On Compressible Navier-Stokes Equations Subject to Large Potential Forces with Slip Boundary Conditions in 3D Bounded Domains

Guocai CAI\textsuperscript{a}, Bin HUANG\textsuperscript{b}, Xiaoding SHI\textsuperscript{b},
\textsuperscript{a} School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China;
\textsuperscript{b} College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, P. R. China

Abstract

We deal with the barotropic compressible Navier-Stokes equations subject to large external potential forces with slip boundary condition in a 3D simply connected bounded domain, whose smooth boundary has a finite number of 2D connected components. The global existence of strong or classical solutions to the initial boundary value problem of this system is established provided the initial energy is suitably small. Moreover, the density has large oscillations and contains vacuum states. Finally, we show that the global strong or classical solutions decay exponentially in time to the equilibrium in some Sobolev's spaces, but the oscillation of the density will grow unboundedly in the long run with an exponential rate when the initial density contains vacuum states.

Keywords: compressible Navier-Stokes equations; slip boundary condition; vacuum; large external forces; global existence; large-time behavior.

1 Introduction

The motion of three-dimensional viscous compressible barotropic flows is governed by the compressible Navier-Stokes equations

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div} u + \nabla P(\rho) &= \rho f,
\end{aligned}
\]  

(1.1)

where \((x, t) \in \Omega \times (0, T], \Omega\) is a domain in \(\mathbb{R}^N\), \(t \geq 0\) is time, and \(x\) is the spatial coordinate. \(\rho \geq 0\), \(u = (u^1, \cdots, u^N)\) and

\[
P(\rho) = a\rho^\gamma (a > 0, \gamma > 1)
\]  

(1.2)

are the unknown fluid density, velocity and pressure, respectively. We mainly consider the case that the external force \(f(x)\) is a gradient of a scalar potential, that is,

\[
f(x) = \nabla \psi(x)
\]  

(1.3)

*Email: gotry@xmu.edu.cn (G.C.Cai); abinhuang@gmail.com (B. Huang); shixd@mail.buct.edu.cn (X. D. Shi)
The constants \( \mu \) and \( \lambda \) are the shear and bulk viscosity coefficients respectively satisfying the following physical restrictions:

\[
\mu > 0, \quad 2\mu + N\lambda \geq 0. \tag{1.4}
\]

In this paper, we assume that \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^3 \), its boundary \( \partial \Omega \) is of class \( C^\infty \) and only has a finite number of 2-dimensional connected components. In addition, the system is studied subject to the given initial data

\[
\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = \rho_0 u_0(x), \quad x \in \Omega, \tag{1.5}
\]

and slip boundary condition

\[
u \cdot n = 0 \text{ and } \text{curl} u \times n = 0 \text{ on } \partial \Omega, \tag{1.6}
\]

where \( n = (n^1, n^2, n^3) \) is the unit outward normal vector on \( \partial \Omega \).

The first condition in (1.6) is the non-penetration boundary condition, while the second one is also known in the form

\[
(D(u)n)_\tau = -\kappa_\tau u_\tau, \tag{1.7}
\]

where \( D(u) = (\nabla u + (\nabla u)^T)/2 \) is the deformation tensor, \( \kappa_\tau \) is the corresponding normal curvature of \( \partial \Omega \) in the \( \tau \) direction. Therefore, in the case that \( \partial \Omega \) is of constant curvature, (1.6) is a special Navier-type slip boundary condition (see [4]), in which there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress. This type of boundary condition was originally introduced by Navier [27] in 1823, which was followed by many applications, numerical studies and analysis for various fluid mechanical problems, see, for instance [8, 15, 32] and the references therein. In fact, all our results of the paper are also valid for more general boundary conditions including Navier-type slip boundary condition (see Remark 1.4 below).

The compressible Navier-Stokes system has been attracted a lot of attention and significant progress has been made in the analysis of the well-posedness and dynamic behavior. We only briefly review some results related to the existence of strong or classical solutions and large-time behavior. The local solvability in classical spaces subject to the various boundary conditions was established by Serrin [31], Nash [26], Solonnikov [34], Tani [33] early and then some further works were given by Cho-Choe-Kim (see [5, 6, 7]), Huang [12]. For the whole space \( \mathbb{R}^3 \) and without external forces, the global classical solutions were first obtained by Matsumura-Nishida [22] for initial data close to a nonvacuum equilibrium in \( H^3 \). It is worth mentioned that their results have been improved by Huang-Li-Xin [14] and Li-Xin [19], in which the global existence of classical solutions to the Cauchy problem for the barotropic compressible Navier-Stokes equations is obtained with smooth initial data that are of small energy but possibly large oscillations with constant state as far field which could be either vacuum or nonvacuum. Very recently, for the barotropic compressible Navier-Stokes equations in a bounded domain with slip boundary conditions, Cai-Li [4] proved that the classical solution of the initial-boundary-value problem in the absence of exterior forces exists globally with vacuum and small energy but possibly large oscillations.

For compressible Navier-Stokes system with external forces, some early research work focused on small external forces (see [22, 24, 33] and the references therein), mainly
because the large external forces have a powerful influence on the dynamic motion of compressible flows and bring some serious difficulties (cf. [9, 17, 25, 37]). On the one hand, there are many results regarding the large-time behavior of weak solutions to the problem (1.1). Feireisl-Petzeltová [9], Novotny-Stráškraba [30] showed that for different boundary conditions, the density of any global weak solution converges to the steady state density in $L^q$ space for some $q > \frac{3}{2}$ as time goes to infinity if there exists a unique steady state. On the other hand, it seems that there are few results on the global existence of classical or strong solutions to the system (1.1) in $\mathbb{R}^3$ with large external forces except for that of Li-Zhang-Zhao [20], where they proved that the Cauchy problem has a unique global strong solution with large oscillations and interior vacuum, provided the initial data are of small energy and the unique steady state is strictly away from vacuum. However, their method depends crucially on the Cauchy problem and cannot be applied directly to the bounded domains. Therefore, our main purpose in the paper is to study the mechanism of the influence of the large potential forces to the compressible system (1.1) with Navier-slip boundary conditions in bounded domains.

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f \, dx = \int_\Omega f \, dx.$$ 

For integer $k$ and $1 \leq q < +\infty$, $W^{k,q}(\Omega)$ is the standard Sobolev spaces and

$$W^{1,q}_0(\Omega) = \{ u \in W^{1,q}(\Omega) : u \text{ is equipped with zero trace on } \partial\Omega \}.$$ 

We also denote

$$H^k(\Omega) = W^{k,2}(\Omega), \quad H^1_0(\Omega) = W^{1,2}_0(\Omega).$$

For simplicity, we denote $L^q(\Omega), W^{k,q}(\Omega), H^k(\Omega), H^1_0(\Omega)$ by $L^q, W^{k,q}, H^k, H^1_0$ respectively.

For two $3 \times 3$ matrices $A = \{a_{ij}\}, B = \{b_{ij}\}$, the symbol $A; B$ represents the trace of $AB$, that is,

$$A; B \triangleq \text{tr}(AB) = \sum_{i,j=1}^3 a_{ij}b_{ji}.$$ 

Finally, for $v = (v^1, v^2, v^3)$, we denote $\nabla_i v \triangleq (\partial_i v^1, \partial_i v^2, \partial_i v^3)$ for $i = 1, 2, 3$, and the material derivative of $v$ by $\dot{v} \triangleq v_t + u \cdot \nabla v$.

It is natural to expect an equilibrium density $\rho_s = \rho_s(x)$ and velocity $u_s = u_s(x)$ to the initial boundary-value problem (1.1)–(1.6), which is a solution of the rest state equations

$$\begin{cases}
\text{div}(\rho_s u_s) = 0 & \text{in } \Omega, \\
-\mu \Delta u_s - (\lambda + \mu)\nabla \text{div} u_s + \nabla P(\rho_s) = \rho_s \nabla \psi & \text{in } \Omega, \\
u_s \cdot n = 0, \quad \text{curl} u_s \times n = 0 & \text{on } \partial\Omega, \\
\int \rho_s dx = \int \rho_0 dx.
\end{cases} \quad (1.8)$$

We have the following conclusion.
We denote the initial total energy of (1.1) as
\[ \psi \in H^2, \quad \int \left( \frac{\gamma - 1}{a} \left( \psi - \inf_{\Omega} \psi \right) \right)^{\frac{1}{\gamma - 1}} dx < \int \rho_0 dx, \] (1.9)
then there exists a unique solution \((\rho_s, 0)\) of (1.8) such that
\[ \rho_s \in H^2, \quad 0 < \underline{\rho} \leq \inf_{\Omega} \rho_s \leq \sup_{\Omega} \rho_s \leq \bar{\rho}, \] (1.10)
where \(\underline{\rho}\) and \(\bar{\rho}\) are positive constants depending only on \(a, \gamma, \inf_{\Omega} \psi\) and \(\sup_{\Omega} \psi\). In addition, if \(\psi \in W^{2,q}\) for some \(q \in (3, 6)\), then
\[ \|\rho_s\|_{W^{2,q}} \leq C, \] (1.11)
where \(C\) is a positive constant depending only on \(a, \gamma, \inf_{\Omega} \psi\) and \(\|\psi\|_{W^{2,q}}\).

**Remark 1.1** It should be noted that the equilibrium density \(\rho_s\) is in fact a solution of the rest state equations
\[
\begin{align*}
\nabla P(\rho_s) &= \rho_s \nabla \psi \quad \text{in } \Omega, \\
\int \rho_s dx &= \int \rho_0 dx.
\end{align*}
\] (1.12)

We denote the initial total energy of (1.1) as
\[ C_0 \triangleq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0, \rho_s) \right) dx, \] (1.13)
where \(G(\rho, \rho_s)\) is the potential energy density given by
\[ G(\rho, \rho_s) \triangleq \rho \int_{\rho_s}^{\rho} \frac{P(\xi) - P(\rho_s)}{\xi^2} d\xi. \] (1.14)

Our first result is concerned with the global existence of a strong solution of the problem (1.1)-(1.6) in a bounded domain.

**Theorem 1.1** Let \(\Omega\) be a simply connected bounded domain in \(\mathbb{R}^3\) and its smooth boundary \(\partial \Omega\) has a finite number of 2-dimensional connected components. For \(\psi \in H^2\) with (1.9), \(\rho_s\) is the steady state density given by (1.12). For given positive constants \(M\) and \(\hat{\rho} > \bar{\rho} + 1\), assume that \((\rho_0, u_0)\) satisfy for some \(q \in (3, 6)\),
\[
(\rho_0, P(\rho_0)) \in W^{1,q}, \quad u_0 \in \left\{ f \in H^1 \mid f \cdot n = 0, \ \text{curl} f \times n = 0 \ \text{on } \partial \Omega \right\},
\] (1.15)
\[
0 \leq \rho_0 \leq \bar{\rho}, \quad \|u_0\|_{H^1} \leq M.
\] (1.16)
Then there exists a positive constant \(\varepsilon\) depending only on \(\mu, \lambda, \gamma, a, \inf_{\Omega} \psi, \|\psi\|_{H^2}, \bar{\rho}, \\Omega, \) and \(M\) such that if the initial total energy \(C_0 < \varepsilon\), then the initial-boundary-value problem (1.1)-(1.6) has a unique strong solution \((\rho, u)\) in \(\Omega \times (0, \infty)\) satisfying
\[
0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad (x, t) \in \Omega \times (0, \infty),
\] (1.17)
and for any \(\tau \in (0, T)\),
\[
\begin{align*}
(\rho, P) &\in C([0, T]; W^{1,q}), \quad \rho_t \in L^\infty(0, T; L^2), \quad \rho u \in C([0, T]; L^2), \quad u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \quad \rho^{1/2} u_t \in L^2(0, T; L^2), \quad (\rho^{1/2} u_t, \nabla^2 u) \in L^\infty(\tau, T; L^2) \quad (1.18)
\end{align*}
\]
Remark 1.2 In this conclusion, we have no restrictions on the potential force $\psi$ expect for (1.9). Moreover, besides the small total energy, large oscillations of the density and vacuum are also allowed.

Remark 1.3 It is clear that $u \in C([\tau, T]; H^1)$ for any $0 < \tau < T$. However, although $u_0 \in H^1$, it seems difficult to derive that $u \in C([0, T]; H^1)$ due to the lack of the compatibility condition (see (1.20)) and the presence of vacuum.

The second goal of this paper is to provide the global existence of classical solutions of (1.1)-(1.6) in a bounded domain as follows:

Theorem 1.2 In addition to the conditions of Theorem 1.1, assume further that $\psi \in H^3$, the initial data $(\rho_0, u_0)$ satisfy

$$ (\rho_0, P(\rho_0)) \in W^{2,q}, \quad (1.19) $$

and the compatibility condition

$$ -\mu \triangle u_0 - (\mu + \lambda)\nabla \text{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g, \quad (1.20) $$

for some $g \in L^2$. Then there exists a positive constant $\varepsilon$ depending only on $\mu, \lambda, \gamma, a, \inf_\Omega \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega,$ and $M$ such that the initial-boundary-value problem (1.1)-(1.6) has a unique classical solution $(\rho, u)$ in $\Omega \times (0, \infty)$ satisfying (1.17) and for any $0 < \tau < T < \infty$,

$$ \begin{cases} 
(\rho, P) \in C([0, T]; W^{2,q}), \\
\nabla u \in C([0, T]; H^1) \cap L^\infty(\tau, T; W^{2,q}), \\
u_t \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1), \\
\sqrt{\rho} u_t \in L^\infty(0, \infty; L^2),
\end{cases} \quad (1.21) $$

provided $C_0 \leq \varepsilon$.

Moreover, there exist positive constants $C$ and $\hat{C}$ depending only on $\mu, \lambda, \gamma, a, \inf_\Omega \psi, \|\psi\|_{H^2}, \hat{\rho}, M, \Omega, p, q$ and $C_0$ such that the following large-time behavior holds for any $q \in [1, \infty)$ and $p \in [1, 6]$,

$$ \left(\|\rho - \rho_s\|_{L^q} + \|u\|_{W^{1,p}} + \|\sqrt{\rho} u\|_{L^2}^2\right) \leq Ce^{-\hat{C}t}. \quad (1.22) $$

Remark 1.4 Similar to what have done in [4], one can get the same conclusion under more relaxed assumption on the initial data and more wide boundary condition (see [4, Theorem 1.1] for the details).

With (1.22) at hand, similar to [4, Theorem 1.2], we can obtain the following large-time behavior of the gradient of the density when vacuum states appear initially.

Theorem 1.3 In addition to the conditions of Theorem 1.2, assume further that there exists some point $x_0 \in \Omega$ such that $\rho_0(x_0) = 0$. Then there exist positive constants $\hat{C}_1$ and $\hat{C}_2$ depending only on $\mu, \lambda, \gamma, a, \inf_\Omega \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega, M, r$ and $C_0$ such that for any $t > 0$,

$$ \|\nabla \rho(\cdot, t)\|_{L^r} \geq \hat{C}_1 e^{\hat{C}_2 t}. \quad (1.23) $$
We now comment on the analysis of this paper. Compared with [4], as indicated by [9, 17, 25, 37], the large external forces will bring some serious difficulties due to its powerful influence on the dynamic motion of compressible flows. To overcome these difficulties, we need some new ideas. More precisely, firstly, introducing

\[ \text{curl}u \triangleq \nabla \times u, \quad F \triangleq (\lambda + 2\mu) \text{div}u - (P - P(\rho_s)), \]

we rewrite (1.12) in the form

\[ \rho \dot{u} - \rho \nabla \psi = \nabla F - \mu \nabla \times \text{curl}u, \]

where \( F \) is called the effective viscous flux and plays an important role in our following analysis. Combining (1.25) with the slip boundary condition (1.6), we obtain the estimates of \( \nabla F \) and \( \nabla \times \text{curl}u \). Furthermore, together with the following inequality

\[ \| \nabla u \|_{W^{k,q}} \leq C(\| \text{div}u \|_{W^{k,q}} + \| \text{curl}u \|_{W^{k,q}}) \]

for any \( q > 1, \ k \geq 0 \), which is shown in [36] when \( \Omega \) is simply connected, it allows us to control \( \nabla u \) by means of \( \text{div}u \) and \( \text{curl}u \). Secondly, since \( u \cdot n = 0 \) and \( \text{curl}u \times n = 0 \) on \( \partial \Omega \), denote \( u^\perp \triangleq -u \times n \), then \( u = u^\perp \times n \), moreover,

\[ u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u, \]

and

\[ (\dot{u} + (u \cdot \nabla n) \times u^\perp) \cdot n = 0 \text{ on } \partial \Omega, \]

which are the key to estimating the integrals on the boundary \( \partial \Omega \). Finally, to deal with the large external potential forces, following [11] (see also [17, 20]), we have

\[ \rho_s^{-1}(\nabla (\rho^\gamma - \rho_s^\gamma) - \gamma(\rho - \rho_s) \rho_s^{\gamma-2} \nabla \rho_s) = \nabla (\rho_s^{-1}(\rho^\gamma - \rho_s^\gamma)) - \gamma - 1 \frac{a}{\gamma} G(\rho, \rho_s) \nabla \rho_s^{-1}, \]

which indeed gives the ‘small’ estimate of \( \| \rho - \rho_s \|_{L^2(\Omega \times [0,T])} \) (see (3.45) and (3.46)).

The rest of the paper is organized as follows. In Section 2, some known facts and elementary inequalities needed in later analysis are collected. Sections 3 and 4 are devoted to deriving the necessary a priori estimates which can guarantee the local strong (or classical) solution to be a global strong (or classical) one. Finally, the main results, Theorems 1.2 and 1.3 will be proved in Section 5.

2 Preliminaries

2.1 Some known inequalities and facts

In this subsection, we will recall some known theorems and facts, which are frequently utilized in our discussion.

First, similar to the proof of [12] Theorems 1.2 and 1.4, we have the local existence of strong and classical solutions.
Lemma 2.1 Let $\Omega$ be as in Theorem 1.2, assume that $(\rho_0, u_0)$ satisfy (1.15). Then there exist a small time $T > 0$ and a unique strong solution $(\rho, u)$ to the problem (1.1) on $\Omega \times (0, T]$ satisfying for any $\tau \in (0, T)$,
\[
\begin{aligned}
(\rho, P) & \in C([0, T]; W^{1,q}), \\
\nabla u & \in L^\infty(0, T; L^2) \cap L^\infty(\tau, T; W^{1,q}), \\
\rho^{1/2}u_t & \in L^2(0, T; L^2) \cap L^\infty(\tau, T; L^2), \\
u_t & \in L^\infty(\tau, T; H^1).
\end{aligned}
\]
Furthermore, if the initial data $(\rho_0, u_0)$ satisfy (1.19) and the compatibility conditions (1.20), then there exist $T > 0$ and a unique classical solution $(\rho, u)$ to the problem (1.1) on $\Omega \times (0, T]$ satisfying for any $\tau \in (0, T),$
\[
\begin{aligned}
(\rho, P) & \in C([0, T]; W^{2,q}), \\
\nabla u & \in C([0, T]; H^1) \cap L^\infty(\tau, T; W^{2,q}), \\
u_t & \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1), \\
\sqrt{\rho}u_t & \in L^\infty(0, T; L^2).
\end{aligned}
\]

Next, the well-known Gagliardo-Nirenberg’s inequality (see [28]) will be used later.

Lemma 2.2 (Gagliardo-Nirenberg) Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$. For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exist generic constants $C_i > 0(i = 1, \ldots, 4)$ which depend only on $p$, $q$, $r$, and $\Omega$ such that for any $f \in H^1(\Omega)$ and $g \in L^q(\Omega) \cap D^{1,r}(\Omega)$,
\[
\begin{aligned}
\|f\|_{L^p(\Omega)} & \leq C_1\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}} + C_2\|f\|_{L^2}, \\
\|g\|_{C(\overline{\Omega})} & \leq C_3\|g\|_{L^q}^{\frac{1(r-3)}{(3r+q)(r-3)}} \|\nabla g\|_{L^r}^{\frac{3r}{3r+q(r-3)}} + C_4\|g\|_{L^2}.
\end{aligned}
\]
Moreover, if either $f \cdot n|_{\partial \Omega} = 0$ or $\bar{f} = 0$, we can choose $C_2 = 0$. Similarly, the constant $C_4 = 0$ provided $g \cdot n|_{\partial \Omega} = 0$ or $\bar{g} = 0$.

We need the following Zlotnik’s inequality, by which we can get the uniform (in time) upper bound of the density $\rho$.

Lemma 2.3 ([38]) Suppose the function $y$ satisfies
\[
y'(t) = g(y) + b(t) \text{ on } [0, T], \quad y(0) = y^0,
\]
with $g \in C(R)$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and
\[
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)
\]
for all $0 \leq t_1 < t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then
\[
y(t) \leq \max\{y^0, \zeta\} + N_0 < \infty \text{ on } [0, T],
\]
where $\zeta$ is a constant such that
\[
g(\zeta) = -N_1 \quad \text{for} \quad \zeta \geq \zeta.
\]
For the Lamé’s system

\[
\begin{aligned}
-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= f, \quad x \in \Omega, \\
u \cdot n &= 0, \quad \text{curl} u \times n = 0, \quad x \in \partial \Omega,
\end{aligned}
\]

(2.5)

where \( u = (u^1, u^2, u^3) \), \( f = (f^1, f^2, f^3) \), \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^3 \), and \( \mu, \lambda \) satisfy the condition (1.4), the following estimate is standard (see [1]).

**Lemma 2.4** Let \( u \) be a solution of the Lamé’s equation (2.5), there exists a positive constant \( C \) depending only on \( \lambda, \mu, q, k \) and \( \Omega \) such that

1. If \( f \in W^{k,q} \) for some \( q \in (1, \infty) \), \( k \geq 0 \), then \( u \in W^{k+2,q} \) and \( \| u \|_{W^{k+2,q}} \leq C(\| f \|_{W^{k,q}} + \| u \|_{L^q}) \).
2. If \( f = \nabla g \) and \( g \in W^{k,q} \) for some \( q \geq 1 \), \( k \geq 0 \), then \( u \in W^{k+1,q} \) and \( \| u \|_{W^{k+1,q}} \leq C(\| g \|_{W^{k,q}} + \| u \|_{L^q}) \).

**Definition 2.1** Let \( \Omega \) be a domain in \( \mathbb{R}^3 \). If the first Betti number of \( \Omega \) vanishes, namely, any simple closed curve in \( \Omega \) can be contracted to a point, we say that \( \Omega \) is simply connected. If the second Betti number of \( \Omega \) is zero, we say that \( \Omega \) has no holes.

The following two lemmas can be found in [36, Theorem 3.2] and [2, Propositions 2.6-2.9].

**Lemma 2.5** Let \( k \geq 0 \) be an integer, \( 1 < q < +\infty \), and assume that \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^3 \) with \( C^{k+1,1} \) boundary \( \partial \Omega \). Then for \( v \in W^{k+1,q} \) with \( v \cdot n = 0 \) on \( \partial \Omega \), there exists a constant \( C = C(q, k, \Omega) \) such that

\[
\| v \|_{W^{k+1,q}} \leq C(\| \text{div} v \|_{W^{k,q}} + \| \text{curl} v \|_{W^{k,q}}).
\]

In particular, for \( k = 0 \), we have

\[
\| \nabla v \|_{L^q} \leq C(\| \text{div} v \|_{L^q} + \| \text{curl} v \|_{L^q}).
\]

**Lemma 2.6** ( [2] ) Let \( k \geq 0 \) be an integer, \( 1 < q < +\infty \). Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) and its \( C^{k+1,1} \) boundary \( \partial \Omega \) only has a finite number of 2-dimensional connected components. Then for \( v \in W^{k+1,q} \) with \( v \times n = 0 \) on \( \partial \Omega \), there exists a constant \( C = C(q, k, \Omega) \) such that

\[
\| v \|_{W^{k+1,q}} \leq C(\| \text{div} v \|_{W^{k,q}} + \| \text{curl} v \|_{W^{k,q}} + \| v \|_{L^q}).
\]

In particular, if \( \Omega \) has no holes, then

\[
\| v \|_{W^{k+1,q}} \leq C(\| \text{div} v \|_{W^{k,q}} + \| \text{curl} v \|_{W^{k,q}}).
\]

The following Beale-Kato-Majda type inequality, which was first proved in [3, 16] when \( \text{div} u \equiv 0 \), and improved in [13], we give a similar result with respect to the slip boundary condition (1.6) to estimate \( \| \nabla u \|_{L^\infty} \) and \( \| \nabla \rho \|_{L^5} \) which have been proven in [4].
Lemma 2.7 ([4]) For $3 < q < \infty$, assume that $u \cdot n = 0$ and $\text{curl} u \times n = 0$ on $\partial \Omega$, $u \in W^{2,q}$, then there is a constant $C = C(q, \Omega)$ such that the following estimate holds

$$\|\nabla u\|_{L^{\infty}} \leq C (\|\text{div} u\|_{L^{\infty}} + \|\text{curl} u\|_{L^{\infty}}) \ln(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C.$$  

Next, consider the problem

\begin{align*}
\text{div} v &= f, \quad x \in \Omega, \\
v &= 0, \quad x \in \partial \Omega.
\end{align*}

(2.8)

One has the following conclusion.

Lemma 2.8 [10] Theorem III.3.1] There exists a linear operator $B = [B_1, B_2, B_3]$ enjoying the properties:

1) $B : \{ f \in L^p(\Omega) \mid \int_{\Omega} f dx = 0 \} \mapsto (W^{1,p}_0(\Omega))^3$

is a bounded linear operator, that is,

$$\|B[f]\|_{W^{1,p}_0(\Omega)} \leq C(p)\|f\|_{L^p(\Omega)}, \text{ for any } p \in (1, \infty),$$

(2.9)

2) The function $v = B[f]$ solve the problem (2.8).

3) if, moreover, $f$ can be written in the form $f = \text{div} g$ for a certain $g \in L^r(\Omega), g \cdot n|_{\partial \Omega} = 0$, then

$$\|B[f]\|_{L^r(\Omega)} \leq C(r)\|g\|_{L^r(\Omega)}, \text{ for any } r \in (1, \infty).$$

(2.10)

2.2 Estimates for $F$, curl $u$ and $\nabla u$

From now on, we always assume $\Omega$ is a simply connected bounded domain in $\mathbb{R}^3$ whose smooth boundary $\partial \Omega$ only has a finite number of 2-dimensional connected components and $\psi \in H^2$ satisfies (1.9). For $F$, curl $u$ and $\nabla u$, we give the following conclusion, which is often used later.

Lemma 2.9 Let $(\rho, u)$ be a smooth solution of (1.11) with slip boundary condition (1.10). Then for any $p \in [2, 6]$, there exists a positive constant $C$ depending only on $p, q, \mu, \lambda, \Omega$ and $\|\psi\|_{H^2}^2$ such that

$$\|\nabla u\|_{L^p} \leq C(\|\text{div} u\|_{L^p} + \|\text{curl} u\|_{L^p}),$$

(2.11)

$$\|\nabla F\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\rho - \rho_s\|_{L^{(6p)/(6-p)}}),$$

(2.12)

$$\|\nabla \text{curl} u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^{(6p)/(6-p)}}),$$

(2.13)

$$\|F\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p}^{(3p-6)/(2p)}(\|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2})^{(6-p)/(2p)})$$

$$+ C(\|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2}),$$

(2.14)

$$\|\text{curl} u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p}^{(3p-6)/(2p)}\|\nabla u\|_{L^2}^{(6-p)/(2p)}) + C(\|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2}).$$

(2.15)

Moreover,

$$\|F\|_{L^p} + \|\text{curl} u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2}),$$

(2.16)

$$\|\nabla u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)}\|\nabla u\|_{L^2}^{(6-p)/(2p)}) + C(\|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2} + \|\rho - \rho_s\|_{L^p}).$$

(2.17)
Remark 2.1 If $p = 6$, then $\|\rho - \rho_s\|_{L^6(\Omega)} \triangleq \|\rho - \rho_s\|_{L^\infty}$.

Proof. The inequality (2.11) is a direct result of Lemma 2.5, since $u \cdot n = 0$ on $\partial \Omega$. By (1.12), one can find that the viscous flux $F$ satisfies

$$\int \nabla F \cdot \nabla \eta dx = \int (\rho \dot{u} - (\rho - \rho_s) \nabla \psi) \cdot \nabla \eta dx, \quad \forall \eta \in C^\infty(\mathbb{R}^3),$$
i.e.,

$$\begin{cases}
\Delta F = \text{div}(\rho \dot{u} - (\rho - \rho_s) \nabla \psi), & x \in \Omega, \\
\frac{\partial F}{\partial n} = (\rho \dot{u} - (\rho - \rho_s) \nabla \psi) \cdot n, & x \in \partial \Omega.
\end{cases}$$

It follows from Lemma 4.27 in [29] that for any $p \in [2, 6]$,

$$\|\nabla F\|_{L^p} \leq C\|\rho \dot{u} - (\rho - \rho_s) \nabla \psi\|_{L^p}$$

$$\leq C(\|\rho \dot{u}\|_{L^p} + \|\rho - \rho_s\|_{L^6(\Omega)} \|\nabla \psi\|_{L^6})$$

$$\leq C(\|\rho \dot{u}\|_{L^p} + \|\rho - \rho_s\|_{L^6(\Omega)}).$$

On the other hand, one can rewrite (1.12) as $\mu \nabla \times \nabla u = \nabla F - \rho \dot{u}$. Notice that $\nabla u \times n = 0$ on $\partial \Omega$ and $\text{div}(\nabla \times \nabla u) = 0$, by Lemma 2.6 we get

$$\|\nabla \nabla \psi\|_{L^p} \leq C(\|\nabla \times \nabla u\|_{L^p} + \|\nabla \psi\|_{L^p})$$

$$\leq C(\|\rho \dot{u}\|_{L^p} + \|\rho - \rho_s\|_{L^6(\Omega)} + \|\nabla \psi\|_{L^p}).$$

By Sobolev’s inequality and (2.19),

$$\|\nabla \nabla \psi\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla \nabla \psi\|_{L^p} + \|\rho - \rho_s\|_{L^6(\Omega)})$$

$$\leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla \psi\|_{L^2} + \|\nabla \nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^6(\Omega)})$$

$$\leq C(\|\rho \dot{u}\|_{L^p} + \|\rho \dot{u}\|_{L^2} + \|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^6(\Omega)}).$$

so that (2.13) holds.

Furthermore, one can deduce from (2.1) and (2.12) that for $p \in [2, 6],$

$$\|F\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^6(\Omega)})$$

$$\leq C(\|\rho \dot{u}\|_{L^2} + \|\rho - \rho_s\|_{L^3})^{(3p-6)/(2p)}(\|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^2})^{(6-p)/(2p)}$$

$$\leq C(\|\rho \dot{u}\|_{L^2} + \|\rho - \rho_s\|_{L^3})^{(3p-6)/(2p)}(\|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^2})^{(6-p)/(2p)}$$

$$\leq C(\|\rho \dot{u}\|_{L^2} + \|\rho - \rho_s\|_{L^3}),$$

which also implies that

$$\|F\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^3}).$$

Similarly,

$$\|\nabla \psi\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^3})^{(3p-6)/(2p)} + C(\|\nabla \psi\|_{L^2}$$

$$\leq C(\|\rho \dot{u}\|_{L^2} + \|\rho - \rho_s\|_{L^3})^{(3p-6)/(2p)}(\|\nabla \psi\|_{L^2} + \|\rho - \rho_s\|_{L^2})^{(6-p)/(2p)}$$

$$\leq C(\|\rho \dot{u}\|_{L^2} + \|\rho - \rho_s\|_{L^3}),$$

which also implies that

$$\|\nabla \psi\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2} + \|\rho - \rho_s\|_{L^3}).$$
which also leads to

\[ \| \text{curl} u \|_{L^p} \leq C(\| \rho \dot{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| \rho - \rho_s \|_{L^3}). \]

Hence, (2.14) and (2.16) are established.

By virtue of (2.11) and (2.14), it indicates that

\[ \| \nabla u \|_{L^p} \leq C(\| \text{div} u \|_{L^p} + \| \text{curl} u \|_{L^p}) \]
\[ \leq C(\| F \|_{L^p} + \| )\|_{L^p} + \| P - P(\rho_s) \|_{L^p}) \]
\[ \leq C(\| \rho \dot{u} \|_{L^2}^{(3p-6)/(2p)} \| \nabla u \|_{L^2}^{(6-p)/(2p)} + \| \nabla u \|_{L^2} + \| \rho - \rho_s \|_{L^3} + \| \rho - \rho_s \|_{L^p}). \]

This completes the proof. \( \square \)

**Remark 2.2** Consider the following Lamé’s system

\[
\begin{aligned}
    -\mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= -\rho \dot{u} - \nabla (P - P(\rho_s)) + (\rho - \rho_s) \nabla \psi, & x \in \Omega, \\
    u \cdot n &= 0 \text{ and curl} u \times n = 0, & x \in \partial \Omega,
\end{aligned}
\]

by Lemma 2.4 and Gagliardo-Nirenberg’s inequality,

\[ \| \nabla^2 u \|_{L^p} \leq C(\| \rho \dot{u} \|_{L^p} + \| P - P(\rho_s) \|_{W^{1,p}} + \| (\rho - \rho_s) \nabla \psi \|_{L^p} + \| u \|_{L^p}) \]
\[ \leq C(\| \rho \dot{u} \|_{L^p} + \| \nabla u \|_{L^2} + \| \nabla P \|_{L^p} + \| P - P(\rho_s) \|_{L^2} + \| P - P(\rho_s) \|_{L^p}) \]
\[ + C(\| (\rho - \rho_s) \nabla \psi \|_{L^p}). \]

and

\[ \| \nabla^3 u \|_{L^p} \leq C(\| \rho \dot{u} \|_{W^{1,p}} + \| P - P(\rho_s) \|_{W^{2,p}} + C(\| (\rho - \rho_s) \nabla \psi \|_{W^{1,p}} + \| u \|_{L^p}) \]
\[ \leq C(\| \rho \dot{u} \|_{L^p} + \| \nabla u \|_{L^2} + \| \nabla (\rho \dot{u}) \|_{L^p} + \| \nabla^2 P \|_{L^p}) + C(\| (\rho - \rho_s) \nabla \psi \|_{W^{1,p}} \]
\[ + C(\| \nabla P \|_{L^p} + \| P - P(\rho_s) \|_{L^2} + \| P - P(\rho_s) \|_{L^p}). \]

### 3 Time-independent a priori estimates

Let \( T > 0 \) be a fixed time and \((\rho, u)\) be a smooth solution to (1.1)-(1.6) on \( \Omega \times (0, T) \) with smooth initial data \((\rho_0, u_0)\) satisfying (1.15) and (1.16). We will derive some necessary a priori bounds for smooth solutions to the problem (1.1)-(1.6) which can extend the local strong or classical solution guaranteed by Lemma 2.4 to be a global one.

Setting \( \sigma = \sigma(t) \triangleq \min \{1, t\} \), we define

\[ A_1(T) \triangleq \sup_{0 \leq t \leq T} \left( \sigma \| \nabla u \|_{L^2}^2 \right) + \int_0^T \sigma^{1/2} \| \dot{u} \|_{L^2}^2 dt, \]  \hspace{1cm} (3.1)

\[ A_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^3 \| \rho^{1/2} \|_{L^2}^2 dx + \int_0^T \sigma^3 \| \nabla u \|_{L^2}^2 dt, \]  \hspace{1cm} (3.2)

and

\[ A_3(T) \triangleq \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2. \]  \hspace{1cm} (3.3)

This section is entirely devoted to prove the following conclusion.
Proposition 3.1 Under the conditions of Theorem 1.2, for \( \delta_0 \triangleq \frac{2\alpha-1}{4\alpha} \in (0, \frac{1}{4}] \), there exist positive constant \( \varepsilon \) and \( K \) depending on \( \mu, \lambda, \gamma, a, \inf_{\Omega} \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega, \) and \( M \) such that if \((\rho, u)\) is a smooth solution of \((1.1)-(1.6)\) on \( \Omega \times (0,T) \) satisfying
\[
\sup_{\Omega \times [0,T]} \rho \leq 2\hat{\rho}, \quad A_1(T) + A_2(T) \leq 2C_0^{1/2}, \quad A_3(\sigma(T)) \leq 2K, \tag{3.4}
\]
then the following estimates hold
\[
\sup_{\Omega \times [0,T]} \rho \leq 7\hat{\rho}/4, \quad A_1(T) + A_2(T) \leq C_0^{1/2}, \quad A_3(\sigma(T)) \leq K, \tag{3.5}
\]
provided \( C_0 \leq \varepsilon \).

Proof. Proposition 3.1 is a consequence of the following Lemmas 3.4–3.6.

One can extend the function \( n \) to \( \Omega \) such that \( n \in C^3(\bar{\Omega}) \), and in the following discussion we still denote the extended function by \( n \).

The first lemma in this section, which depends on \( u\cdot n|_{\partial\Omega} = 0 \), is proven in [4, Lemma 3.2].

Lemma 3.1 ([4]) If \((\rho, u)\) is a smooth solution of \((1.1)\) with slip boundary condition \((1.6)\), then there exists a positive constant \( C \) depending only on \( \Omega \) such that
\[
\|\dot{u}\|_{L^\infty} \leq C(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}^2), \tag{3.6}
\]
\[
\|\nabla \dot{u}\|_{L^2} \leq C(\|\text{div} \dot{u}\|_{L^2} + \|\text{curl} \dot{u}\|_{L^2} + \|\nabla u\|_{L^4}^2). \tag{3.7}
\]

In the following, we will use the convention that \( C \) denotes a generic positive constant depending on \( \mu, \lambda, \gamma, a, \inf_{\Omega} \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega, M, \) and use \( C(\alpha) \) to emphasize that \( C \) depends on \( \alpha \).

We begin with the following standard energy estimate for \((\rho, u)\).

Lemma 3.2 Let \((\rho, u)\) be a smooth solution of \((1.1)-(1.6)\) on \( \Omega \times (0,T) \) satisfying \( \rho \leq 2\hat{\rho} \). Then there is a positive constant \( C \) depending only on \( \mu, \lambda, \gamma, \hat{\rho}, \) and \( \Omega \) such that
\[
\sup_{0 \leq t \leq T} \int (\rho|u|^2 + (\rho - \rho_s)^2) \ dx + \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C(\hat{\rho})C_0. \tag{3.8}
\]

Proof. First, since
\[-\Delta u = -\nabla \text{div} u + \nabla \times \text{curl} u,
\]
using \((1.12)\), we rewrite \((1.1)\) as
\[
\rho \dot{u} - (\lambda + 2\mu)\nabla \text{div} u + \mu \nabla \times \text{curl} u + \nabla P = \frac{\alpha\gamma}{\gamma - 1} \rho \nabla \rho_s^{\gamma-1}. \tag{3.9}
\]

Multiplying \((3.9)\) by \( u \) and integrating the resulting equality over \( \Omega \) lead to
\[
\frac{1}{2} \left( \int \rho|u|^2 \ dx \right)_t + (\lambda + 2\mu) \int (\text{div} u)^2 \ dx + \mu \int |\text{curl} u|^2 \ dx
\]
\[
= \int P \text{div} u \ dx - \frac{\alpha\gamma}{\gamma - 1} \int \rho_s^{\gamma-1} \text{div}(\rho u) \ dx. \tag{3.10}
\]
Multiplying \((1.1)\) by \(G'(\rho)\) with \(G(\rho)\) as in \((1.1)\), integrating over \(\Omega\) and applying slip boundary condition \((1.6)\), we have
\[
\left(\int G(\rho)dx\right)_t + \int P\text{div}\,u\,dx - \frac{\alpha\gamma}{\gamma - 1}\int \rho_s^{\gamma - 1}\text{div}(\rho u)\,dx = 0. \tag{3.11}
\]

Finally, it is easy to check that there exists a positive constant \(C = C(\rho, \hat{\rho})\) such that
\[
C^{-1}(\rho - \rho_s)^2 \leq G(\rho) \leq C(\rho - \rho_s)^2,
\]
which together with \((3.10)\), \((3.11)\), and \((2.11)\) gives \((3.8)\). \(\square\)

The following conclusion shows preliminary \(L^2\) bounds for \(\nabla u\) and \(\rho^{1/2}\dot{u}\).

**Lemma 3.3** Let \((\rho, u)\) be a smooth solution of \((1.1)\) - \((1.6)\) on \(\Omega \times (0, T]\) satisfying \(\rho \leq 2\hat{\rho}\). Then there is a positive constant \(C\) depending only on \(\mu, \lambda, a, \gamma, \inf_{\Omega} \psi\), \(\|\psi\|_{H^2}\), \(\hat{\rho}\) and \(\Omega\) such that
\[
A_1(T) \leq CC_0 + C\int_0^T \int \sigma |\nabla u|^2\,dx\,dt, \tag{3.12}
\]
and
\[
A_2(T) \leq CC_0 + CA_1(T) + C\int_0^T \int \sigma^2 |\nabla u|^4\,dx\,dt. \tag{3.13}
\]

**Proof.** The proof is similar to that of [4, Lemma 3.4] expect for some modifications which are caused by the external force term. For convenience, we still write down the proof completely.

Let \(m \geq 0\) be a real number which will be determined later. Now we start with \((3.12)\), \((1.1)2\) can be rewritten as
\[
\rho \dot{u} - (\lambda + 2\mu)\nabla \text{div}\,u + \mu\nabla \times \text{curl}\,u + \nabla(P - P(\rho_s)) = (\rho - \rho_s)\nabla \psi. \tag{3.14}
\]

Multiplying it by \(\sigma^m \dot{u}\) and then integrating the resulting equality over \(\Omega\) lead to
\[
\int \sigma^m \rho |\dot{u}|^2\,dx = -\int \sigma^m \dot{u} \cdot \nabla(P - P(\rho_s))\,dx + (\lambda + 2\mu)\int \sigma^m \nabla \text{div}\,u \cdot \dot{u}\,dx
- \mu\int \sigma^m \nabla \times \text{curl}\,u \cdot \dot{u}\,dx + \int \sigma^m (\rho - \rho_s)\nabla \psi \cdot \dot{u}\,dx \tag{3.15}
\]
\[
\triangleq I_1 + I_2 + I_3 + I_4.
\]

We will estimate \(I_1, I_2, I_3\) and \(I_4\) one by one. Firstly, by \((1.1)1\), one can check that
\[
P_t + \text{div}(Pu) + (\gamma - 1)P\text{div}\,u = 0, \tag{3.16}
\]
or
\[
P_t + \nabla P \cdot u + \gamma P\text{div}\,u = 0. \tag{3.17}
\]
A direct calculation together with (3.16) gives

\[ I_1 = - \int \sigma^m \dot{u} \cdot \nabla (P - P(\rho_s)) dx \]

\[ = \int \sigma^m (P - P(\rho_s)) \text{div} u_t dx - \int \sigma^m u \cdot \nabla (P - P(\rho_s)) dx \]

\[ = \left( \int \sigma^m (P - P(\rho_s)) \text{div} u dx \right)_t - m \sigma^{-1} \sigma' \int (P - P(\rho_s)) \text{div} u dx \]

\[ + \int \sigma^m P \nabla u : \nabla u dx + (\gamma - 1) \int \sigma^m P \text{div}(\text{div} u)^2 dx \]

\[ + \int \sigma^m u \cdot \nabla u \cdot \nabla P(\rho_s) dx - \int \sigma^m P u \cdot \nabla u \cdot n ds \]

\[ \leq \left( \int \sigma^m (P - P(\rho_s)) \text{div} u dx \right)_t + C \| \nabla u \|^2_{L^2} + C m \sigma^{-1} \sigma' C_0 \]

\[ - \int_{\partial \Omega} \sigma^m P u \cdot \nabla u \cdot n ds \]

\[ \leq \left( \int \sigma^m (P - P(\rho_s)) \text{div} u dx \right)_t + C \| \nabla u \|^2_{L^2} + C m \sigma^{-1} \sigma' C_0. \]

where in the last inequality, we have utilized the fact that

\[ - \int_{\partial \Omega} \sigma^m P u \cdot \nabla u \cdot n ds = \int_{\partial \Omega} \sigma^m P u \cdot \nabla n \cdot u ds \]

\[ \leq C \int_{\partial \Omega} \sigma^m |u|^2 ds \leq C \sigma^m \| \nabla u \|^2_{L^2}, \]

due to (1.26). Similarly, it indicates that

\[ I_2 = (\lambda + 2\mu) \int \sigma^m \nabla \text{div} u \cdot \dot{u} dx \]

\[ = (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \text{div} (\dot{u} \cdot n) ds - (\lambda + 2\mu) \int \sigma^m \text{div} \dot{u} \cdot u dx \]

\[ = -(\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \text{div} (u \cdot \nabla n \cdot u) ds - \frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\text{div} u)^2 dx \right)_t \]

\[ + \frac{\lambda + 2\mu}{2} \int \sigma^m (\text{div} u)^3 dx - (\lambda + 2\mu) \int \sigma^m \text{div} u \cdot \nabla u dx \]

\[ + \frac{m(\lambda + 2\mu)}{2} \sigma^{-1} \sigma' \int (\text{div} u)^2 dx. \]

Notice that

\[ \left| \int_{\partial \Omega} \text{div} (u \cdot \nabla n \cdot u) ds \right| \]

\[ = \frac{1}{\lambda + 2\mu} \left| \int_{\partial \Omega} (F + P - P(\rho_s))(u \cdot \nabla n \cdot u) ds \right| \]

\[ \leq C \left( \int_{\partial \Omega} |F||u|^2 ds + \int_{\partial \Omega} |u|^2 ds \right) \]

\[ \leq C \| F \|_{H^1} \| u \|^2_{H^1} + \| \nabla u \|^2_{L^2} \]

\[ \leq C (\| \nabla F \|_{L^2} + \| \nabla u \|_{L^2} + 1) \| \nabla u \|^2_{L^2} \]

\[ \leq \frac{1}{2} \| \rho^{\frac{1}{2}} \dot{u} \|^2_{L^2} + C (\| \nabla u \|^2_{L^2} + \| \nabla u \|^2_{L^2}). \]
Therefore,
\[
I_2 \leq -\frac{\lambda + 2\mu}{2} \left( \int \sigma^m(\text{div}u)^2 dx \right) + C\sigma^m\|\nabla u\|_{L^3}^3 \\
+ \frac{1}{2}\sigma^m\|\rho^\frac{1}{2}\dot{u}\|_{L^2}^2 + C\sigma^m\|\nabla u\|^4_{L^2} + C\|\nabla u\|^2_{L^2}.
\] (3.21)

By (1.6), we have
\[
I_3 = -\mu \int \sigma^m \nabla \times \text{curl} \cdot \dot{u} dx \\
= -\mu \int \sigma^m \text{curl} \cdot \text{curl} \dot{u} dx \\
= -\mu \left( \left( \int \sigma^m |\text{curl} u|^2 dx \right) - \frac{\mu m}{2} \left( \sigma^m \sigma' \right) \int |\text{curl} u|^2 dx \right) \\
+ \mu \int \sigma^m \text{curl} (u \cdot \nabla u) dx \\
= -\mu \left( \int \sigma^m |\text{curl} u|^2 dx \right) + \mu m \sigma^{-1} \sigma' \int |\text{curl} u|^2 dx \\
- \mu \int \sigma^m (\nabla u \times \nabla u) \cdot \text{curl} u dx + \frac{\mu}{2} \int \sigma^m |\text{curl} u|^2 div u dx \\
\leq -\frac{\mu}{2} \left( \int \sigma^m |\text{curl} u|^2 dx \right) + C\|\nabla u\|^2_{L^2} + C\sigma^m\|\nabla u\|^3_{L^3}.
\] (3.22)

Finally,
\[
I_4 = \int \sigma^m (\rho - \rho_s) \nabla \psi \cdot \dot{u} dx \\
= \left( \left( \int \sigma^m (\rho - \rho_s) \nabla \psi \cdot u dx \right) + m\sigma^{-1} \sigma' \int (\rho - \rho_s) \nabla \psi \cdot u dx \right) \\
+ \int \sigma^m \rho u \cdot \nabla (\nabla \psi \cdot u) dx + \int \sigma^m (\rho - \rho_s) \nabla \psi \cdot (u \cdot \nabla u) dx \\
\leq \left( \left( \int \sigma^m (\rho - \rho_s) \nabla \psi \cdot u dx \right) + Cm\sigma^{-1} \sigma' C_0 + C\|u\|^2_{L^4} \|\nabla^2 \psi\|_{L^2} \right) \\
+ C\|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla \psi\|_{L^6} \\
\leq \left( \int \sigma^m (\rho - \rho_s) \nabla \psi \cdot u dx \right) + C\|\nabla u\|^2_{L^2} + Cm\sigma^{-1} \sigma' C_0.
\] (3.23)

It follows from (3.15) and (3.16)-(3.23) that
\[
\left( \lambda + 2\mu \right) \int \sigma^m(\text{div}u)^2 dx + \mu \int \sigma^m|\text{curl} u|^2 dx \right) + \int \sigma^m \rho^2 |\dot{u}|^2 dx \\
\leq \left( 2 \int \sigma^m (P - P(\rho_s)) \text{div} u dx + 2 \int \sigma^m (\rho - \rho_s) \nabla \psi \cdot u dx \right) \right) \\
+ Cm\sigma^{-1} \sigma' C_0 + C\sigma^m\|\nabla u\|^2_{L^2} + C\|\nabla u\|^2_{L^2} + C\sigma^m\|\nabla u\|^3_{L^3},
\] (3.24)

which together with (2.11), Lemma 3.2 and Young’s inequality, yields that for any \( m \geq 1 \),
\[
\sigma^m\|\nabla u\|^2_{L^2} + \int_0^T \int \sigma^m \rho^2 |\dot{u}|^2 dx dt \\
\leq CC_0 + C \int_0^T \sigma^m\|\nabla u\|^4_{L^2} dt + C \int_0^T \sigma^m\|\nabla u\|^3_{L^3} dt.
\] (3.25)
Choosing \(m = 1\), and by virtue of the assumption (3.14) and (3.18), we obtain (3.12).

It remains to prove (3.13). In the discussion, we will utilize the following facts more than once, which are given (3.6) and (3.7), that is,

\[
\| F \|_{H^1} + \| \text{curl} u \|_{H^1} \leq C(\| \rho \dot{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| \rho - \rho_s \|_{L^1})
\]  

(3.26)

and

\[
\| \nabla F \|_{L^6} + \| \nabla \text{curl} u \|_{L^6} \\
\leq C(\| \dot{u} \|_{H^1} + \| \nabla u \|_{L^2} + \| \rho - \rho_s \|_{L^\infty}) \\
\leq C(\| \nabla \dot{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla u \|_{L^4}^2 + 1).
\]  

(3.27)

Rewrite (3.14) as

\[
\rho \dot{u} = \nabla F - \mu \nabla \times \text{curl} u + (\rho - \rho_s) \nabla \psi.
\]  

(3.28)

Operating \(\sigma^m \ddot{u}^j [\partial / \partial t + \text{div}(u \cdot)]\) to (3.28), summing with respect to \(j\), and integrating over \(\Omega\), together with (1.11), we get

\[
\left(\frac{m}{2} \int \rho |\ddot{u}|^2 dx\right)_t - \frac{m}{2} \sigma^m \sigma' \int \rho |\ddot{u}|^2 dx \\
= \int \sigma^m (\dot{u} \cdot \nabla F_i + \dot{u}^j \text{div}(u \partial_j F)) dx \\
+ \mu \int \sigma^m (-\dot{u} \cdot \nabla \times \text{curl} u - \dot{u}^j \text{div}(\nabla \times \text{curl} u)^j u)) dx \\
+ \int \sigma^m (\rho \dot{u} \cdot \nabla \psi + \dot{u}^j \text{div}((\rho - \rho_s) \partial_j \psi u)) dx \\
\triangleq J_1 + \mu J_2 + J_3.
\]  

(3.29)

For \(J_1\), by (1.16) and (3.17), a direct computation yields

\[
J_1 = \int \sigma^m \dot{u} \cdot \nabla F_i dx + \int \sigma^m \dot{u}^j \text{div}(u \partial_j F) dx \\
= \int_{\partial \Omega} \sigma^m F_i \dot{u} \cdot nds - \int \sigma^m F_i \text{div}\dot{u} dx - \int \sigma^m u \cdot \nabla \dot{u}^j \partial_j F dx \\
= \int_{\partial \Omega} \sigma^m F_i \dot{u} \cdot nds - (2\mu + \lambda) \int \sigma^m (\text{div}\dot{u})^2 dx \\
+ (2\mu + \lambda) \int \sigma^m \text{div}\dot{u} \nabla u : \nabla \dot{u} dx + \int \sigma^m \text{div}\dot{u} u \cdot \nabla F dx \\
- \gamma \int \sigma^m \text{Pdiv}\text{div}\dot{u} dx - \int \sigma^m u \cdot \nabla \dot{u}^j \partial_j F dx \\
\leq \int_{\partial \Omega} \sigma^m F_i \dot{u} \cdot nds - (2\mu + \lambda) \int \sigma^m (\text{div}\dot{u})^2 dx + \frac{\delta}{2} \sigma^m \| \nabla \dot{u} \|_{L^2}^2 \\
+ \sigma^m \| \nabla F \|_{L^2}^\frac{2}{3} \| \nabla F \|_{L^6}^\frac{1}{3} \| \nabla \dot{u} \|_{L^2} \| \nabla u \|_{L^2} + C(\delta) \sigma^m (\| \nabla u \|_{L^4}^4 + \| \nabla u \|_{L^2}^2) \\
\leq \int_{\partial \Omega} \sigma^m F_i \dot{u} \cdot nds - (2\mu + \lambda) \int \sigma^m (\text{div}\dot{u})^2 dx + \delta \sigma^m \| \nabla \dot{u} \|_{L^2}^2 \\
+ C(\delta) \sigma^m (\| \nabla u \|_{L^4}^4 + \| \nabla u \|_{L^4}^4 + \| \nabla u \|_{L^2}^2)
\]  

(3.30)

where in the third equality we have used

\[
F_i = (2\mu + \lambda) \text{div} u_i - P_i \\
= (2\mu + \lambda) \text{div} u - (2\mu + \lambda) \text{div} (u \cdot \nabla u) + u \cdot \nabla P + \gamma P \text{div} u \\
= (2\mu + \lambda) \text{div} u - (2\mu + \lambda) \nabla u : \nabla u - u \cdot \nabla F + \gamma P \text{div} u.
\]
For the first term on the righthand side of (3.30), we have

\[
\int_{\Omega} \sigma^m F_t \hat{u} \cdot nds \\
= - \int_{\partial\Omega} \sigma^m F_t (u \cdot \nabla n \cdot u) ds \\
= - \left( \int_{\Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + m \sigma^{m-1} \sigma' \int_{\partial\Omega} (u \cdot \nabla n \cdot u) F ds \\
+ \sigma^m \int_{\partial\Omega} F \hat{u} \cdot \nabla n \cdot u ds + \sigma^m \int_{\partial\Omega} F u \cdot \nabla n \cdot \hat{u} ds \\
- \sigma^m \int_{\partial\Omega} F (u \cdot \nabla) u \cdot \nabla n \cdot u ds - \sigma^m \int_{\partial\Omega} F u \cdot \nabla n \cdot (u \cdot \nabla) u ds \\
\leq - \left( \int_{\partial\Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + C m \sigma^{m-1} \| \nabla u \|_{L^2}^2 \| F \|_{H^1}^2 \\
+ \delta \sigma^m \| u \|_{H^1}^2 + C(\delta) \sigma^m \| \nabla u \|_{L^2}^2 \| F \|_{H^1}^2 \\
- \sigma^m \int_{\partial\Omega} F (u \cdot \nabla) u \cdot \nabla n \cdot u ds - \sigma^m \int_{\partial\Omega} F u \cdot \nabla n \cdot (u \cdot \nabla) u ds,
\]

where in the last inequality we have used

\[
\left| \int_{\partial\Omega} (u \cdot \nabla n \cdot u) F ds \right| \leq C \| \nabla u \|_{L^2}^2 \| F \|_{H^1}.
\]  

(3.32)

Since \(u \cdot n|_{\partial\Omega} = 0\), we have

\[
u = -(u \times n) \times n = u^\perp \times n \text{ on } \partial\Omega,
\]

(3.33)

with \(u^\perp \triangleq -u \times n\). And then

\[
- \int_{\partial\Omega} F (u \cdot \nabla) u \cdot \nabla n \cdot u ds \\
= - \int_{\partial\Omega} u^\perp \times n \cdot \nabla u_i \nabla_i n \cdot u F ds \\
= - \int_{\partial\Omega} n \cdot (\nabla u_i \times u^\perp) \nabla_i n \cdot u F ds \\
= - \int_{\Omega} \text{div}(\nabla u^\perp \times u^\perp) \nabla_i n \cdot u F dx \\
= - \int_{\Omega} \nabla (\nabla_i n \cdot u F) \cdot (\nabla u_i \times u^\perp) dx + \int_{\Omega} \nabla u^\perp \cdot \nabla \times u^\perp \nabla_i n \cdot u F dx
\]

(3.34)

\[
\leq C \int_{\Omega} \| \nabla F \| \nabla u \| u \|_{L^2}^2 dx + C \int_{\Omega} |F| (|\nabla u| \| u \| + |\nabla u| \| u \|^2) dx \\
\leq C \| \nabla u \|_{L^6} \| \nabla u \|_{L^2}^3 + C \| F \|_{H^1} \| \nabla u \|_{L^2} (\| \nabla u \|_{L^4}^2 + \| \nabla u \|_{L^2}^2) \\
\leq \delta \| \nabla u \|_{L^2}^2 + C(\delta) \| \nabla u \|_{L^2}^6 + C \| \nabla u \|_{L^4}^2 + C \| \nabla u \|_{L^2}^2 \\
+ C\| \rho u \|_{L^2} \| \nabla u \|_{L^2}^2 + 1,
\]

where in the fourth equality we have used

\[
\text{div}(\nabla u^\perp \times u^\perp) = -\nabla u^\perp \cdot \nabla \times u^\perp.
\]
Similarly, we have

\[- \int_{\partial \Omega} F u \cdot \nabla n \cdot (u \cdot \nabla) v ds \leq C \||F||_{L^6} \||\nabla u||_{L^2}^3 + C \||F||_{H^1} \||\nabla u||_{L^2} (\||\nabla u||_{L^2}^2 + ||\nabla u||_{L^2}^2) \]

\[ \leq \delta \||\nabla u||_{L^2}^2 + C(\delta) \||\nabla u||_{L^2}^6 + C \||\nabla u||_{L^2}^4 + C \||\nabla u||_{L^2}^2 \]

\[ + C \||\rho \hat{u}||_{L^2}^2 (\||\nabla u||_{L^2}^2 + 1) . \]  

(3.35)

It follows from (3.30), (3.36), (3.37), (3.31), (3.34), and (3.35) that

\[ J_1 \leq C m \sigma^{-1} \sigma' (\||\rho \hat{u}||_{L^2}^2 + \||\nabla u||_{L^2}^2 + \||\nabla u||_{L^2}^2) \]

\[ - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + C \delta \sigma^m \||\nabla u||_{L^2}^2 \]

\[ + C(\delta) \sigma^m \||\nabla u||_{L^2}^4 + C(\delta) \sigma^m \||\nabla u||_{L^2}^6 + \delta \sigma^m \||\nabla u||_{L^2}^6 \]

(3.36)

Observing that \( \text{curl} u_t = \text{curl} \hat{u} - u \cdot \nabla \text{curl} u - \nabla u^i \times \nabla u \), we get

\[ J_2 = - \int \sigma^m \text{curl} \hat{u}^2 dx + \int \sigma^m \text{curl} \hat{u} \cdot (\nabla u^i \times \nabla u) dx \]

\[ + \int \sigma^m u \cdot \nabla \text{curl} u \cdot \text{curl} \hat{u} dx + \int \sigma^m u \cdot \nabla \hat{u} \cdot (\nabla \times \text{curl} u) dx \]

(3.37)

Finally,

\[ J_3 = - \int \sigma^m (\rho u \cdot \nabla (\hat{u} \cdot \nabla \psi) + (\rho - \rho_s) (u \cdot \nabla \hat{u}) \cdot \nabla \psi) dx \]

\[ \leq \sigma^m \||u||_{L^3} \||\nabla u||_{L^2} \||\nabla \psi||_{L^6} + \sigma^m \||u||_{L^3} \||\nabla \hat{u}||_{L^2} \||\nabla \psi||_{L^6} \]

\[ \leq \delta \sigma^m \||\nabla u||_{L^2}^2 + C(\delta) \||u||_{L^2}^2 . \]  

(3.38)

Putting (3.36)–(3.38) into (3.29) gives

\[ \left( \sigma^m \||\rho \hat{u}||_{L^2}^2 \right)_t + (\lambda + 2\mu) \sigma^m \||\text{div} \hat{u}||_{L^2}^2 + \mu \sigma^m \||\text{curl} \hat{u}||_{L^2}^2 \]

\[ \leq C m \sigma^{-1} \sigma' (\||\rho \hat{u}||_{L^2}^2 + \||\nabla u||_{L^2}^2 + \||\nabla u||_{L^2}^4) + 2 \delta \sigma^m \||\nabla u||_{L^2}^2 \]

\[ - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + C \sigma^m \||\rho \hat{u}||_{L^2}^2 (\||\nabla u||_{L^2}^2 + 1) \]

\[ + C(\delta) \sigma^m \||\nabla u||_{L^2}^4 + \||\nabla u||_{L^2}^6 + \||\nabla u||_{L^2}^4 . \]  

(3.39)

which together with (3.7) leads to

\[ \left( \sigma^m \||\rho \hat{u}||_{L^2}^2 \right)_t + (\lambda + 2\mu) \sigma^m \||\text{div} \hat{u}||_{L^2}^2 + \mu \sigma^m \||\text{curl} \hat{u}||_{L^2}^2 \]

\[ \leq C m \sigma^{-1} \sigma' (\||\rho \hat{u}||_{L^2}^2 + \||\nabla u||_{L^2}^2 + \||\nabla u||_{L^2}^4) \]

\[ - 2 \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + C \sigma^m \||\rho \hat{u}||_{L^2}^2 (\||\nabla u||_{L^2}^2 + 1) \]

\[ + C \sigma^m \||\nabla u||_{L^2}^6 + \||\nabla u||_{L^2}^4 . \]

(3.40)
Combining this, (3.32), and (3.4) gives (3.13) by taking \( m = 3 \) in (3.40). We finish the proof of Lemma 3.3.

**Lemma 3.4** Assume that \((\rho, u)\) is a smooth solution of (1.1) - (1.6) satisfying \( \rho \leq 2\bar{\rho} \) and the initial data condition \( \| u_0 \|_{H^1} \leq M \) in (1.16), then there exist positive constants \( K \) and \( \varepsilon_1 \) depending only on \( \mu, \lambda, \gamma, a, \inf \psi, \| \psi \|_{H^2}, \bar{\rho}, \Omega \) and \( M \) such that

\[
A_3(\sigma(T)) + \int_0^{\sigma(T)} \| \rho^{1/2}\dot{u} \|_{L^2}^2 dt \leq K, \tag{3.41}
\]

provide that \( C_0 < \varepsilon_1 \).

**Proof.** Choosing \( m = 0 \) in (3.24) and integrating over \((0, \sigma(T))\), we deduce from (2.11), (2.17) and (3.8) that

\[
A_3(\sigma(T)) + \int_0^{\sigma(T)} \| \rho^{1/2}\dot{u} \|_{L^2}^2 dxdt \leq \frac{1}{2} C_1 (C_0 + M) + \frac{1}{2} \int_0^{\sigma(T)} \| \rho^{1/2}\dot{u} \|_{L^2}^2 dxdt + \frac{1}{2} C_2 C_0 A_3(\sigma(T))(A_3(\sigma(T)) + 1),
\]

where we have used the fact that

\[
\int_0^{\sigma(T)} \| \nabla u \|_{L^2}^3 dt \leq C \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2} \| \rho^{1/2}\dot{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| \rho - \rho_s \|_{L^2} \right) dt \leq CC_0 + CC_0 A_3(\sigma(T))(A_3(\sigma(T)) + 1) + \frac{1}{2} \int_0^{\sigma(T)} \| \rho^{1/2}\dot{u} \|_{L^2}^2 dxdt.
\]

Hence,

\[
A_3(\sigma(T)) + \int_0^{\sigma(T)} \| \rho^{1/2}\dot{u} \|_{L^2}^2 dxdt \leq C_1 (C_0 + M) + C_2 C_0 A_3(\sigma(T))(A_3(\sigma(T)) + 1).
\]

Now we can choose a positive constant \( K \) such that \( K \geq 2C_1 (M + 1) \), as a result, if \( A_3(\sigma(T)) < 2K \) and \( C_0 < \varepsilon_1 = \min\{1, 1/(8(K + 1)C_2)\} \), then we establish (3.41).

**Lemma 3.5** Assume that \((\rho, u)\) is a smooth solution of (1.1) - (1.6) satisfying (3.4) with \( K \) given by Lemma 3.4 and the initial data condition \( \| u_0 \|_{H^1} \leq M \) in (1.16). Then there exists a positive constant \( C \) depending only on \( \mu, \lambda, \gamma, a, \inf \psi, \| \psi \|_{H^2}, \bar{\rho}, \Omega \) and \( M \) such that

\[
\sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 + \int_0^T \| \rho^{1/2}\dot{u} \|_{L^2}^2 dt \leq C, \tag{3.42}
\]

\[
\sup_{0 \leq t \leq T} \sigma \| \rho^{1/2}\dot{u} \|_{L^2}^2 + \int_0^T \sigma \| \nabla u \|_{L^2}^2 dt \leq C, \tag{3.43}
\]

provide that \( C_0 < \varepsilon_1 \).
Proof. (3.32) is an immediate result of (3.31) and (3.4). It remains to prove (3.33).

Choosing $m = 1$ in (3.40), by (3.4), (3.7), (3.8), (3.20), (3.32) and (3.34),

$$\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 dt$$

$$x \leq C + \frac{1}{2} \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \int_0^T \sigma \|\nabla u\|_{L^2}^2 dt$$

$$\leq C + C \int_0^T \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 \|\nabla u\|_{L^2}^2 dt + \frac{1}{2} \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2$$

$$\leq C + C \sup_{0 \leq t \leq T} (\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2)^{\frac{1}{2}} + \frac{1}{2} \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2,$$

which gives (3.33).

Lemma 3.6 Let $(\rho, u)$ be a smooth solution of (1.11) on $\Omega \times (0, T]$ satisfying (3.3). Then there exists a positive constant $\varepsilon_2$ depending only on $\mu, \gamma, a, \rho, \mathcal{P}$ and $\rho$ and $\Omega$ such that

$$\int_0^T \|\rho - \rho_s\|_{L^2}^2 dt \leq C_0.$$  \hfill (3.44)

Proof. Using (1.27), we can rewrite (1.11) as

$$- \nabla \left( \rho^{-1}_s (P - P(\rho_s)) \right) = \rho^{-1}_s (\rho \dot{u} - (\lambda + \mu) \nabla \text{div} u - \mu \Delta u) - \frac{\gamma - 1}{a} \nabla \rho^{-1}_s G(\rho, \rho_s).$$ \hfill (3.45)

Multiplying (3.45) by $\mathcal{B}[\rho - \rho_s]$ and integrating over $\Omega$, by (2.8), one has

$$\int \rho^{-1}_s (P - P(\rho_s))(\rho - \rho_s) dx$$

$$= \left( \int \rho^{-1}_s \rho u \cdot \mathcal{B}[\rho - \rho_s] dx \right)_t - \int \rho u \cdot \mathcal{B}[\rho_t] dx$$

$$- \int \rho^{-1}_s \rho u^j \partial_j \mathcal{B}_i(\rho - \rho_s) dx - \int \rho u^j \partial_j \rho^{-1}_s \mathcal{B}_j(\rho - \rho_s) dx$$

$$+ \mu \int \rho^{-1}_s \nabla u : \nabla \mathcal{B}(\rho - \rho_s) dx + \mu \int \partial_i u^j \partial_j \rho^{-1}_s \mathcal{B}_j(\rho - \rho_s) dx$$

$$+ (\lambda + \mu) \int \left( \rho^{-1}_s (\rho - \rho_s) + \nabla \rho^{-1}_s \cdot \mathcal{B}(\rho - \rho_s) \right) \nabla u dx$$

$$- \frac{\gamma - 1}{a} \int G(\rho, \rho_s) \nabla \rho^{-1}_s \cdot \mathcal{B}(\rho - \rho_s) dx$$

$$\leq \left( \int \rho^{-1}_s \rho u \cdot \mathcal{B}[\rho - \rho_s] dx \right)_t + C \|\nabla u\|_{L^2}^2 + C \|\rho u\|_{L^6}^6 \|\nabla \rho^{-1}_s\|_{L^6} \|\mathcal{B}[\rho - \rho_s]\|_{L^2}$$

$$+ C \|\nabla u\|_{L^2} \|\mathcal{B}[\rho - \rho_s]\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla \rho^{-1}_s\|_{L^6} \|\mathcal{B}[\rho - \rho_s]\|_{L^3}$$

$$+ C \|\nabla u\|_{L^2} \|\rho - \rho_s\|_{L^2} + C \|G(\rho, \rho_s)\|_{L^3} \|\nabla \rho^{-1}_s\|_{L^6} \|\mathcal{B}[\rho - \rho_s]\|_{L^6}$$

$$\leq \left( \int \rho u \cdot \mathcal{B}[\rho - \rho_s] dx \right)_t + \delta \|\rho - \rho_s\|_{L^2}^2 + C(\delta) \|\nabla u\|_{L^2}^2 + C_3 \|\rho - \rho_s\|_{L^2}^2.$$  \hfill \(\Box\)
Lemma 3.7 Let $(\rho,u)$ be a smooth solution of (1.1)–(1.6) on $\Omega \times (0,T]$ satisfying (3.4) and the initial data condition $\|u_0\|_{H^1} \leq M$ in (1.16). Then there exists a positive constant $\varepsilon_3$ depending only on $\mu, \lambda, \gamma, a, \inf_{\Omega_{\varepsilon}} \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega$ and $M$ such that

$$A_1(T) + A_2(T) \leq C_0^\frac{1}{7},$$

(3.46)

provided $C_0 \leq \varepsilon_3$.

Proof. By (2.17), (3.4), (3.44), and Lemmas 3.2 and 3.5 we get

$$\int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 dt$$

$$\leq C \int_0^T \sigma^3 \|\rho^\frac{3}{2} \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2} dt + C \int_0^T \sigma^3 (\|\nabla u\|_{L^2}^4 + \|\rho - \rho_s\|_{L^4}^4) dt$$

$$\leq C \left( \int_0^T (\sigma^3 \|\rho^\frac{3}{2} \dot{u}\|_{L^2}^3) (\sigma \|\rho^\frac{3}{2} \ddot{u}\|_{L^2}^2) (\sigma \|\nabla u\|_{L^2}^2)^\frac{1}{2} dt \right) + C \left( \int_0^T (\sigma \|\nabla u\|_{L^2}^2) \|\nabla u\|_{L^2}^2 dt + \int_0^T \sigma^3 \|\rho - \rho_s\|_{L^2}^2 dt \right)$$

$$\leq C \left[ (A_1^\frac{1}{7}(T) + C_0^\frac{1}{7}) A_2^\frac{1}{7}(T) A_1(T) + C_0 \right]$$

$$\leq CC_0,$$

(3.47)

which along with (3.12) and (3.13) gives

$$A_1(T) + A_2(T) \leq CC_0 + C \int_0^T \sigma \|\nabla u\|_{L^3}^3 dt.$$  

(3.48)

For the last term on the right-hand side of (3.48), on the one hand, we deduce from (2.17), (3.4), (3.44) and Lemmas 3.2 that

$$\int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^3 dt$$

$$\leq C \int_0^{\sigma(T)} \sigma \|\rho^\frac{3}{2} \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2}^3 dt + C \int_0^{\sigma(T)} \sigma (\|\nabla u\|_{L^2}^4 + \|\rho - \rho_s\|_{L^3}^4) dt$$

$$\leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2}^3 (\sigma \|\rho^\frac{3}{2} \ddot{u}\|_{L^2}^2)^\frac{1}{2} dt + CC_0$$

$$\leq C \left( \sup_{0 \leq \tau \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \right)^\frac{1}{2} \left( \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt \right)^{\frac{1}{4}} \left( \int_0^{\sigma(T)} \sigma \|\rho \ddot{u}\|_{L^2}^2 dt \right)^{\frac{3}{4}} + CC_0$$

$$\leq C (A_1(T))^\frac{3}{4} C_0^\frac{1}{4} + CC_0$$

$$\leq CC_0^\frac{5}{3},$$

(3.49)

provided $C_0 < \hat{\varepsilon}_2 = \min\{\varepsilon_1, \varepsilon_2\}$.

On the other hand, by (3.47) and (3.8),

$$\int_{\sigma(T)}^{T} \sigma \|\nabla u\|_{L^3}^3 dt \leq \int_{\sigma(T)}^{T} \sigma \|\nabla u\|_{L^4}^4 dt + \int_{\sigma(T)}^{T} \sigma \|\nabla u\|_{L^2}^2 dt \leq CC_0.$$  

(3.50)
Hence, by (3.48)-(3.50),
\[ A_1(T) + A_2(T) \leq C_4 C_5^\frac{\hat{\rho}}{4}, \]
which yields that (3.46) holds, provided that \( C_0 < \varepsilon_3 \triangleq \min\{\hat{\varepsilon}_2, (\frac{1}{C_4})^8\} \).

To give a uniform (in time) upper bound for the density, which is crucial to get all the higher order estimates and thus to extend the classical solution globally. We will adopt an approach motivated by the work of [18], see also [14].

**Lemma 3.8** There exists a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \gamma, a, \inf \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega \) and \( M \) such that, if \((\rho,u)\) is a smooth solution of (1.1)-(1.6) on \( \Omega \times (0,T] \) satisfying (3.4) and the initial data condition \( \|u_0\|_{H^s} \leq M \) in (1.16), then
\[ \sup_{0 \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \frac{7\hat{\rho}}{4}, \]
provided \( C_0 \leq \varepsilon \).

**Proof.** Denote
\[ D_t \rho \triangleq \rho_t + u \cdot \nabla \rho, \quad g(\rho) \triangleq -\rho(P - P(\rho_s)) \frac{2\mu}{2\mu + \lambda}, \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_0^t \rho F dt, \]
then (1.1) can be rewritten as
\[ D_t \rho = g(\rho) + b'(t). \]
By Lemma 2.3, it is sufficient to check that the function \( b(t) \) must verify (2.3) with some suitable constants \( N_0, N_1 \). First, it follows from (2.2), (2.12), (2.16), (3.6), (3.4), (3.8) and Lemma 3.5 that for
\[ 0 \leq t_1 \leq t_2 \leq \sigma(T), \]

\[ |b(t_2) - b(t_1)| \leq C \int_0^{\sigma(T)} \|(\rho F)(\cdot, t)\|_{L^\infty} dt \]

\[ \leq C \int_0^{\sigma(T)} \|F\|_{L^4} \|\nabla F\|_{L^4} dt + C \int_0^{\sigma(T)} \|F\|_{L^2} dt \]

\[ \leq C \int_0^{\sigma(T)} (\|\rho^{\frac{1}{3}} \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^4}) (\|\rho \dot{u}\|_{L^4} + \|\rho - \rho_s\|_{L^1}) dt \]

\[ + C \int_0^{\sigma(T)} (\|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2}) dt \]

\[ \leq C \int_0^{\sigma(T)} (\|\rho^{\frac{1}{3}} \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^4}) (\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2}) dt + C\rho^{\frac{1}{3}} \]

\[ \leq C \int_0^{\sigma(T)} \sigma^{\frac{1}{2}} (\|\rho^{\frac{1}{3}} \dot{u}\|_{L^2}) \rho^{\frac{1}{2}} (\|\nabla \dot{u}\|_{L^2}) \|\nabla \dot{u}\|_{L^2} dt + C \rho^{\frac{1}{2}} \]

\[ + C \int_0^{\sigma(T)} (\|\rho^{\frac{1}{3}} \dot{u}\|_{L^2}) \rho^{\frac{1}{2}} (\|\nabla \dot{u}\|_{L^2}) \|\nabla \dot{u}\|_{L^2} dt \]

\[ \leq C \left( \int_0^{\sigma(T)} \sigma^{\frac{1}{2}} (\|\rho^{\frac{1}{3}} \dot{u}\|_{L^2}) \right)^{\frac{2}{3}} + C \left( \int_0^{\sigma(T)} t^{-\frac{3}{2}} (\|\nabla \dot{u}\|_{L^2})^2 dt \right)^{\frac{1}{3}} + C \rho^{\frac{1}{2}} \]

\[ \leq C A_1 (\sigma(T))^{\frac{1}{3}} + C \rho^{\frac{1}{2}} \]

\[ \leq C_5 C_0^{\frac{1}{20}}, \]

provided \( C_0 < \varepsilon_1 \).

By (3.53) and (3.52), choosing \( N_1 = 0, N_0 = C_5 C_0^{\frac{1}{20}}, \bar{\zeta} = \hat{\rho} \) in Lemma 2.3 gives

\[ \sup_{t \in [0, \sigma(T)]} \|\rho\|_{L^\infty} \leq \hat{\rho} + C_5 C_0^{\frac{1}{20}} \leq \frac{3\hat{\rho}}{2}, \]

(3.54)

provided \( C_0 \leq \varepsilon_3 \triangleq \min\{\varepsilon_3, (\frac{\hat{\rho}}{C_0})^{20}\} \).

On the other hand, for \( \sigma(T) \leq t_1 \leq t_2 \leq T \), we deduce from (2.12), (2.14), (3.6),
\[ |b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} \|F\| \, dt \]
\[ \leq a \lambda + 2 \mu (t_2 - t_1) + C \int_{t_1}^{t_2} \left( \|F\|^\frac{6}{5}_{L^6} + \|F\|^\frac{6}{5}_{L^2} \right) \, dt \]
\[ \leq \frac{a}{\lambda + 2 \mu} (t_2 - t_1) + \frac{a}{\lambda + 2 \mu} \int_{t_1}^{t_2} \|\psi\|_{H^1}^2 \, dt \]
\[ \leq \frac{a}{\lambda + 2 \mu} (t_2 - t_1) + C_0 C_1^\frac{1}{2}. \]

Now we choose \( N_0 = C_0 C_1^\frac{1}{2} \), \( N_1 = \frac{a}{\lambda + 2 \mu} \) in (2.3) and set \( \bar{\zeta} = \frac{3 \hat{\rho}}{2} \) in (2.4). Notice that for all \( \zeta \geq \bar{\zeta} = \frac{3 \hat{\rho}}{2} > \hat{\rho} + 1 \),
\[ g(\zeta) = -a \zeta^2 + \lambda (\zeta^\gamma - \rho^\gamma) \leq -a \frac{\lambda + 2 \mu}{\lambda + 2 \mu} = -N_1. \]

Consequently, by (3.52), (3.55) and Lemma 2.3, we have
\[ \sup_{t \in [\sigma(T), T]} \|\rho\|_{L^\infty} \leq \frac{3 \hat{\rho}}{2} + C_0 C_1^\frac{1}{2} \leq \frac{7 \hat{\rho}}{2}, \quad (3.56) \]
provided
\[ C_0 \leq \epsilon \triangleq \min\{\epsilon \frac{3}{2}, \left( \frac{\hat{\rho}}{4C_1} \right)^2 \}. \quad (3.57) \]

The combination of (3.54) with (3.56) completes the proof of Lemma 3.8.

4 Time-dependent higher-order estimates

Let \((\rho, u)\) be a smooth solution of (1.1)-(1.6). This section is devoted to deriving some necessary higher order estimates, which play an important role in proving that the classical or strong solution exists globally in time. We will adopt some ideas of the article [14, 19] with slight modifications. In this section, we always assume that the initial energy \( C_0 \) satisfies (3.57).

4.1 Estimates for strong solutions

Lemma 4.1 For \( q \in (3, 6) \) as in Theorem 1.1, it holds that for \( r = (9q - 6)/(10q - 12) \)
\[ \sup_{0 \leq t \leq T} \left( \|\nabla \rho\|_{L^q} + \|u\|_{H^2} \right) + \int_0^T \left( \|\nabla u\|_{L^\infty}^r + \|\nabla^2 u\|_{L^2}^r \right) \, dt \leq C, \quad (4.1) \]
where and in what follows, \( C \) is a positive constant depending on \( T, \rho_0 - \rho_s, \|\psi\|_{W^{1,q}}, \mu, \lambda, \gamma, a, \inf_{\Omega} \psi, \|\psi\|_{H^2}, \hat{\rho}, \Omega \) and \( M \).
Proof. By \(1\), it is clear that \(|\nabla \rho|^p, p \in [2, 6]\) satisfies

\[
(\|\nabla \rho\|_p)_t + \text{div}(\|\nabla \rho\|_p u) + (p-1)\|\nabla \rho\|_p \text{div}u \\
+ p\|\nabla \rho\|^{p-2}(\nabla \rho)^{tr} \nabla u(\nabla \rho) + p\rho \|\nabla \rho\|^{p-2}\nabla \rho \cdot \nabla u = 0,
\]

where \((\nabla \rho)^{tr}\) is the transpose of \(\nabla \rho\).

Therefore, by \((2.12), (3.6)\) and \((3.42)\),

\[
(\|\nabla \rho\|_p)_t \leq C(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^p} + C\|\nabla F\|_{L^p} \\
\leq C(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^p} + C(\|\rho \dot{u}\|_{L^P} + 1). \quad (4.2)
\]

By Lemma \(2.7\), \((2.21)\) and \((3.6)\), it indicates that

\[
\|\nabla u\|_{L^\infty} \leq C(\|\text{div}u\|_{L^\infty} + \|\text{curl}u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^p}) + C\|\nabla u\|_{L^2} + C \\
\leq C(1 + \|\rho \dot{u}\|_{L^p}) \ln(e + \|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p}). \quad (4.3)
\]

where in the second inequality, we have taken advantage of the fact

\[
\|\text{div}u\|_{L^\infty} + \|\text{curl}u\|_{L^\infty} \\
\leq C(\|F\|_{L^\infty} + \|P - P(\rho_s)\|_{L^\infty}) + \|\text{curl}u\|_{L^\infty} \\
\leq C(\|F\|_{L^2} + \|\nabla F\|_{L^p} + \|\text{curl}u\|_{L^2} + \|\nabla \text{curl}u\|_{L^p} + \|P - P(\rho_s)\|_{L^\infty}) \\
\leq C(\|\rho \dot{u}\|_{L^p} + 1), \quad (4.4)
\]

which is due to Gagliardo-Nirenberg’s inequality, \((3.6), (2.12), (2.13)\) and \((3.42)\).

Consequently,

\[
(e + \|\nabla \rho\|_{L^p})_t \leq C(1 + \|\rho \dot{u}\|_{L^p}) \ln(e + \|\rho \dot{u}\|_{L^p}) \ln(e + \|\nabla \rho\|_{L^p}) (e + \|\nabla \rho\|_{L^p}), \quad (4.5)
\]

which implies that

\[
(\ln(e + \|\nabla \rho\|_{L^p}))_t \leq C(1 + \|\rho \dot{u}\|_{L^p}) \ln(e + \|\rho \dot{u}\|_{L^p}) \ln(e + \|\nabla \rho\|_{L^p}). \quad (4.6)
\]

Notice that, by Lemma \(3.5\),

\[
\int_0^T \|\rho \dot{u}\|_{L^3} dt \\
\leq \int_0^T \|\rho^{1/2} \dot{u}\|_{L^2}^{(6-\eta)r/2q} (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{(3q-6)r/2q} dt \\
\leq \int_0^T \sigma^{-1/2}(\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^{2(6-\eta)r/4q} (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{\frac{3q-6}{4q}} dt \\
\leq (\int_0^T \sigma^{\frac{2q-6}{6q-3q+6r}} dt)^{\frac{4q-3q+6r}{4q}} (\int_0^T \sigma \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) dt)^{\frac{4r}{6q-3q+6r}} \\
\leq C,
\]

which together with \((4.7)\) and Gronwall’s inequality shows

\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^3} \leq C.
\]

Combining this with \((4.3), (4.7), (4.13), (2.21)\) and Lemma \(3.5\) proves \((4.1)\).
Lemma 4.2 There exists a positive constant $C$ such that
\[
\sup_{0 \leq t \leq T} (\| \rho_t \|_{L^2} + \sigma \| \rho^\frac{1}{2} u_t \|_{L^2}) + \int_0^T (\| \rho^\frac{1}{2} u_t \|_{L^2}^2 + \sigma \| \nabla u_t \|_{L^2}^2) dt \leq C. 
\] (4.8)

Proof. First, by (1.1) and (4.1),
\[
\| \rho_t \|_{L^2} \leq \| u \|_{L^6} \| \nabla \rho \|_{L^3} + \| \nabla u \|_{L^2} \leq C.
\]
Next, a direct calculation gives
\[
\| \rho^\frac{1}{2} u_t \|_{L^2} \leq C(\| \rho^\frac{1}{2} \dot{u} \|_{L^2} + \| \nabla u \|_{H^1})
\]
which together with (4.1) leads to
\[
\sup_{0 \leq t \leq T} \sigma \| \rho^\frac{1}{2} u_t \|_{L^2} + \int_0^T \| \rho^\frac{1}{2} u_t \|_{L^2}^2 dt \leq C.
\]
Similarly,
\[
\int_0^T \sigma \| \nabla u_t \|_{L^2}^2 dt \leq C \int_0^T \sigma (\| \nabla \dot{u} \|_{L^2}^2 + \| \nabla (u \cdot \nabla u) \|_{L^2}^2) dt \\
\leq C \int_0^T \sigma (\| \nabla \dot{u} \|_{L^2}^2 + \| \nabla u \|_{H^1}^4) dt.
\]
\[
\square
\]

4.2 Estimates for classical solutions

In this section, we always assume that the initial energy $C_0$ satisfies (3.57), $\psi \in H^3$, and that the positive constant $C$ may depend on
\[
T, \| g \|_{L^2}, \| \nabla u_0 \|_{H^1}, \| \rho_0 - \rho_\ast \|_{W^{2,q}}, \| P(\rho_0) - P(\rho_\ast) \|_{W^{2,q}}, \| \psi \|_{W^{2,q}},
\]
besides $\mu, \lambda, \gamma, a, \inf_\Omega \psi, \bar{\rho}, \Omega$ and $M$, where $q \in (3, 6)$ and $g \in L^2$ is given as in (1.20).

Lemma 4.3 There exists a positive constant $C$, such that
\[
\sup_{0 \leq t \leq T} (\| \rho^\frac{1}{2} \dot{u} \|_{L^2} + \int_0^T \| \nabla \dot{u} \|_{L^2}^2 dt) \leq C, 
\] (4.9)
\[
\sup_{0 \leq t \leq T} (\| \nabla \rho \|_{L^6} + \| u \|_{H^2}) + \int_0^T (\| \nabla u \|_{L^\infty} + \| \nabla^2 u \|_{L^6}^2) dt \leq C. 
\] (4.10)

Proof. First, choosing $m = 0$ in (3.40), by (2.97), we have
\[
\left(\| \rho^\frac{1}{2} \dot{u} \|_{L^2}^2\right)_t + \| \nabla \dot{u} \|_{L^2}^2 \\
\leq - \left(\int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds\right)_t + C \| \rho^\frac{1}{2} \dot{u} \|_{L^2}^2 (\| \nabla u \|_{L^2}^4 + 1) \\
+ C (\| \nabla u \|_{L^2}^2 + \| u \|_{L^2}^6 + \| \nabla u \|_{L^4}) \\
\leq - \left(\int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds\right)_t + C \| \rho^\frac{1}{2} \dot{u} \|_{L^2}^2 (\| \rho^\frac{1}{2} \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^4 + 1) \\
+ C (\| \nabla u \|_{L^2}^2 + \| u \|_{L^2}^6 + \| P - P(\rho_\ast) \|_{L^4}^4 + \| P - P(\rho_\ast) \|_{L^4}^4 + \| \rho - \rho_\ast \|_{L^2}^2). 
\]
By Gronwall’s inequality and the compatibility condition (1.20), we deduce (4.9) from (4.11), (3.42) and (3.32). Furthermore, by (3.6),
\[
\int_0^T \| \rho \dot{u} \|_{L^6}^2 \, dt \leq \int_0^T \| \nabla \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^4 \, dt \leq C.
\] (4.12)
As a result, setting \( p = 6 \) in (4.6), and by Gronwall’s inequality again, we have
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} \leq C.
\] (4.13)
Finally, by (4.3) and (2.21), together with (3.6), (3.42) and (4.9), we have
\[
\int_0^T \| \nabla u \|_{L^\infty} \, dt \leq C, \quad \int_0^T \| \nabla^2 u \|_{L^6}^2 \, dt \leq C \quad \text{and} \quad \sup_{0 \leq t \leq T} \| u \|_{H^2} \leq C.
\]
This completes the proof of Lemma 4.3.

**Lemma 4.4** There exists a positive constant \( C \) such that
\[
\sup_{0 \leq t \leq T} \| \rho \frac{1}{2} u_t \|_{L^2}^2 + \int_0^T \int |\nabla u_t|^2 \, dx \, dt \leq C,
\] (4.14)
\[
\sup_{0 \leq t \leq T} (\| \rho - \rho_s \|_{H^2} + \| P - P(\rho_s) \|_{H^2}) \leq C.
\] (4.15)

**Proof.** By Lemma 4.3, a straightforward calculation shows that
\[
\| \rho \frac{1}{2} u_t \|_{L^2}^2 \leq \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 + \| \rho \frac{1}{2} u \cdot \nabla u \|_{L^2}^2 \leq C + C \| u \|_{L^4}^2 \| \nabla u \|_{L^4}^2 \leq C + C \| \nabla u \|_{L^2}^2 \| u \|_{H^2}^2 \leq C,
\] (4.16)
and that
\[
\int_0^T \| \nabla u_t \|_{L^2}^2 \, dt \leq \int_0^T \| \nabla \dot{u} \|_{L^2}^2 \, dt + \int_0^T \| \nabla (u \cdot \nabla u) \|_{L^2}^2 \, dt \leq C + \int_0^T \| \nabla u \|_{L^4}^4 + \| u \|_{L^\infty}^2 \| \nabla^2 u \|_{L^2}^2 \, dt \leq C + C \int_0^T (\| \nabla^2 u \|_{L^2}^4 + \| \nabla u \|_{H^1}^2 \| \nabla^2 u \|_{L^2}^2) \, dt \leq C,
\] (4.17)
and then (4.14) holds.

Using (3.17), (1.11), (2.21), (2.22) and Lemma 4.3 we have
\[
\frac{d}{dt} (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) \leq C(1 + \| \nabla u \|_{L^\infty}) (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) + C \| \nabla \dot{u} \|_{L^2}^2 + C.
\] (4.18)
Combining this with Gronwall’s inequality and Lemma 4.3 implies that
\[
\sup_{0 \leq t \leq T} (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) \leq C.
\]
Thus we finish the proof of Lemma 4.4. \( \square \)
Lemma 4.5 There exists a positive constant $C$, such that

$$\sup_{0 \leq t \leq T} (\|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2) \, dt \leq C, \quad (4.19)$$

$$\sup_{0 \leq t \leq T} \sigma \|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma \|\rho^2 u_{tt}\|_{L^2}^2 \, dt \leq C. \quad (4.20)$$

**Proof.** By (3.17) and Lemma 4.3,

$$\|P_t\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla P\|_{L^2} + C \|\nabla u\|_{L^2} \leq C. \quad (4.21)$$

Differentiating (3.17) yields

$$\nabla P_t + u \cdot \nabla P + \frac{\gamma}{2} \nabla P \div u + \gamma P \nabla \div u = 0.$$ 

Hence, by Lemmas 4.3 and 4.4,

$$\|\nabla P_t\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla^2 P\|_{L^2} + C \|\nabla u\|_{L^3} \|\nabla P\|_{L^6} + C \|\nabla^2 u\|_{L^2} \leq C, \quad (4.22)$$

which together with (4.21) yields

$$\sup_{0 \leq t \leq T} \|P_t\|_{H^1} \leq C. \quad (4.23)$$

By (3.17) again, we find that $P_{tt}$ satisfies

$$P_{tt} + \gamma P_t \div u + \gamma P \div u_t + u_t \cdot \nabla P + u \cdot \nabla P_t = 0. \quad (4.24)$$

Multiplying (4.24) by $P_{tt}$ and integrating over $\Omega \times [0, T]$, by (4.23), Lemmas 4.3 and 4.4 we obtain that

$$\int_0^T \|P_{tt}\|_{L^2}^2 \, dt$$

$$= -\int_0^T \int \gamma P_{tt} P_t \div u \, dx \, dt - \int_0^T \int \gamma P_{tt} P_{tt} \div u_t \, dx \, dt$$

$$- \int_0^T \int P_{tt} u_t \cdot \nabla P \, dx \, dt - \int_0^T \int P_{tt} u_t \cdot \nabla P_t \, dx \, dt$$

$$\leq C \int_0^T \|P_t\|_{L^2} (\|P_t\|_{L^3} \|\nabla u\|_{L^6} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^3} \|\nabla P\|_{L^6} + \|u\|_{L^\infty} \|\nabla P_t\|_{L^2}) \, dt$$

$$\leq C \int_0^T \|P_t\|_{L^2} (1 + \|\nabla u_t\|_{L^2}) \, dt$$

$$\leq \frac{1}{2} \int_0^T \|P_{tt}\|_{L^2}^2 \, dt + C,$$

which gives

$$\int_0^T \|P_{tt}\|_{L^2}^2 \, dt \leq C.$$

We can deal with $\rho_t$ and $\rho_{tt}$ similarly and get (4.19).

Finally, we will prove (4.20). Introducing the function

$$H(t) = (\lambda + 2\mu) \int (\div u_t)^2 \, dx + \mu \int |\curl u_t|^2 \, dx.$$
using \( u_t \cdot n = 0 \) on \( \partial \Omega \) and Lemma 2.5 one has
\[
\| \nabla u_t \|_{L^2}^2 \leq CH(t). \tag{4.25}
\]

Differentiating (1.1) with respect to \( t \) and multiplying by \( u_{tt} \), we have
\[
\frac{d}{dt} H(t) + 2 \int \rho |u_t|^2 dx = \frac{d}{dt} \left( -\int \rho_t |u_t|^2 dx - 2 \int \rho_t u \cdot \nabla u \cdot u_t dx + \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx - 2 \right) 
\]
\[
\int \rho_t |u_t|^2 dx + 2 \int (\rho_t |u_t|^2 dx - 2 \int \rho u \cdot \nabla u \cdot u_{tt} dx 
\]
\[
- 2 \int \rho u \cdot \nabla u \cdot u_{tt} dx - 2 \int P_t \rho \cdot \nabla \psi \cdot u_t dx 
\]
\[
\Delta = \frac{d}{dt} I_0 + \sum_{i=1}^{6} I_i. \tag{4.26}
\]

We have to estimate \( I_i \) (\( i = 0, 1, \cdots, 6 \)) one by one. It follows from (1.1), (3.6), (3.9), (4.10), (3.32), (4.14), (4.19), (4.25) and Sobolev’s and Poincaré’s inequalities that
\[
|I_0| = \left| -\int \rho_t |u_t|^2 dx - 2 \int \rho_t u \cdot \nabla u \cdot u_t dx + \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx \right| 
\]
\[
\leq \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx + \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx + \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx 
\]
\[
\leq C \int u \rho u_t \cdot \nabla u_t dx + C \| u \|_{L^2} \|
\]
\[
\leq C \| u \|_{L^6} \rho^{1/2} u_t \|_{L^2}^{1/2} \| u_t \|_{L^6}^{1/2} \| \nabla u_t \|_{L^2} + C \| \nabla u_t \|_{L^2} 
\]
\[
\leq C \| \nabla u_t \|_{L^2} \rho^{1/2} u_t \|_{L^2}^{1/2} \| u_t \|_{L^6}^{3/2} + C \| \nabla u_t \|_{L^2} 
\]
\[
\leq \frac{1}{2} H(t) + C, \tag{4.27}
\]
\[
|I_1| = \int \rho u_t \| u_t \|_{L^2}^2 dx = \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx \leq C \left( \| \rho_t \|_{H^1} \| u \|_{H^2} + \| \rho^{1/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} \right) \| \nabla u_t \|_{L^2}^2 \tag{4.28}
\]
\[
\leq C \| \nabla u_t \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2 + C \leq C \| \nabla u_t \|_{L^2}^2 H(t) + C \| \nabla u_t \|_{L^2}^2 + C, \tag{4.29}
\]
\[
|I_2| = 2 \left| \int (\rho u_t \cdot \nabla u)_{tt} \cdot u_t dx \right| 
\]
\[
\leq \| \rho u_t \|_{L^2} \| u \cdot \nabla u_t \|_{L^2} \| u_t \|_{L^6} + \| \rho u_t \|_{L^6} \| \nabla u_t \|_{L^6} \| \nabla u_t \|_{L^6} \| u_t \|_{L^6} 
\]
\[
\leq C \| \rho u_t \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2, \tag{4.29}
\]

\[ I_3 + I_4 = 2 \left| \int \rho u_t \cdot \nabla u \cdot u_t dx \right| + 2 \left| \int \rho u \cdot \nabla u_t \cdot u_t dx \right| \]
\[ \leq C \| \rho^{1/2} u_t \|_{L^2} (\| u_t \|_{L^6} \| \nabla u \|_{L^3} + \| u \|_{L^\infty} \| \nabla u_t \|_{L^2}) \]
\[ \leq \| \rho^{1/2} u_t \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2, \tag{4.30} \]
and
\[ I_5 + I_6 = 2 \left| \int P_t \div u_t dx \right| + 2 \left| \int \rho_t \nabla \psi \cdot u_t dx \right| \]
\[ \leq C \| P_t \|_{L^2} \| \div u_t \|_{L^2} + \| \rho_t \|_{L^3} \| \nabla u_t \|_{L^2} \| \nabla \psi \|_{L^6} \]
\[ \leq C \| P_t \|_{L^2}^2 + C \| \rho_t \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2. \tag{4.31} \]

Consequently, along with (4.28)-(4.31), and by (4.26), we have
\[
\frac{d}{dt}(\sigma H(t) - \sigma I_0) + \sigma \int \rho |u_t|^2 dx \\
\leq C (1 + \| \nabla u_t \|_{L^2}^2) \sigma H(t) + C (1 + \| \nabla u_t \|_{L^2}^2 + \| \rho_t \|_{L^2}^2 + \| P_t \|_{L^2}^2).
\]

By Gronwall’s inequality, (4.14), (4.19) and (4.27),
\[
\sup_{0 \leq t \leq T} (\sigma H(t)) + \int_0^T \sigma \| \rho^{1/2} u_t \|_{L^2}^2 dt \leq C.
\]
which together with (4.25), gives (4.20).

\[ \tag*{□} \]

**Lemma 4.6** There exists a positive constant \( C \) so that for any \( q \in (3, 6) \),
\[
\sup_{t \in [0, T]} \sigma \| \nabla u \|_{H^2}^2 + \int_0^T (\| \nabla u \|_{H^2}^2 + \| \nabla^2 u \|_{W^{1,q}}^p + \sigma \| \nabla u_t \|_{H^1}^2) dt \leq C, \tag{4.32} \]
\[
\sup_{t \in [0, T]} (\| \rho - \rho_s \|_{W^{2,q}} + \| P - P(\rho_s) \|_{W^{2,q}}) \leq C, \tag{4.33} \]

where \( p_0 = \frac{\nu q - 6}{12 \nu q - 6} \in (1, \frac{7}{6}). \)

**Proof.** It follows from Lemma 4.3 Poincaré’s and Sobolev’s inequalities that
\[
\| \nabla (\rho \bar{u}) \|_{L^2} \leq \| \nabla \rho \|_{L^2} |u_t|_{L^2} + \| \rho \nabla u_t \|_{L^2} + \| \nabla \rho \| |u|_{L^2} |\nabla u|_{L^2} \]
\[
+ \| \rho \| |u_t|^2 \|_{L^2} + \| \rho \| |u|^2 \|_{L^2} \| \nabla^2 u \|_{L^2} \]
\[
\leq C + C \| \nabla u_t \|_{L^2}, \tag{4.34} \]
which together with (4.15) and Lemma 4.3 yields
\[
\| \nabla^2 u \|_{H^1} \leq C (\| \rho \bar{u} \|_{H^1} + \| P - P(\rho_s) \|_{H^2} + \| u \|_{L^2}) \]
\[
\leq C + C \| \nabla u_t \|_{L^2}. \tag{4.35} \]

And by (4.35), (4.10), (4.14) and (4.20),
\[
\sup_{0 \leq t \leq T} \sigma \| \nabla u \|_{H^2}^2 + \int_0^T \| \nabla u \|_{H^2}^2 dt \leq C. \tag{4.36} \]
We deduce from Lemma 4.3, (4.15) and (4.19) that

\[
\|\nabla^2 u_t\|_{L^2} \leq C(\|(\rho u_t)\|_{L^2} + \|P_t\|_{H^{1}} + \|u_t\|_{L^2} + \|\rho t \nabla \psi\|_{L^2}) \\
\leq C(\|\rho u_t\| + \rho u_t \cdot \nabla u + \rho u_t \cdot \nabla \rho u + \rho u \cdot \nabla u_t) \\
+ C(\|P_t\|_{L^2} + \|u_t\|_{L^2} + \|\rho t \nabla \psi\|_{L^2}) + C \\
\leq C(\|\rho u_t\|_{L^2} + \|\rho u\|_{L^6} + \|\rho t \nabla \psi\|_{L^6}) + \|u_t\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2} + \|\rho t \nabla \psi\|_{L^2} + C \\
\leq C(\rho^2 u_t\|_{L^2} + \|\nabla u_t\|_{L^2} + C, 
\]

where in the first inequality, we have applied Lemma 2.4 to the system

\[
\begin{aligned}
\mu \Delta u_t + (\lambda + \mu) \nabla \text{div} u_t &= (\rho u_t) + \nabla P_t - \rho t \nabla \psi & & \text{in } \Omega, \\
\rho u_t \cdot n &= 0 \text{ and } \text{curl} u_t \times n = 0 & & \text{on } \partial \Omega.
\end{aligned}
\]

By (4.37) and (4.20), we get

\[
\int_0^T \sigma \|\nabla u_t\|_{H^1}^2 dt \leq C. 
\]

By Sobolev’s inequality, (3.6), (4.10), (4.15) and (4.20), we check that for any \( g \in (3, 6), \)

\[
\|\nabla (\rho u)\|_{L^q} \leq C(\|\nabla \rho\|_{L^q} + \|\nabla \rho u\|_{L^q} + \|\nabla u\|_{L^q}^2 + \|\nabla \rho\|_{L^q}) \\
\leq C(\|\nabla \rho u\|_{L^q} + \|\nabla u\|_{L^q}^2 + \|\nabla \rho\|_{L^q}) \\
\leq C(\|\nabla u_t\|_{L^q}^2 + \|\nabla u\|_{L^q}^2 + \|\nabla \rho\|_{L^q})^{\frac{3(q-2)}{6(q-1)}} + C, 
\]

which along with (4.9) and (4.39), leads to

\[
\int_0^T \|\nabla (\rho u)\|_{L^q}^p_0 dt \leq C. 
\]

On the other hand, notice that, by (2.21), (2.22), (4.9) and (4.15),

\[
\|\nabla^2 u\|_{W^{1, q}} \leq C(\|\rho u\|_{L^q} + \|\nabla (\rho u)\|_{L^q} + \|\nabla^2 P\|_{L^q} + \|\nabla P\|_{L^q} \\
+ \|\nabla u\|_{L^2} + \|P - P(\rho_s)\|_{L^q} + \|P - P(\rho_s)\|_{L^q} + C(\|\rho - \rho_s\|_{W^{1, q}})) \\
\leq C(\|\rho u\|_{L^q} + \|\nabla^2 P\|_{L^q} + \|\nabla^2 P\|_{L^q}), 
\]

which together with (3.17) and (4.15) yields

\[
\|\nabla^2 P\|_{L^q} \leq C \|\nabla u\|_{L^q} + C \|\nabla^2 u\|_{W^{1, q}} \\
\leq C(1 + \|\nabla u\|_{L^q}) \|\nabla^2 P\|_{L^q} + C(1 + \|\nabla u_t\|_{L^2}) + C(1 + \|\nabla u_t\|_{L^2}) + C(1 + \|\nabla u_t\|_{L^2}) \\
+ C(\|\nabla (\rho u)\|_{L^q}), 
\]

Now by Gronwall’s inequality, (4.10), (4.14) and (4.41), we derive that

\[
\sup_{t \in [0, T]} \|\nabla^2 P\|_{L^q} \leq C, 
\]
which along with \((4.14), (4.15), (4.42)\) and \((4.41)\) also gives
\[
\sup_{t \in [0,T]} \|P - P(\rho_s)\|_{W^{2,q}} + \int_0^T \|\nabla^2 u\|_{W^{1,q}}^{p_0} dt \leq C. \tag{4.45}
\]
Similarly,
\[
\sup_{0 \leq t \leq T} \|\rho - \rho_s\|_{W^{2,q}} \leq C,
\]
As a result, we obtain \((4.33)\) and finish the proof of Lemma 4.6 \(\Box\)

**Lemma 4.7** There exists a positive constant \(C\) such that
\[
\sup_{0 \leq t \leq T} \sigma \left( \|\nabla u_t\|_{H^1} + \|\nabla u\|_{W^{2,q}} \right) + \int_0^T \sigma^2 \|\nabla u_t\|^2 dt \leq C, \tag{4.46}
\]
for \(q \in (3, 6)\).

**Proof.** Differentiating \((1.1)\) with respect to \(t\) twice leads to
\[
\rho u_{ttt} + \rho u \cdot \nabla u_{tt} - (\lambda + 2\mu) \nabla \text{div} u_t + \mu \nabla \times \text{curl} u_t
= 2 \text{div}(\rho u) u_{tt} + \text{div}(\rho u) u_t - 2(\rho u)_t \cdot \nabla u_t - (\rho u_t + 2\rho_t u) \cdot \nabla u
- \rho u_t \cdot \nabla u - \nabla P_{tt} + \rho u_t \nabla \psi. \tag{4.47}
\]
Then, multiplying by \(2u_{tt}\) and integrating over \(\Omega\), we have
\[
\frac{d}{dt} \int \rho |u_{tt}|^2 dx + 2(\lambda + 2\mu) \int (\text{div} u_{tt})^2 dx + 2\mu \int |\text{curl} u_{tt}|^2 dx
= -8 \int \rho u_{tt} u \cdot \nabla u_t dx - 2 \int (\rho u)_t \cdot [\nabla (u_t \cdot u_{tt})] + 2 \text{div} u_t \cdot \nabla u_t dx
- 2 \int \rho u_t + 2\rho_t u_t \cdot \nabla u \cdot u_{tt} dx - 2 \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx
+ 2 \int P_{tt} \text{div} u_{tt} dx + 2 \int \rho u_t \nabla \psi \cdot u_{tt} dx \Delta \sum_{i=1}^6 J_i. \tag{4.48}
\]
Due to \((4.10), (4.9), (4.14), (4.19)\) and \((4.20)\), we have
\[
|J_1| \leq C \|\rho^{1/2} u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \|u\|_{L^\infty}
\leq \delta \|\nabla u_{tt}\|^2_{L^2} + C(\delta) \|\rho^{1/2} u_{tt}\|^2_{L^2}. \tag{4.49}
\]
\[
|J_2| \leq C \left( \|\rho u_{tt}\|_{L^3} + \|\rho u\|_{L^3} \right) \left( \|u_{tt}\|_{L^6} \|\nabla u_t\|_{L^2} + \|u_{tt}\|_{L^2} \|u_t\|_{L^6} \right)
\leq C \left( \|\rho^{1/2} u_{tt}\|_{L^2}^{1/2} \|u_{tt}\|_{L^6}^{1/2} + \|\rho u_t\|_{L^6} \|u_t\|_{L^6} \right) \|\nabla u_{tt}\|_{L^2} \|\nabla u_t\|_{L^2}
\leq \delta \|\nabla u_{tt}\|^2_{L^2} + C(\delta) \sigma^{-3/2}, \tag{4.50}
\]
\[
|J_3| + |J_6| \leq C \left( \|\rho u_{tt}\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} + \|\rho u_t\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2} + \|\rho u_{tt}\|_{L^2} \|u_{tt}\|_{L^6} \right)
\leq \delta \|\nabla u_{tt}\|^2_{L^2} + C(\delta) \|\rho u_{tt}\|^2_{L^2} + C(\delta) \sigma^{-1}, \tag{4.51}
\]
and
\[ |J_4| + |J_5| \leq C \| \rho_{tt} \|_{L^2} \| \nabla u \|_{L^3} \| u_{tt} \|_{L^6} + C \| P_{tt} \|_{L^2} \| \nabla u_{tt} \|_{L^2} \]
\[ \leq \delta \| \nabla u_{tt} \|_{L^2}^2 + C(\delta) \| \rho^{1/2} u_{tt} \|_{L^2}^2 + C(\delta) \| P_{tt} \|_{L^2}^2. \]  
(4.52)

Now choosing \( \delta \) small enough, it follows from (4.48) that
\[ \frac{d}{dt} \| \rho^{1/2} u_{tt} \|_{L^2}^2 + \| \nabla u_{tt} \|_{L^2}^2 \]
\[ \leq C(\| \rho^{1/2} u_{tt} \|_{L^2}^2 + \| \rho_{tt} \|_{L^2}^2 + \| P_{tt} \|_{L^2}^2) + C\sigma^{-3/2}, \]  
(4.53)

where we have utilized the fact that
\[ \| \nabla u_{tt} \|_{L^2} \leq C(\| \text{div} u_{tt} \|_{L^2} + \| \text{curl} u_{tt} \|_{L^2}), \]  
(4.54)
due to \( u_{tt} \cdot n = 0 \) on \( \partial \Omega \).

Together with (4.19), (4.20), and by Gronwall’s inequality, we get
\[ \sup_{0 \leq t \leq T} \sigma \| \rho^{1/2} u_{tt} \|_{L^2}^2 + \int_0^T \sigma^2 \| \nabla u_{tt} \|_{L^2}^2 dt \leq C. \]  
(4.55)

Furthermore, by (4.37) and (4.20),
\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{H^1} \leq C. \]  
(4.56)

Finally, by (4.42), (4.40), (4.20), (4.33), (4.32), (4.55) and (4.56), we have
\[ \sigma \| \nabla^2 u \|_{W^{1,q}} \leq C \left( \sigma + \sigma \| \nabla u_t \|_{L^2} + \sigma \| \nabla (\rho u) \|_{L^q} + \sigma \| \nabla^2 P \|_{L^q} \right) \]
\[ \leq C \left( \sigma + \sigma^{1/2} + \sigma \| \nabla u_t \|_{H^2} + \sigma^{1/2} (\sigma \| \nabla u_t \|_{H^1}^2)^{\frac{3(q-2)}{4q}} \right) \]
\[ \leq C \sigma^{\frac{1}{2}} + C \sigma^{\frac{1}{2}} (\sigma^{-1})^{\frac{3(q-2)}{4q}} \]
\[ \leq C, \]
which, together with (4.55) and (4.56) yields (4.46). \( \Box \)

5 Proof of the Main Theorems

With all the a priori estimates in Section 3 and Section 4 at hand, we are going to prove Theorems 1.1-1.2 in this section.

Proof of Theorem 1.1. By Lemma 2.1 there exists a \( T_* > 0 \) such that the system (1.1)-(1.6) has a unique strong solution \((\rho, u)\) on \( \Omega \times (0, T_*) \). In order to extend the local strong solution globally in time, first, by the definition of \( A_1(T) \), \( A_2(T) \) and \( A_3(T) \) (see (3.1), (3.2), (3.3)), the assumption of the initial data (1.10), one immediately checks that
\[ A_1(0) + A_2(0) = 0, \quad 0 \leq \rho_0 \leq \tilde{\rho}, \quad A_3(0) \leq M. \]

Therefore, there exists a \( T_1 \in (0, T_*) \) such that
\[ 0 \leq \rho_0 \leq 2\tilde{\rho}, \quad A_1(T) + A_2(T) \leq 2C\sigma^{\frac{1}{2}}, \quad A_3(\sigma(T)) \leq 2K \]  
(5.1)
hold for \( T = T_1 \).
Next, we set
\[ T^* = \sup\{ T \mid (5.1) \text{ holds} \}. \tag{5.2} \]

Then \( T^* \geq T_1 > 0 \). Therefore, for any \( 0 < \tau < T \leq T^* \) with \( T \) finite, by Lemmas 4.1-4.2, we have
\[
\begin{cases}
(p, P) \in L^\infty([0, T]; W^{1,q}), & \rho_t \in L^\infty(0, T; L^2), \\
u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \\ho^{1/2}u_t \in L^2(0, T; L^2), & (\rho^{1/2}u_t, \nabla^2 u) \in L^\infty(\tau, T; L^2) \\
u_t \in L^2(\tau, T; H^1),
\end{cases}
\]
which immediately implies for any \( r \geq 2 \),
\[
(p, P) \in C([0, T]; L^r), & \rho u \in C([0, T]; L^2),
\]
where we have used the standard embedding \( L^\infty(0, T; H^1) \cap H^1(0, T; H^{-1}) \hookrightarrow C([0, T]; L^q), \) for any \( q \in [2, 6) \).

By (1.1), for any \( T > t > s \geq 0 \),
\[
\| \nabla \rho(t) \|_{L^q} \leq \left( \| \nabla \rho(s) \|_{L^q} + C \int_{s}^{t} \| \nabla^2 u(s) \|_{L^q} \, ds \right) \exp(C \int_{s}^{t} \| \nabla^2 u(s) \|_{L^\infty} \, ds),
\]
which leads to
\[
\limsup_{t \to s^+} \| \nabla \rho(t) \|_{L^q} \leq \| \nabla \rho(s) \|_{L^q}. \tag{5.3}
\]

On the other hand, by (4.1), \( \nabla \rho \in C([0, T]; L^q - \text{weak}) \), together with (5.3), we conclude that for any \( q \in [2, 6) \),
\[
\nabla \rho \in C([0, T]; L^q).
\]

Finally, we claim that
\[ T^* = \infty. \]

Otherwise, \( T^* < \infty \). Then by Proposition 3.1, it holds that
\[
0 \leq \rho \leq \frac{7}{4} \hat{\rho}, & A_1(T^*) + A_2(T^*) \leq C_0^2, & A_3(\sigma(T^*)) \leq K. \tag{5.4}
\]
and \((p(x, T^*), u(x, T^*))\) satisfy the initial data condition (1.15)-(1.17). Thus, Lemma 2.1 implies that there exists some \( T^{**} > T^* \) such that (5.1) holds for \( T = T^{**} \), which contradicts the definition of \( T^* \). As a result, \( 0 < T_1 < T^* = \infty \).

By (2.1) and (1.18), it indicates that \((\rho, u)\) is really the unique strong solution defined on \( \Omega \times (0, T] \) for any \( 0 < T < T^* = \infty \).

**Proof of Theorem 1.2.** By Theorem 1.1, we only need to prove that the unique strong solution is a classical one under the assumption of Theorem 1.2. for any \( 0 < \tau < T \leq T^* \) with \( T \) finite, it follows from Lemmas 4.5-4.7 that
\[
\begin{cases}
\rho - \rho_s \in C([0, T]; W^{2,q}), \\
\nabla u_t \in C([\tau, T]; L^q), & \nabla u, \nabla^2 u \in C([\tau, T]; C(\bar{\Omega})),
\end{cases}
\tag{5.5}
\]
where one has taken advantage of the standard embedding
\[ L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \rightarrow C([\tau, T]; L^q), \quad \text{for any } q \in [2, 6). \]

By Lemmas 2.1 and 4.5-4.7, \((\rho, u)\) is in fact the unique classical solution defined on \(\Omega \times (0, T]\) for any \(0 < T < \infty\).

It remains to prove (1.22). By (3.10) and (3.11), we get
\[
\left( \int \frac{1}{2} \rho |u|^2 + G(\rho, \rho_s) \, dx \right)_t + \phi(t) = 0, \tag{5.6}
\]
where
\[ \phi(t) \triangleq (\lambda + 2\mu)\|\text{div} u\|_{L^2}^2 + \mu\|\text{curl} u\|_{L^2}^2. \]

Notice that there exists a positive constant \(\tilde{C}_1 < 1\) depending only on \(\gamma, \rho\) and \(\bar{\rho}\) such that for any \(\rho \geq 0\),
\[
\tilde{C}_1^2(\rho - \rho_s)^2 \leq \tilde{C}_1 G(\rho, \rho_s) \leq (\rho^\gamma - \rho_s^\gamma)(\rho - \rho_s),
\]
and by (3.46) and (2.11),
\[
a \tilde{C}_1 \int G(\rho, \rho_s) \, dx \leq a \int (\rho^\gamma - \rho_s^\gamma)(\rho - \rho_s) \, dx \leq 2 \left( \int \rho u \cdot B[\rho - \rho_s] \, dx \right)_t + C_7 \phi(t). \tag{5.7}
\]

Introducing the function
\[ W(t) = \int \left( \frac{1}{2} \rho |u|^2 + G(\rho, \rho_s) \right) \, dx - \delta_0 \int \rho u \cdot B[\rho - \rho_s] \, dx, \]
where \(\delta_0 = \min\{\frac{1}{2C_7}, \frac{1}{2C_8}\}\), and noticing that
\[
\left| \int \rho u \cdot B[\rho - \rho_s] \, dx \right| \leq C_8 \left( \frac{1}{2} \|\sqrt{\rho u}\|_{L^2}^2 + \int G(\rho, \rho_s) \, dx \right),
\]
we have
\[
\frac{1}{2} \|\sqrt{\rho u}\|_{L^2}^2 + \int G(\rho, \rho_s) \, dx \leq 2W(t) \leq 4 \left( \frac{1}{2} \|\sqrt{\rho u}\|_{L^2}^2 + \int G(\rho, \rho_s) \, dx \right). \tag{5.8}
\]

Noticing that
\[ \int \rho |u|^2 \, dx \leq C \|\nabla u\|_{L^2}^2 \leq C_3 \phi(t), \]
setting \(\delta_1 = \min\{\frac{\alpha_0 \tilde{C}_1}{2}, \frac{1}{2C_3}\}\) and adding (5.7) multiplied by \(\delta_0\) to (5.6) yields
\[ W'(t) + \delta_1 W(t) \leq 0, \]
which together with (5.8) leads to
\[
\int \left( \frac{1}{2} \rho |u|^2 + G(\rho, \rho_s) \right) \, dx \leq 4C_0 e^{-\delta_1 t} \tag{5.9}
\]
for any \( t > 0 \). Moreover, by (5.6), for any \( 0 < \delta_2 < \delta_1 \),

\[
\int_{0}^{\infty} \phi(t)e^{\delta_2 t} dt \leq C. \tag{5.10}
\]

Choose \( m = 0 \) in (3.24), along with (2.11), (2.17) and (3.42), a direct calculation shows that

\[
\left( \phi(t) - 2 \int (P - P(\rho_s)) \text{div} u dx \right) + \frac{1}{2} \| \sqrt{\rho \dot{u}} \|_{L^2}^2 \leq C(\| \rho - \rho_s \|_{L^2}^2 + \phi(t)) \tag{5.11}
\]

Multiplying (5.11) by \( e^{\delta_2 t} \), and using the fact

\[
\left| \int (P - P(\rho_s)) \text{div} u dx \right| \leq C\| \rho - \rho_s \|_{L^2}^2 + \frac{1}{2} \phi(t),
\]

we get

\[
\left( e^{\delta_2 t} \phi(t) - 2e^{\delta_2 t} \int (P - P(\rho_s)) \text{div} u dx \right) + \frac{1}{2} e^{\delta_2 t} \| \sqrt{\rho \dot{u}} \|_{L^2}^2 \leq C e^{\delta_2 t}(\| \rho - \rho_s \|_{L^2}^2 + \phi(t)),
\]

which, together with (5.9) and (5.10), yields that for any \( t > 0 \),

\[
\| \nabla u \|_{L^2}^2 \leq C e^{-\delta_2 t},
\]

and

\[
\int_{0}^{\infty} e^{\delta_2 t} \| \sqrt{\rho \dot{u}} \|_{L^2}^2 dt \leq C. \tag{5.12}
\]

A similar analysis based on (3.40) and (5.12) shows

\[
\| \sqrt{\rho \dot{u}} \|_{L^2}^2 \leq C e^{-\delta_2 t}.
\]

As a result, (1.22) is established with some \( \tilde{C} \) depending only on \( \mu, \lambda, \gamma, a, \inf_{\Omega} \psi, \| \psi \|_{H^2}, \hat{\rho}, M, \Omega, p, q \) and \( C_0 \) and we complete the proof.

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