On the spectrum of high-energy cosmic rays produced by supernova remnants in the presence of strong cosmic-ray streaming instability and wave dissipation

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Abstract. The cosmic-ray streaming instability creates strong magnetohydrodynamic turbulence in the precursor of a SN shock. The level of turbulence determines the maximum energy of cosmic-ray particles accelerated by the diffusive shock acceleration mechanism. The present consideration continues our work 

The cosmic-ray streaming instability is less efficient as the shock velocity decreases with time and the nonlinear wave interactions reduce the level of turbulence at the late Sedov stage (Völk et al. 1988, Fedorenko 1990). This leads to the fast diffusion and to the corresponding decrease of $E_{\text{max}}$. The effect is aggravated by the possible wave damping on the ion-neutral collisions (Bell 1978, Drury et al. 1993). The acceleration of cosmic rays and their streaming instability in a wide range of shock velo-
ties was considered in our paper Ptuskin & Zirakashvili (2003) (Paper I). The analytical expressions for cosmic ray diffusion coefficient and for the instability growth rate were generalized to the case of arbitrary strong random magnetic field, $\delta B \gtrsim B_0$. The rate of nonlinear wave interactions was assumed to correspond to the Kolmogorov nonlinearity of magnetohydrodynamic waves. The collisional dissipation was also taken into account. The maximum energy of accelerated particles was determined as a function of shock velocity and thus as a function of SNR age. The maximum energy of the particle with charge Ze can be as high as $10^{17}$ Z eV in some very young SNRs and falls down to about $10^{10}$ Z eV at the end of adiabatic (Sedov) stage. The widely accepted estimate of cosmic ray diffusion coefficient at the strong shock that corresponds to the Bohm diffusion value calculated for the interstellar magnetic field strength turns out to be not correct. This result may explain why the SNRs with the age more than a few thousand years are not prominent sources of very high energy gamma-rays (Buckley et al. 1998, Aharonian et al. 2002). The presence of strongly amplified random magnetic field in young SNRs is evidently supported by the interpretation of data on synchrotron X-ray emission from young SNRs, see e.g. Vink (2003) for review.

The main objective of the present work is the calculation of the average spectrum of cosmic rays ejected in the interstellar medium by a SNR in a course of its evolution. Some necessary results of Paper I are presented in the next Section 2, the evolution of SNR shocks is discussed in Section 3, the average cosmic-ray source spectrum is calculated in Section 4 followed by the discussion in Section 5, the conclusion is given in Section 6. Appendix A describes the thin shell approximation used in our calculations.

2. Maximum Energy of Accelerated Particles

In the test particle approximation, the distribution of accelerated particles in momentum for high Mach number shocks has the canonical form $f(p) \sim p^{-4}$ (Krymsky 1977, Bell 1978). In the case of efficient acceleration, the action of cosmic ray pressure on the shock structure causes non-linear modification of the shock that changes the shape of particle spectrum making it flatter at ultra relativistic energies (Eichler 1984, Berezhko et al. 1996, Malkov & Drury 2001). Because of this effect, we assume that the distribution of ultrarelativistic particles at the shock is of the form $f_0(p) \sim p^{-4+a}$ where $0 < a < 0.5$, and the value $a = 0.3$ is used in the numerical estimates below. The normalization of function $f(p)$ is such that the integral $N = 4\pi \int dp p^2 f(p)$ gives the number density of cosmic rays. The differential cosmic ray intensity is $I(E) = f(p) p^2$. We assume that the cosmic ray pressure at the shock is some fraction $\xi_{cr}$ of the upstream momentum flux entering the shock front, so that $P_{cr} = \xi_{cr} \rho u^2_{sh}$ and the equation for the distribution function of relativistic accelerated particles at the shock is

$$f_0(p, t) = \frac{3 \xi_{cr} \rho u^2_{sh} H(p_{\text{max}}(t) - p)}{4\pi c (mc)^a \varphi(p_{\text{max}}) p^{4-a}},$$

where $p_{\text{max}}$ is the maximum momentum of accelerated particles, $H(p)$ is the step function, and $\varphi(p_{\text{max}}) = \int_0^{p_{\text{max}}/mc} \frac{dp}{\sqrt{1+p^2}}$. The approximation of the last integral at $p_{\text{max}} \gg mc$ is $\varphi(p) \approx a^{-1} (p/mc)^a - a^{-1}(1+a)^{-1}$. The value of $\xi_{cr} \approx 0.5$ and the total compression ratio at the shock close to 7 were found in the numerical simulations of strongly modified SN shocks by Berezhko et al. (1996).

Here and below we mainly consider protons as the most abundant cosmic ray component. For ions with charge $Z$, the equations should be written as functions of $p/Z$ instead of $p$. In particular, the nuclei with charge $Z$ reach the maximum momentum a factor of $Z$ larger than protons. We use the notation $m$ for the proton mass. The acceleration in old SNRs ($t \gtrsim 3 \times 10^4 - 10^5$ yr) when $p_{\text{max}}/mc < 10$ are not considered in the present paper because Eq. (1) is not applied at low Mach numbers, see Paper I for detail. [Using the test particle approximation for not modified shock, Drury et al. 2003 found that the spectrum of accelerated particles is somewhat steeper if the diffusion coefficient is increasing with time compared to the case of constant $D$. This effect is not included in our consideration.]

The following steady-state equation determines the energy density $W(k)$ ($k$ is the wave number) of the magnetohydrodynamic turbulence amplified by the streaming instability in the cosmic-ray precursor upstream of the supernova shock:

$$u \nabla W(k) = 2(\Gamma_{cr} - \Gamma_1 - \Gamma_{nl}) W(k).$$

Here the l.h.s. describes the advection of turbulence by highly supersonic gas flow. The terms on the r.h.s. of the equation describe respectively the wave amplification by cosmic ray streaming, the linear damping of waves in background plasma, and the nonlinear wave-wave interactions that may limit the amplitude of turbulence. The equation for wave growth rate at the shock

$$\Gamma_{cr}(k) = \frac{C_{cr}(a) \xi_{cr} u^2_{sh} k^{1-a}}{(1 + A_{tot}^2)^{(1-a)/2}} c V_a \varphi(p_{\text{max}}) r_g^a$$

was suggested in Paper I as the generalization of equation derived for the case of weak random field (Berezhinskii et al. 1990). Here $V_a = B_0/\sqrt{4\pi \rho}$ is the Alfvén velocity ($\rho$ is the gas density), $A = \delta B / B_0$ is the dimensionless wave amplitude, and $r_g = mc^2/2eB_0$. The ion-neutral and electron collisions usually determine the linear damping processes in the thermal space plasma. The Kolmogorov-type nonlinearity with a simplified expression

$$\Gamma_{nl} = (2C_K)^{-3/2} V_a k A(> k) \approx 0.05 V_a k A(> k)$$

at $C_K = 3.6$ (as it follows from the numerical simulations by Verma et al. 1996) was used in Paper I. The wave-particle interaction is of resonant character and the resonance condition is $k_{res} r_g = \sqrt{1 + A_{tot}^2}$, where the Larmor radius is defined through the regular field $B_0$ and $A_{tot}$ is the total amplitude of random field. The particle scattering leads to the cosmic-
The maximum momentum of accelerated particles, and the amplified magnetic field are given then by the approximate equations

\[
\frac{p_{\text{max}}}{mc} \approx 2\alpha C_{\text{cr}}(a) \eta_{\text{sh}} u_{\text{sh}}^2 R_{\text{sh}} (r_{g0} V_{\alpha} c)^{-1},
\]

and

\[
A_{\text{tot}} \approx 2u_{\text{sh}} \frac{\eta_{\text{cr}}}{3V_{\alpha}} C_{\text{cr}}
\]

The cosmic ray diffusion coefficient depends on particle Larmor radius as \(D \propto v_{r}\) at \(p \leq p_{\text{max}}\) in this case.

In the low velocity limit, when \(u_{\text{sh}} \ll 4\pi a C_{\text{cr}}(a) C_{\text{res}}(2C_{K})^{2/3} \xi_{\text{cr}}^{-1}\) and \(u_{\text{sh}} \ll 3V_{\alpha}[2a C_{\text{cr}}(a) C_{\text{res}}(2C_{K})^{2/3} \xi_{\text{cr}}^{-1}\), the nonlinear dissipation term dominates over the advection term in the l.h.s. of Eq. (5) and the wave amplitude is small, \(A_{\text{tot}} \ll 1\). The maximum momentum of accelerated particles and the amplified magnetic field are given then by the approximate equations

\[
\frac{p_{\text{max}}}{mc} \approx 2\alpha C_{\text{cr}}(a) \eta_{\text{sh}} u_{\text{sh}}^2 R_{\text{sh}} (r_{g0} V_{\alpha} c)^{-1},
\]

and

\[
A_{\text{tot}} \approx 2u_{\text{sh}} \frac{\eta_{\text{cr}}}{3V_{\alpha}} C_{\text{cr}}
\]

assuming that this value of amplified field exceeds the value of random interstellar magnetic field. The cosmic ray diffusion coefficient depends on particle Larmor radius as \(D \propto v_{r}\) at \(p \leq p_{\text{max}}\). (Notice the misprint in the numerical coefficient in the first equality of Eq. (19) in Paper I that is analogous to the present Eq. (9).)

Fig. 1 illustrates the results of calculations of \(p_{\text{max}}\) at the Sedov stage of SNR evolution at \(E = 10^{53}\) erg in the warm interstellar gas with the temperature \(T = 8 \times 10^{3}\) K, the average density \(n_{0} = 0.4 \text{ cm}^{-3}\) that includes small interstellar clouds, the intercloud density \(n = 0.1 \text{ cm}^{-3}\), the number density of ions \(n_{i} = 0.03 \text{ cm}^{-3}\), the interstellar magnetic field value \(B_{0} = 5 \mu\text{G}\), see Paper I. The time dependence of the shock radius and the shock velocity are given by the following equations (the Sedov solution, see e.g. Ostriker & McKee [1988]):

\[
R_{\text{sh}} = 4.3 \frac{(E_{51}/n_{0})^{1/5}}{k_{\text{yr}}} \frac{3^{5/2}}{4} \text{ pc},
\]

\[
u_{\text{sh}} = 1.7 \times 10^{3} (E_{51}/n_{0})^{1/5} k_{\text{yr}}^{-3/5} \text{ km/s},
\]

where we assume that the ultrarelativistic gas of cosmic rays mainly determines the pressure behind the shock. The value \(\varkappa = 0.04\) was assumed in the calculations in Fig. 1. Three solid lines correspond to the three cases of wave dissipation considered separately: the nonlinear wave interactions; the damping by ion-neutral collisions at constant gas density; the damping by ion-neutral collisions when the diffuse neutral gas restores its density after complete ionization by the radiation from the SN burst. For the last two curves, the dissipation of wave due to the ion-neutral collisions with damping rate

\[
\Gamma_{1} = \frac{\nu_{\text{in}}}{2} \left(1 + (1 + A_{\text{tot}})^{-1} \left(1 + \frac{n_{i}}{n_{H}} \frac{\nu_{\text{in}}}{k_{\alpha}}\right)^{2}\right)^{-1}
\]

was taken into account whereas the term \(\Gamma_{\text{nl}}\) that describes the nonlinear dissipation was omitted. Here \(\nu_{\text{in}} = n_{H} (v_{\text{in}} / \sigma) \approx 8.4 \times 10^{-9} \text{(K/10}^{4}\text{ K})^{0.4} (n_{H} / 1 \text{ cm}^{-3}) \text{ s}^{-1}\) for the temperature \(T \sim 10^{2} - 10^{5}\) K is the frequency of ion-neutral collisions with the cross section \(\sigma\) averaged over velocity distribution of thermal particles, \(n_{H}\) is the number density of neutral hydrogen. (Notice some correction of the “collisional damping” curve in Fig.1 comparing to corresponding Figure 2 in paper I.) The maximum energy of protons accelerated by SN shocks at the early Sedov stage is close to \(3 \times 10^{14}\) eV that exceeds the Bohm limit calculated for the interstellar magnetic field value by one order of magnitude. The maximum energy decreases to about \(10^{10}\) eV at the end of the Sedov stage that is much less than the Bohm limit calculated for the interstellar magnetic field value. In particular, the particle energy is less than \(10^{13}\) eV at \(t > 3 \times 10^{3}\) yr and this may explain the absence of a TeV \(\gamma\)-ray signal from many SNRs (Buckley et al. [1985]; Aharonian et al. [2002]) where the gamma-rays could in principle be produced through \(\pi^{0}\) decays if sufficiently energetic cosmic rays were present.

With the extreme choice of parameters of the flying apart young SNR envelope, it was found (Bell & Lucek [2001] Paper I) that the maximum particle energy may reach the ultra high energies. The estimate of the highest particle energy according to Paper I is

\[
E_{\text{max}} \approx 2 \times 10^{17} \frac{Z}{u_{\text{sh}}/3 \times 10^{3}\text{ km s}^{-1}} \frac{0.1}{\xi_{\text{cr}}} \frac{M_{e}^{1/3}}{(n_{i} / 1 \text{ cm}^{-3})^{1/6}} \text{ eV}
\]

at the end of a free expansion stage which precedes the Sedov
stage (here $M_{ej}$ is the mass of ejecta in solar masses). We shall see below that this promising estimate is in some sense devaluated by the results of calculations of particle flux - the flux turns out to be low at the highest energies which can be achieved in the process of acceleration.

3. Evolution of SNR Shocks

The typical source of galactic cosmic rays is most probably associated with the core collapse supernova, Type II SNe, that is the final stage of evolution for stars more massive than about 8 solar masses while on the main sequence. The massive star before the explosion goes through the sequence O-star stage, Red Super Giant star stage, and through the Wolf-Rayet stage for the most massive progenitors ($>20M_\odot$) that give the rare in occurrence Type Ib/c SNe. The fast wind of a massive progenitor star on the main sequence produces a big bubble of hot rarefied gas with the temperature about $10^6$ K in the surrounding interstellar medium, see Weaver et al. (1977), Lozinskaya (1992). The typical Type II SN goes through the Red Super Giant phase before the explosion and this process is accompanied by the flow of a low-velocity dense wind. Thus, immediately after the supernova burst, the shock propagates through the wind of a Red Super Giant star then through the hot bubble and finally it enters the interstellar medium. Our calculations will be done for the ejecta mass $M_{ej} = 1M_\odot$ (the solar mass). The spherically
symmetric distribution of gas density in the stellar wind is 
\[ n_w = \frac{M}{(4\pi r_w u_w^2)} \], where \( M = 10^{-5} M_5 \) (solar mass)/yr is the mass loss rate, \( n_w = 1.4 n \) is the mean interstellar atom mass per hydrogen nucleus, the wind velocity \( u_w = 10^9 u_{w,6} \) cm/s. The stellar wind magnetic field has the shape of the Parker spiral similar to the case of interplanetary magnetic field (Parker, 1958). At relatively large distances from the surface of the star that are of interest here, the magnetic field has predominately azimuthal structure and its value is \( B_\theta = B_r r_\theta^2 \Omega \sin\theta / (u_w r) \) where \( B_r \) is the surface magnetic field strength at the star rotation radius \( r_\star \), \( \Omega \) is the angular velocity of star rotation, and \( \theta \) is the polar angle. Hence \( B_\theta(r) = 2 \times 10^{13} u_{w,6}^{-1} G \times \text{cm} \) at \( B_r = 1 \text{ G}, r_\star = 3 \times 10^{13} \text{ cm}, \Omega = 3 \times 10^{-8} \text{ s}^{-1} \) that gives \( B_\theta \approx 6 \mu G \) at the distance \( r = 1 \text{ pc} \) from the star.

Below we shall also use the following set of parameters of the medium surrounding the Type II SN: the radius of spherical Red Super Giant wind \( R_w = 2 \text{ pc} \), the star mass loss \( M_5 = 1 \), and the wind velocity \( u_{w,6} = 1 \). The radius of the spherical bubble of hot gas \( R_0 = 60 \text{ pc} \), the gas density in the bubble \( n_0 = 1.5 \times 10^{-2} \text{ cm}^{-3} \), the magnetic field there \( B_0 = 5 \mu G \). The gas density in the undisturbed interstellar medium around the bubble is assumed to be equal to \( n_0 = 1 \text{ cm}^{-3} \) (physically, the value of \( n_0 \) determines \( n_b \), see Weaver et al. 1977). The hot bubble is separated by the dense thin shell from the interstellar gas. The accepted parameters are close to those assumed by Berezhko & Völk (2000) in their analysis of gamma-ray production in SNRs. The lengthy discussion and the additional references can be found there.

The considerable fraction of cosmic rays is probably accelerated in Type Ia SNe (their explosion rate in the Galaxy is about 1/4 of supernovae Type II rate). These supernovae are caused by the thermonuclear explosions of compact white dwarfs following mass accretion. The characteristic mass of a progenitor star and the mass of ejecta are 1.4 solar mass. The progenitor stars do not appear to have observable amount of mass loss nor do they emit ionizing radiation that could essentially modify the ambient medium around the star. We assume that the SNR shock goes through the uniform weakly ionized interstellar medium with density 1 cm\(^{-3} \), and the magnetic field 7 \( \mu G \).

The two asymptotic regimes of the propagation of SNR shock - the ejecta dominated stage and the adiabatic stage - are instructive to consider.

The adiabatic regime was mentioned earlier, see the Sedov solution (11) for the shock moving in the gas with constant density, and it refers to the stage of SNR evolution when the mass of swept-up gas significantly exceeds the mass of ejecta. This condition is fulfilled in the case of the medium with constant density at \( R_{sh} > R_0 = (3M_{ej}/4\pi n_{0} t_0)^{1/3} = 1.9(M_{ej}/M_5 t_0)^{1/3} \) pc, \( t_0 > R_0/u_0 \approx 190 n_0^{-1/3} \text{ yr} \), where \( u_0 \approx 10^6 \text{ cm/s} \) is the ejecta initial velocity. The adiabatic regime for the SNR shock moving through the progenitor star wind is described by the equations (at \( u_{sh} \gg u_w \)):

\[
R_{sh} = 7.9 \left( \frac{\dot{E}_{51} u_{w,6}}{M_5} \right)^{1/3} t_{Kyr}^{2/3} \text{ pc},
\]

\[
u_{sh} = 5.2 \times 10^3 \left( \frac{\dot{E}_{51} u_{w,6}}{M_5} \right)^{1/3} t_{Kyr}^{-1/3} \text{ km/s},
\]

(13) see Ostriker & McKee (1998). As in Eq. (11), we assume that the ultrarelativistic gas of cosmic rays mainly determines the pressure behind the shock. Eq. (13) is valid when the mass of swept-up gas is relatively large and \( R_{sh} > R_0 = M_{ej} u_w/M \approx 1(M_{ej}/M_5)u_{w,6}/M_5 \) pc.

The quantity \( \rho_{sh} u_{sh}^3 R_{sh}^3 = K \dot{E} \) is conserved for the considered adiabatic shocks. The constant \( K \approx 0.16 \) for the solution (11), and \( K \approx 0.34 \) for the solution (13). In the general case of the power-law gas distribution \( \rho = \rho_0 (r/r_0)^{-s} \), \( s < 5 \), the adiabatic shock evolution is described by the equations \( \dot{E} = \eta(s) (\dot{E}/\rho_0) = \eta(s)\dot{E}/\rho_0 \), where \( \eta \) is constant at fixed \( s \), that gives the general formula \( K = \frac{4 \eta(s) K_0}{(s-3)^2} \) (the values of \( \eta(s) \) were given by Ostriker & McKee, 1988).

The ejecta dominated stage precedes the adiabatic one. As long as the mass of the ejecta is large compared to the swept-up mass, the blast wave is moving with relatively weak deceleration. At this stage shortly after the explosion, the structure of the flying apart envelope of the presupernova star is important for the shock evolution. Actually, the blast wave consists of two shocks, the forward shock and the reverse shock, with the contact discontinuity surface between them. This surface separates the shocked wind or interstellar gas downstream of the forward shock from the shocked envelope gas that fills the downstream region of the reverse shock. The reverse shock lags behind the forward shock and enters the dense internal part of the flying apart star by the time of the beginning of Sedov stage. Though it can not be well justified approximation for the very young SNRs, we ignore below the cosmic ray acceleration at the reverse shock compared to the forward shock and use the notation \( R_{sh} \) for the radius of forward shock (Berezinsky & Ptuskin, 1989) considered the cosmic ray acceleration by both shocks, see also Yoshida & Yanagita (2001)). The outer part of the star that freely expands after the SN explosion has a power law density profile \( \rho_0 \propto r^{-k} \), see e.g. Chevalier & Liang (1989). The value of \( k \) typically lies between 6 and 14. The value \( k \approx 7 \) is characteristic of the SNe Type Ia, and \( k \approx 10 \) is typical of the SNe Type II. The self similar solution for the blast wave at the ejecta dominated stage was found by Chevalier (1982) and Nadyozhin (1983, 1985). It was shown that at the age larger than about one week, the evolution of the shock at the ejecta dominated stage can be approximately described by the power-law dependence \( R_{sh} \propto t^\lambda \) where the expansion parameter \( \lambda = 1.8 \) for the explosion in the uniform medium, and \( \lambda = \frac{k}{k-3} \) for the explosion in the wind of a presupernova star (for \( k > 5 \)).
ejecta). In particular, using the results of two mentioned above papers, one can obtain the following equations

\[ R_{sh} = 5.3 \left( \frac{E_{51} M_{ej}}{n_0 M_{ej}} \right)^{1/7} t_{Kyr}^{4/7} \text{pc}, \]

\[ u_{sh} = 2.7 \times 10^4 \left( \frac{E_{51} M_{ej}}{n_0 M_{ej}} \right)^{1/7} t_{Kyr}^{-3/7} \text{km/s} \] (14)

for the Type Ia SN explosion in the uniform interstellar medium at \( k = 7 \);

\[ R_{sh} = 7.7 \left( \frac{E_{51}^{7/2} u_{w,0} M_{ej}^{5/2}}{M_{ej}^{7/2}} \right)^{1/8} t_{Kyr}^{7/8} \text{pc}, \]

\[ u_{sh} = 6.6 \times 10^3 \left( \frac{E_{51}^{7/2} u_{w} M_{ej}^{5/2}}{M_{ej}^{7/2}} \right)^{1/8} t_{Kyr}^{-1/8} \text{km/s} \] (15)

for the Type II SN explosion in the wind of a presupernova star at \( k = 10 \).

Following the approach of Truelove & McKee (1999), one can describe the shock produced by the Type Ia SN using the continuous solution which coincides with the ejecta dominated equation (14) until the moment \( t_0 = 260(M_{ej}/1.4M_{\odot})^{5/6}E_{51}^{-1/2}n_0^{-1/3} \text{yr} \), and is given by the equations

\[ R_{sh} = 4.3 \left( \frac{E_{51}}{n_0} \right)^{1/5} t_{Kyr}^{2/5} \times \left(1 - \frac{0.06(M_{ej}/E_{51})^{5/6}}{E_{51}^{1/2} n_0^{1/3}} t_{Kyr} \right)^{2/5} \text{pc}, \]

\[ u_{sh} = 1.7 \times 10^3 \left( \frac{E_{51}}{n_0} \right)^{1/5} t_{Kyr}^{-3/5} \times \left(1 - \frac{0.06(M_{ej}/E_{51})^{5/6}}{E_{51}^{1/2} n_0^{1/3}} t_{Kyr} \right)^{-3/5} \text{km/s} \] (16)

at a later time \( t > t_0 \). It is evident from Eq. (16) that the adiabatic asymptotics (11) holds at \( t > t_0 \).

The evolution of the Type II SN shock first follows the ejecta dominated solution (15) in a presupernova wind and then, while still moving in the wind, it enters the adiabatic regime at the distance \( r \sim 1 \text{pc} \). The subsequent evolution proceeds in the medium with a complicated structure described above for the Type II SN. The fairly accurate solution for the SNR evolution during this period can be obtained in the "thin-shell" approximation, e.g. Ostriker & McKee (1988), Bisnovatyi-Kogan & Silich (1985). Using this approximation for the strong shock and assuming the spherically symmetric distribution of the circumstellar gas density \( \rho(r) \), we come to the following equations where the shock velocity \( u_{sh} \) and the SNR age \( t \) are parameterized as functions of the shock radius \( R_{sh} \) (see Appendix for the derivation of these equations):

\[ u_{sh}(R_{sh}) = \frac{\gamma_{ad} + 1}{2} \left[ \frac{12(\gamma_{ad} - 1)E}{(\gamma_{ad} + 1)M^2(R_{sh})R_{sh}^{\gamma_{ad}-1}/(\gamma_{ad}+1)} \right] \]

\[ t(R_{sh}) = \int_0^{R_{sh}} \frac{dr}{u_{sh}(r)} \]

where \( \gamma_{ad} \) is the adiabatic index (\( \gamma_{ad} = 4/3 \) if the pressure downstream of the shock is determined by the relativistic particles), \( M(R) = M_{ej} + 4\pi \int_0^R dr r^2 \rho(r) \) is the mass of the swept up gas. The self similar solution by Chevalier and Nadyozhin is not explicitly reproduced by Eqs. (17). The solutions (15) and (17) are fitted together at the transition from the ejecta dominated regime to the adiabatic regime (at \( r \sim 0.3 \text{pc} \)) in our numerical simulations of cosmic ray acceleration in the Type II SNRs described below.

It is worth noting that the energy loss of SNR in a form of escaping cosmic rays is not taken into account in the solutions for shock evolution that were described in this Sections. In fact the shock evolution is only approximately adiabatic.

4. Average Spectrum of Cosmic Rays Injected in the Interstellar medium

At a given SNR age \( t \), the cosmic rays are accelerated up to maximum momentum \( p_{max}(t) \). Also, particles with \( p > p_{max}(t) \) cannot be confined in the precursor of the shock even if they were accelerated earlier. Thus particles accelerated to the maximum energy escape from a SNR (see also Berezko & Krymsky 1988). Let us estimate the flux of these run-away particles. We consider the simplified approach for maximum energy of accelerated particles and take the dependence of diffusion on momentum in the following simplified form:

\[ D(p) = D_0 << R u_{sh}, \ p \leq p_{max}(t), \]

\[ D(p) = D_m \gg R u_{sh}, \ p > p_{max}(t). \] (18)

The spectrum of accelerated particles in this case has a very steep cut-off at \( p > p_{max} \) (cf. Eq (1)) and the spectrum of run-away particles beyond \( p_{max} \) can be approximated by \( \delta \)-function. To find the equation for these particles, let us integrate the equation for cosmic-ray distribution function

\[ \frac{\partial f}{\partial t} - \nabla D \nabla f + u \nabla f - \frac{\nabla u}{3} \frac{\partial f}{\partial p} = 0 \] (19)

on momentum \( p \) from \( p_{max} \) to \( p_{max} + \Delta p \), where \( \Delta p \ll p_{max} \) and larger then the width of run-away particle spectrum. Denoting \( G = \int_{p_{max}}^{p_{max} + \Delta p} f dp \) one obtains from Eq (19):

\[ \frac{\partial G}{\partial t} + u \nabla G = \]

\[ \nabla D m \nabla G - \frac{\partial p_{max}}{\partial t} f(p_{max} - 0) - \frac{\nabla u}{3} p_{max} f(p_{max} - 0). \] (20)
Since diffusion coefficient of run-away particles is large, the advection terms play no role in this equation and the last two terms can be considered as source of particles. The total source of run-away particles is given by the volume integral of these terms. As a result, the source spectrum of run-away particles has the form

\[ q(p, t) = -\delta(p - p_{\text{max}}) \times \]
\[ \int d^3r \left( \frac{\partial p_{\text{max}}}{\partial t} + \frac{\nabla u}{3} p_{\text{max}} \right) f(p_{\text{max}} - 0, r). \]  

(21)

The integration here is performed over the domain where the integrand is negative. The integral \( 4\pi \int d p^2 q(p, t) \) has dimensions number of particles per unit time.

Below we consider the case of spherically symmetric SN shock with linear velocity profile at \( r < R_{\text{sh}} \):

\[ u = \left( 1 - \frac{1}{\sigma} \right) u_{\text{sh}}(t)r/R_{\text{sh}}(t), \]

(22)

where \( \sigma \) is the total shock compression ratio. It includes a thermal subshock and a cosmic ray precursor. Linear profile of velocity (22) is a good approximation of Sedov’s solution and it can be considered as a very approximate one at the ejecta-dominated stage. Since the shock is partially modified in the presence of cosmic rays, we should not assume any relation between the shock compression ratio \( \sigma \) and the spectral index of accelerated particles \( 4 - a \) (recall that \( 4 - a = 3\sigma/(\sigma - 1) \) for not modified shocks). We accept the value \( \sigma = 7 \) in our calculations.

The preshock at \( r > R_{\text{sh}} \) is created by the cosmic ray pressure gradient. Its width is small in comparison with the shock radius under the conditions given by Eq. (18) and the plane shock approximation can be used. Since the cosmic ray pressure dominates the gas pressure in the preshock region, its gradient is proportional to the velocity gradient \( \partial P_{\text{cr}}/\partial r = \rho u_{\text{sh}} \partial u/\partial r \), where \( \rho \) is the circumstellar medium density. We also use an assumption that cosmic ray pressure at the shock is some fraction \( \xi_{\text{cr}} \) of the upstream momentum flux, see Eq. (1). Now assuming that \( f(p_{\text{max}}) \) is proportional to the cosmic ray pressure we express the expression (21) for the run-away particle source takes the form

\[ q(p, t) = 4\pi \delta(p - p_{\text{max}}) \left( \frac{1}{3} \left( 1 - \frac{1}{\sigma} - \frac{\xi_{\text{cr}}}{2} \right) R^2 u_{\text{sh}} f_0(p) - \int_0^R r^2 dr f(p_{\text{max}}, r) \left( \frac{\partial p_{\text{max}}}{\partial t} + \frac{\nabla u}{\sigma} p_{\text{max}} u_{\text{sh}} \right) \right) \]

(23)

The first term in this expression describes the particles which runs away from the shock front, and the second term describes the particles escaping from the shock interior. In principle, the turbulence downstream the strong shock might be enhanced that would result in the small cosmic ray diffusion coefficient. In this case the particles do not run-away from the downstream and the second term in Eq. (23) should be omitted. If the turbulence downstream is maintained by the same process of the cosmic ray streaming instability as in the upstream region, the downstream diffusion coefficient is comparable to the upstream diffusion coefficient for particles with \( p \sim p_{\text{max}} \).

We shall further assume that particles can run away both from upstream and downstream of the shock. The uncertainty of the efficiency of run-away process in the inner part of SNR does not qualitatively change the conclusion about the average source spectrum of cosmic rays calculated later in this Section and shown in Fig. 2.

The distribution function of particles with \( p \leq p_{\text{max}} \) can be found using the solution of transport equation (19) at \( r < R_{\text{sh}} \) with the boundary condition \( f(p, r = R_{\text{sh}}, t) = f_0 \) by the method of characteristics. As the result

\[ q(p, t) = 4\pi \delta(p - p_{\text{max}}) \left( \frac{1}{3} \left( 1 - \frac{1}{\sigma} - \frac{\xi_{\text{cr}}}{2} \right) R^2 u_{\text{sh}} f_0(p) + \left( -\frac{\partial p_{\text{max}}}{\partial t} - \frac{\sigma - 1}{\sigma} p_{\text{max}} u_{\text{sh}} \left( \frac{u_{\text{sh}}}{R} \right) \right) \int_0^t \frac{dt}{\sigma} R^2(t') u_{\text{sh}}(t') \times \]

\[ f_0 \left( \frac{p}{R(t)} \right)^{1 - \frac{\xi_{\text{cr}}}{2}} \left( \frac{R(t)}{R(t')} \right)^{3 - \frac{\xi_{\text{cr}}}{2}} \right]. \]

(24)

The expression in brackets in front of the integral in Eq. (24) should be positive that means that the particles lose energy adiabatically slower then the maximum energy decreases. For the opposite sign, the adiabatic losses of particles are faster then the decrease of maximum energy and the particles don’t run away from downstream of the shock. They can run away at later time if at that time the decrease of maximum momentum will be faster.

The average source power \( Q(p) \) of run-away cosmic rays per unit volume in the galactic disk is obtained by the integrating \( q \) with respect of \( t \) and by the averaging over many SN explosions: \( Q(p) = \nu_{\text{sn}} \int_{p_{\text{min}}}^{p_{\text{max}}} dt dq(p, t) \), where \( \nu_{\text{sn}} \) is the average frequency of SN explosions per unit volume of the galactic disk. Changing the variable of integration from \( t \) to \( R_{\text{sh}} \) \( (dR_{\text{sh}} = u_{\text{sh}} dt) \) one can derive the following equation:

\[ Q(p) = \frac{3a \xi_{\text{cr}} \nu_{\text{sn}}}{c p^3} \int \frac{d \ln(p_{\text{max}})}{d \ln(R)} \left[ \frac{1}{3} \left( 1 - \frac{1}{\sigma} - \frac{\xi_{\text{cr}}}{2} \right) \left( \frac{R_{\text{sh}}}{R} \right)^3 + \left( \frac{1}{\sigma} - 1 - \frac{d \ln(p_{\text{max}})}{d \ln(R)} \right) \int_{R_{\text{min}}}^R \frac{dR'}{\sigma} \frac{\rho(R') u_{\text{sh}}^2(R')}{\left( \frac{mc}{p_{\text{max}}(R')} \right)^{\alpha}} \times \int_{R_{\text{sh}}}^{R_{\text{max}}(R')} \left( \frac{p}{R'} \right)^{\frac{\alpha}{\alpha - 1}} \left( \frac{R}{R'} \right)^{(\alpha - 1)(\sigma - 1)/\sigma} \right] \]

(25)

Here Eq. (1) for \( f_0 \) with the approximation \( \varphi(p) \approx a^{-(p/mc)^a} - a^{-(1 + a)^{-1}} \) is used, and the condition \( \xi_{\text{cr}} = \text{const} \) is assumed. The function \( R_{\text{sh}}(p) \) in Eq. (25) is defined by the equation \( p_{\text{max}}(R_{\text{sh}} = R_{\text{sh}}(p)) = p \). If
the last equation has multiple solutions at given p, the summation on all these solutions should be performed in (25). The physical meaning of \( R_m(p) \) is that it is the value of shock radius when the maximum energy of accelerating particles is equal to \( p \). The second term in the r.h.s. of Eq. (25) should be omitted if the expression in round parenthesis in front of the integral is negative.

Let us assume that the maximum momentum is a power law function of the shock radius, \( p_{\text{max}} \propto R^{-\delta} \), the particles are ultrarelativistic, \( p \gg mc \), and the compression ratio is constant, \( \sigma = \text{const.} \). The remarkable feature of Eq. (25) is then that the expression in square brackets does not depend on momentum at the adiabatic stage of SNR shock propagation in the medium with a power-law distribution of gas density, because \( \rho u^2_{\text{sh}} R^3_{\text{sh}} = KE \), \( K = \text{const} \) in this case, see Section 3. The average cosmic ray source power is now given by the simple equation

\[
Q(p) = \frac{3K a \xi_{\text{cr}} v_{\text{sh}} E}{\sigma \rho} \left[ \frac{1}{3 \delta} \left( 1 - \frac{1}{\sigma} \right) \right] + \frac{1}{\sigma - 1} \frac{1 - \frac{\sigma - 1}{\sigma}}{\sigma - 1 + a \left( \delta - 1 + \frac{1}{\sigma} \right)}.
\]

Here the factors \( \delta \), and \( 1 - \frac{\sigma - 1}{\sigma} \) should be positive. The first term in the square brackets describes the particles which run-away from the shock and the second term describes the particles which run-away from the SNR interior. Consequently, while in the adiabatic regime, the SNR shock during its evolution produces the run-away particles with the universal power-law overall spectrum \( Q(p) \propto p^{-4} \), whereas the instantaneous spectrum at the shock is more flat and not universal (see Eq. (1)) and the instantaneous spectrum of run-away particles has a delta-function form (see Eq. (24)).

The total source power of ultrarelativistic particles calculated with the use of Eq. (26) is \( W = 4\pi e \int dp Q(p) = C \xi_{\text{cr}} v_{\text{sh}} E \ln(p_{\text{max}2}/p_{\text{max}1}) \), where \( C = 12\pi K a \left( \frac{1}{3 \delta} \right) + \frac{1}{\sigma - 1 + a \left( \delta - 1 + \frac{1}{\sigma} \right)} \cdot p_{\text{max}2} \) and \( p_{\text{max}} \) are the maximum momenta of accelerated particles at the beginning and at the end of the adiabatic stage respectively, thus typically \( \ln(p_{\text{max}2}/p_{\text{max}1}) \approx 10 \). It leads to the estimate \( W \approx 0.5(\xi_{\text{cr}}/0.5) v_{\text{sh}} E \) for the shock moving in the uniform interstellar medium. Hence the considerable part of the total available mechanical energy of SN explosion \( v_{\text{sh}} E \) goes to cosmic rays at \( \xi_{\text{cr}} \sim 0.5 \). As it is well known, the source spectrum \( \propto p^{-4} \) or somewhat more steep, and the efficiency of cosmic ray acceleration at the level \( 10 - 30\% \) are needed to fit the cosmic ray data below the knee in the cosmic ray spectrum at about \( 4 \times 10^{15} \) eV in the empirical model of cosmic ray origin (e.g. Ptuskin (2001), see also discussion below).

At the ejecta dominated stage which precedes the adiabatic stage, the average spectrum of the run-away particles is different from \( p^{-4} \). Let us consider the general case and assume that \( \rho \propto r^{-s}, R_{\text{sh}} \propto t^4 \) and hence \( u_{\text{sh}} \propto t^{\delta-1} \). The maximum momentum of accelerated particles in the high velocity limit (7) has the scaling \( p_{\text{max}} \propto u_{\text{sh}}^2 R_{\text{sh}}^{1/2} \propto t^{(3 - \delta)/2} \propto R_{\text{sh}}^{3 - \delta - 1/2} \) so that \( R_{\text{sh}}(p) \propto p^{1 - \delta - 1/2} \). Now Eq. (25) at \( \lambda < 4/(6 - s) \) gives the following shape of the average spectrum of run-away particles:

\[
Q(p) \propto p^{-4 - \frac{\lambda(s - 1)}{6 - s}}.
\]

A characteristic of the adiabatic regime is the relation \( \lambda = \frac{2}{3 - s} \) and therefore Eq. (27) gives \( Q(p) \propto p^{-4} \) in agreement with Eq. (26). The Chevalier - Nadyozhin solution for the ejecta-dominated stage has \( \lambda = \frac{k - 3}{k - 3} \). With the set of parameters excepted in Section 3, we have then \( Q(p) \propto p^{-6.5} \) for the acceleration at the shock produced by the Type II SN in the presupernova star wind (\( s = 2, k = 10, \lambda = 7/8 \)), and \( Q(p) \propto p^{-7} \) for the acceleration at the shock produced by the Type Ia SN in the uniform interstellar medium (\( s = 0, k = 7, \lambda = 4/7 \)). Thus the cosmic rays accelerated at the ejecta dominated stage have higher energies than at the later adiabatic stage but the average energy spectrum of produced cosmic rays is rather steep at the "canonical" choice of presupernova star parameters.

The results of our numerical calculations of the average spectra for Type II and Type Ia SNe are shown in Fig. 2. The parameter \( \kappa \) is equal to 0.1 in a high-velocity regime (7) and it is equal to 0.04 in a low-velocity regime (9).

The calculations for Type II SN are based on Eqs. (5), (6), (14), (16) and (25). For the set of parameters accepted in the present paper, the Type II supernovae are able to accelerate cosmic ray protons to the maximum energy of the order \( 4 \times 10^{16} \) eV if the acceleration starts one week after the SN explosion when \( u_{\text{sh}} \approx 2.4 \times 10^4 \) km/s. The energy spectrum is close to \( p^{-4} \) at energies less than about \( 6 \times 10^{15} \) eV and it experiences the steepening above this energy. Thus the proton knee lies at about \( 6 \times 10^{15} \) eV in good agreement with the observational data. The sharp dip in the average proton spectrum at \( p/mc \sim 1 \times 10^6 - 3 \times 10^6 \) is caused by the assumed abrupt change of the gas density at the boundary between the dense Red Super Giant wind and the low density bubble. We run calculations up to the maximum shock radius 60 pc (the corresponding SNR age is \( 9 \times 10^4 \) yr) when the Mach number approaches 3 and the use of the particle spectrum (1) characteristic of the strongly modified shocks can not be longer justified. The protons are accelerated to about 20 GeV at this moment.

The calculations for Type Ia SN in Fig. 2 are based on Eqs. (5), (6), (14), (16) and (25). The calculations were made for the shock radius range from 0.2 to 30 pc (the SNR age from 4 yr to 1.3 \times 10^5 yr). The shock velocity is changing during this period from \( 2.8 \times 10^4 \) km/s to 91 km/s. The protons are accelerated from the maximum energy \( 7 \times 10^{15} \) eV to about 10 GeV. The approximate proton spectrum \( p^{-4} \) extends to the knee at about \( 3 \times 10^{15} \) eV.

The average source spectrum produced by Type Ia SNe is multiplied by 1/4 in Fig. 2 that corresponds to their
Fig. 2. The solid line shows the average source spectrum $Q(p) p^4 c$ (given in units of $\xi_{\nu_{sn}} \nu_{sn} E$ per steradian) for protons released into the interstellar medium during SNR evolution after SNII explosion in the wind of RSG progenitor star. The dashed line presents the case of SNIa explosion in the uniform interstellar gas; the average source spectrum is multiplied by 1/4. The dotted line shows the shape of proton source spectrum used by Hörandel (2003) to fit the KASCADE data.

relative burst rate and hence reflects the relative contribution of this type of supernovae to the total production of cosmic rays in the Galactic disk as compared to the contribution of Type II SNe.

5. Consistency with Cosmic Ray Data and Discussion

The spectrum of high-energy cosmic rays in the Galaxy is of the form $f \propto p^{-\gamma}$, $\gamma = \gamma_s + b$ under the steady state conditions when the action of cosmic ray sources (with the source power $Q \propto p^{-\gamma_s}$) is balanced by the escape of energetic particles from the Galaxy (with the escape time $T \propto p^{-b}$). The observed at the Earth all-particle spectrum of cosmic rays is close to $f \propto p^{-4.7}$ at energies $E \gtrsim 10$ GeV/nucleon with a characteristic transition (the knee, Kulikov & Khrustianskii 1958) ranging across less than one decade in the vicinity of $4 \times 10^{15}$ eV to the another power-law $f \propto p^{-5.1}$. The last extends to about $5 \times 10^{17}$ eV where the second knee with the break $\delta \gamma \sim 0.3$ is seen in the cosmic ray spectrum, see Hörandel (2003) for review. This structure is usually associated with a severe decrease of the efficiency of cosmic ray acceleration or/and confinement in the Galaxy. The extragalactic component of cosmic rays probably dominates at $E \gtrsim 3 \times 10^{18}$ eV (Gaisser et al. 1993). In the alternative interpretation (Berezinsky et al. 2004), the Galactic component falls steep (with $\gamma \sim 6$) at $E \gtrsim 10^{17}$ eV and the extragalactic component dominates from energy $\sim 3 \times 10^{17}$ eV and on.

The exponent $b = 0.3...0.7$ was obtained from the data on the abundance of secondary nuclei at energies $10^9$ to $10^{11}$ eV/nucleon. The secondary nuclei are produced in cosmic rays in a course of nuclear fragmentation of more heavy primary nuclei moving through the interstellar gas. The uncertainty in the value of $b$ is mainly due to the choice of specific model of cosmic ray transport in the Galaxy, see Ptuskin (2001). Hence it follows that the source exponent below the first knee lies in the range $\gamma_s = 4.0...4.4$. The value $\gamma_s \approx 4.0$ for the aver-
age source spectrum was obtained above in the consideration of particle acceleration by SNR shocks during their adiabatic evolution (though smaller $b \sim 0.3$ and consequently larger $\gamma_\text{e} \sim 4.4$ would be more favorable for the explanation of high isotropy of cosmic rays observed at $10^{12}$ to $10^{14}$ eV). According to the results of Section 4, the calculated average source spectrum $p^{-4}$ for protons accelerated by a "typical" Type II SNe extends up to about $6 \times 10^{15}$ eV that coincides with the observed position of the knee $\sim 4 \times 10^{15}$ eV within the accuracy of our analysis. The knee position at 3-5 PeV was determined in the recent KASCADE experiment (H. Ulrich et al. 2003). The scaling of the knee position in our model is $p_{\text{knee}} \propto Z x \xi E_{\text{ej}} M_{\text{ej}}^{1/2} M_{\text{ej}}^{-1} n_{\text{ej}}^{-1}$ for the explosion in the stellar wind and $p_{\text{knee}} \propto Z x \xi E_{\text{ej}}^{-2/3} n_{0}^{-1/6}$ for the explosion in the uniform interstellar medium.

As was reminded earlier, the diffusive shock acceleration at the strong nonmodified shock produces the spectrum $f \propto p^{-4}$. The back reaction of efficiently accelerating particles modifies the shock structure that results in a more flat particle spectrum (see references at the beginning of Section 2 and Eq. (1) where $a \sim 0.5$ if the shock modification is very strong). However, the numerical simulation of acceleration by SNR shocks under the standard assumption of Bohm diffusion in the shock precursor (calculated for the interstellar magnetic field strength) and with efficient confinement of accelerated particles during all SNR evolution gives the overall source spectrum that is close to $p^{-4}$ (Berezhko et al. 1996). Berezhko & Völk (2003) pointed out that the last result is in some sense accidental. The late stages of the SNR evolution are important here since relatively weak shock produces steep particle spectrum that has an effect on the overall spectrum. The situation is different in the model discussed in the present paper because the coefficient of diffusion is strongly decreasing with SNR age and the cosmic rays with energies larger than $10 - 30$ GeV/nucleon leave the supernova shell as the run away particles when the shock remains strong. The final average source spectrum of high-energy cosmic rays with energies larger than $10 - 30$ GeV/nucleon is close to $p^{-4}$ provided that the shock evolution is approximately adiabatic and the efficiency of particle acceleration $\xi_{\text{st}}$ is roughly constant. The source spectrum of particles with energies less than $10 - 30$ GeV/n may be more steep because they are accelerated by not very strong shocks. In this connection it should be noted that the source spectrum in the basic empirical model of cosmic ray propagation in the Galaxy is of the form $Q(p) p^a \propto p^{-2.4}$ at $E < 30$ GeV/n, and $Q(p) p^a \propto p^{-2.15}$ at $E > 30$ GeV/n, see Jones et al. 2001.

There are other important differences between the standard and the presented here models of cosmic ray acceleration. As noted before, our model of cosmic ray acceleration with strong increase in time of the diffusion coefficient and the corresponding decrease of maximum particle energy may naturally explain why the SNRs are generally not bright in very high energy gamma-rays at the age larger than a few thousand years. At this period of time, there are no particles with energies needed to generate the very high energy gamma-rays in SNR shell. Another problem is the contribution of gamma-ray emission from numerous unresolved SNRs with relatively flat spectra to the diffuse galactic background at very high energies. Working in the standard model Berezhko & Völk (2003) took the maximum possible energy of protons accelerated in SNRs equal to $10^{14}$ eV, i.e. well below the knee position, and it allowed not to exceed the upper limits on the diffuse gamma-ray emission at $4 \times 10^{11}$ to $10^{13}$ eV obtained in the Whipple, HEGRA, and TIBET experiments. In the model considered in the present work even with the efficient proton acceleration that may go beyond the knee, the expected gamma-ray emission from SNRs at $4 \times 10^{11}$ eV is order of magnitude smaller than in the model with Bohm diffusion. For a similar reason, the standard model compared to the present model predicts larger ratio of fluxes of secondary and primary nuclei formed at very high energies through the reacceleration of secondaries by strong shocks and through the direct production of secondaries by primary nuclei with flat energy spectra inside SNRs, see Berezhko et al. (2003).

The interpretation of energy spectrum beyond the knee in the present model is associated with the cosmic ray acceleration during the ejecta dominated stage of SNR evolution when the protons gain by an order of magnitude larger energy than at adiabatic stage but the number of particles involved in the shock accelerated is relatively small. The average source spectrum of accelerated particles is not universal at this stage. It has a power law high-energy asymptotics with the exponent $\gamma_\text{e}$ which value is very sensitive to the parameter $k$. The last is not well determined from the observations but the typical values accepted in our calculations were $k = 10$ for the Type II SN explosion in the wind of a Red Super Giant progenitor, and $k = 7$ for the Type Ia SN explosion in the uniform interstellar medium (see Chevalier & Liang 1989) that gives $\gamma_\text{e} = 6.5$ and $\gamma_\text{e} = 7$ respectively, see Section 4. To illustrate the range of possible uncertainty, it is worth noting that the value $k = 5.4$ was suggested for the Type Ia SNe by Imshennik et al. (1981). This value of $k$ results in $\gamma_\text{e} = 4.3$ at the ejecta dominated stage.

The breaks and cutoffs in the spectra of ions with different charges should occur at the same magnetic rigidity as for protons, i.e. at the same ratio $p/Z$ (or $E/Z$ for ultrarelativistic nuclei). The data of KASCADE experiment (Ulrich et al. 2003) for the most abundant groups of nuclei (protons, helium, CNO group, and the iron group nuclei) are, in general, consistent with this concept. According to Hörandel (2003) the good fit to the observations is reached if an individual constituent ion spectrum has a gradual steepening by $\delta \gamma_\text{e} \sim 2$ at energy $4 \times 10^{15} Z$ eV. Eq. (27) shows that the value $\delta \gamma_\text{e} = 2$ can be obtained at $k = 9, s = 2$ (the SN explosion in the progenitor wind), or $k = 6.6, s = 0$ (the SN explosion in the uniform interstellar medium) that is not very different from our accepted "typical" values, see Fig. 2.
At present, the main problem of the data interpretation centers around the second knee in the cosmic ray spectrum. The natural assumption that all individual ions has only one knee at \( \sim 4 \times 10^{15} \) Z eV and that the knee in the spectrum of iron \((Z = 26)\) expected at about \(10^{17} \) eV explains the second knee in the all-particle spectrum does not agree with the observed position of the second knee at \(5 \times 10^{17} \) eV. One way out was suggested by Hörandel (2003) who included all elements up to \(Z = 92\) into the consideration and assumed that \(\gamma_s\) decreases with \(Z\) to rise the contribution of ultra heavy nuclei from Galactic sources to the cosmic ray flux at \(\gtrsim 10^{17} \) eV. Of considerable promise is the approach by Svetsnikova (2003) who took into account the dispersion of parameters of SN explosions in her calculations of the knee position and the maximum particle energy. It leads to the widening of the energy interval between the two knees in the overall all-particle spectrum. This analysis should be supplemented by the account of different chemical composition of the progenitor star winds that determines the composition of accelerated cosmic rays (Silberberg et al. 1991). We plan the future work on this topic in the frameworks of the model developed in the present paper and Paper I. It should be noted that the model by Berezinsky et al. (2004) is quite consistent with the falling down of the flux from Galactic sources above \(10^{17} \) eV since the conjunction with the intergalactic cosmic ray flux in their model occurs at relatively low energy.

There is also a very different scenario which assumes the strong reacceleration of cosmic rays above the knee by the collective effect of multiple SNR shocks in the violent regions of Galactic disk (Axford 1968, Bykov & Toptygin 2001, Klepach et al. 2000) or Galactic wind (Volk & Zirakashvili 2004).

Finally, it is worth noting that in principle the knee may arise not in the sources but in the process of cosmic ray propagation in the Galaxy, e.g. as a result of interplay between the ordinary and the Hall diffusion (Ptuskin et al. 1998, Roulet 2003). However, this explanation requires the existence of the power-law source spectrum which extends without essential breaks up to about \(10^{18} \) eV or even further.

6. Conclusion

The accounting for non-linear effects which accompany the cosmic ray streaming instability raises the maximum energy of accelerated particles in young SNRs above the standard Bohm limit by about two orders of magnitude. It also considerably reduces the maximum energy of particles that are present inside SNRs at the late Sedov stage if, as it was assumed in our calculations in Section 4, the cosmic ray diffusion coefficient downstream of the shock is not much smaller than the diffusion coefficient in the cosmic ray precursor of the shock and the energetic particles with \(p \propto p_{\text{max}}\) runs away from the SNR interior. In the present paper we studied the effect of arising strong time dependence of maximum particle momentum \(p_{\text{max}}(t)\) on the average spectrum of cosmic rays injected into interstellar space from many supernova remnants over their lifetime. The instantaneous cosmic ray spectrum at strongly modified shock is flat \((f_0 \propto p^{-4+a}, a > 0, \text{Eq. (1)})\) and the particle energy density is mainly determined by the particles with maximum momentum \(p_{\text{max}}(t)\). The instantaneous source spectrum of the run away particles is close to the delta function \((q_{\text{s}}(t,p) \propto \delta(p_{\text{max}}(t) - p), \text{Eq. (24)}\).

At the same time, the assumption that the constant fraction \(\zeta_{\text{cr}}\) of incoming gas momentum flux goes to the cosmic ray pressure at the shock, and the fact that the supernova remnant evolution is adiabatic lead to the average on ensemble of SNRs source spectrum of ultrarelativistic particles that is close to \(Q_{\text{ra}} \propto p^{-4}\) from energies 10 – 30 GeV/n up to the knee position in the observed cosmic ray spectrum independent of the value of \(a\), see Eq. (26) and Fig 2. This source spectrum is consistent with the empirical model of cosmic ray propagation in the Galaxy. The acceleration at the preceding ejecta-dominated stage of SNR evolution provides the steep power-law tail in the particle distribution at higher energies up to \(\sim 10^{18} \) eV (if the iron nuclei dominate at these energies). The knee in the observed energy spectrum of cosmic rays at \(\sim 4 \times 10^{15} \) eV is explained in our model by the transition from the particle acceleration at the ejecta-doninate stage to the adiabatic stage of SNR shock evolution. In spite of approximate character of our consideration, it seems that the suggested scenario of particle acceleration can explain the energy spectrum of Galactic cosmic rays.

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Appendix A: Thin shell approximation

The thin-shell approximation can be used when the swept-up gas is concentrated in a thin layer behind the shock. In particular, it is applied to the case of a spherical adiabatic shock, see Ostriker J.P. & McKee C.F. (1988) and Bisnovatyi-Kogan & Silich (1993) for detail. The total mass of the gas shell involved in the motion and confined by the shock of radius \(R_{sh}\) in the spherically symmetrical case is

\[
M = M_{ej} + 4\pi \int_0^{R_{sh}} dr r^2 \rho(r),
\]

(A.1)

where \(M_{ej}\) is the ejected mass, \(\rho\) is the density of ambient gas.

The equation of momentum conservation is

\[
\frac{d(M u)}{dt} = 4\pi R_{sh}^2 (P_{in} - P).
\]

(A.2)

Here \(u\) is the gas velocity behind the shock, \(P_{in}\) is the pressure behind the shock, and \(P\) is the pressure of ambient gas. For the adiabatic blastwave, \(u\) is related to the...
shock velocity \( u_{sh} \) by the equation \( u_{sh} = \frac{\gamma_{ad} + 1}{2} u \), where \( \gamma_{ad} \) is the ratio of the specific heats (adiabatic index). The energy of explosion \( E = \mathcal{E}_{th} + \frac{4}{3} M u^2 \) consists of the internal energy \( \mathcal{E}_{th} = \frac{4\pi R_0^3}{3(\gamma_{ad} - 1)} P_{in} \) and the kinetic energy.

Now for the case of very strong shock when \( P_{in} \) can be omitted as negligible compared to \( P \), Eq. (A2) can be presented as:

\[
d\left(\frac{M_u}{M}ight)^2 = \frac{12(\gamma_{ad} - 1)}{(\gamma_{ad} + 1)R_u} \left( \mathcal{E}M - \frac{1}{2}(M_u)^2 \right).
\]

The solution of Eq. (A3) allows finding the shock velocity and the shock age as functions of the shock radius:

\[
u_{sh}(R_u) = \frac{\gamma_{ad} + 1}{2} \left[ \frac{2u\mathcal{E}}{M^2(R_u)R_{sh}} \int_{R_u}^{R_{sh}} dr r^{w-1} M(r) \right]^{1/2},
\]

\[
t(R_u) = \int_{R_u}^{R_{sh}} \frac{dr}{u_{sh}(r)}.
\]

where \( w = \frac{6(\gamma_{ad} - 1)}{\gamma_{ad} + 1} \) that coincides with Eq. (17) in the main text.

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