ON THE K-THEORY OF COORDINATE AXES IN AFFINE SPACE

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1. INTRODUCTION

Let $k$ be a perfect field of characteristic $p > 0$ and let $A_d$ denote the $k$-algebra $k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$, which is the coordinate ring of the coordinate $(x_1, \ldots, x_d)$-axes in affine $d$-space $\mathbb{A}^d = \text{Spec}(k[x_1, \ldots, x_d])$ over $k$. This is an affine curve with a singularity at the origin, the singularity being defined by the ideal $I_d = (x_1, \ldots, x_d)$. Our main result is the computation of the relative algebraic $K$-theory, $K(A_d, I_d)$ of the pair $(A_d, I_d)$. The relative $K$-theory is defined to be the mapping fiber of the map $K(A_d) \to K(k)$ induced by the projection onto $k = A_d/I_d$.

To state the result we introduce some notation. We consider words in $d$ letters $x_1, \ldots, x_d$, i.e. a finite string $\omega = w_1 w_2 \ldots w_m$ where each $w_i$ is one of the letters $x_1, \ldots, x_d$. A word $\omega = w_1 w_2 \ldots w_m$ has no cyclic repetitions if $w_i \neq w_{i+1}$ for all $i = 1, \ldots, m-1$ and if $w_m \neq w_1$. For $d \geq 1$ and $s \geq 1$ let $\text{cyc}_d(s)$ denote the number of cyclic words in $d$ letters, of length $s$, period $s$, having no cyclic repetitions. In Section 7 we give a formula for $\text{cyc}_d(s)$. Suppose $m' \geq 2$ and let $t_{ev} = t_{ev}(p, r, m')$ be the unique positive integer such that $p^{t_{ev} - 1} m' \leq 2r < p^{t_{ev}} m'$, or zero if such $t_{ev}$ does not exist. Let $t_{od} = t_{od}(p, r, m')$ be the unique positive integer such that $p^{t_{od} - 1} m' \leq 2r + 1 < p^{t_{od}} m'$, or zero if such $t_{od}$ does not exist. Let $J_p$ denote the set of positive integers $m' \geq 2$ which are coprime to $p$.

**Theorem 1.1.** Let $k$ be a perfect field of characteristic $p > 0$. Let $A_d$ be the ring $k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$ of coordinate axes, and let $I_d = (x_1, \ldots, x_d)$. Then if $p > 2$

$$K_q(A_d, I_d) \cong \begin{cases} \prod_{m' \in J_p} \prod_{s|m'} \prod_{u \leq t_{ev}} W_{t_{ev} - u}(k)^{\oplus \text{cyc}_d(p^u s')} & q = 2r \\ \prod_{m' \in J_p} \prod_{s|m'} \prod_{u \leq t_{od}} W_{t_{od} - u}(k)^{\oplus \text{cyc}_d(p^u s')} & q = 2r + 1 \end{cases}$$
If $p = 2$ then
\[
K_q(A_d, I_d) \cong \begin{cases} 
\prod_{m' \geq 1} \prod_{s' | m'} \prod_{1 \leq u \leq t_{ev}} W_{t_{ev} - u} (k) \oplus \text{cyc}_d (2^u s') & q = 2r \\
\prod_{m' \geq 1} \prod_{s' | m'} \prod_{0 \leq v \leq t_{ev}} k \oplus \text{cyc}_d (s') & q = 2r + 1
\end{cases}
\]

In both cases $t_{ev} = t_{ev}(p, r, m')$ and $t_{od} = t_{od}(p, r, m')$ are as defined above.

Note that the products appearing in the statement are all finite since for $m'$ large enough $t_{ev} = t_{od} = 0$.

The result extends the calculation by Dennis and Krusemeyer when $q = 2$, [5, Theorem 4.9]. Hesselholt carried out the computation in the case $d = 2$ in [13]. Our strategy follows the one in [13], except that we use the framework for TC set up by Nikolaus and Scholze [20] to which we refer for background on TC and cyclotomic spectra. The computation is achieved through the use of the cyclotomic trace map from $K$-theory to topological cyclic homology, TC. See also [14] for background on cyclotomic spectra and for similar calculations. Recently, Hesselholt and Nikolaus [16] have completed a calculation for $K$-theory of cuspidal curves using similar methods as this paper.

1.1. Overview. In Section 2 we reduce the $K$-theory computation to a TC computation. Section 3 carries out the requisite THH computation. Then Section 4 and Section 5 assemble this to complete the proof. Section 4 contains a new method for computing TP, which makes crucial use of the Nikolaus-Scholze framework, see also [16]. In Section 6 we consider the characteristic zero situation and extend the computation of [8, Theorem 7.1.]. Finally in Section 7 we derived the necessary counting formula for cyclic words.

1.2. Acknowledgements. I am grateful to my advisor Lars Hesselholt for his guidance and support during the production of this paper. Special thanks are due to Fabien Pazuki for encouraging and useful conversations. I thank Benjamin Böhme, Ryo Horiuchi, Joshua Hunt, Mikala Jansen, Manuel Krannich, and Malte Leip for several useful conversations. It is a pleasure to thank Malte Leip for carefully reading a draft version of this paper and providing several corrections and suggestions.
2. Bi-relative K-theory and topological cyclic homology

The normalization of $A_d$ is just $d$ disjoint lines, whose coordinate ring is

$$B_d = k[x_1] \times \cdots \times k[x_d].$$

Gluing these lines together at $x_1 = x_2 = \cdots = x_d = 0$ one obtains $\text{Spec}(A_d)$. Algebraically this is the statement that the following square is a pullback of rings

$$\begin{array}{ccc}
A_d & \longrightarrow & k \\
\downarrow & & \downarrow \\
B_d & \longrightarrow & k^{\times d}
\end{array}$$

Here the horizontal maps take the variables $x_i$ to zero. The left-vertical map is the normalization map. It maps $x_i$ to $(0, \ldots, x_i, \ldots, 0)$ where $x_i$ is in the $i$'th position. The right-vertical map is the diagonal. If K-theory preserved pullbacks then the diagram would give a computation of $K(A_d)$ in terms of $K(B_d)$ and $K(k)$. Using the fundamental theorem of K-theory (since $k$ is regular) one would get a formula for $K(A_d)$ purely in terms of $K(k)$. But K-theory does not preserve pullbacks. However, there is still something to be done. We can form the bi-relative K-theory, $K(A_d, B_d, I_d)$, as the iterated mapping fiber of the diagram

$$\begin{array}{ccc}
K(A_d) & \longrightarrow & K(k) \\
\downarrow & & \downarrow \\
K(B_d) & \longrightarrow & K(k^{\times d})
\end{array}$$

Again, if K-theory preserved pullbacks then $K(A_d, B_d, I_d)$ would be trivial, but (as we shall see) it is not. Since $k$ is regular the fundamental theorem of algebraic K-theory [24, Section 6], and the fact that K-theory does preserve products, shows that the canonical map

$$K(A_d, B_d, I_d) \longrightarrow K(A_d, I_d)$$

is an equivalence. Here $K(A_d, I_d)$ is the relative K-theory spectrum, i.e. the mapping fiber of the map $K(A_d) \to K(A_d/I_d)$ induced by the quotient. Any splitting of the quotient map $A_d \to A_d/I_d$ provides a splitting of
So if we can compute $K_q(A_d, I_d) = K_q(A_d, B_d, I_d)$ we have a computation of $K_q(A_d)$. This is what we will do.

Geisser and Hesselholt [6] have shown that the cyclotomic trace induces an isomorphism

$$K_q(A, B, I, \mathbb{Z}/p^v) \overset{\sim}{\longrightarrow} TC_q(A, B, I; p; \mathbb{Z}/p^v)$$

between the bi-relative $K$-theory and the bi-relative topological cyclic homology, when both are considered with $\mathbb{Z}/p^v$ coefficients. The corresponding statement with rational coefficients (using Connes-Tsygan negative cyclic homology and the Chern character) was proven by Cortiñas in [3].

In fact it suffices to compute $TC(A_d, B_d, I_d; p; \mathbb{Z}_p)$ since the trace map

$$K(A, B, I) \to TC(A, B, I; p; \mathbb{Z}_p)$$

is an equivalence whenever $p$ is nilpotent in $A$, as shown in [3, Theorem C]. Furthermore, since $THH(A_d, B_d, I_d; p)$ is an $Hk$-module, it is in particular $p$-complete, and so $TC(A, B, I, p; \mathbb{Z}_p) \simeq TC(A, B, I, p)$ (see [20, Section II.4]). Thus, to prove Theorem 1.1 it suffices to prove the following result.

**Theorem 2.1.** Let $k$ be a perfect field of characteristic $p > 0$. Let $A_d$ be the ring $k[x_1, \ldots, x_d]/(x_i, x_j)_{i \neq j}$ of coordinate axes, $B_d = k[x_1] \times \cdots \times k[x_d]$ and let $I_d = (x_1, \ldots, x_d)$. Then if $p > 2$,

$$TC_q(A_d, B_d, I_d) \cong \begin{cases} \prod_{m' \in T} \prod_{s' \in T} \prod_{u \leq \nu} W_{t_{ev}}(k) \oplus \text{cyc}_d(p^v s') & q = 2r \\ \prod_{m' \in T} \prod_{s' \in T} \prod_{u \leq \nu} W_{t_{od}}(k) \oplus \text{cyc}_d(p^v s') & q = 2r + 1 \end{cases}$$

If $p = 2$ then

$$TC_q(A_d, B_d, I_d) \cong \begin{cases} \prod_{m' \in T} \prod_{s' \in T} \prod_{u \leq \nu} W_{t_{ev}}(k) \oplus \text{cyc}_d(2^v s') & q = 2r \\ \prod_{m' \in T} \prod_{s' \in T} \prod_{u \leq \nu} W_{t_{od}}(k) \oplus \text{cyc}_d(s') & q = 2r + 1 \end{cases}$$

3. THH and the cyclic bar construction

The spectrum $TC(A_d, B_d, I_d)$ is defined using topological Hochschild homology, so that is where we start. In this section we drop the subscript $d$, so that $A = A_d$, $B = B_d$ and $I = I_d$. The bi-relative topological Hochschild
homology is the spectrum $\text{THH}(A,B,I)$ defined as the iterated mapping fiber of the following diagram.

$$
\begin{array}{ccc}
\text{THH}(A) & \longrightarrow & \text{THH}(A/I) \\
\downarrow & & \downarrow \\
\text{THH}(B) & \longrightarrow & \text{THH}(B/I)
\end{array}
$$

(1)

The ring $A$ is a pointed monoid ring and, by Lemma 3.1 below, we may compute $\text{THH}(A)$ in terms of the cyclic bar construction of the defining pointed monoid.

### 3.1. Unstable cyclotomic structure on the cyclic bar construction.

Let $\Pi$ be a pointed monoid, that is a monoid object in the symmetric monoidal category of based spaces and smash product. The cyclic bar construction of $\Pi$ is the cyclic space $B^{\text{cy}}(\Pi)[-]$ with

$$B^{\text{cy}}(\Pi)[k] = \Pi^{(k+1)}$$

and with the usual Hochschild-type structure maps.

$$
\begin{align*}
\delta_1(\pi_0 \wedge \cdots \wedge \pi_m) &= \begin{cases} 
\pi_0 \wedge \cdots \wedge \pi_i \pi_{i+1} \wedge \cdots \wedge \pi_m & 0 \leq i < m \\
\pi_m \pi_0 \wedge \pi_1 \wedge \cdots \wedge \pi_{m-1} & i = m
\end{cases} \\
\sigma_i(\pi_0 \wedge \cdots \wedge \pi_m) &= \pi_0 \wedge \cdots \wedge \pi_i \wedge \pi_{i+1} \wedge \cdots \wedge \pi_m \\
t_m(\pi_0 \wedge \cdots \wedge \pi_m) &= \pi_m \wedge \pi_0 \wedge \cdots \wedge \pi_{m-1}
\end{align*}
$$

We write $B^{\text{cy}}(\Pi)$ for the geometric realization of $B^{\text{cy}}(\Pi)[-]$. It is a space with $T$-action where $T$ is the circle group (for a proof see for example [19, Theorem 7.1.4]). Furthermore it is an unstable cyclotomic space, i.e. there is a map

$$\psi_p : B^{\text{cy}}(\Pi) \to B^{\text{cy}}(\Pi)^{C_p}$$

which is equivariant when the domain is given the natural $T/C_p$-action. This map goes back to section 2. We briefly sketch the construction. The $C_p$-action on $B^{\text{cy}}(\Pi)$ is not simplicial, but we can make it so by using the edge-wise subdivision functor $sd_p : \Delta \to \Delta$ which is given by the $p$-fold concatenation, $sd_p[m-1] = [m-1] \sqcup \cdots \sqcup [m-1]$ and $sd_p(\theta) = \theta \sqcup \cdots \sqcup \theta$ for morphisms $\theta : [m-1] \to [n-1]$. Given a simplicial set $X[-]$ we
let $sd_p X[-] = X[-] \circ sd^p_\Sigma$. For the topological simplex $\Delta^{m-1} \subset \mathbb{R}^m$ let $d_p : \Delta^{m-1} \to \Delta^{pm-1}$ be the diagonal embedding

$$d_p(z) = \frac{1}{p} z \oplus \cdots \oplus \frac{1}{p} z.$$ 

This induces a (non-simplicial) map on geometric realization

$$D_p : |(sd_p X)[-]| \to |X[-]|$$

by $id \times d_p : X[pm-1] \times \Delta^{m-1} \to X[pm-1] \times \Delta^{pm-1}$. Then [2, Lemma 1.1.] shows that $D_p$ is a homeomorphism – one need only check on the representables $\Delta^k[-]$ where it follows by explicit calculations. In the case $X[-] = B^{\Sigma}(\Pi)[-]$ one has available the simplicial diagonal

$$\bar{\Delta}_p : B^{\Sigma}(\Pi)[-] \to sd_p B^{\Sigma}(\Pi)[-]$$

given by

$$\pi_0 \wedge \cdots \wedge \pi_m \mapsto \pi_0 \wedge \cdots \wedge \pi_m \quad (p \text{ copies})$$

It clearly lands in the $C_p$-fixed points of $sd_p B^{\Sigma}(\Pi)[-]$ and induces an isomorphism $\bar{\Delta}_p : B^{\Sigma}(\Pi)[-] \to sd_p B^{\Sigma}(\Pi)[-]^{C_p}$. We define

$$\psi_p : B^{\Sigma}(\Pi) \to B^{\Sigma}(\Pi)^{C_p}$$

to be the composite

$$B^{\Sigma}(\Pi) = |B^{\Sigma}(\Pi)[-]| \xrightarrow{\bar{\Delta}_p} |sd_p B^{\Sigma}(\Pi)[-]^{C_p}| \xrightarrow{\cong} |sd_p B^{\Sigma}(\Pi)[-]^{C_p}|$$
$$\xrightarrow{D_p} |B^{\Sigma}(\Pi)[-]^{C_p} = B^{\Sigma}(\Pi)^{C_p}|$$

where the middle map is the canonical equivalence witnessing the fact that geometric realization commutes with finite limits.

Passing to suspension spectra now gives us a cyclotomic structure on $\text{THH}(S(\Pi))$ where $S(\Pi) = \Sigma^\infty B^{\Sigma}(\Pi)$. Indeed, there is always a $\mathbb{T}/C_p$-equivariant map $B^{\Sigma}(\Pi)^{C_p} \to B^{\Sigma}(\Pi)^{hC_p}$. Composing with this map gives a $\mathbb{T} \simeq \mathbb{T}/C_p$-equivariant map $\psi^h : B^{\Sigma}(\Pi) \to B^{\Sigma}(\Pi)^{hC_p}$. Thus, we obtain a
\[ \mathbb{T} \cong \mathbb{T}/C_p \text{-equivariant map} \]

\[
\begin{align*}
\text{THH}(S(\Pi)) &= \Sigma^\infty \text{B}^\text{sy}(\Pi) \\
\xrightarrow{\psi h} & \quad \Sigma^\infty \text{B}^\text{sy}(\Pi)^{hC_p} \\
\xrightarrow{\text{can}} & \quad (\Sigma^\infty \text{B}^\text{sy}(\Pi))^{tC_p} = \text{THH}(S(\Pi))^{tC_p}
\end{align*}
\]

where the middle map is the canonical map from the suspension spectrum of a homotopy limit, to the homotopy limit of the suspension spectrum and the last map is the canonical map from the homotopy fixed points to the Tate construction.

### 3.2. THH of monoid algebras

Both of the algebras \( A \) and \( B \) (and their quotients by \( I \)) are pointed monoid algebras, enabling us to use the following splitting result.

**Lemma 3.1.** [14, Theorem 5.1.] Let \( k \) be a ring, \( \Pi \) a pointed monoid, and \( k(\Pi) \) the pointed monoid algebra. Let \( \text{B}^\text{sy}(\Pi) \) be the cyclic bar-construction on \( \Pi \). Then there is a \( \mathbb{T} \)-equivariant equivalence

\[ \text{THH}(k) \otimes \text{B}^\text{sy}(\Pi) \xrightarrow{\sim} \text{THH}(k(\Pi)). \]

Under this equivalence, the Frobenius on \( \text{THH}(k(\Pi)) \) is identified with the tensor product of the Frobenius on \( \text{THH}(k) \) with the one on \( \text{THH}(S(\Pi)) \).

**Proof.** Since \( \text{THH} \) is symmetric monoidal [20, Section IV.2.] we obtain

\[ \text{THH}(k(\Pi)) = \text{THH}(k \otimes S(\Pi)) \cong \text{THH}(k) \otimes \text{THH}(S(\Pi)). \]

Since \( \text{THH}(S(\Pi)) = S(\text{B}^\text{sy}(\Pi)) \) we obtain

\[ \text{THH}(k(\Pi)) \cong \text{THH}(k) \otimes S(\text{B}^\text{sy}(\Pi)) \]

as claimed. \( \square \)

Let \( \Pi^d = \{0, 1, x_1, x_1^2, \ldots, x_2, x_2^2, \ldots, x_d, x_d^2, \ldots\} \) be the multiplicative monoid with base-point 0 and multiplication determined by \( x_i x_j = 0 \) when \( i \neq j \). Then \( A_d \cong k(\Pi^d) \). Let \( \Pi^1 = \{0, 1, t, t^2, \ldots\} \) and \( \Pi^0 = \{0, 1\} \), so \( B = k(\Pi^1) \times \cdots \times k(\Pi^1) \) (with \( d \) products) and \( k = k(\Pi^0) \). The diagram
of cyclotomic spectra $(1)$ is induced by the diagram of pointed monoids

$$
\begin{array}{ccc}
\Pi^d & \xrightarrow{\varepsilon} & \Pi^0 \\
\downarrow (\varphi_1, \ldots, \varphi_d) & & \downarrow \Delta \\
\Pi^1 \times \cdots \times \Pi^1 & \xrightarrow{\epsilon^{x_d}} & \Pi^0 \times \cdots \times \Pi^0
\end{array}
$$

The map $\varphi_j$ takes $x_i$ and to 0 when $i \neq j$ and takes $x_j$ to $t$. The map $\varepsilon$ takes the variables $x_1, \ldots, x_d$ and $t$ to 0. The map $\Delta$ is the diagonal.

The cyclic bar-construction sometimes decomposes, as a pointed $T$-space, into a wedge of spaces simple enough that one can understand their $T$-homotopy type. To do this for $B^\gamma(\Pi^d)$ we need the notion of cyclic words.

We consider the alphabet $S = \{x_1, \ldots, x_d\}$ and words

$$\omega : \{1, 2, \ldots, m\} \to S.$$ 

Here $m$ is the length of $\omega$. The cyclic group $C_m$ acts on the set of words of length $m$. An orbit for this action is called a cyclic word. Such a cyclic word $\overline{\omega}$ has a period namely the cardinality of the orbit. Words may be concatenated to give new, longer words, though concatenation is not well-defined for cyclic words. The empty word $\emptyset \to S$ is the unit for concatenation. It has length 0 and period 1.

We can associate words to non-zero elements of $B^\gamma(\Pi^d)[m]$ as follows. If $\pi \in \Pi^d = B^\gamma(\Pi^d)[0]$ is non-zero then it is of the form $\pi = x_j^l$ for some $1 \leq j \leq d$ and $l \geq 0$. Let $\omega(\pi)$ be the unique word of length $l$ all of whose letters are $x_j$. For example $\omega(x_1^2) = x_1x_1$ and $\omega(1) = \emptyset$. Now for a non-zero element $\pi_0 \wedge \cdots \wedge \pi_m \in B^\gamma(\Pi^d)[m]$ we let

$$\omega(\pi_0 \wedge \cdots \wedge \pi_m) = \omega(\pi_0) \star \cdots \star \omega(\pi_m)$$

be the concatenation of each of the words $\omega(\pi_j)$. Note that $\omega(1)$ is the empty word, which is the unit for concatenation. For a cyclic word $\overline{\omega}$ we define

$$B^\gamma(\Pi^d, \overline{\omega})[m] \subseteq B^\gamma(\Pi^d)[m]$$

to be the subset consisting of the base-point and all elements $\pi_0 \wedge \cdots \wedge \pi_m$ such that

$$\omega(\pi_0 \wedge \cdots \wedge \pi_m) \in \overline{\omega}.$$
The cyclic structure maps preserve this property and so, as \( m \geq 0 \) varies, this defines a cyclic subset
\[
B^\gamma(\Pi^d, \overline{\omega})[-] \subseteq B^\gamma(\Pi^d)[-].
\]
Denote by \( B^\gamma(\Pi^d, \overline{\omega}) \) the geometric realization of this subset. We will also often abbreviate \( B(\overline{\omega}) = B^\gamma(\Pi^b, \overline{\omega}) \). As \( \overline{\omega} \) ranges over all cyclic words every non-zero \( m \)-simplex \( \pi_0 \wedge \cdots \wedge \pi_m \) appears in exactly one such cyclic subset \( B^\gamma(\Pi^d, \overline{\omega}) \). Thus we get a decomposition
\[
B^\gamma(\Pi^d) = \bigvee B^\gamma(\Pi^d, \overline{\omega})
\]
indexed on the set of all cyclic words with letters in \( S = \{x_1, \ldots, x_d\} \).

**Lemma 3.2.** There is a canonical \( \mathcal{T} \)-equivariant equivalence
\[
\bigoplus \text{THH}(k) \otimes B^\gamma(\Pi^d, \overline{\omega}) \overset{\sim}{\to} \text{THH}(A, B, I)
\]
where the sum on the left-hand side is indexed over all cyclic words whose period is \( \geq 2 \).

**Proof.** We consider the diagram induced from (1) using Lemma 3.1.
\[
\begin{array}{ccc}
\text{THH}(k) \otimes B^\gamma(\Pi^d) & \overset{\varepsilon}{\longrightarrow} & \text{THH}(k) \otimes B^\gamma(\Pi^0) \\
\downarrow (\varphi_1, \ldots, \varphi_d) & & \downarrow \Delta \\
(\text{THH}(k) \otimes B^\gamma(\Pi^1)) \times d & \overset{\varepsilon \times d}{\longrightarrow} & (\text{THH}(k) \otimes B^\gamma(\Pi^0)) \times d
\end{array}
\]

\( \text{THH}(A, B, I) \) is the iterated mapping fiber of this diagram. The mapping fiber of the left-hand vertical map consists of two parts; the part of the wedge sum indexed on cyclic words containing at least two different letters from \( S \) and a part on which \( \varepsilon \) is an equivalence. The map \( \varepsilon \) is trivial on this part indexed on cyclic words containing at least two different letters, finishing the claim. \( \square \)

### 3.3. Homotopy type of \( B^\gamma(\Pi^d, \overline{\omega}) \)
In this section we determine the homotopy type of the subspaces \( B^\gamma(\Pi^d, \overline{\omega}) \subseteq B^\gamma(\Pi^d) \).

**Definition 3.1.** We say that a word \( \omega = w_1w_2\ldots w_m \) has no cyclic repetitions if \( w_i \neq w_{i+1} \) for all \( i = 0, 1, \ldots, m-1 \) and if \( w_m \neq w_1 \). If on the other hand this is not satisfied, we say \( \omega \) (or \( \overline{\omega} \)) has cyclic repetitions.
Lemma 3.3. Let \( \omega \) be a cyclic word of period \( s \geq 2 \), with letters in the alphabet \( S = \{ x_1, \ldots, x_d \} \). The homotopy type of the pointed \( \mathbb{T} \)-space \( \mathbb{B}^\mathbb{C}(\Pi^d, \omega) \) is given as follows.

(1) If \( \omega \) has length \( m = si \) and has no cyclic repetitions then a choice of representative word \( \omega \in \omega \) determines a \( \mathbb{T} \)-equivariant homeomorphism

\[
S^{R[C_m]^{-1}} \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor \lor 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ON THE K-THEORY OF COORDINATE AXES IN AFFINE SPACE

Now we use the identification of the $C_m$-space $\Delta^{m-1}/\partial\Delta^{m-1}$ with the one-point compactification of the reduced regular representation to finish the proof of part (1).

In case $\omega$ does have a cyclic repetition, then $\pi_0 \wedge \cdots \wedge \pi_{m-1}$ will have a non-base point face. So $f_\omega$ collapses at least one codimension 1 face (and its $T$-orbit) to the base-point, and leaves at least one codimension 1 face, say $F \subseteq \Delta^{m-1}$, un-collapsed. The cone on $F$ is canonically homeomorphic to $\Delta^{m-1}$ and has a canonical null-homotopy given by shrinking down to the basepoint of the cone. This null-homotopy then induces a null-homotopy on $B^{cy}(\Pi^d, \omega)$. However it may not be a $T$-equivariant null-homotopy. To get this we note that there is a $C_i$-equivariant homeomorphism

$$\Delta^{s-1} \ast \cdots \ast \Delta^{s-1} \to \Delta^{m-1}$$

where $C_i$ acts on the left by cyclically permuting the factors of the join. Again $f_\omega$ will collapse at least one codimension 1 face of $\Delta^{s-1}$ and leave at least one un-collapsed, say $F \subseteq \Delta^{s-1}$. Now the null-homotopy of $cone(F)$ will induce a $T$-equivariant null-homotopy $B^{cy}(\Pi^d, \omega) \wedge [0, 1]_+ \to B^{cy}(\Pi^d, \omega)$.

This completes the proof. □

Let $C(i)$ denote the 1-dimensional complex $T$-representation where $z \in T \subseteq C$ acts through multiplication with the $i$’th power $z^i$. For $i \geq 1$ let $\lambda_i = C(1) \oplus \cdots \oplus C(i)$.

**Lemma 3.4.** The regular representation $R[C_m]$ is isomorphic to $R \oplus \lambda_{\frac{m-2}{2}} \oplus R_-$ if $m$ is even, and $R \oplus \lambda_{\frac{m-1}{2}}$ if $m$ is odd.

**Proof.** By Maschke’s theorem $R[C_m]$ is a semisimple ring. In particular it decomposes as a sum of irreducible sub-representations. Furthermore it contains a copy of every irreducible $C_m$-representation (since for any such $V$ and any non-zero $v \in V$, there is a surjection $R[C_m] \to V$ given by $\sum \lambda g \mapsto \sum \lambda g v$ so, by semisimplicity, $V$ embeds into $R[C_m]$). Since $R$ and $C(i)$, for $1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$, (and $R_-$ in case $m$ is even) are irreducible, pair-wise non-isomorphic, and have real dimensions summing to $\dim_R R[C_m] = m$ this completes the proof. □

**Lemma 3.5.** Let $\omega$ be a cyclic word with no cyclic repetitions, of length $m$, period $s$ and with $i = \frac{m}{s}$ blocks.
(1) If \( s \) is even, then there is a \( \mathcal{T} \)-equivariant homeomorphism

\[
\Sigma B(\omega) \simeq S^{\lambda_{m/2}} \otimes (\mathcal{T}/C_i)_+
\]

(2) If both \( s \) and \( i \) are odd, then there is a \( \mathcal{T} \)-equivariant homeomorphism

\[
B(\omega) \simeq S^{\lambda_{(m-1)/2}} \otimes (\mathcal{T}/C_i)_+
\]

(3) If \( s \) is odd, and \( i \) is even, then there is a \( \mathcal{T} \)-equivariant homeomorphism

\[
B(\omega) \simeq S^{\lambda_{(m-2)/2}} \otimes R^2(i)
\]

where \( R^2(i) \) is by definition the mapping cone of the quotient map \( \mathcal{T}/C_{i/2} \to \mathcal{T}/C_{i+}. \)

Proof. We use Lemma 3.4 in each case.

(1) If \( s \) is even, then \( s = 2k \) is even so is \( m \) and so

\[
R[C_m] - 1 \cong \lambda_{(m-2)/2} \oplus R._-
\]

The restriction along the inclusion \( C_i \subseteq C_m \) (given by \( \sigma_i \mapsto \sigma_m^j \) where \( \sigma_j \) is a generator) turns the sign representation \( R._- \) into a trivial representation. Since \( C(\frac{m}{2}) = C(ki) = R \oplus R \) as \( C_i \)-representations, we have

\[
S^{R[C_m]-1} \wedge C_i \mathcal{T} \cong S^{-1} \wedge S^{\lambda_{m/2}} \wedge C_i \mathcal{T} \cong S^{-1} \wedge S^{\lambda_{m/2}} \wedge (\mathcal{T}/C_i)_+
\]

where the last isomorphism uses that \( \lambda_{m/2} \) extends to a representation of \( \mathcal{T} \) and so allows the \( \mathcal{T} \)-equivariant untwisting map \( (x, z) \mapsto (xz, zC_i) \).

(2) If both \( s \) and \( i \) are odd then so is \( m = si \) and so

\[
R[C_m] - 1 = \lambda_{(m-1)/2}.
\]

Then we proceed as above, using the untwisting map.

(3) If \( s \) is odd and \( i \) is even, then \( m \) is even and so

\[
R[C_m] - 1 = \lambda_{(m-2)/2} \oplus R._-
\]

The restriction along the inclusion \( C_i \subseteq C_m \) leaves the sign representation unchanged. Now repeat the argument of [14, Cor. 7.2].
3.4. Homology of $B^o(\Pi^d;\overline{\omega})$.

**Proposition 3.6.** Let $R$ be any commutative ring. Let $\overline{\omega}$ be a cyclic word with no cyclic repetitions, of length $m$, period $s$ and with $i = m/s$ blocks. The singular homology $\tilde{H}_s(B(\overline{\omega}); R)$ is concentrated in degrees $m-1$ and $m$. If either $s$ is even, or both $s$ and $i$ are odd, then the $R$-modules in degree $m-1$ and $m$ are free of rank $1$. If $s$ is odd and $i$ is even, then the $R$-module in degree $m-1$ is isomorphic to $R/2R$, and the $R$-module in degree $m$ is isomorphic to $2R$.

Furthermore, when $m$ and $s$ have the same parity, Connes’ operator takes a generator $y_{\overline{\omega}}$ of the $R$-module $\tilde{H}_{m-1}(B(\overline{\omega}); R)$ to $i$ times a generator $z_{\overline{\omega}}$ of $\tilde{H}_m(B(\overline{\omega}); R)$, that is $d(y_{\overline{\omega}}) = iz_{\overline{\omega}}$. When $s$ is odd and $i$ is even, Connes’ operator acts trivially.

**Proof.** The homology computations follow directly from Lemma 3.5. We also deduce the behaviour of Connes’ operator from Lemma 3.5. In general for a pointed $T$-space $X$, Connes’ operator $d : \tilde{H}_s(X;k) \to \tilde{H}_{s+1}(K;k)$ is given by taking the cross-product with the fundamental class $[T]$ and then applying the map induced by the action $\mu : T_+ \wedge X \to X$ on reduced homology. We consider two cases.

1. When $X = T/C_i$ we claim that $d : \tilde{H}_0(X;k) \to \tilde{H}_1(X;k)$ is multiplication by $i$ (up to a unit). This follows from the fact that the map

$$S^1 \to T \times T/C_i \to T/C_i$$

is a degree $i$ map.

2. When $X = RP^2(i)$ we claim that $d : \tilde{H}_1(X;k) \to \tilde{H}_2(X;k)$ is trivial. Consider the diagram

$$
\begin{array}{ccc}
T_+ \wedge (T/C_i)_+ & \longrightarrow & (T/C_i)_+ \\
\downarrow{id \wedge q} & & \downarrow{q} \\
T_+ \wedge RP^2(i) & \longrightarrow & RP^2(i)
\end{array}
$$

Now $H_1((T/C_i)_+)$ surjects onto $H_1(RP^2(i))$ and the above diagram commutes. Since $H_2((T/C_i)_+) = 0$ the claim follows.

$\square$

**Remark 3.1.** From this proposition we see that if the characteristic of $k$ is different $2$ then $\tilde{H}_s(B(\overline{\omega};k)$ is trivial when $s$ is odd and $i$ is even. On the
other hand, if $k$ has characteristic 2 then $k/2k = 2k = k$. This explains why the combinatorics of Geller, Reid, and Weibel in [8], avoids cyclic words whose period is not congruent (mod 2) to the length, [8, Remark 3.9.1.], since they work over characteristic zero fields.

3.5. THH of the coordinate axes. We put together the various results of the previous sections to describe $\text{THH}(A, B, I)$ as a cyclotomic spectrum. Again, let $A = A_d, B = B_d$ and $I = I_d$.

We will use the Segal conjecture for $C_p$. This is the statement that the map

$$S \to S^{tC_p}$$

identifies the codomain as the $p$-completion of the domain. For a proof of this see [20, Theorem III.1.7], though the result was originally proved by Lin [18] (for $p = 2$) and Gunawardena [10] (for $p$ odd) in 1980.

**Lemma 3.7.** Let $T$ be a bounded below spectrum with $C_p$-action and $X$ a finite pointed $C_p$-CW-complex. Then the lax symmetric monoidal structure map

$$T^{tC_p} \otimes (\Sigma^\infty X)^{tC_p} \longrightarrow (T \otimes \Sigma^\infty X)^{tC_p}$$

is an equivalence.

**Proof.** Since both $T^{tC_p} \otimes (-)^{tC_p}$ and $(T \otimes -)^{tC_p}$ are exact functors we may reduce to checking the statement for the $C_p$-spectra $S$ and $S \otimes C_{p^+}$. This is because $\Sigma^\infty X$ may be constructed from $S$ and $S \otimes C_{p^+}$ using finitely many cofiber sequences (since $X$ is built by attaching finitely many $C_p$-cells). Replacing $\Sigma^\infty X$ by $S$ the map in question reduces to

$$T^{tC_p} \otimes S_p \longrightarrow T^{tC_p}$$

where we use the Segal conjecture to identify $S^{tC_p} \simeq S_p$. Since $T$ is bounded below it follows from [20, Lemma I.2.9]) that $T^{tC_p}$ is $p$-complete and so the map is an equivalence. Replacing $\Sigma^\infty X$ by $S \otimes C_{p^+}$ instead we see that both the domain and codomain of the map are zero, since $(-)^{tC_p}$ kills induced spectra as well as spectra of the form $T \otimes Z$ where $Z$ is induced (cf. [20, Lemma I.3.8. (i) and (ii)]).

We will describe the Frobenius map on

$$\text{THH}(k(\Pi^d)) \simeq \text{THH}(k) \otimes B^\Sigma(\Pi^d)$$
in terms of the splitting of $B^{\Sigma Y}(\Pi^d)$ into the $T$-equivariant subspaces $B(\overline{\omega})$. The unstable Frobenius $\psi_p : B^{\Sigma Y}(\Pi^d) \to B^{\Sigma Y}(\Pi^d)^C_p$ (defined in Section 3.1) restricts to a homeomorphism

$$\psi_p : B(\overline{\omega}) \to B(\overline{\omega^{\ast p}})^C_p$$

landing in the subspace $B(\overline{\omega^{\ast p}})$ corresponding to the cyclic word $\overline{\omega^{\ast p}}$ which has length $pm$ and period $s$ (if $\overline{\omega}$ has length $m$ and period $s$).

**Proposition 3.8.** There is a $T$-equivariant equivalence of spectra

$$\text{THH}(A, B, I) \simeq \bigoplus \text{THH}(k) \otimes B(\overline{\omega})$$

where the sum is indexed over cyclic words of length $m \geq 2$, having no cyclic repetitions. Under this equivalence the Frobenius map restricts to the map

$$\text{THH}(k) \otimes B(\overline{\omega}) \xrightarrow{\psi_p \otimes \Phi_p} \text{THH}(k)^C_p \otimes B(\overline{\omega^{\ast p}})^C_p \xrightarrow{\text{THH}(k) \otimes B(\overline{\omega^{\ast p}})} (\text{THH}(k) \otimes B(\overline{\omega^{\ast p}}))^C_p$$

where the second map is the lax symmetric monoidal structure of the Tate-$C_p$-construction. This second map is an equivalence, while the restricted Frobenius $\tilde{\Phi} : \Sigma^\infty B(\overline{\omega}) \to (\Sigma^\infty B(\overline{\omega^{\ast p}}))^C_p$ is a $p$-adic equivalence.

**Proof.** Applying Lemma 3.7 with $T = \text{THH}(k)$ and $X = B(\overline{\omega^{\ast p}})$ we get that the map

$$\text{THH}(k)^C_p \otimes B(\overline{\omega^{\ast p}})^C_p \xrightarrow{\psi_p \otimes \Phi_p} (\text{THH}(k) \otimes B(\overline{\omega^{\ast p}}))^C_p$$

is an equivalence. It remains to show that

$$\tilde{\Phi}_p : S \otimes B(\overline{\omega}) \to (S \otimes B(\overline{\omega^{\ast p}}))^C_p$$

is a $p$-adic equivalence. To do this we factor it as follows. To ease notation, let $Y = |sd_p B(\overline{\omega^{\ast p}})|$. By definition $\tilde{\Phi}_p$ factors as

$$S \otimes B(\overline{\omega}) \xrightarrow{\Delta_p} S \otimes Y^C_p \xrightarrow{\gamma} (S \otimes Y)^C_p \xrightarrow{D_p} (S \otimes B(\overline{\omega^{\ast p}}))^C_p$$

where $\Delta_p$ (the space-level diagonal) and $D_p$ are a homeomorphisms as remarked in Section 3.1. The middle map is the composition

$$\tilde{\Phi}'_p : S \otimes Y^C_p \xrightarrow{\gamma} S \otimes Y^{hc}_p \xrightarrow{\text{can}} (S \otimes Y)^{hc}_p \xrightarrow{\text{can}} (S \otimes Y)^C_p$$

This map fits into the following commutative diagram
Here $\Delta_p : S \to \Sigma^{tC_p} S$ is a $p$-adic equivalence by the Segal conjecture. The map labelled (1.) is the equivalence witnessing that $(-)^{tC_p}$ is an exact functor, and $sd_p B(\omega^{tC_p})^{tC_p}$ is finite and has trivial $C_p$-action. The left-most square commutes by construction of the map (1.). Indeed by exactness in the variable $Y^{C_p}$ (a spectrum with trivial $C_p$-action) one reduces to the case $Y^{C_p} = S$ where the square becomes

$$
\begin{array}{c}
S \\
\downarrow \Delta_p \\
\Sigma^{tC_p} S
\end{array} 
\xrightarrow{(1.)} 
\begin{array}{c}
S^{hC_p} \\
\downarrow \text{can} \\
\Sigma^{tC_p} S
\end{array}
$$

The map labelled (2.) is the equivalence witnessing that the inclusion of the $C_p$-singular set $B^{C_p} \subseteq B$ induces an equivalence

$$(T \otimes B^{C_p})^{tC_p} \to (T \otimes B)^{tC_p}$$

for any $C_p$-spectrum $T$ and finite $C_p$-CW-complex $B$, cf. [14, Lemma 9.1]. The right-most triangle (with (2.) as a side) commutes since can is natural.

Finally, in the top triangle, the map $S \otimes Y^{C_p} \to (S \otimes Y^{C_p})^{hC_p}$ arises since $S \otimes Y^{C_p}$ has trivial $C_p$-action. By the universal property of $(S \otimes Y)^{hC_p}$ it follows that the following diagram commutes.

$$
\begin{array}{c}
S \otimes Y^{C_p} \\
\downarrow \\
S \otimes Y^{hC_p} \\
\downarrow \\
(S \otimes Y)^{hC_p}
\end{array} 
\xrightarrow{(\text{2.})} 
\begin{array}{c}
(S \otimes Y)^{hC_p} \\
\downarrow \\
(\Sigma Y^{C_p})^{hC_p}
\end{array}
$$

This completes the proof. \qed
I thank Malte Leip for abundant help with the above argument.

**Corollary 3.1.** Let $\overline{\omega}$ be a cyclic word having no cyclic repetitions and length $m$. The map

$$\varphi : \text{THH}(k) \otimes B(\overline{\omega}) \to (\text{THH}(k) \otimes B(\overline{\omega}^p))^{tC_p}$$

induces an isomorphism on homotopy groups in degrees greater than or equal to $m$.

**Proof.** The Frobenius map $\varphi : \text{THH}(k) \to \text{THH}(k)^{tC_p}$ induces an isomorphism on non-negative homotopy groups. This is shown in [20, Prop. IV. 4.13] for $k = \mathbb{F}_p$, and in [14, Section 5.5] for $k$ a perfect field of characteristic $p$. The result now follows from Proposition 3.6 and Proposition 3.8 as can be shown for example with the Atiyah-Hirzebruch spectral sequence. □

By Proposition 3.6 the homology of $B(\overline{\omega})$ depends only on the length and period of $\overline{\omega}$. For $\overline{\omega}$ of length $m$ and period $s$ we therefore introduce the notation $B(m, s) = B(\overline{\omega})$. The function $\text{cyc}_d(s)$ counts how many cyclic words with no cyclic repetitions, of length $s$, and period $s$, there are. For a fixed $m$ there are $\text{cyc}_d(s)$ many cyclic words with no cyclic repetitions, of length $m$, and period $s$. Since the homology of $B(m, s)$ depends only on the parity of $m$ and $s$ (by Proposition 3.6) we may rewrite Proposition 3.8 as follows; if $p > 2$ then

$$\text{THH}(A, B, I) \simeq \bigoplus_{m \geq 2 \text{ even}} \bigoplus_{s|m \text{ even}} \bigoplus_{d} (\text{THH}(k) \otimes B(m, s)) \oplus \text{cyc}_d(s)$$

$$\bigoplus_{m \geq 2 \text{ odd}} \bigoplus_{s|m \text{ odd}} \bigoplus_{d} (\text{THH}(k) \otimes B(m, s)) \oplus \text{cyc}_d(s).$$

If $p = 2$ then we add the similar double sum indexed over $m$ even and $s$ odd, i.e.

$$\text{THH}(A, B, I) \simeq \bigoplus_{m \geq 2 \text{ even}} \bigoplus_{s|m \text{ even}} \bigoplus_{d} (\text{THH}(k) \otimes B(m, s)) \oplus \text{cyc}_d(s)$$

$$\bigoplus_{m \geq 2 \text{ odd}} \bigoplus_{s|m \text{ odd}} \bigoplus_{d} (\text{THH}(k) \otimes B(m, s)) \oplus \text{cyc}_d(s)$$

$$\bigoplus_{m \geq 2 \text{ even}} \bigoplus_{s|m \text{ odd}} \bigoplus_{d} (\text{THH}(k) \otimes B(m, s)) \oplus \text{cyc}_d(s).$$
Note that we do not need to know the homotopy-type of $B(m, s)$
for the above equivalences, since the homotopy type of $\text{THH}(k) \otimes B(m, s)$
is determined – using the Atiyah-Hirzebruch spectral sequence – from
the homology of $B(m, s)$. It is the simplicity of the homology of $B(m, s)$
that makes it possible for us to compute the Atiyah-Hirzebruch spectral
sequence.

4. Negative- and periodic topological cyclic homology

In this section we compute $\text{TC}^{-}(A, B, I)$ and $\text{TP}(A, B, I)$. We will need the
following general lemma about the Tate $\mathbb{T}$-construction.

**Lemma 4.1.** Let $\{X_i\}_{i \geq 0}$ be a sequence of spectra with a $\mathbb{T}$-action, such that the
connectivity of (the underlying spectrum) $X_i$ is unbounded, as $i$ grows. Then the
canonical map $(\bigoplus_i X_i)^{\mathbb{T}} \rightarrow \prod (X_i)^{\mathbb{T}}$ is an equivalence.

**Proof.** Because of the connectivity assumption the canonical map $\bigoplus_i X_i \rightarrow \prod X_i$ is an equivalence. Consider the norm cofiber sequence

$$(\Sigma X)^{h \mathbb{T}} \rightarrow \prod X^{h \mathbb{T}} \rightarrow \bigoplus_i X^{t \mathbb{T}}$$

The first term commutes with colimits, the second with limits, so we get
the cofiber sequence

$$\bigoplus (\Sigma X_i)^{h \mathbb{T}} \rightarrow \prod X_i^{h \mathbb{T}} \rightarrow (\bigoplus X_i)^{t \mathbb{T}}$$

Now $(-)^{hG}$ preserves connectivity for any group $G$. To see this one may
use the homotopy orbit spectral sequence whose $E^2$-page consists of ordi-
nary group homology. Thus the cofiber sequence becomes

$$\prod (\Sigma X_i)^{h \mathbb{T}} \rightarrow \prod X_i^{h \mathbb{T}} \rightarrow (\bigoplus X_i)^{t \mathbb{T}}$$

and we see that $(\bigoplus X_i)^{t \mathbb{T}} \rightarrow \prod (X_i)^{t \mathbb{T}}$ is an equivalence. \qed

4.1. The Tate spectral sequence. Let $X$ be a $\mathbb{T}$-spectrum. The Tate con-
struction $\text{TP}(X) = X^{t \mathbb{T}}$ is the target of the Tate spectral sequence for the
circle group $\mathbb{T}$, see [15, end of section 4.4], and also [1, section 3]. This
spectral sequence has the form

$$E^2(\mathbb{T}, X) = S(t^{\pm 1}) \otimes \pi_s X \Rightarrow \pi_s \text{TP}(X)$$

Here $t$ has bi-degree $(-2, 0)$ and is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$. This spec-
tral sequence is conditionally convergent. When $X = \text{THH}(R)$ is an $E_\infty$
algebra the spectral sequence is multiplicative, and if $R$ is a $k$-algebra then
\( E'(T, \text{THH}(R)) \) is a module over \( E'(T, \text{THH}(k)) \). By Bökstedt periodicity, when \( k \) is a perfect field of characteristic \( p > 0 \) there is an isomorphism

\[
\text{THH}_*(k) \simeq k[x]
\]

with \( x \) in degree 2. In particular, in this case \( \text{TP}_*(R) \) is periodic. The following is well-known.

**Proposition 4.2.** Let \( k \) be a perfect field of characteristic \( p > 0 \), then there is an isomorphism

\[
\text{TP}_*(k) \simeq W(k)[t^{\pm 1}]
\]

**Proof.** By Bökstedt periodicity the \( E^2 \)-term of the spectral sequence takes the form \( k[t^{\pm 1}] \otimes k[x] \), and since all classes are in even total degree there are no non-trivial differentials on any page. Thus \( E^2 = E^\infty \) and it remains to determine the extensions. Again by periodicity it suffices to show that \( \text{TP}_0(k) = W(k) \). For \( k = \mathbb{F}_p \) this is done in [20, Cor. IV. 4.8]. To conclude the result for \( k \) we use functoriality along the map \( \mathbb{F}_p \to k \). More precisely we argue as follows. From the Tate spectral sequence we have a complete descending multiplicative filtration

\[
\ldots \subseteq \text{Fil}^{i+1}(k) \subseteq \text{Fil}^i(k) \subseteq \ldots \subseteq \text{Fil}^1(k) \subseteq \text{Fil}^0 = \text{TP}_0(k)
\]

with associated graded \( \text{gr}^i(k) \simeq \text{THH}_{2i}(k) \simeq k \). By the universal property of the \( p \)-typical Witt vectors \( W(k) \) [22, Chapter II, paragraph 5] we get a unique multiplicative continuous map \( W(k) \to \text{TP}_0(k) \). By functoriality we have a commuting diagram

\[
\begin{array}{ccc}
W(\mathbb{F}_p) & \xrightarrow{\simeq} & \text{TP}_0(\mathbb{F}_p) \\
\downarrow & & \downarrow \\
W(k) & \longrightarrow & \text{TP}_0(k)
\end{array}
\]

of ring homomorphisms. In particular the map on associated graded induced by \( W(k) \to \text{TP}(k) \) maps 1 to 1 and so must be an isomorphism. It follows that \( W(k) \to \text{TP}(k) \) is itself an isomorphism. \( \square \)

**Proposition 4.3.** Let \( k \) be a perfect field of characteristic \( p > 0 \). The elements \( x \) and \( t^\pm \) from the \( E^2 \)-page of the Tate spectral sequence are infinite cycles.

**Proof.** This is true for degree reasons, by Bökstedt periodicity. \( \square \)
Lemma 4.4. Let $X$ be a $\mathbb{T}$-spectrum such that the underlying spectrum is an $\mathbb{H}\mathbb{Z}$-module. The $d^2$ differential of the $\mathbb{T}$-Tate spectral sequence is given by $d^2(\alpha) = td(\alpha)$ where $d$ is Connes’ operator.

Proof. See [11, Lemma 1.4.2] □

Following Proposition 3.6 we now choose some generators for the homology of the spaces $B(m, s)$. When $m$ and $s$ have the same parity let $z_{(m, s)}$ be a generator for $\tilde{H}_m(B(m, s); k)$ and let $y_{(m, s)}$ a generator for $\tilde{H}_{m-1}(B(m, s); k)$. When $m$ is even and $s$ is odd let $z_{(m, s)}$ be a generator of $\tilde{H}_{m-1}(B(m, s); k)$. Finally, when $p = 2$ (so $\tilde{H}_m(B(m, s); k)$ is free of rank 1 over $k$) let $w_{(m, s)}$ be a generator.

Lemma 4.5. As an element of the $E^2$-page of the Tate spectral sequence, $z_{(m, s)}$ is an infinite cycle.

Proof. By [14, Theorem B] for any perfect field $k$ of positive characteristic, there is an equivalence $\tau_{\geq 0} \mathbb{T}C(k) \simeq \mathbb{Z}_p$. As a result we obtain a map

$$\mathbb{Z}_p \simeq \tau_{\geq 0} \mathbb{T}C(k) \to \mathbb{T}C(k) \to \mathcal{T}C^{-}(k) \to \mathbb{THH}(k).$$

which is $\mathbb{T}$-equivariant, for the trivial $\mathbb{T}$-action on the domain. It thus induces a map of Tate spectral sequences,

$$E^2 = \mathbb{Z}_p[t^{\pm 1}][y_{(m, s)}, z_{(m, s)}] \to \pi_*(\mathbb{Z}_p \otimes B(m, s))^{r \mathbb{T}}$$

For degree reasons $z_{(m, s)}$ is an infinite cycle in the top spectral sequence. It follows that $z_{(m, s)}$, viewed as a class in the bottom spectral sequence, is an infinite cycle. □

4.2. $\mathbb{T}C^{-}$ and TP of coordinate axes. In this section we compute $\mathbb{T}C^{-}$ and TP using the Tate spectral sequence. When $p > 2$. By Lemma 4.1 and Section 3.5 we have

$$\text{TP}(A, B, I) \simeq \prod_{m \geq 2} \prod_{s|m \text{ even}} \prod_{s|m \text{ even}} \left( (\mathbb{THH}(k) \otimes B(m, s))^{r \mathbb{T}} \right)^{\oplus \text{cyc}_{(s)}}$$

$$\oplus \prod_{m \geq 2} \prod_{s|m \text{ odd}} \prod_{s|m \text{ odd}} \left( (\mathbb{THH}(k) \otimes B(m, s))^{r \mathbb{T}} \right)^{\oplus \text{cyc}_{(s)}}$$
and likewise for negative topological cyclic homology, we have

\[ \text{TC}^{-}(A, B, I) \simeq \prod_{m \geq 2 \text{ even}} \prod_{s|m \text{ even}} \left( (\text{THH}(k) \otimes B(m, s))^{ht} \right)^{\oplus \text{cyc}_d(s)} \]

\[ \oplus \prod_{m \geq 2 \text{ odd}} \prod_{s|m \text{ odd}} \left( (\text{THH}(k) \otimes B(m, s))^{ht} \right)^{\oplus \text{cyc}_d(s)} \]

When \( p = 2 \) there is an extra double product indexed over \( m \geq 2 \) even and \( s \mid m \) odd (cf. Proposition 3.6 and Remark 3.1). See Section 4.3 below.

Since THH\((k)\) is \( p \)-complete, it follows from [20, Lemma II. 4.2.1] that we may identify the homotopy \( T \)-fixed points of the Frobenius morphism for \( \text{THH}(A, B, I) \) with the map induced by the product of the maps

\[ \varphi(m, s) : (\text{THH}(k) \otimes B(m, s))^{ht} \to (\text{THH}(k) \otimes B(pm, s))^{ht}. \]

Since the homotopy fixed point functor preserves co-connectivity, it follows from Corollary 3.1 that \( \pi_{\ast} \varphi(m, s) \) is an isomorphism when \( * \geq m \).

Indeed, the fiber of

\[ (\text{THH}(k) \otimes B(m, s)) \to (\text{THH}(k) \otimes B(pm, s))^{C_p} \]

has no homotopy groups above degree \( m - 1 \), hence the same is true of the \( T \)-homotopy fixed point spectrum.

**Proposition 4.6.** Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( m \geq 2 \) and \( s \mid m \). Write \( m = p^vm' \) and \( s = p^us' \) with \( m' \) and \( s' \) coprime to \( p \).

1. If both \( m \) and \( s \) are even then \( \pi_{\ast}((\text{THH}(k) \otimes B(m, s))^{ht}) \) as well as \( \pi_{\ast}((\text{THH}(k) \otimes B(m, s))^{ht}) \) are concentrated in even degrees and given by

\[ \pi_{2r}((\text{THH}(k) \otimes B(m, s))^{ht}) \simeq \mathbb{W}_{v-u}^{r}(k) \]

and

\[ \pi_{2r}((\text{THH}(k) \otimes B(m, s))^{ht}) \simeq \begin{cases} \mathbb{W}_{v-u+1}^{2r}(k) & 2r \geq m \\ \mathbb{W}_{v-u}^{2r}(k) & 2r < m \end{cases} \]

2. If both \( m \) and \( s \) are odd, then \( \pi_{\ast}((\text{THH}(k) \otimes B(m, s))^{ht}) \) as well as \( \pi_{\ast}((\text{THH}(k) \otimes B(m, s))^{ht}) \) are concentrated in odd degrees and given by

\[ \pi_{2r+1}((\text{THH}(k) \otimes B(m, s))^{ht}) \simeq \mathbb{W}_{v-u}(k) \]
and

\[ \pi_{2r+1}(\text{THH}(k) \otimes B(m,s))^{hT} \simeq \begin{cases} W_{v-u+1}(k) & 2r \geq m \\ W_{v-u}(k) & 2r < m \end{cases}. \]

**Proof.** Suppose first that \( m \) and \( s \) are even. We proceed by induction on \( v \geq 0 \). Suppose \( v = 0 \), so \( m = m' \) and \( s = s' \). Consider the Tate spectral sequence (cf. Section 4.1)

\[ E^2 = k[t^{\pm 1}, x] \{y_{(m',s')}, z_{(m',s')}\} \Rightarrow \pi_* (\text{THH}(k) \otimes B(m',s'))^{hT} \]

By Lemma 4.4 and Proposition 4.3, the differential structure is determined by the differentials on \( y_{(m',s')} \). Furthermore

\[ d^2(y_{(m',s')}) = td(y_{(m',s')}) = tiz_{(m',s')} \]

by Lemma 4.4 and Proposition 3.6 where \( i = m'/s' \). Here we use the characterization of the \( d^2 \)-differential from Lemma 4.4. Since \( i \) is a unit in \( k \), \( d^2 \) is an isomorphism. In summary, the \( E^2 \)-page looks as follows (where we have dropped the indices for clarity).

```
  ...   ...   ...   ...
  \( \cdots \) \( \cdots \) \( \cdots \)
  \( \leftrightarrow \) \( \leftrightarrow \) \( \leftrightarrow \)
  \( \leftrightarrow \) \( \leftrightarrow \) \( \leftrightarrow \)
  \( \leftrightarrow \) \( \leftrightarrow \) \( \leftrightarrow \)
  \( \cdots \) \( \cdots \) \( \cdots \)
```

All the arrows indicate isomorphisms. Thus \( E^3 \), hence \( E^\infty \), is trivial as claimed. To determine the \( T \)-homotopy fixed points, we truncate the Tate spectral sequence, removing the first quadrant. Thus the class \( z_{(m',s')} \) and its multiples by \( x^n \), are no longer hit by differentials and so

\[ E^3 = E^\infty = k[x] \{z_{(m',s')}\} \]

where \( z_{(m',s')} \) has degree \( m \). This proves the claim for \( v = 0 \). The same argument works for \( v > 0 \) and \( u = v \), since in this case \( i \) is again coprime to \( p \).
Suppose the claim is known for all integers less than or equal to $v$ and all $u \leq v$. As we saw above, the homotopy fixed points of the Frobenius map

$$\pi_*(\text{THH}(k) \otimes B(p^m, p^s))^{ht} \to \pi_*(\text{THH}(k) \otimes B(p^{m+1}, p^{s'}))^{ht}$$

is an isomorphism when $* \geq p^m$. The induction hypothesis implies that the domain is isomorphic to $W_{v-u+1}(k)$ when $* = 2r \geq p^m$. By periodicity we conclude that

$$\pi_{2r}(\text{THH}(k) \otimes B(p^{m+1}, p^{s'}))^{ht} \simeq W_{v-u+1}(k)$$

for any $r \in \mathbb{Z}$. Considering again the Tate spectral sequence we see that we must have

$$d^{2(v-u)+2}(y(p^{m+1}, p^{s'})) = tz(p^{m+1}, p^{s'})(xt)^{v-u}$$

(see the figure, which is page E8 when $v = 2$ and $u = 0$) and so we conclude that $E^{2(v-u)+3} = E^\infty$.

Truncating the spectral sequence we now see that

$$\pi_{2r}(\text{THH}(k) \otimes B(p^{m+1}, p^{s'}))^{ht} \simeq \begin{cases} W_{v-u+2}(k) & \text{if } 2r \geq p^{m+1} \\ W_{v-u+1}(k) & \text{if } 2r < p^{m+1} \end{cases}$$

This completes the proof of (1).

The arguments in case (2), where $m$ and $s$ are both odd, are very similar. \qed
Remark 4.1. In particular this shows that \( TP(A, B, I) \) is non-trivial. This may be contrasted with the Cuntz-Quillen result that \( HP((A, B, I)/\mathbb{Q}) \) is trivial, i.e. that rational periodic cyclic homology satisfies excision. Similarly it is shown in \([12]\) that \( TP \) does not satisfy nil-invariance. Here again it is a result of Goodwillie \([9]\) that rational periodic cyclic homology does satisfy nil-invariance.

Suppose \( m' \) is even and let \( t_{\text{ev}} \), \( t_{\text{od}} \) be the unique positive integer such that \( p^{t_{\text{ev}}-1}m' \leq 2r < p^{t_{\text{ev}}}m' \) (or zero if such \( t_{\text{ev}} \) does not exist, i.e. if \( m' \) is too big). Then we may restate the \( \text{TC}^- \) calculations as

\[
\pi_{2r}((\text{THH}(k) \otimes B(p^m m', p^s s'))^{hT}) \cong \begin{cases} 
W_{\nu-1}(k) & \nu < t_{\text{ev}} \\
W_{\nu}(k) & \nu \geq t_{\text{ev}}
\end{cases}.
\]

Similarly if \( m' \) and \( s' \) are both odd, let \( t_{\text{od}} \) be the unique positive integer such that \( p^{t_{\text{od}}-1}m' \leq 2r+1 < p^{t_{\text{od}}}m' \) (or zero if such \( t_{\text{od}} \) does not exist, i.e. if \( m' \) is too big). Then

\[
\pi_{2r+1}((\text{THH}(k) \otimes B(p^m m', p^s s'))^{hT}) \cong \begin{cases} 
W_{\nu-1}(k) & \nu < t_{\text{od}} \\
W_{\nu}(k) & \nu \geq t_{\text{od}}
\end{cases}.
\]

4.3. The case \( p = 2 \). We now deal with the case when \( k \) has characteristic two. From Section 3.5 we see that the case \( m \) even and \( s \) odd, is missing from Proposition 4.6.

Proposition 4.7. Assume \( p = 2 \), \( m \) even and \( s \) odd. Write \( m = 2^m m' \) with \( m' \) odd, and let \( s' \mid m' \). Then the homotopy groups of \((\text{THH}(k) \otimes B(m, s))^hT\) and \((\text{THH}(k) \otimes B(m, s))^kT\) are concentrated in odd degrees where they are isomorphic to \( k \).

Proof. When \( v = 0 \) we are in the case (2) of Proposition 4.6 so

\[ \pi_*(\text{THH}(k) \otimes B(m', s'))^{hT} = 0 \]

and

\[ \pi_*(\text{THH}(k) \otimes B(m', s'))^{kT} = k \] when \( * = 2r \geq m' \).

From the high co-connectivity of the Frobenius we conclude that

\[ \pi_*(\text{THH}(k) \otimes B(2m', s'))^{hT} = k \]

when \( * \) is odd, and zero otherwise. Now the Tate spectral sequence

\[ E^2 = k[t^{\pm 1}, x][z, w] \Rightarrow \pi_*(\text{THH}(k) \otimes B(2m', s'))^{hT} \]
must collapse on the $E^2$-page, from which we conclude that $d^2(w) = (tx)z$. Truncating the spectral sequence we see the homotopy fixed point spectral sequence has $E^3 = E^\infty = (k[t,x]/(tx))z$ so

$$\pi_*(\text{THH}(k) \otimes B(2m', s'))^{ht} = k$$

in every odd degree. Now an induction argument shows that this pattern continues for all $v > 1$. □

5. Topological cyclic homology of coordinate axes

In this section we complete the proof of Theorem 2.1, and thus Theorem 1.1. We start with the case where $p > 2$. For $2 \leq m'$ and $s' \mid m'$ let

$$\text{TP}(m', s') := \prod_{0 \leq v \leq v} \prod_{0 \leq u \leq v} \left( (\text{THH}(k) \otimes B(p^v m', p^u s'))^{ht} \right)_{cyc}(p^v s')$$

and

$$\text{TC}^-(m', s') := \prod_{0 \leq v \leq v} \prod_{0 \leq u \leq v} \left( (\text{THH}(k) \otimes B(p^v m', p^u s'))^{ht} \right)_{cyc}(p^v s')$$

Thus, $\text{TC}(A,B,I)$ is identified with the product of the equalizer of the maps $\phi, \text{can} : \text{TC}^-(m', s') \to \text{TP}(m', s')$ as $m'$ and $s'$ vary accordingly. Let us denote by $\text{TC}(m', s')$ the equalizer

$$\text{TC}(m', s') \to \text{TC}^-(m', s') \cong \text{TP}(m', s')$$

A priori the homotopy groups of $\text{TC}(m', s')$ sit in a long exact sequence with those of $\text{TC}^-(m', s')$ and $\text{TP}(m', s')$. However this sequence splits into short exact sequences since $\text{TC}^-(m', s')$ and $\text{TP}(m', s')$ are concentrated in either even or odd degrees, depending on the parity of $m'$ and $s'$, cf. Proposition 4.6.

If $m'$ and $s'$ are even then the Frobenius map

$$\pi_{2r}(\text{THH}(k) \otimes B(p^v m', p^u s'))^{ht} \to \pi_{2r}(\text{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{ht}$$

is an isomorphism for $0 \leq v < t_{ev}$, and the canonical map

$$\pi_{2r}(\text{THH}(k) \otimes B(p^v m', p^u s'))^{ht} \to \pi_{2r}(\text{THH}(k) \otimes B(p^{v+1} m', p^u s'))^{ht}$$
is an isomorphism for $t_{ev} \leq v$. Thus we have a map of short exact sequences

\[
\prod_{t_{ev} \leq v} \prod_{u \leq v} (W_{v-u}(k))^\text{cyc}_d(p^s') \xrightarrow{\text{q-can}} \prod_{t_{ev} \leq v} \prod_{u \leq v} (W_{v-u}(k))^\text{cyc}_d(p^s')
\]

\[
\text{TC}_{2r}(m', s') \xrightarrow{\text{q-can}} \text{TP}_{2r}(m', s')
\]

\[
\prod_{0 \leq v < t_{ev}} \prod_{u \leq v} (W_{v-u+1}(k))^\text{cyc}_d(p^s') \xrightarrow{\text{q-can}} \prod_{0 \leq v < t_{ev}} \prod_{u \leq v} (W_{v-u}(k))^\text{cyc}_d(p^s')
\]

The top horizontal map is an isomorphism, and the bottom horizontal map is an epimorphism so, by the snake lemma, we conclude that

\[
\text{TC}_{2r}(m', s') \simeq \prod_{u \leq t_{ev}} (W_{t_{ev}-u}(k))^\text{cyc}_d(p^s').
\]

Now suppose $m'$ and $s'$ are odd. Then the Frobenius map

\[
\pi_{2r+1}(\text{THH}(k) \otimes B(p^v m', p^u s'))^hT \to \pi_{2r+1}(\text{THH}(k) \otimes B(p^{v+1} m', p^u s'))^T
\]

is an isomorphism for $0 \leq v < t_{od}$, and the canonical map

\[
\pi_{2r+1}(\text{THH}(k) \otimes B(p^v m', p^u s'))^hT \to \pi_{2r+1}(\text{THH}(k) \otimes B(p^v m', p^u s'))^T
\]

is an isomorphism for $t_{od} \leq v$. Thus we have a map of short exact sequences

\[
\prod_{t_{od} \leq v} \prod_{u \leq v} (W_{v-u}(k))^\text{cyc}_d(p^s') \xrightarrow{\text{q-can}} \prod_{t_{od} \leq v} \prod_{u \leq v} (W_{v-u}(k))^\text{cyc}_d(p^s')
\]

\[
\text{TC}_{2r+1}(m', s') \xrightarrow{\text{q-can}} \text{TP}_{2r+1}(m', s')
\]

\[
\prod_{0 \leq v < t_{od}} \prod_{u \leq v} (W_{v-u+1}(k))^\text{cyc}_d(p^s') \xrightarrow{\text{q-can}} \prod_{0 \leq v < t_{od}} \prod_{u \leq v} (W_{v-u}(k))^\text{cyc}_d(p^s')
\]

The top horizontal map is an isomorphism, and the bottom horizontal map is an epimorphism so, by the snake lemma, we conclude that

\[
\text{TC}(m', s') \simeq \prod_{u \leq t_{od}} (W_{t_{od}-u}(k))^\text{cyc}_d(p^s').
\]
This finishes the proof of Theorem 2.1 in the case \( p > 2 \).

5.1. The case \( p = 2 \). If \( p = 2 \) then we must correct slightly the definition of \( TP(m', s') \) and \( TC^-(m', s') \). Suppose \( m' \) and \( s' \) are odd. To deal with the case \( m = p^v m' \) even and \( s = p^u s' \) even, we let

\[
TP(m', s')_{cv} := \prod_{1 \leq v} \prod_{1 \leq u \leq v} \left( (THH(k) \otimes B(2^v m', 2^u s'))^{\text{cyc}_d(2^u s')} \right)
\]

and

\[
TC^-(m', s')_{cv} := \prod_{1 \leq v} \prod_{1 \leq u \leq v} \left( (THH(k) \otimes B(2^v m', 2^u s'))^{\text{cyc}_d(2^u s')} \right)
\]

Both of these spectra are concentrated in even degrees, with their homotopy groups given by Proposition 4.6. As a result we see that, for \( m' \geq 1 \) odd, and \( s' \mid m' \) odd,

\[
TC_{2v}(m', s')_{cv} \simeq \prod_{1 \leq u \leq t_{cv}} W_{c_{v-u}}(k)^{\oplus \text{cyc}_d(2^u s')}
\]

To deal with the case where \( m \) is even and \( s \mid m \) is odd, let

\[
TP(m', s')_{cv, od} := \prod_{0 \leq v} \left( (THH(k) \otimes B(2^v m', s'))^{\text{cyc}_d(s')} \right)
\]

and

\[
TC^-(m', s')_{cv, od} := \prod_{0 \leq v} \left( (THH(k) \otimes B(2^v m', s'))^{\text{cyc}_d(s')} \right)
\]

(note that when \( v = 0 \), we have \( m = m' \) odd, but we must include this case since the Frobenius connects it with the case \( v = 1 \)) By Proposition 4.7, \( TP(m', s')_{cv, od} \) and \( TC^-(m', s')_{cv, od} \) are concentrated in odd degrees, where they are isomorphic to \( k \). Thus

\[
TC_{2r+1}(m', s')_{cv, od} \simeq \prod_{0 \leq v \leq t_{cv}} k^{\oplus \text{cyc}_d(s')}.
\]

This completes the proof of Theorem 2.1 when \( p = 2 \).

6. The characteristic zero case

In this section we compute the relative cyclic homology and the bi-relative K-theory of \( A_d = k[x_1, \ldots, x_d] / (x_i x_j)_{i \neq j} \) in the case that \( k \) is an ind-smooth \( \mathbb{Q} \)-algebra. In the case where \( k / \mathbb{Q} \) is a field extension, the results in this section were found, by different means, already in 1989 by Geller, Reid and Weibel [8, Theorem 7.1].
We proceed as in \cite[Section 3.9]{12}. We will compute the relative groups

\[
\text{HC}_q((A, I)/\mathbb{Z}) \otimes \mathbb{Q} \simeq \text{HC}_q((A, I)/\mathbb{Q})
\]

using our understanding of the $T$-homotopy type of the cyclic bar construction for $\Pi^d$, as found in Lemma \ref{lem:cyclic_bar_construction}. First we need the following analogue of Lemma \ref{lem:cyclic_bar_construction}

**Lemma 6.1.** Let $k$ be a ring, $\Pi$ a pointed monoid, and $k(\Pi)$ the pointed monoid algebra. There is a natural equivalence of $T$-spectra

\[
\text{HH}(k(\Pi)) \xrightarrow{\sim} \text{HH}(k) \otimes \text{B}^\forall(\Pi)
\]

So the arguments from Lemma \ref{lem:cyclic_bar_construction} still work, yielding a description of the relative (and bi-relative) Hochschild homology spectrum of $(A_d, I_d)$ (and $(A_d, B_d, I_d)$). First we note that the relative Hochschild homology differs only slightly from the absolute version. The sole difference is that we “cut out” the part of the space $\text{B}^\forall(\Pi^d)$ given by $\text{B}^\forall(\Pi^d, \emptyset)$, i.e. the part corresponding to the unique cyclic word of length zero. Since the spaces $B(m, s)$ for $m$ even and $s$ odd have torsion integral homology they disappear in the rational case. So we conclude that

\[
\text{HH}(A, I) \simeq \bigoplus_{m \geq 2 \text{ even}} \bigoplus_{s \text{ even}} \left( \text{HH}(k) \otimes B(m, s) \right) \oplus \text{cyc}_d(s)
\]

\[
\oplus \bigoplus_{m \geq 2 \text{ odd}} \bigoplus_{s \text{ odd}} \left( \text{HH}(k) \otimes B(m, s) \right) \oplus \text{cyc}_d(s)
\]

\[
\oplus \bigoplus_{i \geq 1} \left( \text{HH}(k) \otimes (T/C_i) \right) \oplus \text{d}
\]

where $B(m, s)$ is given by Lemma \ref{lem:cyclic_bar_construction}. The bottom summands, with terms $\text{HH}(k) \otimes (T/C_i)_{+}$, corresponds to the spaces $\text{B}^\forall(\Pi^d, x_1^i)$, $\text{B}^\forall(\Pi^d, x_2^i)$ etc. These bottom summands disappear when looking at the bi-relative theory, $\text{HH}(A, B, I)$.

The following theorem is due to Geller, Reid, Weibel, \cite[Corollary 3.12.]{8} when $k$ is a field extension of $\mathbb{Q}$.

**Theorem 6.2.** Let $k$ be any commutative unital ring. There is an isomorphism

\[
\text{HC}_0(A_d/\mathbb{Q}) \simeq \text{HC}_0(k/\mathbb{Q})
\]
and, for $q \geq 1$, an isomorphism
\[
\text{HC}_q((A_d, I_d) / Q) \simeq \bigoplus_{m \geq 2} \bigoplus_{s \mid m \text{ even}} (\text{HH}_{q+1-m}(k) \otimes Q)^{\oplus \text{cyc}_d(s)}
\]
\[
\oplus \bigoplus_{m \geq 2} \bigoplus_{s \mid m \text{ odd}} (\text{HH}_{q+1-m}(k) \otimes Q)^{\oplus \text{cyc}_d(s)}
\]
\[
\oplus \bigoplus_{i \geq 1} (\text{HH}_q(k) \otimes Q)^{\oplus d}
\]

Proof. We must compute $\pi_q(X_{hT})$ where $X$ ranges over the summands in the above decomposition of $B^\Omega(\Pi^d)$. If $X = \text{HH}(k) \otimes S^{\lambda_j} \otimes (\mathbb{T} / C_i)_+$ for some complex $\mathbb{T}$-representation $\lambda_j$ of complex dimension $j$, then since,
\[
(\text{HH}(k) \otimes S^{\lambda_j} \otimes \mathbb{T} / C_i)_{hT} \simeq \left(\text{HH}(k) \otimes S^{\lambda_j}\right)_{hC_i},
\]
we may regard the $C_i$-homology spectral sequence
\[
E^2_{s,t} = H_{s}(C_i, \pi_t(\text{HH}(k) \otimes S^{\lambda_j}) \otimes Q) \Rightarrow \pi_{s+t}(\text{HH}(k) \otimes S^{\lambda_j})_{hC_i} \otimes Q
\]
Since the rational group homology of $C_i$ is concentrated in degree 0, the edge homomorphism
\[
H_0(C_i, \pi_q(\text{HH}(k) \otimes S^{\lambda_j}) \otimes Q) \xrightarrow{\simeq} \pi_q\left(\left(\text{HH}(k) \otimes S^{\lambda_j}\right)_{hC_i}\right) \otimes Q
\]
is an isomorphism. Furthermore, since the $C_i$-action on $\text{HH}(k) \otimes S^{\lambda_j}$ extends to a $\mathbb{T}$-action, the induced action on homotopy groups is trivial. We conclude that
\[
\pi_q\left(\left(\text{HH}(k) \otimes S^{\lambda_j}\right)_{h\mathbb{T}}\right) \cong \text{HH}_{q-2j}(k)
\]
The result follows. \hfill $\square$

The following theorem is due to Geller, Reid, Weibel, \[8\] Theorem 7.1.1) when $k$ is a field extension of $Q$. We use their counting function $c_d(q)$ which we recall in Section 7 Eq. (6).

**Theorem 6.3.** Suppose $k$ is an ind-smooth $Q$-algebra. Let $d \geq 2$ and consider the ring $A_d = k[x_1, \ldots, x_d]/(x_i x_j)_{i \neq j}$, and let $I_d = (x_1, \ldots, x_d)$. Then
\[
K_q(A_d, I_d) \cong k^{\oplus c_d(q)} \oplus (\Omega_{k/Q}^1)^{\oplus c_d(q-1)} \oplus \cdots \oplus (\Omega_{k/Q}^{q-2})^{\oplus c_d(2)}.
\]

Proof. By \[6, Corollary 0.2\] we have $K_n(A_d, I_d) \simeq \text{HC}_{n-1}(A_d, B_d, I_d)$. Here we use that $K_q(A_d, I_d)$ is a rational vector space \[6, Theorem 0.1\], so no further rationalization is necessary. By Theorem \[6,2\] we are reduced to
Hochschild homology calculations. By the Hochschild-Kostant-Rosenberg theorem \([23, \text{Theorem } 9.4.7]\) we have

\[
\HH_n(k/Q) \simeq \Omega_k^n/Q.
\]

Thus,

\[
K_q(A_d, I_d) \simeq \HC_{q-1}(A_d, B_d, I_d) \\
\simeq k^{\oplus c_d(q)} \oplus (\Omega_k^1)^{\oplus c_d(q-1)} \oplus \cdots \oplus (\Omega_k^{q-2})^{\oplus c_d(2)}
\]

This completes the proof. \(\square\)

**Remark 6.1.** If \(k\) is a field extension of \(Q\) for which we know the transcendence degree of \(k\) over \(Q\) then this result completely determines the relative \(K\)-theory, since \(\dim_k \Omega_k^1 = \text{tr.deg}(k/Q)\). For example if \(k/Q\) is algebraic then \(\Omega_k^1 = 0\) and so the calculation reduces to

\[
K_q(A_d, I_d) \simeq k^{\oplus c_d(q)}.
\]

### 7. Appendix: counting cyclic words

In this section we use some counting techniques inspired by [8]. We first deal with all words, then with cyclic words. Let \(A_d(m)\) denote the number of words in \(d\) letters of length \(m\) having no cyclic repetitions.

**Lemma 7.1.** For all \(d \geq 1\) and all \(m \geq 1\) we have

\[
A_d(m) = (d - 1)^m + (-1)^m (d - 1).
\]

**Proof.** Define the auxiliary counter, \(B_d(m)\) counting words \(\omega = w_1 w_2 \ldots w_m\) with no allowed repetitions, except that we require \(w_m = w_1\). Then

\[
A_d(m) + B_d(m) = d(d - 1)^{m-1}.
\]

To see this, consider a word \(\omega = w_1 w_2 \ldots w_m\). There are \(d\) choices for \(w_1\), and \(d - 1\) choices for \(w_2, w_3, \ldots\) and \(w_{m-1}\). Finally for \(w_m\) there are again \(d\) choices since choosing any letter different from \(w_{m-1}\) gives either a type \(A\) \((d - 2\) possibly choices) or a type \(B\) word (one choice, namely \(w_m = w_1\)). The formula for \(A_d(m)\) now follows by induction using the equation \(B_d(m) = A_d(m-1)\) (for \(m \geq 2\)). This last equality is true since deleting the last letter of a type \(B\) word yields a word with no cyclic repetitions. \(\square\)
Let \( \tilde{\text{cyc}}_d(s) \) denote the number of words in \( d \) letters of length \( s \), period \( s \) and having no cyclic repetitions (see Definition 3.1) and let \( \text{cyc}_d(s) \) denote the number of cyclic words in \( d \) letters of length \( s \), period \( s \) and having no cyclic repetitions. Then \( \text{cyc}_d(s) = \frac{1}{s} \tilde{\text{cyc}}_d(s) \). Also we have

\[
A_d(m) = \sum_{s|m} \tilde{\text{cyc}}_d(s)
\]

So using Möbius inversion we have

\[
\tilde{\text{cyc}}_d(s) = \sum_{j|s} \mu\left(\frac{s}{j}\right)A_d(j).
\]

hence we obtain a formula for \( \text{cyc}_d(s) \),

\[
\text{cyc}_d(s) = \frac{1}{s} \sum_{j|s} \mu\left(\frac{s}{j}\right)((d-1)^j + (-1)^j(d-1)).
\]

Below is a table of the first few values.

| \( s \) | \( \text{cyc}_d(s) \) | \( d = 3 \) | \( d = 4 \) |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | \( \frac{1}{2}((d-1)^2 + (d-1)) \) | 3 | 6 |
| 3 | \( \frac{1}{3}((d-1)^3 - (d-1)) \) | 2 | 8 |
| 4 | \( \frac{1}{4}((d-1)^4 - (d-1)^2) \) | 3 | 18 |
| 5 | \( \frac{1}{5}((d-1)^5 - (d-1)) \) | 6 | 48 |
| 6 | \( \frac{1}{6}((d-1)^6 - (d-1)^3 - (d-1)^2 + (d-1)) \) | 9 | 116 |
| 7 | \( \frac{1}{7}((d-1)^7 - (d-1)) \) | 18 | 312 |
| 8 | \( \frac{1}{8}((d-1)^8 - (d-1)^4) \) | 30 | 810 |
| 9 | \( \frac{1}{9}((d-1)^9 - (d-1)^3) \) | 56 | 2184 |
| 10 | \( \frac{1}{10}((d-1)^{10} - (d-1)^5 - (d-1)^2 + (d-1)) \) | 99 | 5880 |
| 11 | \( \frac{1}{11}((d-1)^{11} - (d-1)) \) | 186 | 16104 |
| 12 | \( \frac{1}{12}((d-1)^{12} - (d-1)^6 - (d-1)^4 + (d-1)^2) \) | 335 | 44220 |
If we want to count the number of cyclic words without cyclic repetitions, with length $m$ but with no restrictions on period then we can just sum $\text{cyc}_d(s)$ over all divisors $s$ of $m$. For example there are $3 + 2 + 9 = 14$ cyclic words in $d = 3$ letter, with no repetitions having length $m = 6$.

We may describe the function $c_{d-1}(m)$ that Geller, Reid, and Weibel introduce as follows

$$c_{d-1}(m) = \sum_{\substack{s|m \quad s \equiv m \mod 2}} \text{cyc}_{d-1}(s).$$

That is, $c_{d-1}(m)$ is the number of cyclic words without repetitions, having length $m$ and period $s$ where $s$ and $m$ have the same parity.

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