CONVEXITY OF $\lambda$-HYPERSURFACES

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Abstract. We prove that any $n$-dimensional closed mean convex $\lambda$-hypersurface is convex if $\lambda \leq 0$. This generalizes Guang’s work on 2-dimensional strictly mean convex $\lambda$-hypersurfaces. As a corollary, we obtain a gap theorem for closed $\lambda$-hypersurfaces with $\lambda \leq 0$.

1. Introduction

A hypersurface $M^n$ in $\mathbb{R}^{n+1}$ is called a $\lambda$-hypersurface if it satisfies

\begin{equation}
H - \frac{\langle x, n \rangle}{2} = \lambda
\end{equation}

where $H$ is the mean curvature, $n$ is the outer unit normal of $M$, $x$ is the position vector, and $\lambda$ is a constant. This equation arises in the study of isoperimetric problems in weighted (Gaussian) Euclidean spaces (c.f. [MR15]), which is a long-standing topic studied in various fields in science ([Led94], [Bar01], [Bor03], etc.). Recently, Cheng and Wei [CW18] defined a weighted volume functional, and showed that the critical points of the functional under some weighted volume-preserving variations are exactly $\lambda$-hypersurfaces.

When $\lambda = 0$, $\lambda$-hypersurfaces are exactly self-shrinkers. Self-shrinkers play an important role in the study of mean curvature flow (MCF), since White [Whi97] and Ilmanen [Ilm95] showed that self-shrinkers arise as the tangent flows of MCF based on Huisken’s monotonicity formula [Hui90] and Brakke’s compactness theorem [Bra78]. Many classification results of self-shrinkers were proposed. Abresch and Langer [AL86] showed that the only 1-dimensional closed embedded self-shrinker is the circle $S^1$. Huisken [Hui90] later dealt with the higher-dimensional cases, proving that any closed, embedded, and mean convex (which means $H \geq 0$) $n$-dimensional self-shrinkers are exactly spheres $S^n$. For the non-compact situation, Huisken [Hui93] proved that all smooth, embedded, and mean convex self-shrinkers with polynomial volume growth and bounded second fundamental form are generalized cylinders $S^k \times \mathbb{R}^{n-k}$. This result was later improved by Colding and Minicozzi [CM12], in which they removed the condition of bounded second fundamental form in Huisken’s classification.

When $\lambda \neq 0$, there are relatively few and incomplete classification results so far. In [CW18], Cheng and Wei characterized compact $\lambda$-hypersurfaces with $H - \lambda \geq 0$ and some curvature conditions (c.f. theorem 5.1). Inspired by [SX20], Guang [Gua21] showed that any strictly mean convex (which means $H > 0$) 2-dimensional $\lambda$-hypersurfaces are in fact convex if $\lambda \leq 0$. The main goal of this paper is to generalize Guang’s result to higher-dimensional mean convex $\lambda$-hypersurfaces.

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Theorem 1.2. Let $M^n$ be a smooth, closed, and embedded $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda \leq 0$. If $M$ is mean convex, then it is convex.

This theorem is a generalization of Guang’s result in [Gua21]. Guang used the explicit expressions for the derivatives of the principal curvatures at the non-umbilical points of a surface, which were first derived in [HIMW19]. We follow the same spirit to derive a differential inequality for the sum of a part of principal curvatures at the points where there is a gap among some principal curvatures (c.f. lemma 3.1). Though we could not derive similar explicit expressions, it turns out that the information we derive is sufficient to obtain the higher-dimensional generalization. Also, we use the maximum principle to weaken the assumption of strict mean convexity in [Gua21], which Huisken [Hui90] also applied when classifying closed mean convex self-shrinkers.

A natural further question is whether Huisken-type classification also holds for $\lambda$-hypersurfaces. That is, could we classify all $\lambda$-hypersurfaces given some curvature conditions, like mean-convexity? In the curve case, Guang [Gua18] proved that any smooth embedded 1-dimensional $\lambda$-hypersurface (or $\lambda$-curve) is either a straight line or a circle if $\lambda \geq 0$, which generalized Abresch and Langer’s result. For the higher dimensional case, Heilman [Hei17] proved that convex $n$-dimensional $\lambda$-hypersurfaces are generalized cylinders if $\lambda \geq 0$. However, when $\lambda < 0$, Chang [Cha17] showed that for certain $\lambda < 0$, there are some closed embedded mean convex $\lambda$-curves other than circles. Thus we could not expect Huisken-type results to hold for general $\lambda \in \mathbb{R}$. We hope that theorem 1.2 will shed some light on the higher-dimensional case when $\lambda \leq 0$. In particular, using the curvature condition discovered in [CW18], we can prove the following gap theorem for mean convex $\lambda$-hypersurfaces when $\lambda \leq 0$.

Theorem 1.3. Let $M^n$ be a smooth, closed, and embedded $\lambda$-hypersurface in $\mathbb{R}^{n+1}$. If $\lambda \leq 0$ and the mean curvature of $M$ satisfies

$$0 \leq H \leq \frac{\sqrt{\lambda^2 + 2} + \lambda}{2},$$

then $M$ is a round sphere.

We remark that if we assume $M$ is a convex $\lambda$-hypersurface, then the result of theorem 1.3 could also be derived from the gap theorem proven by Guang [Gua18]. What’s new here is that we only need to assume $H \geq 0$, and then by theorem 1.2 we can get the convexity.

The organization of this paper is as follows. In section 2, we will introduce the Simons-type identities for $\lambda$-hypersurfaces, which was given by Guang in [Gua18]. In section 3, we derive a differential inequality for the sum of a part of principal curvatures at the points where there is a gap among some principal curvatures. In section 4, we use the identities and the inequality in the preceding sections to prove the main theorem 1.2. In section 5, we prove a gap theorem 1.3 by applying Cheng and Wei’s theorem.

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2. Simons-type Identities

On a hypersurface $M$ in $\mathbb{R}^{n+1}$, we consider the drift Laplacian
\[ \mathcal{L} := \Delta - \frac{1}{2} \nabla_{x^T} (\cdot) \]
and also the following linear operator
\[ L := \mathcal{L} + |A|^2 + \frac{1}{2} = \Delta - \frac{1}{2} \nabla_{x^T} (\cdot) + |A|^2 + \frac{1}{2} \]
where $\Delta$ and $A$ denote the Laplacian operator and the second fundamental form of $M$, and $x^T$ is the tangential component (with respect to $M$) of the position vector $x$. These operators were introduced by Colding and Minicozzi to study the stability of self-shrinkers. In fact, the operator $L$ appears in the second variation formula of the $F$-functional (c.f. [CM12]).

Guang [Gua18] established the following Simons-type identities. These identities will play a crucial role in the proof of the main theorem 1.2. We remark that these kinds of identities have been developed in [CM12] and [CM15] for self-shrinkers. For completeness, we include the proof in [Gua18] here.

Lemma 2.1 ([Gua18]). If $M$ is a $\lambda$-hypersurface in $\mathbb{R}^{n+1}$, then
\[ LA = A - \lambda A^2 \] (2.2)
and in particular, taking the trace of (2.2) gives
\[ LH = H + \lambda |A|^2. \] (2.3)

(Proof.) For any fixed $p \in M$, take a local orthonormal frame $\{e_i\}_{i=1,\ldots,n}$ such that $\nabla_{e_i} e_j = 0$ for all $i$ and $j$, where $\nabla^M$ is the Riemannian connection of $M$. Thus we can write $\nabla_{e_i} e_j = a_{ij} n$ where $a_{ij}$ is the component of the second fundamental form $A$. As a result,
\[ \text{Hess}_{(x,n)}(e_i, e_j) = \nabla_{e_j} \nabla_{e_i} \langle x, n \rangle = \nabla_{e_j} \sum_{l=1}^{n} \langle x, -a_{il} e_l \rangle \]
\[ = -a_{ij} + \sum_{l=1}^{n} (-a_{il} \langle x, e_l \rangle - a_{il} \langle x, a_{lj} n \rangle) \]
\[ = -A(e_i, e_j) - (\nabla_{x^T} A)(e_i, e_j) - \langle x, n \rangle A^2(e_i, e_j) \]
where $a_{il,j}$ is the component of $\nabla A$, and we use the Codazzi equation $a_{il,j} = a_{ij,l}$. In conclusion, we derive
\[ \text{Hess}_{(x,n)} = -A - \nabla_{x^T} A - \langle x, n \rangle A^2. \] (2.4)
Plug this into the Simons identity
\[ \Delta A = -|A|^2 A - HA^2 - \text{Hess}_H \] (2.5)
which holds for any hypersurface in \( \mathbb{R}^n \) (c.f. the formula (2.14) in [CM11]), and we get

\[
LA = \Delta A - \frac{1}{2} \nabla_x r(A) + |A|^2 A + \frac{1}{2} A
\]

\[
= A - \left( H - \frac{\langle x, n \rangle}{2} \right) A^2
\]

\[
= A - \lambda A^2
\]

based on the \( \lambda \)-hypersurface equation (1.1). (2.3) follows directly after taking the trace since \( \text{tr} A = -H \).

\[\square\]

3. Estimates of Principal Curvatures

In this section, we let \( M \) be a smooth mean convex hypersurface in \( \mathbb{R}^{n+1} \). Besides, we will write \( k_1 \leq \cdots \leq k_n \) to be the principal curvatures of \( M \) in the ascending order. For \( l \geq 1 \), consider

\[
S_l := \sum_{m=l+1}^{n} k_m,
\]

which is the sum of the largest \( n-l \) principal curvatures. In general, \( S_l \) is just a continuous function on \( M \). However, if \( k_l < k_{l+1} \) at a point \( p \in M \), the inverse function theorem will imply \( k_1 + \cdots + k_l \) and thus \( S_l \) are both differentiable near \( p \). At such a point, we establish the following differential inequality for \( S_l \).

**Lemma 3.1.** Suppose \( k_l < k_{l+1} \) for some \( l \geq 1 \) at a point \( p \in M \). Then at \( p \), we have

\[
(LS_l) \geq \frac{S_l}{2} - |A|^2 S_l + \lambda \sum_{m=l+1}^{n} k_m^2.
\]

**Proof.** We only need to consider those points near which we could take a principal frame \( \{v_1, \cdots, v_n\} \) such that

\[
k_i = -a_{ii} := -A(v_i, v_i)
\]

and

\[
a_{ij} := A(v_i, v_j) = 0 \text{ for } 1 \leq i \neq j \leq n.
\]

Such points form a dense and open set in \( M \) (c.f. [Sin75]), so after proving (3.2) at these points, it follows that (3.2) holds for all \( p \in \{k_l < k_{l+1}\} \) by continuity.

Now assume \( v_1, \cdots, v_n \) form a principal frame near \( p \). For any fixed \( i \), since \( \langle v_i, v_i \rangle = 1 \), we have

\[
\langle \nabla v_i, v_i \rangle = \frac{1}{2} \nabla v \langle v_i, v_i \rangle = 0
\]

for any local vector field \( v \). Hence we can write

\[
\nabla v_i v_m = \sum_{j \neq m} c^{mj}_i v_j
\]
for some smooth functions $c_{ij}^{mj}$ near $p$. Based on (3.10) and (3.6), near the point $p$, we have

$$\nabla_v k_m = -\nabla_v (A(v_m, v_m)) = -(\nabla_v A)(v_m, v_m) + 2A(\nabla_v v_m, v_m) = -(\nabla_v A)(v_m, v_m),$$

so

$$\Delta k_m = -\sum_{i=1}^{n} \nabla_v \nabla_v (A(v_m, v_m)) = -\sum_{i=1}^{n} \nabla_v ((\nabla_v A)(v_m, v_m))$$

$$= -(\Delta A)(v_m, v_m) - 2\sum_{i=1}^{n} (\nabla_v A)(\nabla_v v_m, v_m)$$

(3.7)

$$= -(\Delta A)(v_m, v_m) - 2\sum_{i=1}^{n} \sum_{j\neq m} c_{ij}^{mj} a_{jm,i}$$

by (3.6). To calculate the term involving the derivative of the second fundamental form, notice that for $j \neq m$, $A(v_j, v_m) = 0$, based on which we have

$$0 = \nabla_v (A(v_j, v_m))$$

$$= a_{jm,i} + A(\nabla_v v_j, v_m) + A(v_j, \nabla_v v_m)$$

$$= a_{jm,i} + A(\sum_{l\neq j} c_{lj}^{mj} v_l, v_m) + A(v_j, \sum_{l\neq m} c_{il}^{ml} v_l)$$

(3.8)

$$= a_{jm,i} - c_{ij}^{jm} k_m - c_{ij}^{mj} k_j$$

where we use the decomposition (3.6) and the relations (3.3) and (3.4). To get a more precise form, observe that the orthogonality condition $\langle v_j, v_m \rangle = 0$ implies

(3.9)

$$0 = \nabla_v \langle v_j, v_m \rangle = \langle \nabla_v v_j, v_m \rangle + \langle v_j, \nabla_v v_m \rangle = c_{ij}^{jm} + c_{ij}^{mj}$$

due to the orthonormality and (3.6). Putting (3.9) back into (3.8), we obtain

$$0 = a_{jm,i} + c_{ij}^{mj} (k_m - k_j),$$

with which we could simplify (3.7) as

(3.10)

$$\Delta k_m = -(\Delta A)(v_m, v_m) + 2\sum_{i=1}^{n} \sum_{j\neq m} (c_{ij}^{mj})^2 (k_m - k_j).$$

Now we apply the Simons-type identity (2.2), which gives

$$(\Delta A)(v_m, v_m) = \frac{1}{2}(\nabla_x A)(v_m, v_m) + \frac{1}{2}A(v_m, v_m) - |A|^2 A(v_m, v_m) - \lambda A^2 (v_m, v_m).$$

$$= -\frac{1}{2}\nabla_x k_m - \frac{1}{2}k_m + |A|^2 k_m - \lambda k_m^2.$$

Combining this with (3.10), we derive

$$\Delta k_m = \frac{1}{2}\nabla_x k_m + \frac{1}{2}k_m - |A|^2 k_m + \lambda k_m^2 + 2\sum_{i=1}^{n} \sum_{j\neq m} (c_{ij}^{mj})^2 (k_m - k_j).$$
As a result,
\[ \mathcal{L} k_m = \Delta k_m - \frac{1}{2} \nabla_x^T k_m = \frac{1}{2} k_m - |A|^2 k_m + \lambda k_m^2 + 2 \sum_{i=1}^n \sum_{j \neq m} (c_{ij}^m)^2 (k_m - k_j). \]

Therefore, summing over \( m \) from \( l+1 \) to \( n \) leads to
\[ \mathcal{L} S_l = \sum_{m=l+1}^n \mathcal{L} k_m = \frac{1}{2} \sum_{m=l+1}^n k_m - |A|^2 \sum_{m=l+1}^n k_m + \lambda \sum_{m=l+1}^n k_m^2 + 2 \sum_{m=l+1}^n \sum_{i=1}^n \sum_{j \neq m} (c_{ij}^m)^2 (k_m - k_j) \]
\[ = \frac{1}{2} S_l - |A|^2 S_l + \lambda \sum_{m=l+1}^n k_m^2 + 2 \sum_{m=l+1}^n \sum_{i=1}^n \sum_{j \neq m} (c_{ij}^m)^2 (k_m - k_j) \]
where some of the terms in the large sum get cancelled when \( m \) and \( j \) are switched since (3.9) implies
\[ (c_{ij}^m)^2 = (c_{ji}^m)^2 \]
for all \( j \neq m \). Then the inequality (3.2) follows since by our convention, \( k_m - k_j \geq 0 \) for all \( m > l \geq j \).

\[ \square \]

4. Proof of the Main Theorem

We are in a position to prove the main theorem 1.2 using lemma 2.1 and 3.1. We state the main theorem here again.

**Theorem 4.1.** Let \( M^n \) be a smooth, closed, and embedded \( \lambda \)-hypersurface in \( \mathbb{R}^{n+1} \) with \( \lambda \leq 0 \). If \( M \) is mean convex, then it is convex.

**(Proof.)** The case with \( \lambda = 0 \) directly follows from the classification of closed mean convex self-shrinkers, so we may assume \( M \) is a mean convex \( \lambda \)-hypersurface with \( \lambda < 0 \).

First we show that \( M \) is strictly convex. In fact, (2.3) implies
\[ \Delta H - \frac{1}{2} \nabla_x^T H + \left( |A|^2 - \frac{1}{2} \right) H = \lambda |A|^2 \leq 0. \]

Therefore, if \( H \) vanished at some points, the maximum principle would imply that \( H \equiv 0 \). Thus \( M \) would be planar, contradicting the assumption. Consequently we verify that \( M \) is strictly convex. That is, \( H > 0 \) on \( M \). In particular, \( S_l > 0 \) on \( M \) for all \( l \geq 1 \).

Next, we will prove the conclusion of the theorem by a contradiction argument. That is, assume there existed \( \overline{p} \in M \) such that \( k_1(\overline{p}) < 0 \). Then
\[ \frac{H}{S_1} = 1 + \frac{k_1}{S_1} \]
would attain its minimum at such point, say at \( p \). We can find \( l \geq 1 \) such that at this point \( p \),
\[ k_1 = \cdots = k_l < k_{l+1}. \]
We claim that at $p$, the function $\frac{H}{S_l}$ also attains its minimum. Otherwise, if $\frac{H(q)}{S_l(q)} < \frac{H(p)}{S_l(p)}$ for some $q \neq p$, which means
\[
\sum_{m=1}^{l} k_m(q) \frac{S_l}{H(q)} - \sum_{m=1}^{l} k_m(p) \frac{S_l}{H(p)} < 0,
\]
then after expanding the terms, we get
\[
H(p) \sum_{m=1}^{l} k_m(q) < H(q) \sum_{m=1}^{l} k_m(p) = H(q) \cdot l k_1(p).
\]
This particularly implies
\[
H(p) k_1(q) < H(p) \cdot \frac{1}{l} \sum_{m=1}^{l} k_m(q) < H(q) k_1(p),
\]
which then results in
\[
\frac{H(q)}{S_l(q)} = 1 + \frac{k_1(q)}{S_l(q)} < 1 + \frac{k_1(p)}{S_l(p)} = \frac{H(p)}{S_l(p)},
\]
contradicting the minimality of $\frac{H}{S_l}$ at $p$. Thus we prove that $\frac{H}{S_l}$ attains its minimum at $p$. Consequently, we have
\[
(4.2) \quad \mathcal{L} \left( \frac{H}{S_l} \right) \geq 0 \text{ and } \nabla \left( \frac{H}{S_l} \right) = 0
\]
at $p$, where $S_l$ is differentiable at $p$ since $k_1(p) < k_{l+1}(p)$. Note that \[23\] implies
\[
\mathcal{L} H = \frac{H}{2} - |A|^2 H + \lambda |A|^2.
\]
Combining this with lemma \[3.1\], we obtain that at $p$,
\[
\mathcal{L} \left( \frac{H}{S_l} \right) = \frac{S_l \mathcal{L} H - H L S_l}{S_l^2} - 2 \left\langle \nabla \left( \frac{H}{S_l} \right), \frac{\nabla S_l}{S_l} \right\rangle
\leq \frac{1}{S_l} \left( \frac{H}{2} - |A|^2 H + \lambda |A|^2 \right) - \frac{H}{S_l^2} \left( \frac{S_l}{2} - |A|^2 S_l + \lambda \sum_{m=l+1}^{n} k_m^2 \right)
\leq \frac{\lambda}{S_l} \left( |A|^2 - \frac{H}{S_l} \sum_{m=l+1}^{n} k_m^2 \right)
= \frac{\lambda}{S_l} \left( \sum_{i=1}^{n} k_i^2 - \left( 1 + \frac{l}{S_l} \frac{1}{S_l} \right) \left( \sum_{m=l+1}^{n} k_m^2 \right) \right)
= \frac{\lambda}{S_l} \left( \sum_{i=1}^{l} k_i^2 - \sum_{j=1}^{l} k_j \left( \sum_{m=l+1}^{n} k_m^2 \right) \right)
= \frac{\lambda k_1}{S_l} \left( k_1 - \frac{1}{S_l} \left( \sum_{m=l+1}^{n} k_m^2 \right) \right),
\]
which is negative since \( k_1(p) = \cdots = k_l(p) < 0 \). Thus we derive a contradiction with (1.2), and the conclusion of the theorem follows. \( \square \)

5. Gap theorem for Mean Convex \( \lambda \)-hypersurfaces

In [CW18], Cheng and Wei proved a rigidity theorem for \( \lambda \)-hypersurfaces under some curvature assumptions. Their result is an application of the arguments that Huisken applied in [Hui90] and [Hui93]. (Note that the definition of \( \lambda \)-hypersurfaces in [CW18] is different from that in this article by a constant. The sign convention of the second fundamental form in [CW18] is also different from ours.) We use the maximum principle to give a proof of the theorem here following the ideas in [Hui90]. In the mean convex case, we can use theorem [1.2] to derive a gap theorem when \( \lambda \leq 0 \).

**Theorem 5.1 ([CW18]).** Let \( M^n \) be a smooth, closed, and embedded \( \lambda \)-hypersurface in \( \mathbb{R}^{n+1} \). If \( H - \lambda \geq 0 \) and \( \lambda(2(H - \lambda)\text{tr} A^3 + |A|^2) \leq 0 \), then \( M \) is a round sphere.

**(Proof.)** By the maximum principle, we have \( H - \lambda > 0 \). Using (2.4), (2.5), and the \( \lambda \)-hypersurface equation (1.1), we can derive
\[
\Delta H = \frac{1}{2} H + \frac{1}{2} \nabla_{x^T} H - (H - \lambda)|A|^2
\]
and
\[
\Delta |A|^2 = 2|\nabla A|^2 + |A|^2 - 2|A|^4 + \frac{1}{2} \nabla_{x^T}|A|^2 - 2\lambda \text{tr} A^3.
\]
As a result,
\[
\Delta \left( \frac{|A|^2}{(H - \lambda)^2} \right) = \frac{\Delta |A|^2}{(H - \lambda)^2} - \frac{2|A|^2}{(H - \lambda)^3} \Delta H - \frac{4}{(H - \lambda)^3} \langle \nabla |A|^2, \nabla H \rangle + \frac{6|A|^2}{(H - \lambda)^4} |\nabla H|^2
\]
\[
= \frac{1}{(H - \lambda)^4} \left( 2(H - \lambda)^2|\nabla A|^2 + \frac{1}{2} H^2 \nabla_{x^T}|A|^2 - H|A|^2 \nabla_{x^T} H \right)
\]
\[
+ \frac{1}{(H - \lambda)^4} \left( -\lambda(H - \lambda) \left( 2(H - \lambda)\text{tr} A^3 + |A|^2 \right) - 4(H - \lambda) \langle \nabla |A|^2, \nabla H \rangle + 6|A|^2 |\nabla H|^2 \right).
\]
Plugging in
\[
|a_{ij} \nabla_i H - (H - \lambda) \nabla_i a_{ij}|^2 = |A|^2 |\nabla H|^2 + |\nabla A|^2 (H - \lambda)^2 - (H - \lambda) \langle \nabla H, \nabla |A|^2 \rangle
\]
and
\[
\nabla \left( \frac{|A|^2}{(H - \lambda)^2} \right) = \frac{\nabla |A|^2}{(H - \lambda)^2} - \frac{2|A|^2}{(H - \lambda)^3} \nabla H,
\]
we finally obtain
\[
\Delta \left( \frac{|A|^2}{(H - \lambda)^2} \right) = \frac{2}{(H - \lambda)^2} \left( |a_{ij} \nabla_i H - (H - \lambda) \nabla_i a_{ij}|^2 - \frac{1}{2} \lambda(H - \lambda) \left( 2(H - \lambda)\text{tr} A^3 + |A|^2 \right) \right)
\]
\[
+ \left( -\frac{2}{H - \lambda} \nabla H + \frac{x^T}{2}, \nabla \left( \frac{|A|^2}{(H - \lambda)^2} \right) \right).
\]
By our assumptions, we have
\[
|a_{ij} \nabla_i H - (H - \lambda) \nabla_i a_{ij}|^2 - \frac{1}{2} \lambda(H - \lambda) \left( 2(H - \lambda)\text{tr} A^3 + |A|^2 \right) \geq 0,
\]
so the maximum principle implies $|A|^2 = C(H - \lambda)^2$ for some constant $C$ and that

$$|a_{ij} \nabla_i H - (H - \lambda) \nabla_i a_{ij}|^2 - \frac{1}{2} \lambda (H - \lambda) \left(2(H - \lambda) \text{tr} A^3 + |A|^2\right) = 0.$$ 

In particular, we have

$$|a_{ij} \nabla_i H - (H - \lambda) \nabla_i a_{ij}|^2 = 0.$$ 

This tells us that the anti-symmetric part of this tensor also vanishes, which implies (5.2)

$$|a_{ij} \nabla_i H - a_{il} \nabla_j H|^2 = 0$$

by the Codazzi equation.

Now we assume $M$ is not a round sphere. Then we can find a point $p \in M$ at which $\nabla H \neq 0$. If we take a local frame $e_1, \ldots, e_n$ such that $e_1 = \frac{\nabla H}{|\nabla H|}$ at $p$, then (5.2) implies

$$|\nabla H|^2 \left(|A|^2 - \sum_{i=1}^n a_{1i}^2\right) = 0$$

at $p$. Since $\nabla H(p) \neq 0$, we get $|A|^2 - \sum_{i=1}^n a_{1i}^2 = 0$ at $p$. As a result,

$$\sum_{i=1}^n a_{1i}^2 = |A|^2 = \sum_{j=1}^n \sum_{k=1}^n a_{jk}^2,$$

which implies $a_{jk} = 0$ if $(j,k) \neq (1,1)$. Thus we get $|A|^2 = a_{11}^2 = H^2$. This along with the fact that $|A|/(H - \lambda)$ is constant implies that $H$ is constant, which leads to a contradiction since we assume $M$ is not a round sphere. □

We remark that when $\lambda = 0$, the calculations above reduce to those in [Hui90]. Now we can use theorem 5.1 to prove our gap theorem 1.3 for mean convex $\lambda$-hypersurfaces. We state theorem 1.3 here again.

**Theorem 5.3.** Let $M^n$ be a smooth, closed, and embedded $\lambda$-hypersurface in $\mathbb{R}^{n+1}$. If $\lambda \leq 0$ and the mean curvature of $M$ satisfies

$$0 \leq H \leq \frac{\sqrt{\lambda^2 + 2} + \lambda}{2},$$

then $M$ is a round sphere.

**(Proof.)** By theorem 1.2 we know that $M$ is convex. That is, $k_i \geq 0$ for all $i = 1, \ldots, n$. In particular, this implies

$$H|A|^2 = \left(\sum_{i=1}^n k_i\right) \cdot \left(\sum_{j=1}^n k_j^2\right) \geq \sum_{m=1}^n k_m^3 = -\text{tr} A^3.$$

On the other hand, by the upper bound of $H$, we can conclude that $1 \geq 2(H - \lambda)H$. Combining these gives

$$|A|^2 \geq 2(H - \lambda)H|A|^2 \geq -2(H - \lambda)|A|^2,$$

which implies $\lambda(2(H - \lambda)|A|^2 + |A|^2) \leq 0$. Applying theorem 5.1, the conclusion follows. □
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