Discrete and continuous description of physical phenomena

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Abstract.

The values \( z \) of many observable quantities in the Nature are determined at some discrete set of times \( t_n \), separated by a small interval \( \Delta t \) (which may also represent a coordinate, etc.). Let the \( z \) value in neighbour point \( t_{n+1} = t_n + \Delta t \) be expressed by the evolution equation as

\[
 z(t_{n+1}) \equiv z(t_n + \Delta t) = f(z(t_n)).
\]

This equation gives a discrete description of phenomena. Considering phenomena at \( t \gg \Delta t \) this equation is transformed often into the differential equation allowing to determine \( z(t) \) leading to continuous description. It is usually assumed that the continuous description describes correctly the main features of a phenomenon at values \( t \gg \Delta t \).

In this paper I show that the true behavior of some physical systems can differ strongly from that given by the continuous description. The observation of such effects may lead to the desire to supplement the original evolutionary model by additional mechanisms, the origin of which require special explanation. We will show that such construction may not be necessary — simple evolution model can describe properly different observable effects.

This text contains no new calculations. Most of the discussed facts are well known. New is the treatment of the results.

Let the phenomenon described by quantity \( z_n \equiv z(t_n) \) be determined at some discrete set of times \( t_n \) with steps \( \Delta t \) and \( z(t_{n+1}) = z(t_n + \Delta t) \) can be expressed by the evolution equation

\[
 z(t_{n+1}) \equiv z(t_n + \Delta t) = f(z(t_n)). \tag{1}
\]

This is a discrete description of the phenomenon. This equation is transformed often into the differential equation for \( z(t) \leftarrow z_n \), giving the continuous description of the phenomenon [1]:

\[
 \frac{z(t_{n+1}) - z(t_n)}{\Delta t} \Rightarrow \frac{dz(t)}{dt} = \frac{f(z(t_n)) - z_n}{\Delta t} \equiv f_1(z(t)). \tag{2}
\]

It is usually assumed that the latter equation correctly describes the main features of phenomenon at time \( t \gg \Delta t \).

In this report I show that in many cases the transition to the continuous description results in an incorrect description of reality, described by evolution equation (1). Vice versa, the interpretation of some observations, based on the "standard" continuous description, can result in an incorrect description of elementary mechanisms.
1. Model

For definiteness, we consider the livestock of carps $z_n$ in a pond without pikes ($n$ is the number of year). In this example time spacing is limited from below, $\Delta t = 1$ year (time between spawning). The simplest model consider only natural growth, $z_{n+1} = k z_n$ ($k$ is called reproduction factor). At $k > 1$ this equation describes unlimited growth $z_n = k^n z_0$.

In reality this growth is stopped by competition for food and other effects (pressure of population). We describe this pressure by adding in the right hand of the simplest model the term $-\ell z_n^2$. As a result, our physical model takes the form of the evolution equation $z_{n+1} = k z_n - \ell z_n^2$. The change of variables $x_n = (\ell/k) z_n$ transforms this equation in the more convenient form

$$x_{n+1} = f_1(x_n), \quad f_1(x) = k x (1 - x), \quad x_n \geq 0. \quad (3)$$

We are interested situation when the lifetime of population is not limited, the evolution starts from initial condition $x_0 > 0$ and $k > 1$.

Note that our problem has two equilibrium solutions,

$$x_\infty = f(x_\infty) \Rightarrow \begin{cases} x_{(\infty,1)} = 0 \quad (\text{stable at } k < 1), \\ x_{(\infty,2)} = \frac{k - 1}{k} \quad (\text{stable at } k > 1) \end{cases} \quad (4)$$

2. Continuous description. Differential equation

Ordinarily to describe phenomena at the scales larger than $\Delta t$ (in our case $\Delta t$ is 1 year) the above difference equation is conventionally reduced to the differential equation [1]. To this end, the continuous variable – time $t$ is introduced (in the initial evolution equation the variable $t$ has only integer values $n$). Following (2), at $t \gg 1$ our equation (3) is transformed to continuous limit with the aid of standard sequence of relations:

$$\frac{x_{n+1} - x_n}{\Delta t} = x_n (k - 1 - k x_n) \Rightarrow \frac{dx}{dt} = x (k - 1 - k x), \quad x(0) = x_0. \quad (5)$$

This differential equation is solved easily,

$$x = \frac{(k - 1) x_0}{k x_0 + (k - 1 - k x_0) e^{-(k-1)t}}. \quad (6)$$

This equation describes the evolution up to infinite time for an arbitrary value of $k$ and for an arbitrary initial value $x(0) > 0$. At $t \to \infty$ and $k > 1$ the quantity $x$ tends monotonically to the limiting value $x_{(\infty,2)} = (k - 1)/k$ (4), determined for the incident (discrete) evolution equation.

3. Real evolution. Discrete description

Below we reproduce some known results of the study of evolution model (3) (see e.g [2]). Our task is to present some interpretations, which are not discussed usually.

In contrast to the continuous description the equation (3) describes evolution of system during infinite time only for $0 < x_n < 1$, in particular for

$$0 < x_0 < 1. \quad (7)$$

Indeed, if $x_n > 1$, the process is stopped since eq. (3) gives senseless $x_{n+1} < 0$.

The details of picture are different for different values of the reproduction factor $k$. 

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3.1. The region \( k < 4 \).

Below we consider the asymptotical solutions of eq. (3) at large time \( n \) and their stability.

- The case \( k < 3 \).

  For the study of the asymptotical solution of eq. (3) we substitute in this equation the expression \( x_n = x_\infty + \delta_n \equiv (k - 1)/k + \delta_n \) and, considering \( \delta_n \) to be small at large \( n \), linearize this equation (neglecting terms \( \propto \delta_n^2 \)). It results in the equation for \( x_\infty \) and the recurrence relation for \( \delta_n \),

\[
x_\infty = kx_\infty(1 - x_\infty), \quad \delta_{n+1} = (2 - k)\delta_n.
\]

The equation for \( x_\infty \) has two solutions (4).

A. \( k < 1 \).

At \( k < 1 \) the trivial solution \( x_\infty \equiv x_{(\infty,1)} = 0 \) is realized (by the population going extinct).

B. \( k > 1 \).

At \( k > 1 \) nontrivial solution \( x_\infty \equiv x_{(\infty,2)} = (k - 1)/k \) is realized. The relation for \( \delta_n \) shows that at \( |2 - k| < 1 \) (i.e. at \( 1 < k < 3 \)) the quantity \( \delta_n \) tends to 0 at \( n \to \infty \), and the solution converges to \( x_\infty \equiv x_{(\infty,2)} \):

\[ \diamond \] At \( 1 < k < 2 \) the quantity \( x_n \) tends to grow monotonically to \( x_\infty \) for larger \( n \) just as the solution of differential equation (6). Continuous description works for initial values (7).

\[ \diamond \] At \( 2 < k < 3 \) the quantity \( x_n \) also tends to \( x_\infty \) but in contrast with the solution of the differential equation, this increase is not monotonic but (damped) oscillating. The continuous description describes the main features of evolution (limiting value) but not important details.

- The case \( 3 < k < 4.5 \). Doubling of period.

According to eq. (8) at \( k > 3 \) we have \(|\delta_{n+1}| > |\delta_n|\). Therefore, the asymptotic solution (4) becomes unstable. The point \( k = 3 \) is a branching point. What next?

Let us check "the simplest" hypothesis that at \( n \to \infty \) the number of carps is reproduced through two years (not yearly), with different populations in even and odd years. In other words, we suggest that \( k = 3 \) is the point of the doubling of the period.

The iteration of eq. (3) gives the equation

\[
x_{n+2} = f_2(x_n), \quad f_2(y) = f_1(f_1(y)) \equiv k^2y(1 - y)[1 - ky(1 - y)].
\]

The possible stationary states are given by equation of 4-th order \( x_\infty - f_2(x_\infty) = 0 \). Two solutions of this equation are also known solutions (4) of equation \( x_\infty - f_1(x_\infty) = 0 \). To find other solutions we consider equation \( [x_\infty - f_2(x_\infty)]/[x_\infty - f_1(x_\infty)] = 0 \) which has the form \( k^2y^2 - k(k + 1)y + k + 1 = 0 \). Its solution exists only at \( k > 3 \) (we introduced new asymptotical values \( x_{(\infty,3)} \) and \( x_{(\infty,4)} \) so that \( x_{(\infty,3)} > x_{(\infty,4)} \)):

\[
y = x_{(\infty,3,4)} \equiv k + 1 \pm \sqrt{(k + 1)(k - 3)} \, 2k.
\]

It is easy to check that

\[
f_1(x_{(\infty,3,4)}) = x_{(\infty,4,3)}, \quad x_{(\infty,3)} > x_{(\infty,2)} = (k - 1)/k > x_{(\infty,4)}.
\]

Therefore, at large \( n \) the quantity \( x_n \) oscillates between the values \( x_{(\infty,3)} \) and \( x_{(\infty,4)} \) with a period of 2 years. This phenomenon is called the period doubling.

Now we find the range of validity of this solution. For this goal we repeat above calculations with substitution \( x_n = x_{(\infty,3,4)} + \delta_n \) in Eq. (9). In this case the linearized recurrence relation for \( \delta_n \) has the form

\[
\delta_{n+2} = (4 + 2k - k^2) \delta_n.
\]
Just as above we find that the solutions \( x_{(\infty,3,4)} \) (10) are stable only at \( |4 + 2k - k^2| < 1 \), i.e. for \( 3 < k < 1 + \sqrt{6} \approx 3.45 \).

- The case \( 3.45 < k < 4 \) and general picture at \( k < 4 \).

  With the growth of \( k \) the analytical study of the problem becomes complex. The Fig. 1 (obtained with "Mathematica" package) represents limiting values \( x_\infty \) at different \( k \). Here in the abscissa the reproduction coefficient \( k \) is shown and in the ordinate values \( x_\infty \). At \( 1 < k < 3 \) population tends to the single limiting value given by eq. (4). At \( 3 < k < 3.45 \) the population oscillates (with period 2 year – the period doubling). At \( 3.45 < k < 3.54 \) we have a subsequent doubling of the period; the population varies periodically between four values. With the growth of \( k \) the number of limiting values grows (new period doublings replace each other). At \( k > 3.569 \) zones of chaotic behavior appears. With the growth to \( k = 4 \) these zones spread to entire interval (0, 1).

3.2. The case \( k = 4 \). Chaotic behaviour.

At \( k = 4 \) evolution equation (3) can be solved exactly. We seek a solution in the form

\[
x_n = \frac{1 - \cos 2\pi \alpha_n}{2}.
\]

Now eq. (3) gives a simple sequence of identities:

\[
x_{n+1} = 4 \frac{1 + \cos 2\pi \alpha_n}{2} \frac{1 - \cos 2\pi \alpha_n}{2} = \sin^2(2\pi \alpha_n) = \frac{1 - \cos 4\pi \alpha_n}{2}.
\]

Comparing with (12), we find \( \alpha_{n+1} = 2\alpha_n \Rightarrow \alpha_n = 2^n \alpha_0 \). The adding of any integer number to \( \alpha_n \) does not change \( x_n \). Therefore, in this solution only \( \{\alpha_n\} \) – fractional part of \( \alpha_n \) – makes sense, and

\[
\alpha_n = \left\{ 2^n \alpha_0 \right\}.
\]

Now we discuss the meaning of this solution from different perspectives.

- Let us consider two initial values \( x_0(\alpha_0) \) and \( x'_0(\alpha'_0) \) with \( |\alpha_0 - \alpha'_0| = \varepsilon \ll 1 \) and \( 2^{-r} > \varepsilon > 2^{-(r+1)} \). The "acceptable" solution should be stable, it means that it is naturally to expect that the solutions with initial values \( x_0 \) and \( x'_0 \) are close to each other even after long time. On the contrary, in our case at \( n > r \) (after \( r \) years) the difference between \( x_n \) and \( x'_n \) becomes unpredictably large.

- The variable \( z_n \) is the number of carps averaged over some period. It varies weakly due to natural and accidental deaths. Therefore, the initial value \( z_0 \) (or \( x_0 \) or \( \alpha_0 \) ) is determined with some uncertainty \( \varepsilon \) with \( 2^{-r} > \varepsilon > 2^{-(r+1)} \). Let us present this initial state in the binary-decimal form like \( \alpha_0 = 0.110100110... \) Here the \( r \)-th term is known while the \( r + 1 \)-th term is unknown. Solution (13) means that in each next year \( \alpha_{n+1} \) is obtained from \( \alpha_n \) by shift of point from one digit to the right with elimination of signs before the point.
Therefore, the population of carps is predicted via an initial state (by means of solution (13)) during first \( r \) years (with downward accuracy) but the population in \( r + 1 \)-th year is unpredictable.

- The typical initial value \( a_0 \) is irrational. By definition of irrational, the numbers – consecutive values \( \{a_n\} \) are non correlated. In other words, for almost all randomly chosen \( x_0 \) (or \( a_0 \)) the range of produced values \( \{a_n\} \) is distributed uniformly in the segment \([0, 1] \).

Therefore, at \( k = 4 \) the limiting value \( x_n \to \infty \) does not exist. The values \( x_n \) vary with time unpredictably. Let us underline that even the averaged value \( \langle x_n \rangle \) differs strongly from the value \( 3/4 \) predicted by eq. (5)\(^1\).

3.3. The case \( k > 4 \).
At \( 0 < x_n < 1 \) the maximal value of the quantity \( x_n(1 - x_n) \) is \( 1/4 \). Therefore, to prevent the restraining of process at some stage, it should be \( k \leq 4 \). We expected that the mean time of life of population at \( k > 4 \) is small, but at small values of \( k - 4 \) it is not very small. The numerical experiment shows infinitesimal time of life even for \( k - 4 = 0.01 \). (Certainly, for some rational values \( x(0) \) time of life can be large, but the rounding errors kill this opportunity).

3.4. General picture
In two tables below we compare some features of discrete and continuous description of our system in the regions of parameters allowing evolution during infinite time.

| Table 1. Range of parameters and initial values |
|-----------------------------------------------|
| **discrete description** | **continuous description** |
| range of parameter \( k \) of equation | restricted, \( k \leq 4 \) | unlimited |
| range of initial values \( x(0) \) | restricted, \( x(0) < 1 \) | infinite |

| Table 2. Behaviour at large time, \( t \gg 1 \) |
|-----------------------------------------------|
| \( k \) | **discrete description** | **continuous description** |
| \( k < 1 \) | monotonic approaching to \( x_{\infty,1} = 0 \) | monotonic |
| \( 1 < k < 2 \) | monotonic approaching to \( x_{\infty,2} = 0 \) | monotonic |
| \( 2 < k < 3 \) | non-monotonic approaching to \( x_{\infty,2} = 0 \) | monotonic |
| \( 3 < k < 3.45 \) | 2 year oscillations between 2 values | monotonic |
| \( 3.45 < k < 3.569 \) | \( 2^n \) year oscillations between \( 2^n \) values | monotonic |
| \( 3.569 < k < 4 \) | unpredictable, chaotic behaviour appears | regular evolution |
| \( k = 4 \) | unpredictable, chaotic behaviour appears | regular evolution |
| \( k > 4 \) | population is dying out quickly | regular evolution |

4. Discussion
Most of presented facts are well known [2]. We discuss lessons from this picture for two fields of studies\(^2\).

\(^1\) At \( k = 4 \) the probability \( p(y)dy \) that the quantity \( x_n \) is within interval \( y \), \( y + dy \) is calculated with the aid of eq. (12). For almost all initial values \( x_0 \) we have \( p(y) = [\pi y(1 - y)]^{-1/2} \). For these \( x_0 \) mean value \( \langle x_n \rangle = 1/2 \) and fluctuation \( \langle \Delta x_n \rangle \sim \langle x_n \rangle \).

\(^2\) The discussed discrete approach results in many new physical phenomena. Besides, the considered phenomena produce some difficulties in the computational algorithms at the transition from continuous description to the discrete
4.1. Lessons for using continuous approach for description of a real system

The standard approach in the study of physical phenomena is to change discrete evolution equation to the continuous one and subsequent use of differential equation for the description of long-time evolution of system. Our discussion demonstrates that this approach is valid only for some relatively narrow region of parameters.

- The approximation of discrete variation of some physical quantity \( x_{n+1} = f(x_n) \) by continuous one is justified only at relatively low temp of variation of this quantity \( k = (df/dx)_{x=0} \) in the basic difference equation. With growth of this temp \( k \) the behavior of such system can be changed strong.

- There are the threshold values \( k_0 \) of parameter \( k \). The differential equation describes evolution of physical quantity \( x \) at \( k < k_0 \) while at \( k > k_0 \) such description become incorrect. The quantity \( k_0 \) is different for different physical properties. In our example at \( k < k_{01} = 2 \) continuous approach describes real picture well. At \( k_{01} < k < k_{02} = 3 \) continuous approach gives good description of main picture with incorrect description of important details (it skips non-monotonic variation of \( x(t) \)).

- At \( k > k_{02} = 3 \) the continuous approach absolutely inapplicable for description of the system during long time.

Certainly, these \( k_0 \) are different in different physical problems.

For the validity of continuous approach in the multi-dimensional problems the eigenvalues of reproduction coefficient matrix \( \{k\} \) should be small enough (this limitation can be broken even at small values of separate elements of matrix \( \{k\} \) due to large number of its component.)

4.2. Lessons for construction of model for physical phenomenon.

Next problem is the construction of model for physical phenomenon, based on the preliminary understanding of the process and observations. Typical procedure is construction of evolution equation, its description in the continuous approach, and obtaining of coefficients from observation. If observed behaviour differs from that predicted in this approach not very strongly, we conclude that our model is correct in general, but some additional mechanisms should be taken into account for description of details. If we observe new phenomena, which don’t appear in our continuous approach, we conclude that our incident understanding of problem was incorrect, and new mechanisms are responsible for this phenomenon.

Our analysis shows that such approach can be often incorrect. Simple model describes phenomenon, and only continuous approach can appear wrong.

4.3. Examples

- Let population dies out in finite time. In the traditional approach it would be natural to seek an explanation in the existence of some new force which kills population. We understand that in the discrete approach it will be due to inappropriate initial conditions or parameters (e. g. \( x_n > 1 \) or \( k > 4 \)) with the same simple evolution equation.

- Let we observe periodic behavior. In the traditional approach it would be natural to seek an explanation in the existence of additional mechanism providing periodicity (periodic external force or resilient restoring force). Our example for \( 3 < k < 3,569 \) shows that even complex periodic behavior can be explained without these additional mechanisms.

One. These problems have huge literature, which is beyond our scope here.
Let we observe chaotic behavior. In the traditional approach it would be natural to seek an explanation in the existence of additional mechanisms, like random external force (open system). Our example for \( 4 > k > 3.569 \) shows that even complex chaotic behaviour can be explained without these additional mechanisms.

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References
[1] See e.g. Tricomi F G 1961 Differential equations (BLACKIE & SON LIMITED)
[2] See e.g. Feigenbaum M. J. 1978 J. Stat. Phys. (1978) 19, p. 25; 1979 J. Stat. Phys. (1979) 21, p. 669; 1979 Lecture Notes in Physics (1979) 93, p. 163; 1980 Comm. Math. Phys. (1980) 77, p. 65; 1980 Los Alamos Science (1980) 1 No. 1, pp. 4-27; Chirikov B V 1979 Phys. Rept. (1979) 52, p. 263