Rank Independence
and
Rearrangements of Random Variables *†

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Abstract
We study rearrangements \((Y_1, \ldots, Y_n) = (X_{\sigma_1}, \ldots, X_{\sigma_n})\) (where \(\sigma\) is a random permutation) of an i.i.d. sequence of random variables \((X_1, \ldots, X_n)\) uniformly distributed on \([0, 1]\); in particular we consider rearrangements satisfying the strong rank independence condition, that the rank of \(Y_k\) among \(Y_1, \ldots, Y_k\) is independent of the values of \(Y_1, \ldots, Y_{k-1}\). Nontrivial examples of such rearrangements are the “travellers’ processes” defined by Gnedin and Krengel. We show that these are the only examples when \(n = 2\), and when certain restrictive assumptions hold for \(n \geq 3\); we also construct a new class of examples of such rearrangements for which the restrictive assumptions do not hold.

1 Introduction
A sequence \(\bar{X} = (X_1, \ldots, X_n)\) of numbers can be reordered by means of any permutation \(s\) (thought of as the map \(i \mapsto s_i, i = 1, \ldots, n\)) to obtain the new sequence which we denote
\[
\bar{X}^s := (X_{s_1}, \ldots, X_{s_n})
\]
When the sequence \(\bar{X}\) consists of random points chosen independently according to the uniform distribution on the unit interval \(I = [0, 1]\), for any fixed permutation \(s\) the process \(\bar{X}^s\) has the same distribution as \(\bar{X}\). The situation changes, however, when \(s\) is also allowed to vary.

We shall call a sequence of random variables \(\bar{Y} = (Y_1, \ldots, Y_n)\) a rearrangement of \(\bar{X}\) if there is a random variable \(\sigma\), defined on the same probability space as \(\bar{X}\) with values in the symmetric group \(\mathcal{S}\), such that \(\bar{Y}\) has the same distribution as \(\bar{X}^\sigma\):

\[
(Y_1, \ldots, Y_n) \overset{d}{=} (X_{\sigma_1}, \ldots, X_{\sigma_n}).
\]

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Of course, the distribution of $\vec{X}^\sigma$ is the same as that of $\vec{X}$ when $\vec{X}$ is i.i.d. and $\sigma$ is independent of $\vec{X}$, but in general they can be quite different. Our definition does not require that the entries of $\vec{X}$ be uniformly distributed on $I$ or even i.i.d.; it can be applied to any random process $\vec{X}$. We will focus for the most part on $\vec{X}$ i.i.d. and uniformly distributed (i.u.d.), noting that the transformation technique can be used to reduce the case of $\vec{X}$ any continuously distributed i.i.d. sequence to the i.u.d. case. However, in Lemma 1 ($\S$2) it will be useful to use this idea in a non-i.i.d. setting.

There are three standard rearrangements of $\vec{X}$: the sequence itself is identified with the trivial rearrangement ($\sigma = \text{id}$), and we also have the descending (resp., ascending) rearrangements $\vec{X}_\downarrow$ (resp., $\vec{X}_\uparrow$) obtained by rearranging according to size. In keeping with [GK], where certain applications to games were investigated, we shall focus on the maximal order statistic, hence on the descending order, which we number largest-to-smallest with indices in parentheses:

$$\vec{X}_\downarrow = (X_1, \ldots, X_n) \text{ with } X_i \geq X_{i+1}, \quad i = 1, \ldots, n.$$ 

We are interested in this paper in the consequences of certain conditions on the rank statistics of a rearrangement. Given $\vec{Y}$, we define the initial ranks as

$$\mathcal{R}_k = 1 + \sharp\{i < k : Y_i > Y_k\}, \quad k = 1, \ldots, n. \quad (1)$$

Of course, $\mathcal{R}_1 = 1$ and in general $\mathcal{R}_k \in \{1, \ldots, k\}$; note that $\mathcal{R}_k$ is a relative rank (it gives only the position of $Y_k$ relative to the earlier elements in $\vec{Y}$) and measures positions in descending order: $\mathcal{R}_k = j$ precisely if $Y_k$ is the $j^{th}$ largest of $Y_1, \ldots, Y_k$. Of course, we can ignore ties, since they have probability zero.

We will investigate rearrangements $\vec{Y}$ with the property that

$$\mathcal{R}_{k+1}, \ldots, \mathcal{R}_n \text{ are independent of } (Y_1, \ldots, Y_k) \text{ for } k = 1, \ldots, n - 1$$

which we refer to as strong rank independence. Note that this is strictly stronger than independence of the initial ranks. For example, the trivial rearrangement has independent ranks, with each of the $n!$ possible rank configurations $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ equally likely, but the distribution of $\mathcal{R}_{k+1}$ conditioned on the values $(X_1, \ldots, X_k)$ depends in an essential way on how these points divide the interval. On the other hand, the ascending and descending rearrangements $\vec{X}_\downarrow$ and $\vec{X}_\uparrow$ induce a deterministic sequence of ranks, as does any rearrangement obtained by applying a fixed permutation $s$ to either of these, so that strong rank independence holds for these rearrangements in a trivial way.

A nontrivial family of rearrangements with the strong rank independence property are the “travellers’ processes” constructed in [GK]. One can describe these as follows: imagine the $X_i$’s as giving the locations of various cities; two travellers leave a specified interior point of $I$ (corresponding to $\theta$ below) travelling in opposite directions, toward the two endpoints of $I$, with constant speeds adjusted so that they will reach their respective endpoints simultaneously. The reordering of $(X_1, \ldots, X_n)$ is then given by the order in which the various cities are reached by one or the other traveller. Formally, these processes can be defined as follows:
Example 1 ("Travellers’ process", [GK]) Pick the parameter $\theta \in [0, 1]$ and consider the “V-shaped” function $f_\theta : I \to I$ (Figure 1) defined by

$$f_\theta (x) = \begin{cases} \theta - x & 0 \leq x \leq \theta \\ \frac{x}{1-x} & \theta \leq x \leq 1. \end{cases}$$

With probability one, there exists a unique permutation $\sigma = \sigma \left( \theta, \vec{X} \right) \in \mathcal{S}$ such that

$$f_\theta (X_{\sigma_1}) < f_\theta (X_{\sigma_2}) < \ldots < f_\theta (X_{\sigma_n})$$

and the rearrangement

$$\vec{Y}_\theta := \vec{X}^\sigma$$

using $\sigma = \sigma \left( \theta, \vec{X} \right)$ has the strong rank independence property: given the values of $Y_1, \ldots, Y_k$, we know the value of $f_\theta (Y_k) = \max_{i=1,\ldots,k} f_\theta (Y_i)$ and that $f_\theta (Y_{k+1}) > f_\theta (Y_k)$; this tells us that $Y_{k+1}$ lies in one of two intervals, the ratio of whose lengths is $\theta/(1-\theta)$, one to the left and the other to the right of the interval $\{t : f_\theta (t) \leq f_\theta (Y_k)\}$ ($I_2$ in Figure 3). One can easily check [GK, section 4] that in fact the following hold:

- the $k^{th}$ initial rank can only take the extreme values 1 and $k$;
- the initial rank process $(\mathfrak{r}_1, \ldots, \mathfrak{r}_n)$ can be represented as

$$\mathfrak{r}_k = J_k + k(1 - J_k), \quad k = 1, \ldots, n$$

where

$$J_k = 1_{[\theta,1]}(X_{\sigma_k})$$

are i.i.d. Bernoulli variables with

$$\mathbb{P} \{ J_k = 1 \} = 1 - \theta;$$
• \(J_{k+1}, \ldots, J_n\) are independent of \((Y_1, \ldots, Y_k)\).

This family builds a continuous bridge between the ascending and descending rearrangements, with \(X_↑_k = Y_0\) and \(X_↓_k = Y_1\).

The strong rank independence condition has appeared in various guises in connection with different classes of random variables. While it is known not to hold for exchangeable sequences without ties, its relevance to the problems of Bayesian inference has been discussed in the statistical literature (see e.g., [H1, section 6] and [H2]). A sequence of independent (but not identically distributed) random variables satisfying strong rank independence was constructed in [HK].

In this paper, we investigate the extent to which the strong rank independence property characterizes the travellers’ process of Example 1. §2 gives a framework for thinking about rearrangements in terms of the descending arrangement \(X_↓\). In §3 we will show that when \(n = 2\), the travellers’ processes are the only rearrangements with the strong rank independence property. In §4, we show that the strong rank independence property forces a certain dependence between the set of values taken on by the sequence \(X\) and the rearranging permutation. In §5, we consider the more limited class of binary rearrangements in which the ordering is determined by some real-valued attribute (such as the function \(f_θ\) in Example 1) and show that for all \(n\) the travellers’ processes are the only binary rearrangements with the strong rank independence property. In fact, we show that for binary rearrangements, the strong independence of just a single rank \(R_k\), \(k \in \{2, \ldots, n\}\) already forces the rearrangement to be a travellers’ process. Finally, in §6 we discuss some further examples satisfying the strong rank independence property which share some features with the travellers’ processes and others with the constant rearrangements in which the components of \(X_↓\) are rearranged according to a fixed element of \(S\).

2 Rerrangements and Order Statistics

We shall find it easier to think in terms of the descending rearrangement \(X_↓\) instead of the original sequence \(X\). In this section we set up some machinery to show that this point of view is equivalent to the original one.

Note first some general properties of rearrangements.

Lemma 1 Rearrangement is an equivalence relation; that is, for any three processes \(X, Y, Z\) (with the same number of components) defined on sufficiently rich probability spaces:

1. \(X\) is a rearrangement of \(X\);

2. If \(Y\) is a rearrangement of \(X\), then \(X\) is a rearrangement of \(Y\);

3. If \(Y\) is a rearrangement of \(X\) and \(Z\) is a rearrangement of \(Y\), then \(Z\) is a rearrangement of \(X\).
Proof:
(1) is (literally) trivial.
Note that equality in distribution is preserved by rearrangement, in the sense that if
\( \vec{X} \overset{d}{=} \vec{X}' \)
and \( \sigma \in \mathcal{S} \) is a random permutation defined on the same space as \( \vec{X} \), then there exists a random permutation \( \sigma' \in \mathcal{S} \) defined on the same space as \( \vec{X}' \) so that
\[ \vec{X}^\sigma \overset{d}{=} (\vec{X}')^{\sigma'} . \]
Thus, to see (2) we simply note that for any random permutation \( \sigma \in \mathcal{S} \), the inverse permutation \( \bar{\sigma} \in \mathcal{S} \) is also a random permutation, and
\[ \vec{X} = (\vec{X}^\sigma)^{\bar{\sigma}} . \]
To see (3), we note that if \( \vec{Z} \overset{d}{=} \vec{Y}^\rho \) for some random \( \rho \in \mathcal{S} \) (defined on the space for \( \vec{Y} \)) then by (2) there is \( \rho' \in \mathcal{S} \) (defined on the space for \( \vec{Z} \)) with \( \vec{Z}^{\rho'} \overset{d}{=} \vec{Y} \), and hence, since \( \vec{Y} \overset{d}{=} \vec{X}^\sigma \), we have \( \vec{Z}^{\rho'} \overset{d}{=} \vec{X}^\sigma \). But then again we have \( \rho'' \) (defined on the space for \( \vec{X} \)) with \( \vec{Z} = (\vec{Z}^{\rho'})^{\rho''} = (\vec{X}^\sigma)^{\rho''} . \)
We can apply this reasoning in particular to the descending (resp., ascending) arrangements \( \vec{X}_\downarrow \) (resp., \( \vec{X}_\uparrow \)). Denote by \( I^n_\downarrow \) the simplex of descending \( n \)-tuples in \( I^n \):
\[ I^n_\downarrow := \{(a_1, \ldots, a_n) : 1 \geq a_1 \geq a_2 \geq \ldots \geq a_n \geq 0 \} . \]
There is a “descending” permutation, defined as a map \( \delta : I^n \to \mathcal{S} \), such that for all \( \vec{a} = (a_1, \ldots, a_n) \in I^n \),
\[ \vec{a}^{\delta(\vec{a})} = (a_{\delta_1}, \ldots, a_{\delta_n}) \in I^n_\downarrow . \]
The value of \( \delta \) is uniquely determined at almost every \( \vec{a} \in I^n \), specifically, off the generalized diagonal \( \Delta_n \) in \( I^n \):
\[ \Delta_n := \{(a_1, \ldots, a_n) \in I^n : a_i = a_j \text{ for some } i \neq j \} . \]
Thus, given \( \vec{X} \) whose entries are continuously distributed on \([0, 1]\), there is a canonical random permutation \( \delta \in \mathcal{S} \) defined on the same space as \( \vec{X} \) (and uniquely determined a.e.) so that
\[ \vec{X}_\downarrow = \vec{X}^{\delta} \]
and hence \( \vec{X} \) is a rearrangement of \( \vec{X}_\downarrow \),
\[ \vec{X} = \vec{X}_\downarrow^{\delta} . \]
Similar reasoning applies to the ascending rearrangement. We have, then, as a corollary of Lemma 1,

Proposition 1 Given \( \vec{X} = (X_1, \ldots, X_n) \) and \( \vec{Y} = (Y_1, \ldots, Y_n) \) two sequences of random variables as above, the following are equivalent:
1. \( \vec{Y} \) is a rearrangement of \( \vec{X} \): for some random \( \sigma \in \mathfrak{S} \), \( \vec{Y} \overset{d}{=} \vec{X}^\sigma \);

2. \( \vec{Y} \) and \( \vec{X} \) have equivalent descending rearrangements: \( \vec{Y} \downarrow \overset{d}{=} \vec{X} \downarrow \);

3. \( \vec{Y} \) and \( \vec{X} \) have equivalent ascending rearrangements: \( \vec{Y} \uparrow \overset{d}{=} \vec{X} \uparrow \);

4. \( \vec{Y} \) is a rearrangement of \( \vec{X} \downarrow \): for some random \( \mu \in \mathfrak{S} \), \( \vec{Y} \overset{d}{=} \vec{X} \mu \).

The various formulations in Proposition 1 can be combined in a unified picture of rearrangements. It is easy to see that the “descending” permutation \( \delta : I^n \to \mathfrak{S} \) is constant on each connected component of \( I^n \setminus \Delta_n \). Thus, we can identify \( I^n \) (mod 0) with \( I^n_{\downarrow} \times \mathfrak{S} \), by identifying the point \( \vec{a} \in I^n \setminus \Delta_n \) with the pair \( \vec{a} \delta \in I^n_{\downarrow}, \delta \in \mathfrak{S} \), where \( \delta = \delta (\vec{a}) \) is the “descending” permutation for \( \vec{a} \), and \( \bar{\delta} \) denotes the inverse of \( \delta \) (as a permutation), so that the identification map \( I^n_{\downarrow} \times \mathfrak{S} \to I^n \) is given by \((\vec{a}, \delta) \mapsto \bar{\delta} \).

We define a “state space”

\[ \Sigma := I^n_{\downarrow} \times \mathfrak{S} \]

and note that there are two natural “projections” of \( \Sigma \); given \((\vec{a}, \delta) \in \Sigma ,

\[ \text{proj}_{\downarrow} (\vec{a}, \delta) := \vec{a} \in I^n_{\downarrow} \]

\[ \text{proj}_Y (\vec{a}, \delta) := \vec{a} \delta \in I^n. \]

Now, if \( \vec{Y} \) is a rearrangement of \( \vec{X} \), we can associate to it the \( \Sigma \)-valued random variable

\[ \mathcal{Y} := (\vec{X}_{\downarrow}, \mu) \]

where \( \mu \) is given by Proposition 1(4). We see that in this case \( \vec{Y} \) and \( \vec{X} \downarrow \overset{d}{=} \vec{Y} \downarrow \) can be recovered via the projections:

\[ \vec{Y} \overset{d}{=} \text{proj}_Y (\mathcal{Y}) \]

\[ \vec{X} \downarrow = \text{proj}_{\downarrow} (\mathcal{Y}) \]

Conversely, we have

**Lemma 2** If \( \mathcal{Y} \) is a random variable with values in \( \Sigma := I^n_{\downarrow} \times \mathfrak{S} \), then \( \vec{Y} := \text{proj}_Y (\mathcal{Y}) \) is a rearrangement of \( \vec{X} \downarrow \) (where \( \vec{X} \) is i.u.d.) if and only if for every measurable set \( A \subseteq I^n_{\downarrow} \),

\[ \mathcal{P} \{ \mathcal{Y} \in A \times \mathfrak{S} \} = n! \text{Leb}_n (A). \]  \hspace{1cm} (2)

**Proof:**

The right side of the equation is just the normalized Lebesgue measure on \( I^n_{\downarrow} \), or \( \mathcal{P} \{ \vec{X} \downarrow \in A \} \), while the left side is the same as \( \mathcal{P} \{ \text{proj}_{\downarrow} (\mathcal{Y}) \in A \} \), or equivalently \( \mathcal{P} \{ \vec{Y} \downarrow \in A \} \).

Thus, Equation 2 is simply a restatement of the requirement that \( \vec{Y} \downarrow \overset{d}{=} \vec{X} \downarrow \). \( \square \)

An advantage of representing a rearrangement \( \vec{Y} \) of \( \vec{X} \) in terms of \( \vec{X} \downarrow \) and \( \mu \) is that it separates data about the values taken by the variables \( X_i \) from data about their “arrival
times” in $\vec{Y}$. One can view the “descending” arrangement $\vec{X}_\downarrow$ as a canonical representation of the random (unordered, $n$-point) set of values $\{X_1, \ldots, X_n\}$, and the random permutation $\mu$ as representing the order in which they are arranged in $\vec{Y}$. $\mu_k$ gives the “final” rank of $Y_k$ among all the variables $(Y_1, \ldots, Y_n)$, or equivalently $\bar{\mu}_j$ gives the “arrival time” for the $j^{th}$ largest value in the sequence $(Y_1, \ldots, Y_n)$. We shall sometimes refer to $\vec{X}_\downarrow$ as the “value data” and to $\mu$ as the “arrival data” for the rearrangement $\vec{Y}$.

An event of the form $(Y_1, \ldots, Y_k) \in A \subseteq I^k$ can be viewed as a condition on the first $k$ entries of $\text{proj}_Y(\mathcal{Y})$; since $\text{proj}_Y(\cdot)$ is a fixed arrangement of coordinates on each “level” $I^n_\downarrow \times \{s\}$, $s \in \mathcal{S}$ of $\Sigma$, we can formulate the condition as follows: given $A \subseteq I^k$ measurable and $s \in \mathcal{S}$, let

$$A(s) := \{\vec{a} = (a_1, \ldots, a_n) \in I^n_\downarrow : (a_s, \ldots, a_{s_k}) \in A\} \subseteq I^n_\downarrow,$$

and

$$A^* := \bigcup_{s \in \mathcal{S}} A(s) \times \{s\} = \text{proj}^{-1}_Y \big( A \times I^{n-k} \big) \subseteq \Sigma.$$

Thus, the event $\{(Y_1, \ldots, Y_k) \in A\}$ corresponds in our representation to $\{\mathcal{Y} \in A^*\}$.

Rank conditions can also easily be formulated in terms of $\mathcal{Y} \in \Sigma$. In addition to the initial ranks defined by Equation 1, we will find it useful to consider other (relative) ranks: for any sequence $\vec{Y} = (Y_1, \ldots, Y_n)$ of random variables without ties, we define $n^2$ partial ranks by

$$\mathfrak{R}_{j,k} = \mathfrak{R}_{j,k}(\vec{Y}) := 1 + \#\{i \leq k : Y_i > Y_j\}, \quad j, k \in \{1, \ldots, n\}.$$

The initial ranks are given by the special case $j = k$:

$$\mathfrak{R}_k = \mathfrak{R}_{k,k}, \quad k = 1, \ldots, n;$$

more generally, for $j \leq k$, the numbers $\mathfrak{R}_{j,k}$ are current ranks: if the values of $(Y_1, \ldots, Y_n)$ are displayed consecutively, then for each $k = 1, \ldots, n$ the $k$-tuple $(\mathfrak{R}_{1,k}, \ldots, \mathfrak{R}_{k,k})$ gives the relative ranking of the first $k$ variables displayed, and encodes all the rank information known at the $k^{th}$ stage.

To study the interrelationships between the partial ranks more carefully, we consider their combinatorial analogue, associating to each permutation $s \in \mathcal{S}$ the array $\rho(s)$ of $n^2$ numbers

$$\rho_{j,k}(s) := 1 + \#\{i \leq k : s_i < s_j\}.$$

It is clear that, for $\mathcal{Y} = (\vec{X}_\downarrow, \mu) \in \Sigma$ as above the sequence of variables $\vec{Y} = \text{proj}_Y(\mathcal{Y})$ satisfies

$$\mathfrak{R}_{j,k}(\vec{Y}) = \rho_{j,k}(\mu).$$

The entries on and above the diagonal of $\rho(s)$ ($\rho_{j,k}(s)$, $1 \leq j \leq k \leq n$) will be referred to as the upper entries; they correspond to the current ranks for $\text{proj}_Y(\mathcal{Y})$.

**Lemma 3** For each $s \in \mathcal{S}$, the numbers $\rho_{j,k} = \rho_{j,k}(s)$ defined by Equation 3 satisfy

1. $\rho_{j,k} \in \{1, \ldots, k+1\}$, the upper entries are less than or equal to $k$, and $\rho_{j,n}(s) = s_j$;
2. the upper values in any column, $\rho_{1,k} \ldots \rho_{k,k}$, are distinct.
3. for any \( j < k \) with \( k > 1 \),
\[
\rho_{k,k} < \rho_{j,k} \iff \rho_{k,k} \leq \rho_{j,k-1};
\]

4. for \( j \leq k < k' \),
\[
\rho_{j,k'} = \rho_{j,k} + \#\{\ell : k < \ell \leq k' \text{ and } \rho_{\ell,\ell} \leq \rho_{j,\ell-1}\}. \tag{4}
\]

**Proof:** 
(1) is trivial and (2) is an immediate consequence of the fact that \( s_1, \ldots, s_k \) are distinct. To see (3), note that for any \( j, k \) with \( k \geq 2 \),
\[
\rho_{j,k} = \begin{cases} 
\rho_{j,k-1} & \text{if } s_k > s_j \\
\rho_{j,k-1} + 1 & \text{if } s_k < s_j
\end{cases} \tag{5}
\]
and in either case the two inequalities of (3) are equivalent.

Finally, to see (4), note that Equation 4 with the inequality \( \rho_{\ell,\ell} \leq \rho_{j,\ell-1} \) replaced by \( s_\ell < s_j \) is an easy consequence of (2) and the definitions. \(\blacksquare\)

We can apply Equation 4 recursively to show that any upper entry \( \rho_{j,k'} (1 \leq j \leq k' \leq n) \) of \( \rho \) is determined uniquely by any upper entry to its left in the same row (\( \rho_{j,k} \), \( k \) fixed, \( j \leq k < k' \)) together with the diagonal entries \( \rho_{k,k}, \rho_{k+1,k+1}, \ldots, \rho_{k',k'} \) between. Conversely, the observation that the upper entries in column \( k \) give the ranking of \( s_1, \ldots, s_k \) shows that any upper entry \( \rho_{j,k} (j \leq k) \) is also determined uniquely by the entries in any single column to its right which lie on or above the same row (\( \rho_{i,k'} \), with \( i = 1, \ldots, j \) and \( j \leq k < k' \) fixed). To formalize this, for \( k \leq k' \) set
\[
\mathcal{S}^{(k,k')} := \{(s_1, \ldots, s_k) : s_j \in \{1, \ldots, k'\} \text{ and } s_i \neq s_j \text{ for } i \neq j\}
\]
(so that \( \mathcal{S}^{(k,k)} \) is the set of permutations of \( \{1, \ldots, k\} \)) and for \( m \leq m' \) set
\[
\mathcal{R}^{(m,m')} := \{(r_m, \ldots, r_{m'}) : r_k \in \{1, \ldots, k\} \text{ for } k = m, \ldots, m'\}.
\]

Then we have

**Remark 1** Given \( 1 \leq k < k' \leq n \), there exist functions
\[
\begin{align*}
f_{j,k,k'} : & \mathcal{R}^{(k,k')} \to \{1, \ldots, k'\} \\
g_{j,k,k'} : & \mathcal{S}^{(j,k)} \to \{1, \ldots, k\}
\end{align*}
\]
for \( 1 \leq j \leq k \) such that for each \( s \in \mathcal{S} \), the array \( \rho = \rho(s) \) defined by Equation 3 satisfies
\[
\begin{align*}
1. & \quad \rho_{j,k'} = f_{j,k,k'} (\rho_{j,k}, \rho_{k+1,k+1}, \ldots, \rho_{k',k'}); \\
2. & \quad \rho_{j,k} = g_{j,k,k'} (\rho_{1,k'}, \rho_{2,k'}, \ldots, \rho_{j,k'}). 
\end{align*}
\]

8
Using these functions one easily obtains a bijection for each \( k < k' \) between \( S(k, k) \times R(k + 1, k') \) (the upper entries in column \( k \) followed by the diagonal through column \( k' \)) and \( S(k', k') \) (the upper entries in column \( k' \)). While an explicit formula for these bijections is not particularly useful, we will make use of the (well-known) special case \( k = 1, k' = n \). These bijections also allow us to label the levels of \( \Sigma \) with appropriate \( n \)-tuples of partial ranks, instead of permutations. In particular, we can label these levels with initial ranks \( \rho_{k,k}(s) \). For \( 1 \leq \ell \leq k \leq n \), let

\[ S(k, \ell) := \{ s \in S : \rho_{k,k}(s) = \ell \}, \]

and

\[ \Sigma_{k,\ell} := \{ (\bar{a}, s) \in \Sigma : \mathfrak{R}_{k,k}(\text{proj}_Y((\bar{a}, s))) = \ell \} = I_{\downarrow}^n \times S(k, \ell). \]

Using this notation, we can easily formulate the strong rank independence condition in terms of \( Y \in \Sigma \).

**Remark 2** A random variable \( Y \in \Sigma \) satisfying Equation 2 in Lemma 2 has the strong rank independence property (for \( \bar{Y} \doteq \text{proj}_Y(Y) \)) if and only if there exist constants

\[ p_{k,\ell} \geq 0, \quad 1 \leq \ell \leq k \leq n \]

such that for every \( A \subset I^{k-1} \),

\[ \mathcal{P} \{ Y \in \Sigma_{k,\ell} \cap A^* \} = p_{k,\ell} \mathcal{P} \{ Y \in A^* \}, \quad (6) \]

or equivalently,

\[ \sum_{s \in \mathfrak{S}_{k,\ell}} \mathcal{P} \{ Y \in A(s) \times \{s\} \} = p_{k,\ell} \mathcal{P} \{ Y \in A^* \}. \]

Note that for \( k = 1 \), this forces \( p_{1,1} = 1 \) and for each \( k \in \{1, \ldots, n\} \)

\[ \sum_{\ell=1}^{k} p_{k,\ell} = 1; \]

of course in general, condition (6) is the same as

\[ \mathcal{P} \{ \mathfrak{R}_k = \ell \mid Y_1, \ldots, Y_{k-1} \} = p_{k,\ell}. \quad (7) \]

Henceforth, we use this picture to view the descending arrangement \( \bar{X}_{\downarrow} \) as our primary object (instead of \( \bar{X} \)), using Proposition 1(4) to view any rearrangement \( \bar{Y} \) as \( \bar{X}_{\downarrow}^\mu \) for some random \( \mu \in \mathfrak{S} \), where

\[ (Y_1, \ldots, Y_n) \doteq (X_{(\mu_1)}, \ldots, X_{(\mu_n)}). \]
3 The case $n = 2$

In this section we show that the travellers’ processes in Example 1 are the only rearrangements of two (i.u.d.) random variables with the strong rank independence condition. In this case, the combinatorics is simplified enormously because $\mathcal{S}$ contains only two elements, the identity $id$ and the transposition $\tau$ ($\tau_1 = 2, \tau_2 = 1$). In terms of ranks,

$$\mathcal{S}(2, 1) = \{\tau\}, \quad \mathcal{S}(2, 2) = \{id\};$$

it will be convenient to modify our notation from the previous section slightly and write for each $A \subset I$

$$A\langle 1 \rangle := A(id) = \{\vec{a} \in I^2_+ : a_1 \in A\}$$

$$A\langle 2 \rangle := A(\tau) = \{\vec{a} \in I^2_+ : a_2 \in A\}.$$

Also, since the strong rank independence condition involves only the two constants $p_{2,1}, p_{2,2} \geq 0$ which sum to one, we can express them in terms of a single parameter $\theta \in [0, 1]$, with

$$p_{2,1} = 1 - \theta, \quad p_{2,2} = \theta$$

and the strong rank independence condition is then that for every $A \subset I$,

$$\mathcal{P}\{\mathcal{Y} \in A\langle 2 \rangle \times \{\tau\}\} = \theta \cdot \mathcal{P}\{\mathcal{Y} \in A^*\}.$$

To simplify our manipulations of certain relations arising from this and related conditions, we make the following simple algebraic observation.

**Remark 3** Given $0 \leq \theta \leq 1$, let

$$\alpha := \frac{\theta}{1 - \theta} \quad (\text{if } \theta \neq 1)$$

and

$$\alpha^{-1} := \frac{1 - \theta}{\theta} \quad (\text{if } \theta \neq 0).$$

Then for a given value of $\theta$ and any $a, b \geq 0$, the following are equivalent, provided they make sense (i.e., $\theta \neq 1$ in (3) and $\theta \neq 0$ in (4)):

1. $a = \theta(a + b)$;
2. $(1 - \theta)a = \theta b$;
3. $a = \alpha b$;
4. $\alpha^{-1}a = b$.

The same holds if equality is replaced by “$\leq$” in (1)-(4).
The travellers’ process $\vec{Y}_\theta$ from Example 1 (for $n = 2$) can be characterized in terms of the function $f_\theta$:

$$\vec{Y} := (Y_1, Y_2) \overset{d}{=} \vec{Y}_\theta$$

if and only if

$$\mathcal{P} \left\{ f_\theta (Y_2) < f_\theta (Y_1) \right\} = 0$$

(i.e., almost surely $Y_1$ is the one with the lower $f_\theta$-value). This is equivalent to

$$\forall c \in [0, 1) \quad \mathcal{P} \{ f_\theta (Y_2) \leq c < f_\theta (Y_1) \} = 0 \quad (8)$$

which is what we will prove.

To this end, fix $c \in [0, 1)$ and let

$$\mathcal{P}_c := \{ I_1, I_2, I_3 \}$$

be the partition of $I$ into intervals where

$$I_2 := \{ x : f_\theta (x) \leq c \}$$

and $I_1, I_3$ are the components of $\{ x : f_\theta (x) > c \}$, with $0 \in I_1, 1 \in I_3$ (see Figure 3).

Denote the length of $I_i$ by

$$\ell_i := \text{Leb}_1 (I_i).$$

Using similar triangles in Figure 3, one sees easily that

- $\ell_2 = c$;
- $\ell_1/\theta = \ell_3/(1 - \theta) = 1 - c$
so that in particular (using the notation of Remark 3)
\[ \ell_1 = \alpha \ell_3. \] (9)

The partition \( \mathfrak{P}_c \) of \( I \) gives the product partition \( \mathfrak{P}_c \times \mathfrak{P}_c \) of \( I^2 \), which restricts to \( I^2_i \). The atoms of this restricted partition are
\[ \mathfrak{A}(i, j) := \{ \tilde{a} \in I^2_i : a_1 \in I_i, a_2 \in I_j \} \]
and since \( a_1 \geq a_2 \) in \( I^2_i \), the only atoms of positive measure are the six possibilities for
\[ 3 \geq i \geq j \geq 1 \]
(see Figure 3).

We will also find useful the notation
\[ \mathfrak{A}(i, \ast) := \bigcup_{j=1}^{i} \mathfrak{A}(i, j), \quad \mathfrak{A}(\ast, j) := \bigcup_{i=j}^{3} \mathfrak{A}(i, j). \]
When \( i = j \), \( \mathfrak{A}(i, j) \) is a triangle with area
\[ \text{Leb}_2 (\mathfrak{A}(i, i)) = \frac{\ell_i^2}{2} \quad i = 1, 2, 3 \]
while for \( i > j \), \( \mathfrak{A}(i, j) \) is a rectangle, with
\[ \text{Leb}_2 (\mathfrak{A}(i, j)) = \ell_i \ell_j \quad 3 \geq i > j \geq 1. \]
Using the notation \( m (A) := 2 \cdot \text{Leb}_2 (A) \) for the normalized Lebesgue measure on \( I^2_i \), we see from Equation 9 that in particular
\[ \frac{1}{2} m (\mathfrak{A}(3, 1)) = \alpha^{-1} m (\mathfrak{A}(1, 1)) = \alpha m (\mathfrak{A}(3, 3)), \]
\[ \alpha^{-1} m(x(1,1)) + \alpha m(x(3,3)) = m(x(3,1)). \]  

(10)

**Theorem 1** If a rearrangement \( \vec{Y} = (Y_1, Y_2) \) of \( \vec{X} = (X_1, X_2) \) (i.i.d.) satisfies the strong rank independence condition

\[ P \{ R_2 = 2 \mid Y_1 \in A \} = \theta \quad \text{for all } A \subset I \text{ with } \text{Leb}_1(A) > 0 \]

then \( \vec{Y} \) is equal in distribution to the corresponding travellers’ process of Example 1:

\[ \vec{Y} \overset{d}{=} \vec{Y}_\theta. \]

**Proof:**

Fix \( c \in [0, 1) \); we shall show that the two special cases of the hypothesis with \( A = I_1 \) (resp., \( A = I_3 \))

\[ P \{ R_2 = 2 \mid Y_1 \in I_1 \} = \theta \quad \text{(11)} \]
\[ P \{ R_2 = 2 \mid Y_1 \in I_3 \} = \theta \quad \text{(12)} \]

imply (8).

The first hypothesis (11) can be expressed in terms of \( Y \) and the partition \( \mathfrak{P}_c \) as

\[ P \{ Y \in I_1 \times \{ \tau \} \} = \theta P \{ Y \in I_1^* \}. \]

These sets can be expressed in terms of the partition \( \{ x(i, j) \} \) as follows:

\[
\begin{align*}
I_1(1) &= x(1,*) = x(1,1) \\
I_1(2) &= x(*,1) = x(1,1) \cup x(2,1) \cup x(3,1) \\
I_1^* &= I_1(1) \times \{ id \} \cup I_1(2) \times \{ \tau \}.
\end{align*}
\]

Using this and rewriting (11) in form (2) of Remark 3 gives

\[
(1 - \theta) P \{ Y \in [x(1,1) \cup x(2,1) \cup x(3,1)] \times \{ \tau \} \} = \theta P \{ Y \in x(1,1) \times \{ id \} \}
\]

and, dropping \( x(1,1) \times \{ \tau \} \) from the event on the left, adding it to the event on the right, and dividing by \( 1 - \theta \) gives us

\[
P \{ Y \in [x(2,1) \cup x(3,1)] \times \{ \tau \} \} \leq \alpha P \{ Y \in x(1,1) \times \mathfrak{S} \}. \quad \text{(13)}
\]

Similarly, (12) says

\[ P \{ Y \in I_3 \times \{ \tau \} \} = \theta P \{ Y \in I_3^* \}. \]

This time,

\[
\begin{align*}
I_3(1) &= x(3,*) = x(3,3) \cup x(3,2) \cup x(3,1) \\
I_3(2) &= x(*,3) = x(3,3) \\
I_3^* &= I_3(1) \times \{ id \} \cup I_3(2) \times \{ \tau \}
\end{align*}
\]

13
and form (2) of Remark 3 reads

\[
(1 - \theta)\mathcal{P}\{Y \in x(3, 3) \times \{\tau}\} = \theta\mathcal{P}\{Y \in [x(3, 3) \cup x(3, 2) \cup x(3, 1)] \times \{id\}\}
\]

from which, adding \(x(3, 3) \times \{id\}\) to the event on the left, dropping it from the right and dividing by \(\theta\), we get

\[
\alpha^{-1}\mathcal{P}\{Y \in x(3, 3) \times \{\tau\}\} \geq \mathcal{P}\{Y \in [x(3, 2) \cup x(3, 1)] \times \{id\}\}.
\]

Using Lemma 2 on the right side of (13) and the left side of (14), writing (14) in reverse order, and adding the inequalities gives

\[
\mathcal{P}\{Y \in x(2, 1) \times \{\tau\}\} + \mathcal{P}\{Y \in x(3, 2) \times \{id\}\} + \mathcal{P}\{Y \in x(3, 1) \times \{\tau\}\} \leq \alpha m(x(1, 1)) + \alpha^{-1}m(x(3, 3)).
\]

But by Lemma 2, the last term on the left is just \(m(x(3, 1))\), so that (10) forces

\[
\mathcal{P}\{Y \in x(2, 1) \times \{\tau\}\} = \mathcal{P}\{Y \in x(3, 2) \times \{id\}\} = 0
\]

Finally, we analyze (8):

\[
\{f_\theta(Y_2) \leq c\} = \{Y \in x(*, 2) \times \{id\} \cup x(2, *) \times \{\tau\}\} \quad \text{and} \quad \{c < f_\theta(Y_1)\} = \{Y \in [x(1, *) \cup x(3, *)] \times \{id\} \cup [x(*, 1) \cup x(*, 3)] \times \{\tau\}\}
\]

so that

\[
\{f_\theta(Y_2) \leq c < f_\theta(Y_1)\} = \{Y \in [x(*, 2) \cap (x(1, *) \cup x(3, *))] \times \{id\} \cup [x(2, *) \cap (x(*, 1) \cup x(*, 3))] \times \{\tau\}\}
\]

and hence (15) is precisely the desired conclusion, (8). □

4 Dependence of arrival data on value data

We saw in §2 that a rearrangement \(\vec{Y}\) is, up to equivalence in distribution, a function of its “value data” \(\vec{X}_\downarrow\) and its “arrival data” \(\mu\). It is therefore entirely characterized by the joint distribution of these data. One can consider the extent to which arrival data depends on values; at one extreme the arrival permutation \(\mu\) is independent of \(\vec{X}_\downarrow\), and at the other it is deterministic in the sense that for some function \(u: I_1^n \to \mathcal{G}\) we have \(\bar{\mu} = u(\vec{X}_\downarrow)\). Note the distinction between the arrival data \(\mu\) and the rearranging permutation \(\sigma\) (where \(\vec{X}_\mu = \vec{X}_\sigma\)): in particular, independence of \(\mu\) and \(\vec{X}_\downarrow\) is not equivalent to independence of \(\sigma\) and \(\vec{X}\). The arrival data for the “trivial” rearrangement (\(\vec{X}_\mu = \vec{X}\)) is independent of \(\vec{X}_\downarrow\), since to recover the i.u.d. sequence \(\vec{X}\) from \(\vec{X}_\downarrow\), \(\mu\) must take each of the \(n!\) possible values in \(\mathcal{G}\) independently of \(\vec{X}_\downarrow\) with equal probability. The travellers’ processes of §1 as
well as the examples we will construct in §6 are by definition deterministic. Of course, mixed cases are conceivable.

The intersection of the independent and deterministic classes is the set of constant rearrangements in which \( \mu \) takes a single value in \( \mathcal{S} \) (almost surely). A useful “partial” version of constancy is that of a fixed position. The \( k^{th} \) position in the rearrangement \( \vec{Y} \) is fixed at \( X(\ell) \) if \( Y_k = X(\ell) \) (almost surely). Clearly, the following are equivalent formulations:

\[
\begin{align*}
\mathcal{P} \{ Y_k = X(\ell) \} &= 1; \\
\mathcal{P} \{ \mu_k = \ell \} &= 1; \\
\mathcal{P} \{ \bar{\mu}_\ell = k \} &= 1; \\
\mathcal{P} \{ \mathcal{R}_{k,n}(\vec{Y}) = \ell \} &= 1.
\end{align*}
\]

A constant rearrangement is one in which each position is fixed.

**Lemma 4** Suppose for some rearrangement \( \vec{Y} \) the \( k^{th} \) position is fixed and the initial ranks \( \mathcal{R}_j(\vec{Y}) \), \( j = k, \ldots, n \) are independent. Then each of the partial ranks \( \mathcal{R}_{k,j}(\vec{Y}) \), \( j = k, \ldots, n \) is fixed: that is, it takes a single value (almost surely).

**Proof:**

In terms of the representation \( \vec{Y} \doteq \text{proj}_Y(\vec{X}_\downarrow, \mu) \), our hypotheses are that the diagonal entries \( \rho_{j,j}, j = k, \ldots, n \) of \( \rho := \rho(\mu) \) are independent, and that the last entry \( \rho_{k,n} \) in the \( k^{th} \) row is fixed; we need to show that then every upper entry \( \rho_{k,j}, j = k, \ldots, n \) in the \( k^{th} \) row is fixed.

To this end, let \( m_j \) (resp., \( M_j \)), \( j = k, \ldots, n \) denote the minimum (resp., maximum) of the set \( \{ r : \mathcal{P} \{ \rho_{k,j}(\mu) = r \} > 0 \} \) of essential values for \( \rho_{k,j}(\mu) \). We claim for \( j = k, \ldots, n - 1 \)

\[
M_{j+1} - m_{j+1} \geq M_j - m_j. \tag{16}
\]

To see this, note that Equation 4 (Lemma 3) says for any \( s \in \mathcal{S} \) and any \( j = k, \ldots, n - 1 \) that the following analogue of (5) holds:

\[
\rho_{k,j+1}(s) = \begin{cases} 
\rho_{k,j}(s) & \text{if } \rho_{j+1,j+1}(s) > \rho_{k,j}(s) \\
\rho_{k,j}(s) + 1 & \text{if } \rho_{j+1,j+1}(s) \leq \rho_{k,j}(s).
\end{cases} \tag{17}
\]

In particular, \( M_{j+1} - M_j \) and \( m_{j+1} - m_j \) are both either 0 or 1, and (16) can fail only if (a) \( M_{j+1} = M_j \) and (b) \( m_{j+1} = m_j + 1 \). If (b) occurs, it does so via some particular permutation \( s \in \mathcal{S} \) with \( \mathcal{P} \{ \mu = s \} > 0 \) for which

\[
\rho_{k,j}(s) = m_j \geq \rho_{j+1,j+1}(s).
\]

We will show that in this case (a) fails. Let \( s' \in \mathcal{S} \) with \( \mathcal{P} \{ \mu = s' \} > 0 \) and

\[
\rho_{k,j}(s') = M_j;
\]

by independence of initial ranks, there exists \( \tilde{s} \in \mathcal{S} \) with \( \mathcal{P} \{ \mu = \tilde{s} \} > 0 \) and

\[
\rho_{i,i}(\tilde{s}) = \begin{cases} 
\rho_{i,i}(s') & \text{if } i = k, \ldots, j \\
\rho_{i,i}(s) & \text{if } i = j + 1.
\end{cases}
\]

15
Recursive application of Equation 17 gives

\[ \rho_{k,j}(\tilde{s}) = \rho_{k,j}(s') = M_j, \]

but then

\[ \rho_{j+1,j+1}(\tilde{s}) = \rho_{j+1,j+1}(s) \leq m_j \leq M_j = \rho_{k,j}(\tilde{s}) \]

and another application of Equation 17 then gives \( M_{j+1} = M_j + 1 \), contradicting (a). \( \square \)

As we noted in §1, the strong rank independence condition fails for the trivial arrangement \( \vec{X} \) (where \( \vec{X} \) is i.u.d. and \( \mu \) is equiprobable and independent of \( \vec{X}_\downarrow \)), and holds for any constant arrangement; this generalizes.

**Theorem 2** If a rearrangement of the i.u.d. sequence \( \vec{X} \) satisfies the strong rank independence condition and has \( \vec{X}_\downarrow \) and \( \mu \) independent, then it is a constant rearrangement.

**Proof:**

We will show (by induction) that every position is fixed. Pick \( k \in \{1, \ldots, n\} \) and assume that for every \( \ell, 1 \leq \ell < k \), the \( \ell \)th position is fixed. We will show that the \( k \)th position is fixed. The argument rests on three observations.

The first is that the initial rank \( \mathfrak{R}_k \) is fixed. For \( k = 1 \), this is trivial. For \( k > 1 \), our inductive hypothesis, together with Lemma 4 applied to the \( \ell \)th position, \( \ell = 1, \ldots, k-1 \), implies that \( \mathfrak{R}_k, \ell \) is fixed. But if \( \mathfrak{R}_{k,1}, \ldots, \mathfrak{R}_{k,k-1} \) are fixed then Lemma 3(2) implies that \( \mathfrak{R}_k = \mathfrak{R}_{k,k} \) is also fixed.

The second observation is that, for each \( y \in (0,1) \) and \( r \in \{1, \ldots, k\} \), Remark 1 and the strong rank independence condition formulated as Equation 6 (Remark 2) give us

\[
P \{\mu_k = r \mid Y_k > y\} = \sum_{f_{k,k,n}(r_k, \ldots, r_n) = r} P \{ (\mathfrak{R}_k, \ldots, \mathfrak{R}_n) = (r_k, \ldots, r_n) \mid Y_k > y\}
\]

\[
= \sum_{f_{k,k,n}(r_k, \ldots, r_n) = r} \prod_{j=k}^n p_{j,r_j} \cdot P \{\mathfrak{R}_k = r_k \mid Y_k > y\}.
\]

In view of the first observation, the last factor above depends only on \( r_k \). In particular, the conditional probability at the beginning of this equation is independent of \( y \in (0,1) \), and hence

\[
P \{\mu_k = r \mid Y_k > y\} = P \{\mu_k = r\}. \quad (18)
\]

The third observation is that, if \( \vec{X}_\downarrow \) and \( \mu \) are independent, we have (again for given \( y \) and \( r \) as above)

\[
P \{\mu_k = r, Y_k > y\} = P \{\mu_k = r, X(r) > y\} = P \{\mu_k = r\} \cdot P \{X(r) > y\}
\]

and the standard binomial distribution (for \( \vec{X} \) i.u.d.) gives that

\[
P \{X(r) > y\} = \binom{n}{r} y^{n-r} (1-y)^r + o((1-y)^r) \text{ as } y \to 1.
\]
It follows that for each \( r \in \{1, \ldots, k\} \) we have
\[
P \{ \mu_k = r, Y_k > y \} = P \{ \mu_k = r \} \left( \begin{array}{c} n \\ r \end{array} \right) y^{n-r} (1-y)^r + o((1-y)^r) \quad \text{as } y \to 1
\] (19)
and, letting \( b \) be the minimum value of \( \mu_k \) which appears with positive probability,
\[
P \{ Y_k > y \} = \sum_{r=b}^k P \{ \mu_k = r, Y_k > y \}
= P \{ \mu_k = b \} \left( \begin{array}{c} n \\ b \end{array} \right) y^{n-b} (1-y)^b + o((1-y)^b) \quad \text{as } y \to 1.
\] (20)

Thus, using Equation 19 with \( r = b \) and Equation 20, we have
\[
\lim_{y \to 1} P \{ \mu_k = b \mid Y_k > y \} = \lim_{y \to 1} \frac{P \{ \mu_k = b, Y_k > y \}}{P \{ Y_k > y \}} = 1
\]
which, in view of Equation 18, implies
\[
P \{ \mu_k = b \} = 1.
\]
Hence position \( k \) is fixed. As \( k \in \{1, \ldots, n\} \) was arbitrary, every position is fixed, so the rearrangement is constant and the theorem follows. \( \square \)

5 Binary rearrangements

In general, for the deterministic rearrangement given by a function \( u : I^n_+ \to \mathcal{S} \) (as at the beginning of §4), the position \( \bar{\mu}_i(\bar{a}) \) assigned to the \( i^{th} \) coordinate \( a_i \) of \( \bar{a} \in I^n_+ \) depends not only on the value of \( a_i \), but also on all the other coordinates of \( \bar{a} \). In this section, we consider those deterministic rearrangements for which the relative positions assigned to two coordinates depend only on the values of these two coordinates. We shall call a map \( u : I^n_+ \to \mathcal{S} \) binary if there is a subset \( \mathcal{F} \subset I^2 \) such that for almost all \( \bar{a} \in I^n_+ \) and all \( i \neq j \),
\[
u_i(\bar{a}) < \nu_j(\bar{a}) \text{ iff } (a_i, a_j) \in \mathcal{F}.
\] (21)
It is clear that this condition forces \( \mathcal{F} \) to (almost) satisfy the basic condition for a total ordering, that for (almost) every pair \( (u, v) \in I^2 \), either \( (u, v) \in \mathcal{F} \) or \( (v, u) \in \mathcal{F} \), but not both.

We expect a total ordering to also be transitive. However, this is not forced by Equation 21 when \( n = 2 \), as can be seen from the example
\[
u(u, v) = \begin{cases} id & \text{if } \frac{1}{3} \leq u \leq \frac{2}{3} \\
 \tau \left( \tau_1 = 2, \tau_2 = 1 \right) & \text{otherwise,}
\end{cases}
\]
where \((0.2, 0.5)\) and \((0.5, 0.8)\) but not \((0.2, 0.8)\) belong to \( \mathcal{F} \). (This pathology occurs because our formulation makes every \( u : I^2_+ \to \mathcal{S} \) binary.) However, for \( n \geq 3 \) transitivity is forced: if \( (u, v) \) and \( (v, w) \) both belong to \( \mathcal{F} \) and \( \bar{a} \in I^n_+ \) has (a permutation of) \( (u, v, w) \) as its first three coordinates, then in \( \bar{a}^{\nu(\bar{a})} \), \( u \) (almost surely) precedes \( v \) and \( v \) precedes \( w \), so \( u \) precedes \( w \), hence \( (u, w) \in \mathcal{F} \). Thus we have
**Remark 4** For \( n \geq 3 \), every binary map \( u : I^n \to \mathcal{S} \) is determined by an almost total ordering of \( I \), that is, a binary relation \( \prec \) satisfying:

1. completeness: the set \( \{(u,v) \in I^2 : \text{neither } u \prec v \text{ nor } v \prec u \} \) has measure zero in \( I^2 \);
2. antisymmetry: the set \( \{(u,v) \in I^2 : \text{both } u \prec v \text{ and } v \prec u \} \) has measure zero in \( I^2 \);
3. transitivity: for almost every triple \( (u,v,w) \in I^3 \) with \( u \prec v \) and \( v \prec w \), we also have \( u \prec w \).

One natural way of defining an almost total ordering is by means of a measurable function \( f : I \to \mathbb{R} \), setting \( u \prec v \) if and only if \( f(u) < f(v) \): then properties (1),(2) and (3) follow if we assume \( f \) is nonsingular, that is, each level set has measure zero. Conversely,

**Lemma 5** Every almost total ordering \( \prec \) on \( I \) is generated by some nonsingular measurable function \( f : I \to \mathbb{R} \), and among all such functions (for given ordering \( \prec \)) there is a unique one with values in \( I \) which preserves Lebesgue measure.

**Proof:**

Given the almost total ordering \( \prec \), define the lower sections for \( u \in I \) by

\[
L_u := \{ v \in I : v \prec u \}
\]

and set

\[
f(u) := \text{Leb}_1(L_u).
\]

The transitivity of \( \prec \) implies that for almost all pairs \( (u,v) \in I \),

\[
u \prec v \Rightarrow L_u \subseteq L_v \pmod{0}
\]

(22)

(that is, \( L_u \) is a subset of \( L_u \cup N \) for some null set \( N \)). In particular, almost surely in \( I \times I \) we have

\[
f(u) < f(v) \Rightarrow u \prec v \Rightarrow f(u) \leq f(v);
\]

the second implication is (22) and the first is its contrapositive (with \( u \) and \( v \) reversed). It remains to show that the set

\[
\{(u,v) : u \prec v \text{ and } f(u) = f(v)\}
\]

has measure zero.

To this end, pick \( t \in [0,1] \) and define

\[
C_t := \{ u \in I : f(u) = t \}.
\]

We will show that \( C_t \) has measure zero.
Equation 22 implies that almost surely,

\[ u \prec v, \ u, v \in C_t \Rightarrow L_u = L_v \quad (\text{mod } 0). \]

But then completeness of \( \prec \) insures that for almost every pair \((u, v) \in C_t \times C_t\) we have \(L_u = L_v \pmod{0}\). Fix some \(u' \in C_t\) such that

\[ L_u = L_{u'} \pmod{0} \]

for almost every \(u \in C_t\), and let

\[ D_t := C_t \cap L_{u'}. \]

Then

\[ C_t \times D_t = \{(u, v) \in C_t \times C_t : v \prec u\} \pmod{0} \]

\[ D_t \times C_t = \{(u, v) \in C_t \times C_t : u \prec v\} \pmod{0} \]

and by completeness,

\[ C_t \times D_t \cup D_t \times C_t = C_t \times C_t \pmod{0}. \quad (23) \]

But then

\[ D_t \times D_t = C_t \times D_t \cap D_t \times C_t = \{(u, v) \in C_t \times C_t : u \prec v \text{ and } v \prec u\} \pmod{0} \]

must, by antisymmetry, have measure zero. This implies \(D_t\) has measure zero, and hence by the Cavalieri principle, each of the (product) sets in Equation 23 has measure zero. It follows that \(C_t\) has measure zero, as required.

We have shown that \(\text{Leb}_1 (C_t) = 0\) for each \(t\).

First, this implies that

\[ \{(u, v) \in I \times I : u \prec v \text{ and } f(u) = f(v) = t\} \subset C_t \times C_t \]

has measure zero, so that (by Fubini) almost surely in \(I \times I\)

\[ u \prec v \Rightarrow f(u) < f(v) \Rightarrow u \prec v \]

and second, it implies that \(f\) is nonsingular.

Now, consider the distribution function \(F(t) := \text{Leb}_1 \{u : f(u) \leq t\}\). Note that \(F\) is continuous, and for almost every \(v \in I\)

\[ F(f(v)) = \text{Leb}_1 \{u : f(u) < f(v)\} = \text{Leb}_1 \{u : u \prec v\} = f(v) \]

so that \(F(t) = t\) for all essential values of \(f\); but nonsingularity of \(f\) implies all values are essential, hence \(F(t) = t\ \forall t \in I\). This means \(f\) is measure-preserving.

Finally, suppose \(h : I \to [0, 1]\) is another measure-preserving function such that almost surely \(v \prec u \text{ iff } h(v) < h(u)\); then for all \(u \in I\),

\[ h(u) = \text{Leb}_1 \{v \in I : h(v) < h(u)\} = \text{Leb}_1 (L_u) = f(u). \]
Remark 4 and Lemma 5 justify the following terminology. A rearrangement \( \vec{Y} \) of \( \vec{X} \) is a **binary rearrangement** if \( \vec{Y} \equiv \vec{X}_{\mu}^u \), where \( \mu = u(\vec{X}) \) and \( u : I^n \rightarrow \mathcal{S} \) is a binary mapping determined by some almost total ordering on \( I \). This means (in view of Lemma 5) that the arrival times are determined from the values of a (measure-preserving) function \( f : I \rightarrow I \) via

\[
  f \left( X_{(\mu_1)} \right) < f \left( X_{(\mu_2)} \right) < \ldots < f \left( X_{(\mu_n)} \right).
\]

We will say that the rearrangement is **directed** by \( f \), and refer to the family of sets

\[
  B_t := \{ u : f(u) \leq t \}
\]

as the **filtration** of the rearrangement.

It will be useful for what follows to identify a finite random set with a random measure composed of unit point masses. Suppose \( B \subset I \) is a set of positive measure. Let \( (U_1, \ldots, U_m) \) be an i.u.d. sample from \( B \); we define the uniform \( m \)-point process relative to \( B \), \( \mathcal{N}[m, B] \), by setting, for each Borel set \( A \subset I \),

\[
  \mathcal{N}[m, B](A) := \sharp \{ i \in \{1, \ldots, m \} : U_i \in A \}.
\]

Then the familiar formula for multinomial probabilities gives \( \mathcal{N}[m, B] \): if \( \{A_1, \ldots, A_k\} \) is a (disjoint) partition of \( I \), then for each \( k \)-tuple \( i_1, \ldots, i_k \in \mathbb{N} \) with \( i_1 + \ldots + i_k = m \), we have (using \( \mathcal{N} = \mathcal{N}[m, B] \))

\[
  \mathbb{P} \{ \mathcal{N}(A_j) = i_j, \ j = 1, \ldots, k \} = \frac{m!}{i_1! \cdots i_k!} \prod_{j=1}^{k} \left[ \frac{\text{Leb}_1(A_j \cap B)}{\text{Leb}_1(B)} \right]^{i_j}.
\]  

(24)

The uniform point processes determine the original i.u.d. samples, in the sense that given a point process \( \mathcal{N} \) satisfying (24), we can set up \( (U_1, \ldots, U_m) \) with \( \mathcal{N} \left( \{U_1\} \right) = \ldots \mathcal{N} \left( \{U_m\} \right) = 1 \), \( U_1 > \ldots > U_m \) and then \( (U_1, \ldots, U_m) \equiv (X_{(1)}, \ldots, X_{(m)}) \), where \( X_1, \ldots, X_m \) are i.u.d. in \( B \).

The following properties of \( \mathcal{N}[m, B] \) are straightforward consequences of (24).

**Proposition 2** For any \( m \in \mathbb{N} \) and \( B \subset I \), the uniform \( m \)-point processes satisfy:

1. If \( B_i \) are sets converging to \( B \) in measure, then the processes \( \mathcal{N}[m, B_i] \) converge in distribution to \( \mathcal{N}[m, B] \);

2. If \( \{A_1, A_2\} \) is a partition of \( I \), then the distribution of the restriction

   \[
   (\mathcal{N} = \mathcal{N}[m, B])|_{A_2}
   \]

   conditioned on \( \mathcal{N}|_{A_1} \), coincides with that of

   \[
   \mathcal{N}[m - \mathcal{N}(A_1), B \cap A_2].
   \]

3. For \( B' \subset B \), the distribution of \( \mathcal{N}[m, B]|_{B'} \), conditioned on \( \mathcal{N}(B') = m \) coincides with that of \( \mathcal{N}[m, B'] \).
The following relates the processes $\mathcal{N}[m, B]$ to binary rearrangements. We use $B^c$ to denote the complement of $B \subset I$ in $I$.

**Proposition 3** Suppose $\vec{Y}$ is a binary rearrangement of $\vec{X}$ (i.u.d.) directed by $f$, with filtration $\{B_t, t \in I\}$. Then for any $k = 1, \ldots, n-1$, the distribution of the random set $\{Y_{k+1}, \ldots, Y_n\}$ coincides with that of the random point process $\mathcal{N}[n-k, B^c_{f(Y_k)}]$.

**Proof:**

Let $\mathcal{N} = \mathcal{N}[n, I]$ be the process obtained from $\vec{X}$. By Proposition 2(2) for any fixed $t \in I$, the distribution of $\mathcal{N}|_{B^c_t}$ conditioned on $\mathcal{N}|_{B_t}$ coincides with that of $\mathcal{N}[n - \mathcal{N}(B_t), B^c_t]$. This observation extends in a straightforward way to the stopping time $T = f(Y_k)$ which is the moment at which the filtration encounters a point of the original process $\mathcal{N}$ for the $k^{th}$ time. By definition, the random set $\{Y_{k+1}, \ldots, Y_n\}$ is $\mathcal{N}|_{B^c_t}$. The assertion follows. $\blacksquare$

We turn now to binary rearrangements; in view of Theorem 1 we focus on $n \geq 3$.

**Proposition 4** Suppose $\vec{Y}$ is a binary rearrangement of $\vec{X}$ (i.u.d.), $n \geq 3$, such that some initial rank $\mathfrak{R}_k$, $k \in \{3, \ldots, n\}$ is independent of the random variable $(Y_1, \ldots, Y_{k-1})$. Then almost surely, $\mathfrak{R}_k$ takes only its extreme values, 1 and $k$:

$$P \{1 < \mathfrak{R}_k < k\} = 0.$$
$Y_1, \ldots, Y_{k-1} \in A_\varepsilon$, the probability that $1 < \mathcal{R}_k < k$ is bounded above by the probability that at least one of the points $Y_k, \ldots, Y_n$ belongs to $I_\varepsilon$. Now, using Proposition 3, let

$$\mathcal{N} \overset{\triangle}{=} \mathcal{N}[n - k + 1, B_{j(Y_{k-1})}]$$

be the $(n - k + 1)$-point distribution for $\{Y_k, \ldots, Y_n\}$. Then an easy computation yields

$$\mathbb{P}\{\mathcal{N}(I_\varepsilon) \geq 1\} = \mathcal{O}(\varepsilon).$$

But

$$p_2 + \ldots + p_{n-1} = \mathbb{P}\{1 < \mathcal{R}_k < k \mid Y_1, \ldots, Y_{k-1} \in A_\varepsilon\} \leq \mathbb{P}\{\mathcal{N}(I_\varepsilon) \geq 1\}$$

and so the proposition follows. □

Using proposition 4 we can prove the main result of this section.

**Theorem 3** Suppose $\vec{Y}$ is a binary rearrangement of $\vec{X}$ (i.u.d.), $n \geq 3$, such that for some $k \in \{2, \ldots, n\}$ the initial rank $\mathcal{R}_k$ is independent of the random variable $(Y_1, \ldots, Y_{k-1})$.

Then $\vec{Y}$ is equal in distribution to some travellers’ process:

$$\vec{Y} \overset{d}{=} \vec{Y}_\theta.$$

**Proof:**

By Proposition 4, $\mathcal{R}_k$ takes only its extreme values, 1 and $k$. Hence for some $\theta \in [0, 1]$ our assumption is

$$\mathbb{P}\{\mathcal{R}_k = k \mid Y_1, \ldots, Y_{k-1} \in A_\varepsilon\} = \theta, \quad \mathbb{P}\{\mathcal{R}_k = 1 \mid Y_1, \ldots, Y_{k-1}\} = 1 - \theta.$$

As before, we assume $\vec{Y}$ is directed by the (measure-preserving) function $f$ with filtration $B_t$, $t \in I$, so that $\text{Leb}_1(B_t) = t$.

Fix $t \in (0, 1)$, and let $x'$ (resp., $x''$) be the essential infimum (resp., essential supremum) of the set $B_t \subset I$, and set

$$t' := \lim \text{ess sup}_{x' \uparrow x'} \{f(u) : u \in [x', x] \cap B_t\};$$

this limit exists because ess sup$\{f(u) : u \in [x', x]\}$ decreases with $x$, and $t' \leq t$.

For $\varepsilon > 0$, define

$$A'_\varepsilon := \{x : f(x) \in [t' - \varepsilon, t' + \varepsilon] \cap B_t \cap [x', x' + \varepsilon]\}.$$

It follows from the definition of $x'$ that $A'_\varepsilon$ has positive measure.

Similarly, set

$$t'' := \lim \text{ess sup}_{x'' \downarrow x'} \{f(u) : u \in [x, x''] \cap B_t\},$$

and for $\varepsilon > 0$

$$A''_\varepsilon := \{x : f(x) \in [t'' - \varepsilon, t'' + \varepsilon] \cap B_t \cap [x'' - \varepsilon, x'']\}.$$
so that again \( t'' \leq t \) and \( A''_\varepsilon \) has positive measure. Note that, as \( \varepsilon \to 0 \), we have
\[
\sup_{x \in A'_\varepsilon} |t' - f(x)| \to 0, \quad \sup_{x \in A'_\varepsilon} |t'' - f(x)| \to 0. \tag{25}
\]

Now consider the uniform \((n - k + 1)\)-point process
\[
\mathcal{N}' = \mathcal{N}[n - k + 1, B''_t]
\]
and set \( Z' \) the atom of \( \mathcal{N}' \) minimizing \( f \). As in the proof of Proposition 4, the event \( \{Y_1, \ldots, Y_{k-1} \in A'_\varepsilon \} \) has positive probability. Clearly, if \( (Y_1, \ldots, Y_{k-1}) \in A'_\varepsilon \) then \( Y_k > x' + \varepsilon \) implies \( \mathfrak{r}_k = 1 \), and \( Y_k < x' \) implies \( \mathfrak{r}_k = k \). Thus,
\[
p_k = P \{ \mathfrak{r}_k = k \mid Y_1, \ldots, Y_{k-1} \in A'_\varepsilon \} \\
= P \{ \mathfrak{r}_k = k, Y_k \in [x', x' + \varepsilon] \mid Y_1, \ldots, Y_{k-1} \in A'_\varepsilon \}
+ P \{ \mathfrak{r}_k = k, Y_k \in [0, x'] \mid Y_1, \ldots, Y_{k-1} \in A'_\varepsilon \}.
\]

The first term goes to zero as \( \varepsilon \to 0 \), while by Proposition 2(3) and (25) the second converges to \( P \{ Z' \in [0, x'] \} \) so
\[
p_k = P \{ Z' \in [0, x'] \}.
\]

A similar argument involving conditioning on \( (Y_1, \ldots, Y_{k-1}) \in A''_\varepsilon \) gives
\[
p_k = P \{ Z'' \in [0, x''] \}
\]
where \( Z'' \) is the atom of \( \mathcal{N}'' := \mathcal{N}[n - k + 1, B''_t] \) which minimizes \( f \).

Next, we claim: \( B_t = [x', x''] \) and \( t = \max \{t', t''\} \).

Begin with the case \( t' \geq t'' \), so that \( B''_t \supseteq B''_t \) and \( B''_t \setminus B''_t = B_t \setminus B''_t \), and consider the process \( \mathcal{N}'' \). Whenever all atoms of \( \mathcal{N}'' \) fall into \( B''_t \), Proposition 2(3) tells us that (conditionally) \( \mathcal{N}'' \) agrees in distribution with \( \mathcal{N}' \); thus,
\[
P \{ Z'' \in [0, x''] \mid \mathcal{N}'' (B''_t \setminus B''_t) = 0 \} = P \{ Z' \in [0, x'] \}.
\]

On the other hand, if \( \mathcal{N}'' \) has some atoms in \( B''_t \setminus B''_t \), then, since \( f(x) > t \geq t' \) off \([x', x'']\), we must have \( Z'' \in [x', x''] \), and
\[
P \{ Z'' \in [0, x''] \mid \mathcal{N}'' (B''_t \setminus B''_t) \geq 1 \} = 1.
\]

Hence
\[
p_k = P \{ Z'' \in [0, x''] \} \\
= P \{ Z'' \in [0, x''] \mid \mathcal{N}'' (B''_t \setminus B''_t) = 0 \} \cdot P \{ \mathcal{N}'' (B''_t \setminus B''_t) = 0 \}
+ P \{ Z'' \in [0, x''] \mid \mathcal{N}'' (B''_t \setminus B''_t) \geq 1 \} \cdot P \{ \mathcal{N}'' (B''_t \setminus B''_t) \geq 1 \}
= P \{ Z' \in [0, x'] \} \cdot P \{ \mathcal{N}'' (B''_t \setminus B''_t) = 0 \}
+ P \{ \mathcal{N}'' (B''_t \setminus B''_t) \geq 1 \}
\geq P \{ Z' \in [0, x'] \}
= p_k.
\]
In particular, $\mathcal{P}\{Z' \in [0, x']\} = \mathcal{P}\{Z' \in [0, x']\}$ implies that $\mathcal{P}\{Z' \in [x', x'']\} = 0$, a situation possible iff $\mathcal{N}'$ (almost surely) puts no atoms in $[x', x'']$, or equivalently iff $\text{Leb}_1([x', x''] \cap B_t) = 0$, which in turn means $[x', x''] \subset B_{t'} \pmod{0}$ so that $f(x) \leq t'$ (almost surely) on $[x', x'']$. Again, since $f(x) \geq t$ (almost surely) outside $[x', x'']$ and $f$ preserves measure, we must have $t = t'$ and, since $x', x''$ are the essential bounds on $B_t$, it follows also that $B_t = [x', x'']$, and in particular $x'' - x' = t$.

The argument in case $t' \leq t''$ is similar, involving two computations of $p_1$.

Having established the claim, we now consider each of the endpoints of $B_t$ as a non-increasing (resp., non-decreasing) function $x'(t)$ (resp., $x''(t)$), with

$$x''(t) - x'(t) = t \quad \text{for all } t.$$  

We wish to compute the derivative of $x'(t)$. Let $\mathcal{N}_t$ be a uniform $(n - k + 1)$-point process on $B_t^c$, and $Z_t$ be the atom of $\mathcal{N}_t$ where $f$ is minimized. Arguments like those above give

$$p_k = \mathcal{P}\{Z_t \in [0, x'(t)]\}$$

$$= \mathcal{P}\{Z_t \in [0, x'(t)] \mid \mathcal{N}_t(B_{t+\varepsilon}^c) = n - k + 1\} \cdot \mathcal{P}\{\mathcal{N}_t(B_{t+\varepsilon}^c) = n - k + 1\} + o(\varepsilon)$$

$$= \mathcal{P}\{Z_{t+\varepsilon} \in [0, x'(t) + \varepsilon]\} \cdot \mathcal{P}\{\mathcal{N}_t(B_{t+\varepsilon}^c) = n - k + 1\}$$

$$+ \mathcal{P}\{\mathcal{N}_t([x'(t) + \varepsilon), x'(t)] = 1 \mid \mathcal{N}_t(B_{t+\varepsilon}^c) = 1\} \cdot (n - k + 1)\varepsilon(1 - t)^{-1} + o(\varepsilon)$$

$$= p_k \cdot \mathcal{P}\{\mathcal{N}_t(B_{t+\varepsilon}^c) = n - k + 1\} + \left(\frac{x'(t) - x'(t+\varepsilon)}{\varepsilon}\right) \cdot (n - k + 1)\varepsilon(1 - t)^{-1} + o(\varepsilon).$$

Rearranging terms and letting $\varepsilon \to 0$ we find that the derivative is

$$\frac{dx'(t)}{dt} = -p_k.$$  

This implies

$$x'(t) = -p_k \cdot t + p_k, \quad x''(t) = p_1 \cdot t + (1 - p_1)$$

which in turn forces $f$ to equal $f_\theta$ with $\theta = p_k$ (up to a null set). \hfill \square

6 Further examples

So far, the only examples of rearrangements with the strong rank independence property have been the travellers’ processes of Example 1 and the constant rearrangements $\bar{X}_s$, where $s \in \mathcal{S}$ is a fixed permutation. In this section we construct multiparameter families of deterministic rearrangements with the strong rank independence property which combine features of both the travellers’ processes and constant rearrangements, but are of neither type. The idea is that if the position of some $X_{(k)}$ in $\bar{Y}$ is fixed, then we can use it to partition $I$ into two subintervals $I_1 \cup I_2$, with $n - k$ (resp., $k - 1$) points uniformly
distributed on \( I_1 \) (resp., \( I_2 \)), whatever value \( X_{(k)} \) takes; these two “sub”-processes are independent, and we can rearrange each separately.

Keep in mind the following features of our examples so far:

- for a constant rearrangement, each initial rank \( \mathfrak{R}_k \) almost surely takes a single value;
- for the travellers’ process (or by theorem 3, any binary rearrangement), each initial rank \( \mathfrak{R}_k \) takes only the extreme values 1 and \( k \).

Before giving a general construction, we consider two specific examples:

**Example 2** Take \( n = 3 \) and choose \( \theta \in (0, 1) \). Now set

\[
Y_1 = X_{(1)},
\]

and given \( X_{(1)} \), let \( \gamma : [0, X_{(1)}] \to I \) be the unique linear, order-preserving bijection, \( t \mapsto t/X_{(1)} \). (Of course, \( \gamma \) is a random transformation, since it depends on \( X_{(1)} \).) Then apply the travellers process to \( \gamma(X_{(2)}), \gamma(X_{(3)}) \) to order these: that is,

\[
(Y_2, Y_3) = (X_{(2)}, X_{(3)}) \text{ iff } f_{\theta}(\gamma(X_{(2)})) < f_{\theta}(\gamma(X_{(3)})).
\]

Now, having observed \( Y_1 \), we know that there are two independent points below \( Y_1 \), arranged according to \( \bar{Y}_\theta \) (normalized). Thus the initial ranks are

\[
\mathfrak{R}_1 = 1, \quad \mathfrak{R}_2 = 2,
\]

\[
\mathcal{P}\{\mathfrak{R}_3 = 2 \mid Y_1, Y_2\} = 1 - \mathcal{P}\{\mathfrak{R}_3 = 3 \mid Y_1, Y_2\} = 1 - \theta
\]

so the strong rank independence condition holds.

In the preceding example, \( X_{(1)} \) always has the fixed position \( Y_1 \), and the third initial rank \( \mathfrak{R}_3 \) takes the non-extreme value 2 with positive probability. A more complicated variation is the following:

**Example 3** Take \( n = 5 \), and pick two values \( \theta_1, \theta_2 \in (0, 1) \). Set

\[
Y_3 = X_{(3)},
\]

and given \( X_{(3)} \), let

\[
I_2 = [0, X_{(3)}], \quad I_1 = [X_{(3)}, 1],
\]

and set \( \gamma_i : I_i \to I \) to be the affine order-preserving bijection for \( i = 1, 2 \).

Now, we will “couple” the other positions as follows

\[
\{Y_1, Y_2\} = \{X_{(4)}, X_{(5)}\}, \quad \{Y_4, Y_5\} = \{X_{(1)}, X_{(2)}\}
\]
with the specific order within each pair specified by \( f_{\theta_i \circ \gamma_i} \); thus,

\[
(Y_1, Y_2) = (X(4), X(5)) \quad \text{iff} \quad f_{\theta_1} \left( \gamma_1 \left( X(4) \right) \right) < f_{\theta_1} \left( \gamma_1 \left( X(5) \right) \right)
\]

(else \( (Y_1, Y_2) = (X(5), X(4)) \)) and

\[
(Y_4, Y_5) = (X(1), X(2)) \quad \text{iff} \quad f_{\theta_2} \left( \gamma_2 \left( X(1) \right) \right) < f_{\theta_2} \left( \gamma_2 \left( X(2) \right) \right)
\]

(else \( (Y_4, Y_5) = (X(2), X(1)) \)).

Here, by contrast with Example 2, the point \( X(3) \) at the fixed position \( Y_3 \) is not known until the third observation. However,

\[
\mathcal{P} \{ \mathfrak{g}_3 = 3 \mid Y_1, Y_2 \} = 1
\]

and conditioned on any value of \( Y_3 = X(3) \), we have

\[
\mathcal{P} \{ \mathfrak{g}_2 = 2 \mid Y_1, Y_3 \} = 1 - \mathcal{P} \{ \mathfrak{g}_2 = 1 \mid Y_1, Y_3 \} = \theta_1
\]

so that this is also true if we drop the conditioning on \( Y_3 \). Once having observed \( Y_3 = X(3) \), we know the next two points lie above \( X(3) \), so

\[
\mathcal{P} \{ \mathfrak{g}_4 = 1 \mid Y_1, Y_2, Y_3 \} = 1
\]

and independently of \( Y_1, Y_2 \) we know that \( Y_4, Y_5 \) satisfy the rank condition

\[
\mathcal{P} \{ \mathfrak{g}_5 = 2 \mid Y_1, Y_2, Y_3, Y_4 \} = 1 - \mathcal{P} \{ \mathfrak{g}_5 = 1 \mid Y_1, Y_2, Y_3, Y_4 \} = \theta_2.
\]

In this example, the rearranged positions were coupled to descending positions according to a partition into intervals (26). However, this is easily modified: the reader can check that if for example we couple the positions via

\[
Y_2 = X(3)
\]

\[
\{Y_1, Y_4\} = \{X(4), X(5)\}
\]

\[
\{Y_3, Y_5\} = \{X(1), X(2)\}
\]

but still use the functions \( f_{\theta_i \circ \gamma_i} \) to decide how each pair is ordered, then we obtain a rearrangement with

\[
\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}_3 = 1
\]

\[
\mathcal{P} \{ \mathfrak{g}_4 = 3 \mid Y_1, Y_2, Y_3 \} = 1 - \mathcal{P} \{ \mathfrak{g}_4 = 4 \mid Y_1, Y_2, Y_3 \} = 1 - \theta_1
\]

\[
\mathcal{P} \{ \mathfrak{g}_5 = 4 \mid Y_1, Y_2, Y_3, Y_4 \} = 1 - \mathcal{P} \{ \mathfrak{g}_5 = 5 \mid Y_1, Y_2, Y_3, Y_4 \} = 1 - \theta_2.
\]

One can also increase the number of deterministic positions and/or the number of positions in any “coupled” group.

The general construction involves three types of parameters: fixed positions, switching schemes, and jump probabilities. Suppose we are working with \( n \) variables.
**Fixed positions:** Pick $d$, set $n_0 = 0$, $n_{d+1} = n + 1$ and pick a subsequence $n_1 < n_2 < \ldots < n_d$ from $\{1, \ldots, n\}$ such that $n_{i+1}$ is either adjacent to $n_i$, or there are at least two intermediate values (i.e., $n_{i+1} - n_i \neq 2$). Let $N_i := \{n_{i-1} + 1, \ldots, n_i - 1\}$ (so $\#N_i \geq 2$ if $N_i \neq \emptyset$).

**Switching schemes:** Pick $d$ distinct positions $m_1, \ldots, m_d \in \{1, \ldots, n\}$, and partition the rest of $\{1, \ldots, n\}$ into subsets $M_i, i = 1, \ldots, d + 1$, with $\#M_i = \#N_i$ for $i = 1, \ldots, d + 1$. Our rearranged positions will be coupled to the descending ones via the scheme

$$Y_{m_i} = X_{(n_i)} \quad i = 1, \ldots, d$$
$$\{Y_j : j \in M_i\} = \{X_{(j)} : j \in N_i\} \quad i = 1, \ldots, d + 1$$

**Jump probabilities:** For each $i$ such that $N_i \neq \emptyset$, we pick $\theta_i \in (0, 1)$.

Our map $\mu : I^n_i \to \mathfrak{S}$ defining the rearrangement will then be defined as follows: given $\vec{a} = (a_1, \ldots, a_n) \in I^n_i$, $a_0 = 1$, $a_{n+1} = 0$, $\mu(\vec{a})$ will satisfy

1. $\bar{\mu}_{m_i} = n_i, i = 1, \ldots, d$;
2. $j \in M_i$ iff $\bar{\mu}_j \in N_i$;
3. if $N_i \neq \emptyset$, let $I_i := [a_{n_{i+1}}, a_{n_i}]$, take $\gamma_i : I_i \to I$ the affine orientation-preserving bijection, and set $f_i := f_n \circ \gamma_i$; then if $M_i = \{j_1 < j_2 < \ldots < j_i\}$, we define $\mu : M_i \to N_i$ by the condition

$$f_i \left(X_{(\mu_j)}\right) < f_i \left(X_{(\mu_{j_2})}\right) < \ldots < f_i \left(X_{(\mu_{j_i})}\right).$$

**Proposition 5** Any rearrangement constructed as above has the strong rank independence property.

**Proof:**

We keep the notation of the construction above.

First, we determine the initial rank of $Y_{m_i}$. Consider $h < m_i$. Either $h = m_p$ for some $p \neq i$, and since $a_{n_1} > a_{n_2} > \ldots > a_{n_d}$,

$$Y_{m_p} > Y_{m_i} \text{ iff } p < i$$

or $h \in M_p$ for some $p$, and since $N_p = \{q : n_{p-1} < q < n_p\} = \{q : a_{n_{p-1}} > a_q > a_{n_p}\}$,

$$Y_h > Y_{m_i} \text{ iff } p \leq i.$$ 

Thus, by Equation 1 we have (with probability 1) $\#_{m_i} = s$, with

$$s = 1 + \#\{j < i : m_j < m_i\} + \#\left[\{1, \ldots, m_i\} \cap \bigcup_{p \leq i} M_p\right].$$

(27)

Now, suppose $k \in M_i$, and consider $h < k$. If $h = m_p$ for some $p \in \{1, \ldots, d + 1\}$ or if $h \in M_p$ for some $p \neq i$, then we have

$$Y_h > Y_k \text{ iff } p < i.$$
Thus, the only undetermined relative sizes are those involving $h \in M_i$ and $h < k$. Let

$$r := \sharp \{ h < k : h \in M_i \}.$$  

Then we know

$$P \left\{ \mathcal{R}_k = s \mid Y_1, \ldots, Y_{k-1}, Y_{m_i}, Y_{m_i+1} \right\} = 1 - P \left\{ \mathcal{R}_k = s + r \mid Y_1, \ldots, Y_{k-1}, Y_{m_i}, Y_{m_i+1} \right\} = 1 - \theta_i,$$

where $s$ is given by (27), and hence the conditioning on $Y_{m_i}$ and $Y_{m_i+1}$ can be removed, as in Example 3. $\square$

We pose some unresolved questions concerning the characterization of rearrangements with the strong rank independence property. We use the notation of (6) in Remark 2.

**Question 1** Are rearrangements with the strong rank independence property characterized by the distributions of their rank configurations? That is, if $\vec{Y}$ and $\vec{Y}'$ both satisfy (6) with $p_{k,\ell} = p'_{k,\ell}$ for all $k, \ell$, then does it follow that $\vec{Y} \overset{d}{=} \vec{Y}'$?

Theorem 1 can be viewed as an affirmative answer for $n = 2$; a particular extension would be whether $\vec{Y}_\theta$ is characterized by

$$p_{k,1} = 1 - \theta, \quad p_{k,\ell} = 0 \text{ for } 1 < \ell < k, \quad p_{k,k} = \theta$$

for $k = 2, \ldots, n$. Two other questions are raised by our construction above, and by Theorem 2.

**Question 2** Is every rearrangement with the strong rank independence property necessarily deterministic? That is, is it determined by some map

$$u : I^n \rightarrow \mathcal{S}?$$

We note that while in our most general examples the ranks do not necessarily take extreme values, they still have the property that each rank takes at most two values.

**Question 3** Does there exist a rearrangement with the strong rank independence property for which some initial rank can take three or more values with positive probability?

The following example shows that strong independence for a single rank, as in Theorem 3, does not alone restrict that rank to two values.

28
Example 4 Take \( n = 6 \), and fix
\[
(Y_1, Y_2, Y_3) = (X_{(1)}, X_{(3)}, X_{(5)}).
\]
Then arrange
\[
\{Y_4, Y_5, Y_6\} = \{X_{(2)}, X_{(4)}, X_{(6)}\}
\]
in equiprobable random order.
Now \( R_4 \) is equally likely to equal 2, 4 or 6, independently of the values of \( Y_1, Y_2, Y_3 \).

Note, of course, that this rearrangement is not deterministic.

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