Hawking type radiation from acoustic black holes with time-dependent metric

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Abstract

We consider a time-dependent acoustic metric in 2+1 dimensions having a black hole and we study the Hawking type radiation from such black hole.

We construct eikonals of special form with the support close to the black hole. We compute the average number of created particles where the average is taken with respect to the Unruh type vacuum.

Keywords: Hawking radiation; black holes.

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1 Introduction

In this paper we study the rotating acoustic black holes when the metric is

\begin{equation}
(dx - \frac{A(x_0)}{|x|} \hat{x} dx_0)^2,
\end{equation}

where \(x = (x_1, x_2), \hat{x} = \frac{x}{|x|}, A(x_0) < 0, A(x_0) \rightarrow A(+\infty) \text{ when } x_0 \rightarrow +\infty, A(x_0) \rightarrow A(-\infty) \text{ when } x_0 \rightarrow -\infty.\)

The wave operator \(\Box_g\), corresponding to the metric (1.1), has the form

\begin{equation}
\Box_g u = \left( \frac{\partial}{\partial x_0} + \frac{A(x_0)}{\rho} \frac{\partial}{\partial \rho} \right)^2 - \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2},
\end{equation}

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where \((\rho, \varphi)\) are polar coordinates.

It was shown in [1] that metric \(\text{(1.1)}\) has a black hole \(B = \{0 < |x| < \rho^*(x_0)\}\) when \(A < 0\). In this paper we study the Hawking type radiation from such black holes.

In §2 we introduce elements of the quantum fields theory on curved space-times following mostly the lecture notes of T.Jacobson [6] (see also [4]). We define the number operator of created particles and we use the Unruh type vacuum state (cf. [7], [2]) to compute the average number of created particles.

In §3 we review (cf. [1]) the black holes for the rotating time-dependent acoustic metrics. In §4 we embark on the computation of the average number of particles created by some special wave packets that are similar to the wave packets used in the case of stationary rotating acoustic metrics (cf. [2]).

More precisely, we take

\[
C_0(\rho, x_0, \alpha) = \frac{1}{\sqrt{\rho}} e^{i\alpha \ln(\sigma(\rho, x_0) - \sigma_*) - a(\sigma(\rho, x_0) - \sigma_*)} (\sigma(\rho, x_0) - \sigma_*)^\varepsilon \theta(\sigma(\rho, x_0) - \sigma_*) ,
\]

where \(\alpha > 0\) is a constant, \(\sigma(\rho, x_0)\) is the solution of

\[
\frac{\partial \sigma}{\partial x_0} + \left( \frac{A(x_0)}{\rho} + 1 \right) \frac{\partial \sigma}{\partial \rho} = 0, \quad \sigma \big|_{x_0=0} = \rho ,
\]

and \(\sigma_* = \rho^*(x_0)\) when \(x_0 = 0\).

Note that the cases of black holes in the present paper (as in the paper [2] that treats the stationary acoustic metric with variable velocity) are not accessible by the methods of previous works of S.Hawking, T.Jacobson and many others. We use a new approach in a new situation. Thus we introduce a splitting on positive and negative modes, and we modify the definition of Unruh vacuum state. We construct appropriate wave packets to enhance the Hawking radiation.

The guiding motivation for our new definitions and constructions is that they are natural, and in the case of overlapping with the previous results they give the same (modulo greybody factors) exponential expression for the Hawking radiation.
2 Elements of quantum field theory on a curved spacetime

We introduce elements of quantum fields theory on curved spacetime following the lecture notes of T. Jacobson [6].

Let \( \langle u, v \rangle \) be the Klein-Gordon (KG) inner product of \( u \) and \( v \), i.e.

\[
\langle u, v \rangle = i \int_0^{2\pi} \int_0^{2\pi} \left[ \left( \frac{\partial u}{\partial x_0} - \frac{\partial \pi}{\partial \rho} v \right) + A(x_0) \frac{\partial v}{\partial \rho} - \frac{\partial \pi}{\partial \rho} \right] \rho d\rho d\varphi.
\]

Here \( x_0 \) is fixed. When \( u, v \) are the solutions of the wave equation (1.2) and the metric is independent of \( x_0 \) then \( \langle u, v \rangle \) is independent of \( x_0 \) (cf. [6]).

Denote by \( f^+_{k}(x_0, \rho, \varphi), f^-_{k}(x_0, \rho, \varphi) \) the solutions of (1.2) with the following initial conditions (cf. [2]):

\[
f^\pm_{k}(0, \rho, \varphi) = \gamma_0 e^{i\eta_\rho \rho + im\varphi},
\]

\[
\frac{\partial f^\pm_{k}(0, \rho, \varphi)}{\partial x_0} = i\lambda^\pm(k)\gamma_0 e^{i\eta_\rho \rho + im\varphi},
\]

where \( k = (\eta_\rho, m), m \in \mathbb{Z}, \gamma_0 = \frac{1}{2\pi \sqrt{2}} \frac{1}{\sqrt{(\eta_\rho^2 + 1)^2}} \) is a normalizing factor,

\[
\lambda^\pm(k) = -\frac{A(x_0)}{\rho} \eta_\rho \pm \sqrt{\eta_\rho^2 + 1}.
\]

Note that

\[
f^+_{k}(x_0, \rho, \varphi) = f^-_{-k}(x_0, \rho, \varphi).
\]

As in [2] one can show that

\[
\langle f^+_k(0, \rho, \varphi), f^+_k(0, \rho, \varphi) \rangle = \delta(\eta_\rho - \eta'_\rho)\delta_{mm'},
\]

\[
\langle f^-_k(0, \rho, \varphi), f^-_{k'}(0, \rho, \varphi) \rangle = -\delta(\eta_\rho - \eta'_\rho)\delta_{mm'}, \quad \langle f^+_k, f^+_{k'} \rangle = 0, \quad \forall k, \forall k',
\]

where

\[
k = (\eta_\rho, m), k' = (\eta'_\rho, m').
\]

Thus the following theorem holds:
Theorem 2.1. Let \( \{ f_k^+(0, \rho, \varphi), f_k^-(0, \rho, \varphi) \} \) form an “orthogonal” basis with respect to KG inner product. Let \( C(x_0, \rho, \varphi) \) be smooth. Then \( C(x_0, \rho, \varphi) \) can be expanded in \( \{ f_k^+, f_k^- \} \):

\[
C(x_0, \rho, \varphi) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} C_k^+ f_k^+(x_0, \rho, \varphi) d\eta \rho + \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} C_k^- f_k^-(x_0, \rho, \varphi) d\eta \rho,
\]

where

\[
C_k^+ = \langle f_k^+, C \rangle \big|_{x_0=0}, \quad C_k^- = -\langle f_k^-, C \rangle \big|_{x_0=0}.
\]

Note that the KG inner product is taken for \( x_0 = 0 \).

Let \( \Phi(x_0, \rho, \varphi) \) be the wave operator (cf. [6]):

\[
\Box_g \Phi = 0.
\]

Expanding \( \Phi \) in the basis \( \{ f_k^+, f_k^- \} \) we get

\[
\Phi = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \alpha_k^+ f_k^+(x_0, \rho, \varphi) + \alpha_k^- f_k^-(x_0, \rho, \varphi) \right) d\eta \rho,
\]

where \( \alpha_k^+, \alpha_k^- \) are called the annihilation and creation operators

\[
(\alpha_k^+)^* = \alpha_k^-, \quad \alpha_k^+ = \langle f_k^+, \Phi \rangle \big|_{x_0=0}, \quad \alpha_k^- = -\langle f_k^-, \Phi \rangle \big|_{x_0=0}.
\]

These operators satisfy the following commutation relation (cf. [6]):

\[
[\alpha_k^+, \alpha_{k'}^-] = \delta(\eta_{\rho} - \eta_{\rho}') \delta_{mm'} I, \quad [\alpha_k^+, \alpha_k^+] = 0,
\]

\[
[\alpha_k^-, \alpha_{k'}^-] = 0, \quad k = (\eta_{\rho}, m), \quad k' = (\eta_{\rho}', m'),
\]

\( I \) is the identity operator.

To introduce the Unruh type vacuum space we need to split \( f_k^+(x_0, \rho, \varphi) \) and \( f_k^-(x_0, \rho, \varphi) \) into two parts:

\[
f_k^{++} = f_k^+ \theta(\eta_{\rho}), \quad f_k^{+-} = f_k^+ (1 - \theta(\eta_{\rho})),
\]
where \( \theta(\eta, \rho) = 1 \) when \( \eta, \rho > 0 \), \( \theta = 0 \) when \( \eta, \rho < 0 \). Analogously

(2.15) \[
f^{-+} = f^{-}\theta(\eta, \rho), \quad f^{--} = f^{-}(1 - \theta(\eta, \rho)),
\]

(2.16) \[
\alpha^{++} = \alpha^{+}\theta(\eta, \rho), \quad \alpha^{+-} = \alpha^{+}(1 - \theta(\eta, \rho)), \\
\alpha^{--} = \alpha^{-}\theta(\eta, \rho), \quad \alpha^{---} = \alpha^{-}(1 - \theta(\eta, \rho)).
\]

Using (2.15), (2.16), we have

(2.17) \[
\Phi = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^{++}_{k} f^{++}_{k} + \alpha^{+-}_{k} f^{+-}_{k} + \alpha^{--}_{-k} f^{-+}_{-k} + \alpha^{---}_{-k} f^{-++}_{-k}) d\eta, 
\]

Operator \( N \) of the number of the particles created by packet \( C(x_0, \rho, \varphi) \) is defined as (cf. [6], [4]):

(2.18) \[
N = \langle C, \Phi \rangle^* \langle C, \Phi \rangle.
\]

We shall define the Unruh type vacuum state \( |\Psi\rangle \) by the conditions

(2.19) \[
\alpha^{++}_{k} |\Psi\rangle = 0, \quad \alpha^{+-}_{-k} |\Psi\rangle = 0, \quad \forall k.
\]

The average number of created particles is

(2.20) \[
\langle |\Psi\rangle |N| |\Psi\rangle.
\]

Analogously to [2] we can prove

**Theorem 2.2.** The average number of particles created by the wave packet \( C \) is given by

(2.21) \[
\langle |\Psi\rangle |N| |\Psi\rangle = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} ( - |C^{++}(k)|^2 + |C^{+-}(k)|^2 ) d\eta.
\]

**Proof:** Since \( \alpha^{++}_{k} |\Psi\rangle = 0, \quad \alpha^{+-}_{-k} |\Psi\rangle = 0 \) we have

(2.22) \[
\langle C, \Phi \rangle |\Psi\rangle = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (C^{++}(k)\alpha^{++}_{k} |\Psi\rangle - C^{+-}(k)\alpha^{+-}_{-k} |\Psi\rangle) d\eta.
\]
where $C^{+}(k) = \langle f_{k}^{+}, C \rangle$, $C^{-}(k) = - \langle f_{-k}^{-}, C \rangle$. Since

$$\langle \Psi | (\alpha_{k}^{++})^* = 0, \langle \Psi | (\alpha_{-k}^{-})^* = 0,$$

(2.23) we have

$$\langle \Psi | (C, \Phi)^* = \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} \langle \Psi | C^{+}(\alpha_{k}^{++})^* - \langle \Psi | C^{-}(\alpha_{k}^{-})^* d\eta \rho.$$

(2.24) Therefore, as in [2], we get

$$\langle \Psi | N | \Psi \rangle = \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} (- |C^{+}(k)|^2 + |C^{-}(k)|^2) d\eta \rho.$$

(2.25)

3 Black holes in rotating acoustic time-dependent metric

Consider a rotating fluid with the velocity

$$\bar{v} = \frac{A(x_0)}{\rho} \hat{x},$$

(3.1) where $\hat{x} = \frac{(x_1, x_2)}{|x|}, \rho = |x|, x_0$ is the time coordinate, $A(x_0) \in C^{\infty}, A(x_0) < 0$ for all $x_0$, $\lim_{x_0 \to +\infty} A(x_0) = A(+\infty), \lim_{x_0 \to -\infty} A(x_0) = A(-\infty)$. We assume that there is no dependence on $\varphi$, where $(\rho, \varphi)$ are polar coordinates. The acoustic metric associated with this flow has the form (cf. [9])

$$\left( dx - \frac{A(x_0)}{|x|} \hat{x} dx_0 \right)^2, \ x = (x_1, x_2),$$

(3.2) and the corresponding wave equation is

$$\Box_g u \equiv \left( \frac{\partial}{\partial x_0} + \frac{A(x_0)}{\rho} \frac{\partial}{\partial \rho} \right)^2 u - \frac{\partial^2 u}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} = 0.$$

(3.3) The Hamiltonian (symbol) of $\Box_g$ is

$$H = \left( \xi_0 + \frac{A(x_0)}{\rho} \xi_\rho \right)^2 - \xi_\rho^2 - \frac{1}{\rho^2} \xi_\varphi^2.$$


The case of time-dependent acoustic metric was studied in [1]. It was proven there that when \( A(x_0) < 0 \) for all \( x_0 \) there exists a black hole \( B = \{ 0 < \rho < \rho^*(x_0) \} \) where \( \rho^*(x_0) \) is the solution of

\[
\frac{d\rho}{dx_0} = \frac{A(x_0)}{\rho} + 1,
\]

with some initial data

\[
\rho^*(0) = \sigma^*.
\]

Note (cf. [1]) that all solutions \( \rho = \rho(x_0) \) of (3.4) tends to \( A(-\infty) \) when \( x_0 \to -\infty \). When \( \rho(0) = \sigma > \sigma^* \) then the solution \( \rho = \rho(x_0) \) tends to \( +\infty \) when \( x_0 \to +\infty \) and when \( \rho(0) = \sigma^* < \sigma^* \) then the solution \( \rho = \rho(x_0) \) tends to zero when \( x_0 \to +\infty \). Thus the solution \( \rho = \rho^*(x_0) \) separates the solutions of (3.4) that tend to \( +\infty \) when \( x_0 \to +\infty \) from the solutions of (3.4) that tend to 0 when \( x_0 \to +\infty \).

Note that \( \rho^*(x_0) \to |A(-\infty)| \) when \( x_0 \to -\infty \) and \( \rho^*(x_0) \to |A(+\infty)| \) when \( x_0 \to +\infty \).

In the next section we shall study the Hawking type radiation from the acoustic black hole \( B = \{ 0 < \rho < \rho^*(x_0) \} \).

4 Hawking type radiation from rotating acoustic black hole

We shall use wave packet \( C(x_0, \rho) \) independent of \( \varphi \).

Let \( f^\pm_k(x_0, \rho, \varphi) \) be the same as in (2.2). Since \( C(x_0, \rho) \) is independent of \( \varphi \) we have

\[
\int_0^\infty \int_0^{2\pi} f^\pm_{\eta, m}(x_0, \rho, \varphi) C(x_0, \rho) \rho d\rho d\varphi = \begin{cases} 2\pi \int_0^\infty f^\pm_{\eta, 0}(x_0, \rho) C(x_0, \rho) \rho d\rho & \text{if } m = 0 \\ 0 & \text{if } m \neq 0, \end{cases}
\]

where \( f^\pm_{\eta, 0}(x_0, \rho) \) satisfies

\[
Lf^\pm_{\eta, 0} = \left( \frac{\partial}{\partial x_0} + \frac{A(x_0)}{\rho} \frac{\partial}{\partial \rho} \right)^2 f^\pm_{\eta, 0} - \frac{\partial^2 f^\pm_{\eta, 0}(x_0, \rho)}{\partial \rho^2} = 0.
\]
Note that $f^\pm_{\eta_\rho,0}$ has the following initial conditions (cf. (2.2), (2.3))

\begin{equation}
\left.\frac{\partial f^\pm_{\eta_\rho,0}}{\partial x_0}\right|_{x_0=0} = i\lambda^\pm(\eta_\rho)e^{i\eta_\rho \rho},
\end{equation}

where

\begin{equation}
\lambda^\pm(\eta_\rho) = -\frac{A(0)}{\rho} \eta_\rho \pm \sqrt{\eta_\rho^2 + 1}, \quad \gamma = \frac{1}{\sqrt{2}} \sqrt{\eta_\rho^2 + 1}^{-\frac{3}{4}}.
\end{equation}

Let

\begin{equation}
E = \gamma e^{-i\eta_\rho \sigma(\rho, x_0)}, \quad \eta < 0,
\end{equation}

where

\begin{equation}
\frac{\partial \sigma(\rho, x_0)}{\partial x_0} + \left(\frac{A(x_0)}{\rho} + 1\right) \frac{\partial \sigma}{\partial \rho} = 0, \quad \sigma(\rho, 0) = \rho.
\end{equation}

Note that $\sigma = \sigma(\rho, x_0), \sigma(\rho, 0) = \rho$ is the inverse function to $\rho = \rho(\sigma, x_0)$ where $\rho(\sigma, x_0)$ is the solution of

\begin{equation}
\frac{d\rho}{dx_0} = \frac{A(x_0)}{\rho} + 1, \quad \rho(\sigma, 0) = \sigma.
\end{equation}

Indeed, differentiating identity $\sigma(\rho(\sigma, x_0), x_0) = \sigma$ in $x_0$ we get

\begin{equation}
\frac{\partial \sigma}{\partial x_0} + \frac{\partial \sigma}{\partial \rho} \left(\frac{A}{\rho} + 1\right) = 0.
\end{equation}

Note that

\begin{equation}
\left.\frac{\partial}{\partial x_0} E\right|_{x_0=0} = E(-i\eta_\rho) \left(\frac{A}{\rho} \frac{\partial \sigma}{\partial x_0} + i\eta_\rho \frac{\partial \sigma}{\partial \rho}\right)\bigg|_{x_0=0}.
\end{equation}

Note that $\left.\frac{\partial \sigma}{\partial \rho}\bigg|_{x_0=0}\right. = 1$. Comparing (1.8) and (4.3) we have that factor $\sqrt{\eta^2_\rho + 1}$ in (4.3) is replaced by $|\eta_\rho|$ in (1.8).

For the simplicity of notations we shall denote $f^\pm_{\eta_\rho,0}$ for $\eta_\rho < 0$ by $f^+_0$ and we shall write $\eta$ instead of $\eta_\rho$. 

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We shall construct \( f_0^{+-} \) in the form
\[
(4.9) \quad f_0^{+-} = E + f_1^{+-} + f_2^{+-},
\]
where
\[
(4.10) \quad L(E + f_1^{+-}) = 0, \quad f_1^{+-}\bigg|_{x_0=0} = \frac{\partial f_1^{+-}}{\partial x_0} \bigg|_{x_0=0} = 0,
\]
\( L \) is the same as in (4.2) and \( f_2^{+-} \) satisfies
\[
(4.11) \quad Lf_2^{+-} = 0,
\]
\[
f_2^{+-}\bigg|_{x_0=0} = 0, \quad \frac{\partial f_2^{+-}}{\partial x_0} \bigg|_{x_0=0} = -i\gamma(\sqrt{\eta^2 + 1} - |\eta|)e^{-i\rho} = \frac{-i}{\sqrt{\eta^2 + 1 + |\eta|}} e^{-i\rho}.
\]

We shall estimate \( f_1^{+-} \) and \( f_2^{+-} \) later, but first we compute the contribution to the Hawking radiation of the principal term \( E \).

We shall define the wave packet \( C_0(x_0, \rho) \) as
\[
(4.12) \quad C_0(x_0, \rho) = \frac{1}{\sqrt{\rho}} e^{i\alpha \ln(\sigma(\rho, x_0) - \sigma^*)} e^{-\alpha(\sigma(\rho, x_0) - \sigma^*)} \cdot (\sigma(\rho, x_0) - \sigma^*)^\varepsilon \theta(\sigma(\rho, x_0) - \sigma^*)
\]
where \( \sigma^* \) is the same as in (3.5).

There are following important features of \( C_0(x_0, \rho) \):
\begin{enumerate}
  \item a) when \( \rho \to \rho^* \) then \( \sigma(\rho, x_0) \to \sigma^* \) and vice versa since \( \sigma(\rho, x_0) \) is an inverse function to \( \rho = \rho(\sigma, x_0) \). Therefore, \( \text{supp} \ C_0(\rho, x_0) \) tends to \( \{ \rho = \rho^*(x_0) \} \) when \( a \to \infty \);
  \item b) \( \alpha \ln(\sigma(\rho, x_0) - \sigma^*) \) is singular on the event horizon \( \rho = \rho^*(x_0) \).
\end{enumerate}

To compute the average of the number of created particles we need to compute \( \text{cf. (2.25)} \)
\[
(4.13) \quad \int_{-\infty}^{\infty} (|C^{-+}(\eta)|^2 - |C^{+-}(\eta)|^2) d\eta,
\]
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where

\begin{equation}
C^{-+}(\eta) = - < f_0^{++}, C_0 > \bigg|_{x_0=0} = C_1^{-+} + C_2^{-+},
\end{equation}

\begin{equation}
C^{+-}(\eta) = < f_0^{+-}, C_0 > \bigg|_{x_0=0} = C_1^{+-}(\eta) + C_2^{+-}(\eta).
\end{equation}

We have

\begin{equation}
C_1^{+-}(\eta) = i \int_0^\infty \left( \frac{\partial C_0}{\partial x_0} + \frac{A(x_0)}{\rho} \frac{\partial C_0}{\partial \rho} \right) \rho d\rho,
\end{equation}

\begin{equation}
C_2^{+-} = -i \int_0^\infty \left( \frac{\partial f_0^{+-}}{\partial x_0} C_0 + \frac{A(x_0) \partial f_0^{+-}}{\partial \rho} C_0 \right) \rho d\rho.
\end{equation}

Analogously, we define \( C_1^{-+}(\eta) \) and \( C_2^{-+}(\eta) \). We first consider the case when \( f_0^{+-} \) is replaced by

\[ E = \frac{1}{\sqrt{2\sqrt{\rho}(\eta^2 + 1)^{\frac{3}{2}}} e^{-i\eta \sigma(x_0, \rho)}, \eta < 0. \]

We have

\begin{equation}
C_{10}^{+-}(\eta) = \int_0^\infty \frac{i}{\sqrt{2\eta^2 + 1}} e^{i\eta \sigma(x_0, \rho)} \left( \frac{\partial C_0}{\partial x_0} + \frac{A(x_0) \partial C_0}{\partial \rho} \right) \rho d\rho,
\end{equation}

\( \eta < 0, \) where \( C_{10}^{+-} \) is the same as \( C_1^{+-} \) with \( f_0^{+-} \) replaced by \( E \). It follows from \( 4.12 \) that

\[ C_0 = \frac{1}{\sqrt{\rho}} C_{01}(\sigma(x, x_0)) \text{ where } C_{01}(\sigma) = e^{i\alpha \ln(\sigma-\sigma^*)-a(\sigma-\sigma^*)^\varepsilon}(\sigma-\sigma^*). \]

Therefore,

\begin{equation}
\frac{\partial C_0}{\partial x_0} + \frac{A(x_0) \partial C_0}{\rho} = \frac{A(x_0)}{\rho} \left( \frac{\partial}{\partial \rho} \frac{1}{\sqrt{\rho}} \right) C_0(\sigma(x, x_0)) - \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial \rho} C_{01}(\sigma(x, x_0))
\end{equation}
since
$$\frac{\partial C_{01}}{\partial x_0} + A \frac{\partial C_{01}}{\partial \rho} + \frac{\partial C_{01}}{\partial \rho} = 0.$$ 

Thus
$$C^{+-}_{10} = \dot{C}^{+-}_{10} + I_1,$$

where

(4.20) \(\dot{C}^{+-}_{10} = \int_{\sigma_*}^{\infty} \frac{i}{\sqrt{2(\eta^2 + 1)^{\frac{3}{4}}}} e^{i\eta\sigma(\rho,x_0)} \left(-\frac{\partial}{\partial \rho}\right) C_{01}(\sigma(\rho,x_0)) d\rho,

(4.21) \(I_1 = \int_{\rho^*(x)}^{\infty} \frac{A(x_0)}{\rho} \left(\frac{\partial}{\partial \rho} \frac{1}{\sqrt{\rho}}\right) \frac{1}{\sqrt{2(\eta^2 + 1)^{\frac{1}{4}}}} e^{i\eta\sigma(\rho,x_0)} C_{01}(\sigma(\rho,x_0) - \sigma_*) d\rho.

Making in (4.20) change of variable \(\sigma = \sigma(\rho,x_0)\) we get

(4.22) \(\dot{C}^{+-}_{10} = \int_{\sigma_*}^{\infty} \frac{i}{\sqrt{2(\eta^2 + 1)^{\frac{3}{4}}}} e^{i\eta\sigma} \left(-\frac{\partial}{\partial \sigma}\right) C_{01}(\sigma - \sigma_*) d\sigma.

Performing the Fourier transform in (4.22) and replacing \(\sigma - \sigma_*\) by \(\sigma'\) we get

(4.23) \(\dot{C}^{+-}_{10} = e^{i\sigma\eta} \frac{i}{\sqrt{2(\eta^2 + 1)^{\frac{3}{4}}}} \eta \tilde{C}_{01}(-\eta),

where

(4.24) \(\tilde{C}_{01}(-\eta) = \int_0^{\infty} e^{i\sigma'\eta} e^{\alpha \ln \sigma' - a\sigma'(\sigma')^\varepsilon} d\sigma' = \frac{\Gamma(i\alpha + \varepsilon + 1)}{(\eta + ia)^{\alpha + \varepsilon + 1}} e^{i\frac{\pi}{2}(\alpha + \varepsilon + 1)}

= e^{i\frac{\pi}{2}(\alpha + \varepsilon)} \Gamma_0(i\alpha + \varepsilon + 1) e^{(-i\alpha - \varepsilon - 1) \ln(\eta + ia)} e^{i\frac{\pi}{2}(\alpha + \varepsilon + 1)}

Here (cf. [2])

(4.25) \(\Gamma(i\alpha + \varepsilon + 1) = e^{\frac{i\pi}{2}(\alpha + \varepsilon)} \Gamma_0(i\alpha + \varepsilon + 1), \Gamma_0(i\alpha + \varepsilon + 1) = \int_0^{\infty} e^{(i\alpha + \varepsilon) \ln y - iy} dy.$
Since $\eta < 0$, $a > 0$, we have $\arg(\eta + ia) = \pi - \sin^{-1} \frac{\eta}{\sqrt{\eta^2 + a^2}}$

Analogously, 

$$C_{\pm 20}^{\pm -} = \hat{C}_{20}^{\pm -} - I_1,$$

where

$$(4.26) \quad \hat{C}_{20}^{\pm -} = -i \int_0^\infty \frac{e^{i\sigma \ast}}{\sqrt{2(\eta^2 + 1)}} \left( -\frac{\partial}{\partial \rho} e^{i\sigma(\rho, x_0)} \right) C_{01}(\sigma(\rho, x_0)) d\rho.$$ 

Since $\frac{\partial}{\partial \rho} e^{i\sigma(\rho, x_0)} = i\eta \frac{\partial}{\partial \rho} e^{i\sigma(\rho, x_0)}$ we get, changing variables $\sigma = \sigma(\rho, x_0)$ :

$$(4.27) \quad \hat{C}_{20}^{\pm -} = \frac{ie^{i\sigma \ast}}{\sqrt{2(\eta^2 + 1)}} i\eta \tilde{C}_{01}(-\eta).$$

Analogously to [2] we have

$$(4.28) \quad C_{-1}^{\pm +} = -\hat{C}_{10}^{\pm +} + \hat{C}_{20}^{\pm +},$$

and, therefore, from (4.13), we get

$$(4.29) \quad |C_{-1}^{\pm +}|^2 - |C_{20}^{\pm +}|^2 = -4\Re \hat{C}_{10}^{\pm +} \overline{\hat{C}_{20}^{\pm +}}.$$ 

Thus, 

$$(4.30) \quad \langle \Psi | N | \Psi \rangle = -\int_0^\infty 4\Re \hat{C}_{10}^{\pm +} \overline{\hat{C}_{20}^{\pm +}} d\eta.$$ 

When $f_{0}^{\pm -}$ is replaced by $E$ we have, as in [2],

$$(4.31) \quad -\int_0^\infty 4\Re \hat{C}_{10}^{\pm +} \overline{\hat{C}_{20}^{\pm +}} d\eta = \int_0^\infty \frac{2\eta^2 |\Gamma_0(i\alpha + \varepsilon + 1)|^2 e^{-2\alpha \sin^{-1} \frac{\eta}{\sqrt{\eta^2 + a^2}}}}{\sqrt{\eta^2 + 1}|\eta + ia|^{2\varepsilon + 2}} d\eta.$$ 

Changing $\eta = \eta' a$ and taking the limit when $a \to \infty$ we get

$$(4.32) \quad -\int_0^\infty 4\Re \hat{C}_{10}^{\pm +} \overline{\hat{C}_{20}^{\pm +}} d\eta$$

$$= a^{-2\varepsilon} |\Gamma_0(i\alpha + \varepsilon + 1)|^2 \int_0^\infty \frac{2\eta}{|\eta^2 + 1|^\varepsilon + 1} e^{-2\alpha \sin^{-1} \frac{1}{\sqrt{\eta^2 + a^2}}} d\eta + O(a^{-2\varepsilon - 1}).$$
We used in (4.32) that \( \eta^{-1} \to 1 \) when \( a \to \infty \).

Now we shall normalize \( C_0 \). We have (cf. (3.8) in [2])

\[
\langle C_0, C_0 \rangle = \int_0^\infty \int_0^{2\pi} \rho d\rho d\varphi = 4\pi \alpha \int_0^\infty (\sigma - \sigma_*)^{2\varepsilon-1} e^{-2a(\sigma - \sigma_*)} d\sigma.
\]

Note that we changed variables \( \sigma = \sigma(\rho, \theta) \) in (4.33). Therefore

\[
\langle C_0, C_0 \rangle = 4\pi \alpha \Gamma(2\varepsilon) \frac{(2a)^{2\varepsilon}}{(2\varepsilon)}.\]

Denote

\[
C_n = \frac{C_0}{\langle C_0, C_0 \rangle}, \quad N_n(x_0) = \frac{N}{\langle C_0, C_0 \rangle}.
\]

Then

\[
\langle \Psi | N_n | \Psi \rangle = \frac{\langle \Psi | N | \Psi \rangle}{\langle C_0, C_0 \rangle}.
\]

Thus (cf. [3])

\[
\lim_{a \to \infty} \langle \Psi | N_n | \Psi \rangle = \frac{2^{2\varepsilon} \Gamma_0(i\alpha + \varepsilon + 1)^2}{2\pi \alpha \Gamma(2\varepsilon)} \int_0^\infty \frac{\eta}{|\eta^2 + 1|^{1/2}} e^{-2\alpha \sin^{-1} \frac{1}{\sqrt{\eta^2 + 1}}} d\eta,
\]

where we replaced in (4.32) and (4.37) \( f_0^{+-} \) by \( E \).

Now we shall compute the contribution of \( f_1^{+-} \) and \( f_2^{+-} \), and we will show that they contribute to the lower order terms in \( \frac{1}{a} \), thus (4.32) holds for \( f_0^{+-} \) and not only for \( E^{+-} \). In coordinates \( (\sigma, x_0) \) \( f_1^{+-} \) satisfies

\[
L f_1^{+-} = \frac{\partial^2}{\partial x_0^2} f_1^{+-} + b_1 \frac{\partial^2 f_1^{+-}}{\partial x_0 \partial \sigma} + b_2 \frac{\partial f_1^{+-}}{\partial x_0} + b_3 \frac{\partial f_1^{+-}}{\partial \sigma} = -LE,
\]
(4.39) \[ f_1^{+-} \big|_{x_0=0} = \frac{\partial f_1^{+-}}{\partial x_0} \big|_{x_0=0} = 0. \]

Note that

\[ LE = b_3(x_0, \sigma) \gamma(-i\eta)e^{-i\eta\sigma}, \]

where \( \gamma \) is the same as in (4.4). Let \( G(x_0, \sigma, x'_0, \sigma') \) be the Green function for \( L \)

(4.40) \[ G \big|_{x_0=0} = 0, \quad \frac{\partial G}{\partial x_0} \big|_{x_0=0} = 0. \]

Then

(4.41) \[ f_1^{+-} = \int \int_{D(x_0, \sigma)} G(x_0, \sigma, x'_0, \sigma') b_3(x'_0, \sigma') \gamma(-i\eta) \gamma e^{-i\eta \sigma'} \, dx'_0 \, d\sigma', \]

where \( D(x_0, \sigma) \) is the domain of dependence of \( (x_0, \sigma) \).

Integrating by parts using the identity

\[ e^{-i\sigma\eta} = -\frac{1}{i\eta} \frac{d}{d\sigma} e^{-i\eta\sigma} \]

we get

(4.42) \[ |f_1^{+-}| \leq \frac{C}{1 + |\eta|} \frac{1}{(1 + |\eta|^2)^{\frac{1}{2}}}, \quad \left| \frac{\partial f_1^{+-}}{\partial \sigma} \right| \leq \frac{C|\eta|}{1 + |\eta|} \frac{1}{(1 + |\eta|^2)^{\frac{1}{2}}}. \]

Substituting in (4.13) \( f_1^{+-} \) instead of \( f_0^{+-} \) we get \( C_{11}^{+-} \) and \( C_{12}^{+-} \), where \( C_{11}^{+-} \) and \( C_{12}^{+-} \) are the same as in (4.16), (4.17) with \( f_0^{+-} \) replaced by \( f_1^{+-} \).

Using (4.42) we get

(4.43) \[ |C_{11}^{+-}| \leq \int_0^\infty \frac{C}{(\eta^2 + 1)^{\frac{1}{2}} (1 + |\eta|)} \left| \frac{\partial}{\partial \rho} C_{01} \right| \left| \frac{\partial \sigma}{\partial \rho} \right| d\rho \]

\[ \leq C \int_0^\infty \frac{1}{(\eta^2 + 1)^{\frac{1}{2}} (1 + |\eta|)} \left| \frac{\partial}{\partial \sigma} C_{01}(\sigma) \right| d\sigma. \]

Since

\[ \left| \frac{\partial}{\partial \sigma} C_{01}(\sigma') \right| \leq C e^{-a\sigma'} \left| \frac{\alpha}{\sigma'} + \frac{\varepsilon}{\sigma} + a \right| \sigma', \]
we get, changing variable $a\sigma = t$,

\begin{equation}
|C_{11}^{-+}| \leq \frac{C}{|\eta_0^2 + 1|^{1/2}} \frac{1}{a^{2\epsilon}}.
\end{equation}

Analogously, since $|C_{01}(\sigma')| \leq Ce^{-a\sigma'}(\sigma')^{\epsilon}$, we get

\begin{equation}
|C_{21}^{-+}| \leq \frac{C|\eta|}{|\eta_0^2 + 1|^{1/2}(|\eta| + 1)} \frac{1}{a^{\epsilon+1}}.
\end{equation}

Therefore

\begin{equation}
-\int_0^\infty 4\Re C_{11}^{-+} \overline{C_{21}^{-+}} d\eta \leq C \int_0^\infty \frac{1}{(|\eta| + 1)^2} \frac{1}{a^{\epsilon}} \frac{1}{a^{\epsilon+1}} \leq \frac{C}{a^{2\epsilon+1}}.
\end{equation}

Thus $f_1^{++}$ contributes to a higher order term in $\frac{1}{a}$.

Now we shall estimate the contribution of $f_2^{++}$ where $f_2^{++}$ is the same as in (4.11), i.e.

\begin{equation}
\frac{f_2^{++}}{x_0} = 0, \quad \frac{\partial f_2^{++}}{\partial x_0} \bigg|_{x_0=0} = \frac{\gamma e^{-i\eta\rho}}{\sqrt{\eta^2 + 1 + |\eta|}}.
\end{equation}

Note that $\sigma(\rho, 0) = \rho$. We have, in $(\sigma, x_0)$ coordinates:

\begin{equation}
\frac{f_2^{++}}{x_0} = \frac{-i\gamma}{\sqrt{\eta^2 + 1 + |\eta|}} x_0 e^{-i\sigma} + f_3^{++},
\end{equation}

where

\begin{equation}
\frac{f_3^{++}}{x_0} = 0, \quad \frac{\partial f_3^{++}}{\partial x_0} \bigg|_{x_0=0} = 0.
\end{equation}

\begin{equation}
Lf_3^{++} = -\frac{i\gamma x_0 e^{-i\eta\rho}}{\sqrt{\eta^2 + 1 + |\eta|}} - \frac{b_1 e^{-i\eta\rho}}{\sqrt{\eta^2 + 1 + |\eta|}} + \frac{b_2 \gamma x_0 e^{-i\sigma}}{\sqrt{\eta^2 + 1 + |\eta|}}
\end{equation}
Thus, $f_{3}^{+-}$ satisfies the same estimates as $f_{2}^{+-}$. Therefore,

\begin{equation}
|f_{3}^{+-}| \leq \frac{C}{(\eta^2 + 1)^{1/4}(\sqrt{\eta^2 + 1} + |\eta|)} \quad \text{and} \quad \frac{|\partial f_{3}^{+-}|}{\partial \sigma} \leq \frac{C|\eta|}{(\eta^2 + 1)^{1/4}(\sqrt{\eta^2 + 1} + |\eta|)}.
\end{equation}

Having estimates (4.50) we can prove, as for $f_{1}^{+-}$, that

\begin{equation}
|C_{13}| \leq \frac{C}{(\eta^2 + 1)^{1/4}(\sqrt{\eta^2 + 1} + |\eta|)} \frac{1}{a^{\epsilon + 1}},
\end{equation}

\begin{equation}
|C_{23}| \leq \frac{C|\eta|}{(\eta^2 + 1)^{1/4}(\sqrt{\eta^2 + 1} + |\eta|)} \frac{1}{a^{\epsilon + 1}}.
\end{equation}

Therefore we proved

**Theorem 4.1.** Let $C_{0}$ be the same as in (4.12) and $C_{n}$ be the same as in (4.33), (4.35). Then

\begin{equation}
\lim_{a \to \infty} \langle \Psi | N_{n} | \Psi \rangle = \frac{-2\varepsilon |\Gamma_{0}(i\alpha + \varepsilon + 1)|^{2}}{2\pi \alpha \Gamma(2\varepsilon)} \int_{0}^{\infty} \frac{\eta e^{-\frac{2\sin^{-1}\frac{1}{\sqrt{\eta^2 + 1}}}{\eta^2 + 1}}}{(\eta^2 + 1)^{\varepsilon + 1}} \, d\eta.
\end{equation}

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