Factorized ground state for a general class of ferrimagnets

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We have found the exact (factorized) ground state of a general class of ferrimagnets in the presence of a magnetic field which includes the frustrated, anisotropic and long range interactions for arbitrary dimensional space. In particular cases, our model represents the bond-alternating, ferromagnet-antiferromagnet and also homogenous spin s model. The factorized ground state is a product of single particle kets on a bipartite lattice composed of two different spins (ρ, σ) which is characterized by two angles, a bi-angle state. The spin waves analysis around the exact ground state show two branch of excitations which is the origin of two dynamics of the model. The signature of these dynamics is addressed as a peak and a broaden bump in the specific heat.

Introduction. - Spin models are the building blocks of the theory of quantum magnetism and strongly correlated electron systems. In addition, they have been considered as an effective model to describe the behavior of a system in several disciplines. Recently, the implementation of quantum notions in quantum devices has attracted much attentions both in research labs and demanding applications like nanotechnology, quantum computation[1] and particularly optical lattices[2]. Quantum spin models are prototype realization of many relevant properties of quantum implementation in such devices. Therefore, different aspects of a quantum phase is of utmost importance for scientists and engineers. Quantum phases are characterized by the ground state (GS) properties of the corresponding many body system.[3]

Except of a few particular cases such as 1D bond alternating Heisenberg[2], anisotropic Heisenberg model (XYZ), XXZ in a longitudinal magnetic field and Ising model in transverse field which are exactly solvable[4], the GS of a general spin model is not known. However, at some special values of the model parameters the quantum correlations are vanishing and the GS is a product of single particle states. The factorized state (FS) manifests zero entanglement which is necessary to be identified for reliable manipulating of quantum computing. A FS (unentangled) which is associated with an entanglement phase transition can be also a quantum critical point in certain condition which is discussed in this article. This information is also attractive for the study of quantum phase transitions. Moreover, finding an exact ground state (as a FS) even at particular values of the parameter space of a many body spin model leads to the identification of that phase in addition to more knowledge about the properties of the model close to the factorized point via implementing an approximate method.

In a seminal work, Kurmann, Thomas and Müller[6] identified the factorized state of a homogeneous spin-s XYZ chain at a magnetic field of arbitrary direction. Factorized GS has been also observed in the two dimensional lattice through Quantum Monte Carlo simulation in terms of entanglement estimators.[2] Recently, Giampaolo, Adesso and Illuminati[8] have introduced a general analytic approach to find the factorized ground states in a homogenous translational invariant spin-s quantum spin model for arbitrary long range interaction and any dimensional space. Their study is based on the single-spin unitary operation and the factorized point is determined at the position where the associated entanglement excitation energy becomes zero. More elaborate explanations came up with some generalizations quite recently.[4] The factorized GS of the dimerized XYZ spin chain in a transverse magnetic field has been investigated and reported that the factorized point in the parameter space of the Hamiltonian corresponds to an accidental ground state degeneracy.[10] However, in this article we will present (i) the FS of an inhomogeneous (ferrimagnetic) spin model which is composed of two spins (ρ, σ) in the presence of a magnetic field on a bipartite lattice with arbitrary long range interaction and dimensional space, (ii) the Hamiltonian is not necessarily translational invariant and (iii) the exchange couplings can be competing antiferromagnetic and ferromagnetic arbitrarily between different sublattices to build many practical models such as frustrated, dimerized and tetramerized materials. Moreover, our results recover the previous ones for σ = ρ and a particular configuration of the couplings.[2][4][8][10] In addition, we will address on the existence of two energy scales which lead to a surprising dynamics of the model close to the factorizing point and its fingerprint as a double peak in the specific heat versus temperature. As an enclosure, the results have been ap-
plied to the 1D ferrimagnetic XXZ ($\rho, \sigma$) spin chain in
the presence of a transverse magnetic field which is real-
ized as a bimetallic substance. [11] We will also address
the cases where the factorizing field coincides the critical
point.

Factorized state.- Let us consider a two sites model
which is composed of two spins $\sigma = \frac{1}{2}$ and $\rho = 1$ with
the following Hamiltonian

$$H' = J^x \sigma^x \rho^x + J^y \sigma^y \rho^y + J^z \sigma^z \rho^z + h' (\sigma_z + \rho_z),$$

(1)

where $J^m, m = x, y, z$ are the exchange couplings in dif-
ferent directions and $h'$ is proportional to the magnetic
field. We are looking for a factorized state which is satis-
ified by $H' |\sigma\rangle |\rho\rangle = \epsilon(\sigma) |\rho\rangle$, in which $|\sigma\rangle$ and $|\rho\rangle$ are the single particle states. It is appropriate to choose $|\sigma\rangle$ and $|\rho\rangle$ to be the eigenstates of $\vec{n'} \cdot \nu'$ and $\vec{\rho} \cdot \nu''$ with eigenvalues $+\frac{1}{2}$ and $+1$; respectively, where $\nu'(\theta, \varphi)$ and $\nu''(\beta, \alpha)$ are unit vectors in Bloch sphere. The solution of $H' |\sigma\rangle |\rho\rangle = \epsilon(\sigma) |\rho\rangle$ gives the factorized state at $h' = h'_f$ and
its corresponding energy ($\epsilon$ [2]). Moreover, we found
that the angles $\theta$ and $\beta$ are fixed by the couplings ($J^m, h'$; see Eq.(2)) while $\alpha$ and $\varphi$ are given by one of these choices
(I) $\alpha = 0, \varphi = 0$, (II) $\alpha = 0, \varphi = \pi$, (III) $\alpha = \frac{\pi}{2}, \varphi = -\frac{\pi}{2}$,
(IV) $\alpha = \frac{\pi}{2}, \varphi = \frac{\pi}{2}$. The spins are located in the xz-plane
for choices I and II while they have projections only in the
yz-plane for III and IV. Without loss of generality we can
assume the spins are located in the xz-plane. In fact, the
spins of yz-plane will fall to xz-plane by interchange of
$J^y \leftrightarrow J^w$. Moreover, the coordinates $(\theta, \varphi = 0)$ and
$(-\theta, \varphi = \pi)$ are representing the same direction, there-
fore case (I) $\alpha = 0, \varphi = 0$ is able to describe all possibil-
ities.

The two spin model ($\sigma = \frac{1}{2}, \rho = 1$) is now general-
ized to arbitrary ($\sigma, \rho$) spins. [6] To find the factorized
state of a general two site ferrimagnet we consider a ro-
tation on $\sigma$ and $\rho$ spins such that $\vec{\sigma}$ and $\vec{\rho}$ point in
$(\theta, \varphi = 0)$ and $(\beta, \alpha = 0)$ directions, respectively. The
rotation operator is $D = D^0(0, \theta, 0)D^\rho(0, \beta, 0)$ where
$D^0(0, \beta, 0) = D(\alpha = 0, \beta, \gamma = 0) = D_z(\alpha)D_y(\beta)D_z(\gamma)$
which is defined in terms of Euler angles and a similar expres-
sion is considered for $D^\rho(0, \theta, 0)$. Then, we impose the condition to have a factorized (fully polarized) eigenstate
for this Hamiltonian which fixes the following relations for
the model parameters

$$\cos \theta = \frac{h_f^2 J^y + J^z (J^z - J^\rho)) \rho \sigma + h_f J^z (J^\rho + J^\sigma)}{h_f^2 J^z + J^y (J^z - J^\sigma)) \rho \sigma + h_f J^z (J^\rho + J^\sigma)},$$

(2)

$$\cos \beta = \frac{h_f J^y + J^z (J^z - J^\rho)) \rho \sigma + h_f J^z (J^\rho + J^\sigma)}{h_f J^z + J^y (J^z - J^\rho)) \rho \sigma + h_f J^z (J^\rho + J^\sigma)},$$

$$h_f = \sqrt{\frac{1}{2} (2 J^z J^\rho \rho \sigma + (\rho^2 + \sigma^2)J^z + C J^z)},$$

$$C = \sqrt{4 \rho \sigma (\rho J^\rho + \sigma J^\sigma)(\sigma J^\sigma + \rho J^\rho) + (\rho^2 - \sigma^2)^2 J^2},$$

$$\epsilon = \frac{J^z J^\rho \rho \sigma + h_f^2}{J^z}$$

Therefore, for arbitrary ($\sigma, \rho$) and at the above value for
$h' = h'_f$ we have a fully polarized eigenstate which is a
factorized state. The ordering of this state is defined by
two angles ($\theta, \beta$) which show the orientations of $(\vec{\sigma}, \vec{\rho}$),
respectively.

Now, we intend to find the condition for having a fac-
torized state for a ferrimagnetic state in a magnetic
field. We will then show that the factorized state is the
ground state of lattice implementing a spin wave theory to
study the quantum fluctuations. We consider a general
Hamiltonian of ferrimagnets on a bipartite lattice where
sublattice ($A_\rho$) contains $\sigma$ spins and the other sub-
lattice ($B_\rho$) includes $\rho$ spins. The interaction can be long
ranged between different sublattices but no interaction in
the same sublattice. The ferrimagnetic Hamiltonian for
such case can be written as

$$H = \sum_{i,r} [\zeta_i \hat{C}_{i+r} (J^x \sigma^x \rho^x + J^y \sigma^y \rho^y + J^z \sigma^z \rho^z) + h \sum_i (\sigma_i^z + \rho_i^z)],$$

(3)

where $i = (i_1, i_2, i_3)$ and $r = (r_1, r_2, r_3)$ are repre-
senting the three dimensional index on the lattice and
$\zeta_i, \zeta_{i+r} = \pm 1$ which realize both ferromagnetic (F) and
antiferromagnetic (AF) exchange interactions. A remark
is in order here, the Hamiltonian in Eq.(4) is a sum of
two sites Hamiltonian defined in Eq.(1) where the two
spins can be far from each other. However, the inter-
action between each couple of ($\sigma, \rho$) spins can depend on
distance (r) with different strength and also be F or AF
arbitrarily defined by $\zeta_i, \zeta_{i+r}$. A factorized eigenstate for
the Hamiltonian of Eq.(4) can be written as

$$|FS\rangle = \bigotimes_{i \in A_\rho} |\sigma_i^z\rangle |\rho_i^z\rangle$$

(4)

where $|\sigma_i^z\rangle$ and $|\rho_i^z\rangle$ are the eigenstates of
$\vec{n}_i \cdot \nu'$ and $\vec{\rho}_i \cdot \nu''$ with largest eigenvalue where $\nu_i'$ and $\nu_i''$ are unit vectors pointing in ($\zeta_i \theta, \varphi = 0$) and ($\zeta_i \beta, \alpha = 0$), respectively. However, the factorized state ($|FS\rangle$) is an eigenstate of
the Hamiltonian if the angle $\zeta_i \theta$ ($\zeta_i \beta$) be consistent with all pair of interactions originating from $\sigma, \rho$ on sub-
lattices $A_\rho$ ($B_\rho$). According to Eq.(2) the former condition
is satisfied if the interaction between each pair ($\sigma, \rho; \rho$) is
the same for all directions while depending on distance
($r$), i.e. $J^x = \chi(r) J^x, \mu = x, y, z, \chi(r) > 0$. Under these
constraints the factorized state (Eq.(4)) is an eigenstate of
$H$ with the characteristic angles ($\theta, \beta$) defined in Eq.(2)
and the factorizing field is

$$h_f = h'_f \sum_{r=0}^{N_r} \chi(r)$$

(5)

where $N_r$ is the number of spins on each sublattice. To show
that $|FS\rangle$ is the ground state of $H$ at $h_f$ we first
implement a rotation on the Hamiltonian. The rotated Hamiltonian ($\tilde{H}$) is the result of rotations on all lattice points of $H$,

$$\tilde{H} = \hat{D}^{\dagger} H \hat{D}, \quad \hat{D} = \bigotimes_{i \in A, j \in B_i} D_i^\sigma(0, \zeta, \theta, 0) D_j^\beta(0, \tilde{\zeta}, \beta, 0).$$

(6)

In the next step the rotated Hamiltonian is bosonized using the Holstein-Primakoff (HP) transformation, $\sigma_i^\pm = \sqrt{2\sigma - a_i^\dagger a_i}$ and $\rho_i^\pm = \sqrt{2\rho - b_i^\dagger b_i}$, with $a_i^\dagger a_i$ and $b_i^\dagger b_i$ are two types annihilation (creation) boson operators. Using HP transformations the anisotropic ferrimagnetic spin model is mapped to an interacting system of bosons. The Hamiltonian in the momentum ($k$) space and in the linear spin wave theory is diagonalized via the rotation, $\chi = \cos \eta_k - i \sin \eta_k$, $\psi = e^{i \theta} \cos \eta_k + a_k \sin \eta_k$, and a shift at $k = 0$ where $\delta$ is defined by $\sum_e e^{-i k \cdot J_e} = \sum_e e^{-i k \cdot J_e} f^\delta$. The diagonalized Hamiltonian is

$$\tilde{H} = E_{gs} + \sum_k \left( \omega^-(k) \chi_k^\dagger \chi_k + \omega^+(k) \psi_k^\dagger \psi_k \right),$$

(7)

where $E_{gs}$ is the ground state energy $[12]$ and $\omega^\pm(k) \geq 0$ are normal modes parallel and perpendicular to the field direction.

$$\omega^\pm(k) = D^\pm = \frac{D^- + \sqrt{\rho \tan(2\eta_k)} \sum_e e^{-i k \cdot J_e^\delta}}{\sqrt{1 + \tan^2(2\eta_k)}},$$

$$\tan(2\eta_k) = \sqrt{\frac{\rho \sigma}{\rho \sigma + \sum_e e^{-i k \cdot J_e^\delta}}},$$

(8)

in which

$$D^\pm = \frac{h^2}{2\tau^2} \left( \frac{1}{2} \pm \frac{\rho}{\sigma} \right) + h_f \left( \frac{\sigma}{2\rho} \cos \theta \pm \frac{\rho}{2\sigma} \cos \beta \right) + \frac{\tau^\sigma + \tau^\rho}{2\tau^2} \left( \sigma \pm \rho \right) + \frac{h_f - h}{2} \left( \cos \beta \pm \cos \theta \right),$$

$$\tau^\rho = J^\rho \sum_r \lambda(r).$$

(9)

The bosons number $\langle a_i^\dagger a_i \rangle$ is proportional to $h_f - h$ which states that at $h = h_f$ the bosons number in the ground state is exactly zero $[12]$, i.e., $\langle a_i^\dagger a_i \rangle \mid_{h = h_f} = 0 = \langle b_i^\dagger b_i \rangle \mid_{h = h_f}$. It is also true for $\langle a_i^\dagger b_i \rangle \mid_{h = h_f} = 0$. Thus, the spin wave theory is exact in all orders at the factorizing field ($h = h_f$) and $|FS\rangle$ is the corresponding ground state of $H$.

To visualize the configuration of a factorized ground state of a general frustrated model we have plotted an example in Fig. 1 with the assumption $J_x^f, J_y^f > 0$ and $J_r^f < 0$ where $\zeta_i$ and $\tilde{\zeta}_{i+r}$ define the sign of interactions. The solid lines represent antiferromagnetic interaction and the dash-dotted ones are the ferromagnetic counterparts. As shown in Fig. 1, the interactions can be frustrated and long ranged without a translational invariance. However, the factorized state is defined by two angles ($\theta, \beta$) while each $\sigma (\rho)$ spin is directed in $\theta (\beta)$ or $-\theta (-\beta)$ directions. We call this a bi-angle ordering. In a special case the bi-angle ordered state can configure a ferromagnet ($\theta = \beta$) or antiferromagnet ($\theta = \pi - \beta$) factorized state.

**Discussions.** In a spin model when the magnetic field is strong enough all spins will align in the direction of the magnetic field which characterizes the saturated phase as far as $h \geq h_s$. In our notation, the saturated phase appears when all $\sigma (\rho)$ spins get $\theta = \pi (\beta = \pi)$. Thus, the saturating field ($h_s$) is a factorizing one when $J^x = J^y$. In case of $J^x \neq J^y$ the saturation can only appear at infinite value of the magnetic field while a finite factorizing point ($h_f$) still exists. For $J^x = J^y$, the lower excitation band becomes gapless ($\omega^-(k = 0) = 0$) which confirms that the factorizing point ($h_f = h_s$) is the critical point which separates the non-saturated phase ($h < h_s$) from the saturated one ($h > h_s$). It is worth to mention that at $J^x = J^y$ the rotational symmetry around the magnetic field is restored where the quantum fluctuations around the field axis are suppressed. In the absence of rotational symmetry the saturation in the field direction does not happen at a finite value of magnetic field while the model approaches saturation asymptotically at infinity.

A general feature of our result is that it can simply recover the previous study of homogenous systems by replacing $\sigma = \rho = s$. In that case the restriction of bipartite lattice is promoted to arbitrary lattice and the interaction between any pair of spins can exist. However, our Hamiltonian is not restricted to the translational invariant symmetry or bond-alternating ones which is witnessed by the example given in Fig. 1. This can also be generalized to any dimension. We claim that the general Hamiltonian which can possess a nontrivial factorized ground state should be of the form Eq. 8 with the restriction $J^\mu = \lambda(r) J^\mu, \mu = x, y, z$.

In the absence of magnetic field, the spin-$\sigma$ ferromagnet turns into the spin-$\sigma$ antiferromagnet in the limit $\rho \rightarrow \sigma$, whereas it looks like the spin-$\rho$ ferromagnet in the

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**Fig. 1:** The configuration of a factorized state on a one-dimensional lattice for arbitrary frustrated couplings. Solid lines (dash-dotted) represent antiferromagnetic (ferromagnetic) couplings which are defined by $\zeta_i \tilde{\zeta}_{i+r}$ as depicted by $\pm$ on each site. Each color belongs to equal distance interaction (same $r$).
other extreme limit $\frac{2\pi}{\Delta} \to \infty$. In this sense, the difference $\rho - \sigma$ can be regarded as the ferromagnetic contribution. The analysis of our result shows that the value of $h_f$ is a decreasing function of $\frac{\rho}{\sigma}$ converging to the ferromagnet feature as this ratio becomes large. More investigations on different applications of our approach in several models are in progress.\cite{12}

Let us now be more concrete by concentrating on the one-dimensional nearest neighbor XXZ ferrimagnet in the presence of transverse magnetic field. Suppose that $J^x = J^y = J$ and $J^z = J\Delta$ where $\Delta$ represents the easy axis anisotropy. At zero magnetic field the quantum fluctuations are large and the ground state of the model is strongly entangled. Upon adding the transverse magnetic field the $\text{U}(1)$ symmetry of the XXZ model is lost and the entanglement of the GS is decreased. In the mapped bosonic system the magnetic field is served as a chemical potential, thus the number of bosons ($\langle a^\dag a \rangle$ and $\langle b^\dag b \rangle$) is dependent on the magnetic field. An entanglement of the magnetic field causes deducing of the bosons’ number and the quantum correlations decrease. At factorizing field $h = h_f$, the number of bosons is zero and the quantum fluctuations become completely uncorrelated. Moreover, our calculations show that the factorizing field in ferrimagnetic model depends on the anisotropy parameter ($\Delta$) similar to a homogeneous antiferromagnetic Heisenberg model. For $\Delta = 0$ (ferrimagnetic XY model) the in-plane magnetic field completely breaks the rotational symmetry. Increasing $\Delta$ suppresses the effect of the field and try to evoke the rotational symmetry to the system. Thus, by increasing $\Delta$ the factorizing Neél field approaches to the saturation field. At $\Delta = 1$, the rotational symmetry is completely repayed and $h_f$ is exactly lied on $h_s$.

A benefit of identification a factorized state is that we can work out an approximate method around the factorized point to get some information on the properties of that phase. This help us to calculate the magnetic properties of the ferrimagnetic XXZ model in the presence of a transverse magnetic field. We have implemented the linear spin wave theory around $h = h_f$ for $\sigma = \frac{1}{2}$ and $\rho = 1$. Our results for magnetization ($M_x, M_y$) and staggered magnetization ($SM_x, SM_y$) in both $x$ and $y$ directions are plotted in Fig. (2) where the magnetic field is in $x$ direction. The approach can be extended to derive the thermodynamic of this model which will be done in near future.

It is also worth to mention that our results are applicable to the homogenous XXZ Heisenberg spin-$1/2$ chains in the presence of a transverse magnetic field ($h^x$). This model has been studied intensively in the literature.\cite{13,14,15,16} On the onset of transverse magnetic field a perpendicular anti-ferromagnetic order is stabilized by promoting a spin-flop phase. At the classical (factorizing) field ($h_f < h_c$) in the spin-flop phase the ground state is factorized to single spin states and the staggered magnetization along the $y$ direction has a large value. Very close to the critical field ($h_c$) the anti-ferromagnetic order becomes unstable and the staggered magnetization falls sharply to vanish at the critical point. For $h > h_c$ the spins are almost aligned in field ($x$) direction and the factorized state as a fully polarized phase will be appeared for $h \to \infty$. The excitation energies around the factorizing point have the following form:

$$\omega^\pm (k) = (1 + \Delta)\frac{h_f}{h} + \Delta(\pm \cos(\frac{k}{2}) - 1).$$

Thus, we have two branches of magnon energies as two scales which impose two dynamics in the system. The most interesting feature is that around the factorizing field both scales show up. These dynamics correspond to the coexistence of two different features of the model. In other words, by increasing the transverse magnetic field, spins try to align in the $x$ direction which is the ferromagnetic feature of the system. In the intermediate values of $h$, we have both anti-ferromagnetic and ferromagnetic behaviors since the anti-ferromagnetic order in $y$ direction has already been stabilized. The finger print of these features appear in the thermodynamic functions such as specific heat and internal energy. As it is seen from Ref.\cite{10} the second feature can be seen as a shoulder at the right side of specific heat curve. By further increasing of $h$, the ferromagnetic behavior is seen as a broaden peak in the curve. This point is almost near the classical field where the ground state of the system has been factorized.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The magnetization and staggered magnetization of an anisotropic ferrimagnetic ($\sigma = 1/2, \rho = 1$) spin chain versus transverse field and for $\Delta = 0.25$.}
\end{figure}

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