The category of equilogical spaces and the effective topos as homotopical quotients

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Abstract

We show that the two models of an extensional version of Martin-Löf type theory, those given by the category of equilogical spaces and by the effective topos, are homotopical quotients of appropriate categories of 2-groupoids.

1 Introduction

The category of $T_0$-spaces embeds fully in the category of equilogical spaces; the category of equilogical spaces is locally cartesian closed and the embedding functor preserves products and any exponential available in the original category. Thus the category of equilogical spaces provides a nice extension of the category of $T_0$-spaces. The effective topos is the categorical rendering of Kleene’s realizability model for intuitionistic logic, and is the first interesting example of a non-Grothendieck topos. We show that the category of equilogical spaces is the homotopical quotient of a category of groupoids, and that the effective topos is the homotopical quotient of a category of 2-groupoids of partitioned assemblies.

Groupoids are a main tool in algebraic topology, see [Bro68] and groupoids were the first nontrivial models of the intensional version of Martin-Löf Type Theory in [HS98]. Moreover in recent years the Univalent Foundations Program, see [Uni13], has advocated a strong connection between algebraic topology and type theory.

Since both the category of equilogical spaces and the effective topos are models of an extensional version of Martin-Löf type theory, it is useful to find that each comes from the “extensionalization” of a model of intensional type theory and that such a process is actually a homotopical quotient. We should stop here to point out that the meaning we adopt for an homotopical quotient of a category is in line with a suggestion in [CV98] and is the more naive notion obtained from an interval-like object than that derived from a Quillen model category—the main reason is that one of the two example categories we study

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is neither complete nor cocomplete. So, as a homotopical quotient, we shall consider a category obtained as a quotient category from a category $C$ with finite limits, as follows:

- there is a fixed interval-like object $I$, i.e. it has two global points $0: T \to I$ and $1: T \to I$ whose pushout

\[
\begin{array}{ccc}
T & \xrightarrow{1} & I \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0'} & 0' \\
I & \xrightarrow{1'} & I + T I
\end{array}
\]

exists in $C$ and is stable under products, an arrow $\gamma: I \to I + T I$ and an arrow $\iota: I \to I$ such that the four arrows together with the unique arrow $!: I \to T$ form an equivalence co-span in $C$, i.e. the following diagrams commute

\[
\begin{array}{ccc}
T & \xrightarrow{0} & I & \xrightarrow{1} & T \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{1} & I & \xrightarrow{1'} & I + T I \\
1 & \xleftarrow{\iota} & I & \xleftarrow{!} & I + T I
\end{array}
\]

—note that there is also a necessarily commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{0} & I \\
\downarrow & & \downarrow \\
0 & \xrightarrow{!} & T \\
\end{array}
\]

since $T$ is terminal—;

- two arrows $f, g: X \to Y$ are identified in the quotient if there is an arrow $h: X \times I \to Y$ such that the following diagram commute

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X \times I & \xrightarrow{h} & Y \\
\end{array}
\]

The condition of the structure on $I$ ensures that the identification is an equivalence relation on parallel arrows in $C$. 2
It seems plausible that the categories we analyse in the following sustain suitable notions of fibrations, cofibrations and weak equivalences—in particular, that a map of the kind \((\text{id}_X, i): X \to X \times I, i = 0, 1\), is a weak equivalence. But the categories are certainly not complete, nor cocomplete, and that prevents a direct comparison with standard homotopical quotients. It will be considered in future work.

We introduce the category of equilogical spaces in section 2 and we recall one of the presentations of the effective topos in section 3, reviewing properties which are needed in the following sections. In section 4 we determine a category \(\mathcal{A}\) of topological groupoids and an interval-like topological groupoid \(\mathcal{I}\) such that the homotopical quotient of \(\mathcal{A}\) determined by \(\mathcal{I}\) is equivalent to the category of equilogical spaces. In section 5 we produce a similar result for the effective topos using a category of 2-groupoids on partitioned assemblies.

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2 Equilogical spaces

Recall from [Sco96, BBS04] that an equilogical space \(\mathcal{E} = (S_\mathcal{E}, \tau_\mathcal{E}, \sim_\mathcal{E})\) consists of a \(T_0\)-space \((S, \tau)\) and an equivalence relation \(\sim_\mathcal{E} \subseteq S_\mathcal{E} \times S_\mathcal{E}\) on the points of the space.

A map \([f]: \mathcal{E} \to \mathcal{F}\) of equilogical spaces is an equivalence class of continuous functions \(f: (S_\mathcal{E}, \tau_\mathcal{E}) \to (S_\mathcal{F}, \tau_\mathcal{F})\) preserving the equivalence relations, i.e. if \(x \sim_\mathcal{E} x'\), then \(f(x) \sim_\mathcal{F} f(x')\) for all \(x\) and \(x'\) in \(S_\mathcal{E}\). For two such continuous functions \(f, g: (S_\mathcal{E}, \tau_\mathcal{E}) \to (S_\mathcal{F}, \tau_\mathcal{F})\), one sets \(f\) equal to \(g\) when \(f(x) \sim_\mathcal{F} g(x)\) for all \(x \in S_\mathcal{E}\).

Composition of maps of equilogical spaces \([f]: \mathcal{E} \to \mathcal{F}\) and \([g]: \mathcal{F} \to \mathcal{G}\) is given on (any of) their continuous representatives: \([g] \circ [f] := [g \circ f]\).

The data above determine a category \(\mathcal{Equ}\) of equilogical spaces. There is a full embedding
\[
Y: \mathcal{Top}_0 \hookrightarrow \mathcal{Equ}
\]
which maps a \(T_0\)-space \((S, \tau)\) to the equilogical space on \((S, \tau)\) with the diagonal relation, i.e. the equilogical space \((S, \tau, =_S)\).

The category \(\mathcal{Equ}\) is a locally cartesian closed full extension of the category \(\mathcal{Top}_0\) of \(T_0\)-spaces. In fact, it is the intersection of two other locally cartesian
closed full extensions of

\[
\begin{array}{c}
\text{Top}_{0} \downarrow \quad \text{Equ} \downarrow \quad (\text{Top}_{0})_{\text{ex}} \\
\text{Top} \downarrow \quad \text{Top}_{\text{reg}} \downarrow \quad \text{Top}_{\text{ex}}
\end{array}
\]

The exact completions \((\text{Top}_{0})_{\text{ex}}\) and \(\text{Top}_{\text{ex}}\) are pretoposes, while the regular completion \(\text{Top}_{\text{reg}}\) is a quasitopos, see [Ros00].

The product of equilogical spaces \(E \times F\) is computed as expected taking the topological product \((S_{E}, \tau_{E}) \times (S_{F}, \tau_{F})\) and the equivalence relation

\[
(a, b) \sim_{E \times F} (a', b') \text{ when } a \sim_{E} a' \text{ and } b \sim_{F} b'.
\]

The projections to the factors are obvious.

The construction of the exponential \(F^{E}\) is less direct and we refer the reader to the basic sources [Sco76, Sco96, BBS04] as well as [BR14, BCRS98].

It is useful for the purpose of this paper to point out the strong similarity between the presentation of \(\text{Equ}\) and that of \((\text{Top}_{0})_{\text{ex}}\). So recall from [CC82, Car95, FS91, CV98] that the exact completion \(\mathcal{C}_{\text{ex}}\) of a category \(\mathcal{C}\) with finite limits is a quotient category of the full subcategory \(\text{ES}(\mathcal{C})\) of the category \(\mathcal{C}\) of graphs in \(\mathcal{C}\) on the equivalence spans.

Recall that a (directed) graph in \(\mathcal{C}\) is a parallel pair \(A_{1} \xrightarrow{d_{1}} A_{0}\) of arrows of \(\mathcal{C}\) and a homomorphism from the graph \(A_{1} \xrightarrow{d_{1}} A_{0}\) to the graph \(B_{1} \xrightarrow{e_{1}} B_{0}\) is a pair \((f_{1}: A_{1} \to B_{1}, f_{0}: A_{0} \to B_{0})\) of arrows in \(\mathcal{C}\) such that the following diagram commutes

\[
\begin{array}{ccc}
A_{0} & \xrightarrow{d_{1}} & A_{1} & \xrightarrow{d_{2}} & A_{0} \\
| & = & | & = & | \\
B_{0} & \xrightarrow{e_{1}} & B_{1} & \xrightarrow{e_{2}} & B_{0}.
\end{array}
\]

An **equivalence span** is a graph \(A_{1} \xrightarrow{d_{1}} A_{0}\) in \(\mathcal{C}\) which is reflexive, symmetric, and endowed with a compatible operation on pairs of consecutive arcs, i.e. there are arrows \(r: A_{0} \to A_{1}\), \(s: A_{1} \to A_{1}\), and \(t: A_{1} \times_{A_{0}} A_{1} \to A_{1}\),
where

\[
\begin{array}{c}
A_1 \times_{A_0} A_1 \xrightarrow{d_2'} A_1 \\
\downarrow d_1' \quad \downarrow d_1 \\
A_1 \xrightarrow{d_2} A_0
\end{array}
\]

is a pullback in \( \mathcal{C} \), such that the following diagrams commute:

\[
\begin{array}{c}
A_0 \xrightarrow{id_{A_0}} A_0 \xrightarrow{r} A_1 \\
\downarrow d_1 \downarrow \downarrow \\
A_0 \xrightarrow{s} A_1 \xrightarrow{d_2} A_0
\end{array}
\]

\[
\begin{array}{c}
A_1 \xrightarrow{d_1} A_0 \xrightarrow{t} A_1 \\
\downarrow d_1 \downarrow \downarrow \\
A_0 \xrightarrow{t} A_1 \xrightarrow{d_2} A_0
\end{array}
\]

The quotient category \( \mathcal{C}_{ex} \) is obtained by identifying homomorphisms \((f_1, f_0)\) and \((g_1, g_0)\) from \(A_1 \xrightarrow{d_1} A_0\) to \(B_1 \xrightarrow{e_1} B_0\) if there is an arrow \(h: A_0 \to B_1\) such that

\[
\begin{array}{c}
A_0 \xrightarrow{f_0} A_0 \\
\downarrow h \downarrow \\
B_0 \xrightarrow{g_0} B_0
\end{array}
\]

—nothing is asked of the other component.

The following proposition makes the similarity explicit.

2.1 Proposition. The category \( \mathcal{E}qu \) is equivalent to the full subcategory \( \mathcal{A} \) of \( (\text{Top}_0)_{ex} \) on those equivalence spans \( A_1 \xrightarrow{d_1} A_0 \) of topological spaces and continuous maps such that the pair \( \langle d_1, d_2 \rangle: A_1 \to A_0 \times A_0 \) is a subspace inclusion.

Proof. Consider an equivalence span \( A = A_1 \xrightarrow{d_1} A_0 \) of topological spaces and continuous maps such that the pair \( \langle d_1, d_2 \rangle: A_1 \to A_0 \times A_0 \) is a subspace inclusion. Note that the functions \( r, s \) and \( t \) requested by the definition of
equivalence span are unique, and determine that the subset $|A_1|$ of pairs of points of $|A_0|$ is an equivalence relation. Write $F(A)$ for the equilogical space which consists of the topological space $A_0$ and the equivalence relation $|A_1|$.

For a homomorphism $(f_1, f_0)$ between two such equivalence spans, the component $f_1$ is uniquely determined by the other data as the restriction of the pair $(f_0, f_0)$, and ensures that $f_0$ is a representative of a map of equilogical spaces. Moreover, in the quotient category $(\text{Top}_0)^{ex}$, the homomorphism $(f_1, f_0)$ is identified with $(g_1, g_0)$ precisely when $(f(x), g(x))$ is in $A_1$ for all points $x$ in $A_0$.

Thus the assignment $F([f_1, f_0]) = [f_0]$ is well defined, and determines a functor from $\mathcal{A}$ to $\mathcal{E}_{qu}$ which is full and faithful.

To see that $F$ is also bijective on objects, suppose $E = (S_E, \tau_E, \sim_E)$ is an equilogical space. Consider the subspace topology $\sigma_E$ on $\sim_E \subseteq S_E \times S_E$ and the graph of topological spaces

$$
\begin{array}{c}
\sim_E, \sigma_E \longrightarrow (S_E, \tau_E)
\end{array}
$$

induced by the two projections. It is easy to check that it is an equivalence span and, by construction, the pair $\langle \pi_1, \pi_2 \rangle: (\sim_E, \sigma_E) \longrightarrow (S_E, \tau_E) \times (S_E, \tau_E)$ is a subspace inclusion. It is obvious that the functor $F$ takes that equivalence span of $\mathcal{A}$ to the equilogical space $E$.

In the following, we shall refer to an equivalence span $A_1 \xrightarrow{d_1} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle: A_1 \longrightarrow A_0$ is a subspace inclusion as a subspatial equivalence span.

## 3 The effective topos

The effective topos $\mathcal{E}_{eff}$ was introduced in [HJP80, Hyl82]. It was shown in [RR90] that $\mathcal{E}_{eff}$ is (equivalent to) the exact completion of the category $\mathcal{PAsm}$ of partitioned assemblies, see [CFS88].

A **partitioned assembly** is a function $\xi: X \longrightarrow \mathbb{N}$; a **map** $X \xrightarrow{\xi} Y$ of partitioned assemblies is a function $f: X \longrightarrow Y$ such that there is a partial recursive function $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that the following diagram commutes

$$
\begin{array}{c}
X & \xrightarrow{f} & Y \\
\downarrow \xi & & \downarrow \zeta \\
\mathbb{N} & \xrightarrow{\phi} & \mathbb{N}
\end{array}
$$

In order to make sure that the exact completion introduced in section 2 can be applied to the category $\mathcal{PAsm}$ we recall how finite limits can be obtained in that category.
The product of two partitioned assemblies is obtained by adopting some particular recursive encoding $\langle n, m \rangle$ of pairs of numbers; the product partitioned assembly of $X\downarrow_N$ and $Y\downarrow_N$ is the function

$$(x, y) \mapsto \langle \langle \xi(x), \zeta(y) \rangle \rangle : X \times Y \rightarrow \mathbb{N}$$

with obvious projections.

The equalizer of $X\downarrow_\xi N \xrightarrow{f} \xrightarrow{g} Y\downarrow_\zeta N$ is the partitioned assembly $\xi\upharpoonright E : E \rightarrow N$ where $E := \{ x \in \mathbb{N} \mid f(x) = g(x) \}$ with the obvious inclusion into $\xi\downarrow_N$.

The next result will be useful in the following.

3.1 Lemma. Every equivalence span

$$A_1 \xrightarrow{d_1} A_0 \xleftarrow{d_2} A_0$$

in $\mathcal{PAsm}_{ex}$ is isomorphic to one of the form

$$E \xrightarrow{\epsilon} E \xleftarrow{e_2} A_0$$

such that the triple $(e_1, e_2, \epsilon)$ is monic.

Proof. Consider an arbitrary equivalence span

$$A_1 \xrightarrow{d_1} A_0 \xleftarrow{d_2} A_0$$

in $\mathcal{PAsm}_{ex}$. So there are two partial recursive functions $\phi_1$ and $\phi_2$ such that the following diagram commutes

$$A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} A_0$$

$$\alpha_0 \quad \alpha_1 \quad \alpha_0$$

$$\phi_1 \quad \phi_2 \quad \phi_2$$

Take $E$ to be the image of the function $\langle d_1, d_2, \alpha_1 \rangle : A_1 \rightarrow A_0 \times A_0 \times \mathbb{N}$, let $f : A_1 \rightarrow E$ be the factoring surjection, and let $\epsilon := \pi_3 \upharpoonright E : E \rightarrow \mathbb{N}$. Let
$e_1, e_2: \frac{E}{\mathbb{N}} \xrightarrow{\epsilon} A_0 \xrightarrow{\alpha_0} \frac{\mathbb{N}}{}$ be the first and second projection respectively. It is easy to see that it is an equivalence span.

Clearly $f$ gives rise to a map of partitioned assemblies $\frac{A_1}{\mathbb{N}} \xrightarrow{\alpha_1} \frac{E}{\mathbb{N}}$ since there is a commutative diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f} & E \\
\downarrow{\alpha_1} & & \downarrow{\epsilon} \\
\mathbb{N} & \xrightarrow{\text{id}_\mathbb{N}} & \mathbb{N}.
\end{array}
\]

Moreover any section $s: E \rightarrow A_1$ of $f$ (as a function of sets) is a map of partitioned assemblies $\frac{E}{\mathbb{N}} \xrightarrow{\epsilon} \frac{A_1}{\mathbb{N}}$ and a section of $\frac{A_1}{\mathbb{N}} \xrightarrow{f} \frac{E}{\mathbb{N}}$ in $\mathcal{PAsm}$.

Thus an appeal to the axiom of choice yields the conclusion.

**3.2 Remark.** Note that the axiom of choice was used in a crucial way in 3.1 to determine an equivalence span of the required form and the requested isomorphism, but the proof that

\[
\begin{array}{ccc}
\frac{E}{\mathbb{N}} & \xrightarrow{e_1} & \frac{A_0}{\mathbb{N}} \\
\downarrow{\epsilon} & & \downarrow{\alpha_0} \\
\frac{E}{\mathbb{N}} & \xrightarrow{e_2} & \frac{A_0}{\mathbb{N}}
\end{array}
\]

is an equivalence span does not require the use of the axiom of choice.

We conclude this brief review of the effective topos recalling a diagram of functors considered by Aurelio Carboni in [Car95] which shows how similar the situation is to that of topological spaces. Write $\mathcal{PAsm}_0$ for the full subcategory of $\mathcal{PAsm}$ on those partitioned assemblies which are 1-1 (functions). This is clearly equivalent to the category $\mathcal{PR}$ whose objects are subsets of $\mathbb{N}$ and whose arrows are restriction of partial recursive functions between those, total on the domain.

\[
\begin{array}{ccc}
\mathcal{PAsm}_0 & \xleftarrow{} & \mathcal{PER} & \xrightarrow{} & (\mathcal{PAsm}_0)_{\text{ex}} \\
\downarrow{\bot} & & \downarrow{\bot} & & \downarrow{\bot} \\
\mathcal{PAsm} & \xleftarrow{} & \mathcal{PAsm}_{\text{reg}} & \xrightarrow{} & \mathcal{PAsm}_{\text{ex}}.
\end{array}
\]

In the diagram of full subcategories of $\mathcal{Eff}$, the exact completion $\mathcal{PAsm}_{\text{ex}}$ is itself the effective topos; $\mathcal{PAsm}_{\text{reg}}$ is the full subcategory of $\mathcal{Eff}$ on the $\bot$-separated objects; $(\mathcal{PAsm}_0)_{\text{ex}}$ is the full subcategory of $\mathcal{Eff}$ on the discrete objects—i.e. subquotients of the natural number object of $\mathcal{Eff}$, see [HRR90].
and $\mathcal{PER}_\text{reg}$ is the intersection of the last two, the full subcategory of $\mathcal{Eff}$ on the $\neg\neg$-separated subquotients of the natural number object of $\mathcal{Eff}$, also known as “partial equivalence relations on $\mathbb{N}$”, see [Hyl88]. As is shown in [Car95], this last is not the regular completion of $\mathcal{PR}_\equiv = \mathcal{PAsm}_0$. A similar remark applies to $\mathcal{Equ}$ and $(\mathcal{Top}_0)_{\text{reg}}$ which are not equivalent—this corrects a hastily mistaken, happily irrelevant statement in [BR14].

4 Groupoids

Consider a category $\mathcal{C}$ with pullbacks. A groupoid $G$ in $\mathcal{C}$ is a graph $G_1 \xrightarrow{d_1} G_0$ of objects and arrows in $\mathcal{C}$ together with three more arrows

$$i: G_0 \rightarrow G_1 \quad c: G_1 \times_{G_0} G_1 \rightarrow G_1 \quad s: G_1 \rightarrow G_1$$

where

$$
\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \xrightarrow{d'_2} & G_1 \\
\downarrow^{d'_1} & & \downarrow^{d_1} \\
G_1 & \xrightarrow{d_2} & G_0
\end{array}
$$

is a pullback in $\mathcal{C}$, such that

- the graph $G_1 \xrightarrow{d_1} G_0$ with $i$ and $c$ is a category object in $\mathcal{C}$,
- $s$ is an involution which makes every arrow an isomorphism.

The notions of functor of groupoids in $\mathcal{C}$ is obvious as well as that of natural transformation. It is straightforward to check that a functor between groupoids preserves the involution which makes every arrow an isomorphism.

We have already available a large number of examples as follows from the next property.

4.1 Proposition. Let $G_1 \xrightarrow{d_1} G_0$ be a graph in $\mathcal{C}$ with arrows $r: G_0 \rightarrow G_1$,
\(t: G_1 \times G_0 G_1 \to G_1\), and \(s: G_1 \to G_1\) such that the diagrams commute. If the pair \(G_1 \xrightarrow{\delta_1} \xleftarrow{\delta_2} G_0\) is jointly monic, then

(i) the structure given by

\[
G_2 \xrightarrow{t} G_1 \xleftarrow{d_1} G_0
\]

is a groupoid \(G\) in \(C\),

(ii) for any groupoid \(H\) in \(C\), a graph-homomorphism from the underlying graph \(H_1 \xrightarrow{e_1} H_0\) of \(H\) to \(G_1 \xrightarrow{\delta_1} G_0\) is also a functor from \(H\) to \(G\),

(iii) for any groupoid \(H\) in \(C\), let \((f_1, f_0)\) and \((g_1, g_0)\) be functors from the groupoid \(H\) to the groupoid \(G\). Then an arrow \(a: H_0 \to G_1\) such that

is a natural transformation from \((f_1, f_0)\) to \((g_1, g_0)\).

Proof. Straightforward. \(\Box\)

4.2 Corollary. Every subspatial equivalence span is a groupoid in \(\text{Top}_0\). Every representative of an arrow in \(\mathcal{A}\) is a functor between the groupoids.
Consider the interval-like groupoid
\[ I := \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]
with the discrete topology. A natural transformation as in 4.1(iii) is the same as a functor \( H \times I \rightarrow G \). Thanks to 2.1, we may rephrase Corollary 4.2 as follows.

**4.3 Theorem.** The category \( \mathcal{E} \) of equilogical spaces is the homotopical quotient of the category \( \mathcal{A} \) of topological groupoids.

## 5 2-groupoids

A similar case can be made for the effective topos. We prove in the following that it is the homotopical quotient of a category of higher groupoids in \( \mathcal{P} \).

Consider a category \( C \) with pullbacks. A **2-groupoid** \( G \) in \( C \) is a 2-graph

\[
\begin{array}{ccc}
G_2 & \xrightarrow{d_{21}} & G_1 \\
\downarrow^{d_{22}} & & \downarrow^{d_{11}} \\
G_1 & \xrightarrow{d_{12}} & G_0
\end{array}
\]

of objects and arrows in \( C \) together with arrows

\[
\begin{align*}
i_1 &: G_0 \to G_1 \\
i_2 &: G_1 \to G_2 \\
c_1 &: G_1 \times_{G_0} G_1 \to G_1 \\
c_2 &: G_2 \times_{G_1} G_2 \to G_2 \\
c_2' &: G_2 \times_{G_0} G_2 \to G_2 \\
s_1 &: G_1 \to G_1 \\
s_2 &: G_2 \to G_2
\end{align*}
\]

where

\[
\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \to & G_1 \\
\downarrow^{d_{11}} & & \downarrow^{d_{21}} \\
G_1 & \to & G_0 \\
\downarrow^{d_{12}} & & \downarrow^{d_{11}d_{21}} \\
G_2 \times_{G_1} G_2 & \to & G_2 \\
\downarrow^{d_{22}} & & \downarrow^{d_{11}d_{22}} \\
G_2 & \to & G_0 \\
\downarrow^{d_{12}d_{22}} & & \downarrow^{d_{11}d_{22}}
\end{array}
\]

are pullbacks in \( C \), such that

- the 2-graph \( G_2 \xrightarrow{d_{21}} G_1 \xrightarrow{d_{11}} G_0 \) with \( i_1, c_1, i_2, c_2, c_2' \) is a 2-category object in \( C \),
- \( s_1 \) is an involution which makes every 1-arrow an equivalence via the pair of arrows given by \( q \).
• $s_2$ is an involution which makes every 2-arrow an iso.

The notions of 2-functor of 2-groupoids in $C$ is obvious as well as that of 2-transformation.

Consider the 2-category $\text{Grpd}(\mathcal{PAsm})$ of 2-groupoids in $\mathcal{PAsm}$ with 2-functors and 2-transformations. Clearly, the underlying graph of a 2-groupoid $G$ of $\mathcal{PAsm}$ is an equivalence span in $\mathcal{PAsm}$, thus an object of $\text{Eff}$. This extends directly to a functor $U: \text{Grpd}(\mathcal{PAsm}) \to \text{Eff}$.

5.1 Theorem. The functor $U: \text{Grpd}(\mathcal{PAsm}) \to \text{Eff}$ is essentially surjective.

Proof. Consider an object in $\text{Eff}$, by 5.1 we can assume without loss of generality that it is an equivalence span $\xymatrix{A_1 \ar[d]_{\alpha_1} \ar[r]^{a_1} & A_0 \ar[d]_{\alpha_0}}$ in $\mathcal{PAsm}$ such that the triple $(a_1, a_2, \alpha_0)$ is monic. Take the free dagger category on that graph in $\mathcal{PAsm}$—by a dagger category we mean a category together with an involutive contravariant functor which is the identity on objects. It consists of $A_0 \xymatrix{\ar[d]_{\alpha_0}}$ as objects of objects. The object of 1-arrows is $\xymatrix{A^\wedge \ar[d]_{\alpha_0}}$ where $A^\wedge$ consists of the zigzag paths in the graph $\xymatrix{A_1 \ar[r]^{a_1} & A_0}$. By a zigzag path in the graph we mean a list which is either of the form $(x)$ where $x \in A_0$ or

$\langle x_0, e_1, i_1, x_1, e_2, i_2, x_2, \ldots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle$,

where

• $x_\ell \in A_0$ for $0 \leq \ell \leq n+1$,

• $e_\ell \in A_1$ for $1 \leq \ell \leq n+1$,

• $i_\ell \in \{0,1\}$ for $1 \leq \ell \leq n+1$,

• for $0 \leq \ell \leq n$, if $i_\ell = 0$, then $\langle x_\ell, x_{\ell+1}, e_{\ell+1} \rangle \in A_1$,

• for $0 \leq \ell \leq n$, if $i_\ell = 1$, then $\langle x_{\ell+1}, x_\ell, e_{\ell+1} \rangle \in A_1$.

Intuitively, if one considers a triple $\langle x, x', e \rangle \in A_1$ as an edge $e$ from the source $x$ to the target $x'$ in the graph $\xymatrix{A_1 \ar[r]^{a_1} & A_0}$, then the zigzag

$\langle x_0, e_1, i_1, x_1, e_2, i_2, x_2, \ldots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle$

is a mixed-directional path of edges from the vertex $x_0$ to the vertex $x_{n+1}$ where each edge $e_\ell$ between $x_\ell$ and $x_{\ell+1}$ is marked with either 0 or 1: if the mark is 0, $e_\ell$ goes from $x_\ell$ to $x_{\ell+1}$ in the original graph; if the mark is 1, $e_\ell$ goes from
\(x_{l+1}\) to \(x_{l}\). The function \(\alpha^\wedge\) is defined by mapping a zigzag to the encoding of the list of its numerical components:

\[
\alpha^\wedge((x)) := \langle \langle 0, \alpha_0(x) \rangle \rangle
\]

\[
\alpha^\wedge((x_0, e_1, i_1, \ldots, x_n, e_{n+1}, i_{n+1}, x_{n+1})) := \langle \langle n+1, \langle \langle \alpha^\wedge((x_0, e_1, i_1, \ldots, x_n), \langle \langle e_{n+1}, i_{n+1} \rangle, a_0(x_{n+1}) \rangle \rangle \rangle \rangle \rangle.
\]

The structure of dagger category in \(\mathcal{PAsm}\) is obvious, changing each \(i_\ell\) with \(\sigma(i_\ell)\) where \(\sigma: \{0,1\} \rightarrow \{0,1\}\) swaps 0 with 1. The object of 2-arrows \(A^\wedge\) is formed by taking the total relation on each 1-homset, where \(A^\wedge := A^\wedge \times A_0 A^\wedge\). Explicitly, \(A^\wedge\) consists of all pairs of zigzags

\[
((x_0, e_1, \ldots, x_n), (x_0, e_1', \ldots, x_n))
\]

between each two given vertices \(x\) and \(x'\); clearly all 2-diagrams commute as there is at most one 2-arrow from an 1-arrow to another. In this way, the dagger functor becomes the involution which makes every 1-arrow an equivalence. It is easy to see that that gives a 2-groupoid on the given span in \(\mathcal{PAsm}\) and that the functor \(U\) takes it to a span which is isomorphic to

\[
\begin{array}{ccc}
A_1 & \xrightarrow{e_1} & A_0 \\
\downarrow & & \downarrow \\
\mathbb{N} & \xrightarrow{\alpha_0} & \mathbb{N}
\end{array}
\]

We shall refer to a 2-groupoid like that produced in the proof of 5.1 as a **numeric** 2-groupoid as all edges are denoted by numbers. More precisely, it is a 2-groupoid \(G\) in \(\mathcal{PAsm}\) such that its underlying category in \(\mathcal{PAsm}\)

\[
\begin{array}{ccc}
G_1 & \xrightarrow{d_{11}} & G_0 \\
\downarrow & & \downarrow \\
G_0 & \xrightarrow{d_{12}} & G_0
\end{array}
\]

is a free dagger category and \(G\) embeds, fully at level 2, into the 2-groupoid

\[
G_0 \times G_0 \times \mathbb{N} \times \mathbb{N} \xrightarrow{\pi_{123}} G_0 \times G_0 \times \mathbb{N} \xrightarrow{\pi_{1}} \mathbb{N} \xrightarrow{\sigma_2} G_0 \times G_0 \times \mathbb{N} \xrightarrow{\pi_{124}} G_0
\]

where \(\pi_{123}\) and \(\pi_{124}\) are the projections deleting the fourth and third component, respectively.

**5.2 Theorem.** The functor \(U: \text{Grpd}(\mathcal{PAsm}) \rightarrow \mathcal{E}\text{ff}\) restricts to a homotopical quotient of the full subcategory \(\mathcal{K}\) on the numeric 2-groupoids.

**Proof.** Suppose that \(G\) and \(H\) are numeric groupoids. Since \(G\) is a free dagger category and all 2-diagrams commute in \(H\), it is easy to see that every arrow \([f]: U(G) \rightarrow U(H)\) in \(\mathcal{E}\text{ff}\) has a representative which is a 2-functor \(F: G \rightarrow H\). To see that the functor \(U: \text{Grpd}(\mathcal{PAsm}) \rightarrow \mathcal{E}\text{ff}\) restricted to \(\mathcal{K}\) is indeed a homotopical quotient, consider the interval-like groupoid \(I\): it is the free dagger category on the graph in \(\mathcal{PAsm}\) on \(T + T\) with two (disjoint) nodes and a single
edge $u$ connecting one with the other, with all possible 2-arrows. It is clearly a numeric 2-groupoid. Consider now two functors $F, F': G \to H$ such that $U(F) = U(F')$; in other words, there is a map $k: G_0 \to H_1$ in $\mathcal{PA}sm$ such that

$$F_0 = d^H_{11} \circ k \quad \text{and} \quad F'_0 = d^H_{12} \circ k.$$  

Note that the 1-category underlying the 2-groupoid $G \times I$ is a retract of a free dagger category. Using $k$ to act on the generating arrow of $I$ as follows

$$
\begin{array}{ccc}
(x, 0) & \longrightarrow & F_0(x) \\
\downarrow & & \downarrow \\
\langle(x), u \rangle & \longrightarrow & k(x) \\
\downarrow & & \downarrow \\
(x, 1) & \longrightarrow & F'_0(x)
\end{array}
$$

by freeness it is easy to obtain a functor $K: G \times I \to H$ which gives a homotopy from $F$ to $F'$.

\[\square\]

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