Abstract

We propose a generalization of the classical M/M/1 queue process. The resulting model is derived by applying fractional derivative operators to a system of difference-differential equations. This generalization includes both non-Markovian and Markovian properties which naturally provide greater flexibility in modeling real queue systems than its classical counterpart. Algorithms to simulate M/M/1 queue process and the related linear birth-death process are provided. Closed-form expressions of the point and interval estimators of the parameters of the proposed fractional stochastic models are also presented. These methods are necessary to make these models usable in practice. The proposed fractional M/M/1 queue model and the statistical methods are illustrated using financial data.

Keywords: Transient analysis, Fractional M/M/1 queue, Mittag–Leffler function, Fractional birth-death process, Parameter estimation, Simulation.

1 Introduction

The M/M/1 queue is without a doubt the simplest model for a queue process. It is characterized by arrivals determined by a Poisson process and an independent service time which is negative-exponentially distributed. It is relatively simple and yet the analysis of its transient behavior leads to considerable difficulties. The main source of these difficulties is the presence of a non-absorbing boundary at zero (empty queue). This means that the analysis becomes simpler when we consider models with absorbing boundaries. As a direct result, the state probability of a linear birth-death process, that is the probability that the queue length is \( n \) at a specific time \( t \), has a particularly nice form.

The aim of this paper is to study some related point processes governed by difference-differential equations containing fractional derivative operators. These processes are direct generalizations of the classical M/M/1 queue and the linear birth-death processes. It is well-known that a fractional derivative operator induces a non-Markovian behavior.
into a system [see 19]. Moreover, parameter estimation and path generation algorithms of these new fractional stochastic models are derived. Note that the proposed fractional point models (with Markovian and non-Markovian properties) are parsimonious which makes them desirable for modeling real-world non-Markovian queueing systems.

Observe that fractional point processes driven by fractional difference-differential equations such as the fractional Poisson, the fractional birth, the fractional death, and the fractional birth-death processes have already been gaining attention more recently [see, e.g., 11, 4, 7, 14, 9].

The article is structured as follows. Section 2 presents the explicit construction of the fractional M/M/1 queue and an application which shows that our constructed estimators perform well in a real-world example. This is preparatory for the similar subsequent analysis applied to the M/M/1 queue (Section 4). The article ends with an application which shows that our constructed estimators perform well in a real-world example.

## 2 Results for a fractional process related to M/M/1 queues

The classical M/M/1 queue process $N(t)$, $t \geq 0$, that is the queue length in time can be described by the following difference-differential equations governing the state probabilities $p_k(t) = \Pr\{N(t) = k | N(0) = i\}$, $k \geq 0$:

$$
\begin{align*}
\frac{d}{dt} p_k(t) &= - (\lambda + \mu) p_k(t) + \lambda p_{k-1}(t) + \mu p_{k+1}(t), & k \geq 1, \\
\frac{d}{dt} p_0(t) &= - \lambda p_0(t) + \mu p_1(t), \\
p_k(0) &= \delta_{k,i},
\end{align*}
$$

where $i \in \mathbb{N} \cup \{0\}$ is the initial number of individuals in the queue and $\delta_{k,i}$ is the Kronecker’s delta. In (2.1) $\lambda > 0$ and $\mu > 0$ are the entrance and the service rates, respectively.

To arrive at a possible fractional model we consider the Caputo fractional derivative $D^\alpha_t$, $\alpha \in (0,1]$, with respect to time $t$. If $p_k^\alpha(t) = \Pr\{N^\alpha(t) = k\}$, $k \geq 0$, where $N^\alpha(t)$, is the fractional $M/M/1$ queue with parameter $\alpha$, the generalized difference-differential equations for the state probabilities with arrival rate $\lambda > 0$, service rate $\mu > 0$ and $i \geq 0$ initial customers, read

$$
\begin{align*}
D^\alpha_t p_k^\alpha(t) &= - (\lambda + \mu) p_k^\alpha(t) + \lambda p_{k-1}^\alpha(t) + \mu p_{k+1}^\alpha(t), & k \geq 1, \\
D^\alpha_t p_0^\alpha(t) &= - \lambda p_0^\alpha(t) + \mu p_1^\alpha(t), \\
p_k^\alpha(0) &= \delta_{k,i}.
\end{align*}
$$

First, we will follow Bailey [2, 3] for the derivation of the probabilities $p_k^\alpha(t)$, $k \geq 0$, $t \geq 0$ but adapting the method to take into considerations the presence of the Caputo derivative. The result obtained by Bailey is the so-called classical solution in terms of modified Bessel functions of the first kind. Note however that the derivation of the state probabilities in the classical case $\alpha = 1$ can be carried out in several equivalent ways (see for example Champernowne [8], Parthasarathy [15], Abate and Whitt [1]). In the following we will first treat the solution derived by Bailey and then we will use a simpler but lesser known form due to Sharma [17].

We indicate $G^\alpha(z,t) = \sum_{k=0}^{\infty} z^k p_k^\alpha(t)$ as the probability generating function.

**Theorem 2.1.** The Laplace transform $\hat{G}^\alpha(z,s) = \int_0^\infty e^{-st} G^\alpha(z,t) \, dt$, $\alpha \in (0,1]$, can be written as

$$
\hat{G}^\alpha(z,s) = \frac{s^{\alpha-1} z^{|z|} - (1-z) [a_2(s)]^{|z|} [1 - a_2(s)]^{-1}}{-\lambda [z - a_1(s)] [z - a_2(s)]}, \quad |z| \leq 1, \ \Re(s) > 0.
$$

where $a_1(s)$ and $a_2(s)$ are the zeros of $f(z,s) = z s^\alpha - (1-z)(\mu - \lambda z)$. 

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Proof. From (2.2), we can write

\[
D_t^\alpha [G^\alpha(z, t) - p_0^\alpha(t)] = -(\lambda + \mu) [G^\alpha(z, t) - p_0^\alpha(t)] + \lambda z G^\alpha(z, t).
\]  (2.4)

Using the equation on \( p_0^\alpha(t) \) we have

\[
D_t^\alpha G^\alpha(z, t) = -\lambda G^\alpha(z, t) - \mu G^\alpha(z, t) + \mu p_0^\alpha(t) + \frac{\mu}{z} [G^\alpha(z, t) - p_0^\alpha(t)],
\]  (2.5)

and after simplifying, we obtain, for \(|z| \leq 1\), the Cauchy problem

\[
\begin{cases}
z D_t^\alpha G^\alpha(z, t) = (1 - z) [G^\alpha(z, t)(\mu - \lambda z) - \mu p_0^\alpha(t)], \\
G^\alpha(z, 0) = z^t.
\end{cases}
\]  (2.6)

Applying the Laplace transform \( \tilde{G}^\alpha(z, s) = \int_0^\infty e^{-st} G^\alpha(z, t) \, dt \) to (2.6) leads to

\[
z \left[ s^\alpha \tilde{G}^\alpha(z, s) - s^\alpha G^\alpha(z, 0) \right] = (1 - z) \left[ \tilde{G}^\alpha(z, s)(\mu - \lambda z) - \mu \tilde{p}_0^\alpha(s) \right],
\]  (2.7)

where \( \tilde{p}_0^\alpha(s) = \int_0^\infty e^{-st} p_0^\alpha(t) \, dt \). After some simple algebraic calculations we then have

\[
\tilde{G}^\alpha(z, s) = \frac{s^\alpha - z s^\alpha}{z s^\alpha - (1 - z)(\mu - \lambda z)}, \quad |z| \leq 1, \quad \Re(s) > 0.
\]  (2.8)

As the above function converges in \(|z| \leq 1\), the zeros of the numerator and the denominator should coincide. Let us indicate the zeros of the numerator as

\[
a_{12}(s) = s^\alpha + \lambda + \mu \pm \left[ (s^\alpha + \lambda + \mu)^2 - 4\lambda \mu \right]^{1/2},
\]  (2.9)

with \(|a_{12}(s)| < |a_1(s)|, \Re(s) > 0\). Note that

\[
\begin{cases}
a_1(s) + a_2(s) = (s^\alpha + \lambda + \mu)/\lambda, \\
a_1(s)a_2(s) = \mu/\lambda, \\
-\lambda[1 - a_2(s)]/\lambda = s^\alpha.
\end{cases}
\]  (2.10)

By Rouché theorem [10, Page 168] we have that the only zero in the unit circle is \( a_2(s) \). Therefore it follows that

\[
s^\alpha [a_2(s)]^{i+1} - \mu [1 - a_2(s)] \tilde{p}_0^\alpha(s) = 0,
\]  (2.11)

which gives

\[
\tilde{p}_0^\alpha(s) = \frac{s^\alpha - 1 [a_2(s)]^{i+1}}{\mu[1 - a_2(s)]}.
\]  (2.12)

Now, by considering that

\[
z s^\alpha - (1 - z)(\mu - \lambda z) = -\lambda [z - a_1(s)][z - a_2(s)],
\]  (2.13)

equation (2.8) can be rewritten as

\[
\tilde{G}^\alpha(z, s) = s^\alpha \frac{z^{i+1} - (1 - z) [a_2(s)]^{i+1} [1 - a_2(s)]^{-1}}{-\lambda [z - a_1(s)][z - a_2(s)]}.
\]  (2.14)
In the following Theorem 2.2 we prove a subordination relation for the fractional queue \(N^\alpha(t), t \geq 0, \alpha \in (0, 1]\). This is essential for our next results. Before that, let us introduce some facts on the \(\alpha\)-stable subordinator and its inverse process.

Let us call \(V^\alpha(t), t \geq 0, \alpha \in (0, 1]\) the \(\alpha\)-stable subordinator (see for details Bertoin [5], cap. III) and let us define its inverse process as its hitting time

\[
E^\alpha(t) = \inf\{s > 0; V^\alpha(s) > t\}. \tag{2.15}
\]

The processes \(V^\alpha(t)\) and \(E^\alpha(t)\) are characterized by their Laplace transforms. For the \(\alpha\)-stable subordinator we have

\[
\mathbb{E}e^{-\xi V^\alpha_t} = e^{-t\xi^\alpha}, \quad \alpha \in (0, 1], \tag{2.16}
\]

and for its inverse process the time-Laplace transform reads

\[
\int_0^\infty e^{-\xi t} (\Pr\{E^\alpha(t) \in ds\}/ds) dt = \xi^{\alpha-1} e^{-\xi s}, \quad \alpha \in (0, 1]. \tag{2.17}
\]

**Theorem 2.2.** Let \(N^1(t) = N(t), t \geq 0, \) be the classical \(M/M/1\) queue and let \(E^\alpha(t), t \geq 0, \alpha \in (0, 1]\), be an inverse \(\alpha\)-stable subordinator (2.15) independent of \(N^1(t)\). The fractional \(M/M/1\) queue \(N^\alpha(t), t \geq 0, \alpha \in (0, 1]\), can be represented as

\[
N^\alpha(t) = N^1(E^\alpha(t)), \quad t \geq 0, \alpha \in (0, 1), \tag{2.18}
\]

where the equality holds for the one-dimensional distribution.

**Proof.** Let us consider the initial value problem

\[
\begin{cases}
   zD^\alpha_t G^\alpha(z, t) = (1 - z) [G^\alpha(z, t)(\mu - \lambda z) - \mu \bar{p}_0^\alpha(t)], \\
   G^\alpha(z, 0) = z^t,
\end{cases} \tag{2.19}
\]

which is equivalent to (2.2). Applying the Laplace transform we obtain

\[
z \left[s^\alpha \tilde{G}^\alpha(z, s) - s^{\alpha-1} G^\alpha(z, 0)\right] = (1 - z) \left[\tilde{G}^\alpha(z, s)(\mu - \lambda z) - \mu \bar{p}_0^\alpha(s)\right], \tag{2.20}
\]

Note that if (2.18) holds we can write

\[
\tilde{G}^\alpha(z, s) = \int_0^\infty e^{-st} \left[\sum_{k=0}^\infty z^k \int_0^\infty \Pr\{N^\alpha(y) = k\} \Pr\{E^\alpha(t) \in dy\}\right] dt \tag{2.21}
\]

\[
= \int_0^\infty e^{-st} \left[\int_0^\infty G(z, y) \Pr\{E^\alpha(t) \in dy\}\right] dt
\]

\[
= \int_0^\infty G(z, y)s^{\alpha-1} e^{-ys^\alpha} dy,
\]

and

\[
\bar{p}_0^\alpha(s) = \int_0^\infty e^{-st} \bar{p}_0^\alpha(t) dt \tag{2.22}
\]

\[
= \int_0^\infty e^{-st} \left[\int_0^\infty p_0(y) \Pr\{E^\alpha(t) \in dy\}\right] dt
\]
\[ = \int_0^\infty p_0(y)s^{\alpha-1}e^{-ys^\alpha}dy. \]

We now show that (2.21) and (2.22) satisfy (2.20). Observe that

\[ z \left[ s^\alpha \int_0^\infty G(z,y)e^{-ys^\alpha}dy \mu - z^i \right] = (1 - z) \left[ (\mu - \lambda z) \int_0^\infty G(z,y)e^{-ys^\alpha}dy - \mu \int_0^\infty p_0(y)e^{-ys^\alpha}dy \right]. \quad (2.23) \]

Consider the right hand side of (2.21). We can write

\[ (1 - z) \left[ (\mu - \lambda z) \int_0^\infty G(z,y)e^{-ys^\alpha}dy - \mu \int_0^\infty p_0(y)e^{-ys^\alpha}dy \right] = \int_0^\infty e^{-ys^\alpha}(1 - z) [(\mu - \lambda z)G(z,y) - \mu p_0(y)]dy. \quad (2.24) \]

Considering that \( G(z,y) \) and \( p_0(y) \) satisfy

\[ z \frac{\partial}{\partial y}G(z,y) = (1 - z) \left[ (\mu - \lambda z)G(z,y) - \mu p_0(y) \right], \quad (2.25) \]

we immediately obtain that

\[ (1 - z) \left[ (\mu - \lambda z) \int_0^\infty G(z,y)e^{-ys^\alpha}dy - \mu \int_0^\infty p_0(y)e^{-ys^\alpha}dy \right] = \int_0^\infty e^{-ys^\alpha} \left. \frac{\partial}{\partial y}G(z,y)dy \right|_{y=0} \]

\[ = z \left[ G(z,y)e^{-ys^\alpha} \bigg|_{y=0} + s^\alpha \int_0^\infty G(z,y)e^{-ys^\alpha}dy - z^i \right]. \]

This concludes the proof. \( \square \)

Using the Laplace transform (2.3) and the calculations carried out in Bailey [3] we can gain some insights on the mean value of the process.

**Theorem 2.3.** We have that

\[ \mathbb{E}N^\alpha(t) = i + (\lambda - \mu) \frac{t^\alpha}{\Gamma(\alpha + 1)} + \mu J^\alpha p_0^\alpha(t), \quad (2.27) \]

where

\[ J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (y-t)^{\alpha-1}f(y)dy, \quad t > 0, \quad (2.28) \]

is the Riemann–Liouville fractional integral [16].

**Proof.** By means of the Laplace transform (2.3) of the probability generating function we can write

\[ \mathbb{E}N^\alpha(s) = \frac{d}{dz}G^\alpha(z,s) \bigg|_{z=1} \quad (2.29) \]
\[
\begin{align*}
\mathbb{E}N^\alpha (t) &= i + (\lambda - \mu) \frac{t^\alpha}{\Gamma(\alpha + 1)} + \mu J^\alpha p_0^\alpha (t). \\
\end{align*}
\]  
Note the in the above calculation we have used the relation (2.10). Result (2.29) immediately yields

\[
\mathbb{E}N^\alpha (t) = i + (\lambda - \mu) \frac{t^\alpha}{\Gamma(\alpha + 1)} + \mu J^\alpha p_0^\alpha (t). 
\]  

(2.30)

**Remark 2.1.** The validity of Theorem 2.2 can be checked with the aid of formula (2.27) as follows.

\[
\begin{align*}
\mathbb{E}N^\alpha (t) &= \int_0^\infty \mathbb{E}N^1 (w) \Pr \{ E^\alpha (t) \in dw \} \\
&= i + (\lambda - \mu) \int_0^\infty w \Pr \{ E^\alpha (t) \in dw \} + \mu \int_0^\infty \int_0^t p_1^\alpha (y) dy \Pr \{ E^\alpha (t) \in dw \}. \\
\end{align*}
\]

Therefore the time-Laplace transform, recalling that \( \int_0^\infty e^{-st} dt \Pr \{ E^\alpha (t) \in dw \} = s^{-1} e^{-ws^\alpha} dw \), can be written as

\[
\begin{align*}
\mathbb{E}N^\alpha (s) &= \frac{i}{s} + \int_0^\infty ws^{-1} e^{-ws^\alpha} dw + \mu \int_0^\infty \int_0^t p_1^\alpha (y) dy s^{-1} e^{-ws^\alpha} dw \\
&= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha + 1}} + \mu s^{-1} \int_0^\infty p_1^\alpha (y) dy e^{-ws^\alpha} dw \\
&= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha + 1}} + \mu s^{-1} \int_0^\infty \frac{p_1^\alpha (y)}{s^\alpha} e^{-ys^\alpha} dw \\
&= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha + 1}} + \frac{\tilde{p}_0^\alpha (s^\alpha)}{s}. \\
\end{align*}
\]  

(2.32)

Now, by noticing that \( p_0^\alpha (s^\alpha) / s = \tilde{p}_0^\alpha (s^\alpha) / s^\alpha \) (see formula (2.12)) we arrive at

\[
\begin{align*}
\mathbb{E}N^\alpha (s) &= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha + 1}} + \frac{\tilde{p}_0^\alpha (s)}{s^\alpha}, \\
\end{align*}
\]  

(2.33)

which leads to formula (2.27).
Remark 2.2. A different form of formula (2.29) can be achieved by writing
\[
\hat{E} N^\alpha(s) = \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha+1}} + \frac{s^{\alpha-1} a_2^2}{s^{\alpha}(1 - a_2)} \tag{2.34}
\]
\[
= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha+1}} - \frac{s^{\alpha-1}(1 - a_2)}{s^{2\alpha}} - \frac{s^{\alpha-1} a_2^2}{s^{\alpha}(1 - a_2)},
\]
where we used the fact that $1/(1 - a_2) = -\lambda(1 - a_1)/s^{\alpha}$. Furthermore, after considering
\[
a_1 = \frac{s^{\alpha} + \lambda + \mu}{\lambda} - a_2 = \frac{\mu}{\lambda a_2}
\]
we arrive at
\[
\hat{E} N^\alpha(s) = \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha-1}} + \frac{s^{\alpha-1} a_2^2(\mu - \lambda a_2)}{s^{2\alpha}} \tag{2.36}
\]
\[
= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha-1}} + \frac{s^{\alpha-1} \mu a_2^2 - \lambda a_2^2}{s^{2\alpha}}
\]
\[
= \frac{i}{s} + \frac{\lambda - \mu}{s^{\alpha-1}} + \frac{\mu a_2^2 - \lambda a_2^2}{s^{\alpha+1}}.
\]

Let us now address the problem of finding explicit results for the state probabilities $p_k^\alpha(t) = \Pr \{N^\alpha(t) = k | N^\alpha(0) = i\}$ of the proposed fractional queue model. We start by using the subordination relation stated in Theorem (2.2) with the classical solution of the M/M/1 queue in terms of modified Bessel functions of the first kind. In the non-fractional case ($\alpha = 1$) we have \([3,\ Page\ 154]\)
\[
p_k^1(t) = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(k-2)} e^{-(\lambda+\mu)t} I_{i-k} \left(2(\lambda \mu)^{1/2}t\right) \tag{2.37}
\]
\[
+ \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(k-i)} \int_0^t e^{-(\lambda+\mu)t} \left\{ \lambda I_{i+k+2} \left(2(\lambda \mu)^{1/2}t\right) - 2(\lambda \mu)^{1/2} I_{i+k+1} \left(2(\lambda \mu)^{1/2}t\right) \right\} dt,
\]
where $I_{\nu}(z)$ is the modified Bessel function of the first kind.

The state probabilities $p_k^\alpha(t)$, $t \geq 0$, $k \geq 0$, $\alpha \in (0,1]$ can thus be determined formally by subordination in the following way:
\[
p_k^\alpha(t) = \int_0^\infty p_k^1(y) \Pr \{E^\alpha(t) \in dy\}. \tag{2.38}
\]

Using the time-Laplace transform $\tilde{p}_k^\alpha(s) = \int_0^\infty e^{-st} p_k^\alpha(t) dt$ we have
\[
\tilde{p}_k^\alpha(s) = \int_0^\infty p_k^1(y) s^{\alpha-1} e^{-ys^\alpha} dy \tag{2.39}
\]
\[
= \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(k-2)} \int_0^\infty e^{-(\lambda+\mu)y} I_{i-k} \left(2(\lambda \mu)^{1/2}y\right) s^{\alpha-1} e^{-ys^\alpha} dy
\]
\[
+ \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(k-i)} \int_0^\infty \left[ \int_0^y e^{-(\lambda+\mu)\tau} \left\{ \lambda I_{i+k+2} \left(2(\lambda \mu)^{1/2}\tau\right) - 2(\lambda \mu)^{1/2} I_{i+k+1} \left(2(\lambda \mu)^{1/2}\tau\right) \right\} d\tau \right] s^{\alpha-1} e^{-ys^\alpha} dy
\]
Applying the well-known Laplace transform for \(I_k\) we get

\[
\hat{p}_k^2(s) = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}(k-2)} s^{\alpha-1} \left[ 2(\lambda\mu)^{1/2} \right]^{-i-k} \left[ s^\alpha + \lambda + \mu - \sqrt{(s^\alpha + \lambda + \mu)^2 - 4\lambda\mu} \right]^{i+k}
\]

(2.40)
Theorem 2.4. The state probabilities

\[
   p_k^\alpha(t) = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^k + \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{r-m}{r+m} \lambda^r \mu^{m-1} t^r \alpha(r-m-\alpha) E_{\alpha,\alpha(r+m)-\alpha+1} [-\lambda \mu t^\alpha] \tag{2.41}
\]

where

\[
   E_{\beta,\gamma}^\delta(w) = \sum_{r=0}^{\infty} \frac{(\delta)_r w^r}{r! \Gamma(\beta r + \gamma)} = \sum_{r=0}^{\infty} \frac{w^r \Gamma(\delta + r)}{r! \Gamma(\beta r + \gamma) \Gamma(\delta)}, \quad w, \gamma, \beta, \delta \in \mathbb{C}, \quad \Re(\beta) > 0, \tag{2.43}
\]

is the Generalized Mittag–Leffler function [12].

Proof. Recurring to Theorem 2.2 we can write for \(k \geq 0, \lambda \neq \mu, \)

\[
   p_k^\alpha(t) = \int_0^\infty p_k^\alpha(s) \Pr\{E^\alpha(t) \in ds\} \tag{2.44}
\]

Applying the Laplace transform to both terms on the right-hand side we obtain

\[
   \tilde{p}_k^\alpha(z) = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^k z^{-1} + \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{r-m}{r+m} \lambda^r \mu^{m-1} t^r \alpha(r-m-\alpha) e^{-(\lambda+\mu)s} \cdot \int_0^\infty e^{-(\lambda+\mu)s} s^{r+m-1} \Pr\{E^\alpha(t) \in ds\} \tag{2.45}
\]
Furthermore, it is worth noticing that this geometric distribution coincides with that of the classical case

\[ \alpha \]

We now focus on a related point process which is relatively simpler to treat. Let

\[ \text{Analogous to the preceding section, in order to produce a fractional process related to the classical birth-death process} \]

\[ \text{This has been already carried out in Orsingher and Polito [14]. In the following, we exploit a subordination relation} \]

\[ \text{To invert equation (2.45) we use the Laplace transform (see formula (2.3.24) of Mathai and Haubold [13])} \]

\[ \int_0^\infty e^{-zt}t^{\gamma-1}E_{\beta,\gamma}(zt^\beta)dt = \frac{z^{\beta \delta - \gamma}}{(z^\beta - w)^{\delta}}, \]

(2.46)

which immediately leads to (2.42).

\[ \square \]

Remark 2.3. When \( \alpha = 1 \), formula (2.42) becomes the classical solution (2.41) because

\[ E_{1,\gamma}^\delta(w) = e^w/\Gamma(\delta). \]

Remark 2.4. Result (2.42) is particularly interesting because its first addend contains the steady-state solution

\[ sp^\alpha_k(t) = \lim_{t \to \infty} p^\alpha_k(t) = \left(1 - \frac{\lambda}{\mu}\right)^k \left(\frac{\lambda}{\mu}\right)^k, \quad k \geq 0. \]

(2.47)

Furthermore, it is worth noticing that this geometric distribution coincides with that of the classical case \( \alpha = 1 \). The whole difference between the fractional and the non-fractional case lies in the transient regime.

## 3 Path simulation and parameter estimation for the fractional linear birth-death process

We now focus on a related point process which is relatively simpler to treat. Let \( \mathcal{N}(t), t \geq 0 \) be a classical linear birth-death process with \( \lambda k > 0, \mu k > 0 \) as its birth and death rates, respectively. Furthermore, define \( S_k, k \in \mathbb{N} \cup \{0\} \), as the sojourn time of the process \( \mathcal{N}(t), t \geq 0 \) in state \( k \), i.e., given that the process is in state \( k \), \( S_k \) is the time until the process leaves that state. It is well-known [18, Section 3.2, Chapter VI] that

\[ \Pr\{S_k \geq t\} = \exp\left[-(\lambda + \mu)kt\right], \]

(3.1)

and thus,

\[ \Pr\{S_k \in dt\}/dt = (\lambda + \mu)k \exp\left[-(\lambda + \mu)kt\right], \quad t \geq 0. \]

(3.2)

Analogous to the preceding section, in order to produce a fractional process related to the classical birth-death process it would be natural to substitute the unit-order time-derivative in the governing equations with a fractional derivative. This has been already carried out in Orsingher and Polito [14]. In the following, we exploit a subordination relation similar to that used in Section 2 in order to continue the analysis. In particular our aim is to develop methods
suitable to simulation and parameter estimation that will be also applied to the fractional M/M/1 case in the last section.

Recall thus that a fractional linear birth-death process $N^\alpha(t)$, $t \geq 0$, $\alpha \in (0, 1]$ satisfies the subordination-relation [14]

$$N^\alpha(t) \overset{d}{=} N[E^\alpha(t)], \quad (3.3)$$

where $E^\alpha(t)$ is the right-inverse process to an $\alpha$-stable subordinator defined in the previous section. Using the above relation, we can easily calculate the distribution of the sojourn or holding times $S_k^\alpha$, $k, \in \mathbb{N} \cup \{0\}$ for the fractional linear birth-death process $N^\alpha(t)$, $t \geq 0$, as follows.

$$\Pr\{S_k^\alpha \in dt\}/dt = (\lambda + \mu)k \int_0^\infty \exp[-(\lambda + \mu)ks] \Pr\{E^\alpha(t) \in ds\}$$

$$= (\lambda + \mu)kt^{\alpha-1}E_{\alpha,1}[-(\lambda + \mu)kt^\alpha], \quad t \geq 0, \, k \geq 1, \, 0 < \alpha \leq 1. \quad (3.4)$$

Hence, the holding or sojourn time $S_k^\alpha$ of the fractional linear birth-death process $N^\alpha(t)$ is Mittag–Leffler distributed. Another equivalent way to derive the event time distribution above is to replace the unit-order derivative in equation (3.4) of Taylor and Karlin [18, page 356] by the Caputo’s fractional derivative operator $D_k^\alpha$ used by [14]. That is,

$$D_k^\alpha \Pr\{S_k^\alpha \geq t\} = -(\lambda + \mu)k \Pr\{S_k^\alpha \geq t\}, \quad (3.5)$$

and solving the above equation, we obtain

$$\Pr\{S_k^\alpha \geq t\} = E_{\alpha,1}[-(\lambda + \mu)kt^\alpha], \quad (3.6)$$

which gives equation (3.4).

An interesting observation is that the birth and death sojourn times $B_k$ and $D_k$, respectively are no longer independent for $0 < \alpha < 1$, i.e.,

$$P(S_k \geq t) = P(\min\{B_k, D_k\} \geq t) \quad (3.7)$$

$$= P(B_k > t, D_k > t) \quad (3.8)$$

$$\neq P(B_k > t)P(D_k > t), \quad (3.9)$$

When the process is in state $k$, $k \in \mathbb{N} \cup \{0\}$, it transitions to the neighboring states $k + 1$ and $k - 1$ with probabilities $\lambda/(\lambda + \mu)$ and $\mu/(\lambda + \mu)$, respectively. Following Taylor and Karlin [18, page 358], a standard procedure to simulate trajectories of a fractional linear birth-death process is as follows:

**ALGORITHM:**

i) Fix the birth intensity $\lambda$, the death intensity $\mu$, and the initial population size $N^\alpha(0) = m$.

ii) Simulate $S_1^\alpha \overset{d}{=} \xi_{1/\alpha}T_\alpha$ and $U \overset{d}{=} U(0, 1)$.

iii) If $U < \frac{\lambda}{\lambda + \mu}$ then $N^\alpha(s_1) = m + 1$. Otherwise, $N^\alpha(s_1) = m - 1$.

iv) Continuing in the same fashion and supposing that the current process state is $N^\alpha(s_{k-1}) = k$, generate $S_k^\alpha \overset{d}{=} \xi_{1/\alpha}T_\alpha$ and $U \overset{d}{=} U(0, 1)$. If $U < \frac{\lambda}{\lambda + \mu}$ then $N^\alpha(s_k) = k + 1$. Otherwise, $N^\alpha(s_k) = k - 1$. Repeat iv) until the desired population size is achieved or until extinct.
Note that $\mathcal{E}_k$ is negative exponentially distributed ($\mathcal{E}_k \stackrel{d}{=} \exp((\lambda + \mu)k)$), and $T_\alpha$ is a one-sided $\alpha$-stable and independently distributed random variable. Below are some sample paths of the fractional linear birth-death process. The nonlinear trend, the longer holding times, and the slow or bursting behavior of the fractional linear birth-death process are apparent in Figure 1.

We now provide point estimation algorithms for the parameters $\alpha, \lambda$, and $\mu$. Assume that a sample trajectory of size $n$ corresponding to the $n$ random inter-event times $S_k$’s of the fractional linear birth-death process is observed, where there are $n_B$ births, $n_D$ deaths, and $n = n_B + n_D$. Recall the structural representation of the Mittag–Leffler distributed random sojourn time $S_k \stackrel{d}{=} \mathcal{E}_k^{1/\alpha}T_\alpha$, where $\mathcal{E}_k \stackrel{d}{=} \exp(\theta k)$ is independent of a one-sided $\alpha^+$-stable distributed random variable $T_\alpha$, and $\theta = \lambda + \mu$. Let $S'_k = \ln(S_k)$. Then it is well-known that the mean and variance [see details in 7] of the log-transformed $k$-th random sojourn time of the fractional linear birth-death process are

$$\mu_{S'_k} = \frac{-\ln(\theta)}{\alpha} - \gamma,$$

and

$$\sigma^2_{S'_k} = \frac{\pi^2}{2\alpha^2 - \frac{1}{6}},$$

respectively, where $\gamma \approx 0.5772156649$ is the Euler–Mascheroni’s constant. Following [6], the first two moments above therefore suggest that the simple linear regression model below can be fitted:

$$S'_k = b_0 + b_1 \ln k + \varepsilon_k, \quad k = 1, \ldots, n,$$

where

$$b_0 = \frac{-\ln(\theta)}{\alpha} - \gamma, \quad b_1 = -1/\alpha,$$

Figure 1: Sample paths of the fractional linear birth-death process: (top left) $\alpha = 1, \lambda = 5, \mu = 15$; (top right) $\alpha = 0.8, \lambda = 5, \mu = 15$; (bottom left) $\alpha = 0.8, \lambda = 15, \mu = 5$; (bottom right) $\alpha = 0.8, \lambda = \mu = 5$ with an initial population of $m = 500$. 
and \( \varepsilon_k \overset{iid}{=} (\mu_k, \sigma_k^2 = \sigma_{S_k'0}^2) \) \( \overset{iid}{=} \ln (E^{1/\alpha} T_k) + \gamma, E \overset{d}{=} \exp(1) \). We point out that the error distribution depends only on \( \alpha \) and is independent of the state \( k \) and \( \theta \), and this gives us a simple way of testing the rate fit as follows. Generate, say \( m \) samples (each of sample size \( n \)) from the error distribution using \( \hat{\alpha} \). For a fixed significance level, test equality of two parent populations of the observed residuals and each of the \( m \) simulated errors using the two-sample Kolmogorov–Smirnov test, for instance. The proportion of the null acceptance out of \( m \) tests can then be used to measure model fit.

Letting \( \sigma_k^2 \) or \( \sigma_{S_k'}^2 \) in formula (3.11) equal to its unbiased estimator \( \hat{\sigma}_k^2 = \sum_{j=1}^{n} \hat{\varepsilon}_j^2/(n-2) \), we readily obtain the residual-based point estimators

\[
\hat{\alpha} = 3 \left( \frac{\hat{\sigma}_k^2/\pi^2 + 1/6}{n} \right)^{-1/2}, \quad \hat{\theta} = \exp \left( -\hat{\alpha} \left( \hat{b}_0 + \gamma \right) \right), \tag{3.14}
\]

of the model parameters \( \alpha \) and \( \theta \), correspondingly, where \( \hat{\varepsilon}_k = S_k' - \hat{S}_k' \), and \( \hat{S}_k' = \hat{b}_0 + \hat{b}_1 \ln k \). Notice that the above estimators exploit the residuals to estimate \( \alpha \) instead of the negative inverse of the least squares (LS) estimate of the slope \( b_1 \). Furthermore, the least squares estimators of the slope and intercept are

\[
\hat{b}_1 = \frac{\sum_{j=1}^{n} S_j'}{\sum_{j=1}^{n} (\ln j - \ln k)^2}, \quad \hat{b}_0 = \frac{\bar{S}_k'}{\bar{k}} \cdot \ln \bar{k}, \tag{3.15}
\]

where \( \ln k = \sum_{j=1}^{n} \ln j/n \) and \( \bar{S}_k'/n = \sum_{j=1}^{n} S_j'/n \). Hence, the closed-form point estimators of the intensities \( \lambda \) and \( \mu \) are

\[
\hat{\lambda} = \frac{\# \text{ of births}}{n} \cdot \hat{\theta} = \frac{nB}{n} \cdot \hat{\theta}, \tag{3.16}
\]

and

\[
\hat{\mu} = \frac{\# \text{ of deaths}}{n} \cdot \hat{\theta} = \frac{nD}{n} \cdot \hat{\theta} = \hat{\lambda} - \hat{\lambda} = \frac{nD}{n} \cdot \hat{\theta} = \left( 1 - \frac{nB}{n} \right) \cdot \hat{\theta}, \tag{3.17}
\]

respectively. Table 2 in the appendix shows some test results based on the percent bias

\[
100 \times \frac{|\text{average estimate}-\text{parameter value}|}{\text{parameter value}}
\]

and the coefficient of variation

\[
CV = 100 \times \frac{\text{standard deviation of the estimates}}{\text{average estimate}}
\]

using 1000 simulation runs. Note that we replaced the least squares estimator \( \hat{b}_0 \) by the average of \( S_k' \) \( (1/\hat{\alpha}) \ln k \) to improve small sample performance. Apparently, the proposed point estimators, especially \( \hat{\alpha} \) performed relatively well even if the sample size is as small as 100.

We now provide formulas for the interval estimators of the model parameters. It is worth emphasizing that the explicit expressions of the estimators can be utilized to obtain resampling-based interval estimates especially for relatively small sample sizes. It is shown in [7] that

\[
\sqrt{n} (\hat{\alpha} - \alpha) \overset{d}{\longrightarrow} N \left( 0, \frac{\alpha^2 (32 - 20\alpha^2 - \alpha^4)}{40} \right), \tag{3.18}
\]

and a residual-based \((1 - \epsilon)100\%\) confidence interval for \( \alpha \) directly follows as

\[
\hat{\alpha} \pm z_{\epsilon/2} \sqrt{\frac{\hat{\alpha}^2 (32 - 20\hat{\alpha}^2 - \hat{\alpha}^4)}{40n}}, \tag{3.19}
\]
where $z_{\epsilon/2}$ is the $(1 - \epsilon/2)$th quantile of the standard normal distribution, and $0 < \epsilon < 1$. We will now show the asymptotic normality of the estimators $\hat{\lambda}$ and $\hat{\mu}$.

**Theorem 3.1.** Let $\hat{p} = n_B/n$ and $p = \lambda/\theta$. Then

$$\sqrt{n} \left( \hat{\lambda} - \lambda \right) \xrightarrow{d} N \left( 0, \theta^2 p(1-p) + p^2 \sigma^2_\theta \right) \quad (3.20)$$

as $n \to \infty$ where

$$\sigma^2_\theta = e^{-2\alpha \gamma} \left( b_0 + \gamma \right)^2 \left[ \frac{\alpha^2 (32 - 20 \alpha^2 - \alpha^4)}{40} + n \alpha^2 \sigma^2_z \left( \frac{1}{n} + \frac{\ln k^2}{s} \right) \right], \quad (3.21)$$

and $s = \sum_{j=1}^{n} \left( \ln j - \ln k \right)^2$.

**Proof.** It can be deduced from [6] and the asymptotic property of a Bernoulli/binomial sampled proportion that

$$\sqrt{n} \left( \frac{\hat{p} - p}{\theta - \theta} \right) \xrightarrow{d} N \left( 0, \Sigma \right) \quad (3.22)$$

as $n \to \infty$, where the variance-covariance matrix $\Sigma$ is defined as

$$\Sigma = \begin{pmatrix} p(1-p) & 0 \\ 0 & \sigma^2_\theta \end{pmatrix}. \quad (3.23)$$

Invoking a standard result on asymptotic theory, the two-dimensional Central Limit Theorem implies that

$$\sqrt{n}(\mathbf{h}(\hat{\theta}_n) - \mathbf{h}(\theta)) \xrightarrow{d} N \left( 0, \mathbf{h}(\theta)^T \Sigma \mathbf{h}(\theta) \right), \quad (3.24)$$

where $\hat{\theta}_n = (\hat{\theta}, \hat{\theta})^T$, $\mathbf{h}$ is a mapping from $\mathbb{R}^2 \to \mathbb{R}$, $\mathbf{h}(\mathbf{x})$ is continuous in a neighborhood of $\theta \in \mathbb{R}^2$, $\mathbf{h}(p, \theta) = p \cdot \theta$, and $\mathbf{h}(p, \theta) = (\theta, \theta)^T$. This concludes the proof.

**Theorem 3.2.** Let $\hat{q} = 1 - \hat{p} = n_D/n$, $p = \lambda/\theta$, and $q = \mu/\theta$. Then

$$\sqrt{n} (\hat{\mu} - \mu) \xrightarrow{d} N \left( 0, \theta^2 p(1-p)^2 + (1-p)^2 \sigma^2_\theta \right) \quad (3.25)$$

as $n \to \infty$.

**Proof.** The proof directly follows from the preceding theorem except that here we consider $\mathbf{h}(p, \theta) = (1-p) \cdot \theta$ and $\mathbf{h}(p, \theta) = (-\theta, (1-p))^T$.

We can now approximate the $(1 - \epsilon)100\%$ confidence interval for $\lambda$ and $\mu$ as

$$\hat{\lambda} \pm z_{\epsilon/2} \cdot \tilde{\sigma}_\lambda, \quad \text{and} \quad \hat{\mu} \pm z_{\epsilon/2} \cdot \tilde{\sigma}_\mu, \quad (3.26)$$

respectively, where

$$\tilde{\sigma}_\lambda = \sqrt{\frac{\hat{\theta}^2 \hat{p} \hat{q} + \hat{p}^2 \sigma^2_\theta}{n}}, \quad (3.27)$$
\[
\hat{\sigma}_\mu = \sqrt{\frac{\hat{\theta}^2 \hat{\mu}^2 + (1 - \hat{\mu})^2 \hat{\sigma}^2}{n}}.
\] (3.28)

We now calculate the coverage probabilities using sample sizes \( n = 10^2, 10^3, 10^4 \) and \( 10^3 \) simulations to test our interval estimators of \( \lambda \) and \( \mu \) only. Notice that the interval estimator for \( \nu \) has already been shown to perform well in past related studies [see 7, 6]. Table 3 of the appendix clearly illustrates that the coverage probabilities of the interval estimators are closely approaching the true confidence level when \( n \) is at least 1000. If a narrower interval and a larger coverage are preferred then our simulations suggested that the previous point estimate replacement can be used instead. Note that this replacement and the above simple fit testing schemes can be directly applied to the fractional birth and fractional death processes in [6] to enhance performance of the point and interval estimators as well.

Overall, Tables 2 and 3 provide additional merit to the proposed point and interval estimators of the model parameters. Aside from the computational simplicity of the proposed parameter estimation methods, the rate fits can also be checked straightforwardly.

4 Trajectory generation and parameter estimation for the fractional simple linear birth-death or M/M/1 queue process

Let \( N(t), t \geq 0 \) be a classical simple birth-death process with \( \lambda > 0, \mu > 0 \) as its constant birth and death rates, respectively. Furthermore, define \( S_k, k \in \mathbb{N} \cup \{0\} \), as the sojourn time of the process \( N(t), t \geq 0 \) in state \( k \), i.e. given that the process is in state \( k \), \( S_k \) is the time until the process leaves that state. Then from the preceding sections, it can easily be deduced that the holding/sojourn time \( S^\alpha_k \)’s of the fractional simple birth-death or M/M/1 queue \( N_\alpha(t) \) are independently and identically (IID) Mittag–Leffler distributed, i.e.,

\[
\Pr\{S^\alpha_k \in dt\}/dt = (\lambda + \mu)t^{\alpha-1}E_{\alpha,\alpha}[-(\lambda + \mu)t^\alpha], \quad t \geq 0, \ \alpha \in (0, 1].
\] (4.1)

Note that everything here immediately follows from the previous section except that the \( E_k \)’s of the fractional simple birth-death or M/M/1 process is observed, where there are \( n_B \) births, \( n_D \) deaths, and \( n = n_B + n_D \). From [7], a method-of-moments estimator for \( \alpha \) is

\[
\hat{\alpha} = \frac{\pi}{\sqrt{3(\hat{\sigma}_{S^\alpha_k'} + \pi^2/6)}}
\] (4.2)

and

\[
\hat{\theta} = \exp\left(-\hat{\alpha}(\hat{E}S^\alpha_k' + \gamma)\right)
\] (4.3)

is an estimator for \( \theta \). Recall that the asymptotic normality of \( \hat{\alpha} \) follows from the earlier result \( (3.18) \). The appendix’s Table 4 shows some test results based on the percent bias and CV using 1000 simulation runs. Apparently, the proposed point estimators perform even better than the ones in the linear case.

As in the preceding section, we provide formulas for the interval estimators of the model parameters. The explicit expressions of the estimators can be used to obtain resampling-based interval estimates especially for small sample sizes. A \((1 - \epsilon)100\%\) confidence interval for \( \alpha \) is directly obtained from the previous section by simply replacing the point estimator of \( \alpha \).
Theorem 4.1. Let $\hat{p} = n_B/n$ and $p = \lambda/\theta$. Then
\[
\sqrt{n} \left( \hat{\lambda} - \lambda \right) \xrightarrow{d} N \left( 0, \theta^2 p(1 - p) + p^2 \sigma_\theta^2 \right)
\] (4.4)
as $n \to \infty$ where
\[
\sigma_\theta^2 = \frac{\theta^2 \left[ 20\pi^4 (2 - \alpha^2) - 3\pi^2 (\alpha^4 + 20\alpha^2 - 32)(\ln \theta)^2 - 720\alpha^3 (\ln \theta) \zeta(3) \right]}{120\pi^2},
\] (4.5)
and where $\zeta(3)$ is the Riemann-zeta function evaluated at 3.

Proof. We omit the routine proof as it follows from the previous theorem. \qed

Theorem 4.2. Let $\hat{q} = 1 - \hat{p} = n_D/n$, $p = \lambda/\theta$, and $q = \mu/\theta$. Then
\[
\sqrt{n} \left( \hat{\mu} - \mu \right) \xrightarrow{d} N \left( 0, \theta^2 p(1 - p)^2 + (1 - p)^2 \sigma_\theta^2 \right)
\] (4.6)
as $n \to \infty$.

Proof. This directly follows from the preceding theorem. \qed

We now test our interval estimators for $\lambda$ and $\mu$ by calculating the coverage probabilities using sample sizes $n = 10^2, 10^3$, and $10^4$ simulations. Table 5 of the appendix clearly demonstrates that the coverage probabilities of the interval estimators start to approach the true confidence level when $n$ is at least 1000.

In general, the empirical tests indicate better performance of the proposed point and interval estimators than the procedures for the fractional linear birth-death process.

5 Application

We demonstrate our methods using two real financial datasets: 1) the monthly Standard & Poor’s (S&P) index from January 1, 1980 until August 13, 2013 with 248 positive and 155 negative changes; 2) the semi-annual Dow Jones Industrial Average (DJIA) from 1970 until 2013 with 58 positive and 28 negative changes. The data can be downloaded directly from finance.yahoo.com and http://www.djindexes.com, respectively. In particular, we apply the simple fractional birth-death or M/M/1 queue to model the number of positive-negative index changes.

Table 1 provides the point and 95% interval estimates for the two financial datasets.

| Parameter | S&P Data | DJIA Data |
|-----------|----------|-----------|
|           | Point    | Interval  | Point    | Interval  |
| $\alpha$  | 0.949    | (0.895, 1.002) | 0.897    | (0.780, 1.014) |
| $\lambda$ | 0.032    | (0.024, 0.041) | 0.004    | (0.001, 0.008) |
| $\mu$     | 0.020    | (0.015, 0.026) | 0.002    | (0.000, 0.004) |

The above estimates simply suggest that the monthly S&P and semi-annual DJIA changes are highly likely to be non-standard birth-death processes. We also examined the rate fit by simulating 1000 samples using the point
estimates, and tested the equality of two parent populations using the two-sample Kolmogorov-Smirnov’s test. The proportions of p-values larger than 0.05 are 98.5% and 98.6% for the two datasets, accordingly, which significantly indicate good model fit. These real-world examples clearly demonstrate that the proposed fractional birth-death model is more general and can also be used as a proof-of-concept or a smoothing tool for the standard birth-death process.

6 Appendix

Table 2: Percent bias and dispersion of the proposed point estimators of $\alpha, \lambda$ and $\mu$ in Section 3.

| $(\alpha, \lambda, \mu)$ | Estimator | $n = 10^2$ | $n = 10^3$ | $n = 10^4$ |
|--------------------------|-----------|------------|------------|------------|
|                          |           | Bias CV    | Bias CV    | Bias CV    |
| (0.1, 0.5, 9)            | $\hat{\alpha}$ | 0.989 8.902 | 0.075 2.880 | 0.010 0.903 |
|                          | $\hat{\lambda}$ | 22.389 46.798 | 2.920 25.643 | 0.394 10.729 |
|                          | $\hat{\mu}$ | 24.938 46.696 | 3.568 23.146 | 0.363 9.358 |
| (0.25, 1, 6)             | $\hat{\alpha}$ | 1.125 8.919 | 0.019 2.067 | 0.028 0.909 |
|                          | $\hat{\lambda}$ | 22.480 45.615 | 4.749 21.624 | 0.020 9.309 |
|                          | $\hat{\mu}$ | 23.273 45.997 | 4.309 20.478 | 0.171 9.467 |
| (0.5, 5, 5)              | $\hat{\alpha}$ | 0.709 8.279 | 0.140 2.474 | 0.016 0.822 |
|                          | $\hat{\lambda}$ | 20.249 42.057 | 3.343 21.170 | 0.545 8.655 |
|                          | $\hat{\mu}$ | 20.992 41.142 | 3.767 21.090 | 0.621 8.637 |
| (0.75, 7, 1)             | $\hat{\alpha}$ | 0.316 7.213 | 0.109 2.272 | 0.004 0.679 |
|                          | $\hat{\lambda}$ | 10.372 38.858 | 1.945 17.192 | 0.286 6.991 |
|                          | $\hat{\mu}$ | 10.205 42.200 | 2.443 19.205 | 0.204 7.472 |
| (0.95, 10, 0.5)          | $\hat{\alpha}$ | 0.924 5.386 | 0.077 1.764 | 0.008 0.532 |
|                          | $\hat{\lambda}$ | 9.544 28.672 | 1.539 13.572 | 0.272 5.186 |
|                          | $\hat{\mu}$ | 8.875 41.985 | 1.559 19.671 | 0.468 7.193 |

Table 3: Coverage probabilities of the proposed interval estimators of $\lambda$ and $\mu$ using a 95% confidence level in Section 3.

| $(\alpha, \lambda, \mu)$ | Parameter | $n = 10^2$ | $n = 10^3$ | $n = 10^4$ |
|--------------------------|-----------|------------|------------|------------|
| (0.1, 10, 90)            | $\lambda$ | 0.879 0.936 | 0.954       |
|                          | $\mu$     | 0.888 0.937 | 0.955       |
| (0.25, 70, 30)           | $\lambda$ | 0.892 0.925 | 0.955       |
|                          | $\mu$     | 0.876 0.926 | 0.958       |
| (0.5, 50, 50)            | $\lambda$ | 0.896 0.934 | 0.946       |
|                          | $\mu$     | 0.894 0.933 | 0.947       |
| (0.7, 5, 95)             | $\lambda$ | 0.864 0.922 | 0.947       |
|                          | $\mu$     | 0.882 0.924 | 0.950       |
| (0.95, 20, 80)           | $\lambda$ | 0.900 0.948 | 0.951       |
|                          | $\mu$     | 0.910 0.950 | 0.959       |
Table 4: Percent bias and dispersion of the proposed point estimators of $\alpha, \lambda$ and $\mu$ in Section 4.

| $(\alpha, \lambda, \mu)$ | Estimator | $n = 10^2$ | $n = 10^3$ | $n = 10^4$ |
|--------------------------|-----------|------------|------------|------------|
|                          |           | Bias   | CV     | Bias   | CV     | Bias   | CV     |
| $(0.1, 0.5, 9)$          | $\hat{\alpha}$ | 1.180  | 9.077  | 0.262  | 2.924  | 0.015  | 0.864  |
|                          | $\hat{\lambda}$ | 4.551  | 45.894 | 0.731  | 15.929 | 0.072  | 4.804  |
|                          | $\hat{\mu}$ | 6.017  | 25.210 | 0.964  | 8.649  | 0.122  | 2.728  |
| $(0.25, 1, 6)$           | $\hat{\alpha}$ | 0.973  | 8.836  | 0.044  | 2.786  | 0.022  | 0.916  |
|                          | $\hat{\lambda}$ | 7.344  | 31.717 | 0.270  | 11.330 | 0.054  | 3.501  |
|                          | $\hat{\mu}$ | 5.322  | 22.923 | 0.337  | 7.979  | 0.021  | 2.450  |
| $(0.5, 5, 5)$            | $\hat{\alpha}$ | 0.506  | 8.043  | 0.016  | 2.534  | 0.044  | 0.818  |
|                          | $\hat{\lambda}$ | 6.411  | 24.339 | 0.088  | 8.609  | 0.050  | 2.640  |
|                          | $\hat{\mu}$ | 5.242  | 28.822 | 0.154  | 8.869  | 0.048  | 2.682  |
| $(0.75, 7, 1)$           | $\hat{\alpha}$ | 0.793  | 7.568  | 0.054  | 2.388  | 0.027  | 0.701  |
|                          | $\hat{\lambda}$ | 3.933  | 21.190 | 0.452  | 6.796  | 0.204  | 1.901  |
|                          | $\hat{\mu}$ | 6.330  | 32.114 | 0.497  | 10.516 | 0.098  | 3.254  |
| $(0.95, 10, 0.5)$        | $\hat{\alpha}$ | 0.541  | 5.587  | 0.019  | 1.795  | 0.007  | 0.558  |
|                          | $\hat{\lambda}$ | 1.820  | 13.947 | 0.030  | 4.508  | 0.031  | 1.540  |
|                          | $\hat{\mu}$ | 0.061  | 43.363 | 0.147  | 14.001 | 0.121  | 4.539  |

Table 5: Coverage probabilities of the proposed interval estimators of $\lambda$ and $\mu$ using a 95% confidence level in Section 4.

| $(\alpha, \lambda, \mu)$ | Parameter | $n = 10^2$ | $n = 10^3$ | $n = 10^4$ |
|--------------------------|-----------|------------|------------|------------|
|                          |           | $\lambda$ | $\mu$    | $\lambda$ | $\mu$    | $\lambda$ | $\mu$    |
| $(0.1, 0.10, 0.9)$       | $\lambda$ | 0.921  | 0.950  | 0.951  |
|                          | $\mu$    | 0.932  | 0.959  | 0.948  |
| $(0.25, 0.70, 0.30)$     | $\lambda$ | 0.936  | 0.954  | 0.955  |
|                          | $\mu$    | 0.927  | 0.942  | 0.958  |
| $(0.5, 0.50, 0.50)$      | $\lambda$ | 0.932  | 0.957  | 0.949  |
|                          | $\mu$    | 0.925  | 0.948  | 0.953  |
| $(0.7, 0.75, 0.95)$      | $\lambda$ | 0.869  | 0.944  | 0.947  |
|                          | $\mu$    | 0.931  | 0.955  | 0.950  |
| $(0.95, 0.20, 0.80)$     | $\lambda$ | 0.927  | 0.961  | 0.959  |
|                          | $\mu$    | 0.917  | 0.947  | 0.947  |

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References

[1] Abate, J, Whitt, W. Transient behavior of the M/M/1 queue via Laplace transforms. *Advances in Applied Probability*, 20:145–178, 1988.

[2] Bailey, NTJ. A continuous time treatment of a simple queue using generating functions. *Journal of the Royal Statistical Society. Series B*, 288–291, 1954.

[3] Bailey, NTJ. *The elements of stochastic processes with applications to the natural sciences*, volume 25. Wiley–Interscience, 1990.

[4] Beghin, L, Orsingher, E. Fractional Poisson processes and related planar random motions. *Electronic Journal of Probability*, 14(61):1790–1826, 2009.

[5] Bertoin, J. *Lévy Processes*. Cambridge University Press, 1996.

[6] Cahoy, DO, Polito, F. Parameter estimation for fractional birth and fractional death processes. *Statistics and Computing*, DOI:10.1007/s11222-012-9365-1.

[7] Cahoy, DO, Uchaikin, V, Woyczynski, W. Parameter estimation for fractional Poisson processes. *Journal of Statistical Planning and Inference*, 140(11):3106–3120, 2010.

[8] Champernouwe, DG. An elementary method of solution of the queueing problem with a single server and constant parameters. *Journal of the Royal Statistical Society. Series B*, 18(1):125–128, 1956.

[9] Fedotov, S, Falconer, S. Random death process for the regularization of subdiffusive anomalous equations. *arXiv:1210.8020 [cond-mat.stat-mech]*.

[10] Greene, RE, Krantz, SG. *Function Theory of One Complex Variable*. Wiley–Interscience, 1997.

[11] Laskin, N. Fractional Poisson process. *Communications in Nonlinear Science and Numerical Simulation*, 8(3-4):201–213, 2003.

[12] Kilbas, AA, Srivastava, HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. North–Holland, 2006.

[13] Mathai, AM, Hano, HJ. *Special Functions for Applied Scientists*. Springer, 2008.

[14] Orsingher, E, Polito, F. On a fractional linear birth-death process. *Bernoulli*, 17:114–137, 2011.

[15] Parthasarathy, PR. A transient solution to an M/M/1 queue: a new simple approach. *Advances in Applied Probability*, 19:997–998, 1987.

[16] Samko, SG, Kilbas, AA, Marichev, OI. *Integrals and derivatives of fractional order and some of their applications*. Gordon and Breach, 1987.

[17] Sharma, OP. *Markovian Queues*. Allied Publishers Limited, Mumbai, 1997.

[18] Taylor, HM, Karlin, S. *An Introduction to Stochastic Modeling*. Third edition, Academic Press, London, 1998.

[19] Mark Veillette, M, Taqqu, MS. Numerical Computation of first-passage times of increasing Lévy processes. *Methodology and Computing in Applied Probability*, 12(4):695–729, 2010.