ON EIGENELEMENTS SENSITIVITY FOR COMPACT SELF-ADJOINT OPERATORS AND APPLICATIONS

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Abstract. In this manuscript, we present optimal sensitivity results of eigenvalues and eigenspaces with respect to self-adjoint compact operators. We show that while eigenvalues depend in a Lipschitzian way in compact operators, the eigenspaces are only locally Lipschitz. Our results generalize to arbitrary dimension eigenspaces the results obtained in [19] for one-dimensional eigenspaces sensitivity and thus simplify the celebrate results by Davis and Kahan [6] developed for general Hermitian operator perturbations. Moreover, Proper Orthogonal Decomposition bases sensitivity is carried out in the case of time-interval perturbations, spatial perturbations (Gappy-POD) or parameter perturbations.

1. Introduction. The main objective in proper orthogonal decomposition (POD) is to provide an optimal low dimensional basis, in a sense which will be described below, to represent high-dimensional experimental or numerical data. This low-dimensional basis allows the formulation of reduced-order models of complex flows. POD decomposes a given space-time dependent flow field \( u(x, t) \) into an orthonormal system of spatial modes \( (\Phi_n(x))_{n \in \mathbb{N}} \) and their corresponding temporal amplitudes \( (a_n(t))_{n \in \mathbb{N}} \) as the following

\[
u(x, t) = \sum_{n \geq 0} a_n(t) \Phi_n(x).
\]

The classical mathematical framework for POD is the space \( L^2([0, T]; H) \), where \([0, T]\) and \(H\) denote respectively a time-interval and a Hilbert space of spatial functions. The optimality of the basis \( (\Phi_n(x))_{n \in \mathbb{N}} \) means that any truncated series expansion \( \sum_{n=0}^N a_n(t) \Phi_n(x) \) of \( u(x, t) \) in this basis has the smallest mean-square truncation error among all the other bases in the sense of \( L^2([0, T]; H) \) norm.

Using the spectral theory of compact self-adjoint operators, the POD provides a natural ordering of the spatial modes with respect to the mean-square of their amplitude.

In various problems in fluid mechanics or fluid structure interaction, the POD basis is computed for a given Reynolds number or for other flow parameters. This basis is then used to build a reduced model to predict the flow for other values of these parameters. The question concerning the domain of validity of this reduced
model arises then naturally. In Akkari et al. [3], for quasilinear parabolic partial differential equation, this domain of validity was specified according to the solution regularity and number of modes considered in the reduced model. Similar results were obtained for Navier-Stokes equations [1] and for Burgers equations [2]. Numerical studies on parametric sensitivity for fluid structure interaction problems was performed in [17]. This methods are also used in control theory [14, 20, 12], in medical imaging [7, 18] one where one often seeks to use the POD basis to represent data although time-interval variation, spatial domain variation or parameter variation may occur.

The aim of the present manuscript is the relevant question concerning the sensitivity of POD bases with respect to such variations. In the seminal work of Davis and Kahane [6], the authors gave a general answer about the rotation of eigenspaces with respect to Hermitian perturbations. The theory developed in [6] were simplified in the case of compact and self-adjoint operators by Rousselet and Chenais [19], but only the sensitivity of one-dimensional eigenspaces with respect to compact variations were studied. In the present work, we extend the results of [19] to multi-dimensional eigenspaces and place the POD sensitivity with respect to compact variations were studied. In the present work, we extend the results of [19] to multi-dimensional eigenspaces and place the POD sensitivity with respect to time variation, space variation or parameter variation in the general framework of compact operator perturbations.

2. Preliminaries and notations. Let $H$ be a Hilbert space, endowed with its inner product $\langle \cdot, \cdot \rangle_H$ and its norm $\| \cdot \|_H$. Consider the space of self-adjoint compact operators $K_{sa}(H)$. Recall that the spectrum of any operator $K \in K_{sa}(H)$ consists of isolated real eigenvalues forming a sequence converging towards 0. Moreover, 0 can be itself an eigenvalue if $K$ is not injective.

We denote by $(\mu_n(K))_{n \geq 1}$ the sequence of non-zero eigenvalues of $K$, repeated a number of times equal to their multiplicity and

- $\mu_1^+, \cdots, \mu_n^+, \cdots$ is the sequence of positive eigenvalues sorted in decreasing order $\mu_1^+ \geq \mu_2^+ \geq \cdots \geq \mu_n^+ \geq \cdots > 0$.
- $\mu_1^-, \cdots, \mu_n^-, \cdots$ is the sequence of negative eigenvalues sorted in increasing order $\mu_1^- \leq \mu_2^- \leq \cdots \leq \mu_n^- \leq \cdots < 0$.
- $E(\mu_n)$ is the eigenspace associated to the eigenvalue $\mu_n$ and $m_n \in \mathbb{N}^*$ is its multiplicity.
- $S(\mu_n) = E(\mu_n) \cap S$, where $S$ is the unit sphere of $H$.
- For any $u \in S$, we set $d(u, S(\mu_n)) = \min\{\|u - v\|_H : v \in S(\mu_n)\}$, the “distance” from the point $u$ to the set $S(\mu_n)$.
- For any closed subspace $E$ of $H$, the orthogonal projection on $E$ is denoted by $\Pi_E$.

The Courant-Fisher Theorem provides the following characterization of all eigenvalues:

$$
\mu_n^+ = \min_{V \in G_{n-1}(H)} \max_{v \in V^\perp \cap S} \langle Kv, v \rangle_H, 
$$

where $G_{n-1}(H)$ is the Grassmann analytical manifold of all $(n-1)$-dimensional subspaces of $H$, and $V^\perp$ denotes the orthogonal, in $H$, of the subspace $V$. This min–max value is achieved for $V = E(\mu_1^+) + E(\mu_2^+) + \cdots + E(\mu_{n-1}^+)$ and any vector $v \in S(\mu_n^+)$. In the same way,

$$
\mu_n^- = \max_{V \in G_{n-1}(H)} \min_{v \in V^\perp \cap S} \langle Kv, v \rangle_H.
$$
This max–min value is achieved for \( V = E(\mu_1^-) + E(\mu_2^-) + \cdots + E(\mu_n^-) \) and any vector \( v \in S(\mu_n^-) \), moreover
\[
\|K\| = \max(\mu_1^+, -\mu_1^-) = \max \left( \max_{v \in S} \langle Kv, v \rangle_H, -\min_{v \in S} \langle Kv, v \rangle_H \right), \tag{3}
\]
where \( \| \cdot \| \) denotes the norm in \( L(H) \), the space of linear continuous operators on \( H \).

3. Sensitivity of eigenvalues and eigenspaces on \( K_n(H) \). Let us introduce, for every \( n \in \mathbb{N}^* \), the function
\[
\mu_n : K_n(H) \rightarrow \mathbb{R}
\]
\[
K \mapsto \mu_n(K)
\]

**Lemma 3.1.** The function \( \mu_n \) is Lipschitzian from \( (K_n(H), \| \cdot \|) \) to \( \mathbb{R} \).

**Proof.** For any \( K_1, K_2 \in K_n(H) \) and \( u \in S \), the inequality
\[
|\langle K_2 u, u \rangle_H - \langle K_1 u, u \rangle_H| = |\langle (K_2 - K_1) u, u \rangle_H| \leq \|K_2 - K_1\|,
\]
holds true. It follows that
\[
\langle K_1 u, u \rangle_H - \|K_2 - K_1\| \leq \langle K_2 u, u \rangle_H \leq \langle K_1 u, u \rangle_H + \|K_2 - K_1\|.
\]
Applying the min-max characterization (1), we obtain:
\[
\mu_n^+(K_1) - \|K_2 - K_1\| \leq \mu_n^+(K_2) \leq \mu_n^+(K_1) + \|K_2 - K_1\|.
\]
The same arguments can be used for the negative eigenvalues \( \mu_n^-(K_i), i = 1, 2 \), with the max-min characterization (2), which achieves the proof. \( \square \)

In what follows, we will study the regularity of eigenspaces with respect to self-adjoint compact operators. Before announcing our main result, let us dwell a while on the notion of locally Lipschitz dependence for eigenspaces in \( K_n(H) \).

Let \( K \in K_n(H) \) and \( v \in S \). First, remark that for every \( n \in \mathbb{N}^* \), one has
\[
0 \leq d(v, S(\mu_n(K))) \leq \sqrt{2}. \tag{4}
\]

**Remark 1.** The following equivalences hold true:

i. \( d(v, S(\mu_n(K))) = 0 \iff v \in S(\mu_n(K)) \).

ii. \( 0 < d(v, S(\mu_n(K))) < \sqrt{2} \iff \hat{u} := \Pi_{E(\mu_n(K))}(v) \neq 0 \) and \( u := \frac{v - \hat{u}}{\|\hat{u}\|} \) is the unique point in \( S(\mu_n(K)) \) such that \( d(v, S(\mu_n(K))) = \|v - u\| \).

iii. \( d(v, S(\mu_n(K))) = \sqrt{2} \iff v \perp E(\mu_n(K)) \).

We show now a coerciveness result of the operator \( (K - \mu_n(K)id_H) \) on \( E(\mu_n(K))^\perp \), for any \( K \in K_n(H) \) and \( n \in \mathbb{N}^* \).

**Lemma 3.2.** Let \( K \in K_n(H) \) and \( n \in \mathbb{N}^* \). Then there is \( C(K, n) > 0 \) such that
\[
|\langle (K - \mu_n(K)id_H)w, w \rangle_H| \geq C(K, n) \|w\|^2, \text{ for any } w \in E(\mu_n(K))^\perp. \tag{5}
\]

**Proof.** Consider the subspaces:

- \( E_n^0(K) = \text{Ker}(K) \), the kernel of the operator \( K \).
- \( E_n^+(K) \) is the direct sum of all eigenspaces of \( K \) whose eigenvalues are larger than \( \mu_n(K) \).
- \( E_n^-(K) \) is the direct sum of all eigenspaces of \( K \) whose eigenvalues are lower than \( \mu_n(K) \).
Since the operator $K$ is compact and self-adjoint, the space $E(\mu_n(K))$ can be decomposed as the following:

$$E(\mu_n(K))^\perp = E_n^0(K) \oplus E_{n+}^+(K) \oplus E_{n-}^-(K).$$

Moreover, for any $w \in E(\mu_n(K))^\perp$,
- if $w \in E_n^0(K)$ then $\langle (K - \mu_n(K) \text{id}_H) w, w \rangle_H = -\mu_n(K)\|w\|^2$,
- if $w \in E_{n+}^+(K)$ then $\langle (K - \mu_n(K) \text{id}_H) w, w \rangle_H \geq (\overline{\mu_n}(K) - \mu_n(K))\|w\|^2$, where $\overline{\mu_n} := \min \{\mu_k : \mu_k(K) > \mu_n(K), \ k \in \mathbb{N}^+\}$,
- if $w \in E_{n-}^-(K)$ then $\langle (K - \mu_n(K) \text{id}_H) w, w \rangle_H \leq (\underline{\mu_n}(K) - \mu_n(K))\|w\|^2$, where $\underline{\mu_n} := \max \{\mu_k(K) : \mu_k(K) < \mu_n(K), \ k \in \mathbb{N}^+\}$.

Therefore, the claim follows by setting

$$C(K, n) := \min\{\|\mu_n(K)\|, \overline{\mu_n}(K) - \mu_n(K), \mu_n(K) - \underline{\mu_n}(K)\} > 0.$$  

(6)

Remark 2. The constant $C(K, n)$ defined by (6) satisfies:

$$\lim_{n \to +\infty} C(K, n) = 0,$$ 

(7)

for every operator $K \in \mathcal{K}_s(H)$.

At this stage, we can state and show our main result. We mention here that our result is a generalization of a result obtained by Rousselet and Chenais [19], where only the special case of one-dimensional eigenspaces is considered. Our work provides a direct and simple proof of the celebrate result obtained by Davis and Kahane [6] on the rotation of eigenvectors by perturbations for more general Hermitian operators.

**Theorem 3.3.** Let $K \in \mathcal{K}_s(H)$ and $n \in \mathbb{N}^*$. Given $\delta K \in \mathcal{K}_s(H)$, then

$$d(v, \mathcal{S}(\mu_n(K))) \leq \frac{4}{C(K, n)} \|\delta K\|,$$

holds true for any $v \in \mathcal{S}(\mu_n(K + \delta K))$, where the constant $C(K, n)$ is given by (6).

**Proof.** Let $K, \delta K \in \mathcal{K}_s(H)$ and $n \in \mathbb{N}^*$ and consider an arbitrary $v \in \mathcal{S}(\mu_n(K + \delta K))$. From Remark 1, we can distinguish three situations:

**Case 1.** If $d(v, \mathcal{S}(\mu_n(K))) = 0$, then $v \in \mathcal{S}(\mu_n(K))$ and we get hence the result.

**Case 2.** If $0 < d(v, \mathcal{S}(\mu_n(K))) < \sqrt{2}$, then by classical geometry arguments, $\vec{u} := \prod_{E(\mu_n(K))} \neq 0$ and $u := \frac{\vec{u}}{\|\vec{u}\|}$ is the unique point in $\mathcal{S}(\mu_n(K))$ such that $d(v, \mathcal{S}(\mu_n(K))) = \|v - u\|$. Then

$$v - u = (\|\vec{u}\| - 1)u + (v - \vec{u}), \quad \text{where} \ v - \vec{u} \perp E(\mu_n(K)).$$

(8)

Let us set $\alpha = \|\vec{u}\| - 1 = \langle v - u, u \rangle_H$, then we get on one hand

$$\|v - u\|^2 = \alpha^2 + \|v - \vec{u}\|^2.$$ 

(9)

Since $v = u + (v - u)$, on has on the other hand

$$\|v\|^2 = \|u\|^2 + \|v - u\|^2 + 2 \alpha,$$

that is

$$\alpha = -\frac{1}{2}\|v - u\|^2.$$
Substituting the last identity in (9), we get
\[ \|v - u\|^2 \left(1 - \frac{1}{4}\|v - u\|^2\right) = \|\tilde{v}\|^2. \]

Using the facts that \(\|v - u\| = d(v, S(\mu_n(K))) \leq \sqrt{2}\), and thus \(1 - \frac{1}{4}\|v - u\|^2 \geq \frac{1}{2}\), we obtain
\[ \|v - u\|^2 \leq 2\|\tilde{v}\|^2. \]

Moreover, using the coerciveness inequality (5), it holds
\[ \|\langle (K - \mu_n(K) id_H) (v - \tilde{v}) , v - \tilde{v}\rangle_H \| \geq \frac{C(K, n)}{2} \|v - u\|^2. \]

But from (8) it follows:
\[ \langle (K - \mu_n(K) id_H) (v - \tilde{v}) , v - \tilde{v}\rangle_H = \langle (K - \mu_n(K) id_H) (v - u) , v - u\rangle_H. \]

To lighten the notation, we set \(\delta \mu_n = \mu_n(K + \delta K) - \mu_n(K)\), then a direct computation gives
\[ \|\langle (K - \mu_n(K) id_H) (v - u) , v - u\rangle_H\| = \|\langle (\delta K - \delta \mu_n id_H) v , v - u\rangle_H\|, \]
and therefore by Lemma 3.1
\[ \|\langle (K - \mu_n(K) id_H) (v - u) , v - u\rangle_H\| \leq 2\|\delta K\| \times \|v\| \times \|v - u\|. \]

Finally, we conclude that
\[ \frac{C(K, n)}{2} \|v - u\|^2 \leq 2\|\delta K\| \times \|v - u\|, \]
that is
\[ d(v, S(\mu_n(K))) \leq \frac{4}{C(K, n)} \|\delta K\|, \]
where \(C(K, n)\) is given by (6), which provides they result.

**Case 3.** If \(d(v, S(\mu_n(K))) = \sqrt{2}\) then \(v \perp E(\mu_n(K))\). The situation is similar to Case 2 with \(\alpha = -1\). The conclusion is then straightforward, which achieves the proof.

To simplify the statement of the following corollary, we will assume that all operators are nonnegative, which implies that all their eigenvalues are nonnegative.

**Corollary 1.** Let \(K_1\) and \(K_2\) be two nonnegative operators in \(K_n(H)\) and \(N \in \mathbb{N}^*\). Let \((\Phi_n(K_j))_{1 \leq n \leq N} \in S\) be eigenvectors of \(K_j\), \(j = 1, 2\), corresponding to the eigenvalues \(\mu_1(K_j) \geq \mu_2(K_j) \geq \cdots \geq \mu_N(K_j) > 0\). Define \(V_N(K_j)\) the subspace of \(H\) spanned by the eigenvectors \((\Phi_n(K_j))_{1 \leq n \leq N}\). Then
\[ \left\| \prod_{N(K_j_2)} - \prod_{N(K_j_1)} \right\| \leq 8 \|K_2 - K_1\| \sum_{n=1}^{N} \frac{1}{C(K_1, n)}. \]
Proof.

\[
\left\| \prod_{\nu_n(K_2)} - \prod_{\nu_n(K_1)} \right\| = \sup_{u \in S} \left\| \prod_{\nu_n(K_2)} u - \prod_{\nu_n(K_1)} v \right\|
\]

\[
= \sup_{u \in S} \left\| \sum_{n=1}^{N} (u, \Phi_n(K_2)) \Phi_n(K_2) - \sum_{n=1}^{N} (u, \Phi_n(K_1)) \Phi_n(K_1) \right\|
\]

\[
\leq \sum_{n=1}^{N} \| \Phi_n(K_2) - \Phi_n(K_1) \| (\| \Phi_n(K_1) \| + \| \Phi_n(K_2) \|)
\]

\[
\leq 2 \sum_{n=1}^{N} \| \Phi_n(K_2) - \Phi_n(K_1) \|
\]

\[
\leq 2 \sum_{n=1}^{N} \frac{4}{C(K_1, n)} \| K_2 - K_1 \|
\]

\[
= 8 \| K_2 - K_1 \| \sum_{n=1}^{N} \frac{1}{C(K_1, n)},
\]

which ends the proof.

For a given \( K_1 \), the constant \( C(K_1, n) \) appearing in the estimate

\[
d(v, S(\mu_n(K_1))) \leq \frac{4}{C(K_1, n)} \| K_1 - K_2 \|, \text{ for any } v \in S(\mu_n(K_2)) \text{ and } K_2 \in \mathcal{K}_n(H),
\]

can be of the same order than \( \| K_1 - K_2 \| \). Then we can find a vector \( v^* \in S(\mu_n(K_2)) \) such that the distance \( d(v^*, S(\mu_n(K_1))) \) is maximal, that is \( d(v^*, S(\mu_n(K_1))) = \sqrt{2} \). In such situations the eigenspace \( E(\mu_n(K_2)) \) varies significantly even if the operator’s variation \( \delta K := K_2 - K_1 \) is small.

We give here a small example to illustrate this well-known phenomenon. Consider the special case where \( H = \mathbb{R}^2 \):

\[
K_1 = \begin{pmatrix} 1 + \varepsilon & 0 \\ 0 & 1 - \varepsilon \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 1 + \varepsilon & \varepsilon \sqrt{3} \\ \varepsilon \sqrt{3} & 1 - \varepsilon \end{pmatrix},
\]

where \( \varepsilon \) is a small real parameter. It is clear that \( \| K_2 - K_1 \| = |\varepsilon| \sqrt{3} \ll 1 \). The eigenvalues of \( K_1 \) and \( K_2 \) are respectively

\[
\mu_1(K_1) = 1 + \varepsilon, \quad \mu_2(K_1) = 1 - \varepsilon, \quad \mu_1(K_2) = 1 + 2\varepsilon, \quad \text{and} \quad \mu_2(K_2) = 1 - 2\varepsilon,
\]

whose variations satisfy \( |\mu_i(K_2) - \mu_i(K_1)| = |\varepsilon| \ll 1 \), for \( i = 1, 2 \). However, the eigenvectors of \( K_1 \) and \( K_2 \) are respectively

\[
\Phi_1(K_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Phi_2(K_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

\[
\Phi_1(K_2) = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \Phi_2(K_2) = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix},
\]

whose variations satisfy \( \| \Phi_1(K_2) - \Phi_1(K_1) \| = \| \Phi_2(K_2) - \Phi_2(K_1) \| = \sqrt{2 - \sqrt{3}} \ll 1 \).

This leads to the introduction of the clustering value of an operator \( K_1 \) with respect to an operator variation \( \delta K := K_2 - K_1 \).
Definition 3.4. The clustering index value of the operator $K \in K_s(H)$ with respect to the operator variation $\delta K$ and the threshold $\eta > 0$ is defined by

$$n_c(K, \delta K, \eta) := \max \left\{ n \in \mathbb{N}^* : \frac{\|\delta K\|_{C(K,n)}}{C(K,n)} < \eta \right\}. \quad (10)$$

4. Application to Proper Orthogonal Decomposition methods. One significant question concerning Proper Orthogonal Decomposition methods (POD) is: given a solution to a problem for a set of (physical) parameters, can we control the accuracy of the corresponding POD basis when the parameter set varies? This will be the subject of what follows.

We confine ourselves to spatiotemporal framework, that is, a spatial domain $\Omega \subset \mathbb{R}^N$, a temporal interval $[0,T]$ and possibly (physical) parameters $\lambda \in \Lambda$, where $\Lambda$ is a subset of a normed space (or more generally a metric space) are given. Let $H(\Omega)$ be a spatial Hilbert space, then $L^2(0,T;H(\Omega))$ is the set of all functions $u$ defined on the time interval $[0,T]$ with values in $H(\Omega)$, such that

$$\int_0^T \|u(t)\|_{H(\Omega)}^2 \, dt < +\infty.$$
and this estimate is minimal among all Hilbertian bases of $H(\Omega)$.

Now, consider a perturbation $\delta\lambda$ of $\lambda_1$ and set $\lambda_2 := \lambda_1 + \delta\lambda$, the resultant operator perturbation is thus $\delta K = K_{\lambda_2, T, \Omega} - K_{\lambda_1, T, \Omega}$. The continuous dependence of $u_{\lambda_1, T, \Omega}$ w.r.t $\Omega$ on $L^2(0, T; H(\Omega))$ leads directly to a continuous dependence of $K_{\lambda_1, T, \Omega}$ on $\mathcal{L}(H(\Omega))$ and then

$$
\lim_{||\delta\lambda||_H \to 0} ||K_{\lambda_2, T, \Omega} - K_{\lambda_1, T, \Omega}||_{\mathcal{L}(H(\Omega))} = 0.
$$

Let $(\Phi_{n_\lambda, T, \Omega})_{n \geq 1}$ be the POD basis associated to $u_{\lambda_2, T, \Omega}$, via the operator $K_2$. Similarly, we have

$$
\frac{1}{T} \left\| u_{\lambda_2, T, \Omega} - \sum_{n=1}^{N} \langle u_{\lambda_2, T, \Omega}, \Phi_{n, T, \Omega} \rangle_{H(\Omega)} \Phi_{n, T, \Omega} \right\|_{L^2(0, T; H(\Omega))}^2 = \sum_{n \geq N + 1} \mu_{n, T, \Omega}.
$$

A surprising fact occurs when we represent the solution $u_{\lambda_2, T, \Omega}$ in the first basis $(\Phi_{n_\lambda, T, \Omega})_{n \geq 1}$. Indeed, let $N > n_\varepsilon(K_1, \delta K, \varepsilon)$, for some fixed threshold $\varepsilon > 0$ and set $\beta_n := \langle u_{\lambda_2, T, \Omega}, \Phi_{n, T, \Omega} \rangle_{H(\Omega)}$. It holds thus:

$$
\frac{1}{T} \left\| u_{\lambda_2, T, \Omega} - \sum_{n=1}^{N} \beta_n \Phi_{n, T, \Omega} \right\|_{L^2(0, T; H(\Omega))}^2 
\leq \frac{1}{T} \left\| u_{\lambda_2, T, \Omega} - \sum_{n=1}^{n_\varepsilon(K_1, \delta K, \varepsilon)-1} \beta_n \Phi_{n, T, \Omega} \right\|_{L^2(0, T; H(\Omega))}^2
\leq \frac{1}{T} \left\| u_{\lambda_2, T, \Omega} - \sum_{n=1}^{n_\varepsilon(K_1, \delta K, \varepsilon)-1} \beta_n \Phi_{n, T, \Omega} \right\|_{L^2(0, T; H(\Omega))}^2 + O(\varepsilon),
= \sum_{n \geq n_\varepsilon(K_1, \delta K, \varepsilon)} \mu_{n, T, \Omega} + O(\varepsilon),
$$

where $O(\varepsilon)$ denotes an expression of order $\varepsilon$. The last equality suggests that the approximation of the solution $u_{\lambda_2, T, \Omega}$ in the subspace of $H(\Omega)$ spanned by the eigenvectors $(\Phi_{n_\lambda, T, \Omega})_{1 \leq n \leq N}$ is restricted by the clustering index $n_\varepsilon(K_1, \delta K, \varepsilon)$ even if its dimension is larger than $n_\varepsilon(K_1, \delta K, \varepsilon)$. To illustrate this, we denote by

$$
W_{\lambda_1, T, \Omega}^N := \text{span} \left\{ \Phi_{n, T, \Omega} \mid 1 \leq n \leq N \right\}, \quad i = 1, 2
$$

and

$$
u_{\lambda_1, T, \Omega}^N := \prod_{W_{\lambda_1, T, \Omega}^N} u_{\lambda_1, T, \Omega}, \quad i = 1, 2,
$$
the projection of \( u_{\lambda_i,T,\Omega} \) on the subspace \( W_{N}^{\lambda_i,T,\Omega} \) with respect to the inner product on \( H(\Omega) \), for \( i = 1, 2 \).

Consider the case \( N > n_c(K_1, \delta K, \varepsilon) \). By Theorem 3.3, we know that \( \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_2,T,\Omega} = \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_1,T,\Omega} + O(\varepsilon) \), for every \( n \in \{1, 2, \cdots n_c(K_1, \delta K, \varepsilon)\} \). However, for \( n = n_c(K_1, \delta K, \varepsilon) \), we may have \( \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_2,T,\Omega} = \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_1,T,\Omega} + O(1) [7] \). In such situations, we get:

\[
\prod_{n_c(K_1, \delta K, \varepsilon)} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} = n_c(K_1, \delta K, \varepsilon) \sum_{n=1}^{\varepsilon} \langle u_{\lambda_n^{2,T,\Omega}} , \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_1,T,\Omega} \rangle_{H(\Omega)} \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_2,T,\Omega} + O(\varepsilon) + \alpha,
\]

where

\[
\alpha = \langle u_{\lambda_2,T,\Omega} , \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_1,T,\Omega} \rangle_{H(\Omega)} \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_2,T,\Omega} - \langle u_{\lambda_2,T,\Omega} , \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_2,T,\Omega} \rangle_{H(\Omega)} \Phi_{n_c(K_1, \delta K, \varepsilon)}^{\lambda_2,T,\Omega}.
\]

Hence, we obtain

\[
\prod_{n_c(K_1, \delta K, \varepsilon)} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} = \prod_{n_c(K_1, \delta K, \varepsilon)} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} + O(\varepsilon) + \alpha.
\]

Since \( \alpha \) has no reason to be small w.r.t. \( \| \cdot \|_{H(\Omega)} \), it follows that the distance between the projections \( \prod_{n_c(K_1, \delta K, \varepsilon)} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} \) and \( \prod_{n_c(K_1, \delta K, \varepsilon)} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} \) can be significant.

But as it was pointed above, we have

\[
\prod_{n_c(K_1, \delta K, \varepsilon)-1} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} = \prod_{n_c(K_1, \delta K, \varepsilon)-1} W_{\lambda_n^{1,T,\Omega}} u_{\lambda_n^{2,T,\Omega}} + O(\varepsilon),
\]

which means that when we consider subspaces of POD eigenvectors corresponding to clustered eigenvalues, the approximation of solutions is sensitive to small parameter perturbations. In what follows, we will present other situations where clustering sensitivity can occur. More precisely, we will see that the Gappy POD method or extension of time intervals leads to perturbations of compact operators and the phenomenon of clustered eigenvalues may hold true.

4.2. Spatial domain variation: Gappy POD. The gappy POD method uses a POD basis to reconstruct missing or "gappy" data. This method was introduced by Everson and Sirovich [8] and can be described as follows. Let \( (\Phi_{n}^{\lambda,T,\Omega})_{n \geq 1} \) be a POD basis corresponding to a solution \( u_{\lambda,T,\Omega} \). Let \( \omega \) be a “small” subset of \( \Omega \) and \( u_{\lambda,T,\Omega,\omega} \) be the restriction of \( u_{\lambda,T,\Omega} \) to \( \Omega \setminus \omega \). The aim of the Gappy POD method is to reconstruct \( u_{\lambda,T,\Omega,\omega} \) on the missing subset \( \omega \) by the direct use of the basis \( (\Phi_{n}^{\lambda,T,\Omega})_{n \geq 1} \). To this end, let us write the “repaired” solution on all \( \Omega \) from the incomplete data \( u_{\lambda,T,\Omega,\omega} \) by

\[
\tilde{u}_{\lambda,T,\Omega} := \sum_{n \geq 1} \tilde{a}_n(t) \Phi_{n}^{\lambda,T,\Omega},
\]

where the unknown temporal coefficients \( (\tilde{a}_n(t))_{n \geq 1} \) minimize the error term \( \| \tilde{u}_{\lambda,T,\Omega} - u_{\lambda,T,\Omega,\omega} \|^2_{H(\Omega)|\Omega,\omega} \) and \( \| \cdot \|^2_{H(\Omega)|\Omega,\omega} \) denotes the restriction of the norm of \( H(\Omega) \) to \( \Omega \setminus \omega \).

We will see that the Gappy POD procedure can be placed in the framework of spatial basis perturbation. Indeed, let \( K_1 := K_{\lambda,T,\Omega} \) and \( K_2 := K_{\lambda,T,\Omega,\omega} \) the
correlation compact, self-adjoint and nonnegative operators, associated to \( u_{\lambda,T,\Omega} \) and \( u_{\lambda,T,\Omega,\omega} \) respectively. Then \( \delta K_{\omega} := K_1 - K_2 \) is defined by

\[
\delta K_{\omega} : H(\Omega) \rightarrow H(\Omega)
\]

\[
\varphi \rightarrow \frac{1}{T} \int_0^T \langle u_{\lambda,T,\omega}(t), \varphi \rangle_{H(\Omega)} u_{\lambda,T,\omega}(t) \, dt,
\]

where \( u_{\lambda,T,\omega} \) is the restriction of \( u_{\lambda,T,\Omega} \) to \( \omega \). It is clear that \( \delta K_{\omega} \in K_c(H(\Omega)) \) and due to the Lebesgue’s Dominated Convergence Theorem, we get easily

\[
\lim_{|\omega| \rightarrow 0} \delta K_{\omega} = 0 \text{ in } L(H(\Omega)),
\]

where \( |\omega| \) denotes the measure of \( \omega \). Thus, Gappy-POD method can be seen as a spatial perturbation and the corresponding POD bases as perturbed eigenvectors of compacts self-adjoint operators. Whence, similar sensitivity results may occur in the presence of clustered eigenvalues for the operator \( K_{\lambda,T,0} \).

4.3. Time interval variation. Assume that the solution \( u_{\lambda,T_1,\Omega} \in L^2(0,T_1; H(\Omega)) \) is known on the time interval \([0,T_1]\) Let \( (\Phi_n^{\lambda,T_1,\Omega})_{n \geq 1} \) be the POD basis associated to \( u_{\lambda,T_1,\Omega} \), via the compact, self-adjoint and nonnegative operator

\[
K_{\lambda,T_1,\Omega} : H(\Omega) \rightarrow H(\Omega)
\]

\[
\varphi \rightarrow \frac{1}{T_1} \int_0^{T_1} \langle u_{\lambda,T_1,\Omega}(t), \varphi \rangle_{H(\Omega)} u_{\lambda,T_1,\Omega}(t) \, dt.
\]

Let \( \delta T > 0 \) be a time perturbation and set \( T_2 := T_1 + \delta T \). The consequent perturbation in correlation operator term is given by

\[
\delta K_T : H(\Omega) \rightarrow H(\Omega)
\]

\[
\varphi \rightarrow \int_0^{T_2} \langle u_{\lambda,T_2,\omega}(t), \varphi \rangle_{H(\Omega)} u_{\lambda,T_2,\omega}(t) \left( \frac{1}{T_2} - \frac{1}{T_1} 1_{[0,T_1]} \right) \, dt,
\]

where \( 1_{[0,T_1]} \) denotes the indicator function of the sub-interval \([0,T_1]\) in the interval \([0,T_2]\). As before, the operator \( \delta K_T \in K_c(H(\Omega)) \) and

\[
\lim_{\delta T \rightarrow 0} \delta K_T = 0 \text{ in } L(H(\Omega)).
\]

Thus, time interval perturbation induces compact operator perturbation and thus POD bases perturbations. Therefore, similar sensitivity results are expected in the presence of clustered eigenvalues for the operator \( K_{\lambda,T_1,\Omega} \).

Acknowledgments. The authors are very grateful to the anonymous referee for his interesting remarks and suggestions that improve the quality of the manuscript.

REFERENCES

[1] N. Akkari, A. Hamdouni, E. Liberge and M. Jazar, A mathematical and numerical study of the sensitivity of a reduced order model by POD (ROM–POD), for a 2D incompressible fluid flow, Journal of Computational and Applied Mathematics, 270 (2014), 522–530.

[2] N. Akkari, A. Hamdouni and M. Jazar, Mathematical and numerical results on the sensitivity of the POD approximation relative to the Burgers equation, Applied Mathematics and Computation, 247 (2014), 951–961.
ON EIGENELEMENTS SENSITIVITY

[3] N. Akkari, A. Hamdouni, E. Liberge and M. Jazzar, On the sensitivity of the POD technique for a parameterized quasi-nonlinear parabolic equation, Advanced Modeling and Simulation in Engineering Sciences, 1 (2014), p14.

[4] C. Allery, C. Béghein and A. Hamdouni, On investigation of particle dispersion by a POD approach, Int. Applied Mechanics, 44 (2008), 110–119.

[5] R. Bhatia and L. Elsner, The Hoffman-Wielandt inequality in infinite dimensions, Proc. Indian Acad. Sci. (Math. Sci.), 104 (1994), 483–494.

[6] C. Davis and W. M. Kahan, The rotation of eigenvectors by a perturbation. III, SIAM J. Numer. Anal., 7 (1970), 1–46.

[7] B. Denis de Senneville, A. El Hamidi and C. Moonen, A direct PCA-based approach for real-time description of physiological organ deformations, IEEE Transactions on Medical Imaging, 34 (2014), 974–982.

[8] R. Everson and L. Sirovich, Karhunen-Loeve procedure for gappy data, Journal of the Optical Society of America A: Optics, Image Science and Vision, 12 (1995), 1657–1664.

[9] E. Liberge and A. Hamdouni, Reduced order modelling method via proper orthogonal decomposition (POD) for flow around an oscillating cylinder, Journal of Fluids and Structures, 26 (2010), 292–311.

[10] A. Hay, J. Borggaard and D. Pelletier, Improved low-order modeling from sensitivity analysis of the proper orthogonal decomposition, J. Fluid Mech., 629 (2009), 41–72.

[11] J. Hoffman and H. W. Wielandt, The variation of the spectrum of a normal matrix, Duke Math. J., 20 (1953), 37–39.

[12] D. Hömberg and S. Volkwein, Control of laser surface hardening by a reduced-order approach utilizing proper orthogonal decomposition, Math. Comput. Model., 38 (2003), 1003–1028.

[13] K. Kunisch and S. Volkwein, Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics, SIAM J. Numer. Anal., 40 (2002), 492–515.

[14] K. Kunisch and S. Volkwein, Control of Burgers equation by a reduced order approach using proper orthogonal decomposition, J. Optim. Theory Appl., 102 (1999), 345–371.

[15] T. Lassila and G. Rozza, Parametric free-form shape design with PDE models and reduced basis models, Comput. Methods Appl. Mech. Engrg., 199 (2010), 1583–1592.

[16] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1980.

[17] M. Pomarède, Investigation et Application des Méthodes D’ordre Réduit pour les Calculs D’écoulements dans les Faisceaux Tubulaires D’Échangeurs de Chaleur, PhD thesis, University of La Rochelle, 2012.

[18] S. Roujol, M. Ries, B. Quesson, C. Moonen and B. Denis de Senneville, Real-time MR-thermometry and dosimetry for interventional guidance on abdominal organs, Magnetic Resonance in Medicine, 63 (2010), 1080–1087.

[19] B. Rousselet and D. Chenais, Continuité et différentiabilité d’éléments propres: Application à l’optimisation de structures, Appl. Math. Optim., 22 (1990), 27–59.

[20] S. Volkwein, Optimal control of a phase-field model using the proper orthogonal decomposition, Z. Angew. Math. Mech., 81 (2001), 83–97.

Received May 2015; revised November 2015.

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