ON THE HILBERT FUNCTION OF A FINITE SCHEME
CONTAINED IN A QUADRIC SURFACE

MARIO MAICAN

ABSTRACT. Consider a finite scheme of length $l$ contained in a smooth quadric
surface over the complex numbers. We determine the number of linearly inde-
dependent curves passing through the scheme, of degree at least $l-1$.

1. INTRODUCTION

All schemes considered in this paper will be over $\mathbb{C}$. We shall work on the smooth
quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. We choose homogeneous coordinates $(x : y)$ on the first
$\mathbb{P}^1$ and $(z : w)$ on the second $\mathbb{P}^1$. Given a positive integer $l$, we denote by $\text{Hilb}(l)$
the Hilbert scheme of $l$ points in $\mathbb{P}^1 \times \mathbb{P}^1$. Given non-negative integers $m$ and
$n$, which are not both zero, we denote by $\text{Hilb}(l, (m,n))$ the flag Hilbert scheme
parametrizing pairs $(Z,C)$, where $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bidegree $(m,n)$ and
$Z \subset C$ is a finite subscheme of length $l$. Given $Z \in \text{Hilb}(l)$ we define its Hilbert
function by analogy with the Hilbert function of a finite subscheme of the projective
space, namely

$$Z \geq 0 \times Z \geq 0 \ni (m,n) \mapsto \dim_{\mathbb{C}} H^0(O(m,n)) - \dim_{\mathbb{C}} H^0(\mathcal{I}_Z(m,n)).$$

In this paper we determine the values of the Hilbert function of $Z$ for $m+n \geq l-1$,
see Theorem 5.3. As a consequence, we compute the topological Euler characteristic
of $\text{Hilb}(l, (m,n))$, see Theorem 6.4. The answer is given in terms of the Euler
characteristic of $\text{Hilb}(l)$, which was computed in [2], and in terms of the numbers
$\xi(l,m)$ and $\xi(l,n)$, which can be computed by an easy algorithm. The proof of
Theorem 5.3 rests on the Vanishing Theorem 4.4 which relies on the fact that
there are curves of arithmetic genus zero and arbitrary degree in $\mathbb{P}^1 \times \mathbb{P}^1$. For $Z$
contained in such a curve, an analysis can be performed, as in section 4.

The paper is organized as follows. In section 2 we recall a few basic facts concerning
the cohomology of a finite subscheme of a multiple line. In section 3, following
Ellingsrud and Strømme [2], we review the geometry of the punctual Hilbert scheme
of a surface. This will be needed in section 6 which is devoted to the calculation of
the Euler characteristic of the flag Hilbert scheme. Sections 4 and 5 are occupied by
the Vanishing Theorem, respectively, by the description of the Brill-Noether loci in
$\text{Hilb}(l)$, i.e. the loci on which the dimension of the fibers of the forgetful morphism
$\text{Hilb}(l, (m,n)) \to \text{Hilb}(l)$ jumps.

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loci.
2. Finite schemes contained in a multiple line

In this section we collect several well-known facts about finite subschemes of a multiple projective line. Let \( \nu \geq 1 \) be an integer. We denote by \( \nu L \) the multiple projective line \( \text{Spec} \mathbb{C}[z]/(z^{\nu}) \times \mathbb{P}^1 \). Given an integer \( m \), we write \( \mathcal{O}_{\nu L}(m) = \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(m) \).

**Proposition 2.1.** Let \( Z \subset \nu L \) be a zero-dimensional subscheme. Consider the restricted scheme \( Z_0 = Z \cap L \). We claim that the ideal sheaf of \( Z_0 \) in \( Z \) is isomorphic, as an \( \mathcal{O}_Z \)-module, to the structure sheaf of a finite scheme \( Z' \subset (\nu - 1)L \).

Let \( Z \subset L \) be a finite subscheme. By induction on \( \nu \), we define a string of non-negative integers \( \rho(Z, \nu L) = (r_0, \ldots, r_{\nu-1}) \), as follows. If \( \nu = 1 \), we put \( r_0 = \text{length}(Z) \). If \( \nu \geq 2 \), we consider the schemes \( Z_0 \) and \( Z' \) from Proposition 2.1. We write \( \rho(Z', (\nu - 1)L) = (r_0', \ldots, r_{\nu-2}') \) and we let \( r_0 = \text{length}(Z_0) \) and \( r_i = r_{i-1}' \) for \( 1 \leq i \leq \nu - 1 \).

**Remark 2.2.** Clearly, \( r_0 + \cdots + r_{\nu-1} = \text{length}(Z) \) and \( r_0 \geq \cdots \geq r_{\nu-1} \).

**Proposition 2.3.** Let \( \nu \geq 1 \) and \( m \) be integers. Let \( Z \subset \nu L \) be a zero-dimensional scheme. Let \( \rho(Z, \nu L) = (r_0, \ldots, r_{\nu-1}) \) be as above. We claim that

\[
\dim_{\mathbb{C}} H^1(I_{Z, \nu L}(m)) = \sum_{0 \leq i < \nu-1 \atop \mu + 1 < r_i} (r_i - m - 1).
\]

Let \( Z \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a zero-dimensional scheme. Let \( \mu(Z) \) be the smallest integer \( \mu \) such that there exists a curve \( M \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of bidegree \((\mu, 0)\) containing \( Z \). Let \( \nu(Z) \) be the smallest integer \( \nu \) such that there exists a curve \( N \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of bidegree \((0, \nu)\) containing \( Z \). Note that \( M \) and \( N \) are unique, so they may be denoted by \( M(Z) \), respectively, \( N(Z) \). Let \( D \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a line of bidegree \((1, 0)\). If \( M = \mu D \), then \( \rho(Z, \mu D) \) depends only on \( Z \), so it may be denoted by \( \sigma(Z) \). Let \( E \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a line of bidegree \((0, 1)\). If \( N = \nu E \), then \( \rho(Z, \nu E) \) depends only on \( Z \), so it may be denoted by \( \tau(Z) \). Note that \( \sigma(Z) \) and \( \tau(Z) \) are strings of positive integers.

Let \( \kappa(Z) \) be the number of distinct lines in the reduced support of \( M(Z) \). Write

\[
M(Z) = \bigcup_{1 \leq i \leq \kappa(Z)} \mu_i D_i.
\]

Let \( X_i \) be the subscheme of \( Z \) that is concentrated on \( D_i \). Let \( \lambda(Z) \) be the number of distinct lines in the reduced support of \( N(Z) \). Write

\[
N(Z) = \bigcup_{1 \leq j \leq \lambda(Z)} \nu_j E_j.
\]

Let \( Y_j \) be the subscheme of \( Z \) that is concentrated on \( E_j \).

**Corollary 2.4.** Let \( Z \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a zero-dimensional scheme. Let \( m \geq \mu(Z) - 1 \) and \( n \geq -1 \) be integers. We adopt the above notations.

(i) We claim that \( \dim_{\mathbb{C}} H^1(I_Z(m, n)) = \sum_{1 \leq i \leq \kappa(Z)} \sum_{0 \leq j \leq \mu_i - 1 \atop \sigma(X_i)_j > n} (\sigma(X_i)_j - n - 1) \).

(ii) Assume, in addition, that \( \text{length}(Z \cap D) \leq n + 1 \) for every line \( D \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of bidegree \((1, 0)\). We claim that \( H^1(I_Z(m, n)) = \{0\} \).

**Proof.** (i) From the exact sequence

\[
0 \to \mathcal{O}(m - \mu(Z), n) \to I_Z(m, n) \to I_{Z,M(Z)}(n) \to 0,
\]

and from the vanishing of $H^q(O(m - \mu(Z), n))$ for $q = 1, 2$, we get the isomorphism

$$H^1(I_Z(m, n)) \cong H^1(I_{Z_M}(n)).$$

The claim follows from Proposition 2.2.

(ii) By hypothesis, for all indices $1 \leq i \leq \kappa(Z)$, we have the inequality $\sigma(X_i) = \text{length}(Z \cap D_i) \leq n + 1$. In view of Remark 2.2 for $1 \leq j \leq \mu_i - 1$, we have the inequalities $\sigma(X_i) \leq \sigma(X_i) \leq n + 1$. The r.h.s. in formula (i) vanishes. \qed

**Corollary 2.5.** Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme. Let $n \geq \nu(Z) - 1$ and $m \geq -1$ be integers. We adopt the above notations.

(i) We claim that $\dim_C H^1(I_Z(m, n)) = \sum_{1 \leq j \leq \lambda(Z)} \sum_{0 \leq \tau_j \leq \nu_j - 1} \frac{(\tau_j(Y_j))}{\tau_j(Y_j)} - m - 1$.

(ii) Assume, in addition, that $\text{length}(Z \cap E) \leq m + 1$ for every line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$. We claim that $H^1(I_Z(m, n)) = \{0\}$.

3. An affine decomposition of the punctual Hilbert scheme

Let $p \in \mathbb{P}^1 \times \mathbb{P}^1$ be the point given by the equations $x = 0, z = 0$. Let $l$ be a positive integer. Let $\text{Hilb}_p(l)$ be the punctual Hilbert scheme parametrizing subschemes of length $l$ of $\mathbb{P}^1 \times \mathbb{P}^1$ that are concentrated at $p$. In this section we will give a decomposition of $\text{Hilb}_p(l)$ into locally closed subsets that are isomorphic to affine spaces. This decomposition is due to Ellingsrud and Strømme, see [2]. The following proposition is well-known.

**Proposition 3.1.** Let $P \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a subscheme that is concentrated at $p$. Write $\nu = \nu(P)$ and $\tau(P) = (t_0, \ldots, t_{\nu - 1})$. Let $I_P \subset O_P = \mathbb{C}[x, z]_{(x, z)}$ be the ideal of $P$. We claim that, for $0 \leq k \leq \nu - 2$, there are polynomials of the form

$$f_k = x^k z^k + \sum_{j=\nu-k+1}^{\nu-1} \sum_{i=0}^{j-1} a_{ijk} x^i z^j$$

such that $I_P = (f_0, \ldots, f_{\nu-2}, x^{\nu-1}, z^\nu)$. We claim that the set of polynomials

$$\{x^i z^j \mid 0 \leq j \leq \nu - 1, \ 0 \leq i \leq t_j - 1\}$$

gives a basis of $O_{P, p} = O_p/I_P$ as a $\mathbb{C}$-vector space.

**Corollary 3.2.** We adopt the above notations. We claim that the set of polynomials

$$\{x^i z^j \mid 0 \leq i \leq t_0 - 1, \ 0 \leq j \leq \nu - 1\}$$

generates $O_{P, p}$ as a $\mathbb{C}$-vector space.

Indeed, according to Remark 2.2, $t_j \leq t_0$, so the set of polynomials from the above corollary contains the basis from the above proposition. Given a positive integer $l$, we consider the set of partitions of $l$,

$$\Pi(l) = \{\tau = (t_0, \ldots, t_{\nu-1}) \mid t_0 \geq \cdots \geq t_{\nu-1} > 0, \ t_0 + \cdots + t_{\nu-1} = l\}.$$ 

Given $\tau \in \Pi(l)$, we consider the subset $A(\tau) = \{P \mid \tau(P) = \tau\} \subset \text{Hilb}_p(l)$.

**Proposition 3.3.** The family $\{A(\tau)\}_{\tau \in \Pi(l)}$ constitutes a decomposition of $\text{Hilb}_p(l)$ into locally closed subsets. We equip $A(\tau)$ with the induced reduced structure. We claim that $A(\tau) \cong \mathbb{A}^{l-t_0}$. 


The above proposition is a direct consequence of the proof of [2] Theorem (1.1)(iv). We will sketch the argument here for the sake of the reader. Consider the action of the multiplicative group \( \mathbb{C}^* \) on \( \mathbb{C}[x, z] \) given by \( a.x^iz^j = a^ix^iz^j \). Identifying \( \text{Hilb}_p(l) \) with the set of ideals in \( \mathbb{C}[x, z] \) that have colength \( l \) and that are contained in \( (x, z) \), we obtain an induced action of \( \mathbb{C}^* \) on \( \text{Hilb}_p(l) \). The set of fixed points for this action is \( \{ P_\tau \mid \tau \in \Pi(l) \} \), where \( P_\tau \) is given by the ideal \( (x^{\nu_0}, x^{\nu_1}z, \ldots, x^{\nu_r}z^{\nu-1}, z^\nu) \). Choose an arbitrary \( P \in A(\tau) \) and write \( I_P = (f_0, \ldots, f_{\nu-2}, x^{\nu-1}z^{\nu-1}, z^\nu) \), as in Proposition 3.1. Note that \( \lim_{a \to 0} a.P \) has ideal

\[
\lim_{a \to 0} \left( \ldots, a_kx^kz^k + \sum_{j=k+1}^{\nu-1} \sum_{i=0}^{j-1} a_{ijk} a^jx^iz^j, \ldots, a^\nu-1x^{\nu-1}z^{\nu-1}, a^\nu z^\nu \right)
\]

\[
= \lim_{a \to 0} \left( \ldots, x^kz^k + \sum_{j=k+1}^{\nu-1} \sum_{i=0}^{j-1} a_{ijk} x^jz^j, \ldots, x^{\nu-1}z^{\nu-1}, z^\nu \right)
\]

\[
= (\ldots, x^kz^k, \ldots, x^{\nu-1}z^{\nu-1}, z^\nu),
\]

which is the ideal of \( P_\tau \). Thus, \( \lim_{a \to 0} a.P = P_\tau \). It has now become clear that

\[
A(\tau) = \{ P \in \text{Hilb}_p(l) \mid \lim_{a \to 0} a.P = P_\tau \},
\]

so we may apply [1] Theorem 4.4 to in order to deduce that \( \{ A(\tau) \}_{\tau \in \Pi(l)} \) is a locally closed decomposition of \( \text{Hilb}_p(l) \) and that each \( A(\tau) \) is isomorphic to an affine space. The dimension of these affine spaces was computed in [2] p. 350.

4. The Vanishing Theorem

In this section we prove the Vanishing Theorem 4.4, which will lead us to the description of the Brill-Noether loci in Theorem 5.3. The key technical step towards the proof of Theorem 4.4 is contained in the following lemma.

**Lemma 4.1.** Consider integers \( c \geq 0, \nu \geq 1, m \geq 1 \) and \( n \geq c + \nu - 1 \). Let \( C \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a curve of bidegree \((1, c)\). Let \( E \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a line of bidegree \((0, 1)\), which is not contained in \( C \). Consider a zero-dimensional scheme \( Z \subset C \cup \nu E \) and consider its restriction \( X = Z \cap C \). Assume that \( \text{length}(Z \cap E) \leq m \). Then

\[
H^1(I_Z(m, n)) \simeq H^1(I_X(m, n)).
\]

**Proof.** Consider the ideal \( I_{X,Z} \subset O_Z \) of \( X \) in \( Z \). From the short exact sequence

\[
0 \rightarrow I_Z(m, n) \rightarrow I_X(m, n) \rightarrow I_{X,Z} \rightarrow 0
\]

we obtain the long exact cohomology sequence

\[
H^0(I_X(m, n)) \rightarrow H^0(I_{X,Z}) \rightarrow H^1(I_Z(m, n)) \rightarrow H^1(I_X(m, n)) \rightarrow H^1(I_{X,Z}).
\]

The space on the right vanishes because \( I_{X,Z} \) is supported on a finite set. We have reduced the lemma to proving that the first arrow is surjective. We may assume that \( Z \) is contained in the affine chart \( U = \{ y \neq 0, w \neq 0 \} \). We may assume that \( C \) does not contain the line given by the equation \( w = 0 \). We may further assume that the point \( p = C \cap E \) is given by the equations \( x = 0, z = 0 \). Choose a polynomial \( f(x, z) \) of degree 1 in the variable \( x \) and of degree \( c \) in the variable \( z \), which vanishes on \( C \cap U \). Let \( P \) be the (possibly empty) subscheme of \( Z \) that is concentrated on \( p \). Denote \( e = \text{length}(P \cap E) \). Denote \( Z_{\text{red}} \cap E \setminus \{ p \} = \{ p_1, \ldots, p_r \} \). For \( 1 \leq k \leq r \),
Remark 4.2. Let $\mathfrak{I}_X = \mathfrak{J} \oplus \mathcal{O}_{P_1} \oplus \cdots \oplus \mathcal{O}_{P_r}$.

Let $q \subset \mathcal{O}_P = \mathbb{C}[x, z]_{(x, z)}$ be the annihilator of $\mathfrak{J}_P$. Let $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the finite scheme concentrated at $p$ that is defined by $q$. By hypothesis, $z^e f = 0$ in $\mathcal{O}_P$, hence $z^e \in q$, and hence $Q \subset \nu E$. According to Corollary 3.2, the set of polynomials

$$\{fx^iz^j \mid 0 \leq i \leq e-1, \ 0 \leq j \leq \nu(P)-1\}$$

generates $\mathfrak{J}_P$ as a $\mathbb{C}$-vector space. It follows that the set of polynomials

$$\{x^iz^j \mid 0 \leq i \leq e-1, \ 0 \leq j \leq \nu(P)-1\}$$

generates $\mathcal{O}_{Q,P}$ as a $\mathbb{C}$-vector space, hence $\{x^i \mid 0 \leq i \leq e-1\}$ generates $\mathcal{O}_{Q \cap \nu E,p}$ and hence $\text{length}(Q \cap \nu E) \leq e$. The finite scheme $Y = Q \cup P_1 \cup \cdots \cup P_r$ is contained in $\nu E$. It satisfies the condition $\text{length}(Y \cap E) \leq \text{length}(Z \cap E) \leq m$. According to Corollary 2.3(ii), $H^1(\mathcal{I}_Y(m-1, \nu-1)) = \{0\}$. It follows that the set of polynomials

$$\{x^iz^j \mid 0 \leq i \leq m-1, \ 0 \leq j \leq \nu-1\}$$

generates $H^0(\mathcal{O}_P)$ as a $\mathbb{C}$-vector space. Since $f$ does not vanish at $p_1, \ldots, p_r$, we deduce that the set of polynomials

$$G = \{fx^iz^j \mid 0 \leq i \leq m-1, \ 0 \leq j \leq \nu-1\}$$

generates $H^0(f\mathcal{O}_Y) = H^0(\mathcal{I}_X,Z)$ as a $\mathbb{C}$-vector space. Each $g \in G$ vanishes on $C \cap U$, so it vanishes on $X$. The degree of $g$ in the variable $z$ is at most $c$ (the variable $x$ at most $m$). The degree of $g$ in the variable $z$ is at most $c + \nu - 1 \leq n$. Given $g \in G$, we consider the form $\bar{g} = g(x, z)yz^mw^n \in H^0(\mathcal{I}_X(m, n))$. The set $\{\bar{g} \mid g \in G\}$ maps to a set of generators of $H^0(\mathcal{I}_X,Z)$ as a $\mathbb{C}$-vector space. We deduce that the map $H^0(\mathcal{I}_X(m, n)) \to H^0(\mathcal{I}_X,Z)$ is surjective, which concludes the proof of the lemma.

\[\square\]

**Remark 4.2.** Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible curve of bidegree $(1, c)$. Let $Z \subset C$ be a finite scheme of length $l$. Let $m \geq 0$ and $n \geq c - 1$ be integers. Then

$$H^1(\mathcal{I}_Z(m, n)) \simeq H^1(\mathcal{O}_{\mathbb{P}^1}(cm + n - l)).$$

Indeed, from the exact sequence

$$0 \longrightarrow \mathcal{I}_C(m, n) \simeq \mathcal{O}(m - 1, n - c) \longrightarrow \mathcal{I}_Z(m, n) \longrightarrow \mathcal{I}_{Z,C}(m, n) \longrightarrow 0,$$

and from the vanishing of $H^q(\mathcal{O}(m - 1, n - c))$ for $q = 1, 2$, we obtain the isomorphism

$$H^1(\mathcal{I}_Z(m, n)) \simeq H^1(\mathcal{I}_{Z,C}(m, n)).$$

Note that $\mathcal{I}_{Z,C}(m, n)$ is supported on $C \simeq \mathbb{P}^1$ and $\mathcal{I}_{Z,C}(m, n)|_C \simeq \mathcal{O}_{\mathbb{P}^1}(cm + n - l)$.

**Proposition 4.3.** Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree $(1, c)$, $Z \subset C$ be a finite scheme of length $l$. Let $m \geq 1$ and $n \geq c - 1$ be integers such that $m + n \geq l - 1$. Assume that $\text{length}(Z \cap D) \leq n + 1$ for every line $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 0)$. Assume that $\text{length}(Z \cap E) \leq m$ for every line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$. Then

$$H^1(\mathcal{I}_Z(m, n)) = \{0\}.$$
In what follows we adopt the notations of section 2. Assume that $k$ is irreducible, then, by virtue of Remark 4.2,
$$H^1(I_Z(m, n)) \simeq H^1(O_{\mathbb{P}^1}(cm + n - l)).$$
The r.h.s. vanishes because $cm + n - l \geq -1$. Indeed, if $c = 0$, then this inequality holds by hypothesis. If $c > 0$, then $cm + n - l \geq m + n - l \geq -1$. Assume now that $C = C' \cup \nu E$, where $C'$ is a curve of bidegree $(1, c')$ which does not contain $E$, and $\nu \geq 1$. Denote $X = Z \cap C'$. According to Lemma 4.1,
$$H^1(I_Z(m, n)) \simeq H^1(I_X(m, n)).$$
Since $C'$ has fewer irreducible components than $C$, we may apply the induction hypothesis to $X \subset C'$ in order to deduce that the r.h.s. vanishes. \qed

**Theorem 4.4.** Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a finite scheme of length $l$. Let $m$ and $n$ be non-negative integers such that $m + n \geq 1 - l$. Assume that $\text{length}(Z \cap D) \leq n + 1$ for every line $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 0)$. Assume that $\text{length}(Z \cap E) \leq m + 1$ for every line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$. Then $H^1(I_Z(m, n)) = \{0\}$.

**Proof.** By symmetry, we may assume that $n \geq m$. If $m = 0$, then $n \geq l - 1 \geq \nu(Z) - 1$, and the conclusion follows from Corollary 2.3(ii). Assume that $m \geq 1$. If $\text{length}(Z \cap E) = m + 1$ for some line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$, then $\nu(Z) \leq l - m \leq n + 1$, and we again apply Corollary 2.3(ii). Assume that $\text{length}(Z \cap E) \leq m$ for every line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$. Write $l = 2c$ or $l = 2c + 1$ for an integer $c$. Notice that $n \geq c$ and that $Z$ is contained in a curve of bidegree $(1, c)$.

The conclusion follows from Proposition 4.3. \qed

**5. The Brill-Noether loci**

Consider integers $k \geq 0$, $l \geq 1$, $m \geq 0$ and $n \geq 0$ such that $m$ and $n$ are not both zero. Consider the Brill-Noether locus
$$\text{BN}(k, l, (m, n)) = \{Z \mid \dim_{\mathbb{C}} H^1(I_Z(m, n)) = k\} \subset \text{Hilb}(l).$$

By the semicontinuity theorem, this locus is locally closed, so, for any $m$ and $n$ as above, we have a locally closed decomposition
$$\text{Hilb}(l) = \bigsqcup_{k \geq 0} \text{BN}(k, l, (m, n)).$$

In what follows we adopt the notations of section 2. Assume that $k \geq 1$. Inside $\text{Hilb}(l)$ we consider the subset $S(k, l, n)$ of subschemes $Z$ satisfying the equation
$$k = \sum_{1 \leq i \leq \kappa(Z)} \sum_{0 \leq j \leq \mu_i - 1} (\sigma(X_{ij})_j - n - 1),$$
and the subset $T(k, l, m)$ of subschemes $Z$ satisfying the equation
$$k = \sum_{1 \leq j \leq \lambda(Z)} \sum_{0 \leq i \leq \nu_j - 1} (\tau(Y_{ji})_i - m - 1).$$

Inside $\text{Hilb}(l)$ we consider the subset $\text{Ver}(k, l)$ of schemes $Z$ for which there exists a line $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 0)$ (a vertical line) such that $\text{length}(Z \cap D) = k$.

Inside $\text{Hilb}(l)$ we consider the subset $\text{Hor}(k, l)$ of schemes $Z$ for which there exists a line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$ (a horizontal line) such that $\text{length}(Z \cap E) = k$. 


Note that Ver($l, l$) is the closed subvariety of Hilb($l$) of schemes that are contained in a vertical line, while Hor($l, l$) is the closed subvariety of Hilb($l$) of schemes that are contained in a horizontal line. Both subvarieties are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

**Proposition 5.1.** Consider integers $l \geq 1$ and $n \geq 0$.

(i) If $l \leq n + 1$, then $S(k, l, n) = \emptyset$ for all $k \geq 1$. If $l > n + 1$, then $S(k, l, n)$ is nonempty if and only if $1 \leq k \leq l - n - 1$.

(ii) Assume that $l > n + 1$. Then $S(l - n - 1, l, n) = \text{Ver}(l, l)$.

(iii) Assume that $l \leq 2n + 3$. Then $S(k, l, n) = \text{Ver}(k + n + 1, l)$ for all $k \geq 1$.

**Proof.** (i) Assume that $k \geq 1$ and $Z \in S(k, l, n)$. From equation (2) we deduce that there are indices $i$ and $j$ such that $\sigma(X_i) > n + 1$. From Remark 2.2 we obtain the inequality $\sigma(X_i) \geq \sigma(X_j) > n + 1$. Equation (2) can be rewritten in the form

$$k = \sigma(X_i) - n - 1 + \sum_{1 \leq j \leq l, j \neq i} \sigma(X_j) + \sum_{1 \leq k \leq \ circumference\ of\ a\ point.\ Put\ Z = X_1 \cup \cdots \cup X_{l-k-n} \ and \ notice \ that \ Z \in S(k, l, n).$$

(ii) Note that inequality (4) is strict if $\mu(Z) > 1$. For $Z \in S(l - n - 1, l, n)$, inequality (4) is not strict, hence $\mu(Z) = 1$, that is, $Z$ is contained in a line of bidegree $(1, 0)$. This proves the inclusion $S(l - n - 1, l, n) \subset \text{Ver}(l, l)$. The reverse inclusion is obvious.

(iii) Assume that $Z \in S(k, l, n)$. As noted above, there is an index $i$ such that $\sigma(X_i) > n + 1$. For all indices $1 \leq j \leq \mu_i - 1$ we have the inequality $\sigma(X_j) \leq n + 1$; for all indices $h \neq i$, $1 \leq h \leq \kappa(Z)$, and $j$ we have the inequality $\sigma(X_hj) \leq n + 1$, otherwise $l \geq 2n + 4$. Equation (2) takes the form $k = \sigma(X_i) - n - 1$, that is, length($Z \cap D_i$) = $k + n + 1$. Thus, $Z \in \text{Ver}(k + n + 1, l)$. This proves the inclusion $S(k, l, n) \subset \text{Ver}(k + n + 1, l)$. The reverse inclusion can be proved analogously. □

**Proposition 5.2.** Consider integers $l \geq 1$ and $m \geq 0$.

(i) If $l \leq m + 1$, then $T(k, l, m) = \emptyset$ for all $k \geq 1$. If $l > m + 1$, then $T(k, l, m)$ is nonempty if and only if $1 \leq k \leq l - m - 1$.

(ii) Assume that $l > m + 1$. Then $T(l - m - 1, l, m) = \text{Hor}(l, l)$.

(iii) Assume that $l \leq 2m + 3$. Then $T(k, l, m) = \text{Hor}(k + m + 1, l)$ for all $k \geq 1$.

**Theorem 5.3.** Let $k \geq 1$, $l \geq 2$, $m \geq 0$ and $n \geq 0$ be integers such that $m + n \geq l - 1$. Then we have the locally closed decomposition

$$\text{BN}(k, l, (m, n)) = S(k, l, n) \cup T(k, l, m).$$

**Proof.** Take $Z \in \text{BN}(k, l, (m, n))$. According to Theorem 4.3, length($Z \cap D$) $\geq n + 2$ for some line $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 0)$, or length($Z \cap E$) $\geq m + 2$ for some line $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, 1)$. In the first case, $\mu(Z) \leq l - n - 1 \leq m,$
so we may apply Corollary \(2.4\) in order to obtain formula (2). In the second case, \(\nu(Z) \leq l - m - 1 \leq n\), so we may apply Corollary \(2.5\) in order to obtain formula (3). This proves the inclusion “\(\subset\)”. Take \(Z \in S(k, l, n)\). As mentioned in the proof of Proposition \(5.1\), length(\(Z \cap D_j\)) \(= \sigma(X_i) \geq n + 2\) for an index \(i\). As noted above, this inequality allows us to apply Corollary \(2.4\). We deduce that dim(\(m, n\)) \(= \sigma(Y_i) \geq m + 2\). As noted above, this inequality allows us to apply Corollary \(2.5\). We deduce that \(\dim(\mathcal{H}_i(\mathcal{I}_Z(m, n)) = k\). This proves the reverse inclusion. The union is disjoint because we cannot have the inequalities length(\(Z \cap D_j\)) \(= n + 2\) and length(\(Z \cap E_j\)) \(= m + 2\) at the same time. Indeed, 

\[m + n + 2 \geq l + 1 \geq \text{length}(Z \cap (D_j \cup E_j)) + 1 \geq \text{length}(Z \cap D_j) + \text{length}(Z \cap E_j)\]

According to Proposition \(5.2\), \(T(k, l, l) = \emptyset\), hence \(S(k, l, n) = \text{BN}(k, l, (l, n))\), and hence \(S(k, l, n)\) is locally closed. According to Proposition \(5.1\), \(S(k, l, l) = \emptyset\), hence \(T(k, l, m) = \text{BN}(k, l, (m, l))\), and hence \(T(k, l, m)\) is locally closed.

Combining Theorem \(5.3\), Proposition \(5.1\) and Proposition \(5.2\) we can describe more explicitly the highest Brill-Noether locus.

**Corollary 5.4.** We adopt the assumptions of Theorem \(5.3\). We denote

\[k_{\text{max}} = \max\{l - m - 1, l - n - 1, 0\}\]

For \(k > k_{\text{max}}\), we claim that \(\text{BN}(k, l, (m, n)) = \emptyset\). If \(k_{\text{max}} > 0\), then we claim that

\[
\text{BN}(k_{\text{max}}, l, (m, n)) = \begin{cases} 
\text{Ver}(l, l) & \text{if } m > n, \\
\text{Hor}(l, l) & \text{if } m < n, \\
\text{Ver}(l, l) \cup \text{Hor}(l, l) & \text{if } m = n.
\end{cases}
\]

**Corollary 5.5.** We adopt the assumptions of Theorem \(5.3\). We further assume that \(l \leq 2m + 3\) and \(l \leq 2n + 3\). Then, for every \(k \geq 1\),

\[
\text{BN}(k, l, (m, n)) = \text{Ver}(k + n + 1, l) \cup \text{Hor}(k + m + 1, l).
\]

6. **The Euler characteristic of the flag Hilbert schemes**

For a scheme \(S\) of finite type over \(\mathbb{C}\), we denote by \(\chi(S)\) the topological Euler characteristic of the complex analytic space associated to \(S\). Using straightforward arguments, we reduce the computation of \(\chi(\text{Hilb}(l, (m, n)))\) to the computation of the Euler characteristic of the Brill-Noether loci occurring in decomposition (1).

**Proposition 6.1.** Let \(l \geq 1, m \geq 0\) and \(n \geq 0\) be integers such that \(m\) and \(n\) are not both zero. We claim that

\[
\chi(\text{Hilb}(l, (m, n))) = (mn + m + n + 1 - l) \chi(\text{Hilb}(l)) + \sum_{k \geq 1} k \chi(\text{BN}(k, l, (m, n))).
\]

**Proof.** Consider \(Z \in \text{Hilb}(l)\). From the short exact sequence

\[
0 \rightarrow \mathcal{I}_Z(m, n) \rightarrow \mathcal{O}(m, n) \rightarrow \mathcal{O}_Z \rightarrow 0
\]

we obtain the relation

\[
\dim(\mathcal{H}^0(\mathcal{I}_Z(m, n))) = \dim(\mathcal{H}^1(\mathcal{I}_Z(m, n))) + mn + m + n + 1 - l.
\]
Let $\phi : \text{Hilb}(l, (m, n)) \to \text{Hilb}(l)$ be the forgetful morphism. For $Z \in \text{BN}(k, l, (m, n))$ we have the relation
\[
\chi(\phi^{-1}(Z)) = \dim_k H^0(\mathcal{I}_Z(m, n)) = k + mn + m + n + 1 - l.
\]
Taking into account the decomposition (1), we calculate:
\[
\chi(\text{Hilb}(l, (m, n))) = \sum_{k \geq 0} (mn + m + n + 1 - l) \chi(\text{BN}(k, l, (m, n)))
\]
\[
= (mn + m + n + 1 - l) \sum_{k \geq 0} \chi(\text{BN}(k, l, (m, n))) + \sum_{k \geq 0} k \chi(\text{BN}(k, l, (m, n)))
\]
\[
= (mn + m + n + 1 - l) \chi(\text{Hilb}(l)) + \sum_{k \geq 1} k \chi(\text{BN}(k, l, (m, n))). \quad \square
\]

Let $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a line of bidegree $(0, 1)$. Let $\epsilon \geq 1$ be an integer and let $\epsilon = (e_1, \ldots, e_h)$ be a partition of $\epsilon$. Consider the discriminant locus
\[
\Delta(\epsilon) = \{ e_1p_1 + \cdots + e_hp_h \mid p_i \in E \text{ are mutually distinct} \} \subset |\mathcal{O}_E(\epsilon, 0)| \simeq \mathbb{P}^\epsilon.
\]
Consider the open subvariety
\[
U_h = \{(p_1, \ldots, p_h) \mid p_i \in E \text{ are mutually distinct} \} \subset E^h.
\]
The map $U_h \to \Delta(\epsilon)$ given by $(p_1, \ldots, p_h) \mapsto e_1p_1 + \cdots + e_hp_h$ is a geometric quotient modulo the action of a certain subgroup $\Sigma_\epsilon$ contained in the group of permutations of $h$ elements. For $h \geq 3$ we have $\chi(U_h) = 0$. We obtain the formulas
\[
\chi(\Delta(\epsilon)) = \frac{1}{|\Sigma_\epsilon|} \chi(U_h) = 0 \quad \text{if } h \geq 3,
\]
\[
\chi(\Delta(\epsilon)) = \chi(U_2) = 2 \quad \text{if } \epsilon = (e_1, e_2) \text{ with } e_1 > e_2,
\]
\[
\chi(\Delta(\epsilon)) = \frac{1}{2} \chi(U_2) = 1 \quad \text{if } \epsilon = (e_1, e_2) \text{ with } e_1 = e_2,
\]
\[
\chi(\Delta(\epsilon)) = \chi(U_1) = 2 \quad \text{if } \epsilon = (\epsilon).
\]

Given a partition $\tau = (t_0, \ldots, t_{\nu-1}) \in \Pi(l)$ we consider the locally closed subset
\[
\text{Hilb}_{\nu \epsilon}(\tau) = \{ Z \mid Z \subset \nu E, \, \tau(Z) = \tau \} \subset \text{Hilb}(l).
\]
The number $b(\tau)$ of bipartitions of $\tau$ is the number of pairs $(\tau', \tau'')$ such that $\tau = \tau' + \tau''$, where $\tau'$ and $\tau''$ are non-increasing strings of length $\nu$ of non-negative integers. Notice that
\[
b(\tau) = (t_0 - t_1 + 1) \cdots (t_{\nu-2} - t_{\nu-1} + 1)(t_{\nu-1} + 1).
\]

**Proposition 6.2.** We adopt the above notations. We claim that
\[
\chi(\text{Hilb}_{\nu \epsilon}(\tau)) = b(\tau).
\]

**Proof.** Consider the morphism of schemes
\[
\psi : \text{Hilb}_{\nu \epsilon}(\tau) \longrightarrow \text{Hilb}_{\nu}(t_0) \simeq \mathbb{P}^{t_0}
\]
given on closed points by $\psi(Z) = Z \cap E$. Consider $Z_0 = e_1p_1 + \cdots + e_hp_h$ in $\text{Hilb}_{\nu}(t_0)$. Here $\epsilon = (e_1, \ldots, e_h)$ is a partition of $t_0$ and $p_1, \ldots, p_h$ are mutually distinct points of $E$. Then $\psi(Z) = Z_0$ if and only if $Z = P_1 \cup \cdots \cup P_h$, where $P_i$ is a subscheme of $\nu E$ that is concentrated at $p_i$, such that $\tau(P_i) = e_i$ and $\tau(P_1) + \cdots + \tau(P_h) = \tau$. According to Proposition 3.3 for a fixed partition $\tau_i$, the
family of schemes $P_i \subset \nu E$ that are concentrated at $p_i$ and such that $\tau(P_i) = \tau_i$ is parametrized by an affine space $A(\tau_i)$ of dimension $\sum_{j \geq 1} \tau_{ij}$. It follows that

$$\psi^{-1}(Z_0) = \bigcup_{\tau_1 + \cdots + \tau_h = \tau} A(\tau_1) \times \cdots \times A(\tau_h),$$

hence

$$\chi(\psi^{-1}(Z_0)) = \left| \left\{ (\tau_1, \ldots, \tau_h) \mid \tau_1 + \cdots + \tau_h = \tau, \tau_{10} = e_1, \ldots, \tau_{h0} = e_h \right\} \right|.$$

The r.h.s. is constant when $Z_0$ varies in the discriminant locus $\Delta(\epsilon) \subset \mathbb{P}^t$. We obtain the formulas

$$\chi(\text{Hilb}_{p, \epsilon}(\tau))$$

$$= \sum_{\epsilon \in \Pi(t_0)} \chi(\Delta(\epsilon)) \left| \left\{ (\tau_1, \ldots, \tau_h) \mid \tau_1 + \cdots + \tau_h = \tau, \tau_{10} = e_1, \ldots, \tau_{h0} = e_h \right\} \right|$$

$$= \sum_{(e_1, e_2) \in \Pi(t_0)} 2 \left| \left\{ (\tau_1, \tau_2) \mid \tau_1 + \tau_2 = \tau, \tau_{10} = e_1, \tau_{20} = e_2 \right\} \right| + 2 \left| \left\{ (\tau_1, \tau_2) \mid \tau_1 + \tau_2 = \tau, \tau_{10} \neq 0, \tau_{20} \neq 0 \right\} \right| + 2 = b(\tau). \quad \square$$

Given positive integers $k$, $l$, $\lambda$, $\nu_1, \ldots, \nu_\lambda$ and a non-negative integer $m$, we define the sets

$$\Theta(k, l, m, \nu_1, \ldots, \nu_\lambda) = \left\{ (\rho_1, \ldots, \rho_\lambda) \mid \rho_i = (r_{i,0}, \ldots, r_{i,\nu_i-1}), \right.$$

$$\left. r_{i,0} \geq \cdots \geq r_{i,\nu_i-1} > 0, r_{ij} \in \mathbb{Z}, \right.$$

$$l = \sum_{1 \leq i \leq \lambda} \sum_{0 \leq j \leq \nu_i-1} r_{ij},$$

$$k = \sum_{1 \leq i \leq \lambda} \sum_{0 \leq j \leq \nu_i-1} (r_{ij} - m - 1) \right\},$$

$$\Phi(k, l, m) = \bigcup_{\nu_1 \geq 1} \Theta(k, l, m, \nu_1),$$

$$\Psi(k, l, m) = \bigcup_{\nu_1 \geq 1, \nu_2 \geq 1} \Theta(k, l, m, \nu_1, \nu_2).$$

**Proposition 6.3.** Consider integers $k, l \geq 1$ and $m$, $n \geq 0$. We claim that

$$\chi(S(k, l, n)) = \sum_{\sigma \in \Phi(k, l, n)} 2 b(\sigma) + \sum_{(\sigma_1, \sigma_2) \in \Psi(k, l, n)} b(\sigma_1) b(\sigma_2),$$

$$\chi(T(k, l, m)) = \sum_{\tau \in \Phi(k, l, m)} 2 b(\tau) + \sum_{(\tau_1, \tau_2) \in \Psi(k, l, m)} b(\tau_1) b(\tau_2).$$

**Proof.** The two formulas are analogous, so we prove only the second one. Given $Z \in \text{Hilb}(l)$, write $\nu = \nu(Z)$ and $N(Z) = \nu_1 E_1 \cup \cdots \cup \nu_\lambda E_\lambda$, where $E_1, \ldots, E_\lambda$ are distinct lines of bidegree $(0, 1)$ and $(\nu_1, \ldots, \nu_\lambda) \in \Pi(\nu)$. Let $Y_j$ be the subscheme
of $Z$ whose reduced support is contained in $E_j$. The condition that $Z$ belong to $T(k, l, m)$ is equivalent to the condition

$$(\tau(Y_1), \ldots, \tau(Y_\lambda)) \in \Theta(k, l, m, \nu_1, \ldots, \nu_\lambda).$$

The subset of $T(k, l, m)$ with fixed $N(Z)$ and with fixed

$$(\tau(Y_1), \ldots, \tau(Y_\lambda)) = (\tau_1, \ldots, \tau_\lambda) \in \Theta(k, l, m, \nu_1, \ldots, \nu_\lambda)$$

is parametrized by $\text{Hilb}_{\nu_1, E_1}(\tau_1) \times \cdots \times \text{Hilb}_{\nu_\lambda, E_\lambda}(\tau_\lambda)$. Applying Proposition 6.2, we calculate:

$$\chi(T(k, l, m)) = \sum_{\nu \geq 1} \sum_{(\nu_1, \ldots, \nu_\lambda) \in \Pi(\nu)} \chi(\Delta(\nu_1, \ldots, \nu_\lambda)) \sum_{(\tau_1, \ldots, \tau_\lambda) \in \Theta(k, l, m, \nu_1, \ldots, \nu_\lambda)} b(\tau_1) \cdots b(\tau_\lambda).$$

Given integers $l \geq 1$ and $m \geq 0$, we write

$$\xi(l, m) = \sum_{1 \leq \nu \leq l - m - 1} \left( \sum_{\rho \in \Phi(k, l, m)} 2k b(\rho) + \sum_{(\rho_1, \rho_2) \in \Psi(k, l, m)} k b(\rho_1) b(\rho_2) \right).$$

Notice that $\xi(l, m) = 0$ if $m \geq l - 1$. For the last theorem we combine Proposition 6.1, Theorem 5.3, Proposition 6.3, Proposition 5.1(i) and Proposition 5.2(i).

**Theorem 6.4.** Let $l \geq 2$, $m \geq 0$ and $n \geq 0$ be integers such that $m + n \geq l - 1$. We claim that

$$\chi(\text{Hilb}(l, (m, n))) = (mn + m + n + 1 - l) \chi(\text{Hilb}(l)) + \xi(l, m) + \xi(l, n).$$

At the end, we illustrate the above theorem for schemes of length at most 8. From [2, Theorem (5.1)] or [3, Theorem 0.1] we read the values of $\chi(\text{Hilb}(l))$, see Table 1. The values of $\xi(l, m)$ are easily computed using the definition and are indicated in Table 2. Substituting these into the above formula, yields the values given in Table 3 and Table 4.

**Table 1.** The values of $\chi(\text{Hilb}(l))$ for $2 \leq l \leq 8$.

| $l$ | 2   | 3  | 4  | 5  | 6  | 7  | 8   |
|-----|-----|----|----|----|----|----|-----|
| \chi(\text{Hilb}(l)) | 14  | 40 | 105| 252| 574| 1240| 2580|
| $(l, m)$ | $\xi(l, m)$ | $(l, m)$ | $\xi(l, m)$ | $(l, m)$ | $\xi(l, m)$ | $(l, m)$ | $\xi(l, m)$ | $(l, m)$ | $\xi(l, m)$ |
|--------|-----------|--------|-----------|--------|-----------|--------|-----------|--------|-----------|
| (2, 0) | 6         | (3, 0) | 36        | (3, 1) | 8         | (4, 0) | 152       |
| (4, 1) | 48        | (4, 2) | 10        | (5, 0) | 508       | (5, 1) | 160       |
| (5, 2) | 60        | (5, 3) | 12        | (6, 0) | 1506      | (6, 1) | 652       |
| (6, 2) | 246       | (6, 3) | 72        | (6, 4) | 14        | (7, 0) | 4024      |
| (7, 1) | 1896      | (7, 2) | 812       | (7, 3) | 296       | (7, 4) | 84        |
| (7, 5) | 16        | (8, 0) | 10034     | (8, 1) | 5024      | (8, 2) | 2358      |
| (8, 3) | 980       | (8, 4) | 346       | (8, 5) | 96        | (8, 6) | 18        |

Table 3. The values of $\chi(\text{Hilb}(l, (m, n)))$ for $3 \leq l \leq 8$, $m + n \geq l - 1$ and $1 \leq m \leq n \leq l - 2$.

| $(l, (m, n))$ | $\chi(\text{Hilb}(l, (m, n)))$ | $(l, (m, n))$ | $\chi(\text{Hilb}(l, (m, n)))$ | $(l, (m, n))$ | $\chi(\text{Hilb}(l, (m, n)))$ |
|---------------|---------------------------------|---------------|---------------------------------|---------------|---------------------------------|
| (3, 1, 1)     | 56                              | (4, 1, 2)     | 268                             | (4, 2, 2)     | 545                             |
| (5, 1, 3)     | 928                             | (5, 2, 2)     | 1128                            | (5, 3, 2)     | 1836                            |
| (5, 3, 3)     | 2796                            | (6, 1, 4)     | 2962                            | (6, 2, 3)     | 3762                            |
| (6, 2, 4)     | 5426                            | (6, 3, 3)     | 5884                            | (6, 4, 3)     | 8122                            |
| (6, 4, 4)     | 10934                           | (7, 1, 5)     | 8112                            | (7, 2, 4)     | 10186                           |
| (7, 2, 5)     | 14468                           | (7, 3, 3)     | 11752                           | (7, 4, 3)     | 16500                           |
| (7, 3, 5)     | 21392                           | (7, 4, 4)     | 22488                           | (7, 5, 4)     | 28620                           |
| (7, 5, 5)     | 35992                           | (8, 1, 6)     | 20522                           | (8, 2, 5)     | 28254                           |
| (8, 2, 6)     | 35916                           | (8, 3, 4)     | 32286                           | (8, 3, 5)     | 42356                           |
| (8, 3, 6)     | 52958                           | (8, 4, 4)     | 44552                           | (8, 4, 5)     | 57202                           |
| (8, 4, 6)     | 70024                           | (8, 5, 5)     | 72432                           | (8, 5, 6)     | 87834                           |
| (8, 6, 6)     | 105816                          |               |                                 |               |                                 |

Table 4. The values of $\chi(\text{Hilb}(l, (m, n)))$ for $2 \leq l \leq 8$ and $0 \leq m < l - 2 < n$.

| $(l, (m, n))$ | $\chi(\text{Hilb}(l, (m, n)))$ | $(l, (m, n))$ | $\chi(\text{Hilb}(l, (m, n)))$ |
|---------------|---------------------------------|---------------|---------------------------------|
| (2, 0, n)     | 14$n$ – 8                       | (3, 0, n)     | 40$n$ -- 44                    |
| (3, 1, n)     | 80$n$ – 32                      | (4, 0, n)     | 105$n$ – 163                   |
| (4, 1, n)     | 210$n$ – 162                    | (4, 2, n)     | 315$n$ – 95                    |
| (5, 0, n)     | 252$n$ – 500                    | (5, 1, n)     | 504$n$ – 596                   |
| (5, 2, n)     | 756$n$ – 444                    | (5, 3, n)     | 1008$n$ – 240                  |
| (6, 0, n)     | 574$n$ – 1364                   | (6, 1, n)     | 1148$n$ – 1644                 |
| (6, 2, n)     | 1722$n$ – 1476                  | (6, 3, n)     | 2296$n$ – 1076                 |
| (6, 4, n)     | 2870$n$ – 560                   | (7, 0, n)     | 1240$n$ – 3416                 |
| (7, 1, n)     | 2480$n$ – 4304                  | (7, 2, n)     | 3720$n$ – 4148                 |
| (7, 3, n)     | 4960$n$ – 3424                  | (7, 4, n)     | 6200$n$ – 2396                 |
| (7, 5, n)     | 7440$n$ – 1224                  | (8, 0, n)     | 2580$n$ – 8026                 |
| (8, 1, n)     | 5160$n$ – 10456                 | (8, 2, n)     | 7740$n$ – 10542                |
| (8, 3, n)     | 10320$n$ – 9340                 | (8, 4, n)     | 12900$n$ – 7394                |
| (8, 5, n)     | 15480$n$ – 5064                 | (8, 6, n)     | 18060$n$ – 2562                |

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Institute of Mathematics of the Romanian Academy, Calea Grivitei 21, Bucharest 010702, Romania

Email address: maican@imar.ro