A relaxation result in $BV \times L^p$ for integral functionals depending on chemical composition and elastic strain.

Graça Carita, Elvira Zappale

January 8, 2015

Abstract

An integral representation result is obtained for the relaxation of a class of energy functionals depending on two vector fields with different behaviors which appear in the context of thermochemical equilibria and are related to image decomposition models and directors theory in nonlinear elasticity.

Keywords: relaxation, convexity-quasiconvexity.

MSC2000 classification: 49J45, 74Q05

1 Introduction

In this paper we consider energies depending on two vector fields with different behaviors: $u \in W^{1,1}(\Omega; \mathbb{R}^n)$ and $v \in L^p(\Omega; \mathbb{R}^m)$, $\Omega$ being a bounded open set of $\mathbb{R}^N$.

Let $1 < p \leq \infty$ for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ define the functional

$$J(u, v) := \int_{\Omega} f(v, \nabla u) \, dx$$

(1.1)

where $f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty)$ is a continuous function.

Minimization of energies depending on two independent vector fields have been introduced to model several phenomena. For instance the case of thermochemical equilibria among multiphase multicomponent solids and Cosserat theories in the context of elasticity: we refer to [7, 6] and the references therein for a detailed explanation about this kind of applications.

In the Sobolev setting, after the pioneer works [6, 7], relaxation with a Carathéodory density $f \equiv f(x, u, \nabla u, v)$, and homogenization for density of the type $f(\xi, \nabla u, v)$ have been considered in [4] and [3].

In the present paper we are interested in studying the lower semicontinuity and relaxation of (1.1) with respect to the $L^1$-strong$x\times L^p$-weak convergence. Clearly, bounded sequences $\{u_h\} \subset W^{1,1}(\Omega; \mathbb{R}^n)$ may converge in $L^1$, up to a subsequence, to a $BV$ function.

In the $BV$-setting this question has been already addressed in [5], only when the density $f$ is convex-quasiconvex (see [2.2]) and the vector field $v \in L^\infty$.

Here we allow $v$ to be in $L^p$, $p > 1$ and $f$ is not necessarily convex-quasiconvex. We provide an argument alternative to the one in [5], devoted to clarify some points in the lower semicontinuity result therein.

We also emphasize that under specific restrictions on the density $f$, i.e. $f(x, u, v, \nabla u) \equiv W(x, u, \nabla u) + \varphi(x, u, v)$, such analysis was considered already in [8] in order to describe image decomposition models. In [9] a general $f$ was taken into account only in the case the target $u$ is in $W^{1,1}$.

In the present manuscript we consider $f \equiv f(v, \nabla u)$ and $u \in BV$.

We study separately the cases $1 < p < \infty$ and $p = \infty$. To this end we introduce for $1 < p < \infty$ the functional

$$\mathcal{F}_p(u, v) := \inf \left\{ \liminf_{h \to \infty} J(u_h, v_h) : u_h \in W^{1,1}(\Omega; \mathbb{R}^n), \ v_h \in L^p(\Omega; \mathbb{R}^m), \ u_h \to u \text{ in } L^1, \ v_h \rightharpoonup v \text{ in } L^p \right\},$$

(1.2)
for any pair \((u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)\) and, for \(p = \infty\) the functional

\[
J_{\infty}(u,v) := \inf \left\{ \liminf_{h \to \infty} J(u_h,v_h) : u_h \in W^{1,1}(\Omega;\mathbb{R}^n), \; v_h \in L^\infty(\Omega;\mathbb{R}^m), \; u_h \to u \text{ in } L^1, \; v_h \rightharpoonup v \text{ in } L^\infty \right\},
\]

for any pair \((u,v) \in BV(\Omega;\mathbb{R}^n) \times L^\infty(\Omega;\mathbb{R}^m)\).

Since bounded sequences \(\{u_h\}\) in \(W^{1,1}(\Omega;\mathbb{R}^n)\) converge in \(L^1\) to a \(BV\) function \(u\) and bounded sequences \(\{v_h\}\) in \(L^p(\Omega;\mathbb{R}^m)\) if \(1 < p < \infty\), (in \(L^\infty(\Omega;\mathbb{R}^m)\) if \(p = \infty\)) weakly converge to a function \(v\in L^p(\Omega;\mathbb{R}^m)\), (weakly * in \(L^\infty\)), the relaxed functionals \(J_p\) and \(J_{\infty}\) will be composed by an absolutely continuous part and a singular one with respect to the Lebesgue measure (see (2.12)). On the other hand, as already emphasized in [5], it is crucial to observe that \(v\), regarded as a measure, is absolutely continuous with respect to the Lebesgue measure of the distributional gradient of \(u\), \(D^*u\), is concentrated, thus specific features of the density \(f\) will come into play to ensure a proper integral representation.

The integral representation of (1.2) will be achieved in Theorem 1.1 under the following hypotheses:

(H0) \(f\) is convex-quasiconvex;

(H1) \(p\) there exists a positive constant \(C\) such that

\[
\frac{1}{C} (|b|^p + |\xi|) - C \leq f(b,\xi) \leq C (1 + |b|^p + |\xi|),
\]

for \((b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}\);

(H2) \(p\) there exists \(C^* > 0, L > 0, 0 < \tau \leq 1\) such that

\[
t > 0, \; \xi \in \mathbb{R}^{n \times N}, \; \text{with } t |\xi| > L \implies \left| \frac{f(b,t\xi)}{t} - f^\infty(b,\xi) \right| \leq C^* \left( \frac{|b|^p + 1}{t^\tau} + \frac{|\xi|^{1-\tau}}{t^\tau} \right),
\]

where \(f^\infty\) is the recession function of \(f\) defined for every \(b \in \mathbb{R}^m\) as

\[
f^\infty(b,\xi) := \limsup_{t \to \infty} \frac{f(b,t\xi)}{t}.
\]

In order to characterize the functional \(J_{\infty}\) introduced in (1.3) we will replace assumptions (H1)\(_p\) and (H2)\(_p\) by the following ones:

(H1) \(\infty\) given \(M > 0\), there exists \(C_M > 0\) such that, if \(|v| \leq M\) then

\[
\frac{1}{C_M} |\xi| - C_M \leq f(b,\xi) \leq C_M (1 + |\xi|),
\]

for every \(\xi \in \mathbb{R}^{n \times N}\);

(H2) \(\infty\) given \(M > 0\), there exist \(C'_M > 0, L > 0, 0 < \tau \leq 1\) such that

\[
|b| \leq M, \; t > 0, \; \xi \in \mathbb{R}^{n \times N}, \; \text{with } t |\xi| > L \implies \left| \frac{f(b,t\xi)}{t} - f^\infty(b,\xi) \right| \leq C'_M \left( \frac{|\xi|^{1-\tau}}{t^\tau} \right).
\]

The paper is organized as follows: section 2 will be devoted to notations, preliminaries about measure theory, and results dealing with energy densities. In particular, we stress that we present a series of results devoted to show all the properties and relations among the relaxed energy densities involved in the integral representation and that can be of further use for the interested readers since they often appear in the integral representation context. Section 3 will contain the arguments devoted to prove the main results stated below.
Theorem 1.1 Let $J$ be given by (1.1), with $f$ satisfying $(H_1)_p$ and $(H_2)_p$ and let $\overline{J}_p$ be given by (1.2) then

$$\overline{J}_p(u,v) = \int_\Omega f(v,\nabla u) \, dx + \int_\Omega f^\infty \left(0, \frac{dD^s u}{d|D^s u|} \right) d|D^s u|,$$

for every $(u,v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$.

Theorem 1.2 Let $J$ be given by (1.1), with $f$ satisfying $(H_1)_\infty$ and $(H_2)_\infty$ and let $\overline{J}_\infty$ be given by (1.3) then

$$\overline{J}_\infty(u,v) = \int_\Omega f(v,\nabla u) \, dx + \int_\Omega f^\infty \left(0, \frac{dD^s u}{d|D^s u|} \right) d|D^s u|,$$

for every $(u,v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

Namely Theorem 3.1 contains the lower bound inequality for the case $1 < p < \infty$. The case $p = \infty$ is discussed in subsection 3.2 where also the upper bound in the case $1 < p < \infty$ is in Theorem 3.2.

Furthermore, we observe that Proposition 2.14 in subsection 2.3 is devoted to remove the convexity-quasiconvex assumption on $f$ in theorems 1.1 and 1.2 indeed we can prove that if $f$ does not satisfy $(H_0)$ then $\overline{J}_p$ (resp. $\overline{J}_\infty$) is given by

$$\int_\Omega CQf(v,\nabla u) \, dx + \int_\Omega (CQf)^\infty \left(0, \frac{dD^s u}{d|D^s u|} \right) d|D^s u|,$$

where $CQf$ denotes the convex-quasiconvex envelope of $f$ in (2.6) and $(CQf)^\infty$ represents the recession function of $CQf$, defined according to (2.14), which coincides, under suitable assumptions, (see assumptions (2.0), (2.7), Proposition 2.12 and Remark 2.13), with the convex-quasiconvex envelope of $f^\infty$, $CQ(f^\infty)$, and this allows us to remove the parenthesis.

## 2 Notations preliminaries and properties of the energy densities

In this section, we start by establishing notations, recalling some preliminary results on measure theory that will be useful through the paper and finally we define the space of functions of bounded variation. Then we deduce the main properties of convex-quasiconvex functions, recession functions and related envelopes. If $\nu \in S^{N-1}$ and $\{\nu, \nu_2, \ldots, \nu_N\}$ is an orthonormal basis of $\mathbb{R}^N$, $Q_{\nu}$ denotes the unit cube centered at the origin with its faces either parallel or orthogonal to $\nu, \nu_2, \ldots, \nu_N$. If $x \in \mathbb{R}^N$ and $\rho > 0$, we set $Q(x, \rho) := x + \rho Q_{\nu}$, $Q$ is the cube $\left(-\frac{1}{2}, \frac{1}{2}\right)^N$.

Let $\Omega$ be a generic open subset of $\mathbb{R}^N$, we denote by $\mathcal{A}(\Omega)$, the family of all open subsets of $\Omega$, and by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in $\Omega$ with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $C_0(\Omega)$ of continuous functions on $\Omega$ vanishing on the boundary $\partial \Omega$. The $N$-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$. If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodym derivative of $\mu$ with respect to $\lambda$. By a generalization of the Besicovich Differentiation Theorem (see [1, Proposition 2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho \to 0^+} \frac{\mu(x + \rho C)}{\lambda(x + \rho C)}$$

for all $x \in \text{Supp} \lambda \setminus E$ and any open bounded convex set $C$ containing the origin.

We recall that the exceptional set $E$ above does not depend on $C$. An immediate corollary is the generalization of Lebesgue-Besicovich Differentiation Theorem given below.

Theorem 2.1 If $\mu$ is a nonnegative Radon measure and if $f \in L^1_{\text{loc}}(\mathbb{R}^N, \mu)$ then

$$\lim_{\varepsilon \to 0^+} \frac{1}{\mu(x + \varepsilon C)} \int_{x+\varepsilon C} |f(y) - f(x)| \, d\mu(y) = 0$$

for $\mu$- a.e. $x \in \mathbb{R}^N$ and for every, bounded, convex, open set $C$ containing the origin.
**Definition 2.2** A function $u \in L^1(\Omega; \mathbb{R}^n)$ is said to be of bounded variation, and we write $u \in BV(\Omega; \mathbb{R}^n)$, if all its first distributional derivatives, $D_j u_i$, belong to $M(\Omega)$ for $1 \leq i \leq n$ and $1 \leq j \leq N$.

The matrix-valued measure whose entries are $D_j u_i$ is denoted by $Du$ and $|Du|$ stands for its total variation. We observe that if $u \in BV(\Omega; \mathbb{R}^n)$ then $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega; \mathbb{R}^n)$ with respect to the $L^1_{loc}(\Omega; \mathbb{R}^n)$ topology.

By the Lebesgue Decomposition Theorem we can split $Du$ into the sum of two mutually singular measures $D^a u$ and $D^s u$, where $D^s u$ is the absolutely continuous part of $Du$ with respect to the Lebesgue measure $\mathcal{L}^N$, while $D^a u$ is the singular part of $Du$ with respect to $\mathcal{L}^N$. By $\nabla u$ we denote the Radon-Nikodym derivative of $D^s u$ with respect to the Lebesgue measure so that we can write

$$Du = \nabla u \mathcal{L}^N + D^s u.$$  

**Proposition 2.3** If $u \in BV(\Omega; \mathbb{R}^n)$ then for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{Q(x_0, \varepsilon)} |u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)|^{\frac{N}{N-1}} \, dx = 0. \quad (2.1)$$

For more details regarding functions of bounded variation we refer to [2].

### 2.1 Convex-quasiconvex functions

We start by recalling the notion of convex-quasiconvex function, presented in [5] (see also [6] and [7]).

**Definition 2.4** A Borel measurable function $f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ is said to be convex-quasiconvex if, for every $(b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}$, there exists a bounded open set $D$ of $\mathbb{R}^N$ such that

$$f(b, \xi) \leq \frac{1}{|D|} \int_D f(b + \eta(x), \xi + \nabla \varphi(x)) \, dx, \quad (2.2)$$

for every $\eta \in L^\infty(D; \mathbb{R}^m)$, with $\int_D \eta(x) \, dx = 0$, and for every $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^n)$.

**Remark 2.5**

i) It can be easily seen that, if $f$ is convex-quasiconvex then condition (2.2) is true for any bounded open set $D \subset \mathbb{R}^N$.

ii) A convex-quasiconvex function is separately convex.

iii) the growth condition from above in $(H_1)_p$, ii) and [3, Proposition 2.11], entail that there exists $\gamma > 0$ such that

$$|f(b, \xi) - f(b', \xi')| \leq \gamma \left( |\xi - \xi'| + (1 + |b|^{p-1} + |b'|^{p-1} + |\xi| + |\xi'|)^{\frac{1}{p'}} |b - b'| \right) \quad (2.3)$$

for every $b, b' \in \mathbb{R}^m$, $\xi, \xi' \in \mathbb{R}^{n \times N}$, where $p > 1$ and $p'$ its conjugate exponent.

iv) In case of growth conditions expressed by $(H_1)_\infty$ (see [9, Proposition 4]), ii) entails that, given $M > 0$ there exists a constant $\beta(M, n, M, N)$ such that

$$|f(b, \xi) - f(b', \xi')| \leq \beta (1 + |\xi| + |\xi'|) |b - b'| + \beta |\xi - \xi'| \quad (2.4)$$

for every $b, b' \in \mathbb{R}^m$, such that $|b| \leq M$ and $|b'| \leq M$, for every $\xi, \xi' \in \mathbb{R}^{n \times N}$.

We introduce the notion of convex-quasiconvex envelope of a function, which is crucial to deal with the relaxation procedure.

**Definition 2.6** Let $f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a Borel measurable function bounded from below. The convex-quasiconvex envelope is the largest convex-quasiconvex function below $f$, i.e.,

$$CQf(b, \xi) := \sup \{ g(b, \xi) : g \leq f, \ g \text{ convex-quasiconvex} \}.$$
We recall that, by Theorem 4.16 in [7], the convex-quasiconvex envelope coincides with the so called convex-quasiconvexification

\[
\text{CQ} f(b, \xi) = \inf \left\{ \frac{1}{|D|} \int_D f(b + \eta(x), \xi + \nabla \varphi(x)) \, dx : \eta \in L^\infty(D; \mathbb{R}^m), \int_D \eta(x) \, dx = 0, \varphi \in W^{1,\infty}_0(D; \mathbb{R}^n) \right\}.
\] (2.5)

As for convexity-quasiconvexity, condition (2.5) can be stated for any bounded open set \( D \subset \mathbb{R}^N \).

It can also be showed that if \( f \) satisfies a growth condition of the type \((H_1)_p\) then in [2.2] and [2.5] the spaces \( L^\infty \) and \( W^{1,1}_0 \) can be replaced by \( L^p \) and \( W^{1,1}_0 \), respectively.

The proof of the following proposition, that will be exploited in the sequel, can be found in [9, Proposition 5].

**Proposition 2.7** Let \( f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty) \) be a continuous function satisfying \((H_1)_p\). Then \( \text{CQ} f \) is continuous and satisfies \((H_1)_p\). Consequently \( \text{CQ} f \) satisfies (2.3).

In order to deal with \( v \in L^\infty \) and to compare with the result in \( BV \times L^p, 1 < p < \infty \), one can consider a different setting of assumptions on the energy density \( f \).

Namely, following [9] Proposition 6 and Remark 7, if \( \alpha : [0, \infty) \to [0, \infty) \) is a convex and increasing function, such that \( \alpha(0) = 0 \) and if \( f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty) \) is a continuous function satisfying

\[
\frac{1}{C} (\alpha(|b|) + |\xi|) - C \leq f(b, \xi) \leq C (1 + \alpha(|b|) + |\xi|)
\] (2.6)

for every \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N} \), then \( \text{CQ} f \) satisfies a condition analogous to (2.6). Moreover, \( \text{CQ} f \) is a continuous function.

Analogously, one can assume that \( f \) satisfies the following variant of \((H_2)_\infty\): there exist \( c' > 0, L > 0, 0 < \tau \leq 1 \) such that

\[
t > 0, \xi \in \mathbb{R}^{n \times N}, \text{ with } t |\xi| > L \implies \left| \frac{f(b, t\xi)}{t} - f^\infty(b, \xi) \right| \leq c' \left( \frac{\alpha(|b|) + 1}{t^{\tau}} + |\xi|^{1-\tau} \right).
\] (2.7)

We observe that, if from one hand (2.6) and (2.7) generalize \((H_1)_p\) and \((H_2)_p\), respectively, from the other hand they can be regarded also as a stronger version of \((H_1)_\infty\) and \((H_2)_\infty\) respectively.

**2.2 The recession function**

Let \( f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty[ \), and let \( f^\infty : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty[ \) be its recession function, defined in (1.4).

The following properties are an easy consequence of the definition of recession function and conditions \((H_0)_p\), \((H_1)_p\) and \((H_2)_p\), when \( 1 < p < \infty \).

**Proposition 2.8** Provided \( f \) satisfies \((H_0)_p\), \((H_1)_p\) and \((H_2)_p\), then

1. \( f^\infty \) is convex-quasiconvex;
2. there exists \( C > 0 \) such that

\[
\frac{1}{C} |\xi| \leq f^\infty(b, \xi) \leq C |\xi|;
\] (2.8)
3. \( f^\infty(b, \xi) \) is constant with respect to \( b \) for every \( \xi \in \mathbb{R}^{n \times N} \);
4. in particular, \( f^\infty \) is continuous.

**Remark 2.9** We emphasize that not all the assumptions \((H_0)_p\), \((H_1)_p\) and \((H_2)_p\) on \( f \) in Proposition 2.8 are necessary to prove items above. In particular, one has that:

i) the proof of 2. uses only the fact that \( f \) satisfies \((H_1)_p\).
\textit{Proof.}

1. The convexity-quasiconvexity of \( f^\infty \) can be proven exactly as in \cite{[5]} Lemma 2.1.

2. By definition (1.4) we may find a subsequence \( \{t_k\} \) such that

\[
f^\infty (b, \xi) = \lim_{t_k \to \infty} \frac{f(b, t_k \xi)}{t_k}.
\]

By (H1)\(_p\) one has

\[
f^\infty (b, \xi) \leq \lim_{t_k \to \infty} \frac{C (1 + |b|^p + |t_k \xi|)}{t_k} = C |\xi| \quad \text{and} \quad f^\infty (b, \xi) \geq \lim_{t_k \to \infty} \frac{\frac{1}{C} (|b|^p + |t_k \xi|) - C}{t_k} \geq \frac{1}{C} |\xi|.
\]

Hence (H1)\(_p\) holds for \( f^\infty \).

3. We start by observing that (2.8) and 1. guarantee that \( f^\infty \) satisfies (2.3). Let \( \xi \in \mathbb{R}^{n \times N} \), and let \( b, b' \in \mathbb{R}^m \), up to a subsequence, by (1.4) and (2.3) it results that,

\[
f^\infty (b, \xi) - f^\infty (b', \xi) \leq \lim_{t_k \to \infty} \frac{f(b, t_k \xi) - f(b', t_k \xi)}{t_k} \leq \lim_{t_k \to \infty} \frac{\gamma (1 + |b|^p + |b'|^{p-1} + |t_k \xi|) |b - b'|}{t_k} = 0.
\]

By interchanging the role of \( b \) and \( b' \), it follows that \( f^\infty (\cdot, \xi) \) is constant and this concludes the proof.

\[\square\]

\textbf{Remark 2.10} Under assumptions (H0), (H1)\(_\infty\) and (H2)\(_\infty\), \( f^\infty \) satisfies properties analogous to those at the beginning of subsection In particular in \cite{[3]} Lemma 2.1 and Lemma 2.2] it has been proved that

\begin{enumerate}
\item \( f^\infty \) is convex-quasiconvex;
\item \( \frac{1}{M} |\xi| \leq f^\infty (b, \xi) \leq C_M |\xi|, \) for every \( b, \) with \( |b| \leq M; \)
\item If \( \text{rank} \xi \leq 1, \) then \( f^\infty (b, \xi) \) is constant with respect to \( b. \)
\end{enumerate}

\textbf{Remark 2.11} We observe that, if \( f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty) \) is a continuous function satisfying (H1)\(_p\) and (H2)\(_p\), then the function \( (CQ f)^\infty : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty], \) obtained first taking the convex-quasiconvexification in (2.3) of \( f \) and then its recession through formula (1.4) applied to \( CQ f, \) satisfies the following properties:

\begin{enumerate}
\item \( (CQ f)^\infty \) is convex-quasiconvex;
\item there exists \( C > 0 \) such that \( \frac{1}{C} |\xi| \leq (CQ f)^\infty (b, \xi) \leq C |\xi|, \) for every \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}; \)
\item for every \( \xi \in \mathbb{R}^{n \times N}, \) \( (CQ f)^\infty (\cdot, \xi) \) is constant, i.e. \( (CQ f)^\infty \) is independent on \( v; \)
\item \( (CQ f)^\infty \) is Lipschitz continuous in \( \xi. \)
\end{enumerate}

Under the same set of assumptions on \( f, \) one can prove that the convex-quasiconvexification of \( f^\infty \) satisfies the following conditions:

\begin{enumerate}
\item \( CQ(f^\infty) \) is convex-quasiconvex;
\item there exists \( C > 0 \) such that \( \frac{1}{C} |\xi| \leq CQ(f^\infty)(b, \xi) \leq C |\xi|, \) for every \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}; \)
\item for every \( \xi \in \mathbb{R}^{n \times N}, \) and assuming that \( f \) satisfies (2.3), \( CQ(f^\infty)(\cdot, \xi) \) is constant, i.e. \( CQ(f^\infty) \) is independent on \( b; \)
\item \( CQ(f^\infty) \) is Lipschitz continuous in \( \xi. \)
\end{enumerate}
The above properties are immediate consequences of Propositions 2.7, 2.8, and (2.3). In particular 8. follows from 3. of Proposition 2.8, without requiring (H2)p.

On the other hand, Proposition 2.12 below entails that \( \mathcal{CQ}(f) \) is independent on \( b \), without requiring that \( f \) is Lipschitz continuous, but replacing this assumption with \( (H2)_p \).

We also observe that \( (\mathcal{CQ}f)^\infty \) and \( \mathcal{CQ}(f^\infty) \) are only quasiconvex functions, since they are independent of \( b \). In particular, in our setting, these functions coincide as it is stated below.

**Proposition 2.12** Let \( f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty) \) be a continuous function satisfying \( (H1)_p \) and \( (H2)_p \).

Then

\[
\mathcal{CQ}(f^\infty)(b, \xi) = (\mathcal{CQ}f)^\infty(b, \xi)
\]

for every \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N} \).

**Proof.** The proof will be achieved by double inequality.

For every \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N} \) the inequality

\[
(\mathcal{CQ}f)^\infty(b, \xi) \leq \mathcal{CQ}(f^\infty)(b, \xi)
\]

follows by Definition 2.6 and the fact that \( \mathcal{CQ}f(b, \xi) \leq f(b, \xi) \). In fact, (1.4) entails that the same inequality holds when, passing to \( (\cdot)^\infty \). Finally 1. in Proposition 2.8 guarantees (2.9).

In order to prove the opposite inequality, fix \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N} \) and, for every \( t \geq 1 \), take \( \eta_t \in L^\infty(Q; \mathbb{R}^m) \), with 0 average, and \( \phi_t \in W^{1,\infty}_0(Q; \mathbb{R}^n) \) such that

\[
\int_Q f(b + \eta_t, \xi + \nabla \phi_t(y)) dy \leq \mathcal{CQ}f(b, t\xi) + 1.
\]

By \( (H1)_p \) and Proposition 2.7 we have that \( \|b + \eta_t\|_{L^p(Q)} \), \( \|\nabla(\frac{1}{t}\phi_t)\|_{L^1(Q)} \leq C \) for a constant independent on \( t \). Defining \( \psi_t := \frac{1}{t}\phi_t \), one has \( \psi_t \in W^{1,\infty}_0(Q; \mathbb{R}^n) \) and thus

\[
\mathcal{CQ}(f^\infty)(b, \xi) \leq \int_Q f^\infty(b + \eta_t, \xi + \nabla \psi_t(y)) dy.
\]

Let \( L \) be the constant appearing in condition \( (H2)_p \). We split the cube \( Q \) in the set \( \{ y \in Q : t|\xi + \nabla \psi_t(y)| \leq L \} \) and its complement in \( Q \). Then we apply condition \( (H2)_p \) and (2.8) to get

\[
\mathcal{CQ}(f^\infty)(b, \xi) \leq \int_Q \left( C \frac{1 + \|b + \eta_t\|^p}{t} + C \frac{|\xi + \nabla \psi_t|^{1-\tau}}{t^\tau} + \frac{f(b + \eta_t, t\xi + \nabla \phi_t) + CL}{t} \right) dy.
\]

Applying Hölder inequality and (2.10), we get

\[
\mathcal{CQ}(f^\infty)(b, \xi) \leq \frac{C}{t^\tau} \left( \int_Q |\xi + \nabla \psi_t| dy \right)^{1-\tau} + \frac{\mathcal{CQ}f(b, t\xi) + 1}{t} + \frac{CL}{t} + \frac{C'}{t},
\]

and the desired inequality follows by definition of \( \mathcal{CQ}f \) and using the fact that \( \nabla \psi_t \) has bounded \( L^1 \) norm, letting \( t \) go to \( \infty \). \( \blacksquare \)

**Remark 2.13** It is worth to observe that inequality

\[
(\mathcal{CQ}f^\infty)(b, \xi) \leq \mathcal{CQ}(f^\infty)(b, \xi)
\]

for every \( (b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N} \), has been proven without requiring neither \( (H1)_p \) and \( (H2)_p \) on \( f \), nor \( (H1)_\infty \) and \( (H2)_\infty \).

Furthermore, we emphasize that the proof of Proposition 2.12 cannot be performed in the same way in the case \( p = \infty \), with assumptions \( (H1)_p \) and \( (H2)_p \) replaced by \( (H1)_\infty \) and \( (H2)_\infty \). Indeed, an \( L^\infty \) bound on \( b + \eta_t \) analogous to the one in \( L^p \) cannot be obtained from \( (H1)_\infty \). On the other hand it is possible to deduce the equality between \( \mathcal{CQ}f^\infty \) and \( \mathcal{CQ}(f)^\infty \), when \( f \) satisfies (2.6) and (2.7).
2.3 Auxiliary results

Here we prove that assumption \((H_0)\) on \(f\) is not necessary to provide an integral representation for \(J_p\) in \((1.2)\).

Indeed we can assume that \(f : \mathbb{R}^{m} \times \mathbb{R}^{n \times N} \to [0, \infty]\) is a continuous function and satisfies assumptions \((H_1)_p\) and \((H_2)_p\), \((p \in (1, \infty))\). First we extend, with an abuse of notation, the functional \(J\) in \((1.1)\), to \(L^1(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m), p \in (1, \infty)\), as

\[
J(u, v) := \begin{cases} 
\int_{\Omega} f(v, \nabla u) dx & \text{if } (u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m), \\
\infty & \text{otherwise}.
\end{cases} \tag{2.11}
\]

Then we define, according to Definition \((2.6)\) the convex-quasiconvex envelope of \(f, CQf\), and introduce, in analogy with \((2.11)\) and \((1.2)\), the functional

\[
J_{CQf}(u, v) := \begin{cases} 
\int_{\Omega} CQf(v, \nabla u) dx & \text{if } (u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m), \\
\infty & \text{otherwise},
\end{cases}
\]

\((p \in (1, \infty))\) and,

\[
\overline{J_{CQf}}(u, v) := \inf \left\{ \liminf_{h \to \infty} J_{CQf}(u_h, v_h) : u_h \in W^{1,1}(\Omega; \mathbb{R}^n), \ v_h \in L^p(\Omega; \mathbb{R}^m), \ u_h \rightharpoonup u \ in \ L^1, \ v_h \rightharpoonup v \ in \ L^p \right\},
\]

for any pair \((u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m), p \in (1, \infty)\). Analogously, one can consider

\[
\overline{J_{CQf}}(u, v) := \inf \left\{ \liminf_{h \to \infty} J_{CQf}(u_h, v_h) : u_h \in W^{1,1}(\Omega; \mathbb{R}^n), \ v_h \in L^p(\Omega; \mathbb{R}^m), \ u_h \rightharpoonup u \ in \ L^1, \ v_h \rightharpoonup v \ in \ L^\infty \right\},
\]

for any pair \((u, v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m).\)

Clearly it results that for every \((u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m),\)

\[
J_{CQf}(u, v) \leq J_p(u, v),
\]

but, as in \([9]\) Lemma 8 and Remark 9, the following proposition can be proven.

**Proposition 2.14** Let \(p \in (1, \infty)\) and consider the functionals \(J\) and \(J_{CQf}\) and their corresponding relaxed functionals \(\overline{J}_p\) and \(\overline{J_{CQf}}\). If \(f\) satisfies conditions \((H_1)_p\) and \((H_2)_p\) if \(p \in (1, \infty)\), and both \(f\) and \(CQf\) satisfy \((H_1)_\infty\) and \((H_2)_\infty\) if \(p = \infty\), then

\[
\overline{J}_{CQf}(u, v) = J_p(u, v)
\]

for every \((u, v) \in BV(\Omega, \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m), p \in (1, \infty)\).

**Remark 2.15** The argument has not been shown since it is already contained in \([9]\) Lemma 8 and Remark 9. On the other hand in \([9]\) it is not required that \(f\) satisfies \((H_2)_p\), \((p \in (1, \infty))\). Indeed, the coincidence between the two functionals \(\overline{J}_p\) and \(\overline{J_{CQf}}\) holds independently on this assumption on \(f\), but in order to remove hypothesis \((H_0)\) from the representation theorem we need to assume that \(CQf\) inherits the same properties as \(f\), which is the case as it has been observed in Proposition \((2.7)\). It is also worth to observe that, when \(p = \infty\), \((2.7)\) is equivalent to

\[
|f^\infty(b, \xi) - f(b, \xi)| \leq C(1 + |\alpha(b)| + |\xi|)
\]

for every \((b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N},\) and this latter property is inherited by \(CQf\) and \(CQf^\infty\) as it can be easily verified arguing as in \([8]\) Proposition 2.3. Thus Proposition \((2.14)\) holds when \(p = \infty\) just requiring that \(f\) satisfies \((2.6)\) and \((2.7)\).
In order to provide an integral representation for the functionals in (1.2) and (1.3), we introduce, for every $1 < p \leq \infty$, the energy $J(\cdot, \cdot; \cdot) : W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m) \times A(\Omega) \to \mathbb{R}$, namely,

$$J(u, v; A) = \begin{cases} \int_A f(v, \nabla u) dx & \text{in } W^{1,1}(A; \mathbb{R}^n) \times L^p(A; \mathbb{R}^m) \\ + \infty & \text{otherwise,} \end{cases}$$

and the relaxed ones

$$\overline{J}_p(u, v; A) := \inf \{ \liminf_{n \to \infty} J_p(u_n, v_n; A) : u_n \to u \text{ in } L^1(A; \mathbb{R}^n), v_n \to v \text{ in } L^p(A; \mathbb{R}^m) \}, \quad 1 < p < \infty$$

$$\overline{J}_\infty(u, v; A) := \inf \{ \liminf_{n \to \infty} J_\infty(u_n, v_n; A) : u_n \to u \text{ in } L^1(A; \mathbb{R}^n), v_n \rightharpoonup^* v \text{ in } L^\infty(A; \mathbb{R}^m) \}, \quad p = \infty$$

The following result can be deduced in full analogy with [9, Theorem 12], where it has been proven for $\overline{J}_\infty$.

**Proposition 2.16** Let $\Omega$ be a bounded and open set of $\mathbb{R}^N$ and let $f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a continuous function satisfying $(H_1)_p$ and $(H_2)_p$, $1 < p \leq \infty$. Let $J$ be the functional defined in (1.1), then $\overline{J}_p$ in (1.2) $(1 < p < \infty)$, $\overline{J}_\infty$ $(p = \infty)$ is a variational functional.

By virtue of this result, it turns out that for every $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$, $\overline{J}_p(u, v; \cdot)$, $(1 < p \leq \infty)$ is the restriction to the open subsets in $\Omega$ of a Radon measure on $\Omega$, thus it can be decomposed as the sum of two terms

$$\overline{J}_p(u, v; \cdot) = \overline{J}_p^a(u, v; \cdot) + \overline{J}_p^s(u, v; \cdot),$$

where $\overline{J}_p^a(u, v; \cdot)$ and $\overline{J}_p^s(u, v; \cdot)$ denote the absolutely continuous part and the singular part with respect to the Lebesgue measure, respectively. Next proposition deals with the scaling properties of $\overline{J}_p$.

**Proposition 2.17** Let $f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a continuous and convex-quasiconvex function, let $J$ and $\overline{J}_p$ be the functionals defined respectively by (1.1) and (1.2) when $1 < p < \infty$, (1.3), when $p = \infty$. Then the following scaling properties are satisfied

$$\overline{J}_p(u + \eta, v; \Omega) = \overline{J}_p(u, v; \Omega) \quad \text{for every } \eta \in \mathbb{R}^n,$n

$$\overline{J}_p(u(-x_0), v(-x_0); x_0 + \Omega) = \overline{J}_p(u(\cdot), v(\cdot); \Omega) \quad \text{for every } x_0 \in \mathbb{R}^N,$n

$$\overline{J}_p \left( u_v, v_{\Omega - x_0}, \frac{\Omega - x_0}{q} \right) = q^{-N} \overline{J}_p(u, v; \Omega),$$

where $u_v(y) := \frac{u(x_0 + qy) - u(x_0)}{q}$ and $v_{\Omega}(y) := v(x_0 + qy)$, for $y \in \Omega - x_0$.

The following result will be exploited in the sequel. The proof is omitted since it develops along the lines of [2] Lemma 5.50], the only differences are the presence of $v$ and the convexity-quasiconvexity of $f$.

**Lemma 2.18** Let $f : \mathbb{R}^m \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a continuous and convex-quasiconvex function, and let $J$ and $\overline{J}_p$ be the functionals defined respectively by (1.1) and (1.2). Let $\nu \in S^{N-1}$, $\eta \in S^{n-1}$ and $\psi : \mathbb{R} \to \mathbb{R}$, bounded and increasing. Denoted by $Q$ the cube $Q_p$, let $u \in BV(Q; \mathbb{R}^n)$ be representable in $Q$ as

$$u(y) = \eta \psi(y \cdot \nu),$$

and let $w \in BV(Q; \mathbb{R}^n)$ be such that supp$(w - u) \subset Q$. Let $v \in L^p(Q; \mathbb{R}^m)$. Then

$$\overline{J}_p(w, v; Q) \geq f \left( \int_Q vdy, Dw(Q) \right).$$
3 Main Results

This section is devoted to deduce the results stated in Theorems 1.1 and 1.2. We start by proving the lower bound in the case $1 < p < \infty$. For what concerns the upper bound, we present, for the reader’s convenience a self contained proof in Theorem 3.2. For the sake of completeness we observe that the upper bound, in the case $1 < p < \infty$, could be deduced as a corollary from the case $p = \infty$ (see Theorem 1.2), which, in turn, under slightly different assumptions, is contained in [5].

3.1 Lower semicontinuity in $BV \times L^p$, $1 < p < \infty$

**Theorem 3.1** Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, let $f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty)$ be a continuous function satisfying $(H_0), (H_1)_p$, and $(H_2)_p$, and let $J_p$ be the functional defined in [1, Theorem 11]. Then

$$J_p(u, v; \Omega) \geq \int_\Omega f (v, \nabla u) \, dx + \int_\Omega f^\infty \left(0, \frac{dD^s u}{d|D^s u|} \right) d|D^s u|$$

(3.1)

for any $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$.

**Proof.** The proof will be achieved, in two steps, namely by showing that

$$\lim_{\varepsilon \to 0^+} \frac{J_p(u, v; Q(x_0, \varepsilon))}{\mathcal{L}^N(Q(x_0, \varepsilon))} \geq f(v(x_0), \nabla u(x_0)), \quad \text{for } \mathcal{L}^N - \text{a.e. } x_0 \in \Omega,$$

(3.2)

$$\lim_{\varepsilon \to 0^+} \frac{J_p(u, v; Q(x_0, \varepsilon))}{\mathcal{L}^N(Q(x_0, \varepsilon))} \geq f^\infty \left(0, \frac{dD^s u}{d|D^s u|} \right)(x_0), \quad \text{for } |D^s u| - \text{a.e. } x_0 \in \Omega.$$

(3.3)

Indeed, if (3.2) and (3.3) hold then, by virtue of (2.12), and [2, Theorem 2.56], (3.1) follows immediately.

**Step 1.** Inequality (3.2) is obtained through an argument entirely similar to [2, Proposition 5.53] and exploiting [9, Theorem 11].

For $\mathcal{L}^N$–a.e. $x_0 \in \Omega$ it results that $u$ is approximately differentiable (see (2.1)) and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\mathcal{L}^N(Q(x_0, \varepsilon))} \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)| \, dx = 0.$$

Consequently, given $\varrho > 0$, and defined $u_\varrho$ and $v_\varrho$ as in Proposition 2.17 it results that $u_\varrho \to u_0$ in $L^1(\Omega; \mathbb{R}^n)$, where $u_0 := \nabla u(x_0) x$ and $v_\varrho \to v(x_0)$ in $L^p(\Omega; \mathbb{R}^m)$. Then the scaling properties (2.13), and the lower semicontinuity of $J_p$ entail that

$$\liminf_{\varepsilon \to 0^+} \frac{J_p(u, v; Q(x_0, \varepsilon))}{\varrho^N} = \liminf_{\varepsilon \to 0^+} J_p(u_\varrho, v_\varrho; Q) \geq J_p(u_0, v(x_0); Q).$$

(3.4)

Then the lower semicontinuity result proven in [9, Theorem 11], when $u$ is in $W^{1,1}(\Omega; \mathbb{R}^n)$ and $v \in L^p(\Omega; \mathbb{R}^m)$, allow us to estimate the last term in (3.4) as follows

$$J_p(u_0, v(x_0); Q) \geq f(v(x_0), \nabla u(x_0)),$$

and that provides (3.2).

**Step 2.** Here we present the proof of (3.3). To this end we exploit techniques very similar to [1]. Let $Du = z|Du|$ be the polar decomposition of $Du$ (see [2, Corollary 1.5]), for $z \in S^{N \times n - 1}$, and recall that for $|D^s u|$–a.e. $x_0$, $z(x_0)$ admits the representation $\eta(x_0) \otimes v(x_0)$, with $\eta(x_0) \in S^{n - 1}$ and $v(x_0) \in S^{N - 1}$ (see [2, Theorem 3.94]). In the following we will denote the cube $Q_\varepsilon(x_0, 1)$ by $Q$.

To achieve (3.3) it is enough to show that

$$\lim_{\varepsilon \to 0^+} \frac{J_p(u, v; Q(x_0, \varepsilon))}{|Du(Q(x_0, \varepsilon))|} \geq f^\infty(0, z(x_0))$$
Let us fix \( t \) conditions (where \( Q \) at any Lebesgue point

\[ z(x_0) = \eta(x_0) \otimes \nu(x_0), \quad \lim_{\varepsilon \to 0^+} \frac{|Du|(Q(x_0, \varepsilon))}{\varepsilon^N} = \infty, \tag{3.5} \]

\( 0 = \lim_{\varepsilon \to 0^+} \frac{\int_{Q(x_0, \varepsilon)} |v|^p \, dx}{|Du|(Q(x_0, \varepsilon))} = \lim_{\varepsilon \to 0^+} \frac{\int_{Q(x_0, \varepsilon)} |v|^p \, dx}{\varepsilon^N} \tag{3.6} \]

The above requirements are, indeed, satisfied at \( |D^s u| \)-a.e. \( x_0 \in \Omega \), by Besicovitch’s derivation theorem and Alberti’s rank-one theorem (see \([2, \text{Theorem 3.94}]\)). Set \( \eta = \eta(x_0) \) and \( \nu = \nu(x_0) \), for \( \varepsilon < N^{-\frac{1}{2}} \text{dist}(x_0, \partial \Omega) \), we define

\[ u_\varepsilon(y) := \frac{u(x_0 + \varepsilon y) - \tilde{u}_\varepsilon}{\varepsilon^N |Du|(Q(x_0, \varepsilon))}, \quad y \in Q, \]

where \( \tilde{u}_\varepsilon \) is the average of \( u \) in \( Q(x_0, \varepsilon) \). Analogously we define, as in Proposition \( 2.17 \)

\[ v_\varepsilon(y) := v(x_0 + \varepsilon y), \quad y \in Q. \tag{3.7} \]

Let us fix \( t \in (0, 1) \). By \([2, \text{formula (2.32)}]\), there exists a sequence \( \{\varepsilon_h\} \) converging to 0 such that

\[ \lim_{h \to \infty} \frac{|Du|(Q(x_0, t\varepsilon_h))}{|Du|(Q(x_0, \varepsilon_h))} \geq t^N. \tag{3.8} \]

Denote \( u_{\varepsilon_h} \) by \( u_h \), then \( |Du_h|(Q) = 1 \) and, passing to a not relabelled subsequence, \( \{u_h\} \) converges in \( L^1(Q; \mathbb{R}^n) \) to a \( BV \) function \( \overline{\nu} \). Correspondingly, denote \( v_{\varepsilon_h} \) by \( v_h \). Then, we have

\[ |D\overline{\nu}|(Q) \leq 1 \quad \text{and} \quad |D\overline{\nu}|(Q_T) \geq t^N, \tag{3.9} \]

where \( Q_T := tQ \). It results that \( \overline{\nu}(y) = \eta \psi(y \cdot \nu) \), for some bounded increasing function \( \psi \) in \((-\frac{1}{2}, \frac{1}{2})\). Take \( \varphi \in C^1_c(Q) \) such that \( \varphi = 1 \) on \( Q_T \) and \( 0 \leq \varphi \leq 1 \), and let us define \( w_h := \varphi u_h + (1 - \varphi)\overline{\nu} \). The functions \( w_h \) converge to \( \overline{\nu} \) in \( L^1(Q; \mathbb{R}^n) \) and moreover we have

\[ |D(w_h - u_h)|(Q) \leq |D(u_h - \overline{\nu})||(Q \setminus Q_T) + \int_Q |\nabla \varphi||u_h - \overline{\nu}||dy \]

\[ \leq |Du_h|(Q \setminus Q_T) + |D\overline{\nu}|(Q \setminus Q_T) + \int_Q |\nabla \varphi||u_h - \overline{\nu}||dy. \]

Therefore, by \([6,8] \) and \([8,9] \), one has

\[ \limsup_{h \to \infty} |D(w_h - u_h)|(Q) \leq 2(1 - t^N). \tag{3.10} \]

Similarly,

\[ |Du_h|(Q \setminus Q_T) \leq |D(u_h)||(Q \setminus Q_T) + |D\overline{\nu}|(Q \setminus Q_T) + \int_Q |\nabla \varphi||u_h - \overline{\nu}||dy, \]

consequently

\[ \limsup_{h \to \infty} |Du_h|(Q \setminus Q_T) \leq 2(1 - t^N). \tag{3.11} \]

Setting \( c_h := \frac{|Du|(Q(x_0, \varepsilon_h))}{\varepsilon_h} \), by the scaling properties of \( \mathcal{F}_p \) in Proposition \( 2.17 \) and by the growth conditions \( (H_1)_p \), we have

\[ \frac{\mathcal{F}_p(u, v; Q(x_0, \varepsilon_h))}{|Du|(Q(x_0, \varepsilon_h))} = \frac{\mathcal{F}_p(c_h w_h, v_h; Q)}{c_h} = \frac{\mathcal{F}_p(c_h w_h, v_h; Q)}{c_h} \]

\[ \geq \frac{\mathcal{F}_p(c_h w_h, v_h; Q)}{c_h} - C(c_h^{-1} |Q \setminus Q_T| + |Dw_h|(Q \setminus Q_T) + c_h^{-1} \int_{Q \setminus Q_T} |v_h|^p \, dy). \]
By (3.5), \( c_h \to \infty \), moreover taking into account (3.7) and (3.6), by (3.11), it results that

\[
\lim_{\varepsilon \to 0^+} \frac{J_p(u,v; Q(x_0, \varepsilon))}{|Du| Q(x_0, \varepsilon))} \geq \lim_{h \to \infty} \sup_{h \to \infty} \frac{J_p(c_h u_h, v_h; Q)}{c_h} - 2C(1-t^n)
\]

On the other hand, Lemma 2.18 entails that, for every \( h \in \mathbb{N} \),

\[
J_p(c_h u_h, v_h; Q) \geq f \left( \int_Q v_h dy, c_h Du_h(Q) \right) \geq f \left( \int_Q v_h dy, c_h Du_h(Q) \right) - c_h \gamma |D(w_h - u_h)|(Q),
\]

where \( \gamma \) is the constant appearing in (2.3). Then by (3.10), we have that

\[
\lim_{\varepsilon \to 0^+} \frac{J_p(u,v; Q(x_0, \varepsilon))}{|Du| Q(x_0, \varepsilon))} \geq \lim_{h \to \infty} \sup_{h \to \infty} \frac{f \left( \int_Q v_h dy, c_h Du_h(Q) \right)}{c_h} - 2(C + \gamma)(1-t^n).
\]

By the definition of \( u_h, Du_h(Q) = \frac{Du(Q(x_0, \varepsilon))}{|Du| Q(x_0, \varepsilon))} \), hence \( Du_h(Q) \to z(x_0) \), since \( x_0 \) is a Lebesgue point of \( z \). Now, taking into account (2.3) and (H2), we have

\[
\limsup_{h \to \infty} \frac{f \left( \int_Q v_h dy, c_h Du_h(Q) \right)}{c_h} = \lim_{h \to \infty} \frac{f \left( \int_Q v_h dy, c_h z(x_0) \right)}{c_h} = \lim_{h \to \infty} \left( f^\infty \left( \int_Q v_h dy, z(x_0) \right) + C \frac{\int_Q v_h dy}{c_h} \right) = f^\infty(z(x_0)),
\]

where it has been exploited the fact that \( c_h \to \infty \), 3. of Proposition 2.8, the nondecreasing behaviour of the \( L^p \) norm in the unit cube with respect to \( p \) (i.e., \( \int_Q v_h dy \geq \int_Q |v_h|^p dy \)), and (3.6).

### 3.2 Relaxation

We start observing that Theorem 1.2 is contained in [5] under a uniform coercivity assumption. We do not propose the proof in our setting, since it develops along the lines of Theorems 3.1 and 3.2. On the other hand, several observations about Theorem 1.2 are mandatory:

1. If \( f \) satisfies (H1)_\( p \), and (H2)_\( p \) then \( J_p(u,v) \leq J_\infty(u,v) \) for every \( (u,v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m) \).

2. For the reader’s convenience we observe that the proof of the lower bound in Theorem 1.2 develops exactly as that of Theorem 3.1 using the \( L^\infty \) bound on \( v \) to deduce (3.6) and the uniform bound on \( v_e \) in (3.7), (H2)_\( \infty \) and (2.4) in order to estimate \( \limsup_{h \to \infty} \frac{f \left( \int_Q v_h dy, c_h Du_h(Q) \right)}{c_h} \).

Regarding the upper bound, the bulk part follows from [5] Theorems 12 and 14], while for the singular part we can argue exactly as proposed in the proof of the upper bound in [5] just considering conditions (H1)_\( \infty \) and (H2)_\( \infty \) in place of (H1)_\( p \) and (H2)_\( p \).

3. The above arguments remain true under assumptions (2.0) and (2.7).

We are now in position to prove the upper bound for the case \( BV \times L^p \), for \( 1 < p < \infty \). We emphasize that an alternative proof could be obtained via a truncation argument from the case \( p = \infty \) as the one presented in [5] Theorem 12], but we prefer the self contained argument below.

**Theorem 3.2** Let \( \Omega \) be a bounded open set of \( \mathbb{R}^N \) and let \( f : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, \infty) \) be a continuous function. Then, assuming that \( f \) satisfies (H0), (H1)_\( p \) and (H2)_\( p \),

\[
J_p(u,v) \leq \int_\Omega f(v, \nabla u) dx + \int_\Omega f^\infty \left( 0, \frac{dD^s u}{d|D^s u|} (x) \right) d|D^s u|(x),
\]

for every \((u,v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)\).
Proof. First we observe that Proposition 2.10 entails that $\mathcal{T}_p$ is a variational functional. Thus the inequality can be proved following arguments analogous to 1. For what concerns the bulk part, it is enough to observe that given $u \in BV(\Omega; \mathbb{R}^n)$ and $v \in L^p(\Omega; \mathbb{R}^m)$, taking a sequence of standard mollifiers $\{\varrho_{\varepsilon_k}\}$, where $\varepsilon_k \to 0^+$, it results that $\nabla u_{\varepsilon_k} = \nabla u \ast \varrho_{\varepsilon_k} + D^*u \ast \varrho_{\varepsilon_k}$, where $u_{\varepsilon_k} := u \ast \varrho_{\varepsilon_k}$. The local Lipschitz behaviour of $f$ in (2.3) gives
\[
\int_A f(v, \nabla u_{\varepsilon_k})dx \leq \int_A f(v, \nabla u \ast \varrho_{\varepsilon_k})dx + \gamma |D^*u|(I_{\varepsilon_k}(A))
\]
for every $k \in \mathbb{N}$, where $I_{\varepsilon_k}(A)$ denotes the $\varepsilon_k$ neighborhood of $A$. Then if $|D^*u|(\partial A) = 0$, letting $\varepsilon_k \to 0^+$, we obtain
\[
\mathcal{T}_p(u, v; A) \leq \int_A f(v, \nabla u)dx + \gamma |D^*u|(A),
\]
for every open subset $A$ of $\Omega$. And this gives $\mathcal{T}_p(u, v; B) \leq \int_B f(v, \nabla u)dx$, for every borel set $B \subset \Omega$. Thus we can conclude that $\mathcal{T}_p(u, v; B) \leq \int_B f(v(x), \nabla u(x))dx$ for every $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ and $B$ Borel subset of $\Omega$.

To achieve the result, it will be enough to show that
\[
\mathcal{T}_p(u, v; B) \leq \int_B f^\infty \left(0, \frac{dD^*u}{d|D^*u|}\right) d|D^*u| \quad \text{for every Borel subset of } \Omega.
\]
For every $z \in \mathbb{R}^{n \times N}$ and $b \in \mathbb{R}^m$, define the function
\[
g(b, z) := \sup_{t \geq 0} \frac{f(b, tz) - f(b, 0)}{t}.
\]
Arguments entirely similar to those of Proposition 2.8 guarantee that $g$ is convex-quasiconvex, has linear growth on $z$, and $g$ is constant with respect to $b$.

Observe, as in 5, that $g$ is positively one homogeneous in the second variable and Lipschitz continuous. Also if $z$ is a rank-one matrix $g = f^\infty$.

Then for every open set $A \subset \subset \Omega$ such that $|Du|(A) = 0$, defining for every $h \in \mathbb{N}$, $u_h := u \ast \varrho_{\varepsilon_h}$ and $v_h := v \ast \varrho_{\varepsilon_h}$ where $\{\varrho_{\varepsilon_h}\}$ is a sequence of standard mollifiers and $\varepsilon_h \to 0^+$. Then $u_h \to u$ in $L^1$ and $v_h \to v$ in $L^p$. Also 2 Theorem 2.2 entails that $|Du_h| \to |Du|$ weakly $^*$ in $A$ and $|Du_h|(A) \to |Du|(A)$. Thus
\[
\mathcal{T}_p(u, v; A) \leq \liminf_{h \to \infty} \int_A f(v_h, \nabla u_h)dx \leq \limsup_{h \to \infty} \int_A f(v_h, 0)dx + \liminf_{h \to \infty} \int_A g(v_h, \nabla u_h)dx.
\]
For what concerns the first term in the right hand side, we have that
\[
\frac{1}{C} \int_A |v_h|^p dx - C \leq \int_A f(v_h, 0)dx \leq C \int_A (1 + |v_h|^p)dx
\]
and the term $\int_A |v_h|^p dx$ converges to $\int_A |v|^p dx$ as $h \to \infty$, thus taking the Radon-Nikodým derivative with respect to $|D^*u|$ we obtain 0.

Regarding the second term in the right hand side of (3.12), we can observe that $g$ does not depend on $b$ thus we can rewrite
\[
\liminf_{h \to \infty} \int_A g(v_h, \nabla u_h)dx = \liminf_{h \to \infty} \int_A g(0, \nabla u_h)dx = \int_A g \left(0, \frac{dDu}{d|D^*u|}\right) d|Du|,
\]
where it has been exploited Reshetnyak continuity theorem. Again, the Radon-Nikodým derivative, Alberti’s Theorem (see 2 Theorem 3.94) ensuring that for $|D^*u|$-a.e. $x \in \Omega$, the density $\frac{dDu}{d|D^*u|}$ is a rank-one matrix, and the fact that $g = f^\infty$ on rank-one matrices gives the desired result.

Proof of Theorem 1.1] The result follows from Theorems 3.1 and 3.2.]

Acknowledgements The research of the authors has been partially supported by Fundação para a Ciência e Tecnologia (Portuguese Foundation for Science and Technology) through UTA-CMU/MAT/0005/2009 and CIMA-UE.
References

[1] L. Ambrosio & G. Dal Maso, *On the Relaxation in $BV(\Omega; \mathbb{R}^m)$ of Quasi-convex Integrals*. Journal of Functional Analysis, 109, (1992), 76-97.

[2] L. Ambrosio, N. Fusco & D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Clarendon Press, Oxford, (2000).

[3] G. Carita, A.M. Ribeiro & E. Zappale, *An homogenization result in $W^{1,p} \times L^q$*. J. Convex Anal. 18, n. 4, (2011), 1093-1126.

[4] G. Carita, A.M. Ribeiro & E. Zappale, Relaxation for some integral functionals in $W^{1,p} \times L^q$. Bol. Soc. Port. Mat., (2010), Special Issue, 47-53.

[5] I. Fonseca, D. Kinderlehrer & P. Pedregal, *Relaxation in $BV \times L^\infty$ of functionals depending on strain and composition*. Lions, Jacques-Louis (ed.) et al., Boundary value problems for partial differential equations and applications. Dedicated to Enrico Magenes on the occasion of his 70th birthday. Paris: Masson. Res. Notes Appl. Math., 29, (1993), 113-152.

[6] I. Fonseca, D. Kinderlehrer & P. Pedregal, *Energy functionals depending on elastic strain and chemical composition*. Calc. Var. Partial Differential Equations, 2, (1994), 283-313.

[7] H. Le Dret & A. Raoult, *Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results*. Arch. Ration. Mech. Anal. 154, No. 2, (2000), 101-134.

[8] A. M. Ribeiro & E. Zappale, *Relaxation of certain integral functionals depending on strain and chemical composition*. Chinese Annals of Mathematics Series B, 34B(4) (2013), 491-514.

[9] A. M. Ribeiro & E. Zappale, *Lower semicontinuous envelopes in $W^{1,1} \times L^p$*. Banach Center Publ. 101 (2014), 187-206.