LOG–POISSON HIERARCHICAL CLUSTERING OF COSMIC NEUTRAL HYDROGEN AND Lyα TRANSMITTED FLUX OF QSO ABSORPTION SPECTRUM

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ABSTRACT

In this paper, we study the non-Gaussian features of the mass density field of neutral hydrogen fluid and the Lyα transmitted flux of QSO absorption spectrum from the point of view of self-similar log–Poisson hierarchy. It has been recently shown that, in the scale range, from the onset of nonlinear evolution to dissipation, the velocity and mass density fields of cosmic baryon fluid are extremely well described by the She–Leveque’s scaling formula, which is due to the log–Poisson hierarchical cascade. Since the mass density ratio between ionized hydrogen and total hydrogen is not uniform in space, the mass density field of the neutral hydrogen component is not given by a similar mapping of total baryon fluid. Nevertheless, we show, with hydrodynamic simulation samples of the concordance ΛCDM universe, that the mass density field of neutral hydrogen is also well described by the log–Poisson hierarchy. We then investigate the field of Lyα transmitted flux of QSO absorption spectrum. Due to redshift distortion, Lyα transmitted flux fluctuations no longer show all features of the log–Poisson hierarchy. However, some non-Gaussian features predicted by the log–Poisson hierarchy are not affected by the redshift distortion. We test these predictions with the high resolution and high signal-to-noise ratio data of the quasar’s Lyα absorption spectra. All results given by real data, including β–hierarchy, high-order moments and scale–scale correlation, are found to be consistent with the log–Poisson hierarchy. We compare the log–Poisson hierarchy with the popular log-normal model of the Lyα transmitted flux. The latter is found to yield too strong non-Gaussianity at high orders, while the log–Poisson hierarchy is in agreement with observed data.

Key words: cosmology: theory – large-scale structure of universe

1. INTRODUCTION

Baryon matter of the universe is mainly in the form of an intergalactic medium (IGM), of which the dynamics can be described as a compressible fluid. Luminous objects are formed from baryon matter in the gravitational well of dark matter. Therefore, the dynamical state of baryon fluid in nonlinear regime is crucial to understanding the formation and evolution of the large-scale structures of the universe. In the linear regime, the baryon fluid follows the mass and velocity fields of collisionless dark matter. In the nonlinear regime, however, the baryon fluid statistically decouples from the underlying dark matter field. The statistical behavior of baryon fluid is no longer described by a similar mapping of the underlying dark matter field (e.g., Pando et al. 2004).

It was first pointed out by Shandarin & Zeldovich (1989) that the dynamical behavior of baryon matter clustering on large scales is similar to turbulence. The expansion of the universe eliminates the gravity of uniformly distributed dark matter. Therefore, the dynamical state of baryon fluid in nonlinear regime is crucial to understanding the formation and evolution of the large-scale structures of the universe. In the linear regime, the baryon fluid follows the mass and velocity fields of collisionless dark matter. In the nonlinear regime, however, the baryon fluid statistically decouples from the underlying dark matter field. The statistical behavior of baryon fluid is no longer described by a similar mapping of the underlying dark matter field (e.g., Pando et al. 2004).

Nevertheless, the turbulence-like behavior of cosmic baryon fluid has been gradually noticed. First, the dynamics of growth modes of the cosmic matter are found to be sketched by a stochastic force driven by Burger’s equation (Gurbatov et al. 1989; Berera & Fang 1994). The Burger’s equation driven by the random force of the gravity of dark matter can also sketch the evolution of baryon fluid, if cooling and heating are ignored (Jones 1999; Matarrese & Mohayaee 2002). Later, the Burger’s fluid is found to show turbulence behavior if the Reynolds number is large enough (Polyakov 1995; Laessig 2000; Bec & Frisch 2000; Davoudi et al. 2001). The Reynolds number of IGM at the nonlinear regime is actually large. Therefore, we may expect that, in the scale-free range, the dynamical state of cosmic baryon fluid should be Burger’s turbulence. The turbulence of Burger’s fluid is different from the turbulence of compressible fluid. The latter consists of vortices on various scales, while the former is a collection of shocks.

With the cosmological hydrodynamic simulation based on Navier–Stokes equations in which heating and cooling processes are properly accounted, it has been found that the velocity field of the IGM consists of an ensemble of shocks, and satisfies some scaling relations predicted by Burger’s turbulence (Kim et al. 2005). This result reveals that the turbulence features of cosmic baryon fluid are independent of the details of dissipation (heating and cooling) mechanism, if we consider only the scale-free range, i.e., from the scale of the onset of nonlinear evolution to the scale of dissipation, say, Jeans length.

A new progress is to show that the velocity field of cosmic baryon fluid can be extremely well described by the SL formula (He et al. 2006). The SL formula is considered to be the basic statistical features of the self-similar evolution of fully developed turbulence. Very recently, the
non-Gaussianities of mass density field of the hydrodynamic simulation samples are found to be consistent with the predictions of the log–Poisson hierarchy, which originates from some hidden symmetry of the Navier–Stokes equations. This hierarchical model gives a unified explanation of non-Gaussian features of baryon fluid, including the intermittence, hierarchical relation, and scale–scale correlations (Liu & Fang 2008). These results strongly indicate that, in the scale-free range, the dynamical state of cosmic baryon fluid is similar to fully developed turbulence.

In this paper, we investigate the log–Poisson hierarchy of cosmic baryon fluid with observed data—the Lyα transmitted flux of quasar absorption spectrum, which is due to the absorptions of quasar continuum by the diffusely distributed neutral hydrogen (Bi et al. 1995; Bi & Devidsen 1997; Rauch 1998). These samples offer a unique way to study the non-Gaussian feature of cosmic baryon fluid. The fields of Lyα forests and transmitted flux are highly non-Gaussian. Observation samples of Lyα forests and transmitted flux show scale–scale correlation (Pando et al. 1998), intermittence (Jamkhedkar et al. 2000; Pando et al. 2002; Feng et al. 2003), and nonthermal broadening of H I and He II Lyα absorption lines (Zheng et al. 2004; Liu et al. 2006). We will show that the log–Poisson hierarchy provides a crux to understand the non-Gaussian behavior.

The outline of this paper is as follows. Section 2 gives an introduction of the log–Poisson hierarchy. Section 3 shows that the neutral hydrogen component of cosmic baryon fluid is of log–Poisson hierarchy. In Section 4, we study the log–Poisson hierarchy of the field of Lyα transmitted flux with observed samples of quasar absorption spectra. A comparison between log–Poisson hierarchy and the log-normal model is also presented in Section 4. The conclusion and discussion are given in Section 5.

2. LOG–POISSON HIERARCHY

2.1. Structure Function

To describe the statistical properties of an isotropic and homogenous random field \( \rho(x) \), it generally uses correlation functions of \( \delta \rho(x) = \rho(x) - \bar{\rho}, \bar{\rho} \) being the mean of density. For instance, a two-point correlation function is \( \langle \delta \rho(x) \delta \rho(y) \rangle \). To reveal the turbulence–like behavior, we use variable \( \delta \rho_r = \rho(x + r) - \rho(x) \), where \( r = |r| \). The variable \( \delta \rho_r \) is very different from variable \( \delta \rho(x) \). The latter can be larger than \( \bar{\rho} \), but cannot be less than \( -\bar{\rho} \), and therefore, for a nonlinear field, the distribution of \( \delta \rho(x) \) generally is skewed; however, the distribution of \( \delta \rho_r \) is symmetric with respect to positive and negative \( \delta \rho_r \), if the field is statistically uniform.

With \( \delta \rho(x) \), the clustering and non-Gaussianity of mass density \( \rho(x) \) are measured by two- and multiple-point correlation functions of \( \delta \rho(x) \). With \( \delta \rho_r \), the statistical features are described by the structure function defined as

\[
S_p(r) \equiv \langle |\delta \rho_r|^p \rangle, \tag{1}
\]

where \( p \) is the order of statistics, and the average \( \langle \cdots \rangle \) is taken over the ensemble of the density field. A comparison of the correlation function and structure function has been analyzed in detail by Monin & Yaglom (1975). The second structure function \( S_2 = \langle |\delta \rho_r|^2 \rangle \) as a function \( r \) (scale) actually is the power spectrum of the mass density field \( \rho(r) \) (Fang & Feng 2000).

In the scale-free range of the dynamical equations and initial conditions, the structure function is scale invariant, and therefore, it is generally expressed as a power law of \( r \),

\[
S_p(r) \propto r^{\xi(p)}, \tag{2}
\]

where \( \xi(p) \) is called intermittent exponent. Since the pioneer work of Kolmogorov (1941), it is believed that the relation of \( \xi(p) \) versus \( p \) is related to the scale-covariance of the dynamical equations and initial conditions. For fully developed turbulence of Navier–Stokes fluid, \( \xi(p) \) is a nonlinear function of \( p \). Since then, many hierarchy models for interpreting \( \xi(p) \) have been proposed (Frisch 1995). Finally, the best model is given by the SL scaling formula (She & Leveque 1994), which is yielded from the log–Poisson hierarchy process (Dubrulle 1994).

Although the cosmic baryon fluid is not incompressible, samples of mass and velocity fields of cosmic baryon fluid produced by the cosmological hydrodynamic simulation of the concordance \( \Lambda \)CDM model show good arrangement with the SL scaling and the log–Poisson hierarchy. This is not surprising, because the hierarchical structure model is based mainly on the invariance and symmetry of nonlinear dynamical systems. Therefore, for systems other than the Navier–Stokes, incompressible fluid also will show the SL scaling and the log–Poisson hierarchy if they have the similar invariance and symmetry (She 1997).

2.2. Log–Poisson Hierarchical Cascade

The scenario of hierarchical clustering has been widely used to describe nonlinear evolution of the mass field of cosmic matter. We will first give the basic assumptions of the log–Poisson hierarchy cascade, and then discuss the physics behind this model.

The log–Poisson hierarchy assumes that, in the scale-free range, the variables (density fluctuation) \( |\delta \rho_r| \) on different scales \( r \) are related by a statistical relation as (Dubrulle 1994; She & Waymire 1995),

\[
|\delta \rho_{r_2}| = W_{r_2/r_1} |\delta \rho_{r_1}|, \tag{3}
\]

where

\[
W_{r_2/r_1} = \beta^m (r_1/r_2)^{\gamma}. \tag{4}
\]

In Equations (3) and (4), \( r_1 \geq r_2 \), and \( m \) is a random variable with the Poisson probability distribution function (PDF) as

\[
P(m) = \exp (-\lambda_{r_1,r_2}) \lambda_{r_1,r_2}^m / m!. \tag{5}
\]

To ensure the normalization \( \{W_{r_1/r_2}\} = 1 \), where \( \langle \cdots \rangle \) is over \( m \), the mean \( \lambda_{r_1,r_2} \) of the Poisson distribution should be

\[
\lambda_{r_1,r_2} = \gamma / (\ln(r_1/r_2)) / (1 - \beta). \tag{6}
\]

The model Equation (3) describes how a density fluctuation \( |\delta \rho_{r_1}| \) on larger scale \( r_1 \) is statistically related to fluctuation \( |\delta \rho_{r_2}| \) on smaller scale \( r_2 \). The log–Poisson model depends mainly on the ratio \( r_1/r_2 \). Thus, it is scale invariant. The random variable \( m \) can be considered as the step of the evolution from \( |\delta \rho_{r_1}| \) to \( |\delta \rho_{r_2}| \). When \( \beta \) is smaller, only the evolutionary process with lower steps is important.

For a Gaussian field, variables \( \delta \rho_{r_1} \) and \( \delta \rho_{r_2} \) have to be statistically independent. It is like that the Fourier modes with different wavenumber \( k_1 \propto 1/r_1 \) and \( k_2 \propto 1/r_2 \) are statistically independent. Therefore, a Gaussian field has to be \( \beta = 1 \). Thus, the parameter \( \beta < 1 \) is a measure of the deviation from Gaussian field. The meaning of \( \gamma \) will be given later.
Among hierarchical clustering models, the log–Poisson hierarchy has the following features. First, the relation between $|\delta \rho_r|$ and $|\delta \rho_r|$ given by Equation (3) is a multiplicative random process. A random multiplicative cascade generally yields a non-Gaussian field. That is, even if the field on large scale $r_1$ is Gaussian, it will be non-Gaussian on small scale $r_2$. This is different from an additive random process (e.g., Cole & Kaiser 1988), which generally yields Gaussian field (Pando et al. 1998).

Second, the cascade from scale $r_1$ to $r_2$ and then to $r_3$ is identical to the cascade from $r_1$ to $r_2$. It is because $W_{r_1} = W_{r_2} W_{r_3} = \beta^N (r_1/r_2)^\gamma$, where $N$ is again a Poisson random variable with $\lambda r_2 = \lambda r_3 + \lambda r_3 r$. Therefore, the log–Poisson hierarchy removes an arbitrariness in defining the steps of cascade from $r_1$ to $r_2$ or $r_2$ to $r_3$. This arbitrariness is a major shortcoming of many hierarchical clustering models, one of them, for instance, is the clustering hierarchy models (Soneira & Peebles 1977; Peebles 1980).

Third, although the log–Poisson hierarchical process is discrete in terms of the discrete random number $m$, it is infinitely divisible. That is, there is no lower limit on the difference $r_1 - r_2$. It can be infinitesimal. This is consistent with the continuous variable $r$ used in the hydrodynamic equation of cosmic baryon fluid. The infinite divisibility cannot be modeled with hierarchy of discrete objects with finite size. The log-normal model is also infinitely divisible. However, their asymptotic behavior of $\xi(p)$ at larger $p$ is unbound. We will compare the log-normal model with the log–Poisson in Section 4.4.

2.3. Intermittent Exponent

With the log–Poisson hierarchy Equation (3), the intermittent exponent $\xi(p)$ is found to be (Liu & Fang 2008)

$$\xi(p) = -\gamma [p - \frac{1 - \beta^p}{1 - \beta}].$$

(7)

When $\beta \to 1$, we have $\xi(p) = 0$. Therefore, $\beta = 1$ is a Gaussian field. A field with $\beta < 1$ is called intermittent. Equation (7) requires $\xi(1) = 0$. Therefore, the difference of an intermittent field from Gaussian is given mainly by the term $\gamma (1 - \beta^p)/(1 - \beta)$ of Equation (7).

From Equation (7), the power spectrum $S_2(r) = \text{const.}$ is flat. This is not generally applicable to the cosmic density field. In the scale-free range, the power spectrum of the mass field is of power-law. Thus, we should generalize the log–Poisson hierarchy Equation (3) by replacing $\delta \rho_r$ and $\delta \rho_r$ with $r^{\alpha} \delta \rho_r$ and $r^{\lambda} \delta \rho_r$. In this case, the intermittent exponent $\xi(p)$ is

$$\xi(p) = -\alpha p - \gamma [p - \frac{1 - \beta^p}{1 - \beta}].$$

(8)

Equations (2) and (7) yield $S_2(r) \propto r^{-2\alpha}$, and therefore, the parameter $2\alpha$ is the index of power spectrum. When $\alpha = 0$, Equation (8) is the same as Equation (7).

2.4. $\beta$-Hierarchy

Since $|\delta \rho_r|^p$ is the $p$th moment of the $\delta \rho_r$, for high $p$, one can attribute the structure function $S_p(r)$ to the events located at the tail of the PDF of $\delta \rho_r$. To pick up the structures, which dominate the $p$-order statistics of $\delta \rho_r$, we define

$$F_p(r) = S_{p+1}(r)/S_p(r).$$

(9)

From Equations (2) and (8), Equation (9) gives

$$F_p(r) = A r^{-\alpha - \gamma(1 - \beta^p)},$$

(10)

where the constant $A$ is independent of $r$ and $p$. For an intermittent field $\beta < 1$, we have $F_\infty(r) = A r^{-\gamma(1 - \beta^p)}$. Thus, from Equation (10), one can find

$$\frac{F_p(r)}{F_\infty(r)} = \left[ \frac{F_{p+1}(r)}{F_\infty(r)} \right]^{1/\beta},$$

(11)

Equation (11) is invariant with respect to a translation in $p$. Since $F_p(r)$ measures the structures dominating the $p$-order statistics, the larger the $p$, the larger the contribution of strong clustered structures to $F_p(r)$. Therefore, Equation (11) describes the hierarchical relation between the stronger (or high $p$) and weaker (or low $p$) clustering. In the scale-free range, where $F_{p+1}(r)/F_\infty(r) < 1$, we have $F_p(r)/F_\infty(r) < F_{p+1}(r)/F_\infty(r)$ if $\beta < 1$. That is, for an intermittent field, weak clustering structures are strongly suppressed with respect to the strong clustering; the smaller the $\beta$, the stronger the suppression of weak clustering structures.

From Equation (11), we have

$$\ln F_{p+1}(r)/F_p(r) = \beta \ln F_p(r)/F_2(r).$$

(12)

This, for all $r$ and $p$, $\ln [F_{p+1}(r)/F_p(r)]$ versus $\ln [F_p(r)/F_2(r)]$ should be on a straight line with slope $\beta$. It is called $\beta$ hierarchy. Equation (12) does not contain parameter $\gamma$ and term $F_\infty(r)$, and therefore, it is a testable prediction of the log–Poisson hierarchy.

Therefore, log–Poisson hierarchy links the sizes of fluctuation structures (Equation (3)) as well as their amplitude or intensity (Equation (11)). This is different from hierarchy models, which give only the relationship between the sizes of objects. The hierarchical relation in Equation (11) is the origin of the log–Poisson hierarchy. That is, the hierarchical relation like Equation (11) is the first recognized to be held for dynamical system described by the Naiver–Stokes equations, or equations close to the Naiver–Stokes fluid (Dubrulle 1994; Leveque & She 1997). Therefore, the SL formula has also been successfully applied to describe the mass fields of compressible fluid (Boldyrev et al. 2002; Padoan et al. 2003).

2.5. $\gamma$-Related Non-Gaussianities

The ratio between higher order to second order moments $\langle |\delta \rho_r|^{2p} \rangle / \langle |\delta \rho_r|^2 \rangle$ is a popular tool to measure non-Gaussianity. When $p = 2$, the ratio is kurtosis. For a Gaussian field, the ratio is independent of $r$, and equal to

$$\frac{\langle |\delta \rho_r|^{2p} \rangle}{\langle |\delta \rho_r|^2 \rangle} = (2p - 1)!!.$$  

(13)

For the log–Poisson hierarchy, one can show (Liu & Fang 2008)

$$\ln \frac{\langle |\delta \rho_r|^{2p} \rangle}{\langle |\delta \rho_r|^2 \rangle} = K_p \ln r + \text{const.}$$

(14)

The moment ratio $\ln \langle |\delta \rho_r|^{2p} \rangle / \langle |\delta \rho_r|^2 \rangle$ is linearly dependent on $\ln r$ (scale free) with the slope $K_p$ given by

$$K_p = -\gamma p (1 - \beta^2) - \frac{(1 - \beta^{2p})}{1 - \beta}.$$  

(15)

As expected, for Gaussian field ($\beta \to 1$), $K_p = 0$, i.e., the ratio of moments is independent of $r$ (Equation (14)). That is, $K_p$ only depends on the last term on the right-hand side of Equation (8), regardless the parameter $\alpha$. 
Another useful non-Gaussian detector is the scale–scale correlation defined as
\[ C_{\rho_1,\rho_2} = \frac{\langle \delta \rho_1 \delta \rho_2 \rangle}{\langle \delta \rho_1 \rangle \langle \delta \rho_2 \rangle}, \]
which describes the correlation between the density fluctuations on scales \( r_1 \) and \( r_2 \). Obviously, for a Gaussian field, \( C_{\rho_1,\rho_2} = 1 \). The clustering of cosmic large-scale structure in the nonlinear regime essentially is due to the interaction between the modes of fluctuations on different scales (e.g., Peebles 1980). Therefore, it is important to detect the scale–scale correlation of cosmic baryon matter.

If the ratio \( r_2/r_1 \) is fixed, the log–Poisson hierarchy predicts the scale–scale correlation to be (Liu & Fang 2008)
\[ C_{\rho_1,\rho_2} = B(r_2/r_1)^{\xi(2p)-2\xi(p)}, \]
where the factor \( B(r_2/r_1) \) is constant when the ratio \( r_2/r_1 \) is fixed. This is because the log–Poisson model is invariant with respect to scale dilation. From Equations (7) or (8), we have
\[ \xi(2p) - 2\xi(p) = -\gamma(1 - \beta^p)^2/(1 - \beta). \]
Thus, if \( r_2/r_1 \) remains constant, the relationship of \( \ln C_{\rho_1,\rho_2} \) versus \( \ln r_1 \) should be a straight line with slope \( -\gamma(1 - \beta^p)^2/(1 - \beta) \). This slope is also dependent only on the last term on the right-hand side of Equation (8), regardless the parameter \( \alpha \).

3. LOG–POISSON HIERARCHY OF NEUTRAL HYDROGEN DENSITY FIELD

The absorption spectrum Ly\( \alpha \) depends on the distribution of neutral hydrogen. We first study the mass density field of diffused neutral hydrogen. Since the UV background radiation is uniform and does not introduce a special spatial scale, one may expect that the mass density field of neutral hydrogen is also a field of the log–Poisson hierarchy. However, the ratio of the neutral hydrogen density \( \rho_{HI}(x) \) to total hydrogen density \( \rho_{HI}(x) \) is not spatially constant, because the temperature–density relation is of multiphase (He et al. 2004, 2005). Therefore, the mass field of the neutral hydrogen would not have the same log–Poisson hierarchical features as of \( \rho(x) \).

3.1. Simulation Samples

The simulation samples of the fields of baryon fluid are generated by the same way as Liu & Fang (2008), which is based on the hybrid hydrodynamic N-body code of weighted essentially nonoscillatory (WENO) scheme (Feng et al. 2004). It is an Eulerian scheme and suitable to analyze the fluid in high density scale range \( 0.9 h \) Mpc and less than nonlinear scale. We use the concordance ΛCDM cosmology model with parameters \( \Omega_m = 0.27, \Omega_b = 0.044, \Omega_\Lambda = 0.73, h = 0.71, \sigma_8 = 0.84, \) and spectral index \( n = 1 \). The transfer function is calculated using CMBFAST (Seljak & Zaldarriaga 1996). We take a primordial composition of H and He (\( X = 0.76, Y = 0.24 \)).

The ionization fraction is calculated with ionization–recombination equilibrium under a uniform UV radiative background, of which the intensity is adjusted to fit the mean of observed Ly\( \alpha \) transmitted flux. We produce the distributions of the mass density of hydrogen \( \rho(x) \), the fraction of neutral hydrogen \( f_{HI}(x) \), temperature \( T(x) \), and velocity \( v(x) \) at redshift \( z \simeq 2.5 \), which is the mean redshift of observed samples used in Section 4.

We randomly sample 10,000 one-dimensional (1D) subsamples to simulate the Ly\( \alpha \) transmitted flux (Section 4). To estimate the errors, we divided the 10,000 samples into 10 subsamples, each of which has 1000 line samples.

3.2. Log–Poisson Hierarchical Statistics

The variable \( \delta \rho_r = \rho(x + r) - \rho(x) \) of density field \( \rho(x + r) \) can be calculated by a discrete wavelet transform (DWT) as
\[ \delta \rho_{r,l} = \int \rho_{HI}(x) \psi_{j,l}(x) dx, \]
where \( \psi_{j,l}(x) \) is the set of discrete wavelet transform (e.g., Fang & Thews 1998). For a 1D sample in physical space from \( x = 0 \) to \( L \), the scale index \( j \) is related to the scale \( r = L/2^j \) and the position index \( l \) is for the cell located at \( x = lL/2^j \). Because the DWT bases \( \psi_{j,l}(x) \) are orthogonal, the variables \( \delta \rho_{r} \) do not cause false correlation. They are effective to describe turbulence of fluid (Farge 1992). For a given scale \( r \) or \( j \), the statistical average (\( \ldots \)) of Equation (5) is over all cells \( l \). We will use the Hare wavelet to do the calculation below. We also repeat the calculations with wavelet Daubechies 4.

Figures 1–3 demonstrate that the density field \( \rho_{HI}(x) \) of neutral hydrogen shows the features of the log–Poisson hierarchy. Figure 1 shows the functions \( F_p(r) \) of the field \( \rho_{HI} \) in the physical scale range 0.9 h\(^{-1}\) Mpc (< \( r < 15 h^{-1} \) Mpc and orders of \( p = 0.5 \times n \) with \( n = 1, 2, \ldots, 8 \). For all \( p \), \( \ln F_p(r) \) can be well fitted by a straight line of \( \ln r \). It is consistent with Equation (10). When \( p > 3 \), \( F_p(r) \) given in Figure 1 is almost independent of \( p \). This indicates that \( F_p(r) \rightarrow \infty \) for higher \( p \). Thus, from Equation (10), \( \beta \) should be less than 1, and therefore, the field is intermittent. In this case, the slope of the straight lines with higher \( p \) has to be equal to \(-\alpha - \gamma\). Figure 1 shows \( \alpha + \gamma = 1.0 \).

Figure 2 is the \( \beta \)-hierarchy Equation (12) of the density field \( \rho_{HI} \), in which the error bars are given by the ranges of \( \ln[F_{p+1}(r)/F_{p}(r)] \) and \( \ln[F_{p}(r)/F_{p+1}(r)] \) of the 10 subsamples, each of which consists of 1000 lines. The \( \beta \)-hierarchy is held for all \( r \) in 1 h\(^{-1}\) Mpc (< \( r < 15 h^{-1} \) Mpc and \( p = 0.5 \) to 4. It yields \( \beta = 0.73 \pm 0.19 \).

Figure 3 plots the intermittent exponent \( \xi(p) \) (Equation (8)). It shows that \( \xi(p) \) can be well fitted with Equation (8) with parameters \( \beta = 0.45, \alpha + \gamma = 1.0, \alpha = 0.25 \). These parameters are in agreement with that given by Figures 1 and 2. The error bars of Figure 3 are also in the range of the 10 subsamples. Therefore, the H\( \text{I} \) mass density distribution in the scale range of \( 1 \) to 15 h\(^{-1}\)Mpc can also be described by the log–Poisson hierarchy. However, the parameters \( \beta \) and \( \gamma \) are different from that of density field \( \rho(x) \), which has \( \beta \simeq 0.47 \),
and $\gamma \simeq 1$ (Liu & Fang 2008). That is, the density field $\rho_{HI}(x)$ is weaker intermittent and less singular than that of $\rho(x)$.

4. LOG–POISSON HIERARCHY OF Lyα TRANSMITTED FLUX

4.1. Samples

4.1.1. Observed Data

For observed samples of Lyα transmitted flux, we use the high resolution, high S/N QSO Lyα absorption spectra of Jamkhedkar (2002) and Jamkhedkar et al. (2003). The power spectrum and intermittency of this data set have been extensively and deeply analyzed (Pando et al. 2002; Feng et al. 2004; Jamkhedkar et al. 2003). It is useful for testing the log–Poisson hierarchy.

This observational data set consists of 28 Keck High Resolution (HIRES) QSO spectra (Kirkman & Tytler 1997). The QSO emission redshifts cover a redshift range from 2.19 to 4.11. The resolution is about 8 km s$^{-1}$. For each of the 28 QSOs, the data are given in form of pixels with wavelength, flux, and noise. The noise accounts for the Poisson fluctuations in the photon count, the noise due to the background, and the instrumentation. The continuum of each spectrum is given by IRAF CONTINUUM fitting.

The data are divided into 12 redshift ranges from $z = 1.6 + n \times 0.20$ to $1.6 + (n + 1) \times 0.20$, where $n = 0, \ldots, 11$. In our analysis below, we use only the data in the range $z = 2.4$ to
2.6, which is the same as our simulation data. The scale range is taken to be 1.54 to 12.3 h⁻¹ Mpc, which is also about the same as simulation samples. The mean flux in this redshift range is \( \langle F \rangle \approx 0.75 \) (Jamkhedkar 2002).

We use the same methods of Jamkhedkar et al. (2003) to deal with noises, metal lines, proximity effect, and bad data chunks. On average the S/N of the Keck spectra is high. Most of the regions with low S/N are saturated absorption regions. Although the percentage of pixels within these regions is not large, they may introduce large uncertainties in the analysis. We should reduce the uncertainty given by low S/N pixels. The method is as follows. First, we calculate the SFCs (scaling function coefficients) of both transmission flux field \( F(x) \) and noise field \( n(x) \) with \( \epsilon_j = \int F(x) \phi_j(x) dx \) and \( \epsilon_j^N = \int n(x) \phi_j(x) dx \), where \( \phi_j(x) \) is the scaling function of wavelet on scale \( j \) and at position \( l \). We then identify unwanted mode \( (j, l) \) by using the threshold condition \( |\epsilon_j^N/\epsilon_j^F| < f \). This condition flags all modes with S/N less than \( f \). The parameter \( f \) is taken to be 3. We also flag modes dominated by metal lines. In order to flag data gaps easily, we set the flux at the gaps to be zero and the error to one, and do the same thing for pixels with negative flux. Finally, we skip all the flagged modes when doing statistics. With this method, no rejoining and smoothing of the data are needed.

### 4.1.2. Simulation Samples

To simulate Lyα transmitted flux \( F \), we use the 10,000 1D samples given in Section 3.1. For each 1D sample, the transmitted flux of Lyα, \( F(z) \) in redshift space is calculated with \( F(z) = \exp[-\tau(z)] \), where \( \tau(z) \) is the optical depth defined as

\[
\tau(z) = \frac{\sigma_0 c}{H} \int_{-\infty}^{\infty} n_{H1}(x) V[z - x - v(x), b(x)] dx,
\]

where \( \sigma_0 \) is the effective cross section of the resonant absorption, \( H \) is Hubble constant at the redshift of the sample, and \( n_{H1}(x) \) is the number density of neutral hydrogen atoms. The Voigt function is \( V[z - x - v(x), b(x)] = 1/(\pi^{1/2}b) \exp[-(z - x - v(x))^2/b^2(x)] \), and \( b(x) \) is the thermal broadening. To have a proper comparison between simulation and observation, we add noises \( n_i = G \times A_i \) to each pixel \( i \), where \( G \) is randomly sampled from standard normal distribution and \( A_i \) is the noise level of pixel \( i \). We then take the same data reduction of noised modes as observed data.

### 4.2. Redshift Distortion and \( \beta \)-Hierarchy

The velocity field \( v(x) \) and thermal broadening \( b(x) \) of Equation (20) will lead to the deviation of the statistical properties of \( \ln F(z) = -\tau(z) \) from \( n_{H1}(x) \) or \( \rho_{H1}(x) \). The field \( \ln F(z) \) is no longer to show all features of the log–Poisson hierarchy. We should study which properties of the log–Poisson hierarchy can still be seen with Lyα transmitted flux.

Because the scale-free range is larger than the Jeans length, the effect of thermal broadening would be small in this range. In this case, Equation (20) can be approximated as

\[
-\ln F(z) = z - v(x) \int n_{H1}(x) \delta[z - x - v(x)] dx \quad (21)
\]

This relation is actually a mapping from a physical space field \( n_{H1}(x) \) to redshift space field \( \tau(z) \), which is the same as that used in the redshift distortion of galaxy distribution. It has been shown that, with the DWT variables, the redshift distortion of Equation (21) can be estimated by (Yang et al. 2002)

\[
\delta\tau_{r,l} = \mathcal{R}_r \delta\rho_{r,l} \quad (22)
\]

where the DWT variables \( \delta\tau_{r,l} \) are given by \( \delta\tau_{r,l} = \int \tau(x) \psi_{j,l}(x) dx = \int [-\ln F(x)] \psi_{j,l}(x) dx \). The redshift distortion factor \( \mathcal{R}_r \) depends on the DWT power spectrum of the velocity field \( v(x) \) on scale \( r = L/2^l \).

Because the average of \( \langle \ldots \rangle \) in the structure function Equation (1), or \( S_p(r = L/2^l) = \langle |\delta\rho_{j,l}|^p \rangle \), is only over on modes \( l \), the redshift distortion factor \( \mathcal{R}_r \) does not involve in this average. Thus, the function \( F_p(r) \) of \( \delta\tau_{r,l} \) will be different from the function \( F_p(r) \) of \( \delta\rho_{r,l} \) by a factor \( \mathcal{R}_r \). Thus, both \( F_{p+1}(r)/F_p(r) \) and \( F_p(r)/F_{p-1}(r) \) of \( \delta\tau_{r,l} \) do not contain the redshift distortion factor \( \mathcal{R}_r \). Thus, \( F_{p+1}(r)/F_3(r) \) and \( F_p(r)/F_2(r) \)
Figure 4. $\beta$ hierarchy of observed sample of the Ly$\alpha$ transmitted flux at redshift $z = 2.4 – 2.6$ and physical scale range from $\sim 1$ to $15 \, h^{-1}$ Mpc. The error bars are given by the maximum and minimum of bootstrap resampling.

Figure 5. $\beta$ hierarchy for (a) observed data (square) as Figure 4; (b) simulation sample (circle) of the Ly$\alpha$ transmitted flux at redshift $z = 2.5$ and Gaussian noises are added with the same level as real data.

of $\delta\tau_{r,l}$ should also satisfy the $\beta$ hierarchy Equation (12) as that of field $\rho_{HI}$. That is, the $\beta$ hierarchy is not affected by the redshift distortion.

Figure 4 presents the $\beta$ hierarchy of observed transmitted flux $F(z)$. It is very well fit by a straight line for data on the scale range from $\sim 1$ to $15 \, h^{-1}$ Mpc. It yields $\beta = 0.67 \pm 0.02$. Figure 5 shows the $\beta$ hierarchy of observed transmitted flux, which is the same as Figure 4, and the noised simulation samples $F$. We see that the both real and simulation samples are well coincident. The simulation samples yield $\beta = 0.66 \pm 0.02$. Therefore, both real and simulation samples are well $\beta$ hierarchical, and the numbers of $\beta$ are consistent.

The data points of Figure 4 are scattered along the straight line $F_{p+1}(r)/F_3(r) - F_p(r)/F_2(r)$, while the points of Figure 5 are clustered. This is due to all simulation samples have the same length, while for real samples consisting of quasars with different redshift, we take only the sections of transmitted flux, which are in the redshift range $z \simeq 2.4 – 2.6$.

4.3. High-Order Moments

The statistics of Equation (14) is based on the ratio between high-order moment $\langle \delta\rho^p_r \rangle$ and $\langle \delta\rho^p_{r,l} \rangle$, both of which have the same order of $\delta\rho$. When $p = 2$, the ratio actually is kurtosis. The moment ratio of $\langle \delta\rho^2_{r,l} \rangle$ to $\langle \delta\rho^2_r \rangle$, obviously, is independent of the redshift distortion factor $R_r$. Therefore, the ratio of $\langle \delta\rho^2_{r,l} \rangle$ to $\langle \delta\rho^2_r \rangle$ has to satisfy the same property as $\langle \delta\rho^2_r \rangle$ to $\langle \delta\rho^2_{r,l} \rangle$. Thus, for a given $p$, the relation of $\ln(\langle \delta\tau^2_{r,l} \rangle)$ to $\langle \delta\tau^2_r \rangle$ has to be a straight line with slope $-\gamma[p(1 - \beta^2) - (1 - \beta^2)]/(1 - \beta)$ (Equation (15)). Since the parameter $\beta$ is already determined by the $\beta$ hierarchy, the test here is whether we can find one
parameter $\gamma$ to fit the slopes of $\langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p$ versus $\ln r$ for different $p$.

The result is presented in Figure 6. It shows the relation of $\ln \left( \langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p \right)$ versus $\ln r$ for real data with $p = 2$ and 3, i.e., the statistics are of the order of 4 and 6. The slopes of the fitted straight lines are $0.54 \pm 0.19$ for $p = 2$, and $1.3 \pm 0.4$ for $p = 3$. Thus, using $\beta = 0.67$, we have $\gamma = 0.59 \pm 0.20$ for $p = 2$ and $\gamma = 0.58 \pm 0.22$ for $p = 3$. Although the statistical errors are still large, one can already see that different $p$ lines yield about the same parameter $\gamma$. In other words, the moment statistics passed the test of the log–Poisson hierarchy.

The number of $p$ given by the real data of the transmitted flux is less than the number $\gamma = 0.75$ of $\rho_{HI}$ field (Section 3.2). Therefore, the field of transmitted flux of real samples is less singular than the field $\rho_{HI}$. This is reasonable considering that the real data are noised.

### 4.4. Log–Poisson Hierarchy and Log-Normal Model

In Sections 4.2 and 4.3., we have found that the log–Poisson parameters of the transmitted flux should be $\beta = 0.67 \pm 0.02$ and $\gamma = 0.58 \pm 0.22$. This result can be used to compare the log–Poisson hierarchy and the log-normal model. We now consider the moment ratio $\ln \left( \langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p \right)$ as a function of $p$ on a given scale $r$.

Figure 7 presents the $(\delta \tau_r^2)/(\delta \tau_i^2)^p$ as a function of $p$ for real data on scales $r = 1.54$ (nabla), 3.07 (triangle), 6.14 (circle) and 12.3 (square) h$^{-1}$ Mpc. $p$ is from 1 to 4. The $p$-dependent curves shown in Figure 7 are given by Equations (14) and (15), in which the fitted parameters are $\beta = 0.67 \pm 0.02$ and $\gamma = 0.55 \pm 0.10$. The constant $\gamma$ in Equation (14) is determined by $\ln \left( \langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p \right) = 0$ when $p = 1$.

Although the observed data points of Figure 7 are the same as Figure 6, we see that the observed data points show large scatter in Figure 6, but almost no scatter in Figure 7. This is because Figure 6 gives the moment ratio as an r function, while Figure 7 shows the $p$-dependence of the moment ratio. For a given $r$, the Gaussian noise will yield a moment ratio given by Equation (13), which is a smooth function of $p$. Therefore, a Gaussian noise sample should not cause scatter with respect to $p$. On the other hand, Gaussian noise generally is not a smooth function of $r$. It yields the scatter of Figure 6.

Figure 7 shows that the $p$-dependence of moment ratio $\ln \left( \langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p \right)$ is significantly dependent on scale $r$. Therefore, it cannot be fitted by a Gaussian field, for which the moment ratio is $r$-independent (Equation (13)). The field of $\sigma$ is non-Gaussian. The smaller the scale $r$, the stronger the non-Gaussianity.

The log-normal model of the transmitted flux of QSO Ly$\alpha$ absorption spectrum is very successful in explaining various low-order statistical features of Ly$\alpha$ forests (Bi 1993; Bi & Davidsen 1997). The log-normal model also predicts that the transmitted flux is non-Gaussian and intermittent. For the moment ratio, the log-normal model yields (Pando et al. 2002)

$$\frac{\langle \delta \tau_r^p \rangle}{\langle \delta \tau_i^p \rangle} = \exp \left( 2(p^2 - p)\sigma^2(r) \right),$$

where $\sigma^2(r)$ is the power spectrum of the field. For each $r$, one can fit Equation (24) to observed points with $\sigma^2(r)$. The results are plotted in Figure 8. It shows that, if we try to give a good fitting of Equation (24) with real data at orders $p \leq 2$, the $p$-curves of Equation (24) always give a large deviation from the real data at $p > 2$. This deviation cannot be reduced by selecting $\sigma^2(r)$. This is because the deviation is from the $p^2$-dependence of $\langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p$ when $p$ is large. The $p^2$-dependence cannot be reduced with the parameter $\sigma^2(r)$.

On the other hand, at high $p$, the log–Poisson model gives $\langle \delta \tau_r^2 \rangle / \langle \delta \tau_i^2 \rangle^p \propto p$. The increasing with $p$ is then consistent with observation. In Figures 7 and 8, we show the error bars for data points of $r = 1.54 h^{-1}$ Mpc. Although the errors are large at high $p$, the result is clearly consistent with $p$-dependence, and does not favor the $p^2$-dependence. Therefore, the higher-order statistics of the Ly$\alpha$ transmitted flux is effective in discriminating between the log–Poisson and the log-normal model. The log-normal model yields too strong non-Gaussianity. This point has already been mentioned in the study of turbulence (e.g., Frisch 1995).

## 4.5. Scale–Scale Correlation

The last statistical feature used to test the log–Poisson hierarchy is the scale–scale correlation. Similar to the statistics of high order moment (Section 4.3), the ratio of scale–scale correlation, $\langle \delta \tau_r^2 \delta \tau_i^2 \rangle / \langle \delta \tau_r^p \delta \tau_i^p \rangle$, is independent of the redshift distortion factor $\delta \tau_r$. The field $\tau$ should show the scale–scale correlation as $\rho_{HI}$ Equations (17)–(19).

Before doing this test, we would emphasize that the scale–scale correlation is statistically independent of the statistics of high-order moments. For instance, one can construct a field that shows Gaussian distribution in terms of its one point distribution of $\delta \rho_{HI}$, but high scale–scale correlation between variables $\delta \rho_{HI}$ with different $j$ (Pando et al. 1998; Feng & Fang 2000). The clustering of cosmic large-scale structure in the nonlinear regime essentially is due to the interaction between Fourier modes on different scales. Therefore, cosmic clustering will definitely lead to scale–scale correlation. Scale–scale correlation is also very effective in distinguishing various hierarchy cascade models (Pando et al. 1998).

Figure 9 presents the $p = 2$ scale–scale correlation of observed data, in which the ratio $r_2/r_1$ is fixed to be equal to 2. The slope of the fitting straight line in Figure 9 should be given by Equation (19). Since Equation (19) depends only on the parameters $\beta$ and $\gamma$, both of which have already been determined in Sections 4.2 and 4.3, there is no free parameters in the fitting of Figure 9. With a straight line, we found the parameters $\beta = 0.67$ and $\gamma = 0.43 \pm 0.12$. The value of $\beta$ is the same as that detailed in Section 4.2, while the value of $\gamma$ is smaller than that of Section 4.3, but the deviation is not larger than 1r. The scale–scale correlation is more sensitive to the quality of the data, as is the correlation between modes on different scales. The result of Figure 9 is basically consistent with the scale–scale correlation predicted by the log–Poisson hierarchy.

## 5. DISCUSSION AND CONCLUSION

Nonlinear evolution of mass and velocity fields is a central problem of large scale structure of the universe. The clustering of the cosmic baryon fluid, governed by the Navier-Stokes equation in gravitational field of an expanding universe, has to be self similarly hierarchical in the scale-free range in which the dynamical equations and initial perturbations are scale invariant.

The log–Poisson hierarchical clustering sketches the nonlinear evolution of cosmic baryon fluid in the scale free range. If
the initial density perturbations are Gaussian, and their power spectrum is given by power law $P(k) \propto k^\alpha$, the structure functions initially have to be $S_p(r) \propto r^{-\alpha p/2}$. In the regime of linear evolution, the structure functions will be $S_p(r) \propto r^{\xi_l(p)}$, and the intermittent exponent is $\xi_l(p) = -\alpha p/2$. According to the log–Poisson hierarchy scenario, the nonlinear evolution leads to the hierarchical transfer of the power of clustering from large scales to small scales. The structure function will become $S_p(r) \propto r^\xi(p)^p$, where the nonlinear term of the intermittent exponent is $\xi(p) = -\gamma p(1-\beta)/(1-\beta)$, in which the parameters $\beta$ and $\gamma$ are dimensionless. $\beta$ measures the intermittency of the field, and $\gamma$ measures the singularity of the clustering. For Gaussian field, we have $\beta = 1$, and therefore, $\xi(p) = 0$ for all order $p$. Since the onset of nonlinear evolution, the parameter $\beta$ will gradually decrease, and the field becomes intermittent.

With simulation samples, we found that the parameter $\beta$ is decreasing with the decrease of redshift $z$. It means that the field is weakly intermittent at an earlier time, but strong intermittent at a later time (Liu & Fang 2008). Although $\xi(p) \neq 0$, the nonlinear evolution keeps the cosmic baryon fluid to be scale invariant. We showed that the mass density field of neutral hydrogen fluid in the scale-free range is also well described by the log–Poisson hierarchy despite the neutral hydrogen fraction of the baryon fluid not being constant in space. This is because the UV ionization photon is assumed to be uniform, and it does not violate the scale invariance of this system. However, the number of $\beta$ of neutral hydrogen is found to be less than that
of total baryon fluid. Therefore, the neutral hydrogen is less intermitting.

The Ly\(\alpha\) transmitted flux of the quasar’s Ly\(\alpha\) absorption spectrum is considered to be effective in probing the mass and velocity fields of neutral hydrogen. However, the redshift distortion of the velocity field leads to the deviation of the field of the Ly\(\alpha\) transmitted flux from the neutral hydrogen field. The transmitted flux does not satisfy all predictions of log–Poisson hierarchy. Fortunately, the effect of redshift distortion is approximately negligible for some log–Poisson hierarchical predicted features. Thus, we can test the log–Poisson hierarchy with the quasar’s Ly\(\alpha\) absorption spectrum.

Using high resolution and high S/N data of the quasar’s Ly\(\alpha\) absorption spectrum, we show that all the non-Gaussian features predicted by the log–Poisson hierarchy, including the hierarchical relation, the intermittent exponent, the ratios of different moments, and the scale–scale correlation, are consistent with observed samples. The observed samples of the transmitted flux yield the same intermittence parameter \(\beta\) as that of the neutral hydrogen field produced with hydrodynamic simulation of the concordance \(\Lambda\)CDM universe. Our result shows that the intermittent exponent \(\xi(p)\), or parameters \(\beta\) and \(\gamma\), is effective in discriminating among models of nonlinear evolution.

The log-normal model can well fit the observed data on lower order statistics, but is not good at higher orders. On the other hand, the log–Poisson model gives good fitting on lower order as well as higher order statistics. Therefore, a comparison between the log–Poisson model and the log-normal model on lower order statistics will be able to find the relation between parameters of the log–Poisson and log-normal models. This relation would be useful in explaining the parameters of the log–Poisson model with well known parameters in cosmology, as the parameters of the log-normal model generally are known in cosmology.
Recent studies have shown that the turbulence behavior of baryon gas can be detected by the Doppler-broadened spectral lines (Sunyaev et al. 2003; Lazarian & Pogosyan 2006). Although these works focus on the turbulence of baryon gas in clusters, the result is still applicable, at least, for the warm–hot intergalactic medium (WHIM), which is shown to follow the evolution of Burger’s fluid on large scales (He et al. 2004, 2005). Last but not the least, the polarization of the cosmic microwave background (CMB) is dependent on the density of electrons, and therefore, the map of CMB polarization would provide a direct test on the non-Gaussian features of ionized gas when the data on small scales become available.

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