Signatures of paths transformed by polynomial maps

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Abstract

We characterize the signature of piecewise continuously differentiable paths transformed by a polynomial map in terms of the signature of the original path. For this aim, we define recursively an algebra homomorphism between two shuffle algebras on words. This homomorphism does not depend on the path and behaves well with respect to composition and homogeneous maps. It allows us to describe the relation between the signature of a piecewise continuously differentiable path and the signature of the path obtained by transforming it under a polynomial map. We also study this map as a half-shuffle homomorphism and give a generalization of our main theorem in terms of Zinbiel algebras.

Keywords: Signature tensors, iterated integrals, tensor algebra, shuffle product, polynomial maps.

1 Introduction

In the 1950s, K. T. Chen introduced the iterated-integral signature of a piecewise continuously differentiable path, which up to a natural equivalence relation, determines the initial path. In general, the signature of a path can be seen as a multidimensional time series. When the terminal time is fixed, the signature of a path can be seen as tensors and the calculation of the signature becomes a standard problem in data science. In [PSS19], M. Pfeffer, A. Seigal, and B. Sturmfels study the inverse problem: given partial information from a signature, can we recover the path? They consider signature tensors of order three under linear transformations and establish identifiability results and recovery algorithms for piecewise linear paths, polynomial paths, and generic dictionaries.

Coming from stochastic analysis, the signatures are becoming more relevant in other areas, such as algebraic geometry and combinatorics, and we would like to highlight some recent work. For instance, in [DR19], J. Diehl and J. Reizenstein offer a combinatorial approach to the understanding of invariants of multidimensional time series based on their signature. Another reference is [AFS19], in which C. Améndola, P. Friz, and B. Sturmfels look at the varieties of signatures of tensors for both deterministic and random paths, focusing on piecewise linear paths and polynomials paths, among others. Answering one of their questions, in [Gal19], F. Galuppi looks at rough paths, for which their signature variety shows surprising analogies with the Veronese variety.

In stochastic analysis, the study of the signatures of paths arises in the theory of rough paths, where [FV10, FH14] are textbook references. Iterated integrals and the non-commutative series that encode them have also arisen in a variety of contexts in geometry and arithmetic, including the work of R. Hain in [Hai02], M. Kapranov in [Kap09], and J. Balakrishnan in [Bal13]. The results we derive in this paper have the potential for future applications in all of these contexts.

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Let us now present our problem and our main two results, Theorems 1.1 and 1.2. A piecewise continuously differentiable path \( X \) in \( \mathbb{R}^d \) is a map defined by \( d \) piecewise smooth functions \( X^i(t) \) in a parameter \( t \in [0, L] \), for \( i = 1, \ldots, d \). Its signature stores the collection of all the iterated integrals of the path \( X \), which are of the form

\[
\int_0^L \cdots \int_0^{r_n} dX^i_{r_n} \cdots dX^i_{r_1},
\]

where \( X^i_r := X^i(r) \). The iterated integral (1) is a real number and it is associated to the sequence \((i_1, i_2, \ldots, i_n)\), for which the order is relevant. Therefore, we consider the signature \( \sigma(X) \) as an element of \( T((\mathbb{R}^d)) \), the space of formal power series in words in the alphabet \( \{1, 2, \ldots, d\} \). This space becomes an algebra with the concatenation product, denoted by the symbol \( \cdot \). Its algebraic dual, denoted by \( T(\mathbb{R}^d) \), is the space of non-commutative polynomials in the same set of words. It is a commutative algebra with the shuffle product, which is denoted by \( \shuffle \) and interleaves two words in all order-preserving ways, [Reu93].

We also consider the following duality paring in \( T((\mathbb{R}^d)) \times T(\mathbb{R}^d) \):

\[
\left\langle \sum_{w \in \mathcal{W}_d} a_w w, v \right\rangle = a_v,
\]

where \( \mathcal{W}_d \) denotes the set of words in the alphabet \( \{1, \ldots, d\} \), together with the empty word \( \varepsilon \).

Let \( X \) be a piecewise continuously differentiable path in \( \mathbb{R}^d \) and \( \sigma(X) \) be its signature. Consider a polynomial map \( p \) from \( \mathbb{R}^d \) to \( \mathbb{R}^m \). One can compute the image path \( p(X) \) and ask for its signature, \( \sigma(p(X)) \). Then, the following question comes up:

**How are both signatures, \( \sigma(X) \) and \( \sigma(p(X)) \), related?**

We approach this question from an algebraic point of view. We consider the dual map \( p^* : \mathbb{R}[x_1, \ldots, x_m] \longrightarrow \mathbb{R}[x_1, \ldots, x_d] \), where both sets of variables are commutative. It is natural and common to embed the polynomial ring \( \mathbb{R}[x_1, \ldots, x_m] \) into the tensor algebra \( T(\mathbb{R}^m) \). For that we identify the variable \( x_i \) with the letter \( i \) and we define the embedding, denoted by \( \varphi_m \) (or \( \varphi \)), by sending the monomial \( x_{i_1} \cdots x_{i_t} \) to the shuffle product \( i_{i_1} \shuffle \cdots \shuffle i_{i_t} \), for \( 1 \leq i_1, \ldots, i_t \leq m \), and extending by linearity. By construction, this map is a morphism of commutative algebras, and it is injective but not surjective. For instance, for \( t \geq 2 \), \( \varphi(x_1 \cdot x_2) = 1 \shuffle 2 = 12 + 21 \) and there is no other way to obtain the words \( 12 \) and \( 21 \) as \( \varphi(h) \), for any polynomial \( h \in \mathbb{R}[x_1, \ldots, x_d] \). Therefore, we cannot find a polynomial in \( \mathbb{R}[x_1, \ldots, x_d] \) with image \( 12 \).

Our first step is to define a map \( M_p : T(\mathbb{R}^m) \longrightarrow T(\mathbb{R}^d) \), which is an algebra homomorphism and that is unique in the following sense.

**Theorem 1.1.** There exists an algebra homomorphism \( M_p : (T(\mathbb{R}^m), \shuffle) \longrightarrow (T(\mathbb{R}^d), \shuffle) \) such that its restriction \( M_p|_{Im(\varphi_m)} \) is the unique algebra homomorphism that makes the
We look at 1.2 3.3 3.2 4.7 4

Given a piecewise continuously differentiable path \( X : [0, L] \rightarrow \mathbb{R}^d \) we define the map \( M_p \) and prove its properties in Proposition 3.2. Moreover, we present our main theorems, Theorems 1.1 and 1.2, together with a generalization of the last one, Corollary 3.3. In Section 3.1 we look at \( M_p \) as a half-shuffle homomorphism and give a generalization of Theorem 1.2 in terms of Zinbiel algebras. In Section 4, we also present two examples and a few consequences, Corollaries 4.4–4.7. Finally, Section 5 is dedicated to applications and future work.

2 Signatures of paths and words

Given a piecewise continuously differentiable path \( X : [0, L] \rightarrow \mathbb{R}^d \), for any \( i_1, \ldots, i_n \in \{1, 2, \ldots, d\} \) the following integral is classically well-defined

\[
\int_0^L dX^{i_1} \ldots dX^{i_n} := \int_0^L \int_0^{r_1} \cdots \int_0^{r_n} dX^{i_1}_{r_1} \ldots dX^{i_n}_{r_n} = \int_0^L \int_0^{r_1} \cdots \int_0^{r_n} \dot{X}^{i_1}_{r_1} \ldots \dot{X}^{i_n}_{r_n} dr_1 \ldots dr_n.
\]

We would like to store the collection of all these integrals.

**Definition 2.1.** The signature of \( X \) is defined as the following formal power series

\[
\sigma(X) = \sum_{n \geq 0} \sum_{i_1, \ldots, i_n} \int_0^L \int_0^{r_1} \cdots \int_0^{r_n} dX^{i_1}_{r_1} \ldots dX^{i_n}_{r_n} \cdot i_1 \cdots i_n \in T((\mathbb{R}^d)).
\]

As we mention in the introduction, \( T((\mathbb{R}^d)) \) is the space of formal power series in words in the alphabet \( \{1, \ldots, d\} \), and we denote by \( \emptyset \) the empty word. It is an algebra with the concatenation product, denoted by \( vv \) (or simply \( vw \)), which is well-defined since it respects the grading given by the number of letters appearing in each word. We also consider its algebraic dual \( T(\mathbb{R}^d) \), which is the set of polynomials in words in the same alphabet. The algebra \( T(\mathbb{R}^d) \) has the concatenation product, which is the same as for \( T((\mathbb{R}^d)) \) if we multiply two finite power series. However, we consider \( T(\mathbb{R}^d) \) as an algebra with the shuffle product, which we define recursively as follows.
Definition 2.2. Let \( w, w_1 \) and \( w_2 \) be three words and \( a \) and \( b \) two letters. We define the shuffle product of two words recursively by

\[
e \shuffle w = w \shuffle e = w, \quad \text{and} \quad (w_1 \cdot a) \shuffle (w_2 \cdot b) = (w_1 \shuffle (w_2 \cdot b)) \cdot a + ((w_1 \cdot a) \shuffle w_2) \cdot b.
\]

Note that in the shuffle product, we distinguish duplicated letters. For instance, for a letter \( a \), we have \( a \shuffle a = 2 \cdot a a \). Notice that the concatenation is a non-commutative operation, whereas the shuffle product is commutative.

We also need a few notions on words. The length of a word \( w \) is denoted by \( \ell(w) \) and counts the number of letters in \( w \). We extend this definition by linearity, defining \( \ell(w_1 + w_2) := \max(\ell(w_1), \ell(w_2)) \), for any words \( w_1 \) and \( w_2 \). Therefore, \( \ell(w_1 \cdot w_2) = \ell(w_1) + \ell(w_2) = \ell(w_1 \shuffle w_2) \). As an \( \mathbb{R} \)-vector space, \( T(\mathbb{R}^d) \) is graded by the length of the words:

\[
T(\mathbb{R}^d) = \bigoplus_{n \geq 0} T^n(\mathbb{R}^d),
\]

where \( T^n(\mathbb{R}^d) \) is the vector space spanned by the words of length \( n \). We also denote by \( T^{\leq n}(\mathbb{R}^d) \) the partial direct sum \( \bigoplus_{k \leq n} T^k(\mathbb{R}^d) \). This notation extends to \( T((\mathbb{R})^d) \). The same way, \( \sigma^{(n)}(X) \) denotes the partial sum of \( \sigma(X) \) for which all the appearing words have length exactly \( n \). We are ready to prove the following result.

Proposition 2.3. The shuffle product is associative.

Proof. The associativity is clear for the empty word since \( (e \shuffle e) \shuffle e = e = e \shuffle (e \shuffle e) \). Now, we proceed by induction. Assume that for any words \( w_1, v_1, \) and \( u_1 \) such that \( \ell(w_1) + \ell(v_1) + \ell(u_1) = n \), for some \( n \in \mathbb{N}_0 \), we have that \( (w_1 \shuffle v_1) \shuffle u_1 = w_1 \shuffle (v_1 \shuffle u_1) \). This is our inductive hypothesis.

Let \( w_2, v_2, u_2 \) be arbitrary words with the property that \( \ell(w_2) + \ell(v_2) + \ell(u_2) = n + 1 \). At least one of those words must thus be non-empty. If exactly two of the words are empty, both \( (w_2 \shuffle v_2) \shuffle u_2 \) and \( w_2 \shuffle (v_2 \shuffle u_2) \) are obviously equal to the non-empty word. If exactly one of the words is empty, both \( (w_2 \shuffle v_2) \shuffle u_2 \) and \( w_2 \shuffle (v_2 \shuffle u_2) \) are obviously equal to the shuffle product of the two non-empty words. In the remaining case, if \( w_2, v_2, u_2 \) are all non-empty, there are words \( w, v, u \) and letters \( i, j, k \) such that \( w_2 = wi, v_2 = vj \) and \( u_2 = uk \). Then,

\[
(w_2 \shuffle v_2) \shuffle u_2 = (wi \shuffle vj) \shuffle uk = ((w \shuffle vj) \cdot i + (wi \shuffle v) \cdot j) \shuffle uk
= ((w \shuffle vj) \shuffle uk) \cdot i + ((wi \shuffle v) \shuffle uk) \cdot j + ((wi \shuffle vj) \shuffle u) \cdot k
= ((w \shuffle vj) \shuffle uk) \cdot i + ((wi \shuffle v) \shuffle uk) \cdot j + ((wi \shuffle vj) \shuffle u) \cdot k
\]

Analogously,

\[
w_2 \shuffle (v_2 \shuffle u_2) = (w \shuffle (vj \shuffle uk)) \cdot i + (wi \shuffle (v \shuffle uk)) \cdot j + (wi \shuffle (vj \shuffle u)) \cdot k.
\]

Thus, since

\[
\ell(w) + \ell(vj) + \ell(uk) = \ell(wi) + \ell(v) + \ell(uk) = \ell(wi) + \ell(vj) + \ell(u) = n,
\]

we again get \( (w_2 \shuffle v_2) \shuffle u_2 = w_2 \shuffle (v_2 \shuffle u_2) \) due to the induction hypothesis. \qed
Going back to the signatures, the dual pairing (2) in $T((\mathbb{R}^d)) \times T(\mathbb{R}^d)$ allows us to extract the coefficient of a word in the signature of a path in the following way:

$$\langle \sigma(X), i_1i_2\ldots i_n \rangle = \int_0^L dX^{i_1} \ldots dX^{i_n}.$$

Both operations, the concatenation and the shuffle products, behave nicely with respect to the signature, as the following two known results describe. The first result, known as the shuffle identity, relates the signature of a path with the shuffle product.

**Proposition 2.4 (Shuffle identity, [Ree58]).** Let $X : [0, L] \rightarrow \mathbb{R}^d$ be a piecewise continuously differentiable path. Then, for every $u, v \in T(\mathbb{R}^d)$,

$$\langle \sigma(X), u \rangle \langle \sigma(X), v \rangle = \langle \sigma(X), u \shuffle v \rangle.$$

Another important result, known as Chen’s relation, describes the signature when we concatenate paths. Let us see how the concatenation path is defined.

**Definition 2.5.** Let $X, Y : [0, L] \rightarrow \mathbb{R}^d$ be two piecewise continuously differentiable paths. We define the concatenation of $X$ and $Y$ as the path $X \sqcup Y : [0, 2L] \rightarrow \mathbb{R}^d$ given by $X$ on $[0, L]$ and by $Y_{2L} - Y_0 + X_L$ on $[L, 2L]$ (i.e. take $Y$, move it back to 0 and then move it to the end of $X$).

The concatenation product interplays nicely with the concatenation of paths, as the following proposition shows.

**Proposition 2.6 (Chen’s identity, [Che57]).** Let $X, Y : [0, L] \rightarrow \mathbb{R}^d$ be two piecewise continuously differentiable paths and consider their concatenation $X \sqcup Y : [0, 2L] \rightarrow \mathbb{R}^d$. Then,

$$\sigma(X \sqcup Y) = \sigma(X) \cdot \sigma(Y).$$

We finish this section with an example on how to compute the first terms of the signature of a path.

**Example 2.7.** Consider the path $X : [0, 1] \rightarrow \mathbb{R}^2$ given by $X^1(t) = t$ and $X^2(t) = t^2$. We compute a few terms of its signature.

$$\langle \sigma(X), 1 \rangle = \int_0^1 dX^1_{r_1} = \int_0^1 1 dt = 1 \quad \langle \sigma(X), 2 \rangle = \int_0^1 dX^2_{r_1} = \int_0^1 2 t dt = 1$$

$$\langle \sigma(X), 11 \rangle = \int_0^1 \int_0^{r_2} dX^1_{r_1} dX^1_{r_2} = \int_0^1 r_2 dX^1_{r_2} = \int_0^1 r_2dr = \frac{1}{2}$$

$$\langle \sigma(X), 12 \rangle = \int_0^1 \int_0^{r_2} dX^1_{r_1} dX^2_{r_2} = \int_0^1 r_2 dX^2_{r_2} = \int_0^1 2r^2 dr = \frac{2}{3}$$

$$\langle \sigma(X), 21 \rangle = \int_0^1 \int_0^{r_2} dX^2_{r_1} dX^1_{r_2} = \int_0^1 r_2 dX^1_{r_2} = \int_0^1 r_2 dr = \frac{1}{3}$$

$$\langle \sigma(X), 22 \rangle = \int_0^1 \int_0^{r_2} dX^2_{r_1} dX^2_{r_2} = \int_0^1 r_2^2 dX^2_{r_2} = \int_0^1 2r_2^3 dr = \frac{2}{4} = \frac{1}{2}$$

$$\langle \sigma(X), 222 \rangle = \int_0^1 \int_0^{r_2} \int_0^{r_2} dX^2_{r_1} dX^2_{r_2} dX^2_{r_3} = \int_0^1 \int_0^{r_3} r_2^2 dX^2_{r_2} dX^2_{r_3} = \int_0^1 \int_0^{r_3} r_2^3 dr = \frac{1}{6}$$

Therefore, the signature of $X$ is of the form

$$\sigma(X) = 1 + 2 + \frac{1}{2} \cdot (11 + 22) + \frac{1}{3} \cdot (2 \cdot 12 + 21) + \frac{1}{6} \cdot 222 + \ldots$$
3 Signatures under the action of polynomial maps

Let \( p : \mathbb{R}^d \rightarrow \mathbb{R}^m \) be a polynomial map given by the polynomials \( p_i(x_1, x_2, \ldots, x_d) \), for \( i = 1, 2, \ldots, m \), with the property that \( p(0) = 0 \). The degree of the polynomial map \( p \) is the maximum of the degree of the polynomials that define it, \( \deg(p) = \max_i \deg(p_i) \). Moreover, we say that a polynomial map is homogeneous if the polynomials \( p_i \) are homogeneous of the same degree. Finally, we denote by \( J_p \) the Jacobian matrix of format \( m \times d \) with entries \( J_p^{ij} = \partial_i p_j \), for \( i \in \{1, 2, \ldots, m\} \) and \( j \in \{1, 2, \ldots, d\} \).

Recall the algebra homomorphism \( \varphi_d \) defined by:
\[
\varphi_d : \mathbb{R}[x_1, x_2, \ldots, x_d] \rightarrow T(\mathbb{R}^d, \shuffle)
\]
\[
x_i \mapsto i
\]
\[
x_i \cdots x_i \mapsto i_1 \shuffle \cdots \shuffle i_l
\]

As we mention above, by the properties of the shuffle product in \( T(\mathbb{R}^d) \), this map is injective but is not surjective.

We now consider maps \( M_p, \varphi, \) and \( p^* \) such that they complete the diagram (3) that arises in our main question in the following way:

\[
\begin{array}{ccc}
\mathbb{R}[x_1, \ldots, x_m] & \xrightarrow{p^*} & \mathbb{R}[x_1, \ldots, x_d] \\
\varphi_m & \downarrow & \varphi_d \\
\left(T(\mathbb{R}^m, \shuffle)\right) & \xrightarrow{M_p} & \left(T(\mathbb{R}^d, \shuffle)\right)
\end{array}
\]

Notice that the map \( M_p \) is unique when restricted to the image of \( \varphi_m \), but not the full \( M_p \) on the tensor algebra \( T(\mathbb{R}^m) \).

**Definition 3.1.** For any polynomial map \( p : \mathbb{R}^d \rightarrow \mathbb{R}^m \) such that \( p(0) = 0 \), let \( k_p^{ij} = \varphi_d(J_p^{ij}) \in T(\mathbb{R}^d) \), where \( J_p^{ij} \) is the \((i, j)\)-entry of the Jacobian matrix of \( p \). We define the map \( M_p : T(\mathbb{R}^m) \rightarrow T(\mathbb{R}^d) \) recursively as follows:

\[
M_p(\emptyset) = \emptyset, \text{ for } \emptyset \text{ the empty word, and }
\]
\[
M_p(wi) = \sum_{j=1}^d \left(M_p(w) \shuffle k_p^{ij}\right) \bullet j, \text{ for any word } w \text{ and any letter } i \in \{1, \ldots, m\}.
\]

The following result summarizes a few properties of the map \( M_p \). We will use these properties to show that the map that \( M_p \) as we construct it restricts according to what we need.

**Proposition 3.2.** Consider two polynomial maps \( p : \mathbb{R}^d \rightarrow \mathbb{R}^m \) and \( q : \mathbb{R}^m \rightarrow \mathbb{R}^s \), with \( p(0) = 0 \) and \( q(0) = 0 \), and the algebra homomorphisms \( \varphi_m \) and \( \varphi_d \). Then, we have the following list of properties:

(I) \( M_p : (T(\mathbb{R}^m), \shuffle) \rightarrow (T(\mathbb{R}^d), \shuffle) \) is an algebra homomorphism.

(II) For \( i = 1, \ldots, s \), \( M_p(\varphi_m(q_i)) = \varphi_d(q_i \circ p) \), where the \( q_i \)'s are the polynomials defining the polynomial map \( q \).

(III) \( k_{q \circ p}^{ij} = \sum_{l=1}^m M_p(k_q^{il}) \shuffle k_p^{lj} \).
(IV) \( M_{qop} = M_p M_q \).

(V) If \( p \) is a polynomial map of degree \( n \), then \( M_p \left( T^k(\mathbb{R}^m) \right) \subseteq \mathcal{T}^{SN_k}(\mathbb{R}^d) \).

(VI) If \( p \) is an homogeneous polynomial map of degree \( n \), \( M_p \left( T^k(\mathbb{R}^m) \right) \subseteq \mathcal{T}^{nk}(\mathbb{R}^d) \).

Proof. (I) We need to show that, for any words \( w_1 \) and \( w_2 \) in \( T(\mathbb{R}^m) \),
\[
M_p(w_1 \uplus w_2) = M_p(w_1) \uplus M_p(w_2).
\]
We proceed by induction on \( \ell(w_1) + \ell(w_2) \). For \( \ell(w_1) + \ell(w_2) \leq 1 \), at least one of the two words is the empty word \( e \), and so we assume that \( w_2 = e \). Therefore,
\[
M_p(w_1 \uplus w_2) = M_p(w_1) = M_p(w_1) \uplus M_p(w_2)
\]
Assume now that the statement is true for any pair of words with sum of lengths at most \( n - 1 \). Let \( u \) and \( v \) be two words such that \( \ell(u) + \ell(v) = n - 1 \) and \( a \) and \( b \) two arbitrary letters. Then, using Definitions 2.2 and 3.1, and the inductive hypothesis (IH),
\[
M_p(ua \uplus vb) \overset{\text{Def. } 2.2} = M_p(\left(u \uplus vb\right) \bullet a + (ua \uplus v) \bullet b) \overset{\text{Def. } 3.1} = \sum_{i=1}^{d} \left[ M_p(u \uplus vb) \uplus k_p^{ai} + M_p(ua \uplus v) \uplus k_p^{bi} \right] \bullet i
\]
We proceed by induction on \( p \).

(II) Assume that \( M_p(1) = \varphi_d(p_1) \), for all \( i \). Then, for the monomial \( h(x_1, \ldots, x_m) = x_1^{n_1} \cdots x_d^{n_d} \cdot x_m^{n_m} \), \( \varphi_m(h) = 1^{\mathbb{R}^m} \uplus \cdots \uplus m^{\mathbb{R}^m} \). Therefore, by the property (I),
\[
M_p(\varphi_m(h)) = M_p(1)^{\uplus 1} \uplus \cdots \uplus M_p(m)^{\uplus m} = \varphi_d(p_1)^{\uplus 1} \uplus \cdots \uplus \varphi_d(p_m)^{\uplus m} = \varphi_d(p_1^{n_1} \cdots p_m^{n_m}) = \varphi_d(h \circ p),
\]
and the property (II) follows by linearity.

We prove now the claim \( M_p(1) = \varphi_d(p_i) \), for all \( i \). Since the two maps \( p \mapsto M_p(1) \) and \( p \mapsto \varphi_d(p_i) \) are linear, it is enough to prove the claim for the case when \( p_i \) is a monomial of the form \( p_i = x_1^{n_1} \cdots x_d^{n_d} \), with at least one of the \( n_i \)'s non-zero. In this case,
\[
M_p(1) = \sum_{j=1}^{d} (M_p(1) \uplus k_p^{ij}) \bullet j = \sum_{j=1}^{d} k_p^{ij} \bullet j = \sum_{j=1}^{d} n_j (i^{\mathbb{R}^m(n_1-\delta_j)} \uplus \cdots \uplus a^{\mathbb{R}^m(n_d-\delta_d)}) \bullet j = i^{\mathbb{R}^m} \uplus \cdots \uplus d^{\mathbb{R}^m} = \varphi_d(p_i),
\]
where \( \ast \) follows by applying enough iterations of the recursive definition of the shuffle product and the fact that \( i^{\mathbb{R}^m+1} = (n+1)i^{\mathbb{R}^m} \bullet i \).
(III) By the chain rule, \( J^i_{q} p = \sum_{i=1}^{m} (J^i_{q} \circ p) \cdot J^j_{p} \). For one term of that sum, by (II),
\[
\phi_d(J^i_{q} \circ p) = M_p(\phi_m(J^i_{q}))) = M_p(k^i_{q}) .
\]
Thus,
\[
k^i_{q} = \phi_d(J^i_{q} \circ p) = \phi_d \left( \sum_{i=1}^{m} (J^i_{q} \circ p) \cdot J^j_{p} \right) = \sum_{i=1}^{m} \phi_d(J^i_{q} \circ p) = \sum_{i=1}^{m} M_p(k^i_{q}) \cdot k^j_{p}.
\]

(IV) We proceed by induction on \( \ell(w) \). For a letter \( i \),
\[
M_p \circ M_q(i) = M_p(M_q(i)) = M_p \left( \sum_{j=1}^{m} M_q(\phi_i) \cdot j \right) = \sum_{j=1}^{m} \left( \sum_{i=1}^{d} M_p(k^i_{q} \cdot k^j_{p}) \right) \cdot 1
\]

Now, we assume that the statement is true for all the words of length at most \( n \) and we refer to it as (IH). Let \( w \) be one of these words and \( i \) any letter. Then,
\[
M_p \circ M_q(wi) = M_p \left( M_q(wi) \right) = M_p \left( \sum_{j=1}^{m} M_q(w) \cdot k^j_{q} \right) \cdot j
\]
\[
= \sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j
\]
\[
\sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j = \sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j = \sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j
\]
\[
\sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j = \sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j = \sum_{j=1}^{m} M_p \left( M_q(w) \right) \cdot k^j_{q} \cdot j
\]

(V) We start by noticing that since the polynomial map \( p \) has degree \( n \), then \( \deg(p_i) \leq n \), for all \( i \). Thus, \( \deg(J^i_{q}) \leq n - 1 \) and \( \ell(J^i_{q} \circ J^j_{p}) \leq n - 1 \), for all \( i \) and \( j \).

Now, we proceed by induction on \( k \). For \( k = 1 \), \( T^1(\mathbb{R}^m) \) is the set of letters \{1, \ldots, m\}.

Since for a letter \( i \) in this set \( M_p(i) = \sum_{j=1}^{m} k^j_{p} \cdot j \), then \( \ell(M_p(i)) = \max \{ \ell(k^j_{p} \cdot j) \} \leq n \).

Thus, \( M_p(i) \in T^{n+1}(\mathbb{R}^d) \).

Assume that the statement is true for \( k \). Any word \( w' \in T^{k+1}(\mathbb{R}^m) \) can be written as \( w' = w \circ i \), with \( w \in T^k(\mathbb{R}^m) \) and \( i \) a letter. We analyze the length of \( M_p(w \circ i) \).

Since \( M_p(w \circ i) = \sum_{j=1}^{m} (M_p(w) \cdot k^j_{p}) \cdot j \), it is enough to upper bound the length of the terms appearing in the sum. By the inductive hypothesis, \( \ell(M_p(w)) \leq nk \), and since \( \ell(k^j_{p}) \leq n - 1 \), \( \ell(M_p(w) \cdot k^j_{p}) \leq nk + n - 1 \). Therefore, \( \ell(M_p(w \circ i)) \leq nk + n - 1 + 1 = n(k + 1) \).

(VI) In this case, since \( p \) is homogeneous of degree \( n \), then \( \deg(p_i) = n \), for all \( i \). Moreover, \( \deg(J^i_{q}) = n - 1 \), if the variable \( x_j \) appears in \( p_i \), or zero, otherwise.
We proceed by induction on \( k \). For \( k = 1 \), let \( \mathbf{i} \) be a letter in \( \{1, \ldots, m\} \). Then, \( M_p(\mathbf{i}) = \sum_{j=1}^d k_p^{ij} \cdot j \). This sum contains only terms \( k_p^{ij} \cdot j \), which has length exactly \( n \), otherwise \( k_p^{ij} \) is zero according to our observation about the Jacobian entries above. Therefore, the statement follows.

Now, assume the statement is true for \( k \). Let \( \mathbf{w} \in T^k(\mathbb{R}^m) \) be a word and \( \mathbf{i} \) a letter. In this case, \( M_p(\mathbf{w} \cdot \mathbf{i}) = \sum_{j=1}^d \left( M_p(\mathbf{w}) \sqcup k_p^{ij}\right) \cdot j \). Again, the terms appearing in this sum have length \( nk + n - 1 + 1 = n(k + 1) \), which concludes the proof.

\[ \square \]

Once we have these properties, we recall Theorem 1.1 and prove it.

**Theorem 1.1.** There exists an algebra homomorphism \( M_p : (T(\mathbb{R}^m), \sqcup) \to (T(\mathbb{R}^d), \sqcup) \) such that its restriction \( \left. M_p \right|_{\text{Im}(\varphi_m)} \) is the unique algebra homomorphism that makes the following diagram commute:

\[
\begin{array}{ccc}
\mathbb{R}[x_1, \ldots, x_m] & \xrightarrow{p^*} & \mathbb{R}[x_1, \ldots, x_d] \\
\varphi_m \downarrow & & \varphi_d \\
\text{Im}(\varphi_m) & \xrightarrow{\exists! M_p|_{\text{Im}(\varphi_m)}} & \text{Im}(\varphi_d)
\end{array}
\]

**Proof.** By (I) in Proposition 3.2, the restriction of \( M_p \) to the image \( \text{Im}(\varphi_m) \) is an algebra homomorphism. Moreover, due to (II), we have that \( M_p(\text{Im}(\varphi_m)) \subseteq \text{Im}(\varphi_d) \). Since we restrict to their images, \( \varphi_m \) and \( \varphi_d \) are isomorphisms and the map \( M_p \) is the unique one making the diagram commute. \( \square \)

Let us see now the answer to our main question, which is stated as Theorem 1.2

**Theorem 1.2.** Let \( X : [0, L] \to \mathbb{R}^d \) be a piecewise continuously differentiable path with \( X_0 = 0 \) and let \( p : \mathbb{R}^d \to \mathbb{R}^m \) be a polynomial map with \( p(0) = 0 \). Then, for all \( \mathbf{w} \in T(\mathbb{R}^m) \),

\[ \langle \sigma(p(X)), \mathbf{w} \rangle = \langle \sigma(X), M_p(\mathbf{w}) \rangle. \]

Equivalently, \( \sigma(p(X)) = M_p^* (\sigma(X)) \).

**Proof.** Denote by \( Y = p(X) \) and by \( \dot{Y} = \sum_{j=1}^d J_p^{ij}(X) \cdot \dot{X}^j \). Notice that each component \( X^j \) of the path equals the first signature component, \( \sigma^1(X) = X \), and therefore, the entries of the Jacobian matrix can be seen as coefficients of the signature of \( X \),

\[ J_p^{ij}(X) = \langle \sigma \left( X \right|_{[0,t]} \right), k_p^{ij} \rangle. \tag{4} \]

We proceed by induction on the length of the word \( \mathbf{w} \). For a letter \( \mathbf{i} \), we have that

\[ \langle \sigma(Y), \mathbf{i} \rangle = \int_0^L \langle \sigma \left( Y \right|_{[0,t]} \right), \mathbf{e} \rangle d\dot{Y}_t^i = \int_0^L d\dot{Y}_t^i = \sum_{j=1}^d \int_0^L J_p^{ij}(X) \dot{X}_t^j dt = \sum_{j=1}^d \int_0^L J_p^{ij}(X_t) dX_t^j \tag{4} \]

\[ = \sum_{j=1}^d \int_0^L \langle \sigma \left( X \right|_{[0,t]} \right), k_p^{ij} \rangle dX_t^j = \sum_{j=1}^d \langle \sigma(X), k_p^{ij} \cdot j \rangle = \langle \sigma(X), M_p(\mathbf{i}) \rangle. \]
Now, assume that the statement is true for all the words of length at most \( n \). Let \( w \) be any of these words and \( i \) any letter. By the definition of the signature,

\[
\langle \sigma(Y), w \rangle = \int_0^L \left\langle \sigma(Y|_{[0,t]}), w \right\rangle dY_t^i = \int_0^L \left\langle \sigma(Y|_{[0,t]}), w \right\rangle \dot{Y}_t^i dt = \\
\sum_{j=1}^d \int_0^L \left\langle \sigma(Y|_{[0,t]}), w \right\rangle J_p^{ij}(X_t) \dot{X}_t^j dt = \sum_{j=1}^d \int_0^L \left\langle \sigma(Y|_{[0,t]}), w \right\rangle J_p^{ij}(X_t) dX_t^j = \sum_{j=1}^d \int_0^L \left\langle \sigma(Y|_{[0,t]}), w \right\rangle \sigma(X|_{[0,t]}), k_p^{ij} \right\rangle dX_t^j.
\]

Now, apply the inductive hypothesis to \( \left\langle \sigma(Y|_{[0,t]}), w \right\rangle \) in (5), and then by Chen’s identity, Proposition 2.6,

\[
\langle \sigma(Y), w \rangle = \sum_{j=1}^d \int_0^L \left\langle \sigma(X|_{[0,t]}), M_p(w) \right\rangle \left\langle \sigma(X|_{[0,t]}), k_p^{ij} \right\rangle dX_t^j = \\
\sum_{j=1}^d \int_0^L \left\langle \sigma(X|_{[0,t]}), M_p(w) \right\rangle \left\langle \sigma(X|_{[0,t]}), k_p^{ij} \right\rangle dX_t^j = \sum_{j=1}^d \left\langle \sigma(X^i), (M_p(w) \cup k_p^{ij}) \cdot j \right\rangle = \langle \sigma(X), M_p(w) \rangle.
\]

\[\square\]

We finish this section with a generalization of Theorem 1.2 to polynomial maps that do not satisfy the condition \( p(0) = 0 \) and paths that do not start at the origin.

**Corollary 3.3.** Let \( X : [0, L] \rightarrow \mathbb{R}^d \) be a piecewise continuously differentiable path and let \( p : \mathbb{R}^d \rightarrow \mathbb{R}^m \) be a polynomial map. Consider the map \( \tilde{p} \) given by \( \tilde{p}(y) = p(y + X_0) - p(X_0) \). Then, for all \( w \in T(\mathbb{R}^m) \),

\[
\langle \sigma(p(X)), w \rangle = \langle \sigma(X), M_p(w) \rangle.
\]

**Proof.** The statement follows using the same argument as in the proof of Theorem 1.2 if we take into account that in this case, at the end of the step (4),

\[
J_p^{ij}(X_t) \dot{X}_t^j dt = J_p^{ij}(X_t - X_0) dX_t^j = \left\langle \sigma(X|_{[0,t]}), k_p^{ij} \right\rangle.
\]

\[\square\]

### 3.1 \( M_p \) as a half-shuffle homomorphism

The shuffle product can be seen as the symmetrization of the right half-shuffle, which we define in the following way. Let \( T^{\geq 1}(\mathbb{R}^d) = \bigoplus_{n \geq 1} T^n(\mathbb{R}^d) \) denote the vector space spanned by the non-empty words built from \( d \) letters.

**Definition 3.4.** [Sch58, Eq. (S2)], [FP13, Def. 1], [EM53, Sect. 18] The right half-shuffle \( > : T^{\geq 1}(\mathbb{R}^d) \times T^{\geq 1}(\mathbb{R}^d) \rightarrow T^{\geq 1}(\mathbb{R}^d) \) is recursively given on words as

\[
w > i := wi, \quad w > vi := (w > v + v > w) \cdot i,
\]

where \( w, v \) are words and \( i \) is a letter.
Therefore, for any non-empty words \( w, v \)

\[
    w \shuffle v = w \succ v + v \succ w.
\]

Indeed, for non-empty words \( w, v \) and letters \( i, j \), we have

\[
    i \succ j + j \succ i = ij + ji, \\
    wi \succ j + j \succ wi = wij + (j \succ w + w \succ j) \cdot i, \quad \text{and} \\
    wi \succ vj + vj \succ wi = (wi \succ v + v \succ wi) \cdot j + (vj \succ w + w \succ vj) \cdot i,
\]

in accordance with Definition 2.2. Thus, the second equation in Definition 3.4 can be rewritten as

\[
    w \succ vi = (w \shuffle v) \cdot i. \quad (6)
\]

It turns out that the right half-shuffle is an example of a more general type of algebras, the Zinbiel algebras.

**Definition 3.5.** [Sch58, Eq. (S0)], [FP13, Eq. (4)] A (right) Zinbiel algebra is a vector space \( Z \) together with a bilinear map \( \succ : Z \times Z \to Z \) such that, for all \( a, b, c \in Z \),

\[
    a \succ (b \succ c) = (a \succ b \succ a) \succ c.
\]

We include here the proof of the next result since it is interesting.

**Theorem 3.6.** [Lod95, Proposition 1.8] \((T^{\geq 1}(\mathbb{R}^d), \succ)\) is a Zinbiel algebra, i.e. for any non-empty words \( w, v, \) and \( u \),

\[
    w \succ (v \succ u) = (w \shuffle v) \succ u. \quad (7)
\]

**Proof.** Let \( w, v, \) and \( u \) be non-empty words and \( i \) be an arbitrary letter. By the definition of the half-shuffle and Equation (6), we have

\[
    w \succ (v \succ i) = w \succ vi = (w \shuffle v) \cdot i = (w \shuffle v) \succ i.
\]

Using Equation (6) and associativity of the shuffle product, Proposition 2.3, we obtain that

\[
    w \succ (v \succ ui) = w \succ ((v \shuffle u) \cdot i) = (w \shuffle (v \shuffle u)) \cdot i = ((w \shuffle v) \shuffle u) \cdot i = (w \shuffle v) \succ ui.
\]

\(\square\)

In fact, it is known that \((T^{\geq 1}(\mathbb{R}^d), \succ)\) is free in this case.

**Theorem 3.7.** [Sch58, page 19]/[Lod95, Proposition 1.8] Indeed, \((T^{\geq 1}(\mathbb{R}^d), \succ)\) is the free Zinbiel algebra over \( \mathbb{R}^d \).

This means that for any Zinbiel algebra \((Z, \succ)\) and any linear map \( B : \mathbb{R}^d \to Z \), there is a unique homomorphism \( \Lambda_B : (T^{\geq 1}(\mathbb{R}^d), \succ) \to (Z, \succ) \) such that \( B = \Lambda_B \circ \iota \), where \( \iota : \mathbb{R}^d \to T^{\geq 1}(\mathbb{R}^d) \) is the canonical embedding. This is known as the universal property of the free Zinbiel algebra and is described in Diagram 3.1. We call \( \Lambda_B \) the unique extension of \( B \) to a Zinbiel homomorphism.
Diagram 3.1: Universal property of the free Zinbiel algebra

The following result describes the relation between the map \( M_p \) and the half-shuffle.

**Theorem 3.8.** The restriction of \( M_p \) to \( T^{z_1}(\mathbb{R}^d) \), \( M_p|_{T^{z_1}(\mathbb{R}^d)} \), is the unique half-shuffle homomorphism such that \( M_p(i) = \varphi_d(p_i) \).

**Proof.** Using (6) and the definition of \( M_p \), we get

\[
M_p(u) = \sum_{j=1}^{d} \left( M_p(u) \cdot k_{i}^{j} \right) \cdot j = M_p(u) \cdot \left( \sum_{j=1}^{d} k_{i}^{j} \cdot j \right) = M_p(u) \cdot M_p(i),
\]

and thus the statement follows immediately from the proof of Theorem 3.7.

We finish this section with a generalization of Theorem 1.2 in terms of Zinbiel algebras stated in the following result.

**Theorem 3.9.** Let \( X : [0, L] \to \mathbb{R}^d \) be a piecewise continuously differentiable path and \( B : \mathbb{R}^m \to (T^{z_1}(\mathbb{R}^d), \cdot ) \) be a linear map. Then, the signature of the path

\[
Y : [0, L] \to \mathbb{R}^m, \quad Y^i = \langle \sigma (X|_{[0,t]}), B_i \rangle,
\]

is a linear transformation of the signature of \( X \), namely

\[
\langle \sigma (Y), u \rangle = \langle \sigma (X), \Lambda_B u \rangle,
\]

where \( \Lambda_B \) is the unique extension of \( B \) to a Zinbiel homomorphism.
Before proving this result, we introduce some notation. We denote by \( X^z_{0s} \) the coefficient of \( z \) in the signature of the path \( X \) restricted to the interval \([0, s]\). That is,
\[
X^z_{0s} := \langle \sigma(X I_{[0,s]}), z \rangle.
\]
Note that \( X^i_{0s} = X^i_s - X^i_0 \). Then, the path \( Y \) introduced in Theorem 3.9 is given by \( Y^i_s = X^{B1}_{0s} \). Moreover, for any letter \( i \), we define the maps \( T_i^- : T(\mathbb{R}^d) \to T(\mathbb{R}^d) \) to be the unique linear maps given recursively by \( T_i^- w = \delta_{ij} w \) and \( T_i^- w_j = \delta_{ij} w \) with \( T_i^- e = 0 \), respectively, for any word \( w \).

These two maps allows us to define the right half-shuffle as shown in the following technical result.

**Lemma 3.10.** For any \( x, y \in T \mathbb{R}^d \), we have \( x \ast y = \sum_{i=1}^d T_i^+(x \shuffle i y) \).

**Proof.** This is just a reformulation of (6) in the following way. For any word \( v \), any non-empty word \( w \), and any letter \( j \), we have that
\[
\sum_{i=1}^d T_i^+(w \shuffle i v j) = T_j^+(w \shuffle v) = (w \shuffle v) \cdot j = w \ast v,
\]
where the last equality follows from (6). Then, the general statement for any \( x, y \in T \mathbb{R}^d \) follows from (bi)linearity.

Now we are ready to prove Theorem 3.9.

**Proof of Theorem 3.9.** For better readability, we put \( \Lambda := \Lambda_B \). First note that by the definition of the signature and the fact that \( X \) is continuously differentiable almost everywhere, we have
\[
Y^i_s = X^{B1}_{0s} = X^{A1}_{0s} \sum_{i=1}^d \int_0^s X_{0t}^{T_i^- \Lambda_i} \, dt = \sum_{i=1}^d \int_0^s X_{0t}^{T_i^- \Lambda_i} X^i_t \, dt
\]
and thus, for almost all \( s \in [0, T] \),
\[
Y^i_s = \sum_{i=1}^d X_{0t}^{T_i^- \Lambda_i} X^i_t.
\]
Following an inductive argument, assume now that \( X^{\Lambda w}_{0s} = Y^w_{0s} \) holds for some word \( w \). Then,
\[
\begin{align*}
Y^w_{0s} &= \int_0^s Y^w_{0t} dY^i_t = \left. \int_0^s Y^w_{0t} dX^i_t \right|_{t=0}^{t=s} = \sum_{i=1}^d \int_0^s X_{0t}^{\Lambda w} X_{0t}^{T_i^- \Lambda_i} X^i_t \, dt \\
&= \int_0^s X_{0t}^{\sum_{i=1}^d \Lambda w \shuffle i \Lambda_i} \, dt = X_{0t}^{\sum_{i=1}^d \Lambda w \shuffle i \Lambda_i} = X_{0t}^{\Lambda w} = X^{\Lambda w}_{0t},
\end{align*}
\]
were we used Lemma 3.10 and the fact that \( \Lambda \) is a homomorphism of Zinbiel algebras.

Theorem 3.9 can also be directly shown using
\[
\int_0^s X^x_{0t} dX^y_{0t} = X^{x \ast y}_{0s} \tag{8}
\]
for any \( x, y \in T \mathbb{R}^d \), a relation which is quite fundamental for an algebraic understanding of the signature and was mentioned already for example in [GK08] right after equation (6). Conversely, starting from Theorem 3.9, equation (8) is immediate with the choice \( B1 = x, B2 = y \).

The following example illustrates the results presented in this section.
Example 3.11. For a given path $X : [0, L] \rightarrow \mathbb{R}^3$, let

$$Y = \left( \text{Area}(X^2, X^3), \text{Area}(X^3, X^1), \text{Area}(X^1, X^2) \right)$$

denote the area path of $X$, where $\text{Area}(X^i, X^j)_t = \int_0^t \int_0^s dX^i_u dX^j_u - \int_0^t \int_0^s dX^i_v dX^j_v$. Let us compute $\langle \sigma(Y), 12 - 21 \rangle$ in the special case that $X : [0, 1] \rightarrow \mathbb{R}^3$, $X(t) = (t, t^2, t^3)$. We have $Y^i_s = X^i_{0s}$ where

$$B_1 = 23 - 32, \quad B_2 = 31 - 13, B_3 = 12 - 21$$

We need to compute $\Lambda_B(12 - 21) = B_1 > B_2 > B_2 > B_1$. To this end,

$$(23 - 32) > (31 - 13) = 2 \cdot 2331 - 2 \cdot 3321 - 2313 - 2133 - 1233 + 3213 + 3123 + 1323,$$

$$(31 - 13) > (23 - 32) = 3123 + 3213 + 2313 - 1323 - 2133 - 1233 - 3123 - 3122 - 2 \cdot 3312 + 2 \cdot 1332 + 3132.$$

Thus, $\Lambda_B(12 - 21) = 2 \cdot (-1323 + 3321 + 2313 - 2331 + 3312 - 3321)$. Since in our special case of $X(t) = (t, t^2, t^3)$ it holds that [AFS19, Example 2.2]

$$\int_0^1 \int_0^t \int_0^s dX^i_u dX^j_u dX^k_u dX^l_u = \frac{j \cdot k \cdot l}{(j + i)(k + j + i)(i + k + j + i)},$$

we get that

$$\langle \sigma(Y), 12 - 21 \rangle$$

$$= 2 \cdot \left( \frac{3 \cdot 2 \cdot 3}{4 \cdot 6 \cdot 9} + \frac{3 \cdot 3 \cdot 2}{5 \cdot 6 \cdot 9} - \frac{3 \cdot 1 \cdot 3 - 3 \cdot 3 \cdot 1}{5 \cdot 8 \cdot 9} + \frac{3 \cdot 1 \cdot 2}{6 \cdot 7 \cdot 9} - \frac{3 \cdot 2 \cdot 1}{6 \cdot 8 \cdot 9} \right)$$

$$= -\frac{1}{315} \approx -0.00317$$

as the desired result.

4 Examples and consequences

Let us start with an easy example to illustrate how Theorem 1.2 works, and also the property (V) in Proposition 3.2.

Example 4.1. Consider the polynomial map $p : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by the polynomials $p_1 = x^2$, $p_2 = y^3$ and $p_3 = x - y$. Consider also the path in Example 2.7, $X : [0, 1] \rightarrow \mathbb{R}^2$ given by $X^1(t) = t$ and $X^2(t) = t^2$. We want to compute a few terms in the signature of the path $p(X)$.

We start computing the Jacobian matrix and its image under $\varphi$:

$$(J^i_{j p})_{i,j} = \begin{pmatrix} 2x & 0 \\ 0 & 3y^2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad (k^i_{j p})_{i,j} = (\varphi(J^i_{j p}))_{i,j} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 6 & 22 \\ e & -e \end{pmatrix}$$

Notice that $\varphi(3y^2) = 3 \cdot 2 \parallel 2 = 6 \cdot 22$. We use Definition 3.1 to compute the image of a few words:

$$M_p(1) = 2 \cdot 11, \quad M_p(2) = 6 \cdot 222, \quad M_p(3) = 1 - 2,$$

$$M_p(33) = (1 - 2) \cdot 1 - (1 - 2) \cdot 2 = 11 + 22 - 12 - 21.$$
We observe that for any word \( w \) above, \( \ell(M_p(w)) \leq 3 \cdot \ell(w) \). This is due to property (V) in Proposition 3.2 since \( \deg(p) = 3 \). Now, applying Theorem 1.2 and looking at the signature terms computed in Example 2.7, we obtain that
\[
\langle \sigma(p(X)), 1 \rangle = \langle \sigma(X), M_p(1) \rangle = \langle \sigma(X), 2 \cdot 11 \rangle = 2 \cdot \frac{1}{2} = 1,
\]
\[
\langle \sigma(p(X)), 2 \rangle = \langle \sigma(X), M_p(2) \rangle = \langle \sigma(X), 6 \cdot 222 \rangle = 6 \cdot \frac{1}{6} = 1,
\]
\[
\langle \sigma(p(X)), 3 \rangle = \langle \sigma(X), M_p(3) \rangle = \langle \sigma(X), 1 - 2 \rangle = 1 - 1 = 0, \text{ and}
\]
\[
\langle \sigma(p(X)), 33 \rangle = \langle \sigma(X), M_p(33) \rangle = \langle \sigma(X), 11 + 22 - 12 - 21 \rangle = 0.
\]

This second example is more generic and shows the property (VI) in Proposition 3.2.

**Example 4.2.** Let \( X \) be any piecewise continuously differentiable path in \( \mathbb{R}^2 \). Consider the polynomial map \( p : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) given by \( p(x, y) = (x^2, xy, y^2) \), and fix \( k = 2 \).

By Theorem 1.2, the coefficient of \( w \) in \( \sigma(p(X)) \) is given by the coefficient of \( M_p(w) \) in \( \sigma(X) \). Moreover, by Proposition 3.2 (VI), the words appearing in \( M_p(w) \) have length exactly \( \ell(w) \). One way of storing \( \sigma^{(2)}(p(X)) \) is using a matrix that encodes the coefficients in \( M_p(w) \).

\[
\sigma^{(2)}(p(X)) = \begin{bmatrix}
11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

More generally, by Proposition 3.2 (VI), for any word \( w \) in the alphabet \( \{1, 2, 3\} \) with \( \ell(w) = k \), \( M_p(w) \) is a sum of words in the alphabet \( \{1, 2\} \) of length \( 2k \). Applying Theorem 1.2, the information of \( \sigma^{(k)}(p(X)) \) can be stored in terms of \( \sigma^{(2k)}(X) \). In fact, there exists a matrix \( M \) of format \( 3^k \times 2^{2k} \) that describes the change of coordinates in the following sense. Fix an order on the words, for instance, lexicographic order as above.
The coefficients of $\sigma^{(k)}(p(X))$ and the coefficients of $\sigma^{(2k)}(X)$ are related as

$$
\sigma^{(k)}(p(X)) = \begin{bmatrix} 11 \ldots 1 \\ 1 \ldots 12 \\ \vdots \\ 3 \ldots 32 \\ 3 \ldots 33 \\ k \end{bmatrix} = M \begin{bmatrix} 11 \ldots 1 \\ 1 \ldots 12 \\ \vdots \\ 2 \ldots 21 \\ 2 \ldots 22 \\ 2k \end{bmatrix},
$$

where the row indexed by $w$ in $M$ is given by the coefficients of $M_p(w)$. Moreover, we want to point out the following properties of this homomorphism $M_p$:

- For $w = 1 \ldots 1$, $M_p(w) = \frac{(2k)!}{k!} \cdot w \cdot w$.
- For $w = 2 \ldots 2$, $M_p(w) = k! \cdot 1 \ldots 1 \uplus 2 \ldots 2$.
- For $w = 3 \ldots 3$, $M_p(w) = \frac{(2k)!}{k!} \cdot 2 \ldots 2$.

The rest of this section is dedicated to analyze the consequences of Theorems 1.1 and 1.2 in several particular cases. We start looking at the case in which $X$ is itself a polynomial map and we need the following definition.

**Definition 4.3.** For an element $a \in T(\mathbb{R}^d)$, with zero coefficient for the empty word $e$, we define the concatenation product exponential of $a$ as

$$
\exp_{\bullet}(a) := \sum_{n \geq 0} \frac{a^n}{n!}.
$$

More information on this exponential map and its inverse, the logarithm, can be found in [Reu93].

**Corollary 4.4.** Let $X : [0, L] \rightarrow \mathbb{R}^d$ be a polynomial map, with $L \in \mathbb{R}$, $L \geq 1$. Then, for any $w \in T(\mathbb{R}^d)$,

$$
\langle \sigma(X), w \rangle = \langle \exp_{\bullet}(L \cdot 1), M_{\tilde{X}}(w) \rangle,
$$

where $\tilde{X}(y) = X(y) - X_0$. Equivalently, $\sigma(X) = M^*_{\tilde{X}}(\exp_{\bullet}(L \cdot 1))$.

**Proof.** Let $Y : [0, L] \rightarrow \mathbb{R}$ be the path given by $Y(t) = t$. Then, $\sigma(Y) = \exp_{\bullet}(L \cdot 1)$. The statement follows by Corollary 3.3 applied to the path $Y$ and the polynomial map $X$. \qed

The next result looks at the case when $M^*_{\tilde{X}}(\sigma(X)) = 0$ from the perspective of the polynomial map and of the piecewise continuously differentiable path. We introduce first two concepts. Given a polynomial map $p$, we define the ideal generated by $p$ as the ideal $I_p$ generated by the polynomials that define the map $p$, i.e., $I_p = \langle p_1, \ldots, p_m \rangle$. Moreover, we define a tree-like path as a path $X$ such that $\sigma(X) = 0$. This definition is the characterization given by B.M. Hambly and T. J. Lyons [HL10], and a more general topological definition can be found in [BGLY16, Definition 1.1].
Remark 4.5. A very simple example of a tree-like path is a concatenation of paths $A \sqcup B \sqcup C \sqcup D$ such that the paths $A$ and $B$ (resp. $C$ and $D$) are of the same shape, but parametrized in the opposite direction. When we compute the integrals on such a path, we get cancellations and the signature of the path does not see the $A \sqcup B \sqcup C \sqcup D$ loop, i.e. $\sigma(A \sqcup B \sqcup C \sqcup D) = e$.

The following result describes the situation in which the (image of the) path lies in the zeros of the ideal $I_p$, for some polynomial map $p$.

Corollary 4.6. Let $p : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be a polynomial map with associated ideal $I_p$. We define the polynomial map $\tilde{p}(y) = p(y + X_0) - p(X_0)$, for which $\tilde{p}(0) = 0$.

- If $X : [0, L] \rightarrow \mathbb{R}^d$ is a piecewise continuously differentiable path such that $X(t) \in V(I_p)$ for all $t \in [0, L]$, then $M_p^*(\sigma(X)) = e$.
- Conversely, if $M_p^*(\sigma(X)) = e$, for some piecewise continuously differentiable path $X : [0, L] \rightarrow \mathbb{R}^d$, then $p(X)$ is tree-like.

The last consequence is that the dual map $M_p^*$ behaves nicely with respect to the concatenation of signatures.

Corollary 4.7. Let $p : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be a polynomial map with $p(0) = 0$ and let $X, Y : [0, L] \rightarrow \mathbb{R}^d$ be two piecewise continuously differentiable paths with $X_0 = Y_0 = 0$. Then,

$$M_p^*(\sigma(X) \cdot \sigma(Y)) = M_p^*(\sigma(X)) \cdot M_q^*(\sigma(Y)),$$

where $q(y) = p(y + X_L - X_0) - p(X_L - X_0)$.

5 Applications and future work

The results presented in this paper solve an algebraic question that arises from the signatures of paths, an object commonly studied in stochastic analysis. There are several interesting problems that do not fit on the algebraic flavour of this paper. We summarize them in the following list.

(A) In comparison with the results presented in [PSS19], we would like to explore a nonlinear version of their approach using dictionaries. The idea is that if we have a family of generic paths, $\chi$, for which we know the signature and a polynomial map $p$, then Theorem 1.2 allows us to compute the signature of all the paths in $p(\chi)$.

For instance, Example 4.2 shows that we can compute the signature of $p(X)$ for any piecewise continuously differentiable path $X$ by multiplying the signature of $X$ by a matrix at each level. Therefore, we have the following question:

*Is it possible to understand $\sigma(p(X))$ in terms of $\sigma(X)$ and $M_p$ in the language of tensors?*

(B) Another line of future research is focused on the map $M_p$. Since it is defined from a polynomial map without involving any piecewise continuously differentiable path and gives us the commuting diagram (3), we intuit that it is worth to look for more interesting properties. For that, we should look to the big picture involving the Hopf algebra structures, as well as other constructions.

In this direction, at the end of Example 4.2 we describe combinatorially $M_p(\mathbf{w})$ for some particular words $\mathbf{w}$. A more general question would be the following:
For which words $w$ and polynomial maps $p$ is there a non-recursive combinatorial formula for $M_p(w)$?

Answering this question could be very useful from the computational perspective.

Acknowledgements

The authors would like to thank Bernd Sturmfels, Francesco Galuppi, Joscha Diehl, Mateusz Michałek, and Max Pfeffer for their fruitful conversations and observations. The collaboration between the co-authors would not have been possible without the financial support from the research institute MPI-MiS Leipzig (Germany). R.P. is currently supported by European Research Council through CoG-683164.

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