COMPLETIONS OF INFINITESIMAL HECKE ALGEBRAS OF $\mathfrak{sl}_2$

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Abstract. We relate completions of infinitesimal Hecke algebras of $\mathfrak{sl}_2$ to noncommutative deformations of Kleinian singularities of type $D$ of Crawley-Boevey and Holland. As a consequence, we show an analogue of Bernstein’s inequality and simplicity of generic maximal primitive quotients of these algebras. We also establish Skryabin type equivalence for these algebras.

1. Introduction

Given an associative Noetherian $\mathbb{C}$-algebra $A$ of finite Gelfand-Kirillov dimension, it is natural to ask if (generalized) Bernstein’s inequality holds: Is it true that for any finitely generated $A$-module $M$ one has $\text{GK}_A(M) \geq \frac{1}{2} \text{GK}(A/\text{Ann}(M))$? (here $\text{GK}$ stands for Gelfand-Kirillov dimension). We show that this is the case for infinitesimal Hecke algebras of $\mathfrak{sl}_2$.

Recall that for a reductive Lie algebra $\mathfrak{g}$ and its finite dimensional representation $V$, Etingof, Gan and Ginzburg [EGG] defined a family of PBW deformations of the universal enveloping algebra $U(\mathfrak{g} \ltimes V)$ of the semi-direct product Lie algebra $\mathfrak{g} \ltimes V$, which they called infinitesimal Hecke algebras.

There is a particularly nice family of infinitesimal Hecke algebras for the pair $\mathfrak{g} = \mathfrak{sl}_2, V = \mathbb{C}^2$. These algebras, denoted by $H_z$, depend on a parameter $z$ which is an element of the center of $U\mathfrak{sl}_2$. Algebras $H_z$ were studied by Khare and the author [Kh, KT]. In this paper, we relate noncommutative deformations of Kleinian singularities of type $D$ of Crawley-Boevey and Holland (which are spherical subalgebras of symplectic reflection algebras for $\dim V = 2$) [CBH] to infinitesimal Hecke algebras of $\mathfrak{sl}_2$. Namely, we show that a certain completion of the central quotient of an infinitesimal Hecke algebra of $\mathfrak{sl}_2$ is isomorphic to the tensor product of the completed Weyl algebra in two generators and the completion of an algebra of Crawley-Boevey and Holland. As a consequence, we establish Bernstein’s inequality for these algebras (Theorem 4.1). We also show an equivalence between certain subcategory of modules over $H_z$ and the category of modules over corresponding noncommutative deformations of type $D$ singularities (Theorem 5.1).
2. COMPLETIONS OF ALMOST COMMUTATIVE ALGEBRAS

We will always work over the field of complex numbers $\mathbb{C}$. In this section we will recall some necessary constructions and fix notations related to a slice algebra construction due to Losev [L1].

Throughout we will use the following convention: For a commutative algebra $B$ and a closed point $y \in \text{Spec} B$, $B_y$ will denote the completion of $B$ with respect to the maximal ideal $m_y$ corresponding to $y$. Also, for a symplectic variety $Y$ and $y \in Y$ we will denote by $W_h(Y_y)$ the completed Weyl algebra, a deformation quantization of $\mathcal{O}(Y)_y$, where $h$ is the deformation parameter.

By an almost commutative algebra we will mean an associative $\mathbb{C}$-algebra equipped with an ascending filtration

$$
\mathbb{C} = A_0 \subset A_1 \subset \cdots A_n \subset \cdots, \quad \bigcup_{n \in \mathbb{N}} A_n = A, \quad A_n A_m \subset A_{n+m},
$$

such that the associated graded algebra is a finitely generated commutative ring over $\mathbb{C}$. Recall that in this case $\text{gr} A$ comes equipped with a natural Poisson bracket. The origin of $\text{Spec gr} A$ will be denoted by $\{0\}$.

By a generic subset of an algebraic variety we will mean Weil generic subset, i.e. a set which is the complement of a countable union of proper closed subvarieties.

Let $A$ be an almost commutative algebra equipped with a filtration $A_n$ ($n \geq 0$) over $\mathbb{C}$. Let $Y \subset \text{Spec} \text{gr} A = X$ be an algebraic symplectic leaf $\text{dim} Y = 2d$, and let $y \in Y$. We will recall the slice algebra construction of Losev [L1]. One starts by completing the Rees algebra $R(A) = \bigoplus_n A_n h^n \subset A[h]$ with respect to the ideal $p^{-1}(m_y)$, where $m_y \subset \text{gr} A$ is the maximal ideal corresponding to $y$, and $p : R(A) \rightarrow R(A)/h = \text{gr} A$ is the natural projection. This completed algebra will be denoted by $R(A)_y$. Then according to Losev, $R(A)_y$ is a free $\mathbb{C}[[h]]$-module and $R(A)_y/h = (\text{gr} A)_y$, the completion of $\text{gr} A$ at $m_y$.

According to Kaledin [?], $\text{(gr} A)_y$ is isomorphic to the completed tensor product $\text{Spec} Y_y \otimes B$, where $B$ is a complete Poisson algebra with the origin being a symplectic leaf of $\text{Spec} B$. As proved by Losev [L1], this decomposition can be lifted to $R(A)_y$, meaning that there is a free $\mathbb{C}[[h]]$-subalgebra $A_{\bullet}_y$, a slice algebra, such that $R(A)_y$ is identified with the completed tensor product $W_h(Y_y) \otimes_{\mathbb{C}[[h]]} A_{\bullet}_y$, such that $A_{\bullet}_y/h = B$.

If $M$ is a finitely generated left $A$-module, then $M$ can be equipped with a filtration compatible with the filtration of $A$ such that the corresponding Rees module $R(M)$ is finitely generated over $R(A)$ (a good filtration). As before, one defines $R(M)_y$ as the completion of $R(M)$ with respect to $p^{-1}(m_y)$, so $R(M)_y = R(M) \otimes_{R(A)} R(A)_y$ which is a nonzero finitely generated $R(A)_y$-module when $y \in \text{Supp}(\text{gr} M)$. In which case $R(M)_y/h = (\text{gr} M) \otimes_{\text{gr} A} (\text{gr} A)_y = (\text{gr} M)_y$. We will denote by $SS(M)$
the corresponding characteristic variety $\text{Supp}(\text{gr} M) \subset \text{Spec} \text{gr} A$. Recall the following observation of Losev.

**Proposition 2.1.** Let $I \subset A$ be a two-sided ideal of $A$ such that $Y$ is a connected component of $V(\text{gr} I)$, where as before $Y$ is a symplectic leaf in $\text{Spec} \text{gr} A$. Then for any $y \in Y$, there exists a nonzero left $A_y$-module which is a finitely generated free module over $\mathbb{C}[[h]]$.

We recall the proof for the convenience of the reader.

**Proof.** We have that $\bar{I} = R(I)R(A)_g$ is a two-sided ideal of $R(A)_g$, such that $R(A)_g/\bar{I}$ has support $Y_g$. Therefore, any finitely generated $A_y$-submodule of $R(A)_g/\bar{I}$ is supported at the origin in $\text{Spec} A_y/h$, hence is finite-dimensional. $\square$

For a finitely generated module $M$ over an almost commutative algebra $A$, we will denote its Gelfand-Kirillov dimension by $\text{GK}(M)$, as usual. Recall that in the above setting $\text{GK}(M) = \dim \text{SS}(M)$.

3. Completions of infinitesimal Hecke algebras of $\mathfrak{sl}_2$

We will denote by $\Delta$ the rescaled Casimir element $\Delta = h^2 + 4fe + 2h \in \mathfrak{u}\mathfrak{s}\mathfrak{l}_2$, where $h, e, f$ denote the standard basis elements for $\mathfrak{sl}_2$. For a given $z' \in \mathbb{C}[\Delta]$, the algebra $H_{z'}$ is the quotient $\mathfrak{u}\mathfrak{g} \ltimes TV/([x, y] − z')$, where $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \mathbb{C}x \oplus \mathbb{C}y$ is its standard 2-dimensional representation with relations

$$[e, x] = 0, \quad [f, x] = y, \quad [h, x] = x.$$

It was shown in [KT] that center of $H_{z'}$ is generated by $t_z = ey^2 + \frac{1}{2}h(xy + yx) − f x^2 + z$, where $z \in \mathbb{C}[\Delta]$ is an element uniquely defined by $z'$ up to adding a constant (but $z'$ is uniquely determined by $z$). One has $\deg(z) = \deg(z') + 1$, and the leading coefficient of $z$ as a polynomial in $\Delta$ is $\frac{1}{\deg(z') + 1}$ times the leading coefficient of $z'$. Let $U_z$ be the quotient $H_{z'}/(t_z)$. We will introduce the following filtration on $U_z$:

$$\deg e = \deg f = \deg h = 2, \quad \deg x = \deg y = 2n + 1,$$

where $n = \deg z'$ as a polynomial in $\Delta$. From now on we will assume that $n \geq 2$ and $z$ is monic in $\Delta$. In what follows for a filtered algebra $A$, given an element $a \in A$, we will denote for simplicity $\text{gr} a \in \text{gr} A$ still by $a$ whenever it will not cause a confusion. It follows easily from PBW property that $\text{gr} U_z$ is the quotient of the polynomial algebra $\mathbb{C}[e, f, h, x, y]$ by the principal ideal $(ey^2 + hxy − f x^2 − (\Delta)^{n+1})$, where $\Delta = h^2 + 4ef$. We will denote this Poisson algebra by $B_n$. Thus, the Poisson bracket on $B_n$ is defined as follows

$$\{e, x\} = 0 = \{f, y\}, \{f, x\} = y, \{x, y\} = (2n + 1)\Delta^n,$$

$$\{h, e\} = 2e, \{h, f\} = −2f, \{e, f\} = f, \{e, y\} = x.$$

We note that $SL_2(\mathbb{C})$ acts naturally on $H_{z'}, U_z$ preserving the corresponding filtration. This action gives rise to a natural action of $SL_2(\mathbb{C})$ on $B_n$ preserving the Poisson bracket.
We now describe the symplectic leaves of Spec $B_n$.

**Proposition 3.1.** $B_n$ is a normal integral domain. The singular locus of Spec $B_n$ is $\{x = y = 0\}$, and its smooth locus is symplectic. The symplectic leaves of Spec $B_n$ are the origin $\{0\}$, $V(I) \setminus \{0\}$, and its smooth locus $\text{Spec } B_n \setminus \{x = y = 0\}$.

**Proof.** It is easy to see that $B_n$ is a domain. It is clear that the ideal $I = (x, y, \Delta)$ is a Poisson ideal and $V(I)$ (the zero locus of $I$) belongs to the singular locus of Spec $B_n$. Let us take $a \in \text{Spec } B_n \setminus V(I)$. Let $m_a$ be the corresponding maximal ideal. The $m_a$-adic completion of $B_n$ will be denoted by $\bar{B}_n$ for brevity. We will show that Spec $\bar{B}_n$ a (formal) symplectic variety. Using the action of $SL_2(\mathbb{C})$ on Spec $B_n$ we may assume without loss of generality that $e(a) \neq 0$. It is easily seen that $\bar{B}_n$ is isomorphic to the tensor product of $\mathbb{C}[[e^{-1}h - \frac{|\Delta|}{e(a)}, e - e(a)]]$ and of the completion of $\mathbb{C}[A, C, \Delta]/(C^2 - \Delta(\frac{1}{4}A^2 + \Delta^n))$ at the point $(A(a), C(a), \Delta(a))$, where $A = e^{-\frac{1}{2}}x, C = e^{\frac{1}{2}y} + \frac{1}{2}hA$. The latter is the ring of functions of the Kleinian singularity of type $D_{n+2}$, which is symplectic outside the origin. Thus, Spec $\bar{B}_n \setminus V(I)$ is a symplectic. On the other hand, $V(I)$ considered as a reduced subvariety of Spec $B_n$ is the nilpotent cone of $\mathfrak{sl}_2$. Hence $V(I) \setminus \{0\}$ is a symplectic leaf of Spec $B_n$. Normality of $B_n$ follows Serre’s criterion, since $\bar{B}_n$ is regular in codimension 1.

The following is an analogue of Kostant’s theorem.

**Proposition 3.2.** The algebra $H_2$ is a free module over its center.

**Proof.** It suffices to check that gr $H_2$ is a free module over $\mathbb{C}[\text{gr } t_2]$. Let us introduce another filtration of gr $H_2$ by setting $\text{deg } x = \text{deg } y = 0, \text{deg } e = \text{deg } f = \text{deg } h = 1$. Then under the new filtration, gr(gr $t_2$) = $\Delta^{n+1}$. But by Kostant’s theorem, Sym gr $\mathfrak{g}$ is free over $\mathbb{C}[\Delta^{n+1}]$. Hence we conclude that gr(gr$(H_2)$) = $\mathbb{C}[x, y] \otimes \text{Sym } \mathfrak{g}$ is free over gr(gr $t_2$). Hence gr $H_2$ is free over $\mathbb{C}[\text{gr } t_2]$.

Recall that for a given finite group $\Gamma \in SL_2(\mathbb{C})$, and a central element of the group algebra $\lambda \in \mathbb{C}[\Gamma]$ Crawley-Boevey and Holland [CBH] defined an algebra $\mathcal{O}^{\lambda}$ as $e(T(V)/([x, y] - \lambda))e$, where $e = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$ and $x, y$ is the standard basis of $V = \mathbb{C}^2$.

In what follows we will consider square roots of certain non-central elements. Let us clarify this and fix the appropriate notation. Let $A$ be an associative $\mathbb{C}$-domain, and let $e \in A$ be an element such that $\text{ad}(e)$ acts locally nilpotently on $A$. Moreover, assume that $A$ is separated in $(e - 1)$-adic topology, i.e. $\cap_n (e - 1)^n = 0$. Then we will denote by $A[e^{-\frac{1}{2}}]$ the subalgebra of the completion of $A$ by $(e - 1)$ generated over $A$ by $e^{-\frac{1}{2}} = \exp(-\frac{1}{2} \log e)$, where $\log e$ is understood as a power series in $(e - 1)$.

\footnote{Apoorva Khare has obtained this answer earlier using a different computation.}
Also, by \( W_1(R) \) we will denote the Weyl algebra over \( R \) with 2 generators: 
\[ W_1(R) = R(x, y)/([x, y] - 1). \]

The following result was motivated by Losev’s work on completions of symplectic reflection algebras \([2]\).

**Theorem 3.1.** For any \( a \in Y = V(I) \setminus \{0\} \), the algebra \( R(U_z) \) is isomorphic to the completed tensor product \( W_h(Y_a) \otimes \mathbb{C}[\mathbb{A}] R(\mathcal{O}_x(0)) \), where \( \mathcal{O}_x(0) \) is the noncommutative deformation of the Kleinian singularity of type \( D_{n+2} \) (parameter \( \lambda(z) \) will be determined in the proof). For generic \( z \), the algebra \( U_z \) is simple.

**Proof.** Since \( SL_2(\mathbb{C}) \) acts transitively on \( V(I) \setminus \{0\} \), we may assume without loss of generality that \( a \) is the point with coordinates \( e = 1, h = f = 0 \). It is straightforward to check that the algebra \( U_z[e^{-\frac{1}{2}}] \) is generated by \( e^\frac{1}{2}, e^{-\frac{1}{2}} \) over \( U_z \) subject to the following relations:
\[
[f, e^m] = -mhe^{m-1} + m(m-1)e^{m-1}, \quad [y, e^m] = -me^{m-1}xe^{m-1}, \quad [h, e^m] = 2me^m, \quad [x, e^m] = 0
\]
for \( m \in \frac{1}{2} \mathbb{Z} \). We have that \( [\frac{1}{2}e^{-1}h, e-1] = 1 \), and both \( \frac{1}{2}e^{-1}h, e-1 \) commute with \( A = e^{-\frac{1}{2}}x, C = e\frac{1}{2}y + \frac{1}{2}hA \), \( \Delta = h^2 + 4fe + 2h \). Let us denote the subalgebra of \( U_z[e^{-\frac{1}{2}}] \) generated by \( A, C, \Delta \) by \( U'_z \). It is easy to see that \( U_z[e^{-\frac{1}{2}}] = \mathbb{C}[\frac{1}{2}e^{-1}h, e-1][e^{-\frac{1}{2}}] \otimes U'_z \). Direct computation shows that the following relations hold in \( U'_z \):
\[
[\Delta, C] = \Delta A + (A - C), \quad [A, C] = z' - \frac{1}{2}A^2, \\
[\Delta, A] = 4C - A, \quad z + \frac{1}{4}\Delta A^2 - \frac{1}{2}AC = C^2.
\]

Now we will need to recall the explicit relations for \( \mathcal{O}_x(\mathcal{A}) \) (noncommutative deformations of type \( D_{n+2} \) singularities). Recall that Levy \([4]\) has defined the following algebras \( D(Q, \gamma) \) for a polynomial \( Q(t) \), and \( \gamma \in \mathbb{C} \), with generators \( u, v, w \), and relations:
\[
[u, v] = 2w, \quad [u, w] = 2uv + 2w + \gamma, \quad [v, w] = v^2 + P(u)
\]
and
\[
Q(u) + uv^2 + w^2wvyv = 0,
\]
where \( P \) is the unique polynomial such that
\[
Q(-s(s - 1)) - Q(-s(s + 1)) = (s - 1)P(-s(s - 1)) + (s + 1)P(-s(s + 1)).
\]

Similarly, Boddington has defined an algebra \( D(q) \) depending on a polynomial \( q \), and has showed that \( D(q) \) is isomorphic to \( D(Q, \gamma) \) when \( \gamma = 2q(\frac{1}{2}) \) and \( Q(-u + \frac{1}{2}) = [-\sqrt{u} - \frac{1}{2}p(\sqrt{u})] \), where \( p(x) = \frac{x^2-2q(x)q(-x-1)+\gamma^2}{(1+2y)^2} \) and notation \( [f(\sqrt{v})] \in \mathbb{C}[x] \) for a polynomial \( f(x) \) means the following:
f(\sqrt{x}) = h(x) + \sqrt{x}[f(\sqrt{x})] \text{ for unique polynomials } h(x), [f(\sqrt{x})]. \text{ Boddington [Bo] shows that } D(q) \text{ is isomorphic to } O^\lambda \text{ where } \lambda \text{ is the tuple } (\lambda_a, \lambda_b, \lambda_1, \cdots, \lambda_{n-1}, \lambda_c, \lambda_d) \text{ such that } q(x) = \prod_{i=0}^{n-1}(x + \mu_i), \text{ where }

\mu_0 = \frac{1}{2} \lambda_a - \lambda_b, \mu_1 = \frac{1}{2} (\lambda_a + \lambda_b), \mu_2 = \mu_1 + \lambda_1, \cdots, \mu_{n-1} = \mu_1 + \lambda_1 + \cdots + \lambda_{n-1} + \lambda_c

Direct computation shows that our algebra \( U'_z \) is isomorphic to \( D(Q, 0) \) for \( Q(u) = 3z'(-u - \frac{x}{2}) - z(-u - \frac{x}{2}) \). This can be seen by putting first \( \Delta = -u, C = \frac{1}{2}w, A = -v \), and then replacing \( u, w \) by \( u + \frac{x}{2} \) and \( w + \frac{x}{2} \) respectively. (We denote the corresponding parameter by \( \lambda(z) \).) Then we see that for generic \( z \), \( \lambda(z) \) is the generic parameter such that \( q(\frac{1}{2}) = 0 \), i.e. \( \mu_i = -\frac{1}{2} \) for some \( i \). Thus, \( U'_z \) is isomorphic to \( O^\lambda(z) \), and for generic \( z \), \( O^\lambda(z) \) is simple.

To summarize, \( U_z[e^{1/2}] \cong W_1(C)[e^{-1/2}] \otimes O^\lambda(z) \), where the Weyl algebra \( W_1(C) \) is defined by the following generators and relations: \( W_1(C) = C\langle e, \frac{1}{2}e^{-1}h \rangle /[e, \frac{1}{2}e^{-1}h] - 1 \). Similarly we can establish an isomorphism on the level of Rees algebras

\[ R(U_z)[(h^2e)^{\frac{1}{2}}] \cong R(W_1(C))[\frac{1}{2}e] \otimes_{C[h]} R(O^\lambda(z)) \]

(recall that \( h^2e \in R(U_z) \)). Since, \( R(U_z)_{\alpha} \) is complete with respect to \( (h^2e - 1) \), we have that \( (h^2e)^{\frac{1}{2}} \in R(U_z)_{\alpha} \). This and the above isomorphism yields the desired isomorphism \( R(U_z)_{\alpha} \cong W_1(C)[e^{-1/2}] \otimes_{C[h]} R(O^\lambda(z))_0 \).

For generic \( z \), \( U_z \) has no finite-dimensional representations (this was shown in [KT]). Thus, simplicity of \( U_z \) for generic \( z \) follows from Proposition 2.1 since \( O^\lambda(z) \) has no nontrivial finite-dimensional representations if \( \lambda(z) \cdot \alpha \neq 0 \) for all non-Dynkin roots, according to [CBH].

\[ \square \]

4. Analogue of Bernstein’s inequality

We start by recalling few standard results (whose proofs are given for the convenience of the reader), which will be used for the proofs of Theorem 4.1, Theorem 4.2.

The proof of the following Proposition follows directly a well-known proof of Bernstein’s inequality given by Joseph [GM].

**Proposition 4.1.** Let \( R \) be a Noetherian ring with finite GK-dimension and let \( M \) be a finitely generated \( W_1(R) \)-module. Then for any finite dimensional \( C \)-subspace \( N \subset M \) we have \( GK_R(RN) \leq GK_{W_1(R)}M + GK_{W_1(C)}(W_1(C)N) - 2 \). In particular \( GK_R(RN) \leq 2(GK_{W_1(R)}(M) - 1) \).

**Proof.** Let \( R_i \) be an ascending filtration of \( R \) by finite dimensional \( C \)-spaces, such that \( R_0 = C, R_nR_m = R_{n+m} \). Denote by \( A_n \) the \( n \)-the degree part of the Bernstein filtration on \( W_1(C) \), so \( A_n = \sum_{i+j \leq n} Cx^iy^j \). Put \( U_n = \sum_{i \leq n} A_iR_{n-i} \). Also put \( N^n = U_nN \). It is easy to check that the kernel of the
Let $V$ and $GK$ be a finitely generated $W_1(R)$-module such that $GK_{W_1(R)}(M) = 1$. Then $GK(R/Ann_R(M)) = 0$.

Proof. Let $N \subseteq M$, $\dim N < \infty$ be such that $W_1(R)N = M$. Then by Proposition 4.1, $GK(RN) = 0$, so $GK(R/Ann_R(RN)) = GKR(Ann_R(M)) = 0$.

Lemma 4.1. Let $R$ be an affine Noetherian $\mathbb{C}$-algebra, and let $e \in R$ be an element such that $\text{ad}(e)$ is locally nilpotent on $R$. Then for any finitely generated $R$-module $M$, we have $GK_{R[e^{-\frac{1}{2}}]}[M[e^{-\frac{1}{2}}] \leq GKRM$.

Proof. Without loss of generality we may assume that $M$ is $e$-torsion free. Let $V \subseteq R$ be an $\text{ad}(e)$-stable finite dimensional generating subspace of $R$ such that $1, e \in V$. Let $M_0 \subseteq M$ be a subspace such that $RM_0 = M, \dim \mathbb{C} M_0 < \infty$. Since $e^{-i}(V^{n-i-1}) \subseteq e^{-i(V^{n-i-1})}, \text{we get that}$

$$\dim_{\mathbb{C}}(\sum_{i \leq n} e^{-i}V^{n-i}M_0) \leq \dim_{\mathbb{C}} \sum_{i \leq n} (V^{n-i}M_0/eV^{n-i-1}M_0).$$

Therefore, $\dim_{\mathbb{C}} \sum_{i \leq n} e^{-i}V^{n-i}M_0 = O(n^d)$ where $d = GKRM$. There exists $m > 0$, such that $Ve^{-1} \subseteq \sum_{i \leq m} e^{-i}V, Ve^{\frac{1}{2}} \subseteq e^{\frac{1}{2}} \sum_{i \leq m} e^{-i}V$. Put

$L = \mathbb{C}e^{-1} + \mathbb{C}e^{\frac{1}{2}} + V$. Then $L$ generates $R[e^{-\frac{1}{2}}]$ and

$L^n \subseteq \sum_{i \leq mn} e^{-i}V^{mn-i} + e^{\frac{1}{2}} e^{-i}V^{mn-i}$. Thus we conclude that $\dim L^n M_0 = O(n^d), \text{so} GK_{R[e^{-\frac{1}{2}}]}[M[e^{-\frac{1}{2}}] \leq d$.

Lemma 4.2. Let $M$ be a finitely generated $R$-module which can be filtered with $R$-submodules $M_0 \subseteq \cdots \subseteq M_n = M$, such that $GK(M_i/M_{i-1}) \geq \frac{1}{2}GK(R/Ann(M_i/M_{i-1}))$ for all $i = 1, \cdots, n$.

Then $GK(M) \geq \frac{1}{2}GK(R/Ann(M))$.

Proof. We will proceed by induction on $i$ to show that $GK(M_i) \geq \frac{1}{2}GK(R/Ann(M_i))$. Put $I = Ann(M_{i-1}), J = Ann(M_i/M_{i-1})$ then

$$GK(H/Ann(M_i)) \leq GK(R/IJ) \leq \text{Max}\{GK(R/I), GK(R/J)\}$$

and $GK(M_i) \geq \text{Max}\{GK(M_{i-1}), GK(M_i/M_{i-1})\}$. Thus $GK(M_i) \geq \frac{1}{2}GK(R/Ann(M_i))$. 

\[\square\]
Theorem 4.1. For any $z$ and any finitely generated $U_z$-module $M$, one has $GK(M) \geq \frac{1}{2} GK(U_z/\text{Ann}(M))$.

Proof. Recall that $V(I)$ is the singular locus of Spec gr $U_z =\text{Spec } B_n$, where $I = (x, y, \Delta)$. Let $M$ be a finitely generated $U_z$-module. If $SS(M)$ (the characteristic variety of $M$) intersects with Spec $B_n \setminus V(I)$ (which is the smooth locus of Spec $B_n$), then it follows from Gabber’s theorem that $SS(M) \cap (\text{Spec } B_n \setminus V(I))$ is a coisotropic subvariety of a symplectic variety Spec $B_n \setminus V(I)$, thus $GK(M) \geq \frac{1}{2} GK(U_z)$. On the other hand, if $SS(M) = \{0\}$, then $\dim M$ is finite, so $GK(U_z/\text{Ann}(M)) = 0$ and we are done. Therefore, we may assume that $\{0\} \neq SS(M) \subset V(I)$.

let $a \in SS(M), a \neq 0$. We will proceed by the induction on the number of irreducible components of $SS(M)$. If $SS(M) = V(I)$, then $GK(M) = 2$ and there is nothing to prove. Thus we may assume that $SS(M)$ is a union of finitely many lines in $V(I)$ through the origin, equivalently $GK(M) = 1$. We may assume without loss of generality that $a = (e - 1, f, h)$. In particular, $e \not\in \sqrt{\text{Ann}(\text{gr} M)}$. Denote by $M' = \{m \in M : e^n = 0, n >> 0\}$. Thus, $M'$ is a $U_z$-submodule of $M$ and $M'' = M/M'$ is $e$-torsion free $U_z$-module.

Then, $SS(M') \subset SS(M)$ and $a \notin SS(M')$. Hence the number of irreducible components of $SS(M')$ is less than that of $SS(M)$. Therefore, by the induction assumption $GK(U_z/\text{Ann}(M')) \leq 2$. By Lemma 4.1 we have that $GK_U[e^{-1/2}] (M''[e^{-1/2}]) \leq 1$. Let us denote $C[e^{-1} h, e - 1] \otimes U'_z = W_1(U'_z)$ by $U'_z$. As was shown in the proof of Theorem 3.1 $e \in W_1(O_{\lambda}(e), U'_z[e^{-1/2}] = W_1(O_{\lambda}(e))[e^{-1/2}]$. Let $N \subset M''[e^{-1/2}]$ be a finitely generated $W_1(O_{\lambda}(e))$-module which generates $M''[e^{-1/2}]$ over $U'_z[e^{-1/2}]$. Thus, $N[e^{-1/2}] = M''[e^{-1/2}]$. Then (by Lemma 4.1) $GK_{W_1(O_{\lambda}(e))}(N) \leq GK_{U'_z[e^{-1/2}]}(M''[e^{-1/2}]) \leq 1$. Therefore, $GK(O_{\lambda}(e)/\text{Ann}(N)) = 0$ by Corollary 4.1. Then, there is a nonzero polynomial $g$, such that $g(\Delta)N = 0$. Similarly, there is a nonzero polynomial $\phi$, such that $\phi(e^{-1/2}) \in \text{Ann}(N)$ (recall that $\Delta, e^{-1/2} \in O_{\lambda}(e)$). Hence $e^m \phi(e^{-1/2}) \in \text{Ann}(M'')$ for all $m$. Choosing $m$, such that $e^m \phi(e^{-1/2}) \in U_z$, we conclude that there is a nonzero $\psi \in C[e, x]$, such that $\psi \in \text{Ann}(N)$. Since $g(\Delta), \psi$ commute with $e^{-1/2}$ and $M'' \subset \sum_{i} e^{-i} N$ we conclude that $g(\Delta), \psi \in \text{Ann}_{U_z} M''$. So, $GK(U_z/\text{Ann}(M'')) \leq 2$. Applying Lemma 4.2 we are done.

Theorem 4.2. If $M$ is a finitely generated $H_z$-module, then $GK_H(M) \geq \frac{1}{2} GK(H_z/\text{Ann}(M))$

Proof. Throughout we will suppress index $z$ for simplicity. Let $M$ be a finitely generated $\mathbb{C}[t] \otimes \mathbb{C} F$ be an embedding into an algebraically closed field $F$. Put $H_F = H \otimes_{\mathbb{C}} F$. Then $U_F = H_F/(t - \rho(t)) = H(t) \otimes_{\mathbb{C}} F$ and $M_F = M(t) \otimes_{\mathbb{C}(t)} F$ is a finitely generated module over $U_F$. So, we may apply Theorem 4.1 to $U_F, M_F$ (where instead of $C$ the ground field is $F$). Thus, $2GK_{U_F}(M_F) \geq GK_F(U_F/\text{Ann}(M_F))$. But $\text{Ann}_{U_F} M_F = \text{Ann}_{H(t)}(M(t) \otimes_{\mathbb{C}(t)} F)$, so
$U_F/\text{Ann}(M_F) = (H(t)/\text{Ann}M(t)) \otimes_{\mathbb{C}(t)} F$. Therefore, $2\text{GK}_{H(t)}(M(t)) \geq \text{GK}(H(t)/\text{Ann}(M))$. On the other hand $\text{GK}_{H}(M) - 1 = \text{GK}_{H(t)}(M(t))$ and $\text{GK}_{\mathbb{C}(t)}(H(t)/\text{Ann}(M)) = \text{GK}(H/\text{Ann}(M)) - 1$.

Thus, $\text{GK}_{H}(M) \geq \frac{1}{2}\text{GK}(H/\text{Ann}(M))$.

Let $M$ be an arbitrary finitely generated $H$-modules. Let $M' \subset M$ be the submodule of all $\mathbb{C}[t]$-torsion modules. Thus, $M/M'$ is $\mathbb{C}[t]$-torsion free and $M'$ can be filtered by $H$-submodules, such that each subquotient is annihilated by some $t - \lambda$ for some $\lambda \in \mathbb{C}$. Then $M$ can be filtered by $H$-submodules, such that for each subquotient the conclusion of the theorem holds. By Lemma 4.2 we are done.

\[ \square \]

5. Equivalence

In this section we establish an equivalence between the category of (generalized) Whittaker $U_2$-modules and the category of $\mathcal{O}^{M(z)}$-modules, an analogue of Skryabin’s equivalence \[S\]. This equivalence is a direct consequence of the exactness of the Whittaker functor for $\mathfrak{sl}_2$ (simplest case of Kostant’s result) and the isomorphism $U_2[e^{\frac{t}{2}}] \cong W_1(\mathcal{O}^{M(z)})$.

Let us briefly recall the setup of the quantum hamiltonian reduction of algebras. Let $A$ be an associative $\mathbb{C}$-algebra, and let $n \subset A$ be a finite dimensional nilpotent Lie subalgebra under the commutator bracket of $A$. Suppose that $\text{ad}(n)$ action of $n$ on $A$ is locally nilpotent. Let us denote $(A/An)^n$ by $H(n, A)$. Then $H(n, A) = \text{End}_A(A/An)^{\text{op}}$ is an algebra. The full subcategory of $A$-modules consisting of those $A$-modules on which $n$-acts locally nilpotently (Whittaker modules) will be denoted by $(A, n)$-mod. One has a functor $Wh$ (Whittaker functor) from $(A, n)$-mod to the category of $H(n, A)$-modules defined as follows $Wh_A(M) = M^n = Hom_A(A/An, M)$. There is a functor in the opposite direction $F(N) = A/An \otimes_{H(n, A)} N$. Under these assumptions we have the following standard

**Proposition 5.1.** Let $B \subset A$ be a $\mathbb{C}$-subalgebra containing $n$. If $H^i(n, B/Bn) = 0$ for all $i > 0$, then the functor $Wh_B$ induces an equivalence. Moreover, if $A/An$ is a faithfully flat right $H(n, A)$-module, then the inverse is given by the functor $F$.

**Proof.** At first we check that the functor $Wh_B$ is an exact functor. Of course this will imply the same about $Wh_A$. It suffices to check that for any $B$-module $M$ on which $n$ acts locally nilpotently, one has $H^i(n, M) = 0$ for all $i > 0$. Let $m$ be a nonnegative integer. Assume that for any such $M$ one has $H^i(n, M) = 0$ for all $i > m$. This is obviously true for $m = \dim n$. We will proceed by descending on $m$. Let $C$ be the full subcategory of all $(B, n)$-mod whose objects are $M$, such that $H^m(n, M) = 0$. Clearly $Wh$ is exact on $C$. Also, $C$ is closed under taking quotients, arbitrary direct sums, extensions and contains $B/Bn$. Let $N$ be an object in $(B, n)$. Let $N'$ be the sum of all submodules of $N$ that belong to $C$. Then $N'$ belongs to $C$ and
no nontrivial submodule of $N/N'$ belongs to $C$. This implies that $N/N' = 0$, otherwise it will contain a nonzero quotient of $B/Bn$, a contradiction. Thus, $N$ belongs to $C$, and $Wh_A$ is an exact functor. This implies that $A/An$ is a projective generator of $(A, n)$-mod since $Wh_A = Hom_A(A/An, -)$. Therefore, $(A, n)$-mod is equivalent to $End_A(A/An)^{op} = \mathbb{H}(n, A)$ with $Wh_A$ being an equivalence.

Now suppose that $A/An$ is a flat right $\mathbb{H}(n, A)$-module. Then, $F$ is an exact functor, and so is $Wh_A(F) : \mathbb{H}(n, A)$-mod $\rightarrow \mathbb{H}(n, A)$-mod. Clearly $Id$ is a subfunctor of $Wh(F)$, moreover $Wh(F(\mathbb{H}(n, A))) = \mathbb{H}(n, A)$. Therefore, since $Wh(F)$ preserves direct sums, we get that $Wh(F) = Id$.

Recall that $\mathcal{O}^\lambda$ denoted the noncommutative deformation of Kleinian singularity of type $D_{n+2}$ with parameter $\lambda$.

**Theorem 5.1.** Let $c$ be a nonzero complex number. One has $\mathcal{O}^{\lambda(z)} = (U_z/U_z(e - c))^{c - e}$. The functor $M \rightarrow Wh(M) = \{ m \in M : (e - c)m = 0 \}$ defines an equivalence between the category of $U_z$-modules on which $(e - c)$ acts locally nilpotently and the category of $\mathcal{O}^{\lambda(z)}$-modules, the inverse functor is given by $N \rightarrow F(N) = U_z/U_z(e - c) \otimes_{\mathcal{O}^{\lambda(z)}} N$.

**Proof.** Without loss of generality assume that $c = 1$. It is well-known and easy to check that $H^1(Ce, \mathfrak{sl}_2/\mathfrak{sl}_2(e - 1)) = 0$. Thus, according to Proposition 5.1, it suffices to check that $\mathcal{O}^{\lambda(z)}$ is isomorphic to $(U_z/U_z(e - 1))^{c - 1}$ and $U_z/U_z(e - 1)$ is a free right $(U_z/U_z(e - 1))^{c - 1}$-module.

It was shown in the proof of Theorem 5.1 that $U_z[e^{-h\frac{1}{2}}] = A[e^{-\frac{h}{2}}]$, where $A = W_1(\mathcal{O}^{\lambda(z)})[e^{-\frac{h}{2}}]$. Thus, $U_z/U_z(e - 1) = A/A(e^{-\frac{h}{2}} - 1)$, and $(U_z/U_z(e - 1))^{c} = (A/A(e^{-\frac{h}{2}} - 1))^{c}$. But $A/A(e^{-\frac{h}{2}} - 1) = \mathbb{C}[e^{-1}h] \otimes \mathcal{O}^{\lambda(z)}$ and $(A/A(e^{-\frac{h}{2}} - 1))^{c} = \mathcal{O}^{\lambda(z)}$ and we are done.

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