On the Mareš Cores of Fuzzy Vectors
Cheng-Yong Du, Lili Shen

Abstract—It is known that every fuzzy number has a unique Mareš core and can be decomposed in a unique way as the sum of a skew fuzzy number, given by its Mareš core, and a symmetric fuzzy number. The aim of this paper is to provide a negative answer to the existence of an n-dimensional version of the above theorem. By applying several key tools from convex geometry, we establish a representation theorem of fuzzy vectors through support functions, in which a necessary and sufficient condition for a function to be the support function of a fuzzy vector is provided. Furthermore, symmetric and skew fuzzy vectors are postulated, based on which a Mareš core of each fuzzy vector is constructed through convex bodies and support functions. It is shown that every fuzzy vector over the n-dimensional Euclidean space has a unique Mareš core if, and only if, the dimension n = 1.

Index Terms—fuzzy vector, symmetric fuzzy vector, skew fuzzy vector, Mareš core, Mareš equivalence, convex body, support function.

I. INTRODUCTION

SINCE Zadeh introduced the concept of fuzzy sets [1] in the 1960s, fuzzy numbers, as a special kind of fuzzy subsets of the set R of real numbers, have acquired considerable attention both in the theory and the applications of fuzzy sets ([2]–[9]).

Based on fruitful results of fuzzy numbers, recent works of Qiu-Lu-Zhang-Lan [10] and Chai-Zhang [11] reveal a crucial property of fuzzy numbers regarding their Mareš cores. Explicitly, a fuzzy number u is skew [11] if it cannot be written as the sum of a fuzzy number and a non-trivial symmetric fuzzy number in the sense of Mareš [12]; that is, if

\[ u = v \oplus w \]

and w is symmetric, then w is constant at 0. A fuzzy number v is the Mareš core [13], [14] of a fuzzy number u if v is skew and \( u = v \oplus w \) for some symmetric fuzzy number w. The following theorem combines the main results of [10] and [11]:

Theorem I.1. Every fuzzy number has a unique Mareš core, so that every fuzzy number can be decomposed in a unique way as the sum of a skew fuzzy number, given by its Mareš core, and a symmetric fuzzy number.

Since the notion of fuzzy number may be extended to the n-dimensional Euclidean space \( \mathbb{R}^n \) without obstruction, fuzzy vectors, as the n-dimensional version of fuzzy numbers, have been widely studied as well ([15]–[22]). It is then natural to ask whether it is possible to establish the n-dimensional version of Theorem I.1 for general fuzzy vectors. Unfortunately, a negative answer will be given in this paper (Theorem V.10).

Although it is well known that fuzzy vectors can be characterized through their level sets ([23], [24]), the well-developed theory of convex geometry (see [25], [26]) is not usually perceived by researchers of the fuzzy community. In order to reconcile the two lines of research on fuzzy vectors and on convex geometry, in this paper we take full advantage of the arsenal of geometers, among which the key tools are convex bodies and support functions. Based on the well-known characterization through convex bodies (Proposition II.3), a representation theorem of fuzzy vectors through support functions is established (Theorem II.6), whose proof is the most challenging one of this paper. Explicitly, Theorem II.6 presents a necessary and sufficient condition for a function

\[ h : [0, 1] \times S^{n-1} \rightarrow \mathbb{R} \]

to be the support function of a (unique) fuzzy vector, where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). Moreover, Theorem II.8 describes the sum of fuzzy vectors defined by Zadeh’s extension principle through the Minkowski sum of convex bodies and the sum of their support functions, which is the cornerstone of the results of this paper.

With the preparations in Section II, we carefully postulate the notion of symmetric fuzzy vector in Section III, which is defined as symmetric around the origin in accordance with the case of \( n = 1 \) (cf. [10, Remark 2.1]); using the language of convex bodies and support functions, a fuzzy vector is symmetric whenever its level sets are closed balls centered at the origin, or whenever the support functions of its level sets are constant (Theorem III.4).

The notion of symmetric fuzzy vector allows us to postulate skew fuzzy vectors and Mareš cores of fuzzy vectors naturally in Section V. However, for the purpose of studying their properties we have to be familiar with the inner parallel bodies of convex bodies, and this is the subject of Section IV, in which we characterize inner parallel bodies through support functions, and prove that every convex body can be uniquely decomposed as the Minkowski sum of an irreducible convex body and a closed ball centered at the origin (Theorem IV.4).

The first part of Section V is devoted to the decomposition

\[ u = c(u) \oplus s(u) \]

(1)

of each fuzzy vector \( u \in \mathcal{F}^n \), where \( c(u) \) is a Mareš core of \( u \), and \( s(u) \) is a symmetric fuzzy vector (Theorem V.5). However, unlike Theorem I.1 for the case of \( n = 1 \), Example V.9 reveals that Equation (1) may not be the unique way of decomposing a fuzzy vector. Hence, we indeed obtain a negative answer.

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to the possibility of establishing the $n$-dimensional version of Theorem I.1, which is stated as Theorem V.10.

Finally, we investigate Marčev equivalent fuzzy vectors in Section VI. As we shall see, comparing with Marčev equivalent fuzzy numbers (see Corollary VI.6), the Marčev equivalence relation of fuzzy vectors may behave in quite different ways. As Example VI.7 reveals, the smallest fuzzy vector $k(u)$ of the Marčev equivalence class of a fuzzy vector $u$ may not be a Marčev core of $u$, and different skew fuzzy vectors may be Marčev equivalent to each other.

II. Fuzzy vectors via convex bodies

Throughout, let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. Following the terminology of [22], by a fuzzy vector we mean a fuzzy subset of $\mathbb{R}^n$, i.e., a function

$$ u : \mathbb{R}^n \rightarrow [0, 1], $$

subject to the following requirements:

(V1) $u$ is regular, that is, there exists $t_0 \in \mathbb{R}^n$ with $u(t_0) = 1$;

(V2) $u$ is compactly supported, that is, the closure of $\{ t \in \mathbb{R}^n \mid u(t) > 0 \}$ is compact;

(V3) $u$ is convex, that is, $u(s) \wedge u(t) \leq u(\lambda s + (1 - \lambda) t)$ for all $s, t \in \mathbb{R}^n$ and $\lambda \in [0, 1]$;

(V4) $u$ is upper semi-continuous, that is, $\{ t \in \mathbb{R}^n \mid u(t) \geq \alpha \}$ is closed for all $\alpha \in [0, 1]$.

The set of all fuzzy vectors of dimension $n$ is denoted by $\mathcal{F}^n$, and a canonical embedding of $\mathbb{R}^n$ into $\mathcal{F}^n$ assigns to each $a \in \mathbb{R}^n$ a “crisp” fuzzy vector

$$ \tilde{a} : \mathbb{R}^n \rightarrow [0, 1], \quad \tilde{a}(t) := \begin{cases} 1 & \text{if } t = a, \\ 0 & \text{else} \end{cases} $$

**Remark II.1.** The conditions (V1)–(V4), first appeared in [15] and [16], are originated from the definition of fuzzy numbers, i.e., fuzzy vectors of dimension 1 (see [2], [4]–[6], [27]): so, fuzzy vectors are also called $n$-dimensional fuzzy numbers (see [18]–[21]). It should be reminded that $n$-dimensional fuzzy vectors defined in [21] are different from our fuzzy vectors here.

For each $u \in \mathcal{F}^n$ and $\alpha \in [0, 1]$, the $\alpha$-level sets of $u$ are defined as

$$ u_\alpha := \left\{ t \in \mathbb{R}^n \mid u(t) \geq \alpha \right\} \quad \text{if } \alpha \in (0, 1), $$

$$ \bigcup_{\alpha \in (0, 1)} u_\alpha = \left\{ t \in \mathbb{R}^n \mid u(t) > 0 \right\} \quad \text{if } \alpha = 0. $$

It is easy to see that

$$ u(t) = \bigvee_{\alpha \in [0, 1]} \alpha $$

for each $u \in \mathcal{F}^n$ and $t \in \mathbb{R}^n$.

**Remark II.2.** Since $\alpha$ ranges in the closed interval $[0, 1]$, the supremum in Equation (2) is computed in $[0, 1]$. Hence, if $t \notin u_\alpha$ for all $\alpha \in [0, 1]$, i.e.,

$$ \{ \alpha \mid t \in u_\alpha \} = \emptyset, $$

then $u(t) = 0$ because 0, as the bottom element of $[0, 1]$, is the supremum of the empty subset of $[0, 1]$.

Following the terminologies of convex geometry (see [25], [26]), by a convex body in $\mathbb{R}^n$ we mean a nonempty compact convex subset of $\mathbb{R}^n$; that is, $A \subseteq \mathbb{R}^n$ is a convex body if it is nonempty, compact, and

$$ \lambda s + (1 - \lambda) t \in A $$

whenever $s, t \in A$ and $\lambda \in [0, 1]$. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{C}^n$. It is well known that fuzzy vectors can be characterized via convex bodies as follows:

**Proposition II.3.** ([23], [24]) Let $\{ A_\alpha \mid \alpha \in [0, 1] \}$ be a family of subsets of $\mathbb{R}^n$. Then there exists a (unique) fuzzy vector

$$ u : \mathbb{R}^n \rightarrow [0, 1], \quad u(t) = \bigvee_{\alpha \in A_\alpha} \alpha $$

such that

$$ u_\alpha = A_\alpha $$

for all $\alpha \in [0, 1]$ if, and only if,

(L1) $A_\alpha$ is a convex body in $\mathbb{R}^n$ for each $\alpha \in [0, 1]$;

(L2) $A_\alpha \supseteq A_\beta$ whenever $0 \leq \alpha < \beta \leq 1$;

(L3) $A_\alpha = \bigcap_{k \geq 1} A_{\alpha_k}$ for each increasing sequence $\{ \alpha_k \} \subseteq [0, 1]$ that converges to $\alpha_0 > 0$;

(L4) $A_0 = \bigcup_{\alpha \in (0, 1]} A_\alpha$.

Recall that the support function ([25], [26]) of a convex body $A \in \mathcal{C}^n$ is given by

$$ h_A : S^{n-1} \rightarrow \mathbb{R}, \quad h_A(x) := \bigvee_{a \in A} \langle a, x \rangle, $$

where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and $\langle - , - \rangle$ is the standard Euclidean inner product. Obviously, $h_A$ is bounded on $S^{n-1}$; indeed,

$$ |h_A(x)| \leq \bigvee_{a \in A} ||a|| $$

for all $x \in S^{n-1}$, where $||-||$ refers to the standard Euclidean norm. Conversely, we may associate a subset

$$ A_h := \{ t \in \mathbb{R}^n \mid \forall x \in S^{n-1} : \langle t, x \rangle \leq h(x) \} $$

of $\mathbb{R}^n$ to each function $h : S^{n-1} \rightarrow \mathbb{R}$.

In fact, convex bodies can be fully characterized through support functions as follows, which is a slight reformulation of [25, Theorem 4.3] and [26, Theorem 1.7.1]:

**Proposition II.4.** ([25], [26]) A function $h : S^{n-1} \rightarrow \mathbb{R}$ is the support function of a (unique) convex body $A_h$ if, and only if, $h$ is subadditive in the sense that

$$ h(x) + h(-x) \geq 0 \quad \text{and} \quad h \left( \frac{x + y}{||x + y||} \right) \leq \frac{h(x) + h(y)}{||x + y||} $$

for all $x, y \in S^{n-1}$ with $x + y \neq 0$.

It should be reminded that the proof of Proposition II.4 is highly non-trivial, and we refer to [25, Theorem 4.3] and [26, Theorem 1.7.1] for details.

As an immediate consequence of Proposition II.4, the following characterization of elements of convex bodies will be useful later (cf. [18, Theorem 2.2]):
Proposition II.5. Let $A \in C^n$ be a convex body. Then $t \in A$ if, and only if, $(t, x) \leq h_A(x)$ for all $x \in S^{n-1}$.

Since all the level sets of a fuzzy vector are convex bodies, it makes sense to define the support function \cite{[17]} of $u \in F^n$ as

$$h_u : [0, 1] \times S^{n-1} \to \mathbb{R}, \quad h_u(\alpha, x) := \bigvee_{t \in u_{\alpha}} \langle t, x \rangle;$$

that is, $h_u(\alpha, -) := h_u_{\alpha}$ is the support function of the convex body $u_{\alpha}$ for each $\alpha \in [0, 1]$. In particular, $h_u$ is bounded on $[0, 1] \times S^{n-1}$, because

$$h_u(\alpha, x) \leq \bigvee_{t \in u_{\alpha}} \langle t, x \rangle = h_{u_1}(x)$$

for all $\alpha \in [0, 1], x \in S^{n-1}$, and $h_{u_0}$ is bounded on $S^{n-1}$.

With Propositions II.3 and II.4 we may describe fuzzy vectors through support functions as the following representation theorem reveals, whose proof is the most challenging one of this paper and will be attached in the appendix:

Theorem II.6. A function $h : [0, 1] \times S^{n-1} \to \mathbb{R}$ is the support function of a (unique) fuzzy vector $u : \mathbb{R}^n \to [0, 1]$ given by

$$u(t) = \bigvee_{t \in A_h(\alpha, -)} \alpha = \bigvee \{\alpha \mid \forall x \in S^{n-1} : \langle t, x \rangle \leq h(\alpha, x)\}$$

if, and only if,

(VS1) $h(\alpha, -)$ is subadditive for each $\alpha \in [0, 1]$;

(VS2) for each $x \in S^{n-1}$, $h(\alpha, -) : [0, 1] \to \mathbb{R}$ is non-increasing, left-continuous on $(0, 1]$ and right-continuous at 0.

With the above preparations we are now able to characterize the addition $\oplus$ of fuzzy vectors through their level sets and support functions. Explicitly, the sum

$$u \oplus v \in F^n$$

of fuzzy vectors $u, v \in F^n$ is defined by Zadeh’s extension principle \cite{[2],[28]}, i.e.,

$$(u \oplus v)(t) := \bigvee_{r+s=t} u(r) \wedge v(s)$$

for all $t \in \mathbb{R}^n$.

Recall that the Minkowski sum \cite{[26]} of convex bodies $B, C \in C^n$ is given by

$$B + C := \{b + c \mid b \in B, c \in C\}.$$ 

Note that a direct computation

$$h_{B+C}(x) = \bigvee_{b \in B, c \in C} \langle b + c, x \rangle = \bigvee_{b \in B} \big( \langle b, x \rangle + \langle c, x \rangle \big)$$

for any $x \in S^{n-1}$ shows that $h_{B+C} = h_B + h_C$ (cf. \cite[Theorem 1.7.5]{[26]}), in combination with Proposition II.4 we deduce:

Lemma II.7. For convex bodies $A, B, C \in C^n$, $A = B + C$ if, and only if, $h_A = h_B + h_C$.

Theorem II.8. For fuzzy vectors $u, v, w \in F^n$, the following statements are equivalent:

(i) $u = v \oplus w$.

(ii) $u_{\alpha} = v_{\alpha} + w_{\alpha}$ for all $\alpha \in [0, 1]$.

(iii) $h_u = h_v + h_w$.

Proof. (ii) $\iff$ (iii) is an immediate consequence of Lemma II.7. For (i) $\iff$ (ii), by Proposition II.3 it suffices to observe that $(v \oplus w)_{\alpha} = v_{\alpha} + w_{\alpha}$ for all $\alpha \in [0, 1]$, which is a well-known fact of Zadeh’s extension principle (see \cite[Proposition 3.3]{[29]}).

III. Symmetric fuzzy vectors

Let $O(n)$ denote the orthogonal group of dimension $n$, i.e., the group of $n \times n$ orthogonal matrices.

Definition III.1. A fuzzy vector $u \in F_n$ is symmetric (around the origin) if it is $O(n)$-invariant; that is,

$$u(t) = u(Qt)$$

for all $t \in \mathbb{R}^n$ and $Q \in O(n)$.

We denote by $F^s_n$ the set of all symmetric fuzzy vectors of dimension $n$.

Remark III.2. In the case of $n = 1$, since $O(1) = \{-1, 1\}$, $u \in F_1$ is symmetric if $u(t) = u(-t)$ for all $t \in \mathbb{R}$; that is, $u$ is a symmetric fuzzy number in the sense of Mareš \cite{[12],[13]}. Hence, the symmetry of fuzzy numbers is a special case of Definition III.1. In fact, as indicated by \cite[Remark 2.1]{[10]}, a symmetric fuzzy number actually refers to a fuzzy number that is symmetric around zero.

In order to characterize symmetric fuzzy vectors through convex bodies and support functions, let us first prove a lemma:

Lemma III.3. A convex body $A \in C^n$ is a closed ball centered at the origin if, and only if, its support function $h_A : S^{n-1} \to \mathbb{R}$ is a (nonnegative) constant function.

Proof. Suppose that $A$ is a closed ball of radius $\lambda$ centered at the origin. Then

$$h_A(x) = \bigvee_{a \in A} \langle a, x \rangle = \langle \lambda x, x \rangle = \lambda$$

for all $x \in S^{n-1}$. Conversely, if $h_A(x) = \lambda$ for all $x \in S^{n-1}$, it follows from Proposition II.5 that

$$t \in A \iff \forall x \in S^{n-1} : \langle t, x \rangle \leq \lambda$$

$$\iff ||t|| = \left( t, \frac{t}{||t||} \right) \leq \lambda,$$

which also guarantees the nonnegativity of $\lambda$.

Theorem III.4. For each fuzzy vector $u \in F^n$, the following statements are equivalent:

(i) $u$ is symmetric.

(ii) For each $\alpha \in [0, 1]$, $u_{\alpha}$ is invariant under the action of $O(n)$; that is, $Qt \in u_{\alpha}$ for all $t \in u_{\alpha}$ and $Q \in O(n)$. 


(iii) For each $\alpha \in [0, 1]$, $u_\alpha$ is a closed ball centered at the origin.

(iv) For each $\alpha \in [0, 1]$, $h_u(\alpha, -) : S^{n-1} \rightarrow \mathbb{R}$ is a (nonnegative) constant function.

Proof. Since (iii) \iff (iv) is an immediate consequence of Lemma III.3, it remains to prove that (i) \implies (iv) and (iii) \implies (ii) \implies (i).

(i) \implies (iv): Let $\alpha \in (0, 1]$ and $x \in S^{n-1}$. For each $Q \in O(n)$ and $t \in u_\alpha$, the $O(n)$-invariance of $u$ implies that $Q^{-1}t \in u_\alpha$ since $u(Q^{-1}t) = u(t) \geq \alpha$, and consequently

$$\langle t, Qx \rangle = \langle Q^{-1}t, x \rangle \leq h_u(\alpha, x).$$

Thus

$$h_u(\alpha, Qx) = \bigvee_{t \in u_\alpha} \langle t, Qx \rangle \leq h_u(\alpha, x).$$

Since $Q$ is arbitrary, it also holds that $h_u(\alpha, Q^{-1}x) \leq h_u(\alpha, x)$, and consequently $h_u(\alpha, x) \leq h_u(\alpha, Qx)$. Hence

$$h_u(\alpha, Qx) = h_u(\alpha, x)$$

for all $Q \in O(n)$. Note that the function

$$O(n) \rightarrow S^{n-1}, \quad Q \mapsto Qx$$

is surjective, and thus

$$h_u(\alpha, x) = h_u(\alpha, y)$$

for all $x, y \in S^{n-1}$; that is, $h_u(\alpha, -) : S^{n-1} \rightarrow \mathbb{R}$ is constant.

In this case, to see that the value of $h(\alpha, -)$ is nonnegative, just note that for any $t \in u_\alpha$ with $t \neq 0$, from $\frac{t}{||t||} \in S^{n-1}$ we deduce that

$$h_u(\alpha, -) = h_u(\alpha, \frac{t}{||t||}) \geq \langle \frac{t}{||t||}, Qx \rangle = ||Qx|| \geq 0.$$

(iii) \implies (ii): Let $\alpha \in [0, 1]$. If $u_\alpha$ is a closed ball of radius $\lambda_\alpha$ centered at the origin, then $Qt \in u_\alpha$ whenever $t \in u_\alpha$ since $||t|| \leq \lambda_\alpha$, obviously implies that $||Qt|| \leq \lambda_\alpha$.

(ii) \implies (i): Let $t \in \mathbb{R}^n$ and $Q \in O(n)$. If $u(t) > 0$, then $t \in u_\alpha$ and consequently $Qt \in u_\alpha$, i.e., $u(Qt) \geq u(t)$. As $Q$ is arbitrary, from $u(Q^{-1}t) \geq u(t)$ we immediately deduce that $u(t) \geq u(Qt)$, and thus $u(t) = u(Qt)$.

If $u(t) = 0$, then $t \notin u_\alpha$ for all $\alpha \in (0, 1]$, and consequently $Qt \notin u_\alpha$ for all $\alpha \in (0, 1]$, which forces $u(Qt) = 0$ and completes the proof.

From Theorem III.4 we see that the support function $h_u : [0, 1] \times S^{n-1} \rightarrow \mathbb{R}$ of a symmetric fuzzy vector $u \in \mathcal{F}_n^+$ actually reduces to a single-variable function

$$h_u : [0, 1] \rightarrow \mathbb{R},$$

and conversely:

**Corollary III.5.** A function $h : [0, 1] \rightarrow \mathbb{R}$ is the support function of a symmetric fuzzy vector $u \in \mathcal{F}_n^+$ if, and only if, $h$ is nonnegative, non-increasing, left-continuous on $[0, 1]$ and right-continuous at 0.

**Proof.** Follows immediately from Theorems II.6 and III.4.

As a direct application of Lemma III.3 and Theorem III.4, let us point out the following easy but useful facts:

**Corollary III.6.** Let $A, B \in \mathbb{C}^n$ be convex bodies and $u, v \in \mathcal{F}_n$ be fuzzy vectors.

(i) If $A$ and $B$ are both closed balls centered at the origin, then so is $A + B$, and

$$\lambda_{A+B} = \lambda_A + \lambda_B,$$

where $\lambda_A$, $\lambda_B$ and $\lambda_{A+B}$ are the radii of $A$, $B$ and $A+B$, respectively.

(ii) If $u$ and $v$ are both symmetric, then so is $u \oplus v$.

**Proof.** For (i), just note that the support function of $A+B$ satisfies

$$h_{A+B} = h_A + h_B$$

by Lemma II.7, and thus $h_{A+B}$ is a (nonnegative) constant function since so are $h_A$ and $h_B$. The conclusion then follows from Lemma III.3 and Equation (3) in its proof.

For (ii), since $u$ and $v$ are both symmetric, for each $\alpha \in [0, 1]$, Theorem III.4 ensures that $h_u(\alpha, -)$ and $h_v(\alpha, -)$ are both constant on $S^{n-1}$ which, in conjunction with Theorem II.8, implies that

$$h_{u \oplus v}(\alpha, -) = h_u(\alpha, -) + h_v(\alpha, -)$$

is constant on $S^{n-1}$; that is, $u \oplus v$ is also symmetric.

**IV. INNER PARALLEL BODIES OF CONVEX BODIES**

Let $B_\alpha$ denote the closed ball of radius $\lambda \geq 0$ centered at the origin. Recall that the inner parallel body (see [25], [26]) of a convex body $A \in \mathbb{C}^n$ at distance $\lambda$ is given by

$$A_{-\lambda} = \{ t \in \mathbb{R}^n \mid t + B_{\lambda} \subseteq A \},$$

which is also a convex body as long as $A_{-\lambda} \neq \emptyset$.

**Lemma IV.1.** For convex bodies $A, B \in \mathbb{C}^n$ and $\lambda \geq 0$,

$$A = B + B_{\lambda} \iff h_A = h_B + \lambda \iff B = A_{-\lambda}.$$

**Proof.** The equivalence of $A = B + B_{\lambda}$ and $h_A = h_B + \lambda$ follows immediately from Lemmas II.7 and III.3. In this case, from $A = B + B_{\lambda}$ and the definition of $A_{-\lambda}$ we soon see that $B \subseteq A_{-\lambda}$. For the reverse inclusion, suppose that $\alpha \in A_{-\lambda}$. For each $x \in S^{n-1}$, note that $\lambda x \in B_{\lambda}$, and consequently

$$a + \lambda x \in A.$$  \hspace{1cm} (4)

It follows that

$$\langle a, x \rangle + \lambda = \langle a, x \rangle + \langle \lambda x, x \rangle = \langle a + \lambda x, x \rangle \leq h_A(x),$$

where the last inequality is obtained by applying Proposition II.5 to (4). Therefore,

$$\langle a, x \rangle \leq h_A(x) - \lambda = h_B(x)$$

for all $x \in S^{n-1}$, and thus $a \in B$.

In general, the last implication of Lemma IV.1 is proper; that is, $B = A_{-\lambda}$ does not imply $A = B + B_{\lambda}$. For example, let $A$ and $B$ be the hypercubes of side lengths 4 and 3, respectively, both centered at the origin. Then $B = A_{-4}$, but $A \nsubseteq B + B_3$. 

\[ Q.E.D. \]
We say that an inner parallel body $A_{-\lambda}$ of $A \in \mathbb{C}^n$ is regular if

$$A = A_{-\lambda} + B_{\lambda}.$$  

or equivalently (see Lemma IV.1), if

$$h_A = h_{A_{-\lambda}} + \lambda.$$

For each convex body $A \in \mathbb{C}^n$, let

$$\Lambda_A := \{ \lambda \geq 0 | A_{-\lambda} \text{ is a regular inner parallel body of } A \}.$$

**Proposition IV.2.** $\Lambda_A$ is a closed interval, given by

$$\Lambda_A = [0, \lambda_A],$$

where $\lambda_A := \sqrt{\Lambda_A}$.

**Proof.** **Step 1.** $A_{-\lambda_A}$ is a regular inner parallel body of $A$.

In order to obtain $A_{-\lambda_A} + B_{\lambda_A} = A$, it suffices to show that every $t \in A$ lies in $A_{-\lambda_A} + B_{\lambda_A}$. Since $A = A_{-\lambda} + B_{\lambda}$ for all $\lambda \in \Lambda_A$, for an increasing sequence $\{\lambda_k\} \subseteq \Lambda_A$ that converges to $\lambda_A$, we may find $a_k \in A_{-\lambda_k}$ and $b_k \in B_{\lambda_k}$ with

$$t = a_k + b_k$$

for all $k \geq 1$. Note that both the sequences $\{a_k\}$ and $\{b_k\}$ are bounded, and thus they have convergent subsequences. Without loss of generality we assume that $\lim k \rightarrow \infty a_k = a_0$ and $\lim k \rightarrow \infty b_k = b_0$. Then it is clear that

$$t = a_0 + b_0$$

and $b_0 \in B_{\lambda_A}$, and it remains to prove that $a_0 \in A_{-\lambda_A}$. To this end, we need to show that $a_0 + y \in A$ for all $y \in B_{\lambda_A}$. Indeed, let $\{y_k\} \subseteq B_{\lambda_A}$ be a sequence with $\lim k \rightarrow \infty y_k = y$ and $y_k \in B_{\lambda_k}$ for all $k \geq 1$. Then from $a_k \in A_{-\lambda_k}$ we deduce that $a_k + y_k \in A$, and consequently $a_0 + y \in A$, as desired.

**Step 2.** If $\lambda \in \Lambda_A$ and $0 \leq \lambda' < \lambda$, then $\lambda' \in \Lambda_A$.

In order to obtain $A_{-\lambda'} + B_{\lambda'} = A$, it suffices to show that every $t \in A$ lies in $A_{-\lambda'} + B_{\lambda'}$. Since $A = A_{-\lambda} + B_{\lambda}$, we may find $a \in A_{-\lambda}$ and $b \in B_{\lambda}$ with

$$t = a + b,$$

where $b \neq 0$ since $\lambda > 0$. Then it is clear that

$$t = \left[ a + \left(1 - \frac{\lambda'}{\lambda}\right)b \right] + \lambda' b$$

and $\frac{\lambda'}{\lambda} b \in B_{\lambda'}$, and it remains to prove that $a + \left(1 - \frac{\lambda'}{\lambda}\right)b \in A_{-\lambda'}$. To this end, we need to show that

$$a + \left(1 - \frac{\lambda'}{\lambda}\right)b + y' \in A$$

for all $y' \in B_{\lambda'}$. Indeed, $\left(1 - \frac{\lambda'}{\lambda}\right)b + y' \in B_{\lambda}$ since

$$\left\| \left(1 - \frac{\lambda'}{\lambda}\right)b + y' \right\| \leq \left(1 - \frac{\lambda'}{\lambda}\right) \left\| b \right\| + \left\| y' \right\| \leq \left(1 - \frac{\lambda'}{\lambda}\right) \lambda + \lambda' = \lambda,$$

and together with $a \in A_{-\lambda}$ we deduce that $a + \left(1 - \frac{\lambda'}{\lambda}\right)b + y' \in A$, which completes the proof.

As an immediate consequence of Proposition IV.2, we have the following:

**Corollary IV.3.** For each nonempty subset $\Lambda_0 \subseteq \Lambda_A$, let $\lambda_0 = \sqrt{\Lambda_0}$. Then $A_{-\lambda_0}$ is a regular inner parallel body of $A$, which satisfies

$$A_{-\lambda_0} = \bigcap_{\lambda \in \Lambda_0} A_{-\lambda} \quad \text{and} \quad h_{A_{-\lambda_0}} = \bigwedge_{\lambda \in \Lambda_0} h_{A_{-\lambda}}.$$

**Proof.** Firstly, with Lemma IV.1 we obtain that

$$h_{A_{-\lambda_0}} = h_A - \lambda_0 = h_A - \sqrt{\Lambda_0}$$

$$= \bigwedge_{\lambda \in \Lambda_0} (h_A - \lambda) = \bigwedge_{\lambda \in \Lambda_0} h_{A_{-\lambda}},$$

Secondly, if $t \in A_{-\lambda}$ for all $\lambda \in \Lambda_0$, then Proposition II.5 implies that

$$\langle t, x \rangle \leq h_{A_{-\lambda}}(x) = h_A(x) - \lambda$$

for all $x \in S^{n-1}$ and $\lambda \in \Lambda_0$, and thus

$$\langle t, x \rangle \leq \bigwedge_{\lambda \in \Lambda_0} (h_A(x) - \lambda) = h_{A_{-\lambda_0}}(x)$$

for all $x \in S^{n-1}$, which means that $t \in A_{-\lambda_0}$. Hence $\bigcap_{\lambda \in \Lambda_0} A_{-\lambda} \subseteq A_{-\lambda_0}$, which in fact becomes an identity since the reverse inclusion is trivial. \[\square\]

In particular,

$$A_{-\lambda_A} = \bigcap_{\lambda \in \Lambda_A} A_{-\lambda}$$

is the smallest regular inner parallel body of $A$. We say that a convex body $A \in \mathbb{C}^n$ is irreducible if

$$A = A_{-\lambda_A};$$

that is, if $A$ does not have non-trivial regular inner parallel bodies.

Let $\mathbb{C}^n_i$ denote the set of irreducible convex bodies in $\mathbb{R}^n$, and let $\mathbb{B}^n$ denote the set of closed balls in $\mathbb{R}^n$ centered at the origin. For each convex body $A \in \mathbb{C}^n$, the decomposition

$$A = A_{-\lambda_A} + B_{\lambda_A}$$

is unique in the sense of the following:

**Theorem IV.4.** For each convex body $A \in \mathbb{C}^n$, there exist a unique $B \in \mathbb{C}^n_i$ and a unique $B_{\lambda} \in \mathbb{B}^n$ such that $A = B + B_{\lambda}$. Moreover, the correspondence

$$A \mapsto (A_{-\lambda_A}, B_{\lambda_A})$$

establishes a bijection

$$\mathbb{C}^n \sim \mathbb{C}^n_i \times \mathbb{B}^n,$$

whose inverse is given by $(B, B_{\lambda}) \mapsto B + B_{\lambda}$.
### V. Skew Fuzzy Vectors and Mareš Cores

Motivated by the notion of \textit{skew fuzzy number} in the sense of Chai-Zhang [11], we introduce skew fuzzy vectors:

**Definition V.1.** A fuzzy vector \( u \in \mathcal{F}^n \) is \textit{skew} if it cannot be written as the sum of a fuzzy vector and a non-trivial symmetric fuzzy vector; that is, if

\[
u = v \oplus w
\]

for some \( v \in \mathcal{F}^n \) and \( w \in \mathcal{F}^n_\sigma \), then \( w = 0 \).

Following the terminology from fuzzy numbers ([10], [13], [14]), Mareš cores of fuzzy vectors are defined as follows:

**Definition V.2.** A fuzzy vector \( v \in \mathcal{F}^n \) is a \textit{Mareš core} of a fuzzy vector \( u \in \mathcal{F}^n \) if \( v \) is skew and

\[
u = v \oplus w\]

for some symmetric fuzzy vector \( w \in \mathcal{F}^n_\sigma \).

These concepts are closely related to inner parallel bodies introduced in Section IV:

**Lemma V.3.** For fuzzy vectors \( u, v \in \mathcal{F}^n \), if \( u = v \oplus w \) for some symmetric fuzzy vector \( w \in \mathcal{F}^n_\sigma \), then for each \( \alpha \in [0, 1] \),

(i) \( v_\alpha \) is a regular inner parallel body of \( u_\alpha \),

(ii) \( h_\alpha(\alpha, -) - h_\alpha(\alpha, -) : S^{n-1} \rightarrow \mathbb{R} \) is a (nonnegative) constant function.

**Proof.** Since \( u = v \oplus w \) and \( w \) is symmetric, Theorem II.8 and Corollary III.5 ensure that

\[
h_\alpha(\alpha, -) = h_\alpha(\alpha, -) + h_\alpha(\alpha),
\]

and thus (ii) holds. For (i), by setting \( \lambda = h_\alpha(\alpha) \) and rewriting (6) as \( h_\alpha = h_\alpha + \lambda \) it follows soon that \( v_\alpha = (u_\alpha) - \lambda \) and \( u_\alpha = v_\alpha + B_\lambda \) by Lemma IV.1, and hence \( v_\alpha \) is a regular inner parallel body of \( u_\alpha \).

In order to construct a Mareš core of each fuzzy vector \( u \in \mathcal{F}^n \), we start with the following proposition, in which

\[
\Upsilon_u := \{ v \in \mathcal{F}^n \mid u = v \oplus w, \ w \in \mathcal{F}^n_\sigma \}
\]

clearly is a non-empty set as \( u \in \Upsilon_u \):

**Proposition V.4.** There is a fuzzy vector \( c(u) \in \mathcal{F}^n \) whose level sets are given by

\[
c(u)_\alpha := \begin{cases} \bigcap_{v \in \Upsilon_u} v_\alpha & \text{if } \alpha \in (0, 1], \\ \bigcup_{\beta \in [0, 1]} c(u)_\beta & \text{if } \alpha = 0, \end{cases}
\]

and whose support function \( h_{c(u)} : [0, 1] \times S^{n-1} \rightarrow \mathbb{R} \) is given by

\[
h_{c(u)}(\alpha, x) = \begin{cases} \bigcap_{v \in \Upsilon_u} h_\alpha(\alpha, x) & \text{if } \alpha \in (0, 1], \\ \lim_{\beta \to 0^+} h_{c(u)}(\beta, x) & \text{if } \alpha = 0. \end{cases}
\]

**Proof.** For the existence of \( c(u) \), we show that \( \{ c(u)_\alpha \mid \alpha \in [0, 1] \} \) satisfies the conditions (L1)-(L3) in Proposition II.3, as (L4) trivially holds.

Firstly, (L2) holds since \( v_\alpha \supseteq v_\beta \) for all \( v \in \Upsilon_u \) whenever \( 0 \leq \alpha < \beta \leq 1 \).

Secondly, for (L1), note that for each \( \alpha \in (0, 1] \), Lemma V.3 tells us that \( c(u)_\alpha \) is the intersection of a family of regular inner parallel bodies of \( u_\alpha \), and thus \( c(u)_\alpha \) is itself a regular inner parallel body of \( u_\alpha \) (see Corollary IV.3); in particular, \( c(u)_0 \) is a convex body. It remains to show that \( c(u)_0 \) is a convex body. Indeed, \( c(u)_0 \) is convex since it is the closure of the union of a family of convex bodies linearly ordered by inclusion, and its boundedness follows from \( c(u)_0 \subseteq u_0 \).

Thirdly, in order to obtain (L3), let \( \{ \alpha_k \} \subseteq (0, 1] \) be an increasing sequence that converges to \( \alpha_0 = 0 \). Then \( v_{\alpha_0} = \bigcap_{k \geq 1} v_{\alpha_k} \) for each \( v \in \Upsilon_u \), and thus

\[
c(u)_{\alpha_0} = \bigcap_{v \in \Upsilon_u} v_{\alpha_0} = \bigcap_{k \geq 1} \bigcap_{v \in \Upsilon_u} v_{\alpha_k} = \bigcap_{k \geq 1} \bigcap_{v \in \Upsilon_u} c(u)_{\alpha_k}.
\]

For the support function of \( c(u) \), let \( \alpha \in (0, 1] \). Since \( c(u)_\alpha \) is the intersection of a family of regular inner parallel bodies of \( u_\alpha \), it follows from Corollary IV.3 that

\[
h_{c(u)}(\alpha, -) = h_{c(u)}(\alpha) = \bigwedge_{v \in \Upsilon_u} h_{v_\alpha} = \bigwedge_{v \in \Upsilon_u} h_{v_\alpha}(-, -).
\]

Finally, the value of \( h_{c(u)}(0, -) \) follows from the right-continuity of \( h_{c(u)}(-, -) \) at 0 (see Theorem II.6).

In fact, \( c(u) \) also lies in \( \Upsilon_u \), and it is a Mareš core of each fuzzy vector \( u \in \mathcal{F}^n \):

**Theorem V.5.** \( c(u) \) is skew, and there exists a symmetric fuzzy vector \( s(u) \in \mathcal{F}^n_\sigma \) such that \( u = c(u) \oplus s(u) \). Hence, \( c(u) \) is a Mareš core of \( u \).

**Proof.** Step 1. There exists a symmetric fuzzy vector \( s(u) \in \mathcal{F}^n_\sigma \) such that \( u = c(u) \oplus s(u) \).

Let \( h := h_{c(u)} : [0, 1] \times S^{n-1} \rightarrow \mathbb{R} \). Note that for each \( x \in S^{n-1} \), \( h(-, x) \) is left-continuous on \( [0, 1] \) and right-continuous at 0 since so are \( h_{c(u)}(-, x) \) and \( h_{c(u)}(\cdot, x) \) by Theorem II.6. Moreover, as there exists a symmetric fuzzy vector \( w \in \mathcal{F}^n_\sigma \) such that \( u = v \oplus w \), for all \( v \in \Upsilon_u \), it follows from Theorem II.8 and Proposition V.4 that

\[
h(\alpha, x) = h_\alpha(\alpha, x) - h_{c(u)}(\alpha, x)
\]

\[
= h_\alpha(\alpha, x) - \bigwedge_{v \in \Upsilon_u} h_{v_\alpha}(\alpha, x)
\]

\[
= \bigwedge_{v \in \Upsilon_u} (h_\alpha(\alpha, x) - h_{v_\alpha}(\alpha, x))
\]

\[
= \bigwedge_{\beta \to 0^+} h_{c(u)}(\beta, x)
\]

for all \( \alpha \in (0, 1] \), \( x \in S^{n-1} \). Hence, \( h \) is nonnegative, independent of \( x \in S^{n-1} \) and non-increasing on \( \alpha \in [0, 1] \) because so is each \( h_{v_\alpha} (v \in \Upsilon_u) \); that is, \( h \) satisfies all the conditions of Corollary III.5. Therefore, \( h \) is the support function of a symmetric fuzzy vector \( s(u) \in \mathcal{F}^n_\sigma \), which clearly satisfies \( u = c(u) \oplus s(u) \) by Theorem II.8.

Step 2. \( c(u) \) is skew.
Suppose that \( c(u) = v \oplus w \) and \( w \) is symmetric. Then \( h_v \leq h_{c(u)} \) by Lemma V.3.

Conversely, since \( w \) and \( s(u) \) are both symmetric, so is \( v \oplus s(u) \) by Corollary III.6. Thus, together with

\[
u = c(u) \oplus s(u) = v \oplus w \oplus s(u) = v \oplus (w \oplus s(u))
\]

we obtain that \( v \in \mathcal{Y}_u \), which implies that \( h_{c(u)} \leq h_v \) by Proposition V.4.

Therefore, \( h_{c(u)} = h_v \), and it forces \( w = 0 \), which shows that \( c(u) \) is skew. \( \square \)

An obvious application of Theorem V.5 is to determine whether a fuzzy vector is skew:

**Corollary V.6.** A fuzzy vector \( u \in F^n \) is skew if, and only if, \( u = c(u) \).

**Proof.** The “if” part is already obtained in Theorem V.5. For the “only if” part, just note that \( \mathcal{Y}_u = \{u\} \) if \( u \) is skew, and thus \( u = c(u) \) necessarily follows. \( \square \)

Let \( F^*_k \) denote the set of skew fuzzy vectors of dimension \( n \). Theorem V.5 actually induces a surjective map as follows:

**Corollary V.7.** The assignment \( (v, w) \mapsto v \oplus w \) establishes a surjective map

\[
F^n_k \times F^n_s \longrightarrow F^n.
\]

Unfortunately, as the following Example V.9 reveals, unlike Theorem I.1 of the case \( n = 1 \) or Theorem IV.4 of convex bodies, in general the map given in Corollary V.7 may not be injective. In other words, a fuzzy vector may have many Mareš cores, so that there may be many ways to decompose a fuzzy vector as the sum of a skew fuzzy vector and a symmetric fuzzy vector!

As a preparation, let us present a sufficient condition for a fuzzy vector to be skew that is easy to verify:

**Lemma V.8.** Let \( u \in F^n \) be a fuzzy vector. If the 0-level set \( u_0 \) of \( u \) is an irreducible convex body, then \( u \) is skew.

**Proof.** Suppose that \( u = v \oplus w \) and \( w \) is symmetric. Then \( u_\alpha = v_\alpha + w_\alpha \) for all \( \alpha \in [0, 1] \). In particular, \( u_0 = v_0 + w_0 \).

Since \( u_0 \) is irreducible, \( w_0 \) must be trivial, i.e., \( w_0 = \{0\} \), where \( 0 \) is the origin of \( R^n \). The condition (L2) of Proposition II.3 then forces \( w_\alpha = \{0\} \) for all \( \alpha \in [0, 1] \), which means that \( w = 0 \). \( \square \)

**Example V.9.** Suppose that \( n \geq 2 \). For each \( \alpha \in [0, 1] \), let

\[
A_\alpha = \prod_{i=1}^n [\alpha - 1, 1 - \alpha]
\]

be the hypercube in \( R^n \) centered at the origin whose edge length is \( 2 - 2\alpha \).

For every \( \lambda \in [0, 1] \), we may construct a fuzzy vector \( v_\lambda \in F^n \) whose level sets are given by

\[
(v_\lambda)_\alpha = A_\alpha + B_{\lambda \alpha},
\]

and a non-trivial symmetric fuzzy vector \( w_\lambda \in F^n \) whose level sets are given by

\[
(w_\lambda)_\alpha = B_{2-\lambda \alpha}.
\]

Then there exists a fuzzy vector \( u \in F^n \) with

\[
u_\alpha = A_\alpha + B_2 = A_\alpha + B_{\lambda \alpha} + B_{2-\lambda \alpha} = (v_\lambda)_\alpha + (w_\lambda)_\alpha
\]

for all \( \lambda \in [0, 1] \), where the second equation follows from Corollary III.6. Hence

\[
u = v_\lambda \oplus w_\lambda
\]

for all \( \lambda \in [0, 1] \). As \( (v_\lambda)_0 = A_0 = \prod_{i=1}^n [-1, 1] \) is irreducible, from Lemma V.8 we know that every \( v_\lambda \) is skew, and therefore every \( v_\lambda (0 \leq \lambda \leq 1) \) is a Mareš core of \( u \).

With Theorem I.1 and Example V.9 we can now conclude:

**Theorem V.10.** Every fuzzy vector \( u \in F^n \) has a unique Mareš core if, and only if, the dimension \( n = 1 \).

**VI. Mareš Equivalent Fuzzy Vectors**

The following definition is also originated from fuzzy numbers (see [10], [11], [13]):

**Definition VI.1.** Fuzzy vectors \( u, v \in F^n \) are Mareš equivalent, denoted by \( u \sim_M v \), if there exist symmetric fuzzy vectors \( w, w' \in F^n_s \) such that

\[
u \oplus w = v \oplus w'.
\]

The relation \( \sim_M \) is clearly an equivalence relation on \( F^n \), and we denote by \( [u]_M \) the equivalence class of each \( u \in F^n \).

**Proposition VI.2.** For fuzzy vectors \( u, v \in F^n \), the following statements are equivalent:

(i) \( u \sim_M v \).
(ii) For each \( \alpha \in [0, 1] \), either \( u_\alpha \) is a regular inner parallel body of \( v_\alpha \), or \( v_\alpha \) is a regular inner parallel body of \( u_\alpha \).
(iii) For each \( \alpha \in [0, 1] \), the function \( h_u(\alpha, -) - h_v(\alpha, -) \) is constant on \( S^{n-1} \).

**Proof.** (ii) \( \iff \) (iii) is an immediate consequence of Theorem II.8 and Lemma IV.1, and (i) \( \implies \) (iii) follows soon from Theorems II.8 and III.4. For (iii) \( \implies \) (i), let us fix \( x_0 \in S^{n-1} \), and let \( \eta \) be a common upper bound of \( h_u \) and \( h_v \) on \( [0, 1] \times S^{n-1} \). Then the functions

\[
\begin{array}{ccc}
[0, 1] & \longrightarrow & R \\
\alpha & \mapsto & \eta + h_u(\alpha, x_0)
\end{array}
\]

and

\[
\begin{array}{ccc}
[0, 1] & \longrightarrow & R \\
\alpha & \mapsto & \eta + h_v(\alpha, x_0)
\end{array}
\]

clearly satisfy the conditions of Corollary III.5, and thus they are support functions of symmetric fuzzy vectors \( w, w' \in F^n_s \), respectively.

Since \( h_u(\alpha, -) - h_v(\alpha, -) \) is constant on \( S^{n-1} \) for each \( \alpha \in [0, 1] \), it follows that

\[
h_u(\alpha, x) - h_v(\alpha, x) = h_u(\alpha, x_0) - h_v(\alpha, x_0) = h_w(\alpha) - h_w'(\alpha)
\]

for all \( \alpha \in [0, 1] \), \( x \in S^{n-1} \); that is, \( h_u + h_w = h_v + h_w' \).
and therefore \( u \oplus w = v \oplus w' \) by Theorem II.8, showing that \( u \sim_M v \).

\[ \text{Remark VI.3. For fuzzy vectors } u, v \in F^n, u \sim_M v \text{ does not necessarily imply that } u = v \oplus w \text{ or } v = u \oplus w \text{ for some } w \in F^n. \text{ For example, if } \]

\[
\begin{align*}
    u(t) = \begin{cases} 
        1 & \text{if } t \in B_1, \\
        0 & \text{if } t \not\in B_1
    \end{cases} \quad \text{and} \quad v(t) = \begin{cases} 
        1 - \frac{||t||}{2} & \text{if } t \in B_2, \\
        0 & \text{if } t \not\in B_2
    \end{cases}
\end{align*}
\]

then \( u, v \in F^n \) and it trivially holds that \( u \oplus v = v \oplus u \); hence \( u \sim_M v \). However, there is no \( w \in F^n \) such that \( u = v \oplus w \) or \( v = u \oplus w \). Indeed, the level sets of \( u, v \) are given by

\[
    u_\alpha = B_1 \quad \text{and} \quad v_\alpha = B_2 - 2\alpha
\]

for all \( \alpha \in [0, 1] \); that is, \( u_\alpha \) is a regular inner parallel body of \( v_\alpha \) when \( 0 \leq \alpha \leq \frac{1}{2} \), and \( v_\alpha \) is a regular inner parallel body of \( u_\alpha \) when \( \frac{1}{2} \leq \alpha \leq 1 \).

It is obvious that \( c(u) \sim_M u \) for all \( u \in F^n \). In fact, every Mareš core of \( u \) is Mareš equivalent to \( u \). In what follows we construct another skew fuzzy vector \( k(u) \) that is Mareš equivalent to \( u \) but may not be a Mareš core of \( u \).

\[ \text{Proposition VI.4. There is a skew fuzzy vector } k(u) \in F^n \text{ whose level sets are given by } \]

\[
k(u)_\alpha := \begin{cases} 
    \bigcap_{\beta \in [0,1]} v_\alpha & \text{if } \alpha \in (0, 1], \\
    \bigcup_{\beta \in [0,1]} k(u)_\beta & \text{if } \alpha = 0,
\end{cases}
\]

and whose support function \( h_{k(u)} : [0,1] \times \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) is given by

\[
h_{k(u)}(\alpha, x) = \begin{cases} 
    h_v(\alpha, x) & \text{if } \alpha \in (0, 1], \\
    \lim_{\beta \rightarrow 0^+} h_{k(u)}(\beta, x) & \text{if } \alpha = 0.
\end{cases}
\]

In particular, \( k(u) \sim_M u \).

\[ \text{Proof. The verification of } k(u) \text{ being a fuzzy vector is similar to Proposition V.4 under the help of Proposition V.2, and thus we leave it to the readers. In particular, } k(u) \sim_M u \text{ is an immediate consequence of Proposition VI.2 and the fact that each } k(u)_\alpha \text{ is a regular inner parallel body of } u_\alpha. \]

To see that \( k(u) \) is skew, suppose that \( k(u) = v \oplus w \) and \( w \) is symmetric. Then \( h_v \leq h_{k(u)} \) by Lemma V.3. Conversely, since \( k(u) \sim_M v \), it holds that \( u \sim_M v \), and thus \( h_{k(u)} \leq h_v \). Hence \( h_{k(u)} = h_v \), which forces \( w = 0 \) and completes the proof. \]

\[ \text{Remark VI.5. For each } \alpha \in (0, 1], \text{ the } \alpha \text{-level set } \]

\[
u_\alpha = \{ t \in \mathbb{R}^n \mid u(t) \geq \alpha \}
\]

of a fuzzy vector \( u \in F^n \) has a smallest regular inner parallel body given by Equation (5) below Corollary IV.3. It is then tempting to ask whether \( k(u) \) could be determined by the \( \alpha \)-level sets

\[
k(u)_\alpha = (u_\alpha) - \lambda u_\alpha.
\]

Unfortunately, this is not true since, in general, \( \{ (u_\alpha) - \lambda u_\alpha \mid \alpha \in [0,1] \} \) does not satisfy the conditions of Proposition II.3 even when \( n = 1 \). For example, the level sets of the fuzzy number

\[
u : \mathbb{R} \rightarrow [0, 1], \quad u(t) = \begin{cases} 
1 - \frac{t}{2} & \text{if } t \in [0, 2], \\
0 & \text{else}
\end{cases}
\]

are given by

\[
u_\alpha = [0, 2 - 2\alpha]
\]

for all \( \alpha \in [0, 1] \), but

\[
\{ (u_\alpha) - \lambda u_\alpha \mid \alpha \in [0, 1] \} = \{ 1 - \alpha \mid \alpha \in [0, 1] \}
\]

consists of non-identical singleton sets which obviously violate the condition (1.2).

From the definition it is clear that \( k(u) \) is the smallest fuzzy vector in the Mareš equivalence class of \( u \in F^n \); that is,

\[
h_{k(u)} \leq h_u
\]

for all \( v \in [u]_M \). In fact, if the dimension \( n = 1 \), \( k(u) \) is precisely the (unique) Mareš core of a fuzzy number \( u \) constructed in [11, Proposition 4.2] (cf. [11, Remark 4.4]); that is:

\[ \text{Corollary VI.6. For each fuzzy number } u, \text{ it holds that } \]

\[
k(u) = c(u).
\]

\[ \text{Moreover, } k(u) \text{ is the unique Mareš core of } u \text{ and the unique skew fuzzy number in the Mareš equivalence class } [u]_M. \]

However, if the dimension \( n \geq 2 \), the following continuation of Example V.9 shows that \( k(u) \) may not be a Mareš core of \( u \) and skew fuzzy vectors may be Mareš equivalent to each other:

\[ \text{Example VI.7. For the fuzzy vectors } u, v_\lambda, w_\lambda (0 \leq \lambda \leq 1) \text{ considered in Example V.9, since } \]

\[
v_\lambda \in [u]_M
\]

for all \( \lambda \in [0, 1] \), all skew fuzzy vectors \( v_\lambda \) are Mareš equivalent to each other. Moreover, although

\[
v_0 = k(u) = k(v_\lambda),
\]

\( v_0 \) is not a Mareš core of \( v_\lambda \) whenever \( \lambda \in (0, 1] \); in fact, since \( v_\lambda \) is skew, its Mareš core must be itself.

\[ \text{VII. Conclusion} \]

The Mareš core is a fundamental construction in the theory of fuzzy numbers. By applying the toolkit from convex geometry, we investigate Mareš cores of fuzzy vectors over the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) through convex bodies and support functions, and it is shown that the following statements are equivalent:

(i) The dimension \( n = 1 \).
(ii) Each fuzzy vector has a unique Mareš core.
(iii) Each fuzzy vector can be decomposed in a unique way as the sum of a skew fuzzy vector and a symmetric fuzzy vector.
(iv) There is only one skew fuzzy vector in each Mareš equivalence class of fuzzy vectors.

This paper is a first step towards the study of Mareš cores of fuzzy vectors, and many questions remain unsolved. We end this paper with two questions to be considered in future works:

(i) Is it possible to find all the Mareš cores of a given fuzzy vector?

(ii) Is it possible to find a necessary and sufficient condition to characterize those fuzzy vectors with only one Mareš core?

APPENDIX

PROOF OF THEOREM II.6

Proof. Necessity. Let $u \in F^n$ be a fuzzy vector. Then $h_u$ clearly satisfies (VS1) by Proposition II.4. For (VS2), let us fix $x \in S^{n-1}$. Then $h_u(-, x) : [0, 1] \to \mathbb{R}$ is non-increasing because of (L2).

To see that $h_u(-, x)$ is left-continuous at each $\alpha_0 \in (0, 1]$, let $\{\alpha_k\} \subseteq (0, 1]$ be an increasing sequence that converges to $\alpha_0$. Then $u_{\alpha_0} = \bigcap_{k \geq 1} u_{\alpha_k}$ by (L3). For each $\epsilon > 0$, we claim that there exists a positive integer $k$ such that for all $t \in u_{\alpha_k}$, there exists some $r_t \in u_{\alpha_0}$ such that $|t - r_t| < \epsilon$. Indeed, suppose that we find an $\epsilon_0 > 0$ such that for all positive integers $k$, there exists $t_k \in u_{\alpha_k}$ such that

$$d(t_k, u_{\alpha_k}) := \bigwedge_{r \in u_{\alpha_k}} ||t_k - r|| \geq \epsilon_0.$$ 

Then the sequence $\{t_k\}$ is contained in the compact set $u_{\alpha_1}$, and thus it has a convergent subsequence. Without loss of generality we may suppose that $\lim_{k \to \infty} t_k = t_0$. Then $t_0 \in u_{\alpha_k}$ for all $k \geq 1$ because $\{t_m | m \geq k\} \subseteq u_{\alpha_k}$, and consequently $t_0 \in \bigcap_{k \geq 1} u_{\alpha_k} = u_{\alpha_0}$. But the construction of $t_k$ forces

$$d(t_0, u_{\alpha_0}) \geq \epsilon_0,$$

which is a contradiction.

Suppose that $r_t = t$ for $t \in u_{\alpha_0}$, we obtain

$$|h_u(\alpha_0, x) - h_u(\alpha_0, x)| = \bigwedge_{t \in u_{\alpha_0}} \langle t, x \rangle - \bigwedge_{t \in u_{\alpha_0}} \langle t, x \rangle$$

$$= \bigwedge_{t \in u_{\alpha_0}} \langle t - r_t, x \rangle + \bigwedge_{t \in u_{\alpha_0}} \langle r_t, x \rangle - \bigwedge_{t \in u_{\alpha_0}} \langle t, x \rangle$$

$$\leq ||x|| \cdot ||t - r_t|| < \epsilon.$$ 

Hence $\lim_{k \to \infty} h_u(\alpha_k, x) = h_u(\alpha_0, x)$, which proves the left-continuity of $h_u(-, x)$ at $\alpha_0$.

To see that $h_u(-, x)$ is right-continuous at 0, let $\epsilon > 0$. The compactness of $u_0$ allows us to find $q_1, \ldots, q_k \in u_0$ such that $u_0$ is covered by finitely many balls

$$B(q_1, \frac{\epsilon}{2}), \ldots, B(q_k, \frac{\epsilon}{2})$$

centered at $q_1, \ldots, q_k$, respectively, with radii $\frac{\epsilon}{2}$.

Note that for each $t \in u_0 = \bigcup_{\alpha \in (0,1]} u_\alpha$, there exists $\alpha_t \in (0,1]$ and $s_t \in u_{\alpha_t}$ such that $||t - s_t|| < \frac{\epsilon}{2}$. Let

$$\alpha_q := \min \{\alpha_{q_1}, \ldots, \alpha_{q_k} \} > 0,$$

and let $B\left(q_t, \frac{\epsilon}{2}\right)$ ($q_t \in \{q_1, \ldots, q_k\}$) be the ball containing $t$. Then $s_{q_t} \in u_{\alpha_{q_t}}$, and

$$||t - s_{q_t}|| \leq ||t - q_t|| + ||q_t - s_{q_t}|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Suppose that $s_{q_t} = t$ for all $t \in u_{\alpha_q}$, we obtain

$$|h_u(0, x) - h_u(\alpha_0, x)|$$

$$= \bigwedge_{t \in u_{\alpha_0}} \langle t, x \rangle - \bigwedge_{t \in u_{\alpha_0}} \langle t, x \rangle$$

$$= \bigwedge_{t \in u_{\alpha_0}} \langle t - s_{q_t}, x \rangle + \bigwedge_{t \in u_{\alpha_0}} \langle s_{q_t}, x \rangle - \bigwedge_{t \in u_{\alpha_0}} \langle t, x \rangle$$

$$= \bigwedge_{t \in u_{\alpha_0}} \langle t - s_{q_t}, x \rangle$$

$$\leq ||x|| \cdot ||t - s_{q_t}|| < \epsilon.$$ 

Hence $\lim_{\alpha \to 0^+} h_u(\alpha, x) = h_u(0, x)$, which proves the right-continuity of $h_u(\alpha, x)$ at 0.

Sufficiency. It suffices to show that $\{A_{h(\alpha, -)} \ | \ \alpha \in [0, 1]\}$ satisfies the conditions (L1)–(L4).

Firstly, (L1) is a direct consequence of Proposition II.4. Secondly, (L2) holds since $h(-, x)$ is non-increasing for all $x \in S^{n-1}$.

Thirdly, for (L3), let $\{\alpha_k\} \subseteq (0, 1]$ be an increasing sequence that converges to $\alpha_0 \in (0, 1]$. It remains to show that

$$\bigcap_{k \geq 1} A_{h(\alpha_k, -)} \subseteq A_{h(\alpha_0, -)},$$

since the reverse inclusion is trivial by (L2). Suppose that $t \in A_{h(\alpha_k, -)}$ for all $k \geq 1$. Then $\langle t, x \rangle \leq h(\alpha_k, x)$ for all $x \in S^{n-1}$. Thus the left-continuity of $h(-, x)$ at $\alpha_0$ implies that

$$\langle t, x \rangle \leq \lim_{k \to \infty} h(\alpha_k, x) = h\left(\lim_{k \to \infty} \alpha_k, x\right) = h(\alpha_0, x),$$

and consequently $t \in A_{h(\alpha_0, -)}$.

Finally, it remains to prove (L4) by showing that

$$A_{h(0, -)} \subseteq A := \bigcup_{\alpha \in (0,1]} A_{h(\alpha, -)}$$

as the reverse inclusion is trivial by (L2). We proceed by contradiction. Suppose that $t_0 \in A_{h(0, -)}$ but $t_0 \not\in A$. Note that $A$ is also a convex body, since $A \subseteq A_{h(0, -)}$ and $A$ is the closure of the union of a family of convex bodies linearly ordered by inclusion. Thus

$$\lambda_0 := d(t_0, A) := \bigwedge_{r \in A} ||t_0 - r|| > 0,$$

and there exists $r_0 \in A$ with $||t_0 - r_0|| = \lambda_0$, where $r_0$ must be in the boundary of $A$. Let

$$x_0 := \frac{t_0 - r_0}{||t_0 - r_0||}.$$
By [25, Theorem 4.1] there exists a (unique) support hyperplane $H$ of $A$ passing through $r_0$ with $x_0$ as its exterior normal vector. The hyperplane $H$ divides $\mathbb{R}^n$ into two parts, denoted by $H^-$ and $H^+$, with

$$H^- = \{ t \in \mathbb{R}^n \mid \langle t, x_0 \rangle \leq \langle r_0, x_0 \rangle \} \quad \text{and} \quad H^+ = \{ t \in \mathbb{R}^n \mid \langle t, x_0 \rangle \geq \langle r_0, x_0 \rangle \},$$

which satisfies $t_0 \in H^+$ and $A \subseteq H^-$. It follows that

$$h(\alpha, x_0) = \bigvee_{t \in A_h(\alpha, \cdot)} \langle t, x_0 \rangle \quad \text{(Proposition II.4)}$$

$$\leq \langle r_0, x_0 \rangle - \langle t_0 - r_0, x_0 \rangle = \langle t_0, x_0 \rangle - \|t_0 - r_0\| \langle x_0 = \frac{t_0 - r_0}{\|t_0 - r_0\|} \rangle$$

$$\leq h(0, x_0) - \lambda_0 \quad (t_0 \in A_h(\alpha, \cdot))$$

for all $\alpha \in (0, 1]$, which contradicts to the right-continuity of $h(\cdot, x_0)$ at $0$. The proof is thus completed. \(\square\)

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