Conservation of the Dirac Current in Models with a General Spin Connection

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Abstract

Here I obtain the conditions necessary for the conservation of the Dirac current when one substitutes the assumption $\gamma^A|_B = 0$ for $\gamma^A|_B = [V_B, \gamma^A]$, where the $\gamma^A$s are the Dirac matrices and "\|" represents the components of the covariant derivative. As an application, I apply these conditions to the model used in Ref. [M. Novello, Phys. Rev. D8, 2398 (1973)].

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I. INTRODUCTION

The Dirac equation has been successfully used as a relativistic version of the Schrödinger equation. It has the additional degree of freedom known as spin and, in an inertial frame with Cartesian coordinates, one writes this equation in a very simple and well-established form, namely,

\[ i\gamma^\mu \partial_\mu \psi - m\psi = 0. \]  \hspace{1cm} (1)

On the other hand, when we go to either an accelerated frame or a curved spacetime we need the concept of covariant derivative to change \( \partial_\mu \) for \( \partial_\mu + \Gamma_\mu \), where \( \Gamma_\mu \) is known as the spin connection. This also happens when an electromagnetic field is present. In this case, one substitutes \( \partial_\mu \) for \( \partial_\mu + A_\mu \), where \( A_\mu \) is the electromagnetic 4-potential and \( \partial_\mu + A_\mu \) can be seen as the covariant derivative in a fiber bundle. It turns out that \( \Gamma_\mu \) can simulate \( A_\mu \), even when one takes the standard definition of \( \Gamma_\mu \) (the vanishing of the covariant derivative of the Dirac matrices). A common feature of these covariant derivatives is the conservation of the Dirac current. However, this is not always true when one takes a more general spin connection. In this article I show the conditions needed for the conservation of the current with a general spin connection. In addition, I use this conditions in a model based on the generalization of \( \Gamma_\mu \) that simulates some physical interactions by using only the spin connection. It turns out that these conditions do not affect the model significantly.

This article is organized as follows. In Sec. II I present the notation and conventions adopted here and write down some identities that will be use throughout this paper. Sec. III is devoted to a brief review of the model presented in Ref. [3], while in Sec. IV the conditions for the conservation of the Dirac current are obtained. Some final comments are left to Sec. V.

II. NOTATION AND CONVENTIONS

Throughout this paper capital Latin letters represent tetrad indices, while Greek letters represent coordinate ones. All of them run over 0–3, but the tetrad ones are written between parentheses when numbered (\( A^{(0)} \), for example).

In the tetrad basis, the components of the metric takes the form \( \eta_{AB} = \eta^{AB} = \text{diag}(1, -1, -1, -1) \).
Following the standard notation, I use “\([\ldots]\)” to indicate the antisymmetric part of a tensor. For instance, \(\gamma^{[A}\gamma^{B}\gamma^{C]} = (\gamma^{A}\gamma^{B}\gamma^{C} - \gamma^{C}\gamma^{B}\gamma^{A})/2\). When no vertical bar is present, one must antisymmetrize all indices inside the brackets. For instance, \(\gamma^{[A}\gamma^{B}\gamma^{C]} = (\gamma^{A}\gamma^{B}\gamma^{C} + \gamma^{C}\gamma^{A}\gamma^{B} - \gamma^{A}\gamma^{C}\gamma^{B} - \gamma^{B}\gamma^{A}\gamma^{C} - \gamma^{C}\gamma^{B}\gamma^{A})/6\).

The Levi-Civita alternating symbol will be denoted by \(\epsilon_{ABCD}\), where \(\epsilon_{0123} = \epsilon_{0123} = +1\). Notice that this is just a symbol, not a component of a tensor. Besides, \(\epsilon^{ABCD} \neq \eta^{AL}\eta^{BM}\eta^{CN}\eta^{DO}\epsilon_{LMNO}\). Nonetheless, we can define a pseudo-tensor through the identification \(\epsilon_{ABCD} \equiv \epsilon_{ABCD}\). In this case, we have \(\epsilon^{ABCD} = \eta^{AL}\eta^{BM}\eta^{CN}\eta^{DO}\epsilon_{LMNO} = \eta^{-1}\epsilon^{ABCD} = -\epsilon^{ABCD}\), where \(\eta\) is the determinant of the metric.

There are many ways to represent the generators of the Clifford algebra. Nevertheless, I will stick to \(\{I, \gamma^{A}, \gamma^{[A}\gamma^{B]}, \gamma^{[A}\gamma^{B}\gamma^{C]}, \gamma^{(5)}\}\), where \(\gamma^{(5)} = \gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)}\). The \(\gamma^{A}\)s are the standard Dirac matrices in four dimensions. These matrices satisfy \(\gamma^{A}\gamma^{B} + \gamma^{B}\gamma^{A} = 2\eta^{AB}I\), where the unit matrix \(I\) will be omitted from now on. The generators of the Clifford algebra satisfy many identities, some of them are shown below (for more details, see Ref. [4]).

\[
\gamma^{A}\gamma^{B} = \gamma^{[A}\gamma^{B]} + \eta^{AB}, \quad (2)
\]
\[
\gamma^{E}\gamma^{[A}\gamma^{B]} = \gamma^{[E}\gamma^{A}\gamma^{B]} + \eta^{E}\gamma^{A}\gamma^{B} - \eta^{E}\gamma^{B}\gamma^{A}, \quad (3)
\]
\[
\gamma^{[A}\gamma^{B]}\gamma^{E} = \gamma^{[E}\gamma^{A}\gamma^{B]} - \eta^{E}\gamma^{A}\gamma^{B} + \eta^{E}\gamma^{B}\gamma^{A}, \quad (4)
\]
\[
\gamma^{E}\gamma^{[A}\gamma^{B}\gamma^{C]} = -\epsilon^{EABC}\gamma^{(5)} + \eta^{E}\gamma^{[B}\gamma^{C]} + \eta^{E}\gamma^{[C}\gamma^{A]} + \eta^{E}\gamma^{[A}\gamma^{B]}, \quad (5)
\]
\[
\gamma^{[A}\gamma^{B}\gamma^{C]}\gamma^{E} = \epsilon^{EABC}\gamma^{(5)} + \eta^{E}\gamma^{[B}\gamma^{C]} + \eta^{E}\gamma^{[C}\gamma^{A]} + \eta^{E}\gamma^{[A}\gamma^{B]}, \quad (6)
\]
\[
\gamma^{E}\gamma^{(5)} = -\gamma^{(5)}\gamma^{E} = \frac{1}{3!}\epsilon^{E}\epsilon^{EABC}\gamma^{[A}\gamma^{B}\gamma^{C]} \quad (7)
\]

### III. SELF-INTERACTION OF THE \(\gamma\) FIELD

A usual assumption when obtaining the Dirac equation is \(\gamma^{A}\big|_{B} = 0\) (see, e.g., Ref. [2]), which is sufficient but not necessary to guarantee that \(\eta_{AB|C} = 0\). Some models substitute this assumption for a more general one \(\gamma^{A}\big|_{B} = [V_{B}, \gamma^{A}]\), which leads to the connection

\[
\Gamma_{C} = (1/4)\omega_{ABC}\gamma^{[A}\gamma^{B]} + V_{C}, \quad (8)
\]
where $\omega^A_{BC} \equiv \theta^A, \nabla_{e_B} e_C >$ is the affine connection in the tetrad basis, and $V_C$ belongs to the Clifford algebra. In Ref. [3], the author takes $V_B$ in the following general form:

$$V_B = q_1 A_B(x) + q_2 F^A(x) \gamma_B \gamma_A \gamma_5(x)$$

$$+ q_3 \phi(x) \gamma_B \gamma_5(x) + q_4 B_B(x) \gamma_5(x)$$

$$+ q_5 \xi(x) \gamma_B .$$

(9)

Through the identifications $e \phi_B(x) = q_1 A_B(x) - 3q_2 F_B(x)$, $g_F W_B(x) = 3q_2 F_B(x)$, and the definition $L_D \equiv i (\bar{\psi} A \gamma^A \psi - \bar{\psi} \gamma^A \psi_A) - 2m \bar{\psi} \psi$, the Dirac Lagrangian becomes

$$L_D = L_D^S + i e \phi^A \bar{\psi} A_2 \gamma_A \psi + i g_F W^A \bar{\psi} A_1 \gamma_5 (1 + \gamma_5) \psi$$

$$+ 4q_5 \xi(x) \bar{\psi} \psi,$$

(10)

where $L_D^S$ is the standard Dirac Lagrangian and $\bar{\psi} = \psi^\dagger \gamma^{(0)}$. The first term after $L_D^S$ may be interpreted as a kind of electromagnetic interaction, the second as a weak interaction with a vector boson, and the third term as a mass correction [3]. The Dirac equation for $\psi$ and $\bar{\psi}$ are

$$i \gamma^A \psi_A - m \psi = 0,$$

(11)

$$i \gamma^A \bar{\psi}_A + m \bar{\psi} = 0,$$

(12)

where one has to demand $\gamma^{(0)} \Gamma_\mu^\dagger \gamma^{(0)} = -\Gamma_\mu^\dagger$ to ensure that (10) is a real-valued Lagrangian and guarantee the validity of Eq. (12). This limits the possible values of the coefficients appearing in (9). The possible values of these coefficients will be written down later on.

An intriguing feature of the model adopted in Ref. [3] is the non-conservation of the current $J^A \equiv \bar{\psi} \gamma^A \psi$. It is straightforward to verify that the covariant derivative of this current satisfies $J^A_{\mid A} = \nabla [V_A, \gamma^A] \psi$. This means that this current is conserved only if $[V_A, \gamma^A]$ vanishes. I shall present the conditions needed for the conservation of $J^A$ in the next section.
IV. CONSERVATION OF THE CURRENT

By imposing the condition $\gamma^{(0)} \Gamma^\dagger_{\mu} \gamma^{(0)} = -\Gamma_\mu$, one obtains

\begin{align*}
(q_1 A_A)^* &= -q_1 A_A, \\
(q_5 \xi)^* &= -q_5 \xi, \\
(q_2 F^D)^* &= q_2 F^D, \\
(q_3 \phi)^* &= q_3 \phi, \\
(q_2 F_A + q_4 B_A)^* &= -(q_2 F_A + q_4 B_A).
\end{align*}

But, in general, the term $[V_A, \gamma^A]$ is still nonzero. We need another set of conditions to make this term vanish.

In order to use the identities (2-7), I write $V_A$ in the following form:

\begin{equation}
V_A \equiv \bar{A}_A \mathbb{I} + V_{AB} \gamma^B + V_{ABC} \gamma^{[B} \gamma^{C]} + V_{ABCD} \gamma^{[B} \gamma^{C} \gamma^{D]} + \bar{B}_A \gamma^{(5)}.
\end{equation}

It is easy to verify that

\begin{align*}
\gamma_A \gamma_B \gamma^{(5)} &= \eta_{AB} \gamma^{(5)} + \varepsilon_{ABCD} \gamma^{[C} \gamma^{D]}, \\
\gamma_A \gamma^{(5)} &= -\varepsilon_{ABCD} \gamma^{[B} \gamma^{C} \gamma^{D]}, \\
\gamma_5 (x) &= -\overline{(\varepsilon^{0123})^{-1}(x) \gamma^{(5)}},
\end{align*}

where $\varepsilon^{\mu\nu\alpha\beta}(x) = e_A^\mu e_B^\nu e_C^\alpha e_D^\beta \varepsilon^{ABCD}$. Note also that $\gamma^{(5)} = -\gamma^{(5)}$. Besides, one can easily verify the relations

\begin{align*}
\bar{A}_A &= q_1 A_A, \\
V_{AB} &= q_5 \xi \eta_{AB}, \\
V_{ABC} &= -\frac{q_2}{\varepsilon} \varepsilon_{ABCD} F^D, \\
V_{ABCD} &= \frac{q_3}{\varepsilon} \phi \varepsilon_{ABCD} \\
\bar{B}_A &= \frac{1}{\varepsilon} (q_2 F_A + q_4 B_A),
\end{align*}

where $\varepsilon(x) \equiv \varepsilon^{0123}(x)$.

By using the identities (2-7) into $[V_A, \gamma^A] = 0$, we obtain

\begin{equation}
V_{[AB]} = V^A_{AB} = V_{[ABCD]} = \bar{B}_E = 0.
\end{equation}
It is worth emphasizing that (27) is independent of the model considered here.

From (27) and (22-26), one finds \( q_2 F_A + q_4 B_A = 0 \) and \( q_5 \phi = 0 \), which is compatible with (13-17). This reduces (9) to

\[
V_B = q_1 A_B + q_2 F^A \gamma_B \gamma_A \gamma_5 - q_2 F_B \gamma_5 + q_5 \xi \gamma_B
\]

(28)

with \( q_1 A_B \) and \( q_5 \xi \) being pure imaginary “numbers”, and \( q_2 F_B \) a real one. It is interesting to emphasize that the form of the Lagrangian (10) remains the same.

V. FINAL REMARKS

As we have seen, it is possible to take a more general spin connection \( \Gamma_A \) without changing the conservation of the current \( J^A \). We only need to impose some weak restrictions on \( V_A \). We have also seen that it is possible to keep the requirement \( \gamma^{(0)} \Gamma^i_{\mu} \gamma^{(0)} = -\Gamma_{\mu} \), since (27) is compatible with (13-17).

[1] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. (IOP Publishing, 2003).
[2] Although this identification is deceptive (see, e.g., Ref. [5]).
[3] M. Novello, *Phys. Rev.* D8, 2398 (1973)
[4] J. Formiga, (2012), arXiv:1209.5792 [math-ph]
[5] J. P. Crawford, *Class. Quant. Grav.* 20, 2945 (2003).