Minimal Surfaces with Only Horizontal Symmetries

Márcio Fabiano da Silva, Guillermo Antonio Lobos, and Valério Ramos Batista

1 CMCC, UFABC, Rua Santa Adélia 166, BL.A-2, 09210-170 Santo André, SP, Brazil
2 DM, UFSCar, Rua Washington Luis km 235, 13565-905 São Carlos, SP, Brazil

Correspondence should be addressed to Valério Ramos Batista, valerio.batista@ufabc.edu.br

Received 6 April 2011; Accepted 11 May 2011

Academic Editors: L. V. Bogdanov, G. Martin, and C. Qu

Copyright © 2011 Márcio Fabiano da Silva et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The Schwarz reflection principle states that a minimal surface $S$ in $\mathbb{R}^3$ is invariant under reflections in the plane of its principal geodesics and also invariant under $180^\circ$-rotations about its straight lines. We find new examples of embedded triply periodic minimal surfaces for which such symmetries are all of horizontal type.

1. Introduction

During the Clay Mathematics Institute 2001 Summer School on the Global Theory of Minimal Surfaces, M. Weber introduced the following terminology in his first lecture entitled Embedded minimal surfaces of finite topology:

“A horizontal symmetry is a reflection at a vertical plane or a rotation about a horizontal line. A vertical symmetry is a reflection at a horizontal plane or a rotation about a vertical line.”

With this terminology, he proved that such symmetries induce symmetries in the cone metrics determined by $dh$, $gdh$, and $dh/g$ from a Weierstrass pair $(g, dh)$ of a minimal surface (see [1, 2] for details).

By classifying the symmetries this way, we sort out the space groups that might admit one, both, or none of them. Since minimal surfaces may model some natural structures, like crystals and copolymers, an example within a given symmetry group might fit an already existing compound, or even hint at nonexistent ones. However, several symmetry groups are not yet represented by any minimal surface (see [3, 4] for details and comments).
Restricted to symmetries given by reflections in the plane of principal geodesics and by 180°-rotations about straight lines contained in the surface, outside the triply periodic class it is easy to find complete embedded minimal surfaces in \( \mathbb{R}^3 \) of which these symmetries are either only horizontal or only vertical. For instance, the Costa surface (see [5–7]) has only horizontal symmetries. The doubly periodic examples found by Meeks and Rosenberg in [8] have only vertical symmetries (see also [9] for nice pictures).

In the class of triply periodic minimal surfaces almost all known examples have either both or none of such symmetries, after suitable motion in \( \mathbb{R}^3 \). In fact, this must be true because most of the surfaces in this class have a cubic symmetry group. Examples with only horizontal symmetries do not seem to be well known. Besides the surfaces shown herein, perhaps there are only the “TT-surfaces” as Karcher named them in [10, pp. 297, 328-9] (see also [11]), and a surface from Fischer-Koch [12], which is, however, presented without rigorous proof.

The “TT-surfaces” are generated by an annulus, of which the boundary consists of two twisted equilateral triangles. For edge length \( 2\sqrt{3} \) and height 1, they coincide with the Schwarz P-surface, and hence have further symmetries besides the horizontal ones. Moreover, when a TT-surface has only horizontal symmetries, its translation group cannot be given by an orthogonal lattice.

In the present work, we give existence proofs for examples that are probably the first triply periodic minimal surfaces with only horizontal symmetries, of which the translation group is given by an orthogonal lattice. They are constructed by Karcher’s method [6, 7, 10], although the purpose alone of few symmetries could be accomplished by modern methods introduced, for instance, by Traizet [13] and Fujimori and Weber [14]. However, Traizet’s method is not explicit (in the sense explained in [15]) whereas Fujimori-Weber’s method may turn it hard to analyse the so-called *period problems*. These are equations involving elliptic integrals with interdependent parameters.

Regarding examples with only vertical symmetries, we believe they have not been found yet.

The examples presented herein are inspired in the surfaces \( C_2 \) and \( L_{2,4} \) from [16, 17]. Any of those is generated by a fundamental piece, which is a surface with boundary in \( \mathbb{R}^3 \) with two catenoidal ends. The fundamental piece resembles the Costa surface with its planar end replaced by either symmetry curves or line segments. By suppressing the catenoidal ends, if we pile up several copies of the fundamental piece, we get the pictures in Figures 1 and 2(b). They are also named \( C_2 \) and \( L_{2,4} \).
The reader will notice that the surfaces $C_4$, also described in [16, 17], were not mentioned beforehand. This is because, for them, the “piling up” procedure naturally forces extra symmetries to exist, and one goes back to another famous surface from H. Schwarz (see Figure 2(a)). Notice, for instance, the vertical straight line that comes out in the surface.

We are going to prove the following results:

**Theorem 1.1.** There exists a one-parameter family of triply periodic minimal surfaces in $\mathbb{R}^3$, of which the members are called $C_2$, and for any of them the following holds.

(a) The quotient by its translation group has genus 9.

(b) The whole surface is generated by a fundamental piece, which is a surface with boundary in $\mathbb{R}^3$. The boundary consists of four curves, each contained in a vertical plane. The fundamental piece has a symmetry group generated by reflections in two vertical planes and 180°-rotations about two line segments.

(c) By successive reflections with respect to planes bounding the fundamental domain, and successive vertical translations, one obtains the triply periodic surface.

**Theorem 1.2.** For $k = 2, 4$, there exists a one-parameter family of triply periodic minimal surfaces in $\mathbb{R}^3$, of which the members are called $L_k$, and for any of them the following holds.

(a) The quotient by its translation group has genus $2k + 1$.

(b) The whole surface is generated by a fundamental piece, which is a surface with boundary in $\mathbb{R}^3$. The boundary consists of four line segments. The fundamental piece has a symmetry group generated by reflections in two vertical planes and 180°-rotations about two line segments. Each of these segments makes an angle of $\pi/k$ with the boundary.

(c) By successive rotations about the boundary of the fundamental piece, and successive vertical translations, one obtains the triply periodic surface.

Sections 3 to 7 are devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 follows very similar arguments, and we briefly discuss it in Section 8.
2. Preliminaries

In this section we state some basic definitions and theorems. Throughout this work, surfaces are considered connected and regular. Details can be found in [6, 7, 18–20].

**Theorem 2.1.** Let \( X : \mathbb{R} \rightarrow \mathbb{E} \) be a complete isometric immersion of a Riemannian surface \( R \) into a three-dimensional complete flat space \( \mathbb{E} \). If \( X \) is minimal and the total Gaussian curvature \( \int_R K dA \) is finite, then \( R \) is biholomorphic to a compact Riemann surface \( \overline{R} \) punctured at finitely many points \( p_1, \ldots, p_r \).

**Definition 2.2.** Let \( P = \{p_1, \ldots, p_r\} \) as in Theorem 2.1. An end is the image under \( X \) of a punctured neighbourhood \( V_p \) of a point \( p \in P \) such that \( P \cap V_p \neq \emptyset \). We say that the surface has no ends when \( P = \emptyset \).

**Theorem 2.3** (Weierstraß Representation). Let \( R \) be a Riemann surface, \( g \) and \( dh \) meromorphic function and 1-differential form on \( R \), such that the zeros of \( dh \) coincide with the poles and zeros of \( g \). Suppose that \( X : R \rightarrow \mathbb{E} \), given by

\[
X(p) := \text{Re} \int_X^p (\phi_1, \phi_2, \phi_3), \quad \text{where } (\phi_1, \phi_2, \phi_3) := \frac{1}{2} \left( \frac{1}{g} - g, \frac{i}{g} + ig, 2 \right) dh, \tag{2.1}
\]

is well defined. Then \( X \) is a conformal minimal immersion. Conversely, every conformal minimal immersion \( X : R \rightarrow \mathbb{E} \) can be expressed as (2.1) for some meromorphic function \( g \) and 1-form \( dh \).

**Definition 2.4.** The pair \( (g, dh) \) is the Weierstraß data and \( \phi_1, \phi_2, \phi_3 \) are the Weierstraß forms on \( R \) of the minimal immersion \( X : R \rightarrow X(R) \subset \mathbb{E} \).

**Theorem 2.5.** Under the assumptions of Theorems 2.1 and 2.3, the Weierstraß data \( (g, dh) \) extend meromorphically on \( \overline{R} \).

The function \( g \) is the stereographic projection of the Gauss map \( N : R \rightarrow S^2 \) of the minimal immersion \( X \). It is a covering map of \( \hat{\mathbb{C}} \) and \( \int_S K dA = -4\pi \deg(g) \). These facts will be largely used throughout this work.

3. The Surfaces \( \overline{M} \) and the Functions \( z \)

Consider the surface indicated in Figure 1(a). A reflection in any of its vertical planar curves of the boundary leads to a fundamental piece which represents the quotient of a triply periodic surface \( M \) by its translation group. We are going to denote this quotient by \( \overline{M} \). It is not difficult to conclude that it has genus 9. The fundamental domain of \( \overline{M} \) is the shaded region indicated in Figure 3(a).

The surface \( \overline{M} \) is invariant under 180°-rotations around the directions \( \vec{x}_3 \) and \( \vec{x}_2 \). These rotations we call \( r_v \) and \( r_h \), respectively, (see Figure 3(a)). Based on this picture, one sees that the fixed points of \( r_v \) are \( S, S', L, L', F, F' \) and the images of \( S \) and \( S' \) under the symmetries of \( \overline{M} \). They sum up 8 in total. The quotients by \( r_v \) and \( r_h \) we call \( \rho_v \) and \( \rho_h \), respectively. The surface \( \rho_v(\overline{M}) \) is still invariant under the rotation \( r_h \). In this case, the fixed points of \( r_h \) will
be \( \rho_v(A), \rho_v(A') \) and their images under the symmetries of \( \rho_v(\overline{M}) \). They sum to 8 in total. Because of that,

\[
\chi\left( \rho_h\left( \rho_v \left( \overline{M} \right) \right) \right) = \frac{1 - 9 + 8/2}{2} + \frac{8}{2} = 2. \tag{3.1}
\]

Let us define \( z := \rho_h \circ \rho_v : \overline{M} \to S^2 \cong \hat{\mathbb{C}} \), such that \( z(S) = 0 \), \( z(L) = 1 \) and \( z(B) = i \). The involutions of \( \overline{M} \) are induced by \( \rho_v \) and \( \rho_h \) on \( \hat{\mathbb{C}} \), and since all the involutions of \( \hat{\mathbb{C}} \) are given by Möbius transformations, we can conclude the following: \( z(S') = 0, z(F') = -z(F) = -z(L') = 1 \) and \( z(B') = i \). By applying the symmetries of \( \overline{M} \), one easily reads off the other values of \( z \) at the images (under these symmetries) of \( S, S', L, L', F, F', B, \) and \( B' \). Regarding the points \( A \) and \( A' \), we have \( z(A) = x \in \mathbb{C} \) such that \( |x| < 1 \) and \( \text{Arg}(x) \in (0, \pi/2) \). Consequently, \( z(A') = -\overline{x} \) and one easily gets the other values of \( z \) at the images of \( A \) and \( A' \) under the symmetries of \( \overline{M} \).

4. The \( g \)-Function on \( \overline{M} \) in Terms of \( z \)

First of all, observe that Jorge-Meeks’ formula gives \( \deg(g) = 9 - 1 = 8 \). Let us then consider Figure 3(b). We will have \( g - g^{-1} = \infty \) if and only if \( z - z^{-1} \in \{0, \infty\} \). Moreover, \( g - g^{-1} = 0 \) if and only if \( z \in \{-x, x, x^{-1}, -x^{-1}, ia, ia^{-1}\} \), where \( a \in (0, 1) \). From this point on we introduce the following notation:

\[
Z := z - z^{-1}, \quad X := x^{-1} - x, \quad A := a + a^{-1}. \tag{4.1}
\]
By following Karcher’s method in [6, 7], Figure 3 represents the surfaces whose existence we want to prove. From this picture we read off the necessary conditions for Theorem 1.1 to be valid. Afterwards, these will prove Theorem 1.1. The first condition is an algebraic relation between \( g \) and \( z \). Hence, based on Figure 3 and Karcher’s method, it is not difficult to conclude that

\[
\left( g - \frac{1}{g} \right)^2 = \frac{-ic}{Z^3} \cdot (Z - i\mathcal{A})^2(Z - X)(Z + \overline{X}),
\]

where \( c \) is a positive constant. Now we define \( \overline{M} \) as a member of the family of compact Riemann surfaces given by the algebraic equation (4.2). Later on, we are going to verify that \( \overline{M} \) has genus 9 indeed. But first we derive some conditions on the variables \( a, x, \) and \( c \) in order to guarantee that \( g^2 = -1 \) at \( z = -ia^{\pm 1} \). This will be the case if

\[
c = \frac{a}{a^2 + 2a \text{Im}\{X\} + |X|^2}.
\]

Since \( |X|^2 = \text{Im}\{X\} + \text{Re}\{X\} \), one easily sees that \( c \) is positive.

Now we analyse what happens to (4.2) under the map \( z \to \overline{z} \). In this case we will get \( g \to -ig \) or \( g \to -i\overline{g} \). Therefore

\[
\left( g + \frac{1}{g} \right)^2 = \frac{-ic}{Z^3} \cdot (Z + i\mathcal{A})^2(Z - \overline{X})(Z + X).
\]

At this point we are ready to prove that \( \overline{M} \) has genus 9. The function \( z \) is a four-sheeted branched covering of the sphere. The values \( 0, \infty, \pm 1, \pm x^{\pm 1}, \pm \overline{x}^{\pm 1} \) correspond to the only branch points of \( z \), all of them of order 2, and each of these values is taken twice on \( \overline{M} \). Therefore, from the Riemann-Hurwitz’s formula we have

\[
\text{genus}(\overline{M}) = \frac{12 \cdot (2 - 1) \cdot 2}{2} - 4 + 1 = 9.
\]

Now we are ready to find some relations that the parameters \( a, c, \) and \( x \) will have to satisfy. These relations will make (4.2) and (4.4) consistent with the values of \( g \) on the symmetry curves and lines of \( \overline{M} \).

5. Conditions on the Parameters \( a, c, \) and \( x \)

Consider the curves \( S'L \) and \( F'S \) represented in Figure 3. The same picture shows how we have positioned our coordinate system. On the curve \( S'L \), we expect that \( g \in e^{i\pi/4}\mathbb{R} \), and on \( F'S \) one should have \( g \in e^{-i\pi/4}\mathbb{R} \). Let us now verify under which conditions this will really happen.
On $S'L$, we ought to have $\text{Re}\{(g - g^{-1})^2\} = -2$. By taking $z(t) = t$, $0 < t < 1$, defining $T := t - t^{-1}$, and applying it to (4.2) we get the following equality:

$$
\left. (g - g^{-1})^2 \right|_{z(t)} = \frac{-ic}{T^3} \cdot (T - i\mathcal{A})^2 \left( T^2 - 2i \text{Im}\{X\} \cdot T - |X|^2 \right). \quad (5.1)
$$

Therefore,

$$
\text{Re}\left\{ (g - g^{-1})^2 \right\} = -\frac{2c}{T^2} \cdot \left( \text{Im}\{X\} \cdot T^2 + \mathcal{A}T^2 - \text{Im}\{X\} \cdot \mathcal{A}^2 - \mathcal{A}|X|^2 \right), \quad (5.2)
$$

on the curve $z(t)$. Since we want $\text{Re}\{(g - g^{-1})^2\} = -2$ on this curve, (5.2) will then give rise to the following conditions

$$
c = \frac{1}{\mathcal{A} + \text{Im}\{X\}}, \quad (5.3)
$$

$$
\mathcal{A} = -\frac{|X|^2}{\text{Im}\{X\}}. \quad (5.4)
$$

Equation (5.3) can be deduced from (4.3) and (5.4) by a simple calculation. Equation (5.4) will restrict the definition domain of our parameters. Since $a \in (0, 1)$, then $a > 2$, and by taking $x = |x|e^{i\theta}$ one clearly sees that $\text{Im}\{X\} < 0$ for $\theta \in (0, \pi/2)$. From (5.4) we finally get the following restriction for the $x$-variable

$$
\text{Re}^2\{X\} > -2 \text{Im}\{X\} - \text{Im}^2\{X\}. \quad (5.5)
$$

Figure 4 illustrates the $X$-domain established by (5.5), and we recall that $|x| < 1$ and $\theta \in (0, \pi/2)$. 
Since the surface $M$ has no ends, $dh$ must be a holomorphic differential form on it. The zeros of $dh$ are exactly at the points where $g = 0$ or $g = \infty$, and $\text{ord}(dh) = |\text{ord}(g)|$ at these points. They should sum up 16 in total, which is consistent with $\text{deg}(dh) = -\chi(\overline{M})$. Let us now analyse the differential $dz$. Based on Figure 3, one sees that $dz$ has a simple zero at the points $z^{-1}([0, \pm 1, \pm x^{\pm 1}, \pm x^{-1}])$ and a pole of order 3 at the points $z^{-1}((\infty))$. Let the symbol $\sim$ indicate that two meromorphic functions on $\overline{M}$ differ by a nonzero proportional constant. It is not difficult to conclude that

$$
\left(\frac{dh}{dz}\right)^2 \sim \frac{(z^2 - 1)^2}{(z^2 - x^2)(z^2 - x^{-2}) (z^2 - \overline{x}^2)(z^2 - \overline{x}^{-2})}.
$$

(6.1)

If we had a well-defined square root of the function at the right-hand side of (6.1), then we could get an explicit formula for $dh$ in terms of $z$ and $dz$. This square root exists indeed. By multiplying (4.2) and (4.4) it follows that

$$
(Z - X)\left(Z + \overline{X}\right)\left(Z - \overline{X}\right)(Z + X) = \frac{-Z^6}{c^2(Z^2 + \omega^2)^2}\left(\frac{1}{g^2} - 1\right)^2,
$$

(6.2)
### Table 2

| Symmetry | z-values | $g \in$ | $dh(z) \in$ |
|----------|----------|---------|------------|
| $SB$     | $it, 0 < t < 1$ | $\mathbb{R}_+$ | $i\mathbb{R}$ |
| $BL$     | $e^{it}, \pi/2 > t > 0$ | $\mathbb{R}_+$ | $\mathbb{R}$ |
| $LS'$    | $t, 1 > t > 0$ | $e^{i\pi/4}\mathbb{R}$ | $\mathbb{R}$ |
| $S'B'$   | $it, 0 < t < 1$ | $\mathbb{R}_-$ | $i\mathbb{R}$ |
| $B'F'$   | $e^{it}, \pi/2 > t > 0$ | $\mathbb{R}_-$ | $\mathbb{R}$ |
| $F'S$    | $t, 1 > t > 0$ | $e^{-i\pi/4}\mathbb{R}$ | $\mathbb{R}$ |

which allows us to define

$$
\sqrt{(Z^2 - X^2)(Z^2 - \overline{X}^2)} := \frac{iZ^3}{c(Z^2 + \varphi^2)} \left( g^2 - \frac{1}{g^2} \right). \quad (6.3)
$$

Now we apply (6.3) to (6.1) and obtain

$$
dh = \frac{Z}{\sqrt{(Z^2 - X^2)(Z^2 - \overline{X}^2)}} \cdot \frac{dz}{z}. \quad (6.4)
$$

At (6.4) the equality sign holds because we want Re\{$dh\} = 0$ on the straight line segment $SB$ (see Figure 3(a)). On this segment $z$ is purely imaginary and then we can fix both sides of (6.4) to be equal. Let us now verify if the symmetry curves and lines of $M$ really exist. From Table 1 and (6.4) we write down Table 2.

From Table 2 it follows that $dg/g \cdot dh$ is purely imaginary on $SB$ and $S'B'$. It is real on the other paths, confirming that $M$ will have the expected symmetry curves and lines.

### 7. Solution of the Period Problems

The analysis of the period problems can be reduced to the analysis of the fundamental domain of our minimal immersion. If this fundamental domain is contained in a rectangular prism of $\mathbb{R}^3$, and if the boundary of the former is contained in the border of the latter, we will have that the fundamental piece of our minimal surface will be free of periods.

In order to obtain such a prism, a little reflection will show us that the following two conditions will be enough.

1. The symmetry $\rho_h$ really exists in $\mathbb{R}^3$.
2. After an orthonormal projection of the fundamental domain in the direction $x_3$, we will have $S = S'$ and $B = B'$ (see Figure 5).

The first condition is easy to prove. Take a path $P \to A \to P'$ on $\overline{M}$ as indicated in Figure 6. Consider that $A \rightarrow P'$ with reversed orientation is the image of $P \rightarrow A$ under the
involution \((g, z) \rightarrow (-1/g, z)\). Now we compute in \(\mathbb{R}^3\) what happens to the coordinates of our minimal surface:

\[
(x_1, x_2, x_3)|_{(g, z) \rightarrow (-1/g, z)} = \Re \int_{p=(g, z_0)}^{A=(-i,x)} (\phi_1, \phi_2, \phi_3) = \Re \int_{p=(-1/g_0, z_0)}^{A=(-i,x)} (\phi_1, -\phi_2, \phi_3) = \Re \int_{p=(g, z_0)}^{A=(-i,x)} (-\phi_1, \phi_2, -\phi_3) = (-x_1, x_2, -x_3).
\]

(7.1)

Therefore, our minimal surface is really invariant under 180°-rotations around the \(x_2\)-axis. This proves the existence of the symmetry \(\rho_h\) of our initial assumptions.
Now we are ready to deal with the second condition. Consider Figure 5 with the segments $SB$ and $BL$ on it. The period will be zero if and only if these segments have the same length, or equivalently

$$\text{Re} \int_{SB} \phi_2 = \text{Re} \int_{BL} \phi_1. \tag{7.2}$$

On $SB$ we can take $Z(t) = it$, $2 < t < \infty$. This implies that $dz/z = -dt/\sqrt{t^2 - 4}$. From (4.4) and (6.4) we have

$$\phi_2 \big|_{Z(t)=it} = \frac{c^{1/2}(t + \mathcal{A})}{t^{1/2} \left(t^2 - 2 \text{Im} \{X\} \cdot t + |X|^2\right)^{1/2}} \cdot \frac{dt}{\sqrt{t^2 - 4}}. \tag{7.3}$$

On $BL$ we can take $z(t) = e^{it}$, $0 < t < \pi/2$. From (4.2) and (6.4) it follows that

$$\phi_1 \big|_{z(t)=e^{it}} = \frac{1}{\sqrt{2}} \cdot \frac{c^{1/2}(\mathcal{A} - 2 \sin t)}{\left(4 \sin^2 t + 4 \text{Im} \{X\} \cdot \sin t + |X|^2\right)^{1/2}} \cdot \frac{dt}{\sqrt{\sin t}}. \tag{7.4}$$

Now define $I_1 := (1/\sqrt{2c}) \int_{BL} \phi_1$ and $I_2 := (1/\sqrt{2c}) \int_{SB} \phi_2$. For $I_1$ apply the change of variables $u^2 = \sin t$ and for $I_2$, $t = 2u^2$. A simple reckoning will lead to the following equalities:

$$I_1 = \int_0^1 \frac{\mathcal{A} - 2u^2}{\left(4u^4 + 4 \text{Im} \{X\} \cdot u^2 + |X|^2\right)^{1/2}} \cdot \frac{du}{\sqrt{1 - u^4}}. \tag{7.5}$$

$$I_2 = \int_0^1 \frac{2 + \mathcal{A} u^2}{\left(4 - 4 \text{Im} \{X\} \cdot u^2 + |X|^2 u^4\right)^{1/2}} \cdot \frac{du}{\sqrt{1 - u^4}}. \tag{7.6}$$

The next proposition will solve the period problem given by (7.2).

**Proposition 7.1.** For any fixed positive value of $\text{Re} \{X\}$ one has that the following limit exists and is positive

$$\lim_{\text{Im} \{X\} \to 0} (-\text{Im} \{X\}) \cdot (I_1 - I_2). \tag{7.7}$$

For $\text{Im} \{X\} = -1$ one has that $\lim_{\mathcal{A} \to -2} (I_1 - I_2)$ exists and is negative.
Proof. By recalling (5.4), a simple reckoning will show that

\[
\lim_{\text{Im}\{X\} \to 0} (-\text{Im}\{X\}) \cdot I_1 = \int_0^1 \frac{\text{Re}^2\{X\}}{(4u^4 + \text{Re}^2\{X\})^{1/2}} \cdot \frac{du}{\sqrt{1 - u^4}},
\]

\[
(7.8)
\]

\[
\lim_{\text{Im}\{X\} \to 0} (-\text{Im}\{X\}) \cdot I_2 = \int_0^1 \frac{\text{Re}^2\{X\} \cdot u^2}{(4 + \text{Re}^2\{X\} \cdot u^4)^{1/2}} \cdot \frac{du}{\sqrt{1 - u^4}}
\]

Since

\[
\frac{u^2}{(4 + \text{Re}^2\{X\} \cdot u^4)^{1/2}} < \frac{1}{(4u^4 + \text{Re}^2\{X\})^{1/2}}
\]

(7.9)

for every \(\text{Re}\{X\} > 0\) and \(u \in (0,1)\), from (7.8) the first assertion of Proposition 7.1 follows.

By fixing \(\text{Im}\{X\} = -1\) and recalling (5.5), the convergence \(\mathcal{A} \to 2\) is equivalent to \(\text{Re}\{X\} \to 1\). This means that \(X\) approaches the point \(1 - i\) indicated in Figure 4. An easy calculation will give us

\[
\lim_{\mathcal{A} \to 2} I_1 = \sqrt{2} \int_0^1 (2u^4 - 2u^2 + 1)^{-1/2} \cdot \left(\frac{1-u^2}{1+u^2}\right)^{1/2} \cdot \frac{du}{\sqrt{1 - u^4}},
\]

(7.10)

\[
\lim_{\mathcal{A} \to 2} I_2 = \sqrt{2} \int_0^1 (2 + 2u^2 + u^4)^{-1/2} \cdot \left(\frac{1+u^2}{1-u^2}\right)^{1/2} \cdot \frac{du}{\sqrt{1 - u^4}}.
\]

(7.11)

The integrand of (7.10) can be rewritten as

\[
(\frac{u^4 + (1 - u^2)^2}{1 + u^2})^{-1/2} \cdot \left(\frac{1-u^2}{1+u^2}\right)^{1/2} = \left[\frac{u^4}{(1-u^2)^2} + 1\right]^{-1/2} \cdot \frac{1}{\sqrt{1 - u^4}}.
\]

(7.12)

while one rewriting the integrand of (7.11) as

\[
(\frac{u^4 + (1 + u^2)^2}{1 - u^2})^{-1/2} \cdot \left(\frac{1+u^2}{1-u^2}\right)^{1/2} = \left[\frac{u^4}{(1+u^2)^2} + 1\right]^{-1/2} \cdot \frac{1}{\sqrt{1 - u^4}}.
\]

(7.13)

Since

\[
\frac{u^4}{(1-u^2)^2} > \frac{u^4}{(1+u^2)^2}
\]

(7.14)

for every \(u \in (0,1)\), the last assertion of Proposition 7.1 follows.

Proposition 7.1 provides a family of triply periodic surfaces of which a member is depicted in Figure 1(a). By looking at Figure 4, this family can be represented by the values
of $X$ which belong to a curve $C$ contained in the shaded region. All members of this family will have only three periods, as suggested by Figure 1(a). Nevertheless, a priori there might be some nonembedded members, but it will not be the case. This is the subject of our next section.

8. Embeddedness of the Triply Periodic Surfaces

From now on we will denote our triply periodic surfaces by $M_X$, where $X \in \mathcal{C}$. Figure 6 shows that the projection of the unitary normal on a fundamental domain of $M_X$ is contained in the lower hemisphere of $\hat{\mathbb{C}}$. This means that $(x_1, x_3)$ is an immersion of $\mathcal{G} := \{ z \in \mathbb{C} : |z| < 1 \text{ and } 0 < \text{Arg}(z) < \pi/2 \}$ in $\mathbb{R}^2$. Figure 7 shows a possible image of this map in $\mathbb{R}^2$:

![Figure 7: A possible $x_2$-projection of the fundamental domain on $x_1Ox_3$.](image)

It is not difficult to prove that the contour of the shaded region in Figure 7 is a monotone curve. The $x_1$-coordinate of the curve $BL$ is given by the integral of $-\phi_1$ as in (7.4). The integrand is clearly positive, hence this stretch is monotone. Regarding $LS'$, where we can take $Z(t) = t, 0 > t > -\infty$, a simple reckoning gives us

$$
\frac{dh|_{Z(t)=t}}{\sqrt{t^4 - \text{Re}(X^2)^2 + |X|^4}} \cdot \frac{dt}{\sqrt{t^2 + 4}}
$$

Hence, the stretch $LS'$ is also monotone. By using the symmetry $\rho_h$, it follows that the whole contour indicated in Figure 7 is a monotone curve. Since the third coordinate of $BL$ is increasing, the projections $BL$ and $LS'$ will intersect only at the point $L$. Nevertheless, it can happen that the projection $LS'$ crosses $B'F'$. If we prove that this is not the case, the contour will have no self-intersections. The shaded region will then be simply connected, and we will conclude that the fundamental domain is a graph, hence embedded.

But even so, it can happen that the expanded triply periodic surface will not be embedded. We do not know whether the curve $LS'$ crosses the $x_3$-axis or not. A little reflection will show that, if $g$ does not take the value $-e^{ix/4}$ along $LS'$, then this curve does not intersect the vertical axis. Consequently, its projection will not intersect $B'F'$. In this case, since the triply periodic surface is expanded horizontally by reflections only, and vertically by rotations only, the whole surface will then be embedded.
By using the maximum principle, if we find an embedded member of our family in the curve \( C \), the whole family will then consist of embedded surfaces. The following proposition gives us such a member and will conclude this section.

**Proposition 8.1.** There is an \( X \in -i + (1, 2\sqrt{2}) \) such that \( X \in C \) and \( M_X \) is embedded.

**Proof.** We will prove that \( g \neq -e^{i\pi/4} \) along \( LS' \) for any \( X \in -i + (1, 2\sqrt{2}) \). Moreover, \( (I_1 - I_2)_{\mid X=2\sqrt{2}i} \) will be positive. These two facts together with Proposition 7.1 will conclude Proposition 8.1.

By recalling (5.1), we would have \( g = -e^{i\pi/4} \) for some \( T \in (-\infty, 0) \) if and only if

\[
\left( T^2 - \Re \right) \left( T^2 - |X|^2 \right) = 4\Re \Im \{X\} \cdot T^2. \tag{8.2}
\]

Equation (8.2) will not be fulfilled by any \( T^2 \in (0, \infty) \) providing

\[
\left| \Re + |X|^2 + 4\Re \Im \{X\} \right| < 2\Re |X|, \tag{8.3}
\]

or equivalently

\[
-\frac{\Re |X|}{\Im |X|} < 2\sqrt{2}. \tag{8.4}
\]

We have fixed \( \Im |X| = -1 \), hence \( g \neq -e^{i\pi/4} \) along \( LS' \) for any \( X \in -i + (1, 2\sqrt{2}) \). Let us now verify that \( (I_1 - I_2)_{\mid X=2\sqrt{2}i} > 0 \). From (7.5) we have

\[
I_1_{\mid X=2\sqrt{2}i} = \int_0^1 \frac{9 - 2u^2}{4u^4 - 4u^2 + 9} \frac{du}{\sqrt{1 - u^4}} \tag{8.5}
\]

and from (7.6) it follows that

\[
I_2_{\mid X=2\sqrt{2}i} = \int_0^1 \frac{2 + 9u^2}{4 + 4u^2 + 9u^4} \frac{du}{\sqrt{1 - u^4}} \tag{8.6}
\]

But

\[
\frac{9 - 2u^2}{(4u^4 - 4u^2 + 9)^{1/2}} > 3 - \frac{2}{3}u^2, \quad \forall u \in (0,1), \tag{8.7}
\]

and if we define \( a := 1 - 11/\sqrt{17} \) it is possible to prove that

\[
\frac{2 + 9u^2}{(4 + 4u^2 + 9u^4)^{1/2}} < au^2 - 2au + 1, \quad \forall u \in (0,1). \tag{8.8}
\]
But
\[
\tilde{I}_1 := \int_0^1 \frac{(3 - 2u^2/3)\,du}{\sqrt{1 - u^4}} = \frac{3}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) - \frac{1}{6} B\left(\frac{3}{4}, \frac{1}{2}\right),
\]
\[
\tilde{I}_2 := \int_0^1 \frac{a(au^2 - 2au + 1)\,du}{\sqrt{1 - u^4}} = \frac{a}{4} B\left(\frac{3}{4}, \frac{1}{2}\right) - \frac{a}{2} B\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right). \tag{8.9}
\]

Now we use \(B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m + n)\), \(\Gamma(1/4) = 3.625600\ldots\), \(\Gamma(1/2) = \sqrt{\pi}\) and \(\Gamma(3/4) = 1.225417\ldots\) in order to conclude that
\[
\tilde{I}_1 > \tilde{I}_2. \tag{8.10}
\]

Together with (8.5)–(8.9), (8.10) shows that \((I_1 - I_2)|_{x=2\sqrt{2}-i}\) is positive. \(\square\)

9. The Surfaces \(L_{2,4}\)

In order to prove Theorem 1.2, one follows very similar ideas already explained in Sections 3 to 7. For the surfaces \(L_{2,4}\), consider Figures 8(a) and 8(b). The fundamental piece \(\bar{M}\) has genus 5, and \(Ox_2\) passes through point \(A\). The piece is invariant under \(r_v\) and \(r_h\), with quotient functions \(\rho_v\) and \(\rho_h\), respectively.

Since
\[
\chi\left(\rho_h\left(\rho_v\left(\bar{M}\right)\right)\right) = \frac{1 - 5 + 8/2}{2} + \frac{4}{2} = 2, \tag{9.1}
\]
we may define \(z := \rho_h \circ \rho_v : \bar{M} \to S^2 \approx \hat{C}\), such that \(z(S) = 0, z(B) = 1\) and \(z(L) = \infty\). The symmetries imply \(z(S') = 0, z(B') = 1\), and \(z(L') = \infty\) whereas \(z(A)\) is a certain complex \(x\) in the first quadrant. Moreover, there is a point in the segment \(BS\) at which \(g = 1\). After
analysing the divisors of $z$ and $g$ on $\overline{M}$, together with the behaviour of the unitary normal on symmetry curves and lines, we get

$$\left(\frac{\gamma + 1}{\gamma}\right)^2 = \frac{1/a - a}{|x - a|^2} \cdot \frac{(z - x)(z - \overline{x})(z + a)^2}{z(1 - z^2)}. \quad (9.2)$$

Since there is a point in the segment $FS$ at which $g = -i$, we should also have

$$\left(\frac{\gamma - 1}{\gamma}\right)^2 = \frac{1/a - a}{|x - a|^2} \cdot \frac{(z + x)(z + \overline{x})(z - a)^2}{z(1 - z^2)}. \quad (9.3)$$

In order to have equivalence between (9.2) and (9.3), a necessary and sufficient condition is $\mathcal{A} = a + a^{-1} = (|x|^2 + 1)/\text{Re}(x)$. Now, it is easy to get

$$dh = \frac{idz}{\sqrt{(z^2 - x^2)(z^2 - \overline{x}^2)}}, \quad (9.4)$$

with a well-defined square root in the denominator. One checks the assumed symmetries the same way we did in Tables 1 and 2. The unique period problem is again (7.2), which can be visualised again by Figure 5. Therefore, (7.2) is equivalent to $J_1 = J_2$, where

$$J_1 = \int_0^1 \frac{(t + a)dt}{|t + x|\sqrt{t(1 - t)}}, \quad (9.5)$$

$$J_2 = \int_1^\infty \frac{(t - a)dt}{|t - x|\sqrt{t(t - 1)}}.$$

The change $t \rightarrow 1/t$ for $J_2$ makes clear that $J_1 < J_2$ ($J_1 > J_2$) providing $R_1 < R_2$ ($R_1 > R_2$), where $R_1 = (t + a)/(1 - at)$ and $R_2 = |(t + x)/(1 - xt)|$, $0 < t < 1$. On the one hand, for a fixed $r = \text{Re}(x) > 1$, if $\text{Im}(x) \to 0$ then $a \to 1/r$, and consequently $R_1 < R_2$. On the other hand, by fixing $\text{Im}(x)$ and letting $\text{Re}(x) \to 0$, then $a \to 0$ and so $R_1 > R_2$. In this case, notice that the singularity at $t = 1$ of both integrands in (9.5) is easily removable with a change of variables. This means, no matter if we have $R_1|_{t=1} = R_2|_{t=1}$, it still holds $J_1 > J_2$.

For the surfaces $L_4$, consider Figures 9(a) and 9(b). The fundamental piece $\overline{M}$ has genus 9, and $Ox_2$ passes through point $A$. The piece is invariant under $r_v$ and $r_h$, with quotient functions $\rho_v$ and $\rho_h$, respectively. We will have $g - g^{-1} = \infty$ if and only if $z + z^{-1} \in \{\pm i, 0, \infty\}$. Moreover, $g - g^{-1} = 0$ if and only if $z \in \{-x, \overline{x}, -x^{-1}, \overline{x}^{-1}, ia, -ia^{-1}\}$, where $a \in (0, 1)$.

From this point on we redefine the following:

$$Z := z^{-1} + z, \quad X := x^{-1} + x, \quad \mathcal{A} := a^{-1} - a. \quad (9.6)$$
Based on Figure 9 it is not difficult to conclude that

\[
\left( g - \frac{1}{g} \right)^2 = \frac{ic}{Z^3} \cdot (Z + i\phi)^2(Z + X)(Z - X),
\]

where \( c \) is given by (4.3) again. Moreover, (9.7) is equivalent to

\[
\left( g + \frac{1}{g} \right)^2 = \frac{ic}{Z^3} \cdot (Z - i\phi)^2(Z + \overline{X})(Z - X).
\]

Similar arguments as in Section 5 will give again (5.3) and (5.4), but unlike (5.5) there is no restriction now. Regarding \( dh \), it still holds (6.4), but unlike Figure 5 the period problem is now illustrated by Figure 10.
Integrals $I_1$ and $I_2$ are again given by (7.5) and (7.6), but now the period is solved when $2I_1 = I_2$. This will come with the following.

**Proposition 9.1.** For any fixed positive value of $\text{Re}(X)$ one has that the following limit exists and is positive:

$$\lim_{\text{Im}(X) \to 0} (-\text{Im}(X)) \cdot (2I_1 - I_2).$$

For $\text{Im}(X) = -1$ one has that $\lim_{\text{d} \to 0} (I_1 - 2I_2)$ exists and is negative.

The proof of Proposition 9.1 is quite similar to the proof of Proposition 7.1, and so we will omit it here. The arguments for the embeddedness of $L_{2,A}$ are even easier than the ones used in Section 8 for $C_2$, because now the contours are given by four straight line segments and two curves, pairwise congruent.

**Acknowledgments**

For this present paper, V. R. Batista was supported by the Grants “Bolsa de Produtividade Científica” from CNPq—Conselho Nacional de Desenvolvimento Científico e Tecnológico, and “Bolsa de Pós-Doutorado” FAPESP 2000/07090-5.

**References**

[1] M. Weber, *The genus one helicoid is embedded*, Habilitation thesis, Bonn, Germany, 2000.

[2] M. Weber, “Embedded minimal surfaces of finite topology,” Clay Mathematics Institute 2001 Summer School on the Global Theory of Minimal Surfaces, MSRI, 2001, http://www.msri.org/realvideo/ln/msri/2001/minimal/weber/6/banner/09.html.

[3] G. Hart, “Where are nature’s missing structures?” *Nature Materials*, vol. 6, pp. 941–945, 2007.

[4] E. A. Lord and A. L. Mackay, “Periodic minimal surfaces of cubic symmetry,” *Current Science*, vol. 85, no. 3, pp. 346–362, 2003.

[5] C. J. Costa, “Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends,” *Boletim da Sociedade Brasileira de Matemática*, vol. 15, no. 1-2, pp. 47–54, 1984.

[6] H. Karcher, “Construction of minimal surfaces,” in *Surveys in Geometry*, pp. 1–96, University of Tokyo, 1989.

[7] Lecture Notes, vol. 12, SFB256, Bonn, Germany, 1989.

[8] W. H. Meeks, III and H. Rosenberg, “The global theory of doubly periodic minimal surfaces,” *Inventiones Mathematicae*, vol. 97, no. 2, pp. 351–379, 1989.

[9] J. Pérez, M. M. Rodríguez, and M. Traizet, “The classification of doubly periodic minimal tori with parallel ends,” *Journal of Differential Geometry*, vol. 69, no. 3, pp. 523–577, 2005.

[10] H. Karcher, “The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions,” *Manuscripta Mathematica*, vol. 64, no. 3, pp. 291–357, 1989.

[11] W. H. Meeks, III and B. White, “The space of minimal annuli bounded by an extremal pair of planar curves,” *Communications in Analysis and Geometry*, vol. 1, no. 3-4, pp. 415–437, 1993.

[12] W. Fischer and E. Koch, “A crystallographic approach to 3-periodic minimal surfaces,” in *Statistical Thermodynamics and Differential Geometry of Microstructured Materials*, vol. 51 of *The IMA Volumes in Mathematics and its Applications*, pp. 15–48, Springer, New York, NY, USA, 1993.

[13] M. Traizet, “On the genus of triply periodic minimal surfaces,” *Journal of Differential Geometry*, vol. 79, no. 2, pp. 243–275, 2008.

[14] S. Fujimori and M. Weber, “Triply periodic minimal surfaces bounded by vertical symmetry planes,” *Manuscripta Mathematica*, vol. 129, no. 1, pp. 29–53, 2009.

[15] M. F. da Silva and V. Ramos Batista, “Scherk saddle towers of genus two in $\mathbb{R}^3$,” *Geometriae Dedicata*, vol. 149, pp. 59–71, 2010.
[16] V. Ramos Batista, *Construction of new complete minimal surfaces in \( \mathbb{R}^3 \) based on the Costa surface*, Doctoral thesis, University of Bonn, 2000.

[17] V. Ramos Batista, “The doubly periodic Costa surfaces,” *Mathematische Zeitschrift*, vol. 240, no. 3, pp. 549–577, 2002.

[18] F. J. López and F. Martín, “Complete minimal surfaces in \( \mathbb{R}^3 \),” *Publicacions Matemàtiques*, vol. 43, no. 2, pp. 341–449, 1999.

[19] J. C. C. Nitsche, *Lectures on Minimal Surfaces*, vol. 1, Cambridge University Press, Cambridge, UK, 1989.

[20] R. Osserman, *A Survey of Minimal Surfaces*, Dover Publications Inc., New York, NY, USA, 2nd edition, 1986.

[21] O. Forster, *Lectures on Riemann Surfaces*, vol. 81 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1981.
