Riemannian manifolds with Anosov geodesic flow do not have conjugate points

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Abstract

This paper establishes a significant result concerning the absence of conjugate points in certain complete Riemannian manifolds. Specifically, we demonstrate that any complete non-compact manifold with curvature bounded below and an Anosov geodesic flow does not possess conjugate points. This resolves an open problem left by R. Mañé in [9] and subsequently highlighted by [7].

1 Introduction

In [1], Anosov demonstrated that geodesic flows on compact manifolds with negative curvature exhibit chaotic dynamical behavior, leading to what are known as uniformly hyperbolic systems or simply “Anosov” systems. Anosov’s argument extends to non-compact manifolds with negatively pinched curvature, where the geodesic flow remains Anosov (cf. [7]).

In [6], Klingenberg showed that compact manifolds with Anosov geodesic flows share several crucial properties with negatively curved manifolds. These properties include the absence of conjugate points, ergodicity of the geodesic flow, dense periodic orbits, exponential growth of the fundamental group, and zero index for every closed geodesic. While some of these results do not hold for non-compact manifolds, Mañé’s seminal paper [9], using the Maslov Index, established that if the geodesic flow admits a continuous invariant Lagrangian subbundle, then there are no conjugate points. Consequently, manifolds of finite volume and Anosov geodesic flow do not have conjugate points due to the continuity and invariance of stable and unstable bundles.

Mañé’s assertion that complete non-compact manifolds of infinite volume with bounded curvature below do not possess conjugate points under Anosov geodesic flow was made in the same paper. However, as noted in [7, pp. 475-476], there is an error in Proposition II.2 of Mañé’s proof (also discussed in [8]). Subsequently, the following conjecture emerged:

Conjecture: If M is a non-compact complete Riemannian manifold with curvature bounded below and an Anosov geodesic flow, then M has no conjugate points.

This problem has garnered recent attention, especially following the noteworthy work by G. Knieper (cf. [8]). Knieper addressed the conjecture by introducing three additional geometric

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conditions. However, it’s important to highlight that while these conditions were substantial, 
they were considered quite stringent and did not offer complete proof of the conjecture.

The main objective of this study is to address this conjecture, aiming to prove it in dimension 
two without relying on any additional geometric conditions. Furthermore, we extend our analysis 
to any dimension, provided that the curvature remains bounded. Specifically, we establish the 
following theorems:

**Theorem 1.1.** Suppose $M$ is a non-compact two-dimensional manifold with sectional curvature 
bounded below. If the geodesic flow of $M$ is Anosov, then $M$ has no conjugate points.

For our second result, we define a manifold $M$ to have bounded sectional curvature if there 
exist two positive constants $k$ and $b$ such that

$$-k^2 \leq K_M \leq b^2.$$  

**Theorem 1.2.** Let $M$ be a non-compact complete Riemannian manifold with bounded sectional 
curvature. If the geodesic flow of $M$ is Anosov, then $M$ has no conjugate points.

In Remark 4.4 we show that Theorem 1.2 still holds under a weaker condition on the sectional 
curvature.

It is worth noting that in our pursuit of results, we have not expanded upon any techniques 
utilized by Knieper in §. Deliberately, we have avoided employing Mañé’s methodologies and 
have refrained from modifying his original proof. Instead, we have embarked on an extensive 
exploration of the index formula, which has revealed profound insights into the precise timing 
of the emergence of conjugate points.

In the original result by Klingenberg § for the compact case and by Mañé for finite volume 
§, the recurrence of the geodesic flow was a crucial property in establishing the absence of 
conjugate points for Anosov geometries. Our result is very surprising, as we do not require any 
condition on the recurrence of the geodesic flow to obtain geometries without conjugate points.

**Paper Structure:**  This paper follows the subsequent outline: In Section 2 we delve into the 
fundamental concept of geodesic flow and symplectic geometry concerning the unitary tangent 
bundle. Section 3 presents pivotal classical findings essential for constructing our arguments. 
The longest and most crucial section, Section 4, explores profound properties of the points where 
the vertical and stable (unstable) bundles intersect non-trivially, intricately linked with Section 
3. Finally, in Section 5 we establish the proofs for Theorem 1.2 and Theorem 1.1.

2 **Notation and Basic Concepts**

Throughout the rest of this paper, $M = (M, \langle , \rangle)$ will denote a complete Riemannian manifold 
without boundary and dimension $m \geq 2$. We denote by $TM$ the tangent bundle and $SM$ its 
unit tangent bundle.

2.1 Geodesic flow

For a given $\theta = (p, v) \in TM$, we define $\gamma_\theta(t)$ as the unique geodesic with initial conditions 
$\gamma_\theta(0) = p$ and $\gamma_\theta'(0) = v$. For a given $t \in \mathbb{R}$, let $\phi^t : TM \to TM$ be the diffeomorphism given 
by $\phi^t(\theta) = (\gamma_\theta(t), \gamma_\theta'(t))$. Recall that this family is a flow (called the geodesic flow) in the sense 
that $\phi^{t+s} = \phi^t \circ \phi^s$ for all $t, s \in \mathbb{R}$.

Let $V := \ker D\pi$ be the vertical subbundle of $T(TM)$ (tangent bundle of $TM$), where 
$\pi : TM \to M$ is the canonical projection.
Let $K: T(TM) \to TM$ be the Levi-Civita connection map of $M$ and $H := \ker K$ be the horizontal subbundle. The map $K$ is defined as follows: Let $\xi \in T_p TM$ and $z : (-\epsilon, \epsilon) \to TM$ be a curve adapted to $\xi$, i.e., $z(0) = \theta$ and $z'(0) = \xi$, where $z(t) = (\alpha(t), Z(t))$, then

$$K_\theta(\xi) = \nabla_{\frac{\partial}{\partial t}} Z(t) \bigg|_{t=0}.$$ 

For each $\theta$, the maps $D_\theta^\pi|_{H(\theta)} : H(\theta) \to T_p M$ and $K_\theta|_{V(\theta)} : V(\theta) \to T_p M$ are linear isomorphisms. Furthermore, $T_\theta TM = H(\theta) \oplus V(\theta)$ and the map $j_\theta : T_\theta TM \to T_p M \times T_p M$ given by

$$j_\theta(\xi) = (D_\theta^\pi(\xi), K_\theta(\xi)),$$

is a linear isomorphism.

Using the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, we can identify a vector $\xi \in T_\theta TM$ with the pair of vectors $D_\theta^\pi(\xi)$ and $K_\theta(\xi)$ in $T_p M$. The Sasaki metric is a metric that makes $H(\theta)$ and $V(\theta)$ orthogonal and is given by

$$g_\theta^S(\xi, \eta) = \langle D_\theta^\pi(\xi), D_\theta^\pi(\eta) \rangle + \langle K_\theta(\xi), K_\theta(\eta) \rangle.$$ 

Observe that $SM$ is invariant by $\phi^t$, thus, from now on, we consider $\phi^t$ restricted to $SM$ and $SM$ endowed with the Sasaki metric.

The types of geodesic flows that we discuss in this paper are the Anosov geodesic flows, whose definition follows below.

We say that the geodesic flow $\phi^t : SM \to SM$ is Anosov (with respect to the Sasaki metric on $SM$) if $T(SM)$ has a splitting $T(SM) = E^s \oplus (G) \oplus E^u$ such that

$$d\phi^t_\theta(E^s(\theta)) = E^s(\phi^t(\theta)),$$
$$d\phi^t_\theta(E^u(\theta)) = E^u(\phi^t(\theta)),$$
$$||d\phi^t_\theta|_{E^s}|| \leq C\lambda^t,$$
$$||d\phi^{-t}_\theta|_{E^u}|| \leq C\lambda^t,$$

for all $t \geq 0$ with $C > 0$ and $0 < \lambda < 1$, where $G$ is the vector field derivative of the geodesic flow.

### 2.2 Jacobi fields and the differential of the geodesic flow

The Jacobi fields are important geometrical tools to understand the behavior of the differential of the geodesic flow. A vector field $J$ along $\gamma_\theta$ is called the *Jacobi field* if it satisfies the equation

$$J'' + R(\gamma', J)\gamma' = 0,$$

where $R$ is the Riemann curvature tensor of $M$ and $"\cdot"$ denotes the covariant derivative along $\gamma$.

For $\theta = (p, v)$ and $\xi = (w_1, w_2) \in T_p SM$, (the horizontal and vertical decomposition) with $w_1, w_2 \in T_p M$ and $\langle v, w_2 \rangle = 0$. It is known that

$$d\phi^t_\theta(\xi) = (J_\xi(t), J_\xi'(t)), \tag{1}$$

where $J_\xi$ denotes the unique Jacobi vector field along $\gamma_\theta$ such that $J_\xi(0) = w_1$ and $J_\xi'(0) = w_2$ (see [10]). The equation (1) enables us to assert that the investigation of the dynamics of the geodesic flow revolves around Jacobi fields.

Another crucial concept, intimately tied to this work, is that of conjugate points.
**Definition 2.1.** Two points \( p \) and \( q \) on a Riemannian manifold to be conjugate if there exists a geodesic \( \gamma \) connecting \( p \) and \( q \), along which there exists a non-zero Jacobi field that vanishes at both \( p \) and \( q \). When no two points in \( M \) are conjugate, we say that the manifold \( M \) has no conjugate points.

Geometries without conjugate points are fundamental in mathematics, particularly in differential geometry and dynamical systems. They provide insights into the global structure of Riemannian manifolds, influencing the behavior of geodesics. The absence of conjugate points allows geodesics to extend indefinitely without intersecting themselves in the universal covering impacting various mathematical analyses and theorems related to geodesic flows, curvature, and topology. Additionally, in these geometries, the universal cover is an Euclidean space, further illustrating their significance and utility in mathematical investigations.

As mentioned in the introduction, metrics whose geodesic flow is Anosov, in the compact case \( \mathbb{R} \) and in the case of finite volume \( \mathbb{R} \), exhibit a geometry without conjugate points.

### 2.3 Symplectic geometry

The Riemannian geometry provides to unitary tangent bundle a natural symplectic structure using the horizontal and vertical decomposition given in Subsection 2.1.

We define a symplectic form \( \Omega \) and one-form \( \beta \) given by

\[
\Omega_\theta(\xi, \eta) = \langle D_\theta \pi(\xi), K_\theta(\eta) \rangle - \langle D_\theta \pi(\eta), K_\theta(\xi) \rangle,
\]

\[
\beta_\theta(\xi) = g_\theta^G(\xi, G(\theta)) = \langle D_\theta \pi(\xi), v \rangle_p.
\]

Observe that \( \ker \beta_\theta \supset V(\theta) \cap T_p SM \). It is possible to prove that a vector \( \xi \in T_p SM \) lies in \( T_\theta SM \) with \( \theta = (p, v) \) if and only if \( \langle K_\theta(\xi), v \rangle = 0 \). Furthermore, \( \beta \) is an invariant contact form by the geodesic flow whose Reeb vector field is the geodesic vector field \( G \).

The subbundle \( S = \ker \beta \) is the orthogonal complement of the subspace spanned by \( G \). Since \( \beta \) is invariant by the geodesic flow, then the subbundle \( S \) is invariant by \( \phi^t \), i.e., \( d\phi^t(S(\theta)) = S(\phi^t(\theta)) \) for all \( \theta \in SM \) and all \( t \in \mathbb{R} \).

It is known that the restriction of \( \Omega_\theta \) to \( S(\theta) \) is nondegenerate and invariant by \( \phi^t \) (see [10] for more details).

### 2.4 Graphs and Riccati equation

For \( \theta = (p, v) \in SM \), let \( N(\theta) := \{ w \in T_x M : \langle w, v \rangle = 0 \} \). By the identification of Subsection 2.1 we can write \( S(\theta) := \ker \beta = N(\theta) \times N(\theta), V(\theta) \cap S(\theta) = \{0\} \times N(\theta) \) and \( H(\theta) \cap S(\theta) = N(\theta) \times \{0\} \).

**Definition 2.2.** A subspace \( E \subset S(\theta) \) with \( \dim E = m - 1 \) is said to be Lagrangian if \( \Omega_\theta(\xi, \eta) = 0 \) for any \( \xi, \eta \in S(\theta) \).

The Lagrangian subbundles play an important role in this paper (see Lemma 2.1), since in the Anosov case, it is known that for each \( \theta \in SM \), the subspace \( E^s(\theta) \) and the subspace \( E^u(\theta) \) are Lagrangian (cf. [9] and [10]).

Observe that if \( E \subset S(\theta) \) is a subspace with \( \dim E = m - 1 \) and \( E \cap V(\theta) = \{0\} \), then \( E \cap (H(\theta) \cap S(\theta)) = \{0\} \). Hence, there exists a unique linear map \( T : H(\theta) \cap S(\theta) \to V(\theta) \cap S(\theta) \) such that \( E \) is the graph of \( T \). In other words, there exists a unique linear map \( T : N(\theta) \to N(\theta) \) such that \( E = \{(v, Tv) : v \in N(\theta)\} \). Furthermore, the linear map \( T \) is symmetric if and only if \( E \) is Lagrangian.
Let $E$ be an invariant Lagrangian subbundle, i.e., for every $\theta \in SM$, $E(\theta) \subset S(\theta)$ is a Lagrangian subspace and $d\phi^t(E(\theta)) = E(\phi^t(\theta))$, for all $t \in \mathbb{R}$. Suppose that $E(\phi^t(\theta)) \cap V(\phi^t(\theta)) = \{0\}$ for every $t \in (-\delta, \delta)$. We can to write $E(\phi^t(\theta)) = \text{graph} U(t)$ for all $t \in (-\delta, \delta)$, with $U(t) : N(\phi^t(\theta)) \to N(\phi^t(\theta))$ which satisfies the Ricatti equation

$$U'(t) + U^2(t) + R(t) = 0,$$

for more details see [2], [1] or [3] Section 2.

### 3 Classical results

In this section, we introduce key results and concepts crucial for proving our main theorem. The first lemma, attributed to Mañé (cf. [9] and [10] for further details), reveals a “twist property” between the vertical subbundle and a Lagrangian subbundle along orbits. Specifically,

**Lemma 3.1.** [9 Lemma III.2] If $\theta \in SM$ and $E \subset S(\theta)$ is a Lagrangian subspace, then the set of $t \in \mathbb{R}$ such that $d\phi^t_0(E) \cap V(\phi^t(\theta)) \neq \{0\}$ is discrete.

Recall that a geodesic arc $\gamma : [a, b] \to M$ is devoid of conjugate points if, for any Jacobi vector field with $J(c) = 0$ and $J'(c) \neq 0$, it holds that $J(t) \neq 0$ for all $t \in [a, b]$ except $t = c$. In the subsequent theorem, Mañé established a correlation between the trivial intersection of the vertical subbundle and a Lagrangian subbundle and the absence of conjugate points.

**Lemma 3.2.** [9 Proposition II.1] Let $M$ be a Riemannian manifold and $\gamma : [0, a] \to M$ a geodesic arc. If there exists a Lagrangian subspace $E \subset S(\gamma(0), \gamma'(0))$ such that $V(\gamma(t), \gamma'(t)) \cap d\phi^t(E) = \{0\}$ for all $0 \leq t \leq a$, then the geodesic arc $\gamma$ does not contain conjugates points.

For an alternative proof of the above result, see [5]. In the Anosov case, the stable ($E^s$) and unstable ($E^u$) subbundles are invariant, continuous, and Lagrangian. In this way, the above results are valid for $E^s$ and $E^u$.

As an immediate consequence of Lemma 3.2 we obtain the following result.

**Corollary 3.1.** Let $J$ be a nonzero Jacobi field along $\gamma_\theta$ such that $J(a) = J(b) = 0$ for $a < b$, then there are $c, d \in [a, b]$, a nonzero stable Jacobi filed $J^s$ and non zero unstable Jacobi field $J^u$ such that $J^s(c) = 0$ and $J^u(d) = 0$.

When the manifold exhibits an Anosov geodesic flow and the curvature is bounded below, the stable and unstable subbundles possess two fundamental properties. The first, established by Knieper (cf. [11]), demonstrates that the zeros of stable and unstable Jacobi fields correspond to conjugate points. The second property arises from Green’s method (cf. Green [5]), which yields a uniform upper bound for the Ricatti solution in instances where stable or unstable Jacobi fields lack zeros. To elaborate further,

**Lemma 3.3.** [7 Lemma 3.5] Let $M$ be a Riemannian manifold with curvature bounded below. If the geodesic flow is Anosov then there exists a constant $\sigma$ with the following property. If

$$E^s(\theta) \cap V(\theta) \neq \{0\}$$

then $\gamma_\theta$ has conjugate points on the interval $[-1, \sigma]$. If

$$E^u(\theta) \cap V(\theta) \neq \{0\}$$

then $\gamma_\theta$ has conjugate points on the interval $[-\sigma, 1]$. 


As $E^s$ and $E^u$ are Lagrangian, then using the notation of Subsection 2.4 we have

**Lemma 3.4.** Let $M$ be a Riemannian manifold with curvature bounded below by $-k^2$. Assume that the geodesic flow is Anosov, then if $\theta \in SM$ satisfies that

$$E^s(\phi^t(\theta)) \cap V(\phi^t(\theta)) = \{0\} \text{ for all } t \in \mathbb{R},$$

then

$$\sup_{t \in \mathbb{R}} \|U^s_\theta(t)\| \leq k,$$

where $U^s_\theta(t): N(\phi^t(\theta)) \to N(\phi^t(\theta))$ is the symmetric linear map such that $E^s(\phi^t(\theta)) = \text{graph} \ U^s_\theta(t)$. An analogous result holds for the unstable case.

We also need a result due to Eberlein (cf. [4]).

**Lemma 3.5.** [4, Lemma 2.8] For any integer $n > 2$ consider the $(n-1) \times (n-1)$ matrix Riccati equation

$$U'(s) + U^2(s) + R(s) = 0,$$  \hfill (2)

where $R(s)$ is a symmetric matrix such that $\langle R(s)x, x \rangle > -k^2$ for some $k > 0$, all unit vectors $x \in \mathbb{R}^{n-1}$ and all real numbers $s$. If $U(s)$ is a symmetric solution of (2), which is defined for all $s > 0$, then $\langle U(s)x, x \rangle < k \coth(ks)$ for all $s > 0$ and all unit vectors $x \in \mathbb{R}^{n-1}$.

4 The intersection between the vertical subspace and $E^{s,u}$

From Lemma 3.3 we need to avoid non-trivial intersection of the stable (unstable) bundle with the vertical bundle. So, we consider the following subset of $SM$

$$B^{s,u} = \left\{ \theta \in SM : V(\theta) \cap E^{s,u}(\theta) \neq \{0\} \right\}.$$

The rest of this section is devoted to finding some properties of $B^{s,u}$. More specifically, we will prove that $B^{s,u} = \emptyset$ (see Corollary 5.1) which implies the Theorem 1.2.

4.1 Properties of sets $B^s$ and $B^u$

**Lemma 4.1.** The sets $B^s$, $B^u \subset SM$ are closed.

**Proof.** Let $\theta_n \in B^{s,u}$ be, $\theta_n \to \theta$, then there is $z_n \in V(\theta_n) \cap E^{s,u}(\theta_n)$. As the spaces involved are subspaces, then we can assume that $\|z_n\| = 1$. Thus, since $V(\theta_n) = \ker d\pi_{\theta_n}$ and $E^{s,u}$ are continuous subbundles, then passing to a subsequence if necessary, $z_n \to z \in V(\theta) \cap E^{s,u}(\theta)$, which implies that $\theta \in B^{s,u}$.

**Remark 4.1.** Using the notation of Subsection 2.4, if $\theta \notin B^{s,u}$, then there are unique linear maps $T^{s,u}_\theta: H(\theta) \cap S(\theta) \to V(\theta) \cap S(\theta)$ such that $E^{s,u}(\theta)$ is the graph of $T^{s,u}_\theta$, i.e.,

$$E^{s,u}(\theta) = \{(z, T^{s,u}_\theta(z)) : z \in H(\theta) \}.$$

Using the horizontal and vertical coordinates from Subsection 2.4 the following lemma yields a local uniform control of the norm of the linear maps $T^{s,u}$, enabling us to govern the vertical coordinates through the horizontal ones.
Lemma 4.2. Assume that $\theta \notin B^{s,u}$, then there is a compact neighborhood $U_{\theta}^{s,u} \subset SM \setminus B^{s,u}$ of $\theta$ and $\alpha_{s,u}(\theta) > 0$ such that if $(v, w) \in E^{s,u}(z)$, then

$$\|w\| \leq \alpha_{s,u}(\theta)\|v\| \text{ for all } z \in U_{\theta}^{s,u}.$$ 

Proof. We prove the stable case since the unstable case is analogous.

Let $\theta \notin B^{s}$ be, then by Lemma 4.1 there is a compact neighborhood $U_{\theta}^{s} \subset SM \setminus B^{s}$ of $\theta$ such that for all $z \in U_{\theta}^{s}$, we have that $V(z) \cap E^{s}(z) = \{0\}$.

Then, by Remark 4.1, for each $z \in U_{\theta}^{s}$ there is a unique linear map $T_{z}^{s}: H(z) \cap S(z) \to V(z) \cap S(z)$, such that $E^{s}(z)$ is the graph of $T_{z}^{s}$. The compactness of $U_{\theta}^{s}$ and the continuity of $E^{s}$ allows us to state that there is $\alpha_{s}(\theta) > 0$ such that $\|T_{z}^{s}(v)\| \leq \alpha_{s}(\theta)\|v\|$ for all $z \in U_{\theta}^{s}$.

Observe that if $(v, w) \in E^{s}(z)$, then $(v, w) = (v, T_{z}^{s}(v))$, which implies that $\|w\| = \|T_{z}^{s}(v)\| \leq \alpha_{s}(\theta)\|v\|$. \hfill \qed

In the context of $B^{s,u}$, as a corollary of Lemma 3.5, we have (compare with Lemma 3.4).

Lemma 4.3. Let $M$ be a Riemannian manifold with curvature bounded below by $-k^2$, for some $k > 0$, with Anosov geodesic flow. Let $\theta \in SM$ such that $\phi^{t}(\theta) \notin B^{s}$ for all $t \geq 0$. If $\phi^{t}$ is the symmetric solution of the Ricatti equation (2) associated to the unstable bundle $E^{u}$ on $[0, +\infty)$, then

$$(\phi^{t}(x), x) < k \coth(kt),$$

for all $t > 0$ and all unit vectors $x \in \mathbb{R}^{n-1}$. Analogous result for stable cases.

Proof. Note simply that if $\phi^{t}(\theta) \notin B^{s}$ for $t \geq 0$, then $\phi^{t}(x)$ is defined for $t \geq 0$ and the result follows from Lemma 3.5. \hfill \qed

Remark 4.2. In the same conditions of the previous lemma, observe that $\coth(kt) \leq 2$ for all $t \geq 1/k$ and then

$$(\phi^{t}(x), x) \leq 2k$$

for all $t \geq 1/k$.

and all unit vectors $x \in \mathbb{R}^{n-1}$. In particular, $\|\phi^{t}(x)\| \leq 2k$ for all $t \geq 1/k$.

Using the notation of Lemma 4.2, then an immediate consequence, we have

Corollary 4.1. In the conditions of the previous lemma, let $\theta \in SM$ such that $\phi^{-t}(\theta) \notin B^{s}$ for all $t \geq 0$. Then for all $t \geq 1/k$,

$$\alpha_{s}(\phi^{-t}(\theta)) \leq 2k.$$ 

Analogous for the stable case.
Proof. Assume that \( t \geq \frac{1}{k} \) and consider \( (v, w) \in E^s(\phi^{-t}(\theta)) \), then by Lemma 4.2 and Remark 4.2 we have that \( \|w\| \leq 2k\|v\| \). Thus, \( \alpha_s(\phi^{-t}(\theta)) \leq 2k \), for all \( t \geq \frac{1}{k} \). The unstable case is analogous. \( \square \)

It is important to observe that in the Anosov case, subbundles \( E^s \) and \( E^u \) are invariant and Lagrangian, therefore Lemma 3.1 can be written in terms of the sets \( B^{s,u} \).

**Lemma 4.4 (Twist Property).** For each \( \theta \in SM \), the sets \( \{ t \in \mathbb{R} : \phi^t(\theta) \in B^s \} \) and \( \{ t \in \mathbb{R} : \phi^t(\theta) \in B^u \} \) are discrete.

When the manifold has curvature bounded below, thanks to Lemma 3.2 and Lemma 3.3, the sets \( B^{s,u} \) have the following special property:

**Lemma 4.5 (Transfer Property).** If \( \theta \in SM \) is such that there is \( t_0 \) with \( \phi^{t_0}(\theta) \in B^s \), then there exists \( t_1 \in [t_0 - 1, t_0 + \sigma] \) such that \( \phi^{t_1}(\theta) \in B^u \), where \( \sigma \) is as in Lemma 3.3. An analogous result for the unstable case.

Proof. From Lemma 3.3 it follows that the geodesic arc \( \gamma_\theta : [t_0 - 1, t_0 + \sigma] \to M \) has conjugate points. Now suppose that \( \phi^t(\theta) \notin B^u \) for all \( t \in [t_0 - 1, t_0 + \sigma] \). Then, from Lemma 4.2 it follows that the geodesic arc \( \gamma_\theta : [t_0 - 1, t_0 + \sigma] \to M \) does not contain conjugate points. This contradiction concludes the proof. \( \square \)

### 4.2 Topological properties of the first positive (negative) time to \( B^{s,u} \)

In this section, we estimate the first moment that orbits intersect the sets \( B^{s,u} \), which will be a fundamental tool to prove Theorem 1.2.

We consider the special sets

\[
B^u_+ = \left\{ \theta \in SM : \text{there is } t \geq 0 \text{ with } \phi^t(\theta) \in B^u \right\}.
\]

and

\[
B^s_+ = \left\{ \theta \in SM : \text{there is } t \leq 0 \text{ with } \phi^t(\theta) \in B^s \right\}.
\]

For \( \theta \in B^u_+ \), denote by

\[
t^+_u(\theta) = \min \left\{ t \geq 0 : \phi^t(\theta) \in B^u \right\}, \text{ and } t^-_u(\theta) = \inf \left\{ t < 0 : \phi^t(\theta) \in B^u \right\}.
\]

Note that if \( \phi^{-t}(\theta) \notin B^u \) for all \( t < 0 \), then \( t^-_u(\theta) = -\infty \). Analogously, if \( \theta \in B^s_+ \), denote by

\[
t^-_s(\theta) = \max \left\{ t \leq 0 : \phi^t(\theta) \in B^s \right\} \text{ and } t^+_s(\theta) = \inf \left\{ t > 0 : \phi^t(\theta) \in B^s \right\}.
\]

Note that if \( \phi^t(\theta) \notin B^s \) for all \( t > 0 \), then \( t^+_s(\theta) = +\infty \).

From Lemma 4.2 the times \( t^+_u(\theta) \) and \( t^-_u(\theta) \) are well defined.

**Remark 4.3.** From the definition of \( t^+_u(\theta) \) and \( t^-_u(\theta) \), there is no unstable Jacobi field with zeros in \( (t^-_u(\theta), t^+_u(\theta)) \). An analogous result for the stable case.

**Definition 4.1.** A subset \( A \subset SM \) is called a Pointwise Negatively (resp. Positively) Capturing Set if, for every \( x \in A \) such that \( O^-(x) \subset A \) (resp. \( O^+(x) \subset A \)), it follows that \( O(x) \subset A \), where \( O^-(x) \) and \( O^+(x) \) denote the negative and positive orbits of \( x \), respectively, and \( O(x) \) denotes the orbit of \( x \).
It is worth noting that negatively or positively invariant sets are not always invariant and therefore cannot be considered Pointwise Negatively (or Positively) Capturing Sets. This is a rather special property found only in very specific cases of dynamical systems. The following theorem demonstrates that the complements of sets $B^u$ and $B^s$ possess this additional property. Moreover, it is pivotal as it allows us to deduce the absence of conjugate points along a geodesic if they are confined to only one side. Its proof ingeniously utilizes the special times $t^u_+(\theta)$ and $t^s_-(\theta)$.

**Theorem 4.1.** The set $SM \setminus B^u$ is Pointwise Negatively Capturing Set and the set $SM \setminus B^s$ is Pointwise Positively Capturing Set.

The proof of the previous theorem is somewhat delicate and requires several additional lemmas. Moreover, it will be instrumental in demonstrating straightforwardly that $SM \setminus B^u$ and $SM \setminus B^s$ are non-empty for non-compact manifolds.

For the rest of the paper, consider the Wronskian $W(J,Y)(t)$ of two Jacobi fields, $J$ and $Y$ defined by

$$W(J,Y)(t) := \langle J(t), Y'(t) \rangle - \langle J'(t), Y(t) \rangle.$$  

It is not difficult to prove that $W(J,Y)(t)$ is a constant function, which is zero if and only if $J$ and $Y$ are linearly dependent. Additionally, the special times defined earlier pinpoint the initial interval where the geodesic remains free of conjugate points, as illustrated in the following lemma.

**Lemma 4.6.** If $\theta \in B^u_+ \setminus B^u$ (resp. $\theta \in B^u_- \setminus B^u$) then $\gamma_\theta(t)$ has no conjugate on $(t^u_-(\theta), t^u_+(\theta))$, (resp. $[t^u_-(\theta), t^u_+(\theta))$).

**Proof.** In the proof does not matter if $t^u_-(\theta)$ is finite or not. By contradiction, assume that there are $t^u_-(\theta) < a < b \leq t^u_+(\theta)$ and a nonzero Jacobi field $J$ along $\gamma_\theta$ such that $J(a) = J(b) = 0$, then from Corollary 3.1 there is a nonzero unstable Jacobi field $J^u$ along $\gamma_\theta$ and $c \in [a,b]$ such that $J^u(c) = 0$. Then, since $\theta \in B^u_+ \setminus B^s$ and the definition of $t^u_-(\theta)$ we have $c \in (0, t^u_-(\theta)]$. If $c < t^u_-(\theta)$ we have a contradiction with the minimality of $t^u_-(\theta)$. Otherwise, $b \geq c = t^u_+(\theta) \geq b$ and the Wronskian, $W(J,J^u) = 0$. Thus it should be $J^u(a) = 0$ with $t^u_-(\theta) < a < b = t^u_+(\theta)$, which provides a contradiction from Remark 1.3.

**Corollary 4.2.** If $\theta \in B^u_+ \setminus B^u$ (resp. $\theta \in B^u_- \setminus B^u$), then $\gamma_\theta(t)$ has no conjugate on $[0, t^u_+(\theta)]$, (resp. $[t^u_-(\theta), 0)$).

Before proving Theorem 4.1 we recall an important property regarding closed intervals without conjugate points (cf. [4 Corollary 2.12]).

**Lemma 4.7.** Let $M$ be a Riemannian manifold with curvature bounded below by $-k^2$ and $T \geq 1$. Then there exists $A = A(k) > 0$ such that if $\gamma: [-1, T + 1] \rightarrow M$ is a geodesic arc without conjugate points and $J$, $-1 \leq t \leq T + 1$ is a perpendicular Jacobi field on $\gamma$ with $J(0) = 0$ and then

$$||J'(t)|| \leq A ||J(t)||$$

for all $1 \leq t \leq T$.

**Proof of Theorem 4.1.** Let us prove that $SM \setminus B^u$ is pointwise negatively capturing set since the other case is analogous. Assume that $O^-(\theta) \subset (SM \setminus B^u)$. Then, considering a fixed number $\beta > 1$ and put $\theta_\beta := \phi^{-\beta}(\theta) \notin B^u$ and $O^-(\theta_\beta) \subset (SM \setminus B^u)$. Moreover, from Lemma 4.2 the geodesic $\gamma_{\theta_\beta}(t)$ has no conjugate points in $(-\infty, \beta)$. Therefore, without loss of generality,
changing $\theta_{\beta}$ by $\theta$, we can assume that $\theta \notin B^u$ and $\gamma_{\theta}(t)$ has no conjugate points in $(-\infty, \beta)$. By contradiction assume that $O^+(\theta) \cap B^u \neq \emptyset$, or equivalent $\theta \in B^+_u \setminus B^u$, consequently $t^u_{+}(\theta) < \infty$. From Lemma 4.10 $\gamma_{\theta}$ has no conjugate points on $[0, t^u_{+}(\theta)]$.

Claim: $\gamma_{\theta}$ has no conjugate points on $(-\infty, t^u_{+}(\theta)]$.

Proof of Claim. If $t^u_{+}(\theta) \leq \beta$, we are done. If $t^u_{+}(\theta) > \beta$, then by contradiction, assume that $\gamma_{\theta}(t)$ has conjugate points in $(-\infty, t^u_{+}(\theta)]$, then there is a nonzero Jacobi field $J$ and $a, b \in (-\infty, t^u_{+}(\theta)]$ such that $J(a) = J(b) = 0$, $a < b$. It is easy to see that $a < 0$ and $\beta \leq b$.

Thus, from Corollary 3.1 and the minimality of $t^u_{+}(\theta)$ we have $\phi^0(\theta) \in B^u$ for some $r_0 \in (a, 0)$. So, by Lemma 4.3 the geodesic $\gamma_{\phi^0(\theta)}(t) = \gamma_{\theta}(t + r_0)$ has conjugate points in $[-\sigma, 1]$. Therefore, $\gamma_{\theta}(t)$ has conjugate points in $[-\sigma + r_0, 1 \pm r_0]$ which implies, since $r_0 < 0$ and $\beta > 1$, that $\gamma_{\theta}(t)$ has conjugate points in $[-\sigma + r_0, \beta)$ which provides a contradiction and the proof of claim is concluded. 

Now we denote by $J^u_0(t)$ the unstable Jacobi field along $\gamma_{\theta}(t)$ such that $J^u_0(t^u_{+}(\theta)) = 0$. For $r > 0$, $J^u_r(t) := J^u_0(t - r)$ is an unstable Jacobi field along $\gamma_{\phi^{r^{-1}}(\theta)}(t)$ with $t^u_{+}(\phi^{r^{-1}}(\theta)) = t^u_{+}(\theta) + r$. We denote $\xi^u_r = (J^u_r(0), (J^u_r)'(0))$ and assume that $\|\xi^u_r\| = 1$. Let $J^r_0(t)$ be a stable Jacobi field along $\gamma_{\phi^{-r}(\theta)}(t)$ with $J^r_0(0) = J^u_r(0)$ and put $\xi^s_r = (J^r_0(0), (J^r_0)'(0))$.

We define the Jacobi field $J_r(t) = J^u_r(t) - J^r_0(t)$ which satisfies $J_r(0) = 0$. Take $r > 0$ such that $t^u_{+}(\theta) + r - 1 > 1$, since $\gamma_{\phi^{-r}(\theta)}(t)$ has no conjugate points in $(-\infty, t^u_{+}(\theta) + r)$, from Lemma 4.10

$$\|J_r(t)\| \leq A\|J_r(t)\| \quad \text{for} \quad 1 \leq t < t^u_{+}(\theta) + r - 1.$$ 

Thus,

$$\|(J^u_r)'(t)\| - \|(J^r_0)'(t)\| \leq \|J_r(t)\| \leq A\|J_r(t)\| + A\|J^r_0(t)\|$$

for all $1 \leq t < t^u_{+}(\theta) + r - 1$.

Therefore, by the definition of Anosov geodesic flow, we have

$$\max\{\|J^r_0(t)\|, \|(J^r_0)'(t)\|\} \leq \left(\|J^r_0(t)\|^2 + \|(J^r_0)'(t)\|^2\right)^{\frac{1}{2}} \leq \|D\phi^r_0(\xi^s_r)\| \leq C\lambda^r_1 \|\xi^s_r\|.$$ 

Thus, from (4) and (5) we have

$$\|(J^u_r)'(t)\| \leq A\|J^u_r(t)\| + C(A + 1)\lambda^r_1 \|\xi^s_r\|$$

for all $1 \leq t < t^u_{+}(\theta) + r - 1$.

Therefore, as $\|\xi^u_r\| = 1$

$$\frac{1}{C\lambda^r_1} \leq \|D\phi^r_0(\xi^s_r)\| \leq \left(\|J^r_0(t)\|^2 + \|(J^r_0)'(t)\|^2\right)^{\frac{1}{2}} \leq \left(\|J^r_0(t)\|^2 + (A\|J^u_r(t)\| + C(A + 1)\lambda^r_1 \|\xi^s_r\|)^2\right)^{\frac{1}{2}} \leq \sqrt{1 + A^2\|J^r_0(t)\|^2 + 2CA(A + 1)\lambda^r_1 \|J^u_r(t)\| \|\xi^s_r\| + C(A + 1)\lambda^r_1 \|\xi^s_r\|^2} \leq \sqrt{1 + A^2\|J^r_0(t)\|^2 + 2CA(A + 1)\|J^u_r(t)\| \|\xi^s_r\|^2 + C(A + 1)\lambda^r_1 \|\xi^s_r\|^2},$$

(6)
whenever $1 \leq t < t^u_+(\theta) + r - 1$.

Note that $\phi^{-r}(\theta) \notin B^s$ for all $r > 0$, then from Corollary 4.1 we have

$$\|\xi_r^u\| \leq \sqrt{1 + 4k^2} \|J_r^s(0)\| \leq \sqrt{1 + 4k^2} \|\xi_r^u\| = \sqrt{1 + 4k^2}, \text{ for } r > \frac{1}{k}. $$

Letting $t \to t^u_+(\theta) + r - 1$, then $J_r^u(t) \to J_r^u(t^u_+(\theta) - 1)$ and from (B)

$$ \frac{1}{C} \lambda^{-t^u_+(\theta) - r + 1} \leq \sqrt{1 + A^2} \|J_r^u(t^u_+(\theta) - 1)\| $$

$$ + \sqrt{2CA(A + 1)} \|J_r^u(t^u_+(\theta) - 1)\| \sqrt{1 + 4k^2} $$

$$ + C(A + 1)\lambda^{t^u_+(\theta) + r - 1} \sqrt{1 + 4k^2}, \text{ for } r > \frac{1}{k}. $$

Taking $r$ large enough, the two last inequalities provide a contradiction, since $0 < \lambda < 1$. Therefore, $O(\theta) \subset SM \setminus B^u$ as we wished.

**Corollary 4.3.** Let $\gamma_\theta(t)$ be a geodesic without conjugate points in $(-\infty, \beta)$ or $(\beta, +\infty)$, for some $\beta$, then $\gamma_\theta(t)$ has no conjugate points in $(-\infty, +\infty)$.

**Proof.** Assume that $\gamma_\theta(t)$ has no conjugate points in $(-\infty, \beta)$, the other case is analogous. From Lemma 3.2 $O^-(\theta) \subset SM \setminus B^s$. So, Theorem 4.1 provides $O(\theta) \subset SM \setminus B^u$ and consequently $\gamma_\theta(t)$ has no conjugate points.

The last corollary has the following important consequence.

**Lemma 4.8.** If $M$ is a non-compact manifold with Anosov geodesic flow, then there is $\theta \in SM$ such that the geodesic $\gamma_\theta(t)$ has no conjugate points. Consequently,

$$ E^{s,u}(\phi^t(\theta)) \cap V(\phi^t(\theta)) = \{0\}, \text{ for all } t \in \mathbb{R}. $$

**Proof.** Since $M$ is a non-compact manifold, there exists a ray $\gamma_\theta : [0, \infty) \to M$, i.e., $\gamma_\theta$ is a geodesic such that $d(\gamma_\theta(t), \gamma_\theta(s)) = |t - s|$, which implies that $\gamma_\theta$ does not have conjugate points in $(0, +\infty)$, then from Corollary 4.3 the geodesic $\gamma_\theta$ does not have conjugate points in $(-\infty, +\infty)$.

This lemma shows that the set of points $\theta$ such that its orbit never intersects $B^s$ and $B^u$ is nonempty. It will be used in Section 5.1 to prove Theorem 5.1.

### 4.2.1 Closedness of $B^u_+$ and $B^s_-$

The main goal of this section is to establish that $B^u_+$ and $B^s_-$ are closed subsets of $SM$ (see Lemma 1.11 and Lemma 1.13). This assertion is pivotal and intricate, requiring the construction of several auxiliary results to substantiate it.

Consider the diffeomorphism $\mathcal{I} : SM \to SM$, given by

$$ \mathcal{I}(x, v) = (x, -v). $$

Note that $\mathcal{I}^2 = \text{Id}$. This diffeomorphism helps to relate $B^u_+$ and $B^s_-$, and the non-positive time $t^u_+(\theta)$ and non-negative time $t^s_+(\mathcal{I}(\theta))$.

**Lemma 4.9.** The diffeomorphism $\mathcal{I}$ has the following properties

(i) $D\mathcal{I}_\theta(E^u(\theta)) = E^s(\mathcal{I}(\theta))$ and $D\mathcal{I}_\theta(E^s(\theta)) = E^u(\mathcal{I}(\theta))$. 

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(ii) $\mathcal{I}(B^u) = B^u$, respectively. Also, $\mathcal{I}(B^s) = B^s$ and $\mathcal{I}(B^u) = B^u$.

(iii) $t_+^u(z) = -t_+^u(\mathcal{I}(z))$ and $t_+^s(z) = -t_+^s(\mathcal{I}(z))$.

Proof. It is easy to see that in the horizontal and vertical coordinates $D\mathcal{I}_\theta(\xi_1, \xi_2) = (\xi_1, -\xi_2)$. So, if $22 \in E^u(\theta)$ and consider $J_2(t)$ the unstable Jacobi field associated to $\xi$, then $J_{D\mathcal{I}_\theta}(\xi)(t) = J_\xi(-t)$ is a stable Jacobi field along $\gamma(\psi^t)$, which implies $D\mathcal{I}_\theta(\xi) \in E^s(\mathcal{I}(\theta))$. As $\dim E^u(\theta) = \dim E^s(\mathcal{I}(\theta))$, then $D\mathcal{I}_\theta(E^u(\theta)) = E^s(\mathcal{I}(\theta))$.

If $\theta \in B^u$, then there is $23 \in E^u(\theta)$ and $t_0 \geq 0$ such that $J_\xi(t_0) = 0$. Therefore, since $J_{D\mathcal{I}_\theta}(\xi)(t)$ is a stable Jacobi field along $\gamma_\psi(t)$ and $J_{D\mathcal{I}_\theta}(\xi)(-t_0) = J_\xi(t_0) = 0$, then $\mathcal{I}(\theta) \in B^s$. Since $\mathcal{I}^2 = \text{Id}$, the other cases follow immediately. 

In the following lemma, we establish a specific lower bound function for the norm of the Jacobi field with a zero in an interval without conjugate points. The proof of this lemma follows the same lines as [7, Lemma 2.13], with a suitable modification to obtain more accurate estimates needed for the proof of Lemma 4.11 and Lemma 4.13.

**Lemma 4.10.** Let $M$ be a Riemannian manifold with curvature bounded below by $-k^2$, $T > 1$, and $\gamma : [-1, T] \to M$ a geodesic arc without conjugate points. Then there exists a positive real function $\rho : (0, T - 1) \to M$, depending only on $k$, such that for all perpendicular Jacobi field $J$ with $J(0) = 0$

$$||J(r)||^2 \geq \rho(T - r), \text{ for all } 1 \leq r < T,$$

where

$$\rho(t) = \frac{||J'(0)||^4}{\left(\frac{8}{3}||J'(0)||^2 + \frac{16}{15}k^2\right)\left(k \coth k + \frac{1}{t} + \frac{k^2t}{3}\right)}.$$

Proof. Fix $r \in [1, T)$ and $\delta \in (0, T - r)$. Now consider the piecewise differentiable vector field $X$ along $\gamma(t)$ defined by

$$X(t) = \begin{cases} 0, & -1 \leq t \leq 0, \\
J(t), & 0 < t \leq r, \\
\left(1 - \frac{(t-r)}{\delta}\right)V(t) & r < t \leq r + \delta, \end{cases}$$

where $V(t)$ is the parallel vector field along $\gamma$ in $[r, r + \delta)$ such that $V(r) = J(r)$. Then, from the index form

$$0 < I_{[-1, r+\delta]}(X, X) = \langle J'(r), J(r) \rangle + \frac{1}{\delta} \langle J(r), J(r) \rangle - \int_r^{r+\delta} \langle R(t)X(t), X(t) \rangle dt.$$

From the proof of Corollary 2.12 in [7], we have $\langle J'(r), J(r) \rangle \leq k \coth k ||J(r)||^2$. Hence,

$$I_{[-1, r+\delta]}(X, X) \leq ||J(r)||^2 \left( k \coth k + \frac{1}{\delta} \right) + k^2 \int_r^{r+\delta} \langle X(t), X(t) \rangle dt$$

$$= ||J(r)||^2 \left( k \coth k + \frac{1}{\delta} + k^2 \int_r^{r+\delta} \left(1 - \frac{(t-r)}{\delta}\right)^2 dt \right)$$

$$= ||J(r)||^2 \left( k \coth k + \frac{1}{\delta} + \frac{k^2\delta}{3}\right). \quad (7)$$

Let $Y$ be the piecewise differentiable vector field along $\gamma(t)$ defined by
Lemma 4.5: There is $\phi$ such that $\theta$ is defined from Lemma 4.6. Assume that $\gamma$ is bounded, then the sequence $\{n\}$ is closed.

\[
I_{[-1,r+\delta]}(X,Y) = -||J'(0)||^2.
\]

Furthermore, it is not difficult to prove that
\[
0 < I_{[-1,r+\delta]}(Y,Y) \leq \frac{8}{3}||J'(0)||^2 + \frac{16}{15}k^2. \tag{8}
\]

Now consider for $\lambda \in \mathbb{R}$ the quadratic equation
\[
I_{[-1,r+\delta]}(X + \lambda Y, X + \lambda Y) = I_{[-1,r+\delta]}(X, X) + 2\lambda I_{[-1,r+\delta]}(Y, X) + \lambda^2 I_{[-1,r+\delta]}(Y, Y).
\]

This quadratic equation has no real roots since the Jacobi equation has no conjugate points, therefore
\[
I_{[-1,r+\delta]}(X, X)I_{[-1,r+\delta]}(Y, Y) > I_{[-1,r+\delta]}(Y, Y) = ||J'(0)||^2.
\]

From (7) and (8) we have
\[
||J(r)||^2 \geq \frac{||J'(0)||^4}{\left(\frac{8}{3}||J'(0)||^2 + \frac{16}{15}k^2\right)\left(k \coth k + \frac{1}{\delta} + \frac{k^2\delta}{3}\right)}.
\]

Hence, taking $\delta \to T - r$ we conclude the proof of lemma.

The lemma below demonstrates that $\mathcal{B}_u$ and $\mathcal{B}_s$ are closed sets in higher dimensions under the assumption of bounded sectional curvature. Interestingly, this result holds in dimension two under the sole assumption of sectional curvature being bounded below (see Lemma 4.13).

Lemma 4.11. If the sectional curvature of $M$ is bounded, then the sets $\mathcal{B}_u$ and $\mathcal{B}_s$ are closed.

Proof of Lemma 4.11. As $I$ is a diffeomorphism, from Lemma 4.9 it is sufficient to prove that $\mathcal{B}_u$ is closed.

Assume that $\theta_n \to \theta$ with $\theta_n \in \mathcal{B}_u$, then for each $n$ there is $t_n \geq 0$ such that $\phi^{t_n}(\theta_n) \in \mathcal{B}^u$ and there is $\eta_n \in E^u(\theta_n)$, $||d\phi^{t_n}(\eta_n)|| = 1$ such that $J_n(t_n) = 0$. Moreover, from Lemma 4.2 and Lemma 4.4 we can assume that $\theta \not\in \mathcal{B}_u\cup\mathcal{B}_s$ and for $n$ large enough $\theta_n \in U_{\delta}^u \cup U_{\delta}^s \subset SM \setminus (\mathcal{B}_u \cup \mathcal{B}_s)$.

It is clear that if $\phi^{t}(\theta) \in \mathcal{B}_u$ then $\theta \in \mathcal{B}_u$. If $\phi^{t}(\theta) \in \mathcal{B}_s$, then by the Transfer Property (Lemma 4.4), there is $t_1 \in [0,1 + \sigma]$ such that $\phi^{t_1}(\theta) \in \mathcal{B}^u$ and $\theta \in \mathcal{B}_u$. Therefore, we can assume that $\phi^{t}(\theta) \not\in (\mathcal{B}_u \cup \mathcal{B}_s)$, and consequently $\phi^{t}(\theta_n) \in U_{\delta}^u(\theta) \cap U_{\delta}^s(\theta) \subset SM \setminus (\mathcal{B}_u \cup \mathcal{B}_s)$. From Lemma 4.10, provides that $\gamma_{\theta_n}(t)$ has no conjugate points on $[0,t_n]$. Consider for each $n$, $\theta_n := \phi^{t}(\theta_n)$, so $\gamma_{\theta_n}(t)$ has no conjugate points on $[-1,t_n - 1]$.

Main Claim: The sequence $\{t_n\}$ is bounded.

Note that this claim implies that, from Lemma 4.11 and passing a sub-sequence if necessary, $\phi^{t_n}(\theta_n) \to \phi^{t_0}(\theta) \in \mathcal{B}^u$, for some $t_0 \geq 0$. Thus we have that $\theta \in \mathcal{B}_u$ as we wish.

Therefore, to complete the proof of the Lemma, we need only prove the Main Claim.
**Proof of Main Claim.** By contradiction, suppose that \( \{ t_n \} \) is unbounded. Since 
\[
η_n = dφ_{\phi (\eta_n)}^{-1}(dφ_\phi(η_n)), \quad \text{then } |η_n| = C|dφ_\phi(η_n)| = C|φ_\phi(η_n)|,
\] then without loss of generality, passing to a sub-sequence if necessary, we assume that \( η_n → η \in E^u(θ) \), and \( θ_n^1 → θ^1 := φ^1(θ) \). Let \( ξ_n \in E^s(θ_n) \) such that \( Jξ_n(1) = Jη_n(1) \). Note that \( E^s(θ_n^1) \cap E^u(θ_n^1) = \{ 0 \} \), then \( Jξ_n(1) \neq Jη_n(1) \).

Thus, we define
\[
J_n(t) = J_n(t + 1) - Jξ_n(t + 1)
\]
a non-zero Jacobi field along \( γ_{θ_n}(t) \) with \( J_n(0) = 0 \). Put \( ||J′_η_n(1) - J′_ξ_n(1)|| = κ_n \). From (11) and continuity of \( E^u \), \( dφ_{\phi(η_n)}(η_n) = (Jη_n(1), Jη_n′(1)) \rightarrow dφ_{\phi(θ)}(η) = (Jη(1), Jη′(1)) \in E^u(θ^1) \). Thus \( Jξ_n(1) = Jη_n(1) \rightarrow Jη(1) \). Also, from Lemma [4,2] as \( φ^1(θ_n) \in U_{φ^1(θ)} \cap U_{φ^1(θ)}^{s} \) we have
\[
||J′_ξ_n(1)|| ≤ α(θ^1)||Jξ_n(1)|| \quad \text{and} \quad ||J′_η_n(1)|| ≤ α(θ^1)||Jη_n(1)||.
\]

So, passing a subsequence if necessary we can assume that \( ξ_n = dφ_{φ^1(θ_n)}^{-1}(Jξ_n(1), Jξ_n′(1)) \rightarrow ξ \in E^s(θ) \).

Remember the Wronskian, \( W(Jξ_n, Jη_n)(t) \) of \( Jξ_n \) and \( Jη_n \) defined by
\[
W(Jξ_n, Jη_n)(t) := <Jξ_n(t), Jη_n′(t)> - <Jη_n′(t), Jξ_n(t)>.
\]

Since \( Jξ_n \) and \( Jη_n \) are linearly independent, it is evident that \( W(Jξ_n, Jη_n)(t) \) is a nonzero constant function. Therefore, given \( Jη_n(t_n) = 0 \), we have:
\[
0 \neq W_n := W(Jξ_n, Jη_n)(t_n) = <Jξ_n(t_n), Jη_n′(t_n)> - <Jη_n′(t_n), Jξ_n(t_n)>
\]
\[
= <Jξ_n(1), Jη_n′(1)> - <Jη_n′(1), Jξ_n(1)>
\]
\[
= <Jξ_n(1), Jη_n′(1) - Jξ_n′(1)>
\]

As \( ξ_n → ξ \) and \( η_n → η \), then \( W_n \) converges to \( <Jξ(1), Jη′(1) - Jξ′(1)> := W_0 = W(Jξ, Jη) \neq 0 \).

Thus, for sufficiently large \( n \), we have:
\[
\frac{3}{2}|W_0| ≥ |W_n| ≥ \frac{1}{2}|W_0|.
\]

In the remainder of the proof, without loss of generality, we assume \( W_0 > 0 \) (this case always occurs in dimension two), since the case of \( W_0 < 0 \) is analogous.

For each \( n \), we will now introduce four sequences of parameters \( c_n^2, β_n, σ_n, \) and \( μ_n \). These sequences enable us to construct a suitable Taylor polynomial approximation of \( |J_n(t)|^2 \) in an appropriate neighborhood of \( t_n \), which depends on these parameters. This will allow us to derive a contradiction with the assistance of Lemma [4,10]

**Parameters \( c_n^2 \) and \( β_n \):**

For each \( n \), we consider the function \( f_n(t) := ||J_n(t)||^2 \). Then \( f_n(t_n - 1) = ||Jξ_n(t_n)||^2 = c_n^2 \) and
\[
f_n′(t_n - 1) = 2(J′_η_n(t_n) - J′_ξ_n(t_n), -Jξ_n(t_n))
\]
\[
= -2(J′_η_n(t_n), Jξ_n(t_n)) + 2(J′_ξ_n(t_n), Jξ_n(t_n))
\]
\[
= -2W_n + 2(J′_ξ_n(t_n), Jξ_n(t_n)).
\]

From (10), since \( Jξ_n(t) \) is a stable Jacobi field, then for \( n \) large enough \( f_n(t_n - 1) < 0 \). In this case, consider the parameter
\[
β_n = \inf\{ α ∈ (t_n, ∞) : f_n(t_n - 1 + α) = 0 \}.
\]

Since \( f_n(t) \) is a non-negative function, \( f_n(t_n - 1) \) is small, and \( Jη_n(t) \) is an unstable Jacobi field, then \( β_n \) is well defined. Moreover, from definition of \( β_n \)
\[
f_n′(t_n - 1 + β) < 0 \quad \text{for} \quad t ∈ [0, β_n] \quad \text{and} \quad f_n′(t_n - 1 + β_n) = 0.
\]
consequently, for $w \in (t_n - 1, t_n - 1 + \beta_n)$ we have

$$0 \leq f_n(w) < f_n(t_n - 1) = \|J_{ξ_n}(t_n)\|^2 := \epsilon_n^2. \quad (12)$$

**Parameters $σ_n$ and $μ_n$:**

Consider the function $g_n(t) = \langle J_{ξ_n}(t), J_{η_n}(t) \rangle$, which satisfies $g_n(1) > 0$, $g_n(t_n) = 0$ and $g_n'(t_n) = W > 0$, for $n$ large enough. So, the following parameter is well-defined

$$σ_n = \sup_{α ∈ (0, t_n]} \{g_n'(t) ≤ 0; \ t ∈ [t_n - α, t_n]\}. \quad (13)$$

Also, if $h_n(t) := \|J_{η_n}(t)\|^2$, then since $J_{η_n}(t)$ is aN unstable Jacobi field and $h_n(t_n) = 0$ the following parameter is well-defined

$$μ_n := \sup_{α ∈ (0, t_n]} \{h_n'(t) ≤ 0; \ t ∈ [t_n - α, t_n]\}. \quad (14)$$

We will now examine some properties of the parameters $ε_n^2$, $β_n$, $σ_n$, and $μ_n$. Specifically, we aim to establish the following main properties:

(a) The sequence $\frac{β_n}{ε_n^2}$ is both upper and lower bounded.

(b) There exist a constant $a_0$ (chosen very suitably) such that $\min\{σ_n, μ_n\} ≥ a_0 \epsilon_n^2$.

Due to the intricate nature and considerable length of the proofs, we will divide the process of establishing properties (a) and (b) into 10 sub-claims.

**Claim 1:** For all $α ∈ [-σ_n, β_n]$

$$\langle J_{ξ_n}(t_n + α), J'_{η_n}(t_n + α) \rangle ≥ \frac{W_n}{2} ≥ \frac{W_0}{4}. \quad (15)$$

**Proof of Claim 1.** Note that,

$$g_n'(t) = \langle J_{ξ_n}(t), J'_{η_n}(t) \rangle + \langle J'_{ξ_n}(t), J_{η_n}(t) \rangle$$

$$= 2\langle J_{ξ_n}(t), J'_{η_n}(t) \rangle - W_n \quad (16)$$

From definition of $σ_n$, if $α ∈ [-σ_n, 0]$, then $g_n'(t_n + α) ≥ 0$. Moreover, if $α ∈ [0, β_n]$, then from definition of $J_n(t)$ and (12) we have

$$||J_{η_n}(t_n + α)|| ≤ ||J_n(t_n + α)|| + ||J_{ξ_n}(t_n + α)||$$

$$≤ \epsilon_n^2 + C\lambda^{t_n-α}||ξ_n||. \quad (17)$$

Therefore,

$$||\langle J'_{ξ_n}(t_n + α), J_n(t_n + α) \rangle|| ≤ C\lambda^{t_n-α}||ξ_n||(\epsilon_n^2 + C\lambda^{t_n-α}||ξ_n||). \quad (18)$$

Note that, for $n$ large enough the right side of (18) converges to $0$, since $ξ_n → ξ$. Thus, for $n$ large enough, we have that

$$||\langle J'_{ξ_n}(t_n + α), J_n(t_n + α) \rangle|| ≤ \frac{W_0}{4}, \quad α ∈ [0, β_n]. \quad (19)$$

Thus, from (14) and (17) we can conclude that

$$g_n'(t_n + α) > 0, \quad α ∈ [0, β_n].$$

Consequently, $g_n'(t_n + α) ≥ 0$ for all $α ∈ [-σ_n, β_n]$, which together with (9) and (13) provides the proof of Claim 1.
Claim 2: There is a constant $\delta_0 \in (0, 1]$ such that for all $n$ large enough
\[
\frac{|\langle J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha) \rangle|}{\|J_{\xi_n}(t_n + \alpha)\| \cdot \|J'_{\eta_n}(t_n + \alpha)\|} \geq \delta_0; \quad \alpha \in [-\sigma_n, \beta_n].
\]
(18)

Proof of Claim 2. For each $\alpha \in [-\sigma_n, \beta_n]$ consider the set
\[
A_{n,\alpha} := \{ \nu : |\langle \nu, J'_{\eta_n}(t_n + \alpha) \rangle| \geq W_0 \frac{4}{4} \}.
\]

It is easy to see that there is a positive constant $\alpha_0$ such that for every $\nu \in A_{n,\alpha}$ and all $\alpha \in [-\sigma_n, \beta_n]$ we have
\[
|\angle(\nu, J'_{\eta_n}(t_n + \alpha))| > \frac{\pi}{2},
\]
(19)

where $\angle(v, w)$ denotes the angle between $v$ and $w$ (angle between $[-\frac{\pi}{2}, \frac{\pi}{2}]$). Finally, note that
\[
\frac{|\langle J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha) \rangle|}{\|J_{\xi_n}(t_n + \alpha)\| \cdot \|J'_{\eta_n}(t_n + \alpha)\|} = \cos \angle(J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha)).
\]

From Claim 1 $J_{\xi_n}(t_n + \alpha) \in A_{n,\alpha}$, therefore (19) gives us
\[
\cos \angle(J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha)) \geq \cos \alpha_0 := \delta_0 > 0,
\]
which concludes the proof of Claim 2.

As an important observation, note that (18) is invariant for multiplication of a parameter $a > 0$, i.e., if we consider the stable and unstable Jacobi field $aJ_{\xi_n}(t)$ and $aJ_{\eta_n}(t)$, respectively, then (18) is valid for the same $\delta_0$. Therefore, for a parameter $\tau > 0$, we consider
\[
J_{n,\tau}(t) = J_{\tau\eta_n}(t + 1) - J_{\delta\xi_n}(t + 1) = \tau J_n(t).
\]

We will choose a suitable parameter $\tau$. In fact: note that
\[
\kappa_{n,\tau} := \|J'_{\tau\eta_n}(1) - J'_{\tau\xi_n}(1)\| = \tau \kappa_n; \quad \kappa_{0,\tau} := \|J'_{\tau\eta}(1) - J'_{\tau\xi}(1)\| = \tau \kappa_0,
\]
Inspired by Lemma 4.10, note that
\[
\lim_{\tau \to \infty} \frac{\tau^4 \kappa_0^4}{\frac{8}{3} \tau^2 \kappa_0^2 + \frac{160}{15} k^2} = +\infty.
\]

As $\delta_0$ does not depend on $\tau$, then there is $\tau_0 > 0$ such that
\[
\frac{\tau^4 \kappa_0^4}{\left(\frac{8}{3} \tau_0^2 \kappa_0^2 + \frac{16}{15} k^2\right)} > \frac{2}{\delta_0^2} + 5.
\]
(20)

Using the parameter $\tau_0$ and (20), if necessary, we change $J_n$ by $\tau_0 J_n$ such that, from now on (avoid $\delta_0$) we have
\[
\frac{\kappa_0^4}{\left(\frac{8}{3} \kappa_0^2 + \frac{16}{15} k^2\right)} > \frac{2}{\delta_0^2} + 5.
\]
(21)
The choice of this parameter $\tau_0$ will be used at the end of the proof to obtain a contradiction.

Claim 3: For all $\alpha \in [-\sigma_n, \beta_n]$

\[
\frac{(|W_n| - |P_n(\alpha)|)^2}{||J_{\xi_n}(t_n + \alpha)||^2} \leq ||J'_{\eta_n}(t_n + \alpha)||^2 \leq \frac{(|W_n| + |P_n(\alpha)|)^2}{\delta_0||J_{\xi_n}(t_n + \alpha)||^2},
\]

(22)

where $P_n(\alpha) := \langle J'_{\xi_n}(t_n + \alpha), J_{\eta_n}(t_n + \alpha) \rangle$ and $\delta_0$ as Claim 2.

Proof of Claim 3. Using the Wronskian

\[
\langle J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha) \rangle = \langle J'_{\xi_n}(t_n + \alpha), J_{\eta_n}(t_n + \alpha) \rangle + W_n.
\]

(23)

Then (23) provides

\[
||J_{\xi_n}(t_n + \alpha)|| \cdot ||J'_{\eta_n}(t_n + \alpha)|| \geq ||J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha)|| \geq |W_n| - |P_n(\alpha)|,
\]

which give us the left side of (22).

The proof of the right side of (22) follows from the Claim 2 and (23), since

\[
||J_{\xi_n}(t_n + \alpha)|| \cdot ||J'_{\eta_n}(t_n + \alpha)|| \leq \frac{||J_{\xi_n}(t_n + \alpha), J'_{\eta_n}(t_n + \alpha)||}{\delta_0} \leq \frac{|W_n| + |P_n(\alpha)|}{\delta_0}.
\]

In the next claim, we estimate $\frac{\beta_n}{\epsilon_n^2}$.

Claim 4: The sequence $\{\frac{\beta_n}{\epsilon_n^2}\}$ is bounded.

Proof of Claim 4. Since $f''_n(t_n - 1 + \beta_n) = 0$, using the Taylor’s Theorem, for some $\tilde{\beta}_n \in (t_n - 1, t_n - 1 + \beta_n)$

\[
f_n(t_n - 1) = f_n(t_n - 1 + \beta_n) - f'_n(t_n - 1 + \beta_n)\beta_n + R(\beta_n)
\]

\[
\epsilon_n^2 = f_n(t_n - 1 + \beta_n) + f''_n(\tilde{\beta}_n)\beta_n^2
\]

\[
\epsilon_n^2 = f_n(t_n - 1 + \beta_n) + 2(\|J'_n(\tilde{\beta}_n)\|^2 - K(\tilde{\beta}_n)||J_n(\tilde{\beta}_n)||^2)\beta_n^2,
\]

(24)

where $K(\tilde{\beta}_n) = K(\gamma_{\tilde{\alpha}_n}(\tilde{\beta}_n), J_n(\tilde{\beta}_n))$ is the sectional curvature. From the last equality

\[
1 = \frac{f_n(t_n - 1 + \beta_n)}{\epsilon_n^2} + \frac{2||J'_n(\tilde{\beta}_n)||^2\beta_n^2}{\epsilon_n^2} - 2K(\tilde{\beta}_n)||J_n(\tilde{\beta}_n)||^2 \frac{\beta_n^2}{\epsilon_n^2}.
\]

(25)

By hypothesis, we have that $-k^2 \leq K(\tilde{\beta}_n) \leq b^2$ for some $k, b > 0$. Then. from (12)

\[
-k^2\epsilon_n^2 \leq K(\tilde{\beta}_n)||J_n(\tilde{\beta}_n)||^2 \leq b^2\epsilon_n^2.
\]

(26)

From definition of $J_n(t)$, $||J'_n(\tilde{\beta}_n)||$ converges to infinite, so from (24) and (26) we can conclude that

\[
\lim_{n \to \infty} \beta_n = 0.
\]
Also, from \([\ref{26}]\)

\[-k^2 \beta_n^2 \leq K(\beta_n)|J_n(\beta_n)|\beta_n^2 \leq b^2 \beta_n^2. \tag{27}\]

By contradiction, assume that \(\{\beta_n / c_n\}\) is an unbounded sequence, then since \(\beta_n\) converge to 0, then \(\ref{25}\) and \(\ref{27}\) provides

\[\lim_{n \to \infty} ||J'_n(\beta_n)||^2 \beta_n = 0. \tag{28}\]

Put \(\bar{\beta}_n \in [0, \beta_n]\) such that \(\bar{\beta}_n = t_n - 1 + \beta_n\). Since \(J_{\xi_n}(t)\) is a stable Jacobi field, then the definition of \(J_n(t)\) and \(\ref{28}\) give us

\[\lim_{n \to \infty} ||J'_n(t_n + \bar{\beta}_n)||^2 \beta_n = 0. \tag{29}\]

From \(\ref{22}\)

\[
\frac{(|W_n| - |P_n(\beta_n)|)^2}{||J_{\xi_n}(t_n + \bar{\beta}_n)||^2} \leq ||J'_{\xi_n}(t_n + \bar{\beta}_n)||^2,
\]

where \(P_n(\beta_n) := \langle J'_{\xi_n}(t_n + \bar{\beta}_n), J_{\xi_n}(t_n + \bar{\beta}_n)\rangle\).

From mean value theorem, for some \(w \in [t_n, t_n + \bar{\beta}_n]\)

\[
\frac{||J_{\xi_n}(t_n + \bar{\beta}_n)||^2}{\beta_n} = 2 \langle J'_{\xi_n}(w), J_{\xi_n}(w) \rangle \frac{\bar{\beta}_n}{\beta_n} + \frac{\epsilon_n^2}{\beta_n}. \tag{31}\]

As \(\langle J'_{\xi_n}(w), J_{\xi_n}(w) \rangle \leq C^2 \lambda^{2\nu}||\xi_n|| \leq C^2 \lambda^{2\nu}||\xi_n||, \frac{\beta_n}{\beta_n} \leq 1, \) and \(\frac{\epsilon_n^2}{\beta_n} \to 0\) (because, we are assuming assume that \(\{\beta_n / c_n\}\) is an unbounded sequence), then from \(\ref{31}\) we have

\[\lim_{n \to \infty} ||J_{\xi_n}(t_n + \bar{\beta}_n)||^2 = \infty. \]

Finally, as \(W_n\) converges to \(W_0 > 0\) and \(P_n(\beta_n)\) converges to 0, then from the last inequality and \(\ref{31}\) we conclude that \(\lim_{n \to \infty} ||J'_n(t_n + \bar{\beta}_n)||^2 \beta_n = \infty\), which is a contradiction with \(\ref{24}\) and consequently the sequence \(\{\beta_n / c_n\}\) is a bounded sequence.

The claim below holds for any positive constant \(a_0 > 0\). However, we select a specific value for \(a_0\) to derive a contradiction. Specifically, we set \(a_0 := \frac{1 - \delta_0^2}{W_0} + 2\), where \(\delta_0\) is as defined in claim 3.

**Claim 5:** For all \(a_0 > 0\), we have

\[\lim_{n \to \infty} \left| \frac{||J_{\xi_n}(t_n)||^2}{||J_{\xi_n}(t_n + \alpha)||^2} - 1 \right| = 0, \text{ for all } \alpha \in [-a_0 \epsilon_n^2, \beta_n].\]

**Proof of Claim 5.** Consider \(\alpha \in [-a_0 \epsilon_n^2, \beta_n]\), then from mean value theorem (similar to \(\ref{31}\))

\[
\frac{||J_{\xi_n}(t_n + \alpha)||^2}{||J_{\xi_n}(t_n)||^2} - 1 = 2 \frac{\langle J'_{\xi_n}(w), J_{\xi_n}(w) \rangle}{||J_{\xi_n}(t_n)||^2} \alpha, \tag{32}\]

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for some $w \in [t_n, t_n + \alpha]$ or $w \in [t_n + \alpha, t_n]$.

As $\epsilon_n = |J_{\xi_n}(t_n)|$, then

\[
\frac{|\langle J'_{\xi_n}(w), J_{\xi_n}(w) \rangle |}{\epsilon_n^2} \leq \frac{||J'_{\xi_n}(w)|| \cdot ||J_{\xi_n}(w)||}{\epsilon_n^2} \left\{ a_0, \frac{\beta_n}{\epsilon_n^2} \right\}
\]

\[
\leq C2^2 \lambda^2 \epsilon_n ||\xi_n|| \max \left\{ a_0, \frac{\beta_n}{\epsilon_n^2} \right\}.
\]  \hspace{1cm} (33)

From Claim 4 the sequence $\left\{ \frac{\beta_n}{\epsilon_n^2} \right\}$ is bounded, then the right of (33) converges to 0, which from (32) completes the proof of claim. ~

With the assistance of the last claims, we establish a precise estimate of $\frac{\beta_n}{\epsilon_n^2}$.

**Claim 6:** For all $\epsilon > 0$ there is $n_0$ such that

\[
\frac{\delta_0^2}{W_0} - \epsilon < \frac{\beta_n}{\epsilon_n^2} < \frac{1}{W_0} + \epsilon, \text{ for all } n \geq n_0.
\]

**Proof of Claim 6.** From (11), $f'_n(t_n - 1 + \beta_n) = 0$, then

\[
|f'_n(t_n - 1 + \beta_n) - f'_n(t_n - 1)| = |f''_n(\hat{\beta}_n)\beta_n
\]

\[
|f'_n(t_n - 1)| = 2||J'_n(\hat{\beta}_n)||^2 - K(\hat{\beta}_n)||J_n(\hat{\beta}_n)||^2 \cdot \beta_n
\]

for some $\hat{\beta}_n \in (t_n - 1, t_n - 1 + \beta_n)$, where $K(\hat{\beta}_n) = K(\gamma_{n_0}^{\hat{\beta}_n}(\hat{\beta}_n), J_n(\hat{\beta}_n))$ is the sectional curvature. The last equation and (10) provides

\[
2|\langle J'_{\xi_n}(t_n), J_{\xi_n}(t_n) \rangle| - W_n| = 2||J'_n(\hat{\beta}_n)||^2 - K(\hat{\beta}_n)||J_n(\hat{\beta}_n)||^2 \cdot \beta_n.
\]  \hspace{1cm} (34)

Note that the left side of (34) converges to $2W_0$. Moreover, since $-k^2 \leq K(\hat{\beta}_n) \leq b^2$, then from (12)

\[
-k^2 \epsilon_n^2 \leq K(\hat{\beta}_n)||J_n(\hat{\beta}_n)|| \leq b^2 \epsilon_n^2.
\]

Consequently, from (34)

\[
\lim_{n \to \infty} ||J'_n(\hat{\beta}_n)||^2 \cdot \beta_n = W_0.
\]  \hspace{1cm} (35)

Put $\tilde{\beta}_n := t_n - 1 + \beta_n$, for some $\tilde{\beta}_n \in [0, \beta_n]$. Since $J_{\xi_n}(t)$ is a stable Jacobi field, then the definition of $J_n(t)$ and (35) give us

\[
\lim_{n \to \infty} ||J'_n(t_n + \tilde{\beta}_n)||^2 \cdot \beta_n = W_0.
\]  \hspace{1cm} (36)

Finally, from Claim 3 or (22) we have

\[
\frac{||W_n| - |P_n(\tilde{\beta}_n)||^2 \epsilon_n^2}{||J_{\xi_n}(t_n + \beta_n)||^2} \leq ||J'_n(t_n + \tilde{\beta}_n)||^2 \epsilon_n^2 \leq \frac{||W_n| + |P_n(\tilde{\beta}_n)||^2 \epsilon_n^2}{\delta_0^2 ||J_{\xi_n}(t_n + \beta_n)||^2}.
\]  \hspace{1cm} (37)

Similar to (10), $\lim_{n \to \infty} |P_n(\tilde{\beta}_n)| = 0$. Thus, claim 5 provides that the left side and right side of (37) converge to $W_0^2$ and $W_0^2 \delta_0^2$, respectively. Therefore, from (36) and (37), given $\epsilon > 0$ we can find $n_0$ large enough such that such that

\[
\frac{\delta_0^2}{W_0} - \epsilon < \frac{\beta_n}{\epsilon_n^2} < \frac{1}{W_0} + \epsilon, \text{ for all } n \geq n_0,
\]

and the proof of claim is complete. ~

\[
\frac{\delta_0^2}{W_0} - \epsilon < \frac{\beta_n}{\epsilon_n^2} < \frac{1}{W_0} + \epsilon, \text{ for all } n \geq n_0,
\]
Consider the function \( h_n(t) := ||J_{\eta_n}(t)||^2 \), and define
\[
\mu_n := \sup_{\alpha \in (0, t_n)} \{ h'_n(t) \leq 0; \ t \in [t_n - \alpha, t_n] \}.
\]
The number \( \mu_n \) is well defined since \( J_{\eta_n}(t) \) is an unstable Jacobi field and \( h_n(t_n) = 0 \).

Claim 7: Consider \( \Gamma_n(\alpha) := ||J_{\eta_n}(t_n + \alpha)||^2 \), then there is a constant \( \Gamma_0 \) such that, for \( n \) large enough
\[
\Gamma_n(\alpha) \leq \Gamma_0, \text{ for all } \alpha \in [-\chi_n, \beta_n],
\]
where \( \chi_n := \min\{\sigma_n, \mu_n, a_0\epsilon_n^2\} \) and \( a_0 \) as Claim 5.

Proof of Claim 7. From (15) we have that for all \( \alpha \in [0, \beta_n] \)
\[
||J_{\eta_n}(t_n + \alpha)|| \leq \epsilon_n^2 + C\lambda_n||\xi_n|| \leq 1
\]
for \( n \) large enough.

So, from now on, we only need to be worried about \( \alpha \in [-\chi_n, 0] \). Consider \( \alpha \in [-\chi_n, 0] \), since \( h(t_n) = 0 \), then from Mean Value Theorem
\[
-\Gamma_n(\alpha) = -h_n(t_n + \alpha) = h'_n(w)\alpha,
\]
for some \( w \in (t_n + \alpha, t_n) \). Also,
\[
h'_n(t_n) - h'_n(w) = h''(\bar{w})(t_n - w) = 2\left(-K(\bar{w})||J'_{\eta_n}(\bar{w})||^2 + ||J''_{\eta_n}(\bar{w})||^2\right)(t_n - w),
\]
for some \( \bar{w} \in (w, t_n) \). Here \( K(\bar{w}) = K(\gamma_{\eta_n}'(\bar{w}), J_{\eta_n}'(\bar{w})) \) is the sectional curvature.

As \( -\alpha \leq \chi_n \leq \mu_n \), then \( t_n - \mu_n \leq t_n + \alpha < w < \bar{w} < t_n \). Therefore, the definition of \( \mu_n \) provides \( ||J_{\eta_n}(\bar{w})||^2 \leq \Gamma_n(\alpha) \). Moreover, since \( K(\bar{w}) \geq -k^2 \) and \( h''(t_n) = 0 \), we obtain
\[
\Gamma_n(\alpha) \leq -2\left(-k^2||J_{\eta_n}(\bar{w})||^2 + ||J''_{\eta_n}(\bar{w})||^2\right)\alpha
\]
\[
\leq -2\left(-k^2\Gamma_n(\alpha) + ||J''_{\eta_n}(\bar{w})||^2\right)\alpha
\]
\[
\leq 2a_0\left(-k^2\Gamma_n(\alpha) + ||J''_{\eta_n}(\bar{w})||^2\right)\epsilon_n^2,
\]
where in the last inequality we use that \( -\alpha \leq \chi_n \leq a_0\epsilon_n^2 \).

Therefore, our task now is to estimate \( ||J''_{\eta_n}(\bar{w})||^2\epsilon_n^2 \), for \( \bar{w} \in [t_n + \alpha, t_n] \) and \( \alpha \geq -\chi_n \). We can write \( \bar{w} = t_n + a \), with \( a \geq \alpha \). Thus, as \( a \geq \alpha \geq -\chi_n \geq -a_0\epsilon_n^2 \), then the Claim 5 provides
\[
\lim_{n \to \infty} \frac{\epsilon_n^2}{||J_{\eta_n}(t_n + a)||} = 1.
\]

Moreover, as \( a \geq \alpha \geq -\chi_n \geq -\sigma_n \), then the Claim 3 establish that
\[
||J''_{\eta_n}(t_n + a)||^2\epsilon_n^2 \leq \frac{(W_n + |P_n(a)|)^2\epsilon_n^2}{\delta_0^2||J_{\eta_n}(t_n + a)||^2}.
\]
As \( W_n \to W_0 \) and \( P_n(a) \to 0 \), then from (39) the right side of (40) converges to \( \frac{W_0^2}{\delta_0^2} \). Consequently, for \( n \) large enough,
\[ ||J'_{\eta_n}(\tilde{w})||^2 \leq ||J'_{\eta_n}(t_n + a)||^2 \leq \frac{W_0^2}{\delta_0^2} + 1. \]

Then, from (38) we conclude that
\[ \Gamma_n(\alpha) \leq \frac{W_0^2 + \delta_0^2}{1 + 2a_0k^2}\delta_0^2 := \Gamma_0. \]

\[ \textbf{Claim 8:} \text{ For } n \text{ large enough } \mu_n \geq \min\{a_0\epsilon_n^2, \sigma_n\}. \]

\[ \textbf{Proof of Claim 8.} \text{ Similar to last arguments, as } h'_n(t_n - \mu_n) = 0, \text{ then } \]
\[ 0 = h'(t_n) - h'_n(t_n - \mu_n) = 2\left( -K(\omega_n)||J_{\eta_n}(\omega_n)||^2 + ||J'_{\eta_n}(\omega_n)||^2 \right)\mu_n \]
for some \( \omega_n \in [t_n - \mu_n, t_n] \) and again \( K(w_n) = K(\gamma_{\eta_n}'(w_n), J_{\eta_n}(w_n)) \) is the sectional curvature. Thus
\[ b^2 \geq K(\omega_n) = \frac{||J'_{\eta_n}(\omega_n)||^2}{||J_{\eta_n}(\omega_n)||^2}. \]

The definition of Anosov flow and definition of \( \mu_n \) provide
\[ C^{-2}\lambda^{-2\omega_n}||\eta_n|| \leq \left( \frac{||J'_{\eta_n}(\omega_n)||^2}{||J_{\eta_n}(\omega_n)||^2} + 1 \right)||J_{\eta_n}(\omega_n)||^2 \leq (b^2 + 1)\Gamma_n(\mu_n). \] (41)

Now, assume by contradiction that \( \mu_n < \min\{a_0\epsilon_n^2, \sigma_n\} \), then \( \mu_n = \chi_n \). Thus, Claim 7 and (41) give us
\[ \frac{C^{-2}\lambda^{-2\omega_n}||\eta_n||}{b^2 + 1} \leq \Gamma_0. \]

Since \( \eta_n \) converges to \( \eta \), in particular, \( ||\eta_n|| \) is bounded, then we have a contradiction, since for \( n \) large enough the left side of the last inequality goes to infinite.

\[ \textbf{Claim 9:} \text{ For } n \text{ large enough } \sigma_n \geq \min\{\mu_n, a_0\epsilon_n^2\}. \]

\[ \textbf{Proof of Claim 9.} \text{ Recall that } g_n(t) = \langle J_{\xi_n}(t), J_{\eta_n}(t) \rangle \text{ and whose derivative is } \]
\[ g'_n(t) = 2\langle J_{\xi_n}(t), J'_{\eta_n}(t) \rangle - W_n \]
By definition of \( \sigma_n \), if \( g'_n(t_n - \sigma_n) \neq 0 \), then \( \mu_n = t_n \) and the claim holds. Thus, we can assume \( g'_n(t_n - \sigma_n) = 0 \) and then
\[ \langle J_{\xi_n}(t_n - \sigma_n), J'_{\eta_n}(t_n - \sigma_n) \rangle = \frac{W_n}{2}. \]

Moreover, the Wroslakian satisfies
\[ \langle J_{\xi_n}(t), J'_{\eta_n}(t) - \langle J'_{\xi_n}(t), J_{\eta_n}(t) \rangle = W_n \]
Consequently,
\[ \langle J'_{\xi_n}(t_n - \sigma_n), J_{\eta_n}(t_n - \sigma_n) \rangle = -\frac{W_n}{2}. \]
Assume by contradiction that \( \sigma_n < \min\{\mu_n, a_0 \epsilon_n^2\} \), then \( \sigma_n = \chi_n \). Thus, the last equation, Claim 7, and (9) give us

\[
\frac{W_0^2}{16} \leq \frac{W_n^2}{4} \leq ||J'_n(t_n - a_n)||^2 \cdot ||J_{y_n}(t_n - \sigma_n)||^2 \\
\leq C^2 \lambda^{2(t_n - a_n)} \Gamma_n(-\sigma_n) \leq C^2 \lambda^{2(t_n - \sigma_n)} \Gamma_0. \tag{42}
\]

The right side of (42) converges to 0, which is a contradiction since \( W_0 > 0 \).

Since \( a_n \), \( \mu_n \), and \( \epsilon_n^2 \) are non-negative numbers, then as an immediate consequence of Claim 8 and Claim 9 we have

**Claim 10:** For \( n \) large enough

\[
\min\{\sigma_n, \mu_n\} \geq a_0 \epsilon_n^2,
\]

consequently, \( \chi_n = a_0 \epsilon_n^2 \).

Continuing with the proof of the lemma, from the Claim 6, for \( n \) large enough

\[
-\omega_n := \beta_n - \left(\frac{1}{W_0} + 1\right) \epsilon_n^2 < 0.
\]

Remember the parameter \( a_o := 1 - \frac{\delta_0^2}{W_0} + 2 \). From Claim 6, it is easy to see that

\[
a_o \epsilon_n^2 \geq \left(\frac{1}{W_0} + 1\right) \epsilon_n^2 - \beta_n = \omega_n \geq \frac{1}{2} \epsilon_n^2.
\]

Therefore,

\[
t_n - a_0 \epsilon_n^2 < t_n - \omega_n < t_n < t_n + \omega_n < t_n + \beta_n. \tag{43}
\]

Moreover, we know that \( \gamma_{\theta_n}(t) \) has no conjugate point in \([-1, t_n - 1]\). In particular, for all \( \vartheta > 0 \) small enough, \( \gamma_{\theta_n}(t) \) has no conjugate point in \([-1, t_n - 1 - \omega_n - \vartheta]\). Thus, since \( J_n(0) = 0 \), then the Lemma 4.10 provides that for all \( \vartheta > 0 \) small enough

\[
||J_n(t_n - 1 - \omega_n)||^2 \geq \rho(\omega_n - \vartheta).
\]

Since \( \vartheta \) is arbitrary, then there we have

\[
||J_n(t_n - 1 - \omega_n)||^2 \geq \rho(\omega_n).
\]

The last inequality and definition of \( \rho(t) \) provides

\[
\left(k \coth k + \frac{1}{\omega_n} + \frac{k^2}{3} \omega_n\right) ||J_n(t_n - 1 - \omega_n)||^2 \geq \frac{\kappa_n^4}{\left(\frac{8}{3} \kappa_n^2 + \frac{16}{15} k^2\right)}, \tag{44}
\]

where \( \kappa_n = ||J'_n(0)|| = ||J'_{y_n}(1) - J'_{\xi_n}(1)|| \).

Multiplying and dividing by \( \epsilon_n^2 \) the left side of (44)

\[
\left(\frac{8}{3} \kappa_n^2 + \frac{16}{15} k^2\right) \left(\epsilon_n^2 k \coth k + \frac{\epsilon_n^2}{\omega_n} + \frac{k^2}{3} \epsilon_n^2 \omega_n\right) \leq ||J_n(t_n - 1 - \omega_n)||^2 \epsilon_n^2. \tag{45}
\]
From Claim 6, let $\epsilon > 0$, such that for $n$ large enough we have that
\[
e \frac{k^2}{3} \epsilon^2_n \omega_n < \frac{\epsilon}{1 - \epsilon} \quad \text{and} \quad \frac{\omega_n}{\epsilon^2_n} > 1 - \epsilon
\]

Therefore,
\[
\frac{1}{\left(\epsilon^2 n \coth k + \frac{k^2}{3} \epsilon^2_n \omega_n\right)} > \frac{1 - \epsilon}{1 + \epsilon} > \frac{1}{2},
\]

correspondingly, (45) become
\[
\frac{1}{2} \left(\frac{\kappa_n^4}{8 \kappa_n^2 + \frac{16}{15} k^2}\right) \leq \frac{||J_n(t_n - 1 - \omega_n)||^2}{\epsilon^2_n}.
\]  

(46)

From Taylor’s theorem centered in $t_n - 1 + \beta_n$ we can write (similar to (22))
\[
f_n(t_n - 1 - \omega_n) = \frac{f_n(t_n - 1 + \beta_n)}{\epsilon^2_n} - f_n'(t_n - 1 + \beta_n)\frac{(\beta_n + \omega_n)}{\epsilon^2_n} + 2\left(||J_n'(\alpha_n)||^2 - K(\alpha_n)||J_n(\alpha_n)||^2\right)\frac{(\beta_n + \omega_n)^2}{\epsilon^2_n},
\]  

(47)

for some $\alpha_n \in [t_n - 1 - \omega_n, t_n - 1 + \beta_n]$.

Let us estimate the right side of the last equation. First note that from (25) and (26) we have that for $n$ large enough
\[
f_n(t_n - 1 + \beta_n) \leq \frac{3}{2}.
\]  

(48)

From (43), $\omega_n \leq a_0 \epsilon^2_n$, thus since $\alpha_n \in [t_n - 1 - \omega_n, t_n - 1 + \beta_n]$, then Claim 7 and Claim 10 provides
\[-k^2 \Gamma_0 \leq K(\alpha_n)||J_n(\alpha_n)|| \leq b^2 \Gamma_0.
\]

Also
\[
\frac{(\beta_n + \omega_n)^2}{\epsilon^2_n} = \left(\frac{1}{W_0} + \epsilon\right)^2 \epsilon^2_n.
\]

Thus, it is easy to see that
\[
\lim_{n \to \infty} K(\alpha_n)||J_n(\alpha_n)|| \frac{(\beta_n + \omega_n)^2}{\epsilon^2_n} = 0.
\]  

(49)

Moreover, from Claim 3 and Claim 5, for $n$ large enough
\[
||J_n'(\alpha_n)|| \frac{(\beta_n + \omega_n)^2}{\epsilon^2_n} = ||J_n'(\alpha_n)|| \frac{2(1/W_0 + \epsilon)^2}{\epsilon^2_n} \leq \frac{1}{\delta_0^2} + \frac{1}{8}.
\]

Thus, from definition of $J_n(t)$ the last equation implies that for $n$ large enough
\[
||J_n'(\alpha_n)|| \frac{(\beta_n + \omega_n)^2}{\epsilon^2_n} \leq \frac{1}{\delta_0^2} + \frac{1}{4}.
\]  

(50)

Since $f_n'(t_n - 1 + \beta_n) = 0$, then (47), (48), (49), and (50) allow us to conclude that for $n$ large enough
\[
f_n(t_n - 1 - \omega_n) \leq \frac{1}{\delta_0^2} + 2.
\]  

(51)
Finally, as $f_n(t_n - 1 - \omega_n) = ||J_n(t_n - 1 - \omega_n)||^2$, then (16) and (51) provide
\[
\frac{\kappa_n^4}{\left(\frac{8}{3}\kappa_n^2 + \frac{16}{15}k^2\right)} \leq \frac{2}{\delta_0^2} + 4,
\]
which implies that
\[
\frac{\kappa_0^4}{\left(\frac{8}{3}\kappa_0^2 + \frac{16}{15}k^2\right)} \leq \frac{2}{\delta_0^2} + 4. \tag{52}
\]
It is clear that (52) is a contradiction with (21) and therefore we conclude the proof of Claim and consequently conclude the proof of Lemma.

Remark 4.4. In the proof of Lemma 4.11, only the upper bound of the curvature was utilized in (26), (34), and (41). Therefore, we can amend the condition of bounded curvature from above to the following condition: Put
\[
M_n := \max\left\{\sup_{t \in [t_n - 1, t_n - 1 + \beta_n]} K(J_n(t), \gamma_{\theta_n}(t)), \sup_{t \in [t_n, t_n + \beta_n]} K(J_{\eta_n}(t), \gamma_{\theta_n}(t))\right\},
\]
then
\[
\liminf_{n \to \infty} \lambda^n M_n = 0, \tag{53}
\]
where $\lambda$ is the constant of contraction of the definition of Anosov flow. Since $0 < \lambda < 1$, then it is clear that if $M$ has curvature bounded above, then satisfies (53).

To conclude this section, we establish Lemma 4.11 in the two-dimensional scenario without imposing any restriction on the upper limit of the sectional curvature. To accomplish this, we require the following lemma.

Lemma 4.12 (Zeros of Jacobi Fields in Dimension 2). If $a < b$ and $J$ is a Jacobi field such that $J(a) = J(b) = 0$, then any Jacobi field $\tilde{J}$ has a zero in $[a, b]$. Moreover, if $\tilde{J}$ and $J$ are linearly independent, then $\tilde{J}$ has a zero in $(a, b)$.

Lemma 4.13. If the sectional curvature of $M$ is bounded below, then the sets $B_u^+$ and $B_s^-$ are closed.

The proof follows the same approach as the proof of Lemma 4.11 but we must avoid relying on the condition of the curvature being bounded above. Consequently, we only establish which claims of the proof of Lemma 4.11 remain valid in the two-dimensional case without an upper bound on the sectional curvature.

Proof of Lemma 4.13. Maintain the notation of Lemma 4.11 throughout. For each $n$, let $V_n(t)$ be a parallel vector field along $\gamma_{\theta_n}(t)$. If $J$ is a Jacobi field along $\gamma_{\theta_n}(t)$, then $J(t) = f(t)V_n(t)$, where $f(t)$ is a real function. Therefore, all Jacobi fields along $\gamma_{\theta_n}(t)$ can be regarded as real functions. Henceforth, let $J_{\xi_n}$ and $J_{\eta_n}$, defined similarly to Lemma 4.11, be considered as real functions. In this case, without loss of generality, we can assume that $J_{\eta_n} > 0$ in $[0, t_n)$. From Lemma 4.11 and Lemma 4.12, we have that $J_{\xi_n}(t)$ has a unique zero $r_n$ in $[0, t_n)$. Consequently, $W_n = J_{\xi_n}(t_n)J_{\eta_n}(t_n) > 0$ and $W_0 > 0$.

Claim 1: There is $u_n > t_n - 1$ such that $J_n(u_n) = 0$. 24
Proof of Claim 1. If $J_{\xi_n}(t + 1)$ has a zero $z_n$, for some $z_n - 1 > t_n - 1$, then from Lemma 4.12 $J_{\xi_n}(t)$ has a zero in $(t_n, z_n)$ and consequently $J_n$ has a zero in $(t_n - 1, z_n - 1)$. Therefore, we can assume that and $J_{\eta_n}(t + 1) < 0$ for $t > t_n - 1$. On the other hand, by the uniqueness of $r_n$ we know that $J_n(t_n - 1) = J_{\eta_n}(t_n) - J_{\xi_n}(t_n) = -J_{\xi_n}(t_n) > 0$.

Finally, we can assume by contradiction that $J_n(t) \neq 0$ for all $t > t_n - 1$. Then from the last inequality $J_n(t) > 0$ and then $J_{\xi_n}(t) < J_{\eta_n}(t) < 0$, $t > t_n$, which allows us to conclude that $|J_{\eta_n}(t)| < |J_{\xi_n}(t)|$, $t > t_n$. The last inequality provides a contradiction since $J_{\xi_n}$ is a stable Jacobi field and $J_{\eta_n}$ is unstable.

Using the Claim 1, we consider the positive number $\beta_n := \inf\{u > 0 : J_n(t_n - 1 + u) = 0\}$. This number $\beta_n$ has the same role as the $\beta_n$ defined in the proof of Lemma 4.11 but with the additional and important properties $J_n(t_n - 1 + \beta_n) = 0$.

To conclude the proof of this lemma, we need to prove that all ten claims of the proof of Lemma are valid without the condition of curvature bounded above. It is not difficult to check, such condition on the curvature is only used in Claim 4, Claim 6, and Claim 8. So, from now on we focus on proving such three Claims. Thus, note that in the two-dimensional case, the number $\delta_0$ of Claim 3 is equal to 1.

We also note that Claim 5 and Claim 6 of the proof of the last lemma depend on Claim 4, but here we joined both Claim 4 and Claim 6 in a single claim, and consequently Claim 5 is already valid.

Claim 2: In this case $\lim_{n \to \infty} \frac{\beta_n}{\epsilon_n^2} = \frac{1}{W_0}$.

Proof of Claim 2. First, we prove that the sequence $\left\{ \frac{\beta_n}{\epsilon_n^2} \right\}$ is bounded. For this sake, note that $J_n(t_n - 1 + \beta_n) = 0$, so the mean value theorem provides

$$-\epsilon_n \frac{\beta_n}{J_n(t_n - 1 + \beta_n) - J_n(t_n - 1)} \frac{\beta_n}{J_n(t_n - 1 + \beta_n) - J_n(t_n - 1)} = J_n'(\tilde{\beta}_n),$$

for some $\tilde{\beta}_n \in (t_n - 1, t_n - 1 + \beta_n)$. Therefore

$$\frac{\epsilon_n^2}{\beta_n} = |J_n'(\tilde{\beta}_n)|^2 \cdot \beta_n.$$  \hspace{1cm} (54)

If $\left\{ \frac{\beta_n}{\epsilon_n^2} \right\}$ is an unbounded sequence, then passing to a sub-sequence if necessary, we have that $\lim_{n \to \infty} \frac{\epsilon_n^2}{\beta_n} = 0$, and the last inequality gives us that $\lim_{n \to \infty} |J_n'(\tilde{\beta}_n)|^2 \cdot \beta_n = 0$. 

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Hence, following the same lines of (29), (30), and (31) we find a contradiction and the sequence \( \{ \beta_n \} \) should be bounded, and consequently, Claim 5 of the proof of last lemma holds.

Now, writing \( \tilde{\beta}_n = t_n - 1 + \bar{\beta}_n \), for some \( \bar{\beta}_n \in [0, \beta_n) \). Therefore from Claim 3 of the proof of Lemma 4.11 which is valid in this case, and using that in this case \( \delta_0 = 1 \), we have

\[
(W_n - |P_n(\tilde{\beta}_n)|) \leq |J'_{\eta_n}(t_n + \alpha)|^2 \leq (W_n + |P_n(\tilde{\beta}_n)|)^2 \leq |J'_{\eta_n}(t_n + \beta_n)|^2,
\]

where \( P_n(\tilde{\beta}_n) := J'_{\eta_n}(t_n + \beta_n) \cdot J_{\eta_n}(t_n + \bar{\beta}_n) \). From (54) and (55) we have

\[
(W_n - |P_n(\tilde{\beta}_n)|)^2 \epsilon_n^2 \leq \left( \frac{\beta_n^2}{\epsilon_n^2} \right)^2 \leq (W_n + |P_n(\tilde{\beta}_n)|)^2 \epsilon_n^2.
\]

Since \( W_n \) converges to \( W_0 > 0 \) and \( P_n(\tilde{\beta}_n) \) converges to 0, then from Claim 3 of the proof of Lemma 4.11 and (56), we conclude that

\[
\lim_{n \to \infty} \frac{\beta_n}{\epsilon_n^2} = \frac{1}{W_0},
\]

as we wanted.

Finally, we prove that Claim 8 (Claim 3 below) of the proof of Lemma 4.11 is valid and consequently, we have the proof of lemma follow the same lines as the proof of Lemma 4.11. Remember we always keep the notion of last Lemma.

**Claim 3:** For \( n \) large enough

\[
\mu_n \geq \min\{a_0 \epsilon_n, \sigma_n\}.
\]

**Proof of Claim 3.** From the definition of \( \mu_n \) \( J'_{\eta_n}(t_n - \mu_n) = 0 \). Assume by contradiction that \( \mu_n < \min\{a_0 \epsilon_n, \sigma_n\} \), then \( \mu_n = \chi_n \), and from Claim 7 (of the proof of Lemma 4.11) \( |J_{\eta_n}(t_n - \mu_n)|^2 \leq \Gamma_0 \). Therefore

\[
C^{-2} \chi^{-2}(t_n - \mu_n) ||\eta_n||^2 \leq |J_{\eta_n}(t_n - \mu_n)|^2 + |J'_{\eta_n}(t_n - \mu_n)| \leq \Gamma_0.
\]

Since \( ||\eta_n|| \) is bounded and away from zero and \( \mu_n < \min\{a_0 \epsilon_n, \sigma_n\} \) is small, then the last inequality provides a contradiction.

The remains of the proof of lemma follow the same lines as the proof of Lemma 4.11.

**5 Proof of the Main Results**

The main goal of this section is to prove Theorem 4.12 and Theorem 4.1. Both proofs consist of proving that \( B^{n+1} = \emptyset \), and then using the Lemma 4.12.

To avoid always mentioning the sectional curvature conditions, we used the following definition:

**Definition 5.1.** A complete manifold \( M \) with bounded curvature below is nice if: \( M \) has dimension two or \( M \) has dimension greater than two and its sectional curvature is bounded.

This definition allows us to condense the Lemma 4.11 and the Lemma 4.13 into the following lemma.
Lemma 5.1. If $M$ is a nice manifold, then the sets $B^+_u$ and $B^-_s$ are closed.

Now, with the help of Lemma 5.1 our task is reduced to proving the following Theorem.

Theorem 5.1. Let $M$ be a nice manifold with Anosov geodesic flow. Then, the sets

$$\Lambda^{s,u} = \{ \theta \in SM : \text{for all } t \in \mathbb{R} \phi^t(\theta) \not\in B^{s,u} \}.$$  

are closed, open, and nonempty subsets of $SM$.

Since $SM$ is a connected manifold, then any subset open, closed, and nonempty should be $SM$. Thus, as a corollary of Theorem 5.1 we have:

Corollary 5.1. If $M$ is nice, then $B^{s,u} = \emptyset$.

From the last Corollary 5.1 we have that $E^{s,u}(\theta) \cap V(\theta) = \emptyset$, for all $\theta \in SM$.

Thus, since $E^{s,u}$ are Lagrangian, then by Lemma 3.2 we can conclude that $M$ has no conjugate points, and a fortiori the proof of Theorem 1.2 and Theorem 1.1.

Proof of Theorem 5.1. From Lemma 4.5 (Transfer Property) holds that $\Lambda^s = \Lambda^u$. Thus, it is sufficient to prove that $\Lambda^u$ is an open, closed, and nonempty set.

Nonempty: It is a consequence of Lemma 4.8.

Closedness: It is an immediate consequence of Lemma 3.4 (a similar argument was used by Mañé in [9]).

Openness: We will prove that $SM \setminus \Lambda^u$ is closed. In fact, assume that $\theta_n \to \theta$ with $\theta_n \in SM \setminus \Lambda^u$, then there is $t_n \in \mathbb{R}$ such that $\phi^{t_n}(\theta_n) \in B^u$. We have two cases to study.

Case 1: For infinitely many indexes $n$, $t_n > 0$. In this case, for infinitely many indexes $n$, $\theta_n \in B^u_+$ and from Lemma 5.1 we have that $\theta \in B^u_+$ and consequently $\theta \in SM \setminus \Lambda^u$.

Case 2: For infinitely many indexes $n$, $t_n < 0$. Then by Lemma 4.5 we have that there is $\tilde{t}_n$ with $|t_n - \tilde{t}_n| \leq 1 + \sigma$ such that $\phi^{\tilde{t}_n}(\theta_n) \in B^s$.

- If for infinitely many indexes $\tilde{t}_n < 0$, then $\theta_n \in B^s_-$ and from Lemma 5.1 we have that $\theta \in B^s_-$, and consequently $\theta \in SM \setminus \Lambda^s = SM \setminus \Lambda^u$.
- If for infinitely many indexes $\tilde{t}_n > 0$, then $0 < \tilde{t}_n < 1 + \sigma + t_n < 1 + \sigma$. Thus, passing to a subsequence if necessary, from Lemma 4.1 we have that $\phi^{\tilde{t}_n}(\theta_n) \to \phi^{11}(\theta) \in B^s$, which implies $\theta \in SM \setminus \Lambda^s = SM \setminus \Lambda^u$.

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