The Peierls argument for higher dimensional Ising models

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Abstract

The Peierls argument is a mathematically rigorous and intuitive method to show the presence of a non-vanishing spontaneous magnetization in some lattice models. This argument is typically explained for the $D = 2$ Ising model in a way which cannot be easily generalized to higher dimensions. The aim of this paper is to present an elementary discussion of the Peierls argument for the general $D$-dimensional Ising model.

Keywords: Peierls argument, spontaneous symmetry breaking, phase transition

1. Spontaneous symmetry breaking and the Ising model

Spontaneous symmetry breaking is a unifying theme of modern theoretical physics, with applications ranging from statistical physics [1] to condensed matter [2, 3] and particle physics [4]. Because of its fundamental relevance in our understanding of nature, it is important to study simple systems for which the presence of spontaneous symmetry breaking can be rigorously established. Probably the simplest system which displays spontaneous symmetry breaking is the Ising model in statistical physics, named after E Ising, who first solved the unidimensional version of the problem [5].

This model is defined on a graph, i.e. on a set of points (called sites in the following) equipped with the notion of nearest neighbourhood. A variable $s_i$, which takes a value in the set $\{+1, -1\}$, is associated to each site $i$ of the graph. We will call a ‘configuration’ a given assignment of the variables $\{s_i\}$ to the graph sites and the energy of a configuration is defined by

$$E = -J \sum_{(i,j)} s_is_j - h \sum_i s_i.$$  \hfill (1)
where $J$, $h$ are constants and $\sum_{(i,\cdot)}$ denotes the sum on the first neighbour sites. For the sake of the simplicity in the following we will only consider the model defined on a hypercubic lattice in $D$ dimensions, which is the most studied case.

For $J > 0$ this model describes a uniaxial ferromagnet in an external magnetic field of intensity $h$. Configurations in which most of the nearest neighbour sites are oriented in the same direction are favoured, since they correspond to lower values of the energy. Moreover configurations with the value sign $(h)$ in most sites are favoured by the interaction with the external magnetic field.

The model with $h = 0$ is particularly interesting since the energy is invariant under the transformation

$$ s_i \rightarrow -s_i \quad \forall i. \quad (2) $$

By applying this transformation twice we come back to the original configuration, so the symmetry group of the model for $h = 0$ is $Z_2$. In this paper we will be interested just in this $h = 0$ case.

We now introduce the average magnetization per site of a configuration, defined by

$$ m = \frac{1}{N} \sum_i s_i = \frac{N_+ - N_-}{N}, $$

where $N$ is the total number of sites and $N_{\pm}$ is the number of sites with $s_i = \pm 1$. It is clear from this definition that $m$ is odd under the transformation in equation (2), i.e. $m$ goes to $-m$.

Since the energy of two configurations related by the symmetry in equation (2) is the same, one could think that the statistical average of $m$, denoted by $\langle m \rangle$, identically vanishes. This is true for finite systems, however, in the thermodynamical limit, if $D \geq 2$ the magnetization $\langle m \rangle_\infty$ vanishes only in the high temperature paramagnetic phase. In the low temperature ferromagnetic phase the value of $\langle m \rangle_\infty$ is not well defined and depends on how the thermodynamical limit is performed. In this case the symmetry in equation (2) is said to be spontaneously broken.

The breaking of a symmetry can be thought as a form of thermodynamical instability: the particular value acquired by $\langle m \rangle_\infty$ in the ferromagnetic phase is determined by small perturbations. A conventional way to uniquely define $\langle m \rangle_\infty$ in the broken phase (where it is called spontaneous magnetization) is to use an infinitesimal magnetic field:

$$ \langle m \rangle_\infty = \lim_{h \to 0^+} \lim_{N \to \infty} \langle m \rangle, \quad (3) $$

where it is crucial to perform the thermodynamical limit before switching off the magnetic field. The instability manifests itself in that using $h \to 0^-$ in equation (3) would change the sign of $\langle m \rangle_\infty$.

A different approach to expose the instability is the use of appropriate boundary conditions: we can for example impose in all the sites $i_b$ on the lattice boundary the condition $s_{i_b} = +1$. In the paramagnetic phase the effect of the boundary conditions does not survive the thermodynamical limit, while in the ferromagnetic phase their effect is analogous to that of the infinitesimal magnetic field in equation (3).

On a finite lattice the mean value of the magnetization $m$ can be written in the form

$$ \langle m \rangle = \frac{\langle N_+ \rangle - \langle N_- \rangle}{N} = 1 - 2 \frac{\langle N_- \rangle}{N}, \quad (4) $$

where we used the fact $N_+ + N_- = N$, and in order to show that $\langle m \rangle_\infty > 0$ it is sufficient to show that for every $N$ we have $\langle N_- \rangle/N < 1/2 - \epsilon$ (with $\epsilon > 0$ and $N$-independent). The Peierls argument is a simple geometrical construction that can be used to prove this bound. It was introduced for the first time in [6] and some errors in the estimates used were later
corrected in [7]. The original formulation referred to the two-dimensional Ising model, whose
solution [8] was still not known, but the idea of the argument can be adapted also to the general
$D$-dimensional problem with $D > 2$, which is still an active field of research (see e.g. [9–11]).

The outcome of the Peierls argument for the model in $D$ dimensions is an estimate of the form

$$\langle N^- \rangle \leq N f_D(x)$$

where $x$ is defined by

$$x = 9 e^{-4J/\beta} \quad \beta = 1/(kT)$$

and $f_D(x)$ is a continuous function of $x$ (independent of $N$) such that $\lim_{x \to 0} f_D(x) = 0$. In particular for sufficiently small $T$ we have the bound $\langle N^- \rangle/N < 1/2 - \epsilon$ ($\epsilon > 0$), which ensures that $(m)_{\infty} \geq 2\epsilon$ and the $Z_2$ symmetry is spontaneously broken.

The original $D = 2$ argument is described in most books on statistical mechanics (like e.g. [1], section 14.3), however the construction is presented in such a way that the generalization to higher dimensions is not immediate: the use of ordered paths in $D = 2$ simplifies the proof of some of the estimates but cannot be easily generalized to the higher-dimensional setting. Moreover the combinatorics needed for $D > 2$ appears at first sight to be much more involved than the one required in the two-dimensional case. On the other hand the higher dimensional problem is discussed in specialized books (like [12–14]), but the topic is approached from a different and more abstract point of view, out of reach for most of the students of a first course in statistical mechanics. The consequence could be the (erroneous!) feeling that the Peierls argument can be conveniently applied only in the case in which, strictly speaking, it is no longer necessary, as the two-dimensional Ising model is exactly solvable.

The purpose of this paper is to fill this gap by presenting an elementary discussion of the Peierls argument in its more general $D$-dimensional version. First of all we present the Peierls argument in $D = 2$ using a construction scheme of the domains different from the one typically adopted, which can be easily generalized to the higher-dimensional case, then we go on to show how to solve the additional problems that arise in the higher-dimensional environment. With this aim we discuss in some detail the $D = 3$ problem, where one still has a good geometrical intuition, and finally analyse the general $D$-dimensional case, which is at this point an almost trivial extension of the three-dimensional one.

2. The $D = 2$ problem

We consider a two-dimensional square lattice of size $\sqrt{N} \times \sqrt{N}$ (with lattice spacing $a = 1$) and fix $s_i = +1$ on the boundary sites $i_b$. The Peierls contours are introduced by the following procedure:

(i) draw a unit square on each site $i$ with $s_i = -1$,
(ii) cancel the edges that appear twice (i.e. that separate two neighbour sites $i, j$ with $s_i = s_j = -1$),
(iii) in the case in which four edges meet at the same point, chop off the corner of the squares in order to remove ambiguities.

An example of the application of this procedure is shown in figure 1 and it is immediate to show that the following facts are true:

- every contour is a closed non-intersecting curve,
- every site $i$ with $s_i = -1$ is inside at least one contour,
the set of the admissible contours is in a one-to-one correspondence with the set of the configurations.

The one-to-one relation in the last property depends on the fixed \( s_b = +1 \) boundary conditions: the presence of a contour signals a change of sign of the site variable and the +1 assignment on the boundaries uniquely fix the signs.

In a finite lattice the number of contours of given length \( L \) is finite, let us denote this number by \( \#(L) \). The generic Peierls contour can then be denoted by \( \gamma_i^L \), where \( L \) is the length of the contour and \( 1 \leq i \leq \#(L) \). If \( A(\gamma) \) is the area of the contour \( \gamma \) (that is the number of sites inside the contour) we have the following upper bound for the number \( N_- \) of sites with \( s_i = -1 \) present in a configuration:

\[
N_- \leq \sum_{L \geq 4, \text{even}} \sum_{i=1}^{\#(L)} A(\gamma_i^L)X(\gamma_i^L),
\]

where \( X(\gamma) = 1 \) if \( \gamma \) occurs in the given configuration and \( X(\gamma) = 0 \) otherwise. The lower bound of the sum is four since this is the minimal length possible for a closed contour, moreover the length of a closed contour has to be even, hence the sum extends on even numbers only. Equation (7) is a simple consequence of the fact that every site \( i \) with \( s_i = -1 \) is inside at least one contour: if we sum the area of all the contours that occur in a configuration we get an upper bound for \( N_- \) in that configuration.

The next step is to show that \( A(\gamma_i^L) \) has an upper bound of the form \( A(L) \), i.e. that depend only on the length \( L \) of the contour. This upper bound can be obtained in the following way: draw the smallest rectangle \( R \) (with sides parallel to the boundaries of the lattice) that contain \( \gamma_i^L \) (see figure 2 for an example). The perimeter of \( R \) is not larger than \( L \): if we draw inside \( R \) a line parallel to one of the axes, this line intersects \( R \) at two edges, but this line also has to intersect \( \gamma_i^L \) at at least two edges, otherwise \( \gamma_i^L \) would be separated in two disjoint contours or \( R \) would not be the smallest rectangle containing \( \gamma_i^L \). If we denote the length of the sides of \( R \) by \( x_1 \) and \( x_2 \), we thus have \( 2(x_1 + x_2) \leq L \) and \( A(\gamma_i^L) \leq x_1x_2 \). As a consequence

\[
A(\gamma_i^L) \leq \max_{2(x_1 + x_2) \leq L} x_1x_2.
\]
Figure 2. The first step of the proof of the bound for $A(\gamma_i L)$ (the edges of the rectangle are slightly shifted to improve readability).

It is simple to show that the maximum in the previous equations is reached when $x_1 = x_2 = L/4$ (i.e. of all the rectangles of fixed perimeter the square is the one with the largest area) and we conclude that

$$A(\gamma_i L) \leq A(L) \equiv \frac{L^2}{16}$$

(9)

We can now use equation (9) to modify equation (7) as follows:

$$N_- \leq \sum_{L \geq 4, \text{even}} A(L) \sum_{i=1}^{\#(L)} X(\gamma_i L).$$

In the following we will not be interested in the value of $N_-$ for a single configuration, but on the mean value $\langle N_- \rangle$. From the previous equation we get

$$\langle N_- \rangle \leq \sum_{L \geq 4, \text{even}} A(L) \sum_{i=1}^{\#(L)} \langle X(\gamma_i L) \rangle$$

(10)

and our next task will be to show that $\langle X(\gamma_i L) \rangle \leq X(L)$, where the function $X(L)$ depends only on the length $L$ of the path $\gamma_i L$.

The thermal average $\langle X(\gamma_i L) \rangle$ is defined by

$$\langle X(\gamma_i L) \rangle = \frac{\sum_{c \in \mathcal{R}} X(\gamma_i L) e^{-\beta E(c)}}{\sum_{c \in \mathcal{R}} e^{-\beta E(c)}},$$

where $\beta = 1/(kT)$ and the sum is extended over the set $\mathcal{R}$ of configurations that satisfy the $s_{ii} = +1$ boundary conditions. From the definition of $X(\gamma_i L)$ we immediately see that its mean value can be rewritten in the form

$$\langle X(\gamma_i L) \rangle = \frac{\sum_{c \in \mathcal{R}} e^{-\beta E(c)}}{\sum_{c \in \mathcal{R}} e^{-\beta E(c)}}.$$
where $\mathcal{C}$ is the set of all the configurations in $\mathcal{B}$ which contain the Peierls contour $\gamma'_1$. Let $c$ be a configuration in $\mathcal{C}$ and define the configuration $\tilde{c}$ as the one obtained from $c$ by changing the sign of all the variables associated to sites inside $\gamma'_1$ (an example of such a transformation is given in figure 1). It is clear that $\tilde{\tilde{c}} \in \mathcal{B}$, since the boundary values of the configuration are unchanged, so the set of all the possible $\tilde{c}$ is a subset of the collection $\mathcal{B}$:

$$\mathcal{C} = \{\tilde{c} | c \in \mathcal{C}\} \subset \mathcal{B}$$

and thus

$$\sum_{\tilde{c} \in \mathcal{C}} e^{-\beta E(\tilde{c})} \leq \sum_{c \in \mathcal{B}} e^{-\beta E(c)}.$$

From this relation see that

$$\langle X(\gamma'_1) \rangle = \frac{\sum_{c \in \mathcal{C}} e^{-\beta E(c)}}{\sum_{c \in \mathcal{B}} e^{-\beta E(c)}}$$

and, noting that if $c \in \mathcal{C}$ we have

$$E(c) = E(\tilde{c}) + 2JL,$$

we obtain the upper bound

$$\langle X(\gamma'_1) \rangle \leq X(L) \equiv e^{-2J\beta L}. \tag{11}$$

By using this bound in equation (10) we get

$$\langle N_+ \rangle \leq \sum_{L \geq 4, \text{even}} A(L) \sum_{i=1}^{\#(L)} X(L) = \sum_{L \geq 4, \text{even}} A(L) \#(L) X(L) \tag{13}$$

and to finish we need an upper bound for $\#(L)$, i.e. for the number of closed paths of length $L$. This can be obtained by enumerating the possible ways in which a closed contour of length $L$ can be constructed by using $L$ edges.

We have $2N$ possible choices of where to put the first edge; we will call this choice step 1. Since we have to build a closed curve, each of the two endings of the first edge has to be connected to other edges and each new edge can be connected in three different ways to the previous one. We now proceed by iteration: at step $n \geq 2$ we add two new edges to the two open endings of the curve obtained at step $n - 1$. Once the edge at step 1 is fixed we have at most $3^{2(n-1)}$ ways to build up the curve to step $n$. The length of the curve at step $n$ is $2(n - 1) + 1$ and when we arrive at the step $\bar{n}$ defined by

$$2(\bar{n} - 1) + 1 = L - 1$$

(remember that $L$ is even) we have only one possible way to add the last edge to close the curve, so the number of closed curves of length $L$ has to be smaller than $2N3^{2(\bar{n}-1)} = 2N3^L$. We now note that the same path can be obtained in a similar way by starting from a different edge in the first step, since all the edges of a closed curve are on the same footing, so we arrive to the upper bound

$$\#(L) \leq N(L) \equiv \frac{2N}{9L} 3^L. \tag{14}$$

By using this estimate in equation (13) and remembering the definitions in equations (9) and (12) we finally get

$$\langle N_+ \rangle \leq \sum_{L \geq 4, \text{even}} A(L) N(L) X(L) \leq \sum_{L \geq 4, \text{even}} N 3^L e^{-2J\beta L}. \tag{15}$$

This sum is convergent provided $3e^{-2J\beta} < 1$ and the sum can be performed analytically (see appendix A). The final result is ($x$ is defined in equation (6))

$$\langle N_+ \rangle \leq \frac{N}{36} x^2 \frac{2 - x}{(1 - x)^2}, \tag{16}$$

which is a bound of the form equation (5).
Figure 3. A surface with the maximum $x_1$ value at fixed $S$: $x_1 = (S - 2)/4$.

3. The $D = 3$ problem

We now consider a three-dimensional square lattice of size $N^{1/3} \times N^{1/3} \times N^{1/3}$ with lattice spacing $a = 1$. The Peierls contours of the two-dimensional case now become surfaces, but their construction proceed along the same line as in the two-dimensional case:

(i) draw a unit cube on each site $i$ such that $s_i = -1$,
(ii) cancel the faces that appear twice (i.e. that separate two neighbour sites $i, j$ with $s_i = s_j = -1$),
(iii) if ambiguities are present, chop off the corners of the cubes in order to remove them.

When using the $s_i = +1$ boundary condition we have the following properties

- every Peierls surface is a closed non-intersecting surface,
- every site $i$ with $s_i = -1$ is inside at least one surface,
- the set of the admissible surfaces is in a one-to-one correspondence with the set of the configurations;

which are the natural extension of the properties seen in the two-dimensional case.

The bound in equation (7) becomes now

$$N_- \leq \sum_{S \geq 6, \text{even}} \sum_{i=1}^{\#(S)} A(\gamma^S_i)X(\gamma^S_i),$$

where $\gamma^S_i$ denotes the general Peierls surface composed by $S$ elementary squares, $A(\gamma^S_i)$ is the volume of $\gamma^S_i$ (that is the number of sites it contains) and $\#(S)$ is the number of closed surfaces of area $S$. $X(\gamma^S_i)$ is defined as in the two-dimensional case, the lower extremum of the sum on $S$ is 6 since this is the smallest area of a closed surface in $D = 3$ and the sum extends on even numbers only since the area of a closed surface is always even.

To show that, for sufficiently low temperature, a spontaneous magnetization is present we have to found the three-dimensional analogues of the bounds equations (9), (12) and (14).

To find the three-dimensional version of equation (9) the procedure used in the two-dimensional case has to be slightly modified. Let us consider the smallest rectangular parallelogram $R$ that contains the surface $\gamma^S_i$. In the three-dimensional space it is not true that the area of $R$ is not larger than the area of $\gamma^S_i$ (a simple counterexample is a donut shaped surface of sufficiently large radius), so we need a different constraint to be imposed on the edges $x_1, x_2, x_3$ of $R$. We can for example notice that we must have $x_1, x_2, x_3 \leq S/4$; by using $S$ elementary squares to construct a closed surface, the maximum value we can get for $x_1$ (or $x_2$ or $x_3$) is $(S - 2)/4$, which correspond to the surface shown in figure 3. Thus we get

$$A(\gamma^S_i) \leq \max(x_1,x_2,x_3) \leq (\max x_1)(\max x_2)(\max x_3) = A(S) \equiv \frac{S^3}{4^3},$$

Proceeding in this way in the two-dimensional case we would have obtained $A(L) = L^2/4$, which is weaker than equation (9).
The bound in equation (12) becomes
\[ \langle X(\gamma) \rangle \leq X(S) \equiv e^{-2JS} \] (19)
and the proof is completely analogous to the one given for the two-dimensional case: the transformation \( c \rightarrow \bar{c} \) now flip all the variables associated to the sites inside the surface \( \gamma \) and equation (11) becomes
\[ E(c) = E(\bar{c}) + 2JS. \] (20)

The last ingredient we need is the bound on \( \#(S) \). Again we can proceed analogously to the two-dimensional case, by enumerating the possible ways in which we can put together \( S \) elementary squares to obtain a closed surface. In step number 1 we have \( 3N \) possible choices, and the surface is then built in the following way: at step \( n \) we add \( s_n \) squares to the surface, in such a way to saturate all the free edges present in the step \( n - 1 \).

Here a little complication arises: in the planar case at every step we always have to add two more elements to the construction, while in the three-dimensional case the number of elementary squares to be added is not constant in \( n \), and in fact for some configurations this number is not even uniquely determined (i.e. it depends on the way the squares are added). This complication is however not serious: an elementary square can be connected to a free edge in at most three different ways, so at step \( n \), when \( s_n \) squares are added, we have at most \( 3s_n \) possibilities. As a consequence, if the construction of the surface is completed in \( \bar{n} \) steps, the total number of different possibilities is at most \( 3N^{3\bar{n}} \), where \( m \equiv \sum_{n=1}^{\bar{n}} s_n = S - 1 \), independent of the construction details. As in the two-dimensional case, in this reasoning we have overestimated the total number of different configurations by a factor \( S \), since all the squares of a surface can be used as a starting point its construction. The final bound is thus:
\[ \#(S) \leq N(S) \equiv N^{3S \over S}. \] (21)

From equation (17) we can now obtain an upper bound of the form in equation (15):
\[ \langle N_- \rangle \leq \sum_{S \geq 6, \text{even}} A(S)N(S)X(S) = \frac{N}{64} \sum_{S \geq 6, \text{even}} S^2 (3e^{-2JS})^S. \] (22)

This series is convergent provided \( x = 9e^{-4J/\beta} < 1 \) and the sum can be performed analytically, see appendix B, the final result of the computation being
\[ \langle N_- \rangle \leq N \frac{9 - 11x + 4x^2}{16 \times (1 - x)^3}. \] (23)

4. The \( D \geq 3 \) problem

The general case of \( D \) dimensions, with \( D > 3 \), does not present additional difficulties with respect to the three-dimensional setting studied before. We just need to substitute ‘cube’ with ‘hypercube’, ‘face’ with ‘hyperface’ and ‘edge’ with ‘hyperedge’.

The hyperface and hyperedge terms are not conventional but we will use them in order to make evident the similarity with the three-dimensional case. A hyperface is an elementary \( D - 1 \) surface in the \( D \)-dimensional space, i.e. a hypercube in \( D - 1 \) dimensions. In an analogous way a hyperedge is an elementary \( D - 2 \) surface in the \( D \)-dimensional environment, i.e. a hypercube in the \( D - 2 \) dimensional space.

We will consider a hypercubic lattice of linear size \( N^{1/D} \) and lattice spacing \( a = 1 \). The construction of the Peierls domains proceeds along the same way as in \( D = 3 \) and the Peierls domains will now be closed non-intersecting hypersurfaces. Again each site \( i \) with \( s_i = -1 \)
is inside at least a Peierls hypersurface and, when the $s_i = \pm 1$ conditions are imposed on the lattice boundary, the set of the admissible hypersurfaces is in a one-to-one correspondence with the set of the configurations.

The generalization of equation (17) is

$$N_- \leq \sum_{H \geq 2D, \text{even}} \sum_{i=1}^{\#(H)} A(\gamma^i_H) X(\gamma^i_H),$$

(24)

where $H$ is the hyperarea of the Peierls surface $\gamma^i_H$. Again $H$ has to be even and the smallest possible value for $H$ is $2D$.

To estimate $A(\gamma^i_H)$, as in the $D = 3$ setting, we have to find the ‘more elongated’ closed hypersurface composed of $H$ hyperfaces. This is given by the $D$-dimensional generalization of figure 3, for which:

$$x_1 \leq \frac{H - 2}{2(D - 1)}.$$

As a consequence the bound in equation (18) becomes

$$A(\gamma^i_H) \leq A(H) \equiv \left( \frac{H}{2(D - 1)} \right)^D.$$  

(25)

The proof of the bound equation (19) goes on without significant modifications and the final result is again

$$\langle X(\gamma^i_H) \rangle \leq X(H) \equiv e^{-2JH}.$$  

(26)

To get an estimate of $\#(H)$ we just have to notice that, as in the $D = 3$ case, in order to build a hypersurface, a hyperface can be connected to a given hyperedge in no more than three ways. As a consequence also this estimate goes along the same lines as the one in $D = 3$, the only difference being that in the first step we now have $DN$ possibilities instead of $3N$, obtaining

$$\#(H) \leq \mathcal{N}(H) \equiv DN \frac{2H}{3H}.$$  

(27)

The final bound on $\langle N_- \rangle$ for the $D$-dimensional problem is thus

$$\langle N_- \rangle \leq \sum_{H \geq 2D, \text{even}} A(H) \cdot \mathcal{N}(H) X(H)$$

$$\leq \sum_{H \geq 2D, \text{even}} \left( \frac{H}{2(D - 1)} \right)^D DN \frac{2H}{3H} e^{-2JH},$$  

(28)

which, using the substitution $H = 2k$ and equation (6), becomes

$$\langle N_- \rangle \leq \frac{ND}{6(D - 1)^D} \sum_{k=0}^{\infty} k^{D-1} x^k.$$  

(29)

For general $D$ the sum cannot be performed in a closed rational form, but, with the substitution $k = p + d$, can be rewritten as

$$\sum_{k=0}^{\infty} k^{D-1} x^k = x^D \sum_{p=0}^{\infty} \frac{x^p}{(p + D)^{D-1}} = x^D \Phi(x, 1 - D, D),$$

where $\Phi$ is the Lerch transcendent function (see e.g. [15] section 1.11, [16]). The final result is thus

$$\langle N_- \rangle \leq \frac{ND}{6(D - 1)^D} x^D \Phi(x, 1 - D, D),$$  

(29)

which is of the form of equation (5) since the Lerch transcendent is regular for $x = 0$. 

9
5. Conclusions

We have presented an elementary discussion of the Peierls argument for the case of the $D$-dimensional ($D \geq 2$) Ising model defined on a cubical lattice. The outcome of this argument is an upper bound for $\langle N_- \rangle / N$ (equations (16), (23) and (29)), which implies for low enough temperature a non-vanishing lower bound for spontaneous magnetization and thus the presence of spontaneous symmetry breaking.

By looking back at the previous exposition we see that, apart from the numerical details, what makes the argument sound is the fact that the ‘entropy’ terms $A(H)$ and $N'(H)$ grow as a finite power of $H$, while the ‘energy’ term $X(H)$ is exponentially dumped by $H$. As a consequence the series in equation (28) is convergent and the sum vanishes in the large $\beta$ limit.

The failure of this condition is the reason why the argument cannot be applied to the one-dimensional Ising model: in that case the domains are just segments, $A(H)$ and $N'(H)$ still grow with $H$ but now $X(H)$ is $H$-independent (it is just $e^{-4\beta J}$). The upper bound for $\langle N_- \rangle$ is now a series which is divergent in the thermodynamical limit and thus useless. In fact the one-dimensional Ising model can be analytically solved and no symmetry breaking is found for any positive value of the temperature (see e.g. [1]).

As a last remark we note that from the bound in equation (29) we can get a bound for the critical coupling $\beta_\text{c}$, which is defined as the coupling at which the system switches from a ferromagnetic state to a paramagnetic state. From equation (4) we see that, as far as $\langle N_- \rangle / N < 1/2 - \epsilon$, the system has to be ferromagnetic, so the critical value $x_\text{c} = x(\beta_\text{c})$ must lie outside the region $[0, x_{1/2}]$, where $x_{1/2}$ is defined as the smallest positive solution of the equation

$$
\frac{D}{6(D-1)^D} \Phi(x, 1-D, D) = \frac{1}{2}
$$

From $x_\text{c} > x_{1/2}$ we get $J\beta_\text{c} \leq J\beta_{1/2}$, where

$$
J\beta_{1/2} = \frac{1}{4} \log \left( \frac{9}{x_{1/2}} \right)
$$
and some numerical values for $J\beta_{1/2}$ as a function of $D$ are reported in figure 4. For comparison, the best available determination of the critical point for the three-dimensional Ising model is $J\beta_c(3D) = 0.22165452(8)$ (see [17]), from which we see that, for $D = 3$, $J\beta_{1/2}$ is of the same order of magnitude of $J\beta_c$. For larger $D$ values this is however no longer true.

The behaviour of the critical temperature of the Ising model in the limit of large $D$ is quite well known: both lower [18, 19] and upper [20] bounds are known for $\beta_c$ and, by the combination of these bounds, one gets ([21])

$$J\beta_c = \frac{1}{2D - 1 + O(1/D)},$$

(30)

and thus $\beta_c$ goes to zero as $D$ grows. On the other hand from figure 4 we see that $\beta_{1/2}$ converges to a non-vanishing limit as $D \to \infty$. This is a well known limitation of the Peierls argument (see the introduction of [22] for a discussion), which has to be modified in a non-elementary way to obtain an upper bound for $J\beta_c$ that vanishes in the large $D$ limit [22].

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Appendix A. The sum in equation (15)

By changing the variable from $L$ to $n = L/2$ we get

$$\sum_{L=1, \text{even}}^{\infty} L3^L e^{-2J\beta L} = 2 \sum_{n=2}^{\infty} n(9 e^{-4J\beta})^n.$$

We introduce the variable $x$ defined in equation (6) and, by using $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$, we obtain

$$2 \sum_{n=2}^{\infty} nx^n = 2 \left( \sum_{n=1}^{\infty} nx^n \right) - 2x = 2x \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) - 2x$$

$$= 2x \left( \frac{1}{(1 - x)^2} - 1 \right) = 2x^2 \frac{2 - x}{(1 - x)^2},$$

which is the result used in the text.

Appendix B. The sum in equation (22)

By changing the variable from $S$ to $k = S/2$ we have

$$\sum_{S=6, \text{even}}^{\infty} S^2 (3e^{-2J\beta})^S = 4 \sum_{k=3}^{\infty} k^2 (9e^{-4J\beta})^k = 4 \left( \sum_{k=1}^{\infty} k^2 x^k \right) - 4x - 16x^2,$$

where in the second line we have used the definition equation (6). Moreover

$$\sum_{k=1}^{\infty} k^2 x^k = x \frac{d}{dx} \sum_{k=1}^{\infty} kx^k = \left( x \frac{d}{dx} \right)^2 \sum_{k=0}^{\infty} x^k,$$

and by using $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ we obtain

$$\sum_{k=1}^{\infty} k^2 x^k = \frac{x(1 + x)}{(1 - x)^3}.$$
The final result is thus
\[ \sum_{S=6, \text{even}}^{\infty} S^2 (3e^{-2J\beta})^S = 4x^3 \frac{9 - 11x + 4x^2}{(1-x)^3}. \]

References

[1] Huang K 1987 Statistical Mechanics (New York: Wiley)
[2] Anderson P W 1997 Basic Notions of Condensed Matter Physics (Boulder, CO: Westview Press)
[3] Chaikin P M and Lubensky T C 1995 Principles of Condensed Matter Physics (Cambridge: Cambridge University Press)
[4] Weinberg S 1996 The Quantum Theory of Fields: Modern Applications vol 2 (Cambridge: Cambridge University Press)
[5] Ising E 1925 Beitrag zur Theorie des Ferromagnetismus Z. Phys. 31 253
[6] Peierls R 1936 On Ising’s model of ferromagnetism Proc. Camb. Phil. Soc. 32 477
[7] Griffiths R B 1964 Peierls proof of spontaneous magnetization in a two-dimensional Ising ferromagnet Phys. Rev. 136 A437
[8] Onsager L 1944 Crystal statistics: I. A two-dimensional model with an order-disorder transition Phys. Rev. 65 117
[9] El-Showk S, Paulos M F, Poland D, Rychkov S, Simmons-Duffin D and Vichi A 2012 Solving the 3D Ising model with the conformal bootstrap Phys. Rev. D 86 025022
[10] Bilíó M, Caselle M, Gaiotto D, Gliozzi F, Meineri M and Pellegrini R 2013 Line defects in the 3D Ising model J. High Energy Phys. JHEP07(2013)055
[11] Gliozzi F 2013 Constraints on conformal field theories in diverse dimensions from the bootstrap mechanism Phys. Rev. Lett. 111 161602
[12] Ruelle D 1969 Statistical Mechanics: Rigorous Results (New York: Benjamin)
[13] Sinai Ya G 1982 Theory of Phase Transitions: Rigorous Results (Oxford: Pergamon)
[14] Simon B 1993 The Statistical Mechanics of Lattice Gases vol 1 (Princeton, NJ: Princeton University Press)
[15] Erdélyi A 1953 Higher Trascendental Functions (New York: McGraw-Hill)
[16] Apostol T M 2013 Zeta and Related Functions NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/
[17] Deng Y and Blöte H W J 2003 Simultaneous analysis of several models in the three-dimensional Ising universality class Phys. Rev. E 68 036125
[18] Griffiths R B 1967 Correlations in Ising ferromagnets: III. A mean-field bound for binary correlations Commun. Math. Phys. 6 121
[19] Fisher M E 1967 Critical temperatures of anisotropic Ising lattices: II. General upper bounds Phys. Rev. 162 480
[20] Fröhlich J, Simon B and Spencer T 1976 Infrared bounds, phase transitions and continuous symmetry breaking Commun. Math. Phys. 50 79
[21] Driessler W, Landau L and Fernando Perez J 1979 Estimates of critical length and critical temperatures for classical and quantum lattice systems J. Stat. Phys. 20 123
[22] Lebowitz J L and Mazel A 1998 Improved Peierls argument for high dimensional Ising models J. Stat. Phys. 90 1051