Solutions of systems with the Caputo–Fabrizio fractional delta derivative on time scales

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Abstract

Caputo–Fabrizio fractional delta derivatives on an arbitrary time scale are presented. When the time scale is chosen to be the set of real numbers, then the Caputo–Fabrizio fractional derivative is recovered. For isolated or partly continuous and partly discrete, i.e., hybrid time scales, one gets new fractional operators. We concentrate on the behavior of solutions to initial value problems with the Caputo–Fabrizio fractional delta derivative on an arbitrary time scale. In particular, the exponential stability of linear systems is studied. A necessary and sufficient condition for the exponential stability of linear systems with the Caputo–Fabrizio fractional delta derivative on time scales is presented. By considering a suitable fractional dynamic equation and the Laplace transform on time scales, we also propose a proper definition of Caputo–Fabrizio fractional integral on time scales. Finally, by using the Banach fixed point theorem, we prove existence and uniqueness of solution to a nonlinear initial value problem with the Caputo–Fabrizio fractional delta derivative on time scales.

Key words: Calculus on time scales, Caputo–Fabrizio fractional delta derivatives and integral, Exponential stability, Laplace transform on time scales, Existence and uniqueness of solution.

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1. Introduction

There is a recent interest on the development of fractional (noninteger) calculi on arbitrary nonempty closed subsets of the real numbers. The subject was initiated in 2011 by using the inverse Laplace transform on time scales \cite{10}.
The first local notion of fractional derivative for functions defined on completely arbitrary time scales seems to have been introduced in 2015 [13]. Such results were then extended to the nabla and nonsymmetric cases in [14], while the possibility of a fractional derivative to be a complex number is addressed in [12]. In [7, 8, 9], the authors study the existence of solutions for fractional differential equations including the Caputo–Fabrizio derivative, that is, they concentrate on systems with continuous time. Existence and uniqueness results for fractional initial value problems on arbitrary time scales are investigated in [15, 32] and the so-called conformable case is studied in [11, 16]. Ortigueira et al. introduced fractional derivatives on arbitrary time scales through convolution [30]. Here we are interested to extend the interesting approach of Caputo–Fabrizio from the time scale $\mathbb{T} = \mathbb{R}$ into an arbitrary time scale $\mathbb{T}$. Our results are completely new in the discrete-time or hybrid cases. Since hybrid systems are dynamical systems that exhibit both continuous and discrete dynamic behaviors, we decided to use time scales, which are recognized as a useful tool to the study of continuous- and discrete-time systems. Observe that instead of describing the system by differential or difference equations, we consider here systems with delta and/or nabla derivatives to write systems defined on domains that are partly continuous and partly discrete.

The Caputo–Fabrizio fractional derivative, which is the convolution of the exponential function and the first order derivative, was introduced in 2015 in [13], see also [20], with the purpose of avoiding singular kernels. The new notion is receiving an increasing of interest. For example, in [24] a numerical approach based on optimization theory and Ritz’s method is developed to deal with Caputo–Fabrizio Fokker–Planck equations; applications in linear viscoelasticity are presented in [8]; while practical signal processing problems are investigated in [22]. Moreover, in [33] the authors show that the Caputo–Fabrizio operator may be a simple and efficient way for incorporating different structural aspects into the system, opening new possibilities for modeling and investigating anomalous diffusive processes. Another application of the Caputo–Fabrizio fractional derivative, to describe real-world problems, can be found for instance in [4, 6, 26]. To the best of our knowledge, we are the first to consider Caputo–Fabrizio delta derivatives and integrals on arbitrary time scales. In this way, we extend the deterministic definition.

The paper is organized as follows. In Section 2, we briefly review the necessary concepts from time scales. Our results are then given in Section 3. Firstly, we introduce Caputo–Fabrizio fractional derivatives on time scales in Section 3.1. Then, in Section 3.2 by considering a suitable fractional differential equation and the Laplace transform on time scales, we arrive to a proper definition of Caputo–Fabrizio fractional integral of order $\alpha$. Next, we study the stability of linear equations with such time-scale fractional derivatives in Section 3.3. Finally, by using Banach’s fixed point theorem, we prove in Section 3.4 existence and uniqueness of solution to a nonlinear fractional initial value problem on time scales. We end with Section 4 of conclusions, including some possible directions of future research.
2. Preliminaries on time scales

We recall here basic concepts and facts of the calculus on time scales. For more information, the reader is referred to [17].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. The standard examples of time scales are $\mathbb{R}$, $h\mathbb{Z}$, $h > 0$, $\mathbb{N}$, $\mathbb{N}_0$, $2^{\mathbb{N}_0}$, or $\mathbb{F}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$. A time scale $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. Let us now recall some basic operators defined on time scales. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} \in \mathbb{T}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} \in \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. We say that $t$ is isolated if $\rho(t) < t < \sigma(t)$, that $t$ is dense if $\rho(t) = \sigma(t)$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

In order to define the delta derivative, one introduces the set

$$
\mathbb{T}^\kappa := \begin{cases} 
\mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \text{if } \sup \mathbb{T} < \infty, \\
\mathbb{T}, & \text{if } \sup \mathbb{T} = \infty.
\end{cases}
$$

**Definition 2.1.** Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then the number $f^\Delta(t)$, when it exists, with the property that, for any $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$
||f(\sigma(t)) - f(s)) - f^\Delta(t)\sigma(t) - s|| \leq \varepsilon|\sigma(t) - s|, \forall s \in U,
$$

is called the delta derivative of $f$ at $t$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the delta derivative of $f$ on $\mathbb{T}^\kappa$. We say that $f$ is delta differentiable on $\mathbb{T}^\kappa$, if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

**Remark 2.2.** Let us consider some illustrative time scales.

1. If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if $f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$, i.e., $f$ is differentiable in the ordinary sense at $t$.

2. If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable on $\mathbb{Z}$ with $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\rho(t)} = f(t+1) - f(t)$, for all $t \in \mathbb{Z}$.

3. If $\mathbb{T} = \mathbb{Q}$, then $f : \mathbb{T} \to \mathbb{R}$ is always delta differentiable on $\mathbb{T} \setminus \{0\}$ and $f^\Delta(t) = \frac{f(q^+)-f(t)}{(q-1)t}$, for all $t \in \mathbb{Q}$. Moreover, $f^\Delta(0) = \lim_{s \to 0} \frac{f(s)-f(0)}{s}$, only if this limit exists.

**Definition 2.3.** A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated if its right-side limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at all left-dense points in $\mathbb{T}$.
Definition 2.4. A function $f : T \to \mathbb{R}$ is called rd-continuous if it is continuous at the right-dense points in $T$ and its left-sided limits exist at all left-dense points in $T$.

Definition 2.5. A continuous function $f : T \to \mathbb{R}$ is pre-differentiable with $D$ (the region of differentiation), if $D \subset T^c$, $T^c \setminus D$ is countable and contains no right-scattered elements of $T$ and $f$ is differentiable at each $t \in D$.

If $f$ is regulated, then there exists a function $F$ that is pre-differentiable with the region of differentiation $D$, such that $F^\Delta(t) = f(t)$ for all $t \in D$. Any function $F$ that satisfies $F^\Delta(t) = f(t)$ is called a pre-antiderivative of $f$. Then the indefinite integral of a regulated function $f$ is defined by $\int f(t) dt = F(t) + C$, where $C$ is an arbitrary constant. The Cauchy integral of a regulated function $f$ is defined by $\int_a^b f(t) dt = F(b) - F(a)$, for all $a, b \in T$. A function $F : T \to \mathbb{R}$ is called an antiderivative of $f : T \to \mathbb{R}$ if it satisfies $F^\Delta(t) = f(t)$, for all $t \in T^c$.

It is known that every rd-continuous function has an antiderivative, so it is delta-integrable. The set of rd-continuous functions on the time scale $T$ is denoted by $C_{rd}(T)$.

Example 2.6. If $T = \mathbb{R}$, then $\int_a^b f(t) dt = \int_a^b f(t) dt$, where the integral on the right-hand side is the usual Riemann integral.

Example 2.7. If $T = h\mathbb{Z}$, where $h > 0$, then $\int_a^b f(t) dt = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h \cdot f(kh)$, $a < b$.

Now, let us recall the concept of exponential function on a time scale. The function $p : T \to \mathbb{R}$ is called regressive, if $1 + p(t)\mu(t) \neq 0$ for all $t \in T^c$. The set of all regressive and rd-continuous functions $f : T \to \mathbb{R}$ is denoted by $\mathcal{R}$.

Let us consider the following linear delta differential equation on a given time scale $T$:

$$x^\Delta(t) = p(t)x(t), \quad (1)$$

where $t \in T$ and $x(t) \in \mathbb{R}$. If $p \in \mathcal{R}$, then one defines the exponential function $e_p(\cdot, t_0)$ on the time scale $T$ by

$$e_p(t, t_0) = \begin{cases} 
\exp \left( \int_{t_0}^{t} p(s) ds \right), & \text{for } t \in T, \mu = 0; \\
\exp \left( \int_{t_0}^{t} \frac{\log(1+\mu(s)p(s))}{\mu(s)} ds \right), & \text{for } t \in T, \mu > 0,
\end{cases}$$

where Log is the principal logarithm function.

Theorem 2.8 (See [17]). Let $t_0 \in T$ and $x_0 \in \mathbb{R}$. Then, system (1) with initial condition $x(t_0) = x_0$ has a unique solution $x : [t_0, +\infty) \cap T \to \mathbb{R}$.

Theorem 2.9 (See [17]). Let $t_0 \in T$ and $p \in \mathcal{R}$. Then $e_p(\cdot, t_0)$ is the solution of the initial value problem

$$x^\Delta(t) = p(t)x(t), \quad x(t_0) = 1.$$
3. Main results

We define the Caputo–Fabrizio (CF, for short) fractional delta derivatives on time scales of order \( \alpha \in [0, 1) \): left-sided and right-sided, with the scalar exponential function of time scales as the kernel. For brevity, we restrict ourselves to the delta calculus. However, similar definitions and results follow easily for the nabla case by using the duality of fractional calculus and time scales \([2, 21, 25]\). Analogously, other new notions of fractional delta derivatives with respect to the matrix exponential function, as well as higher-order derivatives, are also introduced.

**Definition 3.1.** Let \( \mathbb{T} \) be a time scale, \( \alpha \in (q^{-1}, q] \), where \( q \in \mathbb{N}_0 \), and \( a, b \in \mathbb{R} \) with \( a < b \). We denote by \( CF^\alpha[a, b] \) the class of functions \( f : [a, b] \rightarrow \mathbb{R} \) for which both delta integrals \( \int_a^t f^\Delta(\tau)e_\beta(t, \sigma(\tau))\Delta \tau \) and \( \int_b^t f^\Delta(\tau)e_\beta(t, \sigma(\tau))\Delta \tau \), where \( \bar{\beta} = \frac{\beta}{q-1}, \beta = \alpha - q + 1 \), exist.

3.1. CF fractional delta derivative of order \( \alpha \in [0, 1) \)

As usual in fractional calculus, we introduce two notions of fractional derivatives: left-sided and right-sided.

**Definition 3.2** (Caputo–Fabrizio fractional delta derivatives on time scales). Let \( \mathbb{T} \) be a time scale, \( \alpha \in [0, 1) \), \( f \in CF^\alpha[a, b] \), and \( M \) a given function satisfying \( M(0) = M(1) = 1 \). The left-sided Caputo–Fabrizio fractional delta derivative is defined by

\[
(CF_a^\alpha \Delta^\alpha t f)(t) := \frac{M(\alpha)}{1 - \alpha} \int_a^t f^\Delta(\tau)e_\alpha(t, \sigma(\tau))\Delta \tau
\] (2)

and the right-sided Caputo–Fabrizio fractional delta derivative is defined by

\[
(CF_t^\alpha \Delta^\alpha b f)(t) := \frac{M(\alpha)}{1 - \alpha} \int_t^b f^\Delta(\tau)e_\alpha(t, \sigma(\tau))\Delta \tau,
\] (3)

where \( \tilde{\alpha} = \frac{\alpha}{q-1} \).

**Remark 3.3.** For \( \alpha = 0 \), the left-sided Caputo–Fabrizio fractional delta derivative defined by (2) has the form of convolution on a time scale:

\[
(CF_0^\alpha \Delta^\alpha t f)(t) = \frac{M(\alpha)}{1 - \alpha} (f^\Delta * e_\tilde{\alpha})(t).
\]

**Example 3.4.** When \( \mathbb{T} = h\mathbb{Z} \), we get a new notion of fractional difference. Namely, for \( a, t \in h\mathbb{Z} \), we have

\[
(CF_a^\alpha \Delta^\alpha t f)(t) = \frac{M(\alpha)}{1 - \alpha} \sum_{k=0}^{\frac{t}{h} - 1} hf^\Delta(kh)(1 + ha)^{\frac{\alpha}{q-1} - k - 1}.
\]
Lemma 3.5. Let $\alpha \in [0, 1)$ and $F(z) = \mathcal{L}\{f(z)\}$. Then,

$$\mathcal{L}\left\{\Delta_t^{(\alpha)} f\right\}(z) = M(\alpha) \frac{z F(z) - f(0)}{(1 - \alpha) z + \alpha}.$$

Proof. Follows by adapting the proofs of the Laplace transform of convolution on time scales found in [18] and [23].

Proposition 3.6. Let $f \in FC^\alpha [a, b]$ for $\alpha \in [0, 1)$.

1. If $\alpha = 0$, then the left-sided Caputo–Fabrizio fractional delta derivative (2) becomes

$$\left(\Delta_t^{(\alpha)} f\right)(t) = f(t) - f(a)$$

and

$$\left(\Delta_b^{(\alpha)} f\right)(t) = f(b) - f(t).$$

2. If $\alpha$ tends to 1 from the left side, then the Caputo–Fabrizio fractional delta derivatives (2) and (3) tend to the usual delta derivative of time scales, that is,

$$\lim_{\alpha \to 1^-} \left(\Delta_t^{(\alpha)} f\right)(t) = f^\Delta(t)$$

and

$$\lim_{\alpha \to 1^-} \left(\Delta_b^{(\alpha)} f\right)(t) = f^\Delta(t).$$

Proof. The first part for $\alpha = 0$ is obvious. For the second part, we compute the limits and make use of Lemma 3.5.

3.2. $CF$ delta integrals of order $\alpha \in [0, 1)$

From now on, let $\mathbb{T}$ be an unbounded time scale. In this section, using Laplace transform methods on time scales, we present the solution to linear scalar equations.

Let $\alpha \in (0, 1)$. Consider the following delta-fractional differential equation:

$$\left(\Delta_t^{(\alpha)} x\right)(t) = u(t), \quad t \in \mathbb{T}. \quad (4)$$

Using the Laplace transform on time scales, we easily get

$$x(t) = x(0) + \frac{1 - \alpha}{M(\alpha)} (u(t) - u(0)) + \frac{\alpha}{M(\alpha)} \int_0^t u(\tau) \Delta \tau, \quad t \geq 0.$$

It also means that any function defined as

$$y(t) = c + \frac{1 - \alpha}{M(\alpha)} u(t) + \frac{\alpha}{M(\alpha)} \int_0^t u(\tau) \Delta \tau, \quad t \geq 0,$$

where $c \in \mathbb{R}$ is a constant, is also a solution of (4). We can rewrite the delta-fractional equation (4) as

$$\int_0^t e\tilde{\alpha}(t, \sigma(\tau)) x^\Delta(\tau) \Delta \tau = \frac{1 - \alpha}{M(\alpha)} u(t), \quad t \geq 0.$$

Moreover, as $e\tilde{\alpha}(t, \sigma(\tau)) = e\tilde{\alpha}(t, 0) e\tilde{\alpha}(0, \sigma(\tau))$ and $e\tilde{\alpha}(0, t) = e\tilde{\alpha}^{-1}(t, 0)$, we can further write that

$$\int_0^t e\tilde{\alpha}(0, \sigma(\tau)) x^\Delta(\tau) \Delta \tau = \frac{1 - \alpha}{M(\alpha)} e\tilde{\alpha}(0, t) u(t), \quad t \geq 0.$$
Then, taking the delta-derivative on both sides, we have

\[ e_{\bar{\alpha}}(0, \sigma(t)) x^{\Delta}(\tau) = \frac{1 - \alpha}{M(\alpha)} (e_{\bar{\alpha}}(0, t) u(t))^{\Delta}. \]

As \((e_{\bar{\alpha}}(0, t) u(t))^{\Delta} = e_{\bar{\alpha}}^{\Delta}(0, t) u(t) + e_{\bar{\alpha}}(0, \sigma(t)) u^{\Delta}(t)\), then

\[ e_{\bar{\alpha}}(0, \sigma(t)) x^{\Delta}(\tau) = \frac{1 - \alpha}{M(\alpha)} (e_{\bar{\alpha}}(0, t) u(t) + e_{\bar{\alpha}}(0, \sigma(t)) u^{\Delta}(t)) \]

and

\[ x^{\Delta}(\tau) = \frac{1 - \alpha}{M(\alpha)} e_{\bar{\alpha}}(0, \sigma(t)) u(t) + \frac{1 - \alpha}{M(\alpha)} u^{\Delta}(t). \]

Using the properties of delta derivatives on time scales, direct calculations show that

\[ \frac{e_{\bar{\alpha}}^{\Delta}(0, t)}{e_{\bar{\alpha}}(0, \sigma(t))} = -\bar{\alpha} = \frac{\alpha}{1 - \alpha}. \]

Hence,

\[ x^{\Delta}(\tau) = \frac{\alpha}{M(\alpha)} u(t) + \frac{1 - \alpha}{M(\alpha)} u^{\Delta}(t), \]

which, after delta integration, gives

\[ x(t) = x(0) + \frac{\alpha}{M(\alpha)} \int_{0}^{t} u(\tau) \Delta \tau + \frac{1 - \alpha}{M(\alpha)} (u(t) - u(0)), \quad t \geq 0. \]

Hence, similarly as in [27], we can formulate the fractional delta-integral of Caputo–Fabrizio type as follows.

**Definition 3.7.** Let \( \alpha \in (0, 1) \). The fractional delta-integral of order \( \alpha \) of a function \( u \) is defined by

\[ (\text{CF} I_{\alpha} u)(t) := \frac{1 - \alpha}{M(\alpha)} u(t) + \frac{\alpha}{M(\alpha)} \int_{0}^{t} u(\tau) \Delta \tau, \quad t \geq 0. \]

We can also consider, as an extension of [27], that the Caputo–Fabrizio fractional delta-integral of order \( \alpha \) of a function \( u \) is an average between function \( u \) and its delta integral of order one:

\[ \frac{1 - \alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)} = 1. \]

From this, we have \( M(\alpha) = 1 \) and formula (2) in the definition of left-sided Caputo–Fabrizio fractional derivative on time scales started at \( t_0 = 0 \) can be stated as

\[ (\text{CF} \Delta_{\bar{\alpha}}^{(\alpha)} f)(t) := \frac{1}{1 - \alpha} \int_{0}^{t} f^{\Delta}(\tau) e_{\bar{\alpha}}(t, \sigma(\tau)) \Delta \tau, \]

where \( \bar{\alpha} = \frac{\alpha}{\alpha - 1} \).
3.3. Solutions of linear equations with CF fractional delta derivatives and their stability

Now, let us consider the following linear scalar equation:

\[
\left(CF\Delta_i^{(\alpha)} x\right)(t) = \lambda x(t) + u(t), \quad \lambda \in \mathbb{R}, \quad t \in \mathbb{T},
\]

with initial condition \(x(0) = x_0 \in \mathbb{R}\). For simplicity, we use \(t_0 = 0\), which is assumed to belong to \(\mathbb{T}\). Here, we also use \(M(\alpha) = 1\), accordingly to results from Section 3.2.

**Definition 3.8.** Let \(\alpha \in [0,1)\). We say that \(\lambda \in \mathbb{R}\) is \(CF(\alpha)\)-fractionally regressive if \(K(\alpha) := 1 - \lambda(1 - \alpha) \neq 0\).

**Theorem 3.9.** Let \(\alpha \in [0,1)\) and \(\mathbb{T}\) be an unbounded time scale with \(t_0 = 0 \in \mathbb{T}\), \(K(\alpha) \neq 0\). Then, equation (5) subject to the initial condition \(x(0) = x_0 \in \mathbb{R}\) has the unique solution

\[
x(t) = x(0) - \frac{1}{K(\alpha)} (1 - e_{p(\alpha)}(t, 0)) x(0) + \frac{1 - \alpha}{K(\alpha)} (u(t) - u(0)) + \frac{\alpha}{K^2(\alpha)} \int_0^t e_{p(\alpha)}(t, \sigma(\tau)) u(\tau) \Delta \tau,
\]

where \(p(\alpha) = \frac{\lambda \alpha}{K(\alpha)}\).

**Proof.** Using the Laplace transform method, its inverse, and the methods of [18], we obtain the formula of solution as

\[
x(t) = \frac{1}{K(\alpha)} e_{p(\alpha)}(t, 0) x(0) + \frac{1 - \alpha}{K(\alpha)} u(t) + \frac{\alpha}{K^2(\alpha)} \int_0^t e_{p(\alpha)}(t, \sigma(\tau)) u(\tau) \Delta \tau,
\]

while the formula

\[
\hat{x}(t) = \frac{1}{K(\alpha)} e_{p(\alpha)}(t, 0) x(0) + \frac{1 - \alpha}{K(\alpha)} u(t) + \frac{\alpha}{K^2(\alpha)} \int_0^t e_{p(\alpha)}(t, \sigma(\tau)) u(\tau) \Delta \tau + C
\]

is also a solution of equation (5) subject to the initial condition \(x(0) = x_0\). To have uniqueness, we need to calculate \(C = \left(1 - \frac{1}{K(\alpha)}\right) x(0) - \frac{1 - \alpha}{K(\alpha)} u(0)\), which together with equation (5) agrees with the form given by (6).

**Corollary 3.10.** Let \(\alpha \in [0,1)\) and \(x(\cdot)\) be a solution of equation (5) with \(\lambda = 0\), i.e., of equation \(\left(CF\Delta_i^{(\alpha)} x\right)(t) = u(t), \quad t \in \mathbb{T}\), subject to the initial condition \(x(0) = x_0 \in \mathbb{R}\). Then, \(\lambda\) is \(CF(\alpha)\)-fractionally regressive for any \(\alpha \in [0,1)\) and \(x(t) = x_0 + (1 - \alpha)(u(t) - u(0)) + \alpha \int_0^t u(\tau) \Delta \tau\). Moreover, for \(u(t) \equiv 0\), we have the constant function \(x(t) \equiv x_0\) for any \(\alpha\).

**Remark 3.11.** Observe that solutions to equation (5) are continuous/bounded, if \(u(\cdot)\) is a continuous/bounded function.
Based on the results of Pötzsche et al. [31], we make an analysis of the exponential stability of equation (5) in case of a constant function \( u(·) \). In this case, in the formula of solution (7), it disappears the part \( u(t) - u(0) \). In view of the definitions given in [31], the following notations for describing the sets of exponential stability are introduced. For simplicity, we take \( t_0 = 0 \in \mathbb{T} \).

**Definition 3.12** (See [31]). Given a time scale \( \mathbb{T} \) unbounded from above, we denote

\[
S_C(\mathbb{T}) := \left\{ \lambda \in \mathbb{C} : \limsup_{T \to \infty} \frac{1}{T-t_0} \int_{t_0}^{T} \lim_{s \to \mu(t)} \frac{\log |1 + s\lambda|}{s} \Delta t < 0 \right\}
\]

and

\[
S_R(\mathbb{T}) := \{ \lambda \in \mathbb{R} : \forall T \in \mathbb{T} \ \exists t \in \mathbb{T}, \ t > T : 1 + \mu(t)\lambda = 0 \}.
\]

The set of exponential stability for the time scale \( \mathbb{T} \) is then defined by

\[
S(\mathbb{T}) := S_C(\mathbb{T}) \cup S_R(\mathbb{T}).
\]

**Remark 3.13.** For an arbitrary time scale \( \mathbb{T} \), we have \( S_R(\mathbb{T}) \subset (-\infty, 0) \) and \( S_C(\mathbb{T}) \subseteq \{ \lambda \in \mathbb{C} : \text{Re} \lambda < 0 \} \).

**Proposition 3.14.** Let \( \mathbb{T} \) be unbounded from above, \( p(\alpha) \) be regressive and \( u(·) \) be a constant function. Then, equation (5) is exponentially stable if and only if \( p(\alpha) \in S_C(\mathbb{T}) \).

**Proof.** The result is obtained following the proof of [31, Theorem 21]. \( \square \)

**Example 3.15.** Let \( \alpha \in (0, 1) \) and \( M(\alpha) = 1 \). We consider here two classical time scales.

1. Let \( \mathbb{T} = h\mathbb{Z} \), where \( h > 0 \). Then \( S_R(h\mathbb{Z}) = \{ -\frac{1}{h} \} \) and \( S(h\mathbb{Z}) = B_{\frac{1}{h}} (-\frac{1}{h}) \), where \( B_{\frac{1}{h}} (-\frac{1}{h}) \) denotes the disc with center at \( (-\frac{1}{h}, 0) \) and radius \( \frac{1}{h} \). Then, the stability region for parameter \( p(\alpha) \) is \( S(h\mathbb{Z}) = B_{\frac{1}{h}} (-\frac{1}{h}) \). Hence, it follows from Proposition 3.14 that for real values of \( p(\alpha) \), a necessary and sufficient condition for the exponential stability of equation (5) is

   (a) \( \lambda \in \left( -\frac{2}{\alpha} - \frac{2}{1-\alpha} \right), \) for \( h > 2 \left( \frac{1}{\alpha} - 1 \right) \);

   (b) \( \lambda \in (-\infty, 0) \cup \left( \frac{2}{2(1-\alpha)-\alpha}, +\infty \right), \) for \( h \leq 2 \left( \frac{1}{\alpha} - 1 \right) \).

   Observe that for \( \alpha \) tending to \( 1 \) only item (a) is possible, and then the condition agrees with \( \lambda \in (-\frac{2}{\alpha}, 0) \).

2. Let \( \mathbb{T} = \mathbb{R} \). Then \( S_R(\mathbb{R}) = \emptyset \) and \( S(\mathbb{R}) = \{ \lambda \in \mathbb{C} : \text{Re} \lambda < 0 \} \). Equation (5) is exponentially stable if and only if \( \lambda < 0 \) or \( \lambda > \frac{1}{\alpha} \).
Example 3.16. Let $\alpha \in (0, 1]$, $K(\alpha) := 1 - \lambda(1 - \alpha) \neq 0$, and consider the initial condition $x(0) = 0 \in T$. Then, the unique solution of the initial value problem has the form

$$x(t) = \frac{1 - \alpha}{K(\alpha)}(u(t) - u(0)) + \frac{\alpha}{K^2(\alpha)} \int_0^t e_{p(\alpha)}(t, \sigma(\tau))u(\tau)\Delta \tau \quad (9)$$

with $p(\alpha) = \frac{\lambda \alpha}{K(\alpha)}$. Considering $T = h\mathbb{Z}$, $t = kh$, we have that

$$\int_0^t e_{p(\alpha)}(t, \sigma(\tau))u(\tau)\Delta \tau = h \sum_{s=0}^{k-1} (1 + hp(\alpha))^{k-s-1} u(sh)$$

holds. We present graphs with the behavior of the exact solution in different situations:

(a) For $T = \mathbb{Z}$, $\lambda = 0.2$, $u(t) \equiv 1$, and $\alpha \in \{0.2, 0.5, 0.9, 1\}$, we get different exponents tending to infinity, what we see for first 30 steps in Figure 1. The solutions are unstable.

(b) If we would like to receive stable solutions, then we can take, for $T = \mathbb{Z}$, negative $\lambda$ or big enough $\lambda$. Taking $\lambda = 4.2$ we have stable solutions for $\alpha \in \{0.2, 0.5\}$ while for $\alpha > 4$ the solution is stable for $\alpha = 0.5$. If $u(t) \equiv 1$, then there is now a first part in summation in the solution’s formula \(9\). In this case, the values of the control function $u(\cdot)$ do not influence the stability. The situation is presented in Figure 2.
Figure 2: Graphs of solutions to the linear equation \( \lambda = 4.2 \), \( x(0) = 0 \), \( u(t) \equiv 1 \), given by (9), for \( \alpha \in \{0.2, 0.5\} \) on \( T = \mathbb{Z} \), for first 30 steps.

(c) In Figure 3, we compare graphs with \( \lambda = 0.2 \) for different steps \( h \in \{0.1, 0.5, 1\} \) and we take order \( \alpha = 0.5 \). If we include the situation when \( h \to 0 \), which in the limit represents \( T = \mathbb{R} \), we will receive the line on dots for \( h = 0.1 \).

Figure 3: Graphs of solutions to the linear equation \( \lambda = 0.2 \), \( x(0) = 0 \), given by (9), with \( u(t) \equiv 1 \) for \( h \in \{0.1, 0.5, 1\} \) and \( \alpha = 0.5 \), for first 30 steps.
3.4. Nonlinear CF delta-fractional differential equations

Under suitable assumptions on function \( f(\cdot, x(\cdot)) \) of equation
\[
\left( \alpha, \Delta^\alpha \right)_x(t) = f(t, x(t)), \quad x(a) = x_0 \in \mathbb{R}, \quad t \in [a, b] \cap \mathbb{T},
\]
we can extend our results to the nonlinear case by using a fixed point theorem.

**Definition 3.17** (Cf. [17]). Let \( \mathbb{T} \) be a time scale. A function \( f : \mathbb{T} \times \mathbb{R} \to \mathbb{R} \) is called

(i) rd-continuous, if \( g \) defined by \( g(t) = f(t, x(t)) \) is rd-continuous for any continuous function \( x : \mathbb{T} \to \mathbb{R} \);

(ii) regressive at \( t \in \mathbb{T}^\kappa \), if the mapping \( 1 + \mu(t)f(t, \cdot) : \mathbb{R} \to \mathbb{R} \) is invertible; and \( f \) is called regressive at each \( t \in \mathbb{T}^\kappa \);

(iii) bounded on a set \( S \subset \mathbb{T} \times \mathbb{R} \), if there exists a constant \( M > 0 \) such that \( |f(t, x)| \leq M \) for all \( (t, x) \in S \);

(iv) Lipschitz continuous on a set \( S \subset \mathbb{T} \times \mathbb{R} \), if there is a constant \( L > 0 \) such that \( |f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2| \) for all \((t, x_1), (t, x_2)\) \in S.

**Theorem 3.18.** Let \( \alpha \in (0, 1), b > a, \) and \( f : [a, b] \cap \mathbb{T} \times \mathbb{R} \to \mathbb{R} \) be a rd-continuous function such that there exists \( L > 0 \) such that
\[
|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2| \quad \text{for all} \ t, x_1, x_2 \in \mathbb{R}.
\]

If \((1 - \alpha) + \alpha(b - a)L < 1\), then the initial value problem (10) has a unique solution on \([a, b] \cap \mathbb{T} \).

**Proof.** Let \( \|f\| = \sup_{t \in [a, b] \cap \mathbb{T}} |f(t)| \) for all rd-continuous functions on \([a, b] \cap \mathbb{T} \), that is, for all \( f \in \mathcal{C}_d([a, b] \cap \mathbb{T}) \). Consider the operator \( \mathcal{N} : \mathcal{C}_d([a, b] \cap \mathbb{T}) \to \mathcal{C}_d([a, b] \cap \mathbb{T}) \) defined by
\[
(\mathcal{N}x)(t) = x(0) + \alpha \int_a^t f(\tau, x(\tau)) \Delta \tau + (1 - \alpha)(f(t, x(t)) - f(0, x(0))),
\]
x \in \mathcal{C}_d([a, b] \cap \mathbb{T}). Because for all \( x_1, x_2 \in \mathcal{C}_d([a, b] \cap \mathbb{T}) \) and all \( t \in [a, b] \cap \mathbb{T} \) we have
\[
|(\mathcal{N}x_1)(t) - (\mathcal{N}x_2)(t)|
\leq (1 - \alpha)|f(t, x_1(t)) - f(t, x_2(t))| + \alpha \int_a^t |f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))| \Delta \tau
\leq (1 - \alpha)L|b - a||x_1 - x_2| + \alpha L(b - a)||x_1 - x_2|
\leq (1 - \alpha + \alpha(b - a))L|x_1 - x_2|,
\]
we conclude that the operator \( \mathcal{N} \) is a contraction. The statement follows from Banach’s fixed point theorem. \( \square \)
4. Conclusion

The theory of fractional differential equations on time scales, specifically the questions of existence, uniqueness of solutions and their stability, is a recent research topic of great importance, see [11, 28] and references therein. Here, we studied deterministic fractional operators on time scales with Caputo–Fabrizio type kernels. The main difference is seen in solutions, where the language of integrals and exponentials on time scales is used. Such non-singular kernels have recently been applied successfully to some real world problems [1, 5, 19, 20, 27]. For this reason, we trust that the general concepts and calculus here introduced will initiate further interest and developments. As possible future work, motivated respectively by the recent techniques and ideas of [29] and [32], we mention: proof of chain rules and inequalities, and existence and uniqueness results of positive solutions.

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