Divergence and convergence of conjugacies in non-Archimedean dynamics

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Abstract
We continue the study in [21] of the linearizability near an indifferent fixed point of a power series $f$, defined over a field of prime characteristic $p$. It is known since the work of Herman and Yoccoz [13] in 1981 that Siegel’s linearization theorem [27] is true also for non-Archimedean fields. However, they also showed that the condition in Siegel’s theorem is ‘usually’ not satisfied over fields of prime characteristic. Indeed, as proven in [21], there exist power series $f$ such that the associated conjugacy function diverges. We prove that if the degrees of the monomials of a power series $f$ are divisible by $p$, then $f$ is analytically linearizable. We find a lower (sometimes the best) bound of the size of the corresponding linearization disc. In the cases where we find the exact size of the linearization disc, we show, using the Weierstrass degree of the conjugacy, that $f$ has an indifferent periodic point on the boundary. We also give a class of polynomials containing a monomial of degree prime to $p$, such that the conjugacy diverges.

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Key words: dynamical system, linearization, non-Archimedean field

1 Introduction
The study of complex dynamical systems of iterated analytic functions begins with the description of the local behavior near fixed points, see [3,9,24].
Recall that, given a complete valued field $K$, a power series $f \in K[[x]]$ of the form
\[
f(z) = \lambda z + a_2z^2 + a_3z^3 \ldots, \quad |\lambda| = 1,
\]
is said to be analytically linearizable at the indifferent fixed point at the origin if there is a convergent power series solution $g$ to the following form of the Schröder functional equation (SFE)
\[
g \circ f(x) = \lambda g(x), \quad \lambda = f'(0),
\]
which conjugates $f$ to its linear part. The coefficients of the formal solution $g$ of (1) must satisfy a recurrence relation of the form
\[
b_k = \frac{1}{\lambda(1-\lambda^{k-1})}C_k(b_1, \ldots, b_{k-1}).
\]
If $\lambda$ is close to a root of unity, the convergence of $g$ then generates a delicate problem of small divisors. In 1942 Siegel proved in his celebrated paper [27] that the condition
\[
|1 - \lambda^n| \geq Cn^{-\beta} \quad \text{for some real numbers } C, \beta > 0, \tag{2}
\]
on $\lambda$ is sufficient for convergence in the complex field case. Later, Brjuno [8] proved that the weaker condition
\[
-\sum_{k=0}^{\infty} 2^{-k} \log \left( \inf_{1 \leq n \leq 2^{k+1} - 1} |1 - \lambda^n| \right) < +\infty, \tag{3}
\]
is sufficient. In fact, for quadratic polynomials, the Brjuno condition is not only sufficient but also necessary as shown by Yoccoz [29].

Meanwhile, since the work of Herman and Yoccoz in 1981 [13], there has been an increasing interest in the non-Archimedean analogue of complex dynamics, see e.g. [1, 2, 4, 6, 14, 18, 19, 21, 23, 25, 26, 28, 29]. Herman and Yoccoz proved that Siegel’s theorem is true also for non-Archimedean fields.

However, for complete valued fields of prime characteristic, which are necessarily non-Archimedean, the problem was still open; in characteristic $p > 0$, the Siegel condition, and even the weaker Brjuno condition, is only satisfied if $\lambda$ is trivial, that is, that $|1 - \lambda^n| = 1 \forall n \geq 1$. If $\lambda$ is non-trivial (e.g. in locally compact fields all $\lambda$ are non-trivial), then $\lambda$ generates a problem of small divisors. One might therefore conjecture, as Herman [12], that for a locally compact, complete valued field of prime characteristics, the formal conjugacy ‘usually’ diverges, even for polynomials of one variable. Indeed, it was proven in [21] that in characteristic $p > 0$, like in complex dynamics, the formal solution may diverge also in the one-dimensional case. On the other hand, in [21] it was also proven that the conjugacy may still converge due to considerable cancellation of small divisor terms. The main theorem
of [21] stated that quadratic polynomials with non-trivial multipliers are linearizable if and only if the characteristic of the ground field char $K = 2$.

In the present paper we present a new class of polynomials that yield divergence. We also note that the conjugacy converges for all power series $f \in K[[x]]$, whose monomials are all of degree divisible by char $K = p$. Furthermore, in case of of convergence, we estimate the radius of convergence for the corresponding semi-disc, i.e. the maximal disc $V$ such that the semi-conjugacy holds for all $x \in V$, and the linearization disc $\Delta$, i.e. the maximal disc $U$, about the origin, such that the full conjugacy $g \circ f \circ g^{-1}(x) = \lambda x$, holds for all $x \in U$. We also give sufficient conditions, related to the Weierstrass degree of the conjugacy, there being a periodic point on the boundary of the linearization disc. The first non-Archimedean results in this direction were obtained by Arrowsmith and Vivaldi [2] who showed that $p$-adic power functions may have indifferent periodic points on the boundary. In fact, we prove the following theorem, see Lemma 6.1.

**Theorem 1.** Let $K$ be a complete algebraically closed non-Archimedean field. Let $f \in K[[x]]$ have a linearization disc $\Delta$ about an indifferent fixed point. Suppose that $\Delta$ is rational open, and that the radius of the corresponding semi-disc of $f$ is strictly greater than that of $\Delta$, then $f$ has an indifferent periodic point on the boundary of $\Delta$.

This theorem and other results of the present paper, stated below, support the idea that the presence of indifferent periodic points on the boundary of a linearization disc about an indifferent fixed point is typical in the non-Archimedean setting.

For a more thorough treatment of the problem and its relation to earlier works on non-Archimedean and complex dynamics the reader is referred to [21]. Estimates of $p$-adic linearization discs were obtained in [20].

## 2 Summary of results

### 2.1 Divergence and convergence

In the present paper we find a new class of polynomials that yield divergence.

**Theorem 2.** Let char $K = p > 0$ and let $\lambda \in K$, $|\lambda| = 1$. Then, polynomials of the form

$$f(x) = \lambda x + a_{p+1}x^{p+1} \in K[x], \quad a_{p+1} \neq 0,$$

are not analytically linearizable at the fixed point at the origin if $|1-\lambda| < 1$.

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1. Here we use the term ‘linearization disc’ rather than ‘Siegel disc’, because in non-Archimedean dynamics the Siegel disc is often refered to as the larger maximal disc on which $f$ is one-to-one.
On the other hand we also prove convergence for all power series \( f \) whose monomials are all of degree divisible by \( \text{char} K = p \).

**Theorem 3.** Let \( \text{char} K = p > 0 \) and let \( \lambda \in K, |\lambda| = 1 \), but not a root of unity. Then, convergent power series of the form

\[
f(x) = \lambda x + \sum_{p|} a_i x^i,
\]

are linearizable at the origin.

These results indicate that the convergence depends mutually on the powers of the monomials of \( f \) and the characteristic \( p \) of \( K \), ‘good’ powers for convergence being those divisible by \( p \), ‘bad’ powers being those prime to \( p \). However, the blend of prime and co-prime powers may sometimes yield convergence, sometimes not, at least for non-polynomial power series as shown in \([21]\). However, there might be a sharp distinction for polynomials:

**Open problem 2.1.** Let \( K \) be of characteristic \( p > 0 \). Is there a polynomial of the form \( f(x) = \lambda x + O(x^2) \in K[x] \), with \( \lambda \) not a root of unity satisfying \( |1 - \lambda^n| < 1 \) for some \( n \geq 1 \), and containing a monomial of degree prime to \( p \), such that the formal conjugacy \( g \) converges?

### 2.2 Estimates of linearization discs and periodic points

Let \( K \) be a field of prime characteristic \( p \). Let \( \lambda \in K \), not a root of unity, be such that the integer

\[
m = m(\lambda) = \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\},
\]

exists. The case in which such an \( m \) does not exist was treated in \([21]\); it was shown that if \( |1 - \lambda^n| = 1 \) for all \( n \geq 1 \), then the linearization disc of a power series

\[
f(x) = \lambda x + a_2 x^2 + a_3 x^3 + \ldots,
\]

is either the closed or open disc of radius \( 1/a \) where \( a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \).

Note that, by Lemma 3.1 below, \( m \) is not divisible by \( p \). Given \( \lambda \) and hence \( m \), the integer \( k' \) is defined by

\[
k' = k'(\lambda) = \min\{k \in \mathbb{Z} : k \geq 1, p|k, m|k - 1\}.
\]

Let \( a > 0 \) be a real number. We will associate with the pair \((\lambda, a)\), the family of power series

\[
\mathcal{F}_{\lambda,a}(K) = \left\{ \lambda x + \sum_{p|} a_i x^i \in K[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\},
\]

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and the real numbers
\[ \rho = \rho(\lambda, a) = \frac{|1 - \lambda^m|^{\frac{1}{mp}}}{a}, \]  
(7)

\[ \sigma = \sigma(\lambda, a) = \frac{|1 - \lambda^m|^{\frac{1}{k' - 1}}}{a}, \]  
(8)

respectively.

As stated in Theorem 3, power series in the family \( F_{\lambda, a}(K) \) are linearizable at the origin. Given \( f \), the corresponding conjugacy function \( g \), will be defined as the unique power series solution to the Schröder functional equation (1), with \( g(0) = 0 \) and \( g'(0) = 1 \).

In Section 5 we use the ansatz of a power series solution to the Schröder functional equation, to obtain estimates of the coefficients of \( g \), and its radius of convergence. Moreover, applying a result of Benedetto (Proposition 3.3 below), we estimate the radius of convergence for the inverse \( g^{-1} \). The main result can be stated in the following way.

**Theorem 4.** Let \( f \in F_{\lambda, a}(K) \). Then, the semi-conjugacy \( g \circ f (x) = \lambda g(x) \) holds on the open disc \( D_{\rho}(0) \). Moreover, the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds on \( D_{\sigma}(0) \).

Under further assumptions on \( f \), the linearization disc may contain the larger disc \( D_{\rho}(0) \).

**Theorem 5.** Let \( f \in F_{\lambda, a}(K) \) be of the form
\[ f(x) = \lambda x + \sum_{i \geq i_0} a_i x^i, \]
for some integer \( i_0 > k' \). Then, the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds on a disc larger than or equal to \( D_{\rho}(0) \) or the closed disc \( \overline{D}_{\rho}(0) \), depending on whether \( g \) converges on \( D_{\rho}(0) \) or not.

Note that the estimate of the linearization disc in Theorem 3 is maximal in the sense that in \( \hat{K} \), the completion of the algebraic closure of \( K \), quadratic polynomials have a fixed point on the sphere \( S_\sigma(0) \) if \( m(\lambda) = 1 \), breaking the conjugacy there. In fact, the estimate is maximal in a broader sense, according to the following theorem.

**Theorem 6.** Let \( f \in F_{\lambda, a}(K) \). Suppose \( a = |a_k|^{1/(k' - 1)} \) and \( |a_i| < a_i^{k' - 1} \) for all \( i < k' \). Then, \( D_{\sigma}(0) \) is the linearization disc of \( f \) about the origin. In \( \hat{K} \) we have \( \text{deg}(g, \overline{D}_{\sigma}(0)) = k' \). Furthermore, \( f \) has an indifferent periodic point in \( \hat{K} \) on the sphere \( S_\sigma(0) \) of period \( \kappa \leq k' \), with multiplier \( \lambda^\kappa \).
Here, \( \text{deg}(g, D) \) denotes the Weierstrass degree of \( g \) on the disc \( D \), as defined in Section 6. The Weierstrass degree is the same as the notion of degree as ‘the number of pre-images of a given point, counting multiplicity’. Since we assume that \( g(0) = 0 \) and \( g'(0) = 1 \), \( \text{deg}(g, D_\sigma(0)) = k' \) means that in the algebraic closure \( \hat{K} \), \( g \) maps the disc \( D_\sigma(0) \) onto itself exactly \( d \)-to-1, counting multiplicity.

The result in Theorem 6 is based on Lemma 6.1 that shows that if there is a shift of the value of the Weierstrass degree from 1 to \( d > 1 \), of the conjugacy function on a sphere \( S \), then there is an indifferent periodic point of period \( \kappa \leq d \), on the sphere \( S \).

## 3 Preliminaries

Throughout this paper \( K \) is a field of characteristic \( p > 0 \), complete with respect to a nontrivial absolute value \( | \cdot |_K \). That is, \( | \cdot |_K \) is a multiplicative function from \( K \) to the nonnegative real numbers with \( |x|_K = 0 \) precisely when \( x = 0 \), and nontrivial in the sense that it is not identically 1 on \( K^* \), the set of all nonzero elements in \( K \). If a field \( L \) is equipped with an absolute value, we say that \( L \) is a \textit{valued} field. In fact, all valued fields of strictly positive characteristic are non-Archimedean. In what follows, we often use the shorter notation \( | \cdot | \) instead of \( | \cdot |_K \).

Recall that a non-Archimedean field is a field \( K \) equipped with a nontrivial absolute value \( | \cdot | \), satisfying the following strong or ultrametric triangle inequality:

\[
|x + y| \leq \max(|x|, |y|), \quad \text{for all } x, y \in K. \tag{9}
\]

One useful consequence of ultrametricity is that for any \( x, y \in K \) with \( |x| \neq |y| \), the inequality \( |x + y| \leq \max(|x|, |y|) \) becomes an equality. In other words, if \( x, y \in K \) with \( |x| < |y| \), then \( |x + y| = |y| \).

For a field \( K \) with absolute value \( | \cdot | \) we define the \textit{value group} as the image

\[
|K^*| = \{|x| : x \in K^*\}. \tag{10}
\]

Note that \( |K^*| \) is a multiplicative subgroup of the positive real numbers. We will also consider the full image \( |K| = |K^*| \cup \{0\} \). The absolute value \( | \cdot | \) is said to be \textit{discrete} if the value group is cyclic, that is if there is is an element \( \pi \in K \) such that \( |K^*| = \{|\pi|^n : n \in \mathbb{Z}\} \). The absolute value \( | \cdot | \) can be extended to an absolute value on the algebraic closure of \( K \). We shall denote by \( \hat{K} \) the completion of the algebraic closure of \( K \) with respect to \( | \cdot | \). The fact that \( \hat{K} \) is algebraically closed and that \( | \cdot | \) is nontrivial forces the value group \( |\hat{K}^*|_{\hat{K}} \) to be dense on the positive real line. In particular, \( | \cdot |_{\hat{K}} \) cannot be discrete.
Standard examples of non-Archimedean fields include the $p$-adics, see for example [17], and various function fields, see for example [10]. The $p$-adics include the $p$-adic integers and their extensions. These fields are all of characteristic zero. The most important function fields include fields of formal Laurent series over various fields. These can be of any characteristic.

**Example 3.1.** Let $F$ be a field of characteristic $p > 0$, and fix $0 < \epsilon < 1$. Let $K = F((T))$ be the field of all formal Laurent series in variable $T$, and with coefficients in the field $F$. Then $K$ is also of characteristic $p$. An element $x \in K$ is of the form

$$x = \sum_{j \geq j_0} x_j T^j, \quad x_{j_0} \neq 0, \quad x_j \in F,$$

for some integer $j_0 \in \mathbb{Z}$. This field is complete with respect to the absolute value for which

$$| \sum_{j \geq j_0} x_j T^j | = \epsilon^{j_0}.$$

Note that $j_0$ is the order of the zero (or negative the order of the pole) of $x$ at $T = 0$. Let us also note that $| \cdot |$ is the trivial valuation on $F$, the subfield consisting of all constant power series in $K$. In this case $|K^*|$ is discrete and consists of all nonzero integer powers of $\epsilon$. The value group of the completion of the algebraic closure $\hat{K}^*$ is not discrete and consists of all nonzero rational powers of $\epsilon$. Moreover, $K$ can be viewed as the completion of the field of rational functions over $F$ with respect to the absolute value (12) (see, e.g. [10]).

Given an element $x \in K$ and real number $r > 0$ we denote by $D_r(x)$ the open disc of radius $r$ about $x$, by $\overline{D}_r(x)$ the closed disc, and by $S_r(x)$ the sphere of radius $r$ about $x$. If $r \in |K^*|$ (that is if $r$ is actually the absolute value of some nonzero element of $K$), we say that $D_r(x)$, $\overline{D}_r(x)$, and $S_r(x)$ are rational. Note that $S_r(x)$ is non-empty if and only if it is rational. If $r \notin |K^*|$, then we will call $D_r(x) = \overline{D}_r(x)$ an irrational disc. In particular, if $a \in K$ and $r = |a|^s$ for some rational number $s \in \mathbb{Q}$, then $D_r(x)$ and $\overline{D}_r(x)$ are rational considered as discs in $\hat{K}$. However, they may be irrational considered as discs in $K$. Note that all discs are both open and closed as topological sets, because of ultrametricity. However, as we will see in Section 3.3 below, power series distinguish between rational open, rational closed, and irrational discs.

Again by ultrametricity, any point of a disc can be considered its center. In other words, if $b \in D_r(a)$, then $D_r(a) = D_r(b)$; the analogous statement is also true for closed discs. In particular, if two discs have nonempty intersection, then they are concentric, and therefore one must contain the other.
The open and closed unit discs, $D_1(0)$ and $\overline{D}_1(0)$, respectively play a fundamental role in non-Archimedean analysis, because of their algebraic properties. In fact, due to the strong triangle inequality, $D_1(0)$ is a ring and $D_1(0)$ is the unique maximal ideal in $\overline{D}_1(0)$. The corresponding quotient field, $k = \overline{D}_1(0)/D_1(0)$ is called the residue field of $K$. The residue field $k$ will also be of characteristic $p$. Hence we always have $k \supseteq \mathbb{F}_p$. The absolute value $| \cdot |$ is trivial on $k$. For $x \in \overline{D}_1(0)$, we will denote by $\overline{x}$ the reduction of $x$ modulo $D_1(0)$.

3.1 The formal solution

The coefficients of the formal solution of (1) must satisfy the recurrence relation

$$b_k = \frac{1}{\lambda(1 - \lambda^{k-1})} \sum_{l=1}^{k-1} b_l \left( \sum_{\alpha_1! \cdot \ldots \cdot \alpha_k!} l! \alpha_1^{\alpha_1} \cdot \ldots \cdot \alpha_k^{\alpha_k} \right)$$

(13)

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are nonnegative integer solutions of

$$\begin{align*}
\alpha_1 + \ldots + \alpha_k &= l, \\
\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k &= k, \\
1 \leq l \leq k - 1.
\end{align*}$$

(14)

The convergence of $g$ will depend mutually on the denominators $|1 - \lambda^{k-1}|$ and the factorial terms in (13). In view of Lemma 3.1, the denominator is small if $k - 1$ is divisible by $m$ and a large power of the characteristic $p$. In fact, the conjugacy may diverge as in Theorem 4.1 below. On the other hand, the factorial term

$$\frac{l!}{\alpha_1! \cdot \ldots \cdot \alpha_k!}$$

is always an integer and hence of modulus zero or one, depending on whether it is divisible by $p$ or not. Accordingly, factorial terms may extinguish small divisor terms as in Theorem 5.1 below.

3.2 Arithmetic of the multiplier

Let $\lambda \in \mathbb{S}_1(0)$, be an element in the unit sphere. The geometry of the unit sphere and the roots of unity in $K$ was discussed in [21]. We are concerned with calculating the distance

$$|1 - \lambda^n|, \quad \text{for } n = 1, 2, \ldots.$$ 

Note that if $x, y \in \overline{D}_1(0)$, then $|x - y| < 1$ if and only if the reductions $\overline{x}, \overline{y}$ belong to the same residue class. Consequently,

$$|1 - \lambda^n| < 1 \iff \lambda^n - 1 = 0 \quad \text{in } k.$$ 

(15)
Hence, the behavior of $1 - \lambda^n$ falls into one of two categories, depending on whether the reduction of $\lambda$ is a root of unity or not. More precisely we have the following lemma that was proven in [21].

**Lemma 3.1** (Lemma 3.2 [21]). Let $\text{char} \ K = p > 0$ and let $\kappa$ be the residue field of $K$. Let $\Gamma(\kappa)$ be the set of roots of unity in $\kappa$. Suppose $\lambda \in S_1(0)$. Then,

1. $\lambda \not\in \Gamma(\kappa) \iff |1 - \lambda^n| = 1$ for all integers $n \geq 1$.
2. If $\lambda \in \Gamma(\kappa)$, then the integer $m = \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\}$ exists. Moreover, $p \nmid m$ and

$$|1 - \lambda^n| = \begin{cases} 1, & \text{if } m \nmid n, \\ |1 - \lambda^m|p^j, & \text{if } n = \text{map}^j, \ p \nmid a. \end{cases}$$

Note that category 2 in Lemma 3.1 is always non-empty since $\kappa \supseteq \mathbb{F}_p$. Moreover, if $\kappa \subseteq \mathbb{F}_p$, then $\Gamma(K) = \kappa^*$ and all $\lambda \in S_1(0)$ falls into category 2. Consequently category 1 is empty in this case. This happens for example when $K$ is locally compact, see e.g. [10].

**Proposition 3.1.** Let $K$ be a non-Archimedean field with absolute value $|\cdot|$. Then $K$ is locally compact (w.r.t. $|\cdot|$) if and only if all three of the following conditions are satisfied: (i) $K$ is complete, (ii) $|\cdot|$ is discrete, and (iii) the residue field is finite.

On the other hand if $K$ is algebraically closed, then $\kappa$ is infinite and $K$ cannot be locally compact. In this case $\kappa \supseteq \mathbb{F}_p$.

We shall only consider the case in which $\lambda$ belongs to category 2. The other case was treated in [21].

### 3.3 Mapping properties

Let $K$ be a complete non-Archimedean field. Let $h$ be a power series over $K$ of the form

$$h(x) = \sum_{k=0}^{\infty} c_k(x - \alpha)^k, \quad c_k \in K.$$

Then $h$ converges on the open disc $D_{R_h}(\alpha)$ of radius

$$R_h = \frac{1}{\limsup |c_k|^{1/k}},$$

and diverges outside the closed disc $D_{R_h}(\alpha)$. The power series $h$ converges on the sphere $S_{R_h}(\alpha)$ if and only if

$$\lim_{k \to \infty} |c_k|R_h^k = 0.$$
The following proposition is useful to estimate the size of a linearization disc, i.e. the maximal disc on which the full conjugacy, \( g \circ f \circ g^{-1} = \lambda x \), holds.

**Proposition 3.2** (Lemma 2.2 [5]). Let \( K \) be algebraically closed. Let \( h(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k \) be a nonzero power series over \( K \) which converges on a rational closed disc \( U = \overline{D}_R(x_0) \), and let \( 0 < r \leq R \). Let \( V = \overline{D}_r(x_0) \) and \( V' = D_r(x_0) \). Then

\[
\begin{align*}
s &= \max \{|c_k|r^k : k \geq 0\}, \\
d &= \max \{k \geq 0 : |c_k|r^k = |c_1|r\}, \quad \text{and} \\
d' &= \min \{k \geq 0 : |c_k|r^k = s\}
\end{align*}
\]

are all attained and finite. Furthermore,

a. \( s \geq |f'(x_0)| \cdot r \).

b. if \( 0 \in f(V) \), then \( f \) maps \( V \) onto \( \overline{D}_s(0) \) exactly \( d \)-to-1 (counting multiplicity).

c. if \( 0 \in f(V') \), then \( f \) maps \( V' \) onto \( D_s(0) \) exactly \( d' \)-to-1 (counting multiplicity).

Benedetto’s proof uses the Weierstrass Preparation Theorem [7, 11, 17].

We will be interested in the special case in that \( c_0 = x_0 = 0 \). In this case, we have the following proposition.

**Proposition 3.3.** Let \( K \) be an algebraically closed complete non-Archimedean field and let \( h(x) = \sum_{k=1}^{\infty} c_k x^k \) be a power series over \( K \).

1. Suppose that \( h \) converges on the rational closed disc \( \overline{D}_R(0) \). Let \( 0 < r \leq R \) and suppose that

\[
|c_k|r^k \leq |c_1|r \quad \text{for all } k \geq 2.
\]

Then \( h \) maps the open disc \( D_r(0) \) one-to-one onto \( D_{|c_1|r}(0) \). Furthermore, if

\[d = \max \{k \geq 1 : |c_k|r^k = |c_1|r\},\]

then, \( h \) maps the closed disc \( \overline{D}_r(0) \) onto \( \overline{D}_{|c_1|r}(0) \) exactly \( d \)-to-1 (counting multiplicity).

2. Suppose that \( h \) converges on the rational open disc \( D_R(0) \) (but not necessarily on the sphere \( S_R(0) \)). Let \( 0 < r \leq R \) and suppose that

\[
|c_k|r^k \leq |c_1|r \quad \text{for all } k \geq 2.
\]

Then \( h \) maps \( D_r(0) \) one-to-one onto \( D_{|c_1|r}(0) \).
Lemma 3.2. Let \( K \) be an algebraically closed complete non-Archimedean field. Let \( f(x) = \lambda x + \sum_{i=2}^{\infty} a_i x^i \in K[[x]], |\lambda| = 1 \), be convergent on some non-empty disc. Then, the real number \( a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \) exists and \( |a_i| \leq a^{i-1} \) for all \( i \geq 2 \). Furthermore, \( R_f \geq 1/a \) and \( f : \overline{D}_{1/a}(0) \to \overline{D}_{1/a}(0) \) is bijective. If \( |a_i| < a^{i-1} \) for all \( i \geq 2 \) and \( f \) converges on the closed disc \( D_{1/a}(0) \), then \( f : \overline{D}_{1/a}(0) \to \overline{D}_{1/a}(0) \) is bijective. Finally, \( f \) cannot be bijective on a (rational) disc greater than \( \overline{D}_{1/a}(0) \).

Proof. As \( f \) is convergent, we have \( \sup |a_i|^{1/i} < \infty \) and hence we have that \( \sup |a_i|^{1/(i-1)} < \infty \):

\[
\sup |a_i|^{1/(i-1)} \leq \sup_{|a_i| \leq 1} |a_i|^{1/(i-1)} + \sup_{|a_i| > 1} \left( |a_i|^{1/i} \right)^{i/(i-1)} \leq 1 + \left( \sup_{|a_i| > 1} |a_i|^{1/i} \right)^2.
\]

Clearly, \( |a_i| \leq \left( \sup_{i \geq 2} |a_i|^{1/(i-1)} \right)^i \). Moreover, \( R_f = (\limsup |a_i|^{1/i})^{-1} \geq 1/a \). That \( f : D_{1/a}(0) \to D_{1/a}(0) \) is bijective follows from the second statement in Proposition 3.3. \( \square \)

Remark 3.1. Proposition 3.3 and Lemma 3.2 also hold when \( K \) is not algebraically closed with the modification that the mappings \( h : D_r(0) \to D_{|r|}r(0) \) and \( f : D_{1/a}(0) \to D_{1/a}(0) \) are one-to-one but not necessarily surjective; the analogous statement is also true for the closed discs.

Remark 3.2. Note that the disc \( D_{1/a}(0) \) in Lemma 3.2 may be irrational. Let \( K \) be a field such that \( \hat{K}^* = \{ e^r : r \in \mathbb{Q} \} \) for some \( 0 < \epsilon < 1 \) (as in Example 7.7). Let \( \beta \) be an irrational number and let \( p_n/q_n \) be the \( n \)-th convergent of the continued fraction expansion of \( \beta \). Let the sequence \( \{a_i \in K \}_{i \geq 2} \) satisfy

\[
|a_i| = \begin{cases} (1/\epsilon)^{p_n}, & \text{if } i - 1 = q_n \text{ and } p_n/q_n < \beta, \\ 0, & \text{otherwise.} \end{cases}
\]

Then,

\[
\sup |a_i|^{1/(i-1)} = (1/\epsilon)^\beta \notin |\hat{K}|.
\]

On the other hand, if the sequence \( \{a_i \} \) is such that \( \max_{i \geq 2} |a_i|^{1/(i-1)} \) exists, then \( a = \max_{i \geq 2} |a_i|^{1/(i-1)} \in |\hat{K}| \) for any \( K \). This is always the case for polynomials.

4 Divergence

As proven in Section 5, power series of the form \( f(x) = \lambda x + O(x^2) \), with monomials of degree divisible by some nonnegative integer power of \( p \) are analytically linearizable at the origin. In the paper [21], it was proven that
quadratic polynomials, \( f(x) = \lambda x + a_2 x^2 \), with \(|1 - \lambda| < 1\), are not analytically linearizable at the origin. We will prove another result in this direction. Polynomials of the form \( f(x) = \lambda x + a_{p+1} x^{p+1} \), with \(|1 - \lambda| < 1\), are not analytically linearizable in characteristic \( p > 0 \).

The key result is Lemma 4.4 below. In that lemma we obtain the exact modulus of a subsequence of coefficients of the conjugacy function \( g \). It turns out that this subsequence contains sufficiently many small divisor terms to yield divergence.

**Lemma 4.1.** Let \( K \) be a complete valued field and let \( f \) be a power series of the form

\[
f(x) = \lambda x + \sum_{i \geq i_0} a_i x^i \in K[[x]],
\]

where \( \lambda \neq 0 \) but not a root of unity, \( i_0 \geq 2 \) is an integer, and \( a_{i_0} \neq 0 \). Then, the associated formal conjugacy function is of the form

\[
g(x) = x + \sum_{k \geq i_0} b_k x^k,
\]

where \( b_{i_0} = a_{i_0}/\lambda(1 - \lambda^{i_0-1}) \).

**Proof.** By definition, the formal conjugacy \( g \) is of the form \( g(x) = x + \sum_{k \geq 2} b_k x^k \). The left hand side of the Schröder functional equation (1) is of the form

\[
g \circ f(x) = \sum_{k = 1}^{i_0-1} b_k (\lambda x + a_{i_0} x^{i_0} + \ldots)^k + O(x^{i_0}) = \sum_{k = 1}^{i_0-1} b_k (\lambda x)^k + O(x^{i_0}).
\]

For the right hand side we have

\[
\lambda g(x) = \lambda \sum_{k = 1}^{i_0-1} b_k x^k + O(x^{i_0}).
\]

Recall that \( \lambda \neq 0 \) is not a root of unity. Identification term by term yields that \( b_k = 0 \) for \( 1 < k < i_0 \). Consequently, for \( k = i_0 \), the recursion formula (13) yields

\[
b_{i_0} = b_1 a_{i_0}/\lambda(1 - \lambda^{i_0-1}).
\]

But by definition, \( b_1 = 1 \). This completes the proof of the lemma.

**Lemma 4.2.** Let char \( K = p > 0 \). Let \( f(x) = \lambda x + a_{p+1} x^{p+1} \in K[x] \), with \(|\lambda| = 1 \) but not a root of unity. Then, the formal solution \( g \) of the SFE (1) has coefficients \( b_k \) of the form

\[
b_{jp+1} = \frac{1}{\lambda(1 - \lambda^{jp})} \left[ \sum_{i = \lceil \frac{j-1}{p+1} \rceil}^{j-1} b_{ip+1} (\left\lceil \frac{ip + 1}{j-i} \right\rceil) \lambda^{ip+1-(j-i)} a_{jp+1}^{j-i} \right], \quad (18)
\]

for all integers \( j \geq 1, \ b_1 = 1, \) and \( b_k = 0 \) otherwise.
Proof. First note that the case \( k \leq p + 1 \) follows by Lemma 4.1.

Now assume that the lemma holds for all \( k \leq (n - 1)p + 1 \) for some \( n \geq 2 \). In particular, this means that \( b_k = 0 \) for all \((i - 1)p + 1 < k < ip + 1\) where \( 1 \leq i \leq n - 1 \). We now consider the case \((n - 1)p + 1 < k \leq np + 1\).

We have regarding the left hand side of the SFE

\[
g \circ f(x) = \sum_{k=1}^{np+1} b_k(\lambda x + a_{p+1}x^{p+1})^k + O(x^{np+2}),
\]

and by hypothesis,

\[
g \circ f(x) = \sum_{i=0}^{n-1} b_{ip+1}(\lambda x + a_{p+1}x^{p+1})^{ip+1} + \sum_{k=(n-1)p+2}^{np+1} b_k(\lambda x + a_{p+1})^k + O(x^{np+2}).
\]

For all integers \( i \geq 0 \), we have the binomial expansion

\[
(\lambda x + a_{p+1}x^{p+1})^{ip+1} = \sum_{\ell=0}^{ip+1} \binom{ip+1}{\ell} (\lambda x)^{ip+1-\ell}(a_{p+1}x^{p+1})^{\ell}.
\]

Consequently, with \( \delta = \min\{ip + 1, n - i\} \),

\[
(\lambda x + a_{p+1}x^{p+1})^{ip+1} = \sum_{\ell=0}^{\delta} \binom{ip+1}{\ell} \lambda^{ip+1-\ell}a_{p+1}^{\ell}x^{(i+\ell)p+1} + O(x^{np+2}).
\]

Also note that for \( k \geq (n - 1)p + 2 \) we have

\[
(\lambda x + a_{p+1}x^{p+1})^k = (\lambda x)^k + O(x^{np+2}).
\]

Combining (19), (20), and (21) we obtain that \( g \circ f(x) \) is of the form

\[
\sum_{i=0}^{n-1} b_{ip+1} \sum_{\ell=0}^{\delta} \binom{ip+1}{\ell} \lambda^{ip+1-\ell}a_{p+1}^{\ell}x^{(i+\ell)p+1} + \sum_{k=(n-1)p+2}^{np+1} b_k(\lambda x)^k + O(x^{np+2}).
\]

The right hand side of the SFE is of the form

\[
\lambda g(x) = \lambda \sum_{k=1}^{np+1} b_k x^k + O(x^{np+2}).
\]

Note that (20) contains no powers of \( x \) in the closed interval \([(n-1)p+2, np]\).

Consequently, for powers of \( x \) in this interval the SFE gives

\[
\sum_{k=(n-1)p+2}^{np} b_k(\lambda x)^k = \lambda \sum_{k=(n-1)p+2}^{np} b_k x^k,
\]

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and by identification term by term

\[ b_k = 0, \quad \text{for all } (n-1)p + 1 < k < np + 1. \]

In the remaining case the \( x^{np+1} \)-term in (20) (if it exists) occur for \( \ell = n - i \). Such an \( \ell \) exits if and only if \( n - i \leq ip + 1 \), that is if \( i \geq \left\lceil \frac{n-1}{p+1} \right\rceil \). Hence, the equation for the \( x^{np+1} \)-term yields

\[
\begin{aligned}
 b_{np+1} &= \frac{1}{\lambda(1 - \lambda^np)} \left[ \sum_{i=\left\lceil \frac{n-1}{p+1} \right\rceil}^{n-1} b_{ip+1} \binom{ip + 1}{n-i} \lambda^{ip+1-(n-i)} \alpha_{p+1}^{n-i-1} \right],
\end{aligned}
\]

as required.

Lemma 4.3. For each summand in the recursion formula (18), the total power of \( a_{p+1} \) is \( j \), i.e each \( b_{jp+1} \) is of the form

\[
b_{jp+1} = c_{jp+1} a_{p+1}^j,
\]

where where \( c_{jp+1} \) is a sum of products with factors of the form

\[
\frac{1}{\lambda(1 - \lambda^j)} \left( \frac{a}{b} \right)^j \lambda^j.
\]

Proof. First note that in view of (18) we have for \( j = 1 \) that the coefficient \( b_{p+1} = a_{p+1}/(\lambda(1 - \lambda^p)) \). Now assume that (22) holds for \( j \leq n \). For \( j = n+1 \) and \( 1 \leq i \leq n \), we then have regarding the right hand side of (18) that

\[
b_{p+1} a_{p+1}^{n+1-i} = c_{p+1} a_{p+1}^i a_{p+1}^{n+1-i} = c_{p+1} a_{p+1}^{n+1},
\]

as required.

Lemma 4.4. Let \( f(x) = \lambda x + a_{p+1} x^{p+1} \), \( |\lambda| = 1 \) but not a root of unity. Suppose \( |1 - \lambda| < 1 \). Then the formal solution \( g \) of the SFE (1) has coefficients \( b_k \) of absolute value

\[
|b_{jp+1}| = \frac{|a_{p+1}|^j}{\prod_{i=1}^{j} |1 - \lambda^i|},
\]

for all integers \( j \geq 1, b_1 = 1, \) and \( b_k = 0 \) otherwise.

Proof. By Lemma 4.3, the total power of \( a_{p+1} \) in the products \( b_{ip+1} a_{p+1}^{j-i} \) of the right hand side of (18) is \( j \). Also recall that \( |\lambda| = 1 \). Furthermore

\[
\left| \binom{ip+1}{1} \right| = 1,
\]

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for all nonnegative integers $i$. Now, since $|1 - \lambda| < 1$, it follows by induction over $j$ that the $b_{(j-1)p+1}$-term in (18) is strictly greater than all the others and by ultrametricity we obtain

$$|b_{jp+1}| = \frac{1}{\lambda(1 - \lambda^p)} b_{(j-1)p+1} \left(\frac{(j-1)p+1}{1}\right) \lambda^{(j-1)p} a_{p+1} = \frac{|a_{p+1}|^2}{\prod_{i=1}^j |1 - \lambda^p|}.$$  

This completes the proof of Lemma 4.4.

Lemma 4.5. Let $\text{char } K = p > 0$ and $\lambda \in K$, $|\lambda| = 1$, but not a root of unity. Suppose $|1 - \lambda| < 1$, then

$$\prod_{i=1}^{N-1} |1 - \lambda^p| = |1 - \lambda|^{p^{N-1} + p^N(\frac{p-1}{p})(N-1)+1},$$  

(25)

for all integers $N \geq 1$.

Proof. First note that for each $n = 1, \ldots, N-1$, the number of elements in $\{1, 2, \ldots, p^N\}$ that are divisible by $p^n$ but not by $p^{n+1}$, are given by the number $p^N/p^n - p^N/p^{n+1}$. Since $|1 - \lambda| < 1$ we can apply Lemma 3.1 in the case $m = 1$ to obtain

$$\prod_{i=1}^{N-1} |1 - \lambda^p| = |1 - \lambda|^{p^N + \sum_{n=1}^{N-1} p^N/p^{n+1} p^n} = |1 - \lambda|^{p^N(1+\frac{p-1}{p})(N-1)},$$

as required.

Theorem 4.1. Let $\text{char } K = p > 0$ and $f(x) = \lambda x + a_{p+1}x^{p+1} \in K[x]$. Suppose $|\lambda| = 1$ and $|1 - \lambda| < 1$. Then $f$ is not analytically linearizable at $x = 0$.

Proof. Let $j = p^{N-1}$ for some integer $N \geq 1$. By Lemma 4.4

$$|b_{p^{N+1}}| = \frac{|a_{p+1}|^{p^{N-1}}}{\prod_{i=1}^{p^N-1} |1 - \lambda^p|}.$$  

But in view of Lemma 4.5 this means that

$$|b_{p^{N+1}}| = \frac{|a_{p+1}|^{p^{N-1}}}{|1 - \lambda|^{p^N(1+\frac{p-1}{p})(N-1)}},$$

and since $|1 - \lambda| < 1$,

$$\lim_{N \to \infty} |b_{p^{N+1}}|^{1/(p^{N+1})} = \infty.$$  

Consequently, $\limsup |b_k|^{1/k} = \infty$ so that the conjugacy diverges.
Further investigation has led us to believe that we have divergence also in the case \( m > 1 \) in Theorem 4.1, that is \( m > 1 \) is the smallest integer such that \( |1 - \lambda^m| < 1 \). However, the proof seems to be more complicated; we do not necessarily have a strictly dominating term in the right hand side of (18) for all \( j \), and consequently, (24) may no longer be valid.

**Conjecture 4.1** (Generalization of Theorem 4.1). Let \( \text{char} \ K = p > 0 \) and \( f(x) = \lambda x + a_{p+1}x^{p+1} \in K[x] \). Suppose \( |\lambda| = 1 \) and that \( m \geq 1 \) is the smallest integer such that \( |1 - \lambda^m| < 1 \). Then \( f \) is not analytically linearizable at \( x = 0 \).

### 5 Estimates of linearization discs

In this section we consider power series in the family

\[
\mathcal{F}_{\lambda,a}^p(K) = \left\{ \lambda x + \sum_{p \nmid i} a_i x^i \in K[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\},
\] (26)

as defined in Section 2.2. Note that each \( f \in \mathcal{F}_{\lambda,a}^p(K) \) is convergent on \( D_{1/a}(0) \) and by Lemma 3.2 \( f : D_{1/a}(0) \to D_{1/a}(0) \) is one-to-one. We will prove in Theorem 5.1 that each \( f \in \mathcal{F}_{\lambda,a}^p(K) \) is linearizable at the fixed point at the origin. We also estimate the region of convergence for the corresponding conjugacy function \( g \), and its inverse.

**Remark 5.1.** By the conjugacy relation \( g \circ f \circ g^{-1}(x) = \lambda x \), \( f \) must certainly be one-to-one on the linearization disc. Consequently, by Lemma 3.2 the full conjugacy relation cannot hold on a disc greater than \( D_{1/a}(0) \).

We begin by proving the following, simple but important fact. Given a power series \( f \in \mathcal{F}_{\lambda,a}^p(K) \), the conjugacy function \( g \) only contains monomials of degree divisible by some nonnegative integer power of \( p \). More precisely, we have the following Lemma.

**Lemma 5.1.** Let \( \text{char} \ K = p > 0 \) and let \( f \in \mathcal{F}_{\lambda,a}^p(K) \), where \( |\lambda| = 1 \) but not a root of unity. Then, the formal conjugacy \( g \) is of the form

\[
g(x) = x + \sum_{p \nmid k} b_k x^k.
\] (27)

**Proof.** Let \( f \in \mathcal{F}_{\lambda,a}^p(K) \) and let \( f_p(x) = \sum_{i(p) > 1} a_i x^i \). Let the conjugacy \( g(x) = x + \sum_{k=2}^{\infty} b_k x^k \). We will prove by induction that if \( g \) is the formal solution of the Schröder functional equation, then \( b_k = 0 \) for all \( k \geq 2 \) such that \( p \nmid k \).
By definition $b_1 = 1$ and by Lemma 4.1 $b_k = 0$ for $1 < k < p$. Assume that $b_k = 0$ for all $k > 1$ such that $(j - 1)p < k < jp$, where $1 \leq j \leq N$. Let $h_N$ be the polynomial
\[
h_N(x) = \lambda x + f_p(x) + \sum_{i=1}^{N} b_i p(\lambda x + f_p(x))^{ip} \mod x^{(N+1)p}.
\]
Note that the terms
\[(\lambda x + f_p(x))^{ip} = \sum_{l=0}^{ip} \binom{ip}{l}(\lambda x)^l(f_p(x))^{ip-l},
\]
do only contain powers of $x$ divisible by $p$. This follows because, in characteristic $p$, $(\binom{ip}{l}) = 0$ if $p \not| l$. Accordingly, $h_N(x) - \lambda x$ contains only powers of $x$ divisible by $p$. By hypothesis,
\[
g \circ f(x) = h_N(x) + \sum_{k=Np+1}^{(N+1)p-1} b_k (\lambda x + f_p(x))^k + O(x^{(N+1)p}).
\]
As noted above the monomials of $f_p$ are of degree greater than or equal to $p$. Consequently,
\[
g \circ f(x) = h_N(x) + \sum_{k=Np+1}^{(N+1)p-1} b_k (\lambda x)^k + O(x^{(N+1)p}).
\]
Recall that, $h_N(x) - \lambda x$ contains only powers of $x$ divisible by $p$. Consequently, identification term by term with the right hand side $\lambda g(x)$ yields that $b_k = 0$ for all $Np < k < (N+1)p$ as required.

In the following we shall estimate the coefficients of the conjugacy (27). As noted in Section 3.1 these coefficients are given by the recursion formula (13). Our main result (Theorem 5.1) of this section is based on the estimate obtained in Lemma 5.3 below.

In preparation, we recall the following definitions from Section 2.2. The integer $m$ is defined by
\[
m = m(\lambda) = \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\}. \quad (28)
\]
Note that, by Lemma 3.1 $m$ is not divisible by $p$. Given $m$, the integer $k'$ is defined by
\[
k' = k'(\lambda) = \min\{k \in \mathbb{Z} : k \geq 1, p|k, m|k-1\}. \quad (29)
\]
Note the following lemma.
Lemma 5.2. Let \( k \geq 2 \) be an integer. Then,

\[
\left\lfloor \frac{(k - k')}{mp} + 1 \right\rfloor
\]

is the number of positive integers \( l \leq k \) that satisfies the two conditions \( p \mid l \) and \( m \mid l - 1 \).

Proof. Let \( k' \) be the smallest positive integer such that \( p \mid k' \) and \( m \mid k' - 1 \). Let \( \mathbb{Z}_m \) be the residue class modulo \( m \). Also recall that by definition \( p \nmid m \).

Accordingly, \( k' = j'p \) where \( j' \) is the unique solution in \( \mathbb{Z}_m \) to the congruence equation

\[
xp - 1 \equiv 0 \mod m.
\]

The integer solutions of (30) are thus of the form

\[
x = j' + mn,
\]

where \( n \) runs over all the integers. It follows that an integer \( l \) satisfies the two conditions \( p \mid l \) and \( m \mid l - 1 \) if and only if it is of the form

\[
l = (j' + mn)p = k' + mpn,
\]

for some integer \( n \). Given \( k \geq 2 \), let \( t \geq 0 \) be the largest integer such that

\[
k' + (t - 1)mp \leq k.
\]

It follows that

\[
t = \left\lfloor \frac{(k - k')}{mp} + 1 \right\rfloor,
\]

as required.

Lemma 5.3. Let \( f \in \mathcal{F}_{\lambda,a}^p(K) \). Then, the coefficients of the corresponding conjugacy function satisfies

\[
|b_k| \leq \frac{a^{k-1}}{|1 - \lambda^m|\left\lfloor (k-k')/mp+1 \right\rfloor}.
\]

for all \( k \geq 2 \).

Proof. Recall that the coefficients \( |b_k| \) are given by the recursion formula (13) where each factorial term \( l!/\alpha_1! \cdot \ldots \cdot \alpha_k! \) is an integer and thus of modulus zero or one, depending on whether it is divisible by \( p \) or not. By Lemma 5.1 \( b_k = 0 \) if \( k \geq 2 \) and \( p \nmid k \). If \( p \mid k \), we have in view of Lemma 3.1 that

\[
|1 - \lambda^{k-1}| = \left\{ \begin{array}{ll} 1, & \text{if } m \nmid k - 1, \\
|1 - \lambda^m|, & \text{if } m \mid k - 1. \end{array} \right.
\]

Also recall that \( |a_i| \leq a^{i-1} \). It follows by Lemma 5.2 that

\[
|b_k| \leq \left| 1 - \lambda^m \right|^{-\left\lfloor (k-k')/mp+1 \right\rfloor} a^k.
\]
for some integer $\alpha$. In view of equation (14) we have

$$\sum_{i=2}^{k} (i-1)\alpha_i = k - l.$$ 

Consequently, since $|a_i| \leq a^{i-1}$, we obtain

$$\prod_{i=2}^{k} |a_i|^{\alpha_i} \leq \prod_{i=2}^{k} a^{(i-1)\alpha_i} = a^{k-l}.$$ 

(33)

Now we use induction over $k$. By definition $b_1 = 1$ and, according to the recursion formula (13), $|b_2| \leq |1 - \lambda m|^{-\frac{(2-k')}{mp+1}} |a|$. Suppose that

$$|b_l| \leq |1 - \lambda m|^{-\frac{(l-k')}{mp+1}} a^{l-1}$$

for all $l < k$. Then

$$|b_k| \leq |1 - \lambda m|^{-\frac{(k-k')}{mp+1}} a^{k-1} \max \left\{ \prod_{i=2}^{k} |a_i|^{\alpha_i} \right\},$$

and the lemma follows by the estimate (33).

The above estimate of $|b_k|$ is maximal in the sense that we may have equality in (31) for $k = k'$ as shown by the following example.

Example 5.1. Let $f$ be of the form

$$f(x) = \lambda x + a_{k'} x^{k'}.$$ 

Then, $a = |a_{k'}|^{1/(k'-1)}$. Also note that by Lemma 4.4

$$b_{k'} = a_{k'}/\lambda (1 - \lambda^{k'-1}),$$

and consequently,

$$|b_{k'}| = a_{k'-1}/|1 - \lambda^m|.$$ 

Thus we have equality in (31) for $k = k'$ in this case.

By the estimates in Lemma 5.3 and application of Proposition 3.3, the radius of convergence of the conjugacy function $g$ and its inverse $g^{-1}$ can now be estimated by

$$\rho = \frac{|1 - \lambda^m|^{\frac{1}{mp}}}{a},$$

(34)

and

$$\sigma = \frac{|1 - \lambda^m|^{\frac{1}{p-1}}}{a},$$

(35)

respectively. In other words, we have the following theorem.
Theorem 5.1. Let \( f \in \mathcal{F}_{p,a}(K) \). Then, \( f \) is analytically linearizable at \( x = 0 \). The semi-conjugacy relation \( g \circ f(x) = \lambda g(x) \) holds on \( D_\rho(0) \). Moreover, the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds on \( D_\sigma(0) \). The latter estimate is maximal in the sense that there exist examples of such \( f \) which have a periodic point on the sphere \( S_\sigma(0) \).

Remark 5.2. Note that \( g : D_\sigma(0) \to D_\sigma(0) \) is bijective if we consider \( D_\sigma(0) \) in the algebraic closure \( \hat{K} \).

Proof. In view of Lemma 5.3, \( g \) converges on the open disc of radius

\[
1/ \limsup_{k \to \infty} |b_k|^{1/k} \geq a^{-1}|1 - \lambda^m|^{1/mp} = \rho.
\]

Given \( m \geq 1 \), by definition \( k' \) must be one of the numbers

\[ p, \ 2p, \ldots, \ mp, \]

and accordingly,

\[
k' - 1 < mp. \quad (36)
\]

To estimate the radius of the maximal disc on which \( g \) is one-to-one we consider

\[
\sigma_0 = a^{-1} \inf_{k \geq 2} |1 - \lambda^m|^{(k-k')/mp + 1}/(k-1) \leq \inf_{k \geq 2} \left( \frac{|b_1|}{|b_k|} \right)^{1/(k-1)}.
\]

We will show that the maximum value of the exponent

\[
\delta(k) = [(k - k')/mp + 1]/(k - 1)
\]

is attained if and only if \( k = k' \). First note that \( \delta(k) = 0 \) for all \( 2 \leq k < k' \).

In view of (36), we have for each integer \( n \geq 1 \) that

\[
\delta(k' + nmp) - \frac{1}{k' - 1} = \frac{n((k' - 1) - mp)}{(k' - 1 + nmp)(k' - 1)} < 0.
\]

Moreover,

\[
\delta(k) < \delta(k' + nmp), \quad \text{if} \ mnp < k - k' < (n + 1)mp.
\]

Hence, the maximum of \( \delta \) is attained if and only if \( k = k' \) so that

\[
\sigma_0 = a^{-1}|1 - \lambda^m|^{1/k' - 1} = \sigma.
\]

Note that in view of (39) \( \sigma < \rho \) so that \( g \) certainly converges on the closed disc \( \overline{D}_\sigma(0) \). Also note that, in terms of \( \delta \), we have by (31) that

\[
|b_k| \leq a^{k-1}|1 - \lambda^m|^{-(k-1)\delta(k)}.
\]
Consequently, according to the derived properties of \( \delta \),

\[
|b_k|\sigma^k \leq a^{-1}|1 - \lambda^m|^{-\frac{k-1}{(k-1)}} \delta(k) \leq \sigma = |b_1|\sigma.
\]  

In view of Proposition 3.3, \( g : D_\sigma(0) \to D_\sigma(0) \) is a bijection if we consider \( D_\sigma(0) \) in \( \hat{K} \). Consequently, \( g : D_\sigma(0) \to D_\sigma(0) \) is one-to-one if we consider \( D_\sigma(0) \) in \( K \).

Recall that by Lemma 3.2, \( f : D_{1/a}(0) \to D_{1/a}(0) \) is one-to-one. Moreover, \( 1/a > \rho > \sigma \). It follows that the semi-conjugacy and the full conjugacy holds on \( D_\rho(0) \) and \( D_\sigma(0) \) respectively.

That this estimate of \( \sigma \) is maximal follows from the following example.

Let \( \text{char} K = 2 \) and \( f(x) = \lambda x + a_2x^2 \in K[x] \), where \( |1 - \lambda| < 1 \). Then \( m = 1 \) and \( k' = p = 2 \) so that \( \sigma = |1 - \lambda|/|a_2| \). But \( \hat{x} = (1 - \lambda)/a_2 \) is a fixed point of \( f \), breaking the conjugacy on \( S_\sigma(0) \). This completes the proof of Theorem 5.1.

If \( b_{k'} = 0 \), then we can extend the estimate of the full conjugacy to the disc \( D_\rho(0) \).

**Lemma 5.4.** Let \( f \in F_{\lambda,a}^p(K) \). Suppose the coefficient \( b_{k'} \) of the conjugacy function \( g \) is equal to zero. Then, the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds on a disc larger than or equal to \( D_\rho(0) \) or \( \overline{D}_\rho(0) \), depending on whether \( g \) converges on the closed disc \( \overline{D}_\rho(0) \) or not.

**Proof.** Recall that \( k' \) is the positive integer defined by (29). Assume that \( b_{k'} = 0 \). Let \( k'' > k' \) be the integer

\[
k'' := \min\{k \in \mathbb{Z} : k > k', b_k \neq 0, p \mid k, m \mid k - 1\}.
\]

In the same way as in Lemma 5.3

\[
|b_k| \leq \frac{a^{k-1}}{|1 - \lambda^m|^{(k-1)\gamma(k)}},
\]

where

\[
\gamma(k) = \begin{cases} 
\frac{|(k - k'')/mp + 1|/(k - 1)}{k \geq k''}, \\
0, & k < k''.
\end{cases}
\]

It follows from the proof of Lemma 5.2 that \( k'' = k' + mpn \), for \( n = 1 \). Hence,

\[
k'' \geq p + mp.
\]

Consequently,

\[
\gamma(k) - \frac{1}{mp} \leq \frac{k - k''}{(k - 1)mp} + \frac{1}{k - 1} - \frac{1}{mp} = \frac{mp - (k'' - 1)}{(k - 1)mp} < 0,
\]

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for \( k \geq 2 \). Moreover,

\[
\lim_{k \to \infty} \gamma(k) = 1/mp
\]

so that \( \sup \gamma(k) = 1/mp \). Accordingly, \( \max \gamma(k) \) does not exist. Also note that for \( \rho = a^{-1}|1 - \lambda^m|^{1/mp} \), we have

\[
|b_k|\rho^k \leq a^{-1}|1 - \lambda^m|^{\frac{1}{mp} + (k-1)(\frac{1}{mp} - \gamma(k))}.
\]

Hence, according to the derived properties of \( \gamma \),

\[
|b_k|\rho^k < \rho = |b_1|\rho,
\]  \( (38) \)

for \( k \geq 2 \). It follows that \( g : D_\rho(0) \to D_\rho(0) \) is one-to-one. If \( g \) converges on the closed disc \( \overline{D}_\rho(0) \), then, since we have strict inequality in \( (38) \), we also have that \( g : \overline{D}_\rho(0) \to \overline{D}_\rho(0) \) is one-to-one.

Recall that \( f : D_{1/a}(0) \to D_{1/a}(0) \) is one-to-one and that \( 1/a > \rho \). Consequently, the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds on a disc larger than or equal to \( D_\rho(0) \) or \( \overline{D}_\rho(0) \), depending on whether \( g \) converges the closed disc \( \overline{D}_\rho(0) \) or not.

Note that a sufficient condition that \( b_{k'} = 0 \) is that \( a_i = 0 \) for all \( 2 \leq i \leq k' \).

In fact, in view of Lemma 4.1 and the previous lemma we have the following result.

**Theorem 5.2.** Let \( f \in F_{\lambda, a}(K) \) be of the form

\[
f(x) = \lambda x + \sum_{i \geq i_0} a_i x^i,
\]

for some integer \( i_0 > k' \). Then, the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds on a disc larger than or equal to \( D_\rho(0) \) or \( \overline{D}_\rho(0) \), depending on whether \( g \) converges on the closed disc \( \overline{D}_\rho(0) \) or not.

### 6 Linearization discs and periodic points

In this section we give sufficient conditions under which the estimate \( \sigma \) in Theorem 5.1 is maximal in the sense that \( D_{\sigma}(0) \) is equal to the linearization disc. In other words, we give sufficient conditions under which \( D_{\sigma}(0) \) is the maximal disc \( U \subset K \) about the fixed point \( x = 0 \), such that the conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds for all \( x \in U \).

In theorem 5.1 we proved that the estimate \( \sigma \) is maximal in the sense that, in the the special case that \( f \) is quadratic and \( |1-\lambda| < 1 \) (\( m = 1 \)), there is a fixed point in the algebraic closure \( \hat{K} \) on the sphere \( S_{\sigma}(0) \), breaking the conjugacy there. Using the notion of Weierstrass degree of the conjugacy
function, defined below, we will give sufficient conditions for the existence of an indifferent periodic point on the boundary $S_\tau(0)$ for more general $f$.

The Weierstrass degree is defined as follows. Let $K$ be an algebraically closed complete non-Archimedean field. Let $U \subset K$ be a rational closed disc, and let $h$ be a power series which converges on $U$. For any disc $V \subseteq U$, the Weierstrass degree or simply the degree $\deg(h, V)$ of $h$ on $V$ is the number $d$ (if $V$ is closed) or $d'$ (if $V$ is open) in Proposition 3.2. Note that if $0 \in h(V)$, then the Weierstrass degree is the same as the notion of degree as ‘the number of pre-images of a given point, counting multiplicity’. More information on the properties of the Weierstrass degree can be found in [5].

The following lemma shows that a shift of the value of Weierstrass degree from 1 to $d > 1$, of the conjugacy function on a sphere $S$, reveals the existence of an indifferent periodic point on the sphere $S$.

**Lemma 6.1.** Let $K$ be a complete non-Archimedean field. Let $f$ be a linearizable power series of the form $f(x) = \lambda x + \sum_{i \geq 2} a_i x^i \in K[[x]]$, such that $|\lambda| = 1$ and $a = \sup_{i \geq 2} |a_i|^{1/(i-1)}$. Let $g$ be the corresponding conjugacy function. Let $\tau < 1/a$. Suppose $\deg(g, D_\tau(0)) = 1$ and $\deg(g, D_\tau(0)) = d > 1$ in the algebraic closure $\bar{K}$. Then, $f$ has an indifferent periodic point in $\bar{K}$ on the sphere $S_\tau(0)$ of period $\kappa \leq d$. In particular, $D_\tau(0)$ is the linearization disc of $f$ about the fixed point at the origin.

**Proof.** Let $\tau < 1/a$, and suppose $\deg(g, D_\tau(0)) = 1$ and $\deg(g, D_\tau(0)) = d > 1$. Note that by definition $\tau \in [\bar{K}^*]$. Hence, $S_\tau(0)$ is rational and non-empty in the algebraic closure $\bar{K}$. The proof that there is a periodic point in $\bar{K}$ on the sphere $S_\tau(0)$, goes as follows. Since the conjugacy $g$ maps the closed disc $D_\tau(0)$ onto itself exactly $d$-to-1 (counting multiplicity), and the open disc $D_\tau(0)$ one-to-one onto itself, there exist at least one point $\hat{x} \in S_\tau(0)$ such that $g(\hat{x}) = 0$. In view of the Schröder functional equation

$$\left|g(f(\hat{x})) - g(f^{\circ n}(\hat{x}))\right| = |g(\hat{x})||\lambda - \lambda^n| = 0,$$

for all $n \geq 1$. Recall that $1/a > \tau$. By Lemma 3.2 $f : D_\tau(0) \to D_\tau(0)$ is bijective in $\bar{K}$. The same is true for all the iterates $f^{\circ n}$, $n \geq 1$. Moreover, $f^{\circ n}(0) = 0$ and consequently $f^{\circ n}$ can have no zeros on the sphere $S_\tau(0)$ for any $n \geq 1$. In particular $f^{\circ n}(\hat{x}) \neq 0$ for all $n \geq 1$.

Let $y = g(f(\hat{x}))$. Then $y \in D_\tau(0)$. The equation $g(x) = y$ have only $d$ solutions on $D_\tau(0)$ and we conclude from (39) that we must have that $f^{\circ (\kappa+1)}(\hat{x}) = f(\hat{x})$ for some $\kappa \leq d$. This shows the existence of a periodic point $\hat{x} \in \bar{K}$ of period $\kappa \leq d$.

Finally, since $f^{\circ \kappa}$ is one-to-one on $D_\tau(0)$, it follows by the first statement of proposition 3.2 that $|(f^{\circ \kappa})'(\hat{x})| = 1$. This proves that $\hat{x}$ is indifferent. $\square$

Let us return to the case $f \in F_{\lambda,a}^0(K)$. By Theorem 5.1, the Weierstrass degree of $g$ on the open disc $\deg(g, D_\tau(0)) = 1$. In the following lemma we
find a necessary and sufficient condition that the Weierstrass degree on the closed disc \(\deg(g, \sigma(0)) > 1\). Again, the integer \(k'\), defined by (29), plays a significant role.

**Lemma 6.2.** Let \(f \in \mathcal{F}^p_{\lambda,a}(K)\). Then, in \(\hat{K}\), \(\deg(g, \sigma(0)) > 1\) if and only if the coefficient, \(b_{k'}\), of \(g\) satisfies
\[
|b_{k'}| = a^{k' - 1}/|1 - \lambda^m|.
\]
Moreover, if (40) holds, then \(\deg(g, \sigma(0)) = k'\).

**Proof.** By the estimate (31) we always have
\[
|b_{k'}| \leq a^{k' - 1}/|1 - \lambda^m|.
\]
If \(|b_{k'}| = a^{k' - 1}/|1 - \lambda^m|\). Then, we have equality in (37) if and only if \(k = 1\) or \(k = k'\). Consequently, in the algebraic closure \(\hat{K}\), \(g\) maps the closed disc \(\sigma(0)\) onto \(\sigma(0)\) exactly \(k'\)-to-1 counting multiplicity. It follows that the Weierstrass degree \(\deg(g, \sigma(0)) = k' > 1\).

On the other hand, if \(|b_{k'}| < a^{k' - 1}/|1 - \lambda^m|\), then we have equality in (37) if and only if \(k = 1\). Consequently, \(\deg(g, \sigma(0)) = 1\) in this case. \(\square\)

If \(f\) is of the form as in Example 5.1 then Lemma 6.2 applies and we have.

**Theorem 6.1.** Let \(f(x) = \lambda x + a_{k'}x^{k'}\), where \(a_{k'} \neq 0\). Then, the linearization disc of \(f\) about the origin is equal to \(D_{\sigma}(0)\). In the algebraic closure \(\hat{K}\), we have \(\deg(g, \sigma(0)) = k'\). Moreover, \(f\) has an indifferent periodic point in \(\hat{K}\) on the sphere \(S_{\sigma}(0)\) of period \(\kappa \leq k'\), with multiplier \(\lambda^\kappa\).

**Proof.** It remains to prove that the multiplier of the periodic point is of the form \(\lambda^\kappa\). Because the degree of the nonlinear monomials of \(f \in \mathcal{F}^p_{\lambda,a}(K)\) are all divisible by char \(K = p\), the derivative \((f^m)'(x) = \lambda^n\) for all \(x \in \hat{K}\) and all \(n \geq 1\). It follows that \(\hat{x}\) is an indifferent periodic point of period \(\kappa \leq d\), with multiplier \(\lambda^\kappa\). \(\square\)

In fact, this result can be generalized according to the following theorem.

**Theorem 6.2.** Let \(f \in \mathcal{F}^p_{\lambda,a}(K)\). Suppose \(a = |a_{k'}|^{1/(k' - 1)}\) and \(|a_i| < a^{i-1}\) for all \(i < k'\). Then, \(D_{\sigma}(0)\) is the linearization disc of \(f\) about the origin. In \(\hat{K}\) we have \(\deg(g, \sigma(0)) = k'\). Furthermore, \(f\) has an indifferent periodic point in \(\hat{K}\) on the sphere \(S_{\sigma}(0)\) of period \(\kappa \leq k'\), with multiplier \(\lambda^\kappa\).

**Proof.** Let \(f \in \mathcal{F}^p_{\lambda,a}(K)\). Suppose \(|a_{k'}| = a^{k' - 1}\) and \(|a_i| < a^{i-1}\) for all \(i < k'\). For \(k = k'\) and \(l = 1\) the equation (13) has the solution \(a_{k'} = 1\), \(a_j = 0\) for \(j < k'\). Hence, for \(k = k'\), the recursion formula (13) contains the term
\[
b_1a_{k'}/(1 - \lambda^{k' - 1}),
\]
(41)
where \( b_1 = 1 \), \( |a'_k| = a^{k'-1} \), and \( |1 - \lambda^{k'-1}| = |1 - \lambda^m| \). The minimality of \( k' \) and the assumption that \( |a_i| < a^{i-1} \) for all \( i < k' \) yields in view of Lemma 5.3 that the term \((11)\) is strictly greater than all the other terms in the recursion formula \((13)\). Hence, by ultrametricity, \( |b'_k| = a^{k' - 1}|1 - \lambda^m|^{-1} \) as required.

**Corollary 6.1.** Let \( f \in \mathcal{F}^p_{\lambda,a}(K) \) be of the form
\[
f(x) = \lambda x + \sum_{i \geq k'} a_i x^i, \quad a = |a_{k'}|^{1/(k' - 1)} > 0.
\]

Then, \( D_\sigma(0) \) is the linearization disc of \( f \) about the origin. In \( \hat{K} \) we have \( \deg(g, D_\sigma(0)) = k' \). Furthermore, \( f \) has an indifferent periodic point in \( \hat{K} \) on the sphere \( S_\sigma(0) \) of period \( \kappa \leq k' \), with multiplier \( \lambda^\kappa \).

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**References**

[1] D. K. Arrowsmith and F. Vivaldi. Some \( p \)-adic representations of the Smale horseshoe. *Phys. Lett. A*, 176:292–294, 1993.

[2] D. K. Arrowsmith and F. Vivaldi. Geometry of \( p \)-adic Siegel discs. *Physica D*, 71:222–236, 1994.

[3] A. F. Beardon. *Iteration of Rational Functions*. Springer-Verlag, Berlin Heidelberg New York, 1991.

[4] R. Benedetto. Reduction dynamics and Julia sets of rational functions. *J. Number Theory*, 86:175–195, 2001.

[5] R. Benedetto. Non-Archimedean holomorphic maps and the Ahlfors Islands theorem. *Amer. J. Math.*, 125(3):581–622, 2003.

[6] J-P. Bézivin. Fractions rationnelles hyperboliques \( p \)-adiques. *Acta Arith.*, 112(2):151–175, 2004.

[7] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis: A systematic approach to rigid analytic geometry*. Springer-Verlag, Berlin, 1984.
[8] A. D. Brjuno. Analytical form of differential equations. Trans. Moscow Math. Soc., 25,26:131–288,199–239, 1971,1972.

[9] L. Carleson and T. Gamelin. Complex Dynamics. Springer-Verlag, Berlin Heidelberg New York, 1991.

[10] J.W.S. Cassels. Local Fields. Cambridge University Press, Cambridge, 1986.

[11] J. Fresnel and M. van der Put. Géométrie analytique rigide et applications. Birkhäuser, Boston, 1981.

[12] M. Herman. Recent results and open questions on Siegel’s linearization theorem of complex analytic diffeomorphisms of C^n near a fixed point. In Proc. VIII-th International Congress on Mathematical Physics 1986, pages 138–184. World Scientific, 1987.

[13] M. Herman and J.-C. Yoccoz. Generalizations of some theorems of small divisors to non archimedean fields. In Geometric Dynamics, volume 1007 of LNM, pages 408–447. Springer-Verlag, 1981.

[14] L. Hsia. Closure of periodic points over a non-archimedean field. J. London Math. Soc., 62(2):685–700, 2000.

[15] A. Yu. Khrennikov. Small denominators in complex p-adic dynamics. Indag. Mathem., 12(2):177–188, 2001.

[16] A. Yu. Khrennikov and M. Nilsson. On the number of cycles for p-adic dynamical systems. J. Number Theory, 90:255–264, 2001.

[17] N. Koblitz. p-adic numbers, p-adic analysis, and zeta-functions. Springer-Verlag, New York, second edition, 1984.

[18] H-C. Li. p-adic dynamical systems and formal groups. Compos. Math., 104:41–54, 1996.

[19] H-C. Li. On heights of p-adic dynamical systems. Proc. Amer. Math. Soc., 130(2):379–386, 2002.

[20] K.-O. Lindahl. Estimates of linearization discs in p-adic dynamics with application to ergodicity. [http://arxiv.org/abs/0910.3312](http://arxiv.org/abs/0910.3312).

[21] K.-O. Lindahl. On Siegel’s linearization theorem for fields of prime characteristic. Nonlinearity, 17(3):745–763, 2004.

[22] K.-O. Lindahl. On the linearization of non-Archimedean holomorphic functions near an indifferent fixed point. PhD thesis, Växjö University, 2007. Introduction and summary available at [http://www.diva-portal.org/](http://www.diva-portal.org/).
[23] J. Lubin. Non-archimedean dynamical systems. *Compos. Math.*, 94:321–346, 1994.

[24] J. Milnor. *Dynamics in One Complex Variable*. Vieweg, Braunschweig, 2nd edition, 2000.

[25] J. Rivera-Letelier. Dynamique des functions rationnelles sur des corps locaux. *Astérisque*, 287:147–230, 2003.

[26] J. Rivera-Letelier. Espace hyperbolique $p$-adique et dynamique des fonctions rationnelles. *Compos. Math.*, 138(2):199–231, 2003.

[27] C. L. Siegel. Iteration of analytic functions. *Ann. of Math.*, 43:607–612, 1942.

[28] S. De Smedt and A. Khrennikov. Dynamical systems and theory of numbers. *Comment. Math. Univ. St. Pauli*, 46(2):117–132, 1997.

[29] J.-C. Yoccoz. Linéarisation des germes de diffeomorphismes holomorphes de $(\mathbb{C}, 0)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 306(1):55–58, 1988.