Towards a physical expansion in perturbative gauge theories by using improved Baker-Gammel approximants

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Abstract

Applicability of the previously introduced method of modified diagonal Baker-Gammel approximants is extended to truncated perturbation series (TPS) of any order in gauge theories. The approximants reproduce the TPS when expanded in power series of the gauge coupling parameter to the order of that TPS. The approximants have the favorable property of being exactly invariant under the change of the renormalization scale, and that property is arrived at by a generalization of the method of the diagonal Padé approximants. The renormalization scheme dependence is subsequently eliminated by a variant of the method of the principle of minimal sensitivity (PMS). This is done by choosing the values of the renormalization-scheme-dependent coefficients ($\beta_2, \beta_3, \ldots$), which appear in the beta function of the gauge coupling parameter, in such a way that the diagonal Baker-Gammel approximants have zero values of partial derivatives with respect to these coefficients. The resulting approximants are then independent of the renormalization scale and of the renormalization scheme.

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I. INTRODUCTION

Ordinary perturbation theory – despite its intuitive physical content as illustrated by the individual Feynman diagrams – is plagued by several unpleasant features which obstruct
a strictly physical interpretation of various terms. The main reason for this lies in the
dependence on unphysical structures like renormalization scale (RScl) and renormalization
scheme (RSch). This dependence represents a certain amount of arbitrariness common to any
finite order expression. Considerable effort had been directed towards finding a pragmatic
solution to a corresponding problem of defining an appropriate RScl and RSch, respectively,
for a given finite order expression.

Recently, a different method, involving modified diagonal Baker-Gammel approximants
dBGAs), has been developed [1] for dealing with truncated perturbation series (TPS’s) in
gauge theories. These approximants reproduce a TPS to which they were applied, when
expanded to the order of that TPS. In addition, these modified dBGAs were shown to be
invariant under the change of the renormalization scale (RScl) $q^2$, i.e., when the evolution
of the coupling parameter $a(q^2)$ in the TPS is determined by the full $\beta$-function (to any
chosen loop-order). These dBGAs represent an improvement of the related method of
diagonal Padé approximants (dPA’s). The latter are RScl-invariant when $a(q^2)$ is taken to
evolve according to the one-loop beta function (large-$|\beta_0|$ limit) [2]. However, two remaining
deficiencies of the dPA method have not been eliminated in this way. Firstly, the method
appeared to be applicable only to the TPS’s of the observables $S$ with an odd number of
nonleading terms. Secondly, the approximants remained dependent on the renormalization-
scheme (RSch), i.e., on the values of the RSch-dependent coefficients $\beta_j$ ($j \geq 2$) appearing
in the beta function of the gauge coupling parameter.

In this paper, we show that the first deficiency can be cured easily, and that the method
of modified dBGAs’s (and also the method of dPA’s) can be applied, in a somewhat modified
form, also to the TPS’s with an even number of nonleading terms. This is of high practical
importance since nowadays – while perturbation series of several QED observables (e.g.,
anomalous magnetic moment of $e$ and $\mu$) are available up to the third nonleading order –
all interesting QCD quantities are only calculated to at most second nonleading order, with
next order corrections not being expected for the near future. We further show how to apply
the method when the leading term in the TPS of an observable is proportional to $a^\ell(q^2)$
with $\ell > 1$.

Also the second deficiency can be at least partially cured, and we can eliminate the
RSch-dependence by applying a variant of the principle of minimal sensitivity (PMS) to
the obtained dBGAs’s. The method consists in finding such (RSch-dependent) coefficients
$\beta_j$ ($j = 2, 3, \ldots$) for which the partial derivatives of the dBGAs acquire the value zero. The
obtained approximants then possess RScl-invariance that was arrived at by a generalization
of the dPA method, and possess RSch-invariance that was arrived at by a variant of the
PMS approach. In this context, we mention that the original version of the PMS [3] was
applied directly to the TPS’s. Since the method of the (d)PA’s has proven to be remarkably
efficient for QCD observables [4], we expect that the method presented here allows some
room for optimism as to its efficiency when compared with the usual PMS, and with the

\[1\] This means that the approximants are the same, whatever RSch-parameters $\beta_j$ ($j \geq 2$) are used
for the TPS under consideration.
method of effective charges (ECH)\footnote{The usual PMS and the ECH methods consist in defining in a pragmatic way an appropriate RScl and RSch for a given finite order expression (TPS).} that is related to the usual PMS.\footnote{We omit superscript \((\ell)\) when \(\ell = 1\).}

\section*{II. EXTENDING THE METHOD TO ANY TRUNCATED SERIES}

An observable \(S\) in a gauge theory can in general be redefined so that it has the following form as a formal perturbation series:

\[ S^{(\ell)} = a^{(\ell)}(q^2)f(q^2) = a^{(\ell)}(q^2) \left[ 1 + r_1(q^2)a(q^2) + \cdots + r_n(q^2)a^n(q^2) + \cdots \right] \quad (\ell \geq 1) , \quad (1) \]

where \(q^2\) is the chosen renormalization scale (RScl). The observable is, of course, independent of the RScl \(q^2\). The series is known in practice only up to an \(n\)’th order in nonleading terms

\[ S_n^{(\ell)}(q^2) \equiv a^{(\ell)}(q^2)f^{(n)}(q^2) = a^{(\ell)}(q^2) \left[ 1 + r_1(q^2)a(q^2) + r_2(q^2)a^2(q^2) + \cdots + r_n(q^2)a^n(q^2) \right] . \quad (2) \]

This TPS explicitly depends on the RScl \(q^2\), due to truncation. In most practical cases we have \(\ell = 1\), and at present we have at most \(n = 3\) in QED and \(n = 2\) in QCD. In our previous work\footnote{In our previous work we constructed modified dBGA’s with kernel \(a(p^2)/a(q^2)\) for TPS (2) when \(\ell = 1\) and \(n = 2M - 1\) \((M = 1, 2, \ldots)\). The latter condition \((n \text{ odd})\) originated from the need to employ diagonal Padé approximants (dPA’s) in the algorithm, since only such PA’s are RScl-independent in the large-\(|\beta_0|\) limit.} we constructed modified dBGA’s with kernel \(a(p^2)/a(q^2)\) for TPS (2) when \(\ell = 1\) and \(n = 2M - 1\) \((M = 1, 2, \ldots)\). The latter condition \((n \text{ odd})\) originated from the need to employ diagonal Padé approximants (dPA’s) in the algorithm, since only such PA’s are RScl-independent in the large-\(|\beta_0|\) limit.

Now we show how to modify the mentioned method so that it is applicable also to the cases of \(n = 2M - 2\) \((M = 2, 3, \ldots)\) and \(\ell = 1\). For this purpose we consider instead of \(S\) the observable \(\tilde{S}\) its square, i.e., we construct the observable

\[ \tilde{S} \equiv S \ast S \equiv a(q^2)F(q^2) = a(q^2) \left[ 0 + 1 \cdot a(q^2) + R_2(q^2)a^2(q^2) + R_3(q^2)a^3(q^2) + \cdots \right] , \quad (3) \]

where coefficients \(R_j(q^2)\) are in general again RScl-dependent and are related to the expansion coefficients \(r_i(q^2)\)’s in the following way:

\[ R_1 = 1 , \quad R_2 = 2r_1 , \quad R_3 = 2r_2 + r_1^2 , \quad \ldots \]

\[ R_{2M-1} = 2r_{2M-2} + 2r_{2M-3}r_1 + 2r_{2M-4}r_2 + \cdots + 2r_M r_{M-2} + r_{M-1}^2 , \quad \ldots \quad (4) \]

When the observable \(S\) is available up to an even nonleading order, say \(S_{2M-2}\) is available \([i.e., r_1(q^2), \ldots, r_{2M-2}(q^2) \text{ are known}]\), then according to \((4)\) all coefficients of \(\tilde{S}\) up to \(R_{2M-1}(q^2)\) are known and consequently we know \(\tilde{S}_{2M-1}(q^2)\)

\[ \tilde{S}_{2M-1}(q^2) = a(q^2)F^{(2M-1)}(q^2) \]

\[ = a(q^2) \left[ 0 + 1 \cdot a(q^2) + R_2(q^2)a^2(q^2) + R_3(q^2)a^3(q^2) + \cdots + R_{2M-1}(q^2)a^{2M-1}(q^2) \right] . \quad (5) \]

This TPS has formally \(\ell = 1\) and the number of nonleading terms is formally odd \((2M - 1)\). Therefore, we now just repeat the procedure of constructing dBGA’s described in \((4)\) for the
TPS's with odd number of nonleading terms – the only difference being now that the formal leading term is zero instead of one, and $R_1(q^2) \equiv 1$ is RScl-independent. First we recall that the gauge coupling parameter $a(p^2) \equiv \alpha(p^2)/\pi$ evolves according to the perturbative renormalization group equation (RGE)

$$\frac{da(p^2)}{d \ln p^2} = - \sum_{j=0}^{\infty} \beta_j a^{j+2}(p^2) \ ,$$

where $\beta_j$ are constants if particle threshold effects are ignored. We now introduce for two different scales $p^2$ and $q^2$ the ratio

$$k(a_q, u) \equiv \frac{a(p^2)}{a(q^2)} \quad \text{where:} \quad a_q = a(q^2) \ , \ u = \ln(p^2/q^2).$$

Formal expansion of this function in powers of $u$ contains coefficients $k_j \sim a^j(q^2)$

$$k(a_q, u) = 1 + \sum_{j=1}^{\infty} u^j k_j(a_q) \ , \quad \text{with:} \quad k_j(a_q) = \frac{1}{j!} \frac{\partial^j}{\partial u^j} k(a_q, u) \bigg|_{u=0} = \frac{1}{j! a(q^2)} \frac{d^j a(p^2)}{d \ln p^2} \bigg|_{p^2=q^2}. \quad (8)$$

These coefficients can be determined from RGE (3) as a series in powers of $a(q^2)$

$$k_j(a_q) = (-1)^j \beta_j^0 a^j(q^2) + \mathcal{O}(a^{j+1}(q^2)) \ , \quad k_0(a_q) = 1 \ , \quad (9)$$

We rearrange the series for $\tilde{S}/a(q^2) \equiv F(q^2)$ of (3) into the series in powers of $k_j(a(q^2))$

$$\tilde{S} \equiv S \ast S \equiv a(q^2)F(q^2) = a(q^2) \left[ 0 + \sum_{j=1}^{\infty} F_j(q^2) k_j(a(q^2)) \right] \quad , \quad (10)$$

and define the corresponding formal series obtained from the above by replacing the coefficients $k_j(a(q^2))$ by their large-|$\beta_0$| limits ($-\beta_0 a(q^2)$) [cf. (3)]

$$a(q^2)F(q^2) \equiv a(q^2) \left[ 0 + \sum_{j=1}^{\infty} F_j(q^2)(-1)^j \beta_0^j a^j(q^2) \right] \ . \quad (11)$$

Now we construct for this expression the diagonal Padé approximant (dPA) of order $2M$, with argument $a(q^2)$

$$a(q^2)[M-1/M]_F(q^2) = a(q^2) \left[ 0 + \sum_{m=1}^{M-1} \tilde{a}_m(q^2)a^m(q^2) \right] \left[ 1 + \sum_{n=1}^{M} \tilde{b}_n(q^2)a^n(q^2) \right]^{-1} \ . \quad (12)$$

By construction, this dPA satisfies the relation

$$a(q^2)F(q^2) = a(q^2)[M-1/M]_F(q^2) + \mathcal{O}(a^{2M+1}(q^2)) \ . \quad (13)$$

\footnote{Note that this dPA, by construction, has a polynomial of order $M$ [in $a(q^2)$] in the nominator and a polynomial of the same order in the denominator.}
The above dPA depends only on the first \((2M-1)\) coefficients \(F_j(q^2)\) \((j=1,\ldots,2M-1)\), as follows from the standard PA relation (13). The latter coefficients are uniquely determined by the coefficients \(R_1(q^2) = 1,\ldots,R_{2M-1}(q^2)\) of the series (12) via relations (11), and these coefficients are determined uniquely by the initial TPS \(S_{2M-2}^{(1)}(q^2) \equiv S_{2M-2}\) of (12), i.e., by the first \(2M-2\) nonleading terms of the observable \(S^{(1)} \equiv S \left[ r_1(q^2),\ldots,r_{2M-2}(q^2) \right]\) as seen from relations (11). Therefore, knowing \(S_{2M-2}\) \((M\geq 2)\), we can uniquely construct the above dPA (12)-(13).

If we don’t have an exceptional situation when the denominator in the dPA (12) has multiple zeros, we can uniquely decompose this dPA into a sum of simple fractions

\[
a(q^2)[M-1/M]_F(q^2) = a(q^2) \sum_{i=1}^{M} \frac{\tilde{\alpha}_i}{[1 + \tilde{u}_i(q^2)\beta_0 a(q^2)]} = \sum_{i=1}^{M} \tilde{\alpha}_i \frac{a(q^2)}{[1 + \tilde{u}_i(q^2)\beta_0 a(q^2)]}. \tag{14}
\]

Here, \([-\tilde{u}_i(q^2)\beta_0]^{-1}\) are the \(M\) zeros of the denominator of the dPA (12). We note that the above sum is a weighed sum of one-loop-evolved gauge coupling parameters \(a_0(q^2)\), where the generally complex scales \(p_i^2\) are determined by relation \(\tilde{u}_i(q^2) = \ln(p_i^2/q^2)\), and the one-loop evolution (i.e., in the large-\(|\beta_0|\) limit) is performed from the RSc \(q^2\) to \(p_i^2\). The dBGA approximant is then obtained by replacing in the above sum the one-loop-evolved gauge coupling parameters by those which are evolved according to the full RGE (16) from the RSc \(q^2\) to \(p_i^2\)

\[
a(q^2)G_F^{[M-1/M]}(q^2) \equiv \sum_{i=1}^{M} \tilde{\alpha}_i a(p_i^2), \quad \text{where:} \quad p_i^2 = q^2 \exp[\tilde{u}_i(q^2)]. \tag{15}
\]

We emphasize that this dBGA is an approximant for the squared observable \(\tilde{S} \equiv S \ast S\). As in Refs. [1], we can show explicitly that the above approximant fulfills two properties:

1. It has the same formal accuracy as the TPS \(\tilde{S}_{2M-1}(q^2)\) of (12):

\[
\tilde{S} = a(q^2)G_F^{[M-1/M]}(q^2) + \mathcal{O}\left(a^{2M+1}(q^2)\right). \tag{16}
\]

2. It is invariant under the change of the renormalization scale (RSc) \(q^2\), where the evolution from one to another RSc is performed according to RGE (16) with any chosen loop precision. Incidentally, the weights \(\tilde{\alpha}_i\) and the scales \(p_i^2 = q^2 \exp[\tilde{u}_i(q^2)]\) are separately independent of the chosen RSc \(q^2\).

The proof of these two statements can be taken over word by word from Ref. [1] (first entry) where the approximated TPS was \(S_{2M-1}\) and not \(\tilde{S}_{2M-1}\). The only formal difference in the dBGA (15) is that the sum of the \(\tilde{\alpha}_i\) parameters is now zero and not 1. This is due to the

\[
\text{[5]} \quad \text{We should keep in mind, however, that only a limited number of the perturbative coefficients } \beta_j \left(\beta_0,\ldots,\beta_3\right), \text{ appearing on the right of RGE (16) have been calculated and are consequently known in QCD (cf. [3], in } \overline{\text{MS}} \text{ scheme) and in QED (cf. [5], in } \overline{\text{MS}}, \text{ MOM and in on-shell schemes). Hence, in practice, RGE (16) has to be truncated at the four-loop level.}
\]

5 We should keep in mind, however, that only a limited number of the perturbative coefficients \(\beta_j\) \((\beta_0,\ldots,\beta_3)\), appearing on the right of RGE (16) have been calculated and are consequently known in QCD (cf. (3), in \(\overline{\text{MS}}\) scheme) and in QED (cf. (5), in \(\overline{\text{MS}},\text{ MOM and in on-shell schemes). Hence, in practice, RGE (16) has to be truncated at the four-loop level.}
fact that the series for \( \bar{S} \), in the form written in (3), has formally the leading term \([\propto a(q^2)]\)
equal to zero, while this term in \( S \) in Ref. [1] has coefficient 1.

The last step is to take simply the square root of this expression, and this gives us an effective modified dBGA approximant to the TPS \( S_{2M-2}^{(1)} \equiv S_{2M-2} \), up to and including terms \(~a^{2M-1}\), and this approximant is, of course, again RScl-invariant

\[
S = \left[ a(q^2)G_F^{M-1/M}(q^2) \right]^{1/2} + O\left( a^{2M}(q^2) \right).
\] (17)

We mention that the parameters \( \tilde{a}_i \) and \( \tilde{u}_i \) can be in general nonreal (complex). Then the modified dBGA (14) could possibly be nonreal, too. However, in this case we would just take under the square root in (17) the real part of the dBGA (15) – it is also RScl-invariant, it also satisfies relation (16) because observable \( \bar{S} \) is real, and it is positive because \( \bar{S} \equiv S \times S \) is positive. Below we will see, however, that in the practically interesting case of \( n \equiv 2M-2 = 2 \) \((M = 2)\) the approximant under the square root in (17) is always real (and positive).

Let us illustrate this somewhat abstract deliberations in the specific case of \( M = 2 \) \((S_{2M-2} = S_2)\) which is at present of particular interest for several QCD observables. The starting point is then the knowledge of the two coefficients \( r_1(q^2) \) and \( r_2(q^2) \) of the TPS \( S_2^{(1)} \equiv S_2 \) of Eq. (3), at a given RScl \( q^2 \) and in a specific renormalization scheme. We then also know the coefficients \( R_1 = 1 \), \( R_2(q^2) \) and \( R_3(q^2) \) of the TPS (3) via relations (4), i.e., we know \( \bar{S}_3 \) there. Explicit form of relations (3), obtained from RGE (3), yields

\[
k_1(a) = -\beta_0 a - \beta_1 a^2 - \beta_2 a^3 - \ldots ,
\]

\[
k_2(a) = +\beta_0 a^2 + (5/2)\beta_0 \beta_1 a^3 + \ldots , \quad k_3(a) = -\beta_0^3 a^3 - \ldots .
\] (18) (19)

We use the short-hand notation \( a \equiv a(q^2) \). Inverting these relations gives us expressions for \( a, a^2 \) and \( a^3 \) in terms of \( k_1,k_2 \) and \( k_3 \), when terms of order \( k_4 \sim a^4 \) are neglected. Inserting the latter relations into the series (3), we obtain the first three coefficients of the rearranged series (10)

\[
F_1(q^2) = -\frac{1}{\beta_0} , \quad F_2(q^2) = -\frac{\beta_1}{\beta_0} + \frac{1}{\beta_0^2} R_2(q^2) ,
\]

\[
F_3(q^2) = \left( -\frac{5/2}{2\beta_0^2} + \frac{\beta_3}{\beta_0^2} \right) + \frac{5\beta_1}{2\beta_0^2} R_2(q^2) - \frac{1}{\beta_0^2} R_3(q^2) .
\] (20) (21)

On the basis of these three coefficients, we can write down the truncated series for \( a(q^2)F(q^2) \) of (14), and can subsequently construct the dPA \( a(q^2)[1/2,F(q^2)] \) of Eq. (13) in the decomposed form (14). We then obtain expressions for parameters \( \tilde{u}_i(q^2) \) and \( \tilde{a}_i \) \((i=1,2)\)

\[
\left( \frac{\tilde{u}_2}{\tilde{u}_1} \right) = \frac{1}{2/\beta_0} \left[ \tilde{b}_1 \pm \sqrt{\tilde{b}_1^2 - 4\tilde{b}_2} \right] , \quad \tilde{a}_1 = \frac{\beta_0 \tilde{u}_1 \tilde{u}_2}{\tilde{b}_2(\tilde{u}_2 - \tilde{u}_1)} = \frac{1}{\sqrt{\tilde{b}_1^2 - 4\tilde{b}_2}} = -\tilde{a}_2 ,
\] (22)

where \( \tilde{b}_1 \) and \( \tilde{b}_2 \) are the two coefficients in the denominator of the dPA (12) (for the case \( 2M-2 = 2 \))

\[
\tilde{b}_1 = \frac{\beta_1}{\beta_0} - 2r_1 , \quad \tilde{b}_2 = \left( -\frac{3\beta_1^2}{2\beta_0^2} + \frac{\beta_2}{\beta_0} \right) + \frac{\beta_1}{\beta_0} r_1 + 3r_1^2 - 2r_2 .
\] (23)

We note that there are basically two cases to be distinguished when \( 2M-2 = 2 \):
1. The discriminant in (22) is nonnegative; then all the above parameters $\tilde{u}_i$, $\tilde{\alpha}_i$ and $p_{i}^{2}=q^2 \exp(\tilde{u}_i)$ are real, and the dBGA (15) is a real (and positive) number.

2. The discriminant in (22) is negative. Then $\tilde{u}_2 = \tilde{u}_1^*$, $p_{2}^{2}=(p_{1}^{2})^*$, $a(p_{2}^{2})=a(p_{1}^{2})^*$; and $\tilde{\alpha}_1$ is imaginary. Therefore, by (22), the dBGA (15) is a product of two imaginary numbers $\tilde{\alpha}_1$ and $a(p_{1}^{2})-a(p_{1}^{2})^*$, and hence is real (and positive).

It is interesting to establish directly RScl-invariance of the dBGA’s (15) and (17) for the case $2M−2=2$. Based on formulas (22)-(23), it is straightforward to show that the parameters $\tilde{\alpha}_i$ and $\ln(p_{i}^{2}/\Lambda^2)$ are functions of only the RScl-invariants $\rho_1 \equiv \tau−r_1(q^2)$. Here, $\tau \equiv \beta_0 \ln(q^2/\Lambda^2)$ was introduced in [3] and is just a dimensionless version of the RScl parameter $q^2$. However, the RSch-invariance, as expected, is not fulfilled in the $2M−2=2$ case, due to appearance of the RSch-dependent coefficient $\beta_2^*$ in the mentioned parameters.

As mentioned earlier, normalized observables $S^{(\ell)}$ appear sometimes in a modified form (1) with $\ell$ larger than 1. In such a case, we can simply introduce another observable $S = (S^{(\ell)})^{1/\ell}$ which is then represented as a power series with effectively $\ell \to 1$, as application of the simple Taylor expansion formula for $(1+x)^{1/\ell}$ in powers of $x$ shows.

We should keep in mind that Ref. [1] presented a dBGA algorithm applicable directly to truncated perturbation series (2) with $\ell = 1$ and $n$ being an odd positive integer. Here we extended application of this algorithm to the case of (2) with $n$ being an even positive integer ($n \equiv 2M−2$) and $\ell = 1$. Therefore, combining the results of [1] and those of the present paper, keeping in mind also the mentioned trick of reducing the $\ell \neq 1$ to the $\ell = 1$ case, we see that we can apply the method of the modified dBGA’s to any available truncated perturbation series of any normalized observable (1). The modified dBGA’s are those with the kernel $k(a_q, u)$ defined via (8), they reproduce the available TPS up to the order to which that TPS is known, and they are globally RScl–invariant.

### III. IMPROVING THE APPROXIMANTS BY A PMS VARIANT

The described method of modified dBGA’s yields RScl-invariant approximants, i.e., the approximants are independent of the choice of the RScl $q^2$ in the TPS under consideration. However, at this stage, the method does not address the question of the RSch-dependence. As already argued by Stevenson [3], the coefficients $\beta_j$ ($j=2,3,\ldots$) which show up in RGE (1) are not just RScl-dependent, but their values in turn characterize an RSch. The sets of values $\{\beta_j; j=2,3,\ldots\}$ thus represent a convenient parametrization of RSch’s. In fact, the obtained dBGA’s are in general dependent on the RSch parameters $\beta_j$ ($j \geq 2$) appearing in RGE (1). However, in principle, we can achieve RSch-independence by requiring that the corresponding partial derivatives of the dBGA’s be zero

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6 We consider the renormalization schemes in which the RScl-parameter $q^2$ has always the same meaning, i.e., the energy parameter $\Lambda$ is the same in all considered schemes. See also discussion later on.

7 The coefficients $\beta_0$ and $\beta_1$ are RSch-independent.
\[
\frac{\partial}{\partial \beta_j} \left( aG_{j}^{[M-1/M]} \right) = 0 \quad (j = 2, 3, \ldots) .
\]  

We should recall that the RSch-invariance of the approximants means that they are independent of the particular choice of the RSch-parameters \( \beta_j \) \((j \geq 2) \) made in the original TPS under consideration. Looking at conditions (24), we should keep in mind that the gauge coupling parameter \( a \equiv \alpha/\pi \) and the coefficients \( r_j \) in (1) depend not just on the RScl \( q^2 \) [i.e., on \( \tau = \beta_0 \ln(q^2/\Lambda^2) \)], but also on the RSch-parameters \( \beta_2, \beta_3, \ldots \), and that the RScl-independent parameters \( \bar{\alpha}_i \) and \( p_i^2 \) appearing in the dBGA’s (15) also depend on these RSch-parameters. As a matter of fact, Eqs. (24) represent just the principle of minimal sensitivity (PMS) introduced by Stevenson \([3]\). The difference now is that the PMS is applied to the dBGA’s which are already RScl-invariant, while the usual PMS \([3]\) is applied to the TPS’s which, at the outset, are by definition not just RSch-dependent, but also RScl-dependent. In a way, we repeat the PMS approach with a different, presumably more favorable, set of functions. In general, the dBGA applied to an available TPS of an appropriately redefined observable \( S \) (with effective \( \ell = 1 \)) is written in the form (15). Therefore, the PMS Eqs. (24) can be rewritten as

\[
\frac{\partial}{\partial c_j} \left( aG_{j}^{[M-1/M]} \right) = \sum_{i=1}^{M} \frac{\partial \bar{\alpha}_i}{\partial c_j} a(p_i^2) + \sum_{i=1}^{M} \bar{\alpha}_i \left[ \frac{\partial a(p_i^2)}{\partial (p_i^2) \frac{\partial}{\partial c_j}} \right] \bigg|_{p_i^2} = 0 ,
\]

where the RS-ch-parameters \( c_j \) are defined in the conventional way

\[
c_j \equiv \frac{\beta_j}{\beta_0} \quad (j \geq 2) , \quad \frac{\partial}{\partial c_j} = \beta_0 \frac{\partial}{\partial \beta_j} .
\]

The derivatives in (25) are partial in the sense that all other \( c_k \)'s \((k \neq j)\) are kept constant, as well as the RScl \( q^2 \) (although the entire expression is independent of \( q^2 \)).

In the case of the TPS (4) with \( n = 1 \), i.e., when only one term beyond the leading term is known, the dBGA depends explicitly only on the RScl \( q^2 \) (or: \( \tau \)) and not on \( c_j \)'s (cf. [1]). Therefore, the PMS approach cannot be applied in this case. As a matter of fact, for the \( n = 1 \) case the dBGA approach gives the same result as the effective charge method (ECH) \([4]\). The PMS improvement of the dBGA method comes into effect when higher order terms \((n = 2, 3, \ldots)\) are available. For example, we can directly apply the PMS improvement in the case \( n = 2M - 1 = 3 \) \((M = 2)\), by using explicit formulas provided in Ref. [1] for this case. In this case, we have coefficients \( f_j \) \((j = 1,2,3)\) instead of \( F_j \)'s of (17), and these \( f_j \)'s are explicitly given in terms of \( r_j \)'s and \( \beta_j \)'s in Ref. [1]. Now, using the RScl- and RSch-invariant quantities \( \rho_1, \rho_2, \rho_3 \) introduced by Stevenson \([3]\) and derived from the available coefficients \( r_1, r_2, r_3 \) of the TPS (1) (with \( \ell = 1 \)), we have

\[
\begin{align*}
 r_1 &= \tau - \rho_1 , \quad r_2 = \rho_2 + (\tau - \rho_1)^2 + c(\tau - \rho_1) - c_2 , \\
 r_3 &= \rho_3 - \frac{1}{2}c_3 + r_1 \left[ 2r_2 - r_1^2 + \frac{c}{2}r_1 + \frac{c^2}{4} + \rho_2 \right] ,
\end{align*}
\]

8 From now on, \( \ell = 1 \) will be assumed.
where \( c \equiv \beta_1/\beta_0 \) is also RScl- and RSch-invariant, and \( \tau \equiv \beta_0 \ln(q^2/\Lambda^2) \) is a dimensionless form of the RScl parameter \( q^2 \). This means that, when having the TPS \( S_3 \) (i.e., \( r_1, r_2, r_3 \)) available in a certain RSch and at a given RScl, we have \( S_3 \) available in any RScl and at any RScl. From (27)-(29) and from explicit expressions for \( f_j \)'s we obtain

\[
\frac{\partial f_1}{\partial c_j} = 0 \quad (j = 2, 3, \ldots) ; \quad \frac{\partial f_2}{\partial c_j} = -\frac{1}{\beta_0} , \quad \frac{\partial f_3}{\partial c_j} = 0 \quad (j = 3, \ldots) ;
\]

using these identities, as well as the explicit expressions for the dBG A parameters \( \tilde{u}_i \) for the PMS improvement of the dBGA’s for the case of truncated perturbation series \( S_n \) and from explicit expressions for these parameters with respect to \( c_j \)'s, in terms of the original TPS coefficients \( r_j \) (or equivalently: \( f_j \))

\[
\frac{\partial \tilde{u}_{2,1}}{\partial c_2} = \frac{1}{p_{2,1}^2} \frac{\partial p_{2,1}^2}{\partial c_2} = \frac{1}{(f_2 - f_1^2)} \left\{ \left( \frac{-1}{\beta_0^2} f_1 - \frac{5\beta_1}{4\beta_0^4} \right) + \frac{3}{\beta_0^2 \sqrt{\text{det}}} \left[ (f_2 - f_1^2)^2 + \frac{5\beta_1}{12\beta_0^2} (f_3 - 3f_1 f_2 + 2f_1^3) \right] \right\} + \frac{\tilde{u}_{2,1}}{\beta_0^2 (f_2 - f_1^2)} ,
\]

\[
\frac{\partial \tilde{u}_{2,1}}{\partial c_3} = \frac{1}{p_{2,1}^2} \frac{\partial p_{2,1}^2}{\partial c_3} = \frac{1}{4\beta_0^4 (f_2 - f_1^2)} \left\{ 1 \pm \left[ f_3 - 3f_1 f_2 + 2f_1^3 \right] \frac{1}{\sqrt{\text{det}}} \right\} ,
\]

\[
\frac{\partial \hat{a}_1}{\partial c_2} = -\frac{\partial \hat{a}_2}{\partial c_2} = \frac{3}{\beta_0^2 (\text{det})^{3/2}} (f_2 - f_1^2) \left( f_3 - 3f_1 f_2 + 2f_1^3 \right) - \frac{5\beta_1}{3\beta_0^2} (f_2 - f_1^2) ,
\]

\[
\frac{\partial \hat{a}_1}{\partial c_3} = -\frac{\partial \hat{a}_2}{\partial c_3} = \frac{1}{\beta_0^2 (\text{det})^{3/2}} (f_2 - f_1^2)^3 .
\]

In addition, we need in (23) also \( \partial a(p_1^2)/\partial (p_1^2) \) which is directly obtained from RGE (3), and we need as well \( \partial a(p_i^2)/\partial c_j \). The latter derivatives were derived by Stevenson [3]

\[
\frac{\partial a}{\partial c_2} = a^3 \left[ 1 + \frac{1}{3} c_2 a^2 + \left( \frac{1}{6} c_2 c_3 + \frac{1}{2} c_3 \right) a^3 + \cdots \right] ,
\]

\[
\frac{\partial a}{\partial c_3} = \frac{1}{2} a^4 \left[ 1 - c_3 a^2 + \frac{c_2^2}{6} a^2 + \left( -\frac{c_3^2}{10} + \frac{c_2 c_3}{15} + \frac{c_3}{5} \right) a^3 + \cdots \right] .
\]

Inserting expressions (31)-(33) and (3) into PMS relations (23), we obtain explicit equations for the PMS improvement of the dBGA’s for the case of truncated perturbation series \( S_n \) of Eq. (2) with \( n = 3 \) (and \( \ell = 1 \))

\[
\frac{\partial}{\partial c_2} \left( a G_f^{[1/2]} \right) = + \frac{3}{\beta_0^2 (\text{det})^{3/2}} A_2^2 \left[ A_1 - \frac{5\beta_1}{3\beta_0^2} A_2 \right] \left[ a(p_1^2) - a(p_2^2) \right]
\]

\[
+ \sum_{i=1}^{2} \hat{a}_i \left\{ - a^2 (p_i^2) \left[ 1 + ca(p_i^2) + c_2 a^2 (p_i^2) + c_3 a^3 (p_i^2) + \cdots \right] \right\} \times
\]

\[
\times \frac{1}{\beta_0 A_2} \left[ -f_1 - \frac{5\beta_1}{4\beta_0^2} - 3(-1)^i \left( A_2^2 + \frac{5\beta_1}{12\beta_0^2} A_1 \right) \frac{1}{\sqrt{\text{det}}} + \tilde{u}_i \right]
\]

9
\[ +a^3(p_i^2) \left[ 1 + \frac{c_2}{3} a^2(p_i^2) + \left( -\frac{c c_2}{6} + \frac{c_2}{2} \right) a^3(p_i^2) + \cdots \right] = 0 , \quad (37) \]

\[ \frac{\partial}{\partial c_3} \left( a G_F^{[1/2]} \right) = + \frac{1}{\beta_3^3(\text{det})^{3/2}} A_2^2 \left[ a(p_i^2) - a(p_2^2) \right] \]

\[ + \sum_{i=1}^2 \tilde{\alpha}_i \left\{ - a^2(p_i^2) \left[ 1 + c a(p_i^2) + c_2 a^2(p_i^2) + c_3 a^3(p_i^2) + \cdots \right] \times \right. \]

\[ \times \left. \frac{1}{4 \beta_2^2 A_2} \left[ 1 + (-1)^i \frac{A_1}{\sqrt{\text{det}}} \right] + \frac{1}{2} a^4(p_i^2) \times \right. \]

\[ \times \left[ 1 - \frac{c}{3} a(p_i^2) + \frac{c_2}{6} a^2(p_i^2) + \left( -\frac{c^2}{10} + \frac{c c_2}{15} + \frac{c_2}{5} \right) a^3(p_i^2) + \cdots \right] \} = 0 , \quad (38) \]

where we denoted

\[ A_1 = (f_3 - 3 f_1 f_2 + 2 f_1^3) , \quad A_2 = (f_2 - f_1^2) . \quad (39) \]

All the parameters \( f_j, \text{det}, \tilde{\alpha}_i, p_i^2 \) appearing in (37)-(38) are given explicitly in Ref. [1] in terms of the three TPS coefficients \( r_1, r_2, r_3 \). The latter coefficients, when known in one RSch, are known in any RSch since their dependence on \( c_j (j \geq 2) \) is determined according to (27)-(28). We stress that \( a(p_i^2) \) in (37)-(38) are determined by evolving \( a(p^2) \) from a chosen RScl \( q^2 \) to \( p_i^2 = p_l^2(c_2, c_3, \ldots) \) via RGE (8) where the coefficients \( \beta_j \equiv \beta_0 c_j (j \geq 2) \) on the right are the ones of the RSch used in Eqs. (37)-(38). In this evolution, the initial value \( a(q^2; c_2, c_3, \ldots) \) is also known (calculable), once it is known in one specific RSch \( a(q^2; c_2(0), c_3(0), \ldots) \) – cf. Eq. (12), or (13).

We note that the obtained system of equations (37)-(38) for PMS improvement is relatively complicated and can be solved only numerically. It results in finding optimal RSch parameters \( c_2, c_3 \) (i.e., \( \beta_2, \beta_3 \)) – optimal in a PMS sense. A more limited goal of achieving local independence of only the RSch-parameter \( c_2 \) leads us to an easier task of solving numerically only Eq. (37), by varying \( \beta_2 \) parameter and keeping \( \beta_3 \) at a fixed value of a specific scheme.

Incidentally, a very analogous kind of explicit PMS improvement equations can be constructed also for the case of the dBGA approximant to a TPS \( S_n \) of Eq. (2) with \( n=2 \) (and \( \ell=1 \)), i.e., for the dBGA’s discussed and constructed in the previous Section. In this case, expressions are simpler. It is straightforward to check that in this case the PMS-improved expression for the dBGA (15) with respect to the RSch-parameter \( c_2 \) is the one satisfying

\[ \frac{\partial}{\partial c_2} \left( a G_F^{[1/2]} \right) = \frac{6}{(b_1^2 - 4 b_2)^{3/2}} \left[ a(p_1^2) - a(p_2^2) \right] - \frac{3}{(b_1^2 - 4 b_2)} \left\{ a^2(p_1^2) + a^2(p_2^2) \right\} + \]

\[ + c \left[ a^3(p_1^2) + a^3(p_2^2) \right] + c_2 \left[ a^4(p_1^2) + a^4(p_2^2) \right] + \cdots \right\} \]

\[ + \frac{1}{\sqrt{b_1^2 - 4 b_2}} \left\{ a^5(p_1^2) - a^5(p_2^2) \right\} + \frac{c_2}{3} \left[ a^5(p_1^2) - a^5(p_2^2) \right] + \cdots \} = 0 . \quad (40) \]

We emphasize that the renormalization scale \( q^2 \) in various schemes in this formalism is defined to have the same meaning, i.e., that the energy parameter \( \tilde{\Lambda} \) appearing in the parameter \( \tau = \beta_0 \ln(q^2/\tilde{\Lambda}^2) \) is the same in all schemes under consideration (cf. [3] and Appendix
A of [3] for details). Stated otherwise, the gauge coupling parameters in various RSch’s behave, with this definition of the RScl $q^2$, in the following way:

$$a(q^2;\tilde{c}_2,\tilde{c}_3,\ldots) = a(q^2;c_2,c_3,\ldots) \left[ 1 + \mathcal{O}(a^2) \right].$$

(41)

Specifically, as implied by Eqs. (33) and (34) by Taylor expansion, we have the connections

$$a(q^2;\tilde{c}_2,\tilde{c}_3,\ldots) = a(q^2;c_2,c_3,\ldots) \left[ 1 + (\tilde{c}_2 - c_2)a^2(q^2;c_2,c_3,\ldots) + \frac{1}{2}(\tilde{c}_3 - c_3)a^3(q^2;c_2,c_3,\ldots) + \cdots \right].$$

(42)

More exactly, the connection between $a \equiv a(q^2;c_2,c_3,\ldots)$ and $\bar{a} \equiv a(q^2;\tilde{c}_2,\tilde{c}_3,\ldots)$ is given by the following equation (Appendix A of Ref. [3]):

$$\frac{1}{a} + c \ln \left( \frac{ca}{1 + ca} \right) + \int_0^a dx \left[ -\frac{1}{x^2(1 + cx + c_2x^2 + c_3x^3 + \ldots)} + \frac{1}{x^2(1 + cx)} \right] =$$

$$= \frac{1}{\bar{a}} + c \ln \left( \frac{\bar{c}a}{1 + \bar{c}a} \right) + \int_0^\bar{a} dx \left[ -\frac{1}{x^2(1 + \bar{c}x + \bar{c}_2x^2 + \bar{c}_3x^3 + \ldots)} + \frac{1}{x^2(1 + \bar{c}x)} \right],$$

(43)

which can be solved numerically for $\bar{a}$. Eq. (12), or (13), determines then the initial value $a(q^2)$ for integration of RGE (2) from $p^2 = q^2$ to $p^2 = p_1^2$ in any RSch ($\tilde{c}_2,\tilde{c}_3,\ldots$), once it is known in one specific RSch ($c_2,c_3,\ldots$).

For example, in the specific case of QED with the on-shell (OS) and the $\overline{\text{MS}}$ schemes, the above relation (12) gives us immediately the connection between the fine structure constant $\alpha_{\text{f.s.}} / \pi \equiv a_{\text{f.s.}} \equiv a(q^2 = -m_e^2)_{\text{on-shell}}$ and the $\overline{\text{MS}}$ constant $\bar{a}(q^2 = -m_e^2)$

$$\bar{a}(q^2 = -m_e^2) = a_{\text{f.s.}} \left[ 1 + 0.9375a_{\text{f.s.}}^2 + (0.07131285\ldots)a_{\text{f.s.}}^3 + \cdots \right],$$

(44)

which agrees with the result of [3]. In order to obtain (44), it was enough to insert into relation (12) the known QED beta coefficients $\beta_j = \beta_j c_j = -c_j/3$ in the two schemes [3], in the convention of Eq. (3)

$$\beta_2(\text{OS})^{(\text{QED})} = +0.420138888\ldots, \quad \beta_3(\text{OS})^{(\text{QED})} = +0.571156328087\ldots,$$

$$\beta_2(\overline{\text{MS}})^{(\text{QED})} = +0.107638888\ldots, \quad \beta_3(\overline{\text{MS}})^{(\text{QED})} = +0.523614426964\ldots,$$

(45)

while $\beta_0^{(\text{QED})} = -1/3$ and $\beta_1^{(\text{QED})} = -1/4$ are RSch-invariant. We mention these relations in order to stress that similar relations (11)-(12) between the on-shell and the (nonmodified) MS schemes are not true, because the meaning of the RScl in MS scheme is different from that of the $\overline{\text{MS}}$ scheme (and hence of the OS scheme) by a constant factor. In a way, MS and $\overline{\text{MS}}$ schemes can be regarded as the same, except that the meaning of the RScl $q^2$ differs in them by a constant factor. Therefore, although conditions (17)-(18) and (19) search for a PMS-improved dBGA approximant only in a certain class of renormalization schemes (those whose RScl $q^2$ has the same meaning), these conditions nonetheless don’t “miss” any relevant scheme.

For practical purposes, we should keep in mind that a given TPS $S_n$ [cf. Eq. (3)] is always given in such a specific RSch (e.g., on-shell or MS) in which only the first four coefficients $\beta_j$ ($j = 0,1,2,3$) on the right of RGE (3) are known. This means that the
changing of \( a(q^2;c_2,c_3,\ldots) \) when RSch and RScl are changed is practically known only up to (and including) \( \sim a^4 \), as seen, e.g., from (12) and Eq. (54) of Ref. [1] (first entry). This, together with relations (27)-(28), implies that we can consistently define the TPS \( S_n \) in various RSch’s only when \( n \leq 3 \), i.e., when only at most three coefficients \( r_j \) \((j = 1,2,3)\) are available. Therefore, the described PMS-improved method of modified dBGA’s can at present be consistent in practice only for such TPS’s (this is the case also in the usual PMS approach). Incidentally, the available TPS’s \( S_n \) at present have \( n \leq 3 \) (in QCD: \( n \leq 2 \)).

Within this context, the general limitations on the available precision of the value of a given coupling parameter \( a(q^2;c_2,\ldots) \), and hence of the values of the obtained approximants, become apparent. In QCD (and QED), only the first four \( \beta_j \) coefficients \((j = 0,\ldots,3)\) are known in a given specific (“original”) RSch (in QCD: MS or \( \overline{\text{MS}} \); in QED: on-shell) in which the value of that coupling parameter is fairly well known at a certain original RScl \( q^2 \). We can then calculate the coupling parameter \( a(q^2;c_2,\ldots) \) with very high precision in the original RSch and at the original RScl, the values of \( a \) in other RSch’s and at other RScl’s would be available only within the limited precision \( \sim a^4 \). However, the PMS-improved modified dBGA approximants would not be so crucially affected by this limited precision of the values of \( a \) in various RSch’s and at various RScl’s. The reason lies in the fact that these approximants (as well as the usual PMS approximants), besides being dependent on the value of \( a(q^2;c_2,\ldots) \), depend solely on the first \( n \) \((\leq 3)\) coefficients \( r_j(q^2;\bar{c}_2,\ldots,\bar{c}_j) \) \((j = 1,\ldots,n)\) in the TPS \( S_n \), and these coefficients are RScl- and RSch-dependent only via the values of the RScl \( q^2 \) \([i.e., \tau^\prime = \ln(q^2/\tilde{\Lambda}^2)]\) and of the first \( n-1 \) RSch-parameters \( \bar{c}_j = \beta_0 \bar{\beta}_j \) \((j = 2,\ldots,n)\), as seen explicitly in (27)-(28).

### IV. CONCLUSIONS

Based on the previously introduced method of modified diagonal Baker-Gammel approximants (dBGA’s) [1] for a specific class of truncated perturbation series (TPS’s) of observables, we showed how to extend the applicability of the method to any TPS of any given observable. The approximants reproduce the TPS, to which they are applied, to the available order of precision, and they are exactly invariant under the change of the renormalization scale (RScl). Furthermore, we constructed equations for these dBGA’s which have to be satisfied in order to have minimal (zero) sensitivity to the local change of the renormalization scheme (RSch) in these dBGA’s. The latter conditions are just a variant of the method of the principle of the minimal sensitivity (PMS) with respect to changing the RSch-parameters \( \beta_j \) \((j \geq 2,\ldots)\), but this time applied to the obtained dBGA’s and not to the TPS’s. The resulting approximants (dBGA’s) are then RScl- and RSch-invariant, i.e., independent of the RScl \( q^2 \) and the RSch \( \beta_j \) \((j \geq 2)\) chosen in the TPS under consideration.

The described dBGA method is an improvement of the method of the diagonal Padé approximants (dPA), the latter being RScl-invariant only in the large-\( |\beta_0| \) limit and RSch-noninvariant. We believe that there is some room for optimism concerning the efficiency of the described dBGA method when compared with the usual PMS method [3], and with the ECH method [5] which in turn is related with the usual PMS. This optimism rests on the fact that the RScl-invariance in the described dBGA approach is ensured via an algorithm which keeps a close contact with the usual (d)PA method, and the latter method has proven to be
reasonably efficient for various QCD observables [4]. The RSch-invariance of the described dBGA’s, i.e., invariance under the change of the $\beta_j$’s ($j \geq 2$) appearing in the original TPS’s, is achieved in a way similar to the usual PMS approach. The latter approach, on the other hand, achieves the RSch- and RScl-invariance by requiring that the truncated perturbation series (TPS) itself have minimal (zero) sensitivity under the local change of the RScl ($q^2$) and the RSch ($\beta_j$’s, $j \geq 2$).

In order to test the quality of the described method of the PMS-improved modified dBGA’s in practical calculations, it would be necessary to compare results of this method with those of the PMS [3] and ECH [5] method – in the cases of various available truncated perturbation series of QCD and QED observables. Further, comparison with several other methods [10]–[15] which eliminate or reduce the RScl- and RSch-dependence would give additional insights into the question of the value of the presented approximants.

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Abbreviations used frequently in the article: (d)BGA – (diagonal) Baker-Gammel approximant; (d)PA – (diagonal) Padé approximant; ECH – effective charge (method); PMS – principle of minimal sensitivity; RSch – renormalization scheme; RScl – renormalization scale; TPS – truncated perturbation series.
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