On the analytical properties and some exact solutions of the Glukhovsky-Dolzhansky system

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Abstract. Non-linear dynamical systems describe many physical processes. In this work we investigate a three-dimensional Lorenz-like system - the Glukhovsky-Dolzhansky system. We consider analytical properties of the studied system. The problem of existence of meromorphic solution is discussed. We perform the Painlevé test and find conditions imposed on parameters of the system for which meromorphic solutions exist. Laurent series locally representing solutions are built. First integrals are obtained. We also find simply periodic solutions with one and two poles in a stripe of periods.

1. Introduction
Dynamical systems are used to describe various physical processes [1, 2]. In this work we study the Glukhovsky–Dolzhansky system. Comparing to well-known Lorenz system it has an additional non-linear term, which leads to essential differences in analytical structure and dynamics of the system. The Glukhovsky–Dolzhansky system describes following physical processes: convective fluid motion in an ellipsoidal rotating cavity, a rigid body rotation in a resisting medium, the forced motion of a gyrostat, a convective motion in harmonically oscillating horizontal fluid layer [1, 2]. Initially, this system was obtained by Glukhovsky and Dolzhansky as a three-mode model of convection for a fluid in an ellipsoidal rotating cavity, which can be interpreted as one of the models of ocean flows. Note that from this point of view system under investigation is significantly different from the Lorenz system. In the Lorenz system the flow of the two-dimensional convection is considered only. In the Glukhovsky–Dolzhansky system, the flow of the three-dimensional convection can be considered.

The Glukhovsky–Dolzhansky system was numerically investigated in detail (see, e.g. [1, 2]). However it has not been considered from the analytical point of view yet. Consequently in this work we apply analytical methods for studying the Glukhovsky-Dolzhansky system.

2. Painlevé analysis
The Glukhovsky–Dolzhansky system takes the following form:

\begin{align}
  x_t &= -\sigma(x - y) - ayz, \\
  y_t &= rx - y - xz, \\
  z_t &= -bz + xy,
\end{align}

where \( \sigma, a, r, b \) are physical parameters. Detailed discussion of physical meaning of the parameters can be found in work [2]. Note that system (2.1) is invariant under translations in
Accordingly, the Fuchs indicies are the coefficients we obtain the following recurrent relation into (2.1), we get

\[
\begin{align*}
\alpha_1 &= i, \beta_1 = \frac{1}{\sqrt{a}}, \gamma_1 = \frac{1}{\sqrt{a}}, \quad \alpha_2 = -i, \beta_2 = -\frac{i}{\sqrt{a}}, \gamma_2 = \frac{1}{\sqrt{a}}, \\
\alpha_3 &= i, \beta_3 = -\frac{i}{\sqrt{a}}, \gamma_3 = -\frac{1}{\sqrt{a}}, \quad \alpha_4 = -i, \beta_4 = \frac{i}{\sqrt{a}}, \gamma_4 = -\frac{1}{\sqrt{a}}.
\end{align*}
\]

(2.2)

Then, substituting the following series

\[
x = \alpha \sum_{j=0}^{\infty} a_j t^{j-1}, \quad y = \beta \sum_{j=0}^{\infty} b_j t^{j-1}, \quad z = \gamma \sum_{j=0}^{\infty} c_j t^{j-1},
\]

into (2.1), we get \( \alpha_0 = \beta_0 = \gamma_0 = 1 \) and \( \alpha_1 = -(\sqrt{a}(b-\sigma+1) \pm (\sigma-ra))/2\sqrt{a}, \beta_1 = -(\sqrt{a}(\sigma+b-1) \mp (\sigma-ra))/2\sqrt{a}, \gamma_1 = -(\sqrt{a}(b-\sigma-1) \pm (\sigma+ra))/2\sqrt{a} \). For further coefficients we obtain the following recurrent relation

\[
\begin{pmatrix} j-1 & 1 & 1 \\ 1 & j-1 & 1 \\ 1 & 1 & j-1 \end{pmatrix} \begin{pmatrix} \alpha_j \\ \beta_j \\ \gamma_j \end{pmatrix} = \begin{pmatrix} -\sum_{k=1}^{j} \beta_{j-k} \gamma_k - \sigma \alpha_{j-1} + \frac{\sigma \alpha}{\sqrt{a}} \beta_{j-1} \\ -\sum_{k=1}^{j} \alpha_{j-k} \gamma_k - b_{j-1} + r \sqrt{a} \alpha_{j-1} - \frac{r \sqrt{a} \alpha}{\sqrt{a}} \beta_{j-1} \\ -\sum_{k=1}^{j} \alpha_{j-k} \beta_k - b \gamma_{j-1} \end{pmatrix}
\]

(2.4)

Accordingly, the Fuchs indicies are \( j = -1, 2, 2 \). For \( j = 2 \) consistency conditions must be imposed. Restrictions inflicted on the parameters of the system are shown in the table below.

| Table 1. Parameters for which consistency conditions for \( j = 2 \) are satisfied. |
|---------------------------------|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| First pair of branches | Second pair of branches | Intersection |
|-------------------|-------------------|-----------------|
| 1. \( b = 0, r = \frac{\sigma - \sqrt{a}(\sigma-1)}{\sqrt{a}} \) | \( b = 0, r = \frac{\sigma + \sqrt{a}(\sigma-1)}{\sqrt{a}} \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) |
| 2. \( b = \sigma + 1, r = \frac{\sigma - 2 \sqrt{a}}{\sqrt{a}} \) | \( b = \sigma + 1, r = \frac{\sigma + 2 \sqrt{a}}{\sqrt{a}} \) | \( b = 1, r = 0, \sigma = 0 \) | \( b = 1, r = 0, \sigma = 0 \) | \( b = 1, r = 0, \sigma = 0 \) | \( b = 1, r = 0, \sigma = 0 \) | \( b = 1, r = 0, \sigma = 0 \) |
| 3. \( b = \frac{\sigma(\sqrt{a} - 1)}{\sqrt{a}}, r = \frac{\sigma - \sqrt{a}(\sigma-1)}{\sqrt{a}} \) | \( b = \frac{\sigma(\sqrt{a} + 1)}{\sqrt{a}}, r = \frac{\sigma + \sqrt{a}(\sigma-1)}{\sqrt{a}} \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) | \( b = 0, r = \frac{1}{\sqrt{a}}, \sigma = 1 \) |

Now we consider the results given in Tab. 1 in more details. Case 1 does not correspond to any known physical processes because \( b = 0 \). In case 2 for second pair of branches positive sets of the parameters are possible. They may describe at least some extreme cases of real physical systems (for example a fluid with huge heat-conduction coefficient, very low viscosity coefficient inside a slowly rotating ellipsoidal cavity). In case 3 there are some sets of positive parameters for both pairs of roots. We can suppose that they may contain some cases that have physical meaning. One can note that if necessary conditions for the existence of Laurent series for all four solutions branches are held simultaneously (see column 3 of Tab. 1) the corresponding value of the parameters have no physical sense. Further coefficients of the Laurent series are not shown in this work, but they can be easily obtained by means of relation (2.4).
3. First integrals

Dynamical system (2.1) is dissipative, if we suppose that physical parameters are positive:

\[
\frac{\partial x_2}{\partial x} + \frac{\partial y_t}{\partial y} + \frac{\partial z_t}{\partial z} = -\sigma - 1 - b < 0
\]

Consequently we have reasons to expect that first integrals will be time-dependent. Now we consider two types of restrictions on the parameters of the system that are given in the last column of Tab. 1.

If \( b = 0, r = \frac{1}{\sigma}, \sigma = 1 \) we can obtain first integral in the form of \( e^{2t}(y^2 + z^2) = C_1 \). Using this integral we can reduce our system to a second order differential equation that has the form

\[
(C_1 e^{-2t} - z^2)z_{tt} + 2z_t^2 + 2C_1 e^{-2t}z_t + az(C_1 e^{-2t} - z^2)^2 + C_1 e^{-2t}z = 0
\]  

(3.1)

In the case of \( b = 0, r = \frac{1}{\sigma}, \sigma = 1 \), we get first integral in the form of \( e^{2t}(x^2 - ay^2) = C_1 \).

By means of this integral initial system can be reduced to the following equation

\[
(C_1 e^{-2t} + ay^2)y_{tt} - ay^2t_y - y\left((C_1 e^{-2t} + ay)^2 + C_1 e^{-2t}\right) = 0
\]

(3.2)

Necessary condition to possess the Painlevé property is satisfied for both equations (3.1) (3.2). However, in general, we can’t transform them to one of the canonical equations of the Painlevé type [4] using the following transformation (similar for \( z(t) \))

\[
y(t) = \frac{k(t)w(s) + l(t)}{m(t)w(s) + n(t)}, \quad s = h(t).
\]

Consequently, we think that equations (3.1), (3.2) don’t have the Painlevé property in general, but can have it in some particular cases (for example if \( C_1 = 0 \)).

4. Simply periodic solutions with one pole in a stripe of periods

It is convenient to look for simply periodic solutions with one pole in a stripe of periods in the following form: \( x(t) = A_{1x} + A_{2x}w(t), y(t) = A_{1y} + A_{2y}w(t), z(t) = A_{1z} + A_{2z}w(t) \), where \( w(t) \) is the solution of the Riccati equation looking like \( w = -w^2 + B \). If \( B \neq 0 \) then \( w(t) = \sqrt{B}\tanh(\sqrt{Bt}) \) otherwise \( w(t) = \frac{1}{2} \). Substituting \( x, y, z \) in this form into (2.1) for \( A_{2x}, A_{2y}, A_{2z} \) we obtain values (2.2) and three possible cases

1) \( B = 0, b = 0, r = \frac{\sigma}{\pm \sqrt{\alpha} - \sqrt{\alpha}}, \quad (4.1) \)

2) \( B = \frac{\sigma}{\pm \sigma + 1} \left( 4\sigma \sqrt{\alpha} \pm \left( \sigma^2 + \sigma + 4\alpha \right) \right), \quad b = 1 + \sigma, \quad r = \frac{4\alpha}{(\pm \sigma + 2\sqrt{\alpha})\sigma}. \quad (4.2) \)

3) \( B = \frac{\sigma^2 - a^{3/2} - \sqrt{\alpha} + 2\sigma a^{3/2} + 4\sigma \sqrt{\alpha} \pm [\sigma + 5\sigma a - 2a]}{4a^{3/2} \left( \frac{\sigma}{\sqrt{\alpha}} \pm \sqrt{\alpha} \right)}, \quad b = \frac{\sigma}{\pm \sqrt{\alpha} + 1}, \quad r = \frac{\sigma}{\pm \sqrt{\alpha} \sigma - 1}. \quad (4.3) \)
If conditions (4.1) are satisfied, we get $A_{1x} = 0, A_{1y} = 0, A_{1z} = \mp \sigma(\sigma+1+\sqrt{\sigma})/2\alpha$.

In case (4.2) coefficients take the following form for the first pair of branches of solution (according to (2.2)):

$$A_{1x} = \frac{1}{2} i\sigma^2 \sqrt{\frac{1}{\alpha}} + i\sigma + i\frac{\sigma}{\alpha} \sqrt{\frac{1}{\alpha}}, A_{1y} = \pm i\sigma(\sigma+1)/2\alpha, A_{1z} = -\frac{\sigma(\sigma+2+\sqrt{\sigma})}{2\alpha},$$

and for the second pair of branches:

$$A_{1x} = \frac{1}{2} i\sigma^2 \sqrt{\frac{1}{\alpha}} - i\sigma - i\frac{\sigma}{\alpha} \sqrt{\frac{1}{\alpha}}, A_{1y} = \pm i\sigma(\sigma+1)/2\alpha, A_{1z} = \frac{\sigma(\sigma+2-\sqrt{\sigma})}{2\alpha}.$$  

In this case $B = 0$ if $\sigma = 0, \sigma = -1$ or $a = \frac{1}{2} \pm (\sigma+1) + 2(\mp \sigma + \sqrt{\sigma})$.

Considering case (4.3) coefficients for the first pair of branches of solution look like $A_{1x} = \frac{1}{2} i\sigma(\sqrt{\sigma}-1)/2\alpha, A_{1y} = \frac{1}{2} i\sigma(\sqrt{\sigma}-1)/2\alpha$, and for the second pair of branches they take form $A_{1x} = \frac{1}{2} i\sigma(\sqrt{\sigma}+1)/2\alpha, A_{1y} = \pm i\sigma(\sqrt{\sigma}+1)/2\alpha, A_{1z} = \frac{\sigma(\sqrt{\sigma}+1)}{a}$. $B$ turns into zero if $a = 1$ only for the first pair of branches of solution and if $\sigma = 0$ for all four branches of solution.

5. Simply periodic solutions with two poles in a stripe of periods

According to [5, 6] we look for simply periodic solutions with two poles in a stripe of periods in the following form

$$x(t) = \sqrt{L} C_{1x} \frac{A_{1x} \cot (\sqrt{L}t) + \sqrt{L}}{A_{1x} - \sqrt{L} \cot (\sqrt{L}t)} + \sqrt{L} C_{1z} \cot (\sqrt{L}t) + h_x,$$  

(5.1)

and similarly for $y(t)$ and $z(t)$. Now we consider situation where $C_{1x}$ and $C_{1z}$ are coefficients of the first and the second branches of solution in (2.2) respectively. To obtain parameters of solutions we use the Laurent series for (5.1) in a neighborhood of $t = 0$. After that we convert cot functions into tanh and put the phase offset of $\pi/2$ into the arbitrary constant $t_0$ (again we will skip it in further notation). As a result, the corresponding solutions can be represented by two following expressions:

$$b = 0, \quad r = -\frac{\sigma\sqrt{\alpha}-\sigma-\sqrt{\alpha}}{\sqrt{\alpha}},$$

$$x = -\frac{1}{2} \sqrt{L} \left( \frac{\tanh(\sqrt{-L}t) - a}{\tanh(\sqrt{-L}t)} \right),$$

$$y = -\frac{1}{2} \sqrt{L} \left( \frac{\tanh(\sqrt{-L}t) - 1}{\tanh(\sqrt{-L}t)} \right),$$

$$z = -\frac{1}{\sqrt{\alpha}} \left( \frac{\tanh(\sqrt{-L}t)}{1 + \tan^2(\sqrt{-L}t)} \right) - \frac{\sigma(\sqrt{\alpha} - 1)}{a},$$

(5.2)

$$b = \frac{\sigma(\sqrt{\alpha} - 1)}{\sqrt{\alpha}}, \quad r = -\frac{\sigma\sqrt{\alpha} - \sqrt{\alpha}}{\sqrt{\alpha}}, \quad L = -\frac{1}{4} \frac{\sigma^2(\sqrt{\sigma}-1)^2}{a},$$

$$x = -\frac{1}{2} \sqrt{L} \left( \frac{1 + \tanh(\frac{b}{2}t)}{1 + \tanh(\frac{b}{2}t)} \right),$$

$$y = -\frac{1}{2} \sqrt{L} \left( \frac{1 + \tanh(\frac{b}{2}t)}{1 + \tanh(\frac{b}{2}t)} \right),$$

$$z = -\frac{b}{2\sqrt{\alpha}} \left( \frac{\tanh(\frac{b}{2}t)}{1 + \tanh(\frac{b}{2}t)} \right) - 1 \pm 2).$$

(5.3)

Solutions in form of (5.3) degenerate to functions with one pole in a stripe of periods. Solutions (5.2) have no physical sense. The parameters of system (2.1) in case (5.3) may describe at least some extreme cases of physical systems. Methods given in [5, 6] allow us to seek out simply
periodic solutions with three and four poles in a stripe of periods. However, for system (2.1) we can’t find such solutions.

In accordance with [5, 6, 7] it is reasonable to look for elliptic solutions of the system under discussion in the form of

\[ x(t) = \sum_{i \in I} C_{-1}^{(i)x}(\psi(t) + B_{ix}) + h_x, \]

and similarly for \( y(t) \) and \( z(t) \). We tried this method for functions with two and four poles in a parallelogram of periods. But unfortunately this type of solutions can’t be obtained for system (2.1).

6. Conclusion
In this work we studied the Glukhovsky-Dolzhansky system (2.1). The Painlevé test for this system of equations has been carried out. Conditions imposed on parameters of the Glukhovsky–Dolzhansky system for meromorphic solutions to exist have been obtained. They are shown in Tab. 1. Solution in the form of Laurent series has been built. We have also found first integrals of system (2.1). We have shown that by means of this integrals the system can be reduced to second order ordinary differential equations (see (3.1), (3.2)). It seems that these equations do not possess the Painlevé property in general, but can have it in some particular cases.

We have obtained some meromorphic partial solutions of the system. We have found simply periodic solutions with one and two poles in a stripe of periods. Solutions with one pole in a stripe of periods are expressed via solution of the Riccati equation. They are given by formulae (4.1), (4.2), (4.3). Solutions with two poles in a stripe of periods are shown in (5.2), (5.3). However solutions (5.3) degenerate to functions with one pole in a stripe of periods. We have not found either periodic solutions with more poles in a stripe of periods or elliptic solutions by the method applied.

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