Mean Field Theory for a driven Granular Gas of Frictional Particles

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Abstract

We propose a mean field (MF) theory for a homogeneously driven granular gas of inelastic particles with Coulomb friction. The model contains three parameters, a normal restitution coefficient $r_n$, a maximum tangential restitution coefficient $r_t^m$, and a Coulomb friction coefficient $\mu$. The parameters can be tuned to explore a wide range of physical situations. In particular, the model contains the frequently used $\mu \to \infty$ limit as a special case. The MF theory is compared with the numerical simulations of a randomly driven monolayer of spheres for a wide range of parameter values. If the system is far away from the clustering instability ($r_n \approx 1$), we obtain a good agreement between mean field and simulations for $\mu = 0.5$ and $r_t^m = 0.4$, but for much smaller values of $r_n$ the agreement is less good. We discuss the reasons of this discrepancy and possible refinements of our computational scheme.

Key words: Kinetic and transport theory of gases, Computational methods in fluid dynamics PACS: 47.50+d, 51.10.+y, 47.11.+j

1 Introduction

Granular gases [1] are usually described as collections of macroscopic particles with rough surfaces and dissipative interactions. In order to study them, kinetic theories [2–4] and numerical simulations [5] were applied for special boundary conditions. The dynamics of the system is assumed to be dominated...
by two-particle collisions, modeled by their asymptotic states: A collision is characterized by the velocities before and after the contact, and the contact is assumed to be instantaneous. In the simplest model, one describes inelastic collisions by normal restitution $r_n$ only. However, surface roughness is important [3,5], since it allows for an exchange of translational and rotational energy. Here we briefly sketch a study of a model where a Coulomb friction law with coefficient $\mu$ and a tangential restitution coefficient $r_t$ account for tangential inelasticity and friction [5,6]. We first introduce the model, then we describe a MF theory for the driven granular gas of rough spheres, and finally, we compare analytical results and numerical simulations. In the conclusions we discuss possible future refinements of the computational scheme.

2 The Model

We consider $N$ 3-dimensional spheres of mass $m$ and diameter $2a$ interacting via a hard-core potential, confined on a 2-dimensional (2D) layer of linear size $L$, with periodic boundary conditions. Inelasticity and roughness are described by a normal restitution $r_n$, a Coulomb friction law with friction $\mu$, and a tangential restitution $r_t$ which depends on $r_n$, $\mu$ and the collision angle $\gamma_c$ for sliding contacts and on a maximum tangential restitution $r_{tm}$ for sticking contacts, when the tangential elasticity becomes important. When two particles 1 and 2 collide, their velocities after collision depend on the velocities before collision through a collision matrix whose elements depend on $r_n$, $\mu$, $\gamma_c$, and $r_{tm}$. Thus we calculate the momentum change using a model that is consistent with experimental measurements [6]. From the momentum conservation laws for linear and angular direction, energy conservation, and Coulomb’s law we get the change of linear momentum of particle 1 as a function of $r_n$, $\mu$, and $r_t$ [5]. The change of the normal component of the relative velocity depends on the normal restitution $r_n$, which is a tunable parameter, while the change of the tangential component of the relative velocity depends on by the tangential restitution coefficient $r_t = \min\left[r_t^C, r_{tm}\right]$, where $r_{tm}$ is the coefficient of maximum tangential restitution, $-1 \leq r_{tm} \leq 1$. The quantity $r_t^C$ is determined using Coulomb’s law such that for solid spheres $r_t^C = -1 - (7/2)\mu(1 + r_n) \cot \gamma_c$ with the collision angle $\pi/2 < \gamma_c \leq \pi$ [5]. Here, we simplified the tangential contacts in the sense that exclusively Coulomb-type interactions, i.e. $\Delta P^{(t)}$ is limited by $\mu \Delta P^{(n)}$, or sticking contacts with the maximum tangential restitution $r_{tm}$ are allowed [5].

3 The mean field Theory

We start from the results of Huthmann and Zippelius [3] for a freely cooling gas of infinitely rough particles. Here we apply their MF theory to the case of a
driven gas of rough particles, which is the most common experimental situation [7]. The MF kinetic theory of Huthmann and Zippelius is formulated for a gas of rough particles with constant tangential restitution \( r_t \), corresponding to the limit \( \mu = \infty \) in our model. It is based on a pseudo–Liouville–operator formalism and on the assumption of a homogeneous state, with a Gaussian probability distribution of translational and rotational energies. The main outcome of this approach is a set of coupled evolution equations for the translational and rotational temperatures \( T_{tr} \) and \( T_{rot} \) [3]. Here, we write down and solve the MF equations for \( T_{tr} \) and \( T_{rot} \) for a granular gas of rough particles in which the translational velocities are subjected to a random uncorrelated Gaussian driving of variance \( \xi_0^2 \). These equations read for a 2D layer of spheres

\[
\frac{d}{dt} T_{tr}(t) = \frac{2}{D} \left[ -GAt_{tr}^{3/2} + GBT_{tr}^{1/2} T_{rot} \right] + m\xi_0^2
\]

\[
\frac{d}{dt} T_{rot}(t) = \frac{2}{2D-3} \left[ GBT_{tr}^{3/2} - GCT_{rot}^{1/2} T_{rot} \right]
\]

\[
G = 4an\sqrt{\frac{m}{\pi}}, \quad A = \frac{1-r_t^2}{4} + \frac{q}{2} (1 - \eta)
\]

\[
B = \frac{\eta}{2q}, \quad C = \frac{\eta}{2q} \left( 1 - \frac{\eta}{q} \right)
\]

where \( \eta = q(1 + r_t)/(2q + 2), q = 2/5 \) for spheres, \( n \) is the gas density and \( \chi \) is the pair correlation function at contact.

We can use the Verlet-Levesque [8] approximation in 2D \( \chi = (1 - 7\phi/16)/(1 - \phi)^2 \) where \( \phi \) is the volume fraction of the gas. For long times the system approaches a steady state. By imposing \( \frac{d}{dt} T_{eq} = 0, \frac{d}{dt} T_{eq}^{rot} = 0 \) we get the equilibrium temperatures

\[
T_{eq}^{tr} = m \left( \frac{\xi_0^2 \sqrt{\pi}}{2\gamma \Omega_D \chi na^{D-1}} \right)^{2/3}
\]

\[
T_{eq}^{rot} = \frac{R}{9 - 5r_t}
\]

where \( \Omega_D = 2\pi^{D/2}/\Gamma(D/2) = 2\pi \) for \( D = 2 \) and \( \gamma = \frac{1-r_t^2}{4} + \frac{49}{49} (1 + r_t)(6 - r_t) - (5/49) (1 + r_t)^2/(9 - 5r_t) \) for spherical particles. By linearizing the set of Eqs. (1) around \( T_{eq}^{tr} \) and \( T_{eq}^{rot} \) we get the final approach to the steady state: \( \delta T_{rot}(t) \simeq R\delta T_{tr}(t) \) and \( T_{eq}^{tr}(t) - T_{eq}^{rot} = \delta T_{tr}(t) \simeq \delta T_{tr}(0) \exp[-3\gamma \omega t] \). The quantity \( \omega = \Omega_D \chi na^{D-1}/\sqrt{\pi m} \) is the Enskog collision frequency for elastic particles at the temperature \( T_{eq}^{tr} \), and \( t_c = (3\gamma \omega)^{-1} \propto \gamma^{-2/3} \) is a characteristic relaxation time [4]. The MF Eqs. (1) can be applied to the three parameter model in the limit \( \mu = \infty \), see Fig. 1.

However, experimental measurements are well reproduced by this model only for finite \( \mu \). Here we investigate the possibility to describe the effect of finite friction \( \mu \) by replacing \( r_t \) in the Eqs. (1) with its average \( \langle r_t \rangle \) over the probability density \( P[\sin(\gamma_i)] \) of the normalized impact parameter \( b/(2a) = \sin(\gamma_i) \), with the condition \( r_t \leq r_t^{\text{m}} \). The used simplifying approach \( (\gamma_c \approx \gamma_r) \) ignores the fluctuations of \( r_t \) due to its dependence on the rotational degree of freedom, and we expect that it is correct only in some trivial limiting cases. Nevertheless, this simple approximation allows us to realize that for \( r_t^{\text{m}} = 0.4, \mu = \ldots \)
0.5, \( r_n \approx 0.9 \), corresponding to many experiments, simulations fit well with the modified MF theory, as we will see below. If the molecular chaos hypothesis is valid \( P[\sin(\gamma_t)] = 1 \), and we get \( \langle r_t \rangle = -1 + \frac{7}{2} \mu (1 + r_n) \ln(\sqrt{1 + c^2} + c) \), with \( c = \frac{2}{7} (1 + r_t^m) / (\mu (1 + r_n)) \).

4 The simulations

Here we compare numerical simulations of a randomly driven monolayer of spheres, performed by using an Event Driven (ED) algorithm [5], with the MF predictions (for details see [4,5,9,10]). Every simulation is equilibrated without driving with \( r_n = 1 \) and \( r_m^t = -1 \). Then inelasticity and driving are switched on. We used a fixed volume fraction \( \phi = 0.34 \), \( N = 11025 \) and different values of \( r_n, r_t^m, \mu \). In Fig. 1a-d, the translational and rotational temperatures for fixed \( r_n = 0.95, \mu = 10^7 \) and different values of \( r_t^m \) are rescaled with the MF equilibrium temperatures and plotted versus the rescaled time \( t/t_c \propto t \gamma^{2/3} \). The high value of \( \mu \) allows to decouple normal and tangential momentum \( (r_t = r_t^m) \). The agreement is good, even for low positive values of \( r_t^m \). For negative \( r_t^m \), agreement with MF is observed for the translational temperature, while the rotational temperature shows deviations from scaling in the transient phase, although the equilibrium value fits well with MF, for \( r_t^m \sim -1 \). This is due to a failure of the approximation \( \delta T_{rot}(t) \approx R \delta T_{tr}(t) \).
collapse is very good although deviations from MF are observed. This is a
we show simulation results for
and finite time. In Fig. 2a-b we plot the same quantities as in Fig. 1 but for weak coupling between rotational and translational degrees of freedom.

Fig. 2. (a)-(b) Rescaled translational and rotational temperatures \(T_{tr}(t)/T_{tr}^{eq}\), \(T_{rot}(t)/T_{rot}^{eq}\) versus the rescaled time \(tγ^{2/3}\), for \(r_n = 0.95, \mu = 0.5\).

Fig. 3. (a)-(b) Rescaled translational and rotational temperatures \(T_{tr}(t)/T_{tr}^{eq}\), \(T_{rot}(t)/T_{rot}^{eq}\) versus the rescaled time \(tγ^{2/3}\), for \(r_n = 0.95, r_t^m = 0.4\) and different \(μ\); (c)-(d) the same as (a)-(b) but for \(μ = 0.5, r_t^m = 0.4\) and different \(r_n\). for weak coupling between rotational and translational degrees of freedom and finite time. In Fig. 2a-b we plot the same quantities as in Fig. 1 but for \(r_n = 0.95, \mu = 0.5\). We obtain for \(r_t^m\) near to unity significant deviations from mean field theory, while for negative \(r_t^m\) the agreement with mean field theory is very good. The explanation for this results is that for \(r_t^m \sim -1\) and high enough \(μ\), one has \(\langle r_t \rangle \sim r_t^m\), and this correspond to have \(μ → \infty\). In Fig. 3a-b we show simulation results for \(r_n = 0.95\) and variable \(μ\) for \(r_t^m = 0.4\). These simulations confirm the previous interpretation. Finally, in Fig. 3c-d we show simulation results for \(μ = 0.5, r_t^m = 0.4\) and different \(r_n\). In this case the data collapse is very good although deviations from MF are observed. This is a
coincidence, since \( \langle r_t \rangle \simeq 0.363 \sim r^m_t \). To improve the MF theory for the finite \( \mu \) case, the effect of a random \( r_t \) must be fully taken into account. We actually are studying the possibility to include a collision angle dependent \( r_t \) in the ensemble average of the energy variation due to collisions [3]. The MF theory we obtain seems to be very promising and gives a good qualitative agreement with simulations. A paper is in preparation with the results of this study [10] and also a three-dimensional analysis is in progress [11].

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