Approximation by $q$-Szász operators

N. I. Mahmudov
Department of Mathematics
Eastern Mediterranean University
Gazimagusa, TRNC, Mersin 10
Turkey
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Abstract

This paper deals with approximating properties of the newly defined $q$-generalization of the Szász operators in the case $q > 1$. Quantitative estimates of the convergence in the polynomial weighted spaces and the Voronovskaja’s theorem are given. In particular, it is proved that the rate of approximation by the $q$-Szász operators ($q > 1$) is of order $q^{-n}$ versus $1/n$ for the classical Szász–Mirakjan operators.

Keywords: Positive linear operators, Szász-Mirakjan operators, Voronovskaja-type asymptotic formula, weighted space, direct results, Korovkin theorem

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1 Introduction

The approximation of functions by using linear positive operators introduced via $q$-Calculus is currently under intensive research. The pioneer work has been made by A. Lupaș [2] and G. M. Phillips [3] who proposed generalizations of Bernstein polynomials based on the $q$-integers. The $q$-Bernstein polynomials quickly gained the popularity, see [4]–[9]. Other important classes of discrete operators have been investigated by using $q$-Calculus in the case $0 < q < 1$, for example $q$-Meyer-König operators [11], [12], [13], $q$-Bleimann, Butzer and Hahn operators [14], [15], [16], $q$-Szász-Mirakjan operators [17], [18], [20], [19], $q$-Baskakov operators [21].

In the present paper, we introduce a $q$-generalization of the Szász operators in the case $q > 1$. Notice that different $q$-generalizations of Szász-Mirakjan operators were introduced and studied by A. Aral and V. Gupta [17], [18], by C. Radu [20] and by N. I. Mahmudov [19] in the case $0 < q < 1$. Since we define $q$-Szász operators for $q > 1$, the rate of approximation by the $q$-Szász operators ($q > 1$) is of order $q^{-n}$, which is essentially better than $1/n$ (rate of approximation for the classical Szász–Mirakjan operators). Thus our $q$-Szász operators have better approximation properties than the classical Szász–Mirakjan operators and the other $q$-Szász-Mirakjan operators.

The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce $q$-Szász operators and evaluate the moments of $M_{n,q}$. In Section 3 we study convergence properties of the $q$-Szász operators in the polynomial weighted spaces. In Section 4, we give the quantitative Voronovskaja-type asymptotic formula.

2 Construction of $M_{n,q}$ and estimation of moments

Throughout the paper we employ the standard notations of $q$-calculus, see [25], [24].
q-integer and q-factorial are defined by

\[ [n]_q := \begin{cases} 1 - q^n & \text{if } q \in R^+ \setminus \{1\}, \\
q^n & \text{if } q = 1 \\
\end{cases} \quad \text{for } n \in N \quad \text{and} \quad [0] = 0, \]

\[ [n]_q! := [1]_q [2]_q \ldots [n]_q \quad \text{for } n \in N \quad \text{and} \quad [0]! = 1. \]

For integers \( 0 \leq k \leq n \) q-binomial is defined by

\[ \binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}. \]

The q-derivative of a function \( f(x) \), denoted by \( D_q f \), is defined by

\[ (D_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad (D_q f)(0) := \lim_{x \to 0} (D_q f)(x). \]

The formula for the q-derivative of a product and quotient are

\[ D_q (u(x)v(x)) = D_q (u(x))v(x) + u(qx)D_q (v(x)). \]

Also, it is known that

\[ D_q x^n = [n]_q x^{n-1}, \quad D_q E(ax) = aE(qax). \]

If \( |q| > 1 \), or \( 0 < |q| < 1 \) and \( |z| < \frac{1}{1-q} \), the q-exponential function \( e_q(z) \) was defined by Jackson

\[ e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}. \]

If \( |q| > 1 \), \( e_q(z) \) is an entire function and

\[ e_q(z) = \prod_{j=0}^{\infty} \left( 1 + (q-1) \frac{z}{q^{j+1}} \right), \quad |q| > 1. \]

There is another q-exponential function which is entire when \( 0 < |q| < 1 \) and which converges when \( |z| < \frac{1}{1-q} \) if \( |q| > 1 \). To obtain it we must invert the base in (3), i.e. \( q \to \frac{1}{q} \):

\[ E_q(z) := e_{1/q}(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{[k]_{1/q}!}. \]

We immediately obtain from (4) that

\[ E_q(z) = \prod_{j=0}^{\infty} \left( 1 + (1-q) zq^j \right), \quad 0 < |q| < 1. \]

The q-difference equations corresponding to \( e_q(z) \) and \( E_q(z) \) are

\[ D_q e_q(az) = ae_q(qz), \quad D_q E_q(az) = aE_q(qaz), \]

\[ D_{1/q} e_q(z) = D_{1/q} E_{1/q}(z) = E_{1/q} (q^{-1}z) = e_q(q^{-1}z), \quad q \neq 0. \]

Let \( C_p \) is the set of all real valued functions \( f \), continuous on \([0, \infty)\) and such that \( w_pf \) is uniformly continuous and bounded on \([0, \infty)\) endowed with the norm

\[ ||f||_p := \sup_{x \in [0, \infty)} w_p(x) |f(x)|. \]

Here

\[ w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \quad \text{if } p \in N. \]
The corresponding Lipschitz classes are given for $0 < \alpha \leq 2$ by
\[
\Delta_l^n f (x) := f (x + 2h) - 2 f (x + h) + f (x), \quad \omega^n (f; \delta) := \sup_{0 < h \leq \delta} \| \Delta_l^n f \|_p, \quad \text{Lip}^p_{\alpha} f := \{ f \in C_p : \omega^n_p (f; \delta) = 0 (\delta^\alpha), \quad \delta \to 0^+ \}.
\]

Now we introduce the $q$-parametric Szász operator.

**Definition 1** Let $q > 1$ and $n \in \mathbb{N}$. For $f : [0, \infty) \to \mathbb{R}$ we define the Szász operator based on the $q$-integers
\[
M_{n,q} (f; x) := \sum_{k=0}^{\infty} f \left( \frac{k}{n} \right) \frac{1}{q^{k(k-1)/2} [k]!} [n]^{k} x^{k} e_q \left( - [n] q^{-k} x \right).
\]  
(5)

Similarly as a classical Szász operator $S_n$, the operator $M_{n,q}$ is linear and positive. Furthermore, in the case of $q \to 1^+$ we obtain classical Szász–Mirakjan operators.

Moments $M_{n,q} (t^m; x)$ are of particular importance in the theory of approximation by positive operators. From (5) we easily derive the following recurrence formula and explicit formulas for moments $M_{n,q} (t^m; x)$, $m = 0, 1, 2, 3, 4$.

**Lemma 2** Let $q > 1$. The following recurrence formula holds
\[
M_{n,q} (t^{m+1}; x) = \sum_{j=0}^{m} \binom{m}{j} x^{j} q^{j} [n]^{m-j} M_{n,q} (t^j; q^{-1} x).
\]  
(6)

**Proof.** The recurrence formula (6) easily follows from the definition of $M_{n,q}$ and $q [k] + 1 = [k + 1]$.

\[
M_{n,q} (t^{m+1}; x) \\
= \sum_{k=0}^{\infty} [n]^{k+1} \frac{1}{q^{k(k-1)/2} [k]!} [n]^{k} x^{k} e_q \left( - [n] q^{-k} x \right) \\
= \sum_{k=1}^{\infty} [n]^{k} \frac{1}{q^{k(k-1)/2} [k-1]!} [n]^{k-1} x^{k} e_q \left( - [n] q^{-k} x \right) \\
= \sum_{k=0}^{\infty} (q [k] + 1) [n]^{m} \frac{1}{q^{k(k+1)/2} [k]!} [n]^{k} x^{k+1} e_q \left( - [n] q^{-k} q^{-1} x \right) \\
= \sum_{k=0}^{\infty} [n]^{m} \sum_{j=0}^{m} \binom{m}{j} q^{j} [k]^{j} \frac{1}{q^{k(k+1)/2} [k]!} [n]^{k} x^{k+1} e_q \left( - [n] q^{-k} q^{-1} x \right) \\
= \sum_{j=0}^{m} \binom{m}{j} x^{j} q^{j} [n]^{m-j} M_{n,q} (t^j; q^{-1} x).
\]

**Lemma 3** The following identities hold for all $q > 1$, $x \in [0, \infty)$, $n \in \mathbb{N}$, and $k \geq 0$:
\[
x D_q s_{nk} (q; x) = [n] \left( \frac{k}{n} - x \right) s_{nk} (q; x),
\]
\[
M_{n,q} (t^{m+1}; x) = \frac{x}{[n]} D_q M_{n,q} (t^m; x) + x M_{n,q} (t^m; x).
\]  
(7)
Proof. The first identity follows from the following simple calculations

\[ xD_q s_{nk} (q; x) = [k]_q \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q (- [n]_q q^{-k} x) - x q^{-k} [n]_q \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k q^k x^k}{[k]_q!} e_q (- [n]_q q^{-k} x) + [n]_q s_{nk} (q; x) = [n] \left( \frac{[k]}{[n]} - x \right) s_{nk} (q; x). \]

The second one follows from the first.

\[ xD_q M_{n,q} (t^m; x) = [n] \sum_{k=0}^\infty \left( \frac{[k]}{[n]} \right)^m \left( \frac{[k]}{[n]} - x \right) s_{nk} (q; x) = [n] \sum_{k=0}^\infty \left( \frac{[k]}{[n]} \right)^{m+1} s_{nk} (q; x) - [n] x \sum_{k=0}^\infty \left( \frac{[k]}{[n]} \right)^m s_{nk} (q; x) = [n] M_{n,q} (t^{m+1}; x) - [n] x M_{n,q} (t^m; x). \]

\[ \_ \_ \_ \]

Lemma 4 Let \( q > 1 \). We have

\[ M_{n,q} (1; x) = 1, \quad M_{n,q} (t; x) = x, \quad M_{n,q} (t^2; x) = x^2 + \frac{1}{[n]} x, \]

\[ M_{n,q} (t^3; x) = x^3 + \frac{2 + q}{[n]} x^2 + \frac{1}{[n]} x, \]

\[ M_{n,q} (t^4; x) = x^4 + \left( 3 + 2q + q^2 \right) \frac{1}{[n]} x^3 + \left( 3 + 3q + q^2 \right) \frac{1}{[n]} x^2 + \frac{1}{[n]} x. \]

Proof. For a fixed \( x \in R_+ \), by the \( q \)-Taylor theorem [25], we obtain

\[ \varphi_n (t) = \sum_{k=0}^\infty \frac{[k]_1}{[k]_1} D_{1/q}^k \varphi_n (x). \]

Choosing \( t = 0 \) and taking into account

\[ (-x)^k q^{-k(k-1)/2} = \varphi_n (x) = e_q (- [n]_q x) \]

we get for \( \varphi_n (x) = e_q (- [n]_q x) \) that

\[ 1 = \varphi_n (0) = \sum_{k=0}^\infty \frac{(-1)^k x^k}{q^{k(k-1)/2} [k]_1} D_{1/q}^k \varphi_n (x) \]

\[ = \sum_{k=0}^\infty \frac{(-1)^k x^k}{[k]_1 q^{-k(k-1)/2}} \left( -1 \right)^k q^{-k(k-1)/2} [n]_q^k e_q (- [n]_q q^{-k} x) \]

\[ = \sum_{k=0}^\infty \frac{[n]_q^k x^k}{[k]_1 q^{-k(k-1)/2}} e_q (- [n]_q q^{-k} x). \]

In other words \( M_{n,q} (1; x) = 1 \).

Calculation of \( M_{n,q} (t^i; x) \), \( i = 1, 2, 3, 4 \), based on the recurrence formula [7] (or [8]). We only calculate \( M_{n,q} (t^3; x) \) and \( M_{n,q} (t^4; x) \):

\[ M_{n,q} (t^i; x) = \frac{x}{[n]} D_q M_{n,q} (t^2; x) + x M_{n,q} (t^2; x) \]

\[ = \frac{x}{[n]} \left( \left[ \frac{2}{[n]} \right] x + \frac{1}{[n]} \right) + x \left( x^2 + \frac{1}{[n]} x \right) \]

\[ = \frac{1}{[n]} x^2 + 2 \frac{1}{[n]} x^2 + x^3. \]
\[ M_{n,q}(t^4; x) = \frac{x}{[n]} D_q M_{n,q}(t^3; x) + x M_{n,q}(t^3; x) \]
\[ = \frac{x}{[n]} \left( \frac{1}{[n]} + \frac{2+q}{[n]} [2] x + \frac{3}{[n]} x^2 \right) + x \left( \frac{1}{[n]} x + \frac{2+q}{[n]} x^2 + x^3 \right) \]
\[ = \frac{1}{[n]^3} x + (3 + 3q + q^2) \frac{x^2}{[n]^2} + (3 + 2q + q^2) \frac{x^3}{[n]} + x^4. \]

\[ \square \]

**Lemma 5** Assume that \( q > 1 \). For every \( x \in [0, \infty) \) there hold

\[ M_{n,q}((t-x)^2; x) = \frac{x}{[n]}, \tag{8} \]
\[ M_{n,q}((t-x)^3; x) = \frac{1}{[n]^2} x + (q-1) \frac{x^2}{[n]}, \tag{9} \]
\[ M_{n,q}((t-x)^4; x) = \frac{1}{[n]^3} x + (q^2 + 3q - 1) \frac{x^2}{[n]^2} + (q-1)^2 \frac{x^3}{[n]}. \tag{10} \]

**Proof.** First of all we give an explicit formula for \( M_{n,q}((t-x)^4; x) \).

\[ M_{n,q}((t-x)^3; x) = M_{n,q}(t^3; x) - 3x M_{n,q}(t^2; x) + 3x^2 M_{n,q}(t; x) - x^3 \]
\[ = x^3 + \frac{2+q}{[n]} x^2 + \frac{1}{[n]^2} x - 3x \left( \frac{x^2}{[n]} + \frac{x}{[n]} \right) + 3x^3 - x^3 \]
\[ = \frac{1}{[n]^2} x + (q-1) \frac{x^2}{[n]} \]

\[ M_{n,q}((t-x)^4; x) = M_{n,q}(t^4; x) - 4x M_{n,q}(t^3; x) + 6x^2 M_{n,q}(t^2; x) - 4x^3 M_{n,q}(t; x) + x^4 \]
\[ = \frac{1}{[n]^3} x + (3 + 3q + q^2) \frac{x^2}{[n]^2} + (3 + 2q + q^2) \frac{x^3}{[n]} + x^4 \]
\[ - 4x \left( \frac{1}{[n]^2} x + \frac{2+q}{[n]} x^2 + x^3 \right) + 6x^2 \left( \frac{x^2}{[n]} + \frac{x}{[n]} \right) - 4x^4 + x^4 \]
\[ = \frac{1}{[n]^3} x + (-1 + 3q + q^2) \frac{x^2}{[n]^2} + (q-1)^2 \frac{x^3}{[n]}. \]

\[ \square \]

Now we prove explicit formula for the moments \( M_{n,q}(t^m; x) \), which a \( q \)-analogue of a result of Becker, see [22] Lemma 3.

**Lemma 6** For \( q > 1 \), \( m \in \mathbb{N} \) there holds

\[ M_{n,q}(t^m; x) = \sum_{j=1}^{m} S_q(m, j) \frac{x^j}{[n]^{m-j}}. \tag{11} \]

where

\[ S_q(m+1, j) = [j] S_q(m, j) + S_q(m, j-1), \quad m \geq 0, \quad j \geq 1, \]
\[ S_q(0,0) = 1, \quad S_q(m,0) = 0, \quad m > 0, \quad S_q(m,j) = 0, \quad m < j. \tag{12} \]

In particular \( M_{n,q}(t^m; x) \) is a polynomial of degree \( m \) without a constant term.
Proof. Because of $M_{n,q}(t;x) = x$, $M_{n,q}(t^2;x) = x^2 + \frac{x}{[n]}$ the representation (11) holds true for $m = 1, 2$ with $S_q(2,1) = 1$, $S_q(1,1) = 1$.

Now assume (11) to be valued for $m$ then by Lemma 3 we have

\[
M_{n,q}(t^{m+1};x) = \frac{x}{[n]} D_q M_{n,q}(t^m;x) + x M_{n,q}(t^m;x)
\]

\[
= \frac{x}{[n]} \sum_{j=1}^{m} [j] S_q(m,j) \frac{x^{j-1}}{[n]^{m-j}} + x \sum_{j=1}^{m} S_q(m,j) \frac{x^j}{[n]^{m-j}}
\]

\[
= \sum_{j=1}^{m} [j] S_q(m,j) \frac{x^j}{[n]^{m-j+1}} + \sum_{j=1}^{m} S_q(m,j) \frac{x^{j+1}}{[n]^{m-j}}
\]

\[
= \frac{x}{[n]^m} S_q(m,1) + x^{m+1} S_q(m,m)
\]

\[
+ \sum_{j=2}^{m} ([j] S_q(m,j) + S_q(m,j-1)) \frac{x^j}{[n]^{m-j+1}}.
\]

Remark 7 For $q = 1$ the formulae (12) become recurrence formulas satisfied by Stirling numbers of the second type.

3 $M_{n,q}$ in polynomial weighted spaces

Lemma 8 Let $p \in N \cup \{0\}$ and $q \in (1, \infty)$ be fixed. Then there exists a positive constant $K_1(q,p)$ such that

\[
\|M_{n,q}(1/w_p;x)\|_p \leq K_1(q,p), \quad n \in N.
\]  

Moreover for every $f \in C_p$ we have

\[
\|M_{n,q}(f)\|_p \leq K_1(q,p) \|f\|_p, \quad n \in N.
\]

Thus $M_{n,q}$ is a linear positive operator from $C_p$ into $C_p$ for any $p \in N \cup \{0\}$.

Proof. The inequality (13) is obvious for $p = 0$. Let $p \geq 1$. Then by (11) we have

\[
w_p(x) M_{n,q}(1/w_p;x) = w_p(x) + w_p(x) \sum_{j=1}^{p} S_q(p,j) \frac{x^j}{[n]^{p-j}} \leq K_1(q,p),
\]

$K_1(q,p)$ is a positive constant depending on $p$ and $q$. From this follows (13). On the other hand

\[
\|M_{n,q}(f)\|_p \leq \|f\|_p \|M_{n,q}(1/w_p)\|_p
\]

for every $f \in C_p$. By applying (13), we obtain (14). ■

Lemma 9 Let $p \in N \cup \{0\}$ and $q \in (1, \infty)$ be fixed. Then there exists a positive constant $K_2(q,p)$ such that

\[
\left\| M_{n,q} \left( \frac{(t - \cdot)^2}{w_p(t)^2}; x \right) \right\|_p \leq K_2(q,p) \frac{1}{[n]}, \quad n \in N.
\]

Proof. The formula (8) imply (15) for $p = 0$. We have

\[
M_{n,q} \left( \frac{(t - x)^2}{w_p(t)^2}; x \right) = M_{n,q} \left( (t - x)^2; x \right) + M_{n,q} \left( (t - x)^2 t^p; x \right),
\]
for \( p, n \in N \). If \( p = 1 \) then we get
\[
M_{n,q} \left( (t - x)^2 (1 + t) ; x \right) = M_{n,q} \left( (t - x)^2 ; x \right) + M_{n,q} \left( (t - x)^2 t ; x \right)
\]
\[
= M_{n,q} \left( (t - x)^3 ; x \right) + (1 + x) M_{n,q} \left( (t - x)^2 ; x \right),
\]
which by Lemma [5] yields (15) for \( p = 1 \).

Let \( p \geq 2 \). By applying (11), we get
\[
w_p (x) M_{n,q} \left( (t - x)^2 t^p ; x \right)
\]
\[
= w_p (x) \left( M_{n,q} \left( t^{p+2} ; x \right) - 2 x M_{n,q} \left( t^{p+1} ; x \right) + x^2 M_{n,q} \left( t^p ; x \right) \right)
\]
\[
= w_p (x) \left( x^{p+2} + \sum_{j=1}^{p+1} S_q (p + 2, j) \frac{x^j}{[n]^{p+2-j}} - 2 x^{p+2} - 2 \sum_{j=1}^{p} S_q (p + 1, j) \frac{x^j}{[n]^{p+1-j}} + x^{p+2} + \sum_{j=1}^{p-1} S_q (p, j) \frac{x^j}{[n]^{p-j}} \right)
\]
\[
= w_p (x) \left( \sum_{j=2}^{p} (S_q (p + 2, j) - 2 S_q (p + 1, j) + S_q (p, j)) \frac{x^j}{[n]^{p+1-j}} + S_q (p + 2, 1) \frac{x}{[n]^{p+1}} \right.
\]
\[
+ (S_q (p + 2, 2) - 2 S_q (p + 2, 1)) \frac{x^2}{[n]^{p}} \bigg) = w_p (x) \frac{x}{[n]} \mathcal{P}_p (q; x),
\]
where \( \mathcal{P}_p (q; x) \) is a polynomial of degree \( p \). Therefore one has
\[
w_p (x) M_{n,q} \left( (t - x)^2 t^p ; x \right) \leq K_2 (q, p) \frac{x}{[n]},
\]
\]

Our first main result in this section is a local approximation property of \( M_{n,q} \) stated below.

**Theorem 10** There exists an absolute constant \( C > 0 \) such that
\[
w_p (x) |M_{n,q} (g; x) - g (x)| \leq K_3 (q, p) \|g''\| \frac{x}{[n]},
\]
where \( g \in C_p^2 \), \( q > 1 \) and \( x \in [0, \infty) \).

**Proof.** Using the Taylor formula
\[
g (t) = g (x) + g' (x) (t - x) + \int_x^t \int_x^s g'' (u) du ds,
\]
we obtain that
\[
w_p (x) |M_{n,q} (g; x) - g (x)| = w_p (x) \left| M_{n,q} \left( \int_x^t \int_x^s g'' (u) du ds ; x \right) \right|
\]
\[
\leq w_p (x) M_{n,q} \left( \left| \int_x^t \int_x^s g'' (u) du ds \right| ; x \right)
\]
\[
\leq w_p (x) M_{n,q} \left( \|g''\|_p \left| \int_x^t \int_x^s (1 + u^n) du ds \right| ; x \right)
\]
\[
\leq w_p (x) \frac{1}{2} \|g''\|_p M_{n,q} \left( (t - x)^2 \left( 1/w_p (x) + 1/w_p (t) \right) ; x \right)
\]
\[
\leq \frac{1}{2} \|g''\|_p \left( M_{n,q} \left( (t - x)^2 ; x \right) + w_p (x) M_{n,q} \left( (t - x)^2 w_p (t) ; x \right) \right)
\]
\[
\leq K_3 (q, x) \|g''\|_p \frac{x}{[n]}.\]
Now we consider the modified Steklov means

$$f_h(x) := \frac{4}{h^2} \int_0^h \int_0^h [2f(x + s + t) - f(x + 2(s + t))] dsdt.$$

$f_h(x)$ has the following properties:

$$f(x) - f_h(x) = \frac{4}{h^2} \int_0^h \int_0^h \Delta_{s+t}^2 f(x) dsdt, \quad f_h''(x) = h^{-2} \left(8\Delta_{h}^2 f(x) - \Delta_{h}^2 f(x)\right)$$

and therefore

$$\|f - f_h\|_p \leq \omega_p^2(f; h), \quad \|f''\|_p \leq \frac{1}{9h^2}\omega_p^2(f; h).$$

We have the following direct approximation theorem:

**Theorem 11** For every $p \in \mathbb{N} \cup \{0\}, f \in C_p$ and $x \in [0, \infty), q > 1$, we have

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq M_p\omega_p^2 \left( f; \sqrt{\frac{x}{|n|}} \right) = M_p\omega_p^2 \left( f; \sqrt{\frac{(q-1)x}{(q^n-1)}} \right).$$

Particularly, if $\operatorname{Lip}_p^{2\alpha}$ for some $\alpha \in (0, 2]$, then

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq M_p \left( \frac{x}{|n|} \right)^{\frac{q}{2}}$$

**Proof.** For $f \in C_p$ and $h > 0$

$$|M_{n,q}(f; x) - f(x)| \leq |M_{n,q}((f - f_h); x) - (f - f_h)(x)| + |M_{n,q}(f_h; x) - f_h(x)|$$

and therefore

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq \|f - f_h\|_p \left( w_p(x)M_{n,q} \left( \frac{1}{w_p(x)}; x \right) + 1 \right)$$

$$+ K_3(q, p) \|f''\|_p \frac{x}{|n|}. $$

Since $w_p(x)M_{n,q} \left( \frac{1}{w_p(x)}; x \right) \leq K_1(q, p)$, we get that

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq M(q, p)\omega_p^2(f; h) \left[ 1 + \frac{x}{|n|h^2} \right]$$

Thus, choosing $h = \sqrt{\frac{x}{|n|}}$, the proof is completed. ■

**Corollary 12** If $p \in \mathbb{N} \cup \{0\}, f \in C_p, q > 1$ and $x \in [0, \infty)$, then

$$\lim_{n \to \infty} M_{n,q}(f; x) = f(x).$$

This convergence is uniform on every $[a, b], 0 \leq a < b$.

**Remark 13** Theorem 11 shows the rate of approximation by the $q$-Szász operators ($q > 1$) is of order $q^{-n}$ versus $1/n$ for the classical Szasz-Mirakjan operators.
4 Convergence of $q$-Szász operators

An interesting problem is to determine the class of all continuous functions $f$ such that $M_{n,q}(f)$ converges to $f$ uniformly on the whole interval $[0, \infty)$ as $n \to \infty$. This problem was investigated by Totik [27, Theorem 1] and de la Cal [23, Theorem 1]. The following result is a $q$-analogue of Theorem 1 [23].

Theorem 14 Assume that $f : [0, \infty) \to \mathbb{R}$ is bounded or uniformly continuous. Let

$$f^*(z) = f(z^2), \quad z \in [0, \infty).$$

We have, for all $t > 0$ and $x \geq 0$,

$$|M_{n,q}(f;x) - f(x)| \leq 2\omega \left( f^*; \sqrt{\frac{1}{n}} \right).$$

(16)

Therefore, $M_{n,q}(f;x)$ converges to $f$ uniformly on $[0, \infty)$ as $n \to \infty$, whenever $f^*$ is uniformly continuous.

Proof. By the definition of $f^*$ we have

$$M_{n,q}(f;x) = M_{n,q} (f^*(\sqrt{x}); x).$$

Thus we can write

$$|M_{n,q}(f;x) - f(x)| = |M_{n,q} (f^*(\sqrt{x}); x) - f^*(\sqrt{x})|$$

$$= \sum_{k=0}^{\infty} \left| f^* \left( \sqrt{\frac{k}{n}} \right) - f^* (\sqrt{x}) \right| s_{n,k} (q;x)$$

$$\leq \sum_{k=0}^{\infty} \left| f^* \left( \sqrt{\frac{k}{n}} \right) - f^* (\sqrt{x}) \right| s_{n,k} (q;x)$$

$$\leq \sum_{k=0}^{\infty} \omega \left( f^*; \sqrt{\frac{k}{n}} - \sqrt{x} \right) s_{n,k} (q;x)$$

$$\leq \sum_{k=0}^{\infty} \omega \left( f^*; \sqrt[2]{\frac{k}{n}} - \sqrt{x} \right) M_{n,q} (\sqrt[2]{\cdot} - \sqrt{x}; x) s_{n,k} (q;x).$$

Finally, from the inequality

$$\omega (f^*; \alpha \delta) \leq (1 + \alpha) \omega (f^*; \delta), \quad \alpha, \delta \geq 0,$$

we obtain

$$|M_{n,q}(f;x) - f(x)| \leq \omega \left( f^*; M_{n,q} (\sqrt[2]{\cdot} - \sqrt{x}; x) \right) \sum_{k=0}^{\infty} \left( 1 + \frac{\sqrt[2]{\frac{k}{n}} - \sqrt{x}}{M_{n,q} (\sqrt[2]{\cdot} - \sqrt{x}; x)} \right) s_{n,k} (q;x)$$

$$= 2\omega \left( f^*; M_{n,q} (\sqrt[2]{\cdot} - \sqrt{x}; x) \right).$$

In order to complete the proof we need to show that we have for all $t > 0$ and $x > 0$,

$$M_{n,q} (\sqrt[2]{\cdot} - \sqrt{x}; x) \leq \sqrt{\frac{1}{n}}.$$
Indeed we obtain from the Cauchy-Schwarz inequality

\[ M_{n,q} \left( \left| \sqrt[k]{r} - \sqrt[n]{q} \right| ; x \right) = \sum_{k=0}^{\infty} \left| \frac{|k|}{|n|} - \sqrt{r} \right| s_{n,k} (q; x) \]

\[ = \sum_{k=0}^{\infty} \frac{|k| - x}{\frac{|n|}{|k|} + \sqrt{r}} s_{n,k} (q; x) \leq \frac{1}{\sqrt{r}} \sum_{k=0}^{\infty} \left| \frac{|k|}{|n|} - x \right| s_{n,k} (q; x) \]

\[ \leq \frac{1}{\sqrt{r}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{|k|}{|n|} - x \right)^2} s_{n,k} (q; x) = \frac{1}{\sqrt{r}} \sqrt{M_{n,q} \left( (\cdot - x)^2 ; x \right)} \]

\[ = \frac{1}{\sqrt{r}} \sqrt{\frac{1}{|n|}} = \frac{1}{\sqrt{|n|}} \]

showing (10), and completing the proof. ■

Next we prove Voronovskaja type result for \( q \)-Szász-Mirakjan operators.

**Theorem 15** Assume that \( q \in (1, \infty) \). For any \( f \in C^2_q \) the following equality holds

\[ \lim_{n \to \infty} [n] (M_{n,q} (f; x) - f (x)) = \frac{1}{2} f'' (x) x \]

for every \( x \in [0, \infty) \).

**Proof.** Let \( x \in [0, \infty) \) be fixed. By the Taylor formula we may write

\[ f (t) = f (x) + f' (x) (t - x) + \frac{1}{2} f'' (x) (t - x)^2 + r (t; x) (t - x)^2, \tag{17} \]

where \( r (t; x) \) is the Peano form of the remainder, \( r (\cdot; x) \in C_p \) and \( \lim_{t \to x} r (t; x) = 0 \). Applying \( M_{n,q} \) to (17) we obtain

\[ [n] (M_{n,q} (f; x) - f (x)) = f' (x) [n] M_{n,q} (t - x; x) \]

\[ + \frac{1}{2} f'' (x) [n] M_{n,q} \left( (t - x)^2 ; x \right) + [n] M_{n,q} \left( r (t; x) (t - x)^2 ; x \right). \]

By the Cauchy-Schwarz inequality, we have

\[ M_{n,q} \left( r (t; x) (t - x)^2 ; x \right) \leq \sqrt{M_{n,q} \left( r^2 (t; x) ; x \right)} \sqrt{M_{n,q} \left( (t - x)^4 ; x \right)}. \tag{18} \]

Observe that \( r^2 (x; x) = 0 \). Then it follows from Corollary 12 that

\[ \lim_{n \to \infty} M_{n,q} \left( r^2 (t; x) ; x \right) = r^2 (x; x) = 0. \tag{19} \]

Now from (18), (19) and Lemma 5 we get immediately

\[ \lim_{n \to \infty} [n] M_{n,q} \left( r (t; x) (t - x)^2 ; x \right) = 0. \]

The proof is completed. ■
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