On estimation of the post-Newtonian parameters in the gravitational-wave emission of a coalescing binary

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Abstract

The effect of the recently obtained 2nd post-Newtonian corrections on the accuracy of estimation of parameters of the gravitational-wave signal from a coalescing binary is investigated. It is shown that addition of this correction degrades considerably the accuracy of determination of individual masses of the members of the binary. However the chirp mass and the time parameter in the signal is still determined to a very good accuracy. The possibility of estimation of effects of other theories of gravity is investigated. The performance of the Newtonian filter is investigated and it is compared with performance of post-Newtonian search templates introduced recently. It is shown that both search templates can extract accurately useful information about the binary.

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1 Introduction

It is currently believed that the gravitational waves that come from the final stages of the evolution of compact binaries just before their coalescence are very likely signals to be detected by long-arm laser interferometers [4]. The reason is that in the case of binary systems we can predict the gravitational waveform very well; and the amplitudes are reasonably high for sources at distances out to 200 Mpc. An estimate based on the number of compact binaries known in our galaxy and extrapolated to the rest of the Universe shows that there should be one neutron star compact binary coalescence per year out to the distance of 200 Mpc [2, 3]. This estimate is a safe lower bound on the rate of binary coalescence. Arguments based on progenitor evolution scenarios suggest that there should be 100 of two neutron star coalescences, 5 neutron star - black hole coalescences, and 0.5 two black hole coalescences out to 200Mpc [4]. The waveform derived using the quadrupole formula has been known for quite some time [5]. A standard optimal method to detect the signal from a coalescing binary in a noisy data set and to estimate its parameters is to correlate the data with the filter matched to the signal and vary the parameters of the filter until the correlation is maximal. The parameters of the filter that maximize the correlation are estimators for the parameters of the signal. The detailed algorithms and the performance of the matched-filtering method in application to coalescing binary gravitational-wave signal has been investigated by several authors, e.g. [6, 7, 8, 9, 10]. It has recently been realized [11] that the correlation is very sensitive even to very small variations of the phase of the filter because of the large number of cycles in the signal. Consequently the addition of small corrections to the phase of the signal due to the post-Newtonian effects decreases the correlation considerably. Thus the post-Newtonian effects in the coalescing binary waveform can be detected and estimated to a much higher accuracy than it was thought before [12]. This opens up new prospects but also considerable data analysis challenges for the LIGO, VIRGO, and GEO600 projects which are rapidly progressing. It was also found [11] that the post-Newtonian series is not converging rapidly for a binary near coalescence. Hence higher post-Newtonian corrections will affect the correlation. Currently three post-Newtonian corrections to the quadrupole formula are already known [13] and the calculation of further ones is in progress. In this article we analyse the estimation of parameters of the 2nd post-Newtonian signal. This part of work complements a recent detailed analysis of the 3/2 post-Newtonian effects performed recently in Ref.[13]. We also examine the detectability of the post-Newtonian signal and estimation of its parameters using the Newtonian waveform as a filter. This filter can be used as the simplest search template. We compare the Newtonian search templates with the post-Newtonian search templates recently investigated in Ref.[14].

The paper is organized as follows. In the first part of Section 2 we present the gravitational wave signal from a binary system to the currently known 2nd post-Newtonian order. In this work we analyse the signal in the “restricted” post-Newtonian approximation (i.e. only the phase of the signal is given to the 2nd post-Newtonian accuracy whereas the amplitude of the signal is calculated from the quadrupole formula), we assume circularized orbits, and we assume the spin parameters to be constant. In the second part we briefly describe the optimal method of detection of such a signal in noise and the maximum likelihood (ML) method to estimate the parameters of the signal. We derive a number of properties of the ML estimators of the parameters of our signal and we examine the bounds on their variances. Our analysis is based on the Cramer–Rao bound. In the third part we give the approximate rms errors of the estimators for the signal at various post-Newtonian orders. In the fourth part of Section 2 we consider the effects of other theories of gravity and their detectability from gravitational-wave measurements. We consider Jordan-Fiertz-Brans-Dicke theory and Damour-Esposito-Farèse biscal tensor theory. In Section 3 we consider the so called “search templates” introduced in Ref.[11]. These are simple filters containing as few parameters as possible to effectively detect the multi-parameter signals. In the first part of Section 3 we analyse the simplest search template - the Newtonian filter which is the waveform of the gravitational signal from a binary in the quadrupole approximation. We examine the Newtonian filter as a tool both to detect the signal and also to determine its nature. In the second part of Section 3 we compare the Newtonian filter with the other search template analysed recently [4] based on the full post-Newtonian signal. In Section 4 we summarize conclusions from our results. A number of results is left to appendices. In Appendix A we examine the first order effects on the phase of the signal due to eccentricity. In Appendix B we give numerical values of the covariance matrices at various post-Newtonian orders. In Appendix C we give certain detailed formulae for the Damour-Esposito-Farèse theory. In Appendix D we briefly review the theory of optimal detection of known signal in noise and we generalize it to non-optimal detection. In Appendix E we give a useful analytic approximation to the correlation integral of
the optimal filter with the signal from a binary. 

The units are chosen such that $G = c = 1$.

# 2 Post-Newtonian effects

## 2.1 Gravitational wave signal from a coalescing binary

Let us first give the formula for the gravitational waveform of a binary with the three currently known post-Newtonian corrections. We make the following approximations. We work within the so called “restricted” post-Newtonian approximation i.e. we only include the post-Newtonian corrections to the phase of the signal keeping the amplitude in its Newtonian form; this is because the effect of the phase on the correlation is dominant. The inclusion of post-Newtonian effects in amplitudes will not qualitatively change our results. Due to the effect of rapid circularization of the orbit by radiation reaction one can assume that the orbit is quasicircular. For example in the case of the gravitational wave signal from the Hulse-Taylor binary pulsar at the characteristic frequency of the detector for such a signal of around 47 Hz the eccentricity $e$ would be $\sim 10^{-6}$. Moreover the first order contribution to the phase of the signal due to eccentricity goes like $e^2$. Nevertheless for completeness we include the first order correction to the signal due to eccentricity in our formulae. We give a detailed derivation of this correction in Appendix A. We neglect the tidal effects. All tidal contributions to the gravitational wave signal from a coalescing binary were estimated to be small $[12, 15]$. There is also a small additional contribution to the phase due to tail effects which detectability has been considered in detail $[16]$ and was found to be small. This correction is formally of the 4th post-Newtonian order and consequently we neglect it in the present analysis.

With these approximations the waveform, as a function of time, is given by the following expression,

$$h(t) = Af(t)^{2/3} \cos[2\pi \int_{t_a}^t f(t')dt' - \phi],$$  

where

$$A = \frac{8}{5} \frac{\pi^{2/3} \mu m^{2/3}}{R}$$

and where $\phi$ is an arbitrary phase, $\mu$ and $m$ are the reduced and the total mass of the binary, respectively; $t_a$ is a time parameter and $R$ is the distance to the source. $A$ is the rms average amplitude over all Euler angles determining the position of the binary on the sky and the inclination angle between the plane of the orbit of the binary and the line of sight. The rms amplitude $A$ is $2/5$ of the maximum possible amplitude.

The characteristic time for the evolution of the binary to the currently known 2nd post-Newtonian order is given by

$$\tau_{2PN} := \frac{f}{df/dt} = \frac{5}{96} \frac{1}{\mu m^{2/3}} \frac{1}{(\pi f)^{8/3}} \times$$

$$\left[1 - \frac{157}{24} \frac{I_e}{f^{19/9}} + \left(\frac{743}{336} + \frac{11}{4} \frac{\mu}{m}\right)(\pi mf)^{2/3} - (4\pi - s_o)(\pi mf) +$$

$$\left(\frac{3058673}{1016064} + \frac{5429}{1008} \frac{\mu}{m} + \frac{617}{144} \left(\frac{\mu}{m}\right)^2 + s_s \frac{\mu}{m}\right)(\pi mf)^{4/3}\right]$$

where $I_e$ is the asymptotic eccentricity invariant,

$$I_e = e_0^2 f_0^{19/9},$$

$e_0$ is the eccentricity of the binary at gravitational frequency $f_0$ (see Appendix A for derivation and explanations). The quantities $s_o$ and $s_s$ are spin-orbit and spin-spin parameters respectively. They are given by the formula

$$s_o = \frac{113}{12} (s_1 + s_2) + \frac{25}{4} \left(\frac{m_2}{m_1} s_1 + \frac{m_1}{m_2} s_2\right),$$

$$s_s = \frac{247}{48} s_1 \cdot s_2 - \frac{721}{48} s_1 s_2$$
where \( \mathbf{L} \) is the total orbital angular momentum and \( \mathbf{S}_1, \mathbf{S}_2 \) are the spin angular momenta of the two bodies. The terms in square brackets in Eq. (8) are respectively: at lowest order, Newtonian (quadrupole); at order \( f^{-19/9} \), lowest order contribution due to eccentricity (see Appendix A); at order \( f^{2/3} \), 1PN \([17]\); at order \( f \), the non-linear effect of "tails" of the wave (\( 4\pi \) term) \([18, 13, 35, 21]\), and spin-orbit effects \([22]\); and at order \( f^{4/3} \), 2PN \([13]\) and spin-spin effects \([22]\).

In general the spin parameters vary with time. It was shown \([10]\) that \( s_o \) is nearly conserved, it never deviates from its average value by more than \( \sim 0.25 \). Moreover the time dependent part of the spin parameter is oscillatory what reduces considerably its influence on the phase of the signal \([11]\). In this work we shall assume that both spin-orbit and spin-spin parameters are constant. We also neglect the effect of the precession of the orbital plane due to spin on the waveform. The effects of the spin on the waveform of the signal from an inspiralling binary have been investigated in detail in \([23]\). If we take the available estimate of the moment of inertia for the pulsar in the Hulse-Taylor binary and assume masses of the neutron stars in the binary of 1.4 solar masses then \( s_o \simeq 4.8 \times 10^{-2} \) and \( s_s \simeq 2.4 \times 10^{-5} \). If such values are typical then spin effects will make negligible contributions to the phase of the signal. However this may not be the case for binaries containing black holes. Moreover if cosmic censorship is violated and black holes rotate at a higher rate than allowed by maximally rotating Kerr black hole the spin effects will significantly affect the gravitational waveform.

In the analysis of the detection of the above signal and estimation of its parameters it is convenient to work in the Fourier domain. The expression for the Fourier transform of our signal in the stationary phase approximation is given by (cf. \([1, 24, 35, 10]\))

\[
\hat{h} = \hat{A} f^{-7/6} \exp i[2 \pi f t_o - \phi - \pi/4 + \frac{5}{48} (a(f; f_o)k + a_c(f; f_o)k_c + a_1(f; f_o)k_1 + a_{3/2}(f; f_o)k_{3/2} + a_2(f; f_o)k_2)],
\]

for \( f > 0 \) and by the complex conjugate of the above expression for \( f < 0 \) where

\[
\hat{A} = \frac{1}{(30)^{1/2} \pi^{2/3}} \frac{\mu^{1/2} m^{1/3}}{R},
\]

\[
k = \frac{1}{\mu m^{2/3}},
\]

\[
k_c = \frac{1}{\mu m^{2/3} c^2} \bar{c}^2 (\pi f_0)^{19/9},
\]

\[
k_1 = \frac{1}{\mu} \left( \frac{743}{336} + \frac{11}{4} \nu \right),
\]

\[
k_{3/2} = \frac{m^{1/3}}{\mu} (4\pi - s_o),
\]

\[
k_2 = \frac{m^{4/3}}{\mu} \left( \frac{3058673}{1016064} + \frac{5429}{1008} \nu + \frac{617}{144} \nu^2 + \frac{8}{3} \frac{s_s}{\nu} \right),
\]

\[
a(f; f_o) = \frac{9}{40} \frac{1}{(\pi f)^{5/3}} + \frac{3}{8} \frac{\pi f}{(\pi f_o)^{8/3}} - \frac{3}{5} \frac{1}{(\pi f_o)^{5/3}},
\]

\[
a_c(f; f_o) = -\frac{157}{24} \left( \frac{81}{1462} \frac{1}{(\pi f)^{34/9}} + \frac{9}{43} \frac{\pi f}{(\pi f_o)^{43/9}} - \frac{9}{34} \frac{1}{(\pi f_o)^{34/9}} \right),
\]

\[
a_1(f; f_o) = \frac{1}{2} \frac{1}{\pi f} + \frac{1}{2} (\pi f_o)^{-2} - \frac{1}{\pi f_o},
\]

\[
a_{3/2}(f; f_o) = -\frac{1}{2} \left( \frac{9}{10} \frac{1}{(\pi f)^{2/3}} + \frac{3}{5} \frac{\pi f}{(\pi f_o)^{5/3}} - \frac{3}{2} \frac{1}{(\pi f_o)^{2/3}} \right),
\]

\[
a_2(f; f_o) = \frac{9}{4} \frac{1}{(\pi f)^{1/3}} + \frac{3}{4} \frac{\pi f}{(\pi f_o)^{4/3}} - \frac{3}{\pi f_o},
\]

hold and where \( f_o = f(t_o) \).
The stationary phase approximation, Eq.\([\text{14}]\), is an excellent approximation of the Fourier transform of the signal for frequencies which are not influenced by the finite time window of the measurement. In the above expressions for the gravitational wave signal from a binary we can make an arbitrary choice of the time parameter and the phase of the signal.

We also point out that going from the time to the frequency domain we have made yet another approximation. Namely we have taken the modulus \(|\tilde{h}|\) of the Fourier transform to be the Newtonian one i.e. \(|\tilde{h}| \sim f^{-7/6}\). In the stationary phase approximation \(|\tilde{h}|\) goes like \(1/\sqrt{f}\) and consequently by Eq.\([\text{153}]\) there would be other powers of frequency due to the post-Newtonian effects. We neglect those additional terms since the post-Newtonian corrections to the phase have the dominant effect. The inclusion of the post-Newtonian amplitudes to the signal will not qualitatively change the results of this work.

A convenient parameter is the chirp mass defined as \(\mathcal{M} := k^{-3/5}\). In the quadrupole approximation the gravitational wave signal from a binary is entirely determined by the chirp mass.

We shall consider three models of binaries: neutron star/neutron star (NS-NS), neutron star/black hole (NS-BH), and black hole/black hole (BH-BH) binaries with parameters summarized in Table I.

**Table I. Numerical values of the parameters of the three fiducial binary systems.** Black holes are of 10 solar masses and neutron stars are of 1.4 solar masses. Spin parameters are assumed to be constant. Spin for neutron stars was calculated from the typical estimate of the moment of inertia \(I\) for a neutron star of \(I = 10^{48}\) kg m\(^2\).

| Binary | \(m_1[M_\odot]\) | \(m_2[M_\odot]\) | \(\mathcal{M}[M_\odot]\) | \(s_1\) | \(s_2\) | \(s_p\) | \(s_q\) |
|--------|-----------------|-----------------|----------------|------|------|-----|-----|
| 1. NS-NS | 1.4 | 1.4 | 1.2 | 1.5\(\times10^{-3}\) | 1.5\(\times10^{-3}\) | 4.8\(\times10^{-2}\) | 2.4\(\times10^{-5}\) |
| 2. NS-BH | 1.4 | 10 | 3.0 | 3.0\(\times10^{-5}\) | 0.38 | 4.0 | 3.5 \(\times10^{-4}\) |
| 3. BH-BH | 10 | 10 | 8.7 | 0.13 | 0.13 | 3.9 | 0.15 |

| Binary | \(k[M_\odot^{-3/4}]\) | \(k_1[M_\odot^{-1}]\) | \(k_{3/2}[M_\odot^{-2/4}]\) | \(k_2[M_\odot^{-1/4}]\) |
|--------|----------------|----------------|----------------|----------------|
| 1. NS-NS | 0.72 | 4.1 | 25 | 13 |
| 2. NS-BH | 0.16 | 2.0 | 16 | 15 |
| 3. BH-BH | 2.7\(\times10^{-2}\) | 0.58 | 4.7 | 7.7 |

where \(M_\odot\) means solar mass.

For neutron stars we calculated the spin using the available estimate of the moment of inertia for the neutron star in the binary pulsar PSR1916+19. We have taken black holes to be spinning at half the maximum rate (i.e. \(s_i = 0.5m_\odot^2/m^2\)). The orbital momenta vectors were assumed to be parallel to the spin vectors.

To have an idea of the size of the post-Newtonian corrections in the gravitational wave signal from a binary when it enters the observation window of the laser interferometer we have evaluated the characteristic time \(\tau_{2PN}\) for the above three models at the frequency \(f_0 = 47\) Hz which is the characteristic frequency of the detector for this signal (see below). We have made explicit the contributions to the characteristic time from the three post-Newtonian corrections.

\[
\begin{align*}
\tau_{2PN}^1 &= 44(1 + 0.046[\text{from 1pn}] - 0.025[\text{from 3/2pn}] + 0.0012[\text{from 2pn}])\text{sec} \\
\tau_{2PN}^2 &= 9.9(1 + 0.10[\text{from 1pn}] - 0.071[\text{from 3/2pn}] + 0.006[\text{from 2pn}])\text{sec} \\
\tau_{2PN}^3 &= 1.7(1 + 0.17[\text{from 1pn}] - 0.12[\text{from 3/2pn}] + 0.018[\text{from 2pn}])\text{sec}
\end{align*}
\]

One concludes from the above numbers that for the earth-based laser interferometers post-Newtonian corrections are significant. Moreover several things are apparent. The quadrupole term is dominant for all the three models. This indicates a very good accuracy of the quadrupole formula even in the regime of strongly gravitating bodies. This has been noticed in other studies for example in the numerical investigation of the gravitational wave emission from the two black hole collisions [4]. The difference
in size between the 1st post-Newtonian correction and the 3/2 post-Newtonian correction (tail term) is rather small. They differ by a factor of 2 for NS-NS binary and only by a factor of around 1.5 for binaries with a black hole. The second post-Newtonian correction is noticeably smaller than the 3/2 post-Newtonian correction. The difference varies form a factor of 20 for a NS-NS binary to a factor of 7 for a BH-BH binary. The convergence of the post-Newtonian series appears to be worst for BH-BH binaries and in this case it would be desirable to have accurate numerical waveforms and not only the ones based on the post-Newtonian approximation. Such waveforms should be available as a result of the numerical projects such as Grand Challenge project currently under way in the United States.

2.2 Detection of the signal and estimation of its parameters

For the purpose of this investigation we shall use a fit to the total spectral density \( S_h(f) \) of the noise in the advanced LIGO detectors, devised in [10]. This fit comprises seismic, thermal, shot, and quantum noises in the detector:

\[
S_h(f) = S_0((f_o/f)^4 + 2(1 + (f/f_o)^2))/5, \tag{25}
\]

where \( f_o = 70 \text{Hz} \) and \( S_0 = 3 \times 10^{-48} \text{Hz}^{-1} \). It is an excellent approximation to the detailed formulae for various noises given in [6]. The sensitivity function \( \text{Sen}(f) \) of the detector is defined as \( 1/S_h(f) \). The sensitivity function has the maximum at frequency \( f_o \) given above and its half width half magnitude (HWMM) \( \sigma_o \) is around 48Hz.

To determine whether or not there is a signal in a noisy data set we use the Neyman-Pearson test (see Appendix D). When the noise in the detector is Gaussian the Neyman-Pearson test is the correlator test. It consists of linear filtering the data with the filter which Fourier transform is the Fourier transform of the signal divided by the spectral density of the noise \( S_h(f) \). The signal-to-noise ratio \( d \) that can be achieved by optimal filtering is given by \( d = (\langle h|h \rangle)^{1/2} \) where following [10] the scalar product \( \langle h|h \rangle \) is defined by

\[
(\langle h|h \rangle) = 4\Re \int_{f_i}^{f_o} \hat{h}^* \hat{h}/S_h(f) df, \tag{26}
\]

where \( \Re \) denotes the real part. Thus we have

\[
d^2 = 4\tilde{A}^2 \int_{f_i}^{f_o} df/S_h(f)f^{7/3}. \tag{27}
\]

We shall call the integrand of the above signal-to-noise integral signal sensitivity function and we denote it by \( \text{Ind}(f) \). This function has the maximum at the frequency \( f_o' \) where \( f_o' = 47 \text{Hz} \) and its HWHM \( \sigma_o' \) is \( \simeq 26 \text{Hz} \), around half of that of the sensitivity function. This is the signal-to-noise ratio after filtering of the data. We see that linear filtering introduces an effective narrowing of the detector bandwidth \( [27] \).

In the case of our chirp signal the linear filtering increases the signal-to-noise ratio by an amount given roughly by the square root of the number \( n(f) \) of cycles spent near the frequency \( f_o' \) where \( n(f) \) is defined by \( [10] \)

\[
n(f) := f\tau \simeq \frac{5}{96\pi} \frac{1}{M^{5/3}} \frac{1}{(\pi f)^{5/3}} \tag{28}
\]

Consequently the effectiveness of matched filtering falls with the chirp mass. On the other hand the amplitude \( \tilde{A} \) of the signal increases with the chirp mass like \( M^{5/3} \) and the overall factor in the signal-to-noise ratio increases as \( M^{5/6} \). This is born out by the amplitude \( \tilde{A} \) of the Fourier transform. Thus the probability of detection of binaries with the same rate of occurrence increases with the chirp mass.

To estimate the parameters of the signal it is proposed to use the maximum likelihood estimation (MLE) \( [28] \). It is by no means guaranteed that this is the best or the ultimate method. It may sometimes fail to give an estimate and other methods may lead to more accurate estimates. The MLE method consists of maximizing the likelihood ratio with respect to the parameters of the filter. In the case of the Gaussian noise the logarithm of the likelihood ratio \( \Lambda \) is given by \( [28] \)

\[
\ln \Lambda = \langle x|h_F \rangle - \frac{1}{2}(h_F|h_F) \tag{29}
\]
where \( h_F \) which we call the filter has the form of the signal but with arbitrary parameters and \( x \) are the data. We assume that the noise \( n \) in the detector is additive i.e. \( x = h + n \). The maximum likelihood (ML) estimators of the parameters of the signal are given by the following set of differential equations providing that one can differentiate under the integration sign of the scalar product defined above.

\[
(x - h_F|h_F, i) = 0, \tag{30}
\]

where \( h_F, i \) is the derivative of \( h_F \) with respect to the \( i \)th parameter. Rarely these equations can be solved analytically. It was shown \[7\] that in the case of the signal from a binary within the stationary phase approximation analytic expressions can be obtained for the maximum likelihood estimators of the amplitude and the phase.

The ML estimators are random variables since they depend on the noise. It is important to know the statistical properties of these estimators and their probability distributions so that we can determine how well they estimate the true values of the parameters. The most important quantities are the expectation value of the estimator and its variance. We would like to have the expectation value of the estimator to be as close as possible to the true value of the parameter and we would like the variance of the estimator to be as small as possible. The difference between the expectation value of an estimator of a parameter and the true value of the parameter is called the bias of the estimator. The ML estimator is not guaranteed to be either unbiased or minimum variance. We have the following useful general inequality called the Cramer-Rao inequality \[29\] that gives lower bound of the variance of estimators. Let \((\theta_i)\) be a set of \( n \) parameters and let \( \hat{\theta}_I \) be one of the parameters then the variance of its estimator \( \hat{\theta}_I \) satisfies the following inequality

\[
\text{Var}[\hat{\theta}_I] \geq (\Gamma^{-1})_{ij} \alpha^i \alpha^j, \tag{31}
\]

where \( \alpha^i \) and \( \Gamma^{ij} \) are given by

\[
\alpha^i = \frac{\partial E[\hat{\theta}_I]}{\partial \theta_i}, \tag{32}
\]

\[
\Gamma^{ij} = E\left[\frac{\partial \ln \Lambda}{\partial \theta_i} \frac{\partial \ln \Lambda}{\partial \theta_j}\right], \tag{33}
\]

where \( E \) is the expectation value. The matrix \( \Gamma \) is called the Fisher information matrix and its inverse is called the covariance matrix. One easily sees from the above inequality that when an estimator \( \theta_I \) is unbiased then the lower bound on its variance is given by the (11) component of the covariance matrix. For this inequality to hold certain mathematical assumption must be fulfilled \[29\].

1. The likelihood ratio must be a differentiable function with respect to all the parameters \( \theta_i \).
2. The order of differentiation with respect to parameters and the integration in the expectation value integral must be interchangeable.
3. The variances of the estimators must be bounded.
4. The Fisher information matrix must be positive definite.

The Cramer-Rao inequality is very general. It holds no matter what is the probability distribution of the data and it applies to any estimator providing the regularity conditions mentioned above are fulfilled. The above inequality guarantees only that the variance of an estimator is greater then a certain amount. It is important for us to know how well the right hand side of the Cramer-Rao inequality approximates the actual variance of an estimator. It was shown \[29, 8\] that in the case of Gaussian noise and in the limit of high signal to noise ratio \( d \) to the first order the maximum likelihood estimators are Gaussian and moreover they are unbiased and their covariances are given by the covariance matrix defined above. In statistical literature there also exists a series of refined Cramer-Rao bounds called Battacharyya bounds \[29\]. However in our case a useful approach to have an idea of the accuracy of the Cramer-Rao lower bound is given in Ref. \[10\] where the maximum likelihood equations were solved iteratively and a formula for the covariance matrix of the ML estimators was derived to one higher order then given by the inverse of the Fisher information matrix. This formula can be treated as an approximation to the variances of the ML estimators by a series in \( 1/d \) where \( d \) is the signal-to-noise ratio. The first order terms given
by the inverse of Fisher matrix go as $1/d^2$ and the correction terms go like $1/d^4$. Consequently one can expect that for signal-to-noise ratios of 10 or so the diagonal elements of the inverse of the Fisher matrix give variances of the ML estimators to an accuracy of few %.

We shall show that the set of parameters that we have chosen for our chirp signal has particularly useful properties. Note that the phase of the Fourier transform is linear in the phase, the time parameter, and the mass parameters $k_i$. We shall call these parameters phase parameters. Moreover the Fourier transform is linear in the amplitude parameter $\hat{A}$. The maximum likelihood estimators are those values of the parameters that maximize the likelihood ratio. The expectation value of the log likelihood is given by

$$E[\ln \Lambda] = (h|h_F) - \frac{1}{2} (h_F|h_F),$$

where $(h|h_F)$ is called the correlation function and is denoted by $H$. Using the stationary phase approximation to the Fourier transform of the signal $H$ is given by the integral

$$H(\Delta t, \Delta \phi, \Delta k, \Delta k_e, \Delta k_1, \Delta k_{3/2}, \Delta k_2) =$$

$$4\hat{A}\hat{F} \int_{f_i}^{\infty} \frac{df}{S_h(f)}, f^{3/3} \cos[2\pi f \Delta t - \Delta \phi + \frac{5}{48}(a(f; f_a)\Delta k + a_e(f; f_a)\Delta k_1)] + a_1(f; f_a)\Delta k_1 + a_{3/2}(f; f_a)\Delta k_{3/2} + a_2(f; f_a)\Delta k_2],$$

where $\Delta t$ means the difference in time parameters of the signal and the filter. The expectation of the log likelihood ratio depends on the phase parameters only through the correlation integral since $(h_F|h_F) = H(0,0,0,0,0,0,0) = d^2$ where $d$ is the signal-to-noise ratio. We see that the correlation function depends only on the differences between the values of the phase parameters in the signal and the filter and it has the maximum when the differences are zero. Moreover the value of the correlation is the same if we move by the same amount in any direction for a given parameter i.e. $H(-\Delta t, 0,0,0,0,0,0) = H(\Delta t, 0,0,0,0,0,0)$ and so on for all phase parameters. This property means that the probability distribution of any estimator of the phase parameter will be an even function of the difference between the estimator and its true value. In other words the probability distributions of the estimators of the phase parameters are symmetric about their true values. Consequently we have

$$m_l := E[(\hat{\theta}_l - \theta_l)^2] = 0 \quad \text{for } l \text{ odd}$$

and moreover for $l$ even the moments $m_l$ are independent of the true values of the phase parameters. Thus the ML estimators of the phase parameters are unbiased (this is immediate from Eq.36 for $l = 1$) and the covariance matrix of the estimators of the phase parameters is independent of their values. The probability distributions of the phase parameters will depend on the signal-to-noise ratio. We know that for large signal-to-noise ratio they will tend to Gaussian probability distributions. The estimator of the amplitude parameter is biased nevertheless by the symmetry property of the probability distributions of the phase parameters its bias is independent of the values of the phase parameters. These properties of the parameters can also be seen explicitly from the first two terms of the series solution of the ML equations (Eq.33) given in ref. [26]. The properties of our chosen set of parameters greatly simplify calculation of the Cramer-Rao bounds. In our case the Fisher information matrix $\Gamma$ is given by

$$\Gamma_{ij} = \frac{\partial H}{\partial \theta_i S F \theta_j F} \theta_i = \theta_i F,$$

where $S$ refers to the parameters of the signal and $F$ to the parameters of the filter. The inverse of the $\Gamma$ matrix is called the covariance matrix and is denoted by $C$. It is easily seen that $\Gamma^{AA}$ components are all equal to zero when $i \neq A$. Thus the amplitude parameter decouples from the phase parameters. Because the phase parameters are unbiased the lower bounds of their variances are given just by the appropriate diagonal elements of the covariance matrix $C$. In the case of the amplitude parameter the Cramer-Rao bound is given by $VarA \geq b'(A)/\Gamma^{AA}$ where $b'(A)$ is the derivative of the bias of amplitude parameter w.r.t. amplitude and $\Gamma^{AA} = d^2/A^2$. Note that $\Gamma^{AA}$ is independent of $\hat{A}$. This is a consequence of the linearity of the signal in the amplitude.

It is clear from the linearity of the function $H$ in the differences $\Delta \theta$ that the $\Gamma$ matrix is independent of the values of the phase parameters. Thus the Cramer-Rao bound on these parameters is also independent of

---

Footnote: We are indebted to Dr. J.A. Lobo for this observation, see also ref. [26] p. 276.
the values of the parameters. From the argument above we know that this holds not only for the bounds on the variances but also for the variances themselves.

To obtain the maximum of the correlation each phase parameter of the filter has to match a corresponding parameter in the signal (see Eq. 34). Thus by linear filtering we shall get estimates of the time parameter \( t_a \), phase, and the mass parameters \( k_i \). In the filter one can always make an arbitrary choice of the time parameter \( t_a \). For example instead of choosing \( t_a \) as the time at which frequency is \( f_a \) one can choose time \( t'_a \) as the time at which the frequency is equal to \( f'_a \). This new choice is equivalent to the following transformation

\[
 t'_a = t_a + \delta_0 k_0 + \delta_1 k_e + \delta_1 k_1 + \delta_3/2 k_3/2 + \delta_2 k_2,
\]

\[
 \phi' = \phi + \delta_0 k_0 + \delta_1 k_e + \delta_1 k_1 + \delta_3/2 k_3/2 + \delta_2 k_2,
\]

where

\[
 \delta_0 = \frac{5}{256} \left( \frac{1}{(\pi f_a)^{8/3}} - \frac{1}{(\pi f'_a)^{8/3}} \right),
\]

\[
 \delta_0 = \frac{1}{16} \left( \frac{1}{(\pi f_a)^{5/3}} - \frac{1}{(\pi f'_a)^{5/3}} \right),
\]

\[
 \delta_e = \frac{785}{11008} \left( \frac{1}{(\pi f_a)^{43/9}} - \frac{1}{(\pi f'_a)^{43/9}} \right),
\]

\[
 \delta_e = \frac{4352}{785} \left( \frac{1}{(\pi f_a)^{34/9}} - \frac{1}{(\pi f'_a)^{34/9}} \right),
\]

\[
 \bar{\delta}_1 = \frac{5}{192} \left( \frac{1}{(\pi f_a)^{2}} - \frac{1}{(\pi f'_a)^{2}} \right),
\]

\[
 \delta_1 = \frac{1}{48} \left( \frac{1}{\pi f_a} - \frac{1}{\pi f'_a} \right),
\]

\[
 \delta_{3/2} = \frac{1}{32} \left( \frac{1}{(\pi f_a)^{5/3}} - \frac{1}{(\pi f'_a)^{5/3}} \right),
\]

\[
 \delta_{3/2} = \frac{5}{32} \left( \frac{1}{(\pi f_a)^{2/3}} - \frac{1}{(\pi f'_a)^{2/3}} \right),
\]

\[
 \delta_2 = \frac{5}{128} \left( \frac{1}{(\pi f_a)^{1/3}} - \frac{1}{(\pi f'_a)^{1/3}} \right),
\]

\[
 \delta_2 = \frac{5}{16} \left( \frac{1}{(\pi f_a)^{1/3}} - \frac{1}{(\pi f'_a)^{1/3}} \right).
\]

The mass parameter frequency functions \( a_i(f; f_a), (i = 0, 1, 3/2, 2) \) in Eq. 11 are then transformed to \( a_i(f; f'_a) \). The mass parameters remain invariant under the above transformations. By linear filtering with the template parametrized by the new time parameter and the new phase given by the above transformation we estimate the new time parameter \( t'_a \) and the new phase \( \phi' \) but the same mass parameters \( k_i \).

There is also a particularly simple parametrization of the signal. Let us rewrite the Fourier transform of the gravitational wave signal from a binary in the following form

\[
 h = \hat{A} f^{-7/6} \exp[i 2\pi f_t c - \phi_c - \pi/4 + \frac{3 k}{128 (\pi f)^{5/3}} - \frac{4239 k_e}{11696 (\pi f)^{34/9}} + \frac{5 k_1}{96 \pi f^2} - \frac{1}{32 (\pi f)^{2/3}} + \frac{15}{64 (\pi f)^{1/3}}].
\]

(for \( f > 0 \) and by the complex conjugate of the above expression for \( f < 0 \)) where \( t_c \) and \( \phi_c \) are coalescence time and phase respectively and they are given by

\[
 t_c = t_a + \frac{5 k}{256 (\pi f_a)^{8/3}} - \frac{785}{110008} \left( \frac{k_e}{(\pi f)^{43/9}} + \frac{k_1}{192 (\pi f_a)^2} - \frac{1}{32 (\pi f_a)^{5/3}} + \frac{5 k_2}{128 (\pi f_a)^{4/3}} \right),
\]

\[
 \phi_c = \phi_a + \frac{1 k}{16 (\pi f_a)^{5/3}} - \frac{785 k_e}{4352 (\pi f_a)^{34/9}} + \frac{5 k_1}{48 (\pi f_a)} - \frac{5 k_3/2}{32 (\pi f_a)^{2/3}} + \frac{5 k_2}{16 (\pi f_a)^{1/3}}.
\]
Coalescence time and coalescence phase are obtained when the time parameter \( t'_a \) is such that the corresponding frequency \( f'_a \) is infinite which occurs when the two point masses coalesce. We can estimate the coalescence time and the coalescence phase of the template if we filter for combinations of the time and phase parameters with the mass parameter given precisely by right hand sides of Eqs.(51) and (52). There is also a transformation of the phase that we shall find useful (see next Section).

\[
\phi' = \phi - 2\pi f_m t_a.
\] (53)

where \( f_m \) is some arbitrary constant frequency. Using the new phase parameter in the filter given by above transformation we shall estimate a new value of the phase shifted by the amount \( 2\pi f_m t_a \). It is not difficult to show that all the above transformations do not change the CR bound on the mass parameters however the transformation Eq.(49) changes the bound for time and phase parameters whereas transformation Eq.(52) changes the bound on the phase. We can use the freedom of these transformations in the filter to obtain better accuracies of estimation of the time and the phase parameters.

### 2.3 Numerical analysis of the rms errors of the estimators

First of all we investigate the influence of the increasing number of post-Newtonian parameters on the accuracy of their estimation. To this end we have calculated the covariance matrices for the signal containing only the quadrupole term, then covariance matrices for 1st post-Newtonian, 3/2 post-Newtonian, and 2nd post-Newtonian signal, and finally for the 2nd post-Newtonian signal with first order contribution due to eccentricity. The results are summarized in Table II where we have given rms errors of the phase parameters. We have given the rms errors for the time and phase of coalescence \( t_c \) and \( \phi_c \) respectively. We have also determined the frequency \( f_{\text{min}} \) for which the error in the time parameter is minimum and we have given the minimum error \( \Delta t_{\text{min}} \) in the time parameter and the corresponding error \( \Delta \phi \) in phase. We have considered a reference binary of \( M = 1 \text{M}_{\odot} \) located at the distance of 100Mpc. We have taken the range of integration from 10Hz to infinity. The signal-to-noise ratio for such a binary is around 25.

**Table II.** The rms errors for the phase parameters at various post-Newtonian orders for a reference binary of chirp mass of 1 solar mass at the distance of 100Mpc. Expected advanced LIGO noise spectral density is assumed and the integration range from 10Hz to infinity is taken giving signal-to-noise ratio of around 25.

| \( \Delta t_{\text{min}} \) [msec] | \( \Delta \phi \) | \( \Delta t_c \) [msec] | \( \Delta \phi_c \) | \( \Delta k_1 [M_{\odot}^{-5/3}] \) | \( \Delta k_{3/2} [M_{\odot}^{-2/3}] \) | \( \Delta k_2 [M_{\odot}^{-1/3}] \) | \( \Delta k_{5/2} [M_{\odot}^{-1/2} 100\text{Hz}^{10/9}] \) |
|---|---|---|---|---|---|---|---|
| 0.14 | 0.073 | 0.17 | 0.10 | \( 8.3 \times 10^{-6} \) | \( 5.8 \times 10^{-3} \) | \( 0.52 \) | \( 1.3 \times 10^{-6} \) |
| 0.15 | 0.087 | 0.27 | 0.33 | \( 4.0 \times 10^{-5} \) | \( 0.70 \times 10^{-1} \) | \( 7.2 \) | \( 1.7 \times 10^{-3} \) |
| 0.18 | 0.14 | 0.54 | 1.9 | \( 1.7 \times 10^{-4} \) | \( 6.6 \times 10^{-4} \) | \( 28 \) | \( 59 \) |
| 0.24 | 0.14 | 1.6 | 24 | \( 2.3 \times 10^{-3} \) | \( 1.3 \times 10^{-3} \) | \( - \) | \( - \) |
| 0.25 | 0.17 | 2.3 | 45 | \( 2.3 \times 10^{-3} \) | \( 1.3 \) | \( - \) | \( - \) |

The above bounds scale exactly as the inverse of the signal-to-noise ratio \( d \) and they do not depend on the numerical values of the phase parameters. From the above table we see that increasing the number of post-Newtonian corrections and parameters we filter for decreases the accuracy of estimation of the parameters independently of the size of the post-Newtonian correction. Thus searching for a negligible correction due to eccentricity increases the rms error in other parameters by over 100%.

For completeness in Appendix B we give the numerical values of covariance matrices for the phase parameters at various post-Newtonian orders and the corresponding values of the frequency \( f_{\text{min}} \).

As we have indicated above the estimator of the amplitude parameter is biased however if one takes the expansion of the variance of the estimator in the inverse powers of the signal-to-noise ratio (see Eq. (54) for a general formula) then the leading term for the variance of the amplitude is just \( 1/\Gamma^{AA} \) where \( \Gamma^{AA} \) is independent of \( A \). The higher order corrections to the CR bounds of the amplitude go like \( 1/d^4 \) and they do depend on the value of the amplitude. As an amplitude parameter we find convenient to choose \( A_\oplus \) given by

\[
A_\oplus = \frac{M_\odot^{5/6}}{r_{100 \text{Mpc}}}.
\] (54)


where $M_\odot$ is the chirp mass in the units of solar masses and $r_{100\text{Mpc}}$ is the distance in the units of 100Mpc. For our reference binary the amplitude $A_{\oplus} = 1$ and thus the approximate rms error in its ML estimator is $A_{\oplus}/d \simeq 1/25 = 0.04$ and as explained above this last number is independent of the true value of the amplitude.

It is important to assess the accuracy of estimation of the physical parameters of the binary, i.e. the two masses of its members and the spin parameters $s_o$ and $s_s$. This means that we have to make a transformation to a different parameter set. A nice property of the ML estimators is the following. Let $\hat{\theta}_i$ be the maximum likelihood estimators of the set of parameters $\theta_i$. Let $f(\hat{\theta}_i)$ be a function of the parameters then $f(\hat{\theta}_i)$ is the maximum likelihood estimator of the function $f$ (see Ref. [28]). However it is not true in general that if estimators of the old parameters are unbiased then the new parameter is unbiased as well. Consequently by transforming the bounds of the old parameters one will not get the Cramer-Rao bound on the new set of parameters. However we know that Cramer-Rao bounds are approximately equal to the true variances in the limit of high signal-to-noise ratio $d$, correction terms being of the order of $1/d^2$. Hence by transforming the C-R bounds one gets the rms errors of the estimators accurate to the order $1/d$. Another important point is that the transformation to the new parameter set may be singular. Then the determinant of the $\Gamma'$ matrix for the new set of parameters is zero and thus $\Gamma'$ is not positive definite, consequently the Cramer-Rao inequality does not hold. A way to get errors of estimators of the new parameters in such a case could be to attempt to calculate the bias and the variance directly from some approximate probability distributions for the estimators (see ref. [10] for such treatment to determine the accuracy of the distance to the binary). It may happen however that the probability density function is such that the expectation value and the variance do not exist (an example is Cauchy probability distribution) and then one may have to use another measure of bias and error, e.g. median and interquartile distance. The other method proposed in [10] is to use confidence intervals. We shall return to this problem in the future work [30, 31].

The transformation from the 4 mass parameters $k_I$ to new parameters - total mass ($m$), reduced mass ($\mu$) and the spin parameters $s_o$ and $s_s$ is regular. Thus we can obtain approximate values of the errors of the estimators of the reduced mass, the total mass and the spin parameters. However the transformation from $m$ and $\mu$ to individual masses $m_1$ and $m_2$ is singular (determinant of the Jacobian of the transformation is zero when masses are equal, see Ref. [10]). Consequently the errors in the determination of the masses cannot be obtained from the C-R bounds calculated above.

In Table III we show the degradation of the accuracy of estimation of the chirp mass, the reduced mass, and the total mass with the increasing number of parameters in the signal for the NS-NS binary at a distance of 200Mpc.

| pN order | $\Delta M/M$ | $\Delta \mu/\mu$ | $\Delta m/m$ |
|----------|--------------|------------------|-------------|
| 1 pN     | 0.0054%      | 0.55%            | 0.81%       |
| 3/2 pN   | 0.023%       | 6.4%             | 9.6%        |
| 2 pN     | 0.080%       | 42%              | 63%         |

For the calculation of the numbers in the table above and all other tables in the remaining part of this Section we have taken the range of integration in the Fisher matrix integrals to be from 10Hz to the frequency $f = (6^{3/2}/\pi m)^{-1}$ corresponding to the last stable orbit of the test particle in Schwarzschild space-time. This may very roughly correspond to the last stable orbit in a binary [32, 33].

In Table IV we give the signal-to-noise ratios and the Cramer-Rao bounds for the mass and the spin parameters in percents of their true values for the 2nd post-Newtonian signal for our three representative
binary systems at the distance of 200Mpc. We have also given the improvement factors in $\sqrt{n}$ in the S/N due to filtering.

**Table IV.** Accuracy of estimation of the parameters of the 2nd post-Newtonian signal for the three fiducial binaries.

| Binary  | S/N | $\sqrt{n}$ | $\Delta M/M$ | $\Delta \mu/\mu$ | $\Delta m/m$ | $\Delta s_o/s_o$ | $\Delta s_s/s_s$ |
|---------|-----|------------|--------------|-----------------|--------------|-----------------|-----------------|
| NS-NS   | 15  | 32         | 0.080%       | 42%             | 63%          | $60 \times 10^6$% | $12 \times 10^6$% |
| NS-BH   | 32  | 15         | 0.26%        | 40%             | 59%          | 11%             | 19% 10^4%       |
| BH-BH   | 77  | 6          | 0.92%        | 150%            | 230%         | 240%            | 890%            |

We see that only the rms error in the chirp mass is small and also the accuracy of the determination of the spin-orbit parameter for NS-BH binary is satisfactory. The errors in reduced and total masses are large.

One can derive simple general formulae for the accuracy of determination of the chirp mass, the reduced mass, and the total mass in terms of rms errors of the mass parameters $k_i$. From the definition of the chirp mass one immediately obtains the following formula for the relative rms error in terms of the rms error in the mass parameter $k$,

$$\Delta M/M = \frac{3}{5} r_{100\text{Mpc}} \Delta k_i M^{5/3}_\odot. \quad (55)$$

For the errors in the reduced and the total mass we obtain the following general formulae using the standard law of propagation of errors

$$\Delta \mu = \left| - \frac{\partial k}{\partial \mu} \sqrt{\Delta k} + \frac{\partial k}{\partial \mu} \sqrt{\Delta k_1} \right|,$$

$$\Delta m = \left| \frac{\partial k}{\mu} \sqrt{\Delta k} - \frac{\partial k}{\mu} \sqrt{\Delta k_1} \right|,$$

$$\text{det} = \frac{\partial k_1}{\partial \mu} - \frac{\partial k_1}{\partial \mu} \frac{\partial k_1}{\partial \mu} \frac{\partial k_1}{\partial \mu} (58)$$

and $\Delta k, \Delta k_1$ are rms error in mass parameters $k$ and $k_1$ respectively. The formula above is the same when the 1st post-Newtonian, the 3/2 post-Newtonian, and the 2nd post-Newtonian corrections are included. We observe that errors in $\mu$ and $m$ depend only on the masses and the rms errors in the parameters $k$ and $k_1$. The other mass parameters influence the errors in $\mu$ and $m$ only through their correlations with the mass parameters $k$ and $k_1$ and only through the functional form of the corrections as the rms error in the mass parameters are independent of their values. The errors in $\mu$ and $m$ are independent of the numerical values of the parameters $k_{3/2}$ and $k_2$. Since in general the rms error $\Delta k$ is considerably smaller than $\Delta k_1$ we get the following simplified expressions for the relative errors in the reduced and the total mass.

$$\Delta \mu/\mu = \frac{1}{a} r_{100\text{Mpc}} \mu_\odot \Delta k_1,$$

$$\Delta m/m = \frac{3}{2a} r_{100\text{Mpc}} \mu_\odot \Delta k_1,$$

where $a = 743/336 - 33/8 \mu/m$. We see that the error in the determination of the reduced mass and the total mass is determined by error in the first post-Newtonian mass parameter $k_1$. Since the ratio $\mu/m$ is $\leq 1/4$ to a fairly good approximation we can take the value of $a$ roughly equal to 1.

If the spin effects could entirely be neglected and we would only have the reduced mass and the total mass as unknown in the mass parameters $k_i$ then we could achieve the accuracies in the parameters of the signal summarized in Table V. We considered three fiducial binary systems and 2nd post-Newtonian signal but with spin-orbit and spin-spin parameters removed. Thus the number of parameters estimated is 2 less than for the signal considered in Table IV.

**Table V** Accuracy of estimation of the parameters for 3 fiducial binary systems and the 2nd post-Newtonian signal but with spin parameters removed.
| Binary | S/N | Δt_{c,ms} | Δφ_{c} | Δμ/µ | Δm/m |
|--------|-----|------------|--------|-------|------|
| NS-NS  | 15  | 0.47       | 0.82   | 0.29% | 0.43%|
| NS-BH  | 32  | 0.32       | 0.47   | 0.19% | 0.28%|
| BH-BH  | 77  | 0.18       | 0.24   | 0.27% | 0.37%|

We see that if spin parameters could be neglected we would have an excellent accuracy of estimation of the reduced and the total mass of the binary.

2.4 The effects of other theories of gravity

We shall consider two alternative theories. One is Jordan-Fiertz-Brans-Dicke (JFBD) theory (see [35] for a detailed discussion) and the other is a multi-scalar field theory recently proposed in [37].

In the JFBD theory in addition to the tensor gravitational field there is also a scalar field. The theory can be characterized by a coupling constant that we denote by ω. General relativity is obtained when ω goes to infinity. The JFBD theory has two effects on gravitational emission. It admits dipole gravitational radiation and secondly there is a modification of the quadrupole emission due to the interaction of the scalar field with gravitating bodies. In the case of binary system the effects of the JFBD theory has been studied in great detail [34] and a general formula for the change of orbital period was derived ([35] eq.(14.22)). From that formula we get the following expression for the characteristic time τ of the evolution of the binary due to radiation reaction in the case of circularized orbits and assuming that the contribution due to the dipole term is small

\[
\tau = \frac{5}{96} \frac{1}{\mu m^{2/3}} \frac{1}{\kappa} \frac{G^{4/3}}{(\pi f)^{8/3}} \times
\]

\[
(1 - \frac{5}{192} k_B \frac{G^{4/3}}{\kappa} \frac{\Sigma^2}{(\pi m^2 f^{2/3})})
\]

where

\[
k_B = \frac{1}{2 + \omega}
\]

\[
G = 1 - \frac{k_B}{2} (C_1 + C_2 - C_1 C_2)
\]

\[
\kappa = G^2 (1 - \frac{k_B}{2} + \frac{k_B}{12} \gamma^2)
\]

\[
\gamma = 1 - \frac{m_1 C_2 + m_2 C_1}{m_1 + m_2}
\]

\[
\Sigma = C_1 - C_2.
\]

C_1 and C_2 are “sensitivities” of the two bodies to changes of the scalar field. For a black hole the sensitivity C is always equal to 1. For a neutron star C depends on the equation of state. For neutron stars the sensitivity has been studied in [37] for a number of equations of state and it was found for a wide range of such equations that it is proportional to the mass of the neutron star with proportionality constant varying from .17 to .31. Here we shall assume that C_i = 0.21 m_{i⊙} for a neutron star of m_{i⊙} solar masses. From the above formulae one sees that the dipole radiation will vanish if the binary system consists of two black holes or the neutron stars in the binary are the same.

The Fourier transform of the signal in the stationary phase approximation including contributions due to JFKB theory is given by (we neglect any contributions due to eccentricity)

\[
\tilde{h} = \tilde{A} f^{-7/6} \exp i [2\pi ft_a - \phi - \pi/4]
\]

\[
+ \frac{5}{48} (a(f; f_a) k' + a_1(f; f_a) k_1 + a_{3/2}(f; f_a) k_{3/2} + a_2(f; f_a) a_d(f; f_a) k_d]
\]

for f > 0 and by the complex conjugate of the above expression for f < 0 where the function a_d(f; f_a) due to dipole radiation has the form

\[
a_d(f; f_a) = -\frac{5}{192} \left( \frac{9}{70} \frac{1}{(\pi f)^{7/3}} + \frac{3}{10} \frac{f}{(\pi f_a)^{10/3}} - \frac{3}{7} \frac{1}{(\pi f_a)^{7/3}} \right)
\]
and where

\begin{align*}
k' &= \frac{1}{\mu m^{2/3}} \frac{G^{4/3}}{\kappa}, \\
k_d &= \frac{1}{\mu m^{4/3}} k_B \frac{G^{8/3}}{k^2} \Sigma^2
\end{align*}

(69)

(70)

Current observational tests constrain \( \omega \) to be greater than 600 and from timing of binary pulsar a lower limit on \( \omega \) of 200 can be set. Thus it is sufficient to keep only the first terms in \( 1/\omega \). Then the two parameters above are approximately given by

\begin{align*}
k' &= \frac{1}{\mu m^{2/3}} (1 - dk_d), \\
k_d &= \frac{1}{\mu m^{4/3}} k_B \Sigma^2
\end{align*}

(71)

(72)

where

\[ dk_d = k_B \frac{1}{3}(C_1 + C_2 - C_1 C_2) + \frac{1}{2} \frac{\gamma^2}{12}. \]

(73)

We thus see the the JFBD theory introduces a new parameter \( k_d \) due to the dipole radiation and modifies the standard chirp mass parameter \( k \) by fraction \( dk_d \).

We have investigated the potential accuracy of estimation of the parameter \( k_d \) assuming that the spin effects are negligible. We have taken neutron star/black hole binary with parameters given in Table I at the distance of 200Mpc. The result is summarized in Table VI.

**Table VI.** The rms error for signal parameters in JFKB theory assuming spins are negligible for the binary of 1.4 solar mass neutron star and 10 solar mass black hole.

| S/N | \( \Delta t_c \) [ms] | \( \Delta \phi_c \) | \( \Delta \mu/\mu \) | \( \Delta m/m \) | \( \Delta k_d [M_\odot^{-5/3}] \) |
|-----|----------------------|-----------------|-----------------|-----------------|-----------------|
| 32  | 0.47                 | 0.97            | 0.57\%          | 0.73\%          | \( 2.3 \times 10^{-5} \) |

The potential accuracy of determination of the dipole radiation parameter \( k_d \) is high. Current observational constraints indicate however that this parameter is small. We have the following numerical values.

\[ k_d = 3.2 \times 10^{-5} \left( \frac{500}{\omega} \right) \left( \frac{\Sigma^2}{0.5} \right) \left( \frac{32}{\mu m^{4/3}} \right) \]

(74)

\[ \frac{\Delta k_d}{k_d} = 0.7 \left( \frac{\omega}{500} \right) \left( \frac{0.5}{\Sigma^2} \right) \left( \frac{\mu m^{4/3}}{32} \right) \]

(75)

We conclude that the gravitational-wave measurement by planned long arm laser interferometers have the potential of testing JFBD to the accuracy comparable to tests in solar system and measurements from the binary pulsars [38].

From the general class of tensor-multi-scalar theories studied recently [37] we shall consider a two-parameter subclass of tensor-bi-scalar theories denoted by \( T(\beta', \beta'') \). Theories in this subclass have two scalar fields and they tend smoothly to general theory of relativity when both parameters \( \beta' \) and \( \beta'' \) tend to zero. The subclass is defined in such a way that the dipole radiation vanishes. From the general formulae [37] one can calculate the characteristic time \( \tau \). For circularized orbits the only modification is an effective change of the chirp mass parameter \( k \) given by the following formula

\[ k'' = k - d_{DF}, \]

(76)

\[ d_{DF} = \frac{5}{144} \kappa_o(m_1, C_1, m_2, C_2) + \frac{1}{6} \kappa_q(m_1, C_1, m_2, C_2) + \frac{5}{48} \kappa_d(m_1, C_1, m_2, C_2) \]

(77)
where coefficients $\kappa_0, \kappa_4, \kappa_d1, \kappa_d2$ are due to contributions from quadrupole helicity zero, corrections to quadrupole helicity two, and dipole radiation respectively. They are complicated functions of the masses and sensitivities. We give the detailed formulae in Appendix C. In all the tensor-multi-scalar field theories whenever one of the component is a black hole corrections to the radiation reaction vanish. We have also found that for a simple model where sensitivities are proportional to masses of neutron stars and the proportionality constant is the same the correction $d_{DF}$ does not depend on the parameter $\beta''$. For a system of two identical neutron stars the correction $d_{DF}$ takes a simple form

$$d_{DF} = 0.21\beta C^2,$$

where $C$ is the sensitivity of the neutron star to changes of the scalar field introduced above. Current observations constrain parameter $\beta$ to be less than 1. For circularized orbits (the case considered above) the bi-scalar theory does not introduce a new mass parameter in the phase of the signal but only a shift in the “Newtonian” mass parameter $k$. We shall consider the possibility of estimating this shift in the next section.

3 Search templates

3.1 The Newtonian filter

We have seen in the previous Section that the accuracy of estimation of the parameters is significantly degraded with increasing number of corrections even though a correction may be small. If we include the 2nd post-Newtonian correction and filter for all unknown parameters then the accuracy of determination of the masses of the binary becomes undesirably low. Moreover we cannot entirely exclude unpredicted small effects in the gravitational-wave emission (e.g. corrections to general theory of gravity) that we present cannot model. Thus there is a need for simple filters or search templates that will enable us to scan the data effectively and isolate stretches of data where the signal is most likely to be [11].

The simplest such filter is just a Newtonian waveform $h_N$ which Fourier transform in stationary phase approximation is given by

$$h_N = \frac{1}{30^{1/2}} \frac{1}{\pi^{2/3}} \frac{\mu^{1/2} m^{1/3}}{R} f^{-7/6} \exp i[2\pi f t_c - \phi_c - \pi/4 + k \frac{3}{128} (\pi f)^{-5/3}].$$

We shall call the Newtonian filter the filter which Fourier transform is given by the above formula and we shall denote it by $N f$. This filter has been investigated by the present authors [9, 39, 40] and also by other researchers [11, 12, 13, 14, 14]. A different search template based on the post-Newtonian signal has recently been introduced in Ref. [14]. We discuss this alternative search template in the next subsection.

In this section we examine the performance of the Newtonian filter. We demonstrate that such a template will perform well in detecting the signal from a binary and it also gives a reasonable idea of the nature of the binary. We shall investigate the performance of the Newtonian filter both analytically and numerically.

Let us consider the correlation of the post-Newtonian signal with the Newtonian filter. Such an integral has the same form as the correlation integral given by Eq.36 in Section 2.1 except that all post-Newtonian mass parameters will be unmatched by the parameters of the filter. The correlation will be high if we can reduce the oscillations due to the cosine function as much as possible. Since the integrand of the correlation integral is fairly sharply peaked ($HWHM \simeq 26Hz$) around its maximum at the frequency $f' \simeq 47Hz$ we can achieve this by making the phase as small as possible around the peak frequency $f'$. The argument $\Phi$ of the cosine in the integrand of the correlation of the post-Newtonian signal with the Newtonian filter including the effects due to eccentricity and dipole radiation takes the form

$$\Phi(f) = 2\pi f \Delta t + \Delta \phi + \frac{5}{48} [a(f; f_a)\Delta k + a_e(f; f_a)k_e + a_1(f; f_a)k_1 + a_{3/2}(f; f_a)k_{3/2} + a_2(f; f_a)k_2 + a_d(f; f_a)k_d].$$

First we note that for all the mass parameter frequency functions $a_i(f; f_a)$ the functions and their first derivatives vanish at the frequency $f_a$. We shall therefore choose $f_a = f'$. Let us also transform the
phase parameter according to transformation given by Eq.53 with \( f_m = f'_o \). In the new parametrization the phase \( \Phi \) takes the form

\[
\Phi(f) = 2\pi(f - f'_o)\Delta t' + \Delta \phi'' + 5/48 \left[ a(f; f'_o)\Delta k - a_e(f; f'_o)k_e + a_1(f; f'_o)k_1 - a_{3/2}(f; f'_o)k_{3/2} + a_2(f; f'_o)k_2 - a_d(f; f'_o)k_d \right].
\]

(81)

Let us examine the functional behaviour of \( \Phi(f) \) around the frequency \( f'_o \). We find

\[
\Phi(f) \simeq 2\pi(f - f'_o)\Delta t' + \Delta \phi'' + 5/96(f/f'_o - 1)^2\left[ \frac{\Delta k}{(\pi f'_o)^{5/3}} - \frac{157}{24}(\pi f'_o)^{19/9} + \frac{k_e}{(\pi f'_o)^{2/3}} - \frac{k_{3/2}}{(\pi f'_o)^{2/3}} + \frac{k_2}{(\pi f'_o)^{1/3}} - \frac{5}{192}(\pi f'_o)^{7/3} \right] + O((f/f'_o - 1)^3).
\]

(82)

We see that in the above approximation we can make the phase \( \Phi \) vanish to the order \((f/f'_o - 1)^3\) when the following conditions hold

\[
\Delta t_{max} = t'_{Fmax} - t' = 0, \quad \Delta \phi' = \phi''_{Fmax} - \phi'' = 0, \quad \Delta k_{max} = k_{Fmax} - k = 0,
\]

(83)

(84)

(85)

where subscript \( Fmax \) means the value of the parameter of the Newtonian filter that maximizes the correlation. Hence we can expect to match the Newtonian template to the post-Newtonian signal with the Newtonian mass parameter \( k \) shifted from the true value by a certain well-defined amount. The shift depends both on the parameters of the two-body system and the noise in the detector through the frequency \( f'_o \). However the value of the shift in the \( k \) parameter is independent of the choice of the time parameter and phase in the Newtonian filter.

In the following table we have given the numerical values of the shift in the parameter \( k \) calculated from Eq.86 for the 3 binary systems considered in the previous section. We have given three values of the shifts including one (\( \delta k_1 \)), two (\( \delta k_{3/2} \)), and finally three (\( \delta k_2 \)) post-Newtonian corrections.

**Table VII.** Numerical values of the shifts in the mass parameter of the Newtonian filter calculated from the analytic formula (Eq.86).

| Binary | \( \delta k_1 \) | \( \delta k_{3/2} \) | \( \delta k_2 \) |
|--------|-----------------|-----------------|-----------------|
| NS-NS  | 0.03328         | 0.01512         | 0.01597         |
| NS-BH  | 0.01641         | 0.005052        | 0.006023        |
| BH-BH  | 0.004660        | 0.001276        | 0.001775        |

We have also investigated the problem numerically and we have found the maxima to be located at the values of the shifts in the phase, the time, and the mass parameter \( k \) given in Tables VIII A (1st post-Newtonian shift), VIIIB (3\( /2 \) post-Newtonian shift), VIIIC (2nd post-Newtonian shift) below. We have also given the factor \( l \) which is defined as

\[
l = \sqrt{\frac{(h|h_N)}{(h|h)}}
\]

(86)

In a previous work by these authors (43, 44) we have claimed the factor \( l \) to be the drop in the signal-to-noise ratio as a result of using non-optimal (Newtonian) filter. However the signal-to-noise ratio falls as square of the factor \( l \) (see Appendix D). We also give the range of integration over which we calculated the correlation. We have found that the we gain very little by extending the integration beyond that range. For the case of a neutron star binary increasing the range of integration up to 800Hz increases the signal-to-noise ratio by less than 1\%. The reason for this is the effective narrowing of the band of the detector by the chirp signal discussed in the previous section.

---

2We are grateful to T. Apostolatos for pointing this to us.
Table VIII. Numerical values of the factor $l$ and shifts in the parameters of the Newtonian filter with respect to the true values for various post-Newtonian orders calculated numerically by maximizing the correlation function.

A

| Binary | $l_1$ | $\delta k_1$ | $\delta t'$ | $\delta \phi''$ | Range          |
|--------|-------|--------------|--------------|----------------|----------------|
| NS-NS  | 0.68  | 0.03721      | $3.0 \times 10^{-3}$ | 0.61           | 30Hz - 200Hz   |
| NS-BH  | 0.76  | 0.01867      | $1.7 \times 10^{-3}$ | -0.53          | 30Hz - 100Hz   |
| BH-BH  | 0.85  | 0.004931     | $-4.1 \times 10^{-3}$ | -0.20          | 30Hz - 100Hz   |

B

| Binary | $l_{3/2}$ | $\delta k_{3/2}$ | $\delta t'$ | $\delta \phi''$ | Range          |
|--------|-----------|------------------|--------------|----------------|----------------|
| NS-NS  | 0.90      | 0.01564          | $-3.5 \times 10^{-3}$ | -0.40          | 30Hz - 200Hz   |
| NS-BH  | 0.87      | 0.004905         | $0.61 \times 10^{-3}$ | 0.068          | 30Hz - 100Hz   |
| BH-BH  | 0.87      | -0.001219        | $0.39 \times 10^{-3}$ | 0.030          | 30Hz - 100Hz   |

C

| Binary | $l_2$ | $\delta k_2$ | $\delta t'$ | $\delta \phi''$ | Range          |
|--------|-------|--------------|--------------|----------------|----------------|
| NS-NS  | 0.85  | 0.01658      | $-5.5 \times 10^{-3}$ | -0.44          | 30Hz - 200Hz   |
| NS-BH  | 0.87  | 0.006014     | $-1.1 \times 10^{-3}$ | -0.024         | 30Hz - 100Hz   |
| BH-BH  | 0.87  | 0.001789     | $-0.51 \times 10^{-3}$ | -0.018         | 30Hz - 100Hz   |

We see that the agreement between the predicted values of the shifts in the parameters and the numerical values given above is very good. In particular the difference between the predicted values and the values of the shifts for the k parameter obtained numerically differ by less then 5%.

The results of the detailed analysis carried out in [14] show that when the amplitude and phase modulations due to the time dependence of the spin parameters are taken into account then in the worst case $l = 0.63$ for the correlation of the Newtonian filter with the $3/2pN$ signal.

We have also performed the correlation using the signal in the time domain and evaluating the correlation using the fast Fourier transform. We kept the amplitude Newtonian. As we have remarked earlier the restricted post-Newtonian approximation are not equivalent in the frequency and the time domain. So the results are not the same.
Table IX. Numerical values of the $l$ factor and the shifts obtained from the correlation of the Newtonian template with the signal in the time domain at various post-Newtonian orders.

| Binary | $l_1$ | $\delta k_1$ | $l_{3/2}$ | $\delta k_{3/2}$ | $l_2$ | $\delta k_2$ | Range       |
|--------|-------|--------------|-----------|-------------------|-------|--------------|-------------|
| NS-NS  | 0.67  | 0.04097      | 0.97      | 0.01576           | 0.88  | 0.01899      | 30Hz - 200Hz|
| NS-BH  | 0.87  | 0.01916      | 1.00      | 0.004889          | 0.93  | 0.01097      | 30Hz - 100Hz|
| BH-BH  | 0.97  | 0.005130     | 1.00      | 0.001896          | 0.94  | 0.005874     | 30Hz - 100Hz|

We therefore conclude that the Newtonian filter will perform reasonably well in detecting the post-Newtonian signal.

Using the Newtonian filter we would not like to loose any signals. We can achieve this by suitably lowering the detection threshold when filtering the data with the Newtonian filter. By this procedure we would isolate stretches of data where correlation has crossed the lowered threshold. The reduced data would contain all the signals that would be detected with the optimal filter but would also contain false alarms which number would be increased comparing to number of false alarms with the optimal filter. This is the effect of lowering the threshold. The next step would be to analyse the reduced set of data with more accurate templates and the initial threshold to make the final detection.

In Table X we have given examples of the performance of the above procedure. We assume the signal-to-noise ratio threshold $d_T = 5$ and we assume we have 1 signal for the optimal signal-to-noise ratio $d$. $N$ is the expected number of detected signals with the optimal filter, $N_F$ is the number of false alarms, $N_N$ is the number of detected signals with the Newtonian filter, $T_N$ is the lowered threshold, $N_L$ is the number of signals with the lowered threshold and $N_{FL}$ is the number of false alarms with the lowered threshold (see Appendix D for definition of these quantities).
Table X. Comparison of number of true events and false alarms obtained with the optimal filter and the Newtonian filter.

| d  | FF | N  | $N_F$ | $N_N$ | $T_N$ | $N_L$ | $N_{FL}$ |
|----|----|----|-------|-------|-------|-------|----------|
| 15 | .81| 27 | 0.055 | 20    | 4.5   | 28    | 0.16     |
| 15 | .36| 27 | 0.055 | 5.6   | 3.225 | 28    | 2.1      |
| 30 | .81| 225| 1.1   | 165   | 4.5   | 230   | 2.2      |
| 30 | .25| 225| 1.1   | 31    | 2.875 | 229   | 32       |

The theory of filtering with a suboptimal filter is outlined in Appendix D and the terms used in this Section are precisely defined.

We have also calculated the covariance matrix for the parameters estimated with the Newtonian filter. Calculating the second derivatives of the correlation function at the maximum given by the numerical values of the parameters in Table VIII one gets the $\Gamma$ matrix. The inverse gives the covariance matrix. The square roots of a diagonal components of the covariance matrix give lower bounds on the accuracy of determination of parameters with the Newtonian filter and they are approximate rms error for high signal-to-noise ratio as explained in Section 2. The results are summarized in Table XI for our three binary systems located at the distance of 200Mpc. The numbers are given for signals with the currently known post-Newtonian corrections but without the eccentricity and the dipole terms.

Table XI. Accuracy of determination of parameters of the Newtonian filter for the three fiducial binaries located at the distance of 200Mpc.

| Binary    | $\Delta t_{an}$ [ms] | $\Delta k_N [M_{\odot}^{3/5}]$ |
|-----------|----------------------|-------------------------------|
| NS-NS     | 2.9                  | $0.37 \times 10^{-3}$         |
| NS-BH     | 0.53                 | $0.051 \times 10^{-3}$        |
| BH-BH     | 0.22                 | $0.021 \times 10^{-3}$        |

One can easily calculate from Table II that the accuracy of determination of the mass parameter $k$ with the Newtonian filter lies between the accuracy of determination of $k$ for 1 and 3/2 post-Newtonian signal.

In Appendix E we have derived a useful formula for the correlation function based on the approximation to the phase $\Phi$ considered above.

We shall next show that the Newtonian filter can also give a useful estimator characterizing the binary system. From the analytic investigation of the Newtonian filter given above it is clear that we can obtain an estimator of an effective mass parameter $k_E$ of the binary system given approximately by (cf. Eq. 86)

$$k_E = k - \frac{157}{24} \frac{k_c}{(\pi f_0)^{19/9}} + k_1 (\pi f_0')^{2/3} - k_{3/2} (\pi f_0') + k_2 (\pi f_0')^{4/3} - \frac{5}{192} \frac{k_d}{(\pi f_0')^{2/3}}$$

and the numerical investigation has shown that the Newtonian filter will determine the effective mass parameter which numerically value is accurately given by the above analytic formula. The $k_E$ parameter can be used to give an estimate of the chirp mass of the binary system. We define generalized chirp mass $M_g$ as

$$M_g = 1/k_E^{3/5}$$

We have calculated numerically the generalized chirp mass using the analytic formula [87] and we have found that it deviates from the true value by less than 4% for the range of masses from 1.4 to 10 solar mass. For the range of masses from 1.01 to 1.64 which is the expected range of neutron star masses given present observations of binary pulsars [45] the generalized chirp mass is always less than the true one by around 4% but with a very small range of .5% around the average value.

Because of the inequality $m \geq 2^{6/5}M$ and the closeness of the generalized chirp mass to the true chirp mass the generalized chirp mass $M_g$ gives a lower bound on the total mass of the system. Thus from its
estimate we can determine what binary system we observe. Also the R.H.S. of the above inequality gives a poor man’s estimate of the total mass. For the range of masses of \((1M_\odot, 10M_\odot)\) it deviates by 50% from the true value of the total mass but for the range of \((1.01M_\odot, 1.64M_\odot)\) acceptable for neutron star binaries it is only 5% smaller than the true mass.

Another application of this estimate is that it can be used as an additional check on whether we are observing the real signal. If our estimate would deviate unusually from the predicted range of \(M_g\) corresponding to the range of individual masses of \((1M_\odot, 10M_\odot)\) we could veto the detection.

An interesting application of the Newtonian filter would be to determine unexpected effects in the binary interaction that we would not be able to model and introduce into multiparameter numerical templates because we do not know their form. The idea is to use the estimates of the effective mass parameter \(k_E\). Particularly useful would be estimates of \(k_E\) in the case of neutron star binaries. Since the range of the neutron star masses in a binary system is rather narrow the range of the allowable values for the generalized chirp mass will also be narrow. From the analysis in Appendix D the range from the least lower bound and to the greatest upper bound is \((1.01M_\odot, 1.64M_\odot)\) and the range from greatest lower bound to least upper bound is as narrow as \((1.34M_\odot, 1.43M_\odot)\). This implies the respective ranges in \(k_E\) to be \((0.57, 1.26)\) and \((0.79, 0.71)\). From the population of estimates of the parameter \(k_E\) we can determine its probability distribution and also the mean, variance or range of observed values of \(k_E\). One can then compare the observed distribution of \(k_E\) and its characteristics with the ones obtained from observations of the neutron star binaries in our Galaxy or from the theoretical analysis and search for differences. As an example we consider Damour-Esposito-Farèse bi-scalar tensor theory described at the end of Section 2.4. The shift in the Newtonian mass parameter \(k\) due to effects of this theory is given by formula (78).

We have calculated this shift numerically and we have found that for the range of neutron star masses \((1.01M_\odot, 1.64M_\odot)\) and the parameter \(\beta = 1\) (current observational bound) the shift is in the range of \((0.018, 0.022)\). This shift is much larger than rms error in estimation of \(k_E\) of 0.00037 (see Table XI). Consequently the effects of the bi-scalar theory could be determined to an accuracy depending on how well we would know the probability distribution of the neutron star masses and the number of available detections of gravitational waves from binaries.

### 3.2 Post-Newtonian search templates

In a recent work \cite{14} different search templates than the Newtonian filter were recommended and extensively analysed. The proposed templates are the post-Newtonian waveforms with all the spin effects and parameters removed. They have four parameters: amplitude, phase, reduced mass, total mass. We shall denote such search templates by 1PNf, 3/2PNf, 2PNf where the number in front refers to the order of post-Newtonian effects included. In Ref. \cite{14} the fitting factor FF \((FF = l^2\) see Appendix D) of the 3/2PNf search template was calculated and it was concluded that this template family works quite well even for signals with both spin-modulational and the nonmodulated 3/2 post-Newtonian effects combined. In this Appendix we investigate the performance of the 2PNf search template for the case of the 2nd post-Newtonian signal in the approximation considered in Section 2. This means that we ignore all post-Newtonian effects in the amplitudes of both the signal and the template and we assume that the spin-orbit and the spin-spin parameters \(s_o\) and \(s_s\) in the signal are constant. In Table XII we give the factor \(l\) and the shift in the time parameter, phase, reduced mass and total mass for the three representative binary systems described in Section 2. We have also given the shifts in the reduced and the total mass parameters in percentages of their true values.

Table XII. Performance of the 2nd post-Newtonian search template for the three fiducial binaries located at the distance of 200Mpc.

| Binary | \(l\)  | \(\delta \mu\)  | \(\delta \mu\) | \(\delta m\)  | \(\delta m\) | \(\delta t[ms]\) | \(\delta \phi\)  |
|--------|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| NS-NS  | 0.98  | 0.0028         | 0.5%           | -0.017         | 0.61%          | -9.5x10^{-3}  | 0.00027        |
| NS-BH  | 0.95  | 0.52           | 42%            | -4.8           | 42%            | -3.0          | -0.28          |
| BH-BH  | 0.98  | 1.9            | 38%            | -7.8           | 39%            | 2.3x10^{-3}   | -0.00053       |

We see that the 2PNf search template fits the signal better than the Newtonian search template Nf investigated in Section 3.1. There are two reasons for this. The 2PNf template has one more parameter
than Nf template and the phase of 2PNf template has all post-Newtonian frequency evolution terms whereas the phase of the Nf template has only Newtonian frequency evolution $f^{-5/3}$. Also in the case of NS-NS binary which has small spin parameters the expectation values of the estimates of the reduced and the total masses are close to their true values.

The advantage of the Newtonian search template might be its simplicity: it has the least possible number of parameters and hence the least computational time is needed to implement such a template in data analysis algorithms. Before the detailed data analysis schemes are developed for the real detectors it is useful to investigate theoretically a wide range of possible search templates.

We have also calculated the covariance matrix for the 2PNf template. The results are summarized in Table XIII where we have given the rms errors in the time, reduced mass and the total mass parameters of this search template for the three binary systems. We have also given the errors in the reduced and the total mass in percentage of their true values.

### Table XIII. The rms errors in the estimators of the parameters of the 2nd post-Newtonian search template for the three fiducial binary systems located at the distance of 200Mpc.

| Binary   | $\Delta t_{a}$[ms] | $\Delta \mu_{PN}[M_{\odot}]$ | $\Delta m_{PN}[M_{\odot}]$ | $\Delta \mu_{PN}$ | $\Delta m_{PN}$ |
|----------|---------------------|-------------------------------|-----------------------------|-------------------|-----------------|
| NS-NS    | 0.80                | 0.0078                        | 1.1%                        | 0.011             | 0.39%           |
| NS-BH    | 0.40                | 0.012                         | 1.0%                        | 0.0068            | 0.06%           |
| BH-BH    | 0.16                | 0.0090                        | 0.2%                        | 0.0050            | 0.03%           |

We see that the rms errors of the parameters of the post-Newtonian search template are comparable to rms errors obtained with optimal filtering of the signal with spin parameters removed.

### 4 Conclusions

The analysis of the accuracy of estimation of parameters of the 2nd post-Newtonian signal (Section 2.3) has shown that main characteristics of this signal: chirp mass and the time parameter can be estimated to a very good accuracy: chirp mass to 0.1% - 1.0% and time parameter to a quarter of a millisecond for typical binaries. A typical binary consists of compact objects of 1.4 to 10 solar masses and is located at the distance of 200Mpc from Earth and the amplitude of its gravitational wave signal is averaged over all directions and orientations. The signal-to-noise ratio of typical binaries varies from 15 to 77 for the planned advanced LIGO interferometers. However the accuracy of determination of post-Newtonian effects is considerably degraded due to large number parameters: 6 parameters in the phase of the 2nd post-Newtonian signal (Table II). Consequently the errors in determination of the reduced mass and the total mass are large and range from 50% to 200% for typical systems (Table IV). If spin effects could be neglected thereby reducing the number of parameters by 2 the rms errors of estimation of reduced and total masses would have a very impressive value of a fraction of a percent (Table V).

Analysis of the accuracy of estimation of the effects of the dipole radiation in the Jordan-Fiertz-Brans-Dicke theory of gravity has shown that the planned laser interferometric gravitational wave detectors should have ability of testing alternative theories of gravity comparable to that of current observations in the solar system and our Galaxy.

The numerical analysis of Section 2 supports the need for the search templates emphasized in Ref.\[11\]. The results of Section 3 show that the Newtonian filter (a search template with only one mass parameter) will perform reasonably well at least for the case of of constant spin parameters. Such a filter can be used to perform an on line scan of the data to search for the candidates for real signals. The measurement of the mass parameter of the Newtonian signal provides an accurate estimate of an effective mass parameter $k_{E}$ of the binary (see Eq.\[87\]). The value of this parameter gives the information about the binary analogous to the chirp mass in the analysis of the signal in the quadrupole approximation. Moreover this parameter contains information about the post-Newtonian effects and it can contain information about the effects that we cannot at present model for example about the effects due to unknown corrections to
general relativity in the strong field regime. Such information can be extracted if we built a probability
distribution of $k_E$ from its estimators by the Newtonian filter. The post-Newtonian search templates
analysed in [14] perform better than Newtonian filters and considering increasing computational capability
they can also be used in the on line analysis of the data. In the case of large spin parameters it would
be useful to obtain relations of the two mass parameters in such templates to the true masses and spins
similar to relation of the effective mass parameter of the Newtonian filter to the other parameters of the
binary (see Eq. 87). For the case of the observed binary systems, binaries consisting of two neutron
stars with small spin parameters the Newtonian filter will provide an accurate estimate of the chirp mass
whereas the post-Newtonian search templates will provide accurate estimates of reduced and total masses.

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Appendix A: The effects of eccentricity

In this appendix we derive the first order correction due to eccentricity in the phase of the gravitational wave signal from a binary system. The derivation is due to N. Wex [38].

Let \(a\) and \(e\) be respectively the semi-major axis and the eccentricity of the Keplerian orbit of a binary. From the quadrupole formula one obtains the following expressions for the secular changes of \(a\) and \(e\) averaged over an orbit [16]:

\[
\frac{\langle da \rangle}{dt} = -\frac{\beta}{a^3} \left( 1 + \frac{73}{27} e^2 + \frac{37}{180} e^4 \right), \quad \frac{\langle de \rangle}{dt} = -\frac{304 \beta}{15} \frac{e (1 + \frac{121}{360} e^2)}{(1 - e^2)^{7/2}}. \tag{89}
\]

where \(\beta = \frac{64}{5} m^2 \mu\). From these equations we get \(\frac{da}{de}\) which can be integrated with respect to \(e\). The result is:

\[
a(e) = a_0 \frac{\xi(e)}{\xi(e_0)}, \quad \xi(e) \equiv e^{12/19} \left( 1 + \frac{121}{360} e^2 \right)^{870/2299} \frac{1}{1 - e^2} \tag{90}
\]

where \(e_0\) is an arbitrary initial eccentricity and \(a_0 = a(e_0)\). From Kepler’s third law \(\pi f = m^{1/2} a^{3/2}\), where \(f\) is gravitational wave frequency we get an analytic expression for \(f\) as a function of \(e\).

\[
f(e) = f_0 \frac{\eta(e_0)}{\eta(e)}, \quad \eta(e) \equiv e^{18/19} \left( 1 + \frac{121}{360} e^2 \right)^{1305/2299} \frac{1}{(1 - e^2)^{3/2}} \tag{91}
\]

where \(f_0 = f(e_0)\). For small eccentricities we find

\[
e = e_0 \left( \frac{f}{f_0} \right)^{-19/18} [1 + \mathcal{O} \left( e_o^2 \right)] \tag{92}
\]

Thus to first order in \(e\) the quantity \(I_e = e_0^2 f_0^{19/9}\) is a constant. We call \(I_e\) the asymptotic eccentricity invariant. The characteristic time for the evolution of the binary system is given by

\[
\tau_e := \frac{f}{\frac{df}{dt}} = f \left( \frac{df}{da} \frac{da}{dt} \right)^{-1}. \tag{93}
\]

From Kepler’s third law we find

\[
\tau_e = \frac{5}{96} \frac{1}{\mu^{2/3} m^{2/3}} \frac{1}{(\pi f)^{8/3}} \left( 1 - \frac{24}{27} e^2 + \frac{37}{180} e^4 \right) \tag{94}
\]

For small eccentricities \(e\) we get

\[
\tau_e = \frac{5}{96} \frac{1}{\mu^{2/3} m^{2/3}} \frac{1}{(\pi f)^{8/3}} \left[ 1 - \frac{157}{24} e^2 + \mathcal{O} \left( e^4 \right) \right] \tag{95}
\]

Therefore using Eq.92 we can express the characteristic time with first order correction due to eccentricity as

\[
\tau_e = \frac{5}{96} \frac{1}{\mu^{2/3} m^{2/3}} \frac{1}{(\pi f)^{8/3}} \left[ 1 - \frac{157}{24} e_0^2 \left( \frac{f}{f_0} \right)^{-19/9} \right]. \tag{96}
\]

The phase of the Fourier transform of the signal in the stationary phase approximation is given by

\[
\varphi[f] = 2\pi ft - \varphi_i - \pi/4 - 2\pi \int_t^f \tau_e(f')(1 - f / f') \, df =
= 2\pi ft_a - \varphi + \frac{1}{128 \mu m^{2/3}} \times \left[ \left( \frac{3}{(\pi f)^{5/3}} + \frac{5\pi f}{(\pi f_a)^{8/3}} - \frac{8}{(\pi f_a)^{5/3}} \right) \right. \times \left. \left( \frac{9}{(\pi f)^{4/3}} + \frac{34\pi f}{(\pi f_a)^{13/3}} - \frac{43}{(\pi f_a)^{4/3}} \right) \right] \tag{97}
\]

where \(f_a = \frac{f_0}{f_0} = \frac{m^{1/2} a^{3/2}}{m^{1/2} a^{3/2}}\).
and consequently the Fourier transform of our signal in the stationary phase approximation has the form

\[ \hat{h}(f) = Af^{-7/6} \exp \left[ 2\pi ft_a - \varphi - \pi/4 + \frac{5}{48} a(f; f_a)k + a_e(f; f_a)k_e \right] , \quad \text{for } f > 0. \]  

(98)

(and by the complex conjugate of the above expression for \( f < 0 \)) where

\[ \hat{A} = \frac{1}{30^{1/2}} \frac{1}{\pi^{2/3}} \frac{\mu^{1/2} m^{1/3}}{R} , \]

\[ k = \frac{1}{\mu m^{2/3}} , \quad k_e = \frac{1}{\mu m^{2/3}} \tilde{e}_0(\pi f_0)^{19/9} , \]

\[ a(f; f_a) = \frac{9}{40} \frac{1}{(\pi f)^{5/3}} + \frac{3}{8} \frac{\pi f}{(\pi f_a)^{8/3}} - \frac{3}{5} \frac{1}{(\pi f_a)^{5/3}} , \]

\[ a_e(f; f_a) = -\frac{157}{24} \left( \frac{81}{1462} (\pi f)^{34/9} + \frac{9}{43} (\pi f_a)^{43/9} - \frac{9}{34} \frac{1}{(\pi f_a)^{44/9}} \right) . \]

We have investigated the accuracy of measurements of parameters of the above signal with first order eccentricity contribution. We have considered neutron star/neutron star binary. The results are summarized in Table XIV.

**Table XIV** The rms errors of the parameters of the signal with first order contribution due to eccentricity for a binary of two neutron stars of 1.4 solar mass each at the distance of 200Mpc.

| S/N | \( \Delta t_e [\text{ms}] \) | \( \Delta \phi_e \) | \( \Delta \mu/\mu \) | \( \Delta m/m \) | \( \Delta k_e [M_\odot^{-5/3}(100\text{Hz})^{19/9}] \) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 15  | 0.56            | 1.2             | 0.50\%          | 0.74\%          | 3.6 \times 10^{-7} |

However for the currently observed binaries the eccentricity invariant \( I_e \) is extremely small. For Hulse-Taylor pulsar \( I_e = 1.8 \times 10^{-13}[M_\odot^{-5/3}100\text{Hz}^{19/9}] \). We have the following numerical values.

\[ k_e = 1.3 \times 10^{-13}(\frac{I_e}{1.8 \times 10^{-13}})(\frac{1.2}{M_\odot})^{5/3}[M_\odot^{-5/3}100\text{Hz}^{19/9}] \]

(103)

\[ \frac{\Delta k_e}{k_e} = 2.8 \times 10^6 r_{200\text{Mpc}}(\frac{I_e}{1.8 \times 10^{-13}})(\frac{M_\odot}{1.2})^{5/3} , \]

(104)

where \( r_{200\text{Mpc}} \) is distance in 200Mpc. Thus for eccentricity effects to be measured one would need extremely short period binaries of high eccentricity. Such binaries could perhaps occur in the center of a galaxy or be created as a result of some supernova explosions.

**Appendix B: Covariance matrices at various post-Newtonian orders**

In this Appendix we give the numerical values of the covariance matrices at various post-Newtonian orders for the reference binary. The reference binary has the chirp mass \( M \) of 1 solar mass and is located at the distance of 100Mpc. We only give reduced covariance matrices i.e. covariance matrices for the phase parameters. As indicated in Section 2 the estimator of the amplitude parameter is uncorrelated with phase parameters. The integration range in the Fisher matrix integrals was taken to be from 10Hz to infinity and the spectral density of advance LIGO detectors was assumed (Eq.25). The frequency \( f_a \) was chosen such that the rms error in the time parameter is minimum. The minimum frequency is denoted by \( f_m \) and its numerical value is given for each covariance matrix. The subscripts N, 1PN, 3/2PN, 2PN, 2PNe refer to signal including quadrupole radiation, 1st post-Newtonian correction, 3/2 post-Newtonian correction, 2nd post-Newtonian correction, and 1st order effect due to eccentricity respectively. The order of parameters in the matrices is the following: \( t_e, \phi_e, k, k_1, k_3/2, k_2, k_e \).

\[ f_m = 70\text{Hz} \]

\[ C_N = \begin{pmatrix} 1.96 \times 10^{-8} & 8.2 \times 10^{-6} & 1.21 \times 10^{-10} \\ 8.2 \times 10^{-6} & 0.00537 & 2.08 \times 10^{-7} \\ 1.21 \times 10^{-10} & 2.08 \times 10^{-7} & 6.65 \times 10^{-11} \end{pmatrix} \]

(105)
\[
f_{m}^{1PN} = 100 \text{Hz}
\]
\[
C_{1PN} = \begin{pmatrix}
2.24 \times 10^{-8} & 9.93 \times 10^{-6} & 8.87 \times 10^{-10} & -6.61 \times 10^{-8} \\
9.93 \times 10^{-6} & 0.00754 & -6.28 \times 10^{-7} & 0.000147 \\
8.87 \times 10^{-10} & -6.28 \times 10^{-7} & 1.4 \times 10^{-9} & -1.97 \times 10^{-7} \\
-6.61 \times 10^{-8} & 0.000147 & -1.97 \times 10^{-7} & 0.0000289
\end{pmatrix}
\] (106)

\[
f_{m}^{3/2PN} = 160 \text{Hz}
\]
\[
C_{3/2PN} = \begin{pmatrix}
3.41 \times 10^{-8} & 0.00002 & -9.65 \times 10^{-9} & 3.36 \times 10^{-6} & 0.0000212 \\
0.00002 & 0.0191 & -9.34 \times 10^{-7} & -0.000403 & -0.007 \\
-9.65 \times 10^{-9} & -9.34 \times 10^{-7} & 2.3 \times 10^{-8} & -8.71 \times 10^{-6} & -0.000622 \\
3.36 \times 10^{-6} & -0.000403 & -8.71 \times 10^{-6} & 0.00349 & 0.0253 \\
0.0000212 & -0.007 & -0.000622 & 0.0253 & 0.185
\end{pmatrix}
\] (107)

\[
f_{m}^{2PN} = 100 \text{Hz}
\]
\[
C_{2PN} = \begin{pmatrix}
5.62 \times 10^{-8} & 0.0000286 & 2.39 \times 10^{-9} & -9.48 \times 10^{-6} & -0.001999 & -0.00102 \\
0.0000286 & 0.0189 & 0.000016 & -0.0162 & -0.26 & -1.11 \\
2.39 \times 10^{-9} & 0.000016 & 2.3 \times 10^{-7} & -0.000162 & -0.00219 & -0.00796 \\
-9.48 \times 10^{-6} & -0.0162 & -0.000162 & 0.116 & 1.59 & 5.86 \\
-0.001999 & -0.26 & -0.00219 & 1.59 & 22 & 81.5 \\
-0.00102 & -1.11 & -0.00796 & 5.86 & 81.5 & 305.
\end{pmatrix}
\] (108)

\[
f_{m}^{2PN_{c}} = 120 \text{Hz}
\]
\[
C_{2PN_{c}} = \begin{pmatrix}
6.36 \times 10^{-8} & 0.0000369 & -3.56 \times 10^{-8} & 4.9 \times 10^{-7} & -0.000135 & -0.000977 & -4.86 \times 10^{-11} \\
0.0000369 & 0.0279 & 0.000035 & -0.0315 & -0.474 & -1.89 & -5.21 \times 10^{-9} \\
-3.56 \times 10^{-8} & 0.000035 & 2.36 \times 10^{-6} & -0.00121 & -0.0143 & -0.0464 & 1.33 \times 10^{-9} \\
4.9 \times 10^{-7} & -0.0315 & -0.00121 & 0.63 & 7.57 & 24.7 & -6.52 \times 10^{-7} \\
-0.000135 & -0.474 & -0.0143 & 7.57 & 91.5 & 301. & -7.59 \times 10^{-6} \\
-0.000977 & -1.89 & -0.0464 & 24.7 & 301. & 999. & -0.000024 \\
-4.86 \times 10^{-11} & -5.21 \times 10^{-9} & 1.33 \times 10^{-9} & -6.52 \times 10^{-7} & -7.59 \times 10^{-6} & -0.000024 & 8.28 \times 10^{-13}
\end{pmatrix}
\] (109)

**Appendix C : Coefficients in the Damour-Esposito-Farése biscalar \(T(\beta', \beta'')\) theory**

The coefficients \(\kappa_{o}, \kappa_{q}, \kappa_{d1}, \kappa_{d2}\) in the shift of the Newtonian mass parameter \(k\) due to the biscalar \(T(\beta', \beta'')\) theory (Eq.78 in Section 3.4) are given by the following formulae

\[
\kappa_{o} = \frac{1}{2} \beta' B(C_{1}^{2} + C_{2}^{2}),
\]

(110)

\[
\kappa_{q} = \beta' B(C_{1}^{2} x_{2} + C_{2}^{2} x_{1}),
\]

(111)

\[
\kappa_{d1} = \frac{1}{2} \beta' B(C_{1}^{2} x_{1} - C_{2}^{2} x_{2})(x_{1} - x_{2}),
\]

(112)

\[
\kappa_{d2} = (ab_{121} - ab_{221})x_{1} + (ab_{212} - ab_{112})x_{2},
\]

(113)

where

\[
x_{1} = \frac{m_{1}}{m},
\]

(114)

\[
x_{2} = \frac{m_{2}}{m},
\]

(115)

and constant \(A\) and \(B\) have the values

\[
A = 2.1569176, \quad B = 1.0261529.
\]

(116)
\( C_1 \) and \( C_2 \) are sensitivities of the two bodies to changes of the scalar field. The functions \( ab \) are given by

\[
\begin{align*}
\text{ab}_{121} &= \beta'(-C_2 - BC_1^2 + (A - 3B)C_2^2 - (A - B)2C_2C_1^2 + (2A^2 - 7AB + 5B^2)C_1^2C_2^2) + \beta^2B^2(-3C_1^3 + 2C_1^2C_2^2 + C_1^4 + \frac{1}{2}C_2C_1^4 + AC_2^2C_1^4) + \frac{1}{2} \beta''BC_2^2,
\text{ab}_{212} &= \beta'(-C_1 - BC_2^2 + (A - 3B)C_1^2 - (A - B)2C_1C_2^2 + (2A^2 - 7AB + 5B^2)C_2^2C_1^2) + \beta^2B^2(-3C_2^3 + 2C_2^2C_1^2 + C_2^4 + \frac{1}{2}C_1C_2^4 + AC_1^2C_2^4) + \frac{1}{2} \beta''BC_1^2,
\text{ab}_{221} &= \beta'(-C_2 - \frac{1}{2}B(C_1^2 + C_2^2) + (A - 3B)C_2^2 - (A - B)2C_2C_1^2 + (\frac{1}{2}(2A^2 - 7AB + 5B^2)C_2^2(C_1^2 + C_2^2)) + \beta^2B^2(-3C_2^3 + C_2^2(C_1^2 + C_2^2)) + C_2^4 + \frac{1}{2}C_1C_2^2 + AC_1^2C_2^2) + \frac{1}{2} \beta''BC_2^2,
\text{ab}_{112} &= \beta'(-C_2 - \frac{1}{2}B(C_1^2 + C_2^2) + (A - 3B)C_1^2 - (A - B)2C_1C_2^2 + (\frac{1}{2}(2A^2 - 7AB + 5B^2)C_1^2(C_1^2 + C_2^2)) + \beta^2B^2(-3C_1^3 + C_1^2(C_1^2 + C_2^2)) + C_1^4 + \frac{1}{2}C_1C_2^2 + AC_1^2C_1^2) + \frac{1}{2} \beta''BC_1^2.
\end{align*}
\]

For a detailed exposition of the theory the reader should consult Ref. [37].

**Appendix D: Detection of the known signal with a non-optimal filter**

Suppose that we would like to know whether or not in a given data set \( x \) there is present a signal \( h \). We assume that the noise \( n \) in the data is additive. There are two alternatives:

\[
\begin{align*}
\text{NO SIGNAL} & : x = n \\
\text{SIGNAL} & : x = h + n
\end{align*}
\]

A standard method to determine which of the two alternatives holds is to perform the Neyman-Pearson test [28]. This test consists in comparing the likelihood ratio \( \Lambda \), the ratio of probability density distributions of the data \( x \) when the signal is present and when the signal is absent, with a threshold. The threshold is determined by the false alarm probability that we can tolerate (the false alarm probability is the probability of saying that the signal is present when there is no signal). The test is optimal in the sense that it maximizes the probability of detection of the signal. In the case of Gaussian noise and deterministic signal \( h \) the logarithm of \( \Lambda \) is given by

\[
\ln \Lambda = (x|h) - \frac{1}{2}(h|h).
\]

Thus in this case the optimal test consists of correlating the data with the expected signal and it is equivalent to comparing the correlation \( G := (x|h) \) with a threshold. The probability distributions \( p_0 \) and \( p_1 \) of \( G \) when respectively the signal is absent and present are given by

\[
\begin{align*}
p_0(G; d) &= \frac{1}{\sqrt{2\pi d^2}} \exp \left[ -\frac{G^2}{2d^2} \right], \\
p_1(G; d) &= \frac{1}{\sqrt{2\pi d^2}} \exp \left[ -\frac{(G - d)^2}{2d^2} \right],
\end{align*}
\]

where \( d \) is the optimal signal-to-noise ratio \( d^2 = (h|h) \) and we assumed that the noise is a zero mean Gaussian process.

Let \( T \) be a given threshold. This means that we say that the signal is present in a given data set if \( G > T \). The probabilities \( P_F \) and \( P_D \) of false alarm and detection respectively are given by

\[
\begin{align*}
P_F(T, d) &= \int_T^\infty p_0(G; d) \, dG, \\
P_D(T, d) &= \int_T^\infty p_1(G; d) \, dG.
\end{align*}
\]
In the Gaussian case they can be expressed in terms of the error functions.

\[ P_F(d_T, d) = \frac{1}{2} \text{erfc}(\frac{d_T^2}{\sqrt{2}d}), \]  
(127)

\[ P_D(d_T, d) = \frac{1}{2}(1 + \text{erf}(\frac{d_T^2 - d^2}{\sqrt{2}d})), \]  
(128)

where \text{erf} and \text{erfc} are error and complementary error functions respectively \[17\]. and we have introduced for convenience the quantity \(d_T := \sqrt{T}\) that we call the \textit{threshold signal-to-noise ratio}. In practice we adopt a certain value of the false alarm probability that we can accept and from formula (127) we calculate the detection threshold \(T\).

Let \(F\) be a linear filter and let \(n\) be the additive noise in data \(x\) then

\[ (x|F) = (s|F) + (n|F). \]  
(129)

The signal-to-noise (S/N) ratio is defined by

\[ (S/N)^2 := \frac{E_1[(s|F)^2]}{E_1[(n|F)^2]} = \frac{(s|F)^2}{(F|F)}. \]  
(130)

where \(E_1\) means expectation value when the signal is present. By Schwartz inequality we immediately see that (S/N) is maximal and equal to \(d\) when the linear filter is matched to the signal i.e. \(F = h\). This is another interpretation of the matched filter - it maximizes the signal-to-noise ratio over all linear filters \[28\]. However when the noise is not Gaussian the matched filter is not the optimal filter; it does not maximize probability of detection of the signal. We see that in the case of Gaussian noise the problem of detecting a known signal by optimal filter is determined by one parameter - the optimal signal-to-noise ratio \(d\).

Suppose that because of certain restrictions of practical nature we cannot afford to use the optimal filter \(h\) and we use a suboptimal one - \(h_N\) which is not perfectly matched to the signal. Thus \((h|h_N) < (h|h)\). We denote \(\sqrt{(h|h_N)}\) by \(d_o\) and we assume that \((h_N|h_N) = (h|h) = d^2\). Our suboptimal correlation function is given by \(G_N = (x|h_N)\) and its probability distributions \(p_{NO}\) and \(p_{N1}\) when respectively the signal is absent and present are given by

\[ p_{NO}(G_N;d) = \frac{1}{\sqrt{2\pi d^2}} \exp \left[ \frac{G_N^2}{d^2} \right], \]  
(131)

\[ p_{N1}(G_N;d, d_o) = \frac{1}{\sqrt{2\pi d^2}} \exp \left[ \frac{(G_N - d_o^2)^2}{d^2} \right]. \]  
(132)

We see that the suboptimal detection problem is determined by two parameters - \(d\) and \(d_o\), square roots of the expectation values of the optimal and suboptimal correlations when the signal is present. The false alarm and detection probabilities as in the optimal case can be expressed in terms of the error functions.

\[ P_F(d_T, d) = \frac{1}{2} \text{erfc}(\frac{d_T^2}{\sqrt{2}d}), \]  
(133)

\[ P_D(d_T, d, d_o) = \frac{1}{2}(1 + \text{erf}(\frac{d_T^2 - d_o^2}{\sqrt{2}d})). \]  
(134)

We see that the probability of false alarm for the suboptimal case is the same as in the optimal case however the probability of detection in the suboptimal case is always less than the probability of detection in the optimal case since \(d_o < d\) and the error function \(\text{erf}(x)\) is an increasing function of the argument \(x\). The signal-to-noise ratio in the case of suboptimal linear filter \(h_N\) is given by

\[ (S/N)^2 = \frac{(h|h_N)^2}{(h_N|h_N)} = d^2 \left(\frac{d_o}{d}\right)^4. \]  
(135)

Let us denote the ratio \(d_o/d\) by \(l\). The ratio \(l\) measures the drop in the expectation value of the correlation function as a result of non-optimal filtering. We see that due to suboptimal filtering the signal-to-noise
ratio decreases by square of the factor $l$. We denote $l^2$ by FF and following Ref.[14] call it the fitting factor.

In our considerations we need to calculate the number of events that will be detected by linear filtering. We shall make a number of simplifying assumptions. We shall assume a Euclidean universe where in the sphere of radius $r_o$ we have one source and that at the distance $r_o$ the optimal signal-to-noise ratio is $d$. Moreover we shall assume that the magnitudes of the signal $h$ and the suboptimal filter $h_N$ are inversely proportional to the distance $r$ from the source. Then the square roots $d_r$ and $d_{or}$ of the expectation values of the optimal and suboptimal correlations at the distance $r$ are given by

\begin{align}
    d_r &= \frac{r_o}{r} d, \\
    d_{or} &= \frac{r_o}{r} d_o.
\end{align}

We assume that the sources are uniformly distributed in space. The expected number of detected real events $N$ and $N_N$ in the optimal and the suboptimal case respectively is given by

\begin{align}
    N(d_T, d) &= 4\pi \int_0^\infty r^2 P_D(d_T, d_r) \, dr = 3 \int_0^\infty x^2 P_D(d_T, d/x) \, dx \quad (138) \\
    N_N(d_T, d, d_o) &= 4\pi \int_{r_o}^\infty r^2 P_{ND}(d_T, d, d_o/r) \, dr = 3 \int_{r_o}^\infty x^2 P_{ND}(d_T, d, d_o/x) \, dx. \quad (139)
\end{align}

The assumptions that led to the above formulae mean that we neglect general relativistic, cosmological and evolutionary effects. Because of the noise even if there is no signal there is always a non zero probability that the correlation function crosses the threshold. Thus there will be a certain number $N_F$ of false events. For a given optimal signal-to-noise ratio and a threshold $d_T$ this number is the same for both the optimal and suboptimal filter and it is given by

\begin{align}
    N_F &= 4\pi \int_{r_o}^\infty r^2 P_F(d_T, d/r) \, dr = 3 \int_{r_o}^\infty x^2 P_F(d_T, d/x) \, dx. \quad (140)
\end{align}

We observe that in the Gaussian case the integrals in the formulae (138), (139), and (141) are convergent even though we integrate over the all infinite Euclidean volume.

**Appendix E: An approximate formula for the correlation function.**

In this Appendix we shall derive an approximate formula for the correlation integral. Let us consider the expression for the correlation function given by (36). The integrand of the correlation integral is the product of the integrand of the signal-to-noise integral $Ind(f)$ considered in Section 2 and oscillating factor. We know that the $Ind(f)$ is a fairly sharply peaked around a certain frequency $f_o'$ consequently to obtain a reasonable approximation we expand the phase around the frequency $f'_o$. Keeping only the terms to the second order we get

\begin{align}
    \Phi(f) &\simeq 2\pi(f - f'_o)\Delta t' - \Delta \phi'' + \frac{5}{96} \frac{(f/f'_o - 1)^2}{(\pi f'_o)^{5/3}} \Delta k_T + + O[(f/f'_o - 1)^3], \quad (141) \\
    \phi''_o &= \phi_o - 2\pi f'_o t'_o \quad (142)
\end{align}
and
\[ k_E = k - \frac{157}{24} \frac{k_e}{(\pi f'_o)^{1/3}} + k_1(\pi f'_o)^{2/3} - k_3(\pi f'_o)^{4/3} - \frac{5}{192} \frac{k_D}{(\pi f'_o)^{2/3}}. \] (143)

We shall call \( k_E \) an effective mass parameter. \( \Delta k_E \) is the difference in the effective mass parameter of the signal and the filter. Thus in the above approximation the post-Newtonian signal can be parametrized by one effective mass parameter \( k_E \). In other words the dimension of the parameter space of the filters is effectively reduced. This last interpretation has been emphasized in [44] where 1st post-Newtonian corrections to the phase were considered. The mass parameter estimated by Newtonian filter considered in Section 3 is just the effective mass parameter. We stress that the parameter \( k_E \) depends not only on the parameters of the two-body system but also on the characteristic frequency \( f'_o \) of the noise in the detector.

The next step is to obtain a manageable approximation to the function \( Ind(f) \). We approximate it by a Gaussian function with the mean equal to the frequency \( f'_o \) and the standard deviation equal to the HWHM \( \sigma'_o \) of the function \( Ind(f) \). We extend the range of integration from \(-\infty\) to \(+\infty\). We introduce a normalization factor such that the integral of the approximate integrand is equal to the optimal signal-to-noise ratio \( d \). It is then useful to introduce a reduced correlation integral \( H' = H/d^2 \) where \( d \) is the S/N ratio. Thus our approximate formula for the reduced correlation integral takes the form
\[
H'_a = \frac{1}{2\pi \sigma_o^2} \int_{-\infty}^{+\infty} \exp[-(f-f'_o)/(2\sigma_o^2)] \cos[2\pi(f-f'_o)\Delta t' + \Delta \phi'' + \frac{5}{96} (f/f'_o - 1)^2 \Delta k_{TE}] d\phi''
\] (144)

The above integral can be done analytically. It is convenient to introduce the following new variables and new parameters
\[
y = \frac{f - f'_o}{\sqrt{2}\sigma'_o}, \quad \vartheta'' = \Delta \phi'' \quad (145)
\]
\[
\vartheta'' = \Delta \phi'' \quad (146)
\]
\[
\tau = 2\pi \Delta t' \sqrt{2}\sigma'_o \quad (147)
\]
\[
\kappa = \frac{5}{96} \frac{\Delta k_E}{f'_o (\pi f'_o)^{5/3}} 2\sigma_o^2 \quad (148)
\]

then our integral takes a simple form
\[
H'_a(\vartheta'', \tau, \kappa) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp[-y^2] \cos[-\vartheta'' + \tau y + \kappa y^2] dy. \quad (149)
\]

We see that in the new variables introduced above the reduced correlation integral is independent of the characteristics of the integrand \( Ind(f) \) i.e. \( f'_o \) and \( \sigma'_o \). The analytic formula for the function \( H'_a(\vartheta'', \tau, \kappa) \) is given by
\[
H'_a(\vartheta'', \tau, \kappa) = \frac{1}{(1 + \kappa^2)^{1/4}} \exp[-\frac{\tau^2}{4(1 + \kappa^2)}] \cos[1/2(\arctan \kappa - \frac{\kappa \tau^2}{2(1 + \kappa^2)}) - \vartheta''] \quad (150)
\]

By appropriate transformations given in Section 2 we can obtain approximate formulae to the correlation integral for an arbitrary choice of the time and the phase parameters. Let us first consider the transformation given by Eq.(133). In the coordinates introduced above it takes the form
\[
\vartheta'' = \vartheta - \frac{1}{\sqrt{2}} \tau \rho \quad (151)
\]

Then the approximate formula for the correlation function is given by
\[
H'_a(\vartheta, \tau, \kappa) = \frac{1}{(1 + \kappa^2)^{1/4}} \exp[-\frac{\tau^2}{4(1 + \kappa^2)}] \cos[1/2(\arctan \kappa - \frac{\kappa \tau^2}{2(1 + \kappa^2)}) + \frac{1}{\sqrt{2}} \tau \rho - \vartheta] \quad (152)
\]

We see that in these new coordinates for \( \kappa = 0 \) the correlation function oscillates with the maxima at the discrete values of \( \tau \) coordinate given by
\[
\tau_{\text{max}} = \frac{2\sqrt{2}\pi}{\rho}, \quad (153)
\]
where

$$\rho = f'_o/\sigma_o$$  \hspace{1cm} (154)$$

and \(n\) is an integer. In the original coordinates Eq.(153) takes the form \(\Delta t = 1/f'_o\). Thus the correlation integral oscillates with the period determined by the characteristic frequency of the noise of the detector \(f'_o\).

The expressions for the correlation function for different choice of the time and the phase parameters can be obtained by the following transformations. These are transformations given by Eqs.(39) and expressed in our dimensionless coordinates.

$$\theta = \theta' + \frac{3}{5} \kappa \rho^2 (1 - \delta^{5/3}),$$  \hspace{1cm} (155)

$$\tau = \tau' + \frac{3}{4\sqrt{2}} \kappa \rho' (1 - \delta^{8/3})$$  \hspace{1cm} (156)

where

$$\delta = f'_o/f_a.$$  \hspace{1cm} (157)

From the approximate formula for the correlation function obtained above we see that the correlation is given by the product of an oscillating cosine function and an envelope. In the cosine function there are oscillations with the period of \(1/f'_o\). The envelope function is exponentially damped if we move away from the maximum at the center except for the direction given by \(\tau = 0\) along which the damping is least. The equation of the ridge \(\tau = 0\) in the primed coordinates is given by

$$\tau' = -\frac{3}{4\sqrt{2}} \kappa \rho' (1 - \delta^{8/3}),$$  \hspace{1cm} (158)

and in the original coordinates it takes the form

$$\Delta t = \frac{5}{256} \frac{1}{(\pi f'_o)^{8/3}} (1 - \left(\frac{f_a}{f'_o}\right)^{8/3}) \Delta k_E.$$  \hspace{1cm} (159)

Consequently we conclude that the general appearance of the correlation function in coordinates \(\Delta t'\) and \(\Delta k\) is a series of peaks aligned along a straight line given by Eq.(158) above and occurring with the period \(1/f'_o\) in the time coordinate. Numerical investigation shows that the correlation integral exhibits these properties and that our analytic formula reproduces qualitatively its behaviour. The approximate formula obtained above may be a useful tool for developing algorithms to recognize the chirp signal in a noisy data set.
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$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

and the complementary error function $erfc(x)$ is defined by

$$erfc(x) = 1 - erf(x).$$