Eisenstein series and Scattering matrices

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1 The extension problem

Let $G$ be a real semisimple Lie group, $G = KAN$ be an Iwasawa decomposition of $G$, $M := Z_K(A)$ be the centralizer of $A$ in $K$ and $P = MAN$ be a minimal parabolic subgroup. Then we have the symmetric space $X = G/K$ and its boundary $\partial X = G/P = K/M$. Let $g = \kappa(g)a(g)n(g)$, $\kappa(g) \in K$, $a(g) \in A$, $n(g) \in N$ be defined with respect to the given Iwasawa decomposition. Let $g = k \oplus a \oplus n$ be the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$. If $\lambda \in \mathfrak{a}_C^*$, then we set $a^\lambda := e^{\langle \lambda, \log(a) \rangle} \in \mathbb{C}$. Corresponding to $n$ there is a positive Weyl chamber $\mathfrak{a}^+$. Let $W = W(\mathfrak{g}, \mathfrak{a})$ be the Weyl group generated by the reflections at walls of $\mathfrak{a}^+$.

We consider a discrete subgroup $\Gamma \subset G$.

Assumption 1.1 We assume that there is a $\Gamma$-invariant partition $\partial X = \Omega \cup \Lambda$ such that $\Gamma$ acts freely and co-compactly on the open non-empty set $\Omega$.

We call $\Lambda$ the limit set of $\Gamma$ though this is a slight abuse of the notion. For $\lambda, \mu \in \mathfrak{a}^*$ we say that $\lambda > \mu$, iff $\lambda - \mu \in \mathfrak{a}^+_+$. Define $\rho \in \mathfrak{a}_+^*$ as usual by $\rho(H) := \frac{1}{2} \text{tr}(\text{ad}(H)|_n)$, $H \in \mathfrak{a}$.

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Definition 1.2 An exponent $\delta_\Gamma \in a^*$ of $\Gamma$ is a minimal element such that
\[ \sum_{g \in \Gamma} a(gkM)^{-\delta_\Gamma - \lambda - \rho} < \infty \]
converges uniformly for the parameters $kM$ and $\lambda$ varying in compact subsets of $\Omega$ and $a^*_\mathbb{C}$, respectively.

$\Gamma \subset G$ is discrete and has a fundamental domain. Since $G$ has at most exponential volume growth we have clearly $\delta_\Gamma < \infty$. Note that we admit $\delta_\Gamma = -\infty$ and $\delta_\Gamma$ need not to be unique.

Let $\sigma$ be a finite dimensional unitary representation of $M$ on $V_\sigma$ and $\lambda \in a^*_\mathbb{C}$. Then we can form the representation $\sigma_\lambda$ of $P$ on $V_{\sigma_\lambda} := V_\sigma$ by
\[ \sigma_\lambda(m) = \sigma(m)a^{\rho - \lambda} . \]

Let $V(\sigma_\lambda) = G \times_P V_{\sigma_\lambda}$ be the associated homogeneous bundle. By $B$ we denote the compact quotient $\Gamma \backslash \Omega$ and let $V_B(\sigma_\lambda) := \Gamma \backslash V(\sigma_\lambda)$.

Let $\theta$ be the Cartan involution associated to $K$ and set $\bar{n} = \theta n$. Then we have the decomposition $g = \bar{n} \oplus p$, and we can identify the tangent bundle of $\partial X$ with $G \times_P \bar{n}$. Note that $(\Lambda_{\text{max}}\bar{n})^{-1/2} = V_{1_0}$, where $1$ denotes the trivial representation of $M$. In particular the bundle of half-densities of $\partial X$ is $V(1_0)$. Hence we can define natural pairings of sections of $V(\sigma_\lambda)$ with sections of $V(\tilde{\sigma}_{-\lambda})$, where $\tilde{\sigma}$ is the dual representation to $\sigma$.

Besides the different globalizations of the principal series representation of $G$
\[ H^*_\sigma = C^*(\partial X, V(\sigma_\lambda)), \quad * = \omega, \infty, -\infty, -\omega \]
we consider the following $\Gamma$-modules:

- $H^*_{\omega, \Lambda} := \{ f \in C^{-\omega}(\partial X, V(\sigma_\lambda)), \text{supp}(f) \subset \Lambda \}$
- $H^*_{\infty}(\Omega) := C^{-\infty}(\Omega, V(\sigma_\lambda))$
- $H^*_{\omega, [\Lambda]} := \{ f \in C^{-\omega}(\partial X, V(\sigma_\lambda)), \text{res}(f) \in H^*_{\infty}(\Omega) \}$

where $\text{res}$ is the restriction to $\Omega$. $-\infty$ stands for distributions and $-\omega$ for hyperfunctions. These spaces come with natural topologies. In fact the first is a Frechet space with the topology induced from its embedding in the Frechet space of all hyperfunctions. The second is the topological dual of the space of smooth functions on $\Omega$ with compact support. Since the hyperfunctions form a flabby sheaf we have the exact sequence
\[ 0 \to H^*_{\omega, \Lambda} \to H^*_{\omega, [\Lambda]} \to H^*_{\infty}(\Omega) \to 0 \]
inducing a natural topology on the middle space.

Problem 1.3 (Extension problem) Let $\phi \in \Gamma H^*_{\infty}(\Omega)$. Define an invariant extension $\text{ext}(\phi) \in \Gamma H^*_{\omega, [\Lambda]}$ as natural as possible such that $\phi = \text{res} \circ \text{ext}(\phi)$. 

1 \ THE EXTENSION PROBLEM
In fact we will define a natural right inverse of \( \text{res} \) for \( \text{Re}(\lambda) \gg 0, \lambda \in a^{\times}_c \). The idea is then to continue this extension meromorphically.

Let us assume that \( \lambda > \delta_\Gamma \). Then we can define a pairing of \( \Gamma H^{\sigma,\lambda}_\infty(\Omega) \) with \( H^{\sigma,\lambda}_\infty \) as follows. Let \( f \in H^{\sigma,-\lambda}_\infty \). Consider \( f \) as a function on \( K \) with values in \( V_\sigma \) being right \( M \)-invariant. Then \( (\pi^{\sigma,-\lambda}(g)f)(k) = a^{-\lambda-\rho}(g^{-1}k)f(\kappa(g^{-1}k)) \). By assumption the average \( \bar{f} := \sum_{g \in \Gamma} \pi^{\sigma,-\lambda}(g)\text{res}(f) \)

converges in \( C^\infty(\Omega, V(\bar{\sigma}_{-\lambda})) \). Of course \( \bar{f} \in C^\infty(B, V_B(\bar{\sigma}_{-\lambda})) \) and the map \( f \mapsto \bar{f} \) is continuous. Let \( \phi \in C^{-\infty}(B, V_B(\sigma_\lambda)) \). Then \( H^{\sigma,\lambda}_\infty \ni f \mapsto \langle \phi, \bar{f} \rangle \) is continuous. Thus we can define a distribution \( \text{ext}(\phi) \in H^{\sigma,\lambda}_\infty \) by

\[
\langle \text{ext}(\phi), f \rangle := \langle \phi, \bar{f} \rangle .
\] (1)

It is easy to see that \( C^{-\infty}(B, V_B(\sigma_\lambda)) \ni \phi \mapsto \text{ext}(\phi) \in H^{\sigma,\lambda}_\infty \) is continuous.

We now discuss the parametrized version. In order to speak of holomorphic families \( \phi_\lambda \in C^{-\infty}(B, V_B(\sigma_\lambda)) \) we must identify the bundles \( V_B(\sigma_\lambda) \) for different \( \lambda \). This can be done as follows. Fix a basis \( \mu_1, \ldots, \mu_r \) of \( a^*_c \) such that \( \mu_i > \delta_\Gamma \). Let \( e_i \in H^{\mu_i}_\infty \) be the element given by the constant function 1 on \( K \). Then \( e_i \) exists in \( C^\infty(B, V_B(1_{\mu_i})) \). Writing \( \lambda = \sum_{i=1}^r \lambda_i \mu_i \) we have an isomorphism \( V_B(\sigma_\lambda) = V_B(\sigma_0) \otimes \prod_{i=1}^r V_B(1_{\mu_i})^{\lambda_i} \). Hence we can write \( \phi_\lambda = \phi(\lambda) \prod e_i^{\lambda_i} \) where \( \phi(\lambda) \in C^{-\infty}(B, V_B(\sigma_0)) \) is now a family of distributions in a constant bundle. We define the family \( \phi_\lambda \) to be holomorphic, if the family \( \phi(\lambda) \) is a holomorphic family of distributions. It is easy to check that this notion of holomorphicity does not depend on the choices. In a similar way we define holomorphic families \( \psi_\lambda \in H^{\sigma,\lambda}_\infty \). We identify the bundles \( V(\sigma_\lambda) \) for different \( \lambda \) using the sections \( e_i \). When we speak of holomorphic families of maps between section spaces of bundles parametrized by \( \lambda \) we will always assume identifications of the bundles as above.

It is not complicated to see that if \( \phi_\lambda \) is holomorphic, then for \( \lambda > \delta_\Gamma \) the extension \( \text{ext}(\phi_\lambda) \) is holomorphic, too. Thus we have shown the following

**Lemma 1.4** For \( \lambda > \delta_\Gamma \) the extension problem has a canonical continuous solution. Moreover, the extension of a holomorphic family is again a holomorphic family.

## 2 The case of negative \( \delta_\Gamma \)

In this section we assume that \( \delta_\Gamma \) is negative, i.e., \( \delta_\Gamma < 0 \). We find a meromorphic continuation of the extension defined in Section \( \Gamma \). The main tool are the scattering matrices which are closely related to the Knapp-Stein intertwining operators.

Any \( w \in W \) can be represented by an element \( m_w \in N(M) \), the normalizer of \( M \). If \( \sigma \) is a representation of \( M \) we define \( \sigma^w \) by \( \sigma^w(m) := \sigma(m_w^{-1}m m_w) \). Depending on the choices of \( m_w \) there are \( G \)-invariant operators

\[
\hat{J}_{w,\sigma,\lambda} : H^{\sigma,\lambda}_* \to H^{\sigma^w,\lambda^w}_*
\]
which form meromorphic families of pseudodifferential operators \([5]\). Let \(\tilde{N} := \exp(\mathfrak{n})\). Consider \(f \in H^{\sigma,\lambda}_\infty\) as a right \(P\)-invariant function on \(G\) with values in \(V_{\sigma,\lambda}\). For \(\lambda < 0\) the intertwining operator is defined by the convergent integral

\[
(J_{w,\sigma,\lambda} f)(g) := \int_{\tilde{N} \cap w^{-1} Nw} f(g m_w \tilde{n}) d\tilde{n}.
\]

For other parameters it is obtained by meromorphic continuation. It is known that \(\hat{J}_{w,\sigma,\lambda}\) is compatible with the choices \(* \in \{\omega, \infty, -\infty, -\omega\}\). Let \(c_{w,\sigma}^{\lambda}(\lambda)\) be the value of \(\hat{J}_{w^{-1},\sigma,\lambda} c_{w,\sigma}^{\lambda}\) on the minimal \(K\)-type (see [4], Ch. XV for all that) of the principal series representation \(H^{\sigma,\lambda}_\ast\). Then \(c_{w,\sigma}^{\lambda}(\lambda)\) is a meromorphic function on \(a^*_C\) and we define the normalized intertwining operators by

\[
J_{w,\sigma,\lambda} := c_{w^{-1},\sigma,\lambda}^{\lambda} \hat{J}_{w,\sigma,\lambda}.
\]

They satisfy the functional equations

\[
J_{w_1 w_2,\sigma,\lambda} = J_{w_1,\sigma,\lambda} w_2 \circ J_{w_2,\sigma,\lambda}, \quad w_1, w_2 \in W.
\]

A special case is

\[
J_{w^{-1},\sigma,\lambda} w \circ J_{w,\sigma,\lambda} = \text{id}, \quad w \in W.
\]

In order to simplify the notation we assume that \(\sigma\) is Weyl invariant, i.e. \(\sigma^w = \sigma\). This can always be achieved by taking the direct sum of all Weyl translates of \(\sigma\).

The scattering matrices are operators

\[
S_{w,\sigma,\lambda} : C^{-\infty}(B, V_B(\sigma_\lambda)) \to C^{-\infty}(B, V_B(\sigma^{\lambda w})).
\]

**Definition 2.1** For \(\lambda > \delta_\Gamma\) we define

\[
S_{w,\sigma,\lambda}(\phi) = \text{res} \circ J_{w,\sigma,\lambda} \circ \text{ext}(\phi).
\]

Then the \(S_{w,\sigma,\lambda}, \ w \in W\), are meromorphic families of pseudodifferential operators. Locally they coincide with \(J_{w,\sigma,\lambda}\) up to smoothing operators. Since \(\delta_\Gamma < 0\) the scattering matrices are defined on a neighbourhood \(C\) of \(\{\lambda \in a^*_C | \text{Re}(\lambda) \geq 0\}\). They satisfy the functional equations

\[
S_{w_1 w_2,\sigma,\lambda} = S_{w_1,\sigma,\lambda} w_2 \circ S_{w_2,\sigma,\lambda}, \quad w_1, w_2 \in W,
\]

when all terms are defined.

We now provide a simultaneous meromorphic continuation of \(\text{ext}\) and the scattering matrices to all of \(a^*_C\). For \(\lambda \in C\) we define

\[
S_{w^{-1},\sigma,\lambda} w := S_{w,\sigma,\lambda}^{-1}.
\]

We claim that this formula defines a meromorphic family of operators.

To see the claim we invoke the Fredholm theory for Frechet spaces \([3]\). Since \(\Gamma\) acts properly on \(\Omega\) there is a \(\Gamma\)-invariant function \(\chi \in C^\infty(\Omega \times \Omega)\) being identically one on a
small neighbourhood of the diagonal and zero outside of a somewhat larger neighbourhood. We assume the latter neighbourhood to be so small that \(\chi(x,gx) = 0\) for all \(x \in \Omega, 1 \neq g \in \Gamma\). Let \(\tilde{J}_{w,\sigma,\lambda}\) be the operator obtained by cutting off the distributional kernel of \(J_{w,\sigma,\lambda}\) with \(\chi\). Then we can consider \(\tilde{J}_{w,\sigma,\lambda}\) as a meromorphic family of operators on \(B\). Since the singularities of the (unnormalized) intertwining operators are local operators we conclude from (4) that \(\tilde{J}_{w^{-1},\sigma,\lambda^{-1}} \circ S_{w,\sigma,\lambda} = \text{id} + R_1(\lambda), S_{w,\sigma,\lambda} \circ \tilde{J}_{w^{-1},\sigma,\lambda^{-1}} = \text{id} + R_2(\lambda)\), where \(R_1, R_2\) are holomorphic families of smoothing operators. Thus

\[
S_{w,\sigma,\lambda}^{-1} = (\text{id} + R_1(\lambda))^{-1} \circ \tilde{J}_{w^{-1},\sigma,\lambda^{-1}} \quad (5)
\]

is meromorphic if \((\text{id} + R_1(\lambda))^{-1}\) is so. The meromorphicity of the latter family follows from the Fredholm theory for Frechet spaces \([3]\) if \((\text{id} + R_1(\lambda))\) is injective for some \(\lambda \in C\). But there is an open subset in \(C \cap C^{\omega^{-1}}\) where (4) holds.

We call \(\lambda \in a^*_C\) bad if

\[
2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \text{for some root } \alpha \text{ of } (g, a),
\]

and good otherwise, where \(\langle ., . \rangle\) is some invariant \(C\)-bilinear scalar product on \(a^*_C\). It is known that the normalized intertwining operators are holomorphic at good \(\lambda\). From (4) we obtain that at good points \(\lambda_0 \in a^*_C\) the scattering matrix \(S_{w,\sigma,\lambda}^{-1}\) has at most finite dimensional singularities in the following sense. There exists a family of operators \(P(\lambda)\), holomorphic near \(\lambda_0\) and having at most finite dimensional kernels for \(\lambda\) near \(\lambda_0\) such that the composition \(P(\lambda) \circ S_{w,\sigma,\lambda}^{-1}\) is holomorphic. In fact take \(P(\lambda) := \text{id} + R_1(\lambda)\). (In this way we will understand the finite dimensionality of the singularities of \(\text{ext}\) and the Eisenstein series later on.)

This way we have continued \(S_{w^{-1},\sigma,\lambda}\) to \(C^\omega\) for all \(w \in W\). Next we define the extension of \(\phi \in C^{-\infty}(B,V_B(\sigma_\lambda))\), \(\lambda \in C^\omega\), by

\[
\text{ext}(\phi) := J_{w,\sigma,\lambda}w^{-1} \circ \text{ext} \circ S_{w^{-1},\sigma,\lambda}(\phi).
\]

Since \(\bigcup_{w \in W} C^\omega = a^*_C\) the extension \(\text{ext}\) becomes a meromorphic family of continuous maps \(\text{ext} : C^{-\infty}(B,V(\sigma_\lambda)) \to \Gamma H^{\omega,\lambda}_\omega\) on all of \(a^*_C\). In order to discuss the singularities of \(\text{ext}\) we consider the representation

\[
S_{w^{-1},\sigma,\lambda} = \tilde{J}_{w^{-1},\sigma,\lambda} \circ (\text{id} + R_2(\lambda))^{-1}.
\]

Then

\[
\text{ext} = J_{w,\sigma,\lambda}w^{-1} \circ \text{ext} \circ \tilde{J}_{w^{-1},\sigma,\lambda} \circ (\text{id} + R_2(\lambda))^{-1}.
\]

The local singularities of the intertwining operators at bad points cancel, and thus \(J_{w,\sigma,\lambda}w^{-1} \circ \text{ext} \circ \tilde{J}_{w^{-1},\sigma,\lambda}\) is holomorphic on all of \(C^\omega\). \((\text{id} + R_2(\lambda))^{-1}\) has finite dimensional singularities, and so has \(\text{ext}\).

Now we can define all scattering matrices on \(a^*_C\) by \([2]\) obtaining meromorphic families of pseudodifferential operators. At good points they have at most finite dimensional singularities. We have shown
Proposition 2.2 Assume $\delta_T < 0$. Then there is a meromorphic continuation of $\text{ext} : C^{-\infty}(B, V(\sigma)) \to \Gamma H^{+\infty}$ to all of $a_{\mathbb{C}}$. The singularities of $\text{ext}$ are finite dimensional. Moreover there are meromorphic continuations of the scattering matrices satisfying the functional equations (3). At good points the singularities are at most finite dimensional.

3 The general case

In this subsection we assume that $G$ belongs to a series of equal real rank. This means that there is a sequence $\ldots \in G^n \subset G^{n+1} \subset \ldots$ of real semisimple Lie groups inducing embeddings of the corresponding Iwasawa constituents $K_n \subset K^{n+1}, N_n \supseteq N^{n+1}, M_n \subset M^{n+1}$ such that $A = A_n = A^{n+1}$. Then we have totally geodesic embeddings of the symmetric spaces $X_n \subset X^{n+1}$ inducing embeddings of their boundaries $\partial X_n \subset \partial X^{n+1}$. If $\Gamma \subset G^n$ satisfies $[\ldots]$ then it keeps satisfying $[\ldots]$ when viewed as a subgroup of $G^{n+1}$. We obtain the embedding $\Omega_n \subset \Omega^{n+1}$ inducing $B_n \subset B^{n+1}$ while the limit set $\Lambda^n$ is identified with $\Lambda^{n+1}$. Let $\rho_n(H) = \frac{1}{2} \text{tr(ad}(H)_{n^2}), H \in a$.

The exponent of $\Gamma$ now depends on $n$ and is denoted by $\delta_n^n$. We have the relation $\delta_{n+1} = \delta_n^n - \rho_n + \rho$. Hence $\delta_n^n \to -\infty$ as $n \to \infty$. In particular if we embed $\Gamma \subset G^n \subset G^{n+m}$ for large enough $m$ we can obtain $\delta_{n+m} < 0$ and solve the extension problem as well as the continuation of the scattering matrices.

Let $\Gamma \subset G^n$ satisfy $[\ldots]$. The aim of the present section is to show how the solution $\text{ext}^{n+1}$ of the extension problem on $\partial X^{n+1}$ leads to the solution $\text{ext}^n$ of the extension problem on $\partial X^n$.

Let $\sigma$ be a Weyl invariant representation of $M^{n+1}$. Then it restricts to a Weyl invariant representation of $M^n$. The representation $\sigma_\lambda$ of $P_n^{n+1}$ restricts to the representation $\sigma_{\lambda+\rho^n-\rho^n+1}$ of $P^n$. This induces an isomorphism of bundles

$$V_{B^{n+1}}(\sigma_\lambda)|_{B^n} = V_{B^n}(\sigma_{\lambda+\rho^n-\rho^n+1}) .$$

We obtain a push forward of distributions

$$i_* : C^{-\infty}(B^n, V_{B^n}(\sigma_\lambda)) \to C^{-\infty}(B^{n+1}, V_{B^{n+1}}(\sigma_{\lambda+\rho^n-\rho^n+1})) .$$

Assume that we have a solution $\text{ext}^{n+1}$ of the extension problem. For $\phi \in C^{-\infty}(B^n, V_{B^n}(\sigma_\lambda))$ the push forward $i_*(\phi)$ has support in $\partial X^n$. Thus $\text{ext}(\phi)$ has support in $\partial X^n$, too. Let $f \in C^{\infty}(\partial X^{n+1}, V(\tilde{\sigma}_{\lambda-\rho^n+\rho^n+1}))$ such that $f_{|\partial X^n} = 0$. We claim that $\langle \text{ext}^{n+1} \circ i_*(\phi), f \rangle = 0$. To see the claim embed $\phi$ into a holomorphic family $\phi_\lambda \in C^{-\infty}(B^n, V_{B^n}(\sigma_\lambda))$. Then $\langle \text{ext}^{n+1} \circ i_*(\phi_\lambda), f \rangle = 0$ for $\lambda >> 0$. Now the claim follows by meromorphic continuation. Thus we can define a pull back $\text{ext}^n(\phi) := i^* \circ \text{ext}^{n+1} \circ i_*(\phi)$ as follows. For $f \in C^{\infty}(\partial X^n, V(\tilde{\sigma}_{-\lambda}))$ let $\tilde{f} \in C^{\infty}(\partial X^{n+1}, V(\tilde{\sigma}_{-\lambda-\rho^n+\rho^n+1}))$ be an arbitrary extension. We set

$$\langle \text{ext}^n(\phi), f \rangle := \langle \text{ext}^{n+1} \circ i_*(\phi_\lambda), \tilde{f} \rangle .$$

Then $\text{ext}^n$ is well defined. The meromorphic continuation of $\text{ext}$ immediately implies the meromorphic continuation of the scattering matrices by $[\ldots]$. 

For any given finite dimensional representation of $M^n$ we can find a Weyl invariant representation $\sigma$ of $M^{n+1}$ such that $\sigma|_{M^n}$ contains the former as a subrepresentation.

**Theorem 3.1** Let $\Gamma \subset G$ satisfy assumption [7.1] and assume that $G$ belongs to a series. Then the extension problem has a unique continuous meromorphic solution

$$
ext : C^{-\infty}(B, V_B(\sigma)) \to \Gamma H^\sigma_{-\infty}$$

having finite dimensional singularities. Moreover, there are meromorphic continuations of the scattering matrices to all of $a^*_C$. They form meromorphic families of pseudodifferential operators locally coinciding with the intertwining operators up to smoothing operators and satisfying the functional equations (3). At good points the singularities of the scattering matrices are at most finite dimensional.

Note that the singular part of $\text{ext}(\phi_\lambda)$ at $\lambda = \lambda_0$ provides $\Gamma$-invariant distributions in $V(\sigma_{\lambda_0})$ with support in $\Lambda$. The singularities of the scattering matrices at good points are often called resonances while at the bad points in general there is a mixing of the topological singularity localized on the diagonal and resonances.

## 4 Eisenstein series

In this section we define the Eisenstein series and discuss its functional equations. We assume that the extension problem for $\partial X$ has a meromorphic solution $\text{ext}$.

Let $\sigma$ be a Weyl-invariant finite dimensional representation of $M$ and $\gamma$ be a finite dimensional representation of $K$. Let $V(\gamma) := G \times_K V_\gamma$ be the associated homogeneous bundle over $X$. The Eisenstein series $E(\lambda, \phi, T)$ depends on the data $\lambda \in a^*_C$, $\phi \in C^{-\infty}(B, V_B(\sigma))$ and $T \in \text{Hom}_M(V_\sigma, V_\gamma)$. There is a Poisson transform

$$P^T_\lambda : H^\sigma_{-\omega} \to C^\infty(X, V(\gamma))$$

defined by

$$(P^T_\lambda f)(g) := \int_K a(g^{-1}k)^{-(\lambda + \rho)} \gamma(k^{-1}g) Tf(k) dk .$$

The range of $P^T_\lambda$ can be characterized by invariant differential equations formally written as $D_\lambda P^T_\lambda = 0$ [5, 4].

**Definition 4.1** The Eisenstein series is defined by

$$E(\lambda, \phi, T) := P^T_\lambda \circ \text{ext}(\phi) .$$

Thus $E(\lambda, \phi, T)$ is a meromorphic family of smooth sections of $V(\gamma)$ satisfying $D_\lambda E(\lambda, \phi, T) = 0$. Using (4) one can check that for $\lambda > \delta_\Gamma$ our definition coincides with other definitions of Eisenstein series in the literature (up to normalizations). The singularities of the Eisenstein series are finite dimensional.
We now discuss the functional equations satisfied by the Eisenstein series. For $\lambda < 0$, $w \in W$ define

$$c_{w,\gamma}(\lambda) := \int_{N \cap w^{-1}Nw} a(\bar{n})^{-(-\lambda + \rho)} \gamma(\kappa(\bar{n})) d\bar{n} \in \text{End}_M(V_\gamma).$$

c_{w,\gamma}(\lambda)$ extends meromorphically to all of $a_C^\times$. We employ the functional equations of the Poisson transform proved in [9]:

$$c_{w,\sigma}(\lambda) P_T^\lambda \circ J_{w^{-1},\sigma,\lambda} w = P_{\lambda w}^\gamma(m_w)c_{w,\gamma}(\lambda) T, \quad \forall w \in W.$$

It follows

**Proposition 4.2** The Eisenstein series satisfies the functional equations

$$E(\lambda, c_{w,\sigma}(\lambda) S_{w^{-1},\sigma,\lambda} w, \phi, T) = E(\lambda^w, \phi, \gamma(m_w)c_{w,\gamma}(\lambda) T), \quad \forall w \in W.$$

In the rank one case observe that for $\lambda_0 > 0$ the singular part of $E(\lambda, \phi, T)$ at $\lambda = \lambda_0$ provides $L^2$-solutions of the system $D_\lambda f = 0$ on $Y = \Gamma \backslash X$. In fact we obtain the singular part by the Poisson transform of the singular part of $\text{ext}(\phi_\lambda)$ which has support in the limit set $\Lambda$.

**Historical remarks**

In the case that $X$ is the real hyperbolic space of dimension $n$ and $\gamma$ is the trivial $K$-type, the meromorphic continuation of the Eisenstein series was previously obtained by Patterson [11], [12], [13], Mandouvalos [6], [7], [8], and Perry [14], [15]. The idea of using the fact that $G$ belongs to a series we learned from Mandouvalos. As Patterson remarked it can be traced back to Reshnikov and even Selberg. That the Fredholm theory can be applied to the meromorphic continuation of the scattering matrices was first observed by Patterson [12]. Note that the techniques invoked in the literature above are much more complicated than the methods presented in the present paper. The former are based on the analysis of the resolvent kernel of the Laplace operator on $Y$. It is a challenging problem to do this analysis for bundles even over rank one spaces. In [2] we considered the extension problem in connection with the $\Gamma$-cohomology of $H_{-\omega,\Lambda}$ for Fuchsian groups of the second kind.

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