Reflected BSDEs with logarithmic growth and applications in mixed stochastic control problems

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ABSTRACT
In this article we study the existence and the uniqueness of a solution for reflected backward stochastic differential equations in the case when the generator is logarithmic growth in the $z$-variable ($|z|\sqrt{|\ln(|z|)|}$), the terminal value and obstacle are an $L^p$-integrable, for a suitable $p > 2$. To construct the solution we use localization method. We also apply these results to get the existence of an optimal control strategy for the mixed stochastic control problem in finite horizon.

1. Introduction
In this paper we study reflected BSDEs with the applications to stochastic control.

We consider a reflected backward stochastic differential equation with a generator $\varphi$, a terminal condition $\xi$ and an obstacle process $(L_t)_{t\leq T}$

$$
\begin{aligned}
Y_t &= \xi + \int_t^T \varphi(s, Y_s, Z_s) \, ds + K_T - K_t - \int_t^T Z_s \, dB_s, \quad t \in [0, T] \\
Y_t &\geq L_t, \quad t \in [0, T] \quad \text{and} \quad \int_0^T (Y_s - L_s) \, dK_s = 0,
\end{aligned}
$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion and the process $K$ is non-decreasing and its role is to push upward $Y$ in order to keep it above the obstacle $L$. The one barrier reflected BSDEs, have been introduced by El Karoui et al. [13].

Reflected BSDEs were studied by several authors (see e.g. [6,13,15–18], and the references therein). The motivations are mainly related to applications especially the pricing of American options in markets constrained or not, mixed control, partial differential variational inequalities, real options, switching optimal (see e.g. [1,4,5,7,9,11,12,14,20,21], and the references therein). Once more under square integrability of the data and Lipschitz property of the coefficient $\varphi$, the authors of [13] showed the existence and the uniqueness of the solution.
There have been a lot of works which dealt with the issue of existence and uniqueness results under weaker assumptions than the ones of Pardoux and Peng [22] or El Karoui et al. [13]. However, for their own reasons, authors focused only on the weakness of the Lipschitz property of the coefficient and not on the square integrability of the data $\xi$ and $\varphi$. Actually, there have been relatively few papers which dealt with the problem of existence and uniqueness of the solution for reflected BSDEs in the case when the coefficients are not square integrable. Nevertheless, we should point out that Briand et al. [8] have proved existence and uniqueness of a solution for the standard BSDEs in the case when the data belong only to $L^p$ for some $p \in ]1,2[$. Recently, Hamadène and Popier [18] considered reflected BSDEs in the case when the data belong only to $L^p$ for some $p \in ]1,2[$. We proved the existence and the uniqueness of the solution for (1). Recently, Bahlali and El Asri [1] have considered the BSDE when $\varphi$ is allowed to have logarithmic growth ($|z|\sqrt{|\ln(|z|)|}$) in the $z$-variable. Moreover, the terminal value is assumed to be merely $L^p$-integrable, with some $p > 2$. The main objective of our paper is to study the existence and the uniqueness of a solution for reflected BSDEs (1) when the generator and the terminal value are as the same as in Bahlali and El Asri [1] and Bahlali et al. [3], respectively. These kind of generators are between the linear growth and the quadratic one. In this case, the square integrability of the terminal datum is not sufficient to ensure the existence of solutions while the exponential integrability seems strong enough and is somehow restrictive. In this paper, one should require some $p$-integrability of the terminal datum $\xi$ with $p > 2$. It should be noted that we do not need the comparison theorem in our proofs. The main motivation of this work is its several applications in: finance, control, games, PDEs,....

**Application in mixed stochastic control problem with finite horizon**

Suppose that we have a system, whose evolution is described by the process $X$, which has an effect on the wealth of a controller. On the other hand the controller has no influence on the system. The process $X$ may represent, for example, the price of an asset on the market and the controller a small share holder or a small investor. The controller acts to protect his advantage by means of $u \in \mathcal{U}$ via the probability $P^u$, here $\mathcal{U}$ is the set of admissible controls. On the other hand he has also the possibility at any time $\tau \in T$ to stop controlling. The control is not free. We define the payoff

$$J(u, \tau) = \mathbb{E}^u \left[ \int_0^\tau h(s, X, u_s) \, ds + g(\tau, X) \mathbb{1}_{\{\tau < T\}} + g_1(X_\tau) \mathbb{1}_{\{\tau = T\}} \right],$$

$h(., X, u)$ is the instantaneous reward for the controller, $g(., X)$ and $g_1(X)$ are, respectively, the rewards if he decides to stop before or until finite time $T$. The problem is to look for an optimal strategy for the controller, i.e. a strategy $(u^*, \tau^*)$ such that

$$J(u, \tau) \leq J(u^*, \tau^*), \quad \forall \,(u, \tau) \in \mathcal{U} \times T.$$

The Hamiltonian associated with this mixed stochastic control problem is

$$H(t, X, z, u_t) := z\sigma^{-1}(t, X)f(t, X, u_t) + h(t, X, u_t) \quad \forall (t, X, z, u_t) \in [0, T] \times \Omega \times \mathbb{R}^d \times \mathcal{A}.$$ 

where $f(t, X, u_t)$ is the drift of dynamics $(X_t)_{t \leq T}$ and $\mathcal{A}$ a compact metric space. The Hamiltonian function attains its supremum over the set $\mathcal{A}$ at some $u^* \equiv u^*(t, X, z) \in \mathcal{A}$, for any
given \((t, X, z) \in [0, T] \times \Omega \times \mathbb{R}^d\), namely,
\[
\sup_{u \in A} H(t, X, z, u) = H(t, X, z, u^*(t, X, z)).
\]

The main objective of mixed stochastic control problem is to show the existence of an optimal strategy for the stochastic control of diffusion. The main idea consists to characterize the value function as the unique solution of this reflected BSDE
\[
\begin{align*}
Y_t &= g_1(X_T) + \int_t^T H(s, X_s, Z_s, u^*(s, X_s, Z_s)) \, ds + K_T - K_t - \int_t^T Z_s \, dB_s, \quad t \in [0, T] \\
Y_t &\geq g(t, X_t), \quad \forall t \in [0, T] \quad \text{and} \quad \int_0^T (Y_s - g(s, X_s)) \, dK_s = 0.
\end{align*}
\]

This problem has been previously studied by Karatzas and Zamfirescu \([20,21]\) with \(f, g_1, g\) and \(h\) are bounded, who applied the martingale methods. Recently, Bayraktar and Yao \([4]\) extended the work of Karatzas and Zamfirescu \([20]\) but allowing the functions \(g_1, g\) and \(h\) to be bounded below. Our aim in this paper is to relax the boundedness assumption and replace it by the linear-growth condition.

The paper is organized as follows: In Section 2, we present the assumptions, we formulate the problem and give some examples. In Section 3, we show priori estimates for solutions of reflected BSDEs. In Section 4, we give estimate between two solutions. In Section 5, we give the main result on the existence and the uniqueness of the solution of reflected BSDEs. Finally, in Section 6, we introduce the mixed stochastic control problem and we give the connection between mixed stochastic control problem and reflected BSDEs, we show the value function as a solution of reflected BSDEs.

2. Assumptions, setting of the problem and examples

2.1. Assumptions

Let \((\Omega, \mathcal{F}, P)\) be a fixed probability space on which is defined a standard \(d\)-dimensional Brownian motion \(B = (B_t)_{0 \leq t \leq T}\) whose natural filtration is \((\mathcal{F}_t^0 := \sigma \{B_s, s \leq t\})_{0 \leq t \leq T}\). Let \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) be the completed filtration of \((\mathcal{F}_t^0)_{0 \leq t \leq T}\) with the \(P\)-null sets of \(\mathcal{F}\). We consider the following assumptions,

\(\text{(H.1)}\) There exists a positive constant \(\lambda > \max(1, \frac{1}{T} (\ln \frac{\beta}{(\alpha - 1)(\beta - 1)} - 1))\) such that
\[
\mathbb{E} \left[ |\xi| e^{\lambda T + 1} \right] < +\infty,
\]
where \(\bar{\alpha} = \min(2, \frac{3}{\alpha})\), for \(0 < \alpha < 2\), and \(\beta \in ]1, \min(3 - \frac{2}{\bar{\alpha}}, 2)[\).

\(\text{(H.2)}\) \quad (i) The process \((L_t)_{t \leq T}\) is continuous,

\(\quad\text{(ii)} \quad \forall p \in [1, 2][, \mathbb{E} \sup_{0 \leq t \leq T} (L_t^p)^{\frac{p}{2}}] < +\infty.
\)

\(\text{(H.3)}\) \quad (i) \(\varphi\) is continuous in \((y, z)\) for almost all \((t, \omega)\).

\(\quad\text{(ii)} \quad \text{There exists a process} \,(\eta_t)_{0 \leq t \leq T}\, \text{satisfying}
\]
\[
\mathbb{E} \left[ \int_0^T |\eta_s| e^{\lambda s + 1} \, ds \right] < +\infty.
\]
(iii) There exists a positive constant $c_0$ such that: for every $t, \omega, y, z$,

$$|\varphi(t, \omega, y, z)| \leq |\eta_t| + c_0|z|\sqrt{\ln|z|}.$$

(H.4) There exist $v \in L^{q'}(\Omega \times [0, T]; \mathbb{R}_+)$ (for some $q' > 0$), a real valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that:

(i) $\forall N > 1, \quad 1 < A_N \leq N')$.

(ii) $\lim_{N \to +\infty} A_N = +\infty$.

(iii) For every $N \in \mathbb{N}$, and every $y, y', z, z'$ such that $|y|, |y'|, |z|, |z'| \leq N$, we have:

$$\left|y - y'\right|\left|\varphi(t, \omega, y, z) - \varphi(t, \omega, y', z')\right| \leq M_2 \left|y - y'\right|^2 \ln A_N + M_2 \left|y - y'\right|\left|z - z'\right| \sqrt{\ln A_N} + M_2 \frac{\ln A_N}{A_N}.$$

Next we define the following spaces: for $p > 1$,

- $\mathcal{S}^p$ is the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted and continuous processes $(Y_t)_{t \in [0, T]}$ such that

$$||Y||_{\mathcal{S}^p} = E\left[\sup_{0 \leq t \leq T} |Y_t|^p\right]^{\frac{1}{p}} < +\infty.$$

- $\mathcal{M}^p$ denote the set $\mathcal{P}$-measurable processes $(Z_t)_{t \in [0, T]}$ with value in $\mathbb{R}^d$ such that:

$$||Z||_{\mathcal{M}^p} = E\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < +\infty.$$

- $\mathcal{H}^p$ be the set of adapted continuous non-decreasing processes $(K_t)_{t \in [0, T]}$ such that $K_0 = 0$ and $E[(K_T)^p] < +\infty$.

Now it will be convenient to define the notion of solution of the reflected BSDE associated with the triple $(\xi, \varphi, L)$ which we consider throughout this paper.

**Definition 2.1:** We say that $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$ is a solution of the reflected BSDE associated with the terminal condition $\xi$, the generator $\varphi$ and the obstacle $L$ if the followings hold:

1. $(Y, Z, K) \in \mathcal{S}^{\xi, T+1} \times \mathcal{M}^2 \times \mathcal{H}^p, \forall p \in [1, 2]$;
2. $Y_t = \xi + \int_t^T \varphi(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \in [0, T]$;
3. $L_t \leq Y_t, \quad t \in [0, T]$;
4. $\int_0^T (Y_s - L_s) dK_s = 0$.

Now we shall present some examples mentioned in [2] of functions $\varphi$ that satisfies our assumptions.
2.2. Examples

Example 2.1: Let \( \phi \in C(\mathbb{R}^d; \mathbb{R}_+) \cap C^1(\mathbb{R}^d - \{0\}; \mathbb{R}_+) \) be such that: for \( \varepsilon \in ]0,1[ \),

\[
\phi(z) = \begin{cases} 
|z|\sqrt{-\ln(|z|)} & \text{if } |z| < 1 - \varepsilon \\
|z|\sqrt{\ln(|z|)} & \text{if } |z| > 1 + \varepsilon.
\end{cases} \tag{2}
\]

It is not difficult to see that \( \phi \) satisfies (H.3).

To verify that \( \phi \) satisfies also (H.4), it is enough to show that for every \( z \) and \( z' \) such that \( |z|,|z'| \leq N \)

\[
|\phi(z) - \phi(z')| \leq c \left( \sqrt{\ln(N)}|z - z'| + \frac{\ln(N)}{N} \right) \tag{3}
\]

for \( N \) large enough and some positive constant \( c \). This can be proved by considering separately the following four cases. \( 0 \leq |z|,|z'| \leq \frac{1}{N}, \frac{1}{N} \leq |z|,|z'| \leq 1 - \varepsilon, 1 - \varepsilon \leq |z|,|z'| \leq 1 + \varepsilon \) and \( 1 + \varepsilon \leq |z|,|z'| \leq N \).

In the first case for \( N > \sqrt{e} \), the map \( x \rightarrow x\sqrt{-\ln x} \) increases for \( x \in ]0, \frac{1}{N}[ \), it follows that

\[
|\phi(z) - \phi(z')| \leq |\phi(z)| + |\phi(z')| \leq 2\frac{1}{N}\ln(N).
\]

To prove the result for the rest of the cases, we apply the finite increment theorem on \( \phi \).

The following example shows that our assumptions enable to treat some reflected BSDEs with stochastic monotone coefficient.

Example 2.2: Let \( \phi \) satisfying (H.3) and

\[
\begin{align*}
\text{(H.4)} \quad & \text{There exist a positive process } (C_t)_{0 \leq t \leq T} \text{ satisfying } \mathbb{E} \left[ \int_0^T e^{\int_0^t C_s \, ds} \right] < +\infty, \\
& \text{(for some } q' > 0) \text{ and } M \in \mathbb{R}_+ \text{ such that:} \\
& (y - y')(\phi(t,\omega,y,z) - \phi(t,\omega,y',z')) \leq M|y - y'|^2(C_t(\omega) + |\ln|y - y'||) \\
& + M|y - y'||z - z'|\sqrt{C_t(\omega) + |\ln|z - z'||}.
\end{align*}
\]

In particular we have for all \( z, z' \)

\[
|\phi(t,\omega,y,z) - \phi(t,\omega,y,z')| \leq M|z - z'|\sqrt{C_t(\omega) + |\ln|z - z'||}.
\]

To see if (H.4) is verified, we consider the following cases

\[
|y - y'| \leq \frac{1}{2N}, \quad \frac{1}{2N} \leq |y - y'| \leq 2N,
\]

and

\[
|z - z'| \leq \frac{1}{2N}, \quad \frac{1}{2N} \leq |z - z'| \leq 2N.
\]
It is not difficult to prove that for some constant $c$ we have:

$$(y - y')(\varphi(t, y, z) - \varphi(t, y', z)) \leq c \ln(N) \left( |y - y'|^2 + \frac{1}{N} \right)$$

$$|\varphi(t, y, z) - \varphi(t, y, z')| \leq c\sqrt{\ln(N)} \left( |z - z'|^2 + \frac{1}{N} \right),$$

defined whenever $v_s := e^{C_s} \leq N$ and $|y|, |y'|, |z|, |z'| \leq N$.

### 3. Apriori estimates

Here, we want to obtain estimates for solutions to reflected BSDEs in the spirit of the work [3] which shows that these estimates are very useful in the study of existence and uniqueness of solutions. Before we do so, we give the following lemma that has already been mentioned and proved in [3], but for the sake of the reader we give it again.

**Lemma 3.1:** Let $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ be such that $y$ is large enough. Then, for every $C_1 > 0$ there exists $C_2 > 0$ such that:

$$C_1 |y| |z| \sqrt{\ln(|z|)} \leq \frac{|z|^2}{2} + C_2 \ln(|y|) |y|^2.$$  

(4)

Now we start by showing how to control the process $Y$ in terms of the data and $K_T$.

**Lemma 3.2:** Let $(Y, Z, K)$ be a solution of the reflected BSDE (1), where $(\xi, \varphi, L)$ satisfies the assumptions $(H.1)$, $(H.2)$ and $(H.3)$. Then, for $\lambda > 1$, there exists a constant $C(\lambda, T)$ such that:

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t| e^{\lambda t} \right] \leq C(\lambda, T) \mathbb{E} \left[ |\xi| e^{\lambda T} + \int_0^T |\eta_s| e^{\lambda s} ds + \sup_{t \in [0, T]} \left( (T_s^+) e^{\lambda t} \right) K_T \right].$$

(5)

**Proof:** Without loss of generality we assume that the $y$-variable is sufficiently large. For some constant $\lambda$ large enough, let us consider the function from $[0, T] \times \mathbb{R}$ into $\mathbb{R}^+$ defined by

$$u(t, x) = |x| e^{\lambda t} + 1.$$ 

Then,

- $u_t = \lambda e^{\lambda t} \ln(|x|) |x| e^{\lambda t} + 1$;
- $u_x = (e^{\lambda t} + 1) |x| e^{\lambda t} \text{sgn}(x)$;
- $u_{xx} = (e^{\lambda t} + 1) e^{\lambda t} |x| e^{\lambda t} - 1$;

where $\text{sgn}(x) = -\mathbb{1}_{x \leq 0} + \mathbb{1}_{x > 0}$. 
Let \((\tau_k)_{k \geq 0}\) be the sequence of stopping times defined as follows:

\[
\tau_k := \inf \left\{ t \geq 0, \left[ \int_0^t (e^{\lambda s} + 1)^2 \ | Y_s \ |^{2\lambda s} \ | Z_s \ |^2 \ ds \right] \vee |Y_t| \geq k \right\} \wedge T.
\]

Now we apply Itô’s formula on the process \(Y\) and the function \(y \mapsto |y|^{e^{s}+1}\) to obtain:

\[
|Y_{t \wedge \tau_k}|^{e^{\lambda(t \wedge \tau_k)}+1} = |Y_{\tau_k}|^{1+e^{\lambda \tau_k}} - \lambda \int_{t \wedge \tau_k}^{\tau_k} e^{\lambda s} \ln(|Y_s|) \ | Y_s \ |^{e^{s}+1} \ ds
- \frac{1}{2} \int_{t \wedge \tau_k}^{\tau_k} |Z_s|^2 (e^{\lambda s} + 1) e^{\lambda s} \ | Y_s \ |^{e^{s}-1} \ ds
+ \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ \text{sgn}(Y_s) \varphi(s, Y_s, Z_s) \ ds
+ \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ \text{sgn}(Y_s) \ dK_s
- \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ \text{sgn}(Y_s) Z_s \ dB_s.
\]

Using (iii) in assumption (H.3) we have:

\[
|Y_{t \wedge \tau_k}|^{e^{\lambda(t \wedge \tau_k)}+1} \leq |Y_{\tau_k}|^{1+e^{\lambda \tau_k}} - \lambda \int_{t \wedge \tau_k}^{\tau_k} e^{\lambda s} \ln(|Y_s|) \ | Y_s \ |^{e^{s}+1} \ ds
- \frac{1}{2} \int_{t \wedge \tau_k}^{\tau_k} |Z_s|^2 (e^{\lambda s} + 1) e^{\lambda s} \ | Y_s \ |^{e^{s}-1} \ ds
+ \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ (|\eta_s| + c_0 |Z_s| \sqrt{\ln(|Z_s|)}) \ ds
+ \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ \text{sgn}(Y_s) \ dK_s
- \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ \text{sgn}(Y_s) Z_s \ dB_s.
\]

Next by Young’s inequality we have:

\[
(e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ |\eta_s| \leq | Y_s \ |^{e^{s}+1} + (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}+1} \ |\eta_s|^{e^{s}+1}.
\]

For \(|Y_s|\ large enough and thanks to the last inequality, it follows that:

\[
|Y_{t \wedge \tau_k}|^{e^{\lambda(t \wedge \tau_k)}+1} \leq |Y_{\tau_k}|^{1+e^{\lambda \tau_k}} - \lambda \int_{t \wedge \tau_k}^{\tau_k} e^{\lambda s} \ln(|Y_s|) \ | Y_s \ |^{e^{s}+1} \ ds
- \frac{1}{2} \int_{t \wedge \tau_k}^{\tau_k} |Z_s|^2 (e^{\lambda s} + 1) e^{\lambda s} \ | Y_s \ |^{e^{s}-1} \ ds
+ \int_{t \wedge \tau_k}^{\tau_k} c_0 (e^{\lambda s} + 1) \ | Y_s \ |^{e^{s}} \ |Z_s| \sqrt{\ln(|Z_s|)} \ ds.
\]
Therefore we have:

\[ \int_{t \wedge \tau_k}^{\tau_k} |Y_s|^{\frac{s}{2}+1} \ln(|Y_s|) \, ds + \int_{t \wedge \tau_k}^{\tau_k} (e^{s})^{\frac{s}{2}+1} |\eta_s|^{\frac{s}{2}+1} \, ds \]

\[ + \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s) \, dK_s \]

\[ - \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s)Z_s \, dB_s. \]

Then we have:

\[ |Y_{t \wedge \tau_k}|^{e^{\int_{(t \wedge \tau_k)}^{\tau_k}}} \leq |Y_{\tau_k}|^{1+e^{t \tau_k}} + \int_{t \wedge \tau_k}^{\tau_k} |Y_s|^{\frac{s}{2}+1} \left[ c_0(e^{s} + 1) |Y_s| |Z_s| \sqrt{\ln(|Z_s|)} \right] \]

\[ - (e^{s} + 1)e^{\frac{2}{2}} - (\lambda e^{\frac{1}{2}} - 1) \ln(|Y_s|) |Y_s|^2 \, ds \]

\[ + \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s) \, dK_s \]

\[ - \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s)Z_s \, dB_s. \]

Since \( \lambda e^{\frac{1}{2}} - 1 > 0 \) for \( \lambda > 1 \), then using Lemma 3.1, we have for \( \lambda \) large enough:

\[ c_0(e^{s} + 1) |Y_s| |Z_s| \sqrt{\ln(|Z_s|)} \leq (e^{s} + 1)e^{\frac{s}{2}} + (\lambda e^{\frac{1}{2}} - 1) \ln(|Y_s|) |Y_s|^2. \]

Therefore, it follows that:

\[ |Y_{t \wedge \tau_k}|^{e^{\int_{(t \wedge \tau_k)}^{\tau_k}}} \leq |Y_{\tau_k}|^{1+e^{t \tau_k}} + \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s) \, dK_s \]

\[ + \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s)Z_s \, dB_s. \]

Next let us deal with the term \( \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s) \, dK_s \). Indeed, the hypothesis related to increments of \( K \) and \( Y - L \) implies that: \( dK_s = 1_{\{Y_s = L_s\}} \, dK_s \), for any \( s \leq T \). Therefore we have:

\[ \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s) \, dK_s = \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}+1} Y_s \, 1_{\{Y_s = L_s\}} \, dK_s \]

\[ = \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |L_s|^{\frac{2}{2}} L_s \, 1_{\{Y_s = L_s\}} \, dK_s. \]

It follows that:

\[ \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1) |Y_s|^{\frac{s}{2}} \text{sgn}(Y_s) \, dK_s \leq \int_{t \wedge \tau_k}^{\tau_k} (e^{s} + 1)(L_s^+) \, dK_s. \]
\[ \leq 2(e^{\lambda T} + 1) \sup_{t \in [0, T]} \left( (L_t^+) e^{\lambda t} \right) K_{\tau_k} \]

\[ \leq 2(e^{\lambda T} + 1) \sup_{t \in [0, T]} \left( (L_t^+) e^{\lambda t} \right) K_T, \]

which means that:

\[ \int_{t \wedge \tau_k}^{\tau_k} (e^{\lambda s} + 1) | Y_s |^{\lambda} \text{sgn}(Y_s) \, dK_s \leq 2(e^{\lambda T} + 1) \sup_{t \in [0, T]} \left( (L_t^+) e^{\lambda t} \right) K_T. \tag{8} \]

We combine (7) and (8) and we take expectation to get:

\[ \mathbb{E} \left[ | Y_{t \wedge \tau_k} |^{\lambda + 1} \right] \leq \mathbb{E} \left[ | Y_{\tau_k} |^{1+e^{\lambda \tau_k}} \right] + (e^{\lambda T} + 1)^{\lambda+1} \mathbb{E} \left[ \int_{t \wedge \tau_k}^{\tau_k} |\eta_s|^{\lambda+1} \, ds \right] \]
\[ + 2(e^{\lambda T} + 1) \mathbb{E} \left[ \sup_{t \in [0, T]} \left( (L_t^+) e^{\lambda t} \right) K_T \right]. \tag{9} \]

Since the sequence of stopping times \((\tau_k)_{k \geq 0}\) is increasing of stationary type, we pass to the limit when \(k \to +\infty\) then we use Fatou’s Lemma to obtain:

\[ \mathbb{E} \left[ | Y_t |^{\lambda+1} \right] \leq \mathbb{E} \left[ | \xi |^{\lambda+1} \right] + (e^{\lambda T} + 1)^{\lambda+1} \mathbb{E} \left[ \int_0^T |\eta_s|^{\lambda+1} \, ds \right] \]
\[ + 2(e^{\lambda T} + 1) \mathbb{E} \left[ \sup_{t \in [0, T]} \left( (L_t^+) e^{\lambda t} \right) K_T \right]. \tag{10} \]

Finally we use Burkholder–Davis–Gundy’s inequality to complete the proof. \(\blacksquare\)

We will now establish an estimate for the process \(Z\). Actually we have:

**Lemma 3.3**: Let \((Y, Z, K)\) be a solution of the reflected BSDE (1), where \((\xi, \varphi, L)\) satisfies the assumptions (H.1), (H.2) and (H.3). Then, there exists a positive constant \(C(\lambda, c_0, T)\) such that:

\[ \mathbb{E} \left[ \int_0^T |Z_s|^2 \, ds \right] \leq C(\lambda, c_0, T) \mathbb{E} \left[ 1 + |\xi|^{\lambda+1} + \int_0^T |\eta_s|^{\lambda+1} \, ds + \sup_{t \in [0, T]} \left( (L_t^+) e^{\lambda t} \right) K_T \right]. \tag{11} \]

**Proof**: Applying Itô’s formula to the process \(Y\) and the function \(y \mapsto y^2\) yields:

\[ |Y_0|^2 + \int_0^T |Z_s|^2 \, ds = |\xi|^2 + 2 \int_0^T Y_s \varphi(s, Y_s, Z_s) \, ds + 2 \int_0^T Y_s \, dK_s - 2 \int_0^T Y_s Z_s \, dB_s \]
\[ \leq |\xi|^2 + 2 \int_0^T |Y_s| (|\eta_s| + c_0 |Z_s| \sqrt{\ln(|Z_s|)}) \, ds + 2 \int_0^T Y_s \, dK_s \]
\[ - 2 \int_0^T Y_s Z_s \, dB_s. \tag{12} \]
Using Lemma 3.1 we can show that there exists a constant \( \tilde{C} \) that depends on \( c_0 \) such that
\[
2c_0|Y_s||Z_s|\sqrt{\ln(|Z_s|)}| \leq \frac{|Z_s|^2}{2} + \tilde{C}\ln(|Y_s|)|Y_s|^2. \tag{13}
\]
From Young’s inequality we have:
\[
2\ | Y_s | | \eta_s | \leq \ | Y_s |^2 + | \eta_s |^2. \tag{14}
\]
Then by combining (13) and (14) with (12) and by using the fact that \( \int_0^T (Y_s - L_s) \, dK_s = 0 \) we obtain:
\[
|Y_0|^2 + \frac{1}{2} \int_0^T |Z_s|^2 \, ds \leq |\xi|^2 + \sup_{0 \leq s \leq T} |Y_s|^2 + \int_0^T |\eta_s|^2 \, ds + \tilde{C} \int_0^T \ln(|Y_s|)|Y_s|^2 \, ds
- 2 \int_0^T Y_s Z_s \, dB_s + 2 \int_0^T L_s \, dK_s.
\]
Since we assume that \( |Y_s| \) is large enough, we have for any \( \varepsilon > 0 \):
\[
|Y_s|^2 |\ln(|Y_s|)| \leq |Y_s|^{2+\varepsilon} \quad \text{and} \quad |Y_s|^2 \leq |Y_s|^{2+\varepsilon}.
\]
Therefore, there exists a positive constant \( \tilde{C}_1 \) that depends on \( T \) and \( c_0 \) such that:
\[
\frac{1}{2} \int_0^T |Z_s|^2 \, ds \leq |\xi|^2 + \tilde{C}_1 \sup_{0 \leq s \leq T} |Y_s|^{2+\varepsilon} + \int_0^T |\eta_s|^2 \, ds + 2 \int_0^T L_s \, dK_s 
- 2 \int_0^T Y_s Z_s \, dB_s. \tag{15}
\]
We choose \( \varepsilon = e^{\lambda T} - 1 \) and we take expectation in both sides of (15). Then, there exists a constant \( \tilde{C}_2 > 0 \) that still depends on \( c_0 \) and \( T \) such that:
\[
\mathbb{E} \left[ \int_0^T |Z_s|^2 \, ds \right] \leq \tilde{C}_2 \mathbb{E} \left[ |\xi|^2 + \sup_{0 \leq s \leq T} |Y_s|^{e^{\lambda T}+1} + \int_0^T |\eta_s|^2 \, ds + \int_0^T L_s \, dK_s \right] 
+ 2 \mathbb{E} \left[ \int_0^T Y_s Z_s \, dB_s \right]. \tag{16}
\]
Now, thanks to Burkholder–Davis–Gundy’s inequality we have for any \( \beta > 0 \)
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_t^T Y_s Z_s \, dB_s \right| \right] \leq \tilde{K} \mathbb{E} \left[ \left( \int_0^T |Y_s|^2 |Z_s|^2 \, ds \right)^{\frac{1}{2}} \right] 
\leq \tilde{K} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y_s| \left( \int_0^T |Z_s|^2 \, ds \right)^{\frac{1}{2}} \right] 
\leq \tilde{K} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y_s|^2 \right] + \frac{\beta \tilde{K}}{2} \mathbb{E} \left[ \int_0^T |Z_s|^2 \, ds \right].
\]
Therefore, choosing $\beta$ small enough, yields to the existence of a positive constant $\tilde{C}_3$ that depends on $c_0$ and $T$ such that:

$$\mathbb{E}
\left[
\int_0^T |Z_t|^2 \, dt
\right]
\leq \tilde{C}_3 \mathbb{E}
\left[
|\xi|^2 + \sup_{t \in [0,T]} |Y_t|^2 + \int_0^T |\eta_s|^2 \, ds + \sup_{t \in [0,T]} (L_t^+) K_T
\right].$$

Now by using Inequality (5), Young’s inequality and Hölder’s inequality, we conclude that there exists a positive constant $C(\lambda, c_0, T)$ such that:

$$\mathbb{E}
\left[
\int_0^T |Z_s|^2 \, ds
\right]
\leq C(\lambda, c_0, T) \mathbb{E}
\left[
1 + |\xi|^2 + \int_0^T |\eta_s|^2 \, ds + \sup_{0 \leq t \leq T} \left((L_t^+) \right)^{\frac{p}{p-1}} K_T
\right].$$

The proof is now complete.

Now we focus on the control of the process $K$.

**Lemma 3.4:** Let $(Y, Z, K)$ be a solution of the reflected BSDE (1), where $(\xi, \varphi, L)$ satisfies the assumptions (H.1), (H.2) and (H.3). Then, there exists a constant $C(\lambda, p, c_0, T) > 0$ such that: $\forall p \in ]1, 2[,$

$$\mathbb{E}
\left[
(K_T)^p
\right]
\leq C(\lambda, p, c_0, T) \mathbb{E}
\left[
1 + |\xi|^2 + \int_0^T |\eta_s|^2 \, ds + \sup_{0 \leq t \leq T} \left((L_t^+) \right)^{\frac{p}{p-1}} K_T
\right].$$

(17)

**Proof:** First we recall that:

$$K_T - K_t = Y_t - \xi - \int_t^T \varphi(s, Y_s, Z_s) \, ds + \int_t^T Z_s \, dB_s.$$  

(18)

Next using the predictable dual projection property (see e.g. [10]) we have: for any $t \leq T,$

$$\mathbb{E}
\left[
(K_T - K_t)^p
\right] = \mathbb{E}
\left[
\int_t^T p (K_T - K_s)^{p-1} \, dK_s
\right]$$

$$= p \mathbb{E}
\left[
\int_t^T \mathbb{E}
\left[
(K_T - K_s)^{p-1} |\mathcal{F}_s
\right] \, dK_s
\right].$$

Since $p \in ]1, 2[,$ then thanks to Jensen’s conditional inequality we have:

$$\mathbb{E}
\left[
(K_T - K_t)^p
\right] \leq p \mathbb{E}
\left[
\int_t^T \mathbb{E}
\left[
(K_T - K_s)^{p-1} |\mathcal{F}_s
\right] \, dK_s
\right].$$

From (18) we obtain:

$$\mathbb{E}
\left[
(K_T - K_t)^p
\right] \leq p \mathbb{E}
\left[
\int_t^T \mathbb{E}
\left[
\left. Y_s - \xi - \int_s^T \varphi(u, Y_u, Z_u) \, du \right| \mathcal{F}_s
\right]^{p-1} \, dK_s
\right].$$
\[
\begin{align*}
\leq p \mathbb{E} \left[ \int_t^T \left\{ \mathbb{E} \left( 2 \sup_{u \in [s,T]} |Y_u| + \int_s^T |\varphi(u, Y_u, Z_u)| \, du \right) \right\}^{p-1} \int_s^T \Gamma_s^p \, dK_s \right] \\
= p \mathbb{E} \left[ \int_t^T \Gamma_s^{p-1} \, dK_s \right] \\
\leq p \mathbb{E} \left[ \left( \sup_{s \in [t,T]} |\Gamma_s| \right)^{p-1} (K_T - K_t) \right]; \\
\end{align*}
\]

with
\[
\Gamma_s = \mathbb{E} \left( 2 \sup_{u \in [s,T]} |Y_u| + \int_s^T |\varphi(u, Y_u, Z_u)| \, du \bigg| \mathcal{F}_s \right).
\]

Once more by using Young's inequality we have:
\[
\begin{align*}
p \mathbb{E} \left[ \left( \sup_{s \in [t,T]} |\Gamma_s| \right)^{p-1} (K_T - K_t) \right] \\
\leq \frac{1}{2} \mathbb{E} [(K_T - K_t)^p] + (p - 1) \frac{1}{2(p-1)} \mathbb{E} \left[ \left( \sup_{s \in [t,T]} |\Gamma_s| \right)^p \right].
\end{align*}
\]

Hence
\[
\mathbb{E} [(K_T - K_t)^p] \leq \frac{1}{2} \mathbb{E} [(K_T - K_t)^p] \\
+ C_p \mathbb{E} \sup_{s \in [t,T]} \left[ \mathbb{E} \left( 2 \sup_{u \in [s,T]} |Y_u| + \int_s^T |\varphi(u, Y_u, Z_u)| \, du \bigg| \mathcal{F}_s \right) \right]^p.
\]

Thus, using Doob's maximal inequality we obtain:
\[
\begin{align*}
\frac{1}{2} \mathbb{E} [(K_T - K_t)^p] &\leq C_p \sup_{s \in [t,T]} \mathbb{E} \left[ \mathbb{E} \left( 2 \sup_{u \in [s,T]} |Y_u| + \int_s^T |\varphi(u, Y_u, Z_u)| \, du \bigg| \mathcal{F}_s \right) \right]^p \\
&\leq C_p' \mathbb{E} \sup_{u \in [t,T]} |Y_u|^p + \left( \int_t^T |\varphi(u, Y_u, Z_u)| \, du \right)^p.
\end{align*}
\]

Then by taking \( t = 0 \), there exists a constant \( C_p'' > 0 \) such that:
\[
\mathbb{E} [(K_T)^p] \leq C_p'' \mathbb{E} \left[ \sup_{s \in [0,T]} |Y_s|^p + \left( \int_0^T |\varphi(s, Y_s, Z_s)| \, ds \right)^p \right]. \quad (19)
\]

Using Hölder's inequality and then assumption (H.3), we obtain that there exists a positive constant \( C(p, T) \) (changes from line to line) such that:
\[
\left( \int_0^T |\varphi(s, Y_s, Z_s)| \, ds \right)^p \leq C(p, T) \int_0^T |\varphi(s, Y_s, Z_s)|^p \, ds.
\]
\[
\leq C(p, T) \left( \int_0^T |\eta_s|^p \, ds + \int_0^T \left( c_0 |Z_s| \sqrt{\ln(|Z_s|)} \right)^p \, ds \right),
\]

and for any \( \varepsilon > 0 \) we have:
\[
\sqrt{2\varepsilon \ln(|Z_s|)} = \sqrt{\ln(|Z_s|^{2\varepsilon})} \leq |Z_s|^\varepsilon.
\]

Therefore,
\[
\left( \int_0^T \left| \varphi(s, Y_s, Z_s) \right| \, ds \right)^p \leq C(p, T) \left( \int_0^T |\eta_s|^p \, ds + \left( \frac{c_0}{\sqrt{2\varepsilon}} \right)^p \int_0^T |Z_s|^{p(1+\varepsilon)} \, ds \right). \tag{20}
\]

We now put \( \varepsilon = \frac{2}{p} - 1 \), then, there exists a constant \( C(p, c_0, T) > 0 \) such that:
\[
\left( \int_0^T \left| \varphi(s, Y_s, Z_s) \right| \, ds \right)^p \leq C(p, c_0, T) \left( \int_0^T |\eta_s|^p \, ds + \int_0^T |Z_s|^2 \, ds \right).
\]

Hence, there exists a positive constant which we still denote \( C(p, c_0, T) \) such that:
\[
\mathbb{E} \left[ (K_T)^p \right] \leq C(p, c_0, T) \mathbb{E} \left[ \sup_{s \in [0,T]} |Y_s|^p + \int_0^T |\eta_s|^p \, ds + \int_0^T |Z_s|^2 \, ds \right]. \tag{21}
\]

Using Inequalities (5) and (11), Young’s inequality and Hölder’s inequality, there exists a positive constant \( C(\lambda, p, c_0, T) \) that changes from line to line such that:
\[
\mathbb{E} \left[ (K_T)^p \right] \leq C(\lambda, p, c_0, T) \mathbb{E} \left[ 1 + |\xi|^{\lambda T + 1} + \int_0^T |\eta_s|^{\lambda s + 1} \, ds + \sup_{t \in [0,T]} \left( (L_t^+)^{\lambda t} \right) K_T \right]
\]
\[
\leq C(\lambda, p, c_0, T) \mathbb{E} \left[ 1 + |\xi|^{\lambda T + 1} + \int_0^T |\eta_s|^{\lambda s + 1} \, ds + \sup_{0 \leq t \leq T} \left( (L_t^+)^{\lambda t} \right)^{\frac{p}{p-1}} \right]
\]
\[
+ \frac{1}{p} \mathbb{E} \left[ (K_T)^p \right].
\]

Finally, there exists a positive constant which we still denote \( C(\lambda, p, c_0, T) \) such that:
\[
\mathbb{E} \left[ (K_T)^p \right] \leq C(\lambda, p, c_0, T) \mathbb{E} \left[ 1 + |\xi|^{\lambda T + 1} + \int_0^T |\eta_s|^{\lambda s + 1} \, ds + \sup_{0 \leq t \leq T} \left( (L_t^+)^{\lambda t} \right)^{\frac{p}{p-1}} \right].
\]

The proof is now complete. \( \square \)

**Proposition 3.1:** Let \( (Y, Z, K) \) be a solution of the reflected BSDE (1), where \( (\xi, \varphi, L) \) satisfies the assumptions (H.1), (H.2) and (H.3). Then, there exists a positive constant \( C_1(\lambda, p, c_0, T) \) such that: \( \forall \, p \in ]1, 2[, \)
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^{\lambda t + 1} + \int_0^T |Z_s|^2 \, ds + (K_T)^p \right]
\]
\[
\leq C_1(\lambda, p, c_0, T) \mathbb{E} \left[ 1 + |\xi|^{\lambda T + 1} + \int_0^T |\eta_s|^{\lambda s + 1} \, ds + \sup_{0 \leq t \leq T} \left( (L_t^+)^{\lambda t} \right)^{\frac{p}{p-1}} \right]. \tag{22}
\]

**Proof:** We combine (5), (11) and (17) to get (22). \( \square \)
4. Estimate between two solutions

4.1. Some useful tools

We begin with an estimate for \( \varphi \), whose proof can be found in [1].

**Lemma 4.1:** If (H.3) holds then,

\[
E \left[ \int_0^T |\varphi(s, Y_s, Z_s)|^{\alpha} \, ds \right] \leq C \left( 1 + E \left[ \int_0^T \eta_s^2 \, ds \right] + E \left[ \int_0^T |Z_s|^2 \, ds \right] \right),
\]

where \( \alpha = \min(2, \frac{2}{d}) \) for \( 0 < \alpha < 2 \), and \( C \) is a positive constant which depends on \( c_0 \) and \( T \).

**Lemma 4.2:** There exists a sequence of functions \( (\varphi_n) \) such that:

(a) For each \( n \), \( \varphi_n \) is bounded and globally Lipschitz in \( (y, z) \) a.e \( t \) and \( P\)-a.s. \( \omega \).

(b) \( \sup_n \| \varphi_n(t, \omega, y, z) \| \leq |\eta_t| + c_0 |z\sqrt{\ln(|z|)}| \), \( P\)-a.s., a.e \( t \in [0, T] \).

(c) \( \forall N, \rho_N(\varphi_n - \varphi) \to 0 \text{ as } n \to +\infty \); where \( \rho_N(\varphi) = E[\int_0^T \sup_{|y|,|z| \leq N} |\varphi(s, y, z)| \, ds] \).

**Proof:** Let \( \alpha_n : \mathbb{R}^2 \to \mathbb{R}_+ \) be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy \( \int \alpha_n(u) \, du = 1 \). Let \( \psi_n \) from \( \mathbb{R}^2 \) to \( \mathbb{R}_+ \) be a sequence of smooth functions such that \( 0 \leq |\psi_n| \leq 1 \), \( \psi_n(u) = 1 \) for \( |u| \leq n \) and \( \psi_n(u) = 0 \) for \( |u| \geq n + 1 \). We put, \( \varepsilon_{q,n}(t, y, z) = \int \varphi(t, (y, z) - u)\alpha_q(u) \, du\psi_n(y, z) \). For \( n \in \mathbb{N}^* \), let \( q(n) \) be an integer such that \( q(n) \geq n + n^d \). It is not difficult to see that the sequence \( \varphi_n := \varepsilon_{q(n),n} \) satisfies all the assertions (a)-(c). \( \blacksquare \)

Using Proposition 3.1, Lemma 4.1, Lemma 4.2 and standard arguments of reflected BSDEs, one can prove the following estimates.

**Lemma 4.3:** Let \( \varphi, \xi \) and \( L \) be as in Proposition 3.1. Let \( (\varphi_n) \) be the sequence of functions associated to \( \varphi \) by Lemma 4.2. Denote by \( (Y^n, Z^n, K^n) \) the solution of equation

\[
\begin{align*}
Y^n_t = \xi + \int_t^T \varphi_n(s, Y^n_s, Z^n_s) \, ds + K^n_T - K^n_t - \int_t^T Z^n_s \, dB_s, \quad &\forall t \in [0, T] \\
L^n_t \leq Y^n_t, \quad &t \in [0, T] \quad \text{and} \quad \int_0^T (Y^n_T - L^n_s) \, dK^n_s = 0,
\end{align*}
\]

where \( (L^n_t)_{t \leq T} \) is supposed to be continuous, increasing with respect to \( n \) and \( \lim_{n \to +\infty} L^n_t = L_t, \quad \forall t \in [0, T] \).

Then, there exist constants \( \tilde{K}_1, \tilde{K}_2, \tilde{K}_3 \) and \( \tilde{K}_4 \) such that:

(a) \( \sup_n E \left[ \int_0^T |Z^n_s|^2 \, ds \right] \leq \tilde{K}_1 \).

(b) \( \sup_n E \left[ \sup_{0 \leq t \leq T} (Y^n_t |e^{nT+1}) \right] \leq \tilde{K}_2 \).
(c) \(\sup_n \mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s)|^\alpha \, ds \right] \leq \tilde{K}_3, \text{ where } \alpha = \min(2, \frac{2}{\beta}).\)

(d) \(\sup_n \mathbb{E} [(K^n_T)^p] \leq \tilde{K}_4, \forall p \in ]1, 2[.\)

### 4.2. Estimate between two solutions

We can now estimate the variation in the solution.

**Lemma 4.4:** For every \(R \in \mathbb{N}, \beta \in ]1, \min(3 - \frac{2}{\beta}, 2)[, \forall \delta' < (\beta - 1) \min(\frac{1}{4M^2}, \frac{3 - \frac{2}{\beta} - \delta}{2\beta M^2})\)

and \(\epsilon > 0\), there exists \(N_0 > R\) such that for all \(N > N_0\) and \(T' \leq T\):

\[
\limsup_{n,m \to +\infty} \mathbb{E} \left[ \sup_{(T' - \delta')^+ \leq t \leq T'} |Y^n_t - Y^m_t|^\beta \right] + \mathbb{E} \left[ \int_{(T' - \delta')^+}^{T'} \frac{|Z^n_s - Z^m_s|^2}{(|Y^n_s - Y^m_s|^2 + v_R)^{\frac{2-\beta}{2}}} \, ds \right] \\
\leq \epsilon + \frac{\ell}{\beta - 1} e^{C_N \delta'} \limsup_{n,m \to +\infty} \mathbb{E} \left[ |Y^n_{T'} - Y^m_{T'}|^\beta \right] \tag{24}
\]

where \(v_R = \sup \{(A_N)^{-1}, N \geq R\}, C_N = \frac{2M^2}{(\beta - 1)^2}\ln A_N, \text{ and } \ell \text{ is a universal positive constant.}\)

**Proof:** Let \(0 < T' \leq T\). It follows from Itô’s formula that for all \(t \leq T'\),

\[
|Y^n_t - Y^m_t|^2 + \int_t^{T'} |Z^n_s - Z^m_s|^2 \, ds \\
= |Y^n_t - Y^m_t|^2 + 2 \int_t^{T'} (Y^n_s - Y^m_s)(\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)) \, ds \\
+ 2 \int_t^{T'} (Y^n_s - Y^m_s)(dK^n_s - dK^m_s) - 2 \int_t^{T'} \langle Y^n_s - Y^m_s, (Z^n_s - Z^m_s) \rangle \, dB_s.
\]

For \(N \in \mathbb{N}^*\) we set, \(\Delta_t := |Y^n_t - Y^m_t|^2 + (A_N)^{-1}.\)

Let \(C > 0\) and \(1 < \beta < \min\{(3 - \frac{2}{\beta}), 2\}.\) Itô’s formula shows that,

\[
e^{Cs} \Delta_t^\frac{\beta}{2} + C \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta}{2} \, ds \\
= e^{Cs} \Delta_t^\frac{\beta}{2} + \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta - 1}{2} (Y^n_s - Y^m_s)(\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)) \, ds \\
- \frac{\beta}{2} \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta - 1}{2} |Z^n_s - Z^m_s|^2 \, ds - \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta - 1}{2} \langle Y^n_s - Y^m_s, (Z^n_s - Z^m_s) \rangle \, dB_s \\
+ \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta - 1}{2} (Y^n_s - Y^m_s)(dK^n_s - dK^m_s) \\
- \beta \left(\frac{\beta - 2}{2}\right) \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta - 2}{2} (Y^n_s - Y^m_s)(Z^n_s - Z^m_s)^2 \, ds.
\]
Put $\Phi(s) = |Y^n_s| + |Y^m_s| + |Z^n_s| + |Z^m_s| + \nu_s$. Then

e^{Ct} \Delta_t^\beta + C \int_t^{T \vee} e^{Cs} \Delta_s^\beta \, ds

= e^{Ct} \Delta_t^\beta - \beta \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} (Y^n_s - Y^m_s, (Z^n_s - Z^m_s) \, dB_s)

- \frac{\beta}{2} \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} |Z^n_s - Z^m_s|^2 \, ds + \beta \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s)

+ \beta \frac{(2 - \beta)}{2} \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-2} \big( (Y^n_s - Y^m_s) (Z^n_s - Z^m_s) \big)^2 \, ds

+ J_1 + J_2 + J_3 + J_4,

where

\begin{align*}
J_1 & := \beta \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} (Y^n_s - Y^m_s) \big( \varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s) \big) \mathbb{1}_{\{\Phi(s) > N\}} \, ds, \\
J_2 & := \beta \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} (Y^n_s - Y^m_s) \big( \varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s) \big) \mathbb{1}_{\{\Phi(s) \leq N\}} \, ds, \\
J_3 & := \beta \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} (Y^n_s - Y^m_s) \big( \varphi(s, Y^n_s, Z^n_s) - \varphi(s, Y^m_s, Z^m_s) \big) \mathbb{1}_{\{\Phi(s) \leq N\}} \, ds, \\
J_4 & := \beta \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} (Y^n_s - Y^m_s) \big( \varphi(s, Y^m_s, Z^m_s) - \varphi_m(s, Y^m_s, Z^m_s) \big) \mathbb{1}_{\{\Phi(s) \leq N\}} \, ds.
\end{align*}

Now we will estimate $J_1, J_2, J_3$ and $J_4$.

Let $\kappa = 3 - \frac{2}{\beta} - \beta$. Since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\beta} = 1$, we use Hölder’s inequality to obtain

\begin{align*}
J_1 & \leq \beta e^{Ct} \frac{1}{N^\kappa} \int_t^{T \vee} \Delta_s^{\beta-1} \Phi^\kappa(s) |\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)| \, ds \\
& \leq \beta e^{Ct} \frac{1}{N^\kappa} \left[ \int_t^{T \vee} \Delta_s \, ds \right]^{\beta-1} \left[ \int_t^{T \vee} \Phi(s)^2 \, ds \right]^\frac{\kappa}{2} \\
& \times \left[ \int_t^{T \vee} |\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)|^{2 \alpha} \, ds \right]^\frac{1}{2 \alpha}.
\end{align*}

Since $|Y^n_s - Y^m_s| \leq \Delta_s^{\frac{1}{2}}$, it is easy to see that

\begin{align*}
J_2 + J_4 & \leq 2 \beta e^{Ct} \left[ 2N^2 + \nu_1 \right]^{\frac{\beta-1}{2}} \left[ \int_t^{T \vee} \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| \, ds \\
& + \int_t^{T \vee} \sup_{|y|,|z| \leq N} |\varphi_m(s, y, z) - \varphi(s, y, z)| \, ds \right] \\
& \leq 2 \beta M_2 \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} \left[ |Y^n_s - Y^m_s|^2 \ln A_N \right].
\end{align*}

Using assumption (H.4), we get

\begin{align*}
J_3 & \leq \beta M_2 \int_t^{T \vee} e^{Cs} \Delta_s^{\beta-1} \left[ |Y^n_s - Y^m_s|^2 \ln A_N \right].
\end{align*}
\[ + \frac{\ln A_N}{A_N} + |Y^n_s - Y^m_s| |Z^n_s - Z^m_s| \sqrt{\ln A_N} \mathbb{I}_{\{\Phi(s) \leq N\}} \, ds \]

\[ \leq \beta M_2 \int_t^{T'} e^{Cs} \Delta_s^{-\beta} \left[ \Delta_s \ln A_N + |Y^n_s - Y^m_s| |Z^n_s - Z^m_s| \sqrt{\ln A_N} \right] \mathbb{I}_{\{\Phi(s) \leq N\}} \, ds \]

Next let us deal with

\[ \beta \int_t^{T'} e^{Cs} \Delta_s^{-\beta} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s). \]

Actually, since \( dK^n_s = \mathbb{I}_{\{Y^n_s = L^n_s\}} \, dK^n_s \) and \( dK^m_s = \mathbb{I}_{\{Y^m_s = L^m_s\}} \, dK^m_s \) we have

\[ \beta \int_t^{T'} e^{Cs} \Delta_s^{-\beta} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \]

\[ = \beta \int_t^{T'} e^{Cs} \left( |Y^n_s - Y^m_s| + (A_N)^{-1} \right)^{\beta-1} (Y^n_s - Y^m_s) \mathbb{I}_{\{Y^n_s = L^n_s\}} \, dK^n_s \]

\[ - \beta \int_t^{T'} e^{Cs} \left( |Y^n_s - Y^m_s| + (A_N)^{-1} \right)^{\beta-1} (Y^n_s - Y^m_s) \mathbb{I}_{\{Y^n_s = L^n_s\}} \, dK^m_s. \]

Then, it follows that

\[ \beta \int_t^{T'} e^{Cs} \Delta_s^{-\beta} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \]

\[ = \beta \int_t^{T'} e^{Cs} \left( |L^n_s - Y^m_s| + (A_N)^{-1} \right)^{\beta-1} (L^n_s - Y^m_s) \mathbb{I}_{\{L^n_s = Y^m_s\}} \, dK^n_s \]

\[ - \beta \int_t^{T'} e^{Cs} \left( |Y^n_s - L^m_s| + (A_N)^{-1} \right)^{\beta-1} (Y^n_s - L^m_s) \mathbb{I}_{\{Y^n_s = L^m_s\}} \, dK^m_s. \]

Now, let \( F \) be the function \( (s, x, y) \mapsto \beta e^{Cs} (|x - y|^2 + (A_N)^{-1} \beta^{-1} (x - y) \mathbb{I}_{\{x - y \neq 0\}}. \)

Therefore,

\[ \beta \int_t^{T'} e^{Cs} \Delta_s^{-\beta} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \]

\[ = \int_t^{T'} F(s, L^n_s, Y^m_s) \, dK^n_s - \int_t^{T'} F(s, Y^n_s, L^m_s) \, dK^m_s. \]

A simple calculation shows that for every \((x, y) \in \mathbb{R}\) the functions \( x \in \mathbb{R} \mapsto F(s, x, y) \)

and \( y \in \mathbb{R} \mapsto F(s, x, y) \) are, respectively, non-decreasing and non-increasing. Since \( L^n_s \leq Y^n_s \) and \( L^m_s \leq Y^m_s \) for every \( s \in [0, T] \), we obtain

\[ \beta \int_t^{T'} e^{Cs} \Delta_s^{-\beta} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \]

\[ \leq \int_t^{T'} F(s, L^n_s, L^m_s) \, dK^n_s - \int_t^{T'} F(s, L^n_s, L^m_s) \, dK^m_s. \]
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It follows that

\[ \beta \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta - 1}{2}} (Y_n^s - Y_s^m) (dK_s^n - dK_s^m) \]

\[ \leq \beta \int_t^{T'} e^{Cs} \left( |L_s^n - L_s^m|^2 + (A_N)^{-1} \right)^{\frac{\beta - 1}{2}} (L_s^n - L_s^m) (dK_s^n - dK_s^m). \]

Put \( \Delta L_s^{n,m} = (|L_s^n - L_s^m|^2 + (A_N)^{-1})^{\frac{\beta - 1}{2}} (L_s^n - L_s^m) \), then from Hölder’s inequality and from Lemma 4.3 there exist \( C_1 \) and \( \tilde{K}_4 \) such that: \( \forall \rho \in ]1, 2[ \)

\[ \mathbb{E} \left[ \beta \int_t^{T'} e^{Cs} \Delta_s^{\frac{\beta - 1}{2}} (Y_n^s - Y_s^m) (dK_s^n - dK_s^m) \right] \leq \mathbb{E} \left[ \beta \int_t^{T'} e^{Cs} \Delta L_s^{n,m} (dK_s^n - dK_s^m) \right] \leq C_1 e^{\tilde{K}_4} \left( \sup_{0 \leq t \leq T} |\Delta L_t^{n,m}|^{\frac{\rho}{p-1}} \right)^{\frac{p-1}{p}}.

We choose \( C = C_N = \frac{2M_N^2}{2 - \beta} \ln A_N \), and we use Lemma 4.6 in [1] and we get:

\[ e^{C_N^t} \Delta_s^{\frac{\beta}{2}} + \frac{\beta (\beta - 1)}{4} \int_t^{T'} e^{C_N s} \Delta_s^{\frac{\beta - 1}{2}} |Z_s^n - Z_s^m|^2 \, ds \]

\[ \leq e^{C_N T'} \Delta_s^{\frac{\beta}{2}} - \beta \int_t^{T'} e^{C_N s} \Delta_s^{\frac{\beta - 1}{2}} (Y_s^n - Y_s^m, (Z_s^n - Z_s^m) \, dB_s) \]

\[ + \beta e^{C_N T'} \frac{1}{N^\kappa} \left[ \int_t^{T'} \Delta_s \, ds \right]^{\frac{\beta - 1}{2}} \times \left[ \int_t^{T'} \Phi(s)^2 \, ds \right]^{\frac{\beta}{2}} \]

\[ \times \left[ \int_t^{T'} |\varphi_n(s, Y_s^n, Z_s^n) - \varphi_m(s, Y_s^m, Z_s^m)|^\alpha \, ds \right]^{\frac{1}{\alpha}} \]

\[ + \beta e^{C_N T'} [2N^2 + v_1]^{\frac{\beta - 1}{2}} \left[ \int_t^{T'} \sup_{|y|, |z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| \, ds \right] \]

\[ + \beta \int_t^{T'} \sup_{|y|, |z| \leq N} |\varphi_m(s, y, z) - \varphi(s, y, z)| \, ds \]

\[ + \beta \int_t^{T'} e^{C_N s} \Delta_s^{\frac{\beta - 1}{2}} (Y_s^n - Y_s^m) (dK_s^n - dK_s^m). \]

Burkholder’s inequality and Hölder’s inequality (since \( \frac{(\beta - 1)}{2} + \frac{\kappa}{2} + \frac{1}{\alpha} = 1 \)) allow us to show that there exists a universal constant \( \ell > 0 \) such that \( \forall \delta' > 0 \),

\[ \mathbb{E} \left[ \sup_{(T' - \delta')^+ \leq t \leq T'} e^{C_N t} \Delta_t^{\frac{\beta}{2}} \right] + \mathbb{E} \left[ \int_{(T' - \delta')^+}^{T'} e^{C_N s} \Delta_s^{\frac{\beta - 1}{2}} |Z_s^n - Z_s^m|^2 \, ds \right] \]

\[ \leq \frac{\ell}{\beta - 1} e^{C_N T'} \left\{ \mathbb{E} \left[ \Delta_t^{\frac{\beta}{T'}} \right] + \frac{\beta}{N^\kappa} \left( \mathbb{E} \left[ \int_0^T \Delta_s \, ds \right] \right)^{\frac{\beta - 1}{2}} \left( \mathbb{E} \left[ \int_0^T \Phi(s)^2 \, ds \right] \right)^{\frac{\beta}{2}} \right\} ^{\frac{1}{\beta - 1}}. \]
\[
\times \left( \mathbb{E}\left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)|^\frac{2}{\beta} \, ds \right] \right)^{\frac{1}{2}} \\
+ \beta[2N^2 + v_1]^{\frac{\beta-1}{2}} \mathbb{E}\left[ \int_0^T \sup_{|y|, |z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| \, ds \right] \\
+ \int_0^T \sup_{|y|, |z| \leq N} |\varphi_m(s, y, z) - \varphi(s, y, z)| \, ds \right] \right) \}
+ C_1 e^{CN^\delta} \left( \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Delta L_{n,m}^n|^{\frac{p-1}{p}} \right] \right)^{\frac{p-1}{p}} \cdot \]

We use Lemma 4.1, Lemma 4.2 and Lemma 4.3 to obtain, \( \forall N > R, \)
\[
\mathbb{E}\left[ \sup_{(T' - \delta')^+ \leq t \leq T'} |Y^n_T - Y^m_T|^\beta \right] + \mathbb{E}\left[ \int_{(T' - \delta')^+}^{T'} \frac{|Z^n_s - Z^m_s|^2}{(|Y^n_s - Y^m_s|^2 + v_R)^{\frac{2}{\beta}}} \, ds \right] \\
\leq \frac{\ell}{\beta - 1} e^{CN^\delta'} \left( (A_N)^{-\frac{\beta}{2}} + \beta [4K_3^\frac{1}{p} (4T\tilde{K}_2 + Tv_R)^{\frac{\beta-1}{2}} (8T\tilde{K}_2 + 8\tilde{K}_1)^{\frac{\epsilon}{2}} \\
+ \mathbb{E}\left[ |Y^n_T - Y^m_T|^\beta \right] + \beta[2N^2 + v_1]^{\frac{\beta-1}{2}} [\rho_N(\varphi_n - \varphi) + \rho_N(\varphi_m - \varphi)] \right) \\
+ C_1 e^{CN^\delta'} \left( \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Delta L_{n,m}^n|^{\frac{p-1}{p}} \right] \right)^{\frac{p-1}{p}} \cdot \]

Hence for \( \delta' < (\beta - 1) \min\left( \frac{1}{4M_2}, \frac{\kappa}{2M_2^2} \beta \right) \) we derive
\[
\lim_{N \to +\infty} A_N^{\frac{2M_2^2 \beta}{\beta-1}} (A_N)^{\frac{\beta}{2}} = 0 \quad \text{and} \quad \lim_{N \to +\infty} A_N^{\frac{2M_2^2 \beta}{\beta-1}} (A_N)^{\frac{\beta}{2}} = 0. \]

Also we have \( (L^n_t)_{t \leq T} \) is a continuous and increasing sequence, moreover, \( \lim_{n \to +\infty} L^n_t = L_t. \) So from Dini’s theorem the convergence of \( L^n \) is uniform. Next by the dominated
convergence, we conclude that

$$\lim_{(n,m) \to +\infty} \left( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |\Delta L_{s}^{n,m}| \right]^{p-1} \right)^{-1/p} = 0.$$ 

Passing to the limits first on $(n,m)$ and next on $N$, and using assertion (c) of Lemma 4.2 to obtain the desired result.

Now we introduce the comparison theorem.

**Theorem 4.1:** Let $(\xi, f, L)$ and $(\xi', f', L')$ be two sets of data that satisfies all the assumptions; (H.1), (H.2), (H.3), and (H.4). And suppose in addition the followings:

(i) $\xi \leq \xi' \text{ P-a.s.}$
(ii) $f(t,y,z) \leq f'(t,y,z) \text{ dP dP a.e., } \forall (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$.
(iii) $L_t \leq L'_t; \text{ } \forall t \in [0,T] \text{ P-a.s.}$

Let $(Y, Z, K)$ be the solution of the reflected BSDE with data $(\xi, f, L)$ and $(Y', Z', K')$ the solution of the reflected BSDE with data $(\xi', f', L')$. Then,

$$Y_t \leq Y'_t, \text{ } 0 \leq t \leq T \text{ P-a.s.}$$

**Proof:** The arguments of this proof are standard. We defer the proof in Appendix.

**Remark 4.1:** If $L = -\infty$, then $dK = 0$ and the comparison theorem holds also in the standard case.

5. Existence and uniqueness

The main result of this section is the following theorem.

**Theorem 5.1:** Assume that (H.1), (H.2), (H.3) and (H.4) are satisfied. Then, Equation (1) has a unique solution.

**Proof:** We divide the proof into two steps.

**Step 1. Existence.**

Taking successively $T' = T, T' = (T - \delta')^+, T' = (T - 2\delta')^+ \cdots$ in Lemma 4.4. Then we obtain, for every $\beta \in ]1, \min(3 - \frac{2}{R}, 2)[$

$$\lim_{n,m \to \infty} \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^\beta \right] + \mathbb{E} \left[ \int_0^T \frac{|Z^n_s - Z^m_s|^2}{(|Y^n_s - Y^m_s|^2 + v_R)^{2-\beta/2}} \, ds \right] \right) = 0. \quad (25)$$

Since $\beta > 1$, Lemma 4.3 allows us to show that

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y_t|^\beta \right] = 0. \quad (26)$$
Next let us prove that
\[
\lim_{n,m \to +\infty} \mathbb{E} \left[ \int_0^T \left| Z^n_s - Z^m_s \right|^2 \, ds \right] = 0. \tag{27}
\]
It follows from Itô’s formula that:
\[
\left| Y^n_0 - Y^m_0 \right|^2 + \int_0^T \left| Z^n_s - Z^m_s \right|^2 \, ds
= 2 \int_0^T (Y^n_s - Y^m_s) (\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)) \, ds
+ 2 \int_0^T (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) - 2 \int_0^T (Y^n_s - Y^m_s) (Z^n_s - Z^m_s) \, dB_s. \tag{28}
\]
First we argue that the third term of the right side in (28) is a martingale. We can deduce from Burkholder–Davis–Gundy’s inequality and Lemma 4.3 that there exists a constant \( c > 0 \) such that:
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (Y^n_s - Y^m_s) (Z^n_s - Z^m_s) \, dB_s \right|^2 \right] \leq c \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y^n_s - Y^m_s|^2 \right] + c \mathbb{E} \left[ \int_0^T \left| Z^n_s - Z^m_s \right|^2 \, ds \right] < +\infty. \tag{29}
\]
Now we deal with the term \( \int_0^T (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \). Actually, since \( dK^n_s = \mathbb{1}_{\{Y^n_s = L^n_s\}} \, dK^n_s \) and \( dK^m_s = \mathbb{1}_{\{Y^m_s = L^m_s\}} \, dK^m_s \) and since \( L^m_s \leq Y^m_s \) and \( L^n_s \leq Y^n_s \) for every \( s \in [0, T] \), we obtain
\[
\int_0^T (Y^n_s - Y^m_s) (dK^n_s - dK^m_s)
= \int_0^T (Y^n_s - Y^m_s) \, dK^n_s - \int_0^T (Y^n_s - Y^m_s) \, dK^m_s
= \int_0^T (L^n_s - Y^n_s) \mathbb{1}_{\{L^n_s \neq Y^n_s\}} \, dK^n_s - \int_0^T (Y^n_s - L^n_s) \mathbb{1}_{\{Y^n_s \neq L^n_s\}} \, dK^n_s
\leq \int_0^T (L^n_s - L^m_s) \mathbb{1}_{\{L^n_s \neq L^m_s\}} \, dK^n_s - \int_0^T (L^m_s - L^n_s) \mathbb{1}_{\{L^m_s \neq L^n_s\}} \, dK^m_s
= \int_0^T (L^n_s - L^m_s) (dK^n_s - dK^m_s). \tag{30}
\]
Combining (28), (29) and (30) to obtain that there exists a constant \( c_1 > 0 \) such that:
\[
\mathbb{E} \left[ \int_0^T \left| Z^n_s - Z^m_s \right|^2 \, ds \right]
\[ \leq c_1 \mathbb{E} \left[ \int_0^T \left| Y^n_s - Y^m_s \right| \left| \varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s) \right| \, ds \right] 
+ c_1 \mathbb{E} \left[ \int_0^T (L^n_s - L^m_s)(dK^n_s - dK^m_s) \right]. \] (31)

Next by Hölder’s inequality we have
\[
\mathbb{E} \left[ \int_0^T |Y^n_s - Y^m_s| |\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)| \, ds \right]
\leq \mathbb{E} \left[ \left( \int_0^T |Y^n_s - Y^m_s|^\frac{2}{\alpha - 1} \, ds \right)^{\frac{\alpha - 1}{\alpha}} \left( \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)|^{\frac{\alpha}{\alpha - 1}} \, ds \right)^{\frac{\alpha - 1}{\alpha}} \right]
\leq \left[ \int_0^T |Y^n_s - Y^m_s|^\beta \, ds \right]^{\frac{1}{\beta}} \mathbb{E} \left[ \int_0^T |Y^n_s - Y^m_s|^{\frac{\beta}{\beta - 1}} \, ds \right]^{\frac{\beta - 1}{\beta}}
\times \mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)|^{\frac{\alpha}{\alpha - 1}} \, ds \right]^{\frac{\alpha - 1}{\alpha}}. \] (32)

We plug the last inequality in (31). Then, Proposition 3.1, Lemmas 4.1, 4.2 and 4.3 shows that there exists a positive constant \( C' \) such that:
\[
\mathbb{E} \left[ \int_0^T |Z^n_s - Z^m_s|^2 \, ds \right] 
\leq C' \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^\beta \right]^{\frac{1}{\beta}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |L^n_t - L^m_t|^{\frac{p}{p - 1}} \right]^{\frac{p - 1}{p}}. \] (33)

From (25) and the fact that \( L^n \) converge uniformly to \( L \) it follows that:
\[
\lim_{n,m \to +\infty} \mathbb{E} \left[ \int_0^T |Z^n_s - Z^m_s|^2 \, ds \right] = 0. \] (34)

Finally, we use Lemma 4.3 to show that
\[
\lim_{n \to +\infty} \mathbb{E} \left[ \int_0^T |Z^n_s - Z_s|^2 \, ds \right] = 0. \] (35)

On the other hand
\[
\mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s)| \, ds \right] 
\leq \mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s)| \cdot \mathbb{1}_{|Y^n_s| + |Z^n_s| \leq N} \, ds \right]
\]
\[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s)| \left( (|Y^n_s| + |Z^n_s|)^{(2-\frac{2}{\beta})} \right) \mathbb{I}_{(1+|Y^n_s|+|Z^n_s| \geq N)} \, ds \]

\[ \leq \rho N(\varphi_n - \varphi) + \frac{2K_1^{\frac{1}{3}} [T\tilde{K}_2 + \tilde{K}_1]^{1-\frac{1}{\beta}}}{N^{2-\frac{2}{\beta}}} \]

Passing to the limit first on \( n \) and next on \( N \) we obtain

\[ \lim_{{n \to +\infty}} \mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s)| \, ds \right] = 0. \quad (36) \]

Finally, we use Lemmas 4.2 and 4.3 to show that,

\[ \lim_{{n \to +\infty}} \mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s)| \, ds \right] = 0. \quad (37) \]

For the existence of the process \( K \), we have

\[ K^n_t - K^m_t = (Y^n_0 - Y^m_0) - (Y^n_t - Y^m_t) + \int_0^t (Z^n_s - Z^m_s) \, dB_s \]

\[ - \int_0^t (\varphi_n(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)) \, ds. \]

Then

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |K^n_t - K^m_t| \right] \leq \mathbb{E} \left[ |Y^n_0 - Y^m_0| + \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t| + \sup_{0 \leq t \leq T} \left| \int_0^t (Z^n_s - Z^m_s) \, dB_s \right| \right] \]

\[ + \mathbb{E} \left[ \int_0^T |\varphi_n(s, Y^n_s, Z^n_s) - \varphi(s, Y^n_s, Z^n_s)| \, ds \right] \]

\[ + \mathbb{E} \left[ \int_0^T |\varphi(s, Y^n_s, Z^n_s) - \varphi_m(s, Y^m_s, Z^m_s)| \, ds \right]. \]

Consequently from Doob’s martingale inequality, (25), (34) and (37) we have that:

\[ \lim_{{n,m \to +\infty}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |K^n_t - K^m_t| \right] = 0. \quad (38) \]

Combining (38) with Lemma 4.3 allows us to show that:

\[ \lim_{{n \to +\infty}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |K^n_t - K_t| \right] = 0. \quad (39) \]

Additionally, since \( \int_0^T (Y^n_s - L^n_s) \, dK^n_s \leq 0 \) it follows from (26), (39) and \( \lim_{{n \to +\infty}} L^n_t = L_t \) that \( \int_0^T (Y_s - L_s) \, dK_s \leq 0 \), and from the fact that \( L_t \leq Y_t \) we can deduce that \( \int_0^T (Y_s - L_s) \, dK_s = 0 \). The existence is proved.
Step 2. Uniqueness.
Let \((Y, Z, K)\) and \((Y', Z', K')\) be two solutions of Equation (1). Arguing as in the proof of Lemma 4.4, one can show that:

for every \(R > 2, \beta \in ]1, \min(3 - \frac{2}{\beta}, 2)[, \delta' < (\beta - 1) \min(\frac{1}{4M_2}, \frac{3 - \frac{2}{\beta} - \beta}{2rM_2})\) and \(\varepsilon > 0\) there exists \(N_0 > R\) such that for all \(N > N_0, \forall T' \leq T\)

\[
\mathbb{E}\left[ \sup_{(T' - \delta')^+ \leq t \leq T'} |Y_t - Y'_t|^\beta \right] + \mathbb{E}\left[ \int_{(T' - \delta')^+}^{T'} \frac{|Z_s - Z'_s|^2}{(|Y_s - Y'_s|^2 + \nu R)^{2-\beta/2}} \, ds \right] \\
\leq \varepsilon + \frac{\ell}{\beta - 1} e^{CN_0} \mathbb{E}\left[ |Y_{T'} - Y'_{T'}|^\beta \right].
\]

Taking successively \(T' = T, T' = (T - \delta')^+, T' = (T - 2\delta')^+ \cdots\), we obtain immediately \(Y_t = Y'_t\) and also \(Z_t = Z'_t\). Finally, the uniqueness of the process \(K\) is deduced from the fact that \(Y\) and \(Z\) are unique.

\[\square\]

6. Application in mixed stochastic control with finite horizon

In this section, we use the result on finite horizon reflected BSDEs with one barrier to deal with the mixed stochastic control problem. In the sequel \(\Omega = C([0, T], \mathbb{R}^d)\) is the space of continuous functions from \([0, T]\) to \(\mathbb{R}^d\).

Let us consider a mapping \(\sigma : (t, \omega) \in [0, T] \times \Omega \mapsto \sigma(t, \omega) \in \mathbb{R}^d \otimes \mathbb{R}^d\) satisfying the following assumptions,

(1.1) \(\sigma\) is \(\mathcal{P}\)-measurable.

(1.2) There exists a constant \(C > 0\) such that \(|\sigma(t, \omega) - \sigma(t, \omega')| \leq C||\omega - \omega'||_t\) and \(|\sigma(t, \omega)| \leq C(1 + ||\omega||_t), \) where for any \((\omega, \omega') \in \Omega^2\) and \(t \leq T, \ ||\omega||_t = \sup_{s \leq t} ||\omega_s||.\)

(1.3) For any \((t, \omega) \in [0, T] \times \Omega\), the matrix \(\sigma(t, \omega)\) is invertible and \(|\sigma^{-1}(t, \omega)| \leq C\) for a positive constant \(C\).

Let \(x_0 \in \mathbb{R}^d\) and \(x = (x_t)_{t \leq T}\) be the solution of the following standard functional differential equation:

\[x_t = x_0 + \int_0^t \sigma(s, x) \, dB_s, \quad t \leq T;\]  

(40)

the process \((x_t)_{t \leq T}\) exists, since \(\sigma\) satisfies (1.1) \(-\) (1.3) (see, [23, p. 375]). Moreover,

\[\mathbb{E}[(||x||_T)^n] < +\infty, \quad \forall \, n \in [1, +\infty] \text{(see, [25, p. 306])}.\]  

(41)

Now let \(\mathcal{A}\) be a compact metric space and \(\mathcal{U}\) be the space of \(\mathcal{P}\)-measurable processes \(u := (u_t)_{t \leq T}\) with value in \(\mathcal{A}\). Let \(f : [0, T] \times \Omega \times \mathcal{A} \mapsto \mathbb{R}^d\) be such that:

(1.4) For each \(a \in \mathcal{A},\) the function \((t, \omega) \mapsto f(t, \omega, a)\) is predictable.

(1.5) For each \((t, \omega),\) the mapping \(a \mapsto f(t, \omega, a)\) is continuous.
(1.6) There exists a real constant $\tilde{K} > 0$ such that:

$$|f(t, \omega, a)| \leq \tilde{K}(1 + ||\omega||_t), \quad \forall 0 \leq t \leq T, \quad \omega \in \Omega, \quad a \in \mathcal{A}. \quad (42)$$

Under the previous assumptions, for a given admissible control strategy $u \in \mathcal{U}$, the exponential process,

$$\Lambda_t^u := \exp\left\{\int_0^t \sigma^{-1}(s,x)f(s,x,u_s) \, dB_s - \frac{1}{2} \int_0^t |\sigma^{-1}(s,x)f(s,x,u_s)|^2 \, ds\right\}, 0 \leq t \leq T,$$

is a martingale. Therefore, $\mathbb{E}[\Lambda_T^u] = 1$ (see [19], pp. 191 and 200). The Girsanov theorem guarantees then that the process

$$B_t^u = B_t - \int_0^t \sigma^{-1}(s,x)f(s,x,u_s) \, ds, \quad 0 \leq t \leq T, \quad (43)$$

is a Brownian motion with respect to the filtration $\mathcal{F}_t$, under the new probability measure

$$P^u(B) = \mathbb{E}[\Lambda_T^u \mathbb{1}_B], \quad B \in \mathcal{F}_T,$$

which is equivalent to $P$. It is now clear from Equations (40) and (43) that

$$x_t = x_0 + \int_0^t f(s,x,u_s) \, ds + \int_0^t \sigma(s,x) \, dB_s^u, \quad 0 \leq t \leq T. \quad (44)$$

(1.7) $h : [0, T] \times \Omega \times \mathcal{A} \mapsto \mathbb{R}$ is measurable and for each $(t, \omega)$ the mapping $a \mapsto h(t, \omega, a)$ is continuous. In addition, there exists a positive constant $\tilde{K}$ such that:

$$|h(t, \omega, a)| \leq \tilde{K}(1 + ||\omega||_t), \quad \forall 0 \leq t \leq T, \quad \omega \in \Omega, \quad a \in \mathcal{A}. \quad (45)$$

(1.8) $g : [0, T] \times \Omega \mapsto \mathbb{R}$ and $g_1 : \Omega \mapsto \mathbb{R}$ are two continuous functions for which there exists a positive constant $C$ such that:

$$|g(t, \omega)| + |g_1(\omega)| \leq C(1 + ||\omega||_t), \quad \forall (t, \omega) \in [0, T] \times \Omega. \quad (46)$$

We define the payoff

$$J(u, \tau) = \mathbb{E}^u\left[\int_0^\tau h(s,x,u_s) \, ds + g(\tau, x) \mathbb{1}_{[\tau \leq T]} + g_1(x_\tau) \mathbb{1}_{[\tau = T]}\right],$$

and let us set

$$H(t,x,z,u_t) := z \sigma^{-1}(t,x)f(t,x,u_t) + h(t,x,u_t) \quad \forall (t,x,z,u_t) \in [0, T] \times \Omega \times \mathbb{R}^d \times \mathcal{A}. \quad (47)$$

The function $H$ is called the Hamiltonian associated with stochastic control such that: \forall $z \in \mathbb{R}^d$, the process $(H(t,x,z,u_t))_{t \leq T}$ is $\mathcal{P}$-measurable.

The Hamiltonian function defined in (47) attains its supremum over the set $\mathcal{A}$ at some $u^* \equiv u^*(t,x,z) \in \mathcal{A}$, for any given $(t,x,z) \in [0, T] \times \Omega \times \mathbb{R}^d$, namely,

$$\sup_{u \in \mathcal{A}} H(t,x,z,u) = H(t,x,z,u^*(t,x,z)). \quad (48)$$

(This is the case, for instance, if the set $\mathcal{A}$ is compact and the mapping $u \mapsto H(t,x,z,u)$ continuous). Then it can be shown (see Lemma 1 in Benes [7]), that the mapping $u^* : [0, T] \times \Omega \times \mathbb{R}^d \mapsto \mathcal{A}$ can be selected to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable.
Now let \( H^*(t,x,z) = \sup_{u \in \mathcal{A}} H(t,x,z,u) \) where \( x \) is the solution of (40).

**Theorem 6.1:** Let \( (Y^*,Z^*,K^*) \) be the solution of the finite horizon reflected BSDE associated with \( (g_1(x_T),H^*,g(t,x)) \) and \( \tau^* = \inf\{t \in [0,T], Y^*_t < g(t,x)\} \wedge T \), then \( Y^*_0 = J(u^*,\tau^*) \) and \( (u^*,\tau^*) \) is the optimal strategy for the controller.

**Proof:** We consider the reflected BSDE associated with \( (g_1(x_T),H^*,g(t,x)) \)

\[
Y^*_t = g_1(x_T) + \int_t^T H^*(s,x,Z^*_s) \, ds + K^*_T - K^*_t - \int_t^T Z^*_s \, dB_s. \tag{49}
\]

Now we will show that the Hamiltonian \( H^* \) satisfies (H.3) and (H.4). The proof is actually similar to the one in [1] but for the sake of the reader we give it again.

We begin by showing that \( H^* \) satisfies (H.3). Actually it is not difficult to see that \( \forall (t,x,z) \in [0,T] \times \Omega \times \mathbb{R}^d \) and \( |z| \) large enough, there exist two constant \( C > 0 \) and \( c_0 > 0 \) such that:

\[
|H^*(t,x,z)| \leq Ce^{\|x\|_1} + c_0|z|\sqrt{\ln(|z|)}.
\]

Next we move on to prove that \( H^* \) also satisfies (H.4). Indeed for every \( |y|,|y'|,|z|,|z'| \leq N \) we have:

\[
(y - y') \left( H^*(t,x,z) - H^*(t,x,z') \right) \mathbb{1}_{\{v_1(\omega) \leq N\}} \\
\leq |y - y'| |H^*(t,x,z) - H^*(t,x,z')| \mathbb{1}_{\{v_1(\omega) \leq N\}} \\
= |\sigma^{-1}(t,x)||y - y'||z - z'| |f(t,x,u^*)| \mathbb{1}_{\{v_1(\omega) \leq N\}}.
\]

Now we take \( v_1 := e^{vf(t,x,u^*)^2} \) and since \( \sigma^{-1} \) is bounded by a constant \( C \), it follows that:

\[
(y - y') \left( H^*(t,x,z) - H^*(t,x,z') \right) \mathbb{1}_{\{v_1(\omega) \leq N\}} \\
\leq C|y - y'||z - z'| |f(t,x,u^*)| \mathbb{1}_{\{e^{vf(t,x,u^*)^2} \leq N\}} \\
\leq C|y - y'||z - z'| \sqrt{\ln(N)}.
\]

It remains to show that \( e^{vf(t,x,u^*)^2} \) belongs to \( L^q(\Omega \times [0,T];\mathbb{R}_+) \) for some \( q \). Actually, there exists a positive constant \( \bar{K} \) such that:

\[
\mathbb{E} \left[ \int_0^T e^{vf(s,x,u^*)^2} \, ds \right] \leq \mathbb{E} \left[ \int_0^T e^{2q\bar{K}^2(1+\|x\|_1^2)} \, ds \right] \\
\leq Te^{2q\bar{K}^2} \mathbb{E} \left[ e^{2q\bar{K}^2\|x\|_1^2} \right].
\]

Since \( \sigma \) is with linear growth, the result follows for \( q \) small enough.

Therefore, by Theorem 5.1, (49) has a unique solution \( (Y^*,Z^*,K^*) \). Now since \( Y^*_0 \) is a deterministic constant we have,

\[
Y^*_0 = \mathbb{E}^u [Y^*_0] = \mathbb{E}^u \left[ g_1(x_T) + \int_0^T H^*(s,x,Z^*_s) \, ds + K^*_T - \int_0^T Z^*_s \, dB_s \right]
\]
\[
= \mathbb{E}^{u^*} \left[ Y_{\tau^*}^* + \int_0^{\tau^*} H^*(s, x, Z^*_s) \, ds + K_{\tau^*}^* - \int_0^{\tau^*} Z^*_s \, dB_s \right] \\
= \mathbb{E}^{u^*} \left[ Y_{\tau^*}^* + \int_0^{\tau^*} h(s, x, u^*_s) \, ds + K_{\tau^*}^* - \int_0^{\tau^*} Z^*_s \, dB^u_s \right].
\]

From the definition of \( \tau^* \) and the properties of reflected BSDEs we know that the process \( K_{\tau^*}^* \) does not increase between 0 and \( \tau^* \), then \( K_{\tau^*}^* = 0 \). Moreover, \( \{ \int_0^t Z^*_s \, dB^u_s, \, t \in [0, T] \} \) is \( P^u^* \)-martingale then
\[
Y^*_0 = \mathbb{E}^{u^*} \left[ Y_{\tau^*}^* + \int_0^{\tau^*} h(s, x, u^*_s) \, ds \right].
\]

Since \( Y_{\tau^*}^* = g_1(x_T) \mathbb{I}_{\{\tau^* = T\}} + g(\tau^*, x) \mathbb{I}_{\{\tau^* < T\}} \), then we get \( Y^*_0 = J(u^*, \tau^*) \). Now, let us consider \( u \) an element of \( \mathcal{U} \) and \( \tau \) a stopping time. Since \( P \) and \( P^u \) are equivalent probabilities on \( (\Omega, \mathcal{F}) \) we obtain
\[
Y^*_0 = \mathbb{E}^u \left[ Y^*_0 \right] \\
= \mathbb{E}^u \left[ Y^*_0 + \int_0^{\tau^*} H^*(s, x, Z^*_s) \, ds + K^* - \int_0^{\tau^*} Z^*_s \, dB_s \right] \\
= \mathbb{E}^u \left[ Y^*_0 + \int_0^{\tau^*} h(s, x, u_s) \, ds + \int_0^{\tau^*} \left( H^*(s, x, Z^*_s) - H(s, x, Z^*_s, u_s) \right) \, ds \\
+ K_{\tau^*}^* - \int_0^{\tau^*} Z^*_s \, dB^u_s \right].
\]

But \( K_{\tau^*}^* \geq 0 \) and from (48) we have, \( H^*(s, x, Z^*_s) - H(s, x, Z^*_s, u_s) \geq 0 \) for any \( s \in [0, T] \). On the other hand
\[
Y^*_\tau = g_1(x_T) \mathbb{I}_{\{\tau = T\}} + Y^*_\tau \mathbb{I}_{\{\tau < T\}} \\
\geq g_1(x_T) \mathbb{I}_{\{\tau = T\}} + g(\tau, x) \mathbb{I}_{\{\tau < T\}}
\]

and \( \{ \int_0^t Z^*_s \, dB^u_s, \, t \in [0, T] \} \) is \( P^u \)-martingale. It follows that
\[
J(u^*, \tau^*) = Y^*_0 \geq \mathbb{E}^u \left[ \int_0^{\tau^*} h(s, x, u_s) \, ds + g(\tau, x) \mathbb{I}_{\{\tau < T\}} + g_1(x_T) \mathbb{I}_{\{\tau = T\}} \right] = J(u, \tau).
\]

\[\blacksquare\]

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Appendix 1. Proof of Theorem 4.1

First let \((X_t)_{t \leq T}\) be an \([-\infty, \infty]\)-valued continuous semimartingale and \(X_t^+ = \max\{X_t, 0\}\). From Tanaka’s formula we have
\[
dX_t^+ = \mathbb{1}_{\{X_t > 0\}} \, dX_t + \frac{1}{2} \, dL_t, \quad t \leq T
\]
where \((L_t)_{t \leq T}\) is an increasing adapted process such that \(\int_0^T X_s \, dL_s = 0, \forall t \leq T\). Next by using Itô’s formula and the fact that \((X_t^+)^2 = X_t X_t^+\) we obtain:
\[
d(X_t^+)^2 = 2X_t^+ \, dX_t + \mathbb{1}_{\{X_t > 0\}} \, d < X, X >, \quad t \leq T. \tag{A1}
\]

Now let \((Y_t, Z_t, K_t)_{t \leq T}\) and \((Y'_t, Z'_t, K'_t)_{t \leq T}\) be the solution of the reflected BSDE with lower barrier associated with \((\xi, f, L)\) and \((\xi', f', L')\), respectively. We put \((\tilde{Y}, \tilde{Z}, \tilde{K}) = (Y - Y', Z - Z', K - K')\). For \(T' \in [0, T]\), it follows from (A1) that for all \(t \leq T'\),
\[
|\tilde{Y}_t^+|^2 + \int_t^{T'} \mathbb{1}_{\{Y_s > Y'_s\}} |\tilde{Z}_s|^2 \, ds = |\tilde{Y}_t^+|^2 + 2 \int_t^{T'} \tilde{Y}_s^+ (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) \, ds
\]
\[
+ 2 \int_t^{T'} \tilde{Y}_s^+ d\tilde{K}_s - 2 \int_t^{T'} \tilde{Y}_s^+ \tilde{Z}_s \, dB_s,
\]
We set, \(\Delta_t := |\tilde{Y}_t^+|^2 + (A_N)^{-1} \mathbb{1}_{\{Y_s > Y'_s\}}\). Then for \(C > 0\) and \(1 < \beta < \min\{3 - \frac{2}{\delta}, 2\}\), Itô’s formula shows that,
\[
e^{Ct} \Delta_t^\beta + C \int_t^{T'} e^{Cs} \Delta_s^\beta \, ds
\]
\[
= e^{Ct'} \Delta_t^\beta + \beta \int_t^{T'} e^{Cs} \Delta_s^\beta (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) \, ds
\]
\[
- \frac{\beta}{2} \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \mathbb{1}_{\{Y_s > Y'_s\}} |\tilde{Z}_s|^2 \, ds + \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} |\tilde{Y}_s^+| \, dB_s
\]
\[
+ \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-2} |\tilde{Y}_s^+|^2 \, ds + J_1 + J_2 + J_3,
\]
Put \(\Phi(s) = |Y_s| + |Y'_s| + |Z_s| + |Z'_s| + v_s\). Then
\[
e^{Ct} \Delta_t^\beta + C \int_t^{T'} e^{Cs} \Delta_s^\beta \, ds
\]
\[
= e^{Ct'} \Delta_t^\beta - \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \tilde{Y}_s^+ \tilde{Z}_s \, dB_s
\]
\[
- \frac{\beta}{2} \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \mathbb{1}_{\{Y_s > Y'_s\}} |\tilde{Z}_s|^2 \, ds + \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \tilde{Y}_s^+ \, dB_s
\]
\[
+ \beta \frac{(2 - \beta)}{2} \int_t^{T'} e^{Cs} \Delta_s^{\beta-2} |\tilde{Y}_s^+|^2 \, ds + J_1 + J_2 + J_3,
\]
where
\[
J_1 := \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \tilde{Y}_s^+ (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) \mathbb{1}_{\{\Phi(s) > N\}} \, ds.
\]
\[
J_2 := \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \tilde{Y}_s^+ (f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)) \mathbb{1}_{\{\Phi(s) \leq N\}} \, ds.
\]
\[
J_3 := \beta \int_t^{T'} e^{Cs} \Delta_s^{\beta-1} \tilde{Y}_s^+ (f'(s, Y'_s, Z'_s) - f'(s, Y'_s, Z'_s)) \mathbb{1}_{\{\Phi(s) \leq N\}} \, ds.
\]
Now we will estimate $J_1$, $J_2$ and $J_3$.

We start with $J_1$. Let $\kappa = 3 - \frac{\beta}{2} - \beta$. Since \(\frac{(\beta - 1)}{4} + \frac{\kappa}{2} + \frac{1}{2} = 1\), we use Hölder’s inequality to obtain

\[
J_1 \leq \beta e^{CT'\frac{1}{N^\kappa}} \int_T^T \Delta_t^{\frac{\beta - 1}{2}} \Phi^k(s) |f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)| \, ds
\]

\[
\leq \beta e^{CT'\frac{1}{N^\kappa}} \left[ \int_T^T \Delta_t \, ds \right]^{\frac{\beta - 1}{2}} \left[ \int_T^T \Phi(s)^2 \, ds \right]^{\frac{\kappa}{2}}
\times \left[ \int_T^T |f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)|^{\frac{\kappa}{2}} \, ds \right]^{\frac{1}{2}}.
\]

Now for $J_2$, since $f(t, y, z) \leq f'(t, y, z)$, we have that $J_2 \leq 0$.

Finally for $J_3$, using assumption (H.4), we get

\[
J_3 \leq \beta M_2 \int_T^T e^{Cs \Delta_t^{\frac{\beta - 1}{2}}} \left[ |\bar{Y}_s^+|^2 \ln A_N + \frac{\ln A_N}{A_N} \right] \mathbb{I}_{\{Y_s > Y'_s\}} \, ds
\]

\[
\leq \beta M_2 \int_T^T e^{Cs \Delta_t^{\frac{\beta - 1}{2}}} \left[ \Delta_s \ln A_N + \bar{Y}_s^+ |\bar{Z}_s| \sqrt{\ln A_N} \right] \mathbb{I}_{\{\Phi(s) < N\}} \, ds.
\]

Next let us deal with $\beta \int_T^T e^{C s \Delta_t^{\frac{\beta - 1}{2}}} \bar{Y}_s^+ (dK_s - dK'_s)$. Since on $\{Y > Y'\}$, $Y > L' \geq L$ then

\[
\int_T^T e^{C s \Delta_t^{\frac{\beta - 1}{2}}} \bar{Y}_s^+ (dK_s - dK'_s) = - \int_T^T e^{C s \Delta_t^{\frac{\beta - 1}{2}}} \bar{Y}_s^+ dK'_s \leq 0.
\] (A2)

We apply Lemma 4.6 in [1] and we choose $C = C_N = \frac{2M_3\beta}{3}\ln A_N$, then

\[
e^{CNt} \Delta_t^{\frac{\beta}{2}} + \frac{\beta(\beta - 1)}{4} \int_T^T e^{CNs \Delta_s^{\frac{\beta - 1}{2}}} |\bar{Z}_s|^2 \mathbb{I}_{\{Y_s > Y'_s\}} \, ds
\]

\[
\leq e^{CNt} \Delta_t^{\frac{\beta}{2}} - \beta \int_T^T e^{CNs \Delta_s^{\frac{\beta - 1}{2}}} \bar{Y}_s^+ \bar{Z}_s \, dB_s
\]

\[
+ \beta e^{CNt} \frac{1}{N^\kappa} \left[ \int_T^T \Delta_s \, ds \right]^{\frac{\beta - 1}{2}} \times \left[ \int_T^T \Phi(s)^2 \, ds \right]^{\frac{\kappa}{2}}
\times \left[ \int_T^T |f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)|^{\frac{\kappa}{2}} \, ds \right]^{\frac{1}{2}}.
\]

Burkholder’s inequality and Hölder’s inequality (since \(\frac{(\beta - 1)}{2} + \frac{\kappa}{2} + \frac{1}{2} = 1\)) allow us to show that there exists a universal constant $\ell' > 0$ independent from $N$ such that: $\forall \delta' > 0$,

\[
\mathbb{E} \left[ \sup_{(T - \delta') \frac{1}{2} \leq t \leq T} \left[ e^{CNt} \Delta_t^{\frac{\beta}{2}} \right] \right]
\]

\[
\leq \ell' e^{CNt} \left\{ \mathbb{E} \left[ \Delta_t^{\frac{\beta}{2}} \right] + \frac{\beta}{N^\kappa} \mathbb{E} \left[ \int_0^T \Delta_s \, ds \right]^{\frac{\beta - 1}{2}} \mathbb{E} \left[ \int_0^T \Phi(s)^2 \, ds \right]^{\frac{\kappa}{2}}
\times \mathbb{E} \left[ \int_0^T |f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s)|^{\frac{\kappa}{2}} \, ds \right]^{\frac{1}{2}} \right\}.
\]
From Lemma 4.3 there exists another universal constant $\ell'' > 0$ that changes from line to line such that: $\forall N > R,$

$$
\mathbb{E}\left[ \sup_{(T' - \delta')^+ \leq t \leq T'} |\tilde{Y}_t^+|^\beta \right] \leq \ell'' e^{C_N \delta'} \left\{ (A_N)^{\frac{\delta}{4}} + \mathbb{E}\left[ |\tilde{Y}_T^+|^\beta \right] + \frac{1}{N^\kappa} \right\}
$$

$$
\leq \ell'' \left\{ e^{C_N \delta'} \mathbb{E}\left[ |\tilde{Y}_T^+|^\beta \right] + \frac{2M^2 \beta}{(A_N)^{\frac{\beta-1}{2}}} + \frac{2M^2 \beta}{(A_N)^{\frac{\beta-1}{2}}} \right\}.
$$

Hence for $\delta' < (\beta - 1) \min\left( \frac{1}{4M^2}, \frac{\kappa}{2M^2 \beta} \right)$ we derive

$$
\lim_{N \to +\infty} \frac{A_N^{\frac{\beta-1}{2}}}{(A_N)^{1}} = 0 \quad \text{and} \quad \lim_{N \to +\infty} \frac{A_N^{\frac{\beta-1}{2}}}{(A_N)^{1}} = 0.
$$

Then it follows that $\forall \varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $N > N_0$

$$
\mathbb{E}\left[ \sup_{(T' - \delta')^+ \leq t \leq T'} |\tilde{Y}_t^+|^\beta \right] \leq \ell'' e^{C_N \delta'} \mathbb{E}\left[ |\tilde{Y}_T^+|^\beta \right] + \varepsilon. \quad \text{(A3)}
$$

Now taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+$, in (A3), we obtain

$$
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |(Y_t - Y_t')^+|^\beta \right] = 0. \quad \text{(A4)}
$$

The proof is now complete.