Dynamics of a hydrogen-like atom bounded by maximal acceleration

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Abstract
The existence of a maximal acceleration for massive objects was conjectured by Caianiello (1981 Lett. Nuovo Cimento 32 65) 30 years ago based on the Heisenberg uncertainty relations. Many of the consequences of this hypothesis have been studied, but until now there has been no evidence that the boundedness of the acceleration may lead to quantum behavior. In previous research, we predicted the existence of a universal maximal acceleration and developed a new dynamics for which all admissible solutions have an acceleration bounded by the maximal one.

We present here a solution to our dynamical equation for a classical hydrogen-like atom and show that this dynamics leads to some aspects of quantum behavior. We show that the position of an electron in a hydrogen-like atom can be described only probabilistically. We also show that in this model, the notion of a 'center of mass' must be modified. This modification supports the non-existence of a magnetic moment in the atom and explains the relevance of the conformal group in the quantum region.

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1. Introduction
The existence of a maximal acceleration for massive objects was conjectured by Caianiello [1] 30 years ago based on the Heisenberg uncertainty relations. His maximal acceleration was dependent on the mass of the particle. The consequences of the existence of a maximal acceleration have been studied by many scientists, and hundreds of papers have been published on it in the physics literature, see, for example, [2–8]. The dynamics under the restriction of the acceleration was studied in [4]. The maximal acceleration correction to the Lamb shift of hydrogen, deuterium and He was obtained in [6]. Note, however, that in all these papers, the maximal acceleration is either mass dependent or, by using Planck’s constant, equal to about \( a_m = 10^{52} \text{m s}^{-2} \). This value is too large to produce non-classical behavior in the quantum region, where the accelerations (classically) are much smaller.

In this paper, on the other hand, we show that the existence of a maximal acceleration may actually explain quantum behavior. The existence of a universal maximal acceleration was predicted in [9] through the following reasoning. It was shown that from the generalized principle of relativity it follows that there are only two possible types of transformations between uniformly accelerated systems. If the clock hypothesis is not valid, then there exists a universal maximal acceleration. By acceleration, we mean the proper acceleration defined [10] as \( g = \frac{d^2x}{dt^2} \), where \( t \) is the proper time. This acceleration is equal to the four accelerations with respect to the comoving frame. Only this acceleration is bounded, not the acceleration \( \frac{d^2x}{d\tau^2} \). In this case, as shown in [11], a Doppler-type shift for an accelerated source will be observed.

Kündig’s experiment [12] measured the transverse Doppler shift for a rotating disc by means of the Mössbauer effect. In [13], it was shown that among other similar experiments, only this experiment measured properly the transverse Doppler shift. This experiment, as reanalyzed [14] by Kholmetskii et al, shows that the Doppler shift observed in the experiment differs from that predicted by special relativity. In [9], we explained that this discrepancy is due to an additional Doppler shift caused by the acceleration, and we estimated the value of the maximal acceleration \( a_m \) to be
of the order \(10^{19} \text{ m s}^{-2}\). A new test for the determination of the maximal acceleration, as described in [15], is in process.

Recall that relativistic dynamics stems from the limitation of the speed of moving massive objects by the speed of light. This dynamics reduces to classical mechanics in the case of motion with speed significantly smaller than the speed of light, while it accurately describes the dynamics of objects moving with a speed close to the speed of light, which is essentially different from the classical one. We will use here the extended relativistic dynamics introduced in [11]. This new dynamics extends relativistic dynamics so that all admissible solutions have a speed bounded by the speed of light and an acceleration bounded by the maximal one. Note that in quantum systems, such as atoms and molecules, the electromagnetic force would generate accelerations above \(10^{19} \text{ m s}^{-2}\). Thus, we raise the following question: can extended relativistic dynamics, which preserves the limitation of the maximal acceleration, also explain quantum phenomena? This paper represents the first indication of a positive answer.

We will calculate the trajectory of the electron in a hydrogen-like atom under extended relativistic dynamics [11] and show that during the typical time of a quantum measurement, the trajectory of the particle covers a whole area. This may provide an explanation of the probabilistic description of particles in quantum mechanics and the uncertainty in the measurement of observable quantities. In addition, our model reveals another non-classical behavior of a hydrogen-like atom. The notion of ‘center of mass’ must be modified. This leads us to a non-quantum explanation of the non-existence of a magnetic momentum for a hydrogen atom.

2. Extended relativistic dynamics equations for a hydrogen-like atom

The main feature of special relativity is that the relativistically allowed velocities are limited by the speed of light. Relativistic dynamics describes motions that preserve this limitation. In special relativity, the magnitude of the acceleration is unlimited. In [11], we extended relativistic dynamics to an extended relativistic dynamics, a dynamics in which the speed of any moving object is limited by \(c\) and the magnitude of its acceleration is limited by \(a_m\).

In this dynamics we use the 3D position \(\mathbf{r}(t)\) and the proper velocity \(\mathbf{u}(t)\) to describe the state of a moving object.

The proper velocity \(\mathbf{u}\) of an object is the derivative of the object’s position with respect to the proper time. In other words, \(\mathbf{u} = \gamma \mathbf{v}\), where \(\mathbf{v}\) is the object’s velocity and \(\gamma(v) = 1/\sqrt{1 - v^2/c^2}\). In extended relativistic dynamics, the evolution of an object of rest mass \(m\) under a 3D force \(\mathbf{F}\) is described by the following Hamilton-type system of equations:

\[
\begin{align*}
\frac{\text{dr}}{\text{dt}} &= \frac{\mathbf{u}}{\sqrt{1 + |\mathbf{u}|^2/c^2}}, \\
\frac{\text{du}}{\text{dt}} &= \frac{\mathbf{F}(\mathbf{r}, \mathbf{u})/m}{\sqrt{1 + |\mathbf{F}(\mathbf{r}, \mathbf{u})|^2/(a_m^2 m^2)}}.
\end{align*}
\]

For example, if the force \(\mathbf{F}\) is a radial one, \(\mathbf{F} = f(|\mathbf{r}|)\mathbf{r}/|\mathbf{r}|\), then defining

\[H(\mathbf{r}, \mathbf{u}) = mc^2 \sqrt{1 + u^2/c^2} - \int_0^{r(t)} f(s)\text{d}s/\sqrt{1 + f(s)^2/(a_m^2 m^2)},\]

we can rewrite our system (1) as

\[
\begin{align*}
\frac{\text{dr}}{\text{dt}} &= \frac{\partial H}{\partial \mathbf{u}}, \\
\frac{\text{du}}{\text{dt}} &= -\frac{\partial H}{\partial \mathbf{r}}.
\end{align*}
\]

Consider a system of two particles, a proton with mass \(m_p = 1.7 \times 10^{-27} \text{ kg}\) and an electron with mass \(m_e = 9 \times 10^{-31} \text{ kg}\). Let us denote the position and the proper velocity of the proton by \(\mathbf{r}_p, \mathbf{u}_p\) and of those the electron by \(\mathbf{r}_e, \mathbf{u}_e\). At this point we will restrict ourselves to the Coulomb force ignoring the interaction of the particles with the fields. The force of the proton acting on the electron is thus \(\mathbf{F}_1 = k(|\mathbf{r}_p - \mathbf{r}_e|/|\mathbf{r}_p - \mathbf{r}_e|^3\), with \(k = 2.3 \times 10^{-28} \text{ N m}^2\), while the electric force of the electron acting on the proton is \(\mathbf{F}_2 = k(|\mathbf{r}_e - \mathbf{r}_p|/|\mathbf{r}_e - \mathbf{r}_p|^3) = -\mathbf{F}_1\).

Typical distances between the proton and the electron are of the order of \(0.5 \lambda = 0.5 \times 10^{-10} \text{ m}\). For both the proton and the electron, we have

\[\frac{F_2}{a_m m_p} \approx 35 \gg 1, \quad \frac{F_1}{a_m m_e} \approx 10^8 \gg 1.\]

Thus, the second equation of system (1) for both particles becomes

\[
\frac{\text{d}u_p}{\text{d}t} \approx a_m \frac{\mathbf{r}_e - \mathbf{r}_p}{|\mathbf{r}_e - \mathbf{r}_p|}, \quad \frac{\text{d}u_e}{\text{d}t} \approx a_m \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|},
\]

which implies that the magnitude of the acceleration of each particle will be close to the maximal one.

To estimate the velocities of the particles, we can use the connection between the acceleration and the velocity in circular motion. Since, in this case, \(v = aR\), in our system we will have \(v \approx 10^4 \text{ m s}^{-1} \ll c\). Thus, in the first equation of system (1), we can ignore the denominator, and this equation becomes \(u \approx \text{d}r/\text{d}t\). Substituting this into the second equation, we get an approximation of the extended relativistic dynamics equation for the particles in a hydrogen-like atom:

\[
\frac{\text{d}^2 \mathbf{r}_p}{\text{d}t^2} \approx a_m \frac{\mathbf{r}_e - \mathbf{r}_p}{|\mathbf{r}_e - \mathbf{r}_p|}, \quad \frac{\text{d}^2 \mathbf{r}_e}{\text{d}t^2} \approx a_m \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|}.
\]

3. Decomposition into a ‘center of mass’ and their relative motion

As usual, for a two-body problem, in order to solve system (4) of two particles, we decompose their motion into the motion of a ‘center of mass’ and the motion of each particle with respect to the ‘center of mass’. For our system, the average of these two equations gives

\[\frac{\text{d}^2(\mathbf{r}_p + \mathbf{r}_e)}{\text{d}t^2}/2 \approx 0.\]
This implies that the point $\mathbf{R} := (\mathbf{r}_p + \mathbf{r}_e)/2$ moves freely, as a ‘center of mass’. Thus, in this model, the definition of the ‘center of mass’ differs from the definition in classical mechanics.

In a classical model for a hydrogen-like atom, the center of mass is positioned at or close to the proton, and the electron moves around the more or less stationary proton. Hence, classically, there should be a significant magnetic field for the atom. In our model, however, both the electron and the proton move around the new ‘center’ with similar trajectories. They thus produce the same magnetic field, but of opposite signs, due to their opposite charges. Thus, the total magnetic moment of our atom will be almost zero.

We introduce $\mathbf{r} = (\mathbf{r}_p - \mathbf{r}_e)/2$, so that $\mathbf{r}_p = \mathbf{R} + \mathbf{r}$ and $\mathbf{r}_e = \mathbf{R} - \mathbf{r}$. Subtracting the second equation of (4) from the first and dividing by 2, we obtain

$$\frac{d^2 \mathbf{r}}{dt^2} = -a_m |\mathbf{r}|,$$

which we call the ‘radial equation’ describing the relative motion of the particles with respect to the center of mass. This is a typical equation for motion in a central field which can be solved by known methods. We will solve this equation by the method of [16, chapter 14].

4. The solution to the radial equation

The solution of equation (6) is in a stable plane generated by the initial vectors $\mathbf{r}(0), \dot{\mathbf{r}}(0)$. Without loss of generality, we may assume that this plane is the $x, y$-plane. We complexify this plane by identifying the point $(x, y)$ with the complex number $\zeta = x + iy = r(t)e^{i\psi(t)}$. With this notation, $\frac{\mathbf{r}}{r} = e^{i\psi}$, and equation (6) becomes

$$\ddot{r} e^{i\psi} + 2i\dot{r}\dot{\phi} e^{i\psi} + i\dot{r}\dot{\phi} r e^{i\psi} - r\dot{\phi}^2 e^{i\psi} = -a_m e^{i\psi}.$$

Dividing by $e^{i\psi}$, we have

$$\ddot{r} + 2i\dot{r}\dot{\phi} + i\dot{r}\dot{\phi} r - r\dot{\phi}^2 = -a_m.$$

The imaginary part of this equation is

$$2\dot{r}\dot{\phi} + \ddot{r} \equiv \frac{d}{dt}(r^2 \dot{\phi}) = 0,$$

implying that the angular momentum $r^2 \dot{\phi}$ in the center-of-mass system is conserved. Let us denote the angular momentum, defined from the initial conditions, by $c_1$. Then, the equation

$$\dot{\phi} = c_1/r^2$$

uniquely defines $\phi(t)$ if we first solve the equation for $r(t)$ and use the initial conditions.

To find $r(t)$, substitute (8) into the real part of equation (7) to obtain

$$\ddot{r} - \frac{2}{r} + a_m = 0.$$

Multiplying this equation by $2\dot{r}$ and integrating by $t$, we obtain

$$\dot{r}^2 = c_2 - c_1^2/r^2 - 2a_m r,$$

with $c_2$ defined by the initial conditions. This equation shows that the radial part can be regarded as motion in one dimension in an effective field

$$U_{\text{eff}} = \frac{c_1^2 m}{2r^2} - ma_m r,$$

where $c_1^2 m/2r^2$ is called the centrifugal energy, and $U(r) = ma_m r$ is the potential energy.

Since

$$\dot{r} = \pm \sqrt{c_2 - c_1^2/r^2 - 2a_m r},$$

in order that solutions exist, the values of $r$ must be restricted by

$$c_2 r^2 - c_1^2 + 2a_m r^3 \geq 0.$$

The solutions of this inequality are $r_1 < r < r_2$, where $r_1$ and $r_2$ are the two positive roots of the cubic polynomial in (10), which always exist since this polynomial is negative at 0, negative towards $\infty$ and has at least one non-negative value at the initial state.

Thus, the solutions of the radial equation (6) of extended relativistic dynamics for a hydrogen-like atom are obtained by solving the first-order differential equation (9) and then (8).

It is known that only for central fields with potential energy proportional to $r^2$ or $1/r$ all finite motions take place in closed paths. The classical electromagnetic field is of this type, but under our dynamics, $U(r) = ma_m r$ is not. Hence, in general, our solution oscillates between the two radial values $r_1$ and $r_2$ and is not a closed path, see figure 1. We can obtain a circular path at the minimum of the effective potential $r = \sqrt{c_1^2/a_m}$ corresponding to the initial velocity $v = \sqrt{r_{\text{eff}}}$, when the velocity is perpendicular to $\mathbf{r}$. For certain discrete values, we can get a closed path after $n$ periods.

The frequency of these oscillations can be estimated from the fact that the magnitude of the acceleration is approximately $a_m$, and in approximately circular motion, we have $a = R\omega^2$. These considerations yield a frequency of

![Figure 1. Electron and proton trajectories in an extended relativistic dynamics hydrogen-like atom.](image-url)
\[ v \approx 10^{14} \text{ s}^{-1}. \] This implies that during one measurement time, the particle will cover a whole area in the annulus \( r_1 < r < r_2 \).

5. Discussion

Extended relativistic dynamics is an extension of relativistic dynamics in which all admissible solutions have a speed bounded by the speed of light and an acceleration bounded by the maximal acceleration. Here we have used our estimate of the uniform maximal acceleration of \( 10^{19} \text{ m s}^{-2} \). The existence of a maximal acceleration was also conjectured by Caianiello [1] based on the Heisenberg uncertainty relation.

We have shown that at quantum system distances, the classical electromagnetic force would generate accelerations above the maximal one. Thus, at the quantum level, extended relativistic dynamics differs significantly from relativistic dynamics. Moreover, the existence of a maximal acceleration might explain why the behavior of quantum systems differs from the behavior of classical systems.

We have shown that a hydrogen-like atom in extended relativistic dynamics differs significantly from a classical two-body system. We obtained the first approximation of the solution for such a system ignoring the interaction of the particles with the field. In a typical time that can be measured, the particle covers a whole area. This may provide an indication of the probabilistic description of particles in quantum mechanics.

We have shown that in our model, the expression for the center of mass differs from the classical one. In our model, the total magnetic moment of a hydrogen atom is almost zero, which is not so in the classical (non-quantum) model. This observation also reveals the importance of the notion of symmetric velocity, which was introduced in chapter 2 of [17]. This velocity is the relativistic half of the regular velocity. In our model, the velocity of both particles with respect to the new ‘center of mass’ is the symmetric velocity of the velocity of the electron in the classical model. It is known that the transformations of the symmetric velocities are conformal. Conformal transformations play an important role in the quantum region.

This is only the first step in solving the hydrogen-like atom by the use of extended relativistic dynamics. We plan to improve our model by: (i) considering the next approximations of the model; (ii) incorporating the interaction of the charges with the fields; and (iii) taking into consideration the spin of the proton and the electron.

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References

[1] Caianiello E R 1981 Is there a maximal acceleration? Lett. Nuovo Cimento 32 65–70
[2] Scarpetta G 1984 Lett. Nuovo Cimento 41 51–8
[3] Toller M 1990 Maximal acceleration, maximal angular velocity and causal influence Int. J. Theor. Phys. 29 963–84
[4] Nesterenko V V, Feoli A and Scarpetta G 1995 Dynamics of relativistic particle with Lagrangian dependent on acceleration J. Math. Phys. 36 5552–64
[5] Capozziello S, Lambiase G and Scarpetta G 2000 Generalized uncertainty principle from quantum geometry Int. J. Theor. Phys. 39 15–22
[6] Lambiase G, Papini G and Scarpetta G 1998 Maximal acceleration corrections to the Lamb shift of hydrogen, deuterium and He+ Phys. Lett. A 244 349–54
[7] Papini G, Scarpetta G, Bozza V, Feoli A and Lambiase G 2002 Phys. Lett. A 300 603–10
[8] Torrome R G 2007 On a covariant version of Caianiello’s model Gen. Rel. Grav. 39 1833–45
[9] Friedman Y and Gofman Ya 2010 A new relativistic kinematics of accelerated systems Phys. Scr. 82 015004
[10] Rindler W 1977 Essential Relativity, Special, General, and Cosmological (New York: Springer)
[11] Friedman Y 2011 The maximal acceleration, Extended Relativistic Dynamics and Doppler type shift for an accelerated source Ann. Phys. 523 408–16
[12] Kündig W 1963 Measurement of the transverse Doppler effect in an accelerated system Phys. Rev. 126 2371–5
[13] Friedman Y and Nowik I 2012 Testing Einstein’s time dilation under acceleration using Mössbauer spectroscopy Phys. Scr. 85 065702
[14] Kholmetskii A L, Yarman T and Mishevitch O V 2008 Kündig’s experiment on the transverse Doppler shift re-analysed Phys. Scr. 77 035302
[15] Friedman Y 2009 Testing Doppler type shift for an accelerated source and determination of the universal maximal acceleration arXiv:0912.4752
[16] Landau L D and Lifshitz E M 1978 Mechanics 3rd edn (Oxford: Pergamon)
[17] Friedman Y 2004 Physical Applications of Homogeneous Balls: Progress in Mathematical Physics vol 40 (Cambridge, MA: Birkhauser Boston)