MODIFIED AVERAGED VECTOR FIELD METHODS PRESERVING MULTIPLE INVARIANTS FOR CONSERVATIVE STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. A novel class of conservative numerical methods for general conservative Stratonovich stochastic differential equations with multiple invariants is proposed and analyzed. These methods, which are called modified averaged vector field methods, are constructed by modifying the averaged vector field methods to preserve multiple invariants simultaneously. Based on the prior estimate for high order moments of the modification coefficient, the mean square convergence order 1 of proposed methods is proved in the case of commutative noises. In addition, the effect of quadrature formula on the mean square convergence order and the preservation of invariants for the modified averaged vector field methods is considered. Numerical experiments are performed to verify the theoretical analyses and to show the superiority of the proposed methods in long time simulation.

1. Introduction

Numerical methods for stochastic differential equations (SDEs) have attracted extensive attention over the past decades, in view of the difficulty of obtaining the explicit solutions of original systems (see e.g. [2, 7, 10]). It is important to construct numerical methods which preserve properties for original systems as much as possible. For conservative SDEs with one invariant, there have been many works related to numerical methods in recent years. On the one hand, aiming at the SDEs with single noise, [12] proposes an energy-preserving difference method for stochastic Hamiltonian systems and analyzes the local errors. Based on the equivalent skew gradient (SG) form for conservative SDEs with one invariant, [6] proposes direct discrete gradient methods and indirect discrete gradient methods, and proves that these two kinds of methods are of mean square order 1. Authors in [4] construct energy-preserving methods for stochastic Poisson systems, and prove that those methods are of mean square order one and preserve quadratic Casimir functions. On the other hand, in the case of SDEs with multiple noises, [3] proposes the averaged vector field (AVF) methods for conservative SDEs. It is shown that the mean square order of AVF method is 1 if noises are commutative and that the weak order is 1 in the general case. For the case of quadratic invariants, [5] constructs stochastic Runge-Kutta (SRK) methods for SDEs with quadratic invariants and [13] gives the order conditions for SRK methods preserving quadratic invariants.

For conservative SDEs with multiple invariants, one difficulty is to preserve multiple invariants simultaneously. One approach is via projection technique, which combines an arbitrary one-step approximation together with a projection onto the invariant submanifold in each step. [14] shows that this approach is feasible in stochastic settings, and the proposed methods could reach high strong order as supporting methods. In this paper, we focus on constructing a new...
class of multi-invariant-preserving methods, which are called modified averaged vector field (MAVF) methods. More precisely, we add modification terms to AVF methods to preserve multiple invariants simultaneously, motivated by the ideas of line integral methods (LIMs) for deterministic conservative ordinary differential equations (ODEs) in [1].

As is seen in (3.2), the modification terms contain a vector-valued random variable \( \alpha = (\alpha_0, \alpha_1) \) which is called modification coefficient, hence a prerequisite to acquire the convergence order of MAVF methods is the boundedness of the high-order moments of \( \alpha \). To this end, one technique is to truncate the Brownian increments, which not only ensures the solvability of MAVF methods, but also makes sure that for sufficiently small stepsize, \( \alpha \) is uniformly small with respect to the sample path \( \omega \). Another technique is the usage of the orthogonality of Legendre polynomials, which makes us get rid of the effect of low-order terms and then acquire the estimate for high-order moments of \( \alpha \). We compare MAVF methods with Milstein method to prove that MAVF methods are of mean square order 1.

When the integrals contained in MAVF methods can not be obtained directly, numerical integration is an option to approximate these integrals. Thus it is necessary to investigate the effect of numerical integration on the mean square convergence order and the preservation of invariants for the proposed methods. It is proved that the induced MAVF methods are still of mean square order 1 provided that the orders of quadrature formula are no less than 2. Generally, the mean square order of invariants conservation of MAVF methods using numerical integration only depends on the order of quadrature formulas.

The rest of this paper is organized as follows. In Section 2 we give some concepts about conservative SDEs with invariants and preliminary theorems and lemmas for numerical analysis. Section 3 proposes MAVF methods for conservative SDEs with single or multiple noises and shows the properties of these methods. Section 4 investigates the MAVF methods using numerical integration only depends on the order of quadrature formulas. Numerical experiments are performed, in Section 5, to verify the theoretical analyses and to show the advantages of MAVF methods in long time simulations.

In the sequel, for convenience, we will use the following notations:

- \(|x|\): The trace norm of vector or matrix \( x \) by \( |x| = \sqrt{\text{Tr}(x^\top x)} \).
- \( C^k(\mathbb{R}^n, \mathbb{R}^m) \): The space of \( k \) times continuously differentiable functions \( f : \mathbb{R}^m \to \mathbb{R}^n \).
- \( C^k_0(\mathbb{R}^n, \mathbb{R}^m) \): The space of \( k \) times continuously differentiable functions \( f : \mathbb{R}^m \to \mathbb{R}^n \) with uniformly bounded derivatives up to order \( k \).
- \( \nabla f \): The gradient of scalar function \( f \in C^1(\mathbb{R}^m, \mathbb{R}) \): \( \nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}) \) or Jacobian matrix of vector function \( f \in C^k(\mathbb{R}^n, \mathbb{R}^m) \): \( \nabla f = (\nabla f_1^\top, \ldots, \nabla f_m^\top)^\top \).
- Gâteaux derivative \( f^{(k)}(x)(\xi_1, \ldots, \xi_k) \): If \( f(x) \in C^k(\mathbb{R}^n, \mathbb{R}) \) and \( \xi_1, \ldots, \xi_k \in \mathbb{R}^n \), then \( f^{(k)}(x)(\xi_1, \ldots, \xi_k) = \sum_{i_1, \ldots, i_k=1}^n \frac{\partial^k f(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \xi_{i_1} \cdots \xi_{i_k} \).

## 2. Preliminary

In this section, we give the definition of invariant for conservative SDEs and introduce some lemmas and theorems for the proof of convergence.

Consider the general \( m \)-dimensional autonomous SDE in the sense of Stratonovich

\[
\mathrm{d}Y(t) = f(Y(t))\,\mathrm{d}t + \sum_{r=1}^{D} g_r(Y(t)) \circ \mathrm{d}W_r(t), \quad 0 \leq t \leq T, \quad Y(0) = Y_0,
\]

(2.1)

where \( W_r(t), r = 1, \ldots, D \), are \( D \) independent one-dimensional Brownian motions defined on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) with \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual
conditions. Assume that $Y_0$ is $\mathcal{F}_0$-measurable with $E|Y_0|^2 < \infty$, and that $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$, $g_r: \mathbb{R}^m \rightarrow \mathbb{R}^m, r = 1, \ldots, D$, are such that (2.1) has a unique global solution. Next we give the definition of invariant.

**Definition 2.1** (see [14]). SDE (2.1) is said to have $\nu$ invariants $L^i(y) \in C^1(\mathbb{R}^m, \mathbb{R}), i = 1, \ldots, \nu$, if

$$
\nabla L^i(y)f(y) = 0, \nabla L^i(y)g_r(y) = 0, \quad r = 1, \ldots, D, \quad i = 1, \ldots, \nu, \quad \forall y \in \mathbb{R}^m.
$$

If we define vector-valued function $L(y) := (L^1(y), \ldots, L^\nu(y))^\top$, then (2.2) can be compactly written as

$$
\nabla L(y)f(y) = \nabla L(y)g_r(y) = 0, \quad r = 1, \ldots, D, \quad \forall y \in \mathbb{R}^m.
$$

Hereafter, we also say that the vector-valued function $L(y)$, which satisfies (2.3), is the invariant of (2.1). According to the definition of invariants, it follows from stochastic chain rule that $dL(Y(t)) = 0$, where $Y(t)$ is the exact solution of (2.1). This implies that $L(Y(t)) = L(Y_0)$, a.s. This is to say, $L(y)$, along the exact solution $Y(t)$, is invariant almost surely.

The following two theorems give the relationship between local errors and global errors of numerical methods for general SDEs. In the sequel, we always assume that the assumptions of these two theorems in [8] hold unless we make additional statement.

**Theorem 2.2** (see [8]). Suppose the one-step approximation $\tilde{X}_{t,x}(t + h)$ has order of accuracy $p_1$ for the expectation of the deviation and order of accuracy $p_2$ for the mean square deviation; more precisely, for arbitrary $t_0 \leq t \leq t_0 + T - h, x \in \mathbb{R}^d$ the following inequalities hold:

$$
|E(X_{t,x}(t + h) - \tilde{X}_{t,x}(t + h))| \leq K \cdot (1 + |x|)^{1/2} h^{p_1},
$$

$$
[E|X_{t,x}(t + h) - \tilde{X}_{t,x}(t + h)|^2]^{1/2} \leq K \cdot (1 + |x|)^{1/2} h^{p_2}.
$$

Also, let

$$
p_2 \geq \frac{1}{2}, \quad p_1 \geq p_2 + \frac{1}{2}.
$$

Then for any $N$ and $k = 0, \ldots, N$ the following inequality holds:

$$
[E|X_{t_0,x_0}(t_k) - \tilde{X}_{t_0,x_0}(t_k)|^2]^{1/2} \leq K \cdot (1 + E|X_0|^2)^{1/2} h^{p_2 - 1/2},
$$

i.e., the mean-square order of accuracy of the method constructed using the one-step approximation $\tilde{X}_{t,x}(t + h)$ is $p = p_2 - 1/2$.

**Theorem 2.3** (see [8]). Let the one-step approximation $\tilde{X}_{t,x}(t + h)$ satisfy the conditions of Theorem 2.2. Suppose that $\tilde{X}_{t,x}(t + h)$ is such that

$$
|E\left(\tilde{X}_{t,x}(t + h) - \tilde{X}_{t,x}(t + h)\right)| = O(h^{p_1}),
$$

$$
\left[E|\tilde{X}_{t,x}(t + h) - \tilde{X}_{t,x}(t + h)|^2\right]^{1/2} = O(h^{p_2}).
$$

with the same $p_1$ and $p_2$. Then the method based on the one-step approximation $\tilde{X}_{t,x}(t + h)$ has the same mean square order of accuracy as the method based on $\tilde{X}_{t,x}(t + h)$, i.e., its order is equal to $p = p_2 - 1/2$.

Generally speaking, when implementing implicit numerical methods, the truncated random variables $\Delta \tilde{W}_r(h)$ for the Brownian increments $\Delta W_r(h) = W_r(t + h) - W_r(t), r = 1, \ldots, D$, need to be introduced (see [8]). For this end, one can represent $\Delta W_r(h) := \sqrt{h}\xi_r, r = 1, \ldots, D$,
where $\xi_r, r = 1, \ldots, D$, are independent $N(0, 1)$-distributed random variables. Then, one can define $\hat{\Delta W}_r(h) = \sqrt{h} \zeta_r h$ as follows:

$$
\zeta_r = \begin{cases} 
  \xi_r, & \text{if } |\xi_r| \leq A_h, \\
  A_h, & \text{if } \xi_r > A_h, \\
  -A_h, & \text{if } \xi_r < -A_h,
\end{cases} \tag{2.9}
$$

with $A_h := \sqrt{2k|\ln h|}$, where $k$ is an arbitrary positive integer. The following properties hold for the truncated Brownian increments.

**Lemma 2.4** (see [8]). Let $A_h := \sqrt{2k|\ln h|}$, $k \geq 1$, and $\zeta_r h$ be defined by (2.9). Then it holds that

$$
E(\zeta_r h - \xi_r)^2 \leq h^k, \quad \tag{2.10}
$$

$$
0 \leq E(\zeta_r^2 - \zeta_{r h}^2) = 1 - E\zeta_r^2 h \leq (1 + 2\sqrt{2k|\ln h|})h^k. \tag{2.11}
$$

Moreover, it is not difficult to obtain the following properties

$$
E \left( |\hat{\Delta W}_r(h)|^{2p} \right)^{1/2p} \leq E \left( |\Delta W_r(h)|^{2p} \right)^{1/2p} \leq c_p h^{1/2}, \quad \forall p \in \mathbb{N}^+, \tag{2.12}
$$

where $c_p$ is a constant independent of $h$.

3. MAVF METHODS FOR STOCHASTIC SDEs

3.1. MAVF methods for conservative SDEs with single noise. In this part, we propose MAVF methods preserving multiple invariants for conservative SDEs with single noise and prove these methods are of mean square order 1.

Consider the following autonomous $m$-dimensional SDE with single noise

$$
dY(t) = f(Y(t))dt + g(Y(t)) \circ dW(t), \quad 0 \leq t \leq T, \quad Y(0) = Y_0, \tag{3.1}
$$

where $f$ and $g$ satisfy the global Lipschitz condition. Let $L(y) : \mathbb{R}^m \to \mathbb{R}^\nu$ be the invariant of (3.1), i.e., $\nabla L(y)f(y) = \nabla L(y)g(y) = 0$, for all $y \in \mathbb{R}^m$.

We consider the numerical approximation for (3.1) in interval $[0, T]$. Let $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ be a partition of interval $[0, T]$, where $t_n = nh$, $n = 0, 1, \ldots, N$. Let $\{y_n\}_{n=0}^N$ be some numerical discretization. We denote $y_{n+1} = y_{n, y_n(t_{n+1})}$, $n = 0, 1, \ldots, N-1$.

For convenience, we write the one-step approximation as $\hat{Y} = \hat{Y}_{t, y}(t + h)$. Next, we give the MAVF method for (3.1). It is the following one-step approximation $\hat{Y}$

$$
\begin{cases} 
  \hat{Y} = y + h \left[ \int_0^1 f(\sigma(\tau))d\tau - \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \alpha_0 \right] + \hat{\Delta W} \left[ \int_0^1 g(\sigma(\tau))d\tau - \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \alpha_1 \right], \\
  \left[ \int_0^1 \nabla L(\sigma(\tau))d\tau \right] \left[ \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \right] \alpha_0 = \int_0^1 \nabla L(\sigma(\tau))d\tau \int_0^1 f(\sigma(\tau))d\tau, \\
  \left[ \int_0^1 \nabla L(\sigma(\tau))d\tau \right] \left[ \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \right] \alpha_1 = \int_0^1 \nabla L(\sigma(\tau))d\tau \int_0^1 g(\sigma(\tau))d\tau,
\end{cases} \tag{3.2}
$$

where $\sigma(\tau) = y + \tau(\hat{Y} - y)$, $\hat{\Delta W} = \sqrt{h} \zeta_1 h$ and $\zeta_1 h$ is defined by (2.9) with $A_h = \sqrt{4|\ln h|}$ (i.e., $k = 2$), and $\alpha_0, \alpha_1$ are $\mathbb{R}^\nu$-valued random variables.
As is seen above, the MAVF method in (3.2) can be regarded as the modification of AVF method in [3]. Here, \( \int_{0}^{1} \nabla L(\sigma(\tau))d\tau \alpha_0 \) and \( \int_{0}^{1} \nabla L(\sigma(\tau))d\tau \alpha_1 \) are called modification terms. Let \( \alpha = (\alpha_0, \alpha_1) \) and we call \( \alpha \) the modification coefficient of the method (3.2). The modification coefficient \( \alpha \) satisfies the second and the third equality in (3.2) to make the MAVF method conservative.

### 3.1.1. The prior estimate of the modification coefficient

In this section, we give the estimate of high-order moments for the modification coefficient \( \alpha \). Firstly we obtain the solvability of MAVF method (3.2) as follows.

**Lemma 3.1.** Suppose that \( f(y), g(y) \in C^1(\mathbb{R}^m, \mathbb{R}^m) \), \( \nabla L(y) \in C^1(\mathbb{R}^m, \mathbb{R}^{n \times m}) \), and that \( \nabla L(y)\nabla L(y)^T \) is invertible for arbitrary \( y \in \mathbb{R}^m \). Then for arbitrary given \( y \in \mathbb{R}^m \), the method (3.2) is uniquely solvable with respect to \( \bar{Y} \) and \( \alpha \), a.s., for sufficiently small stepsize \( h \). Moreover, for every \( \epsilon > 0 \), there exists \( h_0 > 0 \) such that for all \( h \leq h_0 \),

\[
|\alpha| \leq \epsilon, \quad a.s. \tag{3.3}
\]

**Proof.** Define

\[
F(\bar{Y}, \alpha, h, \Delta \hat{W}) = \begin{pmatrix}
F_1(\bar{Y}, \alpha, h, \Delta \hat{W}) \\
F_2(\bar{Y}, \alpha, h, \Delta \hat{W}) \\
F_3(\bar{Y}, \alpha, h, \Delta \hat{W})
\end{pmatrix},
\]

where

\[
F_1 = \bar{Y} - y - h \left[ \int_{0}^{1} f(\sigma(\tau))d\tau - \int_{0}^{1} \nabla L(\sigma(\tau))^T d\tau \alpha_0 \right] \\
- \Delta \hat{W} \left[ \int_{0}^{1} g(\sigma(\tau))d\tau - \int_{0}^{1} \nabla L(\sigma(\tau))^T d\tau \alpha_1 \right],
\]

\[
F_2 = \left[ \int_{0}^{1} \nabla L(\sigma(\tau))d\tau \int_{0}^{1} \nabla L(\sigma(\tau))^T d\tau \right] \alpha_0 - \int_{0}^{1} \nabla L(\sigma(\tau))d\tau \int_{0}^{1} f(\sigma(\tau))d\tau,
\]

\[
F_3 = \left[ \int_{0}^{1} \nabla L(\sigma(\tau))d\tau \int_{0}^{1} \nabla L(\sigma(\tau))^T d\tau \right] \alpha_1 - \int_{0}^{1} \nabla L(\sigma(\tau))d\tau \int_{0}^{1} g(\sigma(\tau))d\tau.
\]

Then (3.2) can be rewritten as \( F(\bar{Y}, \alpha, h, \Delta \hat{W}) = 0 \). Note that \( \sigma(\tau) = y \) provided that \( \bar{Y} = y \). In addition, \( \nabla L(y)f(y) = \nabla L(y)g(y) = 0 \) by the definition of invariant \( L(y) \). In this way, we obtain

\[
F(y, 0, 0, 0) = \begin{pmatrix}
0 \\
-\nabla L(y)f(y) \\
-\nabla L(y)g(y)
\end{pmatrix} = 0. \tag{3.4}
\]

Then, it holds that

\[
\frac{\partial F}{\partial (\bar{Y}, \alpha)} \bigg|_{(y, 0, 0, 0)} = \begin{pmatrix}
\frac{\partial F_1}{\partial \bar{Y}} & \frac{\partial F_1}{\partial \alpha_0} & \frac{\partial F_1}{\partial \alpha_1} \\
\frac{\partial F_2}{\partial \bar{Y}} & \frac{\partial F_2}{\partial \alpha_0} & \frac{\partial F_2}{\partial \alpha_1} \\
\frac{\partial F_3}{\partial \bar{Y}} & \frac{\partial F_3}{\partial \alpha_0} & \frac{\partial F_3}{\partial \alpha_1}
\end{pmatrix} \bigg|_{(y, 0, 0, 0)}
\]
Remark 3.2. As is seen in (3.3), we actually have that for sufficiently small

\[
\alpha
\]

where \(\delta\) exists

\[
\exists h
\]

Legendre polynomial satisfies

\[
\text{obtain some useful lemmas in the stochastic cases.}
\]

\[
\text{polynomials, when dealing with numerical methods for conservative ODEs. Likewise, we}
\]

\[
\text{lemmas. The authors in [1] give Lemma 3}
\]

\[
\text{In the following, we will use the properties of Legendre polynomial to derive some important}
\]

\[
\text{tions. We have, by implicit function theorem, that there exists a continuous implicit function}
\]

\[
(\hat{Y}, \alpha) = \left(\hat{Y}(h, \Delta \hat{W}), \alpha(h, \Delta \hat{W})\right)
\]

\[
\text{defined on some neighbourhood } U(O) \text{ of } O = (0, 0). Moreover, } (\hat{Y}, \alpha)
\]

\[
\text{satisfies that } \hat{Y}(0, 0) = y, \alpha(0, 0) = 0 \text{ and } F(\hat{Y}(h, \Delta \hat{W}), \alpha(h, \Delta \hat{W}), h, \Delta \hat{W}) = 0,
\]

\[
\text{for all } (h, \Delta \hat{W}) \in U(O).
\]

\[
\text{By the definition of } \Delta \hat{W}, |\Delta \hat{W}| \leq \sqrt{h}A_k = \sqrt{4h|\ln h|}, \text{a.s. Noting that } h|\ln h| \to 0, \text{as}
\]

\[
h \to 0, \text{we have that for arbitrary } \delta > 0, \text{there exists } h_0 > 0 \text{ such that for all } h \leq h_0
\]

\[
|h| \leq \delta, |\Delta \hat{W}| \leq \delta, \text{ a.s.} \quad (3.6)
\]

Thus, for sufficiently small \(h\), \((h, \Delta \hat{W}) \in U(O), \text{a.s.}, \text{which indicates that } \hat{Y} \text{ and } \alpha \text{ are}

\[
\text{uniquely determined by (3.3). At last, the continuity of } \alpha(h, \Delta \hat{W}), \alpha(0, 0) = 0 \text{ and (3.6) yield (3.3).}
\]

\[
\square
\]

Remark 3.2. As is seen in [33], we actually have that for sufficiently small \(h\) independent of \(\omega, |\alpha| \leq \epsilon, \text{a.s.}, \text{which is essential to give the high-order-moment estimates of the modification}

\[
\text{coefficient } \alpha. \text{The key to the proof of the boundedness of } \alpha \text{ is the usage of truncated Brownian}
\]

\[
\text{increments } |\Delta \hat{W}|. \text{Otherwise, one can only obtain that for every } \epsilon > 0 \text{ and every } \omega \in \Omega, \text{there}
\]

\[
\text{exists } h_0(\omega) \text{ such that for all } h(\omega) \leq h_0(\omega), |\alpha| \leq \epsilon.
\]

Next we introduce the Legendre polynomial \(\{P_j(t)\}_{j \geq 0}\) defined on the interval \([0, 1]\). The Legendre polynomial satisfies

\[
\text{deg } P_i = i, \quad \int_0^1 P_i(t)P_j(t) \mathrm{d}t = \delta_{ij}, \quad \forall i, j \geq 0,
\]

where \(\delta_{ij}\) is the Kronecker symbol. Here are some terms of \(\{P_j(t)\}_{j \geq 0}:\)

\[
P_0(t) \equiv 1, \quad P_1(t) = \sqrt{3}(2t - 1), \quad P_2(t) = \sqrt{5}(6t^2 - 6t + 1), \ldots
\]

And it is not hard to obtain the following properties

\[
\int_0^1 P_j(t) \mathrm{d}t = 0, \quad \forall j \geq 1, \quad \int_0^1 P_j(t)^k \mathrm{d}t = 0, \quad \forall k < j. \quad (3.7)
\]

In the following, we will use the properties of Legendre polynomial to derive some important lemmas. The authors in [1] give Lemma 3.1 and some facts in Chapter 6 by means of Legendre polynomials, when dealing with numerical methods for conservative ODEs. Likewise, we obtain some useful lemmas in the stochastic cases.
Lemma 3.3. Assume that $f, g$ and $\nabla L$ are continuous, then
\[
\sum_{j \geq 0} \int_0^1 P_j(\tau)\nabla L(\sigma(\tau)) \, d\tau \cdot \int_0^1 P_j(\tau) f(\sigma(\tau)) \, d\tau = 0,
\]
\[
\sum_{j \geq 0} \int_0^1 P_j(\tau)\nabla L(\sigma(\tau)) \, d\tau \cdot \int_0^1 P_j(\tau) g(\sigma(\tau)) \, d\tau = 0.
\] (3.8)

Proof. Since the Legendre polynomial $\{P_j(t)\}_{j \geq 0}$ forms an orthonormal basis of the Hilbert space $L^2[0, 1]$, it follows that $f(\sigma(\tau)) = \sum_{j \geq 0} P_j(\tau) \left[ \int_0^1 P_j(\tau) f(\sigma(\tau)) \, d\tau \right]$. Noting that $\nabla L(y)f(y) = 0$, for all $y \in \mathbb{R}^m$, one has that
\[
0 = \int_0^1 \nabla L(\sigma(\tau)) f(\sigma(\tau)) \, d\tau
\]
\[
= \int_0^1 \nabla L(\sigma(\tau)) \cdot \sum_{j \geq 0} P_j(\tau) \left[ \int_0^1 P_j(\tau) f(\sigma(\tau)) \, d\tau \right] \, d\tau
\]
\[
= \sum_{j \geq 0} \int_0^1 P_j(\tau)\nabla L(\sigma(\tau)) \, d\tau \cdot \int_0^1 P_j(\tau) f(\sigma(\tau)) \, d\tau.
\]
Likewise, we obtain the second equality of (3.8). \qed

In the sequel, we will use a generic constant $K$, dependent on $y$ but independent of $h$, which may vary from one line to another.

Lemma 3.4. Let $G(y)$ be a scalar or vector-valued function defined on $\mathbb{R}^m$ and $(j + 1)$ times continuously differentiable with bounded $(j + 1)$ derivative. If $f, g$ and $\nabla L$ satisfy the global Lipschitz conditions, then there is a representation
\[
\int_0^1 P_j(t)G(\sigma(t)) \, dt = c_j G^{(j)}(y)(\bar{\bar{Y}} - y, \ldots, \bar{\bar{Y}} - y) + M_{j,G}, \quad \forall \ j \geq 0,
\] (3.9)
where $c_j = \frac{1}{j!} \int_0^1 P_j(\tau) \tau^j \, d\tau$, and $[E[M_{j,G}]_{2p}]^{1/2p} = \mathcal{O}(h^{(j+1)/2})$ for all $p = 1, 2, 3, \ldots$

Proof. Denote $F := G \circ \sigma$, then $F$ is $(j + 1)$ times continuously differentiable. So Taylor expansion gives
\[
F(\tau) = \sum_{k=0}^j \frac{F^{(k)}(0)}{k!} \tau^k + \int_0^1 \frac{(1-\theta)^j}{j!} F^{(j+1)}(\theta\tau) \tau^{j+1} \, d\theta.
\]
Noting that $\int_0^1 P_j(\tau) \tau^k \, d\tau = 0$, for all $k < j$, we obtain
\[
\int_0^1 P_j(t)G(\sigma(t)) \, dt = \int_0^1 P_j(t)F(\tau) \, dt
\]
\[
= \frac{F^{(j)}(0)}{j!} \int_0^1 P_j(\tau) \tau^j \, d\tau + \int_0^1 P_j(\tau) \left[ \int_0^1 \frac{(1-\theta)^j}{j!} F^{(j+1)}(\theta\tau) \tau^{j+1} \, d\theta \right] \, d\tau
\]
\[
\triangleq c_j F^{(j)}(0) + M_{j,G},
\]
where $c_j = \frac{1}{j!} \int_0^1 P_j(\tau) \tau^j \, d\tau$ and $M_{j,G} = \int_0^1 P_j(\tau) \left[ \int_0^1 \frac{(1-\theta)^j}{j!} F^{(j+1)}(\theta\tau) \tau^{j+1} \, d\theta \right] \, d\tau$.

Further, $F^{(k)}(\tau) = G^{(k)}(y + \tau(\bar{\bar{Y}} - y))(\bar{\bar{Y}} - y, \ldots, \bar{\bar{Y}} - y)$, $k = 0, 1, \ldots$, leads to (3.4).
It remains to estimate the moments of $M_{j,G}$. It follows from the boundedness of $G^{(j+1)}$ that $|M_{j,G}| \leq K_j |\widehat{Y} - y|^{j+1}$. Recall the first equality of (3.2). Since $f, g$ and $\nabla L$ satisfy globally Lipschitz conditions, one is able to prove that $|\widehat{Y} - y| \leq K(|\Delta \widehat{W} + h|$ (This proof is analogous to that of Lemma 2.4 in [9]). Using the Hölder inequality and (2.12), we have

$$[E|M_{j,G}|^{2p}]^{\frac{1}{p}} = O(h^{(j+1)/2}), \quad \forall p \geq 1.$$  

This completes the proof. \hfill \square

Next, we give the prior estimate of the modification coefficient $\alpha$.

**Lemma 3.5.** Suppose that $f, g, \nabla L \in C_b^1(\mathbb{R}^m)$, $\nabla L \nabla L^\top$ is invertible and $[\nabla L \nabla L^\top]^{-1} \in C_b(\mathbb{R}^m)$. Then $\alpha = (\alpha_0, \alpha_1)$ determined by (3.2) satisfies

$$[E|\alpha_0|^{2p}]^{\frac{1}{p}} = O(h), \quad |E(\Delta \widehat{W} \alpha)| = O(h^2). \quad (3.10)$$

**Proof.** Firstly, according to the assumptions on $f, g$ and $\nabla L$, we have

$$\widehat{Y} = y + R_{1,0}, \quad \text{with} \quad [E|R_{1,0}|^{2p}]^{\frac{1}{p}} = O(h^{1/2}). \quad (3.11)$$

Expanding $f(\sigma(\tau))$

$$f(\sigma(\tau)) = f(y) + \tau \int_0^1 f'(y + \theta \tau (\widehat{Y} - y))(\widehat{Y} - y) d\theta.$$  

This, associated with (3.11) and the assumptions of lemma, yields

$$\int_0^1 f(\sigma(\tau)) d\tau = f(y) + R_{1,f}, \quad \text{with} \quad [E|R_{1,f}|^{2p}]^{\frac{1}{p}} = O(h^{1/2}). \quad (3.12)$$

Similarly, we obtain

$$\int_0^1 \nabla L(\sigma(\tau)) d\tau = \nabla L(y) + R_{1,L}, \quad \text{with} \quad [E|R_{1,L}|^{2p}]^{\frac{1}{p}} = O(h^{1/2}). \quad (3.13)$$

It follows from (3.13) that the second equation of (3.2) can be written as

$$(\nabla L + R_{1,L})(\nabla L^\top + R_{1,L}^\top)\alpha_0 = \int_0^1 \nabla L(\sigma(\tau)) d\tau \int_0^1 f(\sigma(\tau)) d\tau. \quad (3.14)$$

Then, by Lemmas 3.3 and 3.4 it holds that

$$\int_0^1 \nabla L(\sigma(\tau)) d\tau \int_0^1 f(\sigma(\tau)) d\tau$$

$$= - \sum_{j \geq 1} \int_0^1 P_j(\tau) \nabla L(\sigma(\tau)) d\tau \cdot \int_0^1 P_j(\tau) f(\sigma(\tau)) d\tau$$

$$= - \sum_{j \geq 1} \left[ c_j \nabla L(j)(\widehat{Y} - y, \ldots, \widehat{Y} - y) + M_{j,L} \right] \left[ c_j f^{(j)}(y)(\widehat{Y} - y, \ldots, \widehat{Y} - y) + M_{j,f} \right]$$

$$= K_1 \nabla L(y)(\widehat{Y} - y)(f'(y)(\widehat{Y} - y)) + R_{1,\alpha_0}, \quad (3.15)$$

where $K_1 = -c^2$ with $c_1 = \int_0^1 P_1(\tau) \tau d\tau$. And $[E|R_{1,\alpha_0}|^{2p}]^{\frac{1}{p}} = O(h^{1.5})$. Combining (3.14) and (3.15), we have

$$\alpha_0 = -[\nabla L \nabla L^\top]^{-1}(\nabla L R_{1,L}^\top + R_{1,L} \nabla L^\top + R_{1,L} R_{1,L}^\top) \alpha_0$$

$$+ K_1 [\nabla L \nabla L^\top]^{-1} \nabla L(y)(\widehat{Y} - y)(f'(y)(\widehat{Y} - y)) + [\nabla L \nabla L^\top]^{-1} R_{1,\alpha_0}. \quad (3.16)$$
By Lemma 3.1 for sufficiently small \( h \), \( |\alpha_0| \leq 1 \), a.s. From assumptions on \( f \) and \( \nabla L \), it follows that
\[
|\alpha_0| \leq K| R_{1,L} | + K | R_{1,L} |^2 + K| Y - y |^2 + K | R_{1,\alpha_0} |.
\]
This implies that
\[
| E[|\alpha_0|^2]|^{1/2} = \mathcal{O}(h^{1/2}). \tag{3.17}
\]
Then according to (3.10) and (3.17), we obtain
\[
E[|\alpha_0|^2] \leq KE([R_{1,L}]^2|\alpha_0|^2) + KE| R_{1,L} | + KE| Y - y |^4 + KE| R_{1,\alpha_0} |^4
\leq K[E([R_{1,L}]^4|E|\alpha_0|^4)]^{1/2} + Kh^2
\leq Kh^2.
\]
That is to say, \( E[|\alpha_0|^2]^{1/2} = \mathcal{O}(h) \). Similarly, we have
\[
| E[|\alpha_0|^2]|^{1/2} = \mathcal{O}(h). \tag{3.18}
\]
Thus (3.16) can be rewritten as
\[
\alpha_0 = K_1[\nabla L \nabla L^\top]^{-1}\nabla L'(y)(Y - y)(f'(y)(Y - y)) + \tilde{R}_{1,\alpha_0},
\]
with \( E[|\tilde{R}_{1,\alpha_0}|^2]|^{1/2} = \mathcal{O}(h^{1/5}) \).

Because \( Y - y = \Delta \tilde{W} g + \tilde{R}_{1,0} \) with \( E[|\tilde{R}_{1,0}|^2]|^{1/2} = \mathcal{O}(h) \), we obtain
\[
\Delta \tilde{W} \alpha_0 = K_1\Delta \tilde{W}^3[\nabla L \nabla L^\top]^{-1}\nabla L'g'(y'g(\tilde{Y} - y) + \tilde{R}_{1,\alpha_1}, \text{ with }\ E[|\tilde{R}_{1,\alpha_1}|^2]|^{1/2} = \mathcal{O}(h^{1/5}),
\]
\[
\Delta \tilde{W} \alpha_1 = K_1\Delta \tilde{W}^3[\nabla L \nabla L^\top]^{-1}\nabla L'g'(g' \tilde{Y} + \tilde{R}_{1,\alpha_1}, \text{ with }\ E[|\tilde{R}_{1,\alpha_1}|^2]|^{1/2} = \mathcal{O}(h^2).
\]
Thus, \( E[|\alpha_1|^2]|^{1/2} = \mathcal{O}(h) \) and \( |E(\Delta \tilde{W} \alpha_1)| = \mathcal{O}(h^2) \). \( \square \)

3.1.2. Conservative property and convergence of MAVF methods for SDES with single noise.

**Theorem 3.6.** Let \( f \in C^0_b(\mathbb{R}^m, \mathbb{R}^m), g \in C^0_b(\mathbb{R}^m, \mathbb{R}^m) \) and \( \nabla L \in C^1_b(\mathbb{R}^m, \mathbb{R}^{\nu \times m}). \) Assume that \( \nabla L \nabla L^\top \) is invertible and \( \nabla L \nabla L^\top^{-1} \in C^1_b(\mathbb{R}^m, \mathbb{R}^{\nu \times \nu}). \) Then the numerical method (3.2) for SDE (3.1) possesses the following properties:

I) It preserves multiple invariants \( L, i = 1, \ldots, \nu \), of (3.1), i.e., \( L(\tilde{Y}) = L(y). \)

II) It is of mean square order 1.

**Proof.** (I) By Taylor expansion and (3.2), it follows that
\[
L(\tilde{Y}) - L(y)
= \int_0^1 \nabla L(y + \tau(\tilde{Y} - y))(\tilde{Y} - y) \, d\tau
=h \left[ \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \int_0^1 f(\sigma(\tau)) \, d\tau - \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \right] +
\Delta \tilde{W} \left[ \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \int_0^1 g(\sigma(\tau)) \, d\tau - \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \right]
=0. \tag{3.19}
\]
(II) We divide the proof of this part into four steps:

Step 1: Taylor expansions lead to the results which are shown in formulas (3.11), (3.12) and (3.13). Similarly, we have

\[
\int_0^1 g(\sigma(\tau)) d\tau = g(y) + R_{1,g} \quad \text{with} \quad [E|g(\sigma(\tau))|^2]^{\frac{1}{2p}} = O(h^{1/2}), \tag{3.20}
\]

In order to evaluate these remainders more precisely, we make a further expansion:

\[
R_{1,0} = \Delta \widehat{W}g + \widehat{R}_{1,0} \quad \text{with} \quad [E|\widehat{R}_{1,0}|^2]^{\frac{1}{2p}} = O(h). \tag{3.21}
\]

\[
R_{1,f} = \int_0^1 \int_0^1 f'(y + \theta \tau (\bar{Y} - y)) (\bar{Y} - y) \, d\theta d\tau
\]
\[
= \frac{1}{2} f'_{1,1} (\bar{Y} - y) + R_{1,f}^{(1)}
\]
\[
= \frac{1}{2} \Delta \widehat{W} f'_{1,1} g + R_{1,f}^{(2)}, \tag{3.22}
\]

where \([E|R_{1,f}^{(1)}|^2]^{\frac{1}{2p}} = O(h)\) and \([E|R_{1,f}^{(2)}|^2]^{\frac{1}{2p}} = O(h)\).

Similarly, we have

\[
R_{1,0} = \frac{1}{2} g'(y) (\bar{Y} - y) + R_{1,0}^{(1)} = \frac{1}{2} \Delta \widehat{W} g'_{1,1} + R_{1,0}^{(2)}, \tag{3.23}
\]

where \([E|R_{1,0}^{(1)}|^2]^{\frac{1}{2p}} = O(h)\) and \([E|R_{1,0}^{(2)}|^2]^{\frac{1}{2p}} = O(h)\).

Step 2: Formulas (3.11)-(3.13), (3.20)-(3.23), Lemma 3.5 and Hölder inequality imply that the first equation of (3.2) can be written as

\[
\hat{Y} = y + h[f + R_{1,f} - \nabla L(y)\top \alpha_0 - R_{1,1,L}\alpha_0]
\]
\[
+ \Delta \widehat{W} [g + R_{1,g} - \nabla L(y)\top \alpha_1 - R_{1,1,L}\alpha_1]
\]
\[
= y + \Delta \widehat{W} g + hf + R_{2,0}, \tag{3.24}
\]

with

\[
R_{2,0} = \Delta \widehat{W} R_{1,g} + R_{1,1,g}^{(1)} = \frac{1}{2} \Delta \widehat{W}^2 g'_{1,2} + R_{2,0}^{(2)}, \tag{3.25}
\]

where \([E|R_{2,0}^{(1)}|^2]^{\frac{1}{2p}} = O(h^{1.5})\) and \([E|R_{2,0}^{(2)}|^2]^{\frac{1}{2p}} = O(h^{1.5})\).

Applying Taylor expansion to \(g\) gives:

\[
g(\tau) = g + \tau g'(y) (\bar{Y} - y) + \tau^2 \int_0^1 (1 - \theta) g''(y + \theta \tau (\bar{Y} - y)) (\bar{Y} - y, \bar{Y} - y) d\theta. \tag{3.26}
\]

Thus we have

\[
\int_0^1 g(\tau) d\tau = g + \int_0^1 \tau^2 \int_0^1 (1 - \theta) g''(y + \theta \tau (\bar{Y} - y)) (\bar{Y} - y, \bar{Y} - y) d\theta d\tau
\]
\[
= g + \frac{1}{2} \Delta \widehat{W} g'_{1,1} + R_{2,g}. \tag{3.27}
\]
Comparing our method (3.29) with Milstein method, we get methods are of mean square order 1 if noises are commutative.

Thus the proof of (II) is completed by the Theorem 2.3.

3.2. MAVF methods for conservative SDEs with multiple noises. In this section, we propose the MAVF methods for conservative SDEs with multiple noises and prove that these methods are of mean square order 1 if noises are commutative.
We still suppose that $L(y): \mathbb{R}^m \to \mathbb{R}^\nu$ is the invariant of (2.1). Based on the ideas of dealing with single noise, we construct the MAVF method for (2.1) as follows:

\[
\begin{align*}
\dot{Y} &= y + \left[ \int_0^1 f(\sigma(\tau)) d\tau - \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \alpha_0 \right] + \sum_{r=1}^D \Delta \hat{W}_r \left[ \int_0^1 g_r(\sigma(\tau)) d\tau - \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \alpha_r \right], \\
\left[ \int_0^1 \nabla L(\sigma(\tau)) d\tau \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \right] \alpha_0 &= \int_0^1 \nabla L(\sigma(\tau)) d\tau \int_0^1 f(\sigma(\tau)) d\tau, \\
\left[ \int_0^1 \nabla L(\sigma(\tau)) d\tau \int_0^1 \nabla L(\sigma(\tau))^\top d\tau \right] \alpha_r &= \int_0^1 \nabla L(\sigma(\tau)) d\tau \int_0^1 g_r(\sigma(\tau)) d\tau, \quad r = 1, \ldots, D,
\end{align*}
\]

where $\sigma(\tau) = y + \tau(\dot{Y} - y)$ and $\Delta \hat{W}_r = \sqrt{h} \zeta_r h$ is defined (2.9) with $k = 2$. In addition, $\alpha_r, r = 0, 1, \ldots, D$, are $\mathbb{R}^\nu$-valued random variables. And $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_D)$ is called the modification coefficient.

**Theorem 3.7.** Let $f \in C_b^2(\mathbb{R}^m, \mathbb{R}^m)$, $g_r \in C_b^2(\mathbb{R}^m, \mathbb{R}^m)$, $r = 1, \ldots, D$, and $\nabla L \in C_b^1(\mathbb{R}^m, \mathbb{R}^\nu \times \mathbb{R}^m)$.

Assume that $\nabla L \nabla L^\top$ is invertible and $[\nabla L \nabla L^\top]^{-1} \in C_b^1(\mathbb{R}^m, \mathbb{R}^\nu \times \mathbb{R}^\nu)$. If the noises of SDE (2.1) satisfy the commutative conditions, i.e., $g'_r g_i = g'_i g_r$, $i, r = 1, \ldots, D$, then the numerical method (3.35) for SDE (2.1) possesses the following properties:

(I) It preserves multiple invariants $L^i, i = 1, \ldots, \nu$, of (2.1), i.e., $L(Y) = L(y)$.

(II) It is of mean square order 1.

**Proof.** Given that the proof is similar to that of Theorem 3.6, we will only give the sketch. The first property (I) easily comes out as in the proof of Theorem 3.6.

Let us proceed to the proof of (II). First, as is done in Lemma 3.1, we acquire the solvability of (3.35) and have that for every $\epsilon \geq 0$, there exists $h_0$ such that for all $h \leq h_0$, $
abla \alpha \leq \epsilon$ a.s.

Then, analogous to the Lemma 3.3, we have that

$$\left[ E|\alpha|^{2p} \right]^{\frac{1}{2p}} = \mathcal{O}(h), \quad |E(\Delta \hat{W}_r, \alpha)| = \mathcal{O}(h^2), \quad r = 1, \ldots, D.$$ 

Further, using Taylor expansion repeatedly, we acquire

\[
\dot{Y} = y + \sum_{r=1}^D \Delta \hat{W}_r g_r + \frac{1}{2} \sum_{r=1}^D \sum_{i=1}^D \Delta \hat{W}_r \Delta \hat{W}_i g'_i g_i + h f + R,
\]

where

\[
R = \frac{1}{2} h \sum_{r=1}^D \Delta \hat{W}_r f'_r + \frac{1}{2} \sum_{r=1}^D \Delta \hat{W}_r g'_r f - \sum_{r=1}^D \Delta \hat{W}_r \nabla L^\top \alpha_r
\]

$$+ \frac{1}{6} \sum_{r, i, j=1}^D \Delta \hat{W}_r \Delta \hat{W}_i \Delta \hat{W}_j g''_{r i j} (g_i, g_j) + \frac{1}{4} \sum_{r, i, j=1}^D \Delta \hat{W}_r \Delta \hat{W}_i \Delta \hat{W}_j g'_r (g_i, g_j) + \tilde{R},$$

with $\left[ E|\tilde{R}|^{2p} \right]^{\frac{1}{2p}} = \mathcal{O}(h^2)$.

Thus, it follows that

$$\left[ E|R|^{2p} \right]^{\frac{1}{2p}} = \mathcal{O}(h^{1.5}), \quad |ER| = \mathcal{O}(h^2).$$
Note that, in case of the commutative noises, the Milstein method for (2.1) becomes
\[
\bar{Y}^{[M]} = y + \sum_{r=1}^{D} \Delta W_r g_r + \frac{1}{2} \sum_{r=1}^{D} \sum_{i=1}^{D} \Delta W_r \Delta W_i g'_r g_i + hf. \tag{3.37}
\]
Comparing (3.36) and (3.37), we have
\[
|E|\bar{Y} - \bar{Y}^{[M]}|^2 \leq O(h^{1.5}), \quad |E(\bar{Y} - \bar{Y}^{[M]})| = O(h^2). \tag{3.38}
\]
which means that the method (3.35) is of mean square order 1 by the Theorem 2.3.

Remark 3.8. It is noted that, without commutative noises, the mean square order of method (3.35) is only \(\frac{1}{2}\).

4. Numerical integration

When the integrals contained in MAVF methods can not be obtained directly, we need to use numerical integration to approximate the integrals. In this section, we investigate the effect of numerical integration on MAVF methods, including mean square convergence order and preservation of invariants.

Here, we recall some concepts of numerical integration. Consider the quadrature formula \((c_i, b_i)_{i=1}^{M}\) on the interval \([0, 1]\):
\[
\int_0^1 f(x) dx \approx \sum_{i=1}^{M} b_i f(c_i). \tag{4.1}
\]
The quadrature formula (4.1) is said to have order \(q\) if it is exact for polynomials of degree no larger than \(q - 1\), i.e.,
\[
\int_0^1 x^k dx = \sum_{i=1}^{M} b_i c_i^k, \quad k = 0, 1, \ldots, q - 1.
\]
Here are some examples of quadrature formulas:
\[
\int_0^1 f(x) dx \approx \frac{1}{2} [f(0) + f(1)], \tag{4.2}
\]
\[
\int_0^1 f(x) dx \approx \frac{1}{4} [3f(1) + f(1)], \tag{4.3}
\]
\[
\int_0^1 f(x) dx \approx \frac{1}{2} [f(3 - \sqrt{3}) + f(3 + \sqrt{3})], \tag{4.4}
\]
\[
\int_0^1 f(x) dx \approx \frac{1}{18} [5f(5 - \sqrt{15}) + 8f(\frac{1}{2}) + 5f(5 + \sqrt{15})], \tag{4.5}
\]
and their orders are 2, 3, 4, 6, respectively.

As is well known, if \(f^{(q)} \in C_b\) (the set of bounded and continuous functions) and with \(q\) being the order of the quadrature formula (4.1), then it holds that
\[
\int_0^1 f(x) dx = \sum_{i=1}^{M} b_i f(c_i) + \rho_q f^{(q)}(\eta), \tag{4.6}
\]
where \( \eta \in (0,1) \) and \( \rho_q \) is independent of \( f \). Next, we use the numerical integration to approximate the integrals in (3.35). The induced numerical method using the quadrature formula (4.1) is

\[
\begin{align*}
\bar{Y} &= y + h \left[ \sum_{i=1}^{M} b_i f(\sigma(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma(c_i))^\top \alpha_0 \right] + \sum_{r=1}^{D} \Delta \tilde{W}_r \left[ \sum_{i=1}^{M} b_i g_r(\sigma(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma(c_i))^\top \alpha_r \right],
\end{align*}
\]

where \( \alpha(\tau) = y + \tau(\bar{Y} - y) \).

4.1 Mean square convergence order. In this part, we study the mean square convergence order of (4.7). Following the procedure in Section 3, we firstly present the boundedness of \( \alpha \), and the expansion formula of \( \bar{Y} \).

**Lemma 4.1.** Let \( q \geq 1 \) be the order of quadrature formula \((c_i, b_i)_{i=1}^{M} \). Suppose that \( f, g_r \in C_b^1(\mathbb{R}^m, \mathbb{R}^m) \), \( r = 1, \ldots, D \), \( \nabla L \in C_b^1(\mathbb{R}^m, \mathbb{R}^{r \times m}) \), and that \( \nabla L \nabla L^\top \) is invertible. Then for arbitrary given \( y \in \mathbb{R}^m \), the method (4.7) is uniquely solvable with respect to \( \bar{Y} \) and \( \alpha = (\alpha_0, \ldots, \alpha_D) \), a.s., for sufficiently small stepsize \( h \). Moreover, for every \( \epsilon > 0 \), there exists \( h_0 > 0 \) such that for all \( h \leq h_0 \),

\[
|\alpha| \leq \epsilon, \quad \text{a.s.}
\]

In addition, there is a representation

\[
\bar{Y} = y + \sum_{r=1}^{D} \Delta \tilde{W}_r g_r + R_{\bar{Y}},
\]

with \( \left[E|R_{\bar{Y}}|^{2p}\right]^{\frac{1}{2p}} = O(h) \), \( p = 1, 2, \ldots \).

The proof of Lemma 4.1 follows from the fact that \( \sum_{i=1}^{M} b_i = 1 \) for \( q \geq 1 \), and the implicit function theorem, as in the proof of Lemma 3.1. Thus we omit the proof.

The following lemma is used to estimate the accuracy of numerical integration in (4.7).

**Lemma 4.2.** Let \( f, g_r \in C_b^1 \), \( r = 1, \ldots, D \). Let \( q \) be the order of quadrature formula \((c_i, b_i)_{i=1}^{M} \) and \( G \) be an arbitrary scalar or vector-valued function. Assume that \( \nabla L \in C_b^1 \) and \( \nabla L \nabla L^\top \) is invertible. If \( q \geq 2 \) and \( G^{(q)} \in C_b \), then we have

\[
\int_0^1 G(\sigma(\tau)) \, d\tau = \sum_{i=1}^{M} b_i G(\sigma(c_i)) + \Psi_{G,q},
\]

where \( \Psi_{G,q} \) satisfies that

\[
\left[E|\Psi_{G,q}|^{2p}\right]^{\frac{1}{2p}} = O(h^{\frac{q}{2}}), \quad \forall \ p = 1, 2, \ldots
\]

In addition, it holds that

(1) If \( q \) is odd, then

\[
E\left(\Delta \tilde{W}_r \Psi_{G,q}\right) = O(h^{\frac{q+1}{2}}), \quad r = 1, \ldots, D.
\]
Due to (4.9) and boundedness of \( G \), the second equation in (4.7) can be rewritten as using Hölder inequality and (4.9) gives
\[
q \tag{1}
\]
Besides, it follows that
\[
\]
Let Lemma 4.3.
\[
(2) \text{ If } q \text{ is odd, Hölder inequality yields}
\[
\]
\[
(2) \text{ If } q \text{ is even, it holds that}
\[
\]
Next, we give the prior estimate of modification \( \alpha \) in (4.7).

**Lemma 4.3.** Let \( f, g, r = 1, \ldots, D \), \( \nabla L \in \mathbb{C}^{(q+1)}_b \). Let \( q \) be the order of quadrature formula \( (c_i, b_i)_{i=1}^M \) in (4.7). Assume that \( \nabla L \nabla L^\top \) is invertible, and \( \left(\nabla L \nabla L^\top\right)^{-1} \in \mathbb{C}^1_b \). If \( q \geq 2 \), then we have
\[
\left| E[\alpha^{2p}] \right| = o(h^{\frac{q^2}{p}}), \quad p = 1, 2, \ldots \quad \text{ and } \quad \left| E[\Delta \hat{W}_r \alpha] \right| = o(h^2), \quad r = 1, \ldots, D. \tag{4.15}
\]

**Proof.** By (4.10) in Lemma 4.2, the second equation in (4.7) can be rewritten as
\[
\left[ \int_0^1 \nabla L(\sigma(\tau)) \, d\tau - \Psi_{\nabla L,q} \right] \left[ \int_0^1 \nabla L(\sigma(\tau))^\top \, d\tau - \Psi_{\nabla L,q}^\top \right] \alpha_0
\]
\[
= \left[ \int_0^1 \nabla L(\sigma(\tau)) \, d\tau - \Psi_{\nabla L,q} \right] \left[ \int_0^1 f(\sigma(\tau)) \, d\tau - \Psi_{f,q} \right]. \tag{4.16}
\]
By arranging the above formula, we have
\[
\left[ \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \int_0^1 \nabla L(\sigma(\tau))^\top \, d\tau \right] \alpha_0 = \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \int_0^1 f(\sigma(\tau)) \, d\tau + T_{\alpha_0}, \tag{4.17}
\]
where
\[ T_{a_0} = - \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \Psi_{f,q} - \Psi_{\nabla L,q} \int_0^1 f(\sigma(\tau)) \, d\tau + \Psi_{\nabla L,q} \Psi_{f,q} \]
\[ + \left[ \Psi_{\nabla L,q} \int_0^1 \nabla L(\sigma(\tau))^\top \, d\tau + \int_0^1 \nabla L(\sigma(\tau)) \, d\tau \Psi_{\nabla L,q}^\top - \Psi_{\nabla L,q} \Psi_{\nabla L,q}^\top \right] a_0. \]

Using (3.12), (3.13) and (3.15), we write (4.17) as
\[ (\nabla L + R_{1,L})(\nabla L^\top + R_{1,L}^\top) a_0 = K_1 \nabla L'(y)(\bar{Y} - y)\left(f'(y)(\bar{Y} - y)\right) + R_{1,a_0} + T_{a_0}, \]
where \([E|R_{1,a_0}|^{2p}]^{\frac{1}{2p}} = O(h^{1.5}).\)

Further, since \([\nabla L \nabla L^\top]^{-1} \in C_b^1\), we have
\[ a_0 = -[\nabla L \nabla L^\top]^{-1}(\nabla L R_{1,L}^\top + R_{1,L} \nabla L^\top + R_{1,L} R_{1,L}^\top) a_0 + [\nabla L \nabla L^\top]^{-1} T_{a_0} \]
\[ + K_1[\nabla L \nabla L^\top]^{-1}\nabla L'(y)(\bar{Y} - y)\left(f'(y)(\bar{Y} - y)\right) + [\nabla L \nabla L^\top]^{-1} R_{1,a_0}. \tag{4.18} \]

Next we prove the conclusion under the two cases \(q = 2\) and \(q \geq 3\) respectively.

1. Since \(q = 2\) is even and \(f, g_r, \nabla L \in C_b^2\), it follows from Lemma 4.2 that
\[ [E|\Psi_{f,q}|^{2p}]^{\frac{1}{2p}} = O(h), \quad [E|\Psi_{\nabla L,q}|^{2p}]^{\frac{1}{2p}} = O(h), \]
and
\[ E\left|\Delta \hat{W}_r \Psi_{f,q}\right| = O(h^2), \quad [E|\Delta \hat{W}_r \Psi_{\nabla L,q}|] = O(h^2). \]

Hence, \([E|T_{a_0}|^{2p}]^{\frac{1}{2p}} = O(h),\) and \([E|\Delta \hat{W}_r T_{a_0}|] = O(h^2).\) Analogous to the proof of Lemma 3.5, we have
\[ [E|\alpha_r|^{2p}]^{\frac{1}{2p}} = O(h), \quad [E|(\Delta \hat{W}_r \alpha)|] = O(h^2), \quad r = 1, \ldots, D. \]

2. According to Lemma 4.2, we obtain
\[ [E|\Psi_{f,q}|^{2p}]^{\frac{1}{2p}} = O(h^{\frac{3}{2}}), \quad [E|\Psi_{\nabla L,q}|^{2p}]^{\frac{1}{2p}} = O(h^{\frac{3}{2}}). \]

Notice that \(q \geq 1.5\) provided that \(q \geq 3\). Then Hölder inequality yields
\[ [E|T_{a_0}|^{2p}]^{\frac{1}{2p}} = O(h^{1.5}), \quad \text{and} \quad [E|\Delta \hat{W}_r T_{a_0}|] = O(h^2). \]

Combining above formula and (4.18) produces (4.15).

Next we give the result of convergence of method (4.17).

**Theorem 4.4.** Let \(\nabla L \nabla L^\top\) be invertible and \([\nabla L \nabla L^\top]^{-1} \in C_b^1\). Let \(q\) be the order of quadrature formula \((c_i, b_i)_{i=1}^M\) in (4.7). Assume that \(q \geq 2\), and \(f, g_r, r = 1, \ldots, D, \nabla L \in C_b^{q+1}\). If the noises satisfy the commutative conditions, i.e., \(g'_i g_i = g'_i g_r, i, r = 1, \ldots, D\), then the method (4.17) is of mean square order 1.

**Proof.** This proof is analogous to that of Theorem 3.7, we only give the sketch. It follows from Lemma 4.2 that
\[ \bar{Y} = y + h \left[ \int_0^1 f(\sigma(\tau)) \, d\tau - \Psi_{f,q} - \int_0^1 \nabla L(\sigma(\tau))^\top \, d\tau a_0 + \Psi_{\nabla L,q} a_0 \right] \]
\[ + \sum_{r=1}^D \Delta \hat{W}_r \left[ \int_0^1 g_r(\sigma(\tau)) \, d\tau - \Psi_{g_r,q} - \int_0^1 \nabla L(\sigma(\tau))^\top \, d\tau a_r + \Psi_{\nabla L,q} a_r \right]. \]
\[ y = y + h \left[ \int_0^1 f(\sigma(t)) \, dt - \int_0^1 \nabla L(\sigma(t))^{\top} \, d\alpha_t \right] + \sum_{r=1}^{D} \Delta \widehat{W}_r \left[ \int_0^1 g_r(\sigma(t)) \, dt - \int_0^1 \nabla L(\sigma(t))^{\top} \, d\alpha_r \right] + R^{[0]}, \tag{4.19} \]

where

\[ R^{[0]} = -h \Psi_{f,q} + h \Psi_{\nabla L,q} \alpha_0 + \sum_{r=1}^{D} \Delta \widehat{W}_r \left[ -\Psi_{g_r,q} + \Psi_{\nabla L,q} \alpha_r \right]. \]

According to Lemma 4.2 and Lemma 4.3, we have

\[ [E|R^{[0]}|^2]^{1/2} = O(h^{1.5}), \quad |ER^{[0]}| = O(h^2). \]

Using Taylor expansion, analogous to the proof of Theorem 3.7, we obtain

\[ \tilde{Y} = y + \sum_{r=1}^{D} \Delta \widehat{W}_r g_r + \frac{1}{2} \sum_{r=1}^{D} \Delta \widehat{W}_r^2 g'_r g_r + \sum_{r=1}^{D-1} \sum_{i=r+1}^{D} \Delta \widehat{W}_i \Delta \widehat{W}_r g'_r g_i + hf + T^{[0]}, \tag{4.20} \]

where

\[ T^{[0]} = \frac{1}{2} h \sum_{r=1}^{D} \Delta \widehat{W}_r f' g_r + \frac{1}{2} h \sum_{r=1}^{D} \Delta \widehat{W}_r g'_r f - \sum_{r=1}^{D} \Delta \widehat{W}_r \nabla L^{\top} \alpha_r + \frac{1}{6} \sum_{r,i,j=1}^{D} \Delta \widehat{W}_r \Delta \widehat{W}_i \Delta \widehat{W}_j g'_r (g_i, g_j) + \frac{1}{4} \sum_{r,i,j=1}^{D-1} \Delta \widehat{W}_r \Delta \widehat{W}_i \Delta \widehat{W}_j g'_r (g'_i g_j) + \hat{R}^{[0]} + \tilde{R}^{[0]}, \tag{4.21} \]

with \( E|\tilde{R}^{[0]}|^2 = O(h^2) \). Thus, it holds that

\[ [E|T^{[0]}|^2]^{1/2} = O(h^{1.5}), \quad |ET^{[0]}| = O(h^2). \]

Comparing (4.20) with Milstein method for SDE (2.1), we obtain that the method (4.7) is of mean square order 1.

\[ \square \]

### 4.2. Mean square order of invariant conservation

It is worth noting that the method (4.7) does not preserve exactly the invariant of original system generally, due to the usage of quadrature formula, which makes it necessary to study the preservation of invariant of (4.7). In the following, we give the definition of mean square order of invariant conservation.

**Definition 4.5.** A numerical discretization \( \{y_n\}_{n=0}^{N} \) is said to have mean square order \( p \) of invariant conservation, if the invariant \( L(y) \) of SDE (2.1) satisfies

\[ [E|L(y_N) - L(y_0)|^2]^{1/2} = O(h^p). \tag{4.22} \]

Let \( \{y_n\}_{n=0}^{N} \) be the numerical discretization corresponding to the one-step approximation (4.7) with \( y_0 = Y_0 \), and denote \( \tilde{Y}_{n,y}(t_{n+1}) = y_{n+1}, \ n = 0, 1, \ldots, N - 1 \). The numerical
method generated from the one-step approximation \((4.7)\) reads

\[
\begin{align*}
\hat{Y}_{t_n,y_n}(t_{n+1}) &= y_n + h \left[ \sum_{i=1}^{M} b_i f(\sigma_n(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i))^\top \alpha_0 \right] \\
&+ \sum_{r=1}^{D} \Delta \hat{W}_{r,n} \left[ \sum_{i=1}^{M} b_i g_r(\sigma_n(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i))^\top \alpha_r \right],
\end{align*}
\]

where \(\sigma_n(\tau) = y_n + \tau (\bar{Y}_{t_n,y_n}(t_{n+1}) - y_n)\), and \(\Delta \hat{W}_{r,n} = \Delta \hat{W}_r(t_{n+1}) - \Delta \hat{W}_r(t_n)\), \(r = 1, \ldots, D\), \(n = 0, 1, \ldots, N - 1\), are mutually independent truncated Brownian increments.

The following lemma gives the one-step error estimate of invariant conservation of method \((4.23)\).

**Lemma 4.6.** Let \(\nabla L \nabla L^\top\) be invertible and \(\left[\nabla L \nabla L^\top\right]^{-1} \in \mathcal{C}_b^1\). Let \(q\) be the order of quadrature formula \((c_i, b_i)_{i=1}^{M}\) in \((4.7)\). Assume that \(q \geq 2\), and \(f, g_r, r = 1, \ldots, D, \nabla L \in \mathcal{C}_b^{(q+1)}\). Then it holds that

\[
\left[ E \left| L(\hat{Y}_{t_n,y_n}(t_{n+1})) - L(y_n) \right|^2 \right]^{1/2} = \mathcal{O}(h^{2+1}) , \quad n = 0, 1, \ldots, N - 1,
\]

and

\[
\left| E \left[ L(\hat{Y}_{t_n,y_n}(t_{n+1})) - L(y_n) \right| \mathcal{F}_{t_n} \right] = \begin{cases} 
\mathcal{O}(h^{2+2}), & \text{if } q \text{ is even,} \\
\mathcal{O}(h^{2+1}), & \text{if } q \text{ is odd.}
\end{cases}
\]

**Proof.** Let \(y_n\) denote the random variable and \(y\) denote the deterministic variable in this proof. It follows from Taylor expansion that

\[
L(\hat{Y}_{t_n,y_n}(t_{n+1})) - L(y_n) = \int_0^1 \nabla L(\sigma_n(\tau)) \, d\tau (\hat{Y}_{t_n,y_n}(t_{n+1}) - y_n).
\]

By \((4.10)\), we have

\[
\int_0^1 \nabla L(\sigma_n(\tau)) \, d\tau = \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i)) + \Psi \nabla L_{q},
\]

with \(\left[ E|\Psi \nabla L_{q}|^{2p} \right]^{1/2} = \mathcal{O}(h^{2})\).

Submitting \((4.27)\) and the first equation of \((4.23)\) into \((4.26)\) gives

\[
L(\hat{Y}_{t_n,y_n}(t_{n+1})) - L(y_n)
\]

\[
= h \left[ \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i)) + \Psi \nabla L_{q} \right] \left[ \sum_{i=1}^{M} b_i f(\sigma_n(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i))^\top \alpha_0 \right] \\
+ \sum_{r=1}^{D} \Delta \hat{W}_{r,n} \left[ \sum_{i=1}^{M} b_i g_r(\sigma_n(c_i)) + \Psi \nabla L_{q} \right] \left[ \sum_{i=1}^{M} b_i g_r(\sigma_n(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i))^\top \alpha_r \right].
\]
Utilizing the second and third lines of (4.23), we obtain

\[ L(\tilde{Y}_{t_n,y_n}(t_{n+1})) - L(y_n) \]

\[ = h \Psi_{L,q} \left[ \sum_{i=1}^{M} b_i f(\sigma_n(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i))^\top \alpha_0 \right] \]

\[ + \sum_{r=1}^{D} \Delta \tilde{W}_{r,n} \Psi_{L,q} \left[ \sum_{i=1}^{M} b_i g_r(\sigma_n(c_i)) - \sum_{i=1}^{M} b_i \nabla L(\sigma_n(c_i))^\top \alpha_r \right]. \tag{4.28} \]

In order to acquire (4.24), it suffices to estimate the lowest-order term

\[ \sum_{r=1}^{D} \Delta \tilde{W}_{r,n} \Psi_{L,q} \left[ \sum_{i=1}^{M} b_i g_r(\sigma_n(c_i)) \right]. \]

According to assumptions on \( f, g, \nabla L \) and Hölder inequality, we have

\[
E|L(\tilde{Y}_{t_n,y_n}(t_{n+1})) - L(y_n)|^2 \leq K \sum_{r=1}^{D} E \left[ |\Delta \tilde{W}_{r,n}|^2 |\Psi_{L,q}|^2 \right] + Kh^{q+2} \\
\leq Kh \left[ E|\Psi_{L,q}|^4 \right]^{1/2} + Kh^{q+2} \\
\leq Kh^{q+1}. \tag{4.29} \]

Thus this proves (4.24).

If \( y_n \) is replaced by the deterministic variable \( y \), we are able to use Lemma 4.2 and acquire that

\[
|E \Delta \tilde{W}_r \Psi_{L,q}| = \begin{cases} \mathcal{O}(h^{\frac{q+2}{2}}), & \text{if } q \text{ is even}, \\
\mathcal{O}(h^{\frac{q+1}{2}}), & \text{if } q \text{ is odd}. \end{cases} \tag{4.30} \]

Thus, we have

\[
E \left[ L(\tilde{Y}_{t_n,y}(t_{n+1})) - L(y) \right] = \begin{cases} \mathcal{O}(h^{\frac{q+2}{2}}), & \text{if } q \text{ is even}, \\
\mathcal{O}(h^{\frac{q+1}{2}}), & \text{if } q \text{ is odd}. \end{cases} \]

Notice that \( y_n \) is \( \mathcal{F}_{t_n} \)-measurable and that \( \tilde{Y}_{t_n,y}(t_{n+1}) - L(y) \) is \( \mathcal{F}_{t_n} \)-independent. According to the property of conditional expectation (see [10, Chapter 1]), we have

\[
E \left[ L(\tilde{Y}_{t_n,y_n}(t_{n+1})) - L(y_n) \bigg| \mathcal{F}_{t_n} \right] = \left( E \left[ L(\tilde{Y}_{t_n,y}(t_{n+1})) - L(y) \right] \right) \bigg|_{y=y_n}. \tag{4.31} \]

In this way, we obtain (4.25).

We now give the result about mean square order of invariant conversation for (4.23).

**Theorem 4.7.** Let \( \nabla L \nabla L^\top \) be invertible and \( [\nabla L \nabla L^\top]^{-1} \in C_b^1 \). Let \( q \) be the order of quadrature formula \( (c_i, b_i)^M_{i=1} \) in (4.7). Assume that \( q \geq 2 \), and \( f, g_r, r = 1, \ldots, D, \nabla L \in C_b^{(q+1)} \). Then it holds that

\[
E|L(y_N) - L(y_0)|^2 = \begin{cases} \mathcal{O}(h^q), & \text{if } q \text{ is even}, \\
\mathcal{O}(h^{\frac{q+1}{2}}), & \text{if } q \text{ is odd}. \end{cases} \tag{4.32} \]
Submitting (4.34) into (4.33) and using Young’s inequality
\[ ab \]
where
\[ \text{Thus, Gronwall inequality leads to} \]
\[ e \]
It follows from Gronwall inequality (see [10, Lemma 1]).
\[ \text{Note that} \]
\[ y \]
\[ \text{Theorem 4.7 implies that the mean square order of invariants conservation of MAVF methods} \]
\[ n \]
\[ q \]
\[ \text{In this section, we implement numerical experiments to verify our} \]
\[ f \]
Remark 4.8. (1) If \( q \) is odd, then
\[ e_{n+1} \leq e_n + E|L(y_n) - L(y_0)|^2 + h \epsilon_n + h^{-1} \{ E|E[L(y_{n+1}) - L(y_n)|F_{t_n}]|^2 \} \]
\[ \text{Note that} \]
\[ y_{n+1} = \tilde{Y}_{t_n,y_0}(t_{n+1}) \]. Utilizing (4.24) and (4.25) in Lemma 4.6 we have
\[ \text{If} q \text{ is odd, then} \]
\[ e_{n+1} \leq e_n(1 + h) + K h^{q+1} + h^{-1} K h^{q+2} \]
\[ \leq e_n(1 + h) + K h^{q+1} \]
\[ \text{It follows from Gronwall inequality (see [10, Lemma 1.6.]) that} \]
\[ e \]
\[ n \]
\[ \text{In this case, method (4.23) exactly preserves the invariant } L(y). \text{ For general cases,} \]
\[ \text{Theorem 4.7 implies that the mean square order of invariants conservation of MAVF methods} \]
\[ \text{using numerical integration only depends on the order of quadrature formulas.} \]
5. Numerical experiments
5.1. MAVF methods. In this section, we implement numerical experiments to verify our theoretical analyses. And we show the superiority of MAVF methods when applied to conservative SDEs.
5.1.1. **Example 1: Kubo oscillator.** Consider the following stochastic harmonic oscillator in \[ \begin{align*}
    \text{d}X_1(t) &= -aX_2(t) - \sigma X_2(t) \circ \text{d}W(t), \\
    \text{d}X_2(t) &= aX_1(t) + \sigma X_1(t) \circ \text{d}W(t),
\end{align*} \] (5.1)

where \( a \) and \( \sigma \) are constants, and \( W(t) \) is a one-dimensional Brownian motion. The quadratic function \( I(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \) is the invariant of system (5.1). In the numerical test, we take \( a = \sigma = 1 \) and initial value \((X_1(0), X_2(0)) = (1, 0)\).

Figure 1 displays the convergence order of MAVF method (3.2). Here, the reference solution is obtained by Milstein method with stepsize \( h_{\text{ref}} = 2^{-14} \). The mean square errors are computed at the endpoint \( T = 1 \) by adopting five different stepsizes \( h = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9} \).
The expectation is realized by using the average of 1000 independent sample paths. The convergence of order one, as is shown in this figure, is observed for the MAVF method, which is consistent with theoretical analyses of Theorem 3.6.

Figure 2 and Figure 3 show the superiority of our MAVF method in aspect of numerically simulating the Kubo oscillator over a long time. Figure 2 displays the numerical solutions of Milstein method and MAVF method in the phase space along a single sample path. Here, $T = 100$ and $h = 0.01$. We observe that the numerical solutions of MAVF method remain on the unit circle, but the ones of Milstein method do not share this property. Figure 3 displays the errors of invariant $I(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, along one sample, using the Milstein method and MAVF method, respectively. This shows that MAVF method has a better long time stability.

5.1.2. Example 2: Stochastic cyclic Lotka-Volterra system. Consider the following stochastic dynamical system

$$
\begin{align*}
    \frac{d}{dt} & \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} X(t)(Z(t) - Y(t)) \\ Y(t)(X(t) - Z(t)) \\ Z(t)(Y(t) - X(t)) \end{pmatrix} dt + c \begin{pmatrix} X(t)(Z(t) - Y(t)) \\ Y(t)(X(t) - Z(t)) \\ Z(t)(Y(t) - X(t)) \end{pmatrix} \circ dW(t),
\end{align*}
$$

where $c$ is a real-valued constant and $W(t)$ is a one-dimensional Brownian motion. It can be regarded as a cyclic Lotka-Volterra system of competing 3-species in a chaotic environment [14]. It is verified that system (5.2) has two conservative quantities

$$
    I_1(x, y, z) = x + y + z, \quad I_2(x, y, z) = x \cdot y \cdot z.
$$

In this experiment, we set $c = 0.5$ and initial value $(X_0, Y_0, Z_0) = (1, 2, 1)$. Then, the exact solution of system (5.2) remains on the one-dimensional manifold

$$
    \mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 | I_1(x, y, z) = X_0 + Y_0 + Z_0, \quad I_2(x, y, z) = X_0 \cdot Y_0 \cdot Z_0 \},
$$

which is a closed curve in three-dimensional Euclid space. We compare the MAVF method with Milstein method to demonstrate the strengths of the proposed method.

Figure 4 shows the convergence order of MAVF method. The reference solution is obtained by Milstein method with step size $h_{ref} = 2^{-14}$. The mean square errors are computed at the
The numerical sample paths of Milstein method and MAVF method are shown in Figure 5. The interval length $T = 100$ and step size $h = 0.01$. We observe that numerical solutions of the MAVF method, along one sample, lie in the manifold $M$, but those of Milstein method do not. Figure 6 displays the errors of invariants of these two methods. Here the error is denoted by $\max\{|I_1(Y_n) - I_1(Y_0)|, |I_2(Y_n) - I_2(Y_0)|\}$. On the other hand, the MAVF method exactly preserves the two invariants, as is seen in this figure. Although the coefficients of system (5.2) do not satisfy the globally Lipschitz conditions as required in Theorem 3.7, the
MAVF method for original system still works well, which indicates that MAVF methods can be applied to more general system.

5.1.3. Example 3: Stochastic Hamiltonian system with multiple invariants. In this experiment, we consider the following Stochastic Hamiltonian system with commutative noises

\[
d \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \\ Y_4(t) \end{pmatrix} = \begin{pmatrix} Y_2(t) \\ Y_1(t) \\ -Y_4(t) \\ -Y_3(t) \end{pmatrix} (dt + c_1 \circ dW_1(t) + c_2 \circ dW_2(t)),
\]

(5.4)

where \(c_1\) and \(c_2\) are constants, and \(W_1(t)\) and \(W_2(t)\) are two independent Brownian motions. The system (5.4) can be regarded as the extension of Example 3.1 in [11]. One can verify that this system has three invariants

\[
L_1(y_1, y_2, y_3, y_4) = y_1 y_4 - y_2 y_3,
\]

\[
L_2(y_1, y_2, y_3, y_4) = \frac{1}{2}(y_1^2 - y_2^2 + y_3^2 - y_4^2),
\]

(5.5)

\[
L_3(y_1, y_2, y_3, y_4) = y_1 y_2 + y_3 y_4.
\]

In this experiment, we take parameters \(c_1 = 1, c_2 = 0.5\) and initial value \(Y_0 = (-0.5, 0, 0.5, 1)\). We still compare MAVF method with Milstein method for system (5.4).

We can observe from Figure 7 that the mean square convergence of order one when applying the MAVF method to system (5.4). The mean square errors are computed at the endpoint \(T = 1\) by adopting five different stepsizes \(h = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}\). The reference solution is obtained by Milstein method with step size \(h_{ref} = 2^{-14}\). The expectation is evaluated by average of 1000 independent sample paths. This verifies the conclusion about convergence in Theorem 3.7 under the case of commutative noises.

As in previous experiments, Figure 8 displays the errors of invariants \(L_1, L_2\) and \(L_3\), respectively, when applying MAVF method and Milstein method. Here we set \(T = 100\) and \(h = 0.01\). We observe that MAVF method for system (5.4) preserves exactly these three
Figure 7. Mean square errors of MAVF method at $T=1$ for stochastic Hamiltonian system. The dashed reference line has slope 1.

Figure 8. Errors of invariants of Milstein method and MAVF method for stochastic Hamiltonian system with $T = 100$ and $h = 0.01$.

Invariants but Milstein method fails. This shows that MAVF method possesses a better long time stability.

5.2. MAVF methods using numerical integration. In this section, we perform numerical experiments to present the effect of numerical integration on MAVF methods. The MAVF
methods (4.7) using quadrature formula (4.2), (4.3), (4.4) and (4.5) are called MAVF-Q2 method, MAVF-Q3 method, MAVF-Q4 method and MAVF-Q6 method respectively.

Consider the following SDE with commutative noises (see [3])

\[
\begin{align*}
    d\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ \sin(q) \end{pmatrix} \, dt + \begin{pmatrix} 0 & -\cos(q) \\ \cos(q) & 0 \end{pmatrix} \begin{pmatrix} p \\ \sin(q) \end{pmatrix} (c_1 \circ dW_1(t) + c_2 \circ dW_2(t)),
\end{align*}
\]

(5.6)

where \( c_1, c_2 \) are constants, and \( W_1(t), W_2(t) \) are two independent Brownian motions. This system has \( I(p, q) = \frac{1}{2}p^2 - \cos(q) \) as its invariant. We take \( c_1 = 1, c_2 = 0.5 \), and initial value \( (p_0, q_0) = (0.2, 1) \) in this experiment.

Figure 9 shows the convergence order of MAVF-Q2 method. The reference solution is obtained by Milstein method with step size \( h_{\text{ref}} = 2^{-14} \). The mean square errors are computed at the endpoint \( T = 1 \) by adopting five different stepsizes \( h = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9} \). The expectation is approximated using the average of 1000 independent sample paths. It is observed that the conservative method for this system is of mean square order 1, which is consistent with the conclusion of Theorem 4.4.

Figure 10 presents the sample paths of MAVF method, MAVF-Q2 method, MAVF-Q4 method and MAVF-Q6 method. The computational interval is \( T = 10000 \) and stepsize \( h = 0.01 \). Table 1 shows the errors of invariant of these three methods along single sample path and their computation times. As is seen in Figure 10 and Table 1, as the order of quadrature formula enlarges, the invariant is preserved better.

Figure 11 shows mean square orders of invariant conservation of MAVF methods using numerical integration. Here, we use MAVF-Q2 method, MAVF-Q3 method, MAVF-Q4 method to perform numerical experiment. The reference solution is obtained by Milstein method with step size \( h_{\text{ref}} = 2^{-14} \). The mean square orders of invariant conservation are computed at the endpoint \( T = 1 \) by adopting five different stepsizes \( h = 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10} \). The expectation is approximated using the average of 1000 independent sample paths. It is shown that MAVF-Q2 method and MAVF-Q3 method have mean square order 1 of invariant conservation, while MAVF-Q4 method has mean square order 2 of invariant conservation. These results coincide with those of Theorem 4.7.
Figure 10. Numerical sample paths of MAVF method, MAVF-Q2 method, MAVF-Q4 method and MAVF-Q6 method for stochastic pendulum problem with $T = 10000$ and $h = 0.01$.

Table 1. Errors of invariant of MAVF-Q2 method, MAVF-Q4 method, and MAVF-Q6 method at different times along single sample path for stochastic pendulum problem with $T=10000$ and $h=0.01$.

| t    | MAVF-Q2 | MAVF-Q4 | MAVF-Q6 |
|------|---------|---------|---------|
| 100  | 0.0009  | 0.0036  | 0.0037  |
| 500  | 0.0189  | 0.2108E-05 | 0.0170E-05 |
| 1000 | 0.0193  | 0.4764E-05 | 0.0738E-05 |
| 5000 | 0.0101  | 0.8180E-05 | 0.1478E-05 |
| 10000|        | 0.9179E-05 |         |

CPU time

78s
61s
59s

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Figure 11. Mean square orders of invariant conservation of MAVF-Q2 method, MAVF-Q3 method and MAVF-Q4 method at T=1 for stochastic pendulum problem. The two dashed reference lines have slope 1 and 2 respectively.

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