Existence and nonexistence results of polyharmonic boundary value problems with supercritical growth

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ABSTRACT
Consider the following polyharmonic problems under the Dirichlet or the Navier boundary conditions

\[(−\Delta)^mu = f(u) + |u|^{p-1}u, \text{ in } \Omega,\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain and \(p > 1\) is a sub-critical power. We examine the effect of the positive parameter \(\lambda\) to study the existence and the nonexistence of regular solutions. When \(\lambda\) is large enough, we establish an existence result only under suitable growth condition on \(f\) at zero. Our approach is based on truncation argument as well as \(L^\infty\)-bounds. Also, by virtue of Pucci-Serrin’s variational identity [Pucci P, Serrin J. A general variational identity. Indiana Univ Math J. 1986;35:681–703.], we provide a nonexistence result when \(f\) has a supercritical growth at infinity and \(\lambda\) is small enough.

1. Introduction
Consider the following polyharmonic equations

\[(P_m) : (−\Delta)^mu = g(u) \text{ in } \Omega,\]

under the Dirichlet boundary conditions

\[u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 \text{ on } \partial \Omega,\]

or the Navier boundary conditions

\[u = \Delta u = \ldots = \Delta^{m-1} u = 0 \text{ on } \partial \Omega,\]

where \(N \geq 2m + 1\), \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain and \(g \in C(\mathbb{R})\). Problems \((P_m)\) arise in several models describing various phenomena in the applied sciences and have captured a lot of attention in the last decades [1]. The theory of second-order elliptic problems,
i.e. $m = 1$, is quite well developed for semilinear and even for some fully problems (see [2]). In contrast, the failure of the maximum principle is the main reason why the theory of nonlinear polyharmonic problems is far less than the one of second-order problems (see [1] for more details). To tackle $(P_m)$, various type of solutions were introduced and studied such as distributional solutions, weak solutions, classical solutions, and singular solutions. See [1, 3–9] and references therein. Most part of existence results of weak solutions followed from mini-max method which requires some suitable subcritical growth conditions (see [10]). When the variational approach cannot be employed, the question of existence of solutions may be dealt via topological methods, or bifurcation theory, or truncation technique [9, 10–16]. The proof of existence is essentially reduced to deriving $L^\infty$-bounds for a priori solutions or solutions of the corresponding perturbed problems.

Let us define the main $m$-order differential operator $D^m$ by

$$D^m u = \begin{cases} 
\nabla \Delta^{j-1} u & \text{if } r = 2j - 1, \\
\Delta^j u & \text{if } r = 2j.
\end{cases}$$

The appropriate functional spaces of the variational setting to problems $(P_m)$ are, respectively,

$$H^m_0(\Omega) := \{ v \in H^m(\Omega); \nabla^j u = 0 \text{ on } \partial\Omega, \text{ for } j = 0, 1, \ldots, m - 1 \},$$

and

$$H^m_0(\Omega) := \{ v \in H^m(\Omega); \Delta^j v = 0 \text{ on } \partial\Omega, \text{ for } j < \frac{m}{2} \}.$$  

We denote indifferently $H^m_0(\Omega)$ and $H^m_0(\Omega)$ by $H_m$ which is a Hilbert space equipped with the following scalar product (see [1])

$$\int_\Omega D^m u \cdot D^m v \, dx = \int_\Omega \nabla \Delta^{j-1} u \cdot \nabla \Delta^{j-1} v \, dx \quad \text{if } m = 2j - 1, \quad \int_\Omega \Delta^j u \Delta^j v \, dx \quad \text{if } m = 2j.$$  

Set now $G(s) = \int_0^s g(t) \, dt$. Assume that there exists a positive constant $C$ such that

$$|g(s)| \leq C(1 + |s|^{N+2m}), \quad \forall \, s \in \mathbb{R}.$$  

So, the Euler-Lagrange energy functional $I \in C^1(H_m)$ is defined by

$$I(u) = \frac{1}{2} \int_\Omega |D^m u|^2 \, dx - \int_\Omega G(u) \, dx, \quad \forall \, u \in H_m,$$

and we have $u \in H_m$ is a weak solution of $(P_m)$ if and only if $u$ is a critical point of $I$, that is

$$\int_\Omega D^m u \cdot D^m \nu \, dx = \int_\Omega g(u) \nu \, dx, \quad \forall \, \nu \in H_m.$$  

If we assume in addition that $g \in C^{0,a}_{\text{loc}}(\mathbb{R})$ for some $a \in ]0,1[$, then any weak solution of the boundary value problem $(P_m)$ belongs in $C^{2m}_{\text{loc}}(\overline{\Omega})$ (see the appendix of [10] for more details). Note that this regularity is essential to recuperate the last part of the Navier boundary conditions (2).
The Palais-Smale compactness condition plays a central role in the critical points theory and it is satisfied in the most part of literature under the following standard assumptions [17]:

(AR) Ambrosetti-Rabinowitz condition: There exist $\beta > 2$ and $s_0 > 0$ such that

$$g(s)s \geq \beta G(s) > 0 \quad \text{for all } |s| > s_0.$$ 

(SCP) Subcritical polynomial growth condition: There exist $1 < p < \frac{N + 2m}{N - 2m}$ and a positive constant such that

$$|g(s)| \leq C(1 + |s|^p) \quad \text{for all } s \in \mathbb{R}.$$ 

These assumptions have been relaxed into the following large subcritical growth conditions [10]:

(G1): There exist $C > 0$ and $s_0 > 0$ such that

$$C|g(s)|^{\frac{2N}{N + 2m}} \leq sg(s) - 2G(s), \quad \forall |s| > s_0,$$

and

(G2): $\lim_{s \to \infty} \frac{g(s)}{|s|^{\frac{N + 2m}{N - 2m}}} = 0.$

In this paper, we consider

$$(E_{m,\lambda}) \quad \{ (-\Delta)^mu = f(u) + \lambda |u|^{p-1}u, \text{ in } \Omega; \}
\{ u \text{ satisfies (1); or (2)}, \}$$

where $\lambda$ is a positive real parameter and

$$1 < p < \frac{N + 2m}{N - 2m}. \quad (3)$$

We examine the effect of the parameter $\lambda$ to study the existence and the nonexistence of regular solutions of $(E_{m,\lambda})$. In [18], Brezis and Nirenberg considered the problem $(E_{1,\lambda})$ with $f(s) = |s|^{\frac{4}{N-2}} s$. They used a variant of the mountain pass theorem without the Palais-Smale condition to prove the existence of positive solutions for all $\lambda > 0$ if $N \geq 4$, and $N = 3$ with $3 < p < 5$, or $\lambda$ large enough if $N = 3$ and $1 < p \leq 3$. We shall employ truncation procedure together with minimax argument to obtain existence results of problems $(E_{m,\lambda})$, for $\lambda$ large enough where we only assume that $f \in C(\mathbb{R})$ satisfies the following assumption near 0:

(H0) $\lim_{s \to 0} \frac{f(s)}{s} = L \in (-\infty, \lambda_1)$ and there exist $\nu \in (0, 1)$ and $C_1 > 0$ such that

$$|f(s)| \leq C_1 |s|^{1 - \nu}, \quad \forall |s| \leq 1. \quad (4)$$

Note that if $L \in (-\infty, \lambda_1)$, then (4) holds with $\nu = 0$. 

Note: 

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Our main existence result reads as follows:

**Theorem 1.1:** Assume that $p$ verifies (3) and $f \in C(\mathbb{R})$ satisfies $(H_0)$. Then, there exist $\nu = \nu(N, p, m) > 0$ and $\lambda = \lambda(v_0, N, p, f, m) > 0$ such that for all $\nu < \nu$ and $\lambda \geq \lambda$:

1. The problem $(E_{m, \lambda})$ admits a nontrivial solution belonging in $C^{2m-1}(\Omega)$ and $\|u\|_{L^\infty(\Omega)} < 1$. Moreover, if $f \in C^0, a((-1,1))$ for some $a \in (0,1)$, then $u \in C^{2m}(\Omega)$.
2. If we assume in addition that $f(s)s > 0$, $\forall |s| \leq 1$ and $s \neq 0$. Then the problem $(E_{m, \lambda})$ with the Navier boundary conditions (2), admits two nontrivial regular solutions $u_- < 0 < u_+$.

Consider the following Dirichlet boundary value problem

$$(-\Delta)^m u = |u|^{q-1}u + \lambda |u|^{p-1}u \quad \text{in} \; \Omega, \; u \text{ satisfies}.$$ (5)

When $\Omega$ is a star-shaped domain and $m = 1$, the first relevant nonexistence result follows from Pohozaev’s identity for all $q \geq \frac{N+2}{N-2}$ and $\lambda \leq 0$ (see [19, 20]). For more general higher order Dirichlet boundary value problem, Pucci and Serrin [21] established a variational identity allowing further nonexistence results. In particular they proved **Theorem A.**

- Let $u \in C^2(\Omega) \cap C^{2m-1}(\Omega)$ be a solution of the Dirichlet boundary value problem $(P_m)$. Assume that there exists $q > \frac{N+2m}{N-2m}$ such that $g(s)s - (q+1)G(s) \geq 0$ for all $s \in \mathbb{R}$. Then $u \equiv 0$.
- Moreover, if $u$ is a solution of problem (5) with $\lambda < 0$, $q \geq \frac{N+2m}{N-2m}$, or $\lambda = 0$ and $q > \frac{N+2m}{N-2m}$. Then $u \equiv 0$.

The critical growth $q = \frac{N+2m}{N-2m}$ is more difficult even for the case $\lambda = 0$, which has been achieved if $m = 2$ only for positive solutions of (5) [22] or nodal solutions in a ball [23]. When $\lambda > 0$, $m = 1$ and $\Omega$ is a ball, Brezis and Nirenberg [18] proved that nontrivial positive bounded solutions of (5) only exist in the interval $(\frac{\lambda_1}{4}, \lambda_1)$ for $N = 3$ and $(0, \lambda_1)$ for all $N \geq 4$ (see also [24] concerning radial solutions with prescribed number of nodes). For the higher order case $m \geq 2$, similar result was stated in only a so called critical dimension $2m + 1 \leq N \leq 4m - 1$ [25–27]. The reader may consult [1, 28] for further comments including also some nonexistence results involving the Navier boundary value biharmonic problem.

In the following, we assume that

$$1 \leq p < \frac{N+2m}{N-2m},$$ (6)

and

$(H_1)$: There exist $q > \frac{N+2m}{N-2m}$, $s_0 > 0$ and $C_0 > 0$ such that

$$f(s)s - (q+1)F(s) \geq 0, \quad \text{for all} \; s \in \mathbb{R} \quad \text{and} \quad f(s)s \geq C_0 |s|^{q+1}, \quad \text{for all} \; |s| \leq s_0.$$
We exhibit a \( \lambda > 0 \) such that a necessary condition for a nontrivial solution of the Dirichlet boundary value problem \( (E_{m,\lambda}) \) to exist is \( \lambda > \lambda_* \). This extends the nonexistence results stated in Theorem A.

**Theorem 1.2:** Let \( \Omega \) be a smooth star-shaped domain. Assume that \( p \) verifies (6) and \( f \in C(\mathbb{R}) \) satisfies \( (H_1) \). Then, there exists \( \lambda_* = \lambda_*(p, q, m, s, f, N, \Omega) > 0 \) such that for any \( \lambda < \lambda_* \), the problem \( (E_{m,\lambda}) \) with the Dirichlet boundary conditions (1), has no nontrivial regular solutions.

**Remark 1.1:**

- Assume that \( p \) verifies (3) and \( f \in C(\mathbb{R}) \) satisfies \( (H_0) \) with \( L \in [0, \lambda_1) \), and \( (H_1) \).

Thanks to Theorems 1.1 and 1.2, we can find \( \lambda_* > 0 \) and \( \lambda > 0 \) such that the Dirichlet boundary value problem \( (E_{m,\lambda}) \) admits a nontrivial solution if \( \lambda > \lambda_* \) and has no nontrivial solutions if \( \lambda < \lambda_* \).

- The following nonlinearities satisfy \( (H_0) \)-\( (H_1) \)

\[
f(s) := |s|^{q-1} s \exp(as), \quad a \geq 0 \quad \text{and} \quad f(s) := Ls \exp((q+1)s) \quad \text{with} \quad q > \frac{N+2m}{N-2m} \quad \text{and} \quad L \in [0, \lambda_1).
\]

Next, consider the following problem

\[
(Z_{m, \alpha}) \quad \begin{cases} 
(-\Delta)^m u = \alpha^{-1} f(\alpha u) + |u|^{p-1} u \text{ in } \Omega; \\
u \text{ satisfies (1); or (2),}
\end{cases}
\]

where \( \alpha \in (0, 1) \). From Theorems 1.1 and 1.2, we derive the following Corollary.

**Corollary 1.1:** Assume that \( f \in C(\mathbb{R}) \) and \( p \) satisfy, respectively, \( (H_0) \) and (3). Then there exist \( \nu = \nu(N, m, p) > 0 \) and \( \alpha_* = \alpha_* (f, N, m, \Omega) > 0 \) such that for any \( 0 \leq \nu \leq \nu \) and \( 0 \leq \alpha \leq \alpha_* \), we have

1. The problem \( (Z_{m, \alpha}) \) admits a nontrivial solution belonging in \( C^{2m-1}(\Omega) \) and \( \|u\|_{L^\infty(\Omega)} < \alpha_*^{-1} \). Moreover, if \( f \in C^0, a((-1, 1)) \) for some \( a \in (0, 1) \), then \( u \in C^{2m}(\Omega) \).
2. If we assume in addition that \( f(s) > 0, \forall |s| \leq 1 \) and \( s \neq 0 \). Then the Navier boundary value problem \( (Z_{m, \alpha}) \) admits two nontrivial regular solutions \( u_- < 0 < u_+ \).
3. Let \( \Omega \) be a star-shaped domain. Assume that \( f \in C(\mathbb{R}) \) and \( p \) satisfy respectively \( (H_1) \) and (6). Then there exists \( \alpha_* = \alpha_* (p, q, m, f, N, \Omega) > 0 \) such that for any \( \alpha > \alpha_* \), the Dirichlet boundary value problem \( (Z_{m, \alpha}) \) has no nontrivial regular solution.

The proof of Corollary 1.1 follows from a simple scaling argument. In fact, set \( \nu = \alpha_*^{-1} u \) with \( \alpha = \lambda_*^{-1/p} \), it is easy to see that \( \nu \) is a solution of \( (Z_{m, \alpha}) \) if and only if \( u \) is a solution of \( (E_{m, \lambda}) \).

**Remark 1.2:**

- Point 1 of this corollary improves the existence result stated in [11].
Set \( f(s) = -\frac{|s|^{2-v} \exp(\alpha s)}{s} \). Then \( f \) satisfies \((H_0)\) with \( L = -\infty \) and Corollary 1.1 implies that the following problem has a nontrivial solution

\[
(\Delta)^m u = -\frac{|s|^{2-v} \exp(\alpha s)}{\alpha^v s} + |u|^{p-1} u \quad \text{in } \Omega,
\]

for \( v \) and \( \alpha \) small enough. Point out that the above equation cannot be seen as a small perturbation of the following subcritical problem

\[
(\Delta)^m u = |u|^{p-1} u \quad \text{in } \Omega.
\]

This paper is organized as follows: In Section 2, we provide the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In the following, \( C \) denotes always generic positive constants depending on \((N, \Omega, p, m, C_1)\) only, which could be changed from one line to another.

**Proof of point 1.** According to assumption \((H_0)\), we can find \( 0 < \epsilon_0 < \inf(1, \lambda_1) \) and \( 0 < s_0' < 1 \) such that

\[
F(s) \leq \frac{\lambda_1 - \epsilon_0 s^2}{2} \quad \forall \, |s| \leq s_0',
\]

and

\[
|f(s)| \leq C|s|^{2-v}, \quad \text{and} \quad |F(s)| \leq C|s|^{2-v} \quad \forall \, |s| \leq s_0'.
\]

Observe that we may rewrite the Poincaré inequality as follows

\[
\frac{1}{2} \int_{\Omega} |D^m u|^2 \, dx - \frac{\lambda_1 - \epsilon_0}{2} \int_{\Omega} u^2 \, dx \geq \frac{\epsilon_0}{2\lambda_1} \int_{\Omega} |D^m u|^2 \, dx, \quad \forall \, u \in H.m.
\]

Let \( \theta \in C^1(\mathbb{R}) \) be the cut-off function such that \( \theta(s) = 1 \) if \( |s| \leq \frac{s_0'}{2} \), \( \theta(s) = 0 \) if \( |s| \geq s_0' \), and \( 0 \leq \theta \leq 1 \), \( \forall s \in \mathbb{R} \). For \( \alpha \in (0, 1) \), set

\[
f_\alpha(s) = \frac{\theta(\alpha s)}{\alpha} f(\alpha s), \quad F_\alpha(s) = \int_0^s f_\alpha(t) \, dt, \quad g_\alpha(t) = f_\alpha(s) + \frac{|s|^{p-1}}{p+1} \int_0^s g_\alpha(t) \, dt = F_\alpha(s) + \frac{|s|^{p+1}}{p+1}.
\]

and

\[
0 \leq F_\alpha(s) = \frac{F(\alpha s)}{\alpha^2} \leq \frac{\lambda_1 - \epsilon_0 s^2}{2}, \quad \forall \, (\alpha, s) \in (0, 1) \times \mathbb{R},
\]

respectively

\[
|f_\alpha(s)| \leq C\alpha^{-v}, \quad |F_\alpha(s)| \leq C\alpha^{-v} \quad \text{and} \quad |F_\alpha(s)| \leq C\alpha^{-v}, \quad \forall \, (\alpha, s) \in (0, 1) \times \mathbb{R}.
\]

Consider the truncated problems under the Navier or the Dirichlet boundary conditions

\[
(E_{m, \alpha}) \quad \begin{cases} 
(\Delta)^m u = g_\alpha(u) \quad \text{in } \Omega; \\
\text{u satisfies (1); or (2)}.
\end{cases}
\]
The associate energy functional of problems $(E_{m\alpha})$ is

$$I_{\alpha}(u) = \frac{1}{2} \int_{\Omega} |D^m u|^2 \, dx - \int_{\Omega} G_{\alpha}(u) \, dx, \quad u \in H_m.$$ 

As $g_{\alpha}(s) = |s|^{p-1}s$ for all $|s| \geq \frac{\epsilon_0}{\alpha}$, then $g_{\alpha}$ satisfies (AR). Also (8) implies

$$(SCP)_{\alpha} : \quad |g_{\alpha}(s)| \leq C\alpha^{-\nu}(1 + |s|^p) \quad \text{for all } s \in \mathbb{R}.$$ 

Consequently, $I_{\alpha} \in C^1(H_m)$ and satisfies the Palais-Smale condition. From (9) and (10), one concludes

$$I_{\alpha}(u) \geq \epsilon_0 \left( \int_{\Omega} |D^m u|^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx \right).$$

We apply Sobolev’s inequality, there holds

$$I_{\alpha}(u) \geq \frac{\epsilon_0}{2\lambda_1} \int_{\Omega} |D^m u|^2 \, dx - C \left( \int_{\Omega} |D^m u|^2 \, dx \right)^{\frac{p+1}{2}}.$$ 

Hence, there exist two constants $\rho > 0$ such that $(I_{\alpha})|_{\partial B_{\rho}} \geq \beta$. Fix $w_0 \in H_m$, $w_0 \neq 0$ and set $v_{\nu} = b_0 \alpha^{-\frac{1}{p+1}}w_0$, $b_0 > 0$. As $p + 1 > 2$ and $\alpha \in (0, 1)$, from (8), we can find $b_0 > 0$ large enough such that

$$I_{\alpha}(v_{\nu}) \leq \alpha^{-\nu} \left( \frac{b_0^2}{2} \int_{\Omega} |D^m w_0|^2 \, dx - \frac{b_0^{p+1}}{p+1} \int_{\Omega} w_0^{p+1} \, dx + C \right) \leq 0.$$ 

According to the mountain pass theorem, $I_{\alpha}$ admits a nontrivial critical point $u_{\alpha} \in H_m$ satisfying

$$I_{\alpha}(u_{\alpha}) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_{\alpha}(\gamma(t)) \quad \text{where } \Gamma = \{ \gamma \in C([0,1], H_m); \gamma(0) = 0, \gamma(1) = v_{\nu} \}.$$ 

Using again (8), we derive

$$I_{\alpha}(u_{\alpha}) \leq \sup_{t \in [0,1]} I_{\alpha}(tv_{\nu}) \leq \frac{t^2}{2} \int_{\Omega} |D^m v_{\nu}|^2 \, dx - \int_{\Omega} F_{\alpha}(tv_{\nu}) \, dx \leq C\alpha^{-\nu}. \quad (12)$$

As

$$I'_{\alpha}(u_{\alpha})u_{\alpha} = \int_{\Omega} |D^m u_{\alpha}|^2 \, dx - \int_{\Omega} g(u_{\alpha})u_{\alpha} \, dx = 0,$$

we obtain

$$I_{\alpha}(u_{\alpha}) = I_{\alpha}(u_{\alpha}) - \frac{1}{p+1} I'_{\alpha}(u_{\alpha})u_{\alpha}$$

$$= \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\Omega} |D^m u_{\alpha}|^2 \, dx - \int_{\Omega} F_{\alpha}(u_{\alpha}) \, dx + \frac{1}{p+1} \int_{\Omega} u_{\alpha}f_{\alpha}(u_{\alpha}) \, dx \leq C\alpha^{-\nu}.$$
According to (12) and (8), there holds
\[ \int_{\Omega} |D^{m}u_{\alpha}|^{2} \, dx \leq C\alpha^{-v}. \tag{13} \]

Now, we need the following Lemma based on a boot-strap argument:

**Lemma 2.1:** Let \( u_{\alpha} \in C^{2m-1}(\overline{\Omega}) \), there exist \( C > 0 \) and \( \gamma > 0 \) depending, respectively, only on \((\Omega, N, m, p, C)\) and \((N, p, m)\) such that
\[ \|u_{\alpha}\|_{C^{2m-1,\alpha'}(\Omega)} \leq C\alpha^{-v}\gamma. \tag{14} \]

Precisely if \( \frac{2N}{p(N-2m)} < \frac{N}{2m} \), then \( \gamma = \frac{2mp^{3}}{N(p-1)}\left(\frac{2m}{N(p-1)} - \frac{(N-2m)}{2N}\right)^{-1}. \)

**Proof:** Sobolev’s inequality gives
\[ \|u_{\alpha}\|_{L^{\frac{2N}{N-2m}}(\Omega)} \leq A\|D^{m}u_{\alpha}\|_{L^{2}(\Omega)}^{2}, \]
which combined with \((SCP)_{\alpha}\) imply
\[ \|g_{\alpha}(u_{\alpha})\|_{L^{q_{1}}(\Omega)} \leq C\alpha^{-v}\left(1 + \|D^{m}u_{\alpha}\|_{L^{2}(\Omega)}^{p}\right), \quad \text{where } q_{1} = \frac{2N}{p(N-2m)}. \tag{15} \]

We invoke now Rellich-Kondrachov’s theorem and \( L^{q}-W^{2m,q} \) regularity \([1]\). Precisely, if \( g_{\alpha}(u_{\alpha}) \in L^{q}(\Omega) \) for some \( q > 1 \), we have
\[ \|u_{\alpha}\|_{L^{q^{*}}(\Omega)} \leq C\|u_{\alpha}\|_{W^{2m,q}(\Omega)} \leq C\|g_{\alpha}(u_{\alpha})\|_{L^{q}(B_{1})}, \tag{16} \]
where
\[ q^{*} = \frac{qN}{N-2mq} \quad \text{if } 2mq < N, \quad \text{and} \quad q^{*} = \frac{pN + 1}{2m} \quad \text{if } q = \frac{N}{2m}, \]
and
\[ \|u_{\alpha}\|_{C^{2m-1,\alpha'}(\Omega)} \leq C\|u_{\alpha}\|_{W^{2m,q}(\Omega)} \leq C\|g_{\alpha}(u_{\alpha})\|_{L^{q}(B_{1})}, \quad \text{if } 2mq > N. \tag{17} \]

So, if \( q > p \), \((SCP)_{\alpha}\) and (16) imply
\[ \|g_{\alpha}\|_{L^{q^{*}}(B_{\frac{1}{2}})}^{q^{*}} \leq C\alpha^{-v}\left(1 + \|g\|_{L^{q^{*}}(B_{1})}^{p}\right), \quad \text{if } 2mq \leq N. \tag{18} \]

Consequently, if \( 2mq_{1} > N \) (respectively \( 2mq_{1} = N \), the desired estimate (14) follows from (15) and (17) (respectively (18) with \( q = q_{1} \), (17)–(18) with \( q = \frac{pN + 1}{2m} \) and (15)).
Let us now give more attention to the difficult case $2mq_1 < N$. Firstly, we have

$$q_1^* = \frac{q_1 N}{N - 2mq_1} = \frac{2N}{p(N - 2m) - 4m} > \frac{2N}{N - 2m} > p \quad \text{as} \quad p(N - 2m) < N + 2m.$$ 

Set $q_2 = \frac{q_1^*}{p}$, $q_k - \frac{q_k^*}{p}$. We claim that there exists $k_0 \in \mathbb{N}^*$ such that

$$2mq_{k_0 + 1} > N \quad \text{and} \quad 2mq_{k_0} < N.$$ 

(19)

Indeed, working by contradiction and suppose that $2mq_k < N$ for all $k \in \mathbb{N}^*$. Recall that $q^* = \frac{qN}{N - 2mq}$ if $2mq < N$, then $\frac{1}{q_{k+1}} = \frac{p}{q_k} - \frac{2mp}{N}$. From a direct calculation, we obtain

$$\frac{1}{q_{k+1}} = \frac{p^k}{q_1} - 2mp - \sum_{j=0}^{k-1} p^j = p^k \left( \frac{1}{q_1} - \frac{2mp}{N(p - 1)} \right) + \frac{2mp}{N(p - 1)}.$$ 

(20)

As $p$ verifies (3) and $q_1 = \frac{2N}{p(N - 2m)}$, then $\frac{1}{q_1} - \frac{2mp}{N(p - 1)} < 0$. Consequently, $\frac{1}{q_k}$ converges to $-\infty$, so we reach a contradiction since we assume that $\frac{1}{q_k} > \frac{1}{2mN}$. Set

$$\beta = \frac{2mp}{N(p - 1)} \left( \frac{2mp}{(p - 1)} - \frac{1}{q_1} \right)^{-1}.$$ 

From (20), we may see that

$$p^{k_0} < \beta \quad \text{and} \quad p^{k_0 + 1} > \beta.$$ 

(21)

Therefore, iterating (18) and using (21), there holds

$$\|g_\alpha(u_\alpha)\|_{L^{q_{k_0}+1}(\Omega)} \leq C \alpha^{-\nu p^{k_0}} \left( 1 + \|g_\alpha(u_\alpha)\|_{L^{q_1}(\Omega)}^{p^{k_0}} \right) \leq C \alpha^{-2\nu p^{k_0}} \left( 1 + \|D^m u_\alpha\|_{L^2(\Omega)}^{p^{k_0+1}} \right).$$

From (19), we have $mq_{k_0 + 1} > N$, then the last inequality combined with (17), (15) and (13) imply

$$\|u_\alpha\|_{C^{2m-1,\nu'}(\Omega)} \leq \|g_\alpha(u_\alpha)\|_{L^{q_{k_0}+1}(\Omega)} \leq C \alpha^{-2\nu p^{k_0}} \left( 1 + \|D^m u_\alpha\|_{L^2(\Omega)}^{p^{k_0+1}} \right) \leq A \alpha^{-2\nu p^{k_0+1}}.$$ 

Therefore, $u_\alpha \in C^{2m-1,\nu'}(\bar{\Omega})$ and the desired inequality (14) follows from (20).

Set $\nu = \gamma^{-1}$ and take $\nu < \nu$, then from (14), we can choose $\alpha_0 \in (0, 1)$ small enough such that

$$C \alpha^{-\gamma \nu} < \frac{s'}{2} \quad \text{so that} \quad |\alpha u_\alpha| < \frac{s'}{2} \quad \text{and} \quad g_\alpha(u_\alpha) = \frac{1}{\alpha^p} f(\alpha u_\alpha)$$

$$+ |u_\alpha|^{p-1} u_\alpha, \quad \forall 0 < \alpha < \alpha_0.$$ 

Set $\nu_\alpha = \alpha u_\alpha$, $\bar{\alpha} = \alpha_0^{1-p}$ and $\lambda = \alpha^{1-p}$, we obtain

$$(-\Delta)^m v_\alpha = \alpha g_\alpha(u_\alpha) = f(\nu_\alpha) + \lambda |v_\alpha|^{p-1} v_\alpha.$$ 

So, $v_\alpha \in C^{2m-1,\nu'}((\Omega))$ is a solution of problem $(E_{m,\lambda})$ with $\lambda = \alpha^{1-p}$ and $\|v_\alpha\|_{L^\infty((\Omega))} \leq 1$. This means that Theorem 1.1 holds for $\lambda_p = \alpha_0^{1-p}$. Moreover, if $f \in C^{0,\delta}((-1, 1))$
then \( g_\alpha(u) \in C^{0, \text{inf}}(\alpha, \alpha') (\overline{\Omega}) \) and from Schauder regularity [1], we derive that \( v_\alpha \in C^{2, \text{inf}}(\alpha, \alpha') (\overline{\Omega}) \).

Proof of point 2. Recall first the following Lemma [1]:

**Lemma 2.2:** Let \( u \in C^2 (\overline{\Omega}) \) be a solution of

\[
(-\Delta)^m u = h(x), \quad \text{in } \Omega, \ u \text{ satisfies } (2),
\]

where \( h \in C(\overline{\Omega}) \) is a nonnegative function such that \( h(x_0) > 0 \) for some \( x_0 \in \Omega \). Then, \( u(x) > 0 \) for all \( x \in \Omega \).

Set \( s_+ = \max(0, s) \), \( s_- = \max(0, -s) \) and

\[
g^+_\alpha(s) = f_\alpha(s_+) + s_+^p \quad \text{respectively } \quad g^-\alpha(s) = f_\alpha(-s) - s_-^p.
\]

Following the proof of point 1, we can show that there exists \( \bar{\lambda} > 0 \) such that

\[
(-\Delta)^m u = f(u_+) + \lambda u_+, \quad \text{in } \Omega, \ u \text{ satisfies 1},
\]

respectively

\[
(-\Delta)^m u = f(-u_-) - \lambda u_-^p, \quad \text{in } \Omega, \ u \text{ satisfies 1},
\]

admits a nontrivial solution \( u_+ \in C^{2m-1}(\overline{\Omega}) \) with (respectively \( u_- C^{2m-1}(\overline{\Omega}) \) for all \( \lambda \geq \bar{\lambda} \). Since we assume that \( f(s) > 0 \) for all \( s \neq 0 \) then \( g^+_\alpha \geq 0 \) (respectively \( -g^-\alpha \geq 0 \)). According to Lemma 2.2, we deduce that

\[
 u_- < 0 < u_+ \quad \text{on } \Omega.
\]

The proof of Theorem 1.1 is thereby completed. □

### 3. Proof of Theorem 1.2

As above, \( C \) denotes always generic positive constants depending on \((N, \Omega, p, q, m, f)\). According to assumption \((H_1)\), we get

\[
\left(1 - \frac{2N}{(q + 1)(N - 2m)}\right) f(s)s \leq f(s)s - \frac{2N}{N - 2m} F(s), \quad \forall \ s \in \mathbb{R}, \quad (22)
\]

Also, we can find a positive constant \( C = C(s_0, q) \) such that

\[
C|s|^{q+1} \leq f(s)s, \quad \forall \ s \in \mathbb{R}. \quad (23)
\]

In fact, as \( \leq f(s)s, \forall |s| \leq s_0 \), we have \( F(\pm s_0) > 0 \). From \((H_1)\) we have \( \left(\frac{F(s)}{s^{q+1}}\right)' \geq 0 \) for all \( s > 0 \). Therefore,

\[
f(s)s \geq F(s) \geq \frac{F(s_0)}{s_0^{q+1}} s_0^{q+1}, \quad \forall \ s \geq s_0.
\]

Observe that the nonlinearity \(-f(-s)\) satisfies also \((H_1)\). Then inequality (23) follows.
Let $\Omega$ be a star-shaped smooth bounded domain. Then, there exists $y \in \mathbb{R}^n$ such that $(x - y) \cdot \nu(x) \geq 0$, for all $x \in \partial \Omega$. Consider $u \in C^{2m}(\Omega) \cap C^{2m-1}(\Omega)$ a solution of $(P_m)$ and $a \in \mathbb{R}$. Applying Pucci-Serrin’s variational identity (see [21] page 702), we obtain
\[
\int_{\Omega} \left[ \left( \frac{N}{2} - a - m \right) |D^m u|^2 + \text{aug}(u) - NG(u) \right] \, dx = \frac{-1}{2} \int_{\partial \Omega} |D^m u|^2 (\nu \cdot (x - y)) \, ds.
\]

Fix $a = \frac{N - 2m}{2}$, we derive
\[
\int_{\Omega} \left( g(u)u - \frac{2N}{N - 2m} G(u) \right) \, dx \leq 0. \tag{24}
\]

We multiply $(P_m)$ by $u$, then Poincaré and Sobolev inequalities imply
\[
C \left( \int_{\Omega} |u|^{p+1} \, dx \right)^{\frac{2}{p+1}} \leq \int_{\Omega} |D^m u|^2 \, dx = \int_{\Omega} g(u)u \, dx. \tag{25}
\]

If $u \in C^{2m}(\Omega) \cap C^{2m-1}(\Omega)$ is a nontrivial solution of problem $(E_{m,\lambda})$ under the Dirichlet boundary conditions (1), then $u$ is also a solution of $(P_m)$ with $g(s) = f(s) + \lambda |s|^{p-1}s$. Hence, (24) and (22) imply
\[
\left( 1 - \frac{2N}{(q + 1)(N - 2m)} \right) \int_{\Omega} f(u)u \, dx \\
\leq \int_{\Omega} \left( f(u)u - \frac{2N}{N - 2m} F(u) \right) \, dx \leq \lambda \left( \frac{2N}{(p+1)(N - 2m)} - 1 \right) \int_{\Omega} |u|^{p+1} \, dx.
\]

By (23), we get then
\[
C \left( 1 - \frac{2N}{(q + 1)(N - 2m)} \right) \int_{\Omega} |u|^{q+1} \, dx \\
\leq \left( 1 - \frac{2N}{(q + 1)(N - 2m)} \right) \int_{\Omega} f(u)u \, dx \leq \lambda \left( \frac{2N}{(p+1)(N - 2m)} - 1 \right) \int_{\Omega} |u|^{p+1} \, dx.
\]

As $\frac{2N}{(p+1)(N - 2m)} - 1 > 0$, we deduce that $\int_{\Omega} |u|^{q+1} \, dx = 0$ if $\lambda \leq 0$. Then $u \equiv 0$.

If $\lambda > 0$, from Hölder’s inequality and (22), we obtain
\[
\left( \int_{\Omega} |u|^{p+1} \, dx \right)^{\frac{q+1}{p+1}} \leq C \int_{\Omega} |u|^{q+1} \, dx \leq C \int_{\Omega} f(u)u \, dx \leq C \lambda \int_{\Omega} |u|^{p+1} \, dx. \tag{26}
\]

Therefore,
\[
\int_{\Omega} |u|^{p+1} \, dx \leq C_1 \lambda^{\frac{p+1}{p+q}}, \tag{27}
\]

where $C_1$ is a positive constant depending on $(N, \Omega, p, q, m, f)$. In view of (25) and (26), we get
\[
\left( \int_{\Omega} |u|^{p+1} \, dx \right)^{\frac{2}{p+1}} \leq C \int_{\Omega} |D^m u|^2 \, dx \leq C \lambda \int_{\Omega} |u|^{p+1} \, dx. \tag{28}
\]
If $p = 1$, we get $\lambda > \frac{1}{C}$. If $p$ satisfies (3), from (28), one has
\[
C_2 \lambda^{\frac{p+1}{p-1}} \leq \int_{\Omega} |u|^{p+1} \, dx,
\]
where $C_2$ is a positive constant depending on $(N, \Omega, p, q, m, f)$. Combining the last inequality with (27), we deduce
\[
\lambda^{\frac{p+1}{q-p} + \frac{p+1}{p-1}} \geq \frac{C_2}{C_1}.
\]
In conclusion, the problem $(E_{m, \lambda})$ under the Dirichlet boundary conditions (1), has no nontrivial regular solution if $p$ satisfies (3) and $\lambda < \bar{\lambda}$ where $\bar{\lambda} = (\frac{C_2}{C_1})^{\delta}$ with $\delta = (\frac{p+1}{q-p} + \frac{p+1}{p-1})^{-1}$ (respectively if $p = 1$ and $\lambda < \bar{\lambda}$ with $\bar{\lambda} = \frac{1}{C}$).

\[\square\]

**Note**

1. Note that the $L^q - W^{2m,q}$ regularity is also valid under the Navier boundary conditions (see [14]).

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