUS OF VARIATIONS

JACKY CRESSON
Laboratoire de Mathématiques et de leurs Applications
Université de Pau et des Pays de l’Adour
Pau, CNRS 5142, France

FERNANDO JIMÉNEZ*
Departamento de Matemáticas Aplicadas a la Ingeniería Industrial
Universidad Politécnica de Madrid
José Gutiérrez Abascal 2, 28006 Madrid, Spain

SINA OBER-BLÖBAUM
Department of Mathematics, University of Paderborn
Warburger Strasse 100, 33098 Paderborn, Germany

(Communicated by the associate editor name)

Abstract. We prove a Noether’s theorem of the first kind for the so-called restricted fractional Euler-Lagrange equations and their discrete counterpart, introduced in [26, 27], based in previous results [11, 35]. Prior, we compare the restricted fractional calculus of variations to the asymmetric fractional calculus of variations, introduced in [14], and formulate the restricted calculus of variations using the discrete embedding approach [12, 18]. The two theories are designed to provide a variational formulation of dissipative systems, and are based on modeling irreversibility by means of fractional derivatives. We explicit the role of time-reversed solutions and causality in the restricted fractional calculus of variations and we propose an alternative formulation. Finally, we implement our results for a particular example and provide simulations, actually showing the constant behaviour in time of the discrete conserved quantities outcoming the Noether’s theorems.

1. Introduction. Noether’s theorems [33] are a major achievement in mathematical physics in the 20th century. Quoting the comprehensive review [28]: They establish a profound and elegant correspondence, in variational problems, between symmetries and conservation laws or identities, depending on whether the symmetry group is finite (first theorem) or not (second theorem). Naturally, since the variational formulation of physical laws and dynamics is ubiquitous, the extent of applications of Noether’s theorems is barely enbarcable. As a matter of fact, their

2020 Mathematics Subject Classification. Primary: 26A33, 70-XX, 70G65, 37M15, 49-XX.
Key words and phrases. Restricted fractional calculus of variations, Noether’s theorem, discrete embedding.

This work has been funded by the EPSRC project: “Fractional Variational Integration and Optimal Control”; ref: EP/P020402/1.
* Corresponding author: fernando.jimenez.alburquerque@upm.es.
original formulation was made in the context of the mathematical foundations of General Relativity and the associated conservation laws of energy and momenta. However, the application to classical mechanics is direct (see [2] for a geometrical description, where the notion of symmetry is attached to the action of a Lie group over certain spaces), whereas maybe their most notorious consequence in modern physics is the appearance of conserved currents in Gauge Field Theories [37, 38].

Particularly, in this paper we are going to adopt the perspective of the first Noether’s theorem, i.e. the functional integrals of the considered variational problems will be left invariant by continuous finite-dimensional symmetries. This kind of symmetries leads to the so-called “proper” conservation laws, say, those that imply that the conserved quantity has vanishing time derivative “on-shell”, i.e. along the dynamics determined by the Euler-Lagrange equations.

A few years ago, the fractional calculus of variations was introduced in [3], where the state space of Lagrangian functions contains fractional derivatives [36]. This article has induced a large amount of works; we refer to [10] for a review. Its discrete version, say the discrete fractional calculus of variations, was first introduced in [6], and soon after in [12], particularly in the context of discrete embeddings [18]. A huge number of Noether’s like theorems are derived in this framework (see for example [4, 19] and references therein) but we will only refer as main reference to [11], both in the continuous and discrete setting.

The use of fractional derivatives as a possible tool to design a variational framework for dissipative systems was first discussed in the seminal paper [35]. Despite several problems discussed in [15, 16], this article offers an interesting alternative to more classical approaches, for example [7], where the number of variables is doubled by constructing a complementary set of equations. The approach in [35] was discussed and extended in [15], leading to the asymmetric fractional calculus of variations. The main ingredient is the doubling of the state variables with the use of fractional derivatives. However, in order to derive a necessary and sufficient characterization of solutions of dissipative systems, a particular dependence on the doubled variables is used, whose physical interpretation is not simple. This framework gives the first fractional variational formulation of the fractional and linear damping equations [14] as well as for the convection-diffusion equation [16].

Using the same framework, in [26, 27] a novel approach is developed, called restricted fractional calculus of variations. This approach delivers the dynamics of a Lagrangian mechanical system subject to fractional damping, the linear damping being a particular case. As ingredient of relevance, a particular Lagrangian is employed, the dedoubled fractional Lagrangian henceforth. This Lagrangian has a clear physical meaning and can be used to provide variational formulations of some dissipative systems. However, this approach implies that the dynamical equations are only a sufficient condition for the extremals of the involved actions. A variational formulation of the fractional and linear damping equations is given in [26, 27], and for the convection-diffusion equation in [17].

For convenience of the reader, we present a comparison between the asymmetric fractional calculus of variations and the restricted fractional calculus of variations. The role of time reversed solutions in both formalism, as well as causality of the resulting fractional differential equations, is discussed, which leads to an alternative proposal for a variational formulation of dissipative systems.

As for its main goal, the present paper is devoted to the proof of a Noether’s theorem of the first kind in the framework of the restricted fractional calculus of
variations, both in the continuous and discrete cases. We explore more closely the case of dedoubled fractional Lagrangian systems through time reversed solutions. We use the discrete embedding approach [11, 12, 16, 18] to formulate the discrete restricted fractional calculus of variations. Doing so, we show that this formalism provides a nice form of the equations and other objects, for instance, the discrete fractional functional. This point is important since it simplifies the presentation but also the understanding of the computations and allows an easy comparison with the continuous case.

The paper is structured as follows. In §2 we introduce all the necessary preliminaries for further developments, including the definition and relevant properties of fractional derivatives (continuous and discrete), particularly their relationship with the total time derivative of a particular quantity (Lemma 2.2 and Lemma 2.9). Moreover, the statement of the first Noether’s theorem is given in the usual and fractional cases, both in the continuous and discrete settings (Theorem 2.4 and Theorem 2.12). In §3, we give a presentation of the restricted calculus of variations introduced in [26, 27]. This allows us to compare this approach to the asymmetric fractional calculus of variations introduced in [14]. We focus on the properties of the time reversed solutions and their role in the restricted calculus of variations in particular for the so-called dedoubled fractional Lagrangian. We also discuss an alternative fractional variational formalism dealing directly with time-reversed solutions. §4 is devoted to establish the Noether-like theorem of the first kind for the restricted dynamics (REL$^{(\alpha, \beta)}$). This is done in Theorem 4.3. In §5 we display the discretisation of the restricted dynamics, namely the discrete restricted fractional Euler-Lagrange equations (DREL$^{(\alpha, \beta)}$) using the discrete embedding approach. A comparison with the initial formulation provided in [26, 27] is given. We establish the discrete analogue of the Noether’s theorem: Theorem 5.6. Finally, in §6 we treat the example of $O(d)$ symmetry, that is, the invariance under orthogonal transformations of the configuration coordinates. Several simulations support our results.

2. Preliminaries. We remind classical results about fractional derivatives (Riemann-Liouville) and their discrete counterpart given by Grünwald-Letnikov fractional derivatives. We refer to [20, 36] for more details and proofs. The transfer formula and analogous results proved in [11] are given. We also make a short review of results about fractional Lagrangian systems as introduced in [3] and fractional Noether’s theorem in this setting [11].

2.1. Continuous fractional derivatives.

2.1.1. Fractional derivatives. Let us define the Riemann-Liouville $\alpha$-fractional integrals, $\mathbb{R} \ni \alpha > 0$, for $f : [a, b] \rightarrow \mathbb{R}^d$, $d \in \mathbb{N}$, a $AC^2([a, b])$ function and $[a, b] \subset \mathbb{R}$, $0 \leq a < b$:

\begin{align}
I^{-\alpha}_a f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau) \, d\tau, \quad t \in (a, b], \\
I^\alpha_a f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha - 1} f(\tau) \, d\tau, \quad t \in [a, b),
\end{align}

(1)
where $\Gamma$ is the Euler’s Gamma function and we set $I^0_0 f = I^0_+ f = f$. Restricting to $\alpha \in [0,1]$, we further define the Riemann-Liouville $\alpha$-fractional derivatives:

$$D^\alpha_0 f(t) = \frac{d}{dt} I^{1-\alpha}_0 f(t), \quad t \in (a, b],$$  
$$D^\alpha_+ f(t) = -\frac{d}{dt} I^{1-\alpha}_+ f(t), \quad t \in [a, b).$$  

(2)

It is easy to see that $D^0_0 f = D^0_+ f = f$, whereas it can be proven that

$$D^1_0 f = -D^1_+ f = \frac{df}{dt}. $$  

(3)

2.1.2. Properties of fractional derivatives. The fractional calculus of variations is based on the following properties:

- **Integration by parts:**

$$\int_a^b f(t) D^\alpha_0 g(t) dt = \int_a^b \left( D^\alpha_0 f(t) \right) g(t) dt, \quad \lambda \in \{-, +\}. $$  

(4)

- **Semi-group property:**

$$D^\alpha_0 D^\beta_0 = D^{\alpha+\beta}_0, \quad 0 \leq \alpha, \beta \leq 1/2, $$  

(5)

where we assume both functions $f, g \in AC^2([a, b])$.

We refer to [36] for further details and proofs. In particular, when $\alpha = \beta = 1/2$:

$$D^{1/2}_0 D^{1/2}_+ = d/dt, \quad D^{1/2}_+ D^{1/2}_+ = -d/dt. $$  

(6)

2.1.3. The Leibniz and transfer formulas. An important tool when obtaining a total time derivative from fractional quantities is the transfer formula proved in [11].

**Definition 2.1** (Condition C). The functions $f, g \in C^\infty([a, b], \mathbb{R}^d)$ are said to satisfy the condition (C) if the sequences of functions $\left\{ I^{n-\alpha}_+ f \cdot g^{(n)} \right\}_{n \in \mathbb{N}}$ and $\left\{ f^{(n)} \cdot I^{n-\alpha}_+ g \right\}_{n \in \mathbb{N}}$ converge uniformly to 0 on $[a, b]$.

We then have:

**Lemma 2.2** (Transfer formula). Let $f, g \in C^\infty([a, b], \mathbb{R}^d)$ be functions satisfying condition (C), then

$$D^\alpha_0 f \cdot g - f \cdot D^\alpha_0 g = \frac{d}{dt} I_\alpha(f, g), $$  

(7)

where

$$I_\alpha(f, g) = \sum_{n=0}^{\infty} \{ (-1)^n I^{n+1-\alpha}_+ f \cdot g^{(n)} + f^{(n)} \cdot I^{n+1-\alpha}_+ g \}. $$  

(8)

We employ $\cdot$ to denote the inner product between two elements of $\mathbb{R}^d$, while $f^{(n)}$ denotes the $n$-th time derivative of $f$. We refer to ([11], Section 2.4.2) for a discussion about sufficient conditions for the convergence properties to apply (for instance, it is enough that $||f^{(n)}(t)|| \leq M$, $||g^{(n)}(t)|| \leq M$ for $M > 0$ and $t \in [a, b]$).

The previous result is proved thanks to the following result:

**Lemma 2.3.** Let $f, g \in C^\infty([a, b], \mathbb{R}^d)$ be two functions satisfying condition (C). Then, we have for $\lambda \in \{-, +\}$,

$$D^\alpha_\lambda f \cdot g = \frac{d}{dt} J_{\lambda, \alpha}(f, g), $$  

(9)
where
\[
J_{\alpha,\lambda}(f,g) = -\lambda \left( \sum_{r=0}^{\infty} (\lambda)^r T^{r+1-\alpha}_\lambda f \cdot g^{(r)} \right). \tag{10}
\]

We refer to [11] for a proof.

2.2. Continuous fractional Noether’s theorem.

2.2.1. Fractional Lagrangian systems. The fractional calculus of variations is introduced in [3]. Let \( L \) be a \( C^2 \) Lagrangian function \( L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, (x,v) \mapsto L(x,v) \). The fractional action integral \( \mathcal{L}_\alpha : C^2([a,b],\mathbb{R}^d) \to \mathbb{R} \) is defined by
\[
\mathcal{L}_\alpha(q) = \int_a^b L(q(t), D^\alpha q(t)) \, dt. \tag{11}
\]

When \( \alpha = 1 \), we have \( D^1 = -D^1_+ = -d/dt \) so that (11) reduces to the classical action functional. Extremals of \( \mathcal{L}_\alpha \) are solutions of a fractional differential systems (see [3]) called the fractional Euler-Lagrange equation, given by
\[
D^\alpha_+ \left( \frac{\partial L}{\partial v}(q,D^\alpha q) \right) + \frac{\partial L}{\partial x}(q,D^\alpha q) = 0. \tag{12}
\]

When \( \alpha = 1 \), one recovers the classical Euler-Lagrange equation (see [2]):
\[
-\frac{d}{dt} \left( \frac{\partial L}{\partial v}(q,\dot{q}) \right) + \frac{\partial L}{\partial x}(q,\dot{q}) = 0, \tag{13}
\]

In this result, the asymmetric integration by parts (4) plays a crucial role.

2.2.2. Fractional Noether’s theorem. In regards of the Noether’s theorem, we shall restrict ourselves to a Lagrangian scenario with a finite-dimensional configuration space and also a finite-dimensional symmetry. We refer to [2] for a more geometrical point of view. Other relevant references are [4, 21, 22]; whereas we shall follow the main result in [11].

Let \( \varphi_s := \varphi(s,\cdot) : \mathbb{R}^d \to \mathbb{R}^d, s \in \mathbb{R} \), be a one-parameter group of diffeomorphisms. We say that the Lagrangian \( L \) is invariant under the action of \( \varphi_s \) if
\[
L(\varphi_s(q), D^\alpha \varphi_s(q)) = L(q,D^\alpha q). \tag{14}
\]

This corresponds to the fact that \( \varphi_s \) is a variational symmetry (see [34]) for the fractional action functional (11), i.e.
\[
\mathcal{L}_\alpha(\varphi_s(q)) = \mathcal{L}_\alpha(q), \quad \forall \ s \in \mathbb{R}.
\]

The invariance (14) implies the following relation on the solution of the fractional Euler-Lagrange equation:
\[
D_+ \left( \frac{d}{ds} \bigg|_{s=\varphi_s(q)} \right) \frac{\partial L}{\partial v}(q,D^\alpha q,t) - \frac{d}{ds} \bigg|_{s=0} \varphi_s(q) \cdot D^\alpha_+ \left( \frac{\partial L}{\partial v}(q,D^\alpha q,t) \right) = 0. \tag{15}
\]

When \( \alpha = 1 \), we recover the classical Leibniz formula; hence (15) can be rewritten as the total time derivative of the function
\[
J(q) = \frac{d}{ds} \bigg|_{s=0} \varphi_s(q) \cdot \frac{\partial L}{\partial v}(q,\dot{q}). \tag{16}
\]

As a consequence, \( J \) is a constant of motion over the solutions of the classical Euler-Lagrange equations. This accounts for the classical Noether’s theorem [33].

In the fractional case, thanks to Lemma 2.2, we obtain the following analogue:
Theorem 2.4. Given the invariance of the Lagrangian (14), then

\[ J_\alpha(q) = \sum_{n=0}^{\infty} \left\{ (-1)^n I_\alpha^{n+1} \frac{d}{ds} \varphi_s(q) \cdot \frac{d^n}{dt^n} \frac{\partial L}{\partial v}(q, D^n q) \right\}, \]

is a constant of motion, i.e. \( dJ_\alpha/dt = 0 \), along the solutions of (12).

According to the hypothesis in Lemma 2.2, it is assumed that the sequences \( \left\{ I_\alpha^{n-\alpha} \frac{d}{ds} \varphi_s(q) \cdot \frac{d^n}{dt^n} \frac{\partial L}{\partial v}(q, D^n q) \right\} \) and \( \left\{ \frac{d^n}{dt^n} \frac{\partial L}{\partial v}(q, D^n q) \right\} \) converge uniformly to 0 on \([a, b]\).

2.3. Discrete fractional derivatives. Here, we remind classical results about discrete fractional derivatives. We refer to ([29], §20) and ([20], §2.4) for more details. We follow closely [1, 5]. For a different approach considering convolution properties of the fractional derivatives, we refer to [29, 30].

Let \( N \geq 2 \). We denote by \( h = (b-a)/N \) the step size of the discretization and let \( T = \{ t_k \}_{k=0,\ldots,N} = \{ a + kh \}_{k=0,\ldots,N} \) be the usual uniform partition of the interval \([a, b]\). We denote \( T^+ := T \setminus \{ b \} \) and \( T^- := T \setminus \{ a \} \) and by \( C(T, \mathbb{R}^d) \) the set of all functions defined over \( T \) with values in \( \mathbb{R}^d \). For \( t_k \in T \), we denote \( f_k := f(t_k) \) by \( f_k \).

For all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \), we denote by \( x^{(n)} \) and \( (x)_n \) the Pochhammer symbols, where \( x^{(0)} = x \) and \( x^{(1)} = (x)_1 = 1 \) and for \( n \in \mathbb{N}^* \):

\[ x^{(n)} = x(x+1)\cdots(x+n-1), \quad (x)_n = x(x-1)\cdots(x-n+1). \]

Definition 2.5 (Grünwald-Letnikov discrete fractional derivatives). Let \( 0 < \alpha < 1 \). The right discrete fractional derivative is the mapping from \( C(T, \mathbb{R}^d) \) to \( C(T^+, \mathbb{R}^d) \) defined for all \( f \in C(T, \mathbb{R}^d) \) by

\[ \Delta_+^\alpha f(t_k) = \frac{1}{h^\alpha} \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} (\alpha)_n f_{k+n}, \quad \forall \ k = 0, \ldots, N-1. \] (18)

The left discrete fractional derivative is the mapping from \( C(T, \mathbb{R}^d) \) to \( C(T^-, \mathbb{R}^d) \) defined for all \( f \in C(T, \mathbb{R}^d) \) by

\[ \Delta_-^\alpha f(t_k) = \frac{1}{h^\alpha} \sum_{n=0}^{k} \frac{(-1)^n}{n!} (\alpha)_n f_{k-n}, \quad \forall \ k = 1, \ldots, N. \] (19)

Note that when \( \alpha = 1 \), the operators \( \Delta_+^\alpha \) and \( -\Delta_-^\alpha \) correspond to the classical backward and forward Euler operators, namely

\[ \Delta_+^1 f(t_k) = (f_k - f_{k-1})/h, \quad \Delta_-^1 f(t_k) = (f_k - f_{k+1})/h, \] (20)

for \( k = 1, \ldots, N \) and \( k = 0, \ldots, N-1 \), respectively. The relationships (18) and (19) make sense even for \( \alpha < 0 \). Following ([20], §2.4), we denote by \( J_+^\alpha = \Delta_+^{-\alpha} \) and \( J_-^\alpha = \Delta_-^{-\alpha} \) the right and left discrete fractional integrals:

Definition 2.6 (Grünwald-Letnikov discrete fractional integrals). Let \( 0 < \alpha < 1 \). The right discrete fractional integral is the mapping from \( C(T, \mathbb{R}^d) \) to \( C(T^+, \mathbb{R}^d) \) defined for all \( f \in C(T, \mathbb{R}^d) \) by

\[ J_+^\alpha f(t_k) = h^\alpha \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} (\alpha)_n f_{k+n}, \quad \forall \ k = 0, \ldots, N-1. \] (21)
The left discrete fractional integral is the mapping from \( C(T, \mathbb{R}^d) \) to \( C(T_-, \mathbb{R}^d) \) defined for all \( f \in C(T, \mathbb{R}^d) \) by

\[
J_\alpha^0 f(t_k) = h^\alpha \sum_{n=0}^{k} \frac{(-1)^n}{n!} (-\alpha)_n f_{k-n}, \quad \forall k = 1, \ldots, N,
\]  

(22)

We recover at the discrete level the classical relation between the fractional integral and the fractional derivative, i.e.

\[
\Delta_\alpha f = \Delta^-_1 J^1_{-\alpha} f, \quad \Delta^\alpha f = \Delta^+_1 J^1_{+\alpha} f
\]

(23)

This property follows directly from the more general semi-group property of discrete fractional operators.

2.3.1. Properties of the discrete fractional derivatives. Let us define the shift map \( \sigma \) by

\[
\sigma(f) := \sigma(f_k) = f_{k+1}, \quad k = 0, \ldots, N - 1.
\]

(24)

The discrete fractional operators satisfy a semi-group property, which is enclosed in the following result:

**Lemma 2.7** (Semi-group property). For all \( \alpha, \beta > 0 \) and \( \lambda \in \{-, +\} \), we have

\[
\Delta^\alpha J_\lambda^\beta f = \Delta_\lambda^\alpha J^\beta f.
\]

Moreover, we have

\[
\Delta_\lambda^\alpha J_\lambda^\alpha f = f.
\]

(25)

This is in fact valid for arbitrary values of \( \alpha \) and \( \beta \). In particular, it provides directly (23). Of particular importance in many examples is the relationship

\[
\Delta_{1/2}^1 \Delta_{1/2}^1 = \Delta^1_+, \quad \Delta_{1/2}^1 \Delta_{1/2}^1 = \Delta^1_-.
\]

2.3.2. Discrete Leibniz and transfer formulas.

**Lemma 2.8** (Discrete Leibniz formula). Let \( f, g \in C(T, \mathbb{R}^d) \), then the following equality holds

\[
\Delta^1_+ f(t) \cdot g(t) - f(t) \cdot \Delta^1_+ g(t) = \Delta^1_+ (f \cdot (f \sigma(g))(t),
\]

(27)

for all \( t \in T_+ := T_- \cap T^+ \).

The proof is straightforward from (20). The discrete analogue of the transfer formula is obtained thanks to the introduction of a one parameter family of matrices. For each \( N \in \mathbb{N} \), let us denote by \( \{A_n\}_{n=1, \ldots, N-1} \), the family of \( (N+1) \times (N+1) \) matrices \( A_n \) defined by: \( A_1 = -\text{Id}_{N+1} \) and for \( 2 \leq n \leq N - 1 \)

\[
\forall i, j = 0, \ldots, N, \quad (A_n)_{ij} = \begin{cases} 0 & \text{if } i = 0, \\ \delta_{(j=0)} \delta_{(n \leq i)} - \delta_{(i \leq j \leq n-1)} \delta_{(1 \leq j \leq N-n)} & \text{if } 1 \leq i \leq N - 1, \\ (A_n)_{N-1, j} & \text{if } i = N,
\end{cases}
\]

where \( \delta \) is the Kronecker symbol. For instance, if \( N = 5 \), then \( A_1 = -\text{Id}_6 \) and

\[
A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},
\]

In particular, it provides directly (23). Of particular importance in many examples is the relationship
Given the previous elements, we establish the following result:

**Lemma 2.9** (Discrete transfer formula). Let \( f, g \in C(T, \mathbb{R}^d) \), then the following equality holds

\[
\Delta_{\alpha} f(t) \cdot g(t) - f(t) \cdot \Delta_{\alpha} g(t) = \Delta_{\alpha}^1 I_{\alpha,h}(f,g)(t),
\]

for all \( t \in T^+_\downarrow \), where

\[
I_{\alpha,h}(f,g) = \sum_{n=1}^{N-1} \frac{(-1)^n}{n!} (\alpha)_n A_n f \cdot \sigma^n(g).
\]

The proof of (28) can be found in [11].

**Remark 1.** When \( \alpha = 1 \), (28) reduces to (27), since \((\alpha)_n = 0\) for any \( n \geq 2\)

As in the continuous case, we need the following technical result, which will be used in our derivation of the discrete fractional Noether’s theorem:

**Lemma 2.10.** For all \( f, g \in C(T, \mathbb{R}^d) \) and \( \lambda \in \{-, +\} \), we have:

\[
\Delta_{\alpha}^\lambda f(t) \cdot g(t) = \Delta_{\alpha}^\lambda J_{h,\alpha}^\lambda(f,g)(t),
\]

for all \( t \in T^\downarrow \) where the function \( J_{h,\alpha}^\lambda \in C(T, \mathbb{R}^d) \) is defined by

\[
J_{h,\alpha}^\lambda(f,g)(t) = h^{1-\alpha} \sum_{n=0}^{N-1} \sum_{k=0}^{N} \frac{(-1)^n}{n!} (\alpha)_n M_{\alpha}^{\lambda}(t,t_k)((f \cdot \sigma^n(g))_k),
\]

where \( M_{\alpha}^{\lambda} \) is the \((N + 1) \times (N + 1)\) matrix defined by

\[
M_{\alpha}^{-}(t,t_k) = \delta_{t \leq t_k} \delta_{0 \leq t_k \leq t_n - t_k},
\]
and

\[
M_{\alpha}^{+}(t,t_k) = \delta_{0 \leq n \leq N-i-1} \delta_{1 \leq k \leq N-i} + \delta_{N-i \leq n \leq N-1} \delta_{1 \leq k \leq N-n}.
\]

The idea to prove this lemma follows [4], where classical integrals are considered over terms for which an explicit antiderivative can be taken.

**Proof.** Let us consider the quantity \( \Delta_{\alpha} f(t) \cdot g(t) \) for \( t \in T^+_\downarrow \). The function \( J_{h,\alpha}^\lambda(f,g) \) is the discrete integral \( \Delta_{\alpha}^\lambda f(t) \cdot g(t) \) for \( t \in T^\downarrow \), i.e.

\[
J_{h,\alpha}^\lambda(t) = J^\lambda(\Delta_{\alpha} f \cdot g)(t) \in C(T^+, \mathbb{R}).
\]
An explicit computation gives $J^{-}_{h,\alpha}(a) = 0$ and for $i = 1, \ldots, N - 1,$

$$J^{-}_{h,\alpha}(t_i) = \sum_{k=1}^{i} h g_k \cdot \left( \frac{1}{h^\alpha} \sum_{n=0}^{k} \frac{(-1)^n}{n!} (\alpha)_n f_{k-n} \right),$$

$$= h^{1-\alpha} \sum_{n=0}^{i} \frac{(-1)^n}{n!} (\alpha)_n \sum_{k=1}^{i} g_k f_{k-n},$$

$$= h^{1-\alpha} \frac{(-1)^n}{n!} (\alpha)_n \sum_{l=0}^{i-n} g_{l+i} f_l,$n

$$= h^{1-\alpha} \frac{(-1)^n}{n!} (\alpha)_n \sum_{l=0}^{i-n} (\sigma^n(g) \cdot f)(t_l).$$

Regarding the $+\text{ case, we have}$

$$J^{+}_{h,\alpha}(t) = J^{+}_{1} (\Delta^\alpha_n f \cdot g)(t) \in C(T^+, \mathbb{R}).$$

An explicit computation gives $J^{+}_{h,\alpha}(a) = 0$ and for $i = 1, \ldots, N - 1:$

$$J^{+}_{h,\alpha}(t_i) = \sum_{k=1}^{i} h g_k \cdot \left( \frac{1}{h^\alpha} \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} (\alpha)_n f_{k+n} \right),$$

$$= h^{1-\alpha} \left( \sum_{n=0}^{N-1} \sum_{i=0}^{N-i} \sum_{k=1}^{N-n} \sum_{l=1}^{N-n} \frac{(-1)^n}{n!} (\alpha)_n (g \cdot \sigma^n(f))(t_k) \right).$$

We then have

$$J^{+}_{h,\alpha}(t_i) = h^{1-\alpha} \sum_{n=0}^{N-1} \sum_{k=0}^{N} \frac{(-1)^n}{n!} (\alpha)_n M^+_{n}(t_i, t_k)((f \cdot \sigma^n(g))_k),$$

where

$$M^+_{n}(t_i, t_k) = \delta_{0 \leq n \leq N-i-1} \delta_{1 \leq k \leq N-i} + \delta_{N-i-1 \leq n \leq N-1} \delta_{1 \leq k \leq N-n}.$$

2.4. Discrete fractional Noether’s theorem. The study of the discrete analogue of Noether’s theorem for fractional Lagrangian functionals was first introduced in [6, 25]. In [12] variational integrators are developed for fractional Euler-Lagrange equations; the corresponding discrete fractional Noether’s theorem is proved in [11]. The discrete analogue of the continuous objects (functional, equations, etc.) are constructed by means of the discrete embeddings formalism, developed in [11, 12, 16, 18].

2.4.1. Discrete fractional Lagrangian. For $f \in C(T, \mathbb{R})$ a discrete integral is defined by

$$\int_{a}^{b} f(t) \Delta_- t := \sum_{k=1}^{N} f(t_k)(t_k - t_{k-1}).$$

The choice of this particular quadrature is required if one wants to have a discrete analogue of the fundamental theorem of differential calculus, i.e.:

$$\Delta^1 \int_{a}^{t} f(s) \Delta_- s = f(t), \ \forall \ t \in T_-.$$
Let $L$ be a $C^2$ Lagrangian function. The discrete action integral $\mathcal{L}_h : C(\mathbb{T}, \mathbb{R}^d) \to \mathbb{R}$ is defined for $q \in C(\mathbb{T}, \mathbb{R}^d)$ by

$$\mathcal{L}_h(q) = \int_a^b L(q(t), \Delta^1 q(t)) \Delta t.$$ 

Setting $q_k := q(t_k)$, we have

$$\mathcal{L}_h(q) = \sum_{k=1}^N h L \left( q_k, \frac{q_k - q_{k-1}}{h} \right).$$

At the light of the last expression, we observe the link between the discrete embedding formalism and the discrete mechanics approach, as developed in [31]. The functional $\mathcal{L}_h$ corresponds to the classical functional $\mathcal{L}$ by an appropriate choice of a time-scale $\mathbb{T}$, a discrete derivative and a discrete integral. The properties of these functional depends on the algebraic properties of these operators. We refer to [12, 16, 18] for more details.

The previous construction can be generalized to the fractional framework. A discrete fractional action integral $\mathcal{L}^\alpha_h : C(\mathbb{T}, \mathbb{R}^d) \to \mathbb{R}$ is defined for $q \in C(\mathbb{T}, \mathbb{R}^d)$ by

$$\mathcal{L}^\alpha_h(q) = \int_a^b L(q(t), \Delta^\alpha q(t)) \Delta t,$$

which reduces to

$$\mathcal{L}^\alpha_h(q) = \sum_{k=1}^N h L \left( q_k, \Delta^\alpha q(t_k) \right).$$

From (32), the discrete calculus of variations can be directly applied as an exact analogue of the continuous case (see [11, 16, 18]). Extremals of $\mathcal{L}^\alpha_h$ are then solutions of the discrete fractional Euler-Lagrange equation:

$$\Delta^\alpha \left( \frac{\partial L}{\partial v}(q, \Delta^\alpha q) \right) + \frac{\partial L}{\partial x}(q, \Delta^\alpha q) = 0,$$

for $t \in \mathbb{T}^+$. Observe the analogy between the last equation and (12). When $\alpha = 1$, we recover the classical discrete Euler-Lagrange equation:

$$\Delta^1 \left( \frac{\partial L}{\partial v}(q, \Delta^1 q) \right) + \frac{\partial L}{\partial x}(q, \Delta^1 q) = 0,$$

for all $t \in \mathbb{T}^+$; which is the discrete analogue of (13).

2.4.2. Discrete fractional Noether’s theorem ([11]). We define the discrete analogue of a constant of motion based on the following idea:

$$\forall f \in C(\mathbb{T}, \mathbb{R}), \quad \Delta^1 f = 0 \implies \exists c \in \mathbb{R}, \quad f(t) = c, \quad \forall t \in \mathbb{T};$$

from which it follows:

**Definition 2.11.** [Discrete conservation law] We say that the functional $I : C(\mathbb{T}, \mathbb{R}^d) \to C(\mathbb{T}, \mathbb{R})$ is a discrete conservation law for the discrete fractional Lagrangian system (33) if

$$\Delta^1 I(q) = 0,$$

for all solutions $q$ of (33).
Now we look for the correspondence between symmetries and conservation laws at a discrete level. Let \( \varphi_s := \varphi(s, \cdot) : \mathbb{R}^d \to \mathbb{R}^d, s \in \mathbb{R}, \) be a one-parameter group of diffeomorphisms. We say that the Lagrangian \( L \) is invariant under the action of \( \varphi_s \) if
\[
L(\varphi_s(q), \Delta^\alpha \varphi_s(q)) = L(q, \Delta^\alpha q),
\]
along the solutions of (33). The invariance of \( L \) implies that the following equality is satisfied:
\[
\Delta^\alpha \left( \frac{d}{ds} \bigg|_{s=0} \varphi_s(q) \right) \cdot \frac{\partial L}{\partial v}(q, \Delta^\alpha q) - \frac{d}{ds} \bigg|_{s=0} \varphi_s(q) \cdot \Delta^\alpha \left( \frac{\partial L}{\partial v}(q, \Delta^\alpha q) \right) = 0,
\]
along the solutions as well. This expression is the complete discrete analogue of the continuous invariance relation (15).

Using the discrete Leibniz formula (27) when \( \alpha = 1 \), or the discrete transfer equation (28) when \( 0 < \alpha < 1 \), one obtains the discrete counterpart of Theorem 2.4, namely:

**Theorem 2.12** (Discrete fractional Noether’s theorem). Let \( L \) be invariant under the one parameter group of diffeomorphisms \( \varphi_s \). Then, the functional
\[
J^\alpha_d(q) = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} (\alpha)_n A_n \frac{d}{ds} \bigg|_{s=0} \varphi_s(q) \cdot \sigma^{(\alpha)}(\frac{\partial L}{\partial v}(q, \Delta^\alpha q)),
\]
is a discrete conservation law (35) along the solutions of the discrete fractional Euler-Lagrange equation (33).

See [11] for the proof. When \( \alpha = 1 \), one recovers a complete discrete analogue of the continuous conservation law (16), i.e.
\[
J_d(q) = \frac{d}{ds} \bigg|_{s=0} \varphi_s(q) \cdot \sigma \left( \frac{\partial L}{\partial v}(q, \Delta^1 q) \right).
\]

3. **About the restricted fractional calculus of variations.** We recall some basic results on the restricted fractional calculus of variations developed in [26, 27] and show its differences with respect to the asymmetric fractional embedding developed in [14, 16]. Using the time-reversal map, we also give a new version of the restricted calculus of variations using a Lagrangian depending on a function and its time-reversed associate.

3.1. **Doubling of the state variables and fractional derivatives.** The asymmetric fractional embedding and the restricted fractional calculus of variations have a common framework which was first introduced in [14] in the context of constructing non-usual variational formulations of dissipative systems. We refer to ([16], §2) for more details and historical remarks. The origin of this framework is related to two results on dissipative systems:

- Dissipative systems do not possess a classical variational formulation. This follows from the Helmholtz’s Theorem (see [34], Theorem 5.92, p.364-365). A discussion was first provided in [8].
- Variational formulations can be obtained if one adds complementary set of equations, as proved in [7]. Due to the irreversibility of a dissipative system, a given dynamical equation must be considered as physically incomplete as long as the dynamics is not invariant under time-reversing.

To overcome this issues, some proposals were given in [14]:
Helmholtz’s Theorem is strongly based on the framework of the classical differential calculus and is no longer valid in a different setting. The idea of using the fractional calculus as a possible framework to overcome this difficulty was originally introduced in [35]. However, fractional calculus is not enough to recover a variational formulation of dissipative systems. This is due to the duality between the left and right fractional derivative which impose too strong conditions on the set of variations. We refer to [15, 14, 16] where this problem was discussed. As a consequence, the use of fractional derivative must be combined with something else.

Following [7], the classical set of variables used to describe a dissipative system has to be doubled in order to take into account the irreversibility of the dynamics (see [14, 16]). The new variable is understood as encoding the time-reversed dynamics.

The combination of these two remarks leads to the use of a doubled state space coupled with fractional derivatives [14]. Formally, one considers a Lagrangian

$$L(x, y, v_x, v_y, w_x, w_y),$$

and a fractional Lagrangian functionals of the form

$$L_{\alpha, \beta}(x, y) = \int_a^b L(x(t), y(t), \dot{x}(t), \dot{y}(t), D^\alpha_x x(t), D^\beta_y y(t)) \, dt.$$  

Using the previous framework, different strategies have been used to obtain a fractional variational formulation of dissipative systems, as we discuss next.

3.2. Asymmetric fractional embedding versus restricted fractional calculus of variations. The asymmetric fractional embedding approach [14] is based on a particular choice of (38), namely:

$$L(x, y, v_x, v_y, w_x, w_y) = L(x + y, v_x + v_y, w_x + w_y),$$

where $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a given function.

On the other hand, the restricted approach [26, 27] implies

$$L(x, y, v_x, v_y, w_x, w_y) = L(x, v_x) + L(y, v_y) + \rho w_x w_y,$$

where $\rho \in \mathbb{R}$ and $L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ a given function.

The differences between both approaches emerge from the specific structure of the fractional calculus of variations used to obtain the corresponding fractional Euler-Lagrange equation:

- In [14], one focuses on extremals of the form $(x, 0)$ (resp. $(0, y)$) under set of variations of the form $(0, h)$ (resp. $(h, 0)$). This allows to recover causal equations and moreover a necessary and sufficient condition for a function $(x, 0)$ to be an extremal of the functional.

- In [26, 27] a general class of extremals are considered, with variations of the form $(h, h)$. As a consequence, only a sufficient condition is given for a couple of functions $(x, y)$ to be an extremal of the functional.

The two results are very close to each other: the structure of the set of variations in the restricted fractional calculus of variations gives an equal weight to the variations in $x$ and $y$, which allows to mix in a single necessary and sufficient condition the two usual equations of the fractional Euler-Lagrange equations. We refer to §3.6, in particular to equation (57). The reorganization of this quantity as a sum of two causal equations is made by transferring the terms which break causality.
in each part of the initial set of equations. This reorganization is in fact already performed in the asymmetric embedding by the specific structure of the Lagrangian which already mixes the quantities and by considering variations which only acts partially on the variables.

3.3. Interpretation of the variational structure. The question of how natural or physical are the previous construction is in order. In both cases, a variational formulation is obtained thanks to a mixing of past and future in the set of variations allowing to obtain the considered causal equations.

The choice of such variations is precisely related to the irreversibility of the equations that one wants to model (which are unnecessary for reversible equations). From that point of view, the emergence of a special class of variations and fractional derivatives must be seen as strictly induced by the nature of the equations.

Unlike other approaches, particularly those only using classical differential calculus (or the addition of an ad-hoc dual equation in order to obtain a closed system which preserve energy, [7]), the two previous formalisms are associating a specific Lagrangian where the dissipative term is clearly identified and is responsible for the use of fractional derivatives. We believe that such an approach has a deep physical meaning relating the principal physical properties of irreversible systems and the structure of the corresponding calculus of variations (to be explored in future works).

These precision being made, we go further on the description of the restricted fractional calculus of variations by precising in particular the role of time-reversed solutions.

3.4. Restricted calculus of variations. Given two $AC^2([a, b])$ curves $x, y : [a, b] \to \mathbb{R}^d$, let us consider the following notation for the augmented state space, which is contained in the Cartesian product of six copies of $\mathbb{R}^d$:

$$(x, y, \dot{x}, \dot{y}, D_\alpha^x x, D_\beta^y y) \equiv (x, y, v_x, v_y, v_\alpha^x, v_\beta^y) := (x, y)^{(\alpha, \beta)},$$

(42)

with $\alpha, \beta \in (0, 1/2]$. Moreover, define a $C^2$ Lagrangian function

$$L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

$$(x, y, \dot{x}, \dot{y}, D_\alpha^x x, D_\beta^y y) \mapsto L((x, y)^{(\alpha, \beta)}).$$

(43)

The action integral relevant for our purposes is

$$\mathcal{L}_{\alpha, \beta}(x, y) = \int_a^b L(x(t), y(t), \dot{x}(t), \dot{y}(t), D_\alpha^x x(t), D_\beta^y y(t)) dt.$$

(44)

Remark 2. Observe that we define the action (44) as a function of the curves $(x, y)$, whereas the action (11) is a function of the curve and its velocity, $(x, v)$. This can be easily understood by considering that in (11) we set $x$ and $v$ as independent variables, whereas in the phase space (42) the extra variables are the total or fractional derivatives of the curves $(x, y)$. The geometrical meaning of these as the base point of a particular vector bundle is defined in [27].
The restricted fractional Euler-Lagrange equations (REL$(\alpha,\beta)$), which are a sufficient conditions for the extremals of (44), are [26, 27]:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - D^\beta \left( \frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial x} = 0,
\]

(45a)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) - D^\alpha \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial y} = 0.
\]

(45b)

**Remark 3.** The pure variational fractional dynamics, i.e. unrestricted, or in other words, the one outcoming from (44) via the Hamilton’s principle, is given by [26, 27]:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - D^\alpha \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = 0,
\]

(46)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) - D^\beta \left( \frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} = 0.
\]

Dynamics (45) becomes relevant when we pick a particular class of Lagrangian called dedoubled fractional Lagrangian:

**Definition 3.1 (Dedoubled fractional Lagrangian).** Let $L$ be a given Lagrangian, $L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, where $(x,v) \mapsto L(x,v)$ is $C^2$. Let $\rho \in \mathbb{R}_+$. The dedoubled fractional Lagrangian associated to $L$ is defined by

\[
L(x,y,\dot{x},\dot{y}, D^-\alpha - D^\beta) = L(x,\dot{x}) + L(y,\dot{y}) - \rho D^\alpha x \cdot D^\beta y,
\]

(47)

where $\alpha, \beta \in (0, 1/2]$.

In §3.6.2 we will give a detailed justification of (47) based on causal arguments.

**Lemma 3.2 (Dedoubled fractional Euler-Lagrange equations).** Let $L$ be a dedoubled fractional Lagrangian (47). Then, the restricted fractional Euler-Lagrange equations (45) reduce to

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = -\rho D^{\alpha+\beta} x,
\]

(48a)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} = -\rho D^{\alpha+\beta} y.
\]

(48b)

The proof is straightforward, where the crucial property is the semigroup structure of fractional derivatives (5). These equations define the so-called Lagrangian dynamics with fractional damping [27], being the linear damping a particular case when $\alpha = \beta = 1/2$, according to (6).

The significance of this system is not trivial. In the following, we continue the study of [27] giving an interpretation of (48b) thanks to a time reversal map.

### 3.5. Time reversal and the fractional restricted calculus of variations.

As recalled in §3.1, the necessity of using a doubled state space was originally related to the irreversibility of dissipative systems. In this section, we go further on the relationship between reversibility/irreversibility and the use of the restricted calculus of variations by introducing the time reversal map.

**Definition 3.3 (Time reversal).** Let $a, b \in \mathbb{R}, a < b$. We denote by $\star_{a,b}$ the operator defined on $C^0([a,b])$ by

\[
\star_{a,b} : C^0([a,b]) \rightarrow C^0([a,b]),
\]

\[
x(t) \mapsto \star_{a,b}(x)(t) = x(a + b - t), \quad \forall \ t \in [a,b],
\]

(49)
In the following, we simply denote by \( x^\ast \) the function \( \ast_{a,b}(x) \). We note that the time reversal map is an involution, i.e. \( \ast_{a,b}^2 = \text{id} \), where id is the identity.

**Lemma 3.4 (Time reversal duality).** Let \( x \in AC([a,b]) \), the following diagram commutes

\[
\begin{array}{ccc}
  x & \xrightarrow{D_+^\alpha} & D_+^\alpha x \\
  \downarrow{\ast_{a,b}} & & \downarrow{\ast_{a,b}} \\
  x^\ast & \xrightarrow{D_-^\alpha} & D_-^\alpha x^\ast
\end{array}
\]

which implies the equality

\[
D_+^\alpha x^\ast = (D_-^\alpha x)_\ast. \tag{50}
\]

**Proof.** Using the definition of the fractional derivatives (1), (2), we have

\[
D_+^\alpha x^\ast = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} x^\ast(\tau) d\tau,
\]

\[
= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} x(a+b-\tau)(\tau) d\tau.
\]

Making the change of variable \( u = a+b-\tau \), we obtain

\[
D_+^\alpha x^\ast = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{a+b-t} (a+b-t-u)^{-\alpha} x(u) du,
\]

\[
= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{a+b-t} (a+b-t-u)^{-\alpha} x(u) du.
\]

Denoting \( t^\ast = \ast_{a,b}(t) \), we have \( \frac{d}{dt}t^\ast = -\frac{d}{dt}t \) and

\[
D_+^\alpha x^\ast = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{t^\ast} (t^\ast-u)^{-\alpha} x(u) du = D_-^\alpha x(t^\ast) = (D_-^\alpha x)_\ast.
\]

\[\square\]

This result can be used to analyse the relation between the two sides of the dedoubled fractional Euler-Lagrange system (48).

**Remark 4.** The notion of time reversal duality, as introduced in Lemma 3.4 can be found as well in [13] under the name “duality” in a different context. In the framework of this article, we are led to such a notion for dynamical reasons.

**Lemma 3.5.** Let \( x \) be a solution of the dedoubled fractional Euler-Lagrange equation (48a). Then \( x^\ast \) is a solution of the fractional differential equation

\[
-\frac{d}{dt} \left( \frac{\partial L}{\partial v}(x^\ast,-\dot{x}^\ast) \right) - \frac{\partial L}{\partial x}(x^\ast,-\dot{x}^\ast) = -\rho D_+^{\alpha+\beta} x^\ast. \tag{51}
\]

**Proof.** We have

\[
\left( \frac{d}{dt} \left( \frac{\partial L}{\partial v}(x,\dot{x}) \right) - \frac{\partial L}{\partial x}(x,\dot{x}) + \rho D_-^{\alpha+\beta} x \right)_\ast = \\
\left( \frac{d}{dt} \left( \frac{\partial L}{\partial v}(x,\dot{x}) \right) \ast - \frac{\partial L}{\partial x}(x^\ast,\dot{x}^\ast) + \rho \left( D_-^{\alpha+\beta} x \right)_\ast \right).
As \((\dot{x})_\star = -(\dot{x}_\star)\) and using Lemma 3.4, we obtain
\[
\frac{\partial L}{\partial x}(x_\star, (\dot{x})_\star) = \frac{\partial L}{\partial x}(x_\star, -\dot{x}_\star), \quad \rho \left( D^{\alpha+\beta}_- x \right)_\star = \rho D^{\alpha+\beta}_+ x_\star.
\]
In the same way, we have
\[
\left( \frac{d}{dt} \left( \frac{\partial L}{\partial v}(x, \dot{x}) \right) \right)_\star = -\left( \frac{d}{dt} \left( \frac{\partial L}{\partial v}(x_\star, -\dot{x}_\star) \right) \right),
\]
so that the time reversal of equation (48a) reduces to
\[
-\frac{d}{dt} \left( \frac{\partial L}{\partial v}(x_\star, -\dot{x}_\star) \right) - \frac{\partial L}{\partial x}(x_\star, -\dot{x}_\star) + \rho D^{\alpha+\beta}_+ x_\star = 0.
\]
This concludes the proof.

Now, we show that (51) corresponds in some cases to (48b):

**Lemma 3.6.** Assume that \(L\) satisfies the relation
\[
L(x, -v) = L(x, v), \quad \text{for all } (x, v).
\] (52)
If \(x\) is a solution of (48a), then \(x_\star\) satisfies (48b).

**Proof.** The relationship (52) implies that
\[
\frac{\partial L}{\partial x}(x, -v) = \frac{\partial L}{\partial x}(x, v) \quad \text{and} \quad -\frac{\partial L}{\partial v}(x_\star, -\dot{x}_\star) = \frac{\partial L}{\partial v}(x, v).
\]
Replacing the corresponding expressions in equation (51), we obtain equation (48b) for \(x_\star\).

As a consequence, one can provide solutions of the restricted fractional Euler-Lagrange equations only solving one of them. Precisely:

**Corollary 1.** Let \(\mathcal{L}\) be a dedoubled fractional Lagrangian (47), satisfying (52). Let \(x\) be a solution of the equation (48a), then \((x, x_\star)\) is a solution of the restricted fractional Euler-Lagrange equations (45).

This result suggests an alternative way to present the restricted calculus of variations that we explore next.

### 3.6. Time reversed calculus of variations.

Starting from Corollary 1, we try to avoid the dedoubling of state variables by using directly the time reversal of the state variable.

#### 3.6.1. Preliminaries.

Let \(\mathcal{L}\) be a Lagrangian defined by \(\mathcal{L}(x, x_\star, v, v_\star, w, w_\star)\) whose associated functional is defined by
\[
\mathcal{L}_{\alpha,\beta}(x) = \int_a^b \mathcal{L}(x(t), x_\star(t), \dot{x}(t), \dot{x}_\star(t), D^\alpha_- x(t), D^\beta_- x_\star(t)) dt. \quad (53)
\]
A critical point of \(\mathcal{L}_{\alpha,\beta}\) satisfies, for all variation \(h\) such that \(h(a) = h(b) = 0\), the integral relation:
\[
\int_a^b \left[ h \cdot \left( -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial v} \right) + \frac{\partial \mathcal{L}}{\partial x} + D^\alpha_- \left( \frac{\partial \mathcal{L}}{\partial w} \right) \right) + h_\star \cdot \left( -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial v_\star} \right) + \frac{\partial \mathcal{L}}{\partial x_\star} + D^\beta_- \left( \frac{\partial \mathcal{L}}{\partial w_\star} \right) \right) \right] dt = 0.
\]
A sufficient condition ensuring that \( x \) is a critical point of \( \mathcal{L}_{\alpha,\beta} \) is then given by the system
\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial x} &= D_{\alpha}^+ \left( \frac{\partial L}{\partial w} \right), \\
\frac{d}{dt} \left( \frac{\partial L}{\partial v^*} \right) - \frac{\partial L}{\partial x^*} &= D_{\beta}^\gamma \left( \frac{\partial L}{\partial w^*} \right). \tag{54}
\end{align*}
\]

3.6.2. \textit{Causal Lagrangians.} We restrict now our attention to particular functional dependence for our Lagrangian. Precisely, the Lagrangian under consideration must respect the following structural constraints:

S1. The dissipative term must be connected with the emergence of a fractional term in the Lagrangian.

S2. For a reversible dynamical system, the two equations must be the same under time reversal.

We then consider the following class of Lagrangian:

\textbf{Definition 3.7 (Admissible Lagrangian).} A Lagrangian defined on the doubled state space is called admissible if it is of the form
\[
\mathbb{L}(x, x^*, v, v^*, w, w^*) = L(x, v) + L(x^*, v^*) + P(w, w^*). \tag{55}
\]

The separation between terms depending on \( w \) and \( w^* \) and terms depending on \( x, y, v, v^* \) is due to the condition S1. On the other hand, S2 implies that we have a symmetry under the exchange \((x, v) \rightarrow (x^*, v^*)\).

For Lagrangians (55), the sufficient conditions (54) reads
\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial v(x, \dot{x})} \right) - \frac{\partial L}{\partial x(x, \dot{x})} &= D_{\alpha}^+ \left( \frac{\partial P}{\partial w}(D_{\alpha}^x, D_{\alpha}^\gamma x^*) \right), \\
\frac{d}{dt} \left( \frac{\partial L}{\partial v^*(x^*, \dot{x}^*)} \right) - \frac{\partial L}{\partial x^*(x^*, \dot{x}^*)} &= D_{\beta}^\gamma \left( \frac{\partial P}{\partial w^*}(D_{\alpha}^x, D_{\alpha}^\gamma x^*) \right). \tag{56}
\end{align*}
\]

Another set of constraints is related to the interpretation of these equations in terms of causality [14]:

\textbf{Definition 3.8 (Causality condition).} An admissible Lagrangian (55) is said to satisfy the causality condition if:

C1. The dynamics of \( x \) (resp. \( x^* \)) depends only on \( x \) (resp. \( x^* \)) and derivatives of the form \( D_{\alpha}^x \) (resp. \( D_{\alpha}^\gamma x^* \)). In other words: a given equation is associated to a given arrow of time.

Given this, it is apparent that causality can not be recovered from the term
\[
D_{\alpha}^+ \left( \frac{\partial P}{\partial w}(D_{\alpha}^x, D_{\alpha}^\gamma x^*) \right).
\]

As a consequence, following the philosophy of the restricted calculus of variations, we consider a special class of variations called \textit{reversible variations}:

\textbf{Definition 3.9 (Reversible variations).} A function \( h \) is said to be reversible if it is invariant under time reversal, i.e. that \( h = h^* \).

We then have the following theorem:
Theorem 3.10 (Reversed sufficient conditions). Let \( L \) be an admissible Lagrangian \((55)\). A function \( x \) is a critical point of \( L_{\alpha,\beta} \) \((53)\) over the set of reversible variations if and only if it satisfies
\[
\left( \frac{d}{dt} \left( \frac{\partial L}{\partial v}(x, \dot{x}) \right) - \frac{\partial L}{\partial x}(x, \dot{x}) \right) + \left( \frac{d}{dt} \left( \frac{\partial L}{\partial w}(x_*, \dot{x}_*) \right) - \frac{\partial L}{\partial x}(x_*, \dot{x}_*) \right) = D^\alpha_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right) + D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right).
\]

Under the symmetry condition \((52)\), this condition can be rewritten as
\[
(EL(x)) = D^\alpha_+ \left( \frac{\partial P}{\partial w}(D^\alpha_- x, D^\beta_+ x_*) \right) + D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right),
\]
where \( EL(x) \) is the classical Euler-Lagrange quantity \((13)\) associated to \( L \), i.e.
\[
EL(x) = \frac{d}{dt} \left( \frac{\partial L}{\partial v}(x, \dot{x}) \right) - \frac{\partial L}{\partial x}(x, \dot{x}).
\]

In order to recover causality, we have to impose that:
- The quantities
  \[
  D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right), \quad D^\alpha_+ \left( \frac{\partial P}{\partial w}(D^\alpha_- x, D^\beta_+ x_*) \right)
  \]
  can be expressed using only derivatives of the form \( D^\gamma x \) and \( D^\gamma_+ x_+ \) for certain values of \( \gamma \).
- We have
  \[
  \left( D^\alpha_+ \left( \frac{\partial P}{\partial w}(D^\alpha_- x, D^\beta_+ x_*) \right) \right)_* = D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right).
  \]

The second constraints implies that \((58)\) can be rewritten as:
\[
\left( EL(x) - D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right) \right) + \left( EL(x) - D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right) \right)_* = 0,
\]
so that a sufficient condition reads
\[
EL(x) - D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right) = 0.
\]

Given this, C1 implies
- \( P \) must be a polynomial in \( w \) and \( w_* \).
- The quantity \( \frac{\partial P}{\partial w_*} \) depends only on \( w \) and \( \frac{\partial P}{\partial w} \) depends only on \( w_* \).

Consequently, we can establish a rigidity result:

Lemma 3.11 (Causal rigidity). Under the causality condition C1, the function \( P \) must be of the form \( P(w, w_*) = \rho \, w \, w_*\), where \( \rho \in \mathbb{R} \) is a constant.

Proof. As the partial derivatives must be polynomials, \( P \) must be a polynomial. The conditions on the partial derivatives implies that only terms of the form \( w \, w_* \) are allowed.

In the causal case, we then have
\[
D^\beta_+ \left( \frac{\partial P}{\partial w_*}(D^\alpha_- x, D^\beta_+ x_*) \right) = \rho \, D^{\alpha + \beta}_- x, \quad D^\alpha_+ \left( \frac{\partial P}{\partial w}(D^\alpha_- x, D^\beta_+ x_*) \right) = \rho \, D^{\alpha + \beta}_+ x_+.
\]
As \( \left( \rho D^{\alpha+\beta}_- x \right)_* = \rho D^{\alpha+\beta}_+ x_* \), the sufficient condition (57) reads
\[
\left( EL(x) + \rho D^{\alpha+\beta}_- x \right)_* + \left( EL(x) + \rho D^{\alpha+\beta}_+ x \right)_* = 0.
\] (60)
As a consequence, we obtain the following lemma:

**Lemma 3.12.** Let \( L \) be the Lagrangian
\[
L(x, x_*, v, v_*, w, w_*) = L(x, v) + L(x_*, v_*) + \rho w w_*,
\] (61)
where \( \rho \in \mathbb{R} \) is a constant. Then, under the structural assumptions S1, S2 and causality C1, a sufficient condition for \( x \) to be a critical point of the time reversed fractional functional (53) is
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v}(x, \dot{x}) \right) - \frac{\partial L}{\partial x}(x, \dot{x}) + \rho D^{\alpha+\beta}_- x = 0.
\]
Proof. This follows directly from equation (60). \( \square \)

4. Continuous fractional conserved quantity for restricted calculus of variations. We derive a Noether’s theorem in the framework of the restricted fractional Euler-Lagrange equations dynamics REL\((^{\alpha,\beta})\) (45). Under the light of its time-reversed version exposed in §3.6, the conservation law takes a nice form in the case of dedoubled fractional Lagrangian when \( \alpha = \beta \) as it possesses a decomposition as a sum of a given quantity and its time reversal associate.

4.1. Fractional variational symmetries. We refer to ([34],Chapter 4) for a classical revision on variational symmetries. Let us consider a one-parameter, \( s \in \mathbb{R} \), group of diffeomorphisms denoted by \( \{ \phi_s \}_{s \in \mathbb{R}} \) acting on the doubled configuration space \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \) as
\[
\phi_s := \phi(s, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d,
\]
\[
(x, y) \mapsto \phi_s(x, y) := (\varphi_s(x), \varphi_s(y)),
\] (62)
where \( \varphi_s := \varphi(s, \cdot) : \mathbb{R}^d \to \mathbb{R}^d \) is also a one-parameter group of diffeomorphisms.

**Remark 5.** These groups:

- Do not allow a mixing between the \( x \) and \( y \) variables. This means that they are preserving the product structure of the doubled configuration space. They can be called as fibered as they are preserving the fibers ([34],p.93).
- The action on each variable separately is the same. It means that the two variables must be considered as related in some sense; otherwise this assumption is too strong. The time-reversed calculus of variations, exposed in §3.6 as an alternative presentation of the restricted calculus of variations, can be used to justify this point: the action on \( x \) and \( x_* \) is by definition the same. It must be noted that such a result is not true when one wants to consider a general class of transformation depending on time.

The associated infinitesimal group action (see [34], p.51) associated to \( \phi_s \) is denoted by
\[
\varphi_s(x) = x + s \xi(x) + o(s),
\] (63)
where \( \xi : \mathbb{R}^d \to \mathbb{R}^d \), which, naturally, is the infinitesimal symmetry [2, 34]. From this, it is easy to see that \( d \varphi_s(x)/ds|_{s=0} = \xi(x) \). The action over the augmented state space (42) follows from (62)
\[
\phi_s(x, y)^{(\alpha, \beta)} := (\varphi_s(x), \varphi_s(y), d\varphi_s(x)/dt, d\varphi_s(y)/dt, D^\alpha_\varphi_s(x), D^\beta_\varphi_s(y)).
\]
Using this notation, we generalize the notion of a variational symmetry group of a functional (see [34], Definition 4.10, p.253) to define the fractional fibered variational symmetries adapted to functionals defined by (43):

**Definition 4.1** (Fractional fibered variational symmetries). A local group of fibered transformations \( \{ \varphi \}_{s \in \mathbb{R}} \) is called a fibered variational symmetry group of the functional (44) if and only if

\[
\int_{c}^{b} \mathcal{L}(\varphi_{s}(x, y)^{(\alpha, \beta)}) \, dt = \int_{c}^{b} \mathcal{L}((x, y)^{(\alpha, \beta)}) \, dt,
\]

for all subintervals \([c, b] \subset [a, b]\).

**Remark 6.** The previous formulation does not cover transformations depending on time, as is the case when the base point of the fractional derivative is impacted by the transformation group. We refer to ([19], §3.1 p.876-877) for more details.

The infinitesimal criterion of invariance following from (64) (see [34],p.253) is then given by:

\[
\mathcal{L}(\varphi_{s}(x, y)^{(\alpha, \beta)}) = \mathcal{L}((x, y)^{(\alpha, \beta)}).
\]

**4.2. Fractional Noether’s theorem.** The proof of Theorem 4.3 is based on the next lemma, which gives an explicit form of the invariance condition:

**Lemma 4.2.** Let us denote by \( A_{\alpha}(f, g) \) the quantity

\[
A_{\alpha}(f, g) = D_{\alpha}^\alpha f - g \cdot D_{\alpha}^\beta f,
\]

and by \( B_{\alpha, \beta}(f, g) \) the quantity

\[
B_{\alpha, \beta}(f, g) = D_{\alpha}^\alpha f - D_{\beta}^\beta g.
\]

Given that \( \mathcal{L} \) is of the form (43) and admits a fractional fibered variational symmetry (64), then we have

\[
0 = \frac{d}{dt} \bigg|_{s=0} \mathcal{L}(\varphi_{s}(x, y)^{(\alpha, \beta)}) + A_{\alpha} \left( \xi(x), \frac{\partial \mathcal{L}}{\partial v_{x}^\alpha} \right) + B_{\alpha, \beta} \left( \frac{\partial \mathcal{L}}{\partial v_{x}^\alpha}, \frac{\partial \mathcal{L}}{\partial v_{y}^\beta} \right) \cdot \xi(x)
\]

\[
- \left( A_{\beta} \left( \frac{\partial \mathcal{L}}{\partial v_{y}^\beta}, \xi(y) \right) + B_{\alpha, \beta} \left( \frac{\partial \mathcal{L}}{\partial v_{x}^\alpha}, \frac{\partial \mathcal{L}}{\partial v_{y}^\beta} \right) \cdot \xi(y) \right).
\]

**Proof.** We depart from the invariance condition (65), which implies

\[
\left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(\varphi_{s}(x, y)^{(\alpha, \beta)}) = 0.
\]

Expanding the left hand side we get:

\[
\left. \frac{d}{ds} \right|_{s=0} \mathcal{L}(\varphi_{s}(x, y)^{(\alpha, \beta)}) = \frac{\partial \mathcal{L}}{\partial x} \cdot \left. \frac{d}{ds} \right|_{s=0} \varphi_{s}(x) + \frac{\partial \mathcal{L}}{\partial y} \cdot \left. \frac{d}{ds} \right|_{s=0} \varphi_{s}(y)
\]

\[
+ \frac{\partial \mathcal{L}}{\partial v_{x}^\alpha} \cdot \left. \frac{d}{ds} \right|_{s=0} D_{\alpha}^\beta \varphi_{s}(x) + \frac{\partial \mathcal{L}}{\partial v_{y}^\beta} \cdot \left. \frac{d}{ds} \right|_{s=0} D_{\beta}^\beta \varphi_{s}(y).
\]
Replacing $\partial L/\partial x$ and $\partial L/\partial y$ according to REL$^{(\alpha, \beta)}$ (45) and taking into account that $d/ds$ commutes both with $d/dt$ and $D_{\pm}^\alpha$ (according to the definitions (2)), we obtain:

$$\left. \frac{d}{ds} \right|_{s=0} L(\phi, (x, y))^{(\alpha, \beta)} = \frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) + \frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) \cdot \xi(y) + \frac{\partial L}{\partial v_x} \cdot \xi(y) + \frac{\partial L}{\partial v_y} \cdot \xi(y)$$

$$- D_+^\alpha \frac{\partial L}{\partial v_x} \cdot \xi(x) - D_+^\alpha \frac{\partial L}{\partial v_y} \cdot \xi(y) + \frac{\partial L}{\partial v_y} \cdot D_+^\alpha \xi(x) + \frac{\partial L}{\partial v_x} \cdot D_+^\alpha \xi(y)$$

$$= \frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \cdot \xi(x) + \frac{\partial L}{\partial v_y} \cdot \xi(y) \right)$$

$$- D_+^\alpha \frac{\partial L}{\partial v_x} \cdot \xi(x) - D_+^\alpha \frac{\partial L}{\partial v_y} \cdot \xi(y) + \frac{\partial L}{\partial v_y} \cdot D_+^\alpha \xi(x) + \frac{\partial L}{\partial v_x} \cdot D_+^\alpha \xi(y).$$

Next, we manipulate the last line in order to obtain a total time derivative according to Lemma 2.2 and Lemma 9. Let us denote:

$$f_x := \frac{\partial L}{\partial v_x^\alpha}, \quad f_y := \frac{\partial L}{\partial v_y^\beta}.$$

Then, we have

$$- D_+^\alpha f_y \cdot \xi(x) - D_+^\alpha f_x \cdot \xi(y) + f_x \cdot D_+^\alpha \xi(x) + f_y \cdot D_+^\alpha \xi(y)$$

$$= \left( f_x \cdot D_+^\alpha \xi(x) - D_+ f_x \cdot \xi(x) \right) + \left( f_y \cdot D_+^\alpha \xi(y) - D_+ f_y \cdot \xi(y) \right)$$

$$+ D_+^\alpha f_x \cdot \xi(x) + D_+^\alpha f_y \cdot \xi(y)$$

$$= \left( f_x \cdot D_+^\alpha \xi(x) + f_y \cdot D_+^\alpha \xi(y) \right) + \xi(y) \cdot \left( D_+^\alpha f_x - D_+^\beta f_y \right),$$

where in $=1$ we have added and subtracted $\xi(y) \cdot D_+^\beta f_y$ and $\xi(x) \cdot D_+ f_x$. As can be easily checked, the right hand side of the last equality corresponds to

$$A_\alpha \left( \xi(x), \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y^}\beta \right) + B_{\alpha, \beta} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y^\beta} \right) \cdot \xi(x) - B_{\alpha, \beta} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y^\alpha} \right) \cdot \xi(y).$$

\[\square\]

**Theorem 4.3 (Fractional Noether’s theorem).** Let $L$ be a Lagrangian of the form (43) admitting a fractional fibered variational symmetry (64). Then the following quantity

$$I(x, y) = \frac{\partial L}{\partial v_x} \cdot \xi(x) + \frac{\partial L}{\partial v_y} \cdot \xi(y) + I_\alpha \left( \xi(x), \frac{\partial L}{\partial v_x^\alpha} \right) - I_\beta \left( \frac{\partial L}{\partial v_y^\beta}, \xi(y) \right)$$

$$+ \left( J_{\alpha, +} \left( \frac{\partial L}{\partial v_x^\alpha}, \xi(x) \right) - J_{\beta, -} \left( \frac{\partial L}{\partial v_y^\beta}, \xi(x) \right) \right)$$

$$- \left( J_{\alpha, +} \left( \frac{\partial L}{\partial v_x^\alpha}, \xi(y) \right) - J_{\beta, -} \left( \frac{\partial L}{\partial v_y^\beta}, \xi(y) \right) \right)$$

is a constant of motion along the solutions $(x, y)$ of the REL$^{(\alpha, \beta)}$ equations (45).

Above, $I_\alpha$ is defined in (8), whereas $J_{\alpha, \lambda}$ is defined in (10).
Lemma 4.4. For all dedoubled fractional Lagrangian, we have
\[(45)\] where \(x\) and \(B\) form a total derivative using the transfer formula (7). The two other quantities can also be written as a total derivative thanks to Lemma 2.3. The total time derivatives are the quantities forming \(I(x, y)\).

4.3. Fractional Noether theorem and time reversed solutions. The previous result can in fact be precised in situations of interest. We first prove several technical lemmas about the action of the time reversal map on the quantities \(A_\alpha(f, g)\) and \(B_{\alpha, \beta}(f, g)\).

**Lemma 4.5.** For all \(f, g \in AC^2([a, b])\), we have
\[(B_{\alpha, \beta}(f, g))_\ast = -B_{\beta, \alpha}(g_\ast, f_\ast).\]

**Proof.** Using Lemma 3.4, we have
\[(B_{\alpha, \beta}(f, g))_\ast = (D^\alpha f)_\ast - g_\ast \cdot (D^\alpha f)_\ast = D^\alpha g_\ast \cdot f_\ast - g_\ast \cdot D^\alpha f_\ast = -A_\alpha(g_\ast, f_\ast).\]

**Lemma 4.6.** For all \(f, g, h \in AC^2([a, b])\), let us denote by \(\mathcal{A}_{\alpha, \beta}(f, g, h)\) the quantity
\[\mathcal{A}_{\alpha}(f, g, h) = A_\alpha(h, f) + B_{\alpha, \beta}(f, g) h.\]

Then, we have
\[\mathcal{A}_{\alpha, \beta}(f, g, h)_\ast = -A_{\alpha}(f_\ast, h_\ast) - B_{\beta, \alpha}(f_\ast, g_\ast) h_\ast.\]

If the functions in the previous lemma are vector valued, then the term concerning \(B_{\alpha, \beta}\) becomes \(B_{\alpha, \beta}(f, g) \cdot h\). The proof follows directly from Lemma 4.5 and Lemma 4.5. The quantity \(\mathcal{A}_{\alpha, \beta}\) can be used to rewrite the conclusion of Lemma 4.2 as follows:

**Lemma 4.7.** Let \(L\) be a dedoubled fractional Lagrangian (47) satisfying (52). Let \((x, x_\ast)\) be a solution of the REL\((\alpha, \beta)\) equations (45) where \(x\) is a solution of (48a). Then, we have:
\[0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \cdot \dot{\xi}(x) \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}_y} \cdot \dot{\xi}(x) \right)_\ast + \mathcal{A}_{\alpha, \beta} \left( \frac{\partial L}{\partial \dot{v}_y}, \frac{\partial L}{\partial \dot{v}_y}, \xi(x) \right)_\ast + \left( A_{\beta}(\xi(x) , -\rho D^\alpha x_\ast) + B_{\beta, \alpha} \left( \frac{\partial L}{\partial \dot{v}_y}, \frac{\partial L}{\partial \dot{v}_y} \right) \cdot \xi(x) \right)_\ast.\]

**Proof.** The fact that we can consider \((x, x_\ast)\) as a solution of the REL\((\alpha, \beta)\) equations (45) where \(x\) is a solution of (48a) follows from Lemma 3.6. Moreover, as \(L\) is a dedoubled fractional Lagrangian, we have
\[\frac{\partial L}{\partial \dot{v}_y} = D^\beta y\] and \[\frac{\partial L}{\partial \dot{v}_y} = D^\alpha x.\]
Using the duality result of Lemma 3.4, we obtain:
\[
\left( \frac{\partial L}{\partial v_\alpha^\beta} \right)_* = (D_+^\beta x)_* = D_\alpha^\beta x, \quad \left( \frac{\partial L}{\partial v_\alpha^\beta} \right)_* = (D_\alpha^\beta x)_* = D_\alpha^\beta x_*,
\]
As a consequence, we have
\[
- A_\beta \left( \frac{\partial L}{\partial v^\beta_\alpha}, \xi(x) \right) - B_{\alpha,\beta} \left( \frac{\partial L}{\partial v_\alpha^\beta}, \frac{\partial L}{\partial v^\beta_\alpha} \right) \cdot \xi(x) = - A_\beta \left( -\rho D_\alpha^\alpha x, \xi(x) \right) - B_{\alpha,\beta} \left( -\rho D_\alpha^\beta x_*, -\rho D_\alpha^\alpha x \right) \cdot \xi(x) = - A_\beta \left( (-\rho D_\alpha^\alpha x_*)_*, (\xi(x))_* \right) - B_{\alpha,\beta} \left( (-\rho D_\alpha^\beta x)_*, (-\rho D_\alpha^\alpha x_*)_* \right) \cdot (\xi(x))_* = (A_\beta \left( (\xi(x), -\rho D_\alpha^\alpha x_*)_* \right) + B_{\beta,\alpha} \left( -\rho D_\alpha^\beta x_*, -\rho D_\alpha^\beta x \right) \cdot (\xi(x))_*,
\]
By definition of $B_{\alpha,\beta}$, we have
\[
B_{\beta,\alpha} \left( -\rho D_\alpha^\beta x_*, -\rho D_\alpha^\beta x \right) = -\rho D_\alpha^\beta x_*, -\rho D_\alpha^\beta x
\]
\[
= B_{\alpha,\beta} \left( -\rho D_\alpha^\beta x_*, -\rho D_\alpha^\beta x \right) = B_{\alpha,\beta} \left( \frac{\partial L}{\partial v^\alpha_\beta}, \frac{\partial L}{\partial v^\beta_\alpha} \right).
\]
Consequently:
\[
- A_\beta \left( \frac{\partial L}{\partial v^\beta_\alpha}, \xi(x) \right) - B_{\alpha,\beta} \left( \frac{\partial L}{\partial v_\alpha^\beta}, \frac{\partial L}{\partial v^\beta_\alpha} \right) \cdot \xi(x) = (A_\beta \left( (\xi(x), -\rho D_\alpha^\alpha x_*)_* \right) + B_{\beta,\alpha} \left( \frac{\partial L}{\partial v^\alpha_\beta}, \frac{\partial L}{\partial v^\beta_\alpha} \right) \cdot (\xi(x))_* ,
\]
which concludes the proof. □

The expression is not completely symmetric under time reversal due to the fact that we can not express $A_\beta \left( (\xi(x), -\rho D_\alpha^\alpha x_*)_* \right)$ as $A_\beta \left( (\xi(x), \partial L/\partial v^\alpha_\beta) \right)$ in general. This problem disappears when we assume that $\beta = \alpha$, which plays an essential role in many applications (see [17, 27]).

**Theorem 4.8** (Fractional Noether’s theorem, reversed case). Let $L$ be a dedoubled fractional Lagrangian (47) satisfying (52). Let $(x, x_* )$ be a solution of the REL$^{(\alpha,\beta)}$ equations (45) where $x$ is a solution of (48a). Assume moreover that $\alpha = \beta$. Let us denote by $\mathcal{J}_*(x, y)$ the quantity
\[
\mathcal{J}_*(x, y) = \frac{\partial L}{\partial v^\alpha_\beta} \cdot \xi(x) + I_\alpha \left( (\xi(x), \frac{\partial L}{\partial v^\alpha_\beta}) \right) + J_{\alpha,+} \left( \frac{\partial L}{\partial v^\alpha_\beta}, \xi(x) \right) - J_{\alpha,-} \left( \frac{\partial L}{\partial v^\alpha_\beta}, \xi(x) \right).
\]

Then the quantity
\[
I(x, y) = \mathcal{J}_*(x, y) + (\mathcal{J}_*(x_*, y))_* ,
\]
is a constant of motion along the solutions $(x, x_*)$ of the REL$^{(\alpha,\alpha)}$ equations.

**Proof.** This follows directly from Lemma 4.7. Indeed, when $\alpha = \beta$, we have
\[
A_\beta \left( (\xi(x), -\rho D_\alpha^\alpha x_*)_* \right) = A_\beta \left( (\xi(x), \partial L/\partial v^\alpha_\beta) \right),
\]
and the invariance condition reads
\[
0 = \frac{d}{dt} \left( \frac{\partial L}{\partial v^\alpha_\beta} \cdot \xi(x) \right) + \mathcal{J}_*(x_*, y) + \left( \frac{d}{dt} \left( \frac{\partial L}{\partial v^\alpha_\beta} \cdot \xi(x) \right) \right)_* + \left( \mathcal{J}_*(x_*, y) \right)_* .
\]
Using the transfer Lemma 2.2 and Lemma 9 we arrive to the result.

The form of the conserved quantity is interesting by itself, since we have a symmetric decomposition of the conservation law: one part conducted by \((x, x_*)\) and the same quantity in the reversed case. This decomposition is reminiscent of [7], where an associated dynamics is designed in order to close the system and to recover its conservative nature.

5. Discrete fractional conserved quantity for restricted calculus of variations. In this section, we discuss a discrete version of the fractional Noether’s theorem proved in §4, using the discrete embedding approach [12, 18].

5.1. Discrete restricted calculus of variations.

**Definition 5.1** (Discrete fractional action). Let \(L\) be a Lagrangian of the form (43). The discrete fractional action functional associated to \(L\) over \(\mathbb{T}\), denoted by \(\mathcal{L}_{\alpha,\beta}\), is the map \(C(\mathbb{T}, \mathbb{R}^d) \times C(\mathbb{T}, \mathbb{R}^d) \rightarrow \mathbb{R}\) defined for all \((x, y) \in C(\mathbb{T}, \mathbb{R}^d) \times C(\mathbb{T}, \mathbb{R}^d)\) by

\[
\mathcal{L}_{\alpha,\beta}(x, y) = \int_0^b L(x(t), y(t), \Delta^1_x x(t), -\Delta^1_y y(t), \Delta^\alpha_x x(t), \Delta^\beta_y y(t)) \Delta t.
\] (68)

This definition corresponds to a discrete embedding of the continuous functional \(L\) into \(C^{1,0}\) functions, which are equivalent to \(C^0\) functions when \(\alpha, \beta \rightarrow 1\). Using the discrete embedding approach [12, 18], we arrive to the result.

\[
\mathcal{L}_{\alpha,\beta}(x, y) = h \sum_{k=1}^N L(x_k, y_k, \Delta^1_x x_k, -\Delta^1_y y_k, \Delta^\alpha_x x_k, \Delta^\beta_y y_k).
\] (69)

**Remark 7.** The minus sign in front of \(\Delta^1_y\) ensures that the discrete fractional action (69) is an approximation of the continuous functional (64). Indeed, by definition \(\Delta^1_y\) converges to \(-d/dt\) when \(h \to 0\).

The discrete state space is denoted by

\[(x, y)^{(h, \alpha, \beta)} := (x, y, \Delta^1_x x, -\Delta^1_y y, \Delta^\alpha_x x, \Delta^\beta_y y).\] (70)

The discrete restricted Euler-Lagrange equations (DREL) are

\[
\Delta^1_x \left( \frac{\partial L}{\partial v_x} (x, y)^{(h, \alpha, \beta)} \right) + \Delta^\beta \left( \frac{\partial L}{\partial v_y} (x, y)^{(h, \alpha, \beta)} \right) + \frac{\partial L}{\partial x} (x, y)^{(h, \alpha, \beta)} = 0,
\]

\[
-\Delta^1_y \left( \frac{\partial L}{\partial v_y} (x, y)^{(h, \alpha, \beta)} \right) + \Delta^\alpha \left( \frac{\partial L}{\partial v_x} (x, y)^{(h, \alpha, \beta)} \right) + \frac{\partial L}{\partial y} (x, y)^{(h, \alpha, \beta)} = 0.
\] (71)

These equations, which clearly are the discrete counterpart of (45) in the spirit of discrete embeddings [12, 18], are a sufficient condition for the extremals of (69) [26, 27]. When we pick the dedoubled fractional Lagrangian (47), the previous equations read:

\[
\Delta^1_x \left( \frac{\partial L}{\partial v} (x, \Delta^1_x x) \right) + \frac{\partial L}{\partial x} (x, \Delta^1_x x) = \rho \Delta_{-}^{\beta + \alpha} x,
\] (72a)

\[
-\Delta^1_y \left( \frac{\partial L}{\partial v} (y, -\Delta^1_y y) \right) + \frac{\partial L}{\partial x} (y, -\Delta^1_y y) = \rho \Delta_{+}^{\alpha + \beta} y.
\] (72b)
The proof is straightforward and follows from the semigroup property of the discrete fractional derivatives (25). Naturally, these are the discrete counterpart of equations (48).

5.2. Time reversal and discrete restricted calculus of variations. The time reversal map \( *_{a,b} \) introduced in §3.5 naturally acts on \( C(\mathbb{T}, \mathbb{R}^d) \). For each \( x \in C(\mathbb{T}, \mathbb{R}^d) \), we have \( x_* \in C(\mathbb{T}, \mathbb{R}^d) \) defined by \( x_*(t) = x(a + b - t) \) for all \( t \in \mathbb{T} \). As \( t_k = a + k \frac{(b - a)}{N} \), we have \( t_{N-k} = a + b - t_k \), i.e.

\[
x_*(t_k) = x(t_{N-k}).
\] (73)

Lemma 5.2 (Discrete time reversal duality). Let \( x \in C(\mathbb{T}, \mathbb{R}) \), the following diagram commutes

\[
x \xrightarrow{\Delta_-^\alpha} \Delta_-^\alpha x \\
*_{a,b} \downarrow \quad \downarrow *_{a,b} \\
x_* \xrightarrow{\Delta_+^\alpha} \Delta_+^\alpha x_*
\]

which implies

\[
\Delta_-^\alpha x_* = (\Delta_-^\alpha x)_*.
\]

Proof. For all \( t_k \in \mathbb{T} \), we have

\[
(\Delta_-^\alpha x(t_k))_* = \Delta_-^\alpha x(t_{N-k}) = \frac{1}{h^\alpha} \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} (\alpha)_n x(t_{N-k-n}) = \frac{1}{h^\alpha} \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} (\alpha)_n x_*(t_{k+n}) = \Delta_+^\alpha x_*(t_k).
\]

As a consequence, we have a discrete analogue of Corollary 1:

Lemma 5.3. Let \( L \) be a dedoubled fractional Lagrangian (47), satisfying (52). Let \( x \) be a solution of the equation (72a), then \( (x, x_*) \) is a solution of the discrete restricted fractional Euler-Lagrange equations (71).

Proof. We have

\[
\left( \Delta_+^1 \left( \frac{\partial L}{\partial v}(x, \Delta_- x) \right) \right)_* + \frac{\partial L}{\partial x}(x, \Delta_- x) - \rho \Delta_-^{\beta+\alpha} x = 0.
\]

Using Lemma (5.2):

\[
\left( \Delta_+^1 \left( \frac{\partial L}{\partial v}(x, \Delta_- x) \right) \right)_* = \Delta_-^1 \left( \frac{\partial L}{\partial v}(x_*, \Delta_- x)_* \right) = \Delta_-^1 \left( \frac{\partial L}{\partial v}(x_*, \Delta_-^1 x)_* \right) = -\Delta_-^1 \left( \frac{\partial L}{\partial v}(x_*, -\Delta_-^1 x)_* \right),
\]

and

\[
\frac{\partial L}{\partial x}(x_*, (\Delta_-^1 x)_*) = \frac{\partial L}{\partial x}(x_*, \Delta_-^1 x_*) = \Delta_-^1 \left( \frac{\partial L}{\partial v}(x_*, \Delta_-^1 x)_* \right) = 0.
\]
Moreover, $\rho \left( \Delta^{\beta+\alpha}_- x \right)_* = \rho \Delta^{\beta+\alpha}_- x_*$, so that $x_*$ satisfies the equation

$$-\Delta^1 \left( \frac{\partial L}{\partial v} (x_*, -\Delta^1_+ x_*) \right) + \frac{\partial L}{\partial x} (x_*, -\Delta^1_+ x_*) - \rho \Delta^{\beta+\alpha}_- x_* = 0,$$

where we have used the symmetry (52). This corresponds to equation (72b). \hfill \Box

5.3. Discrete fractional variational symmetries. Let us consider the one parameter group of diffeomorphisms $\{ \phi_s \}_{s \in \mathbb{R}}$ defined by (62) and its associated infinitesimal group action (63). Its action over the discrete augmented state space (70) is given by

$$\phi_s(x, y) = (\varphi_s(x), \varphi_s(y), \Delta^1_- \varphi_s(x), -\Delta^1_+ \varphi_s(y), \Delta^\alpha \varphi_s(x), \Delta^\beta \varphi_s(y)).$$

Using this notation, we generalize the notion of fractional fibered variational symmetries to the discrete case.

Definition 5.4 (Discrete fractional fibered variational symmetries). A local group of transformations $\{ \phi_s \}_{s \in \mathbb{R}}$ is called a discrete fractional fibered symmetry of the discrete fractional functional (68) if and only if

$$\int_c^b L \left( \phi_s(x, y)^{(h, \alpha, \beta)} \right) \Delta_- t = \int_c^b L \left( (x, y)^{(h, \alpha, \beta)} \right) \Delta_- t,$$

for all subintervals $[c, b] \subset [a, b]$ with $c, b \in \mathbb{T}$.

The infinitesimal criterion of invariance can then be written as:

$$L \left( \phi_s(x, y)^{(h, \alpha, \beta)} \right) = L \left( (x, y)^{(h, \alpha, \beta)} \right).$$

5.4. Discrete fractional Noether theorem. The proof of Theorem 5.6 is based on the next lemma, which gives an explicit form of the invariance condition:

Lemma 5.5. For all $f, g \in C(\mathbb{T}, \mathbb{R}^d)$, let us denote by $A_{h, \alpha}(f, g)$ the quantity

$$A_{h, \alpha}(f, g) = \Delta^\alpha_- g \cdot f - g \Delta^\alpha_+ f,$$

and by $B_{h, \alpha,\beta}(f, g)$ the quantity

$$B_{h, \alpha,\beta}(f, g) = \Delta^\alpha_+ f - \Delta^\beta_- g.$$

If $L$ admits ad fractional variational symmetry (75), then we have

$$0 = \Delta^1_- \left( \sigma \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) - \frac{\partial L}{\partial v_y} \cdot \sigma(\xi(y)) \right) + A_{h, \alpha} \left( \xi(x), \frac{\partial L}{\partial v_x^\alpha} \right) + B_{h, \alpha,\beta} \left( \frac{\partial L}{\partial v_x^\alpha}, \frac{\partial L}{\partial v_y^\beta} \right) \cdot \xi(x)$$

$$- \left( A_{h, \beta} \left( \frac{\partial L}{\partial v_y^\beta}, \xi(y) \right) + B_{h, \alpha,\beta} \left( \frac{\partial L}{\partial v_x^\alpha}, \frac{\partial L}{\partial v_y^\beta} \right) \cdot \xi(y) \right).$$

Proof. The invariance condition (75) can be written as

$$0 = \frac{d}{ds} \bigg|_{s=0} L \left( \phi_s(x, y)^{(h, \alpha, \beta)} \right) = \frac{\partial L}{\partial x} \cdot \xi(x) + \frac{\partial L}{\partial y} \cdot \xi(y)$$

$$+ \frac{\partial L}{\partial v_x} \cdot \Delta^1_- \xi(x) - \frac{\partial L}{\partial v_y} \cdot \Delta^1_+ \xi(x) + \frac{\partial L}{\partial v_x^\alpha} \cdot \Delta^\alpha_- \xi(x) + \frac{\partial L}{\partial v_y^\beta} \cdot \Delta^\beta_- \xi(y),$$
using the fact that \( d/ds \) commutes with any fractional operator \( \Delta^\alpha_x \). Now, replacing \( \partial L/\partial x \) and \( \partial L/\partial y \) by its corresponding expressions according to (71), we obtain:

\[
\begin{align*}
\frac{d}{ds} \bigg|_{s=0} L(\phi_s (x, y)) &= \Delta^1_x \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) + \frac{\partial L}{\partial v_y} \cdot \Delta^1_x \xi(x) \\
&\quad + \Delta^1_y \left( \frac{\partial L}{\partial v_y} \right) \cdot \xi(y) - \frac{\partial L}{\partial v_x} \cdot \Delta^1_y \xi(y) \\
&\quad - \Delta^\beta_y \left( \frac{\partial L}{\partial v_y} \right) \cdot \xi(x) - \Delta^\alpha_y \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(y) + \frac{\partial L}{\partial v_x} \cdot \Delta^\alpha_x \xi(x) + \frac{\partial L}{\partial v_y} \cdot \Delta^\beta_x \xi(y) \\
&\quad = \Delta^1_x \left( \xi(x) \sigma \left( \frac{\partial L}{\partial v_x} \right) - \sigma (\xi(y)) \cdot \frac{\partial L}{\partial v_y} \right) \\
&\quad - \Delta^\beta_y \left( \frac{\partial L}{\partial v_y} \right) \cdot \xi(x) - \Delta^\alpha_y \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(y) + \frac{\partial L}{\partial v_x} \cdot \Delta^\alpha_x \xi(x) + \frac{\partial L}{\partial v_y} \cdot \Delta^\beta_x \xi(y),
\end{align*}
\]

where in the second equality we use the discrete Leibniz formula (27).

The next step is to exhibit quantities for which the discrete transfer formula (28) or the partial transfer formula (30) can be applied, in order to obtain a total \( \Delta^1 \)-derivative. For that, we employ the notation

\[
F_x := \frac{\partial L}{\partial v_x}, \quad F_y := \frac{\partial L}{\partial v_y},
\]

where it is assumed that the partial derivatives of \( L \) are evaluated at the discrete state space. We have:

\[
\begin{align*}
- \Delta^\beta_x F_x \cdot \xi(x) &- \Delta^\alpha_x F_x \cdot \xi(y) + F_x \cdot \Delta^\alpha_x \xi(x) + F_y \cdot \Delta^\beta_x \xi(y) \\
&= F_x \cdot \Delta^\alpha_x \xi(x) - \Delta^\alpha_x F_x \cdot \xi(x) + F_y \cdot \Delta^\beta_x \xi(y) - \Delta^\beta_x F_y \cdot \xi(y) \\
&\quad - (\Delta^\beta_x F_y - \Delta^\alpha_x F_x) \cdot \xi(x) + (\Delta^\beta_x F_y - \Delta^\alpha_x F_x) \cdot \xi(y),
\end{align*}
\]

where we have added and subtracted \( \Delta^\alpha_x F_x \cdot \xi(x) \) and \( \Delta^\beta_x F_y \cdot \xi(y) \). Now we can check that the quantity of the right hand side can be rewritten as

\[
A_{h, \alpha} (\xi(x), \partial L/\partial v_x^\alpha) + B_{h, \alpha, \beta} \left( \partial L/\partial v_x^\alpha, \partial L/\partial v_y^\beta \right) \cdot \xi(x) \\
- \left( A_{h, \beta} \left( \partial L/\partial v_y^\beta, \xi(y) \right) + B_{h, \alpha, \beta} \left( \partial L/\partial v_x^\alpha, \partial L/\partial v_y^\beta \right) \cdot \xi(y) \right).
\]

This concludes the proof. \( \square \)

**Theorem 5.6** (Discrete fractional Noether theorem). Let \( L \) be a Lagrangian (43) such that the discrete fractional Lagrangian functional (69) admits a fractional fibered variational symmetry \( \{ \phi_s \}_{s \in \mathbb{R}} \) as in (74). Then the following quantity

\[
I_h(x, y) = \sigma \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) - \frac{\partial L}{\partial v_y} \cdot \sigma (\xi(y)) + I_{h, \alpha} \left( \xi(x), \frac{\partial L}{\partial v_x^\alpha} \right) - I_{h, \beta} \left( \frac{\partial L}{\partial v_y^\beta}, \xi(y) \right) \\
+ \left( J_{h, \alpha}^\beta \left( \frac{\partial L}{\partial v_y^\beta}, \xi(x) \right) \right) - J_{h, \beta}^\alpha \left( \frac{\partial L}{\partial v_x^\alpha}, \xi(x) \right) \\
- \left( J_{h, \alpha}^\beta \left( \frac{\partial L}{\partial v_x^\beta}, \xi(y) \right) \right) - J_{h, \beta}^\alpha \left( \frac{\partial L}{\partial v_y^\alpha}, \xi(y) \right)
\]

(77)

is a constant of motion along the solutions \((x, y)\) of the DREL\(^{(\alpha, \beta)}\) equations (71).

We recall that \( I_{h, \alpha} \) is defined in (29), whereas \( J_{h, \alpha}^\beta \) is defined in (31).
Proof. In (76) we recognize two quantities, namely \( A_{h,\alpha} (\xi(x), \partial L/\partial v_x^\alpha) \) and \( A_{h,\beta} (\partial L/\partial v_y^\beta, \xi(y)) \), which can be rewritten as a total discrete derivative using the discrete transfer formula (28). The two other quantities can also be written as a total derivative thanks to Lemma 2.10. The total time discrete derivatives are the quantities forming \( I_h(x, y) \)

5.5. Discrete fractional Noether’s theorem and time reversed solution.

Lemma 5.7. Let \( \mathbb{L} \) be a dedoubled fractional Lagrangian (47) satisfying (52). Let \( (x, x_*) \) be a solution of the DREL\(^{(\alpha, \beta)}\) equations (71) where \( x \) is a solution of (72a). Then, we have:

\[
0 = \Delta^1_- \left( \sigma \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) \right) + \Delta^1_+ \left( \sigma \left( \frac{\partial L}{\partial v_y} \right) \cdot \xi(x) \right) + \omega_{\alpha, \beta} \left( \frac{\partial L}{\partial v_x^\alpha}, \frac{\partial L}{\partial v_y^\beta} \right) \xi(x) + \left( A_{\beta} (\xi(x), -\rho D^\alpha_+ x), B_{\beta, \alpha} \left( \frac{\partial L}{\partial v_x^\alpha}, \frac{\partial L}{\partial v_y^\beta} \right) \cdot \xi(x) \right) + \rho D^\alpha_+ x_*, \xi(x).
\]

(78)

Proof. We can consider \( (x, x_*) \) as a solution of the DREL\(^{(\alpha, \beta)}\) equations (71) following Lemma 5.3. Moreover, as \( \mathbb{L} \) is a dedoubled fractional Lagrangian, we have

\[
\frac{\partial \mathbb{L}}{\partial v_x} \left( (x, y)^{(h, \alpha, \beta)} \right) = \frac{\partial L}{\partial v_x} (x, \Delta^1_- x) \quad \text{and} \quad \frac{\partial \mathbb{L}}{\partial v_y} \left( (x, y)^{(h, \alpha, \beta)} \right) = \frac{\partial L}{\partial v_y} (y, -\Delta^1_- y).
\]

As a consequence, we obtain

\[
-\Delta^1_- \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) + \frac{\partial L}{\partial v_x} \cdot \Delta^1_- \xi(x) = -\Delta^1_- \left( \frac{\partial L}{\partial v_x} (x, \Delta^1_- x) \right) \cdot \xi(x) + \frac{\partial L}{\partial v_y} (x, \Delta^1_- x) \cdot \Delta^1_- \xi(x)
\]

and

\[
\left( -\Delta^1_- \left( \frac{\partial L}{\partial v_x} \right) \cdot \xi(x) + \frac{\partial L}{\partial v_x} \cdot \Delta^1_- \xi(x) \right)
= \left( -\Delta^1_- \left( \frac{\partial L}{\partial v_x} (x, \Delta^1_- x) \right) \cdot \xi(x), + \frac{\partial L}{\partial v_y} (x, \Delta^1_- x) \cdot \Delta^1_- \xi(x) \right)_*,
\]

\[
= -\Delta^1_- \left( \frac{\partial L}{\partial v_y} (x_*, \Delta^1_- x_*) \right) \cdot \xi(x_*) + \frac{\partial L}{\partial v_y} (x_*, \Delta^1_- x_*) \cdot \Delta^1_- \xi(x_*)
= \Delta^1_- \left( \frac{\partial L}{\partial v_y} (x_*, -\Delta^1_- x_*) \right) \cdot \xi(x_*) - \frac{\partial L}{\partial v_y} (x_*, -\Delta^1_- x_*) \cdot \Delta^1_- \xi(x_*)
\]

from which the first line of (78) follows. Moreover

\[
\frac{\partial \mathbb{L}}{\partial v_x} = \Delta^\beta_+ y \quad \text{and} \quad \frac{\partial \mathbb{L}}{\partial v_y} = \Delta^\alpha_- x.
\]

Using the duality result of Lemma 5.2, we obtain

\[
\left( \frac{\partial \mathbb{L}}{\partial v_x^\alpha} \right)_* = \Delta^\alpha_- x, \quad \left( \frac{\partial \mathbb{L}}{\partial v_y^\beta} \right)_* = \Delta^\beta_+ x_*, \quad \left( \frac{\partial \mathbb{L}}{\partial v_y^\alpha} \right)_* = \Delta^\alpha_- x, \quad \left( \frac{\partial \mathbb{L}}{\partial v_x^\beta} \right)_* = \Delta^\beta_+ x_*. \]
As a consequence, we have
\[-A_\beta \left( \frac{\partial L}{\partial v_y}, \xi(x_*) \right) - B_{\alpha, \beta} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y} \right) \cdot \xi(x_*) \]
\[= -A_\beta \left( \rho \Delta_+^\alpha x, \xi(x_*) \right) - B_{\alpha, \beta} \left( \rho \Delta_+^\beta x - \rho \Delta_-^\alpha x \right) \cdot \xi(x_*) \]
\[= -A_\beta \left( \rho \Delta_-^\alpha x, \xi(x_*) \right) - B_{\alpha, \beta} \left( \rho \Delta_-^\beta x, \rho \Delta_+^\alpha x \right) \cdot \xi(x_*) \]
\[= \left( A_\beta \left( \xi(x), -\rho \Delta_-^\alpha x \right) \right)_* + \left( B_{\beta, \alpha} \left( -\rho \Delta_-^\alpha x, -\rho \Delta_+^\beta x \right) \cdot \xi(x) \right)_*. \]

By definition of $B_{\alpha, \beta}$, we have
\[B_{\beta, \alpha} \left( -\rho \Delta_+^\alpha x, -\rho \Delta_-^\beta x \right) = -\rho \Delta_+^{\alpha + \beta} x + \rho \Delta_-^{\alpha + \beta} x \]
\[= B_{\alpha, \beta} \left( -\rho \Delta_-^\beta x, -\rho \Delta_-^\alpha x \right) \]
\[= B_{\alpha, \beta} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y} \cdot \xi(x) \right). \]

As a consequence:
\[-A_\beta \left( \frac{\partial L}{\partial v_y}, \xi(x_*) \right) - B_{\alpha, \beta} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y} \cdot \xi(x_*) \right) \cdot \xi(x_*) \]
\[= \left( A_\beta \left( \xi(x), -\rho \Delta_-^\alpha x \right) \right)_* + \left( B_{\beta, \alpha} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y} \cdot \xi(x) \right) \right)_*. \]

This concludes the proof.

The expression is not completely symmetric under time reversal due to the fact that we can not express $A_\beta \left( \xi(x), -\rho \Delta_-^\alpha x \right)$ as $A_\beta \left( \xi(x), \frac{\partial L}{\partial v_x} \right)$ in general. As in the continuous case, this problem disappears when $\beta = \alpha$.

**Theorem 5.8** (Discrete fractional Noether’s theorem, reversed case). Let $\mathbb{L}$ be a dedoubled fractional Lagrangian (47) satisfying (52). Let $(x, x_*)$ be a solution of the DREL$^{(\alpha, \beta)}$ equations (71) where $x$ is a solution of (72a). Assume moreover that $\alpha = \beta$. Let us denote by $\mathcal{I}_{h, \alpha}(x, y)$ the quantity
\[\mathcal{I}_{h, \alpha}(x, y) = \sigma \left( \frac{\partial L}{\partial v_x} \cdot \xi(x) \right) + I_{\alpha} \left( \xi(x), \frac{\partial L}{\partial v_x} \right) + J_{\alpha, +} \left( \frac{\partial L}{\partial v_x}, \xi(x) \right) - J_{\alpha, -} \left( \frac{\partial L}{\partial v_y} \cdot \xi(x) \right). \]

Then the function
\[I_h(x, y) = \mathcal{I}_{h, \alpha}(x, y) + \left( \mathcal{I}_{h, \alpha}(x, y) \right)_*, \]
is a constant of motion over the solutions $(x, x_*)$ of the DREL$^{(\alpha, \alpha)}$ for all $t \in \mathbb{T}_\dagger$.

**Proof.** This follows directly from Lemma 5.7. Indeed, when $\alpha = \beta$, we have
\[A_\beta \left( \xi(x), -\gamma \Delta_-^\alpha x \right) = A_\beta \left( \xi(x), \frac{\partial L}{\partial v_x} \right), \]
and the invariance condition reads
\[0 = \Delta_-^1 \left( \sigma \left( \frac{\partial L}{\partial v_x}, \Delta_-^1 x \right) \cdot \xi(x) \right) + \mathcal{I}_{\alpha, \alpha} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y} \cdot \xi(x) \right) \]
\[+ \left( \Delta_-^1 \left( \sigma \left( \frac{\partial L}{\partial v_x} \cdot \xi(x) \right) \right) \right)_* + \left( \mathcal{I}_{\alpha, \alpha} \left( \frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y} \cdot \xi(x) \right) \right)_*. \]

Using the transfer Lemma 2.9 and Lemma 2.10, we obtain
\[0 = \Delta_-^1 \mathcal{I}_{h, \alpha}(x, y) + \left( \Delta_-^1 \mathcal{I}_{h, \alpha}(x, y) \right)_*. \]

For all $f \in C(\mathbb{T}, \mathbb{R}^d)$, we have for all $t \in \mathbb{T}_\dagger$ that
\[\Delta_-^1 \sigma(f) = -\Delta_-^1 f. \]
As a consequence, using the discrete duality relation, we obtain for all \( f \in C(\mathbb{T}, \mathbb{R}^d) \) and all \( t \in \mathbb{T}^+_\perp \)

\[
(\Delta^\perp_t f)_* = (-\Delta^\perp_t \sigma(f))_* = -\Delta^\perp_t \sigma(f_*).
\]

Applying this result to equality (79), we finally obtain

\[
0 = \Delta^\perp_t \left( \mathcal{J}_{\hbar, \alpha}(x, x_*) + (\mathcal{J}_{\hbar, \alpha}(x, x_*)_*) \right),
\]

for all \( t \in \mathbb{T}^+_\perp \). This concludes the proof. \( \Box \)

6. Example: \( O(d) \) symmetry. Let us consider (47) such that \( L(x, \dot{x}) \) (equiv. \( y \)) is invariant under the action of the group \( O(d) = \{ R \in \mathbb{R}^{d \times d} \mid R^T R = R R^T = \text{Id}_{d \times d} \} \) of orthogonal \( d \times d \) matrices. As an example, simple mechanical Lagrangians with central potential energy defined for all \( (x, v) \in \mathbb{R}^d \times \mathbb{R}^d \) by

\[
L(x, v) = \frac{m}{2} \| v \|^2 - V(\|x\|),
\]

where \( m \in \mathbb{R}_+ \), \( \| x \|^2 = \sum_{i=1}^{d} x_i^2 \) and \( V \) is a real function are invariant under \( O(d) \).

This comes directly from the fact that orthogonal transformations are isometries of \( \mathbb{R}^d \). This result extends in the fractional case considering dedoubled fractional Lagrangians. We observe that a dedoubled fractional Lagrangian (47) associated to \( L \) defined by (80) is also invariant under \( O(d) \). To see that, we need only to check that \( \langle D^a_\perp (R \cdot x), D^\beta_\perp (R \cdot y) \rangle \), is \( O(d) \) invariant. This follows from the fact that any \( R \) in \( O(d) \) is an isometry and, moreover, it commutes with fractional derivatives as it is linear and time independent. Namely:

\[
\langle D^a_\perp (R \cdot x), D^\beta_\perp (R \cdot y) \rangle = \langle R \cdot D^a_\perp x, R \cdot D^\beta_\perp y \rangle = \langle D^a_\perp x, D^\beta_\perp y \rangle.
\]

For \( (x, y) \in C(\mathbb{T}, \mathbb{R}^d) \times C(\mathbb{T}, \mathbb{R}^d) \), the discrete dynamics (72) reads

\[
\begin{align*}
  m \Delta^\perp_1 \Delta^1_\perp x(t) + \nabla V(x(t)) &= \rho \Delta^{\beta_\perp + \alpha}_\perp x(t), \\
  m \Delta^\perp_1 \Delta^1_\perp y(t) + \nabla V(y(t)) &= \rho \Delta^{\alpha_\perp + \beta}_\perp y(t),
\end{align*}
\]

for all \( t \in \mathbb{T}^+_\perp \). Setting \( d = 2 \) for simplicity, we have \( x = (x_1, x_2) \), \( y = (y_1, y_2) \), while the orthogonal matrix can be written in terms of a single parameter \( s \), i.e.

\[
R(s) = \begin{bmatrix}
\cos (s) & -\sin (s) \\
\sin (s) & \cos (s)
\end{bmatrix}.
\]

Therefore

\[
\frac{d}{ds}\bigg|_{s=0} \varphi_s(x) = \xi(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = (-x_2, x_1)
\]

(equiv. for \( y \)). Applying Theorem 5.8, choosing a potential \( V(x) = \frac{1}{2} \| x \|^2 \) (corresponding in the non-fractional case, i.e. \( \rho = 0 \), to the harmonic oscillator), setting \( \alpha = \beta \) (with \( \alpha \in (0, 1/2) \)) and taking \( m = 1 \), the discrete constant of motion is given for all \( (x, y) = (x, x_*) \) solution of (81) by

\[
\mathcal{J}_{\hbar, \alpha}(x, y) = (I(x) - \rho J(x, y)) + (I(x) - \rho J(x, y*)_*),
\]

where

\[
I(x_1, x_2) = (x_2 \Delta^1_\perp \Delta^1_\perp x_1 + x_1 \Delta^1_\perp \Delta^1_\perp x_2) + (-\Delta^1_\perp x_1 \Delta^1_\perp x_2 + \Delta^1_\perp x_2 \Delta^1_\perp x_1),
\]

and

\[
J(x, y) = (x_2 \Delta^{2\alpha}_\perp x_1 - x_1 \Delta^{2\alpha}_\perp x_2) + (\Delta^{\alpha}_\perp y_1 \Delta^{\alpha}_\perp x_2 + \Delta^{\alpha}_\perp y_2 \Delta^{\alpha}_\perp x_1).
\]
We implement the previous quantities to test the validity of our result for different values of $\alpha$ and $\rho$. In the following, we take $N = 100$, $h = 0.25$, $x_1(0) = 1$, $x_1(1) = 1.25$, $x_2(0) = 0.7$, $x_2(1) = 0.6$ (the initial data at $t = 1$ are required to initialise the integrators due to the second order discrete operator $\Delta_+\Delta_-$ in the discrete equation (81). This phenomenon already appears for classical variational integrators [31]). In all figures, the blue curve corresponds to the dynamics of $x_1$, the rose one to $x_2$ and the magenta to the computed value of the constant of motion (we neglect the $y$-dynamics, since it is proven in Theorem 5.3 that $y = x_*$ is a solution of the fractional Euler-Lagrange equations). The plots are displayed in Figure 1.

**Figure 1.** Top-Left: $\rho = 0$; this is the classical dynamics of the non-fractional harmonic oscillator (without dissipation). Top-Right: $\rho = 0.2$, $\alpha = 0.2$. Bottom-Left: $\rho = 0.2$, $\alpha = 0.4$. Bottom-Right: $\rho = 0.2$, $\alpha = 0.5$; this is the case of the usual linearly damped harmonic oscillator. Naturally, we observe that the dissipation increases as $\alpha$ increases for equivalent $\rho$.

7. **Conclusions.** We have dedicated this paper to the proof of a Noether’s theorem of the first kind for the so-called restricted fractional Euler Lagrange equations $\operatorname{REL}^{(\alpha,\beta)}$ (45) and their discrete counterpart $\operatorname{DREL}^{(\alpha,\beta)}$ (71). We employ the discrete embedding formalism when treating the discrete version. The role of time reversed solutions is of importance. We discuss a typical example and provide
some simulations actually showing the constant behaviour in time of the discrete conserved quantities.

Alongside, and as a major preparation to prove this result, we have established the connection between the restricted calculus of variations and the asymmetric fractional calculus of variations, which is achieved though the study of the time reversed solutions. This discussion suggests a possible alternative framework to deal with dissipative systems.

REFERENCES

[1] T. Abdeljawad and F. W. Atici, On the definition of Nabla fractional operators, Abstract in Applied Analysis, 2012 (2012), Article ID 406757, 13 pages.
[2] R. Abraham and J. E. Marsden, Foundations of Mechanics, Benjamin-Cummings Publ. Co., 1978.
[3] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl., 272 (2002), 368–379.
[4] T. T. Atanackovic, S. Konjik, S. Pilipovic and S. Simic. Variational problems with fractional derivatives: Invariance conditions and Noether’s theorem, Nonlinear Analysis, 71 (2009), 1504–1517.
[5] F. M. Atici and P. W. Eloe, Discrete fractional calculus with the nabla operator, Electronic Journal of Qualitative Theory of Differential Equations, 2009 (2009), 1–12.
[6] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Discrete-time fractional variational problems, Signal Processing, 91 (2011), 513–524.
[7] H. Bateman, On dissipative systems and Related Variational Principles, Phys. Rev., 38 (1931), 815.
[8] P. S. Bauer, Dissipative dynamical systems, Proc. Nat. Acad. Sci., 17 (1931), 311–314.
[9] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhäuser Boston Inc., 2001.
[10] L. Bourdin, Contributions au calcul des variations et au principe du maximum de Pontryagin en calculs time scale et fractionnaire, Ph.D. Thesis, University of Pau and Pays de l’Adour, 2013.
[11] L. Bourdin, J. Cresson and I. Greff, A continuous/discrete fractional Noether’s theorem, Commun. Nonlinear Sci. Numer. Simulat., 18 (2013), 878–887.
[12] L. Bourdin, J. Cresson, I. Greff and P. Inizan, Variational integrator for fractional Euler-Lagrange equations, Appl. Numer. Math., 71 (2013), 14–23.
[13] M. C. Caputo and D. F. M. Torres, Duality for the left and right fractional derivatives, Signal Processing, 107 (2015), 265–271.
[14] J. Cresson and P. Inizan, Variational formulations of differential equations and asymmetric fractional embedding, Journal of Mathematical Analysis and Applications, 385 (2012), 975–997.
[15] J. Cresson, Fractional embedding of differential operators and Lagrangian systems, J. Math. Phys., 48 (2007), 033504, 34 pages.
[16] J. Cresson, Fractional variational embedding and Lagrangian formulations of dissipative partial differential equations, Fractional Calculus in Analysis, Dynamics and Optimal Control, Nova Publishers, New-York, (2013), 65–127.
[17] J. Cresson, F. Jiménez and S. Ober-Blöbaum, Modeling of the convection-diffusion equation through fractional restricted calculus of variations, J. Nonlinear Sci., 31 (2021), Paper No. 46, 43 pp.
[18] J. Cresson and F. Pierret. Continuous versus discrete structures I: Discrete embeddings and ordinary differential equations, preprint, 2014, arXiv:1411.7117.
[19] J. Cresson and A. Szafrański, About the Noether’s theorem for fractional Lagrangian systems and a generalization of the classical Jost method of proof, Fractional Calculus and Applied Analysis, 22 (2019), 871–898.
[20] K. Diethelm, The Analysis of Fractional Differential Equations, Lect. Notes in Maths. Vol. 2004, Springer, 2010.
[21] R. A. C. Ferreira and A. B. Malinowska, A counterexample to Frederico and Torres’s fractional Noether-type theorem, J. Math. Anal. Appl., 429 (2015), 1370–1373.
[22] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, *J. Math. Anal. Appl.*, **334** (2007), 834–846.
[23] C. R. Galley, Classical mechanics of nonconservative systems, *Phys. Rev. Lett.*, **110** (2013), 17430.
[24] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration*, Springer Series in Computational Mathematics, Vol. 31, second edition, 2006.
[25] P. Inizan, Dynamique fractionnaire pour le chaos hamiltonien, Ph. D. Thesis, Observatoire de Paris, 2010.
[26] F. Jiménez and S. Ober-Blöbaum, A fractional variational approach for modelling dissipative mechanical systems: continuous and discrete settings, *IFAC-PapersOnLine*, **51** (2018), 50–55.
[27] F. Jiménez and S. Ober-Blöbaum, Fractional damping through restricted calculus of variations, *J. Nonlinear Sci.*, **31** (2021), Paper No. 46, 43 pp, arXiv:1905.05608.
[28] R. Leone, On the wonderfulness of Noether’s theorems, 100 years later, and Routh reduction, 2018. arXiv:1804.01714.
[29] C. Lubich, Convolution quadrature and discretized operational calculus II, *Numer. Math.*, **52** (1988), 413–425.
[30] C. Lubich, Convolution quadrature and discretized operational calculus I, *Numer. Math.*, **52** (1988), 129–145.
[31] J. E. Marsden and M. West, Discrete mechanics and variational integrators, *Acta Numerica*, **10** (2001), 357–514.
[32] J. Moser and A. P. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomial, *Comm. Math. Phys.*, **139** (1991), 217–243.
[33] E. Noether, Invariante Variationsprobleme, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse 2, 235, the first english translation is due to M. A. Tavel [Transport Theor. Stat. 1, 186 (1971)], (1918).
[34] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, 2nd edition, Springer-Verlag, 1993.
[35] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, *Phys. Rev. E*, **53** (1996), 1890–1899.
[36] S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Beach, Yverdon, 1993.
[37] R. Utiyama, Invariant theoretical interpretation of interaction, *Phys. Rev. Lett.*, **101** (1956), 1597–1607.
[38] C. N. Yang and R. L. Mills, Conservation of isotopic spin and isotopic gauge invariance, *Phys. Rev. Lett.*, **96** (1954), 191–195.

Received for publication February 2021.

E-mail address: jacky.cresson@univ-pau.fr
E-mail address: fernando.jimenez.alburquerque@upm.es
E-mail address: sinaober@math.upb.de