BOX PRODUCT OF MACKEY FUNCTORS
IN TERMS OF MODULES

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Abstract. The box product of Mackey functors has been studied extensively in Lewis’s notes. As shown in Thévenaz and Webb’s paper, a Mackey functor may be identified with a module over a certain algebra, called the Mackey algebra. We aim at describing the box product, in the sense of Mackey algebra modules. For a cyclic $p$-group $G$, we recover a result from Mazur’s thesis. We generalize it to a general finite group $G$ in this article.

Introduction

A Mackey functor is an algebraic structure, related to many natural constructions from finite groups, such as group cohomology and the algebraic K-theory of group rings. The study of Mackey functor in abstract began in 1980s. Dress\cite{Dress} and Green\cite{Green} first gave the axiomatic formulation of Mackey functors. Several equivalent descriptions of Mackey functors were given by Dress\cite{Dress}, Lindner\cite{Lindner}, Lewis\cite{Lewis} and Thévenaz\cite{Thévenaz}. Specifically, Lewis\cite{Lewis} introduces box product and Thévenaz\cite{Thévenaz} describes a Mackey functor as a module over the Mackey algebra $\mu_R(G)$.

In this article, we give an inductive description of the box product of two left $\mu_R(G)$-modules. The main goal is to construct this box product explicitly and to prove that it is equivalent with the box product of two Mackey functors. When $G$ is a cyclic $p$-group, we recover the formula of Mazur\cite{Mazur}.

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1. Mackey functors

Throughout this article, we assume that $G$ is a finite group and that $R$ is a unital, commutative ring. There are several equivalent definitions of Mackey functors and we concentrate on two of them here. Before giving the definitions, we introduce an auxiliary category $\Omega_R(G)$ and the Mackey algebra $\mu_R(G)$.

1.1. The category $\Omega_R(G)$.

We recall the definition of the category $\Omega_R(G)$ from [4]. A finite group $G$ gives rise to a category $\omega(G)$ whose objects are finite $G$-sets and where the morphisms from $X$ to $Y$ are the equivalence classes of diagrams of $G$-sets $X \leftarrow V \rightarrow Y$. Two such diagrams are said to be equivalent if there is a commutative diagram

\[ \begin{array}{ccc}
X & \xleftarrow{\sigma} & Y \\
\downarrow & \downarrow & \downarrow \\
V' \end{array} \]

where $\sigma$ is an isomorphism of $G$-sets. To define the composition of morphisms, we consider a morphism from $X$ to $Y$ represented by a diagram $X \leftarrow V \rightarrow Y$ and a morphism from $Y$ to $Z$ represented by a diagram $Y \leftarrow W \rightarrow Z$. We form the pullback

\[ \begin{array}{ccc}
U & \xleftarrow{f'} & W \\
\downarrow & \downarrow & \downarrow \\
V & \xleftarrow{f} & Y \\
\downarrow & \downarrow & \downarrow \\
X & \xleftarrow{g} & Z \\
\end{array} \]

which defines a diagram $X \leftarrow U \rightarrow Z$, hence a morphism from $X$ to $Z$. Such a pullback always exists and we can express it explicitly as

\[ U = \{ (v, w) \in V \times W : f(v) = g(w) \}, \]

where $f'(v, w) = v$ and $g'(v, w) = w$. By defining addition in $\text{Hom}_{\omega(G)}(X, Y)$ as

\[ (X \xleftarrow{\alpha} V \xrightarrow{\beta} Y) + (X \xleftarrow{\alpha'} V' \xrightarrow{\beta'} Y) := (X \xleftarrow{\alpha + \alpha'} V \sqcup V' \xrightarrow{\beta + \beta'} Y), \]

$\text{Hom}_{\omega(G)}(X, Y)$ becomes a free abelian monoid, as shown in [5]. Extending the scalars to the ring $R$, we get a free $R$-module

\[ \text{Hom}_{\Omega_R(G)}(X, Y) := R \text{Hom}_{\omega(G)}(X, Y) \]
on the same basis as that of free monoid $\text{Hom}_{\omega(G)}(X, Y)$. Let $\Omega_R(G)$ be a category with the same objects as $\omega(G)$ and hom-set $\text{Hom}_{\Omega_R(G)}(X, Y)$ defined as above. $\Omega_R(G)$ is an $R$-additive category.

1.2. The Mackey algebra $\mu_R(G)$. We define the Mackey algebra as

$$\mu_R(G) := \bigoplus_{H, K \leq G} \text{Hom}_{\Omega_R(G)}(G/H, G/K),$$

where the multiplication is defined on the components in the direct sum by composition of morphisms in the category $\Omega_R(G)$, or zero if two morphisms cannot be composed.

For convenience, we recall the notation from [6].

**Definition 1.1.** Let $f : X \to Y$ be a $G$-equivariant map of finite $G$-sets. Then two spans

$$\xymatrix{X \ar@{=}[r] & Y \ar[l]_f} \quad \text{and} \quad \xymatrix{X \ar@{=}[r] & Y \ar[l]_f}$$

are called the restriction $r_f$ and transfer $t_f$, respectively, of $f$.

**Definition 1.2.** Let $K \leq H \leq G$ and $g \in G$. Then define $R^K_H$, $I^K_H$ and $C_{g, H}$ as

$$R^K_H = r_{\pi^K_H} : \left( G/K = G/K \xrightarrow{\pi^K_H} G/H \right)$$

$$I^K_H = t_{\pi^K_H} : \left( G/H \xleftarrow{\pi^K_H} G/K = G/K \right)$$

$$C_{g, H} = t_{c_{g, H}} = r_{c_{g^{-1}, g} H} : \left( G/\rho^g \xrightarrow{c_{g, H}} G/H = G/H \right),$$

where $\pi^K_H : G/K \to G/H$ denotes the canonical quotient map, mapping $gK$ to $gH$, and $c_{g, H} : G/H \to G/\rho^g H$ denotes the conjugation map, mapping $kH$ to $(kg^{-1})^g H$. Observe that $R^K_H, I^K_H$ and $C_{g, H}$ are all elements in Mackey algebra $\mu_R(G)$.

For conjugation, the notation $C_g$ is preferred to $C_{g, H}$ for simplicity when there is no ambiguity. It is easy to check that the following identities hold:

- (0) $R^K_H = I^K_H = C_{h, H}$ for all $H \leq G$ and $h \in H$
- (1) $R^K_H R^K_J = R^K_J$ for all subgroups $J \leq K \leq H$
- (2) $I^K_H I^K_J = I^K_J$ for all subgroups $J \leq K \leq H$
- (3) $C_g C_h = C_{gh}$ for all $g, h \in G$
- (4) $C_g R^K_H = R^K_H C_g$ for all subgroups $K \leq H$ and $g \in G$
- (5) $C_g I^K_H = I^K_H C_g$ for all subgroups $K \leq H$ and $g \in G$
- (6) $R^K_H I^K_J = \sum_{g \in [J \setminus H] \cap [J \setminus K]} I^K_J C_g R^K_K$ for all subgroups $J, K \leq H$
- (7) $\sum_{H \leq G} I^K_H$ serves as the unit in $\mu_R(G)$.

We use $[J \setminus H] / K$ to denote the set of representatives of the double coset $J \setminus H / K$. In fact, the structure of Mackey algebra $\mu_R(G)$ is rather simple:

**Lemma 1.1** (from [5]). Hom-set $\text{Hom}_{\Omega_R(G)}(G/K, G/H)$ is a free $R$-module, with basis represented by the diagrams

$$I^K_L C_{g, L} R^K_H = \left( G/K \xleftarrow{\pi^K_L \circ c_{g, L}} G/L \xrightarrow{\pi^K_H} G/H \right)$$
where $g \in [K \backslash G/H]$ and $L$ is a subgroup of $H \cap K^g$ taken up to $H \cap K^g$-conjugation.

In other words, the Mackey algebra $\mu_R(G)$ is generated by $R^H_K$, $I^H_K$ and $C_{g,H}$'s as an $R$-algebra.

1.3. Definition of Mackey functors.

**Definition 1.3** (from [2]). A Mackey functor is a $R$-additive functor $M : \Omega_R(G)^{\text{op}} \to R$-mod. They form a category with natural transformations as morphisms and we denote this category by $\text{Mack}_R(G)$.

It is shown in [3] that the category $\mu_R(G)$-modules is equivalent to $\text{Mack}_R(G)$ via the following equivalence of categories

$$\Phi : \text{Mack}_R(G) \leftrightarrow \mu_R(G) \text{-mod}$$

$M \mapsto \oplus_{H \leq G} M(G/H)$

$$(G/H \mapsto I^H_H N) \leftrightarrow N.$$ Since $\sum_{H \leq G} I^H_H$ is the unit in $\mu_R(G)$, a left $\mu_R(G)$-module $N$ can be graded into $N = \oplus_{H \leq G} I^H_H N$. Observe that the multiplication by $R^H_K$ maps $I^H_H N$ to $I^K_K N$ and the other grades to zero. Similarly, multiplication by $I^H_H$ maps $I^K_K N$ to $I^K_H N$ and the other grades to zero. Multiplication by $C_{g,H}$ maps $I^K_H N$ to $I^K_H N$ and the other grades to zero. Also note that $I^K_H \Phi M = M(G/H)$ for later use.

2. The Box product in $\text{Mack}_R(G)$

The box product is a symmetric monoidal structure on $\text{Mack}_R(G)$. The box product in $\text{Mack}_R(G)$ has been studied in [4] and we summarize it in this section. The result we are more interested in is that maps from the box product of $M$ and $N$ to $P$ can be characterized by Dress parings.

Given two Mackey functors $M, N \in \text{Mack}_R(G)$, we can form the exterior product

$$M \square N : \Omega_R(G)^{\text{op}} \times \Omega_R(G)^{\text{op}} \to R\text{-mod}$$

$$(X, Y) \mapsto M(X) \otimes N(Y).$$

**Definition 2.1** (Box product in $\text{Mack}_R(G)$). The box product $M \square N$ is defined to be the left Kan extension of $M \square N$ along the Cartesian product functor $\times : \Omega_R(G)^{\text{op}} \times \Omega_R(G)^{\text{op}} \to \Omega_R(G)^{\text{op}}$.

$$\Omega_R(G)^{\text{op}} \times \Omega_R(G)^{\text{op}} \xrightarrow{\times} \Omega_R(G)^{\text{op}} \xrightarrow{M \square N} R\text{-mod}$$

If a Mackey functor $M \in \text{Mack}_R(G)$ is implicit, we use $r_f$ for both a morphism $\left( X = X \xrightarrow{f} Y \right) \in \Omega_R(G)$ and its value $M(Y) \to M(X)$ under $M$. Similarly for $t_f$.

**Lemma 2.1** (from [3]). A map $\theta : M \square N \to P$ determines and is determined by a collection of $R$-module homomorphisms

$$\theta_X : M(X) \otimes N(X) \to P(X)$$

for every finite $G$-set $X$, such that the following three diagrams commute for each $G$-equivariant map $f : X \to Y$. 

A good exposition and proof of this lemma can be found in [6]. The data in this lemma is called a Dress paring. The natural transformations from $M \boxtimes N$ to another Mackey functor $P$ are the same as the Dress parings from $M$ and $N$ to $P$, via the natural bijection

$$\text{Hom}_{\text{Mack}}(M \boxtimes N, P) \cong \text{Dress}(M, N; P),$$

that maps a map $\theta : M \boxtimes N \to P$ to $\theta_X : M(X) \otimes N(X) \to P(X \times X)$. The first map comes from the Kan adjunction and $r\Delta$ is the restriction associated to the diagonal map of $G$-sets $X \to X \times X$.

In $\text{Mack}_R(G)$, the Burnside ring Mackey functor

$$B^G(\cdot) := \text{Hom}_{\Omega_R(G)}(\cdot, G/G)$$

is the unit for box product, as shown in [4]. In this way, $(\text{Mack}_R(G), B^G(\cdot), \boxtimes)$ is a symmetric monoidal category.

### 3. The Box Product in $\mu_R(G)$-mod

Given two left $\mu_R(G)$-modules $M$ and $N$, we can form an $R$-module $A_H$ for each $H \leq G$ by induction on the cardinality of subgroups of $G$

$$A_c := (I^c_e M) \otimes_R (I^c_e N)$$

$$A_H := \left( (I^H_c M) \otimes_R (I^H_c N) \right) \oplus \bigoplus_{K \leq H} A_K.$$

We say $A_H$ is of grade $H$. By combining all the grades together, we get $A := \bigoplus_{H \leq G} A_H$. Note that the component $A_K$ in grade $H$ is distinct from the grade $K$. Since $\mu_R(G)$ is generated by $R, I, C$’s, we can endow $A$ with a left $\mu_R(G)$-module structure by giving actions of $R, I, C$’s on $A$. The maps $C_{g,H}, I^H_c, R^H_j$ map the grade $H$ to grades $g H, L, J$ respectively, and map the other grades to zero. Their action on the grade $H$ is described as follows:
(1) $I^H_R$ action on the grade $H$:
If $H = L$, $I^H_R$ acts as the identity on grade $H$.
If $H < L$, $I^H_R$ maps an element in $A_H$ to its corresponding copy $A_H$ in grade $L$.
To distinguish $A_H$ from its copy in grade $L$, we write its copy in grade $L$ as $I^H_L A_H$ from now on. That is, grade $H$ is written as
$$A_H = \left( (I^H_R M) \otimes_R (I^H_R N) \right) \bigoplus_{K < H} I^H_K A_K.$$ 

(2) $C_{g,H}$ action on the grade $H$:
We define the action of $C_{g,H}$ by induction on the cardinality of $H$ as follows.
For $m \otimes n \in (I^H_R M) \otimes (I^H_R N)$ and $x \in A_K$, where $K < H$,
$$C_{g,H}(m \otimes n) := (C_{g,H} m) \otimes (C_{g,H} n) \in A_K,$$
$$C_{g,H} I^H_K (x) := I^H_K C_{g,H} (x) \in A_K.$$ 

(3) $R^H_J$ action on the grade $H$:
We also define the action of $R^H_J$ by induction on the cardinality of $H$ as follows.
For $m \otimes n \in (I^H_R M) \otimes (I^H_R N)$ and $x \in A_K$, where $K < H$,
$$R^H_J (m \otimes n) := (R^H_J m) \otimes (R^H_J n) \in A_J,$$
$$R^H_J I^H_K (x) := \sum_{g \in [J/K]} I^H_K C_{g,R^K_{K \cap J}} (x) \in A_J.$$ 

**Definition 3.1** (Box product in $\mu_R(G)$-mod). Based on the left $\mu_R(G)$-module
$\oplus_{H \leq G} A_H$, we can define $M \square N$ as
$$M \square N := (\oplus_{H \leq G} A_H) / FR,$$
where $FR$ is a submodule, called the Frobenius reciprocity submodule, generated by elements of the form
$$a \otimes (I^H_K b) - I^H_K ((R^H_K a) \otimes b)$$
and
$$(I^H_K c) \otimes d - I^H_K (c \otimes (R^H_K d))$$
for all $K < H$, $a \in I^H_R M$, $b \in I^H_K N$, $c \in I^H_K M$ and $d \in I^H_K N$.

Naturally, the image of $A_H$ under the quotient is called the grade $H$ of $M \square N$.

**Proposition 3.1.** $\oplus_{H \leq G} \text{Hom}_{\mu_R(G)}(G/H, G/G)$ is the unit for box product in
$\mu_R(G)$-mod.

**Proof.** Let $M$ be a left $\mu_R(G)$-module and $N = \oplus_{H \leq G} \text{Hom}_{\mu_R(G)}(G/H, G/G)$.
Then $I^H_R N$ is an $R$-module generated by $G/H \xrightarrow{\pi^H_L} G/L \to G/G = I^H_L R^H_L n_H$, where
$$n_H := (G/H = G/H \to G/G).$$
Thus, $I^H_R M \otimes I^H_R N$ is a $R$-module generated by $m \otimes I^H_L R^H_L n_H = I^H_L (R^H_L m \otimes n_H)$.

Then we get a natural bijection between each grade of $M$ and $M \square N$
$$F : I^H_R M \longleftrightarrow I^H_R (M \square N)$$
$$m \mapsto m \otimes n_H,$$
$$I^H_L R^H_L m \mapsto I^H_L (R^H_L m \otimes n_H) \in I^H_L M \otimes I^H_R N,$$
$$I^H_K F^{-1} (x) \mapsto I^H_K x \in I^H_K I^H_K (M \square N).$$
Here $F^{-1}$ is defined by induction on the grade. It is easy to check that this is a bijection and that it preserves the $\mu_R(G)$-module structure. Thus, $M \Box N$ is isomorphic to $M$ as a $\mu_R(G)$-module and $N$ is the unit for box product in $\mu_R(G)$-mod. □

In this way, $(\mu_R(G)\text{-mod}, \oplus_{H \leq G} \text{Hom}_{\Omega_R(G)}(G/H, G/G), \Box)$ is a symmetric monoidal category.

4. Equivalence of box product in $\text{Mack}_R(G)$ and $\mu_R(G)$-mod

**Theorem 4.1.** The equivalence of categories $\Phi : \text{Mack}_R(G) \cong \mu_R(G)$-mod is a symmetric monoidal equivalence. In other words, there are natural isomorphism $\Phi_M \Box \Phi_N \cong \Phi(M \Box N)$ for any two Mackey functors $M, N \in \text{Mack}_R(G)$, and a compatible natural isomorphism $B^G(-) \cong \text{Hom}_{\Omega_R(G)}(G/H, G/G)$.

**Lemma 4.2.** For any Mackey functors $M, N, P \in \text{Mack}_R(G)$, there is a natural bijection

$$\text{Dress}(M, N; P) \cong \text{Hom}_{\mu_R(G)}(\Phi M \Box \Phi N, \Phi P).$$

**Proof.** Given $\beta \in \text{Hom}_{\mu_R(G)}(\Phi M \Box \Phi N, \Phi P)$, we map it to $\theta \in \text{Dress}(M, N; P)$ defined as follows. For each $H \leq G$, $\beta$ maps grade $I_H^H(\Phi M \Box \Phi N)$ to grade $I_H^H \Phi P$, because $\beta(x) = \beta(I_H^H x) = I_H^H \beta(x) \in I_H^H \Phi P$ for each $x \in I_H^H(\Phi M \Box \Phi N)$. Since $I_H^H(\Phi M \Box \Phi N) = (M(G/H) \otimes N(G/H)) \otimes \bigoplus_{K < H} I_K \cong FR$ and

$$I_H^H \Phi P = P(G/H),$$

$\beta$ induces an $R$-module homomorphism $\theta_{G/H}$

$$\theta_{G/H} : M(G/H) \otimes N(G/H) \rightarrow P(G/H)$$

by restricting to the first summand. For a general finite $G$-set $X = \sqcup_{i=1}^P G/H$, $\theta_X$ is defined as

$$\theta_X : \bigoplus_{i,j=1}^P M(G/H_i) \otimes N(G/H_j) \rightarrow \bigoplus_{i=1}^P P(G/H_i)$$

$$m_i \otimes n_j \mapsto \begin{cases} \theta_{G/H_i}(m_i \otimes n_i), & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Having constructed $\theta$, we now proceed to show that $\theta \in \text{Dress}(M, N; P)$. Given finite $G$-sets $X, Y$ and a $G$-equivariant map $f : X \rightarrow Y$, it is sufficient to show that the three diagrams in Lemma 2.1 commute.

If both $X$ and $Y$ are orbits, say $X = G/K$ and $Y = G/H$, observe that a $G$-equivariant map $f$ from $G/K$ to $G/H$ must be of the the form $f = \pi^H_{K, g, K}$ for some $g \in G$. By composition of commuting diagrams, we only need to consider the case where $f = c$ and $f = \pi$.

When $f = c_{g, H} : G/H \rightarrow G/^g H$, we have that $r_f = C_{g^{-1}, s H}$ and $t_f = C_g, H$. The first diagram commutes because

$$r_f \theta_Y (m \otimes n) = C_{g^{-1}, s H} \beta (m \otimes n) = \beta (C_{g^{-1}, s H}(m \otimes n))$$

$$= \beta (C_{g^{-1}, s H} m \otimes C_{g^{-1}, s H} n)) = \theta_X (r_f m \otimes r_f n).$$
The second diagram commutes because
\[ t_f \theta_X(m \otimes r_f n) = C_{g,H} \beta(m \otimes C_{g^{-1},s_H} n) = \beta(C_{g,H}(m \otimes C_{g^{-1},s_H} n)) = \beta(C_{g,H} m \otimes n) = \theta_Y(t_f m \otimes n). \]

The third diagram commutes similarly.
When \( f = \pi_K^H : G/K \to G/H \), we have that \( r_f = R_K^H \) and \( t_f = I_K^H \). The first diagram commutes because
\[ r_f \theta_Y(m \otimes n) = R_K^H \beta(m \otimes n) = \beta(R_K^H(m \otimes n)) = \beta(R_K^H m \otimes R_K^H n)) = \theta_X(r_f m \otimes r_f n). \]

The second diagram commutes because
\[ t_f \theta_X(m \otimes r_f n) = I_K^H \beta(m \otimes R_K^H n) = \beta(I_K^H(m \otimes R_K^H n)) = \beta(I_K^H m \otimes n) = \theta_Y(t_f m \otimes n). \]

The third diagram commutes similarly.

Now, if \( Y \) is an orbit \( G/H \) and \( X = \bigcup_{i=1}^p G/K_i \) is a general \( G \)-set, denote the restriction of \( f \) to \( G/K_i \) by \( f_i : G/K_i \to G/H \). Thus, \( r_f : M(G/H) \to \bigoplus_{i=1}^p M(G/K_i) \) is the sum of \( r_{f_i} : M(G/K_i) \to M(G/K_i) \). Similarly, \( t_f : \bigoplus_{i=1}^p M(G/K_i) \to M(G/H) \) is determined by components \( t_{f_i} : M(G/K_i) \to M(G/H) \). The first diagram commutes because
\[ r_f \theta_Y(m \otimes n) = \sum_{i=1}^p r_{f_i} \theta_Y(m \otimes n) = \sum_{i=1}^p \theta_X(r_{f_i} m \otimes r_{f_i} n) = \theta_X(\sum_{i=1}^p r_{f_i} m \otimes \sum_{i=1}^p r_{f_i} n) = \theta_X(r_f m \otimes r_f n). \]

The second diagram commutes because
\[ t_f \theta_Y(m \otimes r_f n) = t_f \theta_X(\sum_{i=1}^p m_i \otimes \sum_{i=1}^p r_{f_i} n) = t_f \theta_X(\sum_{i=1}^p m_i \otimes r_{f_i} n) = \sum_{i=1}^p t_f \theta_X(m_i \otimes r_{f_i} n) = \sum_{i=1}^p t_f \theta_G/K_i(m_i \otimes r_{f_i} n) = \sum_{i=1}^p \theta_Y(t_f m_i \otimes n) = \theta_Y(t_f m \otimes n) \]

for \( m = \sum_{i=1}^p m_i \), where \( m_i \in M(G/K_i) \). The third diagram commutes similarly.

Lastly, the general case: \( X = \bigcup_{i=1}^p X_i, Y = \bigcup_{i=1}^p G/H_i \), and \( f \) maps \( X_i \) to \( G/H_i \). Thus, \( r_f \) maps \( M(G/H_i) \) to \( M(X_i) \) and \( t_f \) maps \( M(X_i) \) to \( G/H_i \). For the first diagram, take \( m \in G/H_i \) and \( n \in G/H_j \). Thus, \( r_f m \in M(X_i) \) and \( r_f n \in M(X_j) \). If \( i \neq j \), then \( \theta_Y(m \otimes n) = 0 \) and \( \theta_X(r_f m \otimes r_f n) = 0 \). If \( i = j \), then this is the case when \( Y \) is an orbit. For the second diagram, take \( m \in M(X_i) \) and \( n \in G/H_j \). Thus, \( t_f m \in M(G/H_i) \) and \( r_f n \in M(X_j) \). If \( i \neq j \), then \( \theta_X(m \otimes r_f n) = 0 \) and \( \theta_Y(t_f m \otimes n) = 0 \). If \( i = j \), it reduces to the case when \( Y \) is an orbit. The third diagram commutes similarly.
Given a Dress pairing $\theta \in \text{Dress}(M, N; P)$, we define $\beta$, a map from $\Phi M \square \Phi N$ to $P$ grade by grade, as follows:

$$
\beta_H : I^H(\Phi M \square \Phi N) \rightarrow I^H(\Phi P) = P(G/H)
$$

$$
M(G/H) \otimes N(G/H) \ni m \otimes n \mapsto \theta_{G/H}(m \otimes n)
$$

$$
I^H A_K \ni I^H(x) \mapsto I^H(\beta_K(x))
$$

Having constructed $\beta$, we now proceed to show that $\beta$ is indeed a map of $\mu_R(G)$-modules. $\beta$ is linear in multiplication by elements in $\mu_R(G)$. It is enough to check this for the generators $R, I, C$’s. Say $K < H \leq G$ and take $m \otimes n \in M(G/H) \otimes N(G/H)$. Then

$$
R^H_K \beta(m \otimes n) = R^H_K \theta_{G/H}(m \otimes n) = \theta_{G/K}(R^H_K m \otimes R^H_K n)
$$

$$
\beta(R^H_K(m \otimes n)) = \beta(R^H_K(m \otimes n))
$$

$$
C_{g,H} \beta(m \otimes n) = C_{g,H} \theta_{G/H}(m \otimes n) = \theta_{G/H}(C_{g,H} m \otimes C_{g,H} n)
$$

$$
\beta(C_{g,H}(m \otimes n)) = \beta(C_{g,H}(m \otimes n)).
$$

Take $x \in A_K$ and $S \in \mu_R(G)$. Since $SI^H_K$ acts on the grade $K$, $SI^H_K \beta(x) = \beta(SI^H_K x)$ by induction. Therefore,

$$
S \beta(I^H_K x) = SI^H_K (x) = \beta(SI^H_K x).
$$

Let us show that $\beta$ maps the Frobenius reciprocity submodule $FR_H$ to zero for each $H \leq G$. Take $K < H$, $a \in M(G/H)$ and $b \in N(G/K)$. By the second commuting diagram in lemma 2.1, we have

$$
\beta(I^H_K (R^H_K a \otimes b)) = I^H_K \beta(R^H_K a \otimes b) = I^H_K \theta_{G/K}(R^H_K a \otimes b)
$$

$$
= \theta_{G/H}(a \otimes I^H_K b) = \beta(a \otimes I^H_K b).
$$

Take $K < H$, $c \in M(G/K)$ and $d \in N(G/H)$. By the third commuting diagram in lemma 2.1, we have

$$
\beta(I^H_K (c \otimes R^H_K d)) = I^H_K \beta(c \otimes R^H_K d) = I^H_K \theta_{G/K}(c \otimes R^H_K d)
$$

$$
= \theta_{G/H}(I^H_K c \otimes d) = \beta(I^H_K c \otimes d).
$$

Lastly, it is easy to see the composition of those two maps above is identity in either way. For instance, the map

$$
\text{Dress}(M, N; P) \rightarrow \text{Hom}_{\mu_R(G)-\text{mod}}(\Phi M \square \Phi N, \Phi P) \rightarrow \text{Dress}(M, N; P)
$$

, which maps $\theta \mapsto \beta \mapsto \theta'$, is identity because $\theta'_{G/H}$ equals to the restriction of $\beta$ to the first summand of the grade $H$, which in turn equals to the maps $\theta_{G/H}$ according to the constructions above. Thus, $\theta' = \theta$. □

**Proof for Theorem 4.1** Fix three Mackey functors $M, N, P \in \text{Mack}_R(G)$. By Lemma 4.2 there is a natural bijection

$$
\text{Dress}(M, N; P) \cong \text{Hom}_{\mu_R(G)-\text{mod}}(\Phi M \square \Phi N, \Phi P).
$$

By Lemma 2.3 there is a natural bijection

$$
\text{Hom}_{\text{Mack}_R(G)}(M \square N, P) \cong \text{Dress}(M, N; P).
$$

Since $\Phi : \text{Mack}_R(G) \xrightarrow{\cong} \mu_R(G)$-mod is an equivalence of categories, there is a natural bijection

$$
\text{Hom}_{\text{Mack}_R(G)}(M \square N, P) \cong \text{Hom}_{\mu_R(G)-\text{mod}}(\Phi (M \square N), \Phi P).
$$
Therefore, we get a natural bijection

$$\text{Hom}_{\mu_R(G)}(\Phi M \square \Phi N, \Phi P) \cong \text{Hom}_{\mu_R(G)}(\Phi (M \square N), \Phi P).$$

Therefore, $\Phi M \square \Phi N$ is naturally isomorphic to $\Phi (M \square N)$ as a left $\mu_R(G)$-module.

Moreover, the equality $\Phi(B^G(-)) = \oplus_{H \leq G} B^G(G/H) = \oplus_{H \leq G} \text{Hom}_{\Omega_R(G)}(G/H, G/G)$ verifies the correspondence of units for box products in Mack$_R(G)$ and $\mu_R(G)$-mod.

Thus, the equivalence $\Phi$ is monoidal.

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