Outliers of random perturbations of Toeplitz matrices with finite symbols

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Abstract
Consider an $N \times N$ Toeplitz matrix $T_N$ with symbol $a(\lambda) := \sum_{\ell=-d_2}^{d_1} a_{\ell} \lambda^\ell$, perturbed by an additive noise matrix $N^{-\gamma} E_N$, where the entries of $E_N$ are centered i.i.d. random variables of unit variance and $\gamma > 1/2$. It is known that the empirical measure of eigenvalues of the perturbed matrix converges weakly, as $N \to \infty$, to the law of $a(U)$, where $U$ is distributed uniformly on $S^1$. In this paper, we consider the outliers, i.e. eigenvalues that are at a positive ($N$-independent) distance from $a(S^1)$. We prove that there are no outliers outside $\text{spec } T(a)$, the spectrum of the limiting Toeplitz operator, with probability approaching one, as $N \to \infty$. In contrast, in $\text{spec } T(a) \setminus a(S^1)$ the process of outliers converges to the point process described by the zero set of certain random analytic functions. The limiting random analytic functions can be expressed as linear combinations of the determinants of finite sub-matrices of an infinite dimensional matrix, whose entries are i.i.d. having the same law as that of $E_N$. The coefficients in the linear combination depend on the roots of the polynomial $P_{z,a}(\lambda) := (a(\lambda) - z)\lambda^{d_2}$ and semi-standard Young Tableaux with shapes determined by the number of roots of $P_{z,a}(\lambda) = 0$ that are greater than one in moduli.

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1 Introduction

Let \( a : \mathbb{C} \mapsto \mathbb{C} \) be a Laurent polynomial. That is,

\[
a(\lambda) := \sum_{\ell = -d_2}^{d_1} a_\ell \lambda^\ell, \quad \lambda \in \mathbb{C},
\]

for some \( d_1, d_2 \in \mathbb{N} \) and some sequence of complex numbers \( \{a_\ell\}_{\ell = -d_2}^{d_1} \), so that \( d_1 > 0 \). Define \( T(a) : \mathbb{C}^N \mapsto \mathbb{C}^N \) to be the Toeplitz operator with symbol \( a \), that is the operator given by

\[
(T(a)x)_i := \sum_{\ell = -d_2}^{d_1} a_\ell x_{\ell + i}, \quad \text{for } i \in \mathbb{N}, \quad \text{where } x := (x_1, x_2, \ldots) \in \mathbb{C}^N,
\]

and we set \( x_i = 0 \) for non-positive integer values of \( i \). For \( N \in \mathbb{N} \), we denote by \( T_N(a) \) the natural \( N \)-dimensional truncation of the infinite dimensional Toeplitz operator \( T(a) \). As a matrix of dimension \( N \times N \), we have (for \( N > \max(d_1, d_2) \)) that

\[
T_N := T_N(a) := \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & \cdots & 0 \\
\vdots & a_{-1} & a_0 & a_1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & a_{-1} & a_0 & a_1 \\
0 & \cdots & \cdots & a_{-2} & a_{-1} & a_0
\end{bmatrix}.
\]

In general, \( T_N \) is not a normal matrix, and thus its spectrum can be sensitive to small perturbations. In this paper, we will be interested in the spectrum of \( M_N := T_N + \Delta_N \), where \( \Delta_N \) is a “vanishing” random perturbation, and especially in outliers, i.e. eigenvalues that are at positive distance from the limiting spectrum.

Let \( L_N \) denote the empirical measure of eigenvalues \( \{\lambda_i\}_{i=1}^N \) of \( M_N \), i.e. \( L_N := N^{-1} \sum_{i=1}^N \delta_{\lambda_i} \), where \( \delta_x \) is the Dirac measure at \( x \). It has been shown in [3] that under a fairly general condition on the (polynomially vanishing) noise matrix \( \Delta_N \), \( L_N \) converges (weakly, in probability) to \( a_* \text{Unif}(\mathbb{S}^1) \), where \( \text{Unif}(\mathbb{S}^1) \) denotes the Haar measure on \( \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} \). That is, the limit is the law of \( a(U) \) where \( U \sim \text{Unif}(\mathbb{S}^1) \) (see also [19] for the case of Gaussian noise). However, simulations (see Fig. 1) suggest that although the bulk of the eigenvalues approach \( a(\mathbb{S}^1) \), as \( N \to \infty \), there are a few eigenvalues of \( M_N \) that wander around outside a small neighborhood of \( a(\mathbb{S}^1) \). Following standard terminology, we call them outliers. The goal of this paper is to characterize these outliers.

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1 We remark that if one is interested in the case where \( d_1 = 0 \) but \( d_2 > 0 \), one may simply consider, when computing spectra, the transpose of \( T_N \) or of \( M_N = T_N + \Delta_N \). For this reason, the restriction to \( d_1 > 0 \) does not reduce generality.
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Fig. 1 The eigenvalues of $T_N(a) + N^{-\gamma}E_N$, with $N = 4000$, $\gamma = 0.75$, $E_N$ a real Ginibre matrix, and various symbols $a$. On the top left, $a(\lambda) = \lambda$. On the top right, $a(\lambda) = \lambda + \lambda^2$. On the bottom, $a(\lambda) = \lambda^{-1} + 0.5i\lambda$.

In simulations depicted in Fig. 1, all eigenvalues of $M_N = T_N(a) + N^{-\gamma}E_N$, where $\gamma = 0.75$ and $E_N$ is a standard real Ginibre matrix, are inside the unit disk, the limaçon, and the ellipse when the symbol is $a(\lambda) = \lambda, \lambda + \lambda^2,$ and $\lambda^{-1} + 0.5i\lambda,$ respectively. It follows from standard results on the spectrum of Toeplitz operators, e.g. [7, Corollary 1.12] that these regions are precisely $\text{spec } T(a)$, the spectrum of the Toeplitz operator $T(a)$ acting on $L^2(N)$. Thus, Fig. 1 suggests that there are no outliers outside $\text{spec } T(a)$. In our first result, Theorem 1.1, we confirm this and prove the universality of this phenomenon for any finitely banded Toeplitz matrix, $\gamma > \frac{1}{2}$, and under a minimal assumption on the entries of the noise matrix.

We introduce the following standard notation: for $\mathbb{D} \subset \mathbb{C}$ and $\varepsilon > 0$, let $\mathbb{D}^\varepsilon$ be the $\varepsilon$-fattening of $\mathbb{D}$. That is, $\mathbb{D}^\varepsilon := \{z \in \mathbb{C} : \text{dist}(z, \mathbb{D}) \leq \varepsilon\}$, where $\text{dist}(z, \mathbb{D}) := \inf_{z' \in \mathbb{D}} |z - z'|$. Further denote $\mathbb{D}^{-\varepsilon} := ((\mathbb{D}^\varepsilon)^c)^c$.

**Theorem 1.1** Let $a$ be a Laurent polynomial. Let $T(a)$ denote the Toeplitz operator on $L^2(N)$ with symbol $a$, and let $T_N(a)$ be its natural $N$-dimensional projection. Assume $\Delta_N = N^{-\gamma}E_N$ for some $\gamma > \frac{1}{2}$, where the entries of $E_N$ are independent (real
or complex-valued) with zero mean and unit variance. Further, let $L_N$ denote the empirical measure of eigenvalues of $T_N(a) + \Delta_N$. Fix $\varepsilon > 0$. Then,

$$P\left( L_N \left( \left( \text{spec} \ T(a) \right)^c \right)^{-\varepsilon} = 0 \right) \to 1 \quad \text{as } N \to \infty. \quad \text{(1.2)}$$

In the terminology of [19], $\mathbb{C} \setminus \text{spec} \ T(a)$ is a zone of spectral stability for $T_N(a)$. The following remarks discuss some generalizations and extensions of Theorem 1.1.

**Remark 1.2** For clarity of presentation, in Theorem 1.1 we assume the entries of $E_N$ to have a unit variance. The proof shows that the same conclusion continues to hold under the assumption that the entries of $E_N$ are jointly independent (possibly having $N$-dependent distributions), and have zero mean and uniformly bounded second moment, i.e.

$$\mathbb{E}[E_N(i, j)] = 0 \quad \forall \ i, j \in \{1, 2, \ldots, N\} \quad \text{and} \quad \sup_{N \in \mathbb{N}} \max_{i, j=1}^{N} \mathbb{E}[E_N(i, j)^2] < \infty, \quad \text{(1.3)}$$

where $E_N(i, j)$ denotes the $(i, j)$th entry of $E_N$.

We emphasize that under the general assumption (1.3) on the entries of $E_N$, one may not have the convergence of the empirical measure of the eigenvalues of $T_N(a) + N^{-\gamma} E_N$ to $a(U)$. Theorem 1.1 shows that even under such perturbations there are no eigenvalues in the complement of $\text{spec} \ T(a)$.

**Remark 1.3** The proof of Theorem 1.1 shows that its conclusion continues to hold if $\Delta_N = a_N E_N$, with any sequence $\{a_N\}_{N \in \mathbb{N}}$ such that $a_N = o(N^{-1/2})$ (the standard notation $a_N = o(b_N)$ means that $\lim_{N \to \infty} a_N/b_N = 0$). For conciseness, we only consider $a_N = N^{-\gamma}$.

**Remark 1.4** Theorem 1.1 shows that, with probability approaching one, all eigenvalues of the random perturbation of $T_N(a)$ are contained in an $\varepsilon$-fattening of the spectrum of the infinite dimensional Toeplitz operator $T(a)$. Here we have chosen to work with a fixed parameter $\varepsilon > 0$. With some additional efforts it can be shown that in (1.2) one can allow $\varepsilon = \varepsilon_N$ to decay to zero slowly with $N$. We do not pursue this direction.

**Remark 1.5** The ideas used to prove Theorem 1.1 also show that the sequence $\{\varrho_N\}_{N \in \mathbb{N}}$ is tight, where $\varrho_N$ is the spectral radius (maximum modulus eigenvalue) of $E_N/\sqrt{N}$, with $E_N$ as in Theorem 1.1. See Proposition A.1. It has been conjectured in [6] that the spectral radius of a matrix with i.i.d. entries of zero mean and variance $1/N$ converges to one, in probability. Thus Proposition A.1 proves a weaker form of this conjecture.

**Remark 1.6** In [19, Proposition 3.13], the authors show that the resolvent of $T_N(a)$ remains bounded in compact subsets of $(\text{spec} \ T(a))^c$. As noted in [19], this implies Theorem 1.1 in the Gaussian case because in that case, the operator norm of $N^{-\gamma} E_N$ is bounded with high probability. For more general perturbations possessing only four or less moments, the operator norm of $N^{-\gamma} E_N$ is in general not bounded, for some $\gamma \in (1/2, 1)$, and a similar argument fails.
We turn to the identification of the limiting law of the random point process consisting of the outliers of $M_N$, which by Theorem 1.1 are contained in $\text{spec } T(a)$. Before stating the results, we review standard definitions of random point processes and their notion of convergence, taken from [10].

For $\mathbb{D} \subset \mathbb{C}$ we let $\mathcal{B}(\mathbb{D})$ denote the Borel $\sigma$-algebra on it. Recall that a Radon measure on $(\mathbb{D}, \mathcal{B}(\mathbb{D}))$ is a measure that is finite for all Borel compact subsets of $\mathbb{D}$.

**Definition 1.1** A random point process $\zeta$ on an open set $\mathbb{D} \subset \mathbb{C}$ is a probability measure on the space of all non-negative integer valued Radon measures on $(\mathbb{D}, \mathcal{B}(\mathbb{D}))$. Given a sequence of random point processes $\{\zeta_n\}_{n \in \mathbb{N}}$ on $(\mathbb{D}, \mathcal{B}(\mathbb{D}))$, we say that $\zeta_n$ converges weakly to a (possibly random) point process $\zeta$ on the same space, and write $\zeta_n \Rightarrow \zeta$, if for all compactly supported bounded real-valued continuous functions $f$,

$$
\int f(z)d\zeta_n(z) \rightarrow \int f(z)d\zeta(z), \quad \text{in distribution, as } n \rightarrow \infty,
$$

when viewed as real-valued Borel measurable random variables.

Next we proceed to describe the limit. We will see below that the limit is given by the zero set a random analytic function, where the description of the limiting random analytic function differs across various regions in the complex plane. This necessitates the following definition.

**Definition 1.2** (**Description of regions**) For any Laurent polynomial $a$ as in (1.1), set $P_{z,a}(\lambda) := \lambda^{d_2} (a(\lambda) - z)$. Writing $d := d_1 + d_2$, let $\{-\lambda_{\ell}(z)\}_{\ell=1}^d$ be the roots of the equation $P_{z,a}(\lambda) = 0$ arranged in an non-increasing order of their moduli. For $d$ an integer such that $-d_2 \leq d \leq d_1$, set

$$
S_d := \{z \in \mathbb{C} \setminus a(S^1) : d_0(z) = d_1 - d, \quad \text{where } d_0(z) \text{ such that } |\lambda_{d_0(z)}(z)| > 1 > |\lambda_{d_0(z)+1}(z)|\},
$$

where for convenience we set $\lambda_{d+1}(z) = 0$ and $\lambda_0(\zeta) = \infty$ for all $z \in \mathbb{C}$.

Note that for $z \in \mathbb{C} \setminus a(S^1)$ all roots of the polynomial $P_{z,a}(\lambda) = 0$ have moduli different from one. Therefore for such values of $z$, $d_0(z)$ is well defined, and hence so is $S_d$.

By construction, $\bigcup_{d=-d_2}^{d_1} S_d = \mathbb{C} \setminus a(S^1)$. Since, by [3], the bulk of the eigenvalues of $M_N$ approaches $a(S^1)$ in the large $N$ limit, to study the outliers we only need to analyze the roots of $\text{det}(T_N(a) + \Delta N - z \text{ Id}_N) = 0$ that are in $\bigcup_{d=-d_2}^{d_1} S_d$.

Before describing the limiting random field let us mention some relevant properties of the regions $\{S_d\}_{d=-d_2}^{d_1}$. As $a(\cdot)$ is a Laurent polynomial satisfying (1.1), it is straightforward to check that for $z \in S_d$ we have $\text{wind}_z(a) = d$, where $\text{wind}_z(a)$ denotes the winding number about $z$ of the closed curve induced by the map $\lambda \mapsto a(\lambda)$ for $\lambda \in S^1$. Thus $\{S_d\}_{d=-d_2}^{d_1}$ splits the complement of $a(S^1)$ according to the winding number. As will be seen later, the description of the law of the limiting random point process differs across the regions $\{S_d\}_{d=-d_2}^{d_1}$. 
Furthermore, from [7, Corollary 1.12] we have that
\[ \text{spec } T(a) = a(S^1) \cup \{ z \in \mathbb{C} \setminus a(S^1) : \text{wind}_{z}(a) \neq 0 \}. \]

It was noted above that wind\(_{z}(a) = 0\) for \( z \in S_0 \). So
\[ S_0 = (\text{spec } T(a))^c. \]  

(1.4)

Hence in light of Theorem 1.1 we conclude that to find the limiting law of the outliers it suffices to analyze the eigenvalues of \( T_N(a) + \Delta_N \) that are in \( \bigcup_{d \neq 0} S_0 \).

Finally, we note that from the continuity of the roots of \( P_{z,a}(\lambda) = 0 \) in \( z \) in the symmetric product topology (see [26, Appendix 5, Theorem 4A]) it follows that the regions \( \{ S_0 \}_{d_{1} = -d_{2}} \) are open, and hence by Definition 1.1 the random point processes on those regions are well defined.

**Remark 1.7** We highlight that one or more of the regions \( \{ S_0 \}_{d_{1} = -d_{2}} \) may be empty. For example, when \( a(\lambda) = \lambda^{-1} + 0.5i\lambda \) the product of the moduli of the roots of \( P_{z,a}(\lambda) = 0 \) is \( 1/0.5 = 2 \). So both roots of \( P_{z,a}(\lambda) = 0 \) cannot be less than one in moduli. Therefore \( S_1 = \emptyset \) in this case. It can be checked that under this same set-up \( S_0 \) and \( S_{-1} \) are the outside and the inside of the ellipse, respectively, in the bottom panel of Fig. 1. Furthermore, if \( a(\lambda) = a_{1}\lambda + a_{-1}\lambda^{-1} \) with \( |a_{1}| = |a_{-1}| \) then both \( S_1 \) and \( S_{-1} \) are empty.

Fix an integer \( d \neq 0 \) such that \( -d_{2} \leq d \leq d_{1} \). As mentioned above, the limiting random point process in \( S_0 \) will be given by the zero set of a random analytic function \( \mathcal{P}_{d}^{\infty}(\cdot) \) to be defined on \( S_0 \). The function \( \mathcal{P}_{d}^{\infty}(\cdot) \) can be written as a linear combination of determinants of \( |d| \times |d| \) sub-matrices of the noise matrix, where the coefficients depend on the roots of the polynomial \( P_{z,a}(\lambda) = 0 \) and semistandard Young Tableaux of some given shapes with certain restrictions on its entries. We recall the definition of semistandard Young Tableaux [21, Section 7.10].

**Definition 1.3** (Semistandard Young Tableaux) Fix \( k \in \mathbb{N} \). A partition \( \mu := (\mu_1, \mu_2, \ldots, \mu_k) \) with \( k \) parts is a collection of non-increasing non-negative integers \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0 \). Given a partition \( \mu \), a semistandard Young Tableaux of shape \( \mu \) is an array \( \tau := (\tau_{i,j}) \) of positive integers of shape \( \mu \) (i.e. \( 1 \leq i \leq k \) and \( 1 \leq j \leq \mu_i \)) that is weakly increasing (i.e. non-decreasing) in every row and strictly increasing in every column.

The limiting random field depends on the following subset of the set of all semistandard Young Tableaux, for which we have not found a standard terminology in the literature.

**Definition 1.4** (Field Tableaux) Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) be a partition with \( k \geq d_2 \) and \( \mu_k > 0 \). For an integer \( d \neq 0 \) such that \( -d_2 \leq d \leq d_1 \), let \( \mathcal{L}(\mu, d) \) denote the collection of all semistandard Young Tableaux \( \tau \) of shape \( \mu \) that are strictly increasing along the southwest diagonals and satisfies the assumption \( \tau_{i,1} = i \) for all \( i \in [d_2] := \{1, 2, \ldots, d_2\} \). See Fig. 2 for a pictorial illustration.
Equipped with the relevant notion of semistandard Young Tableaux we now turn to define the coefficients that appear when the limiting random analytic function is expressed as a linear sum of determinants of \(|\mathfrak{d}| \times |\mathfrak{d}|\) sub-matrices of the noise matrix.

**Definition 1.5** (Field notation) Fix \(\mathfrak{d} \neq 0\) an integer such that \(-d_2 \leq \mathfrak{d} \leq d_1\). Set \(d_0 := d_1 - \mathfrak{d}\). For any finite set \(\hat{\mathcal{X}} := \{x_1 < x_2 < \cdots < x_\ell\}\), we define \(\text{sgn}(\hat{\mathcal{X}})\) to be the sign of the permutation which places all elements of \(\hat{\mathcal{X}}\) before those in \([x_1, x_1 + 1, x_1 + 2, \ldots, x_\ell])\) \(\setminus \hat{\mathcal{X}}\) but preserves the order of the elements within the two sets.

**The case \(\mathfrak{d} > 0\)**. Denote \(\mathcal{L}_1(\mathfrak{d}) := \mathcal{L}(\mu_1, \mathfrak{d})\) and \(\mathcal{L}_2(\mathfrak{d}) := \mathcal{L}(\mu_2, \mathfrak{d})\), where

\[
\mu_1 := (d, d - 1, \ldots, d_0 + 1) \quad \text{and} \quad \mu_2 := (d - d_0, d - d_0 - 1, \ldots, 1),
\]

see Definition 1.4. Given any \(\hat{x} \in \mathcal{L}_1(\mathfrak{d})\) and \(\eta \in \mathcal{L}_2(\mathfrak{d})\), define

\[
\hat{\mathcal{X}} := \hat{\mathcal{X}}(\hat{x}, \eta) := \{\hat{x}_i, 1 \in [\mathfrak{d} + d_2][[d_2]]\}, \quad \hat{\mathcal{Y}} := \hat{\mathcal{Y}}(\hat{x}, \eta) := \{\eta_i, 1 \in [\mathfrak{d} + d_2][[d_2]]\}.
\]

Next define

\[
c(\hat{x}, \eta) := c(\hat{x}, \eta, z) := \prod_{i=1}^{d_0} \lambda_i(z)^{-c_i(\hat{x}, \eta)} \cdot \prod_{i=d_0+1}^{d} \lambda_i(z)^{c_i(\hat{x}, \eta)},
\]

where

\[
c_i(\hat{x}, \eta) := \begin{cases} 
\sum_{j=1}^{d-d_0} (\hat{x}_{j,i+1} - \hat{x}_{j,i} + 1) & \text{for } i \in [d_0] \\
\eta_{1,d+1-i} + \sum_{j=2}^{i-d_0} \eta_{j,d+1-i} - \eta_{j-1,d+2-i} & + \sum_{j=2}^{d+1-i} (\hat{x}_{j,i} - \hat{x}_{j-1,i+1}) & \text{for } i \in [d] \setminus [d_0]
\end{cases},
\]

and \([\lambda_i(z)]_{i=1}^{d}\) are as in Definition 1.2.

**The case \(\mathfrak{d} < 0\)**. Define \(\mathcal{L}_1(\mathfrak{d}) := \mathcal{L}(\mu_1, \mathfrak{d})\) and \(\mathcal{L}_2(\mathfrak{d}) := \mathcal{L}(\mu_2, \mathfrak{d})\), where now

\[
\mu_1 := (\underbrace{d + 1, d + 1, \ldots, d + 1}_\mathfrak{d}, d, d - 1, \ldots, d_0 + 1) \quad \text{and} \quad \\
\mu_2 := (\underbrace{d + 1, d + 1, \ldots, d + 1}_\mathfrak{d}, d_0, d_0 + \mathfrak{d}, d_0 + \mathfrak{d} - 1, \ldots, 1),
\]

respectively. In the special case \(\mathfrak{d} = -d_2\) the definitions of \(\mu_1\) and \(\mu_2\) simplify to

\[
\mu_1 = \mu_2 = (d + 1, d + 1, \ldots, d + 1).
\]

Given any \(\hat{x} \in \mathcal{L}_1(\mathfrak{d})\) and \(\eta \in \mathcal{L}_2(\mathfrak{d})\) further denote

\[
\hat{\mathcal{X}} := \hat{\mathcal{X}}(\hat{x}, \eta) := \{\hat{x}_{i,d+1} \mid i \in [-\mathfrak{d}]\}, \quad \hat{\mathcal{Y}} := \hat{\mathcal{Y}}(\hat{x}, \eta) := \{\eta_{i,d+1} \mid i \in [-\mathfrak{d}]\}.
\]
Definition 1.6. Let $L = 15$. Top and bottom rows are $\emptyset = 1$ and $\emptyset = -1$, respectively. For the top row

and

$$c(x, \eta) := c(x, \eta, z) := \prod_{i=1}^{d_0} \lambda_i(z)^{-c_i(x, \eta)} \cdot \prod_{i=d_0+1}^{d} \lambda_i(z)^{c_i(x, \eta)},$$

where now

$$c_i(x, \eta) := \begin{cases} \frac{2}{d_0} (f_i, i+1 - f_{i, i} + 1) + \sum_{j=1}^{d_0} (f_{j, j+2-i} - f_{j, j+1-i} + 1) & \text{for } i \in [d_0] \\ f_{i-1, i} + f_{i, i} - 1 - d_2 + \emptyset & \\
+ \sum_{j=2}^{i-1} (f_{j, j+1-i} - f_{j+1, j+2-i}) + \sum_{j=2}^{d-d_2-i} (f_{i, j} - f_{i, j+1}) & \text{for } i \in [d] \setminus [d_0] \end{cases}.$$ 

For all values of $\emptyset \neq 0$, set $\lambda(x, \eta) := \text{sgn}(\hat{X}) \cdot \text{sgn}(\hat{Y}).$ 

Figure 2 gives a pictorial illustration of the definition.

Having defined all necessary ingredients we now introduce the limiting random analytic function $\mathcal{P}_0^\infty(\cdot)$.

Definition 1.6 (Description of the random fields) Let $E_\infty$ denote a semi-infinite array of i.i.d. random variables $\{e_{i,j}\}, i, j \in \mathbb{N}$ with zero mean and unit variance. For $\mathcal{X}, \mathcal{Y} \subset \mathbb{N}$, let $E_\infty[\mathcal{X}; \mathcal{Y}]$ denote the sub-matrix of $E_\infty$ induced by the rows and the columns indexed by $\mathcal{X}$ and $\mathcal{Y}$, respectively. With notation for $c(x, \eta), \lambda(x, \eta), \hat{X}$ and $\hat{Y}$ as in Definition 1.5, we set, for $z \in S_0$ and $L \in \mathbb{N} \cup \{\infty\}$,

$$\mathcal{P}_0^L(z) := \sum_{x \in \mathcal{L}_1(\emptyset)} \sum_{\eta \in \mathcal{L}_2(\emptyset)} c(x, \eta) \cdot 1_{\{\max_j c_j(x, \eta) \leq L\}} \cdot (-1)^{\lambda(x, \eta)} \det(E_\infty[\hat{X}; \hat{Y}]). \quad (1.6)$$

It may not be apriori obvious from Definition 1.6 that $\mathcal{P}_0^\infty(\cdot)$ is well defined, as (1.6) is an infinite sum. Lemma 1.9 will establish that it can be expressed as the local uniform limit of the random analytic functions $\{\mathcal{P}_0^L(\cdot)\}_{L \in \mathbb{N}}$ and thus it is indeed a well defined random analytic function. In addition, under an appropriate anti-concentration property of the entries of $E_N$ and $E_\infty$, the random analytic function $\mathcal{P}_0^\infty(\cdot)$ is not
identically zero on a set of probability one, and thus the random point process induced by the zero set of it is a valid random Radon measure.

To describe the required anti-concentration property, we recall Lévy’s concentration function, defined for any (possibly complex-valued) random variable $X$ by

$$L(X, \varepsilon) := \sup_{w \in \mathbb{C}} \mathbb{P}(|X - w| \leq \varepsilon).$$

Equipped with the above definition we now state the additional assumption on the entries of $E_N$ and $E_\infty$.

**Assumption 1.8 (Assumption on the entries of the noise matrix)** Assume that the entries of $E_N$ and $E_\infty$ are either real-valued or complex-valued i.i.d. random variables with zero mean and unit variance, so that, for some absolute constants $\eta \in (0, 2]$ and $C < \infty$,

$$L(e_{1,1}, \varepsilon) \leq C \varepsilon \eta, \quad (1.7)$$

for all sufficiently small $\varepsilon > 0$, where $e_{1,1}$ is the first diagonal entry of $E_N$.

Note that any random variable having a bounded density with respect to the Lebesgue measure on the real line or the complex plane satisfies the bound (1.7) with $\eta = 1$ and 2, respectively. This in particular includes the standard real and complex Gaussian random variable.

Recall that a sequence of complex-valued functions $\{f_L\}_{L \in \mathbb{N}}$, defined on some open set $D \subset \mathbb{C}$, is said to converge locally uniformly to a function $f : D \mapsto \mathbb{C}$, if given any $z \in D$ there exists some open ball $D_z \subset D$ containing $z$ such that $f_L$ converges to $f$ uniformly on $D_z$, as $L \to \infty$. We now have the following.

**Lemma 1.9 (Description of the limit)** Let $d, \{P_L^d\}_{L \in \mathbb{N} \cup \{\infty\}}$, and $E_\infty$ be as in Definition 1.6. We let $\zeta^0_\infty$ be the random point process induced by the zero set of the random field $\{P^\infty_0(z)\}_{z \in S_0}$. That is,

$$\zeta^0_\infty := \sum_{z \in S_0 : P^\infty_0(z) = 0} \delta_z. \quad (1.8)$$

Then we have the following:

(i) The functions $\{P^L_0\}_{L \in \mathbb{N}}$ are random analytic functions.

(ii) The random function $\mathcal{P}_0^\infty$ is well defined on a set of probability one. Furthermore, $\mathcal{P}_0^L$ converges locally uniformly to $\mathcal{P}_0^\infty$, almost surely, as $L \to \infty$, and hence $\mathcal{P}_0^\infty$ is a random analytic function.

(iii) Under the additional Assumption 1.8, the random function $\mathcal{P}_0^\infty$ is almost surely non constant.

Recalling Definition 1.1 we note that the notion of convergence of random point processes defined on $S_0$ is tested against continuous functions supported on compact subsets of $S_0$. Therefore when discussing convergence it is enough to consider continuous functions on sets $S_0^{\infty - \varepsilon}$ for arbitrary $\varepsilon > 0$. 
Remark 1.10 We emphasize that the random point process $\zeta^0_\infty$, although free of the parameter $\gamma$, is not universal. That is, in general its law depends on the law of the entries of the matrix $E_\infty$.

The main result of this paper shows that given an integer $d \neq 0$ such that $-d_2 \leq d \leq d_1$ and $\gamma > \frac{1}{2}$, the random point process induced by the eigenvalues of $T_N(a) + N^{-\gamma}E_N$ that are in $S_d$ converges weakly to the random point process $\zeta^0_\infty$ induced by the zero set of the random analytic function $P^0_\infty$.

Theorem 1.11 Let $a$ be a Laurent polynomial as in (1.1). Let $T(a)$ denote the Toeplitz operator on $L^2(\mathbb{N})$ with symbol $a$, and let $T_N(a)$ be its natural $N$-dimensional projection. Assume $\Delta_N = N^{-\gamma}E_N$ for some $\gamma > \frac{1}{2}$, where the entries of $E_N$ are i.i.d. satisfying Assumption 1.8. Furthermore assume that the entries of $E_\infty$ in Definition 1.6 are i.i.d. of the same law as that of the entries of $E_N$. For any integer $d \neq 0$ such that $-d_2 \leq d \leq d_1$, let $\Xi_N^d$ denote the random point process induced by the eigenvalues of $T_N(a) + \Delta_N$ that are in $S_d$. That is,

$$\Xi_N^d := \sum_{z \in S_d: \det(T_N(a) + \Delta_N - z \text{Id}_N) = 0} \delta_z.$$  

Then, for such $d$, $\Xi_N^d$ converges weakly, as $N \to \infty$, to the random point process $\zeta^0_\infty$ from Lemma 1.9.

Explicit expressions for the fields in the statement of Theorem 1.11 for two of the symbols depicted in Fig. 1 appear in Sect. 1.1, see (1.9) and Remark 1.13.

At a first glance, it may seem counter intuitive for the limit to be expressed as the zero set of certain random analytic function of the form (1.6). To see that it is in fact natural, we note that the determinants of $T_N(a(z)) + N^{-\gamma}E_N$ can be expressed as a linear combination of products of determinants of sub-matrices of $T_N(a(z))$ and of $E_N$, and further that the determinants of (some) sub-matrices of a finitely banded Toeplitz matrix can be expressed as certain skew-Schur polynomials in $\{\lambda_i(z)\}_{i=1}^d$ (see [1,9]), where these polynomials are defined as a sum of monomials with the sum taken over (skew) semistandard Young Tableaux of some given shapes. This leads to (1.6).

Remark 1.12 As before, when discussing convergence it is enough to consider functions supported on the sets $S_d^{\varepsilon}$ for arbitrary $\varepsilon > 0$. Similar to Remark 1.4, one can allow in Theorem 1.11 $\varepsilon = \varepsilon_N$ to go to zero, as $N \to \infty$, sufficiently slowly, and consider functions supported on $S_d^{\varepsilon}$ as test functions. We do not work out the details here.

1.1 Background, related results, and extensions

The fact that the spectrum of non-normal matrices and operators is not stable with respect to perturbations is well known, see e.g. [22] for a comprehensive account and [11] for a recent study. Extensive work has been done concerning worst case perturbations, which are captured through the notion of pseudospectrum. However, beyond some specific examples the pseudospectra of non-normal operators are not well understood.

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characterized. Therefore, there has been recent interest in studying the spectrum of non-normal operators and matrices under small typical perturbations. See the references in [16, Section 1]. We also refer to [2, Section 1.3] for a discussion about the relation between the pseudospectrum and the spectrum under typical perturbation, and an extensive reference list. We add that early examples of the spectrum obtained by noisily perturbing Toeplitz matrices with finite symbols appeared in [23].

As mentioned above, the convergence of the empirical measure of eigenvalues for randomly perturbed finite-symbol Toeplitz matrices has now been established in great generality, see the recent articles [3, 19] and references therein. Our focus in this paper concerns the study of outliers. In Theorem 1.1 we identify the region where no outliers are present (in the terminology of Sjöstrand and Vogel [19], this is the zone of spectral stability). Then, in Theorem 1.11 we find the limit of the random processes induced by the outliers in the interior of the complement of the region identified in Theorem 1.1.

For the Jordan matrix, i.e. the Toeplitz matrix with symbol \( a(\lambda) = \lambda \), [11, Theorem 2] shows that there are no outliers outside the unit disc (centered at zero) in the complex plane, with high probability. In the general Toeplitz case, Theorem 1.1 follows (for Gaussian perturbations) from the resolvent estimates in [19, Proposition 3.13], see Remark 1.6.

Some bounds on the number of outliers inside \( \text{spec } T(a) \) are available in the literature. In the notation of the current paper, for the Jordan matrix perturbed by additive complex Gaussian noise, with \( \gamma > 3 \), a logarithmic in \( N \) bound for the number of outliers appears in [11]. Similar results (with worse error bounds) are given in [16] for non-triangular tridiagonal Toeplitz matrices (i.e. the symbol is \( a(\lambda) = a_1 \lambda + a_{-1} \lambda^{-1} \)), and in [19] for general Toeplitz matrices with finite symbol.

Sharper results concerning outliers for the Jordan matrix and the non-triangular tridiagonal Toeplitz matrix (under complex Gaussian perturbation), are presented in [17, 18]. In both these cases, a sharp \( O(1) \) control on the number of outliers in the regions \( \mathcal{S}_d \) with \( d \neq 0 \) is provided. In the language of the current paper, the authors compute the first intensity measure of the limiting field \( \xi^\mathcal{S}_d \), that is, they compute the function \( \rho_0(\mathcal{D}) = \mathbb{E}[\xi^\mathcal{S}_d(\mathcal{D})] \) for subsets \( \mathcal{D} \subset \mathcal{S}_d \).

For the Jordan matrix, it has been shown in [18, Theorem 1.1] that \( \rho_1(\cdot) \) has a density with respect to the two dimensional Lebesgue measure, given by

\[
\rho_1(dz) := \frac{2}{\pi(1-|z|^2)^2} 1_{|z|<1} dz.
\]

Due to the Edelman–Kostlan formula (see [12, Theorem 3.1]), \( \rho_1(\cdot) \) is the first intensity measure of the random point process induced by the zero set of the hyperbolic Gaussian analytic function (see [13, Chapter 2.3] for a definition), given by

\[
\widehat{\chi}(z) := \sum_{k=0}^{\infty} z^k g_k \sqrt{k+1},
\]

where \( \{g_k\}_{k \in \mathbb{N}} \) are i.i.d. standard complex Gaussian random variables. We now explain how to recover this result from Theorem 1.11: in the case of the Jordan matrix, \( a(\lambda) = \lambda \).
and then \( \lambda_1(z) = z, \vartheta = 1 \) and \( d_0 = 0 \). Substituting in Definition 1.5, one finds that
\[
c(z, \eta) = z^{c_1(z, \eta)}, \quad c_1(z, \eta) = \eta_{1,1} + \xi_{1,1} - 2,
\]
where \( \xi_{1,1}, \eta_{1,1} \) are arbitrary positive integers. In particular, there are precisely \( k + 1 \) choices of such integers that give \( c_1(z, \eta) = k \). Since the entries of \( E_\infty \) are i.i.d. Gaussian, it follows from (1.6) that \( \mathcal{P}_0^\infty(z) \) coincides with (1.9). Together with the Edelman–Kostlan formula, this recovers [18, Theorem 1.1]. Note that the same expression (1.9) also holds for real Gaussian noise, with the limiting field having now real Gaussian coefficients.

**Remark 1.13** Theorem 1.11 allows one to describe all limiting outlier processes for the other cases depicted in Fig. 1. We give one more example, for the symbol \( a(\lambda) = \lambda + \lambda^2 \) (the “limaçon”), with Gaussian noise matrix \( E_N \) consisting of i.i.d. centered entries. There, we have that \( \lambda_i(z) = (\pm \sqrt{1 + 4z} - 1)/2 \), arranged so that \( |\lambda_1(z)| \geq |\lambda_2(z)| \). In the region \( S_1 \), which corresponds to the region enscribed by the limaçon curve with winding number 1, the limiting field is
\[
\sum_{k=0}^{\infty} \lambda_2(z)^k g_k \sqrt{k + 1}, \tag{1.10}
\]
where \( g_k \) are i.i.d. centered Gaussian, compare with (1.9). On the other hand, in the regions \( S_2 \) (which corresponds to the region of winding number 2, i.e. the smaller region in Fig. 1), the limiting field admits a more complicated description, as follows. Let \( \{g_{ij}\} \) be i.i.d. Gaussians, and for \( i, j, k, l \) integers satisfying \( i < j \) and \( k < l \), introduce the random variables
\[
X_{ijkl} := g_{ik} g_{jl} - g_{il} g_{jk}
\]
and the functions
\[
W_{ijkl}(z) := \sum_{s, t : i \leq s < j, k \leq t < l} \left( \frac{\lambda_2(z)}{\lambda_1(z)} \right)^{s-t}.
\]
Then,
\[
\mathcal{P}_0^\infty(z) = \sum_{1 \leq i < j, 1 \leq k < l} \lambda_1(z)^{i+j-3} \lambda_2(z)^{k+l-3} (-1)^{(j-i-1)(l-k-1)} W_{ijkl}(z) X_{ijkl}. \tag{1.11}
\]
In particular, the random coefficients in (1.11) are in general not Gaussian, and there are terms in the sum in (1.11) with non-trivial correlations.

Sjöstrand and Vogel in [17] compute \( \rho_0 \) for the non-triangular tridiagonal Toeplitz matrix. Again by the Edelman–Kostlan formula, they identify \( \rho_0 \) with the first intensity measure of the random point process induced by the zero set of a Gaussian analytic function with some covariance kernel \( \mathbb{K}_0(\cdot, \cdot) \). Our Theorem 1.11, when applied to...
non-triangular tridiagonal Toeplitz matrix, again shows that under complex Gaussian perturbation the limiting random fields are the zero sets of Gaussian analytic functions, and a computation (which we omit) shows that its covariance kernel is given by $K_0(\cdot, \cdot)$.

Thus, Theorem 1.11 again recovers the results of [17].

We also mention the relevant work [14], where local statistics for the eigenvalues of random perturbations of certain pseudo-differential operators are computed and related to local statistics of the zeros of random Gaussian analytic functions.

Based on [14,17,18] one may be tempted to predict that for general finitely banded Toeplitz matrices the limiting random field is the zero set of some Gaussian analytic function. Theorem 1.11 shows that, even under complex Gaussian perturbations, the limit may not be the zero set of Gaussian analytic functions, e.g. consider $a(\lambda) = \lambda + \lambda^2$ and the limit of the random point process induced by the outlier eigenvalues in $S_2$. Furthermore, even in the framework of [17,18], under general perturbation, as already mentioned in Remark 1.10, the limit turns out to be non-universal.

The work of Śniady [20] considers situations where the additive noise is Gaussian of standard deviation $\sigma N^{-1/2}$, and deals with the limit where first $N \to \infty$ and then $\sigma \to 0$. Some of the subsequent work, reviewed e.g. [2, Section 1.4], can be seen as an attempt to modify the order of limits. In this direction and concerning outliers, Bordenave and Capitaine [5] study outliers of deformed i.i.d. random matrices. Namely, for a sequence of deterministic matrices $\{A_N\}_{N \in \mathbb{N}}$ they study the outlier eigenvalues of $M_N^\sigma := A_N + \sigma \frac{1}{\sqrt{N}} E_N$, where the entries of $E_N$ are i.i.d. complex-valued random variables satisfying some assumptions on its moments, and $\sigma > 0$ is a parameter. When $A_N$ is the Jordan matrix, in [5, Corollary 1.10] it is shown that for any $\sigma > 0$ the random point process induced by the outlier eigenvalues of $M_N^\sigma$ converges to the zero set of a Gaussian analytic function with some covariance kernel $K_\sigma(\cdot, \cdot)$. They also noted that, as $\sigma \to 0$, the kernel $K_\sigma(\cdot, \cdot)$ admits a non-trivial limit and the limiting kernel turns out to be the covariance kernel of the hyperbolic Gaussian analytic function given by (1.9). It is striking to see that for the complex Gaussian perturbation of the Jordan matrix the same limit appears in these two rather different frameworks: in [5] $N \to \infty$ is followed by $\sigma \downarrow 0$, whereas in this paper $\sigma^{-1}$ and $N$ are sent to infinity together with $\sigma = N^{-\delta}$ for some $\delta > 0$. However, it should also be noted that, unlike [5], here the limit is non-universal. Based on this observation, we predict that the same phenomenon should continue to hold for general finitely banded Toeplitz matrices.

Next, we discuss possible extensions of our results. A first obvious direction is to consider in Theorem 1.11 the case of $E_N$ without the density assumption. Many steps of the proof go through, except for anti-concentration results of the type discussed in Sect. 4. As will be explained in Sect. 1.2, in Sect. 4 we derive anti-concentration bounds for linear combinations of determinants of sub-matrices of $E_N$. To obtain such a bound we use that there is at least one term in the linear sum with a large coefficient.

We conjecture that it should be possible to dispense of any density assumption on the entries of the noise matrix and the conclusion of Theorem 1.11 should continue to hold under minimal assumptions on the entries $E_N$, e.g. i.i.d. with zero mean and unit variance. At the level of convergence of empirical measures, this has been verified, first in [25] and then in [3]. The non-universality of the limit process for outliers, see Remark 1.10, complicates however the task of proving this.
The next section outlines the proofs of Theorems 1.1 and 1.11.

1.2 Outline of the proof

We remind the reader that the bulk of the eigenvalues of $T_N(a) + \Delta_N$ approach the curve $a(S^{d})$, as $N \to \infty$. Thus, to study the outlier eigenvalues we need to analyze the set $\{z \in \bigcup_{d} S_{\delta} : \det(T_N(a(z)) + \Delta_N) = 0\}$, where for brevity, hereafter we denote $a(z) := a(\cdot) - z$ and recall the definition of $S_{\delta}$ from Definition 1.2.

To this end, a key observation is that for $z \in S_{\delta}$ the dominant term in the expansion of $\det(T_N(a(z)) + \Delta_N)$ is $P_{|z|}(z)$, where for $k \in [N]$, $P_k(z)$ is the homogeneous polynomial of degree $k$ in the entries of the noise matrix $\Delta_N$ in the expansion of the determinant (see (2.1) for a precise formulation, and (2.2) for a decomposition of the determinant in terms of these polynomials). It suggests that, the roots of $\det(T_N(a(z)) + \Delta_N) = 0$ that are in $S_{\delta}$ should be close to those of $P_{|z|}(z) = 0$. This, in turn, indicates that the limit of the random point process induced by the roots of $\det(T_N(a(z)) + \Delta_N) = 0$ that are in $S_{\delta}$ should be the same for the equation $P_{|z|}(z) = 0$. The proof then boils down to identifying the limit induced by the roots of $P_{|z|}(z) = 0$ that are in $S_{\delta}$. The goal of this paper is to make these heuristics precise, leading to the conclusions of Theorems 1.1 and 1.11.

The heuristics described above can be mathematically formulated as below. We fix $\varepsilon > 0$, and consider the region $S_{0}^{-\varepsilon}$. From [3, Lemma A.1] it follows that the determinant of $T_N(a(z)) + \Delta_N$ can be written as a sum of $P_k(z)$, where $k$ runs from zero to $N$. From [3, Lemma A.3], after some preprocessing, it follows that $P_{|z|}(z)$ is a polynomial in $[\lambda_i(z)]_{i=1}^{d}$ such that it is of degree $N$ in each variable [see (2.17)], where we remind the reader that $[-\lambda_i(z)]_{i=1}^{d}$ are the roots of the polynomial $P_{|z|}(\lambda) = 0$, see Definition 1.2. Since for $z \in S_{\delta}$ we have $|\lambda_1(z)| \geq \cdots \geq |\lambda_{d_1-\delta}(z)| > 1 > |\lambda_{d_1-d_1+1}(z)| \geq \cdots \geq |\lambda_{d}(z)|$, it is natural to believe that for large $L \in \mathbb{N}$ the roots of $P_{|z|}(z) = 0$ and of $P_{|z|}^{L}(z) = 0$ should be close to each other, where $P_{|z|}^{L}(z)$ is obtained from $P_{|z|}(z)$ by removing terms having exponents of $[\lambda_i(z)]_{i=1}^{d_1-\delta}$ and $[\lambda_i(z)]_{i=d_1-d_1+1}^{d}$ that are less than $N - O(L)$, and greater than $O(L)$, respectively. Indeed, we show in Sect. 2 that the errors made by replacing the determinant of $T_N(a(z)) + \Delta_N$ (which is an analytic function) by $P_{|z|}^{L}(z)$ are small (in the sense that the supremum of the modulus of the difference over $S_{0}^{-\varepsilon}$, properly normalized, has small second moment when $N \to \infty$ followed by $L \to \infty$). We also prove that $z \mapsto P_{|z|}^{L}(z)$ are analytic functions in $S_{0}^{-\varepsilon}$, see Lemma 5.1.

The advantage of working with (the normalized form of) $P_{|z|}^{L}(z)$ is that, for fixed $L$, it has a law independent of $N$. This fact is a consequence of a combinatorial analysis, presented in Sect. 3. The upshot is that $P_{|z|}^{L}(z)$, properly normalized, can be replaced by the $N$-independent analytic fields $\mathfrak{P}_{0}^{L}$, and these fields in turn possess a well defined analytic limit $\mathfrak{P}_{0}^{\infty}$ as $L \to \infty$.

In order to pass from convergence of the fields to convergence of the process of zeros, we employ a general criteria formulated in [15]. Namely, we need to check that the limit field $\mathfrak{P}_{0}^{\infty}$ is non degenerate, i.e. does not vanish identically. Since $\mathfrak{P}_{0}^{\infty}$ was obtained as a limit, it suffices to check that the pre-limit possesses good enough anti-
concentration properties at a fixed point \( z \in S_0^{-\varepsilon} \). The pre-limit for which we prove anti-concentration is \( \hat{P}_0(z) \), see Corollary 4.2; the latter builds on an anti-concentration result for polynomials in independent variables with maximal degree one in each variable, see Proposition 4.1.

The proof of Theorem 1.1 follows a simpler line of argument. Indeed, we show that the term with \( d = 0 \) is dominant, now for all \( z \in S_0^{-\varepsilon} \), on a set of probability \( 1 - o(1) \). Thus the task reduces to showing that \( P_0(z) \) does not have any root in \( S_0^{-\varepsilon} \). Turning to do the same, using an operator norm bound on the noise matrix an \( N \)-dependent region \( D_N \) can be identified to not have any eigenvalue of \( T_N(a) + \Delta_N \), with high probability. Hence, one only needs to find a uniform lower bound on the modulus of \( P_0(z) = \det T_N(a(z)) \) for \( z \in S_0^{-\varepsilon} \setminus D_N \).

Evaluating the determinant of a finitely banded Toeplitz matrix has a long and impressive history. If the roots of \( P_{z,a}(\cdot) = 0 \) are distinct then the determinant of \( T_N(a(z)) \) is given by Widom’s formula (see [7, Theorem 2.8] and [4]), whereas in the case of double roots there is an analogous result, known as Trench’s formula, see [7, Theorem 2.10] and [24] for a proof. Recently, Bump and Diaconis [9] noted that, irrespective of whether \( P_{z,a}(\cdot) = 0 \) has double roots or not, the determinant of a finitely banded Toeplitz matrix can be expressed as a ratio of certain Schur polynomials in the roots of \( P_{z,a}(\lambda) = 0 \). Since we are interested in finding a uniform lower bound on the modulus of the determinant we work with the formulation of Bump and Diaconis, from which the desired uniform lower bound follows. This finishes the outline of the proof of Theorem 1.1.

### 1.3 Outline of the rest of the paper

In Sect. 2, using the second moment method, we find upper bounds on the non-dominant terms. Section 3 presents the combinatorial analysis leading to controls of the fields \( \mathcal{P}_0^L \) and \( \mathcal{P}_0^\infty \). Section 4 is devoted to deriving the general anti-concentration bounds alluded to above, which is then applied to yield the same for the dominant term. Section 5 is devoted to the proof of Theorem 1.11, while Sect. 6 is devoted to the proof of Theorem 1.1. Finally, as mentioned in Remark 1.5, extending the ideas of proof of Theorem 1.1, in “Appendix A” we prove that the spectral radii of \( \{N^{-1/2} E_N\}_{N \in \mathbb{N}} \) are tight.

### 2 Identification of dominant term and tightness of the scaled determinants

In this section we show that the determinant of \( T_N(a(z)) + \Delta_N \), when scaled appropriately, is uniformly tight, where we remind the reader that \( a(z)(\cdot) := a(\cdot) - z \) and \( \Delta_N = N^{-\gamma} E_N \). This will be shown by deriving uniform upper bounds on the non-dominant terms of the scaled determinant, as well as the same for the dominant term. For later use, we will also derive bounds on the second moment of the tail of the dominant term.
Before proceeding further we introduce relevant definitions. For \( k \in \mathbb{N} \) set

\[
P_k(z) := \sum_{X, Y \subset \mathbb{N}, |X| = |Y| = k} (-1)^{\text{sgn}(\sigma_X)\text{sgn}(\sigma_Y)} \det(T_N(a(z))(X^c; Y^c))N^{-\gamma k} \det(E_N[X; Y]),
\]

(2.1)

where we recall that \( E_N[X; Y] \) denotes the sub-matrix of \( E_N \) induced by the rows in \( X \) and columns in \( Y \), \( X^c := \mathbb{N}\setminus X \), \( Y^c := \mathbb{N}\setminus Y \), and for \( Z \in \{X, Y\}, \sigma_Z \) is the permutation on \( \mathbb{N} \) which places all the elements of \( Z \) before all the elements of \( Z^c \), but preserves the order of the elements within the two sets. Additionally denote \( P_0(z) := \det(T_N(a(z))) \).

From [3, Lemma A.1] it follows that

\[
\det(T_N(a(z)) + N^{-\gamma} E_N) = \sum_{k=0}^{N} P_k(z). \tag{2.2}
\]

Below we will show that for \( z \in \mathcal{S}_d \), the dominant term in the expansion (2.2) is \( P_d(z) \). In Sect. 4, it will be further argued that for \( z \in \mathcal{S}_{d-\varepsilon} \), \(|P_\emptyset(z)|\) is of the order \( \hat{R}(z) \), where

\[
\hat{R}(z) := R(z, \Phi) := a_{d_1}^N \cdot N^{-\gamma |\Phi|} \prod_{i=1}^{d_1-d} \lambda_i(z)^{N+d_2}. \tag{2.3}
\]

(By convention, we set an empty product to equal 1.) Thus, for uniform tightness, we scale the determinants by this factor, setting

\[
\hat{P}_k(z) := \frac{P_k(z)}{\hat{R}(z)}, \quad k = 1, 2, \ldots, N, \tag{2.4}
\]

and

\[
\hat{\det}_N(z) := \frac{\det(T_N(a(z)) + N^{-\gamma} E_N) / \hat{R}(z)}{.} \tag{2.5}
\]

We will show that, for any compact set \( K \subset \mathcal{S}_{d-\varepsilon} \), the sequence of random variables \( \{\|\hat{\det}_N(z)\|_{L^p} \}_{N \in \mathbb{N}} \) is uniformly tight. It follows from [15, Lemma 2.6] that if \( \sup_N \mathbb{E}[|\hat{\det}_N(z)|^p] \) is locally integrable for some \( p > 0 \), then the sequence \( \{\|\hat{\det}_N(z)\|_{L^p} \}_{N \in \mathbb{N}} \) is indeed uniformly tight. Therefore it suffices to derive the local integrability of \( \sup_N \mathbb{E}[|\hat{\det}_N(z)|^p] \). In this section we derive this local integrability with \( p = 2 \). The following are the main results of this section.

The first two results, whose proofs are postponed, derive a bound on the second moments of the supremum (in \( z \)) of the non-dominant terms in the expansion of \( \hat{\det}_N(z) \).

**Lemma 2.1** Fix \( \varepsilon > 0, \gamma > \frac{1}{2} \), and an integer \( \delta \) such that \( -d_2 \leq \delta \leq d_1 \). Let \( T_N(a) \) be an \( N \times N \) Toeplitz matrix with symbol \( a \), where \( a \) as in (1.1). Assume that the entries

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of $E_N$ are independent with zero mean and unit variance. Then, there exists $\eta_0 > 0$ such that

$$\sup_{k=|B|+1} \mathbb{E} \left[ \sup_{z \in S_b^\varepsilon} |\hat{P}_k(z)|^2 \right] \cdot N^{\eta_0 (k-|B|)} \leq \sup_{k=|B|+1} \mathbb{E} \left[ \sup_{z \in S_b^\varepsilon} |\hat{P}_k(z)|^2 \cdot \prod_{i=1}^{d_1-\partial} |\lambda_i(z)|^{2(k+d)} \right] \cdot N^{\eta_0 (k-|B|)} \leq 1, \quad (2.6)$$

for all large $N$.

**Lemma 2.2** Under the same set-up as in Lemma 2.1, there exists an $\varepsilon_* \in (0, 1)$, depending only on $a$ and $\varepsilon > 0$, such that

$$\sup_{k=0}^{|B|-1} \mathbb{E} \left[ \sup_{z \in S_b^\varepsilon} |\hat{P}_k(z)|^2 \right] \leq (1 - \varepsilon_*)^N,$$

for all large $N$.

Building on Lemmas 2.1 and 2.2 we derive the following bound on the second moment of the non-dominant terms.

**Corollary 2.3** Under the same set-up as in Lemma 2.1, for large $N$ it holds that

$$\mathbb{E} \left[ \sup_{z \in S_b^\varepsilon} \left| \det_N(z) - \hat{P}_{|B|}(z) \right|^2 \right] \leq N^{-\frac{\eta_0}{2}}.$$

The next lemma, whose proof is deferred, gives an upper bound on the second moment of the dominant term in the expansion of $\det_N(z)$.

**Lemma 2.4** Under the same set-up as in Lemma 2.1, there exists a constant $C_0$ such that

$$\sup_{N} \mathbb{E} \left[ \sup_{z \in S_b^\varepsilon} |\hat{P}_{|B|}(z)|^2 \right] \leq C_0.$$

The key to the proof of the above results is a representation of $P_k(z)$ as linear combinations of products of determinants of certain bidiagonal matrices with coefficients that are determinants of sub-matrices of $E_N$. Toward this end, we borrow ideas from [3].

If $T_N(a(z))$ is an upper triangular matrix then it is obvious that

$$T_N(a(z)) = a_d \cdot \prod_{i=1}^{d} (J_N + \lambda_i(z) \text{Id}_N),$$
where $J_N$ is the nilpotent matrix given by $(J_N)_{i,j} = 1_{j=i+1}$ for $i,j \in [N]$. Then the desired representation is simply a consequence of Cauchy–Binet theorem. For a general Toeplitz matrix the above product representation does not hold. It was noted in [3] that $T_N(a(z))$ can be viewed as a certain sub-matrix of an upper triangular finitely banded Toeplitz matrix with a slightly larger dimension. (This is related to the Grushin problem discussed by Sjöstrand and Vogel, see e.g. [19], in that one replaces the study in dimension $N$ with a slightly larger dimension. However, the details of the replacement, as well as the goals, are different.) Therefore one can essentially repeat the same product representation and apply the Cauchy–Binet theorem.

To use efficiently this idea, we introduce the following definition.

**Definition 2.1 (Toeplitz with a shifted symbol)** Let $T_N(a)$ be a Toeplitz matrix with finite symbol $a(\lambda) = \sum_{\ell=-d_2}^{d_1} a_\ell \lambda^\ell$. For $M > d$, $z \in \mathbb{C}$ and $\tilde{d}_1, \tilde{d}_2 \in \mathbb{N}$ such that $\tilde{d}_1 + \tilde{d}_2 = d_1 + d_2 = d$, let $T_M(a, z; \tilde{d}_1)$ denote the $M \times M$ Toeplitz matrix with the first row and column

$$(a'_{d_1-\tilde{d}_1}, a'_{d_1-\tilde{d}_1+1}, \ldots, a'_{d_1}, 0, \ldots, 0) \text{ and } (a'_{d_1-\tilde{d}_1}, a'_{d_1-\tilde{d}_1-1}, \ldots, a'_{-d_2}, 0, \ldots, 0)^T,$$

respectively, where $a'_j := a_j - z \delta_{j,0}$, $j = -d_2, -d_2 + 1, \ldots, d_1$.

From Definition 2.1, it follows that

$$T_N(a(z)) = T_N(a, z; d_1) = T_{N+d_2}(a, z; d) [[N]; [N+d_2]\backslash[d_2]],$$

since

$$T_{N+d_2}(a, z; d_1) = \begin{bmatrix}
-\bar{d}_2 & \ldots & -d_2 & a_0 - z & \ldots & a_{d_1} & 0 & \ldots & 0 \\
0 & -\bar{d}_2 & \ldots & -d_2 & a_0 - z & \ldots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & a_{-\bar{d}_2} \\
\end{bmatrix}.$$

Note that $T_{N+d_2}(a, z; d)$ is an upper triangular Toeplitz matrix. As $\{-\lambda_{\ell}(z)\}_{\ell=1}^{d}$ are the roots of the equation $P_{z,a}(\lambda) = 0$ we obtain that

$$T_{N+d_2}(a, z; d) = \sum_{\ell=0}^{d} (a_\ell - z \delta_{\ell,d_2}) J_{N+d_2}^\ell = a_{d_1} \prod_{\ell=1}^{d} (J_{N+d_2} + \lambda_{\ell}(z) \text{Id}_{N+d_2}).$$

Hence, recalling the definition of $\{P_k(z)\}_{k=1}^{N}$ from (2.1), applying the Cauchy–Binet theorem, and writing $S + \ell := \{x + \ell, x \in S\}$ for any set of integers $S$ and an integer $\ell$, we obtain that
\[ P_k(z) = \sum_{X,Y \subset [N]} (-1)^{\text{sgn}(\sigma_X) \text{sgn}(\sigma_Y)} \det(T_{N+d_2}(\mathbf{a}, z; d)[X^c; Y^c + d_2]) \cdot N^{-\gamma_k} \cdot \det(E_{N}[X; Y]) \]

\[
= \sum_{X,Y \subset [N]} \sum_{i=2}^{d-1} \sum_{X_i \subset [N+d_2]} (-1)^{\text{sgn}(\sigma_X) \text{sgn}(\sigma_Y)} a_{d_1} \cdot \prod_{i=1}^{d} \det\left((J_{N+d_2} + \lambda_i(z) \text{Id}_{N+d_2})[\tilde{X}_i; \tilde{X}_{i+1}]\right) \cdot N^{-\gamma_k} \cdot \det(E_{N}[X; Y]),
\]

(2.7)

where

\[ X_1 := X_1(X) := X \cup [N + d_2]\setminus[N], \quad X_{d+1} := X_{d+1}(Y) := (Y + d_2) \cup [d_2], \]

(2.8)

and \( \tilde{Z} := [N + d_2]\setminus Z \) for any set \( Z \subset [N + d_2] \). We emphasize the notational difference between \( \tilde{Z} \) and \( Z^c \). The former will be used to write the complement of \( Z \) when viewed as a subset of \( [N + d_2] \), where for the latter \( Z \) will be viewed as a subset of \( [N] \).

The RHS of (2.7) gives the desired representation of \( P_k(z) \). To prove Lemmas 2.1 and 2.2 we require some preprocessing of the RHS of (2.7). To obtain a tractable expression we express the sums in (2.7) over \( \{X_i\}_{i=1}^{d+1} \) as an iterated sums, see (2.17) below. The inner sum will be over the choices of \( \{X_i\}_{i=1}^{d+1} \) such that the product of the determinants of the bi-diagonal matrices is constant and the outer sum will be over all possible values of the product of the determinants.

We now describe this decomposition. From (2.7) and (2.8) we have that \( |X_i| = k + d_2 \), for \( i \in [d + 1] \). Therefore, we write

\[ X_i := \{x_{i,1} < x_{i,2} < \cdots < x_{i,k+d_2}\}, \quad X_k := (X_1, X_2, \ldots, X_{d+1}). \]

(2.9)

Using [3, Lemma A.3] we note that

\[
\det((J_{N+d_2} + \lambda_i(z) \text{Id}_{N+d_2})[\tilde{X}_i; \tilde{X}_{i+1}]) = \lambda_i(z)^{x_{i+1,1}-1} \cdot \left(\prod_{\ell=2}^{k+d_2} \lambda_i(z)^{x_{i+1,\ell}-x_{i,\ell}-1}\right) \cdot \lambda_i(z)^{N+d_2-x_{i,k+d_2}} \cdot 1_{\{x_{i+1,\ell} \leq x_{i,\ell} < x_{i+1,\ell+1}, \ell \in [k+d_2]\}},
\]

(2.10)

where we have set \( x_{i+1,k+d_2+1} = \infty \) for convenience. In light of (2.10), for any \( \ell := (\ell_1, \ell_2, \ldots, \ell_d) \) with \( 0 \leq \ell_i \leq N + d_2 \) for \( i \in [d] \), and \( k \in [N] \), we define

\[ L_{\ell,k} := \{X_k : 1 \leq x_{i+1,1} \leq x_{i,1} < x_{i+1,2} \leq x_{i,2} < \cdots < x_{i+1,k+d_2} \leq x_{i,k+d_2} \leq N + d_2\}, \]
and \[ x_{i+1,1} + \sum_{j=2}^{k+d_2} (x_{i+1,j} - x_{i,j-1}) + (N + d_2 - x_{i,k+d_2}) \]
\[ = \ell_i + k + d_2, \text{ for all } i = 1, 2, \ldots, d. \]

Note that (2.10) implies that the summand in (2.7) is non-zero only when \( X_k \in L_{\ell,k} \) for some \( \ell \) and in that case

\[
\prod_{i=1}^{d} \det \left( (J_{N+d_2} + \lambda_i(z) \text{Id}_{N+d_2}) [\hat{X}_i; \hat{X}_{i+1}] \right) = \prod_{i=1}^{d} \lambda_i(z)^{\ell_i}. \tag{2.11}
\]

Recall that in Lemmas 2.1 and 2.2 we aim to show that for \( z \in S_{-\varepsilon}^\circ \) and \( k \neq |\partial| \), \( |P_k(z)| \) is small compared to \( \Re(z) \) of (2.3). Thus it would be convenient to pull out this factor from the RHS of (2.11). So, using the observation that

\[
x_{i+1,1} + \sum_{j=2}^{k+d_2} (x_{i+1,j} - x_{i,j-1}) + \sum_{j=1}^{k+d_2} (x_{i,j} - x_{i+1,j}) + (N + d_2 - x_{i,k+d_2}) = N + d_2,
\]

we have the following equivalent representation of \( L_{\ell,k} \):

\[
L_{\ell,k} := \{ X_k : 1 \leq x_{i+1,1} \leq x_{i,1} < x_{i+1,2} \leq x_{i,2} < \cdots < x_{i+1,k+d_2} \leq x_{i,k+d_2} \leq N + d_2, \sum_{j=1}^{k+d_2} (x_{i,j} - x_{i+1,j} + 1) = \ell_i; i = 1, 2, \ldots, d_0, \}
\]

and \( x_{i+1,1} + \sum_{j=2}^{k+d_2} (x_{i+1,j} - x_{i,j-1}) + (N + d_2 - x_{i,k+d_2}) \]
\[ = \ell_i + k + d_2; i = d_0 + 1, d_0 + 2, \ldots, d \}, \tag{2.12}
\]

where

\[
\ell_i := \begin{cases} \ell_i & \text{if } i > d_0, \\ N + d_2 - \ell_i & \text{if } i \leq d_0. \end{cases} \tag{2.13}
\]

and \( d_0 = d_1 - \varnothing \). Since \( x_{i+1,j} \leq x_{i,j} \) for any \( i = 1, 2, \ldots, d \), and \( j = 1, 2, \ldots, k+d_2 \), we note from (2.12) that

\[
\ell_i \geq k + d_2, \text{ for } i = 1, 2, \ldots, d_0. \tag{2.14}
\]

This will be used below in the proof of Lemma 2.1. Equipped with the above notation we find that, for any \( X_k \in L_{\ell,k} \),

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\[
\prod_{i=1}^{d} \det \left( (J_{N+d_2} + \lambda_i(z) I_{N+d_2}) [\tilde{X}_i; \tilde{X}_{i+1}] \right)
= \prod_{i=1}^{d_0} \lambda_i(z)^{N+d_2} \cdot \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \cdot \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i}.
\] (2.15)

Furthermore, the restriction (2.8) and the fact that the outer sum in (2.7) is over \( X, Y \subset [N] \) implies that the summand in (2.7) vanishes unless \( \lambda_k \in \mathcal{L}_{\ell,k} \), where
\[
\mathcal{L}_{\ell,k} := \{ \lambda_k \in L_{\ell,k} : x_{k+1} = N + j; \ j \in [d_2] \ \text{and} \ x_{d_1+1} = j; \ j \in [d_2] \}. (2.16)
\]

Therefore, from (2.7) and (2.15) we deduce that
\[
P_k(z) = a_d^{N-k} \cdot \prod_{i=1}^{d_0} \lambda_i(z)^{N+d_2} \cdot N^{-\gamma k} \sum_{\ell} \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \cdot \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i}
\cdot \sum_{\lambda_k \in \mathcal{L}_{\ell,k}} (-1)^{\text{sgn}(\sigma_x) \text{sgn}(\sigma_y)} \det(E_N[\mathbb{X}; \mathbb{Y}]),
\] (2.17)

with
\[
\mathbb{X} := \mathbb{X}(X_1) := X_1 \cap [N] \ \text{and} \ \mathbb{Y} := \mathbb{Y}(X_{d_1+1}) := (X_{d_1+1} - d_2) \cap [N], (2.18)
\]

where (2.18) is a consequence of (2.8). We introduce further notation. Set
\[
Q_{\ell,k}(z) := \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i} \cdot \hat{Q}_{\ell,k},
\] (2.19)

where
\[
\hat{Q}_{\ell,k} := \sum_{\lambda_k \in \mathcal{L}_{\ell,k}} (-1)^{\text{sgn}(\sigma_x) \text{sgn}(\sigma_y)} \det(E_N[\mathbb{X}; \mathbb{Y}]),
\] (2.20)

does not depend on \( z \). Recalling the definition of \( \hat{P}_k(z) \) [see (2.4)], we have from (2.17) that
\[
\hat{P}_k(z) = a_d^{N-\gamma (k-|\beta|)} \sum_{\ell} Q_{\ell,k}(z)
= a_d^{N-\gamma (k-|\beta|)} \sum_{\ell} \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i} \cdot \hat{Q}_{\ell,k}
\] (2.21)

Having obtained a tractable expression in (2.21) we now proceed to apply the second moment method to prove Lemmas 2.1 and 2.2. So, we next estimate the variance of...
Using that the entries of $E_N$ are independent with zero mean and unit variance it is straightforward to see that
\[
\mathbb{E}\left[ \det(E_N[X_*; Y_*]) \cdot \det(E_N[X'; Y']) \right] = \begin{cases} k! & \text{if } X_* = X' \text{ and } Y_* = Y' \\ 0 & \text{otherwise,} \end{cases}
\]
for any collection of subsets $X_*, Y_*, X', Y' \subset [N]$, each of cardinality $k$. Hence, we deduce that
\[
\mathbb{E}[|\hat{Q}_{\ell,k}|^2] = \text{Var}(\hat{Q}_{\ell,k}) = k! \cdot \mathfrak{N}_{\ell,k},
\]
where
\[
\mathfrak{N}_{\ell,k} := \left| \left\{ X_k = (X_1, X_2, \ldots, X_{d+1}), X'_k = (X'_1, X'_2, \ldots, X'_{d+1}) \in \mathcal{S}_{\ell,k} : X_1 = X'_1, X_{d+1} = X'_{d+1} \right\} \right|.
\]
Thus an estimate on the variance of $\hat{Q}_{\ell,k}$ requires a bound on $\mathfrak{N}_{\ell,k}$. This is done in the lemma below. The proof is postponed to later in the section.

**Lemma 2.5** Fix $k \in \mathbb{N}$, an integer $d$ such that $-d_2 \leq d \leq d_1$, and $\ell = (\ell_1, \ell_2, \ldots, \ell_d)$ such that $0 \leq \ell_i \leq N + d_2$ for all $i \in [d]$. For $|\mathfrak{d}| \leq k \leq N$ and $\mathfrak{N}_{\ell,k}$ as in (2.23), we have
\[
\mathfrak{N}_{\ell,k} \leq \left( \frac{N + d_2}{k - |\mathfrak{d}|} \right) \cdot \prod_{i=1}^{d_0} \left( \hat{\ell}_i - 1 \right)^2 \cdot \prod_{i=d_0+1}^{d} \left( \hat{\ell}_i + k + d_2 \right)^2. \tag{2.24}
\]

One final ingredient needed for the proof of Lemmas 2.1, 2.2 and 2.4, is a uniform separation of the moduli of the roots $\{-\lambda_i(z)\}_{i=1}^{d}$ from one, for all $z \in S_{\mathfrak{d}}^-$. This is formulated in the lemma below.

**Lemma 2.6** Fix $\epsilon \in (0, 1)$ and an integer $\mathfrak{d}$ such that $-d_2 \leq \mathfrak{d} \leq d_1$, and $\ell = (\ell_1, \ell_2, \ldots, \ell_d)$ such that $0 \leq \ell_i \leq N + d_2$ for all $i \in [d]$. Then
\[
\sup_{z \in S_{\mathfrak{d}}^-} \max \left\{ \max_{i=1}^{d_0} \{|\lambda_i(z)|^{-1}\}, \max_{i=d_0+1}^{d} \{|\lambda_i(z)|\} \right\} \leq 1 - \varepsilon_0, \tag{2.25}
\]
for some sufficiently small $\varepsilon_0 > 0$, depending only on $\varepsilon$ and the symbol $a$.

**Proof** Recalling the definition of $S_{\mathfrak{d}}$ (see Definition 1.2) we have that $S_{\mathfrak{d}} \subset (a(S^1))^\epsilon$. This implies that
\[
S_{\mathfrak{d}}^- \cap (a(S^1))^\epsilon = \emptyset. \tag{2.26}
\]
On the other hand, if (2.25) is violated for some $z \in S_{\mathfrak{d}}^-$ then there exists a root $\lambda_0(z)$ of the equation $P_{z,a}(\lambda) = 0$ such that
\[
|\lambda_0(z)| - 1 \leq 2\varepsilon_0.
\]
whenever $\varepsilon_0 < \frac{1}{2}$. Therefore, denoting
\[
z_0 := \sum_{i=-d_2}^{d_2} a_i \cdot \frac{\lambda_0(z)^i}{|\lambda_0(z)|^i} \in a(S^1),
\]
by the triangle inequality it follows that
\[
|z - z_0| \leq \sum_{i=-d_2}^{d_1} |a_i| \cdot |\lambda_0(z)|^i \left( 1 - \frac{1}{|\lambda_0(z)|^i} \right) = \sum_{i=-d_2}^{d_1} |a_i| \cdot ||\lambda_0(z)|^i - 1|.
\]
Now upon choosing $\varepsilon_0$ sufficiently small we note that the above implies that $z \in (a(S^1))^{\varepsilon}$. This yields a contradiction to (2.26), thereby proving the claim (2.25).

We now proceed to the proof of Lemma 2.1.

Proof of Lemma 2.1 (assuming Lemma 2.5) Note that, as $|\lambda_i(z)| \geq 1$ on $S_{\varepsilon}^{-\varepsilon}$ for $i = 1, 2, \ldots, d_0$, the left most inequality in (2.6) is immediate. So we only need to prove the right inequality. To this end, fix $k > |\theta|$ and introduce the notation
\[
\kappa(\ell, z) := a_{d_1}^{-k} N^{-\gamma(k - |\theta|)} \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i} \prod_{i=1}^{d_0} \lambda_i(z)^{k+d_2} \quad (2.27)
\]
and
\[
\tilde{\kappa}(\ell) := \sup_{z \in S_{\varepsilon}^{-\varepsilon}} |\kappa(\ell, z)|. \quad (2.28)
\]
We now have that
\[
\mathbb{E} \left[ \sup_{z \in S_{\varepsilon}^{-\varepsilon}} |\hat{P}_k(z)|^2 \cdot \prod_{i=1}^{d_0} \lambda_i(z)^{2(k+d_2)} \right] = \mathbb{E} \left[ \sup_{z \in S_{\varepsilon}^{-\varepsilon}} \left| \sum_{\ell} \kappa(\ell, z) \cdot \hat{Q}_{\ell,k} \right|^2 \right] \leq \mathbb{E} \left[ \sum_{\ell, \ell'} \sup_{z \in S_{\varepsilon}^{-\varepsilon}} |\kappa(\ell, z) \cdot \hat{Q}_{\ell,k} \cdot \kappa(\ell', z) \cdot \hat{Q}_{\ell',k}| \right] \leq \sum_{\ell} \tilde{\kappa}(\ell) \cdot (\mathbb{E} |\hat{Q}_{\ell,k}|^2)^{1/2} \cdot \tilde{\kappa}(\ell') \cdot (\mathbb{E} |\hat{Q}_{\ell',k}|^2)^{1/2} = \left( \sum_{\ell} \tilde{\kappa}(\ell) \cdot (\mathbb{E} |\hat{Q}_{\ell,k}|^2)^{1/2} \right)^2, \quad (2.29)
\]
where the second inequality is a consequence of the Cauchy–Schwarz inequality. It follows from Lemmas 2.5 and 2.6 and (2.22), that
\[
\sup_{z \in S_{\varepsilon}^{-\varepsilon}} |\kappa(\ell, z)| \cdot (\mathbb{E} |\hat{Q}_{\ell,k}|^2)^{1/2}
\]
\[
\begin{align*}
&\leq |a_d|^{-k} \cdot N^{-\gamma(k-|\mathcal{B}|)} \prod_{i=1}^{d_0} \left(1 - \varepsilon_0\right) \tilde{\ell}_i \cdot \sqrt{k! \cdot \left(\frac{N + d_2}{k - |\mathcal{B}|}\right) \prod_{i=1}^{d_0} \left(\frac{\hat{\ell}_i - 1}{k + d_2 - 1}\right)} \\
&\quad \cdot \prod_{i=d_0+1}^{d} \left(\frac{\hat{\ell}_i + k + d_2}{k + d_2}\right) \\
&\leq |a_d|^{-k} \cdot e^{d_2} \cdot N^{-\gamma(k-1/2)(k-|\mathcal{B}|)} \prod_{i=1}^{d_0} \left(1 - \varepsilon_0\right) \tilde{\ell}_i \cdot k^{1/2} \\
&\quad \cdot \prod_{i=1}^{d_0} \left(\frac{\hat{\ell}_i - 1}{k + d_2 - 1}\right) \cdot \prod_{i=d_0+1}^{d} \left(\frac{\hat{\ell}_i + k + d_2}{k + d_2}\right),
\end{align*}
\]

where
\[
\tilde{\ell}_i := \begin{cases}
\hat{\ell}_i - (k + d_2) & \text{for } i = 1, 2, \ldots, d_0, \\
\hat{\ell}_i & \text{for } i = d_0 + 1, d_0 + 2, \ldots, d.
\end{cases}
\]

So, to find a bound on \(E[\sup_{z \in S_{\theta}} |\hat{P}_k(z)|^2 \cdot \prod_{i=1}^{d_0} \lambda_i(z)^{2(k+d_2)}]\), we need to sum the RHS of (2.30) over the range of \(\ell\). To control this sum effectively we consider the cases of \(k\) small and large separately. First we consider the case when \(k\) is small.

**Case 1** Let \(k \leq N^{1/\log \log N}\). In this case to evaluate the sum of the RHS of (2.30) over \(\ell\) we split the range of \(\ell\) into two further sub-cases. Let us consider the case when \(\max_i \ell_i\) is small. Set
\[
\mathcal{R}_1 := \{\ell : \hat{\ell}_i \leq N^{2/\log \log N}, i = 1, 2, \ldots, d\}.
\]

We find from the above that as by (2.15) \(\tilde{\ell}_i\)'s are non-negative for all \(i = 1, 2, \ldots, d\), there exists \(\eta_0 > 0\) such that
\[
\left[\sum_{\ell \in \mathcal{R}_1} \tilde{k}(\ell) \cdot (E|\hat{Q}_{\ell,k}|^2)^{1/2}\right] \leq \left[|\mathcal{R}_1| \cdot \max_{\ell \in \mathcal{R}_1} \left\{\tilde{k}(\ell) \cdot (E|\hat{Q}_{\ell,k}|^2)^{1/2}\right\}\right] \\
\leq |a_d|^{-k} \cdot e^{d_2} \cdot N^{-\gamma(1/2)(k-|\mathcal{B}|)} \cdot k^{1/2} \cdot N^{5(k+d_2)/d} \\
\leq N^{-\eta_0(k-|\mathcal{B}|)},
\]

for all large \(N\).

Thus, it remains to consider the case when \(\max_i \ell_i\) is large. For any such \(\ell\) we divide the range as follows: we define
\[
\tilde{\mathcal{R}}_{1,j} := \{(k, \ell) : \hat{\ell}_{(j)} \leq N^{2/\log \log N}, j \in [i - 1]; \text{ and } N^{2/\log \log N} \leq \hat{\ell}_{(j)} \leq N + d_2, j \in [d] \setminus [i - 1]\},
\]

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for any \( i \in [d] \), where \( \hat{\ell}(1) \leq \hat{\ell}(2) \leq \cdots \leq \hat{\ell}(d) \) is a non-increasing rearrangement of \( \{\hat{\ell}_i\}_{i=1}^d \).

Now, if \( \ell \in \tilde{R}_1,i \), and as \( k \leq N^{1/\log \log N} \), we note that \( \hat{\ell}(j) \geq k^2 \vee N^{\frac{1}{\log \log N}} \), for any \( j = i, i+1, \ldots, d \). Therefore, for any such \( j \) we have that

\[
(1 - \varepsilon_0) \hat{\ell}(j) \max \left\{ \left( \frac{\hat{\ell}(j) - 1}{k + d_2 - 1} \right), \left( \frac{\hat{\ell}(j) + k + d_2}{k + d_2} \right) \right\} \leq (1 - \varepsilon_0)^{\hat{\ell}(j)/2},
\]

for all large \( N \), where we set

\[
\tilde{\hat{\ell}}(j) := \hat{\ell}_j' \quad \text{if} \quad \hat{\ell}(j) = \hat{\ell}_j' \quad \text{for some} \quad j' \in \{1, 2, \ldots, d\}.
\]

Hence, using the fact that

\[
\sum_{\ell \geq L} (1 - \varepsilon_0)^{\ell} = (1 - \varepsilon_0)^L \cdot \varepsilon_0^{-1}
\]

we observe that

\[
\sum_{\ell \in \tilde{R}_1,i} \prod_{j=1}^d (1 - \varepsilon_0)^{\ell} \prod_{j=1}^{d_0} \left( \frac{\hat{\ell}_j - 1}{k + d_2 - 1} \right) \prod_{j=d_0+1}^{d-1} \left( \frac{\hat{\ell}_j + k + d_2}{k + d_2} \right) \leq N^{\frac{(5k + d_2)i}{123}} \cdot \varepsilon_0^{-d-1} \cdot (1 - \varepsilon_0)^{\hat{\ell}(j)/2} \leq N^{\frac{(5k + d_2)i}{123}} \cdot \varepsilon_0^{-d} \cdot (1 - \varepsilon_0)^{N^{\frac{1}{\log \log N}}}.
\]

Therefore, using (2.31) we deduce that

\[
\left[ \sum_{\ell \in \tilde{R}_1,i} \tilde{\kappa}(\ell) \cdot (\mathbb{E}|\tilde{Q}_{\ell,k}|^2)^{1/2} \right] \leq N^{-\eta_0(k-|\ell|)} \cdot (1 - \varepsilon_0)^{N^{\frac{1}{\log \log N}}}, \tag{2.34}
\]

for all large \( N \) and any \( i \in [d] \). Now summing (2.31), and (2.34) for \( i \in [d] \), we deduce from (2.29) that

\[
\frac{1}{N^{\log \log N}} \sup_{k=|\ell|+1} \mathbb{E} \left[ \sup_{z \in S^2_{\mathcal{F}}} |\tilde{P}_k(z)|^2 \cdot \prod_{i=1}^{d_0} \lambda_i(z) \cdot 2^{(k+d_2)} \cdot N^{\eta_0(k-|\ell|)} \right] \leq 1,
\]

for all large \( N \).

It remains to consider the case when \( k \) is large.

**Case 2** Fix \( k \geq N^{1/\log \log N} \). Similar to the above we split the range of \( \ell \). First we consider the case when \( \max_i \tilde{\ell}_i \) is small compared to \( k \). Set

\[
R_2 := \{ \ell : \tilde{\ell}_i \leq k(\log N)^2 \wedge (N + d_2), i = 1, 2, \ldots, d \}.
\]
Recalling the inequality \( \binom{n}{m} \leq \left( \frac{en}{m} \right)^m \) we note that

\[
\max \left\{ \frac{\hat{\ell} - 1}{k + d_2 - 1}, \frac{\hat{\ell} + k + d_2}{k + d_2} \right\} \leq (2e(\log N)^2)^{(k + d_2)},
\]

for any \( \ell \in \mathbb{R}_2 \). Thus proceeding as before we deduce that

\[
\sum_{\ell \in \mathbb{R}_2} \tilde{\kappa}(\ell) \cdot (\mathbb{E}|\hat{Q}_{\ell,k}|^2)^{1/2} \leq \left[ |R_2| \cdot \max_{\ell \in \mathbb{R}_2} \left\{ \tilde{\kappa}(\ell) \cdot (\mathbb{E}|\hat{Q}_{\ell,k}|^2)^{1/2} \right\} \right]^{1/2}
\leq |a_d| \cdot e^{d_2} \cdot N^{-(\gamma-1/2)(k-|b|)} \cdot k^{|b|/2} \cdot (2e(\log N)^2)^{(k + d_2)d} \cdot (k(\log N)^2)^d \leq N^{-\eta_0(k-|b|)},
\]

(2.35)

for all large \( N \) [notation as in (2.32)].

We finally consider the case when \( \max_i \ell_i \) is large compared to \( k \). For \( i \in [d] \), set

\[
\tilde{R}_{2,i} := \left\{ \ell : \hat{\ell}_j \leq k(\log N)^2 \land (N + d_2), \ j \in [i - 1]; k(\log N)^2 \leq \hat{\ell}_j \right\} \leq N + d_2, j \in [d] \setminus [i - 1].
\]

If \( \ell \in \tilde{R}_{2,i} \), then \( k(\log N)^2 \leq \hat{\ell}_{(j)} \leq N + d_2 \), for \( j = i, i + 1, \ldots, d \), which in turn implies that

\[
\max_{j=i}^d \left\{ \left( \frac{\hat{\ell}_j - 1}{k + d_2 - 1} \right) \lor \left( \frac{\hat{\ell}_j + k + d_2}{k + d_2} \right) \right\} \cdot (1 - \varepsilon_0)^{\hat{\ell}_i/2} \leq 1,
\]

for all large \( N \). Therefore for any \( \ell \in \tilde{R}_{2,i} \) we obtain that

\[
\prod_{j=1}^d (1 - \varepsilon_0)^{\hat{\ell}_j} \prod_{j=1}^{d_0} \left( \frac{\hat{\ell}_j - 1}{k + d_2 - 1} \right) \cdot \prod_{j=d_0+1}^d \left( \frac{\hat{\ell}_j + k + d_2}{k + d_2} \right) \leq \prod_{j=i}^d (1 - \varepsilon_0)^{\hat{\ell}_j/2} \cdot (2e(\log N)^2)^{(k + d_2)(i-1)} \leq (1 - \varepsilon_0)^{N \frac{1}{\log \log N}} \cdot (2e(\log N)^2)^{d(k + d_2)},
\]

where the last step is a consequence of the fact that for any \( j = i, i + 1, \ldots, d \),

\[
N^{1/\log \log N} \leq k \leq \hat{\ell}_{(j)}/(\log N)^2 \leq \hat{\ell}_{(j)} - (k + d_2) \leq \tilde{\ell}_{(j)}.
\]
Therefore, by (2.30),

\[
\sum_{\ell \in \mathcal{R}_{2,i}} |\tilde{\kappa}(\ell) \cdot (\mathbb{E}[\hat{Q}_{\ell,k}]^2)^{1/2}|
\leq |a_{d_1}|^{-k} \cdot e^{d_2} \cdot N^{-(\gamma-1/2)(k-|b|)} \cdot k^{b}/2 \cdot (1 - \varepsilon_0) \cdot N^{\frac{1}{2\log \log N}} \\
\cdot (2\varepsilon(\log N)^2d(k+d_2) \cdot |\tilde{R}_{2,i}| \\
\leq (1 - \varepsilon_0) \cdot N^{\frac{1}{2\log \log N}} \cdot N^{-\eta_0(k-|b|)},
\]

for all large \( N \). Hence, summing (2.36) for \( i \in [d] \), and (2.35) we derive from (2.29)
that

\[
\sup_{k=N^{1/\log \log N}} \mathbb{E}\left[ \sup_{z \in \mathcal{S}^c} |\hat{P}_{k}(z)|^2 \cdot \prod_{i=1}^{d_0} \lambda_i(z)^{2(k+d_2)} \right] \cdot N^{\eta_0(k-|b|)} \leq 1,
\]

for all large \( N \). This completes the proof of the lemma.

Next we proceed to prove Lemma 2.4, that is, to prove an upper bound on the second moment of the modulus of the dominant term \( \hat{P}_{|b|}(z) \). To derive the uniform tightness of the limiting random field it will be also useful to find a bound on the second moment of the tail of \( \hat{P}_{|b|}(z) \), i.e. a sum over the terms appearing in the RHS of (2.21) involving at least one large negative or positive exponent of \{\lambda_i\}_{i=1}^{d_0} and \{\lambda_i\}_{i=d_0+1}^{d}, respectively.

To this end, we introduce the following set of notation.

Fix \( z \in \mathcal{S}_0 \). For any \( L \in \mathbb{N} \) we define

\[
|\hat{P}_{|b|}^L(z)| := |a_{d_1}|^{-|b|} \sum_{\ell : \max_i \hat{\ell}_i \leq L} \prod_{i=1}^{d_0} |\hat{\lambda}_i(z)|^{-\hat{\ell}_i} \prod_{i=d_0+1}^{d} |\lambda_i(z)|^{\hat{\ell}_i} \cdot |\hat{Q}_{\ell,|b|}|.
\]

where we recall the definition of \( \hat{Q}_{\ell,|b|} \) from (2.20). Next, for \( L_1, L_2 \in \mathbb{N} \), such that \( L_1 \leq L_2 \), we set

\[
|\hat{P}|_{|b|}^{L_1,L_2}(z) := |\hat{P}|_{|b|}^{L_2}(z) - |\hat{P}|_{|b|}^{L_1}(z).
\]

Note that for \( L, L_1, L_2 \leq N + d_2 \), with \( L_1 \leq L_2 \), the random functions \( |\hat{P}|_{|b|}^{L_1}(z) \) and \( |\hat{P}|_{|b|}^{L_1,L_2}(z) \) are well defined. The next lemma derives bounds on the second moments of \( |\hat{P}|_{|b|}^{L_1}(z) \) and \( |\hat{P}|_{|b|}^{L_1,L_2}(z) \).

**Lemma 2.7** Under the same set-up as in Lemma 2.1, we have the following:
(i) For any $L \in \mathbb{N}$, there exists a constant $\hat{C}_L < \infty$ such that

$$\sup_N \mathbb{E} \left[ \sup_{z \in S_0^{-\varepsilon}} \left| \hat{P}_0^L(z) \right|^2 \right] \leq \hat{C}_L.$$ 

(ii) There exists a constant $\hat{c} > 0$, such that for any $L_1, L_2 \in \mathbb{N}$ with $L_1 \leq L_2$,

$$\sup_N \mathbb{E} \left[ \sup_{z \in S_0^{-\varepsilon}} \left| \hat{P}_0^{L_1,L_2}(z) \right|^2 \right] \leq \exp(-\hat{c}L_1).$$

For later use, for any $L \in \mathbb{N}$ denote

$$\hat{P}_0^L(z) := a_{d_1}^{-|b|} \sum_{\ell: \max \hat{e}_i \leq L} \prod_{i=1}^{d_1-\ell} \lambda_i(z)^{-\hat{e}_i} \cdot \prod_{i=d_1-\ell+1}^{d} \lambda_i(z)^{\hat{e}_i} \cdot \sum_{\chi_{b}\in \mathcal{L}_{\ell,d}} (-1)^{\text{sgn}(\sigma_x) \text{sgn}(\sigma_y)} \det(E_N[X;Y]),$$

(2.37)

and

$$\hat{P}_0^{\tilde{L}}(z) := \hat{P}_0^L(z) - \hat{P}_0^L(z).$$

Setting $L_2 = N + d_2$, by the triangle inequality it follows that

$$|\hat{P}_0^{L}(z)| \leq |\hat{P}_0^{L,N+d_2}(z)|. $$

Thus, as a consequence of Lemma 2.7(ii) we obtain the following corollary.

**Corollary 2.8** Under the same set-up as in Lemma 2.1, for any $L \in \mathbb{N}$,

$$\sup_N \mathbb{E} \left[ \sup_{z \in S_0^{-\varepsilon}} \left| \hat{P}_0^L(z) \right|^2 \right] \leq \exp(-\hat{c}L),$$

where $\hat{c}$ is as in Lemma 2.7.

**Proof of Lemma 2.7 (assuming Lemma 2.5)** We begin by noting that, for any $L \in \mathbb{N}$,

$$\mathbb{E} \left[ \sup_{z \in S_0^{-\varepsilon}} \left| \hat{P}_0^L(z) \right|^2 \right] \leq \left( \sum_{\ell: \max \hat{e}_i \leq L} \hat{r}(\ell) \cdot (\mathbb{E} |\hat{Q}_{\ell,b}|^2)^{1/2} \right)^2,$$ 

(2.39)
where
\[ \hat{\kappa}(\ell) := \sup_{z \in S^{\epsilon}} |\kappa^*(\ell, z)| \]
and
\[ \kappa^*(\ell, z) := ad^{-k} \left( \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \cdot \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i} \right). \]

The proof of (2.39) is analogous to that of (2.29). Proceeding as in (2.30) we find that
\[
\sup_{z \in S^{\epsilon}} |\kappa^*(\ell, z)| \cdot (E|\hat{Q}_{\ell,k}|^2)^{1/2} \leq |ad_1|^{-|\mathbf{b}|} \cdot |\mathbf{b}|^{b/2} \cdot \prod_{i=1}^{d} (1 - \varepsilon_0)^{\hat{\ell}_i} \cdot \prod_{i=1}^{d_0} (|\mathbf{b}| + d_2 - 1) \cdot \prod_{i=d_0+1}^{d} (|\mathbf{b}| + d_2).
\]
Therefore
\[
\sum_{\ell: \max_i \hat{\ell}_i \leq L} \hat{\kappa}(\ell) \cdot (E|\hat{Q}_{\ell,\mathbf{b}}|^2)^{1/2} \leq |ad_1|^{-|\mathbf{b}|} \cdot |\mathbf{b}|^{b/2} \cdot \sum_{\ell: \max_i \hat{\ell}_i \leq L} \prod_{i=1}^{d} (1 - \varepsilon_0)^{\hat{\ell}_i} \cdot \prod_{i=1}^{d_0} \hat{\ell}_i^{2d} \cdot \prod_{i=d_0+1}^{d} (\hat{\ell}_i + 2d)^{2d} \leq \hat{C}_L^{1/2},
\]
for some constant \( C_L < \infty \). This together with (2.39) yields part (i) of the lemma. To prove the second part we proceed similarly as in (2.39) to deduce that
\[
\mathbb{E} \left[ \sup_{z \in S^{\epsilon}} \left| \hat{P}_{\ell,|\mathbf{b}|,L_2}(z) \right|^2 \right] \leq \left( \sum_{\ell: \max_i \hat{\ell}_i > L_1} \hat{\kappa}(\ell) \cdot (E|\hat{Q}_{\ell,k}|^2)^{1/2} \right)^2,
\]
Since
\[
\sum_{\ell \geq 0} (\ell + 2d)^{2d} \cdot (1 - \varepsilon_0)^{\ell} < \infty \quad \text{and} \quad \sum_{\ell \geq 0} (\ell + 2d)^{2d} \cdot (1 - \varepsilon_0)^{\ell} \leq (1 - \tilde{\varepsilon})^L,
\]
for any \( L \in \mathbb{N} \) and some \( \tilde{\varepsilon} > 0 \), we use (2.30) again to deduce that
\[
\sum_{\ell: \max_i \hat{\ell}_i > L} \hat{\kappa}(\ell) \cdot (E|\hat{Q}_{\ell,k}|^2)^{1/2} \leq \exp(-C/2 \cdot L_1).
\]
Plugging this bound in (2.41) completes the proof of the lemma. \hfill \Box

The proof of Lemma 2.4 is now immediate.

**Proof of Lemma 2.4** As

$$|\hat{P}_{[0]}(z)| \leq |\hat{P}_{[0]}^{L}(z)| + |\hat{P}_{[0]}^{L}(z)|,$$

for any $L \in \mathbb{N}$, the conclusion of Lemma 2.4 is immediate from Lemma 2.7(i) and Corollary 2.8. \hfill \Box

Next we provide the proof of Lemma 2.5 which has been used in the proofs of Lemmas 2.1 and 2.7. Recall that Lemma 2.5 yields a bound on $\mathcal{N}_{k}$ for any $k = |\partial|, |\partial| + 1, \ldots, N$.

**Proof of Lemma 2.5** To prove (2.24) we need to consider the cases $d \geq 0$ and $d < 0$ separately. First let us consider the case $d \geq 0$.

To this end, denote

$$\delta_{i,j} := \delta_{i,j}(X_{k}) := \begin{cases} 
  x_{i,j} - x_{i+1,j} & \text{for } i \in [d] \text{ and } j \in [k + d_{2}] \\
  x_{i+1,j} & \text{for } i \in [d] \setminus [d_{0}], j = 1 \\
  x_{i+1,j} - x_{i,j-1} & \text{for } i \in [d] \setminus [d_{0}], j \in [k + d_{2}] \setminus [1] \\
  N + d_{2} - x_{i,k+d_{2}} & \text{for } i \in [d] \setminus [d_{0}], j = k + d_{2} + 1.
\end{cases}$$

(2.42)

We claim that for $X_{k} \in \mathcal{L}_{k}$, with $k \geq d$, the set of integers $\{\delta_{i,j}(X_{k})\}$ fixes the choices of $\{x_{d+1,j}\}_{j=1}^{d+d_{2}}$. To see this we note that for any pair of integers $k$ and $j$ such that $k \geq d$ and $d_{2} + 1 \leq j \leq d + d_{2}$, we have

$$j \leq k + d_{2} \text{ and } d - j + 1 \geq d - d_{2} - d + 1 = d_{1} - d + 1 = d_{0} + 1.$$

Therefore

$$x_{d+1,j} = \sum_{i=d-j+2}^{d} (x_{i+1,j-(d-i)} - x_{i,j-(d-i)-1}) + x_{d-j+2,1}$$

$$= \sum_{i=d-j+1}^{d} \delta_{i,j-(d-i)}(X_{k}),$$

(2.43)

where the last equality follows from the definition (2.42) of the $\{\delta_{i,j}\}$’s. This proves that $\{\delta_{i,j}(X_{k})\}$ fixes the choices of $\{x_{d+1,j}\}_{j=d+d_{2}+1}^{d_{2}+d}$. As $X_{k} \in \mathcal{L}_{k}$ we also have that

$$x_{d+1,j} = j, \quad j = 1, 2, \ldots, d_{2}.$$

(2.44)

The last two observations prove the claim.
To complete the proof of the bound on $\mathcal{N}_{\ell,k}$, for $d \geq 0$, we note that the remaining indices of $X_{d+1}$, i.e. $\{x_{d+1,j}\}_{j=0}^{d+2}$ can be chosen in $\binom{N+d}{k+\hat{d}}$ ways. The fact that $X_k \in \mathcal{L}_{\ell,k} \subset L_{\ell,k}$ also implies that $\{\delta_{i,j}(X_k)\}$ can be chosen in

$$\prod_{i=1}^{d_0} \binom{\hat{e}_i - 1}{k+d_2 - 1} \cdot \prod_{i=d_0+1}^{d} \binom{\hat{e}_i + k + d_2}{k+d_2}$$

ways.

From (2.42) it is immediate that choosing $\{\delta_{i,j}(X_k)\}$ and $X_{d+1}$ fixes $X_k$. So, to find the bound on $\mathcal{N}_{\ell,k}$ we then need to find the number of choices $X_k' \in \mathcal{L}_{\ell,k} \subset L_{\ell,k}$ such that $X_{d+1}' = X_{d+1}$. This amounts to choosing only $\{\delta_{i,j}(X_k')\}$, and the number of such choices, as already seen above, is bounded by (2.45). Therefore, combining the above bounds we arrive at the desired bound for $\mathcal{N}_{\ell,k}$, when $d \geq 0$.

It remains to prove (2.24) for $d < 0$. To this end, we claim that choosing $\{\delta_{i,j}(X_k)\}_{i,j}$ automatically fixes $\{x_{d+1,j}\}_{j=0}^{d+2}$ for any $X_k \in \mathcal{L}_{\ell,k}$ and $k \geq |d|$. Indeed, for any $X_k \in \mathcal{L}_{\ell,k}$ the indices $\{x_{1,k+\ell}\}_{\ell=1}^{d}$ are fixed. Therefore, choosing $\{\delta_{i,j}(X_k)\}$ fixes the indices $\{x_{d+1,k+\ell}\}_{\ell=1}^{d}$ [recall (2.42)]. Now similar to (2.23) we observe that for any $j$ such that $d - d_0 + 1 \leq j \leq d$

$$x_{d+1,k+j} = \sum_{i=d_0+1}^{d} \delta_{i,k+j-(d-i)} + x_{d+1,k+j-(d-d_0)}.$$

Therefore choosing $\{\delta_{i,j}(X_k)\}$ also fixes $\{x_{d+1,k+j}\}_{j=1}^{d}$ and hence the claim. On the other hand $\{x_{d+1,j}\}_{j=1}^{d}$ are fixed by the definition of $\mathcal{L}_{\ell,k}$. Now repeating the same argument as in the case $d \geq 0$, we arrive at the bound (2.24) for $d < 0$. We omit further details. \qed

Finally we proceed to the proof of Lemma 2.2. We begin with the following lemma that shows that if $k < |d|$ then $\mathcal{L}_{\ell,k} = \emptyset$ unless the sum of the $\hat{e}_i$’s is close to $N$. The proof appears in [3, Proof of Lemma 4.3], see (4.24) there.

**Lemma 2.9** [3, Lemma 4.3] Fix an integer $k < |d|$ where $d$ as in Lemma 2.1. Then for any $\ell := (\ell_1, \ell_2, \ldots, \ell_d)$ such that $0 \leq \ell_i \leq N + d_2$, for all $i \in [d]$,

$$\mathcal{L}_{\ell,k} \neq \emptyset \Rightarrow \sum_{i=1}^{d} \hat{e}_i \geq N - d^2,$$

where $\mathcal{L}_{\ell,k}$ and $\{\hat{e}_i\}_{i=1}^{d}$ are as in (2.16) and (2.13), respectively.

We now prove Lemma 2.2.

**Proof of Lemma 2.2** As in the proofs of Lemmas 2.1 and 2.4, a key step is a bound on $\mathcal{N}_{\ell,k}$ of (2.23). Since $k < |d|$ we cannot use the bound derived in Lemma 2.5. Instead, we argue as follows.
We noted in the proof of Lemma 2.5 that choosing \( \{\delta_{i,j}(X_k)\} \) and \( X_{d+1} \) fixes the choice of \( \lambda_k' \). Therefore it follows that for any \( k \leq d \),

\[
\mathcal{M}_{\ell,k} \leq (N + d_2)^{k+d_2} \cdot \prod_{i=1}^{I_0} \left( \hat{\epsilon}_i - 1 \right)^2 \cdot \prod_{i=I_0+1}^{d} \left( \hat{\epsilon}_i + k + d_2 \right)^2 \leq (N + d_2)^{k+d_2} \cdot (N + d_2)^{2(k+d_2)} \cdot (N + k + 2d_2)^{2(k+d_2)} \leq (2N)^{10d}, \tag{2.46}
\]

for all large \( N \), where the second inequality follows from the fact that \( \hat{\epsilon}_i \leq N + d_2 \) for all \( i \in [d] \). Now Lemma 2.9 yields that

\[
\hat{P}_k(z) = a_d^{-k} N^{-\gamma(k-|\partial|)} \sum_{\mathcal{\Omega}} \prod_{i=1}^{I_0} \lambda_i(z)^{-\hat{\epsilon}_i} \prod_{i=I_0+1}^{d} \lambda_i(z)^{\hat{\epsilon}_i} \cdot \hat{\Omega}_{\ell,k},
\]

where the sum \( \sum_{\mathcal{\Omega}} \) is taken over all \( \mathcal{\Omega} \) such that \( \sum_{i=1}^{d} \hat{\epsilon}_i \geq N - d^2 \), and hence using the Cauchy–Schwarz inequality and arguing similarly to (2.29), we obtain that for any \( k = 0, 1, 2, \ldots, |\partial| - 1 \),

\[
\mathbb{E} \left[ \sup_{z \in \mathcal{S}_0^{-\varepsilon}} |\hat{P}_k(z)|^2 \right] \leq \left( \sum_{\mathcal{\Omega}} k(\mathcal{\Omega}) \cdot (\mathbb{E} |\hat{\Omega}_{\ell,k}|^2)^{1/2} \right)^2. \tag{2.47}
\]

Using Lemma 2.6 and (2.46), and proceeding similarly as in (2.30), for any \( \ell \) such that \( \sum_{i=1}^{d} \hat{\epsilon}_i \geq N - d^2 \), we derive that

\[
\sup_{z \in \mathcal{S}_0^{-\varepsilon}} |k(\mathcal{\Omega}, z)| \cdot (\mathbb{E} |\hat{\Omega}_{\ell,k}|^2)^{1/2} \leq |a_d|^{-k} \cdot N^{-\gamma d} \cdot \sqrt{d!} \cdot (2N)^{5d} \cdot (1 - \varepsilon_0)^{N/2},
\]

for all large \( N \). Hence, from (2.47) it is now immediate that

\[
\sup_{k=0}^{|\partial|-1} \mathbb{E} \left[ \sup_{z \in \mathcal{S}_0^{-\varepsilon}} |\hat{P}_k(z)|^2 \right] \leq (1 - \varepsilon_0)^{N/4},
\]

for all large \( N \). This completes the proof of the lemma. \( \square \)

We end this section with the proof of Corollary 2.3.

**Proof of Corollary 2.3** As \( \hat{\text{det}}_N(z) = \sum_{k=0}^{N} \hat{P}_k(z) \), see (2.2)–(2.5), applying the Cauchy–Schwarz inequality we find that

\[
\mathbb{E} \left[ \sup_{z \in \mathcal{S}_0^{-\varepsilon}} \left| \hat{\text{det}}_N(z) - \hat{P}_{|\partial|}(z) \right|^2 \right] \leq \mathbb{E} \left[ \sup_{z \in \mathcal{S}_0^{-\varepsilon}} \sum_{k \neq |\partial|} \left| \hat{P}_k(z) \cdot |\hat{P}_k(z)| \right| \right].
\]
\[ \leq \sum_{k,k' \neq |\emptyset|} \left\{ \mathbb{E} \left[ \sup_{z \in S_0^-} |\hat{P}_k(z)|^2 \right] \right\}^{1/2} \cdot \left\{ \mathbb{E} \left[ \sup_{z \in S_0^-} |\hat{P}_{k'}(z)|^2 \right] \right\}^{1/2} \]
\[ = \left( \sum_{k \neq |\emptyset|} \left\{ \mathbb{E} \left[ \sup_{z \in S_0^-} |\hat{P}_k(z)|^2 \right] \right\} \right)^{1/2}. \]

(2.48)

The proof of the corollary now completes upon applying Lemmas 2.1 and 2.2. Further details are omitted.

## 3 Tightness of the limiting random field

Our main goal in this section is to derive the tightness of the limiting random field \( \mathcal{P}_0^\infty (\cdot) \). Recall that Lemma 1.9 states that the random fields \( \{ \mathcal{P}_0^L (\cdot) \}_{L \in \mathbb{N}} \) approximate \( \mathcal{P}_0^\infty (\cdot) \). Thus to show the tightness of \( \mathcal{P}_0^\infty (\cdot) \), it will suffice to show the uniform tightness of the random fields \( \{ \mathcal{P}_0^L (\cdot) \}_{L \in \mathbb{N}} \), and to control the convergence. We will further derive an exponential decay of the tail of \( \mathcal{P}_0^\infty (\cdot) \). These results will eventually be used in the proofs of the main result Theorem 1.11 and Lemma 1.9.

We introduce the following notation. For any \( L \in \mathbb{N} \cup \{ \infty \} \), and \( \xi := (c_1, c_2, \ldots, c_d) \), where \( \{ c_i \}_{i=1}^d \) are non-negative integers, we set

\[ |\mathcal{P}_0^L (z)| := \sum_{\xi: \max_i c_i \leq L} \prod_{i=1}^{d_0} |\lambda_i(z)|^{-c_i} \cdot \prod_{i=d_0+1}^d |\lambda_i(z)|^{c_i} \]
\[ \cdot \left| \sum_{x \in \mathcal{L}_1 (\emptyset)} \sum_{\eta \in \mathcal{L}_2 (\emptyset)} (-1)^{k(x, \eta)} \det (E_\infty [\hat{X}; \hat{Y}]) \cdot \prod_{i=1}^d 1_{c_i(x, \eta) = c_i} \right|, \quad (3.1) \]

where we refer the reader to Definitions 1.5 and 1.6 for the notation \( \mathcal{L}_1 (\emptyset), \mathcal{L}_2 (\emptyset), \{ c_i(x, \eta) \}_{i=1}^d, \hat{X}, \hat{Y} \), and \( E_\infty \). For any \( L_1, L_2 \in \mathbb{N} \) such that \( L_1 < L_2 \) we set

\[ \mathcal{P}_0^{L_1, L_2} (z) := \mathcal{P}_0^{L_2} (z) - \mathcal{P}_0^{L_1} (z) \quad \text{and} \quad |\mathcal{P}|_0^{L_1, L_2} (z) := |\mathcal{P}|_0^{L_2} (z) - |\mathcal{P}|_0^{L_1} (z). \]

**Lemma 3.1** Fix \( \varepsilon > 0 \) and \( \emptyset \neq 0 \) an integer such that \(-d_2 \leq \emptyset \leq d_1\). Let \( E_\infty \) be a semi-infinite array of i.i.d. random variables with zero mean and unit variance. Then we have the following:

(i) For any \( L \in \mathbb{N} \),

\[ \mathbb{E} \left[ \sup_{z \in S_0^-} \left( |\mathcal{P}_0^L (z)|^2 \right) \right] \leq \mathcal{C}_L, \]

where \( \mathcal{C}_L \) is as in Lemma 2.7.
(ii) Fix \( L_1 \in \mathbb{N} \). Then

\[
\sup_{L_2: L_2 > L_1} \mathbb{E} \left[ \sup_{z \in \mathcal{S}_{0}^{-\varepsilon}} \left( |\mathbb{P}^{L_1, L_2}_{0}(z)|^2 \right) \right] \leq \exp(-\hat{c}L_1),
\]

where \( \hat{c} \) is as in Lemma 2.7.

Building on Lemma 3.1 we have the next result.

**Corollary 3.2** Under the same setup as in Lemma 3.1 we have the following:

(i) For any \( L \in \mathbb{N} \),

\[
\mathbb{E} \left[ \sup_{z \in \mathcal{S}_{0}^{-\varepsilon}} \left| \mathbb{P}^{L}_{0}(z) \right|^2 \right] \leq \hat{C}_L,
\]

where \( \hat{C}_L \) is as in Lemma 2.7.

(ii) Fix \( L_1 \in \mathbb{N} \). Then

\[
\sup_{L_2: L_2 > L_1} \mathbb{E} \left[ \sup_{z \in \mathcal{S}_{0}^{-\varepsilon}} \left( |\mathbb{P}^{L_1, L_2}_{0}(z)|^2 \right) \right] \leq \exp(-\hat{c}L_1),
\]

where \( \hat{c} \) is as in Lemma 2.7.

(iii) Consequently, \( \mathbb{P}^{\infty}_{0}(\cdot) \) is well defined and \( \mathbb{P}^{L}_{0}(\cdot) \to \mathbb{P}^{\infty}_{0}(\cdot) \) uniformly on \( \mathcal{S}_{0}^{-\varepsilon} \), as \( L \to \infty \), on a set of probability one. Moreover,

\[
\lim_{L \to \infty} \mathbb{E} \left[ \sup_{z \in \mathcal{S}_{0}^{-\varepsilon}} \left| \mathbb{P}^{\infty}_{0}(z) - \mathbb{P}^{L}_{0}(z) \right|^2 \right] = 0, \tag{3.2}
\]

and

\[
\mathbb{E} \left[ \sup_{z \in \mathcal{S}_{0}^{-\varepsilon}} \left| \mathbb{P}^{\infty}_{0}(z) \right|^2 \right] < \infty. \tag{3.3}
\]

Using Lemma 3.1 let us first provide a proof of Corollary 3.2.

**Proof of Corollary 3.2 (assuming Lemma 3.1)** By the triangle inequality we have that

\[
\left| \mathbb{P}^{L}_{0}(z) \right| \leq |\mathbb{P}^{L}_{1, L_2}(z)| \quad \text{and} \quad \left| \mathbb{P}^{L_1, L_2}_{0}(z) \right| \leq |\mathbb{P}^{L_1, L_2}_{0}(z)|.
\]
Thus (i) and (ii) follow from Lemma 3.1. We proceed to prove (iii). For any \( L \in \mathbb{N} \), write

\[
\mathcal{P}_L := \max_{L' : L < L' \leq 2L} \sup_{z \in \mathbb{S}^{-\varepsilon}} \left| \mathcal{P}_{\tilde{q}}^{L, L'}(z) \right|^2.
\]

Note that

\[
\mathcal{P}_L \leq \sum_{L' = L + 1}^{2L} \sup_{z \in \mathbb{S}^{-\varepsilon}} \left| \mathcal{P}_{\tilde{q}}^{L, L'}(z) \right|^2.
\]

Thus, by part (ii)

\[
\mathbb{E}[\mathcal{P}_L] \leq L \cdot \exp(-\tilde{c}L) \leq \exp\left(-\frac{\tilde{c}L}{2}\right),
\]

for all large \( L \). This, upon using the triangle inequality, further yields that

\[
\mathbb{E} \left[ \sup_{L' \geq L} \sup_{z \in \mathbb{S}^{-\varepsilon}} \left| \mathcal{P}_{\tilde{q}}^{L, L'}(z) \right|^2 \right] \leq \sum_{i = 0}^{\infty} \mathbb{E}[\mathcal{P}_{2iL}] \leq \exp\left(-\frac{\tilde{c}L}{4}\right),
\]

(3.4)

for all large \( L \). Now the bound (3.4) together with Markov’s inequality and the Borel–Cantelli Lemma show that on a set of probability one the random functions \( \{\mathcal{P}_{\tilde{q}}(\cdot)\} \) are uniformly Cauchy on \( \mathbb{S}^{-\varepsilon} \). So, the limit \( \mathcal{P}_{\tilde{q}}^\infty(\cdot) \), as \( L \to \infty \), exists and is well defined on that set of probability one. Furthermore, the convergence is uniform on \( \mathbb{S}^{-\varepsilon} \).

Finally, to prove (3.2) we simply note that

\[
\sup_{z \in \mathbb{S}^{-\varepsilon}} |\mathcal{P}_{\tilde{q}}^{\infty}(z) - \mathcal{P}_{\tilde{q}}^{L}(z)|^2 \leq \sup_{L' \geq L} \sup_{z \in \mathbb{S}^{-\varepsilon}} \left| \mathcal{P}_{\tilde{q}}^{L, L'}(z) \right|^2,
\]

whereas (3.3) follows from (3.2) and part (i) of this corollary. This completes the proof of the corollary. \( \square \)

The rest of this section will be devoted to the proof of Lemma 3.1. Recall from Sect. 2 that analogous bounds were proved for the dominant term in the expansion of \( \hat{\det}_N(z) \), see Lemma 2.7. We will prove Lemma 3.1 by establishing the equality of the laws of the random fields \( \{\mathcal{P}_{\tilde{q}}(\cdot)\}_{\tilde{q}} \) and \( \{\mathcal{P}_{\tilde{q}}(\cdot)\}_{\tilde{q}} \) for any \( L \in \mathbb{N} \) and all large \( N \). This is the content of our next result.

**Lemma 3.3** Fix \( L, L_1, L_2 \in \mathbb{N} \) such that \( L_1 < L_2 \), and an integer \( \delta \neq 0 \) such that \( -d_2 \leq \delta \leq d_1 \). Then for all large \( N \) (depending only on \( L \) and \( L_2 \)), we have the following:

(i) The joint laws of the random fields \( \{\hat{\mathcal{P}}_{\tilde{q}}^{L_1}(z) \} \) and \( \{\hat{\mathcal{P}}_{\tilde{q}}^{L_2}(z) \} \) coincide.
The laws of the random fields \( \{ \hat{P}_L^L(z) \}_{z \in S_0} \) and \( \{ \Psi^L_0(z) \}_{z \in S_0} \) coincide.

We note that equipped with Lemma 3.3(ii) the proof of Lemma 3.1 is immediate from Lemma 2.7. Further details are omitted. Thus, it only remains to prove Lemma 3.3.

Before presenting the proof, we recall all relevant notation and provide a sketch. For brevity we sketch the proof of Lemma 3.3(ii). The idea behind the proof of the first part is the same.

From (2.37) we have that

\[
\hat{P}_L^L(z) := \sum_{\ell: \max \ell_i \leq L} \prod_{i=1}^{d_1-\varnothing} \lambda_i(z)^{-\hat{\ell}_i} \cdot \prod_{i=d_1-\varnothing+1}^{d} \lambda_i(z)^{\hat{\ell}_i} \cdot \sum_{x | \varnothing \in \mathcal{L}_L} (-1)^{\text{sgn}(\sigma_x) \text{sgn}(\sigma_y)} \det(E_N[\bar{X} \cdot \bar{Y}]),
\]

where \( X_d := (X_1, X_2, \ldots, X_{d+1}) \), \( d_0 = d_1 - \varnothing \),

\[
X_i := \{ x_{i,1} < x_{i,2} < \cdots < x_{i,|\varnothing|+d_2} \}, \quad i \in [d+1],
\]

\[
\mathcal{L}_L := \{ X | \varnothing \in L, |i| \in [d] \} \quad \text{and} \quad x_{d+1,j} = j; \quad j \in [d_2],
\]

\[
L := \{ X | \varnothing \in L, \sum_{|\varnothing|+d_2}^{d_0} (x_{i,j} - x_{i+1,j} + 1) = \hat{\ell}_i; \quad i = 1, 2, \ldots, d_0, \}
\]

\[
\hat{\ell}_i := \begin{cases} \ell_i & \text{if } i > d_0 \\ N + d_2 - \ell_i & \text{if } i \leq d_0 \end{cases}
\]

and

\[
\bar{X} := \bar{X}(X_1) := X_1 \cap [N] \quad \text{and} \quad \bar{Y} := \bar{Y}(X_{d+1}) := (X_{d+1} - d_2) \cap [N],
\]

see Definitions 1.4, 1.5 and Eqs. (2.9), (2.12) and (2.13). Since \( \hat{P}_L^L(z) \) involves entries of the noise matrix \( E_N \), it is not apriori clear that the distribution of the random field \( \{ \hat{P}_L^L(z) \}_{z \in S_0} \) is free of \( N \), for all large \( N \). We find affine maps that map bijectively the relevant subset of \( \mathcal{X} | \varnothing \) to that of \( (\bar{x}, \bar{\eta}) \), for all large \( N \), where \( x \in \mathcal{L}_1(\varnothing) \) and \( \eta \in \mathcal{L}_2(\varnothing) \) with \( \mathcal{L}_1(\varnothing) \) and \( \mathcal{L}_2(\varnothing) \) as in Definition 1.5. The rational behind the existence of such
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\[
\begin{array}{cccc}
  x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\
  x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\
  x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\
  x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \\
  x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} \\
  x_{6,1} & x_{6,2} & x_{6,3} & x_{6,4} \\
  x_{7,1} & x_{7,2} & x_{7,3} & x_{7,4} \\
\end{array}
\]

\[
\begin{array}{cccc}
  x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\
  x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\
  x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\
  x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \\
  x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} \\
  x_{6,1} & x_{6,2} & x_{6,3} & x_{6,4} \\
  x_{7,1} & x_{7,2} & x_{7,3} & x_{7,4} \\
\end{array}
\]

Fig. 3 A schematic representation of the entries of the set \(X_{|\bar{d}|} \in \mathcal{L}_{\ell,|\bar{d}|}\), for \(d_1 = d_2 = 3\) and \(|\bar{d}| = 1\). The condition \(\max \hat{\ell}_i \leq L\) induces a partition illustrated by the empty boxes, for \(\bar{d} = 1\) (left panel) and \(\bar{d} = -1\) (right panel). For all large \(N\), in both panels the entries in the left block (demarcated by the empty boxes) are \(O(L)\), whereas the entries in the other block are \(N - O(L)\). Furthermore, rotating the left blocks in both panels clockwise and the right blocks anti-clockwise we note that the shapes of tableaux thus produced matches with those appearing in Fig. 2.

affine maps is that as \(X_{|\bar{d}|} \in \mathcal{L}_{\ell,\bar{d}}\), the restriction \(\max \hat{\ell}_i \leq L\) ensures that for all large \(N\), a sub-collection of the array of integers \(\{x_{i,j}\}_{i \in [d+1], j \in [d_2+|\bar{d}|]}\) is \(O(L)\), whereas the rest are \(N - O(L)\). This induces the affine transformations. This observation further leads to a partition of \(X_{|\bar{d}|}\) which then gives the shapes of the tableaux appearing in Definition 1.5. See Fig. 3 for a pictorial description of these observations.

To complete the argument we then confirm that

\[
c_i(\tau, \eta) = \hat{c}_i \quad \text{for all} \quad i \in [d] \quad \text{and} \quad (-1)^{\text{sgn}(\sigma_\tau) \text{sgn}(\sigma_\eta)} = (-1)^{\text{j}(\tau, \eta)}, \quad (3.9)
\]

under those maps, where \(\{c_i(\tau, \eta)\}_{i \in [d]}\) and \(j(\tau, \eta)\) are as in Definition 1.5.

The above mentioned maps also induce mappings between the entries of \(E_N\) and those of \(E_{\infty}\). Since all maps are bijections, using the fact that the entries of \(E_N\) are i.i.d., it follows that joint law of the random variables under the summation in the RHS of (3.5) is the same as that of (1.6). This establishes the equality in the distributions of the random fields \(\{\hat{P}_{|\bar{d}|}(z)\}_{z \in \mathcal{S}_3}\) and \(\{\hat{P}_{\bar{d}}(z)\}_{z \in \mathcal{S}_2}\). Below we carry out in detail these steps.

**Proof of Lemma 3.3** We will only prove (i). The proof of (ii), being similar, will be omitted. Fix \(L_1, L_2 \in \mathbb{N}\) such that \(L_1 < L_2\).

As indicated by Fig. 3 the forms of the maps differ for \(\bar{d} > 0\) and \(\bar{d} < 0\). For \(\bar{d} > 0\), denote

\[
(\mathcal{S}^+_N(X_{|\bar{d}|}))_{i,j} := N + d_2 + 1 - x_{j,\bar{d}+d_2-i+1},
\]

\[
j \leq d + 1 - i \quad \text{and} \quad i = 1, 2, \ldots, \bar{d} + d_2,
\]

and

\[
(\mathcal{S}^-_N(X_{|\bar{d}|}))_{i,j} := x_{d+2-j,i}, \quad j \leq d - d_0 + 1 - i \quad \text{and} \quad i = 1, 2, \ldots, \bar{d} + d_2.
\]
For $\vartheta < 0$, denote
\[
(\mathcal{G}_N^-(\mathcal{X}[0]))_{i,j} := N + d_2 + 1 - x_{j,d_2-\vartheta-i+1},
\]
\[j \leq \min\{d + 1 - \vartheta - i, d + 1\} \text{ and } i = 1, 2, \ldots, d_2\]
and
\[
(\mathcal{F}_N^-(\mathcal{X}[0]))_{i,j} := x_{d+2-j,i},
\]
for $j \leq d + 1$, $i = 1, 2, \ldots, -\vartheta$ and $j \leq d_2 + 1 - i$, $i = -\vartheta + 1, -\vartheta + 2, \ldots, d_2$.

Consider the case $\vartheta > 0$. It is clear from their definitions that the shapes of the tableaux induced by $\mathcal{G}_N^+(\mathcal{X}[0])$ and $\mathcal{F}_N^+(\mathcal{X}[0])$ are given by $\mu_1$ and $\mu_2$, where $\mu_1$ and $\mu_2$ are as in Definition 1.5. Using (3.6) it is immediate that if $\mathcal{X}[0] \in \mathcal{L}_{\ell,|0|}$ then
\[
(\mathcal{G}_N^+(\mathcal{X}[0]))_{i,1} = (\mathcal{F}_N^+(\mathcal{X}[0]))_{i,1} = i, \quad \text{for } i \in [d_2].
\]

To show that $\mathcal{G}_N^+(\mathcal{X}[0]) \in \mathcal{L}_1(\vartheta)$ and $\mathcal{F}_N^+(\mathcal{X}[0]) \in \mathcal{L}_2(\vartheta)$ we need to prove that they are weakly increasing in every row, and strictly increasing in every column and along the southwest diagonals. As $\mathcal{X}[0] \in \mathcal{L}_{\ell,|0|}$, upon recalling the definition of $\mathcal{L}_{\ell,|0|}$ from (3.7) these are also immediate. Now we check (3.9). Recalling the definitions of $\{c_i(\varrho, \eta)\}_{i=1}^d$ from Definition 1.5 we find that for $i \in [d_0]$,
\[
c_i(\mathcal{G}_N^+(\mathcal{X}[0]), \mathcal{F}_N^+(\mathcal{X}[0])) = \sum_{j=1}^{d-d_0} \left( (\mathcal{G}_N^+(\mathcal{X}[0]))_{j,i+1} - (\mathcal{G}_N^+(\mathcal{X}[0]))_{j,i} + 1 \right)
\]
\[= \sum_{j=1}^{d-d_0} (x_{i,\vartheta+d_2-j+1} - x_{i+1,\vartheta+d_2-j+1} + 1)
\]
\[= \sum_{j=1}^{\vartheta+d_2} (x_{i,j} - x_{i+1,j} + 1) = \hat{\ell}_i,
\]
where we have used the fact that $\vartheta + d_2 = d_1 - d_0$. Similarly for $i \in [d]\backslash[d_0]$, recalling that $d_0 = d_1 - \vartheta$ we obtain
\[
c_i(\mathcal{G}_N^+(\mathcal{X}[0]), \mathcal{F}_N^+(\mathcal{X}[0])) = (\mathcal{F}_N^+(\mathcal{X}[0]))_{1,d_1-i} + (\mathcal{G}_N^+(\mathcal{X}[0]))_{1,i} - 1
\]
\[+ \sum_{j=2}^{i-d_0} ((\mathcal{G}_N^+(\mathcal{X}[0]))_{j,d_1-i} - (\mathcal{F}_N^+(\mathcal{X}[0]))_{j-1,d_2-i})
\]
\[+ \sum_{j=2}^{d_1-i} ((\mathcal{G}_N^+(\mathcal{X}[0]))_{j,i} - (\mathcal{G}_N^+(\mathcal{X}[0]))_{j-1,i+1}) - (\vartheta + d_2)
\]
\[= x_{i+1,1} + (N + d_2 - x_{i,\vartheta+d_2}) + \sum_{j=2}^{i-d_0} (x_{i+1,j} - x_{i,j-1})
\]
\[\hat{\ell}\text{ Springer}
\[ d+1-i \\
+ \sum_{j=2}^{d+1}\left(x_{i+1,0+d_2-j+2} - x_{i,0+d_2-j+1}\right) - (0 + d_2) \]

\[ = x_{i+1,1} + \sum_{j=2}^{d+1}\left(x_{i+1,j} - x_{i,j-1}\right) + (N + d_2 - x_{i,0+d_2}) - (0 + d_2) = \hat{\ell}_i. \]

Now we proceed to show that \( z(\mathcal{G}_N^+(\chi|_\emptyset), \delta_N^+(\chi|_\emptyset)) = \text{sgn}(\sigma_X) \text{sgn}(\sigma_Y). \) To this end, recalling Definition 1.5 again, from (3.8), we find that

\[ \hat{X}(\mathcal{G}_N^+(\chi|_\emptyset), \delta_N^+(\chi|_\emptyset)) = N + d_2 - X \quad \text{and} \quad \hat{Y}(\mathcal{G}_N^+(\chi|_\emptyset), \delta_N^+(\chi|_\emptyset)) = Y + d_2. \]

(3.10)

Since \( \chi|_\emptyset \in L_\ell,\emptyset \) (recall (3.7)), we find that for \( j \in [\emptyset] \),

\[ N + d_2 - x_{1,j} \leq N + d_2 - x_{d_0+1,j} \]

\[ = \sum_{i=d_0+1}^{d-j} (x_{i+1,j+d_0+2-i} - x_{i,j+d_0+1-i}) + (N + d_2 - x_{d_0+d_2}) \leq dL_2, \]

where the last step follows under the assumption that \( \max \hat{\ell}_i \leq L. \) Thus \( X \subset [N] \setminus [N - dL_2]. \) Therefore, while computing \( \text{sgn}(\sigma_X) \) one can view \( X \) as a subset of \([N] \setminus [N - dL_2]\) for all large \( N \), which in particular shows that \( \text{sgn}(\sigma_X) \) is free of \( N \). Therefore, together with (3.10) we derive that \( \text{sgn}(\sigma_X) = \text{sgn}(\hat{X}(\mathcal{G}_N^+(\chi|_\emptyset), \delta_N^+(\chi|_\emptyset))) \) for all large \( N \). A similar argument shows that \( \text{sgn}(\sigma_Y) = \text{sgn}(\hat{Y}(\mathcal{G}_N^+(\chi|_\emptyset), \delta_N^+(\chi|_\emptyset))) \) for all large \( N \). Hence the map has all the desired properties.

Finally to complete the proof we further note that the map \( \chi|_\emptyset \mapsto (x, \eta) \) is a non-singular linear transformation and therefore a bijection. Therefore, we deduce that the joint law of the random variables

\[ \left\{ \prod_{i=1}^{d_1-\emptyset} \lambda_i(z) \hat{\ell}_i, \prod_{i=d_1-\emptyset+1}^{d} \lambda_i(z) \hat{\ell}_i, \sum_{\chi|_\emptyset \in \mathcal{L}_\ell,\emptyset} (-1)^{\text{sgn}(\sigma_X) \text{sgn}(\sigma_Y)} \det(E_N[X; Y]) \right\}_{\max \hat{\ell}_i \leq L_2, z \in S_\emptyset} \]

is equal to that of

\[ \left\{ c(x, \eta) \cdot (-1)^{3(f, -\eta)} \det(E_{\infty}[\hat{X}; \hat{Y}]) \right\}_{x \in \mathcal{L}_1(\emptyset), \eta \in \mathcal{L}_2(\emptyset), \max_j c_j(x, \eta) \leq L_2, z \in S_\emptyset}. \]

(3.12)

The proof for the case \( \emptyset < 0 \) is similar and hence omitted.

To finish the proof we now note that the bivariate random field \( \{(|P|_{0}^{L_1}(z), |P|_{0}^{L_2}(z))\}_{z \in S_\emptyset} \) is some function of the random variables in (3.11), and the bivariate random field \( \{(|\hat{P}|_{0}^{L_1}(z), |\hat{P}|_{0}^{L_2}(z))\}_{z \in S_\emptyset} \) is the same function of the random variables in (3.12). Thus these two bivariate random fields are indeed equal in distribution. This completes the proof of the lemma.
4 Anti-concentration bounds

In Sect. 3 we have already identified the limiting random field and derived its tightness. We recall from Sect. 1.2 that to prove Theorem 1.11 we need to establish that the limiting random field is not identically zero on a set of probability one. To do this we will require the anti-concentration property of the entries of the noise matrix as given by Assumption 1.8. The bound on the Lévy concentration function on the entries of the noise matrix will be used to derive an appropriate anti-concentration bound on the dominant term in the expansion of \( \hat{\det}_N(z) \) [see (2.5)] which will be later utilized to obtain the desired conclusion for the limiting random field.

We begin by providing the following general anti-concentration bound for polynomials in independent real or complex-valued random variables, satisfying a bound on their Lévy concentration function given by Assumption 1.8, such that the degree of every variable is at most one.

**Proposition 4.1** Fix \( k, n \in \mathbb{N} \) and let \( \{U_i\}_{i=1}^n \) be a sequence of independent real or complex-valued random variables, whose Lévy concentration functions satisfy the bound (1.7). Let \( Q_k(U_1, U_2, \ldots, U_n) \) be a homogenous polynomial of degree \( k \) such that the degree of each variable is at most one. That is,

\[
Q_k(U_1, U_2, \ldots, U_n) := \sum_{\mathcal{I} \in \binom{[n]}{k}} b(\mathcal{I}) \prod_{i \in \mathcal{I}} U_i,
\]

for some collection of complex-valued coefficients \( \{b(\mathcal{I}); \mathcal{I} \in \binom{[n]}{k}\} \), where \( \binom{[n]}{k} \) denotes the set of all \( k \) distinct elements of \([n]\).

Assume that there exists an \( I_0 \in \binom{[n]}{k} \) such that \( |b(I_0)| \geq c^* \) for some absolute constant \( c^* > 0 \). Then for any \( \varepsilon \in (0, e^{-1}] \) we have

\[
P( |Q_k(U_1, U_2, \ldots, U_n)| \leq \varepsilon ) \leq \bar{C} \cdot (c^* \wedge 1)^{-\eta} \varepsilon^{\eta} \left( \log \left( \frac{1}{\varepsilon} \right) \right)^{k-1},
\]

where \( \eta \in (0, 2] \) is as in (1.7) and \( \bar{C} < \infty \) is some large absolute constant.

When \( \{U_i\}_{i=1}^n \) are independent real valued random variables and have uniformly bounded densities with respect to the Lebesgue measure, an anti-concentration bound analogous to the above was obtained in [3] (see Proposition 4.5 there), with \( \eta = 1 \). The proof of Proposition 4.1 follows from a simple modification of the proof of [3, Proposition 4.5]. We include it for completeness.

**Proof** Since

\[
\mathcal{L}(U_i, \varepsilon) = \sup_{w \in \mathbb{C}} P(|U_i - w| \leq \varepsilon) \leq C \varepsilon^\eta, \quad i \in [n],
\]

where \( C \) and \( \eta \) are as in (1.7), using the joint independence of \( \{U_i\}_{i \in [n]} \) the desired anti-concentration property is immediate for \( k = 1 \). To prove the general case we proceed by induction.
The idea behind the proof is that since $Q_k$ is a polynomial such that the degree of each $U_i$ is at most one, for $i_0 \in I_0$ one can write $Q_k = Q \cdot U_{i_0} + \tilde{Q}$, for some $Q, \tilde{Q}$ independent of $U_{i_0}$. Thus, the anti-concentration bound of $Q_k$ depends on that of $Q$. The advantage of this decomposition is that the degree of $Q$ is $(k - 1)$. So one can iterate the above argument to obtain the desired anti-concentration bound for $Q_k$. To formulate this idea we introduce some notation. Order the elements of $I_0$ and denote them by $i_0^1, i_0^2, \ldots, i_0^k$. For $j \leq k$, define $I_0^j : = \{ i_0^j, i_0^{j+1}, \ldots, i_0^k \}$. Set

$$Q_k^0 := Q_k^0(U_i; i \notin I_k^0) := \sum_{I : I \supset I_k^0} b(I) \prod_{\ell \in I \setminus I_k^0} U_{i_0^\ell} + \tilde{Q},$$

$$Q_k^1 := Q_k^1(U_i; i \notin I_k^0) := \sum_{I : I \cap I_k^0 = \emptyset} b(I) \prod_{\ell \in I} U_{i_0^\ell}.$$

For $1 \leq j \leq k - 1$, we iteratively define

$$Q_j^0 := Q_j^0(U_i; i \notin I_j^0) := \sum_{I : I \supset I_j^0} b(I) \prod_{\ell \in I \setminus I_j^0} U_{i_0^\ell} + \tilde{Q},$$

$$Q_j^1 := Q_j^1(U_i, i \notin I_j^0) := \sum_{I : I \supset I_j^0, i_0^j \notin I} b(I) \prod_{\ell \in I \setminus I_j^0} U_{i_0^\ell}.$$

Equipped with the above notation we see that

$$Q_k(U_1, U_2, \ldots, U_n) =: Q_{k+1}^0 = U_{i_0^1} \cdot Q_k^0 + Q_k^1,$$

$$Q_{j+1}^0 = U_{i_0^j} \cdot Q_j^0 + Q_j^1, \quad j = 1, 2, \ldots, k - 1,$$

and $Q_1^0 = a(I_0)$. We will prove inductively that

$$\mathbb{P} \left( \left| Q_j^0 \right| \leq \varepsilon \right) \leq (3Ce^n)^{j-1}(c_\ast \wedge 1)^{-n} \varepsilon^n \left( \log \left( \frac{1}{\varepsilon} \right) \right)^{j-2},$$

$$j = 2, 3, \ldots, k + 1,$$

(4.1)

from which the desired anti-concentration bound follows by taking $j = k + 1$. Hence, it only remains to prove (4.1).

For $j = 2$, $Q_0^j$ is a homogeneous polynomial of degree 1 in the variables $U_i$, and (4.1) follows from the assumptions on $\{U_i\}_{i=1}^n$ and the fact that $|b(I_0)| \geq c_\ast$. Assuming that (4.1) holds for $j = j_\ast$ and fixing $\delta \in (0, 1)$, we have that with $C_j := (3Ce^n)^{j-1}(c_\ast \wedge 1)^{-n}$,

$$\mathbb{P} \left( \left| Q_{j_\ast+1}^0 \right| \leq \varepsilon \right)$$
\[
\leq \mathbb{P}\left(\left|Q_{js}^0\right| \leq \delta\right) + \mathbb{E}\left[\mathbb{P}\left(\left|U_{i_0} + \frac{Q_{js}^1}{Q_{js}^0}\right| \leq \frac{\varepsilon}{|Q_{js}^0|} \left|U_{i}, i \neq T_{js}^0\right|\right) \cdot 1\left(|Q_{js}^0| \geq \delta\right)\right] \\
\leq C_{js} \delta^n \left(\log \left(\frac{1}{\delta}\right)\right)^{j_s-2} + C e^n \cdot \mathbb{E}\left[|Q_{js}^0|^{-\eta} 1\left(|Q_{js}^0| \geq \delta\right)\right],
\]  
(4.2)

where we have used the fact that \(Q_{js}^1\) and \(Q_{js}^0\) are independent of \(U_{i_0}^{T_{js}^0}\), and the bound on the Lévy concentration function [i.e. the bound (1.7)] for the latter. Using integration by parts, for any probability measure \(\mu\) supported on \([0, \infty)\) we have that
\[
\int_\delta^{e^{-1}} x^{-\eta} d\mu(x) = e^{-\eta} \mu([\delta, 1]) + \eta \int_\delta^{e^{-1}} \frac{\mu([\delta, t])}{t^{1+\eta}} dt.
\]
Therefore, using the induction hypothesis and the fact that \(\eta \in (0, 2)\), we have
\[
\mathbb{E}\left[|Q_{js}^0|^{-\eta} 1\left(|Q_{js}^0| \geq \delta\right)\right] \leq e^n + \mathbb{E}\left[|Q_{js}^0|^{-\eta} 1\left(|Q_{js}^0| \in [\delta, e^{-1}]\right)\right]
\leq 2e^n + 2 \int_\delta^{e^{-1}} \frac{\mathbb{P}(|Q_{js}^0| \leq t)}{t^{1+\eta}} dt \leq 2e^n + 2C_{js} \int_\delta^{e^{-1}} t^{-1} \left(\log \left(\frac{1}{t}\right)\right)^{j_s-2} dt
\leq 2e^n + \frac{2C_{js}}{j_s-1} \left(\log \left(\frac{1}{\delta}\right)\right)^{j_s-1}.
\]

Since for \(\delta \leq e^{-1}\) we have that \(\log(1/\delta) \geq 1\), combining the above with (4.2) and setting \(\delta = \varepsilon\) we establish (4.1) for \(j = j_s + 1\). This completes the proof. \(\square\)

Using Proposition 4.1, we now derive the following corollary which yields an appropriate lower bound on the dominant term.

**Corollary 4.2** Fix \(\varepsilon > 0\) and \(\delta \neq 0\) an integer such that \(-d_2 \leq \delta \leq d_1\). Let the entries of \(E_N\) satisfy the bound (1.7). Then there exists a constant \(C_* := C_*(\varepsilon, a)\) so that, for any \(z \in S_0^{-\varepsilon}\) and \(\varepsilon_0 \in (0, e^{-1}]\),
\[
\mathbb{P}\left(\left|P_{|B|}(z)\right| \leq \varepsilon_0\right) \leq C_* \varepsilon_0^{\eta} \left(\log \left(\frac{1}{\varepsilon_0}\right)\right)^{|B|-1},
\]
where \((-\lambda_{\varepsilon}(z))_{\ell=1}^{d}\) are the roots of the equation \(P_{z,a}(\lambda) = \lambda^{d_2}(a(\lambda) - z) = 0\) arranged in the non-increasing order of their moduli.

To prove Corollary 4.2 we will need the following lemma. Its proof is deferred to Sect. 6.

**Lemma 4.3** Fix \(\varepsilon > 0\) and \(\delta \neq 0\) an integer such that \(-d_2 \leq \delta \leq d_1\). For \(\delta > 0\) set \(X_* := [N]\setminus[N - \delta]\) and \(Y_* := [\delta]\). For \(\delta < 0\) set \(X_* := [-\delta]\) and \(Y_* := [N]\setminus[N + \delta]\). Then, there exists a constant \(c_0' := c_0'(\varepsilon, a) > 0\) so that
\[
\inf_{z \in S_0^{-\varepsilon}} \left|\frac{\det(T_N(a(z))|X_*^c, Y_*^c))}{\lambda_{\varepsilon}^{d_2}(z)^{N+d_2}}\right| \geq c_0', \quad \text{for all large } N,
\]
where \( \{ \lambda_\ell (z) \}_{\ell = 1}^d \) are as in Corollary 4.2.

**Proof of Corollary 4.2 (assuming Lemma 4.3)** We remind the reader that

\[
\hat{P}_{|d|} (z) := \frac{P_{|d|} (z)}{N - \gamma |d|} \prod_{i=1}^{d-|d|} \lambda_i (z)^{N+d-i},
\]

(4.3)

see (2.3) and (2.4). Recalling (2.1) we note that \( \hat{P}_{|d|} (z) \) is a homogeneous polynomial of degree \(|d|\) in the entries of the noise matrix \( E_N \) such that the degree of each entry is one. By Lemma 4.3, there exist \( X, Y \subset [N] \) with \( |X| = |Y| = |d| \) such that \( \det (T_N (z) [X^c; Y^c]) \) is uniformly bounded below for \( z \in S_{0}^{-\varepsilon} \). Thus, using (2.1) again, we may apply Proposition 4.1 to deduce that

\[
\mathbb{P} \left( \left| \hat{P}_{|d|} (z) \right| \leq \varepsilon_0 \right) \leq \tilde{C} (c' \eta) - \eta \varepsilon_0 \left( \log \left( \frac{1}{\varepsilon_0} \right) \right)^{|d|-1}.
\]

This completes the proof.

5 Proof of Theorem 1.11

Using results from Sects. 2–4 in this section we finally complete the proof of our main result Theorem 1.11. We begin with the proof of Lemma 1.9. Turning to do this task, in the result below we derive analyticity of the random fields \( \{ \mathbb{P}_0^L (\cdot) \} \).

**Lemma 5.1** Fix \( L \in \mathbb{N}, \varnothing \neq 0 \) such that \( -d_2 \geq \varnothing \leq d_1 \), and \( \varepsilon > 0 \). Then the maps \( z \mapsto \hat{P}_{|d|}^L (z) \) and \( z \mapsto \mathbb{P}_0^L (z) \) are analytic on \( S_{0}^{-\varepsilon} \).

As a first step we will argue that the map \( z \mapsto \hat{P}_{|d|}^L (z) \) is continuous and then use Riemann’s removable singularity lemma to derive its analyticity. To this end, we have the following lemma.

**Lemma 5.2** Let \( \varepsilon > 0 \) and \( \varnothing \) be an integer such that \( -d_2 \leq \varnothing \leq d_1 \). Let \( \mathcal{f} : \mathbb{C}^d \mapsto \mathbb{C} \) be a continuous map such that

\[
\mathcal{f}(\lambda_1, \lambda_2, \ldots, \lambda_d) = \mathcal{f}(\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(d)}),
\]

(5.1)

for all permutations \( \pi \) on \([d]\) for which \( \pi([d_1 - \varnothing]) = [d_1 - \varnothing] \). Then the map \( z \mapsto \mathcal{f}(\lambda_1 (z), \lambda_2 (z), \ldots, \lambda_d (z)) \) is continuous on \( S_{0}^{-\varepsilon} \).

**Proof** To prove the lemma we need to use continuity properties of the roots of the equation \( P_{z,a} (\lambda) = 0 \). This requires some notation. Let \( \mathbb{C}^d_{\text{sym}} \), the symmetric \( d \)-th power of \( \mathbb{C} \), denote the set of equivalent classes in \( \mathbb{C}^d \), where two points in \( \mathbb{C}^d \) are set to be equivalent if one can be obtained by permuting the coordinates of the other. Given any two points \( (\lambda_1, \lambda_2, \ldots, \lambda_d), (\mu_1, \mu_2, \ldots, \mu_d) \in \mathbb{C}^d_{\text{sym}} \) we define

\[
\text{dist}( (\lambda_1, \lambda_2, \ldots, \lambda_d), (\mu_1, \mu_2, \ldots, \mu_d)) := \inf_{\pi} \sup_{\ell} |\lambda_\ell - \mu_{\pi(\ell)}|.
\]
where the infimum is taken over all permutations \( \pi \) of \([d]\). This induces a metric on \( \mathbb{C}^d_{\text{sym}} \).

Let \( \tau : C \mapsto \mathbb{C}^d_{\text{sym}} \) be the map given by \( \tau(z) = (\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z)) \), where \( \{-\lambda_i(z)\}_{i=1}^{d} \) are the roots of the equation \( P_{z,a}(\lambda) = 0 \). It is well known that the map \( \tau(\cdot) \) is continuous (see [26, Appendix 5, Theorem 4A]).

Using this we now establish the continuity of the function \( f(\lambda_1(z), \ldots, \lambda_d(z)) \).

Consider any sequence \( \{z_n\} \) such that \( z_n \to z \in S^e_0 \), as \( n \to \infty \). Let \( \pi_n^* \) be the permutation such that

\[
\text{dist}(\tau(z), \tau(z_n)) = \sup_{\pi} |\lambda_{\pi}(z) - \lambda_{\pi_n^*(\pi)}(z)|.
\]

We claim that \( \pi_n^*(([d_1 - \delta]) = [d_1 - \delta] \) for all large \( n \), i.e. \( \pi_n^* \) maps \([d_1 - \delta] \) to \([d_1 - \delta] \). If not, then there exists \( \ell_n \in [d_1 - \delta] \) and \( \ell_n' \in [d]\) \([d_1 - \delta] \) such that \( \pi_n^*(\ell_n) = \ell_n' \). On the other hand \( z \in S^e_0 \) implies that \( z_n \in S^{-\varepsilon/2}_0 \), for all large \( n \). Therefore, the last two observations together with Lemma 2.6 imply that

\[
\text{dist}(\tau(z), \tau(z_n)) \\
\geq |\lambda_{\ell_n}(z) - \lambda_{\ell_n'}(z_n)| \\
\geq |\lambda_{\ell_n}(z)| - |\lambda_{\ell_n'}(z_n)| \geq |\lambda_{d_1 - \delta}(z)| \\
-|\lambda_{d_1 - \delta + 1}(z_n)| \geq \varepsilon_0', \quad (5.2)
\]

for some \( \varepsilon_0' > 0 \). Since \( \tau(z_n) \to \tau(z) \) the inequality (5.2) yields a contradiction. Hence, \( \pi_n^*(([d_1 - \delta]) = [d_1 - \delta] \) for all large \( n \), as claimed above. As \( f(\cdot) \) satisfies (5.1) this further implies that

\[
f(\lambda_1(z_n), \lambda_2(z_n), \ldots, \lambda_d(z_n)) = f(\lambda_{\pi_n^*(1)}(z_n), \lambda_{\pi_n^*(2)}(z_n), \ldots, \lambda_{\pi_n^*(d)}(z_n)),
\]

for all large \( n \). Since \( \text{dist}(\tau(z_n), \tau(z)) \to 0 \), as \( n \to \infty \), the desired continuity of the map \( z \mapsto f(\lambda_1(z), \ldots, \lambda_d(z)) \) is immediate. This completes the proof of the lemma.

\[
\square
\]

Building on Lemma 5.2 we now prove that the maps \( z \mapsto \hat{P}^L_{[\delta]}(z) \) and \( z \mapsto \mathcal{P}^L_{0}(z) \) are analytic on \( S^{-\varepsilon}_0 \).

**Proof of Lemma 5.1** Denote

\[
P^L_{[\delta]}(z) := \hat{P}^L_{[\delta]}(z) \cdot d_{\delta_1} |\delta_1| \cdot N^{-\gamma|\delta|} \cdot \prod_{i=1}^{d_1 - \delta} \lambda_i(z)^{N + d_2}.
\]

Recalling (2.7), (2.15), and (2.17), and the definition of \( \hat{\ell}_i \) from (2.13) we therefore note that \( P^L_{[\delta]}(z) \) is the sum of the sets \( (X_1, X_2, \ldots, X_{d+1}) \) such that \( \ell_i \geq N + d_2 - L \) for all \( i = 1, 2, \ldots, d_1 - \delta \), and \( \ell_j \leq L \) for all \( j = d_1 - \delta + 1, \ldots, d \). Since for any permutation \( \pi \) on \([d]\).

\( \square \) Springer
\[
\prod_{i=1}^{d} (J_{N+d_2} + \lambda_i(z) \, \text{Id}_{N+d_2}) = \prod_{i=1}^{d} (J_{N+d_2} + \lambda_i(z) \, \text{Id}_{N+d_2}),
\]

the representation (2.7) of \( P_{[0]}(z) \) further implies that \( P_{[0]}^L(z) \) is invariant under any permutation \( \pi \) on \([d]\) for which \( \pi([d_1 - \delta]) = [d_1 - \delta] \). Hence, by Lemma 5.2 the map \( z \mapsto P_{[0]}^L(z) \) is continuous on \( S_0^{-\varepsilon} \). The same lemma shows that the map \( z \mapsto \prod_{i=1}^{d_i-1} \lambda_i(z)^N \) is continuous on \( S_0^{-\varepsilon} \). Hence, so is the map \( z \mapsto \hat{P}_{[0]}^L(z) \).

Next to show the analyticity of \( \hat{P}_{[0]}^L(\cdot) \) we apply Riemann’s removable singularity theorem. For that it needs to be shown that except on a collection of isolated points \( \hat{P}_{[0]}^L(\cdot) \) is a holomorphic function.

Recall (3.5):

\[
\hat{P}_{[0]}^L(z) = \sum_{\ell: \max \ell \leq L \leq 1} \prod_{i=1}^{d_0} \lambda_i(z)^{-\hat{\ell}_i} \cdot \prod_{i=d_0+1}^{d} \lambda_i(z)^{\hat{\ell}_i} \prod_{X \in \mathbb{L}_\ell} (-1)^{\text{sgn}(\sigma_X) \text{sgn}(\sigma_Y)} \det(E_N[X; Y]).
\]

Let \( \bar{N} \) is the collection of \( z \)’s for which \( P_{z,a}(\cdot) = 0 \) has double roots. By [7, Lemma 11.4], the cardinality of \( \bar{N} \) is finite, and thus all its elements are isolated. Using the implicit function theorem it follows that for \( z \in S_0^{-\varepsilon} \setminus \bar{N} \) the roots of \( P_{z,a}(\cdot) = 0 \) are analytic in \( z \) (for a proof the reader is referred to [8]). Therefore there exists a reordering of the indices of the roots \( \{\lambda_i(z)\}_{i=1}^{d} \), denoted hereafter by \( \{\hat{\lambda}_i(z)\}_{i=1}^{d} \), such that the maps \( z \mapsto \hat{\lambda}_i(z) \) are holomorphic on \( S_0^{-\varepsilon} \setminus \bar{N} \). From its definition it further follows that, for all \( z \in S_0^{-\varepsilon} \), among \( \{\hat{\lambda}_i(z)\}_{i=1}^{d} \) there are exactly \((d_1 - \delta)\) roots that are strictly greater than one in moduli. So reusing the fact that \( \hat{P}_{[0]}^L(z) \) is invariant under any permutation of \( \{\hat{\lambda}_i(z)\}_{i=1}^{d_1-\delta} \) and any permutation of the rest of the \( \hat{\lambda}_i(z) \)’s, without loss of generality we may write

\[
\hat{P}_{[0]}(z) = \sum_{\ell: \max \ell \leq L \leq 1} \prod_{i=1}^{d_0} \hat{\lambda}_i(z)^{-\hat{\ell}_i} \cdot \prod_{i=d_0+1}^{d} \hat{\lambda}_i(z)^{\hat{\ell}_i} \prod_{X \in \mathbb{L}_\ell} (-1)^{\text{sgn}(\sigma_X) \text{sgn}(\sigma_Y)} \det(E_N[X; Y]), \quad z \in S_0^{-\varepsilon} \setminus \bar{N}. \tag{5.3}
\]

This indeed shows that \( \hat{P}_{[0]}^L(z) \) is a holomorphic function on \( S_0^{-\varepsilon} \setminus \bar{N} \). To apply Riemann’s removable singularity theorem we need to show that it is bounded in a neighborhood of \( \bar{N} \). This is immediate, as from the definition of the polynomial \( P_{z,a}(\lambda) = 0 \) we have that for any \( R < \infty \),

\[
\sup_{z \in B_{\mathbb{C}}(0, R)} \max_{i=1}^{d} |\lambda_i(z)| = O(1).
\]
This yields that \( \hat{P}_{L|d}(z) \) is analytically extendable to the whole of \( S_0^{-\varepsilon} \). Finally, since the function \( \hat{P}_{L|d}(z) \) is continuous at \( z \in \bar{N} \cap S_0^{-\varepsilon} \), we conclude that its analytic extension to \( S_0^{-\varepsilon} \) is the function itself. That is, the equality (5.3) continues to hold for \( z \in \bar{N} \cap S_0^{-\varepsilon} \), where by a slight abuse of notation we use \( \{-\hat{\lambda}_i(z)\}_{i=1}^d \) to denote the analytic extensions of the analytic parametrization of the roots of \( P_{z,a}(\lambda) = 0 \). Thus, the map \( z \mapsto \hat{P}_{L|d}(z) \) is indeed an analytic function.

Turning to prove the analyticity of the map \( z \mapsto P_{L|d}(z) \) we recall that the proof of Lemma 3.3 shows that the maps and \( \hat{P}_{L|d}(z) \) and \( P_{L|d}(z) \), when viewed as functions of \( z \) are the same map, albeit the entries of \( E_N \) gets replaced by that of \( E_\infty \) by the affine function defined there. This shows that \( \{P_{L|d}\}_{L \in \mathbb{N}} \) are random analytic functions as well, thereby completing the proof of this lemma. \( \square \)

Equipped with the necessary ingredients, we now proceed to the proof of Lemma 1.9.

**Proof of Lemma 1.9** We begin by noting that part (i) is a consequence of Lemma 5.1, whereas part (ii) is immediate from Corollary 3.2(iii).

So it only remains to prove that \( \mathbb{P}_0^\infty(\cdot) \) is not identically zero on a set of probability one. Without loss of generality we may assume that \( S_0 \) is non-empty. Fix any \( \varepsilon > 0 \) and pick any \( z_\star \in S_0^{-\varepsilon} \). Note that, as \( S_0^{-\varepsilon} \uparrow S_0 \neq \emptyset \), as \( \varepsilon \downarrow 0 \), for \( \varepsilon \) sufficiently small the sets \( S_0^{-\varepsilon} \) are non-empty, and hence a choice of \( z_\star \) is feasible for any sufficiently small \( \varepsilon \). Fix this choice of \( \varepsilon \) for the remainder of the proof.

Now fix any \( \delta > 0 \). From Corollaries 2.8 and 3.2(iii), upon using Markov’s inequality it follows that there exists an \( L \) such that

\[
\max \left\{ \mathbb{P}(|\hat{P}_{L|d}(z_\star)| \geq \delta), \mathbb{P}(|\hat{P}_d(z_\star)| \geq \delta) \right\} \leq \delta,
\]

where for brevity we set

\[
\mathbb{P}_d(\cdot) := \mathbb{P}_0^\infty(\cdot) - \mathbb{P}_0^L(\cdot).
\]

Thus, using the triangle inequality we find that

\[
\mathbb{P}(|\mathbb{P}_d^\infty(z_\star)| \leq \delta) \leq \mathbb{P}(|\hat{P}_d(z_\star)| \leq \delta + |\mathbb{P}_d^L(z_\star)|)
\]

\[
\leq \mathbb{P}(|\mathbb{P}_d^L(z_\star)| \leq 2\delta) + \delta
\]

\[
= \mathbb{P}(|\hat{P}_{L|d}(z_\star)| \leq 2\delta) + \delta \leq \mathbb{P}(|\hat{P}_{|d}(z_\star)| \leq 3\delta) + 2\delta,
\]

where the equality above follows from Lemma 3.3. Now, by Corollary 4.2, we further deduce from above that

\[
\mathbb{P}(|\mathbb{P}_d^\infty(z_\star)| \leq \delta) \leq C_\star(3\delta)^{\eta} \left( \log \left( \frac{1}{3\delta} \right) \right)^{|d|-1} + 2\delta.
\]
As $\delta > 0$ is arbitrary sending $\delta \downarrow 0$ we derive that

$$\mathbb{P}(\mathcal{P}_0^\infty(\cdot) \equiv 0 \text{ on } S_0) \leq \mathbb{P}\left( \sup_{z \in S_0^{-\epsilon}} |\mathcal{P}_0^\infty(z)| = 0 \right) \leq \limsup_{\delta \downarrow 0} \mathbb{P}(|\mathcal{P}_0^\infty(z_*)| \leq \delta) = 0.$$ 

The proof of the lemma is now complete.

\[\Box\]

**Proof of Theorem 1.11** We will use the following proposition from [15].

**Proposition 5.3** [15, Proposition 2.3] Suppose that a sequence of random analytic functions \(\{X_N\}\) converges in law to (an analytic random function) \(X\). Then, the zero process of \(X_N\) converges in law to the zero process of \(X\) provided that \(X \not\equiv 0\) almost surely.

In our case, we will use \(\hat{\text{det}}_N(z)\), see (2.5), for \(X_N\), and \(\mathcal{P}_0^\infty(z)\), see Definition 1.6 for \(X\). (Note that by Lemma 1.9, the latter is almost surely analytic.) Thus, the proof of Theorem 1.11 boils down to checking the conditions of Proposition 5.3, that is to checking the following.

(i) \(\hat{\text{det}}_N(\cdot)\) converges in law to \(\mathcal{P}_0^\infty(\cdot)\) in \(S_0^{-\epsilon}\).

(ii) \(\mathcal{P}_0^\infty(\cdot) \not\equiv 0\) in \(S_0^{-\epsilon}\).

To see (i), we recall the random functions \(\hat{P}_{[a]}(z)\) and \(\hat{P}_{L[\cdot]}(z)\), see (2.4) and (2.37). By Corollary 2.3, it is enough to prove (i) with \(\hat{\text{det}}_N(z)\) replaced by \(\hat{P}_{[a]}(z)\). By (2.38) and Corollary 2.8, for (i) it is then enough to prove that the law of \(\hat{P}_{L[\cdot]}(\cdot)\) converges, as first \(N \to \infty\) and then \(L \to \infty\), to the law of \(\mathcal{P}_0^\infty(\cdot)\). By Lemma 3.3(iii), the law of \(\hat{P}_{[a]}(\cdot)\) coincides, for \(N\) large, with that of \(\mathcal{P}_{L[i]}(\cdot)\), and the latter law is independent of \(N\). One now concludes (i) by noting that by Lemma 1.9(ii), \(\mathcal{P}_{L[i]}(\cdot)\) converges uniformly in \(S_0^{-\epsilon}\), as \(L \to \infty\), to \(\mathcal{P}_0^\infty(\cdot)\).

The point (ii) is a consequence of part (iii) of Lemma 1.9. This completes the proof of the theorem.

\[\Box\]

**6 Proof of Theorem 1.1**

Next we proceed to the proof of Theorem 1.1, i.e. we aim to show that there are no outliers outside the spectrum of the Toeplitz operator \(T(a)\). In the set-up of Theorem 1.1, Lemma 2.1 yields the desired upper bound on the non-dominant terms. In this set-up the dominant term is the non-random unperturbed Toeplitz matrix \(T_N(a(z))\). Hence to complete the proof we need a uniform lower bound on the latter.

**Lemma 6.1** Let \(a\) be a Laurent polynomial given by

$$a(\lambda) := \sum_{\ell = -d_2}^{d_1} a_\ell \lambda^\ell, \quad \lambda \in \mathbb{C},$$
for some $d_1, d_2 \in \mathbb{N}$. Fix $\varepsilon > 0$. Then, there exists a positive constant $c_0 > 0$ such that

$$
\inf_{z \in \mathbb{C} \cap B_{\mathbb{C}}(0, N^{1/2})} \left| \frac{\det(\mathbf{T}_N(a(z)))}{a_N^{d} \prod_{\ell=1}^{d_1} \lambda_{\ell}(z)^N} \right| \geq c_0, \quad \text{for all large } N,
$$

where $\{-\lambda_{\ell}(z)\}_{\ell=1}^{d}$ are the roots of the equation $P_{z,a}(\lambda) = \lambda^d a(\lambda) - z = 0$ arranged in the non-increasing order of their moduli and $d = d_1 + d_2$.

We will later check, see (6.14), that all eigenvalues of $\mathbf{T}_N(a) + \Delta_N$ are contained in $B_{\mathbb{C}}(0, N^{1/2})$ with high probability. Thus the uniform lower bound of Lemma 6.1 is sufficient to complete the proof of Theorem 1.1. For any $z \in \mathbb{C}$ the expression for the determinant of $\mathbf{T}_N(a(z))$ is well known: it follows from Widom’s formula (see [7, Theorem 2.8]) when the roots of roots of $P_{z,a}(\cdot) = 0$ are distinct, while in the other case one can use Trench’s formula [7, Theorem 2.10]. As Lemma 6.1 requires a uniform bound on the determinant for $z \in \mathbb{C} \cap B_{\mathbb{C}}(0, N^{1/2})$, we refrain from using [7, Theorems 2.8 and 2.10] and instead we use the observation by Bump and Diaconis [9], where they noted that irrespective of whether $P_{z,a}(\cdot) = 0$ has double roots or not, the determinant of a finitely banded Toeplitz matrix can be expressed as a certain Schur polynomial.

Before proceeding to the proof of Lemma 6.1 we recall the definition of the Schur polynomials. Given any partition $\nu := (v_1, v_2, \ldots, v_d)$ with $v_1 \geq v_2 \geq \cdots \geq v_d \geq 0$ we define Schur polynomial $S_\nu$ by

$$
S_\nu(\lambda_1, \lambda_2, \ldots, \lambda_d) := \frac{\det V_\nu(\lambda_1, \lambda_2, \ldots, \lambda_d)}{\det V_0(\lambda_1, \lambda_2, \ldots, \lambda_d)},
$$

where for any partition $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_d)$

$$
V_\alpha(\lambda_1, \lambda_2, \ldots, \lambda_d) = \begin{bmatrix}
\lambda_1^{\alpha_1+d-1} & \lambda_2^{\alpha_1+d-1} & \cdots & \lambda_d^{\alpha_1+d-1} \\
\lambda_1^{\alpha_2+d-2} & \lambda_2^{\alpha_2+d-2} & \cdots & \lambda_d^{\alpha_2+d-2} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{\alpha_d} & \lambda_2^{\alpha_d} & \cdots & \lambda_d^{\alpha_d}
\end{bmatrix},
$$

and $0 := (0, 0, \ldots, 0)$ denotes the zero partition. If $\{\lambda_{\ell}\}_{\ell=1}^{d}$ are not all distinct then both the numerator and the denominator of (6.1) are zero. In that case, the quotient needs to be evaluated using L’Hôpital’s rule. Therefore the proof of Lemma 6.1 also splits into two parts: $z \notin \tilde{N}$ and $z \in \tilde{N}$, where we remind the reader that $\tilde{N}$ is the collection of $z$’s for which $\{\lambda_{\ell}(z)\}_{\ell=1}^{d}$ are not all distinct and it is a set of finite cardinality. The first case is handled in the following lemma.

**Lemma 6.2** Under the same set-up as in Lemma 6.1, and in particular with the same $c_0$,

$$
\inf_{z \in \mathbb{C} \cap B_{\mathbb{C}}(0, N^{1/2}) \setminus \tilde{N}} \left| \frac{\det(\mathbf{T}_N(a(z)))}{a_N^{d} \prod_{\ell=1}^{d_1} \lambda_{\ell}(z)^N} \right| \geq c_0, \quad \text{for all large } N.
$$
We use the following continuity properties to derive Lemma 6.1 from Lemma 6.2.

**Lemma 6.3** Fix $\varepsilon > 0$. For any $z \in S_0^{\varepsilon}$, the maps $z \mapsto \det(T_N(a(z)))$ and $z \mapsto \prod_{\ell=1}^{d_1} |\lambda_\ell(z)|^N$ are continuous, where $\{\lambda_\ell(z)\}_{\ell=1}^d$ are as in Lemma 6.1.

It is obvious that Lemma 6.1 follows from Lemmas 6.2 and 6.3. To prove Lemma 6.3 we see that the continuity of the map $z \mapsto \det(T_N(a(z)))$ is obvious. The continuity of the other map is a consequence of Lemma 5.2 (applied with $\delta = 0$). We next prove Lemma 6.2.

**Proof of Lemma 6.2** It was noted in [9, proof of Theorem 1] that

$$\det(T_N(a(z))) = (-1)^{N \cdot d_1} a_{d_1}^N \cdot S_m(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z)), \tag{6.2}$$

where the partition $m$ is given by

$$m := (N, N, \ldots, N, 0, 0, \ldots, 0).$$

To evaluate the RHS of (6.2) we use the representation (6.1). The denominator of (6.1) is the determinant of the standard Vandermonde matrix. Hence, to complete the proof we expand the determinant in the numerator using Laplace’s expansion, find the dominant term, and show that the sum of the other terms is of smaller order.

Fix $z \notin \bar{N}$, implying that the roots $\{\lambda_\ell(z)\}_{\ell=1}^d$ of the polynomial equation $P_{z, a}(\cdot) = 0$ are all distinct. Now applying Laplace’s expansion of the determinant we find that

$$\det V_m(z) = (-1)^{\frac{d_1(d_1+1)}{2}} \sum_{M \in \binom{[d]}{d_1}} (-1)^{(|\sum_{i \in M} i|)} \det(V_m(z)[[d_1]; M])$$

$$\cdot \det(V_m(z)[[d]\setminus[d_1]; \tilde{M}]), \tag{6.3}$$

where we denote $V_m(z) := V_m(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z))$ and $\tilde{M} := [d] \setminus M$. Recalling the definition of $V_m(z)$ it follows that for any $M \in \binom{[d]}{d_1}$,

$$\nabla_m(z, M) := \det(V_m(z)[[d_1]; M]) \cdot \det(V_m(z)[[d]\setminus[d_1]; \tilde{M}])$$

$$= \prod_{\ell \in M} \lambda_\ell(z)^{N+d_2} \cdot \prod_{\ell < \ell' \in M} (\lambda_\ell(z) - \lambda_{\ell'}(z)) \cdot \prod_{\ell < \ell' \in \tilde{M}} (\lambda_\ell(z) - \lambda_{\ell'}(z)). \tag{6.4}$$

Moreover,

$$\det V_0(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z)) = \prod_{\ell < \ell' \in [d]} (\lambda_\ell(z) - \lambda_{\ell'}(z)). \tag{6.5}$$

Setting now $M = [d_1]$ and using (6.4) and (6.5) together with (6.11), we get

$$\frac{\det(V_m(z)[[d_1]; [d_1]]) \cdot \det(V_m(z)[[d]\setminus[d_1]; [d]\setminus[d_1]])}{V_0(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z))}$$
\[
\prod_{\ell \in [d_1]} |\lambda_\ell(z)|^{N+d_2} \cdot \prod_{\ell \in [d_1]} \prod_{\ell' \in [d]\setminus[d_1]} |\lambda_\ell(z) - \lambda_{\ell'}(z)|^{-1} \\
\geq \prod_{\ell \in [d_1]} |\lambda_\ell(z)|^N \cdot \prod_{\ell \in [d_1]} \prod_{\ell' \in [d]\setminus[d_1]} \left|1 - \frac{\lambda_{\ell'}(z)}{\lambda_\ell(z)}\right|^{-1} \\
\geq 2^{-d^2} \cdot \prod_{\ell \in [d_1]} |\lambda_\ell(z)|^N,
\]

(6.6)

uniformly on \(z \in S_0^e \setminus \tilde{N}\), for some \(c > 0\). Hence, in light of (6.1) and (6.2), to obtain a uniform lower bound on \(\det T_N(a(z))\), for \(z \in S_0^e \cap B_C(0, N^{1/2}) \setminus \tilde{N}\), it suffices to show that

\[
\frac{\sum_{M \in (d_1)!\setminus[[d_1]]} (-1)^{\sum_{i \in M} i} \mathcal{Q}_m(z, M)}{\det V_0(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z))} = o \left(\prod_{\ell=1}^{d_1} |\lambda_\ell(z)|^N\right),
\]

(6.7)

uniformly over \(z \in S_0^e \cap B_C(0, N^{1/2}) \setminus \tilde{N}\). Using (6.4) and (6.5) one may try to individually bound each of the terms in the LHS of (6.7). However, it can be seen that because of the division by the determinant of the Vandermonde matrix and due to the presence of double roots for \(z \in \tilde{N}\) some of the terms in LHS of (6.7) blow up as \(z\) approaches the set \(\tilde{N}\).

To overcome this issue we claim that for any \(z \notin \tilde{N}\), the numerator of the LHS of (6.7) contains a factor

\[
\prod_{\ell < \ell' \in [d_1]} (\lambda_\ell(z) - \lambda_{\ell'}(z)) \cdot \prod_{\ell < \ell' \in [d]\setminus[d_1]} (\lambda_\ell(z) - \lambda_{\ell'}(z)).
\]

(6.8)

Turning to prove this claim, fixing any \(\ell_0 < \ell_1 \in [d_1]\) we show that \((\lambda_{\ell_0}(z) - \lambda_{\ell_1}(z))\) is a factor of the numerator of the LHS of (6.7). Then repeating the same argument one can show that the same holds for \(\ell_0 < \ell_1 \in [d]\setminus[d_1]\). This gives the claim (6.8).

To show that \((\lambda_{\ell_0}(z) - \lambda_{\ell_1}(z))\) is a factor we fix any \(M \in (d_1)!\setminus[[d_1]]\). If \(M\) is such that \(\ell_0, \ell_1 \in M\) then by (6.4) it follows that \((\lambda_{\ell_0}(z) - \lambda_{\ell_1}(z))\) is indeed a factor. Same holds if \(\ell_0, \ell_1 \notin M\). So it boils down to showing that \((\lambda_{\ell_0}(z) - \lambda_{\ell_1}(z))\) is a factor of the sum

\[
\sum (-1)^{\sum_{i \in M} i} \mathcal{Q}_m(z, M),
\]

where the sum is taken over all \(M \in (d_1)!\setminus[[d_1]]\) such that exactly one of \(\ell_0\) and \(\ell_1\) are in \(M\). Pick any such \(M\) and without loss of generality assume \(\ell_0 \in M\). Then we define \(M_1\) by replacing \(\ell_0\) by \(\ell_1\) and keeping the other elements as is. Note that this map is a bijection and moreover \(M_1 \neq [d_1]\). We will show that for any \(M\) and \(M_1\) chosen and defined as above \((\lambda_{\ell_0}(z) - \lambda_{\ell_1}(z))\) is a factor of

\[
f(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z)) := (-1)^{\sum_{i \in M_1} i} \mathcal{Q}_m(z, M) + (-1)^{\sum_{i \in M_1} i} \mathcal{Q}_m(z, M_1).
\]
Indeed, the claim is immediate if $d$ depends on $N \ell$ and $\ell$. This follows from the following two observations. First, from the definition of $M_1$ and (6.4) we observe that $\mathcal{U}(z, M)$ equals $\mathcal{U}(z, M_1)$, upto a change in sign, when $\lambda_{\ell_0}(z)$ is replaced by $\lambda_{\ell_1}(z)$. Second, from (6.4) we further note that the change in sign is $\ell_1 - \ell_0 - 1$ which is the sum total of the number of elements in $M$ and $\bar{M}$ between $\ell_0$ and $\ell_1$. Thus we have that $f = 0$ when $\lambda_{\ell_0}(z) = \lambda_{\ell_1}(z)$. So we now have (6.8).

From (6.4) we have that for every $M \in \binom{[d]}{|d|}$ the determinant $\mathcal{U}_m(z, M)$ is a multivariate polynomial in $[\lambda_{\ell}(z)]_{\ell=1}^d$ with coefficients free of $N$ (only the exponents may depend on $N$). Thus, equipped with (6.8) and using (6.5) we next obtain that

\[
\sum_{M \in \binom{[d]}{|d|} \setminus [d_1]} (-1)^{|\ell'|} \mathcal{U}_m(z, M) = \mathcal{U}(\lambda_1(z), \lambda_2(z), \ldots, \lambda_d(z)) \cdot \prod_{\ell \in [d_1]} \prod_{\ell \in [d] \setminus [d_1]} (\lambda_{\ell}(z) - \lambda_{\ell'}(z))^{-1},
\]

for some multivariate polynomial $\mathcal{U}(\cdot)$ with coefficients free of $N$. Since the sum in (6.9) is taken over $M \neq [d_1]$, using (6.4) once more we find that each of the terms in the polynomial $\mathcal{U}(\cdot)$ is bounded by

\[
C \prod_{\ell=1}^{d_1-1} |\lambda_{\ell}(z)|^{N+d_2} \cdot |\lambda_{d_1+1}(z)|^{N+d_2} \prod_{\ell=1}^{d} |\lambda_{\ell}(z)|^d \\
\leq C (1 - \varepsilon_0)^{2N} \prod_{\ell=1}^{d_1} |\lambda_{\ell}(z)|^N \cdot \prod_{\ell=1}^{d} |\lambda_{\ell}(z)|^{2d},
\]

for some constant $C < \infty$ and $\varepsilon_0 > 0$, where the last step follows from Lemma 2.6.

Next we claim that, for some constant $\widetilde{C} < \infty$ and $|z| > 1$, \n
\[
\max_{\ell} |\lambda_{\ell}(z)| \leq \widetilde{C} |z|.
\]

Indeed, the claim is immediate if $d_1 = 0$, for then $\max_{\ell} |\lambda_{\ell}(z)| = O(1/z)$. Assume therefore $d_1 > 0$. Consider any root $\lambda$ of the polynomial equation $P_{z,a}(\lambda) = 0$. Then,

\[
a_{d_1} \lambda^d = - \sum_{i=-d_2}^{d_1-1} (a_i - z\delta_i,0)\lambda^i.
\]

Therefore, assuming without loss of generality that $|\lambda| \geq 1$ and using the triangle inequality, we find that

\[
|a_{d_1}| |\lambda|^d \leq C' |z| \cdot |\lambda|^{d-1},
\]
for some $C' < \infty$, yielding the claim (6.11) also in case $d_1 > 0$. This, in particular implies that the RHS of (6.10) is upper bounded by

$$C(1 - \varepsilon_0)^{2N} \prod_{\ell=1}^{d_1} |\lambda_\ell(z)|^N \cdot (\tilde{C}|z|)^{2d_2}. \quad (6.12)$$

Since there are at most $O(N^d)$ terms in the polynomial $\Psi(\cdot)$ using Lemma 2.6 again, from (6.9) and the bound in (6.12) we deduce that

$$\left| \sum_{M \in \{[d_1]\} \setminus [d_1]} (-1)^{\left(\sum_{i \in M} i\right)} \Psi_m(z, M) \right| \leq (1 - \varepsilon_0)^N \prod_{\ell=1}^{d_1} |\lambda_\ell(z)|^N,$$

uniformly over $z \in S_0^{-\varepsilon} \cap B_C(0, N^{1/2}) \backslash \tilde{\mathcal{N}}$, for all large $N$. This yields (6.7) and thus the proof is complete. \hfill \Box

Before proving Theorem 1.1, we sketch the proof of Lemma 4.3, which is similar to that of Lemma 6.1.

**Proof of Lemma 4.3** Consider the case $d > 0$. Recalling Definition 2.1 we find that

$$T_N(a(z))[X^c_z, Y^c_z] = T_{N-d}(a, z; d_1 - d).$$

It also follows from there that the polynomial associated with the symbol of the Toeplitz matrix $T_{N-d}(a, z; d_1 - d)$ is $P_{z,a}(\cdot)$. Therefore, for $z \notin \tilde{\mathcal{N}}$ it does not have any double roots. Moreover, for $z \in S_0^{-\varepsilon}$ the number of roots of $P_{z,a}(\lambda) = 0$ that are greater than one in moduli is $d_1 - d$ and it equals the maximal positive degree of the Laurent polynomial associated with the Toeplitz matrix $T_{N-d}(a, z; d_1 - d)$. So we are in the set up of Lemma 6.1. Therefore proceeding as in the proof of Lemma 6.1 we deduce that

$$\inf_{z \in S_0^{-\varepsilon} \cap B_C(0, R) \backslash \mathcal{N}} \left| \det(T_N(a(z))[X^c_z, Y^c_z]) \right| \geq c_0^*, \quad \text{for all large } N,$$

and some constant $c_0^* > 0$.

We claim that for $d \neq 0$, the set $S_0^{-\varepsilon}$ is a bounded set. Indeed, from the definition of $\{S_0\}_{d=-d_2}^{d_1}$ we have that $\cup_{d=-d_2}^{d_1} S_0 = \mathbb{C} \setminus a(\mathbb{S})$. Hence, recalling (1.4), we deduce that $S_0 \subset \text{spec } T(a)$, for any $d \neq 0$. As the spectrum of $T(a)$ is contained in a disk of radius at most $\|T(a)\|$ centered at zero, the claim follows.

The last claim together with (6.11) we therefore have that

$$\inf_{z \in S_0^{-\varepsilon} \backslash \mathcal{N}} \left| \det(T_N(a(z))[X^c_z, Y^c_z]) \right| \geq \tilde{c}_0^*, \quad \text{for all large } N,$$

and some other constant $\tilde{c}_0^* > 0$.

\hfill \Box
A similar reasoning as in the proof of Lemma 6.3 shows that the map $z \mapsto \prod_{\ell=1}^{d_1-3} |\lambda_\ell(z)|^{N-|b|}$ is continuous on $S_{0}^{-\epsilon}$. Hence combining the last two observations we derive the desired uniform lower bound for $\varnothing > 0$. The proof for the case $\varnothing < 0$ is similar. \hfill \Box

We finally prove Theorem 1.1.

**Proof of Theorem 1.1** We begin by recalling (1.4), which implies that

$$((\text{spec } T(a))^{-\epsilon} = S_{0}^{-\epsilon}. $$

We write

$$\tilde{P}_k(z) := \frac{P_k(z)}{a_d^N \prod_{\ell=1}^{d_1} \lambda_\ell(z)^N},$$

for $k = 1, 2, \ldots, N$, where we recall the definition of $P_k(z)$ from (2.1). Recalling (2.4) we note that

$$|\tilde{P}_k(z)| \leq |\tilde{P}_k(z)| \cdot \prod_{i=1}^{d_0} |\lambda_i(z)|^{k+d_2},$$

for any $k = 1, 2, \ldots, N$ and $z \in S_0$. Thus, by Lemma 2.1 and the Cauchy–Schwarz inequality, and upon proceeding similarly to (2.48) it follows that

$$\mathbb{E} \left[ \sup_{z \in S_{0}^{-\epsilon}} \left| \sum_{k=1}^{N} \tilde{P}_k(z) \right|^2 \right] \leq \left( \sum_{k=1}^{N} \mathbb{E} \left[ \sup_{z \in S_{0}^{-\epsilon}} |\tilde{P}_k(z)|^2 \right] \right)^{1/2} \leq N^{-\eta_0/2},$$

for all large $N$, where $\eta_0 > 0$ is as in Lemma 2.1. Hence, by Markov’s inequality

$$\mathbb{P} \left( \sup_{z \in S_{0}^{-\epsilon}} \left| \sum_{k=1}^{N} \frac{P_k(z)}{a_d^N \prod_{\ell=1}^{d_1} \lambda_\ell(z)^N} \right| \leq N^{-\eta_0/8} \right) \geq 1 - N^{-\eta_0/4},$$

for all large $N$. Therefore, recalling $P_0(z) = \det(T_N(a(z)))$ and using Lemma 6.1 we derive that on an event of probability at least $1 - N^{-\eta_0/4}$, we have

$$|P_0(z)| \geq 2 \left| \sum_{k=1}^{N} P_k(z) \right|, \quad \text{for all } z \in ((\text{spec } T(a))^{-\epsilon} \cap B_{C}(0, N^{1/2}),$$

for all large $N$. As the maps $z \mapsto \det(T_N(a(z)) + \Delta_N)$ and $P_0(z)$ are analytic, an application of Rouché’s theorem further yields that on the same event, the number of roots of $\det(T_N(a(z)) + \Delta_N) = 0$ and $P_0(z) = \det(T_N(a(z))) = 0$ are same on the interior of the bounded set $((\text{spec } T(a))^{-\epsilon} \cap B_{C}(0, N^{1/2})$. Since $|\lambda_1(z)| \geq |\lambda_2(z)| \geq \cdots$, Springer
\[
\cdots \geq |\lambda_{d_1}(z)| > 1 \text{ on } S_0 \text{ Lemma 6.1 implies that there are no roots of the equation } P_0(z) = 0 \text{ in } S_0. \text{ So}
\]

\[
P \left( L_N \left( \left( \left( \text{spec } T(a) \right)^c \right)^{-\varepsilon} \cap \left( B_{\mathbb{C}}(0, N^{1/2}) \right)^0 \right) = 0 \right) = 1 - o(1). \tag{6.13}
\]

To complete the proof we recall the spectral radius (i.e. the maximum modulus eigenvalue) of a matrix bounded by it operator norm. Therefore using the triangle inequality we find that

\[
\mathbb{E} \| T_N(a) + \Delta_N \| \leq \| T_N(a) \| + N^{-\gamma} \mathbb{E} \| E_N \| \leq \| T_N(a) \| + N^{-\gamma} \mathbb{E} \| E_N \|_{\text{HS}} \\
\leq C'(1 + N^{-\gamma + 1}) \leq N^{1/2 - \varepsilon'},
\]

for some \( \varepsilon' > 0 \), all large \( N \), and any \( \gamma > \frac{1}{2} \), where \( \| \cdot \| \) and \( \| \cdot \|_{\text{HS}} \) denotes the operator norm and the Hilbert–Schmidt norm respectively. So by Markov’s inequality

\[
P \left( L_N \left( B_{\mathbb{C}}(0, N^{1/2 - \varepsilon'})^c \right) > 0 \right) \leq P \left( \| T_N(a) + \Delta_N \| \geq N^{1/2 - \varepsilon'} \right) \leq N^{-\varepsilon'}.
\tag{6.14}
\]

Combining (6.13) and (6.14) the proof is now complete. \( \square \)

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**Appendix A: The spectral radius of \( E_N \)**

In this short section we show that the decomposition (2.2) used in the proofs of Theorems 1.1 and 1.11 can be adapted to prove the following.

**Proposition A.1** Let \( \{E_N\}_{N \in \mathbb{N}} \) be a sequence of \( N \times N \) random matrices with independent complex-valued entries of mean zero and unit variance. Denote \( \varrho_N \) to be the spectral radius of \( N^{-1/2} E_N \), i.e. the maximum modulus eigenvalue of \( N^{-1/2} E_N \). Then the sequence \( \{\varrho_N\}_{N \in \mathbb{N}} \) is tight.

We remark that Proposition A.1 seems to be contained in Theorem 1.1. However, formally the latter cannot be applied since it would require one to take \( a \equiv 0 \), while throughout the paper (and in particular, in the proof of Theorem 1.1), we assume that \( a \) is a nontrivial Laurent polynomial.

If the entries of \( E_N \) are i.i.d. having a finite \((2 + \delta)\)-th moment and possessing a symmetric law then it is known that \( \varrho_N \to 1 \) in probability, see [6], while the operator...
norm of $N^{-1/2} E_N$ blows up as soon as the fourth moment of the entries is infinite. It is conjectured in [6] that in the critical case of finiteness of second moments, the convergence in probability to one still holds. Proposition A.1 is a weak form of the conjecture with elementary proof.

**Proof** Set $\Delta_N = N^{-1/2} E_N$. We decompose

$$\text{det}(\Delta_N - z I_N) = (-z)^N + \sum_{k=1}^{N} (-z)^{N-k} P_k$$

where

$$P_k := \sum_{X \subset [N]} N^{-k/2} \text{det}(E_N[X; X]),$$

(A.1)

compare with (2.1). Note that $\text{Var}(P_k) \leq 1$, while $\mathbb{E} P_k = 0$. Therefore, for a fixed constant $\bar{C}$, we have that $\mathbb{P}(|P_k| > \bar{C}^k) \leq \bar{C}^{-2k}$. So, setting $\mathcal{A}_0 = \bigcup_{k=1}^{\infty} \{|P_k| > \bar{C}^k\}$, it yields that

$$\mathbb{P}(\mathcal{A}_0) \leq \frac{2}{\bar{C}^2}.$$ 

Note that on $\mathcal{A}_0^c$ we have that for $z$ with $|z| > 4\bar{C}$,

$$\frac{|z|^N}{|\sum_{k=1}^{N} (-z)^{N-k} P_k|} \geq \frac{1}{\sum_{k=1}^{N} 4^{-k}} \geq 3.$$ 

This in particular implies that there can be no zero of $\text{det}(\Delta_N - z I_N)$ with modulus larger than $4\bar{C}$. Thus the claim follows. \qed

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