Prepotentials in $N=2$ SU(2) Supersymmetric Yang-Mills Theory with Massless Hypermultiplets

Katsushi Ito and Sung-Kil Yang

Institute of Physics, University of Tsukuba, Ibaraki 305, Japan

ABSTRACT

We calculate the prepotential of the low-energy effective action for $N = 2$ SU(2) supersymmetric Yang-Mills theory with $N_f$ massless hypermultiplets ($N_f = 1, 2, 3$). The precise evaluation of the instanton corrections is performed by making use of the Picard-Fuchs equations associated with elliptic curves. The flavor dependence of the instanton effect is determined explicitly both in the weak- and strong-coupling regimes.
Low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory is described in terms of a single holomorphic function [1]. This type of holomorphic function, called the prepotential, plays a vital role in four-dimensional $N = 2$ theories [2]. Seiberg and Witten discovered that the prepotential for the gauge group $SU(2)$ is completely determined by holomorphic data associated with elliptic curves [3,4]. On the basis of this seminal work, many non-perturbative aspects of the vacuum structure of $N = 2$ supersymmetric gauge theories have been revealed subsequently by extending the gauge group [5] and by introducing matter hypermultiplets [6].

In this paper we study the quantum moduli space of $N = 2$ supersymmetric gauge theories with massless $N = 2$ hypermultiplets of quarks. The gauge group is $SU(2)$ and $N_f$ hypermultiplets are all in the spin one-half representation of $SU(2)$. The theory is well-known to be asymptotically free for $N_f \leq 3$. Our purpose is to calculate the prepotentials of the low-energy effective actions for all the theories with $N_f = 1, 2$ and 3.

Let $\phi$ be a complex scalar field in the $N = 2$ vector multiplet. The theory has a flat direction with non-vanishing $\phi$, along which the gauge group $SU(2)$ is broken to $U(1)$ and all the quarks turn out to be massive. This is the Coulomb branch of the moduli space. For $N_f \geq 2$ the other branch may develop when scalar fields in the hypermultiplets acquire the vacuum expectation value. This branch is called the Higgs branch where the gauge symmetry is completely broken. In what follows we will concentrate on the Coulomb branch.

The quantum moduli space of the Coulomb branch is described by using the gauge invariant order parameter $u = \langle \text{Tr} \, \phi^2 \rangle$. The low-energy effective theory at generic points in the complex $u$-plane contains an $N = 1$ $U(1)$ vector multiplet $W_{\alpha}$ and an $N = 1$ chiral multiplet $A$ whose scalar component is denoted as $a$. The effective action is then governed by a single holomorphic function $F(A)$. We have

$$
\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial F(A)}{\partial A} \bar{A} + \int d^2\theta \frac{\partial^2 F(A)}{\partial A^2} W_{\alpha} W^\alpha \right] 
$$

(1)
in $N = 1$ superspace. Let us define

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}.$$  \hspace{1cm} (2)

Important insight in [3] is to recognize the pair $(a_D, a)$ as a holomorphic section of an $SL(2, \mathbb{Z})$ bundle over the punctured $u$-plane. The $SL(2, \mathbb{Z})$ acts on $(a_D, a)$ as the quantum monodromy matrix around the singularities in the moduli space. At singularities there appear extra massless states in addition to the $N = 2 U(1)$ vector multiplet.

It was found in [3,4] that the exact description of the moduli space is determined through the elliptic curves. The curves are given by

$$y^2 = x^2(x - u) + \frac{1}{4} \Lambda_0^4 x$$ \hspace{1cm} (3)

for $N_f = 0$ and

$$y^2 = x^2(x - u) - \frac{1}{64} \Lambda_{N_f}^{2(4-N_f)}(x - u)^{N_f - 1}$$ \hspace{1cm} (4)

for $N_f = 1, 2, 3$. Here we have put all the bare quark masses to zero and $\Lambda_{N_f}$ is the mass scale generated by the dimensional transmutation.

In a recent paper [7], Klemm et al. calculated the prepotential for $N_f = 0$ explicitly. Their main tool is the Picard-Fuchs equations associated with the $N_f = 0$ elliptic curve which have also been considered in view of special geometry [8]. For the present purpose we shall also employ the technique of the Picard-Fuchs equation whose fundamental solutions are combined so as to yield the exact expressions for $(a_D, a)$ in the $N_f \geq 1$ theories.

In writing (3) and (4) we have followed conventions in [4] so that particles in the hypermultiplets have integral electric charge. Hence the effective coupling
constant

\[ \tau(u) = \frac{da_D}{da} \]  \hspace{1cm} (5)

is expressed as

\[ \tau(u) = \frac{\theta_{\text{eff}}(u)}{\pi} + i \frac{8\pi}{g^2_{\text{eff}}(u)}. \]  \hspace{1cm} (6)

The global symmetry acting on the \(u\)-plane is \(Z_2\) for \(N_f = 0\) and \(Z_{4-N_f}\) for \(N_f \geq 1\). Notice that for \(N_f = 3\) there is no symmetry in the \(u\)-plane. This fact distinguishes the \(N_f = 3\) theory from the others.

Eq.(4) describes a double cover of the \(x\)-plane branched over \(e_0, e_\pm\) and \(\infty\). We take a cut to run from \(e_-\) to \(e_+\) and from \(e_0\) to \(\infty\). Let us start with the \(N_f = 1\) curve. We have for large \(u\)

\[ e_0 \simeq u, \quad e_\pm \simeq \pm i \frac{\Lambda^3_1}{8\sqrt{u}}. \]  \hspace{1cm} (7)

Thus the singularity at \(u = \infty\) is not stable since \(e_0 \to \infty\) and \(e_\pm \to 0\) as \(u\) goes to infinity [4]. Bearing this in mind we make a rescaling \(x \to ux\). We then obtain

\[ e_0 = \frac{1}{3}(1 + \xi + \xi^{-1}), \quad e_\pm = \frac{1}{3}(1 \mp \omega \xi \pm \omega^2 / \xi), \]  \hspace{1cm} (8)

where \(\omega = e^{i\pi/3}\) and

\[ \xi = 2^{1/3}(2 + 27\eta + 3\sqrt{3\eta(4 + 27\eta)})^{-1/3} \]  \hspace{1cm} (9)

with \(\eta = \Lambda^6_1/(64u^3)\). The singularities in the \(u\)-plane are located at \(u = \infty\) and \(u = -3\Lambda^2_1 \omega^j / 2^{8/3}\) with \(j = 0, 1, 2\).
In the $N_f = 2$ and 3 curves the singularities are stable. For $N_f = 2$ we have

$$e_0 = u, \quad e_\pm = \pm \Lambda^2_2/8$$

(10)

and the singularities in the $u$-plane are $u = \infty$ and $u = e_\pm$. For $N_f = 3$ we get

$$e_0 = u, \quad e_\pm = \frac{\Lambda^3_2}{128} \left(1 \pm i \sqrt{256 \Lambda^2_2 u - 1}\right)$$

(11)

and hence the singularities occur at $u = \infty$, $u = 0$ and $u = \Lambda^2_3/256$.

The pair $(a_D, a)$ is now given by period integrals of the meromorphic one-form $\lambda$,

$$a_D = \oint_\beta \lambda, \quad a = \oint_\alpha \lambda.$$  

(12)

Here the cycle $\alpha$ is taken to go around the cut connecting $e_-$ and $e_+$ counterclockwise. The other cycle $\beta$ loops around the branch points at $e_0$ and $e_-$ for $N_f = 1$ and at $e_0$ and $e_+$ for $N_f = 2, 3$. The intersection of $\alpha$ and $\beta$ is $\alpha \cdot \beta = 1$.

The one-form $\lambda$ is determined by the differential equation [4]

$$\frac{d\lambda}{du} = \frac{\sqrt{2}}{8\pi} \lambda_1 + d(*)$$

(13)

where $\lambda_1$ is the holomorphic differential $\lambda_1 = dx/y$ and $d(*)$ stands for the exact form in $x$. Integrating (13) in $u$ we find

$$\lambda = \frac{\sqrt{2}}{8\pi} \frac{2u - (4 - N_f)x}{y} dx$$

(14)

for $N_f \geq 0$. It is interesting to observe the one-loop beta function coefficient $4 - N_f$ in $\lambda$. 

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We wish to derive the Picard-Fuchs equation for the period $\Pi = \oint \lambda$. For this it is convenient first to write down the Picard-Fuchs equation for the period $\Pi_1 = \oint \lambda_1$. After some algebra we obtain

$$p(u) \frac{d^2 \Pi_1}{du^2} + q(u) \frac{d \Pi_1}{du} + \Pi_1 = 0,$$

where the coefficient functions are given by

$$\begin{align*}
N_f &= 1 & p(u) &= 4u^2 + \frac{27\Lambda_1^6}{64u}, & q(u) &= \frac{1}{u^2} \left(8u^3 - \frac{27\Lambda_1^6}{64}\right), \\
N_f &= 2 & p(u) &= 4 \left(u^2 - \frac{\Lambda_2^4}{64}\right), & q(u) &= 8u, \\
N_f &= 3 & p(u) &= u \left(4u - \frac{\Lambda_3^2}{64}\right), & q(u) &= 8u - \frac{\Lambda_3^2}{64}.
\end{align*}$$

Combining (13) and (15) it is immediate to see that

$$\frac{d^3 \Pi}{du^3} + \frac{q(u) d^2 \Pi}{p(u) du^2} + \frac{1}{p(u)} \frac{d \Pi}{du} = 0. \quad (17)$$

Notice that the relation $q(u) = dp(u)/du$ holds for all $N_f$ in (16). Consequently (17) reduce to the second order differential equations

$$p(u) \frac{d^2 \Pi}{du^2} + \Pi = 0 \quad (18)$$

for $N_f \geq 1$. We have checked by verifying this equation directly from (14) without relying on (13).

Now that we have the Picard-Fuchs equations (18), the periods (12) will be expressed by particular linear combinations of the two fundamental solutions to (18). In order to fix the combinations we have to evaluate the contour integrals (12) explicitly. Let us first discuss the behavior at $u = \infty$. For $N_f \geq 1$ the asymptotic
freedom and the instanton contributions lead one to expect that [4]

\[ a_D(u) = i \frac{4 - N_f}{2\pi} a(u) \ln \frac{u}{\Lambda_{N_f}^2} + \sqrt{u} \sum_{n=0}^{\infty} a_{Dn}(N_f) \left( \frac{\Lambda_{N_f}^2}{u} \right)^{n(4-N_f)}, \]

\[ a(u) = \frac{1}{2} \sqrt{2u} \left[ 1 + \sum_{n=1}^{\infty} a_n(N_f) \left( \frac{\Lambda_{N_f}^2}{u} \right)^{n(4-N_f)} \right]. \]  

Here the instantons with even instanton number \(2n\) contribute the terms \((\Lambda_{N_f}^2/u)^{n(4-N_f)}\). The amplitudes for the odd instanton contributions vanish because of the anomalous \(Z_2\) symmetry for \(N_f \geq 1\) [4]. From (19) one can easily read off the monodromy matrix at \(u = \infty\),

\[ M_{\infty} = \begin{pmatrix} -1 & 4 - N_f \\ 0 & -1 \end{pmatrix}. \]  

We compute the lower order expansion of the integrals (12) explicitly. The results for \(a_D(u)\) read

\[ a_D(u) = i \frac{4 - N_f}{2\pi} \frac{1}{2} \sqrt{2u} \left( \ln \frac{u}{\Lambda_{N_f}^2} + c_{N_f} \right) + \cdots, \]  

where \(c_1 = -\frac{1}{3} \ln e^{i\pi} + 4 \ln 2 - 2\), \(c_2 = 6 \ln 2 - 2\) and \(c_3 = -\ln e^{i\pi} + 12 \ln 2 - 2\). For \(a(u)\) we get

\[ a(u) = \frac{1}{2} \sqrt{2u} \left[ 1 + a_1(N_f) \left( \frac{\Lambda_{N_f}^2}{u} \right)^{4-N_f} + \cdots \right], \]  

where \(a_1(1) = 3/2^{10}\) and \(a_1(2) = a_1(3) = -1/2^{10}\). Thus our explicit results are in agreement with (19).

The exact expressions for \((a_D, a)\) near \(u = \infty\) are now obtained from the solutions to the Picard-Fuchs equations. The solutions to the Picard-Fuchs equations are written succinctly by introducing a new variable

\[ z = c(N_f) \left( \frac{u}{\Lambda_{N_f}^2} \right)^{4-N_f}, \quad c(N_f) = \eta_{N_f} 2^8(4 - N_f)^{N_f-4}, \]  

which keeps the discrete \(Z_{4-N_f}\) symmetry in the \(u\)-plane. Here \(\eta_1 = -1, \eta_2 = \eta_3 = 1\). The Picard-Fuchs equations (18) for \(N_f \geq 1\) become the hypergeometric
system with singularities at $z = 0, 1, \infty$

$$z(1 - z)\frac{d^2 \Pi}{dz^2} + (\gamma - (\alpha + \beta + 1)z)\frac{d\Pi}{dz} - \alpha\beta\Pi = 0, \quad (24)$$

where

$$\alpha = \beta = \frac{-1}{2(4 - N_f)}, \quad \gamma = \frac{3 - N_f}{4 - N_f}. \quad (25)$$

The set of fundamental solutions to (24) near $u = \infty$ are written as $w_1, w_2$

$$w_1(z) = z^{\frac{1}{2(4 - N_f)}} F\left(\frac{1}{2(4 - N_f)}, \frac{-1}{2(4 - N_f)}; 1; \frac{1}{z}\right),$$

$$w_2(z) = w_1(z) \ln \frac{1}{z} + z^{\frac{1}{2(4 - N_f)}} F_1\left(\frac{1}{2(4 - N_f)}, \frac{-1}{2(4 - N_f)}; 1; \frac{1}{z}\right), \quad (26)$$

where

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n;$$

$$F_1(\alpha, \beta; 1; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n!)^2} \sum_{r=0}^{n-1} \left( \frac{1}{\alpha + r} + \frac{1}{\beta + r} - \frac{2}{1 + r} \right) z^n, \quad (27)$$

and $(\lambda)_n \equiv \lambda(\lambda + 1) \cdots (\lambda + n - 1)$. Using this set of solutions, we are able to express $a(u)$ and $a_D(u)$ in the form:

$$a(u) = \frac{\Lambda_{N_f}}{\sqrt{2c(N_f) \pi^{4-N_f}}} w_1(z),$$

$$a_D(u) = -\frac{i\Lambda_{N_f}}{2\sqrt{2\pi c(N_f) \pi^{4-N_f}}} (w_2(z) + A_{N_f} w_1(z)), \quad (28)$$

where $A_{N_f} = \ln c(N_f) - (4 - N_f)c_{N_f}$. 

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In order to calculate the prepotential $F$ we first express $u$ as a series expansion in $a/\Lambda_{N_f}$ by inverting $a(u)$. Substituting the result into $a_D(u)$, and then integrating (2) with respect to $a$ we obtain the instanton expansion of $F$,

$$F(a) = \frac{ia^2}{2\pi} \left\{ (4 - N_f) \ln \frac{a^2}{\Lambda_{N_f}^2} + \sum_{k=0}^{\infty} F_k(N_f) \left( \frac{\Lambda_{N_f}^2}{a^2} \right)^{k(4-N_f)} \right\}. \quad (29)$$

The first five coefficients $F_k(N_f)$ are given by

$$F_1(N_f) = -\frac{1}{2^b c(N_f)} \frac{1}{2^b},$$

$$F_2(N_f) = -\frac{1}{2^{2b} c(N_f)^2} \frac{5 - 8b + 4b^2}{2^8 b^4},$$

$$F_3(N_f) = \frac{1}{2^{3b} c(N_f)^3} \frac{(1 - b)(1 - 2b)(-23 + 30b - 16b^2)}{1728 b^6},$$

$$F_4(N_f) = \frac{1}{2^{4b} c(N_f)^4} \frac{(1 - 2b)(-677 + 3910b - 8124b^2 + 8456b^3 - 4672b^4 + 1152b^5)}{294912 b^8},$$

$$F_5(N_f) = (1 - 2b)(1 - b)(-7313 + 64386b - 210756b^2 + 335960b^3 - 301792b^4 + 152704b^5 - 36864b^6)/\left(2^{5b} c(N_f)^5 \right) \frac{1843200 b^{10}}{1843200 b^{10}},$$

where $b = 4 - N_f$. One can obtain the $N_f = 0$ result by replacing $c(N_f)$ with 1 in the $N_f = 2$ case, which recovers that in [7, 9]. Note that the coefficients $F_{2n+1}(3)$ vanish for $n \geq 1$.

Our next task is to study the strong-coupling regime where the singularities in the moduli space appear as a consequence of the extra massless monopole (or dyon) states. Let us investigate the behavior near the singularity at $u = u_0$ where the monopole becomes massless. The position of the singularities is given by $u_0 = -3\Lambda_1^2/2^{8/3}$ for $N_f = 1$, $u_0 = \Lambda_2^2/8$ for $N_f = 2$ and $u_0 = 0$ for $N_f = 3$. In the $N_f \geq 1$ theory the monopole belongs to the spinor representation of the global $SO(2N_f)$ symmetry [4]. Hence, in the vicinity of $u = u_0$, the effective dual $U(1)$ theory consists of the $N = 2 U(1)$ vector multiplet of magnetic photon and the $k$ hypermultiplets of light monopoles. The multiplicity $k$ is given by the dimension
of the spinor representation of $SO(2N_f)$, i.e. $k = 2^{N_f-1}$. Then the perturbation calculations in the dual theory predict [4]

$$a \simeq i \frac{k}{2\pi} a_D \ln a_D,$$

$$a_D \simeq c_0(u - u_0), \quad c_0 \neq 0,$$

from which we find the monodromy matrix at $u = u_0$,

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}.$$ (32)

Let us turn to the explicit evaluation of the periods (12) as $u$ approaches $u_0$. Our results for $a_D(u)$ read

$$a_D(u) = c_D(N_f) \sqrt{2v_{N_f}} \left[ \frac{u - u_0}{v_{N_f}} + b_D(N_f) \left( \frac{u - u_0}{v_{N_f}} \right)^2 + \cdots \right],$$ (33)

where $v_1 = -3\Lambda_1^2/2^{8/3}$, $v_2 = \Lambda_2^2/4$, $v_3 = \Lambda_3^2/64$, $c_D(1) = -i/4$, $c_D(2) = i/4$, $c_D(3) = 1/4$, and $b_D(1) = -1/24$, $b_D(2) = -1/8$, $b_D(3) = 1/2$.

For $a(u)$ we obtain

$$a(u) = \frac{\sqrt{2v_{N_f}}}{2\pi} \left\{ d(N_f) + b(N_f) \frac{u - u_0}{v_{N_f}} \left[ \ln \frac{u - u_0}{v_{N_f}} + d_N \right] + \cdots \right\},$$ (34)

where $d(1) = -3$, $d(2) = 2$, $d(3) = i$, $b(1) = 1/4$, $b(2) = -1/2$, $b(3) = i$ and $d_1 = -4 \ln 2 - 2 \ln 3 - 1$, $d_2 = -4 \ln 2 - 1$, $d_3 = -2 \ln 2 - 1$. Notice that

$$\frac{b(N_f)}{c_D(N_f)} = i2^{N_f-1}. $$ (35)

Thus our results agree with the perturbation result (31) with $k = 2^{N_f-1}$ which is the right number of massless monopole states.
Having checked the leading perturbative behavior we next discuss the exact determination of \((a_D, a)\) with the use of the Picard-Fuchs equations. In terms of the variable \(z\) defined in (23) the massless monopole points are at \(z = 1\) for \(N_f = 1, 2\) and \(z = 0\) for \(N_f = 3\). Define

\[
p(z) = \begin{cases} 
1 - z, & \text{for } N_f = 1, 2 \\
\frac{z}{2}, & \text{for } N_f = 3
\end{cases}
\]  

(36)

The fundamental solutions to the Picard-Fuchs equation (24) near \(p(z) = 0\) are then obtained as

\[
w_1(z) = p(z)F\left(1 - \frac{1}{2b}; 1 - \frac{1}{2b}; 2; p(z)\right),
\]

\[
w_2(z) = w_1(z) \ln p(z) + p(z) F_1\left(1 - \frac{1}{2b}; 1 - \frac{1}{2b}; 2; p(z)\right),
\]

where \(b = 4 - N_f\) and

\[
F_1(\alpha, \beta; 2; z) = \frac{1}{(\alpha - 1)(\beta - 1)} \frac{1}{z} \left[ \sum_{n=1}^{\infty} \frac{(\alpha)(\beta)_n}{n!(2)_n} \sum_{r=0}^{n-1} \left( \frac{1}{\alpha + r} + \frac{1}{\beta + r} - \frac{1}{1 + r} - \frac{1}{2 + r} \right) z^n \right].
\]

(37)

In comparison with (33) and (34) we find

\[
a_D(u) = C(N_f) w_1(z),
\]

\[
a(u) = D(N_f) (w_2(z) + d_{N_f} w_1(z))
\]

(39)

where

\[
C(N_f) = -(-1)^{\frac{N_f-1}{2}} (4 - N_f) \frac{2-N_f}{2} 2^{-\frac{3}{2}(N_f-1)-\frac{1}{4-N_f}} \Lambda_{N_f},
\]

\[
D(N_f) = \frac{i}{2\pi} 2^{N_f-1} C(N_f).
\]

(40)

In the dual U(1) theory the prepotential \(F_D\) is expressed as a holomorphic function of \(a_D\) and takes the form

\[
F_D(a_D) = \frac{ia_D^2}{8\pi} \left\{ \ln \left( \frac{a_D^2}{\Lambda_{N_f}^2} \right) + \sum_{n \geq -1} F_D(n) \left( \frac{a_D}{\Lambda_{N_f}} \right)^n \right\}.
\]

(41)

where \(a = dF_D/da_D\). The expansion coefficients \(F_D(n)\) are obtained explicitly in a
similar way to the weak-coupling case. After some computation we find

\[ F_D 1(N_f) = \frac{2}{3\tilde{c}(N_f)} \frac{(1 - 2b)(-5 + 2b)}{16b^2}, \]
\[ F_D 2(N_f) = \frac{1}{2\tilde{c}(N_f)^2} \frac{(1 - 2b)(-61 + 190b - 92b^2 + 8b^3)}{1152b^4}, \]
\[ F_D 3(N_f) = \frac{2}{5\tilde{c}(N_f)^3} \frac{(1 - 2b)(1379 - 8162b + 14596b^2 - 7288b^3 + 960b^4)}{110592b^6}, \]

where \( \tilde{c}(N_f) = C(N_f)/\Lambda_{N_f} \).

Let us finally discuss other singularities located at \( u = u_* \) in the \( u \)-plane. The \( 2 \times 2 \) quantum monodromy matrices \( M_{u_*} \) have already been obtained explicitly in [4]. The left eigenvalue \( (p, q) \) of \( M_{u_*} \) gives the magnetic charge \( p \) and the electric charge \( q \) of the BPS dyonic state which becomes massless at \( u = u_* \). Except for the case \( u_* = u_0 \) (massless monopole point) we always have \( q \neq 0 \). Then, from the matrix \( M_{u_*} \) one can infer the behavior of \( (a_D, a) \) near \( u = u_* \), obtaining

\[
\begin{align*}
a_D(u) &\simeq \frac{1}{2\pi i} \frac{m}{q} c_0(u - u_*) \ln(u - u_*) + c_1(u - u_*), \\
a(u) &\simeq -\frac{1}{2\pi i} \frac{mp}{q^2} c_0(u - u_*) \ln(u - u_*) + \frac{1}{q}(c_0 - pc_1)(u - u_*),
\end{align*}
\]

where \( c_0, c_1 \) are non-vanishing constants and \( m \) is the \((1, 2)\) component of \( M_{u_*} \). Notice that a good coordinate near \( u = u_* \) is obtained as \( pa_D + qa \simeq c_0(u - u_*) \).

We have checked the asymptotic formula (43) for several cases of interest. For instance, take a singularity at \( u = \Lambda_3^2/256 \equiv u_* \) in \( N_f = 3 \) theory where the monodromy matrix is given by [4]

\[ M_{u_*} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \]  

and \( (p, q) = (2, 1) \). In order to find the desired expression (43) one has to follow carefully how the positions of the branch points (and hence the branch cuts) in
the $x$-plane evolve as $u$ approaches $u_*$ when evaluating the contour integral (12). Letting $u \to u_* + 0$ along the real axis in the $x$-plane it is observed that the contour $\alpha$ is pushed to the left of $u = u_*$ while $e_{\pm}$ approach $\Lambda_3^2/128$ parallel to the imaginary axis. In the limit $e_{\pm} \to \Lambda_3^2/128$, therefore, the evaluation of the period $a$ receives the contribution of the $\alpha$-integral (with the opposite sign) twice in addition to the contour integral along the imaginary axis. We thus arrive at

$$a_D \simeq \frac{1}{2\pi i} c_0 \epsilon \ln \epsilon + c_1 \epsilon + c_2, \quad (45)$$

$$a \simeq -\frac{1}{2\pi i} 2c_0 \epsilon \ln \epsilon + (c_0 - 2c_1) \epsilon - 2c_2,$$

where $\epsilon = 1 - 256u/\Lambda_3^2$ and $c_i$ are calculable constants. This result is in accordance with (43). Other cases are considered in a similar manner.

In summary, we have calculated the low-energy prepotential for $N = 2$ $SU(2)$ supersymmetric Yang-Mills theory with massless hypermultiplets and determined the flavor dependence of the instanton corrections exactly. It will be interesting to study a similar issue when the bare quark masses are turned on. In this case the elliptic curve takes the form

$$y^2 = x^3 + a(u)x^2 + b(u)x + c(u), \quad (46)$$

where $a(u), b(u), c(u)$ are functions of $u$ and the bare quark masses [4]. This leads to the set of Picard-Fuchs equations for the period integrals of the first and second abelian differentials $\Pi_1 = \int dx/y$ and $\Pi_2 = \int xdx/y$:

$$\frac{d\Pi_1}{du} = \frac{p_{11}(u)}{\Delta(u)} \Pi_1 + \frac{p_{12}(u)}{\Delta(u)} \Pi_2, \quad (47)$$

$$\frac{d\Pi_2}{du} = \frac{p_{21}(u)}{\Delta(u)} \Pi_1 + \frac{p_{22}(u)}{\Delta(u)} \Pi_2,$$
where

\[ p_{11}(u) = -p_{22}(u) = a b^2 a' - 4 a^2 c a' + 3 b c a' - 2 b^2 b' + 6 a c b' + a b c' - 9 c c', \]

\[ p_{12}(u) = 2 b^2 a' - 6 a c a' - a b b' + 9 c b' + 2 a^2 c' - 6 b c', \]

\[ p_{21}(u) = -2 b^3 a' + 8 a b c a' - 18 c^2 a' + a b^2 b' - 4 a^2 c b' + 3 b c b' - 2 b^2 c' + 6 a c c', \]

\[ \Delta(u) = 2 \left( -a^2 b^2 + 4 b^3 + 4 a^3 c - 18 a b c + 27 c^2 \right), \]  

(48)

and the prime \(^{'}\) denotes the derivative with respect to \( u \). \( \Delta(u) \) is the discriminant of the elliptic curve (46). From (47) one can write down the third order differential equation for \( \oint \lambda \), which is analogous to (17). According to our preliminary analysis, however, the equation will not reduce to the second order equation. Thus, one has to deal with the monodromy property of the third order differential equation.

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