ON RATIONAL DYCK PATHS AND THE ENUMERATION OF FACTOR-FREE DYCK WORDS

DANIEL BIRMAJER, JUAN B. GIL, AND MICHAEL D. WEINER

ABSTRACT. Motivated by independent results of Bizley and Duchon, we study rational Dyck paths and their subset of factor-free elements. On the one hand, we give a bijection between rational Dyck paths and regular Dyck paths with ascents colored by factor-free words. This bijection leads to a new statistic based on the reducibility level of the paths for which we provide a corresponding formula. On the other hand, we prove an inverse relation for certain sequences defined via partial Bell polynomials, and we use it to derive a formula for the enumeration of factor-free words. In addition, we give alternative formulas for various enumerative sequences that appear in the context of rational Dyck paths.

1. Introduction

In his paper [3] of 1954, Bizley proved a formula (credited to Howard Grossman) for the number \( \phi_n \) of lattice paths from \((0,0)\) to \((\alpha n, \beta n)\) which may touch but never rise above the line \( \alpha y = \beta x \), where \( n, \alpha, \) and \( \beta \) are positive integers with \( \gcd(\alpha, \beta) = 1 \). Such a path is called \( \frac{\beta}{\alpha} \)-Dyck path of length \( (\alpha + \beta)n \). Using the notation \( f_j = \frac{1}{(\alpha + \beta)^j} \binom{\alpha + \beta}{\alpha j} \) for \( j \in \mathbb{N} \), Bizley’s formula reads

\[
\phi_n = \sum f_{k_1}^{k_1} f_{k_2}^{k_2} \cdots, \tag{1.1}
\]

where the sum runs over all \( k_j \in \mathbb{N}_0 \) such that \( k_1 + 2k_2 + \cdots + nk_n = n \).

On a more recent paper, Duchon [5] studied generalized Dyck languages and discussed the particular case of a two-letter language with alphabet \( \{a, b\} \), having valuations \( h(a) = \beta \) and \( h(b) = -\alpha \) with \( \gcd(\alpha, \beta) = 1 \). In this case, associating the letter \( a \) with the step \((1, 0)\) and the letter \( b \) with the step \((0, 1)\), the set of all Dyck words (words with total valuation equal to 0) is in one-to-one correspondence with the set of \( \frac{\beta}{\alpha} \)-Dyck paths.

In [5, Theorem 9], Duchon established a key connection between the set \( D_{\beta/\alpha} \) of generalized Dyck words with slope \( \beta/\alpha \) and its subset of corresponding factor-free words, denoted by \( \tilde{D}_{\beta/\alpha} \). More precisely, he proved the identity

\[
\phi_n = [t^n] D_{\beta/\alpha}(t) = \frac{1}{1 + (\alpha + \beta)n} [t^n] \tilde{D}_{\beta/\alpha}^{1 + (\alpha + \beta)n}(t), \tag{1.2}
\]

where \( D_{\beta/\alpha}(t) \) and \( \tilde{D}_{\beta/\alpha}(t) \) are the generating functions enumerating (by word length) the elements of \( D_{\beta/\alpha} \) and \( \tilde{D}_{\beta/\alpha} \), respectively. The identities (1.1) and (1.2) are the foundation for the results obtained in this paper.

\[1\] A word in a language \( L \) is said to be factor-free if it has no proper factor in \( L \).
On the one hand, as a consequence of a result given in \cite{1} Theorem 3.5, the right-hand side of (1.2) also counts the elements of \( D_n^{\alpha\beta}(\alpha + \beta, 0) \), the set of Dyck words of semilength \((\alpha + \beta)n\) created from strings of the form “d” and “u(\(\alpha + \beta\))i d” for \( j = 1, \ldots, n \), such that each maximal ascent \( u(\(\alpha + \beta\))i \) may be colored in as many different ways as the number \( \theta_j \) of factor-free words of length \((\alpha + \beta)j\). In other words,

there is a bijection between the set of \( \frac{\beta}{\alpha} \)-Dyck paths of length \((\alpha + \beta)n\) and

the set of colored Dyck paths in \( D_n^{\alpha\beta}(\alpha + \beta, 0) \).

In Section 2 we give an explicit bijection that reveals, in geometric terms, the factor-free factorization of a rational Dyck path. Specifically, we use the number of peaks of a path in \( D_n^{\alpha\beta}(\alpha + \beta, 0) \) to define the reducibility level of the associated \( \frac{\beta}{\alpha} \)-Dyck path. As a application, we give a formula (Theorem 2.6) to compute the number of \( \frac{\beta}{\alpha} \)-Dyck paths with a given reducibility level.

On the other hand, by rewriting (1.1) and (1.2) in terms of partial Bell polynomials, and using an interesting inverse relation that we prove in Proposition 3.4,

we obtain a formula for the number \( \theta_n \) of factor-free \( \frac{\beta}{\alpha} \)-Dyck words of length \((\alpha + \beta)n\) in terms of \( f_1, f_2, \ldots \) in the spirit of (1.1).

Our formula, given in Theorem 4.4, appears to be new for cases other than \( \alpha = 2 \) and \( \beta = 3 \). For the case of slope \( 3/2 \), Duchon already observed (cf. \cite{5} Prop. 10) that there are \( \theta_n = C_n + C_{n-1} \) factor-free Dyck words of length \( 5n \), where \( C_n \) is the \( n \)th Catalan number.

We finish the paper with various formulas connecting the sequences \((f_n)\), \((\phi_n)\), and \((\theta_n)\) with the related sequence \((\psi_n)\) that counts the number of \( \frac{\beta}{\alpha} \)-Dyck paths of length \((\alpha + \beta)n\) that never touch the line \( \alpha y = \beta x \) (except at the initial and terminal points).

2. Rational Dyck paths as colored regular Dyck paths

In this paper, we will follow the terminology used in \cite{5} for the study of generalized Dyck words. We consider the alphabet \( U = \{a, b\} \) and assume the valuations \( h(a) = \beta \) and \( h(b) = -\alpha \) for positive integers \( \alpha \) and \( \beta \) with \( \gcd(\alpha, \beta) = 1 \). A Dyck word \( w \) with slope \( \beta/\alpha \) is a string of letters from \( U \) such that \( h(w) = 0 \), and for each left factor \( u \) of \( w \), \( h(u) \geq 0 \). Let \( D_{\beta/\alpha} \) denote the set of all such words. Note that the length \( |w| \) of a word in \( D_{\beta/\alpha} \) is always a multiple of \( \alpha + \beta \).

A word \( w' \in D_{\beta/\alpha} \) is called a factor of \( w \) if there are words \( u \) and \( v \) (possibly empty words) such that \( w = uw'v \) and \( uv \in D_{\beta/\alpha} \). If \( u \) and \( v \) are both not empty words, then \( w' \) is called a proper factor of \( w \). A word \( w \in D_{\beta/\alpha} \) is said to be factor-free if it has no proper factors in \( D_{\beta/\alpha} \). Let \( \varepsilon \) denote the empty word.

Let \( \phi_n \) be the number of elements in \( D_{\beta/\alpha} \) of length \((\alpha + \beta)n\), and let \( \theta_n \) be the number of factor-free words in \( D_{\beta/\alpha} \) of length \((\alpha + \beta)n\). Applying Faà di Bruno’s formula to the right-hand side of (1.2), we get the equivalent representation

\[
\phi_n = \sum_{k=1}^{n} \binom{(\alpha + \beta)n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \ldots),
\]

(2.1)

where \( B_{n,k}(x_1, \ldots, x_{n-k+1}) \) denotes the \((n,k)\)-th partial Bell polynomial.
As shown in [1, Theorem 3.5], the right-hand side of (2.1) gives the number of regular Dyck paths of semilength \((\alpha + \beta)n\) whose ascents have length a multiple of \(\alpha + \beta\) and such that an ascent of length \((\alpha + \beta)j\) may be colored in \(\theta_j\) different ways. We denote this set of Dyck paths/words by \(D_n^{\Theta}(\alpha + \beta, 0)\). The purpose of this section is to provide an explicit bijection between elements in \(D_{\beta/\alpha}^\Theta\) of length \((\alpha + \beta)n\) and elements of \(D_n^{\Theta}(\alpha + \beta, 0)\). To this end, we first introduce some notation and discuss one example.

Every word \(w \neq \varepsilon\) in \(D_{\beta/\alpha}^\Theta\) can be written, uniquely, as \(w = uw_1'v_1\), where \(w_1'\) is the left-most, nonempty factor-free subword of \(w\). In this case, we say that the word \(uv\) is a reduction of \(w\) by \(w_1'\) and write \(w \to w_1'uv\).

Conversely, if \(w\) and \(w_1'\) are Dyck words and \(1 \leq j \leq |w|\), then we can write \(w = uv\) with \(|u| = j\) and insert \(w_1'\) between \(u\) and \(v\) to form the new word \(\hat{w} = uw_1'v\). We call \(\hat{w}\) the extension of \(w\) by \(w_1'\) at the position \(j\) and write \(\hat{w} = (w \leftarrow_j w_1')\).

**Example 2.2.** Consider the word \(w = aabbbaababaabbbb\) in \(D_{3/2}\), which represents the following \(3/2\)-Dyck path of length 20:

![Dyck Path](image)

Factor \(w\) as \(w = u_1w_1'v_1 = aabbbaabab(aabb)bbb\) noting that \(w_1' = aabb\) is the left-most, nonempty factor-free subword of \(w\). Thus the reduction \(w \to w_1'v_1\) gives

\[
aabbbaabab(aabb)bbb \to aabbabaababbbb.
\]

Similarly, \(w_2' = ababb\) is factor-free, so we can reduce \(aabbabbaab(bab)bb \to aabbababb\). Observe that \(w_3' = aabbbabbb\) is factor-free. If we let \(\ell_j\) be the length of the factor \(u_j\) in the \(j\)th reduction of \(w\), then \(\ell_1 = |u_1| = |aabbabbaab| = 12\) and \(\ell_2 = |u_2| = |aabbabba| = 8\).

Finally, we construct the colored Dyck path \(D_w\) associated with \(w\) as follows:

\[
D_w = u|w_1'|d^{\ell_1}u|w_2'|d^{\ell_2}u|w_3'|d^{20-\ell_3} = u^{10}d^8u^5d^4u^5d^8,
\]

where the ascents are colored (from left to right) by \(aabbabbb\), \(ababb\), and \(aabb\).
The above construction is reversible. Note that for a given Dyck path $D$ of semilength 20, colored by $aabbababb$, $abbb$, and $aabb$, one can create a unique word $w_D$ by successively inserting the factor-free words at positions determined by the downs of the Dyck path.

**Theorem 2.3.** Let $\alpha, \beta \in \mathbb{N}$ with $\gcd(\alpha, \beta) = 1$. The algorithm outlined in Example 2.2 provides a bijection between the set of $\frac{\beta}{\alpha}$-Dyck paths of length $(\alpha + \beta)n$ and $\mathcal{D}_n^\Theta(\alpha + \beta, 0)$.

**Proof.** Let $w$ be a word in $\mathcal{D}_{\beta/\alpha}$ of length $(\alpha + \beta)n$. Without loss of generality, we assume that the corresponding lattice path stays strongly below the line $y = \frac{\beta}{\alpha}x$ (except at the endpoints). For a general path, we just look at its connected components separately.

If $w$ is factor-free, then we define

$$D_w = w^{(\alpha + \beta)n} \cdot d^{(\alpha + \beta)n}$$

and color the ascent with $w$. Clearly $D_w \in \mathcal{D}_n^\Theta(\alpha + \beta, 0)$.

If $w$ is not a factor-free word, then it can be factored uniquely as $w = w_1' v_1$, where $w_1'$ is a factor-free word in $\mathcal{D}_{\beta/\alpha}$, $h(u_1) > 0$, and $h(v_1) < 0$. Let $\ell_1 = |u_1|$ and let $u_1 v_1$ be the reduction of $w$ by $w_1'$. If $u_1 v_1$ is factor-free, then we denote it by $w_2'$ and define

$$D_w = u_1^{\ell_1} | w_1' | d^{(\alpha + \beta)n - \ell_1}.$$ 

Otherwise, we write $u_1 v_1$ as $u_2 w_2' v_2$, where $w_2'$ is the left-most factor-free word contained in $u_1 v_1$. We let $\ell_2 = |u_2|$ and look at the reduction $u_2 v_2$. We continue this process inductively until we get a factor-free reduction.

Suppose $w \in \mathcal{D}_{\beta/\alpha}$ is a word of length $(\alpha + \beta)n$ such that after $k - 1$ reductions

$$w \rightarrow u_1 v_1 \rightarrow u_2 v_2 \rightarrow \cdots \rightarrow u_{k-1} v_{k-1}$$

we arrive at a factor-free word $w_k' = u_{k-1} v_{k-1}$. Let $\ell_j = |u_j|$ for $j \leq k - 1$, and define

$$D_w = u_1^{\ell_1} | w_1' | d^{\ell_1 - \ell_2} | w_2' | d^{\ell_2 - \ell_3} \cdots u_{k-1}^{\ell_{k-1}} | d^{(\alpha + \beta)n - \ell_k}.$$ 

By construction, there is a total of $(\alpha + \beta)n$ downs, $|w_1'| + \cdots + |w_k'| = (\alpha + \beta)n$, and for every $j = 1, \ldots, k - 1$:

- $|w_j'| \equiv 0 \pmod{\alpha + \beta}$,
- $\ell_j - \ell_{j+1} \geq 1$ (letting $\ell_k = 0$),
- $\ell_j \leq |w_{j+1}| + \cdots + |w_k'|$.

In other words, $D_w$ represents a Dyck path of semilength $(\alpha + \beta)n$. Coloring each ascent $w_j$ with the factor-free word $w_j'$, we get an element of $\mathcal{D}_n^\Theta(\alpha + \beta, 0)$.

The above algorithm can be easily reversed. Let $D$ be an element of $\mathcal{D}_n^\Theta(\alpha + \beta, 0)$ with $k$ peaks, whose ascents are colored by the factor-free words $w_1, \ldots, w_k$. Thus $D$ must be of the form

$$D = u_1^{\ell_1} | d^{\ell_2} | u_2^{\ell_3} | d^{\ell_4} \cdots u_k^{\ell_{k-1}} | d^{\ell_k},$$

where each $|w_j|$ is a multiple of $\alpha + \beta$, $|w_1| + \cdots + |w_k| = (\alpha + \beta)n$, and for every $i$ we have $|w_1| + \cdots + |w_i| \geq j_1 + \cdots + j_i$. Once again, without loss of generality, we assume that $D$ stays strongly above the x-axis (it has no interior touch point).

We define $w_D$ by repeated insertion of the coloring factor-free words:

$$w_D = \left( \cdots ((w_1 \leftarrow w_2) \leftarrow w_3) \cdots \right) \leftarrow w_k.$$ 

Remark. From the previous proposition, it is obvious that any reducibility level statistic is only meaningful for rational Dyck paths with non-integer slopes. Theorem 2.6. Thus the number of Dyck paths in every \( w \) equal to the number of Dyck paths in

\[ |w| = (\alpha + \beta)n \]

As a direct consequence of the bijection given in Theorem 2.3, we get that if \( \alpha > 1 \), then \( r_l(w) = \theta \) where \( \theta \) is the number of factor-free words in \( D_{\beta/\alpha} \) of length \( (\alpha + \beta)n \). Conversely, if \( \beta > \alpha \) and \( r_l(w) = \theta \) for every \( w \in D_{\beta/\alpha} \) with \( |w| = (\alpha + \beta)n \), then \( \alpha = 1 \).

Proposition 2.5 (Integer slope). If \( \alpha = 1 \), then \( ab^\beta \) is the only factor-free word in \( D_{\beta} \), and every word \( w \in D_{\beta} \) of length \((1 + \beta)n\) has reducibility level \( n \). Conversely, if \( \beta > \alpha \) and \( r_l(w) = n \) for every \( w \in D_{\beta/\alpha} \) with \( |w| = (\alpha + \beta)n \), then \( \alpha = 1 \).

Proof. Let \( w \in D_{\beta} \) be of length \((1 + \beta)n\) with \( n > 1 \). By definition, \( h(a) = \beta \) and \( h(b) = -1 \). Thus the right-most \( a \) in \( w \) must be followed by a string containing \( b^\beta \). Hence \( w \) can be reduced by \( ab^\beta \), so it is not factor-free and \( r_l(w) = n \).

In order to prove the last statement, assume \( \alpha > 1 \) and let \( \alpha = a^\beta b^\beta - 1 a b^\beta \). Then we have \( |w| = 2(\alpha + \beta) \) and \( r_l(w) = 1 \), which contradicts the assumption that \( r_l(w) = 2 \) for every \( w \in D_{\beta/\alpha} \) with \( |w| = 2(\alpha + \beta) \).

Remark. From the previous proposition, it is obvious that any reducibility level statistic is only meaningful for rational Dyck paths with non-integer slopes.

Theorem 2.6. The number \( r_{n,k} \) of \( \frac{\beta}{\alpha} \)-Dyck paths of length \((\alpha + \beta)n\) that have reducibility level equal to \( k \) is given by

\[
r_{n,k} = \frac{(\alpha + \beta)n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \ldots),
\]

where \( \theta_j \) is the number of factor-free words in \( D_{\beta/\alpha} \) of length \((\alpha + \beta)j\).

Proof. As a direct consequence of the bijection given in Theorem 2.3, we get that \( r_{n,k} \) is equal to the number of Dyck paths in \( D_{\beta/\alpha}^{\frac{\beta}{\alpha}}(\alpha + \beta, 0) \) having exactly \( k \) peaks. Thus (2.7) follows from [1] Theorem 3.5.

Example 2.8. If \( \alpha = 1 \), then \( ab^\beta \) is the only factor-free word in \( D_{\beta} \), so \( \theta_1 = 1 \) and \( \theta_j = 0 \) for \( j \neq 1 \). Therefore, the number of words \( w \) of length \((1 + \beta)n\) with \( r_l(w) = k \) is zero unless \( k = n \). In that case (2.7) gives

\[
r_{n,n} = \frac{(1 + \beta)n}{n-1} \frac{(n-1)!}{n!} = \frac{1}{\beta n + 1} \frac{(\beta + 1)n}{n},
\]

which is, as expected, the total number of words of length \((1 + \beta)n\) in \( D_{\beta} \).

Example 2.9. As mentioned in the introduction, for \( \alpha = 2 \) and \( \beta = 3 \), the number of factor-free words of length \( 5n \) is given by the sum of adjacent Catalan numbers \( C_n + C_{n-1} \).

Thus the number of \( \frac{3}{2} \)-Dyck paths of length \( 5n \) having reducibility level \( k \) is given by

\[
r_{n,k} = \frac{5n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!(C_0 + C_1), 2!(C_1 + C_2), \ldots),
\]
which by means of [7, Example 3.2] can be written as

\[ r_{n,k} = \binom{5n}{k-1} \sum_{j=0}^{k} \frac{(-1)^{k-j}(2j-k)}{nk} \binom{k}{j} \binom{2(n+j) - k - 1}{n-1}. \]

For example, among the 23 words of length 10 \((n = 2)\) there are \(r_{2,1} = 3\) factor-free words and \(r_{2,2} = 20\) words \(w\) with \(\text{rl}(w) = 2\). And among the 377 words of length 15 there are 7 factor-free words, 90 words with \(\text{rl}(w) = 2\), and 280 words with \(\text{rl}(w) = 3\).

### 3. An inverse relation involving partial Bell polynomials

The purpose of this section is to prove an inverse relation for a family of sequences defined through partial Bell polynomials. The main ingredients are Faà di Bruno’s formula, expressed in terms of partial Bell polynomials as in [3, Sec. 3.4, Theorem A], together with a substitution formula introduced by the authors in [2]. For convenience, we recall here these results as lemmas.

**Lemma 3.1** (Faà di Bruno). Let \(f\) and \(g\) be two formal power series:

\[ f = f_0 + \sum_{k=1}^{\infty} f_k u^k \quad \text{and} \quad g = \sum_{\ell=1}^{\infty} g_\ell t^\ell. \]

If \(h = \sum_{n=0}^{\infty} h_n t^n\) is the formal power series of the composition \(f \circ g\), then the coefficients \(h_n\) are given by

\[ h_0 = f_0, \quad h_n = \sum_{k=1}^{n} f_k B_{n,k}(g_1, g_2, \ldots). \]

**Lemma 3.2** ([2, Theorem 15]). Let \(a, b \in \mathbb{Z}\). Given any sequence \((x_n)\), define \((y_n)\) by

\[ y_n = \sum_{k=1}^{n} \left( \frac{an + bk}{k-1} \right) \binom{k-1}{n} B_{n,k}(1!x_1, 2!x_2, \ldots) \]

for every \(n \in \mathbb{N}\). Then, for any \(\lambda \in \mathbb{C}\) we have

\[ \sum_{k=1}^{n} (\lambda_{k-1}) (k-1)! B_{n,k}(1!y_1, 2!y_2, \ldots) = \sum_{k=1}^{n} (\lambda_{k-1} + \lambda a n + b k)(k-1)! B_{n,k}(1!x_1, 2!x_2, \ldots). \]

**Corollary 3.3.** If \(y_n\) is given by

\[ y_n = \sum_{k=1}^{n} \left( \frac{an}{k-1} \right) \binom{k-1}{n} B_{n,k}(1!x_1, 2!x_2, \ldots), \]

then

\[ x_n = \sum_{k=1}^{n} \left( \frac{-an}{k-1} \right) (k-1)! B_{n,k}(1!y_1, 2!y_2, \ldots). \]

For the results in Section [3] we also need the following inverse relation.
Proposition 3.4. Let $a \in \mathbb{Z}$. Given any sequence $(x_n)$, define $(y_n)$ by

$$y_n = \sum_{k=1}^{n} \binom{an - 1}{k - 1} \frac{(k - 1)!}{n!} B_{n,k}(1!x_1, 2!x_2, \ldots).$$

Then, for every $n \in \mathbb{N}$ we have

$$x_n = \sum_{k=1}^{n} \frac{(1 - an)^{k-1}}{n!} B_{n,k}(1!y_1, 2!y_2, \ldots).$$

Proof. Let $(z_n)$ be the sequence defined by

$$z_n = \sum_{k=1}^{n} \binom{an}{k - 1} \frac{(k - 1)!}{n!} B_{n,k}(1!x_1, 2!x_2, \ldots).$$

By Lemma 3.2 with $b = 0$, we have

$$\sum_{k=1}^{n} \binom{\lambda - 1}{k-1} (k - 1)! B_{n,k}(1!z_1, 2!z_2, \ldots) = \sum_{k=1}^{n} \binom{\lambda + an - 1}{k-1} (k - 1)! B_{n,k}(1!x_1, 2!x_2, \ldots) \quad (3.5)$$

for any $\lambda \in \mathbb{C}$, and

$$\sum_{k=1}^{n} \binom{-1}{k-1} (k - 1)! B_{n,k}(1!z_1, 2!z_2, \ldots) = \sum_{k=1}^{n} \binom{an - 1}{k-1} (k - 1)! B_{n,k}(1!x_1, 2!x_2, \ldots). \quad (3.6)$$

If we denote $y(t) = \sum_{n=1}^{\infty} y_n t^n$ and $z(t) = \sum_{n=1}^{\infty} z_n t^n$, then (3.6) means (via Faà di Bruno’s formula) that the generating functions $y(t)$ and $z(t)$ are related by the identity

$$y(t) = \log(1 + z(t)).$$

Thus, if we let $f_1(t) = e^{\lambda t}$ and $f_2(t) = (1 + t)^{\lambda}$, then

$$f_1(y(t)) = f_1(\log(1 + z(t))) = f_2(z(t)),$$

and Faà di Bruno’s formula gives

$$(t^n)(f_1 \circ y) = \sum_{k=1}^{n} \lambda^k B_{n,k}(1!y_1, 2!y_2, \ldots),$$

and

$$(t^n)(f_2 \circ z) = \sum_{k=1}^{n} (\lambda)_k B_{n,k}(1!z_1, 2!z_2, \ldots) = \sum_{k=1}^{n} \lambda^{(\lambda - 1)(k - 1)} B_{n,k}(1!z_1, 2!z_2, \ldots).$$

This implies

$$\sum_{k=1}^{n} \lambda^k B_{n,k}(1!y_1, 2!y_2, \ldots) = \sum_{k=1}^{n} \lambda^{(\lambda - 1)(k - 1)} B_{n,k}(1!z_1, 2!z_2, \ldots).$$

Combining this identity with (3.5), we obtain

$$\sum_{k=1}^{n} \lambda^{k-1} B_{n,k}(1!y_1, 2!y_2, \ldots) = \sum_{k=1}^{n} \binom{\lambda + an - 1}{k-1} (k - 1)! B_{n,k}(1!x_1, 2!x_2, \ldots).$$

Finally, if $\lambda = 1 - an$, then most terms on the right-hand side of the equation vanish and the sum reduces to $B_{n,1}(1!x_1, 2!x_2, \ldots) = n! x_n$. This gives the claimed identity. \qed
4. Enumeration of factor-free Dyck words

Let \( \alpha \) and \( \beta \) be positive integers with \( \gcd(\alpha, \beta) = 1 \). As noted in the introduction, Bizley’s formula (1.1) for the number of \( \frac{\beta}{\alpha} \)-Dyck paths of length \( (\alpha + \beta) n \) can be conveniently written using partial Bell polynomials as

\[
\phi_n = \sum_{k=1}^{n} \frac{1}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots),
\]

where \( f_j = \frac{1}{(\alpha + \beta) j} \binom{(\alpha + \beta)j}{\alpha j} \) for \( j \in \mathbb{N} \). Inverting the above identity, we get

\[
f_n = \sum_{k=1}^{n} \left( -1 \right)^{k-1} \frac{(k-1)!}{(k-1)!} B_{n,k}(1!\phi_1, 2!\phi_2, \ldots). \tag{4.1}
\]

On the other hand, as claimed in (2.1), we have the alternative representation

\[
\phi_n = \sum_{k=1}^{n} \binom{(\alpha + \beta)n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \ldots), \tag{4.2}
\]

where \( \theta_j \) is the number of factor-free Dyck words with slope \( \beta/\alpha \) and length \( (\alpha + \beta)j \). Using Lemma 3.2 with \( a = \alpha + \beta, b = 0, \text{ and } \lambda = -1 \), we get

\[
\sum_{k=1}^{n} \binom{-1}{k-1} (k-1)! B_{n,k}(1!\phi_1, 2!\phi_2, \ldots) = \sum_{k=1}^{n} \binom{(\alpha + \beta)n-1}{k-1} (k-1)! B_{n,k}(1!\theta_1, 2!\theta_2, \ldots),
\]

which together with (4.1) gives the identity

\[
f_n = \sum_{k=1}^{n} \binom{(\alpha + \beta)n-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \ldots) \tag{4.3}
\]

for every \( n \in \mathbb{N} \).

As a consequence of the inverse relation given in Proposition 3.4, we obtain:

**Theorem 4.4.** For every \( n \in \mathbb{N} \), the number of factor-free Dyck words with slope \( \beta/\alpha \) and length \( (\alpha + \beta)n \) is given by

\[
\theta_n = \sum_{k=1}^{n} \frac{(1 - (\alpha + \beta)n)^{k-1}}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots),
\]

where \( f_j = \frac{1}{(\alpha + \beta) j} \binom{(\alpha + \beta)j}{\alpha j} \) for every \( j \).

This formula is easy to implement in any of the mainstream computer algebra systems. For example, in Sage, Maple, and Mathematica, partial Bell polynomials are implemented as \texttt{bell polynomial}, \texttt{IncompleteBellB}, and \texttt{BellY}, respectively. The following table shows a few terms of the sequence \( (\theta_n) \) for various slopes \( \beta/\alpha \).
The other class of lattice paths considered by Bizley [3] is the set of $\frac{\beta}{\alpha}$-Dyck paths that stay strongly below the line $y = \frac{\beta}{\alpha}x$. He proved that the number $\psi_n$ of such paths of length $(\alpha + \beta)n$ is given by

$$\psi_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots).$$

Written in terms of $(\psi_n)$, the sequences $(f_n)$, $(\phi_n)$, and $(\theta_n)$ show an interesting pattern.

**Proposition 4.5.** The following identities hold for every $n \in \mathbb{N}$:

$$f_n = \sum_{k=1}^{n} \frac{(k-1)!}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots),$$

(4.6)

$$\phi_n = \sum_{k=1}^{n} \frac{k!}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots),$$

(4.7)

$$\theta_n = \sum_{k=1}^{n} \frac{-(\alpha + \beta)n + k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots).$$

(4.8)

Note that the above expressions are instances of

$$\sum_{k=1}^{n} \frac{(r+k-1)}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!f_1, 2!f_2, \ldots)$$

for $r = 0, 1, 1 - (\alpha + \beta)n$, respectively. It is worth mentioning that the sequence $(\phi_n)$ is the invert transform of the sequence $(\psi_n)$.

**Proof of Proposition 4.5.** Let $f(t)$, $\phi(t)$, and $\psi(t)$ be the generating functions of $(f_n)$, $(\phi_n)$, and $(\psi_n)$, respectively. Since $\psi(t) = 1 - e^{-f(t)}$ and $1 + \phi(t) = e^{f(t)}$, we get

$$f(t) = -\log(1 - \psi(t))$$

and

$$1 + \phi(t) = \frac{1}{1 - \psi(t)},$$

which give identities (4.6) and (4.8) via Faà di Bruno’s formula.
Using (4.7) we now write \( \phi_n \) as
\[
\phi_n = \sum_{k=1}^{n} \left( \binom{k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \ldots) \right)
\]
and use Lemma 3.2 with \( a = 0, b = 1, \) and \( \lambda = -\left(\alpha + \beta\right)n \) to conclude that
\[
\sum_{k=1}^{n} \left( -\frac{(\alpha+\beta)n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\phi_1, 2!\phi_2, \ldots) \right) = \sum_{k=1}^{n} \left( -\frac{(\alpha+\beta)n+k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \ldots) \right).
\]
Finally, because of the representation (4.2), Corollary 3.3 implies that the left-hand side of the above identity is precisely \( \theta_n \). Thus (4.8) holds. \( \Box \)

REFERENCES

[1] D. Birmajer, J. Gil, P. McNamara, and M. Weiner, Enumeration of colored Dyck paths via partial Bell polynomials, preprint arXiv:1602.03550, 2016.
[2] D. Birmajer, J. Gil, and M. Weiner, Some convolution identities and an inverse relation involving partial Bell polynomials, Electron. J. Combin. 19 (2012), no. 4, Paper 34, 14 pp.
[3] M. T. L. Bizley, Derivation of a new formula for the number of minimal lattice paths from \((0,0)\) to \((km, kn)\) having just \( t \) contacts with the line \( my = nx \) and having no points above this line; and a proof of Grossman’s formula for the number of paths which may touch but do not rise above this line, J. Inst. Actuar. 80 (1954), 55–62.
[4] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, D. Reidel Publishing Co., Dordrecht, 1974.
[5] P. Duchon, On the enumeration and generation of generalized Dyck words, Discrete Math. 225 (2000), no. 1-3, 121–135.
[6] The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, 2016.
[7] W. Wang and T. Wang, General identities on Bell polynomials, Comput. Math. Appl. 58 (2009), no. 1, 104–118.

Department of Mathematics, Nazareth College, 4245 East Ave., Rochester, NY 14618
Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601