Mesoscale structure of wrinkle patterns and defect-proliferated liquid crystalline phases

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Thin solids often develop elastic instabilities and subsequently complex, multiscale deformation patterns. Revealing the organizing principles of this spatial complexity has ramifications for our understanding of morphogenetic processes in plant leaves and animal epithelia, and perhaps even the formation of human fingerprints. We elucidate a primary source of this morphological complexity – an incompatibility between an elastically-favored “micro-structure” of uniformly spaced wrinkles and a “macro-structure” imparted through the wrinkle director and dictated by confinement forces. Our theory is borne out of experiments and simulations of floating sheets subjected to radial stretching. By analyzing patterns of grossly radial wrinkles we find two sharply distinct morphologies: defect-free patterns with a fixed number of wrinkles and non-uniform spacing, and patterns of uniformly spaced wrinkles separated by defect-rich buffer zones. We show how these morphological types reflect distinct minima of a Ginzburg-Landau functional – a coarse-grained version of the elastic energy, which penalizes nonuniform wrinkle spacing and amplitude, as well as deviations of their actual director from the axis imposed by confinement. Our results extend the effective description of wrinkle patterns as liquid crystals (H. Aharoni et al., Nat. Commun. 8:15809, 2017), and we highlight a fascinating analogy between the geometry-energy interplay that underlies the proliferation of defects in the mechanical equilibrium of confined sheets and in thermodynamic phases of superconductors and chiral liquid crystals.
FIG. 1. Director and wavelength of wrinkle patterns. (A) Uniaxial confinement of an ultrathin sheet floating on water. The sheet undulates along the director $\hat{n}$ to suppress compression in this axis, forming a parallel array of wrinkles with a wavelength, $\lambda = 2\pi(B/\rho g)^{1/4}$ (black bar). Deviations from this pattern appear only at the edge, where a liquid meniscus affects a wrinkling cascade [from Ref. [16]]. (B) Wrinkle pattern within an ultrathin spherical shell floating on a liquid bath, where the director varies across domains of parallel, uniformly-spaced wrinkles. (C) Simulation of a circular thin sheet attached to a ball of stiff springs of constant $\propto K_{\text{sub}}$, where confinement imposes a purely-bent director ($\hat{n} = \hat{\theta}$), incompatible with wrinkles separated by a constant, locally-favored wavelength, $\tilde{\lambda} = 2\pi(B/K_{\text{sub}})^{1/4}$ [17, 18]. The pattern exhibits defect-rich buffers and pronounced amplitude modulations, characterized by a length $\ell_{\parallel}$ (white bar shows the prediction of our theory, Eq. 17). (D) The Lamé set-up is an annular sheet under coaxial, co-planar tensile loads, $\gamma_{\text{in}}, \gamma_{\text{out}}$. Oriented wrinkles (namely, $\hat{n} \approx \hat{\theta}$). In a “monochromatic” pattern of $m_0$ wrinkles the spacing varies with radial distance, $\lambda(r) \approx (2\pi/m_0)r$, and is thus incompatible with any locally-favored $\lambda$ that is not $\propto r$. We discover that prominent outcomes of this conflict are modulations of the wrinkle amplitude over a mesoscale scale $\ell_{\perp}$, which depends on the locally-favored wavelength and confinement conditions, and the proliferation of defect-rich regions, where the number of wrinkles varies sharply (Figs. 1c, 2b, 2d). The combined effect of defects and amplitude modulations suggests an analogy between mesoscale wrinkling phenomenology and defect-proliferated phases of liquid crystals and superconductors [14, 15].

MODEL SYSTEM

A bent director may appear whenever a confined solid with a “target” (i.e. metrically-favored) Gaussian curvature $G_{\text{tar}}$ is forced to reside near a “substrate” whose shape has a curvature $G_{\text{sub}} > G_{\text{tar}}$ (e.g. a naturally-planar sheet attached to a liquid drop [19] or a rigid sphere [21, 18], Fig. 1c). However, for concreteness and clarity we focus on the classical Lamé set-up, whose numerous variants were studied extensively, allowing us to exploit a host of experimental [20, 22], numerical [23], and analytic techniques [23, 24]. A schematic of the model is shown in Fig. 1d: a thin solid annulus of thickness $t$ and radii $R_{\text{in}} \ll R_{\text{out}}$ is attached to a “Winkler substrate” of stiffness $K_{\text{sub}}$ (e.g. $K_{\text{sub}} = \rho g$ for a liquid bath of density $\rho$), and subjected to tensile loads, $\gamma_{\text{in}}, \gamma_{\text{out}}$ that pull the edges inward and outward, respectively. The stretching and bending moduli of the sheet are $Y = Et \gg \gamma_{\text{in}}, \gamma_{\text{out}}$, and $B = Et^2/12(1 - \Lambda^2)$, where $E, \Lambda$ are the Young’s modulus and Poisson’s ratio, respectively. The problem is governed by three dimensionless groups, to which we refer,
FIG. 2. Characteristic patterns observed in experiment (A-B) and simulation (C-D) for various values of the control parameters (Eq. 1), as listed in the figure. (In panels A,B, we subtracted a background obtained by a uniform Gaussian filter.) In the experiment, the Bond number is controlled primarily by the droplet size, and we use larger sheets to accommodate longer wrinkles. (A) $R_{\text{in}} = 1.4$ mm, $R_{\text{out}} = 6.9$ mm. (B) $R_{\text{in}} = 4.3$ mm, $R_{\text{out}} = 43$ mm. Panels (E-H) show the wrinkle number $m$ versus radial coordinate $r$ for the images above them. Solid lines show the value that would result from a local balance of substrate stiffness and bending: $m(r) = 2\pi r/\bar{\lambda} = r(\rho g/B)^{1/4}$. Note that the boundary condition at $r = R_{\text{in}}$ tends to increase the wrinkle number in its vicinity. (I) A “phase diagram”, spanned by the parameters $Bo$ and $\tau$ is divided into two regimes where wrinkle patterns are defect-rich (blue symbols, where the undulation wavelength $\approx \bar{\lambda}$, panels B,D and F,H), and defect-free (red symbols, where the wrinkle number is constant, panels A,C and E,G). Dashed line: Empirical scaling relation, Eq. (3), with a fitted numerical prefactor.

respectively, as the “confinement”, tensional “bendability”, and a Bond-like parameter (i.e. the ratio between substrate stiffness and tensile stress):

$$\tau = \frac{\gamma_{\text{in}}}{\gamma_{\text{out}}}; \quad \epsilon^{-1} = \frac{\gamma_{\text{in}} R_{\text{in}}^2}{B}; \quad Bo = \frac{K_{\text{sub}} R_{\text{in}}^2}{\gamma_{\text{in}}}. \quad (1)$$

We study thin, highly bendable sheets ($10^4 < \epsilon^{-1} < 10^8$). If $\tau$ exceeds a finite threshold ($\gtrsim 2$), hoop confinement emerges in an annulus, $R_{\text{in}} < r < L$, which expands upon increasing $\tau$, and compression is suppressed through azimuthal undulations, yielding radially-oriented wrinkles. Macro-scale features of the pattern are governed by $\tau$ (see below), whereas the wavelength scales as $\lambda(r) \sim \epsilon^{1/4}$ [10, 20, 22, 24]. Experiments exhibit a largely uniform wavelength when the substrate stiffness is strong ($Bo \gg 1$) [22], and a constant wrinkle number (i.e. $\lambda(r) \propto r$) for a weak substrate [20, 21, 25], suggesting that:

$$\lambda(r)/R_{\text{in}} \approx \epsilon^{1/4} \cdot \begin{cases} 2\pi \cdot Bo^{-1/4} & \text{if } Bo \gg 1 \\ C(\tau) \cdot r/R_{\text{in}} & \text{if } Bo \ll 1, \end{cases} \quad (2)$$

(where $C(\tau)$ is some smooth function).

Our experiments and simulations explore a broad range of $Bo$ values. The simulations employ a finite element method, similarly to [24], with a (locally-hexagonal) disordered mesh to suppress spurious lattice effects on the pattern. The experiments use spin-coated, ultrathin polystyrene sheets (thickness $60 < t < 308$ nm) of circular shape (radius $6.9 < R_{\text{out}} < 44$ mm) that are floated on a liquid bath and subjected to radial tension $\gamma_{\text{out}} = \gamma_{lv} = 70 \pm 2$ mN/m at $r = R_{\text{out}}$. A liquid drop placed at the center of the sheet forms a circular contact line of radius $R_{\text{in}}$, yielding capillary-induced tension, $\gamma_{\text{in}} > \gamma_{lv}$ [26]; see Supplementary Information (SI).
The images in Fig. 2 show two contrasting responses: a spatially-constant wrinkle number (i.e. non-constant wavelength, \( \lambda \propto r \)), and patterns with nearly-constant wavelength (\( \lambda \approx \lambda_0 \)), as shown in Figs. 2a,2b, with measurements of \( m(r) \) in Figs. 2f,2h). The latter patterns are significantly more complex, as they require the proliferation of defects, i.e., points where wrinkles are created away from boundaries. These patterns also exhibit strong modulations of the wrinkle amplitude, such that defect-rich regions occur at amplitude-suppressed zones. Figure 2i collects our observations into a phase diagram. The observed transition between defect-free and defect-rich states forms a curve \( B_0c(\tau) \) in the parameter plane \( (\tau, B_0) \), which we find to scale as:

\[
B_0c(\tau) \sim (\tau - 2)^{-3.4 \pm 0.3}.
\]

**THEORY**

The above findings motivate us to focus on the morphologically-rich regime at large Bond number. This section and a subsequent one describe succinctly our theoretical approach, delegating many technical details to SI. Our starting point is “tension field theory” (TFT), which provides the leading-order Föppl-von Kármán (FvK) elastic energy at the singular limit of infinite bendability [24]. Positing that the ratio between the compressive (hoop) and tensile (radial) components of the stress tensor must vanish as \( \epsilon \to 0 \), in-plane force balance in the wrinkled zone \(( R_{in} < r < L)\), and matching to the purely-tensile (unwrinkled) region yield:

\[
(R_{in} < r < L) : \sigma_{rr} \approx \gamma_{in} R_{in}/r ; \quad L \approx R_{in} \tau/2.
\]

Suppression of hoop compression requires matching the contraction, \( u_r/r \), due to the displacement \( u_r(r) \) underlying (4), to the fraction of latitudinal length \( \Phi^2 \) “wasted” by wrinkles:

\[
\Phi^2(r) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left( \sqrt{1 + \left( \frac{1}{r} \frac{\partial \zeta}{\partial \theta} \right)^2} - 1 \right) \approx \frac{1}{4\pi r^2} \int_0^{2\pi} d\theta (\frac{\partial \zeta}{\partial \theta})^2 = 2(\gamma_{out}/Y)(L/r) \log(L/r) ,
\]

where \( \zeta(r, \theta) \) is the deflection from the plane [24]. Notably, these macro-scale features characterize the singular limit \( \epsilon \to 0 \), being indifferent to the wavelength \( \lambda(r) \). The corresponding limit value of elastic energy (in comparison to a state with uniform strain \( \gamma_{out}/Y \)) does not depend on \( \epsilon \) or \( B_0 \), namely:

\[
U_{dom}(\tau) \approx -\frac{\pi R_{in}^2 \gamma_{out}^2}{Y} \tau^2 (\log \frac{\tau}{2} + \Lambda - \frac{1}{2}) .
\]

In order to remove this shape degeneracy, one must minimize a \( (\epsilon \)-dependent) contribution to the FvK energy that penalizes bending and substrate deformation, by solving the 1st FvK equation (assuming \( |\partial_\theta \zeta| \gg r |\partial_r \zeta| \)):

\[
(L_0 + L_1) \zeta(r, \theta) = 0
\]

where: \( L_0 = B \frac{1}{\tau^4} \frac{\partial^4}{\partial \theta^4} - \sigma_{\theta \theta} \frac{1}{\tau^2} \frac{\partial^2}{\partial \theta^2} + K_{sub} ; \quad L_1 = -\sigma_{rr} \frac{\partial^2}{\partial r^2} .
\]

Here, \( \sigma_{rr} \) is given by Eq. (4) and \( \sigma_{\theta \theta} \) acts as a Lagrange multiplier that enforces the condition (5), analogous to the inextensibility constraint underlying one-dimensional (1D) elastica.

Specializing to a narrow annulus, \( r \in r_0 \pm \ell_\perp \), where \( \ell_\perp \ll r_0 \) and \( R_{in} \ll r_0 < L \), we notice that Eqs. (5,7) describe a collection
of 1D, decoupled elastica rings of length $2\pi r$, subjected to confinement $\Phi^2(r)$, Eq. (5), and substrate stiffness $K_{\text{sub}}$. This motivates constructing a “monochromatic” ansatz from periodic elastica solutions, labeled by an integer $m$:

$$\zeta(r, \theta) \approx \Psi(r) g(\theta)$$

with $g(\theta) = \cos(m\theta)$; $\Psi(r) = 2r\Phi(r)/m$, (8)

The resulting energy of bending and substrate deformation is:

$$U_m \approx \bar{C} + \frac{1}{2} B \int d\theta \int dr \, \Psi(r)^2 r^{-3} m^2 (m - \bar{m}(r))^2$$ 

where : $\bar{m}(r) \equiv 2\pi r/\bar{\lambda}$ ; $\bar{\lambda} \equiv 2\pi \sqrt{B/K_{\text{sub}}}$, (10)

and $\bar{C}$ is a constant $\sim \sqrt{BK_{\text{sub}}}$. Minimizing $U_m$ over integers $m$ yields $m \approx m_a = 2\pi r_a/\bar{\lambda}$, recovering Eq. (2) for $Bo \to \infty$.

The argument thus far is merely a reformulation of previous analyses of the wrinkle wavelength [1, 10], underscoring two difficulties. First, the ansatz (8) does not indicate how transitions occur between distinct values of $m_a$ at adjacent narrow annuli. Second, for finite $Bo \gg 1$ the perturbation imposed by the operator $L_1$ is resonant (i.e. $L_1 \zeta \propto g(\theta) = \cos(m\theta)$ is a zero mode of $L_0$), akin to periodically driving a harmonic oscillator at its resonant frequency. Hence, Eq. (7) is impervious to regular expansion around the ansatz (8) unless $L_1 \zeta = 0 \Rightarrow \Psi''(r) = 0$, which is incompatible with Eq. (5).

Motivated by multi-scale perturbation theory of nonlinear dynamics problems [27], we generalize the ansatz (8) to:

$$\zeta(r, \theta) \approx \text{Re}[\Psi(r, \theta) e^{im_a \theta}]$$

where the complex amplitude $\Psi(r, \theta)$ satisfies $|\partial_\theta \Psi| \sim Bo^{-b} m_a |\Psi| \ll m_a |\Psi|$ for some $b > 0$. For the ansatz (12), the avoidance of resonant effects in a perturbative expansion of Eq. (7) around the limit $Bo = \infty$ implies the equation:

$$\left[\sigma_{rr} \frac{\partial^2}{\partial r^2} + 4|\sigma_{\theta\theta}| \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \Psi(r, \theta) = 0$$

such that the only nontrivial value of the exponent $b$, for which the two terms in Eq. (13) are in balance for $Bo \gg 1$, is $b = 1/2$ (where we used Eqs. (9,11)). In contrast to the ansatz (8), Eq. (13)
admits azimuthally-oscillatory solutions to the wrinkle amplitude. Crucially, this facilitates a mechanism for transitioning between distinct integer values of \( n_b \) at adjacent annuli: defects can nucleate within amplitude-suppressed zones (\( \Psi(r, \theta) \approx 0 \)) at negligible energy cost.

Within a narrow, defect-free annulus around \( r = r_a \), an azimuthally-oscillatory solution to Eq. (13) satisfies: \( \Psi \propto \exp[-2\pi(r-r_a)\ell^*_{BC}/r_a] \). A crude estimate of \( \ell_{\perp} \), over which the amplitude varies azimuthally, may be obtained by requiring \( \partial \Psi / \partial r \sim \Psi / r_a \) to ensure compatibility with Eq. (5), and recalling Eqs. (9,11):

\[
\ell_{\perp} \approx \ell_{BC}^* \cdot r_a / \lambda \quad \text{where:} \quad \ell_{BC}^* \equiv \sqrt{B/\sigma_{rr}(r_a)},
\]

where \( \ell_{BC}^* \) is a (local) “bendo-capillary” length [26].

### COARSE-GRAINED ENERGY AND AMPLITUDE MODULATIONS

Let us now elaborate on the energetic hierarchy imparted by the ansatz [12], and derive a generic, quantitative version of Eq. (14). Exploiting the separation of scales between the wrinkly undulations and their slowly-varying amplitude, we derive in the generic, quantitative version of Eq. (14). Exploiting the separation of scales between the wrinkly undulations and their slowly-varying amplitude, we derive in the

\[
U_{\Psi} \approx \frac{1}{2} \int d\theta \int dr \left( \sigma_{rr} |\partial_r \Psi|^2 + 4 |\sigma_{\theta\theta}| \frac{1}{\sqrt{r_a}} |\partial_{\theta} \Psi|^2 \right),
\]

where \( \Phi, \sigma_{rr}, \sigma_{\theta\theta} \) are given by Eqs. (11,12). The sum \( U_{\text{nonlin}} + U_m + U_{\Psi} \) describes the deviation of the energy of an actual wrinkle pattern from the TFT limit value (6), reflecting a hierarchy of energetic costs. \( U_{\text{nonlin}} \) includes quartic terms in \( \Psi \), which must be considered since they are penalized by a large stretching modulus, \( Y \), reflecting strain in excess of the residual level already accounted for in the limit value (6). The term \( U_m \) (given by Eq. (10) above), reflects a balance between bending and substrate (i.e., liquid gravity) energies, yielding the favored wrinkle wavelength, \( \lambda \). The last term, \( U_{\Psi} \) reflects the energy cost for amplitude modulations and thereby the emergence of defects to enable proximity of the wavelength \( \lambda(r) \) to \( \lambda \) throughout the pattern. Its typical value is \( \mathcal{O}(B_0^{-1}) \) relative to the energy incurred by \( U_m \), and its corresponding Euler-Lagrange equation is precisely Eq. (13).

For a small, defect-free zone of azimuthal and radial extents \( \ell_{\perp}, \ell_{\parallel} \ll r_a \), the second term in Eq. (15) indicates that strain is induced by any deviation of wrinkles from the tension-carrying lines (i.e., radial lines, for which \( \partial \Psi / \partial \theta = 0 \)). Hence, the radial orientation of wrinkles persists in defect-free zones, locally suppressing the smectic order imparted by uniformly-spaced wrinkles (for which \( \partial \Psi / \partial \theta \approx m(r) - m_a \)). In contrast, the first term in Eq. (15) does not vanish for any azimuthally-oscillating amplitude, but its cost can be made negligible (energy density \( \sim Y(\ell_{\perp}/r_a)^4 \)), by requiring the corresponding integrand and its radial derivative to vanish upon integrating over an oscillatory period of \( \Psi \), yielding:

\[
\ell_{\perp} \approx 4\pi^2 \ell_{BC}^* \ell_{\text{bend}}^* / \lambda \quad \text{where} \quad \ell_{\text{bend}}^* = \frac{1}{r_a} + \left| \frac{\Phi'(r_a)}{\Phi(r_a)} \right|.
\]

In addition to replacing the scaling relation in Eq. (14) by a number \( (4\pi^2) \), the length \( r_a \) is replaced by yet another local length \( \ell_{\text{bend}}^* \), which derives from the (planar) curvature of the axis \( \theta \) along which wrinkles suppress an imposed compression.
GENERAL FORM OF THEORETICAL RESULTS

The arguments underlying Eq. (17) apply to a broad class of confinement problems, for which a wavelength \( \lambda(x) \) is favored in the vicinity of a point \( x \) on the sheet by competition of bending resistance and substrate-induced stiffness (akin to Eqs. 2). For given boundary loads and substrate shape, a TFT solution may be found analytically or numerically \[28\], yielding the macro-scale fields – a director \( \hat{n}_{aux}(x) \) along which wrinkles “waste” a fraction of arclength \( \Phi_{aux}^2(x) \), and a tensile stress \( \sigma_{x,aux}(x) \) (akin to Eqs. 5,4, respectively). Here, “aux” refers to an auxiliary state that describes the singular, infinite bendability limit, of a hypothetic body with zero bending modulus, and “\( ||, \perp \)” denote the (curvilinear) planar axes, along and normal to \( \hat{n}_{aux}(x) \). Our analysis predicts that if \( \hat{n}_{aux}(x) \) is bent \( (\nabla \times \hat{n}_{aux} \neq 0) \) then a defect-riddled pattern consists of defect-free domains where \( \lambda(x) \approx \lambda(x) \), whose longitudinal scale, \( \ell_{\perp} \), is given by Eq. (17), with:

\[
\ell_{BC}^\perp = \sqrt{\frac{B}{\sigma_{x,aux}(x)}}, \ell_{\perp}^{-1} = |\nabla \times \hat{n}_{aux}(x)| + \left| \frac{\partial_{\perp} \Phi_{aux}(x)}{\Phi_{aux}(x)} \right|.
\]

The applicability of our result to another axial confinement problem is demonstrated in Fig. 1, which shows wrinkles near the edge of a circular sheet of radius \( W \) attached to a spherical Winkler foundation (a ball of springs of stiffness \( \propto K_{sub} \) and rest length \( R > W \) \[17\] \[18\]). For that problem, the tensile bendability \( \epsilon^{-1} \) and Bond number \( Bo \) are defined similarly to Eq. (1), with \( \gamma_{in} \) replaced by the tensile load \( \gamma \) exerted at the perimeter, and the parameter analogous to \( \tau \) is \( \alpha = \frac{\gamma W^2}{\gamma_{in} R^2} \), which controls the strength of azimuthal confinement. A TFT solution (Sec. IV.A-C in Ref. [17]) yields expressions for \( \sigma_{rr}(r) \) and \( \Phi(r) \) analogous to Eqs. (15), which we substitute in Eq. (18) to compute \( \ell_{\perp} \) at \( r = W \) (scale bar in Fig. 1c).

COMPARISON WITH EXPERIMENTS AND SIMULATIONS

Figures 4 and 4 show the azimuthal profile of a sheet with \( Bo > Bo_c \) at a single radius, which we quantify using the image intensity \( I(\theta) \) in experiments \[10\], and the vertical deflection \( \zeta(\theta) \) in simulations. A pronounced wrinkle wavelength is clearly present, with large amplitude modulations. Figures 4 and 4 show the measured lengthscale of these modulations at several radii (normalized by the measured \( \lambda(x) \)). Figure 4 compares measured and predicted values of \( \ell_{\parallel}/\lambda \) for several values of \( Bo \) and \( \tau \). No fitting parameters are used. The experiments exhibit good agreement with the theory, whereas the simulation results are systematically lower than expected and approach the predicted values only at large \( Bo \) and small \( \tau \), where the extent of the wrinkled zone is relatively small (Eq. 1). We attribute this discrepancy to the numerical difficulty in reaching the most low-lying energy states. In particular, at large \( \tau \) we find many metastable states where defects are scattered throughout the wrinkled region, thereby decreasing the characteristic distance \( \ell_{\perp} \) between the defect-rich zones.

Considering the phase diagram (Fig. 2), we note that a transition between defect-proliferated and defect-free patterns can be rationalized by comparing the energies \( U_m \) \[10\] and \( U_\psi \) \[16\], whose respective scalings with the length of the wrinkled zone \( (L - R_{in} \sim \tau - 2) \) can be found for \( 0 < \tau - 2 \ll 1 \) (see SI). The energy \( U_m \) of a defect-free pattern (described by Eq. (5)) such that \( m(r) = m_0 \neq \bar{m}(r) \) is larger than that of a defect-proliferated pattern (where \( m(r) \approx \bar{m}(r) \)). The relative energy gain for defect proliferation is \( \Delta U_m \sim K_{sub} R_{in}^2 (\tau - 2)^4 \). In contrast, the azimuthal modulations of the amplitude in a defect-proliferated pattern, which are absent from a defect-free pattern, entail a constant relative energy cost \( \Delta U_\psi \sim \gamma_{in} \) as \( \tau \rightarrow 2 \). These estimates thus predict the scaling \( Bo_c(\tau) \sim (\tau - 2)^{-4} \). Although this exponent differs from the measured value of \( -3.4 \pm 0.3 \) in Eq. (3), our theoretical estimates assume \( 0 < \tau - 2 \ll 1 \), whereas the data are for \( (\tau - 2) > 1 \).
FIG. 4. Amplitude modulations and their lateral lengthscale, $\ell_\perp$ (normalized by the wavelength $\lambda$). (A) Image intensity versus $\theta$ along a circle of radius $r = 26$ mm from the experiment in Fig. 2 with $Bo = 0.25$ and $\tau = 10$. The signal oscillates with a wavelength $r/R_\in$ having longer wrinkles), we measure $\ell$ for our theory ($Bo < 1$, the local parameter, $Bo = K_{sub}r_n^2/\sigma_{tr}(r_n)$ whose inverse is the expansion parameter in our theory (SI Appendix), is substantially larger than 1 in most of the wrinkled zone. Inset: The measured wrinkle number agrees with the gravity-dominated value, $\bar{m}(r) = r(\rho g/B)^{1/4}$ (solid line). Data in the inset are sampled at the same radii as in the main panel.

GEOMETRIC CONFLICTS AND DEFECT-PROLIFERATED STATES

To place our study in a broader context, we consider a thermodynamic ensemble of elongated molecules (“nematogen”). Upon cooling or increasing density, this prototypical system transforms from an isotropic liquid to a “nematic” phase, where the molecular axes are parallel on average. The corresponding order parameter, reflecting broken rotational symmetry, is a director field with a uniform ground state ($\hat{n}(x) = \hat{n}_0$). Cooling further leads to a “smectic-A” phase, where molecules form uniformly spaced layers with normals parallel to the molecular director. The broken translational symmetry underlies a complex order parameter with ground state $\Psi(x) = A_0 \exp[i\hat{n}_0 \cdot x/\lambda_0]$, characterizing the average spacing ($\lambda_0$) and favorable magnitude ($A_0$) of smectic order. The Landau-de Gennes energy describes the elastic response of the smectic phase, penalizing $|\Psi(x)| \neq A_0$, as well as $|\nabla \hat{n}(x)| \neq 0$ and $\partial_\parallel^2 \arg \Psi \neq 0$.[14]

Such a simultaneous satisfaction of favorable orientational and translational orders is impossible if the nematogen is chiral. Here, the nematic is replaced by a “cholesteric” phase, of a twisted director ($\hat{n}(x) \cdot [\nabla \times \hat{n}(x)] = q_0$ on average, where $q_0$ derives from the nematogen structure). In contrast to the non-chiral case, it is impossible to stack flat, uniformly spaced layers along a twisted director. Invoking an analogy to the Abrikosov lattice of magnetic flux lines in a type-II superconductor, this geometric conflict was predicted to give rise to a twist grain boundary (TGB) phase[15] – an inhomogeneous ground state where planes populated with screw-type dislocations separate mutually-tilted smectic domains. By suppressing smectic order near localized grain boundaries...
(i.e. $|\Psi(x)| < A_0$), the TGB phase enables the molecular director to attain there its desired twist $(\hat{n}(x) \cdot [\nabla \times \hat{n}(x)] \approx q_0)$. The result is a spatial structure of alternating smectically-satisfied domains separated by cholesterically-satisfied buffer zones.

The crucial role of the director in 3d liquid-crystal phases and 2d wrinkled sheets motivates us to further develop the analogy between these systems [29]. When the energetics favor a constant director, a defect-free layered structure may emerge: the smectic-A phase for non-chiral nematogen, or a parallel array of uniform wrinkles for a uniaxially-confined sheet. In the geometrically-conflicted case of a non-constant director (twisted for chiral nematogen and bent for an azimuthally-confined sheet), one finds an inhomogeneous, defect-proliferated structure that retains parallel layers in separated domains: the TGB phase or a defect-rich wrinkled film.

This analogy is bolstered by contrasting our coarse-grained energy, $U_m + U_\Psi + U_{\text{nonlin}}$ (Eqs. [10, 16, 15]), and that of chiral nematogen (Eq. 2.11 of Ref. [15]). In our 2d wrinkled sheet, the energy $U_m$ favors smectic order, whereas the confining forces favor a bent director, $\hat{n}_{\text{aux}} = \hat{\theta}$, through the energy $U_{\text{nonlin}}$. If the energetic penalty of $U_m$ is small, the director $\hat{n}_{\text{aux}}$ is imposed forcefully, precluding smectic order (i.e. $\lambda(r) \propto r$) in analogy with the cholesteric phase ($B_0 < B_0_c$ in Fig. 2). In contrast, if $B_0 > B_0_c$, proliferation of defect-rich, amplitude-suppressed zones enables a partial recovery of smectic order (i.e., $|\lambda(r) - \bar{\lambda}| \ll \bar{\lambda}$) in defect-free domains, in analogy with the TGB phase. The parameter $B_0$ is thus akin to a Frank modulus, which penalizes deviations of the director’s twist from the value $q_0$, imparted by a nemtogen chirality. Furthermore, the size (“coherence length”) of smectic domains in the TGB phase derives from the minimal energy associated with varying the order parameter $|\Psi| : 0 \rightarrow A_0$ between defect-rich planes and defect-free domains. This is again similar to our system, where the azimuthal extent $\ell_\perp$ of defect-free zones (Eq. [17]) derives from the energy $(U_\Psi + U_{\text{nonlin}})$ required to generate regions where the wrinkle amplitude $|\Psi|$ is suppressed.

**DISCUSSION**

Wrinkle patterns – highly curved periodic undulations that “waste” an excess length – are common in strongly confined thin solids that are forced to reside close to a smooth substrate. For such problems, tension field theory or its recent extension [18] predict a macro-scale thickness-independent director field $\hat{n}_{\text{aux}}(x)$, a micro-scale wavelength $\bar{\lambda}(x)$ and a corresponding stress field. The simplest pattern – a smectic-like array of uniformly-spaced, parallel wrinkles – emerges when both $\hat{n}_{\text{aux}}(x)$ and $\bar{\lambda}(x)$ are constants. However, if either field is spatially-varying, it may be impossible for the confined body to satisfy both fields everywhere, and the ensuing negotiation gives rise to a host of mesoscale morphologies.

Our work addresses confinements characterized by a uniform $\bar{\lambda}$ and a purely bent director, (i.e. $\nabla \cdot \hat{n}_{\text{aux}} = 0, \nabla \times \hat{n}_{\text{aux}} \neq 0$). Studying the Lamé set-up as a prototypical model for such problems, we showed that the pattern may either consist of a fixed number of wrinkles, absent of smectic order, or be characterized by amplitude modulations over a mesoscale $\ell_\perp$ (Eq. [17]) that enable proliferation of defect-rich zones and thereby partial recovery of smectic order (lower and upper parts of Fig. 2, respectively). A different type of confinement with uniform $\bar{\lambda}$ and non-uniform but unbent director ($\nabla \times \hat{n}_{\text{aux}} \approx 0$) yields a qualitatively different mesoscale structure, an example of which was realized by forcing a patch of a spherical shell to reside close to a plane [29, 30] (Fig. 1b). Here, the unbent director is piece-wise constant, containing splay ($\nabla \cdot \hat{n}_{\text{aux}} \neq 0$) only at “domain walls” that separate defect-free, smectically-ordered domains of uniformly-spaced wrinkles [29].

Two other notable confinement types may occur even under uniaxial compression (e.g. $\hat{n}_{\text{aux}} = \hat{y}$), if the locally-favorable wavelength is spatially-varying such that $\nabla \cdot \bar{\lambda} \times \hat{n}_{\text{aux}} \neq 0$, or the wrinkle amplitude itself is forced to vary spatially such that $\nabla \Psi \times \hat{n}_{\text{aux}} \neq 0$. A recent experimental study
that addressed the former case employed a sheet with a non-uniform thickness (where patterns of defects that resemble Fig. 2c,d were observed) \[31\], whereas the latter type, which underlies “wrinkle cascades” (Fig. 1k), was realized through amplitude-suppressing boundary conditions \[16, 32\].

While our theoretical analysis pertains to confinement problems with bent director fields (\(\nabla \times \hat{n}_{\text{aux}} \neq 0\)), we anticipate that the coarse-graining approach initiated in Ref. \[29\] and further developed here may provide a unified framework to analyze mesoscale structures in a broad class of confinement problems \[4\]. At its core there is an energy functional of a slowly-varying complex function, whose magnitude describes the wrinkle amplitude and whose phase describes deviations from an asymptotic (thickness-independent) director field \(\hat{n}_{\text{aux}}\), imposed by the confining forces. Such a Ginzburg-Landau-like functional may be obtained by expanding the full elastic energy around a suitable TFT limit and coarse-graining over the wrinkle micro-scale. Pursuing this approach further may reveal new analogies with liquid crystals and superconductors, and elucidate the remarkable complexity of wrinkle patterns.

**MATERIALS AND METHODS**

**Experiments.** Polymer films were made by spin-coating solutions of polystyrene (\(M_n = 99k, M_w = 105.5k\), Polymer Source) in toluene (99.9%, Fisher Scientific) onto glass substrates following Ref. \[20\]. A white-light interferometer (Filmetrics F3) was used to measure film thickness, which was uniform over each film to within 4%. Methods for determining \(\gamma_{\text{in}}, \gamma_{\text{out}}\), and descriptions of our image analysis routines are provided in the SI Appendix.

**Simulations.** Finite-element simulations were performed in ABAQUS, as detailed in the SI Appendix.

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[1] E. Cerda and L. Mahadevan, “Geometry and physics of wrinkling,” Phys. Rev. Lett. 90, 074302 (2003).
[2] J. Hure, B. Roman, and J. Bico, “Stamping and wrinkling of elastic plates,” Phys. Rev. Lett. 109, 054302 (2012).
[3] N. Stoop, R. Lagrange, D. Terwagne, P.M. Reis, and J. Dunkel, “Curvature-induced symmetry breaking determines elastic surface patterns,” Nat. Mater. 14, 337–342 (2015).
[4] Joseph D Paulsen, “Wrapping liquids, solids, and gases in thin sheets,” Annual Review of Condensed Matter Physics 10, 431–450 (2019).
[5] Y. Klein, S. Venkataramani, and E. Sharon, “Experimental study of shape transitions and energy scaling in thin non-euclidean plates,” Phys. Rev. Lett. 106, 118303 (2011).
[6] N. Bowden, S. Brittain, A. G. Evans, J. W. Hutchinson, and G. M. Whiteside, “Spontaneous formation of ordered structures in thin films of metals supported on an elastomeric polymer,” Nature 393, 146–149 (1998).
[7] D. Breid and A. J. Crosby, “Effect of stress state on wrinkle morphology,” Soft Matter 7, 4490–4496 (2011).
[8] T. Savin, N.A. Kurpios, A.E. Shyer, P. Florescu, H. Liang, L. Mahadevan C., and Tabin, “On the growth and form of the gut,” Nature 476, 57 (2011).
[9] M. Kücken and A.C. Newell, “A model for fingerprint formation,” EuroPhys. Lett. 68, 141–146 (2004).
[10] J. D. Paulsen, E. Hohlfeld, H. King, J. S. Huang, Z. Qiu, T. P. R. Russell, N. Menon, D. Vella, and B. Davidovitch, “Curvature-induced stiffness and the spatial variation of wavelength in wrinkled sheets,” Proc. Nat. Aca. Sci. USA 113, 1144–1149 (2016).
[11] E. H. Mansfield, The Bending and Stretching of Plates (Cambridge University Press, 1989).
[12] M. Stein and J. M. Hedgepeth, Analysis of Partly Wrinkled Membranes, Tech. Rep. (NASA, 1961).
[13] A. C. Pipkin, “The relaxed energy density for isotropic elastic membranes,” IMA J. Appl. Math. 36, 85–99 (1986).
[14] P.G de Gennes and J. Prost, The Physics of Liquid Crystals 2nd Ed. (Oxford University Press, New York, 1993).
[15] S.R. Renn and T.C. Lubensky, “Abrikosov dislocation lattice in a model of the cholesteric to smectic-a transition,” Phys. Rev. A 38, 2132–2147 (1988).
[16] J. Huang, B. Davidovitch, C. D. Santangelo, T. P. Russell, and N. Menon, “Smooth cascade of wrinkles at the edge of a floating elastic film,” Phys. Rev. Lett. 105, 038302 (2010).
[17] E. Hohlfeld and B. Davidovitch, “Sheet on a deformable sphere: Wrinkle patterns suppress curvature-induced delamination,” Phys. Rev. E 91, 012407 (2015).
[18] B. Davidovitch, Y. Sun, and G.M. Grason, “Geometrically incompatible confinement of solids,” Proc. Nat. Aca. Sci. USA 116, 14837–1488 (2019).
[19] H. King, R. D. Schroll, B. Davidovitch, and N. Menon, “Elastic sheet on a liquid drop reveals wrinkling and crumpling as distinct symmetry-breaking instabilities,” Proc. Natl. Acad. Sci. USA 109, 9716–9720 (2012).
[20] J. Huang, M. Juszkiewicz, W. H. de Jeu, E. Cerda, T. Emrick, N. Menon, and T. P. Russell, “Capillary wrinkling of floating thin polymer films,” Science 317, 650–653 (2007).
[21] K. B. Toga, J. Huang, K. Cunningham, T. P. Russell, and N. Menon, “A drop on a floating sheet: boundary conditions, topography and formation of wrinkles,” Soft Matter 9, 8289–8296 (2013).
[22] M. Piñeirua, N. Tanaka, B. Roman, and J. Bico, “Capillary buckling of a floating annulus,” Soft Matter 9, 10985–10992 (2013).
[23] M. Taylor, B. Davidovitch, Z. Qiu, and K. Bertoldi, “A comparative analysis of numerical approaches to the mechanics of elastic sheets,” J. Mech. Phys. Solids 79, 92–107 (2015).
[24] B. Davidovitch, R. D. Schroll, D. Vella, M. Adda-Bedia, and E. Cerda, “Prototypical model for tensional wrinkling in highly bendable films,” Phys. Rev. E 85, 066115 (2012).
[25] B. Davidovitch, R. D. Schroll, D. Vella, M. Adda-Bedia, and E. Cerda, “Nonperturbative model for wrinkling in highly bendable sheets,” Phys. Rev. E 85, 066115 (2012).
[26] B. Davidovitch and D. Vella, “Partial wetting of thin solid sheets under tension,” Soft Matter 14,
4913–4934 (2018).

[36] R. Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems* (Pearson, 2018).

[37] M. Taffetani and D. Vella, “Regimes of wrinkling in pressurized elastic shells,” Phil. Trans. Roy. Soc. London *375* (2017).

[38] P. Bella and R.V. Kohn, “Wrinkles as the result of compressive stresses in an annular thin film,” Comm. Pure App. Math. *67*, 693–747 (2014).
Supplementary Information for

“Mesoscale structure of wrinkle patterns and defect-proliferated liquid crystalline phases”

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In Secs. 1-6 we provide relevant background on the model system used in our paper, and expand on various technical details of the theoretical developments that are described succinctly in the main text. In Sec. 7 we expand on some technical aspects of the experiments and simulations, as well as data analysis.

Equations that are introduced in the SI are labeled as “S[number]”. Equation labels that are not preceded by ”S” refer to the main text.

I. REVIEW OF THE LAMÉ PROBLEM

In this section we briefly review some essential features of the Lamé set-up, emphasizing key aspects related to tension field theory, far-from-threshold analysis, and the energetic hierarchy underlying the wrinkled state.

A. Axisymmetric state and tension field theory

The axisymmetric (unwrinkled) solution of the Lamé problem is described by a classical solution [33]. Considering for simplicity $R_{in} \ll R_{out}$, the displacement field $u_r(r)$, and the radial and hoop stress components are given by:

$$
\sigma_{rr}(r) = \gamma_{in} \cdot \left[ \frac{R_{in}^2}{r^2} \left( 1 - \frac{1}{\tau} \right) + \frac{1}{\tau} \right] ; \quad \sigma_{\theta\theta}(r) = \gamma_{in} \cdot \left[ \frac{R_{in}^2}{r^2} \left( -1 + \frac{1}{\tau} \right) + \frac{1}{\tau} \right] ,
$$

$$
u_r(r) = \frac{\gamma_{in}}{Y} \cdot \left[ \frac{R_{in}^2}{r} \left( -1 + \frac{1}{\tau} \right) + \frac{r}{\tau} \right] , \tag{S1}
$$

and the elastic energy (including the work done on the sheet by the tensile boundary loads),

$$
\frac{1}{2Y} \int r dr d\theta \; \sigma_{ij}^2 - 2 \pi \gamma_{in} \cdot \left[ \tau^{-1} R_{out} u_r(R_{out}) - R_{in} u_r(R_{in}) \right] ,
$$

is given by:

$$
U^{(Lamé)}(\tau) = -\pi R_{in}^2 \gamma_{in}^2 \cdot \left\{ -1 + 2 \left( \frac{1}{\tau} - 1 \right)^2 + \frac{R_{out}^2}{R_{in}^2 \tau^2} \right\} . \tag{S2}
$$

(In the above expressions we use dimensional and dimensionless parameters as defined in the main text. Additionally, we simplified the expression by taking the Poisson’s ratio $\Lambda = 0$, recalling that the Poisson’s ratio does not affect the wrinkle pattern [23, 34].)

For $\tau > 2$, the axisymmetric state is characterized by hoop compression ($\sigma_{\theta\theta}(r) < 0$) in the zone $R_{in} < r < L^{(Lamé)} = \sqrt{\tau - 1} \; R_{in}$. Hence, if the sheet is sufficiently thin, the axisymmetric state is unstable to the formation of radial wrinkles that relieve the compressive stress. More precisely, there is a “threshold curve”, $\tau_w(\epsilon, Bo) > 2$, such that a system characterized by $\epsilon, Bo$ and $\tau > \tau_w(\epsilon, Bo)$ is unstable to the formation of wrinkles. Furthermore, one finds that $\tau_w(\epsilon, Bo) \to 2$
as $\epsilon \to 0$, reflecting the familiar fact that the buckling threshold vanishes with the sheet’s thickness.

Tension field theory (TFT) describes the stress field and radial displacement for a given $\tau > 2$ in the *singular* limit $\epsilon = 0$, at which bending rigidity vanishes and the sheet cannot support any compressive stress. As was described in detail in Refs. [24, 34], in this limit, the stress field is purely tensile, so that the sheet “splits” into two parts:

*(a)* a wrinkled zone, $R_{\text{in}} < r < L(\tau) = \frac{1}{2} \tau R_{\text{in}}$, in which the hoop compression “collapses”, and the radial stress is tensile, fully determined by (radial) force balance and the tensile load exerted on the inner edge:

$$\sigma_{rr}(r) = \frac{\gamma_{\text{in}}}{Y} \cdot \left[ (\gamma_{\text{in}}/\tau) + \frac{1}{2} \log(\tau) \right] < 0 \quad (R_{\text{in}} < r < L(\tau))$$

(S3)

(b) an unwrinkled zone, $L(\tau) < r < R_{\text{out}}$, at which both radial and hoop stress components are tensile (*i.e.* positive), and are described by Eq. (S1) upon substituting $R_{\text{in}} \to L(\tau)$, $\gamma_{\text{in}} \to 2 \gamma_{\text{out}} = 2 \gamma_{\text{in}}/\tau$, and $\tau \to 2$. The radial displacement in the wrinkled zone is given by:

$$u_r(r) = \frac{\gamma_{\text{in}}}{Y} \cdot \left[ (\gamma_{\text{in}}/\tau) + \frac{1}{2} \log(\tau) \right] < 0 \quad (R_{\text{in}} < r < L(\tau))$$

(S4)

which underlies a “slaving condition” on the wrinkled state, namely, that the fractional arclength “wasted” by wrinkly undulations in a latitude of radius $r$ in the wrinkled zone is:

$$\Phi^2(r) = -u_r(r)/r \quad (R_{\text{in}} < r < L(\tau))$$

(S5)

(Eq. 5 of main text). If one totally neglects the energetic cost of bending (*i.e.* imagining a hypothetic sheet with no bending rigidity), the above tension field solution describes the stress and radial displacement in a wrinkled state at mechanical equilibrium, which is energetically favorable in comparison to the axisymmetric (compressed, unwrinkled) state. Namely, considering the energy stored in the (purely tensile) strain and the work done by the boundary loads, one finds the energy:

$$U_{\text{dom}}(\tau) = -\pi R_{\text{in}}^2 \frac{\gamma_{\text{out}}^2}{Y} \tau^2 \cdot \left[ -\frac{1}{2} + \log(\tau) + \frac{R_{\text{out}}^2}{R_{\text{in}}^2} \frac{1}{\tau^2} \right]$$

(S6)

The corresponding expression in the main text, Eq. 6 (from which we omitted the uniform contribution $\propto R_{\text{out}}^2 \gamma_{\text{out}}^2/Y$, that scales with the sheet’s size) is valid also for nonzero Poisson ratio.

According to TFT, for any $\tau > 2$ the dominant contribution to the energy (S6) is lower than its counterpart, $U^{(\text{Lamé})}(\tau)$ (Eq. [S2]), associated with the axisymmetric (compressed, unwrinkled) state.

### B. Far-from-threshold analysis

The basic premise of a “far-from-threshold” (FT) approach is an implementation of tension field theory to study the mechanical equilibrium of highly bendable sheets, namely, physical sheets with very small, but nevertheless nonzero bending modulus. More precisely, defining

$$\tilde{\tau}_{\text{w}}(B_0, \epsilon) = \tau_{\text{w}}(B_0, \epsilon) - 2$$

(S7)

one may distinguish, for given values of $B_0$ and $0 < \epsilon \ll 1$, between the two parameter regimes: $\tau - 2 \gtrsim \tilde{\tau}_{\text{w}}(B_0, \epsilon)$ and $\tau - 2 \gg \tilde{\tau}_{\text{w}}(B_0, \epsilon)$. In the former (near threshold) regime, mechanical equilibrium may be found by standard post-buckling analysis, namely, an expansion around the axisymmetric, unwrinkled state, Eqs. (S1-2), in which the small parameter is the wrinkle amplitude. In the latter, “far-from-threshold” regime, the expansion is around the tension field solution, Eqs. (S3-6), and the small parameter of the expansion is the inverse-bendability, $\epsilon$. Since $\tilde{\tau}_{\text{w}}(B_0, \epsilon) \to 0$ as $\epsilon \to 0$, the far-from-threshold approach is prevalent for highly bendable sheets, and needs to be applied in fact for every $\tau > 2$, except for an extremely narrow sliver in the parameter space.
1. Identifying the far-from-threshold regime

Our current study addresses the FT parameter regime and implements the corresponding theoretical approach, hence it is important to characterize the threshold curve, \( \tau_w(\mathcal{B}_0, \varepsilon) = 2 + \tilde{\tau}_w(\mathcal{B}_0, \varepsilon) \). Previous studies focused on the case \( \mathcal{B}_0 = 0 \) (i.e. no liquid sub-phase), where scaling arguments show that \( \tilde{\tau}_w(\mathcal{B}_0 = 0, \varepsilon) \sim \varepsilon^{1/4} \) \([24, 34]\). Here, our primary interest is in \( \mathcal{B}_0 \gg 1 \), such that the wrinkle wavelength is governed by the substrate stiffness (associated with the liquid g.p.e.). One may estimate the wrinkling threshold by comparing – for a given set of \( \tau, \mathcal{B}_0, \varepsilon \) – the residual hoop compression in the wrinkled state with the bare compressive stress, namely, the “would-be” compressive hoop stress, had wrinkles not been formed. For \( \mathcal{B}_0 \gg 1 \), the former is just \( \left| \sigma_{\theta\theta} \right| \approx 2 \sqrt{K_{\text{sub}} \mathcal{B}} \), whereas the latter – evaluated at \( r = R_{\text{in}} \), can be estimated as \( \approx \gamma_{\text{in}} (\tau/2 - 1) \) \([24]\). Hence, the bare compression exceeds the residual level if \( \tau > 2 + \tilde{\tau}_w \), where:

\[
\tilde{\tau}_w(\mathcal{B}_0, \varepsilon) \approx 4 \sqrt{\mathcal{B} \cdot \varepsilon}.
\]

(S8)

Since in our analytical study we implement the FT methodology for the parameter regime \( \mathcal{B}_0 \gg 1 \), whereas the value of \( \tau \) explored in our experiments and simulations are no larger than 10, Eq. (S8) implies that a meaningful comparison between theoretical predictions and simulations/experiments requires the values of \( \varepsilon \) to be sufficiently small, namely, \( \varepsilon^{-1} \gg \mathcal{B}_0 \gg 1 \). (Specific numerical values are given in Sec. 7).

2. Energetic hierarchy

Since for given values of the parameters \( \tau, \mathcal{B}_0 \), and \( \varepsilon = 0 \), the energy minimum is given by Eq. (S6), the energy minimum for a small but finite \( \varepsilon \) can be expressed through an expansion around this limit value:

\[
U(\tau, \mathcal{B}_0, \varepsilon) = U_{\text{dom}}(\tau) + U_{\text{sub-dom}}(\tau, \mathcal{B}_0, \varepsilon)
\]

(S9)

such that \( U_{\text{sub-dom}}(\tau, \mathcal{B}_0, \varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and the energy minimum approaches the tension field value, \( U(\tau, \mathcal{B}_0, \varepsilon) \to U_{\text{dom}}(\tau) \) in this limit.

Notwithstanding the fact that the value of the sub-dominant energy is negligible in comparison to the dominant term, its mere existence is crucial for understanding the nature of the wrinkle pattern. The reason is that there are many wrinkle states that are consistent with the tension field limit, and it is thus the minimization of the sub-dominant energy, which determines the value of \( U_{\text{sub-dom}}(\tau, \mathcal{B}_0, \varepsilon) \), and thereby selects the physical state. The primary purpose of our article is to characterize the meso-scale structure of the energy-minimizing wrinkle state that corresponds to the value of \( U_{\text{sub-dom}}(\tau, \mathcal{B}_0, \varepsilon) \), where \( \tau > 2, \mathcal{B}_0 \gg 1 \) and \( \mathcal{B}_0 \cdot \varepsilon \ll 1 \). Our theory does this by minimizing a Ginzburg-Landau energy functional \( U_m + U_\psi + U_{\text{nonlin}} \) (Eqs. 10,15,16), which expresses the FvK energy of a wrinkle pattern in the vicinity of \( r = r_a \) through an ansatz:

\[
\zeta(r, \theta) = \text{Re} [\Psi(r, \theta)e^{im_a \theta}]\]

(S10)

(Eq. 12), subject to the condition that the underlying stress field and radial displacement (thereby the fractional wasted arclength of latitude \( \Phi^2(r) \)) are given by the tension field solution (Eqs. S3 S4 S5). The degrees of freedom of this functional are the coarse-grained fields \( \Psi(r, \theta) \) and the number \( m_a = 2\pi a / \lambda(r_a) \), that describe, respectively, the spatial variation of the magnitude and phase of the (complex) wrinkle amplitude, and the periodic rapid oscillations of the pattern. (In principle, the wrinkle wavelength may vary also slowly in the azimuthal direction, namely, \( \partial_\theta \lambda(r_a, \theta) \neq 0 \), but we do not delve here into this possibility).
3. The bending-substrate energy functional

The term $U_m$ (Eq. 10) describes the energy cost due to bending and substrate deformation. In contrast to the other contributions to the coarse-grained energy functional, $U_\psi$ and $U_{\text{nonlin}}$ (Eqs. 15-16), $U_m$ is a quadratic functional of the magnitude of the complex amplitude, $|\Psi|$, which does not involve its gradients. The physical meaning of this mathematical difference is that $U_m$ can be obtained by considering the wrinkled zone as a continuous set of decoupled elastica-like rings (obtained by ignoring the term $\sigma_r \partial_r^2$ in Eq. 7), whereas $U_\psi$ and $U_{\text{nonlin}}$ account for energetic penalties beyond this simplistic picture that emerge when one accounts for the non-zero value of $\sigma_{rr}$. We show in Sec. 3 that all three terms of the functional can be obtained by computing the FvK and substrate deformation energy for the ansatz [S10]. However, it is useful to show how $U_m$ emerges naturally if one adopts the simplified picture of decoupled elastica-like rings.

Consider then a continuous set of decoupled elastica-like rings of radii $R_{\text{in}} < r < L(\tau) = R_{\text{in}} \tau/2$, attached to a substrate of stiffness $K_{\text{sub}}$, each of them subjected to confinement $\Phi^2(r)$ implied by the tension field solution (Eq. [S5]). Furthermore, assume that the shape is given by the variable-separated ansatz (i.e. Eq. 8, or equivalently, Eq. [S10] with $\partial_\theta \Psi = 0$), such that the contribution to the bending energy due to $\partial_\theta \Psi \neq 0$ is incorporated into the energy functional $U_\psi$. For each such ring, the mechanical equilibrium states are given (assuming $\Phi^2(r) \ll 1$) by sinusoidal undulations (Eq. 8), parameterized by the “wavenumber” $m$, and the energetic cost of bending and substrate deformation is then:

$$U_m = \int dr \int_0^{2\pi} r d\theta \left| \Phi \right|^2 \left( \frac{1}{2} B \left( \frac{1}{r^2} \frac{\partial^2 \cos(m\theta)}{\partial \theta^2} \right)^2 + \frac{1}{2} K \cos^2(m\theta) \right) = \int dr \int_0^{2\pi} r d\theta \, \Phi^2 r^2 \left( B r^{-4} m^2 + K m^{-2} \right), \quad (S11)$$

where we used Eq. 5, and retained the integral over $\theta$ for convenience. Minimizing over $m$, one obtains $\bar{m}(r) = r(K_{\text{sub}}/B)^{1/4}$ (Eq. 11). Expanding around this minimal value and exploiting once again Eq. 5, we obtain the energy functional $U_m\{\Psi, m\}$ given by Eq. 10.

4. Beyond a “local-\lambda law”

Previous studies of the far-from-threshold regime have focused on the “micro-structure” of the wrinkle pattern, namely, the energetically-favorable average wavelength, $\bar{\lambda}(r)$, obtained upon ignoring the fact that the parameter $m$ in the variable-separated ansatz (Eq. 8) is an integer, whose spatial variation requires the presence of localized defects. In that approach, the effect of transverse tension $\sigma_{rr}$ (as well as curvature-induced effects in cases where wrinkles form on a non-planar background), is incorporated by replacing the substrate stiffness $K_{\text{sub}}$ with the stiffness of an “effective substrate”, $K_{\text{eff}} \approx K_{\text{sub}} + K_{\text{tens}} + K_{\text{curv}}$, where $K_{\text{tens}} = \sigma_{rr} [\Phi'(r)/\Phi(r)]^2$ accounts for the effect of transverse tension, $\sigma_{rr}$, on suppressing the wrinkle amplitude (and $K_{\text{curv}}$ accounts for a similar amplitude-suppression effect due to transverse curvature, which is not relevant for the Lamé problem) [10]. In this framework, the sub-dominant energy $U_{\text{sub-dom}}(\tau, B_0, \epsilon)$ in Eq. [S9] is approximated by minimizing a renormalized version of the energy functional $U_m$, with $K_{\text{sub}} \rightarrow K_{\text{eff}}$. Such a balance between a locally-determined effective stiffness and bending energy has been called the “local-\lambda law”.

While such an approach is useful for describing the small deviation ($\sim B_0^{-1}$) of the average value of $\bar{\lambda}$ from $2\pi(B/K_{\text{sub}})^{1/4}$ due to the presence of radial tension, it overlooks the strong deviation of the amplitude from the form assumed by the variable-separated ansatz (Eq. 8), and specifically the azimuthal modulations of the amplitude, and the crucial distinction between defect-rich and
defect-free patterns (Figs. 2, 4 of the main text). Our modified ansatz, Eq. (S10), through which the wrinkle amplitude $\Psi(r, \theta)$ becomes a free variable rather than a “slaved” one, along with describing the sub-dominant energy functional explicitly ($U_m + U_\psi + U_{nonlin}$) rather than merely renormalizing $U_m$ by $K_{sub} \to K_{eff}$, constitute a minimal model for describing the meso-scale structure of the wrinkle pattern, which emerges from the conflict between an energetically-favorable micro-scale (i.e. the wavelength $\lambda$) and an incompatible macro-scale geometry (i.e. the director $\hat{n} = \hat{\theta}$).

C. Capillary-induced tension on a floating sheet

Our experimental system consists of a liquid drop placed at the center of a large, ultrathin floating sheet (see main text). While this “drop on sheet” problem has been a subject of intensive studies in recent years [20, 21, 26], the aspect that is most relevant for the current paper is the wrinkle pattern observed in the exterior of the sheet-drop contact line, namely, where the sheet is flat except azimuthal undulations (i.e. $\int d\theta \zeta(r, \theta) \approx 0$). As was shown in previous studies [26, 35] this part of the sheet can be thought of as a Lamé set-up, where the tension $\gamma_{out} = \gamma_{lv}$ is given by the liquid-vapor surface tension that pulls on the sheet’s edge and $\gamma_{in} \propto \gamma_{lv}^{2/3} Y^{1/3}$ is induced by the capillary tension of the drop that pulls the sheet inward at the contact line. (In fact, the coefficient in the last scaling relation is not merely a constant, but rather $|\log(\gamma_{lv}/Y)|^{-1}$, see e.g. Fig. 4 of Ref. [35]). The crucial point is that since $Y \gg \gamma_{lv}$, the ratio $\tau = \gamma_{in}/\gamma_{out} \sim (Y/\gamma_{lv})^{1/3}$ is generally much larger than the critical value $\tau_w = 2 + \tau_w(Bo, \epsilon)$, above which the axisymmetric state of the sheet consists of a hoop-compressed zone, and the system is therefore unstable to the emergence of radial wrinkles.

II. MULTI-SCALE ANALYSIS OF THE FVK EQUATION

In order to perform a multi-scale analysis of the resonant perturbation problem (Eq. 7) in a narrow, defect-free annulus of radius $r_a$, we start by introducing a local Bond number:

$$Bo_a = K_{sub} r_a^2 / \sigma_{rr}(r_a) ,$$

where $\sigma_{rr}(r_a)$ is given by the TFT solution (S3), and assume an expansion of the deflection $\zeta(r, \theta)$ in powers of $Bo_a^{-b}$, where $b > 0$, of the form:

$$\zeta(r, \theta) = \zeta_0(r, \theta_r, \theta_a) + Bo_a^{-b} \zeta_1(r, \theta_r, \theta_a) + Bo_a^{-2b} \zeta_2(r, \theta_r, \theta_a) + \cdots ,$$

where:

$$\theta_r = \theta \; ; \; \theta_a = m_a Bo_a^{-b} \theta ,$$

and:

$$\zeta_0(r, \theta_r, \theta_a) = \text{Re}[\Psi(r, \theta_a) e^{im_a \theta_r}] = \frac{1}{2} [\Psi(r, \theta_a) e^{im_a \theta_r} + c.c.] .$$

The expression (S15) for the leading term in the expansion (S13) is a more sophisticated (but mathematically-equivalent) form of the generalized ansatz we introduced in the main text (Eq. 12), to assist us with the analysis. This is a standard technique in multi-scale analysis, whereby a single variable ($\theta$) is decomposed into two “independent” variables ($\theta_r$ and $\theta_a$) such that the “rapid” and
“slow” variations of the function $\zeta(r, \theta)$ are demarcated and can be analyzed distinctly. Using this approach, differentiation w.r.t. $\theta$ becomes:

$$\frac{\partial \zeta_0}{\partial \theta} = \frac{\partial \zeta_0}{\partial \theta_r} + m_a B a^{-b} \frac{\partial \zeta_0}{\partial \theta_a} = \frac{1}{2} \left[ i m_a \Psi e^{i m_a \theta_r} + m_a B a^{-b} \frac{\partial \Psi}{\partial \theta_a} e^{i m_a \theta_r} + \text{c.c.} \right].$$  \hspace{1cm} (S16)

A crucial point to understand about the multi-scale expansion technique is that we do not seek here to solve explicitly for the next terms in the expansion (S13), i.e. $\zeta_1, \zeta_2$, etc. but merely to find a solvability condition that guarantees the existence of such an expansion. This solvability condition turns out to be precisely the amplitude equation (Eq. 13).

To see this, let us re-express the operator $L_0$ (Eq. 7) through the new coordinates:

$$L_0 = L_r^0 + L_a^0$$

$$L_r^0 = B \frac{1}{r^4} \partial^4_r - \sigma_{\theta \theta} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + K_{\text{sub}}$$

$$L_a^0 = B \frac{1}{r^4} \left( \frac{\partial^4}{\partial \theta^4} - \frac{\partial^2}{\partial \theta^2} \right) - \sigma_{\theta \theta} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \theta^2} \right),$$  \hspace{1cm} (S17)

and use this expression for an expansion in powers of $B a^{-b}$ of the equation:

$$[L_0 + L_1] \zeta = 0,$$  \hspace{1cm} (S18)

where $L_1 = -\sigma_{rr} \partial^2_r$ (Eq. 7), and the dependence of $\zeta$ on the three coordinates $(r, \theta_r, \theta_a)$ is given by Eq. (S13).

At $O(B a^0)$ we obtain:

$$L_0^r[\zeta_0] = \Psi(r, \theta_a) L_0^r[e^{i m_a \theta_r}] + \text{c.c} = 0,$$  \hspace{1cm} (S19)

which is satisfied for any $\Psi(r, \theta_a)$, and the amplitude-wavelength slaving condition in Eq. 5 only imposes a global constraint, which does not exclude $\partial_{\theta_r} \Psi(r, \theta_a) \neq 0$!

At the next orders in $B a^{-b}$, substitution of Eqs. (S13, S17) into Eq. (S18), yields a series of non-homogenous equations for $\zeta_1, \zeta_2, \cdots$ of the form:

$$L_0^r[\zeta_1] = (\text{nonhom})_1$$

$$L_0^r[\zeta_2] = (\text{nonhom})_2,$$

and so on, where the terms “(nonhom)$_i$" originate from operating with $L_0^r + L_1$ on the lower order terms ($\zeta_j$ with $0 \leq j < i$) in the expansion (S13). In order for these equations to be solvable, the “Fredholm alternative" [36] implies the orthogonality of the (nonhom)$_i$ terms and the zero modes (i.e. functions within the kernel) of the adjoint of $L_0^r$, namely:

$$\int r dr \int_0^{2\pi} d\theta_r [e^{i m_a \theta_r} + \text{c.c}] \cdot (\text{nonhom})_i = 0.$$  \hspace{1cm} (S20)

Note that it is sufficient to perform the integral over an infinitesimally narrow annulus, centered at $r = r_a$ (and furthermore – it is sufficient to limit the integration to an azimuthal sector, $0 < \theta_r < 2\pi/m_a$, namely, the period of the rapid wrinkly undulations). This allows us to avoid the radial integration and replace $r \rightarrow r_a$ in the above integral. Focusing on the leading non-
homogenous term, \( (nonhom)_1 \), we find that:

\[
(nonhom)_1 = -(L_0^a + L_1)\{\zeta_0\}
\]

\[
= -\frac{1}{2} \left\{ B^2 \left( 4 \sigma^2 - 4 \right) \left( 4 \sigma^1 \right) \right\}
\]

Substituting the RHS of Eq. (S21) in Eq. (S20), we find that the terms proportional to \( \partial \Psi / \partial \theta_a \) vanish upon integration over \( \theta_4 \), so that for \( \sigma_{\theta \theta} \rightarrow 0 \) Eq. (S20) becomes:

\[
4B_0^{-2b} \frac{1}{r_a^2} \frac{\partial^2 \Psi}{\partial r^2} + B_0^{-1} \frac{\partial^2 \Psi}{\partial r^2} = 0 ,
\]

where we have used \( \sigma_{\theta \theta} = -2\sqrt{BR_{\text{sub}}} \) and \( m_a = r_a(K_{\text{sub}}/B)^{1/4} \) (Eqs. 9,11), and the definition (S14) of \( B_0 \).

Inspection of Eq. (S22) reveals that the only nontrivial value of the exponent \( b \), for which the two terms remain comparable in the limit \( B_0 \rightarrow \infty \), is \( b = 1/2 \). More precisely, if \( b > 1/2 \), then the solvability condition (S20) becomes \( \partial^2 \Psi / \partial r^2 = 0 \), similarly to the monochromatic ansatz, and consequently the hurdles that motivated us to introduce a generalized ansatz with azimuthal dependence of the amplitude are not resolved (see paragraph prior to Eq. 12 in the main text). Conversely, if \( 0 < b < 1/2 \), then the solvability condition (S20) becomes \( \partial^2 \Psi / \partial r^2 = 0 \), which is independent of the actual resonant perturbation (i.e. the operator \( L_1 \) in Eq. 7) that forced us to replace the monochromatic ansatz (Eq. 8, for which \( \Psi \) is independent of \( \theta \)) with a generalized one that allows a \( \theta \)-dependent amplitude (Eq. 12).

Using Eqs. (S12,S14) with \( b = 1/2 \) to transform back from \( \theta_4 \) to \( \theta \), Eq. (S22) reduces to Eq. 13 of the main text.

### III. COARSE-GRAINED ENERGY

In this section, we show how the coarse-grained energy functional, \( U_m + U_\psi + U_{\text{nonlin}} \), whose minimization underlies our approximation of the sub-dominant energy, \( U_{\text{sub-dom}} \) in Eq. (S9) and thereby the meso-scale structure of the wrinkle pattern, is obtained from coarse-graining the FvK energy of the sheet and the substrate energy. We start by briefly reviewing the various parts of the FvK and substrate energies, and then show how their expansion around the tension field limit, with respect to the small parameters \( \epsilon \ll 1 \) and \( B_0^{-1} \ll 1 \), where the ansatz (S15) is assumed, yields the energy functional, \( U_m + U_\psi + U_{\text{nonlin}} \).

#### A. The FvK and substrate energies

The in-plane strain, and consequently the FvK energy of an elastic sheet, cannot be expressed through the out-of-plane displacement component \( \zeta(r, \theta) \) alone. Instead it requires also the in-plane components. As long as the exerted tensile strain is small (\( \gamma_{in}/Y \ll 1 \)), the slopes remain
small even in the fully-developed wrinkled state (i.e. $|\nabla \zeta| \ll 1$), such that the displacement can be expressed through the Monge parameterization:

$$u(r, \theta) = u_r(r, \theta) \hat{r} + u_\theta(r, \theta) \hat{\theta} + \zeta(r, \theta) \hat{z},$$  \hspace{1cm} (S23)$$

and the components of the strain tensor, $\varepsilon$, are:

$$\varepsilon_{rr} = \partial_r u_r + \frac{1}{2} \left( \partial_r \zeta \right)^2, \hspace{1cm} (S24a)$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} \partial_\theta u_\theta + \frac{1}{r} u_r + \frac{1}{2 r^2} \left( \partial_\theta \zeta \right)^2, \hspace{1cm} (S24b)$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2} \left( \frac{1}{r} \partial_\theta u_r + \partial_r u_\theta + \frac{1}{r} \partial_r \zeta \partial_\theta \zeta \right). \hspace{1cm} (S24c)$$

Furthermore, the curvature tensor $\kappa_{ij}$ can be approximated as:

$$\kappa_{rr} = \partial_r^2 \zeta; \hspace{1cm} \kappa_{\theta\theta} = \frac{1}{r} \partial_\theta \zeta + \frac{1}{2 r^2} \partial_{\theta\theta}^2 \zeta; \hspace{1cm} \kappa_{r\theta} = 2 \partial_r \partial_\theta \zeta \hspace{1cm} (S25)$$

The energy, which we denote by a capital $U$, is conveniently expressed through its areal density (denoted by italic lower case, $u = U/\text{area}$):

$$u = u_{\text{FvK}} + u_{\text{subst}}, \hspace{1cm} (S26a)$$

where $u_{\text{FvK}} = u_{\text{strain}} + u_{\text{bend}}$, and:

$$u_{\text{subst}} = \frac{K_{\text{sub}}}{2} \zeta^2,$$

$$u_{\text{strain}} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}; \hspace{1cm} u_{\text{bend}} = \frac{B}{2} T r(\kappa)^2 \approx \frac{B}{2} \left( \frac{1}{r^2} \partial_\theta^2 \zeta \right)^2$$

where $: \sigma_{ij} = Y \varepsilon_{ij} \hspace{1cm} (S26b)$

(As mentioned already, for simplicity of the presentation we take in the above equations a zero Poisson ratio, noting that the analysis for a non-zero Poisson ratio requires more bookkeeping but does not affect any of the results that are relevant for our study.) Furthermore, since the curvature is governed by the wrinkly undulations (rather than the radial variation of their amplitude), we retained in the above expression only the contribution from the component $\kappa_{\theta\theta}$ to the bending energy.

### B. Principles of the expansion

The tension field theory solution, described in Sec.I, yields the asymptotic state of the radial displacement, and correspondingly the confinement function $\Phi^2(r)$ that determines the fraction of the latitudinal arclength wasted by undulations in the limit $\epsilon \to 0$ (for any $Bo_a$). These limit values are obtained by minimizing the energy $U_{\text{strain}}$ along with the work done by the tensile loads $\gamma_{\text{in}}, \gamma_{\text{out}} = \gamma_{\text{in}} / \tau$, that pull on the edges, subject to the compression-free condition, $\sigma_{\theta\theta} \geq 0$. In that (singular) limit, $U_{\text{subst}} = U_{\text{bend}} = 0$.  

Denoting by $\Delta U_{\text{strain}}$ the difference between the strain energy in the actual wrinkled state and the tension field limit, our purpose here is to express the various contributions to the subdominant energy $U_{\text{sub-dom}}(\tau, Bo, \epsilon)$ in Eq. (S9) by evaluating $U_{\text{bend}} + U_{\text{subst}} + \Delta U_{\text{strain}}$, for $0 < \epsilon \ll 1$ and $Bo \gg 1$. We do this by assuming the ansatz (S15) and re-organizing the various terms in $U_{\text{bend}} + U_{\text{subst}} + \Delta U_{\text{strain}}$ into an energy functional, $U_m + U_\psi + U_{\text{nonlin}}$, whose field variables are the wavelength $\lambda$ and complex amplitude $\Psi$, and whose parameters are derived from the known
features of the tension field limit.

**Small parameters of the expansion:** Let us mention that in addition to the control parameters:
\[ \epsilon \ll 1 \text{ and } B_0 a \gg 1 , \]  
there are yet two other emergent ratios, whose assumed smallness is employed in our analysis, namely:
\[ \ell_\parallel / r_a \ll 1 \text{ and } \ell_\perp / r_a \ll 1 , \]  
where \( \ell_\parallel \) and \( \ell_\perp \) define the sizes of a defect-free zone in the hoop and radial directions, respectively (see Fig. 3 of main text). The condition \( \ell_\parallel \ll r_a \) is guaranteed self-consistently by the results of our analysis (Eq. 17 or 18), whereas \( \ell_\perp \ll r_a \) follows from the assumed low cost of defect energy, assisted by amplitude suppression (see the paragraph following Eq. 13 in the main text). We stress that a complete coarse-grained theory of the defect-proliferated wrinkled state should yield also an actual prediction for \( \ell_\perp \), in terms of the various control parameters (e.g. \( \gamma \in / Y, \tau, \epsilon, B_0 a \)), in an analogous manner to the prediction of \( \ell_\parallel \) (Eq. 18). We suspect that such a prediction requires one to consider yet higher orders in the expansion (in \( B_0^{-1}, \epsilon \), and possibly parameters that involve the stretching modulus, e.g. \( \gamma \in / Y \)), and furthermore, must take into consideration explicitly the actual energetic cost of defects. The current version of our theory does not provide tools to evaluate the actual energetic cost of defects, and therefore it is capable of predicting only \( \ell_\parallel \) but falls short from providing an analogous prediction for \( \ell_\perp \).

### C. Strain and energy densities

We consider then a deformation of the sheet in the zone \( R_\in < r < L(\tau) \), whose out-of-plane component is given by the ansatz ([S15]). In addition to the value of \( L(\tau) \), tension field theory implies two other conditions. First, the radial displacement is given, up to corrections that vanish as \( \epsilon \to 0 \), by Eq. ([S4]), and the arc-length wasted by \( \zeta(r, \theta) \) is given by \( \Phi^2(r) \), Eq. ([S5]), up to corrections that vanish as \( \epsilon \to 0 \). In the following, we provide the corresponding expressions for the strain components, \( \varepsilon_{rr} \) and \( \varepsilon_{\theta\theta} \) (in the Lamé problem, the shear strain, \( \varepsilon_{r\theta} \) contributes to the energy only at a higher order in \( \epsilon \) [34]):

\[ \varepsilon_{rr} = \frac{1}{Y} \sigma_{rr}(r) + \frac{1}{4} \left| \frac{\partial \Psi}{\partial r} \right|^2 + \frac{1}{8} \left[ \left( \frac{\partial \Psi}{\partial r} \right)^2 e^{2im_\theta a} + c.c. \right] , \]  
and

\[ \varepsilon_{\theta\theta} = \left( \frac{m_\theta^2}{4r^2} \right) \left| \Psi \right|^2 - \Phi^2(r) \]  
\[ + \frac{m_\theta^2}{4r^2} B_0 a^{-1} \left( \frac{\partial \Psi}{\partial \theta} \right)^2 + \frac{m_\theta^2}{4r^2} \left( i B_0 a^{-1/2} \Psi \frac{\partial \Psi^*}{\partial \theta} + c.c. \right) , \]

where \( \sigma_{rr}(r) \) is given by Eq. ([S3]). Note that, in contrast to the last term in Eq. ([S29]), the expression for \( \varepsilon_{\theta\theta} \) does not include terms that oscillate “rapidly”, such as \( \frac{m_\theta^2}{4r^2} \left| \Psi \right|^2 e^{2im_\theta a} \). The reason is that such terms can be made to cancel out by properly adjusting the azimuthal displacement, \( u_\theta \) (which contributes to the hoop strain through \( r^{-1} \partial_\theta u_\theta \), see Eq. ([S24b]). This observation was already noted in previous developments of the FT expansion [18, 34, 37]. Using Eq. ([S26]) and the above expressions for the displacement ([S15]) and strain components ([S29] [S30]), we can compute a coarse-grained version of the energy by integrating over the
rapid variable, $\theta$. The outcome of this calculation is three energy densities $\tilde{u}_{\text{strain}}$, $\tilde{u}_{\text{bend}}$, $\tilde{u}_{\text{subst}}$, such that $\tilde{u}(r, \theta) \cdot r \Delta r \Delta \theta$ is the corresponding energy in a small annular zone of opening angle $\Delta \theta \approx 2\pi/m_a$ and a small radial width, $\Delta r \ll \ell_\parallel$, around a point $(r, \theta)$ in a defect-free annulus. Below we give a succinct version of these energy densities, omitting various terms which are negligible in comparison with other terms that appear elsewhere in the following expressions, such that ignoring them does not entail further constraints on energy minimization.

The coarse-grained energy density $\tilde{u}_{\text{strain}}$ is:

$$\tilde{u}_{\text{strain}} = \frac{1}{2Y} \sigma_{rr}^2 + \frac{1}{4} \sigma_{rr} \left| \frac{\partial \Psi}{\partial r} \right|^2 + Bo_a^{-1} \frac{1}{4} \sigma_{\theta \theta} m_a^2 \left| \frac{1}{r_a} \frac{\partial \Psi}{\partial \theta_a} \right|^2 + \frac{Y}{2} \left( \left| \frac{m_a}{2r} \Psi \right|^2 - \Phi^2(r) \right)^2 + \frac{1}{16} Y Bo_a^{-1} m_a^4 \frac{1}{r_a} \Im \left\{ \Psi \frac{\partial \Psi^*}{\partial \theta_a} \right\}^2. \quad (S31)$$

The coarse-grained energy density $\tilde{u}_{\text{bend}}$ is:

$$\tilde{u}_{\text{bend}} = \frac{1}{4} B m_a^4 \left| \Psi \right|^2 + \frac{1}{4} B m_a^4 \cdot Bo_a^{-1} \cdot \left( 4 \left| \frac{\partial \Psi}{\partial \theta_a} \right|^2 - 2 \text{Re} \left( \Psi \frac{\partial^2 \Psi}{\partial \theta_a^2} \right) \right). \quad (S32)$$

The coarse-grained energy $\tilde{u}_{\text{subst}}$ is:

$$\tilde{u}_{\text{subst}} = \frac{1}{4} K_{\text{sub}} \left| \Psi \right|^2. \quad (S33)$$

(Since our analysis is based on the assumption that the ansatz $\Psi$ is valid in a defect-free zone whose sizes are $\ell_\perp, \ell_\parallel \ll r_a$, we simplified the above expressions by replacing $\frac{1}{r} \partial_\theta \rightarrow \frac{1}{r_a} \partial_\theta_a$.)

Let us inspect now the various terms in the above equations:

- The first line of Eq. (S31) is already accounted for by tension field theory, and it is thus part of the tension field energy (S6), along with the strain energy in the unwrinkled portion of the sheet and the work of the tensile boundary loads. Hence, this part of the energy is included in the term $U_\text{dom}$ in Eq. (S9), and does not contribute to the sub-dominant energy.

- Eq. (S33), along with the first line of Eq. (S32), are quadratic in $|\Psi|$ and do not involve any gradients. Together, they form the functional $U_m$ (Eq. 10).

- The terms in the last line of Eq. (S31) underlie the functional $U_{\text{nonlin}}$ (Eq. 15). Note that in order to get this we switched back $\theta_a \rightarrow m_a \theta / \sqrt{Bo_a}$.

- Finally, the terms in the second lines of Eqs. (S31, S32) underlie the functional $U_\psi$ (Eq. 16). To see this, note that:
  
  (a) we once again switched back $\theta_a \rightarrow m_a \theta / \sqrt{Bo_a}$.
  
  (b) Since we consider $Bo_a \gg 1$, we approximated $\sigma_{\theta \theta} \approx -2\sqrt{BK_{\text{sub}}}$ and $m_a \approx r_a (K_{\text{sub}}/B)^{1/4}$.

  (c) Anticipating that $\Psi$ can be approximated by a real function (see main text and Sec. 4), the parenthetical term in the second line of Eq. (S32) transforms as follows: $4 \left( \frac{\partial \Psi}{\partial \theta_a} \right)^2 - 2 \text{Re} \left( \Psi \frac{\partial^2 \Psi}{\partial \theta_a^2} \right) = 6 \left( \frac{\partial \Psi}{\partial \theta_a} \right)^2 - 2 \frac{\partial}{\partial \theta_a} \left( \Psi \frac{\partial \Psi}{\partial \theta_a} \right)$.

  (d) The last term in the above expression is an exact derivative (in $\theta_a$) and thus amounts only to a boundary term, which we ignore (assuming that it vanishes upon integration over $\theta_a$).
IV. MINIMIZING THE COARSE-GRAINED ENERGY SUBJECT TO NEAR-INEXTENSIBILITY CONSTRAINTS

When addressing purely developable deformations of thin solid bodies (*i.e.* that do not affect the midplane’s Gaussian curvature), the Euler elastica principle implies that it is possible to consider a purely inextensible limit, where the energy and residual stress do not depend on the stretching modulus, but only on the bending modulus and the energy associated with boundary tensile loads and deformation of a substrate. An analogous idea underlies the variable-separated ansatz (Eq. 8), assumed in previous studies of radial wrinkle patterns in general, and the Lamé problem in particular. Although a pattern of wrinkles with radial orientation is clearly a non-developable deformation of the naturally-planar sheet, the variable-separated ansatz is constructed such that all hoops remain inextensible, namely, satisfying Eq. 5 for every \( R_{in} < r < L \). For such an ansatz, the residual compressive stress would have been \(-2\sqrt{BK_{sub}}\), independent of the stretching modulus, exactly as for a uni-axially compressed elastica supported on a substrate of stiffness \( K_{sub} \).

A. The amplitude equation as an Euler-Lagrange equation of the coarse-grained energy functional

As we noted in the main text (paragraph preceding Eq. 17), a \( \theta \)-dependent amplitude \( \Psi(r,\theta) \) is not compatible with the strict amplitude-wavelength slaving condition (Eq. 5) for every \( r \), amounting to violation of the perfect hoop inextensibility; consequently, the residual compression is larger than \( 2\sqrt{BK_{sub}} \) and depends on the stretching modulus \( Y \). However, assuming the energetic cost of defects in amplitude-suppressed zones is negligible, such that the size of defect-free zones may be very small (Eq. S28), we can overcome this difficulty by imposing suitable conditions on the amplitude. As we noted in the main text (paragraph following Eq. 16), a key element in our analysis is the energetic hierarchy associated with the three components of the coarse-grained energy: \( U_m \) (Eq. 10), \( U_{\text{nonlin}} \) (Eq. 15), and \( U_\psi \) (Eq. 16). Hence, our approach to minimize the sum \( U_{\text{nonlin}} + U_m + U_\psi \) is to consider the first two terms as implying suitable constraints on the minimization of \( U_\psi \). More precisely, we assume that the actual values of the energies \( U_{\text{nonlin}} \) and \( U_m \) in each defect-free zone are much smaller than the value of \( U_\psi \). Focusing on a single defect-free zone, this approach allows us to consider \( m \) and \( \sigma_{rr}, \sigma_{\theta\theta} \) Eq. 16 as fixed parameters, given by Eqs. 4,9,11, since any deviations from these values will incur an explicit energetic cost of \( U_{\text{nonlin}} \) or \( U_m \). Consequently, the only degree of freedom in the energy functional \( U_\psi \) (Eq. 16) is the complex amplitude \( \Psi(r,\theta) \). A standard variational calculus yields a Laplace-like equation (Eq. 13).

As we noted in the main text, in order to determine the energetically-favorable solution of Eq. 13 we must recall the assumed negligibility of \( U_{\text{nonlin}} \), on which consideration we elaborate in the next two subsections.

B. A nearly radial orientation of wrinkles

The level of local smectic order in defect-free zones is governed by a competition between two parts of the coarse-grained energy functional. The first is \( U_m \) (Eq. 10), which favors a local smectic order with fixed spacing of wrinkles (*i.e.* \( \lambda = \bar{\lambda} \)) and hence would favor a local, nonuniform rotation of the bent director \( \hat{n}_{aux}(x) \rightarrow \hat{n}(x) \), such that \( \nabla \times \hat{n}(x) \approx 0 \). Since variation of the the director is related to the phase of the complex amplitude, this means that in each defect-free zone \( U_m \) favors: \( \partial_\theta \arg \Psi = \bar{m}(r) - m_a \). The competing term is the second part in the integrand comprising \( U_{\text{nonlin}} \) (Eq. 15), which penalizes deviations from radial orientation, hence favors \( \partial_\theta \arg \Psi = 0 \). In order to
determine which of the two dominates, we can evaluate the difference in the values taken by each of these two terms upon substituting these two values of \( \partial_\theta \arg \Psi \) in the respective functionals.

(a) For \( U_m \), substituting \( \partial_\theta \arg \Psi = \bar{m}(r) - m_a \) (which is equivalent to \( m = \bar{m}(r) \)) yields a zero value for the integral term in Eq. 10. Substituting \( \partial_\theta \arg \Psi = 0 \) and Taylor-expanding \((m_a - \bar{m}(r))^2\) with \( m_a = \bar{m}(r_a) \), around \( r = r_a \) (analogously to (S34)), we find the energy density \( \sim \sqrt{B K_{\text{sub}}} \Phi^2(r_a) (\ell_s/r_a)^2 \) (where the Taylor expansion procedure allows us to use \( m \approx m_a \) and \( |\Psi(r)| \approx |\Psi(r_a)| \)).

(b) For \( U_{\text{nonlin}} \), substituting \( \partial_\theta \arg \Psi = 0 \) \( (i.e. \) perfectly radial wrinkles) yields a zero value upon integrating the second integrand in Eq. 15. Substituting \( \partial_\theta \arg \Psi = \bar{m}(r) - m_a \) and using similarly a Taylor expansion around \( r = r_a \), we find the energy density \( \sim Y \Phi^4(r_a) (\ell_s/r_a)^2 \sim \gamma_{\text{lin}} \Phi^2(r_a) (\ell_s/r_a)^2 \) (up to a \( \tau \)-dependent prefactor, where we used Eq. 5).

The ratio between these respective energetic costs is thus \( \sqrt{BK_{\text{sub}}} / \gamma_{\text{lin}} \sim \sqrt{Bo} \cdot \epsilon \). However, as was shown in Sec. 1.B.2, this is precisely the parameter whose decreasing value (and asymptotic vanishing) characterizes the transition from the instability-threshold condition to the far-from-threshold regime. Hence, as the wrinkle pattern becomes far from threshold \( (e.g. \) decreasing the bending modulus while keeping all other physical parameters are fixed), we expect the energetic cost of \( U_{\text{nonlin}} \) to be the dominant among the two terms, and hence the wrinkles in the defect-free zones to become more radially-oriented.

In order to understand the implication of this finding on the meso-scale structure of the pattern we recall that our analysis assumes a defect-proliferated state and specifically \( \Psi(0) \), such that the actual number of wrinkles in a defect-free zone is consistent with the “correct” number at the middle of the zone \( (\approx \ell_s/\lambda) \), and the deviation from a parallel array \( (i.e. \) the relative angle between peaks) is constrained by the small ratio \( \ell_s/r_a \).

C. Weakening the amplitude-wavelength slaving constraint

The energetic cost of deviation from the amplitude-wavelength constraint (Eq. 5) is expressed by the first term in the integrand that comprises \( U_{\text{nonlin}} \) (Eq. 15). Considering a defect-free annular zone, \( r \in (r_a - \ell_s, r_a + \ell_s) \), with \( \ell_s \ll r_a \), where \( m(r) = m_a = 2\pi r_a / \lambda \), we can evaluate this term by considering a formal Taylor expansion around the value of the integrand at \( r = r_a \):

\[
\left( \frac{m_a^2 |\Psi|^2}{4r_a^2} - \Phi^2(r) \right)^2 \approx \left( A_0 + A_1 \left( \frac{r - r_a}{r_a} \right) + \frac{1}{2!} A_2 \left( \frac{r - r_a}{r_a} \right)^2 + \cdots \right)^2 ,
\]

where \( A_i \) are the \( i \)-th derivatives of \( \left( \frac{m_a^2 |\Psi|^2}{4r_a^2} - \Phi^2(r) \right) \) with respect to \( r \) (evaluated at \( r = r_a \)), and \( \langle |\Psi|^2 \rangle \) is averaged over azimuthal undulations of the amplitude. The energetic cost of deviations from the perfect slaving condition (Eq. 5) is determined by the first non-vanishing \( A_i \). A nonzero \( A_0 \) would entail an energy areal density \( \sim Y \), which will totally disrupt the TFT limit (Eq. S6). If \( A_0 = 0 \) but \( A_1 \neq 0 \), the energy density \( \sim Y (\ell_s/r_a)^2 \), and so on. Thus, minimization of the energetic penalty due to violation of the perfect slaving condition is tied to the azimuthal average of \( \langle |\Psi|^2 \rangle \) and its radial derivatives.

In order to proceed, we consider the Laplace-like equation (Eq. 13) in a narrow annulus, where natural basis functions are given by magnitude \( c \) and wavelength \( 2\pi/d \):

\[
\Psi(r, \theta) = c e^{\pm \sqrt{\frac{4|\sigma_{\text{sub}}|}{\sigma_{\text{sub}}(r_a)} d(r-r_a)} \cos(d r \theta) .
\]

(35)

Although any superposition of such states solves Eq. 13, the necessity to minimize the energy \( U_{\Psi} \) (Eq. 16) motivates us to focus only on these “minimal amplitude” states, such that elimination of
the two first coefficients \( A_0 = A_1 = 0 \) in Eq. (S34) provides two equations that determine \( c, d \) in Eq. (S35):

\[
\frac{m_a^2}{4} \left( \frac{\Psi^2}{r^2} \right)_{r=r_a} = \Phi(r_a)^2 \quad \Rightarrow \quad c = \sqrt{8} \frac{r_a}{m_a} \Phi(r_a)
\]

\[
\frac{m_a^2}{4} \partial_r \left( \frac{\Psi^2}{r^2} \right)_{r=r_a} = [\Phi(r_a)^2]' \quad \Rightarrow \quad d = \frac{1}{2} \left( \frac{1}{r_a} + \left| \frac{\Phi'(r_a)}{\Phi(r_a)} \right| \right) \sqrt{\frac{\sigma_{rr}(r_a)}{|\sigma_{\theta\theta}|}}.
\]

The scale \( \ell_\parallel \) characterizes the (azimuthal) distance between nearby zeros of the amplitude (S35), namely: \( \ell_\parallel \approx \pi/d \). Equation (S37) together with the definition of \( \ell_{BC}^* \) (Eq. 14), \( \bar{\lambda} \) (Eq. 11), and \( \sigma_{\theta\theta} \) (Eq. 9) yield the expression for \( \ell_\parallel \) (Eq. 17) we report in the main text.

Tension-field-based theory, which provides the basis for our analysis and whose assumed energetic hierarchy (S9) ignores the explicit energetic cost of deviations from Eq. 5, remains valid if the energy \( U_{\text{nonlin}} \) associated with the fact that \( A_2 \neq 0 \) in the Taylor expansion (S34) is small in comparison to the values of \( U_m \) and \( U_\psi \), whose simultaneous minimization comprises the sub-dominant energy \( U_{\text{sub-dom}} \) in (S9). Specifically, considering \( U_\psi \) (Eq. 16, whose minimization underlies the amplitude equation for \( \Psi \), Eq. 13), whose characteristic value in a defect-free zone is characterized by energy density \( \sim \sigma_{rr} \), we find that this inequality is valid as long as the density of defects is sufficiently large, such that \( \ell_\parallel/r_a \ll (\sigma_{rr}(r_a)/Y)^{1/4} \).

V. COVARIANT FORMULA FOR \( \ell_\parallel \)

In deriving Eq. 17 of the main text we focused on the Lamé set-up, which is characterized by axial symmetry of the sheet and the tensile loads, hence it is not surprising that the corresponding prediction for \( \ell_\parallel \) is explicitly dependent on the radial distance \( r_a \) of a defect-free zone from the center of the sheet. One may wonder, however, whether a similar expression for \( \ell_\parallel \) can be obtained for more generic problems, not necessarily characterized by such a global axial symmetry, in which the topography and confining forces give rise to confinement along an axis \( \hat{n}(x) \) that varies across the wrinkled zone. Assuming one finds (analytically or numerically) the TFT solution for such confinement problems, one may readily evaluate \( \ell_{BC}^* \), \( \bar{\lambda} \), and \( \Phi(x) \), but there is no unique point in space that defines “radial distance”, and therefore one must understand what length scale replaces \( r_a \) in Eq. 18.

If the director is bent in the vicinity of a point \( x \), namely \( \nabla \times \hat{n}_{\text{aux}}(x) \neq 0 \), then a pattern of parallel, uniformly spaced wrinkles is not compatible with the confining conditions, and defect-rich patterns are likely to emerge, similarly to those in Fig. 2 of the main text. We note that our theoretical analysis is based on the existence of a small, defect-free zone, in which the sheet undulates “rapidly” along a bent director \( (\hat{\theta}) \), such that the dependence on \( r_a \) enters only through the azimuthal arclength formula, \( ds = r_a d\theta \), which can be rewritten as: \( ds = |\nabla \times \hat{n}_{\text{aux}}(x)|^{-1} d\theta \). Repeating our analysis with \( r_a \rightarrow |\nabla \times \hat{n}_{\text{aux}}(x)|^{-1} \) yields Eq. 18 for confinement problems with locally bent director fields.

VI. THE TRANSITION CURVE \( B_0, (\tau) \)

As mentioned in the main text, it is possible to obtain the scaling relation (Eq. 3) by considering the limit \( B_0 \gg 1 \) (while keeping \( \sqrt{B_0} \cdot \epsilon \ll 1 \), such that our far-from-threshold expansion around the tension field limit is a valid approach), at which the transition from a defect-free to a defect-proliferated pattern is expected to occur when the wrinkled zone is limited to only a narrow annulus
near the inner edge, namely, $0 < \tau - 2 \ll 1$. As we show below, in this limit it is possible to evaluate the difference in energies between these two states, using a similar approach to the one employed in Secs. 3-4 for evaluating energies in a single defect-free zone.

(a) Consider first the term $U_{\text{nonlin}}$ of the energy (Eq. 15). For the defect-free state (i.e., where the variable-separated ansatz is valid), both terms in the integrand of $U_{\text{nonlin}}$ are identically zero. For the defect-proliferated state, we showed in Sec. 4 that both terms scale with the ratio $\ell_\perp/R_{\text{in}}$ with powers that are $\geq 2$. Hence upon integrating over the wrinkled annulus (where the number of defect-free zones is $\propto 1/\ell_\perp$), the overall value of $U_{\text{nonlin}}$ for the defect-proliferated state scales at least as $\ell_\perp/R_{\text{in}}$ (or as a higher power of this ratio). Consequently, if $\ell_\perp$ is sufficiently small in comparison to the radial width of the wrinkled annulus, namely, $\ell_\perp \ll R_{\text{in}}(\tau/2 - 1)$, then we can ignore the effect of $U_{\text{nonlin}}$ on the difference in energy between defect-free and defect-proliferated states.

(b) Let us consider now the term $U_m$ of the energy (Eq. 10). For a defect-proliferated state (assuming $\ell_\perp \ll R_{\text{in}}(\tau/2 - 1)$, such that $m \rightarrow m(r) \approx \bar{m}(r)$), this term is close to its energy minimum, so that the integral term in Eq. 10 is nearly zero. In contrast, for a defect-free state, $m$ is a constant and minimization of the integral is realized by some $m^* = \bar{m}(r^*)$, where $R_{\text{in}} < r^* < R_{\text{in}}\tau/2$, so that $(m - \bar{m}(r))^2 = (m^* - \bar{m}(r))^2 = \sqrt{K_{\text{sub}}/B}(r - r^*)^2$. Considering the rest of the terms in the integrand we obtain, upon using Eqs. 5,11, and expanding to leading order in $(\tau/2 - 1)$, a factor: $\propto (B\gamma_m/Y R^2_{\text{in}})(R_{\text{in}}\tau/2 - r)$. Hence, to leading order in $(\tau/2 - 1)$, the integral in Eq. 10 for the defect-free state scales as: $\propto \left[\gamma_m\sqrt{\sqrt{B K_{\text{sub}}}/R_{\text{in}}^2}\right] \int_{R_{\text{in}}}^{R_{\text{in}}\tau/2}(r - r^*)^2(r - r) dr$. Since $R_{\text{in}} < r^* < R_{\text{in}}\tau/2$, the integral term is: $\propto R^4_{\text{in}}(\tau/2 - 1)^4$, so that the difference in $U_m$ between the defect-free and defect-proliferated states is: $\Delta U_m \sim \left[\gamma_m\sqrt{\sqrt{B K_{\text{sub}}}/R_{\text{in}}^2}\right](\tau/2 - 1)^4$.

(c) Finally, let us consider the term $U_\psi$ of the energy (Eq. 16). Notably, the radial dependence of the excess latitudinal arclength ($\Phi^2(r)$, Eq. 5) implies that the first term in the integrand in Eq. 16 is nonzero for both defect-free and defect-proliferated states. Furthermore, previous studies (that addressed the case $\mathcal{B}_0 = 0$) found that the respective integral is diverging, and must be regularized by a boundary layer [34] or another structure [38] at the vicinity of the wrinkle’s foot ($r \rightarrow L = R_{\text{in}}\tau/2$). Since this subtle regularization problem stems from only the near vicinity of the wrinkle’s foot, it affects equally both defect-free and defect-proliferated states, and consequently we assume that the difference in $U_\psi$ between these two types is associated with the bulk of the wrinkled zone, $R_{\text{in}} < r < R_{\text{in}}\tau/2$, where $|\partial \Psi/\partial r|^2$ is finite (and can be estimated e.g. by its value at the inner edge $r = R_{\text{in}}$). Comparing Eqs. 8 and [36][37] we note that the value of $|\partial \Psi/\partial r|^2$ is comparable in the two types of states (recall that we consider here $\ell_\perp \ll R_{\text{in}}(\tau/2 - 1)$ so that the wrinkled annulus is densely populated by defects and correspondingly small defect-free zones). However, in contrast to the defect-proliferated state, the defect-free state (Eq. 8) does not have azimuthal modulations of the amplitude. Hence, since Eqs. 16 and [35] show that the energetic cost of azimuthal modulations is a finite multiple of the cost of radial variation of the amplitude, we can estimate the difference $\Delta U_\psi$ through the radial width of the wrinkled annulus, $R_{\text{in}}(\tau/2 - 1)$, multiplied by $|\partial \Psi/\partial r|^2$ evaluated at $r = R_{\text{in}}$. Using Eq. 5 of the main text to evaluate $|\partial \Psi/\partial r|_{r=R_{\text{in}}}$ we obtain, to leading order in $(\tau/2 - 1)$, $\Delta U_\psi \sim \left[\gamma_m\sqrt{\gamma_m\sqrt{B}/K_{\text{sub}}\right]}$.

Comparing the above estimates for $\Delta U_m$ (which favors defect-proliferated states) and $\Delta U_\psi$ (which favors defect-free states), and recalling the definitions of the dimensionless parameters (Eq. 1), we obtain the scaling of the transition curve, $\mathcal{B}_0(\tau) \sim (\tau - 2)^{-4}$, whose scaling with $(\tau - 2)$ is close to, but not identical to the scaling relation extracted from our data (Eq. 3 of the main text). As we noted in the main text, a likely reason for the deviation between the predicted and observed exponents is the fact that the data from experiments and simulations are taken at $\tau/2 - 1 \sim O(1)$, whereas the above scaling analysis is focused on $\tau/2 - 1 \ll 1$. 

VII. SIMULATIONS AND EXPERIMENTS

A. Finite element simulations

Finite element (FE) simulations were performed to simulate wrinkles in the Lamé setup with a wide range of values for $Bo$ ($0.07 < Bo < 333$), confinement ratio ($3.0 < \tau < 8.7$) and bendability ($5 \times 10^4 < \epsilon^{-1} < 3 \times 10^7$). Furthermore, in the large $Bo$ regime, which is the primary focus of our study, we fixed the ratio $\sqrt{Bo} \cdot \epsilon = 0.01$, to make sure the wrinkle pattern is well described by a far-from-threshold analysis around the tension field theory (see Sec. 1).

The simulated films were made of elastic material with large Young’s modulus and Poisson’s ratio $\Lambda = 0.3$ to ensure the tension is much smaller than the in-plane stiffness ($\gamma_{in}/Y < 0.01$). The liquid surface tensions that pull on the inner and outer edges were modeled as tensile tractions along the radial direction of the undeformed thin film. The effect of gravity of the liquid substrate was modeled as a pressure on the sheet whose magnitude is proportional to the out-of-plane displacement of the element.

All the simulations were carried out with ABAQUS/Explicit. The 3-node linear shell elements (S3R) were used and geometric nonlinearity was taken into account. The pressure for modeling the liquid substrate was implemented through the user-defined subroutine for load distribution. A fine mesh was adopted at the inner edge with gradually increasing element sizes towards the outer edge. The FE nodes were randomly distributed to avoid any symmetry due to the discretized network. Large scale simulations with $10^6$ to $1.4 \times 10^7$ elements were required to capture the large number of wrinkles. A small initial pressure was applied on the whole thin film to trigger wrinkles. The pressure then quickly decayed to zero and fictitious material damping was added to the model to help convergence to equilibrium. The simulation was then run dynamically until the out-of-plane deflection, $\zeta(r, \theta)$, converged to a stable state.

B. Wrinkle Analysis

Measurements of the wrinkle number, $m(r)$, were obtained using a custom automated image analysis following Refs. [10, 19]. After an initial filtering step to reduce noise and lighting gradients, an autocorrelation of the intensity versus $\theta$ was performed at each radius within a region free of material imperfections, effectively averaging over many wrinkles. The wrinkle number was extracted from this oscillating signal. The same routine was used to analyze the wrinkle wavelength in the simulations by applying it to grayscale color maps of the out-of-plane deflection.

C. Determination of $\gamma_{out}$ and $\gamma_{in}$ in experiments

The liquid-vapor surface tension, $\gamma_{lv}$, was measured with a Wilhelmy plate under typical experimental conditions, yielding $\gamma_{out} = \gamma_{lv}$. For each experimental image, the confinement ratio $\tau$ was deduced from the wrinkle length via a relation that was derived and validated previously for a finite annulus in the far-from-threshold regime [22, 23]:

$$L = \frac{R_{out}}{\tau} \left( \frac{R_{out}}{R_{in}} - \sqrt{\frac{R_{out}^2}{R_{in}^2} - \tau^2} \right). \quad (S38)$$

Solving for $\tau$ yields the simple relation: $\tau = 2LR_{out}^2/[R_{in}(L^2 + R_{out}^2)]$. Then, $\gamma_{in} = \tau \gamma_{out}$. 

D. Determination of $\ell_\parallel$

In the main text, we report values of $\ell_\parallel$ that are extracted from the grayscale images of the sheet, in both experiments and simulations. Figure S1 shows a typical example from simulation, where the shade corresponds to the out-of-plane displacement, with the medium tone in the upper-right corresponding to zero deflection. The wrinkle pattern may be decomposed into defect-free and defect-rich zones, which are alternating angular sectors of various sizes. The defect-rich zones are the amplitude-suppressed regions (i.e., regions with weaker contrast) where new wrinkles appear. The defect-free zones are characterized by nearly (but not perfectly) parallel wrinkles that have larger amplitude, hence, stronger contrast. Some wrinkles in these regions extend all the way from $r = R_{in}$ to $r = L$ without interruption. At a given radius, $\ell_\parallel$ is taken as the average width of the defect-free regions. In practice, several measurements at different locations are averaged together, and the error bar is taken to be the standard deviation of these values. The wrinkle wavelength serves as a natural local “meter-stick”; we thus measure $\ell_\parallel$ in units of the observed $\lambda$ at that location.

In some cases, one may also extract $\ell_\parallel$ from an analysis of the Fourier spectrum of the height function at fixed radius, $\zeta(\theta)$. We consider the simplest possible amplitude-modulated signal,

$$\zeta(\theta) = \cos(m\theta)\cos(n\theta),$$  \hspace{1cm} (S39)

(corresponding to Eq. S35), where $m = 2\pi r/\lambda$ and $n = \pi r/\ell_\parallel$, as pictured in the schematic in Fig. 3b in the main text. (Note the absence of a factor of 2 in the relation between $n$ and $\ell_\parallel$, since the size of a smectic region corresponds to a half-wavelength of the modulation envelope, see Fig. 3 of the main text.) This signal may be decomposed into two equal-amplitude components, $\zeta(\theta) = (1/2)\cos[(m+n)\theta] + (1/2)\cos[(m-n)\theta]$. Therefore, a Fourier spectrum that has two strong peaks at $k_1$ and $k_2$ is suggestive of a high-frequency component (i.e., a wrinkle number) of wavenumber $m = (k_1 + k_2)/2$, with an amplitude that is modulated at a wavenumber $n = (k_2 - k_1)/2$. We thus obtain:

$$\frac{\ell_\parallel}{\lambda} = \frac{k_1 + k_2}{2(k_2 - k_1)}. \hspace{1cm} \text{(S40)}$$

Figure S1 shows the Fourier spectrum for the image in Fig. S1a. The data show a number of spikes of various strength, but the overall trend (i.e., averaging over the noise) is perhaps best described as two strong peaks at $k_1 = 55$ and $k_2 = 92$. (These exact values are the locations of the maxima of $\zeta_k$ in the domains $k < 70$ and $k > 70$, respectively.) Plugging into Eq. S40 yields $\ell_\parallel/\lambda = 2.0$, in agreement with the value obtained visually from the image (double-sided arrow in Fig. S1a).

Other images do not yield to such an analysis. Figs. S1d,e show the height function and spectrum corresponding to Fig. 2d in the main text, at $r/R_{in} = 1.54$. Here the spectrum is broad, and it is difficult to distinguish two dominant peaks from the noise, or for that matter, to determine that the number of significant peaks in the spectrum is exactly 2. What we can say is that the gross shape of the spectrum is consistent with being centered around $k \approx 160$, in agreement with the wrinkle number measured at that radius using the autocorrelation method described in Section 6B above ($m = 172$). Despite this difficulty in obtaining a value of $\ell_\parallel$ from the Fourier spectrum, an amplitude-modulation lengthscales can still be extracted from direct inspection of the image by identifying regions of strong, nearly-parallel wrinkles.
FIG. S1. (A) Wrinkle pattern from simulations, showing strong amplitude modulation ($Bo = 5$, $\tau = 5.2$). Dashed line: $r/R_{in} = 1.8$. (B) Height function versus $\theta$ along a circle of radius $r = 1.8R_{in}$ from the simulation in panel A. (C) Fourier spectrum, $\zeta_k(k)$, for the signal in panel B. The spectrum shows two dominant peaks. (D) Height function versus $\theta$ along a circle of radius $r = 1.54R_{in}$ from the simulation shown in Fig. 2d in the main text ($Bo = 32$, $\tau = 5$). (E) Fourier spectrum for the signal in panel D. In this case, there are no two dominant peaks, although values of $\lambda$ and $\ell_\parallel$ can still be extracted from the image.