On gauge–independence in quantum gravity

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Abstract

We prove gauge-independence of one-loop path integral for on-shell quantum gravity obtained in a framework of modified geometric approach. We use projector on pure gauge directions constructed via quadratic form of the action. This enables us to formulate the proof entirely in terms of determinants of non-degenerate elliptic operators without reference to any renormalization procedure. The role of the conformal factor rotation in achieving gauge-independence is discussed. Direct computations on $CP^2$ in a general three-parameter background gauge are presented. We comment on gauge dependence of previous results by Ichinose.

1 Introduction

A selfconsistent approach to quantization of gauge theories was suggested in pioneering paper by Faddeev and Popov [1] a quarter of century ago. However, equivalence between different forms of path integral is still under investigation (see, for example, [2–9]). In particular, it was demonstrated [10] that under certain conditions different gauge fixings can give inequivalent results. There is also a problem specific for quantum gravity [11] which is related to the choice of time and reduced phase space quantization. Hence the problem of equivalence of different forms of path integral in quantum gravity deserves further investigation.

Gauge-independence of the on-shell effective action in a general gauge theory was demonstrated from different points of view [1,12–15] (for a more complete bibliography see recent monograph [16]). This general statement was supported by explicit calculations [17] in one-loop quantum gravity. Thus the problem was solved at least at perturbative level. However, recently some authors [18,19] claimed that due to non-renormalizability of quantum gravity

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all proofs of gauge-independence are invalid. As far as only the pure Einstein
gravity was concerned, their argumentation was based on a single compu-
tation by Ichinose [18], which was not supported by independent calculations
[20]. According to this point of view, which was expressed in its extreme form
in Ref. [19], only physical quantities should be gauge-independent. But no
such quantities could be derived from a non-renormalizable theory. Hence,
the one-loop on-shell effective action in quantum gravity is non-physical and
can be gauge-dependent. This dependence does not lead to any real problem
since, again, quantum gravity is non-renormalizable and non-physical. From
our point of view this argumentation is at least incomplete. First, the term
"physical" in this context does not mean that a quantity should be directly
measurable in some experiment, it rather means that a physical quantity is one
which can be extracted from on-shell effective action. Hence, the hand-waving
argumentation [19] is not enough, one should find a loop-hole in the proofs
[1,12–15] of gauge-independence. The proof [14], though being quite formal,
does not rely upon renormalizability of a gauge theory. Second, at one-loop or-
der all gauge theories look very much alike. The fact of non-renormalizability
does not show up at this order of perturbation theory.

The aim of the present paper is to re-examine the situation with gauge-
independence on on-shell one-loop quantum gravity.

We suggest a new proof of gauge-independence for a family of one-loop path
integrals in quantum gravity. As in the papers [2], we use geometric approach
[21] to quantization of gauge theories. However, we do not use integration
over the so-called adapted coordinates. Note, that these adapted coordinates
satisfy standard gauge conditions. We integrate over variables satisfying arbi-
trary gauge conditions. Our path integral can be viewed upon as Mazur and
Mottola [22] path integral in arbitrary gauge. This is the first point where our
technique is new. The second one is related to the representation of projector
onto the space of pure gauge fields in terms of quadratic form of the action.
A similar trick was used before in curved space QED [23]. We shall demon-
strate equivalence of path integral in different gauges to the one in the gauge
$\nabla_i h^i_j = 0$. For the sake of completeness we also verify directly that the latter
path integral coincides with gauge-fixed path integral of Taylor and Veneziano
[24]. One can consider this as a by-product or as a consistency check. Our proof
is valid for any dimensionality of space-time. The proof is given in the next
section, where we also discuss some specific features of quantum gravity. The
crucial point is cancellation of two functional determinant of non-degenerate
second order elliptic operators. This determinants are identical at a formal
level. However, to achieve cancellation in actual computations one should use
gauge-invariant regularization. This situation is common for all gauge theories
with finite cut-off. A more subtle difficulty is related to the conformal factor
problem [25]. It appeared that a kind of complex rotation of the conformal
factor is necessary for gauge independence, as well as for existence of one-loop
path integral.

In section 3 we compute the one-loop effective action for quantum gravity on $CP^2$ in a general three-parameter gauge. Some limiting case of this gauge corresponds to the Itchinose gauge condition [18]. We find no gauge dependence. We also suggest an explanation of gauge-dependence of previous result [18]. All necessary information [26] about geometry and harmonic expansion on $CP^2$ is collected in the Appendix.

2 Proof of gauge–independence

Consider metric fluctuations $h_{ij}$ on a background with metric tensor $g_{ik}$. Linearised diffeomorphism transformations act on $h_{ij}$ as follows

$$h_{ij} \rightarrow h_{ij} + (L\xi)_{ij}, \quad (L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i$$

(1)

where $\nabla_j$ is the covariant derivative with respect to background metric $g_{ik}$.

Consider gauge condition

$$(Gh)_j = 0$$

(2)

with some operator $G$ depending on background metric $g_{ik}$. Let the condition (2) be linearly admissible. It means that the operator $GL : \xi \rightarrow \xi$ is invertible for all vector fields except for Killing vectors of $g_{ij}$, and $h_{ij}$ admits unique decomposition

$$h_{ij} = \tilde{h}_{ij} + (Lv)_{ij}, \quad G\tilde{h} = 0$$

(3)

The Euclidean signature Einstein-Hilbert action reads

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} (R + 2\Lambda).$$

(4)

Let the background metric $g_{ik}$ satisfies classical equations of motion

$$R_{ik} = -\Lambda g_{ik}$$

(5)

Let us introduce a covariant ultralocal inner product in the space of metric fluctuations

$$< h, h' > = \int d^4x \sqrt{g} h'_{ik} h^{ik},$$

(6)
which corresponds to \( C = 0 \) in the De Witt’s configuration space metric. The path integral measure can be defined by

\[
\mathcal{D}h = \frac{1}{\text{vol Diff}} \mathcal{D}_G h.
\]  

where \( \mathcal{D}_G \) is the Gaussian measure with respect to the inner product (6). \( \text{vol Diff} = \int \mathcal{D}\xi \) denotes the volume of the gauge group. This is an infinite constant which can be put equal to unity by imposing an appropriate normalization condition.

In a framework of the so-called geometric approach [21,2] to quantum gravity the path integral

\[
Z(g) = \int \mathcal{D}h \exp(-S(g, h))
\]  

can be evaluated (at one loop) by substituting the decomposition (3) in (8), truncating \( S(g, h) \) to quadratic order in \( h \), and integrating over \( v \). Due to gauge invariance, \( S_2(g, h) \) does not depend on \( v \). Hence integration over \( v \) gives the volume of the gauge group which cancels the corresponding multiplier in (7). Finally we obtain

\[
Z^{(1)}(g) = \int \mathcal{D}\tilde{h} J \exp(-S_2(g, \tilde{h}))
\]  

where \( J \) is the Jacobian factor due to the change of variables \( h \rightarrow \{ \tilde{h}, v \} \) (see (3), which could be expressed in a standard manner [2].

We can rewrite

\[
S_2(g, h) = \langle h, K h \rangle
\]  

with some second order self-adjoint operator \( K \). Hence at one loop

\[
Z^{(1)}(g) = (\det \tilde{K})^{-\frac{1}{2}} J
\]  

where \( \tilde{K} \) is the operator \( K \) restricted to the space defined by gauge condition (2).

**Proposition.** The path integral (11) does not depend on gauge condition (2) provided \( G \) is lineary admissible.

To prove the Proposition, consider the gauge \( \tilde{h} = h^\perp \), where

\[
\nabla_i h^\perp_{ik} = 0
\]  

(12)
Due to gauge invariance

$$\nabla_i (K h)^{ik} = 0 \quad (13)$$

for arbitrary $h^{ik}$. Consequently $K$ can be used as a projection operator onto the space defined by Eq. (12)

$$h_{ik} = h^{\perp}_{ik} + (L \xi)_{ik}, \quad h^{\perp}_{ik} = (K^{-1} K h)_{ik}, \quad (14)$$

where $K^{\perp}$ is the operator $K$ restricted to the space (12). The Jacobian factor for this gauge condition (12) is just

$$J = \det(L^\dagger L)^{\frac{1}{2}}, \quad (15)$$

where $L^\dagger$ is hermitian conjugate to $L$. Note, that $\det(L^\dagger L)^{\frac{1}{2}}$ is the volume of the gauge orbit [2].

In general, a change of variables from $\Phi^A$ with metric $H_{AB}$ to $\Psi^a$ leads to the Jacobian

$$J = \det\{H_{AB} \frac{\delta \Phi^A}{\delta \Psi^a} \frac{\delta \Phi^B}{\delta \Psi^b}\}^{\frac{1}{2}} = \left(\int \mathcal{D} \Psi \exp(- < \Phi(\Psi), \Phi(\Psi)>)\right)^{-1} \quad (16)$$

where in the last line we suppose that the change of variables is linear, $<,>$ is the inner product with respect to the metric $H_{AB}$. The measure in the last path integral is the Gaussian one.

Translating the condensed notations in (16) to our case we obtain

$$J^{-1} = \int \mathcal{D} \tilde{h} \mathcal{D} \xi \exp(- < \tilde{h} + L \xi, \tilde{h} + L \xi>). \quad (17)$$

We can integrate over $\xi$ in (17) taking into account mixing between $\tilde{h}$ and $\xi$ and obvious relation $< h^{\perp}, L \xi > = 0$.

$$J^{-1} = (\det L^\dagger L)^{-\frac{1}{2}} \int \mathcal{D} \tilde{h} \exp(- < \tilde{h}^{\perp}, \tilde{h}^{\perp}>) =$$

$$= (\det L^\dagger L)^{-\frac{1}{2}} \int \mathcal{D} \tilde{h} \exp(- < \tilde{h}, K^{\perp -1} K \tilde{h}>) \quad (18)$$

It is easy to see that the last path integral is equal to

$$(\det K^{\perp})^{\frac{1}{2}} (\det \tilde{K})^{-\frac{1}{2}} \quad (19)$$
Substituting (18) and (19) in (11) we obtain
\[ Z^{(1)}(g) = (\det K^\perp)^{-\frac{1}{2}}(\det L^\dagger L)^{\frac{1}{2}} \] (20)

One-loop integral (11) in arbitrary gauge is equal to the one in the gauge (12) and, hence, does not depend on gauge conditions. This completes the proof.

For the sake of completeness let us verify that on four-sphere \( S^4 \) the path integral (20) coincides with the gauge-fixed path integral [24] and with the Hamiltonian path integral [8].

Let us make use of the decomposition
\[ h_{ik} = h^{\dagger T T}_{ik} + (L^T \xi)_{ik} + \nabla_i \nabla_k \sigma - \frac{1}{4} \Delta \sigma + \frac{1}{4} g_{ik} h, \] (21)

where \( h^{\dagger T T} \) is transverse traceless tensor, \( \xi^T \) is transverse vector, \( \sigma \) and \( h \) are scalar fields. Scalars \( \sigma \) and \( h \) can be decomposed in a sum of spherical harmonics \( Y_l \). The eigenvalues of the Laplace operator \( \Delta \) are \(-l(l+3)/r^2\), \( r \) is the radius of \( S^4 \). For \( \sigma \) the sum starts from \( l = 2 \), while for \( h \) - from \( l = 0 \).

The gauge condition (12) means that \( \xi^T = 0 \), \( h(l = 1) = 0 \). Other scalar harmonics \((l \geq 2)\) are related by the equation
\[ h = -3(\Delta + \frac{4}{r^2})\sigma \] (22)

Let us use the field \( \sigma \) as a coordinate in the configuration space. The Jacobian \( \det(L^\dagger L) \) should be multiplied by another factor arising from the change of variables \( h^\perp \rightarrow \{ h^{\dagger T T}, \sigma(l \geq 2), h(l = 0) \} \). After some algebra one obtains the total Jacobian factor
\[ J_{\text{tot}} = \det_{TV}(\Delta \delta^k_i + R^k_i)^{\frac{1}{2}} \det_{S(l \geq 2)}((-\Delta + \frac{R}{4})(-\Delta + \frac{R}{3})^\frac{1}{2}) \det_{S(l = 1)}(-\Delta)^\frac{1}{2}, \] (23)

where the first factor refers to transversal vector fields excluding Killing vectors, the second and the third ones are computed on scalar harmonics. In our notations the Ricci tensor \( R^k_i \) is negative.

Substituting the fields \( \{ h^{\dagger T T}, \sigma(l \geq 2), h(l = 0) \} \) in the Einstein - Hilbert action and truncating to quadratic order we obtain with the help of equation (22)
\[ S = \frac{1}{32\pi G_N} \int d^4 x \sqrt{|g|} \left[ \frac{1}{2} h^{\dagger T T}_{ik} (-\Delta g^{ij} g^{kl} + R^{ijkl}) h^{-1}_{ij} \right] \]
\[ -3\sigma (-\Delta + \frac{R}{3}) (-\Delta + \frac{R}{4})^2 \sigma - \frac{1}{4} h \left( \frac{R}{3} \right) \]  \tag{24} 

As usual, to obtain a convergent path integral the field \( \sigma \) should be rotated to imaginary values [22,25]. After functional integration we obtain in one-loop order

\[ Z = \det_{TT} (-\Delta g^{ij} g_{kl} + R^{ijkl}) \left( -\frac{1}{2} \det_{VT} (-\Delta \delta^k_i + R^k_i) \right) \frac{1}{2} \det_{S(i=1)} (-\Delta)^{\frac{1}{2}} \]  \tag{25} 

where we omitted contributions of classical action and global dilatations, \( TT \) refers to transverse traceless tensor fields. This is exactly the result of Taylor and Veneziano [24] and of Hamiltonian quantum gravity on de Sitter space [8].

Let us formulate main technical features of our approach. We do not use adapted coordinates (except for single gauge (12)). We integrate over fields satisfying arbitrary gauge conditions directly. This leads to another form of path integral compared with standard geometric approach. We construct the projection operator on pure gauge degrees of freedom using the quadratic form of the action. This makes the proof more straightforward. This method can be extended to other gauge theories. Our result indicates that in a framework of geometric approach one can consider selfconsistently the path integral (8) without pre-fixing of gauge condition.

It is more important for quantum gravity that our proof is formulated in terms of one-loop quantities only. We deal with determinants of elliptic operators which are unambiguous in any given regularization. The only formal step is cancellation of two determinants of \( \tilde{K} \). One of them comes from Gaussian integration in a given gauge, the other is produced by the Jacobian factor. In principle, these two determinants can be regularized in different ways. This could give rise to gauge dependence of one-loop counterterms. However, this situation is common for all gauge theories. One-loop counterterms should be gauge-independent if a BRST invariant regularization is applied. One can make use of, for example, the dimensional regularization.

Another subtle point is related with the conformal factor problem. To obtain a convergent path integral one should rotate the trace part of the metric fluctuations to imaginary values. From our proof it is clear that to achieve gauge-independence one should perform this rotation everywhere simultaneously. The dimensional regularization is insensitive to overall constant factor before propagator, thus one can easily overlook the minus sign produced by the conformal factor rotation. However, if one studies an expansion of the path integral with respect to a small parameter which enters gauge conditions, one can obtain a wrong sign of the contribution of trace part of graviton, which
will no longer cancel gauge-dependent part of the ghost contribution. In other words, one should make the conformal factor rotation before expansion in a small parameter. In the next section we show that the gauge-dependence of the computations [18] can originate from reversed order of expansion and rotation.

3 Quantum gravity on $CP^2$ in a general gauge

Consider the complex projective space $CP^2$. All necessary information about geometry of $CP^2$ and the harmonic expansion is collected in the Appendix. On $CP^2$ there are two invariant covariantly constant rank two tensors, the metric $g_{ik}$ and the complex structure $J_{ik}$. The most general linear invariant gauge condition with even powers of $J$ only reads

$$G_i(h) = \nabla^j(h_{ji} + \beta J_{jk}J_{il}h^{kl} + \gamma g_{ji}h^k_k) = 0.$$  \hspace{1cm} (26)

We impose this condition by adding a gauge-breaking term to the action,

$$\mathcal{L}_{gb} = \frac{1}{16\pi G_N} \frac{1}{2\alpha} G_iG^i.$$ \hspace{1cm} (27)

Using an explicit form (49) of the Riemann tensor one can demonstrate that on this background the gauge condition studied by Ichinose [18]

$$G_i = \nabla^j h_{ji} - \frac{1}{2b} \nabla^j h^j_j + \delta R^{kjl}_i \nabla_j h_{kl}$$

$$b - 1 \ll 1, \quad \delta \ll G, \quad \alpha - 1 \ll 1$$ \hspace{1cm} (28)

is a particular case of the gauge (26). In fact, the papers [18] deal with small deviations from the harmonic gauge.

Here the constants $\alpha$, $\beta$ and $\gamma$ are arbitrary. For some values of these parameters negative or zero modes can appear. As was demonstrated by Allen [27], the infra-red divergencies in graviton and ghost propagators cancel each other on the de Sitter space. Here we will show that the ghost contributions to the path integral are canceled by a part of graviton contribution. This means that the result [27] can be extended to $CP^2$.

Total second order on shell action for graviton $h$ and ghosts $\xi$ and $\bar{\xi}$ has the following form
\[
S = \frac{1}{16\pi G_N} \int d^4x \sqrt{|g|} \left[ \frac{1}{4} \tilde{h}^{ik} (-g_{ij}g_{kl}\Delta + 2R_{ijkl})\tilde{h}^{jl} - \frac{1}{2} \nabla^i \tilde{h}_{ik} \nabla^l \tilde{h}_{lj}^l + \frac{1}{2\alpha} G_j(h)G^j(h) + \frac{1}{\sqrt{\alpha}} G_i(L \xi), \quad \tilde{h}_{ij} = h_{ij} - \frac{1}{2} g_{ij} h^k_k \right].
\] (29)

To produce a damping exponential factor in the path integral over pure gauge fluctuations the parameter \(\alpha\) should be positive. The overall factor \((16\pi G_N)^{-1}\) can be absorbed in field redefinition. For the sake of convenience we use slightly unusual normalization of the ghost action. It is inessential in dimensional regularization because it leads to contribution to the effective action proportional to \(\delta^4(0)\). However, with this normalization cancellation of the ghost contributions is seen in a more straightforward way.

Consider first the graviton part. Due to the orthogonality property (51) and \(SU(3)\) invariance of the gauge fixing term the modes belonging to different \(SU(3)\) representations decouple. One can study contributions of real irreducible representations of \(SU(3)\) separately. We begin with \((m, m+6) \oplus (m+6, m)\) harmonics. Denote by \(\phi[(m, m+6) \oplus (m+6, m)]_q\) corresponding coefficients in the harmonic expansion (50). A part of the action (29) depending on these coefficients is

\[
S[(m, m+6) \oplus (m+6, m)] = 
\sum_{m=0}^{\infty} \sum_{q=1}^{d(m,m+6)} \phi[(m, m+6) \oplus (m+6, m)]_q^2 (m^2 + 8m + 15),
\] (30)

where we used expression (49) for the Riemann tensor, eigenvalues of the Laplace operator (61) and the transversality equation (58). An overall constant factor is omitted.

Due to the orthonormality property (51) the path integral measure is just

\[
d\mu[(m, m+6) \oplus (m+6, m)] = \prod_{m,q} d\phi[(m, m+6) \oplus (m+6, m)]_q. \] (31)

Contribution of the \((m, m+6) \oplus (m+6, m)\) modes to the path integral reads

\[
Z[(m, m+6) \oplus (m+6, m)] = \prod_{m=0}^{\infty} \left( \frac{m^2 + 8m + 15}{r^2} \right)^{-d(m,m+6)}. \] (32)

Here we restored dependence on the scale factor \(r\) defining size of \(CP^2\) after rescaling \(g_{ik} \rightarrow r^2 g_{ik}\).
In the \((m, m + 3) \oplus (m + 3, m)\) sector the quadratic form of the action is represented by a non-minimal differential operator. Hence it is more convenient to use the functions (59) instead of standard orthonormal basis.

\[
h_{ij}[(m, m + 3) \oplus (m + 3, m)] = c_{1,q}^m L v_q[(m, m + 3) \oplus (m + 3, m)]_{ij} + c_{2,q}^m J_i^k J_j^l L v_q[(m, m + 3) \oplus (m + 3, m)]_{kl},
\]

where \(v_q[(m, m + 3) \oplus (m + 3, m)]\) are orthonormal vector harmonics. By substituting (33) in (29) with the help of (54), (61) and (62) we obtain

\[
S((m, m + 3) \oplus (m + 3, m)) = \sum_{q=1}^{20} (c_{1,q}^0)^2 \frac{1}{2\alpha} (3 + 3\beta)^2 + \sum_{m=1}^{\infty} \sum_{q=1}^{2d(m,m+3)} (c_{1,q}^m)^T \times \left( \frac{1}{2\alpha} (\lambda + 3\beta)^2 + \frac{1}{2\alpha} (\beta\lambda + 3)(\lambda + 3\beta) - \frac{1}{2\alpha} (\lambda + 3)(\lambda - 3) + \frac{1}{2\alpha} (\lambda \beta + 3)^2 \right) (c_{2,q}^m)^2, \lambda = m^2 + 5m + 3
\]

The tensor harmonics in the right hand side of eq. (33) are not orthonormal. This means that the path integral measure contains a non-trivial Jacobian factor:

\[
d\mu[(m, m + 3) \oplus (m + 3, m)] = J_1 \prod_q d\epsilon_{1,q}^0 \times \prod_{m \geq 1,q} d\epsilon_{1,q}^m d\epsilon_{2,q}^m, \quad J_1 = \prod_q < L v_q[(0, 3) \oplus (3, 0), L v_q[(0, 3) \oplus (3, 0)] >^{1/2} \times \prod_{m=1}^{\infty} \prod_{q=1}^{2d(m,m+3)} \text{det}^{1/2} \left( \begin{array}{cc} < L v_q^m, L v_q^m > & < L v_q^m, J J L v_q^m > \\ < J J L v_q^m, L v_q^m > & < J J L v_q^m, J J L v_q^m > \end{array} \right) = \left( \frac{3}{r^2} \right)^{1/20} \prod_{m=1}^{\infty} \prod_{q=1}^{2d(m,m+3)} \frac{1}{r^4} \text{det}^{1/2} \left( \begin{array}{cc} \lambda & 3 \\ 3 & \lambda \end{array} \right) = \left( \frac{3}{r^2} \right)^{10} \prod_{m=1}^{\infty} \left( \frac{(\lambda - 3)(\lambda + 3)}{r^4} \right)^{d(m,m+3)}, \quad \lambda = m^2 + 5m + 3
\]

where \(<, >\) is the inner product (6). We used abbreviated notations

\[
(J J L v)_{ij} = J_i^k J_j^l (L v)_{kl}.
\]
Total contribution of gravitons belonging to the representations \((m, m+3) \oplus (m+3, m)\) reads

\[
Z[(m, m+3) \oplus (m+3, m)] = \left(\frac{3}{r^2}\right)^{10} \prod_{m=1}^{\infty} \left(\frac{\lambda + 3\beta}{\sqrt{\alpha r^2}}\right)^{-2d(m, m+3)} .
\]  \tag{37}

In the \((m, m)\) sector let us make use of the harmonics (60) taking into account trace part of \(h_{ij}\)

\[
h_{ij}[(m, m)] = \sum_{m=2}^{\infty} \sum_{q} a^m_q \left(J_j^k \nabla_k \nabla_j + J_j^l \nabla_k \nabla_i\right)s(m, m)_q + \sum_{m=1}^{\infty} \sum_{q} b^m_{1,q} \nabla_i s(m, m)_q + \sum_{m=2}^{\infty} \sum_{q} b^m_{2,q} J_j^k \nabla_k \nabla_i s(m, m)_q + \sum_{m=0}^{\infty} \sum_{q} f^m_q g_{ij} s(m, m)_q, \tag{38}
\]

\(s(m, m)_q\) are orthonormal scalar harmonics. The path integral measure now reads

\[
d\mu[(m, m)] = J_2 \prod_{m,q} da^m_q db^m_{1,q} db^m_{2,q} df^m_q .
\]  \tag{39}

The Jacobian factor \(J_2\) can be evaluated in the same way as \(J_1\):

\[
J_2 = \det \begin{pmatrix}
\kappa_1(\kappa_1 - \frac{3}{2}) & -\kappa_1 & \frac{1}{2}d(1,1) \\
-\kappa_1 & 4 & \\
\frac{1}{2}d(m,m) & & \\
\end{pmatrix}
\]

\[\times \prod_{m=2}^{\infty} \det \begin{pmatrix}
\kappa_m(\kappa_m - 3) & 0 & 0 & 0 & \frac{1}{2}d(m,m) \\
0 & \kappa_m(\kappa_m - \frac{3}{2}) & 0 & 0 & \\
0 & 0 & \frac{3}{2}\kappa_m & -\kappa_m & \\
0 & -\kappa_m & -\kappa_m & 4 & \\
\end{pmatrix}
= \left(\frac{3\kappa_1(\kappa_1 - 2)\kappa_1}{r^4}\right)^{\frac{1}{2}d(1,1)} \prod_{m=2}^{\infty} \left(\frac{\kappa_m(\kappa_m - 3)^2}{r^8}\right)^{\frac{1}{2}d(m,m)},
\]

\[\kappa_m = m^2 + 2m = -\Delta(m, m, 0, 0). \tag{40}
\]

In the last expression we restored dependence of the Jacobian factor on the scale parameter \(r\).

Substituting the decomposition \((38)\) in \((29)\) and performing integration over \(a, b_1, b_2\) and \(f\) we obtain the following expression for the contribution of the
modes to the path integral:

\[ Z(m, m) = J_2 \prod_{m=0}^{\infty} z_m, \quad z_0 = \left( \frac{2}{r^2} \right)^{-\frac{1}{2}}, \]

\[ z_1 = \left( -\frac{3(3 + 3\beta + 6\gamma)^2\kappa_1(\kappa_1 - 2)}{4\alpha r^8} \right)^{-\frac{1}{2}d(1,1)}, \]

\[ z_{m \geq 2} = \left[ -\frac{\kappa_m^4(\kappa_m - 3)^4(1 - \beta)^2(\kappa_m + \frac{3}{2}\beta + \gamma \kappa_m - \frac{3}{2})^2}{\alpha^2 r^{20}} \right]^{-\frac{1}{2}d(m,m)}. \quad (41) \]

Note that the eigenvalues in \( z_1 \) and \( z_m \) are negative. This reflects the conformal factor problem [25]. To overcome this difficulty the scalar harmonics \( s(m, m), m \geq 1 \) should be rotated to imaginary values during transition to the Euclidean domain. As it was explained in the paper [22], the Jacobian factor should remain intact. After the rotation we obtain

\[ Z(m, m) = \left( \frac{2}{r^2} \right)^{-\frac{1}{2}} \left( \frac{(3 + 3\beta + 6\gamma)^2}{4\alpha r^8} \right)^{-\frac{1}{2}d(1,1)} \times \prod_{m=2}^{\infty} \left[ -\frac{(\kappa_m - 3)^2(1 - \beta)^2(\kappa_m + \frac{3}{2}\beta + \gamma \kappa_m - \frac{3}{2})^2}{2\alpha^2 r^{28}} \right]^{-\frac{1}{2}d(m,m)}. \quad (42) \]

Now we are ready to study the ghost contribution. The ghosts \( \bar{\xi} \) and \( \xi \) can be expanded over the vector harmonics:

\[ \xi_i = \sum_{m=0}^{\infty} \sum_{q=1}^{2d(m,m+3)} e^m_{1,q} v_i[(m, m + 3) \oplus (m + 3, m)]_q + \leqno{43} \]

\[ + \sum_{m=2}^{\infty} \sum_{q=1}^{d(m,m)} (e^m_{2,q} v^L_i(m, m)_q + e^m_{3,q} v^\perp_i(m, m)_q) + \]

\[ + \sum_{q=1}^{d(1,1)} e^1_{2,q} v^L_i(1, 1)_q. \]

The same expression with \( e \) replaced by \( \bar{e} \) is valid for \( \bar{\xi} \). The harmonics \( v^\perp(1, 1) \) are excluded since they correspond to the Killing vectors of \( CP^2 \) and do not generate any gauge transformations. One can easily check that these harmonics give zero modes of the ghost action.
Here we can assume that the basis of vector harmonics \( v_{i,q} \) is orthonormal. Hence the path integral measure is trivial:

\[
d\mu_{\text{gh}} = \prod de_{1,q}^m de_{2,q}^m de_{3,q}^m de_{1,q}^m de_{2,q}^m de_{3,q}^m
\]

The ghost path integral is easily evaluated giving

\[
Z_{\text{gh}} = \prod_{m=0}^{\infty} \left( \frac{\lambda + 3\beta}{r^2 \sqrt{\alpha}} \right)^{2d(m,m)+3} \prod_{m=2}^{\infty} \left[ \frac{(1 - \beta)(\kappa_m - 3)}{\sqrt{\alpha r^2}} \right]^{d(m,m)} \\
\times \prod_{m=1}^{\infty} \left[ \frac{2\kappa_m + 3 - 3\beta - 2\gamma\kappa_m}{\sqrt{\alpha r^2}} \right]^{d(m,m)}.
\] (44)

Collecting together all the contributions (32), (37), (42) and (32) and neglecting a constant factor \( 2^{\infty} \) which can be discarded in any analytic regularization and which do not contribute to UV-divergent terms, we obtain

\[
Z = \prod_{m=0}^{\infty} \left[ \frac{m^2 + 8m + 15}{r^2} \right]^{-d(m,m)+6} \left( \frac{3}{r^2} \right)^{10} \left( \frac{2}{r^2} \right)^{-\frac{1}{2}}.
\] (45)

This result is gauge-independent. It coincides with the earlier calculations [26] in a particular gauge.

Now it is in order to comment on gauge-dependent result obtained by Ichinose [18]. The path integral was evaluated by expanding in a small parameter which was essentially the same as \( \beta \) here. However, if one expands the path integral (41) before conformal rotation in the propagator, one gets a different sign compared to expansion of the rotated expression (42). This leads to doubling of gauge-dependent term instead of cancellation when one sums up the corresponding ghost contribution.

To clarify this point, consider scalar field action with negative kinetic energy

\[
S_{\varphi} = -\int d^4x \varphi (-\Delta + X + \delta \mathcal{O}) \varphi,
\] (46)

where \( X \) and \( \mathcal{O} \) are some operators, \( \delta \) is a small parameter. To obtain convergent path integral one should rotate \( \varphi \) to imaginary values. This gives the effective action

\[
W_{\text{eff}} = \frac{1}{2} \ln \det (-\Delta + X + \delta \mathcal{O})
\]
\[
= \frac{1}{2} \ln \det(-\Delta + X) + \frac{\delta}{2} \text{tr} \left( \mathcal{O} \frac{1}{-\Delta + X} \right) + O(\delta^2) \tag{47}
\]

If, instead of this, one first expand over \( \delta \) and then perform rotation by changing sign before the propagator \(-\Delta + X\), the effective action takes the form

\[
\tilde{W}_{\text{eff}} = \frac{1}{2} \ln \det(-\Delta + X) - \frac{\delta}{2} \text{tr} \left( \mathcal{O} \frac{1}{-\Delta + X} \right) + O(\delta^2). \tag{48}
\]

We see, that the sign before \( \delta \) is different in \( W_{\text{eff}} \) and \( \tilde{W}_{\text{eff}} \). The expression (47) corresponds to the conformal factor rotation prescription in quantum gravity. The other expression (48) presumably corresponds to the Ichinose procedure. Note, that in dimensional regularization any constant factor before propagator can be neglected, and the conformal factor rotation can be easily overlooked. This ”implicit” rotation by simply neglecting the minus sign before propagator gives exactly the expression (48). To obtain correct result, the vortex term \( \delta \mathcal{O} \) should be rotated together with the propagator.

4 Conclusions

In this paper we suggested a new proof of gauge-independence in on-shell one-loop quantum gravity. This proof is specially tailored for non-renormalizable theories. It is formulated in terms of functional determinants of non-degenerate operators which are well-defined in any suitable regularization scheme. We stress the role of the conformal factor rotation. We calculated the one-loop partition function on \( CP^2 \) in a general \( SU(3) \)-invariant background gauge. All gauge-dependent contributions of graviton fluctuations are canceled by that of ghosts. We also suggested an explanation of gauge dependence of previous computations by Ichinose [18].

Note, that on manifolds with boundaries the problem of gauge-invariance and gauge-independence becomes much more complicated (see, e.g. [28]). It is connected with the problem of formulation of diffeomorphism invariant boundary conditions for gravitational fluctuations. As well as in our case, this has nothing to do with non-renormalizability of quantum gravity.

Another interesting example of apparent gauge dependence of an on-shell quantity is provided by non-Abelian gauge theories at finite temperature. To remove this gauge dependence one should introduce an infrared regulator which is kept non-vanishing until after one goes on-shell [29].
Appendix: Harmonic expansion on $CP^2$

In this appendix we collect all necessary formulae \cite{26} related to the harmonic expansion of gravitational perturbations and ghosts on $CP^2$. Let $J_{ij}$ be the complex structure, $J_{ij} = -J_{ji}$, $J_i^k J^i_k = -\delta^i_j$. If an appropriate normalization is chosen, the Riemann tensor can be expressed in terms of metric and complex structure:

$$R_{ijkl} = \frac{1}{4}(g_{il}g_{jk} - g_{ik}g_{jl}) - \frac{1}{2}J_{ij}J_{kl} + \frac{1}{4}(J_{il}J_{jk} - J_{ik}J_{jl}). \quad (49)$$

The Ricci tensor is $R_{ik} = -\frac{3}{2}g_{ik}$. $CP^2$ is a symmetric space, $CP^2 = SU(3)/SU(2) \times U(1)$. The isotropy representation of $AdSU(2) \times U(1)$ in the tangent space $T_oCP^2$ is $(-1,2) \oplus (1,2)$, where the first number is the $U(1)$ charge, and the second one is the dimension of $SU(2)$-irrep. The isotropy representation is reducible as complex representation and irreducible as real one. The two components correspond to eigenvalues $\pm i$ of the complex structure $J$.

For any homogeneous space $G/H$ a field $\Phi_A$ belonging to an irreducible representation $D(H)$ can be expanded as \cite{30}

$$\Phi_A(x) = V^{-\frac{1}{2}} \sum_{n,\zeta, \xi} \sqrt{d_D} D^{(n)}_{\zeta | \xi}(g_x^{-1}) \phi^{(n)}_{\zeta \xi}, \quad (50)$$

where $V$ is the volume of $G/H$, $d_D = \text{dim}D(H)$. We summarize over representations $D^{(n)}$ of $G$ which give $D(H)$ after reduction to $H$. $\zeta$ labels multiple components $D(H)$ in the branching $D^{(n)} \downarrow H$, $d_n = \text{dim}D^{(n)}$. The matrix elements of $D^{(n)}$ have the following orthogonality property

$$\int d^4x \sqrt{g} D^{(n) *}_{\zeta | \xi}(g_x^{-1}) D^{(n')}_{\zeta' | \xi'}(g_x^{-1}) = V d_n^{-1} d_D \delta_{\zeta \xi} \delta_{\zeta' \xi'} \delta_{nn'} \quad (51)$$

In the case of $CP^2$ the following representations of $G = SU(3)$ contribute to the harmonic expansion (50).

(i) **Scalar fields.** The representation of $SU(2) \times U(1)$ is $D(H) = (0,1)$. The representations of $SU(3)$ which contribute to (50) are

$$D(SU(3)) = (m, m), \quad m \geq 0 \quad (52)$$

The $SU(3)$ representations are labeled by the Dynkin indices, i.e. by the coordinates of a highest weight in the basis of fundamental weights.
The vector harmonics transforming as \((m, m+3) \oplus (m+3, m)\) have the following transversality property

\[
\nabla_j v^j((m, m+3) \oplus (m + 3, m)) = J_{ij} \nabla^i v^j((m, m+3) \oplus (m + 3, m)) = 0
\]

The two remaining \((m, m)\) harmonics can be identified with linear combinations of derivatives of scalar harmonics \(s(m, m)\):

\[
v^L_i(m, m) = \nabla_i s(m, m), \quad v^\perp_i(m, m) = J_{ij} \nabla^j s(m, m).
\]
Note that every representation $D(G)$ of (52), (53) and (57) contains corresponding representation $D(H)$ with unit multiplicity. This means that we can drop out the index $\zeta$ in (50) and (51).

The harmonics $T((m, m + 6) \oplus (m + 6, m))$ satisfy the equations

$$\nabla_i T_k^i((m, m + 6) \oplus (m + 6, m)) = \nabla_j J^j T^k_{jk}((m, m + 6) \oplus (m + 6, m)) = 0. (58)$$

The $(m, m + 3) \oplus (m + 3, m)$ harmonics can be represented as linear combinations of vector harmonics:

$$Lv((m, m + 3) \oplus (m + 3, m))_{ij} \quad \text{and} \quad J^i J^j Lv((m, m + 3) \oplus (m + 3, m))_{kl} (59)$$

The two tensor fields in (59) are independent for $m \geq 1$ and are linearly dependent for $m = 0$. The operator $L$ is given by the eq. (1), $Lv_{ij} = \nabla_i v_j + \nabla_j v_i$. Three $(m, m)$ harmonics can be constructed from scalar fields:

$$L\nabla s(m, m), \quad m \geq 1, \quad L v^\perp(m, m), \quad m \geq 2, \quad J^i J^j L v^\perp(m, m)_{kl}, \quad m \geq 2. (60)$$

Note that the fields $v^\perp(1, 1)$ correspond to the Killing vectors and do not give rise to tensor fields.

The eigenvalues of the Laplace operator $\Delta = \nabla_i \nabla^i$ can be expressed in terms of quadratic Casimir operators:

$$-\Delta(m_1, m_2; S, Y) = C_2(SU(3)) - C_2(SU(2) \times U(1))$$

$$= \frac{1}{3}(m_1^2 + m_2^2 + m_1 m_2 + 3m_1 + 3m_2) - S(S + 1) - \frac{3}{4} Y^2$$

(61)

where $S$ is the $SU(2)$-spin, $Y$ denotes the $U(1)$ eigenvalues. For example, for the vector component $(\pm 1, 2) Y = \pm 1$, $S = \frac{1}{2}$. Degeneracies of the eigenvalues are given by dimensions of corresponding $SU(3)$ representations

$$d(m_1, m_2) = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2) \quad (62)$$

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