Optimal evaluation of generalized Euler angles with applications to classical and quantum control

Domenico D’Alessandro
Department of Mathematics
Iowa State University
Ames, IA 50011, USA
Tel. (+1) 515 294 8130
e-mail: daless@iastate.edu

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Abstract

Given two linearly independent matrices in $so(3)$, $Z_1$ and $Z_2$, every rotation matrix $X_f \in SO(3)$ can be written as the product of alternate elements from the one dimensional subgroups corresponding to $Z_1$ and $Z_2$, namely $X_f = e^{Z_1 t_1} e^{Z_2 t_2} e^{Z_1 t_3} \ldots e^{Z_1 t_s}$. The parameters $t_i$, $i = 1, \ldots, s$ are called generalized Euler angles.

In this paper, we evaluate the minimum number of factors required for the factorization of $X_f \in SO(3)$, as a function of $X_f$, and provide an algorithm to determine the generalized Euler angles explicitly. The results can be applied to the bang bang control with minimum number of switches of some classical control systems and of two level quantum systems.

Keywords: Decompositions of Lie groups, Rigid Body Dynamics, Geometric Control, Two Level Quantum Systems.

1 Introduction

In this paper, we deal with the problem of steering control for bilinear systems of the form

$$\dot{x} = Ax + Bxu,$$

where $x \in \mathbb{R}^3$, $u$ is a control function, and $A$ and $B$ are skew-symmetric $3 \times 3$ matrices, namely matrices in $so(3)$. Several systems in applications have the structure (1). In particular the most common example is given by the dynamics of the rigid body where one
component of the angular velocity is seen as the control \( u \) and the others are held constant. The fundamental matrix of equation (1) represents the orientation of the rigid body. Another example is the lossless electrical network dealt with in [6]. A two level quantum system driven by a single time varying component of an electro-magnetic field also has the structure (1), where \( x \) represents the state \( \in \mathbb{C}^2 \) and the matrices \( A \) and \( B \) are in the Lie algebra \( su(2) \). Because of the connection between the Lie groups \( SO(3) \) and \( SU(2) \) the results presented here can be applied to the latter system as well.

The fundamental matrix of the system (1), \( X \), satisfies

\[
\dot{X} = AX + BXu,
\]

with initial condition equal to the \( 3 \times 3 \) identity matrix. It follows from the results of [8] that if \( A \) and \( B \) are linearly independent (and therefore generate \( so(3) \) which has dimension 3 and no two-dimensional subalgebras) a piecewise constant control is sufficient to steer the state of (2) from the identity to every matrix \( X_f \) in \( SO(3) \) and, as a consequence, the state \( x \) of (1) between two states with equal length. Let us assume now that the control \( u \) is allowed to attain only two values, \( M \) and \( N \). Define \( Z_1 := A + BM \) and \( Z_2 := A + BN \), and assume a factorization of the desired target state \( X_f \) of the type

\[
X_f = e^{Z_1 t_1} e^{Z_2 t_2} e^{Z_1 t_3} \cdots e^{Z_1 t_s},
\]

is known with \( t_1, t_2, \ldots, t_s > 0 \). Then a piecewise constant control equal to \( M \) for time \( t_s \), \( N \) for time \( t_{s-1} \), \( M \) for time \( t_{s-2} \) and so on, drives the state of (2) from the identity to \( X_f \) in (3). This idea, involving Lie group decompositions, has recently been used to prescribe controls for quantum mechanical systems where the underlying Lie group is the group of special unitary matrices of dimension \( n, SU(n) \) (see e.g. [1], [3], [12], [13], [16] and references therein). If the control is bounded in magnitude, namely \( |u| \leq M \) we can choose \( N := -M \) and \( M \) as the two values for the control. From a practical point of view one would like to have a factorization of \( X_f \) in terms of the matrices \( Z_1 \) and \( Z_2 \) that involves the least number of factors, so that the control law has the minimum number of switches. Moreover, an algorithm is needed to evaluate the generalized Euler angles \( t_i, i = 1, \ldots, s \). This paper is devoted to the solution of these two problems. Constructive factorizations of \( SU(2) \) and \( SO(3) \) can be found in the papers [3], [13] that, however, do not consider the problem of minimizing the number of factors.

The paper is organized as follows. In the next section we give some preliminary definitions that will be used in the following and recall some results proved in [10] concerning factorizations of elements of the Lie group \( SO(3) \) of the type (3). We also transform, using a change of coordinates, every pair of linearly independent matrices \( \in so(3) \) into a canonical form, that will be used in the following sections, without loss of generality. In Section 3 we evaluate the minimum number of factors needed in a factorization of matrix \( X_f \in SO(3) \) of the type (3) given \( Z_1 \) and \( Z_2 \). In Section 4 we give an algorithm for the determination of

\footnote{This is done without loss of generality since the one parameter subgroups corresponding to \( Z_1 \) and \( Z_2 \) are closed, namely the functions \( e^{Z_1 x} \) are periodic.}
the generalized Euler angles $t_1, \ldots, t_s$. We discuss applications to the control of classical and quantum systems in Section 5.

## 2 Preliminaries

The inner product $\langle \cdot, \cdot \rangle$ between two elements of $so(3)$, $Z_1$ and $Z_2$ is defined as

$$\langle Z_1, Z_2 \rangle = \text{Trace}(Z_1 Z_2^T).$$  \hfill (4)

If $\langle Z_1, Z_2 \rangle = 0$, the maximum number of factors $s$ needed to express a matrix $X_f$ in $SO(3)$ as in (4) (maximum over $SO(3)$) is three, and the factorization in (4) is the classical Euler resolution of a rotation (see e.g. [14]) (modulo a change of coordinates and a re-scaling of the variables $t$). The parameters $t_i$ are called Euler angles and their calculation is standard matter (see e.g. [14], pg. 297).

In [10], it was shown that, for every pair of matrices $Z_1, Z_2$, the number of factors needed to express an element $X_f \in SO(3)$ is uniformly bounded, over $SO(3)$ (see also [4] and [13] for generalizations to every compact Lie group). The maximum value for $s$ (maximum over $SO(3)$) is called the order of generation of $SO(3)$ with respect to $Z_1$ and $Z_2$. It has been calculated in [10] and it only depends on the value of the cosine of the angle between $Z_1$ and $Z_2$, namely

$$\psi := \frac{\langle Z_1, Z_2 \rangle}{\langle Z_1, Z_1 \rangle^{1/2} \langle Z_2, Z_2 \rangle^{1/2}}.$$  \hfill (5)

If $\psi = 0$, the order of generation is equal to 3 and we obtain the classical Euler resolution of a rotation. Our treatment in the following was inspired by the proof in [10]. However, most of the treatment in [14] is carried out using stereographic projections and translating the problem to the induced subgroup of the Moebius group. We shall treat the factorization of every element in $SO(3)$ by working on the unit sphere in $\mathbb{R}^3$ and looking at $SO(3)$ as a transformation group on the sphere [11].

We now show that there is no loss of generality in assuming that $Z_1$ and $Z_2$ in (4) have a special form which we shall describe. We shall call $S_{hk}$, $h < k$, the matrix in $so(3)$ which has zeros everywhere except in the $h,k$-th ($k,h$-th) entry which is equal to 1 ($-1$). Given a matrix $Z_1$, there exists a matrix $T_1 \in SO(3)$ such that

$$T_1 Z_1 T_1^T = \lambda_1 S_{12},$$  \hfill (6)

$\lambda_1 \neq 0$. This can be easily seen by choosing $T_1 := [v_1, v_2, v_3]^T$, with $v_3$ such that $v_3^T Z_1 = 0$ and with norm equal to one and $v_1$ and $v_2$ such that $\{v_1, v_2, v_3\}$ form an orthonormal basis in $\mathbb{R}^3$. We also set

$$T_1 Z_2 T_1^T := a S_{12} + b S_{13} + c S_{23}. $$  \hfill (7)

Choose now $T_2 := e^{S_{12} \theta}$ with $\theta$ such that $bcos(\theta) + c \sin(\theta) = 0$, with $b$ and $c$ given in (6). Then we have

$$T_2 T_1 Z_1 T_1^T T_2^T = \lambda_1 S_{12}, $$  \hfill (8)
\[ T_2T_1Z_2T_1^TT_2^T = aS_{12} + dS_{23}, \] (9)

for some parameter \( d \neq 0 \). Therefore, we can always assume that, in appropriate coordinates, the matrices \( Z_1 \) and \( Z_2 \) have the form \( Z_1 := \lambda_1 S_{12} \) and \( Z_2 := aS_{12} + dS_{23} \), respectively. Moreover we can divide \( Z_1 \) by \( \lambda_1 \neq 0 \) (this has the only effect that, in the matrices of the form \( e^{Z_1 t} \), \( t \) has to be scaled by a factor \( \lambda_1 \)) and analogously we can divide \( Z_2 \) (in the new coordinates in (9)) by \( d \neq 0 \) and therefore the parameter \( t \) in the subgroup \( e^{Z_2 t} \) has to be scaled by a factor \( d \). Define \( \rho := \frac{a}{d} \). We can assume, without loss of generality, that the matrices \( Z_1 \) and \( Z_2 \) are given by

\[ Z_1 := S_{12}, \quad (10) \]
and

\[ Z_2 := \rho S_{12} + S_{23}, \quad (11) \]
and we shall do so in the following. Notice that the above manipulations do not modify the value of the parameter \( \psi \) in (5) which is given, in terms of \( \rho \), by

\[ \psi = \frac{\rho}{\sqrt{1 + \rho^2}}. \] (12)

### 3 Decompositions with minimum number of factors

Assume now that an element \( X_f \in SO(3) \) is given, to be expressed as in (3), with \( Z_1 \) and \( Z_2 \) given in (10), (11). We give in this section a procedure to determine the minimum number of factors necessary as a function of \( X_f \).

We assume \( \rho \) in (11) different from zero (the case \( \rho = 0 \) corresponds, from (12), to \( Z_1 \) and \( Z_2 \) orthogonal to each-other and therefore the decomposition is the standard Euler decomposition). Define two sequences \( \{z_k\} \) and \( \{f_k\} \) by \( z_0 = f_0 = -1 \)

\[ f_k := \frac{1}{|\rho|} \sqrt{1 - z_k^2} + z_k \] (13)
and

\[ z_{k+1} := \frac{2\rho^2}{1 + \rho^2} f_k - z_k. \] (14)

We have the following Lemma.

**Lemma 3.1** There exists an index \( \bar{k} \geq 1 \) such that \( f_k \) is defined (\(|z_k| \leq 1\)) for every \( k \leq \bar{k} \), \( f_k < 1 \), for every \( k < \bar{k} \) and \( f_k \geq 1 \).

**Proof.** First notice that if \(|\rho| \leq 1\), the Lemma is true with \( \bar{k} = 1 \) since \( 0 \leq z_1 < 1 \) and \( f_1 \geq 1 \). Let us assume \(|\rho| > 1\). We first show that \( f_k \) well defined and \( f_k < 1 \) implies that \( f_{k+1} \) is well defined, namely that \(|z_{k+1}| \leq 1\). Then we show that there exists the first value of \( k, \bar{k} \), such that \( f_k \geq 1 \).

Assume \( f_k < 1 \). From (13), we obtain

\[ \sqrt{1 - z_k^2} < |\rho|(1 - z_k), \] (15)
which gives, taking into account $|z_k| \leq 1$,
\[-1 \leq z_k < \frac{\rho^2 - 1}{1 + \rho^2}.
\] (16)

Consider the expression of $z_{k+1}$ obtained combining (13) and (14),
\[z_{k+1} = \frac{2\rho^2}{1 + \rho^2} \left( \frac{1}{|\rho|} \sqrt{1 - z_k^2} + z_k \right) - z_k.
\] (17)

Consider $z_{k+1}$ as a function of $z_k$ in the interval defined in (16). This function is always increasing from the value $z_{k+1} = 1 - \rho^2 \frac{1}{1 + \rho^2}$ at $z_k = -1$ to the value $z_{k+1} = 1$ at $z_k = \rho^2 - 1 \rho^2 + 1$. In particular we always have $|z_{k+1}| \leq 1$ which implies that $f_{k+1}$ is well defined. To show the existence of a $\bar{k}$ such that $f_{\bar{k}} \geq 1$, we evaluate $z_{k+1} - z_k$ using (13) and (14). We obtain
\[z_{k+1} - z_k = \frac{2}{1 + \rho^2} \left( \frac{2|\rho|}{1 + \rho^2} - z_k \right) > \frac{2}{1 + \rho^2},
\] (18)

Using the second inequality in (16), we obtain
\[\sqrt{1 - z_k^2} > \frac{2|\rho|}{1 + \rho^2},
\] (19)

and plugging this into (18), we obtain
\[z_{k+1} - z_k > \frac{2}{1 + \rho^2} \left( \frac{2\rho^2}{1 + \rho^2} - z_k \right) > \frac{2}{1 + \rho^2},
\] (20)

where, in the last inequality, we used inequality (16) again. Therefore the sequence $\{z_k\}$ is increasing by at least $\frac{2}{1 + \rho^2}$ at each step and since $f_k \geq z_k$ for every $k$, from (13), we must have a value of the index $\bar{k}$ such that $f_{\bar{k}} \geq 1$. This concludes the proof of the Lemma. □

We now relate the finite sequences $\{z_k\}$ and $\{f_k\}$, $k = 0, 1, ..., \bar{k}$ defined in (13) and (14) to the minimum number of factors needed in the factorization (3). Consider a given target matrix $X_f := \{x_{i,j}\}$, $i, j = 1, 2, 3$ to be factorized. We define a function $O(X_f)$ which is equal to 1 if $x_{3,3} = -z_0 = 1$, it is equal to 2 if $z_0 < -x_{3,3} \leq z_1$ and $x_{1,3} = \rho(-x_{3,3} + 1)$ and equal to 3 if $z_0 < -x_{3,3} \leq z_1$ and $x_{1,3} \neq \rho(-x_{3,3} + 1)$. In cases not considered above, let $\bar{k}$ be the highest value of the index $k$ such that
\[z_k < -x_{3,3}
\] (21)

(recall from (20) that $z_k$ is increasing at each step by at least a given amount). Then we have
\[O(X_f) = 2\bar{k} + 2 \text{ if } \text{sign}(\rho)x_{1,3} \geq -|\rho|(x_{3,3} + f_{\bar{k}})
\] (22)
\[O(X_f) = 2\bar{k} + 3 \text{ if } \text{sign}(\rho)x_{1,3} < -|\rho|(x_{3,3} + f_{\bar{k}})
\] (23)
The following Lemma gives the minimum number of factors in the factorization (3) assuming that the first factor on the right is of the form $e^{Z_{1}t_{s}}$ with $t_{s} > 0$. The proof of the Lemma reveals the geometric meaning of the finite sequences $\{z_{k}\}$ and $\{f_{k}\}$ defined in (13), (14). We denote the minimum number of factors needed to express a general matrix $X_{f}$ as in (3) by $MIN(X_{f})$.

**Lemma 3.2** Assume that $X_{f}$ is such that the factorization with minimum number of factors in (3) starts with a nontrivial factor of the type $e^{Z_{1}t}$ on the right. Then

$$MIN(X_{f}) = O(X_{f}).$$  \tag{24}

Before giving the proof of the Lemma, we describe the geometry of the above construction. Considered as a transformation on the sphere of radius 1 centered at the origin, $X_{f}$ transforms the South Pole $P_{s} := [0,0,-1]^{T}$ into a point $P_{f} := [-x_{1},-x_{2},-x_{3}]$ (which is just the negative of the third column of $X_{f}$). Conversely, any matrix $\tilde{X}_{f}$ such that $P_{f} = \tilde{X}_{f}P_{s}$ is equal to $X_{f}$ up to a factor that leaves $P_{s}$ unchanged. Such factor will in general have the form $e^{Z_{1}t}$ (recall (14)) and therefore we have $X_{f} = \tilde{X}_{f}e^{Z_{1}t}$. We would like to find any product with minimum number of factors

$$\tilde{X}_{f} := e^{Z_{1}t_{1}}e^{Z_{2}t_{2}}e^{Z_{3}t_{3}} \ldots e^{Z_{s}t_{s} - 1}e^{Z_{1}t_{s}},$$  \tag{25}

(with $t_{s}$ possibly equal to zero) such that $P_{f} = \tilde{X}_{f}P_{s}$ and then to obtain $X_{f}$ as $X_{f} = \tilde{X}_{f}e^{Z_{1}t_{s}}$. From the assumption that the minimum number of factors for $X_{f}$ is obtained with a nontrivial factor $e^{Z_{1}t_{s}}$ on the right, the minimum number of factors will be given by $s$. This observation can be interpreted in the language of coset spaces and homogeneous spaces (see e.g. [11]). The subgroup $H := \{X \in SO(3)|X = e^{Z_{1}t}, t \in \mathbb{R}\}$ is the isotropy group of the South Pole $P_{s}$, namely the set of all the elements of $SO(3)$ that leave $P_{s}$ fixed. There exists an isomorphism between elements of the coset space $SO(3)/H$ and elements of the sphere $S^{3}$. In the expression (3) we use the last term $e^{Z_{1}t_{s}}$ to move inside a coset while the remaining factors are used to go from one coset to the other, namely from one point on the sphere $S^{3}$ to the other. We now look for a transformation $\tilde{X}_{f}$ in (25) transforming $P_{s}$ to $P_{f}$ with minimum number of factors.

On the sphere $S^{3}$, every element of the form $e^{Z_{1}t}$ corresponds to a rotation about the $z$ axis. Each point on the sphere $S^{3}$ follows a trajectory on a circle which is the intersection of the sphere with a horizontal plane. The value of the $z$ coordinate of the point is not changed by this rotation. Every matrix of the form $e^{Z_{2}t}$ corresponds to a rotation about the axis defined by the vector $\bar{n}_{\rho} := [1,0,\rho]^{T}$ (points on the line through the origin parallel to this vector are left invariant by the rotation). Under the action of this rotation, every point on the sphere $S^{3}$ follows a trajectory on a circle which is the intersection of a plane perpendicular to $\bar{n}_{\rho}$ and the sphere $S^{3}$. Each such plane forms an angle $\theta := \tan^{-1}\frac{\rho}{t}$ with the $x - y$ plane. If we consider a trajectory $e^{Z_{2}t}P_{s} := [x(t), y(t), z(t)]^{T}$, the maximum value for the coordinate $z(t)$ will be obtained at $z_{1}$ defined in (14) (when $t = \pi$). Let us call this
point $P_1$. Following a horizontal trajectory $e^{z_1t}P_1$, for $t = \pi$, we obtain a point which is opposite to $P_1$. Let us denote this point by $Q_1$. Following from $Q_1$ a trajectory $e^{z_2t}Q_1$ again up to $e^{z_2\pi}Q_1$, we obtain a point with $z$ coordinate given by $z_2$ in (14). The value $f_1$ is the $z-$coordinate of the intersection of the plane perpendicular to $\vec{n}_P$, containing the point $Q_1$, and the $z$-axis. Notice that $z_2$ is the maximum value that can be obtained for $z$ starting from $P_1$ and with just one switch from one type of trajectory to the other. Continuing this way one obtains the elements of the sequences $\{z_k\}$, $\{f_k\}$. It follows from this geometric description that $\{z_k\}$ is an increasing sequence and it was proven in Lemma 3.1 that it is a finite sequence (See also the Remark following the proof of Lemma 3.2). Figure 1 describes (in a two-dimensional plot) the trajectories on the sphere. In this Figure, $k$, defined in Lemma 3.1, is equal to 5. We have denoted by $P_k$, $k = 0, 1, \ldots, 5$ the points on the sphere with $z-$coordinate equal to $z_k$, $k = 0, 1, \ldots, 5$. $F_k$ denotes the point whose $z-$coordinate if $f_k$, $k = 1, \ldots, 5$. We now use this Figure to complete the proof of the Lemma.

**Proof of Lemma 3.2.** We shall refer to Figure 1 and the above discussion. Let $P_1 := \{-x_{1,3}, -x_{2,3}, -x_{3,3}\}^T$. If $x_{3,3} = 1 = -z_0$, then $\tilde{X}_f$ is of the type $e^{z_1t}$ and clearly $MIN(X_f) = 1$. If $z_0 < -x_{3,3} \leq z_1$ and $x_{1,3} = \rho(-x_{3,3} + 1)$ then $P_f$ belongs to the intersection of the plane $x + \rho(z + 1) = 0$ with the sphere $S^3$. The point $P_f$ can be reached by (possibly) following a trajectory of the type $e^{z_1t}$ (which leaves $P_s := [0, 0, -1]$ unchanged) following by a trajectory of the type $e^{z_2t}$. In this case, since we have assumed that the last factor on the right in (3) is a nontrivial $e^{z_2t}$ factor, we have $MIN(X_f) = 2$. Analogously, it is easily seen that $MIN(X_f) = 3$ if $z_0 < -x_{3,3} \leq z_1$ and $x_{1,3} \neq \rho(1 - x_{3,3})$. Now notice that to reach a point with $z$ coordinate $\tilde{z}$, with $z_k < \tilde{z} \leq z_{k+1}$ we need to cross the circle $C_k := \{(x, y, z)|z = z_k, x^2 + y^2 + z^2 = 1, (x, y, z) \neq (x_k, y_k, z_k)\}$. In order to cross any point of $C_k$, the minimum number of pieces of trajectory (including possibly the first one of the type $e^{z_1t}$, if assumed nontrivial, and the last one to leave $C_k$), is $2k + 2$. This is clear when $k = 1$ and follow by induction for the other values of $k$, noticing that we must cross $C_{k-1}$ before crossing $C_k$. To reach a point $P_f := \{-x_{1,3}, -x_{2,3}, -x_{3,3}\}^T$ such that $z_k < -x_{3,3}$, with $k \leq \tilde{k}$, we need to cross $C_k$ and the minimum number of factors to do that is $2\tilde{k} + 2$. No other factor is needed if $P_f = \{-x_{1,3}, -x_{2,3}, -x_{3,3}\}^T$ is below the plane with equation $x + \rho(z - f_k) = 0$ while another factor is needed if $P_f$ is above this plane. This accounts for the inequalities (22), (23).

**Remark:** It is possible to show that the sequence $\{z_k\}$ in (3) can be obtained by $z_k = -\cos(k\beta)$, for some angle $\beta$ obtained as $\beta = \cos^{-1}z_1$. From a geometric point of view, $\beta$ is the angle in the $y - z$ plane between the segments $OD_k$ and $OD_{k+1}$, where $O$ denotes the origin and $D_k := e^{z_1\pi}P_k$, $k = 0, 1, 2, \ldots, 5$. The points $D_k$ are the midpoints of the lines representing a circle $C_k$ in Figure 1. This angle is the same for every $k$. This gives a geometric interpretation and an alternative proof of Lemma 3.1.

The above Lemma solves the problem of finding the minimum number of factors to express $X_f$ in the form (3) if we assume that the last term on the right is of the type $e^{z_1t}$. 

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Figure 1: Geometric Construction for Lemmas 3.1 and 3.2
This assumption can be relaxed by considering a change of coordinates $\tilde{T}$,

$$\tilde{T} := \begin{pmatrix} \frac{-\rho}{\sqrt{1+\rho^2}} & 0 & \frac{1}{\sqrt{1+\rho^2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{1+\rho^2}} & 0 & \frac{\rho}{\sqrt{1+\rho^2}} \end{pmatrix}. \quad (26)$$

We have

$$\tilde{T}Z_2\tilde{T}^T = -\sqrt{1+\rho^2}Z_1, \quad (27)$$

$$\tilde{T}Z_1\tilde{T}^T = -\frac{1}{\sqrt{1+\rho^2}}Z_2. \quad (28)$$

Assume that

$$X_f = e^{Z_1t_1}e^{Z_2t_2} \cdots e^{Z_st_s}, \quad (29)$$

with a term of the type $e^{Z_2t}$ first on the right, is the optimal factorization. Then a factorization with a term $e^{Z_1t}$ first on the right and $s$ factors is the optimal factorization for $\tilde{T}X_f\tilde{T}^T$ and vice versa. Therefore according to Lemma 3.2 we have $MIN(X_f) = O(\tilde{T}X_f\tilde{T}^T)$. We conclude with the following Theorem.

**Theorem 3.3**

$$MIN(X_f) = \min\{O(X_f), O(\tilde{T}X_f\tilde{T}^T)\}, \quad (30)$$

with $\tilde{T}$ given in (22).

### 4 Evaluation of the Generalized Euler Angles

The geometric analysis of the previous section gives a method to determine the generalized Euler parameters corresponding to the optimal factorization. Let us assume, without loss of generality, that $MIN(X_f) = O(X_f)$ namely, the optimal factorization has a nontrivial term of the type $e^{Z_1t}$ last on the right. Referring to Figure 2, we have labeled each region with a number denoting the minimum number of factors needed to drive $P_s$ to $P_f$ in that region (including the last nontrivial factor on the right of the type $e^{Z_1t}$). If $P_f := -[x_{1,3}, x_{2,3}, x_{3,3}]^T$ is in an odd region, such as $P_o$ in Figure 2, (namely it is strictly above a plane dividing a region between two planes $z = constant$, except for the Region 3, which includes points strictly below the plane $x + \rho(z + 1) = 0$ as well) then an optimal factorization for $X_f$ is

$$X_f = e^{Z_1t_1}e^{Z_2t_2}e^{Z_3\pi}e^{Z_4\pi} \cdots e^{Z_1\pi}e^{Z_2\pi}e^{Z_3t_s}. \quad (31)$$

We first determine $t_2$ so that, defined $L := e^{Z_2t_2}e^{Z_3\pi}e^{Z_4\pi} \cdots e^{Z_3\pi}e^{Z_4\pi} := \{l_{i,j}\}$, $l_{3,3} = x_{3,3}$. Then we determine $t_1$ so that $e^{Z_1t_1}LP_s = P_f$, where $P_s$ denotes the South Pole $P_s = [0, 0, -1]^T$ and then $t_s$ such that $e^{Z_1t_1}Le^{Z_1t_s} = X_f$. Notice that each step involves the evaluation of just one parameter. Notice also that the optimal factorization is not unique and, in the above factorization, we could have, for example, replaced the term $e^{Z_2t_2}e^{Z_3\pi}$ with a term $e^{Z_2t_2}e^{Z_1t_1}$ for appropriate values $t_1$ and $t_2$ (see the alternative path with bold face lines in Figure 2).
If $MIN(X_f)$ is even ($P_f = P_e$ in Figure 2) then we have that the optimal factorization is given by

$$X_f = e^{Z_2 t_1} e^{Z_1 t_2} e^{Z_2 \pi} e^{Z_1 \pi} \cdots e^{Z_1 \pi} e^{Z_2 \pi} e^{Z_1 t_s}. \quad (32)$$

Let, in the sequence (13), (14), $\bar{z}_k$ be the largest value of $z_k$ such that $z_k < -x_{3,3}$. Then, we consider a point $\bar{P} := [\bar{x}, \bar{y}, \bar{z}]^T$ intersection of the planes $(x + x_{1,3}) + \rho(z + x_{3,3}) = 0$, $z = \bar{z}_k$ and the sphere $x^2 + y^2 + z^2 = 1$. Then we determine $t_2$ so that, defined $L := e^{Z_1 t_2} e^{Z_2 \pi} e^{Z_1 \pi} \cdots e^{Z_1 \pi} e^{Z_2 \pi} P_s$, we have $LP_{\bar{z}} = \bar{P}$. Then we determine $t_1$ so that $e^{Z_2 t_1} \bar{P} = P_f$ and finally we determine $t_s$ so that $e^{Z_2 t_1} Le^{Z_1 t_s} = X_f$. In this case too, the optimal factorization is not unique.

### 5 Applications

The results of this paper can be used to prescribe bang bang type of controls for bilinear systems whose state varies on the Lie group $SO(3)$, with minimum number of switches. This technique of control can be applied to the dynamics of a rigid body where the angular velocity is seen as control. The same technique can also be employed for the control of
switched electrical networks [6], with minimum number of switches.

In recent years there has been a large amount of interest in the control of systems of the form (1) with $A$ and $B$ in the Lie algebra $su(2)$. This Lie algebra is isomorphic to $so(3)$. These systems model the dynamics of two level quantum systems with just one control [5]. Constructive factorizations (3) of elements of the Lie group $SU(2)$ have been given in [3], [13], and used for control. In particular the factorization of [3] gives a worst case number of factors which is greater than the minimum by at most one. The algorithm presented in this paper can be used to determine the optimal factorization for elements $\bar{X}_f$ in $SU(2)$ as well and therefore to prescribe a control algorithm for two level quantum systems with minimum number of switches.

Let $\tilde{\phi}$ denote the isomorphism between $su(2)$ and $so(3)$ which maps the Pauli matrices

$$S_x := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad S_y := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_z := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

(33)

to $2S_{1,3}$, $2S_{2,3}$, $-2S_{1,2}$ respectively. Let $\bar{Z}_1$ and $\bar{Z}_2$ be two linearly independent matrices in $su(2)$. We look for the factorization of $\bar{X}_f$ of the type

$$\bar{X}_f = e^{\bar{Z}_1 t_1} e^{\bar{Z}_2 t_2} \ldots e^{\bar{Z}_s t_s},$$

(34)

with minimum number of factors. The isomorphism $\tilde{\phi}$ between $su(2)$ and $so(3)$ induces a homomorphism $\phi$ between elements of the corresponding Lie groups, $\phi : SU(2) \rightarrow SO(3)$, which is given, if $S = e^V \in SU(2)$, by $\phi(S) := e^{\tilde{\phi}(V)}$. This homomorphism is two to one in that to $\pm S$ in $SU(2)$ corresponds the same element in $SO(3)$ (for a more detailed treatment of the relation between the Lie groups $SU(2)$ and $SO(3)$ see e.g. [17]. See also [4] for applications to control). Let $X_f$ be the element in $SO(3)$ corresponding to $\bar{X}_f$ under this homomorphism and $Z_1$ and $Z_2$ the elements of $so(3)$ corresponding to $\bar{Z}_1$ and $\bar{Z}_2$. If

$$X_f = e^{Z_1 t_1} e^{Z_2 t_2} \ldots e^{Z_s t_s},$$

(35)

is the optimal factorization for $X_f$ then $s$ is the optimal number of factors for $\bar{X}_f$ in (34) as well. The generalized Euler parameters can also be easily determined. If we use the same values for $t_1, ..., t_s$ in (35) and (34) we obtain a matrix which is $\pm$ the desired $\bar{X}_f$. This affects the quantum mechanical state $x$ in (1) by an overall phase factor which has no physical meaning. In any case, the minus sign can be easily eliminated by changing the value of just one of the parameters so as to change one factor $S$ into $-S$. This is always possible since each one dimensional subgroup in $SU(2)$ that contains $S$ also contains $-S$. Therefore, we can find an optimal factorization for any element in $SU(2)$ as well. This can be easily extended to any Lie algebra isomorphic to $su(2)$ and the corresponding Lie group, which is known to be isomorphic to either $SO(3)$ or $SU(2)$ [6].

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