Article

An Oscillation Criterion of Nonlinear Differential Equations with Advanced Term

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Abstract: The aim of the present paper is to provide oscillation conditions for fourth-order damped differential equations with advanced term. By using the Riccati technique, some new oscillation criteria, which ensure that every solution oscillates, are established. In fact, the obtained results extend, unify and correlate many of the existing results in the literature. Furthermore, two examples with specific parameter values are provided to confirm our results.

Keywords: oscillation; fourth-order; damped differential equations

1. Introduction

Fourth-order advanced differential equations have an enormous potential for applications in engineering, medicine, aviation and physics, etc. The oscillation of differential equations contributes to many applications in science and technology and self-excited oscillation phenomena which occur in bridges and in the oscillatory muscle movement model; see [1,2].

In this article, we study some oscillation properties of the solutions to fourth-order advanced differential equations

\[
\begin{cases}
(j(z)\Phi_p[\xi(z)])' + a(z)f(\xi''(z)) + g(z)\varphi(\zeta(c(z))) = 0, \\
(j(z) > 0, j'(z) + a(z) \geq 0, z \geq z_0 > 0,
\end{cases}
\]

where 1 < p < ∞ and p is an even number. Throughout this work, we assume that

L1: \( \Phi_p[s] = |s|^{p-2}s \),

L2: \( j, a, c, q \in C([z_0, \infty), [0, \infty)), q > 0, c(z) \geq z, \lim_{z \to \infty} c(z) = \infty \) and under condition

\[
\int_{z_0}^{\infty} \left[ \frac{1}{j(s)} \exp \left( -\int_{z_0}^{s} \frac{a(y)}{j(y)} \, dy \right) \right]^{1/p-1} \, ds < \infty.
\]

L3: \( f, g \in C(\mathbb{R}, \mathbb{R}) \) such that \( f(w)/|w|^{p-2}w \geq k_f > 0, g(w)/|w|^{p-2}w \geq k_g > 0 \), for \( w \neq 0, k_f \geq 1, k_g \) are constants.

Definition 1. When a solution of (1) has arbitrarily large zeros on \([z_0, \infty)\), then it is termed oscillatory; otherwise, it is termed non-oscillatory.

Definition 2. When all the solutions of the equation in (1) are oscillatory, the equation is called oscillatory.
Definition 3. If condition \( c(z) \geq z \) hold, then the Equation (1) is called an advanced differential equation.

Asymptotic behavior of solutions of differential equations have been the objective of many authors. Oscillation and asymptotic theory, however, has gained particular attention due to its widespread applications in clinical applications, earthquake structures, which involve symmetrical properties; see [3–8]. Nowadays, there has been an increasing interest in studying the asymptotic behavior of differential equations, see [9–21].

Park et al. [22] studied some oscillation properties of the solutions of differential equations with advanced term, by employing the comparison technique. Agarwal et al. [23,24] established the properties of oscillation for advanced equations using integral averaging technique.

Bazighifan et al. [25,26] considered fourth-order differential equations with advanced term

\[
\left\{ \begin{array}{l}
( j(z) |^{(m-1)}_{2}(z) |^{p-2}_{7}(z) )^{' + \sum_{i=1}^{j} q_{i}(z) g(\zeta(\eta_{i}(z))) = 0, } \n j \geq 1, z \geq z_0 > 0,
\end{array} \right.
\]

where \( m \) is even and \( p > 1 \).

The authors in [4], obtained some oscillation conditions for equation

\[
\left\{ \begin{array}{l}
( j(z) \Phi_p \left( \sum_{i=1}^{j} q_{i}(z) g(\zeta(\eta_{i}(z))) = 0, 
\end{array} \right.
\]

where \( m \) is even and \( p > 1 \). Moreover, the authors used the comparison method to obtain oscillation conditions for this equation.

Other work has been done on similar equations with advanced term. Li et al. [3] investigated some oscillation criteria of equation

\[
\left\{ \begin{array}{l}
( j(z) |^{\alpha}_{1}(z) |^{p-2}_{2}(z) )^{' + \sum_{i=1}^{j} q_{i}(z) g(\zeta(\eta_{i}(z))) = 0, 
\end{array} \right.
\]

The purpose of this paper is to establish new oscillation criteria for (1). The methods used in this paper simplify and extend some of the known results that are reported in the literature [4,26]. The authors in [4,26] used a comparison technique that differs from the one we used in this article. Moreover, the authors in [4,26] also studied the equation under the condition \( f_{20}^{\infty} \left[ \frac{1}{|z|} \exp \left( - \int_{20}^{z} \frac{a(y)}{|y|} dy \right) \right]^{1/\alpha} ds = \infty \) which is different from our condition \( f_{20}^{\infty} \left[ \frac{1}{|z|} \exp \left( - \int_{20}^{z} \frac{a(y)}{|y|} dy \right) \right]^{1/\alpha} ds < \infty \).

The organization of this paper is as follows. After this introduction, in Section 2, we propose some preliminary lemmas that are used in the proof of our main theorems. In Section 3, we establish some oscillation criteria for (1) by Riccati technique; our results extend and correlate many of the existing results in the literature. Then, some examples are considered to check the efficiency of our main results.

2. Some Lemmas

These are some of the important Lemmas

Lemma 1. ([27]) Let \( \alpha \geq 1 \). Then

\[
Dy - C y^{(\alpha+1)/\alpha} \leq \frac{\alpha}{(\alpha + 1)^{\alpha+1} C^{\alpha+1}}.
\]
for all positive $y, C > 0$ and $D$ be positive constant

**Lemma 2.** ([28]) Let $\zeta \in C^m([z_0, \infty), (0, \infty))$ and $\zeta^{(m-1)}(z)\zeta^{(m)}(z) \leq 0$ such that $m$ a positive integer, then

$$
\zeta(\theta z) \geq M\lambda^{m-1} \zeta^{(m-1)}(z),
$$

for all $\theta \in (0, 1)$ there exists a constant $M > 0$.

**Lemma 3.** ([29]) Let $\zeta \in C^m([z_0, \infty), (0, \infty))$ and

$$
\zeta^{(m-1)}(z)\zeta^{(m)}(z) \leq 0,
$$

then

$$
\zeta(z) \geq \frac{\lambda}{(m-1)!} \lambda^{m-1} \left|\zeta^{(m-1)}(z)\right|.
$$

**Lemma 4.** ([30]) Let $\zeta$ is a positive solution of (1). Then, there exist two possible cases

- $\mathbf{(D_1)} \; \zeta(z) > 0$, $\zeta'(z) > 0$, $\zeta''(z) > 0$, $\zeta^{(4)}(z) < 0$;
- $\mathbf{(D_2)} \; \zeta(z) > 0$, $\zeta''(z) > 0$, $\zeta'''(z) < 0$.

for $z \geq z_1$ where $z_1 \geq z_0$ is sufficiently large.

3. Oscillation Criteria

The motivation for this section is to create new oscillation criteria, established for (1) by the Riccati technique.

For ease of use, here are some notations.

$$
G(z_0, z) = \exp \left( \int_{z_0}^{z} \frac{a(u)}{j(u)} \, du \right),
$$

$$
\xi(z) = \int_{z}^{\infty} \frac{ds}{(j(s)G(z_0, s))^{1-p}},
$$

$$
\phi(z) = \frac{\delta'(z)}{\delta(z)} - \frac{k_1a(z)}{j(z)},
$$

$$
\varphi(z) = \frac{1}{G^{\frac{1}{p-1}}(z_0, z)} - \frac{\chi(z)a(z)j(z)^{(2-p)/(p-1)}(z)}{p-1},
$$

and

$$
\varphi(z) = \frac{a(z)}{j(z)} + \frac{(p-1)\delta(z)\psi(z)G(z_0, z)}{\xi(z)j^{\frac{1}{p-1}}(z)}.
$$

**Theorem 1.** Let (2) holds. Suppose that $\delta, \theta \in C^1([z_0, \infty), (0, \infty))$ and $M > 0$ and $k_1 > 1$ are constants such that

$$
\limsup_{z \to \infty} \int_{z_0}^{z} \left( k_1\delta(s)q(s) - \left( \frac{2}{Ms^2} \right)^{p-1} j(s)\delta(s)\left( \phi(s) \right)^p \right) ds = \infty.
$$

If

$$
\frac{\theta(z)}{\xi(z)(j(z)G(z_0, z))^{1/(p-1)}} + \theta'(z) \leq 0
$$

and, for some $\mu \in (0, 1),

$$
\limsup_{z \to \infty} \int_{z_0}^{z} \left( k_1q(s) \left( \frac{R^2(s) \theta'(s)}{\delta(s)} \right)^{p-1} \xi' G(z_0, s) - \varphi(s) \right) ds = \infty,
$$

where $R(s) = \max\{\delta(s), \delta''(s), \delta'''(s)\}$.
then Equation (1) is oscillatory.

**Proof.** Let \( \xi \) be a nonoscillatory solution of Equation (1), then \( \xi \geq 0 \). From Lemma 4, let case \((D_1)\) hold. By Lemma 2, we obtain

\[
\xi'(z/2) \geq Mt^2 \xi''(z).
\]

Define

\[
\psi(z) := \delta(z) \frac{j(z) (\xi'''(z))^{p-1}(z)}{\xi^{p-1}(z/2)}
\]

and

\[
\psi'(z) = \frac{\delta'(z)}{\delta(z)} \psi(z) + \delta(z) \frac{(j(z) (\xi'''(z))^{p-1}(z))'}{\xi^{p-1}(z/2)} - (p-1)M^2 \frac{\psi(z)}{2 \xi^{p}(z/2)}.
\]

Using (7) and (6), we find

\[
\psi'(z) \leq \frac{\delta'(z)}{\delta(z)} \psi(z) + \delta(z) \frac{(j(z) (\xi'''(z))^{p-1}(z))'}{\xi^{p-1}(z/2)} - (p-1)M^2 \frac{\psi(z)}{2 \xi^{p}(z/2)}.
\]

From (1), we obtain

\[
\psi'(z) \leq \frac{\delta'(z)}{\delta(z)} \psi(z) - k_f a(z) \frac{\psi(z)}{j(z)} - k_s \delta(z) q(z) \frac{\xi^{p-1}(z)}{\xi^{p-1}(z/2)} - (p-1)M^2 \frac{\psi(z)}{2 (\delta(z))^{1/(p-1)}}
\]

\[
\leq -k_s \delta(z) q(z) + \left( \frac{\delta'(z)}{\delta(z)} - k_f \frac{a(z)}{j(z)} \right) \psi(z) - (p-1)M^2 \frac{\psi(z)}{2 (\delta(z))^{1/(p-1)}}.
\]

So, we find

\[
\psi'(z) \leq -k_s \delta(z) q(z) + \phi(z) \psi(z) - \frac{(p-1)M^2}{2 (j(z) \delta(z))^{1/(p-1)}} \frac{\psi(z)}{2p}.
\]

Using Lemma 1, we set

\[
D = \phi(z), \quad C = (p-1)M^2 / \left(2 (j(z) \delta(z))^{1/(p-1)}\right) \quad \text{and} \quad y = \psi,
\]

we have

\[
\psi'(z) \leq -k_s \delta(z) q(z) + \left( \frac{2}{M^2} \right)^{p-1} \frac{j(z) \delta(z) (\phi(z))^p}{p^p}.
\]

Integrating from \( z_1 \) to \( z \), we obtain

\[
\int_{z_1}^z \left( k_s \delta(s) q(s) - \left( \frac{2}{M^2} \right)^{p-1} \frac{j(s) \delta(s) (\phi(s))^p}{p^p} \right) ds \leq \psi(z_1),
\]

which contradicts (3).

For case \((D_2)\). Since
\[
\left(-j(z)(-\xi'''(z))^{p-1}G(z_0, z)\right)' = \left(-j(z)(-\xi'''(z))^{p-1}\right)G(z_0, z)
+ \left(-j(z)(-\xi'''(z))^{p-1}\right)G(z_0, z) \frac{d}{dz} a(z)
= (-1)^{p-1}(-a(z)f(\xi'''(z)) - q(z)g(\xi''(z)))G(z_0, z)
- a(z)(-\xi'''(z))^{p-1}G(z_0, z)
\leq (-1)^{p-1}(-\xi'''(z))^{p-1} - k_2q(z)\xi''(z)G(z_0, z)
- a(z)(-\xi'''(z))^{p-1}G(z_0, z)
= \left(-a(z)(-\xi'''(z))^{p-1}(1 - \theta) + k_2q(z)\xi''(z)\right)G(z_0, z)
\leq -k_2q(z)\xi''(z)G(z_0, z) < 0,
\]

we deduce that \(-j(z)(-\xi'''(z))^{p-1}G(z_0, z)\) is decreasing. Thus, for \(s \geq z \geq z_1\)

\[
(j(s)G(z_0, s))^{1/(p-1)}\xi'''(s) \leq (j(z)G(z_0, z))^{1/(p-1)}\xi'''(z).
\tag{10}
\]

Dividing both sides of (10) by \((j(s)G(z_0, s))^{1/(p-1)}\) and integrating from \(z\) to \(h\), we get

\[
\int_z^h \xi'''(z) dt \leq (j(z)G(z_0, z))^{1/(p-1)}\xi'''(z) \int_z^h \frac{ds}{(j(s)G(z_0, s))^{1/(p-1)}}.
\]

Easily, we find that

\[
\xi''(h) - \xi''(z) \leq (j(z)G(z_0, z))^{1/(p-1)}\xi'''(z) \int_z^h \frac{ds}{(j(s)G(z_0, s))^{1/(p-1)}}.
\]

Letting \(h \to \infty\), we find that

\[
-\xi''(z) \leq (j(z)G(z_0, z))^{1/(p-1)}\xi'''(z) \int_z^\infty \frac{ds}{(j(s)G(z_0, s))^{1/(p-1)}}.
\]

Therefore, we see that

\[
-\xi''(z) \leq (j(z)G(z_0, z))^{1/(p-1)}\xi'''(z)\xi(z),
\]

which yields

\[
-\frac{\xi'''(z)}{\xi''(z)}\xi(z)(j(z)G(z_0, z))^{1/(p-1)} \leq 1.
\]

Hence,

\[
j(z)(\xi'''(z))^{p-1} \geq \frac{-1}{\xi''(z)G(z_0, z)}.
\tag{11}
\]

Define

\[
B(z) := -\frac{j(z)(-\xi'''(z))^{p-1}(z)}{\xi''(z)^{p-1}},
\tag{12}
\]

we obtain \(B(z) < 0\) also, from (11) and (12), we have

\[
B(z) \geq \frac{-1}{\xi''(z)G(z_0, z)}.
\tag{13}
\]
Using Lemma 3, we find
\[ B'(z) = \frac{-j(z)(-\xi'''(z))^{p-1}}{(\xi''(z))^{p-1}} - (p-1)\frac{-j(z)(-\xi'''(z))^p}{(\xi''(z))^p}. \]

Using (1), we obtain
\[ B'(z) \leq -\frac{a(z)\xi(z)}{(\xi''(z))^{p-1}} - \frac{q(z)\xi''(c(z))}{(\xi''(z))^{p-1}} - (p-1)\frac{-j(z)(-\xi'''(z))^p}{(\xi''(z))^p}. \]

From (12), we find
\[ B'(z) = -k_j\frac{a(z)}{j(z)}B(z) - k_g q(z)\left(\frac{\xi''(c(z))}{(\xi''(z))^{p-1}} - (p-1)\frac{\xi'''(z)}{(\xi''(z))^{p-1}}\right). \]

Using (1), we obtain
\[ B'(z) \leq -k_j\frac{a(z)}{j(z)}B(z) - k_g q(z)\left(\frac{\xi''(c(z))}{(\xi''(z))^{p-1}} - (p-1)\frac{\xi'''(z)}{(\xi''(z))^{p-1}}\right) - \frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{p}{p-1}}(z)}. \]

Using Lemma 3, we find
\[ \xi(z) \geq \frac{\mu}{2}z^2 \xi''(z). \]

From (13)–(15), we obtain
\[ B'(z) \leq \frac{k_j a(z)}{j(z)\xi''(z)G(z_0, z)} - k_g q(z)\left(\frac{\mu}{2}c^2(z)\right)^{p-1} - (p-1)\frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{p}{p-1}}(z)}. \]

From (11), we see
\[ \frac{\xi''(z)}{\xi'(z)} \geq \frac{-1}{\xi'(z)(j(z)G(z_0, z))^{1/(p-1)}}. \]

Using the latter inequality and (4), we see
\[ \left(\frac{\xi''(z)}{\xi'(z)}\right)' = \frac{\xi'''(z)\xi'(z) - \xi''(z)\xi'(z)}{\xi'(z)} \geq \frac{-\xi''(z)}{\xi''(z)} \left(\frac{\theta(z)}{\xi'(z)(j(z)G(z_0, z))^{1/(p-1)}} + \theta'(z)\right) \geq 0, \]

which implies that \( \xi''(z) / \theta(z) \) is nondecreasing. Thus, it follows from \( c(z) \geq z \) that
\[ \frac{\xi''(c(z))}{\xi''(z)} \geq \frac{\theta(c(z))}{\theta(z)}. \]

So, by (14) and (15), we see
\[ B'(z) \leq \frac{k_j a(z)}{j(z)\xi''(z)G(z_0, z)} - k_g q(z)\left(\frac{\mu}{2}c^2(z)\right)^{p-1} - (p-1)\frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{p}{p-1}}(z)}. \]
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which contradicts (5).

Corollary 1. Let (2) holds. If $\delta, \vartheta \in C^1([z_0, \infty), (0, \infty))$ such that

$$\limsup_{z \to \infty} \int_{z_0}^z \left( k_g \delta(s) q(s) - \frac{j(s) \vartheta(s) (\varphi(s))^p}{p^p} \right) ds = \infty,$$

(18)

additionally, (4) is satisfied and

$$\limsup_{z \to \infty} \int_{z_0}^z \left( k_g q(s) \left( \frac{\vartheta(c(s))}{\xi(s)} \right)^{p-1} G(z_0, s) - \varphi(s) \right) ds = \infty,$$

(19)
then equation
\[(j(z)\Phi_{p}(\xi'(z)))' + a(z)f(\xi'(z)) + q(z)g(c(z))) = 0, \quad (20)\]
is oscillatory.

Now, we present two examples that illustrate the applicability of the obtained results. It is worth noting that these examples represent many physical phenomena, such as their application in earthquake structures, mechanical oscillations and clinical applications.

**Example 1.** Consider the differential equation
\[
(z^2(\xi'''(z)))' + \frac{z}{2}\xi'''(z) + \frac{q_0}{z^2}\xi(2z) = 0. \quad (21)
\]
where \(q_0 > 0, \ p = 2, \ z_0 = 1, \ j(z) = z^2, \ a(z) = z/2, \ q(z) = z, \ c(z) = 2z, \) we now set \(\delta(z) = z, \ k_f = k_g = 1, \) then
\[
G(z_0, z) = \exp\left(\int_{z_0}^{z} \frac{a(s)}{j(s)} ds \right) = z^{1/2}, \ \xi(z) = \int_{z_0}^{z} \frac{ds}{(j(s)G(z_0, s))^{1/4}} = \frac{2z^{-3/2}}{3}, \ \phi(z) = -\frac{1}{2z}
\]
and
\[
\hat{\phi}(z) = \frac{a(z)}{j(z)} + \frac{(p-1)^{p-1} j(z) \delta(s) \phi^{p}(z) G(z_0, z)}{\xi(z)^{p-1}} = \frac{2z^{-1/3}}{3}.
\]
Thus, we obtain
\[
\limsup_{z \to \infty} \int_{z_0}^{z} \left( k_s \delta(s) q(s) - \left( \frac{2}{Ms^2} \right) \left( j(z) \delta(s) \phi(s) \right)^{p-1} \right) ds = \infty
\]
and
\[
\hat{\theta}(z) = \frac{\xi(z) (j(z)G(z_0, z))^{1/(p-1)}}{\xi(z)} + \theta'(z) = 0
\]
also, for some \(\mu \in (0, 1),\)
\[
\limsup_{z \to \infty} \int_{z_0}^{z} \left( k_s q(s) \left( \frac{\mu c^2(s) \theta(c(s))}{2} \right) \xi(s) \right)^{p-1} G(z_0, s) - \hat{\phi}(s) ds = \infty.
\]
Using Theorem 1, the Equation (21) is oscillatory if \(q_0 > 0, \)

**Example 2.** Consider the differential equation
\[
(z^2(\xi'(z)))' + \frac{z}{2}\xi'(z) + q_0(2z) = 0. \quad (22)
\]
where \(q_0 > 0, \ p = 2, \ z_0 = 1, \ j(z) = z^2, \ a(z) = z/2, \ q(z) = q_0, \ c(z) = 2z, \) we now set \(\delta(z) = k_f = k_g = 1, \) then \(\xi(z) = \phi(z) = 2z^{-3/2}/3. \)

Using Corollary 1, the Equation (22) is oscillatory if \(q_0 > 2\sqrt{2}. \)

4. Conclusions

In this article, we establish oscillation conditions of advanced nonlinear differential equations of fourth-order with middle term of the form (1). Our approach is different and obtained by using Riccati technique to reduce the main equation into a first-order equation.
The new proposed criteria complement several results in the literature. We provide two examples with specific parameters to illustrate the applicability of our theorems. In future work, we will discuss the oscillatory behavior of these equations by using the integral averaging technique and under the condition

$$\int_{z_0}^{\infty} \left[ \frac{1}{j(s)} \exp \left( - \int_{z_0}^{s} \frac{\dot{a}(y)}{f(y)} dy \right)^{1/\alpha} ds \right] = \infty. \quad (23)$$

**Author Contributions:** Conceptualization, O.B., B.A., A.A. and M.M. These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** The authors received no direct funding for this work.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors thank the reviewers for their useful comments, which led to the improvement of the content of the paper. This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

**Conflicts of Interest:** The authors declare no conflict of interest.

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