GARSIDE CATEGORIES, PERIODIC LOOPS AND CYCLIC SETS

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Abstract. Garside groupoids, as recently introduced by Krammer, generalise Garside groups. A weak Garside group is a group that is equivalent as a category to a Garside groupoid. We show that any periodic loop in a Garside groupoid $G$ may be viewed as a Garside element for a certain Garside structure on another Garside groupoid $G_m$, which is equivalent as a category to $G$. As a consequence, the centraliser of a periodic element in a weak Garside group is a weak Garside group. Our main tool is the notion of divided Garside categories, an analog for Garside categories of Bökstedt-Hsiang-Madsen’s subdivisions of Connes’ cyclic category. This tool is used in our separate proof of the $K(\pi, 1)$ property for complex reflection arrangements.

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Introduction

A classical theorem says that a periodic homeomorphism of the disk is conjugate to a rotation. It was simultaneously announced by Kerékjártó and Brouwer in 1919 and the first undisputed proof was published by Eilenberg in 1934 (see [14, 22, 26]). Our results include some analogs of Kerékjártó-Brouwer-Eilenberg’s theorem, in the context of braid groups of complex reflection groups and, more generally, cyclic Garside groupoids.

First draft, some details and proofs are missing.
Let $X$ be a $S^1$-space, i.e., a topological space together with a continuous action of $S^1 := \{ z \in \mathbb{C} | |z| = 1 \}$. The fundamental group $\pi_1(X, x_0)$ admits a special element, called full-twist at $x_0$ and denoted by $\tau_{x_0}$ (or simply $\tau$), represented by the path $[0, 1] \to X, t \mapsto e^{2i\pi t}x_0$. It is clearly central. The map $x_0 \mapsto \tau_{x_0}$ lies in the “centre” of the fundamental groupoid, in the sense that it is a natural automorphism of the identity functor.

More generally, an element of $\pi_1(X, x_0)$ is a rotation of angle $\theta$ if it is represented by $t \mapsto e^{i\theta}x_0$. This of course requires $x_0$ to be $e^{i\theta}$-invariant. It is enough to restrict one’s attention to rational rotations, those whose angles are rational multiples of $\pi$, because for other values of $\theta$ the basepoint $x_0$ must be $S^1$-fixed, which implies that irrational rotations are trivial. The full-twist is a rotation of angle $2\pi$. Rotations may be composed and their angles add up.

Let $p \in \mathbb{Z}, q \in \mathbb{Z}_{\geq 1}$. An element $\gamma \in \pi_1(X, x_0)$ is $\frac{p}{q}$-periodic (or simply periodic) if $\gamma^q = \tau^p$ (note that this may a priori depend on the actual $p, q$ and not just on the rational number they represent). A rotation of angle $2\pi \frac{p}{q}$ is $\frac{p}{q}$-periodic.

Given any $S^1$-space $X$, one may be interested in the following questions, whose answers obviously depend on $X$:

**Question 1.** Is any periodic element conjugate to a rotation?

**Question 2.** Are two rotations with identical angles conjugate?

In Question 2, “conjugate” should be understood in the groupoid sense: $\rho \in \pi_1(X, x)$ and $\rho' \in \pi_1(X, x')$ are conjugate if there exists $\gamma \in \text{Hom}_{\pi_1(X)}(x, x')$ such that $\rho \gamma = \gamma \rho'$.

There is an easy sufficient condition for Question 2 to have a positive answer: assume that $p \wedge q = 1$; let $\mu_q \leq S^1$ be the subgroup of $q$-th roots of unity and let $X^{\mu_q} \subseteq X$ be the corresponding set of fixed points; if $\pi_1(X^{\mu_q})$ is a connected category (or, equivalently, is $X^{\mu_q}$ is path-connected), then any two rotations $\rho, \rho'$ of identical angles $2\pi \frac{p}{q}$ are conjugate. Indeed, let $x$ be the basepoint of $\rho$, $x'$ be the basepoint of $\rho'$. Because we have assume $p \wedge q = 1$, $x$ and $x'$ are in $X^{\mu_q}$. For any $\gamma$ in the direct image of $\text{Hom}_{\pi_1(X^{\mu_q})}(x, x')$ under the natural functor $\pi_1(X^{\mu_q}) \to \pi_1(X)$, there is an obvious homotopy showing that $\rho \gamma = \gamma \rho'$.

Actually, the restriction to $X^{\mu_q}$ of the $S^1$-structure on $X$ admits a “$q$-th root”

$$[0, 1] \times X^{\mu_q} \longrightarrow X^{\mu_q}$$

$$(t, x) \longmapsto e^{\frac{2i\pi t}{q}}x$$

whose full-twists are preimages via $\pi_1(X^{\mu_q}) \to \pi_1(X)$ of rotations of angle $\frac{2\pi}{q}$. As a consequence, rotations of angle $2\pi \frac{p}{q}$ lie in the centre of $\text{Im}(\pi_1(X^{\mu_q}) \to \pi_1(X))$. This leads to the following questions:

**Question 3** (group version). Let $\rho$ be a rotation of angle $2\pi \frac{p}{q}$, with $p \wedge q = 1$. Let $x_0$ be the basepoint of $\rho$. Does the inclusion $X^{\mu_q} \hookrightarrow X$ induce an isomorphism

$$\pi_1(X^{\mu_q}, x_0) \simeq C_{\pi_1(X, x_0)}(\rho)$$

**Question 3’** (groupoid version). Let $q$ be a positive integer, let $p \in \mathbb{Z}$ such that $p \wedge q = 1$. Is the natural functor $\pi_1(X^{\mu_q}) \to \pi_1(X)$ faithful? Does it identify $\pi_1(X^{\mu_q})$ with the largest subcategory of $\pi_1(X)$ with object set $X^{\mu_q}$ and whose “centre” contains the natural family of rotations of angle $2\pi \frac{p}{q}$?
Example 0.1. Let $X_n$ be the space of unordered configurations of $n$ distinct points in $\mathbb{C}$, whose fundamental group is the classical braid group $B_n$ on $n$ strings. It has a natural structure of $S^1$-space (via the multiplicative action of $S^1$ on $\mathbb{C}$). The theorem of Kerékjártó-Brouwer-Eilenberg implies that Question 1 has a positive answer (to see this, one uses the standard interpretation of the mapping class group of the punctured disk as the quotient of $B_n$ by the full-twist). Because $X_n^{\mu_q}$ is either empty or connected, Question 2 has a positive answer. Questions 3 and 3' also have positive answers, as it was shown in my earlier work with Digne and Michel, [5].

Example 0.2. Let $V$ be a finite dimensional complex vector space. Let $W \subseteq \text{GL}(V)$ be a finite complex reflection group. Let $V^{\text{reg}}$ be the complement of the reflecting hyperplanes. Scalar multiplication on $V$ commutes with $W$-action and preserves $V^{\text{reg}}$. It induces a structure of $S^1$-space on the quotient $W\backslash V^{\text{reg}}$, the regular orbit space of $W$. When $W$ is the symmetric group in its permutation representation, the regular orbit space is $X_n$. The fundamental group of $W\backslash V^{\text{reg}}$ is the (generalised) braid group of $W$. One application of the tools presented here is the proof in [4] that, when $W$ is well-generated, all above questions have positive answers when applied to $W\backslash V^{\text{reg}}$. This theorem should be understood as a braid analog of Springer’s theory of regular elements in complex reflection groups, [31]. It was motivated by a series of questions and conjectures by Broué and Michel, [13, 12], that naturally arose as they studied Deligne-Lusztig varieties with the insight that their cohomology should provide tilting modules verifying Broué’s abelian defect conjecture for finite groups of Lie type. In the particular case when $W$ is the symmetric group, we recover Example 0.1.

Our main result may be phrased in several ways, one of which is the following:

**Theorem 0.3.** If $X$ is the geometric realisation of the Garside nerve of a cyclic Garside groupoid, then Question 1 has a positive answer.

Moreover, in that setting, there are practical ways to tackle Questions 2, 3 and 3'.

Before explaining what Garside nerves and cyclic Garside groupoids are, let us begin by discussing a particular case of Question 3 that admits an elementary solution relying on classical tools.

If $W$ is a complexified real reflection group, it was proved by Brieskorn, [10], that the associated braid group $B(W)$ is isomorphic to the Artin group

$$A(W, S) := \left\langle S \mid \begin{array}{c}
st_s\ldots = t_s\ldots \\
m_{s,t} \text{ terms} & m_{s,t} \text{ terms}
\end{array} \right\rangle,$$

where $S$ is the set of reflections with respect to the walls of a chosen chamber of the real arrangement and $(m_{s,t})$ is the Coxeter matrix. Let $\Delta$ be the image by Tits’ section $W \to A(W, S)$ of the longest element $w_0$. One may choose the isomorphism $B(W) \simeq A(W, S)$ such that $\Delta$ is a rotation of angle $\pi$, corresponding to the case $p = 1, q = 2$ in Question 3. The word problem for $A(W, S)$ was solved independently by Deligne and Brieskorn-Saito, [19, 11]. In modern terms, their solution relies on the fact that $A(W, S)$ is a Garside group with Garside element $\Delta$ (see Section 2 for more details about Garside groups). Because the conjugacy action of $\Delta$ on $A(W, S)$ can be understood through the Garside normal form, the centraliser is easy to compute. Indeed, the conjugacy action of
$w_0$ on $W$ is a diagram automorphism (i.e., it is induced by a permutation of $S$) and the centraliser $W' := C_W(w_0)$ is a Coxeter group with Coxeter generating set $S'$ indexed by $w_0$-conjugacy orbits on $S$. At the level of Artin groups, one shows (see for example [28]) that

$$A(W', S') \simeq C_{A(W, S)}(\Delta),$$

which is an algebraic reformulation of the case $p = 1, q = 2$ of Question 3 (applied to $W \setminus V_{\text{reg}}$, as in Example 0.2).

**Example 0.4.** When $A(W, S)$ is a Artin group of type $E_6$, the $\Delta$-conjugacy action is the non-trivial diagram automorphism and the centraliser is an Artin group of type $F_4$.

The main strategy throughout this article is to construct Garside structures with sufficient symmetries, so that centralisers of periodic elements can be computed as easily as in Example 0.4.

Birman-Ko-Lee showed that the classical braid group $B_n$ admits, in addition to the type $A_{n-1}$ Artin group structure, another Garside group structure where the Garside element is a rotation $\delta$ of angle $2\pi \frac{1}{n}$. In [5], we used this Garside structure to compute the centralisers of powers of $\delta$, which solves Question 3 for $B_n$ and $q|n$. Thanks to some rather miraculous diagram chasing, we were also able to obtain the remaining case $q|n-1$.

Whenever $G$ is a group and $\Delta \in G$ is the Garside element of a certain Garside structure, the centraliser $C_G(\Delta)$ is again a Garside group, and is easy to compute. Note that the notion of periodic element may be extended to this setting: we say that $\rho \in G$ is $\frac{p}{q}$-periodic if

$$\rho^q = \Delta^p.$$ 

This of course is relative to the choice of a particular Garside structure. However, for $B_n$, the Artin Garside element $\Delta$ and the Birman-Ko-Lee Garside element $\delta$ are commensurable

$$\Delta^2 = \delta^n.$$ 

In particular, whether a given element $\rho$ is periodic or not does not depend on the choice between the two standard Garside structures (although the actual $p$ and $q$ may vary).

As the above particular case illustrates, it is very easy to compute centraliser of Garside elements in Garside groups. In particular, when trying to answer Questions 3 and 3’ in a space whose fundamental group is a Garside group, it is very tempting to expect to build a proof on a positive answer to:

**Question 4.** Let $G$ be a Garside group with Garside element $\Delta$. Let $\rho \in G$ be a periodic element with respect to $\Delta$. Does $G$ admit a Garside structure with Garside element $\rho$?

Note that a positive answer to Question 4 would imply that the following question also admits a positive answer:
Question 5. Let $G$ be a Garside group with Garside element $\Delta$. Let $\rho \in G$ be a periodic element with respect to $\Delta$. Is the centraliser $C_G(\rho)$ a Garside group?

Birman-Ko-Lee’s discovery left many people puzzled: why should there be two natural Garside structures on $B_n$, how many more Garside structures on $B_n$ remain to be discovered? Question 4 asks for a natural explanation in terms of periodic elements. Although it was never written down as an official conjecture, there was some initial hope that the answer might be positive. When $(W,S)$ is a finite Coxeter system, $A(W,S)$ admits a dual Garside structure,\cite{2}, generalising Birman-Ko-Lee’s construction and giving partial answers when $q$ divides $h$, the Coxeter number of $W$.

Our answer to Questions 4 and 5 is “Almost!” Although we do not have definite counterexamples, we do not expect them to have positive answers, because we do not think that they are phrased in a natural language. As the current article will illustrate, the notion of Garside groups is artificially restrictive and unstable under several basic operations. One should rather work with Garside groupoids and weak Garside groups. This generalisation was recently introduced by Krammer and a related notion was independently studied by Digne-Michel,\cite{27,21}. Just like Garside groups are groups of fractions of Garside monoids, Garside groupoids are obtained by localising Garside categories. A monoid is a category with a single object, and a Garside monoid is a Garside category with a single object. Rewriting the whole theory of Garside groups into categorical language is a surprisingly pleasant translation exercise: everything works fine at essentially no cost. The additional syntactic constraints are actually helpful, as they provide a quick test for general statements about Garside groups: if a statement cannot be “categorified”, then it is probably false.

A categorical Garside structure is a triple $(\mathcal{C}, \phi, \Delta)$ where $\mathcal{C}$ is a small category, $\phi$ is an automorphism of $\mathcal{C}$ (it replaces the conjugacy action of the Garside element, and should be thought of as a diagram automorphism) and $\Delta$ is a natural transformation from the identity functor to $\phi$ (the family of Garside elements), subject to certain axioms. A Garside category is a category $\mathcal{C}$ which is part of such a triple. It is cancellative and embeds in its Garside groupoid, the category $\mathcal{G}$ obtained by adding formal inverses to all morphisms. A weak Garside group is the automorphism group $\mathcal{G}_x$ of some object $x$ of a Garside groupoid. When the category has a single object, one recovers the usual notion of Garside group.

A Garside groupoid is cyclic if the automorphism $\phi$ has finite order. Our “almost answers” to Questions 4 and 5 are as follows (for a more precise phrasing of Theorem 0.5, see Theorems 9.5 and 10.1):

**Theorem 0.5.** Let $\mathcal{G}$ be a cyclic Garside groupoid, let $\gamma$ be a periodic loop in $\mathcal{G}$. Then there exists a Garside groupoid $\mathcal{G}_q$, together with an equivalence of categories

$$\Theta_q : \mathcal{G} \to \mathcal{G}_q$$

such that $\Theta_q(\gamma)$ is conjugate to a Garside element.

**Corollary 0.6.** The centraliser of a periodic element in a weak Garside group is a weak Garside group.
Note that, even when $G$ is a group, the groupoid $G_q$ may have several objects. For the issues discussed here, it is unavoidable to think in terms of groupoids.

These results have a geometric interpretation. One associates to any Garside groupoid a simplicial classifying space $N_\Delta G$, which we call the Garside nerve. When $\phi$ is trivial, one shows that the Garside nerve is a cyclic set, in the sense of Connes. More generally, when the Garside groupoid is cyclic, the Garside nerve is very close to being a cyclic set: it is a $\Lambda^{op}_k$-object in the category of sets, in the sense of Bökstedt-Hsiang-Madsen. As a consequence, the realisation $|N_\Delta G|$ comes equipped with a natural $S^1$-structure. Theorem 0.3 should be understood in that setting.

Our main construction, in Section 9, is a sort of barycentric subdivision operation for Garside categories. For any Garside category $C$ with groupoid $G$, we construct $m$-divided Garside categories $C_m$ and groupoids $G_m$, one for each integer $m \geq 1$. We have $C_1 = C$, but higher values of $m$ give categories $C_m$ with more objects than $C$. In particular, $m$-divided categories of Garside monoids are usually not monoids. The Garside nerve of $G_m$ is obtained from that of $G$ by applying Bökstedt-Hsiang-Madsen $m$-subdivision functor, $\mathcal{D}_m$, and their geometric realisations are homeomorphic. This implies that there is an equivalence of categories between $G_m$ and $G$.

In other words, when working with weak Garside groups, one may replace $G$ by $G_m$. What is gained in the procedure is that the automorphism group of $G_m$ is larger than that of $G$: the diagram automorphism of $G_m$ can be thought of as an $m$-th root of that of $G$. This, together with earlier ideas of Bestvina, is the main ingredient in the proof of Theorem 0.5.

Applications. As mentioned above (Example 0.2), we use our techniques in a separate article, [4], to obtain a complete description of periodic elements, their conjugacy classes and their centralisers, in braid groups associated with well-generated complex reflection groups. Even in the case of spherical type Artin groups, these results are new. However, the main application so far is as an ingredient in the proof of $K(\pi,1)$ conjecture for complex reflection arrangements, which is the main result in [4]. It is also likely that our results have algorithmic applications to the conjugacy problem in Garside groups, in the spirit of the work of Birman, Gebhardt and González-Meneses, [8].

Structure of this article. The first four sections provide basic terminology about Garside categories, Garside groupoids and Garside germs. They are included for the convenience of the reader, to serve as a travel guide to a newborn theory, rather than as an encyclopaedia. Proofs are omitted and some axioms are stricter than actually needed. A unified toolbox for Garside categories is currently being developed, in collaboration with François Digne, Daan Krammer and Jean Michel, and should eventually provide complete reference for the missing details. The next four sections cover more advanced material on Garside groupoids: conjugacy problem (Section 5), Galois theory (Section 6), classifying spaces (Section 7) and (a very brief account of a corollary of) Bestvina’s approach to non-positively curved aspects (Section 8). Here again, most proofs are omitted, because writing the details would probably take over 30 additional pages and merely consist of straightforward rephrasings of standard pieces from the theory of Garside groups, copy-pasted from the works of Garside, ElRifai-Morton, Picantin, Franco and González-Meneses, Bestvina, Charney-Meier-Whittlesey and others. The real core of this article consists of Sections 9, 10, 11 – this is where genuinely new material is introduced, that
does not resemble anything that can be done without the categorical viewpoint. The last two sections are devoted to easy illustrative examples.

1. Graphs, germs and free categories

Let \( S \) be an oriented graph. For any two vertices \( x \) and \( y \), denote by \( S_{x \to} \) the set of edges with source \( x \), by \( S_{\to y} \) the set of edges of target \( y \), and set \( S_{x \to y} := S_{x \to} \cap S_{\to y} \).

We may think of a (small) category \( \mathcal{C} \) as a graph \( \mathcal{C} \), whose vertices are \( \mathcal{C} \)-objects and edges are \( \mathcal{C} \)-morphisms, together with a composition law. In that respect, our notations

\[ C_{x \to}, \ C_{\to y}, \ C_{x \to y} \]

are synonyms for

\[ \text{Hom}_{\mathcal{C}}(x, -), \ \text{Hom}_{\mathcal{C}}(-, y), \ \text{Hom}_{\mathcal{C}}(x, y). \]

Both notation systems will be used in the sequel.

The notion of germ generalises the notion of category by allowing the composition law to be only partially defined:

**Definition 1.1.** A germ of (small) category (or simply a germ) is a pair \( \mathcal{S} = (S, m) \) where \( S \) is an oriented graph together with partial “composition” maps

\[ m_{x,y,z} : S_{x \to y} \times S_{y \to z} \to S_{x \to z}, \]

one for each triple \((x, y, z)\) of vertices, satisfying the following axioms:

- **(assoc)** let \( x, y, z, t \) be vertices; the two natural partial maps

\[ m_{x,y,t} \circ (1_{S_{x \to y}} \times m_{y,z,t}) : S_{x \to y} \times S_{y \to z} \times S_{z \to t} \to S_{x \to z} \]

and

\[ m_{x,z,t} \circ (m_{x,y,z} \times 1_{S_{z \to t}}) : S_{x \to y} \times S_{y \to z} \times S_{z \to t} \to S_{x \to z} \]

coincide (in particular, we ask for these maps to have the same domain of definition).

- **(unit)** for all vertex \( x \), there exists an element \( 1_x \in S_{x \to x} \) such that, for all vertices \( x, y \), the partial maps

\[ m_{x,x,y}(1_x, \cdot) : S_{x \to y} \to S_{x \to y} \]

and

\[ m_{x,y,y}(\cdot, 1_y) : S_{x \to y} \to S_{x \to y} \]

coincide with the identity map \( 1_{S_{x \to y}} \) (in particular, we ask for these maps to be everywhere defined).

It is an easy check that **(assoc)** implies that if \( s_1, \ldots, s_n \in S \), the choice of a complete bracketing of \( s_1 \ldots s_n \) has no impact in whether the product is defined or not, nor on the occasional value of this product. In **(unit)**, the element \( 1_x \) is unique.

**Example 1.2.** A germ where \( m_{x,y,z} \) is everywhere defined is nothing but a category.

Let \( \mathcal{S} := (S, m) \) be a germ. Let \( S^* \) be the category of walks on \( S \): its objects are the vertices of \( S \), and a morphism from \( x \) to \( y \) is a finite sequence (possibly empty) of \( S \)-edges \((s_1, \ldots, s_k)\) such that \( s_1 \in S_{x \to}, \ s_k \in S_{\to y}, \) and the source of \( s_{i+1} \) is the target of \( s_i \).
Paths are composed by concatenation. Say that two paths \((s_1, \ldots, s_k)\) and \((t_1, \ldots, t_l)\) are *elementarily* \(S\)-equivalent, which is denoted by
\[
(s_1, \ldots, s_k) \sim_1 (t_1, \ldots, t_l),
\]
if one (or both) of the following conditions is satisfied

(I) There exists \(i\) such that \((s_i, s_{i+1})\) is in the domain of definition of \(m\) and
\[
(t_1, \ldots, t_l) = (s_1, \ldots, m(s_i, s_{i+1}), \ldots, s_k).
\]

(II) There exists \(i\) such that \(s_i\) is the unit of some object \(x_i\) and
\[
(t_1, \ldots, t_l) = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k).
\]

**Remark 1.3.**

(i) In both cases, we have \(l = k - 1\). The second case is almost a particular case of the first one, except that it allows \(() \sim_1 (1_X)\) which is not covered by the first case.

(ii) In point (i) above, the notation \(()\) is slightly ambiguous, since it refers to the trivial walk *starting at* \(x\). When needed, we will use the notation \(x()\) to lift ambiguity.

(iii) Section 0 of my earlier paper \[2\] gives a monoid version of \(\sim_1\) but contains a mistake, pointed out by Deligne: case (II) was forgotten, which had the unpleasant effect of adding artificial units.

Let \(\sim\) be the reflexive symmetric transitive closure of \(\sim_1\). It is clear that the concatenation of paths is compatible with \(\sim\). We obtain a category \(S^*/\sim\) whose morphisms are \(\sim\)-equivalence classes of paths.

**Definition 1.4.** The free category on a germ \(S\), denoted by \(\mathcal{C}(S)\), is the quotient category \(S^*/\sim\).

The terminology is justified by the fact that \(S \mapsto \mathcal{C}(S)\) is a left adjoint of the forgetful functor from the category of small categories to the category of germs.

**Notation 1.5.** When there is no ambiguity, we simply denote a germ \(S = (S, m)\) by its underlying graph \(S\). Furthermore, we use the notation \(s \in S\) to mean that \(s\) is an oriented edge of \(S\) or, with equivalent categorical language, that \(s\) is a morphism of \(S\). We say that \(s \in S\) is an object if it is the identity morphism \(1_x\) associated with a vertex \(x\) (as with categories, one may think of objects as particular morphisms). If \(s, t, u \in S\), the notation \(st = u\) means that \((s, t)\) is in the definition domain of \(m\) and \(m(s, t) = u\).

The partial product endows \(S_{x\to y}\) with a poset structure:
\[
s' \leq s \iff \exists s'' \in S, s's'' = s.
\]

Similarly, it endows \(S_{\to y}\) with a poset structure:
\[
s \geq s'' \iff \exists s', s's'' = s.
\]
2. Garside categories

A monoid is a category with a single object. While Garside monoids have been studied for quite a while, Garside categories are a recent invention. Krammer gives a very neat definition in [27]. As he points out, many theorems about Garside monoids, as well as their proofs, may be rewritten with no effort in the context of Garside categories. In fact, the categorical viewpoint is arguably simpler and more natural, even when dealing with Garside monoids. When Dehornoy was illustrating his articles and lectures with pictures where elements of Garside monoids were represented by arrows, he was adopting a categorical viewpoint without realising it\(^1\). The categorical viewpoint is also implicit in Deligne’s article [19] (see Example 3.4 below).

As with Garside monoids, one may define a Garside category by generators and relations (as Krammer does), or by showing that it satisfies certain axioms. Also, it is tempting to relax some of the axioms, to allow for “quasi”-Garside categories (where “quasi” could mean several different things) retaining some of the main properties. It is not yet clear which final optimal axiom set will be retained, and what follows is a simple variant.

The starting kit is a basic triple

\[ (\mathcal{C}, \mathcal{C} \xrightarrow{\phi} \mathcal{C}, 1_{\mathcal{C}} \xrightarrow{\Delta} \phi) \]

where \(\mathcal{C}\) is a small category, \(\phi\) is an automorphism\(^2\) of \(\mathcal{C}\) and \(\Delta\) is a natural transformation from the identity functor to \(\phi\).

Example 2.1. Let \(M\) a Garside monoid with Garside element \(\delta\), view it as a category with one point \(*\) and arrows labelled by elements of \(M\). For any arrow \(m \in M\), take \(\phi(m) := \delta^{-1}m\delta\), and take \(\Delta\) to be the right multiplication by \(\delta\). The naturality of \(\Delta\) is expressed by \(\forall m \in M, \delta \phi(m) = m\delta\).

To be consistent with Dehornoy’s pictures and other classical material, we use some conventions. Arrows in \(\mathcal{C}\) compose like paths in algebraic topology: \(x \xrightarrow{f} y \xrightarrow{g} z\) is composed into \(x \xrightarrow{fg} g\). The functor \(\phi\) is denoted as if it was a “right conjugacy action”: the image of \(x \xrightarrow{f} y\) by \(\phi\) is \(x \xrightarrow{\phi f} y\phi\). When \(x \in \mathcal{C}\), \(\Delta\) gives a morphism \(x \xrightarrow{\Delta(x)} x\phi\) which we like to simply denote by \(\Delta x\) or even \(\Delta\), calling it “the \(\Delta\) of \(x\)”. Any element \(\Delta x\) should be called a Garside element.

Krammer asks for \(\mathcal{C}\) to have a finite number of objects and a finite number of atoms. If one wants to study infinite type Artin groups, infinite number of atoms should be allowed. And since our main construction turns atoms into objects of newer categories, it is natural to allow infinite number of objects. Krammer’s axiom (GA2) exactly expresses that \(\Delta\) is a natural transformation.

Definition 2.2. Let \((\mathcal{C}, \phi, \Delta)\) be a basic triple. A morphism \(x \xrightarrow{f} y\) is simple if there exists a morphism \(y \xrightarrow{f} x\phi\) such that \(ff = \Delta x\). We denote by \(S(\mathcal{C}, \phi, \Delta)\) (or simply \(S\)) the graph of simples, the subgraph of \(\mathcal{C}\) whose edges are simple morphisms.

A category is connected if the underlying graph in connected (in the unoriented sense).

\(^1\)or without saying it...
\(^2\)It is possible to do part of the theory by simply assuming that \(\phi\) is an endofunctor.
A category is *atomic* if for any morphism $f$, there is a bound on the length $n$ of a factorisation $f = f_1 \ldots f_n$, where the $f_i$ are non-identity morphisms. This implies that there are no non-trivial invertible morphisms; in particular, whenever a limit or colimit exists, it is *unique* (really unique, not just up to automorphism). A nontrivial morphism $f$ which cannot be factorised into two nontrivial factors is an *atom*. A category is *(weighted) homogeneous* if there exists a length function $l$ from the set of $\mathcal{C}$-morphisms to $\mathbb{Z}_{\geq 0}$ such that $l(fg) = l(f) + l(g)$ and $(l(f) = 0) \iff (f$ is a unit). It is clear that homogeneous categories are atomic.

**Definition 2.3.** If $\mathcal{C}$ is a category, we denote by $A(\mathcal{C})$ (or simply $A$) the *atom graph* of $\mathcal{C}$, the subgraph of $\mathcal{C}$ whose edges are atoms.

In an atomic category, any morphism is a product of atoms; in other words, $A(\mathcal{C})$ generates $\mathcal{C}$.

A category is *cancellative* if, whenever a relation $afb = agb$ holds between composed morphisms, it implies $f = g$. Very often, this implies that there is at most one way to add a certain arrow to a commutative diagram. In the context of Definition 2.2, this implies that $f$ is unique.

**Definition 2.4.** A *(homogeneous) categorical Garside structure* is a triple $(\mathcal{C}, \phi, \Delta)$ such that:

- $\mathcal{C}$ is a category, $\phi$ an automorphism of $\mathcal{C}$ and $\Delta$ a natural transformation from the identity functor to $\phi$,
- $\mathcal{C}$ is homogeneous \(^3\) and cancellative,
- atoms are simple: $A(\mathcal{C}) \subseteq S(\mathcal{C}, \phi, \Delta)$,
- for all object $x$, $(\mathcal{C}_x, \leq)$ and $(\mathcal{C}_x, \geq)$ are lattices.

It has *finite type* if $S(\mathcal{C}, \phi, \Delta)$ is finite. Let $k \in \mathbb{Z}_{\geq 1}$. We say that $(\mathcal{C}, \phi, \Delta)$ is *$k$-cyclic* if $\phi^k = 1$. It is *cyclic* if it is $k$-cyclic for some $k \geq 1$.

Note that we do not require Garside categories to be connected nor to be non-empty.

**Definition 2.5.** A *Garside category* is a category $\mathcal{C}$ that may be equipped with $\phi$ and $\Delta$ to obtain a categorical Garside structure.

Krammer’s definition of Garside categories corresponds to our *finite type* Garside categories. *Infinite type* Garside categories fail to satisfy property (P7) from Krammer’s Theorem 36 (automaticity), but satisfy the remaining properties.

**Example 2.6.** The basic triple associated to a Garside monoid is a finite type Garside triple. When considering “quasi-Garside” monoids (with infinite number of simples, such as in [20] or [3]), one obtains an infinite type Garside triple. This also provides us with examples of non-cyclic Garside categories.

**Definition 2.7.** A Garside groupoid is a groupoid obtained by adding formal inverses to all morphisms in $\mathcal{C}$, where $\mathcal{C}$ is a Garside category.

---

\(^3\)This condition is too restrictive, but sufficient for most applications, in particular when considering Garside categories naturally associated with braid groups, like in [4]. As in [2] and in [27], one can replace homogeneity by atomicity. This may yet not be the ultimate general condition. To avoid technicalities, we stay with this overly conservative axiom.
Definition 2.8. Let \((\mathcal{C}, \phi, \Delta)\) be a Garside category, with Garside groupoid \(\mathcal{G}\). The structure group of \(\mathcal{G}\) at an object \(x\) is
\[
\mathcal{G}_x := \text{End}_\mathcal{G}(x, x).
\]

A weak Garside group is a group that is isomorphic to a structure group of a Garside groupoid.

The isomorphism type of \(\mathcal{G}_x\) only depends on the connected component of \(\mathcal{C}\) containing \(x\).

Remark 2.9. If \(\mathcal{C}\) has a single object (hence is a Garside monoid) then \(\mathcal{G}_x\) is a Garside group, in the previously traditional sense. Thus Garside groups are weak Garside groups. Conversely, there is no good reason to expect all weak Garside groups to be Garside groups. We give at the end of this paper an example to illustrate this. Because we think that weak Garside groups are much more natural to consider than traditional Garside groups, we hope that the terminology will evolve and that people will eventually call Garside groups what we call weak Garside groups. Until then, it is probably safer to keep the “weak.”

Definition 2.10. Let \((\mathcal{C}, \phi, \Delta)\) be a categorical Garside structure.

The Garside dimension of \((\mathcal{C}, \phi, \Delta)\) is the element of \(\mathbb{Z}_{\geq 0} \cup \{+\infty\}\) defined by
\[
\dim_\Delta(\mathcal{C}, \phi, \Delta) := \sup\{n \in \mathbb{Z}_{\geq 0} \mid \exists\ \text{simples } s_0, \ldots, s_n \text{ such that } s_0 < s_2 < \cdots < s_n\}.
\]

Let \((\mathcal{C}, \phi, \Delta)\) be a categorical Garside structure with associated groupoid \(\mathcal{G}\). Let \(f \in \mathcal{G}\).

The classical arguments from the theory of normal forms in Garside groups are applicable here, and one sees that there is a unique way to write \(f\) as a product
\[
f = s_1 s_2 \ldots s_l \Delta^k,
\]
where \(s_1, \ldots, s_l\) are simples with sources \(x_1, \ldots, x_l, k \in \mathbb{Z}\), and we have
for all \(i, s_i = s_i s_{i+1} \wedge \Delta x_i\)
and
\[
s_1 < \Delta x_1.
\]

In the above formulae, the symbol \(\wedge\) refers to the left gcd, i.e., the inf with respect to \(\leq\). Note that we choose to put \(\Delta^k\) at the end rather than at the beginning, to stay in line with conventions used by Bestvina and others.

Definition 2.11. We say that \(s_1 s_2 \ldots s_l \Delta^k\) is the (left greedy) normal form of \(f\). The integers \(k\) and \(k + l\) are respectively the infimum and supremum of \(f\), denote by \([f]\) and \([f]\). The integer \(l\) is the canonical length of \(f\).

When the Garside structure has finite type, this normal form is part of an automatic structure for \(\mathcal{G}\).

Remark 2.12. Let \(k\) be a positive integer. Any \((\mathcal{C}, \phi, \Delta)\) categorical Garside structure on \(\mathcal{C}\) admits a “\(k\)-th power” \((\mathcal{C}, \phi^k, \Delta^k)\), whose Garside elements are products of \(k\) consecutive \(\Delta\)’s. Our main construction below provides (a sort of) inverse to this power operation on Garside structures.
3. Garside germs

This section explains how to reconstruct a Garside category from a Garside germ, which should be thought of as a tentative “graph of simples”.

While Dehornoy-Krammer’s syntactic approach is very effective at handling abstract Garside categories whose graph of simples is hard to understand, many situations provide us with a natural candidate for the graph of simples, while there may be no natural complemented presentation and the cube axiom may be unpractical to check.

For Garside monoids, a more intrinsic strategy is explained in my joint work with François Digne and Jean Michel, [5]. The material presented here generalises [5] and is in line with the viewpoint of Digne and Michel on locally Garside categories (a weaker notion), [21].

Let \((C, \phi, \Delta)\) be a categorical Garside structure. Let \(A\) be the graph of atoms and \(S\) be the graph of simples of \(C\). The automorphism \(\phi\) induces automorphisms of these graphs.

Krammer explains how to reconstruct \(C\) as a quotient of the path category of the atom graph, via the choice, for each pair \(a, b\) of atoms, of paths \(a \prec b\) and \(b \prec a\) (a path in the atom graph is a formal sequence of atoms), corresponding to atomic decompositions in \(C\) of the right factors \(a \prec b\) and \(b \prec a\) of the colimit \(a \lor b = a(a \prec b) = a(b \prec a)\). He actually goes the other way around: he starts with an abstract graph, together with choices of walks \(a \prec b\) and \(b \prec a\), choices of walks expressing each \(\Delta_x\), and an automorphism of the whole structure; when these choices satisfy certain conditions (Dehornoy’s cube condition, condition for naturality of \(\Delta,\)...), then he declares that the quotient of the path category of the abstract graph modulo the relations \(a(a \prec b) = b(b \prec a)\) is a Garside category.

We prefer to view \(C\) as the free category on the germ \(S := (S, m)\), where \(S\) is the graph of simple and \(m\) is the restriction of the \(C\)-composition law

\[
S_{x \rightarrow y} \times S_{y \rightarrow z} 
\rightarrow \mathcal{C}_{x \rightarrow z}
\]

to the preimage of \(S_{x \rightarrow z}\). We call \(S\) the germ of simples of \((C, \phi, \Delta)\).

**Lemma 3.1.** Let \(S\) be the germ of simples of a categorical Garside structure \((C, \phi, \Delta)\). The natural morphism \(\mathcal{C}(S) \rightarrow C\) is an isomorphism.

Conversely, suppose we are given a germ \(S\). How can we know whether it is the germ of simples of a Garside category? The axioms of Garside categories may be rewritten into a set of axioms characterising such germs.

**Definition 3.2.** A germ \(S = (S, m)\) is (homogeneous) Garside if the following conditions are satisfied:

(i) It is homogeneous\(^4\) and \(C\)-cancellative\(^5\).

(ii) For any vertex \(x\), \((S_{x \rightarrow}, \leq)\) admits a maximal element \(\Delta_x\).

(iii) Denote by \(x^\phi\) the target of \(\Delta_x\). The map

\[
(S_{x \rightarrow}, \leq) \rightarrow (S_{x \rightarrow x^\phi}, \geq)
\]

\[
s \mapsto \overline{s}\text{ such that } s\overline{s} = \Delta_x
\]

\(^4\)See footnote on page 10. Note also that the definitions for atomicity, homogeneity and cancellativity easily generalise from categories to germs (we leave the details to the reader).

\(^5\)This notion is the natural generalisation of the \(M\)-cancellativity from [2] Section 0]
is an isomorphism.

(iv) For all $x$, the poset $(S_{x\to}, \leq)$ is a lattice.

Note that, in (iii), the map $s \mapsto \overline{s}$ is well-defined because $S$ is cancellative. Because of (iii), $\Delta_x$ is also the maximal element of $(S_{\to x^\phi}, \geq)$ and the apparent chirality of the axiom set is only an optical illusion.

For any objects $x, y$, the isomorphism $(S_{x\to}, \leq) \simeq (S_{y\to}, \geq)$ restricts to a bijection $S_{x\to y} \simeq S_{y\to x^\phi}$. Applying this twice, we obtain a bijection $S_{x\to y} \simeq S_{y\to x^\phi} \simeq S_{x^\phi\to y^\phi}$ which may be shown to be part of a global isomorphism $\phi : S \to S$. It extends to an automorphism $\phi : \mathcal{C}(S) \to \mathcal{C}(S)$ such that $\Delta : x \mapsto (\Delta_x)/\sim$ is a natural transformation $1 \xrightarrow{\Delta} \phi$.

**Theorem 3.3.** The germ of simples of a categorical Garside structure is a Garside germ. Conversely, if $S$ is a Garside germ, then $(\mathcal{C}(S), \phi, \Delta)$, where $\phi$ and $\Delta$ are as constructed above, is a categorical Garside structure whose germ of simples is isomorphic to $S$.

**Example 3.4.** Let $\mathcal{A}$ be a finite real reflection arrangement. Let $\text{ch}(\mathcal{A})$ be the set of chambers (connected components of the complement of the reflecting hyperplanes). There is a natural distance on $\text{ch}(\mathcal{A})$ ($d(C, C')$ is the number of walls separating the two chambers). Let $S$ be the germ whose underlying graph is the complete oriented graph on $\text{ch}(\mathcal{A})$ (edges are pairs $(C, C')$ of chambers) and such that $(C, C')(C', C'')$ is defined as equal to $(C, C'')$ when $C'$ lies on a geodesic from $C$ to $C''$ (and not defined otherwise). The category $\mathcal{C}(S)$ is isomorphic to the category denoted by $\text{Gal}_+$ in [19]. Deligne shows that if $\mathcal{A}$ is simplicial, e.g. if it is the reflection arrangement of a finite real reflection group, then $S$ is a Garside germ and $\mathcal{C}(S)$ is a Garside category.

### 4. Automorphisms of Garside categories

**Definition 4.1.** Let $(\mathcal{C}, \phi, \Delta)$ be a categorical Garside structure. An automorphism of $(\mathcal{C}, \phi, \Delta)$ is an automorphism $\psi$ of $\mathcal{C}$ such that $\phi \psi = \psi \phi$

and, for all object $x$,

$\psi(\Delta_x) = \Delta_{\psi x}$.

If $(\mathcal{C}, \phi, \Delta)$ is a categorical Garside structure, the Garside automorphism $\phi$ is an automorphism of $(\mathcal{C}, \phi, \Delta)$: indeed, if $x$ is an object, the naturality $1 \xrightarrow{\Delta_x} \phi$ applied to the morphism $\Delta_x$ gives the commutative diagram

\[
\begin{array}{ccc}
x & \xrightarrow{\Delta_x} & x^\phi \\
\downarrow{\Delta_x} & & \downarrow{(\Delta_x)^\phi} \\
x^\phi & \xrightarrow{\Delta_{x^\phi}} & x^{\phi^2}
\end{array}
\]
and, by cancellativity,

\[(\Delta_x)^\phi = \Delta_{x^\psi}.\]

**Theorem 4.2.** Let \((\mathcal{C}, \phi, \Delta)\) be a categorical Garside structure. Let \(\psi\) be an automorphism of \(\mathcal{C}\). Then \(\mathcal{C}^\psi, \phi|_{\mathcal{C}^\psi}, \Delta|_{\mathcal{C}^\psi}\) is a categorical Garside structure.

By \(\mathcal{C}^\psi\), we mean the subcategory of \(\mathcal{C}\) consisting of morphisms invariant under \(\psi\).

**Proof.** Since \(\mathcal{C}^\psi\) is a subcategory of \(\mathcal{C}\), it is atomic and cancellative.

Let \(x \in \mathcal{C}^\psi\) be an object. Since \(\psi(x) = x\), we have \(\psi(\Delta_x) = \Delta_{\psi(x)} = \Delta_x\). This justifies that \(\Delta|_{\mathcal{C}^\psi}\) is indeed a natural transformation in \(\mathcal{C}^\psi\).

Let \(a\) be an atom of \(\mathcal{C}\), let \(x\) be the source of \(a\). Assume that \(\psi(x) = x\). For any integer \(k \geq 0\), let \(\psi^k(a)\) be the colimit of \(\{a, \psi(a), \psi^2(a), \ldots, \psi^k(a)\}\). By atomicity, since \(\psi_k(a) \leq \Delta_x\), the sequence \(a, \psi_1(a), \psi_2(a), \ldots\) is eventually constant. Its limit \(\psi^*a\) is \(\psi\)-invariant.

Any atom \(b\) of \(\mathcal{C}^\psi\) is obtained this way: if \(a\) is a \(\mathcal{C}\)-atom such that \(a \leq b\), then \(\psi^*a \leq b\), thus \(\psi^*a = b\). Since \(\psi^*a \leq \Delta_x\), atoms are simple.

If \(f, g \in \mathcal{C}^\psi\) have the same source, then they admit a \(\mathcal{C}\)-colimit \(f \lor g\). This colimit divides a certain power of \(\Delta\). As above, this implies that the infinite family \(\{f \lor g, \psi(f \lor g), \psi^2(f \lor g), \ldots\}\) admits a colimit, which clearly is a \(\mathcal{C}^\psi\)-colimit for \(f\) and \(g\). \(\square\)

**Corollary 4.3.** Let \((\mathcal{C}, \Delta, \phi)\) be a categorical Garside structure. Let \(x\) be an object of \(\mathcal{C}\). Let \(p \in \mathbb{Z}_{\geq 0}\) and consider the element \(\Delta^p\) with source \(x\). Assume that this element is a loop, namely that \(\Delta^p \in \mathcal{G}_x\). Then the centraliser \(C_{\mathcal{G}_x}(\Delta^p)\) is a weak Garside group, namely the structure group at \(x\) of the Garside structure \((\mathcal{C}^\phi, \phi|_{\mathcal{C}^\phi}, \Delta|_{\mathcal{C}^\phi})\).

**Proof.** Because \(1 \Rightarrow \phi\), we have \(c\Delta_p = \Delta_p c^\phi\). Thus, for any \(c \in \mathcal{G}_x\), the conditions \(c \in C_{\mathcal{G}_x}(\Delta^p)\) and \(c \in \mathcal{G}^\phi\) are equivalent. \(\square\)

**Remark 4.4.** Even when \(\mathcal{C}\) is connected, the category \(\mathcal{C}^\psi\) may be disconnected.

### 5. Loops and summits

Let \(\mathcal{G}\) be a category. Let \(g, g', c \in \mathcal{G}\). We write

\[g' = gc\]

as a synonym for the relation

\[gc = cg'.\]

By syntactic constraints, this cannot happen unless \(g\) and \(g'\) are loops, i.e., if \(g \in \mathcal{G}_{x \to x}\) and \(g' \in \mathcal{G}_{y \to y}\) for some objects \(x, y\).

**Definition 5.1.** We say that two loops \(g\) and \(g'\) are *conjugate*, and denote this by \(g \sim g'\) if there exists \(c\) such that \(gc = g'\).

If \(\mathcal{G}\) is a groupoid, then \(\sim\) is an equivalence relation.

**Definition 5.2.** Let \(\mathcal{G}\) be a groupoid. The *conjugacy category* of \(\mathcal{G}\) is the category \(\Omega\mathcal{G}\) whose object set is

\[\bigsqcup_{x \in \mathcal{G}\text{-object}} \mathcal{G}_x\]
and such that $\text{Hom}_{\Omega \mathcal{G}}(g, g') := \{ c \in \mathcal{G} | gc = g' \}$.

The composition law is the obvious one:

$$(g \xrightarrow{c} g') \cdot (g' \xrightarrow{c'} g'') := g \xrightarrow{cc'} g''.$$ 

The conjugacy classes of $\mathcal{G}$ are the connected components of $\Omega \mathcal{G}$.

The conjugacy category is clearly a groupoid. If $g \in \mathcal{G} \times x$, we have

$$C_{\mathcal{G} \times x}(g) \cong (\Omega \mathcal{G})_g,$$

where $C_{\mathcal{G} \times x}(g)$ is the centraliser $\{ c \in \mathcal{G} \times x | gc = cg \}$ and $(\Omega \mathcal{G})_g$ is the structure group of $\Omega \mathcal{G}$ at $g$.

Assume now that $\mathcal{G}$ is the Garside groupoid of a categorical Garside structure $(\mathcal{C}, \phi, \Delta)$. Let $g \in \mathcal{G}$. Write $g$ in normal form

$$g = s_1 \ldots s_l \Delta^k.$$ 

Recall that the infimum and supremum of $g$ are, respectively, the integers $\lfloor g \rfloor := k$ and $\lceil g \rceil := k + l$.

**Definition 5.3.** A loop $g$ in $\mathcal{G}$ is a summit if, for all $h$ such that $g \sim h$,

$$\lfloor h \rfloor \leq \lfloor g \rfloor \text{ and } \lceil g \rceil \leq \lceil h \rceil.$$ 

The summit category of $\mathcal{G}$ is the full category $\Omega_0 \mathcal{G}$ of $\Omega \mathcal{G}$ whose objets are the summits.

Any loop is conjugate to a summit. For any loop $g$, a certain procedure called “cycling and decycling” (after [23, 29]) may be applied to obtain a $c$ such that $g^c$ is a summit (we won’t use this procedure; it easily generalises from the earlier settings). The problem of determining whether $g$ and $h$ are conjugate, or computing the centraliser of $g$, can be reduced to the same problem dealing with summits rather than just loops.

The following key lemma traces back to Garside:

**Lemma 5.4.** Let $g \xrightarrow{c} h$ be a morphism in $\Omega_0 \mathcal{G}$. Write $c = c_1 \ldots c_l \Delta^k$ in normal form. Set $g_0 := g, g_1 := g^c, g_2 := g^{c_1 c_2}, \ldots, g_l := g^{c_1 \ldots c_l}$. Then $g_0, \ldots, g_l$ are summits.

**Corollary 5.5.** Let $\mathcal{G}$ be a finite type Garside groupoid. The conjugacy problem in $\mathcal{G}$, and the problem of finding presentations for the centraliser of a given loop, are solvable.

6. Coverings

**Definition 6.1.** Let $\mathcal{G}$ be a connected groupoid, let $x_0$ be an object (“a basepoint”) of $\mathcal{G}$. The universal cover (aka Cayley category) of $\mathcal{G}$ with respect to $x_0$ is the category $\widetilde{\mathcal{G}}_{x_0}$ as follows:

- objects are $\mathcal{G}$-morphisms $x_0 \xrightarrow{f} x$ with source $x_0$;
- for any pair $x_0 \xrightarrow{f} x, x_0 \xrightarrow{g} y$ of objects, there exists a unique $\widetilde{\mathcal{G}}_{x_0}$-morphism from $f$ to $g$; we denote this morphism by the formal symbol $f^{-1}g$.

Morphisms compose the way they should: since $\text{Hom}_{\widetilde{\mathcal{G}}_{x_0}}(f, h)$ contains a single element, we have to set:

$$f^{-1}g \cdot g^{-1}h = f^{-1}h.$$
In particular, there is a “covering” functor
\[ p : \tilde{G}_{x_0} \rightarrow G \]
\[ x_0 \xrightarrow{f} x \mapsto x \]
\[ f^{-1}g \mapsto f^{-1}g \]

Of course, \( f^{-1}g \) is a formal symbol when viewed in the universal cover, and an actual morphism \( x \rightarrow y \) when viewed in \( \tilde{G} \).

The universal cover comes equipped with a natural basepoint \( 1_{x_0} \). It is clear that applying again the universal cover construction to \( \tilde{G}_{x_0} \) does not bring anything new.

Assume now that \( G \) is the Garside groupoid of a Garside category \((\tilde{C}, \phi, \Delta)\). Let \( x_0 \) be an object of \( \tilde{C} \). Let \( \tilde{C} \) be the subcategory of \( \tilde{G}_{x_0} \) consisting of morphism whose image under \( p \) lies in \( \tilde{C} \). For any object \( x_0 \xrightarrow{f} x \) of \( \tilde{C} \), set
\[ f^\tilde{\phi} := f \Delta_x. \]

If \( f \xrightarrow{f^{-1}g} g \) is a \( \tilde{C} \)-morphism, we set
\[ (f^{-1}g)^\tilde{\phi} := (f^\tilde{\phi})^{-1}g^\tilde{\phi}. \]

One checks that \( (f^{-1}g)^\tilde{\phi} \) is indeed a \( \tilde{C} \)-morphism, that \( \tilde{\phi} \) is an automorphism of \( \tilde{C} \), and that
\[ \tilde{\Delta} : (x_0 \xrightarrow{f} x) \mapsto (f \rightarrow f \Delta_x) \]
is a natural transformation from the identity functor to \( \tilde{\phi} \).

**Lemma 6.2.** The triple \((\tilde{C}, \tilde{\phi}, \tilde{\Delta})\) is a Garside category, whose Garside groupoid is \( \tilde{G}_{x_0} \).

More generally, we may construct an intermediate cover for each prescribed isotropy subgroup of \( G_{x_0} \).

### 7. The Garside nerve

Let us begin by recalling the categorical viewpoint on group cohomology. A good introduction may be found in [30].

The **nerve** of a small category \( \tilde{C} \) is the simplicial set \( NC \) whose 0-skeleton is the object set of \( \tilde{C} \) and whose \( n \)-simplices, \( n \geq 1 \), are sequences \((f_1, \ldots, f_n)\) of \( \tilde{C} \)-morphisms composable as follows:
\[ x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3 \xrightarrow{f_{n-1}} x_n. \]

Face maps correspond to removing objects (and composing or dropping morphisms accordingly) and degeneracy maps correspond to inserting identity morphisms at a given object.

**Lemma 7.1.** If \( G \) is a groupoid, the geometric realisation \(|\tilde{N}G_{x_0}|\) is contractible.

**Proof.** This is because the category \( \tilde{G}_{x_0} \) is equivalent to the trivial category with only one arrow, and the functor sending a small category \( \tilde{C} \) to the realisation of its nerve is a functor of 2-categories: it sends small categories to topological spaces, functors to continuous maps and natural transformations of functors to homotopies between continuous maps. \( \square \)
When \( G \) has a single object \( x_0 \), i.e., when it is a group, the \( \mathbb{Z} \)-basis of the bar resolution is indexed by \( N \tilde{\mathcal{G}}_{x_0} \).

In the general situation, \( N \tilde{\mathcal{G}}_{x_0} \) is related to \( N \mathcal{G} \) via the “bar notation”: let \( x_0 \xrightarrow{g_0} x_1 \) be an object of \( \tilde{\mathcal{G}}_{x_0} \). Let \( g := (g_1, \ldots, g_n) \) be a \( n \)-simplex in \( N \mathcal{G} \) such that the source of \( g_1 \) is \( x_1 \):

\[
x_1 \xrightarrow{g_1} x_2 \xrightarrow{g_2} x_3 \xrightarrow{g_3} \cdots \xrightarrow{g_n} x_{n+1}
\]

of \( G \)-morphism. There exists a unique \( n \)-simplex \((h_1, \ldots, h_n)\) in \( N \tilde{\mathcal{G}}_{x_0} \) that satisfies the following conditions:

- \( h_i \) is mapped to \( g_i \) by the covering functor,
- the source of \( h_1 \) is \( g_0 \).

Indeed, the only solution is that each \( h_i \) should be the unique \( \tilde{\mathcal{G}}_{x_0} \)-morphism from \( g_0 \cdots g_{i-1} \) to \( g_0 \cdots g_i \).

Instead of using the “morphisms” notation

\[(h_1, \ldots, h_n),\]

we may represent this \( n \)-simplex by its bar notation

\[g_0[g_1] \cdots [g_n]\]

or its “endpoints” notation

\[(g_0, g_0g_1, \ldots, g_0g_1 \cdots g_n).\]

For \( g \in \mathcal{G}_{x_0} \) and \( g_0[g_1] \cdots [g_n] \in N \tilde{\mathcal{G}}_{x_0} \), we set

\[g \cdot g_0[g_1] \cdots [g_n] := gg_0[g_1] \cdots [g_n].\]

This defines a left action of the structure group \( \mathcal{G}_{x_0} \) on \( N \tilde{\mathcal{G}}_{x_0} \). This action is free. It is natural to think that \( N \tilde{\mathcal{G}}_{x_0} \) is a “bar resolution for groupoids”.

We don’t know if the following terminology is standard – but it is certainly natural for our purposes:

**Definition 7.2.** Let \( X \) be a simplicial set. The 0-skeletal fundamental groupoid of \( X \) is the full subcategory of the fundamental groupoid of the the geometric realisation \( |X| \) whose object set is the image of the 0-skeleton. In other words, we only allow vertices as endpoints for paths.

Let \( \mathcal{G} \) be a groupoid. A simplicial \( K(\mathcal{G}, 1) \) is a simplicial set \( X \), together with a bijection between the object set of \( \mathcal{G} \) and the 0-skeleton of \( X \), inducing an isomorphism between the \( \mathcal{G} \) and the 0-skeletal fundamental groupoid of \( X \), and such that the higher homotopy groups of each component of \( |X| \) vanish.

Note that this makes sense even when \( \mathcal{G} \) is not connected.

**Proposition 7.3.** The nerve \( N \mathcal{G} \) is a simplicial \( K(\mathcal{G}, 1) \).

**Proof.** The simplices of \( \mathcal{G}_{x_0} \setminus N \tilde{\mathcal{G}}_{x_0} \) are indexed by bar symbols \( x[g_1] \cdots [g_n] \). It is readily seen that the map \( N \tilde{\mathcal{G}}_{x_0} \to N \mathcal{G}, g_0[g_1] \cdots [g_n] \mapsto [g_1] \cdots [g_n] \) induces an isomorphism \( \mathcal{G}_{x_0} \setminus N \tilde{\mathcal{G}}_{x_0} \simeq N \mathcal{G} \). \( \square \)
Assume now that \( \mathcal{G} \) is the Garside groupoid of a Garside category \((\mathcal{C}, \phi, \Delta)\). The Garside structure provides us with a substitute for \( \mathcal{N}\mathcal{G} \), the Garside nerve \( \mathcal{G} \).

**Definition 7.4.** Let \((\mathcal{C}, \phi, \Delta)\) be a Garside category with Garside groupoid \( \mathcal{G} \). The **Garside nerve** of \( \mathcal{G} \) is the simplicial set \( \mathcal{N}\Delta \mathcal{G} \) consisting of \( n \)-simplices of \( \mathcal{N}\mathcal{G} \)

\[
x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n.
\]

such that \( f_1 \ldots f_n \leq \Delta x_0 \) (the faces and degeneracy maps are the same as that of \( \mathcal{N}\mathcal{G} \)).

The maximal length of non-degenerate simplices is precisely the Garside dimension of \( \mathcal{C} \).

**Theorem 7.5.** Let \((\mathcal{C}, \phi, \Delta)\) be a Garside category with Garside groupoid \( \mathcal{G} \). The Garside nerve \( \mathcal{N}\Delta \mathcal{G} \) is a simplicial \( K(\mathcal{G}, 1) \text{ of dimension } \dim_{\Delta} \mathcal{C} \). It is finite if and only \((\mathcal{C}, \phi, \Delta)\) has finite type.

The geometric realisation of a simplicial set \( X \) is obtained by glueing geometric simplices corresponding to the non-degenerate simplices of \( X \).

**Corollary 7.6.** Let \((\mathcal{C}, \phi, \Delta)\) be a Garside category with Garside groupoid \( \mathcal{G} \). Let \( x_0 \) be an object of \( \mathcal{G} \). Let \( \Gamma \) be the graph whose vertex set is \( \mathcal{G}(x_0, -) \), two vertices \( f, g \in \mathcal{G}(x_0, -) \) being connected by an edge when \( f \neq g \) and either \( fg^{-1} \) or \( gf^{-1} \) is simple. The realisation of the simplicial complex \( \text{Flag}(\Gamma) \) is contractible.

**Proof.** Let \( \{f_0, \ldots, f_n\} \) be a simplex in \( \text{Flag}(\Gamma) \). For any \( i \neq j \), we have either \( f_i^{-1} f_j \) or \( f_j^{-1} f_i \). Both cannot happen simultaneously, because the product of two non-trivial simples cannot be 1. Write \( f_i < f_j \) if \( f_i^{-1} f_j \) and \( f_j < f_i \) otherwise. Let \( \leq \) be the reflexive closure of \( < \).

The relation \( \leq \) is transitive: let \( i, j, k \) be such that \( f_i < f_j \) and \( f_j < f_k \); we have either \( f_i < f_k \) or \( f_k < f_i \); if \( f_k < f_i \), then there are simple elements \( s, t, u \) such that \( f_i s = f_j \), \( f_j t = f_k \), \( f_k u = f_i \), thus \( f_i stu = f_i \) and, by cancellativity, \( stu = 1 \), which contradicts atomicity.

It follows that \( \leq \) is a total ordering of \( \{f_0, \ldots, f_n\} \). Up to reordering, we may assume that \( f_i \leq f_j \iff i \leq j \).

It is clear that \( \{f_1, \ldots, f_n\} \) is a non-degenerate \( n \)-simplex of the Garside nerve \( \mathcal{N}_{\Delta} \tilde{\mathcal{G}}_{x_0} \).

Conversely, any non-degenerate \( n \)-simplex of \( \mathcal{N}_{\Delta} \tilde{\mathcal{G}}_{x_0} \) corresponds to a unique \( n \)-simplex of \( \text{Flag}(\Gamma) \).

Because the face operators are compatible with this bijection, and because the geometric realisation of a simplicial set is precisely obtained by glueing geometric simplices associated with non-degenerate simplices, the realisations \( |\text{Flag}(\Gamma)| \) and \( |\mathcal{N}_{\Delta} \tilde{\mathcal{G}}_{x_0}| \) are homotopy equivalent. Since \( \tilde{\mathcal{G}}_{x_0} \) is equivalent to the trivial category, the contractibility of \( |\mathcal{N}_{\Delta} \tilde{\mathcal{G}}_{x_0}| \) follows from the theorem. \( \square \)

The corollary is very useful for topological applications, in particular in the proof of the \( K(\pi, 1) \) conjecture for complex reflection arrangements ([4]): by contrast with simplicial sets, actual simplicial complexes may appear as nerves of open coverings.

**Remark 7.7.** Since we have omitted the proof of [7.5], it is all too convenient to present [7.6] as a corollary of [7.5]. However, when actually checking the details of the proof, one
rather goes the other way around: one first shows that $|\text{Flag}(\Gamma)|$ is contractible (using, for example, Bestvina’s techniques), then that $|\mathcal{N}_\Delta \tilde{G}_{x_0}|$ is contractible (using the argument that we have presented as a “proof” of Corollary 7.6), then that $\mathcal{N}_\Delta \tilde{G}$ is a simplicial $K(\mathcal{G}, 1)$ (using Galois theory).

8. Non-positively curved aspects, after Bestvina

Let $A$ is an Artin group of finite type, with Garside element $\Delta$. In his beautiful article [7], Bestvina made the crucial observation that $A/\langle \Delta^2 \rangle$ is very close to being a hyperbolic group. He constructed a simplicial complex $\mathcal{X}$ together with a simplicial action $A/\langle \Delta^2 \rangle$, and equipped $\mathcal{X}$ with a “non-symmetric” distance with non-positive curvature features.

From this, he was able to prove that any periodic element $\gamma \in A$ is conjugate to some element with canonical length one:

$$(\exists p, q \in \mathbb{Z}_{>0}, \gamma^q = \Delta^p) \Rightarrow (\exists s \text{ simple, } k \in \mathbb{Z}, \gamma \text{ is conjugate to } s\Delta^k).$$

This result is an immediate consequence of his Theorem 4.5 and has many consequences: e.g., that, for given $p, q$, there are only a finite number of conjugacy classes of $\frac{p}{q}$-periodic elements.

Charney-Meier-Whittlesey have rewritten Bestvina’s proof in the language of Garside monoids, see [15]. As expected, there is no obstruction to working with categories:

**Theorem 8.1** (after Bestvina and Charney-Meier-Whittlesey). Let $(\mathcal{C}, \phi, \Delta)$ be a cyclic categorical Garside structure, with Garside groupoid $\mathcal{G}$. Let $p, q \in \mathbb{Z}_{>0}$, let $\gamma \in \mathcal{G}$ be a $\frac{p}{q}$-periodic element. There exists a simple morphism $s \in \mathcal{C}$ and an integer $k \in \mathbb{Z}$ such that

$$ss\phi^{-k}s\phi^{-2k} \ldots s\phi^{-(q-1)k} = \Delta$$

and such that $\gamma$ is conjugate (in the groupoid sense) to $s\Delta^k$.

**Proof.** Long but easy translation exercise, left to the reader – sections 2,3,6 from [15] should be rewritten in categorical language, to obtain an analog of their Corollary 6.9. \qed

**Remark 8.2.** (i) For simplicity, we only consider here cyclic Garside categories, because when the category is not cyclic some definitions and arguments from [7] and [15] have no clear analogs. However, it is very likely that something can be said about the non-cyclic case.

(ii) The version for categories of Bestvina’s complex $\mathcal{X}$ should be thought of as a quotient of the Garside nerve $\mathcal{N}_\Delta \mathcal{G}$, where all vertices are written in normal form and the $\Delta$’s at the end are “forgotten”.

(iii) In the next sections, we will improve the above theorem by showing that, up to changing the Garside structure, any periodic element is conjugate to an element with canonical length zero!, i.e., may be viewed as a Garside element.

(iv) A key step in Bestvina’s approach is a Cartan fixed point theorem ([7, Theorem 3.15]) which, adapted to our setting, implies that the action of $\gamma$ on $\mathcal{X}$ “leaves a simplex [...] invariant (and fixes its barycenter).” To prove that any periodic element is conjugate to an element with canonical length zero, we will make use of an algebraic operation which replaces the category $\mathcal{C}$ by a divided Garside category (see next section). At the level of Garside nerves, this corresponds to taking some
sort of barycentric subdivision. A consequence is that barycentres of simplices of $\mathcal{X}$ become vertices in the divided version of $\mathcal{X}$ – one then concludes using the Cartan fixed point theorem. Because it allows for more explicit proofs, the next sections are written in algebraic language; however, readers with good geometric intuition should keep in mind that everything can thought of in terms of Garside nerves and Bestvina complexes. When adding more vertices, one increases the number of objects in the 0-skeletal fundamental groupoid, without changing the fundamental groups – in algebraic terms, this will be naturally phrased in terms of an equivalence of categories between $\mathcal{G}$ and its divided version.

9. Divided Garside categories

We now proceed to our main construction, which is a general procedure, starting with a Garside category $\mathcal{C}$, to obtain a family $(\mathcal{C}_m)_{m \in \mathbb{Z} \geq 1}$ of Garside categories. When $m = 1$, one recovers $\mathcal{C}$. When $\mathcal{C}$ has finite type, all $\mathcal{C}_m$ have finite type (but they usually have more objects than $\mathcal{C}$). When $\mathcal{C}$ is $k$-cyclic, $\mathcal{C}_m$ is $mk$-cyclic. For $m \neq n$, the categories $\mathcal{C}_m$ and $\mathcal{C}_n$ are usually not isomorphic nor equivalent, but their Garside groupoids $\mathcal{G}_m$ and $\mathcal{G}_n$ are equivalent as categories.

**Definition 9.1.** Let $(\mathcal{C}, \phi, \Delta)$ be a Garside triple and $m \in \mathbb{Z} \geq 1$. A $m$-subdivision of $\Delta$ is a sequence $f = (f_1, f_2, \ldots, f_m)$ of composable $\mathcal{C}$-morphisms such that

$$\prod_{i=1}^{m} f_i = \Delta.$$ 

We denote by $D_m(\mathcal{C}, \phi, \Delta)$ (or simply $D_m(\mathcal{C})$, or simply $D_m$) the set of $m$-subdivisions of $\Delta$.

By “$\prod_{i=1}^{m} f_i = \Delta$”, we of course mean that that the $f_i$’s are indeed composable (the target of $f_i$ is the source of $f_{i+1}$) and that their product is $\Delta x_1$, where $x_1$ is the source of $f_1$. This implies that the target of $f_m$ is $x_1^\phi$. This also implies that each $f_i$ is simple, because one property of Garside categories stipulates that factors of simple elements are simple.

Note that, contrary to our earlier notations, we choose to label objects starting at 1 and not 0, according to:

$$x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \xrightarrow{f_3} \cdots \xrightarrow{f_m} x_1^\phi.$$ 

**Convention 9.2.** In the sequel, whenever $(a_1, \ldots, a_m)$ is a sequence of $\mathcal{C}$-objects or $\mathcal{C}$-morphisms, we extend the notation $a_i$ to all $i \in \mathbb{Z} \geq 1$ by recursively setting $a_{m+i} := a_i^\phi$. (Because we have assumed that $\phi$ is invertible, we may actually extend our index set to $\mathbb{Z} \leq 0$, although we won’t use it).

To illustrate this convention, we observe that, in the above commutative diagram, we may say that “the target of $f_i$ is $x_{i+1}$” without worrying about the case $i = m$. 
The object set of the \(m\)-th divided category \(C_m\) will be \(D_m\). We are going to define the category \(C_m\) by means of its germ of simples. We have to define an oriented graph \(S_m\) on the vertex set \(D_m\) and endow it with a partial product structure.

Let \(f = (f_1, \ldots, f_m)\) and \(g = (g_1, \ldots, g_m)\) be two elements of \(D_m\). An element \(s \in S_{m,f \rightarrow g}\) is, by definition, a sequence \(s := (s_1, \ldots, s_m)\) of \(C\)-simples, each \(s_i\) going from the source to \(f_i\) to the source of \(g_i\), forming a simple commutative diagram in \(C\) as follows:

\[
\begin{array}{cccccccc}
  x_1 & \xrightarrow{f_1} & x_2 & \xrightarrow{f_2} & x_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{m-1}} & x_m & \xrightarrow{f_m} & x_1^\phi \\
  s_1 & \downarrow & s_2 & \downarrow & s_3 & \downarrow & \cdots & \downarrow & s_{m-1} & \downarrow & s_m & \downarrow \\
  y_1 & \xrightarrow{g_1} & y_2 & \xrightarrow{g_2} & y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{m-1}} & y_m & \xrightarrow{g_m} & y_1^\phi \\
\end{array}
\]

and such that the above diagram may be completed by diagonal simple \(C\)-morphisms to obtain a commutative diagram:

\[
\begin{array}{cccccccc}
  x_1 & \xrightarrow{f_1} & x_2 & \xrightarrow{f_2} & x_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{m-1}} & x_m & \xrightarrow{f_m} & x_1^\phi \\
  y_1 & \xrightarrow{g_1} & y_2 & \xrightarrow{g_2} & y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{m-1}} & y_m & \xrightarrow{g_m} & y_1^\phi \\
\end{array}
\]

By cancellativity in \(C\), there is at most one way to add the diagonal arrows, so we may as well consider a morphism to be the whole diagram, including those diagonal arrows.

Let us put it in other words. A simple morphism \(f \xrightarrow{s} g\) consists of factorisations \(f_i = s_is_i'\), one for each \(i = 1, \ldots, m\), such that \(g_i = s_i's_{i+1}\) for all \(i \geq 1\).

Let \(f, g, h \in D_m\), let \(s \in S_{m,f \rightarrow g}\) and \(t \in S_{m,g \rightarrow h}\). This corresponds to a commutative diagram with \(C\)-simple arrows:

\[
\begin{array}{cccccccc}
  x_1 & \xrightarrow{x_2} & x_3 & \xrightarrow{x_4} & \cdots & \xrightarrow{x_{m-1}} & x_m & \xrightarrow{x_1^\phi} \\
  y_1 & \xrightarrow{y_2} & y_3 & \xrightarrow{y_4} & \cdots & \xrightarrow{y_{m-1}} & y_m & \xrightarrow{y_1^\phi} \\
  z_1 & \xrightarrow{z_2} & z_3 & \xrightarrow{z_4} & \cdots & \xrightarrow{z_{m-1}} & z_m & \xrightarrow{z_1^\phi} \\
\end{array}
\]

We say that \(s\) and \(t\) are compatible if the diagram may be completed by simple arrows in \(C\) to obtain a commutative diagram:

\[
\begin{array}{cccccccc}
  x_1 & \xrightarrow{x_2} & x_3 & \xrightarrow{x_4} & \cdots & \xrightarrow{x_{m-1}} & x_m & \xrightarrow{x_1^\phi} \\
  y_1 & \xrightarrow{y_2} & y_3 & \xrightarrow{y_4} & \cdots & \xrightarrow{y_{m-1}} & y_m & \xrightarrow{y_1^\phi} \\
  z_1 & \xrightarrow{z_2} & z_3 & \xrightarrow{z_4} & \cdots & \xrightarrow{z_{m-1}} & z_m & \xrightarrow{z_1^\phi} \\
\end{array}
\]
In other words, \( s = (s_1, \ldots, s_m) \) and \((t_1, \ldots, t_m)\) are compatible if and only if each \( s_it_i \) may be multiplied in \( S \) and
\[
(s_1t_1, \ldots, s_mt_m) \in S_m.
\]
We take as partial product structure on \( S_m \) the map sending compatible \( s \) and \( t \) as above to
\[
st := (s_1t_1, \ldots, s_mt_m).
\]
This defines a germ structure
\[
\mathcal{S}_m
\]
with underlying graph \( S_m \).

Given an object
\[
f = (x_1 \overset{f_1}{\longrightarrow} x_2 \overset{f_2}{\longrightarrow} x_3 \overset{f_3}{\longrightarrow} \cdots \overset{f_{m-1}}{\longrightarrow} x_m \overset{f_m}{\longrightarrow} x_1^\phi)
\]
in \( D_m(C) \), the poset \((S_m(C)_{f \to}, \leq)\) is isomorphic to
\[
\prod_{i=1}^m ([1_{x_i}, f_i], \leq),
\]
where \([1_{x_i}, f_i]\) denotes the interval between \( 1_{x_i} \) and \( f_i \) in \((S_{x_i \to}, \leq)\). Since each \((S_{x_i}, \leq)\) is a lattice, \((S_m(C)_{f \to}, \leq)\) is a lattice, whose maximal element is
\[
\begin{array}{c}
x_1 \quad \overset{f_1}{\longrightarrow} \quad x_2 \quad \overset{f_2}{\longrightarrow} \quad x_3 \quad \overset{f_3}{\longrightarrow} \quad \cdots \quad \overset{f_{m-1}}{\longrightarrow} \quad x_m \quad \overset{f_m}{\longrightarrow} \quad x_1^\phi \\
\downarrow f_1 \quad \downarrow f_2 \quad \downarrow f_3 \quad \downarrow f_4 \quad \downarrow f_{m-1} \quad \downarrow f_m \quad \downarrow f_1^\phi \\
x_2 \quad \overset{f_2}{\longrightarrow} \quad x_3 \quad \overset{f_3}{\longrightarrow} \quad x_4 \quad \overset{f_4}{\longrightarrow} \quad \cdots \quad \overset{f_m}{\longrightarrow} \quad x_1^\phi \quad \overset{f_1^\phi}{\longrightarrow} \quad x_2^\phi
\end{array}
\]
(which is to be completed by diagonal trivial \( C \)-morphisms \( 1_{x_i} \)).

We denote by \( \Delta_{m,f} \) this element of \( S_m \). We define an automorphism \( \phi_m \) of \( S_m \) acting on vertices by
\[
(f_1, f_2, \ldots, f_m)^{\phi_m} := (f_2, \ldots, f_m, f_1^\phi)
\]
and on arrows by
\[
(s_1, s_2, \ldots, s_m)^{\phi_m} := (s_2, \ldots, s_m, s_1^\phi).
\]
This is compatible with the partial product and induces an automorphism \( \phi_m \) of \( S_m \), such that \( \Delta_m \) is a natural transformation from the identify functor to \( \phi_m \).

**Definition 9.3.** The \( m \)-divided category associated with \( (C, \phi, \Delta) \) is the free category \( C_m \) on the germ \( S_m \). The groupoid of fractions of \( C_m \) is denoted by \( G_m \).

**Theorem 9.4.** The triple \( (C_m, \phi_m, \Delta_m) \) is a categorical Garside structure.

**Proof.** It is clear from the preceding discussion. \( \square \)

Let \( (C, \phi, \Delta) \) be a Garside triple, let \( m \geq 1 \).

For any object \( x \) of \( C \), we consider the object of \( C_m \) defined by
\[
\Theta_m(x) := \left( x \overset{1_x}{\longrightarrow} x \overset{1_x}{\longrightarrow} \cdots \overset{1_x}{\longrightarrow} x \overset{\Delta}{\longrightarrow} x^\phi \right).
\]
If $x \xrightarrow{s} y$ is a simple $\mathcal{C}$-morphism, we consider the $\mathcal{C}_m$-morphism

$$\Theta_m(x) \xrightarrow{\Theta_m(s)} \Theta_m(y)$$

defined as the composition from top to bottom of the following simple morphisms (note that $\Theta_m(x)$ itself is not simple):

$$
\begin{array}{cccccccccccccccc}
X & 1 & X & 1 & X & \cdots & X & 1 & X & \Delta & X^\phi \\
1 & 1 & 1 & 1 & s & 1 & & & & & \\
X & 1 & X & 1 & X & \cdots & X & s & y & \bar{\tau} & X^\phi \\
1 & 1 & 1 & s & 1 & 1 & & & & & \\
X & 1 & X & 1 & X & \cdots & y & 1 & y & \bar{\tau} & X^\phi \\
1 & 1 & s & y & 1 & 1 & 1 & 1 & \bar{\tau} & X^\phi \\
s & y & 1 & 1 & 1 & 1 & 1 & s^\phi & & & \\
y & 1 & 1 & 1 & 1 & 1 & \Delta & y^\phi & & & \\
\end{array}
$$

**Theorem 9.5.** The map $x \mapsto \Theta_m(x), (x \xrightarrow{s} y) \mapsto (\Theta_m(x) \xrightarrow{\Theta_m(s)} \Theta_m(y))$ extends to a unique functor

$$\Theta_m : \mathcal{C} \to \mathcal{C}_m$$

whose induced functor

$$\mathcal{G} \to \mathcal{G}_m$$

is an equivalence of categories.

**Proof.** To check that the functor $\Theta_m : \mathcal{C} \to \mathcal{C}_m$ is well-defined, one has to check that, whenever $st = u$ holds in the germ of simples of $\mathcal{C}$, one has $\Theta_m(s)\Theta_m(t) = \Theta_m(u)$ in $\mathcal{C}_m$. This is a straightforward computation.

By theorem 7.3, we know that a Garside group is the fundamental groupoid of its Garside nerve (with respect to the 0-skeleton).

A $(k - 1)$-simplex of the Garside nerve of $\mathcal{G}_m$ consists of the following data:

- a $\mathcal{C}_m$-object

$$f = \left( \begin{array}{cccccccc}
x_1 & f_1 & x_2 & f_2 & x_3 & f_3 & \cdots & x_m & f_m & x_1^\phi \\
\Delta_{x_1} & & & & & & & & & \\
\end{array} \right),$$

which we call the basepoint of the simplex (and we say that the basepoint *starts at* $x_1$).
• a totally ordered (for \( \leq \)) sequence of \( k \) simple \( C_m \)-morphisms with source \( f \) or, equivalently, a totally ordered (for \( \leq \)) sequence of \( k + 1 \) simple \( C_m \)-morphisms with source \( f \) and whose last term is \( \Delta_m \) or, again equivalently, a factorisation \( f_i = f_{i,1} \ldots f_{i,k+1} \) of each \( f_i \) into \( k + 1 \) simple \( C \)-morphisms – in the latter description, the successive simple \( C_m \)-morphisms from \( f \) associated to the factorisations are

\[
\begin{align*}
\begin{array}{cccccccc}
| & f_1 & | & f_2 & | & f_3 & \cdots & f_{m-1} & | & f_m \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow \\
\emptyset & u_{1,j} & \emptyset & u_{2,j} & \emptyset & u_{3,j} & \cdots & u_{m-1,j} & \emptyset & u_{m,j} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow \\
v_{1,j} & u_{2,j} & v_{2,j} & u_{3,j} & v_{3,j} & u_{4,j} & \cdots & v_{m-1,j} & u_{m,j} & v_{m,j} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow \\
v_{1,j} & u_{2,j} & v_{2,j} & u_{3,j} & v_{3,j} & u_{4,j} & \cdots & v_{m-1,j} & u_{m,j} & v_{m,j} \\
\end{array}
\end{align*}
\]

where \( u_{i,j} := f_{i,1} \ldots f_{i,j} \) and \( v_{i,j} := f_{i,j+1} \ldots f_{i,k+1} \).

Since \( f_1 \ldots f_m = \Delta_{x_1} \), we see that a \( (k - 1) \)-simplex of \( \mathcal{N}_{\Delta_m} \mathcal{G}_m \) with basepoint starting at \( x_1 \) is nothing but a factorisation of \( \Delta_{x_1} \) into \( km \) terms, that is, a \( (mk - 1) \)-simplex of \( \mathcal{N}_{\Delta} \mathcal{G} \) with basepoint \( x_1 \).

In other words, \( \mathcal{N}_{\Delta_m} \mathcal{G}_m \) is the \( m \)-th edgewise subdivision of \( \mathcal{N}_{\Delta} \mathcal{G} \), in the sense of Bökstedt-Hsiang-Madsen (see section 1 of [9]). By [9] Lemma 1.1, the realisations \( |\mathcal{N}_{\Delta_m} \mathcal{G}_m| \) and \( |\mathcal{N}_{\Delta} \mathcal{G}| \) are homeomorphic. This implies that there exists an equivalence of categories between their 0-skeletal fundamental groupoids, which are precisely \( \mathcal{G}_m \) and \( \mathcal{G} \).

The last thing to check is that \( \Theta_m \) actually provides such an equivalence of categories. Bökstedt-Hsiang-Madsen’s definition of an homeomorphism\(^6\) \( D_m \) between \( |\mathcal{N}_{\Delta_m} \mathcal{G}_m| \) and \( |\mathcal{N}_{\Delta} \mathcal{G}| \) is constructive and, to conclude, it suffices to check that \( D_m(\Theta_m(x)) = x \) for all \( \mathcal{C} \)-object (and that \( D_m \) behaves as expected with respect to the 1-skeleton). A \( 0 \)-simplex \( (f_1, \ldots, f_m) \) in \( \mathcal{N}_{\Delta_m} \mathcal{G}_m \) with basepoint \( x_1 \) corresponds to a \( (m-1) \)-simplex in \( \mathcal{N}_{\Delta} \mathcal{G} \) whose vertices are the sources of the \( f_i \)'s and the edges are partial products of the first \( m - 1 \) \( f_i \)'s. By definition, \( D_m(f_1, \ldots, f_m) \) is taken to be the barycentre of that \( (m - 1) \)-simplex. But the \( (m - 1) \)-simplex associated with \( \Theta_m(x) \) is completely degenerate, all its vertices are \( x_1 \) and all its edges are \( 1_{x_1} \). Therefore it collapses to the single point \( x_1 \) in the realisation. \( \Box \)

Let us conclude this section with some easy basic properties.

**Proposition 9.6.** Let \( (\mathcal{C}, \phi, \Delta) \) be a finite type categorical Garside structure of Garside dimension \( n \). There exists a polynomial \( Z(\mathcal{C}, \phi, \Delta) \), of degree at most \( n \) and with integral coefficients, such that for all \( m \geq 1 \) the number of elements in \( D_m \) is \( Z(\mathcal{C}, \phi, \Delta)(m) \)

**Proof.** This follows from the fact that the number of \( m \)-chains in a finite poset is a polynomial in \( m \). \( \Box \)

**Proposition 9.7.** Let \( (\mathcal{C}, \phi, \Delta) \) be a categorical Garside structure of Garside dimension \( n \). Then

\[
\dim_{\Delta_m} \mathcal{C}_m = \dim_{\Delta} \mathcal{C} \quad \text{and} \quad \dim_{\Delta_m} \mathcal{C}_m^{\phi_m} \leq \frac{\dim_{\Delta} \mathcal{C}^{\phi}}{m}.
\]

\(^6\)Bökstedt-Hsiang-Madsen’s homeomorphism \( D_m \), used in this paragraph, should not be confused with our \( D_m \).
Proposition 9.8. Let \((\mathcal{C}, \phi, \Delta)\) be a categorical Garside structure. Let \(p, q, e\) be positive integers. The natural bijection

\[ \text{D}_{eq}(\mathcal{C}, \phi, \Delta) \cong \text{D}_e(\mathcal{C}_q, \phi_q, \Delta_q) \]

induces an isomorphism

\[ \mathcal{C}_{eq}^{\phi p} \cong (\mathcal{C}_q^{\phi q})_e. \]

10. Periodic elements are Garside elements

Recall that a \(\frac{p}{q}\)-periodic element in a Garside groupoid is a loop \(\gamma\) such that

\[ \gamma^q = \Delta^p. \]

After Bestvina, we observed in Theorem 8.1 that, if the Garside category is cyclic, any periodic element is conjugate to some \(s\Delta^k\), where \(s\) is a simple element such that

\[ ss^{-k}s^{-2k}\cdots s^{-qk} = \Delta. \]

To alleviate notations, we set \(s_i := s^{-i-k}\) and \(\underline{s} := (s_1, \ldots, s_q)\). The above identity expresses that \(\underline{s}\) is an object of \(\mathcal{C}_q\). Using \(1 \Rightarrow \Delta\), we obtain

\[ \Delta^p = (s\Delta)^q = s_1 \ldots s_q \Delta^{qk} = \Delta^{qk+1} \]

thus

\[ p = qk + 1. \]

Theorem 10.1. Let \((\mathcal{C}, \Delta, \phi)\) be a cyclic categorical Garside structure with associated groupoid \(\mathcal{G}\). As above, let \(\rho = s\Delta^k\) be a \(\frac{p}{q}\)-periodic loop in \(\mathcal{G}\). Consider the \(q\)-divided Garside groupoid \(\mathcal{G}_q\) and the functor \(\Theta_q : \mathcal{G} \to \mathcal{G}_q\). Then \(\Theta_q(\rho)\) is conjugate (in the groupoid sense) to the element \(\Delta^p_q\) (product of \(p\) successive Garside elements of \(\mathcal{G}_q\)) with source \(\underline{s} := (s_1, \ldots, s_q)\).

This theorem should be thought of as an algebraic Kerékártó-Brouwer-Eilenberg theorem for Garside categories – this will become clearer in the next section, as we rephrase this in terms of \(S^1\)-spaces. It gives a positive answer to the categorical rephrasing of Question 4.

Proof. Consider the configuration space \(U_q\) of \(q\) unordered distinct points on the unit circle \(S^1\) (or, more intuitively, “beads on a necklace”). Consider the subset \(U_{q,m}\) consisting of configurations in \(S^1 - \mu_m\), where \(\mu_m\) is the group of \(m\)-th roots of unity. For \(i = 1, \ldots, m\), let \(V_i\) be the connected component of \(S^1 - \mu_m\) consisting of points with argument in \((\frac{2\pi(i-1)}{m}, \frac{2\pi i}{m})\).

Because the connected components of \(U_{q,m}\) are contractible, we may consider the fundamental groupoid \(\text{NB}_{q,m}\) of \(U_q\) with respect to these components:

- objects of \(\text{NB}_{q,m}\) are connected components of \(U_{q,m}\); each component is uniquely determined by the sequence \((n_1, \ldots, n_m)\) where \(n_i\) is the number of beads in \(V_i\); conversely, any sequence \((n_1, \ldots, n_m) \in (\mathbb{Z}_{\geq 0})^m\) with sum \(q\) corresponds to an object;
- if \(C\) and \(C'\) are objects, \(\text{Hom}_{\text{NB}_{q,m}}(C, C') := \lim_{x \in C} \lim_{x' \in C'} \text{Hom}_{\pi_1(U_q)}(x, x').\)
Elements of $NB_{q,m}$ are necklace braids with $q$ beads and $m$ sectors, or simply necklace braids. From now on, we identify objects of $NB_{q,m}$ with their associated sequences.

Let $i \in \{1, \ldots, m\}$ and consider a component $(n_1, \ldots, n_m)$. If $n_i > 0$, we define the \textit{i-slide with source $(n_1, \ldots, n_m)$} as the element of $NB_{q,m}$ obtained by “sliding” a single bead from $V_i$ to $V_{i-1}$ ($V_m$ if $i = 1$) by decreasing its argument and crossing once the point with argument $\frac{2\pi(i-1)}{m}$. We denote this element by $\sigma_i$, regardless of its source (in a given formula, the symbol $\sigma_i$ should be interpreted as the \textit{only} possible i-slide whose source is as provided by the context).

It is clear that the necklace braid groupoid is generated by all slides and inverses of slides. One may easily write a presentation (whenever it makes sense, slides commute).

Let $A$ be an alphabet, together with a permutation $A \to A, a \mapsto a\phi$. Let $W_{q,m}$ the set of $m$-tuples $w = (w_1, \ldots, w_m)$ of words in $A^*$ whose concatenation $w_1 \ldots w_m$ has length $q$.

Let $w = (w_1, \ldots, w_m) \in W_{q,m}$ We say that a necklace braid $\beta$ is \textit{compatible with} $w$ if its source is $(l(w_1), \ldots, l(w_m))$. Assume that $\sigma_i$ is compatible with $w$, i.e., that we may write $w_i = aw'_i$ (with $a \in A$ and $w'_i \in A^*$). We define an element $w \cdot \sigma_i \in W_{q,m}$ as follows:

- if $i > 1$, we set $w \cdot \sigma_i := (w_1, \ldots, w_{i-2}, w_{i-1}a, w'_{i-1}, w_{i+1}, \ldots, w_m)$,
- if $i = 1$, we set $w \cdot \sigma_i := (w'_1, w_2, \ldots, w_{m-1}, w_m a\phi)$.

This extends to a right action of $NB_{q,m}$ on $W_{q,m}$. By this, we mean that there is a category $W_{q,m} \circ NB_{q,m}$ whose object set is $W_{q,m}$, such that Hom$_{W_{q,m} \circ NB_{q,m}}(w, -)$ is the set of necklace braids compatible with $w$, and such that, if as above $w$ and $\sigma_i$ are compatible, the corresponding morphism with source $w$ has target $w \cdot \sigma_i$.

Pursuing our trend of convenient abusive notation, when the source in $W_{q,m}$ is specified, we denote by $\sigma_i$ the only possible $W_{q,m} \circ NB_{q,m}$-morphism with this source and that is associated with an i-slide (there is at \textit{most} one such morphism).

To prove the theorem, we apply this to $q = m$ and $A := \{s^{\phi^k} | k \in \mathbb{Z}\}$ and we only look at $O$, the $NB_{q,q}$ orbit of $(\varepsilon, \varepsilon, s_1 \ldots s_q)$, and the corresponding subcategory $O \circ NB_{q,q}$. By evaluating each element of $A^*$ to its product in $C$, we obtain a map $\psi$ from $O$ to the object set of $C_q$. If $\sigma_i$ is a slide and $(w_1, \ldots, w_q) \xrightarrow{\sigma_i} (w'_1, \ldots, w'_q)$ is a morphism in $O \circ NB_{q,q}$, we define $\psi((w_1, \ldots, w_q)) \xrightarrow{\phi} (w'_1, \ldots, w'_q)$ to be the $C_q$-simple morphism from $(f_1, \ldots, f_q) := \psi((w_1, \ldots, w_q))$ to $(f'_1, \ldots, f'_q) := \psi((w'_1, \ldots, w'_q))$ corresponding to the diagram:

\[
\begin{array}{cccccc}
\text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{1} \\
\text{f}_1 & \text{f}_{i-1} & \alpha & \text{f}_i & \text{f}_{i+1} & \text{f}_q \\
\text{f}'_1 & \text{f}'_{i-1} & \text{f}'_i & \text{f}'_{i+1} & \text{f}'_q \\
\end{array}
\]

where $\alpha$ is the first letter of $w_i$. This extends to a functor

$$\psi : O \circ NB_{q,q} \to C_q.$$

If $w \in O$ is of the form $(\varepsilon, \varepsilon, \varepsilon, w_n)$, then $\Delta_qw = \psi((\sigma_n \sigma_{n-1} \cdots \sigma_1)^q)$. If $w \in O$ is of the form $(a_1, \ldots, a_q)$, where $a_1, \ldots, a_q$ are letters, then $\Delta_qw = \psi(\sigma_n \sigma_{n-1} \cdots \sigma_1)$.

Consider the following elements of $O$:

$$w := (\varepsilon, \varepsilon, \varepsilon, s_1 \ldots s_q),$$

$$w' := (s_1, \ldots, s_q)$$
and the following words in the alphabet \( \{ \sigma_1, \ldots, \sigma_q \} \):
\[
\beta_1 := \sigma_n \sigma_{n-1} \ldots \sigma_1
\]
\[
\beta_2 := (\sigma_n \sigma_{n-1} \ldots \sigma_2)(\sigma_n \sigma_{n-1} \ldots \sigma_3)(\sigma_n \sigma_{n-1} \ldots \sigma_4) \ldots (\sigma_n \sigma_{n-1})(\sigma_n).
\]
In the category \( NB_{q,q} \), one easily checks (make a picture!) the relation
\[
\beta_1^{qk+1} \beta_2 = \beta_2 \beta_1^{qk+1},
\]
where both sides are first expanded as words in the \( \sigma_i \)'s and then interpreted as morphisms with source \((0, \ldots, 0, q)\). As a consequence, we obtain in \( O \cap NB_{q,q} \) the relation
\[
\beta_1^{qk+1} \beta_2 = \beta_2 \beta_1^{qk+1},
\]
where both sides are now interpreted as morphisms with source \( w \). By functoriality, we obtain in \( C_q \) the relation
\[
\psi(\beta_1^{qk+1} \beta_2) = \psi(\beta_2 \beta_1^{qk+1}).
\]
From the definition of \( \Theta_q \), it is clear that
\[
\Theta_q(s^\Delta^k) = \psi(\beta_1^{qk+1})
\]
(\( \beta_1^{qk+1} \) interpreted here with source \( w \)).

We note that \( \psi(w') \) coincides with the \( C_q \)-object \( s \) from the theorem’s statement. When interpreting \( \beta_1 \) with source \( w' \), it maps by \( \psi \) to the Garside element \( \Delta_{q,s} \) with source \( s \). When interpreting \( \beta_2 \) with source \( w \), its target is \( w' \) and it maps by \( \psi \) to a certain \( C_q \)-morphism from \( \Theta_q(x) \) (where \( x \) is the source of \( s \)) to \( s \). The relation \( \psi(\beta_1^{qk+1} \beta_2) = \psi(\beta_2 \beta_1^{qk+1}) \) may be interpreted as expressing a conjugacy relation relation between \( \Theta_q(s^\Delta^k) \) and the \( \Delta_q^p = \Delta_q^{qk+1} \) with source \( s \), proving the theorem. \( \square \)

One should not be surprised by the fact that, in the theorem, \( \Theta_q(\rho) \) is conjugate to some \( \Delta_q^p \) that is by definition a \( p \)-periodic (and not \( \frac{p}{q} \)-periodic) element in \( G_q \). The explanation is that \( \Delta_q \), the Garside element of \( G_q \), behaves very much like a “\( q \)-th root” of \( \Delta \).

Another potentially disturbing fact is that not all elements of \( G_q \) of the form \( \Delta_q^p \) are periodic: in our definition of periodic elements, we have required them to be loops. The \( \Delta_q^p \) with source \( s \) appearing in the theorem is a loop. There is actually a very simple criterion to decide whether such a given \( \Delta_q^p \) is a loop:

**Lemma 10.2.** Let \( f = (f_1, \ldots, f_q) \) be an object of \( G_q \). The following conditions are equivalent:

(i) The \( G_q \)-morphism \( \Delta_q^p \) with source \( f \) is a loop, thus a \( p \)-periodic element.

(ii) Its source \( f \) is an object of \( G_q^{\Delta_q^p} \) (the invariant subcategory of \( G_q \) for the \( p \)-power of the diagram automorphism \( \phi_q \)).

(iii) One has, for all \( i = 2, \ldots, q \), \( f_i = f_1^{q(i-1)k} \).

**Proof.** Because \( 1 \overset{\Delta_q^p}{\Rightarrow} \phi_q \), the target of the \( \Delta_q^p \) with source \( f \) is \( f_{q}^{\Delta_q^p} \). This shows the equivalence between (i) and (ii).

To check the equivalence with (iii), one uses the relation \( p = qk+1 \): a direct computation shows that the target of the \( \Delta_q^p \) with source \( f \) is \( (f_2^{\Delta_q^p}, f_3^{\Delta_q^p}, \ldots, f_q^{\Delta_q^p}, f_1^{\Delta_q^p}) \). The element is a loop if and only if, for all \( i = 2, \ldots, q \), \( f_i = f_1^{q(i-1)k} \). (Note that an additional relation
seem to be required, namely that $f_1^{\phi k+1} = f_q$, but as a consequence of the other relations it rewrites as $f_1^{\phi} = f_1$ and comes for free using $1 \xrightarrow{\Delta} \phi$.

As suggested by the above lemma, Theorem 10.1 should be understood as part of a deeper dictionary between the conjugacy category of $\frac{p}{q}$-periodic elements in $G$ and the fixed subcategory $G^\phi_p$. This has many consequences. For example, the categorical rephrasing of Question 5 admits a positive answer:

**Corollary 10.3.** The centraliser of a periodic element in a cyclic Garside groupoid is a weak Garside group.

**Proof.** We apply Theorem 10.1. Because of the equivalence of categories, it suffices to show that the centraliser of a periodic power of a Garside element in a cyclic Garside groupoid is a weak Garside group. This has been done in Corollary 4.3.

The theorem also yields a precise criterion to test Question 2:

**Corollary 10.4.** Let $(\mathcal{C}, \Delta, \phi)$ be a cyclic categorical Garside structure with associated groupoid $G$. Fix positive integers $p, q$.

Let $s$ be a $\mathcal{C}$-simple such that $s\Delta^k$ is a $\frac{p}{q}$-periodic loop (as we have seen, this forces $p = qk + 1$). The $G_q$-object $\underline{s}$ (as in Theorem 10.1) is an object of $G^\phi_p$.

If $s$ and $s'$ are $\mathcal{C}$-simples such that $s\Delta^k$ and $s'\Delta^k$ are conjugate $\frac{p}{q}$-periodic loops, then $\underline{s}$ and $\underline{s'}$ lie in the same connected component of $G^\phi_p$.

In particular, we have a well-defined map from the set of conjugacy classes of $\frac{p}{q}$-periodic loops in $G$ to the set of connected components of $G^\phi_p$ (sending $\rho$ to the connected component of $\underline{s}$, where $s$ is chosen such that $s\Delta^k$ is a summit in the conjugacy class of $\rho$).

This map is a bijection.

**Remark 10.5.** When $p = q = 1$, the corollary describes conjugacy classes of Garside elements.

**Proof.** For any $s$ such that $s\Delta^k$ is $\frac{p}{q}$-periodic, denote by $\nabla(s)$ the $\Delta^p$ with source $\underline{s}$ (see Theorem 10.1).

Because $\nabla(s)$ is a loop, $\underline{s}$ is $\phi_p^q$-invariant (this is Lemma 10.2).

Suppose that $s\Delta^k$ and $s'\Delta^k$ are conjugate in $G$. Using Theorem 10.1 and the functoriality of $\Theta$, one sees that $\nabla(s)$ and $\nabla(s')$ are conjugate in $G_q$. Let $c \in \text{Hom}_{G_q}(\underline{s}, \underline{s'})$ be such that

$$\nabla(s)c = c\nabla(s').$$

Because $\nabla(s)$ and $\nabla(s')$ are products of $p$ successive $\Delta^p$'s, we deduce from $1 \xrightarrow{\Delta} \phi_q$ that

$$\nabla(s)c^\phi_p = c\nabla(s').$$

By cancellativity, $c = c^\phi_p$. In particular, $c$ connects $\underline{s}$ and $\underline{s'}$ in the category $G^\phi_p$.

Conversely, suppose that $\underline{s}$ and $\underline{s'}$ lie in the same component of $G^\phi_p$. Let $c \in \text{Hom}_{G^\phi_p}(\underline{s}, \underline{s'})$. 
We may view $c$ as a $q$-tuple $(c_1, \ldots, c_q)$ of $G$-morphisms such that the diagram

\[
\begin{array}{cccccccc}
\ast & \rightarrow & \ast & \rightarrow & \ast & \rightarrow & \ast & \rightarrow & \ast \\
c_1 & \downarrow & c_2 & \downarrow & c_3 & \downarrow & c_q & \downarrow & c_1^\phi \\
\uparrow & \rightarrow & \uparrow & \rightarrow & \uparrow & \rightarrow & \uparrow & \rightarrow & \uparrow \\
s_1 & \rightarrow & s_2 & \rightarrow & s_3 & \rightarrow & s_{q-1} & \rightarrow & s_q \\
\end{array}
\]

is commutative. Because $c_1^\phi q = c_1$, we have $c_2 = c_1 c_2^\phi$. In $G$, we have the relation

\[s_1 \Delta^k c_1 = s_1 c_2 \Delta^k = c_1 s_1' \Delta^k = c_1 s_1' \Delta^k.
\]

We have proved that $s_1 \Delta^k$ and $s_1' \Delta^k$ are conjugate. □

11. Cyclic structure on the Garside nerve

This section is a sketch. It explains the connection between the algebraic version (Theorem 10.1) and the geometric version (Theorem 0.3) of the “Ke r´ ekj´ art´ o principle” for Garside categories. It may be skipped in a first reading.

We study a special feature of the simplicial structure of the Garside nerve $N\Delta G$ of a Garside groupoid. When the Garside structure is 1-cyclic, we show that the Garside nerve is a cyclic set, in the sense of Connes. When the Garside structure is only $k$-cyclic, then the Garside nerve is very close to being a cyclic set and its realisation may be equipped with a natural structure of $S^1$-space.

In addition to the $n+1$-degeneracy maps

\[s_0, \ldots, s_n : (N\Delta G)_n \rightarrow (N\Delta G)_{n+1}\]

consisting of inserting identity morphisms at any of the $n+1$ objects of the sequence

\[x_0 \rightarrow f_1 \rightarrow x_1 \rightarrow f_2 \rightarrow x_2 \rightarrow f_3 \rightarrow x_3 \rightarrow \cdots \rightarrow f_{n-1} \rightarrow x_n,
\]

there is another natural way to obtain a $n+1$-simplex from a $n$-simplex: there is a unique way of completing $(f_1, \ldots, f_n)$ to a sequence

\[s_{n+1}(f_1, \ldots, f_n) := (f_1, \ldots, f_{n+1})\]

such that $f_1 \ldots f_{n+1} = \Delta$.

Remark 11.1. It is clear that the image of $s_{n+1}$ is precisely $D_{n+1}$.

Definition 11.2. We call special degeneracy operator the degree 1 map

\[s : N\Delta G \rightarrow N\Delta G\]

whose restriction to $(N\Delta G)_n$ is $s_{n+1}$.

We call first face operator the degree $-1$ map

\[d_0 : N\Delta G \rightarrow N\Delta G \quad (f_1, \ldots, f_n) \mapsto (f_2, \ldots, f_n).\]

Lemma 11.3. (i) For all $(f_1, \ldots, f_n) \in D_n$, $s d_0 (f_1, \ldots, f_n) = (f_2, \ldots, f_n, f_1^\phi)$.

(ii) For all $(f_1, \ldots, f_n) \in (N\Delta G)_n$, $(d_0 s)^{n+1} (f_1, \ldots, f_n) = (f_1^\phi, \ldots, f_n^\phi)$.

Proof. (i) follows from $1 \Delta \phi$. (ii) is an easy consequence of (i). □
When \( \phi \) is the identity, i.e., when \( C \) is 1-cyclic, statement (ii) of the lemma says that \( \mathcal{N}_\Delta \mathcal{G} \) is a cyclic set, in the sense of Connes, [16]. More generally, when \( C \) is \( k \)-cyclic, \( \mathcal{N}_\Delta \mathcal{G} \) is a \( \Lambda^\text{op}_k \)-object in the category of sets, in the sense of Bökstedt-Hsiang-Madsen, [9, Definition 1.5].

**Remark 11.4.** Lemma 11.3 can be understood in the general setting when \( \mathcal{G} \) is not cyclic, in terms of an “helicoidal category” generalising Connes’ cyclic category: the maps \((d_0s_{n+1})^{n+1}\) equip each \((\mathcal{N}_\Delta \mathcal{G})_n\) with a \( \mathbb{Z} \)-action, and the faces and degeneracy maps are \( \mathbb{Z} \)-equivariant.

**Theorem 11.5.** Let \((\mathcal{C}, \phi, \Delta)\) be a \( k \)-cyclic categorical Garside structure. Let \( X := |\mathcal{N}_\Delta \mathcal{G}| \) be the realisation of the Garside nerve of the associated groupoid. There is a canonical structure of \( S^1 \)-space on \( X \), with respect to which Question 1 has a positive answer: any periodic loop of \( \pi_1(X) \) is conjugate to a rotation.

12. **Example: 3-divided category of the Artin-Tits monoid of type \( A_2 \)**

The classical Artin-Tits monoid of type \( A_2 \) is a Garside category with one object \((\Delta) = (sts) = (tst)\) (this name for the object is natural: the category is isomorphic to its 1-divided category). The atom graph is as follows:

![Atom graph for Artin-Tits monoid of type \( A_2 \)](image)

It admits an automorphism of order 2.

The atom graph of the 3-divided category is as follows:

![Atom graph for 3-divided category of Artin-Tits monoid of type \( A_2 \)](image)

To improve readability, copies of the vertices \((s,t,s)\) and \((t,s,t)\) have been introduced. Because the graph is better imagined lying on the surface of a cylinder, we have used dotted arrows to represent the “hidden” edges, going behind the cylinder. The graph admits a symmetry of order 3 (rotating the cylinder by one third of a turn). It also admits
a symmetry of order 2 (reflection with horizontal axis). The diagram automorphism has order 6, and is obtained by composing these two symmetries.

13. Example: weak Garside groups vs Garside groups

Let $(\mathcal{C}, \phi, \Delta)$ be a categorical Garside structure with Garside groupoid $G$. Let $x \in \mathcal{C}$ be an object. It is tempting to think that the category $\mathcal{C}_x := \text{Hom}_\mathcal{C}(x, x)$ is a Garside category with group of fractions $G_x$, the structure group at $x$. This is not true, as shown by the following counterexample.

Let $\mathcal{C}$ be the category defined as the quotient of the free category on

```
[ Diagram of categories and arrows ]
```

by the relations $a^3 = b^3$ (whatever the source may be). It is a Garside category with Garside element $\Delta := a^3 = b^3$. The lattice of left divisors of $\Delta_x^2$ looks as follows:

```
[ Diagram of lattice ]
```

The restriction of the lattice to the $x$ lines is not a lattice: $a^2$ and $b^2$ do not have a colimit in $\mathcal{C}_x$, although they have a colimit in $\mathcal{C}$ ($a^3 = b^3$) and they have common multiples in $\mathcal{C}_x$.

Thanks

Inspiration for this article came as I was working on [4] and observed a relation between certain topological computations and a construction of Drew Armstrong, the $m$-divisible non-crossing partitions from [1]. The connection with Armstrong’s work will be explained in a further version of this preprint. Although Drew had already told me about his
construction, its importance became clear to me only after Vic Reiner brought it to my attention, in connection with a joint work in progress. Michel Broué, Daan Krammer and Jean Michel should also be thanked, for pleasant discussions at early stages of this work.

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