Next-to-next-to-leading order relation between \( R(e^+e^- \rightarrow b\bar{b}) \) and \( \Gamma_{sl}(b \rightarrow cl\nu_l) \) and precise determination of \(|V_{cb}|\)

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Abstract

We present the next-to-next-to-leading order relation between the moments of the \( \Upsilon \) system spectral density and the inclusive \( B \)-meson semileptonic width. The perturbative series for the width as an explicit function of the moments is well convergent in three consequent orders in the strong coupling constant that provides solid and accurate theoretical estimate. As a result, the uncertainty of the value of \(|V_{cb}|\) Cabibbo-Kobayashi-Maskawa matrix element is reduced.
The inclusive $B$-meson semileptonic width $\Gamma_{sl}(b \to cl\nu_l)$ is rather a clean place to obtain the value of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{cb}|$ (see refs.\cite{1,2,3} as a review). The main uncertainty of this estimate is related to the strong dependence of the result on the $b$-quark pole mass $m_b$ which has to be known with extremely high accuracy to get a precise estimate of $|V_{cb}|$. On the other hand the high moment sum rules for the system of $\Upsilon$ resonances are very sensitive to $m_b$ and can be used for the precise determination of this parameter after the proper treatment of Coulomb effects \cite{4,5}. In the present paper we construct the direct relation between the moments of the $\Upsilon$ system spectral density and the inclusive $B$-meson semileptonic width up to the next-to-next-to-leading (NNL) order of perturbative expansion in $\alpha_s$. In this way we avoid the strong dependence of $\Gamma_{sl}$ on $m_b$ and essentially reduce the theoretical uncertainty in $|V_{cb}|$. It is known that if the moments for the $\Upsilon$ system and the inclusive semileptonic width are expressed through the pole mass $m_b$ the perturbative expansions in $\alpha_s$ for both quantities seem to be divergent. The heuristic criterium of convergence of perturbative series, i.e. the requirement that next correction is much smaller than the previous one, is violated. However, the explicit dependence of the inclusive semileptonic width on the pole mass $m_b$ can be removed by reexpressing it through the moments. As a result one arrives at the perturbative expansion for the quantity $V_{cb}$ which converges well up to NNL order in $\alpha_s$.

The theoretical expression for the inclusive semileptonic width of $B$-meson reads

$$\Gamma_{sl} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 F \left( \frac{m_c^2}{m_b^2} \right) C_\Gamma(\alpha_s)$$

where $F(x) = 1 - 8x - 12x^2 \ln x + 8x^3 - x^4$. The perturbative coefficient $C_\Gamma(\alpha_s)$ is known up to the second order in $\alpha_s$ \cite{6}

$$C_\Gamma = 1 - 1.67 \frac{\alpha_s(\mu)}{\pi} - (8.4 \pm 0.4) \left( \frac{\alpha_s}{\pi} \right)^2$$
where $\alpha_s(\mu)$ is defined in $\overline{\text{MS}}$ scheme and the normalization point $\mu = \sqrt{m_b m_c}$ is used.

The nonperturbative corrections to eq. (1) decrease the width by approximately 5% [3, 7].

From formula (1) one sees that the semileptonic width depends rather strongly on $m_b$, the b-quark mass, which is the pole mass. This means that if $m_b$ is taken from some other experiment it should be determined with great accuracy for reasonable determination of $|V_{cb}|$. Being analysed independently, the perturbative series in $\alpha_s$ for physical observables expressed in terms of the pole mass $m_b$ seem not to enjoy a fast explicit convergence that can lead to large uncertainty of the numerical value of $m_b$ extracted from different experiments. However $m_b$ is not an observable and has no immediate physical meaning. Therefore it can be safely removed from relations between physical observables. Here we use this strategy for direct determination of the mixing angle $|V_{cb}|$ reducing theoretical uncertainties to a great extent. Our analysis consists in direct relating the factor $m_5^b$ in eq. (1) to the fifth moment of the $\Upsilon$ sum rules

$$m_5^b = \left( \frac{\mathcal{M}_5^{th}}{\mathcal{M}_5^{exp}} \right)^{\frac{1}{2}}. \quad (2)$$

The moments $\mathcal{M}_n^{th}$ are defined as normalized derivatives of the $b$-quark vector current polarization function $\Pi(s)$

$$\mathcal{M}_n^{th} = \frac{12\pi^2}{n!} (4m_b^2)^n \frac{d^n}{ds^n} \Pi(s) \bigg|_{s=0} = (4m_b^2)^n \int_0^\infty \frac{R(s)ds}{s^{n+1}}$$

where

$$(q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2) = i \int dx e^{ix} \langle 0| T j_\mu(x) j_\nu(0) |0 \rangle, \quad j_\mu = \bar{b}\gamma_\mu b \quad (3)$$

and $R(s) = 12\pi \text{Im} \Pi(s+i\epsilon)$. The (dimensionful) experimental moments $\tilde{\mathcal{M}}_n^{exp}$ are generated by the function $R_b(s)$ which is the normalized cross section $R_b(s) = \sigma(e^+e^- \rightarrow \text{hadrons}_{\bar{b}b})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$

$$\tilde{\mathcal{M}}_n^{exp} = \frac{4^n}{Q_b^2} \int_0^\infty \frac{R_b(s)ds}{s^{n+1}}. \quad (4)$$
Here \( Q_b = -1/3 \) is the \( b \)-quark electric charge.

The theoretical moment \( M_b^{th} \) is a dimensionless quantity which depends on \( m_b \) only logarithmically as \( \ln (m_b/\mu) \) (in a finite order in \( \alpha_s \)). Thus, the substitution of relation (2) to eq. (1) substantially reduces its dependence on \( m_b \) though introduces explicit uncertainty due to \( M_b^{exp} \). At the same time the theoretical moment is rather sensitive to the value of \( \alpha_s \).

Let us discuss the choice of the moment in expression (2) for the \( b \)-quark pole mass in more detail. It is well known that for the high moments the ordinary perturbative expansion in the strong coupling constant is not applicable and the Coulomb resummation is required. Recently the expansion of the theoretical moments around the exact Coulomb solution has been obtained in the NNL approximation [8, 9, 10, 11]. In our analysis we use the analytical expression of the polarization function near threshold obtained in refs. [8, 9]. The explicit formulae are too large to be presented here.

The experimental moment entering eq. (2) is given by the integral of the spectral density eq. (4) and is mostly saturated with the contribution of the first six \( \Upsilon \) resonances that leads to the formula

\[
\tilde{M}_n^{exp} = \frac{4^n}{Q_b} \left( \frac{9\pi}{\alpha^2_{QED}(m_b)} \sum_{k=1}^{6} \frac{\Gamma_k}{M_k^{2n+1}} + \int_{s_0}^{\infty} ds \frac{R_b(s)}{s^{n+1}} \right).
\]  

The leptonic widths \( \Gamma_k \) and masses \( M_k \) \((k = 1 \ldots 6)\) of the resonances are known with good accuracy [12], the electromagnetic coupling constant is renormalized to the energy of order \( m_b \) with the result \( \alpha^2_{QED}(m_b) = 1.07\alpha^2 [12] \). The rest of the spectrum beyond the resonance region for energies larger than \( s_0 \approx (11.2 \text{ GeV})^2 \) (continuum contribution) is approximated by the theoretical spectral density multiplied by the parameter \( 0.5 < t < 1.5 \) which accounts for the uncertainty in the experimental data in this energy region. Note that we assume rather large uncertainty of the continuum to be on the safe side however its contribution is essentially suppressed in comparison
with the resonance one and the resulting error of the whole quantity in eq. (3) is of the same order as the uncertainties introduced by the resonance contribution.

Note that \( m_b^5 \) (or \( m_b \)) can be formally extracted from an arbitrary moment of the spectral density. The region of allowed \( n \) however is quite restricted. Indeed, the low moments cannot be used in sum rules because the experimental spectrum is well known experimentally only for energies close to threshold due to existence of sharp resonances while the contribution of the continuum to these low moments is large in comparison with the resonance contribution. On the other hand for \( n > 10 \) the perturbative expansion of the moments around the Coulomb solution is strongly divergent. This can be considered as a signal that for large \( n \) the Coulomb solution is not the best zero order approximation \([13]\). So \( n = 5 \) seems to be the natural and optimal choice. In any case the result of calculation is almost insensitive to the local variation of \( n \) around this value.

From eqs. (1,2) we find for the mixing angle \( |V_{cb}| \)

\[
|V_{cb}| = \left( \frac{192 \pi^3}{G_F^2} \Gamma_{sl} \sqrt{M_5} \frac{K(\alpha_s, m_b)}{F(m_c^2/m_b^2)} \right)^{\frac{1}{2}}
\]

where the functions

\[
K(\alpha_s, m_b) = \frac{1}{C_T(\alpha_s) \sqrt{M_5^{th}(\alpha_s, m_b)}}
\]

and \( F(m_c^2/m_b^2) \) accumulate theoretical information depending on \( m_b, m_c \) and \( \alpha_s \).

Function \( F(m_c^2/m_b^2) \) introduces rather large theoretical uncertainty if masses of \( b \) and \( c \)-quarks are considered as independent variables. However there is almost model independent constraint of the form

\[
(m_b - m_c)(1 + O(1/m_{b,c})) = \bar{m}_B - \bar{m}_D = 3.34 \text{ GeV}
\]

where \( \bar{m}_B = 5.31 \text{ GeV}, \bar{m}_D = 1.97 \text{ GeV} \) denote the spin-average meson masses, e.g. \( \bar{m}_B = \frac{1}{4}(m_B + 3m_{B^*}) \). We take the value of the nonperturbative corrections to
eq. (8) given in ref. [14] and use \( m_b - m_c = 3.47 \) GeV as the central value for numerical estimates. With this constraint the function \( F(m_c^2/m_b^2) \) becomes a function of a single variable \( \tilde{F}(m_b) \). Note that in such a setting the \( m_b \) dependence of the function \( \tilde{F}(m_b) \) partly cancels the large \( m_b^5 \) dependence of the width. Furthermore using the relation (2) to express the \( b \)-quark pole mass in the argument of the function \( \tilde{F}(m_b) \) in terms of the moments of the spectral density results in only logarithmic dependence of the right hand side of eq. (6) on \( m_b \).

Now one can analyze the theoretical factor \( K(\alpha_s, m_b)/\tilde{F}(m_b) \) numerically order by order in \( \alpha_s \). The result reads

\[
\left( \frac{K(\alpha_s, m_b)}{\tilde{F}(m_b)} \right)^{1/2} = 1.234(1 + 0.101 + 0.014).
\]

(9)

In the numerical analysis of extracting \( |V_{cb}| \) we use the “world average” value of the strong coupling constant \( \alpha_s(M_Z) = 0.118 \) [12] and the normalization point \( \mu \sim m_b \) in the expression for the theoretical moment\(^4\). The value of \( b \)-quark pole mass in the fixed order in \( \alpha_s \) with Coulomb resummation is found from eq. (2). For comparison, the perturbative series for \( m_b \) that follows from eq. (2) and the series for \( C_\Gamma \) are

\[
m_b = 4.75(1 - 0.014 + 0.022),
\]

\[
C_\Gamma = 1 - 0.146 - 0.064.
\]

Thus we find that in the expansions of the theoretical moment (as well as \( m_b \) itself) and width expressed in terms of \( m_b \) the NNL corrections are of the order of the next-

\(^4\) For the numerical estimates it is important to fix the allowed range for the normalization point which is present in the explicit formula of the polarization function. The naive estimate of the “physical scale” of the problem \( \mu \sim m_b \alpha_s \) is not acceptable since the direct calculation of the NNL corrections shows that the perturbation theory for the moments blows up there [8, 9]. The relative weight of the NNL order corrections is stabilized at \( \mu \sim m_b \) [13] so in our opinion there is no reason to use the lower normalization scales.
to-leading ones\footnote{There is a hypothesis that this fact is a consequence of the asymptotic character of the series which leads to the intrinsic ambiguity in the heavy quark pole mass \cite{16}.} while the perturbative series for the mixing angle, or the theoretical coefficient eq. (9), converges much better.

As for numerics, we use the following central values for our experimental inputs (see \cite{12} for more detail):

\[
\text{BR}(B \rightarrow X_c l \nu_l) = 10.5\%, \quad \tau_B = 1.55 \text{ ps},
\]

\[
\sqrt{M_5^{\text{exp}}} = 4.51 \times 10^{-4} \text{ GeV}^{-5}.
\]

With these numbers we obtain the value of the matrix element $|V_{cb}|$

\[
|V_{cb}| = 0.0423 \left( \frac{\text{BR}(B \rightarrow X_c l \nu_l)}{0.105} \right)^{\frac{1}{2}} \left( \frac{1.55\text{ps}}{\tau_B} \right)^{\frac{1}{2}} \times
\]

\[
\left( 1 + 0.02 \frac{\sqrt{M_5^{\text{exp}}} - 4.51 \times 10^{-4} \text{ GeV}^{-5}}{0.20 \times 10^{-4} \text{ GeV}^{-5}} \right) \left( 1 - 0.01 \frac{\alpha_s(M_Z) - 0.118}{0.006} \right) (1 \pm \Delta_{\text{npt}})
\]

where the nonperturbative corrections are included according to ref. \cite{3} and $\Delta_{\text{npt}} \sim 0.02$ is the uncertainty in the nonperturbative contribution coming mainly from the uncertainty in the HQET relation between $m_b$ and $m_c$ eq. \cite{8}. The typical scale of uncertainty of key parameters is also indicated. Another important source of the uncertainty is the scale dependence of the theoretical moment and the experimental errors in the value of $\alpha_s(M_Z)$ because of rather high sensitivity of the theoretical moment to $\alpha_s$. In fact these uncertainties are closely related since the scale dependence of $M_5^{\text{th}}(\alpha_s(\mu), \mu)$ is mainly due to the scale dependence of $\alpha_s(\mu)$ while the explicit dependence on $\mu$ is rather weak. The pointed error bars roughly correspond to the interval $\mu = m_b \pm 1 \text{ GeV}$ at fixed $\alpha_s(M_Z) = 0.118$. Note that the result is almost insensitive to the specific number of the moment used for the estimate. For example, the value of $|V_{cb}|$ matrix element changes approximately by 0.2% when the moment number changes in the interval $5 < n < 15$.\footnote{There is a hypothesis that this fact is a consequence of the asymptotic character of the series which leads to the intrinsic ambiguity in the heavy quark pole mass \cite{16}.}
Now the advantage of our approach becomes clear – large errors due to uncertainty in $m_b$ is now partly shifted to more direct experimental data. Furthermore the terms in the perturbative expansion (9) decrease rapidly which is a solid indication that the higher orders corrections to the obtained result are small enough.

The main part of the experimental uncertainty is related to the uncertainty in the experimentally measured inclusive semileptonic width. The uncertainty in $M_{5}^{\text{exp}}$ comes mainly from the continuum contribution to the moments above $s_0$ and the uncertainties in $\Gamma_k$. Experimental situation is rather dynamic and data are improving fast that means that the experimental uncertainties will be smaller (see e.g. [13]).

Our result for the central value of the parameter $|V_{cb}|$ of the CKM mixing matrix eq. (10) is in a good agreement with the previous estimate $|V_{cb}| = 0.0419$ [3]. This value, however, is somewhat larger than the estimate $|V_{cb}| = 0.039 \pm 0.002$ of ref. [2]. There is no much hope to reduce the uncertainty in the nonperturbative contribution. Thus the model independent result presented in the paper provides one with the most reliable and accurate estimate of the CKM matrix element $|V_{cb}|$ from the inclusive $B$-meson semileptonic width.

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