AVERAGING PRINCIPLE FOR SLOW-FAST STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH HÖLDER CONTINUOUS COEFFICIENTS

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Abstract. By using the technique of Zvonkin’s transformation and the classical Khasminkii’s time discretization method, we prove the averaging principle for slow-fast stochastic partial differential equations with Hölder continuous coefficients. An example is also provided to explain our result.

1. Introduction

In this paper, we consider the following stochastic partial differential equation in a Hilbert space $H$:

$$
\begin{align*}
\frac{dX^\varepsilon_t}{dt} &= [AX^\varepsilon_t + B(X^\varepsilon_t, Y^\varepsilon_t)]dt + \sqrt{Q_1}dW^1_t, \\
\frac{dY^\varepsilon_t}{dt} &= \frac{1}{\varepsilon}[AY^\varepsilon_t + F(X^\varepsilon_t, Y^\varepsilon_t)]dt + \frac{1}{\sqrt{\varepsilon}}\sqrt{Q_2}dW^2_t,
\end{align*}
$$

(1.1)

where $\varepsilon > 0$ is a small parameter describing the ratio of the time scales of the slow component $X_t^\varepsilon$ and the fast component $Y_t^\varepsilon$, $A : \mathcal{D}(A) \to H$ is the infinitesimal generator of a linear strongly continuous semigroup $\{e^{tA}\}_{t \geq 0}$, $B$ and $F$ are appropriate continuous functions, $Q_1$ and $Q_2$ are two non-negative selfadjoint bounded operators in $H$, $\{W^1_t\}_{t \geq 0}$ and $\{W^2_t\}_{t \geq 0}$ are $H$-valued mutually independent cylindrical Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

The multiscale system (1.1) has wide applications in material sciences, fluid dynamics, biology, ecology, climate dynamics, see e.g., [1, 13, 20, 23] and the references therein. The averaging principle of multiscale model is essential to describe the asymptotic behavior of slow component as $\varepsilon \to 0$, i.e., the slow component will convergence to the so-called averaged equation. This theory was first developed for the ordinary differential equations (ODEs for short) by Bogoliubov and Mitropolsky [2], and extended to the stochastic differential equations (SDEs for short) by Khasminskii [21], also see [24, 25]. Since the averaging principle for a general class of stochastic reaction-diffusion systems with two time-scales was investigated by Cerrai and Freidlin in [6], the averaging principle for slow-fast stochastic partial differential equations (SPDEs for short) has been drawn much attentions in the past decades, see e.g., [3, 4, 5, 7, 12, 14, 15, 16, 17, 18, 26, 31, 32] and the references therein.

It is easy to find that all the references mentioned above assumed that the coupled coefficients $B$ and $F$ satisfy at least local Lipshitz continuous condition. However, it was shown in [9] that system (1.1) can be strongly well-posed with only Hölder continuous drift coefficients, see also [10, 11] for further generalizations. Thus it is natural to ask that whether

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the averaging principle still holds under such kind of conditions. As far as we know, the averaging principle for stochastic system with irregular coefficients has not been studied much yet. Even in the case of SDEs, there are only few results in this direction. Veretennikov [30] studied the averaging principle for SDEs under the assumptions that the drift coefficient in slow equation is only bounded and measurable with respect to slow variable, and all the other coefficients are global Lipschitz continuous. Röckner and authors [29] studied the strong and weak convergence in the averaging principle for SDEs with Hölder coefficients drift, also see [27, 28] for the study of diffusion approximations for SDEs with singular coefficients.

Hence, the main purpose of this paper is to prove the strong averaging principle for SPDE (1.1) with bounded and Hölder coefficients $B$ and $F$. More precisely, one tries to prove that for any $p \geq 1$,

$$
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^p \right) = 0,
$$

where $\bar{X}_t$ is the solution of the corresponding averaged equation (see Eq. (2.11) below). To the best of our knowledge, this paper seems to be the first research which studies the averaging principle for slow-fast SPDEs with irregular coefficients.

Our argument is inspired from [29, 30]. Two ingredients will play important roles in the proof: the classical Khasminskii’s time discretization and the Zvonkin’s transformation, which is now widely used to study the strong well-posedness for S(P)DEs with singular coefficients, see e.g. [22, 9, 10, 11]. We point out that unlike the nice regularity for the coupled coefficients with respect to the fast component in [30], we here consider the coupled coefficients are irregular with respect to slow and fast variables. As a consequence, the Zvonkin’s transformation is also used to obtain an estimate of the difference between the fast process $Y^\varepsilon$ and its auxiliary processes $\hat{Y}^\varepsilon$. Meanwhile, by a elaborate estimate of the mild solution, we can control the integral of the time increment of solution, which plays an important role in the proof and releases the regularity of the initial value $x$.

The rest of the paper is organized as follows. In Section 2, we first give some notations and suitable assumptions, then we present our main result and give a direct-viewing the idea of the key technique. Section 3 is devoted to proving our main result. In Section 4, we will give an examples to illustrate the applicability of our result.

Throughout the paper, $C, C_p, C_T$ and $C_{p,T}$ denote positive constants which may change from line to line, where the subscript $p, T$ are used to emphasize that the constant only depends on the parameters $p, T$.

2. Notations and main results

2.1. Notations and assumptions. Let us first introduce some notations. The inner product and the norm of $H$, which are denoted by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ respectively. We assume the following conditions throughout the paper:

**A1.** $B, F : H \times H \to H$ are measurable and bounded. Moreover, there exist constants $\alpha, \beta, \gamma, \eta \in (0, 1]$ and $C > 0$ such that for any $x_1, x_2, y_1, y_2 \in H$,

$$
|B(x_1, y_1) - B(x_2, y_2)| \leq C \left( |x_1 - x_2|^\alpha + |y_1 - y_2|^\beta \right);
$$

$$
|F(x_1, y_1) - F(x_2, y_2)| \leq C \left( |x_1 - x_2|^\gamma + |y_1 - y_2|^\eta \right).
$$
A2. A is a selfadjoint operator satisfying \( Ae_k = -\lambda_k e_k \) with \( \lambda_k > 0 \) and \( \lambda_k \uparrow \infty \), as \( k \uparrow \infty \), where \( \{e_k\}_{k \geq 1} \subset \mathcal{D}(A) \) is a complete orthonormal basis of \( H \).

A3. There exists \( \zeta \in (0, 1) \) such that \( \sum_{k \geq 1} \lambda_k^{\zeta - 1} < \infty \).

A4. Define \( Q_i(t) := \int_0^t e^{sA}Q_i e^{sA^*} ds, i = 1, 2 \), which are two trace class operators. Moreover, there exists \( \theta \in (0, 1) \) such that for any \( T > 0 \),

\[
\begin{align*}
\int_0^T r^{-\theta} \|e^{rA} \sqrt{Q_1} \|_{HS}^2 dr &\leq C_T; \quad (2.1) \\
\int_0^T \|(-A)^{\theta/2} e^{rA} \sqrt{Q_i} \|_{HS}^2 dr &\leq C_T; \quad (2.2) \\
\int_0^\infty \|e^{rA} \sqrt{Q_2} \|_{HS}^2 dr &< \infty, \quad (2.3)
\end{align*}
\]

where \( C_T > 0 \) is a constant depending on \( T \) and \( \| \cdot \|_{HS} \) is the norm of the Hilbert-Schmidt operator.

A5. For \( i = 1, 2 \), the well-defined bounded operator \( \Lambda_i(t) := Q_i^{-1/2}(t)e^{tA} \) satisfying

\[
\int_0^\infty e^{-\lambda t} \|\Lambda_i(t)\|^{1+\kappa_1} dt < \infty, \quad \forall \lambda > 0, \quad (2.4)
\]

for some \( \kappa_1 \geq \max\{\alpha \wedge \beta \wedge \gamma \wedge \eta, 1 - \alpha \wedge (\beta \gamma) \wedge \eta\} \). Moreover, there exists \( \kappa_2 \in (0, 1/2) \) such that

\[
\int_0^\infty e^{-\lambda t} \|(-A)^{\kappa_2} \Lambda_i(t)\| dt < \infty, \quad \forall \lambda > 0. \quad (2.5)
\]

Given \( \alpha \in (0, 1] \), let \( C_\alpha^0(H, H) \) denote the space of all functions \( G(x) : H \to H \) with norm

\[
\|G\|_{C_\alpha^0} : = \|G\|_\infty + \sup_{x \neq y \in H} \frac{|G(x) - G(y)|}{|x - y|^{\alpha}},
\]

where \( \|G\|_\infty : = \sup_{x \in H} |G(x)| \).

For any \( s \in \mathbb{R} \), we define

\[
H_s := \mathcal{D}((-A)^{s/2}) := \left\{ u = \sum_k u_k e_k : u_k = \langle u, e_k \rangle \in \mathbb{R}, \sum_k \lambda_k^{s/2} u_k^2 < \infty \right\}
\]

and

\[
(-A)^{s/2} u := \sum_k \lambda_k^{s/2} u_k e_k, \quad u \in \mathcal{D}((-A)^{s/2}),
\]

with the associated norm

\[
\|u\|_s := \|(-A)^{s/2} u\| = \sqrt{\sum_k \lambda_k^{s} u_k^2}.
\]

It is easy to see \( H^0 = H \) and \( H^{-s} \) is the dual space of \( H^s \). The dual action will also be denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) the operator norm without confusion.

Under the above conditions, one can check that for any \( \theta > 0 \), there exists a constant \( C_\theta > 0 \) such that
\[ |e^{tA}x| \leq e^{-\lambda t} |x|, \quad x \in H, t \geq 0; \]  
(2.6)  
\[ \|e^{tA}x\|_\theta \leq C_\theta t^{\frac{\theta}{2}} |x|, \quad x \in H, t > 0; \]  
(2.7)  
\[ |e^{At}x - x| \leq C_\theta t^{\frac{\theta}{2}} \|x\|_\theta, \quad x \in \mathcal{D}((-A)^{\frac{\theta}{2}}), t \geq 0. \]  
(2.8)  
\[ |e^{At}x - e^{As}x| \leq C_\theta (t-s)^{\theta} \|x\|_\theta, \quad x \in H, t > s > 0. \]  
(2.9)

2.2. Main result. Now, we state our main result.

**Theorem 2.1.** Assume that the conditions **A1-A5** hold. Then for any \(x, y \in H\), \(p \geq 1\) and \(T > 0\), we have

\[ \lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^p \right) = 0, \]  
(2.10)

where \(\bar{X}_t\) is the solution of the corresponding averaged equation:

\[ d\bar{X}_t = A\bar{X}_t dt + \bar{B}(\bar{X}_t) dt + \sqrt{Q_1} dW_1^1, \quad \bar{X}_0 = x, \]  
(2.11)

with \(\bar{B}(x) = \int_H B(x, y) \mu^x(dy)\), and \(\mu^x\) is the unique invariant measure of the transition semigroups for the following frozen equation:

\[ dY_t = [AY_t + F(x, Y_t)] dt + \sqrt{Q_2} dW_2^1, \quad Y_0 = y. \]  
(2.12)

2.3. Idea of proof. Note that the coefficients in system (1.1) are singular, we can not use the classical Khasminskii’s time discretization to prove our main result directly. Inspired from [30], we shall use the Zvonkin transformation to change the singular coefficients to regular ones. Such a technique is now well-known in the study of well-posedness of S(P)DEs with singular coefficients. By a similar argument as in the [9, Section 2], let us give a direct-viewing the idea of how to use the Zvonkin transformation, so we do not care about the rigor of the computations.

We consider the following PDE in \(H\):

\[ \lambda U(x) - \mathcal{L} U(x) = \bar{B}(x), \quad x \in H, \]  
(2.13)

where \(\lambda > 0\) and \(\mathcal{L}\) is the infinitesimal generator of averaged equation, i.e.,

\[ \mathcal{L} f(x) = \langle Ax, Df(x) \rangle + \langle \bar{B}(x), Df(x) \rangle + \frac{1}{2} \text{Tr}[D^2 f(x)Q_1]. \]  
(2.14)

If \(U\) is a sufficiently regular solution, by Itô’s formula we have

\[ dU(\bar{X}_t) = \lambda U(\bar{X}_t) dt - \bar{B}(\bar{X}_t) dt + DU(\bar{X}_t) \sqrt{Q_1} dW_1^1. \]

As a result, we get

\[ \bar{B}(\bar{X}_t) dt = \lambda U(\bar{X}_t) dt - dU(\bar{X}_t) + DU(\bar{X}_t) \sqrt{Q_1} dW_1^1. \]

We put this formula in equation (2.11) and get

\[ d\bar{X}_t = A\bar{X}_t dt + \lambda U(\bar{X}_t) dt - dU(\bar{X}_t) + (I + DU(\bar{X}_t)) \sqrt{Q_1} dW_1^1, \]
where $I$ is the identical operator. By variation of constant method and integration by parts formula, we further get
\begin{align*}
\tilde{X}_t &= e^{tA}(x + U(x)) + \int_0^t e^{(t-s)A} \lambda U(\tilde{X}_s) ds - U(\tilde{X}_t) - \int_0^t A e^{(t-s)A} U(\tilde{X}_s) ds \\
&\quad + \int_0^t e^{(t-s)A} (I + DU(\tilde{X}_s)) \sqrt{Q_1} dW^1.
\end{align*}
(2.15)

Then by a similar argument, we also have
\begin{align*}
X_t^\varepsilon &= e^{tA}(x + U(x)) + \int_0^t e^{(t-s)A} \lambda U(X_s^\varepsilon) ds - U(X_t^\varepsilon) - \int_0^t A e^{(t-s)A} U(X_s^\varepsilon) ds \\
&\quad + \int_0^t e^{(t-s)A} (I + DU(X_s^\varepsilon)) \sqrt{Q_1} dW^1 \\
&\quad + \int_0^t e^{(t-s)A} \left( (I + DU(X_s^\varepsilon)) \right) B(X_s^\varepsilon, Y_s^\varepsilon) - \bar{B}(X_s^\varepsilon)) ds.
\end{align*}
(2.16)

Note that the non-regular drift $B$ has been removed in (2.15). Although that the last term in (2.16) is still non-regular, it is possible to be handled by a time discretization method and the exponential ergodicity of the frozen equation.

3. Proof of main result

In this section, we are devoted to prove Theorem 2.1. The proof consists of the following five subsections. In Subsection 3.1, we show the existence and uniqueness of the solution $(X_t^\varepsilon, Y_t^\varepsilon)$ and give some a-priori estimates. In Subsection 3.2, we study the frozen equation and its exponential ergodicity, which will be used in the final proof. The averaged equation and Zvonkin transformation are considered in Subsection 3.3. In Subsection 3.4, we construct an auxiliary processes $(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon) \in H \times H$ and deduce an estimate of the difference process $Y_t^\varepsilon - \bar{Y}_t^\varepsilon$. In Subsection 3.4, the proof of Theorem 2.1 will be done by estimating the difference processes $X_t^\varepsilon - \tilde{X}_t^\varepsilon$ and $\tilde{X}_t^\varepsilon - \tilde{Y}_t^\varepsilon$ separately. We always assume A1-A5 hold and the initial values $(x, y) \in H \times H$ are fixed in this section.

3.1. Some a-priori estimates of $(X_t^\varepsilon, Y_t^\varepsilon)$.

**Lemma 3.1.** The system (1.1) has a unique strong solution $(X^\varepsilon, Y^\varepsilon)$. Moreover, for any $T > 0$ and $p \geq 1$, there exists a constant $C_{p,T} > 0$ such that
\begin{equation}
\sup_{\varepsilon \in (0,1)} \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|^p \right) \leq C_{p,T} \left( 1 + |x|^p \right)
\end{equation}
(3.1)

and
\begin{equation}
\sup_{\varepsilon \in (0,1)} \sup_{t \geq 0} \mathbb{E} |Y_t^\varepsilon|^p \leq C_p \left( 1 + |y|^p \right).
\end{equation}
(3.2)

**Proof.** Let $\mathcal{H} := H \times H$ be the product Hilbert space. Rewrite the system (1.1) for $Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon)$ as
\begin{equation*}
dZ_t^\varepsilon = \tilde{A} Z_t^\varepsilon dt + B^\varepsilon(Z_t^\varepsilon) dt + \sqrt{Q} dW_t, \quad Z_0^\varepsilon = (x, y) \in \mathcal{H},
\end{equation*}
Lemma 3.2. For any $F$ or the first term, the condition (3.1) yields

\[ \mathbb{E} |Y_t^\varepsilon|^p \leq C_p \left[ |y|^p + \int_0^t \left( \frac{1}{\varepsilon} e^{-s\lambda_1/\varepsilon} ds \right)^p + \mathbb{E} \left[ \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon} \sqrt{Q_2} dW_s \right]^p \right] \]

\[ \leq C_p (1 + |y|^p) + C_p \varepsilon^{p/2} \left( \int_0^t \left( e^{(t-s)A/\varepsilon} \sqrt{Q_2} \right)^2_{HS} ds \right)^{p/2} \]

\[ \leq C_p (1 + |y|^p) + C_p \left( \int_0^\infty \left( e^{rA} \sqrt{Q_2} \right)_H^2 dr \right)^{p/2}, \]

which in turn implies the desired result by condition (2.3). The proof is complete. \qed

Lemma 3.2. For any $t \in (0, T]$ and $p \geq 1$, there exists a constant $C_{p,T} > 0$ such that

\[ \sup_{\varepsilon \in (0, 1)} \mathbb{E} \| X_t^\varepsilon \|_{\theta}^p \leq C_{p,T} t^{-\frac{\theta p}{2}} (|x|^p + 1), \]  

(3.4)

where $\theta$ is given in A4.

Proof. We recall that

\[ X_t^\varepsilon = e^{tA} x + \int_0^t e^{(t-s)A} B(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t e^{(t-s)A} \sqrt{Q_1} dW_s. \]

For the first term, the condition (2.7) yields

\[ \| e^{tA} x \|_{\theta}^p \leq C t^{-\frac{\theta p}{2}} |x|^p. \]  

(3.5)

For the second term, by (2.7) and the boundedness of $B$, we can get

\[ \mathbb{E} \left\| \int_0^t e^{(t-s)A} B(X_s^\varepsilon, Y_s^\varepsilon) ds \right\|_{\theta}^p \leq C \left[ \int_0^t (t-s)^{-\frac{\theta}{2}} ds \right]^p \leq C_{p,T}. \]  

(3.6)
For the third term, by Burkholder-Davis-Gundy’s inequality and condition (2.2), we have for any \( t \in [0, T] \),
\[
\mathbb{E} \left\| \int_0^t e^{(t-s)A} \sqrt{Q_1} dW_s^1 \right\|_{\theta}^p \leq C_p \left( \int_0^t \left\| (-A)^{\theta/2} e^{(t-s)A} \sqrt{Q_1} \right\|_{HS}^2 ds \right)^{p/2} \leq C_p \left( \int_0^T \left\| (-A)^{\theta/2} e^{sA} \sqrt{Q_1} \right\|_{HS}^2 ds \right)^{p/2} \leq C_{p,T}. \tag{3.7}
\]
Hence, the proof is completed by combining (3.5)-(3.7).

Usually, the Hölder continuity of \( X_t^\varepsilon \) in time plays an important role in the method of time discretization (see [4, Proposition 4.4], [12, Lemma 3.4] and [16, Proposition 9]), then the initial value \( x \in H^\theta \) will be assumed for some \( \theta > 0 \). However inspired from [26], studying the Hölder continuity can be replaced by studying the integral of the time increment of \( X_t^\varepsilon \), which is weaker than the Hölder continuity but enough for our purpose, and it only needs initial value \( x \in H \) for advantage.

**Lemma 3.3.** For any \( T > 0 \) and \( p \geq 1 \), there exists a constant \( C_{p,T} > 0 \) such that for any \( \varepsilon \in (0, 1) \) and \( \delta > 0 \) small enough,
\[
\mathbb{E} \left[ \int_0^T |X_t^\varepsilon - X_{t_1}^\varepsilon|^2 dt \right] \leq C_T \delta^\theta (1 + |x|^2), \tag{3.8}
\]
where \( t(\delta) := \lfloor \frac{1}{\delta} \rfloor \delta \) and \( \lfloor s \rfloor \) denotes the integer part of \( s \) and \( \theta \) is given in A4.

**Proof.** It is easy to see that
\[
\mathbb{E} \left[ \int_0^T |X_t^\varepsilon - X_{t_1}^\varepsilon|^2 dt \right] = \mathbb{E} \left( \int_0^\delta |X_t^\varepsilon - x|^2 dt \right) + \mathbb{E} \left[ \int_\delta^T |X_t^\varepsilon - X_t^\varepsilon|\right]^2 dt \right] \leq C_T \delta (1 + |x|^2) + 2\mathbb{E} \left( \int_\delta^T |X_t^\varepsilon - X_{t-\delta}^\varepsilon|^2 dt \right) + 2\mathbb{E} \left( \int_\delta^T |X_{t_1}^\varepsilon - X_{t-\delta}^\varepsilon|^2 dt \right), \tag{3.9}
\]
where we use (3.1) in the last inequality. After simple calculations, we have
\[
X_t^\varepsilon - X_{t-\delta}^\varepsilon = (e^{A\delta} - I)X_{t-\delta}^\varepsilon + \int_{t-\delta}^t e^{(t-s)A} B(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_{t-\delta}^t e^{(t-s)A} \sqrt{Q_1} dW_s^1 := I_1(t) + I_2(t) + I_3(t). \tag{3.10}
\]

For the first term \( I_1(t) \), by property (2.8) and Lemma 3.2, there exists a constant \( C > 0 \) such that
\[
\mathbb{E} \left( \int_\delta^T |I_1(t)|^2 dt \right) \leq C \mathbb{E} \int_\delta^T \delta^\theta |X_{t-\delta}^\varepsilon|^2 dt \leq C \delta^\theta \int_\delta^T \left[ C(t - \delta)^{-\theta} |x|^2 + C_T \right] dt \leq C_T \delta^\theta (1 + |x|^2). \tag{3.11}
\]

For the term \( I_2(t) \), by the boundedness of \( B \), it is easy to see
\[
\mathbb{E} \left( \int_\delta^T |I_2(t)|^2 dt \right) \leq C_T \delta^2. \tag{3.12}
\]
For the term $I_3(t)$, by condition (2.1), we get
\[
\mathbb{E} \left( \int_{\delta}^{T} |I_3(t)|^2 dt \right) \leq C \int_{\delta}^{T} \int_{t-\delta}^{t} \|e^{(t-s)A}\sqrt{Q_1}\|_{HS}^2 ds dt \\
\leq C\delta^\theta \int_{\delta}^{T} \int_{0}^{\delta} s^{-\theta} \|e^{sA}\sqrt{Q_1}\|_{HS}^2 ds dt \leq CT\delta^\theta. \quad (3.13)
\]
Combining estimates (3.10)-(3.13), we get that
\[
\mathbb{E} \left( \int_{\delta}^{T} |X_t^\varepsilon - X_{t-\delta}^\varepsilon|^2 dt \right) \leq C\delta^\theta (1 + |x|^2). \quad (3.14)
\]
By a similar argument as above, we have
\[
\mathbb{E} \left( \int_{\delta}^{T} |X_t^\varepsilon - X_{t-\delta}^\varepsilon|^2 dt \right) \leq C\delta^\theta (1 + |x|^2). \quad (3.15)
\]
Hence, (3.9), (3.14) and (3.15) imply (3.8) holds. The proof is complete. □

3.2. The frozen equation and exponential ergodicity. Recall that the frozen equation is given by (2.12). Since $F(x, \cdot)$ is bounded and Hölder continuous, it follows by [9] that for any fixed $x \in H$ and any initial data $y \in H$, equation (2.12) has a unique strong solution $\{Y_t^x,y\}_{t \geq 0}$. Let $P_t^x$ be the transition semigroup of $Y_t^x,y$, that is, for any bounded measurable function $\varphi$ on $H$,
\[
P_t^x \varphi(y) = \mathbb{E} \left[ \varphi (Y_t^x,y) \right], \quad y \in H, \quad t > 0.
\]
We proceed to prove the exponential ergodicity of the transition semigroup $\{P_t^x\}_{t \geq 0}$.

**Proposition 3.4.** For any fixed $x, y \in H$, there exists a unique invariant measure $\mu^x$ for the transition semigroup $\{P_t^x\}_{t \geq 0}$. Furthermore, there exist $C, \omega > 0$ such that for any bounded and measurable function $\varphi : H \to H$,
\[
\left| P_t^x \varphi(y) - \int_H \varphi(z) \mu^x(dz) \right| \leq C(1 + |y|)e^{-\omega t}\|\varphi\|_{\infty}. \quad (3.16)
\]

**Proof.** It sufficient to check the following three properties:
(i) Markov semigroup $P_t^x$ is strongly Feller and irreducible.
(ii) For each $r > 0$, there exists $T_0 > 0$ and a compact set $K$ such that
\[
\inf_{y \in B_r} \mathbb{P}(Y_t^x,y \in K) > 0,
\]
where $B_r = \{y : |y| \leq r\}$.
(iii) There exist $C > 0$ and $\omega > 0$ such that for any $t \geq 0$,
\[
\mathbb{E}|Y_t^x,y| \leq e^{-\omega t}|y| + C.
\]
Then there exists a unique invariant measure $\mu^x$ and the exponential ergodicity (3.16) holds by [19, Theorem 2.5].
Indeed, under the conditions (2.3) and (2.4), by [8, Theorem 3 and Proposition 4], the statement (i) holds directly.
By similar argument as in the proof of Lemma 3.2 and condition (2.2), we have for any $T > 0$, there exists $C_T > 0$ such that
\[
\mathbb{E}\|Y_T^{x,y}\|_\theta \leq CT^{-\theta/2}|y| + C_T.
\]
Then for any \( r > 0, T > 0 \), by taking \( K = \{ y : \| y \|_\theta \leq R \} \), which is a compact set in \( H \), we have for any \( y \in B_r \),
\[
\mathbb{P}(Y^x_T \in K) = \mathbb{P}(\| Y^x_T \|_\theta \leq R) = 1 - \mathbb{P}(\| Y^x_T \|_\theta > R) \\
\geq 1 - \frac{\mathbb{E}\| Y^x_T \|_\theta}{R} \geq 1 - \frac{C T^{-\theta/2 r} + C_T}{R} > 0,
\]
for \( R \) large enough, hence the statement (ii) holds.

By similar argument as in the proof of Lemma 3.1, it is easy to prove that under the condition \( \text{A}4 \), there exists \( \omega, C > 0 \) such that for any \( t \geq 0 \),
\[
\mathbb{E}|Y^x_t| \leq e^{-\omega t}|y| + C,
\]
which implies the statement (iii). The proof is complete. \( \square \)

3.3. Averaged equation and Zvonkin transformation. In this subsection, we first recall the averaged equation, i.e.,
\[
d\bar{X}_t = A\bar{X}_t dt + \bar{B}(\bar{X}_t) dt + \sqrt{Q_1} dW^1_t, \quad \bar{X}_0 = x \tag{3.17}
\]
with
\[
\bar{B}(x) = \int_H B(x, y) \mu^x(dy),
\]
where \( \mu^x \) is the unique invariant measure of the transition semigroup for equation (2.12).

**Lemma 3.5.** For any \( x \in H \), equation (3.17) has a unique strong solution \( \bar{X} \). Moreover, for any \( T > 0 \) and \( p \geq 1 \), there exists a positive constant \( C_{p,T} \) such that
\[
\mathbb{E}\left( \sup_{t \in [0, T]} |\bar{X}_t|^p \right) \leq C_{p,T}(1 + |x|^p). \tag{3.18}
\]

**Proof.** Obviously, \( \bar{B} \) is bounded due to the boundedness of \( B \). Next, if we can check that the averaged coefficient \( \bar{B} \) of Eq. (3.17) satisfies the following property:
\[
|\bar{B}(x_1) - \bar{B}(x_2)| \leq C|x_1 - x_2|^{\alpha(\beta\gamma)}, \quad x_1, x_2 \in H. \tag{3.19}
\]
Then Eq.(3.17) has a unique solution and (3.18) can be easily obtained by [9, Theorem 7] under the assumptions \( \text{A2} - \text{A5} \). In fact, note that
\[
d(Y^{x_1,y}_t - Y^{x_2,y}_t) = A(Y^{x_1,y}_t - Y^{x_2,y}_t) dt + [F(x_1, Y^{x_1,y}_t) - F(x_2, Y^{x_2,y}_t)] dt.
\]
Then there exists a constant \( C > 0 \) such that
\[
\frac{d}{dt}\mathbb{E}|Y^{x_1,y}_t - Y^{x_2,y}_t|^2 = -2\mathbb{E}|Y^{x_1,y}_t - Y^{x_2,y}_t|^2 + 2\mathbb{E}<F(x_1, Y^{x_1,y}_t) - F(x_2, Y^{x_2,y}_t), Y^{x_1,y}_t - Y^{x_2,y}_t>
\]
\[
\leq -2\lambda_1|Y^{x_1,y}_t - Y^{x_2,y}_t|^2 + C|x_1 - x_2|^{\alpha(\beta\gamma)}\mathbb{E}|Y^{x_1,y}_t - Y^{x_2,y}_t|
\]
\[
\leq -\lambda_1|Y^{x_1,y}_t - Y^{x_2,y}_t|^2 + C|x_1 - x_2|^{2\gamma}.
\]
The comparison theorem yields that
\[
\mathbb{E}|Y^{x_1,y}_t - Y^{x_2,y}_t|^2 \leq C \int_0^t e^{-\lambda_1(t-s)} ds |x_1 - x_2|^{2\gamma} \leq C|x_1 - x_2|^{2\gamma}. \tag{3.20}
\]
So we get
\[
|\tilde{B}(x_1) - \tilde{B}(x_2)| = \left| \int_H B(x_1, z) \mu^x_1(dz) - \int_H B(x_2, z) \mu^x_2(dz) \right| \\
\leq \left| \int_H B(x_1, z) \mu^x_1(dz) - \mathbb{E} B(x_1, Y_t^{x_1,0}) \right| + \left| \mathbb{E} B(x_2, Y_t^{x_2,0}) - \int_H B(x_2, z) \mu^x_2(dz) \right| \\
+ \mathbb{E} \left| B(x_1, Y_t^{x_1,0}) - B(x_2, Y_t^{x_2,0}) \right| \\
\leq C e^{-\omega t} + C \mathbb{E} \left| B(x_1, Y_t^{x_1,0}) - B(x_2, Y_t^{x_2,0}) \right|^{\alpha \wedge (\beta \gamma)} \\
\leq C e^{-\omega t} + C |x_1 - x_2|^{\alpha \wedge (\beta \gamma)},
\]
where the second inequality is consequence of boundedness of \( B \) and Proposition 3.4, and the last inequality is consequence of (3.20). Finally, letting \( t \to \infty \), we obtain
\[
|\tilde{B}(x_1) - \tilde{B}(x_2)| \leq C |x_1 - x_2|^{\alpha \wedge (\beta \gamma)}.
\]
The proof is complete. \( \square \)

We consider the following PDE:
\[
\lambda U(x) - \mathcal{L} U(x) = G(x), \quad x \in H, \quad (3.21)
\]
where \( \lambda > 0 \), \( \mathcal{L} \) is given by (2.14) and \( G : H \to H \) is measurable. Note that equation (3.21) is homogeneous. We have the following result.

**Lemma 3.6.** For every \( G \in C_{\beta}^q(H, H) \) with \( \alpha \wedge (\beta \gamma) \wedge \eta < \theta \leq 1 \), there exists a solution \( U \in C_{\beta}^2(H, H) \) to Eq. (3.21). Moreover, we have
\[
\|DU\|_{\infty} \leq D_\lambda \|G\|_{C_{\beta}^q}; \quad (3.22)
\]
\[
\|(-A)^{\kappa_2} DU\|_{\infty} \leq C \|G\|_{C_{\beta}^q}; \quad (3.23)
\]
\[
\|D^2U\|_{\infty} \leq C \|G\|_{C_{\beta}^q}, \quad (3.24)
\]
where \( C > 0 \) is a constant and \( D_\lambda \) is a positive constant satisfying \( \lim_{\lambda \to \infty} D_\lambda = 0 \).

**Proof.** Since \( G \in C_{\beta}^q(H, H) \) \( \alpha \wedge (\beta \gamma) \wedge \eta < \theta \leq 1 \), under the assumptions **A2-A5**, the estimates (3.22) and (3.24) follow by exactly the same arguments as in the proof of [9, Theorem 5]. As for estimate (3.23), thanks to assumption (2.5), it can be proved similarly, see also [33, (5.5)]. We omit the details here. \( \square \)

Now, we prove the following Zvonkin’s transformation.

**Lemma 3.7.** Let \( \bar{X}_t \) be the solution of equation (3.17). Let \( U \) be the solution of Eq. (2.13). Then the formulas (2.15) and (2.16) hold.

**Proof.** Inspired from [9], the idea of proof is the one given in subsection 2.1. The only point is the application of Itô’s formula. On one hand, due to \( \tilde{B} \in C_{\beta}^{\alpha \wedge (\beta \gamma)}(H, H) \), Eq. (2.13) has a solution \( U \in C_{\beta}^2(H, H) \) by Lemma 3.6. On the other hand, we introduce the approximations:
\[
d\bar{X}_t^{m,n} = [A_m \bar{X}_t^{m,n} + \tilde{B}(\bar{X}_t^{m,n})]dt + \sqrt{Q_1} \Pi_n dW_t^1, \quad \bar{X}_0^{m,n} = x,
\]
where $A_m$ are the Yosida approximations of $A$ and $\Pi_n$ is the orthogonal projection of $H$ onto span$\{e_1, ..., e_n\}$. Then the argument in Subsection 2.1 can be done on these approximations and then one can pass to the limit in both sides. We omit the details which are classical. □

3.4. Construction of auxiliary processes. Following the idea in [21], we introduce an auxiliary process $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon) \in H \times H$ and divide $[0, T]$ into intervals of size $\delta$, where $\delta$ is a fixed positive number depending on $\varepsilon$ and will be chosen later.

We construct a process $\hat{Y}_t^\varepsilon$, with $\hat{Y}_0^\varepsilon = Y_0^\varepsilon = y$, and for any $k \in \mathbb{N}$ and $t \in \lfloor k\delta, \min((k + 1)\delta, T) \rfloor$,

$$\hat{Y}_t^\varepsilon = Y_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_{k\delta}^t A\hat{Y}_s^\varepsilon ds + \frac{1}{\varepsilon} \int_{k\delta}^t F(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t \sqrt{Q_2} dW_s^2.$$

(3.25)

We also construct another auxiliary process $\hat{X}_t^\varepsilon \in H$ by

$$\hat{X}_t^\varepsilon = e^{tA}(x + U(x)) + \int_0^t e^{(t-s)A} \lambda U(X_s^\varepsilon) ds - U(X_t^\varepsilon) - \int_0^t A e^{(t-s)A} U(X_s^\varepsilon) ds$$

$$+ \int_0^t e^{(t-s)A} \sqrt{Q_1} dW_s^1 + \int_0^t e^{(t-s)A} D U(X_s^\varepsilon) \sqrt{Q_1} dW_s^1$$

$$+ \int_0^t e^{(t-s)(\delta)} \langle DU(X_{s(\delta)}^\varepsilon), B(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{B}(X_{s(\delta)}^\varepsilon) \rangle ds$$

$$+ \int_0^t e^{(t-s)(\delta)} A \left[ B(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{B}(X_{s(\delta)}^\varepsilon) \right] ds.$$

(3.26)

Now, we give a position to estimate the difference process $Y_t^\varepsilon - \hat{Y}_t^\varepsilon$.

Lemma 3.8. For any $T > 0, \varepsilon \in (0, 1)$ and $p$ large enough, there exists a constant $C_{p,T} > 0$ such that

$$\mathbb{E} \left( \int_0^T |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p dt \right) \leq C_{p,T} \exp \left\{ \frac{C_{p,\lambda} \delta^p}{\varepsilon^p} + \frac{C_{p,\delta^{p/2}}}{\varepsilon^{p/2}} \right\} \delta^{p/2}.$$

(3.27)

Proof. Due to the non-Lipschitz property of $F$, we need to use the Zvonkin transformation again, i.e., consider the following PDE

$$\lambda V(x, y) - \mathcal{L}(x)V(x, y) = F(x, y), \quad y \in H,$$

(3.28)

where $\lambda > 0$ and $\mathcal{L}(x)$ is the infinitesimal generator of the frozen equation (2.12), i.e.,

$$\mathcal{L}(x)V(x, y) = \langle Ay, D_y V(x, y) \rangle + \langle F(x, y), D_y V(x, y) \rangle + \frac{1}{2} \text{Tr}[D_y^2 V(x, y)Q_2].$$

Note that $F(x, \cdot) \in C^p_b(H, H)$, according to Lemma 3.6, (3.28) has a solution $V(x, \cdot) \in C^2_b(H, H)$ and the following estimates hold:

$$\|D_y V(x, \cdot)\|_{\infty} \leq D_\lambda \|F(x, \cdot)\|_{C^p_b};$$

(3.29)

$$\|(-A)^{\kappa_2} D_y V(x, \cdot)\|_{\infty} \leq C \|F(x, \cdot)\|_{C^p_b};$$

(3.30)

$$\|D_y^2 V(x, \cdot)\|_{\infty} \leq C \|F(x, \cdot)\|_{C^p_b},$$

(3.31)

where $D_\lambda$ is a positive constant satisfying $\lim_{\lambda \to \infty} D_\lambda = 0$. 
By similar argument as in the proof of Lemma 3.7, we have for any \( t \in [k\delta, (k + 1)\delta) \),
\[
\hat{Y}_t^\varepsilon = e^{(t-k\delta)A/\varepsilon}(X_t^\varepsilon + V(X_s^\varepsilon, Y_s^\varepsilon)) + \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \frac{1}{\varepsilon} V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) ds - V(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) \\
- \int_{k\delta}^{t} A e^{(t-s)A/\varepsilon} V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} D_y V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) \sqrt{Q_\delta} dW_s \\
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \sqrt{Q_\delta} dW_s^2
\]  
(3.32)

and
\[
Y_t^\varepsilon = e^{(t-k\delta)A/\varepsilon}(X_t^\varepsilon + V(X_s^\varepsilon, Y_s^\varepsilon)) + \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \frac{1}{\varepsilon} V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) ds - V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) \\
- \int_{k\delta}^{t} A e^{(t-s)A/\varepsilon} V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} D_y V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) \sqrt{Q_\delta} dW_s \\
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \sqrt{Q_\delta} dW_s^2
\]  
(3.33)

Then it is easy to see
\[
Y_t^\varepsilon - \hat{Y}_t^\varepsilon = \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \frac{1}{\varepsilon} \left[ V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) - V(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) \right] ds + V(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) - V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) \\
+ \int_{k\delta}^{t} A e^{(t-s)A/\varepsilon} \left[ V(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) - V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) \right] ds \\
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \left[ D_y V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) - D_y V(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) \right] \sqrt{Q_\delta} dW_s \\
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \left[ D_y V(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) - D_y V(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) \right] \sqrt{Q_\delta} dW_s \\
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{t} e^{(t-s)A/\varepsilon} \left[ F(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon) - F(X_{k\delta}^\varepsilon, Y_{k\delta}^\varepsilon) \right] ds.
\]

By estimates (3.29)-(3.31), we have for \( p \) large enough,
\[
\mathbb{E}|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p \leq \left( C_p \lambda^{\delta p-1} + C_p^{\delta p^2-1} + C_p^{\delta p/2-1} \right) \int_{k\delta}^{t} \mathbb{E}|Y_s^\varepsilon - \hat{Y}_s|^p ds + C_p D_\lambda^p \mathbb{E}|Y_t^\varepsilon - \hat{Y}_t|^p \\
+ C_p^{\delta p^2-1} \int_{k\delta}^{t} \mathbb{E}|Y_s^\varepsilon - \hat{Y}_s|^p ds + C_p^{\delta p/2-1} \int_{k\delta}^{t} \mathbb{E}|Y_s^\varepsilon - \hat{Y}_s|^p ds \\
+ C_p^{\delta p-1} \int_{k\delta}^{t} \mathbb{E}|X_s^\varepsilon - X_{k\delta}^\varepsilon|^{\gamma p} ds.
\]

Note that \( \lim_{\lambda \to \infty} D_\lambda = 0 \), we take \( \lambda \) large enough such that \( C_p D_\lambda^p \) is small, then we obtain
\[
\mathbb{E}|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p \leq \left( C_p \lambda^{\delta p-1} + C_p^{\delta p^2-1} + C_p^{\delta p/2-1} \right) \int_{k\delta}^{t} \mathbb{E}|Y_s^\varepsilon - \hat{Y}_s|^p ds \\
+ C_p^{\delta p-1} \int_{k\delta}^{t} \mathbb{E}|X_s^\varepsilon - X_{k\delta}^\varepsilon|^{\gamma p} ds.
\]
The Gronwall’s inequality yields

\[ \mathbb{E}|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p \leq \frac{C_p \delta_p^{-1}}{\varepsilon_p} \int_{k_0}^t \mathbb{E}|X_s^\varepsilon - X_{k_0}^\varepsilon|^\gamma_p ds \exp \left\{ \frac{C_{p,\lambda} \delta_p}{\varepsilon_p} + \frac{C_p \delta_{p,k_0}^2}{\varepsilon_{p,k_0}^2} + \frac{C_p \delta_p^{p/2}}{\varepsilon_p^{p/2}} \right\}. \]

Then by Fubini’s theorem, we have

\[
\mathbb{E} \int_0^T |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p dt = \sum_{k=0}^{(T/\delta)-1} \mathbb{E} \int_{k \delta}^{(k+1) \delta} |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p dt
\leq \frac{C_p \delta_p^{-1}}{\varepsilon_p} \exp \left\{ \frac{C_{p,\lambda} \delta_p}{\varepsilon_p} + \frac{C_p \delta_{p,k_0}^2}{\varepsilon_{p,k_0}^2} + \frac{C_p \delta_p^{p/2}}{\varepsilon_p^{p/2}} \right\} \sum_{k=0}^{(T/\delta)-1} \mathbb{E} \int_{k \delta}^{(k+1) \delta} |X_s^\varepsilon - X_{k_0}^\varepsilon|^\gamma_p ds dt
\leq \frac{C_p \delta_p}{\varepsilon_p} \exp \left\{ \frac{C_{p,\lambda} \delta_p}{\varepsilon_p} + \frac{C_p \delta_{p,k_0}^2}{\varepsilon_{p,k_0}^2} + \frac{C_p \delta_p^{p/2}}{\varepsilon_p^{p/2}} \right\} \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^\gamma_p dt.
\]

Meanwhile, by Lemmas 3.1 and 3.3, we get

\[
\mathbb{E} \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^\gamma_p dt \leq \mathbb{E} \left[ \sup_{t \leq T} |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^\gamma_p^{-1} \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon| dt \right]
\leq \left\lbrack \mathbb{E} \left( \sup_{t \leq T} |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^{2 \gamma_p - 2} \right) \right\rbrack^{1/2} \left\lbrack \mathbb{E} \left( \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon| dt \right) \right\rbrack^{1/2}
\leq C_{p,T} \mathbb{E} \left[ \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2 dt \right]^{1/2} \leq C_{p,T} \delta^{\theta/2}.
\]

Hence, we finally obtain

\[
\mathbb{E} \int_0^T |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^p dt \leq \frac{C_p \delta_p \delta_p^p}{\varepsilon_p} \exp \left\{ \frac{C_{p,\lambda} \delta_p}{\varepsilon_p} + \frac{C_p \delta_{p,k_0}^2}{\varepsilon_{p,k_0}^2} + \frac{C_p \delta_p^{p/2}}{\varepsilon_p^{p/2}} \right\} \delta^{\theta/2}
\leq C_{p,T} \exp \left\{ \frac{C_{p,\lambda} \delta_p}{\varepsilon_p} + \frac{C_p \delta_{p,k_0}^2}{\varepsilon_{p,k_0}^2} + \frac{C_p \delta_p^{p/2}}{\varepsilon_p^{p/2}} \right\} \delta^{\theta/2}.
\]

The proof is complete. \( \square \)

3.5. **Proof of Theorem 2.1.** This section is devoted to giving the proof of our main result. We first give the estimates for the difference process \( X_t^\varepsilon - \hat{X}_t^\varepsilon \).

**Lemma 3.9.** For any \( T > 0 \) and \( p \) large enough, we have

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^p \right) \leq C_{p,T} \delta^{\theta/2} + C_{p,T} \exp \left\{ \frac{C_{p,\lambda} \delta_p}{\varepsilon_p^p} + \frac{C_p \delta_{p,k_0}^2}{\varepsilon_{p,k_0}^2} + \frac{C_p \delta_p^{p/2}}{\varepsilon_p^{p/2}} \right\} \delta^{\theta/2}.
\]
Proof. By (2.16) and (3.26), it is to see that

$$|X_t^\varepsilon - \hat{X}_t^\varepsilon| \leq \left| \int_0^t e^{(t-s)A} \left( DU(X_s^\varepsilon), B(X_s^\varepsilon, Y_s^\varepsilon) - \bar{B}(X_s^\varepsilon) \right) ds \right|$$

$$+ \left| \int_0^t e^{(t-s)A} \left( DU(X_s^\varepsilon), B(X_s^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{B}(X_s^\varepsilon) \right) ds \right|$$

$$- \left| \int_0^t e^{(t-s)(\delta)A} \left( DU(X_s^{\varepsilon(\delta)}), B(X_s^{\varepsilon(\delta)}, \hat{Y}_s^{\varepsilon(\delta)}) - \bar{B}(X_s^{\varepsilon(\delta)}) \right) ds \right|$$

$$+ \left| \int_0^t e^{(t-s)(\delta)A} \left( DU(X_s^{\varepsilon(\delta)}), B(X_s^{\varepsilon(\delta)}, \hat{Y}_s^{\varepsilon(\delta)}) - \bar{B}(X_s^{\varepsilon(\delta)}) \right) ds \right|$$

$$\leq C \int_0^t \|e^{(t-s)A} - e^{(t-s)(\delta)A}\| ds + C \int_0^t \|DU(X_s^\varepsilon) - DU(X_s^{\varepsilon(\delta)})\| ds$$

$$+ C \int_0^t \left| B(X_s^\varepsilon, Y_s^\varepsilon) - \bar{B}(X_s^\varepsilon) - B(X_s^{\varepsilon(\delta)}, \hat{Y}_s^{\varepsilon(\delta)}) - \bar{B}(X_s^{\varepsilon(\delta)}) \right| ds.$$

Then by the boundedness of $\|D^2U(x)\|$ and $|B(x)|$, the Hölder continuous of $B$ and $\bar{B}$, property (2.9), we get

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^p \right) \leq C_p \delta^{\beta_p} \left( \int_0^T s^{-\theta} ds \right)^p + C_{p,T} \mathbb{E} \int_0^T |X_s^\varepsilon - X_s^{\varepsilon(\delta)}|^p ds$$

$$+ C_{p,T} \mathbb{E} \int_0^T |Y_s^\varepsilon - \hat{Y}_s^{\varepsilon(\delta)}|^\beta_p ds$$

$$\leq C_{p,T} \delta^{\beta_p/2} + C_{p,T} \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^p \right) \leq \delta^{\beta_p/2},$$

where the last inequality follows by estimates (3.27) and (3.34). The proof is complete. \(\square\)

Now, we give a position to prove our main result.

**Proof of Theorem 2.1:** We will divide the proof into three steps.

**Step 1:** By (2.15) and (3.26), it is easy to see that

$$\hat{X}_t - X_t = \int_0^t e^{(t-s)A} \lambda(U(X_t^\varepsilon) - U(\bar{X}_t)) ds + U(\bar{X}_t) - U(X_t^\varepsilon)$$

$$+ \int_0^t A e^{(t-s)A} \left( U(\bar{X}_t) - U(X_s^\varepsilon) \right) ds + \int_0^t e^{(t-s)A} \left( DU(X_s^\varepsilon) - DU(\bar{X}_s) \right) \sqrt{Q} dW_s$$

$$- \int_0^t e^{(t-s)(\delta)A} \left( B(X_s^{\varepsilon(\delta)}, \hat{Y}_s^{\varepsilon(\delta)}) - \bar{B}(X_s^{\varepsilon(\delta)}) \right) ds$$

$$+ \int_0^t e^{(t-s)(\delta)A} \left( DU(X_s^{\varepsilon(\delta)}), B(X_s^{\varepsilon(\delta)}, \hat{Y}_s^{\varepsilon(\delta)}) - \bar{B}(X_s^{\varepsilon(\delta)}) \right) ds.$$
Then we have the following estimate:

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |\hat{X}_t^\varepsilon - \tilde{X}_t| \right) \leq C_p \lambda T \mathbb{E} \int_0^T |U(X_s^\varepsilon) - U(\tilde{X}_s)|^p ds + C_p \mathbb{E} \left( \sup_{t \in [0,T]} |U(X_t^\varepsilon) - U(\tilde{X}_t)|^p \right) \\
+ C_p \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t A e^{(t-s)A} \left( U(X_s^\varepsilon) - U(\tilde{X}_s) \right) ds \right|^p \right) \\
+ C_p \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)A} \left[ D U(X_s^\varepsilon) - D U(\tilde{X}_s) \right] \sqrt{Q_1} d W_s^i \right|^p \right) \\
+ C_p \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)A} \left[ B(X_s^\varepsilon, \hat{Y}_s^\varepsilon) - B(X_s^\varepsilon, \tilde{Y}_s^\varepsilon) \right] ds \right|^p \right) \\
+ C_p \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t e^{-sA} \left[ B(X_s^\varepsilon, \hat{Y}_s^\varepsilon) - B(X_s^\varepsilon, \tilde{Y}_s^\varepsilon) \right] ds \right|^p \right) \\
:= \sum_{i=1}^6 J_i(T). \quad (3.35)
\]

For the term \(J_1(T)\), by the Hölder inequality and (3.22) we have

\[
J_1(T) \leq C_p T \mathbb{E} \sup_{t \in [0,T]} |\bar{Y}_t|^p \int_0^T \mathbb{E} |X_s^\varepsilon - \tilde{X}_s|^p ds. \quad (3.36)
\]

Using (3.22) again, it is easy to see that

\[
J_2(T) \leq C_p D \mathbb{E} \sup_{t \in [0,T]} |X_t^\varepsilon - \tilde{X}_t|^p. \quad (3.37)
\]

For the term \(J_3(T)\), using the factorization method, for \(\kappa_3 \in (0, \kappa_2)\) in **A4**, we write

\[
\int_0^t A e^{(t-s)A} (U(X_s^\varepsilon) - U(\tilde{X}_s)) ds = \frac{\sin(\pi \kappa_3)}{\pi} \int_0^t e^{(t-s)A} (t-s)^{\kappa_3-1} f_s ds,
\]

where

\[
f_s := \int_0^s A e^{(s-r)A} (s-r)^{-\kappa_3} (U(X_r^\varepsilon) - U(\tilde{X}_r)) dr.
\]

Choosing \(p > 1\) large enough such that \(\frac{p(1-\kappa_3)}{p-1} < 1\), we get

\[
\left| \int_0^t A e^{(t-s)A} (U(X_s^\varepsilon) - U(\tilde{X}_s)) ds \right| \leq C \left( \int_0^t (t-s)^{-\frac{p(1-\kappa_3)}{p-1}} ds \right)^{\frac{p-1}{p}} |f|_{L^p(0,T;H)} \\
\leq C_p \kappa_3^{-\frac{1}{p}} |f|_{L^p(0,T;H)},
\]

which implies

\[
\sup_{0 \leq t \leq T} \left| \int_0^t A e^{(t-s)A} (U(X_s^\varepsilon) - U(\tilde{X}_s)) ds \right|^p \leq C_p T |f|_{L^p(0,T;H)}. 
\]
Then we can deduce by (3.23) that for \( p \) large enough such that \( \frac{(1-\kappa_2+\kappa_3)p}{p-1} < 1 \),

\[
J_3(T) \leq C_{p,T} \mathbb{E} \int_0^T |f_1|^p dt \\
\leq C_{p,T} \mathbb{E} \int_0^T \left( \int_0^t \left( \int_0^s \left( e^{(t-r)A}(t-r)^{-\kappa_4} \right)^{p-1} \|(-A)^{\frac{p}{2}} DU\|^p_{\infty} \|X^\varepsilon_r - \bar{X}_r\|^p dt \right) dr \right) dt \\
\leq C_{p,T} \mathbb{E} \int_0^T \left( \int_0^t t^{\left( \frac{(1-\kappa_2+\kappa_3)p}{p-1} \right)} dt \right)^{p-1} \mathbb{E} \int_0^T |X^\varepsilon_t - \bar{X}_t|^p dt \\
\leq C_{p,T} \mathbb{E} \int_0^T |X^\varepsilon_t - \bar{X}_t|^p dt. 
\tag{3.38}
\]

For the term \( J_4(T) \), similar as we did in \( J_3(T) \), we can prove for \( p \) large enough,

\[
J_4(T) \leq C_{p,T} \mathbb{E} \int_0^T \left| \int_0^t \left( \int_0^s \left( e^{(t-s)A}(t-s)^{-\kappa_4} \right)^{p-1} \|(-A)^{\frac{p}{2}} DU(X^\varepsilon_s) - DU(\bar{X}_s)\|^p dt \right) \|
\bar{Q}_1\|_{HS}^2 \right)^{p/2} dt,
\]

where \( \kappa_4 \in (0, \zeta/2) \), and \( \zeta \) is given in A4. Note that

\[
\left\| e^{(t-s)A}(t-s)^{-\kappa_4} \left( DU(X^\varepsilon_s) - DU(\bar{X}_s)\right) \right\|^2_{HS} \\
\leq \sum_{i,j} \left( e^{(t-s)A}(t-s)^{-\kappa_4} \left( DU(X^\varepsilon_s) - DU(\bar{X}_s)\right) \right)^2 \\
\leq \sum_{i,j} e^{-2\lambda_j(t-s)}(t-s)^{-2\kappa_4} \left( \left( DU(X^\varepsilon_s) - DU(\bar{X}_s)\right) \right)^2 \\
\leq \sum_{i,j} e^{-2\lambda_j(t-s)}(t-s)^{-2\kappa_4} \left( DU(X^\varepsilon_s) - DU(\bar{X}_s)\right)^2 \\
\leq \|Q_1\| \sum_{j} e^{-2\lambda_j(t-s)}(t-s)^{-2\kappa_4} \left( DU(X^\varepsilon_s) - DU(\bar{X}_s)\right)^2 \\
\leq C \|Q_1\| \|\bar{B}\|^2 \left( \sum_j e^{-2\lambda_j(t-s)}(t-s)^{-2\kappa_4}\right)^2. 
\]

This and assumption A3, we get

\[
J_4(T) \leq C_{p,T} \mathbb{E} \int_0^T \left( \int_0^t \left| X^\varepsilon_s - \bar{X}_s \right|^p \sum_{j} e^{-2\lambda_j(t-s)}(t-s)^{-2\kappa_4} ds \right)^{p/2} dt \\
\leq C_{p,T} \mathbb{E} \left( \sup_{s \in [0,t]} \left| X^\varepsilon_s - \bar{X}_s \right|^p \right) \left( \sum_{j} \int_0^t e^{-2\lambda_j s} s^{-2\kappa_4} ds \right)^{p/2} dt \\
\leq C_{p,T} \left( \sum_j \lambda_j^{2\kappa_4-1} \right)^{p/2} \left( \int_0^t e^{-r_{2\kappa_4} dr} \right)^{p/2} \mathbb{E} \left( \sup_{s \in [0,t]} \left| X^\varepsilon_s - \bar{X}_s \right|^p \right) dt \\
\leq C_{p,T} \mathbb{E} \left( \sup_{s \in [0,t]} \left| X^\varepsilon_s - \bar{X}_s \right|^p \right) dt. \tag{3.39}
\]
Combining (3.35)–(3.39), we get

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) \leq C_p \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) + C_p \mathbb{E} \left( \sup_{t \in [0,T]} |\tilde{X}_t - \tilde{X}_t|^p \right) \\
\leq C_p D_p^\| \tilde{B} \|^p_{C_0^{\alpha \land \beta \gamma}(\mathbb{R})} \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) + C_{p,T,\lambda} \int_0^T \mathbb{E} \left( \sup_{s \in [0,t]} |X_s^e - \tilde{X}_s|^p \right) dt \\
+ C_p \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) + \mathbb{E} J_5(T) + \mathbb{E} J_6(T).
\]

For any fixed \( p, T > 0 \), since \( \lim_{\lambda \to \infty} D_\lambda = 0 \), taking \( \lambda \) sufficient large such that \( C_p D_p^\| \tilde{B} \|^p_{C_0^{\alpha \land \beta \gamma}(\mathbb{R})} \leq 1/2 \), then we have

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) \leq C_{p,T} \mathbb{E} \left( \sup_{s \in [0,t]} |X_s^e - \tilde{X}_s|^p \right) dt \\
+ C_p \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) + \mathbb{E} J_5(T) + \mathbb{E} J_6(T).
\]

Then the Gronwall inequality yields

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) \leq C_{p,T} \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^e - \tilde{X}_t|^p \right) + \mathbb{E} J_5(T) + \mathbb{E} J_6(T) \right]. \tag{3.40}
\]

**Step 2:** In this step, we intend to estimate \( \mathbb{E} J_5(T) \) and \( \mathbb{E} J_6(T) \).

\[
\mathbb{E} J_6(T) \\
\leq C_{p,T} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s(\delta))A} \langle DU(X_{s(\delta)}^e), B(X_{s(\delta)}^e), \tilde{Y}_{s(\delta)}^e \rangle - \tilde{B}(X_{s(\delta)}^e) \rangle ds \right|^2 \right] \\
\leq C_{p,T} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \sum_{k=0}^{t/\delta - 1} \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta - s(\delta))A} \langle DU(X_{s(\delta)}^e), B(X_{s(\delta)}^e), \tilde{Y}_{s(\delta)}^e \rangle - \tilde{B}(X_{s(\delta)}^e) \rangle ds \right|^2 \right] \\
+ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s(\delta))A} \langle DU(X_{s(\delta)}^e), B(X_{s(\delta)}^e), \tilde{Y}_{s(\delta)}^e \rangle - \tilde{B}(X_{s(\delta)}^e) \rangle ds \right|^2 \right] \\
\leq C_{p,T} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \sum_{k=0}^{t/\delta - 1} \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta - s(\delta))A} \langle DU(X_{s(\delta)}^e), B(X_{s(\delta)}^e), \tilde{Y}_{s(\delta)}^e \rangle - \tilde{B}(X_{s(\delta)}^e) \rangle ds \right|^2 \right] \\
+ C_{p,T} \mathbb{E} \left[ \int_0^t e^{(t-s(\delta))A} \langle DU(X_{s(\delta)}^e), B(X_{s(\delta)}^e), \tilde{Y}_{s(\delta)}^e \rangle - \tilde{B}(X_{s(\delta)}^e) \rangle ds \right]^2 \\
:= J_{61}(T) + J_{62}(T).
\]

For the term \( J_{62}(T) \), by the boundedness of \( |B|, |\tilde{B}| \) and \( ||DU||_\infty \), it is easy to see that

\[
J_{62}(T) \leq C_{p,T} \delta^2. \tag{3.41}
\]
For the term $J_{61}(T)$, we have

$$J_{61}(T) \leq C_{p,T} \frac{|T/\delta| - 1}{\delta} \sum_{k=0}^{(k+1)\delta - 1} \mathbb{E} \left| \int_{k\delta}^{(k+1)\delta} e^{(k+1)\delta - k\delta} \langle DU(X_{k\delta}^{\varepsilon}, B(X_{k\delta}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) - \tilde{B}(X_{k\delta}^{\varepsilon}) \rangle ds \right|^2$$

\[
\leq \frac{C_{p,T}}{\delta^2} \max_{0 \leq k \leq |T/\delta| - 1} \mathbb{E} \left| \int_{k\delta}^{(k+1)\delta} B(X_{k\delta}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) - \tilde{B}(X_{k\delta}^{\varepsilon}) ds \right|^2 
\leq \frac{C_{p,T}\varepsilon^2}{\delta^2} \max_{0 \leq k \leq |T/\delta| - 1} \mathbb{E} \left| \int_{0}^{\delta/\varepsilon} B(X_{k\delta}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) - \tilde{B}(X_{k\delta}^{\varepsilon}) ds \right|^2 
= \frac{C_{p,T}\varepsilon^2}{\delta^2} \max_{0 \leq k \leq |T/\delta| - 1} \int_{0}^{\frac{\delta}{\varepsilon}} \int_{r}^{\frac{\delta}{\varepsilon}} \Psi_k(s, r) ds dr, \tag{3.42}
\]

where for any $0 \leq r \leq s \leq \frac{\delta}{\varepsilon}$,

$$\Psi_k(s, r) := \mathbb{E} \left[ (B(X_{k\delta}^{\varepsilon}, \hat{Y}_{r}^{\varepsilon}) - \tilde{B}(X_{k\delta}^{\varepsilon}), B(X_{k\delta}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) - \tilde{B}(X_{k\delta}^{\varepsilon})) \right].$$

Then by Proposition 3.4 and following a standard argument (see [12] for instance), it is easy to see that there exists $\omega > 0$ such that

$$\Psi_k(s, r) \leq C_T(|y|^2 + 1) e^{-\frac{(s-r)\omega}{2}}. \tag{3.43}$$

As a result, it follows from (3.41)-(3.43),

$$J_0(T) \leq C_{p,T} \frac{\varepsilon^2}{\delta^2} \int_{0}^{\frac{\delta}{\varepsilon}} \int_{r}^{\frac{\delta}{\varepsilon}} e^{-\frac{(s-r)\omega}{2}} ds dr + C_{p,T} \delta^2$$

$$= C_{p,T} \frac{\varepsilon^2}{\delta^2} \frac{\delta}{\omega} - \frac{1}{\omega^2} + \frac{1}{\omega^2} e^{-\frac{\omega\delta}{2}} + C_{p,T} \delta^2$$

$$\leq C_{p,T} \frac{\varepsilon^2}{\delta^2} + C_{p,T} \frac{\varepsilon}{\delta} + C_{p,T} \delta^2. \tag{3.44}$$

By the similar argument above, we also obtain

$$J_5(T) \leq C_{p,T} \frac{\varepsilon^2}{\delta^2} + C_{p,T} \frac{\varepsilon}{\delta} + C_{p,T} \delta^2. \tag{3.45}$$

**Step 3:** By the preparation above, we intend to finish the proof in this step, by Lemma 3.9, (3.40), (3.44) and (3.45), we have

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t|^p \right) \leq C_{p,T} \delta^{\theta/2} + C_{p,T} \exp \left\{ \frac{C_{p,\lambda} \delta^{\beta p}}{\varepsilon^{\beta p}} + \frac{C_p \delta^p}{\varepsilon^{\beta p}} + \frac{C_p \delta^{2p/2}}{\varepsilon^{\beta p}} \right\} \delta^{\theta/2}$$

$$+ C_{p,T} \frac{\varepsilon^2}{\delta^2} + C_{p,T} \frac{\varepsilon}{\delta} + C_{p,T} \delta^2.$$  

Then we can first take $\delta = \delta(\varepsilon)$ such that

$$\delta \to 0, \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad \frac{\delta}{\varepsilon} \to \infty, \quad \text{as} \quad \varepsilon \to 0.$$  

As a result,

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t|^p \right) \leq C_{p,T} \left( \delta^{\theta/2} + \frac{\varepsilon}{\delta} \right) + C_{p,T} \exp \left\{ \frac{C_{p,\lambda} \delta^{\beta p}}{\varepsilon^{\beta p}} \right\} \delta^{\theta/2}.$$
Secondly, by taking $\delta = \delta(\varepsilon)$ such that
\[
\exp \left\{ C_p, \lambda \frac{\delta^p}{\varepsilon^p} \right\} \delta^{p/2} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

For instance, by taking $\delta = \varepsilon \left[ \frac{1}{2C_p, \lambda} \ln \left( \frac{1}{\varepsilon^p} \right) \right]^{1/p}$ which satisfies the above properties. Then it is easy to see that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} |X^\varepsilon_t - \bar{X}_t|^p \right) = 0.
\]
The whole proof is complete.

4. Application to example

In this section we will apply our main result to establish the averaging principle for a class of slow-fast SPDEs with Hölder continuous coefficients. i.e., considering the following non-linear stochastic heat equation on $D = [0, \pi]$ with Dirichlet boundary conditions:

\[
\begin{cases}
    dX^\varepsilon(t, \xi) = [\Delta X^\varepsilon(t, \xi) + B(X^\varepsilon(t, \cdot), Y^\varepsilon(t, \cdot))](\xi) \, dt + (-\Delta)^{-r_1/2}dW^1(t, \xi), \\
    dY^\varepsilon(t, \xi) = \frac{1}{\varepsilon} [\Delta Y^\varepsilon(t, \xi) + F(X^\varepsilon(t, \cdot), Y^\varepsilon(t, \cdot))](\xi) \, dt + \frac{1}{\varepsilon^{1/2}}(-\Delta)^{-r_2/2}dW^2(t, \xi), \\
    X^\varepsilon(t, \xi) = Y^\varepsilon(t, \xi) = 0, \quad t > 0, \quad \xi \in \partial D, \\
    X^\varepsilon(0, \xi) = x(\xi), \quad Y^\varepsilon(0, \xi) = y(\xi), \quad \xi \in D, \quad x, y \in H,
\end{cases}
\tag{4.1}
\]

where $\partial D$ the boundary of $D$, $W^1_t$ and $W^2_t$ are two cylindrical Wiener process in $H := L^2(D)$ (with Dirichlet boundary conditions). Put

\[
A x = \Delta x, \quad x \in \mathcal{D}(A) = H^2(D) \cap H^1_0(D);
\]

\[
Q_1 = (-\Delta)^{-r_1}, \quad Q_2 = (-\Delta)^{-r_2}, \quad r_1, r_2 \in (-1/2, 1/7);
\]

\[
B(x, y)(\xi) = C_1 \sin(\sqrt{|x(\xi)|} + \sqrt{|y(\xi)|}), \quad x, y \in H;
\]

\[
F(x, y)(\xi) = C_2 \cos(\sqrt{|x(\xi)|} + \sqrt{|y(\xi)|}), \quad x, y \in H.
\]

Then it is easy to check that $B$ and $F$ are bounded and Hölder continuous with index $\alpha = \beta = \gamma = \eta = 1/2$. So the assumption A1 holds.

The operator $A$ is a self-adjoint operator and possesses a complete orthonormal system of eigenfunctions, namely

\[
\epsilon_k(\xi) = (\sqrt{2/\pi}) \sin(k\xi), \quad \xi \in [0, \pi],
\]

where $k \in \mathbb{N}$. The corresponding eigenvalues of $A$ are $-\lambda_k$ with $\lambda_k = k^2$. As a result, it is easy to see assumption A2 is satisfied. Moreover, assumption A3 holds for any $\xi \in (0, 1/2)$.

For the assumption A4, we first note that for $i = 1, 2$,

\[
Q_i(t) = \int_0^t e^{sA}(-A)^{-r_i} e^{sA^*} ds = \frac{1}{2}(-A)^{-(r_i+1)}(I - e^{2tA}).
\]

So $Q_i(t)$ is a trace class operator if

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{r_i+1}} < \infty,
\]

which holds by $r_i > -1/2$. 

\[
\int_0^t e^{sA}(-A)^{-r_i} e^{sA^*} ds = \frac{1}{2}(-A)^{-(r_i+1)}(I - e^{2tA}).
\]

So $Q_i(t)$ is a trace class operator if

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{r_i+1}} < \infty,
\]

which holds by $r_i > -1/2$. 

\[
\int_0^t e^{sA}(-A)^{-r_i} e^{sA^*} ds = \frac{1}{2}(-A)^{-(r_i+1)}(I - e^{2tA}).
\]
By a straightforward computer, for any $\theta \in (0, r_1 \land r_2 + 1/2)$,
\[
\int_0^T r^{-\theta} \| e^{rA} \sqrt{Q_1} \|^2_{HS} dr \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{1+\theta}} < \infty, \quad \forall T > 0.
\]
\[
\int_0^T \| (-A)^{\theta/2} e^{rA} \sqrt{Q_1} \|^2_{HS} dr \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{1+\theta}} < \infty, \quad \forall T > 0.
\]
\[
\int_0^\infty \| e^{rA} \sqrt{Q_2} \|^2_{HS} dr \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2+\theta}} < \infty,
\]
which imply the conditions (2.1)-(2.3) hold.

To show assumption $A_5$, note that
\[
\Lambda_i(t) = Q_i^{-1/2}(t) e^{tA} = \frac{1}{2} (-A)^{(1+r_i)/2} (I - e^{2tA})^{-1/2} e^{tA}.
\]
Then $\Lambda_i(t)$ is a bounded operator for any $t > 0$. In fact,
\[
\| \Lambda_i(t) \| = \frac{1}{2} (-A)^{(1+r_i)/2} (I - e^{2tA})^{-1/2} e^{tA} \|
\leq Ct^{-1/2} \| (-A)^{-1/2} (I - e^{2tA})^{-1/2} \| \cdot \| (-A)^{r_i/2} e^{tA} \|
\leq Ct^{-(1+r_i)/2},
\]
where we use the fact that the operator $(-tA)^{-1/2} (I - e^{2tA})^{-1/2}$ is uniformly bounded in $t$.
Furthermore, for any $\lambda > 0$
\[
\int_0^\infty e^{-\lambda t} \| \Lambda_i(t) \|^{1+\kappa_1} dt \leq \int_0^\infty e^{-\lambda t} t^{-(1+r_i)(1+\kappa_1)/2} dt < \infty
\]
holds for $\kappa_1 = 3/4$ due to $r_i < 1/7$. Therefore the condition (2.4) holds.

By a similar arguments, we also have for any $\kappa_2 \in \left(0, \frac{(1-r_2)(1-r_2)}{2}\right)$,
\[
\int_0^\infty e^{-\lambda t} \| (-A)^{\kappa_2} \Lambda_i(t) \| dt \leq C \int_0^\infty e^{-\lambda t} t^{-(1+r_i+2\kappa_2)/2} dt < \infty,
\]
which verifies the assumption (2.5) holds. Consequently, by Theorem 2.1, the slow component $X^\varepsilon$ of the stochastic system (4.1) strongly convergence to the solution $\bar{X}$ of the corresponding averaged equation.

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**References**

[1] R. Bertram, J.E. Rubin, *Multi-timescale systems and fast-slow analysis*, Math. Biosci. 287 (2017) 105-121.
[2] N.N. Bogoliubov, Y.A. Mitropolsky, *Asymptotic methods in the theory of Non-linear Oscillations*, Gordon and Breach Science Publishers, New York (1961).
[3] C.E. Bréhier, *Strong and weak orders in averaging for SPDEs*, Stochastic Process. Appl. 122 (2012) 2553-2593.
[4] S. Cerrai, *A Khasminskii type averaging principle for stochastic reaction-diffusion equations*, Ann. Appl. Probab. 19 (2009) 899-948.
[5] S. Cerrai, *Averaging principle for systems of reaction-diffusion equations with polynomial nonlinearities perturbed by multiplicative noise*, SIAM J. Math. Anal. 43 (2011) 2482-2518.
[6] S. Cerrai, M. Freidlin, *Averaging principle for stochastic reaction-diffusion equations*, Probab. Theory Related Fields 144 (2009) 137-177.
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[7] S. Cerrai, A. Lunardi, Averaging principle for nonautonomous slow-fast systems of stochastic reaction-diffusion equations: the almost periodic case, SIAM J. Math. Anal. 49 (2017) 2843-2884.

[8] A. Chojnowska-Michalik, B. Goldys, Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces, Probab. Theory Related Fields 102 (1995) 331-356.

[9] G. Da Prato and F. Flandoli, Pathwise uniqueness for a class of SPDEs in Hilbert spaces and applications, J. Funct. Anal. 259 (2010) 243-267.

[10] G. Da Prato, F. Flandoli, E. Priola and M. Röckner. Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. Ann. Probab. 41 (2013) 3306-3344.

[11] G. Da Prato, F. Flandoli, E. Priola and M. Röckner. Strong Uniqueness for Stochastic Evolution Equations with Unbounded Measurable Term. J. Theor. Prob. 28 (2015) 1571-1600.

[12] Z. Dong, X. Sun, H. Xiao, J. Zhai, Averaging principle for one dimensional stochastic Burgers equation, J. Differential Equations 265 (2018) 4749-4797.

[13] W. E, B. Engquist, Multiscale modeling and computations, Notice of AMS, 50 (2003) 1062-1070.

[14] H. Fu, L. Wan, J. Liu, Strong convergence in averaging principle for stochastic hyperbolic-parabolic equations with two time-scales, Stochastic Process. Appl. 125 (2015) 3255-3279.

[15] H. Fu, L. Wan, J. Liu, X. Liu, Weak order in averaging principle for stochastic wave equation with a fast oscillation, Stochastic Process. Appl. 128 (2018) 2557-2580.

[16] P. Gao, Averaging principle for stochastic Kuramoto-Sivashinsky equation with a fast oscillation, Discrete Contin. Dyn. Syst.-A 38 (2018) 5649-5684.

[17] P. Gao, Averaging principle for the higher order nonlinear Schrödinger equation with a random fast oscillation, J. Stat. Phys. 171 (2018) 897-926.

[18] P. Gao, Y. Li, Averaging Principle for Multiscale Stochastic Klein-Gordon-Heat System, J. Nonlinear Sci. 29 (4) (2019) 1701-1759.

[19] B. Goldys, B. Maslowski, Exponential ergodicity for stochastic reaction-diffusion equations, Stochastic partial differential equations and applications-VII, 115-131, Lect. Notes Pure Appl. Math., 245, Chapman Hall/CRC, Boca Raton, FL, 2006.

[20] E. Harvey, V. Kirk, M. Wechselberger, J. Sneyd, Multiple timescales, mixed mode oscillations and canards in models of intracellular calcium dynamics, J. Nonlinear Sci. 21 (2011) 639-683.

[21] Khasminskii R. Z.: On stochastic processes defined by differential equations with a small parameter, Theory Probab. Appl. 11 (1966) 211-228.

[22] Krylov N. V. and Röckner M.: Strong solutions of stochastic equations with singular time dependent drift, Probab. Theory Related Fields 131 (2005) 154-196.

[23] C. Kuehn, Multiple time scale dynamics, volume 191 of Applied Mathematical Sciences. Springer, Cham, 2015.

[24] D. Liu, Strong convergence of principle of averaging for multiscale stochastic dynamical systems, Commun. Math. Sci. 8 (2010) 999-1020.

[25] W. Liu, M. Röckner, X. Sun, Y. Xie, Averaging principle for slow-fast stochastic partial differential equations with time dependent locally Lipschitz coefficients, J. Differential Equations (2019), https://doi.org/10.1016/j.jde.2019.09.047.

[26] W. Liu, M. Röckner, X. Sun, Y. Xie, Strong averaging principle for slow-fast stochastic partial differential equations with locally monotone coefficients, https://arxiv.org/abs/1907.03260.

[27] E. Pardoux E. and A. Yu. Veretennikov, On the Poisson equation and diffusion approximation. I, Ann. Prob. 29 (2001) 1061-1085.

[28] E. Pardoux E. and A. Yu. Veretennikov, On the Poisson equation and diffusion approximation 2, Ann. Prob. 31 (2003) 1166-1192.

[29] M. Röckner, X. Sun, L. Xie, Strong and weak convergence in the averaging principle for SDEs with Hölder coefficients, https://arxiv.org/abs/1907.09256.

[30] A.Yu. Veretennikov, On the averaging principle for systems of stochastic differential equations, Math. USSR Sborn. 69 (1991) 271-284.

[31] W. Wang, A.J. Roberts, Average and deviation for slow-fast stochastic partial differential equations, J. Differential Equations 253 (2012) 1265-1286.

[32] W. Wang, A.J. Roberts, J. Duan, Large deviations and approximations for slow-fast stochastic reaction-diffusion equations, J. Differential Equations 253 (2012) 3501-3522.
[33] D. Yang: *Pathwise uniqueness for stochastic evolution equations with Hölder drift and stable Lévy noise*, Nonlinear Differ. Equ. Appl. (2018) 25:20.

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