Notes on Direct Products of Group and the Direct Sums of Rings

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Abstract. The group and ring are one of the main contents of algebraic research. As the important tool of natural science research, they have a wide range of applications. In this paper, the direct products of their important properties for a given finite group have been studied. When given a few groups are abelian groups, their direct product naturally becomes the direct sums of commutative rings. So this paper also studied the direct sums of ring. Finally, the important properties of direct product and direct sums under the sense of isomorphism are obtained.

1. Introduction

When given a few finite groups [1-2], we try to construct a new group from them. We use a new group which combines the properties of the research. We call it a new group of their direct product. The direct product put them as the multiplication factor. In the sense of isomorphism [3], direct product has the properties of each factor group. For multiplication commonly used in commutative groups their direct product naturally becomes direct sums of commutative rings [4-5]. By defining the commutative rings operation, the direct sum homomorphic image is obtained. By introducing the concept of homomorphism kernels, direct sums and relationship with the isomorphism between these direct sums factor are finally obtained.

2. Concerning Direct Products

Suppose that finitely many multiplicative groups \( \Theta_1, \Theta_2, \ldots, \Theta_r \) with identity elements are given.

We define equality and multiplication on the set of all \( r \)-tuples

\[
a = (a_1, \ldots, a_r) \quad (a_i \in \Theta_i)
\]

Component-wise:

\[
(a_1, \ldots, a_r) = (a'_1, \ldots, a'_r) \iff a_i = a'_i, \quad a_r = a'_r
\]

(1)

This yields a group \( \Theta \). The isomorphism type of this group \( \Theta \) is uniquely determined by the isomorphism types of the given groups \( \Theta_1, \Theta_2, \ldots, \Theta_r \). It is called the direct products:

\[
\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_r = \prod_{i=1}^{r} \Theta_i
\]

(2)

In particular, if the groups \( \Theta_i \) are of finite orders \( g_i \), then \( \Theta \) is of finite order:

\[
g = g_1 \cdots g_r
\]

(3)

The given groups \( \Theta_1, \ldots, \Theta_r \) are themselves not contained in this group \( \Theta \). However, \( \Theta \) does contain, in the shape of elements of the form \( \overline{a}_1 = (a_1, 1, \ldots, 1), \ldots, \overline{a}_r = (1, \ldots, 1, a_r)\), some subgroups \( \overline{\Theta}_1, \ldots, \overline{\Theta}_r \), which by virtue of the canonical mappings \( a_i \mapsto \overline{a}_i, \ldots, a_r \mapsto \overline{a}_r \) are isomorphic to \( \Theta_1, \ldots, \Theta_r \). With the help of these subgroups \( \overline{\Theta}_1, \ldots, \overline{\Theta}_r \), the original representation of the elements of \( \Theta \) as \( r \)-tuples may be replaced by a representation as products: \( a = \overline{a}_1 \cdots \overline{a}_r \) (\( \overline{a}_i \in \overline{\Theta}_i \)) in which the factors commute with each other. By virtue of the uniqueness of the
representation as \( r \)-tuples, this product representation is also unique. Therefore, the group \( \Theta \) is determined by the subgroups \( \Theta_1, \ldots, \Theta_r \) as follows:

1. Every element \( a \) in \( \Theta \) possesses a unique decomposition into components \( a = a_1 \cdots a_r \) (\( a_i \in \Theta_i \));

2. The elements of \( \Theta_i \) commute with the elements of all the \( \Theta_j \) (for \( j \neq i \)).

Conversely, let the group \( \Theta \) satisfy the following assumption:

\((A^*)\) \( \Theta \) possesses a system of subgroups \( \Theta_1, \ldots, \Theta_r \) satisfying properties (1) and (2). Then, on the one hand, we have the isomorphism \( \Theta \cong \Theta_1 \times \cdots \times \Theta_r \), since, by (1) and (2), comparison and multiplication of the elements of \( \Theta \) are executed component-wise. However, since here the groups \( \Theta_1, \ldots, \Theta_r \) are themselves subgroups of \( \Theta \), it further follows from (1) and (2) that, on the other hand, \( \Theta = \Theta_1 \cdots \Theta_r \) is also their composite. Accordingly, we write more precisely:

\[
\Theta = \Theta_1 \times \cdots \times \Theta_r = \prod_{i=1}^{r} \Theta_i
\]

with the equality sign.

For certain applications, it is expedient to describe the isomorphism types (sometimes also called group types) of the direct factors \( \Theta_i \) of \( \Theta \) in terms of factor groups of \( \Theta \) in the following manner. By (1) and (2), these \( \Theta_i \), are normal subgroups of \( \Theta \) and for them, the mapping \( a \mapsto a_i \), which assigns to element \( a \) in \( \Theta \) its component \( a_i \) in \( \Theta_i \), is a homomorphism of \( \Theta \) onto \( \Theta_i \). Its kernel \( K_i \) is characterized by \( a_i = 1 \), and hence is the normal subgroup of defined by \( K_i = \prod_{j \neq i} \Theta_j \). According to the homomorphism theorem, we then have \( \Theta/K_i \cong \Theta_i \).

Under the assumption \((A^*)\), it thus follows that:

Up to isomorphism, we have the direct-product decomposition for \( \Theta \):

\[
\Theta \cong \Theta_1 \times \cdots \times \Theta_r \quad \text{with} \quad \Theta_i = \Theta/K_i
\]

Where the \( K_i \) are the kernel groups of the component homomorphism (projections) \( a \mapsto a_i \). In fact, we obtain a canonical isomorphism of \( \Theta \) onto \( \prod_{i=1}^{r} \Theta_i \) if we assign to each element \( a \) in \( \Theta \) the residue classes \( aK_i \) in \( \Theta_i \) for its components.

3. Commutative Case and Commutative Rings

In the commutative case, by switching to additive notation, we obtain the concept of the direct sum:

\[
\Theta \cong \Theta_1 + \cdots + \Theta_r = \sum_{i=1}^{r} \Theta_i
\]

of finitely many additive abelian groups \( \Theta_1, \ldots, \Theta_r \). Here, the null elements \( 0, \ldots, 0 \) of \( \Theta_1, \ldots, \Theta_r \) naturally assume the role of the identity elements \( 1, \ldots, 1 \).

If we apply this to the additive groups of commutative rings and define both addition and multiplication by performing the corresponding operations component-wise:

\[
(a_1, \ldots, a_r) + (b_1, \ldots, b_r) = (a_1 + b_1, \ldots, a_r + b_r)
\]

\[
(a_1, \ldots, a_r)(b_1, \ldots, b_r) = (a_1b_1, \ldots, a_r b_r)
\]
We obtain the concept of the direct sum:

\[ M \cong M_1 + \cdots + M_r = \sum_{i=1}^{r} M_i \]  

(11)

of finitely many commutative rings \( M_1, \ldots, M_r \). Then, the elements of the form: \( \bar{a}_i = (a_i, 0_2, \ldots, 0_r) \), \( \bar{a}_r = (0_1, \ldots, 0_{r-1}, a_r) \) constitute some subrings \( \overline{M_i}, \ldots, \overline{M_r} \) of \( M \), which are isomorphic to \( M_i, \ldots, M_r \) under the mappings \( a_i \mapsto \bar{a}_i, \ldots, a_r \mapsto \bar{a}_r \).

If the rings \( M_1, \ldots, M_r \) have identity elements \( 1_1, \ldots, 1_r \), then \( M \) also has an identity element \( 1 \), namely \( 1 = (1_1, \ldots, 1_r) \).

It has the decomposition into components \( 1 = \overline{1_1} + \cdots + \overline{1_r} \), which are the identity elements \( \overline{1_i} = (1_i, 0_2, \ldots, 0_r), \ldots, \overline{1_r} = (0_1, \ldots, 0_{r-1}, 1_r) \) of the subrings \( \overline{M_1}, \ldots, \overline{M_r} \). These identity elements \( 1_i \), are mutually orthogonal idempotent, that is, they satisfy the relations.

\[
\overline{1_i} \overline{1_j} = \begin{cases} 
\overline{1_i} & \text{for } i = j \quad \text{(idempotence)} \\
0 & \text{for } i \neq j \quad \text{(orthogonality)}
\end{cases} 
\]  

(12)

With the aid of these orthogonal idempotent \( \overline{1_i} \), we obtain the components \( \bar{a}_i \) in \( \overline{M_i} \) of an element \( a = \bar{a}_1 + \cdots + \bar{a}_r \).

In \( M \) in the form, \( \bar{a}_i = a \overline{1_i} \). Conversely, suppose that the following assumption is satisfied for a ring \( M \) with identity element \( 1 \): \( (A^*) \) the identity element \( 1 \) possesses a decomposition \( 1 = \overline{1_1} + \cdots + \overline{1_r} \) into mutually orthogonal idempotent \( \overline{1_i} \).

Then, the elements of the form \( \bar{a}_i = a \overline{1_i} \) constitute a subring \( \overline{M_i} \) of \( M \) since the sum, difference, and product of such elements are again of that form: \( a \overline{1_i} \pm b \overline{1_i} = (a \pm b) \overline{1_i} \), \( (a \overline{1_i})(b \overline{1_i}) = (ab) \overline{1_i} \overline{1_i} = (ab) \overline{1_i} \).

The elements \( a \) in \( M \) decompose into components:

\[ a = a1 = a \sum_{i=1}^{r} \overline{1_i} = \sum_{i=1}^{r} a \overline{1_i} = \sum_{i=1}^{r} \bar{a}_i \quad \text{with } \bar{a}_i \in \overline{M_i} \]  

(13)

Furthermore, these decompositions into components are unique [5].

4. Conclusion

In this paper, we research mainly on the fundamental homomorphism theorem applied to direct products of groups and the direct sums of rings in the commutative case. As the derivation and proof, direct product structured by several finite groups is constitutive homomorphism. To introduce the concept of homomorphism kernel, direct products is the isomorphism relationship between itself with these direct products factor, and, direct sums is the isomorphism relationship between itself with these direct sums factor.

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