DENSITY OF SMALL DIAMETER SUBGRAPHS IN $K_r$-FREE GRAPHS

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Abstract. We denote by $\text{ex}(n, H, F)$ the maximum number of copies of $H$ in an $n$-vertex graph that does not contain $F$ as a subgraph. Recently, Grzesik, Győri, Salia, Tompkins considered conditions on $H$ under which $\text{ex}(n, H, K_r)$ is asymptotically attained at a blow-up of $K_{r-1}$, and proposed a conjecture. In this note we disprove their conjecture.

1. Introduction

Let $G$, $H$ and $F$ be graphs. We say that $G$ is $F$-free if it does not contain $F$ as a (not necessarily induced) subgraph. Write $\mathcal{N}(H, G)$ for the number of (unlabelled) copies of $H$ in $G$. We denote by $\text{ex}(n, H, F)$ the generalized Turán number, which is the maximum value of $\mathcal{N}(H, G)$ in an $F$-free graph $G$ on $n$ vertices.

A prominent topic of research in extremal graph theory is the behaviour of $\text{ex}(n, H, F)$. The classical case $H = K_2$ was studied by Turán [7] for $F = K_r$ and then by Erdős, Simonovits and Stone [3, 2] for general $F$. Alon and Shikhelman [1] introduced and initiated the systematic study of $\text{ex}(n, H, F)$ for general choices of $H$ and $F$.

Extending this line of study, Lidický and Murphy [6] conjectured that given a graph $H$ and an integer $r > \chi(H)$ (where $\chi(H)$ denotes the chromatic number of $H$), there exists a complete $(r - 1)$-partite graph with asymptotically as many copies of $H$ as possible in a $K_r$-free graph (which is by definition $\text{ex}(n, H, K_r)$). Grzesik, Győri, Salia and Tompkins [5] recently showed that for every $r \geq 3$ there is a counterexample to the conjecture of Lidický and Murphy. We need some extra definitions to describe their counterexamples.

Let us denote with $P_n$ the path on $n$ vertices and, given a graph $G$ and a positive integer $r$, let us write $G^r$ for the graph obtained from $G$ by joining every pair of vertices at distance at most $k$ in $G$. Moreover, given a graph $H$ and a vertex $v$ in $V(H)$, let the graph $H'$ be obtained from $H$ by replacing $v$ with an independent set $I_v$ of size $a$ and by adding a complete bipartite graph between $N_H(v)$ and $I_v$. We say that $H'$ is obtained from $H$ by blowing up $v$ by a factor of $a$; we also say that $H'$ is a blow-up of $H$.

As mentioned above, Grzesik, Győri, Salia and Tompkins [5] constructed a counterexample to Lidický and Murphy’s conjecture. This was done by building a sequence $(F_r)_{r \geq 3}$ of graphs where $F_r$ is obtained from $P_{2r-2}^r$ by blowing up its two endvertices by a large factor dependent on $r$. Let us note that Gerbner [4] very recently extended the construction of Grzesik, Győri, Salia and Tompkins [5] to general $F$-free graphs. In the same paper, Grzesik, Győri, Salia and Tompkins [5] proposed a new version of the conjecture.

**Conjecture 1** (Grzesik, Győri, Salia and Tompkins [5]). If $G$ is a graph with $\chi(G) < r$ and diameter at most $2r - 2$, then $\text{ex}(n, G, K_r)$ is asymptotically achieved by a blow-up of $K_{r-1}$.

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2. Counterexamples to conjecture 1

Our main result states that for \( r \geq 4 \) there exists a counterexample to Conjecture 1.

**Theorem 1.** For any \( r \geq 4 \) and \( \delta > 0 \) there is a graph \( G \) of diameter 2 such that for all sufficiently large \( n \) and any \((r-1)\)-partite graph \( T \) on \( n \) vertices we have

\[ \delta \cdot \text{ex}(n, G, K_r) > N(G, T). \]

**Proof.** Let \( r \geq 4 \). Let \( H \) be the graph obtained from the disjoint union of a copy \( K \) of \( K_{r-3} \) and a copy \( P \) of \( P_6 \) by inserting all edges between \( V(K) \) and \( V(P) \). Write \( x \) and \( y \) for the two endvertices of \( P \) in \( H \). Several examples of \( H \) are depicted in Figure 1 with the vertices \( x \) and \( y \) marked. Fix a positive constant \( \varepsilon < \frac{1}{100(r+1)} \) and a positive integer \( a \) such that \( \frac{\delta}{2} \varepsilon^{r+1} (1 - \varepsilon (r+1))^2a \geq \frac{1}{20^2} \). Write \( G \) for the graph obtained from \( H \) by blowing up both \( x \) and \( y \) by a factor of \( a \). Let \( X \) and \( Y \) be the independent sets corresponding to \( x \) and \( y \) respectively. Observe that the diameter of \( G \), which is equal to the diameter of \( H \), is exactly 2. Observe that \( \chi(G) < r \).

Let \( T \) be a complete \((r-1)\)-partite graph on \( n \) vertices. Let us count the number of labelled copies of \( G \) in \( T \). Observe that \( G \) has a unique \((r-1)\)-colouring (up to relabelling of the colours), and this colouring has the property that all vertices in \( X \) get the same colour, all vertices in \( Y \) get the same colour, and these two colours are different. Hence, there are at most \( n^{r+1} \left( \frac{n}{2} \right)^{2a} \) labelled copies of \( G \) in \( T \).

On the other hand, let us denote by \( Q \) the graph obtained from the disjoint union of a copy \( K \) of \( K_{r-3} \) and a copy \( C \) of \( C_5 \) by inserting all edges between \( V(K) \) and \( V(C) \); let us note that \( Q \) can be obtained from \( H \) by contracting the vertices \( x \) and \( y \) to a new vertex \( z \). Write \( S \) for the graph obtained from \( Q \) by blowing up the vertex \( z \) by a factor of \( n - (r+1)\lfloor \varepsilon n \rfloor \) and every other vertex by a factor of \( \lfloor \varepsilon n \rfloor \). Several examples of \( S \) are depicted in Figure 2. Since \( C_5 \) is triangle-free, we have that \( S \) is \( K_r \)-free. Now since we have at least \( \lfloor \varepsilon n \rfloor^{r+1} \) choices for the vertices of \( Q \setminus z \) and at least \( n - (r+1)\lfloor \varepsilon n \rfloor \) choices for the vertices in \( X \cup Y \), the number of labelled copies of \( G \) in \( S \) is at least

\[ \lfloor \varepsilon n \rfloor^{r+1} (n - (r+1)\lfloor \varepsilon n \rfloor)^{2a} - o(n^{2a}). \]

Then, by the definition of \( \text{ex}(n, G, K_r) \) and our choice of \( \varepsilon \) and \( a \) we have

\[ \delta \cdot \text{ex}(n, G, K_r) \geq \delta \cdot N(G, S) > N(G, T) \]

as required. \( \square \)

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Figure 2. Graphs $S$ for $r = 4, 7, 11$

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