NORMAL FORMS OF PLANAR SWITCHING SYSTEMS

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Abstract. In this paper we study normal forms of planar differential systems with a non-degenerate equilibrium on a single switching line, i.e., the equilibrium is a non-degenerate equilibrium of both the upper system and the lower one. In the sense of $C^0$ conjugation we find all normal forms for linear switching systems and use them together with switching near-identity transformations to normalize second order terms, showing the reduction of normal forms. We prove that only one of those 19 types of linear normal form decides if the system is monodromic. With the monodromic linear normal form, we compute the second order monodromic normal form, which gives a condition under which exactly one limit cycle arises from a Hopf bifurcation.

1. Introduction. Among those nonsmooth systems ([4, 12, 19]), an important class is the switching system, which contains at least a switching line (or switching curve) such that the system has different definitions of continuous vector field in the two different regions divided by the line. Since a large number of mathematical models of such a form were raised from mechanics ([5, 6]), electrical engineering ([2, 3]) and automatic control ([13, 26, 27]), many contributions have been made to analyze those models for stability ([14, 25]), limit cycles ([11, 15, 18, 23, 28]), isochronicity of centers ([8, 10, 22]) and bifurcations ([16, 21]).

While discussing on bifurcations of a smooth system, one usually reduces the system to its Poincaré normal form ([9, 20]) in the equivalent sense of conjugacy. The idea is to utilize the Jordan form of the linear part, simplified by an invertible linear transformation, to make a near-identity transformation so that the system is changed into the simplest form containing resonant terms only. A similar theory of normal forms is also needed for switching systems but difficulties come from even the linear part, not mentioning the nonlinear terms. In fact, a switching system has more than one linear parts, which exist separately in different regions divided by switching lines. We need to simplify those matrices by either the same invertible linear transformation or a continuous invertible transformation of different linear representations in those regions.

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In this paper we study normal forms for planar piecewise-smooth differential system

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
(X^+(x, y), Y^+(x, y)), & \text{if } y > 0, \\
(X^-(x, y), Y^-(x, y)), & \text{if } y < 0,
\end{cases}
\]

with a single switching line \( y = 0 \), where \( X^\pm(x, y), Y^\pm(x, y) \) are analytic and vanish at \( (0, 0) \). This system with an equilibrium \( O : (0, 0) \), which lies persistently on the switching line, is an important class of switching systems such as the car brake system studied in \([28]\), where continuous piecewise-smooth ODEs were considered and the existence and stability of a cycle were established but no explicit formula was given for the direction of the cycle bifurcation. Some interesting works (see, e.g., \([18, 24]\)) about Hopf bifurcations for piecewise-smooth systems were done in the case that the equilibrium moves apart from the origin as parameters vary. Here we always restrict the equilibrium at the origin and call the first (second) equation, defined for \( y > 0 \) (for \( y < 0 \)), the upper (lower) system. Flows of the switching system are defined piecewise at \( y > 0 \), \( y = 0 \), \( y < 0 \) known as the Filippov structure without sliding on the switching line in \([4, 12, 19]\). We expand the system as

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
A^+ \begin{pmatrix} x \\ y \end{pmatrix} + \left( \frac{a^{+20} x^2 + a^{+11} x y + a^{+02} y^2}{b^{+20} x^2 + b^{+11} x y + b^{+02} y^2} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
A^- \begin{pmatrix} x \\ y \end{pmatrix} + \left( \frac{a^{-20} x^2 + a^{-11} x y + a^{-02} y^2}{b^{-20} x^2 + b^{-11} x y + b^{-02} y^2} \right) + O(|(x, y)|^3), & \text{if } y < 0,
\end{cases}
\]

where

\[
A^+ := \begin{pmatrix} A^{+11} & A^{+12} \\ A^{+21} & A^{+22} \end{pmatrix} \quad \text{and} \quad A^- := \begin{pmatrix} A^{-11} & A^{-12} \\ A^{-21} & A^{-22} \end{pmatrix},
\]

near the origin \( O \) and show our simplification to the linear terms and the second order terms in the case that the equilibrium \( O \) is non-degenerate, i.e., the matrices \( A^\pm \) are both non-singular. We construct continuous piecewise-linear transformations for conjugation and prove in section 2 that system (1) has 19 types of linear normal forms totally. All orbit structures are persistent in normal forms including the possible sliding motions because of topological equivalence. Further, we prove that only one of those 19 types of linear normal form decides system (1) to be monodromic (\([1]\)), i.e., no orbits approaching \( O \) in a definite direction as \( t \to \pm \infty \). In section 3 we show our normalization with second order terms and normalized linear terms, constructing a near-identity transformation with switching to eliminate some second order terms for the second order normal form of system (1). In section 4 we show how the monodromic linear normal form can be used to simplify the above obtained second order normal form further and give the second order monodromic normal form. Since our reductions made in sections 2, 3 and 4 are given with homeomorphisms preserving the switching line invariant, the Filippov structure without sliding on the switching line remains in the reduced normal forms. Such a reduction of normal forms is applied in section 5 to a general monodromic switching system, which was investigated in \([28, \text{Theorem 4.4}]\) for existence of a limit cycle. We give a quantity the sign of which determines the rise of exactly one limit cycle and the direction of the Hopf bifurcation. In section 6 we give an example to display the reduction of normal form and illustrate our discussion of Hopf bifurcations, and another example to show the critical performance of the second order normal form.
2. Normalization of linear terms. We need to simplify the linear part at first, i.e., to give a ‘linear normal form’ for system (1). For this purpose we need to simplify the two matrices $A^{\pm}$ on the two sides of the switching line $y = 0$ by a continuous and invertible piecewise-linear transformation.

\[
\begin{pmatrix} u \\ v \end{pmatrix} = P(x, y) = \begin{cases} 
\begin{pmatrix} P_1^+ & P_2^+ \\ P_4^+ & P_3^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: P^+(x, y), & \text{if } y \geq 0, \\
\begin{pmatrix} P_1^- & P_2^- \\ P_4^- & P_3^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: P^-(x, y), & \text{if } y \leq 0.
\end{cases}
\tag{2}
\]

Without loss of generality, we assume that the transformation $P$ preserves the switching line $y = 0$ invariant. The continuity at points on the line $y = 0$ and the invertibility imply the following:

Lemma 2.1. $P$ defined in (2) is a homeomorphism on $\mathbb{R}^2$ if and only if $P_4^\pm = 0$, $P_1^+ = P_1^- \neq 0$ and $P_3^+ P_3^- > 0$.

Proof. $P$ is continuous at any point $(x, 0)$ on the switching line $y = 0$ if and only if $P^\pm$ satisfy that

\[
P^+(x, 0) = \begin{pmatrix} u \\ 0 \end{pmatrix} = P^-(x, 0)
\]

for a certain $u \in \mathbb{R}$. It follows that $P_4^\pm x = 0$ and $P_1^+ x = P_1^- x$ for all $x$, implying that $P_4^\pm = 0$ and $P_1^+ = P_1^-$ respectively. Therefore,

\[
P^+ = \begin{pmatrix} P_1 & P_2^+ \\ 0 & P_3^+ \end{pmatrix}, \quad P^- = \begin{pmatrix} P_1 & P_2^- \\ 0 & P_3^- \end{pmatrix},
\]

where $P_1 = P_1^\pm$ and $P_1 P_3^+ P_3^- \neq 0$ because of the invertibility. Further, the above expression of $P^\pm$ shows that $v = P_3^\pm y$ when $y > 0$ and $v = P_5^\pm y$ when $y < 0$. If $P_3^+ P_3^- < 0$, then points on the upper half-plane $y > 0$ and points on the lower half-plane $y < 0$ are both mapped by $P$ into the same half-plane, a contradiction to the homeomorphism of $P$. Hence, $P_3^+ P_3^- > 0$.

Having this lemma, we give the following linear normal forms.

Theorem 2.2. The linear part of the switching system (1) has the following 19 types of normal form

\[
N_{ij} : \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} 
N_i^+(x, y), & \text{if } y > 0, \\
N_j^-(x, y), & \text{if } y < 0,
\end{cases} \quad (i, j) \in \Delta,
\]

where $\Delta := \{(1, n) : n = 1, 2, 3, 6, 7\} \cup \{(m, n) : m \leq n, m = 2, 3, 6, 7, n = 2, \ldots, 7\}$,

\[
N_1^\pm := \begin{pmatrix} \lambda^\pm & 0 \\ 0 & \mu^\pm \end{pmatrix}, \quad N_2^\pm := \begin{pmatrix} \lambda^\pm & 1 \\ 0 & \lambda^\pm \end{pmatrix}, \quad N_3^\pm := \begin{pmatrix} \lambda^\pm & 0 \\ 1 & \lambda^\pm \end{pmatrix}, \quad N_4^\pm := \begin{pmatrix} \lambda^\pm & -1 \\ 0 & \lambda^\pm \end{pmatrix},
\]

\[
N_5^\pm := \begin{pmatrix} \lambda^\pm & 0 \\ -1 & \lambda^\pm \end{pmatrix}, \quad N_6^\pm := \begin{pmatrix} \alpha^\pm & \beta^\pm \\ \beta^\pm & \alpha^\pm \end{pmatrix}, \quad N_7^\pm := \begin{pmatrix} \alpha^\pm & \beta^\pm \\ -\beta^\pm & \alpha^\pm \end{pmatrix}
\]

and $\lambda^\pm, \mu^\pm, \alpha^\pm, \beta^\pm \in \mathbb{R}$ with $\beta^\pm \neq 0$. 

Remark 1. As shown in Theorem 2.2, in $N_1^\pm$, $\lambda^+$ and $\mu^+$ are eigenvalues of $A^+$ and $\lambda^-$ and $\mu^-$ are eigenvalues of $A^-$; in $N_i^\pm$ (i = 2, ..., 5), $\lambda^\pm$ are eigenvalues of $A^\pm$ respectively; in $N_6^\pm$, $\alpha^+ \pm i\beta^+$ are eigenvalues of $A^+$ and $\alpha^- \pm i\beta^-$ are eigenvalues of $A^-$. Each matrix in those cases is of a real Jordan canonical form no matter whether it appears in the upper system or the lower one. In contrast, neither of $N_6^\pm$ is of a Jordan canonical form, which comes from the switching because switching requires the transformation matrices $P^+$ and $P^-$ to be upper triangular matrices. Clearly, in case of $N_6^\pm$ the sums $\alpha^+ \pm \beta^+$ are eigenvalues of $A^+$, the sums $\alpha^- \pm \beta^-$ are eigenvalues of $A^-$, $\alpha^\pm = \text{tr}A^\pm/2$ and $\beta^\pm = \sqrt{(\text{tr}A^\pm)^2 - 4\det A^\pm}/2$. If $A^\pm$ has two real eigenvalues but two different eigenvectors correspondingly, then $A^\pm$ can be normalized as $N_1$ (resp. $N_6$) if the entry $A_{21}^\pm = 0$ (resp. $A_{21}^\pm \neq 0$).

Remark 2. $N_2^\pm$ is similar to $N_3^\pm$, i.e., $N_2^\pm M = MN_3^\pm$, by a nonsingular transformation

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$ 

A simple computation shows that $M_{12} = M_{21}$ and $M_{22} = 0$, but, by Lemma 2.1, the restriction to $P^+$ does not allow such a choice of $M$. Therefore, in Theorem 2.2, $N_2^\pm$ and $N_3^\pm$ are regarded as different normal forms in switching systems. For the same reason, we differ $N_4^\pm$ from $N_6^\pm$ and $N_6^\pm$ from $N_1^\pm$ in switching systems.

Proof. By Lemma 2.1, $P_3^+ P_5^- > 0$. Thus, either $P_3^+ > 0$ and $P_5^- > 0$ or $P_3^+ < 0$ and $P_5^- < 0$. First, we give those linear normal forms reduced by transformations defined in (2) with $P_3^+ > 0$ and $P_5^- > 0$, which preserves the upper half plane $y > 0$ and the lower one $y < 0$. The transformation (2) with $P_3^+ > 0$ and $P_5^- > 0$ changes the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^+ \begin{pmatrix} x \\ y \end{pmatrix}, & \text{if } y > 0, \\ A^- \begin{pmatrix} x \\ y \end{pmatrix}, & \text{if } y < 0, \end{cases} \quad (3)$$

the linear part of system (1), into the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{cases} P^+ A^+ (P^+)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} =: B^+ \begin{pmatrix} u \\ v \end{pmatrix}, & \text{for } v > 0, \\ P^- A^- (P^-)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} =: B^- \begin{pmatrix} u \\ v \end{pmatrix}, & \text{for } v < 0, \end{cases}$$

where

$$B^\pm := \frac{1}{P_1 P_3^\pm} \begin{pmatrix} P_1 P_3^\pm A_{11}^\pm + P_2^\pm P_3^\pm A_{41}^\pm & P_1^2 A_{12}^\pm + P_1 P_2^\pm (A_{22}^\pm - A_{11}^\pm) - P_2^\pm A_{22}^\pm A_{41}^\pm \\ P_1^2 A_{12}^\pm + P_1 P_2^\pm (A_{22}^\pm - A_{11}^\pm) - P_2^\pm A_{22}^\pm A_{41}^\pm & P_1 P_2^\pm A_{22}^\pm - P_2^\pm A_{12}^\pm A_{22}^\pm \end{pmatrix}. \quad (4)$$

In order to simplify $B^\pm$ further, we focus on the back-diagonal entries and classify all possibilities of matrix $A^\pm$ as

$$\begin{align*}
A_1^+ & : A_{21}^+ = 0, A_{12}^+ = 0; \\
A_2^+ & : A_{21}^+ = 0, A_{12}^+ \neq 0, A_{11}^+ \neq A_{22}^+; \\
A_3^+ & : A_{21}^+ = 0, A_{12}^+ \neq 0, A_{11}^+ = A_{22}^+; \\
A_4^+ & : A_{21}^+ = 0, A_{12}^+ \neq 0, A_{11}^+ \neq A_{22}^+.
\end{align*}$$
Let \( \Gamma_{ij} \) denote the case that \( A^+ \) satisfies \( \Lambda^+_{ij} \) and \( A^- \) satisfies \( \Lambda^-_{ij} \), where \( i = 1, \ldots, 6 \) and \( j = 1, \ldots, 6 \). When system (3) lies in case \( \Gamma_{11} \), from (4) we get

\[
B^\pm = \frac{1}{P_1 P_3^\pm} \begin{pmatrix} P_1 P_3^\pm A_{11}^+ & P_1 P_3^\pm (A_{22}^+ - A_{11}^+) \\ 0 & P_1 P_3^\pm A_{22}^+ \end{pmatrix},
\]

which are of the form \( N_1^\pm \) (shown on the second row of Table 1) respectively by choosing \( P_2^+ = 0 \) in the transformation. When system (3) lies in case \( \Gamma_{12} \), from (4) we get

\[
B^\pm = \frac{1}{P_1 P_3^\pm} \begin{pmatrix} P_1 P_3^\pm A_{11}^+ & 0 \\ 0 & P_1 P_3^\pm (A_{22}^+ - A_{11}^+) \end{pmatrix},
\]

which are of form \( N_2^\pm \) (shown on the third row of Table 1) respectively by choosing \( P_2^+ = 0 \) and \( P_2^- = P_1 A_{12}/(A_{11} - A_{22}) \) in the transformation. Similarly, we consider cases \( \Gamma_{ij} \) for all \( i = 1, 2, 3, 6 \), \( j = 1, \ldots, 6 \) and for all \( i = 3, 4 \), \( j = 1, 2, 5, 6 \). The obtained normal forms are summarized in Tables 1 and 2.

In contrast, the cases \( \Gamma_{ij} \), where \( i = 3, 4 \) and \( j = 3, 4 \) need additional conditions. In case \( \Gamma_{13} \), a simple computation shows that the upper system of (3) can be normalized to \( N_3^+ \) by choosing \( P_3^+ = P_1 A_{12} > 0 \) and the lower one of (3) can be normalized to \( N_3^- \) by choosing \( P_3^- = P_1 A_{12} > 0 \) separately (see the second row of Table 3). Thus, system (3) can be reduced to the normal form \( N_{22} \), i.e., the upper system is reduced to \( N_3^+ \) meanwhile the lower system is reduced to \( N_3^- \), if and only if both \( P_3^+ > 0 \) are required in the beginning of the proof. To the opposite, i.e., \( A_{12}^+ A_{12}^- < 0 \), \( N_3^+ \) is not compatible with \( N_3^- \) but system (3) can be reduced to the normal form \( N_{24} \) by choosing \( P_3^+ = P_1 A_{12} > 0 \) and \( P_3^- = -P_1 A_{12} > 0 \) in the transformation (see the third row of Table 3).

Similarly to case \( \Gamma_{13} \), in case \( \Gamma_{34} \) a simple computation shows that the upper system of (3) can be normalized to \( N_3^+ \) by choosing \( P_3^+ = P_1 A_{12} > 0 \) and the lower one of (3) can be normalized to \( N_3^- \) by choosing \( P_3^- = P_1 (A_{22}^+ - A_{11})/(2A_{21}) \), \( P_3^- = P_1 A_{21} > 0 \) separately (see the fourth row of Table 3). Thus, system (3) can be reduced to the normal form \( N_{23} \), i.e., the upper system is reduced to \( N_3^+ \) meanwhile the lower system is reduced to \( N_3^- \), if and only if both \( P_3^+ > 0 \) are required in the beginning of the proof. To the opposite, i.e., \( A_{12}^+ A_{21}^- < 0 \), \( N_3^+ \) is not compatible with \( N_3^- \) but system (3) can be reduced to the normal form \( N_{22} \) by choosing \( P_3^+ = P_1 A_{12} > 0 \) and \( P_3^- = P_1 (A_{22}^+ - A_{11})/(2A_{21}) \), \( P_3^- = -P_1 A_{21} > 0 \) in the transformation (see the fifth row of Table 3).

Similarly to case \( \Gamma_{33} \), in case \( \Gamma_{13} \) a simple computation shows that the upper system of (3) can be normalized to \( N_3^+ \) by choosing \( P_3^+ = P_1 (A_{22}^+ - A_{11})/(2A_{21}) \), \( P_3^- = \ldots \)
\( P_i/A_{21}^+ > 0 \) and the lower one of (3) can be normalized to \( N_3^- \) by choosing \( P_3^- = P_iA_{12} > 0 \) separately (see the 6th row of Table 3). Thus, system (3) can be reduced to the normal form \( N_3 \), i.e., the upper system is reduced to \( N_3^+ \) meanwhile the lower system is reduced to \( N_3^- \), if and only if \( A_{21}^- A_{12}^- > 0 \) because both \( P_3^+ \) and \( P_3^- \) are required in the beginning of the proof. To the opposite, i.e., \( A_{21}^- A_{12}^- < 0 \), \( N_3^+ \) is not compatible with \( N_3^- \) but system (3) can be reduced to the normal form \( N_3 \) by choosing \( P_3^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-) \), \( P_3^+ = P_iA_{21}^+ > 0 \) and \( P_3^- = -P_iA_{12}^- > 0 \) in the transformation (see the 7th row of Table 3).

Similarly to case \( \Gamma_{33} \), in case \( \Gamma_{14} \) a simple computation shows that the upper system of (3) can be normalized to \( N_7^+ \) by choosing \( P_2^+ = P_i(A_{22}^+ - A_{11}^-)/(2A_{21}^+), P_2^+ = P_iA_{21}^+ > 0 \) and the lower one of (3) can be normalized to \( N_7^- \) by choosing \( P_2^- = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_2^- = P_iA_{21}^- > 0 \) separately (see the 8th row of Table 3).

| Case | NF of \( A^+ \) | NF of \( A^- \) | Transformation |
|------|-----------------|-----------------|---------------|
| \( \Gamma_{11} \) | \( N_3^+ \) | \( N_3^- \) | \( P_2^+ = 0, P_2^- = P_iA_{12}^-/(A_{11}^- - A_{22}^-) \) |
| \( \Gamma_{12} \) | \( N_3^+ \) | \( N_1^+ \) | \( P_2^+ = 0, P_2^- = P_iA_{12}^-/(A_{11}^- - A_{22}^-) \) |
| \( \Gamma_{13} \) | \( N_3^+ \) | \( N_2^- \) | \( P_2^+ = 0, P_2^- = P_iA_{12}^- > 0 \) |
| \( \Gamma_{14} \) | \( N_3^+ \) | \( N_3^- \) | \( P_2^+ = 0, P_2^- = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_3^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{15} \) | \( N_3^+ \) | \( N_6^+ \) | \( P_2^+ = 0, P_2^- = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_3^- = P_i/(4detA^- - (A_{11}^- + A_{22}^-)^2)/(2A_{21}^-) \) |
| \( \Gamma_{16} \) | \( N_3^+ \) | \( N_7^+ \) | \( P_2^+ = 0, P_2^- = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_3^- = P_i/(4detA^- - (A_{11}^- + A_{22}^-)^2)/(2A_{21}^-) \) |
| \( \Gamma_{21} \) | \( N_3^- \) | \( N_1^- \) | \( P_2^+ = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_3^- = 0 \) |
| \( \Gamma_{22} \) | \( N_3^- \) | \( N_1^- \) | \( P_2^+ = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_3^- = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{23} \) | \( N_3^- \) | \( N_2^- \) | \( P_2^+ = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{24} \) | \( N_3^- \) | \( N_3^- \) | \( P_2^+ = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{25} \) | \( N_3^- \) | \( N_6^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_2^- = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_3^- = P_i/(4detA^- - (A_{11}^- + A_{22}^-)^2)/(2A_{21}^-) \) |
| \( \Gamma_{26} \) | \( N_3^- \) | \( N_7^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_2^- = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_3^- = P_i/(4detA^- - (A_{11}^- + A_{22}^-)^2)/(2A_{21}^-) \) |
| \( \Gamma_{51} \) | \( N_6^- \) | \( N_1^- \) | \( P_2^+ = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{52} \) | \( N_6^- \) | \( N_1^- \) | \( P_2^+ = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{53} \) | \( N_6^- \) | \( N_2^- \) | \( P_2^+ = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_2^- = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{54} \) | \( N_6^- \) | \( N_3^- \) | \( P_2^+ = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{55} \) | \( N_6^- \) | \( N_6^- \) | \( P_2^+ = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{56} \) | \( N_6^- \) | \( N_7^- \) | \( P_2^+ = P_iA_{21}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{21}^- > 0 \) |
| \( \Gamma_{61} \) | \( N_7^- \) | \( N_1^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_4^- = 0 \) |
| \( \Gamma_{62} \) | \( N_7^- \) | \( N_1^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_4^- = 0 \) |
| \( \Gamma_{63} \) | \( N_7^- \) | \( N_2^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_4^- = 0 \) |
| \( \Gamma_{64} \) | \( N_7^- \) | \( N_3^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_4^- = 0 \) |
| \( \Gamma_{65} \) | \( N_7^- \) | \( N_6^- \) | \( P_2^+ = P_i(A_{22}^- - A_{11}^-)/(2A_{21}^-), P_4^- = 0 \) |
| \( \Gamma_{66} \) | \( N_7^- \) | \( N_7^- \) | \( P_2^+ = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_4^- = P_iA_{12}^-/(A_{11}^- - A_{22}^-), P_5^- = P_iA_{21}^- > 0 \) |

Table 1. Normal forms for \( \Gamma_{ij}, i = 1, 2, 5, 6, j = 1, ..., 6 \)
TABLE 2. Normal forms for $\Gamma_{ij}$ and $\Gamma_{4j}$, $j = 1, 2, 5, 6$

Thus, system (3) can be reduced to the normal form $N_{33}$, i.e., the upper system is reduced to $N_3^+$ meanwhile the lower system is reduced to $N_3^-$, if and only if $A_{21}^+ A_{21}^- > 0$ because both $P_3^+ > 0$ are required in the beginning of the proof. To the opposite, i.e., $A_{21}^+ A_{21}^- < 0, N_3^-$ is not compatible with $N_3^+$ but system (3) can be reduced to the normal form $N_{35}$ by choosing $P_2^+ = P_1(A_{22}^+ - A_{11}^-)/(2A_{21}^-), P_2^- = P_1/A_{21}^- > 0$ and $P_3^+ = P_1(A_{22}^- - A_{11}^+)/(2A_{21}^+), P_3^- = -P_1/A_{21}^+ > 0$ in the transformation (see the 9th row of Table 3).

TABLE 3. Normal forms for $\Gamma_{ij}$, where $i = 3, 4$, $j = 3, 4$

As shown in Tables 1, 2 and 3, there are totally 29 normal forms $N_{ij}$, where

$(i, j) \in \{(m, n) : m = 1, 6, 7, n = 1, 2, 3, 6, 7\} \cup \{(m, n) : m = 2, 3, n = 1, \ldots, 7\}$. (5)

Using transformations defined in (2) with $P_4^+ < 0$ and $P_4^- < 0$, which exchange the upper half plane $y > 0$ with the lower one $y < 0$, we obtain the same linear normal forms $N_{ij}$ (for all $(i, j)$ satisfying (5)) as given for $P_3^+ > 0$ and $P_3^- > 0$. Finally we delete $N_{ji}$ for all $j > i$ because it is equivalent to $N_{ij}$ by the transformation $(u, v) := (x, -y)$. In fact,
\[ = \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}, & \text{if } v > 0, \\ \begin{pmatrix} u \\ v \end{pmatrix}, & \text{if } v < 0, \end{cases} \]

which is of form \( N_{ij} \). For this reason, the linear part of the switching system (1) has totally 19 independent normal forms \( N_{ij}, \ (i, j) \in \Delta \).

The equivalent complex form of system (3) is
\[
\dot{z} = \begin{cases} a_{10}^+ z + a_{01}^+ \bar{z}, & \text{if } \Im(z) > 0, \\ a_{10}^- z + a_{01}^- \bar{z}, & \text{if } \Im(z) < 0, \end{cases}
\]
where \( z = x + iy \) and \( a_{ij}^\pm \in \mathbb{C} \). Let \((\alpha_1, \alpha_2) \cdot (z_1, z_2)\) denote the inner product of the complex vectors \((\alpha_1, \alpha_2)\) and \((z_1, z_2)\) in \( \mathbb{C}^2 \). By Theorem 2.2 we also obtain linear normal forms in the complex form.

**Corollary 1.** Equation (6) has 19 normal forms
\[
\tilde{N}_{ij} : \dot{z} = \begin{cases} \tilde{N}_{ij}^+ \cdot (z, \bar{z}), & \text{if } \Im(z) > 0, \\ \tilde{N}_{ij}^- \cdot (z, \bar{z}), & \text{if } \Im(z) < 0, \end{cases}
\]
where \((i, j) \in \Delta, \tilde{N}_{ij}^\pm := ((\lambda^\pm + \mu^\pm)/2, (\lambda^\pm - \mu^\pm)/2), \tilde{N}_{ij}^\pm := (\lambda^\pm - i/2, i/2), \tilde{N}_{ij}^\pm := (\lambda^\pm + i/2, -i/2), \tilde{N}_{ij}^\pm := (\alpha^\pm + i\beta^\pm), \text{ and } \tilde{N}_{ij}^\pm := (\alpha^\pm - i\beta^\pm, 0).\)

**Proof.** Transformation (2) corresponds to the complex transform
\[
z \rightarrow \begin{cases} g_{10}^+ z + g_{01}^+ \bar{z}, & \text{if } \Im(z) \geq 0, \\ g_{10}^- z + g_{01}^- \bar{z}, & \text{if } \Im(z) \leq 0, \end{cases}
\]
where \( g_{ij}^\pm \)s satisfy \( g_{10}^+ + g_{10}^- = g_{01}^+ + g_{01}^- \in \mathbb{R} \setminus \{0\} \) by Lemma 2.1. For each \((i, j) \in \Delta\) one can check that the complex equation with \( \tilde{N}_{ij} \) corresponds to the real system with \( N_{ij} \). Thus, by Theorem 2.2, equation (6) has 19 normal forms \( \tilde{N}_{ij}, \ (i, j) \in \Delta \).

Among the linear normal forms given in Theorem 2.2, we can find the monodromic ones, whose orbits near \( O \) rotate around \( O \).

**Corollary 2.** Switching system (1) is monodromic at equilibrium \( O \) if and only if its linear normal form is \( N_{77}^\pm \) and \( \beta^+ \beta^- > 0 \).

**Proof.** It is known (from e.g. [17]) that for a non-degenerate system without switching there are orbits tending to the origin along a definite direction as \( t \rightarrow +\infty \) or \( t \rightarrow -\infty \) if the Jordan form of its Jacobian matrix at the origin is not of the form \( N_{77}^\pm \). Thus, if a non-degenerate switching system is of a linear normal form other than \( N_{77}^\pm \) on the upper plane or \( N_{77}^- \) on the lower plane, then there is an orbit tending to the origin along a definite direction in upper plane or lower plane as \( t \rightarrow +\infty \) or \( t \rightarrow -\infty \), a contradiction to monodromy. Moreover, it is necessary for \( O \) to be monodromic that \( \beta^+ \beta^- > 0 \); otherwise, \( O \) is not an isolated equilibrium.

Corollary 2 is a result for a non-degenerate isolated equilibrium. It shows that \( O \) is monodromic of FF type ([11]), i.e., system (1) is monodromic at \( O \) only in the case that both the upper system and the lower one are monodromic, i.e., \( A^\pm \) have a pair of complex eigenvalues separately. The monodromic \( O \) can be of neither PP type nor
Suppose that Theorem 3.1.

\( N \) is a non-degenerate equilibrium of (8). In the case that either (1, 1) or (1, 1) = (1, 1) with \( N \) satisfying \((\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 \neq 0\), system (8) has the normal form

\[
\begin{align*}
\begin{cases}
\dot{x} = N_1^+ (x, y) + \left( \frac{\gamma^+ x^2 + \gamma^+ y^2}{\eta^+ x^2 + \eta^+ y^2} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
\dot{y} = N_1^+ (x, y) + \left( \frac{\gamma^- x^2 + \gamma^- y^2}{\eta^- x^2 + \eta^- y^2} \right) + O(|(x, y)|^3), & \text{if } y < 0.
\end{cases}
\end{align*}
\]

where \( N_1^+ \) and \( N_1^- \) are given in Theorem 2.2 and (i, j) \( \in \Delta \). In this section we show the normalization of nonlinear terms by giving the second order normal forms for (8).

**Theorem 3.1.** Suppose that \( O \) is a non-degenerate equilibrium of (8). In the case that either (i, j) \( \in \Delta \setminus \{1, 1\} \) or (i, j) = (1, 1) with \( N_1^\pm \) satisfying \((\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 \neq 0\), system (8) has the normal form

\[
\begin{align*}
\begin{cases}
\dot{x} = N_1^+ (x, y) + \left( \frac{\gamma^+ x^2 + \gamma^+ y^2}{\eta^+ x^2 + \eta^+ y^2} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
\dot{y} = N_1^+ (x, y) + \left( \frac{\gamma^- x^2 + \gamma^- y^2}{\eta^- x^2 + \eta^- y^2} \right) + O(|(x, y)|^3), & \text{if } y < 0.
\end{cases}
\end{align*}
\]

In the case that (i, j) = (1, 1) with \( N_1^\pm \) satisfying \((\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 = 0\), system (8) has the normal form

\[
\begin{align*}
\begin{cases}
\dot{x} = N_1^+ (x, y) + \left( \frac{\gamma^+ x^2 + \gamma^+ y^2}{\eta^+ x^2 + \eta^+ y^2} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
\dot{y} = N_1^+ (x, y) + \left( \frac{\gamma^- x^2 + \gamma^- y^2}{\eta^- x^2 + \eta^- y^2} \right) + O(|(x, y)|^3), & \text{if } y < 0.
\end{cases}
\end{align*}
\]

**Proof.** In order to simplify the vector fields on the upper side and lower side of the switching line \( y = 0 \) separately, we generally consider the following continuous quadratic near-identity transformation

\[
\begin{align*}
\begin{cases}
\dot{x} = \left( \frac{\gamma^+ x^2 + \gamma^+ y^2}{\eta^+ x^2 + \eta^+ y^2} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
\dot{y} = \left( \frac{\gamma^- x^2 + \gamma^- y^2}{\eta^- x^2 + \eta^- y^2} \right) + O(|(x, y)|^3), & \text{if } y < 0.
\end{cases}
\end{align*}
\]

Without loss of generality, we assume that the transformation (11) preserves the switching line \( y = 0 \) invariant. Similarly to Lemma 2.1, the continuity of the transformation on the line \( y = 0 \) and the invertibility of the transformation imply...
that transformation (11) is a homeomorphism on \(\mathbb{R}^2\) if and only if \(\tilde{p}_{20} = \tilde{p}_{20}\) and \(\tilde{q}_{20} = \tilde{q}_{20} = 0\), i.e., transformation (11) is of form

\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \tilde{p}_{20}x^2 + \tilde{p}_{11}xy + \tilde{p}_{02}y^2 \\ \tilde{q}_{11}xy + \tilde{q}_{02}y^2 \end{pmatrix}, & \text{if } y \geq 0, \\ \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \tilde{p}_{20}x^2 + \tilde{p}_{11}xy + \tilde{p}_{02}y^2 \\ \tilde{q}_{11}xy + \tilde{q}_{02}y^2 \end{pmatrix}, & \text{if } y \leq 0. \end{cases} \tag{12} \]

For less number of coefficients in computation, we change the real system (8) into the complex form, which by Corollary 1 can be presented as

\[
\dot{z} = \begin{cases} \tilde{N}_i^+ \cdot (z, \bar{z}) + \frac{h_i^+}{2} z^2 + h_2^+ z \bar{z} + \frac{h_{11}^+}{2} \bar{z}^2 + O(|z|^3), & \text{if } \text{Im}(z) > 0, \\ \tilde{N}_j^- \cdot (z, \bar{z}) + \frac{h_j^-}{2} z^2 + h_2^- z \bar{z} + \frac{h_{11}^-}{2} \bar{z}^2 + O(|z|^3), & \text{if } \text{Im}(z) < 0, \end{cases} \tag{13} \]

where \((i, j) \in \Delta, \tilde{N}_i^+ \text{ and } \tilde{N}_j^- \text{ are given in Corollary 1, and all coefficients } h_i^\pm \text{ in (13) are linearly dependent on coefficients } a_{ij}^\pm, \tilde{a}_{ij}^\pm \text{ of (8). Since the complex equation (13) contains less coefficients than (8), the computation for (13) will be simpler than that for (8). For the same reason, we rewrite the transformation (12) in the complex form}

\[
z \rightarrow \begin{cases} z + \frac{H_{20}^+}{2} z^2 + H_{11}^+ z \bar{z} + \frac{H_{02}^+}{2} \bar{z}^2, & \text{if } \text{Im}(z) \geq 0, \\ z + \frac{H_{20}^-}{2} z^2 + H_{11}^- z \bar{z} + \frac{H_{02}^-}{2} \bar{z}^2, & \text{if } \text{Im}(z) \leq 0, \end{cases} \tag{14} \]

where the sums \(H_{20}^+ + 2H_{11}^+ + H_{02}^+ \) and \(H_{20}^- + 2H_{11}^- + H_{02}^- \) are both real and the same.

First, we consider the case that \((i, j) = (1, 1)\) with \(\tilde{N}_i^+ \) satisfying \((\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 = 0\). We claim that (13) is \(C^0\) conjugate to the form

\[
\dot{z} = \begin{cases} \mu^+(3/2, 1/2) \cdot (z, \bar{z}) + c_{11}^+ z \bar{z} + c_{02}^+ \bar{z}^2 + O(|z|^3), & \text{if } \text{Im}(z) > 0, \\ \mu^-(3/2, 1/2) \cdot (z, \bar{z}) + c_{11}^- z \bar{z} + c_{02}^- \bar{z}^2 + O(|z|^3), & \text{if } \text{Im}(z) < 0. \end{cases} \tag{15} \]

Actually, the transformation (14) which makes the \(C^0\) conjugation between (13) and (15) can be chosen with the coefficient \(H_{11}^+ := h_1^+ / \mu^+ - 3H_{20}^+ / 2 + H_{02}^+ / 2\) and arbitrary complex \(H_{20}^+, H_{02}^+\) such that \(h_1^+ / \mu^+ - H_{20}^+ + \text{Re}(H_{02}^+) = h_1^- / \mu^- - H_{20}^- + \text{Re}(H_{02}^-) \in \mathbb{R}\).

It is easy to check that \(H_{20}^+ + 2H_{11}^+ + H_{02}^+ = H_{20}^- + 2H_{11}^- + H_{02}^- \in \mathbb{R}\).

Next, we consider the case that either \((i, j) \in \Delta \setminus \{(1, 1)\}\) or \((i, j) = (1, 1)\) with \(\tilde{N}_i^\pm \) satisfying \((\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 \neq 0\). We claim that (13) is \(C^0\) conjugate to the form

\[
\dot{z} = \begin{cases} \tilde{N}_i^+ \cdot (z, \bar{z}) + c_{11}^+ z \bar{z} + O(|z|^3), & \text{if } \text{Im}(z) > 0, \\ \tilde{N}_j^- \cdot (z, \bar{z}) + c_{11}^- z \bar{z} + O(|z|^3), & \text{if } \text{Im}(z) < 0. \end{cases} \tag{16} \]

In fact, from Corollary 1 we see that each \(\tilde{N}_i^\pm, i = 1, \ldots, 7\), is a complex vector in \(\mathbb{C}^2\). For any \(i, j = 1, \ldots, 7\), let \(\tilde{N}_i^+ := (a_1^+, a_2^+)\) and \(\tilde{N}_j^- := (a_1^-, a_2^-)\), where \(a_1^+, a_2^+ \in \mathbb{C}\).

In the case that the equilibrium \(O\) of (13) is non-degenerate, one can check that

\[
2a_1^+ \bar{a}_2^+ - |a_1^+|^2 - |a_2^+|^2 \neq 0, \tag{17} \]

where \(a_1^+ \bar{a}_2^+\) denotes \((a_1^+)\) \(2^\pm\). For example, for \(\tilde{N}_1^+\) we have \(a_1^+ = (\lambda^+ + \mu^+) / 2\) and \(a_2^+ = (\lambda^+ - \mu^+) / 2\). One can check that \(2a_1^+ \bar{a}_2^+ - |a_1^+|^2 - |a_2^+|^2 = (\lambda^+ + \mu^+)^2 / 4 - (\lambda^+ - \mu^+)^2 = 0\).
\( \lambda^\pm \mu^\pm \neq 0 \). Substituting the transformation (14) in equation (13), we get

\[
\dot{z} = \begin{cases} \dot{N}_i^+ \cdot (z, \bar{z}) + c_{10}^+ z^2 + c_{11}^+ \bar{z} z + c_{02}^+ \bar{z}^2 + O(|z|^3), & \text{if } \text{Im}(z) > 0, \\ \dot{N}_j^- \cdot (z, \bar{z}) + c_{10}^- z^2 + c_{11}^- \bar{z} z + c_{02}^- \bar{z}^2 + O(|z|^3), & \text{if } \text{Im}(z) < 0, \end{cases}
\]  

(18)

where

\[
\begin{align*}
c_{20}^+ & := h_1^+ / 2 - a_1^+ H_{20}^+ / 2 + a_2^+ H_{11}^+ / 2 - H_{11}^+ a_2^+, \\ c_{11}^+ & := h_1^+ - H_{11}^+ a_1^+ + H_{11}^+ a_2^+ - H_{20} a_2^+ - H_{11}^+ a_2^+, \\ c_{02}^+ & := h_3^+ / 2 + a_1^+ H_{02}^+ / 2 - H_{02}^+ a_1^+ + H_{11}^+ a_2^+, 
\end{align*}
\]

Choosing

\[
\begin{align*}
H_{20}^+ & := \frac{2h_1^+ a_1^+ - h_2^+ a_1^+ + h_4^+ a_2^+ - 2H_1^+ |a_2^+|^2 - 4H_1^+ a_1^+ a_2^+ + 2H_1^+ a_1^+ a_2^+}{2a_1^+ - |a_1^+|^2 - |a_2^+|^2}, \\
H_{02}^+ & := \frac{h_3^+ a_1^+ - h_4^+ a_2^+ - 2H_{11}^+ a_1^+ a_2^+ - 2H_{11}^+ a_2^2}{2a_1^+ - |a_1^+|^2 - |a_2^+|^2},
\end{align*}
\]

(19)

in the transformation, where we note (17) in the denominators, we compute the expressions of the coefficients \( c_{20}^+ \) and \( c_{02}^+ \) in (18) and obtain that \( c_{20}^+ = c_{02}^+ = 0 \).

It implies that for each \( (i, j) \in \Delta \), equation (13) can be changed into (16) by the transformation (14) of the choice (19) of \( H_{20}^+ \) and \( H_{02}^+ \). Since it is required just below (14) that the transformation (14) satisfies that the sums \( H_{20}^+ + 2H_{11}^+ + H_{02}^+ \) are both real and the same, we consider the choice of \( H_{11}^+ := c^e + i d^e \), where \( c^e, d^e \) are undetermined reals. With the choice of \( H_{20}^+ \) and \( H_{02}^+ \) in (19), we compute

\[
H_{20}^+ + 2H_{11}^+ + H_{02}^+ = F_{1}^e + F_{1}^e H_{11}^+ + F_3^e H_{11}^+ \\
= \text{Re}(F_{1}^e) + \text{Re}(F_{1}^e + F_{3}^e) c^e + \text{Im}(F_{3}^e - F_{2}^e) d^e \\
+ i \{ \text{Im}(F_{1}^e) + \text{Im}(F_{2}^e + F_{3}^e) c^e + \text{Re}(F_{2}^e - F_{3}^e) d^e \},
\]

(20)

where

\[
\begin{align*}
F_{1}^e & := \frac{2h_1^+ a_1^+ - h_2^+ a_1^+ + h_4^+ a_2^+}{2a_1^+ - |a_1^+|^2 - |a_2^+|^2}, \\
F_{2}^e & := \frac{2a_1^+ - |a_1^+|^2 - |a_2^+|^2}{2a_1^+ - |a_1^+|^2 - |a_2^+|^2}, \\
F_{3}^e & := \frac{-2a_1^+ - |a_1^+|^2 - |a_2^+|^2}{2a_1^+ - |a_1^+|^2 - |a_2^+|^2}.
\end{align*}
\]

In the case that \( (i, j) = (2, 2) \), the linear normal form of system (13) is \( \tilde{N}_2^+ \). As assumed just before (17), \( \tilde{N}_2^+ := (a_1^+, a_2^+) \) and \( a_1^+ = \lambda^e - i/2, a_2^+ = i/2 \). It follows that

\[ F_{2}^e + F_{3}^e = \frac{2\lambda^e}{\lambda^e + 1} \neq 0, \]

\[
\text{Re}(F_{2}^e - F_{3}^e) = \text{Re} \left( \frac{2\lambda^e + 2\lambda^e + 4\lambda^e i}{(\lambda^e + 1)^2} \right) = \frac{2\lambda^e}{\lambda^e + 1} \neq 0, \]

\[
\text{Im}(F_{2}^e - F_{3}^e) = \text{Im} \left( \frac{2\lambda^e + 2\lambda^e + 4\lambda^e i}{(\lambda^e + 1)^2} \right) = \frac{4\lambda^e}{(\lambda^e + 1)^2} \neq 0.
\]

Thus, choosing

\[
d^e := \frac{\text{Im}(F_{1}^e) + \text{Im}(F_{2}^e + F_{3}^e) c^e}{-\text{Re}(F_{2}^e - F_{3}^e)} = -\frac{\lambda^e + 1}{2\lambda^e} \text{Im}(F_{1}^e),
\]
we can check from (20) and (21) that
\[
H^+_{20} + 2H^+_{11} + H^\pm_{02} = Re(F_1^+) + Re(F_2^+ + F_3^\pm)c^+ + Im(F_3^+ - F_2^+)d^+ \\
= Re(F_1^+) + \frac{2Im(F_1^\pm)}{\lambda^\pm(\lambda^\pm + 1)} + Re(F_2^+ + F_3^\pm)c^+, \\
\]
which are both real. Further, the equality $H^+_{20} + 2H^+_{11} + H^\pm_{02} = H^-_{20} + 2H^-_{11} + H^-_{02}$ holds with the choice
\[
c^+ := \frac{Re(F_1^-) + \frac{2Im(F_1^-)}{\lambda^-(\lambda^- + 1)} + Re(F_2^- + F_3^-)c^- - Re(F_1^+) - \frac{2Im(F_1^\pm)}{\lambda^+(\lambda^+ + 1)}}{-Re(F_2^- + F_3^-)} \\
= \frac{\lambda^+ + 1}{-2\lambda^+} \left( Re(F_1^-) - Re(F_1^+) + \frac{2Im(F_1^-)}{\lambda^-(\lambda^- + 1)} - \frac{2Im(F_1^\pm)}{\lambda^+(\lambda^+ + 1)} + Re(F_2^- + F_3^-)c^- \right).
\]
Similarly to the case $(i, j) = (2, 2)$, one can complete our discussion in other 17 cases $(i, j) \in \Delta \setminus \{(1, 1)\}$ by finding appropriate $H^\pm_{11}$ in the transformation.

The remaining case is that $(i, j) = (1, 1)$ in system (13) with $\bar{N}_1^\pm$ satisfying $(\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 \neq 0$. As assumed just before (17), $\bar{N}_1^+ := (a_1^+, a_2^+)$ and $a_1^+ = (\lambda^+ + \mu^+)/2, a_2^+ = (\lambda^- - \mu^-)/2$. As discussed above, with the choice
\[
d^+ := -\frac{\lambda^-Im(F_1^+)}{2\mu^+},
\]
we compute from (20) that
\[
H^\pm_{20} + 2H^\pm_{11} + H^\pm_{02} = Re(F_1^+) + Re(F_2^+ + F_3^\pm)c^+ + Im(F_3^+ - F_2^+)d^+ \\
= \frac{2(2\mu^+ - \lambda^\pm)}{\mu^\pm}c^+ + Re(F_1^+), \\
\]
which are both real. Further, from (22) we can obtain the equality $H^\pm_{20} + 2H^\pm_{11} + H^\pm_{02} = H^-_{20} + 2H^-_{11} + H^-_{02}$ with the choice
\[
c^+ := \frac{\mu^+(2\mu^+ - \lambda^-)}{\mu^-(2\mu^- + \lambda^+)}c^- + \frac{\mu^+Re(F_1^- - F_1^+)}{2(2\mu^+ - \lambda^+)} - or c^- := \frac{\mu^-(2\mu^+ - \lambda^-)}{\mu^+(2\mu^- + \lambda^+)}c^+ + \frac{\mu^-Re(F_1^- - F_1^+)}{2(2\mu^- - \lambda^-)},
\]
where we note that $(\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 \neq 0$.

Finally, since (13) is exactly the complex form of (8), we obtain the normal forms (9) and (10) by taking $z = x + iy$ in (16) and (15) respectively. \hfill \Box

Remark that the condition $(\lambda^+ - 2\mu^+)^2 + (\lambda^- - 2\mu^-)^2 = 0$ given in Theorem 3.1 implies $\lambda^\pm = 2\mu^\pm$, which can be regarded as the so-called “double resonance”, i.e., the upper system and the lower system, being treated as smooth systems separately on their half planes, are both resonant of second order. However, unlike smooth systems, the remaining terms in the normal form (10) of switching system under this condition include not only those resonant terms $x^2$ and $y^2$, determined by rational dependence between eigenvalues as for smooth systems, but also the term $xy$, determined by the switching line $y = 0$. Actually, the $x$-axis both lies in the eigen-direction of $\lambda^\pm$ and is the switching line, showing that the eigenvalues $\lambda^\pm$ play a role different from eigenvalues $\mu^\pm$. If we consider the case $\lambda^\pm = 2\mu^\pm$ but the
switching line is \( x = 0 \), then the normal form is

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\left( \begin{array}{cc}
2\mu^+ & 0 \\
0 & \mu^+
\end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{array}{c}
\eta^+(x^2 + y^2) \\
\gamma^+(x^2 + y^2)
\end{array} \right) + O((x, y)^3), & \text{if } x > 0,
\end{cases}
\]

\[
\begin{cases}
\left( \begin{array}{cc}
2\mu^- & 0 \\
0 & \mu^-
\end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{array}{c}
\eta^-(x^2 + y^2) \\
\gamma^-(x^2 + y^2)
\end{array} \right) + O((x, y)^3), & \text{if } x < 0,
\end{cases}
\]

where there is not the term \( xy \). (23) is deduced from the system

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\left( \begin{array}{cc}
\mu^+ & 0 \\
0 & 2\mu^+
\end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{array}{c}
\eta^+(x^2 + y^2) \\
\gamma^+(x^2 + y^2)
\end{array} \right) + O((x, y)^3), & \text{if } y > 0,
\end{cases}
\]

\[
\begin{cases}
\left( \begin{array}{cc}
\mu^- & 0 \\
0 & 2\mu^-
\end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{array}{c}
\eta^-(x^2 + y^2) \\
\gamma^-(x^2 + y^2)
\end{array} \right) + O((x, y)^3), & \text{if } y < 0,
\end{cases}
\]

obtained by Theorem 3.1, by the change of variables \( x \to y, y \to x \).

Theorem 3.1 only gives the second order normal forms for the switching system (8) but its proof shows a method to normalize higher order terms. It is worthy mentioning that the normal forms obtained in Theorem 3.1 can be simplified further in some special cases, for example, the monodromy case, which will be discussed in next section for Hopf bifurcations.

4. Monodromic normal forms. Hopf bifurcation ([20]) is one of the most important topics in differential equations. Recently, efforts ([7, 11, 15, 28]) have been made to Hopf bifurcations in switching systems. It is proved in [18, Theorem 2.3] that one or two limit cycles can be produced from an elementary focus of the least order (order 1 for foci of FF or FP type and order 2 for foci of PP type), different from the case of smooth systems. For a normal form approach to Hopf bifurcations, we need to work on systems with monodromy ([1]), i.e., no orbits tending to the considered equilibrium in a definite direction as \( t \to \pm \infty \). Thanks to our Corollary 2, which tells that the switching system is monodromic if and only if it has the linear normal form \( N_{77} \) with \( \beta^+\beta^- > 0 \). In what follows, we apply Theorem 3.1 to give the second order normal form for monodromic switching systems and see how the monodromic linear normal form \( N_{77} \) makes it possible to simplify the obtained second order normal form further.

Theorem 4.1. Suppose system (1) is non-degenerate and monodromic at \( O \), which has eigenvalues \( \alpha^+ \pm i\beta^+ \) in the upper system and eigenvalues \( \alpha^- \pm i\beta^- \) in the lower system, where \( \beta^+\beta^- > 0 \). Then (1) is topologically conjugate to the system

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\left( \begin{array}{cc}
\alpha^+ & \beta^+ \\
-\beta^+ & \alpha^+
\end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{array}{c}
\eta^+(x^2 + y^2) \\
\gamma^+(x^2 + y^2)
\end{array} \right) + O((x, y)^3), & \text{if } y > 0,
\end{cases}
\]

\[
\begin{cases}
\left( \begin{array}{cc}
\alpha^- & \beta^- \\
-\beta^- & \alpha^-
\end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{array}{c}
\eta^-(x^2 + y^2) \\
\gamma^-(x^2 + y^2)
\end{array} \right) + O((x, y)^3), & \text{if } y < 0,
\end{cases}
\]

where

\[
\gamma^+ := \{ \beta^-(a - \beta^2)(a^2 + 9\beta^2)(a^4 + 8a^2\beta^2 + 3\beta^4)\alpha_{20}^+ - 4a^2 + 3\beta^2\beta_{20}^- \}
\]

\[
-\alpha^+(a^2 + 2\beta^2)(a^2 + 9\beta^2)(a^4 + 8a^2\beta^2 + 3\beta^4)\beta_{20}^- + 4\alpha^2 - 3\beta^2\beta_{20}^- 
\]

\[
+2\beta^2(a - \beta^2)(a^2 + 9\beta^2)(a^4 + 8a^2\beta^2 + 3\beta^4)\alpha_{20}^+ + 2a^2 + 3\beta^2\beta_{20}^- 
\]

\[
+\beta^2(a - \beta^2)(a^2 + 9\beta^2)(a^4 + 8a^2\beta^2 + 3\beta^4)\beta_{20}^- + 2a^2 + 3\beta^2\beta_{20}^- 
\]
and the notations $\alpha^{\pm k}$ denote $(\alpha^\pm)^k$ for convenience.

**Proof.** By Theorem 2.2 and the last row of Table 1, the linear system of (1) can be changed into $N_{77}$ by the transformation

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\rightarrow
\begin{cases}
  \begin{pmatrix}
    H_1 & H_2^+ \\
    0 & H_3^+
  \end{pmatrix}
  \begin{pmatrix}
    x \\
    y
  \end{pmatrix}, & \text{if } y \geq 0, \\
  \begin{pmatrix}
    H_1 & H_2^- \\
    0 & H_3^-
  \end{pmatrix}
  \begin{pmatrix}
    x \\
    y
  \end{pmatrix}, & \text{if } y \leq 0,
\end{cases}
$$

(25)

where $H_1 := (2|A_{11}^+|)^{-1}$ and

$$
H_2^+ := \frac{A_{11}^+ + A_{12}^+}{2A_{21}^+ \sqrt{-4A_{12}^+ A_{21}^+ - (A_{11}^+ - A_{22}^+)^2}}, \quad
H_2^- := \frac{(A_{11}^+ - A_{22}^+) \cdot \text{sgn}(A_{21}^+)}{2|A_{21}^+| \sqrt{-4A_{12}^+ A_{21}^+ - (A_{11}^+ - A_{22}^+)^2}},
$$

$$
H_3^+ := \frac{1}{\sqrt{-4A_{12}^+ A_{21}^+ - (A_{11}^+ - A_{22}^+)^2}}, \quad
H_3^- := \frac{|A_{21}^+|}{|A_{21}^+| \sqrt{-4A_{12}^+ A_{21}^+ - (A_{11}^+ - A_{22}^+)^2}}.
$$
Using the same transformation (25), we change system (1) into the form
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases} 
\left( \begin{array}{cc}
\alpha^+ & \beta^+ \\
-\beta^+ & \alpha^+
\end{array} \right) (x, y) + \left( \begin{array}{c}
\tilde{a}_{11} x^2 + \tilde{a}_{12} y^2 + \tilde{a}_{10} y \\
\tilde{b}_{10} x^2 + \tilde{b}_{11} y + \tilde{b}_{02} y^2
\end{array} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
\left( \begin{array}{cc}
\alpha^- & \beta^- \\
\beta^- & \alpha^-
\end{array} \right) (x, y) + \left( \begin{array}{c}
\tilde{a}_{11} x^2 + \tilde{a}_{12} y^2 + \tilde{a}_{10} y \\
\tilde{b}_{10} x^2 + \tilde{b}_{11} y + \tilde{b}_{02} y^2
\end{array} \right) + O(|(x, y)|^3), & \text{if } y < 0,
\end{cases}
\] (26)

where \( \tilde{a}_{1\pm}, \tilde{b}_{k\pm} \) are given just below (24). Further, applying the near-identity transformation (12) to (26), we obtain that
\[
K^+ \left( \begin{array}{c}
\dot{x} \\
\dot{y}
\end{array} \right) = \left( \begin{array}{cc}
\alpha^+ & \beta^+ \\
-\beta^+ & \alpha^+
\end{array} \right) (x, y) + \left( \begin{array}{c}
\alpha^+ \tilde{p}_{20} + \tilde{a}_{10} x + \tilde{a}_{12} y + \tilde{a}_{11} x^2 + \tilde{a}_{12} y^2 + \tilde{a}_{10} y + \tilde{b}_{10} x^2 + \tilde{b}_{11} y + \tilde{b}_{02} y^2
\end{array} \right) + O(|(x, y)|^3), & \text{if } y > 0,
\]
\[
K^- \left( \begin{array}{c}
\dot{x} \\
\dot{y}
\end{array} \right) = \left( \begin{array}{cc}
\alpha^- & \beta^- \\
\beta^- & \alpha^-
\end{array} \right) (x, y) + \left( \begin{array}{c}
\alpha^- \tilde{p}_{20} + \tilde{a}_{10} x + \tilde{a}_{12} y + \tilde{a}_{11} x^2 + \tilde{a}_{12} y^2 + \tilde{a}_{10} y + \tilde{b}_{10} x^2 + \tilde{b}_{11} y + \tilde{b}_{02} y^2
\end{array} \right) + O(|(x, y)|^3), & \text{if } y < 0,
\]
where
\[K^\pm := \left( \begin{array}{cc}
1 + 2 \tilde{p}_{20} x + \tilde{p}_{11} y & \tilde{p}_{3\pm} x + 2 \tilde{p}_{02} y \\
\tilde{q}_{11} y & 1 + \tilde{q}_{11} x + 2 \tilde{q}_{02} y
\end{array} \right).
\]

Expanding \((K^\pm)^{-1}\) as
\[\begin{pmatrix}
1 & 0 \\
-2 \tilde{p}_{20} x - \tilde{p}_{11} y & -\tilde{p}_{11} x - 2 \tilde{p}_{02} y
\end{pmatrix} + O(|(x, y)|^2)
\]
in (27) and (28), we obtain
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases} 
\left( \begin{array}{cc}
\alpha^+ & \beta^+ \\
-\beta^+ & \alpha^+
\end{array} \right) (x, y) + \left( \begin{array}{c}
C_{11} x^2 + C_{12} x y + C_{02} y^2 \\
D_{11} x^2 + D_{12} x y + D_{02} y^2
\end{array} \right) + O(|(x, y)|^3), & \text{if } y > 0, \\
\left( \begin{array}{cc}
\alpha^- & \beta^- \\
\beta^- & \alpha^-
\end{array} \right) (x, y) + \left( \begin{array}{c}
C_{11} x^2 + C_{12} x y + C_{02} y^2 \\
D_{11} x^2 + D_{12} x y + D_{02} y^2
\end{array} \right) + O(|(x, y)|^3), & \text{if } y < 0,
\end{cases}
\] (29)

where
\[
\begin{align*}
C_{11}^\pm & := \beta^\pm (\tilde{q}_{11}^+ - \tilde{q}_{11}^-) + \alpha^\pm \tilde{p}_{11}^+ x + \tilde{a}_{11}^\pm y, \\
C_{12}^\pm & := \beta^\pm (\tilde{a}_{12}^+ + \tilde{a}_{12}^-) + \alpha^\pm \tilde{p}_{11}^+ x + \tilde{a}_{11}^\pm y, \\
C_{02}^\pm & := \beta^\pm (\tilde{a}_{02}^+ + \tilde{a}_{02}^-) + \alpha^\pm \tilde{p}_{11}^+ x + \tilde{a}_{11}^\pm y, \\
D_{11}^\pm & := \beta^\pm \tilde{p}_{11}^\pm + \alpha^\pm \tilde{p}_{11}^\pm, \\
D_{12}^\pm & := \beta^\pm \tilde{p}_{10}^\pm + \alpha^\pm \tilde{p}_{10}^\pm, \\
D_{02}^\pm & := \beta^\pm \tilde{p}_{10}^\pm + \alpha^\pm \tilde{p}_{10}^\pm.
\end{align*}
\]

Corresponding the form (24), we consider the following relations
\[
C_{11}^\pm = 0, D_{11}^\pm = 0, D_{20}^\pm = 0, D_{02}^\pm = 0, C_{20}^\pm - C_{02}^\pm = 0, D_{20}^\pm - D_{02}^\pm = 0,
\] (30)

which is a linear algebra system of unknown \( \tilde{p}_{20}, \tilde{p}_{11}, \tilde{p}_{02}, \tilde{q}_{11}^+, \tilde{q}_{02}^+ \). From the linear system we uniquely obtain
\[
\tilde{p}_{20} := \left\{ \begin{array}{l}
\alpha^- \tilde{b}_{20} + \alpha^- \tilde{b}_{11} + \alpha^\pm \beta^\pm (\tilde{b}_{02}^- - \tilde{a}_{11}^- + 8 \tilde{b}_{20}^-) + \alpha^- \tilde{b}_{20}^- (5 \tilde{b}_{11}^+ + 4 \tilde{a}_{20}^+ - 4 \tilde{a}_{02}^-) + \beta^\pm (3 \tilde{b}_{20}^- + 6 \tilde{b}_{02}^- + 3 \tilde{a}_{11}^-) \\
+ \beta^\pm (3 \tilde{b}_{20}^- + 6 \tilde{b}_{02}^- + 3 \tilde{a}_{11}^-) \end{array} \right\} (\beta^-)^{-1} (\alpha^2 + \beta^2)^{-1} (\alpha^2 + 9 \beta^2)^{-1},
\]
\[\tilde{p}_{11}^+ := \left\{ \alpha^+ a_{11}^+ + \alpha^+ \beta^+ (2a_{20}^+ + \tilde{b}_{11}^+ - 2a_{20}^+ + \beta^+ (4b_{02}^+ + 5a_{11}^+ - 4b_{02}^+) \right. \]
\[+ \beta^+ (6b_{02}^+ - 3b_{11}^+ - 6a_{20}^+) \right\} (\alpha^+ + \beta^+ - \beta^/-1) (\alpha^+ + \beta^+ - 9\beta^/-1), \]
\[\tilde{p}_{11}^- := \left\{ -\alpha^+ a_{11}^+ - \alpha^+ \beta^+ (\tilde{b}_{11}^- - 2a_{20}^- + 2a_{20}^- + \alpha^+ \beta^- (4b_{02}^- + 5a_{11}^- - 4b_{02}^-) \right. \]
\[+ \beta^- (6a_{02}^- - 3b_{11}^+ - 6a_{20}^-) \right\} (\alpha^- + \beta^- - \beta^-/1) (\alpha^- + \beta^- - 9\beta^-/1), \]
\[\tilde{p}_{02}^+ := -2\alpha^+ a_{11}^+ (\beta^+)^{-1} + \left\{ (\beta^+)^{-1} \left[ \alpha^+ b_{20}^- + \alpha^+ \beta^- b_{11}^+ + \alpha^+ \beta^- b_{11}^+ (2b_{02}^+ - a_{11}^- + \beta_{20}^-) \right. \right. \]
\[+ \alpha^+ \beta^+ b_{11}^- (\tilde{b}_{11}^+ - 2a_{20}^- + 4a_{02}^- + 3b_{02}^+ \right\} (\beta^+)^{-1} (\alpha^- + \beta^- - 9\beta^-/1), \]
\[\tilde{q}_{11}^+ := \left\{ \alpha^+ \beta^+ a_{11}^+ + \alpha^+ \beta^+ (2a_{20}^+ - a_{11}^- - 2b_{20}^- + \alpha^+ \beta^+ (4a_{02}^- - 4a_{02}^+ + 5b_{11}^+ \right. \]
\[+ \beta^+ (6b_{02}^- + 3a_{11}^+ - 6a_{20}^+) \right\} (\alpha^+ + \beta^+ - 1) (\alpha^+ + 9\beta^+ - 1), \]
\[\tilde{q}_{11}^- := \left\{ \alpha^- a_{11}^+ + \alpha^- \beta^- (\tilde{b}_{11}^- - 2a_{20}^- + 2a_{20}^- + \alpha^- \beta^- (2b_{02}^- + 4a_{11}^- - 2b_{02}^-) \right. \]
\[+ \beta^- (6a_{02}^+ - 3a_{11}^+ - 6b_{20}^-) \right\} (\alpha^- + \beta^- - 1) (\alpha^- + 9\beta^- - 1), \]
\[\tilde{q}_{02}^+ := \left\{ \alpha^+ b_{20}^- + \alpha^+ \beta^- b_{11}^- + \alpha^+ \beta^- b_{11}^- (2b_{02}^- - a_{11}^- - \beta_{20}^-) \right. \]
\[+ \alpha^+ \beta^+ b_{11}^- (\tilde{b}_{11}^+ - 2a_{20}^- + 4a_{02}^- + 3b_{02}^+ \right\} (\beta^+)^{-1} (\alpha^- + \beta^- - 9\beta^-/1), \]
\[\tilde{q}_{02}^- := \left\{ \alpha^- b_{20}^- - \alpha^- b_{20}^- + \alpha^- b_{20}^- + \alpha^- \beta^- b_{11}^- (2b_{02}^- - a_{11}^- - \beta_{20}^-) + \alpha^- \beta^- b_{11}^- (2b_{02}^- + 4a_{11}^- - 2b_{02}^-) \right. \]
\[+ \beta^- (6a_{02}^- + 3a_{11}^+ - 6b_{20}^-) \right\} (\alpha^- + \beta^- - 1) (\alpha^- + 9\beta^- - 1), \]

Moreover, with such a choice of \(\tilde{p}_{20}, \tilde{p}_{11}, \tilde{p}_{02}, \tilde{q}_{11}, \tilde{q}_{02}\) in the transformation (12) we see that \(C_{20}^+ = C_{20}^- = \gamma, C_{20}^- = C_{02} = \gamma, D_{20}^+ = D_{02}^+ = \eta, \) where \(\gamma^\pm\) and \(\eta^\pm\) are given in (24). Therefore, (26) is \(C^0\) conjugate to (24) by a transformation of form (12).

Finally, similarly to Lemma 2.1, one can check that the composition of transformations (25) and (12), i.e.,

\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p_{11}^+ & p_{11}^- \\ 0 & H_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p_{20} x^2 + p_{11}^+ x y + p_{02}^ y^2 \\ q_{11}^+ x y + q_{02}^ y^2 \end{pmatrix}, \text{ if } y \geq 0, \\
\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} H_1 & H_2 \\ 0 & H_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p_{20} x^2 + p_{11}^- x y + p_{02}^- y^2 \\ q_{11}^- x y + q_{02}^- y^2 \end{pmatrix}, \text{ if } y \leq 0,
\]

where \(p_{20} := H_1 p_{20}, p_{11}^+ := H_1 p_{11}^+ + H_2 q_{11}^+ p_{02}^+, q_{11}^- := H_1 q_{11}^- + H_2 q_{11}^- p_{02}^-, q_{02}^+ := H_3 q_{02}^+, q_{02}^- := H_3 q_{02}^- p_{02}^+, \) and \(H_1, H_2, H_3\) are given just below (25), is a homeomorphism of \(\mathbb{R}^2\) onto itself.

Theorem 4.1 makes a further simplification based on Theorem 3.1. Actually, Theorem 3.1 normalizes system (8), which contains 12 independent coefficients of quadratic terms, into the form (9) of only 4 independent coefficients. Accordingly, in the above proof of Theorem 4.1, system (26) is normalized into (29). One can
check that
\[ C_{11}^+ = 0, \ D_{11}^+ = 0, \ C_{20}^+ - C_{02}^+ = 0, \ D_{20}^+ - D_{02}^+ = 0, \]
which restricts system (29) to be of the form (9). What Theorem 4.1 does is utilizing the monodromy to simplify system (29) further, i.e., eliminate at least one of parameters \( \gamma^\pm, \eta^\pm \) in (9). For this purpose, we need to append at least one of the following equalities
\[ C_{20}^+ = C_{02}^+ = 0, \ C_{20}^- = C_{02}^- = 0, \ D_{20}^+ = D_{02}^+ = 0, \ D_{20}^- = D_{02}^- = 0 \]
to the linear algebra system (30). The linear algebra system (30), given in the proof of Theorem 4.1, is exactly (32) associated with \( D_{20}^- = D_{02}^- = 0 \), which proves true above with a choice of the near-identity transformation and eliminates \( \eta^- \) from (9), giving (24). Similarly, one can consider (32) associated with another one \( D_{20}^+ = D_{02}^+ = 0 \) in (33), which eliminates \( \eta^+ \) from (9), giving the same (24) with the change \( y \to -y \). For the same reason, considering the equality \( C_{20}^- = C_{02}^- = 0 \) (or the one \( C_{20}^+ = C_{02}^+ = 0 \)) in (33), we eliminate \( \gamma^- \) (or \( \gamma^+ \)) from (9) and obtain the form
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\alpha^+ & \beta^+ \\
-\beta^+ & \alpha^+
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
\gamma^+(x^2 + y^2) \\
\eta^+(x^2 + y^2)
\end{pmatrix} + O(||(x,y)||^3), \quad \text{if } y > 0,
\end{equation}
\[
\begin{pmatrix}
\alpha^- & \beta^- \\
-\beta^- & \alpha^-
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
0 \\
\eta^-(x^2 + y^2)
\end{pmatrix} + O(||(x,y)||^3), \quad \text{if } y < 0
\end{equation}
\]
(or the same [(34)](34) with the change \( y \to -y \)). Remark that the above obtained forms (24) and (34) are the simplest ones in the further simplification. In fact, seen from the proof of Theorem 4.1, the uniqueness of \( \dot{p}_{32}, \dot{q}_{11}, \dot{q}_{02}^\pm, \dot{q}_{20}^\pm \) solved from the linear algebra system (30) implies that we can eliminate no more. We further note that (34) is somehow the same as (24) if we exchange \( x \) with \( y \), which however moves the switching line \( y = 0 \) to be the line \( x = 0 \) and converts an upper-lower switching system to a left-right switching one. For convenience, we call (24) the second order monodromic normal form of system (1).

5. Applications to Hopf bifurcations. In what follows, we use the second order monodromic normal form (24) to discuss Hopf bifurcations of the switching system
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
A^+(\lambda) \\
A^-(\lambda)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
\alpha_{20}^+ \lambda x^2 + a_{11}(\lambda)xy + a_{02}(\lambda)y^2 \\
b_{20}^+ \lambda x^2 + b_{11}(\lambda)xy + b_{02}(\lambda)y^2
\end{pmatrix} + O(||(x,y)||^3), \quad \text{if } y > 0,
\end{equation}
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
A^+(\lambda) \\
A^-(\lambda)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
a_{20}^- \lambda x^2 + a_{11}(\lambda)xy + a_{02}(\lambda)y^2 \\
b_{20}^- \lambda x^2 + b_{11}(\lambda)xy + b_{02}(\lambda)y^2
\end{pmatrix} + O(||(x,y)||^3), \quad \text{if } y < 0,
\end{equation}
\]
parameterized by \( \lambda \in U \subset \mathbb{R} \) \( C^1 \)-smoothly, where \( U \) is a given small neighborhood of 0. This system is non-degenerate and monodromic at the origin \( O \) for all \( \lambda \in U \). By Theorem 4.1, system (35) is conjugate by a transformation of form (31) to the form
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\alpha^+(\lambda) & \beta^+(\lambda) \\
-\beta^+(\lambda) & \alpha^+(\lambda)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
\gamma^+(\lambda)(x^2 + y^2) \\
\eta^+(\lambda)(x^2 + y^2)
\end{pmatrix} + O(||(x,y)||^3), \quad \text{if } y > 0,
\end{equation}
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\alpha^-(\lambda) & \beta^-(\lambda) \\
-\beta^-(\lambda) & \alpha^-(\lambda)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
\gamma^-(\lambda)(x^2 + y^2) \\
0
\end{pmatrix} + O(||(x,y)||^3), \quad \text{if } y < 0,
\end{equation}
where \( \alpha^\pm, \beta^\pm, \gamma^\pm, \eta^+ \) are all \( C^1 \) real functions of \( \lambda \) and \( \beta^\pm(\lambda) \neq 0 \). Thus, the problem of Hopf bifurcations for (35) is reduced to appearance of limit cycles from the origin of (36).

**Theorem 5.1.** Suppose that
\[
\varphi(0) = 0, \quad \frac{d}{d\lambda} \varphi(0) \neq 0, \quad \eta^+(0) \neq 0,
\]
where \( \varphi(\lambda) := \alpha^+(\lambda)/\beta^+(\lambda) + \alpha^-(-\lambda)/\beta^-(\lambda) \). Then one limit cycle appears in (36) when \( \lambda \) varies from 0 to a positive (resp. negative) constant if \( \eta^+(0)\beta^+(0)d\varphi(0)/d\lambda < 0 \) (resp. \( \eta^+(0)\beta^+(0)d\varphi(0)/d\lambda > 0 \)).

**Proof.** Applying time rescaling \( t \to -t/\beta^+ \) (resp. \( t \to t/\beta^- \)) to the upper (resp. lower) system of (36) respectively, we get
\[
\begin{pmatrix}
\dot{\rho} \\
\dot{\theta} \\
\dot{\lambda}
\end{pmatrix} =
\begin{cases}
\begin{pmatrix}
\alpha^+(\lambda) \\
-\beta^+(\lambda)
\end{pmatrix} x - \begin{pmatrix}
-1 \\
1
\end{pmatrix} y + \begin{pmatrix}
\gamma^+(\lambda) \\
-\beta^+(\lambda)
\end{pmatrix} (x^2 + y^2) + O(|(x,y)|^3), & \text{if } y > 0, \\
\begin{pmatrix}
\alpha^-(-\lambda) \\
\beta^-(-\lambda)
\end{pmatrix} x + \begin{pmatrix}
1 \\
-1
\end{pmatrix} y + \begin{pmatrix}
\gamma^-(-\lambda) \\
-\beta^-(-\lambda)
\end{pmatrix} (x^2 + y^2) + O(|(x,y)|^3), & \text{if } y < 0,
\end{cases}
\]
but neither of the time rescaling changes the topological structures of orbits. In the polar coordinates \( x = r \cos \theta, y = r \sin \theta \), system (38) can be presented as
\[
\begin{align*}
\dot{r} &= \frac{\alpha^+(\lambda)}{-\beta^+(\lambda)} r + \frac{\gamma^+(\lambda)}{-\beta^+(\lambda)} \cos \theta + \frac{\eta^+(\lambda)}{-\beta^+(\lambda)} \sin \theta + R^+(r, \theta), & \text{for } \theta \in (0, \pi), \\
\dot{r} &= \frac{\alpha^-(-\lambda)}{-\beta^-(-\lambda)} r + \frac{\gamma^-(-\lambda)}{-\beta^-(-\lambda)} \cos \theta + \frac{\eta^-(-\lambda)}{-\beta^-(-\lambda)} \sin \theta + R^-(r, \theta), & \text{for } \theta \in (\pi, 2\pi),
\end{align*}
\]
where \( R^\pm \) and \( \Theta^\pm \) are analytic in \( r, \theta \) and \( |R^\pm(r, \theta)| = O(r^2), \quad |\Theta^\pm(r, \theta)| = O(r^2) \). Eliminating \( t \) from (39) and (40), we obtain the equation of orbits on the phase plane
\[
\begin{align*}
\frac{dr}{d\theta} &= \begin{cases}
\left\{ \frac{\alpha^+(\lambda)}{-\beta^+(\lambda)} + \frac{\gamma^+(\lambda)}{-\beta^+(\lambda)} \cos \theta + \frac{\eta^+(\lambda)}{-\beta^+(\lambda)} \sin \theta \right\} r + R^+(r, \theta), & \text{for } \theta \in (0, \pi), \\
\left\{ \frac{\alpha^-(-\lambda)}{-\beta^-(-\lambda)} + \frac{\gamma^-(-\lambda)}{-\beta^-(-\lambda)} \cos \theta + \frac{\eta^-(-\lambda)}{-\beta^-(-\lambda)} \sin \theta \right\} r + R^-(r, \theta), & \text{for } \theta \in (\pi, 2\pi).
\end{cases}
\end{align*}
\]
Let \( r^+(\rho, \theta, \lambda) \) and \( r^-(-\rho, \theta, \lambda) \) denote the solutions of (41) for \( \theta \in (0, \pi) \) associated with \( r^+(0, 0, \lambda) = \rho \) and \( r^-(-\rho, \pi, \lambda) = \rho \) respectively. Define maps \( \Pi^+ : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) and \( \Pi^- : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) respectively by
\[
\Pi^+(\rho, \lambda) = r^+(\rho, \pi, \lambda) \quad \text{and} \quad \Pi^-(\rho, \lambda) = r^-(-\rho, \pi, \lambda).
\]
By the analyticity of the vector field in (38) on the upper half-plane, we solve the upper equation in (41) for sufficiently small \( |\rho| \) and obtain
\[
\Pi^+(\rho, \lambda) = e^{-\beta^+(\lambda)} \rho + e^{-\beta^+(\lambda)} \eta^+(\lambda) \left( 1 + e^{-\beta^+(\lambda)} \right) \rho^2 + \sum_{k \geq 3} \nu_k^+(\lambda) \rho^k,
\]
where $v_k^+$’s are Taylor’s coefficients. Similarly, we obtain
\[
\Pi^-(\rho, \lambda) = e^{\frac{\pi \alpha^-(\lambda)}{-\pi \beta^-(\lambda)}} \rho + \sum_{k \geq 3} v_k^- (\lambda) \rho^k.
\] (43)

Define a map $\Pi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ for (38) by $\Pi(\rho, \lambda) := \Pi^+(\rho, \lambda) - \Pi^-(\rho, \lambda)$. Clearly, isolated positive zeros of $\Pi$ (a function of $\rho$) correspond to limit cycles of (38). From (42) and (43) we compute
\[
\Pi(\rho, \lambda) = e^{-\frac{\pi \alpha^+(\lambda)}{-\pi \beta^+(\lambda)}} \left(1 - e^{\pi \nu^+(\lambda)}\right) \rho + e^{-\frac{\pi \alpha^+(\lambda)}{-\pi \beta^+(\lambda)}} \frac{\eta^+(\lambda)}{-\beta^+(\lambda)} \left(1 + e^{\frac{\pi \alpha^+(\lambda)}{-\beta^+(\lambda)}}\right) \rho^2 + \sum_{k \geq 3} v_k(\lambda) \rho^k;
\] (44)

where $v_k(\lambda) := v_k^+(\lambda) - v_k^-(\lambda)$. By (37) we further compute with (44) and obtain
\[
\Pi(\rho, 0) = e^{-\frac{\pi \alpha^+(0)}{-\beta^+(0)}} \frac{\eta^+(0)}{-\beta^+(0)} \left(1 + e^{\frac{\pi \alpha^+(0)}{-\beta^+(0)}}\right) \rho^2 + \sum_{k \geq 3} v_k(0) \rho^k.
\] (45)

It implies that there exists a sufficiently small $0 < \rho_0 < 1$ such that either $\Pi(\rho, 0) > 0$ identically or $\Pi(\rho, 0) < 0$ identically for all $\rho \in (0, \rho_0)$. On the other hand,
\[
\frac{d}{d\lambda} \left( e^{-\frac{\pi \alpha^+(\lambda)}{-\beta^+(\lambda)}} \left(1 - e^{\pi \nu^+(\lambda)}\right) \right)_{\lambda=0} = -\pi e^{-\frac{\pi \alpha^+(0)}{-\beta^+(0)}} \frac{d}{d\lambda} \nu(0) \neq 0
\] (46)

by (37). It follows from (44-46) and the continuity that there is a $\lambda$ near 0 such that $\Pi(\rho, \lambda)$ has an isolated zero in $(0, \rho_0)$, implying that a limit cycle of (36) arises near $O$. Moreover, the limit cycle appears when $\lambda$ changes from 0 to a positive (resp. negative) constant if $\eta^+(0)\beta^+(0)d\nu(0)/d\lambda < 0$ (resp. $> 0$).

In terms of fractional order of weak foci ([8]), we call $-\varphi$ and $-\text{sign}(\beta^+)\eta^+$ the first and second order Lyapunov quantities respectively for monodromic switching system (1), where the expression of $\eta^+$ is given in (24). The relation (44) implies that if $-\varphi < 0$ (resp. $> 0$) then $O$ is a stable (resp. an unstable) focus of order 0 (called rough focus); if $\varphi = 0$ and $-\text{sign}(\beta^+)\eta^+ < 0$ (resp. $> 0$) then $O$ is a stable (resp. an unstable) weak focus of order 1/2. The proof of Theorem 5.1 shows that the limit cycle is stable (resp. unstable) if $\eta^+(0)\beta^+(0) > 0$ (resp. $< 0$) because $O$ varies from a stable (resp. an unstable) weak focus of order 1/2 to an unstable (resp. a stable) rough focus.

There have been many discussions (see e.g. [7, 11, 15, 18, 28]) on Hopf bifurcations for switching systems. In [28] periodic orbits are discussed by applying an appropriate version of the implicit function theorem for fixed points of the return map without computing the second order Lyapunov quantity. Unlike [18] the above Theorem 5.1 about Hopf bifurcations does not consider the change of the origin $O$ from an equilibrium to a regular point under a perturbation. Although the same form $N_T$ of linear part is also considered in [7, 11, 15, 28], the second order Lyapunov quantity is computed under the assumption that $\alpha^\pm = 0$ and $\beta^\pm = -1$ in [7, 15] for some special systems, that is, the upper and lower systems are both Bautin form and have a center at $O$ in [7] and the upper and lower systems are both Liénard equation in [15]. Under the assumption that $\beta^\pm = -1$, the second order Lyapunov quantity is computed in [11], where all coefficients of quadratic terms are considered but the expressions of the second order Lyapunov quantity are complicated even in the complex form. In our paper a more general form of switching systems is considered and the computation of the second order Lyapunov
quantity is simplified by reduction to normal forms. Our method is applicable to monodromic (1) without requiring the linear part to be $N_\gamma$.

6. Examples and further discussion. In this section we use two artificial examples to show our method of normal form reduction in Hopf bifurcations for switching systems and the critical performance of the second order normal form. Consider the switching system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{15\lambda + \lambda}{18} + \frac{13\lambda}{12} & \frac{1}{2} + \lambda \\ -\frac{5}{3} & \frac{1}{2} + \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{1 + \lambda + x} \begin{pmatrix} y^2 \\ xy \end{pmatrix}, \quad \text{if } y > 0,$$

$$\begin{pmatrix} \frac{21\lambda + 3\lambda}{30} & \frac{17(1 + 2\lambda)}{10} \\ -\frac{72\lambda}{20(1 + 2\lambda)} & -\frac{11 \lambda}{10} - \frac{7}{2} \frac{\lambda}{10} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{1 + 2 + \lambda + x} \begin{pmatrix} x^2 \\ 1 + 2 + \lambda \end{pmatrix}, \quad \text{if } y < 0,$$

where $\lambda \in \mathbb{R}$ is sufficiently small. The Jacobian matrix of the upper system at $O$, denoted by $A^+(\lambda)$, has eigenvalues $(2 + \lambda) \pm i$ and the Jacobian matrix of the lower system at $O$, denoted by $A^-(\lambda)$, has eigenvalues $-1 \pm i(1/2 + \lambda)$. One can easily reduce $A^\pm(\lambda)$ to the real canonical forms, i.e.,

$$\begin{pmatrix} 5 & 14 \\ 0 & 8 \end{pmatrix} A^+(\lambda) \begin{pmatrix} 5 & 14 \\ 0 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 2 + \lambda & 1 \\ -1 & 2 + \lambda \end{pmatrix},$$

$$\begin{pmatrix} 5 & \frac{155}{58} \\ 0 & \frac{55}{58} \end{pmatrix} A^-(\lambda) \begin{pmatrix} 5 & \frac{155}{58} \\ 0 & \frac{55}{58} \end{pmatrix}^{-1} = \begin{pmatrix} -1 & \frac{1}{2} + \lambda \\ -\frac{1}{2} - \lambda & -1 \end{pmatrix},$$

implying by Corollary 2 that system (47) is monodromic at $O$ since its linear system is transformed into form $N_7$ by a homeomorphism given in Lemma 2.1. Thus, by Theorem 4.1 system (47) can be transformed into form (24) where $\alpha^+(\lambda) = 2 + \lambda$, $\beta^+(\lambda) = 1$, $\alpha^-(\lambda) = -1$, $\beta^-(\lambda) = 1/2 + \lambda$,

$$\eta^+(\lambda) = \{-13097950 - 9007050\lambda - 23563100\lambda^2 - 25826100\lambda^3 - 280550\lambda^4 + 25725350\lambda^5 + 22741600\lambda^6 + 9287600\lambda^7 + 2008800\lambda^8 + 232200\lambda^9 + (1 + 2\lambda)^2 + 1168180 \lambda + 9417360 \lambda^2 + 1543960 \lambda^3 + 1543960 \lambda^4 + 1543960 \lambda^5 + 1543960 \lambda^6 + 1543960 \lambda^7 + 1543960 \lambda^8 + 1543960 \lambda^9 + 313200 \lambda^{10} + 41760 \lambda^{11}\}^{-1} \ell(\lambda)^{-1}$$

and $\ell(\lambda) := \sqrt{100 + 15776\lambda + 15776\lambda^2}$. As defined in Theorem 5.1,

$$\varphi(\lambda) := \alpha^+(\lambda) / \beta^+(\lambda) + \alpha^-(\lambda) / \beta^-(\lambda) = \lambda(2\lambda + 1)/(1 + 2\lambda).$$

One can compute

$$\varphi(0) = 0, \quad \frac{d}{d\lambda} \varphi(0) = 3 \neq 0, \quad \eta^+(0) = -\frac{19987}{18850} 
eq 0,$$

i.e., (37) holds. By Theorem 5.1, one limit cycle appears in a small neighborhood of $O$ as $\lambda$ varies from 0 to a positive constant because $\eta^+(0)\beta^+(0)d\varphi(0)/d\lambda < 0$.

As seen above, we use the second order monodromic normal form (24) to discuss Hopf bifurcations of (1) in the case that $\eta^+ \neq 0$, where the origin $O$ is a weak focus.
of order 1/2. If \( \eta^+ = 0 \), the second order truncation of (24) can be presented as

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\left( \frac{\alpha^+}{\beta^+} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \frac{\gamma^+}{\gamma^-} \right) (x^2 + y^2), & \text{if } y > 0, \\
\left( \frac{\alpha^-}{\beta^-} \right) \begin{pmatrix} x \\ y \end{pmatrix} + \left( \frac{\gamma^-}{\gamma^+} \right) (x^2 + y^2), & \text{if } y < 0,
\end{cases}
\] (48)

which has a center at \( O \) when \( \alpha^+ / \beta^+ + \alpha^- / \beta^- = 0 \). In fact, the first equation of (41) correspondingly takes the form

\[
dr \over d\theta = \frac{\alpha^+}{-\beta^+} r + \sum_{i=1}^{+\infty} \left( \frac{\gamma^+}{-\beta^+} \right)^i \left( \frac{\alpha^+}{-\beta^+} \sin^i \theta + \sin^{i-1} \theta \cos \theta \right) r^{i+1}.
\] (49)

Let \( r^+(\rho, \theta) := e^{\frac{\alpha^+}{-\beta^+}} \rho + \sum_{i=1}^{+\infty} \omega_i e^{\frac{\alpha^+}{-\beta^+}} \sin^i \theta \rho^{i+1} \), where \( \omega_i \)s are undetermined constants, and substitute it in (49), which leads to the equation

\[
\sum_{i=1}^{+\infty} \left( \frac{\gamma^+}{-\beta^+} \right)^i \left( \frac{\alpha^+}{-\beta^+} \sin^i \theta + \sin^{i-1} \theta \cos \theta \right) \left\{ \rho + \sum_{j=1}^{+\infty} \omega_j e^{\frac{\alpha^+}{-\beta^+}} \sin^j \theta \rho^{j+1} \right\}^{i+1}
= \sum_{i=1}^{+\infty} \omega_i e^{\frac{\alpha^+}{-\beta^+}} \left( \frac{\alpha^+}{-\beta^+} \sin^{i-1} \theta \cos \theta \right) \rho^{i+1}.
\] (50)

Thus, those \( \omega_i \)s are determined by comparing coefficients on both sides of (50). Concretely, comparing the coefficients of \( \rho^2 \) on both sides of (50), we get \( \omega_1 = -\gamma^+/\beta^+ \). Comparing the coefficients of \( \rho^{m+1} \) \((m \geq 2)\) on both sides of (50), we get

\[
\omega_m = \frac{1}{m} \left( \frac{\gamma^+}{-\beta^+} \right)^m + \frac{1}{m} \sum_{i=1}^{m-1} \left( \frac{\gamma^+}{-\beta^+} \right)^i H_i,
\]

a recursive formula for \( m \geq 2 \), where the coefficient \( H_i \) of \( \rho^{m-i} \) in \( \left\{ 1 + \sum_{j=1}^{+\infty} \omega_j \rho^j \right\}^{i+1} \) just depends on \( \omega_1, \ldots, \omega_{m-1} \) for all \( i = 1, \ldots, m - 1 \). Therefore, we obtain

\[
\Pi^+(\rho) = r^+(\rho, \pi) = e^{\frac{\alpha^+}{-\beta^+}} \rho.
\]

Similarly, we obtain

\[
\Pi^-(\rho) = r^-(\rho, \pi) = e^{\frac{\alpha^-}{-\beta^-}} \rho.
\]

It implies that \( \Pi(\rho) = \Pi^+(\rho) - \Pi^-(\rho) \equiv 0 \) for sufficiently small \( \rho \), i.e., the origin is a center of (48), when \( \alpha^+ / \beta^+ + \alpha^- / \beta^- = 0 \). For this reason, degenerate Hopf bifurcations may happen and a higher order monodromic normal form is needed if \( \eta^+ = 0 \).

In this paper we always assume that \( O \) is a non-degenerate equilibrium when considering nonlinear terms. In the case that either \( A^+ \) or \( A^- \) is nilpotent, by Theorem 2.2 one can find 29 possibilities of normal forms for the switching system, much more possibilities than smooth systems, for which only one possibility is nilpotent. So, for degenerate equilibria there will be lots of work in the future.

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