DIMENSIONS OF $C^1$–AVERAGE CONFORMAL HYPERBOLIC SETS

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ABSTRACT. This paper introduces the concept of average conformal hyperbolic sets, which admit only one positive and one negative Lyapunov exponents for any ergodic measure. For an average conformal hyperbolic set of a $C^1$ diffeomorphism, utilizing the techniques in sub-additive thermodynamic formalism and some geometric arguments with unstable/stable manifolds, a formula of the Hausdorff dimension and lower (upper) box dimension is given in this paper, which is exactly the sum of the dimensions of the restriction of the hyperbolic set to stable and unstable manifolds. Furthermore, the dimensions of an average conformal hyperbolic set vary continuously with respect to the dynamics.

1. Introduction. The dimension of invariant sets is one of the important characteristics, which plays an important role in various problems in dynamics, see the books [3, 4, 12, 21, 22]. Despite many interesting and non-trivial developments in the dimension theory of dynamical systems, only the case of conformal dynamics is completely understood. Indeed, Bowen [7] and Ruelle [24] found that the Hausdorff dimension of a $C^{1+\gamma}$ conformal repeller was a solution of an equation involving topological pressure. The smoothness is relaxed to $C^1$ in [14]. The study of dimension of

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hyperbolic sets is analogous. Using techniques in thermodynamic formalism, in [18] McCluskey and Manning obtained a formula of the Hausdorff dimension of a two dimensional hyperbolic set of a $C^{1+\gamma}$ diffeomorphism. Using a geometric method, Palis and Viana relaxed the smoothness to $C^1$ in [20]. Takens [25] proved that the same formula also holds for lower and upper box dimensions. Using the techniques of Markov partition and thermodynamic formalism, the same formula was obtained for the $C^{1+\gamma}$ conformal hyperbolic set in higher dimension, see the books [21] and [3] for detailed description.

For the non-conformal case, the study of dimension is substantially more complicated and difficult to approach it. Only upper and lower bounds of dimension of repellers are obtained, see [2, 13, 27] for details, different versions of Bowen’s equation involving topological pressure are useful in estimating the dimensions of a non-conformal repeller. Finally, using thermodynamic formalism for sub-additive potentials developed in [8], Ban et al [1] showed that the zero of the sub-additive topological pressure gives an upper bound of the Hausdorff dimension of repellers. Furthermore, they showed that the upper bounds obtained in the previous works [1, 13, 27] are all equal. See [11] for Bowen’s equation in estimating Hausdorff dimension in the case of very general non-uniform setting. Recently, in [9] the authors introduced the super-additive topological pressure, and showed that the zero of super-additive topological pressure gives a lower bound of the Hausdorff dimension of repellers. We refer the reader to [10] and [5] for a detailed description of the recent progress in dimension theory of dynamical systems.

In [1], the authors introduced a concept of $C^1$ average conformal repellers which possess only one positive Lyapunov exponent for any ergodic measure. An example is given in [28] to show that such a repeller is indeed non-conformal. The dimension of an average conformal repeller is given by the zero of sub-additive topological pressure, see [1] for details.

In this paper, we introduce a concept of $C^1$ average conformal hyperbolic sets in higher dimension. Roughly speaking, an average conformal hyperbolic set admits only one positive and one negative Lyapunov exponents for any ergodic measure. We obtain a dimension formula of such hyperbolic sets, which can be described as the sum the dimensions of the restriction of the hyperbolic set to a stable and unstable manifolds. Furthermore, the dimension of a $C^1$ average conformal hyperbolic set varies continuously with respect to the dynamics.

1.1. Notions and set-up. Let $f : M \to M$ be a $C^1$ diffeomorphism on an $m$-dimensional compact Riemannian manifold. For each $x \in M$, the following quantities

$$
\|D_x f\| = \sup_{0 \neq u \in T_x M} \frac{\|D_x f(u)\|}{\|u\|} \text{ and } m(D_x f) = \inf_{0 \neq u \in T_x M} \frac{\|D_x f(u)\|}{\|u\|}
$$

are respectively called the maximal norm and minimum norm of the differentiable operator $D_x f : T_x M \to T_{f(x)} M$, where $\| \cdot \|$ is the norm induced by the Riemannian metric on $M$.

Now we recall some definitions and well-known results in hyperbolic dynamics. See [17] for more details. A compact invariant subset $\Lambda \subset M$ is called a hyperbolic set if there exists a continuous splitting of the tangent bundle $T\Lambda M = E^s \oplus E^u$, and constants $C > 0, 0 < \lambda < 1$ such that for every $x \in \Lambda$

1. $D_x f(E^s(x)) = E^s(f(x)), \quad D_x f(E^u(x)) = E^u(f(x));$
(2) for all \( n \geq 0 \), \( \| D_x f^n(v) \| \leq C \lambda^n \| v \| \) if \( v \in E^s(x) \), and \( \| D_x f^{-n}(v) \| \leq C \lambda^n \| v \| \) if \( v \in E^u(x) \),

here we may choose an adapted metric such that \( C = 1 \), and \( \lambda \) is called the skewness of the hyperbolicity. Given a point \( x \in \Lambda \), for each small \( \beta > 0 \), the local stable and unstable manifolds are defined as follows:

\[
W^s_\beta(f, x) = \left\{ y \in M : d(f^n(x), f^n(y)) \leq \beta, \forall n \geq 0 \right\},
\]

\[
W^u_\beta(f, x) = \left\{ y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \beta, \forall n \geq 0 \right\}.
\]

The global unstable and stable sets of \( x \in \Lambda \) are given as follows:

\[
W^u(f, x) = \bigcup_{n \geq 0} f^n(W^u_\beta(f, f^{-n}(x))), \quad W^s(f, x) = \bigcup_{n \geq 0} f^{-n}(W^s_\beta(f, f^n(x))).
\]

Let \( d_u \) be the metric induced by the Riemannian structure on the unstable manifold \( W^u \) and \( d_s \) the metric induced by the Riemannian structure on the stable manifold \( W^s \). For any \( \rho > 0 \), let \( B^u(x, \rho) \) (respectively, \( B^s(x, \rho) \)) be the ball in the unstable (respectively, stable) manifold of radius \( \rho \) centered at \( x \), and

\[
B^i_{n+1}(x, \rho) = \left\{ y \in W^i(f, x) : d_i(f^k x, f^k y) < \rho \text{ for } k = 0, 1, 2, \ldots, n \right\},
\]

where \( i \in \{ u, s \} \) and \( n \in \mathbb{N} \). A hyperbolic set is called locally maximal, if there exists a neighbourhood \( U \) of \( \Lambda \) such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \).

Let \( \text{Diff}^1(M) \) be the set of all \( C^1 \) diffeomorphisms from \( M \) to \( M \), and \( \mathcal{U} \subset \text{Diff}^1(M) \) be a neighbourhood of \( f \) such that, for each \( g \in \mathcal{U} \), \( \Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U) \) is a locally maximal hyperbolic set for \( g \) and there is a homeomorphism \( g|_\Lambda : \Lambda \to \Lambda_g \) which conjugates \( g|_\Lambda \) and \( f|_\Lambda \), i.e., \( g \circ h_g = h_g \circ g \), with \( h_g \) \( C^0 \)-close to identity if \( g \) is \( C^1 \)-close to \( f \).

For \( g \in \mathcal{U} \), let \( T\Lambda_g M = E^u_g \oplus E^s_g \) be the hyperbolic splitting of \( \Lambda_g \). The local unstable and stable sets of \( z \in \Lambda_g \) are denoted by \( W^u_\beta(g, z) \) and \( W^s_\beta(g, z) \) respectively. These are embedded \( C^1 \)-disks with \( TzW^u_\beta(g, z) = E^u_g(z) \) and \( TzW^s_\beta(g, z) = E^s_g(z) \). Moreover, for \( i \in \{ u, s \} \), \( \{ W^i_\beta(g, z) : z \in \Lambda_g \} \) is continuous on \( g \) in the following sense: there is \( \{ \theta^i_{g,x} : x \in \Lambda \} \) where \( \theta^i_{g,x} : W^i_\beta(f, x) \to W^i_\beta(g, h_g(x)) \) is a \( C^1 \) diffeomorphism with \( \theta^i_{g,x}(x) = h_g(x) \), such that if \( g \) is \( C^1 \)-close to \( f \) then, for all \( x \in \Lambda \), \( \theta^i_{g,x} \) is uniformly \( C^1 \)-close to the inclusion of \( W^i_\beta(f, x) \) in \( M \). See [17] for more information.

1.2. Dimension of conformal hyperbolic sets. Roughly speaking, a hyperbolic set is called conformal, if the derivative of the map is a multiple of an isometry along the stable and unstable directions (see precise definition in [21]).

If \( \Lambda \) is a hyperbolic set of a \( C^{1+\gamma} \) surface diffeomorphism \( f \), for every \( x \in \Lambda \), McCluskey and Manning [18] proved that

\[
\dim_H(W^s_\beta(f, x) \cap \Lambda) = t^s \quad \text{and} \quad \dim_H(W^u_\beta(f, x) \cap \Lambda) = t^u
\]

(1)

where \( t^s \) and \( t^u \) are the unique solutions of the equations \( P_\lambda(f, t \log \| Df \|_E^s) = 0 \), \( P_\lambda(f, -t \log \| Df \|_E^u) = 0 \) respectively (here \( P(\cdot) \) denotes the topological pressure). Since \( \dim E^s = \dim E^u = 1 \), the local product structure is a Lipschitz homeomorphism with Lipschitz inverse. Therefore

\[
\dim_H \Lambda = t^s + t^u.
\]

(2)

The equality between the Hausdorff dimension and the lower and upper box dimensions is due to Takens [25]. Palis and Viana relaxed the smoothness to \( C^1 \) in [20].
Their proof used Hölder conjugacies between nearby hyperbolic invariant sets and Hölder stable and unstable foliations with Hölder exponents close to one.

In the case of higher dimensional conformal hyperbolic sets, Pesin [21] and Barreira [3] studied the dimension of a locally maximal hyperbolic invariant sets of $C^{1+\gamma}$ conformal dynamical systems. Using techniques in thermodynamic formalism, they proved the Hausdorff dimension, lower and upper box dimensions all agree for the restriction of the hyperbolic invariant set to local stable (unstable) manifolds. In this case, the formula (2) also holds.

1.3. Statement of main result. In this paper, we introduce the concept of average conformal hyperbolic set, i.e., it admits only one positive and one negative Lyapunov exponents for any ergodic measure (see Definition 2.2). Using McCluskey and Manning’s thermodynamic formalism techniques [18], as well as Palis and Viana’s geometric methods [20], a formula of dimension of locally maximal average conformal hyperbolic sets of a $C^1$ diffeomorphism is obtained. We also give the estimations of the dimensions of the restriction of $C^1$ non-conformal hyperbolic invariant set to stable and unstable manifolds (see Lemma 3.8 and Lemma 3.9). Furthermore, the dimension of a $C^1$ average conformal hyperbolic set varies continuously with respect to the dynamics.

The following theorem gives a formula of dimension of locally maximal average conformal hyperbolic sets of a $C^1$ diffeomorphism. It extends Palis and Viana’s result [20] to the case of average conformal hyperbolic sets in higher dimension. It relaxed the smoothness of the results in Pesin’s book [21] (see also [3]) to $C^1$. Of course, it extends McCluskey and Manning’s result in [18] to both higher dimension and $C^1$ diffeomorphisms. Furthermore, it gives the continuity of the dimension of average conformal hyperbolic sets, which implies the continuity of the dimension of conformal hyperbolic sets.

**Theorem A.** Let $\Lambda$ be a locally maximal average conformal hyperbolic invariant set of a $C^1$ diffeomorphism $f$, such that $f$ is transitive on $\Lambda$. Then for every $x \in \Lambda$,

$$\dim_H \Lambda = \dim_H (W^u_\beta (f,x) \cap \Lambda) + \dim_H (W^s_\beta (f,x) \cap \Lambda),$$

$$\dim_B \Lambda = \dim_B (W^u_\beta (f,x) \cap \Lambda) + \dim_B (W^s_\beta (f,x) \cap \Lambda),$$

$$\overline{\dim_B} \Lambda = \overline{\dim_B} (W^h_\beta (f,x) \cap \Lambda) + \overline{\dim_B} (W^s_\beta (f,x) \cap \Lambda)$$

and $\dim_H \Lambda = \dim_B \Lambda$. Moreover, if $g \to f$, then $\dim \Lambda_g \to \dim \Lambda$, here $\dim$ denotes either $\dim_H$ or $\dim_B$ or $\overline{\dim_B}$, and $\Lambda_g$ is a locally maximal hyperbolic set for $g$ which may be not average conformal.

The paper is organized as follows. In Section 2, we recall definitions of dimension, topological pressure, and introduce the concept of average conformal hyperbolic sets. In Section 3, we give the detailed proof of the main result.

2. Definitions and preliminaries. In this section, we recall the definitions of dimension, entropy and topological pressure. Particularly, we give the definition of average conformal hyperbolic sets and some useful preliminary results.

2.1. Dimensions of sets. We recall some notions from dimension theory. See the books [12] and [21] for more detailed description. Given a subset $X \subset M$. A
countable family $\{U_i\}_{i \in \mathbb{N}}$ of open sets is a $\delta-$cover of $X$ if $\text{diam} U_i < \delta$ for each $i$ and their union contains $X$. For any $s \geq 0$, let

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam} U_i)^s : \{U_i\}_{i \geq 1} \text{ is a } \delta - \text{cover of } X \right\}$$

and

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

This limit exists, though the limiting value can be 0 or $\infty$. We call $\mathcal{H}^s(X)$ the $s-$dimensional Hausdorff measure of $X$.

**Definition 2.1.** The following jump-up value of $\mathcal{H}^s(X)$

$$\dim_B X = \inf \{ s : \mathcal{H}^s(X) = 0 \} = \sup \{ s : \mathcal{H}^s(X) = \infty \}$$

is called the Hausdorff dimension of $X$. The lower and upper box dimension of $X$ are defined respectively by

$$\overline{\dim}_B X = \lim \inf_{\delta \to 0} \frac{\log \mathcal{N}(X, \delta)}{-\log \delta} \text{ and } \underline{\dim}_B X = \lim \sup_{\delta \to 0} \frac{\log \mathcal{N}(X, \delta)}{-\log \delta},$$

where $\mathcal{N}(X, \delta)$ denotes the least number of balls of radius $\delta$ that are needed to cover the set $X$.

2.2. **Average conformal hyperbolic sets.** Let $(M, f)$ and $\Lambda$ be the same as in Section 1.1. We say a diffeomorphism $f$ on $\Lambda$ is $u-$conformal (respectively, $s-$conformal) if there exists a continuous function $a_u(x)$ (respectively, $a^s(x)$) on $\Lambda$ such that $D_x f|_{E^u(x)} = a_u(x) \text{Isom}_x$ for every $x \in \Lambda$ (respectively, $D_x f|_{E^s(x)} = a^s(x) \text{Isom}_x$), where $\text{Isom}_x$ denotes an isometry from $E^u(x)$ (respectively, $E^s(x)$) to $E^u(f(x))$ (respectively, $E^s(f(x))$). A diffeomorphism $f$ on $\Lambda$ is called conformal if it is $u-$conformal and $s-$conformal, in this case, we also call $\Lambda$ a conformal hyperbolic set of $f$; otherwise, we say that $\Lambda$ is a non-conformal hyperbolic set of $f$.

Following the idea in [1], we introduce the concept of average conformal hyperbolic sets which may be non-conformal. The average conformal concept was a generalization of quasi-conformal and weakly conformal concept in [2, 21]. By the Oseledec multiplicative ergodic theorem (see [19]), there exists a total measure set $\mathcal{O} \subset \Lambda$ such that, for each $x \in \mathcal{O}$ and each invariant measure $\mu$ supported on $\Lambda$ there exist positive integers $m_1(x), m_2(x), \ldots, m_{p(x)}(x)$, numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{p(x)}(x)$ and a splitting $T_x M = E_1(x) \bigoplus E_2(x) \bigoplus \cdots \bigoplus E_{p(x)}(x)$ satisfies that

1. $D_x f E_i(x) = E_i(f(x))$ for each $i$ and $\sum_{i=1}^{p(x)} m_i(x) = m$;
2. for each $0 \neq v \in E_i(x)$ we have that

$$\lambda_i(x) = \lim_{n \to \infty} \frac{1}{n} \log \| D_x f^n(v) \|.$$

Here we call the numbers $\{\lambda_i(x)\}$ the Lyapunov exponents of $(f, \mu)$. In the case that $\mu$ is an invariant ergodic measure on $\Lambda$, the numbers $p(x), \{m_i(x)\}$ and $\{\lambda_i(x)\}$ are constants. We denote them simply as $p$, $\{m_i\}_{i=1}^{p}$ and $\{\lambda_i(\mu)\}_{i=1}^{p}$.

**Definition 2.2.** A hyperbolic set $\Lambda$ is called average conformal if it has two unique Lyapunov exponents, one positive and one negative. That is, for any invariant ergodic measure $\mu$ on $\Lambda$, the Lyapunov exponents are $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_k(\mu) > 0$ and $\lambda_{k+1}(\mu) = \lambda_{k+2}(\mu) = \cdots = \lambda_p(\mu) < 0$ for some $0 < k < p$.

Following the same proof of Theorem 4.2 in [1], we get the following result.
Lemma 2.3. If Λ is an average conformal hyperbolic invariant set. Let
\[ \phi_u(f, x) := \left| \det(D_x f|_{E^u(x)}) \right|^{\frac{1}{d_1}} \]
and
\[ \phi_s(f, x) := \left| \det(D_x f|_{E^s(x)}) \right|^{\frac{1}{d_2}} \]
where \( d_1 = \dim E^u \), \( d_2 = \dim E^s \). Then for any \( n \in \mathbb{N} \),
\[ m(D_x f^n|_{E^s(x)}) \leq \phi_1(f^n, x) \leq \|D_x f^n|_{E^u(x)}\| \]
and
\[ \lim_{n \to \infty} \frac{1}{n} \left( \log \|D_x f^n|_{E^s(x)}\| - \log m(D_x f^n|_{E^u(x)}) \right) = 0 \]
uniformly on \( \Lambda \), for \( i \in \{u, s\} \).

2.3. Entropy and pressure. We next recall Bowen’s definition of a topological
entropy \( h(f, Y) \) for a subset \( Y \) of a compact metric space \( X \) and a continuous map
\( f : X \to X \) (see [6] for more details). It is defined in a way that resembles Hausdorff
dimension. Let \( A \) be a finite open cover of \( X \) and write \( E \prec A \) if \( E \) is contained in
some member of \( A \). Denote \( n_A(E) \) the largest non-negative integer such that
\( f^k E \prec A \) for \( 0 \leq k < n_A(E) \).

Definition 2.4. Let \( C = \{E_1, E_2, \cdots\} \) be a cover of \( Y \), for any \( s \geq 0 \) set
\[ D_A(C, s) = \sum_{i=1}^{\infty} \exp[-s n_A(E_i)] \]
and
\[ m_{A,s}(Y) = \lim_{\varepsilon \to 0} \inf \left\{ D_A(C, s) : C = \{E_1, E_2, \cdots\}, Y \subset \bigcup_{i=1}^{\infty} E_i \right\} \]
Then define \( h_A(f, Y) = \inf\{s : m_{A,s}(Y) = 0\} \). The following quantity
\[ h(f, Y) = \sup_A h_A(f, Y) \]
is called the topological entropy of \( f \) on the subset \( Y \).

Let \( f : X \to X \) be a continuous transformation on a compact metric space \( (X, d) \),
and \( \phi : X \to \mathbb{R} \) a continuous function on \( X \). In the following, we recall the definition of
topological pressure. A subset \( F \subset X \) is called an \((n, \varepsilon)\)-separated set with respect
to \( f \) if for any \( x, y \in F, x \neq y \), we have \( d_n(x, y) := \max_{0 \leq k < n-1} d(f^k x, f^k y) > \varepsilon \). A
sequence of continuous functions \( \Phi = \{\phi_n\}_{n \geq 1} \) is called sub-additive, if
\[ \phi_{m+n} \leq \phi_m + \phi_n \circ f^n, \forall n, m \in \mathbb{N}. \]
Furthermore, a sequence of continuous functions \( \Phi = \{\phi_n\}_{n \geq 1} \) is called super-
additive if \( -\Phi = \{-\phi_n\}_{n \geq 1} \) is sub-additive.

Definition 2.5. Let \( Z \) be a subset of \( X \), and \( \Phi = \{\phi_n\}_{n \geq 1} \) a sub-additive/super-
additive potential on \( X \), put
\[ P_n(Z, f, \Phi, \varepsilon) = \sup \left\{ \sum_{x \in F} e^{\phi_n(x)} |F \subset Z \text{ is an } (n, \varepsilon) \text{- separated set} \right\} \]
The following quantity
\[ \overline{P}_Z(f, \Phi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(Z, f, \Phi, \varepsilon) \]
(3)
is called the upper sub-additive/super-additive topological pressure of \( \Phi \) (with respect to \( f \)) on the set \( Z \).

**Remark 1.** Consider \( \lim \inf \) instead of \( \lim \sup \) in (3), we get a quantity \( P_Z(f, \Phi) \) which is called lower sub-additive/super-additive topological pressure of \( \Phi \) (with respect to \( f \)) on \( Z \). For any compact invariant set \( Z \subset X \), we have \( P_Z(f, \Phi) = \overline{P}_Z(f, \Phi) \). The common value is denoted by \( P_Z(f, \Phi) \), which is called the sub-additive/super-additive topological pressure of \( \Phi \) (with respect to \( f \)) on \( Z \). See [2, 21] for proofs.

**Remark 2.** If \( \Phi = \{\phi_n\}_{n \geq 1} \) is additive in the sense that \( \phi_n(x) = \phi(x) + \cdots + \phi(f^{n-1}x) \) for some continuous function \( \phi : X \to \mathbb{R} \), we simply denote the topological pressures \( P_Z(f, \Phi), P_Z(f, \phi) \) and \( P_Z(f, \Phi) \) as \( P_Z(f, \phi), P_Z(f, \phi) \) and \( P_Z(f, \phi) \) respectively.

Let \( M(X) \) be the space of all Borel probability measures on \( X \) endowed with the weak* topology. Let \( M_f(X) \) denote the subspace of \( M(X) \) consisting of all \( f \)-invariant measures. Let \( \Phi = \{\phi_n\} \) be a sub-additive/super-additive potential on \( X \). For \( \mu \in M_f(X) \), let \( h_\mu(f) \) denote the entropy of \( f \) with respect to \( \mu \) (see [26] for the precise definition), and let \( \Phi_*(\mu) \) denote the following limit

\[
\Phi_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \phi_n d\mu.
\]

The existence of the above limit follows from a sub-additive argument. The authors in [8] proved the following variational principle.

**Theorem 2.6.** Let \( f : X \to X \) be a continuous transformation on a compact metric space \( X \), and \( \Phi = \{\phi_n\}_{n \geq 1} \) a sub-additive potential on \( X \), we have

\[
P_X(f, \Phi) = \sup \left\{ h_\mu(f) + \Phi_*(\mu) : \mu \in M_f(X), \Phi_*(\mu) \neq -\infty \right\}.
\]

In general, it is still an open question that whether the super-additive topological pressure satisfies the variational principle. However, in the case of average conformal hyperbolic setting, following the same proof of Theorem 5.1 in [1], one can prove the following theorem.

**Theorem 2.7.** Let \( \Lambda \) be a locally maximal average conformal hyperbolic set of a \( C^1 \) diffeomorphism \( f \). Let \( F = \{-t \log \|D_x f^n|_{E^s(x)}\|\}_{n \geq 1} \) for \( t \geq 0 \) be a super-additive potential. Then we have

\[
P_\Lambda(f, F) = \sup \left\{ h_\mu(f) + F_*(\mu) : \mu \in M_f(\Lambda) \right\},
\]

where \( M_f(\Lambda) \) is the space of all \( f \)-invariant Borel probability measures on \( \Lambda \) and

\[
F_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int -t \log \|D_x f^n|_{E^s(x)}\| d\mu.
\]

**Remark 3.** In the case of average conformal hyperbolic setting, it is easy to see \( F_*(\mu) \neq -\infty \) for any \( \mu \in M_f(\Lambda) \). Moreover, it follows from Lemma 2.3 and Theorems 2.6 and 2.7 that

\[
P_\Lambda \left( f, \{-t \log \|D_x f^n|_{E^s(x)}\|\}\right) = P_\Lambda \left( f, \{-t \log m(D_x f^n|_{E^s(x)})\}\right)
\]

for any \( t \geq 0 \).
3. Proof of main result. This section provides the proof of the main result stated in Section 1.3.

The following theorem shows that the conjugacy map $h_g$ in Section 1.1 restricted to local unstable and stable manifolds are Hölder continuous.

**Theorem 3.1.** Let $f : M \to M$ be a $C^1$ diffeomorphism, and $\Lambda \subseteq M$ a locally maximal average conformal hyperbolic set. Then for any $r \in (0,1)$, there is $C > 0$ (depending on $r$) and a neighborhood $\mathcal{U}^f_r$ of $f$ in $\text{Diff}^1(M)$ such that, for any $g \in \mathcal{U}^f_r$ and any $x \in \Lambda$, $h_g|W^u_g(f,x) \cap \Lambda$, $h_g|W^s_g(f,x) \cap \Lambda$ and $(h_g|W^u_g(f,x) \cap \Lambda)^{-1}$, $(h_g|W^s_g(f,x) \cap \Lambda)^{-1}$ are $(C,r)$–Hölder continuous, where $C$ is the Hölder constant, and $r$ is the Hölder exponent.

**Proof.** Let

$$\tau := \inf \{ \phi_u(f,\xi) : x \in W^u(f,x), x \in \Lambda \} > 1.$$

For any $r \in (0,1)$, there exists $\varepsilon > 0$ such that $\tau e^{-4\varepsilon} \geq \tau^r$. Since $f$ is average conformal on $\Lambda$, by Lemma 2.3 there exists a positive integer $N(\varepsilon)$ such that for any $n \geq N(\varepsilon)$ and $x \in \Lambda$

$$1 \leq \frac{\|D_x f^n|E^u(x)\|}{m(D_x f^n|E^u(x))} < e^{n\varepsilon} \text{ and } 1 \leq \frac{\|D_x f^n|E^s(x)\|}{m(D_x f^n|E^s(x))} < e^{n\varepsilon}.$$

Fix any $N \geq N(\varepsilon)$, let $F := f^N$. Since $\Lambda$ is a locally maximal hyperbolic set for $f$, $\Lambda$ is also a locally maximal hyperbolic set for $F$, and the above inequality shows that $F$ satisfies

$$1 \leq \frac{\|D_x F|E^u(x)\|}{m(D_x F|E^u(x))} < e^{N\varepsilon} \text{ and } 1 \leq \frac{\|D_x F|E^s(x)\|}{m(D_x F|E^s(x))} < e^{N\varepsilon} \text{ for all } x \in \Lambda. \quad (4)$$

Recall that $d_u$ denote the metric induced by the Riemannian structure on the unstable foliation $W^u$ and let $D_y F|E^u(y) := D_y F|_{T_y W^u_y(F,x)}$ denote the derivative of $F$ in the unstable direction for any $y \in W^u_y(F,x), x \in \Lambda$.

For the above $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true for all $x \in \Lambda$,

1. for $y, z \in W^u_y(F,x)$, if $d_u(y, z) \leq 4\delta$, then

$$e^{-\frac{\varepsilon}{4}N\varepsilon} \leq \frac{\|D_y F|E^u(y)\|}{\|D_z F|E^u(z)\|} \leq e^{\frac{\varepsilon}{4}N\varepsilon} \text{ and } e^{-\frac{\varepsilon}{4}N\varepsilon} \leq \frac{m(D_y F|E^u(y))}{m(D_z F|E^u(z))} \leq e^{\frac{\varepsilon}{4}N\varepsilon}.$$

Take $\mathcal{U}^F_r$ a small neighborhood of $F$ in $\text{Diff}^1(M)$ such that for all $G \in \mathcal{U}^F_r$ and $x \in \Lambda$, we have

2. $d_u((\theta^n_{G,x})^{-1} \circ h_G(y), y) \leq \delta$ for every $y \in W^u_G(F,x) \cap \Lambda$;

3. $e^{-N\varepsilon} \leq \frac{m(D_y ((\theta^n_{G,F,x})^{-1} \circ G \circ \theta^n_{G,x}))}{m(D_y F|E^u(y))} \leq e^{N\varepsilon}$ for every $y \in W^u_G(F,x)$.

Since $F$ satisfies (4) on $\Lambda$,

$$1 \leq \frac{\|D_y F|E^u(y)\|}{m(D_y F|E^u(y))} \leq e^{2N\varepsilon} \quad (5)$$

for every $y \in W^u_G(F,x), x \in \Lambda$. By the following Lemma 3.2, there exists $C > 0$ (depending only on $r$) such that $h_G|W^u_G(F,x) \cap \Lambda$, $h_G|W^s_G(F,x) \cap \Lambda$, $(h_G|W^u_G(F,x) \cap \Lambda)^{-1}$ and $(h_G|W^s_G(F,x) \cap \Lambda)^{-1}$ are $(C,r)$–Hölder continuous, for any $G \in \mathcal{U}^F_r$. 
Notice that $F = f^N$ and $\Lambda$ is a hyperbolic set of $f$, thus

$$W^s_\beta(F, x) \cap \Lambda = W^s_\beta(f, x) \cap \Lambda \quad \text{and} \quad W^u_\beta(F, x) \cap \Lambda = W^u_\beta(f, x) \cap \Lambda.$$ 

One may choose a sufficiently small open neighborhood $\mathcal{U}^F_\epsilon$ of $f$ in $\text{Diff}^1(M)$ such that each $g \in \mathcal{U}^F_\epsilon$ satisfies that $g^N \in \mathcal{U}^F_\epsilon$. Put $G := g^N$. Note that $h_g = h_G$, $\Lambda_G = \Lambda_g$ and so $W^s_\beta(G, x) \cap \Lambda_G = W^s_\beta(g, x) \cap \Lambda_g$, $W^u_\beta(G, x) \cap \Lambda_G = W^u_\beta(g, x) \cap \Lambda_g$. The above assertions yield that $h_g|W^s_\beta(f, x) \cap \Lambda$, $h_g|W^u_\beta(f, x) \cap \Lambda$ and $(h_g|W^s_\beta(f, x) \cap \Lambda)^{-1}$, $(h_g|W^u_\beta(f, x) \cap \Lambda)^{-1}$ are $(C, r)$–H"older continuous for any $g \in \mathcal{U}^F_\epsilon$ and any $x \in \Lambda$. □

**Lemma 3.2.** For the above $r$, $F$ and $\mathcal{U}^F_\epsilon$, there is $C > 0$ (depending only on $r$) such that $h_g|W^s_\beta(f, x) \cap \Lambda$, $h_g|W^u_\beta(f, x) \cap \Lambda$, $(h_g|W^s_\beta(f, x) \cap \Lambda)^{-1}$ and $(h_g|W^u_\beta(f, x) \cap \Lambda)^{-1}$ are $(C, r)$–H"older continuous, for any $G \in \mathcal{U}^F_\epsilon$.

**Proof.** Let $x \in \Lambda$ and $y, z \in W^s_\beta(F, x) \cap \Lambda$ with $d_u(y, z) \leq \delta$. For any integer $n > 0$,

$$d_u(F^n y, F^n z) \leq d_u(y, z) \cdot \prod_{j=0}^{n-1} \|D_{\xi_j} F|E^u(\xi_j)\|$$

and

$$d_u(F^n y, F^n z) \geq d_u(y, z) \cdot \prod_{j=0}^{n-1} m(D_{\eta_j} F|E^u(\eta_j))$$

where $\xi_j$, $\eta_j$ are between $F^j y$ and $F^j z$. Let $M \geq 0$ be the smallest integer such that

$$d_u(F^M y, F^M z) \leq \delta < d_u(F^{M+1} y, F^{M+1} z).$$

**Case I.** If $M = 0$, then by (2) we have

$$d_u\left(\left(\theta^n_{G,x}\right)^{-1} \circ h_G(y), \left(\theta^n_{G,x}\right)^{-1} \circ h_G(z)\right) \leq d_u\left(\left(\theta^n_{G,x}\right)^{-1} \circ h_G(y), y\right) + d_u(y, z) + d_u\left(z, \left(\theta^n_{G,x}\right)^{-1} \circ h_G(z)\right) \leq \frac{\delta}{2} + \frac{\delta}{2} = 2\delta \leq 2\delta^r < 2d_u(F(y), F(z))^r \leq 2\|D_{\xi_0} F|E^u(\xi_0)\|^r d_u(y, z)^r = 2\|D_{\xi_M} F|E^u(\xi_M)\|^r d_u(y, z)^r$$

**Case II.** If $M \geq 1$, let $\theta^n_j := \theta^n_{G,F^j}(x)$ for $j \geq 0$, and by (2) we have

$$d_u\left(\left(\theta^n_j\right)^{-1} \circ h_G \circ F^j(y), \left(\theta^n_j\right)^{-1} \circ h_G \circ F^j(z)\right) \leq d_u\left(\left(\theta^n_j\right)^{-1} \circ h_G \circ F^j(y), F^j(y)\right) + d_u(F^j(y), F^j(z)) + d_u\left(F^j(z), \left(\theta^n_j\right)^{-1} \circ h_G \circ F^j(z)\right) \leq \frac{\delta}{2} + \frac{\delta}{2} = 2\delta$$
for $j = 0, 1, \cdots, M$. On the other hand,

$$
d_u\left( (\theta_M^u)^{-1} \circ h_G \circ F^M(y), (\theta_M^u)^{-1} \circ h_G \circ F^M(z) \right)
\geq d_u\left( (\theta_0^u)^{-1} \circ h_G(y), (\theta_0^u)^{-1} \circ h_G(z) \right) \cdot \prod_{j=0}^{M-1} m\left( D_{\tau_j} ((\theta_{j+1}^u)^{-1} \circ G \circ \theta_j^u) \right)
$$

(7)

where $\tau_j$ is between $(\theta_j^u)^{-1} \circ h_G \circ F^j(y)$ and $(\theta_j^u)^{-1} \circ h_G \circ F^j(z)$, $0 \leq j \leq M - 1$.

By (2), (6) and the positions of $\xi_j$ and $\tau_j$, we have that

$$
d_u(\xi_j, \tau_j) \leq d_u(\xi_j, F^j(y)) + d_u\left( F^j(y), (\theta_j^u)^{-1} \circ h_G \circ F^j(y) \right) + d_u\left( (\theta_j^u)^{-1} \circ h_G \circ F^j(y), (\theta_j^u)^{-1} \circ h_G \circ F^j(z) \right) \leq \delta + \frac{\delta}{2} + 2\delta
< 4\delta.
$$

It follows from (1), (3) and (5) that

$$
m\left( D_{\tau_j} ((\theta_{j+1}^u)^{-1} \circ G \circ \theta_j^u) \right) \geq m(D_{\tau_j} F|_{E^u(\xi_j)}) e^{-N\varepsilon}
\geq m(D_{\xi_j} F|_{E^u(\xi_j)}) e^{-2N\varepsilon}
\geq \|D_{\xi_j} F|_{E^u(\xi_j)}\| e^{-4N\varepsilon}.
$$

Since $\|D_{\xi_j} F|_{E^u(\xi_j)}\| \geq \phi_u(f^N, \xi_j) = \prod_{i=0}^{N-1} \phi_u(f, f^i(\xi_j)) \geq \tau^N$ and $\tau e^{-4\varepsilon} \geq \tau^r$,

$$
\|D_{\xi_j} F|_{E^u(\xi_j)}\|^{1-r} e^{-4N\varepsilon} \geq \tau^{N(1-r)} e^{-4N\varepsilon} \geq 1.
$$

Hence,

$$
m\left( D_{\tau_j} ((\theta_{j+1}^u)^{-1} \circ G \circ \theta_j^u) \right) \geq \|D_{\xi_j} F|_{E^u(\xi_j)}\| e^{-4N\varepsilon} \geq \|D_{\xi_j} F|_{E^u(\xi_j)}\|^r.
$$

Combine (6) and (7) that

$$
d_u\left( (\theta_0^u)^{-1} \circ h_G(y), (\theta_0^u)^{-1} \circ h_G(z) \right) \cdot \prod_{j=0}^{M-1} \|D_{\xi_j} F|_{E^u(\xi_j)}\|^r
\leq d_u\left( (\theta_0^u)^{-1} \circ h_G(y), (\theta_0^u)^{-1} \circ h_G(z) \right) \cdot \prod_{j=0}^{M-1} m\left( D_{\tau_j} ((\theta_{j+1}^u)^{-1} \circ G \circ \theta_j^u) \right)
\leq d_u\left( (\theta_M^u)^{-1} \circ h_G \circ F^M(y), (\theta_M^u)^{-1} \circ h_G \circ F^M(z) \right) \leq 2\delta \quad \text{(by (7))}
\leq 2\delta^r
\leq 2 \left[ d_u(F^{M+1}(y, F^{M+1}z)) \right]^r
\leq 2 \left[ d_u(y, z) \right]^r \prod_{j=0}^M \|D_{\xi_j} F|_{E^u(\xi_j)}\|^r.
$$

Hence, we have

$$
d_u\left( (\theta_0^u)^{-1} \circ h_G(y), (\theta_0^u)^{-1} \circ h_G(z) \right) \leq 2 \|D_{\xi_M} F|_{E^u(\xi_M)}\|^r \cdot \left[ d_u(y, z) \right]^r
\leq 2 \|D_{\xi_M} F|_{E^u(\xi_M)}\| \cdot \left[ d_u(y, z) \right]^r.
$$
Combing the two cases, we conclude that
\[
d_u(h_G(y), h_G(z)) = d_u(\theta_0^a \circ (\theta_0^a)^{-1} \circ h_G(y), \theta_0^a \circ (\theta_0^a)^{-1} \circ h_G(z)) \leq C[d_u(y, z)]^r
\]
where \( C = 2 \sup \{ \|Df\|_{E^s(x)} : x \in \Lambda \} \cdot \sup \{ \|D\theta_G\|_x : G \in U^F, x \in \Lambda \} \). This shows that the map \( h_G|_{W_\beta^s(F,x) \cap \Lambda} \) is \((G, r)\)-H"older continuous. The H"older continuity of \( h_G|_{W_\beta^u(F,x) \cap \Lambda}, (h_G|_{W_\beta^u(F,x) \cap \Lambda})^{-1} \) and \( (h_G|_{W_\beta^u(F,x) \cap \Lambda})^{-1} \) can be proven in a similar fashion.

\[\square\]

Recall the holonomy maps of unstable and stable foliations which are Lipschitz or H"older continuous. Let \( F^u, F^s \) be the unstable and stable foliations of hyperbolic dynamical system \((f, \Lambda)\). For \( x, y \in \Lambda \) with \( x \) close to \( y \), let \( F^u_{loc}(f, x) \) and \( F^s_{loc}(f, y) \) be the local stable foliations of \( x \) and \( y \). Define the map \( h : F^u_{loc}(f, x) \to F^s_{loc}(f, y) \), sending \( z \to h(z) \) by sliding along the leaves of \( F^u \). The map \( h \) is called the holonomy map of \( F^u \). The map \( h \) is Lipschitz continuous if
\[
d_y(h(z_1), h(z_2)) \leq Ld_x(z_1, z_2),
\]
where \( z_1, z_2 \in F^u_{loc}(f, x) \) and \( d_x, d_y \) are natural metrics on \( F^u_{loc}(f, x) \), \( F^s_{loc}(f, y) \), path metrics with respect to a fixed Riemannian structure on \( M \). The constant \( L \) is the Lipschitz constant, and it is independent of the choice of \( F^u \). The map \( h \) is \((H, \alpha)\)-H"older continuous if
\[
d_y(h(z_1), h(z_2)) \leq H d_x(z_1, z_2)^\alpha,
\]
where \( H \) is the H"older constant. Similarly we can define the holonomy map of \( F^s \).

In [15], authors prove the regularity of foliations for \( C^2 \)-diffeomorphism. Define four quantities:
\[
a_f = \|Df^{-1}|_{E^s}\| < 1, \quad b_f = \|Df|_{E^s}\| < 1,
\]
\[
c_f = \|Df|_{E^u}\| > 1, \quad d_f = \|Df^{-1}|_{E^u}\| > 1.
\]

**Lemma 3.3** (Theorem 6.3, [15]). Let \( f : M \to M \) be a \( C^2 \)-diffeomorphism, and \( \Lambda \subset M \) be a locally maximal hyperbolic set. If \( a_f b_f c_f < 1 \), then the stable foliation is \( C^1 \). If \( a_f b_f d_f < 1 \), then the unstable foliation is \( C^1 \).

**Remark 4.** If the unstable and stable foliations are \( C^1 \), then the corresponding holonomy maps are locally uniformly \( C^1 \) (see [23], pp. 540–541). Thus the corresponding holonomy maps are Lipschitz continuous. For more information about the regularity of unstable and stable foliations, we refer to [15, 16, 23] for detailed description.

The following two results are well-known in the field of fractal geometry, e.g., see Falconer’s book [12] for proofs.

**Lemma 3.4.** Let \( X \) and \( Y \) be metric spaces. For any \( r \in (0, 1) \), \( \Phi : X \to Y \) is an onto, \((c, r)\)-H"older continuous map for some \( c > 0 \). Then \( \dim_H Y \leq r^{-1} \dim_H X \), \( \dim_B Y \leq r^{-1} \dim_B X \) and \( \overline{\dim}_B Y \leq r^{-1} \overline{\dim}_B X \).

**Corollary 1.** Let \( X \) and \( Y \) be metric spaces, and \( \Phi : X \to Y \) is an onto, Lipschitz continuous map. Then
\[
\dim_H Y \leq \dim_H X, \quad \dim_B Y \leq \dim_B X \quad \text{and} \quad \overline{\dim}_B Y \leq \overline{\dim}_B X.
\]

Using Theorem 3.1 and the transitive property, the following theorem holds.
Let \( f : M \to M \) be a \( C^1 \) diffeomorphism, and \( \Lambda \subseteq M \) a locally maximal average conformal hyperbolic set. Then \( \dim_H(W^u_\beta(f,x) \cap \Lambda), \dim_B(W^u_\beta(f,x) \cap \Lambda) \) and \( \dim_B(W^s_\beta(f,x) \cap \Lambda) \) are continuous functions of \( f \in \text{Diff}^1(M) \), independent of \( \beta \) and \( x \). Moreover,

\[
\dim_H(W^s_\beta(f,x) \cap \Lambda) = \dim_B(W^u_\beta(f,x) \cap \Lambda) = \overline{\dim}_B(W^u_\beta(f,x) \cap \Lambda).
\]

The same statements for \( W^s_\beta(f,x) \cap \Lambda \) hold.

To prove Theorem 3.5, we need two lemmas as follow. Take a small number \( \varepsilon > 0 \) with \( \lambda e^{2\varepsilon} < 1 \), where \( \lambda \) is the skewness of the hyperbolicity defined in Section 1.1. Since \( f \) is average conformal on \( \Lambda \), by Lemma 2.3 there exists a positive integer \( N(\varepsilon) \) such that for any \( n \geq N(\varepsilon) \) and \( x \in \Lambda \)

\[
1 \leq \frac{\|D_x f^n|_{E^u(x)}\|}{m(D_x f^n|_{E^u(x)})} < e^{n\varepsilon} \quad \text{and} \quad 1 \leq \frac{\|D_x f^n|_{E^s(x)}\|}{m(D_x f^n|_{E^s(x)})} < e^{n\varepsilon}.
\]

Fixing any \( n \geq N(\varepsilon) \), let \( F := f^n \). Then \( F \) satisfies (4). Since \( \Lambda \) is a locally maximal hyperbolic set for \( f \), \( \Lambda \) is also a locally maximal hyperbolic set for \( F \). Then we have Lemma 3.6 and 3.7.

Lemma 3.6. Let \( f : M \to M \) be a \( C^2 \) diffeomorphism, and \( \Lambda \subseteq M \) a locally maximal average conformal hyperbolic set. For small \( \varepsilon > 0 \) with \( \lambda e^{2\varepsilon} < 1 \), there exists a positive integer \( N(\varepsilon) \) such that any \( n \geq N(\varepsilon) \), the holonomy maps of the stable and unstable foliations for \( F := f^n \) are Lipschitz continuous respectively.

Proof. Since \( F \) satisfies (4), \( \|DF|_{E^i}\| < e^{n\varepsilon} \) for \( i \in \{u,s\} \) and \( \|DF^{-1}|_{E^i}\| \leq \lambda^n \) and \( \|DF^{-1}|_{E^i}\| \leq \lambda^n \), we conclude

\[
a_F b_{FC_F} F = \frac{\|DF|_{E^u}\| \cdot \|DF|_{E^s}\|}{m(DF|_{E^u})} \leq e^{n\varepsilon} \lambda^n < 1,
\]

\[
a_F b_{FC_F} d_F F = \frac{\|DF|_{E^s}\|}{m(DF|_{E^u}) \cdot m(DF|_{E^s})} \leq e^{n\varepsilon} \lambda^n < 1.
\]

The desired result follows from Lemma 3.3 and Remark 4 immediately.

Lemma 3.7. Let \( f : M \to M \) be a \( C^1 \) diffeomorphism, and \( \Lambda \subseteq M \) a locally maximal average conformal hyperbolic set. For small \( \varepsilon > 0 \) with \( \lambda e^{2\varepsilon} < 1 \), there exists a positive integer \( N(\varepsilon) \) such that any \( n \geq N(\varepsilon) \), \( \dim_H(W^s_\beta(F,x) \cap \Lambda), \dim_B(W^s_\beta(F,x) \cap \Lambda) \) and \( \dim_B(W^u_\beta(F,x) \cap \Lambda) \) are independent of \( \beta \) and \( x \). The same statements for \( W^s_\beta(F,x) \cap \Lambda \) hold. (Here \( F := f^n \))

Proof. For any \( r \in (0,1) \), pick a \( C^2 \) diffeomorphism \( G \) that is \( C^1 \)-close to \( F \) such that for all \( x \in \Lambda_G \) (where \( \Lambda_G \) is a locally maximal hyperbolic set of \( G \)),

\[
1 \leq \frac{\|D_x G|_{E^u_G(x)}\|}{m(D_x G|_{E^u_G(x)})} \leq e^{(n+1)\varepsilon}, \quad i \in \{u,s\}.
\]

As in the proof of Lemma 3.6, we can get

\[
a_G b_{GC_G} G = \frac{\|DG|_{E^u_G}\| \cdot \|DG|_{E^s_G}\|}{m(DG|_{E^u_G})} \leq e^{2n\varepsilon} \lambda^n < 1,
\]

\[
a_G b_{GC_G} d_G G = \frac{\|DG|_{E^s_G}\|}{m(DG|_{E^u_G}) \cdot m(DG|_{E^s_G})} \leq e^{2n\varepsilon} \lambda^n < 1.
\]
Therefore the holonomy maps of stable foliation $\mathcal{F}^s$ and unstable foliation $\mathcal{F}^u$ for $G$ are Lipschitz.

Let $x_0 \in \Lambda_G$ be a transitive point. We claim that $\dim_H (W^u_{\beta}(G, G^j x_0) \cap \Lambda_G)$ is independent of $j \geq 0$ and small $\beta > 0$. In fact, since $G^j$ is a $C^2$ diffeomorphism, there exists some small $\beta' > 0$ such that
\[ W^u_{\beta}(G, G^j x_0) \cap \Lambda_G = G^j (W^u_{\beta'}(G, x_0) \cap \Lambda_G). \]
Take $M \geq 1$ such that $G^M(x_0)$ is sufficiently close to $x_0$. Since $G^M$ is Lipschitz and the holonomy map of $\mathcal{F}^s$ is Lipschitz, by Corollary 1,
\[
\dim_H (W^u_{\beta}(G, x_0) \cap \Lambda_G) \leq \dim_H (W^u_{\beta_0}(G, G^M x_0) \cap \Lambda_G) = \dim_H (G^M(W^u_{\beta_0}(G, x_0) \cap \Lambda_G)) \leq \dim_H (W^u_{\beta_0}(G, x_0) \cap \Lambda_G)
\]
for some $\beta_0 > 0$, $\beta'_0 > 0$. Moreover, by taking $M$ arbitrarily large, we can suppose that $\beta_0$ is close to $\beta$ and $\beta'_0$ is arbitrarily small. Therefore $\dim_H (W^u_{\beta}(G, x_0) \cap \Lambda_G)$ is independent of small $\beta > 0$. Since $G^j$ is bi-Lipschitz continuous,
\[ \dim_H (W^u_{\beta}(G, x_0) \cap \Lambda_G) = \dim_H (W^u_{\beta}(G, G^j x_0) \cap \Lambda_G). \]
The claim now immediately follows.

Take any $x \in \Lambda_G$ and choose $j \geq 0$ such that $G^j x_0$ is close to $x$. Since the holonomy map of $\mathcal{F}^s$ is Lipschitz,
\[
\dim_H (W^u_{\beta_1}(G, G^j x_0) \cap \Lambda_G) \leq \dim_H (W^u_{\beta_0}(G, G^j x_0) \cap \Lambda_G) \leq \dim_H (W^u_{\beta_0}(G, G^j x_0) \cap \Lambda_G)
\]
for some $\beta_1 > 0$, $\beta_2 > 0$ close to $\beta$. By the claim above, we have that
\[ \dim_H (W^u_{\beta}(G, x_0) \cap \Lambda_G) = \dim_H (W^u_{\beta}(G, x) \cap \Lambda_G). \]
Hence $\dim_H (W^u_{\beta}(G, x) \cap \Lambda_G)$ is independent of $x$ and $\beta$.

By Lemma 3.2, $h_{\beta}(W^u_{\beta}(F, x) \cap \Lambda) : W^u_{\beta}(F, x) \cap \Lambda \to W^u_{\beta}(G, h_G(x)) \cap \Lambda_G$ and its inverse are $(C, r)$--Hölder continuous for some $C > 0$. Notice that $r$ can be arbitrarily close to 1. By Lemma 3.4 and the above argument, we have that $\dim_H (W^u_{\beta}(F, x) \cap \Lambda)$ is independent of $\beta$ and $x$. Similarly, $\bar{\dim}_H (W^u_{\beta}(F, x) \cap \Lambda)$ and $\underline{\dim}_B(W^u_{\beta}(F, x) \cap \Lambda)$ are independent of $\beta$ and $x$.

Proof of Theorem 3.5. We only prove the statements for dimensions of unstable manifolds, since the other statements for dimensions of stable manifolds can be proven in a similar fashion.

First of all, we prove the continuity of the dimensions with respect to $f$. For any $r \in (0, 1)$, take a $C^1$ diffeomorphism $g$ that is $C^1$--close to $f$, and let $\Lambda_g$ be a locally maximal hyperbolic set for $g$. By Theorem 3.1, the map $h_g : W^u_{\beta}(f, x) \cap \Lambda \to W^u_{\beta}(g, h_g(x)) \cap \Lambda_g$ and its inverse are $(C, r)$--Hölder continuous for some $C > 0$. It follows from Lemma 3.4 that
\[ r \cdot \dim_H (W^u_{\beta}(f, x) \cap \Lambda) \leq \dim_H (W^u_{\beta}(g, h_g(x)) \cap \Lambda_g) \leq r^{-1} \cdot \dim_H (W^u_{\beta}(f, x) \cap \Lambda). \]
Therefore $\dim_H (W^u_{\beta}(f, x) \cap \Lambda)$ is a continuous function of $f$. Similarly we have that $\bar{\dim}_B(W^u_{\beta}(f, x) \cap \Lambda)$ and $\underline{\dim}_B(W^u_{\beta}(f, x) \cap \Lambda)$ are continuous functions of $f \in \text{Diff}^1(M)$.
Similarly, for any small \( t \), we have the unique root of equation

\[
1 \leq \frac{\|D_x F|_{E^u(x)}\|}{m(D_x F|_{E^u(x)})} < e^{2^k t} \quad \text{and} \quad 1 \leq \frac{\|D_x F|_{E^s(x)}\|}{m(D_x F|_{E^s(x)})} < e^{2^k t},
\]

where \( F := f^{2^k} \). In fact \( \Lambda \) is a locally maximal hyperbolic set for \( f \), \( \Lambda \) is also a locally maximal hyperbolic set for \( F \). Notice that

\[
W^u_\beta(f,x) \cap \Lambda = W^u_\beta(f,x) \cap \Lambda \quad \text{and} \quad W^s_\beta(f,x) \cap \Lambda = W^s_\beta(f,x) \cap \Lambda. \tag{8}
\]

By Lemma 3.7, \( \dim_H (W^u_\beta(f,x) \cap \Lambda), \dim_B (W^u_\beta(f,x) \cap \Lambda) \) and \( \dim_B (W^s_\beta(f,x) \cap \Lambda) \) are independent of \( \beta \) and \( x \).

Finally, we prove the last statement that

\[
\dim_H (W^u_\beta(f,x) \cap \Lambda) = \dim_B (W^u_\beta(f,x) \cap \Lambda) = \dim_B (W^s_\beta(f,x) \cap \Lambda).
\]

By (8), Lemmas 3.8 and 3.9 below, for each \( x \in \Lambda \),

\[
t_u^k \leq \dim_H (W^u_\beta(f,x) \cap \Lambda) \leq \dim_B (W^u_\beta(f,x) \cap \Lambda) \leq \dim_B (W^s_\beta(f,x) \cap \Lambda) \leq t_u^k
\]

where \( t_u^k \) is the unique root of the equation \( P_\Lambda(F, -t \log \|D_x F|_{E^u(x)}\|) = 0 \), and \( \bar{t}_u^k \) is the unique root of equation \( P_\Lambda(F, -t \log m(D_x F|_{E^u(x)})) = 0 \). Using the same arguments as in the proof of Theorem 6.2, Theorem 6.3 and Theorem 6.4 in [1], one can prove that the sequences \( \{t_u^k\} \) and \( \{\bar{t}_u^k\} \) are monotone and

\[
\lim_{k \to \infty} t_u^k = \lim_{k \to \infty} \bar{t}_u^k := t_u. \tag{9}
\]

Therefore \( \dim_H (W^u_\beta(f,x) \cap \Lambda) = \dim_B (W^u_\beta(f,x) \cap \Lambda) = \dim_B (W^s_\beta(f,x) \cap \Lambda) = t_u \).

This completes the proof of Theorem 3.5.

**Remark 5.** Using the same arguments as the proof of Theorem 6.3 in [1], one can show that the limit point \( t_u \) in (9) is exactly the unique solution of the following equation

\[
P_\Lambda \left( f, -t \log m(D_x f^n|_{E^u(x)}) \right) = 0.
\]

This implies that the dimensions \( \dim_H (W^u_\beta(f,x) \cap \Lambda), \dim_B (W^u_\beta(f,x) \cap \Lambda) \) and \( \dim_B (W^s_\beta(f,x) \cap \Lambda) \) are give by the unique zero of \( P_\Lambda \left( f, -t \log m(D_x f^n|_{E^u(x)}) \right) \). Similarly, for any small \( \beta > 0 \) and every \( x \in \Lambda \) one can prove that

\[
\dim_H (W^u_\beta(f,x) \cap \Lambda) = \dim_B (W^u_\beta(f,x) \cap \Lambda) = \dim_B (W^s_\beta(f,x) \cap \Lambda) = t_u,
\]

where \( t_u \) is the unique solution of the following equation

\[
P_\Lambda \left( f, t \log m(D_x f^n|_{E^u(x)}) \right) = 0.
\]
Remark 6. Since $t_u$ is the unique zero of $P_\Lambda(f, -t \{ \log m(D_x f^n|_{E^u(x)}) \})$, one can similarly show that

$$t_u = \sup \left\{ \frac{h_\mu(f)}{-\lim_{n \to \infty} \frac{1}{n} \int \log m(D_x f^n|_{E^u(x)}) d\mu(x)} : \mu \in \mathcal{M}_f(\Lambda) \right\} \quad \text{(by Theorem 2.7)}$$

$$= \sup \left\{ \frac{h_\mu(f)}{-\int \log \phi_u(f, x) d\mu(x)} : \mu \in \mathcal{M}_f(\Lambda) \right\} \quad \text{(by Lemma 2.3)}$$

$$= \sup \left\{ \frac{h_\mu(f)}{-\lim_{n \to \infty} \frac{1}{n} \int \log ||D_x f^n|_{E^u(x)}|| d\mu(x)} : \mu \in \mathcal{M}_f(\Lambda) \right\}. \tag{3.8}$$

Lemma 3.8. Let $f : M \to M$ be a $C^1$ diffeomorphism, and $\Lambda \subseteq M$ a locally maximal hyperbolic set. Then

$$\dim_H \left( W^u_g(f, x) \cap \Lambda \right) \geq t_*$$

where $t_*$ is the unique root of $P_\Lambda(f, -t \log ||D_x f||_{E^u(x)}) = 0.$

Proof. Let $\phi^u(x) := \log ||D_x f||_{E^u(x)}||$, and let $\mu$ be an ergodic equilibrium state of the topological pressure $P_\Lambda(f, -t_\lambda \phi^u(x))$ and

$$G_\mu = \left\{ x \in \Lambda : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} \to \mu \text{ as } n \to \infty \right\}.$$

By the variational principle of topological pressure, we have that

$$t_* = \int \log ||D_x f||_{E^u(x)}|| d\mu.$$

Given $\varepsilon > 0.$ Let $A$ be a covering of $\Lambda$ by open sets on each of which $\phi^u(x)$ varies by at most $\varepsilon.$ Denote $\lambda^u := \int \phi^u d\mu.$ Let $l$ be a Lebesgue number for $A.$ Take any ball $W$ in any unstable manifold and choose $m$ so large that

$$f^m W \cap W^u_{\frac{1}{2}l}(f, x) \neq \emptyset$$

for every $x \in \Lambda.$

For each $r \geq 1,$ define

$$G_{\mu, r} = \left\{ x \in G_{\mu} \mid \frac{1}{m} \sum_{i=0}^{m-1} \phi^u(f^i x) - \lambda^u < \varepsilon \text{ for } m \geq r \right\},$$

it is clear that $G_{\mu, r} \subset G_{\mu, r+1}.$ Since $f^m$ is $C^1$ diffeomorphism, by Corollary 1,

$$\dim_H (f^m W \cap G_{\mu, r}) \leq \dim_H (f^m W \cap \Lambda) = \dim_H (W \cap \Lambda) := d$$

for every $r \geq 1,$ and

$$\dim_H \left( f^m W \cap \bigcup_{r=1}^{\infty} G_{\mu, r} \right) \leq \dim_H (f^m W \cap \Lambda) = d.$$
Lemma 3.9. Let \( f : M \rightarrow M \) be a \( C^1 \) diffeomorphism, and \( \Lambda \subseteq M \) a locally maximal hyperbolic set. Then

\[ \dim_B (W^u_{\beta^*}(f, x) \cap \Lambda) \leq t^* \]

where \( t^* \) is the unique real number such that \( P_{\Lambda}(f, -t \log m(D_x f|_{E^u(x)})) = 0 \).
Proof. Denote \( d := \overline{\dim}_B(W^u_\beta(f, x) \cap \Lambda) \), assume that \( d > 0 \), otherwise there is nothing to prove. For a small number \( \eta > 0 \) with \( d - 3\eta > 0 \), from the definition of upper box dimension, for each sufficiently large \( l \), there exists \( 0 < r_l < 1/l \) such that

\[
N(W^u_\beta(f, x) \cap \Lambda, r_l) \geq r_l^{-d-\eta}
\]

where \( N(W^u_\beta(f, x) \cap \Lambda, r_l) \) denotes the minimal number of balls of radius \( r_l \) that are needed to cover \( W^u_\beta(f, x) \cap \Lambda \).

Let \( \{x_1, x_2, \cdots, x_N\} \) be a maximal \((1, r_l)\)-separated subset of \( W^u_\beta(f, x) \cap \Lambda \), then the balls \( \{B^u(x_i, r_l/2)\}_{i=1}^N \) are mutually disjoint and \( W^u_\beta(f, x) \cap \Lambda \subset \bigcup_{i=1}^N B^u(x_i, r_l) \).

This implies that

\[
N \geq N(W^u_\beta(f, x) \cap \Lambda, r_l) \geq r_l^{-d-\eta}.
\]

For a small \( \varepsilon > 0 \), there exists \( \rho > 0 \) such that

\[
e^{-\varepsilon} < \frac{m(D_x f|E^u(x))}{m(D_y f|E^u(y))} < e^{\varepsilon}
\]

provided that \( d_u(x, y) < \rho \). For each \( 1 \leq i \leq N \), there exists a positive integer \( n_i \) so that

\[
B^u_{n_i + 2}(x_i, \rho) \subset B^u(x_i, r_l/2) \text{ but } B^u_{n_i + 1}(x_i, \rho) \not\subset B^u(x_i, r_l/2).
\]

Therefore, there exists \( y \in B_{n_i + 1}(x_i, \rho) \) such that

\[
d_u(x_i, y) > r_l/2 \text{ and } d_u(f^{n_i}x_i, f^{n_i}y) < \rho.
\]

Note that there exists \( \xi_i \in B_{n_i + 1}(x_i, \rho) \) such that

\[
\lambda^{-n_i} \leq m(D_{\xi_i} f^{n_i}|E^u(\xi_i)) \leq \frac{d_u(f^{n_i}x_i, f^{n_i}y)}{d_u(x_i, y)} < \frac{2\rho}{r_l}.
\]

Hence \( n_i \leq \frac{\log 2\rho}{r_l} \).

On the other hand, since \( f^{n_i + 1} : B_{n_i + 2}(x_i, \rho) \rightarrow B^u(f^{n_i + 1}x_i, \rho) \) is a diffeomorphism and \( B_{n_i + 2}(x_i, \rho) \subset B^u(x_i, r_l/2) \), there exists \( z \in B_{n_i + 2}(x_i, \rho) \) so that

\[
d_u(f^{n_i + 1}x_i, f^{n_i + 1}z) = \rho.
\]

Thus we have

\[
\rho = d_u(f^{n_i + 1}x_i, f^{n_i + 1}z) \leq C_1^{n_i + 1}d_u(x_i, z) \leq C_1^{n_i + 1}r_l/2
\]

where \( C_1 = \max_{x \in \Lambda} ||D_x f|E^u(x)|| \). Therefore, we have \( n_i \geq \frac{\log 2\rho}{\log C_1} - 1 \).

Let \( B = \frac{\log 2\rho}{\log C_1} - \frac{\log 2\rho}{\log C_1} + 1 \). We now think of having \( N \) balls and \( B \) baskets.

Then there exists a basket containing at least \( \frac{N}{B} \) balls. This implies that there exists a positive integer \( \frac{\log 2\rho}{\log C_1} - 1 \leq n \leq \frac{\log 2\rho}{\log C_1} \) such that

\[
\text{Card}\{j|n_j = n\} \geq \frac{N}{B} \geq \frac{r_l^{-d+\eta}}{B} \geq r_l^{-d+2\eta}
\]
the last inequality holds since $l$ is sufficiently large. Since $B^n_{n+2}(x_i, \rho) \subset B^n(x_i, r_l/2)$ and the balls $\{B^n(x_i, r_l/2)\}_{i=1}^N$ are disjoint, we have that the set $E := \{x_i : u_i = n\}$ is an $(n+2, \rho)$-separated subset of $W^u_{\beta}(f, x) \cap \Lambda$. Hence

$$P_{n+2}\left(W^u_{\beta}(f, x) \cap \Lambda, f, -(d - 3\eta) \log m(D_x f|_{E^n}), \rho \right)$$

$$\geq \sum_{x_i \in E} \prod_{k=0}^{n+1} m(D_{f^k(x_i)} f|_{E^n(f^k x_i)})^{-(d - 3\eta)}$$

$$\geq \sum_{x_i \in E} C_2^{-d+3\eta} \frac{\epsilon^{-n(d-3\eta)}}{\prod_{k=0}^{n} m(D_{f^k(x_i)} f|_{E^n(f^k x_i)})^{-(d - 3\eta)}}$$

$$\geq C_2^{-d+3\eta} \sum_{x_i \in E} m(D_{\xi_i} f^n|_{E^n(\xi_i)})^{-(d - 2\eta)}$$

$$\geq C_2^{-d+3\eta} \left(\frac{2\rho}{r_l}\right)^{-d - 2\eta}$$

$$= C_2^{-d+3\eta} \left(2\rho\right)^{-(d - 2\eta)} > 0,$$

where $C_2 = \max_{x \in \Lambda} m(D_x f|_{E^n(x)})$. It immediately follows that

$$P_{W^u_{\beta}(f, x) \cap \Lambda}\left(f, -(d - 3\eta) \log m(D_x f|_{E^n(x)})\right) \geq 0.$$

Hence

$$P_{\Lambda}\left(f, -(d - 3\eta) \log m(D_x f|_{E^n(x)})\right) \geq 0.$$

Thus $t^* \geq d - 3\eta$. The arbitrariness of $\eta$ yields that $t^* \geq d$. We have shown that $f$ is stable or unstable.

**Theorem 3.10.** Let $f : M \to M$ be a $C^1$ diffeomorphism, and $\Lambda \subseteq M$ be a locally maximal average conformal hyperbolic set. Let $\pi^u$ and $\pi^s$ be the holonomy maps of stable and unstable foliations for $f$, i.e., for any $x \in \Lambda, x' \in W^u_{\beta}(f, x)$ and $x'' \in W^u_{\beta}(f, x)$ close to $x$,

$$\pi^u : W^u_{\beta}(f, x) \cap \Lambda \to W^u_{\beta}(f, x') \cap \Lambda \text{ with } \pi^u(y) = W^u_{\beta}(f, y) \cap W^u_{\beta}(f, x')$$

and

$$\pi^s : W^s_{\beta}(f, x) \cap \Lambda \to W^s_{\beta}(f, x') \cap \Lambda \text{ with } \pi^s(y) = W^s_{\beta}(f, y) \cap W^s_{\beta}(f, x').$$

Then for any $\gamma \in (0, 1)$, there exists $D_\gamma > 0$ such that $\pi^s(\cdot), (\pi^u)^{-1}, (\pi^u)^{-1}$ are $(D_\gamma, \gamma)$–Hölder continuous.

**Proof.** Since $f$ is average conformal on $\Lambda$, by Lemma 2.3, for any $\varepsilon \in (0, -\log \lambda)$ we choose a positive integer $N \geq N(\varepsilon)$ such that

$$1 \leq \frac{\|D_x f|_{E^n(x)}\|}{m(D_x f|_{E^n(x)})} < e^{N\varepsilon} \text{ and } 1 \leq \frac{\|D_x f|_{E^n(x)}\|}{m(D_x f|_{E^n(x)})} < e^{N\varepsilon}, \forall x \in \Lambda,$$

here $F := f^N$. In fact $\Lambda$ is a locally maximal hyperbolic set for $f$, $\Lambda$ is also a locally maximal hyperbolic set for $F$. Thus

$$W^s_{\beta}(F, x) \cap \Lambda = W^s_{\beta}(f, x) \cap \Lambda, W^u_{\beta}(F, x) \cap \Lambda = W^u_{\beta}(f, x) \cap \Lambda.$$

Therefore $\pi^s$ is also a map from $W^s_{\beta}(F, x) \cap \Lambda$ to $W^s_{\beta}(F, x') \cap \Lambda$, and $\pi^u$ is also a map from $W^u_{\beta}(F, x) \cap \Lambda$ to $W^u_{\beta}(F, x') \cap \Lambda$ as follows:

$$\pi^s(y) = W^s_{\beta}(F, y) \cap W^s_{\beta}(F, x') \text{ and } \pi^u(z) = W^u_{\beta}(F, z) \cap W^u_{\beta}(F, x'),$$
For any $\gamma \in (0,1)$, let $U^F_\gamma$ be a small $C^1$ neighborhood of $F$. Taking $G \in U^F_\gamma \cap \text{Diff}^2(M)$, by Lemma 3.2, Lemma 3.6 and Remark 4 we have

1. $h_G|_{W^u_\beta(F,x) \cap \Lambda}$ and $(h_G|_{W^s_\beta(F,x) \cap \Lambda})^{-1}$ are $(C,\gamma)$–Hölder continuous for some $C,\gamma > 0$.
2. The stable foliation $\{W^s(G,z) : z \in \Lambda\}$ is invariant and $C^1$. Thus, the holonomy map $\pi_G^s : W^s_\beta(G,h_G(x)) \cap \Lambda_G \to W^s_\beta(G,h_G(x')) \cap \Lambda_G$ defined as $\pi_G^s(z) := W^s_\beta(G,z) \cap W^s_\beta(G,h_G(x'))$ is Lipschitz.

Therefore for any $y \in W^u_\beta(F,x) \cap \Lambda$,

$$h_G(\pi^s(y)) = h_G(W^s_\beta(F,y) \cap W^s_\beta(F,x')) = W^s_\beta(G,h_G(y)) \cap W^s_\beta(G,h_G(x')) = \pi_G^s(h_G(y)).$$

For the above $\gamma$, there exists $D_\gamma > 0$ such that $\pi^s = h_G^{-1} \circ \pi_G^s \circ h_G$ is $(D_\gamma,\gamma)$–Hölder continuous. Using the same arguments, one can prove that $(\pi^s)^{-1}$, $\pi^u$ and $(\pi^u)^{-1}$ are also $(D_\gamma,\gamma)$–Hölder continuous.

We proceed to prove Theorem A.

**Proof of Theorem A. Step 1.** We claim that

$$\dim_H A_x = \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right)$$

$$= \dim_B A_x = \dim_B \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_B \left( W^u_\beta(f,x) \cap \Lambda \right)$$

$$= \overline{\dim_B} A_x = \overline{\dim_B} \left( W^u_\beta(f,x) \cap \Lambda \right) + \overline{\dim_B} \left( W^u_\beta(f,x) \cap \Lambda \right)$$

where $A_x = \left( W^u_\beta(f,x) \cap \Lambda \right) \times \left( W^u_\beta(f,x) \cap \Lambda \right)$ is a product space. By the definitions of dimension, we have that

$$\overline{\dim_B} A_x \leq \overline{\dim_B} \left( W^u_\beta(f,x) \cap \Lambda \right) + \overline{\dim_B} \left( W^u_\beta(f,x) \cap \Lambda \right)$$

and

$$\dim_H A_x \geq \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right).$$

See Theorem 6.5 in [21] for proofs. Combining Theorem 3.5 and the fact that $\dim_H A_x \leq \dim_B A_x \leq \overline{\dim_B} A_x$, we have

$$\dim_H A_x = \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right)$$

$$= \dim_B A_x = \dim_B \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_B \left( W^u_\beta(f,x) \cap \Lambda \right)$$

$$= \overline{\dim_B} A_x = \overline{\dim_B} \left( W^u_\beta(f,x) \cap \Lambda \right) + \overline{\dim_B} \left( W^u_\beta(f,x) \cap \Lambda \right)$$

Thus the claim holds.

**Step 2.** We prove for any $x \in \Lambda$,

$$\dim_H \Lambda = \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_H \left( W^u_\beta(f,x) \cap \Lambda \right)$$

$$= \dim_B \Lambda = \dim_B \left( W^u_\beta(f,x) \cap \Lambda \right) + \dim_B \left( W^u_\beta(f,x) \cap \Lambda \right)$$

(10)

Let $\Phi : A_x \to \Lambda$ be given by $\Phi(y,z) = W^u_\beta(f,y) \cap W^u_\beta(f,z)$. It is easy to see $\Phi$ is a homeomorphism onto a neighborhood $V_x$ of $x$ in $\Lambda$. We claim that $\Phi$ and $\Phi^{-1}$ are
\((E, \gamma)\)-Hölder continuous for any \( \gamma \in (0, 1) \) and some \( E_\gamma > 0 \). This yields that

\[
\gamma \cdot \dim_H A_x \leq \dim_H V_x \leq \gamma^{-1} \cdot \dim_B A_x,
\]

\[
\gamma \cdot \dim_B A_x \leq \dim_B V_x \leq \gamma^{-1} \cdot \dim_B A_x.
\]

Letting \( \gamma \to 1 \), we have that

\[
\dim_H V_x = \dim_B V_x = \dim_B A_x \text{ and } \dim_H V_x = \dim_B A_x.
\]

Since \( \{V_x : x \in \Lambda\} \) is an open cover of \( \Lambda \), one can choose a finite open cover \( \{V_{x_1}, V_{x_2}, \cdots, V_{x_k}\} \) of \( \Lambda \). It follows from Theorem 3.5 that

\[
\dim_H \Lambda = \max_{1 \leq i \leq k} \dim_H V_{x_i} = \dim_H V_x = \dim_B A_x,
\]

\[
\dim_B \Lambda = \max_{1 \leq i \leq k} \dim_B V_{x_i} = \dim_B V_x = \dim_B A_x
\]

and

\[
\overline{\dim}_B \Lambda = \max_{1 \leq i \leq k} \overline{\dim}_B V_{x_i} = \overline{\dim}_B V_x = \overline{\dim}_B A_x.
\]

Therefore for any \( x \in \Lambda \),

\[
\dim_H \Lambda = \dim_H (W_\beta^u(f, x) \cap \Lambda) + \dim_H (W_\beta^s(f, x) \cap \Lambda)
\]

\[
= \dim_B \Lambda = \dim_B (W_\beta^u(f, x) \cap \Lambda) + \dim_B (W_\beta^s(f, x) \cap \Lambda)
\]

\[
= \overline{\dim}_B \Lambda = \overline{\dim}_B (W_\beta^u(f, x) \cap \Lambda) + \overline{\dim}_B (W_\beta^s(f, x) \cap \Lambda).
\]

It suffices to prove the claim above. Let \( y_1, y_2 \in W_\beta^u(f, x) \cap \Lambda \) and \( z_1, z_2 \in W_\beta^s(f, x) \cap \Lambda \). Denote \( w_1 = \Phi(y_1, z_1), w_2 = \Phi(y_2, z_2), w = W_\beta^u(f, w_1) \cap W_\beta^s(f, w_2) = \Phi(y_2, z_1) \). By Theorem 3.10, it has

\[
d(w_1, w_2) \leq d_u(w, w_1) + d_s(w, w_2)
\]

\[
\leq D_u d_u(y_1, y_2) + D_s d_s(z_1, z_2) \gamma
\]

\[
\leq 2D_u \max\{d_u(y_1, y_2), d_s(z_1, z_2)\} \gamma.
\]

This proves the Hölder continuity of \( \Phi \). On the other hand, the fact that there exists \( k > 0 \) such that

\[
d(w_1, w_2) \geq k \max\{d_u(w, w_1), d_s(w, w_2)\},
\]

and Theorem 3.10 implies that

\[
\max\{d_u(y_1, y_2), d_s(z_1, z_2)\}
\]

\[
\leq \max\{D_u d_u(w, w_1), D_s d_s(w, w_2) \gamma\}
\]

\[
\leq D_s k^{-1} d(w_1, w_2) \gamma.
\]

So \((\Phi)^{-1}\) is \((D_s k^{-1}, \gamma)\)-Hölder continuous. Taking \( E_\gamma = \max\{2D_s, D_s k^{-1}\} \), thus \( \Phi \) and \((\Phi)^{-1}\) are \((E_\gamma, \gamma)\)-Hölder continuous.

**Step 3.** We prove the last assertion that the dimensions of an average conformal hyperbolic set varies continuous with respect to \( f \). Since \( f \) is average conformal on \( \Lambda \), for any \( \varepsilon \in (0, \frac{1}{2} \log \lambda) \), by Lemma 2.3 we choose a positive integer \( N \geq N(\varepsilon) \) such that for any \( x \in \Lambda \)

\[
1 \leq \frac{\|D_x F|_{E^u(x)}\|}{m(D_x F|_{E^u(x)})} < e^{\frac{k}{2} \varepsilon} \quad \text{and} \quad 1 \leq \frac{\|D_x F|_{E^s(x)}\|}{m(D_x F|_{E^s(x)})} < e^{\frac{k}{2} \varepsilon},
\]
here $F := f^N$. Since $\Lambda$ is a locally maximal hyperbolic set for $f$, $\Lambda$ is also a locally maximal hyperbolic set for $F$. Then there exists a neighborhood $\mathcal{U}_F$ of $F$ in $\text{Diff}^1(M)$ such that for any $G \in \mathcal{U}_F$,

$$1 \leq \frac{\|D_G f|_{E^c(x)}\|}{m(D_G f|_{E^c(x)})} < e^{N\epsilon}$$

and $1 \leq \frac{\|D_G f|_{E^c(x)}\|}{m(D_G f|_{E^c(x)})} < e^{N\epsilon}$ for any $x \in \Lambda_G$,

where $\Lambda_G$ is a locally maximal hyperbolic invariant set of $G$. By Lemma 3.7, $\dim_H(W^s_\beta(G,x) \cap \Lambda_G)$, $\dim_B(W^s_\beta(G,x) \cap \Lambda_G)$ and $\bar{\dim}_B(W^s_\beta(G,x) \cap \Lambda_G)$ are independent of $\beta$ and $x$. The same statements for $W^s_\beta(G,x) \cap \Lambda_G$ hold.

Let $\Phi^G : (W^u_\beta(G,x) \cap \Lambda_G) \times (W^s_\beta(G,x) \cap \Lambda_G) \rightarrow \Lambda_G$ given by $\Phi^G(y,z) = W^u_\beta(G,y) \cap W^s_\beta(G,z)$. It is clear that $\Phi^G$ is a homeomorphism onto a neighborhood $V^G_x$ of $x$ in $\Lambda_G$. Let $\pi^G_x$ and $\pi^G_y$ be the holonomy maps of stable and unstable foliations for $G$. In fact $G$ satisfies (4), for any $r \in (0, 1)$, as in the proof of Theorem 3.10, we have $\pi^G_x$, $(\pi^G_x)^{-1}$, $\pi^G_y$ and $(\pi^G_y)^{-1}$ are $(D_r,r)$–Hölder continuous for some $D_r > 0$. Using the proof of the Hölder continuity of $\Phi$ and $\Phi^{-1}$ as above, one can prove that the map $\Phi^G$ and its inverse $(\Phi^G)^{-1}$ are $(E^r, r)$–Hölder continuous for some $E^r > 0$. This yields that

$$r \cdot \dim \left( (W^u_\beta(G,x) \cap \Lambda_G) \times (W^s_\beta(G,x) \cap \Lambda_G) \right)$$

$$\leq \dim V^G_x$$

$$\leq r^{-1} \cdot \dim \left( (W^u_\beta(G,x) \cap \Lambda_G) \times (W^s_\beta(G,x) \cap \Lambda_G) \right)$$

where $\dim$ denotes either $\dim_H$ or $\dim_B$ or $\bar{\dim}_B$. Letting $r \rightarrow 1$, we have that

$$\dim V^G_x = \dim \left( (W^u_\beta(G,x) \cap \Lambda_G) \times (W^s_\beta(G,x) \cap \Lambda_G) \right)$$

(14)

where $\dim$ denotes either $\dim_H$ or $\dim_B$ or $\bar{\dim}_B$. Since $V^G_x$ is open, similar as the proof of (11), (12) and (13), one can show that

$$\dim_H V^G_x = \dim_H \Lambda_G, \quad \dim_B V^G_x = \dim_B \Lambda_G \quad \text{and} \quad \bar{\dim}_B V^G_x = \bar{\dim}_B \Lambda_G$$

for each $x \in \Lambda_G$. It follows from Theorem 6.5 in [21] and (14) that

$$\dim_H \left( W^u_\beta(G,x) \cap \Lambda_G \right) + \dim_H \left( W^s_\beta(G,x) \cap \Lambda_G \right)$$

$$\leq \dim_H \Lambda_G \leq \dim_B \Lambda_G \leq \bar{\dim}_B \Lambda_G$$

$$\leq \bar{\dim}_B \left( W^u_\beta(G,x) \cap \Lambda_G \right) + \bar{\dim}_B \left( W^s_\beta(G,x) \cap \Lambda_G \right)$$

By Theorem 3.5, for any small $\xi > 0$, there exists an open neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}^1(M)$ such that for any $g \in \mathcal{U}$ and any $x \in \Lambda$ we have

$$\dim \left( W^i_\beta(g, h_g(x)) \cap \Lambda_g \right) - \xi \leq \dim \left( W^i_\beta(f, x) \cap \Lambda \right) \leq \dim \left( W^i_\beta(g, h_g(x)) \cap \Lambda_g \right) + \xi$$

where $i = u, s$ and $\dim$ denotes either $\dim_H$ or $\dim_B$ or $\bar{\dim}_B$. One may choose a sufficiently small open neighborhood $\mathcal{V}_f$ of $f$ in $\text{Diff}^1(M)$ such that each $g \in \mathcal{V}_f$ satisfies that $g^N \in \mathcal{U}_F$. Put $G := g^N$, note that $\Lambda_G = \Lambda_g$ and so

$$W^u_\beta(G,x) \cap \Lambda_G = W^u_\beta(g, x) \cap \Lambda_g \quad \text{and} \quad W^s_\beta(G,x) \cap \Lambda_G = W^s_\beta(g, x) \cap \Lambda_g$$
for each $x \in \Lambda_g$. It follows from (10) that for each $g \in \mathcal{U} \cap \mathcal{V}_f$, we have
\[
\dim_H \Lambda - 2\xi = \dim_H (W^g_{\beta}(f, x) \cap \Lambda) + \dim_H (W^g_{\beta}(f, x) \cap \Lambda) - 2\xi
\leq \dim_H (W^g_{\beta}(g, h_g(x)) \cap \Lambda_g) + \dim_H (W^g_{\beta}(g, h_g(x)) \cap \Lambda_g)
= \dim_H (W^g_{\beta}(G, h_g(x)) \cap \Lambda_G) + \dim_H (W^g_{\beta}(G, h_g(x)) \cap \Lambda_G)
\leq \dim_H \Lambda_g \leq \overline{\dim}_{B} \Lambda_g
\leq \overline{\dim}_{B} (W^g_{\beta}(g, h_g(x)) \cap \Lambda_g) + \overline{\dim}_{B} (W^g_{\beta}(g, h_g(x)) \cap \Lambda_g)
\leq \overline{\dim}_{B} (W^g_{\beta}(f, x) \cap \Lambda) + \overline{\dim}_{B} (W^g_{\beta}(f, x) \cap \Lambda) + 2\xi
= \overline{\dim}_{B} \Lambda + 2\xi.
\]
This means the dimensions of an average conformal hyperbolic set $\Lambda$ vary continuous with respect to $f \in \text{Diff}^1(M)$. It completes the proof of Theorem A. \hfill \Box

**Remark 7.** For a locally maximal average conformal hyperbolic set $\Lambda$ of a $C^1$ diffeomorphism, by Remark 5 and (10), we have
\[
\dim_H \Lambda = \overline{\dim}_{B} \Lambda = t_s + t_u
\]
where $t_u$ and $t_s$ are unique solutions of $P_{\Lambda}(f, -t\{ \log m(D_xf^n|_{E^u(x)})\}) = 0$ and $P_{\Lambda}(f, t\{ \log \|D_xf^n|_{E^s(x)}\|\}) = 0$ respectively.

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