(Anti-)de Sitter Black Hole Thermodynamics and the Generalized Uncertainty Principle

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Abstract

We extend the derivation of the Hawking temperature of a Schwarzschild black hole via the Heisenberg uncertainty principle to the de Sitter and anti-de Sitter spacetimes. The thermodynamics of the Schwarzschild-(anti-)de Sitter black holes is obtained from the generalized uncertainty principle of string theory and non-commutative geometry. This may explain why the thermodynamics of (anti-)de Sitter-like black holes admits a holographic description in terms of a dual quantum conformal field theory, whereas the thermodynamics of Schwarzschild-like black holes does not.

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I. INTRODUCTION

The Heisenberg uncertainty principle of quantum mechanics allows a heuristic derivation of the Hawking temperature \[^{1}\] of a Schwarzschild black hole. The derivation proceeds as follows \[^{2}\]. The uncertainty in the linear position \( x \) of an emitted quantum is approximately equal to the Schwarzschild radius \( r_s \). By modelling the black hole as an object with linear size \( r_s \), and assuming that the radiation satisfies the condition of minimum uncertainty, the uncertainty in the energy of the emitted quanta is

\[
\Delta E \sim c \Delta p \sim \frac{\hbar c}{\Delta x} \sim \frac{\hbar c}{r_s} , \quad \rightarrow \quad \Delta E = \kappa \frac{\hbar c}{r_s} ,
\]

where \( \kappa \) is a proportionality constant. \( \Delta E \) is identified with the temperature \( T \) of the radiation. Setting \( \kappa = (d-3)/4\pi \), Eq. (1) gives the Hawking temperature for a \( d \)-dimensional Schwarzschild black hole

\[
T_H = \frac{d - 3}{4\pi r_s} \hbar c .
\]

The above derivation deserves some comments. Black hole emission is usually regarded as being originated by quantum effects in the region around the black hole horizon, such as semiclassical wave scattering or particle tunnelling. (See, e.g. Ref. \[^{3}\].) The uncertainty principle does not describe the origin of these effects, but only their consequence on the measurement process. Explaining the origin of black hole emission requires the knowledge of the quantum states that describe the black hole, from which the exact form of the uncertainty principle for the black hole can be derived. On the other hand, Eq. (1) seems to suggest that black hole thermodynamics is a generic low-energy effect of small scale physics. Since any quantum theory of gravity must include some kind of uncertainty principle that reduces to Heisenberg principle at low-energy scales, black hole thermodynamics should not depend too much on the details of the quantum gravity theory. This seems to agree in spirit with Visser’s conclusion that the Hawking radiation only requires ordinary quantum mechanics plus a slowly evolving future horizon, and thus the knowledge of quantum gravity is unnecessary to explain the features of black hole thermodynamics \[^{4}\].

The above derivation, although appealing, is only known for the Schwarzschild black hole. The aim of paper is to extend the uncertainty principle derivation of the Hawking temperature to the de-Sitter (dS) and anti-de Sitter (adS) black holes.
The line element of a $d$-dimensional Schwarzschild-(a)dS black hole $(d > 3)$ with mass $M$ is (see, e.g., Refs. [5, 6])

$$ds^2 = - \left(1 \pm \lambda^2 r^2 - \frac{\omega_d G_d M}{c^2 r^{d-3}}\right) c^2 dt^2 + \left(1 \pm \lambda^2 r^2 - \frac{\omega_d G_d M}{c^2 r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 ,$$

where $G_d$ is Newton’s constant, $\lambda = 1/b$ is the inverse of the (a)dS radius, and the $\pm$ sign is for adS and dS, respectively. The constant $\omega_d$ is equal to $16\pi/(d-2)\Omega_{d-2}$, where $\Omega_{d-2}$ is the volume of the unit $d-2$ sphere. The Hawking temperature of the black hole horizon $r_h$ is

$$T_{S(a)\text{dS}} = \frac{d-3}{4\pi} \left(\frac{1}{r_h} \pm \gamma^2 r_h\right) \hbar c ,$$

where $\gamma$ is proportional to the inverse of the curvature radius of the (a)dS spacetime

$$\gamma = b^{-1} \sqrt{(d-1)/(d-3)} .$$

Two limits of the temperature may be realized. In the Schwarzschild limit, the radius of the event horizon is negligible in comparison to the radius of curvature of the (a)dS spacetime. The Schwarzschild-(a)dS solution reduces to the asymptotically Schwarzschild solution with temperature Eq. (2). In the (a)dS limit, the radius of the black hole event horizon is large in comparison to the radius of curvature of the (a)dS spacetime. The temperature of the (cosmological) horizon is

$$T_{(a)\text{dS}} = \frac{(d-3)^2 r_h}{4\pi} \hbar c .$$

Clearly, the Heisenberg uncertainty principle cannot reproduce Eq. (6). However, the (a)dS temperature may be obtained by substituting the standard Heisenberg relation with its generalized version.

### III. GENERALIZED UNCERTAINTY PRINCIPLE

The generalized version of the Heisenberg uncertainty principle is usually given by

$$\Delta x \Delta p \gtrsim \hbar \left[ 1 + \alpha^2 \ell_p^2 \frac{\Delta p^2}{\hbar^2} \right] ,$$

where $\ell_p = (\hbar G_d/c^3)^{1/(d-2)}$ is the Planck length, and $\alpha$ is a numerical constant [2, 7]. Equation (7) is quite generic, and describes the quantum mechanical uncertainty when
the microscopic structure of spacetime is taken into account. Non-commutative quantum mechanics \cite{8} and black hole gedanken-experiments \cite{9} provide heuristic proofs of the generalized uncertainty principle. The two limits of Eq. (7) (see below) have been derived in the context of string theory in Refs. \cite{10,11}.

The generalized uncertainty principle (7) has both low-energy (quantum mechanical) and high-energy (quantum gravity) limits. The quantum mechanical limit is obtained when the second term in the r.h.s. of Eq. (7) is negligible:

\[ \alpha^2 l_p^2 \Delta p^2 \ll \hbar \rightarrow \frac{\Delta p}{M_p c} \ll \frac{1}{\alpha}. \] (8)

where \( M_p = \left[ \hbar^{d-3}/(c^{d-5} G_d) \right]^{1/(d-2)} \) is the Planck mass. From this limit, it follows that \( \alpha = O(1) \). The quantum gravity limit is obtained when

\[ \alpha^2 l_p^2 \Delta p^2 \sim 1 \rightarrow \frac{\Delta p}{M_p c} \sim \frac{1}{\alpha}, \] (9)

Equation (7) implies the existence of a minimum length \( l_{\text{min}} \) of order of the Planck length. This can be seen by inverting Eq. (7):

\[ \frac{\Delta x}{2\alpha l_p^2} \left[ 1 - \sqrt{1 - \frac{4\alpha^2 l_p^2}{\Delta x^2}} \right] \lesssim \frac{\Delta p}{\hbar} \lesssim \frac{\Delta x}{2\alpha l_p^2} \left[ 1 + \sqrt{1 - \frac{4\alpha^2 l_p^2}{\Delta x^2}} \right]. \] (10)

The lower limit on the uncertainty in position is

\[ \Delta x \gtrsim 2\alpha l_p \equiv l_{\text{min}}. \] (11)

The standard Heisenberg uncertainty relation is obtained when \( l_{\text{min}} \) is negligible compared to the scale of the process, i.e. when \( \Delta x \gg l_p \) or \( \alpha \to 0 \). In the opposite limit, i.e. \( \Delta x \sim l_{\text{min}} \), the uncertainty principle reads

\[ \frac{\Delta p}{M_p c} \sim \frac{\Delta x}{2\alpha^2 l_p^2}. \] (12)

Equation (12) holds when strong quantum gravitational effects are present, and can be derived directly from the conformal invariance property of the fundamental string \cite{10,11}.

In the stringy regime, the position uncertainty is proportional to the momentum uncertainty. Equation (7) is obtained by interpolating Eq. (12) with the standard uncertainty principle.

Equation (7) is not the most general form of the generalized uncertainty principle \cite{12}.

The symmetry of the symplectic space suggests to write

\[ \Delta x \Delta p \geq \hbar \left[ 1 + \beta^2 \frac{\Delta x^2}{l_p^2} \right]. \] (13)
where $\beta$ is a constant parameter. Combining Eq. (7) and Eq. (13) we find the general form
\[
\Delta x \Delta p \gtrsim \hbar \left[ 1 + \alpha^2 \ell_p^2 \frac{(\Delta p)^2}{\hbar^2} + \beta^2 \frac{(\Delta x)^2}{\ell_p^2} \right].
\] (14)

Equation (14) possesses identical quantum mechanical limit and quantum gravity limit of Eq. (7). Thus Eq. (14) is consistent with the string theory derivation of the generalized uncertainty principle. Derivation of Eq. (14) in non-commutative quantum mechanics is discussed in Refs. [12].

It is worthwhile to discuss in detail the “dual” form (13) of the generalized uncertainty principle (7). This will make clear why the general form of the uncertainty principle, Eq. (14), has been mostly overlooked in the literature in favor of Eq. (7). Equation (13) gives a different interpolation between the quantum mechanical limit and the quantum gravity limit than Eq. (7). The quantum mechanical limit is obtained when
\[
\beta \frac{\Delta x}{\ell_p} \ll 1 \quad \rightarrow \quad \Delta x \ll \frac{\ell_p}{\beta}.
\] (15)

Therefore, it follows that $\beta \ll 1$. The quantum gravity limit is obtained when
\[
\beta \frac{\Delta x}{\ell_p} \approx 1 \quad \rightarrow \quad \Delta x \approx \frac{\ell_p}{\beta}.
\] (16)

Since $\beta \ll 1$, one obtains the interesting result that quantum gravitational effects manifest themselves at very large distances. When the generalized uncertainty principle was first derived, the idea of modifications of gravity at great distances had not yet been seriously considered in the literature. Thus the interpolation (13) was overlooked. Inverting Eq. (13),
\[
\frac{\Delta p}{2\beta^2 M_p c} \left[ 1 - \sqrt{1 - \frac{4\beta^2 M_p^2 c^2}{\Delta p^2}} \right] \leq \frac{\Delta x}{\ell_p} \leq \frac{\Delta p}{2\beta^2 M_p c} \left[ 1 + \sqrt{1 - \frac{4\beta^2 M_p^2 c^2}{\Delta p^2}} \right],
\] (17)
one obtains a lower bound on the momentum uncertainty. This defines the minimum momentum $P_{\text{min}} = 2\beta M_p c$.

IV. SCHWARZSCHILD-ADS THERMODYNAMICS WITH THE GENERALIZED UNCERTAINTY PRINCIPLE

The Hawking temperature of the adS black hole can be obtained by repeating the derivation of Sect. I with the generalized uncertainty principle. For semiclassical black holes,
\(\Delta x \gg \ell_p\) and \(\Delta p \ll M_p c\), and the form (13) of the generalized uncertainty principle applies. If we identify the parameter \(\beta\) with \(\gamma \ell_p\), Eq. (13) reproduces the Schwarzschild-adS Hawking temperature

\[
T_{\text{adS}} \sim c \Delta p \sim \left( \frac{1}{\Delta x} + \frac{\beta^2 \Delta x}{\ell_p^2} \right) \hbar c \quad \Rightarrow \quad T_{\text{adS}} = \frac{d - 3}{4\pi} \left( \frac{1}{r_h} + \gamma^2 r_h \right) \hbar c.
\]

The two thermodynamical limits of the Schwarzschild-adS black hole follow from the two limiting relations between position and momentum (\(\Delta p \sim \hbar/\Delta x\) and \(\Delta p \sim \hbar \Delta x/\ell_p^2\)) of the generalized uncertainty principle.

The above identification suggests that the Hawking temperature in adS and Schwarzschild spacetimes may have different origins. Since the adS temperature can be derived from the high-energy limit of the generalized uncertainty principle, the adS thermodynamics seems to have a quantum gravitational nature. It is interesting to note that the generalized uncertainty principle is a consequence of string theory, which can be consistently formulated in adS spacetime, whereas there is no consistent formulation of string theory in the Schwarzschild geometry, where the ordinary uncertainty principle suffices to derive the black hole thermodynamics.

A word of explanation is required on the identification of the inverse adS radius with the generalized uncertainty principle parameter. In the context of known generalized uncertainty models, the coefficient of the correction term in the generalized uncertainty principle is proportional either to the fundamental gravitational length or the inverse string tension. Whereas the functional form of the generalized uncertainty principle seems to be rather generic and model-independent, the exact value of the correction depends on the quantum gravity states of the specific geometry. In the stringy derivation of Ref. [11], for instance, the parameter \(\alpha\) of Eq. (7) is inversely proportional to the total momentum uncertainty of the string in a flat background. If the string propagates in a curved background, we expect its momentum uncertainty, and thus \(\alpha\), to be different.

The Schwarzschild-adS geometry is characterized by two length scales (the fundamental Planck length and the adS radius). The existence of the latter allows to set \(\beta \propto \ell_p/b\) and \(\alpha \propto b/\ell_p\). The exact proportionality constants can be obtained by matching the quantum
gravity limits of the generalized uncertainty principle to the black hole temperature in the 
adS regime, Eq. (18). Since $\beta \ll 1$, the first identification applies to the $b \gg \ell_p$ regime, whereas the second identification applies to the $b \sim \ell_p$ regime. If the adS quantum states were known, the exact constant of proportionality between $\alpha$, $\beta$ and $b$ could be formally derived. Unfortunately, in absence of a definite quantum gravity theory, the derivation of the exact geometry-dependent generalized uncertainty principle remains an open issue.

A heuristic argument that illustrates the connection between the generalized uncertainty 
principle parameter in the adS spacetime and the adS radius is the following. Let us suppose 
to measure the momentum of particle by a scattering with a photon. The uncertainty in 
the measurement of the particle momentum is bounded from below by the value of the 
photon momentum, $\Delta p \sim p_{\gamma} \sim \hbar/\lambda$. Since the photon wavelength cannot exceed the radius 
of the spacetime, the minimum uncertainty is $\Delta p \sim \hbar/b$.

The generalized uncertainty principle derivation applies also to the three-dimensional 
Bañados-Teitelboim-Zanelli (BTZ) black hole [13]

$$ds^2 = -\left(-\frac{8G_3M}{c^2} + \frac{r^2}{b^2}\right)c^2dt^2 + \left(-\frac{8G_3M}{c^2} + \frac{r^2}{b^2}\right)^{-1}dr^2 + r^2d\phi^2,$$

where the black hole radius is $r_{BTZ} = 2b\sqrt{2G_3M}/c$. From Eq. (12) we obtain the Hawking 
temperature of the BTZ black hole

$$T_{BTZ} \sim \frac{r_{BTZ}}{2\alpha^2\ell_p}M_p c^2, \quad \rightarrow \quad T_{BTZ} = \frac{r_{BTZ}}{2\pi b^2} \hbar c,$$

where $\alpha = b\sqrt{\pi}/\ell_p$.

V. SCHWARZSCHILD-DS THERMODYNAMICS WITH THE GENERALIZED 
UNCERTAINTY PRINCIPLE

The Hawking temperature of the Schwarzschild-dS black hole can be obtained from 
Eq. (13) by analytical continuation of the parameter $\beta$ into the imaginary plane. This 
can be shown to be consistent with the topological structure of the dS spacetime as follows. 
For the dS spacetime, the analytic continuation of Eq. (13) reads:

$$\Delta x \Delta p \gtrsim \hbar \left[1 - \beta^2 \frac{\Delta x^2}{\ell_p^2}\right].$$
The inverse of Eq. (21) is

$$\frac{\Delta x}{\ell_p} \gtrsim \frac{\Delta p}{2\beta^2 M_p c} \left[ \sqrt{1 + \frac{4\beta^2 M_p^2 c^2}{\Delta p^2}} - 1 \right]$$

(22)

Since $\Delta p$ is a positive-definite quantity the position uncertainty is limited from above by

$$\Delta x \lesssim b \sqrt{\frac{d - 3}{d - 1}}.$$  

(23)

This relation is a statement that the uncertainty in the measurement of position may not exceed the size of the de-Sitter spacetime.

VI. CONCLUSIONS

We have shown that the uncertainty principle derivation of the Hawking temperature can be extended to (a)dS-like black holes, provided that we consider the generalized uncertainty principle instead of the standard Heisenberg relation. The two thermodynamical limits of Schwarzschild-(a)dS follow from the quantum-regime limit and the standard limit of the generalized uncertainty principle. This result seems to indicate different origins for the thermodynamics of Schwarzschild- and (a)dS-like black holes. This could explain why only (a)dS-like black holes seem to admit a holographic description in terms of a dual quantum conformal field theory.

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