INFINITE FAMILIES OF 2-DESIGNS FROM TWO CLASSES OF LINEAR CODES

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ABSTRACT. The interplay between coding theory and t-designs has attracted a lot of attention for both directions. It is well known that the supports of all codewords with a fixed weight in a code may hold a t-design. In this paper, by determining the weight distributions of two classes of linear codes, we derive infinite families of 2-designs from the supports of codewords with a fixed weight in these codes, and explicitly obtain their parameters.

Keywords: Affine-invariant code, cyclic code, exponential sum, linear code, weight distribution, 2-design

1. INTRODUCTION

Throughout this paper, let p be an odd prime and m be a positive integer. Let $\mathbb{F}_q$ denote the finite field with $q = p^m$ elements and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. An $[n,k,d]$ linear code $C$ over $\mathbb{F}_p$ is a $k$-dimensional subspace of $\mathbb{F}_p^n$ with minimum Hamming distance $d$, and is called a cyclic code if each codeword $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C$. Any cyclic code $C$ can be expressed as $C = \langle g(x) \rangle$, where $g(x)$ is monic and has the least degree. The polynomial $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is referred to as the parity-check polynomial of $C$. The code, whose generator polynomial is $x^k h(x^{-1})/h(0)$, is called the dual of $C$ and denoted by $C^\perp$. Note that $C^\perp$ is an $[n,n-k]$ code. Furthermore, we define the extended code $\overline{C}$ of $C$ to be the code

$$\overline{C} = \{(c_0, c_1, \ldots, c_n) \in \mathbb{F}_p^{n+1}: (c_0, c_1, \ldots, c_{n-1}) \in C \text{ with } \sum_{i=0}^n c_i = 0\}.$$

Let $A_i$ be the number of codewords with Hamming weight $i$ in a code $C$. The weight enumerator of $C$ is defined by

$$1 + A_1 z + A_2 z^2 + \ldots + A_n z^n,$$

and the sequence $(1, A_1, \ldots, A_n)$ is called the weight distribution of the code $C$. If the number of nonzero $A_i$'s with $1 \leq i \leq n$ is exactly $w$, then we call
$C$ a \textit{w-weight code}. Let $c = (c_0, c_1, \ldots, c_{n-1})$ be a codeword in the code $C$. The \textit{support} of $c$ is defined by

$$\text{Suppt}(c) = \{0 \leq i \leq n-1 : c_i \neq 0\} \subseteq \{0, 1, \ldots, n-1\}.$$ 

Let $\mathcal{P}$ be a set of $v \geq 1$ elements and $\mathcal{B}$ be a set of $k$-subsets of $\mathcal{P}$, where $k$ is a positive integer with $1 \leq k \leq v$, and the size of $\mathcal{B}$ is denoted by $b$. Let $t$ be a positive integer with $1 \leq t \leq k$. If every $t$-subset of $\mathcal{P}$ is contained in exactly $\lambda$ elements of $\mathcal{B}$, then we call the pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ a $t$-$(v, k, \lambda)$ design, or simply a \textit{t-design}. The elements of $\mathcal{P}$ are called points, and those of $\mathcal{B}$ are referred to as \textit{blocks}. A \textit{t-design} is \textit{simple} when there is no repeated blocks in $\mathcal{B}$. A \textit{t-design} is called \textit{symmetric} if $v = b$ and \textit{trivial} if $k = t$ or $k = v$. Hereafter, we restrict attention to simple \textit{t-designs} with $t < k < v$. When $t \geq 2$ and $\lambda = 1$, we call the $t$-$(v, k, \lambda)$ design a \textit{Steiner system}. With simple counting argument, we have the following identity, which restricts the parameters of a $t$-$(v, k, \lambda)$ design.

$$b \binom{k}{t} = \lambda \binom{v}{t}. \quad (1)$$

Combinatorial \textit{t-designs} have found important applications in coding theory, cryptography, communications and statistics. The interplay between codes and \textit{t-designs} is two-fold: On one hand, a linear code over any \textit{finite field} can be derived from the incidence matrix of a \textit{t-design} and much progress has been made (see for example \cite{1, 10, 21, 22}). On the other hand, linear and nonlinear codes both may hold \textit{t-designs}. For each $i$ with $A_i \neq 0$, let $\mathcal{B}_i$ define the set of the supports of all codewords with weight $i$ in a code $C$, where the coordinates of a codeword are indexed by $(0, 1, 2, \ldots, n-1)$. Let $\mathcal{P} = \{0, 1, \ldots, n-1\}$. Then the pair $(\mathcal{P}, \mathcal{B}_i)$ might be a $t$-$(v, i, \lambda)$ design, where the parameter $\lambda$ can be accordingly determined using (1). In the literature some codes were used to construct 2-designs and 3-designs \cite{1, 6, 16, 17, 21, 22}. Very recently, infinite families of 2-designs and 3-designs were obtained from several different classes of linear codes by Ding and Li \cite{9}, and Ding \cite{8}. Some other constructions of \textit{t-designs} can be found in \cite{2, 3, 19, 20}.

Generally, if $C$ is a cyclic code, the weight of each codeword can be expressed by certain exponential sums so that the weight distribution of $C$ can be determined when the exponential sums could be computed explicitly (see \cite{12, 23} and the references therein). Using this method, Feng and Luo \cite{11} derived the weight distribution of cyclic code $C_1$ with length $n = q - 1$ and parity-check polynomial $h_1(x)h_2(x)h_3(x)$, where $h_1(x), h_2(x)$ and $h_3(x)$ are the minimal polynomials of $\alpha, \alpha^2$ and $\alpha^{p^i+1}$ ($l \geq 1$ and gcd$(m,l) = 1$) over $\mathbb{F}_p$, respectively, and $\alpha$ is a primitive element of $\mathbb{F}_q$. 

The main objective of this paper is to obtain 2-designs from the following two classes of linear codes $C_1^\perp\perp$ and $C_2^\perp\perp$.

(2) $C_1^\perp\perp := \{(\text{Tr}(ax^{p^{\ell}+1} + bx^2 + cx)_{x \in \mathbb{F}_q} + h) : a, b, c \in \mathbb{F}_q, h \in \mathbb{F}_p\}$

and

(3) $C_2^\perp\perp := \{(\text{Tr}(ax^{p^{\ell}+1} + bx)_{x \in \mathbb{F}_q} + h) : a, b \in \mathbb{F}_q, h \in \mathbb{F}_p\}$,

where $\text{Tr}$ denotes the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$, and $C_2$ is the cyclic code with length $n$ and parity-check polynomial $h_1(x)h_3(x)$.

The remainder of this paper is organized as follows. In Section 2, we introduce some notation and preliminary results on exponential sums, cyclotomic fields, affine invariant codes, which will be used in subsequent sections. In Section 3, we determine the weight distributions of two classes of linear codes by explicitly computing certain exponential sums. In Section 4, we then derive infinite families of 2-designs and calculate their parameters from the two classes of codes in Section 3. Section 5 concludes the paper.

2. Preliminaries

In this section, we summarize some standard notation and basic facts on affine-invariant codes, exponential sums and cyclotomic fields.

2.1. Some notation. For convenience, we adopt the following notation unless otherwise stated in this paper.

- $p^* = (-1)^{\frac{p-1}{2}}p$.
- $m, l$ are positive integers with $\gcd(m, l) = 1$, $q = p^m$.
- $\mathbb{P} = \{0, 1, \ldots, n\}$ and $n = p^m - 1$.
- $\zeta_p = e^{2\pi i/p}$ is a primitive $p$-th root of unity, where $i = \sqrt{-1}$.
- $\eta$ and $\eta'$ are the quadratic characters of $\mathbb{F}_q^*$ and $\mathbb{F}_p^*$, respectively. We extend these quadratic characters by setting $\eta(0) = 0$ and $\eta'(0) = 0$.
- $\text{Tr}$ denotes the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$.

2.2. Affine-invariant codes and 2-designs. We begin this subsection by the introduction of affine-invariant codes since the two classes of linear codes we investigate are both affine-invariant and will be proved to hold 2-designs.

The set of coordinate permutations that map a code $C$ to itself forms a group, which is referred to as the permutation automorphism group of $C$.
and denoted by $PAut(C)$. We define the affine group $GA_1(\mathbb{F}_q)$ by the set of all permutations 
$$\sigma_{a,b} : x \mapsto ax + b$$
of $\mathbb{F}_q$, where $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. An affine-invariant code is an extended cyclic code $\overline{C}$ over $\mathbb{F}_p$, such that $GA_1(\mathbb{F}_q) \subseteq PAut(\overline{C})$ [13].

The $p$-adic expansion of each $s \in \mathbb{P}$ is given by
$$s = \sum_{i=0}^{m-1} s_ip^i, \ 0 \leq s_i \leq p-1, \ 0 \leq i \leq m-1.$$For any $r = \sum_{i=0}^{m-1} r_ip^i \in \mathbb{P}$, we say that $r \preceq s$ if $r_i \leq s_i$ for all $0 \leq i \leq m-1$. Clearly, we have $r \preceq s$ if $r \preceq s$.

For any integer $0 \leq j < n$, the $p$-cyclotomic coset of $j$ modulo $n$ is defined by
$$C_j = \{jp^i \pmod{n} : 0 \leq i \leq \ell_j - 1\}$$where $\ell_j$ is the smallest positive integer such that $j \equiv jp^{\ell_j} \pmod{n}$. The smallest integer in $C_j$ is called the coset leader of $C_j$. Let $g(x) = \prod_j \prod_{i \in C_j} (x - \alpha^i)$, where $j$ runs through some coset leaders of the $p$-cyclotomic cosets $C_j$ modulo $n$. The set $T = \bigcup_j C_j$ is referred to as the defining set of $C$, which is the union of these $p$-cyclotomic cosets.

For certain applications, it is important to know whether a given extended primitive cyclic code $C$ is affine-invariant or not. The following lemma given by Kasami, Lin and Peterson [15] answers the question by examining the defining set of the code.

**Lemma 1.** [15] Let $C$ be an extended cyclic code of length $p^m$ over $\mathbb{F}_p$ with defining set $T$. The code $C$ is affine-invariant if and only if whenever $s \in T$ then $r \in T$ for all $r \in \mathbb{P}$ with $r \preceq s$.

**Lemma 2.** [7] The dual of an affine-invariant code $\overline{C}$ over $\mathbb{F}_p$ of length $n+1$ is also affine-invariant.

Affine-invariant codes are very attractive in the sense that they hold 2-designs due to the following theorem.

**Theorem 3.** [7] For each $i$ with $A_i \neq 0$ in an affine-invariant code $\overline{C}$, the supports of the codewords of weight $i$ form a 2-design.

Lemma 2 and Theorem 3 given above are very powerful tools in constructing $t$-designs from linear codes. We will employ them later in this paper. The following theorem given by Ding in [7] reveals the relationship of the codewords with the same support in a linear code $C$, which will be used to calculate the parameters of 2-designs.
Theorem 4. [7] Let $C$ be a linear code over $\mathbb{F}_p$ with minimum weight $d$. Let $w$ be the largest integer with $w \leq n$ satisfying
\[ w - \left\lfloor \frac{w+p-2}{p-1} \right\rfloor < d. \]

Let $c_1$ and $c_2$ be two codewords of weight $i$ with $d \leq i \leq w$ and $\text{Supp}(c_1) = \text{Supp}(c_2)$. Then $c_1 = ac_2$ for some $a \in \mathbb{F}_p^*$.

2.3. Exponential sums. An additive character of $\mathbb{F}_q$ is a nonzero function $\chi$ from $\mathbb{F}_q$ to the set of complex numbers of absolute value 1 such that $\chi(x+y) = \chi(x)\chi(y)$ for any pair $(x, y) \in \mathbb{F}_q^2$. For each $u \in \mathbb{F}_q$, the function $\chi_u(v) = \zeta^{Tr(uv)}$, $v \in \mathbb{F}_q$, denotes an additive character of $\mathbb{F}_q$. Since $\chi_0(v) = 1$ for all $v \in \mathbb{F}_q$, we call $\chi_0$ the trivial additive character of $\mathbb{F}_q$. We call $\chi_1$ the canonical additive character of $\mathbb{F}_q$ and we have $\chi_u(x) = \chi_1(ux)$ for all $u \in \mathbb{F}_q$ [18].

To determine the parameters of codes $C_1^\perp$ and $C_2^\perp$ defined in Eqs. (2) and (3), we introduce the following functions
\[
(4) \quad S(a, b, c) = \sum_{x \in \mathbb{F}_q} \chi(ax^{p^d+1} + bx^2 + cx), \quad a, b, c \in \mathbb{F}_q,
\]
and
\[
(5) \quad S(a, b) = \sum_{x \in \mathbb{F}_q} \chi(ax^{p^d+1} + bx), \quad a, b \in \mathbb{F}_q.
\]

The Gauss sum $G(\eta', \chi_1')$ over $\mathbb{F}_p$ is defined by
\[
G(\eta', \chi_1') = \sum_{v \in \mathbb{F}_p^*} \eta'(v)\chi_1'(v) = \sum_{v \in \mathbb{F}_p} \eta'(v)\chi_1'(v),
\]
where $\chi_1'$ is the canonical additive characters of $\mathbb{F}_p$. The following Lemmas [5][9] are essential to determine the values of Eqs. (4) and (5).

Lemma 5. [13] With the notation above, we have
\[
G(\eta', \chi_1') = \sqrt{(-1)^{(p-1)/2}} \sqrt{p^*} = \sqrt{p^*}.
\]

Lemma 6. [10] For each $y \in \mathbb{F}_p^*$, $\eta(y) = 1$ if $m \geq 2$ is even, and $\eta(y) = \eta'(y)$ if $m \geq 1$ is odd.

Lemma 7. [4] Let $f(x) = a^{p^d}x^{p^{2d}} + ax \in \mathbb{F}_q[x]$, $\gcd(m, l) = 1$ and $b \in \mathbb{F}_q$. There are three cases.

1. If $m$ is odd, then $f(x)$ is a permutation polynomial over $\mathbb{F}_q$ and
\[
S(a, b) = \sqrt{p^*} \eta(a)\chi_1(-ax^{p^d+1}).
\]
(2) If \( m \) is even and \( a^{\frac{q-1}{m+1}} \neq (-1)^{m/2} \), then \( f(x) \) is a permutation polynomial over \( \mathbb{F}_q \) and

\[
S(a, b) = (-1)^{m/2} p^{m/2} \chi_1(-ax^{q+1}) .
\]

(3) If \( m \) is even and \( a^{\frac{q-1}{m+1}} = (-1)^{m/2} \), then \( f(x) \) is not a permutation polynomial over \( \mathbb{F}_q \). We have \( S(a, b) = 0 \) when the equation \( f(x) = -b^{p^j} \) is unsolvable, and

\[
S(a, b) = -(-1)^{m/2} p^{m/2+1} \chi_1(-ax^{q+1})
\]

otherwise. In particular,

\[
S(a, 0) = \begin{cases} 
\sqrt{p^m} \eta(a) & \text{if } m \text{ is odd}, \\
(-1)^{\frac{m}{2}} p^{\frac{m}{2}} & \text{if } m \text{ is even and } a^{\frac{q-1}{m+1}} \neq (-1)^{\frac{m}{2}}, \\
(-1)^{\frac{m}{2}+1} p^{\frac{m}{2}+1} & \text{if } m \text{ is even and } a^{\frac{q-1}{m+1}} = (-1)^{\frac{m}{2}}.
\end{cases}
\]

Notice that \( x^{q+1}_{a,b} \) is a solution to the equation \( f(x) = -b^{p^j} \). Moreover, \( x^{q+1}_{a,b} \) is the unique solution when \( f(x) \) is a permutation polynomial over \( \mathbb{F}_q \).

**Lemma 8.** [5] For \( m \) even and \( \gcd(m, l) = 1 \), the equation \( a^l x^{2l} + ax = 0 \) is solvable for \( x \in \mathbb{F}_q^* \) if and only if

\[
a^{\frac{q-1}{2}} = (-1)^{\frac{m}{2}}.
\]

In such cases there are \( p^2 - 1 \) nonzero solutions.

**Lemma 9.** [11] For \( m \geq 3 \), \( \gcd(m, l) = 1 \), \( \varepsilon = \pm 1 \), \( 0 \leq i \leq 2 \) and \( j \in \mathbb{F}_p^* \), we define

\[
n_{\varepsilon, i, j} = \begin{cases} 
|\{(a, b, c) \in \mathbb{F}_q^3 : S(a, b, c) = \varepsilon^j p^{\frac{m+1}{2}}\}| & \text{if } m - i \text{ is even}, \\
|\{(a, b, c) \in \mathbb{F}_q^3 : S(a, b, c) = \varepsilon^j \sqrt{p^j} p^{\frac{m+1}{2}}\}| & \text{if } m - i \text{ is odd},
\end{cases}
\]

and \( w = |\{(a, b, c) \in \mathbb{F}_q^3 : S(a, b, c) = 0\}| \). Then the value distribution of the multiset \( \{S(a, b, c) : a, b, c \in \mathbb{F}_q^*\} \) is given in Table 5 when \( m \) is odd and in Table 6 when \( m \) is even, respectively (see Tables 5 and 6 in Appendix 1).

For clarity, we denote the multiplicity of the lines \( 1 - 3 \) in Table 5 by \( n_{\pm 1, 0, 0}, n_{1, 0, 1}, n_{1, 1, 0}, n_{1, 1, 1}, n_{1, 0, 1}, n_{1, 0, 1}, n_{1, 1, 0}, n_{1, 1, 1}, n_{1, 0, 1}, n_{1, 0, 1}, n_{1, 2, 0}, n_{0, 1, 2}, n_{1, 2, 1}, n_{1, 2, 1}, n_{1, 2, 1}, n_{1, 2, 1}, \) respectively.
Theorem 11. Let \( m \geq 3 \). The weight distribution of the code \( C_1^{\perp\perp} \) over \( \mathbb{F}_p^{\perp\perp} \) with length \( n + 1 \) and \( \dim(C_1^{\perp\perp}) = 3m + 1 \) is given in Table 1 when \( m \) is odd and in Table 2 when \( m \) is even, respectively.
Example 1. If $(p, m) = (3, 3)$, then the code $C_{1,1}^⊥$ has parameters $[27, 10, 9]$ and weight enumerator $1 + 78z^9 + 1404z^{12} + 14040z^{15} + 27300z^{18} + 15444z^{21} + 702z^{24} + 80z^{27}$, which confirms the results in Theorem 11.

Example 2. If $(p, m) = (3, 4)$, then the code $C_{1,1}^⊥$ has parameters $[81, 13, 36]$ and weight enumerator $1 + 1440z^{36} + 60120z^{45} + 189540z^{48} + 291600z^{51} + 464640z^{54} + 379080z^{57} + 145800z^{60} + 61200z^{63} + 900z^{72} + 2z^{81}$, which confirms the results in Theorem 11.

Theorem 12. Let $m \geq 3$. The weight distribution of the code $C_{1,1}^⊥$ over $\mathbb{F}_p$ with length $n + 1$ and $\dim(C_{1,1}^⊥) = 2m + 1$ is given in Table 3 when $m$ is odd and in Table 4 when $m$ is even, respectively.

One can see that the code is eight-weight if $m$ is odd and ten-weight if $m$ is even. The proof of Theorem 11 is put in Appendix II.

| Weight | Multiplicity |
|--------|--------------|
| $0$    | $1$          |
| $p^{m-1}(p - 1)$ | $p(p^{2m-1} - p^{2m-2} + 2p^{2m-3} - p^{m-2} + 1)(p^m - 1)$ |
| $p^{\frac{m-2}{2}}(p^{\frac{m+2}{2}} - p^\frac{m}{2} - p + 1)$ | $p^{m+2}(p^m - 1)(p^{m-1} - p^{m-2} + p^{\frac{m}{2}} - p^{\frac{m-2}{2}} + 1)$ $2(p^2 - 1)$ |
| $p^{\frac{m}{2}}(p^{\frac{m}{2}} - p^{\frac{m-2}{2}} - p + 1)$ | $rac{1}{2}p^{m-2}(p^m - 1)(p^{\frac{m}{2}} - 1)(p^{\frac{m-2}{2}} + 1)/(p^2 - 1)$ |
| $p^{\frac{m-2}{2}}(p^{\frac{m+2}{2}} - p^{\frac{m}{2}} + 1)$ | $p^{m+2}(p^m - 1)(p^{m-1} - p^{m-2} - p^{\frac{m}{2}} + p^{\frac{m-2}{2}} + 1)$ $2(p^2 - 1)$ |
| $p^{\frac{m}{2}}(p^m - p^{\frac{m-2}{2}} + 1)$ | $p^{m+2}(p^m - 1)(p^{m-1} + p^{\frac{m}{2}} - p^{\frac{m-2}{2}} - 1)$ $2(p^2 - 1)$ |
| $p^{\frac{m-2}{2}}(p^{\frac{m+2}{2}} - p^\frac{m}{2} - 1)$ | $p^{m+2}(p^m - 1)(p^{m-1} - p^{m-2} + p^{\frac{m}{2}} - p^{\frac{m-2}{2}} + 1)$ $2(p^2 - 1)$ |
| $p^{\frac{m}{2}}(p^{\frac{m}{2}} - p^{\frac{m-2}{2}} - 1)$ | $p^{m+2}(p^m - 1)(p^{m-1} - p^{m-2} + p^{\frac{m}{2}} - p^{\frac{m-2}{2}} - 1)$ $2(p^2 - 1)$ |
| $p - 1$ | $p - 1$ |

Table 2. The weight distribution of $C_{1,1}^⊥$ when $m$ is even.
We will prove the conclusion by Lemma 1. The defining set $T = C_1 \cup C_2 \cup C_{p' + 1}$. Since $0 \not\in T$, the defining set $\overline{T}$ of $\overline{C}_1$ is given by $\overline{T} = C_1 \cup C_2 \cup C_{p' + 1} \cup \{0\}$. Let $s \in \overline{T}$ and $r \in \mathcal{P}$. Assume that $r \preceq s$. We need to prove that $r \in \overline{T}$ by Lemma 1.

Example 3. If $(p, m) = (5, 3)$, then the code $\overline{C}_2$ has parameters $[125, 7, 95]$ and weight enumerator $1 + 31000z^{95} + 16120z^{100} + 31000z^{105} + 4z^{125}$, which confirms the results in Theorem 12.

Example 4. If $(p, m) = (3, 4)$, then the code $\overline{C}_2$ has parameters $[81, 9, 45]$ and weight enumerator $1 + 360z^{45} + 4860z^{48} + 4560z^{54} + 9720z^{57} + 180z^{72} + 2z^{81}$, which confirms the results in Theorem 12.

4. Infinite Families of 2-Designs

In the following, we derive 2-designs from the codes presented in Section 3. To this end, we first prove that these codes are both affine-invariant.

Lemma 13. The extended codes $\overline{C}_1$ and $\overline{C}_2$ are affine-invariant.

Proof. We will prove the conclusion by Lemma 1. The defining set $T$ of the cyclic code $C_1$ is $T = C_1 \cup C_2 \cup C_{p' + 1}$. Since $0 \not\in T$, the defining set $\overline{T}$ of $\overline{C}_1$ is given by $\overline{T} = C_1 \cup C_2 \cup C_{p' + 1} \cup \{0\}$. Let $s \in \overline{T}$ and $r \in \mathcal{P}$. Assume that $r \preceq s$. We need to prove that $r \in \overline{T}$ by Lemma 1.
If \( r = 0 \), then obviously \( r \in \overline{\mathcal{T}} \). Consider now the case \( r > 0 \). If \( s \in C_1 \cup C_2 \), then the Hamming weight \( wt(s) = 1 \). Since \( r \leq s \), \( wt(r) = 1 \). Consequently, \( r \in C_1 \cup C_2 \subset \overline{\mathcal{T}} \). If \( s \in C_{p+1} \), then the Hamming weight \( wt(s) = 2 \). Since \( r \leq s \), either \( wt(r) = 1 \) or \( r = s \). In both cases, \( r \in \overline{\mathcal{T}} \). The desired conclusion then follows from Lemma 1.

Similarly, we can prove that \( C_2 \perp \) is affine-invariant.

Thus the proof is completed. \( \square \)

By Lemmas 2 and 13 we know both \( C_1 \perp \) and \( C_2 \perp \) are affine-invariant. Thus we have the following result by Theorems 3.

**Theorem 14.** Let \( m \geq 3 \) be a positive integer. Then the supports of the codewords of weight \( i > 0 \) in \( C_1 \perp \) or \( C_2 \perp \) form a 2-design, provided that \( A_i \neq 0 \).

The parameters of the 2-designs derived from \( C_1 \perp \) and \( C_2 \perp \) are given in Theorems 15, 16, and 17 respectively. We only give the proof of Theorem 15 since Theorems 16 and 17 can be proved with similar arguments.

**Theorem 15.** Let \( m \) be an odd integer and \( \mathcal{B} \) be the set of the supports of the codewords of \( C_1 \perp \) with weight \( i \), where \( A_i \neq 0 \). Then for \( m \geq 5 \), \( C_1 \perp \) holds 2-(\( p^m, i, \lambda \)) designs for the following pairs:

- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2}(p^{m-2} - p^{m-3})1(p^{m-1} - 1)(p^m - p^{m-1} - p^{m-1} - 1)/2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2}(p^{m-1} - 1)(p^m - p^{m-1} - p^{m-1} - p^{m-1} - 1)/2)\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + p^{m-2} + 2)/2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + 2)p^2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + 2)/2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + 2)p^2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + 2)/2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + 2)p^2(p^2 - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, p^{m-2} - p^{m-1} - p^{m-2} - p^{m-2} - p^{m-2} + p^{m-2} + 2)/2(p^2 - 1))\).

For \( m = 3 \), \( C_1 \perp \) also holds 2-(\( p^m, i, \lambda \)) designs for the following pairs:

- \((i, \lambda) = (p^3 - 2p^2, (p - 2)(p - 2)((p^3 - 2p^2 - 1)/2(p^2 - 1)))\).
- \((i, \lambda) = (p^3 - 2p^2 + p, p(p^2 - 1)(p^3 - 2p^2 + p - 1)/2)\).
\[ (i, \lambda) = (p^3 - 2p^2 - p, p(p^2 - p - 1)(p^3 - p^2 - p - 1)(p^5 - p^4 - p + 1)/2(p^2 - 1)). \]

**Proof.** By Theorem 4, one can prove that the number of supports of all codewords with weight \( i \neq 0 \) in the code \( C_1^\perp \) is equal to \( A_i/(p - 1) \) for each \( i \), where \( A_i \) is given in Table 1. Then the desired conclusions follow from Theorem 14 and Eq. 1. The proof is then completed. \( \square \)

**Example 5.** If \( (p, m) = (3, 3) \), then the code \( C_1^\perp \) has parameters \([27, 10, 9]\) and the weight distribution is given in Example 7. It holds \( 2\)-\((27, i, \lambda) \) designs with the following pairs \( (i, \lambda) : \)

\[ (9, 4), (12, 132), (15, 2100), \]

which confirms the results in Theorem 15.

**Theorem 16.** Let \( m \) be an even integer and \( B \) be the set of the supports of the codewords of \( C_1^\perp \) with weight \( i \), where \( A_i \neq 0 \). Then for \( m \geq 6 \), \( C_1^\perp \) holds \( 2\)-\((p^m, i, \lambda) \) designs for the following pairs:

- \( (i, \lambda) = (p^m - p^{m-1}, (p^m - p^{m-1} - 1)(p^{2m-1} - p^{2m-2} + 2p^{2m-3} - p^{m-2} + 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2}, p^{m-1} + p^{m-2}(p - 1))/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)
- \( (i, \lambda) = (p^m - p^{m-1} + p^{m-2} + p^{m-1} + 1)(p^{2m-1} - p^{2m-2} + p^{m-2} - p^{m-3} + 1)/2(p^2 - 1)). \)

Moreover, for \( m = 4 \), \( C_1^\perp \) also holds \( 2\)-\((p^4, i, \lambda) \) designs for the first eight pairs as above except for the last one.
Example 6. If \((p, m) = (3, 4)\), then the code \(C_2^{\perp -}\) has parameters \([81, 13, 36]\) and the weight distribution is given in Example 2. It gives \(2-(81, i, \lambda)\) designs with the following pairs \((i, \lambda)\):

\[
(60, 39825), (54, 102608), (48, 32994), (36, 140),
(57, 93366), (63, 18445), (51, 57375), (45, 9185),
\]
which confirms the results in Theorem 16.

Theorem 17. Let \(m \geq 3\) be an integer and \(B\) be the set of the supports of the codewords of \(C_2^{\perp -}\) with weight \(i\), where \(A_i \neq 0\). Then for \(m \geq 3\) odd, \(C_2^{\perp -}\) gives \(2-(p^m, i, \lambda)\) designs for the following pairs:

- \((i, \lambda) = (p^m - p^{m-1}, (p^m - p^{m-1} - 1)(p^m - 1)).\)
- \((i, \lambda) = (p^m - p^{m-1} + p^{m-1}, 1/2 p^{m-1} (p^{m-1} - p^{m-1} + 1)(p^m - p^{m-1} - p^{m-1} - 1)).\)
- \((i, \lambda) = (p^m - p^{m-1} - p^{m-1}, 1/2 p^{m-1} (p^{m-1} - p^{m-1} - 1)(p^m - p^{m-1} - p^{m-1} - 1)).\)

For \(m \geq 4\) even, it holds \(2-(p^m, i, \lambda)\) designs for the following pairs:

- \((i, \lambda) = (p^m - p^{m-1}, (p^m - p^{m-1} - 1)(p^m - 1)).\)
- \((i, \lambda) = (p^m - p^{m-1} - 1, 1/2 p^{m-1} (p^{m-1} - 1)(p^m - 1)(p^m - 1))\).
- \((i, \lambda) = (p^m - p^{m-1} - 1, 1/2 p^{m-1} (p^{m-1} - 1)(p^m - 1)).\)
- \((i, \lambda) = (p^m - p^{m-1} - 1, 1/2 p^{m-1} (p^{m-1} - 1)(p^m - 1)).\)
- \((i, \lambda) = (p^m - p^{m-1} - 1, 1/2 p^{m-1} (p^{m-1} - 1)(p^m - 1)).\)

Example 7. If \((p, m) = (3, 3)\), then the code \(C_2^{\perp -}\) has parameters \([27, 7, 15]\). It gives \(2-(27, i, \lambda)\) designs with the following pairs \((i, \lambda)\):

\[
(15, 105), (18, 170), (21, 210),
\]
which confirms the results in Theorem 17.

Example 8. If \((p, m) = (3, 4)\), then the code \(C_2^{\perp -}\) has parameters \([81, 9, 45]\) and the weight distribution is given in Example 4. It gives \(2-(81, i, \lambda)\) designs with the following pairs \((i, \lambda)\):

\[
(45, 55), (54, 1007), (48, 846), (57, 2394), (72, 71),
\]
which confirms the results in Theorem 17.
TABLE 5. The value distribution of $S(a,b,c)$ when $m$ is odd

| Value | Multiplicity |
|-------|--------------|
| $\sqrt{p^j p^m}p^{-\frac{m-1}{2}}$ | $\frac{1}{2}p^{m+1}(p^m-p^{m-1}-p^{m-2}+1)(p^m-1)/(p^2-1)$ |
| $\xi_j \sqrt{p^j p^m}p^{-\frac{m-1}{2}}$, for $1 \leq j \leq p-1$ | $\frac{1}{2}p^\frac{m-1}{2}(-\frac{j}{p}) (p^m-p^{m-1}-p^{m-2}+1) \frac{4^m}{p^2-1}$ |
| $-\sqrt{p^j p^m}p^{-\frac{m+1}{2}}$, for $1 \leq j \leq p-1$ | $\frac{1}{2}p^\frac{m+3}{2}(p^{m-1} -\sqrt{p}) (p^m -p^{m-1} -p^{m-2}+1) \frac{4^m}{p^2-1}$ |
| $p^\frac{m+1}{2}$ | $n_{1,0} = \frac{1}{2}p^{m-2}(p-1) (p^m+1)(p^m-1)$ |
| $\zeta_j p^\frac{m+1}{2}$, for $1 \leq j \leq p-1$ | $n_{1,1} = \frac{1}{2}p^{m-2}(p^m-1)(p^{m-1}+1)(p^m-1)$ |
| $\sqrt{p^j p^m}p^{\frac{m+1}{2}}$, for $1 \leq j \leq p-1$ | $\frac{1}{2}p^{m-1} -\frac{1}{2}p^\frac{m-1}{2}(\xi p^\frac{m+3}{2} -\sqrt{p}) (p^m-1) \frac{4^m}{p^2-1}$ |
| $-\sqrt{p^j p^m}p^{\frac{m+1}{2}}$, for $1 \leq j \leq p-1$ | $n_{1,2,1} = \frac{1}{2}p^{m-3}(p^m-1) (p^m+1) \frac{4^m}{p^2-1}$ |
| $0$ | $w = (p^m-1)(p^{2m-1} - p^{2m-2} + p^{2m-3} - p^{m-2}+1)$ |
| $p^m$ | $n_p = 1$ |

5. Concluding Remarks

In this paper, we first determined the weight distributions of two classes of linear codes derived from the duals of extended cyclic codes. Using the properties of affine-invariant codes, we then found that both $C_1^\perp$ and $C_2^\perp$ hold $2$-designs and explicitly determined their parameters. However, for $C_1^\perp$, it seems hard to determine the parameters of the $2$-designs derived from the supports of all codewords with weight $i = p^2(p-1), p(p^2-p-1), p^3, p(p^2-1)$ for $m = 3$, and weight $i = p^2(p^2-1)$ for $m = 4$, respectively. This may constitute a challenge for future work.

APPENDIX I

APPENDIX II

Proof of Theorem 1.1 For each nonzero codeword $\mathbf{c}(a,b,c,h) = (c_0, c_1, \ldots, c_n)$ in $C_1^\perp$, the Hamming weight of $\mathbf{c}(a,b,c,h)$ is

$$w_H(\mathbf{c}(a,b,c,h)) = n + 1 - T(a,b,c,h) = p^n - T(a,b,c,h),$$

where

$$T(a,b,c,h) = |\{x : Tr(ax^{p+1} + bx^2 + cx) + h = 0, x, a, b, c \in \mathbb{F}_q, h \in \mathbb{F}_p\}|.$$
Then

\[
T(a, b, c, h) = \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} \zeta_p^{y(Tr(ax^{\ell+1} + bx^2 + cx) + h)}
\]

\[
= \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y} \sum_{x \in \mathbb{F}_q} \zeta_p^{y Tr(ax^{\ell+1} + bx^2 + cx)}
\]

\[
= p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y} \sigma_y(S(a, b, c)).
\]

By Lemma\[ for \varepsilon = \pm 1, j \in \mathbb{F}_p^* \text{ and } 0 \leq i \leq 2, \text{ we have } \ell = \frac{m+i}{2} \text{ if } m - i \text{ is even, and } \ell = \frac{m+i-1}{2} \text{ if } m - i \text{ is odd, and then}

\[
S(a, b, c) = \{ \varepsilon p^\ell, \varepsilon \sqrt{p} p^\ell, 0, \varepsilon p^\ell \zeta_p^j, \varepsilon \sqrt{p} \zeta_p^j p^\ell, p^m \}.
\]
Hence from Lemma 10 we get

\[ \sigma_y(S(a,b,c)) = \begin{cases} 
0 & \text{if } S(a,b,c) = 0, \\
\varepsilon p^\ell & \text{if } S(a,b,c) = \varepsilon p^\ell, \\
\varepsilon p^\ell \sqrt{p^\ell} \eta'(y) & \text{if } S(a,b,c) = \varepsilon p^\ell \sqrt{p^\ell}, \\
\varepsilon p^\ell \xi_p^j & \text{if } S(a,b,c) = \varepsilon p^\ell \xi_p^j, \\
\varepsilon p^\ell \sqrt{p^\ell} \eta'(y) \xi_p^j & \text{if } S(a,b,c) = \varepsilon \sqrt{p^\ell} p^\ell \xi_p^j, \\
p^m & \text{if } S(a,b,c) = p^m. 
\end{cases} \]

That is,

\[ T(a,b,c,h) = \begin{cases} 
p^{m-1} & \text{if } S(a,b,c) = 0, \\
p^{m-1} + \varepsilon p^{\ell-1}(p-1) & \text{if } S(a,b,c) = \varepsilon p^\ell \text{ and } h = 0, \\
p^{m-1} + \varepsilon p^{\ell-1}(p-1) & \text{if } S(a,b,c) = \varepsilon p^\ell \text{ and } h \neq 0, \\
p^{m-1} + \varepsilon p^{\ell-1} \sqrt{p^\ell} \eta'(y) G(\eta', \chi'_1) & \text{if } S(a,b,c) = \varepsilon \sqrt{p^\ell} p^\ell, \\
p^{m-1} + \varepsilon p^{\ell-1}(p-1) & \text{if } S(a,b,c) = \varepsilon p^\ell \text{ and } h + j = 0, \\
p^{m-1} + \varepsilon p^{\ell-1}(p-1) & \text{if } S(a,b,c) = \varepsilon p^\ell \text{ and } h + j \neq 0, \\
p^{m-1} + \varepsilon p^{\ell-1} \sqrt{p^\ell} \eta(h+j) G(\eta', \chi'_1) & \text{if } S(a,b,c) = \varepsilon \sqrt{p^\ell} p^\ell \xi_p^j, \\
p^m & \text{if } S(a,b,c) = p^m \text{ and } h = 0, \\
0 & \text{if } S(a,b,c) = p^m \text{ and } h \neq 0. 
\end{cases} \]

Obviously, when \( m \) is odd, by Lemmas 5, 6, 9 and Eq. 6 we have

\[ w_1 = p^m - p^{m-1}, \]
\[ A_{w_1} = pw + 2n_{1,0,0} + 2n_{1,2,0} + (p-1)[(n_{1,0,1}+n_{1,1,0})+(n_{1,2,1}+n_{1,1,1})], \]
\[ w_2 = p^m - [p^{m-1} + p^{m-1}(p-1)], \]
\[ A_{w_2} = n_{1,0,1} + (p-1)n_{1,1,1}, \]
\[ w_3 = p^m - [p^{m-1} - p^{m-1}(p-1)], \]
\[ A_{w_3} = n_{1,1,0} + (p-1)n_{1,1,1}, \]
\[ w_4 = p^m - (p^{m-1} - p^{m-1}), \]
\[ A_{w_4} = (p-1)n_{1,1,0} + (p-1)^2 n_{1,1,1} + \frac{p-1}{2} (n_{1,0,0} + n_{1,0,1}) + \frac{p-1}{2} (p-1)(n_{1,0,1} + n_{1,0,0}), \]
\[ w_5 = p^m - (p^{m-1} + p^{m-1}), \]
\[ A_{w_5} = (p-1)n_{1,1,0} + (p-1)^2 n_{1,1,1} + 2 \cdot \frac{p-1}{2} n_{1,0,0} + 2 \cdot \frac{(p-1)^2}{2} n_{1,0,1}, \]
\[ w_6 = p^m - (p^{m-1} - p^{m-1}), \]
\[ A_{w_6} = 2 \cdot \frac{p-1}{2} n_{1,2,0} + 2 \cdot \frac{(p-1)^2}{2} n_{1,2,1}, \]
\[ w_7 = p^m - (p^{m-1} + p^{m-1}), \]
\[ A_{w_7} = A_{w_6}, \]
\[ A_{w_8} = p - 1. \]

When \( m \) is even, by Lemmas 5, 6, 9 and Eq. 6 we get

\[ w_1 = p^m - p^{m-1}, \]
\[ A_{w_1} = pw + 2n_{1,1,0} + 2(p-1)n_{1,1,1}, \]
\[ w_2 = p^m - [p^{m-1} + p^{m-1}(p-1)], \]
\[ A_{w_2} = n_{1,0,0} + (p-1)n_{1,0,1}, \]
\[ w_3 = p^m - [p^{m-1} + p^{m-1}(p-1)], \]
\[ A_{w_3} = n_{1,2,0} + (p-1)n_{1,2,1}, \]
\[ w_4 = p^m - (p^{m-1} - p^{m-1}(p-1)), \]
\[ A_{w_4} = n_{1,0,0} + (p-1)n_{1,0,1}, \]
\[ w_5 = p^m - [p^{m-1} - p^{m-1}(p-1)], \]
\[ A_{w_5} = n_{1,2,0} + (p-1)n_{1,2,1}, \]
\[ w_6 = p^m - [p^{m-1} - p^{m-1}(p-1)], \]
\[ A_{w_6} = (p-1)n_{1,0,0} + (p-1)^2n_{1,0,1}, \]
\[ w_7 = p^m - (p^{m-1} - p^{m-1}), \]
\[ A_{w_7} = (p-1)n_{1,2,0} + (p-1)^2n_{1,2,1} + 2 \cdot p^{m-1}n_{1,1,0} + 2 \cdot \frac{(p-1)^2}{2}n_{1,1,1}, \]
\[ w_8 = p^m - (p^{m-1} + p^{m-1}), \]
\[ A_{w_8} = (p-1)n_{1,0,0} + (p-1)^2n_{1,0,1}, \]
\[ w_9 = p^m - (p^{m-1} + p^{m-1}), \]
\[ A_{w_9} = (p-1)n_{1,2,0} + (p-1)^2n_{1,2,1} + 2 \cdot p^{m-1}n_{1,1,0} + 2 \cdot \frac{(p-1)^2}{2}n_{1,1,1}, \]
\[ A_{p^m} = p-1. \]

Thus we complete the proof of Theorem 11.

**Proof of Theorem [12]** For each nonzero codeword \( c(a, b, h) = (c_0, \ldots, c_n) \) in \( \mathbb{C}_2^{m-1} \), the Hamming weight of \( c(a, b, h) \) is

\[
\text{w}_H(c(a, b, h)) = p^m - T(a, b, h),
\]

where

\[
T(a, b, h) = |\{x : \text{Tr}(ax^{p+1} + bx) + h = 0, x, a, b \in \mathbb{F}_q, h \in \mathbb{F}_p\}|.
\]

Then

\[
T(a, b, h) = \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} c_{\text{Tr}}(ax^{p+1}+bx)+hy
\]

\[
= p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} c_{\text{Tr}}(ax^{p+1}+bx).
\]

If \( a = b = h = 0 \), then \( c(a, b, h) \) is the zero codeword.

If \( a = b = 0, h \neq 0 \), then \( T(a, b, h) = p^{m-1} + p^{m-1} \sum_{y \in \mathbb{F}_p} c_{hy} = 0. \)

If \( a = 0, b \neq 0 \), then \( T(a, b, h) = p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} c_{\text{Tr}}(bx) = p^{m-1}. \)
If $a \neq 0$, then
\[
T(a, b, h) = p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} r_{\mathbb{F}_p}^h \sigma_y(S(a, b))
\]
\[
= p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} r_{\mathbb{F}_p}^h \sigma_y(S(a, b))
\]
\[
= p^{m-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} r_{\mathbb{F}_p}^h S(ay, by).
\]

When $m$ is odd, by Lemmas 5-7 we have
\[
T(a, b, h) = \begin{cases} 
p^{m-1} & \text{if } h = Tr(ax_{a,b}^{p+1}), \\
p^{m-1} + p^{\frac{m-1}{2}}(-1)^{(p-1)(m+1)}\eta(1)\eta'(h-Tr(ax_{a,b}^{p+1})) & \text{if } h \neq Tr(ax_{a,b}^{p+1}).
\end{cases}
\]

Obviously, for $a \in \mathbb{F}_q^*$ we get that $T(a, b, h) = p^{m-1}$ appears $p^{m}(p^m-1)$ times, both $T(a, b, h) = p^{m-1} + p^{\frac{m-1}{2}}$ and $T(a, b, h) = p^{m-1} - p^{\frac{m-1}{2}}$ appear $\frac{p^{m}(p^m-1)(p^m-1)}{2}$ times, respectively.

When $m$ is even and $a^{\frac{q-1}{p+1}} \neq (-1)^\frac{m}{2}$, by Lemma 7 we have
\[
T(a, b, h) = \begin{cases} 
p^{m-1} + (-1)^\frac{m}{2}p^{\frac{m-1}{2}}(p-1) & \text{if } h = Tr(ax_{a,b}^{p+1}), \\
p^{m-1} - (-1)^\frac{m}{2}p^{\frac{m-1}{2}} & \text{if } h \neq Tr(ax_{a,b}^{p+1}).
\end{cases}
\]

Clearly, there exist $p^m - 1 - p^{\frac{m-1}{2}} = \frac{p(p^m-1)}{p+1}$ elements $a \in \mathbb{F}_q^*$ such that $a^{\frac{q-1}{p+1}} \neq (-1)^\frac{m}{2}$. Then
\[
T(a, b, h) = p^{m-1} + (-1)^\frac{m}{2}p^{\frac{m-1}{2}}(p-1)
\]
appears $\frac{p^{m+1}(p^m-1)}{p+1}$ times, and
\[
T(a, b, h) = p^{m-1} - (-1)^\frac{m}{2}p^{\frac{m-1}{2}}
\]
appears $\frac{p^{m+1}(p^m-1)}{p+1}$ times.

When $m$ is even and $a^{\frac{q-1}{p+1}} = (-1)^\frac{m}{2}$, from Lemma 7 we get
\[
T(a, b, h) = \begin{cases} 
p^{m-1} - (-1)^\frac{m}{2}p^{\frac{m-1}{2}}(p-1) & \text{if } f(x) = -b^p \text{ is solvable and } h = Tr(ax_{a,b}^{p+1}), \\
p^{m-1} + (-1)^\frac{m}{2}p^{\frac{m-1}{2}} & \text{if } f(x) = -b^p \text{ is solvable, } b \neq 0 \text{ and } h \neq Tr(ax_{a,b}^{p+1}), \text{ or } b = 0 \text{ and } h \neq 0, \\
p^{m-1} & \text{if } f(x) = -b^p \text{ is no solvable.}
\end{cases}
\]
By Lemma 8, there are $\frac{q-1}{p+1}$ elements $a \in \mathbb{F}_q^*$ such that $a\frac{q-1}{p+1} = (-1)^{\frac{m}{2}}$, and $p^{m-2}$ elements $b \in \mathbb{F}_q$ such that $f(x) = -b^p$ is solvable. Therefore,

$$T(a, b, h) = p^{m-1} + (-1)^{\frac{m}{2}+1}p^m(p-1)$$

appears $\frac{p^{m-2}(p^{m-1}-1)}{p+1}$ times,

$$T(a, b, h) = p^{m-1} + (-1)^{\frac{m}{2}+1}p^m$$

appears $\frac{p^{m-2}(p^{m-1}-1)(p-1)}{p+1}$ times, and $T(a, b, h) = p^{m-1}$ appears $(p^m-1)p^{m-1}(p-1)$ times.

By all the discussions above, the proof is completed. \[\square\]

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