so(2, 1) scattering via Euclidean Connection for the Natanzon Hypergeometric Potentials

Hermann Albrecht and Sebastián Salamó
Universidad Simón Bolívar, Departamento de Física,
A. P. 89000, Valle de Sartenejas, Miranda 1080-A, Venezuela.

Abstract
The so(2, 1) Euclidean Connection formalism is used to calculate the S matrix for the Hypergeometric Natanzon Potentials (HNP).

1 Introduction
Algebraic scattering has been an active research field for many years. Pioneer works in the late 1960’s and 1970’s by Zwanziger [3] and Barut, et al. [4] dealing with the Coulomb system founded the basic ideas of this area. More recently, Alhassid et al. [1] developed an algebraic formalism, known as the Euclidean connection, to calculate the S matrix for systems whose Hamiltonian may be written as a function of the Casimir operator of an so(2, 1) algebra.

In this work we study the application of the Euclidean connection to obtain in a straightforward manner the S matrix for the HNP, by extending the treatment developed in [5].

We start with a brief review of [5] and then introduce the basic ideas of the Euclidean connection for so(2, 1) [1]. Then we study its application to the scattering sector of the HNP, obtaining the S matrix for this family of potentials.

2 Bound states
As is well known, the Hypergeometric Natanzon Potentials (HNP) [2] are given by:

\[ V(z) = \frac{f z^2 - (h_0 - h_1 + f) z + h_0 + 1}{R(z)} \]
\[ + \left( a + \frac{a + (c_1 - c_0) (2z - 1)}{z (z - 1)} \frac{5 \Delta}{4 R(z)} \right) z^2 (1 - z)^2 \]

where

\[ R(z) = a z^2 + \tau z + c_0 \quad \Delta = \tau^2 - 4 a c_0 \quad \tau = c_1 - c_0 - a \]
where \( f, h_0, h_1, a, c_0, c_1 \) are the Natanzon Parameters and \( z(r) \) is a real arbitrary function that satisfies
\[
\frac{dz}{dr} \equiv z' = \frac{2z(z-1)}{\sqrt{R(z)}} \quad z(r) \in [0,1] \tag{3}
\]
and the transformation \( r(z) \) is assumed to take \( r \to \infty \) to \( z = 1 \). In [5] the bound state sector of the \( HNP \) was studied by means of an \( so(2,1) \) algebra, obtaining an algebraic description of it. The realization used is

\[
J_0 = -i\partial_\phi
\]
\[
J_\pm = \pm e^{\pm i\phi} \left\{ \pm \frac{\sqrt{z(z-1)}}{z'} \partial_r - \frac{1}{2} \frac{i(z+1)}{\sqrt{z}} \partial_\phi \right\}
\]
\[
= \pm e^{\pm i\phi} \left\{ \pm \frac{1}{2} (z-1) \left[ \frac{1 + p_\nu}{\sqrt{z}} - \frac{z^n}{z^{1/2}} \right] \right\}
\]
\[
Q = \frac{z(z-1)^2}{z'^2} \partial_r^2 + \frac{1}{4} \frac{(z-1)^2}{4z} \partial_\phi^2 + \frac{1}{2} \frac{i p_\nu (z^2 - 1)}{z} \partial_\phi
\]
\[
+ \frac{(z-1)^2}{4} \left[ z^2 (2z'' z' - 3z^n) - z'^4 (p_\nu^2 - 1) \right]
\]  \tag{4}

where \( p \) is a parameter, which allows the use of the \( so(2,1) \) algebra to completely describe the \( HNP \). The algebraic treatment makes use of

\[
(Q - q)\Psi(r, \phi) = G(r)(E - H)\Psi(r, \phi) \tag{5}
\]

where \( H \) is the standard radial Hamiltonian, \( q \) is the eigenvalue of the Casimir operator \( Q \) and \( G(r) \) is a function of \( r \). \( \Psi(r, \phi) \) is simultaneously an eigenfunction of \( Q \) and \( J_0 \) as well as of \( H \), and is given by \( \Psi(r, \phi) = e^{im\phi}\Phi(r) \). For the bound states the \( D^{(+)} \) representation is used, therefore the compact generator \( J_0 \) has eigenvalues [6]

\[
m = \nu + \frac{1}{2} + \sqrt{q_\nu + \frac{1}{4}} \quad \nu = 0, 1, 2, ..., \nu_{\text{max}} \tag{6}
\]

The energy spectrum is obtained from [5]:

\[
2\nu + 1 = \alpha_\nu - \beta_\nu - \delta_\nu \tag{7}
\]

where
\[
\alpha_\nu \equiv \sqrt{-a E_\nu + f + 1} = p_\nu + m_\nu
\]
\[
\beta_\nu \equiv \sqrt{-c_0 E_\nu + h_0 + 1} = p_\nu - m_\nu
\]
\[
\delta_\nu \equiv \sqrt{-c_1 E_\nu + h_1 + 1} = \sqrt{4q_\nu + 1}
\]

This are the basic equations of the algebrization derived in [5].
3 Euclidean connection

The main motivation for Euclidean connection [1] is that the asymptotic states (regular solution) are written as [7]:

\[ |\phi\rangle^{(\infty)} = A e^{ikr} + B e^{-ikr} \]  

where \( A \) and \( B \) are the Jost functions and \( e^{\pm ikr} \) are eigenfunctions of an Euclidean algebra \( e(2) \):

\[ [L_z, P_x] = i P_y \quad [L_z, P_y] = -i P_x \quad [P_x, P_y] = 0 \]  

(9)

A realization in polar coordinates of the ladder and Casimir operators is as follows:

\[
P_{\pm} = e^{\pm i\phi} \left[ -i \partial_r + \frac{1}{r} \left( \pm i \partial_\phi + \frac{1}{2} \right) \right]
\]

\[
P^2 = -\partial^2_r - \frac{1}{r^2} \left( \partial^2_\phi + \frac{1}{4} \right)
\]

(10)

where \( P_{\pm} \equiv P_x \pm i P_y \) and \( P^2 \equiv P_+ P_- \). Taking the limit \( r \to \infty \), the asymptotic generators are obtained:

\[
P_{\pm}^{(\infty)} = -ie^{\pm i\phi} \partial_r \quad P_{z}^{(\infty)} = -i \partial_\phi
\]

\[
P^2(\infty) = -\partial^2_r
\]

(11)

that also closes an \( e(2) \) algebra. The eigenstate basis \( \{ | \pm k, m \rangle^{(\infty)} \} \) is

\[
| \pm k, m \rangle^{(\infty)} = e^{\pm ikr} e^{im\phi}
\]

(12)

and the action of the asymptotic generators upon it is given by:

\[
P_{\pm}^{(\infty)} | (\pm)k, m \rangle^{(\infty)} = (\pm)k | (\pm)k, m \pm 1 \rangle^{(\infty)}
\]

\[
P^2 | \pm k, m \rangle^{(\infty)} = k^2 | \pm k, m \rangle^{(\infty)}
\]

(13)

As is well known, we can expand the ladder operators of \( so(2,1) \) in terms of the \( e(2) \) generators [8]. Therefore, given an \( so(2,1) \) algebraization of the bound states, it is natural to study its extension to the scattering sector by means of the mentioned expansion. The expanded ladder operator \( J_+^{(\infty)} \) of \( so(2,1)^{(\infty)} \) in the continuous representation (where \( j = -\frac{1}{2} + i f(k) \)) obtained in [1] is:

\[
J_+^{(\infty)}(\pm k) = e^{i\gamma_{\pm}^{(k)}} \left[ \left( -\frac{1}{2} + i f(k) \right) P_+^{(\infty)} + L_z^{(\infty)} P_+^{(\infty)} \right]
\]

(14)

where \( f(k) \) is defined by:

\[
k^2 = h \left( -f^2(k) \right) \quad E_j = h \left[ q + \frac{1}{4} \right]
\]

(15)
where $h(\eta)$ is the function that connects the Energy $E_j$ with the Casimir eigenvalue $q$ or, equivalently, the Hamiltonian with the Casimir operator $Q$.

A recurrence relation may be calculated and solved for the coefficients $A_m$ and $B_m$ by applying (14) directly to (8), with $|\phi(\infty)\rangle = e^{-im\phi}|j, m(\infty)\rangle$, and equating it to the action of $J_+^{(\infty)}$ upon $|j, m(\infty)\rangle$. Since $S_m(k) = A_m/B_m$, then:

$$S_m(k) = e^{im[\gamma_+ + \gamma_-]} \frac{\Gamma [m + \frac{1}{2} - if(k)]}{\Gamma [m + \frac{1}{2} + if(k)]} \Delta(k) \quad (16)$$

where $\Delta(k)$ is a constant factor to be determined.

4 S matrix for the HNP

Starting from (8), the asymptotic algebra is readily obtained by taking the limit $r \to \infty$ (or equivalently $z = 1$).

$$J_0^{(\infty)} = -i \partial_\phi \quad (17)$$

$$J_\pm^{(\infty)} = e^{\pm i\phi} \left[ \mp \sqrt{c_1} \partial_r - i \partial_\phi \pm \frac{1}{2} \right]$$

$$Q^{(\infty)} = \frac{c_1}{4} \partial^2_r - \frac{1}{4}$$

Since these generators form also an $so(2, 1)$ algebra, the Euclidean connection may be applied directly. Equation (17) corresponds to the $so(2, 1)^{(\infty)}$ algebra mentioned before. We only need to determine the expansion coefficients $f(k)$ and $\gamma_{\pm}(k)$. This is accomplished by equating the action of (14) with the one of (17), obtaining

$$\gamma_{\pm}(k) = 0 \quad f(k) = \frac{k\sqrt{c_1}}{2} \quad (18)$$

$$\Rightarrow \quad h'(\eta) = -\frac{4}{c_1} \eta$$

From (8) we know that

$$E_j = -\frac{4q - h_1}{c_1} = h \left[ q + \frac{1}{4} \right]$$

$$\Rightarrow \quad h(\eta) \equiv -\frac{4}{c_1} \eta + \frac{h_1 + 1}{c_1} \quad (19)$$

Since we are dealing with the scattering sector and $V_{PNH} \propto h_1 + 1$ as $r \to \infty$ [2], $h_1 = -1$. Therefore $h'(\eta) = h(\eta)$ and the $S$ matrix is given by (10):

$$S_m(k) = \frac{\Gamma (m + \frac{1}{2} - \frac{1}{2} ik\sqrt{c_1})}{\Gamma (m + \frac{1}{2} + \frac{1}{2} ik\sqrt{c_1})} \Delta_m(k) \quad (20)$$

where $\Delta_m(k)$ is a regular function of $k$ derived by Natanzon [2]. It can easily be verified that the poles are in agreement with the bound state energy spectrum.
5 Conclusions

We have successfully extended the algebraic treatment for the bound states developed in [5] to the scattering sector via the $so(2,1)$ Euclidean connection, therefore obtaining a purely algebraic description of the $HNP$ using an $so(2,1)$ algebra. This can also be accomplished with a methodology analogous to the one used by Frank, et al. [9]. This was done in detail and it is to be published soon [10]. Meanwhile, a short review may be found in this volume [11]. Even though Alhassid, et al. had obtained an algebraic description of the $HNP$ in [12] by means of an $so(2,2)$ algebra, it is important to notice that our treatment is simpler.

Acknowledgments

This work has been partially supported by grants USB-61D30 and FONACIT No. 6-2001000712.

References

[1] Alhassid, Y., Gürsey, F., Iachello, F. Ann. Phys. 167, (1986) 181.
[2] Natanzon, G. A., Vestnik Leningradskogo Universiteta, 10, (1971) 22-8, physics/9907032 Theor. Math. Phys., 38, (1979) 146.
[3] Zwanziger, D., J. Math. Phys., 8, (1967) 1858.
[4] Barut, A. O., Rasmussen, W., J. Phys. B: Atom. Molec., 6, (1973) 1965; Barut, A. O., Rasmussen, W., Salamó, S., Phys. Rev. D, 10, (1974) 622; Phys. Rev. D, 10, (1974) 630; Rasmussen, W., Salamó, S., J. Math. Phys., 20, (1979) 1064.
[5] Cordero, P., Salamó, S., Found. Phys., 23, 675 (1993). J. Math. Phys., 35, (1994) 3301.
[6] Barut, A.O., Fronsdal, C. Proc. R. Soc. London, Ser. A, 287, (1965) 532.
[7] Taylor, J. R., Scattering Theory: The Quantum Theory on Nonrelativistic collisions, John Wiley & Sons., 1972.
[8] Gilmore, R. Lie Groups, Lie algebras, and some of their applications, J. Wiley & Sons., 1974.
[9] Frank, A., Wolf, Kurt B., Phys. Rev. Lett., 52(20), 1737 (1984).
[10] Albrecht, H., Salamó, S., to be published.
[11] Salamó, S., to appear in Rev. Mex. Fís., (2002).
[12] Alhassid, Y., Wu, J., Gürsey, F., Ann. Phys., 196 (1989) 163-191.