A positive proportion of elliptic curves over $\mathbb{Q}$ have rank one

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Abstract

We prove that, when all elliptic curves over $\mathbb{Q}$ are ordered by naive height, a positive proportion have both algebraic and analytic rank one. It follows that the average rank and the average analytic rank of elliptic curves are both strictly positive.

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1 Introduction

Any elliptic curve $E$ over $\mathbb{Q}$ has a unique Weierstrass model of the form $E_{A,B} : y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$ and such that for all primes $\ell$: $\ell^6 \nmid B$ whenever $\ell^4 \mid A$. The (naive) height of the elliptic curve is then defined by

$$H(E) = H(E_{A,B}) := \max\{4|A^3|, 27B^2\}.$$

When all elliptic curves are ordered by their heights (or indeed, by their discriminants or conductors), it is a well-known conjecture of Goldfeld [16] and of Katz and Sarnak [18] that the average rank of all elliptic curves is $1/2$. In fact, they conjectured that $50\%$ of all elliptic curves should have rank 0 and $50\%$ should have rank 1, with a negligible proportion having rank $\geq 2$. However, as far as proofs, it was not previously known whether a positive proportion of curves have rank 0 or 1, whether the lim inf of the average rank is $> 0$, or whether the lim sup of the average rank is $< \infty$! (See [1] for a nice survey.)

In this direction, in recent papers [6, 8] (using crucial input also from the works [10] and [27]) it was shown that, when all elliptic curves are ordered by height, a positive proportion have rank 0, and in fact analytic rank 0; moreover, the lim sup of the average rank of all elliptic curves is finite, and in fact less than 1.

The purpose of this article is to prove the analogous positive proportion result for rank 1:
Theorem 1 When all elliptic curves over \( \mathbb{Q} \) are ordered by height, a positive proportion have rank 1.

In other words, Theorem 1 states that a positive proportion of elliptic curves over \( \mathbb{Q} \) have infinitely many rational points.

In fact, we prove that a positive proportion of elliptic curves over \( \mathbb{Q} \) have both algebraic and analytic rank 1. More precisely, for any elliptic curve \( E \) over \( \mathbb{Q} \), let us use \( \text{rk}(E) \) and \( \text{rk}_{\text{an}}(E) \) to denote the algebraic and analytic rank of \( E \), respectively. Then we prove

\[
\liminf_{X \to \infty} \frac{\# \{ E : \text{rk}(E) = \text{rk}_{\text{an}}(E) = 1 \text{ and } H(E) < X \}}{\# \{ E : H(E) < X \}} > 0. \tag{1}
\]

In order to keep the arguments as transparent as possible, in this paper we simply prove positivity in (1), and we have not tried to maximize the lower bound on this limit obtainable via the methods employed to prove Theorem 1.

As a consequence we also obtain, for the first time, a positive lower bound on the \( \lim \inf \) of the average rank of elliptic curves:

Corollary 2 When all elliptic curves over \( \mathbb{Q} \) are ordered by height, the average rank and the average analytic rank are both strictly positive.

Finally, we note that Theorem 1 also implies, via rank 1 curves, that a positive proportion of elliptic curves satisfy the Birch and Swinnerton-Dyer rank conjecture. The corresponding result via rank 0 curves was proven in [6].

In the next section, we describe the method of proof, which uses in an essential way a number of recent developments in the arithmetic of elliptic curves: the Gross–Zagier formula (in the general form proved by Yuan, Zhang, and Zhang [29]); a \( p \)-adic variant of the Gross–Zagier formula (due to Bertolini, Darmon, and Prasanna [2] and Brooks [9]); work on an Iwasawa Main Conjecture for \( \text{GL}_2 \) (by Wan [28], building on the earlier work [27]); a converse to a theorem of Gross, Zagier, and Kolyvagin [26]; and the determination of the average orders of \( p \)-Selmer groups of elliptic curves (particularly for \( p = 5 \)) [8].

2 Method of proof

To prove Theorem 1 (and Corollary 2), we first establish two convenient sets of \( p \)-adic criteria that can be used to deduce the existence of rank one curves. The first of these results provides sufficient conditions for an elliptic curve \( E \) to have (algebraic and analytic) rank one; the second provides analogous criteria for the quadratic twist \( E^D \) of \( E \) by an imaginary quadratic field \( \mathbb{Q}(\sqrt{D}) \) (also called the \( D \)-twist of \( E \)) to have rank one.

Theorem 3 Let \( E/\mathbb{Q} \) be an elliptic curve and let \( p \geq 5 \) be a prime. Suppose that:

(a) the conductor \( N_E \) of \( E \) is squarefree and has at least two odd prime factors;
(b) \( E \) has good, ordinary reduction at \( p \);
(c) \( E[p] \) is an irreducible \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-module;
(d) \( \text{Sel}_p(E) \cong \mathbb{Z}/p\mathbb{Z} \).
(c) the image of $\text{Sel}_p(E)$ in $E(\mathbb{Q}_p)/pE(\mathbb{Q}_p)$ under the restriction map at $p$ is not contained in the image of $E(\mathbb{Q}_p)[p]$.

Then the algebraic rank and analytic rank of $E$ are both equal to 1.

**Theorem 4** Let $E/\mathbb{Q}$ be an elliptic curve and let $p \geq 5$ be a prime. Let $K/\mathbb{Q}$ be an imaginary quadratic field of odd discriminant $D$ such that 2 and $p$ split in $K$. Suppose that:

(a) the conductor $N_E$ of $E$ is squarefree with at least two odd prime factors, and $(N_E, D) = 1$;
(b) $E$ has good, ordinary reduction at $p$;
(c) $E[p]$ is an irreducible $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module and ramified at some odd prime $q$ that is inert in $K$;
(d) $\text{Sel}_p(E) = 0$ and $\text{Sel}_p(E^D) \cong \mathbb{Z}/p\mathbb{Z}$;
(e) The image of $\text{Sel}_p(E^D)$ in $E^D(\mathbb{Q}_p)/pE^D(\mathbb{Q}_p)$ under the restriction map at $p$ is not contained in the image of $E^D(\mathbb{Q}_p)[p]$.

Then the algebraic rank and analytic rank of $E^D$ are both equal to 1.

Both of these results will be proven in Section 3 using Theorem B of [26], which gives sufficient $p$-adic criteria for an elliptic curve to have algebraic rank and analytic rank 1 over an imaginary quadratic field. The proof of [26, Thm. B] in turn relies crucially on [2], [9], and [28].

We employ Theorems 3 and 4 with $p = 5$, $K = \mathbb{Q}[\sqrt{-39}]$ (so $D = -39$), $q = 7$, and $E$ belonging to a suitable “large” family of elliptic curves. (A large family of elliptic curves is one that is defined by congruence conditions and consists of a positive proportion of all elliptic curves; see Section 4 for a more precise definition.) The large family we use is the set $\mathcal{F}$ of elliptic curves $E = E_{A,B}$ such that

- $2^3 || A$ and $2^4 || B$;
- $\Delta(A, B) := -4A^3 - 27B^2$ equals $2^8 \Delta_1(A, B)$ with $\Delta_1(A, B)$ positive and squarefree (and necessarily odd), $(\Delta(A, B), 5 \cdot 39) = 1$, and $\Delta(A, B)$ is a square modulo 39;
- $E$ has non-split multiplicative reduction at 7;
- $E$ has good, ordinary reduction at 5.

For these curves, the discriminant of $E$ is $16\Delta(A, B) = 2^{12}\Delta_1(A, B)$ and the conductor is just $\Delta_1(A, B)$, which is squarefree and odd.

Since $7 | \Delta_1(A, B)$ but 7 is not a square modulo 39, $\Delta_1(A, B)$ must be divisible by at least two odd primes. Hence with our above choices of $p = 5$ and $K = \mathbb{Q}(\sqrt{-39})$, conditions (a) and (b) of Theorems 3 and 4 hold for all of the curves in $\mathcal{F}$. Since all the curves $E$ in $\mathcal{F}$ are semistable with non-split reduction at 7, $E[5]$ is an irreducible $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module and ramified at 7, which is inert in $K = \mathbb{Q}[\sqrt{-39}]$; hence the conditions (c) of Theorems 3 and 4 also hold, with $q = 7$ in the case of Theorem 4 for all curves in $\mathcal{F}$.

To show that a positive proportion of elliptic curves also satisfy condition (d) of either Theorem 3 or 4, we use the following result on the average order of the 5-Selmer group in large families of elliptic curves, obtained in [8]:
Theorem 5 ([8, Thm. 31]) When elliptic curves $E$ over $\mathbb{Q}$ in any large family are ordered by height, the average order of the 5-Selmer group $\text{Sel}_5(E)$ is equal to 6.

Theorem guarantees the existence of many curves with 5-Selmer rank 0 or 1; however, this alone is not sufficient to guarantee the existence of curves satisfying either of the conditions (d) in Theorems 3 and 4. In order to deduce positive proportion statements for rank 1 curves, we make use of information regarding the distribution of the parity of the 5-Selmer ranks of these curves and their $-39$-twists. First, we note that the definition of $\mathcal{F}$ implies that for all $E \in \mathcal{F}$, the curves $E$ and $E_{-39}$ have root numbers with opposite signs. It follows that all curves $E \in \mathcal{F}$ have the property that either $E$ or $E_{-39}$ have root number $-1$; in particular, for all elliptic curves of height at most $X$, either at least 50% of curves in $\mathcal{F}$ have root number $-1$ or at least 50% of their $-39$-twists have root number $-1$. The following theorem of Dokchitser–Dokchitser [10] (see also Nekovář [20]) then allows us to relate these root numbers with $p$-Selmer ranks.

Theorem 6 (Dokchitser–Dokchitser) Let $E$ be an elliptic curve over $\mathbb{Q}$ and let $p$ be any prime. Let $s_p(E)$ and $t_p(E)$ denote the rank of the $p$-Selmer group of $E$ and the rank of $E(\mathbb{Q})[p]$, respectively. Then the quantity $r_p(E) := s_p(E) - t_p(E)$ is even (resp. odd) if and only if the root number of $E$ is $+1$ (resp. $-1$).

Let $\mathcal{F}^D$ be the set of $D$-twists of curves in $\mathcal{F}$ (recall that $D = -39$). Since $E[5]$, and hence $E^D[5]$, is irreducible for all $E \in \mathcal{F}$, by Theorem we see that either at least half of the curves in $\mathcal{F}$ of height at most $X$ have odd 5-Selmer rank, or at least half of the curves in $\mathcal{F}^D$ of height at most $39X$ have odd 5-Selmer rank. Of these odd 5-Selmer rank curves in $\mathcal{F}$ or $\mathcal{F}^D$, not all could be of 5-Selmer rank $\geq 3$, or the average size of the 5-Selmer group of the curves in $\mathcal{F}$ and $\mathcal{F}^D$ could not be 6; indeed, at least a proportion of 19/20 of these odd 5-Selmer rank curves must have 5-Selmer rank 1. This is enough to deduce that at least a proportion of 19/20 of the elliptic curves in $\mathcal{F}$ also satisfy (d) for either Theorem 3 or Theorem 4.

To show that a positive proportion of these latter curves also satisfy the corresponding condition (e) of Theorem 3 or 4 we need the following equidistribution result, whose proof is discussed in Section 4:

Theorem 7 Let $F$ be a large family of elliptic curves over $\mathbb{Q}$, and let $E = E_{A_0,B_0}$ be any elliptic curve in $F$. There exists a 5-adic neighborhood $W \subset \mathbb{Z}_5^2 \setminus \{\Delta = 0\}$ of $(A_0,B_0)$ such that the large subfamily $F(W)$ of $F$ containing all the curves $E_{A,B}$ in $F$ with $(A,B) \in W$ has the property that:

(i) For each $E' \in F(W)$, $E'(\mathbb{Q}_5)/5E'(\mathbb{Q}_5)$ is naturally identified with $E(\mathbb{Q}_5)/5E(\mathbb{Q}_5)$, and via this identification the image of the natural map $E'(\mathbb{Q}_5)[5] \to E'(\mathbb{Q}_5)/5E'(\mathbb{Q}_5)$ is identified with that of $E(\mathbb{Q}_5)[5] \to E(\mathbb{Q}_5)/5E(\mathbb{Q}_5)$.

(ii) When the elliptic curves $E' \in F(W)$ are ordered by height, the images of the non-identity 5-Selmer elements under the natural restriction map

$$\text{Sel}_5(E') \to E'(\mathbb{Q}_5)/5E'(\mathbb{Q}_5) = E(\mathbb{Q}_5)/5E(\mathbb{Q}_5)$$

are equidistributed in $E(\mathbb{Q}_5)/5E(\mathbb{Q}_5)$. 

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We use Theorem 7 to deduce that a proportion of at least 1/2 of the elliptic curves in $F$, which automatically satisfy (a)–(c), also satisfy conditions (d) and (e) of either Theorem 3 or Theorem 4. Theorem 1 then follows. Details of this latter argument are discussed in Section 5.

We note that an analogue of the equidistribution result of Theorem 7 for elements of the 2-Selmer groups of Jacobians of odd degree hyperelliptic curves $y^2 = x^{2g+1} + \cdots$ of genus $g$ over $\mathbb{Q}$ was established in [3]. This equidistribution result was used in the work of Poonen and Stoll [21] to show that a positive proportion of odd degree hyperelliptic curves of genus $g \geq 3$ have no rational points other than the obvious one at infinity. It is interesting that, in contrast, we use here the above equidistribution result for 5-Selmer elements to prove the existence of a non-trivial rational point on a positive proportion of elliptic curves.

3 \textit{p-adic criteria for an elliptic curve over $\mathbb{Q}$ to have rank one}

In this section, we prove Theorems 3 and 4, which provide criteria for an elliptic curve $E$ or certain quadratic twists $E^D$ of $E$ to have both algebraic and analytic rank 1; the criteria, in particular, involve the $p$-Selmer groups of $E$ or $E^D$. The proofs involve showing that the hypotheses of these theorems satisfy those of Theorem B of [26].

3.1 Selmer groups

Let $E$ be an elliptic curve with conductor denoted $N_E$ and let $p$ be an odd prime. Let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. For each prime $\ell$, let $\bar{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$ and fix an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell$; the latter realizes $G_{\mathbb{Q}_\ell} = \text{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ as a decomposition subgroup for $\ell$ in $G_{\mathbb{Q}} = \text{Gal}(\mathbb{Q}/\mathbb{Q})$.

Let $p$ be a prime. Recall that the $p^r$-Selmer group of $E$ is

$$\text{Sel}_{p^r}(E) = \ker\{H^1(\mathbb{Q}, E[p^r]) \to \prod_{\ell} H^1(\mathbb{Q}_\ell, E(\bar{\mathbb{Q}}_\ell))\}$$

and that the $p^\infty$-Selmer group of $E$ is

$$\text{Sel}_{p^\infty}(E) = \lim_{r \to \infty} \text{Sel}_{p^r}(E) \subseteq H^1(\mathbb{Q}, E[p^\infty]).$$

The local conditions on classes in $\text{Sel}_{p^r}(E)$ (resp. $\text{Sel}_{p^\infty}(E)$), can also be expressed as the restriction at each prime $\ell$ being in the subgroup $E(\mathbb{Q}_\ell)/p^rE(\mathbb{Q}_\ell) \to H^1(\mathbb{Q}_\ell, E[p^r])$ (resp. $E(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(\mathbb{Q}_\ell, E[p^\infty])$, where the injection is just the usual Kummer map.

Let $T = T_p E$, $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $A = V/T = E[p^\infty]$ (the last identification being given by $(x_n) \otimes \frac{1}{p^m} \mapsto x_m$). The subgroup $E(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(\mathbb{Q}_\ell, A)$ is also the image $H^1_J(\mathbb{Q}_\ell, V)$ in $H^1(\mathbb{Q}_\ell, A)$, where $H^1_J(\mathbb{Q}_\ell, V) = \ker\{H^1(\mathbb{Q}_\ell, V) \to H^1(\mathbb{Q}_p, B_{cris} \otimes \mathbb{Q}_p V)\} \cong \mathbb{Q}_p$, where $B_{cris}$ is the ring of crystalline periods, and $H^1_J(\mathbb{Q}_\ell, V) = H^1(\mathbb{F}_\ell, \lambda)_{\ell \neq p}$ if $\ell \neq p$. For $\ell \neq p$, $H^1_J(\mathbb{Q}_\ell, A)$ (which is in fact 0) is contained in $H^1(\mathbb{F}_\ell, A^{\ell})$ with finite index in general and with equality if $E$ has good reduction at $\ell$. In particular, if $S$ is a finite set of primes containing all those that divide $pN_E$ and if $G_{\mathbb{Q}, S}$ is the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside $S$, then $\text{Sel}_{p^\infty}(E) \subseteq H^1(G_{\mathbb{Q}, S}, A)$ consists of those classes with restriction to $H^1(\mathbb{Q}_\ell, A)$ belonging to $H^1_J(\mathbb{Q}_\ell, A)$ for all $\ell \in S$. As the image of $H^1(G_{\mathbb{Q}, S}, V)$ in $H^1(G_{\mathbb{Q}, S}, A)$ is the maximal divisible
subgroup (and has finite index), it follows that the maximal divisible subgroup of $\text{Sel}_{p^\infty}(E)$ is the image in $H_1^1(\mathbb{Q}, A)$ of the characteristic zero Bloch–Kato Selmer group

$$H_1^1(\mathbb{Q}, V) = \ker\{H_1^1(\mathbb{Q}, V) \to \prod_{\ell} H_1^1(\mathbb{Q}_\ell, V)\}.$$  

Also,

$$\text{Sel}_{p^\infty}(E) \to E(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p \iff H_1^1(\mathbb{Q}, V) \to H_1^1(\mathbb{Q}_p, V).$$

Properties of the $p^\infty$-Selmer group $\text{Sel}_{p^\infty}(E)$ and of the Bloch–Kato Selmer group $H_1^1(\mathbb{Q}, V)$ can sometimes be deduced from knowledge of just the $p$-Selmer group $\text{Sel}_p(E)$, as we do in the following lemma.

**Lemma 8** Suppose $E$ has good reduction at $p$ and $E(\mathbb{Q})[p] = 0$.

(i) If $\text{Sel}_p(E) = 0$, then $\text{Sel}_{p^\infty}(E) = 0$ and $H_1^1(\mathbb{Q}, V) = 0$.

(ii) The $\mathbb{F}_p$-dimension of $\text{Sel}_p(E)$ is even (resp. odd) if and only if $w(E) = +1$ (resp. $w(E) = -1$), where $w(E)$ is the root number of $E$.

(iii) If $\text{Sel}_p(E) \cong \mathbb{Z}/p\mathbb{Z}$ then $\text{Sel}_{p^\infty}(E) \cong \mathbb{Q}_p/\mathbb{Z}_p$, and if furthermore the image of $\text{Sel}_p(E)$ in $E(\mathbb{Q}_p)/pE(\mathbb{Q}_p)$ is not contained in the image of $E(\mathbb{Q}_p)[p]$, then the restriction map $\text{Sel}_{p^\infty} \to E(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is an isomorphism; in particular, $H_1^1(\mathbb{Q}, V) \cong H_1^1(\mathbb{Q}_p, V)$.

**Proof:** Since $E(\mathbb{Q})[p] = 0$, $\text{Sel}_p(E) = \text{Sel}_{p^\infty}(E)[p]$. Part (i) is then immediate, and Part (ii) is a special case of the result of Dokchitser and Dokchitser stated in Theorem [3]

Cassels proved that $\text{Sel}_{p^\infty}(E) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^r \oplus F \oplus F$ for some finite group $F$, so if $\text{Sel}_{p^\infty}(E)[p] = \text{Sel}_p(E) \cong \mathbb{Z}/p\mathbb{Z}$, then it follows that $r = 1$ and $F = 0$ and hence that $\text{Sel}_{p^\infty}(E) \cong \mathbb{Q}_p/\mathbb{Z}_p$. As $E$ has good reduction at $p$, the reduction map induces an injection $E(\mathbb{Q}_p)[p^\infty] \hookrightarrow E(\mathbb{F}_p)[p^\infty]$. By the Riemann Hypothesis for $E$, $p^2$ does not divide the order of $E(\mathbb{F}_p)$, so $E(\mathbb{Q}_p)[p] = E(\mathbb{Q}_p)[p^\infty]$. From the exact sequence

$$0 \to E(\mathbb{Q}_p)[p^\infty]/pE(\mathbb{Q}_p)[p^\infty] \to E(\mathbb{Q}_p)/pE(\mathbb{Q}_p) \to (E(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p] \to 0$$

it then follows that if the image of the restriction map $\text{Sel}_p(E) \to E(\mathbb{Q}_p)/pE(\mathbb{Q}_p)$ is not contained in the image of $E(\mathbb{Q}_p)[p] (= \text{the image of } E(\mathbb{Q}_p)[p^\infty]/pE(\mathbb{Q}_p)[p^\infty])$, then $\text{Sel}_p(E)$ maps isomorphically onto $(E(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p] \cong \mathbb{Z}/p\mathbb{Z}$. That the restriction map $\text{Sel}_{p^\infty}(E) \to E(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$ is an isomorphism then follows from this injectivity at the level of $p$-torsion. Part (iii) follows. □

### 3.2 Proofs of Theorems [3] and [4]

Let $E$ be an elliptic curve over $\mathbb{Q}$. Let $K/\mathbb{Q}$ be an imaginary quadratic field with discriminant denoted $D$, and suppose $p \geq 5$. Theorem B of [26] asserts that if

(I) $E$ has good, ordinary reduction at $p$;

(II) $E[p]$ is an irreducible $G_K$-module and ramified at some odd prime $q \neq p$ that is inert in $K$;

(III) both 2 and $p$ split in $K$;

(IV) $(D, N_E) = 1$;

(V) $\dim_{\mathbb{Q}_p} H_1^1(K, V) = 1$ and the restriction $H_1^1(K, V) \to \prod_{p \mid q} H_1^1(K, V)$ is an injection, then $E(K)$ has rank 1 and $\text{ord}_{s=1} L(E/K, s) = 1$.  

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We recall that [26 Thm. B] is proved by showing that the formal logarithm \( \log_p P_K \) of a suitable Heegner point \( P_K \in E(K) \) for a prime \( p \mid p \) of \( K \) does not vanish. The proof of this is via Iwasawa Theory. Under hypotheses (I)–(IV) there exists a \( p \)-adic \( L \)-function \( L_p(E/K, \chi) \) that is a function of certain anticyclotomic Hecke characters \( \chi \) and that \( p \)-adically interpolates the algebraic parts of the \( L \)-values \( L(E, \chi, 1) = L(V \otimes \sigma_\chi, 0) \) for those \( \chi \) having infinity-type \( z^{-n}z^n \) with \( n > 0 \), where \( \sigma_\chi \) is the \( p \)-adic avatar of \( \chi \). The connection with \( \log_p P_K \) comes in a remarkable formula for the value of this \( p \)-adic \( L \)-function at the trivial character (which does not belong to the set of characters \( \chi \) for which \( L_p(E/K, \chi) \) is interpolating the value of a complex \( L \)-function) that was proved by Bertolini, Darmon, and Prasanna [2 Main Thm.] and Brooks [3 Thm. IX.11] (see also [9 §7.3]):

\[
L_p(E/K, 1) = (1 - a_p(E) + p)^2(\log_p P_K)^2.
\]

The Main Conjecture of Iwasawa Theory, as formulated by Greenberg, would identify \( L_p(E/K, \chi) \) as the generator of the characteristic ideal of a certain \( p \)-adic Selmer group, one consequence of which would be

\[
L_p(E/K, 1) = 0 \implies H_p^1(K, V) = 0,
\]

where \( H_p^1(K, V) \subset H^1(K, V) \) consists of those classes that are trivial at the primes of \( K \) other than \( p \) and are unrestricted at \( p \). Under the hypotheses (I)–(IV), Wan [28 Thm. 1.1] has proved enough in the direction of this conjecture to deduce this implication. Furthermore, the hypothesis (V) implies \( H_p^1(K, V) = 0 \), hence \( L_p(E/K, 1) \neq 0 \), and so \( \log_p P_K \) is non-zero. In particular, \( P_K \in E(K) \) is non-torsion. From (V) it then follows that the rank of \( E(K) \) is 1, and it follows from the general Gross–Zagier formula proved by Yuan–Zhang–Zhang [29] that \( \text{ord}_{s=1} L(E/K, s) = 1 \).

**Proof of Theorem 3.** By hypothesis, \( E \) has good, ordinary reduction at \( p \) and \( E[p] \) is an irreducible \( \mathbb{G}_Q \)-module. From hypotheses (d) and (e) and Lemma 5 it then follows that the root number of \( E \) is \( w(E) = -1 \), \( \text{Sel}_{p, \infty}(E) \cong \mathbb{Q}_p/\mathbb{Z}_p \), and \( H^1_f(\mathbb{Q}, V) \sim H^1_f(\mathbb{Q}^p, V) \). It also follows from Ribet’s level-lowering result [22] that \( E[p] \) is ramified at some odd prime \( q \neq p \) (otherwise, the Galois representation \( E[p] \) would arise from some cuspidal eigenform of weight 2 and level 1, of which there are none). Let \( K/\mathbb{Q} \) be an imaginary quadratic field such that

- the discriminant \( D \) of \( K \) is odd;
- \( 2 \) and \( p \) split in \( K \);
- both the prime \( q \) and one other odd prime divisor of \( N_E \) are inert in \( K \) and all other prime divisors of \( N_E \) split in \( K \) (this is possible as \( N_E \) is assumed to have at least two odd prime divisors), in which case we have \( w(E^D) = w(E)\chi_D(−N_E) = +1 \) (see, e.g., [23 §3.10]);
- \( L(E^D, 1) \neq 0 \).

Such a quadratic field \( K \) exists by a theorem of Friedberg and Hoffstein [14 Thm. B]. Here, \( E^D \) is the \( D \)-twist of \( E \) and \( \chi_D \) is the quadratic Dirichlet character of conductor \( D \) associated with \( K \). We also use \( \chi_D \) to denote the quadratic character of \( \mathbb{G}_Q \) having kernel \( \text{Gal}(\mathbb{Q}/K) \).

We now appeal to Theorem B of [29] by verifying that hypotheses (I)–(V) hold for \( E \) and \( K \). That (I)–(IV) hold is immediate. Since \( L(E^D, 1) \neq 0 \), by [17 Thm. 14.2] or [19 Cor. B], we have \( H^1_f(\mathbb{Q}, V_p E^D) = H^1_f(\mathbb{Q}, V \otimes \chi_D) = 0 \). In particular, \( E^D(\mathbb{Q}) \) is finite. Then

\[
H^1_f(K, V) \cong H^1_f(\mathbb{Q}, V) \oplus H^1_f(\mathbb{Q}, V \otimes \chi_D) \cong H^1_f(\mathbb{Q}, V) \cong H^1_f(\mathbb{Q}^p, V) \Rightarrow \prod_{p \mid p} H^1_f(K_p, V),
\]
Theorem 5. The proof uses the representation also large. Any large family makes up a positive proportion of all elliptic curves \[5, \text{Thm. 3.17}\] by finitely many congruence conditions on \(A\) skew-symmetric matrices. The group \(\text{GL}_5\) whether \\

\[\text{GL}_F\] \\

whence (V) also holds. Therefore, \(E(K)\) has rank 1 and \(\text{ord}_{s=1} L(E/K, s) = 1\). Since the rank of \(E(K)\) is the sum of the ranks of \(E(\mathbb{Q})\) and \(E^D(\mathbb{Q})\) and since \(E^D(\mathbb{Q})\) is finite, \(E(\mathbb{Q})\) must have rank 1. As \(L(E/K, s) = L(E, s)L(E^D, s)\) and \(L(E^D, 1) \neq 0\), \(\text{ord}_{s=1} L(E/K, s) = 1\) implies \(\text{ord}_{s=1} L(E, s) = 1\).

\[\square\]

Proof of Theorem 4 We show that (I)–(V) hold. Note that (I)–(IV) are immediate from the hypotheses of Theorem 4. From Lemma 8 it follows that \(\text{Sel}_{p, \infty}(E) = 0\) (so \(E(\mathbb{Q})\) is finite and \(H^1_E(\mathbb{Q}, V) = 0\)), \(w(E) = +1\), \(w(E^D) = -1\), \(\text{Sel}_{p, \infty}(E) \cong \mathbb{Q}_p/\mathbb{Z}_p\), and \(H^1(E, V \otimes \chi_D) \sim H^1_{\ell}(\mathbb{Q}_p, V \otimes \chi_D)\). It follows that

\[H^1_E(K, V) \cong H^1_E(\mathbb{Q}, V \otimes \chi_D) \sim H^1_{\ell}(\mathbb{Q}_p, V \otimes \chi_D) \rightarrow \prod_{p \neq \ell} H^1_{\ell}(K_p, V),\]

whence (V) also holds. We then conclude, as in the proof of Theorem 3 but reversing the roles of \(E\) and \(E^D\), that \(E^D(\mathbb{Q})\) has rank 1 and \(\text{ord}_{s=1} L(E^D, s) = 1\).

\[\square\]

4 Equidistribution of Selmer elements

In this section, we prove Theorem 7 namely, that for any “large” family \(F\) of elliptic curves \(E\) over \(\mathbb{Q}\) lying in a sufficiently small \(\nu\)-adic disc so that all \(E(\mathbb{Q}_v)/5E(\mathbb{Q}_v)\) for \(E \in F\) are naturally identified, the non-identity elements of the 5-Selmer group become equidistributed in \(E(\mathbb{Q}_v)/5E(\mathbb{Q}_v)\). We also make precise what we mean by “naturally identified”.

4.1 Counting Selmer elements

We begin by recalling from [8] what is meant by a large family of elliptic curves. For each prime \(\ell\), let \(\Sigma_{\ell}\) be a closed subset of \(\{(A, B) \in \mathbb{Z}_\ell^2 : \Delta(A, B) := -4A^3 - 27B^2 \neq 0\}\) with boundary of measure 0. To such a collection \(\Sigma = (\Sigma_{\ell})_{\ell}\), we associate the set \(F_\Sigma\) of elliptic curves over \(\mathbb{Q}\), where \(E_{A,B} \in F_\Sigma\) if and only if \((A, B) \in \Sigma_{\ell}\) for all \(\ell\). We then say that \(F_\Sigma\) is a family of elliptic curves over \(\mathbb{Q}\) that is defined by congruence conditions. We can also impose “congruence conditions at infinity” on \(F_\Sigma\), by insisting that an elliptic curve \(E_{A,B}\) belongs to \(F_\Sigma\) if and only if \((A, B) \in \Sigma_{\infty}\), where \(\Sigma_{\infty}\) consists of all \((A, B)\) with \(\Delta(A, B)\) positive, or negative, or either.

If \(F\) is a family of elliptic curves over \(\mathbb{Q}\) defined by congruence conditions, then let \(\text{Inv}(F)\) denote the set \(\{(A, B) : E_{A,B} \in F\}\). We define \(\text{Inv}_p(F)\) to be the set of those elements \((A, B)\) in the \(p\)-adic closure of \(\text{Inv}(F) \subset \mathbb{Z}_p^2\) such that \(\Delta(A, B) \neq 0\). We define \(\text{Inv}_{\infty}(F)\) to be \(\{(A, B) \in \mathbb{R}^2 : \Delta(A, B) > 0\}\), \(\{(A, B) \in \mathbb{R}^2 : \Delta(A, B) < 0\}\), or \(\{(A, B) \in \mathbb{R}^2 : \Delta(A, B) \neq 0\}\) in accordance with whether \(F\) contains only curves of positive discriminant, negative discriminant, or both, respectively.

Then a family \(F\) of elliptic curves defined by congruence conditions is said to be large if, for all sufficiently large primes \(\ell\), the set \(\text{Inv}_\ell(F)\) contains all pairs \((A, B) \in \mathbb{Z}_\ell^2\) such that \(\ell^2 | \Delta(A, B)\). For example, the family of all elliptic curves \(E_{A,B}\) defined by finitely many congruence conditions on \(A\) and \(B\). The family of all semistable elliptic curves is also large. Any large family makes up a positive proportion of all elliptic curves [5 Thm. 3.17].

To explain the proof of Theorem 7 we briefly outline the strategy of the proof from [8] of Theorem 5. The proof uses the representation \(V = 5 \otimes \wedge^2 5\) of quintuples \((A_1, \ldots, A_5)\) of \(5 \times 5\) skew-symmetric matrices. The group \(\text{GL}_5 \times \text{GL}_5\) acts naturally on \(V\), by

\[(g_1, g_2) \cdot (A_1, A_2, A_3, A_4, A_5) := (g_1 A_1 g_1^t, g_1 A_2 g_1^t, g_1 A_3 g_1^t, g_1 A_4 g_1^t, g_1 A_5 g_1^t) \cdot g_2^t.\]

(2)
Let the determinant of an element \((g_1, g_2) \in \text{GL}_5 \times \text{GL}_5\) by defined by \(\det(g_1, g_2) := (\det g_1)^2 \det g_2\), and let \(G\) denote the algebraic group

\[
G := \{(g_1, g_2) \in \text{GL}_5 \times \text{GL}_5 : \det(g_1, g_2) = 1\}/\{\lambda I_5, \lambda^{-2} I_5\},
\]

where \(I_5\) denotes the identity element of \(\text{GL}_5\) and \(\lambda \in \mathbb{G}_m\). Then the action of \(\text{GL}_5 \times \text{GL}_5\) on \(V\) induces an action of \(G\) on \(V\). The ring of polynomial invariants for the action of \(G(\mathbb{C})\) on \(V(\mathbb{C})\) turns out to have two independent generators, having degrees 20 and 30, which we may denote by \(A(v)\) and \(B(v)\), respectively.

Now let \(K\) be a field of characteristic prime to 2, 3, and 5. If \(v = (A_1, \ldots, A_5) \in V(K)\), then let \(v(t_1, \ldots, t_5) := A_1 t_1 + \cdots + A_5 t_5\) denote the corresponding matrix of linear forms in \(t_1, \ldots, t_5\). Then the determinant of \(v(t_1, \ldots, t_5)\) is zero, since the determinant of any odd-dimensional skew-symmetric matrix is zero. So instead we consider its principal \(4 \times 4\) sub-Pfaffians (i.e., canonical square-roots of the principal \(4 \times 4\) minors of the matrix \(v(t_1, \ldots, t_5)\)). This yields five quadrics in five variables, which (whenever \(\Delta(v) = \Delta(A(v), B(v)) := -4A(v)^3 - 27B(v)^2 \neq 0\)) cut out a smooth genus one curve \(C(v)\) in \(\mathbb{P}^4\) over \(K\) that is embedded by a complete linear system of degree 5. We have chosen our generators \(A = A(v)\) and \(B = B(v)\) of the invariant ring so that the Jacobian \(E(v)\) of this genus one curve is the elliptic curve given by

\[
E_{A,B} : y^2 = x^3 + Ax + B.
\]

The discriminant \(\Delta(v)\) on \(V(K)\) (whose nonvanishing detects stable orbits on \(V(K)\)) thus coincides with the discriminant of the associated Weierstrass equation \([4,1]\) of \(E(v)\). Conversely, given any genus one curve \(C\) over \(\mathbb{Q}\) in \(\mathbb{P}^4\) embedded by a compete linear system of degree 5 and with Jacobian \(E_{A,B}\), there exists an element of \(V(\mathbb{Q})\) having invariants \(A\) and \(B\), unique up to the action of \(G(\mathbb{Q})\), that cuts out the curve \(C\) in \(\mathbb{P}^4\) in this way. Moreover, if the curve \(C\) has a point at every place of \(\mathbb{Q}\) (i.e., if \(C\) is \textit{locally soluble}), then there exists an element of \(V(\mathbb{Z})\) with invariants that cuts out \(C\) (see [13, Thm. 2.1])!

We say that an element \(v \in V(\mathbb{Z})\) (or \(V(\mathbb{Q})\)) is \textit{locally soluble} if the curve \(C(v)\) has a point locally at every place of \(\mathbb{Q}\). We say that \(v\) is \textit{soluble} if \(C(v)\) has a rational point. The \(G(\mathbb{Q})\)-orbits of locally soluble elements in \(V(\mathbb{Q})\) having invariants \(A\) and \(B\) with \(\Delta(A, B) \neq 0\) are then in natural bijection with the elements of the 5-Selmer group \(\text{Sel}_5(E_{A,B})\) of \(E_{A,B}\), and each such locally soluble \(G(\mathbb{Q})\)-orbit contains an element in \(V(\mathbb{Z})\). The analogous statements remain true when \(\mathbb{Q}\) and \(\mathbb{Z}\) are replaced by \(\mathbb{Q}_\ell\) and \(\mathbb{Z}_\ell\), and the 5-Selmer group of \(E_{A,B}(\mathbb{Q})\) is replaced by \(E_{A,B}(\mathbb{Q}_\ell)/5E_{A,B}(\mathbb{Q}_\ell)\).

The correspondence of a soluble \(v \in V(\mathbb{Q}_\ell)\) with an element of the local 5-Selmer group can be made explicit through a 5-cover \(\phi_v : C(v) \to E_{A,B}\) given in terms of certain covariants for the action of \(G\) on \(V\). All these latter facts and much more information on this representation can be found in the works of Fisher [11,12,13] (see also [4,5]).

In [8], the \(G(\mathbb{Q})\)-equivalence classes of locally soluble elements \(v \in V(\mathbb{Z})\) having bounded height and \(\text{Jac}(C(v)) \in F\) are counted asymptotically. By dividing through by the asymptotic number of elliptic curves over \(\mathbb{Q}\) of bounded height in \(F\), the average size of the 5-Selmer group of all elliptic curves in \(F\) is obtained, as in Theorem [5]. Specifically, if \(F = F_{\Sigma}\), then in [8] Prop. 33 and §4.2] (using [8, Theorem 29]), it is first proved that

\[
\lim_{X \to \infty} \frac{\sum_{E \in F, H(E) < X} (\#\text{Sel}_5(E) - 1)}{\sum_{E \in F, H(E) < X} 1} = \tau(G) \frac{M_\infty(V, F; X)}{M_\infty(F; X)} \prod_{\ell} \frac{M_\ell(V, F)}{M_\ell(F)}
\]

\[
(4)
\]
where

\[ M_\ell(V, F) := \int_{(A,B) \in \text{Inv}_\ell(F)} \frac{1}{\# E_{A,B}(\mathbb{Q}_\ell)[5]} \sum_{\sigma \in E_{A,B}(\mathbb{Q}_\ell)/5E_{A,B}(\mathbb{Q}_\ell)} 1 \ dAdB, \]

\[ M_\ell(F) := \int_{(A,B) \in \text{Inv}_\ell(F)} dAdB, \]

\[ M_\infty(V, F; X) := \int_{(A,B) \in \text{Inv}_\infty(F)} \frac{1}{\# E_{A,B}(\mathbb{R})[5]} \sum_{\sigma \in E_{A,B}(\mathbb{R})/5E_{A,B}(\mathbb{R})} 1 \ dAdB, \]

\[ M_\infty(F; X) := \int_{(A,B) \in \text{Inv}_\infty(F)} dAdB, \]

and \( \tau(G) \) denotes the Tamagawa number of \( G \) as an algebraic group over \( \mathbb{Q} \). The right hand side of Equation (4) then simplifies to \( \tau(G) = 5 \).

### 4.2 Proof of Theorem 7

We may extend the arguments of §4.1 to also prove equidistribution of 5-Selmer elements in \( E(\mathbb{Q}_\nu)/5E(\mathbb{Q}_\nu) \). We first use the representation of \( G(\mathbb{Q}_\nu) \) on \( V(\mathbb{Q}_\nu) \) to naturally identify the quotients \( E_v(\mathbb{Q}_\nu)/5E_v(\mathbb{Q}_\nu) \) corresponding to elements \( v \) in sufficiently small neighborhoods in \( V(\mathbb{Q}_\nu) \). With this identification, we then obtain the following version of Theorem 7:

**Theorem 9** Fix a place \( \nu \) of \( \mathbb{Q} \). Let \( F = F_\Sigma \) be a large family of elliptic curves \( E \) such that

- (a) the cardinality of \( E(\mathbb{Q}_\nu)/5E(\mathbb{Q}_\nu) \) is a constant \( k \) for all \( E \) in \( F \); and

- (b) the set

  \[ U_\nu(F) := \{ \text{soluble elements in } V(\mathbb{Z}_\nu) \text{ having invariants } (A,B) \text{ s.t. } (A,B) \in \Sigma_\nu \} \]

  can be partitioned into \( k \) open sets \( \Omega_1, \ldots, \Omega_k \) such that:

  (i) for all \( i \), if two elements in \( \Omega_i \) have the same invariants \( A, B \), then they are \( G(\mathbb{Q}_\nu) \)-equivalent; and

  (ii) for all \( i \neq j \), \( (G(\mathbb{Q}_\nu) \cdot \Omega_i) \cap (G(\mathbb{Q}_\nu) \cdot \Omega_j) = \emptyset \).

Then for \( E \in F \), the elements of \( E(\mathbb{Q}_\nu)/5E(\mathbb{Q}_\nu) \) are in bijection with the sets \( \Omega_i \). (In particular, the groups \( E(\mathbb{Q}_\nu)/5E(\mathbb{Q}_\nu) \) are identified for all \( E \) in \( F \).) When the elliptic curves \( E \) in \( F \) are ordered by height, the images of the nonidentity 5-Selmer elements under the restriction map

\[ \text{Sel}_5(E) \xrightarrow{\text{res}} E(\mathbb{Q}_\nu)/5E(\mathbb{Q}_\nu) \leftrightarrow \{ \Omega_1, \ldots, \Omega_k \} \]

are equidistributed.

To prove Theorem 7 on the right hand side of Equation (11) (which comes from [8, Prop. 33 and §4.2]), we replace the sum over all \( \sigma \) in \( E_{A,B}(\mathbb{Q}_\nu)/5E_{A,B}(\mathbb{Q}_\nu) \) in the expression for \( M_\nu(V, F) \) with the corresponding sum over \( \sigma \) in any subset \( S \subset E_{A,B}(\mathbb{Q}_\nu)/5E_{A,B}(\mathbb{Q}_\nu) \). By property (b), we
are still counting elements in a weighted subset of $V(\mathbb{Z})$ defined by an “acceptable” set of congruence conditions, so \[8\] Theorem 29 again applies. This gives us the average number of nonidentity 5-Selmer elements that map to $S$, and we see that the result is proportional to the size of $S$, proving Theorem \[9\].

Remark 10 Note that the same argument also allows one to show equidistribution of non-identity 5-Selmer elements in $\prod_{\nu \in S} E(Q_{\nu})/5E(Q_{\nu})$ for any finite set $S$ of places of $\mathbb{Q}$, provided that our large family $F$ of elliptic curves lies in the intersection of sufficiently small $\nu$-adic discs ($\nu \in S$) so that both (a) and (b) are satisfied for all $\nu \in S$.

To deduce Theorem \[10\] we combine Theorem \[9\] with two propositions. The first shows that if the elliptic curves in a large family $F$ all have invariants in a sufficiently small $\nu$-adic neighborhood, then the local $p$-Selmer groups are naturally identified so that the images of the torsion groups $E(Q_{\nu})[p]$ are also identified. The second proposition states that for a family of elliptic curves with invariants $A, B$ in a similarly small neighborhood, conditions (a) and (b) of Theorem \[9\] hold. Comparing these two propositions and their proofs then shows that for small enough neighborhoods, the identification of local 5-Selmer groups from the first of these propositions agrees with that coming from Theorem \[9\] from which Theorem \[7\] follows.

Proposition 11 Let $p \geq 5$ and $\ell$ be primes. Let $E \subset \mathbb{P}^2$ be an elliptic curve over $\mathbb{Q}_{\ell}$ given by the Weierstrass equation

$$y^2 + a_1 xy + a_3 = x^3 + a_2 x^2 + a_4 x + a_6, \quad a_i \in \mathbb{Q}_{\ell}. \quad (5)$$

There exists $h = h(E) \geq 2$ such that if $E' \subset \mathbb{P}^2$ is another elliptic curve over $\mathbb{Q}_{\ell}$ given by a Weierstrass equation

$$y^2 + a'_1 xy + a'_3 = x^3 + a'_2 x^2 + a'_4 x + a'_6, \quad a'_i \in \mathbb{Q}_{\ell}, \quad (6)$$

then $E(\mathbb{Q}_{\ell})/pE(\mathbb{Q}_{\ell})$ and $E'(\mathbb{Q}_{\ell})/pE'(\mathbb{Q}_{\ell})$ are identified so that

1. if $P = (x, y) \in E(\mathbb{Q}_{\ell})$ and $P' = (x, y) \in E'(\mathbb{Q}_{\ell})$ are such that $|x - x'|_{\ell} \leq \ell^{-h}, |y - y'|_{\ell} \leq \ell^{-h}$, then the images of $P$ and $P'$ are identified;
2. the natural images of $E(\mathbb{Q}_{\ell})[p] \to E(\mathbb{Q}_{\ell})/pE(\mathbb{Q}_{\ell})$ and $E'(\mathbb{Q}_{\ell})[p] \to E'(\mathbb{Q}_{\ell})/pE'(\mathbb{Q}_{\ell})$ are identified.

Proof: We first assume that (5) is a minimal Weierstrass equation. In particular, $a_i \in \mathbb{Z}_{\ell}$, Equation (5) defines a closed $\mathbb{Z}_{\ell}$-subscheme of $\mathbb{P}^2$ that we also denote by $E$, and the open subscheme $E_0$ obtained by removing the singular points on the closed fiber of $E$ is a group scheme over $\mathbb{Z}_{\ell}$ ($E_0(\mathbb{Z}_{\ell}) = E(\mathbb{Z}_{\ell})$ is just the subgroup of points of $E(\mathbb{Z}_{\ell})$ having nonsingular reduction modulo $\ell$). Suppose also that $E$ does not have split multiplicative reduction. Then the component group $E(\mathbb{Q}_{\ell})/E_0(\mathbb{Q}_{\ell})$ has order at most 4, hence, as $p \geq 5$, $E_0(\mathbb{Q}_{\ell})/pE_0(\mathbb{Q}_{\ell}) \cong E(\mathbb{Q}_{\ell})/pE(\mathbb{Q}_{\ell})$. As $E_0(\mathbb{Q}_{\ell}) = E_0(\mathbb{Z}_{\ell})$, it is easy to see that the reduction modulo $\ell^2$ map (modulo $\ell$ suffices if $\ell \neq p$) induces an isomorphism $E_0(\mathbb{Q}_{\ell})/pE_0(\mathbb{Q}_{\ell}) \sim E_0(\mathbb{Z}_{\ell}/\ell^2\mathbb{Z}_{\ell})/pE_0(\mathbb{Z}_{\ell}/\ell^2\mathbb{Z}_{\ell})$.
Suppose (4) holds. If \( h \geq 2 \) is sufficiently large (in terms of the \( a_i \)'s), then (6) will also be a minimal equation and \( E' \) will have the same reduction type as \( E \), and so, since \( E_0 \) and \( E'_0 \) have the same reduction modulo \( \ell^2 \), there are natural identifications

\[
E(\mathbb{Q}_\ell)/pE(\mathbb{Q}_\ell) \cong E_0(\mathbb{Z}_\ell/\ell^2\mathbb{Z}_\ell)/pE_0(\mathbb{Z}_\ell/\ell^2\mathbb{Z}_\ell) = E'_0(\mathbb{Z}_\ell/\ell^2\mathbb{Z}_\ell)/pE'_0(\mathbb{Z}_\ell/\ell^2\mathbb{Z}_\ell) = E'(\mathbb{Q}_\ell)/pE'(\mathbb{Q}_\ell).
\]

Then it is clear that (i) also holds. If \( E \) (and hence \( E' \)) has split multiplicative reduction, then a similar identification for which (i) also holds can be deduced from the Tate uniformization of \( E(\mathbb{Q}_\ell) \) and \( E'(\mathbb{Q}_\ell) \). As this case is not needed in this paper, we omit the details.

For a general Weierstrass equation (5), we note that if \( E \) is sufficiently close (that is, if \( h \) is sufficiently large) then the coefficients of these division polynomials are so close that

\[
\psi_i(x,y) \rightarrow \frac{x^{p^i} - y^{p^i}}{y} \quad (i = 1, \ldots, p^2 - 1)
\]

and satisfy

\[\left| x - x'_i \right|_\ell, \left| y - y'_i \right|_\ell \leq \ell^{-h_0}\]

so that if \( E' \) (provided that (7) holds) to a minimal equation. Furthermore, if \( h \) is sufficiently large (with respect to a fixed such transformation), then the coefficients of the resulting minimal equations will be close enough that the preceding arguments yield the desired identification and that (i) holds.

We now show that (ii) also holds for \( h \) large enough. Let \( h_0 \) be such that for a Weierstrass equation (6), if \( |a_i - a_i'|_\ell \leq \ell^{-h_0} \) then \( E(\mathbb{Q}_\ell)/pE(\mathbb{Q}_\ell) \) and \( E'(\mathbb{Q}_\ell)/pE'(\mathbb{Q}_\ell) \) are naturally identified so that if \( P = (x,y) \in E(\mathbb{Q}_\ell) \) and \( P' = (x',y') \in E'(\mathbb{Q}_\ell) \), then \( |x - x'|_\ell, |y - y'|_\ell \leq \ell^{-h_0} \), then the images of \( P \) and \( P' \) are identified (we have just proved the existence of such an \( h_0 \)).

The \( x \)-ordinates of the \( p^2 - 1 \) non-trivial \( p \)-torsion points \( P_i = (x_i, y_i) \) and \( P'_i = (x'_i, y'_i) \), \( i = 1, \ldots, p^2 - 1 \), of \( E(\mathbb{Q}_\ell) \) and \( E'(\mathbb{Q}_\ell) \), respectively, are the roots of polynomials of degree \( (p^2 - 1)/2 \) (the division polynomials denoted \( \psi_p \) in [24]) whose coefficients are given by universal polynomials over \( \mathbb{Z} \) in the coefficients of the Weierstrass equations (5) and (6). It then follows, say from Krasner’s lemma, that if the Weierstrass equations (5) and (6) are sufficiently close (that is, if \( |a_i - a_i'|_\ell \leq \ell^{-h_1} \) with \( h_1 > h_0 \) sufficiently large) then the coefficients of these division polynomials are so close that the \( P_i \) and \( P'_i \) can be ordered so that for each \( i \), \( P_i \) and \( P'_i \) are defined over the same extension of \( \mathbb{Q}_\ell \) and satisfy \( |x_i - x'_i|_\ell \leq \ell^{-h_0} \) and \( |y_i - y'_i|_\ell \leq \ell^{-h_0} \), and so, by the choice of \( h_0 \), the points in \( E(\mathbb{Q}_\ell)[p] \) and \( E'(\mathbb{Q}_\ell)[p] \) are identified in \( E(\mathbb{Q}_\ell)/pE(\mathbb{Q}_\ell) = E'(\mathbb{Q}_\ell)/pE'(\mathbb{Q}_\ell) \). Thus the conclusions of the proposition hold with \( h(E) = h_1 \).

As explained in the proof of this proposition, if the Weierstrass equation (5) is minimal and \( E \) does not have split multiplicative reduction at \( \ell \), then Part (i) of the proposition holds with \( h = 2 \) in (7).

**Proposition 12** Fix a place \( \nu \) of \( \mathbb{Q} \). For any given \( (A,B) \in \mathbb{Z}_\nu^2 \setminus \{ \Delta = 0 \} \), there exists a \( \nu \)-adic neighborhood \( W \) of \( (A_0, B_0) = (A,B) \) in \( \mathbb{Z}_\nu^2 \) such that the corresponding (large) family \( F = F(W) \), consisting of all elliptic curves \( E_{A,B} \) with \( (A,B) \in W \), satisfies both (a) and (b) of Theorem 9.

**Proof:** The proposition is trivial for \( \nu = \infty \), as we may let \( W \) equal the set of all \( (A,B) \) having discriminant positive, negative, or either. If \( \nu \) is a finite prime \( p \), let \( k = \text{the cardinality of } E_{A_0,B_0}(\mathbb{Q}_p)/5E_{A_0,B_0}(\mathbb{Q}_p) \). Then the set \( Y \) of soluble elements in the inverse image of \( (A,B) \) under the map \( \pi : V(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^2 \) given by sending \( v \) to the invariants \( (A(v), B(v)) \), is the disjoint union of \( k \) nonempty compact sets \( Y_1, \ldots, Y_k \), namely, the \( k \) \( G(\mathbb{Q}_p) \)-equivalence classes in \( V(\mathbb{Z}_p) \) comprising \( Y \).

Let \( Z_1, \ldots, Z_k \subset V(\mathbb{Z}_p) \setminus \{ \Delta = 0 \} \) be disjoint neighborhoods of \( Y_1, \ldots, Y_k \), respectively, in \( V(\mathbb{Z}_p) \) such that each \( Z_i \) consists of soluble elements and is the union of \( G(\mathbb{Q}_p) \)-equivalence classes in \( V(\mathbb{Z}_p) \). Such \( Z_i \) can be constructed by noting that if \( \epsilon \) is sufficiently small, then the \( \epsilon \)-neighborhoods \( B_\epsilon(Y_i) \) of the \( Y_i \)'s are disjoint and consist only of elements that have nonzero discriminant and are
soluble. The set \( \{g \in G(\mathbb{Q}_p) \mid gB_\varepsilon(Y_j) \cap V(\mathbb{Z}_p) \neq \emptyset \} \) is then compact. Indeed, for a single stable element \( v \in V(\mathbb{Z}_p) \) (i.e., \( v \) has nonzero discriminant \( \Delta(v) \)), the set

\[
\{g \in G(\mathbb{Q}_p) \mid g \cdot v \in V(\mathbb{Z}_p)\}
\]

is compact and constant in a neighborhood of \( v \). This may be seen from the decomposition of \( G(\mathbb{Q}_p) \) as \( G(\mathbb{Z}_p)T(\mathbb{Q}_p)G(\mathbb{Z}_p) \), where \( T(\mathbb{Q}_p) \) is the torus consisting of the diagonal elements of \( G(\mathbb{Q}_p) \); since the set of elements of \( T(\mathbb{Q}_p) \) taking a stable element \( v' \in V(\mathbb{Z}_p) \) to \( V(\mathbb{Z}_p) \) is compact and constant in a neighborhood of \( v' \), it follows that \( \{\} \) is also compact and constant in a neighborhood of \( v \).

The compactness of

\[
\{g \in G(\mathbb{Q}_p) \mid gS \cap V(\mathbb{Z}_p) \neq \emptyset\}
\]

now follows for any compact set \( S \) of stable elements (by covering \( S \) with neighborhoods where \( \{\} \) is constant, and then taking a finite subcover). Hence \( (G(\mathbb{Q}_p) \cdot B_\varepsilon(Y_j)) \cap V(\mathbb{Z}_p) \) is both open and compact, and so is a bounded distance away from \( Y_j \) for all \( j \neq i \). By shrinking \( \varepsilon \) if necessary, we can then ensure that \( (G(\mathbb{Q}_p) \cdot B_\varepsilon(Y_i)) \cap B_\varepsilon(Y_j) = \emptyset \) for all \( j \neq i \), and therefore \( (G(\mathbb{Q}_p) \cdot B_\varepsilon(Y_i)) \cap V(\mathbb{Z}_p) = \emptyset \) for all \( i \neq j \). We then set \( Z_i = (G(\mathbb{Q}_p) \cdot B_\varepsilon(Y_i)) \cap V(\mathbb{Z}_p) \).

Let \( W' = \{(A, B) \in \mathbb{Z}_p^2 \mid (A, B) \in \cap_i \pi(Z_i)\} \). Since \( \pi \) is an open mapping on \( V(\mathbb{Z}_p) \setminus \{\Delta = 0\} \), we see that \( W' \) is an open set in \( \mathbb{Z}_p^2 \) containing \( (A, B) \). Let \( W \subset W' \) be an open neighborhood of \( (A, B) \) small enough so that for all elliptic curves \( E = E_{A,B} \) with \( (A, B) \in W \), we have \( \#(E(\mathbb{Q}_5)/5E(\mathbb{Q}_5)) = k \). Such a neighborhood \( W \) exists because the size of \( E(\mathbb{Q}_5)/5E(\mathbb{Q}_5) \) is locally constant (see Proposition 11). Then \( F(W) \) satisfies both (a) and (b), with \( \Omega_i = Z_i \cap \pi^{-1}\{(A, B) \in W\} \).

We now complete the proof of Theorem 7. We set \( \nu = p = 5 \) in Proposition 12. For \( i = 1, \ldots, k \) in Theorem 9, fix \( v_i \in Y_i \) and \( P_i \in C(v_i)(\mathbb{Q}_5) \) (so \( Y_i \) is identified with the image of \( \phi_v(P_i) \in E_{A_0,B_0}(\mathbb{Q}_5)/5E_{A_0,B_0}(\mathbb{Q}_5) \)). If \( \varepsilon \) is small enough, then for any \( v \in B_\varepsilon(Y_i) \) we have:

- \( (A, B) = (A(v), B(v)) \) are close enough 5-adically to \( (A, B) \) so that for \( E = E_{A_0,B_0} \) and \( E' = E_{A,B} \), the conclusions of Proposition 11 hold;

- the quadratic equations defining \( C(v) \) are close enough 5-adically to those defining \( C(v_i) \) to ensure that there is a point \( P' \in C(v)(\mathbb{Q}_5) \) close enough to \( P_i \) so that the hypotheses of Proposition 11(i) hold for \( \phi_{v_i}(P_i) \in E(\mathbb{Q}_5) \) and \( \phi_v(P) \in E'(\mathbb{Q}_5) \); therefore, the images of \( \phi_{v_i}(P_i) \) and \( \phi_v(P) \) agree under the identification \( E(\mathbb{Q}_5)/5E(\mathbb{Q}_5) = E'(\mathbb{Q}_5)/5E'(\mathbb{Q}_5) \) given by Proposition 11.

It follows that the identification of \( E(\mathbb{Q}_5)/5E(\mathbb{Q}_5) \) with \( E'(\mathbb{Q}_5)/5E'(\mathbb{Q}_5) \) given by the sets \( \Omega_i \) is the same as the identification in Proposition 11. In particular, by Proposition 11(ii), the \( W \) in Proposition 12 can be taken so that the identification of local Selmer groups in Theorem 9 also identifies the images of the \( p \)-torsion subgroups. Theorem 7 then follows from the conclusions of Theorem 9 for the large set \( F(W) \).

**Remark 13** The analogues of Theorems 7 and 9 for the equidistribution of elements in 2-, 3-, and 4-Selmer groups (instead of the 5-Selmer group) may be proven by analogous arguments, using the results in 5, 6, and 7, respectively (instead of 8).
5 Counting curves: Proof of Theorem 1

We recall the definition of the set $\mathcal{F}$ from Section 2: $\mathcal{F}$ consists of those elliptic curves $E = E_{A,B}$ such that

- $2^3||A$ and $2^4||B$;
- $\Delta(A,B) := -4A^3 - 27B^2$ equals $2^8\Delta_1(A,B)$ with $\Delta_1(A,B)$ positive and squarefree (and necessarily odd), $(\Delta(A,B), 5 \cdot 39) = 1$, and $\Delta(A,B)$ is a square modulo 39;
- $E$ has non-split multiplicative reduction at 7;
- $E$ has good, ordinary reduction at 5.

The discriminant of such an $E$ is $16\Delta(A,B)$ and its conductor is just $\Delta_1(A,B)$, which is squarefree and odd. In particular, $E$ is a semistable curve. The set $\mathcal{F}$ is a large family of elliptic curves defined by congruence conditions in the sense of $[5, \S 3]$ (see also Subsection 4.1).

We begin by verifying that all elliptic curves $E \in \mathcal{F}$ satisfy properties (a)–(c) in both Theorems 3 and 4 (with $p = 5$, $K = \mathbb{Q}(\sqrt{-39})$, and $q = 7$):

Lemma 14 Let $E \in \mathcal{F}$. The $G_{\mathbb{Q}}$-module $E[5]$ is irreducible and ramified at every prime factor of the conductor $N_E$, so in particular at 7.

Proof: If $E[5]$ were reducible, then its semisimplification would be a sum of two characters: $E[5]^G \cong \mathbb{F}_5(\chi\omega) \oplus \mathbb{F}_5(\chi^{-1})$, with $\omega$ the mod 5 cyclotomic character. As $E$ has semistable reduction, the conductor of $E[5]^G$ at a prime $\ell \neq 5$, which divides the conductor of $E$, can be at most $\ell$, from which it follows that $\chi$ is unramified at all primes different from 5. But since $E$ also has good, ordinary reduction at 5, it must be that either $\chi^{-1}$ or $\chi\omega$ is unramified at 5 and so is unramified everywhere. It then follows that either $\chi = 1$ or $\chi = \omega$, whence $E[5]^G \cong \mathbb{F}_5(\omega) \oplus \mathbb{F}_5$. Since $E$ is also assumed to have non-split, multiplicative reduction at 7, one of the eigenvalues of a Frobenius at 7 on $E[5]^G$ must be $-1$ modulo 5, a contradiction (as $7 \not\equiv -1 \pmod{5}$). It follows that $E[5]$ is irreducible. The condition that $E[5]$ be unramified at a prime $\ell \neq 5$ of multiplicative reduction is that the discriminant $\Delta_\ell$ of a minimal Weierstrass model at $\ell$ satisfy $\text{ord}_\ell(\Delta_\ell) \equiv 0 \pmod{5}$ $[25, \text{Chap. V, Prop. 6.1 & Ex. 5.13}]$. But the Weierstrass model $E_{A,B}$ of $E$ is clearly a minimal model at each odd prime $\ell$ since $\Delta_1(A,B)$ is squarefree by hypothesis, and so $\text{ord}_\ell(\Delta_\ell) = \text{ord}_\ell(\Delta(A,B)) = 1$. \hfill $\square$

We now turn to counting various families of elliptic curves, in order to establish that a positive proportion of elliptic curves $E \in \mathcal{F}$ also satisfy properties (d)–(e) in either Theorem 3 or 4 (with the same choices of $p$ and $K$). We recall $([5, \text{Thm. 3.17}])$:

Lemma 15 For any large set $F$ of elliptic curves, there exists a constant $c(F) > 0$ such that

$$\# \{E \in F : H(E) < X\} = c(F)X^{5/6} + o(X^{5/6}).$$

Thus, in particular, the elliptic curves in our large family $\mathcal{F}$ have positive density in the family of all elliptic curves over $\mathbb{Q}$, when ordered by height.
Our aim now is to count the number of curves in \( \mathcal{F} \) that satisfy either the hypotheses of Theorem \([3]\) with \( p = 5 \), or the hypotheses of Theorem \([4]\) with \( p = 5 \), \( K = \mathbb{Q}[\sqrt{-39}] \) (so \( D = -39 \)), and \( q = 7 \). Let

\[
N(X) = \# \{ E \in \mathcal{F} : H(E) < X \} \\
N_i(X) = \# \{ E \in \mathcal{F} : H(E) < X \text{ and } \mathrm{Sel}_p(E) \cong (\mathbb{Z}/p\mathbb{Z})^i \} \\
N_{\text{even}}(X) = \# \{ E \in \mathcal{F} : H(E) < X \text{ and } \mathrm{Sel}_p(E) \cong (\mathbb{Z}/p\mathbb{Z})^{2j} \text{ for some } j \} \\
N_{\text{odd}}(X) = \# \{ E \in \mathcal{F} : H(E) < X \text{ and } \mathrm{Sel}_p(E) \cong (\mathbb{Z}/p\mathbb{Z})^{2j+1} \text{ for some } j \}
\]

Then \( N(X) = N_{\text{even}}(X) + N_{\text{odd}}(X) = \sum_{i=0}^{\infty} N_i(X) \). Note that, by Lemma \([8]\)(ii), we may also write

\[
N_{\text{even}}(X) = \# \{ E \in \mathcal{F} : H(E) < X \text{ and } w(E) = 1 \} \\
N_{\text{odd}}(X) = \# \{ E \in \mathcal{F} : H(E) < X \text{ and } w(E) = -1 \}.
\]

Similarly, let

\[
N_i^D(X) = \# \{ E \in \mathcal{F} : H(E) < X \text{ and } \mathrm{Sel}_p(E^D) \cong (\mathbb{Z}/p\mathbb{Z})^i \}.
\]

By Lemmas \([11]\) and \([8]\)(ii), the curves counted by \( N_0(X) \) and \( N_1^D(X) \) (resp. by \( N_1(X) \) and \( N_0^D(X) \)) are a subset of those counted by \( N_{\text{even}}(X) \) (resp. \( N_{\text{odd}}(X) \)).

We also need to count the number \( \tilde{N}_1(X) \) of curves in \( \mathcal{F} \) with height \( < X \) and \( \mathrm{Sel}_p(E) \cong \mathbb{Z}/p\mathbb{Z} \) whose restriction to \( E(\mathbb{Q}_p)/pE(\mathbb{Q}_p) \) lies in the image of \( E(\mathbb{Q}_p)[p] \), and the number \( \tilde{N}_1^D(X) \) of curves in \( \mathcal{F} \) with height \( < X \) and \( \mathrm{Sel}_p(E^D) \cong \mathbb{Z}/p\mathbb{Z} \) whose restriction to \( E^D(\mathbb{Q}_p)/pE^D(\mathbb{Q}_p) \) lies in the image of \( E^D(\mathbb{Q}_p)[p] \).

**Lemma 16** We have

\[
\tilde{N}_1(X) \leq \frac{1}{p-1} N(X) - (N_{\text{even}}(X) - N_0(X)) - (p+1)(N_{\text{odd}}(X) - N_1(X)) + \tilde{\varepsilon}_1(X)
\]

and

\[
\tilde{N}_1^D(X) \leq \frac{1}{p-1} N(X) - (N_{\text{odd}}(X) - N_0^D(X)) - (p+1)(N_{\text{even}}(X) - N_1^D(X)) + \tilde{\varepsilon}_1^D(X),
\]

where both \( \tilde{\varepsilon}_1(X) \) and \( \tilde{\varepsilon}_1^D(X) \) are \( o(X^{5/6}) \).

**Proof:** By Theorem \([7]\), the large family \( \mathcal{F} \) can be partitioned into a finite union of large subfamilies for each of which the groups \( E(\mathbb{Q}_p)/pE(\mathbb{Q}_p) \) as well as the images of \( E(\mathbb{Q}_p)[p] \) have been identified. Furthermore, as each \( E \in \mathcal{F} \) has good, ordinary reduction at \( p \), we have that \( \#E(\mathbb{Q}_p)/pE(\mathbb{Q}_p) = p \cdot \#E(\mathbb{Q}_p)[p] \) is equal to \( p \) or \( p^2 \). By Theorem \([5]\) Lemma \([13]\) and the equidistribution result of Theorem \([7]\) for the curves in each of these subfamilies, the number of non-trivial Selmer elements in \( \mathrm{Sel}_p(E) \) for some \( E \in \mathcal{F} \) with \( H(E) < X \) that restrict to an element in the image of \( E(\mathbb{Q}_p)[p] \) is

\[
\frac{\#E(\mathbb{Q}_p)[p]}{\#E(\mathbb{Q}_p)/pE(\mathbb{Q}_p)} pN(X) + o(X^{5/6}) = N(X) + \tilde{\varepsilon}_1(X)
\]

where \( \tilde{\varepsilon}_1(X) = o(X^{5/6}) \).

Now for \( E \in \mathcal{F} \), if \( w(E) = 1 \) but \( \mathrm{Sel}_p(E) \neq 0 \), then \( \#\mathrm{Sel}_p(E) \geq p^2 \) by Lemma \([8]\)(ii). Since the restriction map is a homomorphism, there are at least \( p - 1 \) non-trivial elements in \( \mathrm{Sel}_p(E) \)
whose restriction lies in the image of \( E(\mathbb{Q}_p)[p] \). Similarly, if \( w(E) = -1 \) but \( \text{Sel}_p(E) \neq 0 \), then \( \#\text{Sel}_p(E) \geq p^3 \), and there are at least \( p^2 - 1 \) non-trivial elements in \( \text{Sel}_p(E) \) with restriction in the image of \( E(\mathbb{Q}_p)[p] \). It follows that

\[
\widetilde{N}_1(X)(p-1) + (p-1)(N_{\text{even}}(X) - N_0(X)) + (p^2 - 1)(N_{\text{odd}}(X) - N_1(X)) \leq N(X) + \varepsilon_1(X)
\]

where \( \varepsilon_1(X) = o(X^{5/6}) \), yielding the inequality for \( \widetilde{N}_1(X) \) in the statement of the lemma. An identical argument applies to \( \widetilde{N}_1^D(X) \).

Let \( \mathcal{F}_{\text{sat}} \subset \mathcal{F} \) be the subset of curves that satisfy the hypotheses of Theorem 3 with \( p = 5 \), and let \( \mathcal{F}^D_{\text{sat}} \subset \mathcal{F} \) be the subset of curves that satisfy the hypotheses of Theorem 4 with \( p = 5 \), \( K = \mathbb{Q}[\sqrt{-39}] \), and \( q = 7 \). These are disjoint subsets of \( \mathcal{F} \). Let

\[
N_{\text{sat}}(X) = \# \{ E \in \mathcal{F}_{\text{sat}} : H(E) < X \}
\]

and similarly

\[
N^D_{\text{sat}}(X) = \# \{ E \in \mathcal{F}^D_{\text{sat}} : H(E) < X \}.
\]

**Proposition 17** We have

\[
N_{\text{sat}}(X) + N^D_{\text{sat}}(X) \geq N(X) \left( 1 - \frac{2}{p - 1} \right) + \varepsilon_{\text{sat}}(X)
\]

with \( \varepsilon_{\text{sat}}(X) = o(X^{5/6}) \).

**Proof:** The definition of \( \mathcal{F} \) together with Lemma 14 and the observation that 7 is not a square modulo 39 (or the observation that there are no elliptic curves of conductor 7) shows that any curve in the count \( N_1(X) \) satisfies (a)–(d) of Theorem 3 so

\[
N_{\text{sat}}(X) = N_1(X) - \widetilde{N}_1(X).
\]

Similarly, the number of curves in \( \mathcal{F} \) of height \( < X \) that satisfy (a)–(d) of Theorem 4 is at least \( N_1^D(X) + N_0(X) - N_{\text{even}}(X) \), so

\[
N^D_{\text{sat}}(X) \geq N_1^D(X) - \widetilde{N}_1^D(X) + N_0(X) - N_{\text{even}}(X).
\]

Thus, by Lemma 16

\[
N_{\text{sat}}(X) \geq N_1(X) - \frac{N(X)}{p - 1} + (N_{\text{even}}(X) - N_0(X)) + (p + 1)(N_{\text{odd}}(X) - N_1(X)) - \varepsilon_1(X)
\]

\[
\geq N_{\text{odd}}(X) - \frac{N(X)}{p - 1} + N_{\text{even}}(X) - N_0(X) - \varepsilon_1(X),
\]

and, similarly,

\[
N^D_{\text{sat}}(X) \geq N_1^D(X) - \widetilde{N}_1^D(X) + N_0(X) - N_{\text{even}}(X)
\]

\[
\geq N_{\text{even}}(X) - \frac{N(X)}{p - 1} + N_0(X) - N_{\text{even}}(X) - \varepsilon_1^D(X).
\]

Combining these two inequalities yields the lower bound in the proposition. \( \square \)
Proof of Theorem 1: The number of elliptic curves of height < $D^6X$ that have both analytic and algebraic rank one is at least the number of curves $E \in \mathcal{F}$ of height < $X$ satisfying the hypotheses of Theorem 3 with $p = 5$ plus the number of elliptic curves $E^D$ with $E \in \mathcal{F}$ of height < $X$ (since $H(E^D) = D^6H(E)$) satisfying the hypotheses of Theorem 4 with $p = 5$, $K = \mathbb{Q}[\sqrt{-39}]$, and $q = 7$—that is, at least $N_{\text{sat}}(X) + N_{\text{sat}}^D(X)$. Therefore,

$$\liminf_{X \to \infty} \frac{\# \{ E : \text{rk}(E) = \text{rk}_{\text{an}}(E) = 1 \text{ and } H(E) < D^6X \}}{\# \{ E : H(E) < D^6X \}} \geq \liminf_{X \to \infty} \frac{N_{\text{sat}}(X) + N_{\text{sat}}^D(X)}{\# \{ E : H(E) < D^6X \}}. \quad (9)$$

Let $\mathcal{E}$ be the set of all elliptic curves over $\mathbb{Q}$. As $\# \{ E : H(E) < D^6X \} = c(\mathcal{E})D^5X^{5/6} + o(X^{5/6})$ for some $c(\mathcal{E}) > 0$ by Lemma 15 it follows by Proposition 17 that the right hand side of (9) is at least

$$\liminf_{X \to \infty} \frac{N(X)(1 - \frac{2}{p+1})}{\# \{ E : H(E) < D^6X \}} = \frac{c(\mathcal{F})}{c(\mathcal{E})} \cdot \frac{1}{2} \cdot \frac{1}{39^5} > 0.$$ 

Remark 18 For the counting argument given, it is crucial that we were able to work with $p = 5$. For $p = 3$ the upper bound on $\tilde{N}_1(X)$ and $\tilde{N}_1^D(X)$ would be $\frac{1}{2}N(X)$, which means we would have had to potentially exclude 50% of the curves in $\mathcal{F}$ from our counts due to the restriction conditions (e) in Theorems 3 and 4 (instead of only 25% for $p = 5$); however, we only expect 50% of the curves in $\mathcal{F}$ and $\mathcal{F}^D$ to have root number $-1$. This is reflected in the lower bound for $N_{\text{sat}}(X) + N_{\text{sat}}^D(X)$ in Proposition 17 which would be trivial if $p = 3$.

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