SOBOLEV EMBEDDING FOR $M^{1,p}$ SPACES IS EQUIVALENT TO A LOWER BOUND OF THE MEASURE.

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1. Introduction

A metric-measure space $(X, d, \mu)$ is a metric space $(X, d)$ with a Borel measure $\mu$ such that $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and all $r > 0$. Sobolev spaces on metric-measure spaces, denoted by $M^{1,p}$, have been introduced in [10], and they play an important role in the so called analysis on metric spaces [4, 5, 20, 16]. Later, many other definitions have been introduced in [7, 11, 12, 23], but in the important case when the underlying metric-measure space supports the Poincaré inequality, all the definitions are equivalent [9, 19]. One of the features of the theory of $M^{1,p}$ spaces is that, unlike most of other approaches, they do not require the underlying measure to be doubling in order to have a rich theory. In this paper we will focus on understanding the relation between the Sobolev embedding theorems for spaces $M^{1,p}$ and the growth properties of the measure $\mu$.

Let $(X, d, \mu)$ be a metric-measure space. We say that $u \in M^{1,p}(X, d, \mu)$, $0 < p < \infty$, if $u \in L^p(\mu)$ and there is a non-negative function $0 \leq g \in L^p(\mu)$ such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \quad \text{for almost all } x, y \in X. \quad (1)$$

More precisely, there is a set $N \subset X$ of measure zero, $\mu(N) = 0$, such that inequality (1) holds for all $x, y \in X \setminus N$. By $D(u)$ we denote the class of all functions $0 \leq g \in L^p(\mu)$ for which the above inequality is satisfied, and we set

$$D(u) \setminus \{0\} := \{g \in D(u) : g(x) \neq 0 \text{ for } \mu\text{-almost every } x \in X\}.$$

This space is equipped with a ‘norm’

$$\|u\|_{M^{1,p}} = \|u\|_p + \inf_{g \in D(u)} \|g\|_p.$$

We put the word ‘norm’ in inverted commas, because it is a norm only when $p \geq 1$. In fact, if $p \geq 1$, The space $M^{1,p}$ is a Banach space.

If $\Omega \subset X$ is an open set, then $(\Omega, d, \mu)$ is a metric measure space and hence $M^{1,p}(\Omega, d, \mu)$ is well defined. In other words, $u \in M^{1,p}(\Omega, d, \mu)$ if $u \in L^p(\Omega)$ and there is $0 \leq g \in L^p(\Omega)$ such that (1) holds for almost all $x, y \in \Omega$.

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1The set $X$ is tacitly assumed to have cardinality $\geq 2$.  

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The space $M^{1,p}$ is a natural generalization of the classical Sobolev space, because if $p > 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with the $W^{1,p}$-extension property, then

$$M^{1,p}(\Omega, d_{\mathbb{R}^n}, \mathcal{L}^n) = W^{1,p}(\Omega)$$

and the norms are equivalent, see [9]. Here we regard $\Omega$ as a metric-measure space with the Euclidean metric $d_{\mathbb{R}^n}$, and the Lebesgue measure $\mathcal{L}^n$. When $p = 1$, the space $M^{1,1}$ in the Euclidean setting is equivalent to the Hardy-Sobolev space [22]. While the spaces $M^{1,p}$ for $0 < p < 1$ do not have an obvious interpretation in terms of classical Sobolev spaces, they found applications to Hardy-Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces [22, ?].

The classical Sobolev embedding theorems for $W^{1,p}(\mathbb{R}^n)$ have different character when $p < n$, $p = n$ or $p > n$. Therefore, in the metric-measure context, in order to prove embedding theorems, we need a condition that would be the counterpart of the dimension of the space. It turns out that such a condition is provided by the lower bound for the growth of the measure

$$\mu(B(x, r)) \geq br^s.$$  \hspace{1cm} (2)

With this condition one can prove Sobolev embedding theorems for $M^{1,p}$ spaces and the embedding has different character if $0 < p < s$, $p = s$ or $p > s$. For a precise statement see Theorem 4. The purpose of the paper is to prove that condition (2) is actually equivalent to the existence of the embeddings listed in Theorem 4. Precise statements are given in Theorem 1. Partial or related results have been obtained in [6, 8, 13, 14, 15, 21].

The first main result of this paper highlights the fact that the lower measure condition in (2) characterizes certain $M^{1,p}$-Sobolev embeddings. See Theorem 11, Theorem 15, and Theorem 20 in the body of the paper for a more detailed account of the following theorem.

**Theorem 1.** Suppose that $(X, d, \mu)$ is a uniformly perfect\(^2\) measure metric space and fix parameters $\sigma \in (1, \infty)$, and $s \in (0, \infty)$. Then the following statements are equivalent.

1. There exists a finite constant $\kappa > 0$ such that
   $$\kappa \, r^s \leq \mu(B(x, r)) \text{ for every } x \in X$$
   $$\text{and every finite } r \in (0, \text{diam}(X)].$$  \hspace{1cm} (3)

2. There exist $p \in (0, s)$ and $C \in (0, \infty)$ such that for every ball $B_0 := B(x_0, R_0)$ with $x_0 \in X$ and finite $R_0 \in (0, \text{diam}(X)],$ one has
   $$\left( \frac{1}{B_0} \int_{B_0} |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C \left( \frac{\mu(B_0)}{R_0^s} \right)^{1/p} \left[ R_0 \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} |u|^{\sigma p} \, d\mu \right)^{1/p} \right],$$
   \hspace{1cm} (4)

whenever $u \in M^{1,p}(\sigma B_0)$ and $g \in D(u)$. Here, $p^* = sp/(s - p)$.

3. There exist $p \in (0, s)$ and $c \in (0, \infty)$ such that for every ball $B_0 := B(x_0, R_0)$ with $x_0 \in X$ and finite $R_0 \in (0, \text{diam}(X)],$ one has
   $$\inf_{\gamma \in \mathbb{R}} \left( \frac{1}{B_0} \int_{B_0} |u - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \leq c \left( \frac{\mu(B_0)}{R_0^s} \right)^{1/p} R_0 \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} g^p \, d\mu \right)^{1/p},$$
   \hspace{1cm} (5)

\(^2\)See (50), below.
whenever \( u \in M^{1,p}(\sigma B_0, d, \mu) \) and \( g \in D(u) \).

(4) There exist constants \( c_1, c_2, \gamma \in (0, \infty) \) such that
\[
\int_{B_0} \exp \left( c_1 \frac{|u - u_{B_0}|}{\|g\|_{L^s(\sigma B_0)}} \right)^\gamma \, d\mu \leq c_2, \tag{6}
\]
whenever \( B_0 \subseteq X \) is a ball (with radius at most \( \diam(X) \)), \( u \in M^{1,s}(\sigma B_0) \) and \( g \in D(u) \setminus \{0\} \).

(5) There exist \( p \in (s, \infty) \) and constant \( c \in (0, \infty) \) such that
\[
|u(x) - u(y)| \leq c d(x, y)^{1-s/p} \|g\|_{L^p(X, \mu)}, \quad \forall x, y \in X, \tag{7}
\]
Hence, every function \( u \in M^{1,p}(X) \) has Hölder continuous representative of order \((1 - s/p)\) on \( X \).

**Remark 2.** Theorem 1 asserts that the lower measure bound in (3) is equivalent to existence of a Sobolev embedding (in each of the cases) for some \( p \). However, in light of Theorem 4, we may conclude that if one of the Sobolev embeddings (4)-(7) holds for some \( p \), then they hold for every \( p \) (in each of the cases).

Given a metric-measure space, \((X, d, \mu)\), the measure \( \mu \) is said to be **doubling** provided there exists a constant \( C \in (0, \infty) \) such that
\[
\mu(2B) \leq C \mu(B) \quad \text{for all balls } B \subseteq X. \tag{8}
\]
The smallest constant playing the role of \( C \) in (8) will be denoted by \( C_\mu \). It follows from (8) that if \( X \) contains at least two elements then \( C_\mu > 1 \) (see [3, Proposition 3.1, p. 72]). Moreover, as is well-known, the doubling property in (8) implies the following quantitative condition: for each \( s \in \left[ \log_2(C_\mu), \infty \right) \), there exists \( \kappa \in (0, \infty) \) satisfying
\[
\kappa \left( \frac{r}{R} \right)^s \leq \frac{\mu(B(x, r))}{\mu(B(y, R))}, \tag{9}
\]
whenever \( x, y \in X \) satisfy \( B(x, r) \subseteq B(y, R) \) and \( 0 < r \leq R < \infty \). Conversely, any measure satisfying (9) for some \( s \in (0, \infty) \) is necessarily doubling. Note that if the space \( X \) is bounded then the above quantitative doubling property implies the lower measure bound in (3).

The following theorem, which constitutes the second main result of our paper, is an analogue of Theorem 1 for doubling measures. The reader is referred to Theorem 13 and Theorem 21.

**Theorem 3.** Suppose that \((X, d, \mu)\) is a uniformly perfect measure metric space and fix parameters \( \sigma \in (1, \infty) \), and \( s \in (0, \infty) \). Then the following statements are equivalent.

(1) There exists a constant \( \kappa \in (0, \infty) \) such that for each ball \( B_1 := B(x_1, R_1) \) with \( x_1 \in X \) and \( R_1 \in (0, \infty) \), the measure \( \mu \) satisfies
\[
\kappa \left( \frac{r}{R_1} \right)^s \leq \frac{\mu(B(x, r))}{\mu(\sigma B_1)} \quad \text{whenever } x \in X \text{ and } \quad r \in (0, \sigma R_1) \quad \text{are such that } B(x, r) \subseteq \sigma B_1. \tag{10}
\]
(2) There exist \( p \in (0, s) \) and \( C \in (0, \infty) \) such that for every ball \( B_0 := B(x_0, R_0) \) with \( x_0 \in X \) and \( R_0 \in (0, \infty) \), one has

\[
\left( \frac{1}{\sigma B_0} \int_{\sigma B_0} |u|^p d\mu \right)^{1/p} \leq CR_0 \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} g^p d\mu \right)^{1/p} + C \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} |u|^p d\mu \right)^{1/p} ,
\]

whenever \( u \in M^{1,p}(\sigma B_0, d, \mu) \) and \( g \in D(u) \).

(3) There exist \( p \in (0, s) \) and \( C_1 \in (0, \infty) \) such that for every ball \( B_0 := B(x_0, R_0) \) with \( x_0 \in X \) and \( R_0 \in (0, \infty) \), one has

\[
\inf_{\gamma \in \mathbb{R}} \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} |u - \gamma|^p d\mu \right)^{1/p} \leq C_1 R_0 \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} g^p d\mu \right)^{1/p} ,
\]

whenever \( u \in M^{1,p}(\sigma B_0, d, \mu) \) and \( g \in D(u) \).

(4) There exist \( p \in (s, \infty) \) and \( c \in (0, \infty) \) with the property that for each \( u \in M^{1,p}(X) \) and \( g \in D(u) \), and each ball \( B_0 := B(x_0, R_0) \) with \( x_0 \in X \) and \( R_0 \in (0, \text{diam}(X)] \), finite, one has there holds

\[
|u(x) - u(y)| \leq c d(x,y)^{1-s/p} R_0^{s/p} \left( \frac{1}{\sigma B_0} \int_{\sigma B_0} g^p d\mu \right)^{1/p}
\]

for every \( x, y \in B_0 \).

Hence, every function \( u \in M^{1,p}(\sigma B_0) \) has Hölder continuous representative of order \((1 - s/p)\) on \( B_0 \).

1.1. Notation. Open balls in a metric space \((X,d)\) will be denoted by \( B(x,r) = \{ y : d(x,y) < r \} \) while notation \( \overline{B} = \{ y \in X : d(x,y) \leq r \} \) will be used for closed balls. As a sign or warning, note that in general \( \overline{B}(x,r) \) is not necessarily equal to the closure of \( B(x,r) \). If \( r = 0 \), then \( B(x,r) = \emptyset \), but \( \overline{B}(x,r) = \{ x \} \). By \( C \) we will denote a general constant whose value may change within a single string of estimates. By writing \( C(s,p) \) we will mean that the constant depends on parameters \( s \) and \( p \) only. The integral average will be denoted by

\[
u_E = \frac{1}{\mu(E)} \int_E u d\mu.
\]

2. Sobolev embedding on metric-measure spaces

The next result from [9] provides a general embedding theorem for Sobolev spaces \( M^{1,p} \) defined on balls in a metric measure space \( X \). While this result has been proven in [9] we decided to include a proof for the following reasons. The paper [9] does not include the inequality (15). While in the case \( p^* \geq 1 \), inequality (15) easily follows from (16) (proven in [9]) we do not know how to conclude it from (16) when \( p^* < 1 \). Also some of the arguments given in [9] are somewhat unclear and hard to follow so we decided that the result needs a complete and a detailed proof. At last, but not least, this result plays a fundamental role in the current paper and proving it here makes the paper more complete and easier to comprehend. To facilitate the formulation of the result we introduce the following piece of
notation. Given constants \( s, b \in (0, \infty) \), \( \sigma \in [1, \infty) \) and a ball \( B_0 \subseteq X \), the measure \( \mu \) is said to satisfy the \( V(\sigma B_0, s, b) \) condition\(^3\) provided
\[
br s \mu(B(x, r)) \leq \mu(B(x, r)) \leq \nu(B(x, r)) \text{ whenever } x \in X \text{ and } r \in (0, \sigma R_0] \quad (14)
\]

**Theorem 4.** Let \( u \in M^{1,p}(\sigma B_0, d, \mu) \) and \( g \in D(u) \), where \( 0 < p < \infty \), \( \sigma > 1 \) and \( B_0 \) is a ball of radius \( R_0 \). Assume that the measure \( \mu \) satisfies the condition \( V(\sigma B_0, s, b) \). Then there exist constants \( C, C_1 \) and \( C_2 \) depending on \( s \), \( p \) and \( \sigma \) only such that

(a) If \( 0 < p < s \), then \( u \in L^{p^*}(B_0) \), where \( p^* = sp/(s-p) \) and the following inequalities are satisfied.
\[
\left( \int_{B_0} |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C \left( \frac{\mu(\sigma B_0)}{bR_0^s} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + C \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p},
\]

\[
\inf_{\gamma \in \mathbb{R}} \left( \int_{B_0} |u - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \leq C \left( \frac{\mu(\sigma B_0)}{bR_0^s} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}.
\]

(b) If \( p = s \) and \( g \in D(u) \setminus \{0\} \), then
\[
\int_{B_0} \exp \left( C_1 b^{1/s} \frac{|u - u_{B_0}|}{\|g\|_{L^s(\sigma B_0)}} \right) \, d\mu \leq C_2.
\]

(c) If \( p > s \), then
\[
\|u - u_{B_0}\|_{L^\infty(B_0)} \leq C \left( \frac{\mu(\sigma B_0)}{bR_0^s} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}.
\]

In particular \( u \) has a Hölder continuous representative on \( B_0 \) and
\[
|u(x) - u(y)| \leq C b^{-1/p} d(x, y)^{1-s/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} \text{ for all } x, y \in B_0.
\]

**Remark 5.** If \( p^* \geq 1 \), then Hölder’s inequality yields
\[
\left( \int_{B_0} |u - u_{B_0}|^{p^*} \, d\mu \right)^{1/p^*} \leq \inf_{\gamma \in \mathbb{R}} \left( \int_{B_0} |u - \gamma|^{p^*} \, d\mu \right)^{1/p^*}
\]
and hence we can replace the expression on the left hand side of (16) with the one on the left hand side of (20). Then, also (15) easily follows from this new version of (20). Although this reasoning fails when \( p^* < 1 \), we prove that (15) is equivalent to (16) in Theorem 13 by showing that both inequalities are equivalent to the lower measure bound in (14).

\(^3\)This condition is a slight variation of the one in [9, p. 197].
Remark 6. If \( p \geq s \), and \( u \in M^{1,p} \), then \( u \in M^{1,q} \) for all \( q < s \) (at least locally). Taking \( q \) sufficiently close to \( s \) we will get \( q^* \geq 1 \) so \( u \) is locally integrable and we can use inequality (20) with \( p^* \) replaced by \( q^* \). This allows to subtract

Proof. Throughout the proof by \( C \) we will denote a generic constant that depends on \( p, s \) and \( \sigma \) only. The dependence of other quantities like \( b, R_0, u \) or \( g \) will be given in an explicit form. By writing \( A \approx B \) we will mean that the quantities \( A \) and \( B \) are non-negative and there is a constant \( C \geq 1 \) (depending on \( p, s \) and \( \sigma \) only) such that \( C^{-1}A \leq B \leq CA \).

Clearly, we can assume that \( 0 < \int_{\sigma B_0} g^p \, d\mu < \infty \). Indeed, if the integral is infinite the result is obvious and if it equals zero, \( u \) is constant and again the result is obvious.

By replacing, if necessary, \( g \) with \( \tilde{g} = g + \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} \), we may further assume that
\[
g(x) \geq 2^{-(1+1/p)} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} > 0 \quad \text{for all } x \in \sigma B_0.
\] (21)

Let \( N \subset X \) be a set of measure zero such that the pointwise inequality (1) holds for all \( x, y \in X \setminus N \).

Define the sets
\[
E_k = \{ x \in \sigma B_0 : g(x) \leq 2^k \} \setminus N, \quad k \in \mathbb{Z}
\]
Clearly, \( E_k \subset E_{k+1} \) and it follows from the pointwise inequality (1) that \( u \) restricted to \( E_k \) is \( 2^{k+1} \)-Lipschitz,
\[
|u(x) - u(y)| \leq 2^{k+1}d(x, y) \quad \text{for all } x, y \in E_k.
\] (22)

Also, the measure of the complement of each of the sets \( E_k \) can be easily estimated from Chebyshev’s inequality
\[
\mu(\sigma B_0 \setminus E_k) = \mu(\{ x \in \sigma B_0 : g(x) > 2^k \}) \leq 2^{-kp} \int_{\sigma B_0} g^p \, d\mu.
\] (23)

Fix \( \gamma \in \mathbb{R} \) arbitrarily. Note that the integrals of \( g^p \) and \( |u - \gamma|^{p^*} \) can be estimated in terms of the sets \( E_k \) as follows.

\[
\int_{\sigma B_0} g^p \, d\mu \approx \sum_{k=-\infty}^{\infty} 2^{kp} \mu(E_k \setminus E_{k-1}).
\] (24)

Let \( a_k = \sup_{E \cap E_k \cap B_0} |u - \gamma| \). Clearly, \( a_k \leq a_{k+1} \) and for any \( 0 < p < s \) we have
\[
\int_{B_0} |u - \gamma|^{p^*} \, d\mu \leq \sum_{k=-\infty}^{\infty} a_k^{p^*} \mu(B_0 \cap (E_k \setminus E_{k-1})).
\] (25)

The idea of the proof in the case \( 0 < p < s \) is to estimate the series at (25) by the series in (24). Similar ideas and also used in other cases \( p = s \) and \( p > s \).

Let \( \tilde{Z} \subset \mathbb{Z} \) be the set of all integers \( k \in \mathbb{Z} \) such that \( \mu(E_k \setminus E_{k-1}) > 0 \). Since the disjoint sets \( \{ E_k \setminus E_{k-1} \}_{k \in \tilde{Z}} \) cover the set \( \sigma B_0 \) up to a set of measure zero, we also have
\( \mu(\sigma B_0 \setminus E) = 0 \), where \( E = \bigcup_{k \in \mathbb{Z}} (E_k \setminus E_{k-1}) \).

Thus for all \( x \in E \), and hence for almost all \( x \in \sigma B_0 \), there is a unique \( k \) such that
\[
x \in E_k \setminus E_{k-1}, \quad \mu(\sigma B_0 \setminus E_{k-1}) > 0.
\]

Let \( k_0 \) be the least integer such that
\[
2^{k_0} \geq \left( \frac{2^{1/s}}{(1 - 2^{-p/s})(\sigma - 1)} \right)^{s/p} \left( \frac{\int_{\sigma B_0} g^p \, d\mu}{b R_0^s} \right)^{-1/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}.
\]  

(26)

Then
\[
2^{k_0} \approx (b R_0^s)^{-1/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}.
\]

(27)

Condition (27) is equivalent to
\[
2^{-k_0 p/s} \frac{2^{1/s} b^{1/s}}{1 - 2^{-p/s}} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} \leq (\sigma - 1) R_0.
\]

(28)

For \( k > k_0 \) we define
\[
r_k = 2^{1/s} b^{1/s} \mu(\sigma B_0 \setminus E_{k-1})^{1/s}.
\]

We claim that
\[
r_k < (\sigma - 1) R_0 \quad \text{for all} \; k > k_0.
\]

(30)

Indeed, \( 2^{-(k-1)p/s} \leq 2^{-k_0 p/s} \) so (30) follows from (23) and (29)
\[
r_k \leq 2^{1/s} b^{1/s} 2^{-(k-1)p/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} < (\sigma - 1) R_0.
\]

(31)

Now we will prove that
\[
\mu(E_{k_0}) \geq \frac{\mu(\sigma B_0)}{2}. \quad \text{(32)}
\]

Suppose to the contrary that \( \mu(E_{k_0}) < \mu(\sigma B_0)/2 \). Then \( \mu(\sigma B_0 \setminus E_{k_0}) > \mu(\sigma B_0)/2 \) so \( r_{k_0+1} > b^{1/s} \mu(\sigma B_0)^{1/s} \).

Let \( z_0 \) be the center of the ball \( B_0 \). Since \( B(z_0, r_{k_0+1}) \subset \sigma B_0 \) by (30), the volume condition \( V(\sigma B_0, s, b) \) yields
\[
\mu(\sigma B_0) \geq \mu(B(z_0, r_{k_0+1})) \geq b r_{k_0+1}^s > \mu(\sigma B_0)
\]
which is a contradiction. This proves (32).

Recall that the set \( E \) has property (26) so if an element of \( E \) belongs to \( E_k \setminus E_{k-1} \), then necessarily \( \mu(E_k \setminus E_{k-1}) > 0 \).

Given \( k > k_0 \) and \( x \in E \cap E_k \cap B_0 \), we want to find a sequence
\[
x_k = x \in E \cap E_k, \; x_{k-1} \in E \cap E_{k-1}, \ldots, x_{k_0} \in E \cap E_{k_0}
\]

such that
\[
x_{k_0} \in E \cap E_{k_0} \quad \text{and} \quad \mu(x_{k_0} \setminus x_{k_0-1}) > 0.
\]

(33)

Indeed, \( x_{k_0} \setminus x_{k_0-1} \subset E_k \setminus E_{k-1} \) and \( \mu(x_{k_0} \setminus x_{k_0-1}) > 0 \).
such that the distances $d(x_i, x_{i-1})$ are relatively small. We will define the sequence by induction.

We set $x_k = x$.

If $x_k \in E_{k-1}$, then we set $x_{k-1} = x_k$ so $d(x_k, x_{k-1}) = 0$.

If $x_k \in E_k \setminus E_{k-1}$, then $0 < r_k < (\sigma - 1)R_0$ by (26) and (30). Since $x_k \in B_0$, it follows that $B(x_k, r_k) \subseteq \sigma B_0$ and hence the volume condition $V(\sigma B_0, s, b)$ yields

$$\mu(B(x_k, r_k)) > \mu(\sigma B_0 \setminus E_{k-1}).$$

Hence the set $B(x_k, r_k) \cap E_{k-1}$ has positive measure so we can find $x_{k-1} \in E \cap E_{k-1}$ such that

$$d(x_k, x_{k-1}) < r_k \leq 2^{1/s} b^{-1/s} 2^{-(k-1)p/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s},$$

where we used (31) in order to estimate $r_k$.

Estimate (33) is also true in the first case when $x_{k-1} = x_k$.

Note that while $x_k \in B_0$, it may happen that $x_{k-1} \notin B_0$, but certainly $x_{k-1} \in \sigma B_0$.

Suppose that we have already selected points

$$x_k \in E \cap E_k \cap B_0, \quad x_{k-1} \in E \cap E_{k-1} \cap \sigma B_0, \ldots, x_{k-i} \in E \cap E_{k-i} \cap \sigma B_0$$

for some $1 \leq i \leq k - k_0 - 1$ in such a way that

$$d(x_{k-j}, x_{k-(j+1)}) \leq 2^{1/s} b^{-1/s} 2^{-(k-(j+1))p/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} \quad \text{for } j = 0, 1, 2, \ldots, i - 1. \quad (34)$$

Note that condition (34) with $i = 1$ is the same as (33) so we already verified the base case of the induction argument. Now we show how to select

$$x_{k-(i+1)} \in E \cap E_{k-(i+1)} \cap \sigma B_0$$

so that the estimate (34) is true for $j = i$.

If $x_{k-i} \in E_{k-(i+1)}$, then we set $x_{k-(i+1)} = x_{k-i}$ and hence (34) is satisfied.

If $x_{k-i} \in E_{k-i} \setminus E_{k-(i+1)}$, then we define

$$r_{k-i} = 2^{1/s} b^{-1/s} \mu(\sigma B_0 \setminus E_{k-(i+1)})^{1/s}$$

so (26) and (23) imply

$$0 < r_{k-i} \leq 2^{1/s} b^{-1/s} 2^{-(k-(i+1))p/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s}.$$
The triangle inequality yields that for any \( y \in B(x_{k-i}, r_{k-i}) \) we have
\[
d(x_k, y) \leq d(x_k, x_{k-1}) + \ldots + d(x_{k-(i-1)}, x_{k-i}) + r_{k-i}
\]
\[
\leq 2^{1/s}b^{-1/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} \sum_{j=0}^{i} 2^{-(k-(j+1))p/s}
\]
\[
< 2^{-kp/s} 2^{1/s}b^{-1/s} \frac{1}{1 - 2^{-p/s}} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} \leq (\sigma - 1)R_0.
\]

Since \( x_k \in B_0 \) it follows that \( B(x_{k-i}, r_{k-i}) \subset \sigma B_0 \). Again, the volume condition \( V(\sigma B_0, s, p) \) yields
\[
\mu(B(x_{k-i}, r_{k-i})) > \mu(\sigma B_0 \setminus E_{k-(i+1)})
\]
so the set \( B(x_{k-i}, r_{k-i}) \cap E_{k-(i+1)} \) has positive measure and hence we may find \( x_{k-(i+1)} \in E \cap E_{k-(i+1)} \) satisfying
\[
d(x_{k-i}, x_{k-(i+1)}) < r_{k-i} \leq 2^{1/s}b^{-1/s} 2^{-(k-(i+1))p/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s}
\]
which satisfies (34) for \( j = i \).

Recall that \( a_k = \sup_{E \cap E_k \cap B_0} |u - \gamma| \). For \( k > k_0 \) and \( x \in E \cap E_k \cap B_0 \) we choose a sequence of points \( x_k = x, x_{k-1}, \ldots, x_{k_0} \) as above. From (22) we have
\[
|u(x) - \gamma| \leq \left( \sum_{i=0}^{k-k_0-1} |u(x_{k-i}) - u(x_{k-(i+1)})| \right) + |u(x_{k_0}) - \gamma|
\]
\[
\leq \left( \sum_{i=0}^{k-k_0-1} 2^{k-i}d(x_{k-i}, x_{k-(i+1)}) \right) + |u(x_{k_0}) - \gamma|.
\]
Taking the supremum over all \( x \in E_k \cap B_0 \) and using (35) we obtain
\[
a_k \leq 4 \cdot 2^{1/s}b^{-1/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} \sum_{j=k_0}^{k-1} 2^{(1-p/s)} + \sup_{E_{k_0} \cap \sigma B_0} |u - \gamma| \quad \text{for all } k \in \mathbb{Z}.
\]
(36)

Indeed, the above estimates establish (36) for \( k > k_0 \), but \( a_k \leq \sup_{E_{k_0} \cap \sigma B_0} |u - \gamma| \) for \( k \leq k_0 \). Note that the last supremum in the above inequality might be larger than \( a_{k_0} \) since we are taking now the supremum over the set \( E_{k_0} \cap \sigma B_0 \) which is larger than the set used in the definition of \( a_{k_0} \).

Since \( \mu(E_{k_0}) > 0 \), we can take \( y \in E_{k_0} \cap E \). If \( \gamma = u(y) \), then the Lipschitz continuity (22) and (28) yield
\[
\sup_{E_{k_0} \cap \sigma B_0} |u - \gamma| \leq 2^{k_0+1} \text{diam}(\sigma B_0) \leq 2^{k_0+2} \sigma R_0 \leq CR_0(bR_0^s)^{-1/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}.
\]
(37)
Proof of (a). First we will prove inequality (16). Let $\gamma = u(y)$ as in (37) and let $a_k = \sup_{E \cap E_0 \cap B_0} |u - \gamma|$. Since $2^{1-p/s} > 1$, we can estimate the finite sum at (36) by the convergent geometric series $\sum_{j=-\infty}^{k-1} s_j$ so (36) gives

$$a_k \leq C b^{-1/s} \left( \int_{E_0} g^p \, d\mu \right)^{1/s} 2^{k(1-p/s)} + \sup_{E_0 \cap E_0 \cap B_0} |u - \gamma| \quad \text{for all } k \in \mathbb{Z}$$

and hence (25) and (24) and then (37) yield

$$\int_{B_0} |u|^p \, d\mu \leq C b^{-p/s} \left( \int_{E_0} g^p \, d\mu \right)^{p^*/p} \sum_{k=-\infty}^{\infty} 2^{kp} \mu(E_k \setminus E_{k-1}) + C \left( \sup_{E_0 \cap E_0 \cap B_0} |u - \gamma| \right)^{p^*} \mu(B_0)$$

$$\leq C b^{-p/s} \left( \int_{E_0} g^p \, d\mu \right)^{p^*/p} + C \left( \sup_{E_0 \cap E_0 \cap B_0} |u - \gamma| \right)^{p^*} \mu(B_0)$$

$$\leq C b^{-p/s} \left( 1 + \frac{\mu(B_0)}{bR_0^s} \right) \left( \int_{E_0} g^p \, d\mu \right)^{p^*/p} \leq C b^{-p/s} \frac{\mu(B_0)}{bR_0^s} \left( \int_{E_0} g^p \, d\mu \right)^{p^*/p}.$$

In the last inequality we used the condition $V(\sigma B_0, s, b)$ to estimate $1 + \mu(B_0)/(bR_0^s) \leq 2 \mu(B_0)/(bR_0^s)$. The above estimate easily imply inequality (16).

Now it remains to prove inequality (15). Take $\gamma = 0$. Let $b_{k_0} = \inf_{E_0 \cap \sigma B_0} |u|$. Since

$$b_{k_0} \chi_{E_{k_0}} \leq |u|^p \chi_{\sigma B_0},$$

inequality (32) yields

$$\frac{\mu(\sigma B_0)}{2} b_{k_0}^p \leq b_{k_0}^p \mu(E_{k_0}) \leq \int_{\sigma B_0} |u|^p$$

so

$$b_{k_0} \leq 2^{1/p} \left( \int_{\sigma B_0} |u|^p \right)^{1/p}.$$

The Lipschitz continuity (22) and (28) yield

$$\sup_{E_0 \cap \sigma B_0} |u| \leq 2^{k_0+1} \text{diam}(\sigma B_0) + b_{k_0} \leq 2^{k_0+2} \sigma R_0 + 2^{1/p} \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p}$$

$$\leq C \left( R_0 (bR_0^s)^{-1/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p} \right)$$

(38)
Hence a similar calculation as above gives

\[
\int_{B_0} |u|^{p^*/s} \, d\mu \leq C b^{-p^*/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{p^*/p} \sum_{k=-\infty}^{\infty} 2^{kp} \mu(E_k \setminus E_{k-1}) + C \left( \sup_{E_{k_0} \subset \sigma B_0} |u| \right)^{p^*} \mu(B_0)
\]

\[
\leq C b^{-p^*/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{p^*/p} + C \left( \sup_{E_{k_0} \subset \sigma B_0} |u| \right)^{p^*} \mu(B_0)
\]

\[
\leq C b^{-p^*/s} \left( 1 + \frac{\mu(B_0)}{bR_0^p} \right) \left( \int_{\sigma B_0} g^p \, d\mu \right)^{p^*/p} + C \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{p^*/p} \mu(B_0)
\]

\[
\leq C b^{-p^*/s} \frac{\mu(B_0)}{bR_0^p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{p^*/p} + C \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{p^*/p} \mu(B_0).
\]

This estimate easily imply inequality (15).

**Proof of (b).** Let \( \gamma = u(y) \) as in (37). For \( a > 0 \), Jensen’s inequality and convexity of \( e^t \) yield

\[
\int_{B_0} e^{a|u(x) - u_{B_0}|} \, d\mu(x) \leq \int_{B_0} \exp \left( \int_{B_0} a|u(x) - u(y)| \, d\mu(y) \right) \, d\mu(x)
\]

\[
\leq \int_{B_0} \int_{B_0} e^{a|u(x) - u(y)|} \, d\mu(y) \, d\mu(x) \leq \int_{B_0} e^{a|u(x) - \gamma|} \, d\mu(x) \int_{B_0} e^{a|u(y) - \gamma|} \, d\mu(y)
\]

\[
= \left( \int_{B_0} e^{a|u(x) - \gamma|} \, d\mu(x) \right)^2.
\]

Hence

\[
\int_{B_0} \exp \left( C_1 b^{1/s} \frac{|u - u_{B_0}|}{\|g\|_{L^s(\sigma B_0)}} \right) \, d\mu \leq \left( \int_{B_0} \exp \left( C_1 b^{1/s} \frac{|u - \gamma|}{\|g\|_{L^s(\sigma B_0)}} \right) \, d\mu \right)^2
\]

and thus it suffices to estimate the right hand side of (39).

Since \( s = p \), inequality (37) reads as

\[
\sup_{E_{k_0} \subset \sigma B_0} |u - \gamma| \leq C b^{-1/s} \left( \int_{\sigma B_0} g^s \, d\mu \right)^{1/s}.
\]

(40)

Since \( 2^{j(1-s/p)} = 1 \), (36) and (40) yield

\[
a_k \leq C b^{-1/s} \left( \int_{\sigma B_0} g^s \, d\mu \right)^{1/s} (k - k_0) \text{ for } k > k_0.
\]

(41)
Note that
\[
\frac{C_1 b^{1/s} |u(x) - \gamma|}{\|g\|_{L^s(\sigma B_0)}} \leq C C_1 \quad \text{for } x \in E_{k_0}
\] (42)
and
\[
\frac{C_1 b^{1/s} |u(x) - \gamma|}{\|g\|_{L^s(\sigma B_0)}} \leq \tilde{C} C_1 (k - k_0) \quad \text{for } k > k_0, \ x \in E_k.
\] (43)

Take a constant \( C_1 \) in such a way that \( \exp(\tilde{C} C_1) = 2^s \).

Let us split the integral that we need to estimate into two integrals
\[
\int_{B_0} \exp \left( \frac{C_1 b^{1/s} |u - \gamma|}{\|g\|_{L^s(\sigma B_0)}} \right) d\mu = \frac{1}{\mu(B_0)} \int_{B_0 \cap E_{k_0}} + \frac{1}{\mu(B_0)} \int_{B_0 \setminus E_{k_0}} = I_1 + I_2.
\]

Estimate (42) gives
\[
I_1 \leq \frac{\mu(B_0 \cap E_{k_0})}{\mu(B_0)} \exp(CC_1) \leq \exp(CC_1)
\]
while estimate (43) and the fact that \( \exp(\tilde{C} C_1) = 2^s \) yield
\[
I_2 \leq \frac{1}{\mu(B_0)} \sum_{k=0}^{\infty} \exp(\tilde{C} C_1 (k - k_0)) \mu(B_0 \cap (E_k \setminus E_{k-1})) \leq \frac{2^{-s k_0}}{\mu(B_0)} \sum_{k=-\infty}^{\infty} 2^{s k} \mu(E_k \setminus E_{k-1}) \leq C \frac{2^{-s k_0}}{\mu(B_0)} \int_{\sigma B_0} g^s \, d\mu \leq C \frac{b \rho_s^{s}}{\mu(B_0)} \leq C
\]
where the last two estimates follow from (28) and the volume condition \( V(\sigma B_0, s, b) \) respectively. The proof in the case \( p = s \) is complete.

**Proof of (c).** Let \( \gamma = u(y) \) be as in (37) and \( a_k = \sup_{E_k \cap E_{k_0} \cap B_0} |u - \gamma| \). Since \( 2^{1-p/s} < 1 \), we can estimate the finite sum at (36) for \( k \geq k_0 \), by the convergent geometric series \( \sum_{j=k_0}^{\infty} = C 2^{k_0 (1-s/p)} \) so
\[
a_k \leq C b^{-1/s} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/s} 2^{k_0 (1-p/s)} + \sup_{E_{k_0} \cap \sigma B_0} |u - \gamma| \quad \text{for } k \geq k_0.
\] (44)

Then (44), (28) and (37) yield
\[
a_k \leq C (b \rho_0^{s})^{-1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} = C \left( \frac{\mu(\sigma B_0)}{b \rho_0^{s}} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} \quad \text{for } k \geq k_0.
\]

Since the right hand side is a constant that does not depend on \( k \), we conclude that \( |u - \gamma| \) is bounded on \( B_0 \). More precisely, \( |u - \gamma| \) equals almost everywhere to a function that is bounded by \( B_0 \) and (18) follows from the estimate
\[
\|u - u_{B_0}\|_{L^\infty(B_0)} \leq 2 \|u - \gamma\|_{L^\infty(B_0)}.
\]
It remains to prove Hölder continuity of \( u \) along with the estimate (19).
If \( x, y \in B_0 \) and \( R_1 := 2d(x, y) \leq (\sigma - 1)R_0/\sigma \), then \( x, y \in B_1 := B(x, R_1) \), \( \sigma B_1 \subset \sigma B_0 \) so estimate (18) applied to \( B_1 \) in place of \( B_0 \) yields

\[
|u(x) - u(y)| \leq 2\|u - u_{B_1}\|_{L^\infty(B_1)} \leq C \left( \frac{\mu(\sigma B_1)}{bR_1^p} \right)^{1/p} R_1 \left( \int_{\sigma B_1} g^p \, d\mu \right)^{1/p} = Cb^{-1/p}d(x, y)^{1-s/p} \left( \int_{\sigma B_1} g^p \, d\mu \right)^{1/p}
\]

If \( 2d(x, y) > (\sigma - 1)R_0/\sigma \), then (18) gives

\[
|u(x) - u(y)| \leq 2\|u - u_{B_0}\|_{L^\infty(B_0)} \leq C \left( \frac{\mu(\sigma B_0)}{bR_0^p} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} \leq Cb^{-1/p}d(x, y)^{1-s/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}
\]

The proof is complete. \( \square \)

3. Auxiliary Results

An open set \( \Omega \subset \mathbb{R}^n \) is a metric-measure space with the Euclidean metric and Lebesgue measure. If \( x \in \Omega \) and \( r \in (0, \infty) \), then we can always find a radius \( r_x < r \) such that \( |B(x, r_x) \cap \Omega| = \frac{1}{2}|B(x, r) \cap \Omega| \). However, in general metric-measure space \( (X, d, \mu) \) it is not always possible to find a concentric ball with half of the measure of the original ball, but for \( x \in X \) and \( r \in [0, \infty) \) we still can define

\[
\varphi_x(r) = \sup \left\{ s \in [0, r] : \mu(B(x, s)) \leq \frac{1}{2} \mu(B(x, r)) \right\}.
\]

(45)

The basic properties of \( \varphi_x(r) \) are listed in the next lemma. The reader is reminded that \( \overline{B}(x, r) := \{ y \in X : d(x, y) \leq r \}, x \in X, r \in [0, \infty) \).

Lemma 7. Suppose that \( (X, d, \mu) \) is a metric-measure space and fix \( x \in X, r \in [0, \infty) \). Then the following statements hold.

1. \( \varphi_x(\cdot) \) is nondecreasing, i.e., \( \varphi_x(s) \leq \varphi_x(t) \) whenever \( 0 \leq s \leq t < \infty \).

2. One has that

\[
\mu(B(x, \varphi_x(r))) \leq \frac{1}{2} \mu(B(x, r)) \leq \mu(\overline{B}(x, \varphi_x(r))),
\]

(46)

3. \( \varphi_x(r) \in [0, r] \) where \( \varphi_x(r) = r \) if and only \( r = 0 \).

4. If \( \mu(\{x\}) = 0 \) and \( r > 0 \), then \( \varphi_x^j(r) > 0 \) for every \( j \in \mathbb{N}_0 \), where

\[
\varphi_x^0(r) := r \quad \text{and} \quad \varphi_x^j(r) := \varphi_x(\varphi_x^{j-1}(r)), \ j \in \mathbb{N}.
\]
Moreover, the sequence \( \{ \varphi_{x}^{j}(r) \}_{j \in \mathbb{N}_0} \) is strictly decreasing, i.e.,
\[
r > \varphi_{x}(r) > \varphi_{x}^{2}(r) > \cdots > \varphi_{x}^{j}(r) > \varphi_{x}^{j+1}(r) > \cdots > 0,
\]
and \( \mu(B(x, \varphi_{x}^{j}(r))) \leq 2^{-j} \mu(B(x, r)) \). Consequently, \( \lim_{j \to \infty} \varphi_{x}^{j}(r) = 0 \).

Proof. Given that (1) follows immediately from the definition of \( \varphi_{x} \), we begin by establishing the second inequality in (46). Observe
\[
\frac{1}{2} \mu(B(x, r)) < \mu(B(x, s)), \quad \forall s \in (\varphi_{x}(r), \infty).
\]
As such, passing to the limit in (48) as \( s \in (\varphi_{x}(r), \infty) \) tends to \( \varphi_{x}(r) \) yields the second inequality in (46).

Regarding the first inequality in (46), we use the definition of \( \varphi_{x}(r) \) to obtain a nondecreasing sequence \( \{r_{j}\}_{j \in \mathbb{N}} \) of points in \([0, r]\) which satisfy
\[
\lim_{j \to \infty} r_{j} = \varphi_{x}(r) \quad \text{and} \quad \mu(B(x, r_{j})) \leq \frac{1}{2} \mu(B(x, r)), \quad \forall j \in \mathbb{N}.
\]
Then
\[
\mu(B(x, \varphi_{x}(r))) = \mu\left( \bigcup_{j \in \mathbb{N}} B(x, r_{j}) \right) = \lim_{j \to \infty} \mu(B(x, r_{j})) \leq \frac{1}{2} \mu(B(x, r)),
\]
which gives the first inequality in (46). This completes the proof of (2).

Regarding the claim in (3), observe first that the definition of \( \varphi_{x}(r) \) immediately gives \( \varphi_{x}(r) \in [0, r] \). Hence, if \( r = 0 \) then \( \varphi_{x}(r) = 0 = r \). On the other hand, if \( \varphi_{x}(r) = r \) and \( r > 0 \) then the first inequality in (46) gives
\[
\mu(B(x, r)) = \mu(B(x, \varphi_{x}(r))) \leq \frac{1}{2} \mu(B(x, r)).
\]
Combining this with the fact that open balls (with positive radius) have strictly positive \( \mu \)-measure yields a contradiction which completes the proof of (3).

As concerns (4), it is clear that \( r > \varphi_{x}(r) > 0 \) given (3) and (46). Then (47) can now be justified using an inductive argument. Finally, repeatedly calling upon (46) we have
\[
\mu(B(x, \varphi_{x}^{j}(r))) \leq 2^{-j} \mu(B(x, r)),
\]
from which it follows that \( \lim_{j \to \infty} \varphi_{x}^{j}(r) = 0 \). This finishes the proof of the lemma.

In what follows, we will need the technical lemma presented below. Recall that a metric space \((X, d)\) is said to be \textbf{uniformly perfect} if there exists a constant \( \lambda \in (0, 1) \) with the property that for each \( x \in X \) and each \( r \in (0, \infty) \) one has
\[
B(x, r) \setminus B(x, \lambda r) \neq \emptyset \quad \text{whenever} \quad X \setminus B(x, r) \neq \emptyset.
\]
Note that every connected space is uniformly perfect; however, the class of uniformly perfect spaces contain very disconnected sets such as the Cantor set. Moreover, observe that if (50) holds for some \( \lambda \in (0, 1) \) then it holds for every \( \lambda' \in (0, \lambda] \). With this definition in mind, we now present the aforementioned lemma.
Lemma 8. Suppose that \((X, d, \mu)\) is a uniformly perfect metric measure space and let \(\lambda \in (0, 1)\) be as in (50). Then the following statements are valid.

1. For each \(x \in X\) and each finite \(r \in (0, \text{diam}(X)]\), one can find a point \(y \in B(x, r)\) so that \(\varphi_y(r/2) > 0\) and \(B(y, r/2) \subseteq B(x, r)\).

2. If \(\lambda < 1/5\) and if there exists a finite constant \(C > 0\) such that \(C \varphi_x^s \leq \mu(B(x, r))\) for every \(x \in X\) and every \(r \in (0, \min\{3 \varphi_x(r)/\lambda^2, \text{diam}(X)\}]\), then there exists a finite constant \(\tilde{C} > 0\) (depending only on \(C, s,\) and \(\lambda\)) such that \(C \varphi_x^s \leq \mu(B(x, r))\) for every \(x \in X\) and every finite \(r \in (0, \text{diam}(X)]\).

Remark 9. The constant \(\tilde{C}\) appearing in part (2) of Lemma 8 may be chosen to be

\[
\tilde{C} := C \lambda^{2s}/2,
\]

where \(C\) is the same constant in the statement of part (2) in Lemma 8.

Proof. We begin by proving (1). Fix a point \(x \in X\) and a finite radius \(r \in (0, \text{diam}(X)]\), and note that \(X \setminus B(x, r/4) \neq \emptyset\). Since \(X\) is uniformly perfect we can select a sequence \(\{x_j\}_{j \in \mathbb{N}}\) of distinct points such that

\[
x_j \in B(x, \lambda^{-1} r/4) \setminus B(x, \lambda^j r/4) \subseteq B(x, r/4), \quad \forall j \in \mathbb{N}.
\]

For such a choice of points, we necessarily have

\[
\infty > \mu(B(x, r/4)) \geq \mu\left(\bigcup_{j \in \mathbb{N}} \{x_j\}\right) = \sum_{j \in \mathbb{N}} \mu(\{x_j\}) \geq 0,
\]

from which it follows that \(\mu(\{x_j\}) \to 0\) as \(j \to \infty\). Next, choose \(j_0 \in \mathbb{N}\) large enough so that \(\mu(\{x_{j_0}\}) < \frac{1}{2} \mu(B(x, r/4))\) and set \(y := x_{j_0}\). Clearly, \(B(x, r/4) \subseteq B(y, r/2) \subseteq B(x, r)\). Moreover,

\[
\mu(\{y\}) < \frac{1}{2} \mu(B(x, r/4)) \leq \frac{1}{2} \mu(B(y, r/2)),
\]

which further implies \(\varphi_y(r/2) > 0\). This finishes the proof of (1).

In order to prove (2), fix a point \(x \in X\) and a finite radius \(r \in (0, \text{diam}(X)]\). From what has been proven in part (1) of this lemma, we can assume that \(\varphi_x(r) > 0\). Otherwise we would consider a smaller ball \(B(y, r/2) \subseteq B(x, r)\) with \(\varphi_y(r/2) > 0\). If \(r \leq 3 \varphi_x(r)/\lambda^2\) then we are done by assumption. Thus, in what follows we will assume that \(r > 3 \varphi_x(r)/\lambda^2\) and \(\varphi_x(r) > 0\). Moving forward with this in mind, since \((X, d)\) is uniformly perfect and \(X \setminus B(x, \varphi_x(r)/\lambda + 2\lambda r) \neq \emptyset\) (given that \(\lambda < 1/5\)), we may choose a point

\[
x' \in B(x, \varphi_x(r)/\lambda + 2\lambda r) \setminus B(x, \varphi_x(r) + 2\lambda^2 r).
\]

With

\[
R := 2 \varphi_x(r)/\lambda + 2\lambda r,
\]

we claim that

\[
\overline{B}(x, \varphi_x(r)) \subseteq B(x', R) \subseteq B(x, r) \quad \text{and} \quad B(x', 2^{-1} \lambda R) \subseteq B(x, r) \setminus \overline{B}(x, \varphi_x(r)).
\]
For the inclusion $\overline{B}(x, \varphi_x(r)) \subseteq B(x', R)$, observe that if $z \in \overline{B}(x, \varphi_x(r))$ then
\[
d(z, x') \leq d(z, x) + d(x, x') < \varphi_x(r) + [\varphi_x(r)/\lambda + 2\lambda r] < R,
\]
given that $1/\lambda > 1$. To prove $B(x', R) \subseteq B(x, r)$, observe that for $z \in B(x', R)$, we have
\[
d(z, x) \leq d(z, x') + d(x', x)
< [2\varphi_x(r)/\lambda + 2\lambda r] + [\varphi_x(r)/\lambda + 2\lambda r]
= 3\varphi_x(r)/\lambda + 4\lambda r < 5\lambda r < r.
\]

(54)

Note that, in obtaining the estimate in (54), we have used the fact that we are currently assuming $3\varphi_x(r)/\lambda^2 < r$ and $\lambda < 1/5$. To finish the proof of (53) we need to show that $B(x', 2^{-1}\lambda R) \subseteq B(x, r \setminus \overline{B}(x, \varphi_x(r)))$. Since $2^{-1}\lambda < 1$ the inclusion $B(x', 2^{-1}\lambda R) \subseteq B(x, r)$ follows from the first line in (53). To show that $B(x', 2^{-1}\lambda R) \subseteq X \setminus \overline{B}(x, \varphi_x(r))$, fix $z \in B(x', 2^{-1}\lambda R)$ and write
\[
\varphi_x(r) + 2\lambda^2 r \leq d(x, x') \leq d(x, z) + d(z, x') \leq d(x, z) + 2^{-1}\lambda R.
\]

(55)

Thus, looking at the extreme most sides of the inequality in (55) we have (keeping in mind $3\varphi_x(r)/\lambda^2 < r$
\[
d(z, x) \geq \varphi_x(r) + 2\lambda^2 r - 2^{-1}\lambda R = \lambda^2 r > 3\varphi_x(r),
\]
which in turn implies the desired inclusion. This finishes the proof of (53).

It follows from (53) that $\overline{B}(x, \varphi_x(r))$ and $B(x', 2^{-1}\lambda R)$ are disjoint subsets of $B(x', R)$ and
\[
\mu(B(x', 2^{-1}\lambda R)) = \frac{1}{2} \left[ \mu(B(x', 2^{-1}\lambda R)) + \mu(B(x', 2^{-1}\lambda R)) \right]
\leq \frac{1}{2} \left[ \mu(B(x, r) \setminus \overline{B}(x, \varphi_x(r))) + \mu(B(x', 2^{-1}\lambda R)) \right]
\leq \frac{1}{2} \left[ \mu(\overline{B}(x, \varphi_x(r))) + \mu(B(x', 2^{-1}\lambda R)) \right] \leq \frac{1}{2} \mu(B(x', R)),
\]

(56)

where, in obtaining the second inequality in (56), we have used Lemma 7. As such, we have that $2^{-1}\lambda R \leq \varphi_x'(R) \leq 3\varphi_x'(R)/\lambda^2$ which, under the current assumption, further implies that the lower measure condition is satisfied for the ball $B(x', 2^{-1}\lambda R)$, i.e.,
\[
\mu(B(x', 2^{-1}\lambda R)) \geq C(2^{-1}\lambda R)^s.
\]
Consequently, combining this with (52) gives
\[
\mu(B(x, r)) \geq \mu(B(x', 2^{-1}\lambda R)) \geq C(2^{-1}\lambda R)^s \geq C\lambda^{2s} r^s,
\]
finishing the proof of the lemma.

In the sequel, we will also need the following well-known result.

**Lemma 10.** Given $x \in X$ and $0 < r < R < \infty$, there exists a $(R-r)^{-1}$-Lipschitz function $\varphi_{r,R}: X \to [0, 1]$ such that $\varphi_{r,R} \equiv 1$ on $B(x, r)$ and $\varphi_{r,R} \equiv 0$ on $X \setminus B(x, R)$. Consequently, one has $(R-r)^{-1} \chi_{B(x, R)} \in D(\varphi_{r,R})$, where $\chi_{B(x, R)}$ denotes the characteristic function of the ball $B(x, R)$.
Proof. Fix a point $x \in X$, numbers $0 < r < R < \infty$, and define $\varphi_{r,R} : X \to [0, 1]$ by setting for each $y \in X$,

$$\varphi_{r,R}(y) := \begin{cases} 
1 & \text{if } y \in B(x, r), \\
\frac{R - d(x, y)}{R - r} & \text{if } y \in B(x, r) \setminus B(x, R), \\
0 & \text{if } y \in X \setminus B(x, R).
\end{cases} \tag{57}$$

Then the claims follow from straightforward computations. \hfill \square

4. THE CASE $p < s$

Theorem 11. Fix $\sigma \in (1, \infty)$, $s \in (0, \infty)$, $p \in (0, s)$, and let $p^* = sp/(s - p)$. Then the following statements are equivalent.

(1) There exists a constant $\kappa \in (0, \infty)$ such that

$$\kappa r^s \leq \mu(B(x, r)) \text{ for every } x \in X$$

and every finite $r \in (0, \operatorname{diam}(X)]$. \tag{58}

(2) There exists a constant $C_S \in (0, \infty)$ such that for every ball $B_0 := B(x_0, R_0)$ with $x_0 \in X$ and finite $R_0 \in (0, \operatorname{diam}(X)]$, one has

$$\left(\frac{\int_{B_0} |u|^{p^*} \, d\mu}{\mu(S)}\right)^{1/p^*} \leq C_S \left(\frac{\mu(S)}{R_0^{n/p}}\right)^{1/p} \left[R_0 \left(\frac{\int_{S} g^p \, d\mu}{\mu(S)}\right)^{1/p} \right] + \left(\frac{\int_{B_0} |u|^p \, d\mu}{\mu(S)}\right)^{1/p}, \tag{59}
$$

whenever $u \in M^{1,p}(\sigma B_0, d, \mu)$ and $g \in D(u)$.

If, in addition, $(X, d)$ is assumed to be uniformly perfect (cf. (50)) then (1) (hence, also (2)) is further equivalent to:

(3) There exists a constant $C_P \in (0, \infty)$ such that for every ball $B_0 := B(x_0, R_0)$ with $x_0 \in X$ and finite $R_0 \in (0, \operatorname{diam}(X)]$, one has

$$\inf_{\gamma \in \mathbb{R}} \left(\frac{\int_{B_0} |u - \gamma|^{p^*} \, d\mu}{\mu(S)}\right)^{1/p^*} \leq C_P \left(\frac{\mu(S)}{R_0^{n/p}}\right)^{1/p} \left[R_0 \left(\frac{\int_{S} g^p \, d\mu}{\mu(S)}\right)^{1/p} \right], \tag{60}
$$

whenever $u \in M^{1,p}(\sigma B_0, d, \mu)$ and $g \in D(u)$.

Remark 12. As the proof of the implication (2) $\implies$ (1) in Theorem 11 will reveal, one can take the constant $\kappa$ in (87) to be $\left[\kappa_p 2^{3p+s}\right]^{-1} \in (0, \infty)$.

Proof. We will first show that (1) implies both (2) and (3) (without the additional assumption that $X$ is uniformly perfect. Fix a ball $B_0$ having finite radius $R_0 \in (0, \operatorname{diam}(X)]$. Then the inequality displayed in (58) implies that the measure $\mu$ satisfies the $V(\sigma B_0, s, c')$
condition (see (14)) with $c' := \kappa \sigma^{-s} \in (0, \infty)$. As such, it follows from (15) in Theorem 4 that

$$
\left( \int_{B_0} |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C \left( \frac{\mu(\sigma B_0)}{c' R_0^s} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + C \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p}
$$

$$
= C \left( \frac{\mu(\sigma B_0)}{c' R_0^s} \right)^{1/p} R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + C \left( \frac{\mu(\sigma B_0)}{\mu(\sigma B_0)} \right)^{1/p} \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p}
$$

$$
\leq C(c')^{-1/p} \left( \frac{\mu(\sigma B_0)}{R_0^s} \right)^{1/p} \left[ R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p} \right]. \tag{61}
$$

Note that, in obtaining the last inequality in (61), we have used the $V(\sigma B_0, s, c')$ measure condition and the fact that $\sigma^{-s} \leq 1$. Hence, (2) is valid. Given that (60) is an immediate consequence of (16), this finishes the proof of the fact that (1) implies both (2) and (3). Note we do not require the uniformly perfect property to establish these implications.

We now focus on proving that (2) implies (1). To this end, fix a ball $B := B(x, r)$ with $x \in X$ and $r \in (0, \text{diam}(X)]$, finite. Specializing (59) to the case when $B_0 := B$ (and simplifying the expression) implies that

$$
\left( \int_{B} |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C s r \left( \frac{1}{r^s} \int_{\sigma B} g^p \, d\mu \right)^{1/p} + C_s \left( \frac{1}{r^s} \int_{\sigma B} |u|^p \, d\mu \right)^{1/p}, \tag{62}
$$

whenever $u \in M^{1,p}(\sigma B)$ and $g \in D(u)$. We define a collection of functions $\{u_j\}_{j \in \mathbb{N}}$ as follows: for each $j \in \mathbb{N}$, let $r_j := (2^{-j} + 2^{-1})r$ and set $B^j := B(x, r_j)$. Then

$$
\frac{1}{2} < r_{j+1} < r_j \leq \frac{3}{4} r, \quad \forall j \in \mathbb{N}. \tag{63}
$$

Then for each $j \in \mathbb{N}$, define $u_j : X \to [0, 1]$ by setting

$$
u_j(y) := \varphi_{r_{j+1}, r_j}(y) \quad \text{for every } y \in X,
$$

where the function $\varphi_{r_{j+1}, r_j}$ is as in Lemma 10. Noting that $(r_j - r_{j+1})^{-1} = 2^{j+2}r^{-1}$, we have that $u_j$ is $2^{j+2}r^{-1}$-Lipschitz on $X$ supported in $B^j$, and that the function $g_j := 2^{j+2}r^{-1}\chi_{B^j}$ is an upper gradient of $u_j$. In particular, we have that $u_j \in M^{1,p}(\sigma B)$. As such, the functions $u_j$ and $g_j$ satisfy (62). Observe that for each fixed $j \in \mathbb{N}$, we have (keeping in mind $\sigma > 1$)

$$
C_s r \left( \frac{1}{r^s} \int_{\sigma B} g_j^p \, d\mu \right)^{1/p} = C_s 2^{j+2} \frac{1}{r^{s/p}} \mu(B^j)^{1/p}, \tag{64}
$$

and

$$
C_s \left( \frac{1}{r^s} \int_{\sigma B} |u_j|^p \, d\mu \right)^{1/p} \leq C_s \frac{1}{r^{s/p}} \mu(B^j)^{1/p}. \tag{65}
$$
Moreover, since \( u_j \equiv 1 \) on \( B^{j+1} \) we may estimate

\[
\left( \int_B |u_j|^{p^*} \, d\mu \right)^{1/p^*} \geq \left( \frac{\mu(B^{j+1})}{\mu(B)} \right)^{1/p^*}. \tag{66}
\]

In concert, (64)-(66) and (62), give

\[
\left( \frac{\mu(B^{j+1})}{\mu(B)} \right)^{1/p^*} \leq C_s \left( \frac{2j+2}{r^{s/p}} + \frac{1}{r^{s/p}} \right) \mu(B^{j+1})^{1/p} \\
\leq \frac{C_s 2j+3}{r^{s/p}} \mu(B^j)^{1/p}, \quad \forall j \in \mathbb{N}. \tag{67}
\]

Therefore,

\[
\mu(B^{j+1})^{1/p^*} \leq \frac{C_s 2j+3}{r^{s/p}} \mu(B)^{1/p^*} \mu(B^j)^{1/p}, \quad \forall j \in \mathbb{N}. \tag{68}
\]

With \( \alpha := p^*/p \in (1, \infty) \) we raise both sides of the inequality in (68) to the power \( p/\alpha^{j-1} \) in order to obtain

\[
\mu(B^{j+1})^{1/\alpha^j} \leq 2^{p(j+3)/\alpha^{j-1}} \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p/\alpha^{j-1}} \mu(B^j)^{1/\alpha^{j-1}}, \quad \forall j \in \mathbb{N}. \tag{69}
\]

If we let \( P_j := \mu(B^{j+1})^{1/\alpha^{j-1}} \), then the inequality in (69) becomes

\[
P_{j+1} \leq 2^{p(j+3)/\alpha^{j-1}} \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p/\alpha^{j-1}} P_j, \quad \forall j \in \mathbb{N}, \tag{70}
\]

which, together with an inductive argument and the fact that \( P_1 \leq \mu(B) \), implies

\[
P_{j+1} \leq P_1 \prod_{k=1}^j \left[ 2^{p(k+3)/\alpha^{k-1}} \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p/\alpha^{k-1}} \right] \\
\leq \mu(B) \prod_{k=1}^j \left[ 2^{p(k+3)/\alpha^{k-1}} \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p/\alpha^{k-1}} \right], \quad \forall j \in \mathbb{N}. \tag{71}
\]

We claim that the product in (71) converges as \( j \to \infty \). Indeed, observe that

\[
\prod_{k=1}^\infty \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p/\alpha^{k-1}} = \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p \sum_{k=1}^\infty \alpha^{1-k}} = \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{p \alpha/(\alpha-1)} = \left( \frac{C_s \mu(B)^{1/p^*}}{r^{s/p}} \right)^{s}, \tag{72}
\]

and

\[
\prod_{k=1}^\infty \left( 2^{p(k+3)} \right)^{1/\alpha^{k-1}} = 2^{\sum_{k=1}^\infty p(k+3) \alpha^{1-k}} =: A(p, s) \in (0, \infty). \tag{73}
\]

On the other hand, it follows from (63) that

\[
0 < \mu(2^{-1}B)^{1/\alpha^{j-1}} \leq P_j = \mu(B^{j+1})^{1/\alpha^{j-1}} \leq \mu(B)^{1/\alpha^{j-1}} < \infty, \tag{74}
\]
which, in turn, further implies \( \lim_{j \to \infty} P_j = 1 \). Consequently, passing to the limit in (71) yields
\[
1 \leq \mu(B) \left( \frac{C_S \mu(B)^{1/p^*}}{r^{s/p}} \right)^s A(p, s) = C_S^s A(p, s) \left( \frac{\mu(B)}{r^s} \right)^{s/p}.
\]
Hence
\[
[C_S^s A(p, s)]^{-p/s} \leq \mu(B).
\]
Going further, one can compute the constant \( A(p, s) \) by observing that
\[
\sum_{k=1}^{\infty} p(k + 3)\alpha^{1-k} = p \sum_{k=1}^{\infty} \frac{k}{\alpha^{k-1}} + 3p \sum_{k=1}^{\infty} \frac{1}{\alpha^{k-1}} = \frac{p}{(1 - 1/\alpha)^2} + \frac{3p\alpha}{\alpha - 1} = \frac{s^2}{p} + 3s.
\]
Therefore, \( A(p, s) = 2^{\frac{s^2}{p} + 3s} \) and the inequality in (76) can be written as
\[
[C_S^s 2^{\frac{s^2}{p} + 3s}]^{-p/s} r^s \leq \mu(B).
\]
Hence, (58) holds with \( \kappa := [C_S^p 2^{3p+s}]^{-1} \). Given that \( \kappa \in (0, \infty) \) is independent of the ball \( B \), this finishes the proof of the implication (2) \( \implies \) (1).

There remains to prove that (3) implies (1) under the additional assumption that \( (X, d) \) is uniformly perfect. To this end, fix \( x \in X \) and a finite radius \( r \in (0, \text{diam}(X)] \). Also, let \( \lambda \in (0, 1) \) be as in (50) and recall that there is no loss in generality in assuming that \( \lambda < 1/5 \) (see discussion following (50)). As such, if we appeal to part (2) in Lemma 8, then it suffices to only consider the case when \( r \leq 3\varphi_x(r)/\lambda^2 \). In particular, we have \( 0 < \varphi_x(r) < r \) (see part (3) of Lemma 7 for the second inequality).

Following a similar line of reasoning as earlier in the proof, we will define a collection of Lipschitz functions \( \{\tilde{u}_j\}_{j \in \mathbb{N}} \) by first considering radii \( \tilde{r}_j := (2^{-j-1} + 2^{-1})\varphi_x(r), \quad j \in \mathbb{N} \) which satisfy
\[
\frac{1}{2} \varphi_x(r) < \tilde{r}_{j+1} < \tilde{r}_j \leq \frac{3}{4}\varphi_x(r).
\]
Here, \( \varphi_x(r) \) is as in (45). For each \( j \in \mathbb{N} \), let \( u_j : X \to [0, 1] \) be the function
\[
\tilde{u}_j(y) := \varphi_{\tilde{r}_{j+1}, \tilde{r}_j}(y) \quad \text{for every} \quad y \in X,
\]
where the function \( \varphi_{\tilde{r}_{j+1}, \tilde{r}_j} \) is as in Lemma 10. Then each \( \tilde{u}_j \) is \( 2^{j+2}\varphi_x(r)^{-1} \)-Lipschitz on \( X \), and \( \tilde{g}_j := 2^{j+2}\varphi_x(r)^{-1} \chi_{\tilde{B}^j} \in D(\tilde{u}_j), \) where \( \tilde{B}^j := B(x, \tilde{r}_j) \). In particular, \( \tilde{u}_j \in M^{1,p}(\sigma B) \), and the functions \( \tilde{u}_j \) and \( \tilde{g}_j \) satisfy (60) (used here with \( B_0 := B \), i.e.,
\[
\inf_{\gamma \in \mathbb{R}} \left( \int_B |\tilde{u}_j - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \leq C_{p r} \left( \frac{1}{r^s} \int_{\sigma B} (\tilde{g}_j)^p \, d\mu \right)^{1/p}, \quad \forall j \in \mathbb{N}.
\]
\( \text{(*)} \) is just a algebraic simplification of (60).
Observe that for each fixed \( j \in \mathbb{N} \), we have (keeping in mind \( \sigma \geq 1 \) and \( r \leq 3\varphi_x(r)/\lambda^2 \))

\[
C_{Pr} \left( \frac{1}{r^s} \int_{\partial B} (\tilde{g}_j)^p \, d\mu \right)^{1/p} = \frac{2^{j+2}C_{Pr}}{\varphi_x(r)} \left( \frac{\mu(B_j)}{r^s} \right)^{1/p} \\
\leq \frac{2^{j+2}C_P}{\lambda^2 r^{s/p}} \mu(B_j)^{1/p}.
\]

On the other hand, since \( \tilde{u}_j \equiv 1 \) on \( B_{j+1} \) and \( \tilde{u}_j \equiv 0 \) on \( B \setminus B_j \) it follows that for each \( \gamma \in \mathbb{R} \) we have \( |\tilde{u}_j - \gamma| \geq \frac{1}{2} \) on at least one of the sets \( B_{j+1} \) and \( B \setminus B_j \). Observe that by combining (46) in Lemma 7 and (79), we have

\[
\mu(B_{j+1}) \leq \mu(B(x, \varphi_x(r))) \leq \frac{1}{2} \mu(B),
\]

and

\[
\mu(B \setminus B_j) = \mu(B) - \mu(B_j) \geq \mu(B) - \mu(B(x, \varphi_x(r))) \geq \frac{1}{2} \mu(B).
\]

Therefore,

\[
\min \{ \mu(B_{j+1}), \mu(B \setminus B_j) \} = \mu(B_{j+1}).
\]

Combining this with (80) and (81), gives

\[
\frac{1}{2} \left( \frac{\mu(B_{j+1})^{1/p^*}}{\mu(B)} \right)^{1/p^*} \leq 2^{j+2} \frac{C_P}{\lambda^2 r^{s/p}} \mu(B_j)^{1/p}.
\]

Hence,

\[
\mu(B_{j+1})^{1/p^*} \leq 2^{j+2} \frac{C_P}{\lambda^2 r^{s/p}} \mu(B_j)^{1/p^*}, \quad \forall j \in \mathbb{N}.
\]

Observe that this inequality is analogous to the one displayed in (68). Thus, arguing as in (69)-(78) will yield the desired result. This finishes the proof of the theorem.

**Theorem 13.** Fix \( \sigma \in (1, \infty), s \in (0, \infty), p \in (0, s) \), and let \( p^* = sp/(s-p) \). Then the following statements are equivalent.

(1) There exists a constant \( \kappa \in (0, \infty) \) such that for each ball \( B_1 := B(x_1, R_1) \) with \( x_1 \in X \) and \( R_1 \in (0, \infty) \), the measure \( \mu \) satisfies

\[
\kappa \left( \frac{r}{R_1} \right)^{s} \mu(B(x, r)) \leq \mu(B_1) = \mu(\partial B_1) \quad \text{whenever } x \in X \quad \text{and}
\]

\[
r \in (0, \sigma R_1] \text{ are such that } B(x, r) \subseteq \sigma B_1.
\]

(2) There exists a constant \( C_S \in (0, \infty) \) such that for every ball \( B_0 := B(x_0, R_0) \) with \( x_0 \in X \) and \( R_0 \in (0, \infty) \), one has

\[
\left( \int_{B_0} |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C_S R_0 \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p} + C_S \left( \int_{\sigma B_0} |u|^p \, d\mu \right)^{1/p},
\]

whenever \( u \in M^{1,p}(\sigma B_0, d, \mu) \) and \( g \in D(u) \).
If, in addition, \((X,d)\) is assumed to be uniformly perfect (cf. \((50)\)) then \((1)\) (hence, also \((2)\)) is further equivalent to:

\[(3) \text{ There exists a constant } C_p \in (0,\infty) \text{ such that for every ball } B_0 := B(x_0,R_0) \text{ with } x_0 \in X \text{ and } R_0 \in (0,\infty), \text{ one has} \]

\[
\inf_{u \in g B_0} \left( \frac{\int_{B_0} |u - \gamma|^p \, d\mu}{\sigma} \right)^{1/p^*} \leq C_p R_0 \left( \frac{\int_{\sigma B_0} g^p \, d\mu}{\sigma} \right)^{1/p}, \tag{89}
\]

whenever \(u \in M^{1,p}(\sigma B_0, d, \mu)\) and \(g \in D(u)\).

**Remark 14.** As the proof of Theorem 13 will reveal, the implication \((1) \implies (3)\) holds in metric measure spaces which are not necessarily uniformly perfect.

**Proof.** We begin proving the implication \((1)\) implies both \((2)\) and \((3)\). Consider a ball \(B_0 := B(x_0,R_0)\) with \(x_0 \in X\) and \(R_0 \in (0,\infty)\), the inequality displayed in \((87)\) (used here with \(B_1 := B_0\)) implies that the measure \(\mu\) satisfies the \(V(\sigma B_0, s, b)\) condition with the choice \(b := \kappa \mu(\sigma B_0) R_0^{-s} \in (0,\infty)\) (see \((14)\)). As such, for this value of \(b\) the inequalities displayed in \((88)\) and \((89)\) follow immediately from \((15)-(16)\) in Theorem 4. Note that these implications are valid without the additional uniformly perfect property.

We prove next that \((1)\) follows from \((2)\). Fix a ball \(B_1 := B(x_1,R_1)\) with \(x_1 \in X\) and \(R_1 \in (0,\infty)\). Specializing \((88)\) to the case when \(B_0 := \sigma B_1\) implies that

\[
\left( \frac{\int_{\sigma^2 B_1} |u|^p \, d\mu}{\sigma} \right)^{1/p^*} \leq C_S \sigma R_1 \left( \frac{\int_{\sigma^2 B_1} g^p \, d\mu}{\sigma} \right)^{1/p} + C_S \left( \frac{\int_{\sigma^2 B_1} |u|^{p^*} \, d\mu}{\sigma} \right)^{1/p}, \tag{90}
\]

whenever \(u \in M^{1,p}(\sigma^2 B_1)\) and \(g \in D(u)\). Moving on, suppose \(x \in X\) and \(r \in (0,\sigma R_1]\) are such that \(B := B(x,r) \subseteq \sigma B_1\). For each \(j \in \mathbb{N}\), let \(r_j := (2^{-j} - 2^{-1}) r\) and set \(B_j := B(x,r_j)\). Then

\[
\frac{1}{2} r < r_{j+1} < r_j \leq \frac{3}{4} r, \quad \forall j \in \mathbb{N}. \tag{91}
\]

Next, let \(u_j : X \to [0,1]\) be the function

\[
u_j(y) := \varphi_{r_{j+1},r_j}(y) \quad \text{for every } y \in X,
\]

where \(\varphi_{r_{j+1},r_j}\) is as in Lemma 10. Then each \(u_j\) is \(2^{j+2} r^{-1}\)-Lipschitz on \(X\) satisfying

\[
u_j \equiv 1 \text{ on } B_{j+1} \quad \text{and} \quad \nu_j \equiv 0 \text{ on } X \setminus B_j.
\]

Moreover, the function \(g_j := 2^{j+2} r^{-1} \chi_{B_j}\) is an upper gradient of \(u_j\). In particular, since \(\sigma > 1\) we have that the functions \(u_j\) and \(g_j\) satisfy \((90)\). Observe that for each fixed \(j \in \mathbb{N}\), we have (keeping in mind \(\sigma > 1\))

\[
C_S \sigma R_1 \left( \frac{\int_{\sigma^2 B_1} g_j^p \, d\mu}{\sigma} \right)^{1/p} = C_S \sigma 2^{j+2} R_1 \left( \frac{\mu(B_j)}{\mu(\sigma^2 B_1)} \right)^{1/p} \leq C_S \sigma 2^{j+2} R_1 \left( \frac{\mu(B_j)}{\mu(\sigma B_1)} \right)^{1/p}, \tag{92}
\]
and

\[ C_S \left( \int_{\sigma B_1} |u_j|^p d\mu \right)^{1/p} \leq C_S \left( \frac{\mu(B)}{\mu(\sigma B_1)} \right)^{1/p}. \]  

(93)

Moreover,

\[ \left( \int_{\sigma B_1} |u_j|^{p^*} d\mu \right)^{1/p^*} \geq \left( \frac{\mu(B)}{\mu(\sigma B_1)} \right)^{1/p^*}. \]  

(94)

In concert, (92)-(94), (90), and the fact that \( r \leq \sigma R_1 \), gives

\[ \left( \frac{\mu(B)}{\mu(\sigma B_1)} \right)^{1/p^*} \leq C_S \left( \frac{\mu(B)}{\mu(\sigma B_1)} \right)^{1/p}, \quad \forall j \in \mathbb{N}. \]  

(95)

Therefore,

\[ \mu(B^{j+1})^{1/p^*} \leq \frac{C_S \sigma^{2j+3} R_1}{r \mu(\sigma B_1)^{1/s}} \mu(B)^{1/p}, \quad \forall j \in \mathbb{N}. \]  

(96)

With \( \alpha := p^*/p \in (1, \infty) \) we raise both sides of the inequality in (96) to the power \( p/\alpha^{j-1} \) in order to obtain

\[ \mu(B^{j+1})^{1/\alpha^j} \leq 2^{p(j+3)/\alpha^j-1} \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{p/\alpha^{j-1}} \mu(B)^{1/\alpha^j-1}, \quad \forall j \in \mathbb{N}. \]  

(97)

If we let \( P_j := \mu(B)^{1/\alpha^j-1} \), then the inequality in (97) becomes

\[ P_{j+1} \leq 2^{p(j+3)/\alpha^j-1} \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{p/\alpha^{j-1}} P_j, \quad \forall j \in \mathbb{N}, \]  

(98)

which, together with an inductive argument and the fact that \( P_1 \leq \mu(B) \), implies

\[ P_{j+1} \leq P_1 \prod_{k=1}^j \left[ 2^{p(k+3)/\alpha^{k-1}} \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{p/\alpha^{k-1}} \right] \]

\[ \leq \mu(B) \prod_{k=1}^j \left[ 2^{p(k+3)/\alpha^{k-1}} \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{p/\alpha^{k-1}} \right], \quad \forall j \in \mathbb{N}. \]  

(99)

Granted the calculations in (72), (73), and (77), we have

\[ \prod_{k=1}^\infty \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{p/\alpha^{k-1}} = \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{p \sum_{k=1}^\infty \alpha^{1-k}} \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^{s}, \]  

(100)

and

\[ \prod_{k=1}^\infty \left( 2^{p(k+3)} \right)^{1/\alpha^{k-1}} = 2^{\sum_{k=1}^\infty p(k+3) \alpha^{1-k}} = 2^{\frac{2}{p+3}s}. \]  

(101)
In addition, based on (91), we may use the estimates in (74) in order to conclude
\[ \lim_{j \to \infty} P_j = 1. \]
Consequently, passing to the limit in (99) yields
\[ 1 \leq \mu(B) \left( \frac{C_S \sigma R_1}{r \mu(\sigma B_1)^{1/s}} \right)^s 2^{s^2 + 3s}. \]  
(102)

Therefore,
\[ \left[ (C_S \sigma^s 2^{s^2 + 3s})^{-1} \left( \frac{r}{R_1} \right)^s \right] \leq \frac{\mu(B)}{\mu(\sigma B_1)}, \]  
(103)

which implies that (87) holds with \( \kappa := \left[ (C_S \sigma^s 2^{s^2 + 3s})^{-1} \right] \). Given that \( \kappa \in (0, \infty) \) is independent of the balls \( B \) and \( B_1 \), this finishes the proof of the implication (2) \( \implies \) (1).

There remains to prove that (3) implies (1) under the additional assumption that \( (X, d) \) is uniformly perfect. To this end, fix balls \( B_1 := B(x_1, R_1), x_1 \in X, R_1 \geq 0, \infty \), and \( B := B(x, r), x \in X, r \in (0, \sigma R_1] \), such that \( B \subseteq \sigma B_1 \). If \( B(x, r) = X \) then \( \sigma B_1 = X \) since \( B \subseteq \sigma B_1 \). Thus, in this case, (87) trivially holds with any \( \kappa \in (0, \sigma^{-1}) \). As such, in what follows we will assume that \( r < \text{diam}(X) \). In light of part (2) in Lemma 8, we may assume that \( r \leq 3\varphi_x(r)/\lambda^2 \) where \( \lambda \in (0, 1) \) as in (50). In particular, we have \( 0 < \varphi_x(r) < r \) (see part (3) of Lemma 7 for the second inequality).

As in the proof of (3) \( \implies \) (1) in Theorem 11, we consider radii \( \tilde{r}_j := (2^{-j-1} + 2^{-1})\varphi_x(r), j \in \mathbb{N} \), where \( \varphi_x(r) \) is as in (45). Then
\[ \frac{1}{2} \varphi_x(r) < r_{j+1} < \tilde{r}_j \leq \frac{3}{4} \varphi_x(r). \quad \forall j \in \mathbb{N}. \]  
(104)

Define \( \tilde{u}_j : X \to [0, 1] \) by setting for each \( y \in X \),
\[ \tilde{u}_j(y) := \varphi_{\tilde{r}_{j+1}, \tilde{r}_j}(y), \]
where the function \( \varphi_{\tilde{r}_{j+1}, \tilde{r}_j} \) is as in Lemma 10. Then each \( \tilde{u}_j \) is \( 2^{j+2} \varphi_x(r)^{-1} \)-Lipschitz on \( X \), and \( \tilde{g}_j := 2^{j+2} \varphi_x(r)^{-1} \chi_{\tilde{B}_j} \in D(\tilde{u}_j) \), where \( \tilde{B}_j := B(x, \tilde{r}_j) \). In particular, the functions \( u_j \) and \( g_j \) satisfy (89) (used here with \( B_0 := \sigma B_1 \)). Observe that for each fixed \( j \in \mathbb{N} \), we have (keeping in mind \( \sigma > 1 \))
\[ C_P \sigma R_1 \left( \frac{\int_{\sigma^2 B_1} (\tilde{g}_j)^p \, d\mu}{\sigma^2 B_1} \right)^{1/p} = \frac{2^{j+2} C_P \sigma R_1}{\varphi_x(r)} \left( \frac{\mu(\tilde{B}_j)}{\mu(\sigma^2 B_1)} \right)^{1/p} \]
\[ \leq \frac{2^{j+2} 3 C_P \sigma R_1}{\sigma^2} \left( \frac{\mu(\tilde{B}_j)}{\mu(\sigma B_1)} \right)^{1/p}. \]  
(105)

On the other hand, granted the size and support conditions for \( \tilde{u}_j \), for each \( \gamma \in \mathbb{R} \) we have \( |\tilde{u}_j - \gamma| \geq \frac{1}{2} \) on at least one of the sets \( \tilde{B}^{j+1} \) and \( B \setminus \tilde{B}^j \). Making use of the estimates in (82)-(83) there holds
\[ \min \left\{ \mu(B^{j+1}), \mu(B \setminus B^j) \right\} = \mu(B^{j+1}), \]  
(106)

\(^5\)Recall that we can assume \( \lambda < 1/5 \) so that we may make use of Lemma 8.
which, when considered in concert with (89), (105), and (106), gives
\[
\left( \frac{\mu(B_j^{j+1})}{\mu(B_j)} \right)^{1/p^*} \leq 3C_p \sigma R_1 \frac{2^{j+3}}{\lambda^2 r} \left( \frac{\mu(B_j)}{\mu(B_1)} \right)^{1/p}.
\]  
(107)

Therefore,
\[
\mu(B_j^{j+1})^{1/p^*} \leq 3C_p \sigma R_1 \frac{2^{j+3}}{\lambda^2 r} \mu(B_1)^{1/p} \mu(B_j)^{1/p}.
\]  
(108)

Recognizing that the inequality in (108) is analogous to the one displayed in (96), an argument along the lines of the one presented in (97)-(103) will yield the desired result. This finishes the proof of the theorem. □

5. The Case \( p = s \)

**Theorem 15.** Suppose that \((X, d, \mu)\) is a uniformly perfect measure metric space. Then for each fixed \( s \in (0, \infty) \) and \( \sigma \in (1, \infty) \), the following two statements are equivalent.

1. There exists a finite constant \( \kappa > 0 \) such that
   \[
   \kappa r^s \leq \mu(B(x, r)) \quad \text{for every } x \in X \\
   \text{and every finite } r \in (0, \text{diam}(X)].
   \]  
(109)

2. There exist constants \( c_1, c_2, \gamma \in (0, \infty) \) such that
   \[
   \int_{B_0} \exp \left( c_1 \frac{|u - u_{B_0}|}{\|g\|_{L^\infty(B_0)}} \right)^\gamma \, d\mu \leq c_2,
   \]  
(110)

whenever \( B_0 \subseteq X \) is a ball (with radius at most \( \text{diam}(X) \)), \( u \in M^{1,s}(\sigma B_0) \) and \( g \in D(u) \setminus \{0\} \).

**Remark 16.** In the proof of \((2) \implies (1)\), one can choose the constant \( \kappa \) to be
\[
\kappa := \frac{c_1^s \lambda^{4s}}{2 \sqrt{c_2} 96^s (2s \gamma^{-1})^{s/\gamma}}.
\]  
(111)

**Proof.** Fix \( s \in (0, \infty) \) and \( \sigma \in (1, \infty) \). For the implication \((1) \implies (2)\), observe that if \( B_0 \) is a ball having finite radius \( R_0 \in (0, \text{diam}(X)] \), then (109) implies that \( \mu \) satisfies the \( V(\sigma B_0, s, b) \) condition (cf. (14)) with \( b := \kappa \sigma^{-s} \in (0, \infty) \), where \( \kappa \) is as in (109). As such, the desired conclusion now follows from part (b) in Theorem 4 with \( \gamma = 1 \).

Regarding the opposite implication, suppose that (110) holds for some \( c_1, c_2, \gamma \in (0, \infty) \), and fix parameters \( x \in X, r \in (0, \text{diam}(X)] \), finite. We will first consider the case when \( r \leq 3\varphi_x(r)/\lambda^2 \), where \( \lambda \in (0, 1) \) is as in (50). Then \( 0 < \varphi_x(r) < r \) (see part (3) in Lemma 7 for the inequality \( \varphi_x(r) < r \)).

\(^6\)Recall that we can assume \( \lambda < 1/5 \) so that we may make use of Lemma 8
We now define a collection of functions \( \{ u_j \}_{j \in \mathbb{N}} \) as follows: for each fixed \( j \in \mathbb{N} \), let \( r_j := (2^{-j-1} + 2^{-1})\varphi_x(r) \), where \( \varphi_x(r) \) is as in (45), and set \( B^j := B(x, r_j) \). By design, we have
\[
\frac{1}{2} \varphi_x(r) < r_{j+1} < r_j \leq \frac{3}{4} \varphi_x(r), \quad \forall j \in \mathbb{N}.
\] (112)

Now, for each \( j \in \mathbb{N} \), let \( u_j : X \to [0, 1] \) be the function defined by setting for each \( y \in X \),
\[
u_j(y) := \varphi_{r_{j+1}, r_j}(y),
\]
where \( \varphi_{r_{j+1}, r_j} \) is as in Lemma 10. It follows that \( u_j \) is \( 2^{j+1} \varphi_x(r)^{-1} \)-Lipschitz on \( X \) and that the function \( g_j := 2^{j+2} \varphi_x(r)^{-1} \chi_{B^j} \) is a nonzero upper gradient of \( u_j \). In particular, we have that \( u_j \) and \( g_j \) satisfy (110) (used here with \( B_0 := B \), i.e.,
\[
\int_B \exp \left( c_1 \frac{|u_j - (u_j)_B|}{g_j \|L^s(\sigma B)\|} \right) \gamma d\mu \leq c_2, \quad \forall j \in \mathbb{N}.
\] (113)

Here, \( (u_j)_B := [\mu(B)]^{-1} \int_B u_j d\mu, \quad j \in \mathbb{N} \).

In order to bound the left-hand side of (113) from below, first note that we immediately have (given that \( \sigma > 1 \) and \( r \leq 3\varphi_x(r)/\lambda^2 \))
\[
g_j \|L^s(\sigma B)\| = \frac{2^{j+2}}{\varphi_x(r)} \mu(B^j)^{1/s} \leq \frac{3 \lambda^2}{\varphi_x(r)} \mu(B^j)^{1/s}.
\] (114)

Secondly, in light of the properties of the function \( \varphi_{r_{j+1}, r_j} \), for each \( a \in \mathbb{R} \) we can conclude that \( |u_j - a| \geq \frac{1}{2} \) on at least one of the sets \( B^{j+1} \) and \( B \setminus B^j \). Then recycling the estimates in (82)-(83) yields
\[
\min \{ \mu(B^{j+1}), \mu(B \setminus B^j) \} = \mu(B^{j+1}),
\] (115)

which, when considered in concert with, (113), (114), and (115), gives
\[
c_2 \geq \int_B \exp \left( c_1 \frac{|u_j - (u_j)_B|}{g_j \|L^s(\sigma B)\|} \right) \gamma d\mu \geq \frac{\mu(B^{j+1})}{\mu(B)} \exp \left( \frac{c_1 \lambda^2 r}{2^{j+3} \mu(B^j)^{1/s}} \right)^{1/s}.
\] (116)

Without loss of generality we can assume that \( c_2 > 1 \). Then it follows from (116) that
\[
\frac{c_1 \lambda^2 r}{2^{j+3} \mu(B^j)^{1/s}} \leq \frac{\log \left( \frac{c_2 \mu(B)}{\mu(B^{j+1})} \right)^{1/s}}{\left( 2 s \gamma^{-1} \right)^{1/s} \left( \frac{\mu(B)}{\mu(B^{j+1})} \right)^{1/(2s)}}.
\] (117)

In obtaining the last inequality in (117), we have used the fact that \( \log(y) \leq q y^{1/q} \), for every \( y, q \in (0, \infty) \) (applied here with \( q = 2 s \gamma^{-1} \)). Therefore,
\[
\mu(B^{j+1})^{1/(2s)} \leq 2^{j+2} \left( \frac{2 s \gamma^{-1} c_2^{1/(2s)} \mu(B)^{1/(2s)}}{c_1 \lambda^2 r} \right) \mu(B^j)^{1/s}.
\] (118)

Next, we raise both sides of the inequality in (118) to the power \( s/2^{j-1} \) in order to obtain
\[
\mu(B^{j+1})^{1/2^{j-1}} \leq 2^{s j/2^{j-1}} \left( \frac{2 s \gamma^{-1} c_2^{1/(2s)} \mu(B)^{1/(2s)}}{c_1 \lambda^2 r} \right)^{s/2^{j-1}} \mu(B^j)^{1/2^{j-1}}, \quad \forall j \in \mathbb{N}.
\] (119)

If we let
\[
P_j := \mu(B^j)^{1/2^{j-1}} \quad \text{and} \quad Q := \frac{24 (2 s \gamma^{-1} c_2^{1/(2s)} \mu(B)^{1/(2s)})}{c_1 \lambda^2 r},
\] (120)
then the inequality in (119) becomes
\[ P_{j+1} \leq 2^{s_j/2^{j-1}} Q^{s_j/2^{j-1}} P_j, \quad \forall j \in \mathbb{N}, \]
which, together with an inductive argument and the fact that \( P_1 \leq \mu(B) \) (cf. (112)), implies
\[ P_{j+1} \leq P_1 \prod_{k=1}^{j} \left[ 2^{s_k/2^{k-1}} Q^{s_k/2^{k-1}} \right] \leq \mu(B) \prod_{k=1}^{j} \left[ 2^{s_k/2^{k-1}} Q^{s_k/2^{k-1}} \right], \quad \forall j \in \mathbb{N}. \]

Recall that we can use (112) and the estimates in (74) to show that \( \lim_{j \to \infty} P_j = 1 \). Moreover,
\[ \prod_{k=1}^{\infty} Q^{s_k/2^{k-1}} = Q^s \sum_{k=1}^{\infty} 2^{1-k} = Q^{2s}, \]
and
\[ \prod_{k=1}^{\infty} 2^{s_k/2^{k-1}} = 2^{\sum_{k=1}^{\infty} s_k 2^{1-k}} =: A(s) \in (0, \infty). \]

Consequently, passing to the limit in (122) as \( j \to \infty \) yields,
\[ 1 \leq A(s) Q^{2s} \mu(B). \]
\[ = A(s) \left( \frac{24(2s\gamma^{-1})^{1/\gamma} c_2^{1/(2s)} \mu(B)^{1/(2s)}}{c_1 \lambda^2 r} \right)^{2s} \mu(B) \]
\[ = c_2 A(s) \left( \frac{24(2s\gamma^{-1})^{1/\gamma}}{c_1 \lambda^2 r} \right) \mu(B)^2 \]
\[ \text{(125)} \]

Hence,
\[ \left( c_2 A(s) \right)^{-1/2} \left( \frac{c_1 \lambda^2}{24(2s\gamma^{-1})^{1/\gamma}} \right)^s r^s \leq \mu(B). \]
\[ \text{(126)} \]

Going further, one can compute the constant \( A(s) \) by observing that
\[ \sum_{k=1}^{\infty} sk 2^{1-k} = s \sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = \frac{s}{(1-1/2)^2} = 4s. \]
\[ \text{(127)} \]

Therefore, \( A(s) = 2^{4s} \) and the inequality in (126) can be written as
\[ \left( \frac{c_1 \lambda^{2s}}{\sqrt{c_2} 96^s (2s\gamma^{-1})^{1/\gamma}} \right) r^s \leq \mu(B). \]
\[ \text{(128)} \]

At this stage, we have shown that (2) implies that the measure condition in (109) holds whenever \( B \) is a ball with radius at most \( \min \{ 3\varphi_2(r)/\lambda^2, \text{diam}(X) \} \). The fact that (109) holds for balls with radius \( \leq \text{diam}(X) \) now follows from (2) in Lemma 8, given that \( (X, d) \) is uniformly perfect. Finally, noting that the constant appearing in (111) comes from the extra factor of \( \lambda^{2s}/2 \) from (51), this finishes the proof of the second implication and, in turn, the proof of the theorem. \( \square \)

In the case of doubling measures we have the following characterization which is a consequence of Theorem 4 and Theorem 19, below.
Theorem 17. Suppose that \((X, d, \mu)\) is a uniformly perfect measure metric space and fix \(\sigma \in (1, \infty)\). Then the following two statements are equivalent.

1. The measure \(\mu\) is doubling.
2. There exist constants \(c_1, c_2, s, \gamma \in (0, \infty)\) such that

\[
\int_{B_0} \exp \left( c_1 \frac{\mu(B_0)^{1/s} |u - u_{B_0}|}{\|g\|_{L^s(B_0)}} \right)^\gamma \, d\mu \leq c_2,
\]

whenever \(B_0 \subseteq X\) is a ball, \(u \in M^{1,s}(\sigma B_0)\) and \(g \in D(u) \setminus \{0\}\).

Remark 18. For the implication \((1) \implies (2)\), one can take \(s := \log_2(C_\mu)\) where

\[
C_\mu := \sup_{x \in X, r \in (0, \infty)} \frac{\mu(B(x, r))}{\mu(B(x, 2r))} \in (1, \infty)
\]

is the doubling constant for \(\mu\).

Theorem 19. Let \((X, d, \mu)\) be a uniformly perfect measure metric space and suppose that there exist \(s, c_1, c_2, \sigma \in [1, \infty)\) such that

\[
\int_{B_0} \exp \left( c_1 \frac{\mu(B_0)^{1/s} |u - u_{B_0}|}{\|g\|_{L^s(B_0)}} \right)^\gamma \, d\mu \leq c_2,
\]

whenever \(B_0 \subseteq X\) is a ball, \(u \in M^{1,s}(\sigma B_0)\) and \(g \in D(u) \setminus \{0\}\).

Then for every \(\varepsilon \in (0, \infty)\), there exists a constant \(\kappa \in (0, \infty)\) such that for each ball \(B_1 := B(x_1, R_1)\) with \(x_1 \in X\) and \(R_1 \in (0, \infty)\), the measure \(\mu\) satisfies

\[
\kappa \left( \frac{r}{R_1} \right)^{s+\varepsilon} \leq \frac{\mu(B(x, r))}{\mu(\sigma B_1)} \quad \text{whenever } x \in X \text{ and } r \in (0, \sigma R_1)
\]

are such that \(B(x, r) \subseteq \sigma B_1\).

Proof. The justification of this result follows a similar reasoning as in the proof of Theorem 17. Fix \(x_1, x \in X\) and \(r, R_1 \in (0, \infty)\) with \(r \leq \sigma R_1\), and suppose that \(B_1 := B(x_1, R_1)\) and \(B := B(x, r)\) satisfy \(B \subseteq \sigma B_1\). If \(B(x, r) = X\) then \(\sigma B_1 = X\) since \(B \subseteq \sigma B_1\). Thus, in this case, (132) trivially holds with any \(\kappa \in (0, \sigma^{-(s+\varepsilon)})\). As such, in what follows we will assume that \(X \setminus B(x, r) \neq \emptyset\). In light of part (2) in Lemma 8, we may assume that \(r \leq 3\varphi_x(r)/\lambda^2\) where \(\lambda \in (0, 1)\) is as in (50)\footnote{Recall that we can assume \(\lambda < 1/5\) so that we may make use of Lemma 8}. Combining this with part (2) in Lemma 7 we have \(0 < \varphi_x(r) < r\).

Moving on, let \(\{u_j\}_{j \in \mathbb{N}}, \{r_j\}_{j \in \mathbb{N}}, \text{ and } \{B_j\}_{j \in \mathbb{N}}\) be as in the proof of Theorem 15. Given that each \(u_j\) is \(2^{j+2}\varphi_x(r)^{-1}\)-Lipschitz on \(X\) and that \(g_j := 2^{j+2}\varphi_x(r)^{-1}\chi_{B_j}\) is a nonzero upper gradient of \(u\), we have that \(u_j\) and \(g_j\) satisfy (131) (used here with \(B_0 := \sigma B_1\)), i.e.,

\[
\int_{\sigma B_1} \exp \left( c_1 \frac{\mu(\sigma^2 B_1)^{1/s} |u_j - (u_j)_{\sigma B_1}|}{\sigma R_1 \|g_j\|_{L^s(\sigma^2 B_1)}} \right)^\gamma \, d\mu \leq c_2, \quad \forall j \in \mathbb{N}
\]
where \((u_j)_{B_1} := [\mu(\sigma B_1)]^{-1} \int_{\sigma B_1} u_j \, d\mu\). Fix \(\beta \in (1, \infty)\). Then by arguing as in (113)-(117) (and using the fact that \(\sigma \geq 1\), we can deduce that
\[
\frac{c_1 \lambda^2 r}{2^{j+3} 3 \sigma R_1} \cdot \frac{\mu(\sigma B_1)^{1/s}}{\mu(B_j^{1/s})} \leq \left[ \log \left( \frac{c_2 \mu(\sigma B_1)}{\mu(B_j^{1/s})} \right) \right]^{1/\gamma}
\leq (\beta s \gamma^{-1})^{1/\gamma} c_2^{1/(\beta s)} \left( \frac{\mu(\sigma B_1)}{\mu(B_j^{1/s})} \right)^{1/(\beta s)}.
\]
Here, we have used the estimate \(\log(y) \leq q y^{1/q}\) with \(q = \beta s \gamma^{-1}\). Rearranging the factors in (134) yields
\[
\mu(B_j^{1/s})^{1/(\beta s)} \leq 2^s \frac{24 \sigma R_1 (\beta s \gamma^{-1})^{1/\gamma} c_2^{1/(\beta s)}}{c_1 \lambda^2 r \mu(\sigma B_1)^{(\beta - 1)/(\beta s)} - \mu(B_j^{1/s})}. \tag{135}
\]
Next, we raise both sides of the inequality in (135) to the power \(s/\beta j - 1\) in order to obtain
\[
\mu(B_{j+1}^{1/\beta}) \leq 2^{s j/\beta j - 1} \left( \frac{24 \sigma R_1 (\beta s \gamma^{-1})^{1/\gamma} c_2^{1/(\beta s)}}{c_1 \lambda^2 r \mu(\sigma B_1)^{(\beta - 1)/(\beta s)}} \right)^{s/\beta j - 1} \mu(B_j^{1/\beta})^{-1}, \quad \forall j \in \mathbb{N}. \tag{136}
\]
Estimates similar to those in (120)-(125) give
\[
1 \leq A(s, \beta) \left( \frac{24 \sigma R_1 (\beta s \gamma^{-1})^{1/\gamma} c_2^{1/(\beta s)}}{c_1 \lambda^2 r \mu(\sigma B_1)^{(\beta - 1)/(\beta s)}} \right)^{s/(\beta - 1)} \mu(B). \tag{137}
\]
Hence,
\[
\kappa(\beta) \left( \frac{r}{R_1} \right)^{s/(\beta - 1)} \leq \frac{\mu(B)}{\mu(\sigma B_1)}, \tag{138}
\]
where
\[
\kappa(\beta) := \left[ A(s, \beta) \right]^{-1} \left( \frac{c_1 \lambda^2}{24 \sigma (\beta s \gamma^{-1})^{1/\gamma}} \right)^{s/(\beta - 1)}. \tag{139}
\]
Finally, given \(\varepsilon \in (0, \infty)\), if we choose \(\beta \in (1, \infty)\) sufficiently large then the desired conclusion in (132) will follow from (138). This finishes the proof of the theorem. \(\square\)

6. The Case \(p > s\)

**Theorem 20.** Suppose that \((X, d, \mu)\) is a uniformly perfect metric measure space and fix \(s \in (0, \infty)\). Then for each \(p \in (s, \infty)\), the following two statements are equivalent.

1. There exists a finite constant \(\kappa > 0\) such that
   \[
   \kappa r^s \leq \mu(B(x, r)) \quad \text{for every } x \in X
   \]
   and every finite \(r \in (0, \text{diam}(X)]\).

2. There exists a constant \(c \in (0, \infty)\) with the property that for each \(u \in M^1_p(X)\) and \(g \in D(u)\), there holds
   \[
   |u(x) - u(y)| \leq c \, d(x, y)^{1-s/p} \|g\|_{L^p(X, \mu)}, \quad \forall x, y \in X\]
   Hence, every function \(u \in M^1_p(X)\) has Hölder continuous representative of order \((1 - s/p)\) on \(X\).
Proof. For the implication \((1) \implies (2)\), observe that if \(p \in (s, \infty)\) and \(B_0\) is any ball, then \((140)\) implies that \(\mu\) satisfies the \(V(2B_0, s, \kappa)\) condition (cf. \((14)\)) for some \(\kappa \in (0, \infty)\) which depends only on \(\kappa\) and the space \(X\). As such, part \((c)\) in Theorem \(4\) guarantees the existence of a constant \(C \in (0, \infty)\) (independent of \(B_0\)) with the property that for each \(u \in M^{1,p}(X)\) and \(g \in D(u)\), there holds

\[
|u(x) - u(y)| \leq C\kappa^{-1/p}d(x,y)^{1-s/p} \left(\int_{2B_0} g^p \, d\mu\right)^{1/p} \quad \text{for all } x, y \in B_0.
\]

Given that the constants \(C\) and \(\kappa\) are independent of the arbitrarily chose \(B_0\), we have that \((142)\) implies \((141)\), finishing the proof of \((1) \implies (2)\).

For the reverse implication, fix \(x \in X\) and \(r \in (0, \text{diam}(X)]\), finite. If \(B(x, r) = X\) then \(r = \text{diam}(X) \in (0, \infty)\) and

\[
\mu(B(x, r)) = \mu(X)[\text{diam}(X)]^{-s} r^s.
\]

Thus, in what follows, we may assume that \(X \setminus B(x, r) \neq \emptyset\).

Let \(\lambda \in (0, 1)\) be as in \((50)\) and define \(u : X \to [0, 1]\) by setting for each \(y \in X\),

\[
u(y) := \begin{cases} 
\frac{\lambda r - d(x, y)}{\lambda r} & \text{if } y \in B(x, \lambda r), \\
0 & \text{if } y \in X \setminus B(x, \lambda r).
\end{cases}
\]

A straightforward computation will show that \(u\) is \((\lambda r)^{-1}\)-Lipschitz on \(X\) and that the function \(g(y) := (\lambda r)^{-1} \chi_{B(x, r)}(y)\) is a gradient of \(u\). Moreover, since \((X, d)\) is assumed to be uniformly perfect, we may select a point \(y \in B(x, r) \setminus B(x, \lambda r)\). Then by \((141)\) (used here with \(u\) as in \((144)\)), we have

\[
1 = |u(x) - u(y)| \leq c d(x, y)^{1-s/p} \|g\|_{L^p(X, \mu)}
\leq c \lambda^{-1} r^{-s/p} \mu(B(x, r))^{1/p},
\]

from which \((140)\) follows with \(C := (\lambda/c)^p \in (0, \infty)\). This finishes the proof of the reverse implication and, in turn, the proof of the theorem. \(\Box\)

**Theorem 21.** Suppose that \((X, d, \mu)\) is a uniformly perfect metric measure space and fix \(s \in (0, \infty), \sigma \in (1, \infty)\). Then for each \(p \in (s, \infty)\), the following two statements are equivalent.

1. There exists a constant \(\kappa \in (0, \infty)\) such that for each ball \(B_1 := B(x_1, R_1)\) with \(x_1 \in X\) and \(R_1 \in (0, \infty)\), the measure \(\mu\) satisfies

\[
\kappa \left(\frac{r}{R_1}\right)^s \leq \frac{\mu(B(x, r))}{\mu(\sigma B_1)} \quad \text{whenever } x \in X \text{ and } r \in (0, \sigma R_1) \text{ are such that } B(x, r) \subseteq \sigma B_1.
\]

2. There exists a finite constant \(c > 0\) such that for each ball \(B_0 := B(x_0, R_0)\) with \(x_0 \in X\) and \(R_0 \in (0, \text{diam}(X)]\), finite, and each \(u \in M^{1,p}(\sigma B_0)\) and \(g \in D(u)\),
there holds
\[ |u(x) - u(y)| \leq c d(x, y)^{1-s/p} R_0^{s/p} \left( \int_{\sigma B_0} g^p \, d\mu \right)^{1/p}, \quad \forall x, y \in B_0, \tag{147} \]

Hence, every function \( u \in M^{1,p}(\sigma B_0) \) has Hölder continuous representative of order \((1 - s/p)\) on \(B_0\).

**Proof.** We begin proving the implication \((1) \implies (2)\). Given a ball \( B_0 := B(x_0, R_0) \) with \( x_0 \in X \) and \( R_0 \in (0, \infty) \), the inequality displayed in (146) (used here with \( B_1 := B_0 \)) implies that \( \mu \) satisfies the \( V(\sigma B_0, s, b) \) condition with \( b := \kappa \mu(\sigma B_0) R_0^{s/p} \in (0, \infty) \) (see (14)). As such, for this value of \( b \) the inequality displayed in (147) follows immediately from (19) in Theorem 4.

In order to prove the implication \((2) \implies (1)\), fix a ball \( B_1 := B(x_1, R_1) \) with \( x_1 \in X \) and \( R_1 \in (0, \infty) \). Specializing (147) to the case when \( B_0 := \sigma B_1 \) implies that
\[ |u(x) - u(y)| \leq c \sigma^{s/p} d(x, y)^{1-s/p} R_1^{s/p} \left( \int_{\sigma^2 B_1} g^p \, d\mu \right)^{1/p}, \quad \forall x, y \in \sigma B_1, \tag{148} \]

whenever \( u \in M^{1,p}(\sigma^2 B_1) \) and \( g \in D(u) \). Moving on, fix a point \( x \in X \) and a radius \( r \in (0, \sigma R_1] \) such that \( B := B(x, r) \subseteq \sigma B_1 \). If \( B(x, r) = X \) then \( \sigma B_1 = X \) since \( B \subseteq \sigma B_1 \). Thus, in this case, (146) trivially holds with any \( \kappa \in (0, \sigma^{-s}] \). As such, in what follows we will assume that \( X \setminus B(x, r) \neq \emptyset \).

Let \( \lambda \in (0, 1) \) be as in (50) and define \( u : X \to [0, 1] \) by setting for each \( y \in X \),
\[ u(y) := \begin{cases} \frac{\lambda r - d(x, y)}{\lambda r} & \text{if } y \in B(x, \lambda r), \\ 0 & \text{if } y \in X \setminus B(x, \lambda r). \end{cases} \tag{149} \]

A straightforward computation will show that \( u \) is \((\lambda r)^{-1}\)-Lipschitz on \( X \) and that the function \( g(y) := (\lambda r)^{-1} \chi_{B}(y) \) is a gradient of \( u \). Moreover, since \((X, d)\) is assumed to be uniformly perfect, we may select a point \( y \in B(x, r) \setminus B(x, \lambda r) \). Then by (148) (used here with \( u \) as in (149)), we have (keeping in mind \( \sigma > 1 \))
\[ 1 = |u(x) - u(y)| \leq c \sigma^{s/p} d(x, y)^{1-s/p} R_1^{s/p} \left( \int_{\sigma^2 B_1} g^p \, d\mu \right)^{1/p} \]
\[ \leq c \sigma^{s/p} \lambda^{-1} \left( \frac{R_1}{r} \right)^{s/p} \left( \frac{\mu(B)}{\mu(\sigma B_1)} \right)^{1/p}, \tag{150} \]

from which (146) follows with \( \kappa := (\lambda/c \sigma^{s/p})^{p} \in (0, \infty) \). This finishes the proof of the reverse implication and, in turn, the proof of the theorem. \( \square \)
7. Global Embeddings

In this section we investigate the relationship between the lower measure bound in (2) and global Sobolev and Sobolev-Poincaré inequalities (in the case \( p < s \)), i.e., estimates of the form

\[
\left( \int_X |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C_1 \left( \int_X g^p \, d\mu \right)^{1/p} + C_1 \left( \int_X |u|^p \, d\mu \right)^{1/p},
\]

and

\[
\inf_{\gamma \in \mathbb{R}} \left( \int_X |u - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \leq C_2 \left( \int_X g^p \, d\mu \right)^{1/p},
\]

where \( u \in M^{1,p}(X, d, \mu) \) and \( g \in D(u) \). It was shown in [8] that if the measure \( \mu \) is doubling then the Sobolev embedding in (151) implies the measure \( \mu \) satisfies the lower bound in (2). In Theorem 22, we prove that the assumption \( \mu \) is doubling is not necessary.

**Theorem 22.** Suppose \( (X, d, \mu) \) is a metric measure space. Fix \( s, p \in (0, \infty) \) such that \( p < s \) and set \( p^* := sp/(s - p) \). Then the following statements are valid.

1. If there exists a finite constant \( C_1 > 0 \) satisfying

\[
\left( \int_X |u|^{p^*} \, d\mu \right)^{1/p^*} \leq C_1 \left( \int_X g^p \, d\mu \right)^{1/p} + C_1 \left( \int_X |u|^p \, d\mu \right)^{1/p},
\]

whenever \( u \in M^{1,p}(X, d, \mu) \) and \( g \in D(u) \), then there exists a finite constant \( \kappa > 0 \) such that

\[
\kappa r^s \leq \mu(B(x, r)), \quad \forall x \in X \text{ and } \forall r \in (0, 1].
\]

2. If the metric space \( (X, d) \) is uniformly perfect and there exists a finite constant \( C_2 > 0 \) satisfying

\[
\inf_{\gamma \in \mathbb{R}} \left( \int_X |u - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \leq C_2 \left( \int_X g^p \, d\mu \right)^{1/p},
\]

whenever \( u \in M^{1,p}(X, d, \mu) \) and \( g \in D(u) \), then the measure \( \mu \) satisfies the lower bound displayed in (154) (with a possibly different constant \( \kappa \)).

**Proof.** We begin by proving statement (1). Fix a point \( x \in X \) and a radius \( r \in (0, 1] \). Let \( \{u_j\}_{j \in \mathbb{N}} \) be the collection of functions defined in Lemma 10, and set \( B^j := B(x, r_j) \), where \( r_j := (2^{-j-1} + 2^{-1})r \). Recall that each \( u_j \) is \( 2^{j+2}r^{-1} \)-Lipschitz on \( X \) with \( u_j \equiv 1 \) on \( B^{j+1} \) and \( u_j \equiv 0 \) on \( X \setminus B^j \). Moreover, the function \( g_j := 2^{j+2}r^{-1} \chi_{B^j} \) belongs to \( D(u_j) \). Hence, \( u_j \in M^{1,p}(X, d, \mu) \). As such, the functions \( u_j \) and \( g_j \) satisfy (153) and we have the following estimates:

\[
C_1 \left( \int_X g_j^p \, d\mu \right)^{1/p} + C_1 \left( \int_X |u_j|^p \, d\mu \right)^{1/p} \leq C_1 \left( \frac{2^{j+2}}{r} + 1 \right) \mu(B^j)^{1/p},
\]

where

\[
\int_X |u_j|^p \, d\mu \leq C_1 \left( \int_X g_j^p \, d\mu \right)^{1/p} \leq C_1 \left( \frac{2^{j+2}}{r} + 1 \right) \mu(B^j)^{1/p},
\]
and
\[
\left( \int_X |u_j|^{p^*} \, d\mu \right)^{1/p^*} \geq \mu(B^{j+1})^{1/p^*}
\]
As such, we may deduce from (153) (keeping in mind \( r \leq 1 \)) that
\[
\mu(B^{j+1})^{1/p^*} \leq C_1 \left( \frac{2^{j+2}}{r} + 1 \right) \mu(B^j)^{1/p} \leq \frac{C_1 2^{j+2}}{r} \mu(B^j)^{1/p} \quad \forall j \in \mathbb{N}. \tag{156}
\]
Following along the lines of the argument made in the proof of (2) implies (1) in Theorem 11, we raise the extreme most sides of the inequality in (156) to the power \( p/\alpha^{j-1} \) (where \( \alpha := p^*/p \in (1, \infty) \)) in order to obtain
\[
\mu(B^{j+1})^{1/\alpha} \leq 2^{p(j+3)/\alpha^{j-1}} \left( \frac{C_1}{r} \right)^{p/\alpha^{j-1}} \mu(B^j)^{1/\alpha}, \quad \forall j \in \mathbb{N}. \tag{157}
\]
Setting \( P_j := \mu(B^j)^{1/\alpha^{j-1}} \), the inequality in (157) can be rewritten as
\[
P_{j+1} \leq 2^{p(j+3)/\alpha^{j-1}} \left( \frac{C_1}{r} \right)^{p/\alpha^{j-1}} P_j, \quad \forall j \in \mathbb{N}.
\]
Then an inductive argument (along with the fact that \( P_1 \leq \mu(B(x, r)) \)) will yield
\[
P_{j+1} \leq P_1 \prod_{k=1}^j \left[ 2^{p(k+3)/\alpha^{k-1}} \left( \frac{C_1}{r} \right)^{p/\alpha^{k-1}} \right] \leq \mu(B) \prod_{k=1}^j \left[ 2^{p(k+3)/\alpha^{k-1}} \left( \frac{C_1}{r} \right)^{p/\alpha^{k-1}} \right], \quad \forall j \in \mathbb{N}. \tag{158}
\]
Recycling the calculations in (72) and (73), passing to the limit in (158) gives
\[
1 \leq \mu(B(x, r)) \left( \frac{C_1}{r} \right)^s A(p, s). \tag{159}
\]
Hence
\[
[C_1^s A(p, s)]^{-1} r^s \leq \mu(B(x, r)), \tag{160}
\]
which completes the proof of the statement in (1).

Turning our attention to proving (2), we will proceed by arguing as in the proof of (3) implies (1) in Theorem 11. Fix \( x \in X \) and \( r \in (0, 1] \). As noted in the proof of Theorem 11, it suffices to consider the case when \( X \setminus B(x, r) \neq \emptyset \) and \( r \leq 3 \varphi_x(r)/\lambda^2 \) where \( \lambda \in (0, 1) \) is as in (50). With this in mind, consider the collection of function \( \{u_j\}_{j \in \mathbb{N}} \) defined in (??), and set \( B^j := B(x, r_j) \), where \( r_j := (2^{-j-1} + 2^{-1}) \varphi_x(r) \). Then each \( u_j \) is \( 2^{j+2} \varphi_x(r)^{-1} \)-Lipschitz on \( X \) with \( u_j \equiv 1 \) on \( B^{j+1} \) and \( u_j \equiv 0 \) on \( X \setminus B^j \). Moreover, the function \( g_j := 2^{j+2} \varphi_x(r)^{-1} \chi_{B^j} \) belongs to \( D(u_j) \). Hence, the functions \( u_j \) and \( g_j \) satisfy (155). Going further, by making use of the estimates in (82)-(83) we have for each \( j \in \mathbb{N} \),
\[
\inf_{\gamma \in \mathbb{R}} \left( \int_X |u_j - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \geq \frac{1}{2} \left( \min\{\mu(B^{j+1}), \mu(B(x, r) \setminus B^j)\} \right)^{1/p^*} = \frac{1}{2} \mu(B^{j+1})^{1/p^*}.
\]
Combining this with the estimate
\[ C_2 \left( \int_X g_j^p \, d\mu \right)^{1/p} \leq \frac{2^{j+2}}{\varphi_2(r)} \mu(B^j)^{1/p} \leq 2^{j+2} \frac{3C_2}{\lambda^2} \mu(B^j)^{1/p}, \]
it follows from (155) that
\[ \frac{1}{2} \mu(B^{j+1})^{1/p^*} \leq 2^{j} \frac{12C_2}{\lambda^2} \mu(B^j)^{1/p}. \] (161)

At this stage we have an inequality which is analogous to the one displayed in (68). Thus, arguing as in (69)-(78) will yield the desired result. This finishes the proof of the theorem.

**Corollary 23.** Let \((X, d, \mu)\) be a uniformly perfect metric measure space where \(\mu(X) < \infty\), and fix \(s, p \in (0, \infty)\) such that \(p < s\). With \(p^* := sp/(s - p)\), the following statements are equivalent.

1. There exists a finite constant \(\kappa > 0\) such that
   \[ \kappa r^s \leq \mu(B(x, r)) \text{ for every } x \in X \]
   and every finite \(r \in (0, \text{diam}(X)]\). (162)

2. One has \(M^{1,p}(X, d, \mu) \subseteq L^{p^*}(X, \mu)\) and there exists a finite constant \(C_1 > 0\) satisfying
   \[ \|u\|_{p^*} \leq C_1 \|u\|_{M^{1,p}}, \quad \forall u \in M^{1,p}(X, d, \mu). \] (163)

3. There exists a finite constant \(C_2 > 0\) satisfying
   \[ \inf_{\gamma \in \mathbb{R}} \left( \int_X |u - \gamma|^{p^*} \, d\mu \right)^{1/p^*} \leq C_2 \left( \int_X g^p \, d\mu \right)^{1/p}, \] (164)
   whenever \(u \in M^{1,p}(X, d, \mu)\) and \(g \in D(u)\).

Consequently, in the context of uniformly perfect metric measure spaces having finite total measure, the global estimates in (163)-(164) are equivalent to the local estimates in (4)-(6) as well as the global Hölder condition in (7).

**Proof.** It is clear from Theorem 22 that both (2) and (3) (considered individually) imply that there exists \(\kappa \in (0, \infty)\) such that
\[ \kappa r^s \leq \mu(B(x, r)) \text{ for every } x \in X \text{ and every finite } r \in (0, 1]. \] (165)
The fact that (165) holds for every \(r \in (0, \text{diam}(X)]\) follows from the observation that if \(x \in X\) and \(1 < r \leq \text{diam}(X)\), then
\[ \mu(B(x, r)) \geq \mu(B(x, 1)) \geq \kappa \geq \frac{\kappa}{[\text{diam}(X)]^s} r^s. \]
Hence, the statement in (1) holds.
On the other hand, if \( \mu \) satisfies the lower measure bound in (162), then we necessarily have \( \text{diam}(X) < \infty \). Indeed, if on the contrary \( \text{diam}(X) = \infty \), then for any fixed \( x \in X \), there holds
\[
\infty = \lim_{n \to \infty} (\kappa n^s) \leq \lim_{n \to \infty} \mu(B(x, n)) = \mu\left(\bigcup_{n=1}^{\infty} B(x, n)\right) = \mu(X),
\]
which contradicts the fact that we are currently assuming that \( \mu(X) < \infty \). Given that \( \text{diam}(X) < \infty \), we can find a constant \( \tilde{\kappa} \in (0, \infty) \) (depending only on \( \kappa, s \), and the space \( X \)) so that \( \mu \) satisfies the \( V(2B_0, s, \tilde{\kappa}) \) condition (cf. (14)), where \( B_0 \subseteq X \) is any ball of radius \( R_0 := 2\text{diam}(X) \). Consequently, (163) and (164) now follow immediately from (15) and (16) in Theorem 4. This finishes the proof of the corollary. \( \square \)

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