A DYNAMICAL CONSTRUCTION OF LIOUVILLE DOMAINS

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ABSTRACT. We first present a general construction of Liouville domains as partial mapping tori. Then we study two examples where the (partial) monodromies exhibit certain hyperbolic behavior in the sense of Dynamical Systems. The first example is based on Smale’s attractor, a.k.a., solenoid; and the second example is based on certain hyperbolic toral automorphisms.

A Liouville domain \((W^{2n}, \omega, X)\) is a triple where \(W\) is a compact manifold with (nonempty) boundary, \(\omega\) is a symplectic form and \(X\) is a vector field, called the Liouville vector field, such that \(L_X \omega = \omega\) and \(X\) is outward pointing along \(\partial W\). Define the Liouville form \(\lambda := i_X \omega\). Then by Cartan’s formula, the symplectic form \(\omega\) is exact. A Liouville domain \((W, \omega, X)\) is called Weinstein if, in addition, \(X\) is gradient-like with respect to a Morse function on \(W\).

It turns out that the Weinstein condition is rather restrictive on the topology of \(W\). Indeed, any Weinstein domain admits a handle decomposition which contains only handles of index at most \(n\). On the other hand, since the first example of McDuff [McD91], it has been known that general Liouville manifolds are not subject to such topological constraints. See [Gei95, Mit95, Gei94, MNW13] for more constructions of Liouville, but not Weinstein, domains. Unfortunately, no good methods are currently available to distinguish between Liouville and Weinstein structures besides the obvious topological distinction.

The goal of this note is produce a few (exotic) examples of Liouville domains where the dynamics of the Liouville vector field \(X\) can be explicitly described. It turns out that the dynamics of \(X\) is only interesting when restricted to the skeleton of \((W, \omega, X)\) which we now introduce.

Given a Liouville domain \((W, \omega, X)\), the skeleton \(\text{Sk}(W, \omega, X)\) is defined by

\[
\text{Sk}(W, \omega, X) := \bigcap_{t > 0} \phi_t^X(W),
\]

where \(\phi_t^X\) denotes the time-\(t\) flow of \(X\). Clearly \(\text{Sk}(W, \omega, X)\) contains all the information of the symplectic structure, but it is, in general, not invariant under Liouville homotopies. It is a very interesting problem to understand how \(\text{Sk}(W, \omega, X)\) and \(\text{Sk}(W, \omega, X')\) are related to each other if \(X, X'\) are two different Liouville vector fields on the same symplectic manifold. In the case of Weinstein domains, an on-going project of Alvarez-Gavela, Eliashberg and Nadler aims at simplifying \(\text{Sk}(W, \omega, X)\), up to Weinstein homotopy, such that it contains only aboreal singularities introduced by Nadler [Nad17]. In the following we sometimes simply write \(\text{Sk}(W)\) for the skeleton if there is no risk of confusion.

Now let’s present the main construction of this note: Liouville domains as partial mapping tori. Let \(M^{2n-1}\) be a compact manifold with boundary and \(\alpha\) be a contact form on \(M\). For the rest of this note, every contact manifold comes with a chosen contact form.

**Definition 0.1.** A compact contact manifold \((M, \alpha)\) admits a contraction if there exists a map \(\phi : M \to M\), which satisfies the following properties:

1. \((D1)\) \(\phi(M) \subset \text{int}(M)\);
2. \((D2)\) \(\phi\) is a diffeomorphism onto its image;
3. \((D3)\) \(\phi^*(\alpha) = e^{-g} \alpha\), where \(g : M \to \mathbb{R}_{>0}\) is a positive function.

Note that \((D3)\) implies \(\partial M \neq \emptyset\) for volume considerations. We start with a not so interesting example.
Example 0.2. Let $Y$ be a closed manifold and consider the 1-jet space $J^1Y$ equipped with the standard contact form $\alpha = dz - pdq$. Let $M \subset J^1Y$ be a closed tubular neighborhood of the 0-section $Y \subset J^1Y$. Then the map $\phi : M \to M$ defined by $\phi(z, q, p) = (z/2, q, p/2)$ is clearly a contraction. In this example, $\phi(M)$ is a deformation retract of $M$, but this is not necessarily the case in general.

We construct a Liouville domain $W_{(M, \phi)}$, as a partial mapping torus, in three steps as follows. Firstly, let $\mathbb{R} \times M$ be the symplectization of $(M, \alpha)$ with Liouville form $\lambda = e^s\alpha$, where $s \in \mathbb{R}$; secondly, let $G : M \to \mathbb{R}_{>0}$ be a smooth extension of the function $g \circ \phi^{-1} : \phi(M) \to \mathbb{R}_{>0}$; finally, define the partial mapping torus

$$W_{(M, \phi)} := \{(s, x) \in \mathbb{R} \times M \mid 0 \leq s \leq G(x)\}/(0, x) \sim (G(x), \phi(x)).$$

We claim that $\lambda$ descends to a 1-form on $W_{(M, \phi)}$. Indeed, define $\Phi : \mathbb{R} \times M \to \mathbb{R} \times M$ by $\Phi(s, x) := (s + G(x), \phi(x))$, then

$$\Phi^*(\lambda) = e^{s+G} \phi^* \phi^* \alpha = e^{s+g} e^{-g} \alpha = \lambda$$

as desired. Abusing notations, we also write $\lambda$ for the descendent 1-form on $W_{(M, \phi)}$. See Figure 1 for an illustration of $W_{(M, \phi)}$, where $G$ appears to be constant.

![Figure 1](image_url)

**Figure 1.** The blue regions on the bottom and the top represent $M$ and $\phi(M)$, respectively, and are identified in $W_{(M, \phi)}$.

Define the vertical boundary of $W_{(M, \phi)}$ by

$$\partial_v W_{(M, \phi)} := \{(s, x) \in \mathbb{R} \times \partial M \mid 0 \leq s \leq G(x)\},$$

and the horizontal boundary $\partial_h W_{(M, \phi)}$ as the closure of $\partial W_{M, \phi} \setminus \partial_v W_{(M, \phi)}$. It follows from the construction that the Liouville vector field $X$ is outward-pointing along $\partial_h W_{(M, \phi)}$ and is tangent to $\partial_v W_{(M, \phi)}$. By slightly tilting $\partial_h W_{(M, \phi)}$, we can assume that $X$ is everywhere outward-pointing along $\partial W_{(M, \phi)}$. Strictly speaking, the Liouville domain $W_{(M, \phi)}$, as constructed, has corners along $\partial_h W_{(M, \phi)} \cap \partial_v W_{(M, \phi)}$. But the corners can be canonically rounded and we denote the resulting smooth Liouville domain, again, by $W_{(M, \phi)}$. We conclude our general construction with an obvious lemma which describes the skeleton of $W_{(M, \phi)}$.

**Lemma 0.3.** The skeleton of $W_{(M, \phi)}$ is given by the mapping torus

$$\text{Sk}(W_{(M, \phi)}) = \{(s, x) \in \mathbb{R} \times K \mid 0 \leq s \leq G(x)\}/(0, x) \sim (G(x), \phi(x)),$$

where $K := \bigcap_{i \geq 0} \phi^i(M) \subset M$.

It is clear that if the input $(M, \phi)$ is as in Example 0.2, then the resulting $W_{(M, \phi)}$ is Weinstein and the skeleton is a smooth Lagrangian submanifold $S^1 \times Y$. In the following, we give two explicit examples of $(M, \phi)$ of very different nature, such that the resulting Liouville domains $W_{(M, \phi)}$ are “more interesting”.
Example 0.4 (Smale’s attractor/Solenoid [KH95, Section 17.1]). Consider $M = S^1 \times D^2$ equipped with the contact form $\alpha = dx + yd\theta$, where $\theta \in S^1$ and $(x, y) \in D^2 \subset \mathbb{R}^2$. Define $\phi : M \to M$ by

$$\phi(\theta, x, y) = \left(2\theta, \frac{1}{10}x + \frac{1}{2}\cos \theta, \frac{1}{2}\left(\frac{1}{10}y + \frac{1}{2}\sin \theta\right)\right).$$

Clearly $\phi$ satisfies (D1)–(D2). But $\phi(M)$ is not a deformation retract of $M$. Instead, it winds around $M$ twice along the $S^1$-factor. It is straightforward to compute that $\phi^2\alpha = \frac{1}{10}\alpha$, and therefore $\phi$ satisfies (D3). Hence $\phi$ is a contraction and we have a well-defined Liouville domain $W_{SA} := W_{(M, \phi)}$.

Let’s examine the skeleton $\text{Sk}(W_{SA})$. First observe that $K = \bigcap_{i \geq 0} \phi^i(M)$ is itself a mapping torus of a Cantor set. Namely, for each fixed $\theta_0 \in S^1$, the intersection $K \cap (\{\theta_0\} \times D^2) \subset D^2$ is a Cantor set. Such $K$ is known as a solenoid in Dynamical Systems, and is a hyperbolic attractor. In particular, it is stable under $C^\infty$-small perturbations. We refer the interested readers to the comprehensive monograph [KH95] for more details. In light of Lemma 0.3, $\text{Sk}(W_{SA})$ is a mapping torus of the solenoid $K$.

It follows from the construction that, as a smooth 4-manifold, $W_{SA}$ can be built by handles of index at most 2. On the other hand, the Hausdorff dimension of $\text{Sk}(W_{SA})$ is strictly greater than 2. Hence it is a very intriguing question to ask whether $W_{SA}$ is Liouville homotopic to a Weinstein domain. More generally, one can ask whether $W_{SA} \times T^*Y$ is Liouville homotopic to Weinstein for any smooth manifold $Y$.

Example 0.5 (Anosov map). Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the $n$-dimensional torus and $M = D^{n-1} \times T^n$ where $D^{n-1} \subset \mathbb{R}^{n-1}$ denotes the unit ball. Suppose $A \in SL(n, \mathbb{Z})$ has real eigenvalues $\lambda_1, \cdots, \lambda_n$ such that $0 < \lambda_i < |\lambda_i|$ for all $1 \leq i \leq n - 1$. In particular $0 < \lambda_n < 1$. The existence of such $A$ will be established in the Appendix. View $A$ as an automorphism of $T^n$. Then there exists linear 1-forms $\beta_i, 1 \leq i \leq n$, on $T^n$ such that $A^*(\beta_i) = \lambda_i \beta_i$. In particular $\beta_i, 1 \leq i \leq n$, are linearly independent. Define the contact form

$$\alpha := \beta_n + \sum_{1 \leq i \leq n-1} y_i \beta_i$$

on $M$, where $(y_1, \cdots, y_{n-1}) \in D^{n-1}$. The map $\phi_A : M \to M$ defined by

$$\phi_A(y_1, \cdots, y_{n-1}, x) = \left(\frac{\lambda_1}{\lambda_n} y_1, \cdots, \frac{\lambda_n}{\lambda_{n-1}} y_{n-1}, Ax\right)$$

is clearly a contraction. Indeed, we have $\phi_A^2(\alpha) = \lambda_n \alpha$. We denote the resulting Liouville domain by $W_A$. Then the skeleton $\text{Sk}(W_A)$ is a smooth $(n + 1)$-manifold given by the mapping torus of $A : T^n \to T^n$.

For $n = 2$, we recover the examples of Mitsumatsu [Mit95].

APPENDIX A. TORAL AUTOMORPHISM WITH REAL SPECTRUM

The goal of this appendix is to construct hyperbolic toral automorphisms with real spectrum. The material presented here came from a joyful discussion with Luis Diogo, who deserves every bit of this wonderful (but maybe trivial) result, in the summer of 2019 in Uppsala.

Proposition A.1. Fix $n \geq 2$. For any $\epsilon > 0$ and any tuple $(\mu_1, \ldots, \mu_{n-2}) \in \mathbb{R}^{n-2}$, there exists a matrix $A \in SL(n, \mathbb{Z})$ which is diagonalizable in $SL(n, \mathbb{R})$ such that the eigenvalues $\lambda_i, 1 \leq i \leq n$, satisfy

1. $|\lambda_i - \mu_i| < \epsilon$ for $1 \leq i \leq n - 2$.
2. $|\lambda_{n-1}| > 1/\epsilon$ and $|\lambda_n| < \epsilon$.

Proof. Using the Frobenius companion matrix, it suffices to find infinitely many tuples $k := (k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}$ such that the polynomial

$$P_k := x^n - k_{n-1}x^{n-1} + \cdots + (-1)^{n-1}k_1x + (-1)^n$$
has \( n \) roots \( \lambda_1 \ldots \lambda_n \) which satisfy the conditions in the Proposition. Indeed \( P_k \) is the characteristic polynomial of the matrix

\[
A_k := \begin{pmatrix}
0 & 0 & \cdots & 0 & (-1)^{n+1} \\
1 & 0 & \cdots & 0 & (-1)^{n}k_1 \\
0 & 1 & \cdots & 0 & (-1)^{n-1}k_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & k_{n-1}
\end{pmatrix} \in SL(n, \mathbb{Z}),
\]

which is exactly what we look for. Let \( \sigma_i^{(n)} = \sigma_i^{(n)}(\lambda_1, \ldots, \lambda_n), 1 \leq i \leq n \), be the \( i \)-th elementary symmetric polynomial in \( n \) variables, i.e.,

\[
\sigma_i^{(n)} := \sum_{1 \leq j_1 < \cdots < j_i \leq n} \lambda_{j_1} \cdots \lambda_{j_i}.
\]

Then it suffices to find \( k \in \mathbb{Z}^{n-1} \) such that the system of equations

\[
\sigma_i^{(n)}(\lambda_1, \ldots, \lambda_n) = k_i, \quad 1 \leq i \leq n,
\]

is satisfied, where \( k_n := 1 \). Roughly speaking, the strategy consists of three steps: first, eliminate \( \lambda_{n-1}, \lambda_n \) from Eq. (1); second, prescribe a "generic" sequence of numbers \( \lambda_i \) with \( |\lambda_i| < 1, 1 \leq i \leq n-2 \), and find \( k \) using ergodic theory such that Eq. (1) (without \( \lambda_{n-1}, \lambda_n \) is approximately satisfied; third, argue that an exact solution to Eq. (1) exists by a suitable choice of \( |\lambda_{n-1}| < 1, |\lambda_n| > 1 \), and a small perturbation of \( \lambda_i, 1 \leq i \leq n-2 \).

**STEP 1. Elimination of \( \lambda_{n-1}, \lambda_n \) from Eq. (1).**

Let us rewrite Eq. (1) as follows:

\[
\begin{align*}
\sigma_1^{(n-2)} + \lambda_{n-1} + \lambda_n &= k_1; \\
\sigma_j^{(n-2)} + (\lambda_{n-1} + \lambda_n)\sigma_{j-1}^{(n-2)} + \lambda_{n-1}\lambda_n\sigma_{j-2}^{(n-2)} &= k_j, \quad 2 \leq j \leq n-1; \\
\sigma_{n-2}^{(n-2)}\lambda_{n-1}\lambda_n &= 1. 
\end{align*}
\]

Here \( \sigma_{n-1}^{(n-2)} \equiv 0 \) and \( \sigma_0^{(n-2)} \equiv 1 \) by convention. Plugging Eq. (2) and Eq. (4) into Eq. (3), we have

\[
\sigma_j^{(n-2)} + (k_1 - \sigma_1^{(n-2)})\sigma_{j-1}^{(n-2)} + \sigma_{j-2}^{(n-2)}/\sigma_{n-2}^{(n-2)} = k_j, \quad 2 \leq j \leq n-1,
\]

which is a system of equations without \( \lambda_{n-1} \) and \( \lambda_n \). Clearly \( \lambda_{n-1}, \lambda_n \) can be solved easily from Eq. (2) and Eq. (4) once we determine the values of \( \lambda_i, 1 \leq i \leq n-2 \).

**STEP 2. Passing to a discrete dynamical system.**

The idea is that for any fixed \( \lambda_1, \ldots, \lambda_{n-2} \), we can view the left-hand side of Eq. (5) as a discrete dynamical system as \( k_1 \) runs through the integers, while the right-hand side \( (k_2, \cdots, k_{n-1}) \in \mathbb{Z}^{n-2} \) forms a lattice in \( \mathbb{R}^{n-2} \). Then the existence of approximate solutions to Eq. (5) is, roughly speaking, a consequence of the ergodicity of such dynamical system.

The technical heart of this argument is a theorem due to Weyl and von Neumann which we now recall. See [Sin76, Lect. 3] for an excellent exposition on this topic. Let \( T^m = \mathbb{R}^m / \mathbb{Z}^m \) be an \( m \)-dimensional torus. Fix a vector \( r = (r_1, \ldots, r_m) \in \mathbb{R}^m \). Define the translation \( \tau_r : T^m \to T^m \) by \( \tau([x]) = [x + r] \).

**Theorem A.2** (Weyl-von Neumann). The translation \( \tau_r \) is ergodic if and only if \( r \) is irrational, i.e., the components of \( r \) are linearly independent over \( \mathbb{Z} \).
In order to apply Theorem A.2 to our case, choose \( \lambda_i, 1 \leq i \leq n - 2 \), such that the vector
\[
\mathbf{r} := (\sigma_1^{(n-2)}, \ldots, \sigma_{n-2}^{(n-2)}) \in \mathbb{R}^{n-2}
\]
is irrational. For example, it suffices to choose \( \lambda_i, 1 \leq i \leq n - 2 \), to be algebraically independent. In particular \( \lambda_i \neq \lambda_j \) whenever \( i \neq j \). Let’s rewrite Eq. (5) as
\[
(\mathbf{x}(\mathbf{r}) + k_i \mathbf{r}) = \mathbf{k}',
\]
where \( \mathbf{k}' := (k_2, \ldots, k_{n-1}) \in \mathbb{Z}^{n-2} \) and \( \mathbf{x}(\mathbf{r}) := (x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) with
\[
x_j := \sigma_j^{(n-2)} - \sigma_1^{(n-2)} \sigma_j^{(n-2)} + \sigma_j^{(n-2)} / \sigma_j^{(n-2)},
\]
for \( 2 \leq j \leq n - 1 \). It follows from Theorem A.2 that for any \( \epsilon > 0 \) and \( K > 0 \), there exists \( k_1 > K \) and \( \mathbf{k}' \in \mathbb{Z}^{n-2} \) such that \( |x + k_1 \mathbf{r} - \mathbf{k}'| < \epsilon \). Switching point of view, one can think of the prescribed tuple \( (\lambda_1, \ldots, \lambda_{n-2}) \) as an approximate solution to Eq. (5) with suitable choices of \( k_i, 1 \leq i \leq n - 1 \).

**STEP 3. From approximate solutions to exact solutions.**

Observe that the tuple \( (\lambda_1, \ldots, \lambda_{n-2}) \) uniquely determines the vector \( \mathbf{r} \) which approximately solves Eq. (6) with \( k_1 > K \). By choosing \( K \) sufficiently large, there exists an exact solution \( \mathbf{r}' \) of Eq. (6) which is close to \( \mathbf{r} \). It remains to argue that \( \mathbf{r}' \) corresponds to a tuple \( (\lambda_1', \ldots, \lambda_{n-2}') \) which is close to \( (\lambda_1, \ldots, \lambda_{n-2}) \). Indeed, consider the map \( \Pi: \mathbb{R}^{n-2} \to \mathbb{R}^{n-2} \) defined by
\[
\Pi(\lambda_1, \ldots, \lambda_{n-2}) = (\sigma_1^{(n-2)}, \ldots, \sigma_{n-2}^{(n-2)}).
\]
The Jacobian \( \text{Jac}(\Pi) \neq 0 \) if \( \lambda_i, 1 \leq i \leq n - 2 \), are algebraically independent. By the Inverse Function Theorem, \( \Pi^{-1}(\mathbf{r}') \) exists and is close to \( (\lambda_1, \ldots, \lambda_{n-2}) \). Abusing notations, let us write \( \Pi^{-1}(\mathbf{r}') = (\lambda_1, \ldots, \lambda_{n-2}) \).

To wrap up the proof, let us rewrite Eq. (2) and Eq. (4) as follows:
\[
(\sigma_1^{(n-2)} + \cdots + \sigma_{n-2}^{(n-2)} + \cdots + \sigma_{n-2}^{(n-2)}) + 1 = 1 / \sigma_{n-2}^{(n-2)}.
\]
Since both \( \sigma_1^{(n-2)} \) and \( \sigma_{n-2}^{(n-2)} \) are finite numbers, Eq. (7) admits a solution \( (\lambda_{n-1}, \lambda_n) \) with \( |\lambda_{n-1}| > 1 / \epsilon \) and \( |\lambda_n| < \epsilon \) as long as \( k_1 > K \) is sufficiently large.

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