Exponential growth and Gaussian–like fluctuations of solutions of stochastic differential equations with maximum functionals

J A D Appleby and H Wu
Edgeworth Centre for Financial Mathematics, School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland
E-mail: john.appleby@dcu.ie, huizhong.wu4@mail.dcu.ie

Abstract. In this paper we consider functional differential equations subjected to either instantaneous state–dependent noise, or to a white noise perturbation. The drift of the equations depend linearly on the current value and on the maximum of the solution. The functional term always provides positive feedback, while the instantaneous term can be mean–reverting or can exhibit positive feedback. We show in the white noise case that if the instantaneous term is mean reverting and dominates the history term, then solutions are recurrent, and upper bounds on the a.s. growth rate of the partial maxima of the solution can be found. When the instantaneous term is weaker, or is of positive feedback type, we determine necessary and sufficient conditions on the diffusion coefficient which ensure the exact exponential growth of solutions. An application of these results to an inefficient financial market populated by reference traders and speculators is given, in which the difference between the current instantaneous returns and maximum of the returns over the last few time units is used to determine trading strategies.

1. Introduction
This paper introduces a class of stochastic functional differential equations whose structure is motivated by three ubiquitous forms of heuristic investment strategy in financial markets: the comparison of current prices with a reference level; trading on noise (or the latest news); and trading based on a comparison of the local maximum of prices with the current price. It is the presence of the last category of speculative behaviour which makes it reasonable to incorporate a maximum functional of the process on the right hand of the stochastic differential equation. Accordingly, the equations studied are stochastic functional differential equations of Itô–type with maximum functionals, and, in order to gain an understanding of the impact of such functionals in simple market models, we establish a number of results concerning the growth rate and size of large fluctuations of solutions. For the quasilinear equations studied we are able to identify two comprehensive and non—overlapping parameter regions in which the equation has either transient or recurrent solutions. The parameters measure the degree of positive feedback from the delayed term and the degree and sign of the positive or negative feedback from the instantaneous term. The transient case, which results from the presence of sufficient positive feedback from the delayed term, may be interpreted as a runaway asset price bubble or crash. The manner in which these dynamics form is consistent with the phenomenon of
mimetic contagion [33]. The recurrent case, which results from the presence of sufficient negative feedback from the instantaneous term, may be interpreted as a conventional fluctuating market. In this case we study the asymptotic rate of growth of the partial maxima of the equation, which measures the asymptotic rate of growth of the largest fluctuations. These results allow us to show that the instantaneous returns are governed by fluctuations consistent with a stationary Gaussian process.

The last few years has seen an increasing interest in modelling financial markets which exhibit informational inefficiency and which therefore violate some form of the Efficient Market Hypothesis (EMH). A classical and seminal collection of papers summarising the rationale behind the EMH is [15]. From the standpoint of stochastic dynamics, the validity of the EMH precludes the dynamics of stock returns satisfying a stochastic functional differential equation; prices (and other market indicators) should possess the Markov property in the natural finite–dimensional state space of the model, because according to the EMH, new information about the price should automatically be incorporated into the current price. Therefore the evolution of prices in the future (after time $t$ say) should not depend on the price history, but should depend only on the current price and any “news” relevant to the asset price evolution, which is usually considered independent of all events that have taken place up to time $t$. Mathematically, “news” is often modelled as increments of a Brownian motion or a more general Lévy process, and we follow this convention in this work, where the driving semimartingale is a standard one–dimensional Brownian motion.

The picture of returns evolution painted by the EMH can be subjected to some general criticism. More particularly, econometric evidence of market returns (cf., e.g. [30]), survey evidence of the history–dependent trading strategies of real financial market agents (cf., e.g. [32]), and the manner in which human psychology tends to shape preferred investment strategies (cf. e.g. [24]) point to the EMH being regularly violated. If we presume that at least some agents base their investment decisions on past price behaviour, and that that a disequilibrium between demand and supply determines the movement of prices, then the price (or return) dynamics can be described by a stochastic functional differential equation. A number of papers using stochastic functional differential or difference equations to model risky asset price dynamics include [1, 2, 7, 6, 13, 23].

A feature of many of these stochastic functional equation models is that traders who base their demand on the price history use moving averages of past prices or returns. This leads to stochastic Volterra equations or to equations using linear functionals of past prices or returns. These equations are quite tractable analytically, but do not allow for the inclusion of agents who use the maximum of the last several periods of returns as a trading indicator. If we wish to include such agents in our model, it will first be necessary to understand and deduce some properties of stochastic functional differential equations with maxima. A body of literature on deterministic equations with maxima has begun to mature in the last ten years, building on original work on the stability of functional differential inequalities with maxima found in [22]. The main result in this direction is referred to as Halanay’s inequality. Current research on deterministic functional differential equations with maxima covers results on existence, oscillation and asymptotic behaviour. A selection of important and representative recent papers is [26, 28, 29]. On the other hand, Halanay’s inequality has been employed in numerical analysis [9], and in the numerical analysis of stochastic functional differential equations [10, 11, 12] in particular. Despite the analysis in [10, 11, 12], it seems that very limited information about stochastic functional differential equations with maxima has appeared in the mathematical literature, apart from existence and uniqueness results and basic exponential moment and pathwise growth bounds on solutions (see [31]). For this type of SFDE, it is usually not possible to express the solution explicitly in terms of an underlying deterministic differential equation. However, by employing a constructive comparison technique similar to that developed in the
study of almost sure asymptotic behaviour of SFDEs in [3, 5], we find it possible to determine quite sharp estimates on the rate of growth of both the partial maxima and of the solutions themselves in the recurrent and transient cases respectively.

The equations studied in this paper do not appear to be amenable to analysis using the techniques of hysteresis, owing to the nature of the interval on which maxima are computed. However, it is clear that close cousins of the equations studied here would have hysteretic properties. Examples of such equations might include those with a whole line maximum functional, or those involving a combination of maximum and minimum functionals evaluated on time intervals defined by a Preisach sequence. A selection of recent papers exploring applications of hysteresis functional differential equation models in economics and finance include [16, 17, 18, 19, 20, 21].

The paper is organised as follows: Section 2 presents and discusses the main mathematical results of the paper. Section 3 shows how the equations presented in Section 2 can be applied to model inefficient markets, and the mathematical results stated in Section 2 are used to interpret the behaviour of the financial market model. Section 4, which concludes the paper, contains the proofs of the results stated in Section 2.

2. Discussion of main results

In this paper we prove results concerning the asymptotic growth and large fluctuations of solutions of stochastic functional differential equations with both state-dependent and state-independent (or additive) noise. This means that the diffusion coefficient in the Itô-type equation can be free of the state, or can depend on the state.

Although both types of equation share the same drift term, we prefer to introduce and label them separately. This is because the state-dependent equations studied often enjoy a positivity property which will not necessarily be satisfied by the additive noise equations.

We therefore first turn our attention equations with additive noise; after this, we consider equations with state-dependent noise. The following section is devoted to economic modelling and the economic interpretation of the results, with proofs of the main results following in the final section.

In what follows, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $B = \{B(t), \mathcal{F}^B(t); 0 \leq t < \infty\}$ be a one-dimensional Brownian motion on it. Here the filtration is the natural one, viz., $\mathcal{F}^B(t) = \sigma(B(s) : 0 \leq s \leq t)$. Thus $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t \geq 0}, \mathbb{P})$ is a complete probability space. Almost sure events referred to in this paper are always $\mathbb{P}$-almost sure. For results from stochastic analysis, we refer the reader to e.g., Revuz and Yor [34] for the Burkholder-Davis-Gundy inequalities [34, Theorem IV.4.1], the martingale convergence theorem [34, Proposition IV.1.26], and Laws of the Iterated Logarithm for martingales [34, Exercise V.1.15] and for standard Brownian motion [34, Theorem II.1.9]. The conventional Landau big O and little o notation is used throughout the paper.

2.1. Max-type equations with additive noise

We suppose that $\zeta \neq 0$ is a real number, let $\tau > 0$ and suppose that $\psi \in C([-\tau, 0]; (0, \infty))$ be a deterministic function. We consider the stochastic functional differential equation of Itô type

\begin{align}
X(t) &= \psi(0) + \int_0^t \left( aX(s) + b \sup_{s-\tau \leq u \leq s} X(u) \right) ds + \int_0^t \zeta dB(s), \quad t \geq 0; \\
X(t) &= \psi(t), \quad t \in [-\tau, 0].
\end{align}
Then (1) has a unique global strong solution. We name the equation the \textit{generalized Langevin equation with maximum delay} which follows from the equation

\begin{align}
X(t) &= \psi(0) + \int_0^t (aX(s) + bX(s - \tau)) \, ds + \int_0^t \varsigma \, dB(s), \quad (2a) \\
X(t) &= \psi(t), \quad t \in [-\tau, 0]. \quad (2b)
\end{align}

We presume that $b > 0$ hereinafter. It is often convenient to write (1) in the equivalent differential shorthand form

\[dX(t) = \left( aX(t) + b \sup_{t-\tau \leq s \leq t} X(s) \right) \, dt + \varsigma \, dB(t),\]

and we will employ this shorthand for this formula and others throughout the paper.

If $b > 0$, we will show that the solutions of (1) grow exponentially in the case when $a + b > 0$ and will fluctuate with a given growth rate in the case when $a + b < 0$. To motivate the result in the first case, we show that when $a + b > 0$, solutions of the underlying deterministic equation grow exponentially, and are asymptotic to $e^{(a+b)t}$ as $t \to \infty$. The deterministic equation is

\[x'(t) = ax(t) + b \sup_{t-\tau \leq s \leq t} x(s), \quad t > 0; \quad x(t) = \psi(t), \quad t \in [-\tau, 0], \quad (3)\]

where once again we request that $\psi \in C([-\tau, 0]; \mathbb{R})$. In this case, equation (3) has a unique continuous solution, which is moreover positive in the case that $\psi(t) \geq 0$ (but not $\psi \neq 0$) for all $t \in [-\tau, 0]$ (we rule out the case $\psi \equiv 0$, because in this case $x(t) = 0$ for all $t \geq -\tau$).

In the case $x(0) > 0$, we have

\[\frac{d}{dt} \left( x(t)e^{-(a+b)t} \right) = be^{-(a+b)t} \left( \max_{t-\tau \leq s \leq t} x(s) - x(t) \right) \geq 0, \quad (4)\]

so $e^{-(a+b)t}x(t) \geq x(0)$, and so $x(t) > 0$ for all $t \geq 0$ and $t \mapsto e^{-(a+b)t}x(t)$ is nondecreasing.

In the case when $x(0) = 0$, we must proceed more carefully. By the continuity of $\psi$ there is an an interval $I = (-t_1, -t_2)$ such that $\psi$ is positive on $I$, and so $x'(0) = b \sup_{t-\tau \leq s \leq 0} \psi(s) > 0$. Therefore, there is a $t_3 > 0$ such that $x(t) > 0$ for all $t \in (0, t_3)$. Hence for $t \in (0, t_3)$ we have $x'(t) \geq (a + b)x(t)$, and so $x$ is positive and increasing on $(0, t_3)$. On $[t_3, \infty)$, $x$ again satisfies $x'(t) \geq (a + b)x(t)$ with $x(t_3) = \xi_3 > 0$, and so $x(t) > 0$ for all $t > 0$. By (4), $t \mapsto e^{-(a+b)t}x(t)$ is non–decreasing.

Since $t \mapsto e^{-(a+b)t}x(t)$ is non–decreasing whether $x(0) > 0$ or $x(0) = 0$, and $t \mapsto e^{(a+b)t}$ is increasing, we have that $t \mapsto x(t)$ is non–decreasing on $[0, \infty)$. Hence for $t > \tau$ we have $x(t) = \max_{t-\tau \leq s \leq t} x(s)$, and so by (4) we have $\frac{d}{dt} \left( x(t)e^{-(a+b)t} \right) = 0$ for $t \geq \tau$. Hence

\[x(t)e^{-(a+b)t} = x(\tau)e^{-(a+b)\tau} > 0, \quad t \geq \tau,\]

and so $\lim_{t \to \infty} x(t)e^{-(a+b)t} = \Lambda_1 > 0$.

\textbf{Theorem 1} Let $\psi \in C([-\tau, 0]; \mathbb{R})$. Suppose that $a + b > 0$, $b > 0$ and $\varsigma \neq 0$. Let $X$ be the unique continuous adapted process which satisfies (1) Then there exists an almost surely finite $\mathcal{F}^B(\infty)$–measurable random variable $\Gamma$ such that

\[\lim_{t \to \infty} X(t)e^{-(a+b)t} = \Gamma, \quad a.s. \quad (5)\]
An explicit formula for $\Gamma$ in (5) is not available, although a formula for $\Gamma$ depends on a functional of $X$ and $B$ is established in (40) in the proof of Theorem 1 below. Perusal of the formula for $\Gamma$ reveals that $\Gamma \geq \psi(0) + \int_0^\infty e^{-(a+b)s} dB(s)$. Therefore $\mathbb{P}[\Gamma > 0] > 0$. If we temporarily emphasise the dependence on $\psi$, by writing $\Gamma = \Gamma(\psi)$ we see that $\lim_{\psi(0) \to \infty} \mathbb{P}[\Gamma(\psi) > 0] = 1$. Therefore, an increasingly large initial condition increases the probability that $X(t) \to \infty$ as $t \to \infty$ rather than $X(t) \to -\infty$. Moreover, $X(t) \to +\infty$ as $t \to \infty$ is the favoured limit: when $\psi(0) = 0$, we have $\Gamma \geq \int_0^\infty e^{-(a+b)s} dB(s)$, and so $\mathbb{P}[\Gamma > 0] \geq 1/2$. These comments are of particular interest from the perspective of financial modelling, as we will see in the next Section.

In the case when $a + b < 0$, we can show that the largest fluctuations of the process are of order $\sqrt{\log t}$ as $t \to \infty$ a.s. To do this, it convenient to introduce the differential resolvent

$$r'(t) = ar(t) + br(t - \tau), \quad t > 0; \quad r(0) = 1, \quad r(t) = 0 \quad t \in [-\tau, 0).$$

(6)

We note that when $a + b < 0$ and $b > 0$ that $r \in L^2(0, \infty)$.

**Theorem 2** Let $\tau > 0$ and $\psi \in C([-\tau, 0], \mathbb{R})$, $b > 0$ and $a + b < 0$. Let $X$ be the unique continuous adapted process which satisfies (1). Then there is a positive constant $C_1$ such that

$$-C_1 \log \frac{-a}{b} - a - b(\frac{-a}{b})^{rC_1} = 0,$$

(7)

and $X$ obeys

$$\sqrt{\int_0^\infty r^2(s) \, ds} |\varsigma| \leq \limsup_{t \to \infty} \frac{|X(t)|}{\sqrt{2 \log t}} \leq ((C_1 \log \frac{-a}{b})^{-1} + 1) \frac{|\varsigma|}{\sqrt{-2a}}.$$

(8)

where $r$ is given by (6).

The fact that the rate of growth is $O(\sqrt{\log t})$ can be motivated by considering (1) in the limiting case when $\tau = 0$, in which case $X = X_0$ obeys $dX_0(t) = (a + b)X_0(t) \, dt + \varsigma dB(t)$. $X_0$ obeys

$$e^{-(a+b)t} X_0(t) = X_0(0) + \int_0^t \varsigma e^{-(a+b)s} dB(s), \quad t \geq 0.$$

(9)

The Law of the iterated logarithm applied to the martingale on the righthand side leads to

$$\limsup_{t \to \infty} \frac{e^{-(a+b)t} |X_0(t)|}{\sqrt{2 \int_0^t \varsigma^2 e^{-2(a+b)s} \, ds \cdot \log_2 \int_0^t \varsigma^2 e^{-2(a+b)s} \, ds}} = 1, \quad a.s.,$$

which simplifies to give

$$\limsup_{t \to \infty} \frac{|X_0(t)|}{\sqrt{2 \log t}} = \frac{\varsigma}{\sqrt{2|a + b|}}, \quad a.s.$$

(10)

This shows that it is the linear leading order growth in the functionals on the righthand side that gives the size of the fluctuations, rather than the Gaussian properties of solutions: in the case of (1) the process is not Gaussian, and does not easily admit a closed form formula in terms of the driving Brownian motion $B$, while the process $X_0$, which is a Gaussian process, admits such a variation of constants representation.

The representation (9) also shows that the exact exponential rate of growth seen in Theorem 1 for the functional equation in the case when $a + b > 0$ is not unreasonable. Since the righthand side of (9) is a martingale with quadratic variation which tends to a finite limit at infinity, by the martingale convergence theorem, the righthand side of (9) converges a.s. as $t \to \infty$, giving

$$\lim_{t \to \infty} e^{-(a+b)t} X_0(t) = X_0(0) + \int_0^\infty \varsigma e^{-(a+b)s} dB(s), \quad a.s.$$
2.2. Max-type equations with state-dependent noise

Before considering asymptotic behaviour of the stochastic delay equation with state-dependent noise, we give assumptions under which solutions are guaranteed to exist, be unique, and be positive, almost surely. Let $\sigma : [0,\infty) \rightarrow \mathbb{R}$ obey the following local Lipschitz condition

$$
\left\{
\begin{array}{l}
\text{For every integer } n \geq 1, \text{ there is } K_n > 0 \text{ such that} \\
|\sigma(x_1) - \sigma(x_2)| \leq K_n|x_1 - x_2| \quad \text{for all } x_1, x_2 \geq 0 \text{ with } |x_1| \lor |x_2| \leq n.
\end{array}
\right.
$$

(11)

We also request that

$$
\sigma(0) = 0,
$$

(12)

and that $\sigma$ obeys a global linear bound of the form

$$
\sigma(x) \leq K|x| \quad \text{for all } x \geq 0.
$$

(13)

Let $\tau > 0$ and $\psi \in C([-\tau,0);(0,\infty))$ be a deterministic function. We consider the stochastic functional differential equation of Itô type

$$
X(t) = \psi(0) + \int_0^t \left( aX(s) + b \sup_{s-\tau \leq u \leq s} X(u) \right) ds + \int_0^t \sigma(X(s)) dB(s), \quad t \geq 0;
$$

(14a)

$$
X(t) = \psi(t), \quad t \in [-\tau,0].
$$

(14b)

Then (14) has a unique global strong solution. Moreover, if $\psi(t) = 0$, $t \in [-\tau,0]$, then $X(t) = 0$ for all $t \geq 0$, a.s. If $b > 0$, $\psi > 0$ implies $X(t) > 0$, $t \geq 0$, a.s.

To see this, let $\tilde{\sigma}(x) = \sigma(x)/x$ for $x > 0$ and suppose to the contrary that there is a set of positive probability on which $t^* := \inf\{t > 0 : X(t) = 0\} < \infty$. Let $\varphi$ be the process obeying

$$
\varphi(t) = 1 + \int_0^t a\varphi(s) ds + \int_0^{t^*} \tilde{\sigma}(X(s))\varphi(s) ds, \quad 0 \leq t < t^*; \quad \varphi(t) = 1, \quad t \in [-\tau,0]
$$

The process $y(t) = X(t)\varphi(t)^{-1}$ may be defined for $t \in [-\tau,t^*]$; then we see that $y'(t) = b\varphi(t)^{-1}\sup_{t-\tau \leq s \leq t} X(s)$, $t \in (0,t^*)$, and because $X(t) > 0$ for $t \in [-\tau,t^*)$, $y(t) > 0$ for $t \in [-\tau,t^*)$. The minimality of $t^*$ implies $y'(t^*) \leq 0$. On the other hand, $t^* - \tau < t^*$, so $X(t^* - \tau) > 0$. Since $b > 0$, $\varphi(t^* - \tau) > 0$, we have $y'(t^*) > 0$, whence a contradiction.

The asymptotic growth of solutions of (14) are studied both in first mean and almost surely.

The first mean behaviour is particularly meaningful in this case because $X(t) > 0$ for all $t \geq 0$ a.s., and so $E[X(t)] = E[|X(t)|]$.

We start by showing in the case when $a + b > 0$ that the Liapunov exponent $a + b$ of the deterministic equation (3) is preserved, provided that the diffusion coefficient $\sigma$ is $o(x)$ as $x \rightarrow \infty$. This is true for both the first mean of the solution, as well as in an almost sure sense.

**Theorem 3** Let $X$ be the unique continuous adapted process which satisfies (14) with $\psi(t) > 0$ for $t \in [-\tau,0]$. Suppose that $a + b > 0$, $b > 0$ and $\sigma$ obeys (11), (12) and (13). If

$$
\lim_{x \rightarrow \infty} \frac{\sigma(x)}{x} = 0,
$$

(15)

then

$$
\lim_{t \rightarrow \infty} \frac{\log E[X(t)]}{t} = a + b, \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{E[\sup_{t-\tau \leq s \leq t} X(s)]}{E[X(t)]} = 1.
$$

(16)

If in addition

$$
\limsup_{x \rightarrow 0^+} \frac{\sigma^2(x)}{x^2} < 2(a + b),
$$

(17)

then

$$
\lim_{t \rightarrow \infty} \frac{\log X(t)}{t} = a + b, \quad \text{a.s.}
$$

(18)
It is now a natural question to ask: when \( a + b > 0 \), under what conditions on the unbounded coefficient \( \sigma \) is it still true that the solution \( X \) of the stochastic equation (14) tends to infinity at exactly the same (exponential) rate as the solution \( x \) of the deterministic equation (3)? We can show that provided that the noise intensity does not grow too rapidly (at a sublinear rate characterised by condition (19) below), then \( \lim_{t \to \infty} X(t)/x(t) \) exists and is positive almost surely.

**Theorem 4** Suppose that \( a + b > 0 \), \( b > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Let \( \sigma \) obey (11), (12), and (13). Suppose further that \( \sigma \) obeys

\[
\limsup_{x \to \infty} \frac{\sigma^2(x)}{x^2/\log^2 x} < +\infty \text{ for some } c > 1.
\]

Let \( X \) be the unique continuous adapted process which obeys (14). Then there exists an almost surely finite and non-negative \( \mathcal{F}^B(\infty)\)-measurable random variable \( \Lambda \) such that

\[
\lim_{t \to \infty} X(t)e^{-(a+b)t} = \Lambda, \quad \text{a.s.}
\]

Moreover, if (17) holds, \( \Lambda \) is positive, a.s.

The growth condition (19) on \( \sigma \) suffices to ensure that solutions of (14) inherit the asymptotic behaviour of (3). Given this condition, it is reasonable to ask whether it is necessary to guarantee the asymptotic equivalence of solutions of the stochastic and deterministic equations. In the next result, we establish a necessary condition on \( \sigma \) such that \( \lim_{t \to \infty} X(t)/x(t) \) exists and is positive on a set of positive probability.

**Theorem 5** Suppose that \( a + b > 0 \), \( b > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Let \( \sigma \) obey (11), (12), and (13) and suppose that

There exists a non-decreasing \( \sigma_0 \) such that \( \liminf_{x \to \infty} \sigma^2(x) / \sigma_0^2(x) = 1 \).

Let \( X \) be the unique continuous adapted process which obeys (14). Suppose that there exists a finite and strictly positive \( \mathcal{F}^B(\infty)\)-measurable random variable \( \Lambda \) such that \( X \) obeys (20) on an event of positive probability. Then \( \sigma_0 \) obeys

\[
\int_1^{\infty} \frac{\sigma_0^2(x)}{x^3} \; dx < +\infty.
\]

It is an immediate corollary of Theorem 5 that the condition

\[
\sigma^2 \text{ obeys (21) and } \int_1^{\infty} \frac{\sigma_0^2(x)}{x^3} \; dx = +\infty
\]

implies

\[
\mathbb{P} \left[ \omega : \lim_{t \to \infty} X(t, \omega) / e^{(a+b)t} \in (0, \infty) \right] = 0.
\]

We now see that there is a “gap” between the sufficient condition (19) and the necessary condition (23) on the growth rate of \( \sigma^2 \) needed to retain exact exponential asymptotic behaviour, but that this gap is only around \( O(\log x) \) as \( x \to \infty \). More precisely, if \( \sigma \) obeys (17), by applying Theorems 4 and 5, we get the following case distinction:

If \( \sigma^2 \) obeys \( \limsup_{x \to \infty} \frac{\sigma^2(x)}{x^2/\log^2 x} < +\infty \) for some \( c > 1 \), then \( \mathbb{P} \left[ \omega : \lim_{t \to \infty} X(t, \omega) / e^{(a+b)t} \in (0, \infty) \right] = 1. \)

If \( \sigma^2 \) obeys \( \liminf_{x \to \infty} \frac{\sigma^2(x)}{x^2/\log x} > 0 \), then \( \mathbb{P} \left[ \omega : \lim_{t \to \infty} X(t, \omega) / e^{(a+b)t} \in (0, \infty) \right] = 0. \)
This shows that we have achieved a relatively good estimate on the rate at which $\sigma^2$ can grow at infinity and still preserve the exact exponential asymptotic rate of growth of the deterministic equation (3).

The final result in this section shows that the asymptotic behaviour of the deterministic equation (3) is preserved in first mean by both $X$ and $\sup_{-\tau \leq u \leq t} X(u)$ provided $\sigma$ does not grow too rapidly.

**Theorem 6** Let $X$ be the unique continuous adapted process which satisfies (14) with $\psi(t) > 0$ for $t \in [-\tau, 0]$. Suppose $a + b > 0$, $b > 0$ and $\sigma$ obeys (11), (12), (13) and (15). Suppose further that there exists a non-decreasing continuous concave function $\varphi : [0, \infty) \to (0, \infty)$ such that

$$|\sigma(x)| \leq \varphi(x), \quad \text{for } x \geq 0;$$

$$x \mapsto \frac{\varphi(x)}{x} \text{ is non-increasing on } (0, \infty);$$

$$\int_1^\infty \frac{\varphi(x)}{x^2} \, dx < +\infty. \quad (24c)$$

Then there exists $c_0 > 0$ such that

$$\lim_{t \to \infty} \mathbb{E}[X(t)] e^{-(a+b)t} = c_0 \quad \text{and} \quad \lim_{t \to \infty} \mathbb{E} \left[ \sup_{t-\tau \leq s \leq t} X(s) \right] e^{-(a+b)t} = c_0. \quad (25)$$

The theorem relies on the existence of an auxiliary function $\varphi$ obeying (24). In the next result we show that such an auxiliary function can be constructed providing $\sigma$ obeys (19). It is interesting to note that this is precisely the condition required to guarantee the almost sure asymptotic equivalence of solutions of (3) and (14).

**Theorem 7** Let $X$ be the unique continuous adapted process which satisfies (14) with $\psi(t) > 0$ for $t \in [-\tau, 0]$. Suppose $a + b > 0$, $b > 0$ and $\sigma$ obeys (11), (12), (13) and (19). Then there exists $c_1 > 0$ such that $X$ obeys (25).

### 3. Economic modelling and interpretation

We now build a simple model of the instantaneous returns of a risky asset whose dynamics can be understood and interpreted in terms of (1). We define the *instantaneous return* $r(t)$ at time $t$ in terms of the stock price $S(t)$ at time $t$ by

$$S(t) = S(0) \exp \left( \int_0^t r(u) \, du \right), \quad t \geq 0.$$ 

If $r$ is a process with continuous sample paths, then $S$ has differentiable sample paths. This is in contrast to most stochastic stock price models (e.g., Geometric Brownian Motion). Despite this undesirable property, we show in this section how the dynamics of the instantaneous return give rise to other nice properties of the price and the *cumulative return* $X(t)$ (which is defined by $X(t) = \log(S(t)/S(0))$ for $t \geq 0$). We hope by doing so we can demonstrate that there are interesting features created in the returns dynamics by the presence of max-type functionals, while at the same time showing that the mathematical complexity incurred by the inclusion of such functionals can be surmounted. This gives us hope for developing more sophisticated yet analytically tractable max-type SFDE market models in the future.

Notice that $r \in C([0, \infty), \mathbb{R})$ implies that $S'(t) = r(t)S(t)$ for all $t > 0$ so that $r(t_0) > 0$ implies that $S$ is increasing at $t = t_0$ and $r(t_0) < 0$ implies that $S$ is decreasing at $t = t_0$.

We suppose that there are $M_1$ *reference level* traders and $M_2$ *technical* traders. We assume that these traders do not change their investment strategies over time, and have infinite lives.
We may interpret this latter assumption as allowing for the replacement of a trader with a finite lifecycle by another with the same investment strategy. Trading takes place continuously in time.

Reference traders believe that instantaneous returns should either (a) revert towards a mean level, or (b) will depart from that level. The latter case reflects the idea that if the returns are currently at a high (resp. low) level this is a signal of higher (resp. lower) returns to come and so it is advantageous to buy (resp. sell) in advance of the increase (resp. decrease) in prices. The mean level \( r_1 \) chosen is idiosyncratic to the \( j \)-th trader, and the planned instantaneous excess demand is proportional to the deviation of the return from \( r_1 \). Therefore, there is \( \alpha_l \in \mathbb{R} \) such that the planned instantaneous excess demand of reference trader \( l \) over the time interval \((t, t+dt)\) is \( \alpha_l(r(t) - r_1) dt \), where we let \( dt \) be a positive infinitesimal. The planned instantaneous excess demand of all reference traders over the time interval \((t, t+dt)\) is therefore \( \sum_{l=1}^{M_1} \alpha_l(r(t) - r_1) dt \).

Some reference traders are contrarians: such traders buy if the daily return is below the reference level, and sell if it is above this level. For this type of trader we set \( \alpha_l < 0 \). Other traders are positive feedback traders: such traders buy if the daily return is above the reference level, and sell if it is below the reference level. This type of trader is modelled by setting \( \alpha_l > 0 \).

Technical traders believe that patterns in the returns are significant and should be traced. We suppose that there are \( M_2 \) such traders and consider for concreteness the \( j \)-th trader. He believes that if the instantaneous return is significantly (meaning more than a tolerance of \( \tau_j \) units) below the maximum instantaneous return over the past \( \tau \) time periods, then this is a signal that the market will advance. Trader \( j \) has planned instantaneous excess demand over the time interval \((t, t+dt)\) proportional to the strength of the signal \( \max_{t-\tau \leq s \leq t} r(s) - r(t) - \tau_j \). The tolerance \( \tau_j \) is idiosyncratic to the trader. However, we assume for mathematical convenience that all traders have the same length of memory \( \tau \) time periods. The planned instantaneous excess demand of trader \( j \) over the time interval \((t, t+dt)\) is therefore \( \beta_j \left( \max_{t-\tau \leq s \leq t} r(s) - r(t) - \tau_j \right) dt \) where \( \beta_j > 0 \) means that a positive signal leads to buying, while a negative signal leads to selling.

We note that if \( r(t) = \max_{t-\tau \leq s \leq t} r(s) \), then the signal is negative, and trader \( j \) sells. Therefore, if the market is currently experiencing very strong returns relative to the recent past, the trader sells, expecting a reversal of the market in the near future. Therefore the planned instantaneous excess demand of all technical traders before trading at time \( t \) is

\[
\sum_{j=1}^{M_2} \beta_j \left( \max_{t-\tau \leq s \leq t} r(s) - r(t) - \tau_j \right) dt.
\]

Speculators react to other random stimuli—"news"—which is independent of past returns. For specifier \( j = 1, \ldots, M_1 + M_2 \) this unplanned excess demand over the time interval \((t, t+dt)\) is \( \int_t^{t+dt} \varsigma_j dB(s) \), where \( B \) is a standard one–dimensional Brownian motion. Let \( \varsigma = \sum_{j=1}^{M_1+M_2} \varsigma_j \).

We suppose that the instantaneous return will be positive (resp. negative) if there is excess demand (resp. supply), with the gain (resp. loss) being larger the greater the excess demand (resp. supply). Hence, the price adjustment for a market with \( M_1 \) reference traders and \( M_2 \) technical traders is given by

\[
dr(t) = \left\{ \sum_{l=1}^{M_1} \alpha_l (r(t) - r_1) + \sum_{j=1}^{M_2} \beta_j \left( \max_{t-\tau \leq s \leq t} r(s) - r(t) - \tau_j \right) \right\} dt + \varsigma dB(t), \quad t \geq 0, \quad (26)
\]

where \( B \) is a standard one–dimensional Brownian motion. If we define \( r(t) = \phi(t) \) for \( t \in [-\tau, 0] \) and \( \phi \in C([-\tau, 0]; \mathbb{R}) \), (26) is guaranteed to possess a strong unique adapted continuous solution.

We now show how to reduce (26) to the equation (1). Suppose that

\[
\sum_{l=1}^{M_1} \alpha_l \neq 0. \quad (27)
\]
Define
\[
a' = \sum_{i=1}^{M_1} \alpha_i, \quad b = \sum_{j=1}^{M_2} \beta_j, \quad r^* = \frac{1}{a'} \left( \sum_{i=1}^{M_1} \alpha_i r_i + \sum_{j=1}^{M_2} \beta_j \tau_j \right),
\]
and \(X(t) = r(t) - r^*\). Then we have
\[
dX(t) = \left\{ (a' - b)X(t) + b \max_{t-\tau \leq s \leq t} X(s) + \left( a' r^* - \sum_{i=1}^{M_1} \alpha_i r_i - \sum_{j=1}^{M_2} \beta_j \tau_j \right) \right\} dt + \varsigma dB(t).
\]
By using the definition \(a := a' - b\) and (28) we get
\[
dX(t) = \left\{ aX(t) + b \max_{t-\tau \leq s \leq t} X(s) \right\} dt + \varsigma dB(t)
\]
where \(b > 0\). It is readily seen that (29) is nothing other than the equation (1) which was discussed in Section 2.1, and whose asymptotic properties were given in Theorems 1 and 2.

It is reasonable to ask how one might reformulate the model to allow for non-differentiable stock prices. Here is one possibility. If we let \(C(t)\) be the \(t\)-time cumulative return, then \(C(t) - C(t-1)\) is the one-period return at time \(t\). Using this as a proxy for the instantaneous return, we consider by analogy to (26) the stochastic functional differential equation
\[
dC(t) = \left\{ \sum_{i=1}^{M_1} \alpha_i (C(t) - C(t-1) - r_i) \right. \\
\left. + \sum_{j=1}^{M_2} \beta_j \left( \max_{t-\tau \leq s \leq t} C(s) - C(s-1) - (C(t) - C(t-1)) - \tau_j \right) \right\} dt + \varsigma dB(t), \quad t \geq 0.
\]
By appending an appropriate initial condition, this equation has a unique strong solution, and moreover \(S(t) = S(0) \exp(C(t))\) is non-differentiable. Letting \(c(t) = C(t) - C(t-1)\) we see that
\[
c(t) = \int_{t-1}^{t} \left\{ \sum_{i=1}^{M_1} \alpha_i (c(s) - r_i) + \sum_{j=1}^{M_2} \beta_j \left( \max_{s-\tau \leq u \leq s} c(u) - c(s) - \tau_j \right) \right\} ds + \varsigma (B(t) - B(t-1)), \quad t \geq 1.
\]
Define \(a', b, a\) as before, let \(a' \neq -1\), and define \(r' = a'r' - \sum_{i=1}^{M_1} \alpha_i r_i - \sum_{j=1}^{M_2} \beta_j \tau_j\). Letting \(x(t) = c(t) - r'\) we get
\[
x(t) = \int_{t-1}^{t} \left\{ a x(s) + b \max_{s-\tau \leq u \leq s} x(u) \right\} ds + \varsigma (B(t) - B(t-1)), \quad t \geq 1.
\]
The methods of the paper could then be applied to (31) in the cases \(a + b \in (-1, 1), \ a + b > 1\).

3.1. Interpretation of Main Results to the Economic Model
Since \(\beta_j > 0\) for each \(j = 1, \ldots, M_2\) by (28) we have \(b > 0\) as required throughout. We note that the key parameter \(a + b\) is given by \(\sum_{i=1}^{M_1} \alpha_i\). Therefore we have \(a + b > 0\) if and only if \(\sum_{i=1}^{M_1} \alpha_i > 0\).
If the feedback traders are overall of positive feedback type (i.e., if $\sum_{l=1}^{M_1} \alpha_l > 0$), by Theorem 1 the market experiences a bubble or crash according to

$$\lim_{t \to \infty} e^{-\left(\sum_{l=1}^{M_1} \alpha_l\right) t} r(t) = r(0) - \frac{\sum_{l=1}^{M_1} \alpha_l r_l + \sum_{j=1}^{M_2} \beta_j T_j}{\sum_{l=1}^{M_1} \alpha_l} \quad \text{as } t \to \infty$$

$$+ \sum_{j=1}^{M_2} \beta_j \cdot \int_0^{\infty} e^{-\sum_{l=1}^{M_1} \alpha_l s} \left( \sup_{s-r \leq u \leq s} r(u) - r(s) \right) ds + \int_0^{\infty} e^{-\sum_{l=1}^{M_1} \alpha_l s} dB(s)$$

It should be noted that the presence of the technical traders neither prevents nor promotes the creation of this runaway event.

Examining the limit on the righthand side of (32), we see that it tends to infinity a.s. as $r(t) \to -\infty$. Therefore, the larger the initial instantaneous return, the greater the probability that $r(t) \to \infty$ as $t \to \infty$. This explains at least in part the manner in which this bubble forms: if initially the stock performs well, this encourages positive feedback traders to take this good performance as a signal that informed investors believe the stock will do well in future, so they buy the stock. This then forces prices up further, encouraging further buying. This upward spiral continues, and a bubble ensues.

Conversely, if the initial value of $r(0)$ is negative, but $|r(0)|$ is large, this tends to make the limit on the righthand side of (32) negative, and makes the event $\{r(t) \to -\infty \text{ as } t \to \infty\}$ more probable. In this situation, this helps to explain the crash dynamics: an initially poor performance by the stock convinces positive feedback traders that informed traders believe the stock will perform poorly in future, so they sell (or short sell) the stock. This then forces prices lower, encouraging further selling, and the result of this downward spiral is a crash. In both cases, and when $\alpha_l > 0$ for each $l$, we see that it is the level of the market $r(0)$ relative to the “weighted consensus” return $r_c := \frac{\sum_{l=1}^{M_1} \alpha_l r_l}{\sum_{l=1}^{M_1} \alpha_l}$ of the reference traders that is particularly important; the greater the difference $r(0) - r_c$, the more probable a bubble.

We next ask what is the impact of a sequence of “good news stories” about the stock at the time shortly after trading begins. Speaking very loosely, we can interpret this as a “majority” of the (infinitesimal) increments of $B$ being positive. Since the integrand of the Itô integral on the righthand side of (32) diminishes rapidly as time increases, it is the sign of these “initial” increments of $B$ that largely determines whether the integral assumes a positive or negative value. Therefore, initial good news about the stock tends to result in a positive value of the Itô integral, while initial bad news about the stock tends to lead to a negative value of the Itô integral. Therefore, if there is good initial news about the stock, the price of the stock tends to increase, and the positive feedback traders force the price higher by misperceiving this increase as arising from demand from informed speculators. As before, this induces further buying, and the stock price undergoes a bubble. Similarly, initial bad news tends to precipitate a crash.

Finally, consider the penultimate term on the righthand side of (32). First, we note that it is always positive, and that the main contribution to the overall value of the integral is from the time shortly after trading begins. This contribution is relatively large if the returns are relatively low, because in this case $\sup_{s-r \leq u \leq s} r(u)$ will tend to strictly exceed $r(s)$, and to do so on an open interval. If the returns are running below their maximum during the initial period of trading, the technical traders will tend to force the returns upwards; this trend will then be extrapolated by the positive feedback traders, increasing the probability of a bubble. On the other hand, the contribution of the penultimate term in (32) is smaller if the returns are generally increasing and therefore at or close to their $\tau$–day running maximum. However, in this case, the contributions of the first two terms on the righthand side are quite likely to be positive, so the additional bubble–promoting impact of the penultimate term, although modest, is likely to be unimportant. Hence, the penultimate term tends to have its greatest bubble–
promoting impact when other bubble-promoting factors (such as strong initial returns relative to the reference levels of the feedback traders, or a sequence of good news stories about the stock) are not so strong. Therefore, it seems that the technical traders can also “seed” a bubble in a market which is naturally prone to generate a bubble. Hence, the interaction of such traders with the positive feedback traders can make bubbles more likely.

These remarks suggest that the mechanisms by which bubbles form in this model are consistent with the notion of mimetic contagion introduced by Orléan (cf. e.g., [33]). In mimetic contagion, we may think of the market as comprising of two forms of traders, with new entrants choosing the trading strategy which tends to dominate at a given time. In the long-run, the proportion of traders in each category settles down to a value which is random but which depends quite strongly on what happens in the first trading periods. The similarities with mimetic contagion are as follows: in (32), the right-hand side depends crucially on the market behaviour in the first few time periods; once a dominant trend becomes apparent, the positive feedback traders will tend to extrapolate that trend; and the long-run behaviour (either a bubble or crash) is not known in advance.

If the feedback traders are, on the whole, of negative feedback type (in which case \( \sum_{l=1}^{M_1} \alpha_l < 0 \)), the market experiences large fluctuations whose size is intimately connected with the distribution of independent increments of the Brownian motion, which we use to assess the impact of various pieces of “news”. For example, by Theorem 2 and the definition of \( X \) in terms of \( r \) we find that \( \sum_{l=1}^{M_1} \alpha_l < 0 \) implies

\[
0 < c_3 \leq \limsup_{t \to \infty} \frac{|r(t)|}{\sqrt{\log t}} \leq c_4, \quad \text{a.s.} \tag{33}
\]

for some pair of deterministic positive constants \( 0 < c_3 \leq c_4 < +\infty \). This property is consistent with a market model which is a non-trivial but degenerate case of the model (26), but which shares many important properties with Geometric Brownian Motion (GBM). We compare GBM and this degenerate model extensively and also point out commonalities between these models and the full SFDE model (26).

For GBM the stock price \( S \) is given by

\[
dS(t) = \mu S(t) dt + \sigma S(t) dB(t), \quad t \geq 0, \quad S(0) = s_0 > 0. \tag{34}
\]

In this case, the price \( S \) obeys

\[
\lim_{t \to \infty} \frac{1}{t} \log S(t) = \mu - \frac{1}{2} \sigma^2, \quad \text{a.s.}
\]

The salient features of the returns associated with (34) are (i) the cumulative return \( X(t) \) is Gaussian distributed with mean and variance which increase linearly with \( t \); (ii) the increments of \( X \) are independent and Gaussian distributed and grow according to

\[
\limsup_{t \to \infty} \frac{|X(t) - X(t - \Delta)|}{\sqrt{2 \log t}} = \sigma \Delta, \quad \text{a.s.}
\]

and (iii) the maximum departure of the cumulative return from the trend growth rate obeys the law of the iterated logarithm

\[
\limsup_{t \to \infty} \frac{\log S(t) - (\mu - \sigma^2/2)t}{\sqrt{2t \log \log t}} = \sigma, \quad \liminf_{t \to \infty} \frac{\log S(t) - (\mu - \sigma^2/2)t}{\sqrt{2t \log \log t}} = -\sigma, \quad \text{a.s.}
\]
All these properties (except those of independence of increments) are enjoyed by the cumulative returns generated by the model without the presence of the technical traders (viz., with $\beta_j = 0$ for all $j$). If, as in (28), we set $a = \sum_{l=1}^{M_j} \alpha_l$ and $r^* = \sum_{l=1}^{M_j} \alpha_l r_l$, then $r$ obeys
\[
dr(t) = a(r(t) - r^*) dt + \varsigma dB(t).
\]
We have $a < 0$ and the cumulative returns are given by $X(t) = \int_0^t r(s) ds$. Writing (35) in integral form and rearranging gives $X(t) = \frac{1}{a} r(t) - \frac{1}{a} r(0) + r^* t - \frac{\varsigma}{a} B(t)$. The Strong Law of Large Numbers for Brownian motion implies $\lim_{t \to \infty} X(t)/t = r^*$ a.s., and $\lim_{t \to \infty} \log S(t)/t = r^*$ a.s. Since $\log S(t) = X(t)$, the Law of the Iterated Logarithm for standard Brownian motion gives
\[
\limsup_{t \to \infty} \frac{\log S(t) - r^* t}{\sqrt{2t \log t}} = \frac{\varsigma}{|a|} \quad \text{a.s.}
\]
Using the fact that $r(t) = r^* + e^{at}(r(0) - r^*) + c e^{at} \int_0^t e^{-as} dB(s)$, we get
\[
X(t) = r^* t + (r(0) - r^*) \frac{1}{a} (e^{at} - 1) + \varsigma \int_0^t \int_0^s e^{a(s-u)} ds dB(u),
\]
so $X(t)$ is Gaussian distributed with $E[X(t)]/t \to r^*$, $\text{Var}[X(t)]/t \to \varsigma^2/a^2$ as $t \to \infty$. Also,
\[
X(t) - X(t-\Delta) = r^* \Delta + (r(0) - r^*) \int_{t-\Delta}^t e^{as} ds + \varsigma \int_{t-\Delta}^t \int_0^s e^{a(s-u)} dB(u) ds, \quad t \geq \Delta.
\]
Define $\rho(t) = 1$ for $t \leq 0$ and $\rho(t) = e^{at}$ for $t > 0$. Then we have
\[
X(t) - X(t-\Delta) = r^* \Delta + \frac{1}{a} (r(0) - r^*)(e^{at} - e^{a(t-\Delta)}) + \varsigma \int_0^t (\rho(t-u) - \rho(t-u-\Delta)) dB(u),
\]
so the $\Delta$–increments of $X$ are Gaussian distributed and asymptotically stationary with mean $r^* \Delta$ and variance $\varsigma^2/a^2 \int_0^\infty (\rho(s) - \rho(s-\Delta))^2 ds$. By [4], we have
\[
\limsup_{t \to \infty} \frac{\int_0^t (\rho(t-u) - \rho(t-u-\Delta)) dB(u)}{\sqrt{2 \log t}} = \sqrt{\int_0^\infty (\rho(s) - \rho(s-\Delta))^2 ds}, \quad \text{a.s.},
\]
with a symmetric result holding for the liminf. Therefore
\[
\limsup_{t \to \infty} \frac{X(t) - X(t-\Delta)}{\sqrt{2 \log t}} = -\liminf_{t \to \infty} \frac{X(t) - X(t-\Delta)}{\sqrt{2 \log t}} = \frac{|\varsigma|}{|a|} \sqrt{\int_0^\infty (\rho(s) - \rho(s-\Delta))^2 ds}, \quad \text{a.s.}
\]
These calculations can be used to show that the cumulative returns associated with (26) also obey a logarithmic growth bound of the form
\[
c_5(\Delta) \leq \limsup_{t \to \infty} \frac{|X(t) - X(t-\Delta)|}{\sqrt{\log t}} \leq c_4 \Delta, \quad \text{a.s.}
\]
The upper limit follows from (33). The lower limit is obtained by noting that $r(t) - r^* > r_0(t)$ for all $t \geq 0$ provided that $r(t) - r^* > r_0(t)$ for $t \in [-\tau, 0]$ and $dr_0(t) = (a+b) r_0(t) dt + \varsigma dB(t)$, $t \geq 0$. This gives
\[
X(t) - X(t-\Delta) = \int_{t-\Delta}^t r(s) ds = \int_{t-\Delta}^t (r(s) - r^*) ds + r^* \Delta \geq \int_{t-\Delta}^t r_0(s) ds + r^* \Delta.
\]
With $X_0(t) := \int_0^t r_0(s) ds$, $\gamma(t) := e^{(a+b)t}$, $t \geq 0$ and $\gamma(t) := 1$ for $t < 0$, by arguing as in (36) we get
\[
\limsup_{t \to \infty} \frac{\int_{t-\Delta}^t r(s) ds}{\sqrt{2 \log t}} \geq \limsup_{t \to \infty} \frac{X_0(t) - X_0(t-\Delta)}{\sqrt{2 \log t}} = \frac{|\varsigma|}{a+b} \sqrt{\int_0^\infty (\gamma(s) - \gamma(s-\Delta))^2 ds}, \quad \text{a.s.}
\]
Setting $c_5(\Delta)$ equal to the righthand side concludes the proof.
4. Proofs

4.1. Proof of Theorem 1

Define the process \( Y = \{ Y(t) : t \geq -\tau \} \) by \( Y(t) = 0 \) for \( t \in [-\tau, 0] \) and by \( dY(t) = -(a + b)Y(t)\,dt + \xi\,dB(t) \) for \( t \geq 0 \). Therefore, by (10) we have

\[
\limsup_{t \to -\infty} \frac{|Y(t)|}{\sqrt{2\log t}} = \frac{|\xi|}{\sqrt{2(a + b)}}, \text{ a.s.} \tag{37}
\]

Define \( Z(t) = X(t) - Y(t) \) for \( t \geq -\tau \). Then \( Z(t) = \psi(t) \) for \( t \in [-\tau, 0] \) and \( Z \in C^1(0, \infty) \) obeys

\[
Z'(t) = a(Z(t) + Y(t)) + b \sup_{t-\tau \leq s \leq t} (Z(s) + Y(s)) + (a + b)Y(t), \quad t > 0.
\]

Also, define \( U(t) = e^{-(a+b)t} Z(t) \) for \( t \geq -\tau \). Then

\[
U'(t) = -bU(t) + be^{-(a+b)t} \sup_{t-\tau \leq s \leq t} (Z(s) + Y(s)) + (2a + b)e^{-(a+b)t}Y(t)
\]

\[
\geq -bU(t) + be^{-(a+b)t} (Z(t) + Y(t)) + (2a + b)e^{-(a+b)t}Y(t) = 2(a + b)e^{-(a+b)t}Y(t).
\]

Hence for any fixed pair \( t_2 > t_1 > 0 \) we have

\[
e^{-(a+b)t_2}Z(t_2) - e^{-(a+b)t_1}Z(t_1) = U(t_2) - U(t_1) \geq 2(a + b) \int_{t_1}^{t_2} e^{-(a+b)s}Y(s)\,ds.
\]

Therefore

\[
e^{-(a+b)t_2}X(t_2) \geq e^{-(a+b)t_2}Y(t_2) + e^{-(a+b)t_1} (X(t_1) - Y(t_1)) + 2(a + b) \int_{t_1}^{t_2} e^{-(a+b)s}Y(s)\,ds.
\]

By (37), \( Y(t)e^{-(a+b)t} \to 0 \) as \( t \to \infty \) and \( \int_0^\infty |Y(t)|e^{-(a+b)t}\,dt < +\infty \) a.s. Therefore

\[
\liminf_{t_2 \to \infty} e^{-(a+b)t_2}X(t_2) \geq e^{-(a+b)t_1}X(t_1) - e^{-(a+b)t_1}Y(t_1) + 2(a + b) \int_{t_1}^{\infty} e^{-(a+b)s}Y(s)\,ds.
\]

Now, appealing as before to (37)

\[
\limsup_{t_1 \to -\infty} e^{-(a+b)t_1}X(t_1) - e^{-(a+b)t_1}Y(t_1) + 2(a + b) \int_{t_1}^{\infty} e^{-(a+b)s}Y(s)\,ds = \limsup_{t_1 \to -\infty} e^{-(a+b)t_1}X(t_1),
\]

so \( \liminf_{t_2 \to \infty} e^{-(a+b)t_2}X(t_2) \geq \limsup_{t_1 \to -\infty} e^{-(a+b)t_1}X(t_1) \) and \( \lim_{t \to \infty} e^{-(a+b)t}X(t) \) exists a.s.

We first observe that this limit is bounded below. Using integration by parts we obtain

\[
e^{-(a+b)t}X(t) = X(0) + \int_0^t be^{-(a+b)s} \left( \sup_{s-\tau \leq u \leq s} X(u) - X(s) \right) \,ds + \int_0^t e^{-(a+b)s} \xi\,dB(s). \tag{38}
\]

The third term approaches a finite limit almost surely, by virtue of the martingale convergence theorem. The second term tends to a limit which is either finite or infinite; therefore, we have

\[
\liminf_{t \to -\infty} e^{-(a+b)t}X(t) \geq \psi(0) + \int_0^\infty e^{-(a+b)s} \xi\,dB(s). \tag{39}
\]

It remains to show that the limit \( \lim_{t \to -\infty} e^{-(a+b)t}X(t) \) is finite. Suppose we can show that \( U \) is bounded above. Then \( t \mapsto Z(t)e^{-(a+b)t} \) is bounded above. Since \( Y(t)e^{-(a+b)t} \to 0 \) as
\( t \to \infty, \ t \mapsto X(t) e^{-(a+b)t} \) is bounded above. Taking this alongside (39) and the existence of \( \lim_{t \to \infty} X(t) e^{-(a+b)t} \), the limit must be finite and, by (38), equal to
\[
\Gamma = X(0) + \int_0^\infty be^{-(a+b)s} \left( \sup_{s-\tau \leq u \leq s} X(u) - X(s) \right) \, ds + \int_0^\infty e^{-(a+b)s} \tau \, dB(s).
\] (40)

We now show that \( U \) is bounded above. First, we recall the identity
\[ U' = -bU + be^{-(a+b)t} \max_{t-\tau \leq s \leq t} X(s) + (2a + b)e^{-(a+b)t}Y, \quad t > 0, \]
which leads to
\[ U(t) \leq -bU(t) + be^{-(a+b)t} \max_{t-\tau \leq s \leq t} X(s) + (2a + b)e^{-(a+b)t}Y(t), \quad t > 0. \]

Let \( \varepsilon > 0 \) and define \( \beta \) by
\[
\beta(t) = \begin{cases} 
be^{-(a+b)t} \max_{t-\tau \leq s \leq t} |X(s)| + |2a + b|e^{-(a+b)t}|Y| + \varepsilon e^{-(a+b)t}, & t > 0, \\
\varepsilon e^{-(a+b)t}, & t \in [-\tau, 0].
\end{cases}
\]

Then \( \beta(t) > 0 \) for all \( t \geq -\tau \) and \( \beta \) is continuous on \([-\tau, \infty)\) because \( Y \equiv 0 \) on \([-\tau, 0)\). Let \( \alpha = \varepsilon + \max \left( 0, \sup_{t \in [-\tau, 0]} e^{-(b+a)t} \psi(t) \right) > 0 \), and set \( U_u(t) = \alpha + \int_{-\tau}^t \beta(s) \, ds, \quad t \geq -\tau. \) Note that \( U'(t) < -bU(t) + be^{-(a+b)t} \max_{t-\tau \leq s \leq t} Z(s) + \beta(t) \) for \( t > 0 \) and therefore
\[ U'(t) < -bU(t) + be^{-(a+b)t} \max_{t-\tau \leq s \leq t} U(s) + \beta(t), \quad t > 0. \] (41)

Next for \( t > 0 \) we have
\[
U'_u(t) + bU_u(t) - be^{-(a+b)t} \max_{t-\tau \leq s \leq t} e^{(b+a)s}U_u(s) - \beta(t)
= b \left( \alpha + \int_{-\tau}^t \beta(s) \, ds - e^{-(a+b)t} \max_{t-\tau \leq s \leq t} e^{(b+a)s}U_u(s) \right).
\]

Since \( U'_u(t) = \beta(t) > 0, U_u(t) > 0 \) for \( t \geq -\tau, \frac{d}{dt} \left( e^{(a+b)t}U_u(t) \right) = e^{(a+b)t}(U'_u(t) + (a+b)U_u(t)) > 0. \)
Hence for \( t > 0, e^{-(a+b)t} \max_{t-\tau \leq s \leq t} e^{(b+a)s}U_u(s) = U_u(t) = \alpha + \int_{-\tau}^t \beta(s) \, ds, \) which implies
\[ U'_u(t) = -bU_u(t) + be^{-(a+b)t} \max_{t-\tau \leq s \leq t} e^{(b+a)s}U_u(s) + \beta(t), \quad t > 0. \] (42)

For \( t \in [-\tau, 0] \) we have \( U_u(t) \geq \inf_{t \in [-\tau, 0]} U_u(t) = \alpha. \) On the other hand for \( t \in [-\tau, 0] \)
\[ U(t) \leq \sup_{t-\tau \leq s \leq t} U(s) = \sup_{t-\tau \leq s \leq t} e^{-(a+b)s}\psi(s) < \varepsilon + \max \left( 0, \sup_{t \in [-\tau, 0]} e^{-(b+a)t}\psi(t) \right) = \alpha \leq U_u(t). \]

Hence
\[ U(t) < U_u(t), \quad t \in [-\tau, 0]. \] (43)

We will now use (41), (42) and (43) to prove that \( U(t) < U_u(t) \) for all \( t \geq 0. \) By (43), we have either that \( U(t) < U_u(t) \) for all \( t \geq 0, \) or the existence of a minimal \( t_0 > 0 \) such that
U(t_0) = U_u(t_0). This minimality of t_0 > 0 implies that U_u(t) > U(t) for all \( t \in [-\tau, t_0) \) and \( U_u(t_0) \leq U'(t_0) \). Hence using (41) in the first inequality and (42) in the last we obtain

\[
U'(t_0) < -bU(t_0) + be^{-(a+b)t_0} \sup_{t_0-\tau \leq s \leq t_0} e^{(a+b)s}U(s) + \beta(t_0)
\]

\[
= -bU_u(t_0) + be^{-(a+b)t_0} \sup_{t_0-\tau \leq s \leq t_0} e^{(a+b)s}U(s) + \beta(t_0)
\]

\[
\leq -bU_u(t_0) + be^{-(a+b)t_0} \sup_{t_0-\tau \leq s \leq t_0} e^{(a+b)s}U_u(s) + \beta(t_0) = U'_u(t_0),
\]

or \( U'(t_0) < U'_u(t_0) \), which contradicts \( U'_u(t_0) \leq U'(t_0) \). Hence \( U(t) < U_u(t) \) for all \( t \geq -\tau \). Finally, as \( t \mapsto e^{-(a+b)t}|Y(t)| \) is in \( L^1(0, \infty) \), we have \( \beta \) is in \( L^1([-\tau, \infty); [0, \infty)) \) a.s. Thus

\[
U(t) < \alpha + \int_{-\tau}^{t} \beta(s) \, ds \leq \alpha + \int_{-\tau}^{\infty} \beta(s) \, ds, \quad \text{for all } t \geq 0, \text{ a.s.}
\]

This establishes that \( U \) is uniformly bounded above, and hence the proof is complete.

4.2. Proof of Theorem 2

Let \( Z(t) = X(t) - Y(t) \) for \( t \geq -\tau \), where \( Y \) is an Ornstein–Uhlenbeck process governed by

\[
dY(t) = aY(t) + \sigma dB(t), \quad t > 0, \quad Y(t) = \phi(t), \quad t \in [-\tau, 0].
\]

Therefore

\[
Z'(t) = aZ(t) + b \sup_{t-\tau \leq s \leq t} (Z(s) + Y(s)), \quad t > 0
\]

(44)

with \( Z(t) = 0 \) for \( t \in [-\tau, 0] \). By taking the upper Dini derivative of \( |Z| \) it can be shown that

\[
D_+|Z(t)| \leq a|Z(t)| + b \sup_{t-\tau \leq s \leq t} |Z(s)| + b \sup_{t-\tau \leq s \leq t} |Y(s)|.
\]

Consider the process \( V \) defined by \( V'(t) = aV(t) + b \sup_{t-\tau \leq s \leq t} V(t) + f(t) \) where \( V \in C([-\tau, \infty); \mathbb{R}^+) \), and \( f \in C([-\tau, \infty); \mathbb{R}^+) \). Let \( \beta \in C([-\tau, \infty); \mathbb{R}^+) \) be such that \( \lim_{-\tau - \rightarrow \infty} \beta(t) = C_1 \). Then

\[
\text{there exists } T_{1,\epsilon} > 0 \text{ such that } C_1 - \epsilon < \beta(t) < C_1 + \epsilon, \quad \text{for all } t > T_{1,\epsilon}.
\]

(45)

So

\[
\text{for all } t > T_{2,\epsilon} > T_{1,\epsilon} + \tau, \quad \int_{t-\tau}^{t} \beta(s) \, ds < \int_{t-\tau}^{t} (C_1 + \epsilon) \, ds = (C_1 + \epsilon) \tau.
\]

(46)

Let \( 0 < C_2 < C_2(\epsilon) \in (0, \log \frac{\alpha}{\epsilon}) \), where \( C_2(\epsilon) \) is a real root of \( g_\epsilon(x) = -(C_1 + \epsilon)x - a - b \exp ((C_1 + \epsilon)\tau x) \). The existence of such a \( C_2(\epsilon) \) is guaranteed due to the facts that (i) \( g_\epsilon \) is continuous and decreasing; (ii) \( g_\epsilon(0) > 0 \); and (iii) \( g_\epsilon(\log \frac{\alpha}{\epsilon}) < 0 \), from the definition of \( C_1 \).

Define the constant

\[
V_\epsilon^* := 2 \max_{-\tau \leq l \leq T_{2,\epsilon}} \{ V(t)e^{C_2 \int_{0}^{l} \beta(u) \, du} - \int_{0}^{l} e^{C_2 \int_{0}^{u} \beta(u) \, du} f(s) \, ds \}
\]

(47)

Note that \( V_\epsilon^* > V(0) > 0 \). Define

\[
V_{U,\epsilon}(t) := e^{-C_2 \int_{0}^{l} \beta(u) \, du} \left( V_\epsilon^* + \int_{0}^{l} e^{C_2 \int_{0}^{u} \beta(u) \, du} f(s) \, ds \right).
\]

(48)
Clearly $V_{U,\epsilon}(t) > 0$ for all $t \in [-\tau, \infty)$. Since $Y$ obeys $\limsup_{t \to \infty} |Y(t)|/\sqrt{2 \log t} = |\varsigma|/\sqrt{2a}$ a.s., there exists $T_{3,\epsilon}(\omega) > 0$, such that for all $t > T_{3,\epsilon}(\omega)$, $|Y(t)| \leq (\varsigma(1 + \epsilon)/\sqrt{a})\sqrt{\log t}$. By considering $f(t)$ as $(\varsigma(1 + \epsilon)/\sqrt{a})\sqrt{\log t}$, together with L'Hôpital's rule, and e.g., [25, Theorem 8.1.4, volume II], we therefore get

$$
\limsup_{t \to \infty} \frac{|Z(t)|}{\sqrt{2 \log t}} \leq \limsup_{t \to \infty} \frac{V(t)}{\sqrt{2 \log t}} \leq \limsup_{t \to \infty} \frac{V_{U,\epsilon}(t)}{\sqrt{2 \log t}} \leq \limsup_{t \to \infty} \frac{\varsigma(1 + \epsilon) \int_{T_{3,\epsilon}}^{t} e^{C_{2} f_{0} \beta(u) du} \sqrt{\log s} ds}{\sqrt{2aC_{1}C_{2}}} \Rightarrow a.s.
$$

Letting $C_{2} \uparrow C_{2}(\epsilon)$, and $\epsilon \downarrow 0$, we get

$$
\limsup_{t \to \infty} \frac{|Z(t)|}{\sqrt{2 \log t}} \leq (C_{1} \log \frac{-a}{b})^{-1} \frac{|\varsigma|}{\sqrt{-2a}}, \quad a.s.
$$

Recalling that $|X(t)| \leq |Y(t)| + |Z(t)|$, the upper bound follows. Finally, the lower bound can be obtained from (6) by comparing (1) with equation (2) (see [4]).

### 4.3. Proof of Theorem 3

We have already shown in Section 2.2 that under the conditions of the theorem $X(t) > 0$ for $t \geq 0$ a.s. Since for $0 \leq t \leq s$ we have

$$
X(s) = X(t) + \int_{t}^{s} (aX(u) + b \sup_{u-t \leq u \leq s} X(u)) du + \int_{t}^{s} \sigma(X(u))dB(u), \quad (49)
$$

we get

$$
X(s) = X(t) + \int_{t}^{s} \sup_{u-t \leq u \leq s} X(u) du + \int_{t}^{s} \sigma(X(u))dB(u),
$$

and $X(t)$ obeys

$$
\frac{dX(t)}{dt} = aX(t) + b \sup_{u-t \leq u \leq t} X(u) + \sigma(X(t))dB(t).
$$

Thus, we have checked all assumptions of [25, Theorem 8.1.4, volume II] for $X(t)$, and we may now apply the theorem.
the fact that \( b \sup_{u - \tau \leq v \leq u} X(v) + aX(u) \geq (b + a)X(u) > 0 \) for \( u \geq 0 \) implies

\[
\sup_{t \leq s \leq t + \tau} X(s) \leq X(t) + \int_{t}^{t + \tau} (aX(u) + b \sup_{u - \tau \leq v \leq u} X(v)) \, du + \sup_{t \leq s \leq t + \tau} \left| \int_{t}^{s} \sigma(X(u)) \, dB(u) \right|.
\]

Defining \( x(t) := \mathbb{E}[X(t)] \), \( x_*(t) := \mathbb{E}[\sup_{t - \tau \leq s \leq t} X(s)] \), we obtain

\[
x_*(t + \tau) \leq x(t) + \int_{t}^{t + \tau} (ax(u) + bx_*(u)) \, du + \mathbb{E} \left[ \sup_{t \leq s \leq t + \tau} \left| \int_{t}^{s} \sigma(X(u)) \, dB(u) \right| \right].
\]

By the Burkholder-Davis-Gundy inequality there exists a \( t- \) and \( X- \) independent positive constant \( C \) such that

\[
\mathbb{E} \left[ \sup_{t \leq s \leq t + \tau} \left| \int_{t}^{s} \sigma(X(u)) \, dB(u) \right| \right] \leq C \mathbb{E} \left[ \left( \int_{t}^{t + \tau} \sigma^2(X(u)) \, du \right)^{1/2} \right] \leq C \sqrt{\tau} \mathbb{E} \left[ \sup_{t \leq u \leq t + \tau} |\sigma(X(u))| \right].
\]

Taking expectations across (49) with \( s = t + \tau \), we get

\[
x(t + \tau) = x(t) + \int_{t}^{t + \tau} (ax(u) + bx_*(u)) \, du,
\]

so \( x_*(t + \tau) \leq x(t + \tau) + C \sqrt{\tau} \mathbb{E}[\sup_{t \leq s \leq t + \tau} |\sigma(X(s))|] \) or

\[
x_*(t) \leq x(t) + C \sqrt{\tau} \mathbb{E} \left[ \sup_{t - \tau \leq s \leq t} |\sigma(X(s))| \right], \quad t \geq \tau.
\]

Since \( \sigma \) obeys (15), for every \( \epsilon > 0 \), there exists \( L(\epsilon) > 0 \) such that \( |\sigma(x)| < L(\epsilon) + \epsilon x \) for \( x \geq 0 \). Hence for all \( t \geq 0 \), \( \mathbb{E}[\sup_{t - \tau \leq s \leq t} |\sigma(X(s))|] \leq L(\epsilon) + \epsilon x_*(t) \), and so \( x_*(t) \leq x(t) + C \sqrt{\tau} L(\epsilon) + \epsilon C \sqrt{\tau} x_*(t) \). Let \( \epsilon C \sqrt{\tau} < 1/2 \), so that \( (1 - \epsilon C \sqrt{\tau}) x_*(t) < x(t) + C \sqrt{\tau} L(\epsilon) \).

Hence

\[
x(t) \leq x_*(t) < \frac{1}{1 - \epsilon C \sqrt{\tau}} x(t) + \frac{C \sqrt{\tau}}{1 - \epsilon C \sqrt{\tau}} L(\epsilon).
\]

Now due to (49), for every \( h > 0 \), we have \( x(t + h) \geq x(t) + (a + b) \int_{t}^{t + h} x(s) \, ds \). Since \( t \mapsto \mathbb{E}[X(t)] = x(t) \) is continuous (this can be inferred from e.g., [27, Problem 5.3.15]), we have \( x^*(t) \geq (a + b)x(t) \). Also since \( x(0) > 0 \), \( x(t) e^{(a+b)t} \geq x(0) \) and \( x(t) \to \infty \) as \( t \to \infty \). Returning to (52) yields

\[
1 \leq \liminf_{t \to \infty} \frac{x_*(t)}{x(t)} \leq \limsup_{t \to \infty} \frac{x_*(t)}{x(t)} \leq \frac{1}{1 - \epsilon C \sqrt{\tau}},
\]

and letting \( \epsilon \downarrow 0 \) yields

\[
\lim_{t \to \infty} \frac{x_*(t)}{x(t)} = 1.
\]

This proves the second part of (16). For the first part, since \( x'(t) = ax(t) + bx_*(t) \) for \( t > \tau \) and \( x(\tau) > 0 \), by defining \( b(t) := bx_*(t)/x(t) \) for \( t > \tau \), we have \( x'(t) = ax(t) + b(t)x(t) \). We have shown that \( b(t) \to b \) as \( t \to \infty \). Thus \( \log x(t) = \log x(\tau) + \int_{\tau}^{t} (a + b(s)) \, ds \), \( t \geq \tau \). Therefore \( \lim_{t \to \infty} \log x(t)/t = \lim_{t \to \infty} \log x_*(t)/t = a + b \), proving the first part of (16). This limit implies that for all \( \epsilon > 0 \), there is a \( K(\epsilon) > 0 \) such that \( \mathbb{E}[\sup_{(n-1)\tau \leq s \leq n\tau} X(s)] \leq K(\epsilon) e^{(a+b+2\epsilon)\tau} \). Thus

\[
P \left[ \sup_{(n-1)\tau \leq s \leq n\tau} X(s) > e^{(a+b+2\epsilon)\tau} \right] \leq e^{-(a+b+2\epsilon)\tau} \mathbb{E} \left[ \sup_{(n-1)\tau \leq s \leq n\tau} X(s) \right] \leq K(\epsilon) e^{-\epsilon \tau}.
\]
Thus by the Borel-Cantelli lemma, there exists \( N(\epsilon, \omega) \in \mathbb{N} \) such that for \( n > N(\epsilon, \omega) \), \( \sup_{n(t) - 1 \leq s \leq n(t) + 1} X(s) \leq e^{(a+b+2\epsilon)n}\). Let \( k > N(\epsilon, \omega) \). Then there exists \( n(t) \in \mathbb{N} \) such that \( (n(t) - 1)\tau < t < n(t)\tau \), such that \( X(t) \leq \sup_{n(t) - 1 \leq s \leq n(t) + 1} X(s) \) and \( n(t) > N(\epsilon, \omega) \). Hence

\[
X(t) \leq \sup_{(n(t) - 1)\tau \leq s \leq n(t)\tau} X(s) < e^{(a+b+2\epsilon)n(t)\tau} \leq e^{(a+b+2\epsilon)(n(t)\tau + e^{(a+b+2\epsilon)\tau})}.
\]

Therefore \( \limsup_{t \to \infty} \log X(t)/t \leq a + b \), a.s. By the comparison principle, we know that \( X(t) \geq Z(t) \) where \( Z \) obeys \( Z(0) = \psi(0) > 0 \) and

\[
dZ(t) = (a + b)Z(t) dt + \sigma(Z(t)) dB(t).
\]

Since \( a + b > 0 \) and (17) holds, by e.g., [27, Remark 5.5.19, Proposition 5.5.22], we have \( Z(t) \to \infty \) as \( t \to \infty \) and \( Z(t) > 0 \) for \( t > 0 \). Hence

\[
\log Z(t) = \log Z(0) + (a + b)t - \frac{1}{2} \int_0^t \frac{\sigma(Z^2(s))}{Z^2(s)} ds + \int_0^t \frac{\sigma(Z(s))}{Z(s)} dB(s).
\]

(54)

By (15), we have \( \lim_{t \to \infty} \sigma(Z(t))/Z(t) = 0 \), a.s. Thus \( \lim_{t \to \infty} \log Z(t)/t = a + b \), a.s., and so \( \liminf_{t \to \infty} \log X(t)/t \geq a + b \), a.s. Therefore (18) is proven.

### 4.4. Proof of Theorem 4

Define \( Y \) by \( Y(0) = 0 \) and \( dY(t) = aY(t) dt + \sigma(X(t)) dB(t) \) for \( t > 0 \). Then

\[
Y(t) = e^{at} \int_0^t e^{-as} \sigma(X(s)) dB(s), \quad t > 0.
\]

(56)

Let \( Z(t) := X(t) - Y(t) \). Then \( Z(t) = aZ(t) + b\sup_{t - \tau \leq s \leq t}[Z(s) + Y(s)] \). Hence \( D_+|Z(t)| \leq a|Z(t)| + b\sup_{t - \tau \leq s \leq t}|Z(s) + Y(s)| \). Let \( Z_*(\tau) \) be defined as

\[
Z_*(\tau) = e^{(a+b)(t-\tau)}Z_*(\tau) + \int_\tau^t be^{-(a+b)(s-\tau)} \left\{ \sup_{s-\tau \leq u \leq s} |Y(u)| + 1 \right\} ds, \quad t \geq 0.
\]

Then

\[
e^{-b(a+b)\tau}Z_*(\tau) + \int_\tau^t be^{-(a+b)(s-\tau)} \left( 1 + \sup_{s-\tau \leq u \leq s} |Y(u)| \right) ds, \quad t \geq 0.
\]

Hence \( \sup_{t - \tau \leq s \leq t} Z_*(s) = Z_*(t) \) for \( t \leq \tau \), and so for \( t > \tau \),

\[
Z_*(t)' = (a + b)Z_*(t) + b \sup_{t - \tau \leq s \leq t} |Y(s)| + aZ_*(t) + b \sup_{t - \tau \leq s \leq t} Z_*(s) + b \sup_{t - \tau \leq s \leq t} |Y(s)|.
\]

By making \( Z_*(\tau) \) sufficiently large we can ensure that \( Z_*(t) > 0 \) for all \( t > 0 \) and also that \( Z_*(t) \geq |Z(t)| \) for \( t \in [0, \tau] \). This can be arranged taking \( Z_*(\tau) \) such that

\[
e^{-b(a+b)\tau}Z_*(\tau) + \int_\tau^t be^{-(a+b)(s-\tau)} \left( 1 + \sup_{s-\tau \leq u \leq s} |Y(u)| \right) ds > \max_{t \in [0,\tau]} |Z(t)|.
\]

Hence \( Z_*(t) \geq |Z(t)| \) for \( t \geq \tau \), and so for \( t > \tau \),

\[
|X(t)| \leq K_1 + |Y(t)| + e^{(a+b)(t-\tau)}Z_*(\tau) + be^{(a+b)t} \int_\tau^t \sup_{s-\tau \leq u \leq s} |Y(u)| e^{-(a+b)s} ds.
\]
This implies

\[ e^{-(a+b)t}|X(t)| \leq Z^*(\tau)e^{-(a+b)\tau} + K_1e^{-(a+b)t} + e^{-(a+b)t}|Y(t)| + b\int_0^t \sup_{s-\tau \leq u \leq s} |Y(u)|e^{-(a+b)s} \, ds. \]

Now by the martingale law of the iterated logarithm applied to the martingale \( e^{-at}Y(t) \) in (56),

\[ e^{-2at}Y^2(t) \leq 4 \log t \int_0^t e^{-2as} \sigma^2(X(s)) \, ds, \quad t > T_{1,\omega}. \]

By Theorem 3, \( \liminf_{t \to \infty} \log X(t)/t \geq a + b \). Hence for all \( t > T_{2,\omega}, (a+b)t/2 < \log X(t) \). Moreover, since \( \sigma \) obeys (19), there is a \( K_2 > 0 \) such that \( \sigma^2(x) < K_2x^2/(\log x)^{2c}, x > x_1 \). Let \( e^{(a+b)T_{1,\omega}} > x_1 \), and take \( T_{4,\omega} := \max\{T_{1,\omega}, T_{2,\omega}, T_3\} \). For \( t > T_{4,\omega}, X(t) > e^{(a+b)t/2} > x_1 \), so

\[ \sigma^2(X(t)) < K_2 \frac{X^2(t)}{(\log X(t))^{2c}} < K_2 \left( \frac{X^2(t)}{(a+b)t/2)^{2c}} : = K_3 \frac{X^2(t)}{t^{2c}}, \quad t > T_4. \]

Now for \( t > T_5 \) and \( T_5 > T_4 \) we have

\[ e^{-2at}Y^2(t) \leq 4 \log t \int_0^{T_5} e^{-2as} \sigma^2(X(s)) \, ds + 4 \log t \int_{T_5}^t e^{-2as} \sigma^2(X(s)) \, ds \]

\[ \leq 4 \log t \cdot K_4(T_5) + 4 \log t \int_{T_5}^t K_3e^{-2as} \frac{X^2(s)}{s^{2c}} \, ds \]

\[ \leq 4 \log t \cdot K_4(T_5) + \sup_{T_5 \leq s \leq t} \frac{X^2(s)}{e^{2(a+b)s}} \cdot 4K_3 \log t \int_{T_5}^t \frac{e^{2bs}}{s^{2c}} \, ds. \]

Thus

\[ e^{-2(a+b)t}Y^2(t) \leq 4e^{-2bt} \log t \cdot K_4(T_5) + \sup_{T_5 \leq s \leq t} \frac{X^2(s)}{e^{2(a+b)s}} \cdot 4K_3 \log t \int_{T_5}^t \frac{e^{2bs}}{s^{2c}} \, ds \]

which gives

\[ e^{-(a+b)t}|Y(t)| \leq 2e^{-bt}\sqrt{\log t/K_4(T_5)} + 2\sqrt{K_2} \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}} \cdot \sqrt{\frac{\log t}{t^c}} \sqrt{t^{2c}e^{-2bt} \int_{T_5}^t \frac{e^{2bs}}{s^{2c}} \, ds.} \]

Now for \( t > T_5 + \tau =: T_6, \)

\[ \frac{|X(t)|}{e^{(a+b)t}} \leq K_5(T_6) + \frac{|Y(t)|}{e^{(a+b)t}} + b\int_{T_6}^t \sup_{s-\tau \leq u \leq s} \frac{|Y(u)|}{e^{(a+b)u}}e^{(a+b)(u-s)} \, ds. \]

Hence

\[ \frac{|X(t)|}{e^{(a+b)t}} \leq K_5(T_6) + \frac{|Y(t)|}{e^{(a+b)t}} + b\int_{T_6}^t \sup_{s-\tau \leq u \leq s} \frac{|Y(u)|}{e^{(a+b)u}} \, ds, \quad t \geq T_6. \]

By L’Hôpital’s rule, we have \( \lim_{t \to \infty} \int_{T_5}^t \frac{e^{2bs}}{s^{2c}} \, ds / (e^{2bt}/t^{2c}) = (2b)^{-1} \). Let \( M \) be so big that \( t \to e^{-bt}\sqrt{\log t} \) and \( t \to -t^{-\tau}\sqrt{\log t} \) are monotone on \([M - \tau, \infty)\). Choose \( T_5 = \max\{T_4, M\} \), then

\[ \sup_{t \geq T_5} \int_{T_5}^t \frac{e^{2bs}}{s^{2c}} \, ds \leq \sup_{t \geq T_5} \int_{T_5}^M \frac{e^{2bs}}{s^{2c}} \, ds \leq \sup_{t \geq M} \int_{M}^t \frac{e^{2bs}}{s^{2c}} \, ds =: K_6(M). \]
Note that $K_6(M) > 0$ is deterministic. Thus by (57), for $t > T_5$,

$$e^{-(a+b)t} |Y(t)| \leq 2 \sqrt{K_4(T_5)} e^{-bt} \log t + 2 \sqrt{K_3} \sqrt{\log t} \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}}. \quad (59)$$

Now let $s > T_6$, so $s - \tau > T_5$. Hence

$$\sup_{s-\tau \leq u \leq s} \frac{|Y(u)|}{e^{(a+b)u}} \leq 2 \sqrt{K_4(T_5)} \sup_{s-\tau \leq u \leq s} e^{-bu} \sqrt{\log u}$$

$$+ 2 \sqrt{K_3} \sqrt{K_6(M)} \sup_{s-\tau \leq u \leq s} \frac{\sqrt{\log t}}{t^c} \cdot \sup_{s-\tau \leq u \leq T_5 \leq \leq u} \frac{|X(v)|}{e^{(a+b)v}},$$

$$\leq 2 \sqrt{K_4(T_5)} e^{-b(s-\tau)} \log (s - \tau) + 2 \sqrt{K_3 K_6(M)} \sqrt{\log (s - \tau)} \cdot \sup_{T_5 \leq \leq s} \frac{|X(v)|}{e^{(a+b)v}}.$$.

Inserting (59), (60) into (58) yields for $t > T_6$,

$$\frac{|X(t)|}{e^{(a+b)t}} \leq K_5(T_6) + 2 \sqrt{K_4(T_5)} e^{-bt} \log t + 2 \sqrt{K_3} \sqrt{\log t} \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}}$$

$$+ b \int_{T_6}^{\infty} 2 \sqrt{K_4(T_5)} e^{-b(s-\tau)} \log (s - \tau) ds + 2b \sqrt{K_3 K_6(M)} \int_{T_6}^{t} \sup_{T_6 \leq v \leq \leq s} \frac{|X(v)|}{e^{(a+b)v}} \sqrt{\log (s - \tau)} \cdot \frac{(s - \tau)^c}{e^{(a+b)v}} ds.$$

Taking the first, second and fourth terms together for $t > T_6$, we have

$$\frac{|X(t)|}{e^{(a+b)t}} \leq K_7(T_6) + 2 \sqrt{K_3} \sqrt{K_6(M)} \sqrt{\log t} \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}}$$

$$+ 2b \sqrt{K_3} \sqrt{K_6(M)} \sup_{T_5 \leq v \leq t} \frac{|X(v)|}{e^{(a+b)v}} \int_{T_6}^{t} \frac{\sqrt{\log (s - \tau)}}{(s - \tau)^c} ds$$

$$= K_7(T_6) + \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}} \left( 2 \sqrt{K_3} \sqrt{K_6(M)} \frac{\sqrt{\log t}}{t^c} ight.$$

$$+ 2b \sqrt{K_3} \sqrt{K_6(M)} \int_{T_6}^{\infty} \frac{\sqrt{\log (s - \tau)}}{(s - \tau)^c} ds \big).$$

Now $T_6 = T_5 + \tau \geq M + \tau$, so for $t > T_6$, and by the definition of $M$, we get

$$2 \sqrt{K_3} \sqrt{K_6(M)} \frac{\sqrt{\log t}}{t^c} + 2b \sqrt{K_3} \sqrt{K_6(M)} \int_{T_6}^{\infty} \frac{\sqrt{\log (s - \tau)}}{(s - \tau)^c} ds$$

$$\leq 2 \sqrt{K_3} \sqrt{K_6(M)} \frac{\sqrt{\log M}}{M^c} + 2b \sqrt{K_3} \sqrt{K_6(M)} \int_{M}^{\infty} \frac{\sqrt{\log s}}{s^c} ds =: K_8(M).$$

Since $M \rightarrow K_6(M)$ is non-increasing, for $c > 1$ there is a $M > 0$ such that $K_8(M) < 1/2$. Hence

$$\frac{|X(t)|}{e^{(a+b)t}} \leq K_7(T_6) + K_8(M) \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}}$$

$$< K_7(T_6) + \frac{1}{2} \sup_{T_5 \leq s \leq T_6} \frac{|X(s)|}{e^{(a+b)s}} + \frac{1}{2} \sup_{T_6 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}} =: K_9(T_6) + \frac{1}{2} \sup_{T_5 \leq s \leq t} \frac{|X(s)|}{e^{(a+b)s}}, \quad t \geq T_6.
Thus for $T \geq T_b$ we have

$$\sup_{T_b \leq s \leq T} |X(s)| \leq K_0(T_b) + \frac{1}{2} \sup_{T_b \leq s \leq T} |X(s)|$$

so $\sup_{T_b \leq t \leq T} |X(t)|/e^{(a+b)t} \leq 2K_0(T_b)$ for $T \geq T_b$, and $\limsup_{t \to \infty} |X(t)|e^{-(a+b)t} < \infty$ a.s. Next

$$e^{-(a+b)t}X(t) = X(0) + \int_0^t e^{-(a+b)s} \left( \sup_{s-\tau \leq u \leq s} X(u) - X(s) \right) ds + \int_0^t e^{-(a+b)s} \sigma(X(s)) dB(s), \quad t \geq 0. \quad (61)$$

Since $t \to e^{-(a+b)t}|X(t)|$ is bounded, $\log X(t) > (a+b)t/2$ for $t > T_2$ and $\sigma^2(x) < Kx^2/\log x$ for $x > x_1$, we have $\int_0^\infty \sigma^2(X(s)) e^{2(a+b)s} ds < \infty$ a.s. Hence the third term on the right-hand side of (61) has a limit as $t \to \infty$, and so

$$\lim_{t \to \infty} e^{-(a+b)t}X(t) = \Lambda(\omega) \quad \text{exists a.s.} \quad \text{(62)}$$

$\Lambda$ must be finite and non-negative, as $\limsup_{t \to \infty} e^{-(a+b)t}X(t) < \infty$ a.s. and $X$ is positive.

Finally, we need to show that the limit in (62) is positive, a.s. To do so, let $Z$ be the process defined in (54). Since $Z(t) \to \infty$ a.s. as $t \to \infty$ and $\sigma(x)/x \to 0$ as $x \to \infty$ we have that $\log Z(t)/t \to a + b$ as $t \to \infty$. As shown in the proof of Theorem 5 below, the condition $\int_0^\infty \sigma^2(x)/x^3 dx < +\infty$ together with $\log Z(t)/t \to a + b$ as $t \to \infty$ implies that $\int_0^\infty \sigma^2(Z(s))/Z^2 ds < +\infty$, a.s. which by the martingale convergence theorem, ensures the existence of the finite limit $\lim_{t \to \infty} \int_0^t \sigma(Z(s))/Z(s) dB(s)$. Using (55) it now follows that $\lim_{t \to \infty} \log Z(t) - (a + b)t = \Lambda''$ a.s., where $\Lambda''$ is a.s. finite. Letting $\Lambda' := \exp(\Lambda'') > 0$, $\lim_{t \to \infty} Z(t)e^{-(a+b)t} = \Lambda'$ a.s. Since $X(t) \geq Z(t)$ and $\lim_{t \to \infty} X(t)e^{-(a+b)t}$ is finite and non-negative a.s., we have $\lim_{t \to \infty} X(t)e^{-(a+b)t} = \Lambda \geq \Lambda' > 0$ a.s., as required.

4.5. Proof of Theorem 5

Let $A = \{ \omega : \lim_{t \to \infty} X(t, \omega)e^{-(a+b)t} = \Lambda(\omega) > 0 \}$ and $\mathbb{P}[A] > 0$. Rearranging (61) gives

$$\int_0^t e^{-(a+b)s} \sigma(X(s)) dB(s) = e^{-(a+b)t}X(t) - X(0) - b \int_0^t e^{-(a+b)s} \left( \sup_{s-\tau \leq u \leq s} X(u) - X(s) \right) ds.$$ 

Since $\sup_{t-\tau \leq u \leq s} X(s) \geq X(t)$, and $X(t)e^{-(a+b)t} \to \Lambda$ as $t \to \infty$, the righthand side has a limit as $t \to \infty$ on $A$. Define $C = \{ \omega \in A : \int_0^\infty \sigma^2(X(s, \omega)) e^{-2(a+b)s} ds = +\infty \}$. Then by e.g., [34, Proposition V.1.8]

$$\limsup_{t \to \infty} \int_0^t \sigma(X(s))e^{-(a+b)s} dB(s) = +\infty, \quad \liminf_{t \to \infty} \int_0^t \sigma(X(s))e^{-(a+b)s} dB(s) = -\infty, \quad \text{on } C.$$ 

This contradicts the existence of $\lim_{t \to \infty} \int_0^t \sigma(X(s))e^{-(a+b)s} dB(s)$, which means we must have $\int_0^\infty \sigma^2(X(s)) e^{-2(a+b)s} ds < +\infty$ a.s. on $A$. For notational simplicity, we suppress $\omega$-dependence hereinafter. By (21) there is an $x^* \geq 0$ such that $\sigma_0^2(x)/2 \leq \sigma^2(x)$, $x \geq x^*$. Since $X(t)e^{-(a+b)t} \to \Lambda$ as $t \to \infty$, $\Lambda/2 \cdot e^{(a+b)t} < X(t)$ for all $t > T_1$. Define $T_2 > 0$ by $\Lambda e^{(a+b)T_2}/2 = x^* > 0$. If $T_3 = \max(T_1, T_2)$, $X(t) > \Lambda/2 \cdot e^{(a+b)t}$ for $t > T_3$. Since $x \to \sigma_0^2(x)$ is non-decreasing on $[x^*, \infty)$, and $\sigma^2(x) \geq \frac{1}{2} \sigma_0^2(x)$ for $x \geq x^*$, $\sigma^2(X(t)) \geq \sigma_0^2(X(t))/2 \geq \sigma_0^2(\Lambda/2 \cdot e^{(a+b)t})/2$ for $t > T_3$. Hence

$$\int_{T_3}^\infty \sigma_0^2(\Lambda/2 \cdot e^{(a+b)t}) e^{-2(a+b)t} ds \leq \int_{T_3}^\infty \sigma_0^2(X(s)) e^{-2(a+b)t} ds < +\infty.$$
The result follows from the identity
\[ \int_{T_3}^\infty \sigma_0^2(\Lambda/2 \cdot e^{(a+b)s}) e^{-2(a+b)s} \, ds = \frac{\Lambda^2}{a+b} \int_{\frac{\Lambda e(a+b)}{T_3}}^\infty \frac{\sigma_0^2(u)}{u^3} \, du. \]

4.6. Proof of Theorem 6

Due to (24a) and the fact that \( \varphi \) is non-decreasing, we have
\[
E \left[ \sup_{t-\tau \leq s \leq t} |\sigma(X(s))| \right] \leq E \left[ \sup_{t-\tau \leq s \leq t} \varphi(X(s)) \right] = E \left[ \varphi \left( \sup_{t-\tau \leq s \leq t} X(s) \right) \right].
\]

Also since \( \varphi \) is concave, \( E[\varphi(x(t))] \leq \varphi(x(t)) \) where \( x_s \) is defined in Theorem 3. Since (51) still holds, we have \( x(t) \leq x_s(t) \leq x(t) + C\sqrt{T} \varphi(x_s(t)) \), with \( x \) as defined in Theorem 3. By (24b) and (24c), we have \( \varphi(x)/x \rightarrow 0 \) as \( x \rightarrow \infty \). Thus \( \lim_{x \rightarrow \infty} x_s(t)/x(t) = 1 \).

Therefore for every \( \epsilon > 0 \), there is \( T_{1,\epsilon} > \tau \) such that \( x_s(t) < (1 + \epsilon)x(t) \) for \( t > T_{1,\epsilon} \). Thus \( x_s(t) < x(t) + C\sqrt{T} \varphi(x_s(t)) < x(t) + C\sqrt{T} \varphi(1 + \epsilon)x(t) \) for \( t > T_{1,\epsilon} \).

Since \( x^\prime(t) = ax(t) + bx_s(t) \), we have \( x^\prime(t)/x(t) < a + b + bC\sqrt{T}(1 + \epsilon) \varphi((1 + \epsilon)x(t))/x(t) \), \( t > T_{1,\epsilon} \).

By Theorem 3 we have \( \lim_{x \rightarrow \infty} \log x_s(t)/t = a + b \). Therefore, for every \( \epsilon > 0 \) there exists \( T_\epsilon > 0 \) such that \( (1 + \epsilon)x(t) > e^{(a+b)\epsilon/2} \) for \( t > T_\epsilon \). Let \( T_{2,\epsilon} := \max\{T, T_{1,\epsilon}\} \). Then for \( t > T_{2,\epsilon} \),
\[
\log x(t) - \log x(T_{2,\epsilon}) < (a + b)(t - T_{2,\epsilon}) + bC\sqrt{T}(1 + \epsilon) \int_{T_{2,\epsilon}}^t \frac{\varphi(x_s(s))}{x(s)} \, ds
\]
where \( x_s(t) := (1 + \epsilon)x(t) \). By (24b) we have \( \varphi(x_s(t))/x_s(t) < \varphi(e^{(a+b)\epsilon/2})/e^{(a+b)\epsilon/2} \) for \( t > T_{2,\epsilon} \), because \( x_s(t) > e^{(a+b)\epsilon/2} \), \( t > T_{2,\epsilon} \).

For \( t > T_{2,\epsilon} \), by (24c), we have
\[
\int_{T_{2,\epsilon}}^t \frac{\varphi(x_s(s))}{x_s(s)} \, ds \leq \int_{T_{2,\epsilon}}^t \frac{\varphi(e^{(a+b)\epsilon/2})}{e^{(a+b)\epsilon/2}} \, ds \leq \frac{2}{a + b} \int_{e^{(a+b)\epsilon/2}T_{2,\epsilon}}^\infty \frac{\varphi(u)}{u^2} \, du.
\]

Therefore
\[
\log x(t) - (a + b)t \leq \log x(T_{2,\epsilon}) - (a + b)T_{2,\epsilon} + \frac{2(1 + \epsilon)bC\sqrt{T}}{a + b} \int_{e^{(a+b)\epsilon/2}T_{2,\epsilon}}^\infty \frac{\varphi(u)}{u^2} \, du,
\]
and there is a \( c_1 < +\infty \) such that \( \limsup_{t \rightarrow \infty} \log x(t) - (a + b)t \leq c_1 < \infty \). Hence
\[
\limsup_{t \rightarrow \infty} x(t)e^{-(a+b)t} \leq e^{c_1}.
\]

As \( x(\tau) > 0 \), there is a \( c_2 > 0 \) such that \( \liminf_{t \rightarrow \infty} x(t)e^{-(a+b)t} \geq c_2 \). Integration by parts gives
\[
e^{-(a+b)t}X(t) = e^{-(a+b)t}X(\tau) + \int_{\tau}^t be^{-(a+b)s} \left( \sup_{s-\tau \leq u \leq s} X(u) - X(s) \right) \, ds
\]
\[\quad + \int_{\tau}^t e^{-(a+b)s}\sigma(X(s)) \, dB(s), \]
for \( t \geq \tau \). Hence \( e^{-(a+b)t}x(t) = e^{-(a+b)t}x(\tau) + \int_{\tau}^t be^{-(a+b)s}(x_s(s) - x(s)) \, ds \). As the integrand on the right-hand side is non-negative, \( \lim_{t \rightarrow \infty} \int_{\tau}^t be^{-(a+b)s}(x_s(s) - x(s)) \, ds =: \lambda_\tau \). Thus \( \lambda_\tau = \infty \) contradicts (63).

Hence \( \lambda_\tau < +\infty \). Therefore there is \( c_3 > 0 \) such that \( \lim_{t \rightarrow \infty} x(t)e^{-(a+b)t} = c_3 \).

Since \( x_s(t)/x(t) \rightarrow 1 \) as \( t \rightarrow \infty \), we get the second part of (25). This completes the proof.
4.7. Proof of Theorem 7
Let $c > 1$ be the number in the logarithmic term in (19). By (19), there is a $\lambda > 0$ such that
\[ |\sigma(x)| \leq \lambda \frac{x + e^{c+1}}{\log^{c+1}(x + e^{c+1})}, \quad x > e^{c+1}. \]
Let $L = 1 + \max_{0 \leq x \leq e^{c+1}} |\sigma(x)| \geq |\sigma(x)|$ for all $x \in [0, e^{c+1}]$. Hence for all $x \geq 0$ we have
\[ |\sigma(x)| \leq L + \lambda \frac{x + e^{c+1}}{\log^{c+1}(x + e^{c+1})} =: \varphi(x). \]
Clearly $\varphi$ is continuous. The fact that $\varphi'(x) > 0$, $\varphi''(x) < 0$ for all $x \geq 0$ follows from
\[ \varphi'(x) = \frac{x + e^{c+1}}{\log^{c+1}(x + e^{c+1})} - \frac{c}{\log^{c+1}(x + e^{c+1})} \quad \text{and} \quad \varphi''(x) = -\lambda c \frac{1}{x + e^{c+1}} \cdot \frac{\log(x + e^{c+1}) - (c + 1)}{\log^{c+2}(x + e^{c+1})}. \]
$\varphi$ also satisfies (24c), and (24a) holds by construction. Finally, if $\varphi_1(x) := \varphi(x)/x$, then
\[ \varphi_1'(x) = -\frac{L}{x^2} + \lambda \cdot \frac{-cx \log^{c-1}(x + e^{c+1}) - e^{c+1} \log^c(x + e^{c+1})}{x^2 \log^{c+1}(x + e^{c+1})}, \]
from which (24b) follows. Hence all the properties of $\varphi$ in Theorem 6 have been verified.

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