\[ N = 4 \text{ super KdV equation} \]

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**Abstract**

We construct \(N = 4\) supersymmetric KdV equation as a hamiltonian flow on the \(N = 4\) \(SU(2)\) super Virasoro algebra. The \(N = 4\) KdV superfield, the hamiltonian and the related Poisson structure are concisely formulated in 1D \(N = 4\) harmonic superspace. The most general hamiltonian is shown to necessarily involve \(SU(2)\) breaking parameters which are combined in a traceless rank 2 \(SU(2)\) tensor. First nontrivial conserved charges of \(N = 4\) super KdV (of dimensions 2 and 4) are found to exist if and only if the \(SU(2)\) breaking tensor is a bilinear of some \(SU(2)\) vector with a fixed length proportional to the inverse of the central charge of \(N = 4\) \(SU(2)\) algebra. After the reduction to \(N = 2\) this restricted version of \(N = 4\) super KdV goes over to the \(a = 4\) integrable case of \(N = 2\) super KdV and so is expected to be integrable. We show that it is bi-hamiltonian like its \(N = 2\) prototype.
1. Introduction. The Korteweg-de Vries (KdV) hierarchy, being the text-book example of 2D integrable system, now proved to be intimately related to many areas of modern theoretical physics, including conformal field theories, 2D gravity, matrix models, etc. As was observed in [1, 2], it admits a nice interpretation as a hamiltonian flow on the Virasoro algebra. In this approach the KdV field is identified with a 2D conformal stress-tensor and the KdV equation, as well as next equations from the KdV hierarchy, come out as hamiltonian equations with respect to the Poisson brackets forming a classical Virasoro algebra.

In [3-7] integrable $N = 1$ and $N = 2$ super KdV equations have been deduced as hamiltonian flows on the $N = 1$ and $N = 2$ super Virasoro (superconformal) algebras (SCA in what follows). An intriguing peculiarity of the $N = 2$ case is that there exists a one-parameter family of the super KdV equations associated with $N = 2$ SCA, but only for three selected values of the parameter, $a = -2, 4, 1$, these equations are integrable, i.e. possess an infinite number of the conserved quantities in involution and a Lax representation [6-8].

A natural minimal extension of $N = 2$ SCA is the $N = 4$ $SU(2)$ SCA [8]. Like the former it is generated by the currents with canonical spins $1, 3/2, 2$, the spinor currents forming a complex doublet of the internal symmetry group $SU(2)$. This is to be contrasted, e.g., with the $N = 3$ SCA, the odd sector of which involves a real $SO(3)$ triplet of the spin $3/2$ currents and a singlet spin $1/2$ current [9]. In the context of conformal field theory the $N = 4$ $SU(2)$ SCA was treated, e.g., in [10, 11]. An obvious interesting problem is to construct the $N = 4$ super KdV associated with the $N = 4$ $SU(2)$ SCA and to see whether the three integrable $N = 2$ super KdV equations can be promoted to integrable $N = 4$ ones [1]. This could shed more light on the origin of integrability in super KdV hierarchies and on the meaning of the parameter $a$.

In the present letter we solve the first part of this problem and report on some partial results regarding the second one. We construct, in a manifestly covariant $N = 4$ superfield form, the most general supersymmetric KdV equation for which $N = 4$ $SU(2)$ SCA provides the second hamiltonian structure and argue, by examining the question of existence of higher order conserved charges, that at least one integrable $N = 2$ super KdV equation, with $a = 4$, generalizes to $N = 4$ while preserving integrability. The parameter $a$ reveals an unexpected meaning within the $N = 4$ framework: the $N = 4$ super KdV equation necessarily contains $SU(2)$ breaking parameters which are combined into a traceless rank 2 $SU(2)$ tensor and $a$ is recognized as a component of this tensor. The requirement of the existence of non-trivial conserved charges in involution beyond the hamiltonian (we explicitly construct the dimension 2 and dimension 4 charges) restricts the components of the $SU(2)$ breaking tensor to be bilinear in components of some vector having a fixed length and so parametrizing a sphere $S^2 \sim SU(2)/U(1)$. The length (radius of the sphere) turns out to be proportional to the inverse of the central charge of the $N = 4$ $SU(2)$ SCA. After performing the reduction $N = 4 \Rightarrow N = 2$ one ends up with the integrable $N = 2$ super KdV equation corresponding to $a = 4$. All the conserved charges of the latter (an infinite tower of them) seem to have proper $N = 4$ counterparts thus implying the $N = 4$ super KdV with the aforementioned restrictions on the parameters to be integrable. One

\footnote{Super KdV equations associated with the $N = 3$ and $N = 4$ $SO(4)$ super Virasoro algebras were discussed in ref. [12, 13, 14].}
more argument in favour of its integrability is based on its bi-hamiltonian nature: we present the first hamiltonian structure for it.

Throughout the paper we systematically use the 1D $N = 4$ version of the harmonic superspace approach [14]. This allows us to formulate all our results in a concise form and to avoid some technical difficulties inevitable when working in ordinary 1D $N = 4$ superspace.

2. N=4 SU(2) superconformal algebra in harmonic superspace. In ordinary 1D $N = 4$ superspace with the coordinates $Z^M ≡ (x, θ^i, \bar{θ}^j)$, where $i, j = 1, 2$ are SU(2) doublet indices, the currents generating $N = 4$ SU(2) SCA are accomodated into a dimension 1 superfield $V^{ij}(Z^M)$, $V^{ij} = V^{ji}$, satisfying the constraints (see, e.g., ref. [17]):

$$D(iV^{jk}) = 0, \quad \bar{D}(iV^{jk}) = 0.$$  (1)

Here

$$D_i = \frac{∂}{∂θ^i} - \frac{i}{2} \bar{θ}^i \frac{∂}{∂x}, \quad \bar{D}^i = -\frac{∂}{∂\bar{θ}^i} + \frac{i}{2} θ^i \frac{∂}{∂x}, \quad \{D_i, \bar{D}^j\} = iδ^j_i \partial, \quad \{D_i, D_j\} = 0 \quad \text{and} \quad \{D_i, \bar{D}^j\} = iδ^j_i \partial,$$  (2)

and the SU(2) indices are raised and lowered by the antisymmetric tensors $ε^{ij}, ε_{ij}$ ($ε^{ij}ε_{jk} = δ^i_k, ε_{12} = -ε^{12} = 1$). It is straightforward to check that the constraints (1) leave in $V^{ij}$ only the following independent superfield projections

$$w^{ij} = iV^{ij}, \quad ξ^k = D^iV^k_i, \quad T(x) = \frac{i}{3} \bar{D}D^kV^k,$$  (3)

where the numerical factors are inserted for further convenience. The $θ$ independent parts of these projections, $w^{ij}(x), ξ^l(x), T(x)$, up to unessential rescalings coincide with the currents of $N = 4$ SU(2) SCA: the SU(2) triplet of the spin 1 currents generating SU(2) Kac-Moody subalgebra, a complex doublet of the spin 3/2 currents and the spin 2 conformal stress-tensor, respectively.

Let us see how the same supercurrent is represented in the harmonic 1D $N = 4$ superspace, an extension of $\{Z^M\}$ by the harmonic variables $u^±_i$ describing a 2-sphere $∼ SU(2)/U(1)$

$$\{Z^M\} \Rightarrow \{Z^M, u^+_i, u^-_j\}, \quad u^+_iu^-_i = 1, \quad u^+_iu^-_j - u^-_iu^+_j = ε_{ij}.$$  (4)

(see [15, 16] for the basics of the harmonic superspace approach).

In what follows we will need the derivatives in harmonic variables which are given by

$$D^{++} = u^+_i \frac{∂}{∂u^-_i}, \quad D^{--} = u^-_i \frac{∂}{∂u^+_i}, \quad D^0 = [D^{++}, D^{--}] = u^+_i \frac{∂}{∂u^+_i} - u^-_i \frac{∂}{∂u^-_i}.$$  (5)

The role of $D^0$ is to count the $U(1)$ charge of functions on the harmonic superspace defined as the difference between the numbers of the indices $+$ and $−$. The strict preservation of this $U(1)$ charge is one of the basic postulates of the harmonic superspace approach. It expresses the fact that the harmonic variables belong to the sphere $S^2$ and the harmonic
superfields are functions on this sphere as well. Also, instead of \( D^i \), \( \bar{D}^j \) we will use their projections on \( u^{\pm i} \)

\[
D^\pm = D^i u^\pm_i, \quad \bar{D}^\pm = \bar{D}^i u^\pm_i, \quad (6)
\]
onvanishing (anti)commutators of which with themselves and the harmonic derivatives \( D^{++}, D^{--} \) are

\[
\{ D^-, D^+ \} = i \partial, \quad \{ D^+, D^- \} = -i \partial, \quad (7)
\]

\[
[D^{++}, D^-] = D^+, \quad [D^{--}, D^+] = D^-. \quad (8)
\]

Define now the 1D \( N = 4 \) harmonic superfield \( V^{++}(Z, u) \) subject to the constraints

\[
D^+ V^{++} = 0, \quad D^{++} V^{++} = 0 \quad (9)
\]

\[
D^{++} V^{++} = 0. \quad (10)
\]

It follows from the harmonic constraint (10) that \( V^{++} \) is a homogeneous function of degree 2 in \( u^{+i} \)

\[
V^{++}(Z, u) = V^{ij}(Z) \ u^+_i u^+_j. \quad (11)
\]

Then, in view of the arbitrariness of \( u^{+i}, u^{+j} \), the constraints (9) imply for \( V^{ij} \) the original constraints (3). Thus the superfield \( V^{++} \) obeying (9), (10) represents the \( N = 4 \) \( SU(2) \) conformal supercurrent in the harmonic 1D \( N = 4 \) superspace.

The constraints (9) can be viewed as Grassmann analyticity conditions covariantly eliminating in \( V^{++} \) the dependence on half of the original Grassmann coordinates, namely, on their \( u^- \) projections \( \theta^- = \theta^i u^-_i, \quad \bar{\theta}^- = \bar{\theta}^i u^-_i \). So \( V^{++} \) is an analytic harmonic superfield living on an analytic subspace containing only the \( u^+ \) projections of \( \theta^i, \bar{\theta}^i \). This harmonic analytic superspace is closed under the action of 1D \( N = 4 \) supersymmetry (and actually under the transformations of the whole \( N = 4 \) \( SU(2) \) SCA), so one may construct additional superinvariants as integrals over this superspace. This opportunity will be exploited in next Sections. In what follows we will never actually need to know the explicit coordinate structure of the analytic superspace and how \( V^{++} \) is expressed there. We will only make use of the constraints (9), (10) and some important consequences of them, e.g.

\[
(D^-)^3 V^{++} = 0 , \quad D^- (D^-)^2 V^{++} = \bar{D}^- (D^-)^2 V^{++} = 0 , \quad etc. \quad (12)
\]

After we have represented the \( N = 4 \) \( SU(2) \) supercurrent as a harmonic superfield \( V^{++} \), it remains to write the Poisson bracket between two \( V^{++} \)'s which yields the \( N = 4 \) \( SU(2) \) SCA Poisson brackets for the component currents. Surprisingly, it is almost uniquely determined by dimensionality and compatibility with the constraints (9), (10). It reads

\[
\{V^{++}(1), V^{++}(2)\} = D^{ (++|++)} \Delta(1-2)
\]

\[
D^{ (++|++)} = (D^+_1 \bar{D}^+_1)(D^+_2 \bar{D}^+_2) \left( \left[ \frac{1}{2} D^-_1 - \left( \frac{u^-_1 u^+_2}{u^+_1 u^+_2} \right) \right] V^{++}(1) + \frac{k}{4} \partial_1 \right), \quad (13)
\]

\[\text{footnote}{\text{2}}\text{The harmonic superspace form of the } N = 4 \text{ } SU(2) \text{ conformal supercurrent has been earlier given in ref. [18].}\]
where $\Delta(1-2) = \delta(x_1-x_2) (\theta^1-\theta^2)^4$ is the ordinary 1D $N=4$ superspace delta function, and we refer to [4] for more details on harmonic distributions. Using the algebra of spinor and harmonic derivatives and also the completeness condition (1), one can check that the r.h.s of (13) is consistent with the constraints (9), (10) with respect to both sets of arguments and is antisymmetric under the interchange 1 $\leftrightarrow$ 2. To be convinced that it gives rise to the correct Poisson brackets for the component currents, let us, e.g., deduce from (13) the Poisson brackets of $SU(2)$ Kac-Moody currents. After simple algebraic manipulations we obtain for $w^a \equiv \sigma^a_{ij} w^i_j$ the familiar relation:

$$\{w^a(1), w^b(2)\} = \epsilon^{abc} w^c(1) \delta(1-2) + \frac{k}{2} \delta^{ab} \partial_1 \delta(1-2).$$

(14)

All other currents can also be checked to satisfy the relations needed to constitute $N=4$ $SU(2)$ SCA.

Finally, we point out that it is straightforward to rewrite the Poisson structure (13) in ordinary 1D $N=4$ superspace. But there it looks much more complicated: it involves intricate combinations of $SU(2)$ indices, etc.

3. **N=4 super KdV equation from N=4 SU(2) SCA.** Our aim is to deduce the most general super KdV equation with the second hamiltonian structure given by the $N=4$ $SU(2)$ SCA in the form (13). The only requirement we impose beforehand is rigid 1D $N=4$ supersymmetry. The most general dimension 3 $N=4$ supersymmetric hamiltonian one may construct out of $V^{++}$ consists of two pieces

$$H = \int [dZ] V^{++}(D^-)^2V^{++} - i \int [d\zeta^{-2}] c^{-4}(u) (V^{++})^3.$$

(15)

Here $[dZ] = dx[du] D^- D^+ \bar{D}^+$ is the integration measure of the full harmonic superspace and $[d\zeta^{-2}] = dx_A[du] D^- \bar{D}^-$ is the integration measure of the analytic subspace. The integral over harmonic variables is defined so that $\int [du] 1 = 1$ and integral of any symmetrized product of harmonics is vanishing [15]. We see that, to balance the $U(1)$ charges, the integral over analytic subspace should inevitably include the harmonic monomial $c^{-4}(u) = c^{ijkl} u_i^j u_j^k u_l^k$ which explicitly breaks $SU(2)$ symmetry. The coefficients $c^{ijkl}$ belong to the dimension 5 spinor representation of $SU(2)$, i.e. form a symmetric traceless rank 2 tensor, and completely breaks the $SU(2)$ symmetry (case (A) in what follows), unless $c^{-4}$ takes the special form $c^{-4}(u) = (a^{-2}(u))^2$, $a^{-2}(u) = a^{ij} u_i^j u_j^i$ (case (B)). In case (B), the symmetry breaking parameter belongs to the dimension 3 (vector) representation of $SU(2)$, and thus has $U(1)$ as a little group. We point out that the presence of the trilinear term in the hamiltonian is unavoidable if one hopes to eventually obtain an integrable super KdV equation (it should somehow contain the $N=2$ super KdV family which is integrable only providing the relevant hamiltonian contains a trilinear term). Thus, one necessary condition for the integrability of $N=4$ super KdV we are going to derive is that $SU(2)$ is broken, at least down to its $U(1)$ subgroup.

Using the hamiltonian (13), we construct an evolution equation:

$$V_t^{++} = \{H, V^{++}\}$$

(16)

\[\text{Actually, the harmonic singularity in the r.h.s. of (13) is fake: it is cancelled after decomposing the harmonics } u^{\pm i}_2 \text{ over } u^{\pm i}_1 \text{ with making use of the completeness relation (1).}\]
which, after a bit tedious but straightforward computations may be cast into the following form:

\[
V^{++} = iD^+ D^+ \left\{ \frac{k}{2} D^- V^{xx} + \left[ V^{++} (D^-)^2 V^{++} - \frac{1}{2} (D^- V^{++})^2 \right]_x - \frac{3}{20} kA^{-4} (V^{++})^2_x + \frac{1}{2} A^{-6} (V^{++})^3 \right\}. \tag{17}
\]

Here \(A^{-4}\) and \(A^{-6}\) are differential operators on the 2-sphere \(\sim SU(2)/U(1)\)

\[
\begin{align*}
A^{-4} &= \sum_{N=1}^{4} (-1)^{N+1} c^{2(N-2)} \frac{1}{N!} (D^-)^N, \\
A^{-6} &= \frac{1}{5} \sum_{N=1}^{5} (-1)^{N+1} c^{2(N-3)} \frac{(6-N)}{N!} (D^-)^N
\end{align*}
\tag{18}
\]

and we have used the notation:

\[
\frac{c^{2N-4}}{4!} = (4-N)! (D^{++})^N c^{-4}, \quad N = 0 \ldots 4.
\tag{19}
\]

The equation (17) is the sought \(N = 4\) super KdV equation. It is easy to check that its r.h.s satisfies the same constraints (1), (2) as the l.h.s. One might bring (17) into a more explicit form using the algebra (7), (8) (e.g., the first term takes then the familiar form \(-\frac{k}{2} V^{xx}\)), however, for many reasons it is convenient to keep the analytic subspace projector \(D^+ D^+\) before the curly brackets in (17). Let us also note that the hamiltonian (13) and eq. (17) can be rewritten in ordinary \(N = 4\) superspace, but they look there, like the Poisson bracket (13), very intricate. For instance, the second term in (17) would involve explicit \(\theta\)'s so it would be uneasy to see that it is supersymmetric. Thus the harmonic superspace seems to provide the most appropriate framework for formulating \(N = 4\) super KdV equation. The last comment concerns the presence of the \(N = 4\) \(SU(2)\) SCA central charge \(k\) in (17). Making in (17) the rescalings \(t \rightarrow bt\), \(V^{++} \rightarrow b^{-1} V^{++}\), \(c \rightarrow bc\) we can in principle fix this parameter at any non-zero value. However, in order to have a clear contact with the original \(N = 4\) \(SU(2)\) Poisson structure (13) we prefer to leave \(N = 4\) super KdV in its original form.

Before going further we present the bosonic core of our super KdV equation. It consists of two coupled equations for the fields \(T\) and \(w^{ij}\), first of which is an extension of the KdV equation and the second is a three-component generalization of the mKdV equation

\[
T_t = -\frac{k}{2} T_{xxx} + 3 \left\{ T^2 + \frac{3}{10} T c_{ijkl} w^{ij} w^{kl} \right\}_x + \left( w^{ij}_{xx} w_{ij} - \frac{3k}{20} c_{ijkl} w^{ij} w^{kl} \right)_x - \frac{1}{2} (w^{ij}_{xx} w_{ij})_x + \frac{3k}{10} c_{ijkl} w^{ij} w^{kl} w_{xx} - \frac{3i}{5} c_{ijkl} (w^{ij} w^{k} w^{lm})_x \tag{20}
\]

\[
\begin{align*}
w_{ij} = & \frac{k}{2} w^{ij}_{xx} + 2 \left( T w^{ij} - \frac{3k}{20} T c_{ijkl} w^{kl} \right)_x + 2i \left( w^{ij}_{xx} w^{kl} \right)_x - \frac{3k}{40} c^{(i} klm (w^{j)} w^{lm})_x + \frac{3ik}{20} c^{ijkl} (w^{km} w^{lm})_x - \frac{2i}{5} c^{ijkl} (T w^{kl} w^{lm})_x + \frac{1}{10} c^{ijklm} (w^{ij} w^{kl} w^{mn} + 4 w^{ik} w^{jl} w^{mn})_x
\end{align*}
\]
4. Reduction to N=2 super KdV. As a first step in analyzing the integrability properties of (17) we perform now a reduction to \( N = 2 \) supersymmetry in order to see which kind of \( N = 2 \) super KdV arises thereupon.

Schematically, this reduction goes as follows. One passes to the \( N = 2 \) notation for \( N = 4 \) spinor derivatives

\[
D \equiv D_1, \quad \bar{D} \equiv \bar{D}_1, \quad d \equiv D_2, \quad \bar{d} \equiv \bar{D}_2,
\]

and identifies \( D, \bar{D} \) with the spinor derivatives of \( N = 2 \) supersymmetry. Respectively, \( 1D \ N = 4 \) superspace is represented as a product of the \( 1D \ N = 2 \) superspace with coordinates \( \{ x, \theta \equiv \theta^1, \bar{\theta} \equiv \bar{\theta}_1 \} \) and the pure Grassmann quotient \( \{ \theta^2, \bar{\theta}_2 \} \). Then one rewrites (17) in ordinary \( 1D \ N = 4 \) superspace using the above notation (we do not know how to perform the reduction directly in the harmonic superspace). Further, in all places where the covariant derivatives \( d \) and \( \bar{d} \) of \( V^{ij} \) are encountered, one expresses them through \( D, \bar{D} \) using the constraints (1).

The crucial observation is that the whole set of currents of \( N = 2 \) SCA is contained in the \( \theta^2, \bar{\theta}_2 \) independent part of the \( N = 4 \) superfield \( V^{12} \), hence the identification

\[
\Phi \equiv V^{12}| , \quad \text{where } \Phi(x, \theta, \bar{\theta}) \text{ is the } N = 2 \text{ conformal supercurrent and } | \text{ means restriction to the } \theta^2, \bar{\theta}_2 \text{ independent part.}
\]

Finally, in the \( N = 4 \) super KdV equation prepared as explained above one puts

\[
\theta^2 = \bar{\theta}_2 = 0, \quad V^{11} = V^{22} = 0 .
\]

As the result of this reduction one gets the \( N = 2 \) super KdV equation in the form

\[
\Phi_t = -\frac{k}{2} \Phi_{xxx} - 3i \left[ (D\bar{D} - D\bar{D})\Phi \right]_x - \frac{i}{2} (a - 1) \left[ (D\bar{D} - D\bar{D})\Phi^2 \right]_x - \frac{2a}{k} \Phi^2 \Phi_x
\]

with

\[
a \equiv \frac{3}{5} c^{1212} .
\]

This equation can be brought precisely to the form given in [6, 7], redefining

\[
\partial_t = i \frac{k}{2} \tilde{\partial}_1 , \quad \partial_x = -i \tilde{\partial}_x , \quad \Phi = \frac{k}{2} \tilde{\Phi} , \quad D = \frac{1}{2} (\tilde{D}_2 + i \tilde{D}_1) , \quad \bar{D} = \frac{1}{2} (\tilde{D}_2 - i \tilde{D}_1) .
\]

The \( N = 4 \) \( SU(2) \) Poisson structure (13) is reduced to the \( N = 2 \) one

\[
\{ \Phi(1), \Phi(2) \} = \frac{1}{4k} \left( \tilde{D}_1 \tilde{D}_2 \tilde{\partial} - \tilde{D}_1 \tilde{\Phi} \tilde{D}_1 - \tilde{D}_2 \tilde{\Phi} \tilde{D}_2 + 2 \tilde{\Phi} \tilde{\partial} + 2 \tilde{\partial} \tilde{\Phi} \right) \left( d \tilde{d} \Delta (1-2) \right) .
\]
well-known one-parameter family of the $N = 2$ super KdV equations. A new point is that the parameter $a$ turns out to be related via the equation (25) to the component $c^{ijkl}$ of the $SU(2)$ breaking tensor present in the $N = 4$ super KdV equation. Thereby, this parameter acquires a group-theoretical meaning.

5. Conserved charges. As was mentioned in Introduction, $N = 2$ super KdV equation is integrable only for $a = 4, -2, 1$. Then, in view of the relation (25) one may expect that the $N = 4$ super KdV equation is integrable only under certain restrictions on the $SU(2)$ breaking tensor $c^{ijkl}$. To see, which kind of restrictions arises, we analyze here the question of existence of non-trivial conserved charges for (17) which are in involution with the hamiltonian (13).

Before all, it is clear that after the reduction to $N = 2$ such charges should go over to those of the integrable $N = 2$ super KdV equations. Any such charge of dimension, say, $l$, is known to contain in the integrand the term $\sim (\Phi)^l$. These terms can be obtained only from analytic integrals of the form

$$\sim \int [d\zeta^{-2}] b^{-2(l-1)} (V^{++})^l ,$$

where

$$b^{-2(l-1)} = b^{i_1 \cdots i_{2(l-1)}} u_{i_1} \cdots u_{i_{2(l-1)}} .$$

Then, if the corresponding charge is required to be conserved, the highest order contribution to the time derivative of (28) (coming from the 3-d order term in the r.h.s. of (17)) should vanish in its own right. A simple analysis shows that it is possible if and only if

$$b^{-2(l-1)} \sim (c^{-4})^n , \ l = 2n + 1; \ b^{-2(l-1)} \sim (a^{-2})^{2n-1} , \ c^{-4} = (a^{-2})^2 , \ l = 2n .$$

In other words, the odd dimension conserved charges can exist only provided $b^{-2(l-1)}$ is a power of $c^{-4}$ while the necessary condition for the existence of the even dimension conserved charges is more restrictive: $c^{-4}$ should be square of some $a^{-2} = a^{ij} u_i^{-} u_j^{-}$, where $a^{ij}$ is a $SU(2)$ vector, i.e. we meet the situation called in Sect. 3 the case (B).

Let us now explicitly construct several first charges. Conservation of the dimension 1 charge :

$$H_1 = \int [d\zeta^{-2}] V^{++}$$

imposes no condition on the parameters of the hamiltonian.

A charge with dimension 2 exists only in the case (B) and reads:

$$H_2 = \int [dz^{-2}] a^{-2} (V^{++})^2 .$$

Already for this charge we find that it is conserved only under an additional non-trivial condition on the $SU(2)$ breaking vector $a^{ij}$: the square of the latter should be proportional to the inverse of the central charge:

$$s \equiv a^{i+2} a_i^{-2} - (a^0)^2 = \frac{1}{2} a^{ij} a_{ij} = -\frac{10}{k} ,$$

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where
\[ a^{+2} = D^{++} a^0 = \frac{1}{2} (D^{++})^2 a^{-2}. \]

Note that the central charge \( k \) is integer (if we restrict ourselves to unitary representations of the \( SU(2) \) Kac-Moody algebra [19]), so eq. (33) means that \( a^{ij} \) parametrize a sphere \( S^2 \sim SU(2)/U(1) \), such that the reciprocal of its radius is quantized. Substituting this relation into eq. (25), where now \( c^{1212} = \frac{2}{3} a^{12} a^{12} \), one finds that it yields \( a = 4 \), i.e. one of the three integrable cases of \( N = 2 \) super KdV.

Dimension 3 charge is the hamiltonian itself.

In the case (B) one can construct a dimension 4 conserved charge. It exists under the same restriction (33) on the square of \( a^{ij} \) and reads:

\[
H_4 = \int [dZ] a^{-2} V^{++}(D^{--}V^{++})^2 + \frac{i}{6} \int [d\zeta^{-2}] \left[ \frac{7}{6} (a^{-2})^3 (V^{++})^4 - k a^{-2} (V^{++}_x)^2 \right].
\]

(34)

Thus we conclude that at least in the case (B) with the restriction (33) our \( N = 4 \) super KdV equation is expected to give rise to an integrable hierarchy. Clearly, in order to prove this one should either find the relevant Lax pair or prove the existence of an infinite number of the conserved charges of the type given above. We will address the problem of existence of the Lax representation for eq. (17) in the nearest future. In order to learn whether the other two integrable cases of \( N = 2 \) super KdV equation, with \( a = -2 \) and 1, possess integrable \( N = 4 \) counterparts, one should consider the dimension 5 conserved charge. In the \( N = 2 \) case it exists for all three integrable super KdV equation. A work on checking its existence in the \( N = 4 \) case is now in progress.

Finally, we wish to show that the \( N = 4 \) super KdV equation with \( c^{-4} = (a^{-2})^2 \) and the restriction (33) is bi-hamiltonian like its \( a = 4 \) \( N = 2 \) prototype [6]. This is one more argument in favour of its integrability.

The first hamiltonian structure of the \( N = 4 \) super KdV equation is associated with a Poisson bracket which is obtained from the original one (13) by shifting the superfield \( V^{++} \) as follows

\[ V^{++} \rightarrow V^{++} + \beta a^{+2}(u), \quad \beta = \text{const}, \]

and so it is proportional to the \( SU(2) \) breaking parameter \( a^{ij} \)

\[ \{V^{++}(1), V^{++}(2)\}_{(1)} = \beta \left( a^0(1) - a^{+2}(1) \frac{u_1^2 u_2^2}{u_1^2 + u_2^2} \right) (D_1^+ \bar{D}_1^+)(D_2^+ \bar{D}_2^+) \Delta(1 - 2). \]

(36)

Then, we take as a hamiltonian the conserved charge \( H_4 \):

\[ H_{(1)} = \frac{9k}{4\beta} H_4. \]

(37)

The hamiltonian flow:

\[ V^{++}_t = \{H_{(1)}, V^{++}\}_{(1)} \]

(38)

again yields the equation (17). Now this comes about in a rather non-trivial way, since both the new Poisson bracket (36) and the new hamiltonian (37) are proportional to the \( SU(2) \) breaking parameter \( a^{ij} \), while the super KdV equation (17) includes terms
containing no dependence on $a^{ij}$. The key point is that these terms appear in (38) multiplied by the factor

$$-\frac{k}{10} s = -\frac{k}{20} a^{ij} a_{ij},$$

(39)

which is independent of harmonic coordinates $u^\pm$ and is constrained to be 1 from the requirement of conservation of $H_2$ and $H_4$ (see eq. (33)).

6. Conclusions. To summarize, starting with 1D $N = 4$ harmonic superspace we have defined the supercurrent which generates $N = 4$ SU(2) super Virasoro algebra via an appropriate Poisson structure (13) and have constructed the $N = 4$ SU(2) super KdV equation (17) as an evolution equation for the supercurrent, with the $N = 4$ SU(2) Poisson structure as a second hamiltonian structure. We gave necessary criteria for integrability of this equation and argued that it is integrable at least for one special choice of the SU(2) breaking parameters in the hamiltonian, such that these parameters describe a sphere $S^2$ with the radius subject to the quantization condition (33). This particular $N = 4$ super KdV equation is bi-hamiltonian and, upon the reduction $N = 4 \Rightarrow N = 2$, yields the $a = 4$ integrable case of the $N = 2$ super KdV equation. In order to prove its integrability and to learn whether two other integrable $N = 2$ super KdV equations (with $a = -2$ and 1) allow a generalization to integrable $N = 4$ ones, it is of crucial necessity to construct the relevant Lax representations and (or) to inspect in more detail the issue of existence of higher order conserved charges for eq.(17) (beginning with $H_5$).

Besides these purely technical problems, the present study poses a number of other interesting questions. For instance, one may wonder what are the $N = 4$ analogs of Miura map and mKdV equation. This question could hopefully be answered using free field representations of the $N = 4$ SU(2) supercurrent, e.g. of the kind given in ref. [17, 14]. Another problem is to re-derive eq.(17) from the self-duality constraints of some higher-dimensional supergauge theory along the lines of refs. [20].

The harmonic superspace approach suggests, in its own right, some new directions of extending the results presented here. An intriguing possibility is to discard the harmonic constraint (14) for the supercurrent $V^{++}$, still keeping the analyticity conditions (1). Then $V^{++}$ will be an unconstrained analytic $N = 4$ superfield generating some infinite dimensional higher isospin extension of $N = 4$ SU(2) SCA. It is interesting to inquire what are the relevant superfield Poisson bracket and super KdV equation. Also, it seems that the harmonic superspace language ideally suits for constructing the $W$ type nonlinear extensions of $N = 4$ SU(2) SCA and related generalized super KdV hierarchies.

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