A LAW OF ITERATED LOGARITHM ON LAMPLIGHTER DIAGONAL PRODUCTS

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Abstract. We prove a Law of Iterated Logarithm for random walks on a family of diagonal products constructed by Brieussel and Zheng [2]. This provides a wide variety of new examples of Law of Iterated Logarithm behaviours for random walks on groups. In particular, it follows that for any $\frac{1}{2} \leq \beta \leq 1$ there is a group $G$ and random walk $W_n$ on $G$ with $\mathbb{E}|W_n| \simeq n^\beta$ such that

$$0 < \limsup \frac{|W_n|}{n^\beta (\log \log n)^{1-\beta}} < \infty$$

and

$$0 < \liminf \frac{|W_n| (\log \log n)^{1-\beta}}{n^\beta} < \infty.$$

1. Introduction

Let $G$ be some finitely generated group, with a finite symmetric generating set $S$. Let $W_n$ be a random walk on $G$, with finitely supported symmetric step measure such that the support of $W_1$ generates $G$. A central object of study in the theory of random walks on groups is the distribution of the distance $|W_n|$ of the random walk from its origin point, and its connection to other geometric and algebraic properties of $G$. One is interested both in understanding the behaviour of $|W_n|$ for specific families of groups and in understanding the set of possible behaviours of the distance function for random walks on groups in general (often referred to as the “inverse problem”). Usually, fine understanding of the underlying metric structure of $G$ is lacking, and beyond some examples of polynomial growth groups and some non-amenable groups, most works focused on understanding the expected distance on some families of groups, with few works also looking at some moderate and large deviation regimes (See subsections 1.1 and 1.2).

The main contribution of this manuscript is proving a Law of Iterated Logarithm type result for random walks on a family of diagonal products.
products studied in [2]. These groups were used to capture a variety of behaviours of random walks on groups, in particular in terms of the expected distance. Our result provides a wide variety of Law of Iterated Logarithm type behaviours for random walks on groups.

1.1. The expected distance $\mathbb{E}|W_n|$. Much advancement has been done in the last years regarding the expected value $\mathbb{E}|W_n|$, also known as the speed or rate of escape of the random walk. It is trivial to see that the expected distance is sub-additive, and in particular that $\mathbb{E}|W_n| \lesssim n$ (Where $\lesssim, \simeq$ denote (in)equalities up to an absolute constant that may depend on the choice of group and random walk measure but not on $n$). By [10], for any random walk $W_n$ one has $\mathbb{E}|W_n| \gtrsim \sqrt{n}$. For many “small” groups, such as virtually nilpotent groups, the random walk is always diffusive, that is $\mathbb{E}|W_n| \simeq \sqrt{n}$, while by a theorem of Kaimanovich and Vershik (see [12]) $\mathbb{E}|W_n| \simeq n$ for non-Liouville measures. For a long time all known examples of random walks on groups exhibited one of the two extreme behaviours above.

This lead to the following natural question, attributed to Vershik.

**Question 1.1.** What functions can be realized as the speed function of a random walk on some group?

The first “non-classical” examples towards this question were given by Erschler [6], who proved that if a random walk on a group $G$ satisfies $\mathbb{E}|W_n| \asymp n^\alpha$, then the random walk on $G \wr \mathbb{Z}$ satisfies $\mathbb{E}|W_n| \asymp n^{1+\frac{\alpha}{2}}$. Starting with $G = \mathbb{Z}$ and using this result recursively, one can realize all of the functions $f(n) = n^{1-2^{-k}}$ as speed functions of iterated wreath product.

The next step was taken by Amir and Virág [1], who constructed groups where $W_n \simeq n^\beta$ for $\frac{3}{4} \leq \beta < 1$ (and more generally with $\mathbb{E}|W_n| \simeq f(n)$ for any “nice enough” function $f$ within that range). Their construction uses permutational wreath product over the natural action of the mother groups $\mathcal{M}_m$ on the boundary of their tree with $\mathbb{Z}$-valued lamps.

In [2] Brieussel and Zheng constructed random walks on groups that, up to some regularity condition, capture the whole range of possible behaviours of the expected distance:

**Theorem 1.2 (Theorem 1.1).** Let $f : [1, \infty) \to [1, \infty)$ be a continuous function with $f(1) = 1$ such that $\frac{f(x)}{\sqrt{x}}$ and $\frac{x}{f(x)}$ are non-decreasing. Then there exists a group $\Delta$ such that the random walk on $\Delta$ satisfies $\mathbb{E}|W_n| \simeq f(n)$. 
The examples constructed and analyzed in Brieussel and Zheng also gave new behaviours of other geometric quantities such as the entropy of the random walk, the return probabilities and the isoperimetric profile.

1.2. The Law of Iterated Logarithm. Since the possible behaviour of the expected distance is well understood in many cases, one may try and study further questions regarding the behaviour of the distance function of the random walk $W_n$. One such direction is to study the times in which the random walk is atypically far from its origin. This can be formulated by the Law of Iterated Logarithm.

The classical Law of Iterated Logarithm on $\mathbb{Z}$ originates from the works of Khintchine [14] and Kolmogorov [15], and states that for any random walk $W_n$ with zero mean and unit variance satisfies

$$\limsup_{n \to \infty} \frac{|W_n|}{\sqrt{2n \log \log n}} = 1$$

almost surely.

Unlike the expected distance which was well studied, there are much fewer examples where the Law of Iterated Logarithm of random walks on groups is understood. There are several ways to phrase a Law of Iterated Logarithm for a random walk on a general group. Perhaps the strongest one is finding a function $g(n)$ such that

$$\limsup_{n \to \infty} \frac{|W_n|}{g(n)} = 1$$

almost surely. However, finding such a function requires a tight estimation for the distance $|W_n|$, which is in many cases hard to achieve.

We say that $g(n)$ is an upper scaling function for $W_n$, if

$$0 < \limsup_{n \to \infty} \frac{|W_n|}{g(n)} < \infty$$

almost surely, and that $h(n)$ is a lower scaling function for $W_n$, if

$$0 < \liminf_{n \to \infty} \frac{|W_n|}{h(n)} < \infty$$

almost surely.

These definitions can also be rephrased in terms of inner and outer radii. We say that a function $R(n)$ is an outer radius for $W_n$ if $|W_n| \geq R(n)$ finitely often with probability 1, and a function $r(n)$ is an inner radius for $W_n$ if $|W_n| \leq r(n)$ finitely often with probability 1. It follows that $g(n)$ is an upper scaling function if $Cg(n)$ is an outer radius for some constant $C > 0$, but $cg(n)$ is not an outer radius for
Let us briefly review previous works in this direction. The classical results for $\mathbb{Z}$ and $\mathbb{Z}^d$ can be found e.g. in [5], where Dvoretzky and Erdős give a characterization for the inner and outer radii of a simple random walk on $\mathbb{Z}^d$. Hebisch and Saloff-Coste generalize this theorem for arbitrary groups with polynomial volume growth of order $d \geq 3$, see [7, Theorem 9.2].

In [18], Revelle studies groups of the form $G \wr \mathbb{Z}$, where the random walk on $G$ has $\alpha$-tight degree of escape, and proves that $G \wr \mathbb{Z}$ has $\alpha' = \frac{1+\alpha}{2}$-tight degree of escape; he also shows that $n^{\alpha'}(\log \log n)^{1-\alpha'}$ is an upper scaling function for $G \wr \mathbb{Z}$, whereas $n^{\alpha'}/(\log \log n)^{1-\alpha'}$ is a lower scaling function for this group. Revelle studies in addition several Baumstagn-Soliter groups, proving they have an inner radius of order $\sqrt{n/\log \log n}$ and an outer radius of order $\sqrt{n \log \log n}$.

Finally, in [20] Thompson proves Laws of Iterated Logarithm for certain polycyclic and metaabelian groups. He shows that these groups are all diffusive, have $\frac{1}{2}$-tight degree of escape (which is a form of control over the tail of the distance function), and have an upper scaling function $g(n) = \sqrt{n \log \log n}$.

1.3. Main results. In this article, we consider the groups constructed in [2]. These groups are diagonal products of lamplighter groups with finite lamp groups. We focus on the case where the lamp groups are expanders, and prove a Law of Iterated Logarithm on these groups. Our main theorem is the following:

**Theorem 1.3.** Let $f : [1, \infty) \to [1, \infty)$ be a continuous function such that $f(1) = 1$ and such that $\frac{x}{f(x)}$ and $\frac{f(x)}{\sqrt{x}}$ are non-decreasing. Let $\Delta$ be the group from [2] (in the expander case) for which the speed function is equivalent to $f(n)$. Write $W_n$ for the random walk on $\Delta$ with the appropriate generators.

1. If $\frac{f(x)}{\sqrt{x \log \log \log x}}$ is non-decreasing, then

\[
0 < \limsup_{n \to \infty} \frac{|W_n|}{\log \log n} f\left(\frac{n}{\log \log n}\right) < \infty
\]

almost surely.

2. If $\frac{f(x)}{\sqrt{x (\log \log x)^{1+\varepsilon}}}$ is non-decreasing for some $\varepsilon > 0$, then

\[
0 < \liminf_{n \to \infty} \frac{1}{\log \log n} f(n \log \log n) < \infty
\]
almost surely.

**Example 1.4.** Let $\frac{1}{2} < \alpha < 1$, and suppose that the speed function we choose is $f(n) = n^{\alpha}$. Then the theorem states that

$$P \left( 0 < \limsup_{n \to \infty} \frac{|W_n|}{n^\alpha (\log \log n)^{1-\alpha}} < \infty \right) = 1$$

and

$$P \left( 0 < \liminf_{n \to \infty} \frac{|W_n| (\log \log n)^{1-\alpha}}{n^\alpha} < \infty \right) = 1.$$

**Remark 1.5.** One could try to construct examples of groups satisfying a Law of Iterated Logarithm as in the example above by considering the wreath products $G \wr \mathbb{Z}$ where $G$ is taken to be a group with speed $n^\beta$, and then using Revelle’s results [17]. However, there are two caveats to this approach. First, the wreath product $G \wr \mathbb{Z}$ will have a rate of escape $n^{(\beta+1)/2}$, thus it could only provide examples with $\alpha \geq \frac{3}{4}$. Second, in order to apply Revelle’s theorems, one must know that the random walk on $G$ satisfies a tight degree of escape which amounts to proving tail bounds on the distance of the random walk on $G$, which requires the same kind of analysis done in our paper.

**Remark 1.6.** Throughout this article, we did not optimize the constants. Any unnumbered constant is some universal constant, but its value may change between claims. However, numbered constants keep their values for the rest of the paper.

### 2. Realizing Speed Functions with Diagonal Products

In this section we describe the groups from [2], and give an outline of the technique used in this article to estimate the speed of the random walk on these groups, proving Theorem 1.2. These groups are diagonal products of a sequence of lamplighter groups with finite lamp groups, where the lamp groups are chosen to be expanders or diffusive groups. In this article we focus on the expander case, although we first give the general description for the groups.

**2.1. Diagonal product.**

**Definition 2.1.** Let $X = \{x_1, \ldots, x_{|X|}\}$ be a set, and let $\{\Delta_s\}_{s \geq 0}$ be a sequence of groups. Suppose that each $\Delta_s$ is generated by a set $X(s)$ which we identify as a copy of $X$ in $\Delta_s$, i.e. $X(s) = \{x_1(s), \ldots, x_{|X|}(s)\}$. The **diagonal product of** $\{\Delta_s\}_{s \geq 0}$ **with respect to** $\{X(s)\}_{s \geq 0}$, which will be denoted $\Delta$, is the subgroup of the direct product $\prod_{s \geq 0} \Delta_s$ generated by the diagonal elements $(x_i(s))_{s \geq 0}$. 
Alternatively, we can construct $\Delta$ as follows: let $F(X)$ be the free group generated by $X$, and let $\pi_s : F(X) \to \Delta_s$ denote the natural projections. Then $\Delta = F(X)/\bigcap_{s \geq 0} \ker \pi_s$.

Note that $\Delta$ has a natural generating set, given by the diagonal elements.

**Remark 2.2.** If $X$ has a natural algebraic structure, for example a union of several groups, it is natural to require that the above construction will be compatible with that structure. This can be done by requiring that the identifications of $X(s)$ respect that structure.

### 2.2. The lamp groups.

Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$ be two groups. Let $\{\Gamma_s\}_{s \geq 0}$ be a sequence of groups, where each $\Gamma_s$ is generated by a set of the form $A(s) \cup B(s)$, where $A(s)$ and $B(s)$ are subgroups of $\Gamma_s$ isomorphic to $A$ and $B$ respectively.

Fix a sequence of strictly increasing integers $\{k_s\}$. For each $s$ let $\Delta_s = \Gamma_s \rtimes \mathbb{Z}$, with a generating set given by $\tau(s) = (e, +1)$, $\alpha_i(s) = (a_i(s) \delta_{k_s}, 0)$ for all $1 \leq i \leq |a|$, and $\beta_j(s) = (b_j \delta_{k_s}, 0)$ for all $1 \leq j \leq |B|$.

Finally, we take $\Delta$ to be the diagonal product of the groups $\Delta_s$ with respect to the above generating sets, marked with the generating set $T = (\tau, \alpha_1, \ldots, \alpha_{|A|}, \beta_1, \ldots, \beta_{|B|})$. If $U_\alpha$ and $U_\beta$ are the uniform measures on the subgroups $A = \{\alpha_1, \ldots, \alpha_{|A|}\}$ and $B = \{\beta_1, \ldots, \beta_{|B|}\}$, and $\mu$ is the uniform measure on $\{\tau, \tau^{-1}\}$, we use the “switch-walk-switch” measure for our random walk on $\Delta$, that is

$$q = (U_\alpha * U_\beta) * \mu * (U_\alpha * U_\beta).$$

Denote by $W_n$ the random walk on $\Delta$ with step distribution $q$. By the choice of $q$, we may write $W_n = (f^{W_n}_s, S_n)$, where $S_n$ is a simple random walk on $\mathbb{Z}$, and $f^{W_n}_s$ denotes the lamp configuration at the layer $\Delta_s$ at time $n$.

Following [2], we make the following assumptions on the groups $\Gamma_s$:

**Assumption 2.3.**

1. $k_0 = 0$ and $\Gamma_0 = A(0) \times B(0) \cong A \times B$;
2. Let $[A(s), B(s)]^{\Gamma_s}$ denote the normal closure of the subgroup generated by commutators $[a_i(s), b_j(s)]$. Then we assume $\Gamma_s/[A(s), B(s)]^{\Gamma_s} \cong A(s) \times B(s) \cong A \times B$.
3. Letting $l_s = \text{diam}(\Gamma_s)$, we assume that $k_s$ and $l_s$ grow at least exponentially.

In [2] the authors treat two cases of families of groups $\Gamma_s$: one of them is when the random walks on $\Gamma_s$ are diffusive, and the other one is when the groups $\{\Gamma_s\}$ satisfy the following linear speed assumption:
Definition 2.4 ([2 Definition 3.1]). Let \( \{ \Gamma_s \} \) be a sequence of finite groups where each \( \Gamma_s \) is marked with a generating set \( A(s) \cup B(s) \). Let \( \eta_s = U_{A(s)} \ast U_{B(s)} \ast U_{A(s)} \) where \( U_{A(s)} \), \( U_{B(s)} \) are uniform distribution on \( A(s) \), \( B(s) \). We say \( \{ \Gamma_s \} \) satisfies the \((\sigma, T_s)\)-linear speed assumption if in each \( \Gamma_s \),

\[
L_{\eta_s}(t) = \mathbb{E} \left| X_t^{(s)} \right|_{\Gamma_s} \geq \sigma t \text{ for all } t \leq T_s
\]

where \( X_t^{(s)} \) has distribution \( \eta_s^t \).

Note that for the definition to be meaningful, we must have \( \sigma \leq 1 \).

In this paper, we focus only on the case where the groups satisfy the linear speed assumption. Such groups can be constructed for instance using Lafforgue super expanders or with lamplighter groups over a \( d \)-dimensional infinite dihedral group, see [2 Examples 3.2 and 3.3].

2.3. Excursions and the speed in a single layer. In a single layer \( \Delta_s = \Gamma_s \wr \mathbb{Z} \), the lamp value at \( x \in \mathbb{Z} \) can only be changed when the random walk reaches \( x \) or \( x - k_s \). As \( A(s) \) is a subgroup and the measure chosen on it is uniform, a product of random elements from \( A(s) \) is again a uniform element from \( A(s) \) (and the same holds for \( B(s) \)). Therefore, conditioning on the trajectory of the simple random walk \( S_t \), the distribution of the lamp value at \( x \in \mathbb{Z} \) depends only on the number of \( k_s \)-excursions:

Definition 2.5. Let \( S_t \) be a random walk on \( \mathbb{Z} \), and let \( k > 0 \). We say that the random walk begins a \( k \)-excursion in time \( j \), if it visits \( S_j - k \) before its next visit in \( S_j \). We denote by \( T(k, x, n) \) the number of \( k \)-excursions the random walk \( S_t \) performs before time \( n \) starting at \( x \).

To estimate the speed of the random walk on \( \Delta_s \), Brieussel and Zheng give bounds on the speed in terms of the number of \( k_s \)-excursions and the diameter \( l_s = \text{diam}(\Gamma_s) \). By dividing the range into intervals of length \( k_s \) and working in each one separately one after the other, one can show that for some universal constant \( C > 0 \),

\[
|W_n|_{\Delta_s} \leq \frac{C k_s}{2} \sum_{j \in \mathbb{Z}} T \left( \frac{k_s}{2}, \frac{k_s}{2}, j, n \right) + C |\text{range}(S_n)|,
\]

which has an expected value of \( \frac{C'}{k_s} \cdot \frac{n}{k_s} \leq \frac{C'}{k_s} \cdot \frac{n}{l_s} \) if \( k_s \leq \sqrt{n} \). Also, since \( l_s = \text{diam}(\Gamma_s) \), we also have the bound

\[
|W_n|_{\Delta_s} \leq |\text{range}(S_n)| l_s,
\]
which has an expected value $\lesssim \sqrt{n} l_s$. Together we get

$$\mathbb{E} |W_n|_{\Delta_s} \lesssim \sqrt{n} l_s + \frac{n}{k_s}.$$  

Brieussel and Zheng show that this also gives a matching lower bound because of the linear speed assumption, so one gets the asymptotic behavior of the speed of the random walk on $\Delta_s$.

### 2.4. Estimating the Speed of the Random Walk

The random walk on $\Delta$ can be thought of as parallel random walks on each $\Delta_s$, so in order to estimate the speed in $\Delta$ one needs to use the estimates for the speed in each layer $\Delta_s$. In [2], Brieussel and Zheng show that there is some universal constant $C > 0$ such that

$$(2) \quad |W_n|_{\Delta_s} \leq |W_n|_{\Delta} \leq C \sum_{s: k_s l_s \leq \sqrt{n}} |W_n|_{\Delta_s}.$$  

As the speed in each layer $\Delta_s$ is of order $\sqrt{n} l_s + \frac{n}{k_s}$ and the sequences $\{k_s\}$ and $l_s$ grow at least exponentially, it follows that for each $n$ there is a layer $s_1(n) = \max \{s \mid k_s l_s \leq \sqrt{n}\}$ that determines the speed of the random walk, i.e. $\mathbb{E}[|W_n|_{\Delta}]$ is equivalent to $\mathbb{E}[|W_n|_{\Delta_{s_1(n)}}]$ up to universal constants. This proves the following:

**Proposition 2.6** ([2, Proposition 3.6]).

$$\mathbb{E}[|W_n|_{\Delta}] \simeq \mathbb{E}[|W_n|_{\Delta_{s_1(n)}}] \simeq \sqrt{n} l_{s_1(n)} + \frac{n}{k_{s_1(n)} + 1}.$$  

This proposition allows one to realize various functions as the speed functions of such groups. To do so, given a function $f : \mathbb{N} \to \mathbb{N}$ one must choose appropriate sequences $\{k_s\}$ and $\{l_s\}$. This is done in the following way:

**Proposition 2.7** ([2, Corollary B.3]). Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\frac{f(n)}{\sqrt{n}}$ and $\frac{n}{f(n)}$ are non-decreasing, and let $m_0 > 1$. Then one can find sequences $\{k_s\}$ and $\{l_s\}$ of positive integers such that $k_{s+1} \geq m_0 k_s$, $l_{s+1} \geq m_0 l_s$, and also such that

$$f(n) \simeq \tilde{f}(n)$$  

where

$$\tilde{f}(x) = \sqrt{x} l_s + \frac{x}{k_{s+1}} \quad \text{if} \quad (k_s l_s)^2 \leq x < (k_{s+1} l_{s+1})^2.$$  

This concludes the proof of Theorem 1.2.
3. Sketch of the proof

As mentioned in the introduction, in order to show that
\[
0 < \limsup_{n \to \infty} \frac{|W_n|_{\Delta}}{a_n} < \infty
\]
almost surely, we need to prove that there are constants \( C, c > 0 \) such that \( Ca_n \) is an outer radius but \( ca_n \) is not an outer radius. For the analogous claim
\[
0 < \liminf_{n \to \infty} \frac{|W_n|_{\Delta}}{b_n} < \infty
\]
we need to prove that there are constants \( C', c' > 0 \) such that \( c'b_n \) is an inner radius for \( |W_n|_{\Delta} \), but \( C'b_n \) is not an inner radius.

Using a similar strategy as in [2], we first study the distance of the random walk on a single layer \( \Delta_s \), and then deduce a bound for the distance of the random walk on \( \Delta \). For a single layer, Brieussel and Zheng provide an upper bound for the distance of the random walk in terms of \( k \)-excursions \( (1) \), and a lower bound on the speed of the random walk
\[
(3) \quad \mathbb{E}[|W_n|_{\Delta_s}] \geq c \mathbb{E} \left[ \sum_{x \in \mathbb{Z}} \min \{ T(k_s, x, n), l_s \} \right].
\]

To estimate the fluctuations of the distance of the random walk, we use the same upper bound \( (1) \), and prove that \( (3) \) holds also for the distance and not only for the expectation, i.e. for large enough \( n \)
\[
|W_n|_{\Delta_s} \geq c \sum_{x \in \mathbb{Z}} \min \{ T(k_s, x, n), l_s \}.
\]

As such, we study the tail behavior of \( T(k, n) = k \sum_{x \in \mathbb{Z}} T(k, kx, n) \) and of \( \sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \) for all \( k, l \). This gives a bound on the distance in a single layer in terms of \( k_s, l_s \), which we then add up by \( (2) \) to bound the distance of the random walk on \( \Delta \).

Because of the Law of Iterated Logarithm on \( \mathbb{Z} \), we know that there are times in which the range is small or large compared to its expected value. We show that in these times, the distance reaches its \( \lim \inf \) and \( \lim \sup \) values, respectively. As in the case of the speed of the random walk, we prove that there is one layer \( \Delta_s \) that is equivalent to the distance in \( \Delta \); however, since we consider the times in which the range is extremal, we need to choose different copies rather than \( s_1(n) \).

The article is constructed as follows. In Section 4 we associate another random walk to \( S_s \), so that the numbers of \( k \)-excursions of \( S_s \) at all points correspond to the local times of the new random walk. To do
so we use the results in Appendix A, which study the induced random walk of $S_t$ on $k\mathbb{Z}$. In Section 5 we study the tail behavior of $T(k, n)$, showing that under appropriate assumptions it is highly concentrated around its mean $n/2$. In Sections 6 and 7 we study the tail behavior of $\sum_{x \in \mathbb{Z}} \min\{T(k, x, n), l\}$, for the two critical layers. These results rely on the tail behavior of the maximal local time of a random walk, which we study in Appendix B.

We then return to the random walk on $\Delta$. In Section 8 we prove the lower bound for the distance of the random walk in terms of the $k$-excursions. Section 9 is devoted to deduce the bounds for the distance in a single layer, and in Section 10 we use the bounds for a single layer to bound the distance of the random walk on $\Delta$. Finally, in Section 11 we express our results in terms of the speed function, and conclude the proof of Theorem 1.3.

Remark 3.1. In Theorem 1.3 we demand that the random walks are slightly super-diffusive. These conditions stem from the fact that the more diffusive the random walk is, the harder it is to get strong concentration inequalities for the number of $k$-excursions.

For the lim sup statement (Theorem 1.3 (1)), this manifests in not having a precise enough bound for the contribution of the layers near the lim sup critical layer. It may be that by having a better understanding of the joint behaviour of the distance in those layers one could improve the bounds.

For the lim inf statement (Theorem 1.3 (2)), the difficulty arises from the sequence of parameters $k_s, l_s$ used in the approximation of the speed function. When $f$ is almost diffusive these sequences may grow extremely fast. One can still write an explicit formula for the lim inf in terms of $k_s, l_s$ (see Proposition 10.2), however this may not coincide with the expression in the theorem.

4. The $k$-induced random walk

Let $S_t$ be a simple random walk on $\mathbb{Z}$. Recall that $T(k, x, n)$ denotes the number of $k$-excursions starting from $x$ that are completed until time $n$. To study the distribution of $T(k, x, n)$ for each $k, x, n$, we associate a new simple random walk on $\mathbb{Z}$ to $S_t$, so that $k$-excursions in $S_t$ can be approximated by the local times of the new random walk.

Definition 4.1. Let $S_t$ be a simple random walk on $\mathbb{Z}$, and let $k \geq 0$. The $k$-induced random walk of $S_t$ is the simple random walk $Y^{(k)}_n$ on $\mathbb{Z}$ defined in the following manner. We first set $n_0^{(k)} = 0$, and for
any \( j \geq 1 \) define inductively
\[
n_{j}^{(k)} = \min \left\{ t > n_{j-1}^{(k)} \left| S_{t} - S_{n_{j-1}^{(k)}} = k \right. \right\}.
\]

The random walk \( Y_{t}^{(k)} \) is then given by \( Y_{t}^{(k)} = S_{n_{t}^{(k)}}/k \). We also write \( N_{k}(n) = \max \left\{ j \left| n_{j}^{(k)} \leq n \right. \right\} \) for the number of steps \( Y_{n}^{(k)} \) made until time \( n \).

It is easy to see that \( Y_{n}^{(k)} \) is indeed a simple random walk on \( \mathbb{Z} \). In addition, as the expected time to reach \( \pm k \) starting from 0 is \( k^{2} \), we have \( \mathbb{E}[N_{k}(n)] = \frac{n}{k^{2}} \).

**Proposition 4.2.** There are universal constants \( d_{1}, c, C > 0 \) such that for all \( n \) and for all \( k \leq \frac{d_{1}\sqrt{n}}{\sqrt{\log \log n}} \),
\[
\mathbb{P} \left( N_{k}(n) \notin \left[ \frac{cn}{k^{2}}, \frac{Cn}{k^{2}} \right] \right) \leq \frac{2}{\log^{3} n}.
\]

**Proof.** By choosing appropriate \( c, c' > 0 \), Appendix A shows that
\[
\mathbb{P} \left( N_{k}(n) \notin \left[ \frac{cn}{k^{2}}, \frac{Cn}{k^{2}} \right] \right) \leq 2 \exp \left( -\frac{n}{20k^{2}} \right).
\]
Note that if \( k \leq \frac{\sqrt{n}}{8\sqrt{\log \log n}} \), we have \( \frac{n}{k^{2}} \geq 64 \log \log n \), so
\[
\exp \left( -\frac{n}{20k^{2}} \right) \leq \exp \left( -3 \log \log n \right) = \frac{1}{\log^{3} n}
\]
completing the proof. \( \square \)

We turn to give bounds on the \( k \)-excursions of \( S_{t} \) by means of the induced random walks. Denote the local time of \( Y_{t}^{(k)} \) at \( x \) by
\[
L^{(k)}(x, n) = \left| \left\{ 0 \leq t \leq n \left| Y_{t}^{(k)} = x \right. \right\} \right|,
\]
and let
\[
\ell^{(k)}(x, n) = \left| \left\{ 0 \leq t < n \left| Y_{t}^{(k)} = x, Y_{t+1}^{(k)} = x - 1 \right. \right\} \right|.
\]

**Proposition 4.3.** For any \( x \in \mathbb{Z} \),
\[
\ell^{(2k)} \left( \left\lfloor \frac{x}{2k} \right\rfloor, n \right) - 1 \leq T(k, x, n) \leq L^{(k/2)} \left( \left\lfloor \frac{x}{k/2} \right\rfloor, n \right).
\]

We remark that when \( k \) is not an integer, it can be replaced with \( \left\lfloor \frac{k}{2} \right\rfloor \) and the proposition still holds. However, we treat \( \frac{k}{2} \) as an integer for convenience.
Proof. For the upper bound, we write $x = \frac{k}{2} j + r$ with $0 \leq r < \frac{k}{2}$. Any $k$-excursion starting at $x$ must include at least one $\frac{k}{2}$-excursion starting at $\frac{k}{2} j$, so $T(k, x, n) \leq T \left( \frac{k}{2}, \frac{k}{2} j, n \right) \leq L(\frac{k}{2})(j, n)$.

For the lower bound, write $x = 2kj' + r'$ with $0 \leq r' < 2k$. A similar argument shows that $T(2k, 2k(j' + 1), n) \leq T(k, x, n)$. We note that $f^{(2k)}(2k(j' + 1), n)$ counts the number of $2k$-excursions starting at $2k(j' + 1)$ which are completed before time $n$, with the possibility that we don’t finish the last $2k$-excursion before time $n$. This proves the lower bound.

5. The total number of $k$-excursions

We turn to give upper and lower bounds for

$$T(k, n) = \sum_{x \in \mathbb{Z}} kT(k, kx, n).$$

Note that this also provides bounds on the total number of $k$-excursions, by the following proposition:

**Proposition 5.1.** For any $n, k$,

$$\frac{1}{2} T(2k, n) \leq \sum_{x \in \mathbb{Z}} T(k, x, n) \leq T \left( \frac{k}{2}, n \right).$$

**Proof.** For the lower bound, note that any $2k$-excursion starting at $2kx$ must contain at least one $k$-excursion from each of $2kx - k, \ldots, 2kx$. Therefore

$$T(2k, 2kx, n) \leq \frac{1}{k} \sum_{j=0}^{k-1} T(k, 2kx - j, n)$$

and we have

$$T(2k, n) = 2k \sum_{x \in \mathbb{Z}} T(2k, 2kx, n) \leq 2 \sum_{x \in \mathbb{Z}} T(k, x, n).$$

For the upper bound, notice that any $k$-excursion from $\frac{k}{2} x + j$ for some $0 \leq j < \frac{k}{2}$ contains at least one $\frac{k}{2}$-excursion from $\frac{k}{2} x$. Therefore

$$\sum_{x \in \mathbb{Z}} T(k, x, n) = \sum_{x \in \mathbb{Z}} \sum_{j=0}^{k/2-1} T \left( k, \frac{k}{2} x + j, n \right) \leq$$

$$\leq \frac{k}{2} \sum_{x \in \mathbb{Z}} T \left( \frac{k}{2}, \frac{k}{2} x, n \right) = T \left( \frac{k}{2}, n \right).$$

$$\square$$
Before studying the asymptotic behavior of $T(k, n)$, we prove an (almost) monotonicity result for $T(k, n)$:

**Lemma 5.2.** Suppose $2k \leq k'$. Then $T(k', n) \leq 3T(k, n)$.

**Proof.** For any $x \in \mathbb{Z}$, each $k'$-excursion from $k'x$ contains at least $\left\lfloor \frac{k'}{k} \right\rfloor - 1$ many $k$-excursions from points $k\mathbb{Z}$. Therefore

$$T'(k', n) = k' \sum_{x \in \mathbb{Z}} T(k', k'x, n) \leq k' \left\lfloor \frac{k'}{k} \right\rfloor - 1 \sum_{x \in \mathbb{Z}} T(k, kx, n) = \frac{k'}{k} - 1 \sum_{x \in \mathbb{Z}} T(k, kx, n) \leq 3T(k, n).$$

For specific values of $n$ and $k$ of order at most $\sqrt{\frac{n}{\log \log n}}$, we use the $k$-induced random walks defined in Section 4 to get a concentration result:

**Lemma 5.3.** There are universal constants $d_2, c, C > 0$ such that the following holds: For large enough $n$, if $k \leq \frac{d_2 \sqrt{n}}{\log \log n}$,

$$P \left( T(k, n) \notin \left[ \frac{cn}{k}, \frac{Cn}{k} \right] \right) \leq \frac{6}{\log^3 n}.$$

**Proof.** Using Proposition 4.2, let $d_1, c, C > 0$ be the constants such that

$$P \left( N_k(n) \notin \left[ \frac{cn}{k^2}, \frac{Cn}{k^2} \right] \right) \leq \frac{2}{\log^3 n}$$

for all $k \leq \frac{d_1 \sqrt{n}}{\log \log n}$. Then for any $k \leq \frac{d_1 \sqrt{n}}{\log \log n}$, by Proposition 4.3

$$P \left( T(k, n) > \frac{4cn}{k} \right) \leq P \left( k \sum_{x \in \mathbb{Z}} L^{(k/2)}(x, n) > \frac{4cn}{k} \right) = P \left( N_{k/2}(n) > \frac{Cn}{(k/2)^2} \right) \leq \frac{2}{\log^3 n}.$$

For the lower bound, we use again Proposition 4.3

$$P \left( T(k, n) < \frac{k}{8} N_{2k}(n) \right) \leq P \left( k \sum_{x \in \mathbb{Z}} (\ell^{(2k)}(x, n) - 1) < \frac{k}{8} N_{2k}(n) \right) \leq P \left( \sum_{x \in \mathbb{Z}} (\ell^{(2k)}(x, n) - 1) < \frac{1}{8} N_{2k}(n) \right).$$
Let $Z = \{|x : L^{(2k)}(x, n) > 0\}|$. This is the range of a simple random walk on $\mathbb{Z}$ after $N_{2k}(n)$ steps. We thus have

$$P\left(Z \geq \frac{1}{8}N_{2k}(n) \bigg| N_{2k}(n)\right) \leq \exp\left(-\frac{(N_{2k}(n)/8)^2}{2N_{2k}(n)}\right) = \exp\left(-\frac{N_{2k}(n)}{128}\right),$$

so

$$P\left(T(k, n) < \frac{k}{8}N_{2k}(n) \bigg| N_{2k}(n)\right) \leq
\leq P\left(\sum_{x \in \mathbb{Z}} \ell^{(2k)}(x, n) < \frac{1}{8}N_{2k}(n) + Z \bigg| N_{2k}(n)\right) \leq
\leq P\left(\sum_{x \in \mathbb{Z}} \ell^{(2k)}(x, n) < \frac{1}{4}N_{2k}(n) \bigg| N_{2k}(n)\right) + \exp\left(-\frac{N_{2k}(n)}{128}\right).$$

Conditioning on $N_{2k}(n)$, $\sum_{x \in \mathbb{Z}} \ell^{(2k)}(x, n)$ is a binomial random variable, with parameters $N_{2k}(n)$ and $\frac{1}{2}$. Therefore, by Chernoff’s inequality,

$$P\left(\sum_{x \in \mathbb{Z}} \ell^{(2k)}(x, n) < \frac{1}{4}N_{2k}(n) \bigg| N_{2k}(n)\right) \leq \exp\left(-\frac{1}{16}N_{2k}(n)\right).$$

We therefore have

$$P\left(T(k, n) < \frac{k}{8}N_{2k}(n) \bigg| N_{2k}(n)\right) \leq 2\exp\left(-\frac{1}{128}N_{2k}(n)\right),$$

so

$$P\left(T(k, n) < \frac{cn}{16k}\right) \leq
\leq P\left(N_{2k}(n) < \frac{cn}{2k^2}\right) + P\left(T(k, n) < \frac{cn}{16k} \bigg| N_{2k}(n) \geq \frac{cn}{2k^2}\right) \leq
\leq \frac{2}{\log^3 n} + P\left(T(k, n) < \frac{k}{8}N_{2k}(n) \bigg| N_{2k}(n) \geq \frac{cn}{2k^2}\right) \leq
\leq \frac{2}{\log^3 n} + 2\exp\left(-\frac{cn}{256k^2}\right).$$

Let $d_2 > 0$ be a constant such that $d_2^2 \leq \sqrt{c/768}$. For any $k \leq \frac{d_2\sqrt{n}}{\log \log n}$,

$$\frac{n}{k^2} \geq \frac{1}{d_2^2} \log \log n \geq \frac{768}{c} \log \log n,$$
and thus
\[ P \left( T(k, n) < \frac{cn}{16k} \right) \leq \frac{4}{\log^3 n} \]
concluding the proof. \(\square\)

We are now ready to get a concentration result for all values of \(n, k\) simultaneously:

**Proposition 5.4.** There are universal constants \(d_2, c_1, C_1 > 0\) such that the following holds almost surely: for all but finitely many \(n\) and for all \(k \leq \frac{d_2 \sqrt{tn}}{2 \log \log n}\),
\[ \frac{c_1 n}{k} \leq T(k, n) \leq \frac{C_1 n}{k} . \]

**Proof.** We choose an exponential sequence of times \(t_m = 2^m\), and for each \(m\) an exponential sequence \(k_{m,i} = 2^i\) defined for all \(i\) such that \(k_{m,i} \leq \frac{d_2 \sqrt{tn}}{2 \log \log t_m}\). Note that for each \(m\) there are at most \(K \log t_m\) such indices for some constant \(K\).

By Lemma 5.3 there are constants \(c, C > 0\) such that for each \(m, i\),
\[ P \left( \frac{ct_m}{k_{m,i}} \leq T(k_{m,i}, t_m) \leq \frac{Ct_m}{k_{m,i}} \right) \leq \frac{6}{\log^2 t_m} . \]
Taking a union bound over \(i\), we have
\[ P \left( \exists i : \frac{ct_m}{k_{m,i}} \leq T(k_{m,i}, t_m) \leq \frac{Ct_m}{k_{m,i}} \right) \leq \frac{6K}{\log^2 t_m} = \frac{6K}{m^2 \log^2 2} . \]
As the latter expression is summable over \(m\), by the Borel-Cantelli lemma we have that for all but finitely many \(m\), for all \(i\)
\[ \frac{ct_m}{k_{m,i}} \leq T(k_{m,i}, t_m) \leq \frac{Ct_m}{k_{m,i}} . \]

We extend this result to general \(n, k\) in two steps.

First, let \(m\) be such that (4) holds for all \(i\), and let \(k \leq \frac{d_2 \sqrt{tn}}{2 \log \log t_m}\). Take \(i\) such that \(k_{m,i} \leq k < k_{m,i+1}\). By Lemma 5.2.
\[ T(k, t_m) \leq 3T(k_{m,i-1}, t_m) \leq \frac{3Ct_m}{k_{m,i-1}} \leq \frac{6Ct_m}{k} . \]
and
\[ T(k, t_m) \geq \frac{1}{3} T(k_{m,i+1}, t_m) \geq \frac{ct_m}{3k_{m,i+1}} \geq \frac{ct_m}{6k} . \]
Finally, take any \(n\), and let \(m\) be such that \(t_m \leq n < t_{m+1}\). Suppose \(n\) is large enough so that (4) holds. Then
\[ T(k, n) \leq T(k, t_{m+1}) \leq \frac{6Ct_{m+1}}{k} \leq \frac{12Cn}{k} . \]
and
\[ T(k, n) \geq T(k, t_m) \geq \frac{ct_m}{6k} \geq \frac{cn}{12k} \]
as required.

6. The truncated sum of \( k \)-excursions, lower layer

We turn to study the sum \( \sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \) for \( k, l > 0 \). Our two cases of interest are when \( kl \sim \sqrt{n \log \log n} \) (which will turn out to be the lim inf layer) and when \( kl \sim \frac{\sqrt{n}}{\sqrt{\log \log n}} \) (which will be the lim sup layer). We begin with the lim inf layer; for this we study the maximal number of \( k \)-excursions from a given point, i.e. \( \max_{x \in \mathbb{Z}} T(k, x, n) \).

6.1. The maximal number of \( k \)-excursions. Similarly to the way we estimated \( T(k, n) \), we get tail bounds on the maximal number of \( k \)-excursions in two steps. We first use the induced random walks to get a tail bound for given values of \( n \) and \( k \), and then use exponential scales to combine these bounds into a bound for the maximal number of \( k \)-excursions.

**Lemma 6.1.** There is a universal constant \( c > 0 \) such that for all large enough \( n \) and for all \( k \leq \frac{d_1 \sqrt{n}}{\sqrt{\log \log n}} \),
\[
P\left( \max_{x \in \mathbb{Z}} T(k, x, n) \geq \frac{c \sqrt{n \log \log n}}{k} \right) \leq \frac{3}{\log^3 n}.
\]

**Proof.** We use the induced random walks from Section 4. By Proposition 4.3
\[
P\left( \max_{x \in \mathbb{Z}} T(k, x, n) \geq \frac{c \sqrt{n \log \log n}}{k} \right) \leq \leq P\left( \max_{x \in \mathbb{Z}} L^{(k/2)}(x, n) \geq \frac{c \sqrt{n \log \log n}}{k} \right).
\]
From Appendix B
\[
P\left( \max_{x \in \mathbb{Z}} L(x, n) \geq xn^{1/2} \right) \leq Cx^2 \exp\left( -C'x^2 \right)
\]
for large enough \( n, x \). Let \( c' > 0 \) be a constant such that for \( k \leq \frac{d_1 \sqrt{n}}{\sqrt{\log \log n}} \)
\[
P\left( N_{k/2}(n) \geq \frac{c' n}{k^2} \right) \leq \frac{2}{\log^3 n}
\]
Using Proposition 4.2,

\[
P \left( \max_{x \in \mathbb{Z}} L^{(k/2)}(x, n) \geq \frac{c \sqrt{n \log \log n}}{k} \right) \leq \\
\leq P \left( N_{k/2}(n) \geq \frac{c' n}{k^2} \right) + P \left( \max_{x \in \mathbb{Z}} \left( x, \frac{c' n}{k^2} \right) \geq \frac{c \sqrt{n \log \log n}}{k} \right) \leq \\
\leq \frac{2}{\log^3 n} + \frac{c^2}{c'} \log \log n \exp \left( -\frac{c^2}{c'} \log \log n \right).
\]

Finally, choosing \( c = 2\sqrt{c'} \), we have

\[
\frac{c^2}{c'} \log \log n \exp \left( -\frac{c^2}{c'} \log \log n \right) = \frac{2 \log \log n}{\log^4 n} \leq \frac{1}{\log^3 n}
\]

for large enough \( n \), as required. \( \square \)

**Proposition 6.2.** There are universal constants \( d_1, C > 0 \) such that the following holds almost surely: For all but finitely many \( n \) and for all \( k \leq \frac{d_1 \sqrt{n}}{2 \sqrt{\log \log n}} \),

\[
\max_{x \in \mathbb{Z}} T(k, x, n) \leq \frac{C \sqrt{n \log \log n}}{k}.
\]

**Proof.** We choose an exponential sequence of times \( t_m = 2^m \), and for each \( m \) an exponential sequence \( k_{m,i} = 2^i \) defined for all \( i \) such that \( k_{m,i} \leq \frac{\sqrt{t_m}}{2 \sqrt{\log \log t_m}} \). Note that for each \( m \) there are at most \( \log t_m \) such indices.

By Lemma 6.1, there is a constant \( c > 0 \) such that for all \( m, i \) we have

\[
P \left( \max_{x \in \mathbb{Z}} T(k_{m,i}, x, t_m) \geq \frac{c \sqrt{t_m \log \log t_m}}{k_{m,i}} \right) \leq \frac{3}{\log^3 t_m}.
\]

Taking a union bound,

\[
P \left( \exists i : \max_{x \in \mathbb{Z}} T(k_{m,i}, x, t_m) \geq \frac{c \sqrt{t_m \log \log t_m}}{k_{m,i}} \right) \leq \frac{3}{\log^2 t_m} = \frac{3}{m^2 \log^2 m}.
\]

The latter expression is summable over \( m \), so by Borel-Cantelli lemma we have that for all but finitely many \( m \), for all \( i \) we have

\[
\max_{x \in \mathbb{Z}} T(k_{m,i}, x, t_m) \leq \frac{c \sqrt{t_m \log \log t_m}}{k_{m,i}}.
\]

Let \( n, k \) be such that \( k \leq \frac{d_1 \sqrt{n}}{2 \sqrt{\log \log n}} \). Take \( m \) such that \( t_m \leq n < t_{m+1} \), and suppose \( n \) is large enough so that (5) holds. Take \( i \) such that
\( k_{m,i} \leq k < k_{m,i+1} \). Then

\[
\max_{x \in \mathbb{Z}} T(k, x, n) \leq \max_{x \in \mathbb{Z}} T(k_{m,i}, x, t_{m+1}) \leq \frac{c \sqrt{t_{m+1} \log \log t_{m+1}}}{k_{m,i}} \leq \frac{4c\sqrt{n \log \log n}}{k}
\]

as required. \( \square \)

### 6.2. The truncated sum

We are now ready to get an asymptotic bound for the truncated sum of \( k \)-excursions.

**Proposition 6.3.** There are universal constants \( d_2, c_2 > 0 \) such that the following holds almost surely: for all but finitely many \( n \), for all \( k \leq \frac{d_2 \sqrt{n}}{2 \sqrt{\log \log n}} \) and for all \( l \),

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq c_2 \min \left\{ \frac{n}{k}, \frac{\sqrt{n}}{\sqrt{\log \log n}} l \right\}.
\]

**Proof.** We first assume that \( l \geq \frac{\sqrt{n \log \log n}}{k} \). By Proposition 6.2, there is a constant \( C > 0 \) such that for large enough \( n \) we have

\[
\max_{x \in \mathbb{Z}} T(k, x, n) \leq \frac{C \sqrt{n \log \log n}}{k} \leq C l,
\]

so by Proposition 5.4

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq \frac{1}{C} \sum_{x \in \mathbb{Z}} T(k, x, n) \geq \frac{1}{2C} T(2k, n) \geq \frac{c'n}{k}
\]

for large enough \( n \) and a universal constant \( c' \).

Now suppose \( l < \frac{\sqrt{n \log \log n}}{k} \). Let

\[
A_1 = \{ x \in \mathbb{Z} \mid T(k, x, n) < l \}, \quad A_2 = \{ x \in \mathbb{Z} \mid T(k, x, n) \geq l \}.
\]

Note that by Proposition 5.1 and Proposition 5.4, for large enough \( n \) we have

\[
\frac{c_1 n}{4k} \leq \frac{1}{2} T(2k, n) \leq \sum_{x \in \mathbb{Z}} T(k, x, n) = \sum_{x \in A_1} T(k, x, n) + \sum_{x \in A_2} T(k, x, n).
\]

We split into two cases:

1. If \( \sum_{x \in A_1} T(k, x, n) \geq c_1 n \frac{8k}{8k} \), then clearly

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq \sum_{x \in A_1} T(k, x, n) \geq c_1 n \frac{8k}{8k} \geq \frac{c_1 \sqrt{n}}{8 \sqrt{\log \log n}} l.
\]
Otherwise, we must have \( \sum_{x \in A_2} T(k, x, n) \geq \frac{c_1 n}{8k} \). By Proposition 6.2,

\[
\frac{c_1 n}{8k} \leq \sum_{x \in A_2} T(k, x, n) \leq \frac{c \sqrt{n \log \log n}}{k} \cdot |A_2|,
\]

so \( |A_2| \geq \frac{c_1 \sqrt{n}}{8c \sqrt{\log \log n}} \), showing

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq l \cdot |A_2| \geq \frac{c_1 \sqrt{n}}{8c \sqrt{\log \log n}} l.
\]

The proposition is thus proved. \( \square \)

7. The truncated sum of \( k \)-excursions, upper layer

For the remaining case, we switch strategy. We first need the exact distribution of \( T(k, 0, n) \), which can be found e.g. in [8]. The main idea is the following: We use the reflection principle successively, reflecting the random walk whenever we first visit any \( jk \) for any \( j < 0 \). In this manner, the number of \( k \)-excursions starting from 0 has the same distribution (up to a factor of 2\( k \)) as the minimum (or maximum) of the random walk. This proves the following:

**Lemma 7.1** ([8]). For any \( a \in \mathbb{N} \),

\[
P \left( T(k, 0, n) \geq a \right) = P \left( \min_{0 \leq t \leq n} S_t \leq -2ka \right) = P \left( \max_{0 \leq t \leq n} S_t \geq 2ka \right).
\]

**Lemma 7.2.** There are universal constants \( c, c' > 0 \) such that for large enough \( n \) and for all \( k \leq \frac{c \sqrt{n}}{\sqrt{\log \log n}} \),

\[
P \left( T(k, 0, n) \leq \frac{c \sqrt{n}}{k \sqrt{\log \log n}} \right) \leq \frac{c'}{\log^3 n}.
\]

**Proof.** By the reflection principle (see Lemma 7.1),

\[
P \left( T(k, 0, n) \leq \frac{c \sqrt{n}}{k \sqrt{\log \log n}} \right) \leq P \left( T(k, 0, n) \leq \left\lfloor \frac{c \sqrt{n}}{k \sqrt{\log \log n}} \right\rfloor \right) =
\]

\[
= P \left( \max_{0 \leq t \leq n} S_t \leq 2k \left\lfloor \frac{c \sqrt{n}}{k \sqrt{\log \log n}} \right\rfloor \right) \leq
\]

\[
\leq P \left( \max_{0 \leq t \leq n} S_t \leq \frac{2c \sqrt{n}}{\sqrt{\log \log n}} + 2k \right) \leq
\]

\[
\leq P \left( \max_{0 \leq t \leq n} S_t \leq \frac{4c \sqrt{n}}{\sqrt{\log \log n}} \right).
\]
where the last inequality follows from \( k \leq \frac{\sqrt{n}}{\log \log n} \). To bound the latter probability, we use [9, Lemma 2]. Following their proof, one can choose \( c = \frac{\pi}{4} \) and get
\[
P \left( \max_{0 \leq t \leq n} S_t \leq \frac{\pi \sqrt{n}}{4 \sqrt{\log \log n}} \right) \leq \frac{c'}{\log^3 n}
\]
for large enough \( n \), concluding our proof.

\[\square\]

**Proposition 7.3.** There exists a universal constant \( c > 0 \) such that the following holds almost surely: For all but finitely many values of \( n \) and for all \( k \leq \frac{\sqrt{n}}{2 \sqrt{\log \log n}} \),
\[
\sum_{x \in \mathbb{Z}} \min \left\{ T(k, x, n), \frac{\sqrt{n}}{k \sqrt{\log \log n}} \right\} \geq \frac{c \sqrt{n}}{k \sqrt{\log \log n}} \cdot |\text{range}(S_{n/4})|.
\]

**Proof.** For convenience, write \( l = \frac{c \sqrt{n}}{k \sqrt{\log \log n}} \). We first note that
\[
\sum_{x \in \mathbb{Z}} \min \left\{ T(k, x, n), l \right\} \geq \sum_{x \in \text{range}(S_{n/2})} \min \left\{ T(k, x, n), l \right\}.
\]

Writing
\[
Z_{n,k} = \sum_{x \in \text{range}(S_{n/2})} \max \left\{ l - T(k, x, n), 0 \right\},
\]
we have
\[
\sum_{x \in \text{range}(S_{n/2})} \min \left\{ T(k, x, n), l \right\} = l \cdot |\text{range}(S_{n/2})| - Z_{n,k},
\]
so we turn to upper bound \( Z_{n,k} \).

Note that for any \( x \in \text{range}(S_{n/2}) \), by Lemma 7.2
\[
P \left( T(k, x, n) \leq l \right) \leq P \left( T(k, 0, n/2) \leq l \right) \leq \frac{c'}{\log^3 n}
\]
for large enough \( n \). Therefore, for any \( x \in \text{range}(S_{n/2}) \),
\[
\mathbb{E}[\max \left\{ l - T(k, x, n), 0 \right\}] \leq l \cdot P \left( T(k, x, n) \leq l \right) \leq \frac{c'}{\log^3 n} l,
\]
and thus
\[
\mathbb{E}[Z_{n,k}] \leq \frac{c'}{\log^3 n} l \cdot \mathbb{E} \left[ \text{range}(S_{n/2}) \right] \leq \frac{c' \sqrt{n}}{\log^3 n} l.
\]

By Markov’s inequality,
\[
P \left( Z_{n,k} \geq \frac{\sqrt{n}}{4 \sqrt{\log \log n}} l \right) \leq \frac{\sqrt{n}}{4 \sqrt{\log \log n}} l \leq \frac{4c''}{\log^2 n}
\]
for large enough \( n \).

Let \( t_m = 2^m \) and let \( k_{m,i} = \frac{c\sqrt{t_m}}{2^{i+1} \log \log t_m} \) (so \( l_{m,i} = \frac{c\sqrt{t_m}}{k_{m,i} \log \log t_m} = 2^i \)) for all \( i \) such that \( k_{m,i} \geq 1 \). As there are at most \( \log t_m \) such indices, we have

\[
\Pr \left( \exists i : Z_{t_m,k_{m,i}} \geq \frac{\sqrt{t_m}}{4\sqrt{\log \log t_m}} l_{m,i} \right) \leq \frac{4c''}{m^{1.5}\log^{1.5} 2}.
\]

The latter expression is summable over \( m \), so by Borel-Cantelli Lemma we have that for all but finitely many \( m \) and for all \( i \) as above,

\[
Z_{t_m,k_{m,i}} \leq \frac{\sqrt{t_m}}{4\sqrt{\log \log t_m}} l_{m,i}.
\]

From the lim inf Law of Iterated Logarithm (see \[3, 9\]), we know that for large enough \( m \), \(|\text{range}(S_{t_m/2})| > \frac{c\sqrt{t_m}}{2^{i+1} \log \log t_m}\), and thus (6) yields

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k_{m,i}, x, t_m), l_{m,i} \} \geq \frac{l_{m,i}}{4} \cdot |\text{range}(S_{t_m/2})|)
\]

for all but finitely many \( m \) and for all \( i \) as above.

Now, let \( n \), let \( k \leq \frac{c\sqrt{n}}{2^{i+1} \log \log n} \), and write \( l = \frac{c\sqrt{n}}{k \log \log n} \). Let \( m \) such that \( t_m \leq n < t_{m+1} \), and let \( i \) such that \( k_{m,i} \leq k < k_{m,i+1} \). Assume \( n \) is large enough so that (7) holds. Then

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq \sum_{x \in \mathbb{Z}} \min \{ T(k_{m,i+1}, x, t_m), l_{m,i+1} \} \geq \frac{l_{m,i+1}}{4} \cdot |\text{range}(S_{t_m/2})| \geq \frac{l}{8} \cdot |\text{range}(S_{t_m/2})| = \frac{c\sqrt{n}}{8k \log \log n} \cdot |\text{range}(S_{t_m/2})|.
\]

\[ \square \]

**Corollary 7.4.** There are universal constant \( d_3, c_3 > 0 \) such that the following holds almost surely: for infinitely many \( n \) and for all \( k, l \) with \( k \leq \frac{d_3\sqrt{n}}{2^{i+1} \log \log n} \),

\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq c_3 \min \left\{ \frac{n}{k}, \sqrt{n \log \log n} l \right\}.
\]

**Proof.** By the usual Law of Iterated Logarithm, there are infinitely many \( n \) such that \(|\text{range}(S_{n/4})| \geq \frac{c_3}{4} \sqrt{n \log \log n} \). Using Proposition 7.3, we have two cases:
(1) If \( l \geq \frac{\sqrt{n}}{k \sqrt{\log \log n}} \), then
\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq \sum_{x \in \mathbb{Z}} \min \left\{ T(k, x, n), \frac{\sqrt{n}}{k \sqrt{\log \log n}} \right\} \geq \frac{c \sqrt{n}}{4k \sqrt{\log \log n}} \sqrt{n \log \log n} = \frac{cn}{4k}.
\]

(2) If \( l < \frac{\sqrt{n}}{k \sqrt{\log \log n}} \), let \( k' = \frac{\sqrt{n}}{l \sqrt{\log \log n}} > k \). Then
\[
\sum_{x \in \mathbb{Z}} \min \{ T(k, x, n), l \} \geq \sum_{x \in \mathbb{Z}} \min \{ T(k', x, n), l \} \geq \frac{c \sqrt{n}}{4k' \sqrt{\log \log n}} \sqrt{n \log \log n} = \frac{c}{4} \sqrt{n \log \log n} l.
\]

This concludes the proof. \( \square \)

8. Bounds on the distance in \( \Delta \)

We return to the construction of [2]. Recall that our group \( \Delta \) is a diagonal product of lamplighter groups \( \Delta_s = \Gamma_s \wr \mathbb{Z} \), where each \( \Gamma_s \) is generated by a set of the form \( A(s) \cup B(s) \), and the diagonal product is taken with respect to the generating set \( \alpha_i(s) = (a_i(s) \delta_0, 0) \), \( \beta_i(s) = (b_i(s) \delta_k, 0) \) and \( \tau(s) = (e, +1) \).

As explained in Subsection 2.3, for any given layer \( s \), the number of \( \Gamma_s \)-steps the random walk makes at \( x \) in \( \Delta_s \) depends only on whether we reached \( x \) and on the number of \( k_s \)-excursions from \( x \). In this section we provide bounds for the distance of a simple random walk on \( \Delta \) and on \( \Delta_s \) in terms of the \( k_s \)-excursions.

For the upper bound on the distance, we use the bound found in [2, Proposition 2.14 and proof of Lemma 3.4]:

**Proposition 8.1.** For all \( s \geq 0 \),
\[
|W_n|_{\Delta_s} \leq 11 \min \left\{ \sum_{j \in \mathbb{Z}} k_s T \left( \frac{k_s}{2}, j \frac{k_s}{2}, n \right), |\text{range}(S_n)|, |\text{range}(S_n)| l_s \right\}.
\]

In addition, writing \( s_0(n) = \max \{ s \geq 0 \mid k_s \leq |\text{range}(S_n)| \} \), we have
\[
|W_n|_{\Delta} \leq 500 \sum_{s \leq s_0(n)} |W_n|_{\Delta_s}.
\]
We now turn to the lower bound. For all \( s \geq 0 \) we have

\[
|W_n|_{\Delta} \geq |W_n|_{\Delta_s} \geq \sum_{x \in \mathbb{Z}} |f_n(x)|_{\Gamma_s}
\]

To get a lower bound for the RHS, we use the following lemma:

**Lemma 8.2.** Let \( \{\Gamma_s\} \) be a sequence of finite groups, let \( l_s = \text{diam}(\Gamma_s) \), and suppose that \( \{\Gamma_s\} \) satisfies \((\sigma, c_0 l_s)\)-linear speed assumption for some constants \( 0 < \sigma, c_0 < 1 \) with \( \sigma c_0 l_s \geq 4 \). Then for all \( t \geq 0 \),

\[
P \left( \left| X_t^{(s)} \right|_{\Gamma_s} \geq \frac{\sigma}{8} \min \{t, c_0 l_s\} \right) \geq \frac{\sigma c_0}{8}.
\]

**Proof.** Assume first that \( t \leq c_0 l_s \). Let \( p = P \left( \left| X_t^{(s)} \right|_{\Gamma_s} \geq \frac{\sigma}{2} t \right) \). Then

\[
\sigma t \leq E \left| X_t^{(s)} \right|_{\Gamma_s} \leq (1 - p) \frac{\sigma}{2} t + pt \leq \frac{\sigma}{2} t + pt,
\]

so \( p \geq \frac{\sigma t}{2} \), and the assertion follows.

Now, let \( t \geq c_0 l_s \). By \[10\] Lemma 4.1, we have

\[
E \left| X_t^{(s)} \right|_{\Gamma_s} \geq \frac{1}{2} E \left| X_{c_0 l_s}^{(s)} \right|_{\Gamma_s} - 1 \geq \frac{\sigma}{2} c_0 l_s - 1.
\]

If \( \sigma c_0 l_s \geq 4 \), we therefore have

\[
E \left| X_t^{(s)} \right| \geq \frac{\sigma c_0 l_s}{4}.
\]

Let \( p = P \left( \left| X_t^{(s)} \right| \geq \frac{\sigma c_0 l_s}{8} \right) \). Then

\[
\frac{\sigma c_0 l_s}{4} \leq E \left| X_t^{(s)} \right| \leq (1 - p) \frac{\sigma c_0 l_s}{8} + pl_s \leq \frac{\sigma c_0 l_s}{8} + pl_s.
\]

We therefore have \( p \geq \frac{\sigma c_0}{8} \), as required. \( \square \)

We are now ready to prove our lower bound on the metric in a single layer:

**Proposition 8.3.** Let \( \{\Gamma_s\} \) be a sequence of finite groups, let \( l_s = \text{diam}(\Gamma_s) \), and suppose that \( \{\Gamma_s\} \) satisfies \((\sigma, c_0 l_s)\)-linear speed assumption for some constants \( \sigma, c_0 > 0 \). Also suppose there is \( m_0 > 1 \) such that

\[
k_{s+1} > 2k_s \quad \text{and} \quad l_{s+1} \geq m_0 l_s
\]

for all \( s \). Then, almost surely, for large enough \( n \), for all \( s \) such that

\[
k_s \leq \frac{d_{k_s} \sqrt{n}}{\sqrt{\log \log n}}
\]

we have

\[
|W_n|_{\Delta_s} \geq \frac{\sigma c_0}{16} \sum_{x \in \mathbb{Z}} \min \{T(k_s, x, n), c_0 l_s\}.
\]
Proof. We use a similar approach to Revelle [17]. It is trivial that for all \( n, s \),
\[
|W_n|_{\Delta_s} \geq \frac{1}{2} \sum_{x \in \mathbb{Z}} |f_n(x)|_{\Gamma_s}.
\]
Let \( I_{x,n} \) denote the indicator of whether the random walk on \( \mathbb{Z} \) visited \( x \) before time \( n \). Conditioning on \( T(k_s, x, n) \) and \( I_{x,n} \) for all \( x \), the random variables \( |f_n(x)|_{\Gamma_s} \) are independent.

Let \( \theta_x = \frac{\sigma}{8} \min \{ T(k_s, x, n), c_0 l_s \} \), and write \( Z_x = |f_n(x)|_{\Gamma_s} \theta_x^{-1} \). By Lemma 8.2, \( P(Z_x \geq 1) \geq \frac{\sigma c_0}{8} \), so we may apply [17, Lemma 2] to get

\[
(8) \quad P \left( \sum_{x \in \mathbb{Z}} |f_n(x)|_{\Gamma_s} \leq \frac{\sigma c_0}{16} \sum_{x \in \mathbb{Z}} \theta_x \middle| T(k_s, x, n), I_{x,n} \right) \leq \exp \left( -\frac{\sigma c_0}{96} \sum_{x \in \mathbb{Z}} \theta_x \right).
\]

Let \( d_2, c, c_2 > 0 \) be constants such that Proposition 6.2 and Proposition 6.3 hold. For each \( m \), let \( A_m \) denote the event that for all \( n \geq m \) and for all \( s \) with \( k_s \leq \frac{d_2 \sqrt{n}}{2\sqrt{\log \log n}} \),
\[
\max_{x \in \mathbb{Z}} \theta_x \leq \min \left\{ \frac{c \sqrt{n \log \log n}}{k_s}, l_s \right\}
\]
and
\[
\sum_{x \in \mathbb{Z}} \theta_x \geq c_2 \min \left\{ \frac{n}{k_s}, \frac{\sqrt{n}}{\log \log n} l_s \right\} \geq \frac{c' \sqrt{n}}{\sqrt{\log \log n}} \max_{x \in \mathbb{Z}} \theta_x.
\]
We have \( P(\bigcup_{m=1}^{\infty} A_m) = 1 \). Also, let \( B_n \) denote the event that for all \( s \) with \( k_s \leq \frac{d_2 \sqrt{n}}{2\sqrt{\log \log n}} \),
\[
\sum_{x \in \mathbb{Z}} |f_n(x)|_{\Gamma_s} \leq \frac{\sigma}{4} \sum_{x \in \mathbb{Z}} \theta_x.
\]
Conditioning on \( A_m \), (8) shows that
\[
P \left( \sum_{x \in \mathbb{Z}} |f_n(x)|_{\Gamma_s} \leq \frac{\sigma}{4} \sum_{x \in \mathbb{Z}} \theta_x \middle| A_m \right) \leq \exp \left( -\frac{c' \sqrt{n}}{\sqrt{\log \log n}} \right).
\]
for all $n \geq m$. As $\{k_s\}$ increases exponentially, we may take a union bound to see
\[
P(B_n \mid A_m) \leq c'' \log n \exp \left( -\frac{c' \sqrt{n}}{\sqrt{\log \log n}} \right)
\]
for all $n \geq m$ and a constant $c''$. The latter expression is summable, so the Borel Cantelli Lemma shows that for each $m$, conditioning on $A_m$ we have that $P(\lim \inf B_n^c \mid A_m) = 1$. But as $P(\bigcup_{m=1}^{\infty} A_m) = 1$, we must have $P(\lim \inf B_n^c) = 1$, as required. \hfill \Box

9. Bounds on the distance in a single layer

We use the results obtained in the previous section to bound the distance of the random walk in each layer separately. As explained before, for a given $s$, the number of steps that the random walk on $\Delta_s = \Gamma_s \wr \mathbb{Z}$ makes in the copy of the lamp group $\Gamma_s$ at any given point $x \in \mathbb{Z}$ until time $n$ is $T(k_s, x, n)$, and the distance in each copy of $\Gamma_s$ is also bounded from above by $l_s$.

When $s$ is small, we expect to reach saturation in $\Gamma_s$ relatively fast, so the effective bound should be $l_s$. However, when $s$ is big, the number of excursions can be small compared to $l_s$, so the bound should depend on $T(k_s, x, n)$. We are therefore looking for the layers which will separate these two cases.

To do this, let $d_1, d_2, d_3$ be the constants from Proposition 5.4, Proposition 6.3 and Corollary 7.4. Take a constant $r \leq \frac{1}{2} \min \{d_1, d_2, d_3\}$, and consider the following layers:
\[
s_0'(n) = \max \left\{ s \geq 0 \mid k_s \leq \frac{r \sqrt{n}}{\sqrt{\log \log n}} \right\},
\]
\[
s_2(n) = \max \left\{ s \geq 0 \mid k_s l_s \leq \frac{r \sqrt{n}}{\sqrt{\log \log n}} \right\},
\]
\[
s_3(n) = \max \left\{ s \geq 0 \mid k_s l_s \leq \sqrt{n \log \log n} \right\},
\]
\[
\tilde{s}_3(n) = \min \{ s_0'(n), s_3(n) \}.
\]
We will prove that the dominating layer for the lim sup is $s_2(n)$, whereas the dominating layer for the lim inf is $\tilde{s}_3(n)$.

9.1. Upper bounds.
Proposition 9.1. There is a universal constants $C > 0$ such that almost surely, for large enough $n$ and all $s \geq 0$,

$$|W_n|_{\Delta_s} \leq \begin{cases} C \sqrt{n \log \log n} l_s, & s \leq s_2(n) \\ \frac{C_n}{k_s}, & s_2(n) < s \leq s'_0(n) \\ C \sqrt{n \log \log n}, & s > s'_0(n) \end{cases}$$

Proof. By Proposition 8.1, we have

$$|W_n|_{\Delta_s} \leq 11 \min \left\{ T \left( \frac{k_s}{2}, n \right) + |\text{range}(S_n)|, |\text{range}(S_n)| l_s \right\}$$

for all $s \geq 0$. Also, by the classical Law of Iterated Logarithm, for large enough $n$ we have

(9) $|\text{range}(S_n)| \leq 2 \sqrt{n \log \log n}.$

The case $s \leq s_2(n)$ follows from $|W_n|_{\Delta_s} \leq 11 \cdot |\text{range}(S_n)| l_s$ and (9).

We move to the case where $s_2(n) < s \leq s'_0(n)$. By Proposition 5.4, almost surely for large enough $n$ and for all $s \leq s'_0(n)$,

$$T \left( \frac{k_s}{2}, n \right) \leq C_1 n \frac{r}{k_s / 2} = \frac{2 C_1 r}{k_s} n.$$

Also, by (9), for large enough $n$ we have

$$|\text{range}(S_n)| \leq 2 \sqrt{n \log \log n} \leq \frac{2 r n}{k_s},$$

proving the second case.

For the case where $s > s'_0(n)$, as $k_s > \frac{r \sqrt{n}}{\sqrt{\log \log n}}$ we may apply Lemma 5.2 and Proposition 5.4

$$T(k_s, n) \leq 3 T \left( \frac{r \sqrt{n}}{2 \sqrt{\log \log n}}, n \right) \leq \frac{3 C_1 r}{r \sqrt{n}} = \frac{6 C_1}{r \sqrt{n}}.$$

Now the bound on $|W_n|_{\Delta_s}$ follows from (9). \(\square\)

Proposition 9.2. There is a universal constant $C > 0$ such that almost surely, for all large enough $n$ for which $|\text{range}(S_n)| \leq \frac{2 \sqrt{n}}{\sqrt{\log \log n}}$ and all $s \geq 0$,

$$|W_n|_{\Delta_s} \leq \begin{cases} \frac{C \sqrt{n}}{\sqrt{\log \log n}} l_s, & s \leq \tilde{s}_3(n) \\ \frac{C_n}{k_s}, & \tilde{s}_3(n) < s \leq s_0(n) \end{cases}$$

Proof. By Proposition 8.1, we have

$$|W_n|_{\Delta_s} \leq 11 \min \left\{ T \left( \frac{k_s}{2}, n \right) + |\text{range}(S_n)|, |\text{range}(S_n)| l_s \right\}$$
for all \( s \geq 0 \). Also, by our assumption on \( n \), \(|\text{range}(S_n)| \leq \frac{2\sqrt{n}}{\sqrt{\log \log n}} \), so the case \( s \leq \tilde{s}_3(n) \) follows automatically.

We move to the case where \( \tilde{s}_3(n) < s \leq \tilde{s}_0'(n) \). By Proposition 5.4, almost surely for large enough \( n \) and for all \( s \leq \tilde{s}_0'(n) \),

\[
T\left(\frac{k_s}{2}, n\right) \leq \frac{C_1 n}{k_s/2} = \frac{2C_1 n}{k_s}.
\]

In addition,

\[
|\text{range}(S_n)| \leq \frac{2\sqrt{n}}{\sqrt{\log \log n}} \leq \frac{2rn}{k_s},
\]

proving part of the second case.

For the case where \( s'_0'(n) < s \leq s_0'(n) \), as \( \frac{r\sqrt{n}}{\log \log n} < k_s \leq \frac{2\sqrt{n}}{\sqrt{\log \log n}} \) we may apply Lemma 5.2 and Proposition 5.4:

\[
T(k_s, n) \leq 3T\left(\frac{r\sqrt{n}}{2\sqrt{\log \log n}}, n\right) \leq \frac{C_1 n}{r\sqrt{n \log \log n}} = \frac{2C_1 n}{r \sqrt{n \log \log n}} < \frac{2rn}{k_s}.
\]

The conclusion follows. \( \square \)

9.2. Lower bounds.

**Proposition 9.3.** There is a universal constant \( c > 0 \) such that almost surely, for all but finitely many \( n \),

\[
|W_n|_{\Delta_s} \geq \begin{cases} \frac{c\sqrt{n}}{\sqrt{\log \log n}} l_s, & s \leq \tilde{s}_3(n) \\ \frac{cn}{k_s}, & \tilde{s}_3(n) < s \leq s'_0(n) \end{cases}
\]

**Proof.** Using Proposition 8.3, for large enough \( n \) we have

\[
|W_n|_{\Delta_s} \geq c \sum_{x \in \mathbb{Z}} \min \{T(k_s, x, n), l_s\}
\]

for all \( s \leq s'_0(n) \). The result follows from Proposition 6.3. \( \square \)

**Proposition 9.4.** There is a universal constant \( c > 0 \) such that almost surely, for infinitely many \( n \),

\[
|W_n|_{\Delta_s} \geq \begin{cases} \frac{c\sqrt{n \log \log n}}{\sqrt{k_s}} l_s, & s \leq s_2(n) \\ \frac{cn}{k_s}, & s_2(n) < s \leq s'_0(n) \end{cases}
\]

**Proof.** Using Proposition 8.3, for large enough \( n \) we have

\[
|W_n|_{\Delta_s} \geq c \sum_{x \in \mathbb{Z}} \min \{T(k_s, x, n), l_s\}
\]

for all \( s \leq s'_0(n) \). The result follows from Corollary 7.4. \( \square \)
10. Bounds on the Total Distance

We now use the bounds on each layer achieved in the previous section to find a bound on the distance of the random walk on $\Delta$ in terms of the sequences $\{k_s\}$ and $\{l_s\}$. Recall the following bounds from [2]:

$$|W_n|_\Delta \leq C \sum_{s \leq s_0(n)} |W_n|_s$$

where $s_0(n) = \min \{s \geq 0 \mid k_s \leq |\text{range}(S_n)|\}$, and

$$|W_n|_\Delta \geq |W_n|_s$$

for any $s \geq 0$.

For the lim sup, recall the critical layer

$$s_2(n) = \max \left\{ s \geq 0 \mid k_s l_s \leq \frac{r \sqrt{n}}{\log \log n} \right\}$$

and that

$$s'_0(n) = \max \left\{ s \geq 0 \mid k_s \leq \frac{r \sqrt{n}}{\log \log n} \right\}$$

where $r > 0$ is some universal constant.

**Proposition 10.1.** Suppose $\{\Gamma_s\}$ satisfy the $(\sigma, c_0 l_s)$-linear speed assumption and $\text{diam}(\Gamma_s) \leq C_0 l_s$. Suppose there exists a constant $m_0 > 1$ such that

$$k_{s+1} > 2 k_s, \quad l_{s+1} \geq m_0 l_s$$

for all $s$. Let

$$g(n) = \frac{n}{k_{s_2(n)+1}} + \sqrt{n \log \log n l_{s_2(n)}}.$$

Then almost surely,

$$0 < \limsup_{n \to \infty} \frac{|W_n|_\Delta}{g(n)} \quad \text{and} \quad \limsup_{n \to \infty} \frac{|W_n|_\Delta}{g(n) + \sqrt{n \log \log n \log \log \log n}} < \infty.$$

**Proof.** For the lower bound, we use (11) and Proposition 9.4 showing that for infinitely many $n$ we have

$$|W_n|_\Delta \geq \frac{1}{2} \left( |W_n|_{\Delta_{s_2(n)}} + |W_n|_{\Delta_{s_2(n)+1}} \right) \geq \frac{c}{2} \left( \sqrt{n \log \log n l_{s_2(n)}} + \frac{n}{k_{s_2(n)+1}} \right) = \frac{c}{2} g(n),$$

so $\limsup_{n \to \infty} \frac{|W_n|_\Delta}{g(n)} > 0$ almost surely.

For the upper bound, we use (10). We divide the interval $0 \leq s \leq s_0(n)$ into three parts, and use Proposition 9.1 to analyze the contribution of each of them:
For $0 \leq s \leq s_2(n)$,
\[
\sum_{s \leq s_2(n)} |W_n|_{\Delta_s} \leq \sum_{s \leq s_2(n)} C_1 \sqrt{n \log \log n} \leq \frac{C_1}{1 - 1/m_0} \sqrt{n \log \log n} l_{s_2(n)}
\]
where the last inequality follows from $l_{s+1} \geq m_0 l_s$.

For $s_2(n) + 1 \leq s \leq s_0(n)$,
\[
\sum_{s_2(n) + 1 \leq s \leq s_0(n)} |W_n|_{\Delta_s} \leq \sum_{s_2(n) + 1 \leq s \leq s_0(n)} \frac{C_1 n}{k_s} \leq \frac{2C_1 n}{k_{s_2(n) + 1}}
\]
where the last inequality follows from $k_{s+1} > 2k_s$.

For $s_0(n) + 1 \leq s \leq s_3(n)$, recall that for large enough $n$ we have $|\text{range}(S_n)| \leq 2\sqrt{n \log \log n}$. As $k_{s+1} > 2k_s$, the number of layers $s_0(n) + 1 \leq s \leq s_3(n)$ is at most $c \log \log \log n$ for some constant $c > 0$, so
\[
\sum_{s_0(n) + 1 \leq s \leq s_3(n)} |W_n|_{\Delta_s} \leq C_1 \sqrt{n \log \log n} \log \log \log n.
\]
Summing the above yields
\[
|W_n|_{\Delta} \leq C g(n) + C \sqrt{n \log \log n} \log \log \log n,
\]
proving the second lim sup is finite.

For the liminf, recall that
\[
s_3(n) = \max \left\{ s \geq 0 \mid k_s l_s \leq \sqrt{n \log \log n} \right\}
\]
and that $\tilde{s}_3(n) = \min \{ s_0(n), s_3(n) \}$.

**Proposition 10.2.** Suppose $\{\Gamma_s\}$ satisfy the $(\sigma, c_0 l_s)$-linear speed assumption and $\text{diam}(\Gamma_s) \leq C_0 l_s$. Suppose there exists a constant $m_0 > 1$ such that
\[
k_{s+1} > 2k_s, \quad l_{s+1} \geq m_0 l_s
\]
for all $s$. Let
\[
h(n) = \begin{cases} \frac{n}{k_{s_3(n) + 1}} + \frac{\sqrt{n}}{\sqrt{\log \log n}} l_{s_3(n)}, & s_3(n) < s_0(n) \\ \frac{\sqrt{n}}{\sqrt{\log \log n}} l_{s_0(n)}, & \text{otherwise.} \end{cases}
\]
Then almost surely
\[
0 < \liminf_{n \to \infty} \frac{|W_n|_{\Delta}}{h(n)} < \infty.
\]
Proof. For the upper bound we use again (10). Take \( n \) such that
\[
|\text{range}(S_n)| \leq \frac{2\sqrt{n}}{\log \log n},
\]
and assume \( n \) is large enough such that Proposition 9.2 holds. We divide the interval \( 0 \leq s \leq s_0(n) \) into several parts and analyze the contribution of each of them:

- For \( 0 \leq s \leq \tilde{s}_3(n) \),

\[
\sum_{s \leq \tilde{s}_3(n)} |W_n|_{\Delta_s} \leq \sum_{s \leq \tilde{s}_3(n)} \frac{C\sqrt{n}}{\log \log n} l_s \leq \frac{C}{1 - 1/m_0} \sqrt{n} \frac{\sqrt{n}}{\log \log n} l_{\tilde{s}_3(n)}
\]

where the last inequality follows from \( l_{s+1} \geq m_0l_s \).

- If \( s_3(n) < s_0'(n) \), then for \( s_3(n) + 1 \leq s \leq s_0'(n) \) we have

\[
\sum_{s_3(n)+1 \leq s \leq s_0'(n)} |W_n|_{\Delta_s} \leq \sum_{s_3(n)+1 \leq s \leq s_0'(n)} \frac{Cn}{k_s} \leq \frac{2Cn}{k_{s_0}(n)}
\]

where the last inequality follows from \( k_{s+1} > 2k_s \).

- For \( s_0'(n) + 1 \leq s \leq s_0(n) \), we have

\[
|W_n|_{\Delta_s} \leq T(k_s, n) + |\text{range}(S_n)| \leq 3T(k_{s_0}(n), n) + |\text{range}(S_n)| \leq \frac{Cn}{k_{s_0}(n)}.
\]

Since \( \{k_s\} \) grows at least exponentially, and by our assumption on \( n \) \( |\text{range}(S_n)| \leq \frac{2\sqrt{n}}{\log \log n} \) the number of such layers is bounded above by a constant, so

\[
\sum_{s_0'(n)+1 \leq s \leq s_0(n)} |W_n|_{\Delta_s} \leq \frac{Cn}{k_{s_0}(n)}.
\]

Summing the above, we have

\[
|W_n|_{\Delta} \leq \begin{cases} \frac{C\sqrt{n}}{\log \log n} l_{s_3(n)} + \frac{Cn}{k_{s_3(n)+1}} + \frac{Cn}{k_{s_0}(n)} & s_3(n) < s_0'(n) \\ \frac{C\sqrt{n}}{\log \log n} l_{s_0'(n)} + \frac{Cn}{k_{s_0}(n)} & \text{otherwise.} \end{cases}
\]

In the first case, \( k_{s_0}(n) \geq k_{s_3(n)+1} \), so the latter term is inessential; in the second case, \( s_0'(n) \leq s_3(n) \), so \( k_{s_0}(n)l_{s_0'(n)} \leq \sqrt{n} \log \log n \), and again the latter term is inessential. We therefore get \( |W_n|_{\Delta} \leq 2Ch(n) \) for each such \( n \), proving the above \( \lim \inf \) is finite.

To prove that the \( \lim \inf \) is positive, we use (11). Take \( n \) large enough so that Proposition 9.3 holds. If \( s_3(n) < s_0'(n) \), we have

\[
|W_n|_{\Delta} \geq \frac{1}{2} \left( |W_n|_{\Delta_{s_3(n)}} + |W_n|_{\Delta_{s_3(n)+1}} \right) \geq \frac{c}{2} \left( \frac{\sqrt{n}}{\log \log n} l_{s_3(n)} + \frac{n}{k_{s_3(n)+1}} \right) = \frac{c}{2} h(n).
\]
Finally, if \( s'_0(n) \leq s_3(n) \), we have

\[
|W_n|_\Delta \geq |W_n|_{\Delta_{s'_0(n)}} \geq \frac{c\sqrt{n}}{\sqrt{\log \log n}} l_{s'_0(n)} = c h(n).
\]

This concludes the proof. \( \square \)

11. Proof of the main theorem

We return to the idea of [2] for approximating a given function. Let \( f : \mathbb{N} \to \mathbb{N} \) be a function such that \( \frac{f(n)}{\sqrt{n}} \) and \( \frac{n}{f(n)} \) are non-decreasing. Choose sequences \( \{k_s\} \) and \( \{l_s\} \) as in Proposition 2.7 and let \( \Delta \) be the corresponding diagonal product. Denote by \( W_n \) the random walk on \( \Delta \) with respect to the “switch-walk-switch” measure.

**Theorem 11.1.** If \( \frac{f(n)}{\sqrt{n \log \log \log n}} \) is non-decreasing, then almost surely

\[
0 \leq \limsup_{n \to \infty} \frac{|W_n|_\Delta}{\log \log n \cdot f\left(\frac{n}{\log \log n}\right)} < \infty.
\]

**Proof.** We first note that, by Proposition 2.7,

\[
f\left(\frac{rn}{\log \log n}\right) \sim \sqrt{\frac{rn}{\log \log n}} l_{s_2(n)} + \frac{rn}{\log \log n} k_{s_2(n)+1} \sim \frac{1}{\log \log n} \left(\sqrt{n \log \log n} l_{s_2(n)} + \frac{n}{k_{s_2(n)+1}}\right).
\]

Therefore, Proposition 10.1 shows that almost surely,

\[
0 < \limsup_{n \to \infty} \frac{|W_n|_\Delta}{\log \log n \left( f\left(\frac{n}{\log \log n}\right) + \sqrt{n \log \log n \log \log \log n}\right)} < \infty.
\]

As \( \frac{f(n)}{\sqrt{n \log \log \log n}} \) is non-decreasing,

\[
\frac{\sqrt{n}}{\log \log n} \log \log \log n \leq f\left(\frac{n}{\log \log n}\right),
\]

and the conclusion follows. \( \square \)

**Theorem 11.2.** If \( \frac{f(n)}{\sqrt{n \log \log n}} \) is non-decreasing for some \( \varepsilon > 0 \), and the sequences \( \{k_s\} \) and \( \{l_s\} \) are chosen so that \( \log \log k_s \leq l_s \) (as in Lemma C.1), then almost surely

\[
0 < \liminf_{n \to \infty} \frac{1}{\log \log n} |W_n|_\Delta \cdot f(n \log \log n) < \infty.
\]
Proof. We first note that, by Proposition 2.7,
\[
 f ( c_1 n \log \log n ) \simeq \sqrt{c_1 n \log \log n} l_{s_3(n)} + \frac{c_1 n \log \log n}{k_{s_3(n)+1}} \simeq \\
 \log \log n \left( \frac{\sqrt{n}}{\sqrt{\log \log n}} l_{s_3(n)} + \frac{n}{k_{s_3(n)+1}} \right).
\]

Also, the assumption that \( \log \log k_s \leq l_s \) is non-decreasing shows that \( s_3(n) \leq s_0'(n) \) for large enough \( n \). Therefore, Proposition 10.2 shows that almost surely
\[
0 < \lim \inf_{n \to \infty} \frac{|W_n|}{f(n \log \log n)} < \infty.
\]
□

Appendix A. \( k \)-jumps of a simple random walk on \( \mathbb{Z} \)

Let \( S_t \) be a simple random walk on \( \mathbb{Z} \), and fix \( k > 0 \). Recall from Definition 4.1 the sequence \( \{n_j^{(k)}\}_{j=0}^\infty \): We set \( n_0^{(k)} = 0 \), and
\[
n_j^{(k)} = \inf \left\{ t > n_{j-1}^{(k)} \mid S_t \in \{S_{n_{j-1}^{(k)}} \pm k\} \right\}.
\]
Recall also \( N_k(n) = \max\{j \mid n_j^{(k)} \leq n\} \). Let \( X_j = n_j^{(k)} - n_{j-1}^{(k)} \). Our goal is to estimate \( N_k(n) \).

**Proposition A.1.** Let \( 0 < r < 1 \). There exists \( c > 0 \) such that for any \( n, k \geq 1 \),
\[
P \left( N_k(n) > \frac{cn}{k^2} \right) \leq \exp \left( -\frac{rn}{k^2} \right).
\]

**Proof.** For \( k = 1 \), we may choose \( c = 1 \) and the inequality will hold; so we assume \( k > 1 \). Note that
\[
P \left( X_j > k^2 \right) = P \left( \max_{n_{j-1} \leq t \leq n_{j-1}+k^2} |S_t - S_{n_{j-1}}| < k \right) = \\
P \left( \max_{0 \leq t \leq k^2} |S_t| < k \right) = 1 - P \left( \max_{0 \leq t \leq k^2} |S_t| \geq k \right) \geq \\
1 - \frac{\mathbb{E}|S_{k^2}|}{k},
\]
where the last step follows from Doob’s maximal inequality. Since \( \mathbb{E}|S_n| \sim \sqrt{\frac{2}{\pi} n} \), there is a constant \( p_0 > 0 \) such that \( P(X_j > k^2) \geq p_0 \).
for all $k$. As the random variables $X_1, X_2, \ldots$ are i.i.d., for all $c > 1$ we have
\[
P \left( N_k(n) > \frac{cn}{p_0 k^2} \right) = P \left( \sum_{j=1}^{cn/p_0 k^2} X_j \leq n \right) \leq P \left( \sum_{j=1}^{cn/p_0 k^2} k^2 \cdot 1 \{ X_j > k^2 \} \leq n \right) \leq P \left( \text{Bin} \left( \frac{cn}{p_0 k^2}, p_0 \right) \leq \frac{n}{k^2} \right).
\]
Chernoff’s inequality shows that
\[
P \left( \text{Bin} \left( \frac{cn}{p_0 k^2}, p_0 \right) \leq \frac{n}{k^2} \right) \leq \exp \left( - \left( 1 - \frac{1}{c} \right)^2 \frac{n}{2k^2} \right).
\]
Now, let $0 < r < 1$. Choosing a large enough $c$ so that $r \leq \left( 1 - \frac{1}{c} \right)^2$, we have
\[
P \left( N_k(n) > \frac{cn}{p_0 k^2} \right) \leq \exp \left( - \frac{rn}{k^2} \right)
\]
as required. \hfill \square

**Proposition A.2.** Let $0 < r < \frac{1}{10}$. Then there exists $c' > 0$ such that for all $n, k \geq 1$,
\[
P \left( N_k(n) < \frac{c'n}{k^2} \right) \leq \exp \left( - \frac{rn}{k^2} \right).
\]

**Proof.** First, for $k = 1$ we may take $c' = \frac{1}{2}$, and the desired probability is 0; so we assume $k > 1$. Note that
\[
P \left( X_i \geq k^2 \right) \leq P \left( |S_{k^2}| \leq k \right),
\]
and the latter converges from below to $P \left( -1 \leq N(0, 1) \leq 1 \right) \approx 0.683$. One can check that by taking $p_1 = 0.9$ we have $P \left( X_i \geq k^2 \right) \leq p_1$ for all $k \geq 2$.

Now, for any $m \geq 1$,
\[
P \left( X_j \geq (m + 1)k^2 \right) = P \left( X_j \geq (m + 1)k^2 \mid X_j \geq k^2 \right) \cdot P(X_j \geq k^2) \leq P \left( X_j \geq mk^2 \right) P \left( X_j \geq k^2 \right),
\]
and induction on $m$ shows that $P(X_j \geq mk^2) \leq P(X_j \geq k^2)^m$ for all $m$. Therefore random variable $\frac{1}{k^2} X_j$ is stochastically dominated by
a geometric random variable $G_j \sim \text{Geo}(1 - p_1)$, so for any $c' \leq 1 - p_1$,

$$P \left( N_k(n) < \frac{c'n}{k^2} \right) = P \left( \sum_{j=1}^{c'n/k^2} X_j > n \right) \leq P \left( \sum_{j=1}^{c'n/k^2} G_j > \frac{n}{k^2} \right).$$

By [11, Theorem 2.1],

$$P \left( \sum_{j=1}^{c'n/k^2} G_j > \frac{n}{k^2} \right) \leq \exp \left( -\frac{c'n}{k^2} (\lambda - 1 - \ln \lambda) \right)$$

for $\lambda = \frac{1 - p_1}{c'}$.

Finally, let $0 < r < \frac{1}{10}$. As $c'(\lambda - 1 - \ln \lambda) = 1 - p_1 - c' - c' \ln \frac{1 - p_1}{c'} \to 1 - p_1$ as $c' \to 0$, by choosing small enough $c'$ we have $c'(\lambda - 1 - \ln \lambda) > r$, and thus

$$P \left( N_k(n) > \frac{c'n}{k^2} \right) \leq \exp \left( -\frac{rn}{k^2} \right).$$

\[ \square \]

**Appendix B. On the maximal local time of a simple random walk**

For a simple random walk $S_t$ on $\mathbb{Z}$, recall that

$$L(x, n) = \left| \{ 0 \leq k \leq n \mid S_k = x \} \right|$$

denotes the local time of $S_t$ at $x$ until time $n$. We are interested in the tail behavior of $L(n) = \max_{x \in \mathbb{Z}} L(x, n)$. The main aim of this section is to prove an upper tail bound on the maximal local time of a simple random walk on $\mathbb{Z}$ (Corollary B.3). This seems like a classical result, however we could not find the exact statement we needed in the literature, and therefore provide the statement with proof and background for completeness.

**B.1. Brownian motion and the Skorokhod embedding.** Let $B_t$ be a Brownian motion on $\mathbb{R}$ starting at 0. Define a sequence of stopping times by $\tau_0 = 0$ and, inductively,

$$\tau_k = \inf \left\{ t > \tau_{k-1} \mid |B_t - B_{\tau_{k-1}}| = 1 \right\}.$$

Then $S_k = B_{\tau_k}$ is a simple random walk on $\mathbb{Z}$, and $\tau_k - \tau_{k-1}$ are i.i.d.r.v. with $\mathbb{E}[\tau_k - \tau_{k-1}] = 1$ and $\sigma^2 = \mathbb{E}\left[ (\tau_k - \tau_{k-1})^2 \right] < \infty$. We therefore have

$$P \left( |\tau_n - n| \geq \sqrt{n} u \right) \leq 2 \exp \left( -\frac{u^2}{2\sigma^2} \right).$$
B.2. **Brownian local time.** Given a Brownian motion $B_t$ on $\mathbb{R}$, a theorem of Trotter [21] shows that almost surely there exists a function $\eta(x,t)$, jointly continuous in $x$ and $t$, such that

$$\eta(x,t) = \frac{d}{dx} \int_0^t 1_{(-\infty,x)}(B_s) \, ds.$$ 

$\eta(x,t)$ is called the **local time** of $B_t$ at $x$. The distribution of $\eta(0,t)$ is well known: for all $u > 0$,

$$P\left( \frac{\eta(0,t)}{\sqrt{t}} \leq u \right) = 2\Phi(u) - 1 \quad (\text{see e.g. [13]})$$

(13)

Let $\eta(t) = \sup_{x \in \mathbb{R}} \eta(x,t)$. By the scaling property of Brownian motion, $\frac{\eta(t)}{\sqrt{t}}$ has the same distribution as $\eta(1)$, which, by [13], satisfies

$$P\left( \eta(t) \geq \sqrt{tu} \right) = P(\eta(1) \geq u) \leq \exp\left( -\frac{u^2}{4} \right)$$

for large enough $u$.

B.3. **Strong approximation for the local time.** We give a strong approximation theorem for the maximal local time of a random walk and a Brownian motion, using the Skorokhod embedding. This is similar to other results (such as in [19, 4, 10]), but with a more quantitative flavor.

We use the notations of [19]. Fix $x \in \mathbb{Z}$. Define

$$\nu_1 = \inf \{ k \geq 0 \mid B_{\tau_k} = S_k = x \}$$

and, inductively,

$$\nu_j = \inf \{ k > \nu_{j-1} \mid B_{\tau_k} = S_k = x \}.$$ 

For each $j$, let

$$a_j(x) = \eta(x, \tau_{\nu_{j+1}}) - \eta(x, \tau_{\nu_j-1}).$$

From [19], $a_1(x), a_2(x), \ldots$ are i.i.d.r.v. with

$$P(\eta(0, \tau_1) \geq u) = P(\eta(0, \tau_1) \geq u) \leq C \exp\left( -\frac{\pi}{4} u \right)$$

(for some universal constant $C > 0$), and $\mathbb{E}[a_j(x)] = 1$. In addition, the random matrices $L = (L(x,n))_{x \in \mathbb{Z}, n \geq 0}$ and $A = (a_j(x))_{x \in \mathbb{Z}, j \geq 1}$ are independent.

**Lemma B.1.** There is a universal constant $c_1 > 0$ such that for any $x \in \mathbb{Z}$ and $u \geq 1$,

$$P\left( |a_1(x) + \cdots + a_{L(x,n)}(x) - L(x,n)| \geq \sqrt{nu} \right) \leq c_1 \exp\left( -\frac{n^{1/3}u^{4/3}}{2} \right)$$
Proof. By the above, we have
\[
\begin{align*}
P \left( |a_1(x) + \cdots + a_{L(x,n)}(x) - L(x,n)| \geq \sqrt{nu} \right) &= \\
&= \sum_{k=0}^{n} P \left( |a_1(x) + \cdots + a_k(x) - k| \geq \sqrt{nu} \right) P \left( L(x,n) = k \right) = \\
&= \sum_{k=0}^{n} P \left( \left| \frac{a_1(x) + \cdots + a_k(x) - k}{\sqrt{k}} \right| \geq \frac{\sqrt{nu}}{\sqrt{k}} \right) P \left( L(x,n) = k \right) \leq \\
&\leq \sum_{k=0}^{(nu)^{2/3}} \left( n^{1/6} u^{2/3} \right)^{2/3} P \left( \left| \frac{a_1(x) + \cdots + a_k(x) - k}{\sqrt{k}} \right| \geq n^{1/6} u^{2/3} \right) P \left( L(x,n) = k \right) + \\
&+ \mathbb{P} \left( L(x,n) > (nu)^{2/3} \right) \leq \\
&\leq 2 \exp \left( -\frac{n^{1/3} u^{4/3}}{2} \right) + C \exp \left( -\frac{n^{1/3} u^{4/3}}{2} \right) \leq \tilde{C} \exp \left( -\frac{n^{1/3} u^{4/3}}{2} \right).
\end{align*}
\]
\(\square\)

Lemma B.2. There is a universal constant \(C_1 > 0\) such that for any \(x \in \mathbb{Z}\) and \(u \geq 1\),
\[
P \left( |a_1(x) + \cdots + a_{L(x,n)}(x) - \eta(x, \tau_n)| \geq \sqrt{nu} \right) \leq C_1 \exp \left( -\frac{\pi \sqrt{n} u}{4} \right)
\]

Proof. By \cite{[19]}, equation (2.6),
\[
|a_1(x) + \cdots + a_{L(x,n)}(x) - \eta(x, \tau_n)| \leq \gamma + a_{L(x,n)}(x)
\]
for some constant \(\gamma \geq 0\). Therefore, by \cite{[15]},
\[
P \left( |a_1(x) + \cdots + a_{L(x,n)}(x) - \eta(x, \tau_n)| \geq \sqrt{nu} \right) \leq \\
\leq P \left( \gamma + a_{L(x,n)}(x) \geq \sqrt{nu} \right) \leq C \exp \left( -\frac{\pi}{4} (\sqrt{n} u - \gamma) \right)
\]
as required. \(\square\)

Corollary B.3. There are universal constants \(C, C' > 0\) such that for all \(n, u \geq 1\),
\[
P \left( L(n) \geq \sqrt{n} u \right) \leq C u^2 \exp \left( -C' u^2 \right).
\]
Proof. If \( u \geq \sqrt{n} \), the claim is trivial. So we assume \( u < \sqrt{n} \). By combining Lemma B.1 and Lemma B.2 for any \( x \in \mathbb{Z} \) we have
\[
P \left( |L(x, n) - \eta(x, \tau_n)| \geq 2\sqrt{n}u \right) \leq
\leq c_1 \exp \left( -\frac{n^{1/3}u^{4/3}}{2} \right) + C_1 \exp \left( -\frac{\pi \sqrt{nu}}{4} \right).
\]
Taking a union bound, we have
\[
P \left( \sup_{|x| \leq n} |L(x, n) - \eta(x, \tau_n)| \geq 2\sqrt{n}u \right) \leq
\leq Cn \left( c_1 \exp \left( -\frac{n^{1/3}u^{4/3}}{2} \right) + C_1 \exp \left( -\frac{\pi \sqrt{nu}}{4} \right) \right).
\]
The function \( n \mapsto n \exp \left( n^{\alpha}u^{\beta} \right) \) for fixed \( u, \alpha, \beta \geq 1 \) is decreasing for \( n \geq 1 \), so we may use \( n > u^2 \) and get
\[
P \left( \sup_{|x| \leq n} |L(x, n) - \eta(x, \tau_n)| \geq 2\sqrt{n}u \right) \leq Cu^2 \exp \left( -\frac{u^2}{2} \right)
\]
for an appropriate constant \( C > 0 \). The claim now follows from (12) and (14).

Appendix C. Approximation of functions

In [2, Appendix B], Brieussel and Zheng show how to approximate a function \( f: [1, \infty) \to [1, \infty) \) such that \( f(1) = 1 \) and \( x f(x) \) and \( f(x) \sqrt{x} \) are non-decreasing for some \( \varepsilon > 0 \), and let \( m_0 > 1 \). Then one can find sequences \( \{k_s\} \) and \( \{l_s\} \) for appropriate sequences of nonnegative integers \( \{k_s\} \) and \( \{l_s\} \). Here we prove the following lemma:

**Lemma C.1.** Let \( f: [1, \infty) \to [1, \infty) \) be a continuous function such that \( f(1) = 1 \) and \( \frac{x f(x)}{f(x)} \) and \( \frac{f(x)}{\sqrt{x} \log \log x} \) are non-decreasing for some \( \varepsilon > 0 \), and let \( m_0 > 1 \). Then one can find sequences \( \{k_s\} \) and \( \{l_s\} \), possibly finite with last value infinity in one of them, such that:

- \( k_{s+1} \geq m_0 k_s \) and \( l_{s+1} \geq m_0 l_s \) for all \( s \);
- \( f(x) \simeq_m 2m_0 f(x) \);
- \( \log \log k_s \leq l_s \) for large enough \( s \).

**Proof.** Writing \( g(x) = f(x^2) \), we have \( \frac{g(x)}{(\log \log x)^{1+\varepsilon}} \) and \( \frac{x}{g(x)} \) are non-decreasing. We can now use [2] Lemma B.1 to find sequences \( \{k_s\} \) and \( \{l_s\} \) so that the first two conditions of the lemma are satisfied.
To prove the last assertion, we recall the construction of \( \{k_s\} \) and \( \{l_s\} \). They are defined inductively, according to the following procedure: Define first \( k_0 = l_0 = 1 \). Assuming that \( k_s \) and \( l_s \) were defined and that 
\[ g(k_s l_s) = l_s, \]
we have 
\[ l_s \leq g(x) \leq \frac{x}{k_s} \]
for all \( x \geq k_s l_s \). Take the minimal \( y \geq m_0 k_s l_s \) such that 
\[ m_0 l_s \leq g(y) \leq \frac{y}{m_0 k_s}. \]
If such \( y \) exists, we have two cases:

1. \( g(y) = \frac{y}{k_s} \) - in which case we take \( k_{s+1} = m_0 k_s \) and \( l_{s+1} = \frac{y}{m_0 k_s} \geq m_0 l_s \);
2. \( g(y) = m_0 l_s \) - in which case we take \( l_{s+1} = m_0 l_s \) and \( k_{s+1} = \frac{y}{m_0 l_s} \geq m_0 k_s \).

If such \( y \) does not exist, the assumption 
\[ \frac{g(x)}{(\log \log x)^{1+\varepsilon}} \]
shows that we have 
\[ g(x) \geq \frac{x}{m_0 k_s} \]
for all \( x \geq k_s l_s \), in which case we take \( k_{s+1} = m_0 k_s \) and \( l_{s+1} = \infty \).

To ensure the last condition, we only need to check that
\[
(16) \quad g(m_0 l_s \exp(\exp(m_0 l_s))) \geq m_0 l_s.
\]
Indeed, if \( (16) \) holds, then in the second case we have \( y \leq m_0 l_s \exp(\exp(m_0 l_s)) \), so \( k_{s+1} \leq \exp(\exp(m_0 l_s)) = \exp(\exp(l_{s+1})) \) as required. To prove \( (16) \), note that
\[
g(m_0 l_s \exp(\exp(m_0 l_s))) \geq g(k_s l_s) \cdot \left( \frac{\log \log (m_0 l_s \exp(\exp(m_0 l_s)))}{\log \log (k_s l_s)} \right)^{1+\varepsilon} \geq l_s \cdot \left( \frac{m_0 l_s}{\log \log k_s + \log \log l_s} \right)^{1+\varepsilon} \geq m_0^{1+\varepsilon} l_s \left( \frac{l_s}{l_s + \log \log l_s} \right)^{1+\varepsilon}
\]
and the last parenthesis tends to 1 as \( l_s \) tends to \( \infty \).

\[ \square \]

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