On the Well-posedness of a Nonlinear Fourth-Order Extension of Richards’ Equation

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Abstract

We study a nonlinear fourth-order extension of Richards’ equation that describes infiltration processes in unsaturated soils. We prove the well-posedness of the fourth-order equation by first applying Kirchhoff’s transformation to linearize the higher-order terms. The transformed equation is then discretized in time and space and a set of a priori estimates is established. These allow, by means of compactness theorems, extracting a unique weak solution. Finally, we use the inverse of Kirchhoff’s transformation to prove the well-posedness of the original equation.

Keywords: Richards’ equation, Nonlinear fourth-order extension, Weak solutions, Existence, Uniqueness, Kirchhoff’s transformation

1. Introduction

The process of fluid infiltration through unsaturated soil is an important part of the hydrological cycle as it represents many crucial examples, such as the flow of rain water or waste fluids into water aquifers and the flow of salt-water into coastal aquifers. These infiltration processes are usually described using Richards’ model \cite{3}. Recent experiments on fluid infiltration show that, even in homogeneous porous media, an initially planar front does not remain planar. The fluids infiltrate in preferential flow paths taking the shape of fingers with different widths and velocities. As most of the fluid channelizes in the fingers with high velocity, this may have crucial effects on the
environment as it reduces the time needed for a contaminant to reach the underground water. Experiments show also that constant flux infiltration into homogeneous porous media leads to higher saturation at the wetting front than behind the front. This natural behavior is called saturation overshoots and is believed to cause the gravity-driven fingering [5, 6].

Richards’ model is unable to describe saturation overshoots, because it is a second-order parabolic differential equation fulfilling the maximum principle. Moreover, it is unable to predict fingered flows, as nonlinear stability analysis shows that the model is unconditionally stable [8, 13]. Therefore, many approaches have been suggested to modify Richards’ model [4, 10, 15].

In this paper, we propose a nonlinear fourth-order extension of Richards’ equation. This extension is related to the fourth-order model in [4], while having the benefit that both second- and fourth-order terms can be simultaneously linearized using Kirchhoff’s transformation, which is more convenient for the well-posedness analysis later.

We prove in this paper the well-posedness of the proposed nonlinear fourth-order extension of Richards’ equation. The paper has the following structure: Section 2 presents Richards’ equation and our proposed nonlinear fourth-order extension. In Section 3, Kirchhoff’s transformation is applied to the fourth-order model as a preparation step for the analysis in the following section, then a list of assumptions is provided. In Section 4 we prove the well-posedness of the transformed fourth-order model. In Section 5 we improve the regularity of the weak solution. Finally, we prove the well-posedness of the nonlinear fourth-order model in Section 6.

2. Modeling in Unsaturated Soil

This section presents two models that describe fluid flows in unsaturated soils: the classical Richards’ model and a nonlinear fourth-order extension of it.

2.1. Richards’ model

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ in the zone of unsaturated soil, where gas occupies most of the pores. Since gas in this zone is naturally connected to the atmospheric air, its pressure is constant and equals the atmospheric air pressure. Assuming
that water infiltrates through the domain $\Omega$ under the effect of gravity and capillary forces, the two-phase flow model for the infiltrating water is a combination of the mass conservation equation and Darcy’s law

$$\phi \partial_t S + \nabla \cdot \mathbf{v} = 0,$$

$$\mathbf{v} = -K_f(S) \left( \frac{\nabla p}{\rho g} - e_3 \right),$$

respectively. Here, $S = S(x, t) \in [0, 1]$ is saturation, $\mathbf{v} = \mathbf{v}(x, t) \in \mathbb{R}^3$ is averaged velocity and $p = p(x, t) \in \mathbb{R}$ is pressure of the infiltrating water phase. The porosity $\phi$ is assumed to be constant, $\rho = 1$ is water density, $g$ is the gravitational acceleration, and $e_3 = (0, 0, 1)^T$. We also consider the closure relation

$$p_c = p - p_g,$$

where $p_g = p_{\text{air}}$ is constant. Then, using the van Genuchten parameterization [16] of the capillary pressure $p_c = p_c(S)$, equation (1) simplifies to Richards’ equation

$$\phi \partial_t S + \nabla \cdot \left( K_f(S) \left( e_3 + \frac{\nabla p_c(S)}{g} \right) \right) = 0.$$  

2.2. The Nonlinear Fourth-Order Extension

We propose a fourth-order extension of Richards’s equation (3) by adding a third-order regularizing term to Darcy’s equation, i.e.

$$\mathbf{v} = K_f(S) \nabla \left( z + \frac{1}{g} p_c(S) \right) - \epsilon \frac{1}{g} \nabla \left( \nabla \cdot (K_f(S) \nabla p_c(S)) \right),$$

where $\epsilon$ is a small parameter. Substituting (4) into the continuity equation in (1) yields the nonlinear fourth-order model

$$\partial_t S + \nabla \cdot \left( K_f(S) \left( e_3 + \frac{1}{g} \nabla p_c(S) \right) \right) - \frac{\epsilon}{g} \Delta \nabla \cdot \left( K_f(S) \nabla p_c(S) \right) = 0.$$  

Since capillary pressure $p_c$ is a strictly monotone decreasing function of saturation $S$, its inverse is well-defined. Thus, we can write saturation $S$ as an increasing function of $p := -\frac{p_c}{g}$ such that

$$S(p) = \begin{cases} S(-\frac{p_c}{g}), & p \leq 0, \\ 1, & p > 0, \end{cases}$$
as shown in Figure 1. The Figure shows also the conductivity \( K_f = K_f(S(p)) \), which is a monotone increasing function of \( S \). Using the inverse function \( p \), the fourth order model (5) can be written as

\[
\partial_t S(p) + \nabla \cdot \left( K_f(S(p)) e_3 \right) - \nabla \cdot \left( K_f(S(p)) \nabla p \right) + \gamma \Delta \nabla \cdot \left( K_f(S(p)) \nabla p \right) = 0, \quad (6)
\]

in \( \Omega \times (0, T) \) with pressure \( p \) is the unknown and \( \gamma := \frac{\xi}{g} \). Since we are interested in the existence of weak solutions in the space \( L^2(0, T; H^2_0(\Omega)) \), equation (6) is augmented with the initial and boundary conditions

\[
\begin{align*}
p(., 0) &= p^0 \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
\nabla p \cdot n &= 0 \quad \text{on } \partial \Omega \times [0, T],
\end{align*}
\]

where \( n \) is the outer normal vector at the boundary \( \partial \Omega \).

3. Preliminaries and Assumptions

In this section, we apply Kirchhoff’s transformation to the fourth-order model (6) to linearize the second- and the fourth-order terms. Then, we summarize all assumptions that are required throughout the paper.

Kirchhoff’s transformation is a continuous monotone increasing map defined as

\[
\psi := \left\{ \begin{array}{ll} \mathbb{R} & \to \mathbb{R} \\
p & \mapsto \psi(p) = \int_0^p K_f(S(\tau)) d\tau \end{array} \right\
\]

where \( \psi(p) \) is the transformed pressure. We set \( u := \psi(p) \). Then, as Figure 2 shows, we have \( u = p \) for \( p \geq 0 \), because \( K_f(S(p)) = 1 \). Moreover, there exists a lower bound...
As the inverse function \( \psi^{-1} : (u_l, \infty) \to \mathbb{R} \) is well-defined, we define the function
\[
b(u) := S(\psi^{-1}(u)),
\]
such that
\[
b'(u) = \frac{S'(p)}{K_f(S(p))}.
\]
Then, the transformed fourth-order model is given as:
\[
\partial_t b(u) + \nabla \cdot \left( K_f(b(u))e_3 \right) - \Delta u + \gamma \Delta^2 u = 0,
\]
with the transformed initial and boundary conditions
\[
\begin{align*}
u(\cdot, 0) &= u^0 \quad \text{in } \Omega \times \{0\}, \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\nabla u \cdot n &= 0 \quad \text{on } \partial \Omega \times (0, T).
\end{align*}
\]
For \( \gamma = 0 \), the wellposedness of (9) is proved in [1, 12]. The wellposedness of other fourth-order parabolic equations describing thin film growth is investigated in
Due to the nonlinearity of the first term on the left side of equation (9), we follow [1] and define the Legendre transform $B$ for the primitive of $b$,

$$B := \begin{cases} \mathbb{R} \to \mathbb{R}^+ \\ z \mapsto B(z) = \int_0^z b(z) - b(s) \, ds \end{cases}$$

(11)

The map $B$ satisfies the following properties:

**Lemma 1.** If $b$ is a continuous and monotone increasing function, then the Legendre transform $B$, defined in (11), satisfies

$$B(z) - B(z_0) \geq \left( b(z) - b(z_0) \right) z_0,$$

$$B(z) - B(z_0) \leq \left( b(z) - b(z_0) \right) z,$$

for any $z, z_0 \in \mathbb{R}$.

**Proof.** The continuity and the monotonicity of $b$ imply the existence of a convex function $\phi \in C^1(\mathbb{R}, \mathbb{R})$ such that $b = \phi' := \frac{d\phi}{du}$. The definition of $B$ and the property that $b = \phi'$ give

$$B(z) = \int_0^z (b(z) - \phi'(s)) \, ds = b(z)z - \left( \phi(z) - \phi(0) \right).$$

(12)

Then, we have

$$B(z) - B(z_0) = b(z)z - b(z_0)z_0 - \left( \phi(z) - \phi(z_0) \right).$$

To prove the first inequality, we add $\pm b(z_0)z_0$ to the right side of the above equation, then we have

$$B(z) - B(z_0) = \left( b(z) - b(z_0) \right) z_0 - b(z_0)(z_0 - z) - \left( \phi(z) - \phi(z_0) \right).$$

The Taylor expansion and the convexity of $\phi$ imply that $M > 0$, which proves the inequality. The second inequality follows similarly by adding $\pm b(z_0)z$. $\square$

We summarize all assumptions that are required throughout the paper:

**Assumption 2.** 1. The domain $\Omega \subset \mathbb{R}^3$ is an open bounded connected region with boundary $\partial \Omega \in C^5$ and $0 < T < \infty$. 6
2. The initial condition $u^0 \in H^2_0(\Omega)$ satisfies $u^0$, $b(u^0)$, $B(u^0) \in L^\infty(\Omega)$.
3. The function $b : (u_1, \infty) \rightarrow (0, 1]$ is strictly positive, monotone increasing and Lipschitz continuous.
4. The conductivity function $K_f : (u_1, \infty) \rightarrow (0, 1]$ is Lipschitz continuous, strictly positive, and there exists a constant $\beta > 0$ such that, for all $z \in \mathbb{R}$, the following growth condition holds
   \[
   \left( K_f(b(z)) \right)^2 \leq \beta \left( 1 + B(z) \right).
   \]

4. Well-posedness of the Transformed Fourth-Order Model

In this section, we prove the well-posedness of the transformed fourth-order model (9) with the initial and boundary conditions (10). In Section 4.1, we approximate the time derivative in the model using backward differences producing a series of elliptic equations. Then, we apply Galerkin’s method to these equations and prove the existence of weak solutions for the discrete problem. In Section 4.2, we prove a set of a priori estimates on the sequence of discrete solutions. These are used in Section 4.3 to conclude a weak convergence of the sequence. Then, we prove that the limit is a weak solution for the transformed problem. Finally, we prove in Section 4.4 the uniqueness of the weak solution.

4.1. An Approximate Model

Let $N > 0$ be an integer and $h = T/N$. Approximating $\partial_t b(u)$ in (9) using the backward difference $\frac{b(u(t)) - b(u(t-h))}{h}$ yields for almost all $t \in [0, T]$ the biharmonic equation

\[
\frac{b(u(\cdot,t)) - b(u(\cdot,t-h))}{h} + \nabla \cdot \left( K_f(b(u(\cdot,t)))e_3 \right) - \Delta u(\cdot,t) + \gamma \Delta^2 u(\cdot,t) = 0.
\]

For any arbitrary but fixed $t \in [0, T]$, we consider weak solutions of (13) in the Hilbert space $V(\Omega) = H^2_0(\Omega)$. Let $\{w_i\}_{i \in \mathbb{N}}$ be a countable orthonormal basis of $V$. By applying Galerkin’s method to equation (13), the solution space $V(\Omega)$ is projected into a finite dimensional space $V_M(\Omega)$ spanned by a finite number of the orthonormal
functions $w_i, i = 1, \ldots, M$. For $h > 0$ and a positive integer $M$, we search the coefficients 
$\alpha^h_{Mi} \in L^\infty((0, T))$, $i = 1, \ldots, M$ defining the function

$$u^h_M(t) := \sum_{i=1}^{M} \alpha^h_{Mi}(t) w_i.$$  \hspace{1cm} (14)

These coefficients are chosen such that, for almost all $t \in [0, T]$, the equation

$$\frac{1}{h} \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t - h)) \right) w_i dx + \int_{\Omega} \nabla u^h_M(t) \cdot \nabla w_i + \gamma \Delta u^h_M(t) \Delta w_i dx = \int_{\Omega} K_f (b(u^h_M(t))) e_3 \cdot \nabla w_i dx$$  \hspace{1cm} (15)

holds for all $i = 1, \ldots, M$. The discrete initial condition is defined as

$$u^h_M(t) = u^0_M, \quad \text{for } t \in (-h, 0],$$  \hspace{1cm} (16)

where, $u^0_M$ is the $L^2$-projection of the initial data $u_0$ into the finite dimensional space $V_M(\Omega)$.

To prove the existence of solutions for the discrete problem (15) and (16), we need the below stated technical lemma on the existence of zeros of a vector field [9].

**Lemma 3.** Let $r > 0$ and $v : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, which satisfies $v(x) \cdot x \geq 0$ if $|x| = r$. Then, there exists a point $x \in B(0, r)$ such that $v(x) = 0$.

**Lemma 4.** For any $M \in \mathbb{N}$, $h > 0$, and almost any $t \in [0, T]$, let $u^h_M(t-h) \in V_M(\Omega)$ and

$$h \leq \frac{1}{\beta},$$  \hspace{1cm} (17)

where $\beta > 0$ is given as in Assumption (14). Then, equation (15) has a solution $u^h_M(t) \in V_M(\Omega)$.

**Proof.** We define the vector field $f: \mathbb{R}^M \to \mathbb{R}^M$ such that $f = (f_1, \ldots, f_M)$ and the vector $\alpha^h_M = (\alpha^h_{M1}, \ldots, \alpha^h_{MM})$ of the unknown coefficients of $u^h_M(t)$ in equation (14). Then, we have

$$f_i(\alpha^h_M) := \frac{1}{h} \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t - h)) \right) w_i dx - \int_{\Omega} K_f (b(u^h_M(t))) e_3 \cdot \nabla w_i dx$$

$$+ \int_{\Omega} \left( \nabla u^h_M(t) \cdot \nabla w_i + \gamma \Delta u^h_M(t) \Delta w_i \right) dx,$$  \hspace{1cm} (18)
for all \(i = 1, \cdots, M\). Note here that \(u_M(t - h)\) for \(t \in (0, \bar{h}]\) is well-defined by the choice of the initial condition (16). Using Assumption (2.3) and (2.4), the vector field \(\mathbf{f}\) is continuous. Moreover, we have

\[
\mathbf{f}(\alpha_M^h) \cdot \alpha_M^h = \frac{1}{h} \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t - h)) \right) u^h_M(t) \, dx - \int_{\Omega} K_f(b(u^h_M(t)))e_3 \cdot \nabla u^h_M \, dx + \int_{\Omega} \nabla u^h_M \cdot \nabla u^h_M \, dx + \gamma \int_{\Omega} \Delta u^h_M \Delta u^h_M \, dx.
\]

Applying Lemma 1 on the first term of the right side and Cauchy’s inequality on the second term yield

\[
\mathbf{f}(\alpha_M^h) \cdot \alpha_M^h \geq \frac{1}{h} \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t - h)) \right) \, dx - \frac{1}{2} \int_{\Omega} \left( K_f(b(u^h_M(t))) \right)^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla u^h_M|^2 \, dx + \gamma \int_{\Omega} \Delta u^h_M \Delta u^h_M \, dx.
\]

The growth condition in Assumption (2.4), equation (14), and the orthonormality of the basis functions \(w_i, i = 1, \cdots, M\), imply that

\[
\mathbf{f}(\alpha_M^h) \cdot \alpha_M^h \geq \frac{1}{h} \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t - h)) \right) \, dx - \beta \int_{\Omega} \left( 1 + B(u^h_M(t)) \right) \, dx + \frac{1}{2} \int_{\Omega} \left[ \sum_{i=1}^{M} \sum_{\gamma} \alpha_M^h \nabla w_i \right]^2 \, dx + \gamma \int_{\Omega} \left( \sum_{\gamma} \alpha_M^h \Delta w_i \right)^2 \, dx \geq \left( \frac{1}{h} - \beta \right) \int_{\Omega} B(u^h_M(t)) \, dx - \beta |\Omega| + \frac{1}{h} \int_{\Omega} B(u^h_M(t - h)) \, dx + \frac{1}{2} \int_{\Omega} \left( \sum_{\gamma} \alpha_M^h \Delta w_i \right)^2 \, dx.
\]

The first term of the right side of above inequality is nonnegative using condition (17). Noting that \(u^h_M(t - h) \in V_M(\Omega)\) is known and setting \(r = |\alpha_M^h(t)|\) yields that \(\mathbf{f}(\alpha_M^h(t)) \cdot \alpha_M^h(t) \geq 0\) provided that \(r\) is large enough. Thus, Lemma 3 implies the existence of a vector \(\alpha_M^h(t) \in \mathbb{R}^M\) satisfying \(\mathbf{f}(\alpha_M^h) = 0\). Now, using (18), we obtain the existence of a function \(u_M^h(t)\) that satisfies the discrete equation (15).

4.2. A Priori Estimates

We proved already the existence of a sequence \(\{S_M^h\}_{M \in \mathbb{N}, h > 0} \subset V_M(\Omega)\) of discrete solution of the discrete problem (15) and (16). In the following, we prove a set of a priori estimates on the sequence that are essential for the convergence analysis in the next subsection.
Lemma 5. There exists a constant $c > 0$ such that
\[
\text{ess sup}_{t \in [0, T]} \int_{\Omega} (B(u_M^h(t))) \, dx + \int_0^T \int_{\Omega} |\nabla u_M^h|^2 + \gamma (\Delta u_M^h)^2 \, dx \, dt \leq c \int_0^T \int_{\Omega} B(u_M^0) \, dx,
\]
for all $h > 0$ and $M \in \mathbb{N}$.

Proof. Multiplying equation (15) by $a_{Mi}^h$, summing for $i = 1, \cdots, M$, and then integrating from 0 to an arbitrary time $\tau \in [0, T]$ yields
\[
\frac{1}{h} \int_0^\tau \int_{\Omega} \left( (b(u_M^h(t)) - b(u_M^h(t-h))) \right) u_M^h(t) \, dx \, dt + \int_0^\tau \int_{\Omega} |\nabla u_M^h|^2 + \gamma (\Delta u_M^h)^2 \, dx \, dt
\]
\[
+ \gamma \int_0^\tau \int_{\Omega} (\Delta u_M^h)^2 \, dx \, dt = \int_0^\tau \int_{\Omega} K_f (b(u_M^h)) e_j \cdot \nabla u_M^h \, dx \, dt. \tag{19}
\]
Applying the first inequality in Lemma 1 to the first term on the left side of equation (19) and Cauchy’s inequality to the right side yield
\[
\frac{1}{h} \int_0^\tau \int_{\Omega} \left( (b(u_M^h(t)) - b(u_M^h(t-h))) \right) u_M^h(t) \, dx \, dt + \int_0^\tau \int_{\Omega} |\nabla u_M^h|^2 + \gamma (\Delta u_M^h)^2 \, dx \, dt
\]
\[
\leq \frac{1}{2} \int_0^\tau \int_{\Omega} (K_f (b(u_M^h)))^2 \, dx \, dt + \frac{1}{2} \int_0^\tau \int_{\Omega} |\nabla u_M^h|^2 \, dx \, dt.
\]
Applying the growth condition in Assumption (24) to the first term on the right side of the above equation gives
\[
\frac{1}{h} \int_0^\tau \int_{\Omega} \left( (b(u_M^h(t)) - b(u_M^h(t-h))) \right) u_M^h(t) \, dx \, dt + \frac{1}{2} \int_0^\tau \int_{\Omega} |\nabla u_M^h|^2 \, dx \, dt
\]
\[
+ \gamma \int_0^\tau \int_{\Omega} (\Delta u_M^h)^2 \, dx \, dt \leq \beta \int_0^\tau \int_{\Omega} (1 + B(u_M^h(t))) \, dx \, dt.
\]
Applying summation by parts to the first term on the left side of the above equation, and noting that $u_M^h$ is a step function in time, leads to
\[
\int_{\Omega} B(u_M^h(\tau)) \, dx + \frac{1}{2} \int_0^\tau \int_{\Omega} |\nabla u_M^h|^2 \, dx \, dt + \gamma \int_0^\tau \int_{\Omega} |\Delta u_M^h|^2 \, dx \, dt
\]
\[
\leq \beta |\Omega| T + \int_{\Omega} B(u_M^0) \, dx + \beta \int_0^\tau \int_{\Omega} B(u_M^h(t)) \, dx \, dt.
\]
Note that $B(u_M^h)$ is nonnegative and summable on $[0, T]$, where the summability results from substituting $z_0 = 0$ into the second inequality in Lemma 1, the boundedness of $b$, and the choice that the coefficients $\alpha_{Mi}^h \in L^\infty([0, T])$. Hence, Gronwall’s inequality is applicable and implies the existence of a constant $c > 0$ depending on $\beta, |\Omega|$, and $T$. 

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such that
\[
\text{ess sup}_{t \in [0,T]} \int_{\Omega} \left( B(u_h^M(t)) \right) dx + \int_0^T \int_{\Omega} \left( \frac{1}{2} |\nabla u_h^M|^2 + \gamma (\Delta u_h^M)^2 \right) dx dt \leq c \int_{\Omega} B(u_h^M) dx.
\]

\[\square\]

**Corollary 6.** It holds that \(u_h^M \in L^2(0,T;H_0^2(\Omega))\) for all \(M \in \mathbb{N}\) and \(h > 0\).

**Proof.** Lemma 5 and Poincaré’s inequality imply the existence of a constant \(C > 0\) such that
\[
\int_0^T \int_{\Omega} (u_h^M)^2 dx dt \leq C,
\]
for all \(M \in \mathbb{N}\) and \(h > 0\). Moreover, the biharmonic operator \(L: \Delta u + \Delta^2 u\) can be written as a combination of two second-order elliptic operators
\[L_1 w = \Delta w + w,
\]
\[L_2 u = \Delta u.
\]

Hence, the basis functions \(w_i\) of the biharmonic operator \(L\) can be chosen as a combination of the eigenfunctions of the operators \(L_1\) and \(L_2\). These eigenfunctions belong to the space \(C^3(\Omega)\), whenever the boundary \(\partial \Omega \in C^5\), [9]. Hence, using Gauss’ theorem and Cauchy’s inequality, we obtain
\[
\int_0^T \int_{\Omega} (\partial_{ij} u_h^M)^2 dx dt = -\int_0^T \int_{\Omega} \partial_i u_h^M \partial_j u_h^M \partial_{ij} dx dt = \int_0^T \int_{\Omega} \partial_i u_h^M \partial_j u_h^M \partial_{ij} dx dt
\]
\[
\leq \frac{1}{2} \int_0^T \int_{\Omega} (\partial_{ij} u_h^M)^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} (\partial_{j,ij} u_h^M)^2 dx dt
\]
\[
= \frac{1}{2} \|\Delta u_h^M\|_{L^2(\Omega \times (0,T))}.
\]

for all \(i, j \in \{1, \cdots, d\}\). Thus we have \(D^\sigma u \in L^2(\Omega \times (0,T))\) for all index vectors \(\sigma \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\) with \(|\sigma| = 2\).

In the following lemma, we prove an a priori estimate on the backward difference quotient \(\frac{h(b(u_h^M(t)) - b(u_h^M(t-h)))}{h}\).

**Lemma 7.** There exists a constant \(c > 0\) such that
\[
\frac{1}{h} \int_0^T \int_{\Omega} \left( b(u_h^M(t)) - b(u_h^M(t-h)) \right) \phi dx dt \leq c,
\]
for any \(\phi \in L^2(0,T;V_M(\Omega))\), \(M \in \mathbb{N}\) and \(h > 0\).
Proof. Let $m \leq M$ be a positive integer and choose a function $\phi \in L^\infty(0,T;H_0^2(\Omega))$ such that for almost all $t \in [0,T]$

$$\phi(t) = \sum_{i=1}^{m} \alpha_M^i(t)w_i,$$

(20)

where $\alpha_M^i \in L^\infty((0,T)), i = 1, \ldots, m, are given functions and $w_i \in H_0^2(\Omega), i = 1, \ldots, m, belong to the orthonormal basis of the subspace $V_M(\Omega)$. Multiplying equation (15) by $\alpha_M^i(t),$ summing for $i = 1, \ldots, M,$ and then integrating from 0 to $T$ yields

$$\frac{1}{h} \int_0^T \int_\Omega \left( b(u_M^h(t)) - b(u_M^h(t-h)) \right) \phi(t) \, dx \, dt = \int_0^T \int_\Omega \left( K_j(b(u_M^h(t)))e_3 \cdot \nabla \phi(t) \right) \, dx \, dt$$

$$- \int_0^T \int_\Omega \nabla u_M^h(t) \cdot \nabla \phi(t) \, dx \, dt$$

$$- \gamma \int_0^T \int_\Omega \Delta u_M^h(t) \Delta \phi(t) \, dx \, dt.$$

Applying Cauchy’s inequality on the terms on the right side of the above equation then using the growth condition in Assumption 2(4) gives

$$\frac{1}{h} \int_0^T \int_\Omega \left( b(u_M^h(t)) - b(u_M^h(t-h)) \right) \phi(t) \, dx \, dt$$

$$\leq \frac{1}{2} \int_0^T \int_\Omega \left( K(b(u_M^h(t))) \right)^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla \phi(t)|^2 \, dx \, dt$$

$$+ \frac{1}{2} \int_0^T \int_\Omega |\nabla u_M^h(t)|^2 \, dx \, dt$$

$$+ \frac{\gamma}{2} \int_0^T \int_\Omega |\Delta \phi(t)|^2 \, dx \, dt + \frac{\gamma}{2} \int_0^T \int_\Omega |\Delta u_M^h(t)|^2 \, dx \, dt.$$

Then, Lemma 5 and the choice that $\phi \in L^\infty(0,T;H_0^2(\Omega))$ implies the existence of a constant $c > 0$ such that

$$\frac{1}{h} \int_0^T \int_\Omega \left( b(u_M^h(t)) - b(u_M^h(t-h)) \right) \phi(t) \, dx \, dt \leq c.$$

Corollary 8. There exist constants $\delta_0, c > 0$ such that

$$\frac{1}{\delta} \int_\delta \int_\Omega \left( b(u_M^h(t)) - b(u_M^h(t-\delta)) \right) \left( u_M^h(t) - u_M^h(t-\delta) \right) \, dx \, dt \leq c,$$

for any $M \in \mathbb{N}, h > 0$ and $\delta \in (0, \delta_0).$
Proof. Choosing $\phi = u_M^h(t) - u_M^h(t - h)$ in Lemma 7 yields
\[ \frac{1}{h} \int_0^T \int_\Omega \left( b(u_M^h(t)) - b(u_M^h(t - h)) \right) \left( u_M^h(t) - u_M^h(t - h) \right) \, dx \, dt \leq c. \]

Noting that $u_M^h$ is a step function in time, we obtain
\[ \frac{1}{\delta} \int_0^T \int_\Omega \left( b(u_M^h(t)) - b(u_M^h(t - \delta)) \right) \left( u_M^h(t) - u_M^h(t - \delta) \right) \, dx \, dt \leq c, \]
for any $\delta > 0$ such that $|\delta - h|$ is small enough.

4.3. Convergence Results

In this subsection, we show the convergence of the sequence $\{u_M^h\}_{M \in \mathbb{N}, h > 0}$ of discrete solutions of equation (15) to a weak solution of the transformed fourth-order problem (9) and (10). This result is summarized in Theorem 10. The proof of the theorem depends on the a priori estimates in Section 4.2 and the following proposition by Alt and Luckhaus [1].

**Proposition 9** (Alt and Luckhaus [1]). Assume that $z_\varepsilon \rightharpoonup z$ in $L^2(0, T; H^1(\Omega))$ as $\varepsilon \to 0$ and there exists a constant $C > 0$ such that
\[ \frac{1}{\delta} \int_0^{T-\delta} \int_\Omega \left( b(z_\varepsilon(t + \delta) - b(z_\varepsilon(t)) \right) \left( z_\varepsilon(t + \delta) - z_\varepsilon(t) \right) \, dx \, dt \leq C, \]
holds for any small $\delta > 0$ and
\[ \int_\Omega B(z_\varepsilon(t)) \, dx \leq C, \quad \text{for } 0 < t < T. \]

Then, $b(z_\varepsilon) \to b(z)$ in $L^1(\Omega \times (0, T))$ and $B(z_\varepsilon) \to B(z)$ almost everywhere.

Before we state and prove the first main theorem in this chapter, we remind that the Sobolev space $L^2(0, T; H^2(\Omega))$ and its dual $L^2(0, T; H^{-2}(\Omega))$ are equipped with the norms
\[
\|u\|_{L^2(0, T; H^2(\Omega))} = \int_0^T \int_\Omega \left( u^2 + |\nabla u|^2 + |D^2 u|^2 \right) \, dx \, dt, \\
\|L\|_{L^2(0, T; H^{-2}(\Omega))} = \sup \left\{ L(u) \mid u \in L^2(0, T; H^2(\Omega)), \|u\|_{L^2(0, T; H^2(\Omega))} \leq 1 \right\}.
\]

In addition, we state Cauchy’s inequality that will be repeatedly used throughout the coming sections
\[ ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \quad \forall a, b \in \mathbb{R}, \varepsilon > 0. \]
Theorem 10. Let Assumption 2 be satisfied and $h \leq \frac{1}{T}$. Then, problem (9), (10) has a weak solution $u \in L^2(0, T; H_0^2(\Omega))$ that satisfies

1. $K_f(b(u)) \in L^2(\Omega \times (0, T))$, $\partial_t b(u) \in L^2(0, T; H^{-2}(\Omega))$, and
   \[ \int_0^T \int_\Omega \left( \partial_t b(u) \phi - K_f(b(u)) e_3 \cdot \nabla \phi + \nabla u \cdot \nabla \phi + \gamma \Delta u \Delta \phi \right) d\Omega dt = 0, \]  
   for every test function $\phi \in L^2(0, T; H_0^2(\Omega))$.

2. $b(u) \in L^\infty(0, T; L^1(\Omega))$, $\partial_t b(u) \in L^2(0, T; H^{-2}(\Omega))$, and
   \[ \int_0^T \int_\Omega \partial_t b(u) \phi d\Omega dt = \int_0^T \int_\Omega (b(u) - b_0) \partial_t \phi d\Omega dt, \]  
   holds for all test functions $\phi \in L^2(0, T; H_0^2(\Omega))$ with $\partial_t \phi \in L^1(0, T; L^\infty(\Omega))$ and $\phi(\cdot, T) = 0$.

Proof. Using Corollary 6 and the Weak Compactness theorem, there exists a function $u \in L^2(0, T; H_0^2(\Omega))$ such that, up to a subsequence,

\[ u^h \rightharpoonup u \quad \text{in} \quad L^2(0, T; H_0^2(\Omega)), \]  

as $M \to \infty$ and $h \to 0$. The next step in the proof is to show that the function $u \in L^2(0, T; H_0^2(\Omega))$ fulfills the conditions (23) and (24). Thus, we consider an arbitrary test function $\phi \in L^2(0, T; V_m(\Omega))$ such that for a fixed integer $m$ and for almost all $t \in (0, T)$ is given as

\[ \phi(t) = \sum_{i=1}^m \alpha_i(t) w_i, \]  

where $\alpha_i^h \in L^\infty(0, T)$, $i = 1, \cdots, m$, are given functions and $w_i \in H_0^2(\Omega)$, $i = 1, \cdots, m$, belong to the orthonormal basis of the subspace $V_m(\Omega)$. Choosing $m < M$, multiplying equation (15) by $\alpha_i^h(t)$, summing for $i = 1, \cdots, m$, and then integrating with respect to time yields

\[ \frac{1}{h} \int_0^T \int_\Omega \left( b(u^h_M(t)) \partial_t (u^h_M(t) - h) \right) \phi(t) d\Omega dt + \int_0^T \int_\Omega \nabla u^h_M \cdot \nabla \phi d\Omega dt \]
\[ + \gamma \int_0^T \int_\Omega \Delta u^h_M \Delta \phi d\Omega dt = \int_0^T \int_\Omega K_f(b(u^h_M)) e_3 \cdot \nabla \phi d\Omega dt. \]  

(27)
In the following we show that equation (27) converges as \( m \to \infty \) and \( h \to 0 \) to equation (23). The weak convergence (25), Corollary 8, and Proposition 9 imply the strong convergences,

\[
b(u_h^M) \to b(u) \quad \text{in} \ L^1(\Omega \times (0, T)),
\]

and

\[
B(u_h^M) \to B(u) \quad \text{almost everywhere.}
\]

The strong convergence of \( B(u_h^M) \) and the estimate in Lemma 5 leads to

\[
B(u) \in L^\infty(0, T; L^1(\Omega)).
\]

Hence, Assumption 2(2) and the first inequality in Lemma 1 with \( z_0 = u^0 \) imply

\[
b(u) \in L^\infty(0, T; L^1(\Omega)).
\]

The Lipschitz continuity of the flux function and the strong convergence (28) imply

\[
K_f(b(u_h^M)) \to K_f(b(u)) \quad \text{in} \ L^1(\Omega \times (0, T)),
\]

and consequently, we have

\[
K_f(b(u_h^M)) \to K_f(b(u)) \quad \text{almost everywhere.}
\]

However, we need to prove at least a weak convergence of \( K_f(b(u_h^M)) \) in \( L^2(\Omega \times (0, T)) \). For this, we use the growth condition on \( K_f \) and (29). Then, we have

\[
(K_f(b(u)))^2 \leq \beta (1 + B(u)) \in L^\infty(0, T; L^1(\Omega)).
\]

This implies the existence of a constant \( C > 0 \) such that

\[
\|K_f(b(u))\|_{L^2(\Omega \times (0, T))} \leq C.
\]

This estimate, the almost everywhere convergence in (31), the boundedness of the domain \( \Omega \times (0, T) \), and Egorov’s theorem imply the weak convergence

\[
K_f(b(u_h^M)) \rightharpoonup K_f(b(u)) \quad \text{in} \ L^2(\Omega \times (0, T)).
\]
The last step in the proof is to show that
\[
\frac{b(u^h_M(t)) - b(u^h_M(t-h))}{h} \to \partial_t b(u) \quad \text{in } L^2(0,T;H^{-1}_0(\Omega)).
\]
To do this, we consider the estimate in Lemma \[7\]
\[
\int_0^T \int_{\Omega} \frac{b(u^h_M(t)) - b(u^h_M(t-h))}{h} \phi(t) \, dx \, dt \leq C,
\]
for any \(\phi \in L^2(0,T;V_m)\). This uniform estimate implies the existence of a sequence of functionals \(v^N_m\) in the dual space \(L^2(0,T;V_m^*(\Omega))\) such that
\[
\int_0^T \langle v^N_m, \phi \rangle \, dt = \int_0^T \int_{\Omega} \frac{b(u^h_M(t)) - b(u^h_M(t-h))}{h} \phi \, dx \, dt \leq C.
\]
Hence, there exists a limit \(v \in L^2(0,T;H^{-1}_0(\Omega))\) such that
\[
\int_0^T \langle v^N_m, \phi \rangle \, dt \to \int_0^T \langle v, \phi \rangle \, dt
\]
for all \(\phi \in L^2(0,T;V_m(\Omega))\) as \(m \to \infty\) and \(h \to 0\). Since \(\bigcup_{m \in \mathbb{N}} V_m\) is dense in \(H^1_0(\Omega)\), the convergence result in (36) holds also for all \(\phi \in L^2(0,T;H^1_0(\Omega))\). To identify the limit \(v\), we consider the test function \(\phi \in L^2(0,T;H^1_0(\Omega))\) with \(\partial_t \phi \in L^1(0,T;L^\infty(\Omega))\) and \(\phi(t) = 0\) for all \(t \in (T-h,T]\). Applying summation by parts to the left side of (34) yields
\[
\int_0^T \int_{\Omega} \frac{b(u^h_M(t)) - b(u^h_M(t-h))}{h} \phi \, dx \, dt
\]
\[
= -\frac{1}{h} \int_{-h}^0 \int_{\Omega} b(u^h_M) \phi \, dx \, dt - \int_0^T \int_{\Omega} b(u^h_M(t)) \frac{\phi(t) - \phi(t-h)}{h} \, dx \, dt
\]
\[
= \int_0^T \int_{\Omega} \left( b(u^h_M) - b(u^h_M(t)) \right) \frac{\phi(t) - \phi(t-h)}{h} \, dx \, dt,
\]
where we get the last equality using \(\frac{1}{h} \int_{-h}^0 \phi \, dt = -\int_0^T \frac{\phi(t) - \phi(u-h)}{h} \, dt\). Letting \(m \to \infty\) and \(h \to 0\) and using the strong convergence (28), we have
\[
\int_0^T \int_{\Omega} \psi \phi \, dx \, dt = \int_0^T \int_{\Omega} (b(u^0) - b(u)) \partial_t \psi \, dx \, dt,
\]
for all \(\phi \in L^2(0,T;H^1_0(\Omega))\) with \(\partial_t \phi \in L^1(0,T;L^\infty(\Omega))\) and \(\phi(T) = 0\). The right side of (39) corresponds to the definition of the time derivative of \(b(u)\) in the distributional sense. Hence, we have \(v = \partial_t b(u)\) and we conclude
\[
\frac{b(u^h_M(t)) - b(u^h_M(t-h))}{h} \to \partial_t b(u) \quad \text{in } L^2(0,T;H^{-1}_0(\Omega)).
\]
The existence of a function \( u \in L^2(0,T;H_0^1(\Omega)) \), the convergence results \((33)\) and \((40)\) imply that equation \((27)\) converges as \( m \to \infty \) and \( h \to 0 \) to equation \((23)\) for all test function \( \phi \in L^2(0,T;H_1^0(\Omega)) \). Hence, the function \( u \) satisfies the first condition in Theorem \((10)\). Clearly, the second condition in Theorem \((10)\) is also satisfied using equations \((39)\) and \((40)\).

4.4. Uniqueness

In this section, we prove the uniqueness of the weak solution of the transformed problem \((9), (10)\).

Theorem 11. Let Assumption \((2)\) be satisfied and the transformed saturation \( b \) be strictly monotone increasing, i.e. there exists a constant \( a > 0 \) such that

\[
\min(b'(\cdot)) > a > 0.
\]

Then, problem \((9), (10)\) has a unique weak solution that satisfies the properties \((23)\) and \((24)\).

Proof. Assume that \( u_1 \) and \( u_2 \) are two weak solutions of problem \((9)\) with the initial and boundary conditions \((10)\) that satisfy the properties \((23)\) and \((24)\). Define also

\[
g := b(u_1) - b(u_2). \tag{41}
\]

Then, property \((24)\) implies that \( g \in L^\infty(0,T;H_0^{-2}(\Omega)) \) and, consequently, we obtain \( g \in L^2(0,T;H_0^{-2}(\Omega)) \). Thus, Riesz Representation theorem implies the existence of a unique function \( w \in L^2(0,T;H_0^2(\Omega)) \) such that for any time \( \tau \in [0,T] \)

\[
\int_0^\tau (g,\phi) \, dt = \int_0^\tau \langle w,\phi \rangle \, dt, \tag{42}
\]

for all \( \phi \in L^2(0,T;H_0^2(\Omega)) \), where

\[
\langle w,\phi \rangle := \int_\Omega \nabla w \cdot \nabla \phi \, dx + \gamma \int_\Omega \Delta w \Delta \phi \, dx. \tag{43}
\]

Substituting the solutions \( u_1 \) and \( u_2 \) into equation \((23)\), using the test function \( w \in L^2(0,T;H_0^2(\Omega)) \), then subtracting the two equations and using \((41)\) gives

\[
\int_0^\tau \int_\Omega \partial_t g \, w \, dx \, dt + \int_0^\tau \int_\Omega \left( (\nabla (u_1 - u_2)) \cdot \nabla w + \gamma \Delta (u_1 - u_2) \Delta w \right) dx \, dt = \int_0^\tau \int_\Omega \left( K_f(b(u_1)) - K_f(b(u_2)) \right) e_3 \cdot \nabla w \, dx \, dt. \tag{44}
\]
Approximating the first term on the left side of (44) using backward differences then applying summation by parts yields
\[
\int_0^\tau \int_\Omega \frac{g(t) - g(t-h)}{h} w(t) \, dx \, dt = -\int_0^\tau \int_\Omega \frac{g(t) - g(t-h)}{h} \frac{w(t) - w(t-h)}{h} \, dx \, dt \\
+ \frac{1}{h} \int_{\tau-h}^{\tau} \int_\Omega g(t)w(t) \, dx \, dt - \frac{1}{h} \int_0^{\tau-h} \int_\Omega g(t)w(t) \, dx \, dt.
\] (45)

Using equations (42) and (43), the first term on the right side of (45) satisfies
\[
\int_0^\tau \int_\Omega \frac{g(t) - g(t-h)}{h} \frac{w(t) - w(t-h)}{h} \, dx \, dt = \int_0^\tau \int_\Omega \nabla w(t) - \nabla w(t-h) \frac{h}{h} \, dx \, dt \\
+ \gamma \int_0^\tau \int_\Omega \Delta w(t) - \Delta w(t-h) \frac{h}{h} \, dx \, dt.
\]

Applying summation by parts to the right side of the above equation yields
\[
\int_0^\tau \int_\Omega \frac{g(t) - g(t-h)}{h} \, dx \, dt = \frac{1}{2h} \int_{\tau-h}^{\tau} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt \\
- \frac{1}{2h} \int_0^{\tau-h} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt.
\] (46)

The second term on the right side of (45), using equations (42) and (43), satisfies
\[
\frac{1}{h} \int_{\tau-h}^{\tau} \int_\Omega g(t)w(t) \, dx \, dt = \frac{1}{h} \int_{\tau-h}^{\tau} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt.
\] (47)

Similarly, the third term on the right side of (45) satisfies
\[
\frac{1}{h} \int_0^{\tau-h} \int_\Omega g(t)w(t) \, dx \, dt = \frac{1}{h} \int_0^{\tau-h} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt.
\] (48)

Substituting equation (46), (47), and (48) into equation (45) gives
\[
\int_0^\tau \int_\Omega \frac{g(t) - g(t-h)}{h} w(t) \, dx \, dt = \frac{1}{2h} \int_{\tau-h}^{\tau} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt \\
- \frac{1}{2h} \int_0^{\tau-h} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt.
\] (49)

Using equation (42) and the initial choice (16), the second term on the right side of (49) satisfies
\[
\int_0^{\tau-h} \int_\Omega |\nabla w|^2 + \gamma (\Delta w)^2 \, dx \, dt = \int_0^{\tau-h} \int_\Omega g \, dx \, dt = 0.
\]

Hence, letting \( h \to 0 \) in equation (49), we get that for almost all \( \tau \in [0, T] \),
\[
\int_0^\tau \int_\Omega \partial_t g \, dx \, dt = \frac{1}{2} \int_\Omega |\nabla w(\tau)|^2 + \gamma (\Delta w(\tau))^2 \, dx.
\] (50)
Using (42) with $\phi = u_1 - u_2$, the second term on the left side of (44) satisfies
\[
\int_0^\tau \int_\Omega \nabla (u_1 - u_2) \cdot \nabla w + \gamma \Delta (u_1 - u_2) \Delta w \, dx \, dt \\
= \int_0^\tau \int_\Omega (u_1 - u_2) g \, dx \, dt = \int_0^\tau \int_\Omega (u_1 - u_2) (b(u_1) - b(u_2)) \, dx \, dt.
\] (51)

The Lipschitz continuity of $K_f$ and $b$ imply the existence of a constant $L > 0$ such that
$\max(b'(\cdot)), \max(K'_f(\cdot)) \leq L$. Using this property and Cauchy’s inequality (22), with
$\varepsilon = \frac{1}{2L^2}$, the first term on the right side of equation (44) simplifies to
\[
\int_0^\tau \int_\Omega (K(b(u_1)) - K(b(u_2))) e_3 \cdot \nabla w \, dx \, dt \\
\leq L \int_0^\tau \int_\Omega |(b(u_1) - b(u_2)) e_3 \cdot \nabla w| \, dx \, dt \\
\leq \frac{1}{2L} \int_0^\tau \int_\Omega |b(u_1) - b(u_2)|^2 \, dx \, dt + \frac{L^3}{2} \int_0^\tau \int_\Omega |\nabla w|^2 \, dx \, dt \\
\leq \frac{1}{2} \int_0^\tau \int_\Omega |(b(u_1) - b(u_2))(u_1 - u_2)| \, dx \, dt + \frac{L^3}{2} \int_0^\tau \int_\Omega |\nabla w|^2 \, dx \, dt.
\] (52)

As the function $b$ is monotone increasing, it follows that
\[
\int_0^\tau \int_\Omega (K(b(u_1)) - K(b(u_2))) e_3 \cdot \nabla w \, dx \, dt \\
\leq \frac{1}{2} \int_0^\tau \int_\Omega (b(u_1) - b(u_2))(u_1 - u_2) \, dx \, dt + \frac{L^3}{2} \int_0^\tau \int_\Omega |\nabla w|^2 \, dx \, dt.
\] (53)

Substituting (50), (51), and (52) into (44) yields, for almost all $\tau \in [0, T]$,
\[
\frac{1}{2} \int_\Omega |\nabla w(\tau)|^2 \, dx + \frac{1}{2} \int_\Omega (\Delta w(\tau))^2 \, dx + \frac{1}{2} \int_\Omega (b(u_1) - b(u_2))(u_1 - u_2) \, dx \, dt \\
\leq \frac{L^3}{2} \int_0^\tau \int_\Omega |\nabla w|^2 \, dx \, dt.
\] (53)

Since $b$ is a monotone increasing function, the third term on the left side of equation (53) is nonnegative. Thus, applying Gronwall’s inequality to the first term on the left side gives
\[
\int_\Omega |\nabla w(\tau)|^2 \, dx = 0,
\] (54)
for any $\tau \in [0, T]$. Substituting (54) in equation (53) yields
\[
\int_0^\tau \int_\Omega (b(u_1) - b(u_2))(u_1 - u_2) \, dx \, dt = 0.
\] (55)

Using the strict monotonicity of $b$, equation (55) implies that $u_1 = u_2$. \qed
5. Regularity

In this section, we improve the regularity of the weak solution from \( u \in L^2(0, T; H^1_0(\Omega)) \) to \( u \in H^1(\Omega \times (0, T)) \cap L^2(0, T; H^2_0(\Omega)) \). For this, it is sufficient to prove that \( \partial_t u \in L^2(\Omega \times (0, T)) \).

**Lemma 12.** Let Assumption $A_0$ be satisfied and the transformed saturation \( b \) be strictly monotone increasing, i.e. there exists a constant \( a > 0 \) such that \( \min(b'(\cdot)) > a > 0 \). Then, the weak solution \( u \in L^2(0, T; H^2_0(\Omega)) \) of the transformed problem \((56)\) and \((10)\) satisfies the property that \( \partial_t u \in L^2(\Omega \times (0, T)) \).

**Proof.** Multiplying equation \((15)\) by \( \frac{u^h_M(t) - u^h_M(t-h)}{h} \), summing for \( i = 1, \ldots, M \), integrating from 0 to \( T \), and using the Gauss theorem yields

\[
\frac{1}{h^2} \int_0^T \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t-h)) \right) \left( u^h_M(t) - u^h_M(t-h) \right) \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} \nabla u^h_M(t) \cdot \nabla u^h_M(t) \, dx \, dt + \gamma \int_0^T \int_{\Omega} \Delta u^h_M(t) \cdot \Delta u^h_M(t) \, dx \, dt
\]

\[
= - \int_0^T \int_{\Omega} \nabla \cdot \left( K_f(b(u^h_M(t))) \right) \, dx \, dt.
\]

Using the strict positivity of \( b' \), the first term on the left side of \((56)\) satisfies

\[
\frac{1}{h^2} \int_0^T \int_{\Omega} \left( b(u^h_M(t)) - b(u^h_M(t-h)) \right) \left( u^h_M(t) - u^h_M(t-h) \right) \, dx \, dt
\]

\[
\geq a \int_0^T \int_{\Omega} \left( \frac{u^h_M(t) - u^h_M(t-h)}{h} \right)^2 \, dx \, dt.
\]

Applying summation by parts to the second term on the left side of \((56)\), we have

\[
2 \int_0^T \int_{\Omega} \nabla u^h_M \cdot \nabla u^h_M(t) \, dx \, dt = \frac{1}{h} \int_{T-h}^T \int_{\Omega} |\nabla u^h_M(t)|^2 \, dx \, dt.
\]

Then, as the discrete solution is a step function in time, we get

\[
\int_0^T \int_{\Omega} \nabla u^h_M \cdot \nabla u^h_M(t) \, dx \, dt = \frac{1}{2} \int_0^T \int_{\Omega} |\nabla u^h_M(t)|^2 \, dx \, dt.
\]

Similarly, the third term on the right side of \((56)\) simplifies to

\[
\int_0^T \int_{\Omega} \Delta u^h_M \cdot \Delta u^h_M(t) \, dx \, dt = \frac{1}{2} \int_0^T \int_{\Omega} |\Delta u^h_M(T)|^2 \, dx \, dt.
\]
The Lipschitz continuity of \( K_f \) and \( b \) implies the existence of a constant \( L > 0 \) such that \( \max(b'(\cdot)), \max(K'_f(\cdot)) \leq L \). Using this property and Cauchy’s inequality (22), with \( \varepsilon = \frac{L^2}{a} \), the right side of (56) gives

\[
\int_0^T \int_\Omega \left| \nabla \cdot (K_f(b(u^h_M(t))) \left( u^h_M(t) - u^h_M(t-h) \right) \right| \, dx \, dt \\
\leq L^2 \int_0^T \int_\Omega \frac{|\nabla u^h_M(t)|}{h} \, dx \, dt \\
\leq \frac{L^4}{a} \int_0^T \int_{\Omega} \left| \nabla u^h_M \right|^2 \, dx \, dt + \frac{a}{4} \int_0^T \int_{\Omega} \left( \frac{u^h_M(t) - u^h_M(t-h)}{h} \right)^2 \, dx \, dt.
\]

Substituting (57), (58), (59), and (60) into inequality (56) gives

\[
\frac{3a}{4} \int_0^T \int_{\Omega} \left( \frac{u^h_M(t) - u^h_M(t-h)}{h} \right)^2 \, dx \, dt + \frac{1}{2} \int_{\Omega} |\nabla u^h_M(T)|^2 + \gamma (\Delta u^h_M(T))^2 \, dx \\
\leq \frac{1}{2} \int_{\Omega} |\nabla u^h_M|^2 + \gamma (\Delta u^h_M)^2 \, dx + \frac{L^4}{a} \int_0^T \int_{\Omega} \left| \nabla u^h_M \right|^2 \, dx \, dt.
\]

Then, Lemma 5 implies the existence of a constant \( c > 0 \) such that

\[
\int_0^T \int_{\Omega} \left( \frac{u^h_M(t) - u^h_M(t-h)}{h} \right)^2 \, dx \, dt \leq c.
\]

This uniform estimate implies that, up to a subsequence,

\[
\frac{u^h_M(t) - u^h_M(t-h)}{h} \to \partial_t u \quad \text{in} \ L^2(\Omega \times (0,T)).
\]

\( \square \)

**Corollary 13.** Let the assumptions of Theorem 11 be satisfied. Then, we have the strong convergence

\[
u^h_M \to u \quad \text{in} \ L^2(\Omega \times (0,T)).
\]

**Proof.** The proof follows using the estimates in Lemma 5 and 12 together with Rellich Kondrachov Compactness theorem with dimension \( n = 4 \) of the domain \( \Omega \times (0,T) \). \( \square \)

**Corollary 14.** Let the assumptions of Theorem 11 be satisfied. Then, the transformed saturation \( b \) satisfies

\[
b(u) \in C([0,T];L^2(\Omega)),
\]

21
and the initial condition satisfies

\[ b(u(0)) = b(u^0) \quad \text{almost everywhere.} \]

**Proof.** The Lipschitz continuity of the transformed saturation \( b \) and Lemma 12 imply

\[ \partial_t b(u) = b'(u) \partial_t u \in L^2(\Omega \times (0, T)). \] (62)

This yields also that

\[ b(u) \in C([0, T]; L^2(\Omega)). \] (63)

To prove that \( b(u(0)) = b(u^0) \) almost everywhere, we choose a test function \( \phi \in C^1([0, T], H^2_0(\Omega)) \) in equation (23) such that \( \phi(T) = 0 \). Then, Gauss theorem gives

\[
\int_0^T \int_\Omega \left( b(u) \partial_t \phi - K_f(b(u))e_3 \cdot \nabla \phi + \nabla u \cdot \nabla \phi + \gamma \Delta u \Delta \phi \right) dx dt = \int_\Omega b(u(0))\phi(0) dx.
\] (64)

Applying summation by parts to the first term in equation (27) yields

\[
\frac{1}{h} \int_0^T \int_\Omega b(u)(t)(\phi(t) - \phi(t - h)) dx dt + \int_0^T \int_\Omega \nabla u_M^h \cdot \nabla \phi dx dt \\
+ \gamma \int_0^T \int_\Omega \Delta u_M^h \Delta \phi dx dt - \int_0^T \int_\Omega K_f(b(u)_M^h) e_3 \cdot \nabla \phi dx dt \\
= \int_\Omega b(u(0))\phi(0) dx dz.
\] (65)

Letting \( M \to \infty \) and \( h \to 0 \) in equation (65) yields, up to a subsequence, that

\[
\int_0^T \int_\Omega \partial_t \phi b(u(t)) dx dt + \int_0^T \int_\Omega \nabla u \cdot \nabla \phi dx dt + \gamma \int_0^T \int_\Omega \Delta u \Delta \phi dx dt \\
- \int_0^T \int_\Omega K_f(b(u))e_3 \cdot \nabla \phi dx dt = \int_\Omega b(u^0)\phi(0) dx dz.
\] (66)

since \( u_M^h \to u^0 \) in \( L^2(\Omega) \) as \( M \to \infty \). As \( \phi(0) \) is arbitrarily chosen, comparing equation (64) and (66) yields that \( b(u(0)) = b(u^0) \) almost everywhere. \( \square \)

### 6. Well-posedness of the Fourth-Order Model

In this section, we utilize the well-posedness of the transformed problem (9) and (10) to prove the well-posedness of the fourth-order model (6) and (7). For this, we
stress that the coefficients $S, S', K_f$ are strictly positive. Then, we apply the inverse of
Kirchhoff’s transformation to the weak solution of the transformed problem (9) and (10).

**Definition 15.** Let the function $S = S(p)$ be Lipschitz continuous and $K_f = K_f(S(p))$
be bounded. We call $p \in H^1(\Omega \times (0, T))$ a weak solution of the fourth-order problem
(6) and (7) if it satisfies the conditions

1. $\partial_t S(p) \in L^2(\Omega \times (0, T))$ and $\nabla \cdot (K_f(S(p))\nabla p) \in L^2(\Omega \times (0, T))$ such that
   \[
   \int_0^T \int_{\Omega} \left( \partial_t S(p) \phi - K_f(S(p)) e_3 \cdot \nabla \phi + K_f(S(p)) \nabla p \cdot \nabla \phi \right) d\mathbf{x} dt \\
   + \gamma \int_0^T \int_{\Omega} \nabla \cdot (K_f(S(p))\nabla p) \Delta \phi d\mathbf{x} dt = 0,
   \]
   for every test function $\phi \in L^2(0, T; H^2_0(\Omega))$.

2. $S(p(0)) = S(p^0)$ almost everywhere.

**Theorem 16.** Assume that the initial condition in (7) satisfies $p^0 \in H^2_0(\Omega)$ and the
saturation function $S \in C^1(\mathbb{R})$ is Lipschitz continuity, strictly positive, and strictly monotone increasing. Assume also that the conductivity function $K_f \in C^1(\mathbb{R})$ is strictly positive, bounded, and monotone increasing. Let $u \in H^1(\Omega \times (0, T)) \cap L^2(0, T; H^2_0(\Omega))$ be
the weak solution of the transformed problem (9) and (10). Then $p = \psi^{-1}(u)$, where
$\psi$ is Kirchhoff’s transformation, is the unique weak solution of the fourth-order problem
(6) and (7) according to Definition 15.

**Proof.** Using equation (8), Lemma 12, the boundedness and the strict positivity of $K_f$,
we have

\[
\nabla p = \frac{\nabla u}{K_f(S(\psi^{-1}(u)))} \in L^2(\Omega \times (0, T)), \\
\partial_t p = \frac{\partial_t u}{K_f(S(\psi^{-1}(u)))} \in L^2(\Omega \times (0, T)),
\]
where there exists a constant $\delta > 0$ such that $K_f > \delta$. These estimates and Poincaré’s
inequality implies that $p \in H^1(\Omega \times (0, T))$. Consequently, we have

\[
p \in C([0, T]; L^2(\Omega)).
\]
In addition to this, we have
\[ \nabla \cdot (K_f(S(p))\nabla p) = \Delta u \in L^2(\Omega \times (0,T)). \] (69)

The Lipschitz continuity of the saturation $S$ and the second equation in (67) imply
\[ \partial_t S(p) = S'(p)\partial_t p \in L^2(\Omega \times (0,T)). \]

These estimates imply that $p$ satisfies the conditions in Definition 15 and, thus, is a weak solution of the fourth-order model (6) and (7). In the same way, if $p \in H^1(\Omega \times (0,T))$ is a weak solution of the fourth-order problem (6) and (7) as in Definition 15, then the Kirchhoff-transformed $u = \psi(p) \in L^2(0,T;H^2_0(\Omega))$ is a weak solution of the transformed fourth-order problem (9) and (10). This implies that the fourth-order problem (6), (7) and the transformed fourth-order problem (9) and (10) are equivalent. This equivalency, the uniqueness of the weak solution $u$ of the transformed problem by Theorem 11, and the strict monotonicity of Kirchhoff’s transformation imply the uniqueness of the weak solution $p$ of the fourth-order problem (6) and (7).

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