How to fairly share a watermelon

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Geometry, calculus and in particular integrals, are too often seen by young students as technical tools with no link to the reality. This fact generates into the students a loss of interest with a consequent removal of motivation in the study of such topics and more widely in pursuing scientific curricula [1–5]. With this note we put to the fore a simple example of practical interest where the above concepts prove central; our aim is thus to motivate students and to inverse the dropout trend by proposing an introduction to the theory starting from practical applications [6, 7]. More precisely, we will show how using a mixture of geometry, calculus and integrals one can easily share a watermelon into regular slices with equal volume.

I. INTRODUCTION TO THE MAIN QUESTION

What better than a fresh watermelon to fight the heat wave in these sunny and hot summer days? In Italy it is very common to buy watermelons weighting as much as 15 Kg for few euros. Sharing the watermelon is hence an opportunity for a party with many friends. A recurring question is then: how can one slice the watermelon into equal volume parts to have a fair sharing among friends? This question is particularly interesting if the slices are not cut along the longitudinal direction, namely the longer one. Cuts along the transversal direction are in fact often performed to return slices which can be more straightforwardly manipulated. In the following we will show that symmetry helps: for the first cuts up to a certain number of friends, one can indeed proceed by dividing each portion into two identical parts. To go further, when the number of guests is large, we propose a rule of the thumb which follows a simple mathematical analysis: this will be referred as the “2/3 rule”.

A) Longitudinal view

B) Lateral view

C) Top view

FIG. 1: Watermelon. Different views of a watermelon (panels A, B and C). Bottom right: a (spheroid) ellipsoid with equal median and minor semi axes, $b$, and major semi axis, $a > b$ whose cartesian equation is given by $x^2/a^2 + y^2/b^2 + z^2/b^2 = 1$. The geometrical shape depicted in the bottom right panel corresponds to the choice $a = 2$ and $b = 1$ in arbitrary units.
The watermelon can be accurately represented as an ellipsoid (see Fig. 1); more precisely, it can be assimilated to a spheroid with two equal semi axes, in our case the median and the minor ones, with length \( b \). The major axis measures \( a \), with \( a > b \). The total volume of the watermelon is thus \( V_{\text{tot}} = \frac{4\pi b^2 a}{3} \). Our goal is to equally share it among the friends. The greengrocer straightforwardly carries out the first few cuts following the symmetries of the watermelon / spheroid, i.e. cutting along the semi major axis (longitudinal cut) and then the semi minor axis (transversal cut) (see Fig. 2).

![FIG. 2: First few cuts.](image)

Arrived at home we make more slices from each fourth of the watermelon to serve all the friends. How to proceed? Cutting each piece along the longitudinal directions is not very easy with more than 4 friends (see panel a) Fig. 3). As a viable alternative, one generally proceeds by cutting the watermelon along the transversal direction, but where to cut? Equally spaced cuts do not provide the solution of the problem because the slices closer to the central face (see panel b) Fig. 3) will have a larger volume. Assume we want to equally share one fourth of the watermelon between \( n \) friends: to obtain \( n \) slices we need to perform \( n - 1 \) cuts. We have to determine the cuts positions \( 0 < \lambda_1 a < \lambda_2 a < \cdots < \lambda_{n-1} a < a \), along the semi major axis where we need to slice the watermelon in such a way that each part has the same volume and it equals \( \frac{1}{n} \) of the volume of one fourth of the watermelon, \( V = \frac{\pi b^2 a}{3} = \frac{V_{\text{tot}}}{4} \) (see Fig. 3 panels b) and c) in the case \( n = 4 \):

**Problem**: find \( \lambda_i \in (0, 1), \ i = 1, \ldots, n - 1 \), such that \( \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} \) and \( V_i = V/n \), where \( V_i \) is the volume of the slice between the cuts \( \lambda_i \) and \( \lambda_{i-1} \).

![FIG. 3: Where to cut next?](image)
II. COMPUTATIONS NEEDED TO SOLVE THE PROBLEM.

Let us consider $0 < \lambda < 1$ and compute the volume $V(\lambda a)$ of the portion of spheroid contained between the central face and the cut at distance $\lambda a$ from this face. Consider a very tiny slice of opening angle $\pi$ at position $x$ with thickness $dx$ (see Fig. 4), then its volume is given by $dV = \pi / 2 \times \ell^2(x) dx$, where $\ell(x) = b \sqrt{1 - x^2/a^2}$ is the radius of the sector of the disk. The slice is indeed a portion of a cylinder of height $dx$ and whose base is a sector of a disk of opening $\pi$ and radius $\ell(x)$; the radius depends on the position of the slice, it equals $b$ (the semi minor and medium axis) for $x = 0$ and it vanishes at $x = a$ (see panel b) of Fig. 3. In conclusion we get for the volume $V(\lambda a)$

$$V(\lambda a) = \frac{\pi}{2} \int_0^{\lambda a} \ell^2(x) dx = \frac{\pi}{2} b^2 \int_0^{\lambda a} \left(1 - \frac{x^2}{a^2}\right) dx = \frac{\pi}{2} b^2 a \lambda \left(1 - \frac{\lambda^2}{3}\right).$$

(1)

![Fig. 4: Geometry for the computation of the volume of the slice. We ideally slice the large piece of watermelon into arbitrarily tiny slices of width $dx$. The latter can be considered as a very tiny sector of disk, with opening $\alpha$ and radius $\ell(x)$, the latter depending on the position $x$ of the slice. Such an arbitrary thin slice has a volume $dV = \alpha / 2 \times \ell^2(x) dx$. Indeed it can be considered as an infinitesimally thin cylinder (with height $dx$), whose base is a sector of disk.](image)

Given the integer $n \geq 2$ and $\lambda_i \in (0, 1)$ such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < 1$, then we get for the volumes $V_i$

$$V_i = V(\lambda_1 a), V_{i+1} = V(\lambda_{i+1} a) - V(\lambda_i a) \quad \forall i = 1, \ldots, n-2 \text{ and } V_n = V(a) - V(\lambda_{n-1} a).$$

(2)

Our problem is thus equivalent to solve the equations

$$\frac{1}{n} = \frac{V_1}{V} = 3 \frac{\lambda_1}{2} \left(1 - \frac{\lambda_1^2}{3}\right),$$

$$\frac{1}{n} = \frac{V_i}{V} = 3 \frac{\lambda_i}{2} \left(1 - \frac{\lambda_i^2}{3}\right) - 3 \frac{\lambda_{i-1}}{2} \left(1 - \frac{\lambda_{i-1}^2}{3}\right) \quad \forall i = 2, \ldots, n-1 \text{ and }$$

$$\frac{1}{n} = \frac{V_n}{V} = \frac{V(a)}{V} - 3 \frac{\lambda_{n-1}}{2} \left(1 - \frac{\lambda_{n-1}^2}{3}\right).$$

(3)

Stated differently, let $p(\lambda) = \frac{3}{2} \lambda \left(1 - \frac{\lambda^2}{3}\right)$, the first equation of [3] is equivalent to looking for a root of $p(\lambda_1) = 1/n$. Then given $\lambda_1$ one can plug this value in the next equation for $i = 2$ and solve for $\lambda_2$ the equation $p(\lambda_2) = 1/n + p(\lambda_1) = 2/n$, where we used that fact that $p(\lambda_1) = 1/n$. We can iterate the above steps and get:

$$\forall i = 2, \ldots, n-1 \quad \frac{i}{n} = p(\lambda_i).$$

The last equation involving $V_n$ results into the identity $p(1) = 1$, i.e. the volume of the whole piece is $V$.

In conclusion given $n$ we have to solve $n - 1$ problems that we compactly write as

$$p(\lambda) = \frac{3}{2} \lambda \left(1 - \frac{\lambda^2}{3}\right) = f_n,$$

(4)
where $f_n \in \{1/n, \ldots, (n-1)/n\}$. Before to proceed, let us observe that the above equation always admits one and only one solution in the interval $(0, 1)$. Indeed, the polynomial $p(\lambda)$ vanishes at $\lambda = 0$, its first derivatives vanishes at $\lambda = \pm 1$. Further, it can be easily proven that it has a maximum at $\lambda = 1$ where it reaches the value $p(1) = 1$ and a minimum at $\lambda = -1$. The polynomial is thus steadily increasing from 0 up to 1. Hence, any constant horizontal line set at $f_n \in (0, 1)$ intersects the polynomial in just one point belonging to the interval $(0, 1)$ (see Fig. 5 for a graphical representation of this claim). One can of course solve analytically the above equations, by making use of the explicit formulae for roots of the third order polynomials. Notice however that a good approximation can be obtained for the first slices by setting $\lambda_{app} = f_n \times 2/3$. In fact, $p(\lambda_{app}) = f_n - f_n^3 \times 4/27$ which is very close to the required value, $f_n$, since the cubic term $f_n^3 \times 4/27$ can be safely neglected.

Consider as an example $n = 2$. We need to share every one fourth of watermelon into 2 equal volume parts. According to the above recipe, we perform the cut at $\lambda_{app}a = a \times 2/3 \times 1/2 = a/3$, namely assuming a 1/3-2/3 ratio of the semi major axis. The above approximated criterion yields two slices with almost identical volumes, the relative difference in volume being as small as 4%.

Similarly, for $n = 3$, the first slice can be obtained with a cut at $\lambda_{app}^1a = a \times 2/3 \times 1/3 = 2a/9$ and its volume will differ from the correct one, i.e. one third of the watermelon piece, by a small amount that we quantify in 1.6%. By using the same approximation to perform the second cut at $\lambda_{app}^2a = a \times 2/3 \times 2/3 = 4a/9$, we obtain a slice whose volume differ by 14% with respect to the desired one. The error is getting worse because the approximated cuts follow the linear law $\lambda_{app} = 2/3 \times f_n$, while the curvature of the watermelon is responsible for the cubic term in $p(\lambda)$.

For a generic $n \geq 2$ one can compute the errors in volumes which follow the approximated cuts $\lambda_{app}^i = a \times 2/3 \times i/n$ for $i = 1, \ldots, n-2$. More precisely let us consider the ratio of the $i$-th approximated volume $V_{app}^i$ with the volume of the watermelon piece:

$$\frac{V_{app}^i}{\bar{V}} = p(\lambda_{app}^i) - p(\lambda_{app}^{i-1}) = \frac{i}{n} - \frac{4}{27} \frac{i^3}{n^3} - \frac{i-1}{n} + \frac{4}{27} \frac{(i-1)^3}{n^3} = \frac{1}{n} - \frac{4}{27} \frac{3i^2 + 3i + 1}{n^3}.$$ 

One can immediately realise that the relative error grows with $i$ and gets largest for the last cut, $i = n - 1$.

The adequacy of the proposed approximation can be evaluated upon inspection of Fig. 6 where we report the approximated volumes for several values of $n$ as compared to their exact homologues (horizontal dashed lines). One

![FIG. 5: Existence and uniqueness of the solution $p(\lambda) = f_n$. We show the polynomial $p(\lambda)$ for $\lambda \in (0, 1)$ (black curve) and a generic horizontal line set at $f_n \in (0, 1)$ (blue dashed line).](image-url)
can see that for all $n$ the first few approximated volumes are quite accurate, while the disagreement increases for further cuts at fixed $n$.

![Graph showing approximated volumes for different $n$ values](image)

**FIG. 6: Approximated volumes.** Each column represents the relative approximated volumes $V_{i}^{app}/V$ for a choice of $n$. The horizontal dashed lines represent the correct equal share for the corresponding $n$, i.e. $1/n$.

### III. THE EXPERIMENT

To challenge our findings and the underlying hypotheses, (i.e. the watermelon can be correctly approximated by a spheroid and the approximated cuts provide equal volumes), we performed the experiment of cutting pieces of watermelon into $n$ portions. In line with the spirit of the note, we decided to use only tools available in our kitchen [9][10] so as to make our experiment easily reproducible by everyone. The experiment has been filmed and a movie is available at [11].

For the experiment we used a small watermelon weighting approximately 4 Kg. Of course the results are independent from the watermelon size. The first step is to measure the watermelon volume. To accomplish this task we make use of the Archimedes principle [12][13]: a body immersed in water, will displace a volume of water equal to the volume of the body. We cut the watermelon into two equal parts (longitudinal cut as shown in Fig. 2), we insert each part in a sufficiently large bowl and we fill it with water until a reference mark is reached. We then remove the watermelon: the volume of water in the bowl is hence lower (for a sake of completeness we refer the reader to the discussion at the end of this section). To measure this volume we use a measuring cup to fill again the bowl. Once the water level has reached again the initial reference mark, the added volume of water is equal to the volume of the watermelon (See Fig. 7). In our case we got $\sim 1.9$ l for each piece.

We are now ready to prepare the $n$ slices. To this end we cut the two half of watermelon into two parts each, yielding 4 quarters. The first quarter is sliced into $n = 2$ equal volume portions, from the second $n = 3$, from the third $n = 4$ and finally $n = 5$ slices from the last one. In each case the positions of the cuts have been determined by using the approximation $\lambda_{app}^{(i)} = a \times 2/3 \times i/n$ for $i = 1, \ldots, n - 2$ (see Fig. 8).

The volumes of the cut slices have been measured by using the Archimedes principle; the results reported in the Table I confirm the correctness of the 2/3-rule in the computation of equal volume slices.

During the experiment we could obtain an accessory result: a measure of the density of the watermelon, $\rho$, namely
FIG. 7: Measure a volume by using the Archimedes principles. Panel a): insert in the bowl the watermelon whose volume is to be measured. Panel b): fill the bowl with water up to a reference mark. Panel c): remove the watermelon. Panels d)-e): to measure the drop in volume, fill again the bowl with water up to the reference mark. Panel f): the volume of the watermelon equals the added volume of water.

### TABLE I: Volumes of the slices

| Slice number | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|--------------|---------|---------|---------|---------|
| 1            | 490     | 300     | 230     | 170     |
| 2            | 510     | 290     | 220     | 170     |
| 3            | *       | 330     | 230     | 160     |
| 4            | *       | *       | 240     | 140     |
| 5            | *       | *       | *       | 190     |

We report the volumes of the watermelon slices (expressed in cm$^3$) obtained in the experiment. Each column corresponds to a choice of $n$, while the rows refer to the slice number. The symbol * means that the slice is not allowed for the choice of $n$.

the quantity of matter contained in the (watermelon) volume. To do this we use a kitchen scale to weight the watermelon. This first piece weights $\sim 1.89$ Kg for a volume of $\sim 1.95$ l; for the second, $\sim 1.73$ Kg and a volume of $\sim 1.81$ l. The density results thus (we take the average of the two measures)

$$
\rho \sim \frac{1}{2} \left( \frac{1.89 \text{ Kg}}{1.95 \text{ l}} + \frac{1.73 \text{ Kg}}{1.81 \text{ l}} \right) = 0.97 \text{ Kg/l} = 0.97 \text{ g/cm}^3.
$$

Observe that the density is smaller than 1 g/cm$^3$ which can be assumed to be the density of the water at the experimental conditions. We can conclude that the watermelon is less dense than water and thus it (almost) floats. Indeed the watermelon is composed for a large fraction by water and the weight of the few fibres (heavier than water) is compensated by the air (way lighter than water) trapped into the watermelon pulp. Let us observe that in [14] the density of the watermelon has been estimated to 0.94 g/cm$^3$, assuming a spherical watermelon and measuring its buoyancy.

Let us conclude this section with an observation about the application of the Archimedes principles. As we have seen the watermelon is slightly lighter than water and thus it can not be completely immersed; to measure the volume of the watermelon part we are interested in, we have thus to force it into the water.

### IV. CONCLUSION

We have provided a straightforward recipe to (transversally) cut a (spheroid) watermelon into an arbitrary number of pieces with equal volume. Interestingly enough, after the first few cuts dictated by the symmetry, i.e. divide by 2 along the semi major and minor axes, a “new rule” of the thumb emerges where the ratio 1/2 is replaced by $2/3 \times 1/n$ where $n$ identifies the number of (equal volume) slices that one wants to recover.

Let us conclude that the same result, and thus an identical rule, holds true for a spherical watermelon. In the latter cases one could always invoke the spherical symmetry and thus cut slices with an angle of $2\pi/n$ (sort of apple wedges); however for large $n$ it can be difficult to make such thin angles, while cutting vertically is normally easier.
FIG. 8: The approximated cuts realised in the experiment. We report the approximated cuts realised in the experiment using the formula $\lambda_{\text{app}}^{(i)} = a \times \frac{2}{3} \times \frac{i}{n}$, where $a$ is the semi major axis of the spheroid (the watermelon) for several values of $n$: $n = 2$ (top left panel), $n = 3$ (top right panel), $n = 4$ (bottom left panel) and $n = 5$ (bottom right panel).

The theory has been complemented with an experiment (a movie describing the experiment is also available [11]), realised by using tools available in our kitchen and thus hopefully reproducible in the students houses.

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