Research Article

On the Exact Values of HZ-Index for the Graphs under Operations

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1.Introduction

A structural formula of a chemical compound is represented by a molecular graph, where atoms and bonds between atoms are represented by the vertices and edges of the molecular graphs, respectively. A topological index (TI) is a mathematical tool which associates a real number to a graph under certain conditions. For two graphs, a TI remains constant if the graphs are isomorphic (see [1–3]). These are used to study different physical attributes, biological activities, and chemical reactivities such as viscosity, critical temperatures (boiling, freezing, melting, and flash points) [4, 5], vapor pressure, surface tension, stability, weight, density, solubility, and connectivity [6–8] in the field of chemical engineering, pharmaceutical industries, and drugs discoveries. TIs are also used in the subject of cheminformatics to study the quantitative structural activity and property relationships (see [9–11]).

In 1947, the very first TI is introduced by Winer to check the critical temperature of paraffin [12]. Trinajstić and Gutman (1972) [13] defined the first and second Zagreb indices that are used to compute the different structure base characteristics of the molecular graphs. After that, many degree, distance, and polynomials based TIs came into existence but the degree-based indices got more attention of the researchers (see [14–16]). For various results on TIs of different graphs, see [17–20]. In 2008, Zhou and Trinajstić defined the general sum connectivity (GSC) index and discussed its various properties [21]. Shirdel et al. [22] studied the concept of hyper-Zagreb index (HZ – index) as a particular case of the GSC index. In addition, the results for the index HZ under the operation of Cartesian, composition, join, and disjunction of graphs can be found in [23–25].

On the other hand, for the studies of the complex graphs, operations for graphs play a key role. Yan et al. (2007) defined four types of operations related to the subdivision of G and computed the Wiener indices of the derived graphs $D_1(G)$, where $D_1 \in \{S_1, R_1, Q_1, T_1\}$ [26]. Taeri et al. (2009) gave the construction of the $D_1$-sum graphs $GD_1 + H$ (Cartesian product of $F_1(G)$ and $H$) and computed their Wiener indices, where $H$ and $G$ are assumed to be two connected graphs [27]. Furthermore, Deng et al. [28], Akhter and Imran [29], Chu et al. [30], and Liu et al. [31] computed the various indices of these graphs with the help of the Cartesian product.

Liu et al. (2019) [32] extended these operations for any integral value of $k$ and obtained the generalized derived graphs $D_k(G)$ of the graph $G$, where $D_k \in \{S_k, R_k, Q_k, T_k\}$. Moreover, using the concept of Cartesian product of graphs, they constructed the generalized sum graphs or $D_k$-sum graphs (denoted by $GD_k + H$) and computed their first and second Zagreb indices.
Javaid et al. (2021) [33] redefined these graphs using strong product and computed their Zagreb indices (first and second). In this development, we compute hyper-Zagreb indices (HZ-index) for these graphs in terms of various degree-based TIs of their factor graphs, where these generalized sum graphs are obtained with the help of strong product. The remaining paper is settled as follows. Section 2 contains the notations and key concepts which are utilized in methodology, Section 3 deals main results, and Section 4 covers examples and conclusion.

2. Preliminaries

This section explains the basic definitions and terminologies.

Definition 1. Let \( G = (V(G), E(G)) \) be a (molecular) graph with \( V(G) \) and \( E(G) \) as sets of vertices and edges, respectively. The degree of a vertex \( v \in V(G) \) is the number of edges which are incident on \( v \) and denoted by \( d(v) \).

Definition 2 (see [13, 34]). For a graph \( G \), the first, second, and forgotten Zagreb indices are defined as follows:

\[
M_1(G) = \sum_{x \in V(G)} d(x) + d(y) \text{ and } M_2(G) = \sum_{x \in V(G)} d^2(x),
\]

\[
F(G) = \sum_{x \in V(G)} d^3(x) + \sum_{x \in E(G)} d^2(x).
\]

These indices have been used to find the various properties of molecular graphs such as entropy, \( \pi \)-electron energy, and heat capacity. These are also used in the studies of the molecular structural relationships such as QSPR and QSAR [13, 35–37]. However, the hyper-Zagreb index of a graph \( (G) \) (given below) is studied by Shirdel et al. in 2013 [22]:

\[
HZ(G) = \sum_{y \in E(G)} [d(y) + d(z)]^2. \tag{1}
\]

Definition 3 (see [32]). For some integral value of \( k \geq 1 \), the graphs obtained by the generalized subdivision-related operations are defined as follows:

(i) \( S_k(G) \) is a graph that is obtained by inserting \( k \) vertices in each edge of \( G \)

(ii) \( R_k(G) \) is a graph obtained from \( S_k(G) \) by joining the vertices which are adjacent in \( G \)

(iii) \( Q_k(G) \) is a graph obtained from \( S_k(G) \) by joining the new vertices which are on the incident edges in \( G \) for each of its vertex

(iv) \( T_k(G) \) is obtained from \( S_k(G) \) after using both \( R_k \) and \( Q_k \), respectively

For \( k = 3 \), see Figure 1.

Definition 4 (see [33]). Let \( G_1 \) and \( G_2 \) be two graphs, \( D_k \in \{S_k, R_k, Q_k, T_k\} \) be generalized subdivision-related operations, and \( D_k(G_1) \) be a graph obtained using \( D_k \) on \( G_1 \) having edge-set \( E(D_k(G_1)) \) and vertex-set \( V(D_k(G_1)) \). The generalized sum graph \( G_1 \otimes G_2 \) under the operation of strong product is a graph having vertex-set \( V(G_1 \otimes G_2) = V(D_k(G_1)) \times V(G_2) = (V(G_1) \cup k(E(G_1))) \times V(G_2) \) such that two vertices \((r_1, s_1) \) and \((r_2, s_2) \) of \( G_1 \otimes G_2 \) are adjacent iff \( r_1 = r_2 \) in \( G_1 \) and \( s_1 \) is adjacent to \( s_2 \) in \( E(G_2) \) or \( s_1 = s_2 \) in \( V(G_2) \) and \( s_1 \) is adjacent to \( s_2 \) in \( E(G_1) \) or \( r_1 \) is adjacent to \( r_2 \) in \( E(D_k(G_1)) \) and \( s_2 \) is adjacent to \( s_1 \) in \( E(G_1) \), where \( k \geq 1 \) is a positive integer. For more explanation, see Figures 2 and 3.

3. Main Results

The main developments are covered by this section.

Theorem 1. For \( k \geq 1 \), the HZ-index of \( G_1 \otimes G_2 \) is

\[
HZ(G_1 \otimes G_2) = 8e_{G_1}M_1(G_2) + n_{G_1}HZ(G_2) + 4e_{G_1}M_1(G_1) + 4e_{G_1}HZ(G_2) + M_1(G_1)HZ(G_2) + 4M_1(G_1)M_1(G_2) + n_{G_1}HZS_1(G_1) + 4e_{G_1}M_1S_1(G_1) + 2e_{G_1}M_1(G_1) + 4e_{G_1}HZS_1(G_2) + 2M_1(G_1)M_1S_1(G_1) + M_1(G_2)HZS_1(G_1) + 16(k - 1)e_{G_1}[n_{G_2} + M_1(G_2) + 4e_{G_1}] + HZG_1F(G_2) + 2M_1(G_1)HZS_1(G_1) + 2M_1(G_2)M_1(S(G_1)) + 2F(G_2)M_1(S(G_1)) + 16e_{G_1}e_{G_2} + 4e_{G_1}F(G_2) + 2e_{G_1}M_1(G_1) + 4(k - 1)e_{G_1}[8e_{G_2} + 2HZ(G_2) + 8M_1(G_1)].
\]

Proof. Let the degree of a vertex \((r, s) \in G_1 \otimes G_2 \) be denoted by \( d(r, s) \):

\[
HZ(G_1 \otimes G_2) = \sum_{(r_1, s_1) (r_2, s_2) \in E(G_1 \otimes G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]

\[
= \sum_{r \in V(G_1)} \sum_{s \in E(G_2)} [d(r, s)]^2 + \sum_{r_1, s_1 \in V(G_1)} \sum_{s_2 \in E(G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]

\[
+ \sum_{r_1, s_1 \in V(G_1)} \sum_{s_2 \in E(G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]

\[
= \sum_A + \sum_B + \sum_C.
\]
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) $G_1 \cong P_4$, (b) $S_3(P_4)$, (c) $R_3(P_4)$, (d) $Q_3(P_4)$ and (e) $T_3(P_4)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{(a) $P_3 \bowtie P_3$ and (b) $P_3 \bowtie P_3$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{(a) $P_3 \bowtie Q_3$ and (b) $P_3 \bowtie T_3$.}
\end{figure}
Consider

\[
\sum_{A} = \sum_{r \in V(G_1)} \sum_{s_1, s_2 \subseteq E(G_1)} [d(r, s_1) + d(r, s_2)]^2
\]

\[
= \sum_{r \in V(G_1)} \sum_{s_1, s_2 \subseteq E(G_1)} [2d(r) + d(s_1) + d(s_2) + d(r)(d(s_1) + d(s_2))]^2
\]

\[
= \sum_{r \in V(G_1)} \sum_{s_1, s_2 \subseteq E(G_1)} [4d(r)(d(s_1) + d(s_2)) + (d^2(s_1) + d(s_2)) + 2d(s_1)d(s_2) + 4d^2(r) + 2d(r)(d^2(s_1) + d^2(s_2)) + 2d(s_1)d(s_2)]
\]

\[
+ 2d(s_1)d(s_2) + d^2(r)(d^2(s_1) + d^2(s_2) + 2d(s_1)d(s_2)) + 4d^2(r)(d(s_1) + d(s_2))]
\]

\[
= 8e_{G_2}M_1(G_2) + n_{G_2}HZ(G_2) + 4e_{G_1}M_1(G_1) + 4e_{G_1}HZ(G_2) + M_1(G_1)HZ(G_2) + 4M_1(G_1)M_1(G_2).
\]

\[
\sum_{B} = \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{s \subseteq V(G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
= \sum_{r_1 \in V(G_1)} \sum_{r_2 \in V(G_1)} \sum_{s \subseteq V(G_1)} [d(r_1, s) + d(r_2, s)]^2 = \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1))} \sum_{s \subseteq V(G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
+ \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{s \subseteq V(G_1)} [d(r_1, s) + d(r_2, s)]^2 = \sum_{B_1} + \sum_{B_2}
\]

\[
\sum_{B_1} = \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1)) \cup (G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
= \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1)) \cup (G_1)} [(d^2(r_1) + d^2(r_2) + 2d(r_1)d(r_2)) + 2d(s)(d(r_1) + d(r_2)) + d^2(s)]
\]

\[
+ 2d(s)(d^2(r_1) + d^2(r_2) + 2d(r_1)d(r_2)) + 2d^2(s)(d(r_1) + d(r_2)) + d^2(s)(d^2(r_1) + d^2(r_2))]
\]

\[
= n_{G_2}HZS_1(G_1) + 4e_{G_2}M_1S_1(G_1) + 2e_{G_1}M_1(G_2) + 4e_{G_1}HZS_1(S_k(G_1)) + 2M_1(G_2)M_1S_1(G_1)
\]

\[
+ M_1(G_2)HZS_1(G_1),
\]

\[
\sum_{B_2} = \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1)) \cup (G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
= \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1)) \cup (G_1)} [d(r_1) + d(r_2)]d(s) + d(r_2) + d(r_2)d(s)]^2
\]

\[
= \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1)) \cup (G_1)} [4 + 4d(s)]^2 = \sum_{r_1, r_2 \subseteq E(S_k(G_1))} \sum_{r_1 \in V(G_1)} \sum_{r_2 \in (S_k(G_1)) \cup (G_1)} [16 + 16d^2(s) + 32d(s)].
\]
Since in this case $|E(S_{k}(G_1))| = (k-1)|E(G_1)|$, we have

$$
\begin{align*}
\sum_{G_1} &= \sum_{r, s \in E(S_{k}(G_1))} [d(r_1, s_1) + d(r_2, s_2)]^2 = \sum_{r, s \in E(S_{k}(G_1))} [d(r_1, s_1) + d(r_2, s_2)]^2 \\
&\quad + \sum_{r, s \in E(S_{k}(G_1))} \sum_{s_1, s_2 \in E(S_{k}(G_1))} [d(r_1, s_1) + d(r_2, s_2)]^2 = \sum_{G_1} + \sum_{G_2},
\end{align*}
$$

(5)

$$
\begin{align*}
\sum_{G_2} &= \sum_{r, s \in E(S_{k}(G_1))} \sum_{s_1, s_2 \in E(S_{k}(G_1))} [d(r_1, s_1) + d(r_2, s_2)]^2 \\
&\quad + \sum_{r, s \in E(S_{k}(G_1))} \sum_{s_1, s_2 \in E(S_{k}(G_1))} ([d(r_1) + d(r_2)] + d(s_1) + d(r_1)d(s_1) + d(r_2)d(s_2)]^2 \\
&= HZG_1F(G_2) + 2M_1(G_2)HZS_1(G_1) + 2M_1(G_2)M_1(S(G_1)) \\
&\quad + 2F(G_2)M_1(S(G_1)) + 16G_1, e_{G_1} + 4e_{G_1}F(G_2) + 2e_{G_1}M_1(G_1),
\end{align*}
$$

(6)

Hence, we obtained our required result.

**Theorem 2.** For $k \geq 1$, the HZ-index of $G_1 \otimes G_2$ is

$$
\begin{align*}
HZ(G_1 \otimes G_2) &= 8[n_G + 6e_G]F(G_1) + [n_G + 20e_G]F(G_2) + 8F(G_1)F(G_2) + 24e_GM_1(G_1) + 36e_GM_1(G_2) \\
&\quad + 24M_1(G_1)M_1(G_2) + 24F(G_1)M_1(G_2) + 8n_G e_G + 8(k-1) e_G [n_G + F(G_2) + 4e_G + 3M_1(G_2)] \\
&\quad + 48e_G e_G + 12F(G_2)M_1(G_1) + 2[M_2(G_2)]4e_G + n_G + k[n_G + 6e_G + 3M_1(G_2) + 2M_2(G_2)] \\
&\quad \times \left[ \frac{1}{2} \sum_{v \in V(G_1)} d^4_{G_1}(v) - d^3_{G_1}(v) + \sum_{v \in V(G_1)} rd_{G_1}(u)d_{G_1}(v) + \sum_{v \in V(G_1)} d^2_{G_1}(v) \sum_{u \in V(G_1)} d_{G_1}(u) - 2M_2(G_1) \right] \\
&\quad + M_1(G_1)[5e_2 + 5M_1(G_2) + 5M_2(G_2)] + k[M_3(G_1) + 2M_2(G_1)] [6e_G + 3M_1(G_2) + 2M_2(G_2) + n_G]
\end{align*}
$$

+ $2e_GM_1(G_2)$.
Proof. Let the degree of a vertex \((r, s) \in G_1 \otimes R_0 G_2\) be denoted by \(d(r, s)\):

\[
\text{HZ}(G_1 \otimes R_0 G_2) = \sum_{(r_1, s_1) \in E(G_1 \otimes R_0 G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]

\[
\sum_{s \in V(G_1)} \sum_{r \in V(R_0 G_2)} [d(r, s_1) + d(r, s_2)]^2 + \sum_{s \in V(R_0 G_2)} \sum_{r \in V(G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
+ \sum_{r, r \in V(R_0 G_2)} \sum_{s \in V(G_1)} [d(r_1, s) + d(r_2, s)]^2 = \sum_{A} + \sum_{B} + \sum_{C}
\]

\[
\sum_{A} = \sum_{s \in V(G_1)} \sum_{r, r \in V(R_0 G_2)} [d(r, s_1) + d(r, s_2)]^2
\]

\[
= \sum_{s \in V(G_1)} \sum_{r, r \in V(R_0 G_2)} [2 d(r) + d(s_1) + 2 d(r) d(s_1) + 2 d(r) + d(s_2) + 2 d(r) d(s_2)]^2
\]

\[
= \sum_{s \in V(G_1)} \sum_{r, r \in V(R_0 G_2)} [4 d(r) + d(s_1) + d(s_2) + 2 d(r) (d(s_1) + d(s_2))]^2
\]

\[
= \sum_{s \in V(G_1)} \sum_{r, r \in V(R_0 G_2)} \left[4d^2(r) + 1 + 4 d(r) \right] \left[4d^2(s_1) + d^2(s_2) + 2 d(s_1) d(s_2) \right] + \left(8 d(r) + 16d^2(r) \right)
\]

\[
x [d(s_1) + d(s_2)] + 16d^2(r)\]
\[
\begin{align*}
\sum_{C_3} = & \sum_{r, r' \in E (R_i (G_i)) \mid r \neq V (G_i), r' \neq V (G_i)} \sum_{s \in V (G_i)} \left[ d (r_1, s) + d (r_2, s) \right]^2 \\
& + \sum_{r, r' \in E (R_i (G_i)) \mid r \neq V (G_i), r' \neq V (G_i)} \sum_{s \in V (G_i)} \left[ d (r_1, s) + d (r_2, s) \right]^2 \\
= & \sum_{r, r' \in E (R_i (G_i)) \mid r \neq V (G_i), r' \neq V (G_i)} \sum_{s \in V (G_i)} \left[ (d (r_1) + d (s)) + d (r_2) \right] d (s) \\
& + 2 \left[ (d (r_1) + d (s)) + d (r_2) \right] ^2 + 2 \left[ d (r_1) + d (s) \right] d (s) + 2 \left[ d (r_2) \right] d (s) \\
= & 8 e_{G_i} F (G_i) + 2 e_{G_i} F (G_2) + 4 F (G_1) F (G_i) + 4 M_1 (G_i) M_1 (G_i) + 4 M_1 (G_1) F (G_i) + 8 M_1 (G_2) F (G_i) \\
& + 2 \left[ 8 M_2 (G_i) e_{G_i} + 2 M_1 (G_i) M_1 (G_i) + 2 M_2 (G_i) e_{G_i} + 8 M_2 (G_i) [ M_2 (G_i) + M_1 (G_i) ] + 4 M_1 (G_1) M_2 (G_i) \right],
\end{align*}
\]
\[\sum_{C_2} = \sum_{r \in V(G_1), t \in V(G_2)} \sum_{s \in V(G_1), t \in V(G_2)} [d(r, s_1) + d(r, s_2)]^2\]
\[= \sum_{r \in V(G_1), t \in V(G_2)} \sum_{s \in V(G_1), t \in V(G_2)} [d(r) + d(s) + d(r) d(s) + d(r_2) d(s_2)]^2\]
\[= \sum_{r \in V(G_1), t \in V(G_2)} \sum_{s \in V(G_1), t \in V(G_2)} [(d(r) + d(s) + d(r_1) d(s_1)]^2 + (d(r_2) + d(r_2) d(s_2))]^2\]
\[+ 2[(d(r) + d(s) + d(r) d(s))] (d(r_2) + d(r_2) d(s_2))]\]
\[= \sum_{r \in V(G_1), t \in V(G_2)} \sum_{s \in V(G_1), t \in V(G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2\]
\[= \sum_{r \in V(G_1), t \in V(G_2)} \sum_{s \in V(G_1), t \in V(G_2)} [d(r_1) + d(r_2) + d(r_1) d(s_1) + d(r_2) d(s_2)]^2\]
\[= [(2 + 2 d(s))]^2 (2 + 2 d(s)_2)]^2 + [2 + 2 d(s)] (2 + 2 d(s))\]
\[= 8 (k - 1) e_G [2 e_G + F(G) + 2 M_1 (G)] + 16 (k - 1) e_G [e_G + M_1 (G) + M_2 (G)].\]

Hence, we reached at our required result. \[\Box\]  

**Theorem 3.** For \(k \geq 1\), the HZ-index of \(G_1 \otimes G_2\) is

\[
HZ(G_1 \otimes G_2) = 2 (k - 1) [F(G_1) + 2 M_2 (G_1)] [3 n_G + 5 M_1 (G_1) + 14 e_G + F(G_2)]
+ k [n_G + 6 e_G + 3 M_1 (G_2) + F(G)]
\]

\[
\left[ M_4 (G_1) - 2 F(G_1) + 2 M_2 (G_1) - 4 M_2 (G_1) + \sum_{u \in V(G_1)} d^2 (u) \sum_{v \in N (u)} d (v) \right] + 6 e_G M_1 (G_1)
\]

\[
+ 10 e_G F(G_2) + 3 F(G_1) F(G_2) + 6 M_1 (G_1) M_1 (G_2) + F(G_2) [n_G + 3 M_1 (G_1) + 6 e_G + 4 M_2 (G)]
\]

\[
+ 2 M_2 (G_1) [n_G + 7 M_1 (G_2)] + 6 e_G M_2 (G_1) + 8 M_2 (G_1) [e_G + M_1 (G_2)] + 2 [k [n_G + 6 e_G + 3 M_1 (G_2)]]
\]

\[
+ 2 M_2 (G_1) \left[ \sum_{u \in V(G_1)} \frac{1}{2} d_G^2 (v) - d_G^3 (v) + \sum_{u \in V(G_1)} t d_G (u) d_G (v) + \sum_{v \in V(G_1)} d_G^2 (v) \sum_{u \in V(G_1)} d_G (u) \right]
\]

\[
- 2 M_2 (G_2) + M_2 (G_2) [4 e_G + n_G] + 2 e_G M_1 (G_2) + M_2 (G_1) [5 e_G + 5 M_1 (G_2) + 5 M_2 (G_2)]
\]

\[
+ k [M_1 (G_1) + 2 M_2 (G_1)] [6 e_G + 3 M_1 (G_2) + 2 M_2 (G_2) + n_G].\]
Proof. Let the degree of a vertex \((r, s) \in G_1 \otimes Q_2 G_2\) be denoted by \(d(r, s)\):

\[
HZ(G_1 \otimes Q_2 G_2) = \sum_{(r, s_1), (r, s_2) \in E(G_1 \otimes Q_2 G_2)} [d(r, s_1) + d(r, s_2)]^2
\]

\[
= \sum_{r \in V(G_1)} \sum_{s_1, s_2 \in E(G_1)} [d(r, s_1) + d(r, s_2)]^2 + \sum_{Q \in V(G_2)} \sum_{r_1, r_2 \in E(Q_2 G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
+ \sum_{r_1, r_2 \in E(Q_2 G_1)} \sum_{s_1, s_2 \in E(G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2 = A + B + C.
\]

\[
A = \sum_{r \in V(G_1)} \sum_{s_1, s_2 \in E(G_1)} [d(r, s_1) + d(r, s_2)]^2
\]

\[
B = 8e_1 M_1 (G_1) + n_1 HZ (G_2) + 4e_2 M_1 (G_1) + 4e_1 HZ (G_2) + M_1 (G_1) HZ (G_2) + 4 M_1 (G_1) M_1 (G_2)
\]

OR

\[
= 2 |E(H_2)| M_1 (H_2) + |V(H_2)| F(H_2) + M_1 (H_2) F(H_2)
\]

\[
+ 4 |E(H_1)| M_1 (H_2) + 2 M_1 (H_1) M_1 (H_2) + 4 |E(H_2)| F(H_2)
\]

\[
+ 2 \{ M_1 (G_1) e_{G_1} + M_1 (G_1) [M_1 (G_2) + M_2 (G_2)] + 2 e_{G_1} [M_1 (G_2) + M_2 (G_2)] + M_2 (G_2) n_{G_1} \},
\]

\(9\)

\[
B = \sum_{r \in V(G_1)} \sum_{s_1, s_2 \in E(Q_2 G_1)} [d(r, s_1) + d(r, s_2)]^2
\]

\[
+ \sum_{s \in V(G_2)} \sum_{r_1, r_2 \in E(Q_2 G_1)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
+ \sum_{r_1, r_2 \in E(Q_2 G_1)} \sum_{s_1, s_2 \in E(Q_2 G_2)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]

\[
\sum_{B_1} \sum_{r_1, r_2 \in E(Q_2 G_1) \cap E(Q_2 G_2)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
= \sum_{r_1, r_2 \in E(Q_2 G_1) \cap E(Q_2 G_2)} [d(r_1, d(s) + d(r_1) d(r) + d(r_2) d(s)]^2
\]

\[
+ 2 [d(r_1) d(s) + d(r_1) d(s)] [d(r_2) d(s) + d(r_2) d(s)]
\]

Consider \(r_1 \in V(G_1)\) and \(d^2 (r_1)\) occurs \(d(r_1)\) times. Thus, Let

\[
D_1 = \sum_{r_1 \in V(G_1) \cap E(Q_2 G_1)}, d^3 (r_1) = F(G_1).
\]

\[
D_2 = \sum_{r_1 \in V(G_1) \cap E(Q_2 G_2)}, d^2 (r_2),
\]

\(10\)

\(11\)
as \( s_2 = uveE(G_1) \) and \( d^2(s_2) \) occurs two times. Therefore,

\[
D_2 = 2 \sum_{s_2 = uveE(Q(G_i))} [d(u) + d(v)]^2 = 2 \sum_{uveE(G_i)} [d^2(u) + d^2(v) + 2d(u)d(v)] = 2[F(G_i) + 2M_2(G_i)],
\]

\[
\sum_{\mathbb{B}_2} = n_{G_2}F(G_1) + 2e_{G_2}M_1(G_2) + M_1(G_2)F(G_1) + 4e_{G_2}[M_1(G_1) + F(G_i)] + 2M_1(G_1)M_1(G_2)
\]

\[
+ 2n_{G_2}[F(G_1) + 2M_2(G_i)] + 2M_1(G_1)[F(G_1) + 2M_2(G_1)] + 8e_{G_2}[F(G_i) + 2M_2(G_1)]
\]

\[
+ 2[(M_3(G_1) + 2M_2(G_1))[n_{G_2} + 4e_{G_2} + M_1(G_2)] + 2M_1(G_1)[2e_{G_2} + M_1(G_2)]],
\]

\[
\sum_{\mathbb{B}_2} = \sum_{r_1, r_2 \in E(Q(G_i))} \sum_{\mathbb{X} \in V(G_i)} [d(r_1, s) + d(r_2, s)]^2.
\]

Now, assume \( \Sigma_{\mathbb{B}_2} = \Sigma_{\mathbb{B}_3} + \Sigma_{\mathbb{B}_4} \) as follows:

\[
\sum_{\mathbb{B}_3} = \sum_{r_1, r_2 \in E(Q(G_i))} \sum_{\mathbb{X} \in V(G_i)} [d(r_1) + d(r_2)d(s)]^2 + (d(r_2) + d(r_2)d(s)]^2 + 2(d(r_1) + d(r_2)d(s)]^2
\]

\[
= 2(k - 1)\left[F(G_1) + 2M_2(G_i)\right]\left[n_{G_2} + M_1(G_2) + 4e_{G_2}\right] + (M_3(G_1) + 2M_2(G_1))\left[n_{G_2} + 4e_{G_2} + M_1(G_2)\right],
\]

\[
\sum_{\mathbb{B}_4} = \sum_{r_1, r_2 \in E(Q(G_i))} \sum_{\mathbb{X} \in V(G_i)} [d(r_1, s) + d(r_2, s)]^2
\]

\[
= \sum_{r_1, r_2 \in E(Q(G_i))} \sum_{\mathbb{X} \in V(G_i)} [d(r_1) + d(r_2)d(s) + d(r_2) + d(r_2)d(s)]^2
\]

\[
= \sum_{r_1, r_2 \in E(Q(G_i))} \sum_{\mathbb{X} \in V(G_i)} [d(r_1)^2 + d(r_2)^2 + d(s)^2(d(r_1)^2 + d(r_2)^2) + 2d(s)(d(r_1)^2 + d(r_2)^2)]
\]

\[
+ 2[(d(r_1) + d(r_2)d(s))(d(r_2) + d(r_2)d(s))],
\]

\[
D_3 = \sum_{r_1, r_2 \in E(Q(G_i))} \sum_{\mathbb{X} \in V(G_i)} [d^2(r_1) + d^2(r_2)].
\]

In \( D_3 \), coefficient of

\[
d^2(u) = 2\left(\frac{2}{d_{G_1}(u)}\right) + \sum_{v \in N(u)} d(v) = d^2(u) - 2d(u) + \sum_{v \in N(u)} d(v).
\]

Therefore,

\[
\sum_{u \in V(G_i)} d^2(u) = M_4(G_1) - 2F(G_1) + \sum_{u \in V(G_i)} d^2(u) \sum_{v \in N(u)} d(v).
\]
For coefficient of \( d(u)d(v) \), let \( r_1, r_2 \in E(Q(G_1)) \) with \( r_1 = \nu \) and \( r_2 = \omega z \). As \( r_1, r_2 \in E(Q(G_1)) \), we have either \( v = \omega \) or \( z = \nu = \omega \). So, \( \nu \) is adjacent to all those vertices in \( G_1 \) which are adjacent to \( u \) and \( v \). Consequently, the number of such \( d(u)d(v) \) is \( (d(u) + d(v) - 2) \). Therefore,

\[
2 \sum_{uv \in E(G_1)} d(u)d(v) = 2 \sum_{uv \in E(G_1)} (d(u) + d(v) - 2)du dv = 2 \sum_{uv \in E(G_1)} (d(u) + d(v))d(u)d(v) - 4 \sum_{uv \in E(G_1)} d(u)d(v) = 2M_2(G_1) - 4M_2(G_1),
\]

so

\[
D_3 = M_4(G_1) - 2F(G_1) + \sum_{u \in V(G_1)} d^2(u) \sum_{v \in N(u)} d(v) + 2M_2(G_1) - 4M_2(G_1),
\]

\[
\sum_{\pi_4} = (k) \left[ n_{G_1} + 4e_{G_2} + M_4(G_2) \right] \left[ M_4(G_1) - 2F(G_1) + 2M_2(G_1) - 4M_2(G_1) + \sum_{v \in V(G_1)} d^2(u) \sum_{v \in N(u)} d(v) \right]
\]

\[
+ 2 \left[ (k) \left[ n_{G_2} + 4e_{G_2} + M_1(G_2) \right] \left[ \frac{1}{2} \sum_{v \in V(G_1)} \left( d^4_{G_1}(v) - d^3_{G_1}(v) \right) + \sum_{u \in V(G_1)} td_{G_1}(u)d_{G_1}(v) \right] \right]
\]

\[
+ \sum_{v \in V(G_1)} d^2_{G_1}(v) \sum_{u \in E(G_1)} d_{G_1}(u) - 2M_2(G_1) \right] \right],
\]

where \( t \) is the number of neighbors which are common vertices of \( u \) and \( v \) in \( G_1 \).
\[
+ 2[d(r_1, s_1)d(r_2, s_2)]
\]
\[
= 6\left[e_{G_1} + M_1(G_2)\right]F(G_1) + 3F(G_1)F(G_2) + 2M_1(G_1)M_1(G_2) + 2\left[e_{G_1} + M_1(G_1) + 2M_2(G_1)\right]F(G_2) + 8M_2(G_1) (18)
\]
\[
\times \left[e_{G_2} + M_1(G_2)\right] + 2\left[\left[M_3(G_1) + 2M_2(G_1)\right]\left[2e_{G_1} + 2M_1(G_2) + 2M_2(G_2)\right] + 2M_1(G_1) \left[2M_2(G_2) + M_1(G_2)\right]\right].
\]

Now, assume \( \Sigma_{C_2} = \Sigma_{C_3} + \Sigma_{C_4} \) as follows:

\[
\sum_{C_3} = \sum_{s_1s_2 \in V(G_1)} \sum_{r_1r_2 \in E(Q_k(G_1))} \sum_{r \in V(G_1)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]
\[
= \sum_{s_1s_2 \in V(G_1)} \sum_{r_1r_2 \in E(Q_k(G_1))} \sum_{r \in V(G_1)} [d(r_1) + d(r_2)d(s_1) + d(r_2) + d(r_2)d(s_2)]^2
\]
\[
= \sum_{s_1s_2 \in V(G_1)} \sum_{r_1r_2 \in E(Q_k(G_1))} \sum_{r \in V(G_1)} [(d(s_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2]
\]
\[
+ 2[(d(r_1) + d(r_1)d(s))(d(r_2) + d(r_2)d(s))]
\]
\[
= 2(k - 1)\left[(F(G_1) + 2M_2(G_1))\right] \left[2e_{G_1} + F(G_2) + 2M_1(G_2)\right] + 2(k - 1)\left[2e_{G_1} + 2M_1(G_2) + 2M_2(G_2)\right] \left[M_3(G_1) + 2M_2(G_1)\right].
\]

\[
\sum_{C_4} = \sum_{s_1s_2 \in V(G_1)} \sum_{r_1r_2 \in E(Q_k(G_1))} \sum_{r \in V(G_1)} [d(r_1, s_1) + d(r_2, s_2)]^2
\]
\[
= \sum_{s_1s_2 \in V(G_1)} \sum_{r_1r_2 \in E(Q_k(G_1))} \sum_{r \in V(G_1)} [d(r_1) + d(r_2) + d(r_1)d(s_1) + d(r_2)d(s_2)]^2
\]
\[
= \sum_{s_1s_2 \in V(G_1)} \sum_{r_1r_2 \in E(Q_k(G_1))} \sum_{r \in V(G_1)} [(d(r_1) + d(r_1)d(s_1))^2 + (d(r_2) + d(r_2)d(s_2))^2]
\]
\[
+ 2[(d(r_1) + d(r_1)d(s))(d(r_2) + d(r_2)d(s))]
\]
\[
= (k)\left[2e_{G_2} + F(G_2) + 2M_1(G_2)\right] \left[M_4(G_1) - 2F(G_1) + 2M_2(G_1) - 4M_2(G_1) + \sum_{u \in V(G_1)} d^2(u) \sum_{v \in N(u)} d(v)
\]
\[
+ (2k)\left[2e_{G_1} + 2M_1(G_2) + 2M_2(G_2)\right] \left[\frac{1}{2} \sum_{u \in V(G_1)} d^2_{G_1}(u) - d^2_{G_1}(v) + \sum_{u \in V(G_1)} [td_{G_1}(u) + d_{G_1}(v)]
\]
\[
+ \sum_{u \in V(G_1)} d^2_{G_1}(v) \sum_{u \in V(G_1)} d_{G_1}(u) - 2M_2(G_1)\right].
\]

(19)
where \( t \) is the number of neighbors which are common vertices of \( u \) and \( v \) in \((G_1)\).

Thus, we arrive at our desired result.

Theorem 4. For \( k \geq 1 \), the HZ-index of \( G_1 \bowtie_k G_2 \) is

\[
\text{HZ}(G_1 \bowtie_k G_2) = 2(k-1)[F(G_1) + 2M_2(G_1)][n_{G_2} + 3M_1(G_2) + 6e_{G_2} + F(G_2)] + k[n_{G_2} + 3M_1(G_2) + 6e_{G_2} + F(G_2)] + 2M_1(G_2)
\]

\[
+ 6e_{G_2} + 3M_1(G_2) + F(G_2)] [M_4(G_1) - 2F(G_1) + 2M_2(G_1) - 4M_2(G_1)]
\]

\[
+ \sum_{u \in V(G_1)} d^2(u) \sum_{v \in N(u)} dv + [F(G_1) + 2M_2(G_1)] [2n_{G_2} + 6M_1(G_2) + 12e_{G_2} + 2F(G_2)]
\]

\[
+ 4F(G_1)[2n_{G_2} + 6M_1(G_2) + 12e_{G_2} + 2F(G_2)] + F(G_2)[n_{G_2} + 12M_1(G_1)]
\]

\[
+ 12e_{G_2} + 12e_{G_2}M_1(G_2) + 16e_{G_2}M_1(G_1) + 20M_1(G_1)M_1(G_2)]
\]

\[
+ \sum_{v \in V(G_1)} d^2_G(v) \sum_{u \in V(G_1)} d_G(u) - 2M_2(G_1)
\]

\[
+ M_2(G_1)[4n_{G_2} + 24e_{G_2} + 8M_2(G_2) + 12M_1(G_2)]
\]

\[
+ k[M_3(G_1) + 2M_3(G_1)][6e_{G_2} + n_{G_2} + 3M_1(G_2) + 2M_2(G_2)] + 5M_1(G_2)e_{G_2}
\]

\[
+ M_2(G_2)[10e_{G_2} + n_{G_2}] + M_1(G_1)[10e_{G_2} + 11M_1(G_2) + 10M_2(G_2)].
\]

4. Applications and Discussion

(i) \( S_1 \)-sum:

Using \( k = 1 \), in Theorems 1–4, the results are obtained for the generalized \( D_1 \)-sum graphs as follows:

\[
\text{HZ}(G_1 \bowtie_1 G_2) = [n_{G_2} + 3M_1(G_2) + 6e_{G_2}] F(G_1) + [n_{G_2} + 3M_1(G_2) + 14e_{G_2}] F(G_2) + 6e_{G_2}M_1(G_1) + 30e_{G_2}M_1(G_2)
\]

\[
+ 6M_1(G_1)M_1(G_2) + 48e_{G_2}e_{G_2} + F(G_1)F(G_2) + 8n_{G_2}e_{G_2} + 2[4M_2(G_1) + 4e_{G_2}] [5e_{G_2} + 3M_1(G_2)]
\]

\[
+ 2M_2(G_2) + n_{G_2}] + 14e_{G_2}M_1(G_2) + M_2(G_2) [12e_{G_2} + n_{G_2}] + M_1(G_1) [e_{G_2} + M_1(G_2) + M_2(G_2)]
\]

\[
+ 8e_{G_2} e_{G_2}].
\]
Table 1: Hyper-Zagreb index of $F_1$-sum path graphs.

| $[n_1, n_2]$ | $\text{HZ}(P_{n_1} \circledast G_{n_1}, P_{n_2})$ | $\text{HZ}(P_{n_1} \circledast R_{n_1}, P_{n_2})$ | $\text{HZ}(P_{n_1} \circledast Q_{n_1}, P_{n_2})$ | $\text{HZ}(P_{n_1} \circledast F_{n_1}, P_{n_2})$ |
|-----------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| (3, 3)    | 4144                        | 11870                       | 8834                        | 16168                       |
| (4, 4)    | 11040                       | 30260                       | 24500                       | 47180                       |
| (5, 5)    | 21232                       | 62122                       | 46516                       | 89230                       |
| (6, 6)    | 33696                       | 99344                       | 75416                       | 145204                      |
| (7, 7)    | 51384                       | 147638                      | 110384                      | 212318                      |

(ii) $R_1$-sum:

$$\text{HZ}(G_1 \circledast R_{G_2}) = 8 \left[ n_{G_1} + 6e_{G_2} \right] F(G_1) + \left[ n_{G_1} + 20e_{G_1} \right] F(G_2) + 8F(G_1)F(G_2) + 24e_{G_1}M_1(G_1) + 36e_{G_2}M_1(G_2) + 24M_1(G_1)M_1(G_2) + 24F(G_1)M_1(G_2) + 8n_{G_1}e_{G_1} + 48e_{G_1}e_{G_2} + 12F(G_2)M_1(G_1) + 2M_1(G_1) \left[ 4e_{G_1} + 6M_1(G_2) + 2M_2(G_2) \right] + 4e_{G_1} \left[ 3M_1(G_2) + 3M_2(G_2) + 4e_{G_2} \right] + M_2(R_1(G_1)) - 4M_2G_1 \left[ 6e_{G_1} + 3M_1(G_2) + 2M_2(G_2) + n_{G_1} \right] + e_{G_1}M_1(G_2) + 4M_2(G_1) \left[ n_{G_2} + 6e_{G_2} + 3M_1(G_2) \right] + M_2(G_2) \left[ n_{G_1} + 2e_{G_1} \right].$$

(iii) $Q_1$-sum:

$$\text{HZ}(G_1 \circledast Q_{G_2}) = 2 \left[ n_{G_2} + 6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) \right] \left[ \frac{1}{2} \sum_{v \in V(G_2)} \left( d^4_{G_1}(v) - d^2_{G_1}(v) \right) + \sum_{u \in E(G_1)} td_{G_1}(u)d_{G_1}(v) \right]$$

$$+ \sum_{v \in V(G_1)} \sum_{u \in E(G_1)} d_{G_1}(v)d_{G_1}(u) - 2M_2(G_1)$$

$$+ M_2(G_2) \left[ 4e_{G_1} + n_{G_1} \right] + 2e_{G_1}M_1(G_2) + M_1(G_1)$$

$$\times \left[ 5e_2 + 5M_1(G_2) + 5M_2(G_2) \right] + M_3(G_1) + 2M_2(G_1) \left[ 6e_{G_1} + 3M_1(G_2) + 2M_2(G_2) + n_{G_1} \right]$$

$$+ \left[ n_{G_2} + e_{G_2} + 3M_1(G_2) + F(G_2) \right] \left[ M_4(G_1) - 2F(G_1) + 2M_2(G_1) - 4M_2(G_1) + \sum_{u \in V(G_1)} d^2(u) \right]$$

$$\times \sum_{v \in N(u)} d(v)$$

$$+ 6e_{G_2}M_1(G_1) + 10e_{G_2}F(G_2) + 3F(G_1)F(G_2) + 6M_1(G_1)M_1(G_2) + F(G_2) \left[ n_{G_1} + 3M_1(G_1) + 6e_{G_2} + 4M_2(G_1) \right] + F(G_1) \left[ n_{G_2} + 7M_1(G_2) \right] + 6e_{G_1}M_2(G_1) + 8M_2(G_1) \left[ e_{G_1} + M_1(G_2) \right].$$
(iv) $T_1$-sum:

\[
\text{HZ}(G_1 \boxtimes T_1 \ G_2) = \left[ n_{G_1} + 6e_{G_2} + 3M_1(G_2) + F(G_2) \right] \left[ M_4(G_1) - 2F(G_1) + 2M_2(G_1) - 4M_2(G_1) + \sum_{u \in V(G_1)} d^2(u) \right] \\
\times \sum_{v \in N(u)} d(v) + \left[ F(G_1) + 2M_2(G_1) \right] \left[ 2n(G_2) + 6M_1(G_2) + 12e_{G_2} + 2F(G_2) \right] + 4F(G_1) \left[ 2n_{G_2} + 6M_1(G_2) + 12e_{G_2} + 2F(G_2) \right] + F(G_2) \left[ n_{G_1} + 12M_1(G_1) + 12e_{G_1} \right] + 12e_{G_1}M_1(G_2) + 16e_{G_1}M_1(G_1) \\
+ 20M_1(G_1)M_1(G_2) + 2 \left[ n_{G_2} + 6e_{G_1} + 3M_1(G_2) + 2M_2(G_2) \right] \left[ \frac{1}{2} \sum_{v \in V(G_1)} \left( d_{G_1}^1(v) - d_{G_1}^1(v) \right) \right] \\
+ \sum_{u \in V(G_1)} t d_{G_1}(u) d_{G_1}(v) + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{u \in V(G_1)} d_{G_1}(u) - 2M_2(G_1) + \sum_{v \in V(G_1)} d_{G_1}(u) \sum_{u \in V(G_1)} d_{G_1}(v) - 2M_2(G_1) \right] + \left[ M_3(G_1) + 2M_2(G_1) \right] \\
\times \left[ 6e_{G_2} + n_{G_2} + 3M_1(G_2) + 2M_2(G_2) \right] + 5M_1(G_2)e_{G_1} + M_1(G_1) \left[ 10e_{G_1} + 11M_1(G_2) + 10M_2(G_2) \right] \\
+ M_2(G_2) \left[ 10e_{G_1} + n_{G_1} \right] + M_2(G_1) \left[ 4n_{G_1} + 24e_{G_2} + 8M_2(G_2) + 12M_1(G_2) \right].
\]

Figure 4: Graphical representation of $\text{HZ}(P_n \boxtimes \Pi_m)$, $\text{HZ}(P_n \boxtimes \Pi_m)$, $\text{HZ}(P_n \boxtimes \Theta_m)$, and $\text{HZ}(P_n \boxtimes \Delta_m)$ in red, green, orange, and purple colour, respectively.
Now, we present tabular form in Table 1 and graphical representation in Figure 4 of path graphs for \( k = 1 \).

Finally, we close this section with the comment that the problem is still open for other topological indices and product of graphs, in particular the general randic index of \( F_k \)-sum graphs under corona product.

**Data Availability**

The data are included within this paper and are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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