QED in Finite Volume and Finite Size Scaling Effect on Electromagnetic Properties of Hadrons

Shumpei Uno and Masashi Hayakawa

Department of Physics, Nagoya University, Nagoya 464-8602, Japan

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On account of the computation of the virtual photon corrections to the hadronic properties by means of lattice QCD simulation, we study the finite size scaling effect on the QED correction using low energy effective theory of QCD with QED. For this purpose, we begin with formulating a new QED on the space with finite volume. By adapting this formalism to the partially quenched QCD with electromagnetism, we compute the finite size scaling effect on the electromagnetic correction to the masses of pseudo-Nambu-Goldstone bosons and observe qualitative features.

§1. Introduction

The recent progress in the lattice QCD simulation enables us to “measure” the hadronic properties through the world with more and more realistic QCD realized in computers. It is not a far future that the lattice QCD simulation will become an effective approach to explore nonperturbative dynamics of QCD in a precise manner. One of the important applications of lattice QCD simulation would be the determination of quark masses. The light quark masses have been determined using dynamical quarks with two flavors$^{1)-6)}$ and (2 + 1) flavors.$^{7)-10)}$

However, the forthcoming precise measurement through the lattice QCD simulation reminds us that quarks are also electrically charged and that all of the hadronic properties suffer from electromagnetic (EM) radiative corrections. Because the EM interaction is other source of explicit breaking of isospin symmetry than the difference between the masses $m_u$, $m_d$ of up- and down-quarks, the determination of $m_u - m_d$ requires us to grasp the size of EM correction in the hadronic observables quantitatively. Among the literatures,$^{1)-10)}$ Ref. 7) is the only work that presented individual values of $m_u$ and $m_d$, though it does not seem to make the data fit using a function that parametrizes the EM splitting in the kaon masses as undetermined constant.

In the context of lattice simulation, the first attempt to incorporate EM correction to pseudoscalar meson masses was done by Duncan et al.$^{11)}$ They also put QED on the lattice and simulate QED to get the EM splittings of pions and kaons from the quenched simulation with Wilson fermion. Recently, the same technique has been applied to investigate the EM splittings with use of dynamical domain wall fermion with two-flavors$^{12)}$ and in the quenched approximation with the renormalization group improved gauge action.$^{13)}$ Reference 14) computed the difference between correlation functions of vector currents and axial-vector currents, and obtained the leading-order pion mass difference, $\Delta m_\pi^2 \equiv m_{\pi^+}^2 - m_{\pi^0}^2$, through the sum rule.$^{15)}$
In practice, the simulation has to be done in the virtual world with finite volume. Since the quarks with colors and electric charges must live in such a finite volume, it is inevitable that hadronic properties suffer from finite size scaling effect no matter what computational method one may choose. It is plausible that two pseudoscalar mesons in a common isospin multiplet share the same QCD finite size correction. This is not the case for the finite size scaling effect on QED. Moreover, electromagnetic force is long-ranged. Therefore, unless the QED finite size scaling is adequately quantified, ignorance on it remains as a source of systematic uncertainty in the quark mass difference \((m_u - m_d)\) derived using lattice simulation.

Thus far, there are diametrical opposite views on the relevance of QED finite size scaling effect. The authors in Ref. 13) performed the direct (quenched) lattice QCD measurement of the EM splitting in pion masses using two different sizes of four-volumes, \(T \times L^3 = 2a \times (12a)^3\) and \(2a \times (16a)^3\), where \(a\) is the lattice spacing. They observed no significant difference between the values of EM splitting measured in these two volumes and concluded that a linear size \(L = 2.4\) fm is sufficient to compute the EM splitting without correcting the measured values within available statistical uncertainty. Contrastingly, Refs. 11) and 12) estimated the QED finite size scaling according to the vector meson dominance (VMD) model, which is equivalent to the one-pole saturation approximation to the sum rule with the momentum integral replaced with the sum

\[
\Delta m^2_\pi \bigg|_{\alpha,\text{VMD}} (T, L) = \frac{3\alpha}{4\pi} \left(\frac{4\pi}{2}\right) \frac{1}{T L^3} \sum_{k \in (\tilde{T}_4 - \{0\})} \frac{m_V^2 m_A^2}{k^2 (k^2 + m_V^2) (k^2 + m_A^2)}, \tag{1.1}
\]

where \(\alpha \equiv e^2/(4\pi)\) is the electromagnetic fine structure constant, \(m_V \simeq 770\) MeV and \(m_A \simeq 970\) MeV is the mass of \(A_1\) in the chiral limit, and

\[
\tilde{T}_4 \equiv \left\{ k = (k_0, k_1, k_2, k_3) \Bigg| k_0 \in \frac{2\pi}{T} \mathbb{Z}, k_j \in \frac{2\pi}{L} \mathbb{Z} \right\}, \tag{1.2}
\]

is a lattice on the Euclidean four-momentum space. For the lattice geometry \(T \times L^3 = 32a \times (16a)^3\) with \(1/a \simeq 1.66\) GeV in Ref. 12), Eq. (1.1) leads

\[
\frac{\Delta m^2_\pi |_{\alpha,\text{VMD}} (T, L)}{\Delta m^2_\pi |_{\alpha,\text{VMD}} (\infty, \infty)} \simeq 0.9, \tag{1.3}
\]

which is not negligible.

Under such circumstances, we give the first study of the finite size scaling effect on the QED contribution to light pseudoscalar meson masses using the chiral perturbation theory including electromagnetism. Although the final quantitative determination of the finite size correction must resort to the first principle computation as in Ref. 13), qualitative understanding to be shown here is also necessary in order to find a good fitting function to deduce the value in infinite volume.

For the study of QED finite size scaling effect, we begin with constructing QED in finite volume, called QED\(_L\). This is required for the following reason, whose more detail exposition is given in §2.1. The system is put on the space with finite volume.
by adopting some boundary condition to fields. For example, periodic boundary condition is usually imposed on quark fields along spatial directions in lattice QCD. Because the generation of dynamical configuration requires huge computational resources, we must reuse available QCD configurations for the study of EM splittings. Thus, the meson fields obey periodic boundary condition in the low energy effective theory on the three dimensional torus $\mathbb{T}^3 \equiv S^1 \times S^1 \times S^1$. Our interest is the effect of finiteness of volume on the properties of a single charged particle. However, a single charged particle cannot be put on the space $\mathbb{T}^3$. Here we propose QED$_L$ as an alternative quantum theory of electromagnetism in finite volume, and explain how it avoids this inconsistency in §2.2. If the lattice simulation is also performed with use of QED$_L$, the macroscopic study of finite size scaling corrections to be done here will provide a consistent framework to estimate the finite size scaling effect involved in the lattice data and to determine EM splittings and quark mass difference $m_u - m_d$.

In §3, we apply the partially quenched chiral perturbation theory$^{34)–37}$ including electromagnetism$^{32)}$ whose electromagnetic part is described by QED$_L$ to evaluate the finite size scaling effect on EM corrections to pseudoscalar meson masses. We find substantial finite size scaling effect on the spaces with volumes available for lattice QCD configurations. We also investigate the dependence on the masses of valence and sea quarks. Section 4 is devoted to conclusion. Appendix A collects the formulae for the basic sums which appear in the evaluation of finite size scaling effect.

§2. QED in finite volume

The aim of this section is to present a new QED on the space with finite volume, which enables us to investigate the properties of a single charged particle. We first clarify in §2.1 the problem itself that confronts us in the QED obtained by the naive compactification procedure. In §2.2, we define a new QED in finite volume and explain how it solves this problem. Throughout this paper, the topology of the space is the three-dimensional torus $\mathbb{T}^3$ with a common circumference $L$ for all $S^1$. A point on $\mathbb{T}^3$ is specified by the coordinates $x \equiv (x^1, x^2, x^3)$ obeying periodicity $x^j \equiv x^j + L$ ($j = 1, 2, 3$). As in the analysis of finite size scaling of QCD,$^{16)–18}$ the temporal direction $t = x^0$ is taken to be infinite, $t \in \mathbb{R}$, for a single particle state to develop a pole in the energy space. We adopt the convention $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ for the signature of the metric.

2.1. Problem

An electromagnetic theory on $\mathbb{T}^3$ can be obtained by applying $\mathbb{Z}^3$-orbifolding procedure on all fields. We call this theory QED$_{\mathbb{Z}^3}$. In QED$_{\mathbb{Z}^3}$, $U(1)$ gauge potential $A_\mu(x)$ ($x^0 = (t, x)$) obeys periodic boundary condition in every spatial direction. This type of boundary condition is required because the boundary condition assumed for the quarks explained in §1 together with momentum conservation at the QED vertex implies that every component of photon momentum must be an integer
multiple of $\frac{2\pi}{L}$. The action for the gauge kinetic term in QED$_{Z^3}$ takes the usual form

$$S_\gamma = \int dt \int_{T^3} d^3x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),$$

with the field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

To elucidate the practical problem for our purpose, we go back to Eq. (1.1) for the pion mass difference. The application of QED$_{Z^3}$ to the electromagnetic sector of the effective Lagrangian including the vector and axial-vector mesons in Ref. 28) immediately leads the same form of the Lagrangian written in terms of spatially periodic fields. This effective theory reproduces Eq. (1.1) with $T \to \infty$

$$\Delta m_{\pi}^2|_{\alpha,\text{VMD}} (\infty, L) = \frac{3\alpha}{4\pi} (4\pi)^2 \int_{-\infty}^{\infty} \frac{dk^0}{2\pi^3} \frac{1}{L^3} \sum_{k \in \Lambda_3} \frac{m_{V}^2 m_{A}^2}{k^2 (k^2 + m_{V}^2) (k^2 + m_{A}^2)},$$

(2.2)

where

$$\Lambda_3 \equiv \left\{ k = (k_1, k_2, k_3) \mid k_j \in \frac{2\pi}{L} \mathbb{Z} \right\}.$$

(2.3)

We can see that the quantity (2.2) contains infrared (IR) divergence coming from the contribution of $k = 0$. The expression (2.2) can be interpreted as the sum of the contribution of Kaluza-Klein modes in one-dimensional field theory. The IR divergence appearing in Eq. (2.3) is attributed to that of a massless mode from this point of view. We recall that $\Delta m_{\pi}^2|_{\alpha,\text{VMD}}$ computed directly in infinite volume is IR-finite. Therefore, $\lim_{L \to \infty} \Delta m_{\pi}^2|_{\alpha,\text{VMD}} (\infty, L) \neq \Delta m_{\pi}^2|_{\alpha,\text{VMD}}$. Likewise, the same pathology always emerges in theory irrespective of the details of matter fields and action where electromagnetic part is given by QED$_{Z^3}$. Thus QED$_{Z^3}$ cannot be used to study the finite size scaling effect on the EM splittings.

The origin of the above pathology will be traced back to the inconsistency of a single charged particle with the classical equation of motion. Though it may be well-known, the detail observation of this point will help us to grasp the essence of our new QED in §2.2. Due to the periodicity of the gauge potential, the electromagnetic current $j^\mu(x) \equiv \delta S_{\text{matter}} / \delta A_\mu(x)$ derived from the matter part $S_{\text{matter}}$ of the action is also periodic along the spatial directions. The current $j^\mu(x)$ is assumed to be conserved under the equations of motion derived from the variation of matter fields. The classical equation of motion derived from the variation of $A_\mu(x)$ is hence

$$\partial_\nu F^{\mu\nu}(x) = j^\mu(x).$$

(2.4)

This contains the Gauss’ law constraint

$$\nabla \cdot \mathbf{E}(x) = \rho(x),$$

(2.5)

where the electric field $E^j(x)$ and the charge density $\rho(x)$ are given by $E^j(x) = F^0j(x)$ and $\rho(x) = j^0(x)$, respectively. For simplicity, we consider an infinitely
heavy charged particle. When it is at rest initially, the charge density and the profile of electric field are both constant in time so that Eq. (2.5) becomes

$$\nabla \cdot E(x) = e \delta^3(x).$$

(2.6)

This equation involves inconsistency; when both sides of this equation are integrated over the whole $T^3$, the left-hand side vanishes while the right-hand side does not. Likewise, we can see that any single charged particle cannot live on $T^3$.

2.2. New QED on $R \times T^3$

The observation in §2.1 shows that the IR divergence in Eq. (2.2) is a manifestation of inconsistency of a single charged particle with the classical equation of motion. The aim of this section is to introduce an alternative QED in finite volume that solves this problem for the sake of our study of the finite size scaling effect on the EM splitting.

On the space $T^3$, three-momenta take discrete values in $\tilde{T}_3$ in Eq. (2.3). Accordingly, the gauge potential is decomposed in the Fourier series with respect to the spatial dimension

$$A_\mu(t, x) = \frac{1}{L^3} \sum_{k \in \tilde{T}_3} e^{i k \cdot x} \tilde{A}_\mu(t, k).$$

(2.7)

The new QED on $R \times T^3$, referred to as QED$_L$, is the theory without variables $\tilde{A}_\mu(t, k = 0)$ ab initio. In other words, we do not incorporate the Wilson lines $U_\mu(t) \sim \exp \left[ ie \int dt \frac{1}{L^3} \tilde{A}_\mu(t, 0) \right]$ as dynamical variables. In terms of such a gauge potential, the action of pure electromagnetism is given by $S_\gamma$ in Eq. (2.1). Below, we observe the various features possessed by QED$_L$.

First, we see how QED$_L$ solves the problems in §2.1. The equation of motion is modified as follows. Since the modes $\tilde{A}_\mu(t, 0)$ are absent, the variation of the full action $S = S_\gamma + S_{\text{matter}}$ with respect to the gauge potential becomes

$$\delta_A S = \int_{-\infty}^{\infty} dt \int_{T^3} d^3x \left[ -\frac{1}{2} \delta F_{\mu\nu} F^{\mu\nu} + \delta A_\mu j^\mu \right]$$

$$= \int_{-\infty}^{\infty} dt \sum_{k \in \tilde{T}_3} \left[ \delta A_\mu(t, k) \right.$$  

$$\left. \times \int d^3x \cos (k \cdot x) \left( -\partial_\nu F^{\mu\nu}(t, x) + j^\mu(t, x) \right) \right],$$

(2.8)

where

$$\tilde{T}_3' \equiv \tilde{T}_3 - \{0\}.$$  

(2.9)

As the result, the extremum condition gives the equations of motion only for $k \neq 0$;

$$\int d^3x \cos (k \cdot x) \left\{ \partial_\nu F^{\mu\nu}(t, x) - j^\mu(t, x) \right\} = 0. \quad (k \neq 0)$$

(2.10)
The inconsistency explained at the end of §2.1 is therefore circumvented. We also note that Eq. (2.10) instead of Eq. (2.5) no longer yields the equality between the charges contained in a domain $V$ in $\mathbb{T}^3$ and the electric flux penetrating the surface $\partial V$.

The quantum field theory is defined in the path integral framework by defining the measure of the gauge potential as usual. Let $\delta \tilde{A}_\mu(t, k)$ be an infinitesimal variation of the mode $\tilde{A}_\mu(t, k)$ ($t \in \mathbb{R}$, $k \in \tilde{\Gamma}_3^\prime$). The functional measure is defined corresponding to the norm in the space of gauge configurations in QED

$$
||\delta A||^2 = \int dt \sum_{k \in \tilde{\Gamma}_3^\prime} \delta \tilde{A}_\mu(t, k) \delta \tilde{A}^\mu(t, k).
$$

As we see later this norm turns out to be gauge-invariant. From the point of view of field theory on $\mathbb{R}$, $\tilde{A}(t, 0)$ are massless fields. Due to the absence of these modes in QED$_L$, the IR divergence encountered in Eq. (2.3) no longer appears.

We next observe the gauge symmetry of QED$_L$. The transformation of the gauge potential is given by

$$
A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda,
$$

while a matter field $\Phi(x)$ with charge $Q_\Phi e$ is transformed as

$$
\Phi(x) \mapsto \Phi'(x) = \exp[iQ_\Phi e \Lambda(x)] \Phi(x).
$$

Here we assume that there is at least one matter field which has the minimum charge $e$. The easiest way to identify the full gauge symmetry is to look at the full gauge symmetry of QED$_{\mathbb{T}^3}$ first. There, the general form of $\Lambda(x)$ that keeps the periodic boundary condition for the matter fields takes the form

$$
\Lambda(x) = \Lambda_P(x) + \frac{2\pi}{eL} \sum_{j=1}^3 m_j x^j, \quad (\text{QED}_{\mathbb{T}^3})
$$

where $m_j \in \mathbb{Z}$, and $\Lambda_P(x)$ is periodic along $\mathbb{T}^3$.

Denoting the Fourier components of $\Lambda_P(t, x)$ by $\tilde{\Lambda}_P(t, k)$, the gauge transformation for the gauge potential becomes in the three-momentum space

$$
\tilde{A}'_j(t, k) = \tilde{A}_j(t, k) + \left(ik^j \tilde{\Lambda}_P(t, k) + L^3 \delta_{k, 0} \frac{2\pi}{eL} m_j \right). \quad (\text{QED}_{\mathbb{T}^3})
$$

In our new QED, $\tilde{A}(t, 0)$ no longer exists. There are hence no redundancy corresponding to $m \in \mathbb{Z}^3$ and time-dependent but spatially homogeneous part of $\Lambda_P(x)$, that is,

$$
\tilde{A}'_j(t, k) = \tilde{A}_j(t, k) + ik^j \tilde{\Lambda}_P(t, k), \quad \text{(QED}_L)
$$

where

$$
\partial_t \tilde{\Lambda}_P(t, 0) = 0. \quad (2.17)
$$
One can readily see that the set of all functions that fulfill Eq. (2.17) forms an abelian group. From Eq. (2.16), it is also easy to see that this gauge group is exactly the redundancy that allows us to take the Coulomb gauge fixing condition

$$\partial_j A_j(t, x) = 0.$$ (2.18)

We recall that in QED$_{Z^3}$ a residual gauge symmetry survives even after imposing the condition (2.18), which should be fixed by the additional condition $\tilde{A}_0(t, 0) = 0$. In contrast, the condition (2.18) suffices to fix redundancy in QED leaving only the global symmetry. Obviously, the norm (2.11) in the space of gauge configurations is gauge-invariant. We can also make a BRST complex by introducing the ghost fields corresponding to the gauge parameters of the form (2.17) in the standard manner.

We close this section with a remark. We consider the scattering of a charged particle and its anti-particle in the center of mass frame, leaving aside the issue whether the scattering may not be well-defined in the presence of a long-ranged force in finite volume. Due to the absence of the modes $\tilde{A}_\mu(t, 0)$, the $s$-channel process mediated by a single virtual photon does not occur. However, we recall that Lorentz invariance, in particular, the symmetry related to the Lorentz boosts, is explicitly violated on the space $\mathbb{R} \times T^3$. Thus, once we consider the collision say, of an incident charged particle with three-momentum $(p + \frac{2\pi}{L})e_x$, where $p \in \frac{2\pi}{L}\mathbb{Z}$ and $e_x$ is a unit three-vector along $x$-direction, and an anti-particle with momentum $(-p)e_x$, the $s$-channel process occurs. In the limit $L \to \infty$, the cross section will approach that in the center of mass frame in infinite volume.

§3. Finite size scaling in meson mass

Now we apply QED$_{L}$ to the study of finite size effect on the EM correction to the pseudoscalar meson masses in the chiral perturbation theory including the electromagnetism. Looking at the practical application to the lattice simulation, we adopt the partially quenched chiral perturbation theory including the electromagnetism and compute the leading-order finite size correction to the EM splitting in this theory. For that purpose, we begin with summarizing our notations for partially quenched chiral perturbation theory including electromagnetism to the next-to-leading order. We derive the formulas for the next-to-leading order correction to the pseudoscalar meson mass in finite volume, and evaluate them numerically to investigate relevance of finite size correction to the EM splitting.

3.1. Partially quenched chiral perturbation theory with electromagnetism

To carry out the computation in finite volume in §3.2, we here fix the notations, in particular, of the low-energy constants at the next-to-leading order, and dictate the free meson propagators necessary for the one-loop calculation. To take the application to the partially quenched system with two-flavors into account, the super-trace of the EM charge matrix $\bar{Q}$ is not assumed here to vanish, unlike in Ref. 32). There then appear more local terms at the next-to-leading order (i.e., $O(p^4)$, $O(e^2p^2)$ and $O(e^4)$) than those found in Ref. 32). We list up all of them by drawing upon Appendix of Ref. 30) where one-loop UV divergences were computed.
for generic flavor number in the unquenched chiral perturbation theory including electromagnetism. After that, we compute the UV divergences that should be absorbed by the coefficients of these local terms.

In what follows, all of the fields and parameters are written in the “flavor” basis

$$\mathbf{Q} \equiv (q_1^V, \cdots, q_{N_V}^V, q_1^S, \cdots, q_{N_S}^S, g_1, \cdots, g_{N_V})^T, \quad (3.1)$$

where $q^S_r$ ($r = 1, \cdots, N_S$) denote the sea quark fields, $q^V_\alpha$ ($\alpha = 1, \cdots, N_V$) the valence quark fields, and $g_\alpha$ ($\alpha = 1, \cdots, N_V$) the ghost quark fields.\(^\text{34}\) The chiral symmetry is a graded Lie group $G = SU(N_S + N_V|N_V)_L \times SU(N_S + N_V|N_V)_R$. It breaks down spontaneously to its vector-like subgroup $H = SU(N_S + N_V|N_V)_V$. The associated Nambu-Goldstone bosons are represented by an $(N_S + N_V|N_V) \times (N_S + N_V|N_V)$ supermatrix $\Pi$. Using

$$u[\Pi(x)] = \exp \left( i \frac{\Pi(x)}{\sqrt{2} F_0} \right), \quad (3.2)$$

$\Pi$ transforms nonlinearly under $(g_L, g_R) \in G$ through

$$u[\Pi] \mapsto u[\Pi'] = g_R u[\Pi] h((g_L, g_R); \Pi)^\dagger = h((g_L, g_R); \Pi) u[\Pi] g_L^\dagger, \quad (3.3)$$

where $h((g_L, g_R); \Pi) \in H$. We follow the convention for the chiral Lagrangian which can be read off from Ref. 32) with minor modification.

The external fields $R_\mu(x), L_\mu(x)$ that couple to the right-handed and left-handed chiral components of vector currents are incorporated in the partially quenched QCD action for the purpose of calculating the connected Green functions of vector and axial-vector currents. They are defined to transform under local $(g_L, g_R)$ as

$$L_\mu \mapsto L'_\mu = g_L L_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger, \quad R_\mu \mapsto R'_\mu = g_R R_\mu g_R^\dagger + i g_R \partial_\mu g_R^\dagger. \quad (3.4)$$

The field strengths

$$L_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu - i[L_\mu, L_\nu], \quad R_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu - i[R_\mu, R_\nu], \quad (3.5)$$

hence transform covariantly. The symmetry breaking parameters are promoted to the spurnion fields with the appropriate transformation laws. For instance, let $\mathcal{M}(x)$ be the spurnion field corresponding to the quark mass matrix $M$ and generalize the mass term in the partially quenched QCD to the form

$$- \overline{\mathbf{Q}}_R \mathcal{M} \mathbf{Q}_L - \overline{\mathbf{Q}}_L \mathcal{M}^\dagger \mathbf{Q}_R. \quad (3.6)$$

The spurnion field is assumed to transform under the local $(g_L, g_R) \in G$ as

$$\mathcal{M} \mapsto \mathcal{M}' = g_R \mathcal{M} g_L^\dagger. \quad (3.7)$$

We write up the low-energy effective Lagrangian in terms of Nambu-Goldstone boson fields whose generating functional of connected Green functions exhibits the same
transformation as that of the microscopic theory.\textsuperscript{39,40} The parameters are then inserted at the positions compatible with the way how the chiral symmetry is broken by them in the Feynman diagrams given by the low energy effective theory. In our context the $U(1)_{em}$-charge matrix $Q$ must also be promoted to a pair of spurion fields $Q_L, Q_R$ which transform respectively as

$$Q_L \mapsto Q'_L = g_L Q_L g_L^\dagger, \quad Q_R \mapsto Q'_R = g_R Q_R g_R^\dagger.$$  

The trace of any integral multiple of $Q_L (Q_R)$ is chirally invariant. It is thus possible to impose the chirally invariant condition

$$\text{str} (Q_R) = \text{str} (Q_L).$$  

After writing up the Lagrangian to the order of our interest, $M$ is set to the diagonal quark mass matrix

$$M_d = \text{diag} (m_1^V, \ldots, m_{N_V}^V, m_1^S, \ldots, m_{N_S}^S, m_1^V, \ldots, m_{N_V}^V),$$

and $Q_L, Q_R$ are set to the diagonal EM charge matrix

$$Q = \text{diag} (Q_1^V, \ldots, Q_{N_V}^V, Q_1^S, \ldots, Q_{N_S}^S, Q_1^V, \ldots, Q_{N_V}^V).$$

The substitution

$$L_\mu \mapsto L_\mu + eQA_\mu, \quad R_\mu \mapsto R_\mu + eQA_\mu.$$  

introduces the coupling of photons to the meson fields.

In practice, in order to write down chirally invariant operators, it is convenient to use the building blocks $O$ which transform as $O \mapsto h((g_L, g_R); II)O h((g_L, g_R); II)^\dagger$. A set of the building blocks $O$, each of which is also the eigenstates of charge conjugation and intrinsic parity transformation, is

\begin{align*}
u_{\mu} &\equiv i \left\{ u^\dagger (\partial_{\mu}u - iR_{\mu}u) - u \left( \partial_{\mu}u^\dagger - iL_{\mu}u^\dagger \right) \right\}, \\
\chi_{\pm} &\equiv u^\dagger \chi u^\dagger \pm u\chi^\dagger u, \quad \chi \equiv 2B_0 M, \\
\tilde{Q}_L &\equiv uQ_L u^\dagger, \quad \tilde{Q}_R \equiv u^\dagger Q_R u, \\
\tilde{S}_{\pm} &\equiv u L_{\mu\nu} u^\dagger \pm u^\dagger R_{\mu\nu} u,
\end{align*}

where $B_0$ is the mass scale characterizing the size of the chiral condensate\textsuperscript{40} and their covariant derivatives with respect to the Maurer-Cartan form

$$\nabla_\mu O \equiv \partial_\mu O - i[I_\mu, O],$$

$$I_\mu \equiv -\frac{1}{2} \left\{ u^\dagger (\partial_{\mu}u - iR_{\mu}u) + u \left( \partial_{\mu}u^\dagger - iL_{\mu}u^\dagger \right) \right\}.  \quad (3.14)$$

Using the quantities described above, the chiral Lagrangian at the leading-order, $O(p^2) \sim O(e^2)$, takes the similar form as in the unquenched case

$$\mathcal{L}_2 = \frac{F_0^2}{4} \text{str} (u_\mu u^\mu + \chi_+)$$

$$+ e^2 C \text{str} \left( \tilde{Q}_L \tilde{Q}_R \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_{\mu}A^\mu)^2.$$  

\( (3.15) \)
Using such a normalization of the decay constant $F_0$ that $F_0 \simeq 90$ MeV, $\Pi$ contains $\pi^+, \pi^3/\sqrt{2}$ in its matrix element. The coefficient $C$ parametrizes the EM correction induced from the short distance dynamics less than the length scale $\sim 1/\mu$ above which the present effective description is valid. Since we adopt QED$_L$ constructed in §2 in case space is finite, the gauge potential $A_\mu$ appearing in Eq. (3.15) does not possess the components $\tilde{A}_\mu(t, 0)$.

The (bare) intrinsic parity even Lagrangian density at the next-to-leading order consists of $\mathcal{O}(p^4)$-, $\mathcal{O}(e^2 p^2)$- and $\mathcal{O}(e^4)$-terms

$$\mathcal{L}_4 = \sum_{j=0}^{12} L_j \mathcal{X}_j + \sum_{j=1}^{25} e^2 F_0^2 K_j \mathcal{Y}_j$$

(3.16)

Here, $\mathcal{X}_j$ ($j = 0, 1, \ldots, 12$) are the terms of $\mathcal{O}(p^4)$

$$\begin{align*}
\mathcal{X}_0 &\equiv \text{str} (u_\mu u_\nu u^\mu u^\nu), \\
\mathcal{X}_1 &\equiv (\text{str} (u_\mu u^\mu))^2, \\
\mathcal{X}_2 &\equiv \text{str} (u_\mu u_\nu) \text{str} (u^\mu u^\nu), \\
\mathcal{X}_3 &\equiv \text{str} \left( (u_\mu u^\mu)^2 \right), \\
\mathcal{X}_4 &\equiv \text{str} (u_\mu u^\mu) \text{str} (\chi^+), \\
\mathcal{X}_5 &\equiv \text{str} (u_\mu u^\mu \chi^+), \\
\mathcal{X}_6 &\equiv (\text{str} (\chi^+))^2, \\
\mathcal{X}_7 &\equiv (\text{str} (\chi^-))^2, \\
\mathcal{X}_8 &\equiv \frac{1}{2} \text{str} \left( \chi^+ \chi^2 + \chi^- \chi^2 \right), \\
\mathcal{X}_9 &\equiv -\frac{i}{2} \text{str} \left( [u^\mu, u^\nu] \tilde{\mathcal{F}}_{\mu\nu} \right), \\
\mathcal{X}_{10} &\equiv \frac{1}{4} \text{str} \left( \tilde{\mathcal{F}}_{\mu\nu} \tilde{\mathcal{F}}^\mu_{\sigma} \tilde{\mathcal{F}}^\nu_{\sigma} - \tilde{\mathcal{F}}_{\mu\nu} \tilde{\mathcal{F}}_{\sigma\tau} \tilde{\mathcal{F}}^{\mu\nu}_{\sigma\tau} \right), \\
\mathcal{X}_{11} &\equiv \text{str} \left( R_{\mu\nu} R^{\mu\nu} + L_{\mu\nu} L^{\mu\nu} \right), \\
\mathcal{X}_{12} &\equiv \frac{1}{4} \text{str} \left( \chi^+ \chi^2 - \chi^- \chi^2 \right) = \text{str} \left( \chi \chi^+ \right).
\end{align*}$$

(3.17)

$L_{11,12}$ are known as $H_1 = L_{11}$ $H_2 = L_{12}.^{40}$ Under the condition (3.9) and the leading-order equations of motion, $\mathcal{Y}_j$ ($j = 1, \ldots, 25$) describe all possible EM corrections at $O(e^2 p^2)$ or $O(e^4)$ in terms of Nambu-Goldstone boson fields

$$\begin{align*}
\mathcal{Y}_1 &\equiv \frac{1}{2} \text{str} \left( \left( \tilde{Q}_L \right)^2 + \left( \tilde{Q}_R \right)^2 \right) \text{str} (u_\mu u^\mu), \\
\mathcal{Y}_2 &\equiv \text{str} \left( \tilde{Q}_L \tilde{Q}_R \right) \text{str} (u_\mu u^\mu), \\
\mathcal{Y}_3 &\equiv -\text{str} \left( \tilde{Q}_R u_\mu \right) \text{str} \left( \tilde{Q}_R u^\mu \right) - \text{str} \left( \tilde{Q}_L u_\mu \right) \text{str} \left( \tilde{Q}_L u^\mu \right), \\
\mathcal{Y}_4 &\equiv \text{str} \left( \tilde{Q}_R u_\mu \right) \text{str} \left( \tilde{Q}_L u^\mu \right),
\end{align*}$$
\[ \mathcal{Y}_5 = \text{str} \left[ \left\{ \left( \mathcal{Q}_L \right)^2 + \left( \mathcal{Q}_R \right)^2 \right\} u_\mu u^\mu \right], \]

\[ \mathcal{Y}_6 = \text{str} \left( \left( \mathcal{Q}_R \mathcal{Q}_L + \mathcal{Q}_L \mathcal{Q}_R \right) u_\mu u^\mu \right), \]

\[ \mathcal{Y}_7 = \frac{1}{2} \text{str} \left( \left( \mathcal{Q}_L \right)^2 + \left( \mathcal{Q}_R \right)^2 \right) \text{str} (\chi_+), \]

\[ \mathcal{Y}_8 = \text{str} \left( \mathcal{Q}_L \mathcal{Q}_R \right) \text{str} (\chi_+), \]

\[ \mathcal{Y}_9 = \text{str} \left[ \left\{ \left( \mathcal{Q}_L \right)^2 + \left( \mathcal{Q}_R \right)^2 \right\} \chi_+ \right], \]

\[ \mathcal{Y}_{10} = \text{str} \left( \left( \mathcal{Q}_R \mathcal{Q}_L + \mathcal{Q}_L \mathcal{Q}_R \right) \chi_+ \right), \]

\[ \mathcal{Y}_{11} = \text{str} \left( \left( \mathcal{Q}_R \mathcal{Q}_L - \mathcal{Q}_L \mathcal{Q}_R \right) \chi_- \right), \]

\[ \mathcal{Y}_{12} = i \text{str} \left( \left[ \nabla_\mu \mathcal{Q}_R, \mathcal{Q}_R \right] u^\mu - \left[ \nabla_\mu \mathcal{Q}_L, \mathcal{Q}_L \right] u^\mu \right), \]

\[ \mathcal{Y}_{13} = \text{str} \left( \nabla_\mu \mathcal{Q}_R \nabla^\mu \mathcal{Q}_L \right), \]

\[ \mathcal{Y}_{14} = \text{str} \left( \nabla_\mu \mathcal{Q}_R \nabla^\mu \mathcal{Q}_R + \nabla_\mu \mathcal{Q}_L \nabla^\mu \mathcal{Q}_L \right), \]

\[ \mathcal{Y}_{15} = e^2 F_0^2 \left( \text{str} \left( \mathcal{Q}_R \mathcal{Q}_L \right) \right)^2, \]

\[ \mathcal{Y}_{16} = e^2 F_0^2 \text{str} \left( \mathcal{Q}_R \mathcal{Q}_L \right) \text{str} \left( \left( \mathcal{Q}_R \right)^2 + \left( \mathcal{Q}_L \right)^2 \right), \]

\[ \mathcal{Y}_{17} = e^2 F_0^2 \left( \text{str} \left( \left( \mathcal{Q}_R \right)^2 + \left( \mathcal{Q}_L \right)^2 \right) \right)^2, \]

\[ \mathcal{Y}_{18} = \text{str} \left( \mathcal{Q}_R u_\mu \mathcal{Q}_R u^\mu + \mathcal{Q}_L u_\mu \mathcal{Q}_L u^\mu \right), \]

\[ \mathcal{Y}_{19} = \text{str} \left( \mathcal{Q}_R u_\mu \mathcal{Q}_L u^\mu \right), \]

\[ \mathcal{Y}_{20} = e^2 F_0^2 \text{str} \left( \left( \mathcal{Q}_R \right)^2 \left( \mathcal{Q}_L \right)^2 \right), \]

\[ \mathcal{Y}_{21} = e^2 F_0^2 \text{str} \left( \mathcal{Q}_R \mathcal{Q}_L \mathcal{Q}_R \mathcal{Q}_L \right), \]

\[ \mathcal{Y}_{22} = e^2 F_0^2 \left\{ \text{str} \left( \left( \mathcal{Q}_R \right)^2 - \left( \mathcal{Q}_L \right)^2 \right) \right\}^2, \]

\[ \mathcal{Y}_{23} = e^2 F_0^2 \left\{ \text{str} \left( \mathcal{Q}_R \right) \text{str} \left( \mathcal{Q}_R \left( \mathcal{Q}_L \right)^2 \right) + \text{str} \left( \mathcal{Q}_L \right) \text{str} \left( \mathcal{Q}_L \left( \mathcal{Q}_R \right)^2 \right) \right\}, \]

\[ \mathcal{Y}_{24} = \text{str} \left( \mathcal{Q}_R \right) \text{str} \left( \mathcal{Q}_L u_\mu u^\mu \right) + \text{str} \left( \mathcal{Q}_L \right) \text{str} \left( \mathcal{Q}_R u_\mu u^\mu \right), \]

\[ \mathcal{Y}_{25} = \text{str} \left( \mathcal{Q}_R \right) \text{str} \left( \mathcal{Q}_L \chi_+ \right) + \text{str} \left( \mathcal{Q}_L \right) \text{str} \left( \mathcal{Q}_R \chi_+ \right). \]
The values of $l_j$ ($j = 1, \cdots, 12$) are available in Ref. 41) for the generic number of flavors. The values of $k_j$ ($j = 1, \cdots, 14, 18, 19$) were computed in Ref. 32) for $N_S = 3$. Table I lists the values of $k_j$ computed using the heat kernel method and evaluating the fermionic meson contribution explicitly for general $N_S$ without setting the quantities in Eq. (3.9) to zero. The computation is performed only for Feynman gauge $\lambda = 1$. The dimensionless quantity $Z$ in Table I is defined by

$$Z \equiv \frac{C}{F_0^3}.$$ (3.21)

In the momentum space, the Feynman rule in the partially quenched chiral perturbation theory in finite volume is the same as that in infinite volume. In particular, the form of propagators that allows us to carry out the computation efficiently is obtained by introducing the super-traceless component called super-$\eta'$, deriving the propagators in the matrix element basis $\Phi^I_J$ ($I, J = 1, \cdots, (N_S + 2N_V)$), and taking the decoupling limit of super-$\eta'$.\(^{36,37}\)

The entry $\chi_{IJ}$ of the matrix of meson mass squared including the leading-order EM correction is given by

$$\chi_{IJ} = \frac{\chi_I + \chi_J}{2} + \frac{2e^2C}{F_0^2} (q_I - q_J)^2,$$ (3.22)

where $\chi_I$ and $q_I$ are the eigenvalues of $\chi_{|M\to M_d}$ and $Q$ in Eq. (3.11), respectively. For instance, $q_j = Q_j^V$ ($1 \leq j \leq N_V$), $q_{r+N_V} = Q_r^S$ ($1 \leq r \leq N_S$). The propagators are shown here in the case that the eigenvalues $\chi_{(r)} \equiv \chi_{N_V+r}$ ($1 \leq r \leq N_S$) corresponding to sea quarks as well as the mass squared $\chi_x$ ($x = 1, \cdots, N_S - 1$)

\(^*\) The normalization of our $k_j$’s differs from those in Ref.32) such that $(k_j)_{ours} = -2(k_j)_{BD}$ for $j = 1, \cdots, 10, 12, 13, 14, 18, 19$, and $(k_{11})_{ours} = 2(k_{11})_{BD}$. 

| $j$ | $k_j$ | $j$ | $k_j$ |
|-----|-------|-----|-------|
| 1   | 0     | 14  | 0     |
| 2   | $Z$   | 15  | $\frac{3}{2} + 8Z^2$ |
| 3   | 0     | 16  | $\frac{3}{2}$ |
| 4   | $2Z$  | 17  | $\frac{3}{2}Z^2$ |
| 5   | $-\frac{3}{4}$ | 18 | $\frac{1}{4}$ |
| 6   | $\frac{N_S}{2}Z$ | 19 | 0 |
| 7   | 0     | 20  | $2NSZ^2 - 3Z$ |
| 8   | $Z$   | 21  | $2NSZ^2 + 3Z$ |
| 9   | $-\frac{1}{8}$ | 22 | $-Z^2$ |
| 10  | $\frac{1}{4} + \frac{N_S}{2}Z$ | 23 | $-8Z^2$ |
| 11  | $-\frac{1}{8}$ | 24 | $-Z$ |
| 12  | $\frac{1}{4}$ | 25 | $-Z$ |
| 13  | 0     | 26  | 0     |

The coefficients $L_j$, $K_k$ absorb the UV divergences that arise from the one-loop correction

$$L_j + l_j \Delta_\epsilon = L_j^R(\mu),$$
$$K_j + k_j \Delta_\epsilon = K_j^R(\mu).$$ (3.19)

Here we employ the dimensional regularization where the spatial dimension is analytically continued to $d$. Denoting the full space-time dimension as $D = d + 1 = 4 - 2\epsilon$, $\Delta_\epsilon$ in Eq. (3.19) is given by

$$\Delta_\epsilon \equiv \frac{1}{32\pi^2} \left\{ \frac{1}{\epsilon} - \ln \left( \frac{\mu^2}{4\pi} \right) - \gamma_E + 1 \right\}.$$ (3.20)
of the diagonal meson eigenstates in the sea-meson sub-sector, referred to as sea mesons, differ from all $\chi_j$ ($j = 1, \ldots, N_V$). The propagators of bosonic mesons (1 $\leq$ $I, J, K, L$ $\leq$ $(N_V + N_S)$ or (1 + $N_V + N_S$) $\leq$ $I, J, K, L$ $\leq$ $(2N_V + N_S)$) are denoted by

$$i G_{I, J; K, L}(p^2) \equiv \int d^4 x e^{ip \cdot x} \langle \Pi_{I, J}(x) \Pi_{K, L}^0(0) \rangle.$$  \hspace{1cm} (3.23)

The off-diagonal meson fields have the usual form of propagators

$$i G_{I, J; K, L}(p^2) = \delta_{I, K} \delta_{J, L} \frac{i}{p^2 - \chi_{IJ}} \left( I \neq J, K \neq L \right) \hspace{1cm} (3.24)$$

The propagators of the fermionic mesons $\Xi_{i, j} \equiv \Pi_{i+N_S+N_V}^{j}$ ($1 \leq i \leq N_V, 1 \leq j \leq (N_S + N_V)$) also have simple forms

$$i S_{I, J; K, L}(p^2) \equiv \int d^4 x e^{ip \cdot x} \left\langle \Xi_{I, J}(x) \Xi_{K, L}^0(0) \right\rangle$$

$$= \delta_{I, K} \delta_{J, L} \frac{i}{p^2 - \chi_{IJ}} \left( 1 \leq i, l \leq N_V, 1 \leq j, k \leq (N_S + N_V) \right).$$ \hspace{1cm} (3.25)

The one-loop calculation needs the propagators of diagonal mesons only for $i, j = 1, \ldots, N_V$;

$$G_{I, J; K, L}(p^2) = -\frac{1}{N_S} \left( \frac{R_{ij}}{p^2 - \chi_i} + \frac{R_{ij}}{p^2 - \chi_j} + \sum_x^{sm} \frac{R_{ij}}{p^2 - \chi_x} \right)$$

for $\chi_i \neq \chi_j$,

$$G_{I, J; K, L}(p^2) = \frac{1}{p^2 - \chi_i}$$

$$- \frac{1}{N_S} \left( \frac{R_{ij}^{(d)}}{(p^2 - \chi_i)^2} + \frac{R_{ij}^{(s)}}{p^2 - \chi_i} + \sum_x^{sm} \frac{R_{ij}^x}{p^2 - \chi_x} \right)$$

for $\chi_i = \chi_j$,

$$\hspace{1cm} (3.26)$$

where $\sum_x^{sm}$ denotes the sum over all sea mesons, and

$$R_{ij}^{i} \equiv \prod_{r=1}^{N_S} \left( \chi_i - \chi_r \right) \prod_x^{sm} \left( \chi_i - \chi_x \right) = R_{ji}^{i},$$

$$R_{ij}^{x} \equiv \prod_{r=1}^{N_S} \left( \chi_x - \chi_r \right) \prod_{y \neq x}^{sm} \left( \chi_x - \chi_y \right).$$
\[ R_i^d \equiv \prod_{r=1}^{N_S} \prod_{x}^{s_{m}} (\chi_i - \chi_r) (\chi_i - \chi_x), \]

\[ R_i^s = \prod_{r=1}^{N_S} \prod_{x}^{s_{m}} (\chi_i - \chi_r) \left( \sum_{s=1}^{N_S} \frac{1}{\chi_i - \chi(s)} - \sum_{x}^{s_{m}} \frac{1}{\chi_i - \chi_x} \right). \quad (3.27) \]

### 3.2. Electromagnetic correction to meson mass in finite volume

We are ready to compute the next-to-leading order correction \( m_{ij}^2|_4(L) \) to the off-diagonal pseudoscalar meson mass squared in finite volume, which can be obtained from the self-energy function \( \Sigma^i_j (p^2)|_L \) within our approximation as

\[ m_{ij}^2|_4(L) = \Sigma^i_j (\chi_{ij})|_L, \quad (3.28) \]

and investigate the relevance of finite size correction to the EM splitting. As recalled in §2.2 Lorentz boost symmetry is violated in finite volume. Here \( m_{ij}^2|_4(L) \) is defined in the rest frame, \( p^\mu = (\sqrt{\chi_{ij}}, 0) \).

Before carrying on the explicit calculation further, we mention the limitation inherent to our approach on the ability to capture the dynamics. While the low energy effective theory enables us to evaluate the effect of finiteness of volume on the virtual quanta with low frequencies, it provides no knowledge about the effect on the short distance dynamics packed in the low energy constants \( L_j(\mu) \) and \( K_j(\mu) \). The finite size scaling effect on the low energy constants \( L_j(\mu) \) carrying the information on QCD less than \( 1/\mu \), is insignificant so as to affect to the properties of Nambu-Goldstone bosons. The situation is different in QED where there is no intrinsic scale \( 1/\mu \) that separates long and short distances.\(^{31}\) For this reason, though the finite size scaling effect on QED is dominated by the loop contribution whose finite part represents the long distance physics, the low energy constants \( K_j(\mu) \) possibly suffer from sub-dominant but non-negligible finite size scaling effect. For the purpose explained in §1, even qualitative features of the finite size scaling effect such as the size and sign are worthwhile to investigate.

With the above remark in mind, we proceed to calculate \( m_{ij}^2|_4(L) \) in finite vol-

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Fig. 1. The Feynman diagrams that contribute to \( m_{ij}^2|_4(L) \). A wavy line and a thin line represent the photon and meson propagation, respectively. The vertex in (b) comes from the term with coefficient \( C \) in Eq. (3.15) while the quartic coupling in (a) comes from the pure QCD part.
volume. For the purpose of (1) getting $m_{ij}^2|_4(L)$ for $N_S = 2, 3$ simultaneously, and (2) demonstrating that QED$_L$ shares the common UV divergent structure as QED in infinite volume, we describe the computation in some detail. As the probes are valence quarks, it suffices to put $1 \leq i \neq j \leq N_F$. Straightforward calculation of four types of Feynman diagrams in Fig. 1 yields

$$m_{ij}^2|_4(L) = \frac{1}{6 F_0^2} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{1}{L^d} \sum_{k \in \Gamma_d} \left[ 2 \left( k^2 + 2 \chi_{ij}^{QCD} \right) G_{i; j}^i (k^2) + (\chi_i - k^2) \left( G_{i; i}^i (k^2) - S_{i; i}^i (k^2) \right) 
+ (\chi_j - k^2) \left( G_{j; j}^j (k^2) - S_{j; j}^j (k^2) \right) \right]$$

$$+ 2 e^2 \sum_{n} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{1}{L^d} \sum_{k \in \Gamma_d} \left\{ q_{ij} q_{in} G_{n; i}^i (k^2) + q_{ij} q_{nj} G_{n; j}^j (k^2) \right\}$$

$$+ \left( q_{ij} \right)^2 e^2 \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{1}{L^d} \sum_{k \in \Gamma_d} \left\{ (D - 1) \frac{1}{-k^2} + 2 \frac{p \cdot k}{-k^2 (\chi_{ij} - (k + p)^2)} + 4 \chi_{ij} \frac{1}{-k^2 (\chi_{ij} - (k + p)^2)} \right\}$$

$$+ m_{C, ij}^2,$$  \hspace{1cm} (3.29)

where $\sum_n S$ represents the sum over the indices $n$ of sea quark flavors only, $\tilde{\Gamma}_d \equiv \tilde{\Gamma}_d - \{0\}$ with $\tilde{\Gamma}_d$ in Eq. (A.2), and

$$\chi_{ij}^{QCD} \equiv \frac{\chi_i + \chi_j}{2},$$

$$q_{ij} \equiv q_i - q_j.$$  \hspace{1cm} (3.30)

$m_{C, ij}^2$ represents the contribution of the next-to-leading order local terms in Eqs. (3.17) and (3.18) to the pseudoscalar meson mass squared

$$m_{C, ij}^2 = m_{C, ij}^{QCD} + m_{C, ij}^{EM},$$  \hspace{1cm} (3.31)

where $m_{C, ij}^{QCD}$ and $m_{C, ij}^{EM}$ are given by QCD low energy constants $L_j$ and EM low energy constants $K_j$, respectively

$$m_{C, ij}^{QCD} = \frac{1}{F_0^2} \left[ - 4 \{ 2 L_4 N_S \chi_S + L_5 (\chi_i + \chi_j) \} \chi_{ij} \right].$$

*) In (3.29), the terms in the first, second and third lines come from the diagram (a) in Fig. 1, the ones in the fourth and fifth lines from (b), the one in the sixth line from (c), and the ones in the seventh and eighth lines from (d).
\[ m_{C,ij}^{EM} = e^2 \left\{ -4N_S \bar{Q}^2 (K_1 + K_2) - 4 \left( q_i^2 + q_j^2 \right) (K_5 + K_6) \\
-4q_iq_j (2K_{18} + K_{19}) - 4N_S \bar{Q} (q_i + q_j) K_{24} \right\} \chi_{ij} \]

\[ +2N_S \bar{Q}^2 (\chi_i + \chi_j) K_7 \]

\[ +2N_S \left\{ \bar{Q}^2 (\chi_i + \chi_j) + 2\bar{Q} S (q_i - q_j)^2 \right\} K_8 \]

\[ +4 \left( q_i^2 \chi_i + q_j^2 \chi_j \right) K_9 \]

\[ +4 \left\{ q_i^2 \chi_i + q_j^2 \chi_j + (q_i - q_j)^2 (\chi_i + \chi_j) \right\} K_{10} \]

\[ -4(q_i - q_j)^2 (\chi_i + \chi_j) K_{11} + 4N_S \bar{Q} (q_i \chi_i + q_j \chi_j) K_{25} \]

\[ +e^4 F_0^2 \left[ 4N_S \bar{Q}^2 (q_i - q_j)^2 (K_{15} + K_{16}) + 2(q_i^2 - q_j^2) K_{20} \right] \]

\[ +4(q_i - q_j)^2 (q_i^2 + q_j^2) K_{21} \]

\[ +4N_S \bar{Q} (q_i - q_j)^2 (q_i + q_j) K_{23} \]. \tag{3.32} \]

In the above,

\[ \bar{Q} \equiv \frac{1}{N_S} \sum_{r=1}^{N_S} q(r) , \]

\[ \bar{Q}^2 \equiv \frac{1}{N_S} \sum_{r=1}^{N_S} q(r)^2 , \tag{3.33} \]

with use of electric sea quark charges \( q(r) = q_{N_V+r} \). We note that the expression in Eq. (3.29) is independent of the value of the gauge fixing parameter \( \lambda \).

Equation (3.29) can be rewritten in terms of four basic functions, \( (A\cdot3) \), \( (A\cdot20) \), \( (A\cdot26) \) and \( (A\cdot30) \), defined in Appendix A. The function \( I_1(m^2; L) \) in Eq. (A\cdot30) emerges in the QCD finite scaling effect. \( ^{23} \) In Appendix A, we derive the expression written in terms of a Jacobi theta function for the other three functions. Applying the identities

\[ (2\chi_i + \chi_j) R_{ij} i - \frac{1}{2} R_{i}^{(d)} \chi_i = \frac{3}{2} (\chi_i + \chi_j) R_{ij} i , \]

\[ (\chi_i + 2\chi_j) R_{ij} j - \frac{1}{2} R_{j}^{(d)} \chi_j = \frac{3}{2} (\chi_i + \chi_j) R_{ij} j , \]

\[ (\chi_x + \chi_i + \chi_j) R_{ij} x + \frac{\chi_i - \chi_x}{2} R_{xi} x + \frac{\chi_j - \chi_x}{2} R_{jj} x = \frac{3}{2} (\chi_i + \chi_j) R_{ij} x , \tag{3.34} \]

to the terms in Eq. (3.29) that are written by \( I_1(m^2; L) \), we get a compact expression

\[ m_{ij}^{2\cdot4}(L) = m_{QCD,ij}^{2\cdot4}(L) + m_{EM,ij}^{2\cdot4}(L) . \tag{3.35} \]

Here \( m_{QCD,ij}^{2\cdot4}(L) \) and \( m_{EM,ij}^{2\cdot4}(L) \) are QCD and EM contributions, respectively

\[ m_{QCD,ij}^{2\cdot4}(L) = \frac{1}{2N_SF_0^2} (\chi_i + \chi_j) \]
This expression is free from UV divergence, confirming that QED results in Appendix A immediately leads

\[ m_{\text{EM},ij}^2(L) = -2e^2 Z \sum_{n}^{S} \{ q_{ij} q_{in} I_1(\chi_{in}; L) + q_{ij} q_{nj} I_1(\chi_{nj}; L) \} \]

\[ + (q_{ij})^2 e^2 \left\{ (D - 1) I_1^0(L) + 2 J_{11}(\chi_{ij}; L) + 4 \chi_{ij} I_{11}(\chi_{ij}; L) \right\} \]

\[ + m_{c,ij}^2 |_{\text{EM}} \]  

From Eq. (3.36), the finite size scaling correction is obtained as

\[ \Delta m_{ij}^2(L) = m_{ij}^2 |_{4}(L) - m_{ij}^2 |_{4}(\infty), \]  

using the quantity \( m_{ij}^2 |_{4}(\infty) \) evaluated directly in infinite volume. The use of the results in Appendix A immediately leads

\[ \Delta m_{ij}^2(L) = \Delta m_{\text{QCD},ij}^2(L) + \Delta m_{\text{EM},ij}^2(L), \]  

where \( \Delta m_{\text{QCD},ij}^2(L) \) and \( \Delta m_{\text{EM},ij}^2(L) \) represent the finite size scaling corrections on QCD and QED, respectively.

\[ \Delta m_{\text{QCD},ij}^2(L) = \frac{1}{(4\pi)^2} \frac{\chi_i + \chi_j}{2N_s F_0^2} \frac{1}{L^2} \left\{ R_{ij}^i \mathcal{M}(\sqrt{\chi_i} L) + R_{ij}^j \mathcal{M}(\sqrt{\chi_j} L) \right\} \]

\[ + \sum_{x}^{s_m} R_{ij}^x \mathcal{M}(\sqrt{\chi_x} L) \right\}, \]

\[ \Delta m_{\text{EM},ij}^2(L) = -\frac{2e^2 Z}{(4\pi)^2} \frac{1}{L^2} \sum_{n}^{S} \{ q_{ij} q_{in} \mathcal{M}(\sqrt{\chi_{in}} L) + q_{ij} q_{nj} \mathcal{M}(\sqrt{\chi_{nj}} L) \} \]

\[ -3 \frac{(q_{ij})^2 e^2}{4\pi} \frac{\kappa}{L^2} \]

\[ + \frac{(q_{ij})^2 e^2}{(4\pi)^2} \left\{ \frac{\mathcal{K}(\sqrt{\chi_{ij}} L)}{L^2} - 4 \sqrt{\chi_{ij}} \frac{\mathcal{H}(\sqrt{\chi_{ij}} L)}{L} \right\}. \]  

This expression is free from UV divergence, confirming that QED\(_L\) shares the common UV divergent structure as QED. Noting Eq. (3.38) and

\[ m_{ij}^2 |_{4}(L) = m_{ij}^2 |_{4}(\infty) + \Delta m_{\text{QCD},ij}^2(L), \]

\[ m_{ij}^2 |_{4}(L) = m_{ij}^2 |_{4}(\infty) + \Delta m_{\text{EM},ij}^2(L), \]  

\( m_{ij}^2 |_{4}(L) \) in Eq. (3.35) hence becomes finite by carrying out renormalization for the QCD part \( m_{\text{QCD},ij}^2 |_{4}(\infty) \) and the part including EM correction \( m_{\text{EM},ij}^2 |_{4}(\infty) \)

\[ m_{\text{QCD},ij}^2 |_{4}(\infty) = m_{\text{QCD},ij}^2 |_{\text{loop}} + m_{c,ij}^2 |_{\text{QCD}}, \]

\[ m_{\text{EM},ij}^2 |_{4}(\infty) = m_{\text{EM},ij}^2 |_{\text{loop}} + m_{c,ij}^2 |_{\text{EM}}, \]  

(3.41)
where the terms \( m_{QCD,ij}^2(\infty) \)\(_{\text{loop}}\) and \( m_{EM,ij}^2(\infty) \)\(_{\text{loop}}\) are those involving the chiral logarithms

\[
\begin{align*}
m_{QCD,ij}^2(\infty) |_{\text{loop}} &= \frac{1}{(4\pi)^2} \frac{X_i + X_j}{2N SF_0^2} \left\{ R_{ij}^i X_i \ln \left( \frac{X_i}{\mu^2} \right) + R_{ij}^j X_j \ln \left( \frac{X_j}{\mu^2} \right) \right. \\
&\quad + \left. \sum_x R_{ij}^x X_x \ln \left( \frac{X_x}{\mu^2} \right) \right\}, \\
\end{align*}
\]

\[
\begin{align*}
m_{EM,ij}^2(\infty) |_{\text{loop}} &= -\frac{2e^2 Z}{(4\pi)^2} \sum_n^S \left\{ q_{ij} q_{in} \chi_{in} \ln \left( \frac{\chi_{in}}{\mu^2} \right) + q_{ij} q_{nj} \chi_{nj} \ln \left( \frac{\chi_{nj}}{\mu^2} \right) \right\} \\
&\quad - \frac{(g_{ij})^2}{(4\pi)^2} \chi_{ij} \left\{ 3 \ln \left( \frac{\chi_{ij}}{\mu^2} \right) - 4 \right\},
\end{align*}
\]

while the terms \( m_{C,ij}^2 |_{QCD} \) and \( m_{C,ij}^2 |_{EM} \) are given by the expression in Eq. (3.32) but now written in terms of renormalized low energy constants. For the sake of simplifying expressions, the superscript “\( R \)” is omitted from every renormalized parameter that appears in what follows. Now, all the quantities on the right-hand sides of Eq. (3.40) are UV-finite. In terms of these quantities, the full next-to-leading order correction \( m_{ij}^2 |_{4}(L) \) to meson squared in finite volume is given by Eq. (3.35).

The formulas derived thus far applies both to \( N_S = 3 \) and \( N_S = 2 \) if the sum over sea flavors and the one over sea mesons are appropriately understood. The expressions for \( m_{QCD,ij}^2 |_{4}(\infty) \) and \( m_{EM,ij}^2 |_{4}(\infty) \) will reduce to the result found in Ref. 32) if one sets \( N_S = 3 \) and \( \overline{Q} = 0 \) and discards the terms of order \( e^4 \). A close examination shows that pseudoscalar meson mass depends on 11 combinations of EM low energy constants \( K_j \). If QED is quenched, only 6 is relevant. It is difficult to determine all of the values of \( K_j \)'s. However, it is possible to fit the simulation data with \( Z \) and these combinations of parameters, and to determine the value of \( m_u - m_d \).

Before turning to the numerical analysis, we observe a few features that can be read off from Eq. (3.39). First, the asymptotic behavior for \( L/\sqrt{X_{ij}} \gg 1 \) is determined by the term including the function \( H(x) \). This term behaves like \( 1/L \) times the function \( H(\sqrt{X_{ij}} L) \). As can be seen from Fig. 2, \( H(x) \) increases in the region \( x \leq 10 \) which is important for sizes of available lattices. The presence of this term possibly causes the finite size scaling effect to decrease slower than \( 1/L \).

Secondly, the terms in Eq. (3.39) all vanish for \( \sqrt{X_{ij}} \) except for the one proportional to the constant \( \kappa \) defined in Eq. (A.29).\(^*\) This term seems to appear whatever is used as the low effective field theories, because the contribution leading to this term originates from the diagrams in scalar QED theory. For instance, that contribution is also involved in Eq. (1.1) which is derived from the model including the vector and axial-vector mesons. The presence of this term indicates that \( 1/L \) should be regarded as being the same order as the pseudo-Nambu-Goldstone mass

\(^*\) Here we use the fact that \( \frac{N(x)}{x} \) is finite for all \( x \). This point, however, has been checked only through our numerical investigation which gives Eq. (A.19).
and the elementary charge $e$ in order for the chiral perturbation to remain systematic. In fact, the computation done thus far implicitly assumes the $p$-regime for the relative magnitude of pseudo-Nambu-Goldstone mass and $L$, i.e., $m_\pi \sim \frac{1}{L} \sim p$ for pure QCD correction. Our result suggests that there is a $p$-regime for the systematic chiral perturbation theory including electromagnetism in finite volume, $m_\pi \sim \frac{1}{L} \sim p \sim e$. The issue examining whether this is actually true at higher orders is beyond the scope of this paper.

### 3.3. Numerical investigation

We turn to the numerical evaluation of the EM correction in the next-to-leading order approximation

$$m_{EM,ij}^2(L) = \frac{2e^2C}{F_0^2}(q_i - q_j)^2 + m_{EM,ij}^2\big|_4(L),$$

(3.43)

to study the qualitative features of the QED finite size scaling effect. For that purpose the values of various low energy constants found in Ref.42) are used as reference

\begin{align*}
F_0 &= 87.7 \text{ MeV}, \quad C = 4.2 \cdot 10^{-5} \text{ (GeV)}^4, \\
L_4 &= 0, \quad L_5 = 0.97 \cdot 10^{-3}, \quad L_6 = 0, \quad L_8 = 0.60 \cdot 10^{-3}, \\
K_5 &= 2.85 \cdot 10^{-3}, \quad K_9 = 1.3 \cdot 10^{-3}, \quad K_{10} = 4.0 \cdot 10^{-3}, \\
K_{11} &= -1.25 \cdot 10^{-3}. \quad (3.44)
\end{align*}

The others are set to zero. We employ the formula for $N_S = 3$ and $N_V = 3$ and set

\begin{align*}
\chi_3 = \chi_6 = \chi_9 &= (500 \text{ MeV})^2, \\
q_1 = q_4 = q_7 &= \frac{2}{3}, \quad q_2 = q_3 = q_5 = q_6 = q_8 = q_9 = -\frac{1}{3}. \quad (3.45)
\end{align*}
Fig. 3. Linear volume size \( (L) \) dependence of the electromagnetic correction in the charged pion mass squared throughout the analysis. In what follows, it is always understood that the value of the mass of a ghost quark is set equal to that of the valence quark with the same flavor.

We plot the dependence of \( m_{EM,12}^2(L)/m_{EM,12}^2(\infty) \) on \( L \) in Fig. 3 for the values of quark masses corresponding to \( \chi_1 = \chi_2 = \chi_4 = \chi_5 = (150 \text{ MeV})^2 \) and \( (300 \text{ MeV})^2 \). The horizontal axis denotes the linear size of volume, \( L \), normalized in unit of \( 1/M_\rho \approx 1/(770 \text{ MeV}) \). For instance, for the size \( L = 16a \approx 0.72 \times 1/M_\rho \) used in Ref. 12), \( m_{EM,12}^2(L)/m_{EM,12}^2(\infty) \) are 0.144 and 0.421 for \( \chi_1 = (150 \text{ MeV})^2 \) and \( \chi_1 = (300 \text{ MeV})^2 \), respectively. Thus, our calculation indicates that the finite size effect is significant to available lattice geometries. The difference between two quark masses emerges for small \( L \), and the finite size effect is smaller for larger quark mass.

We next study the EM splitting \( \Delta m_K^2(L) \) in the (valence) kaon mass squared. Using \( m_{EM,i,j}^2(L) \) in Eq. (3.43), it is given by

\[
\Delta m_K^2(L) \equiv m_{EM,13}^2(L) - m_{EM,23}^2(L).
\]

Figure 4 shows the \( L \)-dependence of \( \Delta m_K^2(L)/\Delta m_K^2(\infty) \) for the two sets of quark masses corresponding to the same values of \( \chi_1 = \chi_2 = \chi_4 = \chi_5 \) used in Fig. 3 with \( \chi_3 = (500 \text{ MeV})^2 \) fixed. The \( L \)-dependence in Fig. 4 is almost similar to that in Fig. 3 for each set of masses. For small \( L \) and \( \chi_1 = (150 \text{ MeV})^2 \), the size of finite size scaling effect is smaller than the electromagnetic correction to the charged pion mass. For instance, for \( L = 16a \), \( \Delta m_K^2(L)/\Delta m_K^2(\infty) \) is 0.299 for \( \chi_1 = (150 \text{ MeV})^2 \) and 0.415 for \( \chi_1 = (500 \text{ MeV})^2 \). Figures 3 and 4 show that the values of EM correction depends on the quark masses for \( L = 16a \). This observation indicates that the sizes...
of the terms depending on the quark masses in Eq. (3.39) are comparable to or more important than that of the term proportional to $\kappa$ for the quark masses used in the present analysis.

Figure 5 compares the $L$-dependence of the electromagnetic splitting of $m^2_K(L)$ in partially quenched QCD with that in (unquenched) QCD. The two sets of plots are drawn in Fig. 5 for

$$\text{set 1} \Leftrightarrow \chi_1 = \chi_2 = (150 \text{ MeV})^2, \quad \chi_4 = \chi_5 = (300 \text{ MeV})^2,$$
$$\text{set 2} \Leftrightarrow \chi_1 = \chi_2 = (300 \text{ MeV})^2, \quad \chi_4 = \chi_5 = (150 \text{ MeV})^2. \quad (3.47)$$

As can be seen, these two sets of plots in partially quenched QCD coincide with each other. By evaluating the $L$-dependence for various sets of the values of quark masses, we find that the $L$-dependence for a set of the values of quark masses is almost the same as that for the set with the valence and sea quark masses interchanged. This observation together with the comparison of the relative sizes of four sets of plots in Fig. 5 shows that the size of the finite size correction is roughly determined by the average value of the quark masses involved irrespective of whether the system is unquenched or partially-quenched. To elucidate this point, we show in Figs. 6 and 7 the quark mass dependence of the finite size scaling effect on $m^2_K(L)$ on $m^2 \equiv \chi_1 = \chi_2 = 2B_0m_u$ for fixed $L = 16a$ and $L = 32a$ ($a \simeq 1/(1.66 \text{ GeV})^{12}$), respectively. In the partially quenched case, the masses of up and down sea quarks are fixed to be 300 MeV. We can see that, at $L = 16a$, the size of the finite size effect changes rapidly for $m^2 \lesssim 0.05 \text{ (GeV)}^2$ in the unquenched case, while no such significant change is observed in the partially quenched case. For larger $m^2$, they
Fig. 5. Finite size scaling effect on the EM splitting ($\Delta m_k^2$) in kaon mass squared in partially quenched QCD, where open squares stand for the $L$-dependence for $\chi_1 = \chi_2 = 150$ MeV, $\chi_4 = \chi_5 = 300$ MeV, and dark squares with cross marks stand for the one for $\chi_1 = \chi_2 = 300$ MeV, $\chi_4 = \chi_5 = 150$ MeV. No significant change is observed between them. They differ from the other plots corresponding to unquenched QCD in the small volume region.

For $L = 32a$ and larger $L$, the finite size effect is determined by the relative size of $L$ and $\chi_3 = \chi_6 = (500 \text{ MeV})^2$.

§4. Conclusion

In this paper, we presented the first study of the finite size scaling effect on the electromagnetic (EM) correction to the pseudoscalar meson masses from the low energy effective field theory of QCD including electromagnetism. For that purpose, we began with constructing a new QED in finite volume, QED$_L$. The application of QED$_L$ both to lattice QCD simulation with electromagnetism and to the low energy effective theory allows us to extract the prediction of EM corrections in infinite volume and to determine 11 (or 6 for quenched QED) linear combinations of low energy constants that enter in pseudoscalar meson masses.

Taking the practical application to the lattice simulation into account, we adapted QED$_L$ to the partially quenched chiral perturbation theory including electromagnetism. We computed the electromagnetic correction to the pseudoscalar meson mass squared at the next-to-leading order on both of the spaces with finite and infinite volumes for generic number $N_S$ of sea quarks. Through numerical investigation for $N_S = 3$, we found that the finite size scaling effect on the EM correction is sizable on the space with the volume available in the lattice simulation. By investigating
its dependence on the quark masses in unquenched and partially quenched systems, we pointed out that the finite size correction is determined by the averaged values of masses of quarks involved in the system. Though the current study was restricted to the pseudoscalar meson masses, we can study the EM corrections to the other hadronic observables such as decay constants in finite volume.

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Appendix A

—— Formulae for Several Sums ——

The one-loop corrections to meson masses in chiral perturbation theory for QCD plus QED system with finite volume are described in terms of several basic functions. Each of these functions takes the form of the one-dimensional integral of the sum over three-dimensional momenta of some function. We employ dimensional regularization
Fig. 7. Dependence of the finite size scaling effect on the electromagnetic splitting ($\Delta m^2_K$) in kaon mass squared on $m^2 = \chi_1$ at the tree level for $L = 32a$. 

and define

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi i} \frac{1}{V} \sum_{k \in \tilde{\Gamma}_d} = 0. \quad (A.1)$$

Here, $d \equiv D - 1$, $V \equiv L^d$ and the sum runs over all $k \in \tilde{\Gamma}_d$, where

$$\tilde{\Gamma}_d \equiv \left\{ k = (k^1, \ldots, k^d) \mid k^j \in \frac{2\pi}{L} \mathbb{Z} \right\}. \quad (A.2)$$

The aim of this Appendix is to get compact expressions for the four functions that can be evaluated by MATHEMATICA and so forth by following the strategy in Ref. 23).

We first consider the function

$$I_{11}(m^2; L) \equiv (\mu^2)^{2 - \frac{D}{2}} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi i} \frac{1}{V} \sum_{k \in \tilde{\Gamma}_d} \left( \frac{1}{-k^2 - i\epsilon} \right) \left\{ \frac{1}{m^2 - (k + p)^2 - i\epsilon} \right\}. \quad (A.3)$$

In the above, $p$ is assumed to be on-shell, $p^2 = m^2$ and $\tilde{\Gamma}_d' \equiv \tilde{\Gamma}_d - \{0\}$.

By introducing the Feynman parameters as usual, Eq. (A.3) becomes

$$I_{11}(m^2; L) \equiv \int_0^1 dy \int_{-\infty}^{\infty} \frac{dk^0}{2\pi i} \frac{1}{V} \sum_{k \in \tilde{\Gamma}_d'} \left\{ \frac{1}{y^2 m^2 - (k + yp)^2 - i\epsilon} \right\}. \quad (A.4)$$
The sum of a function \( \tilde{F}(k) \) over \( k \in \tilde{\Gamma}'_d \) can be written as an integral

\[
\frac{1}{V} \sum_{k \in \tilde{\Gamma}'_d} \tilde{F}(k) = \int d^d k' \tilde{F}(k') \frac{1}{V} \sum_{k \in \Gamma_d} \delta^d(k' - k) - \frac{1}{V} \tilde{F}(0). \tag{A.5}
\]

Using the Poisson resummation

\[
\frac{1}{V} \sum_{k \in \Gamma_d} \delta^d(k' - k) = \frac{1}{(2\pi)^d} \sum_{x \in \Gamma_d} e^{ix \cdot k'},
\]

where

\[
\Gamma_d \equiv \left\{ x \equiv (x^1, \ldots, x^d) \mid x^j \in \mathbb{LZ} \right\}, \tag{A.7}
\]

the sum (A.5) can be written as

\[
\frac{1}{V} \sum_{k \in \tilde{\Gamma}'_d} \tilde{F}(k) = \sum_{x \in \Gamma_d} F(x) - \frac{1}{V} \int d^d x F(x), \tag{A.8}
\]

where \( F(x) \) is the Fourier transform of \( \tilde{F}(k) \)

\[
F(x) = \int \frac{d^d k}{(2\pi)^d} e^{i x \cdot k} \tilde{F}(k). \tag{A.9}
\]

In the present context \( \tilde{F}(k) \) is

\[
\tilde{F}(k) = \frac{1}{\left\{ y^2 m^2 - (k + y_p)^2 - i\epsilon \right\}^2} = \frac{1}{\Gamma(2)} \int_0^\infty \frac{d \lambda}{\lambda} \frac{1}{(i \lambda)^2} \exp \left[ -i\lambda \left\{ y^2 m^2 - (k + z_p)^2 - i\epsilon \right\} \right]. \tag{A.10}
\]

A given \( d \)-dimensional vector \( x \) defines a \( D \)-dimensional space-like vector \( x^\mu = (0, x) \). A one-dimensional integral in Eq. (A.4) and a \( d \)-dimensional integral in Eq. (A.9) combine to become an integral over \( D \)-dimensional momenta, which can be readily carried out;

\[
\int_0^\infty d \lambda \frac{1}{\lambda} \left( i \lambda \right)^2 \int \frac{d^D k}{i (2\pi)^D} \exp \left[ -i\lambda \left\{ (k + y_p)^2 - i\epsilon \right\} - ix \cdot k \right]
\]

\[
= \int_0^\infty d \lambda \frac{\lambda^2}{\lambda \left( 4\pi \lambda \right)^{D/2}} \exp \left[ -\lambda \left\{ \left( \frac{i x}{2\lambda} - y_p \right)^2 - (yp)^2 \right\} \right]. \tag{A.11}
\]

For \( p^\mu = (m, 0), \left( i \frac{x}{2\lambda} - y_p \right)^2 - (yp)^2 = \frac{|x|^2}{4\lambda^2} \). Equation (A.4) thus becomes

\[
I_{11}(m^2; L) = \left( \mu^2 \right)^{2 - \frac{D}{2}} \int_0^1 dy \left( \sum_{n \in \mathbb{Z}^d} - \int d^d n \right)
\]

\[
\times \int_0^\infty d \lambda \frac{\lambda^2}{\lambda \left( 4\pi \lambda \right)^{D/2}} \exp \left( -\lambda y^2 m^2 - \frac{L^2}{4\lambda} |n|^2 \right). \tag{A.12}
\]
In the sum appearing above, the contribution of \( n = 0 \) is exactly \( I_{11}(m^2) = I_{11}(m^2; L \to \infty) \). \( I_{11}(m^2; L) - I_{11}(m^2) \) is hence free of UV divergence. Letting \( D \to 4 \) for this difference and rescaling \( \lambda \to \frac{L^2}{4\pi} \lambda \) leads

\[
I_{11}(m^2; L) - I_{11}(m^2) = -\frac{1}{16\pi^2} \frac{1}{mL} \mathcal{H}(mL).
\]

Here

\[
\mathcal{H}(mL) \equiv \pi \int_0^\infty \frac{d\lambda}{\lambda^2} \text{erf} \left( mL \sqrt{\frac{\lambda}{4\pi}} \right) S(\lambda),
\]

\[
S(\lambda) \equiv -\left( \sum_{n \in \mathbb{Z}^3 - \{0\}} - \int d^3n \right) \exp \left( -\frac{\pi}{\lambda} |n|^2 \right)
\]

\[
= - \left\{ \left( \vartheta_3 \left( 0, \frac{1}{\lambda} \right) \right)^3 - 1 - \lambda^2 \right\},
\]

where \( \text{erf}(x) \) is the error function

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x ds e^{-s^2},
\]

and \( \vartheta_3(v; \tau) \) is a Jacobi-theta function

\[
\vartheta_3(v; \tau) = \sum_{n=-\infty}^{\infty} \exp \left( \pi \tau n^2 + 2\pi v n \right).
\]

We recall that the term \( \lambda^{\frac{3}{2}} \) in \( S(\lambda) \) arises in our new QED, QED\(_L\). Because

\[
\vartheta_3 \left( 0, \frac{1}{\lambda} \right) \to \lambda^{\frac{1}{2}},
\]

in the infrared limit \( \lambda \to \infty \), the presence of that term indeed ensures IR-finiteness of the integral over \( \lambda \) in the expression (A.14). The numerical study shows that

\[
\lim_{x \to 0} \frac{\mathcal{H}(x)}{x} \simeq 10.4.
\]

Next we consider the function

\[
J_{11}(m^2; L) \equiv (\mu^2)^2 \frac{D}{2} \int_{-\infty}^\infty \frac{dk^0}{2\pi i} \frac{1}{V} \sum_{k \in \Gamma_d} \left\{ \frac{p \cdot k}{(k^2 - i\epsilon)} \right\} \left\{ \frac{1}{m^2 - (k + p)^2 - i\epsilon} \right\}.
\]

The same manipulation as done for \( I_{11}(m^2; L) \) in Eq. (A.10) leads

\[
J_{11}(m^2; L) = \int_0^1 dy \left( \sum_{x \in \Gamma_d} -\frac{1}{V} \int d^d x \right)
\]
We follow Appendix of Ref. 43) to carry out such integrals with 

\[
J \quad \text{over} \quad y
\]

it can be expressed as

\[
\text{Carrying out the integral of } \kappa \text{ and performing a } \rho\text{-derivative gives}
\]

\[
J_{11}(m^2; L) = \int_0^1 dy \left( \sum_{x \in \Gamma_d} -\frac{1}{V} \int d^d x \right) \int_0^\infty d\lambda \frac{\lambda^2}{(4\pi\lambda)^{D/2}} \times p \cdot (\frac{x}{2\lambda} - yp) \exp \left[ -\lambda \left( \frac{x}{2\lambda} - yp \right)^2 \right]
\]

\[
= -\frac{1}{2} \left( \sum_{x \in \Gamma_d} -\frac{1}{V} \int d^d x \right) \int_0^\infty d\lambda \frac{\lambda}{(4\pi\lambda)^{D/2}} \left( 1 - e^{-\lambda m^2} \right) e^{-\frac{\rho^2}{4\lambda}},
\]

(A-23)

where the second equality follows by taking \( p^\mu = (m, 0) \) and performing the integral over \( y \). In Eq. (A-23), UV divergence is contained in the term with \( x = 0 \), which is exactly \( J_{11}(m^2) = J_{11}(m^2, L \to \infty) \). Therefore, \( J_{11}(m^2; L) - J_{11}(m^2) \) is UV-finite. In terms of the function

\[
\mathcal{K}(x) \equiv 4\pi \int_0^\infty d\lambda \frac{1}{\lambda} \left( 1 - e^{-\frac{\lambda^2}{4\pi}} \right) S(\lambda), \quad \text{(A-24)}
\]

it can be expressed as

\[
J_{11}(m^2; L) - J_{11}(m^2) = \frac{1}{32\pi^2} \frac{1}{L^2} \mathcal{K}(mL). \quad \text{(A-25)}
\]

The quantity

\[
I_1^0(L) \equiv (\mu^2)^{2-D/2} \int_{-\infty}^\infty \frac{dk^0}{2\pi i} \frac{1}{V} \sum_{k \in \Gamma_d^e} \frac{1}{-k^2 - i\epsilon}, \quad \text{(A-26)}
\]

needs a special care. In dimensional regularization, we put

\[
I_1^0(\infty) = (\mu^2)^{2-D/2} \int \frac{d^Dk}{i (2\pi)^D} \frac{1}{-k^2 - i\epsilon} = 0.
\]

However, we cannot set \( I_1^0(L) = 0 \). For instance, if the integral

\[
(\mu^2)^{2-D/2} \int_{-\infty}^\infty \frac{dk^0}{2\pi i} \frac{1}{V} \sum_{k \in \Gamma_d^e} \frac{1}{(-k^2 - i\epsilon)(m^2 - k^2 - i\epsilon)}
\]
\[
= \frac{1}{m^2} \left( \mu^2 \right)^{2 - \frac{D}{2}} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi i} \frac{1}{L^d} \sum_{k \in \Gamma_d} \left( \frac{1}{m^2 - k^2 - i\epsilon} - \frac{1}{m^2 - k^2 - i\epsilon} \right), \quad (A.27)
\]

were evaluated in two different ways; (1) direct evaluation of the left-hand side by introducing a Feynman parameter, and (2) evaluation of the right-hand only with the second term kept, inconsistency would arise. A straightforward calculation yields

\[
I_1^0(L) - I_1^0(\infty) = -\frac{\kappa}{4\pi} \frac{1}{L^2}, \quad (A.28)
\]

where \(\kappa\) is a constant defined by

\[
\kappa \equiv \int_0^{\infty} \frac{d\lambda}{\lambda^2} S(\lambda) \approx 2.837. \quad (A.29)
\]

All the one-loop contributions induced by quartic couplings are described by a function

\[
I_1(m^2; L) \equiv \left( \mu^2 \right)^{2 - \frac{D}{2}} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi i} \frac{1}{V} \sum_{k \in \Gamma_d} \frac{1}{m^2 - k^2 - i\epsilon}. \quad (A.30)
\]

The expression for \(I_1(m^2; L)\) in terms of Jacobi-theta function was obtained in Ref. 23)

\[
I_1(m^2; L) - I_1(m^2) = \frac{1}{(4\pi)^2} \frac{M(mL)}{L^2}, \quad (A.31)
\]

where

\[
M(x) \equiv 4\pi \int_0^{\infty} \frac{d\lambda}{\lambda^2} \exp \left( -\frac{x^2}{4\pi} \lambda \right) T(\lambda), \quad (A.32)
\]

and

\[
T(\lambda) \equiv \sum_{n \in (\mathbb{Z}^3 - \{0\})} \exp \left( -\frac{\pi}{\lambda} |n|^2 \right) = \left( \theta_3 \left( 0, i \frac{1}{\lambda} \right) \right)^3 - 1. \quad (A.33)
\]

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