Dynamical Symmetries in Supersymmetric Matrix Models

V. Bach1  J. Hoppe2  D. Lundholm3

Abstract
We reveal a dynamical SU(2) symmetry in the asymptotic description of supersymmetric matrix models. We also consider a recursive approach for determining the ground state, and point out some additional properties of the model(s).

1 Introduction
Ten years ago, a lot of effort was put into the question of zero-energy states in SU(N)-invariant supersymmetric matrix models. While the attempt to explicitly construct such a state mainly used a space-independent (but manifest SO(d)-invariance-breaking) decomposition of the fermions into creation and annihilation operators, the asymptotic form of the wave function was determined with the help of space-dependent fermions.

In this paper we would like to point out a dynamical SU(2) symmetry and the formation of 'Cooper pairs' that arise in the SO(d)-breaking formulation when diagonalizing certain ingredients of the fermionic part of the Hamiltonian. We start by considering the asymptotic SU(2) theory but note that several features extend to the non-asymptotic and general SU(N) cases.

2 Asymptotic form of the Hamiltonian
The bosonic configuration space is a set of \( d = 2, 3, 5, \) or 9 traceless hermitian matrices \( X_s \), corresponding to the Lie algebra of the gauge group SU(N). For simplicity, we start by taking \( N = 2 \). Coordinatizing regions where the bosonic potential,

\[
V = -\frac{1}{2} \sum_{s \neq t=1} \text{tr} \ [X_s, X_t]^2,
\]

is zero by (cp. e.g. [3, 4])

\[
X_t = r \cos \theta \tilde{E}_t \left( \frac{1}{2} e_A \sigma_A \right), \quad t = 1, \ldots, d - 2
\]

\[
X_{d-1} + iX_d = r \sin \theta e^{i\phi} \left( \frac{1}{2} e_A \sigma_A \right),
\]

\[1, \text{Supported by the Swedish Research Council}
\[2\text{vbach@mathematik.uni-mainz.de}
\[3\text{hoppe@math.kth.se}
\[4\text{dogge@math.kth.se}
where \( e^2 = 1 \), \( \sum_{i=1}^{d-2} \hat{E}_i^2 = 1 \) and \( \sigma_A \) are the Pauli matrices, \( \frac{1}{\varrho} \) times the effective asymptotic Hamiltonian (cf. [1]) becomes

\[
H^\infty = H^\infty_B + 2 \cos \theta \left( -i e \epsilon_{ABC} \right) \Gamma_{\alpha\beta} \lambda_\alpha A \partial_{\lambda_\beta B} + \sin \theta \ e^{i\varphi} \left( e \epsilon_{ABC} \right) \lambda_\alpha A \lambda_\alpha B + \sin \theta \ e^{-i\varphi} \left( e \epsilon_{ABC} \right) \partial_{\lambda_\alpha B} \partial_{\lambda_\alpha A}.
\]

The last three terms are the leading ones in the fermionic part of the Hamiltonian (as \( r \to \infty \)) while \( H^\infty_B \), which arises from \( -\Delta + V \) in that limit, denotes an independent set of harmonic oscillators on \( \mathbb{R}^d \) with ground state energy \( s_d = 2, 4, 8, \) or \( 16 \). \( \Gamma := \sum_{i=1}^{d-2} \hat{E}_i \Gamma_i \) is a purely imaginary, antisymmetric, and hence self-adjoint \( \frac{d}{2} \times \frac{d}{2} \) matrix squaring to unity. \( \lambda_\alpha A \) and \( \partial_{\lambda_\alpha A} = \lambda_\alpha A^\dagger \) are space-independent fermion creation resp. annihilation operators satisfying

\[
\{ \lambda_\alpha A, \partial_{\lambda_\beta B} \} = \delta_{\alpha\beta} \delta_{AB},
\]

\[
\{ \lambda_\alpha A, \lambda_\beta B \} = \{ \partial_{\lambda_\alpha A}, \partial_{\lambda_\beta B} \} = 0
\]

and acting on the fermionic vacuum state \( |0\rangle \), defined by \( \partial_{\lambda_\alpha A} |0\rangle = 0 \) \( \forall \alpha, A \).

We define space-dependent fermion creation operators (for \( d = 3, 5, 9 \))

\[
\lambda_{\alpha j} := (\tilde{\epsilon}_{\alpha j})_\alpha (n_{\tau}) A \lambda_\alpha A,
\]

where \( \sigma, \tau \) denote + or - , and \( n_{\pm} \in \mathbb{C}^3 \) resp. \( \tilde{\epsilon}_{\alpha j} \in \mathbb{C}^{s_d/2} \) are eigenvectors of \( (-i e \epsilon_{ABC}) \) resp. \( \Gamma_{\alpha\beta} \).

\[
\frac{i e \times n_\pm}{\pm} = n_\pm
\]

\[
\Gamma \tilde{\epsilon}_{\pm j} = \pm \tilde{\epsilon}_{\pm j}, \quad j = 1, \ldots, s_d.
\]

We choose these to depend continuously on \( e \) and \( \hat{E} \), as well as to be orthonormal and such that the complex conjugates \( (n_{\pm})^* = n_{\mp} \) and \( (\tilde{\epsilon}_{\pm j})^* = \tilde{\epsilon}_{\mp j} \). The asymptotic Hamiltonian \( H^\infty \), when acting on the ground state of \( H^\infty_B \), can then be written as

\[
H = H_0 + H_+ + H_-,
\]

\[
H_0 = s_d + 2 \cos \theta \sum_j (N_{A_j} - N_{B_j}),
\]

\[
H_+ = 2 \sin \theta \sum_j (A_j + B_j),
\]

\[
H_- = 2 \sin \theta \sum_j (A_j^\dagger + B_j^\dagger),
\]

where

\[
A_j := i e^{i\varphi} \lambda_{+j} + \lambda_{-j},
\]

\[
B_j := i e^{i\varphi} \lambda_{-j} + \lambda_{+j},
\]

satisfy

\[
[A_j, A_j^\dagger] = N_{A_j} - 1 := \lambda_{+j} + \partial_{\lambda_{+j}} + \lambda_{-j} \partial_{\lambda_{-j}} - 1,
\]

\[
[B_j, B_j^\dagger] = N_{B_j} - 1 := \lambda_{-j} + \partial_{\lambda_{-j}} + \lambda_{+j} \partial_{\lambda_{+j}} - 1.
\]

For the \( d=2 \) case we instead of (5) define \( \lambda_{\pm} := (n_{\pm})_A \lambda_A \) and the corresponding expressions for the asymptotic Hamiltonian are simply

\[
H_0 = 2, \quad H_+ = 2C, \quad H_- = 2C^\dagger, \quad C := i e^{i\varphi} \lambda_+ \lambda_-.
\]
3 Dynamical symmetry

Let us now restrict to d=9 (for definiteness). Denoting $A := \sum_j A_j$ by $J_+ \otimes 1$, $A^\dagger = \sum_j A_j^\dagger$ by $J_- \otimes 1$, $1/2(N_A - 4) := 1/2(\sum_j N_{A_j} - 4)$ by $J_3 \otimes 1$, and similarly $1 \otimes J_+, 1 \otimes J_- 1 \otimes J_3$ for the Bs, with

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \quad J_\pm = J_1 \pm iJ_2,$$  

(13)
eqs. (8), (9) can be written as

$$\frac{1}{4}H = (2 + \cos \theta J_3 + \sin \theta J_1) \otimes 1 + 1 \otimes (2 - \cos \theta J_3 + \sin \theta J_1),$$  

(14)
thus exhibiting the dynamical symmetry mentioned above. The relevant SU(2) representations are the tensor product of four spin $\frac{1}{2}$ representations, i.e. direct sums of two singlets (note that both $(A_1A_3 + A_2A_4 - A_1A_4 - A_2A_3)|0\rangle$ and $(A_1A_2 + A_3A_4 - A_1A_4 - A_2A_3)|0\rangle$ are annihilated by $A_i A_i^\dagger$, and $\frac{1}{2}(N_A - 4)$, three spin 1 representations, and (most importantly, as providing the zero-energy state of $H$) one spin 2 representation acting irreducibly on the space spanned by the orthonormal states

$$|0\rangle, \quad \frac{1}{\sqrt{2}}A|0\rangle, \quad \frac{1}{\sqrt{2}}A^2|0\rangle, \quad \frac{1}{\sqrt{2}}A^3|0\rangle, \quad \frac{1}{\sqrt{2}}A^4|0\rangle.$$  

(15)

Restricting to that space (correspondingly for the Bs), we can write

$$J_3 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 \end{bmatrix}, \quad J_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

(16)

Since the spectrum of $\sin \theta J_1 \pm \cos \theta J_3$ is the same as that of $J_3$, the spectrum of $\frac{1}{4}H$ clearly consists of all integers between zero and eight, with the unique zero-energy state $\Psi$ most easily obtained by solving individually, for each $A_j$ resp. $B_j$ degree of freedom,

$$\left(1 + \cos \theta \sigma_3^{(j)} + \sin \theta \sigma_1^{(j)}\right) \Psi = e^{\pm \frac{i}{2} \theta i \sigma_1^{(j)}(1 \pm \sigma_3^{(j)})} e^{\pm \frac{i}{2} \theta i \sigma_2^{(j)}(1 \pm \sigma_3^{(j)})} \Psi = 0,$$  

(17)

where we identify $2J_k = \sigma_k^{(1)} \otimes 1 \otimes 1 \otimes \ldots + 1 \otimes 1 \otimes \sigma_k^{(4)} = \sum_{j=1}^4 \sigma_k^{(j)}$.

In our notation $\sigma_3^{(j)}|0\rangle = -|0\rangle$ and $\sigma_3^{(j)}A_j|0\rangle = +A_j|0\rangle$, and we easily find the solution to (17) as

$$\Psi = \left(\prod_j e^{-\frac{i}{2} \sigma_1^{(j)}}\right) \left(\prod_j e^{\frac{i}{2} \sigma_2^{(j)}} B_j\right)|0\rangle = e^{-\theta i (J_2 \otimes 1 - 1 \otimes J_2) B_j^4} |0\rangle.$$  

(18)

Using the nilpotency of $A_j$ and $B_j$ for

$$e^\alpha (A_j - A_j^\dagger)|0\rangle = \cos \alpha e^{\sin \alpha} A_j|0\rangle \quad \text{and} \quad e^{-\alpha (B_j - B_j^\dagger)} B_j|0\rangle = \sin \alpha e^{\cot \alpha} B_j|0\rangle,$$  

(19)

the ground state can also be written as

$$\Psi = \frac{1}{\pi e^{-4i\varphi}(\sin \theta)^4} \prod_j \left(\sin \theta - (1 - \cos \theta) A_j\right) \left(\sin \theta - (1 + \cos \theta) B_j\right)|0\rangle$$  

$$= \frac{1}{\pi e^{-4i\varphi}(\sin \theta)^4} e^{-\frac{1 - \cos \theta}{\sin \theta} A - \frac{1 + \cos \theta}{\sin \theta} B} |0\rangle \sim e^{-C_0}|0\rangle,$$  

(20)
with \( C_\theta := \frac{1 - \cos \theta}{\sin \theta} (J_+ \otimes 1) + \frac{1 + \cos \theta}{\sin \theta} (1 \otimes J_+) \). Alternatively, one can solve the \( 2 \times 2 \) matrix eigenvector equations resulting from \( (17) \),

\[
\begin{align*}
(1 + \cos \theta (N_{A_j} - 1) + \sin \theta (A_j + A_j^\dagger)) \Psi &= 0, \\
(1 - \cos \theta (N_{B_j} - 1) + \sin \theta (B_j + B_j^\dagger)) \Psi &= 0,
\end{align*}
\]  

(21)

to obtain \( (20) \).

For \( d=2 \) the asymptotic ground state is easily found from \( (12) \),

\[
\Psi = \frac{1}{\sqrt{2}} e^{-C} |0\rangle = \frac{1}{\sqrt{2}} (1 - C) |0\rangle.
\]  

(22)

An interesting feature of the form \( (18) \) for the ground state is that it expresses it as a spin-rotation by an angle \( \theta \) applied to some reference state \( B^0 |0\rangle \) (which itself also varies in the first \( d - 2 \) directions in space according to \( (2), (5), (7) \)).

### 4 Graded chain of Hamiltonians

Consider the grade- resp. fermion number-ordered equations

\[
H_0 \Psi_0 + H_- \Psi_2 = 0,
\]

\[
H_+ \Psi_0 + H_0 \Psi_2 + H_- \Psi_4 = 0,
\]

\[
H_+ \Psi_2 + H_0 \Psi_4 + H_- \Psi_8 = 0,
\]

\[
\vdots
\]

\[
H_+ \Psi_{12} + H_0 \Psi_{14} + H_- \Psi_{16} = 0,
\]

\[
H_+ \Psi_{14} + H_0 \Psi_{16} = 0,
\]

implied by \( H \Psi = (H_0 + H_+ + H_-) (\Psi_0 + \Psi_2 + \ldots + \Psi_{16}) = 0 \). (We have dropped the eight non-dynamical parallel fermions \( \lambda^\parallel = \lambda_{\alpha A} e_A \).) The following method to construct the ground state we believe to be relevant also for the fully interacting, non-asymptotic theory. Use the first equation in \( (23) \) to express \( \Psi_0 \) in terms of \( \Psi_2 \),

\[
\Psi_0 = -H_0^{-1} H_- \Psi_2.
\]  

(24)

\( H_0 \) is certainly invertible on the zero-fermion subspace, even in the full theory, where (cf. \( [11] \))

\[
H_0 = -\Delta + V - 2i x_j C f_{C A B} \Gamma^j_{\alpha \beta} \lambda^\dagger_{\alpha A} \lambda^\dagger_{\beta B}.
\]  

(25)

Using \( (24) \), the second equation in \( (23) \) can be written as

\[
H_2 \Psi_2 + H_- \Psi_4 = 0, \quad \text{with} \quad H_2 := H_0 - H_+ H_0^{-1} H_-,
\]  

(26)

yielding

\[
\Psi_2 = -H_2^{-1} H_- \Psi_4,
\]  

(27)

provided \( H_2 \) is invertible on \( H_- \Psi_4 \), resp. the two-fermion sector of the Hilbert space. Continuing in this manner, denoting

\[
\mathcal{H}_{2k} := \text{Span}\{A^m B^n |0\rangle\}_{m,n=0,1,2,3,4, m+n=k}
\]  

(28)

4
for the considered 2\(k\)-fermion subspace, we find that if we assume invertibility of \(H_{2k}\) on \(\hat{\mathcal{H}}_{2k}\) we can form

\[
H_{2(k+1)} := H_0 - H_+ H_{2k}^{-1} H_-
\]

on \(\hat{\mathcal{H}}_{2(k+1)}\) and solve for \(\Psi_{2k}\) in terms of \(\Psi_{2(k+1)}\). The final equation for \(\Psi_{16}\) is \(H_{16} \Psi_{16} = 0\).

For concreteness, denote an orthonormal basis of \(\hat{\mathcal{H}} = \oplus_k \hat{\mathcal{H}}_{2k}\) by \(|k, l\rangle := |k\rangle \otimes |l\rangle\), where, as in (15),

\[
|k\rangle := \frac{1}{k! \sqrt{\binom{k}{k}}} j_k^k |0\rangle.
\]

Then \(H_+ H_0^{-1} H_-\), e.g., acts on \(\hat{\mathcal{H}}\) ‘tridiagonally’ according to

\[
\frac{1}{\sin \theta} H_+ H_0^{-1} H_- |k, l\rangle = \left(\frac{k(5-k)}{4+(k-1) \cos \theta} + \frac{l(l-1)}{4+(k+l) \cos \theta}\right) |k, l\rangle
+ \sqrt{\frac{k(k+1)(4-k)}{4+(k+l) \cos \theta}} |k+1, l-1\rangle
+ \sqrt{\frac{k(k-1)(4-l)}{4+(k+l) \cos \theta}} |k-1, l+1\rangle.
\]

Calculating the spectra of \(H_{2k}\) on \(\hat{\mathcal{H}}_{2k}\) (e.g., with the help of a computer) one can verify the invertibility of all \(H_{2k}\) on \(\hat{\mathcal{H}}_{2k}\) for \(k < 8\), while \(H_{16}\) is identically zero on \(\hat{\mathcal{H}}_{16}\). Hence, one can also start with the state \(\Psi_{16} \sim A^4 B^4 |0\rangle\) (with correct normalization in \(\theta\)) and generate the lower grade parts of the full ground state \(\Psi\) using the relations (24), (27), etc.

Let us finish this section by noting a simple consequence of the graded form (30) of the ground state equation \(H \Psi = 0\) (for general \(d\) and \(N\)). Taking the inner product of the grade 2\(k\)-equation with \(\Psi_{2k}\) yields

\[
\langle \Psi_{2k}, H_- \Psi_{2(k+1)} \rangle = -\langle H_0 \rangle_{2k} - \langle \Psi_{2k}, H_+ \Psi_{2(k-1)} \rangle
= -\langle H_0 \rangle_{2k} - \langle \Psi_{2(k-1)}, H_- \Psi_{2k} \rangle^*,
\]

where \(\langle H_0 \rangle_{2k} := \langle \Psi_{2k}, H_0 \Psi_{2k} \rangle\). The first equation reads \(\langle \Psi_0, H_- \Psi_2 \rangle = -\langle H_0 \rangle_0\) which is real. The second then becomes \(\langle \Psi_2, H_- \Psi_4 \rangle = -\langle H_0 \rangle_2 + \langle H_0 \rangle_0\), and so on, so that in the last step one obtains

\[
\sum_{k=0}^{\Lambda} (-1)^k \langle H_0 \rangle_{2k} = 0,
\]

where \(\Lambda\) is the total number of fermions in the relevant Fock space.

It is instructive to verify (33) for the asymptotic \(N = 2\) case studied above, since there all relevant terms can be calculated explicitly. Using the basis (30) and the notation \(\alpha := 1 - \cos \theta, \beta := 1 + \cos \theta\), we find

\[
\Psi \sim e^{-\zeta \theta} |0\rangle = \sum_k \frac{(-1)^k \sqrt{\binom{k}{k}}}{(\sin \theta)^k} \alpha^k |k\rangle \otimes \sum_l \frac{(-1)^l \sqrt{\binom{l}{l}}}{(\sin \theta)^l} \beta^l |l\rangle.
\]

Hence,

\[
\langle \Psi_{2n}, H_0 \Psi_{2n} \rangle = \frac{1}{64} (\sin \theta)^{8-2n} \sum_{k+l=n} \binom{4}{k} \binom{4}{l} (4 + (k - l) \cos \theta) \alpha^{2k} \beta^{2l}.
\]
5 General SU($N$)

Let us now derive, for general $N \geq 2$, the ground state energy of

$$H_F = i\gamma^{\alpha}_\beta f_{ABC} x_{iC} \theta_\alpha A \theta_\beta B$$

(36)

in regions of the configuration space where the potential $V$ is zero. (As in (25), $f_{ABC}$ denote the structure constants of SU($N$) in an orthonormal basis.) By (1) this means that all $X_s$ are commuting, hence can be written $X_s = U D_s U^\dagger$ where $U$ is unitary and independent of $s$ and the $D_s$ are diagonal. If we look into a particular direction (corresponds to fixing $e$ in the SU(2) case) and choose a basis \{$T_A$\} accordingly we may write $X_s = D_s = x s_A T_A = x s_k T_k$ and $x s_a = 0$, where $k = 1, \ldots, N - 1$ are indices in the Cartan subalgebra and $a, b = N, \ldots, N^2 - 1$ denote the remaining indices.

Denoting the eigenvalues of $X_t$ by $\mu_t^i$, i.e. $X_t = \text{diag}(\mu_1^i, \ldots, \mu_N^i)$, then the eigenvectors \{e_{ki}\}$_k \neq l$ of $M_t^{kl} := -i f_{abc} x_{iC} = -i f_{abk} x_{ik}$ satisfy (cf. e.g. [5])

$$M_t^{kl} e_{kl} = (\mu_k^i - \mu_l^i) e_{kl} =: \mu_{kl}^i e_{kl}, \quad (e_{kl}^a)^* = e_{lk}^a.$$  \hspace{1cm} (37)

The crucial observation is that these eigenvectors are independent of $t$. Now,

$$H_F = -\gamma^{\alpha}_\beta M_t^{ab} \theta_\alpha a \theta_\beta b = W_{\alpha a, \beta b} \theta_\alpha a \theta_\beta b,$$  \hspace{1cm} (38)

where $W := -\sum_t \gamma^t \otimes M^t$. From the above observations we have the ansatz $E_{\mu kl} := v_{\mu} \otimes e_{kl}$ for the eigenvectors of $W$, giving

$$WE_{\mu kl} = -\gamma^{\mu}_\nu \sum \gamma^t v \otimes M^t e_{kl} = \gamma(k, l)v_{\mu} \otimes e_{kl},$$  \hspace{1cm} (39)

where $\gamma(k, l) := -\sum_t \mu_k^t \gamma^t$ squares to $\sum_t (\mu_k^t)^2$. Letting $v_{\mu} = v_{+jk l}$ denote the corresponding 16 eigenvectors of $\gamma(k, l)$, we find

$$WE_{\pm jk l} = \pm \sqrt{\sum (\mu_k^t)^2} E_{\pm jk l}$$  \hspace{1cm} (40)

and $H_F$ therefore has $E_0 := -16 \sum_{k<l} \sqrt{\sum_{t=1}^9 (\mu_k^t - \mu_l^t)^2}$ as its lowest eigenvalue.

This agrees with the following two previously known cases: [5], where only $X_9$ is assumed to have large eigenvalues so that $E_0 \to -16 \sum_{k<l} |\mu_k^9 - \mu_l^9|$; as well as the SU(2)-case studied above, where $\sum \gamma^t$ with e.g. $e_3 = \delta_{33}$ gives $E_0 = -16r$.

Acknowledgements

One of us (J.H.) would like to thank Choonkyu Lee for his hospitality, and Ki-Myeong Lee for useful discussions.

References

[1] J. Hoppe, On the construction of zero energy states in supersymmetric matrix models I, II, III, [hep-th/9709132] 9709217, 9711033.
[2] M.B. Halpern, C. Schwartz, *Asymptotic search for ground states of SU(2) matrix theory*. Int. J. Mod. Phys. A 13 (1998) 4367, hep-th/9712133.

[3] G.M. Graf, J. Hoppe, *Asymptotic ground state for 10-dimensional reduced supersymmetric SU(2) Yang-Mills theory*, hep-th/9805080.

[4] J. Fröhlich, G.M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, *Asymptotic form of zero energy wave functions in supersymmetric matrix models*, Nucl. Phys B 567 (2000), 231-248.

[5] D. Hasler, J. Hoppe, *Asymptotic factorisation of the ground-state for SU(N)-invariant supersymmetric matrix-models*, hep-th/0206043.

V. Bach, FB Mathematik, Universität Mainz, DE-55099 Mainz, Germany

J. Hoppe and D. Lundholm, Department of Mathematics, Royal Institute of Technology, SE-10044 Stockholm, Sweden