We consider glass states of several disordered systems: vortices in impure superconductors, amorphous magnets, and nematic liquid crystals in random porous media. All these systems can be described by the random-field or random-anisotropy $O(N)$ model. Even arbitrarily weak disorder destroys long range order in the $O(N)$ model. We demonstrate that at weak disorder and low temperatures quasi-long range order emerges. In quasi-long-range-ordered phases the correlation length is infinite and correlation functions obey power dependencies on the distance. In pure systems quasi-long range order is possible only in the lower critical dimension and only in the case of Abelian symmetry. In the presence of disorder this type of ordering turns out to be more common. It exists in a range of dimensions and is not prohibited by non-Abelian symmetries.

1. Introduction

Solids resist small perturbations and possess some (e.g. crystal) order. Liquids and gases are not ordered and offer no resistance to a static shear stress. An intermediate class of substances which possess ordering but respond strongly to small external disturbances is known as soft matter. The fourth and last possibility, no ordering and weak response to weak perturbations, is represented by glass phases of strongly disordered systems. Where should weakly disordered systems be put in this classification? The answer depends on their symmetry with respect to transformations of the order parameter. If it is discrete, weak disorder is irrelevant and the system belongs to the class of its pure analog (e.g. the ground state of the pure and weakly disordered Ising ferromagnets is the same 'spin solid'). On the other hand, if the symmetry group is continuous then even arbitrarily weak disorder may lead to the formation of a glass state. However, recent studies suggest that in many cases glass phases of weakly disordered systems are qualitatively different from glass states at strong disorder and can be considered as a new type of soft condensed matter (i.e. combine ordering and strong response to weak disturbances). A common feature of these novel glass phases is quasi-long range order (QLRO).
As disorder is strong, the order parameter depends only on the local potential of impurities and hence ordering is absent. In the case of weak disorder there is a competition between internal interactions that tend to establish long range order at low temperatures and randomness that works in the opposite direction. Naively one could expect that stronger internal interactions win and weak disorder has no pronounced effect on the system. This point of view is supported by the mean field theory and for many years was generally accepted in the field of amorphous magnets. However, it is not true and even arbitrarily weak disorder may be sufficient to destroy long range order. This fact was first understood in the context of vortex lattices in impure superconductors but is valid in any system of continuous symmetry. The explanation of the effect is based on the low energy cost of large-scale collective excitations (Goldstone modes). In three dimensions this energy cost turns out to be lower than the energy gain due to interaction of the large-scale excitations with disorder. Here the continuous symmetry which is a prerequisite for the existence of Goldstone modes is crucial. In the systems of discrete symmetry weak disorder is indeed not important. Note that if disorder is weak the correlation length is large. This sometimes led to a wrong interpretation of experimental results about weakly disordered systems as evidences of long range order.

Two questions about impure systems of continuous symmetry immediately arise: Whether the phase transition between high- and low-temperature phases survives in the presence of disorder, and if yes, what is the nature of the low-temperature state? Recently these questions have attracted a renewed experimental and theoretical interest. On the one hand, it was stimulated by new types of disordered systems: superfluid Helium and liquid crystals in low-density aerogels. On the other hand, an important progress was achieved in studies of vortex lattices in impure superconductors after high $T_c$ superconductivity was discovered. In particular, it was found that there are two different glass states in disordered superconductors (Vortex glass and Bragg glass). The Bragg glass state is observed at weak disorder and the properties of this phase are much closer to the pure Abrikosov state than those of Vortex glass. It turns out that in contrast to Vortex glass, Bragg glass possesses topological order. Topologically ordered glass phases are predicted in some other weakly disordered systems and are the subject of the present review.

Since there is no long-range order in impure systems of continuous symmetry, low-temperature phases fall into the class of glass states which are known to be a difficult subject for theory. A usual theoretical approach in condensed matter physics is based on an exact solution of a simple related problem. Then one can develop e.g. a perturbative expansion to take into account the ingredients missed in the exactly solvable problem. A standard source of simple and useful exact solutions is mean field models. However, for strongly disordered systems the mean field approximation is often neither simple nor very useful. An example is the classical solution of the mean-field spin glass by Parisi. If this very nontrivial result does capture physics of real short range spin glasses is still an open question.

Much effort was devoted to theoretical understanding of strongly disordered
magnets, liquid crystals, vortex states of impure superconductors, and other related problems, but the lack of useful exactly solvable models has limited progress in the field. On the other hand, one could hope that for weakly disordered systems an appropriate solvable model is just an analogous system without disorder, and impurities can be taken into account with a perturbation theory. Apparently, the ability of weak randomness to destroy long range order, which is present in pure systems, is an argument against such an optimistic view. On a more rigorous level, the exact solution of the spherical model of amorphous magnets suggests that there is no qualitative difference between weakly and strongly disordered systems with continuous symmetry. Only recently has it been realized that this insensitivity to the disorder strength is an artifact of the spherical approximation, and a similarity between ordering in weakly disordered continuous systems and their pure analogs does exist. Long range order is prohibited in the systems of continuous symmetry in the presence of impurities, but instead quasi-long range order can emerge. This means that the average value of the order parameter over the volume is zero, but the correlation length is infinite and correlation functions obey power dependencies on the distance. In other words, long range order is only weakly broken.

The concept of quasi-long range order (QLRO) was first introduced in the pure two-dimensional XY model. Due to Abelian symmetry the degeneracy space of the XY model (i.e. the manifold each point of which represents a ground state) is locally indistinguishable from the degeneracy space of the free field. Thus, neglecting topological defects (this is legitimate at low temperatures) one can map the pure XY model onto the free field. This mapping gives an easy way to understand the low-temperature behavior of the XY model. The result is that long range order is present in spatial dimensions $D > 2$ and absent in lower dimensions. In two dimensions there is an intermediate situation of QLRO in the low-temperature phase. Our experience with pure systems shows that QLRO is a rare phenomenon. It is possible only in Abelian systems and only in the lower critical dimension that separates the regime of long range order in all higher dimensions from the regime with neither long nor quasi-long range order in all lower dimensions.

Recent theoretical studies of impure systems show however that QLRO can be much more common in the problems with disorder than in homogeneous systems. First, in a given system it can occur in a whole range of dimensions and not only at the lower critical dimension. Second, it is not prohibited by non-Abelian symmetry. The possibility of QLRO was predicted in the Abrikosov state of impure superconductors, in amorphous magnets, and in uniaxial nematics confined in random porous matrices. Although more work is needed to completely understand glass states of weakly disordered systems, and in particular numerical and experimental results are contradictory, we believe that this is only a small portion of the list of disordered systems which could possess QLRO phases. Note that QLRO is possible only in weakly disordered systems. The nature of glass states at strong disorder and disorder driven phase transitions from the weak-disorder to strong-disorder regime is an important open problem which is beyond
the scope of the present review. A very recent discussion of this question in the context of disordered superconductors can be found in Ref. \(^{18}\). Note also that the existence of a QLRO state is sensitive to details of the system. For example, it exists in the random-anisotropy Heisenberg model but is absent in the very similar random-field Heisenberg model that describes relaxor ferroelectrics \(^{19}\).

From the technical point of view the theoretical prediction of QLRO is based on the renormalization group applied to the weak-disorder fixed point. It turns out that an infinite set of relevant operators emerges in the problem and dealing with them requires some care. This technical problem takes its origin in the complicated structure of the energy landscape that contains a plethora of energy minima. The latter is certainly not surprising for a glass state. Early attempts to apply the renormalization group to this class of problems (e.g. Ref. \(^{20}\)) failed due to an incorrect treatment of the complicated energy landscape: Theory incorrectly predicted the dimensional reduction by 2 in comparison with corresponding pure systems and suggested that QLRO is possible only in 4-dimensional random systems of Abelian symmetry.

In this paper we discuss recent theoretical results on the nature of glass states of weakly disordered systems with continuous symmetry. Since the problem of the Bragg glass state of the vortex lattice in impure superconductors was recently reviewed in excellent papers \(^{3}\), \(^{4}\), we consider impure superconductors only briefly and concentrate on systems of non-Abelian symmetry: disordered magnets and nematic liquid crystals. The central point is that in many cases the Larkin-Imry-Ma picture of uncorrelated domains breaks down and quasi-long range order emerges.

The article is organized as follows. In the next section we introduce models of some relevant systems: amorphous magnets, relaxor ferroelectrics, vortex lattices in disordered superconductors and uniaxial nematics in random porous media, and map them onto the random-field (RF) and random-anisotropy (RA) \(O(N)\) models. Section 3 contains a qualitative discussion of the glass phases. In section 4 we derive the functional renormalization group equations. Their solution is given in section 5. First we discuss a simple approximate solution and then proceed with a systematic approach. Some details of the systematic solution are summarized in the Appendix. In section 6 we compare the predictions with experiments and numerical experiments. In that section we also describe the rich phase diagram of disordered nematic. This phase diagram includes, in particular, two QLRO glass states. Section 7 contains conclusions.

### 2. Disordered Systems of Continuous Symmetry

In this section we introduce several disordered systems of continuous symmetry. Albeit physically very different they all can be described by the random-field or random-anisotropy \(O(N)\) models (or slight modifications of these models). Note that the problems of nematics and vortex lattices in superconductors can be mapped onto the \(O(N)\) models in the low-temperature phases only. Mapping becomes in-
valid near the phase transitions to the high-temperature phases.

2.1. Amorphous magnet

If a solid is obtained with a fast freezing from a liquid the atoms may have not enough time to form a crystal lattice. In this case an amorphous solid is formed. One can imagine it as a liquid in which the positions of all particles are suddenly fixed. Similar to crystals amorphous solids have special directions in each point but these directions are different in different places and uncorrelated for distant points. In a magnetic system, special directions lead to the appearance of easy magnetic axes or planes. In contrast to crystals, in amorphous magnets the directions of easy axes are random. The simplest model of amorphous ferromagnets is hence the random-anisotropy (RA) Heisenberg model. It has the following Hamiltonian:

\[ H = -J \sum_{\langle ij \rangle} n_i n_j - D \sum_i (n_i h_i)^2, \tag{1} \]

where \( J \) is the exchange strength, \( D \) the anisotropy strength, \( n_i \) the spin in site \( i \), \( \langle ij \rangle \) denotes a pair of neighboring sites, and the unit vector \( h_i \) describing the direction of random anisotropy in site \( i \) varies from site to site.

At low temperatures amorphous magnets possess a spin-glass-like state. This fact can be easily understood as random anisotropy is strong since in this case the spins are frozen along their local anisotropy axis. A review on strongly disordered amorphous magnets can be found in Ref. In this paper we concentrate on the case of weak anisotropy. We shall see that in the latter case there are strong correlations between spins and the system possesses QLRO. An important consequence of this prediction is divergence of the magnetic susceptibility in the low-temperature phase.

2.2. Nematic in aerogel

Quenched disorder is inevitably present even in the most pure solids. This explains a lot of phenomena, e.g. the residual resistance of metals. On the other hand, liquids are usually homogeneous and introducing quenched disorder in them requires special efforts. One of the approaches consists in pouring a liquid into a randomly interconnected network of pores. Such liquid-porous-matrix systems emerge in many natural and technological processes giving rise to a lasting scientific activity. The recent surge of interest in the field is due to a new micropore material: low-density silica aerogel. Its density can be varied in a wide range up to more than 99% void volume fraction. This allows the investigation of both strongly and weakly confined fluids. The most interesting situation emerges in systems with many degrees of freedom, e.g. He-3 and liquid crystals. In these substances the porous matrix not only geometrically confines the liquid but also induces a random orienting field that fixes the direction of the order parameter near the surface of the matrix. The random-field disorder is known to cause spin-glass effects and such phenomena were indeed observed experimentally in liquid-crystal-aerogel systems.
In particular, a slow glassy dynamics was reported in Refs. 25, 26. Our aim is to describe the glass state of nematics in random porous media in the case of weak disorder. This corresponds e.g. to a weak interaction with the surface of a porous matrix. We postpone the discussion of the disorder strength to section 6 and formulate the simplest model of disordered uniaxial nematics. This is a slight complication of the RA Heisenberg model (1). The free energy density

\[ F = F_d + F_{pm} \]  

(2)
of the nematic in the porous matrix includes the Frank distortion energy \[ F_d = \frac{[K_1(\text{div}\mathbf{n})^2 + K_2(\mathbf{n} \cdot \mathbf{curl}\mathbf{n})^2 + K_3(\mathbf{n} \times \mathbf{curl}\mathbf{n})^2]}{2}, \]  

(3)
where \( \mathbf{n} \) is the director, and the interaction \( F_{pm} \) with the surface of the random matrix. The interaction tends to align the director parallel to the surface. We model the interaction as

\[ F_{pm} = (h \mathbf{n})^2, \]  

(4)
where \( h \) is a random vector representing the normal to the surface. This is the simplest choice compatible with the equivalence of the opposite orientations of the director. Due to the universality the model captures all large-scale physics. The average amplitude of the random vector \( h \) is a measure of the disorder strength. It is a phenomenological parameter which depends on the pore size, anchoring energy and fractal structure of the porous matrix.

We demonstrate that the model (2-4) possesses QLRO in its low-temperature phase at weak disorder. As a consequence the light-scattering cross-section diverges at small angles. We also consider effects of external electric and magnetic fields, and mechanical deformations of the porous matrix. It turns out that at weak disorder there are several phases including two glass states.

### 2.3. Relaxor ferroelectric

Relaxor ferroelectrics are interesting because of their diffuse phase transitions extending over a finite range of temperatures. Most relaxor ferroelectrics are disordered ionic structures, in particular, solid solutions. In the best known relaxor ferroelectric, \( \text{PbMg}_{1/3}\text{Nb}_{2/3}\text{O}_3 \), the spontaneous polarization vanishes at all temperatures and the phase transition is believed to be destroyed by the randomness. The randomness can be described as quenched random electric fields. This leads to the random-field Heisenberg model of the relaxor ferroelectric. Its Hamiltonian reads

\[ H = - \sum_{ij} J_{ij} \mathbf{n}_i \mathbf{n}_j - \sum_i (\mathbf{n}_i \cdot \mathbf{h}_i), \]  

(5)
where \( \mathbf{n}_i \) is the local dipole moment and \( \mathbf{h}_i \) the random electric field. This model is certainly a simplification. In particular, it misses nonrandom anisotropy (which
is cubic in $\text{PbMg}_{1/3}\text{Nb}_{2/3}\text{O}_3$). Nonrandom anisotropy leads to a discrete symmetry group and hence can stabilize long range (ferroelectric) order which is destroyed by random fields. However, it is expected that in $\text{PbMg}_{1/3}\text{Nb}_{2/3}\text{O}_3$ the anisotropy is weak. We demonstrate that the model with the interaction of the nearest neighbors only does not possess QLRO. The effect of the long range dipole interactions which are present in relaxor ferroelectrics is an interesting open question.

2.4. Vortices in disordered superconductor

Disordered superconductors were recently considered in excellent reviews so in this subsection we only provide a basic background and briefly discuss mapping onto the random-field XY model. Our discussion of that model in the subsequent sections is also brief. More details can be found in the abovementioned reviews.

In the absence of disorder, vortices in type II superconductors form a lattice. Small fluctuations of the vortices about their equilibrium positions can be described with elastic theory. Let us denote by $R_i$ the equilibrium position of the vortex in lattice site $i$. The vortex displacement $u(R_i, z) \equiv u_i(z)$ relative to its equilibrium position is a two-component vector. In the absence of dislocations we can assume that the displacement field is slowly varying on the scale of the lattice spacing $a$. We will see in section 5 that topological defects are indeed irrelevant at weak disorder and low temperatures. Hence, a continuous description can be used. The continuous elastic Hamiltonian has the following form

$$H_{el} = \frac{1}{2} \sum_{\alpha, \beta} \int \frac{d^3 q}{(2\pi)^3} \Phi_{\alpha \beta}(q) u_\alpha(q) u_\beta(-q),$$

(6)

where $\alpha, \beta = x, y$ label the coordinates, and the structure of the matrix $\Phi_{\alpha \beta}$ for the triangular Abrikosov lattice can be found in Ref. For simplicity we consider below a one-component field $u$. This corresponds to an anisotropic superconductor such that the vortices can be displaced in one direction only. It turns out that this simple model allows to obtain a good quantitative description of some aspects of the large-scale behavior of disordered isotropic superconductors. A discussion of a more accurate model taking into account two components of the displacement field and the triangular symmetry of the Abrikosov lattice can be found in Refs. We briefly discuss the results of Refs. in section 6. The following elastic Hamiltonian describes the one-component displacement field:

$$H_{el} = \frac{c}{2} \int d^3 x (\nabla u)^2.$$

(7)

This is the Hamiltonian of the XY model without dislocations.

The next step is to include disorder. The pinning energy reads

$$H_{pin} = \int d^3 x \rho(x) V(x),$$

(8)

where the vortex density
\[ \rho(x) = \sum_i \delta(x - R_i - u_i), \]  
\[ \text{and } V \text{ is the random pinning potential with the following statistics:} \]
\[ \langle V(x) \rangle = 0; \langle V(x)V(y) \rangle = \Delta \delta(x - y). \]  

Mapping on the random field XY model is a bit tricky. We rewrite the vortex density in the following way

\[ \rho(x) = \sum_i \int d^3x' \delta(x - x' - u_i(x'))\delta(R_i - x') \]
\[ = \rho_0 \int d^3x' \delta(x - x' - u_i) \sum_Q \exp(iQx') \]
\[ = \rho_0 \det^{-1}[^{\alpha\beta} + \partial_\alpha u_\beta] \sum_Q \exp(iQ[x - u]) \]
\[ \sim \rho_0[1 - \partial_\alpha u_\alpha] \sum_Q \exp(iQ[x - u]), \]  

where \( \rho_0 \) is the average vortex density, \( Q \) are reciprocal lattice vectors. Substituting (11) into (8) we get an infinite set of random contributions to the energy of the form
\[ \rho_0[1 - \partial_\alpha u_\alpha] \sum_Q \exp(iQ[x - u]), \]

where \( h \) and \( \alpha \) are random. This is the Hamiltonian of the random-field XY model.

It turns out that a QLRO state (Bragg glass) emerges in impure superconductors at weak disorder and low temperatures. The name 'Bragg glass' is due to sharp Bragg peaks which are observed in this state because long range order is only weakly broken. Note that the random field \( h \) depends on the vortex density \( \rho_0 \) and hence on the applied magnetic field along which the vortices are directed. Since the density is inversely proportional to the magnetic field, the effective disorder strength is small at weak magnetic fields. Indeed, in a given sample (i.e. at a fixed pinning potential \( V(x) \)) the Bragg glass state is observed at low magnetic fields.

3. Glass States
In this section we qualitatively discuss glass states of systems with continuous symmetry. We demonstrate that disorder, which breaks the continuous symmetry, makes long range order impossible and discuss a possibility of QLRO. A more general question is the existence of a phase transition to a glass state with or without QLRO. There is a simple argument in favor of such transition in weakly disordered systems whose pure analogs have a first order transition between high- and low-temperature phases. Note that in homogeneous nematics the phase transition is of the first order. The phase transition from vortex liquid to vortex solid in pure superconductors is also believed to have first order.

First, let us show that even at arbitrarily weak disorder there is no long range order in disordered systems of continuous symmetry. We employ the Imry-Ma argument which can be used in any system in which disorder breaks the continuous symmetry, but we restrict our discussion by the random-field and random-anisotropy $O(N)$ models which are considered in more details in the following sections.

We start from the the classical nonlinear $\sigma$-model with the Hamiltonian

$$H = \int d^D x [J \sum_\mu \partial_\mu \mathbf{n}(x) \partial_\mu \mathbf{n}(x) + V_{\text{imp}}(x)],$$

where $\mathbf{n}(x)$ is the unit vector of the $N$-component magnetization, $V_{\text{imp}}(x)$ the random potential. In the RF case it has the form

$$V_{\text{imp}} = -\sum_\alpha h_\alpha(x) n_\alpha(x); \quad \alpha = 1, ..., N,$$

where the random field $h(x)$ has a Gaussian distribution and $\langle h_\alpha(x) h_\beta(x') \rangle = A^2 \delta(x - x') \delta_{\alpha\beta}$. In the RA case the random potential is given by the equation

$$V_{\text{imp}} = -\sum_{\alpha,\beta} \tau_{\alpha\beta}(x) n_\alpha(x) n_\beta(x); \quad \alpha, \beta = 1, ..., N,$$

where $\tau_{\alpha\beta}(x)$ is a Gaussian random variable, $\langle \tau_{\alpha\beta}(x) \tau_{\gamma\delta}(x') \rangle = A^2 \delta_{\alpha\gamma} \delta_{\beta\delta} \delta(x - x')$. The random potential corresponds to the same symmetry as the more conventional choice $V_{\text{imp}} = -\langle h(x) n \rangle^2$ used in section 2 but is more convenient for the further discussion.

We assume that the temperature is low and thermal fluctuations are negligible. The Imry-Ma argument suggests that in our problem long-range order is absent at any dimension $D < 4$. Let us assume that the magnetization changes by order $1$ at scale $L$. This costs the exchange energy of order $E_e \sim JV/L^2$, where $V$ is the volume. It should be compared with a possible energy gain due to disorder potential $E_d \sim AV/L^D \times L^{D/2} = AV/L^{D/2}$. One finds that at $D < 4$ the exchange energy $E_e < E_d$ as $L$ is large. Hence, it is favorable to destroy long range order.

One can estimate the Larkin length, up to which there are strong ferromagnetic correlations, with the following qualitative RG approach. Let one remove fast modes and rewrite the Hamiltonian in terms of the block spins, corresponding to the scale $L = ba$, where $a$ is the ultraviolet cut-off, $b > 1$. Then let one make rescaling such
that the Hamiltonian would restore its initial form with new constants $A(L), J(L)$. Dimensional analysis provides estimations

$$J(L) \sim b^{D-2} J(a); \quad A(L) \sim b^{D/2} A(a).$$

To estimate the typical angle $\phi$ between neighboring block spins, one notes that the effective field, acting on each spin, has two contributions: the exchange contribution and the random one. The exchange contribution of order $J(L)$ is oriented along the local average direction of the magnetization. The random contribution of order $A(L)$ may have any direction. This allows one to write at low temperatures that $\phi(L) \sim A(L)/J(L)$. The Larkin length corresponds to the condition $\phi(L) \sim 1$ and equals $L \sim (J/A)^{(4/D)}$, in agreement with the Imry-Ma argument. If Eq. (16) were exact the Larkin length could be interpreted as the correlation length. However, there are two sources of corrections to Eq. (16). Both of them are relevant already at the derivation of the RG equation for the pure system in $2 + \epsilon$ dimensions. The first source is the renormalization due to the interaction and the second one results from the magnetization rescaling which is necessary to ensure the fixed length condition $n^2 = 1$. The leading corrections to Eq. (16) are proportional to $\phi^2 J, \phi^2 A$. Thus, the RG equation for the combination $(A(L)/J(L))^2$ reads

$$\frac{d}{d \ln L} \left( \frac{A(L)}{J(L)} \right)^2 = \epsilon \left( \frac{A(L)}{J(L)} \right)^2 + c \left( \frac{A(L)}{J(L)} \right)^4, \quad \epsilon = 4 - D$$

If the constant $c$ in Eq. (17) is positive the Larkin length is the correlation length indeed. But if $c < 0$ the RG equation has a fixed point, corresponding to the phase with an infinite correlation length. As seen below, both situations are possible, depending on the number $N$ of the magnetization components. The case of the infinite correlation length corresponds to QLRO. We shall see that QLRO is possible as $N < N_c$, where $N_c$ is a critical number. This critical number is larger in the RA model, since the fluctuations of the magnetization are stronger in the RF case. Indeed, in the RF model the magnetization tends to be oriented along the random field, whereas in the RA case there are two preferable local magnetization directions so that the spins tend to lie in the same semispace.

The above scaling argument does not allow to understand if the low-temperature phase is different from the high-temperature paramagnetic state in case when QLRO is absent. The answer turns out to be positive, at least if disorder is weak and the system would have a first order phase transition in the absence of disorder.

Near a first order phase transition a pure system has two phases: a disordered phase with a finite correlation length and an ordered one with an infinite correlation length. In the presence of impurities the ordered state is broken into a set of Imry-Ma domains of size $L \sim (J/A)^{(4/D)}$. This lowers the free energy that would equal $F_1(T)$ in the pure system by $E_1 \sim -A(V/L^D)L^{D/2} \sim -VA^{4/(4-D)}$. As disorder is weak the correlation length $l$ of the disordered phase is less than the Larkin length $L$ and we can estimate the free energy gain in the presence of disorder as $E_2 \sim -V \chi A^2$, where $\chi$ is the susceptibility. The condition of the first order transition is that the
free energies of two phases $F_1(T) + E_1$ and $F_2(T) + E_2$ are equal. One finds the following shift of the phase transition temperature

$$\Delta T = \frac{E_2 - E_1}{d(F_1 - F_2)/dT}. \quad (18)$$

In 3 dimensions we obtain $\Delta T \sim A^2$. This argument does not work for systems with second order transitions since near the phase transition the correlation lengths of both phases are infinite. It is also valid only at $D > 2$. In two dimensions first order transitions are prohibited in disordered systems: They are destroyed due to roughening of domain walls.\[44, 45\]

The last remark is that QLRO is not expected in strongly disordered systems. This agrees with the structure of the phase diagram of impure superconductors.\[44\]

4. Functional Renormalization Group

In the previous section the RG equations are discussed from the qualitative point of view. Eq. (17) corresponds to the simple Migdal-Kadanoff approach of the section 5.1. In the present section we derive the RG equations in a systematic way.

The one-loop RG equations for the $N$-component RF and RA models in $4 + \epsilon$ dimensions were derived in Ref.\[20\] Their solution allowed to describe the phase transition from the ferromagnetic to paramagnetic state of the RF $O(N)$ model above 4 dimensions.\[46\] The RG equations in dimensions $D < 4$ can be obtained by just changing the sign of $\epsilon$ in the RG equations in $D > 4$.

In our derivation of RG equations we follow Ref.\[15\] We use the method, suggested by Polyakov\[42, 43\] for the pure system in $2 + \epsilon$ dimensions. This method is technically simpler and closer to the intuition than the other approaches. A disadvantage of the method is the difficulty of the generalization for the higher orders in $\epsilon$. This generalization requires the field-theoretical approach.\[47\]

The same consideration as in the XY\[12\] and random manifold\[48, 49\] models suggests that near a zero-temperature fixed point in $4 - \epsilon$ dimensions there is an infinite set of relevant operators. Let us show that after replica averaging\[5\] the relevant part of the effective replica Hamiltonian can be represented in the form

$$H_R = \int d^D x \left[ \sum_a \frac{1}{2T} \sum_\mu \partial_\mu n_a \partial_\mu n_a - \sum_{ab} \frac{R(n_a n_b)}{T^2} \right], \quad (19)$$

where $a, b$ are replica indices, $R(z)$ is some function, $T$ the temperature. We ascribe to the field $n$ the scaling dimension 0. We also assume that the coefficient before the gradient term in (19) is $1/(2T)$ at any scale. Then in the $(4 - \epsilon)$-dimensional space the scaling dimension of the temperature $\Delta T = -2 + O(\epsilon)$. Any operator $A$ containing $m$ different replica indices is proportional\[50\] to $1/T^m$. Indeed, before replica averaging any term of the Hamiltonian contains one replica index and the temperature in the minus first power. If we expand the partition function in Taylor
series any $m$-replica term of the expansion contains $T^{-m}$. This property conserves after disorder averaging. To obtain the effective replica Hamiltonian one reexponentiates the disorder-average series. Then one easily sees that the power of $1/T$ in any term of the replica Hamiltonian equals the number of the replica indices in it. Hence, the scaling dimension $\Delta_A$ of the operator $A$ satisfies the relation $\Delta_A = 4 - n + m\Delta_T + O(\epsilon)$, where $n$ is the number of the spatial derivatives in the operator. The relevant operators have $\Delta_A \geq 0$. Hence, the relevant operators cannot contain more than two different replica indices. A symmetry consideration shows that all possible relevant operators are included into Eq. (19). The function $R(z)$ is arbitrary in the RF case. In the RA case $R(z)$ is even due to the symmetry with respect to changing the sign of the magnetization.

The one-loop RG equations for the $N$-component model in $4-\epsilon$ dimensions are obtained by a straightforward combination of the methods of Ref. [49] and Refs. [42, 43]. We express each replica $n^a(x)$ of the magnetization as a combination of fast fields $\phi^a_i(x)$, $i = 1, \ldots, N - 1$ and a slow field $n'^a(x)$ of the unit length. We use the representation

$$n^a(x) = n'^a(x) \sqrt{1 - \sum_i (\phi^a_i(x))^2} + \sum_i \phi^a_i(x)e^a_i(x), \quad (20)$$

where the unit vectors $e^a_i(x)$ are perpendicular to each other and the vector $n'^a(x)$. The slow field $n'^a(x)$ can be chosen in different ways. For example, one can cut the system into blocks of size $L \gg a$, where $a$ is the ultra-violet cut-off. In the center $x$ of a block the vector $n'^a(x)$ should be parallel to the total magnetization of the block. In other points the field $n'^a(x)$ should be interpolated. We assume that the fluctuations of the magnetization are weak, that is $\langle \phi^a_i \rangle \ll 1$. Then the fluctuations of the field $n^a$ are orthogonal to the vector $n'^a$ because of the fixed length constraint $(n'^a)^2 = 1$.

To substitute the expression [20] into the Hamiltonian we have to differentiate the vectors $e^a_i$. We use the following definition

$$\frac{\partial e^a_i}{\partial x_\mu} = -c^a_{\mu i} n'^a + \sum_k f^a_{\mu i k} e^a_k, \quad (21)$$

It is easy to show that $\sum_i (c^a_{\mu i})^2 = \sum_\mu (\partial_\mu n'^a)^2$. With the accuracy up to the second order in $\phi$ the replica Hamiltonian (19) can be represented as follows

$$H_R = \int d^D x \left[ \frac{1}{2T} \sum_a \{ (\partial_\mu n'^a)^2 (1 - (\phi^a_i)^2) + c^a_{\mu i} c^a_{\mu k} \phi^a_i \phi^a_k + (\partial_\mu \phi^a_i - f^a_{\mu i k} \phi^a_k)^2 \} - \frac{1}{T^2} \sum_{a b} \{ R(n'^a n'^b) + A^{a b} (\phi^a_i)^2 + B^{a b} \phi^a_i \phi^b_j + C^{a b} \phi^a_i \phi^b_j \} \right], \quad (22)$$

where the summation over the repeated indices $i, j, k, \mu$ is assumed and
\[ A_{ij} = - (n_i a_i n_j b_j) R'(n_i a_i n_j b_j); B_{ij} = (n_i a_i e_i^j e_j^b) R''(n_i a_i n_j b_j); C_{ij} = (e_i^a e_j^b) R'(n_i a_i n_j b_j) + (n_i a_i e_j^b) R''(n_i a_i n_j b_j). \] 

(23)

In the last formula \( R' \) and \( R'' \) denote the first and second derivatives of the function \( R(z) \). We have omitted the terms of the first order in \( \phi \) in Eq. (22). These terms are proportional to the products of the fast field \( \phi \) and some slow fields. Hence, they are non-zero only in narrow shells of the momentum space. One can show that their contributions to the RG equations are negligible.

To obtain the RG equations we have to integrate out the fast variables \( \phi_i^a \). Near a zero-temperature fixed point the Jacobian of the transformation \( n \rightarrow (n', \phi_i) \) can be ignored, since the Jacobian does not depend on the temperature. The integration measure is determined from the condition that the field \( n_i^a \) is a slow part of the magnetization. This condition imposes restrictions on the fields \( \phi \). The expression (22) depends on the choice of the vectors \( e_i^a \) (21). However, after integrating out the fields \( \phi \) the Hamiltonian can depend only on the slow part \( n_i^a \) of the magnetization. One can make the calculations simpler, considering special realizations of the field \( n_i^a \). To find the renormalization of the disorder-induced term \( R(z) \) (19) we can assume that the field \( n_i^a \) does not depend on the coordinates. The renormalization of the gradient energy can be determined, assuming that the vectors \( n_i^a(x) \) depend on one spatial coordinate only and lie in the same plane. In both cases the vectors \( e_i^a \) can be chosen such that in Eq. (21) \( f_{ijk}^a = 0 \). At such a choice the integration measure can be omitted and the fields \( \phi_i^a \) can be considered as weakly interacting fields with the wave vectors from the interval \( 1/a > q > 1/L \).

To derive the one-loop RG equations we express the free energy via the Hamiltonian (22). Then we expand the exponent in the partition function up to the second order in \( (H_R - \int d^D x \sum_{\mu i} (\partial_\mu \phi_i)^2/(2T)) \) and integrate over \( \phi_i^a \). Finally, we make a rescaling. The vectors \( e_i^a \) can be excluded from the final expressions with the relation \( \sum_i (ae_i^a)(be_i^a) = (ab) - (an^a)(bn^a) \). In a zero-temperature fixed point the RG equations are

\[ \frac{d \ln T}{d \ln L} = -(D - 2) + 2(N - 2) R'(1) + O(R^2, T); \]

(24)

\[ 0 = \frac{d R(z)}{d \ln L} = \epsilon R(z) + 4(N - 2) R(z) R'(1) - 2(N - 1) z R'(1) R'(z) + 2(1 - z^2) R'(1) R''(z) + (R'(z))^2 (N - 2 + z^2) - 2 R'(z) R''(z) z (1 - z^2) + (R''(z))^2 (1 - z^2)^2, \]

(25)

where the factor \( 1/(8\pi^2) \) is absorbed into \( R(z) \) to simplify notations. The RG equations become simpler after the substitution of the argument of the function \( R(z) \): \( z = \cos \phi \). In terms of this new variable one has to find even periodic solutions \( R(\phi) \). The period is \( 2\pi \) in the RF case and \( \pi \) in the RA case due to the additional symmetry of the RA model. The one-loop equations get the form...
\[ \frac{d \ln T}{d \ln L} = -(D - 2) - 2(N - 2)R''(0) + O(R^2, T); \quad (26) \]

\[ 0 = \frac{dR(\phi)}{d \ln L} = \epsilon R(\phi) + (R''(\phi))^2 - 2R''(\phi)R''(0) - 
\]

\[(N - 2) \left[ 4R(\phi)R''(0) + 2\text{ctg}^2(\phi)R''(0) - \left( \frac{R'(\phi)}{\sin(\phi)} \right)^2 \right] + O(R^3, T) \quad (27) \]

Eq. (26) provides the following result for the scaling dimension \( \Delta_T \) of the temperature

\[ \Delta_T = -2 + \epsilon - 2(N - 2)R''(0). \quad (28) \]

The two-spin correlation function is given in the one-loop order by the expression

\[ \langle n^a(x)n^a(x') \rangle = \langle n'^a(x)n'^a(x') \rangle \left( 1 - \left( \sum_i (\phi_i^a)^2 \right) \right). \quad (29) \]

Hence, in the fixed point \( \langle n(x)n(x') \rangle \sim |x - x'|^{-\eta} \), where

\[ \eta = -2(N - 1)R''(\phi = 0) \quad (30) \]

Let us find the magnetic susceptibility in the weak uniform external field \( H \). We add to the Hamiltonian (19) the term \( -\sum_a \int d^Dx H n^a_z/T \) (the field is directed along the z-axis). The renormalization of the field \( H \) is determined by the renormalization of the temperature \( \[24\] \) and the field \( n \). In the zero-loop order the renormalized magnetic field \( h(L) \) depends on the scale as \( h(L) = H \times (L/a)^2 \). Hence, the correlation length \( R_c \sim H^{-1/2} \). The magnetization, averaged over a block of size \( R_c \), is oriented along the field. The absolute value of this average magnetization is proportional to \( R_c^{-\eta/2} \). This allows us to calculate the critical exponent \( \gamma \) of the magnetic susceptibility \( \chi(H) \sim H^{-\gamma} \) in a zero-temperature fixed point:

\[ \gamma = 1 + (N - 1)R''(\phi = 0)/2. \quad (31) \]

In Ref.\[24\] Eqs. (24, 25) were derived with a different method. In that paper the critical behavior in \( 4 + \epsilon \) dimensions was studied by considering analytical fixed point solutions \( R(z) \). In the Heisenberg model, analytical solutions are absent and they are unphysical for \( N \neq 3 \). In \( 4-\epsilon \) dimensions appropriate analytical solutions are absent for any \( N \). To demonstrate this let us differentiate Eq. (24) over \( z \) at \( z = 1 \). For any analytical \( R(z) \) we obtain the following flow equation

\[ \frac{dR'(z = 1)}{d \ln L} = \epsilon R'(z = 1) + 2(N - 2)(R'(z = 1))^2. \quad (32) \]
At $N > 2$ the fixed point of this equation $R'(z = 1) = -\epsilon/[2(N - 2)] < 0$. It corresponds to the negative critical exponent $\eta$ and hence is unphysical (two-spin correlation functions are limited and cannot grow up to infinity as $R \to \infty$).

However, we shall see that in the RA model some appropriate non-analytical fixed points $R(z)$ appear. In these fixed points $R''(z = 1) = \infty$. In Ref. the RG charges are the derivatives of the function $R(z)$ at $z = 1$. Thus, in a non-analytical fixed point these charges diverge. In the systems with a finite number of the charges their divergence implies the absence of a fixed point. However, if the number of the RG charges is infinite such a criterion does not work and is even ambiguous. Indeed, the set of charges can be chosen in different ways and e.g. the coefficients of the Taylor expansion about $z = 0$ remain finite in our problem.

Nonanalicity of the fixed point solution is related with a complicated structure of the energy landscape. This has important consequences for dynamics. An interested reader may refer to Ref.

5. Solution of the Renormalization Group Equations

In this section we solve the RG equations of the preceding section. The main aim is to understand in which cases there is QLRO and when it is absent. The RG equations are the same for the RF and RA $O(N)$ models. The difference between the models is the symmetry of the fixed point solution. In the RF case we are looking for the solution which is stable to arbitrary perturbations and describes the low-temperature phase of a weakly disordered system. In the RA case one has to respect the symmetry to changing the sign of the magnetization. As discussed above in the RA case the function $R(z)$ must be even. The solution should be stable only to the perturbations that do not break this property.

We obtain the following results: In the RF model, QLRO exists at $N = 2$ only. In the RA case, QLRO is present in $4 - \epsilon$ dimensions for $N < 10$. The critical exponents describing the QLRO phases of the RA $O(N)$ model are given in Table 1.

The section is organized as follows: In the first subsection we discuss a simple approximate solution of the RG equations. In two next subsections we provide a systematic analysis of the RF and RA models. Since our RG approach ignores topological defects we have to check that they are irrelevant. This is done in the fourth subsection.

5.1. Migdal-Kadanoff renormalization group

This subsection contains a simple approximate version of the renormalization group. The results for the critical exponents of the XY and Heisenberg models have a very good accuracy. The value of the magnetization component number $N_c$, at which QLRO disappears in the RF model, is probably exact. However, the critical number of the components in the RA model is underestimated.

5.1.1. Random field
We use the following ansatz for the disorder-induced term in the Hamiltonian (19):
\[ R(n_a, n_b) = \alpha n_a n_b + \beta, \]
where \( \alpha \) and \( \beta \) are constants. This expression corresponds to the Gaussian RF disorder (14). Below we ignore the generation of other contributions to the function \( R(z) \). The missed contributions are related to random anisotropies of different orders. In terms of the angle variable \( \phi \) \[ R(\phi) = \alpha \cos \phi + \beta. \] (33)

To ensure consistency we have to truncate the RG equation (27). We substitute the ansatz (33) into Eq. (27) and retain only the terms, proportional to \( \cos \phi \) or independent of \( \phi \). This leads to the following RG equation for the constant \( \alpha \) (33)
\[ \frac{d\alpha}{d \ln L} = \epsilon \alpha + 2\alpha^2(N - 3). \] (34)

For \( N < 3 \) Eq. (34) has a stable solution \( \alpha = \epsilon / [2(3-N)] \). This solution corresponds to a QLRO state. The critical exponent (35) equals
\[ \eta = \frac{(N-1)\epsilon}{(3-N)}. \] (35)

At \( N = 2 \) this result has less than ten percent difference from the systematic approach. For \( N > 3 \) a fixed point exists in \( 4 + \epsilon \) dimensions. It describes the transition between the ferromagnetic and paramagnetic phases. In this fixed point the critical exponent (33) satisfies the modified dimensional reduction hypothesis. However, this is an artifact of the Migdal-Kadanoff approximation, since the correct value of the critical exponent differs from Eq. (33). The detailed discussion of the critical exponents of the RF \( O(N) \) model in \( 4 + \epsilon \) dimensions is beyond the scope of the present article. Some details can be found in Ref. 46.

5.1.2. Random anisotropy

In this case we use the ansatz \( R(n_a, n_b) = A(n_a n_b)^2 + B \). In terms of the variable \( \phi \) Eqs. (26\textsuperscript{27}) \( R(\phi) = \alpha \cos 2\phi + \beta \). We again substitute our ansatz into Eq. (27) and retain the terms, proportional to \( \cos 2\phi \), and the terms, independent of \( \phi \). The RG equation for the constant \( \alpha \) has the form
\[ \frac{d\alpha}{d \ln L} = \epsilon \alpha + 8(N - 6)\alpha^2. \] (36)
The fixed point solution of this equation is \( \alpha = \epsilon / [8(6-N)] \). It describes the QLRO phase at \( N < 6 \). At \( N = 3 \) the function \( R(\phi) = \alpha \cos 2\phi + \beta \) is just \( R_{\omega=0} \) of section 5.3.3. The critical exponent of the two-spin correlation function is given by the following equation
\[ \eta = \frac{\epsilon(N-1)}{6-N}. \] (37)
At $N = 2, 3$ this value is close to the results of the systematic approach (Table 1).

The approximate analysis suggests that QLRO disappears at $N = 6$. This is different from the result of the systematic approach $N_c = 10$.

5.2. Random field

5.2.1. $N = 2$

For the RF XY model the one-loop RG equations (26,27) can be solved analytically. The solution is a periodic function $R(\phi)$ with period $2\pi$. In interval $0 < \phi < 2\pi$ the fixed point solution $R(\phi)$ is given by the formula

$$R(\phi) = \frac{\pi^3 \epsilon}{9} \left[ \frac{1}{36} - \frac{(\phi/2\pi)^2}{(1 - (\phi/2\pi))^2} \right].$$

This is a stable fixed point. This can be verified with the linearization of the flow equation (27) for small deviations from the fixed point.

The solution (38) corresponds to QLRO with the critical exponents (30,31)

$$\eta = \frac{\pi^2}{9\epsilon}, \gamma = 1 - \frac{\pi^2}{18\epsilon}.$$ 

In the first order in $\epsilon$ the exponent $\eta$ equals the prefactor $C$ before the logarithm in the correlation function $\langle \phi(x) \rangle$ between the spins $n(x)$ and some fixed direction: $\langle (\phi(x_1) - \phi(x_2))^2 \rangle = C \ln |x_1 - x_2|$. We expect that this coincidence does not extend to the higher orders.

5.2.2. $N > 2$

If $N \neq 2$ the RG equation (27) is more complex. Fortunately, at $N > 3$ there is still a simple method to study the large-distance behavior. The method is based on the Schwartz-Soffer inequality and shows that QLRO is absent.

In Ref.53 the inequality is proven for the Gaussian distribution of the random field. It can also be proved for the arbitrary RF distribution. We prove the inequality in the Appendix.

Let us demonstrate the absence of physically acceptable fixed points in the RF case at $N > 3$. We derive some inequality for critical exponents. Then we show that the inequality has no solutions. We use a rigorous inequality for the connected and disconnected correlation functions

$$\langle \langle n(q)n(-q) \rangle \rangle = \langle n_a(q)n_b(-q) \rangle - \langle n_a(q)n_b(-q) \rangle \leq \text{const} \sqrt{\langle n_a(q)n_a(-q) \rangle}, \quad (39)$$

where $n(q)$ is a Fourier-component of the magnetization, $a, b$ are replica indices. The disconnected correlation function is described by the critical exponent (30). The large-distance behavior of the connected correlation function in a zero-temperature fixed point can be derived from the expression

$$\chi \sim \int \langle \langle n(0)n(x) \rangle \rangle d^D x \quad (40)$$
and the critical exponent of the susceptibility (31). The integral in the right hand side of Eq. (40) is proportional to \( R_{D}^{-\eta_{1}} \), where \( R_{c} \) is the correlation length in the external field \( H \), \( \eta_{1} \) the critical exponent of the connected correlation function. For the calculation of the exponent \( \gamma \) (31) we used the zero-loop expression of \( R_{c} \) via \( H \). Now we need the one-loop accuracy. In this order \( R_{c} \sim H^{-1/[2-(N-3)R''(0)]} \). This allows us to get the following equation for the exponent \( \eta_{1} \)

\[
\eta_{1} = D - 2 - 2R''(0) \tag{41}
\]

In a fixed point Eq. (39) provides an inequality for the critical exponents of the connected and disconnected correlation functions. The inequality has the form

\[
2(2 - D + \eta_{1}) \geq 4 - D + \eta \tag{42}
\]

This allows us to obtain the following relation

\[
4 - D \leq \frac{3 - N}{N - 1} \eta + o(R) \tag{43}
\]

where \( \eta \) is given by Eq. (30). The two-spin correlation function cannot increase up to infinity as the distance increases. Hence, the critical exponent \( \eta \) is positive. At \( N > 3 \) this is incompatible with Eq. (43) at small \( \epsilon \). Thus, there are no appropriate fixed points for \( N > 3 \). This suggests the strong coupling regime with a presumably finite correlation length. We see that QLRO disappears at \( N > N_{c} \leq 3 \). Numerical analysis of the RG equations supports \( N_{c} < 3 \).

Certainly, in the RF XY model \cite{12,13} Eq. (43) is satisfied. However, the unstable fixed points of the RG equations \cite{12,13} do not satisfy the inequality.

5.3. **Random anisotropy**

In this subsection we investigate a possibility of QLRO in the RA \( O(N) \) model. The first subsubsection is devoted to the simplest case of the XY model. The second subsubsection contains an inequality for the critical exponent \( \eta \). The derivation of the inequality is analogous to the derivation of Eq. (43). This inequality is applied in the next subsubsections. The third subsubsection contains the results for the Heisenberg model. In the last subsubsection we consider the case \( N > 3 \).

5.3.1. \( N = 2 \)

This case is studied analogously to the RF XY model \cite{14} At \( N = 2 \) the RG equation (27) can be solved analytically. Its solution is a periodic function with period \( \pi \). In interval \( 0 < \phi < \pi \) the fixed point solution \( R(\phi) \) is given by the formula

\[
R(\phi) = \frac{\pi^{4} \epsilon}{144} \left[ 1/36 - (\phi/\pi)^{2} (1 - (\phi/\pi))^{2} \right] \tag{44}
\]

It is a stable fixed point. This can be verified with the linearization of the flow equation (27) for small deviations from the fixed point. Another proof of stability is based on the inequality of the next subsubsection.
The stable fixed point corresponds to the QLRO phase at low temperatures and weak disorder. The critical exponents $\eta = \pi^2 \epsilon/36$, $\gamma = 1 - \pi^2 \epsilon/72$.

The solution (44) is non-analytical at $\phi = 0$, since $R^IV(\phi = 0) = \infty$. Hence, the Taylor expansion over $\phi$ is absent. However, a power expansion over $|\phi|$ exists. We shall see below that the same behavior at small $\phi$ conserves also at other $N$.

5.3.2. An inequality for a critical exponent

We use the same approach as in the RF model. Since in the RA case the random field is conjugated with a second order polynomial of the magnetization, the Schwartz-Soffer inequality should be applied to correlation functions of the field $m(x) = (n_z(x))^2 - 1/N$, where $n_z$ denotes one of the magnetization components, $1/N$ is subtracted to ensure the relation $\langle m \rangle = 0$.

To calculate the critical exponent $\mu$ of the disconnected correlation function we use the representation (20) and obtain the relation

$$\langle m^a(x)m^a(x') \rangle = \langle m^{a'}(x)m^{a'}(x') \rangle \left( 1 - \frac{2N \sum \langle (\phi^a_i)^2 \rangle}{N-1} \right),$$

where $a$ is a replica index, $m' = (n'_z)^2 - 1/N$ the slow part of the field $m$. One finds $\mu = -4NR''(0)$.

The critical exponent $\mu_1$ of the connected correlation function is determined analogously to the RF case. We apply a weak uniform field $\tilde{H}$, conjugated with the field $m$, and calculate the susceptibility $dm/d\tilde{H}$ in two ways. The result for the critical exponent is $\mu_1 = D - 2 - 2(N+2)R''(0)$.

The Schwartz-Soffer inequality provides a relation between the exponents $\mu$ and $\mu_1$. It has the same structure as Eq. (42). Finally, we obtain the following equation

$$\eta \geq \frac{4-D}{4} (N-1) + o(R).$$

In terms of the RG charge $R(\phi)$ this inequality can be rewritten in the form

$$R''(0) \leq -\epsilon/8 + o(R).$$

As discussed in Ref. 15, any solution of the RG equations that does not satisfy Eq. (17) describes an unstable fixed point.

5.3.3. $N = 3$

In this case we solve Eq. (27) numerically. Since coefficients of Eq. (27) are large as $\phi \to 0$, it is convenient to use a series expansion of the fixed-point solution $R(\phi)$ at small $\phi$. At larger $\phi$ the equation can be integrated with the Runge-Kutta method.

The following expansion over $t = \sqrt{(1 - z)/2} = |\sin(\phi/2)|$ holds

$$R(\phi)/\epsilon = \frac{(N-1)a^2}{1-4(N-2)a} + 2a \sin^2 \frac{\phi}{2} \pm \frac{4\sqrt{2}}{3} \sqrt{\frac{-a + 2(N-2)a^2}{N+2}} |\sin^3 \frac{\phi}{2}|$$
\[ a = R''(\phi = 0)/\epsilon. \] We see that the RG charge \( R(\phi) \) is non-analytical at small \( \phi \). Similar to the random manifold\(^{27,49}\) and random-field XY\(^{12}\) models \( R^{IV}(0) = \infty \).

Numerical calculations show that at any \( N \) the solutions, compatible with the inequality \( |47| \), have sign "+" before the third term of Eq. \((48)\). The solutions to be found are even periodical functions with period \( \pi \). Hence, their derivative is zero at \( \phi = \pi/2 \). At \( N = 3 \) there is only one solution that satisfies Eq. \((47)\). It corresponds to

\[ R''(\phi = 0) = -0.1543\epsilon. \] (49)

If this solution is stable Eqs. \((30, 31)\) provide the following results for the critical exponents

\[ \eta = 0.62\epsilon; \gamma = 1 - 0.15\epsilon. \] (50)

All the other solutions of Eq. \((27)\) do not satisfy Eq. \((47)\) and hence are unstable.

We have still to test the stability of the solution found. For this aim we use an approximate method. First, we find an approximate analytical solution of Eq. \((27)\).

We rewrite Eq. \((27)\), substituting \( \omega(R''(\phi))^2 \) for \( (R''(\phi))^2 \). The case of interest is \( \omega = 1 \) but at \( \omega = 0 \) the equation can be solved exactly. The solution at \( \omega = 1 \) can then be found with the perturbation theory over \( \omega \). The exact solution at \( \omega = 0 \) is \( R_{\omega=0}(\phi) = \epsilon(\cos 2\phi/24 + 1/120) \). The corrections of order \( \omega^k \) are trigonometric polynomials of order \( 2(k + 1) \). The first correction is

\[ R_1(\phi) = -\frac{2\omega\epsilon}{99} \cos 2\phi + \frac{\omega\epsilon}{264} \cos 4\phi + \text{const} \] (51)

After the calculation of the corrections we can write an asymptotic series for the critical exponent \( \eta \) \((30)\): \( \eta = \epsilon(0.67 - 0.08\omega + 0.14\omega^2 - \ldots) \). The resulting estimation \( \eta = \epsilon(0.67 \pm 0.08) \) agrees with the numerical result \((30)\) well. This allows us to expect that the stability analysis of the solution \( R_{\omega=0} \) of the equation with \( \omega = 0 \) provides information about stability of the solution of Eq. \((27)\).

To study stability of the exact solution of the equation with \( \omega = 0 \) is a simple problem. We introduce a small deviation \( r(\phi) \): \( R(\phi) = R_{\omega=0}(\phi) + r(\phi) \) and write the flow equation for this deviation:

\[ \frac{dr(\phi)}{d\ln L} = (5r(\phi) + r''(\phi) + r''(0) \cos 2\phi)/3 + \text{const} \times r''(0). \] (52)

It is convenient to use the Fourier expansion \( r(\phi) = \sum_m a_m \cos 2m\phi. \) The flow equations for the Fourier harmonics can be easily integrated. We see that \( a_m \to 0 \) as \( L \to \infty \) for any \( m > 0 \). The solution is unstable with respect to the constant shift \( a_0 \). However, this instability has no interest for us, since correlation functions do not change at such shifts.\(^{27}\) Indeed, a constant shift corresponds to the addition of just a random term, independent of the magnetization, to the Hamiltonian \((13)\).
Thus, the RG equation possesses a stable fixed point. This fixed point describes the QLRO phase with the critical exponents (50).

As usual in critical phenomena, in 4 dimensions the one-loop RG equations allow one to obtain the exact large-distance asymptotics of the correlation function. In the 4-dimensional case \( R(\phi) = \tilde{R}(\phi)/\ln L \), where \( \tilde{R}(\phi) \) satisfies Eq. (27) at \( \epsilon = 1 \). We obtain the following result for the two-spin correlation function with Eq. (29)

\[
\langle \mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}') \rangle \sim \ln^{-0.62} |\mathbf{x} - \mathbf{x}'|.
\] (53)

5.3.4. \( N > 3 \)

Numerical analysis of Eq. (27) shows that solutions, compatible with Eq. (47), are absent at \( N \geq 10 \). Hence, QLRO is absent for any \( N \geq 10 \). This agrees with the previous results for the spherical model. For each integer \( N < 10 \) the RG equation (27) has exactly one solution satisfying the inequality (47). These solutions are described in Table 1. In the table, \( \eta \) is the critical exponent of the two-spin correlation function, \( \Delta_T \) the scaling dimension of the temperature (28).

Unfortunately, it is not clear if the fixed points, found at \( N > 3 \), survive in 3 dimensions. A zero-temperature fixed point can exist only if the scaling dimension of the temperature is negative. Table 1 shows that scaling dimension is positive in the one-loop approximation at \( \epsilon = 1 \) and \( N \geq 5 \). In the 3-dimensional O(4) model the one-loop correction to the scaling dimension \( -2(N - 2)R''(0) \approx 0.7\epsilon \) is close to the zero-loop approximation \(-2 + \epsilon\). Thus, the next orders of the perturbation theory are crucial to understand what happens in 3 dimensions.

In the \( O(2) \) model the scaling dimension \( \Delta_T = -2 + \epsilon \) is exact. \( N \geq 3 \) Hence, QLRO disappears in 2 dimensions. In systems with larger numbers of magnetization components fluctuations become stronger. Thus, one expects the absence of QLRO in all the two-dimensional \( O(N) \) models.

At zero temperature Eq. (27) is valid independently of the scaling dimension \( \Delta_T \). It is tempting to assume that at zero temperature, QLRO still exists in the RA \( O(N > 3) \) models below the critical dimension, in which \( \Delta_T = 0 \). However, the experience of the two-dimensional RF XY model does not support such an expectation. Recent numerical simulations show that QLRO is absent even in the ground state of that model.

| \( N \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|--------|---|---|---|---|---|---|---|---|
| \( \eta \) | \( \pi^2/36 \) | 0.62\epsilon | 1.1\epsilon | 1.6\epsilon | 2.1\epsilon | 2.6\epsilon | 3.1\epsilon | 3.6\epsilon |
| \( \Delta_T \) | \( -2 + \epsilon \) | \( -2 + 1.3\epsilon \) | \( -2 + 1.7\epsilon \) | \( -2 + 2.3\epsilon \) | \( -2 + 3.2\epsilon \) | \( -2 + 4.8\epsilon \) | \( -2 + 8.7\epsilon \) | \( -2 + 30\epsilon \) |

5.4. Topological defects

Our RG procedure is based on the decomposition (20) which makes sense only if the magnetization change is slow at the microscopic scale \( \mathbf{a} \). This condition is broken.
in the core of a topological defect. The nature of defects depends on the particular model but we consider different types of defects in the same way. In the XY model the defects are dislocation loops. In the Heisenberg model they are point defects. Two types of defects are possible in nematic liquid crystals: disclination loops and point defects. A disclination loop is a line the rotation by $2\pi$ around which reverses the director: $\mathbf{n} \rightarrow -\mathbf{n}$. The structure of a point defect is analogous to the structure of a hedgehog in the Heisenberg model.

Topological defects are irrelevant at small $\epsilon = 4 - D$ for weak disorder. This can be understood from the consideration of the contribution of disclination (dislocation) loops and pairs of point defects of size $l \gg a$ to the RG equations at the scale $l$. After averaging over the small-scale fluctuations the size $l$ of topological excitations plays the role of the ultra-violet cut-off. The renormalized temperature is small: $T(l) \ll 1$. Hence, thermal fluctuations are irrelevant. In the random-field problem the disorder-induced term $R(\mathbf{n}_a \mathbf{n}_b) \sim \epsilon$ in the renormalized replica Hamiltonian (19) is of order $\langle h^2 \rangle$, where the random vector $\mathbf{h}$ describes the (renormalized) random field $E_{\text{dis}} = -h \mathbf{n}$, $\langle ... \rangle$ denotes the average over the realizations of disorder. In the RA problem the disorder-induced term $R(\mathbf{n}_a \mathbf{n}_b) \sim \epsilon$ is of order $\langle h^4 \rangle$, where the random vector $\mathbf{h}$ describes the (renormalized) random anisotropy $E_{\text{dis}} = (h \mathbf{n})^2$. Inside a defect the director change is of order 1 at the cut-off scale. Hence, the elastic excitation energy determined by the renormalized Hamiltonian $H(l)$ (13) can be compensated by the interaction with disorder only in the positions where $h \sim 1$. The concentration of such positions is exponentially small $\sim \exp(-1/\epsilon)$. Thus, defects produce corrections of order $\exp(-1/\epsilon)$ to the RG equations and do not modify the results of the paper qualitatively. The concentration of the topological excitations of size $l$ is not more than of order $l^{-D} \exp(-1/\epsilon)$. The above discussion is valid, if disorder is weak. In the case of strong disorder, topological defects are present at the microscopic scale $a$ and QLRO is absent. Thus, topological defects may drive the system into another glass state in which the orientation of the director is determined only by the local random potential. The critical strength of disorder at which QLRO disappears can be estimated by comparison of the elastic and random contributions to the energy.

The irrelevance of topological defects for weak disorder can also be demonstrated with the energy argument modified to take into account the scale dependence of the interaction. A possible fractal structure of large dislocation loops can lead to their strong suppression.

Recently the role of defects in the RF XY model was a subject of intensive discussions. For a review see Ref.

6. Comparison with Experiments and Numerical Experiments

In this section we discuss experimental and numerical results about QLRO in disordered superconductors, amorphous magnets and nematics in random porous media. In the first subsection we consider superconductors. Our discussion is very brief. More details can be found in Refs. The second subsection is devoted to amor-
phous magnets. In the last subsection we consider nematics. Since nematics are described by the model (2-4) which differs from the random-anisotropy Heisenberg model, we have to derive RG equations for all three Frank constants $K_i$. It turns out that at large scales the difference from the RA Heisenberg model is irrelevant and QLRO phase exists at low temperatures and weak disorder. We then make predictions about light scattering in the QLRO state and discuss several new phases that emerge in the presence of external electric or magnetic fields, or mechanical stresses. At the end of the subsection we consider existing experimental and numerical data concerning QLRO in disordered nematics.

6.1. Disordered superconductors

There is a huge amount of literature about vortex states of disordered superconductors. For example, in the latest issue of Nature before this review was submitted two important experimental observations were published: inverse melting of the vortex lattice in BSCCO and discovery of a second vortex liquid phase in YBCO. A particular question of QLRO in disordered superconductors has also received much attention.

The most important theoretical prediction is the emergence of Bragg peaks in scattering experiments with disordered superconductors at weak applied magnetic fields. The vortex density $\rho(x)$ is given by Eq. (9). In scattering experiments the square of the modulus $|\rho_k|^2$ of the Fourier transform of the density is measured. For a perfect lattice the Fourier transform is non-zero only for $k$ equal to reciprocal vectors of the lattice. For a non-ideal lattice we get

$$|\rho_k|^2 \sim \sum_{ij} \exp(ik[R_i - R_j]) \exp(ik[u_i - u_j]).$$

(54)

After disorder and thermal averaging one sees that the scattering measures the Fourier transform of the correlation function

$$C_k(x) = \langle \exp(ik[u(x) - u(0)]) \rangle = \langle \cos(k[u(x) - u(0)]) \rangle.$$

(55)

In terms of the Hamiltonian (12) for the vortex displacements from the regular positions in the Abrikosov lattice, the correlation function (29) is $C_k(x)$ for a particular value of $k$. We have seen that the correlation function (29) obeys a power dependence on the distance. The generalization of this property for an arbitrary $k$ is straightforward. In terms of the scattering experiment this leads to the situation which is intermediate between an ideal lattice and short-range order: There are Bragg peaks but they have finite widths. These Bragg peaks were indeed observed in agreement with theory.

Numerical experiments also support QLRO in the RF XY model of disordered superconductors (see Ref.66).

Finally, we note that the RF XY model is a simplification. A more realistic model that takes into account the triangular symmetry of the Abrikosov lattice...
gives slightly different predictions. An interesting point is nonuniversal QLRO: critical exponents are different in different points of the phase diagram. However, they vary in a very limited range.

6.2. Amorphous magnets

A consequence of QLRO, predicted in the RA Heisenberg model, is divergence of the magnetic susceptibility (31). Earlier analysis of Arrott plots, which show the field dependence of the magnetization, suggested that the susceptibility does diverge at low applied fields. Later it become clear that in strongly disordered amorphous magnets the zero-field susceptibility is actually finite (e.g. Ref. 68). What happens at weak disorder is a more difficult experimental question. There is an evidence of a finite susceptibility at weak random anisotropy. This could be interpreted as an experimental argument in favor of the absence of QLRO in the RA Heisenberg model in $4 - \epsilon$ dimensions at $\epsilon = 1$. However, in such case the Imry-Ma argument and the RG analysis in the spirit of section 3 would predict the following scaling for the susceptibility: $\chi \sim (J/A)^4$, where $J$ is the exchange strength, $A$ is the anisotropy. This scaling was not observed and in contrast to the theoretical expectations it turned out that $\chi \to \text{const}$ as $A \to 0$. The authors of Ref. 69 interpreted this as an effect of dipole forces. However, the magnetic susceptibility of the pure Heisenberg ferromagnet is expected to be infinite even in the presence of dipole forces. This suggests that the experimental system used in Ref. 69 cannot be described as a realization of the RA Heisenberg model even with dipole forces. Besides, the effect of dipole forces on the existence of a QLRO state in the RA Heisenberg model is an open question.

From the numerical side the existence of QLRO is supported by a recent paper. However, numerical results support also QLRO in the RF Heisenberg model in contradiction with our predictions. We believe that this is a finite-size effect. More work is needed for a better numerical understanding of the RA and RF Heisenberg models.

6.3. Disordered nematics

In the one-constant approximation $K_1 = K_2 = K_3$ the energy $F = F_{\text{d}} + F_{\text{pm}}$ (3) reduces to the Hamiltonian of the RA Heisenberg model. Since that model possesses QLRO the same ordering is expected for the randomly confined nematic. However, in all nematics $K_1, K_3 > K_2$ and this could change the critical exponents of the correlation functions in the QLRO state in comparison with the random Heisenberg model. Below we demonstrate that this is not the case, i.e. the nematic in the porous matrix belongs to the universality class of the RA Heisenberg model.

To get a simple idea why it occurs we first consider a two-dimensional nematic film with the director $\mathbf{n} = (n_x, n_y, n_z) = (\cos \phi, \sin \phi, 0)$ confined in the plane $xy$ of the film in the absence of disorder. The Frank energy is $F_{\text{d}} = (K_1 + K_3)(\nabla \phi)^2/2 + (K_3 - K_1)\{\cos 2\phi[(\partial_x \phi)^2 - (\partial_y \phi)^2]/2 + \sin 2\phi \partial_x \phi \partial_y \phi\}$. The low-temperature phase
of this system possesses QLRO, only the term \((K_1 + K_3)(\nabla \phi)^2/2\) being relevant at large scales since \(\langle \sin 2\phi \rangle = \langle \cos 2\phi \rangle = 0\).

The systematic consideration is based on the RG equations in \(4 - \epsilon\) dimensions. Our method follows the line of section 4. All relevant operators of the appropriate symmetry are included in the following replica Hamiltonian

\[
H_R = \int d^3r \left[ \frac{1}{2} \sum_a (\lambda_1 \partial_\alpha n^a_\beta \partial_\beta n^a_\alpha + \lambda_2 \partial_\alpha n^a_\alpha \partial_\beta n^a_\beta + \lambda_3 \partial_\alpha n^a_\gamma \partial_\beta n^a_\gamma) - \sum_{ab} \frac{R(n^a n^b)}{T} \right],
\]

where \(a, b\) are replica indices, \(\alpha, \beta = x, y, z\) label the spatial coordinates, \(\lambda_1 = K_2, \lambda_2 = K_1 - K_2, \lambda_3 = K_3 - K_2, T\) is the temperature, the function \(R(z)\) describes disorder, and the summation over the repeated indices \(\alpha\) and \(\beta\) is assumed. Due to the symmetry \(n^a_\alpha \leftrightarrow -n^a_\alpha\) the function \(R(z)\) is even. Below we measure the temperature in units of \(K_2\), and hence set \(\lambda_1 = 1\). To define the energy in \(4 - \epsilon\) dimensions we add to the Hamiltonian (56) the term \(\lambda_0 \sum_{\alpha\beta} \partial_\alpha n^a_\alpha \partial_\beta n^a_\beta / 2\), where \(\alpha\) labels the coordinates in the \((1 - \epsilon)\)-dimensional subspace, \(\beta = x, y, z\). The stability conditions \(K_1, K_3 > 0\) lead to the inequality

\[
\lambda_2, \lambda_3 > -1.
\]

At each step of the RG procedure which is exactly the same as in section 4 we require that \(\lambda_1 = 1\) is unchanged. We get two additional RG equations in comparison with the RA and RF Heisenberg models. These equations describe the renormalization of the elastic constants \(\lambda_2\) and \(\lambda_3\). The RG equations in the first order in \(\epsilon = 4 - D\) read

\[
\begin{aligned}
\frac{dT}{d\ln L} &= -(D - 2)T + (1 - \lambda_3)C_\phi T; \\
\frac{d\lambda_2}{d\ln L} &= -\lambda_2(1 + \lambda_3)C_\phi; \\
\frac{d\lambda_3}{d\ln L} &= -(3\lambda_3 + \lambda_2^2 - \lambda_2)C_\phi,
\end{aligned}
\]

where the constant

\[
C_\phi = \frac{dR(z = 1)/dz}{8\pi^2 \sqrt{\lambda_0(1 + \lambda_3)}} \left[ 1 + \frac{1}{1 + \lambda_2} \right]
\]

describes the fluctuations of the small-scale fields \(20\)

\[
\langle \phi^2 \rangle = \langle \phi^2_2 \rangle = C_\phi \ln(L/a).
\]

We omit the RG equations for \(\lambda_0\) and \(R(z)\) since their structure is irrelevant below. Eqs. \(58\) have the only fixed point compatible with the stability conditions \(57\). In

\[25\]
this fixed point $T = \lambda_2 = \lambda_3 = 0$ and Eq. (50) reduces to the Hamiltonian of the RA Heisenberg model which thus describes the large-distance physics of the randomly confined nematic. Since that model possesses QLRO in its low-temperature phase for weak disorder, QLRO is also possible in confined nematics. For strong disorder or high temperature the ordering disappears. Thus, there are three phases: the high-temperature isotropic phase and two low-temperature glass phases with and without QLRO. In both glass phases the local orientation of the director is fixed by the random potential. The disorder driven transition between the glass phases is related with topological defects.

The large-scale correlations of the director lead to strong small-angle light scattering. We determine its intensity in the limit of the weak optical anisotropy, i.e. assuming that in the dielectric tensor $\epsilon_{\alpha\beta} = \epsilon_{\perp}\delta_{\alpha\beta} + \epsilon_a n_\alpha n_\beta$ the anisotropic term $\epsilon_a \ll \epsilon_{\perp}$. In this case the scattering cross-section can be found with the Born approximation. The scattering cross-section with the change of the wave vector by $q$ is given by the expression:

$$\sigma(q) = |\omega^2/(4\pi c^2)i_\alpha \epsilon_{\alpha\beta}(q)f_b|^2,$$

where $\omega$ is the light frequency, $i$ and $f$ are the unit vectors specifying the initial and final polarizations, $\epsilon_{\alpha\beta}(q)$ is the Fourier transform of the dielectric tensor. Hence, $\sigma(q) \sim \langle Q_{\alpha\beta}(q)Q_{\alpha\beta}(-q) \rangle$, where $Q_{\alpha\beta} = n_\alpha n_\beta - \delta_{\alpha\beta}/3$ is the order parameter and the angular brackets denote the disorder and thermal averages. In contrast to the bulk nematic the scattering is caused not by the thermal fluctuations but by the frozen configuration of the director. The cross-section $\sigma(q)$ is proportional to the Fourier transform of the correlation function $G(r) = \langle Q_{\alpha\beta}(0)Q_{\alpha\beta}(r) \rangle$. In the QLRO state this correlator obeys a power dependence on the distance $G(r) \sim r^{-\eta}$. To calculate the exponent $\eta$ we decompose $Q_{\alpha\beta}$ into small-scale and large-scale parts with Eq. (23) and average over the small-scale fluctuations with Eq. (56):

$$\langle Q_{\alpha\beta}(0)Q_{\alpha\beta}(r) \rangle_\phi = \{n'_\alpha(0)n'_\beta(0)(1 - \sum_i \langle \phi_i^2 \rangle) + \sum_{ij} e_\alpha^i(0)e_\beta^j(0)(\phi_i \phi_j) - \delta_{\alpha\beta}/3 \} \times \{n'_\alpha(r)n'_\beta(r)(1 - \sum_i \langle \phi_i^2 \rangle) + \sum_{ij} e_\alpha^i(r)e_\beta^j(r)(\phi_i \phi_j) - \delta_{\alpha\beta}/3 \} = Q'_{\alpha\beta}(0)Q'_{\alpha\beta}(r)[1 - 6C_\phi \ln L/a],$$

where $Q'_{\alpha\beta} = n'_\alpha n'_\beta - \delta_{\alpha\beta}/3$. $\langle \ldots \rangle$ denotes the average over the fluctuations of $\phi$, and the relation $\langle \phi_i \phi_j \rangle \sim \delta_{ij}$ which is valid in the RA Heisenberg fixed point is used. The constant $C_\phi = -2R''(0) = 0.309\epsilon$ Eq. (50) is the same as in the fixed point of the RA Heisenberg model (see Eq. (10)). The exponent $\eta$ can be found with the iterative use of Eq. (22) at each RG step until the scale $L = r$ is reached. At the scale $r$ the values of the renormalized director field $n'$ are the same at the points 0 and $r$. Hence, $Q'_{\alpha\beta}(0)Q'_{\alpha\beta}(r) \sim 1$ and $r^{-\eta} \sim (1 - 6C_\phi \ln L/a)^K$, where
\[ K = \ln(r/a) / \ln(L/a) \] is the number of the RG steps. Thus, \( \eta = 6C_0 \). The small-angle scattering cross-section is given by the expression

\[ \sigma(q) \sim q^{-D+\eta} = q^{-4+2.9\epsilon}. \] (63)

The uniaxial stress modifies the large-distance behavior. The compression along the z-axis can be described by adding to the Hamiltonian the term \( F_S = An_z^2 \), where \( A > 0 \), since the deformation tends to make the pore surfaces parallel to the \( xy \) plane and hence favors the planar configuration of the director. The uniaxial stretch is described by \( F_S = An_z^2 \) with a negative \( A \). In both cases \( A \) is proportional to the deformation. The effect of the electric field is analogous to the effect of the stress but the sign of the electric energy \( F_e = -\epsilon_a(nE)^2/8\pi \) is fixed for a given substance. The RG flow is unstable with respect to the perturbation \( F_s \) and new regimes emerge at the scale \( R = R_c \) at which the renormalized \( A(R) \sim 1 \). The critical length \( R_c \) can be found analogously to the correlation length of the RA Heisenberg model in the uniform magnetic field (section 5). At small \( A \) the result is \( R_c \sim |A|^{-1/(2-2C_0)} = |A|^{-0.5-0.15\epsilon} \). The stretched system is long-range-ordered at the scales \( R > R_c \). The nematic order parameter can be calculated analogously to the magnetization of the RA Heisenberg model in the uniform magnetic field and is given by the formula \[ Q = \langle n_\alpha n_\beta - \delta_\alpha\beta/3 \rangle \sim R_c^{-3C_0} \sim |A|^{0.46\epsilon}. \] Long range order can also be achieved by applying an arbitrarily weak external magnetic field to the confined nematic since the magnetic contribution to the energy \( F_m = -\chi_a\langle nH \rangle^2/2 \) has the same structure as the energy related with the uniaxial stretch. A more interesting situation emerges under the compression. The director averaged over a scale \( R > R_c \) is confined in the \( xy \)-plane. The system is thus described by the RA XY model. It possesses QLRO but the critical exponents are different from the exponents of the Heisenberg model. Thus, at the scale \( R_c \) the cross-over from one QLRO state to another occurs. Using the RA XY fixed point found in section 5 and repeating the derivation of Eq. (63) one finds the Born light-scattering cross-section for \( q < 1/R_c \): \[ \sigma(q) \sim q^{-4+(1+\pi^2/9)}. \] In the RA XY regime the cross-section is anisotropic: The small-angle scattering is suppressed, if the incident or scattered light is polarized along the compression direction. The phase diagram in the presence of a uniaxial deformation is shown in Fig. 1.

The only experimental attempt to test the existence of QLRO \( \text{did not provide any decisive evidence. While a power decay of correlation functions was not observed, only distances less than the experimentally determined Larkin length (section 3) were probed.} \] However, as discussed in section 3, behavior at the scales, which are less than the Larkin length, is the same both with and without QLRO.

Besides, the experiment \( \text{was done with nematic confined in aerosil.} \) \( \text{This system cannot be described by the model since disorder is only partially quenched and elasticity-mediated nonlocal interactions can exist.} \) \( \text{The same problem concerns a possible application of our results to nematic elastomers.} \)

One more problem is that no way to determine the disorder strength in experimental systems was suggested. In Ref. \( \text{it was claimed that the effective disorder} \)
strength is less in low-density porous media. This claim however ignores the dependence of the disorder strength on the structure of the porous media. For example, it is evident that no ordering is possible in the case of disconnected pores even if the density of the porous media is arbitrarily low.

Results of numerical experiments are controversial: Ref.\textsuperscript{80} supports QLRO but Ref.\textsuperscript{31} does not. We believe that one should take the existing numerical results with care. In particular, there is a question about finite-size effects in Ref.\textsuperscript{80} while in Ref.\textsuperscript{31} disorder is not weak. Randomness is introduced only in a small fraction of sites, but the random anisotropy on each site is assumed to be infinitely strong. This is equivalent to the application of a still-not-weak random field in all six neighboring sites, and the random fields in these six sites are correlated. The latter also makes effective disorder stronger. Hence, the effective disorder strength in the corresponding continuous model is not small. Thus, more experimental and numerical work is needed.

7. Conclusions

In conclusion, we have demonstrated that the random-field and random-anisotropy $O(N)$ models possess quasi-long range order at low temperatures and weak disorder. In the random-field $O(N)$ model, quasi-long range order is possible at $N = 2$. In the random-anisotropy model, QLRO exists at $N < 10$. These results can be applied
to vortex states of disordered superconductors, amorphous magnets and nematic liquid crystals in random porous media.

There are many other related systems which we did not consider in this paper. As the only examples we mention weakly disordered two-dimensional XY ferromagnets with both exchange and dipole-dipole interactions and charge density waves with long-range Coulomb interactions in two dimensions in the presence of disorder. These systems can be described as super-quasi-long-range-ordered: The correlation functions depend on the distance logarithmically slow similar to the correlation function of the four-dimensional random-anisotropy Heisenberg model.

The results about glass states of weakly disordered systems of continuous symmetry are a part of a wider development. We have good understanding of weakly disordered systems, if disorder breaks only the translational symmetry. Random fields and random anisotropy also break the symmetry with respect to transformations of the order parameter. In this case much has still to be investigated. Even the simplest problem, random-field Ising model, is far from being completely understood. Recent results make it hopeful that a satisfactory theory of phase transitions in the random-field Ising model will be developed soon. Such theory should include new insights in comparison with the existing theory of phase transitions in homogeneous systems and may open important new perspectives in statistical mechanics in the presence of disorder.

Acknowledgments

Useful discussions with T. Bellini, E. Domany, Y. Gefen, S.E. Korshunov, V.V. Lebedev, V.P. Mineev, T. Nattermann, V.L. Pokrovsky, M. Schwartz, B. Spivak and V. Steinberg are gratefully acknowledged. I thank R. Whitney for the critical reading of the manuscript. This work was supported by the Koshland fellowship and RFBR grant 00-02-17763.

Appendix A

In this appendix we derive an inequality for the correlation functions of the disordered systems. We consider a system with the Hamiltonian

$$H = \int dx^D [H_1(\phi(x)) - h(x)m(\phi(x))]$$

(A.1)

where \(\phi\) is the order parameter, \(h\) the random field with short range correlations, \(H_1\) may depend on some other random fields. We prove the inequality for the Fourier components of the field \(m\):

$$G_{\text{con}}(q) \leq \text{const} \sqrt{G_{\text{dis}}(q)}$$

(A.2)

where

$$G_{\text{dis}}(q) = \langle m(q)m(-q) \rangle, G_{\text{con}}(q) = \langle m(q)m(-q) \rangle - \langle m(q)\rangle \langle m(-q) \rangle,$$

the angular brackets denote the thermal averaging, the bar denotes the disorder averaging.
Let $P(h)$ be the distribution function of the field $h$. Then

$$G_{con}(q) = \int \left( P(h) \frac{d}{dh(q)} m_q(h) \right) D\{h\} =$$

$$- \int P(h) \left( \frac{d\ln P(h)}{dh(q)} m_q(h) \right) D\{h\}, \quad (A.3)$$

where $\int D\{h\}$ denotes the integration over the realizations of the random field, $m_q(h) = \int D\{\phi\} \exp(-H/T)m(q) / \int D\{\phi\} \exp(-H/T)$. Applying the Cauchy-Bunyakovsky inequality to Eq. (A.3) one gets Eq. (A.2) where the const $= \max_q \sqrt{\int D\{h\} |dP(h)/dh(q)|^2/P(h)}$.

For systems in the critical domain there is a simple way to understand why the inequality is valid not only in the Gaussian case but also in a general situation. This is just a consequence of the universality.

References

[1] A.I. Larkin, Sov. Phys. JETP 31, 784 (1970) [Zh. Eksp. Teor. Fiz. 58, 1466 (1970)].
[2] Y. Imry and S.K. Ma, Phys. Rev. Lett. 35, 1399 (1975).
[3] T. Giamarchi and P. Le Doussal, in Spin Glasses and Random Fields, edited by A.P. Young (World Scientific, Singapore, 1998) p. 321; e-print cond-mat/9708034.
[4] T. Nattermann and S. Scheidl, Adv. Phys. 49, 607 (2000).
[5] M. Mezard, G. Parisi, and M. Virasoro, Spin Glass Theory and Beyond (World Scientific, Singapore, 1987).
[6] S.L. Ginzburg, Zh. Eksp. Teor. Fiz. 80, 244 (1981).
[7] Y.Y. Goldshmidt, Nucl. Phys. B225, 123 (1983).
[8] D. Boyanovsky, Nucl. Phys. B225, 523 (1983).
[9] V.L. Berezinskii, Sov. Phys. JETP 32, 493 (1970) [Zh. Eksp. Teor. Fiz. 59, 907 (1970)].
[10] T. Nattermann, Phys. Rev. Lett. 64, 2454 (1990).
[11] S.E. Korshunov, Phys. Rev. B48, 3969 (1993).
[12] T. Giamarchi and P. Le Doussal, Phys. Rev. Lett. 72, 1530 (1994).
[13] T. Giamarchi and P. Le Doussal, Phys. Rev. B52, 1242 (1995).
[14] D.E. Feldman, JETP Lett. 70, 135 (1999).
[15] D.E. Feldman, Phys. Rev. B61, 382 (2000).
[16] D.E. Feldman, Phys. Rev. Lett. 84, 4886 (2000).
[17] D.E. Feldman, Phys. Rev. B62, 5364 (2000).
[18] N. Avraham et al., Nature 411, 451 (2001).
[19] V. Westphal, W. Kleemann, and M.D. Glinchuk, Phys. Rev. Lett. 68, 847 (1992).
[20] D.S. Fisher, Phys. Rev. B31, 7233 (1985).
[21] R. Harris, M. Plischke, and M.J. Zuckermann, Phys. Rev. Lett. 31, 160 (1973).
[22] D.J. Sellmyer and M.J. O’Shea, in Recent Progress in Random Magnetism, ed. D. Ryan (World Scientific, Singapore, 1992) p. 71.
[23] J. Fricke, T. Tillotson, Thin Solid Films, 297, 212 (1997).
[24] J.V. Porto III and J.M. Parpia, Phys. Rev. Lett. 74, 4667 (1995).
[25] X-I. Wu, W.I. Goldburg, and M.X. Liu, Phys. Rev. Lett. 69, 470 (1992).
[26] A. Martelj and M. Copic, Phys. Rev. E55, 504 (1997).
[27] T. Bellini et al., Phys. Rev. Lett. 69, 788 (1992).
[28] Z. Kutnjak and C.W. Garland, Phys. Rev. E55, 488 (1997).
[29] L. Wu, B. Zhou, C.W. Garland, T. Bellini, and D.W. Shafer, *Phys. Rev.* E51, 2157 (1995).
[30] H. Zeng, B. Zalar, G.S. Iannacchione, and D. Finotello, *Phys. Rev.* E60, 5607 (1999).
[31] T. Bellini et al., *Phys. Rev. Lett.* 85, 1008 (2000).
[32] For a review see T. Nattermann, in *Spin Glasses and Random Fields*, edited by A.P. Young (World Scientific, Singapore, 1998), p. 277; e-print cond-mat/9705295.
[33] P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Claredon Press, Oxford, 1993).
[34] Other types of anchoring do not modify the results of the paper.
[35] T. Nattermann, in *Spin Glasses and Random Fields*, edited by A.P. Young (World Scientific, Singapore, 1998), p. 277; e-print cond-mat/9705295.
[36] P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Claredon Press, Oxford, 1993).
[72] R. Fisch, *Phys. Rev.* **B58**, 5684 (1998).
[73] R. Fisch, *Phys. Rev.* **B57**, 269 (1998).
[74] Note that Fig. 1 in Ref. [16] contains a typo.
[75] T. Bellini, private communication.
[76] M. Kreuzer and R. Eidenschink, in *Liquid Crystals in Complex Geometries*, edited by G.P. Grawford and S. Zumer (Taylor & Francis, London, 1996).
[77] S.V. Fridrikh and E.M. Terentjev, *Phys. Rev. Lett.* **79**, 4661 (1997).
[78] N. Uchida and A. Onuki, *Europhys. Lett.* **45**, 341 (1999).
[79] L. Radzihovsky and J. Toner, *Phys. Rev.* **B60**, 206 (1999).
[80] J. Chakrabaty, *Phys. Rev. Lett.* **81**, 385 (1998).
[81] D.E. Feldman, *JETP Lett.* **65**, 114 (1997).
[82] D.E. Feldman, *Phys. Rev.* **B56**, 3167 (1997).
[83] R. Chitra, T. Giamarchi, and P. Le Doussal, *Phys. Rev.* **B59**, 4058 (1999).
[84] M. Mezard and A.P. Young, *Europhys. Lett.* **18**, 653 (1992).
[85] M. Gofman *et al.*, *Phys. Rev.* **B53**, 6362 (1996).