Stochastic Vs Worst-case Condition Numbers

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December 25, 2008

Abstract

We compare Stochastic and Worst-case condition numbers and loss of precision for general computational problems. We show an upper bound for the ratio of Worst-case condition number to the Stochastic condition number of order $O(\sqrt{m})$. We show an upper bound for the difference between the Worst-case loss of precision and the Stochastic loss of precision of order $O(\ln m)$. The results hold if the perturbations are measured norm-wise or componentwise.

1 Introduction

Let $x \in \mathbb{R}^m$ and $f(x) \in \mathbb{R}^m$ be the input and output of a computational problem respectively. We assume $f(x)$ is differentiable. Condition numbers are real numbers measuring the sensitivity of the output $f(x)$ to the input $x$ of the problem. But there are many different versions of condition numbers. Below are the definitions of Worst-case Norm-wise, Worst-case Component-wise, Stochastic Norm-wise and Stochastic Component-wise condition numbers. For any set $S$, we write $x \sim S$ if $x$ is a random variable (or vector) uniformly distributed in $S$. And we denote by $\mathbb{E}_{x \sim S} f(x)$ the expected
value of \( f(x) \) when \( x \sim \mathcal{S} \).

\[
\text{WNC}(x) = \lim_{\delta \to 0} \sup_{x' \in \mathcal{P}(x,\delta)} \frac{\|f(x') - f(x)\|}{\delta\|f(x)\|},
\]

\[
\text{WCC}_j(x) = \lim_{\delta \to 0} \sup_{x' \in \mathcal{C}\mathcal{P}(x,\delta)} \frac{|f_j(x') - f_j(x)|}{\delta|f_j(x)|},
\]

\[
\text{SNC}(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x,\delta)} \frac{\|f(x') - f(x)\|}{\delta\|f(x)\|} \quad \text{and}
\]

\[
\text{SCC}_j(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{C}\mathcal{P}(x,\delta)} \frac{|f_j(x') - f_j(x)|}{\delta|f_j(x)|},
\]

where

\[
\mathcal{P}(x,\delta) = \{x' \in \mathbb{R}^m : \|x' - x\| \leq \delta\|x\|\} \quad \text{and}
\]

\[
\mathcal{C}\mathcal{P}(x,\delta) = \{x' \in \mathbb{R}^m : |x'_i - x_i| \leq \delta|x_i| \text{ for } i = 1, \ldots, m\}.
\]

There are two reasons why researchers study condition numbers. First, when we input a real number \( x_1 \) in a computer, the computer can never store \( x_1 \) with 100\% accuracy. Instead, an approximate value \( x'_1 \) will be stored. How accurate we can store a number depends on the data type chosen for storing the number \( x_1 \). Suppose we store this number with the data type \texttt{double}. The relative error

\[
\frac{|x'_1 - x_1|}{|x_1|} \leq \frac{1}{2^{52}}, \quad \text{which means } x' \in \mathcal{C}\mathcal{P} \left( x, \frac{1}{2^{52}} \right).
\]

So, if \( \text{WCC}_1(x) = 4 \), we can ensure

\[
\frac{|f_1(x') - f_1(x)|}{|f_1(x)|} \leq 4 \times \frac{1}{2^{52}} = \frac{1}{2^{50}}.
\]

Similarly, if \( \text{SCC}_1(x) = 4 \), we can expect

\[
\frac{|f_1(x') - f_1(x)|}{|f_1(x)|} = 4 \times \frac{1}{2^{52}} = \frac{1}{2^{50}}.
\]

The second reason of studying condition numbers is related to the stability of algorithms. Even if we can store the input \( x \) accurately, we still cannot find the output \( f(x) \) with 100\% accuracy. It is because errors appear and accumulate after every operation (addition, subtraction and etc.) done in a computer. How accurate we can compute \( f(x) \) depends on the algorithm

\[
\text{f}(x)
\]
applied. We say that an algorithm is **backward stable** if the computed output \( f' \) satisfy the following.

\[
f' = f(x') \text{ for some } x, \text{ s.t. } \|x' - x\| \leq \|x\| O(\varepsilon_{\text{machine}}),
\]

where \( \varepsilon_{\text{machine}} \) is the upper bound for the relative error occurring after one operation done. As a result, applying a backward stable algorithm, one can **ensure** the computed solution \( f' \) satisfy the following.

\[
\frac{\|f' - f(x)\|}{\|f(x)\|} \leq \text{WNC}(x)O(\varepsilon_{\text{machine}}).
\]

Or one can **expect**

\[
\frac{\|f' - f(x)\|}{\|f(x)\|} = \text{SNC}(x)O(\varepsilon_{\text{machine}}).
\]

Unless specified, \( \log(x) \) refer to the logarithm with base 2. \( \log(|x| - \log|x' - x|) \) is called the precision of \( x' \). Roughly speaking, it is the number of trustable (or accurate) bits. \( \log \text{WNC}(x) \) is called the **Worst-case Norm-wise Loss of Precision** since \( \log \text{WNC}(x) = \lim_{\delta \to 0} \sup_{x' \in \mathcal{P}(x,\delta)} \left( \log|x| - \log|x' - x| - (\log|f(x)| - \log|f(x') - f(x)|) \right) \)

\[
= \lim_{\delta \to 0} \sup_{x' \in \mathcal{P}(x,\delta)} \text{Precision of input } x' - \text{Precision of output } f(x').
\]

Similarly, \( \log \text{WCC}_j(x) \) is called the **Worst-case Component-wise Loss of Precision**. Besides, we define **Stochastic Norm-wise Loss of Precision** and **Stochastic Component-wise Loss of Precision** as follows.

\[
\text{SNLP}(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x,\delta)} \log \frac{\|f(x') - f(x)\|}{\delta\|f(x)\|} \quad \text{and} \quad \text{SCLP}_j(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{CP}(x,\delta)} \log \frac{|f_j(x') - f_j(x)|}{\delta|f_j(x)|}.
\]

In short, both condition numbers and Loss of Precision are numbers telling us how trustable is the computed output when there is round-off errors. If these numbers are large, the output is not accurate and we should not trust the output. Otherwise, the computed output should be accurate and we can trust it.
The main goal of this paper is to compare the worst-case condition numbers with the stochastic condition numbers, i.e.

\[ \text{WNC}(x) \text{ Vs } \text{SNC}(x), \quad \text{WCC}(x) \text{ Vs } \text{SCC}(x), \]
\[ \log \text{WNC}(x) \text{ Vs } \text{SNLP}(x) \quad \text{and} \quad \log \text{WCC}(x) \text{ Vs } \text{SCLP}(x). \]

The theorem below is one of our main results. It compares SNC and WNC. Denote by \( e \) the base of the logarithm of \( \ln(\cdot) \).

**Theorem 1.** For any general computational problem with input \( x \in \mathbb{R}^m \) and output \( y \in \mathbb{R}^n \), let \( k = \min \{m, n\} \) then

\[
\frac{1}{e\sqrt{m}} \leq \frac{\text{SNC}(x)}{\text{WNC}(x)} \leq \sqrt{\frac{k}{m + 2}} \quad \text{and} \quad -\frac{\log m}{2} - \log e \leq \text{SNLP}(x) - \log \text{WNC}(x) \leq \frac{\log k - \log(m + 2)}{2}.
\]

Similar results can be found in [5] and [6]. In this paragraph, we explain the differences between our theorem and results in other papers. We write \( x \sim N(\mu, \Sigma) \) if \( x \) follows the multivariate normal distribution with mean \( \mu \) and variance-covariance matrix \( \Sigma \). In [6], the following quality (which is different from our definitions of condition numbers) was studied.

\[
\mathbb{E}_{(x' - x) \sim N(0, \Sigma)} \left\| \left[ \nabla f_1(x), \ldots, \nabla f_n(x) \right]^T (x' - x) \right\|,
\]

where \( \Sigma \) can be any variance-covariance matrix and \( \nabla f_j(x) \) is the gradient of \( f_j(x) \). So, their results are completely different from ours and depend on \( \Sigma \). The above quantity was studied since, by Taylor expansion, when \( x' \) is close to \( x \),

\[
\left\| \left[ \nabla f_1(x), \ldots, \nabla f_n(x) \right]^T (x' - x) \right\| \approx \left\| f(x') - f(x) \right\|.
\]

The results in [5] hold only for the problem of solving system of linear equations. Our theorem holds for general computational problems. Besides, the output \( f(x) \) in [5] was considered to be a real number in \( \mathbb{R} \). In this paper, the output is considered as a real vector in \( \mathbb{R}^n \).

Similar to theorem above, the corollary below compares SNC(\( x \)) and WNC(\( x \)). Comparing with theorem corollary is less explicit and less general (only holds when \( n = 1 \)). But, it provides equality result instead of inequality. Both theorem and corollary will be proved in section 2.
Corollary 1. For any general computational problem with input \( x \in \mathbb{R}^m \) and output \( y \in \mathbb{R} \),
\[
\frac{\text{SNC}(x)}{\text{WNC}(x)} = \begin{cases} 
\frac{(m)(m-2)...1}{(m+1)(m-1)...1} & \text{if } m \text{ is odd} \\
\frac{(m)(m-2)...2}{(m+1)(m-1)...1} \left( \frac{2}{e} \right) & \text{if } m \text{ is even}
\end{cases}
\]
\[
\frac{\text{SNLP}(x) - \log \text{WNC}(x)}{\log e} = \begin{cases} 
-\frac{1}{m} - \frac{1}{m-2} - ... - \frac{1}{3} - 1 & \text{if } m \text{ is odd} \\
-\frac{1}{m} - \frac{1}{m-2} - ... - \frac{1}{2} - \ln 2 & \text{if } m \text{ is even}
\end{cases}.
\]

The theorem 2 below compares \( \text{WCC}(x) \) and \( \text{SCC}(x) \).

Theorem 2. For any general computational problem with input \( x \in \mathbb{R}^m \) and output \( y \in \mathbb{R}^n \), if \( m > 1 \),
\[
-\log(m-1) - \frac{\log 3}{2} - (1 + \varepsilon_m) \log e < \text{SCLP}_j(x) - \log \text{WCC}_j(x) \leq -1 \quad \text{and}
\]
\[
\frac{e^{-(1+\varepsilon_m)}}{\sqrt{3(m-1)}} < \frac{\text{SCC}_j(x)}{\text{WCC}_j(x)} \leq \frac{1}{2} \quad \text{where } \varepsilon_m = \frac{2 + 2 \ln m}{\sqrt{m-1}}.
\]

Note: It can be easily shown that, \( \text{SCLP}_j(x) = \log \text{WCC}_j(x) - 1 \) and \( \text{SCC}_j(x) = 0.5 \times \text{WCC}_j(x) \) when \( m = 1 \).

Similar results can also be found in [5]. Just like our theorem 2, the result in [5] depends on a constant \( \varepsilon'_m \) which approach to 0 as \( m \to \infty \). But, the speed of convergency was not discussed. Let alone a formula computing the value (or bound) of \( \varepsilon'_m \) for general \( m \). Our theorem 2 only depends on the size of the input and output (\( m \) and \( n \)). Once again, in [5], only one problem (solving system of linear equations) was considered. In this paper, we consider general computational problems. Theorem 2 will be proved in section 3.

Note: In practice, \( \frac{\log m}{2} \) is not very large (\( \frac{\log m}{2} = 20 \) when the number of data input \( m = 1.0995... \times 10^{12} \)). So, from the theorems above, we claim that the value difference between worst case and stochastic loss of precision are normally very small in practice.

2 Proof of Corollary 1 and Theorem 1

For any \( c \in \mathbb{R}^m \) and \( r \in \mathbb{R} \), let the ball centered at \( c \) and with radius \( r \) be
\[
B^m(c, r) = \{ u \in \mathbb{R}^m : \| u - c \| \leq r \}.
\]
and let the sphere be
\[ S^{m-1}(c, r) = \{ u \in \mathbb{R}^m : \| u - c \| = r \}. \]

**Lemma 1.** If \( u \sim B^m(0, 1) \), then
\[
\mathbb{E}(\| u \|) = \frac{m}{m + 1}, \quad \mathbb{E}(\| u \|^2) = \frac{m}{m + 2}, \quad \text{and} \quad \mathbb{E}(\ln \| u \|) = -\frac{1}{m}.
\]

**Proof.** Since \( u \sim B^m(0, 1) \),
\[
\text{Prob}(\| u \| < r) = \frac{\text{The volume of } B^m(0, r)}{\text{The volume of } B^m(0, 1)} = r^m.
\]
So, the p.d.f. (probability density function) of \( \| u \| \) is
\[
f_{\| u \|}(r) = \frac{d}{dr}(r^m) = mr^{m-1}.
\]
By the definition of expectation and integration by parts,
\[
\mathbb{E}(\| u \|) = \int_0^1 mr^m dr = \frac{m}{m + 1}.
\]
\[
\mathbb{E}(\| u \|^2) = \int_0^1 mr^{m+1} dr = \frac{m}{m + 2}.
\]
\[
\mathbb{E}(\ln \| u \|) = \int_0^1 mr^{-1} \ln(r) dr = -\frac{1}{m}.
\]

For any vectors \( u, v \in \mathbb{R}^m/\{0\} \), let the angle between \( u \) and \( v \) be
\[
\vartheta(u, v) = \arccos \left( \frac{u^T v}{\| u \| \| v \|} \right) \in [0, \pi].
\]
Suppose \( u \) is fixed and \( v/\| v \| \sim S^{m-1}(0, 1) \). From [2], the p.d.f. (probability density function) of \( \vartheta(u, v) \) is
\[
f_{\vartheta(u,v)}(t) = \frac{(\sin(t))^{m-2}}{I_{m-2}(\pi)}, \quad \text{where} \quad I_m(T) = \int_0^T (\sin(t))^m dt.
\]
By integration by part, it can be shown that

\[ I_m \left( \frac{\pi}{2} \right) = \left( \frac{m-1}{m} \right) I_{m-2} \left( \frac{\pi}{2} \right). \]  

(1)

It is easy to check that

\[ I_0 \left( \frac{\pi}{2} \right) = \frac{\pi}{2} \quad \text{and} \quad I_1 \left( \frac{\pi}{2} \right) = 1. \]  

(2)

Combining equations (1) and (2), for \( m \geq 2 \),

\[ I_m \left( \frac{\pi}{2} \right) = \begin{cases} \frac{(m-1)(m-3)...2}{m(m-2)...1} & \text{if } m \text{ is odd} \\ \frac{(m-2)(m-4)...2}{(m-1)(m-3)...1} \left( \frac{2}{\pi} \right) & \text{if } m \text{ is even} \end{cases}. \]  

(3)

Lemma 2. Suppose \( u \) is a fixed vector in \( \mathbb{R}^m \) and \( v/\|v\| \sim S^{m-1}(0, 1) \). Then, for \( m \geq 3 \),

\[ E(|\cos \vartheta(u, v)|^2) = \frac{1}{m}, \]

\[ E(|\cos \vartheta(u, v)|) = \begin{cases} \frac{(m-2)(m-4)...1}{(m-1)(m-3)...2} & \text{if } m \text{ is odd} \\ \frac{(m-2)(m-4)...2}{(m-1)(m-3)...1} \left( \frac{2}{\pi} \right) & \text{if } m \text{ is even} \end{cases}. \]

\[ E(\ln |\cos \vartheta(u, v)|) = \begin{cases} -\frac{1}{m-2} - \frac{1}{m-4} - \cdots - \frac{1}{3} - 1 & \text{if } m \text{ is odd} \\ -\frac{1}{m-2} - \frac{1}{m-4} - \cdots - \frac{1}{2} - \ln 2 & \text{if } m \text{ is even} \end{cases}. \]

Proof. By the definition of Expectation,

\[ E(|\cos \vartheta(u, v)|) = \int_0^\pi |\cos(t)| f_{\vartheta(u,v)}(t) dt = \int_0^\pi |\cos(t)| \frac{(\sin(t))^{m-2}}{I_{m-2}(\pi/2)} dt \]

\[ = \int_0^{\pi/2} \cos(t) \frac{(\sin(t))^{m-2}}{I_{m-2}(\pi/2)} dt = \int_0^{\pi/2} (\sin(t))^{m-2} \frac{1}{I_{m-2}(\pi/2)} d\sin(t) \]

\[ = \frac{1}{(m-1)I_{m-2}(\pi/2)}. \]

By equation (3), for \( m \geq 3 \),

\[ E(|\cos \vartheta(u, v)|) = \begin{cases} \frac{(m-2)(m-4)...1}{(m-1)(m-3)...2} & \text{if } m \text{ is odd} \\ \frac{(m-2)(m-4)...2}{(m-1)(m-3)...1} \left( \frac{2}{\pi} \right) & \text{if } m \text{ is even} \end{cases}. \]
Similarly,
\[
\mathbb{E}(\cos \vartheta(u, v)^2) = \int_0^\pi (\cos(t))^2 f_{\vartheta(u, v)}(t) dt = \int_0^\pi (\cos(t))^2 \frac{(\sin(t))^{m-2}}{I_{m-2}(\pi)} dt
\]
\[
= \int_0^\pi \left(1 - (\sin(t))^2\right) \frac{(\sin(t))^{m-2}}{I_{m-2}(\pi)} dt
\]
\[
= \int_0^\pi \frac{(\sin(t))^{m-2} - (\sin(t))^m}{I_{m-2}(\pi)} dt = \frac{I_{m-2}(\pi) - I_m(\pi)}{I_{m-2}(\pi)}
\]
\[
= \frac{I_m(\pi/2) - I_m(\pi/2)}{I_{m-2}(\pi/2)} = 1 - \frac{m-1}{m} = \frac{1}{m}.
\]

The second equality above is due to equation (1). Besides, let
\[J_m(T) = \int_0^T (\sin(t))^m \ln |\cos(t)| dt.\]

Then,
\[
J_m\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} (\sin(t))^m \ln |\cos(t)| dt = -\int_0^{\pi/2} (\sin(t))^{m-1} \ln |\cos(t)| d(\cos(t))
\]
\[
= \int_0^{\pi/2} \cos(t) d((\sin(t))^{m-1} \ln |\cos(t)|)
\]
\[
= \int_0^{\pi/2} -(\sin(t))^m + (m-1)(\cos(t))^2(\sin(t))^{m-2} \ln |\cos(t)| dt
\]
\[
= -I_m(\pi/2) + (m-1)J_{m-2}(\pi/2) - (m-1)J_m(\pi/2).
\]

Combining the above and equation (1),
\[
\frac{J_m(\pi/2)}{I_m(\pi/2)} = \frac{J_{m-2}(\pi/2)}{I_{m-2}(\pi/2)} - \frac{1}{m}. \tag{4}
\]

Besides, it is easy to check that
\[
J_0\left(\frac{\pi}{2}\right) = -\frac{\pi \ln 2}{2} \quad \text{and} \quad J_1\left(\frac{\pi}{2}\right) = -1. \tag{5}
\]

Combining equations (2), (4) and (5), we have
\[
\frac{J_m(\pi/2)}{I_m(\pi/2)} = \begin{cases} 
-\frac{1}{m} - \frac{m-1}{m-2} = ... = -\frac{1}{3} - 1 & \text{if } m \text{ is odd} \\
-\frac{1}{m} - \frac{m-1}{m-2} = ... = -\frac{1}{2} - \ln 2 & \text{if } m \text{ is even} 
\end{cases}
\]

The proof is completed since \(\mathbb{E}(\ln |\cos \vartheta(u, v)|) = \frac{J_{m-2}(\pi/2)}{I_{m-2}(\pi/2)}. \) □
2.1 Proof of corollary 1

Proof. Denote by $\nabla f(x)$ the gradient of $f$. By Taylor expansion,

$$f(x') = f(x) + (x' - x)^T \nabla f(x) + O(\|x' - x\|^2).$$  \hspace{1cm} (6)

Combining the definitions of $\text{WNC}(x)$ and $\mathcal{P}(x, \delta)$ and equation (6),

$$\text{WNC}(x) = \lim_{\delta \to 0} \sup_{x' \in \mathcal{P}(x, \delta)} |(x' - x)^T \nabla f(x)| = \frac{\|x\| \|\nabla f(x)\|}{|f(x)|}. \hspace{1cm} (7)$$

Combining the definitions of $\text{SNC}(x)$ and $\mathcal{P}(x, \delta)$ and equation (6),

$$\text{SNC}(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} \left( \frac{(x' - x)^T \nabla f(x)}{\delta |f(x)|} \right)$$

$$= \|\nabla f(x)\| \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} \left( \frac{\|x' - x\| \times \cos \vartheta(\nabla f(x), x' - x)}{\delta |f(x)|} \right). \hspace{1cm} (8)$$

Since $x' \sim \mathcal{P}(x, \delta) = \text{Ball}(x, \delta \|x\|)$, by lemma 1

$$\mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} \|x' - x\| = \frac{m \delta \|x\|}{m + 1}. \hspace{1cm} (10)$$

By lemma 2

$$\mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} |\cos \vartheta(\nabla f(x), x' - x)| = \begin{cases} \frac{(m-2)(m-4)...1}{(m-1)(m-3)...2} & \text{if } m \text{ is odd} \\ \frac{(m-2)(m-4)...2}{(m-1)(m-3)...1} \left( \frac{2}{\pi} \right) & \text{if } m \text{ is even} \end{cases}. \hspace{1cm} (11)$$

Combining equations (9), (10), (11) and (7), we have

$$\frac{\text{SNC}(x)}{\text{WNC}(x)} = \begin{cases} \frac{(m)(m-2)...1}{(m+1)(m-1)...2} & \text{if } m \text{ is odd} \\ \frac{(m)(m-2)...2}{(m+1)(m-1)...1} \left( \frac{2}{\pi} \right) & \text{if } m \text{ is even} \end{cases}. \hspace{1cm} (12)$$

Similarly, applying lemmas 1 and 2 it can be shown that

$$\frac{\text{SNLP}(x) - \ln \text{WNC}(x)}{\ln e} = \begin{cases} -\frac{1}{m} - \frac{1}{m-2} - \ldots - \frac{1}{2} - 1 & \text{if } m \text{ is odd} \\ -\frac{1}{m} - \frac{1}{m-2} - \ldots - \frac{1}{2} - \ln 2 & \text{if } m \text{ is even} \end{cases} \hspace{1cm} (13)$$

\qed
2.2 Proof of Theorem 1

Proof. Denote by \( f_j(x) \) the \( j \)th entry of \( f(x) \). Denote by \( \nabla f_j(x) \) the gradient of \( f_j \). By Taylor Expansion,

\[
    f_j(x') = f_j(x) +(x' - x)^T \nabla f_j(x) + O(\|x' - x\|^2).
\]

Let \( G = [\nabla f_1(x), \nabla f_2(x), ..., \nabla f_n(x)] \in \mathbb{R}^{m \times n} \). So,

\[
    f(x') = f(x) + G^T(x' - x) + O(\|x' - x\|^2). \quad (12)
\]

Combining the definitions of \( \text{WNC}(x) \) and \( \mathcal{P}(x, \delta) \) and equation (12),

\[
    \text{WNC}(x) \overset{\text{def}}{=} \lim_{\delta \to 0} \sup_{x' \in \mathcal{P}(x, \delta)} \frac{\|G^T(x' - x)\|}{\delta \|f(x)\|} = \|x\| \|G\| \frac{\|f(x)\|}{\|f(x)\|} = \|x\| \|G\|. \quad (13)
\]

Combining the definition of \( \text{SNC}(x) \) and equation (12),

\[
    \text{SNC}(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} \left( \frac{\|G^T(x' - x)\|}{\delta \|f(x)\|} \right). \quad (14)
\]

Let \( UDV \) be the singular value decomposition of \( G^T \), i.e. \( U, V \in \mathbb{R}^{n \times n} \) are orthogonal matrices, \( D \in \mathbb{R}^{n \times m} \) is a diagonal matrix with entries \( \sigma_1, \sigma_2, ..., \sigma_k \) on its diagonal where \( k = \min\{m, n\} \),

\[
    G^T = UDV \quad \text{and} \quad \|G\| = \sigma_1 \geq \sigma_2 \geq ... \sigma_k \geq 0. \quad (15)
\]

Since \( U \) is orthogonal, by equations (14) and (15),

\[
    \text{SNC}(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} \left( \frac{\|UDV(x' - x)\|}{\delta \|f(x)\|} \right) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{P}(x, \delta)} \left( \frac{\|DV(x' - x)\|}{\delta \|f(x)\|} \right).
\]

Let \( x'' = V(x' - x) \). By the definition of \( \mathcal{P}(x, \delta) \), \( x'' \sim B^m(0, \delta \|x\|) \). So,

\[
    \text{SNC}(x) = \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \left( \frac{\|Dx''\|}{\delta \|f(x)\|} \right). \quad (16)
\]

\[
    \text{SNC}(x)^2 \leq \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \left( \frac{\sigma_1^2 x''_1^2 + \sigma_2^2 x''_2^2 + ... + \sigma_k^2 x''_k^2}{\delta^2 \|f(x)\|^2} \right) \leq \sigma_1^2 \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \left( \frac{x''_1^2 + x''_2^2 + ... + x''_k^2}{\delta^2 \|f(x)\|^2} \right). \quad (17)
\]
Let $v$ be the vector in $\mathbb{R}^m$ with the first $k$ entries equal to 1 and 0 elsewhere.

$$
\text{SNC}(x)^2 \leq \sigma_1^2 \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \left( \frac{(v^T x'')^2}{\delta^2 \|f(x)\|^2} \right)
$$

(19)

$$
= \sigma_1^2 \|v\|^2 \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \left( \frac{(\cos \theta(v, x''))^2 \|x''\|^2}{\delta^2 \|f(x)\|^2} \right)
$$

(20)

$$
= \left( \frac{\sigma_1^2 \|v\|^2 \|x\|^2}{(m + 2) \|f(x)\|^2} \right) \quad \text{By lemmas 1 and 2}
$$

(21)

Since $\sigma_1 = \|G\|$ and $\|v\| = \sqrt{k}$, by equations (13) and (21)

$$
\frac{\text{SNC}(x)}{\text{WNC}(x)} \leq \sqrt{\frac{k}{m + 2}}.
$$

Since $\log(\cdot)$ is concave function,

$$
\frac{\text{SNLP}(x) - \log \text{WNC}(x)}{\log e} \leq \frac{\log k - \log(m + 2)}{2}.
$$

On the other hand, by equation (16)

$$
\text{SNLP}(x) \geq \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \log \left( \frac{\sigma_1 x''^T}{\|f(x)\|} \right).
$$

Let $e_1$ be the vector in $\mathbb{R}^m$ with the first entry to 1 and 0 elsewhere.

$$
\text{SNLP}(x) \geq \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \log \left( \frac{\sigma_1 e_1^T x''}{\|f(x)\|} \right)
$$

$$
= \lim_{\delta \to 0} \mathbb{E}_{x'' \sim B^m(0, \delta \|x\|)} \log \left( \frac{\sigma_1 \|x''\| \cos \theta(e_1, x'')}{\|f(x)\|} \right).
$$

By equation (13) and lemmas 1 and 2,

$$
\frac{\text{SNLP}(x) - \log \text{WNC}(x)}{\log e} \geq \left\{ \begin{array}{ll}
\frac{1}{m} - \frac{1}{m - 1} - \cdots - \frac{1}{2} - 1 & \text{if } m \text{ is odd} \\
\frac{1}{m} - \frac{1}{m - 1} - \cdots - \frac{1}{2} - \ln 2 & \text{if } m \text{ is even}
\end{array} \right.
$$

Since $\ln m$ is the area of the region $\{(x, y) : 1 \leq x \leq n, 0 \leq y \leq 1/x\}$,

$$
\frac{\text{SNLP}(x) - \log \text{WNC}(x)}{\log e} \geq \left\{ \begin{array}{ll}
-\frac{1}{2} \ln m - 1 & \text{if } m \text{ is odd} \\
-\frac{1}{2} (\ln m - \ln 2) - \frac{1}{2} - \ln 2 & \text{if } m \text{ is even}
\end{array} \right.
$$

$$
\geq -\frac{1}{2} \ln m - 1.
$$

$$
\frac{\text{SNC}(x)}{\text{WNC}(x)} \geq \frac{1}{e \sqrt{m}} \quad \text{since } \log(\cdot) \text{ is a concave function.}
$$

$\Box$
3 Proof of Theorem 2

We write $Z \sim N(0, 1)$ if $Z$ is random variable following standard normal distribution. Below is the well-known Berry-Esseen theorem (See [4]).

\textbf{Theorem 3.} Let $u_1, \ldots, u_m$ be i.i.d. random variables with $E(u_1) = 0$, $E(u_1^2) = \sigma$ and $E(|u_1|^3) = \rho < \infty$. Then, for any real number $a$,

$$\left| \text{Prob} \left( a < \frac{u_1 + \ldots + u_m}{\sigma \sqrt{m}} \right) - \text{Prob}(a < Z) \right| \leq \frac{c \rho}{\sigma^3 \sqrt{m}}.$$

where $Z \sim N(0, 1)$ and $c$ is a universal constant (independent of $m$).

Calculated values of the constant $c$ have decreased markedly over the years, from 7.59 (Esseen’s original bound) to 0.7975 in 1972 (by P. van Beeck). The best current bound is 0.7655 (by I. S. Shiganov in 1986). The lemma below follows Berry-Esseen theorem immediately.

\textbf{Lemma 3.} Let $Z \sim N(0, 1)$ and $u_1, \ldots, u_m \sim [-1, 1]$ be i.i.d. random variables. Then, for any real number $a$,

$$\left| \text{Prob} \left( a < \frac{u_1 + \ldots + u_m}{\sqrt{m/3}} \right) - \text{Prob}(a < Z) \right| \leq \frac{1}{\sqrt{m}}.$$

\textit{Proof.} Since $u_m \sim [-1, 1]$, $\rho = 1/4$ and $\sigma^2 = 1/3$. By Berry-Esseen theorem,

$$\left| \text{Prob} \left( a < \frac{u_1 + \ldots + u_m}{\sqrt{m/3}} \right) - \text{Prob}(a < Z) \right| \leq \frac{3^{1/5} c}{4 \sqrt{m}} \approx 0.9555 c \sqrt{m}.$$

\hfill \Box

\textbf{Lemma 4.} Let $\delta, b$ be positive numbers, s.t. $b > 1$. Let $Z \sim N(0, 1)$. Then

$$\delta \ln \delta + \int_0^b \text{Prob}(Z > z) \ln \left| \frac{z + \delta}{z - \delta} \right| \, dz > 0.$$

\textit{Proof.} Let

$$F(\delta) = \int_0^b (z + \delta) \ln |z + \delta| - (z - \delta) \ln |z - \delta| \, d \text{Prob}(Z > z).$$
Since \( \text{Prob}(Z > 0) = 1/2 \), by integration by parts,

\[
F(\delta) = \left[ (b + \delta) \ln |b + \delta| - (b - \delta) \ln |b - \delta| \right] \text{Prob}(Z > b)
- \delta \ln \delta - \int_0^b \text{Prob}(Z > z) d \left[ (z + \delta) \ln |z + \delta| - (z - \delta) \ln |z - \delta| \right].
\]

Since \((b + \delta) \ln |b + \delta| = 2 \delta \ln |b + \delta| + (b - \delta) \ln |b + \delta| > (b - \delta) \ln |b - \delta|\), \( F(\delta) \)

\[
> - \delta \ln \delta - \int_0^b \text{Prob}(Z > z) [ (z + \delta) \ln |z + \delta| - (z - \delta) \ln |z - \delta| ]
\]  \hspace{1cm} (22)

= - \delta \ln \delta - \int_0^b \text{Prob}(Z > z) \ln \frac{z + \delta}{z - \delta} dz. \hspace{1cm} (23)

Obviously, \( F(0) = 0 \). Besides,

\[
\frac{dF(\delta)}{d\delta} = \int_0^b \frac{d(z + \delta) \ln |z + \delta|}{d\delta} - \frac{d(z - \delta) \ln |z - \delta|}{d\delta} d \text{Prob}(Z > z)
\]

\[
= \int_0^b \ln |z + \delta| - \ln |z - \delta| d \text{Prob}(Z > z)
\]

\[
= - \int_0^b \ln \left| \frac{z + \delta}{z - \delta} \right| f_Z(z) dz < 0 \hspace{1cm} \text{since} \hspace{0.5cm} \left| \frac{z + \delta}{z - \delta} \right| \geq 1, \forall z \geq 0.
\]

So, \( F(\delta) \leq 0 \). Together with equation (23), the proof is completed. \( \square \)

**Lemma 5.** Let \( \delta \leq \sqrt{3m} \) be a positive number. Let \( u_1, ..., u_m \sim [-1, 1] \) be i.i.d. random variables. Then

\[
\mathbb{E} \left( \left( \frac{u_1 + ... + u_m}{\sqrt{m/3}} + \delta \right) \ln \left| \frac{u_1 + ... + u_m}{\sqrt{m/3}} + \delta \right| \right)
> -\frac{2\delta}{\sqrt{m}} \left( \ln \left( 1 + \frac{\sqrt{3m}}{\delta} \right) + 1 \right).
\]

**Proof.** Let \( W = (u_1 + ... + u_m)\sqrt{3/m} \) and \( f_W(w) \) be the probability density
function of $W$. By definition of expectation,

$$\mathbb{E}[(W + \delta) \ln |W + \delta|] = \int_{-\sqrt{3m}}^{\sqrt{3m}} (w + \delta) \ln |w + \delta| f_W(w) \, dw$$

$$= \int_{-\sqrt{3m}}^{\sqrt{3m}} (w + \delta) \ln |w + \delta| f_W(w) \, dw + \int_{0}^{\sqrt{3m}} (w + \delta) \ln |w + \delta| f_W(w) \, dw$$

$$= \int_{0}^{\sqrt{3m}} (w + \delta) \ln |w + \delta| f_W(w) \, dw - \int_{\sqrt{3m}}^{0} (\delta - w) \ln |\delta - w| f_W(-w) \, dw$$

$$= \int_{0}^{\sqrt{3m}} [(w + \delta) \ln |w + \delta| - (w - \delta) \ln |w - \delta|] f_W(w) \, dw$$

$$= -\int_{0}^{\sqrt{3m}} (w + \delta) \ln |w + \delta| - (w - \delta) \ln |w - \delta| \, d \text{Prob}(W > w).$$

Note that: when $w = 0$, $\text{Prob}(W > w) = 0.5$ and $(w + \delta) \ln |w + \delta| = -(w - \delta) \ln |w - \delta| = \delta \ln \delta$. So, by integration by parts, $\mathbb{E}[(W + \delta) \ln |W + \delta|] - \delta \ln \delta$

$$= \int_{0}^{\sqrt{3m}} \text{Prob}(W > w) \, dw [(w + \delta) \ln |w + \delta| - (w - \delta) \ln |w - \delta|]$$

$$= \int_{0}^{\sqrt{3m}} \text{Prob}(W > w) \ln \left| \frac{w + \delta}{w - \delta} \right| \, dw.$$

Since $|w + \delta| \geq |w - \delta|$ for all $w > 0$, by lemma 4, $\mathbb{E}[(W + \delta) \ln |W + \delta|$

$$\geq \delta \ln \delta + \int_{0}^{\sqrt{3m}} \text{Prob}(Z > z) \ln \left| \frac{z + \delta}{z - \delta} \right| \, dz - \frac{1}{\sqrt{m}} \int_{0}^{\sqrt{3m}} \ln \left| \frac{w + \delta}{w - \delta} \right| \, dw$$

$$> -\frac{1}{\sqrt{m}} \int_{0}^{\sqrt{3m}} \ln \left| \frac{w + \delta}{w - \delta} \right| \, dw \quad \text{by lemma 4}$$

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So, \(-\sqrt{m} \mathbb{E}[(W + \delta) \ln |W + \delta|] \leq 1\),
\[
\begin{align*}
\int_0^{\sqrt{3m}} \ln \left| \frac{w + \delta}{w - \delta} \right| dw &= \int_0^{\sqrt{3m}} \ln(w + \delta) dw - \int_0^{\sqrt{3m}} \ln |w - \delta| dw \\
&= [w \ln w - w]_0^{\sqrt{3m} + \delta} - \int_0^{\sqrt{3m}} \ln (\delta - w) dw - \int_\delta^{\sqrt{3m}} \ln (w - \delta) dw \\
&= (\sqrt{3m} + \delta) \ln(\sqrt{3m} + \delta) - (\sqrt{3m} - \delta) \ln(\sqrt{3m} - \delta) - 2\delta \ln \delta \\
&= 2\delta \ln(\sqrt{3m} + \delta) + 2\delta - 2\delta \ln \delta = 2\delta \left( \ln \left( 1 + \frac{\sqrt{3m}}{\delta} \right) + 1 \right).
\end{align*}
\]

Lemma 6. If \(u_1, \ldots, u_{m+1} \sim [-1, 1]\) are i.i.d.,
\[
\mathbb{E}(\ln |u_1 + \ldots + u_{m+1}|) = \mathbb{E}((u_1 + \ldots + u_m + 1) \ln |u_1 + \ldots + u_m + 1|) - 1.
\]

Proof. For any fixed \(a \in \mathbb{R}\),
\[
\begin{align*}
\int_{-1}^{1} \ln |a + u| du &= \int_{a-1}^{a+1} \ln |v| dv \quad \text{(let } v = a + u) \quad (24) \\
&= [v \ln |v|]_{a-1}^{a+1} = (a + 1) \ln |a + 1| - (a - 1) \ln |a - 1| - 2. \quad (25)
\end{align*}
\]
So, by the definition of expectation, \(\mathbb{E}(\ln |u_1 + \ldots + u_{m+1}|)\)
\[
\begin{align*}
&= \frac{1}{2^{m+1}} \int_{-1}^{1} \cdots \int_{-1}^{1} \ln |u_1 + \ldots + u_{m+1}| du_{m+1} \cdots du_1 \\
&= \frac{1}{2^{m+1}} \int_{-1}^{1} \cdots \int_{-1}^{1} (u_1 + \ldots + u_{m+1} + 1) \ln |u_1 + \ldots + u_{m+1} + 1| du_{m+1} \cdots du_1 \\
&= \frac{1}{2^{m+1}} \int_{-1}^{1} \cdots \int_{-1}^{1} (u_1 + \ldots + u_{m} + 1) \ln |u_1 + \ldots + u_{m} + 1| du_{m+1} \cdots du_1 \\
&= \frac{1}{2^{m+1}} \int_{-1}^{1} \cdots \int_{-1}^{1} 2du_m \cdots du_1 \quad \text{by equation (25)}.
\end{align*}
\]
So, \( \mathbb{E}(\ln |u_1 + \ldots + u_{m+1}|) + 1 \)

\[
= \frac{1}{2} \mathbb{E}((u_1 + \ldots + u_m + 1) \ln |u_1 + \ldots + u_m + 1|) \quad (26)
\]

\[- \frac{1}{2} \mathbb{E}((u_1 + \ldots + u_m - 1) \ln |u_1 + \ldots + u_m - 1|). \quad (27)
\]

Since \( u \) and \(-u\) follow the same distribution,

\[
\mathbb{E}[(u_1 + \ldots + u_{m+1} - 1) \ln |u_1 + \ldots + u_{m+1} - 1|] \quad (28)
\]

\[
= \mathbb{E}[-(u_1 - \ldots - u_{m+1} - 1) \ln |-u_1 - \ldots - u_{m+1} - 1|] \quad (29)
\]

\[- \mathbb{E}[(u_1 + \ldots + u_{m+1} + 1) \ln |u_1 + \ldots + u_{m+1} + 1|]. \quad (30)
\]

Combining equations (26) and (30), the proof is completed. \(\square\)

In this section, we follow the definition of \(\varepsilon_m\) given in theorem 2,

\[
\varepsilon_m = \frac{2 + 2 \ln m}{\sqrt{m-1}}.
\]

**Corollary 2.** If \(u_1, \ldots, u_{m+1} \sim [-1, 1]\) are i.i.d. then

\[
\mathbb{E}(\ln |u_1 + \ldots + u_{m+1}|) > \frac{\ln m}{2} - \frac{\ln 3}{2} - 1 - \varepsilon_{m+1}.
\]

**Proof.** Let \(W = (u_1 + \ldots + u_m)\sqrt{3/m}\).

\[
\mathbb{E}((u_1 + \ldots + u_m + 1) \ln |u_1 + \ldots + u_m + 1|)
\]

\[
= \mathbb{E} \left( \left( W + \sqrt{3/m} \right) \ln \left| W + \sqrt{3/m} \right| \right)
\]

\[
= \sqrt{\frac{m}{3}} \mathbb{E} \left( \left( W + \sqrt{\frac{3}{m}} \right) \ln \left| W + \sqrt{\frac{3}{m}} \right| \right)
\]

\[
= \sqrt{\frac{m}{3}} \mathbb{E} \left( \left( W + \sqrt{\frac{3}{m}} \right) \ln \left| W + \sqrt{\frac{3}{m}} \right| \right) + \frac{\ln(m/3)}{2} \sqrt{\frac{m}{3}} \mathbb{E} \left( W + \sqrt{\frac{3}{m}} \right)
\]

\[
> \frac{\ln(m/3)}{2} \sqrt{\frac{m}{3}} \mathbb{E} \left( W + \sqrt{\frac{3}{m}} \right) - \frac{2}{\sqrt{m}} (\ln (m+1) + 1) \quad \text{by lemma 5}
\]

\[
= \frac{\ln(m/3)}{2} - \frac{2}{\sqrt{m}} (\ln (m+1) + 1) \quad \text{Since } \mathbb{E}(W) = 0.
\]

Applying lemma 5 the proof is completed. \(\square\)
Denote by \( e_n \) the vector in \( \mathbb{R}^n \) with all entries 1. For any \( a \in \mathbb{R}^m \), denote by \( \|a\|_1 = |a_1| + \ldots + |a_m| \) the 1-norm of \( a \). Below is a lemma from [5],

**Lemma 7.** For any increasing function \( \varphi : \mathbb{R} \to \mathbb{R} \), \( a \in \mathbb{R}^m \) and \( b \in \mathbb{R} \), s.t. \( \|a\|_1 = m \),

\[
\text{Prob}(|a^T u| > b) \geq \text{Prob}(|e_m^T u| > b) \quad \text{and} \quad \mathbb{E}(\varphi(|a^T u|)) \geq \mathbb{E}(\varphi(|e_m^T u|)).
\]

### 3.1 Proof of Theorem 2

**Proof.** Denote by \( f_j(x) \) the \( j \)-th entry of \( f(x) \). Denote by \( \nabla f_j(x) \) the gradient of \( f_j \). By Taylor Expansion,

\[
f_j(x') = f_j(x) + (x' - x)^T \nabla f_j(x) + O(\|x' - x\|^2).
\]

(31)

Combining the definition of WCC \((x)\) and equation (31),

\[
\text{WCC}_j(x) = \lim_{\delta \to 0} \sup_{x' \in \mathcal{CP}(x, \delta)} \frac{|(x' - x)^T \nabla f_j(x)|}{\delta f_j(x)}.
\]

For \( i = 1, \ldots, m \), let

\[
g_i = x_i \times \text{the \( i \)-th component of } \nabla f_j(x).
\]

Then, by the definition of \( \mathcal{CP}(x, \delta) \),

\[
\text{WCC}_j(x) = \sup_{u \in [-1,1]^m} \frac{|u^T g|}{|f_j(x)|} = \frac{\|g\|_1}{|f_j(x)|}.
\]

(32)

Combining the definition of SCC \((x)\), \( \mathcal{CP}(x, \delta) \) and \( g \) and equation (31),

\[
\text{SCC}_j(x) = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{CP}(x, \delta)} \frac{|(x' - x)^T \nabla f_j(x)|}{\delta f_j(x)} = \mathbb{E}_{u \sim [-1,1]^m} \frac{|u^T g|}{f_j(x)}. \quad (33)
\]

\[
\frac{\text{SCC}_j(x)}{\text{WCC}_j(x)} = \mathbb{E}_{u \in [-1,1]^m} \left( \frac{|u^T g|}{\|g\|_1} \right) \quad \text{by equation (32)}.
\]

(34)

Obviously,

\[
\mathbb{E}_{u \in [-1,1]^m} |u^T g| = \mathbb{E}_{u \in [-1,1]^m} |u_1 g_1 + \ldots + u_m g_m| \quad (35)
\]

\[
\leq \mathbb{E}_{u \in [-1,1]^m} (|u_1 g_1| + \ldots + |u_m g_m|) \quad (36)
\]

\[
= |g_1| \mathbb{E}_{u \in [-1,1]^m} |u_1| + \ldots + |g_m| \mathbb{E}_{u \in [-1,1]^m} |u_m| \quad (37)
\]

\[
= 0.5 (|g_1| + \ldots + |g_m|) = 0.5\|g\|_1.
\]

(38)
Combining equations (34) and (38), \( \frac{\text{SCC}_j(x)}{\text{WCC}_j(x)} \leq \frac{1}{2} \). Since \( \log(\cdot) \) is a concave function, \( \text{SCLP}_j(x) - \log C_j(x) \leq -1 \). Combining the definition of \( \text{SCLP}_j(x) \), \( \mathcal{CP}(x, \delta) \) and \( g \) and equation (31),

\[
\text{SCLP}_j(x) = \log e = \lim_{\delta \to 0} \mathbb{E}_{x' \sim \mathcal{CP}(x, \delta)} \ln \left| (x' - x)^T \nabla f_j(x) / \delta f_j(x) \right|
\]

\[
= \ln \frac{\|g\|}{m|f_j(x)|} + \mathbb{E}_{u \sim [-1,1]^m} \ln \left| u^T mg / \|g\| \right|
\]

\[
\geq \ln \frac{\|g\|}{m|f_j(x)|} + \mathbb{E}_{u \sim [-1,1]^m} \ln |u^Te| \quad \text{by lemma 7}
\]

\[
> \ln \frac{\|g\|}{|f_j(x)|} - \frac{\ln (m-1)}{2} - \frac{\ln 3}{2} - 1 - \varepsilon_m \quad \text{by corollary 2}
\]

\[
= \ln \text{WCC}_j(x) - \frac{\ln (m-1)}{2} - \frac{\ln 3}{2} - 1 - \varepsilon_m \quad \text{by equation (32)}.
\]

That is, \( \text{SCLP}_j(x) > \log \text{WCC}_j(x) - \frac{\ln (m-1)}{2} - \frac{\ln 3}{2} - (1 + \varepsilon_m) \log e \). Since \( \log(\cdot) \) is a concave function, \( \text{SCC}_j(x) > \frac{\text{WCC}_j(x)}{\sqrt{3(m-1)}} e^{-(1+\varepsilon_m)} \).

\[
\square
\]

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