ON SOME CLASSICAL PROBLEMS CONCERNING \(L_{\infty}\)-EXTREMAL POLYNOMIALS WITH CONSTRAINTS

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Abstract. First we consider the following problem which dates back to Chebyshev, Zolotarev and Achieser: among all trigonometric polynomials with given leading coefficients \(a_0, \ldots, a_l, b_0, \ldots, b_l \in \mathbb{R}\) find that one with least maximum norm on \([0, 2\pi]\). We show that the minimal polynomial is on \([0, 2\pi]\) asymptotically equal to a Blaschke product times a constant where the constant is the greatest singular value of the Hankel matrix associated with the \(\tau_j = a_j + ib_j\). As a special case corresponding statements for algebraic polynomials follow. Finally the minimal norm of certain linear functionals on the space of trigonometric polynomials is determined. As a consequence a conjecture by Clenshaw from the sixties on the behavior of the ratio of the truncated Fourier series and the minimum deviation is proved.

1. Introduction

In 1858 Chebyshev discovered that the polynomial

\[
2^{-n+1}T_n(x) = 2^{-n+1}\cos n \arccos x
\]

deviates least from zero with respect to the maximum norm on \([-1, 1]\) among all polynomials with leading coefficient one. Then he posed the following problem to his circle: Let \(l\) real numbers \(A_0, A_1, \ldots, A_l\) be given. Among all polynomials of degree less or equal \(n\) with leading coefficients \(A_0, A_1, \ldots, A_l\), i.e., of the form \(\sum_{j=0}^{l} A_j x^{n-j} + q(x)\), \(q \in \mathbb{P}_{n-l-1}\), find that one which has least max-norm on \([-1, 1]\), that is, find the unique polynomial \(\tilde{q} \in \mathbb{P}_{n-l-1}\) such that

\[
\min_{q \in \mathbb{P}_{n-l-1}} \left\| \sum_{j=0}^{l} A_j x^{n-j} + q(x) \right\| = \left\| \sum_{j=0}^{l} A_j x^{n-j} + \tilde{q}(x) \right\|
\]

where \(\|f\| = \max_{x \in [-1,1]} |f(x)|\). This was the begin of a long story. Indeed, ten years later Zolotarev, a student of Chebyshev, determined the minimal polynomial in terms of elliptic functions when the first and second
coefficient is given. In 1930 Achieser gave a description of the minimal polynomial in terms of automorphic functions when three leading coefficients are given. In the words of Bernstein [16, p. 156] “Akhieser treated the more difficult problem and arrived at three algebraic equations containing automorphic Schottky functions, whose solutions let to the determination of the minimum deviation. Unfortunately these equations are so complicated that it seems to be quite difficult to obtain simple and sufficiently accurate inequalities”. Kolmogorov, Krein at al. also mentioned in [3, p. 233] that the solution of the problem was one of the significant contributions of Akhieser. Since in the explicit representations there appear parameters given implicitly (as in Zolotarev’s representation the module of the elliptic functions) even for these two cases there was (and is) still a demand for an asymptotic description in elementary functions. Already in 1913 Bernstein himself attacked the problem. For Zolotarev’s case Bernstein succeeded in finding an asymptotic solution of the minimum deviation, that is, of \( E_n(A_0 x^n + A_1 x^{n-1}) = \inf_{q \in P_{n-2}} \| A_0 x^n + A_1 x^{n-1} + q(x) \| \) in terms of elementary functions and also upper and lower bounds. In the sequel he [1] and later Achieser [] obtained asymptotics of the minimum deviation for some other special cases, for more recent results on estimates of the minimum deviation see Gutknecht and Trefethen [13], where the error function is studied also, and Haussman and Zeller [14]. But neither Bernstein nor Achieser gave asymptotics for the minimal polynomials in contrast to the \( L_2 \)-norm where both were main contributors in the development of an asymptotic theory. Interesting enough the same holds for Szegő the other great master in asymptotics of orthogonal polynomials.

In the sixties N. N. Meiman [20, 21, 22] attacked the problem to describe the minimal polynomial, called \( Z_n \) in the following, when \( l, l' \in \mathbb{N} \), coefficients are given.

By the Alternation Theorem \( Z_n \) has at least \( n-l \) alternation (abbreviated a-) points on \([-1,1]\). Since every a-point from \((-1,1)\) is a critical point it follows that the inverse image of \([-1,1]\) under \( Z_n \) consists of at most \( l' \), \( 1 \leq l' \leq l + 1 \), analytic arcs \( \Gamma_j \), one of them, say \( \Gamma_0 \) is the interval \([-1,1]\). Denoting the endpoints of the arcs by \( \alpha_{2,n}, \beta_{2,n}, \ldots, \alpha_{l',n}, \beta_{l',n} \) it follows by the equioscillating property that the normed Zolotarev polynomial \( \tilde{Z}_n = Z_n/\|Z_n\| \) satisfies

\[
\tilde{Z}_n^2 - 1 = \left( \frac{\tilde{Z}_n'(x)}{\prod_{j=1}^{l'} (x - \gamma_{j,n})} \right)^2 \prod_{j=1}^{l'} (x - \alpha_{j,n})(x - \beta_{j,n})
\]

that is,

\[
\frac{\tilde{Z}_n^2}{Z_n^2 - 1} = \frac{\prod_{j=1}^{l'} (x - \gamma_{j,n})^2}{\prod_{j=1}^{l'} (x - \alpha_{j,n})(x - \beta_{j,n})}
\]
where \( \gamma_{j,n} \)'s are such that, \( j = 0, ..., l \),

\[
\int_{\alpha_{j,n}}^{\beta_{j,n}} \frac{\prod'_{j=1} (x - \gamma_{j,n})}{\sqrt{\prod_{j=1}^{l} (x - \alpha_{j,n})(x - \beta_{j,n})}} dx = \frac{k_j \pi}{n}
\]

where \( k_j \) is the number of alternation points on the arc \( \Gamma_j \).

\[
\tilde{Z}_n(t) = \pm \cosh(n \int_{-1}^{t} \frac{\prod'_{j=1} (x - \gamma_{j,n})}{\sqrt{\prod_{j=1}^{l} (x - \alpha_{j,n})(x - \beta_{j,n})}} dx).
\]

Roughly speaking Meiman investigated in detail the precise number \( l' \) of arcs and gave a (more) detailed geometric description of the arcs \( \Gamma_j \). For an explicit representation of the polynomial \( \tilde{Z}_n \) explicit expressions for the endpoints \( \alpha_{j,n}, \beta_{j,n} \) and the zeros of the derivative \( \gamma_{j,n} \) would be needed. To find such explicit expressions is extremely unlikely because one has to solve the system of hyperelliptic integrals (5) and to find out how the endpoints of the arcs are related to the given leading coefficients. In \([20, 21, 22]\) no way of solution is offered to this fundamental open question.

But let us observe that there is an interesting property of these points. Since \( \tilde{Z}_n \) has a finite number of zeros, precisely at most \( l \) zeros, outside of \([-1, 1]\) and since \( \tilde{Z}_n \) is a minimal polynomial on \( \cup \Gamma_j \), the length of each arc \( \Gamma_j, j = 1, \ldots, l \), has to shrink to a point in the limit. Thus if we are interested in asymptotics the problem reduces to find the connection between the \( l \) given coefficients and the accumulation points of the \( l' \)-arcs or, in other words, the \( l' \)-zeros of \( \tilde{Z}_n \) lying outside \([-1, 1]\); recall that every \( \Gamma_j \) is a component of \( \tilde{Z}_n^{-1}([-1, 1]) \), for a description of inverse polynomial images see \([25]\). To find this connection we proceed as follows. As usual we transform the problem by the Joukowski-map to the complex plane such that the interval \([-1, 1]\) corresponds to the unit circle. Then we approximate the polynomial of degree \( l \) with the given \( l \) coefficients by functions from \( H_\infty \) (so-called Caratheodory-Fejer approximation). This yields a Blaschke product. Reflecting the Blaschke product at the unit circle it turns out that its real part represents asymptotically the polynomial \( Z_n \) and, in particular, the zeros of the reflected Blaschke product are the limits of the \( l' \)-arcs.

Roughly speaking we have shown that asymptotically there is a unique correspondence between polynomials with \( l \) fixed leading and of least maximum norm on \([-1, 1]\) and polynomials which vanish outside \([-1, 1]\) at \( l \) given points and are minimal on \([-1, 1]\). Since we may expect that outside \([-1, 1]\) the polynomial grows exponentially fast we may conclude that the minimal polynomial which vanishes at given \( l \) points represents asymptotically (up to a multiplication constant) every polynomial satisfying in each of the \( l \) given points any interpolation condition (not depending on \( n \)). Indeed in this way we obtain asymptotic representations of polynomials satisfying interpolation constraints including constraints on the derivative. So far for special cases
asymptotics for the minimum deviation (but not for the minimal polynomial) have been found by Bernstein [], see also [], \( n \)-th root asymptotics has been derived by Fekete and Walsh [11], see also [].

Mostly it is more convenient to formulate the problem in terms of Chebyshev polynomials, that is, to use the representation

\[
\sum_{j=0}^{l} A_j x^{n-j} = \sum_{j=0}^{l} a_j T_{n-j}(x) + q(x)
\]

\( q \in \mathbb{P}_{n-l-1} \), where the first \( l \) coefficients \( a_j \) are given by the \( A_j \)'s. In fact we will even study the more general problem of minimal trigonometric polynomials with fixed leading coefficients; more precisely, denote by \( T_m = \{ \sum_{k=0}^{m} a_k \cos k \varphi + b_k \sin k \varphi : a_k, b_k \in \mathbb{R} \} \) the set of trigonometric polynomials of degree less or equal \( m \), and let \( a_0, ..., a_l, b_0, ..., b_l \in \mathbb{R} \) be given: find the unique trigonometric polynomial \( Z_n(\varphi; a_0, ..., a_l, b_0, ..., b_l) \) for which

\[
\min_{a_l, b_l} \| \sum_{j=0}^{n} a_j \cos (n-j) \varphi + b_j \sin (n-j) \varphi \|_{[0,2\pi]} = \\
= \| Z_n(\varphi; a_0, ..., a_l, b_0, ..., b_l) \|_{[0,2\pi]}
\]

is attained; or in other words: given \( \tau_0 = a_0 - ib_0, ..., \tau_l = a_l - ib_l \in \mathbb{C} \) find the unique polynomial \( \mathcal{Z}_n(z; \tau_0, ..., \tau_l) \) of degree \( n \) for which

\[
\min_{\tau_j \in \mathbb{C}} \| \text{Re}\{ e^{i(n-j) \varphi} \} \| = \| \text{Re}\{ \mathcal{Z}_n(e^{i\varphi}; \tau_0, ..., \tau_l) \} \|
\]

is attained. Naturally

\[
\mathcal{Z}_n(\varphi; \tau_0, ..., \tau_l) := Z_n(\varphi; a_0, ..., a_l, b_0, ..., b_l) = \text{Re}\{ \mathcal{Z}_n(e^{i\varphi}; \tau_0, ..., \tau_l) \}
\]

Obviously when the \( b_j \)'s are zero then, using (7), we are back in the algebraic case.

In the second part we consider linear functionals on the space of truncated trigonometric polynomials and their applications, that is, given \( \mu_0, ..., \mu_l \in \mathbb{C} \) how large can be the linear functional \( | \sum_{j=0}^{l} \mu_{l-j} \tau_j | \) if \( | \text{Re}\{ \sum_{j=0}^{n} \tau_j e^{i(n-j) \varphi} \} | \leq 1 \). Note that the first \( l + 1 \) leading coefficients of the trigonometric polynomial are given by \( \tau_0, \tau_1, ..., \tau_l \) and the remaining \( n-l+1 \) coefficients are free available. We will determine the least upper bound of \( | \sum_{j=0}^{l} \mu_{l-j} \tau_j | \) for all \( n \in \mathbb{N} \).

With the help of the solution of the problem just discussed we are able to solve an old problem, see [] whether truncated Fourier series can be used as a substitute for best approximations. A justification of the method resulted in the conjecture of Clenshaw that \( | \sum_{j=0}^{l} \tau_j \cos j \varphi | / E_n(f) \) is bounded by Landau’s constant as \( n \to \infty \). The articles by Clenshaw, Lam and Elliot and Talbot [] are devoted to show numerically that Clenshaw’s conjecture hold, at least for small \( l \), i.e. for \( l = 1, 2, 3, 4 \) respectively by giving algorithm for larger \( l \), for details concerning open and solved questions of Clenshaw’s conjecture, see [] p. 275. In contrast to p. 275 the last but
one paragraph in the introduction in [37] is misleading with this respect, partly incorrect, as the statement “The first published proof of Clenshaw’s conjecture was given by Lam and Elliot [17] in 1972.” There is no proof in [17] as mentioned in [37, p. 275] also. The conjecture will follow as an easy consequence of our derivations.

2. Main Theorem

**Theorem 2.1.** Let \( p_l(z) = z^l + \ldots \) be a polynomial of degree \( l \) which has all its zeros in \(|z| < 1\) and the expansion at \( z = 0 \)

\[
\frac{p_l(z)}{p_l^*(z)} = \tau_0 + \tau_1 z + \tau_2 z^2 + \ldots + \tau_m z^m + O(z^{m+1})
\]

where \( m \geq l \). Then on \([0,2\pi]\) the trigonometric Zolotarev polynomials of degree \( n \) with leading coefficients \( \bar{\tau}_0, \ldots, \bar{\tau}_k, k = l, \ldots, m, k \leq n \), are given asymptotically by

\[
Z_n(\varphi; \bar{\tau}_0, \bar{\tau}_1, \ldots, \bar{\tau}_k) = \text{Re}\{z^n \frac{p_l^*(z)}{p_l(z)}\} + O(\tilde{r}^n),
\]

where \( \tilde{r} > r := \max\{|z_j| : z_j \text{ zero of } p_l\} \) and the constant in the \( O(\ ) \) term does not depend on \( n \). Moreover

\[
||Z_n(\varphi; \bar{\tau}_0, \bar{\tau}_1, \ldots, \bar{\tau}_k)|| \sim 1
\]

with geometric convergence.

Next let us show that condition (10) is satisfied always, because of the following Theorem which goes back to Caratheodory and Fejer [36] and Schur [35].

**Theorem 2.2.** (35) Let \( \tau_0, \ldots, \tau_m \in \mathbb{C} \) be given. Then there exists a \( l \in \mathbb{N}_0 \), \( 0 \leq l \leq m \), a polynomial \( p_l(z) \) of degree \( l \) and a \( \gamma \in \mathbb{C} \) such that \( p_l(z) = z^l + \ldots \)

\[
\gamma \frac{p_l(z)}{p_l^*(z)} = \tau_0 + \ldots + \tau_m z^m + O(z^{m+1}),
\]

where \( \gamma \) is the zero of largest modulus of

\[
D_{l+1}(\lambda) := \begin{vmatrix}
\lambda & 0 & \ldots & 0 & \tau_0 & \tau_1 & \ldots & \tau_l \\
0 & \lambda & \ldots & 0 & \tau_0 & \tau_l & \ldots & \tau_{l-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda & 0 & 0 & \ldots & \tau_0 \\
\bar{\tau}_0 & 0 & \ldots & 0 & \lambda & 0 & \ldots & 0 \\
\bar{\tau}_1 & \bar{\tau}_0 & \ldots & 0 & 0 & \lambda & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{\tau}_l & \bar{\tau}_{l-1} & \ldots & \bar{\tau}_0 & 0 & 0 & \ldots & \lambda
\end{vmatrix}
\]

and \( D_j(\gamma) \neq 0 \) for \( j = 1, \ldots, l \). If \( \tau_0, \ldots, \tau_l \in \mathbb{R} \) then

\[
D_{l+1}(\lambda) = \Delta(\lambda)\Delta(-\lambda)
\]
where
\[
\Delta(\lambda) := \begin{vmatrix}
\tau_l - \lambda & \tau_{l-1} & \tau_{l-2} & \cdots & \tau_1 & \tau_0 \\
\tau_{l-1} & \tau_{l-2} - \lambda & \tau_{l-3} & \cdots & \tau_2 & \tau_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tau_0 & \cdots & -\lambda & \cdots & \cdots & \cdots \\
\end{vmatrix}
\]
is the characteristic polynomial of the associated Hankel matrix.

Combining Theorem 2.1 and Theorem 2.2 we obtain that for given \(\tau_0, ..., \tau_m \in \mathbb{C}\) there exists a \(l \in \mathbb{N}_0\), \(0 \leq l \leq m\) and \(\gamma \in \mathbb{C}\) such that for \(\varphi \in [0, 2\pi]\) and for \(k = l, ..., m\)
\[
Z_n(\varphi; \bar{\tau}_0, ..., \bar{\tau}_k) = |\gamma| \text{Re} \left\{ e^{i\arg \gamma} z^n \frac{p_l^*(z)}{p_l(z)} \right\} + O(\tilde{r}_n),
\]
where the constant in the \(O(\ )\) term does not depend on \(n\), and
\[
||Z_n(\varphi; \bar{\tau}_0, ..., \bar{\tau}_k)|| \sim |\gamma|
\]
where the convergence is geometric. \(l, p_l\) and \(\gamma\) are determined by the above Theorem.

Thus the problem of determining the trigonometric and algebraic Zolotarev polynomials with an arbitrary given number of leading coefficients with respect to the maximum norm is completely solved asymptotically by the above Theorems.

We mention that by Fejér [] the condition \(\tau_l \geq \tau_{l-1} \geq ... \geq \tau_1 \geq \tau_0 > 0\) implies that \(l = m\) in Theorem [?] and that the coefficients of \(p_l(z) = \sum_{\nu=0}^l a_\nu z^{l-\nu}\) satisfy \(a_0 \geq a_1 \geq ... \geq a_l \geq 0\).

For given \(\tau\) only a few explicit values of \(|\gamma(\tau)| =: \tilde{\gamma}(\tau)\) are known, for instance
\[
\tilde{\gamma}((1, ..., 1)) = 1/2 \sin \left( \frac{\pi}{2(2l+3)} \right)
\]
which gives by (13) that
\[
Z_n(\varphi; (1, ..., 1)) = \frac{1}{2 \sin \left( \frac{\pi}{2(2l+3)} \right)} \text{Re} \left\{ e^{i\arg \gamma} z^n \frac{p_l^*(z)}{p_l(z)} \right\} + O(\tilde{r}_n),
\]

3. PROOFS

We will prove the following more general version of Theorem 2.1.

**Theorem 3.1.** Let \((n_\nu)\) be a subsequence of \(\mathbb{N}\) and let \(p_{l,n_\nu}(z) = z^l + ...\) be such that
\[
\frac{p_{l,n_\nu}(z)}{p_{l,n_\nu}^*(z)} = \tau_{0,n_\nu} + \tau_{1,n_\nu} z + ... + \tau_{m,n_\nu} z^m + O(z^{m+1}),
\]
where \( m \) is independent of \( n_\nu \) and \( m \leq l \), and that

\[
\text{(15)} \quad p_{l,n_\nu}(z) \xrightarrow{n_\nu \to \infty} p_l(z), \text{ where } p_l(z) \text{ has all zeros in } |z| \leq r < 1.
\]

Then for \( \varphi \in [0, 2\pi] \) and \( k = l, \ldots, m \),

\[
\begin{align*}
Z_{n_\nu}(\varphi; \bar{\tau}_0, n_\nu, \ldots, \bar{\tau}_{k,n_\nu}) &= \text{Re}\{z_{n_\nu} p_{l,n_\nu}^*(z) / p_{l,n_\nu}(z)\} + O(\tilde{r}^n) \\
&= \text{Re}\{z_{n_\nu} p_l^*(z) / p_l(z)\} + O(\tilde{r}^n)
\end{align*}
\]

where \( \tilde{r} > r \) and the constant in the \( O(\ ) \) term does not depend on \( n \). Furthermore,

\[
\text{(17)} \quad ||Z_{n_\nu}(\varphi; \bar{\tau}_0, n_\nu, \ldots, \bar{\tau}_{k,n_\nu})|| \sim 1.
\]

Notation: Let \( l, n \in \mathbb{N} \) and suppose that \( p_{l,n}(z) = z^l + \ldots \) has no zero on \( |z| = 1 \). In the following let, \( z = e^{i\varphi} \),

\[
\begin{align*}
\text{(18)} \quad R_n(\varphi) &= \text{Re}\{z^n p_{l,n}^*(z) / p_l(z)\} = \text{Re}\{z^n - l(p_{l,n}^*(z))^2\} / |p_{l,n}(z)|^2 \\
S_n(\varphi) &= \text{Im}\{z^n p_{l,n}^*(z) / p_l(z)\} = \text{Im}\{z^n - l(p_{l,n}^*(z))^2\} / |p_{l,n}(z)|^2
\end{align*}
\]

Obviously \( R_n(\varphi) \) and \( S_n(\varphi) \) are rational trigonometric functions with a trigonometric polynomial of degree \( n + l \) in the numerator and a positive trigonometric polynomial of degree \( l \) in the denominator. Note that

\[
\text{(20)} \quad R_n^2(\varphi) + S_n^2(\varphi) = 1
\]

**Lemma 3.2.** Let \( l, n \in \mathbb{N} \) with \( n > l \) and suppose that \( p_{l,n}(z) \) has all zeros in \( |z| \leq \tilde{r} < 1 \) for all \( n \geq n_0 \). Then the following statements hold for every \( n \geq n_1 : a) \) Both \( R_n \) and \( S_n \) have exactly \( 2(n - l) \) simple zeros in \([0, 2\pi]\) and their zeros strictly interlace. Furthermore, \( R_n(S_n) \) has \( 2(n - l) \) \( \pm \)-points at the zeros of \( S_n(R_n) \). In particular, 0 is a best approximation to \( R_n \) and \( S_n \) with respect to \( \bigcup_{n-i-1} \).

b) The numerators in (18) and (19) can be represented as follows, \( z = e^{i\varphi} \),

\[
\begin{align*}
\text{(21)} \quad &\text{Re}\{z^n - l(p_{l,n}^*(z))^2\} = t_{n-l,n}(\varphi)|\hat{r}_{2l,n}(e^{i\varphi})|^2 \\
&\text{and}
\end{align*}
\]

\[
\begin{align*}
\text{(22)} \quad &\text{Im}\{z^n - l(p_{l,n}^*(z))^2\} = u_{n-l,n}(\varphi)|\hat{s}_{2l,n}(e^{i\varphi})|^2
\end{align*}
\]

where \( t_{n-l}(\varphi), u_{n-l}(\varphi) \) are trigonometric polynomials of degree \( n - l \) which have all their \( 2(n - l) \) zeros in \([0, 2\pi]\) and their zeros strictly interlace. Furthermore \( \hat{r}_{2l}(z) \) and \( \hat{s}_{2l}(z) \) are monic polynomials of degree \( 2l \) which have all their zeros in \( |z| < 1 \).
Lemma 3.3. Under the assumptions of Theorem 2.1 the polynomials \( \hat{r}_{2l,n}(z) \) and \( \hat{s}_{2l,n}(z) \) associated with \( p^*_n(z) \) by (21) and (22), respectively, satisfy on any compact subset of \( \mathbb{C} \)
\[
\hat{r}_{2l,n}(z) = (p_l(z))^2 + O(r^n) \\
\hat{s}_{2l,n}(z) = (p_l(z))^2 + O(r^n)
\]
where \( p_l \) is given by (15) and where \( 0 < r < 1 \).

Proof. Let us consider the zeros of the polynomial
\[
P_{2n+2}(z) = z^{2n}(p^*_n(z))^2(z) + (p_l,n)^2(z).
\]

Note that, \( z = e^{i\varphi} \),
\[
P_{2n+2}(e^{i\varphi}) = z^{n+l} \text{Re} \{z^{n-l}(p^*_n(z))^2\} = z^{n-l}v_{n-l,n}(\varphi)r_{2l,n}(z)r^*_m(z)
\]
Thus by Lemma 3.2 \( P_{2n+2}(z) \) has \( 2n - 2l \) zeros on \( |z| = 1 \) and by the self-reciprocal property \( 2l \) zeros in \( |z| < 1 \) and in \( |z| > 1 \) which are the zeros of \( \hat{r}_{2l,n}(z) \) and \( \hat{s}_{2l,n}(z) \), respectively. By assumption (15), (21) and Rouché’s theorem it follows that for \( n \geq n_0 \) \( \hat{r}_{2l,n}(z) \) has (exactly) two zeros in each neighborhood of a zero of \( p_l \). More precisely, if \( v_{j,n} \) is a zero of \( P_{2n+2l} \) from the neighborhood of a zero \( z_j \) of \( p_l \) we have, by (24) again, that \( |v_{j,n} - z_j| = O(r^n) \) for some \( r, 0 < r < 1 \), where \( r \) is independent of \( j \) and \( n \). Hence
\[
\hat{r}_{2l,n}(z) = p^2_l(z) + O(r^n).
\]
Analogously the statement for \( \hat{s}_{2l,n} \) is proved. \( \square \)

Lemma 3.4. Under the assumption of Theorem 2.1 \( R_n \) defined in (18) is of the form
\[
R_n(\varphi) = V_n(\varphi) + \psi_n(\varphi)
\]
where \( V_n \in \mathfrak{S}_n \) with \( V_n(\varphi) = \text{Re} \{\sum_{j=0}^{m} j, z^{n-j}\} + \ldots \) and \( \psi_n(\varphi) \) is a rational trigonometric function with
\[
||\psi_n|| = O(\bar{r}^n)
\]
where \( \max |z_j| < \bar{r} < 1 \), the \( z_j \)’s are the zeros of \( p_l \) and the constant in the \( O(\cdot) \) term does not depend on \( n \).
Proof. By Euclid $R_n$ from (18) can be written in the form

$$R_n(\varphi) = \frac{V_n(\varphi)|p_{l,n}(e^{i\varphi})|^2}{|p_{l,n}(e^{i\varphi})|^2} + t(\varphi),$$

where $V_n \in \mathcal{I}_n$ and $t \in \mathcal{I}_{l-1}$. Putting, $z = e^{i\varphi}$,

$$e^{i\varphi}V_n(\varphi) = z^n P_n^*(z) + P_n(z)$$

we obtain by (19) and partial fraction expansion, assuming that $p_{l,n}$ has simple zeros, that

$$\frac{z^{2n}(p_{l,n}^*)^2(z) + p_{l,n}^2(z)}{p_{l,n}(z)p_{l,n}^*(z)} = z^n P_n^*(z) + P_n(z) + z^n \left( \sum_{j=1}^{l} \frac{\lambda_{j,n}}{z - z_{j,n}} + \sum_{j=1}^{l} \frac{\beta_{j,n}}{z - \bar{z}_{j,n}} \right),$$

where

$$\lambda_{j,n} = z_{j,n}^n p_{l,n}(z_{j,n}) p_{l,n}^*(z_{j,n}) = O(\tilde{r}^n)$$

and analogously

$$\beta_{j,n} = z_{j,n}^n p_{l,n}(z_{j,n}) p_{l,n}^*(z_{j,n}) = O(\tilde{r}^n)$$

where for the last equalities we took (15) into consideration. By (28) and (30) it follows that

$$z^{2n} \frac{p_{l,n}^*}{p_{l,n}} + \frac{p_{l,n}}{p_{l,n}^*} = z^n P_n^* + P_n + O(z^{n+1}),$$

hence by (14)

$$P_n(z) = \sum_{j=0}^{m} \tau_{j,n} z^j + ...$$

which gives by (29) the assertion on the leading coefficients of $V_n$. □

Notation 3.5. Let $G$ be a linear space of $C[a,b]$ and $f \in C[a,b]$ with 0 as a best approximation. Denote by $\gamma(f,G)$ the strong unicity constant of $f$ with respect to $G$ i.e.,

$$\gamma(f,G) = \inf \frac{||f - g|| - ||f||}{||g||}$$

$$= \inf_{||g|| = 1} \max_{y \in E(f)} \text{sgn}(f(y))g(y)$$

where $E(f)$ denotes the set of extremal points. If $G$ is a Haar space then by [?]

$$\gamma(f,G) = 1/\max_{1 \leq k \leq n} ||g_k||$$
where \( g_k \in G \) is the polynomial given by \( g_k(y_j) = (-1)^j, j = 1, \ldots, n+1; j \neq k \) and the \( y_j \) denote the \( a \)-points of \( f \).

**Lemma 3.6.**

\[
\frac{1}{\gamma}(R_n, \Sigma_{n-l-1}) = O(n)
\]

**Proof.** Let

\[
\mathcal{G}_{2(n-l)-1} := \left\{ q(\varphi) \frac{\hat{r}_{2l,n}(e^{i\varphi})^2}{|p_{l,n}(e^{i\varphi})|^2} : q \in \Sigma_{n-l-1} \right\}
\]

where \( \hat{r}_{2l,n} \) is the polynomial associated with \( p_{l,n} \) by (21). By Lemma 3.2 \( t_{n-l,n} \) can be written in the form, \( z = e^{i\varphi} \),

\[
t_{n-l,n}(\varphi) = \text{Re} \left\{ c_n z^{n-l} + \ldots \right\} = \frac{c_n}{2} z^{-(n-l)} \prod_{\nu=1}^{2(n-l)} (z - e^{i\psi_{\nu,n}})
\]

(34)

\[
t_{n-l,n}(\varphi) = |c_n| (2i)^{2(n-l)} \prod_{\nu=1}^{2(n-l)} \sin \frac{\varphi - \varphi_{\nu,n}}{2}
\]

where we used the fact that

\[
e^{i \arg c_n} = e^{-i \sum_{\nu=1}^{2(n-l)} \frac{\psi_{\nu,n}}{2}}
\]

(35)

Note that by (34) and (35)

\[
t_{n-l,n}(\varphi) = |c_n| \left( \cos((n-l)\varphi - \sum_{\nu=1}^{2(n-l)} \frac{\psi_{\nu,n}}{2}) + q(\varphi) \right)
\]

(36)

where \( q \in \Sigma_{n-l-1} \). Analogously we obtain

\[
u_{n-l,n}(\varphi) = \text{Im} \left\{ d_n z^{n-l} + \ldots \right\} = \frac{d_n}{2} z^{-(n-l)} \prod_{\nu=0}^{2(n-l)} (z - e^{i\psi_{\nu,n}})
\]

(37)

\[
u_{n-l,n}(\varphi) = |d_n| (2i)^{2(n-l)} \prod_{\nu=1}^{2(n-l)} \sin \frac{\varphi - \varphi_{\nu,n}}{2}
\]

where

\[
e^{i \arg d_n} = e^{i \left( \frac{\pi}{2} - \sum_{\nu=1}^{2(n-l)} \frac{\psi_{\nu,n}}{2} \right)}
\]

moreover

\[
u_{n-l,n}(\varphi) = |d_n| \sin((n-l)\varphi + \frac{\pi}{2} - \sum_{\nu=1}^{2(n-l)} \frac{\varphi_{\nu,n}}{2})
\]

Taking into consideration (23) we have

\[
\lim_n d_n = \lim_n c_n \neq 0
\]
Next let us put for \( k = 1, \ldots, 2n - 2l, k \neq 1, \)

\[
q_{k,n}(\varphi) = t_{n-l,n}(\varphi) + \\
\frac{|c_n|}{|d_n|} u_{n-l,n}(\varphi) \left( \sin \left( \varphi + \sum_{\nu=1,\nu \neq 1,k}^{2(n-l)} \frac{\varphi_{\nu,n}}{2} - \sum_{\nu=1}^{2(n-l)} \frac{\psi_{\nu,n}}{2} - \frac{\pi}{2} \right) - e_{k,n} \right) \\
\frac{(-2) \sin \left( \frac{\varphi - \varphi_{1,n}}{2} \right) \sin \left( \frac{\varphi - \varphi_{k,n}}{2} \right)}
\]

where

\[
e_{k,n} = \sin \left( \varphi_{1,n} + \sum_{\nu=1,\nu \neq 1,k}^{2(n-l)} \frac{\varphi_{\nu,n}}{2} - \sum_{\nu=1}^{2(n-l)} \frac{\psi_{\nu,n}}{2} - \frac{\pi}{2} \right)
\]

i.e. the second factor of the numerator is of the form \( \text{const} \sin \left( \frac{\varphi - \varphi_{1,n}}{2} \right) \sin \left( \frac{\varphi - \varphi_{k,n}}{2} \right) \). Moreover the second expression at the right hand side in (24) is from \( \mathfrak{S}_{n-l} \). Since

\[
\frac{u_{n-l,n}(\varphi)}{(2i)^2 \sin \left( \frac{\varphi - \varphi_{1,n}}{2} \right) \sin \left( \frac{\varphi - \varphi_{k,n}}{2} \right)} = |d_n| \left\{ \text{Im} \left\{ e^{i \left( (n-l-1) \varphi + \frac{\pi}{2} - \sum_{\nu=1,\nu \neq 1,k}^{2(n-l)} \frac{\varphi_{\nu,n}}{2} \right) } \right\} \right\}
\]

it follows by straightforward calculation that the second expression at the right hand side of (24) is of the form \(-|c_n| \cos \left( (n-l) \varphi - \sum_{\nu=1}^{2(n-l)} \frac{\psi_{\nu,n}}{2} \right) + q, q \in \mathfrak{S}_{n-l-1} \); hence it follows by (36) that \( q_{k,n} \in \mathfrak{S}_{n-l-1} \) and therefore

\[
g_{k,n}(\varphi) = q_{k,n}(\varphi) \left| \hat{r}_{2l,n}(e^{i\varphi}) \right|^2 \\
\left| p_{l,n}(e^{i\varphi}) \right|^2
\]

has the properties that \( g_{k,n} \in \mathfrak{S}_{2(n-l)-1} \) and, by (24) and Lemma 3.2 a)

\[
\pm g_{k,n}(\varphi_{\nu,n}) = (-1)^{\nu} \quad \nu = 1, \ldots, 2(n-l), \nu \neq k.
\]

Furthermore,

\[
h_{k,n}(\varphi) = \sin \left( \frac{\varphi - \varphi_{k,n}}{2} \right) g_{k,n}(\varphi)
\]

is uniformly bounded on \([0, 2\pi]\) with respect to \( n \). Indeed, \( t_{n-l,n}|\hat{r}_{2l,n}|^2/|p_{l,n}|^2 = R_n(\varphi) \) and \( u_{n-l,n}|\hat{s}_{2l,n}|^2/|p_{l,n}|^2 = S_n(\varphi) \) are bounded by one. Further \( |\hat{r}_{2l,n}/p_{l,n}| \) as well as, recall (23), \( |\hat{s}_{2l,n}/|p_{l,n}| \) are uniformly bounded on \(|z| = 1\), since \( p_{l}(e^{i\varphi}) \) is, by assumption, bounded away from zero on \(|z| = 1\), the boundedness of \( h_{k,n} \) follows by the choice (40) of \( e_{k,n} \) in (24) and (33). Now \( h_{k,n}(\varphi) \) is a trigonometric polynomial of half argument for which Bernstein’s
inequality for the derivative still holds, therefore
\[
\|g_{k,n}\| = \left\| \frac{h_{k,n}(\varphi) - h_{k,n}(\varphi_k)}{\varphi - \varphi_k} \sin \left( \frac{\varphi - \varphi_k}{2} \right) \right\|
\]
(44)
\[
\leq \|h_{k,n}\| \left\| \frac{\varphi - \varphi_k}{\sin \left( \frac{\varphi - \varphi_k}{2} \right)} \right\| \leq \text{const}. n
\]
Thus
\[
1/\gamma(R_n; \mathfrak{G}_{2(n-t)-1}) \leq \text{const}. n
\]
Using the simple fact that 0 is a best approximation to \(1 = \text{const}. n\) as well as from \(\mathfrak{T}_{n-t-1}\) we obtain by (??) and (33) that
\[
\gamma(R_n; \mathfrak{G}_{2(n-t)-1}) \leq \|\hat{g}_{2,n}/\hat{p}_{l,n}\|^2 \gamma(R_n; \mathfrak{T}_{n-t-1})
\]
hence
\[
1/\gamma(R_n; \mathfrak{T}_{n-t-1}) \leq \text{const}. n
\]
\[
\square
\]
4. Proof of Theorem 2.1 and Theorem 3.1

**Proposition 4.1.** Let \(n, j(n) \in \mathbb{N}\). Let \(\hat{t}_{j(n)}(\varphi)\) be a best approximation to \(f_n \in C_{2\pi}\) on \([0, 2\pi]\) with respect to the linear subspace \(G_j(n)\) and let \(0 = \text{const}. n\) be a best approximation from \(G_{j(n)}\) to \(h_n \in C_{2\pi}\) on \([0, 2\pi]\). Suppose that \(\|h_n\| = 1\) and that
\[
f_n(\varphi) - \hat{t}_{j(n)}(\varphi) = h_n(\varphi) + \epsilon_n(\varphi),
\]
where \(t_{j(n)} \in G_{m(n)}\). If \(\gamma(h_n; G_{j(n)})\|\epsilon_n\| \xrightarrow{n \to \infty} 0\) then for \(\varphi \in [0, 2\pi]\)
\[
f_n(\varphi) - \hat{t}_{j(n)}(\varphi) = h_n(\varphi) + O(\|\epsilon_n\| \gamma(h_n; G_{j(n)}))
\]
and in particular
\[
t_{j(n)}(\varphi) - \hat{t}_{j(n)}(\varphi) = O(\|\epsilon_n\|(1 + \gamma(h_n; G_{j(n)})))
\]
**Proof.**
\[
|E_{j(n)}(f_n) - 1| \leq \|\epsilon_n\|
\]
since
\[
E_{j(n)}(f_n) \leq \|f_n - t_{j(n)}\| \leq \|h_n\| + \|\epsilon_n\| = 1 + \|\epsilon_n\|
\]
and
\[
1 = E_{j(n)}(h_n) = \|h_n\| \leq \|f_n - t_{j(n)} - \epsilon_n - (\hat{t}_{j(n)} - t_{j(n)})\| \leq E_{j(n)}(f_n) + \|\epsilon_n\|
\]
Next let us show that
\[
\|\hat{t}_{j(n)} - t_{j(n)}\| \leq (1 + 2\gamma(h_n; G_{j(n)}))\|\epsilon_n\|
\]
Indeed, by the definition (31) of the strong uniqueness constant
\[
\|\hat{t}_{j(n)} - t_{j(n)}\| \leq \gamma(h_n; G_{j(n)}) \left(\|h_n - (\hat{t}_{j(n)} - t_{j(n)})\| - \|h_n\|\right)
\]
\[
\leq \gamma(h_n; G_{j(n)}) (E_{j(n)}(f_n) + \|\epsilon_n\| - \|h_n\|) \leq 2\gamma(h_n; G_{j(n)})\|\epsilon_n\|
\]
where we have used (45) and the fact that $||h_n|| = 1$ and (47) in the second and third inequality, respectively.

\[ \tag{47} \]

of Theorem 2.1 Put $k(n) = n - k - 1, G_{k(n)} = \Xi_{n-k-1}$ and $k \in \{l, \ldots, m\}$ fixed, and

\[ h_n(\varphi) = R_n(\varphi) \quad \text{and} \quad f_n(\varphi) = \text{Re} \left\{ \sum_{j=0}^{k} \bar{\tau}_{j,n} z^{n-j} \right\}. \]

Recall that 0 is a best approximation with respect to $T_{n-l-1}$ hence with respect to $T_{n-k-1}$ for $k \in \{l, \ldots, n-1\}$. By Lemma 3.4

\[ V_n = f_n - t_{k(n)} \quad \text{and} \quad R_n = f_n - t_{k(n)} + \psi \]

where $t_{k(n)} \in \Xi_{n-j-1}$ and $||\psi_n|| = O(r^n)$. Since $f_n - \tilde{t}_{k(n)} = Z_n(x; \bar{\tau}_0,n, \ldots, \bar{\tau}_j,n)$

Theorem 3.1 follows by Proposition 4.1 in conjunction with Lemma 3.2 and Lemma 3.6.

\[ \tag{49} \]

5. Extremal problems on coefficients and Clenshaw’s conjecture

First let us recall how to solve the following classical problem in complex function theory. Let $\mu_0, \ldots, \mu_l \in \mathbb{C}$ be given. Among all $f \in H^\infty$ with

\[ f(z) = \sum_{j=0}^{\infty} a_j z^j \quad \text{and} \quad |f(z)| \leq 1 \quad \text{for} \quad |z| < 1 \]

find

\[ \eta_l := \max_{a_j} |\mu_0 a_0 + \mu_1 a_1 + \ldots + \mu_l a_l|. \]

By [? , ?] for given $\mu_0, \ldots, \mu_l \in \mathbb{C}$ there exist polynomials $s_{l-\nu}(z)$ and $r_{\nu}(z)$ of degree $l - \nu$ and $\nu$ respectively, which have all zeros in $|z| < 1$ such that

\[ \mu_0 + \mu_1 z + \ldots + \mu_l z^l + \ldots = (s_{l-\nu}(z))^2 r_{\nu}(z)r_{\nu}(z) =: h_{2l}^*(z) \]

F. Riesz [28] has shown that among all functions $g \in H^1$ with $g(z) = \mu_0 + \mu_1 z + \ldots + \mu_l z^l + \ldots$ the mean modulus $\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\varphi})|d\varphi$ is minimal for $h_{2l}^*(z)$. With the help of this result Szasz [] derived that

\[ \eta_l = \frac{1}{2\pi} \int_0^{2\pi} |h_{2l}^*(e^{i\varphi})|d\varphi \]

and that equality in (49) is attained by

\[ f(z) = e^{i\gamma} \frac{s_{l-\nu}(z)}{s_{l-\nu}^*(z)} = c_0 + c_1 z + \ldots + c_l z^l + \ldots \]

We mention that (51) holds, since Szasz’s problem (48)-(49) and the problem considered by F. Riesz are dual problems, see [12].
The simple case $\nu = 0$ appears if the expansion in $z = 0$

$$\sqrt{\mu_0 + \mu_1 z + \ldots + \mu_l z^l} = \sum_{j=0}^{\infty} \lambda_j z^j$$

is such that

$$s_l^*(z) = \sum_{j=0}^{l} \lambda_j z^\nu$$

has no zero in $|z| < 1$. Then

$$\eta_l = \sum_{j=0}^{l} |\lambda_j|^2 = \frac{1}{2\pi} \int_0^{2\pi} |s_l(e^{i\varphi})|^2 d\varphi$$

and

$$f(z) = e^{i\gamma} \frac{s_l(z)}{s_l^*(z)}$$

is the function for which the maximum (49) is attained.

The special case $\mu_j = 1, j = 0, \ldots, l$, i.e. the determination of the maximum of the sum of coefficients (which fits into the case just considered), has been first considered and solved by an ad hoc method in the celebrated paper by Landau [9]. Landau has shown that

$$|a_0 + a_1 + \ldots + a_l| \leq 1 + \sum_{j=1}^{l} \left( \frac{1.3, \ldots, (2j - 1)}{2.4, \ldots, 2j} \right)^2$$

and that equality holds only for

$$F(z) = e^{i\gamma} \frac{\sum_{\nu=1}^{l} (-1)^\nu \left( \frac{-\frac{1}{\nu}}{\nu} \right) z^{l-\nu}}{\sum_{\nu=1}^{l} (-1)^\nu \left( \frac{-\frac{1}{\nu}}{\nu} \right) z^\nu}$$

where $\gamma \in \mathbb{R}$.

The other simple case $\nu = l$ appears if the $\mu_j$'s are such that

$$\text{Re} \left\{ \mu_0 e^{i\varphi} + \ldots + \mu_{-1} e^{i\varphi} + \frac{\mu_l}{2} \right\} \geq 0 \text{ on } [0, 2\pi]$$

Then

$$\eta_l = \mu_l$$

and equality is attained in (49) for $f(z) = \varepsilon, |\varepsilon| = 1$.

Here we study the following problem: Let $\mu_0, \ldots, \mu_l \in \mathbb{C}$ be given. How large can be $|\sum_{j=0}^{l} \mu_{l-j} \tau_j|$ if $||\text{Re} \left\{ \sum_{j=0}^{n} \tau_j e^{i(n-j)\varphi} \right\}|| \leq 1$ and $n$ is large. In other words: among all upper bounds $L$ such that for each $n \in \mathbb{N}$ and for every $(\tau_0, \ldots, \tau_l, \ldots, \tau_n) \in \mathbb{C}^n$

$$|\sum_{j=0}^{l} \mu_{l-j} \tau_j| \leq L||\text{Re} \left\{ \sum_{j=0}^{n} \tau_j e^{i(n-j)\varphi} \right\}||$$




find the least upper bound.

We point out that by (48) and (49) $\eta_l$ is such an upper bound $L$ in (56).
As we show in the next theorem it is even the least upper bound.

**Theorem 5.1.** The least upper bound in (56) is given by $\eta_l(\mu_0, ..., \mu_l)$.

**Proof.** First we note that by (48) for every $n \in \mathbb{N}$

$$|\sum_{j=0}^{l} \mu_{l-j} \tau_j| \leq \eta_l \text{ if } ||\text{Re} \left\{ \sum_{j=0}^{n} \tau_j e^{i(n-j)\varphi} \right\}|| \leq 1$$

Thus we have to show that the upper bound $\eta_l$ cannot be improved. By (52) we know that $\max_{\tau_j} |\sum_{j=1}^{l} \mu_{l-j} \tau_j| = \eta_l$ is attained for a Blaschke product

$$e^{i\kappa s_l - \nu(z)} = c_0 + c_1 z + c_2 z^2 + ... + c_l z^l + ...$$

Thus it follows by Theorem 2.1

$$Z_n(\varphi; \bar{c_0}, ..., \bar{c_l}) = \text{Re} \left\{ z^n e^{i\kappa s_l - \nu(z)} \right\} + O(r^n)$$

with, setting $c = (c_0, ..., c_l)$,

$$1 - \varepsilon_n \leq ||Z_n(\varphi; \bar{c})|| \leq 1 + \varepsilon_n, \text{ i.e. } \frac{1}{1 - \varepsilon_n} \leq E_n(c) \leq \frac{1}{1 + \varepsilon_n}$$

and $\varepsilon_n \to 0$ geometrically. Hence

$$\left| \sum_{j=0}^{l} \mu_{l-j} \tau_j \right| \leq \frac{\eta_l}{1 + \varepsilon_n} \text{ with } \varepsilon_n \to 0 \text{ geometrically}$$

which proves the theorem. □

**Corollary 5.2.** Let $\mu_0, ..., \mu_l \in \mathbb{C}$ be given. Then for every $(\tau_0, ..., \tau_l) \in \mathbb{C}^l$
and for every $n \in \mathbb{N}$

$$||\text{Re} \left\{ \sum_{j=0}^{l} \mu_{l-j} \tau_j e^{i(n-j)\varphi} \right\}|| \leq \eta_l ||\text{Re} \left\{ \sum_{j=0}^{n} \tau_j e^{i(n-j)\varphi} \right\}||$$

and the constant $\eta_l$ cannot be improved.

**Proof.** For any $\psi \in [0, 2\pi]$ and for every $n \in \mathbb{N}$

$$\sum_{j=0}^{l} \mu_{l-j} \tau_j e^{i(n-j)\psi} \leq \eta_l \min_{\tau_j} \left| \tau_j \right| \text{ if } \text{Re} \left\{ \sum_{j=0}^{n} \tau_j e^{-i(n-j)\psi} \right\}$$

$$= \eta_l \min_{\tau_j} \left| \text{Re} \left\{ \sum_{j=0}^{n} \tau_j e^{i(n-j)\varphi} \right\} \right|$$

where in the first inequality the upper bound $\eta_l$ is best possible by Theorem 5.1 and the last equality follows by $2\pi$-periodicity. □
As a consequence of Corollary 5.2 and Landau's results (??) we obtain a proof of Clenshaw's conjecture [] from the sixties of the last century.

**Notation 5.3.** For given \( \tau = (\tau_0, \ldots, \tau_l) \in \mathbb{C}^{l+1} \) let

\[
E_n(\tau) = ||Z_n(\varphi; \tau)||_{[0,2\pi]}
\]

the minimum deviation.

**Theorem 5.4.** Clenshaw's conjecture holds, that is, for any \( \tau \in \mathbb{R}^{l+1} \)

\[
\lim_{n \to \infty} \left( \left| \sum_{j=0}^{l} \tau_j \cos(n-j)\varphi \right|/E_n(\tau) \right) \leq 1 + \sum_{j=1}^{l} \left( \frac{1.3 \cdots (2j-1)}{2.4\cdots(2j)} \right)^2
\]

where in (59) equality is attained.

**Proof.** Put \( \mu = \mu - 1 = \cdots = \mu_0 = 1 \). Then, as in the proof of Corollary 5.2 we have for \( \psi \in [-\pi, \pi] \) or equivalently for any \( -\psi \in [-\pi, \pi] \)

\[
\left| \text{Re} \left\{ \sum_{j=0}^{l} \bar{\tau}_j e^{i(n-j)\psi} \right\} \right| \leq \left| \sum_{j=0}^{l} \tau_j e^{-i(n-j)\psi} \right| \leq \eta_l((1, \ldots, 1)) \min_{l+1 \leq j \leq n} \left| \text{Re} \left\{ \sum_{j=0}^{n} \bar{\tau}_j e^{i(n-j)\varphi} \right\} \right| = \eta_l((1, \ldots, 1)) E_n(\bar{\tau})
\]

\( \square \)

**Theorem 5.5.** For any \( \varepsilon > 0 \) and all \( n, n \geq n_0(\varepsilon) \), any \( \mu_0, ..., \mu_n, \tau_0, ..., \tau_n \in \mathbb{C} \)

\[
4||\text{Re} \left\{ \sum_{j=0}^{l} \mu_{l-j} \bar{\tau}_j e^{i(n-j)\varphi} \right\} || \leq \left( \left| \text{Re} \left\{ \sum_{j=0}^{n} \mu_{l-j} \bar{\tau}_j e^{i(n-j)\varphi} \right\} \right|_1 + \varepsilon \right)
\]

(60)

where \( ||f(\varphi)||_1 = \int_0^{2\pi} |f(\varphi)|d\varphi \).

If the \( \mu_j \)'s and \( \tau_j \)'s are real for \( j = 0, \ldots, l \) then the estimate (60) cannot be improved in the sense, that for given \( \mu(\tau) \) there exists a \( \tau(\mu) \) such that equality is attained \( n \to \infty \).

**Proof.** Proof of Theorem 5.5

**Case 1.** \( s_l(z) \) satisfies condition ??.

By ??

\[
z^{n-2l}s_l^2(z) = \sum_{j=0}^{l} \mu_j z^{n-j} + O(z^{n-l-1}).
\]
We claim that
\[
\min_{t \in \Sigma_{n-l-1}} \int_0^{2\pi} \left| \operatorname{Re}\left\{ \sum_{j=0}^l \mu_j z^{n-j} \right\} + t(\varphi) \right| d\varphi = \int_0^{2\pi} \left| \operatorname{Re}\left\{ z^{n-2l} s_l^2(z) \right\} \right| d\varphi
\]
(62)
\[
= \frac{2}{\pi} \int_0^{2\pi} |s_l(z)|^2 d\varphi = 4 \sum_{j=0}^l |\lambda_j|^2
\]
from which the assertion follows by recalling (??). \( \operatorname{Re}\left\{ z^{n-2l} s_l^2(z) \right\} \) deviates least from zero with respect to \( L_1 \)-norm on \([0, 2\pi]\) among all trigonometric polynomials of the form \( \operatorname{Re}\left\{ \sum_{j=0}^l \mu_j z^{n-j} + t(\varphi) \right\}, \ t \in \Sigma_{n-l-1} \). As it is well known it suffices to show that
\[
\int_0^{2\pi} e^{ik\varphi} \operatorname{sgn} \operatorname{Re}\left\{ e^{i(n-2l)\varphi} s_l^2(e^{i\varphi}) \right\} d\varphi = 0 \ 	ext{for} \ k = 0, \ldots, n-l-1.
\]
(63)
Obviously
\[
\operatorname{sgn} \operatorname{Re}\left\{ e^{i(n-2l)\varphi} s_l^2(e^{i\varphi}) \right\} = \operatorname{sgn} \operatorname{Re}\left\{ e^{i(n-l)\varphi} \frac{s_l(e^{i\varphi})}{s_l^*(e^{i\varphi})} \right\}
\]
\[
= \operatorname{sgn} \cos((n-l)\varphi + \Phi(\varphi)) = \frac{4}{\pi} \operatorname{Re} \arctan e^{i((n-l)\varphi+\Phi(\varphi))}
\]
\[
= \frac{4}{\pi} \operatorname{Re} \arctan z^{n-l} \frac{s_l(z)}{s_l^*(z)} = \frac{4}{\pi} z^{n-l} \frac{s_l(z)}{s_l^*(z)} + O(z^n),
\]
where we used the fact that, \( z = e^{i\varphi} \),
\[
\frac{4}{\pi} \operatorname{Re} \arctan z = \frac{4}{\pi} \arg \frac{i - z}{i + z} = \operatorname{sgn} \cos \varphi;
\]
hence (63) follows and thus the first equality in (62) is proved. Next we observe that
\[
\left| \operatorname{Re}\left\{ z^{n-2l} s_l^2(z) \right\} \right| = |s_l(z)|^2 \left| \operatorname{Re}\left\{ z^{n-l} \frac{s_l(z)}{s_l^*(z)} \right\} \right|
\]
\[
= |s_l(z)|^2 \left| \cos((n-l)\varphi + \Phi(\varphi)) \right| = |s_l(z)|^2 \left| \frac{2}{\pi} + O(z^{2(n-l)}) \right|
\]
where the last equality follows by Fourier-expansion. \( \square \)

References
[1] N. I. Achieser, Sur les propriétés asymptotiques de quelques polynomes, Comptes Rendues Acad. Sci. Paris 191 (1930), 908-910.
[2] N. I. Akhiezer, Theory of Approximation, Ungar, New York, 1956.
[3] N. I. Achieser, Über einige Funktionen, die in gegebenen Intervallen am wenigsten von Null abweichen, Bull. Phys. Math. Kasan, Ser. III 3(1929), 1-69.
[4] N. I. Akhiezer and M. G. Krein, Some Questions in the Theory of Moments, Translations of mathematical Monographs, Vol. 2, Amer. Math. Soc., providence, R.I., 1962.
[5] Yu. M. Berezanskii, A. N. Kolmogorov, M. G. Krein, B. Ya. Levin, B. M. Levitan and V. A. Marchenko, *Naum Il’ich Akhieser (on his seventieth birthday)*, Russ. Math. Surv. 26 (1971), 233-237.

[6] S. N. Bernstein, *Lecons sur les propriétés extremales et la meilleure approximation des fonctions analytiques d’une variable réelle*, Gauthier-Villars, Paris (1926), reprinted Chelsea Publ. Co., New York (1970).

[7] C. W. Clenshaw, *A comparison of “best” polynomial approximation with truncated Chebyshev series expansions*, J. SIAM Numer. Anal. 1 (1964), 26-37.

[8] D. Elliot and B. Lam, *An estimate of $E_n(f)$ for large $n$*, SIAM J. Numer. Anal. 10 (1973), 1091-1102.

[9] E. Egerváry, *Über gewisse Extremumprobleme der Funktionentheorie*, Math. Annalen 99 (1928), 542-561.

[10] L. Fejér, *Über gewisse Minimumprobleme der Funktionentheorie*, Math. Annalen 97 (1926), 104-123.

[11] M. Fekete and J. L. Walsh, *Asymptotic behaviour of restricted extremal polynomials and of their zeros*, Pac. J. Math. 7 (1957), 1037-1064.

[12] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, 96. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[13] M. H. Gutknecht, L. N. Trefethen, *Real polynomial Chebyshev approximation by the Carathéodory-Fejér method*, SIAM J. Numer. Anal. 19 (1982), no. 2, 358-371.

[14] W. Haussmann and K. Zeller, *Approximation of Zolotarev type*, Rocky Mountain J. Math. 19 (1989), 181-187.

[15] A. N. Kolmogorov and A. P. Yushkevich, *Mathematics of the 19th century. Constructive Function Theory, Ordinary Differential Equations, Calculus of Variation, Theory of Finite Differences*, editors Birkhäuser Basel, 1998.

[16] M. G. Krein and B. Ya. Levin, *Naum Il’ich Akhieser (on his sixtieth birthday)*, Russ. Math. Surveys 16 (1961), 129-141.

[17] B. Lam and D. Elliot, *On a conjecture of C. W. Clenshaw*, SIAM J. Numer. Anal. 9 (1972), 44-52.

[18] E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, reprinted by Chelsea Publ. Co., New York, 1946.

[19] W. Markoff, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. 77 (1916), 213-258.

[20] N. N. Meiman, *Polynomials deviating least from zero with an arbitrary number of given coefficients*, Math. Doklady 1 (1960), 72-75

[21] N. N. Meiman, *Solution of the fundamental problems of the theory of polynomials and entire functions least deviating from zero*, Trudy Moskov. Mat. Obšč. 9 (1960), 507-535 (in russian)

[22] N. N. Meiman, *On the theory of polynomials deviating least from zero*, Soviet. Math. Dokl. 1 (1960), 41-44

[23] F. Peherstorfer, *Trigonometric polynomial approximation in the $L^1$-Norm*, Math. Z. 169 (1979), 261-269.

[24] F. Peherstorfer, *Deformation of minimal polynomials and approximation of several intervals by an inverse polynomial mapping*, J. Approx. Theory 111 (2001), 180-195.

[25] F. Peherstorfer, *Inverse images of polynomial mappings and polynomials orthogonal on them*, J. Comput. Appl. Math. 153 (2003), 371-385.

[26] Q.I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Math. Soc. Monographs New series 26, Oxford University Press, 2002

[27] A. R. Reddy, *On certain problems of Chebyshev, Zolotarev, Bernstein and Akhieser*, Invent. Math. 45 (1978), 83-110.

[28] F. Riesz, *Über Potenzreihen mit vorgeschriebenen Anfanggliedern*, Acta Mathematica 42 (1919), 145-171.
[29] W.W. Rogosinski, *Extremum problems for polynomials and trigonometrical polynomials*, Journal London Math. Soc. 29 (1954), 259-275.

[30] Yu. Ryzakov, *An analogue of a problem of E. I. Zolotarev*, Dokl. Akad. Nauk USSR 160 (1965), 552-554.

[31] O. Szász, *Über beschränkte Potenzreihen*, Sitzungsberichte der III. Klasse der Ungarischen Akademie der Wissenschaften, 1926, 503-520 (in hungarian).

[32] O. Szász, *Über die Koeffizienten beschränkte Potenzreihen*, Sitzungsberichte der III. Klasse der Ungarischen Akademie der Wissenschaften, 1926, 488-502 (in hungarian).

[33] O. Szász, *Ungleichungen für die Koeffizienten einer Potenzreihe*, Math. Z. 1 (1918), 163-183.

[34] A. Schönhage, *Approximationstheorie*, Berlin-New York: Walter de Gruyter and Co., 1971.

[35] I. Schur, *Über Potenzreihen, die im Inneren des Einheitskreises beschränkt sind*, Journal für die reine und angewandte Mathematik, 147 (1917), 205-232, 148 (1918), 122-145.

[36] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 3rd ed., 1967.

[37] A. Talbot, *The uniform approximation of polynomials by polynomials of lower degree*, J. Approximation Theory 17 (1976), 254-279.

[38] E. V. Voronovskaja, *The functional method and its applications* Trans. Russ. Math. Monographs 28, Providence, RI: Amer. Math. Soc. 1970.

[39] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Adv. Math. 3 (1969), 127-232.

[40] E.I. Zolotarev, *Applications of elliptic functions to problems of functions deviating least and most from zero*, Zapiski St-Petersburg Akad. Nauk 30 (1877), Oeuvres de E. I. Zolotarev, Volume 2, Izdat. Akad. Nauk SSSR, Leningrad, 1932, pp. 1-59 (in Russian).

[41] S.I. Zuhovichkii, *On the approximation of real functions in the sense of P.L. Čebyšev*, Uspehi Mat. Nauk. (N.S.) 11 (1956), no. 2 (68), 125-159; Amer. Math. Soc. Transl. 19 (2) (1962), 221-252.