On the determination of a function from cylindrical Radon transforms

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Abstract

This paper is devoted to a Radon-type transform arising in Photoacoustic Tomography that uses integrating line detectors. We consider two situations: when the line detectors are tangent to the boundary of a cylindrical domain and when the line detectors are located on a plane. We present the analogue of the Fourier slice theorems for each case of the Radon-type transforms. Also, we provide several new inversion formulas, a support theorem, and stability estimate and necessary range condition results.

1 Introduction

Photoacoustic Tomography (PAT) is the best-known example of a hybrid imaging method. It has applications to functional brain imaging of animals, early cancer diagnostics, and imaging of vasculature [8]. In 1880, A.G. Bell discovered the photo-acoustic effect [2]. This effect enabled one to combine advantages of pure optical and ultrasound imaging, providing both high optical contrast and ultrasonic resolution. Nevertheless, PAT has rather low cost.

In PAT, one induces an acoustic pressure wave inside of an object of interest by delivering optical energy [13, 20]. The acoustic wave on a surface is measured outside of the object. Mathematically, in this model we study, the problem boils down to recovering the initial data of the three dimensional wave equation from the values of the solution observed at all times on the surface. This initial pressure field contains diagnostic information.

Various types of detectors have been considered for measuring the acoustic data: point-like detectors, line detectors, planar detectors, cylindrical detectors, and circular detectors. While point-like detectors approximately measure the pressure at a given point, other types of detectors are integrating. More specifically, the line detector renders the integral of the pressure along its length. We obtain this integration value at different moments of time. This data is equivalent to measuring the surface integral over the cylinders with central axis corresponding to a detector line and whose radii are arbitrary.

Various configurations of line detectors were considered in [3, 4, 8, 9]. In this article, we deal with two basic geometries: the line detectors are tangent to a cylinder, and the line detectors are
located on a plane. We call these the **cylindrical version** and the **planar version**, respectively. Some inversion formulas for the first version were found in [9]. In this text, we address other issues of importance in tomography [16] [17]: a support theorem, a stability estimate, and necessary range conditions. We also consider an \( n \)-dimensional case of this model. In the planar version, Haltmeier [8] provided a two-step procedure of image reconstruction. In this text, we define a cylindrical Radon transform and present an analogue of the Fourier slice theorem as well as a stability estimate, and necessary range conditions.

Two different versions of cylindrical Radon transforms are discussed in sections 2 and 4. Various inversion formulas of a cylindrical version of a cylindrical Radon transform different from [9] are provided in section 2.1. Section 2.2 is devoted to a support theorem for this version of the transform. In sections 2.3 and 2.4, we provide a stability estimate of the transform and the necessary range conditions of the transform. We also provide inversion formulas, a stability estimate, and the necessary range conditions of a plane version of a cylindrical Radon transform in sections 4.1, 4.2, and 4.3 respectively. In two sections 3 and 5 we study \( n \)-dimensional cases of above cylindrical Radon transforms.

## 2 Cylindrical geometry

We explain first the mathematical model arising in PAT with line detectors as introduced in [9]. Let \( B^k_R \) be the ball in \( \mathbb{R}^k \) centered at the origin with radius \( R > 0 \). Then \( B^2_R \times \mathbb{R} \) is the cylinder in \( \mathbb{R}^3 \) with radius \( R \). For fixed \( p \in \mathbb{R} \) and \( \theta \in S^1 \), let

\[
L_C(\theta, p) = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \cdot \theta = R, z = p\}
\]

be the line occupied by a detector. Detector lines \( L_C(\theta, p) \) are tangent to the cylinder \( B^2_R \times \mathbb{R} \) (see Figure 1).

**Definition 1.** The cylindrical Radon transform \( R_C \) maps a function \( f \in C^\infty_c(B^2_R \times \mathbb{R}) \) to

\[
R_C f(\theta, p, r) = \frac{1}{2\pi r} \iint_{d(L_C(\theta, p), (x, y, z)) = r} f(x, y, z) d\omega,
\]

for \((\theta, p, r) \in S^1 \times \mathbb{R} \times [0, \infty)\). Here \( d\omega \) is the area measure on the cylinder

\[
\{(x, y, z) \in \mathbb{R}^3 : d(L_C(\theta, p), (x, y, z)) = r\}
\]

and

\[
d(L_C(\theta, p), (x, y, z)) := \sqrt{(R - (x, y) \cdot \theta)^2 + (p - z)^2}
\]

denotes the Euclidean distance between the line \( L_C(\theta, p) \) and the point \((x, y, z)\).

**Remark 2.** When one fixes \( \theta \) and restricts the cylindrical Radon transform \( R_C f \) to a plane \( \{t(\theta, z) : t \in \mathbb{R}, z \in \mathbb{R}\} \), \( R_C f \) turns into the 2-dimensional circular Radon transform whose centers are located at \((R\theta, z)\) (see Figure 1 (b)).
Figure 1: (a) the integral of $f$ supported in the cylinder $B_R^2 \times \mathbb{R}$ over cylindrical surfaces and (b) the restriction to the $\{(t\theta, z) : t \in \mathbb{R}, z \in \mathbb{R}\}$ plane.

By definition, we have

$$ R_C f(\theta, p, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t\theta) + (R - r \cos \psi)\theta, p + r \sin \psi) d\psi dt $$

where $p$ and $r$ are the height and radius, $\theta$ is the direction from the $z$-axis to the central axis of the cylinder, $t$ is a parameter along the central axis of the cylinder, and $\psi$ is the polar angle of the circle that is the intersection of plane $\{(t\theta, z) : t \in \mathbb{R}, z \in \mathbb{R}\}$ and the cylinder.

### 2.1 Inversion formulas

We have two integrals in the definition formula of $R_C f$. For fixed $\theta$, the inner integral is a circular Radon transform with centers at the space $\{R\theta, z : z \in \mathbb{R}\}$ (see Figure 2(b)). Also, the outer integral can be thought of as the 2-dimensional regular Radon transform for a fixed $z$ variable [9].

We start by applying the inversion of the circular Radon transform for fixed $\theta$.

To obtain inversion formulas, we define the operator $R_C^*$ for $g \in C^\infty_c(S^1 \times \mathbb{R} \times [0, \infty))$ by

$$ R_C^* g(\theta, z, \rho) = \int_{\mathbb{R}} g(\theta, p, \sqrt{(z-p)^2 + \rho^2}) dp, $$

for $z \in \mathbb{R}$ and $\rho \in \mathbb{R}$. 

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We have an analogue of the Fourier slice theorem.

**Theorem 3.** Let \( f \in C^\infty_c(B_R^2 \times \mathbb{R}) \). If \( g = R_C f \), then we have for \((\theta, \sigma, \xi) \in S^1 \times \mathbb{R} \times \mathbb{R}\),

\[
\hat{f}(\sigma, \xi) = \pi^{-1} \hat{R}_C^* g(\theta, \xi, \sigma)e^{iR_\sigma|\sigma|},
\]

where \( \hat{f} \) is the 3-dimensional Fourier transform of \( f \) and \( \hat{R}_C^* g \) is the 2-dimensional Fourier transform of \( R_C^* g \) with respect to \((z, \rho)\).

**Remark 4.** We remind readers the Fourier slice theorems for the circular and regular Radon transforms.

When \( Rf(\theta, s) = \int_{\theta=x=s} f(x)dx \) for \((\theta, s) \in S^1 \times \mathbb{R}\) is the regular Radon transform, we have \( \hat{R}f(\theta, \sigma) = f(\sigma \theta) \). Also, when \( Mf(u, r) = \int_{S^1} f((u, 0) + r\alpha)d\alpha \) for \((u, r) \in \mathbb{R} \times [0, \infty)\) is the circular Radon transform, we have \( \hat{f}(\xi) = M^* Mf(\xi)\xi_2 \), where \( M^* g(x, y) = \int_{\mathbb{R}} g(u, \sqrt{(u - x)^2 + y^2})du \) for a function \( g \) on \( \mathbb{R} \times [0, \infty) \). Equation (1) can be thought of as the combination of two Fourier slice theorems: for the circular and regular Radon transforms.

**Proof of theorem.** Taking the Fourier transform of \( R_C f \) with respect to \( p \) yields

\[
\hat{R}_C f(\theta, \xi, r) = \frac{1}{2\pi} \int_\mathbb{R} \int_0^1 \hat{f}(t \theta^1 + (R - r \sqrt{1 - s^2}) \theta, \xi) e^{irs\xi} \frac{ds}{\sqrt{1 - s^2}} dt,
\]

where \( \hat{f} \) and \( \hat{R}_C f \) are the 1-dimensional Fourier transforms of \( f \) and \( R_C f \) with respect to \( z \) and \( p \), respectively. Taking the Hankel transform of order zero of \( \hat{R}_C f \) with respect to \( r \), we have

\[
H_0 \hat{R}_C f(\theta, \xi, \eta) = \frac{1}{2\pi} \int_0^\infty \int_\mathbb{R} \int_0^1 \hat{f}(t \theta^1 + (R - r \sqrt{1 - s^2}) \theta, \xi) e^{irs\xi} \frac{ds}{\sqrt{1 - s^2}} dt J_0(r\eta)r dr
\]

\[
= \frac{1}{2\pi} \int_0^\infty \int_\mathbb{R} \int_0^1 \hat{f}(t \theta^1 + (R - r \sqrt{1 - s^2}) \theta, \xi) \cos(rs\xi) \frac{ds}{\sqrt{1 - s^2}} J_0(r\eta)r dr dt dr
\]

\[
= \frac{1}{2\pi} \int_\mathbb{R} \int_0^\infty \int_0^1 \hat{f}(t \theta^1 + (R - b) \theta, \xi) \cos(\rho\xi) J_0(\eta\sqrt{\rho^2 + b^2}) d\rho db dt,
\]

where in the last line, we changed variables \((r, s) \rightarrow (b, \rho)\) where \( r = \sqrt{\rho^2 + b^2} \) and \( s = \rho/\sqrt{\rho^2 + b^2} \). We will use the following identity: for \( a, b > 0 \)

\[
\int_0^\infty J_0(a \sqrt{\rho^2 + b^2}) \cos(\rho\xi) d\rho = \begin{cases}
\frac{1}{\sqrt{a^2 - \xi^2}} \cos(b\sqrt{a^2 - \xi^2}) & \text{if } 0 < \xi < a, \\
0 & \text{otherwise}
\end{cases}
\]

(3)
Applying this identity to equation (2), we get
\[
H_0 R_C f(\theta, \xi, \eta) = \begin{cases} 
\frac{1}{2\pi} \int_0^\infty \frac{1}{\sqrt{\eta^2 - \xi^2}} \cos(b\sqrt{\eta^2 - \xi^2}) db dt & \text{if } 0 < \xi < \eta, \\
0 & \text{otherwise}.
\end{cases}
\]
Substituting \(\eta = \sqrt{\xi^2 + \sigma^2}\) yields
\[
H_0 R_C f(\theta, \xi, |(\xi, \sigma)|) = \frac{1}{2\pi} \int_0^\infty \frac{1}{\sqrt{\eta^2 - \xi^2}} \cos(b\sigma) \sigma db dt.
\]
The inner integral in the right hand side is the Fourier cosine transform with respect to \(b\), so taking the Fourier cosine transform of the above formula, we get
\[
\int_\mathbb{R} \hat{f}(t \theta^\perp + (R - b)\theta, \xi) dt = 4 \int_0^\infty H_0 R_C f(\theta, \xi, |(\xi, \sigma)|) \cos(s \sigma) d\sigma,
\]
where \(\hat{f}\) is the 1-dimensional Fourier transform of \(f\) with respect to the last variable \(z\). For a fixed \(\xi\), one recognizes the Radon transform in the left side. We, thus, can apply an inversion of the Radon transform.

Before doing that, we change the right side of equation (4) into a term containing the backprojection operator \(R^*_C\). Taking the Fourier transform of \(R^*_C g\) on \(S^1 \times \mathbb{R}^2\) with respect to the last two variables \((z, \rho)\) yields
\[
\hat{R}^*_C g(\theta, \xi, \sigma) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z,\rho) \cdot (\xi,\sigma)} R^*_C g(\theta, z, \rho) dz d\rho
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z,\rho) \cdot (\xi,\sigma)} g(\theta, p, \sqrt{(z - p)^2 + \rho^2}) dp dz d\rho
\]
\[
= e^{-i\xi \rho} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z,\rho)^2 \cdot (\xi,\sigma)} g(\theta, p, |(z, \rho)|) dp dz d\rho
\]
\[
= 2\pi e^{-i\xi \rho} H_0 g(\theta, p, |(\xi, \sigma)|) dp
\]
\[
= 2\pi H_0 \hat{g}(\theta, \xi, |(\xi, \sigma)|),
\]
where \(\hat{R}^*_C g\) is the 2-dimensional Fourier transform of \(g\) in \((z, \rho)\). Combining this equation with
formula \((4)\), we have for \(g = R_C f\),

\[
\int_{\mathbb{R}} \hat{f}(t\theta + s\theta, \xi) dt = \frac{2}{\pi} \int_0^\infty \hat{R}_C g(\theta, \xi, \sigma) \cos((R - s)\sigma) \sigma d\sigma
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} \hat{R}_C g(\theta, \xi, \sigma) e^{i(R-s)\sigma} |\sigma| d\sigma,
\]

since \(\hat{R}_C g\) is even in \(\sigma\) by the evenness in \(\rho\) of \(\hat{R}_C g\). Taking the Fourier transform of equation \((6)\) with respect to \(s\) completes the proof. \(\square\)

**Theorem 5.** Let \(f \in C^\infty_c(B_2^R \times \mathbb{R})\). If \(g = R_C f\), then we have

\[
f(x, y, z) = \frac{1}{4\pi^2} \int_{S^1} \mathcal{I}_2 R_C g(\theta, z, \rho)|_{\rho = (x, y) - R} d\theta,
\]

where we use the Riesz potential \(\mathcal{I}_2 h(\theta, \xi, \sigma) = |\sigma|^2 h(\theta, \xi, \sigma)\) for a function \(h(\theta, z, \rho)\) on \(S^1 \times \mathbb{R}^2\) with its 2-dimensional Fourier transform \(\hat{h}(\theta, \xi, \sigma)\) with respect to real variables.

**Proof.** Using Theorem 3 we have

\[
f(x, y, z) = \frac{1}{(2\pi)^3} \int_0^\infty \int_{S^1} \int_{\mathbb{R}} \hat{f}(\sigma \theta, \xi, \sigma) e^{i(\sigma(x, y) - \theta + z \xi)} d\sigma d\theta d\xi
\]

\[
= \frac{1}{(2\pi)^3} \int_0^\infty \int_{S^1} \int_{\mathbb{R}} \hat{R}_C g(\theta, \xi, \sigma) e^{iR\sigma |\sigma|^2} e^{i(\sigma(x, y) - \theta + z \xi)} d\sigma d\theta d\xi
\]

\[
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}} \int_{S^1} \int_{\mathbb{R}} \hat{R}_C g(\theta, \xi, \sigma) e^{iR\sigma |\sigma|^2} e^{i(\sigma(x, y) - \theta + z \xi)} d\sigma d\theta d\xi.
\]

\(\square\)

**Remark 6.** Inversion formula \((13)\) is the same as that of \([9]\). There M. Haltmeier obtained it combining two inversion formulas for the circular Radon transform and the 2-dimensional Radon transform. We obtain it instead through an analogue of the Fourier slice theorem.

The equation \((6)\) hints that it is natural to try to use another inversion of the Radon transform, the one using circular harmonics. Let \(f(t, \varphi, z)\) be the image function in cylindrical coordinates. Then the Fourier series of \(f(t, \varphi, z)\) and \(g(\theta, p, r) := R_C f(\theta, p, r)\) with respect to their angular variables \(\varphi\) and \(\theta\) can be written as follows:

\[
f(t, \varphi, z) = \sum_{l=-\infty}^{\infty} f_l(\rho, z) e^{il\varphi} \quad \text{and} \quad g(\theta, p, r) = \sum_{l=-\infty}^{\infty} g_l(p, r) e^{il\theta}.
\]
where $\theta = (\cos \vartheta, \sin \vartheta)$ and the Fourier coefficients are given by

$$f_l(t,z) = \frac{1}{2\pi} \int_0^{2\pi} f(t,\varphi,z) e^{-il\varphi} d\varphi$$

and

$$g_l(p,r) = \frac{1}{2\pi} \int_{S^1} g(\theta,p,r) e^{-ipl\theta} d\theta,$$

where $\theta = (\cos \vartheta, \sin \vartheta)$. Consider the $l$-th Fourier coefficient of the right hand side of formula (6). Then we have

$$\int_{S^1} \int_{R^2} \hat{R}_C g_l(\theta,\xi,\sigma) e^{iR(u-s)\sigma} |\sigma| e^{-il\vartheta} d\sigma d\theta = \int_{R} \hat{R}_C g_l(\xi,\sigma) e^{iR(u-s)\sigma} |\sigma| d\sigma,$$

where $\hat{R}_C g_l$ is the 2-dimensional Fourier transform of $R_C g_l$ with respect to $(z,\rho)$ and

$$R_C^* g_l(z,\rho) = \int_R g_l(p,\sqrt{(z-p)^2 + \rho^2}) dp.$$

According to [16], when $g = \mathcal{R} f$ for the regular 2-dimensional Radon transform $\mathcal{R}$ and $g_l$ and $f_l$ are the Fourier coefficients of $g$ and $f$, we have for $t > 0$,

$$f_l(t) = -\pi^{-1} \int_{\rho}^{\infty} \cosh \left( l \arccosh \frac{s}{\rho} \right) \frac{\partial}{\partial s} g_l(s) \frac{ds}{\sqrt{s^2 - t^2}}. \quad (9)$$

Hence, we get a different type of inversion formula.

**Theorem 7.** Let $f \in C_c^\infty(B_R^2 \times \mathbb{R})$. Then we have for $t > 0$

$$f_l(t,z) = -4i \int_{\rho}^{\infty} \cosh \left( l \arccosh \frac{s}{t} \right) I_2^2 R_C^* g_l(z,\sigma) \frac{ds}{\sqrt{s^2 - t^2}}.$$

**Proof.** Applying equation (14) to equation (8) gives

$$\hat{f}_l(\rho,\xi_1) = -i \frac{1}{\pi^2} \int_{t}^{\infty} \cosh \left( l \arccosh \frac{s}{t} \right) \int_{\mathbb{R}} \hat{R}_C g_l(\xi) e^{i(u-s)\sigma} |\sigma|^2 d\sigma \frac{ds}{\sqrt{s^2 - t^2}},$$

where $\hat{f}_l$ is the Fourier transform of $f_l$ with respect to $z$. \hfill $\Box$

The regular Radon transform can be obtained from the cylindrical Radon transform.

**Theorem 8.** Let $f \in C_c^\infty(B_R^2 \times \mathbb{R})$. Then we have

$$\int_{\mathbb{R}} f(t\theta^+ + (R-s)\theta, z) dt = \frac{2}{\pi} \int_0^{\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{R}_C f(\theta, -\eta, \rho) e^{-i\rho^2 \xi} e^{-i(2\eta \rho + (z^2 + s^2) + \eta^2) \xi} d\rho d\eta d\xi.$$
We notice that the expression in the left hand side is the standard 2-dimensional Radon transform for a fixed $z$ variable. Hence, applying different Radon transform inversions, one gets different inversions of the cylindrical Radon transform $R_C$.

**Proof.** Let $G$ be defined by

$$G(\theta, p, \xi) := \int_0^\infty r R_C f(\theta, p, r) e^{-ir^2 \xi} dr.$$ 

Then we have

$$G(\theta, p, \xi) = \frac{1}{2\pi} \int_0^\infty \int_0^\pi \int \cos \psi (R - r \cos \psi) \theta, p + r \sin \psi) e^{-ir^2 \xi} d\psi dr d\theta,$$

$$= \frac{1}{2\pi} \int_0^\infty \int f(t^\perp + (R - y) \theta, p + z) e^{-i(y^2 + z^2) \xi} dy dz dt,$$

$$= \frac{1}{2\pi} \int_0^\infty \int f(t^\perp + (R - y) \theta, z) e^{-i(y^2 + z^2) \xi} e^{2ipz \xi} dy dz dt,$$

where in the second line, we switched from the polar coordinates $(r, \psi)$ to Cartesian coordinates $(y, z)$. Making the change of variable $r = y^2 + z^2$ yields

$$G(\theta, p, \xi) = \frac{e^{-ip^2 \xi}}{2\pi} \int_0^\infty \int f(t^\perp + (R - \sqrt{r - z^2}) \theta, z) e^{-ir \xi} e^{2ipz \xi} dr dz dt,$$

where we do not need to care about $f(t^\perp + (R + \sqrt{r - z^2}) \theta, z)$ because $f$ is compactly support on $B^2 \times \mathbb{R}$. Let us define the function

$$k_\theta(t, z, r) := \left\{ \begin{array}{ll} f(t^\perp + (R - \sqrt{r - z^2}) \theta, z)/\sqrt{r - z^2} & \text{if } 0 < z^2 < r, \\ 0 & \text{otherwise.} \end{array} \right.$$ 

Then we have

$$G(\theta, p, \xi) = \frac{e^{-ip^2 \xi}}{4\pi} \int_{\mathbb{R}^3} k_\theta(t, z, r) e^{-i \xi z} e^{2ipz \xi} dr dz dt = \frac{e^{-ip^2 \xi}}{4\pi} \int_{\mathbb{R}} \hat{k}_\theta(t, -2p \xi, \xi) dt,$$

where $\hat{k}_\theta$ is the 2-dimensional Fourier transform of $k_\theta$ with respect to the last two variables $(z, r)$. 

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Also, we have

\[
\int_{\mathbb{R}} f(t\theta^1 + (R-s)\theta, z) dt = \int_{\mathbb{R}} sk_\theta(t, z, z^2 + s^2) dt
\]

\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} sk_\theta(t, \eta, \xi) e^{-i(\eta_1(z^2 + s^2) + \eta_2)\xi} dt d\eta d\xi
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} s \hat{G}(\theta, -\eta, \xi) e^{-i(\eta_1(z^2 + s^2) + \eta_2)\xi} d\eta d\xi
\]

where in the last line, we changed variables \( \eta \to 2\xi\eta \). \( \square \)

### 2.2 Support theorem

By a support theorem, we mean a statement that claims that if integrals of \( f \) over all surfaces not intersecting a set \( A \) are equal to zero, then \( f \) is equal to zero outside \( A \).

**Lemma 9.** Let \( p_0 \in \mathbb{R}, \epsilon > 0, B > 0, \) and \( \theta \in S^1 \) be given. Let \( f \in C^\infty(B^2_R \times \mathbb{R}) \) and suppose that \( g = R_C f \) is equal to zero on the open set \( U_{B, \epsilon} = \{(p, r) : |p - p_0| < \epsilon, 0 \leq r < B\} \). Then \( R_\theta f(p, s) \) is equal to zero on the open set \( V_B = \{(p, s) : |p - p_0|^2 + (R - s)^2 < B^2, s > R\} \) where

\[
R_\theta f(p, s) = \int_{\mathbb{R}} f(t\theta^1 + s\theta, p) dt.
\]

We will follow the idea suggested in [1] to prove this lemma.

**Proof.** Without loss of generality, we may assume \( p_0 = 0 \). Let \( G(\theta, p, r) \) be defined by

\[
G(\theta, p, r) = \int_0^r g(\theta, p, s) ds = \frac{1}{2\pi} \int_{|\eta| \leq r} \int_{\mathbb{R}} f(t\theta^1 + (R - \eta_1)\theta, p + \eta_2) dt d\eta,
\]

where \( \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \). Differentiating \( G \) with respect to \( p \) yields

\[
\frac{\partial}{\partial p} G(\theta, p, r) = \frac{1}{2\pi} \int_{|\eta| \leq r} \int_{\mathbb{R}} \frac{\partial}{\partial p} f(t\theta^1 + (R - \eta_1)\theta, p + \eta_2) dt d\eta
\]

\[
= \frac{1}{2\pi} \int_{|\eta| \leq r} \int_{\mathbb{R}} \frac{\partial}{\partial \eta_2} f(t\theta^1 + (R - \eta_1)\theta, p + \eta_2) dt d\eta
\]

\[
= \frac{1}{2\pi r} \int_{|\eta| = r} \int_{\mathbb{R}} f(t\theta^1 + (R - \eta_1)\theta, p + \eta_2) d\eta dt.
\]
where in the last line, we used the divergence theorem. Now we have

\[
R_C(pf)(\theta, p, r) = \frac{1}{2\pi r} \int_{\mathbb{R}} \int_{|\eta|=r} (p + \eta_2)f(t\theta + (R - \eta_1)\theta, p + \eta_2)d\eta dt
\]

\[
= pg(\theta, p, r) + \frac{1}{2\pi} \frac{\partial}{\partial p} G(\theta, p, r) = pg(\theta, p, r) + \frac{\partial}{\partial p} \int_0^r g(\theta, p, s)ds.
\]

Let the linear operator \( D \) be defined by \( Dg(\theta, p, r) := pg(\theta, p, r) + \frac{\partial}{\partial p} \int_0^r g(\theta, p, s)ds \). Then \( R_C(pf) \) is equal to \( Dg \). By iteration, we have \( R_C(\mathcal{P}(p)f) = \mathcal{P}(D)g \) where \( \mathcal{P} \) is any polynomial. If \( g = 0 \) in \( U_{B,\epsilon} \), then \( \mathcal{P}(D)g = 0 \) in \( U_{B,\epsilon} \). Also, we have for any point \((p, r) \in U_{B,\epsilon}\),

\[
R_C(\mathcal{P}(p)f)(\theta, p, r) = \frac{1}{2\pi r} \int_{\mathbb{R}} \int_{|\eta|=r} \mathcal{P}(p + \eta_2)f(t\theta + (R - \eta_1)\theta, p + \eta_2)d\eta dt
\]

\[
= \frac{1}{2\pi} \int_{-r}^r \mathcal{P}(p + \eta_2)f(t\theta + (R - \sqrt{r^2 - \eta_2^2})\theta, p + \eta_2) \frac{d\eta dt}{\sqrt{r^2 - \eta_2^2}} = 0.
\]

For fixed \( \theta \in S^1 \), \( r > 0 \), and \( p \in \mathbb{R} \), choose a sequence of polynomials \( \mathcal{P}_i \) such that \( \mathcal{P}_i(p + \eta_2) \) converges to \( \int_{\mathbb{R}} f(t\theta + (R - \sqrt{r^2 - \eta_2^2})\theta, p + \eta_2)dt \) uniformly for \( |\eta_2| < r \). It follows that \( R_\theta f(p, s) = 0 \) in \( V_B \).

**Theorem 10.** Let \( p_0 \in \mathbb{R} \) and \( B > 0 \). Let \( f \in C^\infty(B_R^2 \times \mathbb{R}) \) and suppose that \( g = R_C f \) is equal to zero on the open set \( U_B = \{(\theta, p_0, r) : 0 \leq \theta < 2\pi, 0 \leq r < B\} \). Then \( f \) is equal to zero on the set \( \{(x, y, z) : |(x, y)| > R - \sqrt{B^2 - (p - p_0)^2}, z = p\} \).

**Proof.** Let \( \epsilon > 0 \) be arbitrary. Then \( g \) vanishes on the open set \( U_{B-\epsilon,\epsilon} \) and by Lemma 9, \( R_\theta f \) vanishes on the open set \( V_{B-\epsilon,\epsilon} \). Let \( p \in \mathbb{R} \) be arbitrary. Then by the support theorem of the regular Radon transform \( \mathcal{F} \), \( f \) is equal to zero on the set \( \{(x, y, z) : |(x, y)| > R - \sqrt{B - \epsilon^2 - (p - p_0)^2}, z = p\} \).

**Corollary 11.** Let \( A \subset B_R^2 \times \mathbb{R} \) be a closed set invariant under rotation around z-axis and let \( f \in C^\infty(B_R^2 \times \mathbb{R}) \). Suppose that for any point \( (x, y, z) \in \mathbb{R}^3 \setminus A \), there are \((p(x, y, z), r(x, y, z)) \in \mathbb{R} \times (0, \infty) \) such that a sphere centered at \((Rx/(|x, y|), Ry/(|x, y|), p(x, y, z)) \) with radius \( r(x, y, z) \) separates the point \((x, y, z) \) and \( A \). If \( g = R_C f \) vanishes on \( \{(\theta, p, r) : p = p(x, y, z), 0 \leq r < r(x, y, z) \} \), for any \((x, y, z) \in \mathbb{R}^3 \setminus A \), then \( f \) vanishes on \( \mathbb{R}^3 \setminus A \).

## 2.3 A stability estimate

In this subsection, we discuss the stability estimate of the cylindrical Radon transform \( R_C f \). For next section, we define spaces we need as \( n \)-dimension. For \( \gamma \geq 0 \), let \( \mathcal{H}^\gamma(\mathbb{R}^n) \) be the regular
Sobolev space with the norm \( || \cdot ||_\gamma \). Let \( L^2_{n-k}(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)) \) be the set of a function \( g \) on 
\( S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty) \) with 
\[
||g||^2 := \int_{S^{k-1}} \int_{\mathbb{R}^{n-k}} \int_{0}^{\infty} |g(\theta, p, r)|^2 r^{n-k} dr dp d\theta < \infty.
\]
Then \( L^2_{n-k}(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)) \) is a Hilbert space. Also, by the Plancherel formula, we have 
\[
||g|| = (2\pi)^{2k-2n-1}||\tilde{g}||,
\]
where 
\[
\tilde{g}(\theta, \xi, |\zeta|) := \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k+1}} g(\theta, p, |w|) e^{-i(p,w) \cdot (\xi, \zeta)} dp dw.
\]

Let \( \mathcal{H}^\gamma(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)) \) be the set of a function \( g \in L^2_{n-k}(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)) \) with 
\[
||g||_\gamma < \infty,
\]
where 
\[
||g||^2_\gamma := \int_{S^{k-1}} \int_{\mathbb{R}^{n-k}} \int_{0}^{\infty} |\tilde{g}(\theta, \xi, \eta)|^2 (1 + |\xi|^2 + |\eta|^2)^\gamma |\eta|^{n-k} d\eta d\xi d\theta.
\]

**Theorem 12.** For \( \gamma \geq 0 \), we have 
\[
||f||_\gamma \leq 4\pi^{-1} ||R_C f||_{\gamma+1},
\]
for \( f \in \mathcal{H}^\gamma(\mathbb{R}^3) \) supported in \( B^2_R \times \mathbb{R} \) (i.e., \( n = 3 \) and \( k = 2 \)).

**Remark 13.** As mentioned before, \( R_C \) can be thought of as the composition of the circular Radon transform and the regular Radon transform. We know that the regular Radon transform maps \( H^\gamma(\mathbb{R}^2) \) into \( H^{\gamma+1/2}(S^1 \times \mathbb{R}) \) and a circular Radon transform maps \( H^\gamma(\mathbb{R}^2) \) into \( H^{\gamma+1/2}(\mathbb{R} \times [0, \infty)) \) which is defined by the norm 
\[
\int_{\mathbb{R}} \int_{0}^{\infty} |\tilde{\phi}(\xi, \rho)|^2 (1 + |\xi|^2 + \rho^2)^\gamma^{1/2} \rho d\rho d\xi < \infty
\]
in [1, 10]. Hence, the estimate in Theorem 12 looks reasonable.

**Proof.** Let \( g = R_C f \). Note that from equation (5), we have 
\[
\tilde{R_C g}(\theta, \xi, \sigma) = \int_{\mathbb{R}} e^{-i\xi p} \int_{\mathbb{R}} e^{-i(|z, \rho| \cdot (\xi, \sigma))} g(\theta, p, |(z, \rho)|) dz dp = \tilde{g}(\theta, \xi, |(\xi, \sigma)|).
\]
Combining this equation and equation (11), we have 
\[
\hat{f}(\sigma \theta, \xi) = 4\pi^{-1} \tilde{g}(\theta, \xi, |(\xi, \sigma)|) e^{i\sigma |\xi, \sigma|}.
\]
Hence, we have

$$
||f||^2_{\gamma} = \int \int (1 + |\xi|^2)^2 |\hat{f}(\xi)|^2 d\xi = \int \int_\mathbb{R}^3 |\sigma|^2 (1 + |\xi|^2)^\gamma |\hat{f}(\sigma, \xi)|^2 d\sigma d\xi
$$

$$
= \frac{16}{\pi^2} \int \int_\mathbb{S} \int_0^\infty |\sigma|^3 (1 + |(\xi, \sigma)|^2)^\gamma |\hat{g}(\theta, \xi, ||(\xi, \sigma)||)|^2 d\sigma d\xi d\theta
$$

$$
\leq \frac{16}{\pi^2} \int \int_\mathbb{S} \int_0^\infty \int_0^\infty (1 + |(\xi, \sigma)|^2)^\gamma |(\xi, \sigma)|^2 |\hat{g}(\theta, \xi, ||(\xi, \sigma)||)|^2 |\sigma| d\sigma d\xi d\theta
$$

$$
= \frac{16}{\pi^2} \int \int_\mathbb{S} \int_0^\infty \int_0^\infty (1 + |\eta|^2)^\gamma |\eta|^2 |\hat{g}(\theta, \xi, \eta)|^2 \eta d\eta d\xi d\theta,
$$

where in the last line, we changed the variable $|(\xi, \sigma)|$ to $\eta$.

2.4 Range conditions

In this subsection, we describe the necessary range conditions of the cylindrical Radon transform $R_C f$.

**Theorem 14.** If $g = R_C f$ for $f \in C^\infty(B^2_R \times \mathbb{R})$, then we have

1. $$
\int_\mathbb{R} g(\theta, p, \sqrt{(p-z)^2 + (\rho-R)^2}) dp = \int_\mathbb{R} g(-\theta, p, \sqrt{(p-z)^2 + (\rho+R)^2}) dp.
$$

2. For $m = 0, 1, 2, \cdots$, $P_2(\theta)$ is a homogeneous polynomial of degree $m$ in $\theta$, where

$$
P_2(\theta) = \int_\mathbb{R} I_2^{-1} R_C^* g(\theta, z, R-s) s^m ds,
$$

and

3. $\text{supp} \, \tilde{g} \subset \{(\theta, \xi, \eta) : \eta \geq |\xi|\}$.

**Proof.**

1. From equation (1), $I_2^{-1} R_C^* g(\theta, z, R-\rho)$ should be equal to $I_2^{-1} R_C^* g(-\theta, z, R+\rho)$.

2. This follows from equation (6) and the range condition of the regular Radon transform.
3. Let a function $\phi$ on $S^1 \times \mathbb{R} \times [0, \infty)$ satisfy $\text{supp} \ 2\phi(\theta, \xi, \eta) \subset \{(\theta, \xi, \eta) : 0 \leq \eta < |\xi|\}$. Note that from equation (5), we have

$$\hat{R}_C^* \phi(\theta, \xi, \sigma) = \int_{S^1} \int_\mathbb{R} e^{-i(\tau + (R - s)\theta, z, P)} e^{-i\tau(\xi, \sigma)} \phi(\theta, \tau, |(z, P)|) d\tau dP = \tilde{\phi}(\theta, \xi, |(\xi, \sigma)|).$$

Then we have

$$\int_{S^1} \int_\mathbb{R} \tilde{g}(\theta, \xi, \eta) \tilde{\phi}(\theta, \xi, \eta) \eta d\eta d\theta = \int_{S^1} \int_\mathbb{R} \tilde{g}(\theta, p, r) \phi(\theta, p, r) r d\eta d\theta$$

$$= \int_{S^1} \int_\mathbb{R} \int_{\mathbb{R}} f(t\theta + (R - s)\theta, z) dt R_C^* \phi(\theta, z, s) ds dz d\theta$$

$$= \int_{S^1} \int_\mathbb{R} \int_\mathbb{R} \hat{f}(\tau, \xi, \sigma) e^{iR\tau} \tilde{R}_C^* \phi(\theta, \tau, |(\xi, \sigma)|) d\tau d\sigma d\theta$$

$$= \int_{S^1} \int_\mathbb{R} \int_\mathbb{R} \hat{f}(\tau, \xi, \sigma) e^{iR\tau} \tilde{\phi}(\theta, \xi, |(\xi, \sigma)|) d\tau d\sigma d\theta = 0.$$

\[\square\]

### 3 An $n$-dimensional case of $R_C$

In this section, we consider the cylindrical Radon transform $R_C$ of a function $f \in C^\infty_c(B^R_k \times \mathbb{R}^{n-k})$ where $n \geq 3$ is arbitrary. As mentioned before (see also [9]), in 3-dimension $R_C f$ can be decomposed into the circular Radon transform and the usual 2-dimensional Radon transform. A natural $n$-dimensional analog of the cylindrical Radon transform would split into composition of the $n-k+1$-dimensional spherical Radon transform and the usual $k$-dimensional Radon transform. We consider a more general possibility. Namely, we define $R_{C, n,k}$ of a function $f \in C^\infty_c(B^R_k \times \mathbb{R}^{n-k})$ that decomposes into the $n-k+1$-dimensional spherical Radon transform and the usual $k$-dimensional Radon transform. We define $R_{C, n,k}$ for $1 < k \leq n-1$ and $(\theta, p, r) \in S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)$ as follows:

$$R_{C, n,k} f(\theta, p, r) = \frac{1}{|S^{n-k}|} \int_{\theta^\perp} \int_{S^{n-k}} f(\tau + (R - r\alpha_1)\theta, p + r\alpha') d\alpha d\tau,$$

where $\alpha = (\alpha_1, \alpha') \in S^{n-k}$ and $\theta^\perp$ refers to $\theta^\perp \cap \{(x, z) \in \mathbb{R}^{n-k} \times \mathbb{R}^k : z = p\}$. Then we have an analogue of the Fourier slice theorem, similar to theorem [3]

**Theorem 15.** Let $f \in C^\infty_c(B^R_k \times \mathbb{R}^{n-k})$. If $g = R_{C, n,k} f$, then we have for $(\theta, \xi, \sigma) \in S^{k-1} \times \mathbb{R}^{n-k} \times \mathbb{R},$

$$\hat{f}(\sigma, \xi, \sigma) = 2|S^{n-k}|(2\pi)^{-n+k-1} R_{C, n,k}^* g(\theta, \xi, \sigma) e^{iR\sigma} |(\xi, \sigma)|^{n-k-1} |\sigma|.$$
where \( \hat{f} \) is the Fourier transform of \( f \) and \( \hat{R}_{C_{n,k}}^* g \) is the \( n - k + 1 \)-dimensional Fourier transform of \( g \) in \((z, \rho)\). Here

\[
R_{C_{n,k}}^* g(\theta, z, \rho) = \int_{\mathbb{R}^{n-k}} g(\theta, p, \sqrt{|z - p|^2 + \rho^2}) dp,
\]

for \( g \in C_c^\infty(S^k \times \mathbb{R}^{n-k} \times [0, \infty)) \) and \((z, \rho) \in \mathbb{R}^{n-k} \times \mathbb{R} \).

This theorem is natural in view of remark 4. The proof of this theorem is the almost same as that of theorem 3. Instead of taking the Hankel transform in \( r \), one take the radial Fourier transform, i.e.,

\[
\int_0^\infty r^{n-k} J_{\frac{n-k-2}{2}}(r \eta) \hat{R}_{C_{n,k}}^* f(\theta, \xi, r) dr.
\]  

Also, we need the following identity:

\[
\int_{S^{n-1}} e^{i \xi \cdot \theta} d\theta = (2\pi)^{n/2} |\xi|^{(2-n)/2} J_{(n-2)/2}(\|\xi\|)
\]

(see [5]). Lastly instead of the identity [3], we need the following identity: for \( a, b > 0 \) and \( \mu > \nu > -1 \),

\[
\int_0^\infty \rho^{\nu+\frac{1}{2}} (\rho^2 + \beta^2)^{-\frac{1}{2}} J_\mu(a \sqrt{\rho^2 + \beta^2} J_\nu(\rho|\xi|)(\rho|\xi|)^1/2 d\rho
\]

\[
= \begin{cases} 
  a^{-\mu} |\xi|^{\nu+\frac{1}{2}} b^{-\mu+\nu+1} (a^2 - |\xi|^2)^{\frac{1}{2}} J_{\mu-\nu-1}(b \sqrt{a^2 - |\xi|^2}) 
  & \text{if } 0 < |\xi| < a, \\
  0 & \text{otherwise} 
\end{cases}
\]

[5] p.59 (18) vol.2 or for \( n = 2 \) p.55 (35) vol.1. The other steps are the same as in the proof of theorem 3.

For \( \gamma < n - k + 1 \), we define the linear operators \( I^\gamma \) and \( I^\gamma_2 \) by

\[
I^\gamma \hat{h}(\theta, \xi, \sigma) = |(\xi, \sigma)|^{-\gamma} \hat{h}(\theta, \xi, \sigma) \quad \text{and} \quad I^\gamma_2 \hat{h}(\theta, \xi, \sigma) = |\sigma|^{-\gamma} \hat{h}(\theta, \xi, \sigma),
\]

for a function \( h(\theta, z, \rho) \) on \( S^{k-1} \times \mathbb{R}^{n-k+1} \) with its \( n - k + 1 \)-dimensional Fourier transform \( \hat{h} \) with respect to \((z, \rho)\). Then we have the inversion similar to Theorem 5.

**Theorem 16.** Let \( f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k}) \). If \( g = R_{C_{n,k}} f \), then we have for \((x, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \),

\[
f(x, z) = \frac{|S^{n-k}|}{(2\pi)^n} \int_{S^{k-1}} I_2^{-k} I_1^{1-n+k} R_{C_{n,k}}^* g(\theta, z, \rho) \big|_{\rho = x - R} d\theta.
\]
To obtain inversion formula similar to Theorem 7, let \( f(t, \varphi, z) \) be the image function in cylindrical coordinates where \( t = |x| \) and \( \varphi = \pi/|x| \in S^{k-1} \). Then the Fourier series of \( f(\rho, \phi, z) \) and \( g(\theta, p, r) \) with respect to their angular variables can be written as follows:

\[
f(t, \varphi, z) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(k,l)} f_{lj}(t, z)Y_{lj}(\varphi) \quad \text{and} \quad g(\theta, p, r) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(k,l)} g_{lj}(p, r)Y_{lj}(\theta),
\]

where \( Y_{lj} \) is a spherical harmonic and

\[
N(k, l) = \frac{(2l + k - 2)(k + l - 3)!}{l!(k - 2)!}, \quad N(k, 0) = 1.
\]

According to [16], when \( g = \mathcal{R}f \) for the regular \( k \)-dimensional Radon transform \( \mathcal{R} \) and \( g_{lj} \) and \( f_{lj} \) are the Fourier coefficients of \( g \) and \( f \), we have for \( t > 0 \),

\[
f_{lj}(t) = \frac{(-1)^{k-1}}{2\pi^{k/2}} \frac{\Gamma((k-2)/2)}{\Gamma(k-2)} t^{2-k} \int_{t}^{\infty} (s^2 - t^2)^{(k-3)/2} C_l^{(k-2)/2}(\frac{s}{t}) \partial_{s}^{k-1} g_{lj}(s) ds,
\]

where \( C_l^{(k-2)/2} \) is the (normalized) Gegenbauer polynomial of degree \( l \). From Theorem 15, we have

\[
\int_{\theta \perp} \hat{f}(\tau + s\theta, \xi) d\tau = \frac{2|S^{n-k}|}{(2\pi)^{n-k+1}} \int_{\mathbb{R}} \hat{R}_{\mathcal{R}_{n,k}}^{*} g(\theta, \xi, \sigma) e^{i(R-s)\sigma} |\sigma|^{n-k-1} d\sigma.
\]

Consider the \( lj \)-th Fourier coefficient of the right hand side of formula (15). Then we have

\[
\int_{S^{k-1}} \int \hat{R}_{\mathcal{R}_{n,k}}^{*} g(\theta, \xi, \sigma) e^{i(R-s)\sigma} |\sigma|^{n-k} Y_{lj}(\theta) d\sigma d\theta = \int_{0}^{\infty} \hat{R}_{\mathcal{R}_{n,k}}^{*} g_{lj}(\xi, \sigma) e^{i(R-s)\sigma} |\sigma|^{n-k} Y_{lj}(\xi, \sigma) d\sigma.
\]

**Theorem 17.** Let \( f \in C_{c}^{\infty}(B_{R}^{k} \times \times \mathbb{R}^{n-k}) \). Then we have for \( \rho > 0 \)

\[
f_{lj}(t, z) = \frac{c_{k}}{(2\pi)^{n-k}} t^{2-k} \int_{t}^{\infty} (s^2 - t^2)^{(k-3)/2} C_l^{(k-2)/2}(\frac{s}{t}) I_{2-k} I_{1+n-k} R_{\mathcal{R}_{n,k}}^{*} g_{lj}(z, R - s) ds,
\]

where

\[
c_{k} = \frac{2\pi^{n-k-1}}{\Gamma((k-2)/2)} \frac{|S^{n-k}| \Gamma((k-2)/2)}{\Gamma(k-2)}.
\]

**Proof.** Applying equation (14) to equation (16) gives

\[
\hat{f}_{lj}(t, \xi) = c_{k} t^{2-k} \int_{t}^{\infty} (s^2 - t^2)^{(k-3)/2} C_l^{(k-2)/2}(\frac{s}{t}) \int_{0}^{\infty} \hat{R}_{\mathcal{R}_{n,k}}^{*} g_{lj}(\xi, \sigma) e^{i(R-s)\sigma} |\sigma|^{k} |(\xi, \sigma)|^{n-k-1} d\sigma ds,
\]

where \( \hat{f}_{lj} \) is the \( n - k \)-dimensional Fourier transform of \( f_{lj} \) with respect to \( z \). \( \square \)
Also, we can get the following theorem similar to theorem 8.

**Theorem 18.** Let \( f \in C^\infty_c(B^k_R \times \mathbb{R}^{n-k}) \). Then we have

\[
\int_{\theta} f(\tau + s\theta, z)d\tau = \frac{4|S^{n-k}|}{(2\pi)^{n-k+1}} \int_{\mathbb{R}^{n-k}} \int_0^\infty (R-s)rR_{C_{n,k}} f(\theta,-p,r)e^{-iv\cdot(p+(R-s)^2)}|p|^2|\xi|d\tau drd\xi.
\]

As in section 2, we can obtain a stability estimate and necessary range conditions for \( R_{C_{n,k}} \).

**Theorem 19.** We have

\[
||f||_\gamma \leq \frac{2|S^{n-k}|(2\pi)^{-n+k-1}}{\gamma+(n-1)/2} ||R_{C_{n,k}} f||_\gamma+(n-1)/2,
\]

for \( f \in \mathcal{H}^\gamma(\mathbb{R}^n) \) supported in \( B^k_R \times \mathbb{R}^{n-k} \).

Also, we have a similar support theorem to Theorem 10.

**Theorem 20.** Let \( p_0 \in \mathbb{R}^{n-k} \) and \( B > 0 \). Let \( f \in C^\infty(B^k_R \times \mathbb{R}^{n-k}) \) and suppose that \( g = R_{C_{n,k}} f \) is equal to zero on the open set \( U_B = \{(\theta, p_0, r) : 0 \leq \theta < 2\pi, 0 \leq r < B\} \). Then \( f \) is equal to zero on the set \( \{(x, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x| > R - \sqrt{B^2 - |p - p_0|^2}, z = p_0\} \).

**Remark 21.** We can obtain the same result as Theorem 14 for an \( n \)-dimensional case using Theorem 15 instead of Theorem 3.

4 Planar geometry

Let us first explain the mathematical model arising in PAT with line detectors introduced in 8. Let \( L_P(\theta,p) = \{(0, y, z) \in \mathbb{R}^3 : (y, z) \cdot \theta = p\} \) for \( p > 0 \) and \( \theta \in S^1 \) be a line detector. Then we have \( L_P(\theta,p) = L_{-\theta,-p} \) and a detector line \( L_P(\theta,p) \) is located on \( yz \)-plane.

**Definition 22.** Let a function \( f \) be even in \( x \). The cylindrical Radon transform \( R_P \) maps \( f \in C^\infty_c(\mathbb{R}^3) \) into

\[
R_P f(\theta,p,r) = \frac{1}{2\pi r} \iint_{d(L_P(\theta,p),(x,y,z)) = r} f(x,y,z)d\varpi,
\]

for \( (\theta, p, r) \in S^1 \times \mathbb{R} \times [0, \infty) \). Here \( d\varpi \) is the area measure on the cylinder

\[
\{(x, y, z) \in \mathbb{R}^3 : d(L_P(\theta,p),(x,y,z)) = r\}
\]

and

\[
d(L_P(\theta,p),(x,y,z)) := \sqrt{x^2 + (p - (y, z) \cdot \theta)^2}
\]

denotes the Euclidean distance between the line \( L_P(\theta,p) \) and the point \( (x, y, z) \).
Remark 23. We have $R_P f(\theta, p, r) = R_P f(-\theta, -p, r)$.

By definition, we have

$$R_P f(\theta, p, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}} f(r \cos \psi, t\theta + (p - r \sin \psi)\theta) d\psi dt,$$

where $r$ is the radius, $p$ and $\theta$ are the distance and the direction from the origin to the central axis of the cylinder, $t$ is a parameter variable of the central axis of the cylinder, and $\psi$ is the polar angle of the circle that is the intersection of plane $\{(x, t\theta) : t \in \mathbb{R}, x \in \mathbb{R}\}$ and the cylinder.

![Figure 2: (a) the integral area cylinder and (b) the restriction to the $\{(x, t\theta) : x \in \mathbb{R}, t \in \mathbb{R}\}$ plane](image)

4.1 Inversion formulas

We have two integrals in the definition formula of $R_P f$. Like $R_C f$, the inner integral is a circular Radon transform with centers on the line for fixed $\theta$ and the outer integral can be thought of as the 2-dimensional regular Radon transform for a fixed $x$-coordinate $[8]$. Similarly, we start to apply the inversion of the circular Radon transform for a fixed $\theta$.

Theorem 24. Let $f \in C_c^\infty(\mathbb{R}^3)$ be even in $x$. If $g = R_P f$, then we have

$$\hat{f}(\xi, \sigma \theta) = 4|\xi|\hat{R_P^* g}(\theta, \sigma, \xi),$$

(17)

where $\hat{f}$ and $\hat{R_P^* g}$ are the Fourier transforms of $f$ and $R_P^* g := R_C^* g$ with respect to $(x, y, z) \in \mathbb{R}^3$ and $(z, \rho) \in \mathbb{R}^2$. 

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Remark 25. The evenness of \( g \) in \((\theta, p)\) implies the evenness of \( \hat{R}_P^*g \) in \((\theta, \sigma)\).

Remark 26. Equation (17) can be thought of as the combination of two Fourier slice theorems: for the circular and regular Radon transforms, too.

Proof of Theorem. Taking the Fourier transform of \( R_P f \) with respect to \( p \) yields

\[
\hat{R}_P f(\theta, \sigma, r) = \frac{1}{\pi} \int_{-1}^{1} \hat{f}(r\sqrt{1-s^2}, \sigma \theta) e^{irs\sigma} \frac{ds}{\sqrt{1-s^2}},
\]

where \( \hat{f} \) and \( \hat{R}_P f \) are the Fourier transforms of \( f \) and \( R_P f \) with respect to \((y, z) \in \mathbb{R}^2 \) and \( p \in \mathbb{R} \), respectively. Taking the Hankel transform \( H_0 \) of \( \hat{R}_P f \) with respect to \( r \), we have

\[
H_0 \hat{R}_P f(\theta, \sigma, \eta) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-1}^{1} \hat{f}(r\sqrt{1-s^2}, \sigma \theta) e^{irs\sigma} J_0(r\eta) r dr ds
\]

where in the last line, we made a change of variables \((r, s) \rightarrow (\rho, b)\) where \( r = \sqrt{\rho^2 + b^2} \) and \( s = \rho/\sqrt{\rho^2 + b^2} \). Applying the identity (3) to equation (18), we get

\[
H_0 \hat{R}_P f(\theta, \sigma, \eta) = \begin{cases} 
\frac{2}{\pi} \int_{0}^{\infty} \hat{f}(b, \sigma \theta) \frac{1}{\sqrt{\eta^2 - \sigma^2}} \cos(b \sqrt{\eta^2 - \sigma^2}) db & \text{if } 0 < \sigma < \eta, \\
0 & \text{otherwise.}
\end{cases}
\]

Substituting \( \eta = \sqrt{\xi^2 + \sigma^2} \) yields

\[
H_0 \hat{R}_P f(\theta, \sigma, |(\sigma, \xi)|) = \frac{2}{\pi} \int_{0}^{\infty} \hat{f}(b, \sigma \theta) \frac{\cos(b \xi)}{\xi} db = \frac{1}{\pi} \hat{f}(\xi, \sigma \theta) |\xi|^{-1}.
\]

As in the proof of theorem 3, we change the right side of equation (19) into a term containing the backprojection operator \( R_P^* \). We have \( \hat{R}_P^* g(\theta, \sigma, \xi) = 2\pi H_0 \hat{g}(\theta, \sigma, |(\sigma, \xi)|) \), so we get equation (17).

Let the linear operator \( I_i \) for \( h \in C^\infty(S^1 \times \mathbb{R}^2) \) be defined by \( \hat{I}_i^{-1} \hat{h}(\theta, \xi) = |\xi| \hat{h}(\theta, \xi) \) where \( \hat{h} \) is the 2-dimensional Fourier transform of \( h \) in the last two dimensional variable. Then we have the following inversion formula.
**Theorem 27.** Let \( f \in C^\infty_c(\mathbb{R}^3) \) be even in \( x \). Then we have

\[
f(x, y, z) = 4\pi^{-1} \int_{S^1} I_1^{-1} I_2^{-1} R_P^* g(\theta, \theta \cdot (y, z), x) d\theta,
\]
for \( g = R_P f \).

From equation (17), we have

\[
\int_{\mathbb{R}} f(x, t\theta^\perp + s\theta) dt = 4I_2^{-1} R_P^*(\theta, s, x).
\]

As in below Remark 16, let \( f(x, t, \phi) \) be the image function in cylindrical coordinates where \((y, z) = t(\cos \varphi, \sin \varphi)\). Consider the \( l \)-th Fourier coefficient of the right hand side of formula (24). Then we have

\[
\int_{S^1} I_1^{-1} I_2^{-1} R_P^* g(\theta, s, x) e^{-il\theta} d\theta d\theta = I_2^{-1} R_P^* g_l(s, x),
\]
where \( \theta = (\cos \varphi, \sin \varphi) \). Similar to Theorem 7, we have

**Theorem 28.** Let \( f \in C^\infty_c(\mathbb{R}^3) \) be even in \( x \). Then we have for \( t > 0 \)

\[
f_l(x, t) = \frac{-2\pi}{2\pi} \int_{0}^{\infty} (s^2 - t^2)^{-1/2} \cos \left( l \arccos \left( \frac{s}{t} \right) \right) \frac{\partial}{\partial s} I_2^{-1} R_P^* g_l(s, x) ds.
\]

Also, we have another relation between the Radon transform and \( R_P \) similar to Theorem 8.

**Theorem 29.** Let \( f \in C^\infty_c(\mathbb{R}^3) \) be even in \( x \). Then we have

\[
\int_{\mathbb{R}} f(x, t\theta^\perp + z\theta) dt = 2\pi \int_{\mathbb{R}} \int_{0}^{\infty} zr R_P f(\theta, -p, r) e^{-ir^2\sigma} e^{-i(2xp + (z^2 + x^2) + p^2)\sigma} \sigma dr dp d\sigma.
\]

We can prove this in a way similar to Theorem 8.

**4.2 A stability estimate**

In this subsection, we discuss the stability estimate of the cylindrical Radon transform \( R_P \).

For \( \gamma \geq 0 \), let us define \( \mathcal{H}_\gamma^o(\mathbb{R}^n) := \{ f \in \mathcal{H}^o(\mathbb{R}^n) : f \text{ is even in } x \} \). As in subsection 2.3 let \( L_1^2(S^{n-2} \times \mathbb{R} \times [0, \infty)) \) be the set of a function \( g \) on \( S^{n-2} \times \mathbb{R} \times [0, \infty) \) with

\[
||g||^2 := \int_{S^{n-2}} \int_{\mathbb{R}} \int_{0}^{\infty} |g(\theta, p, r)|^2 r dr dp d\theta < \infty.
\]
Then $L^2(S^{n-2} \times \mathbb{R} \times [0, \infty))$ is a Hilbert space. Also, by the Plancherel formula, we have $||g|| = (2\pi)^{-3}||\hat{g}||$, where

$$
\hat{g}(\theta, \sigma, |\zeta|) := \int_{\mathbb{R}^2} g(\theta, p, |w|) e^{-i(p,w)-(\sigma,\zeta)} \, dp \, dw.
$$

Let $H^\gamma(S^{n-2} \times \mathbb{R} \times [0, \infty))$ be the set of a function $g \in L^2_1(S^{n-2} \times \mathbb{R} \times [0, \infty))$ with $||g|| \leq \infty$, where

$$
||g||_\gamma := \int_{S^{n-2}} \int_{\mathbb{R}} ||\hat{g}(\theta, \sigma, |\zeta|)||^2 (1 + |\sigma|^2 + |\eta|^2)^\gamma \, \eta \, d\eta \, d\sigma \, d\theta.
$$

**Theorem 30.** For $\gamma \geq 0$, there exists a constant $c$ such that for $f \in H^\gamma_0(\mathbb{R}^3)$,

$$
||f||_\gamma \leq c ||R_P f||_{\gamma+1}.
$$

**Proof.** Let $g = R_P f$. Similar to equation (5), we have

$$
\hat{R_P} g(\theta, \sigma, \xi) = \int_{\mathbb{R}} e^{-i \sigma p} \int_{\mathbb{R}^2} e^{-i(\xi, \rho)-(\sigma, \xi)} g(\theta, p, |(z, \rho)|) \, dz \, dp = \hat{g}(\theta, \sigma, |(\sigma, \xi)|).
$$

Combining this equation and equation (17), we have

$$
\hat{f}(\xi, \sigma \theta) = 4|\xi| \hat{g}(\theta, \sigma, |(\sigma, \xi)|).
$$

Hence, we have

$$
||f||_\gamma^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2 + |\xi|^2)^\gamma |\hat{f}(\xi, \xi)|^2 d\xi = 2^{-1} \int_{S^1} \int_{\mathbb{R}} |\sigma| (1 + |\sigma|^2 + |\xi|^2)^\gamma |\hat{f}(\xi, \sigma \theta)|^2 \, d\xi \, d\sigma \, d\theta
$$

$$
= 8 \int_{S^1} \int_{\mathbb{R}} |\sigma| (1 + |(\sigma, \xi)|^2)^\gamma |\xi|^2 |\hat{g}(\theta, \sigma, |(\sigma, \xi)|)|^2 \, d\xi \, d\sigma \, d\theta
$$

$$
= 16 \int_{S^1} \int_{0}^{\infty} |\sigma| (1 + |(\sigma, \xi)|^2)^\gamma |\xi|^2 |\hat{g}(\theta, \sigma, |(\sigma, \xi)|)|^2 \, d\xi \, d\sigma \, d\theta
$$

$$
= 16 \int_{S^1} \int_{0}^{\infty} \sqrt{\eta^2 - \sigma^2} |\sigma| (1 + \eta^2)^\gamma |\hat{g}(\theta, \sigma, \eta)|^2 \eta \, d\eta \, d\sigma \, d\theta,
$$

where in the last line, we changed the variable $|(\sigma, \xi)|$ to $\eta$. Continuing the computation yields

$$
||f||_\gamma^2 \leq c \int_{S^1} \int_{0}^{\infty} (1 + \sigma^2 + \eta^2)^{\gamma+1} |\hat{g}(\theta, \sigma, \eta)|^2 \eta \, d\eta \, d\sigma \, d\theta.
$$

□
4.3 Range conditions

From Theorem 32, we have necessary range conditions for $R_p$ as follows:

**Theorem 31.** If $g = R_P f$ for a even function $f \in C^\infty(\mathbb{R}^3)$, then we have

1. $g(\theta, p, r) = g(-\theta, -p, r)$,
2. For $m = 0, 1, 2, \cdots$, $P_x(\theta)$ is a homogeneous polynomial of degree $m$ in $\theta$, where
   \[ P_x(\theta) = \int_{\mathbb{R}} g(\theta, p, \sqrt{(s - p)^2 + x^2})s^m ds, \]
   and
3. $\text{supp} \, \tilde{g}(\theta, \sigma, \eta) \subset \{ (\theta, \sigma, \eta) : \eta \geq |\sigma| \}.$

**Proof.**

2. From equation (17) and the range description of the regular Radon transform, we have that for fixed $x$, the polynomial $\int_{\mathbb{R}} I_{-1} R^*_P g(\theta, s, x)s^m ds$ is homogeneous of degree $m$ in $\theta$, which implies that $P_x(\theta)$ is a homogeneous polynomial of degree $m$ in $\theta$.

3. We can prove this proof in way similar to that of 3 in Theorem 14.

5 An $n$-dimensional case of $R_P$

As in section 3, we consider the cylindrical Radon transform $R_P$ of a function $f \in C^\infty_c(\mathbb{R}^n)$. Assume $n \geq 3$. We define $R_{P_n}$ of a function $f \in C^\infty_c(\mathbb{R}^n)$ even in $x \in \mathbb{R}$ by

\[ R_{P_n} f(\theta, p, r) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^n} f(r \cos \psi, \tau + (p - r \sin \psi)\theta) d\psi d\tau, \]

for $(\theta, p, r) \in S^{n-2} \times \mathbb{R} \times [0, \infty)$. Here $\theta^\perp$ actually refers to $\theta^\perp \cap \{ (0, z) : z \in \mathbb{R}^{n-1} \}$. We still have $R_{P_n} f(\theta, p, r) = R_{P_n} f(-\theta, -p, r)$.

**Theorem 32.** Let $f \in C^\infty_c(\mathbb{R}^n)$ be even in $x \in \mathbb{R}$. If $g = R_{P_n} f$, then we have

\[ \hat{f}(\xi, \sigma\theta) = 4|\xi| \hat{R^*_P g}(\theta, \sigma, \xi), \] (23)

where $\hat{f}$ and $\hat{R^*_P g}$ are the Fourier transforms of $f$ and $R^*_P g$ with respect to $(x, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $(\zeta, \rho) \in \mathbb{R}^2$. Here for a function $g$ on $S^{n-2} \times \mathbb{R} \times [0, \infty)$,

\[ R^*_P g(\theta, \zeta, \rho) = \int_{\mathbb{R}} g(\theta, p, \sqrt{(\zeta - p)^2 + \rho^2}) dp. \]
This proof is similar to that of Theorem 32. The only difference is that one takes the radial Fourier transform (11) and use equation (12) as in the proof of Theorem 15.

**Theorem 33.** Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in $x$. Then we have

$$f(x,z) = 2(2\pi)^{2-n}\int_{S^{n-2}} I_1^{2-n} I_2^{-1} R_{P_n}^* g(\theta, \theta \cdot z, x) d\theta,$$

for $g = R_{P_n} f$ and $(x, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Let $f(x, t, \phi)$ be the image function in cylindrical coordinates where $t = |z|$ and $\phi = z/|z| \in S^{n-2}$. Then the Fourier series of $f(x, t, \phi)$ and $g(\theta, p, r)$ with respect to their angular variables can be written as follows:

$$f(x, t, \phi) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(n-1, l)} f_{lj}(x, t) Y_{lj}(\phi) \quad \text{and} \quad g(\theta, p, r) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(n-1, l)} g_{lj}(p, r) Y_{lj}(\theta).$$

From equation (23), we have

$$\int_{\theta \perp} f(x, \tau + s\theta) d\tau = 4I_2^{-1} R_{P_n}^* (\theta, s, x). \quad (24)$$

Consider the $lj$-th Fourier coefficient of the right hand side of formula (24). Then we have

$$\int_{S^{n-2}} I_2^{-1} R_{P_n}^* g(\theta, s, x) Y_{lj}(\theta) d\theta = I_2^{-1} R_{P_n}^* g_{lj}(s, x). \quad (25)$$

Applying equation (25) to equation (14), we have the following theorem.

**Theorem 34.** Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in $x$. Then we have for $t > 0$

$$f_{lj}(x, t) = 4c_{n-1} t^{3-n} \int_0^{\infty} (s^2 - t^2)^{(n-4)/2} C_{l}^{(n-3)/2} \left( \frac{s}{t} \right) \frac{\partial^{n-2}}{\partial s^{n-2}} I_2^{-1} R_{P_n}^* g_{lj}(s, x) ds,$$

where

$$c_{n} = \frac{(-1)^{n-1} \Gamma((n-2)/2)}{2\pi^{n/2} \Gamma(n-2)}.$$

Also, as in subsection 3, we have the following theorem.

**Theorem 35.** Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in $x$. Then we have

$$\int_{\theta \perp} f(x, \tau + s\theta) d\tau = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{\infty} s t R_{P_n}^* f(\theta, -p, r) e^{-ir^2} e^{-i(2xp + (s^2 + x^2) + p^2)\sigma} dr dp d\sigma.$$
As in section 4, we can obtain a stability estimate and necessary range conditions for $R_P^n$.

**Theorem 36.** For $\gamma \geq 0$, there exists a constant $c_n$ such that for $f \in H_\gamma^r(\mathbb{R}^n)$,

$$||f||_\gamma \leq c_n ||R_P^n f||_{\gamma+n-2}.$$ 

**Proof.** Let $g = R_C f$. As in the proof of Theorem 30 using equation (23), we have

$$\hat{f}(\xi, \sigma \theta) = 4|\xi| \hat{g}(\theta, \sigma, |(\sigma, \xi)|),$$

so

$$||f||^2_\gamma = 16 \int \int \int_{|\sigma|}^{\infty} \sqrt{\rho^2 - \sigma^2} |\sigma|^{n-2} (1 + \rho^2)^\gamma |\hat{g}(\theta, \sigma, \rho)|^2 \rho d\rho d\sigma d\theta.$$ 

Here, we changed the variable $|(\sigma, \xi)|$ to $\rho$. Hence, we have

$$||f||^2_\gamma \leq c_n \int \int \int_{|\sigma|}^{\infty} |\sigma|^{n-2} (\rho^2 - \sigma^2)^{(n-2)/2} (1 + \rho^2)^\gamma |\hat{g}(\theta, \sigma, \rho)|^2 \rho d\rho d\sigma d\theta$$

$$\leq c_n \int \int \int_{|\sigma|}^{\infty} (1 + \sigma^2 + \rho^2)^{\gamma+n-2} |\hat{g}(\theta, \sigma, \rho)|^2 \rho d\rho d\sigma d\theta.$$ 

$\square$

**Remark 37.** The Theorem 31 holds for $R_P^n$ for $n \geq 3$.

### 6 Conclusion

In this article, we study two different versions of cylindrical Radon transforms arising in PAT. We describe some inversion formulas of these transforms and discuss their stability estimate and necessary range conditions.

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