ISOSPECTRAL GRAPHS VIA SPECTRAL BRACKETING

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Abstract. In this article, we develop a perturbative technique to construct families of non-isomorphic discrete graphs which are isospectral for the standard (also called normalised) Laplacian and its signless version. We use vertex contractions as a graph perturbation and spectral bracketing with auxiliary graphs which have certain eigenvalues with high multiplicity. There is no need to know explicitly the eigenvalues or eigenfunctions of the corresponding graphs. We illustrate the method by presenting several families of examples of isospectral graphs including fuzzy complete bipartite graphs and subdivision graphs obtained from the previous examples. All the examples constructed turn out to be also isospectral for the standard (Kirchhoff) Laplacian on the associated equilateral metric graph.

1. Introduction

Perturbative techniques in the study of linear Hilbert space operators (or matrices) have a long and fertile history (see, e.g., [K95, B85]). From a general point of view, they provide a method of investigation based on the knowledge of the spectrum of a free (non-perturbed) operator (typically a Laplacian on some domain of \( \mathbb{R}^n \) or on a graph) and the analysis of its stability or variation under a controlled perturbation expressed in terms of a potential or a geometrical manipulation of the underlying spaces (see, e.g., [Se99, LP07]). In [FLP22a] we presented a systematic study of the spectral effects of graph perturbations (like edge deletion, vertex contraction, vertex virtualisation etc.) for arbitrary weights and including arbitrary magnetic potentials. These effects can be quantified in terms of a natural spectral preorder. In this article, we present a perturbative method leading to the construction of isospectral graphs which are interpreted as perturbations (via vertex merging) of auxiliary graphs which have certain eigenvalues with high enough multiplicity.

Spectral graph theory studies the relationship between the structure of a graph and the spectrum of operators naturally associated with the graph which are usually represented by matrices with respect to a suitable orthonormal basis. In this article, we consider only simple finite graphs, i.e., there are no multiple edges nor loops. For a graph \( G = (V, E) \), one can define different natural operators associated with it. Typically, one considers the adjacency matrix \( A_G \) with respect to a numbering of the vertices \( V = \{v_1, \ldots, v_n\} \) having entries 1 and 0 depending on whether there is an edge between two vertices or not. The combinatorial Laplacian is given by \( D_G - A_G \), where \( D_G \) is the diagonal matrix with the degree of each vertex on the diagonal. The signless combinatorial Laplacian is defined by \( D_G + A_G \). Another well known choice is the standard Laplacian (also called geometric or normalised in the literature, cf. Remark 2.3), represented by

\[
\Delta_G = D_G^{1/2}(D_G - A_G)D_G^{-1/2} = I_n - D_G^{-1/2}A_G D_G^{-1/2},
\]

where \( I_n \) is the identity \((n \times n)\)-matrix (see also [Ch97] for additional references and motivation). The signless standard Laplacian is defined as \( \Delta_{G^+} = I_n + D_G^{-1/2}A_G D_G^{-1/2} \) (see also Subsection 2.2). Both standard Laplacians have their spectrum contained in the interval \([0, 2]\).

An important question with many applications is whether the spectrum of a certain operator on a graph determines the graph up to isomorphism or not. Two non-isomorphic graphs having the same spectrum for a certain operator are called isospectral graphs with respect to that operator. Two non-isomorphic graphs such that their adjacency matrices have the same spectrum are often referred to as cospectral cf. e.g., [CDG88], i.e., isospectral with respect to the adjacency matrix in our notation. Often, classical problems related to isospectrality are considered for the combinatorial Laplacian, see e.g., [CDS95, BrH12, DH16]. Much less is known for the standard Laplacian, as most previous works concentrate on the adjacency matrix or the combinatorial Laplacian. We refer in this article to isospectral graphs as isospectral with respect to the standard Laplacian (or its signless version). In recent time, though, results and constructions of isospectral graphs for the standard Laplacian have been considered by several...
authors. In [BuG11], the authors construct from given isospectral bipartite graphs (e.g., unions of complete bipartite graphs) new isospectral graphs. The proof is based on a concrete construction of eigenfunctions of the new graphs from the old ones and checking that all eigenvalues are exhausted. Examples where isospectral graphs with different number of edges are present are constructed in [Bu15], based on so-called twin vertices and simultaneously scaling the edge and vertex weights. Other recent references describing isospectral graphs for the standard Laplacian are given in [DP16, L22, Bu10].

In this article we present a new perturbative technique to construct isospectral graphs of the standard Laplacian (including its signless version). This new philosophy is based on the elementary perturbation of a graph and the control of the effect of this perturbation on some of its eigenvalues. From known multiplicity of certain eigenvalues of some auxiliary graphs (and no knowledge of the corresponding eigenfunctions), we construct isospectral graphs using spectral bracketing, i.e., enclosing unknown eigenvalues between the corresponding eigenvalues of the auxiliary graphs. Concretely, a spectral bracketing of a graph $G$ with another graph $G'$ means that the eigenvalues interlace, i.e., that for some $t \in \mathbb{N}$ the relations

$$\lambda_1(G') \leq \lambda_1(G) \leq \lambda_1+t(G'), \quad \lambda_2(G') \leq \lambda_2(G) \leq \lambda_2+t(G'), \quad \ldots, \quad \lambda_n(G') \leq \lambda_n(G) \leq \lambda_n+t(G')$$

(1.1)

hold, where $n = |G|$ and $n + t = |G'|$ are the orders of $G$ and $G'$, respectively. Symbolically, we denote this as $G' \lesssim G \lesssim t$ for this eigenvalue interlacing, where $t \in \mathbb{N}_0$ denotes the spectral shift in the discrete label counting the eigenvalues that fixes the bracketing, i.e., $\lambda_k(G) \in [\lambda_k(G'), \lambda_k+t(G')]$, $k = 1, \ldots, n$. If $G$ now has an eigenvalue of multiplicity $\mu > t$, then $G'$ also has this eigenvalue with multiplicity at least $\mu - t > 0$. In Remark 3.3 we also give a diagrammatic visualisation of the spectral bracketing method. See also [CDHLP84] for an interlacing result under edge deletion.

This technique is more efficient the smaller the spectral shift $t$ is, since then, the bracketing becomes tighter. Ideally one looks for elementary perturbations of the graph that have $t = 1$ in only one of the spectral relations leading to the classical interlacing of eigenvalues. In this sense, the spectral shift quantifies the spectral cost of the perturbation. This cost depends much on which weights are used on the graph. For example, an edge deletion has a minimal spectral cost for combinatorial weights but not for standard weights while a single vertex contraction is a convenient perturbation of the graph with standard weights but not for combinatorial weights (see [FLP22] Corollaries 4.2 and 4.7). For this reason we will focus in this article on perturbations where the graph $\tilde{G}$ is obtained from $G$ by contracting vertices of $G$ such that the difference of the number of vertices is given by $t = |G| - |\tilde{G}|$ and the spectral bracketing of this operation is given by

$$G \preceq \tilde{G} \preceq G.$$ 

The task is now to find suitable auxiliary graphs $\tilde{G}$ having eigenvalues with high multiplicity such that one can exhaustively determine eigenvalues $\tilde{G}$ by the respective bracketings. This can be done, for example, if the different eigenvalue sets of the auxiliary graphs are disjoint (and hence the corresponding eigenspaces must be orthogonal). If two non-isomorphic graphs $G_1$ or $G_2$ can be sandwiched by the same auxiliary graphs as above and if all eigenvalues are exhausted, then they are isospectral by Theorem 3.7. In the final section we apply this method to different classes of examples where one knows the spectrum of the corresponding auxiliary graphs and, hence, one can also determine the spectrum of the corresponding isospectral graphs.

But the construction technique of isospectral graphs proposed in this article only requires to know the multiplicity of certain eigenvalues of the auxiliary graphs.

This article is structured as follows: Section 2 contains a brief introduction of the notation and results on graphs with standard weights and their Laplacians. In the following section the main tools for the analysis are presented, the spectral bracketing of graphs and vertex contraction as the main geometrical perturbation of the graph. It is proved that vertex contraction has small spectral cost for the standard weights. Theorem 3.7 is the main result in this section and shows how the multiplicity of certain eigenvalues of the auxiliary graphs may lead to the determination of the spectrum of the sandwiched graph. This idea is exploited in Section 4 to present different families of isospectral and non-isomorphic graphs. These examples include fuzzy ball and fuzzy complete graph families. Finally, new examples obtained as subdivision graphs are shown to be isospectral using directly the diagrammatic picture associated to different spectral bracketings. All the examples of isospectral discrete graphs can be naturally turned into equilateral metric graphs which, again, are isospectral for the corresponding (unbounded) self-adjoint standard Laplacian (cf. [BK13]).

2. Discrete graphs and their standard Laplacians

In this section, we introduce briefly the discrete structures and operations needed for the construction for families of isospectral graphs. We will only consider here discrete, simple and unoriented finite graphs together with its
Laplacian and signless Laplacian with standard weights. We refer to [FLP22a] for a general analysis in the context of directed multigraphs with an additional magnetic potential described as a $S^1$-valued function on the edges.

2.1. Discrete graphs. A (simple) graph $G$ consists of a finite vertex set $V$ and an adjacency relation on $V$ (i.e., a symmetric and non-reflexive relation on $V$): two vertices being in this relation are called adjacent and $\{u, v\}$ is called an edge joining $u$ and $v$. The order of a finite graph $G$ is the cardinality of its vertex set which we denote by $|G| := |V|$. We define the neighbourhood of a vertex $v$ as $$N_v := \{ u \in V \mid u \text{ and } v \text{ are adjacent} \}$$ and the degree of $v$ is the number of vertices adjacent with $v$ which we denote as $$\deg v := |N_v|.$$ Since we focus in this article on Laplacians with standard weights we exclude isolated vertices (i.e., vertices of degree 0 or, alternatively, we assume $\deg_G v > 0$ for all $v \in V$). A vertex of degree 1 is called pendant. Let $G$ be a graph with $n$ vertices labelled as $v_1, v_2, \ldots, v_n$ and such that $\deg v_1 \leq \deg v_2 \leq \cdots \leq \deg v_n$. The list $\deg G := (\deg v_1, \deg v_2, \ldots, \deg v_n)$ is called the degree list of $G$ and provides a graph invariant, i.e., if the degree sequence or the degree list of two graphs differ, then the graphs cannot be isomorphic. A graph $G$ is called connected if for any two vertices $u, v$ there exists a path from $u$ to $v$, i.e., there is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$ such that $v_{i-1}$ and $v_i$ are adjacent for $1 \leq i \leq n$ and such that $u = v_0$ and $v = v_n$. If the explicit dependence of the graph $G$ is needed, we write $V = V(G)$ or $\deg^G v$ etc.

A central operation needed in this article is the contraction of vertices (also the name gluing or merging is used in the literature). Recall that the choice of a subset of vertices $V_0 \subset V$ naturally specifies an equivalence relation of vertices. In this article the result of contracting vertices must give, again, a simple graph. We refer to Definition 2.1 and Remark 2.2 [FLP22a] for a general description of contractions in the context of multiple directed graphs.

**Definition 2.1** (Contracting vertices). Let $G$ be a graph with vertex set $V$, consider an equivalence relation $\sim$ on the vertex set $V$ and denote by $[v]$ the class of vertices related with $v$. The quotient graph $G/\sim$ is the graph with vertex set $V/\sim = \{ [v] \mid v \in V \}$ and keeping all edges.

We say that $G/\sim$ is obtained from $G$ by contracting the vertices (with respect to $\sim$). Two vertices $[u]$ and $[v]$ in $G/\sim$ are adjacent if $[u] \neq [v]$ and if some vertex in the class of $[v]$ is adjacent in $G$ to some vertex in the class of $[u]$.

The shrinking number $r$ of the equivalence relation $\sim$ is defined by $r := |G| - |G/\sim| = |V| - |V/\sim|$ and quantifies the reduction of vertices in the quotient graph.

In this article we will only allow contractions that respect the category of simple graphs. In particular, the quotient graph must be simple as well. For example, adjacent vertices in $G$ can not be contracted since the quotient graph is not allowed to have loops in this article.

2.2. Standard Laplacians. For a finite graph $G$, we associate the following natural weighted Hilbert space $L_2(V, \deg) := \left\{ f : V \rightarrow \mathbb{R} \mid \|f\|_{L_2(V, \deg)}^2 = \sum_{v \in V} |f(v)|^2 \deg v \right\}$.

The standard Laplacian is defined as follows (for an approach using discrete exterior derivatives, more general weights and magnetic potentials we refer to [FLP22a] Sec. 3 and references therein):

**Definition 2.2** (standard Laplacian and standard signless Laplacian). The **standard (discrete) Laplacian** of a graph $G$ is the self-adjoint operator $\Delta_G : L_2(V, \deg) \rightarrow L_2(V, \deg)$ defined for $v \in V$ and $f \in L_2(V, \deg)$ by $$(\Delta_G f)(v) = \frac{1}{\deg v} \sum_{u \in N_v} (f(v) - f(u)) = f(v) - \frac{1}{\deg v} \sum_{u \in N_v} f(u). \quad (2.1a)$$

If $G$ has order $n$, then we write the spectrum of the corresponding Laplacian $\Delta_G$ as the list $$\sigma(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)),$$

where the eigenvalues $\lambda_1(G), \ldots, \lambda_n(G)$ are written in ascending order and repeated according to their multiplicities.

Similarly, we define the **signless standard Laplacian** $\Delta_{G^+}$ for $v \in V$ and $f \in L_2(V, \deg)$ by $$(\Delta_{G^+} f)(v) = \frac{1}{\deg v} \sum_{u \in N_v} (f(v) + f(u)) = f(v) + \frac{1}{\deg v} \sum_{u \in N_v} f(u). \quad (2.1b)$$

Note that in [FLP18] and [FLP22a] Sec. 5.3] we reserved the superscript $^+$ to denote Dirichlet Laplacians. Since in this article we do not need the operation of vertex virtualisation we use the superscript $^+$ in a different way: the results in the last section can be written in a much more convenient way using $^-$ for the Laplacian and $^-$ for the signless Laplacian.
Its eigenvalues are denoted by \( \sigma(G^+) = (\lambda_1(G^+), \lambda_2(G^+), \ldots, \lambda_n(G^+)) \).

We refer to \( \text{LP08a} \) for a description of the unoriented homology related to the signless Laplacian and to \( \text{FLP22a} \) Sec. 2.4 and Sec. 3.1] for the use of the magnetic potential as an interpolation parameter connecting both Laplacians. More applications of the magnetic potential to analyse combinatorial properties of the graph are given in \( \text{FLP22b} \).

**Remark 2.3.**

(a) Some authors call the Laplacian with degree weight geometric, as some results referring to this Laplacian are closer to the Laplacian on a manifold or a metric graph. Sometimes, the standard Laplacian is also called normalised, but we find that this name should better be reserved for edge weights \( w_e \) with edge \( e = \{u, v\} \) and associated vertex weights \( \text{deg}_e^v := \sum_{u \in N_v} w_{(u, v)} \).

(b) Recall that the spectra of the previous Laplacians is contained in the interval \([0, 2]\), i.e., \( \sigma(G^+) \subset [0, 2] \). Moreover, we always have \( 0 \in \sigma(G) \), and 0 is simple if and only if \( G \) is connected. Similarly, we have \( 2 \in \sigma(G^+) \). The eigenfunction of both eigenvalues is a constant function on \( V \) as one easily sees from \( 2.1a \) and \( 2.1b \).

(c) With respect to the orthonormal basis \( \{ \delta_v | v \in V \} \) with \( \delta_v(u) = (\text{deg} v)^{-1/2} \) if \( u = v \) and \( \delta_v = 0 \) if \( u \neq v \), the standard Laplacian denoted with \((-)\) as \( \Delta_G = \Delta^{-} \) and the standard signless Laplacian denoted with \((+)\) as \( \Delta_G^{+} \) are represented by the matrices with entries

\[
(\Delta_G^{\pm})_{u,v} = \begin{cases} 1, & u = v, \\ \mp((\text{deg} u)(\text{deg} v))^{-1/2}, & u \text{ and } v \text{ are adjacent}, \\ 0, & \text{otherwise}. \end{cases}
\]

3. **Spectral bracketing and graph perturbations**

We present in this section our original idea of how to construct isospectral graphs based on certain graph perturbations whose spectral effect we can control. We will sandwich the original graphs between two auxiliary graphs with high symmetry which will specify the spectral bracketing. Our method here is based on the concept of “spectral preorder” in relation to vertex contraction (see also \( \text{FLP22a} \) Section 4) for a general spectral analysis of elementary perturbations of multidigraphs).

### 3.1. Spectral preorder.

In this section we introduce a spectral preorder in the class of simple graphs with standard weights based on the order relation of consecutive lists of eigenvalues of the corresponding Laplacians written in increasing order and repeated according to their multiplicities. From an algebraic viewpoint the spectral preorder is a quite flexible generalisation of the eigenvalue interlacing known for matrices (see e.g. \( \text{Haji13} \) Theorem 4.3.28) and \( \text{BrH12} \) Section 2.5 and 3.2). This relation can be generalised to include a shift \( t \in \mathbb{N}_0 \) in the list of eigenvalues. We refer to \( \text{FLP18} \) \text{FLP22a} for proofs and additional motivation. We will use here this preorder to control the spectral effect of vertex contraction.

**Definition 3.1** (Spectral preorder \( \preceq \)). Let \( G \) and \( G' \) be two graphs and denote the corresponding increasing lists of the eigenvalues of the respective standard Laplacians, repeated according to their multiplicity, by

\[
(\lambda_1(G), \lambda_2(G), \ldots, \lambda_{|G|}(G)) \quad \text{and} \quad (\lambda_1(G'), \lambda_2(G'), \ldots, \lambda_{|G'|}(G')).
\]

We say that \( G \) is spectrally smaller than \( G' \) with shift \( t \in \mathbb{N}_0 \), and we denote this by

\[
G \preceq_t G',
\]

if the following two conditions hold:

\[
|G| \geq |G'| - t \quad \text{and} \quad \lambda_k(G) \leq \lambda_{k+t}(G') \quad \text{for all} \quad 1 \leq k \leq |G'| - t.
\]

If \( t = 0 \) we write simply \( G \preceq G' \).

The same definition applies to the signless Laplacian, i.e., we write \( G^+ \preceq_t (G')^+ \) if the eigenvalues of the signless Laplacians fulfill \( \lambda_k(G^+) \leq \lambda_{k+t}((G')^+) \) for all \( 1 \leq k \leq |G'| - t \).

**Remark 3.2** (spectral preorder visualised). We present here the following useful diagrammatic representation of the relation \( G \preceq G' \).

- For each eigenvalue \( \lambda_i \) of \( G \), we draw a box below the corresponding eigenvalue \( \lambda_i' \) of \( G' \). This indicates that \( \lambda_i \leq \lambda_{i+t} \) for some shift \( t \).

\[
\begin{array}{cccccccc}
\lambda_1 & \lambda_2 & \ldots & \lambda_{n-1} & \lambda_n \\
G' & \lambda_1 & \ldots & \lambda_{n-1+t} & \lambda_n \\
G & \lambda_1 & \ldots & \lambda_{n-1} & \lambda_n \\
\end{array}
\]
We list the eigenvalues $\lambda_k = \lambda_k(G)$, $n = |G|$ and $\lambda'_k = \lambda_k(G')$ of $G$ and $G'$ in the first and second row, respectively. The condition $|G| \geq |G'|-t$ just means that the last box of the second row can not exceed the last box of the first row. Moreover, eigenvalues in the same column increase when going down the column and one can easily visualise the spectral relation just looking at the successive columns.

Let $G$, $G'$ and $G''$ be finite graphs. We denote next some useful conventions and easy consequences of the preceding definition. We write $G \preceq G'$ and $G' \preceq G''$ simply by $G \preceq G' \preceq G''$, and call such estimates spectral bracketing or eigenvalue interlacing.

Remark 3.3 (spectral bracketing visualised). Later we will often consider the graph sandwiching $G' \preceq G \preceq G''$ which implies that $|G| + t = |G'|-t \geq |G'|$. If $|G| = n$, then $G' \preceq G \preceq G''$ is equivalent with the interlacing of the spectra corresponding to the Laplacians of $G$ and $G'$ similarly as in [BH12, Section 2.5], i.e., (1.1) holds. Using the diagrammatic representation of Remark 3.2, we write

$$G'$$

| $\lambda_1'$ | $\lambda_2'$ | $\lambda_3'$ | $\ldots$ | $\lambda_{n-1}'$ | $\lambda_n'$ | $\lambda'_{n+1}$ | $\ldots$ | $\lambda'_{n+t}$ |
|---|---|---|---|---|---|---|---|---|

$G$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\ldots$ | $\lambda_{n-1}$ | $\lambda_n$ | $\lambda'_{n+1}$ | $\ldots$ | $\lambda'_{n+t}$ |

$G'$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\ldots$ | $\lambda_{n-1}$ | $\lambda_n$ | $\lambda'_{n+1}$ | $\ldots$ | $\lambda'_{n+t}$ |

The condition $|G| = |G'|-t$ means here that the second and third columns are aligned on the right hand side. In particular, if $t = 1$ it becomes the usual eigenvalue interlacing (justifying also the name)

$$\lambda_1(G') \leq \lambda_1(G) \leq \lambda_2(G') \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G') \leq \lambda_n(G) \leq \lambda_{n+1}(G').$$

The following result is a direct consequence of the definition of spectral preorder and quantifies the stability of eigenvalues with high multiplicity under vertex contraction perturbation.

Proposition 3.4. Consider two graphs $G, G'$ satisfying the spectral relation $G' \preceq G \preceq G''$. If $\lambda$ is an eigenvalue of the Laplacian on $G'$ with multiplicity $\mu > t' + t''$, then $\lambda$ is an eigenvalue of the Laplacian on $G$ with multiplicity at least $\mu - (t' + t'')$.

3.2. Spectral bracketing and vertex contraction. The following result will be central to the class of examples considered in the next section. An important observation for our method is that multiple eigenvalues remain eigenvalues in the quotient graph provided the shrinking number of the quotient is small enough (cf. Definition 2.1). We present next vertex contraction as a perturbation with low spectral cost for standard weights; see Theorem 3.14 in [LP22a] for a proof.

Proposition 3.5 (contracting vertices and spectral bracketing). Let $G$ be a finite discrete graph. If $\tilde{G} = G/\sim$ is a graph obtained from $G$ by contracting vertices according to an equivalence relation $\sim$ on $V(G)$, then we have the following spectral relation between the corresponding standard Laplacians

$$G \preceq \tilde{G} \preceq G,$$

where $t = |G| - |\tilde{G}| \geq 0$ is the shrinking number of $\sim$ (cf. Definition 2.1). The same result is true for the signless Laplacians, i.e., we have $G^+ \preceq \tilde{G}^+ \preceq G^+$.

Note that Proposition 3.5 can be immediately applied to the graph perturbation introduced before. In fact, the total spectral shift $t$ is the shrinking number of the relation $\sim$ (cf. Proposition 3.5).

3.3. Spectral bracketing and determination of the spectrum. We tacitly understand subsets $\Lambda$ of a spectrum $\sigma(G)$ as multisets without formally introducing them. A simple workaround is to think of $\Lambda$ as an ordered list $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and multiple members are repeated according to their multiplicity. The notation $\lambda \in \Lambda$ then means that there is $j \in \{1, \ldots, n\}$ such that $\lambda = \lambda_j$. Similarly, we understand $\Lambda' \subset \Lambda$, i.e., each $\lambda \in \Lambda'$ appearing $\mu$ times appears at least $\mu$ times also in $\Lambda$.

Definition 3.6 (multiplicity and subsets). We say that a subset of non-negative numbers $\Lambda$ has multiplicity $\mu$ in $\sigma(G)$ if each element $\lambda \in \Lambda$ having multiplicity $s$ in $\Lambda$ has at least multiplicity $s\mu$ in $\sigma(G)$. We denote this relation by $\Lambda^{(\mu)} \subset \sigma(G)$.

Next, we present the key result on which we base our construction of isospectral graphs in Section 4. Namely, we specify conditions that guarantee that the spectrum of a graph $G$ is obtained by spectrally bracketing the original graph with the auxiliary graphs $G_j$ having eigenvalues of high multiplicity.
Theorem 3.7. Let $J$ be a finite index set and consider for each $j \in J$ the finite set of non-negative numbers $\Lambda_j \subset [0,2]$. Assume that the graphs $G$ and $G_j \\ (j \in J)$ with standard Laplacians satisfy the following conditions:

(a) $G_j \preceq G \preceq G_j$, $j \in J$ and denote the total spectral shift by $t_j := t'_j + t''_j$;

(b) $\Lambda_j$ has multiplicity $\mu_j$ in $\sigma(G_j)$, $j \in J$, and $\mu_j > t_j = t'_j + t''_j$;

(c) the sets $\Lambda_j$, $j \in J$ are pairwise disjoint;

(d) the order of $G$ is

$$|G| = \sum_{j \in J} (\mu_j - t_j)|\Lambda_j|.$$  \hspace{1cm} (3.1)

Then the spectrum of $G$ is determined by the subsets $\Lambda_j$, namely we have $\sigma(G) = \biguplus_{j \in J} \Lambda_j^{(\mu_j - t_j)}$, where $\biguplus$ means the disjoint union of multisets.

Proof. From Proposition 3.4 and conditions (3.1) above we conclude that each $\Lambda_j$ has multiplicity at least $(\mu_j - t_j)$ in $\sigma(G)$, where the total shift is $t_j := t'_j + t''_j$. We denote this relation as $\Lambda_j^{(\mu_j - t_j)} \subset \sigma(G)$ (as multiset). Moreover, since $\Lambda_j \cap \Lambda_{j'} = \emptyset$ for all $j, j' \in J$ with $j \neq j'$ condition (c) implies that an eigenvalue of $G$ is contained in only one of the sets $\Lambda_j$. Finally, condition (d) guarantees that all eigenvalues of $G$ are necessarily in one of the sets $\Lambda_j$ and, therefore, the spectrum is exhausted. \hfill \Box

Remark 3.8.

(a) The disjointness condition (c) of Theorem 3.7 is necessary to guarantee that one obtains all different eigenvalues, i.e., that eigenvalues from different sets $\Lambda_j$ and $\Lambda_{j'}$ for $j, j' \in J$ with $j \neq j'$ correspond to different eigenfunctions.

(b) Sometimes the condition (d) can be relaxed. For example, recall that 0 (resp. 2) is always an eigenvalue of the Laplacian (resp. signless Laplacian) with constant eigenfunction; also, some eigenvalues may possibly be known because the graph is bipartite etc.

- It is enough that (3.1) adds up to $n - 1$, where we put $n := |G|$. In fact, the missing eigenvalue can be recovered from the remaining ones by the following trace relation for the the standard or signless Laplacians

$$\sum_{j=1}^{n} \lambda_k(G) = \text{tr} \Delta_G = n.$$  

The latter equality follows from the fact that the matrix representation of $\Delta_{G^\pm}$ has only 1 on its diagonal, cf. (2.2).

- Any known eigenvalues of $G$ can be put in a (multi-)set $\Lambda_{j_0}$ for some index $j_0 \in J$. Choosing $G_{j_0} = G$, $t'_{j_0} = t''_{j_0} = 0$ and $\mu_{j_0} = 1$, Theorem 3.7 can formally applied and only $|G| - |\Lambda_{j_0}|$ eigenvalues need to be recovered. We apply this idea sometimes to the eigenvalue 0 or to $\{0,2\}$ if the graph is bipartite. Recall finally, that if $G$ is bipartite, one can also use the fact that $\lambda \in \sigma(G)$ if and only if $2 - \lambda \in \sigma(G)$ (cf. [Cl97] Lemma 1.8 or [LP08a] Proposition 2.3)).

4. A perturbative construction of isospectral graphs

We base our discrete perturbative approach (spectral bracketing technique) on the result presented in Theorem 3.7 and using auxiliary graphs having suitable symmetry. Note that in the following classes of examples of isospectral graphs (see e.g. Theorem 4.2 we can determine the spectrum explicitly due to the fact that we know the spectra of the auxiliary graphs. But our technique only requires to know that the multiplicity of certain eigenvalues of the auxiliary graphs is high enough to guarantee isospectrality. Our bracketing method is, in this sense, based on multiplicities of certain eigenvalues and not on explicit computation of the spectrum or eigenfunctions of the auxiliary graphs. In [FLP22c] we develop a new geometrical construction of families of isospectral magnetic graphs. This method starts with several copies of an arbitrary building block graph $G$ which are glued following a specific pattern. To prove isospectrality in the mentioned reference we lift eigenfunctions on $G$ and, also, eigenfunctions with certain Dirichlet conditions on the merged vertices to the constructed graph. We mention that the perturbative method developed in this article can also be applied to alternatively prove isospectrality of some families of graphs constructed in [FLP22c].

Our construction of non-isomorphic graphs in some examples is based on the combinatorial notion of partition of length $s \in \mathbb{N}$ of a given natural number $r \in \mathbb{N}$ (which we call an $s$-partition of $r$ for short). Recall that an $s$-partition of $r$ is given by a multiset $A = \{a_1, a_2, \ldots, a_s\}$ (i.e. taking multiplicities into account) with $a_i \in \mathbb{N}$ and $a_1 + \cdots + a_s = r$. We will show that different $s$-partitions of $r$ lead to isospectral non-isomorphic graphs. As an
example, the smallest natural number having two different partitions is \( r = 4 \): the length is \( s = 2 \) and the partitions are \( \{1,3\} \) and \( \{2,2\} \) (since \( 4 = 2 + 2 = 1 + 3 \)).

4.1. The fuzzy ball construction. We revisit and generalise the fuzzy ball construction of [BuG11 Section 4], proof isospectrality using spectral bracketing technique and without computing eigenfunctions. Let \( K_r \) be the complete graph with \( r \) vertices and denote by \( \hat{K}_r \) the graph with \( 2r \) vertices obtained from \( K_r \) by attaching a pendant edge to each vertex of \( K_r \).

We first calculate the eigenvalues with high multiplicity since we only need this information to prove isospectrality of the family of graphs in Theorem 4.2.

**Lemma 4.1** (Spectrum of auxiliary graphs).

(a) If \( K_r \) is the complete graph with \( r \) vertices, then

\[
1 \pm \frac{1}{r-1} \in \sigma(K_r^+) \]

has multiplicity \( r - 1 \) in the spectrum of the (signless) standard Laplacian (here and in the sequel, the upper sign is for the standard Laplacian, the lower sign for the signless Laplacian).

(b) Let \( \hat{K}_r \) be the complete graph with a pendant edge decoration at each vertex of \( K_r \). Then

\[
\frac{2r \pm 1 - \sqrt{4r+1}}{2r}, \quad \frac{2r \pm 1 + \sqrt{4r+1}}{2r} \in \sigma(\hat{K}_r^+) \]

with multiplicity \( r - 1 \).

**Proof.** Part (a) is standard. To show part (b) we use a factorisation result of characteristic polynomials in [Hey19]. In fact, the decorated complete graph \( \hat{K}_r \) can be seen as a rooted product of \( K_r \) with edges \( K_2 \) on each vertex. Since \( K_r \) a is simple \((r-1)\)-regular graph and since \( K_2 \) has degree 1 we obtain from Corollary 1 of [Hey19] after some straightforward manipulations that the spectrum is as claimed in (b).

\[\Box\]

**Figure 1.** The figures above correspond to the auxiliary graph \( \hat{K}_6 \), the isospectral graphs \( \hat{K}_6(A) \) and \( \hat{K}_6(B) \) for \( A = (2,2,2) \) and \( B = (1,2,3) \), and finally the auxiliary graph \( K_7 \). Note that \( \hat{K}_6(A) \) and \( \hat{K}_6(B) \) are obtained from \( \hat{K}_6 \) by contracting the pendant vertices according to \( A \) and \( B \), and that \( K_7 \) is obtained from either \( \hat{K}_6(A) \) or \( \hat{K}_6(B) \) by contracting all formerly pendant vertices into one vertex.

We now construct the first example class (the so-called fuzzy balls). Note that we can also treat the signless version without any modification of the proof and that the signless spectrum cannot be calculated from the unsigned one as the graphs are not regular. We start with the decorated complete graph \( \hat{K}_r \). Let \( A = \{ a_1, \ldots, a_s \} \) be an \( s \)-partition of \( r \in \mathbb{N} \) and label the pendant vertices as \( \{ v_1, \ldots, v_r \} \). Consider the equivalence relation \( \sim_A \) which contracts the set of vertices \( \{ v_1, \ldots, v_{a_1} \}, \{ v_{a_1+1}, \ldots, v_{a_1+a_2} \}, \ldots, \{ v_{a_1+\ldots+a_{s-1}+1}, \ldots, v_r \} \) into \( s \) vertices and denote the corresponding quotient graph by \( \hat{K}_r(A) = G/\hat{K}_r/\sim_A \). In order to have two different \( s \)-partitions of \( r \) we necessarily need \( r \geq 4 \) and \( s \in \{ 2, \ldots, r-1 \} \).

**Theorem 4.2** (“The fuzzy ball theorem”). Assume that \( A \) and \( B \) are two different \( s \)-partitions of \( r \). Then the graphs \( \hat{K}_r(A) \) and \( \hat{K}_r(B) \) constructed above are non-isomorphic and isospectral for the standard and signless standard Laplacian.
Proof. We apply Theorem 5.7 and show that the spectrum of $G := \tilde{K}_r(A)$ depends only on $r$ and the length $s$ of the partition. We choose as first bracketing graph the decorated complete graph $G_1 = \tilde{K}_r$ and as spectral set

$$\Lambda_1 = \left\{ \frac{2r \pm 1 - \sqrt{4r + 1}}{2r}, \frac{2r \pm 1 + \sqrt{4r + 1}}{2r} \right\}.$$

From Lemma 4.1 we obtain $\Lambda_1 \cap (r-1) \subset \sigma(G^+)$ and hence, $\mu_1 = r - 1$. Since $\tilde{K}_r(A) = \tilde{K}_r / \sim_A$ has shrinking number $t_1 := r - s$ we obtain from Proposition 5.3 that

$$\tilde{K}_r^\top \lesssim \tilde{K}_r(A)^\top \lesssim \tilde{K}_r^\top.$$

For the second bracketing graph we consider $G_2 = K_{r+1}$ and the spectral set $\Lambda_2 := \{1 \pm \frac{1}{r}\}$ which has multiplicity $\mu_2 := r$ in $\sigma(G^+_2)$. Note that $K_{r+1}$ can be obtained from $\tilde{K}_r(A)$ by contracting the $s$ vertices obtained by the contraction of the pendant vertices of $\tilde{K}_r$ given by the relation $\sim_A$. The shrinking number of this process is $t_2 = s - 1$ and hence $\tilde{K}_r(A)^\top \lesssim K_{r+1}^\top \lesssim G_{r+1}(A)^\top$, or, equivalently,

$$K_{r+1}^\top \lesssim \tilde{K}_r(A)^\top \lesssim K_{r+1}^\top.$$

It is clear that the spectral subsets $\Lambda_1$ and $\Lambda_2$ are disjoint. Moreover, we have already detected

$$(\mu_1 - t_1)|\Lambda_1| + (\mu_2 - t_2)|\Lambda_2| = \frac{(r - 1) - (r - s)}{s - 1} \cdot 2 + \frac{(r - (s - 1)) \cdot 1}{r - s + 1} = r + s - 1 = |\tilde{K}_r(A)| - 1$$

eigenvalues, hence the spectrum of the Laplacian $\Delta_{\tilde{K}_r(A)}$ is determined by Theorem 5.7 and Remark 5.8 (b).

Since the values in $\Lambda_1$ and $\Lambda_2$ as well as the multiplicities depend only on $r$ and the length $s$ and not on the concrete partition $A$ or $B$ we obtain $\sigma(\tilde{K}_r(A)^\top) = \sigma(\tilde{K}_r(B)^\top)$ as claimed.

Finally, the degree lists of $\tilde{K}_r(A)$ and $\tilde{K}_r(B)$ are given respectively by

$$(r^{(r)}) \not\subset A \quad \text{and} \quad \{r^{(r)}\} \not\subset B,$$

where the union of multisets takes multiplicities into account. Since the partitions $A$ and $B$ are different we conclude that the corresponding graphs cannot be isomorphic. 

\[\square\]

**Corollary 4.3.** The spectrum of $\tilde{K}_r(A)^\top$ and $\tilde{K}_r(B)^\top$ is given by

$$\sigma(\tilde{K}_r(A)^\top) = \sigma(\tilde{K}_r(B)^\top) = \left\{ \frac{2r \pm 1 - \sqrt{4r + 1}}{2r}, \frac{2r \pm 1 + \sqrt{4r + 1}}{2r} \right\}^{(s-1)} \cup \{1 \pm \frac{1}{r}\}^{(r-s+1)} \cup \{\lambda^\pm\},$$

where powers $(\cdot)^{(m)}$ indicate multiplicity. The remaining eigenvalue is $\lambda^\pm = 1 \pm 1$, i.e., $0$ for the standard and $2$ for the signless Laplacian.

For the next result we need to mention the notion of metric graph. Recall that any discrete graph $G$ has an associated equilateral metric graph which we denote here by $G$ (see LP08a, BK13, KM21 and references cited therein). Recall that the standard Laplacian (sometimes also called “Kirchhoff”) is an unbounded second order differential operator on the metric graph with standard conditions on the vertices (the sum of the in-derivatives equals the sum of the out-derivatives) which guarantee that the operator is self-adjoint.

We observe that the construction of isospectral graphs as in Theorem 4.2 produces also isospectral equilateral metric graphs for the corresponding Kirchhoff Laplacians. The proof is based on a beautiful relation between the spectra of the standard Laplacian and the Kirchhoff Laplacian (see, e.g. BK13, LP08a and references therein).

**Corollary 4.4.** Let $A$ and $B$ be two different $s$-partitions of $r$ and denote by $\tilde{K}_r(A)$ and $\tilde{K}_r(B)$ the equilateral metric graph associated with the discrete graphs $\tilde{K}_r(A)$ and $\tilde{K}_r(B)$. Then the graphs $\tilde{K}_r(A)$ and $\tilde{K}_r(B)$ are non-isomorphic and isospectral for the standard metric Laplacian.

**Sketch of the proof.** We prove in FLP22c that $G_1$ and $G_2$ are isospectral with respect to the standard metric Laplacian if and only if $G_1$ and $G_2$ are isospectral with respect to the standard discrete Laplacian and if $G_1$ and $G_2$ have the same number of edges. A similar statement is stated in KM21 Proposition 1. (Note that isospectrality guarantees that the number of vertices is the same and, therefore, preservation of the number of edges guarantees that the corresponding Betti numbers are also the same.) Note that this follows for eigenvalues $\lambda \in (0, 2)$ by e.g. LP08a Proposition 4.1, 4.7 and 5.2. Recall that the bipartiteness of a graph can be seen from the spectrum of the standard discrete Laplacian (a connected graph is bipartite if and only if 2 is in its spectrum).

The claim on isospectral metric graphs follows from the fact that in all our constructions of isospectral graphs using vertex contraction, the number of edges is the same. 

\[\square\]
Remark 4.5. Note that the splitted complete graphs $K_3$ in Figure 1 of [KM21] fall into the class of fuzzy balls. In fact, they correspond to $\tilde{K}_4(A)$ and $\bar{K}_4(B)$ with partitions of $r = 4$ given by $A = (2, 2)$ and $B = (1, 3)$. Therefore the corresponding equilateral metric graphs $\overline{K}_4(A)$ and $\overline{K}_4(B)$ will also be isospectral for the corresponding Kirchhoff Laplacian.

Example 4.6. We illustrate here in a concrete example with $r = 6$ and $s = 3$ (cf. Figure 1) how to apply directly and with the help of diagrams the spectral bracketing technique developed in the proof Theorem 4.2. Recall that the lowest eigenvalue $0$ is always included in all spectra.

The first bracketing here arises from

$$\tilde{K}_6 \leq \tilde{K}_6(A) \leq \tilde{K}_6,$$

with shrinking number $t_1 := r - s = 3$. Hence, for the eigenvalues we have

\[
\begin{array}{c|cccccccc}
K_6 & 0 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & \ast & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 \\
K_6(A) & 0 & 2/3 & 2/3 & ? & ? & ? & ? & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 \\
K_6 & 0 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & \ast & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 \\
\end{array}
\]

where the simple eigenvalue $\ast = \frac{7}{6}$ of $K_6$ is useless for this bracketing and the question marks are the four eigenvalues that still need to be determined. Note that $2/3$ and $3/2$ are eigenvalues of $\tilde{K}_6$ both with multiplicity $\mu_1 = (r-s) = 5 - (6 - 3) = 2$ and determine through enclosure four eigenvalues of $\tilde{K}_6(A)$ (light grey in the diagram).

To determine the remaining eigenvalues we consider a second bracketing by choosing $G_2 = K_s$ and the spectral set $\Lambda_2 := \{\frac{7}{6}\}$ which has multiplicity $\mu_2 := 6$ in $\sigma(G_2)$. The shrinking number is $t_2 = 2$ and hence

$$\tilde{K}_7 \leq \tilde{K}_6(A) \leq \tilde{K}_7.$$

These relations can be represented diagrammatically for the eigenvalues as

\[
\begin{array}{c|cccccccc}
K_7 & 0 & 2/3 & 2/3 & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} \\
\tilde{K}_6(A) & 0 & 2/3 & 2/3 & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} \\
K_7 & 0 & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} \\
\end{array}
\]

Therefore the remaining four eigenvalues are equal to $\frac{7}{6}$ since multiplicity is given by $\mu_2 = (s - 1) = 6 - (3 - 1) = 4$ (dark grey). Altogether we have that both bracketings determine the complete spectrum of $\tilde{K}_6(A)$:

\[
\begin{array}{c|cccccccc}
0 & 2/3 & 2/3 & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} & \frac{7}{6} \\
\end{array}
\]

Note that in the reasoning only $r = 6$ and the length of the partition $s = 3$ matter and so the two different partitions $A = (2, 2, 2)$ and $B = (1, 2, 3)$ lead to isospectral graphs $\tilde{K}_6(A)$ and $\overline{G}^-\tilde{K}_6(B)$. Moreover, the corresponding degree lists

$$\text{deg } \tilde{K}_6(A) = (2, 2, 2, 6, 6, 6, 6, 6) \quad \text{and} \quad \text{deg } \tilde{K}_6(B) = (1, 2, 3, 6, 6, 6, 6, 6)$$

are different and, hence, the graphs are not isomorphic.

Finally, for the signless Laplacian one can reason similarly choosing the same auxiliary graphs and replacing the eigenvalues $0$ by $2$ (constant eigenfunction), $2/3$ by $1/2$, $3/2$ by $1/3$ and $7/6$ by $5/6$, respectively.

4.2. The fuzzy complete bipartite construction. In this section we will show a different class of examples leading to isospectral graphs. Instead of following the lemma-theorem-structure of the preceding subsection we will directly reason using spectral diagrams in concrete examples that can be generalised in an obvious way.

For $p, r \in \mathbb{N}$, let $\tilde{K}_{p,r}$ be the complete bipartite graph with $p + r$ vertices and denote by $\tilde{K}_{p,r}$ the graph with $p + 2r$ vertices obtained from $K_{p,r}$ by attaching a pendant edge to each of the $r$ vertices in the partition of $K_{p,r}$. Note since the resulting classes of graphs in this subsection will be bipartite there is no need to distinguish between Laplacian and signless Laplacian as they have the same spectrum. This is due to the fact that on a bipartite graph, the signatures 1 and $-1$ on each edge are gauge-equivalent (see e.g. [FLP18] Sec. 3.1]. We now construct the isospectral graphs from an $s$-partition $A$ of the natural number $r$, i.e., $A = (a_1, \ldots, a_r)$. We start with the decorated complete bipartite graph $\tilde{K}_{p,r}$ (decorated with $r$ pendant edges numbered by $\{v_1, \ldots, v_r\}$). Let $A = (a_1, \ldots, a_s)$ be an $s$-partition of $r \in \mathbb{N}$. Consider the equivalence $\sim_A$ which contracts the set of vertices $\{v_1, \ldots, v_{a_1}\}, \{v_{a_1+1}, \ldots, v_{a_1+a_2}\}, \ldots, \{v_{a_1+\cdots+a_{r-1}+1}, \ldots, v_r\}$ into $s$ vertices and denote the corresponding quotient graph by $\tilde{K}_{p,r}(A) = \tilde{K}_{p,r}/\sim_A$.

To illustrate the bracketing technique directly in terms of diagrams we will consider a concrete example of this class by choosing the case $p = 2$, $r = 5$ and $s = 2$ (see Figure 2). The method works similarly for other cases of in this class.
For the first bracketing we use \( G_1 = \hat{K}_{2,5} \) as auxiliary graph which, by construction, give the relations

\[
\hat{K}_{2,5} \cong \hat{K}_{2,5}(A) \cong \hat{K}_{2,5}.
\]

with shrinking number \( t_1 := r - s = 3 \). Diagrammatically, the eigenvalues satisfy

\[
\begin{array}{cccccccc}
\hat{K}_{2,5} & 0 & 1/2 & 1/2 & 1/2 & 1/2 & * & * & 3/2 & 3/2 & 3/2 & 3/2 & 2 \\
\hat{K}_{2,5}(A) & 0 & 1/2 & ? & ? & ? & ? & ? & 3/2 & 2 \\
\hat{K}_{2,5} & 0 & 1/2 & 1/2 & 1/2 & * & * & 3/2 & 3/2 & 3/2 & 2 \\
\end{array}
\]

where now the double eigenvalue \(* = 1\) is irrelevant for this bracketing. Note that we are able to enclose the two eigenvalues \( 1/3 \) and \( 3/2 \), both with multiplicity \( \mu_1 = (r - s) = 4 - (6 - 3) = 1 \) (light grey in the diagram).

To determine the remaining eigenvalues we consider a second bracketing with the auxiliary graph \( G_2 = K_{3,5} \). Note that the eigenvalue \( 1 \) has multiplicity \( \mu_2 := (p + 1) + r - 2 = 6 \) in \( \sigma(G_2) \) and the shrinking number is \( t_2 = s - 1 = 1 \). Therefore we obtain the spectral relations

\[
K_{3,5} \cong \hat{K}_{2,5}(A) \cong K_{3,5}.
\]

which diagrammatically can be represented by

\[
\begin{array}{cccccccc}
K_{3,5} & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
\hat{K}_{2,5}(A) & 0 & 1/2 & 1 & 1 & 1 & 1 & 3/2 & 2 \\
K_{3,5} & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\
\end{array}
\]

We have now fixed the eigenvalue \( 1 \) with multiplicity \( \mu_2 = t_2 = 6 - 1 = 5 \) (dark grey in the diagram).

Altogether, the spectrum of \( \hat{K}_{2,5}(A) \) is given by

\[
\begin{array}{cccccccc}
0 & 1/2 & 1 & 1 & 1 & 1 & 1 & 3/2 & 2 \\
\end{array}
\]

Again, the reasoning only depends on the values \( p = 2, r = 6 \) and the length of the partition \( s = 2 \) and so the two different partitions \( A = \{2, 3\} \) and \( B = \{1, 4\} \) lead to isospectral graphs \( \hat{K}_{2,5}(A) \) and \( \hat{K}_{2,5}(B) \). Since their degree lists

\[
\deg \hat{K}_{2,5}(A) = (2, 3, 3, 3, 3, 3, 3, 5) \quad \text{and} \quad \deg \hat{K}_{2,5}(B) = (1, 3, 3, 3, 3, 4, 5, 5)
\]

are different the graphs are not isomorphic. To make contact with the main result in Theorem 3.7 recall that in the two bracketings used in this class of examples we have for the first auxiliary graph \( G_1 = \hat{K}_{2,5} \) the spectral set \( \Lambda_1 = \{1/2, 3/2\} \) with multiplicity \( \mu_1 = 5 - 1 = 4 \) and for the second auxiliary graph \( G_2 = K_{3,5} \) the spectral set is \( \Lambda_2 = \{1\} \) with \( \mu_2 = 7 - 1 = 6 \).

As in Corollary 4.4 we can extend isospectrality to the metric graph scenario. Let \( A \) and \( B \) are two different \( s \)-partitions of \( r \) and denoting by \( \overline{K}_{p,r}(A) \) and \( \overline{K}_{p,r}(B) \) the equilateral metric graph associated with the discrete graphs \( \hat{K}_{p,r}(A) \) and \( \hat{K}_{p,r}(B) \). Then the graphs \( \overline{K}_{p,r}(A) \) and \( \overline{K}_{p,r}(B) \) are non-isomorphic and isospectral for the Kirchhoff Laplacian.
4.3. Edge subdivision. For presenting the following class of examples we first need to recall an important operation on a graph $G = (V,E)$. The edge subdivision of $e = \{u,v\} \in E$ consists in the deletion of $e = \{u,v\}$ from $G$, the addition of a new vertex $w$ and two new edges $e_1 = \{u,w\}$ and $e_2 = \{w,v\}$. We denote by $S(G)$ the graph where all edges of $G$ are subdivided. Let $\tilde{K}_5$ be the complete graph with a pendant edge decoration at each vertex of $K_5$ as in Figure 1 and consider the corresponding edge subdivision graph shown in Figure 3. Since all graphs involved are bipartite and connected we will always have 0 and 2 as simple eigenvalues in the corresponding spectra.

![Figure 3](image)

**Figure 3.** The figures above correspond to the auxiliary graph $S(\tilde{K}_6)$, the isospectral graphs $S(\tilde{K}_5)(A)$ and $S(\tilde{K}_5)(B)$ for $A = (2,3)$ and $B = (1,4)$, and finally the auxiliary graph $S(K_6)$. Note that $S(\tilde{K}_6)(A)$ and $S(\tilde{K}_6)(B)$ are obtained from $S(\tilde{K}_5)$ by contracting the pendant vertices according to $A$ and $B$, and that $S(K_6)$ is obtained from either $S(\tilde{K}_5)(A)$ or $S(\tilde{K}_5)(B)$ by contracting all formerly pendant vertices once again into one vertex.

We will show using spectral diagrams that $S(\tilde{K}_5)(A)$ and $S(\tilde{K}_5)(B)$ are isospectral. As before we consider the first graph and to determine the nontrivial 20 eigenvalues of $S(\tilde{K}_5)(A)$ we begin with the auxiliary graph $S(\tilde{K}_5)$. By construction we have the spectral relations

$$S(\tilde{K}_5) \preceq S(\tilde{K}_5)(A) \preceq S(\tilde{K}_5),$$

with shrinking number $t_1 := r - s = 3$ using Proposition 3.3. Hence, for the corresponding eigenvalues we have the following spectral diagram

| $S(\tilde{K}_5)$ | $S(\tilde{K}_5)(A)$ | $S(K_6)$ |
|-----------------|-------------------|----------|
| 0 w_1 w_2 w_3 w_4 w_5 | 0 w_1 w_2 w_3 w_4 w_5 | 0 w_1 w_2 w_3 w_4 w_5 |
| 0 w_2 w_3 w_4 w_5 | 0 w_2 w_3 w_4 w_5 | 0 w_2 w_3 w_4 w_5 |
| 0 w_3 w_4 w_5 | 0 w_3 w_4 w_5 | 0 w_3 w_4 w_5 |
| 0 w_4 w_5 | 0 w_4 w_5 | 0 w_4 w_5 |
| 0 w_5 | 0 w_5 | 0 w_5 |

where we denote for simplicity $w_\pm = 1 \pm \left( \frac{1}{2} \sqrt{\frac{1}{5}} (9 + \sqrt{21}) \right)$, $\bar{w}_\pm = 1 \pm \left( \frac{1}{2} \sqrt{\frac{1}{5}} (9 - \sqrt{21}) \right)$ and $z_\pm = 1 \pm \sqrt{\frac{2}{5}}$. This bracketing determines six eigenvalues (highlighted in light gray in the diagram) where we exploit the fact that $w_\pm$ and $\bar{w}_\pm$, both have multiplicity 4 ($6 - 3 = 1$ and 1 has multiplicity $5 - (6 - 3) = 2$.

To determine the remaining 14 eigenvalues we take $G_2 = S(K_6)$ as an auxiliary graph. In this case the shrinking number is $t_2 = 1$ and we obtain the relation

$$S(K_6) \preceq S(\tilde{K}_5)(A) \preceq S(K_6)$$

which can be represented by the following spectral diagram that fixes the remaining eigenvalues which are highlighted in dark gray. Again, only the number pendant paths and the length of the partitions are relevant. Therefore, the partitions $A = \{2,3\}$ and $B = \{1,4\}$ give isospectral graphs $S(\tilde{K}_5)(A)$ and $S(\tilde{K}_5)(B)$ which are not isomorphic because the degree lists are different:

$$\deg S(\tilde{K}_5)(A) = (2, 3, 4, 5, 5, 5, 5) \quad \text{and} \quad \deg S(\tilde{K}_5)(B) = (2, 3, 4, 5, 5, 5, 5).$$
Example 4.7. In a similar vein, the discrete fuzzy bipartite graphs below are also isospectral and, clearly, non-isomorphic using the edge subdivision for all the edges on the graph (see Figure 4).

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{S(K_2.5(A))}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{S(K_2.5(B))}
\end{subfigure}
\caption{Examples of isospectral graphs $S(\tilde{K}_{2.5}(A))$ and $S(\tilde{K}_{2.5}(B))$, obtained by edge subdivision of the fuzzy complete bipartite construction $\tilde{K}_{2.5}(A)$ and $\tilde{K}_{2.5}(B)$ for $A = (4, 1)$ and $B = (2, 3)$.}
\end{figure}

Finally, as in Corollary 4.4, we can extend isospectrality to the metric graph scenario since the subdivision graphs are connected, and the number of edges and vertices are preserved in the merging along the different partitions. Hence all the graphs constructed in this section are also isospectral as equilateral metric graphs for the Kirchhoff Laplacian.

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