A confined system with Rashba coupling in a constant magnetic field

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Abstract

We study a two dimensional system of electrons with Rashba coupling in a constant magnetic field, $B$, and its confining potential. We algebraically diagonalize the corresponding Hamiltonian to obtain solutions of the energy spectrum. In terms of two kinds of operator, we construct two symmetries and discuss the filling of the shells with electrons for both strong and weak $B$. Subsequently, we show that our system shares some common features with quantum optics, where the exact operator solutions for the basic Jaynes–Cummings variables are derived from our results. An interesting limit is studied and the corresponding quantum dynamics is recovered.

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1. Introduction

Spin–orbit coupling, which couples electron spin with its orbital motion, has been the subject of several theoretical and experimental studies [1]. This facilitates the development of a new generation of electronic spin (spintronics) and presents a new branch of the physics of semiconductors. It puts the spin of the electron at the centre of interest and exploits the spin-dependent electronic properties of magnetic materials and semiconductors. The underlying basis for this new electronics is the intimate connection between the electron charge and spin. A crucial implication of this relationship is that one can have access to spin through the spin property of the electron orbital in the solid. The link between the electron charge and spin is expressed by the spin–orbit interaction in semiconductors.

Novel spin properties arise from the interplay between Rashba spin splitting [2] and further confinement of two dimensional (2D) electrons in quantum wires [3–6], rings [7, 8] or dots [9–15]. Spin–orbit coupling has also been shown to affect the statistics of energy
levels, eigenfunctions as well as current distribution [16, 17]. The interplay between spin–orbit coupling and external magnetic fields has been analysed theoretically using random matrix theory [18]. In this respect, Schliemann [19] studied cyclotron motion and magnetic focusing in semiconductor quantum wells with spin–orbit coupling. More precisely, the ballistic motion of electrons in III–V semiconductor quantum wells, with Rashba spin–orbit coupling in a perpendicular magnetic field, was investigated. Taking into account the full quantum dynamics of the problem, the modification of classical cyclotron orbits, due to spin–orbit interaction, was explored and the analogy to the Jaynes–Cummings model was established. Also we refer to studies that have contributed to the understanding of magnetic effects and spin–orbit coupling in quantum wires [20].

Moreover, the Jaynes–Cummings model describes the basic interaction of a two-level atom and a quantized field, which is also the cornerstone for the treatment of the interaction between light and matter in quantum optics [21]. It can be used to explain many quantum phenomena, such as the collapses and revivals of atomic population inversions, the squeezing of the quantized field and atom-cavity entanglement. Recent experiments have shown that the Jaynes–Cummings model can be implemented in quantum-state engineering and quantum information processing, e.g. the generation of Fock states [22] and entangled states [23], and the implementation of quantum logic gates [24], etc. Originally, the Jaynes–Cummings model was physically implemented with a cavity quantum electrodynamic system, see, for instance, [25]. Certainly, there has also been interest in realizing the Jaynes–Cummings model with other physical systems. Such a system is a cold ion, trapped in a Paul trap and driven by classical laser beams [26, 27], where the interaction between two selected internal electronic levels and the external vibrational mode of the ion can be induced.

Motivated by the studies cited above and in particular [19, 28], we develop our proposal to deal with different issues. For this, we consider a 2D system in the presence of an external magnetic field $B$ and study its quantum dynamics. But, we include the parabolic potential to confine our system and the Rashba interaction to suggest a link with quantum optics. Through the Weyl–Heisenberg symmetries, we obtain the solutions of the energy spectrum and construct the algebra $su(2)$, as well as $su(1, 1)$. By considering a strong and a weak $B$, we show that our system reduces to the Landau problem for the first case. By using the Heisenberg picture, we derive two copies of the Jaynes–Cummings model, oscillating with different frequencies. Finally, we recover the results of a ‘without confinement case’ [19] in a simple way and conclude that our findings are general and have different potential extensions.

The present paper is organized as follows. In section 2, we formulate our problem by setting the Hamiltonian and choosing a convenient gauge. In section 2.2 we introduce a series of annihilation and creation operators to diagonalize our Hamiltonian, which serves to determine, explicitly, the exact eigenvalues and eigenstates. We then construct two symmetries and analyse the system behaviour by distinguishing strong and weak magnetic field cases in section 2.3. We establish a link with the Jaynes–Cummings model and make different comments in section 3. Moreover, to show the relevance of our results we study a limiting case. Finally, we give our conclusions and suggest possible areas for further study in section 4.

2. Solutions of the energy spectrum

We start by formulating our problem to find the appropriate Hamiltonian which describes the system under consideration. Subsequently, we use the algebraic approach to determine, explicitly, the eigenvalues as well as the eigenstates. These will be used to discuss the possibility of filling the shells with electrons when the magnetic field is strong and is weak.
2.1. Hamiltonian formalism

We consider a system of electrons in the presence of a constant magnetic field \( \vec{B} = B\hat{e}_z \) and its confining potential. By taking into account the Rashba spin–orbit coupling and the Zeeman effect, the Hamiltonian for a single electron reads

\[
H = \frac{\vec{p}^2}{2m} + \frac{1}{2} \hbar \omega_c^2 (x^2 + y^2) + \lambda \vec{\sigma} \cdot \vec{A} \frac{\vec{p}}{\hbar} + \frac{1}{2} \mu_B B \vec{\sigma} \cdot \vec{z},
\]

where \( \vec{p} = \vec{p}^0 + \xi \vec{A} \) is the conjugate momentum and \( \vec{A} \) is the vector potential. \( \lambda \) is the Rashba coupling parameter, \( \mu_B \) is the Bohr magneton and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices. We then choose the symmetric Landau gauge

\[
\vec{A} = \frac{B}{2} (-y, x, 0)
\]

and write the Hamiltonian (1) as

\[
H = \frac{1}{2m} \left[ \left( p_x - \frac{eB}{2c} y \right)^2 + \left( p_y + \frac{eB}{2c} x \right)^2 \right] + \frac{1}{2} \hbar \omega_c^2 (x^2 + y^2) + \lambda \left[ \sigma_y \left( p_x - \frac{eB}{2c} y \right) - \sigma_x \left( p_y + \frac{eB}{2c} x \right) \right] + \frac{1}{2} \mu_B B \vec{\sigma} \cdot \vec{z}.
\]

The algebraic structure of the above Hamiltonian is easily displayed if we adopt the method of separation of Cartesian variables. This process suggests decomposing (3) into four parts

\[
H = H_F + H_R + \frac{1}{2} \omega_c L_z + \frac{1}{2} \mu_B B \vec{\sigma} \cdot \vec{z},
\]

such that the free part takes the form

\[
H_F = \left( \frac{p_x^2}{2m} + \frac{1}{8} \hbar \omega_c^2 x^2 \right) + \left( \frac{p_y^2}{2m} + \frac{1}{8} \hbar \omega_c^2 y^2 \right)
\]

and the Rashba coupling in the magnetic field is given by

\[
H_R = \lambda \left[ \sigma_y \left( p_x - \frac{eB}{2c} y \right) - \sigma_x \left( p_y + \frac{eB}{2c} x \right) \right]
\]

where \( \omega_c = \frac{eb}{mc} \) is the cyclotron frequency, \( L_z = xp_y - yp_x \) is the angular momentum and we have set the new frequency as \( \omega = \sqrt{\omega_c^2 + 4\omega_0^2} \). Finally, we emphasize that (4) splits into two independent harmonic oscillator Hamiltonians, supplemented by the angular momentum and the Rashba spin–orbit coupling added to the Zeeman term. This convenient form of the Hamiltonian will help us to obtain its diagonalization in a simple way.

2.2. Solution through Weyl–Heisenberg symmetries

We introduce standard techniques to obtain solutions of the energy spectrum of the Hamiltonian (4). Instead of directly using the oscillator annihilation operators

\[
a_x = \frac{1}{\sqrt{2}} \left( \frac{x}{l_0} + \frac{i}{\hbar} p_x \right), \quad a_y = \frac{1}{\sqrt{2}} \left( \frac{y}{l_0} + \frac{i}{\hbar} p_y \right)
\]

we work with two new ones, which are the linear superposition of \( a_x \) and \( a_y \), so that

\[
ad = \frac{1}{\sqrt{2}} (a_x - ia_y), \quad ag = \frac{1}{\sqrt{2}} (a_x + ia_y)
\]

where \( l_0 = \sqrt{\frac{2\hbar}{m \omega_c}} \) is the magnetic length. Note that \( ad \) and \( ag \) are bosonic operators and satisfy the relation commutations

\[
[a_d, a_d^\dagger] = [a_g, a_g^\dagger] = 1 = [a_e, a_e^\dagger]
\]
and that the other relations vanish. From the above operators, one can obtain useful identities for the conjugate momentum

\[ \pi_x = \frac{\hbar}{2 l_0} [l_1 (a_d - a_d^\dagger) + l_2 (a_s - a_s^\dagger)], \quad \pi_y = \frac{\hbar}{2 l_0} [l_1 (a_d^\dagger + a_d) - l_2 (a_s + a_s^\dagger)] \] (10)

as well as for the positions

\[ x = \frac{l_0}{2} (a_d + a_d^\dagger + a_s + a_s^\dagger), \quad y = \frac{l_0}{2i} (-a_d + a_d^\dagger + a_s - a_s^\dagger) \] (11)

where we have set \( l_1 = (1 + \frac{i}{2}) \), \( l_2 = (1 - \frac{i}{2}) \) and \( l^2 = \hbar \varepsilon \). These algebraic structures will play a crucial role in solving different issues and, more precisely, in diagonalizing different Hamiltonians which are produced.

We start by writing the Rashba Hamiltonian (6) in terms of the annihilation and creation operators introduced above. So, we have

\[ H_R = H_R^E + H_R^d \] (12)

where these two parts are given by

\[ H_R^E = \lambda \sqrt{\frac{\hbar \omega}{2}} \begin{pmatrix} 1 - \frac{\omega_c}{\omega} & 0 \\ 0 & 1 + \frac{\omega_c}{\omega} \end{pmatrix} \begin{pmatrix} a_d^\dagger & a_d \\ a_s & a_s^\dagger \end{pmatrix} \] \quad \[ H_R^d = -\lambda \sqrt{\frac{\hbar \omega}{2}} \begin{pmatrix} 1 + \frac{\omega_c}{\omega} & 0 \\ 0 & 1 - \frac{\omega_c}{\omega} \end{pmatrix} \begin{pmatrix} a_d & a_d^\dagger \\ a_s^\dagger & a_s \end{pmatrix}. \] (13)

In the same way, we can diagonalize the free Hamiltonian and the angular momentum which gives a new form of the Hamiltonian (4). This is

\[ H = \frac{\hbar \omega}{2} (a_d^\dagger a_d + a_s^\dagger a_s + 1) + \frac{\hbar \omega}{2} (a_d^\dagger a_d - a_s^\dagger a_s) + H_R^E + H_R^d + \frac{1}{2} g \mu_B B \sigma_z. \] (14)

To determine the solutions of the energy spectrum in the above problem, we solve the eigenvalue equation

\[ H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \] (15)

which gives the eigenvalues

\[ E_{n_s, n_d} = \hbar \omega^+ n_d + \hbar \omega^− n_s \pm \frac{1}{2} \hbar \omega^\dagger \sqrt{8 \frac{m \lambda^2}{\hbar \omega} n_s + 1} \pm \frac{1}{2} \hbar \omega^\dagger \sqrt{8 \frac{m \lambda^2}{\hbar \omega} n_d + \left( 1 + \frac{g \mu_B B}{\hbar \omega} \right)^2} \] (16)

where the new frequencies are \( \omega^\pm = \frac{1}{2} (\omega \pm \omega_c) \). The corresponding eigenstates read

\[ |n_s, n_d, \sigma \rangle = u^+_n |n_s, n_d, \uparrow \rangle + v^+_n |n_s - 1, n_d - 1, \downarrow \rangle \] (17)

and we show that the amplitudes parameterizing these states are given by

\[ u^+_n = \frac{1}{\sqrt{2}} \left( 1 \pm \frac{\hbar \omega + g \mu_B B}{\hbar \omega^\dagger \sqrt{8 \frac{m \lambda^2}{\hbar \omega} n_s + 1} + \hbar \omega^\dagger \sqrt{8 \frac{m \lambda^2}{\hbar \omega} n_d + \left( 1 + \frac{g \mu_B B}{\hbar \omega} \right)^2}} \right)^\frac{1}{4} \] (18)

\[ v^+_n = \frac{\pm i}{\sqrt{2}} \left( 1 \mp \frac{\hbar \omega + g \mu_B B}{\hbar \omega^\dagger \sqrt{8 \frac{m \lambda^2}{\hbar \omega} n_s + 1} + \hbar \omega^\dagger \sqrt{8 \frac{m \lambda^2}{\hbar \omega} n_d + \left( 1 + \frac{g \mu_B B}{\hbar \omega} \right)^2}} \right)^\frac{1}{4}. \] (19)

Having obtained the solutions of the energy spectrum, let us briefly discuss how to recover an interesting case from what we have generated so far. Indeed, in studying cyclotron motion and magnetic focusing in semiconductor quantum wells with spin–orbit coupling, Schliemann...
[19] introduced the Hamiltonian type (1) without the confining potential. Therefore, to recover the corresponding solutions of the energy spectrum, we consider the case of $\omega = \omega_c$ or $\omega_0 = 0$ in our previous equations. This gives the eigenvalues

$$\varepsilon_{n_d} = \hbar \omega_c n_d \pm \sqrt{2m\lambda^2 \hbar \omega_c n_d + \frac{1}{4} (\hbar \omega_c + g \mu_B B)^2}$$

as well as the eigenstates

$$|n_g, n_d, \sigma\rangle = u_n^\pm |n_g, n_d, \uparrow\rangle + v_n^\pm |n_g - 1, n_d - 1, \downarrow\rangle$$

where the amplitudes are given by

$$u_n^\pm = \left( \frac{1}{2} \pm \frac{\frac{1}{4} (\hbar \omega_c + g \mu_B)}{\sqrt{2m\lambda^2 \hbar \omega_c n_d + \frac{1}{4} (\hbar \omega_c + g \mu_B B)^2}} \right)^{\frac{1}{2}}$$

$$v_n^\pm = (\pm i) \left( \frac{1}{2} \mp \frac{\frac{1}{4} (\hbar \omega_c + g \mu_B)}{\sqrt{2m\lambda^2 \hbar \omega_c n_d + \frac{1}{4} (\hbar \omega_c + g \mu_B B)^2}} \right)^{\frac{1}{2}}.$$

We note that Schliemann [19] used the notation $\alpha = \hbar \lambda$ and the quantum number $n = n_d$. These show clearly that our findings are general and are different to those obtained in [19]. This will play a crucial role in the forthcoming analysis.

### 2.3. Symmetries and shells

Now we introduce two symmetries which we compare with [28], they deal with a confined 2D system in a magnetic field. These dynamical symmetries, $su(2)$ and $su(1, 1)$, can be realized in terms of their shell operators. We start with $su(2)$ where the corresponding generators can be realized as

$$S_+ = a_d^\dagger a_g, \quad S_- = a_g^\dagger a_d, \quad S_z = \frac{L_z}{2\hbar}$$

which verify the commutation relations

$$[S_+, S_-] = 2S_z, \quad [S_z, S_{\pm}] = \pm S_{\pm}.$$  

These give an invariant Casimir operator

$$C = \frac{1}{2} (S_+ S_- + S_- S_+) + S_z^2 = \frac{H_F^2}{\hbar^2 \omega_c^2} - \frac{1}{4}.$$  

Therefore, to a fixed value $\mu = (n_d + n_g)/2$ of the operator $\frac{H_F}{\hbar \omega_c} - \frac{1}{2}$, there corresponds a $(2\mu + 1)$-dimensional unitary irreducible representation (UIR) of $su(2)$, in which the operator $S_z = \frac{L_z}{2\hbar}$ assumes its spectral values in the range $-\mu \leq g = (n_d - n_g)/2 \leq \mu$. In the case of the second symmetry, $su(1, 1)$, we consider the generators

$$T_+ = a_d^\dagger a_g^\dagger, \quad T_- = a_g a_d, \quad T_0 = \frac{H_F}{\hbar \omega_c}$$

satisfying the relations

$$[T_+, T_-] = -2T_0, \quad [T_0, T_{\pm}] = \pm T_{\pm}.$$  

The Casimir operator then is given by

$$D = \frac{1}{2} (T_+ T_- + T_- T_+) - T_0^2 = -\frac{1}{4} \left( \frac{L_z^2}{\hbar^2} - 1 \right).$$
As in the previous case, with reference to a fixed value \( \eta \equiv (n_d - n_g)/2 + 1/2 \geq 1/2 \), with \( n_d - n_g = \alpha \geq 0 \), of the operator \( \frac{1}{2}(\lambda - 1) \), there corresponds a UIR of \( su(1, 1) \) in the discrete series, in which the operator \( T_0 = \frac{1}{2}(\lambda - 1) + N_g \) assumes its spectral values in the infinite range \( \eta, \eta + 1, \eta + 2, \ldots \). Alternatively, with reference to a fixed value \( \varrho = -(n_d - n_g)/2 + 1/2 \geq 1/2 \), with \( n_g - n_d = -\alpha \geq 0 \) of the operator \( \frac{1}{2}(\lambda + 1) \), there corresponds a UIR of \( su(1, 1) \) in the discrete series, in which the operator \( T_0 = \frac{1}{2}(\lambda + 1) + N_d \) assumes its spectral values in the infinite range \( \varrho, \varrho + 1, \varrho + 2, \ldots \).

We will see the importance of both these symmetries when we analyse the two interesting cases below. With these, we will underline the system behaviour with respect to different limits of the magnetic field, \( B \). We start our analysis by considering the case where \( B \) is strong, which is equivalent to the limit \( \omega_c \gg \omega_0 \), and gives the frequencies \( \omega^+ \simeq \omega_c \) and \( \omega^- \simeq 0 \). These tell us that the total energy can be approximated by

\[
E_{\eta_0} \simeq \hbar \omega_c \left( n_d \pm \frac{1}{2} \right) \pm \frac{1}{2}\hbar \mu_4 B
\]

where the quantum number is \( n_d = 0, 1, \ldots \). Now, by scaling the energy as

\[
E_{\eta_0} \pm \frac{1}{2} g \mu_4 B \simeq \hbar \omega_c \left( n_d \pm \frac{1}{2} \right) = \bar{E}_{n_d}
\]

one can immediately see that our system behaves like a harmonic oscillator in 2D (the Landau problem) and for a given \( n_d \) there is an infinite degeneracy of Landau levels. We notice that there are two types of quantum numbers \( n_d + \frac{1}{2} \), and \( n_d - \frac{1}{2} \), which means that we have two independent sectors of Hilbert space. However, both sectors are connected via a linear transformation, \( n = n_d + 1 \), that allows movement from one sector to another and vice versa. From a symmetry point of view, this behaviour can be regarded as our system having ladder states for a discrete series of representations of the algebra \( su(1, 1) \) labelled by \( \frac{1}{2}(-\alpha \pm 1) \) where \( \alpha = n_d - n_g \leq n_d \) for \( \alpha \leq 0 \).

For the weak magnetic field, which corresponds to the limit \( \omega_c \ll \omega_0 \), we can approximate the frequencies by \( \omega^+ = \omega^- \simeq \omega_0 \) and therefore write the total energy as

\[
E_{\eta_0, n_d} \simeq \hbar \omega_0 (n_d + n_g \pm 1) \pm \frac{1}{2} g \mu_4 B.
\]

To interpret this result let us rearrange it as follows

\[
E_{\eta_0, n_d} \pm \frac{1}{2} g \mu_4 B \simeq \hbar \omega_0 (2\lambda \pm 1) = \bar{E}_\lambda,
\]

which shows clearly that our system now becomes invariant under the algebra \( su(2) \) and therefore each Landau level has a degeneracy of an order \( (2\lambda \pm 1) \). Note that here, we also have two UIRs with dimensions of \( (2\lambda + 1) \) and \( (2\lambda - 1) \) for the same algebra, where the transition between them can be obtained by defining \( \rho = \lambda \pm 1 \). Furthermore, from (32) we see that we need \( 2(\lambda_0 \pm 1)(2\lambda_0 \pm 1) \) to fill the shells up to the value \( \lambda_0 \).

3. Link with the Jaynes–Cummings model

Very recently, interesting connections between different areas of physics have appeared. Among them, we cite the extraordinary connection between condensed matter physics and high energy physics through the link of graphene with quantum electromagnetics [30]. Also, a connection between massless Dirac electrons and quantum optics has been established [31]. This inspired us to look for other links and establish more connections between different systems. Thus, we make a connection with another area of physics by showing how our system can be linked to quantum optics through a mapping between the corresponding Hamiltonian and the Jaynes–Cummings model. This may help to strengthen our knowledge of different aspects of quantum optics.
3.1. Equivalence between models

To show the relevance of the results obtained so far, we study the presence and absence of the confining potential cases in the Hamiltonian system. For the first case, we show that our Hamiltonian (3) is formally equivalent to two copies of the Jaynes–Cummings model for the atomic transition in a radiation field, but oscillating with different frequencies. To do this, we adopt the same method used by Ackerhalt and Rzazewski [29] in analysing the operator perturbation theory in the Heisenberg picture. For the second case, we derive the corresponding results in a simple way from our findings.

To proceed further, we need to rearrange our Hamiltonian in order to deal with each part separately and establish the associated link. For this, we start by splitting (14) into two parts

$$H = H_g + H_d$$

where the first part is

$$H_g = \hbar \omega - a^\dagger g a_g + \frac{\hbar \omega}{4} + H^d_g + \frac{1}{4} g \mu_B B \sigma_z$$

and the second reads

$$H_d = \hbar \omega + a^\dagger d a_d + \frac{\hbar \omega}{4} + H^d_d + \frac{1}{4} g \mu_B B \sigma_z.$$}

Clearly, these two parts are different because they involve different frequencies and therefore different oscillations.

Let us consider the Hamiltonian (35) and link it to the Jaynes–Cummings model. In doing so, we show that (35) can be written as

$$H_g = \hbar \omega - \gamma^- \sigma_z + \xi(\omega) \left( a^\dagger_g \sigma^- + a_g \sigma^+ \right)$$

where the two operators $M_g$ and $\sigma^\pm$ are given by

$$M_g = N_g + \sigma^+ \sigma^- + \frac{\omega_c}{2(\omega + \omega_c)} I$$

$$\sigma^\pm = \frac{1}{2}(\sigma_x \pm i \sigma_y)$$

and we have set the constants $\gamma^-$ and $\xi(\omega)$ as

$$\gamma^- = \frac{1}{4}(2\hbar \omega - g \mu_B B)$$

$$\xi(\omega) = \lambda \sqrt{\frac{\hbar \omega}{2}} \left( 1 - \frac{\omega_c}{\omega} \right).$$

It is convenient to define an operator as

$$C_g = -\gamma^- \sigma_z + \xi(\omega) \left( a^\dagger_g \sigma^- + a_g \sigma^+ \right)$$

for use later, which verifies the commutation relation $[M_g, C_g] = 0$. It tells us that there are two constants of motion corresponding to the Hamiltonian (37). This helps in studying the different dynamics of the operators involved.

To study the dynamics related to the Hamiltonian (37), we introduce the Heisenberg equation of motion for the operators $a^\dagger_g$ and $\sigma^+$. These are

$$\frac{d}{dt} a^\dagger_g = \frac{i}{\hbar} [H_g, a^\dagger_g]$$

$$\frac{d}{dt} \sigma^+ = \frac{i}{\hbar} [H_g, \sigma^+].$$
A straightforward calculation gives
\[
\left( i \hbar \frac{d}{dt} + \hbar \omega \right) a_g^+ = -\zeta(\omega) \sigma^+.
\] (45)
\[
\left( i \hbar \frac{d}{dt} + \hbar \omega - 2\gamma \right) \sigma^+ = \zeta(\omega) a_g^+ \sigma_z.
\] (46)

Using the relations \( \sigma^+ \sigma^+ = \sigma^- \sigma^- = 0, \sigma^+ \sigma_z = -\sigma^- \sigma_z = \sigma^- \), we show that (46) can be written in terms of the constant of motion, \( C_\sigma \), as
\[
\left( i \hbar \frac{d}{dt} + \hbar \omega - 2C_\sigma \right) \sigma^+ = \zeta(\omega) a^+_g.
\] (47)

From (45) and (47), we derive the same second order differential equation for \( \sigma^+(t) \) and \( a^+_g(t) \). This is given by
\[
\left( i \hbar \frac{d}{dt} + \hbar \omega - 2C_\sigma \right) \sigma^+ = \zeta(\omega) \left( a^+_g \sigma_z \right).
\] (48)

If we are looking for the quantum dynamics, then we need to solve (45) and (47). One way of doing this is to find solutions of the form
\[
\sigma^+ (t) = e^{i\beta_g t / \hbar \omega} s^+_g + e^{i\beta_g t / \hbar \omega} l^-_g
\] (49)
\[
a^+_g (t) = e^{i\beta_g t / \hbar \omega} s^+_g + e^{i\beta_g t / \hbar \omega} l^-_g
\] (50)
where \( \beta_g^\pm, l^+_g \) and \( s^+_g \) are the initial time operators. From the shape of the decomposition (45) and (47), one can see that they have an analogy with the solution of an ordinary harmonic oscillator. Therefore, the solution will be in the form of \( e^{i\beta_g t / \hbar \omega} \) and thus, after substitution into (48) gives a second order equation for \( \beta_g \). This is
\[
\beta_g^2 - 2(\hbar \omega - C_\sigma) \beta_g + \hbar \omega (\hbar \omega - 2C_\sigma) + \zeta^2(\omega) = 0.
\] (51)

By requiring the condition \([\beta_g, C_\sigma] = 0\), we show that the corresponding solutions under the decomposition forms
\[
\beta_g^\pm = h \omega - \alpha_g^\pm
\] (52)
where \( \alpha_g^\pm \) are given by
\[
\alpha_g^\pm = h \omega - C_\sigma \pm \sqrt{C_\sigma^2 - \zeta^2(\omega)}.
\] (53)

From the decomposition of \( a^+_g(t) \) and \( \sigma^+(t) \), one can see that the operator’s constants, \( s^+_g \) and \( l^+_g \), can be obtained by fixing \( t = 0 \) in (49) and (50). These give the relations
\[
\sigma^+(0) = s^+_g + s^-_g
\] (54)
\[
a^+_g(0) = l^+_g + l^-_g.
\] (55)

Injecting the forms (49) and (50) into (45) and (47) we have
\[
\alpha_g^\pm l^+_g + \alpha_g^\pm l^-_g = \zeta(\omega)(s^+_g + s^-_g)
\] (56)
\[
\alpha_g^\pm s^+_g + \alpha_g^\pm s^-_g = \zeta(\omega)(l^+_g + l^-_g).
\] (57)
These can be solved to obtain the initial operators as

\[ s_\pm^g = \frac{\pm \zeta(\omega) a_\pm^g(0)}{\alpha_g^+ - \alpha_g^-} \]  

\[ l_\pm^g = \frac{\mp \zeta(\omega) \sigma^+ (0) \pm \alpha_g^2 a_\pm^g(0)}{\alpha_g^- - \alpha_g^+}. \]

Combining all, we have the final solutions of (45) and (47). These are

\[ \sigma^+(t) = \frac{e^{i\omega t}}{\alpha_g^+ - \alpha_g^-} \left\{ (\zeta(\omega) a_\pm^g(0) - \alpha_g^2 \sigma^+(0)) e^{i\omega t \sqrt{2}} / h + (\zeta(\omega) a_\pm^g(0) + \alpha_g^2 \sigma^+(0)) e^{i\omega t \sqrt{2}} / h \right\} \]

\[ a_\pm^g(t) = \frac{e^{i\omega t}}{\alpha_g^+ - \alpha_g^-} \left\{ (-\zeta(\omega) \sigma^+(0) + \alpha_g^2 a_\pm^g(0)) e^{i\omega t \sqrt{2}} / h + (\zeta(\omega) \sigma^+(0) - \alpha_g^2 a_\pm^g(0)) e^{i\omega t \sqrt{2}} / h \right\}. \]

They constitute the exact operator solutions for the basic Jaynes–Cummings variables, which were obtained in [29].

As far as the second part (34) is concerned, we apply the same process as before to derive similar results. Indeed, by introducing the operator

\[ M_d = N_d + \sigma^+ \sigma^- - \frac{\omega_c}{2(\omega_c + \omega_c)} \]

and the two constants

\[ \gamma^+ = \frac{1}{4} (2 \omega^+ - g \mu B) \]

\[ \zeta'(\omega) = -\frac{\sqrt{m \hbar \omega}}{2} \left( 1 + \frac{\omega_c}{\omega} \right) \]

we write the Hamiltonian \( H_d \) (36) as

\[ H_d = \hbar \omega^+ M_d - \gamma^+ \sigma_+ + \zeta'(\omega) (a_\pm^d \sigma^- - a_\pm^d \sigma^+). \]

In the same way, we find the required solutions

\[ \sigma^+(t) = \frac{e^{i\omega t}}{a_\pm^d - a_\pm^d} \left\{ (\zeta'(\omega) a_\pm^d(0) - \alpha_d^2 \sigma^+(0)) e^{i\omega t \sqrt{2}} / h + (\zeta'(\omega) a_\pm^d(0) + \alpha_d^2 \sigma^+(0)) e^{i\omega t \sqrt{2}} / h \right\} \]

\[ a_\pm^d(t) = \frac{e^{i\omega t}}{a_\pm^d - a_\pm^d} \left\{ (-\zeta'(\omega) \sigma^+(0) + \alpha_d^2 a_\pm^d(0)) e^{i\omega t \sqrt{2}} / h + (\zeta'(\omega) \sigma^+(0) - \alpha_d^2 a_\pm^d(0)) e^{i\omega t \sqrt{2}} / h \right\}. \]

where different quantities are given by

\[ \alpha_d^\pm = C_d \pm \sqrt{C_d^2 - \zeta^2(\omega)} \]

\[ C_d = -\gamma^+ \sigma_+ + \zeta'(\omega) (a_\pm^d \sigma^- - a_\pm^d \sigma^+). \]
These solutions show how to obtain the second copy of the Jaynes–Cummings model from our findings. We conclude that our Hamiltonian is equivalent to the two copies of the Jaynes–Cummings model but oscillating with different frequencies.

Finally, we note that we can easily obtain the dynamics of the complex position from the above solutions. Indeed, using (11) to write

$$z(t) = l_0 (a(t) + a(t))$$

and therefore summing up the adjoint time evolution operator of (61) and (67), we obtain the dynamics of $z(t)$. The established link shows clearly that our system shares some common features with quantum optics. Thus, one can use the present system to handle different issues related to the Jaynes–Cummings model and vice versa.

### 3.2. The limiting case

The above results show the analogy with the Jaynes–Cummings model and therefore allow us to establish a relation with work already published [29]. Now we study the case where the confining potential is absent, which should lead naturally to the Schliemann results [19] for the Jaynes–Cummings model. These will be derived in a simple way from our findings to show clearly that our work is general and can be extended to deal with different issues.

To recover the dynamics obtained in [19] for the Jaynes–Cummings model, we start by fixing the frequency as $\omega_0 = 0$ in the previous results. This requirement leads to the constraints

$$\omega_0^+ = 0, \quad l_1 = 2, \quad l_2 = 0$$

and therefore, according to the dynamical equation (45) or (61), we end up with a time independent operator $a_g^+$

$$a_g^+(t) = a_g^+(0) = a_g^+$$

However, (67) gives the time dependent $a_g^+(t)$

$$a_g^+(t) = \frac{e^{-i\omega_0 t}}{r_d^+ - r_d^-} \left\{ \left( \zeta'(\omega_0)\sigma^+(0) - r_d^- a_g^+(0) \right) e^{-i\frac{\omega_0}{2} t} + \left( -\zeta'(\omega_0)\sigma^+(0) + r_d^+ a_g^+(0) \right) e^{-i\frac{\omega_0}{2} t} \right\}$$

where the roots are

$$r_d^\pm = C_d \pm \left( C_d^2 + \zeta^2(\omega_0) \right)^{\frac{1}{2}}$$

and all involved functions are now in terms of the cyclotron frequency $\omega_c$ instead of $\omega$. Returning to the definition of different operators to obtain the time evolution of the position operators in the Heisenberg picture, this is simply

$$x(t) + iy(t) = l_0 (a_g^+(t) + a_g^-)$$

Since the operator $a_g$ is time independent, then at $t = 0$ we have

$$x(0) + iy(0) = l_0 a_g$$

and then, after replacing it, we find

$$x(t) + iy(t) = x(0) + iy(0) + l_0 a_g^+(t).$$

From (10) we can express $a_g^+$ in terms of the conjugate momentum as

$$a_g^+ = \frac{l_0}{2i\hbar} (\pi_+ + i\pi_-)$$
which leads to the final form of the complex position

\[
x(t) + i y(t) = x(0) + i y(0) + \frac{e^{-i \omega \tau \frac{r}{r_d} + \frac{1}{2} \frac{C^2}{r_d}}}{r_d} \left( \frac{r_d}{\omega_c} \pi_x + i \pi_y \right) m - \frac{i 2 \lambda \hbar}{m} \sigma^+ \\
- \frac{1}{r_d} \left( \frac{r_d}{\omega_c} \pi_x + i \pi_y \right) m - \frac{i 2 \lambda \hbar}{m} \sigma^+
\]

(79)

where the operators valued \( r^\pm_d \) are given by

\[
r^\pm_d = C_d \pm \left( C_d^2 + 2 \lambda^2 m \hbar \omega_c \right)^{\frac{1}{2}}.
\]

(80)

This is the same as the result obtained by Schliemann [19] in dealing with the same system without confinement. Thus, it really shows that our findings are important in a general sense and that we can also derive other results.

4. Conclusion

We have investigated the basic features of a confined two-dimensional system with Rashba spin–orbit interaction in the presence of an external magnetic field \( B \). This allowed us to obtain a confining potential, along \( x \)- and \( y \)-directions, that has been used to deal with different issues. In particular, it has been used to split the corresponding Hamiltonian into two parts. This decomposition was useful in that different spectra are obtained and it leads to total solutions of the energy spectrum. We have shown that the solutions obtained by Schliemann [19] can be derived in a simple way from our solutions.

Using the different operators involved in the Hamiltonian, we have realized two dynamical symmetries \( su(2) \) and \( su(1, 1) \). These, together with the strength of the magnetic field \( B \), allowed us to discuss the filling of the shells with electrons. For strong \( B \), we have concluded that our system behaves like a harmonic oscillator in 2D with an infinite degeneracy of the Landau levels. However, for weak \( B \), our system becomes invariant under the algebra \( su(2) \) and then each Landau level has a finite degeneracy.

To make a connection with quantum optics, we have elaborated a method based on building two Hamiltonians from the original one. Indeed, by splitting the Hamiltonian into two parts, we have shown that it is possible to recover the Jaynes–Cummings model, which describes a system with two states. This has been done by using the Heisenberg dynamics to find the dynamics of the rising Pauli operator \( \sigma^+ \) and the creation operators \( (a_d^\dagger, a^\dagger_g) \). After solving different equations, we have ended up with the exact operator solutions for the basic Jaynes–Cummings variables, those which were obtained in [29]. To show the validity of our results, we have derived those obtained by Schliemann [19] as particular cases.

The present work can be extended to deal with different issues. For instance, we can use the route used by Schliemann [19] to explicitly study the full quantum dynamics. This is based on expanding the initial state of the system in terms of its eigenstates. Another possibility is to use the obtained results to study different issues related to graphene and the spin Hall effect.

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