More Examples of Non-Rational Adjoint Groups

Nivedita Bhaskhar
Department of Mathematics & Computer Science, Emory University, Atlanta, GA 30322, USA.
nbhaskh@emory.edu

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Abstract

In this paper, we give a recursive construction to produce examples of quadratic forms \( q_n \) in the \( n^{th} \) power of the fundamental ideal in the Witt ring whose corresponding adjoint groups \( \text{PSO}(q_n) \) are not stably rational. Computations of the \( \mathcal{R} \)-equivalence classes of adjoint classical groups by Merkurjev are used to show that these groups are not \( \mathcal{R} \)-trivial. This extends earlier results of Merkurjev and Gille where the forms considered have non-trivial and trivial discriminants respectively.

1 Introduction

Rationality of varieties of connected linear algebraic groups has been a topic of great interest. It is well known that in characteristic 0 every such group variety is rational over the algebraic closure of its field of definition. However, rationality over the field of definition itself is a more delicate question and examples of Chevalley and Serre of non-rational tori and semisimple groups respectively indicate its subtle nature. Platonov’s famous example of non-rational groups of the form \( \text{SL}_1(D) \) settled negatively the long standing question of whether simply-connected almost simple \( k \) groups were rational over \( k \), shifting the focus to adjoint groups.

Platonov himself conjectured ([7], pg 426) that adjoint simple algebraic \( k \)-groups were rational over any infinite field. Some evidence of the veracity of this conjecture is found in [2] where Chernousov establishes that \( \text{PSO}(q) \) is a stably rational \( k \) variety for the special quadratic form \( q = \langle 1, 1, \ldots, 1 \rangle \) where \( k \) is any infinite field of characteristic not 2. Note that the signed discriminant of the quadratic form in question is \( \pm 1 \).
However Merkurjev in [6] constructs a quadratic form $q$ of dimension 6 with non-trivial signed discriminant over a base field $k$ of characteristic not 2 and cohomological dimension 2 such that $\text{PSO}(q)$ is not a $k$-stably rational variety. This example is obtained as a consequence of his computations of $R$-equivalence classes of adjoint classical groups which relates to stable-rationality via the following elementary result:

*If $X$ is a $k$-stably rational variety, then $X(K)/R$ is trivial for any extension $K/k$."

In fact, Merkurjev shows that if $q$ is a quadratic form of dimension $\leq 6$, then $\text{PSO}(q)(K)/R$ is not trivial for some extension $K/k$ if and only if $q$ is a *virtual Albert form*. Bruno Kahn and Sujatha give a cohomological description of $\text{PSO}(q)(k)/R$ ([4], Thm 4) for any virtual Albert form $q$ over fields of characteristic 0.

It is to be noted that Merkurjev’s example uses the non-triviality of the signed discriminant of the quadratic form in a crucial way.

Let $W(k)$ denote the Witt ring of quadratic forms defined over $k$ and $I(k)$ denote the fundamental ideal of even dimensional forms. One can ask if there are examples of quadratic forms $q_n$ defined over fields $k_n$ satisfying the following two properties:

1. $q_n \in I^n(k_n)$, the $n$-th power of the fundamental ideal.
2. $\text{PSO}(q_n)$ is not $k_n$-stably rational.

Gille answers the question for $n = 2$ very precisely in [3] and produces a quadratic form of dimension 8 with trivial discriminant over a field of cohomological dimension 3. He also shows that the dimension of the quadratic form has to be at least 8 and that the base field should have cohomological dimension at least 2.

This paper produces pairs $(k_n, q_n)$ for every $n$ in a recursive fashion such that $\text{PSO}(q_n)(k_n)/R \neq \{1\}$. This implies that $\text{PSO}(q_n)$ is not stably rational.

## 2 Notations and Conventions

All fields considered are assumed to have characteristic 0. Let $W(k)$ denote the Witt ring of quadratic forms defined over $k$ and $I(k)$, the fundamental ideal of even dimensional forms. $P_n(k)$ is the set of isomorphism classes of anisotropic $n$-fold Pfister forms and $I^n(k)$ denotes the $n$-th power of the fundamental ideal. Let us fix the convention that $\langle\langle a \rangle\rangle$ denotes the 1-fold Pfister form $\langle 1, a \rangle$. A generalized Pfister form is any scalar multiple of a Pfister form.
3 A formula for R-equivalence classes

Let $G$ be a connected linear algebraic group over $k$. The following relation defined on $G(k)$ is an equivalence relation, called the R-equivalence relation.

$$g_0 \sim g_1 \iff \exists g(t) \in G(k(t)), \ g(0) = g_0, \ g(1) = g_1$$

The induced equivalence classes are called R-equivalence classes of $G(k)$.

An algebraic variety $X$ over $k$ is said to be $k$-stably rational if there exist two affine spaces $\mathbb{A}^n_k, \mathbb{A}^m_k$ and a birational map defined over $k$ between $\mathbb{A}^n_k \times_k X$ and $\mathbb{A}^m_k$.

A $k$ algebraic group $G$ is said to be R-trivial if $G(K)/R = \{1\}$ for every extension $K$ of $k$.

Recall the fact that stably $k$ rational varieties are R-trivial. The following formula, a special case of Merkurjev’s computations of R-equivalence classes of classical adjoint groups, is a key ingredient for our construction of nonrational adjoint groups.

**Theorem 3.1** (Merkurjev, [6], Thm 1). $\text{PSO}(q)(k)/R \cong G(q)/\text{Hyp}(q)k^\times 2$ where

1. $q$ is an even dimensional (say of dimension $2b$) non-degenerate quadratic form over $k$ of characteristic $\neq 2$.
2. $G(q) = \{a \in k^\times | aq \cong q\}$, the group of similarities.
3. $\text{Hyp}(q) = \langle N_{l/k}(l^\times) | [l : k] < \infty, q_l \cong \mathbb{H}^l \rangle$.
4. $k^\times 2 := \{a^2 | a \in k^\times\}$.

Thus to check that $\text{PSO}(q_n)$ is not $k_n$-stably rational, it is enough to check that $\text{PSO}(q_n)(k_n)/R \neq \{1\}$ using the above formula.

4 Lemmata

This section collects a list of lemmas which come in handy whilst constructing nonrational adjoint groups.

**Lemma 4.1** (Odd extensions). Let $q$ be a quadratic form over $k$. Let $k'/k$ be an odd degree extension. Then,

$$\text{PSO}(q_{k'})(k')/R = \{1\} \implies \text{PSO}(q)(k)/R = \{1\}.$$
Proof. Suppose that $x \in G(q)$. Clearly $G(q) \subseteq G(q_{k'}) = \text{Hyp}(q_{k'}) k'^{\times 2}$ as $\text{PSO}(q_{k'}) (k')/R = \{1\}$.

The definition of Hyp groups and the transitivity of norms immediately yield the fact that $N_{k'/k} (\text{Hyp}(q_{k'}) k'^{\times 2}) \subseteq \text{Hyp}(q) k^{\times 2}$.

If $2n + 1$ is the degree of $k'$ over $k$, it follows that $x^{2n+1} = N_{k'/k}(x) \in \text{Hyp}(q) k^{\times 2}$. Hence $x \in \text{Hyp}(q) k^{\times 2}$.

Let $p$ be a Pfister form over $k$. Its pure-subform $	ilde{p}$ is defined uniquely up to isometry via the property that $	ilde{p} \perp \langle 1 \rangle \cong p$. The following useful result connects the values of pure-subforms and Pfister forms:

**Lemma 4.2** ([9], Chap 4, Thm 1.4). If $D(q)$ denotes the set of non-zero values represented by the quadratic form $q$, then, for $p \in P_n(k)$,

$$b \in D(\tilde{p}) \iff p \cong \langle \langle b, b_2, \ldots, b_n \rangle \rangle,$$

for some $b_2, \ldots, b_n \in k^{\times}$.

The next lemma is a useful tool for converting an element which is a norm from two different quadratic extensions into a norm from a biquadratic extension of the base field up to squares.

**Lemma 4.3** (Biquadratic-norm trick, [5], Lemma 1.4). If $l_1$ and $l_2$ are two quadratic extensions of a field $l$, then

$$N_{l_1/l} (l_1^{\times}) \cap N_{l_2/l} (l_2^{\times}) = N_{l_1 \otimes l_2/l} (l_1 \otimes l_2^{\times}) / l^{\times 2}.$$  

**Lemma 4.4** (Folklore). Let $k(u)$ be a finite separable extension of $k$ generated by $u$ of degree $p^gh$ where $p$ is a prime not dividing $h$ and $g \geq 1$. Then there exist finite separable extensions $M_1/M_2/k$ such that the following conditions hold:

1. $k(u) \subset M_1$ and $M_1 = M_2(u)$.
2. $[M_1 : M_2] = p$ and $p \nmid [M_1 : k(u)]$

Proof. Let $M/k$ be any finite Galois extension containing $k(u)$ and let $S$ be any $p$-Sylow of $\text{Gal}(M/k(u))$. Since $\text{Gal}(M/k(u))$ is a subgroup of $\text{Gal}(M/k)$, there is a $p$-Sylow subgroup $T$ of $\text{Gal}(M/k)$ containing $S$.  

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Let $M^S$ and $M^T$ denote the fixed fields of $S$ and $T$ in $M$ respectively. Note that $M^S \supseteq M^T$ and since $S$ and $T$ are appropriate $p$-Sylow subgroups, we have $p \nmid [M^S : k(u)]$ and $p \nmid [M^T : k]$. Comparing degrees yields $[M^S : M^T] = p^g$. Also note that $u \notin M^T$ and $k(u) \subset M^S$.

In fact, $M^T(u) = M^S$ because $[M^S : M^T]$ and $[M^S : k(u)]$ are coprime.

$S$ is a proper subgroup of its normalizer $N_T(S)$ because $T$ is nilpotent and $S \neq T$. Thus, you can find a subgroup $V$ such that $S \subseteq V \subseteq T$ and index of $S$ in $V$ is $p$. Set $M_2$ to be the fixed field of $V$ in $M$.

Thus $M_1 = M_2(u)$ is of degree $p$ over $M_2$ and satisfies the other conditions given in the Lemma.

\begin{center}
\begin{tikzpicture}
  \node (M) at (0,0) {$M$};
  \node (MS) at (0,-1) {$M^S = M_1$};
  \node (k) at (0,-2) {$k(u)$};
  \node (py) at (0,-3) {$M^V = M_2$};
  \node (M2) at (0,-4) {$M^T$};

  \draw[->] (M) to (MS);
  \draw[->] (k) to (MS);
  \draw[->] (MS) to (py);
  \draw[->] (M2) to (MS);
  \draw[->] (k) to (py);
  \draw[->] (py) to (M2);

  \draw[->] (k) to (M2);

  \node at (0,-5) {$p \nmid [M^S : k(u)]$};
  \node at (0,-3) {$p$};
  \node at (0,-4) {$p \nmid [M^T : k]$};

\end{tikzpicture}
\end{center}

The following lemma tells us that Pfister forms yield R-trivial varieties. Note that in fact more is true, namely that $\text{PSO}(q)$ is stably-rational for any generalized Pfister form $q$ [6, Prop 7).

\textbf{Lemma 4.5.} If $q$ is an $n$-fold Pfister form over a field $k$, then

$$\text{PSO}(q)(k)/R = \{1\}.$$ 

\textit{Proof.} If $q$ is isotropic, it is hyperbolic and hence $G(q) = \text{Hyp}(q) = k^\times$. Therefore assume without loss of generality that $q$ is anisotropic.

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Case \( n = 1 \): Let \( q = \langle 1, -a \rangle \). Then \( G(q) = N_{k(\sqrt{a})/k}(k(\sqrt{a})^\times) \). Further, \( q \) splits over a finite field extension \( L \) of \( k \) if and only if \( a \) is a square in \( L \). Therefore \( q_L \) splits if and only if \( L \supseteq k(\sqrt{a}) \supseteq k \) and hence clearly \( \text{Hyp}(q) = G(q) \).

General case: Recall that Pfister forms are round, that is \( D(q) = G(q) \) for any Pfister form \( q \). Let \( \tilde{q} \) be the pure-subform of \( q \). If \( b \in D(\tilde{q}) \subseteq D(q) \), then by Lemma 4.2

\[
 b \in D(\langle 1, b \rangle) = \text{Hyp}(\langle 1, b \rangle) k^\times \subseteq \text{Hyp}(q) k^\times. 
\]

Note that any \( x \in G(q) = D(q) \) can be written (upto squares from \( k^\times \)) as either \( b \) or \( 1 + b \) for some \( b \in D(\tilde{q}) \). Since \( x = b \in D(\tilde{q}) \) has just been taken care of, it is enough to note that for \( b \in D(\tilde{q}) \),

\[
 1 + b \in D(\langle 1, b \rangle) \subseteq \text{Hyp}(q) k^\times. 
\]

\[
 \square
\]

5 Comparison of some Hyp groups

Let \( q \) be an anisotropic quadratic form over a field \( k \) of characteristic 0. Let \( p \) be an anisotropic Pfister form defined over \( k \) and let \( Q = q \perp tp \) over the field of Laurent series \( K = k((t)) \). Note that \( K \) is a complete discrete valued field with uniformizing parameter \( t \) and residue field \( k \). Recall the exact sequence in Witt groups:

\[
 0 \rightarrow W(k) \xrightarrow{\text{Res}} W(K) \xrightarrow{\delta_{2,t}} W(k) \rightarrow 0
\]

where \( \text{Res} \) is the restriction map and \( \delta_{2,t} \) denotes the second residue homomorphism with respect to the parameter \( t \).

Remark 5.1. \( Q \) is anisotropic and \( \dim(Q) > \dim(p) \).

This can be shown with the aid of the above exact sequence. Let the anisotropic part of \( Q \) be \( Q_{an} \cong q_1 \perp tq_2 \) for quadratic forms \( q_i \) defined over \( k \). Then each \( q_i \) is anisotropic. The following equality in \( W(k) \) is in fact an isometry because the forms are anisotropic:

\[
 \delta_{2,t}(Q) = p = q_2.
\]
This immediately implies \( q \cong q_1 \). The inequality between dimensions of \( Q \) and \( p \) follow immediately.

**Proposition 5.2.** \( \text{Hyp}(Q)K^{\times 2} \subseteq \text{Hyp}(q_K)K^{\times 2} \) if \( \text{PSO}(q)(k)/R \neq \{1\} \).

**Proof.** Let \( L/K \) be a finite field extension which splits \( Q \). There is a unique extension of the discrete valuation on \( K \) to \( L \) which makes \( L \) into a complete discrete valued field. Let \( l \) denote the residue field of \( L \). Since the characteristic of \( k \) is 0, \( k \subseteq K \) and \( l \subseteq L \). Let \( K_{nr} \) denote the maximal non-ramified extension of \( K \) in \( L \) and \( \pi \) be a uniformizing parameter of \( L \). Let \( f = [l : k] \), the degree of the residue field extensions and \( e \) be the ramification index of \( L/K \). Let \( v_X \) denote the corresponding valuation on fields \( X = K, K_{nr}, L \) and \( O_X \), the corresponding discrete valuation rings.

Since \( L/K_{nr} \) is totally ramified, the minimal polynomial of \( \pi \) (which is also its characteristic polynomial) over \( K_{nr} \) is an \textit{Eisenstein} polynomial \( x^e + a_{e-1}x^{e-1} + \ldots + a_1x + a_0 \) in \( K_{nr}[x] \), where \( v_{K_{nr}}(a_0) = 1 \) and \( v_{K_{nr}}(a_i) \geq 1 \forall 1 \leq i \leq e - 1 \) ([1], Chap 1, Sec 6, Thm 1). Note that \( N_{L/K_{nr}}(\pi) = (-1)^e a_0 \).

\( K_{nr} = l((t)) \) and \( L = l((\pi)) \) ([8], Chap 2, Thm 2). Let \( a_0 = -ut(1 + u_1t + \ldots) \) in \( O_{K_{nr}} = l[[t]] \). By Hensel’s lemma, \( 1 + u_1t + \ldots = w^2 \) for some \( w \) in \( K_{nr} \). The relation given by the Eisenstein polynomial can be rewritten by applying Hensel’s lemma again as follows:

\[
\pi^e = utv^2, \ u \in l^\times, \ v \in L^\times.
\]

Hence the norm of \( \pi \) can be computed up to squares. That is,

\[ N_{L/K}(\pi) = N_{K_{nr}/K}((-1)^e a_0) \in (-1)^e (-t)^e N_{l/k}(u)K^{\times 2}. \]

The problem is subdivided into two cases depending on the parity of the ramification index \( e \) of \( L/K \).

**Case I :** \( e \) is odd

We show that \( L \) also splits \( q \) in this case. Let \( \delta_{2,\pi} : W(L) \rightarrow W(l) \) be the second Milnor residue map with respect to the uniformizing parameter \( \pi \) chosen above. Note that \( Q_L = q + \pi up \) in \( W(L) \). Then
\[ Q_L = 0 \implies \delta_{2,\pi}(Q) = 0 \in W(l) \]
\[ \implies u_p = 0 \in W(L) \]
\[ \implies q = 0 \in W(L) \]

**Case II :** \( e \) is even

Now \( Q_L = q + u_p = 0 \) in \( W(L) \). Since any element of \( L^\times \) is of the form \( \alpha \pi b^2 \) or \( \alpha b^2 \) for some \( \alpha \in l^\times \) and \( b \in L \), the norm computation of \( \pi \) done before yields the following:

\[ N_{L/K}(L^\times) \subseteq \langle N_{l/k}(u)(-t)^f \rangle K^{\times 2}. \]

So it is enough to show that \( f \) is even and \( N_{l/k}(u) \) is in \( \text{Hyp}(q) k^{\times 2} \).

**Claim :** \( f \) is even.

If \( f \) is odd, then \( \text{PSO}(q_l)(l)/R \neq \{1\} \) by Lemma 4.1. But \( q_l = -u_p \) is a form similar to a Pfister form. Hence by Lemma 4.5, \( \text{PSO}(q_l)(l)/R = \{1\} \) which is a contradiction.

**Claim :** \( N_{l/k}(u) \in \text{Hyp}(q) k^{\times 2} \)

Look at \( l \supseteq k(u) \supseteq k \). If \([l : k(u)]\) is even, then \( N_{l/k}(u) = N_{k(u)/k}(u[k(u)]) \in k^{\times 2} \) which proves the claim.

Otherwise \( r : W(k(u)) \to W(l) \) is injective and hence \( q + u_p = 0 \) in \( W(k(u)) \). It remains to show that \( N_{k(u)/k}(u) \in \text{Hyp}(q) k^{\times 2} \).

Suppose that \([k(u) : k]\) is odd. Then Lemma 4.1 implies that \( \text{PSO}(q_{k(u)})/R \neq \{1\} \). On the other hand, \( q_{k(u)} \) is similar to Pfister form \( p_{k(u)} \). This contradicts Lemma 4.5. Therefore \([k(u) : k]\) is even.

Let \([k(u) : k] = 2^g h \) where \( h \) is odd and \( g \geq 1 \). Lemma 4.4 gives us a quadratic extension \( M_1 = M_2(u) \) over \( M_2 \) such that \( M_1 \) is an odd extension of \( k(u) \).

Since \([M_1 : k(u)]\) is odd, there is a \( w \in k^\times \) such that

\[ N_{M_1/k}(u) = N_{k(u)/k}(N_{M_1/k(u)}(u)) = N_{k(u)/k}(u)w^2. \]

Hence it suffices to show that \( N_{M_1/k}(u) \in \text{Hyp}(q) k^{\times 2} \). Using transitivity of norms and the definition of \( \text{Hyp} \) groups, showing \( N_{M_1/M_2}(u) \in \text{Hyp}(q_{M_2}) M_2^{\times 2} \) proves the claim.
Let $\eta := N_{M_1/M_2}(u)$. By using Scharlau’s transfer and Frobenius reciprocity ([9], Chap 2, Lemma 5.8 and Thm 5.6),

$$p \otimes \langle 1 \rangle = -\langle u \rangle q \in W(M_1) \implies p \otimes \langle 1, -\eta \rangle = 0 \in W(M_2).$$

Hence $\eta \in G(p_{M_2}) = D(p_{M_2})$ since $p$ is a Pfister form.

Let $s$ be the pure subform associated with $p_{M_2}$. We can assume (upto squares from $M_2$) that $\eta = b$ or $1 + b$ for some $b \in D(s)$. In either case, $\eta \in N_{M_2(\sqrt{-b})/M_2}\left((M_2(\sqrt{-b}))^\times\right)$. By Lemma 4.2, $p = \langle\langle b, \ldots \rangle\rangle$. Note that if $-b$ is already a square in $M_2$, then the above reasoning shows that $q$ splits over $M_1$ which shows that $\eta \in \text{Hyp}(q_{M_2})M_2^\times$. If $-b$ is not a square, then $p$ splits in $M_2(\sqrt{-b})$ and hence $q = -up$ splits in $M_1(\sqrt{-b})$.

The introduction of subfield $M_2$ is useful because the biquadratic norm trick can be used!

More precisely, since

$$\eta \in N_{M_2(\sqrt{-b})/M_2}\left((M_2(\sqrt{-b}))^\times\right) \cap N_{M_1/M_2}(M_1^\times),$$

Lemma 4.3 shows that $\eta$ is upto squares a norm from $M_1(\sqrt{-b})$ and $M_1(\sqrt{-b})$ splits $q$. Thus $\eta \in \text{Hyp}(q_{M_2})M_2^\times$ as claimed. \qed

**Proposition 5.3.** $\text{Hyp}(q_K)K^\times \subseteq \text{Hyp}(q)K^\times$.

**Proof.** Using the exact sequence associated to the second Milnor residue map again, it is clear that if $q$ is split by a finite field extension $L$ of $K$, then it is also split by $l$, the residue field of $L$. Thus $\text{Hyp}(q_K)$ is generated by $N_{L/K}(L^\times)$ where $L$ runs over finite unramified extensions of $K$ which split $q$. By Springer’s theorem, $[l : k]$ has to be even. And characteristic of $k = 0$ implies that $L \cong l((t))$. To conclude, it is enough to observe that

$$N_{l((t))/k((t))}(l((t)))^\times \subseteq N_{l/k}(l^\times)K^\times.$$ \qed
6 A recursive procedure

All fields have characteristic 0. We say a quadruple \((n, \lambda, L, \phi)\) has property \(\star\) if the following holds:

\(\phi\) is an anisotropic quadratic form over \(L\) in \(I^n(L)\) such that the scalar \(\lambda\) is in \(G(\phi)\) but not in \(\text{Hyp}(\phi)L^\times\) and there exists a decomposition of \(\phi\) into a sum of generalized \(n\)-fold Pfister forms in the Witt ring \(W(L)\), each of which is annihilated by \(\langle 1, -\lambda \rangle\). More precisely, in \(W(L)\),

\[
\phi = \sum_{i=1}^{m} \alpha_i p_{i,n}, \text{ where } \alpha_i \in L^\times, p_{i,n} \in P_n(L)
\]

\[\langle 1, -\lambda \rangle \otimes p_{i,n} = 0 \forall i.\]

Assume that \((n, \lambda, k_n, q_n)\) has property \(\star\) with \(q_n = \sum_{i=1}^{m} \alpha_i p_{i,n}\) for \(p_{i,n} \in P_n(k_n)\) and \(\alpha_i \in k_n^\times\) such that each \(p_{i,n}\) is annihilated by \(\langle 1, -\lambda \rangle\). Let \(K_0\) denote the field \(k_n\). Define the fields \(K_i\) recursively as follows:

\[K_i := K_{i-1}((t_i)) \forall 1 \leq i \leq m\]

Let \(Q_0\) denote the quadratic form \(q_n\) defined over \(K_0\). Define the quadratic forms \(Q_i\) over fields \(K_i\) recursively as follows:

\[Q_i := Q_{i-1} \perp t_i p_{i,n} \forall 1 \leq i \leq m\]

Note that \(\lambda \in G(Q_i)\) for each \(1 \leq i \leq m\) since \(\lambda \in G(q_n)\) and \(G(p_{i,n})\) for each \(i\).

**Theorem 6.1.** Let \((n, \lambda, k_n, q_n)\) has property \(\star\). Then for \((K_m, Q_m)\) as above, the following hold:

1. \(Q_m \in I^{n+1}(K_m)\)

2. \(\lambda \in G(Q_m) \setminus \text{Hyp}(Q_m) K_m^\times\). In particular, \(\text{PSO}(Q_m)\) is not \(K_m\)-stably rational.

3. \((n + 1, \lambda, K_m, Q_m)\) has property \(\star\).

**Proof.** In the Witt ring \(W(K_m)\),

\[
Q_m = q_n + \sum_{i=1}^{m} t_ip_{i,n} = \sum_{i=1}^{m} \alpha_i p_{i,n} + t_i p_{i,n} = \sum_{i=1}^{m} p_{i,n} \otimes \langle \alpha_i, t_i \rangle \in I^n I \subseteq I^{n+1}(K_m) \quad (1)
\]
We now prove by induction that $\lambda \in G (Q_i) \setminus \text{Hyp} (Q_i) K_i^{x^2}$ for each $i \leq m$.

The base case $i = 0$ is given, namely the pair $(k_n, q_n)$. Assume as induction hypothesis that this statement holds for all $i \leq j$. The proof of the statement for $i = j + 1$ follows:

The following notations are introduced for convenience.

\[
(Q, K) := (Q_{j+1}, K_{j+1}) \\
(q, k) := (Q_j, K_j) \\
t := t_{j+1} \\
p := p_{j+1, n} \in P_n (k)
\]

Thus $Q = q + tp \in W (K)$.

Since $\lambda \in k^\times$ and not in $\text{Hyp}(q)^{x^2}$, it is not in $\text{Hyp}(q) K^{x^2}$. By Proposition 5.3, $\lambda \not\in \text{Hyp}(q_K) K^{x^2}$ and by Proposition 5.2, $\lambda \not\in \text{Hyp}(Q) K^{x^2}$. By construction, $\lambda \in G (Q)$ as $\lambda \in G (p) \cap G (q)$. Hence $\lambda \in G (Q) \setminus \text{Hyp}(Q) K^{x^2}$.

It is clear that $(n + 1, \lambda, K_m, Q_m)$ has property $\star$ by Equation (1).

\[\square\]

7 Conclusion

**Theorem 7.1.** For each $n$, there exists a quadratic form $q_n$ defined over a field $k_n$ such that $\text{PSO} (q_n)$ is not $k_n$-stably rational.

**Proof.** Let $q$ be an anisotropic quadratic form of dimension 6 over a field $F$ of characteristic 0. If the discriminant of $q$ is not trivial and $C_0(q)$ is a division algebra, then there exists a field extension $E$ of $F$ such that $\text{PSO}(q)(E) / R \neq \{1\}$ ([6], Thm 3).

Define $k_1 := E$, $q_1 := q_E$ and pick a $\lambda \in G (q_1) \setminus \text{Hyp}(q_1) k_1^{x^2}$.

We can write $q_1 = \sum_{i=1}^t \alpha_i f_i$ in the Witt ring $W (k_1)$ for some scalars $\alpha_i \in k_1^\times$ and 1-fold Pfister forms $f_i$ which are annihilated by $\langle 1, -\lambda \rangle$ ([9], Chap 2, Thm 10.13).

Therefore Theorem 6.1 can be applied repeatedly to produce pairs $(k_n, q_n)$ such that $\text{PSO} (q_n) (k_n) / R \neq \{1\}$.

This implies that $\text{PSO} (q_n)$ is not $k_n$-stably rational. \[\square\]

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