Gridless Multidimensional Angle of Arrival Estimation for Arbitrary 3D Antenna Arrays

Matilde Sánchez-Fernández, Senior Member, IEEE, Vahid Jamali, Member, IEEE Jaime Llorca, Senior Member, IEEE, Antonia Tulino, Fellow, IEEE

Abstract

A full multi-dimensional characterization of the angle of arrival (AoA) has immediate applications to the efficient operation of modern wireless communication systems. In this work, we develop a compressed sensing based method to extract multi-dimensional AoA information exploiting the sparse nature of the signal received by a sensor array. The proposed solution, based on the atomic $\ell_0$ norm, enables accurate gridless resolution of the AoA in systems with arbitrary 3D antenna arrays. Our approach allows characterizing the maximum number of distinct sources (or scatters) that can be identified for a given number of antennas and array geometry. Both noiseless and noisy measurement scenarios are addressed, deriving and evaluating the resolvability of the AoA propagation parameters through a multi-level Toeplitz matrix rank-minimization problem. To facilitate the implementation of the proposed solution, we also present a least squares approach regularized by a convex relaxation of the rank-minimization problem and characterize its conditions for resolvability.

Index Terms

Multi-dimensional AoA, atomic norm, 3D antenna array, multi-level Vandermonde decomposition.

I. INTRODUCTION

Optimizing the performance of wireless communication systems critically relies on extracting relevant information from the wireless propagation channel. Array processing techniques have been extensively used to reveal, extract, and exploit key propagation parameters such as angle of arrival (AoA) and angle of departure (AoD) in order to tailor and optimize the transmission to individual users in systems with multiple antennas [1].

In particular, AoA (and analogously AoD) estimation refers to the process of retrieving multi-dimensional direction of propagation parameters (e.g., azimuth, elevation) associated with possibly multiple electromagnetic sources (e.g., transmitters, scatters), from the observation of a receiving vector $y$, whose elements represent the signal outputs at each of the receiving antennas that form a sensor array. The received signal associated with a given source $k \in \{1, \ldots, K\}$ impinging on an $N$-antenna array from direction $(\theta_k, \phi_k)$ can be represented by a so-called steering vector $r_N(\theta_k, \phi_k) \in \mathbb{C}^N$, characterized
by the relative phase shifts at each of the antennas. The observable y is hence a noisy linear combination of such steering vectors, one for each source. The goal is to identify each steering vector and extract their associated multi–dimensional AoA parameters, represented by multi–dimensional frequencies, from the observation of y.

Existing methods for angular characterization of signal propagation include initial beamforming–based approaches [2] (highly limited in their resolution), covariance-based subspace methods such as Capons beamformer, multiple signal classification (MUSIC), estimation of signal parameters via rotational invariance techniques (ESPRIT), and their extensions [3], [4], and (deterministic) maximum likelihood (ML) methods based on nonlinear least squares (NLS) optimization problems [3]. However, all aforementioned methods suffer from well–known limitations. Subspace and NLS based methods require a priori knowledge of the number of sources, which may be difficult to obtain in practice. Capons beamformer, MUSIC, and ESPRIT need accurate estimates of the observable covariance matrix, which in turn requires a slow time-varying environment, a large number of observations (to build the sample covariance matrix), and a low level of source correlation (to avoid rank deficiency in the sample covariance matrix). In addition, NLS algorithms require very accurate initialization since their objective function has a complicated multimodal shape with a sharp global minimum.

On the other hand, an important observation is that AoA estimation can be seen as a sparse data representation/separation problem, where compressed sensing methods [5] can be used to significantly enhance accuracy and robustness. In particular, compressed sensing does not require any statistical knowledge of the parameters to be estimated or associated observations. Hence, it avoids the need to estimate the observable covariance matrix or even having prior knowledge of the number of sources to identify, while still providing strong performance guarantees. Such obliviousness to the number of sources and the statistics of the received signals makes compressed sensing techniques notably suitable to overcome the limitations of previous approaches in terms of robustness to mismatches with respect to prior knowledge.

The estimation of AoA information via compressed sensing techniques can be classified into three main categories: on–grid, off–grid, and gridless estimation. Despite AoA being an intrinsically continuous variable, initial approaches for AoA estimation using sparse methods were based on sampling the angular space, forcing the angles to be estimated to lay on a discrete grid [6], [7]. The problem with this approach is clear – even with very fine gridding, there is always a mismatch between grid and real values. Off–grid solutions try to overcome this mismatch by means of adaptive grids [8] or by estimating and compensating the grid offset [9]. Finally, gridless approaches do not rely on any discretization of the angular space – they directly operate in the continuous domain, completely avoiding the grid mismatch
problem, at the expense of increased computational complexity [10], [11].

A number of recent works have studied the use of gridless compressed sensing techniques for AoA estimation. However, most of them have only been applied to the estimation of 1D AoA parameters in [10]–[13], and only more recently, to multidimensional scenarios, with applications to uniform antenna arrays in [14], [15], and allowing non-uniform deployments in [16], [17].

It is important to note that non–uniform 3D deployments are becoming increasingly relevant in upcoming 5G/6G communication paradigms such as enhanced mobile broadband (EMB), massive machine type (MMT), or ultra-reliable low-latency (URLL) communications, where high rates, low latency, massive connectivity, and extremely accurate localization require a level of multi–dimensional space awareness that can only be achieved with arrays of matching dimensionality. Importantly, 3D arrays rarely deploy antennas uniformly, since they typically follow cubic [18] or cylindrical [19] geometries without antennas deployed inside the volume.

A. Contributions

In this work, we focus on the design of robust an efficient techniques for i) full–dimensional AoA estimation in systems with ii) arbitrary antenna deployments. We leverage compressed sensing techniques that exploit the sparse nature of signal measurements impinging on arbitrary multi–dimensional antenna arrays. The approach undertaken is based on sparse approximation of the signal impinging on a 3D antenna array and the use of the Vandermonde decomposition to enable gridless extraction of multi–dimensional propagation parameters that are intrinsically continuous variables (see [11], [12], [14]). It is worth highlighting that while the application scenario considered in this paper focuses on 3D array deployments where the goal is the gridless extraction of $d$-dimensional parameters with $d \leq 3$, the results derived in Section III hold for multi–dimensional parameters of arbitrary dimensionality $d$.

Our main contributions can be summarized as follows:

In the noiseless setting:

- Given a receiving $N$-antenna array following an arbitrary $d$-dimensional configuration, we characterize the resolvable region, i.e., the maximum number of resolvable sources (e.g., scatters) $K$ and their associated $d$-dimensional propagation parameters, referred to as frequencies. We show that the size of the resolvable region grows linearly with the number of antennas, not only in the uniform setting, but also with non–uniform arrays. Specifically, the maximum number of resolvable scatters is given by $\left\lfloor Sc - \frac{(d-1)}{2} \right\rfloor$, where $Sc$ is the sum of the number of antennas in each dimension of the largest uniform array embedded in the original structure. We further strongly conjecture, as validated by our numerical results, that the maximum number of resolvable scatters is in fact given by $\left\lceil \frac{Nc}{2} - 1 \right\rceil$, where $Nc$ is the total number of antennas in the largest uniform array embedded in
the original structure. Importantly, the resolvable region characterized in this work \(i\) is larger than the region already available in literature \([14]\) and \(ii\) has more general applicability, since it can be used with measurements coming from non-uniform sampled observations.

- For the case of non-uniform deployments with large number of antennas (e.g., massive multiple-input multiple-output (MIMO) systems), we show that the large-scale nature of the array deployment allows guaranteeing resolvability with probability \(1 - \epsilon\) of up to \(K\) scatters with \(N = O(K \log(K/\epsilon))\) independently of the specific array geometry.

- The aforementioned characterization follows a novel constructive approach that allows not only identifies the resolvable scatters, but also extracts the associated \(d\)-dimensional propagation parameters. Such approach consists of three steps: First, the \(N\)-dimensional observable vector \(y\) is represented as a sampled version of an enlarged \(\bar{N}\)-dimensional vector \(s\) composed of a linear combination of uniform steering vectors associated with the smallest uniform array covering each of the original antennas, referred to as virtual uniform array. In the second step, we find the minimum number of steering vectors that compose \(s\) by reformulating the minimization of the \(\ell_0\)-atomic norm (AN) of \(s\) as the search for a min-rank \(d\)-dimensional multi–level Toeplitz (MLT) matrix. A distinctive feature of our approach is to restrict this search to a specific positive semidefinite (PSD) matrix set, termed canonical (see Definition \([4]\)), that is shown to preserve optimality, while significantly reducing the required dimension of the virtual uniform array \(\bar{N}\). Finally, in the third step, we show that the multilevel Vandermonde decomposition of the resulting min-rank \(d\)-MLT canonical matrix allows the extraction of the \(d\)-dimensional propagation parameters.

- The proposed rank minimization based approach is NP-hard in general, and its traditional convex counterpart based on \(\ell_1\)-AN and trace minimization is known to exhibit significant limitations \([11]\). Most prominently, the resulting \(\ell_1\)-AN and trace minimizer is not guaranteed to be sparse. To circumvent this problem, we resort to a weighted \(\ell_2 + \ell_1\)-AN minimization method. We numerically show the advantage of the proposed approach in overcoming the drawbacks of the \(\ell_1\)-AN convex approximation.

- Noting that the complexity of the proposed rank and trace minimization based methods is driven by the size of the virtual uniform array, our approach allows finding the \(d\)-dimensional propagation parameters of any number of scatters in the resolvable region \(K\) using a virtual uniform array with at least \(K\) antennas along its largest dimension, leading to a complexity that scales (exponentially for \(\ell_0\)-AN and polynomially for \(\ell_1\)-AN) with \(K\). This result significantly outperforms the existing best known result in \([14]\) that requires a virtual uniform array with at least \(K\) antennas along its smallest dimension, resulting in a complexity that scales (exponentially or polynomially) with
\(K^d\). The reduced requirement on the dimension of the virtual uniform array is a consequence of a novel generalization of the Carathodory-Fejr theorem, which allows enlarging the set of PSD \(d\)-MLT matrices that admit a unique Vandermonde decomposition (see Lemma 1).

In the noisy setting:
- For the case of noisy measurements, we resort to the weighted \(\ell_2+\ell_1\)-AN optimization approach and provide a closed-form expression for the weight as a function of the signal-to-noise ratio SNR, which is shown to well approximate the numerically computed optimal value. We evaluate the performance of our approach for moderate and high SNR, showing an estimation error very close to the associated Cramér-Rao lower bound (CRLB).

General array design guidelines:
- Our results suggest that if the goal is identifying up to \(K\) scatters (and extract their associated \(d\)-dimensional propagation parameters), we should design an antenna array whose smallest uniform embedded structure has at least \(2K\) antennas, where most of them are lying along a dominate dimension, while maintaining at least 2 antennas in the other dimensions. While \(2(K+1)\) antennas, independent of geometry, allows resolving \(K\) scatters, a design with a dominant dimension will help resolving the \(K\) scatters with complexity \(O(K)\) instead of \(O(K^d)\).

B. Related work

We now describe results and limitations of the compressed sensing based studies that are more closely related to our work. We emphasize that the limitations of such previous results arise from the fact of requiring more stringent conditions for frequency recovery and/or lacking applicability to general array structures.

In terms of the resolvable region, the best known result, provided in [14], bounds the number of resolvable scatters \(K\) by the minimum number of antennas in each of the \(d\) dimensions of the antenna array. In contrast, we significantly enlarge the resolvable region by bounding the number of resolvable scatters by the maximum number of antennas among each dimension. In terms of the number of antennas to resolve \(K\) scatters, our approach improves the result in [14] by requiring \(O(K)\) antennas instead of \(O(K^d)\). In addition, while the recovery conditions in [14] are only applicable to uniform antenna deployments, our approach applies to arbitrary 3D antenna arrays. Finally, different from [14], we ensure uniqueness of the solution to the \(\ell_0\) optimization problem by introducing a condition on the dimension of the measurement vector (see Theorem 2) or imposing a specific array geometry (see Corollary 1).

In terms of non-uniform array deployments, the works of [11], [16], and [17] initiated such analyses by considering non-uniform geometries as random samples of uniform array structures in the asymptotic regime. Specifically, [11] focused on the one-dimensional setting (i.e., linear array) and
provided recovery conditions that allow recovering $K$ scatters with high probability (i.e., $1 - \epsilon$) given that there is a minimum separation among the frequencies and a number of active antennas $N = O \left( K \log \left( K / \epsilon \right) \log \left( \bar{N} / \epsilon \right) \right)$ with $\bar{N}$ the number of antennas in the uniform array structure. The works of [16] and [17] extend the approach in [11] to the 2-dimensional setting (i.e., square planar array). While [16] provides recovery conditions that are exactly the extensions to the 2-dimensional setting of the ones provided in [11], [17] formulates a low complexity implementation of the recovery algorithm, albeit less practically verifiable recovery conditions. The major drawback of the results in [11], [16], and [17] is that for most common applications, non-uniform array deployments cannot be seen as the result of sampling uniform array structures uniformly at random. Our work extends and generalizes the results in [11] and [16] considering a completely arbitrary non-uniform deployment, which can be either deterministic or arbitrarily random, it does not require a minimum separation among the frequencies, and it requires a number of active antennas that grows as $O \left( K \log \left( K / \epsilon \right) \right)$ (see Theorems 2 and 4).

The authors in [7] extended the recovery conditions stated in [11] and [16] to a more general set of sensing matrices provided that they obey isotropy and incoherence conditions. However, differently from [11] and [16], [7] resorts to on-grid approaches to identify the propagation parameters or frequencies, resulting in potentially significant gridding errors.

**Notation:** $[\cdot]^\top$ is the transpose and $[\cdot]^{\dagger}$ is the Hermitian. The operator $\text{diag}(\mathbf{x})$, returns a diagonal matrix with diagonal given by $\mathbf{x}$. Also, $\mathbf{X}_{(m)} = \mathbf{x}_m$ and $\mathbf{X}^{(m)} = \mathbf{x}^m$ are respectively the $m$-th column/row of matrix $\mathbf{X}$, $x_{nm}$ is the matrix element in row $n$ and column $m$, and $\mathbf{X}^{(I)}$ and $\mathbf{x}^{(I)}$ are the submatrix of $\mathbf{X}$ and the subvector of $\mathbf{x}$ given respectively by the rows and elements in the index set $I$. For a given integer $K \in \mathbb{Z}$, $[K] = \{1, \ldots, K\}$. $\mathbb{T}$ denotes the unit circle $[0, 1]$ by identifying the beginning and the ending points. $\| \cdot \|_\rho$ represents the $\ell_\rho$ norm and $\| \mathbf{X} \|_{2 \to 2}$ represents the squared-root of the largest eigenvalue of the Hermitian matrix $\mathbf{X}^{\dagger} \mathbf{X}$. The $y$–modulus of value $x$ is given by $\text{mod} (x, y)$. A $N \times M$ all-zeroes matrix is given by $\mathbf{0}_{N \times M}$ and $\mathbf{1}_n$ is the all–ones vector of dimension $n \times 1$. The different products of vectors and matrices are the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ represented with $\mathbf{x} \cdot \mathbf{y}$, the Kronecker product represented by $\otimes$ and the Khatri–Rao product represented with $\odot$. $\mathbf{A}^g$ denotes the generalized inverse of matrix $\mathbf{A}$; and $|c|$ and $\angle c$ are the absolute value and angle of complex number $c$ in polar coordinates, respectively.

**II. Characterization of the steering vectors in the unit circle frequency domain**

**A. General model for steering vectors**

Let us consider a propagation scenario with a receiving $N$-antenna array, partly characterized by the response of the steering vector $\mathbf{r}_N (\theta, \phi) \in \mathbb{C}^N$ in the direction $(\theta, \phi)$, identified as the azimuth and
The vector depends on the number of antennas elevation angles, respectively \( \theta \in [0, 2\pi] \) and \( \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \). The structure of the measuring steering vector depends on the number of antennas \( N \) and on their normalized relative positions in the sensor array \( \mathbf{p}_n = [p^x_n, p^y_n, p^z_n]^{\top} \) with \( n \in [N] \). The phase at the \( n \)-th antenna is given by \( \Phi_n = -2\pi(p^x_n \sin \theta \cos \phi + p^y_n \sin \theta \sin \phi + p^z_n \cos \theta) \). Then, the measuring steering vector can be represented as

\[
\mathbf{r}_N(\theta, \phi) = \frac{1}{\sqrt{N}} [e^{j\Phi_1}, \ldots, e^{j\Phi_N}]^{\top}.
\]

(1)

In the following subsection, we first simplify the steering vector in (1) for uniform arrays. Subsequently, we show that the steering vector of a general non-uniform array can be equivalently represented by a sampled version of the steering vector of a virtual uniform array via a proper sampling (or sensing) matrix.

**B. Steering vector for a uniform 3D array**

Consider a uniform 3D array deployment (3D-UD) with \( N = XYZ \) antenna elements, where \( X, Y, \) and \( Z \) are the number of elements deployed in each spatial dimension of the Cartesian coordinate system. We define the \( d = 3 \) dimensional vector \( \mathbf{N} \triangleq [X, Y, Z] \) identifying the number of antenna elements, and deploy the antennas with normalized spacing \( \delta^x, \delta^y \) and \( \delta^z \) in each of the dimensions. The \( n \)-th antenna normalized position is given in this case by \( \mathbf{p}_n = [x_n\delta^x, y_n\delta^y, z_n\delta^z]^{\top} \) with \( x_n \in \{0, \ldots, X-1\}, y_n \in \{0, \ldots, Y-1\}, \) and \( z_n \in \{0, \ldots, Z-1\} \). Then, the phase simplifies to \( \Phi_n = -2\pi(x_n\delta^x \sin \theta \cos \phi + y_n\delta^y \sin \theta \sin \phi + z_n\delta^z \cos \theta) \).

Let us define a normalized frequency vector \( \mathbf{f} = [f^x, f^y, f^z]^{\top} \in \mathbb{T}^3 \) that contains the information on both azimuth and elevation, with \( f^x = \text{mod} \ (\delta^x \sin \theta \cos \phi, 1) \), \( f^y = \text{mod} \ (\delta^y \sin \theta \sin \phi, 1) \), and \( f^z = \text{mod} \ (\delta^z \cos \theta, 1) \) and the normalized position of antenna \( n \)-th as \( \mathbf{n}_n = [x_n, y_n, z_n]^{\top} \). Then, we obtain the phase \( \Phi_n = -2\pi(x_nf^x + y_nf^y + z_nf^z) + 2m\pi = -2\pi \mathbf{f} \cdot \mathbf{n}_n + 2m\pi \), where \( m \) is an integer. The receive antenna steering vector \( \mathbf{r}_N(\mathbf{f}) \) is obtained as

\[
\mathbf{r}_N(\mathbf{f}) = \frac{1}{\sqrt{N}} [e^{j2\pi f^x \mathbf{n}_1}, e^{j2\pi f^x \mathbf{n}_2}, \ldots, e^{j2\pi f^x \mathbf{n}_N}]^{\top} = \mathbf{r}_X(f^x) \otimes \mathbf{r}_Y(f^y) \otimes \mathbf{r}_Z(f^z).
\]

(2)

where \( \mathbf{r}_X(f^x) = \frac{1}{\sqrt{X}} [e^{j2\pi x_1 f^x}, e^{j2\pi x_2 f^x}, \ldots, e^{j2\pi x_X f^x}]^{\top} \), and \( \mathbf{r}_Y(f^y), \mathbf{r}_Z(f^z) \) are defined equivalently. We assume a \( d \)-dimensional frequency vector \( \mathbf{f} \in \mathbb{T}^d \), where \( d \in \{1, 2, 3\} \). In the most general case, if the antenna array is deployed along all dimensions of the Cartesian system, we have \( d = 3 \). From (2), 3D-UD antenna deployments lead to a regular structure of the steering vector which we refer to as uniform steering vector.
C. Arbitrary array deployments

Many antenna deployments, particularly those using cubic or cylindrical geometries, [18], [19], cannot be described, in terms of their associated steering vectors, following the regular structural property mentioned in Section II-B. In the following we refer to such types of arbitrary deployment (AD) as an 3D-AD array.

Given any arbitrary 3D-AD array, the associated steering vector \( r_N (f) \), even though not characterized by any regular structure, can be obtained starting from an encompassing steering vector of a virtual 3D-UD as follows. Let \( X\delta^x, Y\delta^y \) and \( Z\delta^z \) be the maximum normalized antenna coordinates along the directions of the 3D-AD array, where here \( \delta^x, \delta^y, \delta^z \) represents the minimum normalized spacing between the array elements in each direction. Next, consider the virtual 3D-UD with \( \bar{N} = [\bar{X}, \bar{Y}, \bar{Z}] \) elements, encompassing the 3D-AD array, such that \( \bar{X} \geq X, \bar{Y} \geq Y \) and \( \bar{Z} \geq Z \) and \( \bar{N} = \bar{X}\bar{Y}\bar{Z} \geq N \), with associated steering vector \( r_{\bar{N}} (f) \):

\[
    r_{\bar{N}} (f) = r_X(f^x) \otimes r_Y(f^y) \otimes r_Z(f^z). \tag{3}
\]

Then, \( r_N (f) \) can be obtained starting from \( r_{\bar{N}} (f) \) using a binary sensing matrix \( A \in \{0, 1\}^{N \times \bar{N}} \), which effectively removes some elements from \( r_{\bar{N}} (f) \) (see Fig. 1(a) and Fig. 1(b)), as:

\[
    r_N (f) = Ar_{\bar{N}} (f) \tag{4}
\]

with \( A \) being a fat matrix with elements \( a_{ij} = 0 \) if component \( j \in [\bar{N}] \) in the generating virtual 3D-UD array is not included as component \( i \in [N] \) in the 3D-AD array, and \( a_{ij} = 1 \) otherwise. Note that in order to ensure that the sensing matrix removes elements from the 3D-UD and that one particular element in the 3D-UD is not considered more than once in the 3D-AD the sensing matrix \( A \in \{0, 1\}^{N \times \bar{N}} \) should

\[\text{Note that } r_N (f) \text{ refers to a steering vector of dimension } N \text{ without any specific structure, while } r_{\bar{N}} (f) \text{ and } r_N (f) \text{ represent uniform arrays with a deployment respectively given by } \bar{N} = [X, Y, Z] \text{ and } \bar{N} = [X, Y, Z] \text{ elements.} \]
meet both following conditions $A_N = 1_N$ and $A_N^\top 1_N = 1_N$. This is equivalent to impose that the sensing matrix $A \in \{0, 1\}^{N \times \bar{N}}$ belongs to the sensing set $\mathcal{A} \subset \{0, 1\}^{N \times \bar{N}}$ defined as:

$$\mathcal{A} = \{ A \in \{0, 1\}^{N \times \bar{N}} : [I_N | 0_{N \times (\bar{N} - N)}] \Pi , \Pi \in \mathcal{P} \}$$

with $\mathcal{P}$ the set of all $\bar{N} \times \bar{N}$ permutation matrices, and $N \leq \bar{X}YZ = \bar{N}$. Each 3D-AD array can be then described using the associated virtual 3D-UD array and sensing matrix $A$. Note that there are multiple pairs of virtual 3D-UD array and sensing matrix $A$ that can be associated to a given 3D-AD array.

Remark 1: Note that the ordering of the Kronecker product in (3), induces, via the 3D-UD steering vector $r_N$, an indexing of the virtual array antenna elements. Such Kronecker ordering is arbitrary and it will be in the following properly chosen to guarantee the maximum system capability in terms of number of resolvable directions $k$. In the following, to refer to the Kronecker ordering given in (3) we use the notation $\bar{Z} \rightarrow \bar{Y} \rightarrow \bar{X}$.

III. GRIDLESS PARAMETER EXTRACTION

Starting from an arbitrary $d$D-AD array with $d \leq 3$ and its associated general steering vector structure described in Sec. II-C, equations (3)-(4), in this section, we focus on the problem of extracting key multidimensional propagation parameters, such as AoA, from the signal received at the antennas elements of the $d$D-AD array. Specifically, given a $d$D-AD array, under the assumption of $K$ multiple sources incident on the array from $K$ AoAs (or equivalently $K$ local scatters reflecting the impinging incoming signal from $K$ AoDs), the received signal, measured at the antennas elements, $y \in \mathbb{C}^N$, can be written as:

$$y = A R_N (f^*_1 : K) u^* + w = A s^* + w,$$

where $A \in \mathcal{A}$ and $R_N (f^*_1 : K) = [r_N(f^*_1), \ldots, r_N(f^*_K)] \in \mathbb{C}^{N \times K}$ are, respectively, the sensing matrix and the steering vector matrix of the encompassing virtual 3D-UD associated to the $d$D-AD array as defined in Sec. II-C $u^* = [u^*_1, \ldots, u^*_K] \in \mathbb{C}^K$ is the unknown incoming signal vector incident on the array from the $K$ AoAs, and $w \in \mathbb{C}^N$ is the additive white Gaussian noise (AWGN) noise vector whose elements are independent with zero mean and variance $\sigma^2$.

The objective is to retrieve from the measurement $y \in \mathbb{C}^N$, in both noiseless (i.e. $w = 0_{N \times 1}$) and noisy scenarios (i.e $w \neq 0_{N \times 1}$), the $K$-scatter vector $s^*$, the number of sources/scatters, $K$, and the associated set of frequencies, $f^*_1 : K = \{f^*_1, f^*_2, \ldots, f^*_K\}$ that characterizes the $d$D AoAs of each source/scatter, with $f^*_k \in \mathbb{T}^d$ and $d \leq 3$. All our frequency recovery conditions are derived under the following assumption:
Assumption 1: The dD frequency vectors, $f^*_k$ with $k \in [K]$, associated to the dD AoAs of the K sources/scatters are modeled as independent and identically distributed random vectors whose components are independent and uniformly distributed on $[0,1)$, i.e. $f^*_k \sim \mathcal{U}[0,1)$, for $k \in [K]$, and $\alpha \in \{x,y,z\}$.

Note that, given our work’s application scenario, this assumption is not restrictive. In fact, in wireless propagation environments, sources/scatters are typically modeled to be uniformly random placed in the surroundings of the receiver.

Furthermore, even though the derivations of the frequency recovery conditions are conducted for $d \leq 3$ given the application scenario of our work, they are applicable also to $d > 3$.

A. Signal and frequency recovery in noiseless scenarios

1) Problem statement and previous results: In this section, we focus on the noiseless setting, i.e:

$$y = AR_N(f^*_{1:K}) u^* = As^*, \quad (7)$$

with $A \in A \subset \mathbb{C}^{N \times \tilde{N}}$. As already stated our objective is to retrieve $s^*$, $K$, and $f^*_{1:K}$. As first step, let us provide the following definitions.

**Definition 1:** Given a $\tilde{N}$-dimensional vector $s$, the $\ell_0$ AN ($\ell_0$-AN) of $s$ in $\mathcal{R} = \{r_N(f) : f \in T^d\}$ is defined as $\|s\|_{\mathcal{R},0} = \inf_{f_k \in T^d, u_k \in \mathbb{C}} \{K : s = \sum_{k=1}^{K} u_k r_N(f_k)\}$. □

**Definition 2:** A $K$-scatter vector $s$ is defined as a vector whose atomic $\ell_0$ norm is equal to $K$. □

It is well known that for a given measurement $y$ and a given sensing matrix $A \in A$ as in (5), the $K$-scatter vector $s^* = R_N(f^*_{1:K}) u^*$ can be reconstructed, from the noiseless measurement $y$ in (7), as the unique solution of:

$$\min_{s \in \mathbb{C}^N} \|s\|_{\mathcal{R},0} \quad \text{s.t.} \quad y = As \quad \text{(P.1)}$$

if $AR_N(f^*_{1:2K})$ is injective as a map from $\mathbb{C}^{2K} \to \mathbb{C}^N$. For the problem of identifying the frequencies, $f^*_{1:K}$, associated to $s^*$, several methods have been proposed in literature [6], [8], [9], [11], [12], [14]. In the following we focus on the so-called gridless approaches [8], [14]. The gridless approach for $d = 1$ is based on the Carathodory-Fejr theorem. Such approach and the associated Carathodory-Fejr theorem has been recently generalized to the case of $d > 1$ in [14] by introducing a PSD $d$-Level Toeplitz ($d$-LT) matrix, whose definition is provided in the following for the specific case of $d = 3$.

**Definition 3:** Let $d = 3$. A $\tilde{N} \times \tilde{N}$ matrix $V \in \mathbb{C}^{\tilde{N} \times \tilde{N}}$ is a $d$-Level Toeplitz ($d$-LT) matrix with nesting ordering $\tilde{Z} \to \tilde{Y} \to \tilde{X}$ if it is a $\tilde{X} \times \tilde{X}$ block Hermitian Toeplitz matrix defined as $V = V_{XYZ}$ in (8).a). Furthermore, the $\tilde{Y} \tilde{Z} \times \tilde{Y} \tilde{Z}$-dimensional generic block $V_{aYZ}$ with $-\tilde{X} + 1 \leq a \leq -\tilde{X} - 1$ is a $\tilde{Y} \times \tilde{Y}$ block Hermitian Toeplitz matrix of the form given in (8).b) such that $V_{aYZ} = V_{-aYZ}^\dagger$. Finally, the
generic block $V_{abZ}$ with $-X + 1 \leq a \leq X - 1$ and $-Y + 1 \leq b \leq Y - 1$ is a $Z \times Z$ Toeplitz matrix given in (8). (c) such that $V_{abZ} = V^t_{-a-bZ}$, and defined from vector $v = [v_{ab(-Z+1)}, \ldots, v_{ab(Z-1)}] \in \mathbb{C}^{(2Z-1)}$ with $v_{abc} = v^t_{-a-b-c}$ for $-X + 1 \leq a \leq X - 1$, $-Y + 1 \leq b \leq Y - 1$ and $-Z + 1 \leq c \leq Z - 1$.

\[
V_{XYZ} = \begin{bmatrix}
V_{0YZ} & \cdots & V_{(X-1)YZ} \\
V_{(-1)YZ} & \cdots & V_{(X-2)YZ} \\
\vdots & \ddots & \vdots \\
V_{(-X+1)YZ} & \cdots & V_{0YZ}
\end{bmatrix} \quad V_{dYZ} = \begin{bmatrix}
V_{aYZ} & \cdots & V_{a(Y-1)Z} \\
V_{a(-1)Z} & \cdots & V_{a(Y-2)Z} \\
\vdots & \ddots & \vdots \\
V_{a(-Y+1)Z} & \cdots & V_{a0Z}
\end{bmatrix} \quad V_{abZ} = \begin{bmatrix}
v_{ab0} & \cdots & v_{ab(Z-1)} \\
v_{ab(-1)} & \cdots & v_{ab(Z-2)} \\
\vdots & \ddots & \vdots \\
v_{ab(-Z+1)} & \cdots & v_{ab0}
\end{bmatrix}
\]

\[(a)(b)(c)\]

For $d = 2$, we define a $d$-LT matrix using (8).(b) and (8).(c) by fixing $X = 1$ so that $a = 0$ and $V = V_{0YZ}$. Analogously, for $d = 1$, we define a $d$-LT matrix from (8).(c) by fixing $X = 1$ and $Y = 1$ so that $a = b = 0$ and $V = V_{00Z}$.

We further introduce the following definition:

**Definition 4:** Given a $\bar{N} \times \bar{N}$ 3-LT matrix $V$ with nesting ordering $\bar{Z} \to Y \to \bar{X}$, we say that $V$ has a canonical ordered structure (or equivalently $V$ is a canonical $d$-LT matrix) if there is a descending ordering in the component dimensions i.e. $\bar{X} \leq \bar{Y} \leq \bar{Z}$.

**Remark 2:** There is an interesting connection, that we will later explore, between the Kronecker ordering of a steering vector and the nesting ordering of a 3-LT matrix. Given a set of $K$ steering vectors $r_N(f_k)$ with $k \in [K]$ with Kronecker ordering $\bar{Z} \to \bar{Y} \to \bar{X}$ (see Remark 1), the $\bar{N} \times \bar{N}$ matrix defined as $V = \sum_{k=1}^{K} p_k r_N(f_k)r_N(f_k)^\dagger$ is a PSD 3-LT matrix with nesting ordering $\bar{Z} \to \bar{Y} \to \bar{X}$.

Using the previous definitions, we now overview some previous results. In [14] Theorem 3 the authors show that given a 3D-UD array with $N = [X, Y, Z]$ elements, and its associated pair of sensing matrix $A_U$ and virtual 3D-UD array of $\bar{N} = [\bar{X}, \bar{Y}, \bar{Z}]$ elements, the received signal $s^*$ (the incoming signal $u^*$) and its associated frequencies $f^*_1:K$ can be uniquely and perfectly reconstructed from the noiseless measurement $y$ in (7), solving the following rank minimization problem [14, Theorem 3, Remark 4]:

\[
\min_{r, s \in \mathbb{C}^{\bar{N}}, S_{XYZ}} \text{rank}\{S_{XYZ}\} \quad \text{s.t.} \quad \begin{bmatrix} S_{XYZ} & s \\ s^\dagger & r \end{bmatrix} \succeq 0, \quad A_Us = y, \quad \text{(P.2)}
\]

if $K < \min\{\bar{X}, \bar{Y}, \bar{Z}\}$. In (P.2) $S_{XYZ}$ is a PSD $d$-LT matrix with arbitrary nesting order and $A_U = [I_N|0_{N \times (\bar{N} - N)}] \Pi_U$ with $\Pi_U$ a proper permutation matrix such that only the first $X \leq \bar{X}$, $Y \leq \bar{Y}$, and $Z \leq \bar{Z}$ antennas are sensed. However, the results in [14] Theorem 3 only hold for 3D-UD arrays and rely (for the frequency recovery) on the uniqueness of the Vandermonde decomposition of the resulting min-rank PSD $d$-LT matrix, $S_{XYZ}^o$, which can only be guaranteed if $\text{rank}\{S_{XYZ}^o\} < \min\{\bar{X}, \bar{Y}, \bar{Z}\}$ (see [14, Theorem 1]).
In the next section, we derive recovery results for arbitrary \( d \)-D-AD arrays and, exploiting the structure of our problem, we are able to reformulate [14, Theorem 1] and [14, Theorem 3] and effectively enlarge the frequency recovery region, under much less restrictive conditions on the rank of a PSD \( 3 \)-LT matrix and on the dimensions of the virtual array.

2) Main results on recovery conditions: In this section, we provide the exact characterization of the frequency recovery region for arbitrary \( d \)-D-AD array, with \( d \leq 3 \) by identifying the conditions on the system parameters that guarantee, from the measurement \( y \) in (7), perfect and unique recovery of the \( K \)-scatter vector \( s^* \) and its associated frequencies \( f^*_{1:K} \in \mathbb{T}^{d \times K} \).

Before stating our main results, let us present the following lemma, which will be used to prove our frequency recovery conditions and which also extends the result provided in [14, Theorem 1], deriving a less restrictive sufficient condition for the uniqueness of the Vandermonde decomposition of a \( d \)-level Toeplitz matrix.

**Lemma 1:** Let \( S \in \mathbb{C}^{\bar{N} \times \bar{N}} \) be an \( \bar{N} \times \bar{N} \) PSD \( d \)-LT matrix with rank \( r < \max \{ \bar{X}, \bar{Y}, \bar{Z} \} \), and with ordered canonical structure (as per Definition 4). Denoting \( W = \max \{ \bar{X}, \bar{Y}, \bar{Z} \} \), if the rank of the \( W \times W \) upper-left corner\(^2\) of \( S \) also equals \( r \) then \( S \) can be uniquely decomposed, via Algorithm 1, as \( S = R_{\bar{N}}(f_{1:r})P_{\bar{N}}(f_{1:r}) \), with \( f_{1:r} = \{ f_1, f_2, \ldots, f_r \} \in \mathbb{T}^{d \times r} \) being a unique set of frequencies, \( R_{\bar{N}}(f_{1:r}) = [r_{\bar{N}}(f_1), r_{\bar{N}}(f_2), \ldots, r_{\bar{N}}(f_r)] \in \mathbb{C}^{\bar{N} \times r} \) being the steering vector matrix associated to the vector frequencies \( f_{1:r} \), and \( P = \text{diag}(\{p_1, \ldots, p_r\}) \), \( p_k \in \mathbb{R}^+ \) with \( k \in [r] \).

**Proof:** The proof is given in Appendix A, where in a constructive way we prove existence and consequently uniqueness of the Vandermonde decomposition of \( S \). Following the proof we are also able to develop an algorithm (Alg. 1) that can be applied now with less stringent conditions on the rank than in [14].

Note that while Lemma 1 is stated for \( d \in \{1, 2, 3\} \), it is worth to underline that the result holds even for \( d > 3 \).

**Remark 3:** Lemma 1 admits a more general formulation where denoting by \( S \in \mathbb{C}^{\bar{N} \times \bar{N}} \) an \( \bar{N} \times \bar{N} \) PSD \( d \)-LT matrix with rank \( r \), and denoting by \( W \) the dimension of the largest 1-LT upper-left corner of \( S \). If the rank of such block also equals \( r \) and \( r < W \), then \( S \) admits a unique Vandermonde decomposition.

**Remark 4:** Lemma 1 shows that while the sufficient condition for the uniqueness of the Vandermonde decomposition \( (r < \min \{ \bar{X}, \bar{Y}, \bar{Z} \}) \) in [14, Theorem 1] is tight for \( d = 1 \), this would not hold any more for a PSD \( d \)-LT matrix \( S \in \mathbb{C}^{\bar{N} \times \bar{N}} \) with \( d \geq 2 \). In fact if \( S \in \mathbb{C}^{\bar{N} \times \bar{N}} \) follows Lemma 1 conditions,

---

\(^2\)The \( W \times W \) upper-left corner of \( S \), is the \( W \times W \) sub block of \( S \) obtained considering the first \( W \) rows and the first \( W \) columns of \( S \).
then a sufficient condition for \( S \) to admit a unique Vandermonde decomposition is the less restrictive condition \( r < \max \{ \bar{X}, \bar{Y}, \bar{Z} \} \), compared to \( r < \min \{ \bar{X}, \bar{Y}, \bar{Z} \} \) [14, Theorem 1]. This interesting result essentially stems from the fact that the proposed decomposition approach relies on the decomposing the upper-left block matrix in \( S \), cf. lines 17 and 18 of Algorithm 1. Therefore, the canonical ordering required in Lemma 1 poses the base to allow the less restrictive condition \( r < \min \{ \bar{X}, \bar{Y}, \bar{Z} \} \) on the uniqueness of the decomposition and hence enlarges the set of PSD \( d \)-LT matrices for which we can guarantee unique decomposition. However, if the ordering does not follow Definition 4 as required in Lemma 1, one has to enforce the condition stated in [14, Theorem 1], i.e. \( r < \min \{ \bar{X}, \bar{Y}, \bar{Z} \} \), to ensure the decomposability of the upper-left block matrix. To the best of the authors’ knowledge, the result stated in Lemma 1 is the tightest condition for the uniqueness decomposition of PSD \( d \)-LT matrices in the literature.

In addition to Lemma 1, in order to state our recovery conditions, we also need to provide some few useful definitions and assumptions.

Specifically, in the following, we always consider as Kronecker ordering for the virtual 3D-UD

| Algorithm 1 Algorithm for Lemma 1 |
|-----------------------------------|
| **Input:** \( S = S_{XYZ} \) with \( \text{rank}(S_{XYZ}) = r \). |
| **Step 1: Decomposing in the \( \bar{X} \) dimension** |
| 1: Obtain the Cholesky decomposition of \( S_{XYZ} = C_{XYZ} C_{XYZ}^\dagger \). |
| 2: Split \( C_{XYZ} \) row-wise in \( X \) blocks as \( C_{XYZ} = [C_{012}^\dagger, \ldots, C_{1(\bar{X}-1)2}^\dagger] \). |
| **Step 2: Decomposing in the \( \bar{Y} \) dimension** |
| 3: Obtain the eigen-decomposition \( U_{\bar{X}} = K_{\bar{X}} X K_{\bar{X}}^\dagger \) and we have that \( \bar{X} = \text{diag}(e^{2\pi i f_1}, \ldots, e^{2\pi i f_r}) \). |
| 4: **if** \( \bar{X} > 1 \) **then** |
| 5: Find the \( U_{\bar{X}} \) unitary matrix such that \( [C_{012}^\dagger, \ldots, C_{1(\bar{X}-1)2}^\dagger] \) \( \bar{U}_{\bar{X}} = [C_{012}^\dagger, \ldots, C_{1(\bar{X}-1)2}^\dagger] \). |
| 6: **else** |
| 7: **end if** |
| **Step 3: Decomposing in the \( \bar{Z} \) dimension** |
| 8: Set \( f_k^X = 0 \) \( \forall k = [1, \ldots, r] \). |
| **Step 4: Obtain the set of frequencies** \( f_{1:r} \) |
| 9: **Output:** The recovered frequencies \( f_{1:r} = [f_{1:r}^X, f_{1:r}^Y, f_{1:r}^Z]^\top \), the full rank matrix \( R_n(\{f_1, \ldots, f_r\}) = \{r_n(f_1), \ldots, r_n(f_r)\} \) and \( P = R_n^S(\{f_1, \ldots, f_r\}) S_{XYZ} R_n^S(\{f_1, \ldots, f_r\}) \). |
streering vectors $\bar{Z} \rightarrow \bar{Y} \rightarrow \bar{X}$ and we always assume without loss of generality $\bar{X} \leq \bar{Y} \leq \bar{Z}$. Note that in case this assumption would not initially hold, we can always operate a proper rotation of the Cartesian system (which corresponds to a permutation of the elements of the measurement vector $y$, or a different reading of the antenna elements of the $d$-AD). This will ensure that we can always enforce the canonical ordering of Definition 4 on the set of PSD matrices, in the subsequent proposed min-rank optimization.

Furthermore, being $\mathcal{A}$ the sensing set defined in (5), let us introduce the following definition:

**Definition 5:** $\mathcal{A}^{(K)} \subseteq \mathcal{A}$ denotes the subset of all binary $N \times \bar{N}$ sensing matrices $\mathcal{A}$ for which it exists a subset of rows of $\mathbf{A} \in \mathcal{A}$, say $\mathcal{L} \subseteq \{N\}$ with $2K < |\mathcal{L}| \leq N$ such that:

1. **C** under Assumption 1, $\mathbf{A}^{(C)}$ is injective as map from $\mathbb{C}^{2K} \rightarrow \mathbb{C}^{|\mathcal{L}|}$;
2. **C** denoted by $f_{1:K}^* \mathbf{R}_N^c(f_{1:2K}^*)$ is injective as map from $\mathbb{C}^{2K} \rightarrow \mathbb{C}^{|\mathcal{L}|}$;
2. **C** denoted by $f_{1:K}^*$ a set of $K$ frequencies satisfying Assumption 1 and by $f_{1:r}^*$ arbitrary set of frequencies satisfying Assumption 1 such that $r^o \leq K$ frequencies such that for any $k \in [K]$ and $j \in [r^o]$, $f_j^o \neq f_k^o$, rank $\{\mathbf{A}^{(C)} \mathbf{R}_N^c(f_{1:K+r^o}^o)\} = K + \max\{\mathbf{A}^{(C)} \mathbf{R}_N^c(f_{1:K}^*)\}$, with $f_{1:K+r^o}^o = [f_{1:K}^* f_{1:r}^o]$.

In the following we provide, through Lemma 5 and Lemma 4 (see Appendices C and D), a class of sensing matrices of practical interest in $\mathcal{A}^{(K)}$. Specifically, in Lemma 5 (see Appendix D), we show that, provided that the number of elements in the $d$D-AD array is sufficiently large, $\mathcal{A}^{(K)} \equiv \mathcal{A}$ with high probability. In Lemma 4 we identify a second class of matrices, say $\mathcal{A}^{(K)}_c \subseteq \mathcal{A}$, for which we prove that $\mathcal{A}^{(K)}_c \subset \mathcal{A}^{(K)}$. Such class is identified by the so-called well structured sensing matrices as in the next two definitions:

**Definition 6:** A permutation matrix $\Pi$ belongs to the well structured permutation class $\mathcal{P}_c$, and note it as $\Pi^c$ if:

i) denoting by $\mathcal{I}$ the set of indices for the columns of $\Pi^c$ that have a 1 among its first $N$ rows, i.e., $j \in \mathcal{I}$ if $\exists n \in [N]$ such that $\pi_{nj} = 1$,

ii) denoting by $\mathcal{I}_c \subseteq \mathcal{I}$ the largest subset of ordered indices in $\mathcal{I}$ not necessarily consecutive, such that for any arbitrary vector $\mathbf{f} = [f^x, f^y, f^z]^\top \in \mathbb{T}^d$, the $N_c$ dimensional vector containing the elements of $\mathbf{r}_N^c(\mathbf{f})$ whose indices belong to $\mathcal{I}_c$ can be written as

$$
\mathbf{r}_N^c(\mathbf{f}) = e^{12\pi i (\Delta_x f^x + \Delta_y f^y + \Delta_z f^z)} \mathbf{r}_N^c(\mathbf{f}^c)
$$

where $\mathbf{r}_N^c(\mathbf{f}^c)$ corresponds to a $d$D-UD with structure given by $\mathbf{N}_c = [X_c, Y_c, Z_c]$, and $\mathbf{f}^c = [\ell_x f^x, \ell_y f^y, \ell_z f^z]^\top$ is a $d$-dimensional vector with $\ell_\alpha$ a proper positive integer, $\alpha \in \{x, y, z\}$, $\Delta_x \in [\bar{X}], \Delta_y \in [\bar{Y}], \Delta_z \in [\bar{Z}]$ and finally $S_c = X_c + Y_c + Z_c$, $N_c = |\mathcal{I}_c| = X_c Y_c Z_c$.

we have that $S_c \geq (d + 1)$ or $N_c \geq 2$.

---

Note that $\mathcal{I}$ identifies the components of the steering vector $\mathbf{r}_N^c(\mathbf{f})$ that are going to be sensed.
From Definition 6, it follows that:

**Definition 7:** The set of well structured sensing matrices is defined as $\mathcal{A}_c = \{ A \in \{0, 1\}^{N \times K} : [I_N | 0_{N \times (N - N)}] \Pi_c, \Pi_c \in \mathcal{P}_c \}$, with $\mathcal{P}_c$ as in Definition 6.

**Example 1:** Let’s assume the planar array from Fig. 1, whose encompassing virtual array is given by the 2D-UD structure characterized by $X = 1, Y = 3$ and $Z = 4$, and where the three antennas in red are removed by means of the sensing matrix. In this configuration we have that $N = 12, K = 9$ and we can find $I = \{2 : 5, 8 : 12\}, I = \{2 : 4, 10 : 12\}$, $N_c = [1, 2, 3], f^c = [f_x, 2f_y, f_z]^T$ and $\Delta_x = \Delta_y = 0, \Delta_z = 1$, such that $S_c = 6$ and $N_c = 6$. Then we say that the array in Fig. 1a, is a well structured array.

**Example 2:** A special subset of $\mathcal{P}_c$ is the set of all permutations matrices, say $\Pi_U$, whose associated sensing matrix $A_U = [I_N | 0_{N \times (N - N)}] \Pi_U$ is the one where only the first $X \leq X, Y \leq Y$, and $Z \leq Z$ antennas are sensed.

We are now ready to state our main results:

**Theorem 1:** Consider a linear measurement model as in (7). Under Assumption 1, provided that $K < \max\{X, Y, Z\}$, if $A \in \mathcal{A}(K)$ as in Definition 5, the $K$-scatter vector $s^*$ and its associated frequencies $f^*_{s:K} \in \mathbb{T}^{d \times K}$ can be uniquely identified by solving the optimization problem:

$$
\min_{r,s \in \mathbb{C}^N : S_{XYZ}, S_{XYZ}^\top \in \mathcal{T}_{XYZ}} \text{rank } \{S_{XYZ}\} \quad \text{s.t.} \quad \begin{bmatrix} S_{XYZ} & s \\ s^\top & r \end{bmatrix} \succeq 0, \; As = y. \quad \text{(P.3)}
$$

where $\mathcal{T}_{XYZ} \subseteq \mathbb{C}^{N \times N}$ denotes the set of all canonical PSD $d$-LT matrices (see Definition 4). The frequencies $f^*_{s:K} \in \mathbb{T}^{d \times K}$ can be obtained by Vandermonde Decomposition, via Algorithm 1, of the $d$-LT matrix $S_{XYZ}$ solution of (P.3).

**Proof:** The proof is provided in Appendix B.

**Remark 5:** Compared to the result provided in [14, Theorem 3], Theorem 1 is able to extend the region of resolvable scatters. In fact, differently from [14], where a sufficient condition for recovery is provided consisting of $K < \min\{X, Y, Z\}$, we significantly enlarge this condition to $K < \max\{X, Y, Z\}$. To do this, we first i) restructure the $N$-dimensional observable vector as the sampled version of a linear combination of steering vectors with Kronecker ordering $Z \rightarrow Y \rightarrow X$ such that $Z \geq Y \geq X$, ii) we search for the min-rank matrix $S_{XYZ}^c$, solution of (P.3), among all canonical $d$-LT matrices and finally iii) we decompose the canonical matrix $S_{XYZ}^c$, via Algorithm 1, to extract relevant frequency parameters. This procedure ensures a significant enlargement in the number of scatters $K$ that can be resolved, together with the error-free identification of the associated frequencies $f^*_{s:K} \in \mathbb{T}^{d \times K}$ providing information on the multi-dimensional propagation parameters of the aforementioned scatters.

**Corollary 1:** Theorem 1 holds for the special class of well structured sensing matrices (as per Definition 7), such that $S_c - (d - 1) \geq 2K$. We denote such class of sensing matrices by $\mathcal{A}_c^{(K)} \subset \mathcal{A}_c$. 
**Proof:** The proof is provided in Appendix C.

Corollary 1 refers to the class, $\mathcal{A}^{(K)}_c \subset \mathcal{A}_c$, of well structured sensing matrices for which $S_c \geq 2K + (d-1)$, and states that if $A \in \mathcal{A}^{(K)}_c$ the region of resolvable scatters is

$$K < \min\{\frac{S_c-(d-1)}{2}, \max\{\bar{X}, \bar{Y}, \bar{Z}\}\}.$$ 

An enlarged class of well structured sensing matrices which contains $\mathcal{A}^{(K)}_c \subset \mathcal{A}_c$ is the class of well structured sensing matrices such that $N_c > 2K$. For this enlarged class we conjecture that the region of resolvable scatters is enlarged to

$$K < \min\{\frac{N_c}{2}, \max\{\bar{X}, \bar{Y}, \bar{Z}\}\},$$

including the region stated in Corollary 1. More formally:

**Conjecture 1:** Theorem 1 holds for the subset of well structured sensing matrices (as per Definition 7), such that $N_c > 2K$.

Conjecture 1 validity will be shown by simulation in Section IV-A.

The previous results consider a class of sensing matrices with a specific structure which in some practical applications could not be satisfied or even difficult to verify. A natural question arises on how previous results extend to the case of sensing matrices with arbitrary measuring structures, if they do so. Next theorem answers to this question showing that frequency recovery is still possible, even in the presence of arbitrary sensing matrices, if the dimension of the measurement $N$ is larger than a certain threshold which linearly grows with the number of scatters we want to resolve. This implies that we can have a linear trade-off between hardware cost and resolvable capability. This result is of significant practical relevance specially in the context of intelligent smart surfaces with a massive distribution of antennas [20].

**Theorem 2:** Given a linear measurement model as in (7) with $A \in \mathcal{A}$ being an arbitrary binary sensing matrix as defined in (5), under Assumption 1 there exist with probability $1 - \epsilon$ a unique $K$-scatter vector $s^*$ as the solution to problem (P.3), and a unique frequency set $f^*_{1:K} \in \mathbb{T}^{d \times K}$ satisfying (7), provided that, i) $K < \max\{\bar{X}, \bar{Y}, \bar{Z}\}$ and ii) $N \geq 2KC \log (2K\epsilon^{-1})$ with $C$ being a proper constant which is not larger than 12. The frequency set $f^*_{1:K} \in \mathbb{T}^{d \times K}$ is obtained by Vandermonde Decomposition, via Algorithm 1 of the $d$-LT matrix $S_{XYZ}$ solution of (P.3).

**Proof:** The proof is provided in Appendix D.

3) **Reconstruction via Convex Relaxation:** In the previous section, we focused on the atomic $\ell_0$-AN reconstruction. However, the $\ell_0$-AN minimization, as well as its equivalent reformulation in terms of a rank minimization problem, are NP-hard [21]. Nonetheless, keeping in mind that the atomic $\ell_q$-norm approaches the atomic $\ell_0$-norm as $q$ tends to zero, the usual approach is to approximate (P.1), or equivalently (P.3), via the following optimization problem:

$$\min_{s \in \mathbb{C}^N} \|s\|_{\mathcal{R},1} \quad \text{s.t.} \quad As = y,$$

(P.4)
where \( \| s \|_{\mathcal{R}, 1} \) denotes the atomic \( \ell_1 \)-norm (\( \ell_1 \)-AN) of a vector \( s \) in \( \mathcal{R} = \{ \mathbf{r}_N(\mathbf{f}) : \mathbf{f} \in \mathbb{T}^d \} \). Given by \( \| s \|_{\mathcal{R}, 1} = \inf_{\mathbf{f}_k \in \mathbb{T}^d, u_k \in \mathbb{C}} \left\{ \sum_k |u_k| : \mathbf{s} = \sum_k u_k \mathbf{r}_N(\mathbf{f}_k) \right\} \).

In [11], a semidefinite characterization of the \( \ell_1 \) atomic norm was introduced for \( d = 1 \) and generalized to arbitrary \( d \) dimensions in [14] as follows:

\[
\min_{t, s \in \mathbb{C}^N, T_{\mathbf{x} \mathbf{y} \mathbf{z}} \in \mathbb{T}_{\mathbf{x} \mathbf{y} \mathbf{z}}} \frac{1}{2} t + \frac{1}{2} \text{Tr} \{ T_{\mathbf{x} \mathbf{y} \mathbf{z}} \} \quad \text{s.t.} \quad \begin{bmatrix} T_{\mathbf{x} \mathbf{y} \mathbf{z}} & s \\ s^\dagger & t \end{bmatrix} \succeq 0, \quad A \mathbf{s} = \mathbf{y}. \quad (P.5)
\]

In the next two theorems, we provide recovery results for the convex optimization problem (P.5), both for the sensing set \( \mathcal{A}_c^{(K)} \) and for an arbitrary sensing set where the dimension of the measurement \( N \) is larger than a threshold. These scenarios would be analogous respectively to Corollary 1 and Theorem 2 for the non-convex problem (P.3).

**Theorem 3:** Consider a linear measurement model as in (7). Under Assumption 1, if \( \mathbf{A} \in \mathcal{A}_c^{(K)} \) as defined in Corollary 1 there exists a unique \( K \)-scatter vector \( \mathbf{s}^* \) as the solution to problem (P.5) and a unique frequency set \( \mathbf{f}^{*,K}_1 \in \mathbb{T}^{d \times K} \) satisfying \( \mathbf{y} = \mathbf{A} \sum_k u_k^* \mathbf{r}_N(\mathbf{f}^*_k) \) provided that, i) \( K < \max\{X, Y, Z\} \) and \( K \leq (S_c - (d - 1))/2 \), ii) the optimal solution to (P.5), denoted by \( (t^o, s^o_t, T^o_{\mathbf{x} \mathbf{y} \mathbf{z}}) \), also satisfies \( r^o_{\ell_1} < \min\{X, Y, Z\} \) and \( r^o_{\ell_1} \leq (S_c - (d - 1))/2 \) with \( r^o_{\ell_1} \triangleq \text{rank} \{ T^o_{\mathbf{x} \mathbf{y} \mathbf{z}} \} \).

**Proof:** Theorem 3 is easily proved following the same derivations of Theorem 1 and making use of the rank condition of the solution in (P.5). ■

Analogously to what considered for Corollary 1, we have numerically verified the following conjecture:

**Conjecture 2:** Theorem 3 holds for the subset of \textit{well structured} sensing matrices (as per Definition 7), provided that i) \( K < \min\{\frac{N}{T}, \max\{X, Y, Z\}\} \), and ii) \( r^o_{\ell_1} < \min\{\frac{N}{T}, \max\{X, Y, Z\}\} \).

**Theorem 4:** Given a linear measurement model as in (7) with \( \mathbf{A} \in \mathcal{A} \) being an arbitrary binary sensing matrix as defined in (5), under Assumption 1 there exist with probability \( 1 - \epsilon \) a unique \( K \)-scatter vector \( \mathbf{s}^* \) as the solution to problem (P.5), and a unique frequency set \( \mathbf{f}^{*,K}_1 \in \mathbb{T}^{d \times K} \) satisfying (7), provided that, i) \( K < \min\{\frac{N}{T}, \max\{X, Y, Z\}\} \), ii) \( N \geq 2KC \log(2K \epsilon^{-1}) \) with \( C \) being a proper constant which is not larger than 12, and iii) the optimal solution to (P.5) noted as \( (t^o, s^o_t, T^o_{\mathbf{x} \mathbf{y} \mathbf{z}}) \) with rank \( \{ T^o_{\mathbf{x} \mathbf{y} \mathbf{z}} \} \triangleq r^o_{\ell_1} \) also satisfies \( r^o_{\ell_1} < \min\{\frac{N}{T}, \max\{X, Y, Z\}\} \).

**Proof:** The proof is similar to the proof of Theorem 2 except that following Lemma 5 we exploit that \( \mathbf{A} \in \mathcal{A}^{(K)} \) holds with probability \( 1 - \epsilon \) under condition ii) \( N \geq 2KC \log(2K \epsilon^{-1}) \). ■

**B. Improved convex relaxation via \( \ell_2 + \ell_1 \)-AN optimization**

In this section, we highlight some shortcomings of the convex recovery approach in (P.5) and formulate an alternative convex optimization to address these issues. i) Recall that the recovery problem (P.5) is
formulated under a noise-free assumption. ii) The conditions in Th. 3 guarantee an error–free recovery under a rank condition on the solution to the optimization problem which ensures a unique Vandermonde decomposition of $T^s_{XYZ}$ that assures the sparse nature of the minimizer $s^*_\ell_1$ solution to (P.5)⁴.

To address the aforementioned challenges, we formulate a convex recovery problem as the extension of (P.5) by formulating a convex combination, parameterized in a regularization parameter $\tau$, of the the distance between $A$s and $y$ (i.e. $\ell_2$-norm of the difference $y - As$) and the $\ell_1$-norm of $s$, i.e.,

$$
\min_{t,s \in \mathbb{C}^N, T_{XYZ} \in \mathbb{T}_{S^2}} \left( (1 - \tau) \|As - y\|_2^2 + \tau \left( \frac{1}{2} t + \frac{1}{2} \text{Tr} \{ T_{XYZ} \} \right) \right) \text{ s.t. } \begin{bmatrix} T_{XYZ} & s \end{bmatrix} \begin{bmatrix} s \end{bmatrix} \geq 0. \tag{P.6}
$$

where $\tau$ in (P.6), can be properly optimized to minimize the average frequency recovery error. Let $\tau_o$ be such optimal value. Clearly $\tau_o$ is function of the SNR defined as $\text{SNR} = \frac{E[|u^*|^2]}{\sigma^2}$. Unfortunately, an explicit expression for $\tau_o$ is hard to derive. However, upper and lower bounds of the optimal $\tau$ have been provided in [10] as a function of SNR:

$$
\tau_u = \frac{2 \sigma \left( 1 + \frac{1}{\log(N)} \right) \sqrt{N \log(N) + N \log(4\pi \log(N))}}{1 + 2 \sigma \left( 1 + \frac{1}{\log(N)} \right) \sqrt{N \log(N) + N \log(4\pi \log(N))}} \quad \text{and} \quad \tau_l = \frac{2 \sigma \sqrt{N \log(N) - \frac{N}{2} \log(4\pi \log(N))}}{1 + 2 \sigma \sqrt{N \log(N) - \frac{N}{2} \log(4\pi \log(N))}}
$$

with $\tau_l \leq \tau_o \leq \tau_u$.

Using classical tools from noise sparse representation, bounds on the frequency recovery errors can be derived [21]. Due to space constraints, we do not include these generalizations in this paper and leave this topic as a potential direction for future research. Instead, in Section IV, we investigate the performance of the convex recovery problem in (P.6) via simulation.

IV. APPLICATION SCENARIOS AND FREQUENCY RECOVERY PERFORMANCE

The definition of well structured sensing matrices (i.e., Definition 6) provides an operational characterization of Definition 5 in the sense that Definition 6 identifies simple parameters that allow establishing rules that facilitate verifying the conditions needed for signal recovery uniqueness (conditions 1.C and 2.C in Section III-A). The way in which the definition of well structured sensing matrices translates into the physical structure of an arbitrary non-uniform 3D array (referred from now on as physical array) is in finding an underlying uniform antenna structure, embedded in the non-uniform array (see examples in sections III-A2 and IV-A) and its associated geometric parameters whose relationship with the number of scatters guarantee the possibility of signal recovery. More precisely, given an arbitrary non-uniform physical array, in order to identify its resolvable region, i.e., the number of scatters that can be resolved, the first step is to find a uniform sub-array which corresponds to the largest uniform structure embedded

⁴Recall, in fact, that $\ell_1$-atomic norm minimizers are not always sparse.
in the original non-uniform deployment, characterized by \( N_c = [X_c, Y_c, Z_c] \), \( S_c = X_c + Y_c + Z_c \) and \( N_c = X_c Y_c Z_c \), where \( X_c, Y_c, Z_c \) represent the number of antennas in each dimension. Based on these physical parameters, we prove (see Corollary 1 and Theorem 3) that any number of scatters smaller or equal than \((S_c-(d-1))/2\) can be resolved. We strongly conjecture, as validated by our numerical results (see Section IV-A), that the resolvable region is in fact \( N_c/2 \) (see Conjectures 1-2). While evaluating the resolvable region requires identifying an embedded uniform sub-array, in order to recover the multidimensional propagation parameters, we also need to identify an encompassing virtual uniform array, i.e., an expanded virtual uniform structure that contains the original physical structure as a sub-array, characterized by \( \overline{N} = [\overline{X}, \overline{Y}, \overline{Z}] \) and \( \overline{N} = \overline{X} \overline{Y} \overline{Z} \) with \( \overline{X}, \overline{Y}, \overline{Z} \) denoting the number of antennas in each dimension, with \( \overline{Z} \) the dominant dimension by convention. A proper identification of the virtual uniform array allows recovering the propagation parameters of a given number of possible scatters up to the physical limit \( K \leq (S_c-(d-1))/2 \) (or \( K < N/2 - 1 \) as we conjecture) if \( \overline{Z} > K \). Since the dimensions of the virtual array influence the computational complexity of the proposed recovery algorithms, they can be seen as tunable parameters that enable trading computational complexity with frequency-recovery performance. Furthermore, as already pointed in Section I-A our results provide guidelines for the design of the non-uniform array geometry. In fact, setting a geometric deployment of the array such that \( Z_c = 2K \), \( Y_c = 2 \) and \( X_c \leq 2 \), and given that the complexity of (P.5) and (P.6) scales as \( \gamma = 3.5 \) with the dimension of the problem variables, we have that up to \( K = (S_c-(d-1))/2 \) scatters can be resolved with \( O(K^{\gamma}) \) complexity, significantly improving existing results that require \( O(K^{d\gamma}) \) complexity.

### A. Resolvable region and recovery performance

We provide next some numerical results to show the resolvable region and the frequency recovery performance of \( \ell_0\)-AN and \( \ell_1\)-AN for different 2D and 3D physical antenna deployments. Given a set of measurements following (6), the error \( 1/d \|\tilde{f}^{\circ}_{1:K} - \tilde{f}^*_{1:K}\|_1 \) between the recovered \( d \)-dimensional frequencies \( \tilde{f}^{\circ}_{1:K} \) and the real frequencies \( \tilde{f}^*_{1:K} \) is computed and presented both for noiseless and noisy scenarios. In all the results following, the number of scatters \( K \) is assumed to be known.

1) Noiseless measurements: We consider a 2D-UD physical planar array (uniform structure) and a 3D-AD physical cubic deployment (non-uniform structure). The 2D-UD array is characterized by \( N = [1, 3, 6] \) and here the largest underlying uniform subarray coincides with the physical array, i.e., \( \overline{N} = N_c \), then \( S_c = 10 \) and \( N_c = 18 \). In order to reach all the points in the resolvable region, the 2D-UD is enlarged with three virtual 2D-UD: \( \overline{N} = \{ [1, 3, 6], [1, 3, 10], [1, 8, 10] \} \). Similarly the 3D-AD physical cubic deployment has \( 4 \times 4 \) active antennas deployed uniformly in each face of a cube and
Physical array  |  Physical parameters  |  Corollary 1  |  Conjecture 1
--- | --- | --- | ---
2D-UD  |  \( N = N_c = [1, 3, 6] \), \( S_c = 10 \), \( N_c = 18 \)  |  \( K \leq \left\lfloor \frac{S_c - (d - 1)}{2} \right\rfloor \)  |  \( K \leq \left\lceil \frac{N_c}{2} - 1 \right\rceil \)
3D-AD cubic  |  \( N_c = [2, 4, 4] \), \( S_c = 10 \), \( N_c = 32 \)  |  \( K \leq 4 \)  |  \( K \leq 15 \)

TABLE I
RESOLVABLE REGION UNDER COROLLARY 1 AND CONJECTURE 1 FOR THE PHYSICAL ARRAY CONFIGURATIONS EXPLORED.

no antennas are deployed inside the cube. The 3D-AD has a total of \( N = 56 \) active antennas and its largest underlying 3D-UD subarray embedded in the non-uniform structure, is given by \( N_c = [2, 4, 4] \), with \( S_c = 10 \) and \( N_c = 32 \). Two 3D-UD virtual deployments are considered for the cubic deployment: \( \bar{N} = [4, 4, 4] \) with \( \bar{N} = 64 \) and \( \bar{N} = [6, 6, 6] \) with \( \bar{N} = 216 \).

For the aforementioned physical deployments we evaluate, over 1000 measurements, i) the recovery performance and uniqueness of the decomposition proposed in Lemma 1, ii) the resolvable region and error free recovery conditions of the \( \ell_0 \)-AN, and iii) the performance of the full \( \ell_1 \)-AN optimization (P.5). The results are shown in Fig. 2(a-b). While the plots confirm Corollary 1 resolvability region, they also validate Conjecture 1. Refer to Table I for more detail on the resolvable region. In Fig. 2(a-b), the plots labelled as Lm. 1 correspond to the frequency error between the real frequencies \( f^*_{1:K} \) and the recovered \( d \)-dimensional frequencies \( \hat{f}_{1:K} \) obtained via Vandermonde decomposition applying Algorithm 1 directly to \( S_{XYZ} = \sum_{k=1}^{K} |u_k|^2 r_N(f_k) r_N(f_k)^\dagger \). Fig. 2(a-b) show that, as stated in Lemma 1 as long as the condition \( K < \max\{X, Y, Z\} \) holds (i.e., the Vandermonde decomposition is unique), the
frequencies can be error-free recovered. The performance of $\ell_0$-AN in (P.3) is evaluated in Fig. 2(a-b) labelled as $\ell_0$-AN. It is shown in all the cases that we have error-free recovery as long as the number of scatters falls within the physical resolvable region (see Table I) and $K < \max\{\bar{X}, \bar{Y}, \bar{Z}\}$. Finally, the performance of the frequency recovery based on $\ell_1$-AN optimization (P.5) is also given in Fig. 2. Enlarging the virtual array, in both uniform and non-uniform settings, improves the recovery performance of the $\ell_1$-AN without additional hardware cost but with an increase of the computational complexity.

2) $\ell_2 + \ell_1$-AN optimization for noiseless measurements: Next, we evaluate the performance of the $\ell_2 + \ell_1$-AN optimization problem in (P.6) for noiseless measurements. The results are shown in Fig. 3(a) for $A = A_U$, $N = N_c = [1, 6, 6]$, $\bar{N} = \{[1, 6, 6], [1, 6, 8]\}$ and $K = \{4, 5, 6\}$. Note that the number of scatters in all cases lay within the lower bound of the resolvable region given in Table I and also $K < \max\{\bar{X}, \bar{Y}, \bar{Z}\} = \bar{Z}$. It is observed, that as the $\ell_1$-AN error increases the joint $\ell_2 + \ell_1$-AN approach significantly outperforms $\ell_1$-AN, providing a smaller error than the one provided for the virtually enlarged array if $\tau$ is properly optimized. For example for $K = \{5, 6\}$ the $\bar{N} = \{[1, 6, 8]\}$ scenario significantly outperforms the $\ell_1$-AN.

3) Noisy measurements: Finally, we study the noisy scenario frequency recovery performance of (P.6) for different SNR scenarios and the 3D-AD cubic array. First, it is explored the behavior with respect to the regularization parameter $\tau$ for $SNR = \{0, 5, 10, 15\}$ (Fig. 3(b)). As expected, the regularization
parameter $\tau_0$ that minimizes the error depends on SNR. Furthermore, for a given SNR, the error achieved by $\tau_0$ is almost coincident for the different virtual configurations, i.e., at $\tau_0$, there is no improvement in recovery performance by enlarging the virtual array. From the plots, we can see that $\tau_u$ well approximate the optimal $\tau_0$ for a large range of SNR. Regarding the performance of the different virtual configurations, it is observed that at low SNR values there is a gain provided by the virtual enlarging. Finally we compare the best recovery performance given by $\tau_0$ with the error performance for the upper bound $\tau_u$, the noiseless $\ell_1$-AN and the CRLB in Fig. 3(c). The plots show that the performance achieved by the proposed $\ell_2 + \ell_1$-AN approach are very close to the associated CRLB.

V. Conclusions

There is a growing need for multi-dimensional characterization of key performance parameters such as the signal AoAs in order to enable the provisioning of future wireless services. In this work, we have shown under which conditions it is possible to recover a set of $K_d$-dimensional parameters from a noiseless/noisy linear parametric measurement model that contains a $K$-sparse mixture of the array steering vectors particularized in each of the AoAs. Finding the $K$ AoAs is formulated in terms of a multi–level Toeplitz matrix rank–minimization problem with a pertinent convex relaxation procedure that ensures the extraction of the relevant parameters in polynomial time.

APPENDIX A

PROOF OF LEMMA 1

To prove Lemma 1, let first observe that for $d = 1$ (i.e., $\bar{X} = \bar{Y} = 1$), the result is immediate and well known (see [12], [14], [16]). For $2 \leq d \leq 3$, we will use a result that follows from [14, Lemma 2] and [24, Proposition 1] that we will state in a slightly different way which is more useful for our purposes.

Lemma 2 ([14, Lemma 2], [24, Lemma in Proposition 1]): Let $S_{QT} \in \mathbb{C}^{qt \times qt}$ be a $q \times q$ PSD block Toeplitz matrix with rank equal to $r < t$, where each block is a $t \times t$ matrix. Then, it exists a matrix $G_t = [g_1, \ldots, g_r]$ and a set of $r$ frequencies $f_j \in \mathbb{T}$, with $j \in [r]$, such that $S_{QT} = \sum_{j=1}^{r} r_q(f_j) r_q(f_j) \hat{\otimes} g_j g_j^\dagger = \sum_{j=1}^{r} (r_q(f_j) \otimes g_j) (r_q(f_j) \otimes g_j)^\dagger$, and $S_{aT} = G_t \text{diag}(e^{j2\pi a f_1}, \ldots, e^{j2\pi a f_r}) G_t^\dagger$, where $S_{aT}$, with $-q + 1 \leq a \leq q - 1$, denotes the matrix on the $a$-th block of $S_{QT}$.

We are now ready to prove Lemma 1 for $d = 2$, i.e., $\bar{X} = 1$ and $1 < \bar{Y} \leq \bar{Z}$. In this case, $S = S_{0YZ}$ is a PSD 2-LT matrix and its upper-left block $S_{00Z}$ is a PSD 1-LT matrix of dimension $\bar{Y}\bar{Z}$ and $\bar{Z}$, respectively, with $\text{rank} \{S_{0YZ}\} = \text{rank} \{S_{00Z}\} = r < \bar{Z}$ by assumption. Furthermore, note that since $S_{0YZ}$ is a PSD 2-LT matrix, then it is also a $\bar{Y} \times \bar{Y}$ PSD block Toeplitz matrix with rank equal to $r < \bar{Z}$.

Note that while the expression of $S_{aT}$ is not explicitly stated in [14, Lemma 2], it is stated as part of the corresponding proof in [14, Eq. (19)].
by assumption, and with each block being a $\bar{Z} \times \bar{Z}$ matrix. Hence, applying Lemma 2 with $q = Y$ and $t = \bar{Z}$, we have that there exists a matrix $G_{\bar{Z}} = [g_{ij}^z, \ldots, g_{ij}^x]$ and a set of $r$ frequencies $f_j^y \in \mathbb{T}$, with $j \in [r]$, such that $S_{0Y\bar{Z}} = \sum_{j=1}^r r_j (f_j^y) r_j(f_j^y)^\dagger \otimes g_j^y g_j^x = \sum_{j=1}^r (r_j(f_j^y) \otimes g_j^y) (r_j(f_j^y) \otimes g_j^x)^\dagger = C_{Y\bar{Z}} G_{Y\bar{Z}}^\dagger$ and

$$S_{0b\bar{Z}} = \sum_{j=1}^r e^{j 2 \pi b f_j^y} g_j^y g_j^x = G_{Z} \text{diag}(e^{j 2 \pi b f_1^y}, \ldots, e^{j 2 \pi b f_r^y}) G_{Z}^\dagger = G_{Z} Y^b C_{Z}^\dagger$$

where $S_{0b\bar{Z}}$, with $-Y + 1 \leq b \leq Y - 1$, denotes the generic $b$-block diagonal of $S_{0Y\bar{Z}}$, and $Y = \text{diag}(e^{2 \pi f_1^y}, e^{2 \pi f_2^y}, \ldots, e^{2 \pi f_r^y})$. In particular, if we set $b = 0$, we have that $S_{00\bar{Z}} = G_{Z} G_{Z}^\dagger$. Furthermore, since by assumption $S_{00\bar{Z}}$ is PSD $1$-LT matrix of dimension $\bar{Z}$ whose rank $r$ is strictly smaller than $\bar{Z}$, then $S_{00\bar{Z}}$ admits a unique Vandermonde decomposition of order $r$ (see [12], [14], [16]), i.e., $S_{00\bar{Z}} = R_{Z}(f_{1,r}^z) P R_{Z}^\dagger(f_{1,r}^z)$, where $R_{Z}(f_{1,r}^z) = [r_1(f_1^z), \ldots, r_1(f_r^z)]$. Given that $S_{00\bar{Z}} = G_{Z} G_{Z}^\dagger = R_{Z}(f_{1,r}^z) P R_{Z}^\dagger(f_{1,r}^z)$, we can always find a unitary matrix, $O_{Z}$ such that $G_{Z} = R_{Z}(f_{1,r}^z) P^{1/2} O_{Z}$ and $S_{0b\bar{Z}}$ can be written as $S_{0b\bar{Z}} = R_{Z}(f_{1,r}^z) P^{1/2} O_{Z} Y^b O_{Z}^\dagger P^{1/2} R_{Z}^\dagger(f_{1,r}^z)$. Since for all $b$ with $-Y + 1 \leq b \leq Y - 1$, $S_{0b\bar{Z}}$ is a $1$-LT matrix, using [14] Lemma 3 for $d = 1$, it follows immediately that the matrix $D_{Y} = O_{Z} Y^b O_{Z}^\dagger$ has to be diagonal. Furthermore, letting $D_{Y} = O_{Z} Y O_{Z}^\dagger$, it is immediate to verify that $D_{Y} = O_{Z} Y^b O_{Z}^\dagger$ and that $D_{Y} D_{Y} = I_r$. Therefore, $D_{Y}$ is a diagonal matrix whose diagonal elements are complex number with modulo 1, i.e., $D_{Y} = \text{diag}(e^{2 \pi f_1^y}, \ldots, e^{2 \pi f_r^y})$, with $f_i^y \in \mathbb{T}$. Then we have that $S_{0b\bar{Z}} = R_{Z}(f_{1,r}^z) P^{1/2} \text{diag}(e^{2 \pi f_1^y}, \ldots, e^{2 \pi f_r^y}) P^{1/2} R_{Z}^\dagger(f_{1,r}^z)$ and after some algebraic manipulations, it is easy to show that:

$$S_{0Y\bar{Z}} = \sum_{j=1}^r p_j (r_j(f_j^y) \otimes r_j(f_j^y)^\dagger) (r_j(f_j^y) \otimes r_j(f_j^y)) G_{Y\bar{Z}} Y^b C_{Z}^\dagger = R_{Y\bar{Z}}(f_{1,r}^z) P R_{Y\bar{Z}}^\dagger(f_{1,r}^z).$$

Now that we have proved the decomposition for $d = 2$, we can proceed in proving it for $d = 3$. To this end, let us now assume that $S = S_{X\bar{Y}Z}$ is a PSD $3$-LT matrix, i.e., $1 \leq X \leq Y \leq \bar{Z}$ and hence the sub-matrix $S_{0Y\bar{Z}}$ is a PSD $2$-LT matrix of dimension $\bar{Y} \bar{Z}$ and the a-block $S_{aY\bar{Z}}$ with $-X + 1 \leq a \leq X - 1$ is also a $2$-LT matrix. By assumption we have that the rank $r$ of $S_{X\bar{Y}Z}$ satisfies $r = \text{rank } \{S_{X\bar{Y}Z}\} = \text{rank } \{S_{00\bar{Z}}\} < \bar{Z}$. Furthermore, since $S_{X\bar{Y}Z}$ is a PSD $3$-LT matrix, then it is $\bar{X} \times \bar{X}$ PSD block Toeplitz matrix whose blocks are $2$-LT matrices. Therefore, we can apply Lemma 3 with $q = \bar{X}$ and $t = \bar{Y} \bar{Z}$, from which it follows that it exists a matrix $G_{Y\bar{Z}} = [g_{ij}^{yz}, \ldots, g_{ij}^{yz}]$ and $f_j^z \in \mathbb{T}$, with $j \in [r]$, such that $S_{X\bar{Y}Z} = \sum_{j=1}^r r_j (f_j^z) r_j(f_j^z)^\dagger \otimes g_j^{yz} g_j^{yz} = C_{X\bar{Y}Z} C_{X\bar{Y}Z}^\dagger$, and the generic a-block of $S_{X\bar{Y}Z}$ with $-X + 1 \leq a \leq X - 1$, can be written as

$$S_{aY\bar{Z}} = \sum_{j=1}^r e^{j 2 \pi a f_j^z} g_j^{yz} g_j^{yz} = G_{Y\bar{Z}} \text{diag}(e^{j 2 \pi a f_1^z}, \ldots, e^{j 2 \pi a f_r^z}) G_{Y\bar{Z}}^\dagger = G_{Y\bar{Z}} X^a G_{Y\bar{Z}}^\dagger$$

with $X = \text{diag}(e^{2 \pi f_1^z}, \ldots, e^{2 \pi f_r^z})$. Furthermore, its $\bar{Y} \bar{Z} \times \bar{Y} \bar{Z}$ upper-left corner, say $S_{0Y\bar{Z}}$, is a $2$-LT matrix, composed by the first $\bar{Y} \bar{Z}$ rows and the first $\bar{Y} \bar{Z}$ columns of $S_{X\bar{Y}Z}$, whose $\bar{Z} \times \bar{Z}$ upper-left block of
such that \( S_{0YZ} \), say \( S_{00Z} \), is a 1-LT matrix. It is immediate to prove that rank \( \{S_{0YZ}\} = \text{rank} \{S_{00Z}\} = r < \bar{Z} \). In fact rank \( \{S_{00Z}\} = r \leq \text{rank} \{S_{0YZ}\} = \text{rank} \{S_{XYZ}\} = r \). Hence, the assumption of Lemma \([1]\) proved previously for \( d = 2 \), are satisfied by \( S_{0YZ} \). Therefore, \( S_{0YZ} \) admits a Vandermonde decomposition of order \( r \), i.e \( S_{0YZ} = R_{YZ}(f_{1:r}^x)PR_{YZ}^{\dagger}(P_{1:r}) \). Following steps and arguments very similar to the case \( d = 2 \) and using again \([14, \text{Lemma 3}]\) to the generic 2-LT matrix block \( S_{YZ} \), it is straightforward, after some algebraic manipulations, to verify that:

\[
S_{XYZ} = \sum_{j=1}^r p_j r_N(f_j)r_N(f_j)^\dagger = R_N(f_{1:r})PR_N^{\dagger}(f_{1:r}).
\]  

(13)

In order complete the proof, both for \( d = 2 \) and \( d = 3 \), we need to show uniqueness of the Vandermonde decomposition given in \([13]\). To do this let assume that it exists, it is unique. This completes the proof.

\[
\{r_N(f_j)e\}_{j=1}^r \subset \{r_N(f_j)e\}_{j=1}^r.
\]

This implies that the \( r+1 \) steering vectors \( \{\{r_N(f_j)e\}_{j=1}^r \cup r_N(f'_k)e\} \) are linearly dependent.

Next, denoting by \( R_N(f_{1:r}, f'_k) = [r_N(f_{1:r}), r_N(f'_k)] \) the \( N \times (r+1) \) matrix, we have that \( R_N(f_{1:r}, f'_k) = R_X(f_{1:r}, f'_k) \oplus R_Y(f_{1:r}, f'_k) \oplus R_Z(f_{1:r}, f'_k) \) from which it follows that the rank of \( R_N(f_{1:r}, f'_k) \) satisfies the following inequality (see \([25, \text{Lemma 1}]\)):

\[
\text{rank} \{R_N(f_{1:r}, f'_k)\} \geq \min \{\text{rank} \{Z(f_{1:r})\} + \text{rank} \{Y(f_{1:r})\} + \text{rank} \{X(f_{1:r})\} - 1 + a_x, r+1\}
\]

(15)

where \( r_Z = \text{rank} \{Z(f_{1:r})\} \), \( r_Y = \text{rank} \{Y(f_{1:r})\} \), \( r_X = \text{rank} \{X(f_{1:r})\} \), while \( a_x = 1 \) if \( f'_{k} \notin \{f_{j}\}_{j=1}^r \) and \( \beta^x > r \) and zero otherwise, with \( \{x, y, z\} \) and \( \beta^x = X, \beta^y = Y \) and \( \beta^z = Z \). From \([15]\), it follows that in order for \([14]\) to be valid for any \( k \in \{1, \ldots, r\} \), it is necessary that for any \( \{f'_k\}_{k=1}^r \subset \{f_j\}_{j=1}^r \), i.e \( a_x + a_y + a_x = 0 \). By similar argument we can prove that \( \{f'_k\}_{k=1}^r \supset \{f_j\}_{j=1}^r \), from which it follows immediately that also \( p'_k = p_k \) for all \( k \in \{1, \ldots, r\} \) and consequently the decomposition is unique. Hence if Vandermonde decomposition in \([11]\) for \( d = 2 \) or \([13]\) for \( d = 3 \) exists, it is unique. This completes the proof.
APPENDIX B
PROOF OF THEOREM 1

In the following, we will prove that given the assumption of Theorem 1, i.e., $K < \max \{\bar{X}, \bar{Y}, \bar{Z}\}$, denoting by $(r^o, s^o, S^o_{\overline{X}Y\overline{Z}})$ the optimal solution to (P.3) is given by

$$
(r^o, s^o, S^o_{\overline{X}Y\overline{Z}}) = \left( K, \sum_{k=1}^{K} u_k^r r_N(f_k^r), \sum_{k=1}^{K} |u_k^r|^2 r_N(f_k^r)r_N(f_k^r) \right)
$$

and is unique in terms of $s^o = s^* = \sum_{k=1}^{K} u_k^r r_N(f_k^r)$ and in terms of the frequencies $f_k^{1:K}$ identified using Algorithm 1.

To prove (16), let first observe that $\left( K, \sum_{k=1}^{K} u_k^r r_N(f_k^r), \sum_{k=1}^{K} |u_k^r|^2 r_N(f_k^r)r_N(f_k^r) \right)$ is a feasible solution for (P.3), therefore $\text{rank} \{S^o_{\overline{X}Y\overline{Z}}\} \geq r^o \leq K < \max \{\bar{X}, \bar{Y}, \bar{Z}\}$.

On the other hand, it will be proved next, 1) that $K \leq r^o$, 2) that $s^o$ is unique and equal to $s^* = \sum_{k=1}^{K} u_k^r r_N(f_k^r)$ and finally 3) that $S^o_{\overline{X}Y\overline{Z}}$ has also a unique decomposition in terms of the $f_k^{1:K}$ identified using Algorithm 1.

In the following we assume $d = 3$, nevertheless the proof can be easily particularized for $d \leq 2$ and $d > 3$. By formulation of (P.3), the optimal solution $S^o_{\overline{X}Y\overline{Z}}$ belongs to $G_{\overline{X}Y\overline{Z}} \subseteq \mathbb{C}^{\bar{X} \times \bar{Y} \times \bar{Z}}$ which denotes the set of all PSD 3-LT matrices of dimension $\bar{X} \times \bar{Y} \times \bar{Z}$, with canonical ordered structure. Consequently, its $\bar{Y} \times \bar{Y}$ upper block $S^o_{\overline{Y}Z}$ is a PSD 2-LT matrix and the $\bar{Z} \times \bar{Z}$ upper block $S^o_{\overline{Z}0Z}$ is a PSD 1-LT matrix. We denote the rank of each of these matrices as $r^o = \text{rank} \{S^o_{\overline{X}Y\overline{Z}}\}$, $r^o_{\overline{Y}Z} = \text{rank} \{S^o_{\overline{Y}Z}\}$ and $r^o_{\overline{Z}0Z} = \text{rank} \{S^o_{\overline{Z}0Z}\}$ and we have that $r^o_{\overline{Y}Z} \leq r^o_{\overline{Z}0Z} \leq r^o < \max \{\bar{X}, \bar{Y}, \bar{Z}\} = \bar{Z}$. By Lemma 2, we have that it exists a matrix $G_{\overline{Y}Z} = [g_{1,\bar{Y}}^{\bar{Y}}, \ldots, g_{\bar{Y},\bar{Y}}^{\bar{Y}}]$ and $f_{j}^{y\bar{Y}} \in \mathbb{T}$, with $j \in [r^o]$ such that $S^o_{\overline{X}Y\overline{Z}} = \sum_{j=1}^{r^o} (f_{j}^{y\bar{Y}} \otimes g_{j}^{y\bar{Y}})(f_{j}^{y\bar{Y}} \otimes g_{j}^{y\bar{Y}})^\dagger = C_{\overline{X}Y\overline{Z}} C_{\overline{Y}Z}^\dagger$, where

$$
C_{\overline{X}Y\overline{Z}} = [r_{X}(f_{1}^{y\bar{Y}} \otimes g_{1}^{y\bar{Y}}, \ldots, r_{X}(f_{r^o}^{y\bar{Y}} \otimes g_{r^o}^{y\bar{Y}}) = R_{X}(f_{1}^{y\bar{Y}}) \otimes G_{\overline{Y}Z}.
$$

Also denoting $S^o_{\overline{Y}Z}$ as the $a$-th block of $S^o_{\overline{X}Y\overline{Z}}$ for $-\bar{X} + 1 \leq a \leq \bar{X} - 1$ we have that $S^o_{\overline{Y}Z} = \sum_{j=1}^{r^o} e^{j2\pi f_{j}^{y\bar{Y}} \otimes g_{j}^{y\bar{Y}}}$, $g_{j}^{y\bar{Y}} = G_{\overline{Y}Z} X_{j}^{a} G_{\overline{Y}Z}^{\dagger}$ with $X = \text{diag}(e^{j2\pi f_{1}^{y\bar{Y}}}, e^{j2\pi f_{2}^{y\bar{Y}}}, \ldots, e^{j2\pi f_{r^o}^{y\bar{Y}}})$. Similarly, applying Lemma 2 to $S^o_{\overline{Z}0Z}$ it exists a matrix $G_{\overline{Z}} = [g_{1,\bar{Z}}^{\bar{Z}}, \ldots, g_{\bar{Z},\bar{Z}}^{\bar{Z}}]$ and $f_{j}^{y\bar{Z}} \in \mathbb{T}$, with $j \in [r^o_{\overline{Z}0Z}]$, such that $S^o_{\overline{Y}Z} = \sum_{j=1}^{r^o_{\overline{Z}0Z}} (f_{j}^{y\bar{Z}} \otimes g_{j}^{y\bar{Z}})(f_{j}^{y\bar{Z}} \otimes g_{j}^{y\bar{Z}})^\dagger = C_{\overline{Y}Z} C_{\overline{Z}}^{\dagger}$ where

$$
C_{\overline{Y}Z} = [r_{Y}(f_{1}^{y\bar{Z}} \otimes g_{1}^{y\bar{Z}}, \ldots, r_{Y}(f_{r^o_{\overline{Z}0Z}}^{y\bar{Z}} \otimes g_{r^o_{\overline{Z}0Z}}^{y\bar{Z}}) = R_{Y}(f_{1}^{y\bar{Z}}) \otimes G_{\overline{Z}}.
$$

Similarly, the generic $b$-block of $S^o_{\overline{Y}Z}$ for $-\bar{Y} + 1 \leq b \leq \bar{Y} - 1$ is given by $S^o_{\overline{Y}Z} = \sum_{j=1}^{r^o_{\overline{Y}Z}} e^{j2\pi b f_{j}^{y\bar{Z}} \otimes g_{j}^{y\bar{Z}}}$, $g_{j}^{y\bar{Z}} = G_{\overline{Z}} Y_{b}^{a} G_{\overline{Z}}^{\dagger}$ with $Y = \text{diag}(e^{j2\pi f_{1}^{y\bar{Z}}}, e^{j2\pi f_{2}^{y\bar{Z}}}, \ldots, e^{j2\pi f_{r^o_{\overline{Y}Z}}^{y\bar{Z}}})$. Furthermore, $S^o_{\overline{Y}Z}$ admits a Vandermonde decomposition of order $r^o_{\overline{Y}Z} < \bar{Z}$, $S^o_{\overline{Y}Z} = C_{\overline{Z}} C_{\overline{Y}Z}^{\dagger} = R_{\overline{Z}}(f_{1}^{y\bar{Z}}) \text{PR}_{\overline{Z}}(f_{1}^{y\bar{Z}})^\dagger$, where

$$
C_{\overline{Z}} = [r_{Z}(f_{1}^{y\bar{Z}}) p_{1/2}^{1/2}, \ldots, r_{Z}(f_{r^o_{\overline{Y}Z}}^{y\bar{Z}}) p_{1/2}^{1/2}] = R_{Z}(f_{1}^{y\bar{Z}}) \text{PR}_{\overline{Z}}(f_{1}^{y\bar{Z}})^\dagger,
$$
Combining $S_{00z}^o = C_Z G^\dagger_Z$ with $S_{0bz}^o$ evaluated for $b = 0$, and $S_{0yz}^o = C_{YZ} G^\dagger_{YZ}$ with $S_{aYZ}^o$ evaluated for $a = 0$ we have that $G_Z G^\dagger_Z = C_Z C^\dagger_Z$ and $G_{YZ} G^\dagger_{YZ} = C_{YZ} C^\dagger_{YZ}$ and therefore, we can always find a $r^{o}_Z \times r^{o}_{YZ}$ unitary matrix, $O_Z$ and a $r^{o}_{YZ} \times r^{o}$ unitary matrix, $O_{YZ}$, such that

$$G_Z = C_Z O_Z, \quad G_{YZ} = C_{YZ} O_{YZ}, \quad (19)$$

Replacing (19) in (18) and (17) we have that

$$\text{Lemma 3:} \quad \text{Next, by being solution of (P.3),}\$$

$$\text{steering vectors with a Kronecker ordering}\,$$

$$\text{therefore, there exists a set of coefficients }\{\alpha_{1}^{o}, \ldots, \alpha_{r^{o}}^{o}\} \text{ such that:}$$

$$s^o = \sum_{j=1}^{r^{o}} \alpha_{j}^{o} c_{j}^{xzy} = \sum_{j=1}^{r^{o}} \alpha_{j}^{o} r_{X}(f_{j}^{x}) \otimes \sum_{i=1}^{r^{2}} o_{ji}^{y_{z}} r_{Y}(f_{j}^{y}) \otimes \sum_{l=1}^{r^{2}} p_{l}^{1/2} o_{il}^{z} r_{Z}(f_{l}^{z}),$$

$$= \sum_{j=1}^{r^{o}} \sum_{i=1}^{r^{2}} \sum_{l=1}^{r^{2}} \alpha_{j}^{o} o_{ji}^{y_{z}} p_{l}^{1/2} o_{il}^{z} r_{X}(f_{j}^{x}) \otimes r_{Y}(f_{i}^{y}) \otimes r_{Z}(f_{l}^{z}),$$

where we have $r^{o}_{Z} r^{o}_{Y} r^{o}_{Z} \tilde{N}$-dimensional vectors $r_{X}(f_{j}^{x}) \otimes r_{Y}(f_{i}^{y}) \otimes r_{Z}(f_{l}^{z})$ with $j \in [r^{o}], i \in [r^{2}_{Z}]$ and $l \in [r^{2}]$ that are necessarily linearly dependent given that rank $\{S_{XYZ}^o\} = r^{o}$. We define $M$ as the set of indexes $(j_{m}, i_{m}, l_{m})$ such that the vectors $r_{X}(f_{j_{m}}^{x}) \otimes r_{Y}(f_{i_{m}}^{y}) \otimes r_{Z}(f_{l_{m}}^{z})$ are linearly independent, note that $|M| = r^{o}$ and $m \in [r^{o}]$. Then, we can rewrite $s^o$ as a linear combination of those $r^{o}$ linearly independent $r_{X}(f_{j_{m}}^{x}) \otimes r_{Y}(f_{i_{m}}^{y}) \otimes r_{Z}(f_{l_{m}}^{z})$ vectors, now indexed in $m \in [r^{o}]$, $s^o = \sum_{m=1}^{r^{o}} u_{m}^{o} r_{X}(f_{m}^{x}) \otimes r_{Y}(f_{m}^{y}) \otimes r_{Z}(f_{m}^{z}) = \sum_{m=1}^{r^{o}} u_{m}^{o} r_{N}(f_{m})$. It follows then that $s^o$ can be expressed as a linear combination of $r^{o}$ steering vectors with a Kronecker ordering $\tilde{Z} \rightarrow \tilde{Y} \rightarrow \tilde{X}$ given by the nesting ordering of $S_{XYZ}^o$ which by formulation of (P.3) is assumed to have canonical ordered structure.

Next, by being solution of (P.3), $s^o$ satisfies $A s^o = A s^*$. Hence, using the injectivity property of $A \in A^{(K)}$ (see Definition 5), we will prove that, if $f_{1}^{*;K}$ satisfies Assumption [1], $r^{o} = K$ and that it exists a unique set of $r^{o} = K$ frequencies that gives as linear combination $s^o$ and this set coincides with $f_{1}^{*;K}$. To this end let us recall a classical result:

**Lemma 3:** Given a vector $s^* = \sum_{k=1}^{K} u_{k}^{*} r_{N}(f_{k})$, with $f_{1}^{*;K}$ satisfying Assumption [1] and given a vector $s^o = \sum_{j=1}^{r^{o}} u_{j}^{o} r_{N}(f_{j})$ with $f_{1}^{*;r^{o}}$ not necessarily satisfying Assumption [1] for any given matrix $A \in A^{(K)}$ (see Definition 5), if $A s^o = A s^*$, there exists a unique set of frequencies that gives as linear combination $s^o$, provided that $r^{o} \leq K$, and this set coincides with $f_{1}^{*;K}$. 


we also have that $\sum_{i=1}^{r_o} r_i^o = K$, $f_{1:r_o}^* = f_{1,K}^*$. From this, it also follows immediately that the span of $S_{XYZ}^o$ admits as generating vectors the set of steering vectors with nesting ordering $\bar{Z} \to \bar{Y} \to \bar{X}$ associated to the set of $K$ frequencies $f_{1,K}^*$, i.e., the columns of $R_N(f_{1,K}^*)$. Hence, to complete the theorem we need now to prove that $S_{XYZ}^o$ admits a unique Vandermonde decomposition that can be obtained via Algorithm 1 in order to uniquely identify the frequencies $f_{1,K}^*$.

With this aim, let us consider the first $\bar{Z}$ components of $s^o = s^*$. They identify a $\bar{Z}$-dimensional vector that we denote by $A_2s^o$. Analogously, we denote by $A_2s^*$ the first $\bar{Z}$ components of $s^* = \sum_{k=1}^{K} u_k^* r_N(f_k^*)$.

From (21), we have that $A_2s^o = \sum_{i=1}^{r_1} r_1^o f_i^o (f_i^o)^{\bar{Z}-1} \sum_{j=1}^{r_1} \sum_{k=1}^{K} o_j^o o^{\bar{Z}}_k f_i^o p_i^o$ and also since $A_2s^o = A_2s^*$, we also have that $\sum_{i=1}^{r_1} r_1^o f_i^o (f_i^o)^{\bar{Z}-1} \sum_{j=1}^{r_1} \sum_{k=1}^{K} o_j^o o^{\bar{Z}}_k f_i^o p_i^o = \sum_{k=1}^{K} r_1^o (f_k^o)^{\bar{Z}} u_k^*$. This combined with the fact that $K = r_o = r_2^o$. Finally, since we had that $r_o^* \geq r_1^o \geq r_2^o$, it is true then that $K = r_o^* \geq r_1^o \geq r_2^o$. From this it follows that $\text{rank}\{S_{XYZ}^o\} = \text{rank}\{S_{002}^o\} = K$. This combined with the fact that $K \leq \max\{X, Y, Z\}$ and $S_{XYZ}^o$ belongs to $T_{XYZ}$, by Lemma 1 $S_{XYZ}^o$ admits a unique Vandermonde decomposition of order $K$ from which the set of frequencies $f_{1,K}^*$ can be uniquely determined.

APPENDIX C

PROOF OF COROLLARY 1

To prove Corollary 1 it is enough to prove the following lemma:

**Lemma 4:** Under Assumption 1, any sensing matrix $A$ belonging to the subset $\mathcal{A}_{c}(K)$ of the well structured sensing matrices set (see Definition 6), belongs to $\mathcal{A}(K)$ as in Definition 5 provided that $S_c \geq 2K + (d - 1)$.

**Proof:** Starting from Definition 6, the proof uses [25, Th. 3] and [25, Lemma 1] to prove conditions 1.C and 2.C respectively.

APPENDIX D

PROOF OF THEOREM 2

Recall that in Theorem 1 the condition for frequency recovery has been given assuming $A \in \mathcal{A}(K)$. Hence, the proof of Theorem 2 can be identical to that given in Appendix B for Theorem 1 except that in Theorem 2 we will exploit the fact that $\mathcal{A}(K) \equiv \mathcal{A}$ with high probability. Hence, to prove Theorem 2 is it sufficient to prove the following Lemma.

**Lemma 5:** Let $\mathcal{A}$ be the sensing set defined in (5). Then, for any binary sensing matrix $A \in \mathcal{A}$, and under Assumption 1 $\mathcal{A}(K) \equiv \mathcal{A}$ with probability $1 - \epsilon$ provided that $N \geq 2KC \log(2K\epsilon^{-1})$, with $C$ that is not larger than 12.
To prove Lemma 5 we need then to verify that conditions 1.C and 2.C in Definition 5 hold with probability $1 - \epsilon$ for any $A \in \mathcal{A}$ provided that $N \geq 2KC\log(2K\epsilon^{-1})$, with $C$ that is not larger than 12. Let us first state the following lemma,

**Lemma 6:** Consider a set of $L$ frequencies, $f_{1:L}$, satisfying Assumption I. For any $A \in \mathcal{A}$, and for any $p \times N$, random matrix $B = [b_1, \ldots, b_N]$ with $p \leq N$ and $p = O(N)$, such that $\mathbb{E}[B^{\dagger}B] = I_N$, $\mathbb{E}[b_{ij}b_{jm}] = 0$, $\forall i \neq j$ or $\forall l \neq m$ and $\forall i \neq j$

$$
\mathbb{E}[b_{ij}^{*}b_{jl}b_{im}b_{jl}^{*}] = \begin{cases} 
\mathbb{E}[|b_{il}|^2]\mathbb{E}[|b_{jl}|^2] & \text{if } l = m = \ell \\
0 & \text{otherwise}
\end{cases},
$$

we have with probability $1 - \epsilon$ that:

$$
\delta_L = \|\Xi^{\dagger}\Xi - I_L\|_{2 \rightarrow 2} < 1
$$

provided that $L < p$, and $N \geq LC\log(L\epsilon^{-1})$, where $C$ is an appropriate universal constant, $\Xi = c\text{BAR}_N(f_{1:L}) \in \mathbb{C}^{p \times L}$ and $c$ is an arbitrary non-zero constant.

**Proof:** Lemma 6 proof uses the classical definition of $\| \cdot \|_{2 \rightarrow 2}$ norm, and typical tools from concentration inequalities, in particular [21] Prop. 8.16.

Note that when $\delta_L \in (0, 1)$, then the largest and smallest singular values of $\text{BAR}_N(f_{1:L})$ satisfy $\sigma_{\min}(\text{BAR}_N(f_{1:L})) \geq \sqrt{1 - \delta_L}/c$ and $\sigma_{\max}(\text{BAR}_N(f_{1:L})) \leq \sqrt{1 + \delta_L}/c$ which implies the injectivity of $\text{BAR}_N(f_{1:L})$. Note that if $\text{BAR}_N(f_{1:L})$ is injective and if $B$ is a full rank $N \times N$ square matrix then the $N \times L$ matrix $\text{AR}_N(f_{1:L})$ is also injective [26].

To prove 1.C, let us consider a set of $2K$ frequencies, $f_{1:2K}$, satisfying Assumption I. Setting $L = 2K$ and $B = \text{diag}([d_1, \ldots, d_N])$ with $d_i, i \in [N]$ modelled as i.i.d. zero mean and unit variance random variables, we have that the square diagonal matrix $B$ is almost surely full rank. Furthermore, $\mathbb{E}[B^{\dagger}B] = I_N$, $\mathbb{E}[b_{ij}b_{jm}] = 0$, $\forall i \neq j$ or $\forall l \neq m$ while (22) trivially holds. Hence, from Lemma 6 it follows immediately that $\text{BAR}_N(f_{1:2K})$ is injective with probability $1 - \epsilon$ provided that $N \geq LC\log(L\epsilon^{-1})$ with $C$ an appropriate universal constant not larger than 12. Finally, due to the full rank property of the square matrix $B$, the injectivity result also holds for $\text{AR}_N(f_{1:2K})$, which concludes the proof of condition 1.C.

Let us now prove condition 2.C. To this end, consider a set of $K$ frequencies, $f_{1:K}$, satisfying Assumption I and an arbitrary set of $r^o \leq K$ frequencies, $f_{1:r^o}$, such that for any $k \in [K]$ and $j \in [r^o]$, $f_{j}^o \neq f_k^\dagger$. By Lemma 5 we can show that $\text{rank}\{\text{AR}_N(f_{1:K})\} = K$. Furthermore, denote by $r$ the rank of $\text{AR}_N(f_{1:r^o})$, and, with no loss of generality, assume that the first $r$ columns of $\text{AR}_N(f_{1:r^o})$ are linearly independent and denote such columns by $f_{1:r}^o$. To prove condition 2.C. we have to prove that $\text{rank}\{\text{AR}_N(f_{1:K+r})\} = K + r$, where $f_{1:K+r}^o = [f_{1:K}^o, f_{1:r}^o]$. To this end, it
is also full rank. Denoting by \(X^o = QAR_N(f_{1,K}^o)\), observing that \(X^o\) is full rank and using the determinant of a block matrix, we have that to prove condition 2.C it is sufficient to prove that 
\[
\det\{R_N^+(f_{1,K}^o)A^\dagger Q^d(I_N - X^o(X^o\dagger X^o)^{-1}X^o\dagger)QAR_N(f_{1,K}^o)\} \neq 0.
\]

Next, let \(U\Sigma V^\dagger\) be the singular value decomposition of \(AR_N(f_{1,r}^o)\) with \(\Sigma\), \(V^\dagger\) and \(U\) denoting the square diagonal \(r \times r\) singular-value matrix, the \(r \times r\) right-singular eigenvector matrix and the \(N \times r\) left-singular eigenvector matrix of \(AR_N(f_{1,r}^o)\). Furthermore, set \(Q = \text{diag}(d)U^\dagger\) with \(d = \{d_1, \ldots, d_N\}\) a possibly complex vector with i.i.d. zero mean and unit variance components and with \(U_e = [UE]\) a unitary matrix whose first \(r\) columns are the left-singular vectors of \(AR_N(f_{1,r}^o)\), \(U\), while the remaining \(N - r\) columns are uniformly distributed over the manifold \(E^\dagger E = I_{N-r}\) such that \(EE^\dagger\) is a projector on the orthogonal complement to the subspace described by the columns of \(U\), i.e., \(\text{span}\{U\}\).

Based on the above definitions, after some algebraic manipulations, we have that 
\[
Q^\dagger(I_N - X^o(X^o\dagger X^o)^{-1}X^o\dagger)Q = E \text{diag}(|d_{r+1}|^2, \ldots, |d_N|^2)E^\dagger
\]
Defining \(B = \text{diag}([d_{r+1}, \ldots, d_N])E^\dagger\), if we set \(L = K\) and \(p = N - r\), since \(K < N - r\), by Lemma 6 we have that \(\text{BAR}_N(f_{1,K}^o)\) is injective and consequently \(\det\{R_N^+(f_{1,K}^o)A^\dagger B^\dagger \text{BAR}_N(f_{1,K}^o)\} \neq 0\) if \(N \geq LC\log(Lc^{-1})\) with \(C\) an appropriate universal constant not larger than 12. This completes the proof of 2.C and also Lemma 5.

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