Localization Control of Spatiotemporal Chaos.

Roman O. Grigoriev\textsuperscript{1}, Sanjay G. Lall\textsuperscript{2} and Geir E. Dullerud\textsuperscript{3}

\textit{1 Condensed Matter Physics 114-36, California Institute of Technology, Pasadena, CA 91125, USA}
\textit{2 Control and Dynamical Systems 107-81, California Institute of Technology, Pasadena, CA 91125, USA}
\textit{3 Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1}

\textbf{Abstract}—A linear output feedback control scheme is developed for a coupled map lattice system. $H_\infty$ control theory is used to make the scheme local: both the collection of information and the feedback are implemented through an array of locally coupled control sites. Robustness properties of the control scheme are discussed.

\section{I. Introduction}
Learning to tame spatiotemporal chaos in spatially extended nonlinear systems is very attractive due to a large number of potential applications. Some of these are continuous, such as turbulence \textsuperscript{1}, plasma instabilities \textsuperscript{2} and chemical reaction systems \textsuperscript{3}, some are discrete: neural networks \textsuperscript{4} and distributed memory systems are only a few examples. The main objective is usually to stabilize some suitable unstable periodic orbit (UPO), or a group of orbits, embedded in the chaotic attractor of the system.

Although spatially extended homogeneous systems could be treated as a special case of the high-dimensional chaotic systems, some of the practical issues, that arise in the control problem are quite specific and could be best handled by taking into account the spatiotemporal structure of the system and the controlled state in general and their symmetry properties in particular \textsuperscript{3}.

In the present paper we will illustrate the control algorithm applying it to the general coupled map lattice (CML), originally introduced by Kaneko \textsuperscript{4}:

$$z_{i}^{t+1} = f(z_{i}^{t}) + \epsilon (f(z_{i-1}^{t}) - 2f(z_{i}^{t}) + f(z_{i+1}^{t})), \quad (1)$$

and considered to be one of the simplest models, possessing the essential properties of an extended spatiotemporally chaotic system.

There are many ways to achieve the stabilization of a non-chaotic trajectory. However, the requirements imposed by different control algorithms and their performance could vary widely. For instance, it was shown \textsuperscript{1}, that a number of UPOs of the CML \textsuperscript{4} could be stabilized with feedback applied through a periodic array of controllers. Although limited knowledge of the system state was required, the density of controllers had to be extremely high for the control to work. Re-arranging the controllers, one can significantly reduce their density and improve the robustness characteristics of the control scheme \textsuperscript{5} at the expense of requiring additional information about the system state. In the present paper we will show how the CML can be controlled using low density of controllers and requiring very limited information about the system state.

\section{II. The system}
Rewrite eq. (1), adding to it the uncorrelated random noise $\langle w_{i}^{t}w_{i}^{t'} \rangle = \sigma^2 \delta_{t,t'} \delta_{ii'}$ and applying control perturbations $u_{i}^{t} = G_{k}(z_{i}^{t}, z_{i}^{t-1}, \cdots)$ at sites $i_k, k = 1, \cdots, m$:

$$z_{i}^{t+1} = \epsilon f(z_{i-1}^{t}) + (1 - 2\epsilon) f(z_{i}^{t}) + \epsilon f(z_{i+1}^{t}) + w_{i}^{t} + \sum_{k} \delta_{ii_k} u_{k}^{t}, \quad (2)$$

assuming, that the lattice is finite, $i = 1, 2, \cdots, n$, and periodic boundary conditions $z_{i+n}^{t} = z_{i}^{t}$ are imposed.

Due to the translational symmetry of the CML \textsuperscript{4} additional parameters can only enter the evolution equation through the nonlinear local map function, which we choose as $f(z) = az(1 - z)$, emphasizing, that the only result affected by this particular choice is the set of existing periodic trajectories. In particular, for any choice of $f(z)$, the homogeneity of the system response to the perturbation of any internal parameter ($a$ and $\epsilon$ in our case) makes it impossible to use either internal parameter for control.

Linearizing equation (2) around the period-$\tau$ target UPO $\hat{z}^{1}, \hat{z}^{2}, \cdots, \hat{z}^{\tau}$ and denoting the displacement $\tilde{x}^{t} = x - \hat{x}$, we obtain

$$\tilde{x}_{i}^{t+1} = A_{i}^{t} \tilde{x}_{i}^{t} + B_{N}^{t} w_{i}^{t} + B_{u}^{t} u_{i}^{t}, \quad (3)$$

where $A_{i}^{t} = \partial_{z} f(z_{i}^{t+1} \hat{z})$ is the Jacobian and the matrices $B_{N}^{t} = I_{n \times n}$ and $B_{u}^{t} = \sum_{k} \delta_{ii_k} \delta_{ii_k}$ specify the response of the system to the external noise $w_{i}^{t}$ and the applied feedback $u_{i}^{t}$ (also called the input).

Finally, assume that only $q$ functions $\eta_{i}^{t} = H_{i}(\tilde{x}_{i}^{t})$ (called the output) of the system state are accessible to measurement. Denoting $C_{i}^{t} = \partial_{\tilde{x}^{t}} H_{i}(\hat{z})$ we obtain for the linearized output:

$$\tilde{y}_{i}^{t} = \eta_{i}^{t} - H_{i}(\hat{z}) = C_{ij}^{t} \tilde{x}_{j}^{t}, \quad (4)$$
III. The control scheme

The algorithm presented below allows one to determine whether the feedback $u^t$ stabilizing the chosen UPO can be obtained as a function of the output $y^t$, and determines the solution, which minimizes the noise amplification factor or induced-power-norm

$$
\gamma = \max_{\|w\|_{p} < \infty} \frac{\|z\|_{p}}{\|w\|_{p}},
$$

where the $r$-dimensional performance vector

$$
z^t = C_N x^t + D_N u^t
$$

gives the deviation of the system from the target state, and the power norm is defined as

$$
\|z\|_p = \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} |z_t|^2 \right]^{1/2}.
$$

The solution to the time-periodic output feedback problem (1) can be obtained using the generalization of the results of $H_\infty$ control theory (8) for linear time invariant (LTI) systems. In particular, Dullerud and Lall have shown (9), that if a locally stabilizing linear feedback $u^t$ exists, it could be written as

$$
v^{t+1} = A_C^t v^t + B_C^t y^t,
$$
$$
u^t = C_C^t v^t + D_C^t y^t,
$$

where $A_C^t$, $B_C^t$, $C_C^t$ and $D_C^t$ are matrices with the same periodicity $\tau$ as the target orbit $\hat{z}^t$, and $v^t$ is the $p$-dimensional internal state of the controllers. The standard state feedback law $u^t = K^t x^t$ used in (8), is seen to be just a special case of this general setup.

Construct constant block diagonal matrices $A, B, C, B_N, C_N$ and $D_N$ according to the following rule:

$$
Q = \begin{bmatrix} Q^1 & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & Q^r \end{bmatrix}.
$$

For $\tau > 1$ define a $\tau n \times \tau n$ cyclic shift matrix

$$
Z = \begin{bmatrix}
0_{n \times n} & \cdots & 0_{n \times n} & I_{n \times n} \\
I_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & \cdots & I_{n \times n} & 0_{n \times n} \\
\cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}.
$$

In the time-invariant case ($\tau = 1$) set $Z = I_{n \times n}$. Also introduce the notations $Q > 0$ for positive definite, $Q \geq 0$ for semi-positive definite matrices and $Q^\dagger$ for the transpose of $Q$.

It can be shown (10), that a stabilizing solution (1) with $p \geq n$ such that $\gamma < 1$ for the system (8) exists, if and only if there exist block-diagonal matrices $R > 0$ and $S > 0$, satisfying

$$
\begin{bmatrix} R & I \\ I & S \end{bmatrix} \succeq 0, \quad O_S^\dagger P_S O_S < 0, \quad O_R^\dagger P_R O_R < 0
$$

where $P_R, P_S, O_R$ and $O_S$ are given by

$$
\begin{align*}
P_S &= \begin{bmatrix} A^t Z^t S Z A - S & A^t Z^t S Z B_N & C_N^\dagger \\
B_N^\dagger Z^t S Z A & B_N^\dagger Z^t S Z B_N - I & 0 \\
C_N & 0 & -I \end{bmatrix} \\
P_R &= \begin{bmatrix} A R A^\dagger - Z^t R Z & A R C_N^\dagger & B_N \\
C_N R A^\dagger & C_N R C_N - I & 0 \\
B_N^\dagger & 0 & -I \end{bmatrix} \\
O_S &= \begin{bmatrix} N_S & 0 \\
0 & I \end{bmatrix}, \quad O_R = \begin{bmatrix} N_R & 0 \\
0 & I \end{bmatrix}
\end{align*}
$$

and the unitary matrices $N_R$ and $N_S$ satisfy

$$
\begin{align*}
\text{Im} N_R &= \ker \begin{bmatrix} B_N^\dagger \\
D_N^\dagger \end{bmatrix} \\
\text{Im} N_S &= \ker \begin{bmatrix} C \\
0_{p \times n} \end{bmatrix}.
\end{align*}
$$

To minimize $\gamma$, rescale $C_N^\dagger$ and $D_N^\dagger$, such that the above condition tests for $\gamma < \gamma_0$ instead of $\gamma < 1$ and decrease $\gamma_0$ until the test fails; standard software exists to do this. If there is any linear stabilizing controller, we can therefore find it using this algorithm.

If $R = \text{diag}(R^1, \ldots, R^r)$ and $S = \text{diag}(S^1, \ldots, S^r)$ are determined, one can find the matrices in (8) using the following procedure. First, construct nonsingular matrices $M_t$ and $N_t$, such that

$$
M_t N_t^\dagger = I - R^t S^t.
$$

Determine the matrix $X_t$ as the unique solution of

$$
\begin{bmatrix} S_t & I \\
N_t^\dagger & 0 \end{bmatrix} = X_t \begin{bmatrix} I & R^t \\
0 & M_t^\dagger \end{bmatrix}.
$$

Next, define the matrices

$$
\tilde{A}_t = \begin{bmatrix} A_t & 0_{n \times n} \\
0_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad \tilde{B}_t = \begin{bmatrix} B_N^\dagger \\
0_{0 \times n} \end{bmatrix},
$$
$$
\tilde{C}_t = \begin{bmatrix} C_N^t & 0_{n \times n} \\
0_{n \times n} & I_{n \times n} \end{bmatrix}, \quad \tilde{D}_t = \begin{bmatrix} D_N^\dagger \\
0_{0 \times n} \end{bmatrix},
$$

and then define

$$
H_t = \lfloor -X_{t+1}^{-1} \tilde{A}_t \tilde{B}_t \tilde{C}_t \tilde{D}_t \rfloor,
$$
$$
Q_t = \begin{bmatrix} 0_{n+p \times 2n} & \tilde{C}_t \\
\tilde{B}_t & 0_{n+p \times n} \end{bmatrix}, \quad P_t = \begin{bmatrix} \tilde{B}_t^\dagger \\
0_{n+m \times n} \end{bmatrix}
$$

Finally, the matrices $A_t^C, B_t^C, C_t^C$ and $D_t^C$ are extracted from the solution

$$
J_t = \begin{bmatrix} A_t^C \\
B_t^C \\
C_t^C \\
D_t^C \end{bmatrix}
$$

to the linear matrix inequality

$$
H_t + Q_t^\dagger J_t^\dagger P_t + P_t^\dagger J_t Q_t < 0.
$$

Linear matrix inequalities (LMI) like (11) and (12) can be conveniently solved using the tools of convex optimization theory. The big practical advantage of this technique is the guaranteed convergence.
subdomains, by adding appropriate perturbations to as well as correct for interactions between adjacent subdomains. Specifically, we will need to export the minimal number of controllers required is odic in space as well as time.

We impose periodic boundary conditions on each subdomain of limited length, that the minimal number of controllers required is reduced to the problem of controlling an isolated subdomain of length \( n_p \ll n \), each interacting with two adjacent subdomains. The original problem is thus reduced to the problem of controlling an isolated subdomain of limited length \( n_p \) (we drop the index below). We impose periodic boundary conditions on each subdomain to allow the existence of unstable orbits periodic in space as well as time.

The symmetry properties of the CML determine that the minimal number of controllers required is two. Placing them at the boundaries of the subdomain allows one to change the boundary condition at will, as well as correct for interactions between adjacent subdomains, by adding appropriate perturbations to

IV. Control of large lattices

Although, using the above algorithm, we can in principle obtain the stabilizing feedback for a system of arbitrary size, solving matrix inequalities involving large matrices requires considerable computational resources.

This problem could be avoided using distributed control approach. The idea is to subdivide the complete system into a number of weakly interacting subsystems, and learn to control each of the subsystems independently, neglecting interactions with other subsystems. Finally, the control can be adjusted to correct for interactions by introducing coupling between formerly independent controllers.

Since the coupling in our model is local, we can partition the whole lattice into a number of identical subdomains of length \( n_p \ll n \), each interacting with two adjacent subdomains. The original problem is thus reduced to the problem of controlling an isolated subdomain of length \( n_p \) (we drop the index below). We impose periodic boundary conditions on each subdomain to allow the existence of unstable orbits periodic in space as well as time.

The symmetry properties of the CML determine that the minimal number of controllers required is two. Placing them at the boundaries of the subdomain allows one to change the boundary condition at will, as well as correct for interactions between adjacent subdomains, by adding appropriate perturbations to

\[
\begin{align*}
B_{ij}^t &= \delta_{ii} \delta_{j1} + \delta_{in} \delta_{j2}. \\
C_{ij} &= B_{ji}^t, \quad q = m = 2, \quad \text{which we use below.}
\end{align*}
\]

V. Comparison of \( H_2 \) and \( H_\infty \) approaches

In order to compare the results of the proposed approach with those, obtained using linear quadratic (\( H_2 \)) theory for the state feedback, we select a similar optimization criterion. Specifically, we take

\[
\begin{align*}
C_N &= \begin{bmatrix} I_{n \times n} & 0_{m \times n} \end{bmatrix}, \\
D_N &= \begin{bmatrix} 0_{n \times m} \\ I_{m \times m} \end{bmatrix}, \\
\end{align*}
\]

such that \( r = n + m \), \( z^t = \{x^t; u^t\} \) and, consequently,

\[
||z||^2_P = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} (x^t x^t + u^t u^t). 
\]

We demonstrate the \( H_\infty \) approach by stabilizing a number of UPOs of the noisy CML with \( n = 8 \) sites, \( a = 4.0 \) and \( \epsilon = 0.33 \). The feedback is calculated using the algorithm outlined above. Figure 1 shows the process of capturing and controlling the steady homogeneous state (S1T1), the time-period-2 space-period-8 (S8T2), and the time-period-4 space-period-8 (S8T4) orbits.

The real power of the \( H_\infty \) approach, however, can be full appreciated only in application to orbits of very high periodicity, where the accurate treatment of the effects of noise is of ultimate importance. Any method based on the reduction of periodic trajectories to steady states will fail for orbits of sufficiently long period. The \( H_\infty \) approach does not suffer from this limitation. Indeed, we have observed stabilization of a number of periodic orbits with period \( \tau > 10 \). One such example is presented in Fig. 2.
Noise limits our ability to control arbitrarily large systems with local interactions, using just two controllers. Rather simple arguments show [8], that the size of the largest system, that could be stabilized in the presence of noise under control (8). The results are presented in Fig. 3. One can see that the estimate (25) approximates the actual results rather well. This work was partially supported by the NSF through grant no. DMR-9013984.

References

[1] C. Lee, J. Kim, D. Bobcock and R. Goodman, “Applications of neural networks to turbulence control for drag reduction,” Physics of Fluids, V 9, N 6, pp.1740-1747, 1997.

[2] A. Pentek, J. B. Kadlke and Z. Toroczkai, “Stabilizing chaotic vortex trajectories - An example of high-dimensional control,” Phys. Lett. A, V 224, N 1-2, pp.85-92, 1996.

[3] V. Petrov, M. J. Crowley and K. Showalter, “Tracking unstable periodic orbits in the Belousov-Zhabotinsky reaction,” Phys. Rev. Lett., V 72, N 18, pp.2955-2958, 1994.

[4] C. Lourenco and A. Babloyantz, “Control of spatiotemporal chaos in neuronal networks,” Int. J. Neu. Sys, V 7, N 4, pp.507-517, 1996.

[5] R. O. Grigoriev and M. C. Cross, “Controlling physical systems with symmetries,” submitted to Phys. Rev. E.

[6] K. Kaneko, “Period-doubling of kink-antikink patterns, quasiperiodicity in antiferro-like structures and spatial intermittency in coupled logistic lattice - towards a prelude of a field theory of chaos,” Prog. Theor. Phys., V 72, N 3, pp.480-486, 1984.

[7] G. Hu and Z. Qu, “Controlling spatiotemporal chaos in coupled map lattice systems”, Phys. Rev. Lett., V 72, N 1, pp.68-71, 1994.

[8] R. O. Grigoriev, M. C. Cross and H. G. Schuster, “Pinning control of spatiotemporal chaos,” Phys. Rev. Lett., V 79, N 15, pp.2795-2798, 1997.

[9] K. Zhou, J. C. Doyle and K. Glover, “Robust and optimal control,” Prentice Hall, 1996, chapter 16.

[10] G. E. Dullerud and S. G. Lall, “A new approach for analysis and synthesis of time varying systems,” in Proc. 1997 IEEE/CDC.