Multi-photon Scattering Theory and Generalized Master Equations

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We develop a scattering theory to investigate the multi-photon transmission in a one-dimensional waveguide in the presence of quantum emitters. It is based on a path integral formalism, uses displacement transformations, and does not require the Markov approximation. We obtain the full time-evolution of the global system, including the emitters and the photonic field. Our theory allows us to compute the transition amplitude between arbitrary initial and final states, as well as the $S$-matrix of the asymptotic in- and out- states. For the case of few incident photons in the waveguide, we also re-derive a generalized master equation in the Markov limit. We compare the predictions of the developed scattering theory and that with the Markov approximation. We illustrate our methods with five examples of few-photon scattering: (i) by a two-level emitter, (ii) in the Jaynes-Cummings model; (iii) by an array of two-level emitters; (iv) by a two-level emitter in the half-end waveguide; (v) by an array of atoms coupled to Rydberg levels. In the first two, we show the application of the scattering theory in the photon scattering by a single emitter, and examine the correctness of our theory with the well-known results. In the third example, we analyze the condition of the Markov approximation for the photon scattering in the array of emitters. Finally, we highlight the interplay between the phenomenon of electromagnetic-induced transparency and the Rydberg interaction, and show how this results in a rich variety of possibilities in the quantum statistics of the scattering photons.

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I. INTRODUCTION

The exploration of quantum optical systems characterized by strong photon-photon interactions has inspired a lot of research and extensive studies recently [1,2]. Those systems provide us with a versatile platform to investigate the generation and transport of non-classical light, as well as the behavior of single-photon sources [3,4] and switches [5,6]. Those are the basic ingredients in quantum-optical [7] and quantum information devices [8,9].

The manipulation of non-classical light typically requires devices displaying either strong nonlinearity [10], or quantum interference effects [11,12]. Among many other phenomena, they give rise to peculiar quantum statistical behavior of the emitted or scattered photons, like anti-bunching [13] in the generation of single photons or photon pairs [14,15]. Such devices are being investigated in different incarnations, including cavity QED [1,2,6,16,17,20,21], solid state [18,19,25,28], and circuit QED systems [29,30]. At the many-particle level, the strong interaction between the nonlinear devices and the photons results in the generation of many-body states, which can be studied in terms of dissipative versions of quantum spin models [31]. In particular, the investigations of the atomic steady state and the photon transmission properties reveal a rich variety of quantum phases [32,33] and photon statistics.

In order to characterize how atomic (or any other) non-linear devices can be used to create and manipulate photonic states, one can analyze the transmission spectra and the photon statistics, for instance, in terms of the second order correlation function of the photons emitted under the presence of weak driving light. This analysis is typically addressed through an input-output theory relating the correlation functions of the emitted photons to those of the atomic system in steady state [17]. Those can be determined using a master equation approach, based on the Born-Markov approximation. This approach has proven to be very successful in most of the experimentally relevant situations. Even though that is a very good approximation for most models in quantum optics, in the presence of several emitters, or in certain regimes its use may not be justified.

In order to analyze the transmission properties exactly, several elegant approaches [20,21,34,42] have been developed for the few photon scattering process, where the Born-Markov approximation is not involved. The exact analysis of single and two photon transmissions was first addressed through the Bethe ansatz approach [20], which is equivalent to the Lippmann-Schwinger scattering theory. This approach establishes the exact scattering matrix ($S$-matrix) between the in- and out- asymptotic states of photons, which determines the transmission spectrum and the second order correlation function of outgoing photons. It turns out that the Bethe ansatz approach is very successful in the two-photon scattering by a single emitter with simple structure, e.g., the two-level emitter, however, the generalization to the photon transmission by several emitters is difficult. The approach [34] based on the input-output theory is able to provide the exact $S$-matrix for the two photon scattering by two emitters. Here, since a closed set of motion equations for emitter operators are required, the exact $S$-
matrix can only be obtained in some very limited cases, e.g., two emitters with simple structures.

The systematic approach to the exact $S$-matrix in the multi-photon scattering process is based on quantum field theory. Through either the Lehmann-Symanzik-Zimmerman (LSZ) reduction formalism or the input-output theory, the exact $S$-matrix is related to the time ordered correlation function of emitter operators, which can be obtained by the functional integration of the photonic bath modes. The functional integration provides an efficient approach to analyze the photon statistical properties in the multi-photon scattering by emitters with complicated structures, e.g., the Jaynes-Cummings (JC) system and the atom-coupled whispering gallery resonator. However, the exact $S$-matrix is only able to describe the evolution of photonic bath in the asymptotic limit, thus, it fails to depict the transient dynamics of emitters, e.g., the single-photon detection in the superconducting edge sensor. Recently, a generalized input-output theory was established by the subtle combination of the input-output theory and the quantum regression theorem, which allowed us to investigate both the transient dynamics and the photon statistics in the asymptotic limit. Here, a generalized master equation is obtained to describe the transient dynamics of several emitters under the presence of few incident photons. The generalized input-output theory and master equation involve the Markov approximation, thus, they are not able to provide the exact results. The use of the Markov approximation in the presence of several emitters need to be justified.

In this paper, based on the path integral formalism, we develop the scattering theory to characterize the full time evolution of the global system (including the emitters and the photonic bath) exactly for the few-photon scattering by several emitters with complicated structures. Our theory provides the transition amplitudes from the arbitrary initial state to the corresponding final state without the Markov approximation, where both the quantum statistics of scattering photons and the transient dynamics of emitters can be analyzed exactly. In the Markovian limit, the exact results from our theory perfectly agree with those from the quantum regression approach, which justifies the validity of the Markov approximation. Similarly, the generalized master equation is re-produced to describe the dynamics of emitters in the presence of few incident photons. Here, we emphasize that the generalized master equation establishes the close relation between the properties of photons emitted under the present of weak driving light and those in the few-photon scattering by the emitters.

Using the two paradigmatic examples, i.e., the few-photon transmission to the two-level emitter and the JC system, we examine the correctness of our theory in the photon scattering by the single emitter. Here, the developed scattering theory re-produces the well-known results for the transmission spectra and the second order correlation function. For the few-photon scattering by several emitters, we investigate the condition of the Markov approximation in detail. In particular, we show that the Markov approximation is valid under certain conditions relating the bandwidth of the dynamics and the distance of separation between emitters. In the non-Markovian regime, the exact results exhibit peculiar features associated with retardation of pulses between emitters.

By the developed scattering theory, we explore some novel phenomena in two new situations. For the photon scattering by a single two-level emitter in the half-end waveguide, we show how the two-level emitter can generate entanglement of out-going photons. For the photon transmission in the array of atoms under conditions of electromagnetically induced transparency (EIT) and coupled to Rydberg levels, we highlight the EIT phenomenon and the Rydberg interaction. Here, the second order correlation functions of emitted photons, the propagation of dark polaritons and the collision of the counter-propagating polaritons in the transient process are analyzed, which show a rich variety of the quantum statistics of photons and Rydberg excitations. The developed scattering theory is proven to be a very efficient and systematical approach to investigate the quantum statistics of photons in the array of the interacting emitters with complicated structures.

The paper is organized as follows. In Sec. II, the five models are introduced, which are the two-level emitter and the JC system coupled to the photonic waveguide, the two-level emitter in the half-end waveguide, and the photon scattering by an array of two-level atoms and EIT atoms coupled to Rydberg-levels. In Sec. III, the developed scattering theory is established, where the exact $S$-matrix relates the asymptotic incident and final states. For the transient process, the generalized master equation is derived to characterize the time evolution of the quantum emitters. In Sec. IV, two paradigmatic examples, i.e., the photon transmission to the two-level emitter and the JC system, are used to examine our theory and illustrate how our method works in the photon scattering by the single emitter. In Sec. V, we analyze the validity of the Markov approximation for the photon scattering by several emitters. In Sec. VI, we show that the generation of entanglement between two scattering photons by a single two-level emitter in the half-end waveguide. In Sec. VII, the photon transmission to an array of EIT atoms coupled to Rydberg levels is investigated. In Sec. VIII, the results are summarized with the outlook.

II. SCATTERING MODELS

In this section, we introduce the models we are going to use in order to investigate the transmission of waveguide photons interacting with quantum systems. Those could be single emitters (e.g., a multi-level atom, an atom coupled to a cavity mode), or an array of emitters coupled, for instance, to Rydberg levels.
The annihilation operator \( O \) mode, and the coupling constant is \( g \) spacing describes a single two-level system, with the energy level \( \omega \) mode directly couples to the waveguide, i.e., \( O \) where the collective operators \( \sum \) consist of a two-level system with the energy \( \omega \) will be explicitly given latter for different examples; (c) The interaction between the waveguide photons and the emitters is given by the Hamiltonian

\[
H_T = \sum_k \left[ \sqrt{\frac{\Gamma}{L}} d^\dagger(x_i) + \sqrt{\frac{\Gamma}{L}} d(x_i) \right] O_k + \text{H.c.},
\]

where the left- and right-moving photon fields \( r(x_i) = \sum_k r_k e^{i(k+\frac{\pi}{2})x_i}/\sqrt{L} \) and \( l(x_i) = \sum_k l_k e^{i(k-\frac{\pi}{2})x_i}/\sqrt{L} \) couple to the operator \( O_i \) of the i-th emitter at the position \( x_i \) with coupling strengths \( \Gamma_{r/l,i} \). In momentum space, the interaction Hamiltonian reads

\[
H_T = \sum_k \left[ r_k O_{k,+} + l_k^\dagger O_{k,-} \right] + \text{H.c.},
\]

where the collective operators \( O_{k,\pm} = \sum_i \sqrt{\Gamma_i/\omega_i} e^{-i(k\pm\omega)x_i}/\sqrt{L} \), and we focus on the symmetric coupling case, i.e., \( \Gamma_{r,i} = \Gamma_{l,i} = \Gamma_i \). Without loss of generality, we choose the position of the first emitter \( i = 1 \) at the origin \( x_1 = 0 \); (d) Apart from the decay to the waveguide, there may exist other decay channels to free space (spontaneous emission outside the waveguide). We describe this through the Hamiltonian \( H_T \). Here, each emitter couples to some system operator \( \hat{O} \) with a JC-type coupling \( (2) \).

We now present some relevant examples for the system Hamiltonian \( H_{\text{sys}} \), and which will be analyzed in detail in the following sections. As shown in Fig. 1b, the simplest example consists of a two-level system with the energy level spacing \( \omega_c = k_0 \) between states \( |e\rangle \) and \( |g\rangle \). The Hamiltonian is \( H_{\text{sys}} = \omega_c |e\rangle \langle e| \) and \( O_{g=1} = \hat{O} = \hat{O} = |g\rangle \langle e| \).

The second example is the JC system shown in Fig. 1c. The Hamiltonian

\[
H_{\text{sys}} = \omega_c a^\dagger a + \omega_c |e\rangle \langle e| + g(a^\dagger |g\rangle \langle e| + \text{H.c.})
\]

describes a single two-level system, with the energy level spacing \( \omega_c \), coupled to a cavity mode of frequency \( \omega_c \). The annihilation operator \( a \) corresponds to the cavity mode, and the coupling constant is \( g \). Here, the cavity mode directly couples to the waveguide, i.e., \( O_{g=1} = \hat{O} = a \). The excitations of cavity field and the two-level system can decay into free space; thus, \( \hat{O} \) would be a and \( |g\rangle \langle e| \).

We analyze the validity of the Markov approximation and the retardation effect using the third example. Here, as shown in Fig. 1d, the waveguide photons couple to an array of two-level emitters with frequency \( \omega_e = k_0 \) and lattice spacing \( d \). The emitter Hamiltonian is

\[
H_{\text{sys}} = \sum_i (\omega_e b^\dagger_i b_i + \frac{1}{2} U_0 b^\dagger_i b^\dagger_j b_i b_j),
\]

where the hardcore boson \( b_i \) is introduced to describe a two-level emitter, and the hardcore behavior is obtained in the limit \( U_0 \to \infty \). The waveguide photons couple to the collective modes \( \hat{O}_{k,\pm} = \sum_i \sqrt{\Gamma_i/L} e^{-i(k\pm\omega)x_i} b_i \), where the decay rates \( \Gamma_{r/l,i} \) are taken to be the constant \( \Gamma \). The excitation decays into the free space, i.e., \( \hat{O}_{l} = b_i \).

The fourth example is a single emitter in front of a mirror, as shown in Fig. 1b. The system Hamiltonian is \( H_{\text{sys}} = H_{\text{emitter}} + \omega_e b^\dagger b \), and the collective operators are

\[
O_{k,+} = \sqrt{\frac{\Gamma}{L}} e^{-i(k+\omega)x_0} b + \sqrt{\frac{\Gamma}{L}} b^\dagger, \\
O_{k,-} = \sqrt{\frac{\Gamma}{L}} e^{-i(k-\omega)x_0} b + \sqrt{\frac{\Gamma}{L}} b^\dagger,
\]

where the boson mode \( b \) is introduced to describe the mirror in the limit \( \Gamma_b \to \infty \). The decay of the emitter excitation into free space is characterized by \( \hat{O}_l = |g\rangle \langle e| \).

The fifth example is an array of EIT atoms coupled to Rydberg levels, with lattice spacing \( d \), as it is schematic.
cally shown in Fig. III. The emitter Hamiltonian is

\[ H_{\text{sys}} = \sum_i \left[ \omega_e e_i^\dagger e_i + \omega_s s_i^\dagger s_i + (\Omega e^{-i\omega_c t} e_i^\dagger s_i + \text{H.c.}) \right] + H_{\text{HC}} + \frac{1}{2} \sum_{ij} U_{ij} s_i^\dagger s_j^\dagger s_j s_i. \]  
(7)

Here, we use hardcore bosons \(e\) and \(s\) to describe a Rydberg-EIT atom. There are three atomic levels: the ground state \(|g\rangle\), the excited state \(|e\rangle\) with frequency \(\omega_e\), and the Rydberg state \(|s\rangle\) with frequency \(\omega_s\). The hardcore behavior is obtained through

\[ H_{\text{HC}} = \frac{U_0}{2} \sum_i (e_i^\dagger e_i + s_i^\dagger s_i)(e_i^\dagger e_i + s_i^\dagger s_i - 1) \]  
(8)

in the limit \(U_0 \to \infty\), which projects out the states \(|ee\rangle\), \(|es\rangle\), and \(|ss\rangle\) of each atom with double occupation. The atoms in the Rydberg state \(|s\rangle\) experience long-range interactions, \(U_{ij}\). For instance, we can take the Van-der-Waals interaction \(U_{ij} = C_6 / |x_i - x_j|^6\), the dipolar interaction \(U_{ij} = C_3 / |x_i - x_j|^3\), or the uniform interaction \(U_{ij} = C_0\) [58]. The transition between states \(|e\rangle\) and \(|s\rangle\) is induced by a classical field of frequency \(\omega_d\) and corresponding Rabi frequency \(\Omega\).

The waveguide photons couple to the collective modes \(O_{k,\pm} = \sqrt{\Gamma} \sum \epsilon^{-i(k \pm k_0)z} e_i / \sqrt{\mathcal{L}}\), where the decay rates \(\Gamma_{r(t),i}\) are taken to be the constant \(\Gamma_{r(t),i} = \Gamma\). The excitation decays into the free space, i.e., \(\tilde{O}_i = e_i\).

For convenience, we transform the Hamiltonian in the rotating frame \(r_k \to r_k e^{-ik_0 t}\) and \(l_k \to l_k e^{-ik_0 t}\). The system operators are also transformed correspondingly such that the total Hamiltonian is time-independent. As a result, in the rotating frame, \(H_\omega = \sum_k k(r_k^\dagger r_k - l_k^\dagger l_k)\), and the frequencies in the system Hamiltonian are replaced by the detunings. This will be used in the five examples presented in the following sections.

### III. GENERAL FORMALISM

In this section, we show the general formalism to study the full dynamics of the global system during the few-photon scattering process. In Secs. IIIA and IIIB, we briefly review the results based on the Markov approximation in Ref. [59]. Here, a different method, i.e., the displacement transformation, is introduced to relate the transition amplitude between arbitrary initial and final states with the correlators of emitter operators. Based on the quantum regression theorem, these correlators are obtained by the effective Hamiltonian of the emitters, where the quantum regression theorem requires the Markov approximation. In order to obtain the exact result and examine the validity of the Markov approximation, in Sec. IIIIC we use the path integral formalism to derive the exact transition amplitude, where the non-Markovian effects are taken into account.

In both approaches, the \(S\)-matrix of the asymptotic in- and out-states and the transition amplitudes in the transient process are obtained, which agree with each other in the Markov limit. It turns out that for the single emitter coupled to the photon with linear dispersion, the Markovian result is proven to be exact by the path integral approach. For the photon scattering by several emitters, the validity of the Markov approximation is analyzed in Sec. V in detail. In the Markov limit, both approaches give rise to the generalized master equation [59] governing the dynamics of the emitters during the scattering process, where the generalized master equation establishes a close relation between the transient behaviors of emitters in the presence of few incident photons and the dynamics of emitters under the weak driving light.

### A. \(S\)-matrix by quantum regression theorem

In this section, based on the quantum regression theorem, the displacement transformation is used to derive the transition amplitude between arbitrary initial and final states, where the quantum regression theorem involves the Markov approximation.

We introduce the transition amplitude

\[ A(T) = \langle \text{sys} | \varphi_{\text{out}} \rangle \langle \text{out} | e^{-iH(t_f - t_i)} | \varphi_{\text{in}} \rangle_{\text{sys}} \]  
(9)

from the initial state \(|\varphi_{\text{in}}\rangle_{\text{sys}} = |0\rangle_{\text{free}} |\psi_{\text{in}}\rangle_{\text{w}} |\varphi_{\text{in}}\rangle_{\text{sys}}\) to the final state \(|\varphi_{\text{out}}\rangle_{\text{sys}} = |0\rangle_{\text{free}} |\psi_{\text{out}}\rangle_{\text{w}} |\varphi_{\text{out}}\rangle_{\text{sys}}\) during the time \(T = t_f - t_i\). At the instant \(t_i\), the waveguide photons are in the state \(|\psi_{\text{in}}\rangle_{\text{w}}\), and the initial state of the emitters is \(|\varphi_{\text{in}}\rangle_{\text{sys}} = \gamma_{\text{in}}^\dagger |G\rangle_{\text{sys}}\), where \(|G\rangle_{\text{sys}}\) and \(\gamma_{\text{in}}^\dagger\) denote the ground state and some creation operator of the emitters, respectively. Similarly, at the instant \(t_f\), the corresponding final states are \(|\varphi_{\text{out}}\rangle_{\text{sys}} = \gamma_{\text{out}}^\dagger |G\rangle_{\text{sys}}\) and \(|\psi_{\text{out}}\rangle_{\text{w}}\), respectively. Here, we focus on the transition process without excitations leaking to the free space, and the initial and final states of the free space are the vacuum state \(|0\rangle_{\text{free}}\).

The initial and final states of waveguide photons can be generally written as

\[ |\psi_{\text{in}}\rangle_{\text{w}} = \sum_{\{n_{k}\}} |\psi_{\text{in}}\rangle \prod_k |n_{k}\rangle, \]
\[ |\psi_{\text{out}}\rangle_{\text{w}} = \sum_{\{m_{k}\}} |\psi_{\text{out}}\rangle \prod_k |m_{k}\rangle, \]  
(10)

where \(\{n_{k}\} = \{n_{k_1, \alpha}, n_{k_2, \alpha}, \ldots\}\) and \(\{m_{k}\} = \{m_{k_1, \alpha}, m_{k_2, \alpha}, \ldots\}\) are the number distribution of photons with different momenta \(k\) in the initial and final states, respectively, and \(\alpha = r, l\) denote the right- and left-moving modes. We notice that the relation

\[ |n_{k}\rangle = \lim_{J_{k \to 0}} \frac{1}{\sqrt{n_{k}}} \frac{\partial n_{k}}{\partial J_{k \alpha}} |J_{k}\rangle \]  
(11)
between the Fock state and the unnormalized coherent state $|J_{k\alpha}\rangle = \sum_{n_{k\alpha}} J_{n_{k\alpha}}^{n_{k\alpha}} |n_{k\alpha}\rangle / \sqrt{n_{k\alpha}}$, leads to the coherent representation of the initial and final states

\[
|\psi_{in}\rangle = F_{in} |\{J_{k\alpha}\}\rangle,
\]

\[
|\psi_{out}\rangle = F_{out} |\{J_{k\alpha}\}\rangle,
\]

(12)

where

\[
F_{in} = \lim_{\{J_{k\alpha}\} \rightarrow 0} \sum_{\{n_{k\alpha}\}} \psi_{in}(\{n_{k\alpha}\}) \prod_{\alpha} \frac{1}{\sqrt{n_{k\alpha}}} \delta^{n_{k\alpha}} J_{k\alpha},
\]

(13)

\[
F_{out} = \lim_{\{J_{k\alpha}\} \rightarrow 0} \sum_{\{m_{k\alpha}\}} \psi_{out}(\{m_{k\alpha}\}) \prod_{\alpha} \frac{1}{\sqrt{m_{k\alpha}}} \delta^{m_{k\alpha}} J_{k\alpha}.
\]

In terms of Eq. (12), the transition amplitude reads

\[
\mathcal{A}(T) = F_{out} F_{in} A_J(T),
\]

(14)

where the transition amplitude

\[
A_J(T) = \text{sys} \langle \varphi_{out} |_{\text{b}} \langle \{J_{k\alpha}\} | e^{-iHT} | \{J_{k\alpha}\} |_{\text{b}} | \varphi_{in}\rangle_{\text{sys}},
\]

(15)

and $|\{J_{k\alpha}\}\rangle_{\text{b}} = |\text{0}\rangle_{\text{free}} |\{J_{k\alpha}\}\rangle$. The transition amplitude (15) can be evaluated by either the quantum regression theorem or the path integral approach.

We first show the result from the quantum regression theorem, where the Markov approximation is required. In the interacting picture, the time evolution operator $\hat{U} = \mathcal{T} \exp[-i \int_{t_f}^{t_i} dt H(t)]$ is determined by the Hamiltonian $H(t) = H_{\text{sys}} + H_{\text{F}} + H_{\text{T}}(t)$, where $\mathcal{T}$ is the time-ordering operator and the interaction part $H_{\text{T}}(t) = e^{H_{\text{eff}} t} \mathcal{H}_e e^{-i H_{\text{eff}} t}$ is

\[
H_{\text{T}}(t) = \sum_k (r_k^i O_{k,+} e^{ikt} + l_k^i O_{k,-} e^{-ikt}) + H.c.
\]

(16)

In terms of $\mathcal{U}$, the amplitude (15) reads

\[
A_J(T) = \text{sys} \langle \varphi_{out} |_{\text{b}} \langle \{J_{k\alpha,\text{out}}\} | \hat{U} | \{\langle J_{k\alpha,\text{in}}\} |_{\text{b}} | \varphi_{in}\rangle_{\text{sys}},
\]

(17)

where $J_{k\alpha,\text{in}} = J_{k\alpha} e^{i e_{k\alpha} \cdot t_f}$, $J_{k\alpha,\text{out}} = J_{k\alpha} e^{i e_{k\alpha} \cdot t_i}$, the dispersion relations $e_{k\alpha} = \sigma_{k\alpha} k$, and $\sigma_{k\alpha} = \pm$ for the right- and left- moving modes, respectively.

The displacement transformation $U$: $|\{J_{k\alpha}\}\rangle = e^{\sum_k |J_{k\alpha}|^2 / 2 U} |\text{0}\rangle_{\text{sys}}$ is introduced to rewrite the amplitude

\[
A_J(T) = \exp(\sum_{k,\alpha=r,l} |J_{k\alpha}|^2 e^{-i e_{k\alpha} \cdot T}) \times \text{sys} \langle \varphi_{out} |_{\text{b}} \langle \{0\}\rangle | \hat{U} | \{\langle 0\}\} |_{\text{b}} | \varphi_{in}\rangle_{\text{sys}},
\]

(18)

where the state of the waveguide photon is transformed to the vacuum state, and the time evolution operator $\hat{U} = \mathcal{T} e^{-i \int_{t_f}^{t_i} dt [H(t) + H_{\text{F}}(t)]}$ is determined by $H(t)$ and the driving term

\[
H_{\text{F}}(t) = \sum_{k,\alpha=r,l} J_{k\alpha}^\dagger O_{k,\alpha} e^{-i \sigma_{k\alpha} (t_f - t)} + J_{k\alpha} O_{k,\alpha} e^{-i \sigma_{k\alpha} (t - t_i)}.
\]

(19)

It follows from Eq. (14) that the functional derivative of $A_J(T)$ determines the transition amplitude $\mathcal{A}(T)$, which is composed of Fourier transforms of time ordered correlators

\[
\mathcal{G}(T) = \left\langle T \gamma_{\text{out}}(t_f) O_1(t_1) ... O_n(t_n) \gamma_{\text{in}}^\dagger(t_i) \right\rangle
\]

(20)

on the ground state $|\{0k\}\rangle_b |G\rangle_{\text{sys}}$. Here, $O_j(t_j) = \mathcal{U}^\dagger(t_i) O_j \mathcal{U}(t_i)$ is given by the emitter operators $O_j = O_{k,j,\pm}$ and $O_{k,j,\pm}$.

By the quantum regression theorem, as shown in Ref. 36, the bath degree of freedom can be traced out and the correlator

\[
\mathcal{G}(T) = \text{sys} \langle G | T \gamma_{\text{out}}(t_f) O_{\text{eff},1}(t_1) ... O_{\text{eff},n}(t_n) \gamma_{\text{in}}^\dagger(t_i) | G \rangle_{\text{sys}}
\]

becomes the average value of emitter operators $O_{\text{eff},j}(t) = \mathcal{U}_{\text{eff}}^\dagger(t) O_j \mathcal{U}_{\text{eff}}(t)$ on the ground state $|G\rangle_{\text{sys}}$. The time-evolution operator $\mathcal{U}_{\text{eff}}(t) = \exp(-iH_{\text{eff}}t)$ is given by the non-Hermitian effective Hamiltonian $H_{\text{eff}} = H_{\text{sys}} + H_{\text{decay}}$, where

\[
H_{\text{decay}} = -i \sum_i \Gamma_{f,i} \hat{O}_{i}^\dagger \hat{O}_{i} - i \sum_{ij} \sqrt{\Gamma_{f,i} \Gamma_{f,j}} O_{j} e^{i\theta_{j}} |x_i - x_j|,
\]

(21)

and $\Gamma_{f,i}$ is decay rate to the free space of the emitter at the position $x_i$.

Based on the quantum regression theorem, each term in $\mathcal{A}(T)$ is related to the correlator of emitter operators $O_{\text{eff},j}(t)$ governed by the effective Hamiltonian $H_{\text{eff}}$. The re-summation of these terms results in $\mathcal{A}(T) = F_{out} F_{in} A_J(T)$ with the compact form

\[
A_J(T) = \text{sys} \langle \varphi_{out} | \mathcal{U}_{\text{eff}} | \varphi_{in}\rangle_{\text{sys}} e^{\sum_{k,\alpha=r,l} |J_{k\alpha}|^2 e^{-i e_{k\alpha} \cdot T}},
\]

(22)

where $\mathcal{U}_{\text{eff}} = \mathcal{T} e^{-i \int_{t_f}^{t_i} dt [H_{\text{eff}} + H_{\text{in}}]}$. The equations (14) and (22) establish the relation between the transition amplitude $\mathcal{A}(T)$ and the time ordered correlators of emitters governed by the effective Hamiltonian $H_{\text{eff}}$.

By different choices of $\gamma_{\text{in}}, \gamma_{\text{out}}, t_f, t_f, \mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$, the dynamics of the global system can be fully characterized. Hereafter, we refer the choice of $\gamma_{\text{in(out)}}, t_i(t_f)$, and $\mathcal{F}$(in(out)) as the boundary condition. For instance, $\mathcal{A}(T)$ with the boundary condition $\gamma_{\text{in}} = \gamma_{\text{out}} = I$ (identity operator), $t_f \to -\infty$, and $t_f \to \infty$ leads to the photonic S-matrix. For the boundary condition $\gamma_{\text{in}} = I$, $\gamma_{\text{out}} \neq I$, $t_f = 0$, and $t_f = T$, $\mathcal{A}(T)$ describes how the incident photons transform to the emitter excitations and propagate in the transient regime. For the boundary condition $\gamma_{\text{in}} \neq I$, $\gamma_{\text{out}} = I$, $t_i = 0$, and $t_f = T$, the behaviors of spontaneous and stimulated emissions can be investigated by $\mathcal{A}(T)$. Using Eqs. (14), (21), and (22), together with different boundary conditions, we shall study the few photon scattering process in the five models in Secs. IV-VII.
B. Generalized master equation

In order to study the dynamics of emitters during the scattering, one can either use the transition amplitude \[ |J(0)|^2 \] with proper boundary conditions or derive the master equation of emitter reduced density operator by tracing out the photonic degree of freedom. In quantum optics, the initial states of the photonic bath are usually the vacuum state, the thermal state, and Gaussian states, where the conventional master equation is obtained based on the Born-Markov approximation.

During the scattering process, the initial state of photons is dramatically changed. For instance, the single incident photon resonant with the transition energy of the two-level emitter is totally reflected, as predicted by the single photon scattering theory \[ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \], where the photon in the initial asymptotic state \( r_k^\dag |0 \rangle \) is totally scattered to the photon in the out-going asymptotic state \( l_k^\dag |0 \rangle \). However, the conventional master equation assumes that the initial state of the photonic bath is unchanged, and the state of the global system is always the product state of the emitter and the photonic bath \[ |J(k)\rangle \]. This assumption contradicts with the exact result of the single photon scattering theory, thus, the conventional master equation breaks down. The generalized master equation is required to investigate the dynamics of emitters in the few-photon scattering process.

For the initial state of few photons in the waveguide, the evolution of the system state is described by the reduced density matrix \( \rho_s(T) = T_{\text{rath}}[U|\rho(0)|U^\dag] \), where in terms of the coherent state \( |J(k)\rangle \)

\[
\rho(0) = F_{\text{in}}^\ast F_{\text{in}} |\rho_{\text{sys}}(0)\rangle \otimes \{\{J(k)\}\}_b \{\{J(k)\}\}_a.
\] (23)

The displacement transformation \( U \) relates the density matrix

\[
\rho_s(T) = F_{\text{in}}^\ast \sum_{k,a} |J(k)\rangle \langle J(k)| \rho_s(T)
\] (24)

to the generating density matrix

\[
\rho_s(T) = T_{\text{rath}}[U|\rho_{\text{sys}}(0)|U^\dag] \langle J(k)|_b \{\{J(k)\}\}_a,
\] (25)

where \( U_J = U^\dag U \) describes the evolution of emitters under the driving field. In \( \rho_s(T) \), the initial state of photons is transformed to the vacuum state, thus, the conventional master equation can be used to describe the time evolution of \( \rho_s(T) \). Under the Markov approximation, the master equation governing the evolution of \( \rho_s(T) \) reads

\[
\partial_T \rho_s(T) = -i[H_{\text{sys}}, \rho_s(T)] + \mathcal{L} \rho_s(T),
\] (26)

where the initial condition is \( \rho_s(0) = \rho_{\text{sys}}(0) \). The displacement transformation induces the driving term in

\[
H_{\text{sys}} = H_{\text{sys}} + \sum_{i,j} \sqrt{\Gamma_i \Gamma_j} O_i^\dag O_j \sin(k_0 |x_i - x_j|)
+ \sum_{k,a} (J_{k,a}^* O_{k,a} e^{i \sigma_a kT} + J_{k,a} O_{k,a}^\dag e^{-i \sigma_a kT}).
\] (27)

and the Lindblad operator is

\[
\mathcal{L} \rho_s(T) = 2 \sum_{ij} \sqrt{\Gamma_i \Gamma_j} O_i^\dag O_j \cos[k_0 (x_i - x_j)]
- \sum_{ij} \sqrt{\Gamma_i \Gamma_j} \cos[k_0 (x_i - x_j)] \{O_i^\dag O_j, \rho_s(T)\}
+ \sum_i \Gamma_f,i [2 O_i^\dag \rho_s(T) O_i - \{O_i^\dag O_i, \rho_s(T)\}].
\] (28)

The generalized master Eqs. (24) and (26) are the main results in this section, which lead to some significant results: (a) In principle, once \( \rho_s(T) \) is obtained, \( \rho_s(T) \) is determined through Eq. (26), which describes the transient dynamics of emitters during the scattering processes. In practice, the evolution of emitters for the few incident photons can be studied by the perturbative expansion of the classical sources \( J_{k,a} \) and \( J_{k,a}^* \). For instance, for the single incident photon, Eq. (24) contains at most the second order derivative \( \delta^2 \rho_s(T)/\delta J_{\text{sys}}^2 \), the second order perturbative expansion of \( J_{k,a} \) and \( J_{k,a}^* \) in \( \rho_s(T) \) determines \( \rho_s(T) \). Here, the zero order and second order contributions lead to the emitter reduced density matrix for the single incident photon. The zero order contribution \( \rho_{\text{sys}}(0) \) describes the unchanged emitter state without interacting with the photon, while the second order contribution \( F_{\text{in}}^\ast F_{\text{in}} \rho_s(T) \) describes the dynamics of emitters responding to the single incident photon. Similarly, for two incident photons, the reduced density matrix can be determined by the forth order expansion of \( J_{k,a} \) and \( J_{k,a}^* \).

(b) In quantum optics, we are interested in the transmission properties of weak probe light to emitters, i.e., the transmission spectrum and the quantum statistics. We notice that Eq. (26) just describes the emitters under the classical probe light with the strength \( J_{\text{sys}} \). If the driving field is weak, we can solve Eq. (26) by the perturbative expansion of \( J_{k,a} \) and \( J_{k,a}^* \). As we discussed above, the second and forth order perturbative expansions describe the transient dynamics of emitters in the single and two photon scattering processes. As a result, the quantum properties of out-going photons and emitters under the weak driving light can be studied by the few photon scattering theory, where only the time evolution in the forward path is involved, as shown in Eq. (26). We shall show in Sec. VII that for the emitters with complicated structures the scattering theory is still able to provide the analytic results, which perfectly agree with the numerical results from the master equation approach. The few photon scattering theory enables us to analytically investigate the quantum properties of photons from emitters under the weak driving field.

(c) If we set the external source to be zero, Eq. (26) agrees with the master equation \[ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \] for the photonic bath initially in the ground state.
C. Exact S-matrix by path integral approach

The equations (14), (21), and (22) in Sec. IIIA relate the transition amplitude to the correlators of emitters governing by the effective Hamiltonian, where under the Markov approximation the quantum regression theorem leads to the instantaneous effective Hamiltonian. In this section, we show the exact result of the transition amplitude (14), from which the result obtained by the quantum regression theorem is proven to exact for the single emitter coupled to the waveguide. The validity of the Markov approximation and some non-Markov effects for several emitters coupled to the waveguide will be investigated by the exact transition amplitude in Sec. VI.

Following the procedure of path integral formalism, we discretize the evolution time $T$ by $N \rightarrow \infty$ instants $\{t_1, \ldots, t_N\}$ and insert coherent state basis at each instant $t_i$. By integrating out the free space modes we obtain

$$A_J(T) = \int D[\text{system}] \gamma_{\text{out}}(t_f) \gamma^*_{\text{in}}(t_i) e^{iS_{\text{sys}}} \times \int D[\alpha_k, \alpha^*_k] e^{\sum_{k, \alpha} J_k, \alpha(t_f) - i\Gamma} e^{iS},$$

(29)

where the action

$$S = \int_{t_i}^{t_f} dt \left\{ \sum_k \left[ r_k^* (i\partial_t - k) r_k + l^*_k (i\partial_t + k) l_k \right] - H_{\Gamma} \right\}$$

(30)

describes free propagation of the right- and left- moving photons in the waveguide and the interaction with the emitters, and $\int D[\text{system}]$ is the integral over the emitter field.

The action of the emitter is

$$S_{\text{sys}} = S_{\text{sys}}^{(0)} + i \int_{t_i}^{t_f} dt \sum_i \Gamma_{f,i} \hat{O}^\dagger_i \hat{O}_i,$$

(31)

where the first term is the action of emitters, and the second term describes the decay to the free space.

By $\delta S/\delta r^*_k = 0$ and $\delta S/\delta l^*_k = 0$, the classical motion equations read

$$(i\partial_t - k) r_{k,cl} - O_{k,+} = 0,$$

$$(i\partial_t + k) l_{k,cl} - O_{k,-} = 0,$$

(32)

which give the classical paths

$$r_{k,cl}(t) = J_{k,e} e^{-ik(t-t_i)} - i \int_{t_i}^{t} ds O_{k,+}(s) e^{-ik(t-s)},$$

$$l_{k,cl}(t) = J_{k,e} e^{ik(t-t_i)} - i \int_{t_i}^{t} ds O_{k,-}(s) e^{ik(t-s)}.$$  

(33)

Following the saddle point method, we expand the photon fields $r_k = r_{k,cl} + \delta r_k$ and $l_k = l_{k,cl} + \delta l_k$ by the quantum fluctuation fields $\delta r_k$ and $\delta l_k$ around the classical paths, and integrate out the fluctuation fields in $A_J(T)$. Due to the quadratic form of the action (30),

$$A_J(T) = \exp \left( \sum_{k, \alpha} J_{k,\alpha}^2 e^{-i\sigma_k, kT} \right)$$

$$\times \int D[\text{system}] \gamma_{\text{out}}(t_f) \gamma^*_{\text{in}}(t_i) e^{iS_{\text{sys}} + iS_{\text{re}}},$$

(34)

is obtained exactly, where the nonlocal effective action $S_{\text{eff}} = S_{\text{sys}} + S_{\text{re}}$ of emitters is determined by

$$S_{\text{re}} = i \int_{t_i}^{t_f} dt \int_{t_i}^{t} ds \left\{ O_{k,+}^*(t) O_{k,+}(s) e^{-ik(t-s)} + O_{k,-}^*(t) O_{k,-}(s) e^{ik(t-s)} \right\},$$

(35)

and the source terms $S_J = - \int_{t_i}^{t_f} dt H_{\text{eff}}(t)$.

The equations (34) and (35) are the central results in this section. It follows from Eq. (14) that the functional derivatives of $A_J(T)$ lead to the transition amplitude $A(T)$. We also notice that, apart from the phase factor, $A_J(T)$ is the generating functional (53) of the emitter correlators, namely, the functional derivatives of $A_J(T)$ give the correlation functions of emitter operators. As a result, the transition amplitude $A(T)$ is determined by the correlation functions of emitter fields, where the effective action $S_{\text{eff}}$ is generally time nonlocal.

For the single emitter coupled to the waveguide with the linear dispersion, the action $S_{\text{re}}$ turns out to be time-local, and the effective Hamiltonian $H_{\text{eff}}$ is deduced. Here, the renormalized part (35) has the time-local form

$$S_{\text{re}} = i \Gamma \int_{t_i}^{t_f} dt O^*(t) O(t),$$

(36)

and effective Hamiltonian

$$H_{\text{eff}} = H_{\text{sys}} - i \Gamma f \hat{O}^\dagger \hat{O} - i \Gamma \hat{O}^\dagger O$$

(37)

of emitters is obtained. In fact, the functional integral in Eq. (31) is just the amplitude $\langle \varphi_{\text{out}} | U_{\text{eff}} | \varphi_{\text{in}} \rangle$ in Eq. (22), thus, the result based on the quantum regression theorem agrees with the exact result (34).

For several emitters, the renormalization part

$$S_{\text{re}} = i \int_{t_i}^{t_f} dt \sum_{ij} \sqrt{T_i} O^*_i(t) O_j(t-|x_i - x_j|) e^{ik_0|x_i - x_j|}.$$  

(38)

is time nonlocal. If the wavelength of frequency fluctuations around $k_0$ is much larger than the size $Nd$ of the emitter array, the time delay effect can be neglected, which leads to the time-local action

$$S_{\text{re}} = i \int_{t_i}^{t_f} dt \sum_{ij} \sqrt{T_i} O^*_i(t) O_j(t) e^{ik_0|x_i - x_j|},$$

(39)

and the effective Hamiltonian (21). In this limit, the result (35) agrees with the exact result (34) for several emitters.

In the end of this section, we briefly summarize the results. We use two approaches to obtain the transition amplitude and the generalized master equation for the photon scattering process, where the derivation of the generalized master equation (20) from the path integral approach is left in the Appendix. The relation between the few photon scattering by emitters and the emitters...
under weak driving light is established. For the single emitter case, the result [22] based on the quantum regression theorem is proven to be exact. For several emitters, in the Markovian limit, the result [22] agrees with the exact result from the path integral approach.

IV. PARADIGMATIC EXAMPLES

In this section, as paradigmatic examples, the few-photon scattering by the single emitter is studied by the scattering theory developed in Sec. III. The emitter is chosen to be the two-level emitter or the JC system, where the decay to the free space is neglected, i.e., $\Gamma_f = 0$. For the two-level emitter, we investigate the transmission spectra and the quantum statistics of outgoing photons for the single and two incident photons. For the emitter initially in the excited state without the incident photon, the spontaneous emission of the emitter is studied. The transient process, i.e., the response of the two-level emitter in the ground state to the single photon wave-packet, is analyzed by the exact transition amplitude of the emitter in the excited state. For the stimulated emission of the two-level emitter initially in the excited state, the wave-packet shape of the single incident photon is designed, such that the probability of stimulated emission is maximal. For the JC system, we show the transmission spectra and the quantum statistics of scattering photons. By these two examples, we examine our theory by the comparison with the well-known results in Refs. [20–22, 47], which justifies the correctness of the developed scattering theory.

A. Two-level emitter

For the two-level emitter, $O = \sigma^-$ and the effective Hamiltonian $H_{\text{eff}} = H_{\text{sys}} - i\Gamma\sigma^+\sigma^-$ follows from Eq. (37), where $H_{\text{sys}}$ vanishes in the rotating frame since the transition frequency $\omega_c = k_0$ is resonant with the central frequency of the waveguide. It follows from Eq. (19) that the external source term is

$$ S_J = -\sqrt{\frac{\Gamma}{\pi}} \int_{t_1}^{t_f} dt \sum_{k,\alpha=r,t} [J^*_{k,\alpha}\sigma^-(t)e^{-i\sigma_\alpha k(t_f-t)} + J_{k,\alpha}\sigma^+(t)e^{-i\sigma_\alpha k(t-t_i)}]. $$

(39)

For the scattering of single right-moving photon with the momentum $k$ by the emitter in the ground state, the boundary condition in the asymptotic limit $t_i \to -\infty$ is $\gamma_{\text{in}} = I$ and $F_{\text{in}} = \lim_{(J_k) \to 0} \delta/\delta J_{k,r}$. By the boundary condition $\gamma_{\text{out}} = I$ and $F_{\text{out}} = \lim_{(J_k) \to 0} \delta/\delta J_{p,(r,t)}$ in the asymptotic limit $t_f \to \infty$, Eqs. (14) and (44) lead to the reflection and transmission coefficients

$$ R_k \delta_{p,-k} = \frac{-i\Gamma}{k_0 + i\Gamma} \delta_{p,-k}, $$

(40)

and

$$ T_k \delta_{p,k} = \frac{k_0}{k_0 + i\Gamma} \delta_{p,k}, $$

(41)

in the asymptotic limits $t_i \to -\infty$ and $t_f \to \infty$. For the scattering of two right-moving photons with momenta $k_1$ and $k_2$ by the emitter in the ground state, the boundary condition in the asymptotic limit $t_i \to -\infty$ is $\gamma_{\text{in}} = I$ and $F_{\text{in}} = \lim_{(J_k) \to 0} \delta^2/\delta J_{p_1}\delta J_{p_2}$. By the boundary condition $\gamma_{\text{out}} = I$ and $F_{\text{out}} = \lim_{(J_k) \to 0} \delta^2/\delta J_{p_1}\delta J_{p_2}$ in the asymptotic limit $t_f \to \infty$, it follows from Eqs. (14) and (44) that the $S$-matrix

$$ S_{p_1p_2;k_1k_2} = \Gamma^2 \int_{-\infty}^{+\infty} dt_1dt_2 \langle \sigma^- (t_1') \sigma^- (t_2') | e^{i\sigma_1(t_1) \sigma_1^+(t_2)} \rangle, $$

(42)

of two reflected photons with momenta $-p_1$ and $-p_2$ is given by the time ordered correlator of spin operators. Here, the time evolution $\sigma_{\pm} (t) = \sigma_{\pm} e^{i\Gamma t}$ governed by $H_{\text{eff}}$ determines the four-point correlator, which leads to the $S$-matrix

$$ S_{p_1p_2;k_1k_2} = R_{k_1}R_{k_2} (\delta_{p_1k_1}\delta_{p_2k_2} + \delta_{p_1k_2}\delta_{p_2k_1}) $$

$$ + \frac{i\Gamma^2}{\pi} (p_1 + i\Gamma)(p_2 + i\Gamma)(k_1 + 2\Gamma)(k_2 + 2\Gamma). $$

(43)

The wavefunction of two reflected photons is obtained directly from the Fourier transform of Eq. (43) as

$$ \psi(x_1, x_2) = \frac{1}{2} \int dp_1dp_2 e^{i p_1x_1 + i p_2x_2} S_{p_1p_2;k_1k_2} $$

$$ = e^{iEx}\ R_{k_1}R_{k_2}(\cos(kx) - e^{i(E-E')|x|}) $$

(44)

where the total momentum $E = k_1 + k_2$, the relative momentum $k = (k_1-k_2)/2$, the center of mass coordinate $x_c = (x_1 + x_2)/2$, and the relative momentum $x = x_1 - x_2$. The results for $S_{p_1p_2}$ agree with those in Ref. [20–22].

For two photons with the resonant frequency $k_1 = k_2 = 0$, the reflection coefficients are $R_{k_1} = R_{k_2} = 1$ and the wavefunction

$$ \psi(x_c, x) = 1 - e^{-|x|} $$

(45)

In Sec. III, we prove that the quantum statistics of outgoing photons from the emitter under the weak driving light can be characterized by the two photon wavefunction $\psi(x_c, x)$. More precisely, for the weak driving light with the zero detuning $\Delta_d = \omega_d - \omega_c$, the second order correlation function $g^{(2)}(x) = |\psi(x_c, x)|^2$ of the emitted photons can be obtained by the two photon wavefunction $\psi(x_c, x)$ from the scattering theory. Here, $g^{(2)}(x) = (1 - e^{-|x|})^2$ display the anti-bunching behavior.

The spontaneous decay of the two-level emitter in the excited state can be investigated by the boundary conditions $\gamma_{\text{in}} = \sigma^+$, $F_{\text{in}} = 1$ and $\gamma_{\text{out}} = I$, $F_{\text{out}} = \lim_{(J_k) \to 0} \delta/\delta J_{p,(r,t)}$ at the initial and the final instants.
The response of the two-level emitter in the ground state to the single photon wavepacket is given by the equations (14) and (34) result in the amplitude

$$A(T) = \sqrt{\frac{\Gamma}{\Gamma - \gamma}} e^{-\Gamma(x_0 + T)} - e^{-\gamma(x_0 + T)}|\theta(x_0 + T). \quad (50)$$

In Fig. 2b, we show $|A(T)|^2$ of the emitter in the excited state for different widths $\gamma$.

In the stimulated emission, the two-level emitter is initially prepared in the excited state at $t_i = 0$ and the wavepacket of single right-moving photon is designed to realize the maximal probability of emitting two right-moving photons in the asymptotic limit $t_f \to \infty$.

With the initial and final boundary conditions $\gamma_{in} = \sigma^-, \mathcal{F}_{in} = \lim_{(J_k) \to 0} \int dk f_{in}(k) \delta/\delta J_{k,r}$ and $\gamma_{out} = 1 \mathcal{F}_{out} = \lim_{(J_k) \to 0} \delta^2/\delta J_{p1,r} \delta J_{p2,r}$, Eqs. (14) and (34) result in the $S$-matrix

$$S_{p1,p2,k} = \sqrt{\frac{\Gamma}{2\pi}} \int dk f_{in}(k) [\delta(kp_1 - 1/p_2 + i\Gamma)]$$

$$+ \frac{1}{2\pi} (\frac{1}{p_1 + i\Gamma} - \frac{1}{p_1 - k + i0^+} + \frac{1}{p_2 - k + i\Gamma})$$

$$+(p_1 \leftrightarrow p_2) \quad (51)$$

of two right-moving photons with momenta $p_1$ and $p_2$.

The Fourier transform of $S_{p1,p2,k}$ results in the wavefunction

$$\psi(x_1,x_2) = \int \frac{dp_1 dp_2}{2\pi} S_{p1,p2,k} e^{ip_1 x_1 + ip_2 x_2} \quad (52)$$

in the coordinate space, where

$$\mathcal{B}(x_1,x_2) = -i\sqrt{\Gamma} e^{\Gamma x_2} \theta(-x_2) \delta(x - x_1)$$

and

$$f_{in}(x) = \int \frac{dk}{2\pi} f_{in}(k) e^{ikx} \quad (54)$$

describes the incident wavepacket in the coordinate space.

We design the shape of the wavepacket $f_{in}(x)$ to maximize the stimulate emission probability

$$P_{st} = \frac{1}{2} \int dx_1 dx_2 |\psi(x_1,x_2)|^2 \quad (55)$$

The amplitude (52) leads to

$$P_{st} = \int dx dy f^*_m(x) W(x,y) f_{in}(y) \quad (56)$$

where

$$W(x,y) = \int dx_1 dx_2 B^*(x_1,x_2;x) \times$$

$$\{\mathcal{B}(x_1,x_2;y) + \mathcal{B}(x_2,x_1;y)\} \quad (57)$$
The function $f(x)$ gives

$$W(x, y) = \frac{1}{2} \delta(y-x) + \frac{\Gamma}{4} (3e^{\Gamma(x+y)} - e^{-\Gamma|x-y|}) \theta(-x) \theta(-y).$$

The manifest effect of two photons to the system is realized by the initial wavepacket $f_{in}(x)$ localized at $x < 0$. The largest eigenvalue of $W(x, y)$ gives the maximal probability of stimulated emission, and the corresponding eigenstate determines the shape of single photon wavefunction $f_{in}(x)$.

Fortunately, the eigen-equation, i.e., the integral equation

$$\int_{-\infty}^{0} dy W(x, y) f_{in}(y) = \lambda f_{in}(x) \quad \text{(58)}$$

can be solved exactly. In Eq. (58), the second order derivative to $x$ leads to the differential equation

$$\partial^2_x f_{in}(x) = \frac{\lambda}{\lambda - \frac{\Gamma}{2}} f_{in}(x), \quad \text{(59)}$$

where the solution is

$$f_{in}(x) = \left( \frac{4\Gamma^2 \lambda}{\lambda - \frac{\Gamma}{2}} \right)^{1/4} \exp\left[ \frac{\lambda}{\lambda - \frac{\Gamma}{2}} \Gamma x \right] \theta(-x). \quad \text{(60)}$$

The eigenvalue $\lambda = 2/3$ is obtained by the fact that $f_{in}(x)$ is the solution of Eq. (58). Finally, we conclude that the largest probability of stimulated emission in the two-level emitter is $P_{st}^{\max} = 2/3$, and the corresponding incident wavepacket is $f(x) = 2 \sqrt{\Gamma} e^{\Gamma x} \theta(-x)$.

We notice that this result agrees with that in Ref. [73], where two types of the incident wavepackets, i.e., the Gaussian type and the Lorentzian shape are considered. By tuning the width of the wavepacket, the maximal probability $2/3$ of the stimulated emission was found for the initial wavepacket with the Lorentzian shape. Here, we proved the exact result $P_{st}^{\max} = 2/3$ by solving the integral Eq. (58) analytically.

### B. JC system

In this section, we consider the JC system as the single emitter coupled to waveguide photons, and apply the scattering theory to study the different single processes. The emitter Hamiltonian in the rotating frame reads

$$H_{sys} = \Delta_e a^\dagger a + \Delta_e \langle e | e \rangle + g \langle a^\dagger | g \rangle \langle e | H_c \rangle, \quad \text{(61)}$$

where the detunings are $\Delta_e = \omega_e - \omega_0$ and $\Delta_e = \omega_e - \omega_0$. For convenient, we focus on the resonant case $\Delta_e = \Delta_e = 0$. The effective Hamiltonian $H_{eff} = H_{sys} - i \Gamma a^\dagger a$ follows from Eq. (55). The external source term is

$$S_J = -\sqrt{\Gamma L} \int_{t_i}^{t_f} dt \sum_{k, \alpha = r,l} [J_{k, \alpha} a(t) e^{-i\sigma_\alpha k(t_f - t)} + J_{k, \alpha} a^\dagger(t) e^{-i\sigma_\alpha k(t(t_i - t))}]. \quad \text{(62)}$$

For the single incident photon with the momentum $k$ and the emitter initially in the ground state, the boundary condition is $\gamma_{in} = I$ and $\mathcal{F}_{in} = \lim_{\delta \Delta f_{in} \to 0} \delta / \delta J_{k, r}$ in the asymptotic limit $t_i \to -\infty$. By the boundary condition $\gamma_{out} = I$ and $\mathcal{F}_{out} = \lim_{\delta \Delta f_{out} \to 0} \delta / \delta J_{k, r}$ in the asymptotic limit $t_f \to \infty$, Eqs. (14) and (54) lead to the reflection and transmission coefficients

$$R_k \delta_{p,-k} = \frac{i \Gamma k}{(k + i\Gamma)k - g^2} \delta_{p,-k},$$
$$T_k \delta_{p,k} = \frac{k^2 - g^2}{(k + i\Gamma)k - g^2} \delta_{p,k}. \quad \text{(63)}$$

For the two incident photons with momenta $k_1$ and $k_2$ to the emitter initially in the ground state, the boundary condition is $\gamma_{in} = I$ and $\mathcal{F}_{in} = \lim_{\delta \Delta f_{in} \to 0} \delta / \delta J_{k_1, r} \delta J_{k_2, r}$ in the asymptotic limit $t_i \to -\infty$. By the boundary condition $\gamma_{out} = I$ and $\mathcal{F}_{out} = \lim_{\delta \Delta f_{out} \to 0} \delta / \delta J_{k_1, r} \delta J_{k_2, r}$ in the asymptotic limit $t_f \to \infty$, Eqs. (14) and (54) lead to the S-matrix element

$$S_{p_1 p_2; k_1 k_2} = \Gamma^2 \int_{-\infty}^{+\infty} \frac{dt_1 dt_2 dt_1 dt_2}{(2\pi)^2} e^{ip_1 t_1 + ip_2 t_2 - i k_1 t_1 - i k_2 t_2} \times \left\langle T \{ a(t'_1) a(t'_2) a^\dagger(t_1) a^\dagger(t_2) \} \right\rangle, \quad \text{(64)}$$

of two reflected photons with momenta $-p_1$ and $-p_2$. Here, the correlator for operators $a(t) = e^{iH_{st} t} a e^{-iH_{st} t}$ and $a^\dagger(t) = e^{iH_{st} t} a^\dagger e^{-iH_{st} t}$ leads to the S-matrix element

$$S_{-p_1 -p_2; k_1 k_2} = R_{k_1} R_{k_2} \delta_{p_1 k_1} \delta_{p_2 k_2} + \delta_{p_1 k_2} \delta_{p_2 k_1} + \frac{\Gamma^2 g^4 (E + i\Gamma) \delta_{p_1 + p_2, k_1 + k_2}}{\pi (E + i\Gamma)(E + 2i\Gamma) - 2g^2} \times \frac{E(E + 2i\Gamma) - 4g^2}{\prod_{l=1,2} |k_l(k_l + i\Gamma) - g^2| \prod_{l=1,2} |p_l(p_l + i\Gamma) - g^2|}. \quad \text{(65)}$$

The Fourier transform of Eq. (65) leads to the wavefunction

$$\psi(x, x) = \frac{1}{2} \sum_{p_1 p_2} S_{p_1 p_2; k_1 k_2} e^{ip_1 x_1 + ip_2 x_2} e^{iE x} \{ R_{k_1} R_{k_2} \cos(k x) - \frac{\Gamma^2 g^4}{\lambda_+ - \lambda_-} \sum_{s = \pm} s(E - 2\lambda_s) e^{i(\Delta_\pm - \lambda_\pm) x} \times \prod_{l=1,2} |k_l(k_l + i\Gamma) - g^2| \} \quad \text{(66)}$$

of two photons, where $\lambda_s$ is the solution of $k^2 + i\Gamma k - g^2 = 0$. As we discussed in Sec. III, under the weak driving light with frequency $\omega_d$, the quantum statistics of photons emitting from the JC system can be described by the second order correlation function $g^{(2)}(x) = \langle \psi^2(x, x) \rangle$, where $\psi(x, x)$ is the two photon wavefunction for the incident photons with frequency $k_1 = k_2 = \omega_d - \omega_0$. The above results agree with those in Ref. [22].

As a summary, in this section, we use the single emitter case as the example to show how our theory works in the
V. VALIDITY OF MARKOV APPROXIMATION

In this section, we use the simple example to study the condition of the Markov approximation in the array of two-level emitters. As shown in Secs. III and IV, the dynamics of the single emitter coupled to the waveguide with the linear dispersion can be exactly described by the effective Hamiltonian \( H \) and the generalized master equation \( \dot{S}_{\text{eff}} \). For the array of emitters, the dynamics is exactly characterized by the effective time-nonlocal action \( S_{\text{eff}} \), where the effective Hamiltonian \( H \) and the generalized master equation \( \dot{S}_{\text{eff}} \) only describe the emitter evolution under the Markov approximation.

In order to justify the validity of Markov approximation, we compare the exact result given by the path integral approach and the approximate result based on the quantum regression theorem. This comparison shows that the exact result and Markovian limit coincide when the bandwidth of the dynamics is sufficiently small compared to the distance between emitters.

We focus on the single- and two-photon scattering processes, where the effective action

\[
S_{\text{eff}} = \int d\omega \sum_{ij} b_{ij}^\dagger(\omega)[i\omega I - H_0(\omega)]_{ij} b_{ij}(\omega) - \frac{U_0}{2} \int dt \sum_j b_j^\dagger(t)b_j^\dagger(t)b_j(t)b_j(t)
\]

of emitters is given by the matrix with elements \([H_0(\omega)]_{ij} = -i\Gamma f \delta_{ij} - i\Gamma e^{i(k_0 + \omega)|x_i - x_j|} \). The driving action \( S_J = -\int_{t_i}^{t_f} dt H_0(t) \) is given by Eq. (19) with \( O_{k,\sigma} \) and \( I_0 \) lead to the exact results of the transmission spectrum and the second order correlation function by the effective action \( S_{\text{eff}} \) and \( S_J \). Under the Markov approximation, Eqs. (14) and (34) give the approximate results by the effective Hamiltonian \( H_0(\omega) \sim H_0(0) \equiv H_0^M \) and \( O_{k,\sigma} \sim O_{0,\sigma} \).

A. Single photon processes

For the single incident photon with the momentum \( k \) and the emitters initially in the ground state, the boundary condition is \( \gamma_{\text{in}} = I \) and \( \mathcal{F}_{\text{in}} = \lim_{J_{\text{in}} \to 0} \delta/\delta J_{k,x} \) in the asymptotic limit \( t_i \to -\infty \). It follows from Eqs. (14) and (34) that the reflection coefficient

\[
R_k = -i\Gamma \sum_{ij} [G(k)]_{ij} e^{i(k + k_0)(x_i + x_j)}
\]

of the photon with momentum \( p = -k \) is determined by the boundary condition \( \gamma_{\text{out}} = I \) and \( \mathcal{F}_{\text{out}} = \lim_{J_{\text{in}} \to 0} \delta/\delta J_{k,x} \) in the asymptotic limit \( t_f \to \infty \), where the Green function matrix \( G(k) = [k - H_0(k)]^{-1} \). In the Markovian limit, the reflection coefficient

\[
R_k^M = -i\Gamma \sum_{ij} [G^M(k)]_{ij} e^{i(k_0 + \omega)|x_i - x_j|}.
\]

is determined by the approximate Green function \( G^M(k) = [k - H_0(0)]^{-1} \).

The difference of Eqs. (65) and (66) is that the phase factor \( e^{ik_0x} \) in \( H_0(k) \) is neglected in \( G^M(k) \), which results in the condition \( kNd \ll 1 \) of the Markov approximation in the single photon scattering. In Fig. 3, we show the reflection probabilities \( |R_k|^2 \) and \( |R_k^M|^2 \) of the scattering photon by two atoms with lattice spacing \( d \), where \( k_0d = 2\pi n \) and \( n \) is an integer. The approximate result \( |R_k^M|^2 \) does not depend on \( d \), which agrees with the exact result \( |R_k|^2 \) very well for small \( kNd \ll 1 \). In the non-Markovian regime \( kNd > 1 \), the dotted (black) curve \( d = 1 \) shows that the reflection probability has the peaks localized around \( n_0\pi/d \) and \( n_0 \) is an integer. The positions of these peaks are the resonant frequencies of eigenmodes in the “cavity” formed by the two emitters.

The behavior of single excitation propagation in the emitter array can also be analyzed by Eqs. (14) and (34), where the exact result gives the condition of the Markov approximation. For the first emitter initially in the excited state and the rest emitters in the ground state, the boundary conditions are \( \mathcal{F}_{\text{in}} = I \) and \( \gamma_{\text{in}} = b_1 \) at the initial instant \( t_i = 0 \). The propagation of single excitation in the emitter array is described by the boundary condition \( \mathcal{F}_{\text{out}} = I \) and \( \gamma_{\text{out}} = b_1 \) at the final instant \( t_f = T \). The equations (14) and (34) lead to the amplitude

\[
A(j, T) = i \int \frac{d\omega}{2\pi} \frac{1}{\omega - H_0(\omega)} j_1 e^{-i\omega T}
\]

of detecting the excitation at the \( j \)-th emitter. The single photon emitted by the excitation of the first emitter is
described by the amplitude
\[
A_\alpha(x, T) = -i\sqrt{T} \sum_j e^{-i\alpha k_0 x_j} \theta(\sigma_\alpha(x - x_j)) \times \theta(T - \sigma_\alpha(x - x_j)) A(j, T - \sigma_\alpha(x - x_j))
\] (71)
of detecting a single photon at the position \(x\) in the waveguide, where \(\alpha\) denotes the right- and left-moving modes, and the boundary condition is \(F_{\text{out}} = \lim_{t_f \to T} \delta / \delta J_{k_e(x, i)}\) and \(\gamma_{\text{out}} = I\) at the final instant \(t_f = T\). In the Markovian limit, the amplitudes become
\[
A^M(j, T) = i \int \frac{d\omega}{2\pi} \left[ \frac{1}{\omega - \mathcal{H}_0(0)} \right] j_1 e^{-i\omega T},
\] (72)
and
\[
A^M_\alpha(x, T) = -i\sqrt{T} \sum_j e^{-i\alpha k_0 x_j} \theta(\sigma_\alpha(x - x_j)) \times \theta(T - \sigma_\alpha(x - x_j)) A^M(j, T - \sigma_\alpha(x - x_j)).
\] (73)

To investigate the non-Markov effects and justify the validity of the Markov approximation, we compare the exact results and the approximate results. It follows from the residue theorem that the poles \(\xi_\lambda\) and the corresponding residues \(Z_\lambda\) of \(1/\omega - \mathcal{H}_0(\omega)\) determine the time evolution \(A(j, T) \sim \sum \xi_\lambda e^{-ix_\lambda^T}\). Similarly, \(A^M(j, T) \sim \sum \xi_\lambda^M e^{-ix_\lambda^{MT}}\) is given by the poles \(\xi_\lambda^M\) and the corresponding residues \(Z_\lambda^M\) of \(1/\omega - \mathcal{H}_0(0)\). Under the condition \(\xi_\lambda^M N d \ll 1\), the phase factor \(e^{i\xi_\lambda^M|x - x_j|} \sim 1\), and \(\xi_\lambda^M\) and \(Z_\lambda^M\) are approximately the pole \(\xi_\lambda\) and the corresponding residue \(Z_\lambda\) of the exact propagator \(1/\omega - \mathcal{H}_0(\omega))\). As a result, the condition of the Markov approximation is \(\xi_\lambda^M N d \ll 1\).

To understand this condition, we consider two emitters in the waveguide, where the first emitter is placed at the origin, and the effective action is determined by the \(2 \times 2\) matrix
\[
\mathcal{H}_0(\omega) = \begin{pmatrix}
-i\Gamma_f - i\Gamma & -i\Gamma e^{ik_0 + \omega d} \\
-i\Gamma e^{ik_0 + \omega d} & -i\Gamma - i\Gamma_f
\end{pmatrix}.
\] (74)
It follows from Eqs. (70) and (71) that the exact amplitudes are
\[
A(1, T) = \frac{1}{2} e^{-i(\Gamma_f + \Gamma)T} [C_+ (T) + C_- (T)],
\]
\[
A(2, T) = \frac{1}{2} e^{-i(\Gamma_f + \Gamma)T} [C_- (T) - C_+ (T)],
\] (75)
and
\[
A_\alpha(x, T) = -i\sqrt{T} \theta(\sigma_\alpha x) \theta(T - \sigma_\alpha x) A(1, T - \sigma_\alpha x) + e^{-i\alpha k_0 x} \theta(\sigma_\alpha x - d) \theta(T - \sigma_\alpha x - d) \times A(2, T - \sigma_\alpha x + \sigma_\alpha d)],
\] (76)
where
\[
C_{\pm}(T) = \sum_{n=0}^{\infty} \frac{1}{n!} (\pm i\Gamma e^{ik_0dT + (\Gamma_f + \Gamma)T} (T - nd))^{n} \theta(T - nd).
\] (77)
A by approximation, the probability $P_A = 0$, and Markovian results are shown by the solid (blue) curves. Here, $\Gamma_f = 0$ and $\Gamma$ is taken as the unit. $\Gamma_f = 0$, both the exact and Markovian results show that in the steady state $T \to \infty$ the emitters have the probability $P_e$ to form the entangle state. Under the Markov approximation, the probability $P_e = 1/2$ is determined by $A^M(1, \infty) = -A^M(2, \infty) = 1/2$. The exact result $A(1, \infty) = -A(2, \infty) = 1/(2 + 2\Gamma d)$ gives the probability $P_e = 1/[2(1 + \Gamma d)^2]$ to form the entangle state. In the limit $\Gamma d \ll 1$, the Markov approximation works perfectly, i.e., $A(j, \infty) \sim A^M(j, \infty) = 1/2$.

During the formation of the entangled state, the dynamics of the waveguide photon is described by the amplitudes $A_r(t, x, T)$. In Fig. 3 we show the propagation of the right- and left- moving wavepackets by $A_r(t, x, T)$ for the distance $d = 1$. In the steady state $T \to \infty$, the entangle state is established, where the standing wave in the regime $0 < x < d$ forms to mediate the interaction of two emitters.

B. Two photon processes

In this section, we study the scattering process of two incident photons with momenta $k_1$ and $k_2$. By comparing the exact result and the approximate result, we investigate the condition of the Markov approximation in the two-photon processes.

The initial boundary condition is $\gamma_{in} = I$ and $F_{in} = \lim_{t_i \to -\infty} \delta^3 / \delta x_i r \delta k_{1r} \delta k_{2r}$ in the asymptotic limit $t_i \to -\infty$. It follows from Eqs. (14) and (51) that the $S$-matrix

$$S_{p_1p_2,k_1k_2} = R_{k_1} R_{k_2} (\delta_{p_1k_1} \delta_{p_2k_2} + \delta_{p_2k_1} \delta_{p_1k_2}) \quad (79)$$

of two reflected photons with momenta $-p_1$ and $-p_2$ are determined by the boundary condition $\gamma_{out} = I$ and $F_{out} = \lim_{t_f \to \infty} \delta^3 / \delta J_{-p_1} \delta J_{-p_2}$ in the asymptotic limit $t_f \to \infty$. Here,

$$\bar{G}_i(p_1, p_2) = \sum_{i_1 i_2} G_{i_1 i_2}^2(p_1) G_{i_2 i_1}(p_2) e^{i p_1 x_{i_1} + i p_2 x_{i_2}} , \quad (80)$$

and the $T$-matrix $T(E) = -\Pi^{-1}(E)$ is given by the Dyson Expansion with the bubble

$$\Pi_{ij}(E) = i \int \frac{d\omega}{2\pi} G_{ij}(\omega) G_{ij}(E - \omega) \quad (81)$$

and $E = k_1 + k_2$.

Under the Markov approximation, $S_{p_1p_2,k_1k_2}$ is given by the same form of Eq. (79), where $R_{k_1} \sim R_{k_2}$.

$$\bar{G}_i(p_1, p_2) \sim \sum_{i_1 i_2} G_{i_1 i_2}^M(p_1) G_{i_2 i_1}^M(p_2) e^{i k_0 (x_{i_1} + x_{i_2})} \quad (82)$$

and the bubble (51) are determined by the approximate Green function $G^M(\omega) \sim [\omega - \hbar \omega(0)]^{-1}$. The exact result shows that the Markov approximation is valid in the limit $k_i N d, p_i N d$, and $\Gamma N d \ll 1$.

In order to understand the condition, we investigate the two-photon scattering processes explicitly by considering the photon scattering by two emitters, where $k_0 d = 2\pi n$. We first compare the exact $T$-matrix $T(E)$ with the $T$-matrix $T^M(E)$ under the Markov approximation, where the elements $\Pi^M_{ij}(E) = \Pi^M_{ij}(E) = \Pi^M_{ij}(E) + \Pi^M_{ij}(E)$ and $\Pi^M_{ij}(E) = \Pi^M_{ij}(E) - \Pi^M_{ij}(E)$ are given by

$$\Pi^M_{ii}(E) = \frac{1}{4} \sum_{\sigma = \pm 1} \frac{1}{E + 2i \Gamma f + 2i \Gamma (1 + \sigma e^{ik_0 d})} ,$$

$$\Pi^M_{ij}(E) = \frac{1}{2} \frac{1}{E + 2i \Gamma f + 2i \Gamma} . \quad (83)$$

The absolute values of the exact $T$-matrix elements are shown in Figs. 4 and 5 for $k_0 d = 2\pi n$, which illustrate the perfect agreement between the Markovian result $T^M(E)$ and the exact result in the limit $Ed, \Gamma d \ll 1$.

The Fourier transformation of $S$-matrix leads to the wavefunction

$$\psi(x, x) = \frac{1}{2} \int dp_{1} dp_{2} S_{p_1p_2,k_1k_2} e^{i p_1 x_1 + i p_2 x_2} \quad (84)$$

of two reflected photons, where the center of mass coordinate $x_c = (x_1 + x_2)/2$ and the relative coordinate $x = x_1 - x_2$. For two incident photons with the same
momentum $k_1 = k_2 = k$, the photon statistics is characterized by the second order correlation function

$$g^{(2)}(\tau) = \frac{\langle \psi(x, \tau) \psi(x, 0) \rangle^2}{\langle \psi(x, \tau) \psi(x, 0) \rangle}.$$  \hfill (85)

In Figs. [5] and [6], we show $g^{(2)}(0)$ as the function of $E = 2k$ and $g^{(2)}(\tau)$ for the resonant frequency $k = 0$, where the Markovian results perfectly agree with the exact result in the small $kd$ and $\Gamma d$ limit. When $d$ is increasing, e.g., $\Gamma d = 0.1$ and 1, the Markovian result deviates from the exact one.

As the summary in this section, we use the exact result to study the non-Markovian effects and examine the condition of the Markov approximation. We conclude that in the limit $\Gamma N d, k N d \ll 1$, the Markov approximation works perfectly. For the large $\Gamma N d$ and $k N d$, some non-Markov effects emerge, e.g., the retardation effect. In the following sections, we assume the system length $N d$ is small enough and the Markov approximation is always valid.

VI. ENTANGLED PHOTON PAIRS BY SINGLE EMITTER

In this section, we use the scattering theory to study the generation of entangled photons in the scattering process. In order to realize the deterministic generation of reflected photons, the two-level emitter is placed at the left hand side of a perfect mirror, i.e., in the half-end waveguide, as shown in Fig. [7]. Here, the mirror is put at the origin, and the position of the emitter is $x_0 < 0$.

It follows from Eq. (33) that the effective action is

$$S_{\text{eff}} = S_{\text{sys}} + i \int dt \Gamma^\dagger e(t) e(t) + i \int dt \Gamma b^\dagger(t) b(t)$$

$$+ i \sqrt{\Gamma_0} \int dt b^\dagger(t) e(t + x_0) e^{-i k_0 x_0}$$

$$+ i \sqrt{\Gamma_0} \int dt e^\dagger(t) b(t + x_0) e^{-i k_0 x_0},$$  \hfill (86)

where the emitter action $S_{\text{sys}}$ is determined by the Hamiltonian $H_{\text{sys}} = H_{\text{emitter}}$. By the creation (annihilation) operator $e^\dagger(e)$, the hardcore boson is introduced to describe the two-level emitter, where the energy level spacing is $\omega_e = k_0$ and $H_{\text{emitter}} = U_0 e^\dagger e / 2$ in the limit $U_0 \to \infty$. The external source term $S_{J} = - \int dt H_\Lambda(t)$ is given by Eqs. (19) and (6) with the jump operator $O = e$.

A. Single- and two- photon scattering

For the single incident photon with the momentum $k$ and the emitter initially in the ground state, the boundary condition is $\gamma_{\text{in}} = I$ and $F_{\text{in}} = \lim_{t_1 \to 0} \delta / \delta J_{k,r}$ in the asymptotic limit $t_1 \to -\infty$. By the boundary condition $\gamma_{\text{out}} = I$ and $F_{\text{out}} = \lim_{t_2 \to 0} \delta / \delta J_{p,l}$ in the asymptotic limit $t_2 \to \infty$, Eq. (14) and (34) result in the reflection coefficient

$$R_{k} \delta_{p,-k} = -i \delta_{p,-k} [\Gamma G_{ee}(k) e^{2i(k+k_0)x_0} + \Gamma G_{eb}(k)$$

$$+ 2 \sqrt{\Gamma \Gamma_0} G_{eb}(k) e^{(k+k_0)x_0}],$$  \hfill (87)

where the Green functions $G_{ss'}(k) = -i \int dt e^{ikt} (s(t) s^\dagger(t))$ with $s, s' = e, b$. The effective action determines the single particle Green functions

$$\left( \begin{array}{cc} G_{ee}(k) & G_{eb}(k) \\ G_{be}(k) & G_{bb}(k) \end{array} \right)$$

$$= \left( \begin{array}{cc} k + i \Gamma & i \sqrt{\Gamma \Gamma_0} e^{-(i(k+k_0)x_0)} \\ i \sqrt{\Gamma \Gamma_0} e^{i(k+k_0)x_0} & k + \Gamma_0 \end{array} \right)^{-1}$$  \hfill (88)

which give the reflection coefficient

$$R_{k} \delta_{p,-k} = \frac{k - i \Gamma + i \Gamma e^{-2i(k+k_0)x_0}}{k + i \Gamma - i \Gamma e^{2i(k+k_0)x_0}} \delta_{p,-k}$$  \hfill (89)

and $|R_{k}| = 1$ in the limit $\Gamma_0 \to \infty$. The phase shift $\arg(R_{k})$ is shown in Figs. [10] for $d = 10^{-4}$, where $\theta_0 = k_0 |x_0|$. For two incident photons with momenta $k_1$ and $k_2$ and the emitter initially in the ground state, the boundary condition is $\gamma_{\text{in}} = I$ and $F_{\text{in}} = \lim_{t_1 \to 0} \delta^2 / \delta J_{k_1,r} \delta J_{k_2,r}$. In the asymptotic limit $t_1 \to -\infty$. By the boundary condition $\gamma_{\text{out}} = I$ and $F_{\text{out}} = \lim_{t_2 \to 0} \delta^2 / \delta J_{-p_1} \delta J_{-p_2}$ in the asymptotic limit $t_2 \to -\infty$, the S-matrix element

$$S_{-p_1-p_2;k_1,k_2} = R_{k_1} R_{k_2} \left( \delta_{p_1,k_1} \delta_{p_2,k_2} + \delta_{p_1,k_2} \delta_{p_2,k_1} \right)$$

$$- \frac{16}{\pi} \Gamma^2 \delta_{p_1+p_2,k_1+k_2} T(E) \times$$

$$\prod_{i=1,2} \sin[(k_i + k_0) |x_0|] \sin[(p_i + k_0) |x_0|] |p_i + i \Gamma - i \Gamma e^{2i(k_0+k_0)|x_0|} |$$

of two reflected photons with momenta $p_1$ and $p_2$ is given by Eqs. (14) and (34) in the asymptotic limit $t_f \to \infty$. Here, the $T$-matrix element

$$T(E) = \frac{1}{U_0^{-1} - \Pi(E)}$$  \hfill (91)

is determined by the vacuum bubble

$$\Pi(E) = i \int \frac{d\omega}{2\pi} G_{ee}(\omega) G_{ee}(E - \omega)$$  \hfill (92)

and $E = k_1 + k_2 = p_1 + p_2$.

B. Entangled photon pairs

For the state $\sum_{k_1,k_2} f(k_1) f(k_2) r_{k_1}^\dagger r_{k_2}^\dagger |0\rangle / \sqrt{2}$ of two independent incident photons, the $S$-matrix (60) leads to the asymptotic state

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}} \sum_{p_1,p_2} \psi_{\text{out}}(p_1,p_2) |l_{p_1}^\dagger r_{p_2}^\dagger |0\rangle$$  \hfill (93)
VII. RYDBERG-EIT SYSTEM

In this section, we consider the application of the scattering theory in the photon transmission in the EIT atoms coupled to the Rydberg level. We shall show that the developed scattering theory is an efficient approach to the photon transmission in the array of interacting emitters with complicated structures. Here, we highlight the interplay between the EIT phenomenon and the Rydberg interaction, and show a rich variety of quantum statistics of the scattering photons and polariton excitations.

In the rotating frame, the Hamiltonian (7) becomes

\[
H_{\text{sys}} = \sum_i \Delta_e e_i^\dagger e_i + \Delta_s s_i^\dagger s_i + (\Omega e_i^\dagger s_i + \text{H.c.}) + H_{\text{HC}} + \frac{1}{2} \sum_{ij} U_{ij} s_i^\dagger s_j^\dagger s_j s_i.
\]  

(97)

where \(\Delta_e = \omega_e - k_0\), \(\Delta_s = \omega_s - k_0 + \omega_d\), and we focus on the two photon resonance case \(\Delta_e = 0\). The waveguide photons couple to the \(N\) atoms collectively through the operator \(O_{k,\pm} = \sqrt{\Gamma} \sum_{i=1}^N e^{-i(k \pm k_0) x_i} e_i / \sqrt{L}\), and the free space modes couple to the atom operator \(e_i\) locally, which induces the decay of the excited state \(e\). The effective action

\[
S_{\text{eff}} = \int dt \{ \sum_i (e_i^\dagger(t) (i \partial_t + i \Gamma_f) e_i(t) + s_i^\dagger(t) i \partial_t s_i(t)) - H_{\text{sys}} \} + S_{\text{re}},
\]

(98)
of EIT atoms is given by

\[
S_{\text{re}} = i \Gamma \int dt \sum_{ij} e_i^\dagger(t) e_j(t - |x_i - x_j|) e^{i k_0 (x_j - x_i)}.
\]

(99)

The source term \(S_f = - \int_{t_i}^{t_f} dt H_4(t)\) is determined by Eq. (17). Here, the lattice spacing \(d\) satisfies \(k_0 d = (2n + 1/2)\pi\).

Based on the general result (13) and (34), we first study the single- and two-photon scattering, and show the transmission spectrum and the second order correlation function of out-going photons. In the second part, we investigate the transient processes and show how the wavepackets of single and two incident photons transfer to the atom excitations, propagate in the Rydberg-EIT atom array, and finally emit back to the waveguide. For the single incident photon, we show the free propagation of the dark polariton. For two incident photons, we show the counter-propagation and co-propagation of two polaritons.

A. Single and two photon scatterings

For the single incident photon with the momentum \(k\) and the atoms initially in the ground state, the boundary condition is \(\gamma_{in} = I\) and \(F_{in} = \lim_{J_k \to 0} \delta / \delta J_{k,r}\) in the asymptotic limit \(t_i \to -\infty\). By the boundary condition \(\gamma_{out} = I\) and \(F_{out} = \lim_{J_k \to 0} \delta / \delta J_{k,r}\) in the asymptotic limit \(t_f \to \infty\), Eqs. (13) and (34) lead to the reflection and transmission coefficients

\[
R_k \delta p, -k = -i \Gamma \delta p, -k \sum_{ij} |G_0(k)|^2 e^{i (k + k_0) (x_i - x_j)},
\]

(100)

\[
T_k \delta p, k = \delta p k \sum_{ij} |G_0(k)|^2 e^{-i (k + k_0) (x_i - x_j)},
\]

of the photon with momentum \(p\).
FIG. 7: (Color Online) The single photon transmission spectra, where \( \Gamma = 1 \), \( \Delta s = 0 \), \( \Omega = 1 \), and the atom number is 20. (a) \( \Gamma_f = 0 \) and \( d = 10^{-4} \); (b) \( \Gamma_f = 1 \) and \( d = 10^{-4} \); (c) \( \Gamma_f = 0 \) and \( d = 10^{-2} \); (d) \( \Gamma_f = 1 \) and \( d = 10^{-2} \).

The free Green function

\[
G_0(\omega) = \frac{1}{\omega - \mathcal{H}_0(\omega)}
\]

of atoms is determined by the block form

\[
\mathcal{H}_0(\omega) = \begin{pmatrix}
(\Delta_e - i\Gamma_j)\delta_{ij} - i\Gamma_e e^{i(k_0 + \omega)x_j} & \Omega\delta_{ij} \\
\Omega\delta_{ij} & \Delta_s\delta_{ij}
\end{pmatrix}
\]

in the basis \( |e_i\rangle \) and \( |s_i\rangle \), where each matrix element is the \( N \)-dimensional matrix in the coordinate basis. In the notation \( [G_0(k)]_{ij}^{\sigma\sigma'} \), \( \sigma, \sigma' = e, s \) denotes the different “spin” blocks and \( i, j \) denote the coordinate in the block. In the Markov limit \( \omega d \ll 1 \), \( \mathcal{H}_0(\omega) \) is approximated by the frequency independent effective Hamiltonian

\[
\mathcal{H}_0^M = \begin{pmatrix}
(\Delta_e - i\Gamma_j)\delta_{ij} - i\Gamma_e e^{i(k_0 + \omega)x_j} & \Omega\delta_{ij} \\
\Omega\delta_{ij} & \Delta_s\delta_{ij}
\end{pmatrix}
\]

In Fig. 7, the exact transmission probability is compared with that under the Markov approximation. As shown in Figs. 7a and 7b, for the small lattice spacing \( d = 10^{-4} \), the Markov approximation works very well, which leads to the same result as that from Eq. 102. When the lattice spacing \( d = 10^{-2} \) is larger, Fig. 7c shows the difference between the results under the Markov approximation 103 and the exact result 102. In the realistic case, the small length of the array justifies the validity of the Markov approximation. Henceforth, we focus on the Markov limit. In Figs. 7a and 7d, we have taken into account the decay of the excited state \( |e_i\rangle \) to the free space, i.e., \( \Gamma_f \neq 0 \). The single photon transmission shows the EIT nature of atoms, where the total transmission appears at the resonant frequency \( k = 0 \) in the EIT window. The reason for the total transmission is that the resonant photon transforms to the free-propagating dark polariton excitation, which is only the superposition of states \( r_{k'}^{e}(g_i) \) and \( |s_i\rangle \). As a result, the free space decay \( \Gamma_f \) of the state \( |e_i\rangle \) does not affect the total transmission of the resonant dark polariton with \( k = 0 \).

For the two incident photons with momenta \( k_1 \) and \( k_2 \) and the atoms initially in the ground state, the initial boundary condition is \( \gamma_{in} = I \) and \( \mathcal{F}_{in} = \lim_{t_1 \to -\infty} \delta^2/\delta J_{k_1,r}\delta J_{k_2,r} \) in the asymptotic limit \( t_1 \to -\infty \). By the boundary condition of the final state \( \gamma_{out} = I \) and \( \mathcal{F}_{out} = \lim_{t_1 \to -\infty} \delta^2/\delta J_{p_1,r}\delta J_{p_2,r} \) in the asymptotic limit \( t_1 \to \infty \), Eqs. (14) and (34) lead to the two photon \( S \)-matrix

\[
S_{p_1p_2;k_1k_2} = T_{k_1} T_{k_2} (\delta_{p_1k_1}\delta_{p_2k_2} + \delta_{p_1k_2}\delta_{p_2k_1}) + \frac{1}{(2\pi)^2} \sum_{i_1i_2j_1j_2} e^{ik_0(x_{i_1} + x_{j_2} - x_{i_2} - x_{j_1})} \times C_{i_1i_2;j_1j_2}^{\sigma\sigma'} (p_1, p_2; k_1, k_2)
\]

(104)

for two transmitted photons with momenta \( p_1 \) and \( p_2 \). Here, the Green function

\[
C_{i_1i_2;j_1j_2}^{\sigma\sigma'} (p_1, p_2; k_1, k_2)
\]

(105)

is the Fourier transformation of the four-point connected Green function.

The Dyson expansion in terms of the two-body interaction \( H_{HC} + \sum_{ijkl} U_{ijkl} |s_i\rangle \langle s_j|s_i\rangle \langle s_j|/2 \) leads to

\[
S_{p_1p_2;k_1k_2}^{(0)} = \sum_{ijkl} [w(p_1, p_2)_{ij}]_{\sigma\sigma'} [T(E)]_{ij,j'j}^{\sigma\sigma'} [w(k_1, k_2)]_{j'j}^{\sigma\sigma'}
\]

(106)

where we define the independent scattering part \( S_{p_1p_2;k_1k_2}^{(0)} = T_{k_1} T_{k_2} \delta_{p_1k_1}\delta_{p_2k_2} \), and the function

\[
[w(k_1, k_2)]_{ij,j'j'}^{\sigma\sigma'} = \sum_{j_ij_2} e^{ik_0(x_{j_1} + x_{j_2})} [G_0(k_1)]_{ij,j_1}^{\sigma\sigma'} [G_0(k_2)]_{j_2,j'j}^{\sigma\sigma'}
\]

(107)

The \( T \)-matrix, depicted by the ladder diagram in Fig. 8, satisfies the Lippmann-Schwinger equation

\[
[T(E)]_{ij,j'j'}^{\sigma\sigma'} = U_{ij}^{\sigma\sigma'} \delta_{i'j'} \delta_{\sigma\sigma'} + \sum_{ijkl} [\Pi(E)]_{ij,j_1j_2}^{\sigma\sigma',\mu\mu'} [T(E)]_{l_1l_2,j'j'}^{\sigma\sigma',\mu\mu'}
\]

(108)

where the vacuum bubble is

\[
[\Pi(E)]_{ij,j'j'}^{\sigma\sigma',\mu\mu'} = \frac{1}{2\pi} \frac{d\omega}{d\omega} [G_0(\omega)]_{i'j'}^{\sigma\sigma'} [G_0(E - \omega)]_{j'j}^{\mu\mu'}
\]

(109)
In the matrix form, the Lippmann-Schwinger equation is formally solved as

\[ T(E) = \frac{1}{U - \Pi(E)} , \]  

where the vacuum bubble \( \Pi_0(E) = (E - \mathcal{H}_2)^{-1} \) is given by \( \mathcal{H}_2 = \mathcal{H}_0^\text{M} \otimes \mathcal{I}_2 + \mathcal{I}_2 \otimes \mathcal{H}_0^\text{M} \), and the interaction matrix \( U \) has the diagonal element \( U^{ss}_{ij} = U^{ss}_{ij} = U_0 \to \infty \) and \( U^{ps}_{ij} = U_{ij} \) in the basis \( \{ | s_i e_j \rangle, | e_i s_j \rangle \} \).

The wavefunction

\[
\psi(x_c, x) = \int \frac{dp_1 dp_2}{2\pi} S_{p_1 p_2; k_1 k_2} e^{ip_1 x_1 + ip_2 x_2} \\
\times \frac{e^{iEx_c}}{2\pi} \left( 2T_{\frac{E}{2} + k} T_{\frac{E}{2} - k} \cos(kx) \right) - i\Gamma^2 \sum_{\sigma \sigma'} \left[ F(x) \right]_{i\sigma j \sigma'}^{i\sigma' j \sigma'} \\
\times \left[ T(E) \right]_{i\sigma j \sigma'}^{i\sigma' j \sigma'} \left[ w(E/2 + k - E) - k x \right] \right] , \tag{111}
\]

of two transmitted photons is the Fourier transform of \( S_{p_1 p_2; k_1 k_2} \), where

\[
[F(x)]_{i\sigma j \sigma'}^{i\sigma' j \sigma'} = -i \sum_{i\sigma i' \sigma'} \left( \begin{array}{c}
\chi_i \chi_{i'} \chi_l \chi_{l'} \chi_e \chi_{e'}
\end{array} \right) \left( \begin{array}{c}
\delta_{\sigma \sigma' \mu \mu'}
\end{array} \right) \left( \begin{array}{c}
\delta_{\sigma \sigma' \mu \mu'}
\end{array} \right).
\]

is determined by \( x_{c,12} = x_{i1} + x_{i2} \) and the eigenstates \( | \chi_i \rangle \) \((| \chi_i \rangle) \) of \( \mathcal{H}_0^\text{M} \) \((\mathcal{H}_0^\text{M}) \) with the corresponding eigenenergies \( \varepsilon_i \) \((\varepsilon_i) \). Here, \( \langle \sigma | \chi_i \rangle = \chi_i(\sigma \varepsilon_i) \) and \( \langle i \sigma | \chi_i \rangle = \chi_i(i \sigma) \).

By the wavefunction (111), we show the normalized second order correlation function \( g^{(2)}(x) = \left| \frac{\chi_i \chi_i | \chi_i \rangle \langle \chi_i | \chi_i \rangle}{T_{E/2}^2} \right|^2 \) of out-going photons for the two incident photons with the same momentum \( k_1 = k_2 = E/2 \) in Figs. 8b and 8c. For the resonant case \( E = 0 \), Fig. 8b shows the second order correlation functions for the uniform interaction \( U^{ss}_{ij} = C \), the van-der Waals interaction \( U^{ss}_{ij} = C_6 / i - j \), and the dipolar interaction \( U^{ss}_{ij} = C_3 / i - j \), which exhibit the anti-bunching behavior of out-going photons.

It can be understood in the following way. As shown in the left panel of Fig. 8d, if two Rydberg excitations are close to each other, the Rydberg energy levels \( | s_i \rangle \) are shifted. The photons with finite momentum \( E/2 \) realize the two photon resonance with the shifted energy level \( | s_i \rangle \), and can be transmitted. In order to achieve the transmission of photons by the shifted Rydberg levels \( | s_i \rangle \), the two Rydberg excitations prefer to stay next to each other, which induces the bunching behavior of the transmitted photons.

As discussed in Sec. III, the second order correlation function \( g^{(2)}(x) \) obtained by the scattering theory can characterize the quantum statistics of photons emitted by the Rydberg-EIT atoms under the weak driving field. In Ref. 36, the result from the scattering theory and the numerical solution of the master equation for the effective spin model under the weak driving light are compared, where the two results agree with each other perfectly.

B. Propagations of single and two excitations

In this section, we investigate how the single and two incident photons transform to the excitations of the Rydberg atoms, and the propagation of excitations.

For the single incident photon with the wavepacket \( f(k) \) and the atoms initially in the ground state,
the boundary condition is \( \gamma_{in} = I \) and \( \mathcal{F}_{in} = \sum_{k} f_{\gamma}(k) \lim_{|J_\gamma| \to 0} \delta_{J_{k,r}} \delta_{J_{k,s}} \) at the instant \( t_i = 0 \), where the wavepackets of the right- and left- moving photons are

\[
f(k) = f_{\gamma}(k) = \sqrt{\frac{\gamma}{\pi k + i\gamma}}, \quad (113)
\]

and

\[
f(k) = f_{\gamma}(k) = \sqrt{\frac{\gamma e^{-ikx_N}}{\pi k - i\gamma}}, \quad (114)
\]

with the width \( 1/\gamma \). By the final boundary condition \( \mathcal{F}_{out} = 1 \) and \( \gamma_{out} = \mu_i = e_i,s_i \) at the instant \( t_f = T \), Eqs. (14) and (34) lead to the amplitude

\[
A^{(}\gamma_{in},\mu_i(T) = i\sigma_\alpha \sqrt{2\gamma\Gamma} \sum_{l_j} e^{i\sigma\alpha k_{0} x_{l_j}} \theta(T_\alpha - \sigma_\alpha x_{l_j}) \times (115)
\]

\[
\chi_l(i\mu_i(x_{l_j}) = \epsilon_l + i\gamma_l) \left[ e^{-\gamma(T_\alpha - \sigma_\alpha x_{l_j})} - e^{-i\gamma_l(T_\alpha - \sigma_\alpha x_{l_j})} \right],
\]

of the excitation \( \mu = e, s \) at the position \( i \), where \( T_\alpha = T - z_\alpha \) and \( z_\alpha = 0, z_i = x_N \). For this situation, the dark polariton forms, with the probability of the occupation in the excited state \( |e_i\rangle \) is quite small \( \sim 10^{-4} \). In Fig. 9(b) we show the dark polariton propagation for the single right moving incident photon. To order to show the slow propagation of the dark polariton, we choose the small ratio \( \Omega/\Gamma = 0.1 \).

For the wavepacket of two incident photons and the atoms initially in the ground state, we consider both the co-propagation and counter-propagation cases. For the co-propagation, the initial boundary condition at the instant \( t_i = 0 \) is \( \gamma_{in} = I \) and

\[
\mathcal{F}_{in} = \sum_{k_1,k_2} f_{r}(k_1)f_{r}(k_2) \lim_{|J_\gamma| \to 0} \delta_{J_{k_1,r}} \delta_{J_{k_2,t}} \quad (116)
\]

while for the counter-propagation, the initial boundary condition at the instant \( t_i = 0 \) is \( \gamma_{in} = I \) and

\[
\mathcal{F}_{in} = \sum_{k_1,k_2} f_{r}(k_1)f_{l}(k_2) \lim_{|J_\gamma| \to 0} \delta_{J_{k_1,r}} \delta_{J_{k_2,t}} \quad (117)
\]
is shown. When the first photon transforms into the dark polariton in the atom array, it blocks the transmission of the second photon. This blockade occurs over a characteristic distance \( r_b \sim \left( C_2 \Gamma / \Omega_2 \right)^{1/3} \). For an incoming two-photon wavepacket whose size is larger than \( r_b \), the blockade manifests itself as a suppression of the probability of two photons to overlap with each other as they propagate through the medium (see Fig. 10 d-f). During the propagation in the Rydberg-EIT atoms, the two polaritons keep away from each other such that the energy level \( |s_i \rangle \) are not shifted, which results in the transmission of two anti-bunched dark polaritons.

VIII. CONCLUSION

We summarize our results in this section. We developed the scattering theory to investigate propagation of photons through an array of quantum emitters using the path integral approach. The exact transition amplitude for arbitrary initial and final states is obtained to describe the quantum statistics of scattering photons and the dynamics of emitters in the transient process. The exact result justifies the correctness of the Markov approximation for the single emitter coupled to the waveguide photons with linear dispersion. The exact and Markovian results coincide when the bandwidth of the dynamics is sufficiently small compared to the distance between emitters. Here, the generalized master equation for the few-photon scattering process is obtained to describe the transient dynamics of emitters. The generalized master equation establishes the relation between two equivalent systems, i.e., few photon scattering by the emitters and the emitters under weak driving light.

For the single emitter case, two paradigmatic examples, i.e., the two-level emitter and the JC system, are used to show the correctness of our theory by comparison with the well-known results. For an array of emitters, the validity of the Markov approximation is examined in the system with two-level emitters coupled to waveguide photons. Here, the dynamical evolution of emitters shows some non-Markovian effects, i.e., the retardation effect.

The generation of entangled photons by the single emitter in front of the mirror is also analyzed by the exact result from our theory. Finally, the photon transmission in an array of EIT atoms coupled to the Rydberg level is investigated by the scattering theory. We highlight the interplay between the EIT phenomenon and the Rydberg interaction, and show how this results in the bunching and anti-bunching behaviors of the scattering photons and the dark polaritons propagating in the array.

The scattering theory also provides the way to explore some non-Markovian effects of emitters coupled to the waveguide with nonlinear spectrum, where the multiphoton bound states \( |54, 55 \rangle \) may form. The general result of the transition amplitude and the generalized master equation enable us to study the photon transmission in more complicated quantum optics systems and the dissipative many-body systems.

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Appendix A: Generalized master equation by path integral

In this Appendix, we derive the generalized master equation \( (20) \) by the path integral approach. To simplify the notation, we neglect the term \( H_{\Gamma} \) describing the independent decay of each emitter to the free space, and the derivation including the free space mode follows the same procedure. In the end, we show the result by considering the effect of the free space decay.

Here, we need to introduce the emitter field explicitly. In quantum optics systems, the emitter operators are usually the annihilation (creation) operators of bosonic modes and the ladder operators. If one use the hardcore boson to describe the emitters, all the operators are bosonic operators \( (b_l) \), where \( l \) denotes the different modes of the emitters. In the coherent basis, the reduced density operator \( \rho_s(T) = T_{\text{harm}} e^{-iH T} \rho(0) e^{iH T} \) becomes \( \rho_s(T) = F_{\text{in},-} F_{\text{in},+} e^\sum \langle \{ \beta_{l,+} \}, \{ \beta_{l,-} \}, \{ \beta_{l,+}^* \}, \{ \beta_{l,-}^* \} \rangle \),

\[
\rho_f(T) = \int d\mu(\beta_{l,+}, \beta_{l,-}^*) d\mu(\beta_{l,-}, \beta_{l,+}^*) e^{-\sum \langle \{ \beta_{l,+} \}, \{ \beta_{l,-} \}, \{ \beta_{l,+}^* \}, \{ \beta_{l,-}^* \} \rangle \times \{ \{ \beta_{l,+} \}, \{ \beta_{l,-} \}, \{ \beta_{l,+}^* \}, \{ \beta_{l,-}^* \} \rangle \right),
\]

where \( \alpha = r, l \) denotes the right- and left- moving photon modes,

\[
F_{\text{in},\pm} = \lim_{\{ J_{\alpha} \} \to 0} \sum_{\{ n_{\alpha} \}} \psi_{\text{in}}(\{ n_{\alpha} \}) \prod_{\alpha} \frac{1}{\sqrt{n_{\alpha}! \delta_{n_{\alpha}}}},
\]

the measure for the unnormalized coherent state \( |\{ \beta \} \rangle \) is \( d\mu(\beta_{l,+}) = \prod_l (e^{-|\beta_{l,+}|^2} d^2 \beta_{l,+} / \pi) \), and the element

\[
\rho_f(\beta_{l,+}, \beta_{l,-}; T) = \int d\mu(\beta_{l,+}, \beta_{l,-}^*) \times \exp \left( - \frac{i}{\hbar} \int dt \left( \beta_{l,+} \dot{\beta}_{l,-} + \beta_{l,-} \dot{\beta}_{l,+} \right) \right)
\]

is given by the propagator

\[
Z_f(\beta_{l,+}, \beta_{l,-}, \beta_{l,+}, \beta_{l,-}) = \int d\mu(\beta_{l,+}, \beta_{l,-}) \times \left( \langle \{ J_{\alpha} \}, \{ \beta_{l,+} \}, \{ \beta_{l,-} \} | e^{-iH T} | \{ J_{\alpha} \}, \{ \beta_{l,+} \}, \{ \beta_{l,-} \} \rangle \right).
\]
in the closed time path. Here, we define \( \rho_0(\beta^*_{i,+}; \beta_{i,-}) = \{ \{ \beta_{i,+} \} \} \rho_{\text{sys}}(0) \{ \{ \beta_{i,-} \} \} \).

The propagator (A3) is obtained by the saddle point method shown in Sec. IIIC, which is

\[
Z_J(\beta^*_{i,+}, \beta_{i,-}; T) = e^{iS_b} \times \int d\mu(J_{k,\alpha}, J_{k,\alpha}^*) \exp\left\{ \sum_{l, \alpha \in \sigma} \left[ J_{k,\alpha} J_{k,\alpha}^* e^{-i\sigma a_c kT} + J_{k,\alpha}^* J_{k,\alpha} e^{i\sigma a_c kT} \right] \right\} \times \int D[\text{system}] e^{iS_{\text{eff},+} + iS_{\text{eff},-} + iS_{\text{jump}}}.
\]

Here, \( iS_b = \sum_{l}[\beta^*_{i,+} \beta_{i,-}(T) + \beta^*_l(T) \beta_{i,-}(T)] \) is the boundary term for the emitters, and \( S_{\text{eff},\pm} \) are the effective actions in the forward and backward time-evolution paths, which are given by substituting the fields \( \beta_{i,\pm} \) for the emitter fields in the effective action \( S_{\text{eff}} \). The external source term is

\[
S_{J,\pm} = - \int_0^T dt \sum_{k,\alpha} [J_{k,\alpha}^* O_{k,\sigma_c,\pm}(t) e^{-i\sigma a_c kT} + J_{k,\alpha} O_{k,\sigma_c,\pm}(t) e^{-i\sigma a_c kT}].
\]

Finally, the Gaussian integral over \( J_{k,\alpha}, J_{k,\alpha}^* \) leads to the propagator

\[
Z_J(\beta^*_{i,+}, \beta_{i,-}; T) = e^{\sum_{k,\alpha} J_{k,\alpha}^* J_{k,\alpha}} \int D[\text{system}] e^{iS_{\text{eff},+} + S_{\text{jump}}}
\exp\left\{ \sum_l \left[ \beta^*_{i,+} \beta_{i,-}(T) + \beta^*_l(T) \beta_{i,-}(T) \right] \right\},
\]

where

\[
S_{\text{eff},+} = S_{\text{eff},-} - \int_0^T dt \sum_{k,\alpha} [J_{k,\alpha}^* O_{k,\sigma_c,\pm}(t) e^{i\sigma a_c kT} + J_{k,\alpha} O_{k,\sigma_c,\pm}^*(t) e^{-i\sigma a_c kT}],
\]

\[
S_{\text{jump}} = -i \int dt dt' \sum_{k,\alpha} e^{i\sigma a_c k(t-t')} O_{k,\sigma_c,\pm}(t) O_{k,\sigma_c,\pm}^*(t').
\]

Under the Markov approximation, the actions read

\[
S_{\text{eff},\pm} = S_{\text{sys},\pm} + \int dt \sum_{ij} \sqrt{\Gamma_i \Gamma_j} 
O_{i,\pm}(t) O_{j,\pm}(t) e^{i\sigma a_c [k_0(x_i - x_j)]},
\]

\[
S_{\text{jump}} = -2i \int_0^T dt \sum_{ij} \sqrt{\Gamma_i \Gamma_j} \cos[k_0(x_i - x_j)] O_{i,\pm}(t) O_{j,\pm}^*(t).
\]

By taking the time derivative of Eq. (A1) and using Eq. (A6), we obtain the motion equation

\[
\partial_t \rho_J(T) = -i[H_{\text{sys}}(T), \rho_J(T)] + L\rho_J(T)
\]

for the generating density matrix \( \rho_J(T) \) with the initial condition \( \rho_J(0) = \rho_{\text{sys}}(0) \), where the Lindblad term is

\[
L\rho_J(T) = 2 \sum_{ij} \sqrt{\Gamma_i \Gamma_j} O_{i,\pm}(T) O_{j,\pm}^*(T) \cos[k_0(x_i - x_j)] - \sum_{ij} \sqrt{\Gamma_i \Gamma_j} \cos[k_0(x_i - x_j)] [O_{i,\pm}^*(T), O_{j,\pm}(T)]
\]

Finally, by adding the Lindblad term describing the free space decay, we reproduce the results [20], [27], and [28].

[1] A. Imamoglu, H. Schmidt, G. Woods, and M. Deutsch, Phys. Rev. Lett. 79, 1467 (1998).
[2] K. M. Birnbaum, A. Boca, R. Miller, A. D. Boozer, T. E. Northup and H. J. Kimble, Nature 436, 87 (2005).
[3] D. E. Chang, A. S. Sorensen, E. A. Demler and M. D. Lukin, Nat. Phys. 3, 807 (2007).
[4] D. E. Chang, V. Vuletić, and M. D. Lukin, Nat. Photon. 8, 685 (2014).
[5] Z. Yuan, B. E. Kardynal, R. M. Stevenson, A. J. Shields, C. J. Lobo, K. Cooper, N. S. Beattie, D. A. Ritchie, and M. Pepper, Science 295, 102 (2002).
[6] B. Darquié, M. P. A. Jones, J. Dingjan, J. Beugnon, S. Bergamini, Y. Sortais, G. Messin, A. Browaeys, and P. Grangier, Science 309, 454 (2005).
[7] A. J. Shields, Nat. Photon. 1, 215 (2007).
[8] P. Michler, A. Kiraz, C. Becher, W. V. Schoenfeld, P. M. Petroff, L. Zhang, E. Hu, A. Imamoglu, Science 290, 2282 (2000).
[9] T. G. Tiecke, J. D. Thompson, N. P. de Leon, L. R. Liu, V. Vuletić, and M. D. Lukin, Nature 508, 241 (2014).
[10] T. Volz, A. Reinhard, M. Winger, A. Badolato, K. J. Hemmsey, E. L. Hu, and A. Imamoglu, Nat. Photon. 6, 605 (2012).
[11] L. Zhou, Z. R. Gong, Y. X. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. 101, 100501 (2008).
[12] T. G. Walker, Nature 488, 39 (2012).
[13] I. A. Walmsley and M. G. Raymer, Science 307, 1733 (2005).
[14] H. J. Kimble, Nature 453, 1023 (2008).
[15] T. C. H. Liew and V. Savona, Phys. Rev. Lett. 104,
[16] A. Majumdar, M. Bajcsy, A. Rundquist, and J. Vučković, Phys. Rev. Lett. **108**, 183601 (2012).

[17] M. O. Scully and M. S. Zubairy, “Quantum Optics”, *Cambridge University Press, Cambridge, United Kingdom*, 1997.

[18] C. S. Muñoz, E. d. Valle, A. G. Tudela, K. Müller, S. Lichtmannecker, M. Kaniber, C. Tejedor, J. J. Finley, and F. P. Laussy, Nat. Photon. **8**, 550 (2014).

[19] Y. Chang, A. G. Tudela, C. S. Muñoz, C. N. Benlloch, and T. Shi, *in preparation*.

[20] J. T. Shen and S. Fan, Phys. Rev. Lett. **98**, 153003 (2007).

[21] T. Shi and C. P. Sun, Phys. Rev. B **79**, 205111 (2009).

[22] T. Shi, S. Fan, and C. P. Sun, Phys. Rev. A **84**, 063803 (2011).

[23] M. Bamba, A. Imamoğlu, I. Carusotto, and C. Ciuti, Phys. Rev. A **83**, 021802(R) (2011).

[24] T. Shi and S. Fan, Phys. Rev. A **87**, 063818 (2013).

[25] C. Kurtsiefer, S. Mayer, P. Zarda, and H. Weinfurter, Phys. Rev. Lett. **85**, 290 (2000).

[26] R. Brouri, A. Beveratos, J. P. Poizat, and P. Grangier, Opt. Lett. **25**, 1294 (2000).

[27] K.-M. C. Fu, C. Santori, P. E. Barclay, I. Aharonovich, S. Frawer, N. Meyer, A. M. Holm, and R. G. Beausoleil, Appl. Phys. Lett. **93**, 234107 (2008).

[28] T. M. Babinec, B. J. M. Hausmann, M. Khan, Y. Zhang, J. R. Maze, P. R. Hemmer, and M. Lončar, Nat. Nanotechnol. **5**, 195 (2010).

[29] A. A. Houck, D. I. Schuster, J. M. Gambetta, J. A. Schreier, B. R. Johnson, J. M. Chow, J. Majer, L. Frunzio, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Nature **449**, 328 (2007).

[30] Stojan Rebić, J. Twamley, and G. J. Milburn, Phys. Rev. Lett. **103**, 150503 (2009).

[31] D. E. Chang, J. I. Cirac, and H. J. Kimble, Phys. Rev. Lett. **110**, 113606 (2013).

[32] M. J. Hartmann, F. G. S. L. Brandao, and M. B. Plenio, Nat. Phys. **2**, 849 (2006).

[33] A. D. Greentree, C. Tahan, J. H. Cole, and L. C. L. Hollenberg, Nat. Phys. **2**, 856 (2006).

[34] E. Rephaeli and S. Fan, IEEE. J. Sel. Top. Quant. **18**, 1754 (2012).

[35] S. Xu and S. Fan, Phys. Rev. A **91**, 043845 (2015).

[36] T. Caneva, M. T. Manzoni, T. Shi, J. S. Douglas, J. I. Cirac, and D. E. Chang, *arXiv:1501.04127*, 2015.

[37] H. X. Zheng, D. J. Gauthier, and H. U. Baranger, Phys. Rev. Lett. **111**, 090502 (2013).

[38] H. X. Zheng, D. J. Gauthier, and H. U. Baranger, Phys. Rev. A **85**, 043832 (2012).

[39] H. X. Zheng, D. J. Gauthier, and H. U. Baranger, Phys. Rev. A **82**, 063816 (2010).

[40] D. Roy, Phys. Rev. Lett. **106**, 053601 (2011).

[41] D. Roy, Phys. Rev. A **83**, 043823 (2011).

[42] B. Q. Baragiola, R. L. Cook, A. M. Branczyk, and J. Combes, Phys. Rev. A **86**, 013811 (2012).

[43] M. Pletyukhov and V. Gritsev, *arXiv:1504.00330*.

[44] A. J. Miller, S. W. Nam, and J. M. Martinis, and A. V. Sergienko, Appl. Phys. Lett. **83**, 791 (2003).

[45] C. M. Natarajan, M. G. Tanner, and R. H. Hadfield, Supercond. Sci. Technol. **25**, 063001 (2012).

[46] K. D. Irwin and G. C. Hilton, Topics Appl. Phys. **99**, 63 (2005).

[47] E. Rephaeli and S. Fan, Phys. Rev. Lett. **108**, 143602 (2012).

[48] T. Peyronel, O. Firstenberg, Q. Liang, S. Hofferberth, A. V. Gorshkov, T. Pohl, M. D. Lukin, and V. Vuletić, Nature **488**, 57 (2012).

[49] J. F. Huang, T. Shi, C. P. Sun, and F. Nori, Phys. Rev. A **88**, 013836 (2013).

[50] Y. Chang, Z. R. Gong, and C. P. Sun, Phys. Rev. A **83**, 013825 (2011).

[51] J. F. Huang, J. Q. Liao, and C. P. Sun, Phys. Rev. A **87**, 023822 (2013).

[52] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”, *Addison-Wesley Publishing Company*, 1995.

[53] T. Shi and CP Sun, *arXiv:0907.2776*.

[54] C. Zhu, S. Endo, P. Naidon, and P. Zhang, Few-Body syst. **54**, 1921 (2013).