Injective Presentations of Induced Modules over Cluster-Tilted Algebras

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Abstract Every cluster-tilted algebra $B$ is the relation extension $C \ltimes \text{Ext}^2_C(DC, C)$ of a tilted algebra $C$. A $B$-module is called induced if it is of the form $M \otimes_C B$ for some $C$-module $M$. We study the relation between the injective presentations of a $C$-module and the injective presentations of the induced $B$-module. Our main result is an explicit construction of the modules and morphisms in an injective presentation of any induced $B$-module. In the case where the $C$-module, and hence the $B$-module, is projective, our construction yields an injective resolution. In particular, it gives a module theoretic proof of the well-known 1-Gorenstein property of cluster-tilted algebras.

Keywords Cluster-tilted algebra · Induction · Coinduction · Relation extension

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1 Introduction

Cluster-tilted algebras are finite dimensional associative algebras which were introduced in [15] and, independently, in [19] for the type $A$.

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One motivation for introducing these algebras came from Fomin and Zelevinsky’s cluster algebras [24]. To every cluster in an acyclic cluster algebra one can associate a cluster-tilted algebra, and the indecomposable rigid modules over the cluster-tilted algebra correspond bijectively to the cluster variables outside the chosen cluster. Generalizations of cluster-tilted algebras, the Jacobian algebras of quivers with potentials, were introduced in [23], extending this correspondence to the non-acyclic types. Many people have studied cluster-tilted algebras in this context, see for example [12, 15–18, 20, 21, 26].

The second motivation came from classical tilting theory. Tilted algebras are the endomorphism algebras of tilting modules over hereditary algebras, whereas cluster-tilted algebras are the endomorphism algebras of cluster-tilting objects over cluster categories of hereditary algebras. This similarity in the two definitions lead to the following precise relation between tilted and cluster-tilted algebras, which was established in [2].

There is a surjective map

\[
\begin{array}{ccc}
\text{tilted algebras} & \longrightarrow & \text{cluster-tilted algebras}, \\
C & \mapsto & B = C \ltimes E,
\end{array}
\]

where \( E \) denotes the \( C\)-\( C \)-bimodule \( E = \text{Ext}^2_C(DC, C) \) and \( C \ltimes E \) is the trivial extension.

This result allows one to define cluster-tilted algebras without using the cluster category. It is natural to ask how the module categories of \( C \) and \( B \) are related, and several results in this direction have been obtained, see for example [3–5, 11, 13, 22].

The Hochschild cohomology of the algebras \( C \) and \( B \) has been compared in [6–9, 27].

In [30], we initiated a new approach to study the relation between the module categories of a tilted algebra \( C \) and its cluster-tilted algebra \( B = C \ltimes E \), namely induction and coinduction.

The induction functor \( - \otimes_C B \) and the coinduction functor \( \text{Hom}_C(B, -) \) from \( \text{mod} \ C \) to \( \text{mod} \ B \) are defined whenever \( C \) is a subring of \( B \) which has the same identity. If we are dealing with algebras over a field \( k \), we can, and usually do, write the coinduction functor as \( D(B \otimes_C D-), \) where \( D = \text{Hom}(-, k) \) is the standard duality.

Induction and coinduction are important tools in classical Representation Theory of Finite Groups. In this case, \( B \) would be the group algebra of a finite group \( G \) and \( C \) the group algebra of a subgroup of \( G \) (over a field whose characteristic is not dividing the group orders). In this situation, the algebras are semi-simple, induction and coinduction are the same functor, and this functor is exact.

For arbitrary rings, and even for finite dimensional algebras, the situation is not that simple. In general, induction and coinduction are not the same functor and, since the \( C \)-module \( B \) is not projective (and not flat), induction and coinduction are not exact functors.

However, the connection between tilted algebras and cluster-tilted algebras is close enough so that induction and coinduction are interesting tools for the study of the relation between the module categories.

In this paper, we use induction and coinduction to construct explicit injective presentations of induced modules over cluster-tilted algebras. Since the induction functor sends projective \( C \)-modules \( P_C \) to projective \( B \)-modules \( P_B = P_C \otimes_C B \), and the coinduction functor sends injective \( C \)-modules \( I_C \) to injective \( B \)-modules \( I_B = D(B \otimes_C D_I_C) \), we are able to construct injective presentations in \( \text{mod} \ B \) from corresponding injective presentations in \( \text{mod} \ C \).

Our main result is the following. Here \( \nu \) denotes the Nakayama functor and \( \Omega \) the first syzygy.