The Quasi-exact models in two-dimensional curved space based on the generalized CRS Harmonic Oscillator

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Abstract

In this paper, by searching the relation between the radial part of Higgs harmonic oscillator in the two-dimensional curved space and the generalized CRS harmonic oscillator model, we can find a series of quasi-exact models in two-dimensional curved space based on this relation.

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I. INTRODUCTION

The quasi-exactly solvable quantum problems was a remarkable discovery in last century[1]. This kind of problem can be solved by Lie algebra[2] or the analytical method[3]. Meanwhile, the quantum nonlinear harmonic oscillator (QNHO) has been studied with great interest [6–12]. Wang and Liu[6] generalized a class of QNHO which is called CRS model[7, 8] by the factorization method, whose Hamiltonian reads

\[
H' = \epsilon \left( -K \frac{d^2}{dx^2} - \lambda_Q x \frac{d}{dx} \right) + V'(x), \quad (\epsilon = \frac{\hbar^2}{2m}),
\]

where \(K = 1 + \lambda_Q x^2\), \(\lambda_Q\) is a real number, \(m\) is the mass for the particle and

\[
V'(x) = \epsilon \frac{(\beta X + \gamma)^2 + (\beta X + \gamma)(AX + B)}{K(\frac{dX}{dx})^2} + C,
\]

where \(\beta, \gamma\) and \(C\) are arbitrary real numbers; \(X = X(x)\) is a function which is analytic nearby \(x = 0\) here; the parameters \(A\) and \(B\) need to satisfy the equation

\[
K \frac{d^2X}{dx^2} + \lambda_Q x \frac{dX}{dx} = AX + B.
\]

It is easily proved that the solutions of Hamiltonian (1) can be solved exactly with the potential (2) by the factorization method.

On the other hand, Higgs [4] and Leemon [5] introduced a generalization of the hydrogen atom and isotropic harmonic oscillator in a space with constant curvature. On 2-dimensional curved sphere, the Hamiltonian can be written as

\[
H = \frac{1}{2m} \left( \pi^2 + \lambda_G L^2 \right) + V(r),
\]

where \(\pi = p + \frac{1}{2}\lambda_G [x(x \cdot p) + (p \cdot x)x], \ L^2 = \frac{1}{2} L_{ij} L_{ij}, \ r = |\sqrt{x^2}|\) and \(\lambda_G\) is the curvature of the 2-dimensional curved sphere.

In this work, by studying the relation between the generalized CRS harmonic oscillator model[6] and the radial part of Higgs harmonic oscillator[4] in the two-dimensional curved space, we can find a series of quasi-exact models in two-dimensional curved space based on this relation. The paper is organized as follows. In Sec. 2, the link between a special
generalized CRS model and the Higgs model will be given; in Sec. 3, the generalized Higgs models which are quasi-exactly solvable will be shown; in Sec. 4, there will be a conclusion finally.

II. THE RELATION BETWEEN THE GENERALIZED CRS HARMONIC OSCILLATOR AND THE RADIAL PART OF HIGGS OSCILLATOR

A. The exactly solvable Higgs oscillator

Considering the two-dimensional Hamiltonian (4), we substitute it into the stationary Schrödinger equation, which \( V(r) = \frac{1}{2} m \omega^2 r^2 \). The partial differential equation can be written as

\[
\frac{-\hbar^2}{2m} \left[ (1 + \lambda_G r^2)^2 \frac{\partial^2}{\partial r^2} + \left( 1 + \lambda_G r^2 \right) \frac{1 + 5 \lambda_G r^2}{r} \frac{\partial}{\partial r} + \left( 3 \lambda_G + \frac{15 \lambda_G^2 r^2}{4} \right) \right] \Psi(r, \theta) = \left( E_G - \frac{1}{2} m \omega^2 r^2 \right) \Psi(r, \theta).
\]

which \( E_G \) is the stationary energy eigenvalue. If we make \( \Psi(r, \theta) = e^{i m G \theta} \psi(r) \) and \( m_G \) is the angular parameter, it gives the radial part of above equation

\[
\frac{-\hbar^2}{2m} \left[ (1 + \lambda_G r^2)^2 \frac{d^2}{dr^2} + \left( 1 + \lambda_G r^2 \right) \frac{d}{dr} + \left( 3 \lambda_G - \lambda_G m_G^2 + \frac{15 \lambda_G^2 r^2}{4} - \frac{m_G^2}{r^2} \right) \right] \psi(r) = \left( E_G - \frac{1}{2} m \omega^2 r^2 \right) \psi(r).
\]

Considering the work of Higgs [4], we know that the harmonic oscillator (4) on the 2-dimensional curve sphere with constant curvature \( \lambda_G \) has the radial wave function

\[
\psi(r)_{N, m_G} = r^{|m_G|} \left( \frac{1}{1 + \lambda_G r^2} \right)^{|m_G|^2 + \frac{m_G^2}{2 \lambda_G}} F(-N, N + |m_G| + 1, \frac{m_G^2}{\lambda_G \hbar}, |m_G| + 1; 1 + \frac{\lambda_G r^2}{1 + \lambda_G r^2})
\]

and the energy spectrum

\[
E_G(N, m_G) = \hbar \omega'_G (2N + |m_G| + 1) + \frac{\lambda_G \hbar^2}{2m} (2N + |m_G| + 1)^2,
\]

which \( \omega'_G = \sqrt{\omega^2 + \frac{\hbar^2 \lambda_G^2}{4m^2}} \), \( N \) and \( m_G \) are both integer number here.
B. The exactly solvable generalized CRS harmonic oscillator

For the generalized CRS model[6], if we set the function and parameters in the potential (2) as

\[ X(x) = \cos(2\Theta(x)), \quad \beta = 2\lambda_Q(m_Q + 1) + \sqrt{\lambda_Q^2 + \frac{4m_Q^2\omega^2}{\hbar^2}}, \quad \gamma = 2\lambda_Qm_Q - \sqrt{\lambda_Q^2 + \frac{4m_Q^2\omega^2}{\hbar^2}}, \quad A = -4\lambda_Q, \quad B = 0 \text{ and } C = \epsilon \left( \lambda_Q(m_Q^2 - 1) + m_Q\sqrt{\lambda_Q^2 + \frac{4m_Q^2\omega^2}{\hbar^2}} \right), \]

we get

\[ V'(x) = \frac{1}{2}m\omega^2 \left( \frac{\tan(\Theta(x))}{\sqrt{\lambda_Q}} \right)^2 - \frac{\lambda_Q\hbar^2}{8m} \left( 1 + (1 - 4m_Q^2) \csc^2(\Theta(x)) \right), \tag{9} \]

where \( \Theta(x) = \arcsinh(\sqrt{\lambda_Q}x) \) and \( m_Q \) is a real number. With the potential above, by solving the generalized CRS eigen-equation

\[ \left[ \epsilon \left( -K \frac{d^2}{dx^2} - \lambda_Q x \frac{d}{dx} \right) + V'(x) \right] \phi(x) = E_Q \phi(x), \tag{10} \]

we have the wavefunction

\[ \phi(x) = (-\sin^2(2\Theta(x)))^{-\frac{3}{4}} \sin^2(\Theta(x)) \left( \frac{\tan(\Theta(x))}{\sqrt{\lambda_Q}} \right)^{|m_Q|} \]

\[ \left( \cos(\Theta(x)) \right)^{\frac{|m_Q|+2}{2} + \frac{m_Q^2}{2\pi\lambda_Q}} F\left( -N, N + |m_Q| + 1 + \frac{m_Q^2}{\lambda_Q \hbar}, |m_Q| + 1; \sin(\Theta(x)) \right). \tag{11} \]

and the energy spectrum

\[ E_{Q(N,m_Q)} = \hbar \omega_Q' (2N + |m_Q| + 1) + \frac{\lambda_Q \hbar^2}{2m} (2N + |m_Q| + 1)^2, \tag{12} \]

which \( \omega_Q' = \sqrt{\omega^2 + \frac{\hbar^2\lambda_Q}{4m^2}} \), \( N \) and \( m_Q \) are both integer number here.

C. The transformation from generalized CRS harmonic oscillator to radial Higgs model

Comparing the energy spectrum (8) and (12), if \( \lambda_G = \lambda_Q = \lambda \) and \( m_G = m_Q = m \), it is obviously that they are exactly same. With the transformation

\[ \Theta(x) = \arcsinh(\sqrt{\lambda}x) = \Theta(x(r)) = \Upsilon(r) = \arctan(\sqrt{\lambda}r) \tag{13} \]
and separating the wave function (11)

$$\phi(x) = \phi(x(r)) = g(r)\psi(r), \quad g(r) = (\sin^2(2\Upsilon(r)))^{-\frac{3}{4}}\sin^2(\Upsilon(r)),$$

we get the same differential equation as (6) and the same wave-function as (7).

Thus, we find the transformation relation here. If the Hamiltonian (1) with potential

$$V(x)$$

can be solved exactly with the wave function $$\phi(x)$$, the radial part of Hamiltonian (4)

with $$V(r)$$ above also can be solved exactly with the following wave function

$$\psi(r) = \csc(\Upsilon(r))^2 (-\sin(2\Upsilon(r))^2)^{\frac{1}{4}}\phi(x(r)),$$

which $$V(x)$$ and $$V(r)$$ satisfies the relation

$$V(r) = V(x(r)) + \frac{\lambda \hbar^2}{8m} (1 + (1 - 4m^2_Q) \csc^2(\Upsilon(r))).$$

III. THE QUASI-EXACT MODEL IN TWO-DIMENSIONAL CURVED SPACE

For the transformation from generalized CRS harmonic oscillator to radial Higgs model, it can be easily found that the potential $$V(r)$$ in two-dimensional Higgs model can only be solved exactly while the angular parameter $$m_G$$ equals to the real number $$m_Q$$. For $$m_Q \neq m_G$$ case, the 2-dimensional Higgs model is a quasi-exact model for angular part of this model can not be exactly solved.

Here, we would like to give some explicit examples, which satisfy $$\lambda_G = \lambda_Q = \lambda$$.

e.g: (1) $$X(x) = \cos(2\Theta(x)), \Theta(x) = \arcsinh(\sqrt{\lambda x}), \beta = 2\lambda(m_Q + 1) + \frac{2m\omega'}{\hbar}, \gamma = 2\lambda m_Q - \frac{2m\omega'}{\hbar}, A = -4\lambda, B = 0, C = \epsilon \left(\lambda(m^2_Q - 1) + \frac{2m\omega'}{\hbar}m_Q\right), \omega' = \sqrt{\omega^2 + \frac{b^2\lambda^2}{4m^2}}$$. Thus, we have $$V(r) = \frac{1}{2}m\omega^2r^2$$. However, the wavefunction is

$$\Psi(r; \theta; N, m_G, m_Q) = e^{im_G\theta}\psi(r; N, m_Q)$$

and

$$\psi(r; N, m_Q) = r^{|m_Q|} \left( \frac{1}{1 + \lambda r^2} \right)^{\frac{|m_Q| + 2 + \frac{m\omega'}{\hbar}}{2}} F(-N, N + |m_Q| + 1 + \frac{m\omega'}{\hbar}, |m_Q| + 1; \frac{\lambda r^2}{1 + \lambda r^2})$$
From equation (17), it is obviously that this is a quasi-exact model.

\( X(x) = x \), which means \( A = \lambda, B = 0 \). \( \beta \) is an arbitrary real numbers about parameter \( m_Q \). \( \gamma \) and \( C \) equals to 0. Thus, we have

\[
V(r) = \frac{\beta m_Q (\beta m_Q + \lambda)}{2\lambda} \tanh^2(\Upsilon(r)) + \frac{\lambda \hbar^2}{8m} \left( 1 + (1 - 4m_Q^2) \csc^2(\Upsilon(r)) \right).
\]

The ground state of wave function is

\[
\Psi(r, \theta; 0, m_Q, m_Q) = e^{im_Q \theta} \psi(r; 0, m_Q)
\]

and

\[
\psi(r; 0, m_Q) = \csc(\Upsilon(r))^2 \left( -\sin(2\Upsilon(r))^2 \right)^{\frac{3}{4}} \text{sech}\left( \frac{\beta m_Q}{\lambda} \right).
\]

From equation (18), it says that this quasi-exact model can be built by the transformation (13).

IV. CONCLUSION

From the transformation (13), we can establish lots of quasi-exact models in two-dimensional curved space. This is a progress about quasi-exact theory and also a connection between quantum nonlinear harmonic oscillator (QNHO) theory and curved space model.

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