ON THE REAL DIFFERENTIAL OF A SLICE REGULAR FUNCTION

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Abstract. In this paper we show that the real differential of any injective slice regular function is invertible everywhere. To obtain the goal we introduce a new tool: slice differential forms. For the set of slice differential forms we prove some foundational theorems and we show an example of application. Other instruments regarding the coefficients of the spherical expansion and the slice derivative of a slice regular function will be described.

1. Introduction

In [23] and [8], the authors start an interesting investigations about the real differential of a slice regular function. Let $\mathbb{H}$ be the algebra of quaternions. A slice regular function $f$ is a differentiable quaternionic function of one quaternionic variable such that, for any complex structure $I \in \mathcal{S} := \{x \in \mathbb{H} \mid x^2 = -1\} \subset \mathbb{H}$, the restriction of $f$ to the complex line generated by 1 and $I$, is a holomorphic map.

The theory founded on this notion of regularity, introduced by Cullen in a paper of 1965 ([6]), is rapidly growing in the last years, thanks firstly to the authors of [3, 13, 14], that have set the groundwork.

The main purpose of this paper is to extend the next theorem stated in [8].

Theorem 1. Let $f : \Omega \to \mathbb{H}$ be an injective slice regular function with $\Omega \cap \mathbb{R} \neq \emptyset$. Then its real differential is everywhere invertible.

The extension that we have in mind is about the domain of definition of the function. Indeed, as the reader can see in many papers (see e.g. [1, 16, 17, 18]), the results about slice regular functions defined over domains that intersects the real line, does not extend in a simple way to functions defined over domains that do not have real points. In particular in [17], the authors show that if one takes a quaternionic function of quaternionic variable and imposes regularity without imposing sliceness, the class of function that results is not very manageable: is too big. This blemish does not appear if one consider regular functions over domain that intersects $\mathbb{R}$, because they are all slice for free. The good news is that theorem 1 extends to all slice regular functions as is stated in [22]. However, the path to this result was not so easy. Indeed the proof of theorem 1 contained in [8] does not work properly in our setting. To obtain the goal, we had to work on the contents in [8, 18, 23], taking into account the differences just mentioned.

In particular, we firstly realized that a part of the theory could be formalized in a new way considering the slice factor of the real differential of a slice function: we introduce here the concept of slice differential forms. After giving an application of this theory to slice regular functions (showing a Morera’s type theorem), we connect it with the theory of real differential. To do this, we remember the notion of spherical analyticity introduced in [23] for functions with domain intersecting the real axis, and in [18] for slice regular functions defined over a general real alternative algebra. In this context, we give some new informations about the coefficients of the spherical expansion of a slice regular function, showing among other things a new way to compute slice derivatives (see formula 10). Finally, starting from some results about the rank of the real differential, we show that the real differential of any injective slice regular function is invertible everywhere.

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1We will see later the definition.
differential of a slice regular function, we show the main theorem using moreover a new result that generalize in our context a classical theorem of complex analysis (theorem 41).

The structure of the present work is the following.

In section 2 we states the main preliminary results needed for the reader to understand the theory. We mention as general references for this part the book [12] [5] and the paper [16].

In section 3 we introduce the concept of slice form. With the help of some examples we put the basis for this theory which we think could be useful for the future. In particular we point out the presence of a Poincaré type lemma.

Section 4 is devoted to an application of slice form’s theory. In this section we give a new proof of Morera’s theorem that works also for slice regular functions defined over domains that do not intersects the real axis.

Section 5, the richest one, is divided into 3 subsections: an introductory one in which we try to describe the real differential of a slice regular function in terms of functions defined over domains that do not intersects the real axis.

In all the paper we try, where possible, to give explanatory examples.

2. Preliminary results

In $\mathbb{H}$ we will denote with $x^c$ the usual conjugation, i.e.: if $x = x_0 + ix_1 + jx_2 + kx_3$ then $x^c = x_0 - ix_1 - jx_2 - kx_3$. Let $\mathbb{H}_C = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathbb{H}$. An element $x$ in $\mathbb{H}_C$ will be of the form $x = p + \sqrt{-1}q$, where $p$ and $q$ are quaternions. In $\mathbb{H}_C$ we have two commuting antinvolution acting on the element $x$ in the following ways:

- $x^c = (p + \sqrt{-1}q)^c = p^c + \sqrt{-1}q^c$,
- $\overline{x} = (p + \sqrt{-1}q) = p - \sqrt{-1}q$.

Given now $D$ be a connected open set in $\mathbb{C}$, we remember the following definition.

**Definition 1.** A function $F : D \to \mathbb{H}_C$ is called a stem function on $D$ if it is complex intrinsic, i.e.: if the condition $F(z) = \overline{F(\overline{z})}$ holds for each $z \in D$ such that $\overline{z} \in D$. Moreover we call $F$ continuous or differentiable if the two components of $F = F_1 + \sqrt{-1}F_2$ are respectively continuous or differentiables.

The last definition means that if $F_1, F_2 : D \to \mathbb{H}$ are the quaternionic components of $F = F_1 + \sqrt{-1}F_2$ then $F_1$ is even w.r.t. the imaginary part of $z$ ($F_1(\overline{z}) = F(z)$), while $F_2$ is odd ($F_2(\overline{z}) = -F_2(z)$).

**Definition 2.** Given any set $D \subset \mathbb{C}$ we define the circularization of $D$ in $\mathbb{H}$ as the subset of $\mathbb{H}$ defined by:

$$\Omega_D := \{\alpha + J\beta \in \mathbb{H} \mid \alpha + i\beta \in D, J \in \mathbb{S}\}.$$  

The following notations will be very useful: let $D$ be a subset of the complex plane $\mathbb{C}$, then we will denote with $D_J$ and with $D_J^+$ the following sets:

$$D_J := \Omega_D \cap \mathbb{C}_J, \quad D_J^+ := \Omega_D \cap \mathbb{C}_J^+,$$

where $\mathbb{C}_J := \{\alpha + i\beta \in D \mid J \in \mathbb{S}\}$ and $\mathbb{C}_J^+ := \{\alpha + J\beta \in D_J \mid \beta \geq 0\}$.

**Definition 3.** A function $f : \Omega_D \to \mathbb{H}$ is called a (left) slice function if it is induced by a stem function $F = F_1 + \sqrt{-1}F_2$ on $D$, $f = I(F)$, in the following way:

$$f(\alpha + J\beta) := F_1(\alpha + i\beta) + JF_2(\alpha + i\beta), \quad \forall x = \alpha + J\beta \in \Omega_D.$$  

We will denote by $S(\Omega_D)$ and by $S^1(\Omega_D)$ the real vector spaces of slice functions on $\Omega_D$ induced respectively by continuous and differentiable stem functions. Thanks to definition any slice function is well defined. Indeed, if $f = I(F) : \Omega_D \to \mathbb{H}$ is a slice function induced by $F$, then $f(\alpha + (-J)(-\beta)) = F_1(\alpha + \sqrt{-1}(-\beta)) - JF_2(\alpha + \sqrt{-1}(-\beta)) = F_1(\alpha + \sqrt{-1}\beta) - J(-F_2(\alpha + \sqrt{-1}\beta)) = F_1(\alpha + \sqrt{-1}\beta) + JF_2(\alpha + \sqrt{-1}\beta) = f(\alpha + J\beta)$.
For slice functions we have the following representation theorem that morally says that if we know the values of a slice functions over two different half complex planes then we can reconstruct the whole function. More precisely

**Theorem 2.** Let \( f \) be a slice function on \( \Omega_D \). Let \( J \neq K \in \mathbb{S} \) then the following formula holds:

\[
f(x) = (I - K)((J - K)^{-1}f(\alpha + J\beta)) - (I - J)((J - K)^{-1}f(\alpha + K\beta)),
\]

for every \( x = \alpha + I\beta \in \Omega_D \). In particular, for \( K = -J \), we get the formula

\[
f(x) = \frac{1}{2}(f(\alpha + J\beta) + f(\alpha - J\beta) - JJ(f(\alpha + J\beta) - f(\alpha - J\beta))).
\]

Representation formulas for quaternionic slice regular functions appeared in [2, 3], while the case of continuous slice functions can be found in [16].

**Definition 4.** Given a slice function \( f \), we define its spherical derivative in \( x \in \Omega_D \setminus \mathbb{R} \) as,

\[
\partial_s f(x) := \frac{1}{2} Im(x)^{-1}(f(x) - f(x^c)).
\]

**Remark 1.** We have that \( \partial_s f = \mathcal{I}(\frac{F_{\alpha}(y)}{Im(x)}) \) on \( \Omega_D \setminus \mathbb{R} \). Obviously this function is constant on every sphere \( S_x = \{ y \in \mathbb{H} | y = \alpha + \beta I, I \in \mathbb{S} \} \). In other terms:

\[
\partial_s(\partial_s(f)) = 0,
\]

moreover \( \partial_s f = 0 \) if and only if \( f \) is constant on \( S_x \). If \( \Omega_D \cap \mathbb{R} \neq \emptyset \), under some mild regularity hypothesis on \( F \) (see [10] for more details), \( \partial_s f \) can be extended continuously as a slice function on \( \Omega_D \).

Let \( D \subset \mathbb{C} \) be an open set. Given a differentiable stem function \( F = F_1 + \sqrt{-1}F_2 : D \to \mathbb{H}_\mathbb{C} \). The two functions

\[
\frac{\partial F}{\partial z}, \frac{\partial F}{\partial \bar{z}} : D \to \mathbb{H}_\mathbb{C},
\]

are stem functions. Explicitly:

\[
\frac{\partial F}{\partial z} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha} - i \frac{\partial F}{\partial \beta} \right) = \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - \sqrt{-1} \left( \frac{\partial F_1}{\partial \beta} - \frac{\partial F_2}{\partial \alpha} \right) \right),
\]

and

\[
\frac{\partial F}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha} + i \frac{\partial F}{\partial \beta} \right) = \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + \sqrt{-1} \left( \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha} \right) \right).
\]

The previous stem functions induces the continuous slice derivatives:

\[
\frac{\partial f}{\partial x} = \mathcal{I} \left( \frac{\partial F}{\partial z} \right), \quad \frac{\partial f}{\partial x^c} = \mathcal{I} \left( \frac{\partial F}{\partial \bar{z}} \right).
\]

**Definition 5.** A slice function \( f \in S^1(\Omega_D) \) is called slice regular if the following equation holds:

\[
\frac{\partial f}{\partial x^c}(\alpha + J\beta) = 0, \quad \forall \alpha + J\beta \in \Omega_D.
\]

We denote by \( \text{SR}(\Omega_D) \) the real vector space of all slice regular functions on \( \Omega_D \).

Clearly a slice regular function is a differentiable slice function induced by a holomorphic stem function. The next theorem gives a characterization of slice regular functions, but before state it we need the following notation: let \( f = \mathcal{I}(F) : \Omega_D \to \mathbb{H} \) then we denote the restrictions over a complex slice or a complex semi-slice, respectively, as

\[
f_J := \left. f \right|_{D_J} : D_J \to \mathbb{H}, \quad f^+_J := \left. f \right|_{D^+_J} : D^+_J \to \mathbb{H}.
\]

**Proposition 3.** Let \( f = \mathcal{I}(F) \in S^1(\Omega_D) \), then the following facts are equivalents:

- \( f \in \text{SR}(\Omega_D) \);
- the restriction \( f^+_J \) is holomorphic for every \( J \in \mathbb{S} \) with respect to the complex structures on \( D_J \) and \( \mathbb{H} \) defined by left multiplication by \( J \).
two restrictions \( f^+_J, f^+_K \) (\( J \neq K \)) are holomorphic on \( D^+_J \) and \( D^+_K \) respectively (the possibility \( K = -J \) is not excluded).

The proof of the previous theorem can be found in [16] and implies that, if the set \( D \) has nonempty intersection with the real line, then \( f \) is slice regular on \( \Omega_D \) iff it is Cullen regular in the sense introduced by Gentili and Struppa in [13][14].

The pointwise product of two slice functions is not, in general, a slice function but, if one consider the function induced by the pointwise product of the two stem functions then, the result is a slice function and also regularity is preserved. So, to be more precise, we declare the following definition.

**Definition 6.** Let \( f = \mathcal{I}(F), g = \mathcal{I}(G) \) be two slice functions on \( \Omega_D \). The slice product of \( f \) and \( g \) is the slice function defined by

\[
f \cdot g := \mathcal{I}(FG).
\]

**Remark 2.** In the previous definition, if the components of the first stem function \( F = F_1 + \sqrt{-1}F_2 \) are real valued then \( (f \cdot g)(x) = f(x)g(x) \) for each \( x \in \Omega_D \).

**Definition 7.** A slice function \( f = \mathcal{I}(F) \) is called real, if the two components \( F_1 \) and \( F_2 \) are real-valued.

The next, announced, proposition says that this notion of product is the good one.

**Proposition 4.** If \( f, g \in \mathcal{SR}(\Omega_D) \), then \( f \cdot g \in \mathcal{SR}(\Omega_D) \).

The proof of this proposition can be found in [16] where, it is also pointed out that the regular product introduced in [3][9] is generalized by this one if the domain \( \Omega_D \) does not have real points.

Some notion about the zeros of a slice regular function will be useful in the next parts. We will state, then, the principal results knowing yet in the literature.

Let \( F \) be a stem function, then also \( F^c \) is. We will denote by \( f^c \) the slice function induced by \( F^c \). The next definition given in [16] generalize the one given in [9] for power series.

**Definition 8.** Let \( f \) be a slice function over \( \Omega_D \). Then we define the normal function of \( f \) (or symmetrization of \( f \)) the slice function \( N(f) := f \cdot f^c \in \mathcal{S}(\Omega_D) \).

**Remark 3.** Let \( f \) be a slice function. The following facts are contained, as always, in [16].

- If \( f \) is a slice regular function, then also \( f^c \) and \( N(f) \) are slice regular functions.
- The following equations hold true:
  \[
  (f \cdot g)^c = g^c \cdot f^c,
  \]
  and so \( N(f) = N(f)^c \),

while, in general, \( N(f^c) \neq N(f) \).
- The next equality hold true:
  \[
  N(f \cdot g) = N(f)N(g).
  \]

Let now \( V(f) \) be the zero set of \( f : \Omega_D \to \mathbb{H} \):
\[
V(f) := \{ x \in \Omega_D \mid f(x) = 0 \}.
\]

The next proposition, proved in [16], says that either \( S_x \subset V(f) \) or \( S_x \cap V(f) \) is a singleton.

**Proposition 5.** Let \( f \in \mathcal{S}(\Omega_D) \), then, for any \( x \in \Omega_D \setminus \mathbb{R} \), the restriction of \( f \) to \( S_x \) is injective or constant.

The structure of \( V(f) \) is showed in the next theorem (see [16] for the proof).

**Theorem 6.** Let \( f \in \mathcal{S}(\Omega_D) \) and let \( x = \alpha + J\beta \in \Omega_D \). Then one of the following mutually exclusive statements holds:

1. \( S_x \cap V(f) = \emptyset \).
2. \( S_x \subset V(f) \). In this case \( x \) is called a real or spherical zero of \( f \) if, respectively, \( x \in \mathbb{R} \) or \( x \notin \mathbb{R} \).
3. \( S_x \cap V(f) = \{ y \} \), with \( y \notin \mathbb{R} \). In this case \( x \) is called an \( S \)-isolated non-real zero of \( f \).
Corollary 7. If $f$ is a real slice function then $f$ does not has $S$-isolated non-real zeros. Moreover, for any slice function $f$, it holds:

$$V(N(f)) = \bigcup_{x \in V(f)} S_x.$$ 

In the next theorem we add regularity property.

Theorem 8. Let $\Omega_D$ be connected and let $f$ be a slice regular function such that $N(f)$ does not vanish identically, then $V(f) \cap D_J$ is closed and discrete in $D_J$ for every $J \in \mathbb{S}$. 

In particular in [11] [14] [22] is stated an identity principle.

Theorem 9. Let $\Omega_D$ be a connected open set of $\mathbb{H}$. Given $f = I(F) : \Omega_D \to \mathbb{H} \in \mathcal{SR}(\Omega_D)$. If there exists $K \neq J \in \mathbb{S}$ such that both $D_K^+ \cap V(f)$ and $D_J^+ \cap V(f)$ contain accumulation points, then $f \equiv 0$ on $\Omega_D$. In particular if $\Omega_D \cap \mathbb{R} \neq \emptyset$, then $J$ could be equal to $K$.

The distinction between the two cases in the previous theorem is underlined by the next example.

Example 1. Let $J \in \mathbb{H}$ be fixed. The slice regular function defined on $\mathbb{H} \setminus \mathbb{R}$ by 

$$f(x) = 1 - J I, \quad x = \alpha + \beta I \in \mathbb{C}_J^+$$ 

is induced by a locally constant stem function and its zero set $V(f)$ is the half plane $\mathbb{C}_J^+ \setminus \mathbb{R}$. The function can be obtained by the representation formula in theorem [2] by choosing the constant values $2$ on $\mathbb{C}_j^+ \setminus \mathbb{R}$ and $0$ on $\mathbb{C}_J^+ \setminus \mathbb{R}$.

The notion of slice constant function was introduced in [11] to isolate the class of functions for which the previous example is a representative.

Definition 9. Let $f = I(F) \in \mathcal{S}(\Omega_D)$. $f$ is called slice constant if the stem function $F$ is locally constant.

Proposition 10. Let $f \in \mathcal{S}(\Omega_D)$ be a slice constant function, then $f$ is slice regular. Moreover $f$ is slice constant if and only if 

$$\frac{\partial f}{\partial x} = I \left( \frac{\partial F}{\partial z} \right) \equiv 0.$$ 

The next definition is needed for defining the multiplicity of a slice function on a point.

Definition 10. For each $x \in \mathbb{H}$, the characteristic polynomial of $y$ is the slice regular function $\Delta_y(x) : \mathbb{H} \to \mathbb{H}$ defined by: 

$$\Delta_y(x) := \Delta x - y = (x - y) \cdot (x - y^c) = x^2 - x(y + y^c) + yy^c.$$

Remark 4. The following facts about the characteristic polynomial are quite obvious. If the reader need more details we refer again in [16].

- $\Delta_y$ is a real slice function.
- Two characteristic polynomials $\Delta_y, \Delta_y'$ coincide iff $S_y = S_y'$.
- $V(\Delta_y) = S_y$.

Theorem 11. Let $f \in \mathcal{SR}(\Omega_D)$ and $x_0 \in V(f)$. Then the following statements are true.

- If $x_0 \in \mathbb{R}$, then there exists $g \in \mathcal{SR}(\Omega_D)$ such that $f(x) = (x - x_0)g(x)$.
- If $x_0$ is not real, then there exists $h \in \mathcal{SR}(\Omega_D)$ and $a, b \in \mathbb{H}$ such that $f(x) = \Delta_{x_0}(x)h(x) + xa + b$, where,

  - $S_{x_0} \subset V(f)$ iff $a = b = 0$;
  - $S_{x_0} \cap V(f)$ is a singleton iff $a \neq 0$ (in this case $x_0 = -ba^{-1}$).

Corollary 12. Let $f \in \mathcal{SR}(\Omega_D)$ and $x_0 \in V(f)$ then there exists $g \in \mathcal{SR}(\Omega_D)$ such that $f(x) = (x - x_0) \cdot g(x)$.

Corollary 13. Let $f \in \mathcal{SR}(\Omega_D)$ and $x_0 \in V(f)$ then $\Delta_{x_0}(x)$ divides $N(f)$.

Thanks to the last corollary, we are able to give the following definition.
Definition 11. Let \( f \in \mathcal{SR}(\Omega_D) \) such that \( N(f) \) does not vanish identically. Given \( n \in \mathbb{N} \) and \( x_0 \in V(f) \), we say that \( x_0 \) is a zero of \( f \) of multiplicity \( n \) if \( \Delta^n_{x_0} | N(f) \) and \( \Delta^{n+1}_{x_0} \nmid N(f) \). We will call the integer \( n \) as multiplicity of \( x_0 \) and we will denote it by \( m_f(x_0) \).

If \( m_f(x_0) = 1 \), then \( x_0 \) is called a simple zero of \( f \).

The last definition stated in [16] is equivalent to the one of total multiplicity stated in [15, 12].

Proposition 14. Let \( f, g \in \mathcal{S}(\Omega_D) \). Then \( V(f) \subset V(f \cdot g) \). Moreover it holds:

\[
\bigcup_{x \in V(f \cdot g)} S_x = \bigcup_{x \in V(f) \cap V(g)} S_x.
\]

The following definition is a notion of reciprocal in the context of slice functions. Was firstly introduced in [3, 9, 10] and then in [1] if the domain of definition has empty intersection with the real line. The proof of the subsequent proposition can be found in [1].

Definition 12. Let \( f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D) \). We call the slice reciprocal of \( f \) the slice function

\[
f^{-} : \Omega_D \setminus V(N(f)) \to \mathbb{H}
\]

defined by

\[
f^{-} = \mathcal{I}(FF^c)^{-1}F^c
\]

From the previous definition it follows that, if \( x \in \Omega_D \), then

\[
f^{-}(x) = (N(f)(x))^{-1}f(x).
\]

Proposition 15. Let \( f \in \mathcal{SR}(\Omega_D) \) such that \( V(f) = \emptyset \), then \( f^{-} \in \mathcal{SR}(\Omega_D) \) and \( f \cdot f^{-} = f^{-} \cdot f = 1 \).

We end this preliminary section with the open mapping theorem. Let \( D_f \) be the degenerate set of \( f \in \mathcal{S}(\Omega_D) \):

\[
D_f := \{ S_x \subset \Omega_D \ | \ f|_{S_x} \text{ is constant} \} = Ker(\partial_x f),
\]

and denote with \( K_f \) the set of semislices where \( f \) is constant:

\[
K_f := \{ D_f^+ \subset \Omega_D \ | \ f_f^+ \text{ is constant} \},
\]

then we have the following theorem.

Theorem 16. Let \( f : \Omega_D \to \mathbb{H} \) be a slice regular function. Then

\[
f : \Omega_D \setminus (D_f \cup K_f) \to \mathbb{H}
\]

is open.

Of course, in the previous theorem, which can be found originally in [10] and extended then in [1], the set \( K_f \) can be equal either to the emptyset, to a single semislice or to the whole \( \Omega_D \). In the particular case in which \( \Omega_D \cap \mathbb{R} \neq \emptyset \) the options are only two: either \( K_f = \emptyset \) or \( K_f = \Omega_D \).

We are now ready to start with some new results.

3. Slice differential forms and slice differential

Now, following the idea that was used by Ghiloni and Perotti in [14] to construct slice functions via stem functions, we want to introduce the notion of slice differential forms.

The differential forms \( \omega : \mathbb{H} \to \mathbb{H}^* \) we want to construct, are such that their restriction to a complex line \( \mathbb{C}_J \) behave in the following way:

\[
\omega(\alpha + J\beta) = (\omega_\alpha^J(\alpha, \beta)d\alpha + \omega_\beta^J(\alpha, \beta)d\beta) + J(\omega_\alpha^J(\alpha, \beta)d\alpha + \omega_\beta^J(\alpha, \beta)d\beta).
\]

Before giving the formal definition, we must control the behaviour of such form in an arbitrary point \( p \), expressed in his two forms: \( p = \alpha + J\beta = \alpha + (-J)(-\beta) \).

So, first of all, thinking of \( \mathbb{C}_J \) as a two dimensional real vector space, we are giving implicitly an orientation, in particular we are distinguishing the basis \( \{1, J\} \) as positive. Any other basis obtained from this one by a linear transformation with positive determinant then is likewise
positive, and the remaining bases are negative. Then the basis \( \{1, -J\} \), which is positive for \( \mathbb{C}_J \), is negative for \( \mathbb{C}_J \), with base changing matrix as follow:

\[
A = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

and we must taking into account this duality. Let, thus, denote, for any \( K \in \mathbb{S} \), by \( \{d\alpha_K, d\beta_K\} \) the standard and positive basis for \( \mathbb{C}^*_K \). Then, since the change of basis from \( \{1, J\} \) to \( \{1, -J\} \) is determined by the matrix \( A \) in \([1]\), it is clear that,

\[
da_{-J} = d\alpha_J \quad d\beta_{-J} = -d\beta_J.
\]

So, the form \( \omega \), evaluated in \( \alpha + (-J)(-\beta) \) is equal to

\[
\omega(\alpha + (-J)(-\beta)) = (\omega^1_\alpha(\alpha, -\beta)d\alpha_{-J} + \omega^1_\beta(\alpha, -\beta)d\beta_{-J}) + (-J)(\omega^2_\alpha(\alpha, -\beta)d\alpha_{-J} + \omega^2_\beta(\alpha, -\beta)d\beta_{-J})
\]

\[
= (\omega^1_\alpha(\alpha, -\beta)d\alpha_J - \omega^1_\beta(\alpha, -\beta)d\beta_J) + J(-\omega^2_\alpha(\alpha, -\beta)d\alpha_J + \omega^2_\beta(\alpha, -\beta)d\beta_J),
\]

and this must be equal to:

\[
\omega(\alpha + J\beta) = (\omega^1_\alpha(\alpha, \beta)d\alpha_J + \omega^1_\beta(\alpha, \beta)d\beta_J) + J(\omega^2_\alpha(\alpha, \beta)d\alpha_J + \omega^2_\beta(\alpha, \beta)d\beta_J),
\]

we then obtain the following system:

\[
\begin{align*}
\omega^1_\alpha(\alpha, -\beta) &= \omega^1_\alpha(\alpha, \beta) \\
\omega^1_\beta(\alpha, -\beta) &= -\omega^1_\beta(\alpha, \beta) \\
\omega^2_\alpha(\alpha, -\beta) &= -\omega^2_\alpha(\alpha, \beta) \\
\omega^2_\beta(\alpha, -\beta) &= \omega^2_\beta(\alpha, \beta)
\end{align*}
\]

Justified by the previous considerations we give the following two definitions.

**Definition 13.** A one-form \( \omega : \Omega_D \to \mathbb{H}^+ \) such that

\[
\omega(\alpha + J\beta) = (\omega^1_\alpha(\alpha, \beta)d\alpha + \omega^1_\beta(\alpha, \beta)d\beta + J(\omega^2_\alpha(\alpha, \beta)d\alpha + \omega^2_\beta(\alpha, \beta)d\beta),
\]

with coefficients \( \omega^i_\gamma : D_J \to \mathbb{H}, \gamma \in \{\alpha, \beta\}, i = 1, 2 \), such that equations \([\Box]\) holds true, will be called slice differential one-form.

**Definition 14.** Let \( f = \mathcal{I}(F = F_1 + \sqrt{-1}F_2) \in \mathcal{S}^1(\Omega_D) \). We define the slice differential \( d_{sl}f \) of \( f \) as the following slice form:

\[
d_{sl}f : \Omega_D \to \mathbb{H}^+,
\]

\[
\alpha + J\beta \mapsto dF_1 + JdF_2.
\]

**Proposition 17.** The previous definition is well posed, i.e.:

\[
d_{sl}f(\alpha + J\beta) = d_{sl}f(\alpha + (-J)(-\beta)), \quad \forall \alpha + J\beta \in \Omega_D
\]

**Proof.** Let \( x = \alpha + J\beta \in \Omega_D \) and \( z = \alpha + \sqrt{-1}\beta \), then,

\[
d_{sl}f(\alpha + (-J)(-\beta)) = dF_1(z) - JdF_2(z)
\]

\[
= \frac{\partial F_1}{\partial \alpha}(z)d\alpha_{-J} + \frac{\partial F_1}{\partial \beta}(z)d\beta_{-J} - J\left( \frac{\partial F_2}{\partial \alpha}(z)d\alpha_{-J} + \frac{\partial F_2}{\partial \beta}(z)d\beta_{-J} \right)
\]

\[
= \frac{\partial F_1}{\partial \alpha}(z)d\alpha_J + \frac{\partial F_1}{\partial \beta}(z)d\beta_J - J\left( -\frac{\partial F_2}{\partial \alpha}(z)d\alpha_J - \frac{\partial F_2}{\partial \beta}(z)d\beta_J \right)
\]

\[
= dF_1(z) + JdF_2(z) = d_{sl}f(\alpha + J\beta),
\]

where the third equality holds thanks to the even-odd character of the couple \( (F_1, F_2) \) and to the change of basis between \( \{d\alpha_{-J}, d\beta_{-J}\} \) and \( \{d\alpha_J, d\beta_J\} \).

\[\Box\]
To avoid ambiguity, in the following we will consider always \( \beta \geq 0 \), so, to be more clear, the point \( p = \alpha - J\beta \) will be intended as \( p = \alpha + (-J)\beta \). Having chosen this convention, there will be no ambiguity in writing \( dx \) and \( d\beta \) without subscripts.

It is clear from the definition that, if we choose the usual coordinate system, where \( x = \alpha + J\beta \), then \( d_s x = dx + Jd\beta \) and \( d_s x^c = dx - Jd\beta \). We can now state the following theorem.

**Theorem 18.** Let \( f \in \mathcal{S}^1(\Omega_D) \). Then the following equality holds:

\[
d_{s^1} x \frac{\partial f}{\partial x} + d_{s^1} x^c \frac{\partial f}{\partial x^c} = d_{s^1} f.
\]

**Proof.** The thesis is obtained after the following explicit computations:

\[
d_{s^1} x \frac{\partial f}{\partial x} + d_{s^1} x^c \frac{\partial f}{\partial x^c} = \frac{1}{2} \left( (\partial_x + Id_\beta) \left( \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - I \left( \frac{\partial F_1}{\partial \beta} - \frac{\partial F_2}{\partial \alpha} \right) \right) + \\
+ (\partial_x - Id_\beta) \left( \frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + I \left( \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha} \right) \right) \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - I d_\alpha \frac{\partial F_1}{\partial \beta} + I d_\alpha \frac{\partial F_2}{\partial \alpha} + \\
+ Id_\beta \frac{\partial F_1}{\partial \alpha} + Id_\beta \frac{\partial F_2}{\partial \beta} + d_\beta \frac{\partial F_1}{\partial \beta} - d_\beta \frac{\partial F_2}{\partial \alpha} + \\
+ Id_\beta \frac{\partial F_1}{\partial \alpha} + Id_\beta \frac{\partial F_2}{\partial \beta} + d_\beta \frac{\partial F_1}{\partial \beta} + d_\beta \frac{\partial F_2}{\partial \alpha} \right)
\]

\[
= \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} + d_\beta \frac{\partial F_1}{\partial \beta} + d_\beta \frac{\partial F_2}{\partial \alpha} = \\
= \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} + \frac{\partial F_1}{\partial \beta} + I \left( \frac{\partial F_2}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} \right)
\]

\[
= d_{s^1} f.
\]

We have then the obvious corollary:

**Corollary 19.** Let \( f \in \mathcal{SR}(\Omega_D) \). Then the following equality holds:

\[
d_{s^1} x \frac{\partial f}{\partial x} = d_{s^1} f.
\]

For any \( J \in \mathcal{S} \) and any slice form \( \omega : \Omega_D \to \mathbb{H}^* \) we denote by \( \omega_J \) the restriction

\[
\omega_J := \omega |_{D^+_J} : D^+_J \to \mathbb{H}^*.
\]

We will use also the following notation: if \( \omega \) is a slice form then

\[
\omega(\alpha + J\beta) = \omega^{a_1}(\alpha, \beta) + J(\omega^{a_2}(\alpha, \beta))d\alpha + (\omega^{b_1}(\alpha, \beta) + J(\omega^{b_2}(\alpha, \beta))d\beta =
\]

\[
= (\omega^{a_1}(\alpha, \beta)d\alpha + \omega^{b_1}(\alpha, \beta)d\beta) + J(\omega^{a_2}(\alpha, \beta)d\alpha + \omega^{b_2}(\alpha, \beta)d\beta) =
\]

\[
= \omega^1(\alpha, \beta) + J\omega^2(\alpha, \beta),
\]

where, of course,

\[
\omega^{a_1}d\alpha + \omega^{b_1}d\beta = \omega^1, \quad \omega^{a_2}d\alpha + \omega^{b_2}d\beta = \omega^2.
\]

As for slice functions also for slice forms we have a representation formula.

**Theorem 20. (Representation Formula)** Let \( J \neq K \in \mathcal{S} \). Every slice form \( \omega : \Omega_D \to \mathbb{H}^* \) is uniquely determined by its values on the two distinct semisllices \( D^+_J \) and \( D^-_K \). In particular the following formula holds:

\[
\omega(\alpha + I\beta) = (I - K)((J - K)^{-1}\omega(\alpha + J\beta)) = (I - J)((J - K)^{-1}\omega(\alpha + K\beta)),
\]

for every \( \alpha + I\beta \in \Omega_D \).
Before passing through the proof of the previous theorem, we observe that, if $K = -J$, the previous formula [5], can be written as:

$$\omega(\alpha + I\beta) = \frac{1}{2}(\omega(\alpha + J\beta) + \omega(\alpha - J\beta) - IJ(\omega(\alpha + J\beta) - \omega(\alpha - J\beta))).$$

**Proof.** To prove the theorem, we will show how to derive $\omega^1$ and $\omega^2$ from $\omega_J$ and $\omega_K$. First of all we have,

$$\omega_J(\alpha + J\beta) - \omega_K(\alpha + K\beta) = \omega^1(\alpha, \beta) + J\omega^2(\alpha, \beta) - \omega^1(\alpha, \beta) - K\omega^2(\alpha, \beta) = (J - K)\omega^2(\alpha, \beta),$$

for all $\alpha + i\beta \in D$ with $\beta \geq 0$. Moreover, if $D$ is symmetric w.r.t. the real axis, then $\omega^2(\alpha, \beta) = \omega^0(\alpha, \beta) d\alpha + \omega^2(\alpha, \beta) d\beta$ is determined also for $\beta < 0$ imposing equations [2]. To determine $\omega^1$ it is sufficient to consider the difference between $\omega_J$ and $J\omega^2$:

$$\omega_J - J\omega^2 = \omega^1(\alpha, \beta) + J\omega^2(\alpha, \beta) - J\omega^2.$$

Passing now to prove the formula, we already know that

$$\omega^2(\alpha, \beta) = (J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)), \quad \omega^1(\alpha, \beta) = \omega(\alpha + J\beta) - J\omega^2,$$

and so

$$\omega^1(\alpha, \beta) = \omega(\alpha + J\beta) - J(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)).$$

Let’s now $I \in \mathbb{S}$, then,

$$\omega(\alpha + I\beta) = \omega^1(\alpha, \beta) + J\omega^2(\alpha, \beta) = \omega(\alpha + J\beta) - J(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)) + I(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta))$$

$$= \omega(\alpha + J\beta) + (I - J)(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta))$$

$$= ((J - K) + (I - J)(J - K)^{-1}\omega(\alpha + J\beta) + (I - J)(J - K)^{-1}\omega(\alpha + K\beta))$$

$$= (I - J)(J - K)^{-1}\omega(\alpha + J\beta) + (I - J)(J - K)^{-1}\omega(\alpha + K\beta).$$

\[ \square \]

At this point a natural question arise: does all the slice forms are the slice differential of a slice function?

Of course the answer, in general, is not. As a counterexample we can consider the slice form defined over the whole $\mathbb{H}$ in the following way:

$$\omega : \alpha + J\beta \mapsto J\beta d\alpha + -Jd\beta.$$

The previous slice form is not the slice differential of a slice function since

$$\frac{\partial(J\beta)}{\partial\beta} = J \neq -J = \frac{\partial(-\alpha J)}{\partial\alpha}.$$

**Definition 15.** A slice differential form $\omega : \Omega_D \to \mathbb{H}^*$ is called slice-exact (s-exact) in $\Omega_D$ if there exists a differentiable slice function $f : \Omega_D \to \mathbb{H}$ such that $df = \omega$ for any $x \in \Omega_D$.

The previous counterexample suggests a necessary condition for a slice form to be s-exact. The condition is the following.

**Definition 16.** A slice differential form $\omega : \Omega_D \to \mathbb{H}^*$ is called slice-closed (s-closed) in $\Omega_D$ if, written in coordinates

$$\omega(\alpha + J\beta) = \omega(\alpha + J\beta)d\alpha + \omega(\alpha + J\beta)d\beta,$$

one has

$$\frac{\partial \omega_\alpha}{\partial \beta} = \frac{\partial \omega_\beta}{\partial \alpha}, \quad \forall \alpha + J\beta \in \Omega_D.$$

Thanks to theorem 20 we have the following proposition.

**Proposition 21.** Let $\omega : \Omega_D \to \mathbb{H}^*$ be a slice differential form.
(1) If \( \Omega_D \cap \mathbb{R} \neq \emptyset \) and there exists \( J \in \mathbb{S} \) such that the following equation holds:

\[
\frac{\partial \omega^\alpha_J}{\partial \beta} = \frac{\partial \omega^\beta_J}{\partial \alpha}, \quad \forall \alpha + J\beta \in D^+_J,
\]

then \( \omega \) is s-closed.

(2) If \( \Omega_D \cap \mathbb{R} = \emptyset \) and there exists \( J \neq K \in \mathbb{S} \) such that the following equations holds:

\[
\begin{cases}
\frac{\partial \omega^\alpha_J}{\partial \beta} = \frac{\partial \omega^\beta_J}{\partial \alpha}, & \forall \alpha + J\beta \in D^+_J, \\
\frac{\partial \omega^\alpha_K}{\partial \beta} = \frac{\partial \omega^\beta_K}{\partial \alpha}, & \forall \alpha + K\beta \in D^+_K,
\end{cases}
\]

then \( \omega \) is s-closed.

**Proof.** The proof of the theorem follow applying formula 5 and deriving in case 1 and formula 6 and deriving in case 2. \( \square \)

The previous theorem simplifies the controls needed to see if a form is closed. Moreover, in the case in which the domain does not intersects the real line, the result is sharp as the following examples will show.

**Example 2.** Let \( J \in \mathbb{S} \) a fixed imaginary unit and let \( \omega : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}^* \) the following slice form:

\[
\omega(\alpha + I\beta) = (J\beta + 1 + I(\beta + J))d\alpha + (1 - J\alpha + I(J - \alpha))d\beta.
\]

This slice form is constructed with formula 6 where

\[
\omega_J(\alpha, \beta) = 2J\beta d\alpha - 2Jd\beta,
\]

and so

\[
\frac{\partial \omega^\alpha_J}{\partial \beta} = \frac{\partial \omega^\beta_J}{\partial \alpha}, \quad \text{but,} \quad \frac{\partial \omega^\alpha_J}{\partial \beta} = J \neq -J = \frac{\partial \omega^\beta_J}{\partial \alpha}.
\]

Indeed, if we compute the partial derivatives of \( \omega_J \), for a generical \( I \in \mathbb{S} \), what we obtain is the following:

\[
\frac{\partial \omega^\alpha_J}{\partial \beta} = J + I, \quad \frac{\partial \omega^\beta_J}{\partial \alpha} = -J - I,
\]

which are equal if and only if \( I = -J \).

The next is an obvious and natural result.

**Proposition 22.** Let \( \Omega_D \) be a circular open set in \( \mathbb{H} \) and let \( \omega : \Omega_D \rightarrow \mathbb{H}^* \) a slice form. If \( \omega \) is s-exact then \( \omega \) is s-closed.

Again, as in the general case, a s-closed form may not be s-exact. An example for such a case of this type is given by following slice form:

\[
\omega : \mathbb{H} \setminus \{0\} \rightarrow \mathbb{H}^* \quad \alpha + J\beta \mapsto \frac{J\beta}{\alpha^2 + \beta^2}d\alpha + \frac{-J\alpha}{\alpha^2 + \beta^2}d\beta.
\]

This is of course a slice form which is s-closed but, as the general theory of calculus says, it is not s-exact.

The next theorem is a characterization for s-exact slice form.

**Theorem 23.** Let \( D \subset \mathbb{C} \) be an open and connected set and \( \omega : \Omega_D \rightarrow \mathbb{H}^* \) a slice differential form. The following facts are equivalent:

1. \( \omega \) is s-exact;
2. for any \( J \in \mathbb{S} \) and for any couple of path \( \gamma_1 \) and \( \gamma_2 \) contained in \( D_J \), with same extremal points, the following equation hold,

\[
\int_{\gamma_1} \omega = \int_{\gamma_2} \omega;
\]
(3) for any $J \in \mathbb{S}$ and for any circuit $\gamma$ contained in $D_J$, the following equation hold,

$$\int_\gamma \omega = 0.$$

**Proof.** Following the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$, the only non-trivial implication to prove, and for which we give an argument, is $(2) \Rightarrow (1)$. Since $D$ is connected, then, for any $J \in \mathbb{S}$, any couple of points $x$ and $y$ in $D_J$ can be connected with a curve. Let then $x_0$ be a fixed point in $D_J^+$ and define $f : D_J^+ \to \mathbb{H}$ as $f(x) := \int_{\gamma} \omega_J$, where $\gamma$ is any curve from $x_0$ to $x$. The function $f$ is well defined thanks to the hypothesis $(2)$. It is clear then, using a little bit of standard analysis that $df = \omega_J$. The thesis is obtained, at worst repeating the same argument in another different semislice $D_K$, using the representation formula $\square$.

At this point, as the reader can imagine, we are going to state a Poincaré type lemma.

**Theorem 24.** Let $D$ be a starshaped open subset of $\mathbb{C}$ and let $\Omega_D$ its circularization. If a slice differential form $\omega : \Omega_D \to \mathbb{H}^*$ is $s$-closed then $\omega$ is $s$-exact.

**Proof.** Let $x_0 = \alpha_0 + J_0 \beta_0 \in D_J^+$. We are going to show that the function $f_{J_0}^+ : D_J^+ \to \mathbb{H}$ defined by

$$f_{J_0}^+(\alpha + J_0 \beta) := \int_0^1 [\omega_{J_0}^\alpha(\alpha_0 + t(\alpha - \alpha_0), \beta_0 + t(\beta - \beta_0)) + \omega_{J_0}^\beta(\alpha_0 + t(\alpha - \alpha_0), \beta_0 + t(\beta - \beta_0)) \beta_0] \, dt,$$

is a primitive of $\omega_{J_0}$. To obtain the result it is sufficient to show that the partial derivatives of $f_{J_0}^+$ w.r.t. $\alpha$ and $\beta$ are equal to $\omega_{J_0}^\alpha$ and $\omega_{J_0}^\beta$ respectively, i.e.

$$\frac{\partial f_{J_0}^+}{\partial \alpha}(\alpha + J_0 \beta) = \omega_{J_0}^\alpha(\alpha + J_0 \beta), \quad \frac{\partial f_{J_0}^+}{\partial \beta}(\alpha + J_0 \beta) = \omega_{J_0}^\beta(\alpha + J_0 \beta), \quad \forall \alpha + J_0 \beta \in D_J^+.$$

We have:

$$\frac{\partial f_{J_0}^+}{\partial \alpha}(\alpha + J_0 \beta) = \int_0^1 \frac{\partial \omega_{J_0}^\alpha}{\partial \alpha}(\alpha_0 + t(\alpha - \alpha_0), \beta_0 + t(\beta - \beta_0)) t(\alpha - \alpha_0) + \omega_{J_0}^\alpha(\alpha_0 + t(\alpha - \alpha_0), \beta_0 + t(\beta - \beta_0)) \beta_0 + t(\beta - \beta_0)) \, dt,$$

and, since $\omega$ is $s$-closed ($\frac{\partial \omega}{\partial \beta} = \frac{\partial \omega}{\partial \alpha}$), the last is equal to:

$$\int_0^1 t \frac{\partial \omega_{J_0}^\alpha}{\partial \alpha}(\alpha_0 + t(\alpha - \alpha_0), \beta_0 + t(\beta - \beta_0)) (\alpha - \alpha_0) + \frac{\partial \omega_{J_0}^\alpha}{\partial \beta}(\alpha_0 + t(\alpha - \alpha_0), \beta_0 + t(\beta - \beta_0)) (\beta - \beta_0) \, dt.$$
and so
\[ \frac{\partial f^+_J}{\partial \alpha}(\alpha + J_0 \beta) = \omega^\alpha_J(\alpha + J_0 \beta). \]

Obviously, the computations for \( \frac{\partial f^+_J}{\partial \beta} \) are the same as before and so, if the slice \( D_J \subset \Omega_D \) is connected (i.e.: if \( \Omega_D \cap \mathbb{R} \neq \emptyset \)), the theorem is proved thanks to representation formula for slice functions. If the domain \( \Omega_D \) does not intersects the real axis is sufficient to repeat the same computations on another different semislice, say \( D^+_K \), with \( J_0 \neq K_0 \in \mathbb{S} \), and then apply the representation formula.

\[ \square \]

4. Morera’s theorem

We remember the statements by Giacinto Morera regarding functions of a complex variable.

**Theorem 25.** *(Morera, 1886)* If \( f(z) \) is defined and continuous in an open and connected set \( D \), and if
\[ \int_D f(z) \, dz = 0, \]
for all closed and picewise differentiable curves \( \gamma \) in \( D \), then \( f(z) \) is analytic in \( D \).

In [14] the authors states the following theorem for slice functions of a quaternionic variable.

**Theorem 26.** *(Gentili, Struppa, 2007)* Let \( f : B(0, R) \rightarrow \mathbb{H} \) be a differentiable function. If, for every \( I \in \mathbb{S} \), the differential form \( dx f(x) \), \( x = \alpha + I \beta \), defined on \( B \cap \mathbb{C} \) is closed, then the function \( f \) is regular.

The previous statements could be rewritten as follows:

**Theorem 27.** Let \( f : B(0, R) \rightarrow \mathbb{H} \) be a differentiable function. If there exist \( I \in \mathbb{S} \) such that
\[ \int_{\gamma} d_{sl}x f(x) = 0, \]
for all closed and picewise differentiable curves \( \gamma : [0,1] \rightarrow D_I \), then \( f(z) \) is analytic in \( B(0, R) \).

Indeed, considering the representation formula, as the intersection \( B(0, R) \cap \mathbb{R} \) is non-empty, one can restore the regularity of the function only by the regularity over a slice. As in the case of the identity principle or the maximum modulus principle, this theorem cannot be generalized to the set of slice functions defined over a circular domain that does not intersect the real line. Indeed one can construct a slice function \( f : \Omega_D \rightarrow \mathbb{H} \), via the representation formula, such that over a fixed semislice \( D^+_J \) is regular, and over the opposite semislice \( D^-_J \) is not. Now, following the idea that allow us to obtain the identity principle for slice regular functions, we state the following:

**Theorem 28.** Let \( \Omega_D \subset \mathbb{H} \) be a connected circular domain that does not intersect the real line in \( \mathbb{H} \) and \( f : \Omega_D \rightarrow \mathbb{H} \) be a continuous slice function. If there are \( I \neq J \in \mathbb{S} \) such that
\[ (7) \quad \int_{\gamma} d_{sl}x f(x) = 0, \]
for all closed and picewise differentiable curves \( \gamma : [0,1] \rightarrow D_I \cup D^+_J \), then \( f(x) \) is slice regular in \( \Omega_D \).

It must be noticed that the case \( I = -J \) is not excluded. We underline the meaning of the integral in the previous theorem: If \( f = \mathcal{I}(F) \), \( \gamma(t) = \alpha(t) + \beta(t)I \), then
\[
\int_{\gamma} d_{sl}x f(x) = \int_{0}^{1} (d\alpha + Id\beta)(F_1(z) + IF_2(z)) = \\
= \int_{0}^{1} F_1(\alpha(t) + i\beta(t))\gamma'(t)dt + I \int_{0}^{1} F_2(\alpha(t) + i\beta(t))\gamma'(t)dt
\]

**Proof.** The proof of the theorem can be done modifying a little the result in the case in which the domain has real points. However this is a good chance to show a proof with slice differential
form’s techniques. Now, since \( d_{sl}xf(x) \) is a slice form, then equation 7 with theorem 23 says that it is also s-exact. And so there exist a slice differential function \( g = I(G) \in S^1(\Omega_D) \) such that,

\[
d_{sl}xf = d_{sl}g,
\]

but this entails that,

\[
dG + IdG_2 = \frac{\partial G_1}{\partial \alpha} d\alpha + \frac{\partial G_1}{\partial \beta} d\beta + I \left( \frac{\partial G_2}{\partial \alpha} d\alpha + \frac{\partial G_2}{\partial \beta} d\beta \right)
\]

\[
= F_1 d\alpha - F_2 d\beta + I(F_2 d\alpha + F_1 d\beta), \quad \forall \alpha + I\beta \in \Omega_D
\]

and so,

\[
\begin{align*}
\frac{\partial G_1}{\partial \alpha} &= F_1 = \frac{\partial G_2}{\partial \beta} \\
\frac{\partial G_1}{\partial \beta} &= -F_2 = -\frac{\partial G_2}{\partial \alpha},
\end{align*}
\]

that means that \( g \) is regular and so also \( f \).

\[\blacksquare\]

5. The Real Differential of a Slice Function

In this section we’ll describe the real differential of a slice function. For this purpose, in addition to using what we already discussed in the previous pages, we will remember some results and constructions due to Caterina Stoppato [23]. Moreover, we will also use the concept of spherical differential that will be introduced right now.

Let \( f \in S^1(\Omega_D) \) be a differentiable slice function. We have seen that is possible to define its slice differential, considering, roughly speaking, the restriction of the real differential to each semislice. It is clear that this object does not exhaust the description of the real differential. What we are going to define is the missed part.

**Definition 17.** Let \( f \in S^1(\Omega_D) \) be a differentiable slice function. We define its spherical differential as the following differentiable form:

\[
d_{sp}f : \Omega_D \to \mathbb{H}^*,
\]

\[
d_{sp}f(\alpha + J\beta) := d_{sl}f(\alpha + J\beta) - d_{sl}f(\alpha + J\beta),
\]

where \( d_{sl}f(\alpha + J\beta) \) denote the real differential of \( f \).

In the next pages we will give a more explicit description of the spherical differential of a slice function.

Let \( x \in \mathbb{H} \simeq \mathbb{R}^4 \), \( x = (x_0, x_1, x_2, x_3) \).

When we talk about slice function we implicitly use the following change of coordinates:

\[
(x_0, x_1, x_2, x_3) \mapsto (\alpha, \beta, J),
\]

where \( \alpha \in \mathbb{R}, \beta \in \mathbb{R}^+ \) and \( J = J(\vartheta, \varphi) \in \mathbb{S} \) with the following equalities:

\[
\begin{align*}
\alpha &= x_0 \\
\beta &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\
\vartheta &= \arccos(\frac{x_3}{\beta}) \\
\varphi &= \arctan(\frac{x_2}{x_1}).
\end{align*}
\]
Let now $f : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^4$ be any differentiable function, then its differential in this new variable is of the following form:

$$\text{d}f = \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta + \frac{1}{\beta} \left( \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \varphi} d\varphi \right),$$

where:

$$\begin{cases}
  d\alpha = dx_0 \\
  d\beta = \sin \vartheta \cos \varphi dx_1 + \sin \vartheta \sin \varphi dx_2 + \cos \vartheta dx_3 \\
  d\vartheta = \cos \vartheta \cos \varphi dx_1 + \cos \vartheta \sin \varphi dx_2 - \sin \vartheta dx_3 \\
  d\varphi = -\sin \varphi dx_1 + \cos \varphi dx_2.
\end{cases}$$

If now $f : \Omega_D \to \mathbb{H}$ is a differentiable slice function $f = I(F) \in S^1(\Omega_D)$, $f(\alpha, \beta, J) = F_1(\alpha, \beta) + JF_2(\alpha, \beta)$ then:

$$\text{d}_{\mathbb{H}} f = \left( \frac{\partial F_1}{\partial \alpha} + J \frac{\partial F_2}{\partial \alpha} \right) d\alpha + \left( \frac{\partial F_1}{\partial \beta} + J \frac{\partial F_2}{\partial \beta} \right) d\beta + dJ F_2,$$

where

$$\begin{cases}
  d\alpha = dx_0 \\
  d\beta = \sin \vartheta \cos \varphi dx_1 + \sin \vartheta \sin \varphi dx_2 + \cos \vartheta dx_3 \\
  d\vartheta = \cos \vartheta \cos \varphi dx_1 + \cos \vartheta \sin \varphi dx_2 - \sin \vartheta dx_3 \\
  d\varphi = -\sin \varphi dx_1 + \cos \varphi dx_2.
\end{cases}$$

It is now clear that, if $f \in \mathcal{SR}(\Omega_D)$, then its real differential satisfies the following equation:

$$\text{d}f = d_s x \frac{\partial f}{\partial x} + d_s p \partial_s f.$$

The fact that the elements of the basis of the dual space appear to the left of the coefficients is not a coincidence but depends on the choice we made, at the beginning of the theory when we chose to study left slice functions. Moreover they prescribe the side of application of the vectors. As the reader could object, the previous are only formal considerations but, in the next subsection, all will be proved, at least adding slice-regularity hypothesis, but firstly, we remember the notion of spherical analyticity and its consequences.

### 5.1. Coefficients of the spherical expansion.

In [18, 23], the authors introduce, in slightly different contexts, a spherical series of the form

$$(8) \quad f(x) = \sum_{n \in \mathbb{N}} S_{y,n}(x)s_n,$$

that gave some interesting results. To be more precise, for each $n \in \mathbb{N}$ we remember the definition, given in [18, 23], of the slice regular polynomial functions

$$S_{y,n} := \begin{cases}
  \Delta_y(x)^m & \text{if } n = 2m \\
  \Delta_y(x)^m(x - y) & \text{if } n = 2m + 1,
\end{cases}$$

where $\Delta_y(x)$ are the characteristic polynomials defined in [10]. The sets of convergence of series of type $\sum_{n \in \mathbb{N}} S_{y,n}(x)s_n$ are always open w.r.t. the euclidean topology. Indeed these sets are $u$-balls, where $u$ denote the Cassini pseudo-metric:

$$u(x, y) := \sqrt{||\Delta_y(x)||}, \quad \forall x, y \in \mathbb{H}.$$
Definition 18. Given a function \( f : \Omega_D \to \mathbb{H} \) from a non-empty circular open domain into \( \mathbb{H} \), we say that \( f \) is spherical analytic, if, for all \( y \in \Omega_D \), there exists a non-empty \( u \)-ball \( U \) centered at \( y \) and contained in \( \Omega_D \), and a series \( \sum_{n \in \mathbb{N}} S_{y,n}(x)s_n \) with coefficients in \( \mathbb{H} \), which converges to \( f(x) \) for each \( x \in U \cap \Omega_D \).

Theorem 29. Let \( \Omega_D \) be connected and let \( f : \Omega_D \to \mathbb{H} \) be any function. The following assertions holds true.

- If \( D \cap \mathbb{R} = \emptyset \) then \( f \) is a slice regular function iff \( f \) is slice and spherically analytic.
- If \( D \cap \mathbb{R} \neq \emptyset \) then \( f \) is a slice regular function iff \( f \) is spherically analytic.

Corollary 30. Given a slice regular function \( f : \Omega_D \to \mathbb{H} \) then it satisfies the following equation

\[
\frac{\partial}{\partial q} \Delta f(q) = 0,
\]

where \( q = \alpha + J\beta = x_0 + x_1i + x_2j + x_3k \), \( \Delta \) denotes the Laplacian in the four variables \( x_0, x_1, x_2, x_3 \), and \( \frac{\partial}{\partial q} \) denotes the standard Cauchy-Fueter operator:

\[
\frac{\partial}{\partial q} := \frac{1}{4} \left( \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \right).
\]

Moreover, since \( f \) satisfies equation (9) then it also satisfies the following equation:

\[
\Delta \Delta f = 0.
\]

For more details about the previous corollary and about the theory of quaternionic holomorphic functions, we remind the reader to [7, 20, 21].

Given a slice regular function \( f \in \mathcal{SR}(\Omega_D) \) it is possible to construct its spherical coefficients \( \{s_n\} \) (see [23, 18]), but the methods described in the cited papers allow a correct explanation and interpretation only for the first two coefficients which can be easily written as\(^2\)

\[
s_1 = \frac{1}{2} Im(y) -1(f(y) - f(y^c)) = \partial_s f(y)
\]

\[
s_2 = \frac{1}{2} Im(y)^2(2Im(y)\frac{\partial f}{\partial x}(y) - f(y) + f(y^c)),
\]

and in particular

\[
s_1 + 2Im(y)s_2 = \frac{\partial f}{\partial x}(y).
\]

The following proposition, which have an independent interest, allow us to understand better the nature of \( s_2 \) and also of \( s_3 \).

Proposition 31. Let \( f \in \mathcal{SR}(\Omega_D) \) be a slice regular function, then the following formula holds:

\[
\frac{\partial f}{\partial x}(x) = 2J\beta \left( \frac{\partial}{\partial x} \partial_s f \right)(x) + \partial_s f(x), \quad \forall x = \alpha + J\beta \in \Omega_D.
\]

Proof. Let \( F = F_1 + \sqrt{-1}F_2 \) the inducing stem function of \( f \) and let \( x = \alpha + J\beta \in \Omega_D \), then,

\[
\frac{\partial f}{\partial x}(x) = \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha} + J \frac{\partial F_2}{\partial \alpha} - J \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \beta} \right)(\alpha + i\beta) = \Phi.
\]

Using the slice regularity we have,

\[
\Phi = \frac{\partial F_2}{\partial \beta} + J \frac{\partial F_2}{\partial \alpha}(\alpha + i\beta) = 2J \left( \frac{\partial F_2}{\partial x} \right)(x) = \Phi.
\]

Now, since by definition, \( F_2(\alpha + i\beta) = \beta \partial_s f(x) \), then,

\[
\frac{\partial F_2}{\partial x}(x) = \frac{1}{2} \left( \beta \frac{\partial (\partial_s f)}{\partial \alpha}(x) - J \partial_s f(x) - J \beta \frac{\partial (\partial_s f)}{\partial \beta}(x) \right) = \beta \left( \frac{\partial}{\partial x} \partial_s f \right)(x) - \frac{1}{2} J \partial_s f(x),
\]

\(^2\text{see [18]}\)
and so, finally,
\[ \otimes = 2J\beta \left( \frac{\partial}{\partial x} \partial_s f \right)(x) + \partial_s f(x). \]

\[ \square \]

**Corollary 32.** Let \( f \in \mathcal{SR}(\Omega_D) \) be a slice regular function, then we have the following formula:
\[
\left( \frac{\partial}{\partial x} \right)^{(n)} f(x) = \left( 2J\beta \left( \frac{\partial}{\partial x} \partial_s \right)^{(n)} f \right)(x) + \partial_s \left( 2J\beta \left( \frac{\partial}{\partial x} \partial_s f \right) \right)(x), \quad \forall x = \alpha + J\beta \in \Omega_D,
\]
for any \( n \in \mathbb{N} \), where the apex \((A)^{(n)}\) denote the composition of the operator \( A \) with itself \( n \) times.

**Proof.** The proof is by induction. We yet know that \( \frac{\partial f}{\partial x}(x) = 2J\beta \left( \frac{\partial}{\partial x} \partial_s f \right)(x) + \partial_s f(x) \), but then, using the representation formula in theorem 2, we have that, \( \forall x \in \Omega_D \),
\[
\frac{\partial^2 f}{\partial x^2}(x) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x}(x) \right) = 2J\beta \left( \frac{\partial}{\partial x} \partial_s \left( 2J\beta \left( \frac{\partial}{\partial x} \partial_s f \right) \right)(x) + \partial_s f(x) \right) + \partial_s \left( 2J\beta \left( \frac{\partial}{\partial x} \partial_s f \right) \right)(x), \quad \forall x \in \Omega_D,
\]
and so, again,
\[
\text{and since } \partial_s(\partial_s f) \equiv 0 \text{ (see remark 1), we obtain the thesis for } n = 2. \text{ With an analogous argument it is possible to complete the induction.}
\]

\[ \square \]

**Corollary 33.** Let \( f \in \mathcal{SR}(\Omega_D) \) be a slice regular function with spherical expansion \( f(x) = \sum_{n \in \mathbb{N}} s_{y,n}(x)s_n \) centered in \( x_0 \in \Omega_D \) then,
\[
s_2 = \frac{\partial}{\partial x}(\partial_s f)(x_0).
\]
Passing now to the fourth coefficient, \( s_3 \), we remember how originally were constructed spherical coefficients. Given a slice regular function \( f : \Omega_D \to \mathbb{H} \) and a point \( x_0 = \alpha + J\beta \in \Omega_D \), then, thanks to corollary 12 we have
\[
f(x) - f(x_0) = (x - x_0) \cdot R_{x_0} f(x), \quad \forall x \in \Omega_D,
\]
where \( R_{x_0} f : \Omega_D \to \mathbb{H} \) is a slice regular function which will be called remainder of \( f \) in \( x_0 \).

It can be easily seen that,
\[
\begin{align*}
R_{x_0} f(x_0^\alpha) &= \partial_s f(x_0), \\
R_{x_0} f(x_0) &= \frac{\partial f}{\partial x}(x_0),
\end{align*}
\]
but then, using the representation formula in theorem 2 we have that,
\[
R_{x_0} f(\alpha + I\beta) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) - IJ \left( \frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0) \right) \right), \quad \forall \alpha + I\beta \in \mathbb{S}_{x_0}.
\]
Now, since the remainder of \( f \) in \( x_0 \) is a slice regular function, then we can repeat the previous argument but, this time, we will develop in \( x_0^\alpha \),
\[
R_{x_0} f(x) - R_{x_0} f(x_0^\alpha) = (x - x_0^\alpha) \cdot R_{x_0^\alpha} R_{x_0} f(x), \quad \forall x \in \Omega_D,
\]
and so, again,
\[
\begin{align*}
R_{x_0^\alpha} R_{x_0} f(x_0^\alpha) &= \frac{\partial R_{x_0} f}{\partial x}(x_0^\alpha), \\
R_{x_0^\alpha} R_{x_0} f(x_0) &= \partial_s R_{x_0} f(x_0).
\end{align*}
\]
Proceedings in this way, we obtain that the coefficients in the expansion (8) are equal to:

\[
    s_n = \begin{cases} 
        (R_{x_0}^m R_{x_0})^m f(x_0), & \text{if } n = 2m \\
        R_{x_0}(R_{x_0}^m R_{x_0})^m f(x_0), & \text{if } n = 2m + 1,
    \end{cases}
\]

and so, since \( s_2 = R_{x_0}^m R_{x_0} f(x_0) = \partial_s R_{x_0} f(x_0) = \frac{\partial}{\partial s}(\partial_s f)(x_0) \), we need to know the values of \( R_{x_0}^m R_{x_0} f \) in \( x_0 \) and to apply the representation formula to obtain \( s_3 \). Let’s pass to computations:

\[
    R_{x_0}^m R_{x_0} f(x_0) = \frac{\partial R_{x_0} f}{\partial x}(x_0) = \frac{\partial}{\partial x}((x - x_0)^{-1} \cdot (f(x) - f(x_0)))(x_0) = \frac{\partial f}{\partial x}(x_0),
\]

whereby,

\[
    (x - x_0) \cdot R_{x_0}^m R_{x_0} f(x_0) = -(x - x_0)^{-1} \cdot (f(x) - f(x_0)) + \frac{\partial f}{\partial x}(x_0)
\]

At this point both the right and left hand sides of the equation are well defined in \( x_0 \), and so,

\[
    (x_0^c - x_0) R_{x_0}^m R_{x_0} f(x_0) = -\partial_s f(x_0) + \frac{\partial f}{\partial x}(x_0),
\]

therefore, thanks to formula [10]

\[
    R_{x_0}^m R_{x_0} f(x_0) = (2J\beta)^{-1}(-\partial_s f(x_0^c) + \frac{\partial f}{\partial x}(x_0^c)) = \frac{\partial}{\partial x}(\partial_s f)(x_0).
\]

Summing the results, we have,

\[
    s_3 = \partial_s (R_{x_0}^m R_{x_0} f)(x_0^c) = \partial_s \left( \frac{\partial}{\partial x}(\partial_s f) \right)(x_0),
\]

where the last equality follows applying representation formula in theorem 2 to \( R_{x_0}^m R_{x_0} f \), knowing two of its values on the sphere \( \mathbb{S}_{x_0} \). One can think now that the n-th coefficient \( s_n \) could be equal to

\[
    \begin{cases} 
        \left( \frac{\partial}{\partial x} \right)^m f(x_0) \text{ if } n = 2m, \\
        \partial_s \left( \frac{\partial}{\partial x} \right)^m f(x_0) \text{ if } n = 2m + 1,
    \end{cases}
\]

but this is not true. Actually it is yet false for \( s_4 \) which is equal to

\[
    s_4 = \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 f(x_0) + (2J\beta)^{-3} \partial_s f(x_0),
\]

but since in this paper we will not need other information about the spherical coefficients, we remand the discussion to future works and we end this discussion here.

5.2. Rank of the real differential of a slice regular function. In [8, 23], the authors shows the following theorem.

**Theorem 34.** Let \( f \in \mathcal{SR} (\Omega_D) \) and \( x = \alpha + J\beta \in \Omega_D \). For all \( v \in \mathbb{H}, \|v\| = 1 \), it holds

\[
    \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = vs_1 + (xv - vx^c)s_2,
\]

where \( s_1 \) and \( s_2 \) are the first two coefficients of the spherical expansion of \( f \).

The previous theorem has an important corollary.

\[\text{see formula (30) in [15].}\]
Corollary 35. Let \( f \in \mathcal{SR}(\Omega_D) \) and let \((df)_x\) denote the real differential of \( f \) at \( x = \alpha + J\beta \in \Omega_D \). If we identify \( T_x\mathbb{H} \) with \( \mathbb{H} = \mathbb{C}_J \oplus \mathbb{C}_J^+ \), then for all \( v_1 \in \mathbb{C}_J \) and \( v_2 \in \mathbb{C}_J^+ \),
\[
(df)_x(v_1 + v_2) = v_1(s_1 + 2Im(y)s_2) + v_2s_1 = v_1 \frac{\partial f}{\partial x}(x) + v_2\partial_s f(x).
\]

We will not give a proof of the previous theorem and corollary since the ones in [23] does not use the additional hypothesis of nonempty intersection between the domain and the real axis.

We now want to study the rank of a slice regular function. In [8] the authors proves that an injective slice regular function defined over a circular domain with real points, has invertible differential. The aim of the following pages is to extend this result to all slice regular functions.

Let’s start with a general result.

Proposition 36. Let \( f \in \mathcal{SR}(\Omega_D) \) and \( x_0 = \alpha + J\beta \in \Omega_D \setminus \mathbb{R} \).

- If \( \partial_s f(x_0) = 0 \) then:
  - \( df_{x_0} \) has rank 2 if \( \frac{\partial f}{\partial x}(x_0) \neq 0 \);
  - \( df_{x_0} \) has rank 0 if \( \frac{\partial f}{\partial x}(x_0) = 0 \).
- If \( \partial_s f(x_0) \neq 0 \) then \( df_{x_0} \) is not invertible at \( x_0 \) iff \( 1 + 2Im(x_0)s_2s_1^{-1} = \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} \) belongs to \( \mathbb{C}_J^+ \).

Let now \( \alpha \in \Omega_D \cap \mathbb{R} \). \( df_{x_0} \) is invertible at \( \alpha \) iff its rank is not 0 at \( x_0 = \alpha + J\beta \). This happens iff \( \partial_s f(x_0) \neq 0 \).

The proof of the previous statement can be found (with the appropriate change of notation), on [8] or in [12].

Definition 19. Let \( f : \Omega \to \mathbb{H} \) any quaternionic function of quaternionic variable. We define the singular set of \( f \) as
\[
N_f := \{ x \in \Omega \mid df \text{ is not invertible at } x \}.
\]

The following theorem will characterize the set \( N_f \) of singular points of \( f \).

Theorem 37. Let \( f \in \mathcal{SR}(\Omega_D) \) and \( x_0 = \alpha + J\beta \in \Omega_D \). Then \( x_0 \in N_f \) iff there exists a point \( \tilde{x}_0 \in S_{x_0} \) and a function \( g \in \mathcal{SR}(\Omega_D) \) such that the following equation hold:
\[
f(x) = f(x_0) + (x - x_0) \cdot (x - \tilde{x}_0) \cdot g(x).
\]
Equivalently, \( x_0 \in N_f \) iff the function \( f - f(x_0) \) has multiplicity \( n \geq 2 \) in \( S_{x_0} \).

The proof of the last theorem is analogous to the one in [8]. However, we will rewrite the proof in our setting with our notations. Before proving the last theorem we recall the following remark

Remark 5. For all \( x_0 = \alpha + J\beta \in \mathbb{H} \setminus \mathbb{R} \), setting \( \Psi(x) := (x - x_0)(x - x_0^\dagger)^{-1} \) defines a stereographic projection of \( \alpha + \mathbb{S}\beta \) onto the plane \( \mathbb{C}_J^+ \) from the point \( x_0^\dagger \). Indeed, if we choose \( K \in \mathbb{S} \) with \( K \perp J \) then for all \( x = \alpha + \beta L \) with \( L = tJ + uK + vJK \in \mathbb{S} \) we have \( \Psi(x) = (L - J)(L + J)^{-1} = \frac{u + vJ}{1 + t} JK \) and \( \mathbb{C}_J \cdot K = (\mathbb{R} + \mathbb{R}J)JK = \mathbb{C}_J^+ \).

We are now able to pass to the proof of the theorem.

Proof. If \( x_0 \in \Omega_D \setminus \mathbb{R} \) then it belongs to \( D_f \) iff \( f \) is constant on the sphere \( S_{x_0} \), i.e. there exists a slice regular function \( g : \Omega_D \to \mathbb{H} \) such that
\[
f(x) - f(x_0) = \Delta_{x_0}(x)g(x).
\]
This happens iff the coefficient \( s_1 = \partial_s f(x_0) \) in the spherical expansion vanishes.

Let now pass to the case \( x_0 \in \Omega_D \setminus \mathbb{R} \), \( x_0 \notin D_f \). Thanks to proposition [36], \( x_0 \in N_f \) iff \( 1 + 2Im(x_0)s_2s_1^{-1} = p \in \mathbb{C}_J^+ \). Thanks to the previous remark, \( p \in \mathbb{C}_J^+ \) iff there exists \( \tilde{x}_0 \in S_{x_0} \setminus \{ x_0^\dagger \} \) such that \( p = (\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0^\dagger)^{-1} \). The last formula is equivalent to
\[
2Im(x_0)s_2s_1^{-1} = (\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0^\dagger)^{-1} - (\tilde{x}_0 - x_0^\dagger)(\tilde{x}_0 - x_0^\dagger)^{-1} = (\tilde{x}_0 - x_0^\dagger + x_0^\dagger)(\tilde{x}_0 - x_0^\dagger)^{-1} = -2Im(x_0)(\tilde{x}_0 - x_0^\dagger)^{-1},
\]
that is \( s_1 = (x_0^r - \bar{x}_0)s_2 \). Writing then the first terms of the spherical development of \( f \) around \( x_0 \) we have:

\[
\begin{align*}
f(x) &= s_0 + (x - x_0)s_1 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x)(x - x_0)h(x) \\
&= s_0 + (x - x_0)(x_0^r - \bar{x}_0)s_2 + \Delta_{x_0}(x)s_2 + (x - x_0)\cdot \Delta_{x_0}(x)(x - x_0)h(x) \\
&= s_0 + (x - x_0)(x_0^r - \bar{x}_0)s_2 + \Delta_{x_0}(x)s_2 + (x - x_0)\cdot (x - x_0)h(x) \\
&= s_0 + (x - x_0)\cdot (x - x_0)(x_0^r - \bar{x}_0)s_2 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x)(x - x_0)h(x) \\
&= s_0 + (x - x_0)\cdot (x - x_0)(x_0^r - \bar{x}_0)s_2 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x)(x - x_0)h(x) \\
&= f(x) + (x - x_0)\cdot (x - x_0)(x_0^r - \bar{x}_0)s_2 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x)(x - x_0)h(x),
\end{align*}
\]

for some slice regular function \( h : \Omega_D \to \mathbb{H} \), where we used the following facts:

- \( (x - x_0)(x_0^r - \bar{x}_0) = (x - x_0)\cdot (x_0^r - \bar{x}_0) \) because the second factor is constant;
- \( \Delta_{x_0}(x)(x - x_0) = \Delta_{x_0}(x) \cdot (x - x_0) \) because the first factor is a real slice function;
- \( (x - x_0^r)\cdot (x - x_0) = \Delta_{x_0}(x) \) because if \( f \) is a slice function then \( N(f) = N(f)^r \) and \( \Delta_{x_0}(x) = N(x - x_0); \)
- \( \Delta_{x_0}(x) = \Delta_{\bar{x}_0}(x) \) because \( \bar{x}_0 \in S_{x_0} \).

Finally, if \( x_0 \in \Omega_D \cap \mathbb{R} \) then \( s_1 = 0 \) iff

\[
f(x) = f(x_0) + (x - x_0)^2\cdot l(x) = f(x_0) + (x - x_0)\cdot (x - x_0)\cdot l(x),
\]

for some slice regular function \( l : \Omega_D \to \mathbb{H} \).

For the main result we need, now, two lemmas.

**Lemma 38.** Let \( f : \Omega_D \to \mathbb{H} \in \text{SR}(\Omega_D) \) be non-slice-constant. Then its singular set \( N_f \) has empty interior.

**Proof.** Thanks to remark 1 we yet know that the degenerate set of \( f \) \( D_f := \{ x \in \Omega_D \mid \partial_s f(x) = 0 \} \) is closed in \( \Omega_D \). So, since \( D_f \subseteq N_f \), then the thesis is that \( N_f \setminus D_f \) has empty interior. But if \( N_f \setminus D_f \) contains a nonempty open ball \( B \), therefore, by the constant rank theorem, its image \( f(B) \) could not be open contradicting the open mapping theorem.

**Lemma 39.** Let \( f = I(F) : \Omega_D \to \mathbb{H} \) be an injective slice function. Then for all \( x = \alpha + \beta j \in \Omega_D \setminus \mathbb{R} \), \( \partial_s f(x) \neq 0 \).

**Proof.** We know that \( \partial_s f(x) = 0 \) iff \( f \) is constant on the sphere \( S_x \) (see remark 1). But then if \( f \) is injective then \( \partial_s f(x) \neq 0 \) for all \( x \in \Omega_D \setminus \mathbb{R} \).

Now we have that every injective slice regular function has real differential with rank at almost equal to 2. The next step is to prove that for every injective slice regular function \( f \) the slice derivative \( \frac{\partial f}{\partial z} \) is everywhere different from 0. To do that we need to introduce some tools from complex analysis. The main reference for the following is [19].

**Definition 20.** Given a holomorphic function \( f : D \subseteq \mathbb{C} \to \mathbb{C} \) we define the multiplicity of \( f \) at a point \( x \in D \) as the number:

\[
n(x; f) := \inf \{ k \in \mathbb{N} \setminus \{ 0 \} \mid f^{(k)}(x) \neq 0 \},
\]

\( f^{(k)}(x) \) denoting the \( k \)th derivative of \( f \) w.r.t. \( z \) evaluated in \( x \).

**Definition 21.** Given a holomorphic function \( f \) defined over a region \( D \) we define the valence of \( f \) at \( w \in \mathbb{C} \cup \{ \infty \} \) as

\[
\nu_f(w) := \begin{cases} 
+\infty & \text{if the set } \{ f(z) = w \} \text{ is infinite;} \\
\sum_{f(z) = w} n(z; f) & \text{otherwise.}
\end{cases}
\]
If $f$ does not take the value $w$, then $v_f(w)$ is obviously equal to zero. It turns out that, for any $r > 0$ the valence at $w$ of $f|_{D(x;r)}$ is constant on each component of $(\mathbb{C} \cup \{\infty\}) \setminus f(\partial D(x;r))$, where $D(x;r)$ denote the disc centered in $x$ of radius $r$. Now we can pass to the quaternionic setting. We recall that any slice regular function admit a splitting into two complex holomorphic function as the following lemma claims [3] [18]

**Lemma 40.** Let $f \in \mathcal{SR}(\Omega_D)$ and $J \perp K$ two elements of $\mathbb{S}$. Then there exists two holomorphic functions $f_1, f_2 : D_J^+ \to \mathbb{C}_J$ such that

$$f_J = f_1 + f_2K.$$

We can now state the following theorem.

**Theorem 41.** Let $f = \mathcal{T}(F) : \Omega_D \to \mathbb{R}$ be an injective slice regular function. Then its slice derivative $\frac{\partial f}{\partial x}$ is always different from zero.

**Proof.** What we want to prove is that, for any $x_0 = \alpha + J\beta \in \Omega_D$

$$\frac{\partial f}{\partial x}(x_0) \neq 0.$$

First of all, thanks to the identity principle [9] applied to the slice derivative of $f$, if $\partial f/\partial x$ is equal to zero in $y \in D^+_J \subset \Omega_D$, for some $I \in \mathbb{S}$, then $y$ is isolated in $D^+_J$. Since $f$ is slice regular there exists two holomorphic functions $f_1, f_2 : D^+_J \to \mathbb{C}_J$ such that $f^+_J = f_1 + f_2K$, with $K$ an opportune element orthogonal to $J$ in $\mathbb{S}$. We have then that

$$\frac{\partial f}{\partial x}(x_0) = \frac{\partial F}{\partial z} = \frac{\partial f_1}{\partial z} + \frac{\partial f_2}{\partial z}K,$$

and so the thesis became that at least one of the two derivatives $\frac{\partial f_1}{\partial z}, \frac{\partial f_2}{\partial z}$ is different from zero. Moreover, since $f$ is injective, then also $f^+_J = f|_{D^+_J}$ is injective. So, if one between $f_1$ and $f_2$ is constant, then the other one must be injective, and so we will have an injective holomorphic function and the thesis will follow trivially. Let’s suppose then that both $f_1$ and $f_2$ are non-constant functions and fix the following notations:

$$n(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid \frac{\partial^k f}{\partial x^k}(x) \neq 0\},$$

$$n_1(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid f^{(k)}_1(x) \neq 0\},$$

$$n_2(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid f^{(k)}_2(x) \neq 0\}.$$

It is easy to see that, for every $x \in D^+_J$, $n(x; f) = \min(n_1(x; f), n_2(x; f))$. Moreover, since $f$ is non constant then the null set of its slice derivative restricted to the semislice $D^+_J$ is discrete. Let now $B_1 := B_1(x_0; r_1), B_2 := B_2(x_0; r_2)$ be two open ball in $D^+_J$ such that $f_1$ take the value $f_1(x_0)$ on $B_1$ only at $x_0$ and such that $f_1^{(k)}(z) \neq 0$ for any $z \in B_1 \setminus \{x_0\}$. Let now $B = B_1 \cap B_2$, then the valence $v_{f_1}(f(z))$ of $f_1|_B$ is constant and equal to $n_1(z; f)$ in the component of $(\mathbb{C}_J \cup \{\infty\}) \setminus f(\partial B)$. Since $n(x; f) = \min(n_1(x; f), n_2(x; f))$ and $n(x; f) = 1$ a.e. suppose that $\exists y \in B$ such that $1 = n(y; f) = n_1(y; f)$. Then $n_1$ is constant and equal to $1$ in $B$ and so we have the thesis.

**Remark 6.** The proof of the previous statement works also to prove that a slice regular function $f : \Omega_D \to \mathbb{H}$ injective on a semislice $D^+_J \subset \Omega_D$ has slice derivative nonzero over the same semislice $D^+_J$. We choose to formalize the theorem in the previous less general hypothesis only to simplify the reading.

**Theorem 42.** Let $f$ be an injective slice regular function, then $N_f = \emptyset$.

**Proof.** If, by contradiction, there exists $x_0 = \alpha + J\beta \in N_f \neq \emptyset$, then, thanks to theorem [37] the function $f - f(x_0)$ must have multiplicity $n$ greater or equal to $2$ in $\mathbb{S}_{x_0}$. This means that

$$f(x) - f(x_0) = (x - x_0) \cdot g(x),$$

with $g \in \mathcal{SR}(\Omega_D)$ such that $g(x_1) = 0$ for some $x_1 \in \mathbb{S}_{x_0}$. Since $f$ is injective, then $g(x_0) = \frac{\partial f}{\partial x}(x_0) \neq 0$ and $g(x_0) = \partial_s f(x_0) \neq 0$, and so $x_1 \neq x_0, x_0$. Now, whereas we know the values of $g$
at \( x_0 \) and at \( x_0^c \), we can apply the representation formula in theorem 2 to analyze the behaviour over the sphere \( S_{x_0} \). The result is the following,

\[
g(\alpha + I\beta) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) - IJ \left( \frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0) \right) \right), \quad \forall I \in \mathbb{S}.
\]

So, if there exist \( I \in \mathbb{S} \) such that \( g(\alpha + I\beta) = 0 \), then,

\[
\frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) = IJ \left( \frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0) \right)
\]

\[
\Leftrightarrow \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} + 1 = IJ \left( \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} - 1 \right)
\]

\[
\Leftrightarrow \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} = -(1 + IJ)(1 - IJ)^{-1},
\]

with \( I \neq J, -J \), but then \( \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} \) does not belong to \( \mathbb{C}_r^+ \) and this is in contradiction with proposition 36.

\[\square\]

**Example 3.** Let \( J \in \mathbb{S} \) be a fixed imaginary unit and \( f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H} \) be the following slice regular function:

\[f(\alpha + I\beta) = (\alpha + I\beta)(1 - IJ)\]

This function which was already discussed in \( \mathbb{H} \) is constructed, with the representation formula, to be equal to zero over the semislice \( \mathbb{C}_{r,J}^+ \) and to be equal to \( 2r \) over the opposite semislice \( \mathbb{C}_{r,J}^- \).

What we want to show is that the restriction \( f|_{\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}_{r,J}^+)} \) is injective. So, first of all we write in a more explicit way the function \( f \):

\[f(x) = \alpha(1 + I \cdot J) + \beta I + J\beta - \alpha I \wedge J\]

where \( I \cdot J \) and \( I \wedge J \) denote the scalar and the vectorial products respectively in \( \mathbb{R}^3 \). But then again,

\[f(x) = \alpha(1 + I \cdot J) + \sqrt{2\beta^2 + \alpha^2}\eta^2 \left( \frac{\beta I + J\beta - \alpha \eta \wedge J}{\sqrt{2\beta^2 + \alpha^2}\eta^2} \right),\]

where \( \eta = |I \wedge J| \). Now first of all we show that the application \( A : \mathbb{R}^3 \to \mathbb{R}^3 \), where

\[A(v) = \beta v + J\beta - \alpha v \wedge J\]

is an injective map for any \( \alpha \in \mathbb{R} \) and any \( \beta \in \mathbb{R}^+ \). But this is obvious once the operator \( A - J\beta \) has been written in coordinates.\(^4\) In particular the subspace \( \text{span} \langle -J \rangle \) is preserved by \( A - J\beta \). So, for any \( I_1 \neq I_2 \in \mathbb{S} \setminus \{ \pm J \} \), the images of the two semislices \( \mathbb{C}_{r,I_1}^+ \), \( \mathbb{C}_{r,I_2}^+ \) via \( f \) are contained in two disjoint semislices.

So the goal now is to understand the image of a fixed semislice via \( f \), say \( f(\mathbb{C}_{r,K}^+) \), with \( K \neq \pm J \). Formally

\[f(\alpha + K\beta) = \gamma + \delta L, \quad \text{with } \alpha \in \mathbb{R}, \beta \in \mathbb{R}^+, \]

and so we obtain the following system,

\[
\begin{aligned}
\gamma &= \alpha(1 + K \cdot J) \\
\delta &= \sqrt{2\beta^2 + \alpha^2}\eta^2,
\end{aligned}
\]

\(^4\text{Here we used the ‘scalar-vector’ notation.}\)

\(^5\text{Indeed the reader could see that } A - J\beta \text{ is a non degenerate linear operator.}\)
which says that, for every $K \neq \pm J$, the function $f_K^+$ is injective. The image of the semislice $\mathbb{C}^+_K$ is the following set,
\[
f(\mathbb{C}^+_K) = \left\{ \gamma + \delta L \in \mathbb{H} \mid L = \left( \frac{\beta K + J \beta - \alpha \eta K \wedge J}{\sqrt{2|\beta|^2 + \alpha^2 \eta^2}} \right), \gamma \in \mathbb{R}, \delta \in \left( \frac{|\gamma \eta - 1 + K \cdot J|}{1 + K \cdot J}, \infty \right) \right\}
\]
\[
= \left\{ \gamma + \delta L \in \mathbb{H} \mid L = \left( \frac{\beta K + J \beta - \alpha \eta K \wedge J}{\sqrt{2|\beta|^2 + \alpha^2 \eta^2}} \right), \delta > \frac{|\eta - \gamma|}{1 + K \cdot J}, \gamma \in \mathbb{R} \right\},
\]
which is an open cone in $\mathbb{C}^+_L \simeq \mathbb{R}^2$ with vertex at $(0, 0)$ and angular coefficient equal to $\frac{\eta}{1 + K \cdot J}$.

**Figure 1.** Behaviour of $f(\mathbb{C}^+_K)$, for $K \perp J$, near to $-J$ and near to $J$

Since this function, with the proper restriction, is slice regular and injective then theorem \[42\] says that its real differential is always invertible. This fact could also be seen computing the slice and the spherical derivative. Indeed, since
\[
\partial_s f(\alpha + I \beta) = \frac{\beta - \alpha J}{\beta},
\]
is always different from zero, we need only to control that the product $\frac{\partial f(\alpha + I \beta)}{\partial_s f(\alpha + I \beta)^{-1}}$ does not belong to $\mathbb{C}^+_I$. Now,
\[
\frac{\partial f}{\partial x}(\alpha + I \beta)(\partial_s f(\alpha + I \beta))^{-1} = (1 - IJ) \left( \frac{\beta - \alpha J}{\beta} \right)^{-1} = \frac{\beta(1 - IJ)(\beta + \alpha J)}{\beta^2 + \alpha^2},
\]
and so, whenever $I \neq -J$, the previous product as a real part and so not belong to $\mathbb{C}^+_I$.

**Remark 7.** The reader could ask why we didn’t follow the way of proving theorem \[42\] by Gentili, Salamon and Stoppato in \[8\]. The answer is that, of course, that proof doesn’t work in the case in which the domain of the function does not have real points. This fact, rather than being a mere observation, give space to interesting considerations that are not studied in this paper. To be precise, the theorem that fails is the following:

**Theorem 43.** Let $f : \Omega_D \to \mathbb{H}$ be a nonconstant regular function, and let $\Omega_D \cap \mathbb{R} \neq \emptyset$. For each $x_0 = \alpha + I \beta \in N_f$, there exists an $n > 1$, a neighborhood $U$ of $x_0$ and a neighborhood $T$ of $\mathbb{S}_{x_0}$ such that for all $x_1 \in U$, the sum of the total multiplicities of the zeros of $f - f(x_1)$ in $T$ equals $n$.

A counter example, if the domain does not has real points, is given by the function,
\[
f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}
\]
\[
\alpha + I \beta \mapsto (\alpha + I \beta)(1 - IJ),
\]
for a fixed $J \in \mathbb{S}$. As we have seen, this function is injective over $\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}^+_J)$, and so, if we take $x_0 = -J \in N_f$, for any neighborhood $U$ of $-J$ and any neighborhood $T$ of $\mathbb{S}_{-J}$ the sum of total multiplicities of the zeros of $f - f(x_1)$, for any $x_1 \in U \setminus \mathbb{C}^+_J$ is equal to 1. The previous function is constructed to be equal to 0 over $\mathbb{C}^+_J$ and equal to $2x$ over $\mathbb{C}^+_J$, but other more complex examples can be build in this way, for example considering a function equal to some monomial $x^m$ on a
semislice and equal to another different monomial $x^n$ on the opposite. This feature will certainly be a starting point for future investigations.

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