Positive Unknown Input Observer Design for Positive Linear Systems

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Abstract—Positive systems are important class of dynamic systems with impressive properties. The response of such systems to positive initial conditions and positive inputs remain in the nonnegative orthant of the state space. Although positive observers have been designed for positive systems, they are unable to estimate the states when unknown inputs or disturbances are present in the systems. This paper is a first attempt to design positive unknown input observers (PUIO) for positive linear systems. The structural constraints on observer parameters make the design task cumbersome. However, with the aid of a positive stabilization scheme via LMI and by imposing conditions on positivity of the generalized inverse associated with a certain design matrix, we provide a reliable procedure for the design of PUIOs.

I. INTRODUCTION

Observers have found proven advantages in many different scenarios. The development of observer theory has rapidly evolved over the past several decades according to system classification (linear, nonlinear, time varying, etc) and various observers have been considered for special classes of systems (delay, description, positive, etc). Observers were also classified based on the goals of the designer or the requirements imposed by the design objectives. The most important use of observers was to realize observer-based controllers in optimal and robust control due to the inevitable presence of uncertainty in the system. A subset of the contributions in theory and applications of observers can be found in [11], [2], [3] (see also the references therein).

The application of observers was also evident in disturbance estimation and fault detection [4]. Among different observer structures, the unknown input observer (UIO) and proportional integral observer (PIO) were commonly employed for this purpose. Although the design procedures of PIO and UIO for general linear time invariant systems are reported by many researchers [5], [6], [7], [8], [9], [10] it is not clear how these observers should be designed for the important class of positive systems. There are many physical systems belonging to this class whose trajectories remain in the nonnegative orthant of the state space when excited by positive inputs. Positive systems offer nice stability properties that can be used in positive stabilization of general unstable systems by imposing positivity constraints. It is well known that positive stabilization by state and output feedback for linear systems (regardless of being positive or not) is possible and various design techniques based on LP and LMI are proposed [11], [12], [13], [14], [15], [16]. On the other hand, the design of positive observers to estimate the states of a non-positive system is meaningless. Thus, the observer design requires compatibility between the original system states and the observer states. Consequently, the state estimate of a positive system requires a positive observer [17], [18], [19], [20]. This means that we are facing the challenging problems of designing positive PIO and PUIO for positive systems. The design of positive PIO has been tackled in [21]. However, the design of PUIO has not yet been investigated. This paper is concerned with the design of PUIO based on the obvious motivation that disturbances and faults on positive systems are generic as in the regular systems. Furthermore, the published results on positive observer designs cannot be used to estimate the states of positive systems with unknown disturbances. Section II of the paper is devoted to an overview of positive systems and previous results on positive observer designs. Although some of the results are not new, we provide connections and draw conclusions. Section III presents the main contribution of the paper, namely the design of PUIO, and Section IV includes numerical examples. Before closing this section we define the notation regarding the positivity and the positive definiteness of matrices. We prefer to use the notation $\mathbf{P} \succeq 0$ for a positive definite (positive semi-definite) matrix. On the other hand, we use the notation $\mathbf{P} > 0$ ($\mathbf{P} \succeq 0$) for positive (nonnegative) numbers, vectors and matrices. Thus $A > 0$ ($A \geq 0$) represents a positive (nonnegative) matrix. On the other hand, we use the notation $P > 0$ ($P \succeq 0$) for a positive definite (positive semi-definite) symmetric matrix $P = P^T$. In a similar manner one can define the negativity and the negative definiteness by reversing the associated signs.

II. PRELIMINARY DEVELOPMENT AND PREVIOUS RESULTS

A. Positive Systems

Consider the general linear time invariant system

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\
y(t) &= Cx(t)
\end{align}

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $d \in \mathbb{R}^r$ are state, input, output and disturbance vectors; respectively and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $E \in \mathbb{R}^{n \times r}$ are associated system matrices with $\{A, B\}$ controllable and $\{A, C\}$ observable pairs. The following definition is a slight generalization of the
well known definition of positive systems and the subsequent results are standard [22, 23, 24].

Definition 1: The system in Equations (1), (2) is called internally positive or Metzlerian system if every initial condition, \( x_0 \in \mathbb{R}^n_+ \), and all inputs \( u(t) \in \mathbb{R}^p_+ \), \( d(t) \in \mathbb{R}^r_+ \), \( t \geq 0 \) we have \( x(t) \in \mathbb{R}^n_+ \) and \( y \in \mathbb{R}^r_+ \) for \( t \geq 0 \).

Lemma 2.1: The system in Equations (1), (2) is internally positive if and only if \( A \in \mathbb{R}^{n \times n}_+ \) is a Metzler matrix and \( B \in \mathbb{R}^{p \times n}_+ \), \( C \in \mathbb{R}^{1 \times p}_+ \), and \( E \in \mathbb{R}^{1 \times r}_+ \) are nonnegative matrices.

Note that a matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is called a Metzler matrix if all of its off-diagonal elements are nonnegative, i.e. \( a_{ij} \geq 0, \forall i \neq j, i, j = 1, \ldots, n \). Furthermore, it is called strictly Metzler if in addition \( a_{ii} < 0 \) \( \forall i = 1, \ldots, n \). Every Metzler matrix \( A \) has a real eigenvalue \( \mu = \lambda_{\text{max}}(A) = \max \Re(\lambda_i) \) \( \forall i = 1, \ldots, n \) and a corresponding eigenvector \( v_{\text{max}} \geq 0 \). If \( \mu < 0 \) then \( \Re(\lambda_i) < 0 \) \( \forall i = 1, \ldots, n \) where \( \lambda_i \) are the eigenvalues of \( A \).

Lemma 2.2: Let the system in Equations (1), (2) be Metzlerian. Then it is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All eigenvalues of the \( A \) have negative real parts.
2. All coefficients of the characteristic polynomial \( \det(\lambda I - A) = \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 \) are positive, i.e. \( a_i > 0 \) \( \forall i = 0, 1, \ldots, n-1 \).
3. All leading principal minors of \( -A \) are positive.
4. The matrix \( A \) is nonsingular and \( -A^{-1} > 0 \).
5. There exists a positive definite (possibly diagonal) matrix \( P \) such that \( A^T P + PA < 0 \).
6. There exists a vector \( v \in \mathbb{R}^n_+ \) such that \( Av < 0 \).

B. Previous Design Approaches for Positive Observers

When \( d = 0 \), a full order Luenberger observer for the above system is defined by

\[
\dot{x} = (A - LC)\dot{x} + Ly + Bu
\]  

(3)

Since the pair \( \{A, C\} \) is observable, it is always possible to find \( L \in \mathbb{R}^{n \times p} \) such that all the eigenvalues of \( A - LC \) are arbitrarily assigned in the left half of the complex plane. If one defines the error vector \( e = x - \dot{x} \), then the error dynamics are governed by \( \dot{e} = (A - LC)e \) and \( \lim_{t \to \infty} e(t) = 0 \) is guaranteed with the stable matrix \( A - LC \).

The situation with positive observers is not trivial due to the structural constraint of the matrix \( A - LC \) to be Metzlerian. To design a positive observer it is imperative that Eqs (1), (2) constitute a positive system.

Lemma 2.3: Let \( A \) be a Metzler matrix and let \( Z_1, Z_2 \in \mathbb{R}^{n \times n}_+ \) be such that \( Z_1 \leq Z_2 \). Then \( \lambda_{\text{max}}(A - Z_1) \geq \lambda_{\text{max}}(A - Z_2) \) if \( A - Z_1 \) and \( A - Z_2 \) are Metzlerian matrices.

Using the above definition and lemmas, the authors of [17] established the following result for single output systems.

Theorem 2.1: Given a Metzler matrix \( A \in \mathbb{R}^{n \times n}_+ \) and a nonnegative vector \( C \in \mathbb{R}^{1 \times n}_+ \), let \( L_0 = [l_1, l_2, \ldots, l_n]^T \in \mathbb{R}^{n \times 1}_+ \) be a nonnegative vector defined by

\[
\begin{align*}
n_i &> \frac{a_{ii}}{c_i}, & \text{if } c_j = 0, \forall j \neq i \\
li &\geq \min_{j \neq i, c_j \neq 0} \{ a_{ij} \}, & \text{otherwise}
\end{align*}
\]

(4)

then there exists a nonnegative vector \( L \in \mathbb{R}^{n \times 1}_+ \) such that \( A - LC \) is a stable Metzler matrix if and only if \( \lambda(A - L_0C) < 0 \).

The same authors provided a generalization of the above theorem for multi-output systems. However, their procedure of constructing positive observers is not trivial and inconvenient for high order systems. Therefore, a more systematic approach was proposed in [25] for positive time delay systems, which is adopted here for regular positive systems.

Theorem 2.2: Given a positive system of the form in Eqs (1), (2) without the disturbance term \( d(t) \). Then there exists a positive observer of the form in Eq (3) if and only if one of the following equivalent conditions is satisfied

1. \( A - LC \) is a Metzler stable matrix and \( LC \geq 0 \).
2. The following LP problem in the variable \( w \in \mathbb{R}^n \) and \( Z \in \mathbb{R}^{p \times n}_+ \) is feasible

\[
\begin{align*}
A^T w - C^T Z &\geq 0 \\
A^T \text{diag}(w) - C^T Z + I &\geq 0 \\
C^T Z &\geq 0 \\
w &> 0
\end{align*}
\]

(5)

Moreover, the gain matrix \( L \) for the observer can be obtained as

\[
L = [\text{diag}(w)]^{-1} Z^T
\]

(6)

where \([\text{diag}(w)]\) is a diagonal matrix whose diagonal is formed by the components of the vector \( w \).

3. The following LMI in the variables \( P \in \mathbb{R}^{n \times n}_+ \) and \( Y \in \mathbb{R}^{n \times p}_+ \) is feasible

\[
\begin{align*}
A^T P + PA - C^T Y^T - Y C &< 0 \\
A^T P - C^T Y^T + I &\geq 0 \\
YC &\geq 0 \\
P &\geq 0
\end{align*}
\]

(7)

Moreover, the gain matrix \( L \) for the observer can be obtained as

\[
L = P^{-1} Y
\]

(8)

Two other distinct approaches to the design of positive observers have been reported in [18] and [19]. The first approach is based on the fact that even though an observer is not positive with respect to a coordinate system, it can
be positive with respect to a different coordinate system. The authors of [13] define the following observer structure

\[ \dot{\hat{x}} = F\hat{x} + Gy + Hu \]

where \( F \) is a stable Metzler matrix, \( G \) and \( H \) are matrices, and \( T \) is a solution of the Sylvester equation \( TA - FT = GC \).

Theorem 2.3: Consider a positive system (1), (2) without the disturbance term \( d(t) \). Then there exists a positive observer of the form (9), if there exists \( F, G, H \) and \( T \) such that the following conditions are satisfied

1) \( F \in \mathbb{R}^{n \times n} \) is a stable Metzler matrix, \( G \in \mathbb{R}^{n \times p} \), and \( H \in \mathbb{R}^{n \times m} \).
2) \( T \) is a solution of the Sylvester equation \( TA - FT = GC \). (10)
3) \( T \) is invertible and \( T^{-1} \in \mathbb{R}^{n \times n} \).

Based on the above theorem one can apply different techniques to design the positive observers. They all depend on how to solve the Sylvester equation (10). One simple procedure is readily available from standard textbooks, which has also been suggested in [18]. The procedure starts with a choice of controllable pair \( \{F, G\} \) such that \( F \) is a stable Metzler matrix and \( G \) under the condition that \( A \) and \( F \) do not have common eigenvalues. Then (10) is solved for \( T \) such that \( T^{-1} \in \mathbb{R}^{n \times n} \). If \( T \) does not meet this requirement, then one should repeat the process with another pair of matrices \( \{F, G\} \). Although the Sylvester approach enlarges the possibility of designing positive observers, it has two main drawbacks. First, the nonsingularity of \( T^{-1} \) should be realized and the second obstacle is the inverse eigenvalue problem associated to \( F \), which requires construction of a stable Metzler matrix \( F \) with prescribed set of eigenvalues. Fortunately, the inverse eigenvalue problem, which is known to be a hard problem, has been resolved by the first author of this paper and will be reported in a separate publication. To overcome the first obstacle, one may ignore solving (10) and set \( T = I \). It is obvious that with this \( T \), the problem reduces to the construction of Luenberger type observers as previously discussed. However, if we define (9) with the general parameter matrices \( F, G, H \) with \( \hat{x} = \hat{x} \), and \( H = B \), then we have

\[ \dot{\hat{x}}(t) = F\hat{x}(t) + Gy(t) + Bu(t) \]

whereby the following Theorem is applicable.

Theorem 2.4: Given a positive system of the form (1), (2) without the disturbance term \( d(t) \). Then there exists a positive observer of the form (11) if and only if there exist diagonal matrices \( P > 0 \), \( Q > 0 \), a Metzler matrix \( M \), and a nonnegative matrix \( N \geq 0 \) such that the following LMI has a feasible solution

\[
\begin{bmatrix}
PA + ATP & A^T Q - C^T N T - MT \\
QA - NC - M & M + MT \\
QA - NC - M & MN - M \geq 0.
\end{bmatrix} < 0
\]

\[ (12) \]

Moreover, the observer matrices \( F \) and \( G \) are obtained by

\[ F = Q^{-1} M, \ G = Q^{-1} N. \]

Proof: Note that theorem 2.2 is established based on the stability of the error dynamics and the positivity of the augmented system with the aid of Lemma 2.2 as applied to \( A - LC \). However, using the above structure instead of the Luenberger observer (3), the augmented system with the state \( x(t) \) and the error \( e(t) \) is given by

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
-A - GC & F
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\]

\[ (15) \]

Then by imposing the stability of (15) along with the positivity of (11) i.e. \( F \) to be a Metzler matrix and \( G \geq 0 \), one can establish the existence of the positive observer. Applying the Lyapunov inequality with blockdiag\{P,Q\} to the augmented system matrix in (15), we have

\[
\begin{bmatrix}
PA + AT P & (A^T - C^T N T - MT)Q \\
QA - NC - M & QF + F^T Q
\end{bmatrix} < 0
\]

\[ (12) \]

Denoting \( QF = M \) as a Metzler matrix and \( QG = N \geq 0 \) as a nonnegative matrix we obtain the LMI in (12). It is not difficult to see that the feasibility of the LMI in (12) with the side constraint (13) is a necessary and sufficient condition for the existence of positive observer.

Remark 1: An immediate connection between Theorem (2.4) and Theorem (2.3) is the fact that while in the Sylvester approach one seeks to find appropriate pairs \( \{F, G\} \) with trial and error, Theorem (2.4) searches for the feasible pairs \( \{F, G\} \) systematically by solving LMI (12). (13)

Finally, the authors of [19] provide yet another approach based on positive realization. The basic idea behind this formulation is the fact that if we regard the observer dynamics by a transfer function whose input is the system output and its output is the state estimate, then a positive realization of this transfer function is the solution of the problem. The existence of positive realization and the procedure of constructing positive observers are important steps. One may refer to [18], [19] for further details.

III. Positive Unknown Input Observer (PUIO)

A. Structure of Positive Unknown Input Observers

The previous section discussed various positive observer structures and design strategies for them when \( d(t) = 0 \). We also established certain connections between two approaches and drew interesting conclusions. Unfortunately, when \( d(t) \neq 0 \), none of the above approaches are applicable to design a positive observer to estimate the states of a positive system (1), (2) decoupled from the unknown input \( d(t) \).

From the existing literature, it is well known that an UIO can be designed under the conditions \( rank(C) = p \), \( rank(E) = r \), where \( p \geq r \) and \( rank(CE) = rank(E) \). Let us define the UIO structure as it is usually the case for regular systems, i.e.

\[ \dot{z} = Fz + Gy + Hu \]

\[ \dot{\hat{x}} = Mz + Ny \]
where for simplicity we assume $M = I$. To facilitate the derivation of the design, we decompose $G = G_1 + G_2$ and let $H = TB$ with $T$ being a design parameter, which will be defined later. Note that the UIO only uses the available input and output in order to estimate the states.

**Lemma 3.1:** Let the UIO (10) be an internal positive or Metzlerian system (PUIO), i.e. for every initial state $z(0) \in \mathbb{R}_{+}^{n}$ and all inputs $u(t) \in \mathbb{R}_{+}^{n}$, $t \geq 0$ we have $z(t) \in \mathbb{R}_{+}^{n}$, $\hat{x}(t) \in \mathbb{R}_{+}^{n}$, and $y \in \mathbb{R}_{+}^{n}$ for all $t \geq 0$. Then the PUIO is internally positive if and only if $F \in \mathbb{R}^{n \times n}$ is a Metzler matrix, $G \in \mathbb{R}^{q \times n}$, $H \in \mathbb{R}^{q \times m}$, and $N \in \mathbb{R}^{q \times p}$ are nonnegative matrices.

**Proof:** It is a trivial consequence of definition (1) and lemma (2.1). 

**B. Generalized Inverse of a Nonnegative Matrix**

The concept of generalized inverse of a nonnegative matrix plays a key role in the design of PUIO as will be shown later. If $\tilde{A}$ is an arbitrary $m \times n$ real matrix, then the Moore-Penrose generalized inverse of $\tilde{A}$ is the unique $n \times m$ real matrix, $\tilde{A}^{g}$, satisfying the equalities (i) $\tilde{A}^{g} \tilde{A} = \tilde{A} \tilde{A}^{g}$, (ii) $\tilde{A} = \tilde{A}^{g} \tilde{A}$, (iii) $(\tilde{A}^{g})^{T} = \tilde{A} \tilde{A}^{g}$, and $(\tilde{A}^{g} \tilde{A})^{T} = \tilde{A} \tilde{A}^{g}$. The properties and applications of $\tilde{A}^{g}$ are discussed in most linear algebra and matrix theory text books. Depending on being a fat or tall matrix with full rank assumption in an algebraic equation $\tilde{A} x = b$, one can obtain right or left inverses leading to the solutions of $\tilde{x}$.

If $\tilde{A}$ is nonnegative, then $\tilde{A}^{g}$ is not necessarily nonnegative. For a square nonsingular and nonnegative matrix $\tilde{A}$, we have $\tilde{A}^{g} = \tilde{A}^{-1} \geq 0$ if and only if $\tilde{A}$ is monomial (or generalized permutation matrix) i.e. there is exactly one nonzero entry in each row and each column. Note that in a permutation matrix (i.e. there is exactly one nonzero entry in each row and each column), $\tilde{A}$ is a diagonal matrix with full rank assumption in an algebraic equation $\tilde{A} x = b$, one can obtain right or left inverses leading to the solutions of $\tilde{x}$.

**Lemma 3.2:** Let $\tilde{A}$ be an $m \times n$ nonnegative matrix of rank $r$. Then the following statements are equivalent:

1) $\tilde{A}^{g}$ is nonnegative.

2) There exists a permutation matrix $\tilde{P}$ such that $\tilde{P} \tilde{A}$ has the form

$$\tilde{P} \tilde{A} = \begin{bmatrix}
\tilde{B}_1 \\
\vdots \\
\tilde{B}_r \\
0
\end{bmatrix}$$

where each $\tilde{B}_i$ has rank 1 and the rows of $\tilde{B}_i$ are orthogonal to the rows of $\tilde{B}_j$ for $i \neq j$.

3) $\tilde{A}^{g} = D \tilde{A}^{T}$ for some diagonal matrix $D$ with positive diagonal elements.

The proof of the above Lemma can be found in [26]. However, with the aid of this result one can construct $\tilde{A}^{g}$. Assuming 2) holds, let $\tilde{B} = \tilde{P} \tilde{A}$ have the form specified above. Then for $1 \leq i \leq r$, there exists column vectors $x_i, y_i$ such that $\tilde{B}_i = x_i y_i^{T}$. Furthermore, $\tilde{B}_i^{g}$ is the nonnegative matrix:

$$\tilde{B}_i^{g} = (\|x_i\|^{2} \|y_i\|^{2})^{-1} \tilde{B}_i^{T}$$

and moreover $\tilde{B}^{g} = (\tilde{B}_1^{g}, \ldots, \tilde{B}_r^{g}, 0)$, since $\tilde{B}_i \tilde{B}_j = 0$ for $i \neq j$. In particular, $\tilde{B}^{g} = D \tilde{B}^{T}$ where $D$ is a diagonal matrix with positive diagonal elements and thus $A^{g} = D \tilde{A}^{T}$.

**C. Design of PUIO**

Using Lemma (3.1) and Lemma (3.2), we are now ready to state the main result of the paper, namely the design of PUIO.

**Theorem 3.1:** Given a positive system of the form in Eq (1), with the unknown disturbance term $d(t) \neq 0$ such that the generalized left inverse of $CE$ is nonnegative. Then there exists a PUIO of the form (10) if and only if $CE$ is a stable Metzler matrix and the following conditions are satisfied

$$\begin{align*}
F &= A - NCA - G_1 C \\
T &= I - NC \geq 0 \\
G_2 &= FN \\
G &= G_1 + G_2 \geq 0 \\
H &= TB \\
(NC - I)E &= 0, \quad N \geq 0
\end{align*}$$

**Proof:** Using (1), (2) and (10) the error dynamics of the PUIO is easily derived as

$$\dot{e} = (A - NCA - G_1 C)e + [F - (A - NCA - G_1 C)]z + [G_2 - (A - NCA - G_1 C)N]y + [T - (I - NC)]Bu + (NC - I)Ed$$

It is clear that if the conditions of the theorem are satisfied, then $\dot{e} = Fe$ and $\lim_{t \to \infty} e(t) = 0$. Equivalently, choosing a Lyapunov function $V(t) = e(t)^{T} Pe(t)$ we have

$$\dot{V}(t) = e^{T}(FT P + PF)e$$

and $FT P + PF < 0$ implies that $e(t)$ tends to zero asymptotically for any initial value $e(0)$. Note that when $F$ is a stable Metzler matrix, then there exists a positive definite (possibly diagonal) matrix $P > 0$, which is also positive, satisfying the Lyapunov inequality.

The existence conditions of the PUIO depends on the positive solution of (22). This equation is solvable if and only if the condition $\text{rank}(CE) = \text{rank}(E) = r$ is satisfied provided that a nonnegative left inverse of $CE$ exists and guarantees the nonnegativity of $N$ from

$$N = E(CE)^{g} + S[I - (CE)(CE)^{g}]$$

where $S \in \mathbb{R}^{q \times p}$ is an arbitrary matrix. For simplicity we assume $S = 0$ and require that

$$N = E(CE)^{g} \geq 0$$

Since it is assumed that $(CE)^{g} \geq 0$, the matrix $N$ becomes nonnegative. Furthermore, it is required that $N$ satisfies the
nonnegativity of $T$ in (18). If (24) fails to achieve this, (23) can be employed with the aid of free parameter matrix $S$. Accordingly, the design of PUIO amounts to solving for the remaining unknown matrices $F, G,$ and $H$. This can be done based on a design procedure outlined below.

Design Procedure A:

1) Check if $rank(CE) = rank(E) = r$. If not, then an unknown input observer does not exist. Otherwise, go to the next step.

2) Check the condition of Lemma (3.2). If a nonnegative left inverse of $CE$ exists, then compute $N$ from (23) or (24) such that (18) is satisfied and continue to the next step, otherwise stop.

3) Define $A_1 = A - NCA$ in (17) such that $\{A_1, C\}$ is observable.

4) Solve the following LMI for the variables $P \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{n \times p}$ with respect to $F = A_1 - G_1 C$

$$
\begin{align*}
A_1^T P + PA_1 - C^T Y^T - YC &\prec 0 \\
A_1^T P - C^T Y^T + I &\succeq 0 \\
YC &\succeq 0 \\
P &\succ 0
\end{align*}
$$

(25)

and obtain

$$
G_1 = P^{-1} Y \quad F = A_1 - G_1 C
$$

(26)

5) Compute $T$ and $G_2$ from (18) and (19).

6) Repeat the above steps until (20) is satisfied and specify the observer parameters. Otherwise a PUIO does not exist.

In order to design the PUIO from theorem (3.1), it is clear that we only need to find $N \succeq 0$, $G_1$, and $P$ such that

$$
F^T P + PF \prec 0 \quad (27)$$

$$
(NC-I)E = 0 \quad (28)
$$

under the constraint that $F = A_1 - G_1 C$ is a stable Metzler matrix, where from (18) and (19) $A_1 = (I - NC)A = TA$. Substituting the solution of (28) given by (24) into (27), we get

$$
A_1^T P + PA_1 - C^T Y^T - YC \prec 0
$$

(29)

where $Y = PG_1$. Imposing the condition of a stable Metzler matrix leads to

$$
A_1^T P - C^T Y^T + I \succeq 0.
$$

(30)

Finally (20) can alternatively be written as

$$
G = G_1 + A_1 N - G_1 CN \succeq 0
$$

(31)

which we may also rewrite it with the aid of $PG \succeq 0$ as

$$
Y + PA_1 N - YCN \succeq 0
$$

(32)

Consequently, based on the above development, we state an alternative theorem for the existence of PUIO in terms of an extended LMI.

**Theorem 3.2:** Given a positive system of the form (1), (2) with the unknown disturbance term $d(t) \neq 0$ such that $rank(CE) = rank(E) = r$ and the generalized left inverse of $CE$ is nonnegative. Then there exists a PUIO of the form (15) if and only if the following extended LMI has a feasible solution for $Y$ and a symmetric positive definite matrix $P$

$$
\begin{align*}
A_1^T P + PA_1 - C^T Y^T - YC &\prec 0 \\
A_1^T P - C^T Y^T + I &\succeq 0 \\
Y + PA_1 N - YCN &\succeq 0 \\
YC &\succeq 0 \\
P &\succ 0
\end{align*}
$$

(33)

Note that once $P$ and $Y$ are obtained from the above LMI, the remaining observer matrices can be determined. So the design procedure outlined before can be modified based on the above theorem as follows.

**Design Procedure B:**

1) Perform steps 1 & 2 of design procedure A and compute $N$ from (23) or (24) such that (18) is satisfied. If the conditions in these steps are not met, then stop.

2) Compute $T = I - NC$ and define $A_1 = TA$ such that $\{A_1, C\}$ is observable.

3) Solve the LMI (33) for the variable $P$ and $Y$, and compute $G_1 = P^{-1} Y$.

4) Obtain the parameter matrices of PUIO from

$$
F = A_1 - G_1 C \\
G = G_1 + A_1 N - G_1 CN \\
H = TB
$$

Note that $M = I$ and $N$ is specified by (23) or (24).

**Remark 2:** The matrix $T$ in both design procedures is required to be positive by proper selection of $N$. If this requirement is relaxed, then one should impose $H = TB \succeq 0$. Relaxation of $T$ allows one to have more degrees of freedom for selecting $A_1$, which is not required to be Metzler or any special structure. As long as $\{A_1, C\}$ is an observable pair one can find $G_1$ such that $A_1 - G_1 C$ is a stable Metzler matrix if the associated LMI is feasible. This fact will further be explored in the numerical example.

IV. ILLUSTRATIVE EXAMPLES

**Example 1:** Consider the following controllable and observable system

$$
\begin{align*}
\dot{x} &= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d \\
y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x
\end{align*}
$$

Since $rank(CE) = rank(E) = r = 1$ it is possible to design an UIO. The condition of Lemma (3.2) in step 2 is satisfied and
one can trivially construct $N$, which guarantees the positivity of $T$ as

$$N = E(CE)^9 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T = I - NC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Defining $A_1$ using step 3, we obtain

$$A_1 = (I - NC)A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Solving the LMI in step 4 of the design procedure, we can obtain $G_1$ & $F$ from (26)

$$G_1 = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ -0.5 & 0 \end{bmatrix}$$

$$F = A_1 - G_1C = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -2 & 0 \\ 0.5 & 1 & -1 \end{bmatrix}$$

Then $G_2$ is computed from (19)

$$G_2 = FN = \begin{bmatrix} -3 & 0 \\ 2 & 0 \\ 0.5 & 0 \end{bmatrix}$$

and we obtain the matrices $G$ and $H$ of the observer

$$G = G_1 + G_2 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad H = TB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the desired PUIO is given by

$$\dot{z} = Fz + Gy + Hu$$

$$\dot{x} = Mz + Ny, \quad \text{where} \quad M = I$$

It is interesting to point out that if we use (23) instead of (24) with the free parameter matrix $S = [s_{ij}]$, which is a $3 \times 2$ matrix in this example, then all its elements $s_{ij}$ can be chosen arbitrary whereby the positivity of matrix $T$ is relaxed. However, the positivity requirement is shifted to $H = TB$ in (21) which is automatically satisfied. On the other hand, if we require $T \geq 0$, then simple calculations show that $s_{11}$ for $i = 1, 2, 3$ can be chosen arbitrary and the remaining elements should satisfy $s_{12} \leq 0$, $0 \leq s_{22} \leq 1$, $s_{32} \leq 0$.

Additional examples will be provided in the final version of the paper.

V. Conclusion

We have provided results to design positive unknown input observers (PUIO) for positive linear systems. The structural constraints on observer matrices that make the design task challenging were addressed by an LMI formulation along with conditions on positivity of the generalized inverse associated with a certain design matrix. We provided a reliable procedure for the design of PUIOs and gave an illustrative example.