GRADINGS BY GROUPS ON GRADED CARTAN TYPE
LIE ALGEBRAS

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Abstract. In this paper we describe all gradings by abelian groups without
elements of order $p$, where $p > 2$ is the characteristic of the base field, on
the simple graded Cartan type Lie algebras.

1. Introduction

Let $A$ be an algebra, $G$ a group and let $\text{Aut} A$, $\text{Aut} G$ be the automorphism
groups of $A$ and $G$, respectively.

Definition 1.1. A grading by a group $G$ on an algebra $A$, also called a $G$-grading,
is a decomposition $A = \bigoplus_{g \in G} A_g$ where each $A_g$ is a subspace such that $A_{g'} A_{g''} \subseteq A_{g' \cdot g''}$ for all $g', g'' \in G$. For each $g \in G$, we call the subspace $A_g$ the homogeneous space of degree $g$. A subspace $V$ of $A$ is called graded if $V = \bigoplus_{g \in G} (V \cap A_g)$. The set $\text{Supp} A = \{ g \in G \mid A_g \neq 0 \}$ is called the support of the grading.

For a grading by a group $G$ on a simple Lie algebra $L$, it is well known that the
subgroup generated by the support is abelian [5, Lemma 2.1]. For the remainder of
the paper we always assume without loss of generality that the group is generated
by the support. If $A$ is finite-dimensional, this assumption implies that $G$ is finitely
generated.

Definition 1.2. Two gradings $A = \bigoplus_{g \in G} A_g$ and $A' = \bigoplus_{h \in G} A'_h$ of an algebra $A$ are
called equivalent if there exist $\Psi \in \text{Aut}(A)$ and $\theta \in \text{Aut}(G)$ such that
$\Psi(A_g) = A'_{\theta(g)}$ for all $g \in G$. If $\theta$ is the identity, we call the gradings isomorphic.

Definition 1.3. Let $A = \bigoplus_{g \in G} A_g$ be a grading by a group $G$ on an algebra $A$ and
$\varphi$ a group homomorphism of $G$ onto $H$. The coarsening of the $G$-grading induced
by $\varphi$ is the $H$-grading defined by $A = \bigoplus_{h \in H} A_h$ where

$$A_h = \bigoplus_{g \in G, \varphi(g) = h} A_g.$$

The task of finding all gradings on simple Lie algebras by abelian groups in the
case of algebraically closed fields of characteristic zero is almost complete — see [9]
and also [3, 4, 5, 6, 7, 8, 10]. In the case of positive characteristic, a description
of gradings on the classical simple Lie algebras, with certain exceptions, has been
obtained in [11, 2]. In the case of simple Cartan type Lie algebras, the gradings by $\mathbb{Z}$
have been described in [13]. All of them, up to isomorphism, fall into the category of
what we call standard gradings (which are coarsenings of the canonical $\mathbb{Z}_k$-gradings)
— see Definitions [2,7] and [2,7]. This paper will deal with gradings on the simple
graded Cartan type Lie algebras by arbitrary abelian groups without $p$-torsion
in the case where the base field $F$ (which is always assumed to be algebraically closed) has characteristic $p > 2$. We restrict ourselves to the graded Cartan type Lie algebras (i.e., those that have canonical $\mathbb{Z}$-gradings).

We use the notation of [13], which is our standard reference for the background on Cartan type Lie algebras.

Our main result is the following.

**Theorem 1.4.** Let $L$ be a simple graded Cartan type Lie algebra over an algebraically closed field of characteristic $p > 2$. If $p = 3$, assume that $L$ is not isomorphic to $W(1; \underline{1})$ or $H(2; (1, n_2))^{(2)}$. Suppose $L$ is graded by a group $G$ without elements of order $p$. Then the grading is isomorphic to a standard $G$-grading.

The correspondence between the gradings on an algebra by finite abelian groups of order coprime to $p$ and finite abelian subgroups of automorphisms of this algebra is well known. Using the theory of algebraic groups, this extends to infinite abelian groups. Namely, a grading on an algebra $L = \bigoplus_{g \in G} L_g$ by a finitely generated abelian group without elements of order $p$ gives rise to an embedding of the dual group $\hat{G}$ into $\text{Aut} L$ using the following action:

$$\chi \ast y = \chi(g)y, \quad \text{for all } y \in L_g, \quad g \in G, \quad \chi \in \hat{G}.$$ 

We will denote this embedding by $\eta : \hat{G} \to \text{Aut} L$, so

$$\eta(\chi)(y) = \chi \ast y.$$ 

If $L$ is finite-dimensional, then $\text{Aut} L$ is an algebraic group, and the image $\eta(\hat{G})$ belongs to the class of algebraic groups called quasi-tori. Recall that a quasi-torus is an algebraic group that is abelian and that consists of semisimple elements. Conversely, given a quasi-torus $Q$ in $\text{Aut} L$, we obtain the eigenspace decomposition of $L$ with respect to $Q$, which is a grading by the group of characters of $Q$, $G = X(Q)$.

In this paper, $L$ is a simple graded Cartan type Lie algebra, i.e., one of the following algebras: $W(m; \underline{1})$, $S(m; \underline{1})^{(1)}$, $H(m; \underline{1})^{(2)}$, $K(m; \underline{1})^{(1)}$ where $m$ is a positive integer and $n = (n_1, \ldots, n_m)$ is an $m$-tuple of positive integers — see the definitions in the next section. We will denote the type of $L$ by $X(m; \underline{n})^{(\infty)}$. The automorphism group of $L$ can be regarded as a subgroup of $\text{Aut}_c O(m; \underline{n})$, the group of continuous automorphisms of the commutative divided power algebra $O(m; \underline{n})$. Since $\eta(\hat{G})$ is a quasi-torus, a result by Platonov [12] tells us that $\eta(\hat{G})$ is contained in the normalizer of a maximal torus. Starting from this result, we show that, in fact, $\eta(\hat{G})$ is conjugate to a subgroup of the standard maximal torus $T_X$ (specific for each type $X(m; \underline{n})^{(\infty)}$ of simple graded Cartan type Lie algebra), which is responsible for the standard $\mathbb{Z}_k$-grading on $L$ where $k$ depends on the type $X$.

Unless it is stated otherwise, we denote by $a$ and $b$ some $m$-tuples of non-negative integers and by $i, j, k, l, q, r$ some integers. Any subset denoted by a calligraphic letter, for example $W(m; \underline{a})$, is a subset of $\text{Aut}_c O(m; \underline{n})$.

2. **Cartan Type Lie Algebras and Their Standard Gradings**

In this section we introduce some basic definitions, closely following [13, Chapter 2]. We start by defining the graded Cartan type Lie algebras $W(m; \underline{1})$, $S(m; \underline{1})^{(1)}$, $H(m; \underline{1})^{(2)}$, $K(m; \underline{1})^{(1)}$. 


Definition 2.1. Let $O(m;\underline{n})$ be the commutative algebra

$$O(m;\underline{n}) := \left\{ \sum_{0 \leq a \leq \tau(\underline{n})} \alpha(a)x^{(a)} \mid \alpha(a) \in F \right\}$$

over a field $F$ of characteristic $p$, where $\tau(\underline{n}) = (p^n - 1, \ldots, p^n - 1)$, with multiplication

$$x^{(a)}x^{(b)} = \left( \frac{a + b}{a} \right)x^{(a+b)},$$

where $\frac{a + b}{a} = \prod_{i=1}^{m} \left( \frac{a_i + b_i}{a_i} \right)$.

For $1 \leq i \leq m$, let $\epsilon_i := (0, \ldots, 0, 1, 0 \ldots, 0)$, where the 1 is at the $i$-th position, and let $x_i := x^{(\epsilon_i)}$.

There are standard derivations on $O(m;\underline{n})$ defined by $\partial_i(x^{(a)}) = x^{(a - \epsilon_i)}$ for $1 \leq i \leq m$.

Definition 2.2. Let $W(m;\underline{n})$ be the Lie algebra

$$W(m;\underline{n}) := \left\{ \sum_{1 \leq i \leq m} f_i \partial_i \mid f_i \in O(m;\underline{n}) \right\}$$

with the commutator defined by

$$[f \partial_i, g \partial_j] = f(\partial_i g) \partial_j - g(\partial_j f) \partial_i, \quad f, g \in O(m;\underline{n}).$$

The Lie algebras $W(m;\underline{n})$ are called Witt algebras. $W(m;\underline{n})$ is a subalgebra of Der $O(m;\underline{n})$, the Lie algebra of derivations of $O(m;\underline{n})$.

The remaining graded Cartan type Lie algebras are subalgebras of $W(m;\underline{n})$. When dealing with Hamiltonian and contact algebras in $m$ variables (types $H(m;\underline{n})$ and $K(m;\underline{n})$ below), it is useful to introduce the following notation:

$$i' = \begin{cases} i + r, & \text{if } 1 \leq i \leq r \\ i - r, & \text{if } r + 1 \leq i \leq 2r, \end{cases}$$

$$\sigma(i) = \begin{cases} 1, & \text{if } 1 \leq i \leq r \\ -1, & \text{if } r + 1 \leq i \leq 2r, \end{cases}$$

where $m = 2r$ in the case of $H(m;\underline{n})$ and $2r + 1$ in the case of $K(m;\underline{n})$. Note that we do not define $m'$ or $\sigma(m)$ if $m = 2r + 1$. We will also need the following differential forms --- see [13 Section 4.2].

$$\omega_S := dx_1 \wedge \cdots \wedge dx_m, \quad m \geq 3,$$

$$\omega_H := \sum_{i=1}^{r} dx_i \wedge dx_{i'}, \quad m = 2r,$$

$$\omega_K := dx_m + \sum_{i=1}^{2r} \sigma(i)x_i dx_{i'}, \quad m = 2r + 1.$$

Definition 2.3. We define the special, Hamiltonian and contact algebras as follows:

$$S(m;\underline{n}) := \{ D \in W(m;\underline{n}) \mid D(\omega_S) = 0 \}, \quad m \geq 3,$$

$$H(m;\underline{n}) := \{ D \in W(m;\underline{n}) \mid D(\omega_H) = 0 \}, \quad m = 2r,$$

$$K(m;\underline{n}) := \{ D \in W(m;\underline{n}) \mid D(\omega_K) \in O(m;\underline{n})\omega_K \}, \quad m = 2r + 1.$$
Conversely, if $Q$ is a quasi-torus in $T_W$, it defines a standard grading on $L$ by $G = \chi(Q)$, the group of characters of $Q$.

The algebras of Definitions 2.2 and 2.3 as well as their derived subalgebras, are collectively referred to as \textit{graded Cartan type Lie algebras}.

It is known that the Lie algebras $W(m; \underline{n})$ are simple, but $S(m; \underline{n})$ and $H(m; \underline{n})$ are not simple, and $K(m; \underline{n})$ are simple if and only if $p$ divides $m + 3$. The first derived algebras $S(m; \underline{n})^{(1)}$ and $K(m; \underline{n})^{(1)}$, and second derived algebras $H(m; \underline{n})^{(2)}$ are simple. The Cartan type Lie algebras defined above are called graded, because they have a canonical $\mathbb{Z}$-grading, defined by declaring

$$\text{deg}(x(a)\partial_j) = a_1 + \cdots + a_m - 1$$

for types $W(m; \underline{n})$, $S(m; \underline{n})$, $H(m; \underline{n})$, and $K(m; \underline{n})$. The $\mathbb{Z}$-grading on $W(m; \underline{n})$ is a coarsening of the following $\mathbb{Z}^m$-grading.

\textbf{Definition 2.4.} The $\mathbb{Z}^m$-gradings on $O(m; \underline{n})$ and $W(m; \underline{n})$,

$$O(m; \underline{n}) = \bigoplus_{a \in \mathbb{Z}^m} O(m; \underline{n})_a,$$

$$W(m; \underline{n}) = \bigoplus_{a \in \mathbb{Z}^m} W(m; \underline{n})_a,$$

where

$$O(m; \underline{n})_a = \text{Span}\{x(a)\},$$

$$W(m; \underline{n})_a = \text{Span}\{x^{(a+\epsilon_k)}\partial_k \mid 1 \leq k \leq m\},$$

are called the \textit{canonical $\mathbb{Z}^m$-gradings on $O(m; \underline{n})$ and $W(m; \underline{n})$, respectively.}

Note that in the above grading on $W(m; \underline{n})$ the support includes tuples with negative entries. For example $W(m; \underline{n})_{-\epsilon_i} = \text{Span}\{\partial_i\}$. The algebras $S(m; \underline{n})$ and $S(m; \underline{n})^{(1)}$ are graded subspaces in the canonical $\mathbb{Z}^m$-grading on $W(m; \underline{n})$, so they inherit the \textit{canonical $\mathbb{Z}^m$-grading}.

\textbf{Definition 2.5.} Let $G$ be an abelian group, $L = W(m; \underline{n})$ or $S(m; \underline{n})^{(1)}$, and $\varphi$ a homomorphism $\mathbb{Z}^m \to G$. The decompositions $O(m; \underline{n}) = \bigoplus_{g \in G} O_g$, $L = \bigoplus_{g \in G} L_g$, given by

$$O_g = \text{Span}\{x(a) \mid \varphi(a) = g\},$$

$$L_g = \text{Span}\{x^{(a)}\partial_k \mid 1 \leq k \leq m, \varphi(a - \epsilon_k) = g\} \cap L,$$

are $G$-gradings on $O(m; \underline{n})$ and $L$, respectively. We call them the \textit{standard $G$-gradings induced by $\varphi$ on $O(m; \underline{n})$ and $L$, respectively.} We will refer to the standard $G$-grading induced by any $\varphi$ as a \textit{standard $G$-grading} when $\varphi$ is not specified.

Let $L = W(m; \underline{n})$ and let $L = \bigoplus_{g \in G} L_g$ be the standard $\mathbb{Z}^m$-grading induced by $\varphi$. Let $\varphi(\epsilon_i) = g_i \in G$. The corresponding action of $\widehat{G}$ on $L$ is defined by

$$\chi \ast (x(a)\partial_i) = \chi(\varphi(a - \epsilon_i))x(a)\partial_i = \chi(g_1)^{a_1} \cdots \chi(g_m)^{a_m} \chi(g_i)^{-1}x(a)\partial_i,$$

for all $\chi \in \widehat{G}$. Hence $\eta(\widehat{G})$ is a subgroup of the torus $T_W$.

$$T_W := \{\Psi \in \text{Aut}W(m; \underline{n}) \mid \Psi(x(a)\partial_k) = t_1^{a_1} \cdots t_m^{a_m} t_k^{-1}x(a)\partial_k, t_j \in F^\times\}.$$
In particular, the standard \( \mathbb{Z}^m \)-grading on \( W(m; n) \) corresponds to \( Q = T_{W} \). Note that \( T_{W} \) preserves the subalgebra \( S(m; n)^{(1)} \), and the restriction of \( T_{W} \) to \( S(m; n)^{(1)} \) is an isomorphic torus in \( Aut \ S(m; n)^{(1)} \).

**Lemma 2.6.** [3] Section 7.4] The following are maximal tori of \( Aut \ W(m; n) \), \( Aut \ S(m; n)^{(1)} \), \( Aut \ H(m; n)^{(2)} \) and \( Aut \ K(m; n)^{(1)} \), respectively:

\[
T_{W} = T_{S} = \{ \Psi \in Aut \ W(m; n) \mid \Psi(x^{(a)} \partial_i) = t_{i}^{a_{1}} \cdots t_{m}^{a_{m}} t_{i}^{-1} x^{(a)} \partial_i, \ t_{i} \in F^{\times} \},
\]

\[
T_{H} = \{ \Psi \in Aut \ W(m; n) \mid \Psi(x^{(a)} \partial_i) = t_{i}^{a_{1}} \cdots t_{m}^{a_{m}} t_{i}^{-1} x^{(a)} \partial_i, \ t_{i} \in F^{\times}, \ t_{i} t_{j} = t_{j} t_{i} \ \text{and} \ \ t_{i} t_{j} = t_{m}, \ 1 \leq i, j \leq r \},
\]

\[
T_{K} = \{ \Psi \in Aut \ W(m; n) \mid \Psi(x^{(a)} \partial_i) = t_{i}^{a_{1}} \cdots t_{m}^{a_{m}} t_{i}^{-1} x^{(a)} \partial_i, \ t_{i} \in F^{\times}, \ t_{i} t_{j} = t_{j} t_{i} \ \text{and} \ \ t_{i} t_{j} = t_{m}, \ 1 \leq i, j \leq r \}.
\]

\[\Box\]

We are now ready to define standard \( G \)-gradings on \( H(m; n)^{(2)} \) and \( K(m; n)^{(1)} \). They are obtained by coarsening the canonical \( \mathbb{Z}^m \)-grading on \( W(m; n) \). The homomorphism \( \varphi : \mathbb{Z}^m \to G \) must satisfy certain conditions in order for \( \eta(\hat{G}) \) to preserve \( H(m; n)^{(2)} \) and \( K(m; n)^{(1)} \), respectively.

The **canonical \( \mathbb{Z}^{r+1} \)-grading on** \( H(2r; n) \) is the restriction of a coarsening of the \( \mathbb{Z}^m \)-grading on \( W(2r; n) \), defined as follows. Let \( m = 2r \) and let

\[
\phi_{H} : \mathbb{Z}^m / \langle \varepsilon_{i} + \varepsilon_{j} = \varepsilon_{i} + \varepsilon_{j} \mid 1 \leq i < j \leq r \rangle
\]

be the quotient map. Then the coarsening of the canonical \( \mathbb{Z}^m \)-grading on \( W(m; n) \) induced by \( \phi_{H} \) is a \( \mathbb{Z}^{r+1} \)-grading, which restricts to the subalgebras \( H(m; n) \), \( H(m; n)^{(1)} \) and \( H(m; n)^{(2)} \).

Similarly, the **canonical \( \mathbb{Z}^{r+1} \)-grading on** \( K(2r + 1; n) \) is the restriction of a coarsening of the \( \mathbb{Z}^m \)-grading on \( W(2r + 1; n) \), defined as follows. Let \( m = 2r + 1 \) and

\[
\phi_{K} : \mathbb{Z}^m / \langle \varepsilon_{i} + \varepsilon_{j} = \varepsilon_{i} + \varepsilon_{j} \mid 1 \leq i \leq r \rangle
\]

be the quotient map. Then the coarsening of the canonical \( \mathbb{Z}^m \)-grading on \( W(m; n) \) induced by \( \phi_{K} \) is a \( \mathbb{Z}^{r+1} \)-grading, which restricts to the subalgebras \( K(m; n) \) and \( K(m; n)^{(1)} \).

**Definition 2.7.** Let \( L = H(m; n)^{(2)} \) or \( K(m; n)^{(1)} \) and \( X = H \) or \( K \), respectively. Let \( G \) be an abelian group, and \( \theta \) a group homomorphism from \( \varphi_{X}(\mathbb{Z}^m) \) to \( G \). The decomposition \( L = \bigoplus_{g \in G} L_{g} \), given by

\[
L_{g} = \text{Span}\{ x^{(a)} \partial_{k} \mid 1 \leq k \leq m, \ \theta \varphi_{X}(a - \varepsilon_{k}) = g \} \cap L,
\]

is a \( G \)-grading on \( L \). We call it the **standard \( G \)-grading on** \( L \) **induced by** \( \theta \), or just a standard \( G \)-grading on \( L \) if \( \theta \) is not specified.

We can summarize the above discussion as follows.

**Lemma 2.8.** Let \( L = W(m; n), S(m; n)^{(1)}, H(m; n)^{(2)} \) or \( K(m; n)^{(1)} \). A grading by a group \( G \) on \( L \) is a standard \( G \)-grading if and only if we have \( \eta(\hat{G}) \subset T_{X} \) where \( X = W, S, H \) or \( K \), respectively. \[\Box\]

The goal of Section 4 is to show that if \( G \) has no elements of order \( p \) then \( \eta(\hat{G}) \) is always contained in a maximal torus. Hence, a conjugate of \( \eta(\hat{G}) \) will be contained
Theorem 3.2. \cite{13, Theorem 7.3.2} the case of $H$ to a subgroup of the automorphism group of $O$. The above groups are subgroups of $W$.

Let Definition 3.1. $T$ (isomorphism of groups provided that $O$ is the automorphism groups of the Cartan type Lie algebras as subgroups of the automorphism group of $W$.

We start by introducing so-called continuous automorphisms of $O(m; n)$. Definition 3.1. Let $A(m; n)$ be the set of all $m$-tuples $(y_1, \ldots, y_m) \in O(m; n)^m$ for which $\det(\partial_i(y_j))_{1 \leq i, j \leq m}$ is invertible in $O(m; n)$ and also

$$y_i = \sum_{0 < a \leq \tau(n)} \alpha_i(a)x^{(a)} \quad \text{with} \quad \alpha_i(y^i e_j) = 0 \text{ if } n_i + l > n_j.$$ 

The group of continuous automorphisms of $O(m; n)$, $\text{Aut}_c O(m; n)$, is defined as follows \cite[Theorem 6.32]{13}. For any $(y_1, \ldots, y_m) \in A(m; n)$ we define a map $O(m; n) \to O(m; n)$ by setting

$$\varphi \left( \sum_{0 \leq a \leq \tau(n)} \alpha(a)x^{(a)} \right) = \sum_{0 \leq a \leq \tau(n)} \alpha(a) \prod_{i=1}^m y_i^{(a_i)},$$

where $y^{(q)}$ denotes the $q$th divided power of $y$ \cite[Chapter 2]{13}.

We have a map $\Phi$ from $\text{Aut}_c O(m; n)$ to $\text{Aut} W(m; n)$ defined by

$$\Phi(\psi) \left( \sum_{1 \leq i \leq m} f_i \partial_i \right) = \psi \circ \left( \sum_{1 \leq i \leq m} f_i \partial_i \right) \circ \psi^{-1},$$

where the elements of $W(m; n)$ are viewed as derivations of $O(m; n)$.

**Theorem 3.2.** \cite[Theorem 7.3.2]{13} The map $\Phi : \text{Aut}_c O(m; n) \to W(m; n)$ is an isomorphism of groups provided that $(m; n) \neq (1, 1)$ if $p = 3$. Also, except for the case of $H(m; n)^{(2)}$ with $m = 2$ and $\min(n_1, n_2) = 1$ if $p = 3$,

- $\text{Aut} S(m; n)^{(1)} = \Phi(\{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_S) \in F^\times \omega_S\}),$
- $\text{Aut} H(m; n)^{(2)} = \Phi(\{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_H) \in F^\times \omega_H\}),$
- $\text{Aut} K(m; n)^{(1)} = \Phi(\{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_K) \in O(m; n)^{\times} \omega_K\}).$

**Remark 3.3.** The map of the tangent Lie algebras corresponding to $\Phi$ is a restriction of $\text{ad} : W \to \text{Der} W$, and hence injective. It follows that $\Phi$ is an isomorphism of algebraic groups.

We will use the following notation

- $S(m; n) = \{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_S) \in F^\times \omega_S\},$
- $H(m; n) = \{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_H) \in F^\times \omega_H\},$
- $K(m; n) = \{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_K) \in O(m; n)^{\times} \omega_K\}.$

The above groups are subgroups of $W(m; n) := \text{Aut}_c O(m; n)$. We will refer to them collectively by $\mathcal{X}(m; n)$ where $\mathcal{X} = W, S, H$ or $K$. 

4. Gradings by groups without \( p \)-torsion

As mentioned in Section 2, to prove our Theorem 1.4 we need to show that \( \eta(G) \) is conjugate in \( \text{Aut} X(m; \underline{n})(\infty) \) to a subgroup of the maximal torus \( TX \) where \( X = W, S, H, K \). We are going to use an important general result following from [12, Corollary 3.28].

**Proposition 4.1.** A quasi-torus of an algebraic group is contained in the normalizer of a maximal torus.

This brings us to the necessity of looking at normalizers of maximal tori in \( \text{Aut} X(m; \underline{n})(\infty) \). Using the isomorphism \( \Phi \) described in Section 3, we are going to work inside the groups \( W(m; \underline{n}) = \text{Aut}_c O(m; \underline{n}) \).

### 4.1. Normalizers of maximal tori.

We denote by \( \text{Aut}_0 O(m; \underline{n}) \) the subgroup of \( \text{Aut}_c O(m; \underline{n}) \) consisting of all \( \psi \) such that \( \psi(x_i) = \sum_{j=1}^{m} \alpha_{i,j} x_j, \alpha_{i,j} \in F, 1 \leq j \leq m \).

The group \( \text{Aut}_0 O(m; \underline{n}) \) is canonically isomorphic to a subgroup of \( \text{GL}(m) \), which we denote by \( \text{GL}(m; \underline{n}) \). If \( n_i = n_j \) for \( 1 \leq i, j \leq m \) then \( \text{GL}(m; \underline{n}) = \text{GL}(m) \), otherwise \( \text{GL}(m; \underline{n}) \neq \text{GL}(m) \). The condition for a tuple \((y_1, \ldots, y_n)\) to be in \( A(m; \underline{n}) \),

\[
y_i = \sum_{0 < a} \alpha_{i}(a)x^{(a)} \quad \text{with} \quad \alpha_{i}(p' \epsilon_j) = 0 \quad \text{if} \quad n_i + l > n_j,
\]

imposes a flag structure on the vector space \( V = \text{Span}\{x_1, \ldots, x_m\} \).

**Definition 4.2.** Given \( \underline{n} = (n_1, \ldots, n_m) \), with \( m > 0 \), we set \( \Xi_0 = \emptyset \) and then, inductively,

\[
\Xi_i = \Xi_{i-1} \cup \{ j \mid n_j = \max_{k \in \Xi_{i-1}} \{n_k\} \}.
\]

Set \( V_i = \text{Span}\{x_j \mid j \in \Xi_i\} \) for \( i \geq 0 \). Then \( 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V \) is a flag in \( V \) (i.e., an ascending chain of subspaces). We denote this flag by \( \mathcal{F}(m; \underline{n}) \) and say that an automorphism \( \psi \) of \( O(m; \underline{n}) \) respects \( \mathcal{F}(m; \underline{n}) \) if \( \psi(V_i) = V_i \) for all \( i \).

Condition (1) implies that \( \text{GL}(m; \underline{n}) \) consists of all elements of \( \text{GL}(m) \) that respect \( \mathcal{F}(m; \underline{n}) \).

According to [13, Section 7.3], \( \text{Aut}_0 O(m; \underline{n}) \cap \mathcal{S}(m; \underline{n}) = \text{Aut}_0 O(m; \underline{n}) \), i.e., in the case of special algebras we have to deal with the same subgroup of \( \text{GL}(m) \) as in the case of Witt algebras.

In the Hamiltonian case, \( V = \text{Span}\{x_1, \ldots, x_r\} \oplus \text{Span}\{x_{r+1}, \ldots, x_{2r}\} \), and \( \omega_H \) induces a nondegenerate skew-symmetric form on \( V \), given by \( \langle x_i, x_j \rangle = \delta_{i,j} \sigma(i) \delta_{i,j'} \), for all \( i, j = 1, \ldots, 2r \). The image of \( \text{Aut}_0 O(m; \underline{n}) \cap \mathcal{H}(m; \underline{n}) \) in \( \text{GL}(m) \), \( m = 2r \), is the product of the subgroup of scalar matrices and the subgroup \( \text{Sp}(m; \underline{n}) := \text{Sp}(m) \cap \text{GL}(m; \underline{n}) \). This product is almost direct: the intersection is \{±Id\}.

The maximal tori \( TX \) in \( \text{Aut} X(m; \underline{n})(\infty) \) described in Lemma 2.6 correspond, under the algebraic group isomorphism \( \Phi \), to the following maximal tori in \( \mathcal{X}(m; \underline{n}) \):

- \( T_W = \{ \psi \in W(m; \underline{n}) \mid \psi(x_i) = t_i x_i, t_i \in F^\times \} \),
- \( T_H = \{ \psi \in W(m; \underline{n}) \mid \psi(x_i) = t_i x_i, t_i \in F^\times, t_i t_{j'} = t_j t_{j'} \} \),
- \( T_K = \{ \psi \in W(m; \underline{n}) \mid \psi(x_i) = t_i x_i, t_i \in F^\times, t_i t_{j'} = t_j t_{j'} = t_m, 1 \leq i, j \leq r \} \).
A convenient way to view the elements of the above tori is to view them as m-tuples of nonzero scalars. Define \( \lambda : (F^x)^m \to \text{Aut}_0 O(m; n) \) where \( \lambda(t)(x_i) = t_i x_i \) for \( 1 \leq i \leq m \). Then \( \lambda((F^x)^m) = T_W \).

**Definition 4.3.** We will say that \( t \in (F^x)^m \) is \( X \)-admissible if \( \lambda(t) \in T_X \), where \( X = W, S, H \) or \( K \).

An important subgroup which we use for the description of the normalizer \( N_{X(m; n)}(T_X) \) for \( X = W, S, H \) or \( K \), is the subgroup \( \mathcal{M}(m; n) \) of \( \mathcal{W}(m; n) \).

**Definition 4.4.** Let \( \mathcal{M}(m; n) \) be the subgroup of \( \mathcal{W}(m; n) \) that consists of \( \psi \) such that, for each \( 1 \leq i \leq m \), we have \( \psi(x_i) = \alpha_i x_{j_i} \), where \( \alpha_i \in F^x \) and \( 1 \leq j_i \leq m \).

Thus \( \mathcal{M}(m; n) \subset \text{Aut}_0 O(m; n) \) is isomorphic to the group of monomial matrices that respect the flag.

**Lemma 4.5.** The subgroups \( N_{W(m; n)}(T_W), N_{S(m; n)}(T_S) \) and \( N_{H(m; n)}(T_H) \) are contained in \( \mathcal{M}(m; n) \).

**Proof.** We will show that \( N_{W(m; n)}(T_X) \subset \mathcal{M}(m; n) \) for \( X = W, S, H \). Since \( X(m; n) \subset W(m; n) \) we have \( N_X(T_X) \subset N_{W(m; n)}(T_X) \).

Let \( \psi \in N_{W(m; n)}(T_X) \). For any \( 1 \leq i \leq m \) the element \( x_i \) is a common eigenvector of \( T_X \) so \( \psi(x_i) \) is also a common eigenvector of \( T_X \). Also, since \( \psi \in \text{Aut}_0 O(m; n) \), \( \psi(x_i) = \sum_{0 < a \leq \tau(n)} \alpha_i(a)x^{(a)} \) where, among other conditions, \( \alpha_i(\epsilon_{j_i}) \neq 0 \) for some \( 1 \leq j_i \leq m \).

First we consider the case \( X = W \) and \( X = S \) (recall that \( T_W = T_S \)).

It is easy to see that the eigenspace decomposition of \( O(m; n) \) with respect to \( T_W \) is the canonical \( \mathbb{Z}^m \)-grading on \( O(m; n) \). The homogeneous space \( O_a = \text{Span}\{x^{(a)}\} \) is the eigenspace with eigenvalue \( \sum_{0 < a \leq \tau(n)} \alpha_i(a)x^{(a)} \) with respect to \( \lambda(t) \in T_W \). It follows that \( \psi(x_i) \in O_a \) for some \( 0 \leq a \leq \tau(n) \) since \( \psi(x_i) \) is an eigenvector of \( T_X \). The condition that \( \alpha_i(\epsilon_{j_i}) \neq 0 \) for some \( 1 \leq j_i \leq m \) forces \( a = \epsilon_{j_i} \). Hence \( \psi \in \mathcal{M}(m; n) \).

We continue with the case of \( X = H \). The torus \( T_H \) is contained in \( T_W \). In order for \( \lambda(t) \in T_W \) to belong to \( T_H \), the \( m \)-tuple \( t \) must be \( H \)-admissible, i.e., satisfy \( t_i t_j = t_j t_i \) for \( 1 \leq i, j \leq r \) where \( m = 2r \). The eigenspace decomposition of \( O(m; n) \) with respect to \( T_H \) is a coarsening of the canonical \( \mathbb{Z}^m \)-grading. The eigenspace with eigenvalue \( t^a \) with respect to \( \lambda(t) \in T_H \) is \( Q_a := \bigoplus O_b \) where the direct sum is over the set of all \( b \) such that \( t^a = t^b \) for all \( H \)-admissible \( t \). If \( O_b \neq 0 \) and \( t_k = t^b \) for all \( H \)-admissible \( t \), then \( b = \epsilon_k \) since all the entries in the \( m \)-tuple \( b \) are non-negative. This implies that \( Q_{\epsilon_k} = \text{Span}\{x_k\} \). Now \( \psi(x_i) \in Q_{\epsilon_k} \) for some \( 0 \leq a \leq \tau(n) \) since \( \psi(x_i) \) is an eigenvector of \( T_H \). The condition that \( \alpha_i(\epsilon_{j_i}) \neq 0 \) for some \( 1 \leq j_i \leq 2r \) again forces \( a = \epsilon_{j_i} \). Hence \( \psi \in \mathcal{M}(m; n) \). \( \square \)

For the case of contact algebras, similar arguments do not give us that \( N_{W(m; n)}(T_K) \) is in \( \mathcal{M}(m; n) \).

**Lemma 4.6.** If \( \psi \in N_{K(m; n)}(T_K) \), \( m = 2r + 1 \) then

\[
\psi(x_m) = \alpha_m(\epsilon_m)x_m + \sum_{l=1}^{r} \alpha_m(\epsilon_l + \epsilon_{r'})x_l x_{l'}
\]

and, for \( 1 \leq i \leq 2r \), we have \( \psi(x_i) = \alpha_i(\epsilon_{j_i})x_{j_i} \) where \( 1 \leq j_i \leq 2r \).
Proof. We will prove that any \( \psi \in N_{W(m;\underline{n})}(T_K) \) has the form above. Since \( x_i \) is a common eigenvector of \( T_K \) we have that \( \psi(x_i) \) is also a common eigenvector of \( T_K \). Since \( \psi \in \text{Aut}_c O(m;\underline{n}) \), we have \( \psi(x_i) = \sum_{0 < a \leq \tau(\underline{n})} \alpha_i(a)x^{(a)} \) where, among other conditions, \( \alpha_i(e_{j_i}) \neq 0 \) for some \( 1 \leq j_i \leq m \).

The torus \( T_K \) is contained in \( T_W \). In order for a \( \lambda(\underline{t}) \) to be in \( T_K \), the \( m \)-tuple \( \underline{t} \) must be \( K \)-admissible, i.e., \( t_i t_{i'} = t_m \) for \( 1 \leq i \leq r \). The eigenspace decomposition of \( O(m;\underline{n}) \) with respect to \( T_K \) is a coarsening of the canonical \( \mathbb{Z}^m \)-grading. The eigenspace with eigenvalue \( t^\alpha \) with respect to \( \lambda(\underline{t}) \) in \( T_K \) is \( R_{a} = \bigoplus O_b \) where the direct sum is over the set of \( b \) such that \( x_i \) has the form above. Since \( O_b \) is \( K \)-admissible \( t_i \) for \( 1 \leq k \leq 2r \), then \( b = \varepsilon_k \) since all entries of the \( m \)-tuple \( b \) are non-negative. This implies that \( R_{\varepsilon_k} = \text{Span}(x_i) \) for \( 1 \leq k \leq 2r \). Also, \( R_{\varepsilon_m} = \bigoplus \text{Span}(x_m, x_ix_{i'} | 1 \leq i \leq r) \). Now \( \psi(x_i) \in R_{a} \) for some \( 0 \leq a \leq \tau(\underline{n}) \). The condition that \( \alpha_i(e_{j_i}) \neq 0 \) for some \( 1 \leq j_i \leq m \) forces \( a = \varepsilon_j \). Note that the dimension of \( R_{\varepsilon_i} \) is 1 for \( 1 \leq i \leq 2r \) and the dimension of \( R_{\varepsilon_m} \) is \( r + 1 \). Hence, for \( 1 \leq i \leq 2r \), we have \( \psi(x_i) = \alpha_i(e_{j_i})x_{j_i} \) for some \( 1 \leq j_i \leq 2r \). Also, \( \psi(x_m) \in R_{\varepsilon_m} \) which implies (2).  

4.2. Diagonalization of quasi-tori. In this subsection we prove that any quasi-torus in \( X(m;\underline{n})^{(\infty)} \) is conjugate to a subgroup of the maximal torus \( T_X \), where \( X \) is \( W, S, H \) or \( K \). As before we pass from \( \text{Aut} X(m;\underline{n})^{(\infty)} \) to \( X(m;\underline{n}) \) by virtue of the isomorphism \( \Phi \) described in Section 3.

Proposition 4.7. Let \( Q \) be a quasi-torus contained in \( N_{X(m;\underline{n})}(T_X) \) where \( X = W \) or \( S \). Then there exists \( \psi \in X(m;\underline{n}) \) such that \( \psi Q \psi^{-1} \subset T_X \).

Proof. Recall the flag \( F(m;\underline{n}) \),

\[
V_0 \subset V_1 \subset \cdots \subset V.
\]

Let \( U_i = \text{Span}(x_j | x_j \in V_i, x_j \notin V_{i-1}) \). Then \( V_i = \bigoplus_{j=1}^i U_j \). Since \( Q \subset M(m;\underline{n}) \) we have \( Q(U_i) = U_i \). (Here, as before, we identify \( \text{Aut}_0 O(m;\underline{n}) \) with a subgroup of \( \text{GL}(m) \).)

Restricting the action of \( Q \) to \( U_i \), we obtain a subgroup of \( \text{GL}(U_i) \). Since \( Q|_{U_i} \) is a quasi-torus, there exists a \( P_i \in \text{GL}(U_i) \) such that \( P_i(Q|_{U_i})P_i^{-1} \) is diagonal. We can extend the action of \( P_i \) to the whole space \( V \) by setting \( P_i(y) = y \) for all \( y \in U_j, i \neq j \). These extended \( P_i \) respect \( F(m;\underline{n}) \). The product of these transformations, \( P = P_1 \cdots P_r \), is an element of \( \text{Aut}_0 O(m;\underline{n}) \), which diagonalizes \( Q \).  

In order to obtain the analog of Proposition 4.7 in the case of Hamiltonian algebras, we consider the canonical skew-symmetric inner product \( \langle \cdot, \cdot \rangle \) on \( V \) given by \( \langle x_j, x_k \rangle = \sigma(j)\delta_{j,k}, \) for all \( j, k = 1, \ldots, 2r \).

Lemma 4.8. Let \( Q \) be a quasi-torus contained in \( \text{Sp}(m;\underline{n}) \). Then there is a basis \( \{e_i\}_{i=1}^{2r} \) of \( V \) such that \( \langle e_j, e_k \rangle = \sigma(j)\delta_{j,k}, \) all \( e_j \) are common eigenvectors of \( Q \), and \( V_i = \text{Span}(e_j | j \in \mathbb{Z}_i) \) for all \( i \).

Proof. We can decompose \( V = \bigoplus V^\gamma \) where \( V^\gamma \) are the eigenspaces in \( V \) with respect to \( Q \), indexed by \( \gamma \in \mathfrak{X}(Q) \) where \( \mathfrak{X}(Q) \) is the group of characters of \( Q \).
Since $Q(V_i) = V_i$ for any $i$, there is a basis $\{y_j\}_{j=1}^{2r}$ such that $y_j$ are eigenvectors of $Q$ and each $V_i = \text{Span}\{y_j \mid j \in \Xi_i\}$.

We have $\langle x_j, x_k \rangle = \sigma(j)\delta_{j,k}$. We will show by induction on $r$ that there is a basis $\{e_j\}_{j=1}^{2r}$ such that $\langle e_j, e_k \rangle = \sigma(j)\delta_{j,k}$, all $e_j$ are common eigenvectors of $Q$, and $V_i = \text{Span}\{e_j \mid j \in \Xi_i\}$. The base case $r = 1$ is obvious.

We have a basis $\{y_j\}_{j=1}^{2r}$ such that $y_j$ are eigenvectors of $Q$ and each $V_i = \text{Span}\{y_j \mid j \in \Xi_i\}$. We apply a process similar to the Gram–Schmidt process to find a new basis of common eigenvectors for $Q$ that satisfies the desired conditions.

Since $V_1 \neq 0$ and $\langle \cdot, \cdot \rangle$ is nondegenerate, $(V_1, V_1) \neq 0$ for some $l$. Let $l$ be minimal, i.e., $(V_1, V_1) \neq 0$ and $(V_1, V_i) = 0$ for $i < l$. So there exist $y_s, y_t \in V_1$ and $y_s, y_t \in V_l$ such that $(y_s, y_t) \neq 0$ and $(y_s, V_1) = 0 = (y_t, V_1)$ if $i < l$ by the minimality of $l$. Let $\gamma_j \in \mathfrak{x}(Q)$ be the eigenvalue of $y_j$. Since $Q$ consists of symplectic transformations, we have $\gamma_s = \gamma_t^{-1}$.

For $j \neq s, t$, let

$$z_j = \langle y_s, y_t \rangle y_j - \langle y_s, y_j \rangle y_t + \langle y_t, y_j \rangle y_s.$$

The $z_j$ with $y_s$ and $y_t$ form a basis of $V$. They also satisfy

$$\langle y_s, z_j \rangle = \langle y_s, y_t \rangle \langle y_s, y_j \rangle - \langle y_s, y_j \rangle \langle y_t, y_j \rangle + \langle y_t, y_j \rangle \langle y_s, y_s \rangle = 0,$$

and similarly $\langle y_t, z_j \rangle = 0$.

We also have the property that $z_j \in V_i$ if and only if $y_j \in V_i$. Indeed, for $i < l$ and $y_s, y_t \in V_i$, we have $\langle y_s, y_t \rangle = 0$ by the minimality of $l$. This shows that $z_j$ is in $V_1 + V_i$ and hence $z_j \in V_i$. For $i > l$ and $y_s, y_t \in V_i$ we have $y_s, y_t, y_s, y_t \in V_i$ which implies $z_j \in V_i$. Finally, we want to verify that $z_j$ are common eigenvectors of $Q$.

There are three cases to consider.

Case 1: $\gamma_j \neq \gamma_s^{\pm 1}$

Recall that $\gamma_s = \gamma_t^{-1}$. Since $\gamma_j \neq \gamma_s^{\pm 1}$, we have $\langle y_s, y_j \rangle = \langle y_t, y_j \rangle = 0$. This means $z_j = \langle y_s, y_t \rangle y_j$, which is an eigenvector with eigenvalue $\gamma_j$.

Case 2: $\gamma_j = \gamma_s$ or $\gamma_t^{-1}$, and $\gamma_s \neq \gamma_s^{-1}$.

Suppose $\gamma_j = \gamma_s$. Since $\gamma_j \neq \gamma_s^{-1}$, we have $\langle y_s, y_j \rangle = 0$. This means $z_j = \langle y_s, y_t \rangle y_j + \langle y_t, y_t \rangle y_s$, which is an eigenvector with eigenvalue $\gamma_j = \gamma_s$. A similar argument applies if $\gamma_j = \gamma_s^{-1}$.

Case 3: $\gamma_j = \gamma_s = \gamma_s^{-1}$.

Since $z_j = \langle y_s, y_t \rangle y_j - \langle y_s, y_j \rangle y_t + \langle y_t, y_j \rangle y_s$ and $\gamma_s = \gamma_s^{-1} = \gamma_t$, we see that $z_j$ is an eigenvector with eigenvalue $\gamma_j$.

In order to use the induction hypothesis we relabel our basis as follows: Pick $x_q \in V_1$ with $\langle x_q, V_i \rangle \neq 0$. Since $(V_1, V_{l-1}) = 0$, we have $x_q \in V_l$ and $x_q \notin V_{l-1}$. Set $w_q = y_s, w_q' = y_t, w_s = z_q, w_t = z_q'$ and $w_j = z_j$ for $j \neq q, q', s, t$. The relabelled basis still satisfies $V_i = \text{Span}\{w_j \mid j \in \Xi_i\}$ since $z_q', y_t \in V_1$, and $z_q, y_s \in V_1$. Also, $\langle w_q, w_j \rangle = \langle w_q', w_j \rangle = 0$ for $j \neq q, q', \langle w_q, w_q' \rangle \neq 0$, and $w_j$ are eigenvectors of $Q$.

Let $V' = \text{Span}\{w_j \mid 1 \leq j \leq 2r, j \neq q, q'\}$. Then $V'$ is invariant under $Q$, and the quasi-torus $Q|_{V'}$, satisfies the conditions of the lemma with the flag

$$V'_i = \text{Span}\{w_j \mid j \in \Xi_i \setminus \{q, q'\}\} = V_i \cap V'.$$

Since $\dim V' < \dim V$, we can apply the induction hypothesis and find a basis $\{e_j\}_{j \neq q, q', 1 \leq j \leq 2r}$ such that $\langle e_j, e_k \rangle = \sigma(j)\delta_{j,k}$, where $e_j$ are eigenvectors of $Q|_{V'}$, and $V'_i = \text{Span}\{e_j \mid j \in \Xi_i \setminus \{q, q'\}\}$.
In order to have a complete basis for $V$ we set $e_q = \frac{\sigma(q)}{\langle w_q, w_{q'} \rangle} w_q$ and $e_{q'} = w_{q'}$. Then the basis $\{e_j\}_{j=1}^{2r}$ is a basis with the desired properties, and the induction step is proven. \hfill \Box

In the following proofs we will deal with the differential forms $\omega_H$ and $\omega_K$.

**Proposition 4.9.** Let $Q$ be a quasi-torus contained in $N_{H(m;\mathfrak{n})}(T_H)$. Then there exists $\psi \in \text{Sp}(m;\mathfrak{n})$ such that $\psi Q \psi^{-1} \subset T_H$.

**Proof.** By Lemma 4.5, $N_{H(m;\mathfrak{n})}(T_H) \subset \mathcal{M}(m;\mathfrak{n})$, which we regard as a subgroup of $\text{GL}(m)$. Recall that any element of $\text{Aut}_0 O(m;\mathfrak{n}) \cap \mathcal{H}(m;\mathfrak{n})$ can be written as $\alpha S$ where $\alpha \in F^\times$ and $S \in \text{Sp}(m;\mathfrak{n})$. Let

$$Q' = \{S \in \text{Sp}(m;\mathfrak{n}) \mid \text{there exists } \alpha \in F^\times \text{ such that } \alpha S \in Q\}.$$  

$Q'$ is a quasi-torus since $Q$ is a quasi-torus.

Let $\{e_j\}_{j=1}^{m}$ be a basis as in Lemma 4.8 with respect to $Q'$, and define $\psi : V \to V$ by $\psi(x_j) = e_j$ for $1 \leq j \leq m$. Since $\langle \psi(x_j), \psi(x_k) \rangle = \langle e_j, e_k \rangle = \sigma(j)\delta_{j,k'} = \langle x_j, x_k \rangle$, we have $\psi \in \text{Sp}(m)$. Since $V_i = \text{Span}\{e_j \mid j \in \Xi_i\} = \text{Span}\{x_j \mid j \in \Xi_i\}$, we have $\psi(V_i) = V_i$. Hence $\psi \in \text{Sp}(m;\mathfrak{n})$. Since $e_j$ are common eigenvectors of $Q'$, we have $\psi^{-1}Q'\psi \subset T_H$. Since every element of $Q$ has the form $\alpha S$ with $\alpha \in F^\times$ and $S \in Q'$, we have $\psi^{-1}Q\psi \subset T_H$. Replacing $\psi$ with $\psi^{-1}$, we get the result. \hfill \Box

In order to get a similar result for the contact algebras, we use the Hamiltonian algebras contained in them. Let $m = 2r + 1$, $\mathfrak{n} = (n_1, \ldots, n_{2r+1})$ and $\mathfrak{n}' = (n_1, \ldots, n_{2r})$.

**Lemma 4.10.** Let $\psi \in \text{Aut}_0 O(2r;\mathfrak{n}')$. If $\psi(\omega_H) = \alpha \omega_H$ then there exists $\bar{\psi} \in \mathcal{K}(2r+1,\mathfrak{n})$ such that $\bar{\psi}|_{O(2r;\mathfrak{n}')} = \psi$ and $\bar{\psi}(x_{2r+1}) = \alpha x_{2r+1}$.

**Proof.** Suppose $\psi \in \text{Aut}_0 O(2r;\mathfrak{n})$ given by $\psi(x_i) = \sum_{j=1}^{2r} \alpha_{i,j} x_j$ has the property $\psi(\omega_H) = \alpha \omega_H$. Since

$$\psi(\omega_H) = \psi \left( \sum_{i=1}^{r} dx_i \wedge dx_{i+r} \right) = \sum_{i=1}^{r} d(\psi(x_i)) \wedge d(\psi(x_{i+r}))$$

$$= \sum_{i=1}^{r} d \left( \sum_{j=1}^{2r} \alpha_{i,j} x_j \right) \wedge d \left( \sum_{k=1}^{2r} \alpha_{i+r,k} x_k \right)$$

$$= \sum_{1 \leq j < k \leq 2r} \left( \sum_{i=1}^{r} \alpha_{i,j} \alpha_{i+r,k} - \alpha_{i,k} \alpha_{i+r,j} \right) dx_j \wedge dx_k$$

we have $\sum_{i=1}^{r} \alpha_{i,j} \alpha_{i+r,k} - \alpha_{i,k} \alpha_{i+r,j} = \delta_{k,j+r} \alpha$ for $1 \leq j \leq r$. 

Define $\overline{\psi} \in \text{Aut}_e O(2r + 1, \mathfrak{n})$ by setting $\overline{\psi}(x_i) = \psi(x_i)$, $1 \leq i \leq 2r$, and $\overline{\psi}(x_m) = \alpha x_m$. Then

$$\overline{\psi}(\omega_K) = \overline{\psi}(dx_m) + \overline{\psi}\left(\sum_{i=1}^{r} (x_i dx_{i+r} - x_{i+r} dx_i)\right)$$

$$= d(\alpha x_m) + \sum_{i=1}^{r} \left(\sum_{j=1}^{2r} \alpha_{i,j} x_j \right) d\left(\sum_{k=1}^{2r} \alpha_{i+r,k} x_k\right) - \left(\sum_{k=1}^{2r} \alpha_{i+r,k} x_k\right) d\left(\sum_{j=1}^{2r} \alpha_{i,j} x_j\right)$$

$$= \alpha dx_m + \sum_{1 \leq j < k \leq 2r} \left(\sum_{i=1}^{r} \alpha_{i,j} \alpha_{i+r,k} - \alpha_{i,k} \alpha_{i+r,j}\right) x_j dx_k$$

$$+ \sum_{1 \leq j < k \leq 2r} \left(\sum_{i=1}^{r} \alpha_{i,k} \alpha_{i+r,j} - \alpha_{i,j} \alpha_{i+r,k}\right) x_k dx_j$$

$$= \alpha dx_m + \sum_{j=1}^{r} \alpha x_j dx_{j+r} + \sum_{j=1}^{r} (-\alpha)x_{j+r} dx_j$$

$$= \alpha \omega_K$$

Therefore, $\overline{\psi} \in K(2r + 1, \mathfrak{n})$. $\square$

**Proposition 4.11.** Let $\mathcal{Q}$ be a quasi-torus contained in $N_{K(m; \mathfrak{n})}(T_K)$. Then there exists $\psi \in K(m; \mathfrak{n})$ such that $\psi \mathcal{Q} \psi^{-1} \subset T_K$.

**Proof.** Let $\mu \in \mathcal{Q} \subset N_{K(m; \mathfrak{n})}(T_K)$. By Lemma 4.6, we have $\mu(x_i) = \alpha_i x_j$, for $1 \leq i \leq 2r$ and $\mu(x_m) = \alpha_m x_m + \sum_{i=1}^{r} \beta_i x_i x_{i'}$. Since $\mu \in K(m; \mathfrak{n})$, we must have $\mu(\omega_K) \in O(m; \mathfrak{n}) \times \omega_K$. On the other hand,

$$\mu(\omega_K) = \mu\left(dx_m + \sum_{i=1}^{r} (x_i dx_{i'} - x_{i'} dx_i)\right)$$

$$= d\left(\alpha_m x_m + \sum_{i=1}^{r} \beta_i x_i x_{i'}\right) + \sum_{i=1}^{r} \alpha_i \alpha_{i'} (x_j dx_{j'} - x_{j'} dx_j)$$

$$= \alpha_m dx_m + \sum_{i=1}^{r} \beta_i (x_i dx_{i'} + x_{i'} dx_i) + \sum_{i=1}^{r} \alpha_i \alpha_{i'} (x_j dx_{j'} - x_{j'} dx_j)$$

It follows that $\mu(\omega_K) = \alpha_m \omega_K$ since the only term with $dx_m$ is $\alpha_m dx_m$.

We want to show that $\mu|_{O(2r; \mathfrak{n}')} \in \mathcal{H}(2r; \mathfrak{n}')$. Indeed,

$$d\omega_K = d\left(dx_m + \sum_{i=1}^{r} (x_i dx_{i'} - x_{i'} dx_i)\right) = \sum_{i=1}^{r} (d(x_i dx_{i'}) - d(x_{i'} dx_i))$$

$$= \sum_{i=1}^{r} (dx_i \wedge dx_{i'} - dx_{i'} \wedge dx_i) = 2 \omega_H,$$

and hence

$$2\mu(\omega_H) = \mu(2\omega_H) = \mu(d\omega_K) = d\mu(\omega_K) = d(\alpha_m \omega_K) = 2 \alpha_m \omega_H,$$
so \( \mu|_{O(2r; \mathbb{H}')} \in \mathcal{H}(2r; \mathbb{H}') \).

We have shown that \( N_K(m; \mathbb{N}) \subset T_K \). Moreover, since the restriction of \( T_K \) to \( O(2r; \mathbb{H}') \) is \( T_H \), where \( T_H \) is with respect to \( \mathcal{H}(2r; \mathbb{H}') \), we have \( N_K(m; \mathbb{N}) \subset N(2r; \mathbb{H}')(T_H) \). By Proposition 4.9 there exists \( \psi \in \text{Sp}(2r; \mathbb{H}') \) such that \( \psi(\mathcal{O}(2r; \mathbb{H}')) \psi^{-1} \subset T_H \).

By Lemma 1.10 we can extend \( \psi \) to an automorphism \( \overline{\psi} \) in \( K(m; \mathbb{N}) \) such that \( \overline{\psi}(x_i) = \psi(x_i) \) for \( 1 \leq i \leq 2r \) and \( \overline{\psi}(x_m) = x_m \) (since \( \overline{\psi}(\omega_H) = \omega_H \)). Let \( \mu \in \mathcal{Q} \) as before and set \( \rho = \overline{\psi} \mu \overline{\psi}^{-1} \). Then \( \rho(x_i) = \gamma_i x_i \) for \( 1 \leq i \leq 2r \), \( \gamma_i \in F^x \), and \( \rho(x_m) = \alpha_m x_m + y \) where \( y = \psi \left( \sum_{t=1}^r \beta_t x_t x_{t'} \right) \). Furthermore,

\[
\rho(\omega_H) = \overline{\psi} \mu \overline{\psi}^{-1}(\omega_H) = \overline{\psi}(\alpha_m \omega_H) = \alpha_m \omega_H
\]

and also

\[
\rho(\omega_H) = \rho \left( \sum_{l=1}^r dx_l \wedge dx_{l'} \right) = \sum_{l=1}^r d\rho(x_l) \wedge d\rho(x_{l'}) = \sum_{l=1}^r \gamma_l \gamma_l' dx_l \wedge dx_{l'}.
\]

Hence we conclude that \( \gamma_l \gamma_l' = \alpha_m \) for \( 1 \leq l \leq r \) and

\[
\rho(\omega_K) = \alpha_m dx_m + dy + \sum_{l=1}^r \gamma_l \gamma_l' (x_j dx_{j'} - x_{j'} dx_j)
\]

\[
= \alpha_m dx_m + dy + \alpha_m \sum_{l=1}^r (x_l dx_{l'} - x_{l'} dx_l) = \alpha_m \omega_K + dy.
\]

On the other hand, \( \rho(\omega_K) = \overline{\psi} \mu \overline{\psi}^{-1}(\omega_K) = \overline{\psi}(\alpha_m \omega_K) = \alpha_m \omega_K \). Since \( dy = d \left( \psi \left( \sum_{t=1}^r \beta_t x_t x_{t'} \right) \right) = \psi \left( d \left( \sum_{t=1}^r \beta_t x_t x_{t'} \right) \right) \), we obtain:

\[
0 = d \left( \sum_{l=1}^r \beta_l x_l x_{l'} \right) = \sum_{l=1}^r \beta_l (x_l dx_{l'} + x_{l'} dx_l),
\]

which implies \( \beta_l = 0 \) for \( 1 \leq l \leq r \). It follows that \( \rho \in T_K \).

\[\square\]

**Proposition 4.12.** Let \( L = W(m; \mathbb{N}), S(m; \mathbb{N})^{(1)}, H(m; \mathbb{N})^{(2)} \) or \( K(m; \mathbb{N})^{(1)} \) and let \( Q \) be a quasi-torus in \( \text{Aut } L \). Then there exists \( \Psi \in \text{Aut } L \) such that \( \Psi Q \Psi^{-1} \subset T_X \) where \( X = W, S, H \) or \( K \), respectively.

**Proof.** Let \( Q \) be a quasi-torus in \( \text{Aut } L \). It follows that \( Q := \Phi^{-1}(Q) \) is a quasi-torus in \( \mathcal{X}(m; \mathbb{N}) \). By Proposition 4.11 \( Q \) is contained in the normalizer of a maximal torus in \( \mathcal{X}(m; \mathbb{N}) \). Since all maximal tori are conjugate, without loss of generality we may assume that \( Q \subset N_{\mathcal{X}(m; \mathbb{N})}(T_X) \). Now Propositions 1.7 3, 1.11 show that there exists \( \psi \in \mathcal{X}(m; \mathbb{N}) \) such that \( \psi Q \psi^{-1} \subset T_X \). Hence

\[
\Phi(\psi)Q(\Phi(\psi))^{-1} = \Phi(\psi)Q(\Phi(\psi))^{-1} = \Phi(\psi Q \psi^{-1}) \subset \Phi(T_X) = T_X.
\]

\[\square\]

We can now prove Theorem 1.3.

**Proof.** Let \( L = W(m; \mathbb{N}), S(m; \mathbb{N})^{(1)}, H(m; \mathbb{N})^{(2)} \) or \( K(m; \mathbb{N})^{(1)} \). Suppose \( L = \bigoplus_{g \in G} L_g \) is a G-grading where \( G \) is a group without elements of order \( p \). Without loss of generality, we assume that the support of the grading generates \( G \). Let \( \eta : \hat{G} \to \text{Aut } L \) be the corresponding embedding and \( Q := \eta(\hat{G}) \). Then \( Q \) is a quasi-torus in \( \text{Aut } L \). By Proposition 4.12 we can conjugate \( Q \) by an automorphism \( \Psi \) of
so that $\Psi Q \Psi^{-1} \subset T_X$. It follows from Lemma 2.8 that the grading $L = \bigoplus_{g \in G} L'_g$, where $L'_g = \Psi(L_g)$, is a standard grading.

\[ \square \]

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