短文沟通

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关于不带pendent vertex的二部图的最大ABC指数

Abstract: 对于有向图G，原子–键连接指数 (ABC) 为G定义为

\[ ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}} \]

其中d(v)表示顶点v的度数。在这个问题中，我们发现对于任意二部图G，若阶数\( n \geq 6 \)，大小2n−3，且有δ(G) ≥ 2，\( ABC(G) \leq \sqrt{2(n-6)} + 2\sqrt{\frac{3(n-2)}{n-3}} + 2 \)以及我们证明所有极端二部图的性质。

Keywords: 度数(顶点); 原子–键连接指数; 二部图; 极端图; 发生能量。

1 介绍

令G为一个简单连通图，顶点集为V = V(G)，边集为E = E(G)，其中简单图不包含环和重复的边，在两个顶点之间有两个边。图G的阶数由n = |V(G)|表示，图G的大小由m = |E(G)|表示。对于每个顶点v ∈ V，打开的邻居N(v) = {u ∈ V(G) | uv ∈ E(G)}。对于一个图G，度数δ(G)表示为d(v)。一个顶点v ∈ V的度数为d_G(v) = d_G(v) = |N(v)|。对于一个图G，最小度数为δ(G)。一个图G的叶或pendent vertex是一个顶点度数为1。对于一个子集S的顶点G，我们表示为G[S]的子图，由S诱导的子图。一个二部图图G没有奇数环。

图连通性指数χ，是一种图的连通性指数，定于1975年，由米兰 RANDIC [1] 所提出，该指数作为 Randic 创立的。许多物理-化学性质的性质都依赖于因子而不是不同的分支。为了将这个信息与 Randic 箱

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Lemma 1. ([28]) If $(x, y)$ is strictly decreasing for $x$ if $y \geq 3$.

Lemma 2. ([15]) If $G$ is a graph with $\delta(G) \geq 2$ and $xy \in E(G)$, then $f(x, y) \leq \sqrt{2}$ with equality if and only if $d(x) = 2$ or $d(y) = 2$ (or both).

The proof of the next result is straightforward and therefore omitted.

Lemma 3. ([28]) For $x, y \geq 2$, if $h_x(x, y) = \frac{\partial f(x, y)}{\partial x}$ and $h_y(x, y) = \frac{\partial f(x, y)}{\partial y}$, then $h_x(x, y) < 0$.

Similarly, we can see that $h_y(x, y) > 0$.

Let $K_{3,t}$ be the complete bipartite graph with bipartite sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, \ldots, y_t\}$. For $s \geq 6$, assume $H_{3,t}$ is the bipartite graph obtained from $K_{3,t}$ by adding $s$ vertices $v_1, \ldots, v_s$ and joining them to $x_1$ and $x_3$ (see Figure 1). If $s = 0$, we define $H_{3,t}^3$ to be the graph $K_{3,t}$. The proof of next result is easy to verify by direct calculation.

Observation 4. If $n \geq 6$, then $ABC(H_{3,3}^{3,n-6}) = \sqrt{2}(n-6) + 2\sqrt{3(n-2)} + 2$.

In the rest of the paper, we employ the following notation defined in [15].

$$V_1^G = \{v \in V(G) | d(v) = 1\},$$

$$V_3^G = \{v \in V(G) | d(v) \geq 3\} \text{ and } pk = |V_3^G|,$$

$$E_{s,t}^G = \{uv \in E(G) | d(u) = s \text{ and } d(v) = t\},$$

$$E_{s,t}^{3,3} = \{uv \in E(G) | d(u) \geq 3 \text{ and } d(v) \geq 3\}.$$

and

$$E^G = \{uv | uv \in E(G)\}.$$ (7)

In the following, we will omit the superscript where no confusion can arise.

For an edge $e = uv \in E(G)$, we define $ABC(e(G)) = \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}$. When no confusion can arise, we simplify $ABC(e(G))$ to $ABC(e)$.

2 Main results

In this section, we present an upper bound on the ABC index of a bipartite graph and characterize all extreme bipartite graphs. For any bipartite graph $G$, we let $(X_G, Y_G)$ denote its bipartition.

Theorem 5. Let $G$ be a bipartite graph of order $n \geq 6$, size $2n-3$ with $\delta(G) \geq 2$. Then $ABC(G) \leq \sqrt{2n-6} + 2\sqrt{3(n-2)} + 2$ with equality if and only if $G \approx H_{3,3}^{3,n-6}$.

Proof: We first perform the following steps:

Step 1: Using the software geng in package nauty [29], we generate the set of all graphs of order $n \in \{6, 7, 8, 9\}$, size $2n-3$ with $\delta(G) \geq 2$.

Step 2: For each graph $G$ in the obtained graphs, we compute $ABC(G)$ according to formula (1). It turns out that the result holds for each $n \in \{6, 7, 8, 9\}$.

Now, assume that $n \geq 10$. Let $G_n$ be the family of graphs $G$ of order $n$, size $2n-3$, minimum degree $\delta(G) \geq 2$, the maximum $ABC$ index and different from $H_{3,3}^{3,n-6}$. We will show that $G_n = \emptyset$. Suppose, to the contrary, that $G_n \neq \emptyset$ for some $n \geq 10$. We further assume that $n$ is as small as possible such that $G_n \neq \emptyset$. Let $G \in G_n$. We proceed with establishing several claims.

Claim 1. $k \geq 6, |E_{2,2}| = 0$ and $2k - 3 = |E_{3,3}^3|$.  

Proof of Claim 1. Since every edge is incident with two vertices, an edge contributes 2 to the sum of the degrees of the vertices and so

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 4m(G) = 4n - 6.$$ (8)

Since $|V_1^k| = k$, we have

$$\sum_{v \in V_2} d(v) = 2|V_2| = 2(n - k).$$ (9)

Combining (8) and (9), we obtain

$$\sum_{v \in V_2} d(v) = 2n + 2k - 6.$$ (10)
Since
\[ \sum_{v \in V_3} d(v) = 2|E_{3,3}^1| + |E_{2,3}^1| \] (11)
and \( m(G) = 2n - 3 = |E_{2,2}| + |E_{2,3}^1| + |E_{3,3}^1| \), we conclude from (10) that
\[ |E_{3,3}^1| - |E_{2,2}| = 2k - 3, \] (12)
and so
\[ |E_{3,3}^1| \geq 2k - 3. \] (13)

Suppose \( x = |X_3 \cap V_{3,3}^1| \). Then \( k - x = |Y_3 \cap V_{3,3}^1| \) and we have
\[ |E_{3,3}^1| \leq x(k - x) \leq \frac{k^2}{4}. \] (14)

By (13) and (14), we have \( k^2 - 8k + 12 \geq 0 \), this implies that \( k \leq 2 \) or \( k \geq 6 \). If \( k = 0 \), then (11) and (12) imply that \( |E_{3,3}^1| = |E_{2,3}^1| = 0 \) and \( |E_{2,2}| = 3 \) yielding \( G \cong C_3 \), which is a contradiction. If \( k = 1 \), then we deduce from (14) and (12) that \( |E_{3,3}^1| = 0 \) and \( |E_{2,2}| = 1 \). Let \( E_{2,2} = \{ uv \} \) and \( V_{3,3}^1 = \{ w \} \). Since \( d(u) = d(v) = 2 \) and \( |E_{2,2}| = 1 \), we must have \( uw \in E, vw \in E \), which leads to a contradiction. If \( k = 2 \), then (14) and (12) imply that \( |E_{3,3}^1| = 1 \) and \( |E_{2,2}| = 0 \). It follows that any vertex of \( V_2 \) along with two vertices in \( V_3^1 \) forms a triangular in \( G \), which is a contradiction. Therefore,
\[ k \geq 6. \] (15)

Now, we show that \( |E_{2,2}| = 0 \). Suppose, to the contrary, that \( |E_{2,2}| \geq 1 \) and let \( uv \in E_{2,2} \). Since \( k \geq 6 \), it follows from (12) that there exists an edge \( ts \in E_{3,3} \) such that \( \{s, t\} \cap \{u, v\} = \emptyset \). Assume, with out loss of generality, that \( u, t \in X \) and \( v, s \in Y \). We consider two cases.

Case 1. \( u \notin N(s) \) and \( v \notin N(t) \).

Let \( G' \) be the graph obtained from \( G \) by removing two edges \( uv, st \) and adding the edges \( us, tv \). Clearly, \( G' \) is a bipartite graph of order \( n \), size \( 2n - 3 \) with \( \delta(G') \geq 2 \). By Lemma 2, we have
\[
ABC(G') - ABC(G) = ABC(us \mid G') + ABC(tv \mid G') - ABC(uv \mid G) - ABC(ts \mid G) = \sqrt{\frac{d(u(G') + d(s(G') - 2)}{2}} - \sqrt{-\frac{d(u(G') + d(s(G') - 2}{2d(u(G))d(v(G))}} - \sqrt{-\frac{d(t(G') + d(v(G') - 2}{2d(u(G))d(v(G))}} + \sqrt{\frac{d(t(G') + d(v(G') - 2}{2d(u(G))d(v(G))}}}
\]
which is a contradiction.

Case 2. \( u \in N(s) \) (the case \( v \in N(t) \) is similar).

We distinguish two subcases.

Subcase 2.1. \( V_2 \cap N(v) \neq \emptyset \).

Let \( w \in V_2 \cap N(v) \). Since \( k \geq 6 \), \( |E_{3,3}^1| \geq 2k - 3 \) and \( |E_3 \cap E_{3,3}^1| \leq k - 1 \), there exists an edge \( pq \in E_{3,3}^1 \) such that \( s \notin \{p, q\} \). Assume without loss of generality that \( p \in X_3 \) and \( q \in Y_3 \). We deduce from \( d(u) = d(v) = 2 \), \( N(u) = \{v, s\} \) and \( N(v) = \{u, w\} \), we have \( u \notin N(q) \) and \( v \notin N(p) \). As in Case 1, we obtain a contradiction again.

Thus \( |E_{2,2}| = 0 \) and by (12) we have \( |E_{3,3}^1| = 2k - 3 \). This completes the proof of Claim 1.

\[ 2. \] \( |E_{2,3}| = 0. \)

Proof of Claim 2. Suppose, to the contrary, that \( E_{2,3} \neq \emptyset \) and let \( uv \in E_{2,3} \). Assume without loss of generality that \( d(v) = 2 \) and \( d(u) = 3 \). Since \( E_{2,2} = \emptyset \), the other neighbor of \( v \), say \( t \), is in \( V_3 \). We distinguish the following cases.

(2.a). \( N(u) \cap V_3 \neq \emptyset \).

Let \( w \in N(u) \cap V_3 \) and let \( G' = G - v \). Clearly, \( G' \) is a bipartite graph of order \( n(G') = n - 1 \), size \( m(G') = (2n - 3) - 3 = 2(n - 1) - 3 \) and \( \delta(G') \geq 2 \). By the choice of \( n \), we have \( ABC(G') \leq \sqrt{\frac{3(n - 3)}{2}} + 2 \). We also have \( ABC(uw \mid G') = \sqrt{\frac{2}{2}} \), \( ABC(uv \mid G) = ABC(vt \mid G) = \sqrt{\frac{2}{2}} \) and \( ABC(uw \mid G) = \sqrt{\frac{d(uw)}{2d(uw)}} \). Using inequality
\[
2\sqrt{\frac{3(n - 3)}{n - 3} - 2 \sqrt{\frac{3(n - 2)}{n - 3} < \sqrt{\frac{2}{2} - \frac{2}{3}}},
\]
we obtain
\[
ABC(G) \geq ABC(G') - ABC(uw \mid G') + ABC(uw \mid G) + ABC(uv \mid G) + ABC(vt \mid G)
\]
\[ \sqrt{2(n - 7)} + 2 \sqrt{\frac{3(n - 3)}{n - 4}} + 2 - \sqrt{\frac{2}{3}} + 2 + \sqrt{\frac{2}{3}} + \sqrt{\frac{2}{3}} = \sqrt{2(n - 6)} + 2 + 2 \sqrt{\frac{3(n - 2)}{n - 3}} = ABC(H_{n-6}^3), \]

which is a contradiction.

(2.b). \( N(u) \cap V_3 = \emptyset. \)

Since \( |E_{3,3} \cap V_3| = 2k - 3 \geq 9 \), there is an edge \( sh \in E_{3,3} \). Assume without loss of generality that \( u, s \in Y \) and \( v, h \in X \). First let \( s \notin N(v) \). Suppose \( G' \) is the graph obtained from \( G \) by removing the edges \( v, sh \) and adding the edges \( vs \) and \( uh \). Clearly, \( G' \) is bipartite of order \( n \), size \( 2n - 3 \) and \( \delta(G') \geq 2 \). Since \( ABC(uh|G') \geq ABC(sh|G) \) and \( ABC(vs|G') = ABC(vu|G) \), we have \( ABC(G') \geq ABC(G) \) yielding \( ABC(G') = ABC(G) \). Hence, \( G' \in G_n \). By Claim 1, we have \( k \geq 6, |E_{3,3}'| = 0 \) and \( 2k - 3 = |E_{3,3}'| \). If \( w \in N_G(u) \setminus \{v\} \), then by assumption we have \( uw \in E_{2,3}' \cap E_{G'}^3 \). Considering the edge \( uw \in G' \), we are in situation (a) and we obtain a contradiction.

Now let \( s \notin N(v) \). Using an argument similar to that described in Case 2 of Claim 1, we deduce that there exists an edge \( pq \in E_{3,3} \) such that \( s \notin \{p, q\} \). Using the proof of Case (b), we obtain a contradiction again.

This completes the proof of Claim 2. \( \square \)

Claim 3. There exists a bipartite graph \( G' = (X_{G'}, Y_{G'}) \) of order \( n \), size \( 2n - 3 \) and \( \delta(G') \geq 2 \), which satisfies the following conditions:

(i) \( ABC(G') \geq ABC(G) \),
(ii) \( \bigcup_{u \in V_{G'} \cap X_G} N(u) = \emptyset \) or \( V_{G'} \cap X_G = \emptyset \),
(iii) \( \bigcup_{u \in V_{G'} \cap Y_G} N(u) = \emptyset \) or \( V_{G'} \cap Y_G = \emptyset \).

Proof of Claim 3. For any bipartite graph \( G_1 \) of order \( n \), size \( 2n - 3 \) and \( \delta(G_1) \geq 2 \), we define and \( L(G_1) = |A(G_1)| \). Let \( H \) be the family of bipartite graphs \( H \neq H_{3,3} \) such that \( |V(H)| = n, m(H) = 2n - 3, \delta(H) \geq 2 \) and \( ABC(H) \geq ABC(G) \). Since \( G \in H \), we establish that \( H \neq \emptyset \). Choose a graph \( G' \in H \) such that \( L(G') \) is as small as possible. Clearly, \( G' \) satisfies the conditions (i). Then \( G' \in G_n \) and so \( G' \) satisfies Claims 1 and 2. If \( G' \) satisfies the conditions (ii) and (iii) of Claim 3, then we are done. Assume that \( G' \) does not satisfy the condition in (ii) or (iii). Assume without loss of generality that \( G' \) does not satisfy condition (ii). Then \( |V_{G'} \cap X_G| \notin \{0, 1\} \). It follows that \( \bigcup_{u \in V_{G'} \cap X_G} N(u) \geq 3 \). Then there are two vertices \( u_1, u_2 \in V_{G'} \cap X_G \) such that \( N_{G'}(u_1) \neq N_{G'}(u_2) \). Let \( G_1 = \{v_1, v_2\} \). Since \( E_{2,2} = \emptyset \), we have \( d(v_1|G') \geq 3 \) and \( d(v_2|G') \geq 3 \). We consider the following cases.

Case A. \( |N_{G'}(u_1) \cap N_{G'}(u_2)| = 1 \).

Suppose without loss of generality that \( N_{G'}(u_2) = \{v_2, v_3\} \) and that \( d(v_1|G') \leq d(v_3|G') \). Let

\[ U_1 = \{v \in V_{G'}^{\prime}\cap X_{G'}: |N_{G'}(v) = \{v_1, v_2\}\} = \{a_1 = u_1, \ldots, a_k\}, \]

\[ U_2 = \{v \in V_{G'}^{\prime}: \{v_3 \in N_{G'}(v) \land v \notin N_{G'}(v)\} = \{b_1, b_2, \ldots, b_k\}, \]

and \( |U_1| = k_1, |U_2| = k_2 \). Clearly, \( k_1 \geq 1, k_2 \geq 0 \). We distinguish the following sub-cases.

Subcase A.1. \( k_2 \geq 1 \).

Let \( G_2 \) be the graph obtained from \( G' \) by removing the edges \( v_1a_i, v_3b_i (1 \leq i \leq k_1) \) and adding the edges \( v_3a_i, v_1b_i (1 \leq i \leq k_1) \). Clearly, \( G_2 \) is bipartite, and \( ABC(a_1v_1|G_1) = ABC(a_1v_3|G_1) \) for \( 1 \leq i \leq k_1 \). Since \( d(v_1|G_1) \leq d(v_3|G_1) \), we have \( d(v_1|G_2) \leq d(v_3|G_2) \), and this implies that \( ABC(b_1v_3|G_2) \leq ABC(b_1v_1|G_2) \) for \( 1 \leq i \leq k_1 \). Hence \( ABC(G_2) \geq ABC(G) \). By definition of \( H_{3,3}^3 \), we have \( G_2 \neq H_{3,3}^3 \) and so \( G_2 \in H \). This contradicts the choice of \( G' \) since \( L(G_2) = L(G') - 1 \).

Subcase A.2. \( k_2 = 0 \) and \( d(v_1|G_1) \leq k_1 + 1 \).

In this case we have \( 1 \leq d(v_1|G_1) - 2 \leq k_1 \). Let \( G_3 \) be the graph obtained from \( G' \) by removing the edges \( v_1a_i (1 \leq i \leq d(v_1|G_1) - 2) \) and adding the edges \( v_3a_i (1 \leq i \leq d(v_1|G_1) - 2) \). Clearly, \( G_3 \) is a bipartite graph of order \( n \), size \( 2n - 3 \) with \( \delta(G_3) \geq 2 \) and \( G_3 \neq H_{3,3}^3 \) because \( deg_{G_3}(a_1) = deg_{G_3}(u_2) = 2 \) and \( |N_{G_3}(a_1) \cap N_{G_3}(u_2)| = 1 \). Since \( d(v_1|G_1) \leq k_1 + 1 \), we have \( |N_{G_3}(v_3) \cap V_{G'}^{\prime}| \leq 1 \).

First let \( |N_{G_3}(v_3) \cap V_{G'}^{\prime}| = 0 \). Then \( d(v_1|G_3) = 2 \) and \( ABC(a_1v_1|G_3) = ABC(a_1v_3|G_3) \) for \( 1 \leq i \leq d(v_1|G_1) - 2 \), and this implies that \( ABC(G_3) \geq ABC(G') \geq ABC(G) \). It follows that \( G_3 \in G_n \) which contradicts Claim 1.

Now let \( |N_{G_3}(v_3) \cap V_{G'}^{\prime}| = 1 \) and let \( c \in N_{G_3}(v_3) \cap V_{G'}^{\prime} \).

Clearly, \( ABC(a_1v_1|G_3) = ABC(a_1v_3|G_3) \) for any \( 1 \leq i \leq d(v_1|G_1) - 2, d(v_1|G_3) = 2 \) and \( d(c|G') = d(c|G) \). We write \( d(c) = d(c|G') \). Let

\[ h_1 = f(2, d(c)) - f(d(v_1|G'), d(c)) = \sum_{i=1}^{d(v_1|G')-1} (f(i, d(c)) - f(i + 1, d(c))) \]

and

\[ h_2 = f(d(v_3|G'), d(c)) - f(d(v_3|G') + d(v_1|G') - 2, d(c)) = \sum_{i=d(v_3|G')+(d(v_1|G')-2)}^{d(v_1|G')-1} (f(i, d(c)) - f(i + 1, d(c))) \]

Thus,

\[ ABC(G) - ABC(G') \geq ABC(cv_1|G_3) + ABC(cv_3|G_3) \]
Let \( G_s \) be the graph obtained from \( G' \) by removing the edges \( v_1 a_i (1 \leq i \leq k_1) \) and adding the edges \( v_3 a_i (1 \leq i \leq k_1) \). Clearly, \( G_s \) is a bipartite graph of order \( n \), size \( 2n - 3 \) with \( \delta(G_s) \geq 2 \).

First let \( N_G(v_3) \cap V_{G'}^2 = \emptyset \). Then \( d(v_1|G_s) < d(v_1|G') \), \( ABC(a_iv_1|G) = ABC(a_iv_1|G_s) \) for each \( 1 \leq i \leq k_1 \), and \( ABC(e|G) = ABC(e|G_s) = \sqrt{2} \) for any \( e \in E_{G'} \). This implies that \( ABC(G_s) \geq ABC(G') \). Because \( L(G_s) = L(G') - 1 \), we obtain a contradiction with the above choice of \( G' \).

Now let \( N_G(v_3) \cap V_{G'}^2 \neq \emptyset \) and let \( N_G(v_3) \cap V_{G'}^2 = \{c_1, c_2, \ldots, c_i\} \). Since \( d(c_i|G') = d(c_i|G_s) \) for \( i \in \{1, 2, \ldots, t\} \), we write \( d(c_i) = d(c_i|G') \). For \( i \in \{1, 2, \ldots, t\} \), let

\[
h_1^i = f(d(v_1|G') - k_1, d(c_i)) - f(d(v_1|G'), d(c_i)) = \sum_{j=d(v_1|G')-k_1}^{d(v_1|G')-k_1+1} (f(j, d(c_i)) - f(j+1, d(c_i)))
\]

and

\[
h_2^i = f(d(v_1|G'), d(c_i)) - f(d(v_1|G') + k_1, d(c_i)) = \sum_{j=d(v_1|G')-k_1}^{d(v_1|G')-k_1+1} (f(j, d(c_i)) - f(j+1, d(c_i)))
\]

Assume that \( l = ABC(G_s) - ABC(G') \). We have

\[
l \geq \sum_{i=1}^{i=1} \sum_{j=d(v_1|G')-k_1}^{d(v_1|G')-k_1+1} (f(j, d(c_i)) - f(j+1, d(c_i)))
\]

We conclude from Lemma 1 and the fact \( d(v_3|G') - d(v_1|G') + k_1 \geq 1 \) that \( l = ABC(G_s) - ABC(G') > 0 \), which leads to a contradiction as above.

**Subcase A.3.** \( k_2 = 0 \) and \( d(v_1|G') \geq k_1 + 2 \).

Let \( G_s \) be the graph obtained from \( G' \) by removing the edges \( v_1 a_i (1 \leq i \leq k_1) \) and adding the edges \( v_3 a_i (1 \leq i \leq k_1) \). Clearly, \( G_s \) is a bipartite graph of order \( n \), size \( 2n - 3 \) with \( \delta(G_s) \geq 2 \).

First let \( N_G(v_3) \cap V_{G'}^2 = \emptyset \). Then \( d(v_1|G_s) < d(v_1|G') \), \( ABC(a_iv_1|G) = ABC(a_iv_1|G_s) \) for each \( 1 \leq i \leq k_1 \), and \( ABC(e|G) = ABC(e|G_s) = \sqrt{2} \) for any \( e \in E_{G'} \). This implies that \( ABC(G_s) \geq ABC(G') \). Because \( L(G_s) = L(G') - 1 \), we obtain a contradiction with the above choice of \( G' \).

Now let \( N_G(v_3) \cap V_{G'}^2 \neq \emptyset \) and let \( N_G(v_3) \cap V_{G'}^2 = \{c_1, c_2, \ldots, c_i\} \). Since \( d(c_i|G') = d(c_i|G_s) \) for \( i \in \{1, 2, \ldots, t\} \), we write \( d(c_i) = d(c_i|G') \). For \( i \in \{1, 2, \ldots, t\} \), let

\[
h_1^i = f(d(v_1|G') - k_1, d(c_i)) - f(d(v_1|G'), d(c_i)) = \sum_{j=d(v_1|G')-k_1}^{d(v_1|G')-k_1+1} (f(j, d(c_i)) - f(j+1, d(c_i)))
\]

and

\[
h_2^i = f(d(v_1|G'), d(c_i)) - f(d(v_1|G') + k_1, d(c_i)) = \sum_{j=d(v_1|G')-k_1}^{d(v_1|G')-k_1+1} (f(j, d(c_i)) - f(j+1, d(c_i)))
\]

Assume that \( l = ABC(G_s) - ABC(G') \). We have

\[
l \geq \sum_{i=1}^{i=1} \sum_{j=d(v_1|G')-k_1}^{d(v_1|G')-k_1+1} (f(j, d(c_i)) - f(j+1, d(c_i)))
\]
Thus \( ABC(G_3) \geq ABC(G') \). Clearly, \( L(G_3) \leq L(G') \) and we deduce from the choice of \( G' \) that \( L(G_3) = L(G') \). By definition, we have \( G_3 \neq H_{n-6}^{3,3} \). Thus \( G_3 \in H \) and we obtain a contradiction as in Case A.

**Subcase B.2.** \( q_2 = 0 \) and \( d(v_2|G') \geq q_1 + 1 \).

In this case, we have \( d(v_2|G') - 2 \leq q_1 - 1 \) and \( |N(v_4|G') \cap V_{G'}| \leq 1 \). Let \( G_3 \) be the graph obtained from \( G' \) by removing the edges \( v_2a_i(1 \leq i \leq d(v_2|G') - 2) \) and adding the edges \( v_4a_i(1 \leq i \leq d(v_2|G') - 2) \). Clearly, \( G_3 \) is a bipartite graph of order \( n \), size \( 2n - 3 \) with \( \delta(G_3) \geq 2 \) and \( G_3 \neq H_{n-6}^{3,3} \).

First let \( |N_G(v_3) \cap V_{G'}| = 1 \). We have \( ABC(a_1v_2|G') = ABC(a_1v_4|G_3) \) for each \( 1 \leq i \leq d(v_2|G') - 2 \) and \( ABC(e_{G_3}) = ABC(e_{G'}) = \sqrt{2} \) for any \( e \in E_{G'} \). Hence \( ABC(G_3) \geq ABC(G') \geq ABC(G) \) yielding \( G_3 \in G_n \), contradicting Claim 1, since \( q_1v_2 \in E_{G_{2,2}} \).

Now let \( |N_G(v_4) \cap V_{G'}| = 1 \) and let \( N_G(v_4) \cap V_{G'} = \{c\} \). Clearly, we have \( ABC(a_1v_2|G') = ABC(a_1v_4|G_3) \) for each \( 1 \leq i \leq d(v_2|G') - 2 \) and \( d(v_2|G_3) = 2 \). Since \( d(c|G') = d(c|Gs) \), we write \( d(c) = d(c|G_3) \). Assume that

\[
\begin{align*}
\h_1 &= f(2, d(c)) - f(d(v_2|G'), d(c)) = \\
&= \sum_{j=2}^{d(v_2|G')-1} (f(i,d(c)) - f(i+1,d(c))).
\end{align*}
\]

\[
\begin{align*}
\h_2 &= f(d(v_4|G'), d(c)) - f(d(v_4|G') + d(v_2|G') - 2, d(c)) = \\
&= \sum_{i=d(v_4|G') + d(v_2|G')-2}^{d(v_4|G') + d(v_2|G')-1} (f(i,d(c)) - f(i+1,d(c))).
\end{align*}
\]

and \( l = ABC(G_3) - ABC(G') \). Then we have

\[
\begin{align*}
l &\geq ABC(cv_2|G_3) + ABC(cv_4|G_3) - (ABC(cv_2|G') \\
&+ ABC(cv_4|G')) = f(d(v_2|G_3), d(c)) + f(d(v_4|G_3), d(c)) - f(d(v_2|G'), d(c)) \\
&+ f(d(v_4|G'), d(c)) = f(2, d(c)) - f(d(v_2|G'), d(c)) - f(d(v_4|G'), d(c)) \\
&- f(d(v_4|G') + d(v_2|G') - 2, d(c)) = h_1 - h_2 = \\
&= \sum_{i=2}^{d(v_2|G')-1} (h(i,d(c)) - h(i+1,d(c))).
\end{align*}
\]

We deduce from Lemma 1 that \( ABC(G_3) - ABC(G') > 0 \). This implies that \( ABC(G_3) > ABC(G') \), which is a contradiction.

**Subcase B.3.** \( q_2 = 0 \) and \( d(v_2|G') \geq q_1 + 2 \).

Let \( G_3 \) be the graph obtained from \( G' \) by removing the edges \( v_2a_i(1 \leq i \leq q_1) \) and adding the edges \( v_4a_i(1 \leq i \leq q_1) \). Clearly, \( G_3 \) is a bipartite graph of order \( n \), size \( 2n - 3 \) with \( \delta(G_3) \geq 2 \) and \( G_3 \neq H_{n-6}^{3,3} \).

First let \( N_G(v_4) \cap V_{G'} = \{c\} \). Then we have \( ABC(a_1v_2|G') = ABC(a_1v_4|G_3) \) for any \( 1 \leq i \leq q_1 \), \( d(v_2|G_3) < d(v_2|G') \), and \( ABC(e_{G_3}) = ABC(e_{G'}) = \frac{\sqrt{2}}{2} \) for any edge \( e \in E_{G_3} \). Hence \( ABC(G_3) \geq ABC(G') \) and \( L(G_3) \leq L(G') \). We deduce from the choice of \( G' \) that \( L(G_3) = L(G') \). Now, we consider \( G_3 \) instead of \( G' \) and proceed as Case A to obtain a contradiction.

Now let \( N_G(v_4) \cap V_{G'} \neq \{c\} \) and let \( N_G(v_4) \cap V_{G'} = \{c_1,c_2,\ldots,c_l\} \). Since \( d(c_1|G') = d(c_1|G_3) \) for each \( 1 \leq i \leq l \), we write \( d(c) = d(c_1|G_3) \). Let

\[
\begin{align*}
h_1^1 &= f(d(v_2|G') - k_1, d(c_1)) - f(d(v_2|G'), d(c)) = \\
&= \sum_{j=d(v_2|G')-k_1}^{d(v_2|G')} (f(j, d(c_1)) - f(j + 1, d(c_1))) \\
&= \sum_{j=d(v_2|G')-k_1}^{d(v_2|G')} h(j, d(c_1)),
\end{align*}
\]

and \( l = ABC(G_3) - ABC(G') \). Then we have

\[
\begin{align*}
l &\geq \sum_{i=1}^{l-1} (ABC(v_2c_1|G_3) + ABC(v_4c_1|G_3)) - ABC(cv_2|G') \\
&+ ABC(cv_4|G')) = \sum_{i=1}^{l-1} (ABC(v_2c_1|G_3) - ABC(v_4c_1|G')) \\
&- \sum_{i=1}^{l-1} (ABC(v_2c_1|G_3) - ABC(v_4c_1|G_3)) = \sum_{i=1}^{l-1} (f(d(v_2|G') - k_1, d(c_1)) - f(d(v_2|G'), d(c_1))) \\
&- \sum_{i=1}^{l-1} (f(d(v_2|G'), d(c_1)) - f(d(v_4|G') + k_1, d(c_1))) = \sum_{i=1}^{l-1} h_1^1 - \sum_{i=1}^{l-1} h_2^1 \\
&= \sum_{i=1}^{l-1} (h(i,d(c_1)) - h(i + d(v_4|G') - 2, d(c_1))).
\end{align*}
\]

It follows from Lemma 1 and the fact \( d(v_4|G') - d(v_2|G') + k_1 \geq 1 \) that \( ABC(G_3) - ABC(G') > 0 \). Thus \( ABC(G_3) > ABC(G') \), which is a contradiction.

**Subcase B.4.** \( 0 < q_2 < q_1 \).

Let \( G_3 \) be the graph obtained from \( G' \) by removing the edges \( v_2a_i, v_4b_i(1 \leq i \leq q_2) \) and adding the edges
Let $G' \neq H^{3,3}_{n-6}$ be the graph satisfying the conditions of Claim 3. Then $G' \in G_n$. By exchanging $G$ and $G'$, we may assume that the graph $G$ satisfies the conditions of Claim 3. Let

$$A_X = \{\{u, v\} \mid \text{there is a vertex } w \in Y \text{ such that } N(w) = \{u, v\}\},$$

$$A_Y = \{\{u, v\} \mid \text{there is a vertex } w \in X \text{ such that } N(w) = \{u, v\}\}.$$ 

By Claim 3, $|A_X| \leq 1$ and $|A_Y| \leq 1$. If $A_X \neq \emptyset$ (resp. $A_Y \neq \emptyset$), then let $A_X = \{(u_x, v_x)\}$ (resp. $A_Y = \{(u_y, v_y)\}$).

Claim 4. (a) If $A_X \neq \emptyset$, $|E_u \cap E_{3,3}^1| + |E_{v} \cap E_{3,3}^1| \leq 5$.
(b) If $A_Y \neq \emptyset$, $|E_u \cap E_{3,3}^1| + |E_{v} \cap E_{3,3}^1| \leq 5$.

Proof of Claim 4. We only prove (a). Let $A_X \neq \emptyset$. Suppose, to the contrary, that $|E_u \cap E_{3,3}^1| + |E_v \cap E_{3,3}^1| \geq 6$. Since $G$ is a bipartite graph of size $2n-3$, we conclude that $d(v) \leq n-3$ for any $v \in V(G)$. Assume, without loss of generality, that $|E_u \cap E_{3,3}^1| = 1$. Then there are two vertices $t, s \in Y$ such that $tu \in E_{3,3}^1$, and $s \in N(u) \cap N(v) \cap V_2$. Let $E_1 = E_u \cap E_{3,3}^1$ and $E_2 = E_v \cap E_{3,3}^1$. Suppose $G' = G - s$. Clearly, $G'$ is a bipartite graph of order $n-1$, size $2(n-1) - 3$ with $\delta(G') \geq 2$. By the choice of $n$, we have $ABC(G') \leq \sqrt{2}(n-7) + 2 + \frac{3(n-3)}{n-4} + 2$. Also, we have

$$ABC(tv_k(G') - ABC(tu_k)(G) = f(d(u_k)) - 1, d(t)) - f(d(u_k), d(t)) = h(d(u_k) - 1, d(t)) \geq h(n-4, d(t)) \geq h(n-4, 3).$$

Similarly, for any edge $e \in E_1 \cup E_2$, we have $ABC(e(G')) - ABC(e(G)) \geq h(n-4, 3)$. Hence

$$ABC(G') + \sqrt{2} - ABC(G) = \sum_{e \in E_1 \cup E_2} (ABC(e(G')) - ABC(e(G))) \geq h(n-4, 3) \geq 6h(n-4, 3) = ABC(H^{3,3}_{n-7}) + \sqrt{2} - ABC(H^{3,3}_{n-6}).$$

Since $ABC(G) \geq ABC(H^{3,3}_{n-6})$, we have

$$ABC(H^{3,3}_{n-7}) + \sqrt{2} - ABC(H^{3,3}_{n-6}) \geq ABC(G') + \sqrt{2} - ABC(G) \geq ABC(H^{3,3}_{n-7}) + \sqrt{2} - ABC(H^{3,3}_{n-6})$$

and so the inequalities occurring above become equality, that is, $ABC(G) = ABC(H^{3,3}_{n-6})$ and $ABC(G') = ABC(H^{3,3}_{n-7})$. It follows that $G' \cong H^{3,3}_{n-7}$ and $ABC(e(G')) - ABC(e(G)) = h(n-4, 3)$ for any $e \in E_1 \cup E_2$. Hence $d(u_k) = d(v_k) = n-3$ and so $G \cong H^{3,3}_{n-6}$, a contradiction.

Claim 5. $k \leq 9$.

Proof of Claim 5. Suppose, to the contrary, that $k \geq 10$. Let $q = |A_X| + |A_Y|$. By Claim 4, we have $q \in \{0, 1, 2\}$. Let $B = \{u_x, v_x, u_y, v_y\}$ (possibly $B = \emptyset$). Then $|B| = 2q$. We consider the following cases.

Case C.I. There exists a vertex $v \not\in B$ such that $d(v) \geq 6$. Then

$$ABC(G) = \sum_{e \in E_1 \cup E_2} ABC(e) + \sum_{e \in E_i} ABC(e) \leq (n-k)\sqrt{2} + d(v) \sqrt{\frac{d(v) + 1}{3d(v)}} + (2k-3 - d(v)) \frac{2}{3}$$

$$= (n-k)\sqrt{2} + (2k-3) \frac{2}{3} - \sqrt{\frac{d(v) + 1}{3d(v)}}$$

$$\leq (n-k)\sqrt{2} + (2k-3) \frac{2}{3} - \sqrt{\frac{7}{18}}$$

$$\leq (n-10)\sqrt{2} + (20-3) \frac{2}{3} - \sqrt{\frac{7}{18}}$$

$$\leq (n-6)\sqrt{2} + 2 + 2\sqrt{3}$$

$$< (n-6)\sqrt{2} + 2 + 2\sqrt{\frac{3(n-2)}{n-3}}$$

$$= ABC(H^{3,3}_{n-6}),$$

a contradiction.

Case C.II. There exist two vertices $v_1, v_2 \in B$ such that $d(v_1) = 5$ and $d(v_2) \geq 4$.

Let $t_1 = |E_{v_1} \cap E_{v_2}|$. Then $t_1 \in \{0, 1\}$ and

$$h_1 = t_1 \left( \sqrt{\frac{3 + d(v_2)}{5d(v_2)}} - \sqrt{\frac{6}{15}} \right) \frac{d(v_2) + 1}{3d(v_2)} \frac{2}{3}$$

$$= t_1 (f(3, 3) - f(d(v_2), 3) - (f(3, 5) - f(d(v_2), 5)))$$

$$= t_1 \left( \sum_{j=1}^{|E_{v_1} \cap E_{v_2}|} h(j, 3) - \sum_{j=1}^{|E_{v_1} \cap E_{v_2}|} h(j, 5) \right) \leq 0.$$
It follows that 

\[ ABC(G) = \sum_{e \in E_{2,1}} ABC(e) + \sum_{e \in E_1} ABC(e) \]

\[ + \sum_{e \in E_{1,1} \setminus E_1} ABC(e) + \sum_{e \in \{E_{1,1} \setminus (E_1 \cup E_2)\}} ABC(e) \]

\[ \leq (n-k)\sqrt{2} + t_1 \left[ \frac{3 + d(v_2)}{5d(v_2)} + (5 - t_1) \right] \sqrt{\frac{6}{15}} \]

\[ + (d(v_2) - t_1) \sqrt{\frac{3d(v_2)}{2d(v_2)}} + (2k - 8 - d(v_2) + t_1)^2 \]

\[ = h_1 + (n-k)\sqrt{2} + 5 \sqrt{\frac{6}{15}} + (d(v_2) + 1) \sqrt{\frac{3d(v_2)}{2d(v_2)}} \]

\[ + (2k - 8) \frac{2}{3} \]

\[ \leq (n-10)\sqrt{2} + 5 \sqrt{\frac{6}{15}} - 4(2 \frac{2}{3} \sqrt{\frac{4 + 1}{12}}) \]

\[ + (2k - 8) \frac{2}{3} \leq (n-6)\sqrt{2} + 2 + 2\sqrt{3} \]

\[ < (n-6)\sqrt{2} + 2 + 2 \sqrt{\frac{3(n-2)}{n-3}} \]

\[ = ABC(H_{n-6}^{3,3}) \]

which is a contradiction.

**Case C.III.** There exist two vertices \( u, v \notin B \) such that \( d(v) = 5 \) and \( d(x) = 3 \) for any \( x \in V_3 \setminus \{v\} \).

By Claim 4, we have 

\[ \sum_{v \in V_3} d(v|G[V_3]) = \sum_{v \in V_3 \setminus B} d(v|G[V_3]) + \sum_{v \in B} d(v|G[V_3]) \]

\[ \leq 5 + 3(k - 2q - 1) + 5q \]

\[ = 2 + 3k - q \]

\[ \leq 2 + 3k. \]

By Claim 1, we have \( \sum_{v \in V_3} d(v|G[V_3]) = \sum_{v \in V_3} d(v) = 2(2k - 3) \leq 2 + 3k \) yielding \( k \leq 8 \), which is a contradiction.

**Case C.IV.** There are three vertices \( v_1, v_2, v_3 \notin B \) such that \( d(v_1) = d(v_2) = d(v_3) = 4 \).

Let \( W \) be the set of edges with two ends in \( \{v_1, v_2, v_3\} \), \( t_1 = |W| \) and \( W_1 = (E_1 \cup E_2 \cup E_v) \setminus W \). Clearly, \( t_1 \in \{0, 1, 2\} \) and we have 

\[ ABC(G) = \sum_{e \in E_{2,1}} ABC(e) + \sum_{e \in W} ABC(e) + \sum_{e \in W_1} ABC(e) \]

\[ + \sum_{e \in E_{1,1} \setminus W_1} ABC(e) \]

\[ \leq (n-k)\sqrt{2} + t_1 \sqrt{\frac{6}{16}} + (12 - 2t_1) \sqrt{\frac{5}{12}} \]

\[ + (2k - 3 - 12 + t_1)^2 \frac{2}{3} \]

\[ = t_1 \left[ \frac{6}{16} - 2 \sqrt{\frac{5}{12}} + \frac{2}{3} \right] + (n-k)\sqrt{2} \]

\[ + 12 \sqrt{\frac{5}{12} + (2k - 15) \frac{2}{3}} \leq (n-k)\sqrt{2} + 12 \sqrt{\frac{5}{12} + (20 - 15) \frac{2}{3}} \]

\[ \leq (n-6)\sqrt{2} + 2 + 2\sqrt{3} \]

\[ < (n-6)\sqrt{2} + 2 + 2 \sqrt{\frac{3(n-2)}{n-3}} \]

\[ = ABC(H_{n-6}^{3,3}), \]

a contradiction.

**Case C.V.** For any vertex \( v \notin B \), \( d(v) \leq 4 \) and \( |\{v|d(v) = 4\} \cap (V_3 \setminus B) \leq 2 \).

By Claim 4, we have 

\[ \sum_{v \in V_3} d(v|G[V_3]) = \sum_{v \in V_3 \setminus B} d(v|G[V_3]) + \sum_{v \in B} d(v|G[V_3]) \]

\[ \leq 8 + 3(k - 2q - 2) + 5q \]

\[ = 2 + 3k - q \]

\[ \leq 2 + 3k. \]

Claim 1 implies that \( \sum_{v \in V_3} d(v|G[V_3]) = \sum_{v \in V_3} d(v) = 2(2k - 3) \leq 2 + 3k \) yielding \( k \leq 8 \), a contradiction. This completes the proof of Claim 5.

Now, for any graph \( G'' \), we define \( d'(u) = \left\{ \begin{array}{ll}
\frac{d(u)}{4} & u \geq 3 \\
2 & u \leq 2
\end{array} \right. \)

\[ ABC'(uv) = \sqrt{\frac{d'(u)d'(v) - 2}{d'(uv)}} \]

and \( ABC'(G'') = \sum_{uv \in E(G'')} ABC'(uv) \).

By Claims 1 and 2, \( |E_{1,2,2}| = |E_{1,2,3}| = 0 \). We have \( d(v) \geq 4 \) with \( d(v|G[V_3]) \leq 2 \) for any \( v \in V_3 \). Thus \( ABC(G) \leq (n - k)\sqrt{2} + ABC'(G[V_3]) \).

**Claim 6.** \( k = 6 \).

**Proof of Claim 6.** Suppose, to the contrary, that \( k \geq 7 \).

First let \( k = 9 \). Then \( ABC(G) \leq (n - 9)\sqrt{2} + ABC'(G[V_3]) \).

Since \( ABC(G) \geq ABC(H_{n-9}^{3,3}) = (n - 6)\sqrt{2} + 2 + 2\sqrt{3} + 2 \), we obtain \( ABC'(G[V_3]) > 3\sqrt{2} + 2\sqrt{3} + 2 \). Hence, \( |V(G[V_3])| = 9 \), \( |E(G[V_3])| = 15 \) and \( ABC'(G[V_3]) > 3\sqrt{2} + 2\sqrt{3} + 2 \). A computer search shows that there are exactly two bipartite
graphs $G''$ satisfying the conditions $|V(G'')| = 9, |E(G'')| = 15$ and $ABC(G'') > 3\sqrt{2} + 2\sqrt{3} + 2$ (these two graphs $G_1^3$ and $G_2^3$ illustrated in Figure 2). It is easy to verify that $d(u) + d(v) \geq 6$ for every two vertices $u, v \in V(G_2^3)$ or $u, v \in V(G_3^3)$, a contradiction with Claim 4.

Now let $k = 8$. Then $ABC(G) \leq (n - 8)\sqrt{2} + ABC(G[V_1])$. Since $ABC(G) \geq ABC(H_{n-6}^{3,3}) > (n - 6)\sqrt{2} + 2\sqrt{3} + 2$, we have $ABC(G[V_1]) > 2\sqrt{2} + 2\sqrt{3} + 2$. Thus $G[V_1]$ satisfies the following $|V(G[V_1])| = 8, |E(G[V_1])| = 13$ and $ABC(G[V_1]) > 2\sqrt{2} + 2\sqrt{3} + 2$. A computer search shows that there are exactly two bipartite graphs $G''$ satisfying the conditions $|V(G'')| = 8, |E(G'')| = 13$ and $ABC(G'') > 2\sqrt{2} + 2\sqrt{3} + 2$ (these two graphs $G_2^3$ and $G_3^3$ illustrated in Figure 2). If $G[V_1] = G_2$, then it is easy to verify $d(u) + d(v) \geq 6$ for every two vertices $u, v \in V(G_2^3)$, a contradiction with Claim 4. If $G[V_1] = G_3$, then let $u$ be the vertex of degree 2, and assume without loss of generality that $u \in X$. Suppose $v \in X$ be a vertex of degree 3. By Claim 4, we conclude that $G$ is a graph obtained from $G_2^3$ by adding vertices $a_1, a_2, \ldots, a_{n-8}$ and the edges $u_a, u_b$, for each $i \in \{1, 2, \ldots, n - 8\}$ (see graph $F_8$ in Figure 3). It is not hard to verify that $ABC(G) < ABC(H_{n-6}^{3,3})$, a contradiction.

Finally let $k = 7$. A computer search shows that there are exactly two bipartite graphs $G''$ satisfying the conditions $|V(G'')| = 7, |E(G'')| = 11$. We deduce from Claim 4 that $G$ is a graph obtained from $G''$ by adding new vertices $v_1, v_2, \ldots, v_{n-7}$ and joining them to two vertices $u, v \in X_{G''}$ (see the graph $F_7$ illustrated in Figure 3). It is easy to verify that $ABC(F_7) < ABC(H_{n-6}^{3,3})$, a contradiction. This completes the proof of Claim 6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Graphs used in the proof of Claim 6.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Graphs constructed in the proof of Claim 6.}
\end{figure}

**Claim 7.** For any graph $G_s$ with $|V(G_s)| = n$, $|E(G_s)| = 2n - 3$, $\delta(G_s) \geq 2$, and $ABC(G_s) \geq ABC(G)$, we have $G_s[V_1] \cong K_{3,3}$ and $\bigcup_{u \in V(G_s) \cap V_1} N_{G_s}(u) = \{0, 2\}$.

**Proof of Claim 7.** Note that $E_{2,2}^{G_s} = E_{2,2}^{G_1} = \emptyset$. Then by Claim 6, we have $G_s[V_1] \cong K_{3,3}$. By above Claim, we only need to show $\bigcup_{u \in V(G_s) \cap V_1} N_{G_s}(u) = \{0, 2\}$. Since $E_{2,2}^{G_s} = E_{2,2}^{G_1} = \emptyset$, we have $N_{G_s}(u) \cap V_{G_s}^0 \neq \emptyset$ for any $u \in V(G_s)$. Since $G_s[V_1] \cong K_{3,3}$, we have $N_{G_s}(u) \cap V_{G_s}^0 = N_{G_s}(v) \cap V_{G_s}^0 \neq \emptyset$ for any $u, v \in X \cap V_{G_s}^0$. Then, by a close look at the proof of Case A in Claim 3, we have $\bigcup_{u \in V(G_s) \cap X} N_{G_s}(u) = \{0, 2\}$.

Let $t = |\bigcup_{u \in V(G_s) \cap X} N_{G_s}(u)| + |\bigcup_{u \in V(G_s) \cap Y} N_{G_s}(u)|$. We conclude from Claim 7 and the fact $n \geq 10$ that $t \in \{2, 4\}$. Assume that $|\{u_x, v_x\} = \bigcup_{u \in V(G_s) \cap X} N_{G_s}(u), \{u_y, v_y\} = \bigcup_{u \in V(G_s) \cap Y} N_{G_s}(u), U_1 = \{v(N(v) = \{u_x, v_x\}) \cup \{u_y, v_y\}, U_2 = \{v(N(v) = \{u_x, v_x\}) \cup \{u_y, v_y\}) \cup \{u_x, v_x\}, \{u_x\} = X - (\{u_x, v_x\} \cup V_2), \{u_y\} = Y - (\{u_x, v_x\} \cup V_2)\}$ and $d(u_x) \leq d(u_y)$. Let $t \in U_2$, and let $G'$ be the bipartite graph obtained from $G - \{t u_x, t v_y\}$ by adding the edges $t u_x$ and $t v_x$. Let $h_1 = 4ABC(u_x u_y G') - 4ABC(u_x u_y G)$ and $h_2 = 2ABC(u_x u_y G') - 2ABC(u_x u_y G) + 2ABC(u_x u_y G') - 2ABC(u_x u_y G')$

Then $h_1 = 4\left(\frac{d(u_x) + d(u_y) + 2}{d(u_x) + d(u_y) - 1}\right)$ and $h_2 = 2ABC(u_x u_y G') - 2ABC(u_x u_y G)$

$$= 2f(d(u_x) - 1, 3) - f(d(u_y), 3) + f(d(u_x), 3)$$

$$- f(d(u_x) + 1, 3))$$

$$= 2(h(d(u_x) - 1, 3) - h(d(u_x), 3)).$$
Since \( d(u_2) \leq d(u_4) \), we have \( h_1 > 0 \) and \( h_2 > 0 \). Thus

\[
ABC(G') - ABC(G) = 4ABC(u_2u_3|G') - 4ABC(u_2u_3|G) + 2ABC(u_1u_3|G') - 2ABC(u_1u_3|G) + 2ABC(u_2u_3|G') - 2ABC(u_2u_3|G) = h_1 + h_2 > 0,
\]
a contradiction.

This completes the proof. \( \square \)

By applying the approach used above, we also obtain the following result:

**Theorem 6.** Let \(-3 \leq l \leq -1\) and let \( G \) be a bipartite graph of order \( n \geq l + 9 \), size \( 2n + l \) and \( \delta(G) \geq 2 \). Then \[ABC(G) \leq \sqrt{2(n - 9 - l)} + \sqrt{\frac{(l+6)(l+7)}{3}} + 2(l + 6)\sqrt{\frac{n-3}{3(n-3)}},\]
if and only if \( G \cong H^{3,l+6}_{n-9} \).

## 3 Discussions

We studied the atom–bond connectivity (ABC) index on bipartite graphs, in which the ABC index has an important application in rationalizing the stability of linear and branched alkanes as well as the strain energy of cycloalkanes. All extremal graphs attained at these extremal values are characterized. Our method is to use monotonic functions and combine them with graph operations. Some special graphs and their values are obtained by computational searches. Our results extend the previous outcomes and deduce all bounds.

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**References**

[1] Randić M. Characterization of molecular branching. Journal of the American Chemical Society. 1975;97(23):6609–15.

[2] Estrada E, Torres L, Rodríguez L, Gutman I. An atom–bond connectivity index: modelling the enthalpy of formation of alkanes. 1998.

[3] Estrada E. Atom–bond connectivity and the energetic of branched alkanes. Chemical Physics Letters. 2008;463(4-6):422–5.

[4] Das KC, Trinajstić N. Comparison between first geometric–arithmetic index and atom–bond connectivity index. Chemical physics letters. 2010;497(1-3):149–51.

[5] Gutman I, Tošović J, Randeković S, Marković S. On atom-bond connectivity index and its chemical applicability. 2012.

[6] Chen J, Guo X. Extreme atom-bond connectivity index of graphs. MATCH. Commun. Math Comput Chem. 2011;65(3):713–22.

[7] Furtula B, Graovac A, Vukičević D. Atom–bond connectivity index of trees. Discrete Applied Mathematics. 2009;157(13):2828–35.

[8] Furtula B, Gutman I, Dehmer M. On structure-sensitivity of degree-based topological indices. Applied Mathematics and Computation. 2013;219(17):8973–8.

[9] Gan L, Hou H, Liu B. Some results on atom-bond connectivity index of graphs. MATCH. Commun. Math Comput Chem. 2011;66(2):669–80.

[10] Goubko M, Magnant C, Nowbandegani PS, Gutman I. ABC index of trees with fixed number of leaves. MATCH. Commun. Math Comput Chem. 2015;74(1):697–702.

[11] Gutman I, Furtula B, Ahmad MB, Hosseini SA, Nowbandegani PS, Zarrinderakht M. The abc index conundrum. Filomat. 2013;27(6):1075–83.

[12] Gutman I. Degree-based topological indices. Croatica Chemica Acta. 2013;86(4):351–61.

[13] Xing R, Zhou B. Extremal trees with fixed degree sequence for atom-bond connectivity index. Filomat. 2012;26(4):638–88.

[14] Xing R, Zhou B, Dong F. On atom–bond connectivity index of connected graphs. Discrete Applied Mathematics. 2011;159(15):1617–30.

[15] Shao Z, Wu P, Gao Y, Gutman I, Zhang X. On the maximum ABC index of graphs without pendant vertices. Applied Mathematics and Computation. 2017;315:298–312.

[16] Dimitrov D. Efficient computation of trees with minimal atom-bond connectivity index. Applied Mathematics and Computation. 2013;224:663–70.

[17] Dimitrov D. On structural properties of trees with minimal atom-bond connectivity index. Discrete Applied Mathematics. 2016;172:28–44.

[18] Dimitrov D. On structural properties of trees with minimal atom-bond connectivity index II: Bounds on b1- and b2-branches. Discrete Applied Mathematics. 2016;204:90–116.

[19] Dimitrov D, Du Z, da Fonseca CM. On structural properties of trees with minimal atom-bond connectivity index III: Trees with pendant paths of length three. Applied Mathematics and Computation. 2016;282:276–90.

[20] Gao Y, Shao Y. The smallest abc index of trees with n pendant vertices. MATCH Commun Math Comput Chem. 2016;76(1):141–58.

[21] Lin W, Chen J, Ma C, Zhang Y, Chen J, Zhang D et al. On trees with minimal ABC index among trees with given number of leaves. MATCH Commun Math Comput Chem. 2016;76(1):131–40.

[22] Lin W, Li P, Chen J, Ma C, Zhang Y, Zhang D. On the minimal ABC index of trees with k leaves. Discrete Applied Mathematics. 2017;217:622–7.

[23] Ashrafi A, Dehghan-Zadeh T, Habibi N, John P. Maximum values of atom–bond connectivity index in the class of tricyclic graphs. Journal of Applied Mathematics and Computing. 2016;50(1-2).
2):511–27.

[24] Bianchi M, Cornaro A, Palacios JL, Torrieri A. New upper bounds for the ABC index. MATCH. Commun Math Comput Chem. 2016;76(1):117–30.

[25] Dehghan-Zadeh T, Ashrafi A, Habibi N. Maximum values of atom–bond connectivity index in the class of tetracyclic graphs. Journal of Applied Mathematics and Computing. 2014;46(1-2):285–303.

[26] Zhang XM, Yang Y, Wang H, Zhang XD. Maximum atom-bond connectivity index with given graph parameters. Discrete Applied Mathematics. 2016;215:208–17.

[27] Shao Z, Wu P, Zhang X, Dimitrov D, Liu JB. On the maximum ABC index of graphs with prescribed size and without pendant vertices. IEEE Access. 2018;6:27604-16.

[28] Xing R, Zhou B, Du Z. Further results on atom-bond connectivity index of trees. Discrete Applied Mathematics. 2010;158(14):1536–45.

[29] McKay B, Piperno A. Nauty User’s Guide (version 2.7). Computer Science Department, Australian National University. 2019. http://users.cecs.anu.edu.au/bdm/nauty/.