Rational pencils of cubics and configurations of six or seven points in $\mathbb{R}P^2$

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Abstract

Let six points $1, \ldots, 6$ lie in general position in the real projective plane and consider the pencil of nodal cubics based at these points, with node at one of them, say $1$. This pencil has five reducible cubics. We call combinatorial cubic a topological type (cubic, points) and combinatorial pencil the cyclic sequence of five combinatorial reducible cubics. Up to the action of the symmetric group $S_5$ on $\{2, \ldots, 6\}$, there are seven possible combinatorial pencils with node at $1$. Consider now the set of six pencils obtained, making the node to be $1, \ldots, 6$. Up to the action of $S_6$ on $\{1, \ldots, 6\}$, there are four possible lists of six combinatorial pencils. Let seven points $1, \ldots, 7$ lie in general position in the plane. Up to the action of $S_7$ on $\{1, \ldots, 7\}$, there are fourteen possible lists of seven nodal combinatorial cubics passing through the seven points, with respective nodes at $1, \ldots, 7$.

1 Introduction

Let $1, \ldots, 6$ be six points in the real projective plane and consider the pencil of nodal cubics determined by these points with node at one of them, say $1$. This pencil has five reducible cubics plus, possibly, some cuspidal cubics. We ignore the latter and consider the sequence of reducible cubics. Let us call combinatorial cubic a topological type (cubic, points), up to the following identification: a loop passing through no other point than the node will be assimilated to an isolated node. Let us call combinatorial pencil the cyclic sequence of five combinatorial reducible cubics, they are of the form
1m ∪ 1ijkl, where \(\{i, j, k, l, m\} = \{2, 3, 4, 5, 6\}\). A set of six or seven points is in generic position if no three are aligned and no six are conic. Consider first six points, five of them determine 10 lines and one conic, dividing \(\mathbb{R}P^2\) in 36 zones. The arrangement of 1, . . . 6 is any one of the six equivalent pieces of information: cyclic ordering of five chosen points on their conic and zone containing the sixth one. Another equivalent information is the list of six combinatorial pencils of cubics based at 1, . . . 6, with respective nodes at 1, . . . 6. A configuration of six points is an equivalence class of arrangements for the action of the group \(S_6\). Let us consider now seven points 1, . . . 7. For a point \(n\) among the seven, we denote by \(\hat{n}\) the arrangement of the other six points. An arrangement of seven points is the set \(\hat{1}, . . . \hat{7}\) of arrangements realized by six of the points. A configuration of seven points is an equivalence class of arrangements for the action of \(S_7\).

**Theorem 1** Let 1, . . . , 6 be six generic points in \(\mathbb{R}P^2\). Up to the action of \(S_5\) on \(\{2, . . . , 6\}\), there are seven possible combinatorial pencils of nodal cubics based at these six points with node at 1.

Up to the action of \(S_6\) on \(\{1, . . . , 6\}\), there are four possible lists of six combinatorial pencils of nodal cubics based at these six points with respective nodes at 1, . . . 6. Otherwise stated, six generic points may realize four different configurations.

**Theorem 2** Seven generic points 1, . . . , 7 in \(\mathbb{R}P^2\) may realize fourteen different configurations. Up to the action of \(S_7\), there are correspondingly fourteen possible lists of seven combinatorial nodal cubics through the seven points, with respective nodes at 1, . . . 7.

Theorem 1 is proved in section 2.1, Theorem 2 is proved in sections 3.1 and 3.2. The same results have been obtained independently by Arzu Zabun with a different method (private communication).

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2 Configurations of six points

2.1 Rational pencils of cubics

Proof of Theorem 1: Up to the action of $S_5$ on $\{2, \ldots, 6\}$, we may assume that the six points are disposed as shown in Figure 1, where 2, 5, 3, 6, 4 lie in this ordering on a conic, and 1 is in one of the zones $A, \ldots, G$.

Let $1 \in A$. We perform a cremona transformation $cr : (x_0; x_1; x_2) \rightarrow (x_1 x_2; x_0 x_2; x_0 x_1)$ with base points 1, 4, 5. Let us denote the respective images of the lines 14, 15, 45 by 5, 4, 1. For the other points, we keep the same notation as before $cr$. The three conics 14365, 31452, 61542 are mapped onto three lines 36, 23 and 26, see upper part of Figure 2. After $cr$, let us consider the pencil of conics 1236, and five particular conics of this pencil: the three double lines and the two conics passing respectively through 4 and 5. The cyclic ordering of these conics in the pencil is easily determined. Perform the cremona transformation back. The pencil of conics 1236 is mapped onto the pencil of nodal cubics based at 1, $\ldots$, 6 with node at 1. The five particular conics are mapped onto the five reducible cubics. We repeat this procedure.
for the various positions of $1 \in B, \ldots G$, with a cremona transformation based either in 145 or in 136, see Figures 2, 5. The sequences of five particular conics are displayed in Figure 6. The sequences of five reducible cubics for each case $A, \ldots G$ are shown in Figure 7, the pencils are drawn in Figures 8, 11. More precisely, we have represented the successive types of combinatorial cubics in the five portions bounded by the reducible cubics. The upper middle pictures of each pencil $B, C, G$ are not quite correct: in the actual pencils, the portion starts and finishes with cubics having a loop containing no base point other than 1. Inbetween, there is a pair of cuspidal cubics, and in the middle, cubics with an isolated node. We have represented only the latter.

Consider any one of these seven pencils. We observe the following properties: the cyclic ordering of the points $2, \ldots 6$ is the same on each of the five types of non-reducible cubic, it is given by the pencil of lines based at 1, this follows from Bezout’s theorem. The cyclic ordering of $2, \ldots 6$ given by the lines of the successive reducible cubics is the same as the cyclic ordering of these points on the conic that they determine. The mutual cyclic orderings of $2, \ldots 6$ given on one hand by the conic 25364, on the other hand by the pencil of lines based at 1, are displayed in the marked diagrams of Figure 12, where the circles represent the conics. Let us now consider together all six pencils of cubics with nodes at $1, \ldots 6$. If we take 1 in one of the zones $E, F, G$ exterior to the conic 25364, then we note that 2 lies inside of the conic determined by the other five points. So, up to the action of $S_6$, we may assume that 1 is interior to a conic 25364. There are four lists of six pencils, giving rise to four configurations $\alpha, \beta, \gamma, \delta$, see Figure 13. For a given list, each choice of a point $1, \ldots 6$ gives rise to a diagram, it turns out that all six unmarked diagrams of a list are identical, and each list corresponds to a different diagram as shown in the upper part of Figure 14. Note that the unmarked diagrams remain also unchanged if the roles of the circle and of the polygonal line are swapped.

The space $(\mathbb{R}P^2)^6/S_6$, enhanced with the natural stratification given by the alignment of three points, has four cameras and three walls, [5] and [6]. The cameras correspond to the four configurations. Refine the stratification with the conics, one supplementary wall appears inside of the camera $\beta$. The adjacency graph is obtained assigning a vertex to each camera and an edge to each wall, see Figure 14.
Figure 2: Cremona transformations for zones $A$ and $B$
Figure 3: Cremona transformations for zones C and D
Figure 4: Cremona transformations for zones $E$ and $F$
Figure 5: Cremona transformation for zone \( G \)

\[
1 \in A, cr(145) \quad 12 \cup 36 \quad 13462 \quad 13 \cup 26 \quad 16 \cup 23 \quad 16532
\]

\[
1 \in B, cr(145) \quad 12 \cup 36 \quad 31426 \quad 13 \cup 26 \quad 16 \cup 32 \quad 25163
\]

\[
1 \in C, cr(136) \quad 12 \cup 45 \quad 15 \cup 42 \quad 41562 \quad 41523 \quad 14 \cup 25
\]

\[
1 \in D, cr(145) \quad 12 \cup 36 \quad 31246 \quad 13 \cup 26 \quad 16 \cup 32 \quad 16352
\]

\[
1 \in E, cr(136) \quad 12 \cup 45 \quad 15 \cup 24 \quad 41562 \quad 41523 \quad 14 \cup 25
\]

\[
1 \in F, cr(145) \quad 12 \cup 36 \quad 34126 \quad 13 \cup 26 \quad 16 \cup 23 \quad 32156
\]

\[
1 \in G, cr(136) \quad 12 \cup 45 \quad 15 \cup 24 \quad 41562 \quad 41532 \quad 14 \cup 25
\]

Figure 6: The seven pencils of conics

\[
1 \in A \quad 12 \cup 51436 \quad 15 \cup 31264 \quad 13 \cup 51624 \quad 16 \cup 31452 \quad 14 \cup 21653
\]

\[
1 \in B \quad 12 \cup 31645 \quad 15 \cup 31642 \quad 13 \cup 51642 \quad 16 \cup 31425 \quad 14 \cup 31625
\]

\[
1 \in C \quad 12 \cup 15364 \quad 15 \cup 31264 \quad 13 \cup 51264 \quad 16 \cup 21453 \quad 14 \cup 21653
\]

\[
1 \in D \quad 12 \cup 31654 \quad 15 \cup 31624 \quad 13 \cup 51642 \quad 16 \cup 31425 \quad 14 \cup 31652
\]

\[
1 \in E \quad 12 \cup 15364 \quad 15 \cup 12346 \quad 13 \cup 12546 \quad 16 \cup 12453 \quad 14 \cup 12653
\]

\[
1 \in F \quad 12 \cup 16453 \quad 15 \cup 31642 \quad 13 \cup 16425 \quad 16 \cup 13524 \quad 14 \cup 13526
\]

\[
1 \in G \quad 12 \cup 14635 \quad 15 \cup 12436 \quad 13 \cup 12456 \quad 16 \cup 12453 \quad 14 \cup 12635
\]

Figure 7: The seven pencils of cubics
Figure 8: Pencils with node in 1 for zones A and B
Figure 9: Pencils with node in 1 for zones $C$ and $D$
Figure 10: Pencils with node in 1 for zones $E$ and $F$
Figure 11: Pencil with node in 1 for zone G

Figure 12: Marked diagrams for $1 \in A, \ldots G$, the circle represents the conic
\[ \alpha : 1 \in A \]
\[ 1 < 25364 \quad 12 \cup 51436 \quad 15 \cup 31264 \quad 13 \cup 51624 \quad 16 \cup 31452 \quad 14 \cup 21653 \]
\[ 2 < 51436 \quad 25 \cup 31264 \quad 21 \cup 25364 \quad 24 \cup 21653 \quad 23 \cup 51624 \quad 26 \cup 31452 \]
\[ 3 < 51624 \quad 35 \cup 31264 \quad 31 \cup 25364 \quad 36 \cup 31452 \quad 32 \cup 51436 \quad 34 \cup 21653 \]
\[ 4 < 21653 \quad 42 \cup 51436 \quad 41 \cup 25364 \quad 46 \cup 31452 \quad 45 \cup 31264 \quad 43 \cup 51624 \]
\[ 5 < 31264 \quad 53 \cup 51624 \quad 51 \cup 25364 \quad 52 \cup 51436 \quad 56 \cup 31452 \quad 54 \cup 21653 \]
\[ 6 < 31452 \quad 63 \cup 51624 \quad 61 \cup 25364 \quad 64 \cup 21653 \quad 65 \cup 31264 \quad 62 \cup 51436 \]

\[ \beta : 1 \in B \]
\[ 1 < 25364 \quad 12 \cup 31645 \quad 15 \cup 31642 \quad 13 \cup 51642 \quad 16 \cup 31425 \quad 14 \cup 31625 \]
\[ 2 > 31645 \quad 23 \cup 51642 \quad 21 \cup 25364 \quad 26 \cup 31425 \quad 24 \cup 31625 \quad 25 \cup 31642 \]
\[ 3 > 51642 \quad 35 \cup 31642 \quad 31 \cup 25364 \quad 36 \cup 31425 \quad 34 \cup 31625 \quad 32 \cup 31645 \]
\[ 4 < 31625 \quad 43 \cup 51642 \quad 41 \cup 25364 \quad 46 \cup 31425 \quad 42 \cup 31645 \quad 45 \cup 31642 \]
\[ 5 < 31642 \quad 53 \cup 51642 \quad 51 \cup 25364 \quad 56 \cup 31425 \quad 54 \cup 31625 \quad 52 \cup 31645 \]
\[ 6 > 31425 \quad 63 \cup 51642 \quad 61 \cup 25364 \quad 64 \cup 31625 \quad 62 \cup 31645 \quad 65 \cup 31625 \]

\[ \gamma : 1 \in C \]
\[ 1 < 25364 \quad 12 \cup 15364 \quad 15 \cup 31264 \quad 13 \cup 51264 \quad 16 \cup 31453 \quad 14 \cup 21653 \]
\[ 2 > 15364 \quad 23 \cup 25364 \quad 25 \cup 31264 \quad 23 \cup 51264 \quad 26 \cup 31453 \quad 24 \cup 21653 \]
\[ 3 > 51264 \quad 35 \cup 31264 \quad 31 \cup 25364 \quad 32 \cup 51264 \quad 36 \cup 31453 \quad 34 \cup 21653 \]
\[ 4 < 21653 \quad 43 \cup 51364 \quad 41 \cup 25364 \quad 46 \cup 31453 \quad 45 \cup 31264 \quad 43 \cup 51264 \]
\[ 5 < 31264 \quad 53 \cup 51264 \quad 51 \cup 25364 \quad 52 \cup 15364 \quad 56 \cup 21453 \quad 54 \cup 21653 \]
\[ 6 > 21453 \quad 62 \cup 31364 \quad 61 \cup 25364 \quad 64 \cup 21653 \quad 65 \cup 31264 \quad 63 \cup 51264 \]

\[ \delta : 1 \in D \]
\[ 1 < 25364 \quad 12 \cup 31654 \quad 15 \cup 31624 \quad 13 \cup 51624 \quad 16 \cup 31425 \quad 14 \cup 31652 \]
\[ 2 < 31654 \quad 23 \cup 51642 \quad 21 \cup 25364 \quad 26 \cup 31425 \quad 25 \cup 31624 \quad 24 \cup 31652 \]
\[ 3 > 51642 \quad 35 \cup 31624 \quad 31 \cup 25364 \quad 36 \cup 31425 \quad 34 \cup 31652 \quad 32 \cup 31654 \]
\[ 4 > 31652 \quad 43 \cup 51642 \quad 41 \cup 25364 \quad 46 \cup 31425 \quad 45 \cup 31624 \quad 42 \cup 31654 \]
\[ 5 > 31624 \quad 53 \cup 51642 \quad 51 \cup 25364 \quad 56 \cup 31425 \quad 52 \cup 31654 \quad 54 \cup 31652 \]
\[ 6 > 31425 \quad 63 \cup 51642 \quad 61 \cup 25364 \quad 64 \cup 31652 \quad 62 \cup 31654 \quad 65 \cup 31624 \]

Figure 13: The four lists of six pencils
2.2 Diagrams

The seven arrangements from Figure 1 corresponding to the choices $A, \ldots G$ of the zone containing 1 may be encoded by refining the marked diagrams from Figure 12 as shown in Figure 15. First, we indicate whether 1 lies inside or outside of the conic 25364 using a dotted polygonal line if 1 is inside, and a plain polygonal line if 1 is outside. Once this is done, the marked diagrams of $B, F$ and $G$ still correspond to several zones. The diagram of $B$ fits to the other four $B$-like zones, each of these five zones may be characterized by the point of the conic situated opposite to it, for $B$ this point is 2. Let us add to the diagram of $B$ a dot at the point 2. The same argument applies to $F$. Let us now consider the zone $G$, it is a triangle having 2 as vertex and whose sides are supported by the lines 24, 26, 35. There are in total 10 $G$-like zones and the diagram of $G$ fits also to the one having 4 as vertex and whose sides are supported by the lines 24, 54, 36. In order to differentiate these two zones, we provide the edge 24 in the diagram of $G$ with an arrow from 2 to 4. The extra point 1 is indicated inside of the diagrams in Figure 15. Let us call $n$-diagram a diagram having $n$ as extra point. Each arrangement may actually be encoded by six diagrams, making $n = 1, \ldots 6$. See Figure 16, where all six $n$-diagrams in a row are equivalent. For each of the four arrangements, let one point $n$ move until it crosses a wall: a line determined by two others or a conic determined by five others. Figures 17-20 show the corresponding changes of the $n$-diagrams for $n = 1, \ldots 6$. We deduce the following:
Proposition 1 Each $\beta$-arrangement is adjacent to another $\beta$-arrangement via a conic-wall, and to three $\delta$-arrangements via line-walls. Each $\delta$-arrangement is adjacent to two $\beta$-arrangements and to four $\gamma$-arrangements via line-walls. Each $\gamma$-arrangement is adjacent to six $\delta$-arrangements and to one $\alpha$-arrangement via line-walls. Each $\alpha$-arrangement is adjacent to ten $\gamma$-arrangements via line-walls.

It will be convenient to have a simple encoding for arrangements at our disposal, so our next concern is to define a new kind of diagram, or code, replacing the set of six equivalent $n$-diagrams. Let us say shortly that a point $n$ is interior for the arrangement if $n$ is interior to the conic determined by the other five (the $n$-diagram is dotted). The upper part of Figure 21 shows four codes corresponding to the four arrangements of Figure 16. Let us explain how we defined them. In the case $\beta$, we represent the six points with six dots disposed on a circle in the natural cyclic ordering given by the convex position. The dots are colored alternatively in black and white, the white dots correspond to the interior points. The case $\delta$ offers no such evident solution, so one has to make a choice that is not entirely satisfactory. The interior points in the $\delta$-arrangement of Figure 16 are 1 and 2. The two polygonal lines of their diagrams may be seen as closed paths intercepting successively five points. Let us embed these two paths in a different way, draw now one of them with a dotted and the other with a plain line. We get a graph with vertices 1, ..., 6, having two different kinds of edges. To recover the remaining four $n$-diagrams from this graph, note that the polygonal lines of these diagrams are oriented, with a starting point, 1 for the 3 and 6-diagrams, 2 for the 4 and 5-diagrams. The orderings with which the five points are met by the polygonal lines of these four diagrams are shown with arrows describing paths in the graph, as indicated in Figure 21. Finally, note that this encoding is not unique: the second graph drawn in the bottom of the figure represents the same arrangement. Two graphs obtained from one another swapping pairs of points with a vertical symmetry represent of course also the same arrangement. For the case $\gamma$, note that the 1-diagram is dotted and has 2 as bottom point, whereas the 2-diagram is plain and has 1 as bottom point, we say that 1 and 2 form a pair. The other four points may be similarly distributed in two pairs. Let us encode this with a graph having the six points as vertices, and three edges endowed with arrows as shown in Figure 21. Each arrow connects an interior point to its associated exterior point. The original $n$-diagrams may be deduced easily from this graph. For
the case \( \alpha \), let us simply observe that the \( n \)-diagrams may be deduced easily from one another. For example, start with the 1-diagram. In this diagram, 1 and 2 are separated by the branch 54 of the star. To get the 2-diagram, it suffices to swap 1 with 2 and 5 with 4. To encode this arrangement, we will simply use any one of the \( n \)-diagrams, with the circle removed.

Figures 22-25 show all possible crossings of walls, starting from the four arrangements, using now codes instead of diagrams. Near each arrow corresponding to a wall we indicate the line passing through three points or the conic passing through six points. For the line-walls, we indicate also the (cyclic) ordering with which the line meets: the three points and the three lines determined by the other three points. In Figure 25, the list of ten lines is written, but for place reason, we drew the codes of only two adjacent \( \gamma \)-arrangements.

3 Configurations of seven points

3.1 Fourteen configurations

An arrangement or a configuration of seven points 1,\ldots 7 will be encoded with the list of codes for the subarrangements or subconfigurations 1,\ldots 7. Assume that 1,\ldots 6 are disposed in this ordering on a conic. One can encode the non-generic arrangement 7 with a circle passing successively through six points 1,\ldots 6, it is a \( \beta \)-code from which the colors black and white were
Figure 16: Four arrangements, encoded each by six equivalent refined diagrams
Figure 17: Crossing of walls starting from an $\alpha$-arrangement, using diagrams
Figure 18: Crossings of walls starting from a $\beta$-arrangement, using diagrams.
Figure 19: Crossings of walls starting from a $\gamma$-arrangement, using diagrams
Figure 20: Crossings of walls starting from a $\delta$-arrangement, using diagrams
Figure 21: Codes for the four arrangements from Figure 16

Figure 22: Crossings of walls starting from a $\beta$-arrangement, using codes
Figure 23: Crossings of walls starting from a $\delta$-arrangement, using codes
Figure 24: Crossings of walls starting from a $\gamma$-arrangement, using codes
Figure 25: Crossing of walls starting from an $\alpha$-arrangement, using codes removed. The mutual cyclic orderings of $1, \ldots, 6$ given respectively by the conic and the pencil of lines based at $7$ may be described with a conic-diagram: a closed polygonal line with six vertices, inscribed in a circle. It is easily seen that there are eleven admissible such unmarked conic-diagrams.

**Proposition 2** Seven points $1, \ldots, 7$ in $\mathbb{R}P^2$ with six of them on a conic, but otherwise generic, may realize eleven different configurations.

Seven generic points $1, \ldots, 7$ in $\mathbb{R}P^2$ that lie in convex position may realize eleven different configurations. Up to the action of $S_7$, there are corresponding eleven lists of seven nodal combinatorial cubics.

*Proof:* Let us consider for a start seven points $1, \ldots, 7$ such that $1, \ldots, 6$ lie in this ordering on a conic, and draw this conic as an ellipse in some affine plane. The point $7$ lies in a zone bounded by some of the 15 lines determined by $1, \ldots, 6$. The lines $36, 14, 25$ give rise to six sectors, containing each one edge of the hexagonal convex hull of the six points. Up to cyclic permutation of $1, \ldots, 6$, we may assume that $7$ lies in the sector containing the edge $61$. Let us move the six points keeping them coconic until three triplets of lines become concurrent, as shown in the upper part of Figure 26. Move $7$ along so that it does not cross any of the 15 lines. Note that the zone containing $7$ may be a triangle that shrinks in the end to a triple point. If $7$ is not in such a vanishing zone, one may assume up to the symmetry $(61)(52)(43)$
Figure 26: The eleven configurations with six coconic points
that 7 lies in the end in one of the zones $B, \ldots J$ of Figure 26. Note that the arrangement realized by the seven points is preserved all along the motion. If 7 is in a vanishing triangle, one may up to $(61)(52)(43)$ assume that this triangle is one of the two drawn in the bottom part of Figure 26 denote them by $A$ and $K$. The eleven zones $A, \ldots K$ give rise to eight unmarked conic-diagrams, the other three that turn out to be unrealizable are shown in Figure 27. Figure 28 shows for each zone $A, \ldots K$: the unmarked conic-diagram (refined with either a dotted or a plain polygonal line depending on whether 7 lies inside or outside of the conic), and the six codes $\hat{1}, \ldots \hat{6}$.

Let now $1, \ldots, 7$ be seven generic points, such that six of them lie in convex position. We may assume that the seventh point (extra point) is either inside or outside of all six conics determined by the first six. Let indeed $1, \ldots, 6$ lie in convex position, 6 being outside of the conic 12345, and let 7 lie between two of the six conics. Three possibilities arise: the seven points lie in convex position (the position of 7 in the cyclic ordering is arbitrary), consider one of the two conics adjacent to 7, let $n \in 1, \ldots 6$ be the point that doesn’t lie on this conic, $n$ may be taken as extra point; or $1, 2, 3, 4, 5, 7$ lie in convex position and 6 is outside of all six conics they determine; or $2, 3, 4, 5, 6, 7$ lie in convex position and 1 lies inside of all six conics they determine. Let now $1, \ldots 7$ lie in convex position, and 7 be either inside or outside of all six conics determined by $1, \ldots 6$. One may move the seven points until the first six become coconic, preserving the arrangement all along. So, up to the action of $S_7$, any arrangement of seven points with six of them in convex position may be obtained from one of the eleven non-generic arrangements $A, \ldots K$ by moving 6 away from the conic 12345. Let $X$ be one of the zones in Figure 26. Denote by $(X, 6)$ the arrangement obtained from $X$ moving 6 to the outside of the conic 12345, and $(X, 6')$ the arrangement obtained moving 6 to the inside. Some elements of $S_7$ map pairs of arrangements one onto the other: $(X, 6)$ and $(X, 6')$ with $X \in \{A, B, D, E, F, J\}$ are swapped by the symmetry $(61)(52)(43)$, this symmetry swaps also $(C', 6)$ with $(C, 6')$. The symmetry $(42)(76)(15)$ swaps $(D, 6)$ with $(G, 6')$, and $(C, 6)$ with $(H, 6')$. The symmetry $(63)(14)(52)$ swaps $(I, 6)$ with $(I, 6')$, and $(K, 6)$ with $(K, 6')$. The
Figure 28: Conic-diagrams and codes $\hat{1}, \ldots, \hat{6}$ for $A, \ldots, K$
cyclic permutation \((1234567)\) maps \((E, 6)\) onto \((F, 6')\). There are thus eleven different configurations of seven points with six in convex position. Let us name for first each configuration after some representant for the equivalence class. Later on, we will introduce a more canonical encoding. Each of the zones \(X \in \{A, B, I, J, K\}\) gives rise to one configuration, say \((X, 6)\), the pair \(E, F\) gives rise to one, say \((E, 6)\), the pair \(D, G\) gives rise to two, say \((D, 6)\) and \((G, 6)\) and the pair \(C, H\) gives rise to three, say \((C, 6)\), \((C', 6)\) and \((H, 6)\).

Figure 43 displays the lists of seven nodal cubics with nodes at 1, \ldots, 7 for the arrangements \((E, 6), (D, 6), (C, 6), (B, 6), (A, 6), (C', 6)\). Figure 44 displays the lists for the arrangements \((G, 6), (H, 6), (K, 6), (I, 6), (J, 6)\). Note that \((E, 6)\) is the list denoted by 6− in [4]. Let us explain how we find out these lists of cubics. To get a cubic with node at a given point, say 1, it suffices to perturb the reducible cubic \(17 \cup 123456\), moving one point, say 6, away from the conic 12345 in the appropriate direction. Note that crossing a line \(lnm\) induces the change of the four subcodes \(\hat{p}, p \neq l, n, m\), and of the three cubics with respective node at \(l, n, m\). □

Three configurations with no six points in convex position may be obtained taking 7 in one of the zones \(R, T\) and \(V\) of Figure 29. The codes of these configurations are shown in Figure 30. We have thus constructed 14 configurations, to complete the proof of Theorem 2, we need to grant that there exist no others. This will be done in the next section. The lists of seven nodal cubics corresponding to the three new configurations are shown in Figure 45. Each cubic may be obtained in several ways perturbing reducible cubics. For example, the cubic with node at 7 of the first configuration may be obtained from \(176 \cup 57423\) moving 6 to the left of the line 17, from \(274 \cup 56173\) moving 4 to the top from the line 24. . . . A simple invariant of configurations is obtained counting the numbers of types \(\beta, \delta, \gamma\) and \(\alpha\) realized by the subcodes \(\hat{1}, \ldots, 7\). An encoding for the configurations, using the quadruples \((n_\beta, n_\delta, n_\gamma, n_\alpha)\), is defined in Figure 31. Note that two configurations have the same quadruple if and only if they are adjacent via a conic-wall.

### 3.2 End of the proof of Theorem 2

The space of seven unordered points \((\mathbb{RP}^2)^7/S_7\) endowed with the stratification given by the alignment only has 11 cameras and 27 line-walls, the combinatorial type of a configuration determines its rigid isotopy type, see [5], [6]. In these papers, S. Finashin considered actually configurations of
Figure 29: Three new configurations: $7 \in R, 7 \in T, 7 \in V$

Figure 30: Codes of the three new configurations
Figure 31: Encodings for the 14 configurations, using the quadruples $(n_β, n_δ, n_γ, n_α)$

| arrangement | configuration |
|-------------|---------------|
| $(E, 6)$    | $(7,0,0,0)$   |
| $(D, 6)$    | $(3,4,0,0)_1$|
| $(G, 6)$    | $(3,4,0,0)_2$|
| $(C, 6)$    | $(2,2,3,0)_1$|
| $(C', 6)$   | $(2,2,3,0)_2$|
| $(H, 6)$    | $(2,2,3,0)_3$|
| $(B, 6)$    | $(1,2,2,2)$   |
| $(A, 6)$    | $(1,0,6,0)$   |
| $(K, 6)$    | $(1,6,0,0)$   |
| $(I, 6)$    | $(1,2,4,0)$   |
| $(J, 6)$    | $(1,4,2,0)$   |
| $R$         | $(0,4,3,0)$   |
| $T$         | $(0,3,3,1)$   |
| $V$         | $(0,6,1,0)$   |

lines dual to the configurations of points. For each generic configuration, consider the set of triangles, the adjacent walls are represented by the orbits of the symmetric group acting on these triangles. In our notation, the eleven cameras correspond to the eleven quadruples. Let us explain hereafter how we describe the walls. To this purpose, consider first the possible positions of a line $L$ with respect to four points 1, 2, 3, 4, and add afterwards three last points 5, 6, 7 on $L$. Choose three points among 1, ... 4, they give rise to four triangles in the plane, we call principal triangle the one containing the fourth point. Add a line $L$ passing through none of the four points, $L$ cuts either three or four principal triangles, see upper and lower part of Figure 32. The four points give rise to six lines, let us consider the six intersection points of these lines with $L$. The cyclic ordering of these intersections on $L$ allows to recover the mutual position of $L$ with 1, 2, 3, 4. Let us encode this information using a circle with six marked points, two points have the same color if they play symmetric roles. In the first case, where $L$ does not cut the principal triangle 123, we need two colors (red and blue). In the second case, we need four colors (pale blue, dark blue, red and green). Let us consider the space of seven unordered points stratified by the alignment only. To get the walls, distribute three unmarked points (colored black) in all possible ways
on the two circles from Figure 32 that stay for first marked. For the second circle, note that if an interval between a blue point, say 24, and a red point, say 13, contains no black point, then one can move the line \( L \) until it crosses the intersection \( 24 \cap 13 \) without leaving the wall. On the circle, the positions of 13 and 24 are swapped, and the colorings of all six points change, see second circle in Figure 33. The third and fourth circle show the colorings obtained with the last two possible positions of 34, 12, 13, 24. We can thus reduce the number of colors to two (red and green), see Figure 33, bottom circle. Remove now the markings on the circles, one gets in total 27 line-walls \( W_1, \ldots W_{27} \), splitting in three groups according to the distribution of colors for the six intersections with the line \( L \), see the 27 circles in Figure 34 (where the markings should be ignored, they will be used in the next section).

The following statement is similar to Proposition 1, for seven points in-
Proposition 3 For each of the 14 arrangements \((E, 6), \ldots V\), the configurations realized by the adjacent arrangements are shown in Figures 35-38, along with the corresponding walls.

Proof: For each arrangement \((E, 6), \ldots V\), we find out the list of adjacent line-walls in form of triples: the rule from Figures 22-25 allows to get all of the admissible triples for each subcode \(\tilde{1}, \ldots \tilde{7}\), we get thus seven lists of admissible triples. The relevant triples \(lmm\) are those appearing four times in total (in all lists except for \(\tilde{l}, \tilde{n}, \tilde{m}\)). Draw the code of the arrangement obtained after the crossing. Note that we need the markings only in the case that the arrangement obtained is of type \((3, 4, 0, 0)\) or \((2, 2, 3, 0)\). Each crossing induces the change of four subcodes. To determine the type of the line-wall among \(W_1, \ldots W_{27}\), we need to determine the cyclic ordering of nine points on the line \(L\). Write for each of the four subcodes, the cyclic ordering of the six relevant points on \(L\) using the rule of Figures 22-25. We get thus four circles marked with six points each. The partial cyclic orderings given by these four circles allow to recover the cyclic ordering of the nine points without ambiguity, except for pairs of adjacent red points. For example, start from \((D, 6)\) and take the crossing 456, the partial cyclic orderings are: 4, 5, 6, 27, 37, 23 (change of \(\tilde{1}\)), 4, 5, 17, 6, 37, 13 (\(\tilde{2}\)), 4, 5, 17, 6, 27, 12 (\(\tilde{3}\)), 4, 5, 6, 12, 13, 23 (\(\tilde{7}\)). The resulting cyclic ordering is 4, 5, 17, 6, 27, \{37, 12\}, 13, 23, the wall is \(W_{18}\). The informations obtained are gathered in Figures 35, 37. We know from
Figure 34: The 27 line-walls \( W_1, \ldots, W_{27} \), the 14 line-conic-subwalls
Proposition 2 that there are eleven configurations $A, \ldots, K$ with six coconic points. Each arrangement $(X, 6)$ is adjacent to $(X, 6')$ via the conic-wall $X$; $(X, 6) \simeq (X, 6')$ for $X \in \{A, B, I, J, K\}$; $(D, 6) \simeq (D, 6') \simeq (G, 6')$; $(E, 6) \simeq (E, 6') \simeq (F, 6) \simeq (F, 6')$; and $(C, 6) \simeq (H, 6')$. We get thus the informations of Figure 38. Proposition 3 is proved. It follows from this Proposition that there exist no other configurations than the 14, this finishes the proof of Theorem 2.

In [5, 6], S. Finashin constructed the adjacency graph of the space $(\mathbb{R}P^2)^7/\mathbb{S}_7$ endowed with the stratification given by the alignment only. Using Figures 35-37, we recover this graph, see Figure 39.

3.3 Configurations with three aligned points

**Proposition 4** Seven points in $\mathbb{R}P^2$, with three of them aligned, but otherwise generic, may realize 38 different configurations.

**Proof**: Let us call refined line-walls the configurations with three aligned points, otherwise generic. To find them, we need first to determine, for each of the 27 line-walls, all sets of six points that may be coconic. For each line-wall with some admissible conic, we mark the black points with names 5, 6, 7, and indicate the conic(s) inside of the circle, see Figure 34. Let us explain how these conics are spotted. We say that a non-generic arrangement or configuration is line-conic if it has a conic through six points and a line through three points, one of them not on the conic. The 11 configurations with six coconic points are represented by the arrangements $A, \ldots, K$, with conic 123456. We find for each of these arrangements all triples of points with 7 that may become aligned, see Figure 40. Most of these triples are directly visible in Figure 26 but the safest way to find them without forgetting one is the following: write the list of all admissible triples for each subarrangement $\hat{1}, \ldots, \hat{6}$ using the rules explicated in Figures 22-25; the relevant triples $nm7$ are those appearing four times in total (in all lists but $\hat{n}, \hat{m}$). The line-conic arrangements obtained may be denoted by a letter followed by a triple. We find in total 36 combinations letter-triple. Note that a line-conic arrangement may be described with two letter-triples, for example, $A357 = B357$, see Figure 26. Note also that some of the arrangements give obviously rise to the same configuration: the cyclic permutation $(165432)$ maps $A367$ onto $A257$ and $A257$ onto $A147$; the symmetry $s = (16)(25)(34)$ maps $Xij7$ onto $Xs(i)s(j)7$ for $X \in \{A, B, D, E, F, J\}$. 35
\[
\begin{array}{cccc}
(E, 6) & (7, 0, 0, 0) & 167 & W_{12} \quad (3, 4, 0, 0)_{1} \\
& & 456 & W_{12} \quad (3, 4, 0, 0)_{2} \\
(D, 6) & (3, 4, 0, 0)_{1} & 267 & W_{4} \quad (2, 2, 3, 0)_{1} \\
& & 157 & W_{4} \quad (2, 2, 3, 0)_{2} \\
& & 456 & W_{18} \quad (1, 4, 2, 0) \\
& & 167 & W_{12} \quad (7, 0, 0, 0) \\
(C, 6) & (2, 2, 3, 0)_{1} & 267 & W_{4} \quad (3, 4, 0, 0)_{1} \\
& & 157 & W_{8} \quad (1, 2, 2, 2) \\
& & 367 & W_{16} \quad (2, 2, 3, 0)_{1} \\
& & 126 & W_{27} \quad (1, 6, 0, 0) \\
& & 456 & W_{19} \quad (1, 2, 4, 0) \\
(B, 6) & (1, 2, 2, 2) & 157 & W_{8} \quad (2, 2, 3, 0)_{1} \\
& & 267 & W_{8} \quad (2, 2, 3, 0)_{2} \\
& & 257 & W_{23} \quad (1, 0, 6, 0) \\
& & 367 & W_{21} \quad (1, 2, 2, 2) \\
& & 234 & W_{13} \quad (1, 2, 2, 2) \\
& & 456 & W_{20} \quad (0, 3, 3, 1) \\
& & 147 & W_{21} \quad (1, 2, 2, 2) \\
& & 126 & W_{25} \quad (0, 4, 3, 0) \\
(A, 6) & (1, 0, 6, 0) & 257 & W_{23} \quad (1, 2, 2, 2) \\
& & 367 & W_{23} \quad (1, 2, 2, 2) \\
& & 234 & W_{14} \quad (0, 4, 3, 0) \\
& & 456 & W_{14} \quad (0, 4, 3, 0) \\
& & 147 & W_{23} \quad (1, 2, 2, 2) \\
& & 126 & W_{14} \quad (0, 4, 3, 0) \\
(C''(6)) & (2, 2, 3, 0)_{2} & 157 & W_{4} \quad (3, 4, 0, 0)_{1} \\
& & 267 & W_{8} \quad (1, 2, 2, 2) \\
& & 234 & W_{11} \quad (2, 2, 3, 0)_{2} \\
& & 456 & W_{17} \quad (0, 6, 1, 0) \\
& & 147 & W_{16} \quad (2, 2, 3, 0)_{2} \\
\end{array}
\]

Figure 35: Adjacencies via line-walls
| $(G,6)$ | $(3,4,0,0)_2$ | 234 W1 | (1,6,0,0) |
|---------|-------------|--------|----------|
|         |             | 567 W12 | (7,0,0,0) |
|         |             | 467 W4  | (2,2,3,0) |
|         |             | 127 W18 | (1,4,2,0) |

| $(H,6)$ | $(2,2,3,0)_3$ | 467 W4  | (3,4,0,0) |
|---------|---------------|--------|----------|
|         |               | 457 W27 | (1,6,0,0) |
|         |               | 234 W6  | (0,4,3,0) |
|         |               | 367 W16 | (2,2,3,0) |
|         |               | 127 W19 | (1,2,4,0) |

| $(K,6)$ | $(1,6,0,0)$   | 456 W1  | (3,4,0,0) |
|---------|---------------|--------|----------|
|         |               | 234 W5  | (0,4,3,0) |
|         |               | 367 W22 | (1,2,4,0) |
|         |               | 457 W27 | (2,2,3,0) |
|         |               | 127 W27 | (2,2,3,0) |

| $(I,6)$ | $(1,2,4,0)$   | 457 W19 | (2,2,3,0) |
|---------|---------------|--------|----------|
|         |               | 234 W7  | (0,3,3,1) |
|         |               | 467 W9  | (1,4,2,0) |
|         |               | 367 W22 | (1,6,0,0) |
|         |               | 137 W9  | (1,4,2,0) |
|         |               | 126 W2  | (1,4,2,0) |
|         |               | 127 W19 | (2,2,3,0) |

| $(J,6)$ | $(1,4,2,0)$   | 467 W9  | (1,2,4,0) |
|---------|---------------|--------|----------|
|         |               | 234 W2  | (1,2,4,0) |
|         |               | 567 W18 | (3,4,0,0) |
|         |               | 137 W9  | (1,2,4,0) |
|         |               | 126 W3  | (0,6,1,0) |
|         |               | 127 W18 | (3,4,0,0) |

Figure 36: Adjacencies via line-walls, continued
|       |       |       |       |
|-------|-------|-------|-------|
| $R$   | $(0,4,3,0)$ | 236   | $W_{25}$ | $(1,2,2,2)$ |
|       |         | 457   | $W_{6}$  | $(2,2,3,0)_3$ |
|       |         | 247   | $W_{14}$ | $(1,0,6,0)$ |
|       |         | 357   | $W_{26}$ | $(0,4,3,0)$ |
|       |         | 167   | $W_{15}$ | $(0,4,3,0)$ |
|       |         | 156   | $W_{5}$  | $(1,6,0,0)$ |
|       |         | 134   | $W_{24}$ | $(0,3,3,1)$ |
|       |         | 125   | $W_{26}$ | $(0,4,3,0)$ |
| $T$   | $(0,3,3,1)$ | 247   | $W_{20}$ | $(1,2,2,2)$ |
|       |         | 356   | $W_{7}$  | $(1,2,4,0)$ |
|       |         | 236   | $W_{20}$ | $(1,2,2,2)$ |
|       |         | 357   | $W_{24}$ | $(0,4,3,0)$ |
|       |         | 467   | $W_{7}$  | $(1,2,4,0)$ |
|       |         | 145   | $W_{7}$  | $(1,2,4,0)$ |
|       |         | 137   | $W_{10}$ | $(0,6,1,0)$ |
|       |         | 167   | $W_{24}$ | $(0,4,3,0)$ |
|       |         | 125   | $W_{20}$ | $(1,2,2,2)$ |
| $V$   | $(0,6,1,0)$ | 247   | $W_{17}$ | $(2,2,3,0)_2$ |
|       |         | 356   | $W_{3}$  | $(1,4,2,0)$ |
|       |         | 236   | $W_{17}$ | $(2,2,3,0)_2$ |
|       |         | 467   | $W_{3}$  | $(1,4,2,0)$ |
|       |         | 145   | $W_{3}$  | $(1,4,2,0)$ |
|       |         | 137   | $W_{10}$ | $(0,3,3,1)$ |
|       |         | 125   | $W_{17}$ | $(2,2,3,0)_2$ |

Figure 37: Adjacencies via line-walls, end
(E, 6) (7, 0, 0, 0) E (7, 0, 0, 0)
F (7, 0, 0, 0)
(D, 6) (3, 4, 0, 0) D (3, 4, 0, 0)
G (3, 4, 0, 0)
(C, 6) (2, 2, 3, 0) C (2, 2, 3, 0)
H (2, 2, 3, 0)
(B, 6) (1, 2, 2, 2) B (1, 2, 2, 2)
(A, 6) (1, 0, 6, 0) A (1, 0, 6, 0)
(C', 6) (2, 2, 3, 0) C (2, 2, 3, 0)
(G, 6) (3, 4, 0, 0) G (3, 4, 0, 0)
(H, 6) (2, 2, 3, 0) H (2, 2, 3, 0)
(K, 6) (1, 6, 0, 0) K (1, 6, 0, 0)
(I, 6) (1, 2, 4, 0) I (1, 2, 4, 0)
(J, 6) (1, 4, 2, 0) J (1, 4, 2, 0)

Figure 38: Adjacencies via conic-walls

Figure 39: Adjacency graph for seven unordered points, stratification by lines only
Whenever several equivalent arrangements appear, keep only one. With help of Figure 26 write down, for each arrangement left, the cyclic ordering with which $L$ meets the three aligned points, and the six lines determined by the other four points. We find in total 14 line-conic configurations. Perform an appropriate symmetry on each of the 14 arrangements representing these configurations, so as to get the conic-subwalls of the 27 line-wall arrangements of Figure 34. In this figure, there are in total 15 conics, distributed in 11 line-walls: the two conics 125743 and 126743 of $W_{12}$ determine the same configuration via the symmetry $(13)(24)(75)$. Choose one representant for each set of equivalent letter-triples. Figure 41 shows the symmetries mapping this representant onto the corresponding line-conic-wall from Figure 34. Let us explicit some more symmetries in the case of the less obvious equivalences $H_{127} \cong I_{457}$; $H_{457} \cong K_{127}$; and $I_{367} \cong I_{137} \cong K_{367}$. One has: $I_{457}$ (1463) $W_{19}, 124356$; $K_{127}$ (154263) $W_{27}, 124356$; $I_{137}$ (3654) or (13524) $W_{22}, 154362$; $K_{367}$ (12)(35) or (142365) $W_{22}, 154362$.

The refined line-walls may all be obtained from the set of 14 letter-triples (left column of Figure 41) moving 6 away from the conic 12345 in either direction. The adjacencies between configurations via refined line-walls are indicated in the left column of Figure 42, the right column displays one representant for each refined line-wall. The 11 line-walls from Figure 34 that admit conic-subwalls give rise to 22 refined line-walls. The total number of refined line-walls is $22 + (27 - 11) = 38$. □

Rational cubics and pencils of rational cubics were used in [1]-[3] to study the topology of real algebraic $M$-curves of degree 9. The combinatorics of generic pencils of cubics was studied in [4], for the particular case where eight of the base points lie in convex position.
| arrangement | adjacent line-walls          |
|-------------|-----------------------------|
| A           | 367, 257, 147               |
| B           | 367, 267, 257, 147, 157     |
| C           | 367, 267, 157               |
| D           | 157, 267, 167               |
| E           | 167                         |
| F           | 127, 567                    |
| G           | 467, 567, 127               |
| H           | 367, 457, 467, 127          |
| I           | 367, 457, 467, 137, 127     |
| J           | 467, 567, 137, 127          |
| K           | 457, 367, 127               |

Figure 40: Adjacent line-walls for the 11 arrangements with six coconic points

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| letter-triple | symmetries   | line-conic-walls | equivalent letter-triples |
|--------------|-------------|------------------|--------------------------|
| C267         | (1325)      | W4, 142536       | D267, D157               |
| G467         | (2453)(67)  | W4, 142537       | H467                     |
| B157         | (16342)     | W8, 142536       | B267, C157               |
| I467         | (245)       | W9, 143526       | J467, J137               |
| D167         | (176524)    | W12, 125743      | E167                     |
|              | (1543)(67)  |                  | F567, G567               |
| F127         | (16275)(34) | W12, 126743      |                          |
|              | (164)(253)  | W12, 125643      |                          |
| C367         | (354)       | W16, 164352      |                          |
|              | (14235)     |                  |                          |
| H367         | (354)(67)   | W16, 174352      |                          |
|              | (14235)(67) |                  |                          |
| G127         | (16)(25)    | W18, 124356      | J127, J567               |
| H127         | (16)(25)    | W19, 124356      | I127, I457               |
| B367         | (23654)     | W21, 154263      | B147                     |
|              | (14365)     |                  |                          |
| I367         | (13524)     | W22, 154362      | K367, I137               |
|              | (3654)      |                  |                          |
| A367         | (136524)    | W23, 164352      | A257, A147, B257         |
|              | (14235)     |                  |                          |
| H457         | (12456)     | W27, 124356      | K127, K457               |

Figure 41: The 14 line-conic-walls
(3,4,0,0)\_1 & W_{41} & (2,2,3,0)\_1 & (D,6)267 \\
\quad & W_{42} & (2,2,3,0)\_2 & (C',6)157 \\
(3,4,0,0)\_2 & W_{43} & (2,2,3,0)\_3 & (G,6)467 \\
(1,2,2,2) & W_{81} & (2,2,3,0)\_1 & (B,6)157 \\
\quad & W_{82} & (2,2,3,0)\_2 & (B,6)267 \\
(1,2,4,0) & W_{91} & (1,4,2,0) & (I,6)467 \\
\quad & W_{92} & (1,4,2,0) & (I,6)137 \\
(7,0,0,0) & W_{121} & (3,4,0,0)\_1 & (E,6)167 \\
\quad & W_{122} & (3,4,0,0)\_2 & (E,6)456 \\
(2,2,3,0)\_1 & W_{161} & (2,2,3,0)\_1 & (C,6)367 \\
(2,2,3,0)\_2 & W_{162} & (2,2,3,0)\_2 & (C',6)147 \\
(2,2,3,0)\_3 & W_{163} & (2,2,3,0)\_3 & (H,6)367 \\
(1,4,2,0) & W_{181} & (3,4,0,0)\_1 & (D,6)456 \\
\quad & W_{182} & (3,4,0,0)\_2 & (G,6)127 \\
(1,2,4,0) & W_{191} & (2,2,3,0)\_1 & (C,6)456 \\
\quad & W_{192} & (2,2,3,0)\_3 & (H,6)127 \\
(1,2,2,2) & W_{211} & (1,2,2,2) & (B,6)367 \\
\quad & W_{212} & (1,2,2,2) & (B,6)147 \\
(1,6,0,0) & W_{22} & (1,2,4,0) & (I,6)367 \\
(1,0,6,0) & W_{23} & (1,2,2,2) & (A,6)367 \\
(1,6,0,0) & W_{271} & (2,2,3,0)\_1 & (C,6)126 \\
\quad & W_{272} & (2,2,3,0)\_3 & (H,6)457 \\

Figure 42: Adjacencies via refined line-walls
Figure 43: Lists of seven nodal cubics
Figure 44: Lists of seven nodal cubics, continued
Figure 45: Lists of seven nodal cubics, end

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