Adaptive Algorithms, Tacit Collusion, and Design for Competition∗

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Abstract

Empirical and experimental evidence shows artificial intelligence algorithms learn to charge supracompetitive prices. In this paper we develop a theoretical model to study collusion by adaptive learning algorithms. With a fluid approximation technique, we characterize the learning outcomes in continuous time for general games and identify collusion’s main driver: the coordination bias. In a simple dominant strategy game, we show how correlation between algorithms’ estimates leads to persistent bias, sustaining collusive actions in the long run.

We prove that algorithms using counterfactual returns to inform their updates avoid this bias and converge to dominant strategies. We design a mechanism with feedback: the designer reveals ex-post information to help counterfactual computations. We show that this mechanism implements the social optimum. Finally, we apply our framework to two simulations of price competition and auctions studied in the literature and rationalize analytically the results.

Keywords: Artificial Intelligence, Learning, Cooperation, Continuous Time

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1 Introduction

Scholars and practitioners alike have expressed concerns that automated pricing and bidding software facilitates collusion. Simulations show how such software can learn sophisticated stick-and-carrot strategies (Calvano et al. (2020)), while empirical evidence shows how prices in retail markets suffer a sharp increase after the introduction of automated pricing (Assad et al. (2021)). Motivated by these concerns, various national agencies have begun studying these phenomena in order to regulate and tune online and offline markets (OECD (2017), Competition Bureau (2018)).

Regulatory actions generally rely on theoretical models to evaluate counterfactuals and guide policy interventions. However, the theory surrounding markets with algorithmic agents is scattered and incomplete. While many algorithms enjoy sensational success in stationary environments, the outcomes of their interactions in strategic environments are poorly understood. Policymakers lack a thorough understanding of the key drivers of algorithmic collusion. Similarly, market designers cannot identify which levers will improve market outcomes, and disruptions and inefficiencies remain first-order considerations. The literature so far has shown that the existing theory developed for rational agents is not suitable for learning algorithms, which adapt and evolve over time.

In this paper we develop a theoretical framework for analyzing algorithmic collusion. By applying our techniques, we find that collusion can be driven by coordination between independent algorithmic agents. We show how learning about different prices at different speeds may induce a coordination bias: algorithms tend to act symmetrically, thus introducing bias in their estimates. Furthermore, we show how communication in economic mechanisms can equalize learning speeds and effectively eliminate coordination. We design a mechanism, price-VCG, which implements the social optimum by helping the algorithms in their counterfactual evaluations.

We focus on learning algorithms, which adapt rapidly to market conditions. Both in retail pricing and in institutional settings, pricing and bidding algorithms need to react quickly to changes in demand and competition, making learning algorithms an ideal tool for decision making. After years of improvements in algorithmic prediction models, software programs capable of autonomous decisions are becoming increasingly common: Accenture’s solutions.ai claims “AI optimizes pricing levers in real time”, Vendavo offers dynamic pricing with competitive intelligence, and scibids.com “build customizable AI that dramatically improves Paid Media ROI”.

In this paper we characterize the behavior of such adaptive algorithms in strategic environments through a continuous-time approximation. We get around the complexity
of such algorithms by analyzing a system of ODEs, arising from an appropriate fluid limit of the discrete system. To characterize equilibria and attractive regions of the resulting non-smooth dynamical system, we adopt the differential inclusion formalism of Filippov (1988), whose boundary laws of motion fully approximate the algorithmic market.

Even in environments with the simplest strategic structure, such as dominant-strategy games, we show that algorithms can learn to collude. We provide an example where two AI algorithms (Q-learning) learn to cooperate in a Prisoner’s Dilemma. The results of the simulation prove that, although collusion seems unlikely because strictly dominated, algorithms may fail to compete. Strategic dominance is a strong concept: independently of the opponents’ behaviors, each player strictly prefers one action to the others available. Nonetheless, we show analytically how algorithms may learn to play dominated strategies. Moreover, we find that cooperation is imperfect: equilibria consist of recurrent cycles between dominated and dominant actions.

A novel effect, the coordination bias, sustains dominated strategies. By simultaneously choosing a dominated action, each algorithm overestimates the value of cooperation. When trying the dominant action, the high estimate attached to coordination will attract the algorithms away from defection again. We call this bias in the algorithm’s estimation the coordination bias, to highlight how the nearly simultaneous deviations sustain dominated outcomes.

The intuitions developed in the Prisoner’s Dilemma prove more general: the coordination bias arises when algorithms learn about certain actions faster than others. A single parameter for each action, the relative speed of learning, determines the presence of coordination bias and consequently of collusion. Actions with high relative speed adapt quickly to changing environments, while actions with slow relative speed take much longer to be re-assessed. We establish that for any Prisoner’s Dilemma game and for any adaptive algorithm there exist some relative speeds of learning that lead to collusive behavior.

Fortunately, the coordination bias disappears if adaptive algorithms are able to compute the counterfactual returns with respect to alternative actions. We make the case for a more general observation: if an algorithm can compute counterfactuals, it learns about every action simultaneously. Computing counterfactuals in a pricing game means that algorithms need to be able to compute the profits they would have obtained had they posted a different price: counterfactual reasoning is the core requirement of Nash equilibrium. The ability to compute counterfactuals is a characteristic of the feedback provided by a market, not of the algorithm. While demand estimation might be challenging in a pricing game, auctions and allocation mechanisms can be augmented to help
counterfactual evaluations. As it turns out, there is a scope for market design in the world of algorithmic agents.

We show that a mechanism designer can implement any socially optimal outcome in an algorithmic market by offering a price mechanism: personalized prices for each algorithm that correspond to the counterfactual price for misreporting. A market designer can always provide enough information to the algorithmic participants so that they can compute counterfactuals, and thus converge to the truthful equilibrium. Price-VCG adopts the allocation rule of traditional VCG, but requires additional communication: each participant receives personalized prices for all possible outcomes that could have been reached.

Finally, we show that experimental results found in the literature can be derived analytically within our framework. We apply our model to the recent paper by Asker et al. (2022). In a simulated Bertrand oligopoly the authors find that different learning rules can lead to either competition or collusion. As we proved, their algorithms reach competition only when they have access to counterfactual profits. We then repeat the same analysis in the context of auctions studied in Banchio and Skrzypacz (2022).

The paper is structured as follows: in the next section we review the related literature; in Section 2 we introduce the theoretical framework and derive the approximation results. In Section 3.1 we fully work out a Prisoner’s Dilemma, and in Section 4 we broaden these intuitions to a general setting. Finally, in Section 5 we apply our main results to a Bertrand pricing game and auctions.

1.1 Literature Review

Q-learning and Artificial Intelligence have recently sparked some interest in Economics, often through experimental work (see e.g. Klein (2021), Hansen et al. (2021), Banchio and Skrzypacz (2022)) or empirical work (see e.g. Musolff (2021), Assad et al. (2021)). The work of Calvano et al. (2020) draws attention to strategies as a proxy for collusion: they argue that simply looking at outcomes of learning might be insufficient, as collusion might arise as a “mistake” by poorly designed algorithms. Our work allows to delve deeper into the dynamics of learning, and through comparative statics and convergence analysis determine whether collusion is systemic. Particularly relevant is the work by Asker et al. (2022), which analyzes the impact of algorithm design on collusion in a Bertrand pricing game, and by Banchio and Skrzypacz (2022), who find that additional feedback in first-price auctions restores competition. We contribute to these questions with analytical tools that allow us to generalize their intuition and prove that counterfactual information
guarantees competition in the games they consider.

Some papers analyze models of collusion between algorithms, for example Brown and MacKay (2021), Leisten (2022), and Lamba and Zhuk (2022). In these papers, algorithms choose prices based on the opponent’s last quoted price: the strategies are Markov, which rules out many plausible learning strategies, including most AI algorithms. We focus on learning algorithms, which adapt their decisions along the whole history. Contributions from the bandit literature include Aouad and Van den Boer (2021), who prove tacit collusion schemes arise with classical multi-armed bandits algorithms, and Hansen et al. (2021), which finds supra-competitive prices are sustained by coordinated experiments when bidders use the Upper Confidence Bound algorithm. We prove our results for general classes of algorithms, allowing for heterogeneity in parameters as well as algorithms themselves.

Some work has examined more generally Reinforcement Learning in games, for example Erev and Roth (1998) or Mertikopoulos and Sandholm (2016), but with some notable differences. Firstly, many have analyzed systems experimentally (Erev et al. (1999), Lerer and Peysakhovich (2017)). Our approach is complementary: with the aid of our framework, one can tell apart experimental findings from agent design considerations. On the other hand, there is some theoretical work on convergence of learning procedures. For example, learning through reinforcement has been associated with evolutionary game theory by Börgers and Sarin (1997). Others have formally analyzed some of the simpler models, as Hopkins and Posch (2005). These results have then been extended to more complex systems, but with a focus on the connection with replicator dynamics as their core. Our approach is particularly relevant because instead it examines the workhorse model in applied work, $\varepsilon$-greedy Q-learning. We obtain a tool valuable for regulation and design, but we trade it off against the complete understanding possible in a simpler system. Conveniently, the approach described in Section 2 includes these earlier results under a general structure: Examples 1 and 2 show how our results apply to Fictitious Play and a well-known implementation of regret minimization.

The idea of approximating Q-learning in continuous time is not new, particularly in the single-agent setting. Related to ours is the work of Tuyls et al. (2005): the authors examine a continuous-time approximation of multi-agent Q-learning with Boltzmann exploration, and show a link with the Replicator Dynamics from the Evolutionary Game Theory (EGT) literature. Building on their work, Leonardos and Piliouras (2022) characterize the tradeoff between exploration and exploitation in the same setting. Our approximations and results hold in more general settings: we analyze a general class of adaptive algorithms, and our results leverage their discontinuities. Both Gomes and Kowalczyk
(2009) and Wunder et al. (2010) propose a continuous-time approximation of Q-learning in a multi-agent setting with \(\varepsilon\)-greedy algorithms. Their approximations are mutually inconsistent and require additional conditions on the parameter. Most importantly, those approximations remain model-dependent. Our method applies to general adaptive algorithms. The result is a recipe to analyze equilibria through the lens of dynamical systems, abstaining from heuristic modeling choices. The work of Benaim (1996) often serves as a foundation for stochastic approximations in learning. With respect to our approach, those tools have a harder time handling general learning procedures, including AI techniques that fall under the umbrella of Temporal Difference Learning. Additionally, we adopt the formalism of differential inclusions to analyze points of non-differentiability, generally new to the theory of stochastic approximations.\(^1\)

Fluid models and other forms of approximation have enjoyed great popularity in the fields of applied probability and Operations Research. Starting from the seminal contributions of Iglehart (1965), Kurtz (1970), Halfin and Whitt (1981) and Harrison and Reiman (1981), approximations enabled formal analysis of systems otherwise deemed intractable: see, e.g., Wein (1992) for a classic application to inventory management. More recently, Mitzenmacher (2001) applied fluid models to the analysis of parallel computing systems, while Wager and Xu (2021) employ diffusion approximations to study sequential experiments. Taken together, this literature shows the power of approximate models for deriving insights about intractable settings.

2 Model

While many learning algorithms have been developed and analyzed in stationary environments, their behavior in strategic domains often lacks formal guarantees. First, we formally define the class of adaptive learning algorithms we consider. Because of the complexity of their dynamics, we approximate these algorithms with continuous-time differential equations. The analytical tractability of differential equations allows us to understand and explain collusion as well as provide foundations to market designers for the original discrete model.

\(^1\)One exception is the paper by Wunder et al. (2010), which however abandons this route in favor of simulations.
2.1 Adaptive Algorithms

We define a class of learning algorithms that includes the most common automated decision-making procedures. These algorithms often offer performance guarantees in stationary environments that fail in strategic domains.

Definition 1. An adaptive algorithm is a $d$–dimensional stochastic process $\theta$ which evolves according to

$$\theta_k = \theta_{k-1} + \alpha T(\theta_{k-1}, Y_k, k),$$

where $\theta \in K \subset \mathbb{R}^d$, $\alpha \in \mathbb{R}$, and $(Y_k)_k$ are i.i.d. random variables in $\mathbb{R}^e$, for some $e \in \mathbb{N}$.

We denote the distribution of $Y_k$ over $\mathbb{R}^e$ by $\nu(Y)$, so that $Y_k$ can be thought of as any randomization device used to perform updates. We require the function $T$ to be extendable over the positive real line $[0, +\infty)$ in its last component.

Many popular procedures such as Fictitious Play (Brown (1951)) and the reinforcement learning model of Erev and Roth (1998) can be formulated as adaptive algorithms.

Example 1. Fictitious Play is a learning procedure that requires agents to best respond to the empirical distribution of the opponent’s strategies. Suppose there are two players, $A$ and $B$, and let $\beta(\cdot)$ be the multi-valued best-response function. In our language, the empirical distribution is the relevant stochastic process, and its evolution for player $A$ is guided by the following:

$$\theta_k^A = \theta_{k-1}^A + \frac{\beta(\theta_{k-1}^B) - \theta_{k-1}^A}{k}.$$

Fictitious Play has been well studied and thoroughly understood. However, it is seldom implemented in practice, mostly because of its model-dependent update rule which requires knowledge of the strategic structure of the game. Similarly, regret minimization algorithms require the agent to know the ex-post optimal payoff.

Example 2. A simple adaptive implementation of regret minimization is the Polynomial Weights algorithm proposed by Cesa-Bianchi et al. (2005). The algorithm maintains weights $w_k^i$ for each action $i$ and updates the action chosen according to

$$w_k^i = w_{k-1}^i - \lambda l_k^i w_{k-1}^i,$$

where $l_k^i = r_k^* - r_k^i$ is the regret from obtaining payoff $r^i$ instead of the optimal payoff $r^*$ in iteration $k$ and $\lambda$ is a normalization parameter. The weights are then aggregated in a probability of choosing action $i$ in each period with $\theta_k^i = \frac{w_k^i}{\sum_j w_k^j}$. Some algebra shows that one can write the
update of the probabilities as

\[ \theta^i_t = \theta^i_{t-1} + \left( \frac{\theta^i_{t-1}(1 - \lambda l^i_t)}{\sum_j \theta^j_{t-1}(1 - \lambda l^j_t)} - \theta^i_{t-1} \right) \]

Klos et al. (2010) show that such procedure shares the structure of the Replicator Dynamics from the evolutionary game theory literature.

In Examples 1 and 2 we analyzed settings where algorithms are deployed to learn in static games, which are repeated over time. However, in many applications the environment is ever changing, and optimal play evolves accordingly. The simplest framework to capture this feature is the Markov game defined in Shapley (1953). To accommodate these environments, we introduce the class of adaptive algorithm with states.

Definition 2. Let \( S \) be the set of states of a time-homogeneous Markov Chain \( (X_k)_k \), whose transitions are independent of each \( \theta^s_k \) and with stationary distribution \( (\pi_s)_{s \in S} \). Let \(|S| = f\). An adaptive algorithm with states is a collection \( \theta^s = (\theta^s_s)_{s \in S} \) of \( d \)-dimensional stochastic processes such that each \( \theta^s \) evolves according to

\[ \theta^s_k = \theta^s_{k-1} + \alpha T^s(\theta_{k-1}, X_{k-1}; X_k, Y_k, k), \]

For \( s, s' \in S \), we denote by \( P_{ss'} \) the transition probability from \( s \) to \( s' \). Intuitively, for each state of the Markov chain there is a learning statistic, which is updated (potentially) using the values of the statistics for all other states. We will maintain the following assumptions:

Assumption A1. \( T^s(\theta_{k-1}, X_{k-1}; X_k, Y_k, k) = 0 \) if \( X_{k-1} \neq s \).

Assumption A1 requires that at iteration \( k - 1 \) the algorithm updates only those components of the learning statistic that correspond to state of the Markov chain at iteration \( k - 1 \), i.e., \( X_{k-1} \).\(^2\) The next assumption is chiefly technical, and it will be employed only to ensure the existence of a solution to the system of ODEs that characterize our fluid approximation. In Section 3 we show how it can be relaxed to deal with some discontinuities.

Assumption A2. The function \( T^s(\theta, X_{k-1}; X_k, Y, k) \) is Lipschitz-continuous in \( \theta \) and \( k \) for all \( s \in S \).

\(^2\)This assumption excludes from our analysis algorithms that use information collected when in state \( s \) at iteration \( k \) to update the learning statistic of state \( s' \) at the same time; however, notice that it does not preclude this information to be employed in future iterations.
Definition 2 is rather general: most of temporal-difference-based algorithms belong to this class, with popular choices such as Q-learning, SARSA and Actor Critic methods. Notice also that Definition 2 encompasses Definition 1 as a particular case in which \( f = 1 \), i.e., there is just one state.

Below we show how Q-learning, the building block of many Artificial Intelligence algorithms, belongs to the class of adaptive algorithms with states. The popularity of Q-learning-based methods combined with its interpretability lead us to adopt it as a running example in what follows.

**Example 3.** Q-learning approximates the optimal action-value function in a Markov Decision Process. It maintains estimates \( Q_k(s,a) \) of the value of an action \( a \) at state \( s \), and in each iteration the Q-vector \( (Q_k(s,a))_{(s,a)} \) gets updated as

\[
Q_k(s,a) = \begin{cases} 
Q_{k-1}(s,a) + \alpha [r_{k-1} + \gamma \max_{a'} Q_{k-1}(s_k, a') - Q_{k-1}(s,a)] & \text{if } (s,a) = (s_{k-1},a_{k-1}) \\
Q_{k-1}(s,a) & \text{else}
\end{cases}
\tag{1}
\]

where \( r_{k-1} \) is the reward obtained in iteration \( k \).

Q-learning is an adaptive algorithm, which in a given state and action evolves according to a Lipschitz function of the reward and its estimate of the continuation value. Intuitively, the Q-vector estimates continuation values according to a Bellman equation. In iteration \( k \), the value of an action at a certain state is a convex combination of its previous estimate (with weight \( 1 - \alpha \)) and of a new Bellman estimate (with weight \( \alpha \)). The weight \( \alpha \) serves as rate of learning, and can be thought of as a measure of persistence: higher \( \alpha \) make the algorithm faster in forgetting the past.

When multiple agents employ adaptive algorithms, the system can be represented as a single algorithm that belongs to the class of Definition 2 by stacking the stochastic processes together, for example as \( \theta_k = (\theta_k^A, \theta_k^B) \) if there are two agents, Alice and Bob. Throughout the next sections, \( \theta_k \) will denote this collection of learning statistics for all states and all players unless otherwise specified.

A common feature of adaptive algorithms is that while rather simple to analyze in a stationary environment, they become unpredictable when learning to play against each other. Specific algorithms carry guarantees in some classes of games (e.g. fictitious play and potential games, regret minimization and zero-sum games, etc.) but, when such
guarantees are missing, predicting outcomes proves a difficult task. The random component $Y$, the Markov state $X$, as well as the discreteness of the jumps in the process make equilibrium analysis intractable even for the simplest strategic environments. In order to overcome these difficulties, we turn our attention to a continuous-time approximation, which we construct in the next section.

2.2 Approximation in Continuous Time

The discreteness and randomness present in general adaptive algorithms results in complex dynamics.\textsuperscript{5} In order to derive analytical results, we simplify and characterize the behavior of algorithmic systems through the use of a continuous-time approximation. The limiting fluid ODE system provides tractability and (often) solutions in closed form.

The cornerstone of our continuous-time framework is the following theorem, which guarantees that for any adaptive procedure of the type of Definition 2 there exists a continuous, deterministic process, that approximates the original system.

**Theorem 1.** Let $\theta$ be an adaptive algorithm with states that satisfies Assumptions A1 and A2, and let $K \subset \mathbb{R}^d \times \mathbb{R}^f$ be a compact set. The collection of Cauchy problems

$$\begin{cases}
\frac{d\Theta^s(t)}{dt} = \alpha \pi_s \sum_{x \in S} P_{sx} T^s(\Theta(t), s; x, Y, t) d\nu(Y) \\
\Theta^s(0) = y^s_0
\end{cases}$$

has a solution $\Theta$ over $K$ for all $y^s_0 \in K$ and for all $s \in S$, and there exists a sequence of processes $\{\theta^n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} P \left\{ \sup_{t \leq T} \left\| \theta^n(t) - \Theta(t) \right\| > \eta \right\}$$

for all $T \geq 0$ and $\eta > 0$ such that $\{\Theta(t)\}_{t \leq T} \subset K$.

Here we first offer some interpretation for Theorem 1, and then we provide an intuitive sketch of the proof, which can be found in Appendix A. The process $\Theta^s(t)$ for $t \geq 0$ is the fluid approximation to $\theta^s_k$: it is a deterministic process whose time-derivative is the expected update that the discrete process $\theta^s_k$ would incur over one unit of time. Theorem 1 states that it is possible to construct a sequence $\{\theta^n\}$ of discrete processes that draw closer and closer (in probability) to the continuous process. The theorem relies on a law-of-large-numbers argument: if the updates occur with high frequency, but each update’s contribution is relatively small, the process behaves similarly to its expectation. The proof is constructive, in the sense that we first define this sequence of processes, and

\textsuperscript{5}Refer to Section 3 for a fully worked-out example of these difficulties.
then show that we can apply the fundamental theorem of fluid approximations by Kurtz (1970) to conclude convergence in probability.

**Outline of the proof.** The proof proceeds in three steps. The first step is to turn the discrete-time process into a continuous-time one. This transformation is carried out by means of a *Poissonization* argument: given a Poisson clock with unit rate, we assume that the transitions of the Markov state $X$ and the updates of $\theta$ occur at each tick of the clock. Mathematically it is equivalent to constructing a compound Poisson process with unit rate. Notice in particular that this construction makes sure that, at time $t_k \in \mathbb{R}_+$ such that the clock ticked $k$ times, the continuous-time process thus obtained is equal to $\theta_k$.

The second step consists in finding an appropriate rescaling of time and of the updates; in particular, one usually accelerates time and downsizes the jumps at every update. Formally, it boils down to taking an increasing sequence of rates $\lambda_n$ of the Poisson clock, while dividing the learning parameter $\alpha$ by $n$. Intuitively, by accelerating time the learning statistic $\theta$ is updated more often and more observations of the random quantities occur in the same unit of time; thus, downsizing the extent of the jump is necessary to avoid divergence of the process. When combining these two effects, one should expect a law of large numbers to apply, so that approximating a system with very frequent updates with one whose derivative is the expected update is correct.

Our setting assumes that, if at iteration $k-1$ the state is $X_{k-1} = s$, only the components of $\theta_k$ that correspond to state $s$ are updated. In the continuous-time limit, all components are updated continuously: for each state $s$, the time-derivative of the corresponding components is scaled by a factor of $\pi_s$. Recall that $\pi$ is the stationary distribution of the Markov chain $(X_k)_k$: this factor takes into account that, when updates occur very quickly, some states are visited “more often” than others, and therefore the components of $\theta_k$ that refer to them update more frequently; hence, in the fluid limit, the rate of update of the components corresponding to more “frequent” states is revised upwards, and the rate for the states that are visited seldom is downsized.

**3 An Illustrative Example**

With our continuous time approach, we can now analyse collusion in some simple environments: we focus on games with an equilibrium in dominant strategy where incentives for competition are strong and classic theory predicts collusion should be hard. We show

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6 More precisely, the rates will be different for each state $s$ of the Markov chain. For more details, the reader is referred to the proof in Appendix A.
how algorithms sustain cooperation in a Prisoner’s Dilemma, despite cooperation being strictly dominated. We explain these findings by analytically identifying a bias that arises from coordination: the algorithms cooperate and defect symmetrically, therefore they overestimate the value of cooperation.

Consider the following family of Prisoner’s Dilemma games, with payoffs given by Figure 1. The family is parameterized by $g$, which models the attractiveness of joint cooperation: the larger $g$, the more attractive cooperation becomes. However, for all $g \in (1, 2)$ the dominant strategy, and the only Nash equilibrium, is to always play “defect”.

\[
\begin{array}{ccc}
 & C & D \\
A & 2g, 2g & g, 2 + g \\
B & 2 + g, g & 2, 2 \\
\end{array}
\]

Figure 1: Payoffs of the stage game, $1 < g < 2$.

**Simulations.** We simulate the repeated play of a pair of algorithms, Alice and Bob, of the above Prisoner’s Dilemma. We choose to analyze Q-learning, which maintains independent estimates of the value of each action, $Q_k(C)$ and $Q_k(D)$. In this static game the algorithms have no states, that is they compress all past play in their estimates of the values $Q_k$. We assume that both Alice and Bob choose their actions according to a $\varepsilon$-greedy policy: at each iteration $k$ they play the action with the highest estimated value, i.e., $\operatorname{argmax}_{a \in \{C, D\}} Q_k(a)$, with probability $1 - \varepsilon$, and with probability $\varepsilon$ they choose an action uniformly at random.

Figure 2a shows the results of these numerical experiments. The algorithmic agents learn the dominant strategy equilibrium $\{D, D\}$ only for low values of the parameter $g$. Instead, for high values of $g$ the agents cooperate, albeit imperfectly. In fact, both cooperation and defection appear in “quasi-recurrent cycles”, as shown in Figure 2b: the value of collaboration is generally above that of defection, but at somewhat regular intervals it drops below $Q_k(D)$, so that agents switch to playing $D$. The value of defection then decreases almost immediately, and players revert to cooperation. This analysis raises a few issues: (i) the parameter $g$ should have no effect on strategic decisions, and instead it leads to stark differences in outcomes; (ii) collusion seems to consist of cycles, but since agents cannot condition on the past action of the opponent, it is hard to impute these to “retaliatory” strategies. The continuous-time approximation transparently identifies the causes of these patterns.
(a) Fraction of runs where the agents learn the Nash Equilibrium \(\{D,D\}\).

(b) Cycles in the discrete system, obtained with \(g = 1.8\).

Figure 2: We initialized 100 independent runs of Q-learning with \(T = 100,000\) time steps each, and we consider a profile learned if it is played consistently for the last 100 time periods of a run. In all cases, \(\varepsilon = 0.1\), \(\alpha = 0.05\) and \(\gamma = 0.9\). The initialization is optimistic, i.e. all Q-values are larger than the maximum value they could ever achieve. This leads to a phase of intense exploration at the outset.

### 3.1 Theoretical results

In \(\varepsilon\)-greedy Q-learning the agents learn only about the actions they take, which depend on the values of \(Q_k\). In the parlance of Definition 1, here the function \(T\) depends on whether \(Q_k(D) \geq Q_k(C)\) or not. To sidestep this discontinuity, we first apply Theorem 1 to \(Q_k\) over \(T\)’s maximal continuity domains. In particular, \(T\) is continuous over values of \(Q_k\) where \(\max\{Q_k(C), Q_k(D)\}\) is unique for both players. Let these sets be \(\omega_{C,C}\), \(\omega_{C,D}\), \(\omega_{D,C}\) and \(\omega_{D,D}\): the subscripts indicate which action is the one currently (strictly) preferred by Alice and Bob respectively.

Over \(\omega_{C,C}\) the greedy action for both players is \(C\), so that in every period Alice cooperates with probability \(1 - \frac{\varepsilon}{2}\) and defects with probability \(\frac{\varepsilon}{2}\). Hence, with probability \((1 - \frac{\varepsilon}{2})^2\) she collects reward \(2g\) — similarly for other profiles. Therefore, the fluid limit solves

\[
\begin{align*}
\frac{dQ^i_t}{dt}(C) &= \alpha \left(1 - \frac{\varepsilon}{2}\right) \left(\frac{1 - \varepsilon}{2}2g + \frac{\varepsilon}{2}g + (\gamma - 1)Q^i_t(C)\right) \\
\frac{dQ^i_t}{dt}(D) &= \alpha \frac{\varepsilon}{2} \left(1 - \frac{\varepsilon}{2}\right)(2 + g) + 2\frac{\varepsilon}{2} + (\gamma Q^i_t(C) - Q^i_t(D))
\end{align*}
\]

\[\text{(2)}\]

[^7]: I.e., \(C\) is selected with probability \(1 - \varepsilon\) if the randomization device instructed the agent to be greedy, and with probability \(\frac{\varepsilon}{2}\) if the agent was instructed to play an action at random.
Following the same logic we can derive a similar system for $\omega_{D,D}$. We focus on the symmetric case so $Q_k$ will always evolve in $\omega_{C,C}$ or $\omega_{D,D}$. Proposition 1 below then guarantees that we can suitably “paste” together the fluid limits to obtain a global solution.

**Proposition 1.** Let $F_j$ be the field defined as above over $\omega_j$ for all $j = \{C,C\}, \{D,D\}$. There exists a global solution in the sense of Filippov (1988) to the differential inclusion

$$\frac{dQ_t}{dt} = F_j(Q_t) \quad \text{over } \omega_j, \forall j$$

$$\frac{dQ_t}{dt} \in \text{co}\{F_k(Q_t) \mid \forall k \in K \subset J\} \quad \text{when } Q_t \in \bigcap_{k \in K} \omega_k, \forall K \subset J$$

where $\text{co}\{\cdot\}$ denotes the convex hull of a set, and $\overline{\omega}$ is the closure of $\omega$.

The solution to both systems of ODEs possesses a well-defined time derivative also on the boundary between $\omega_{C,C}$ and $\omega_{D,D}$. We call the vector field on the boundary *sliding field*, defined as the convex combination of the vector fields on $\omega_{C,C}$ and $\omega_{D,D}$, such that the component normal to the boundary vanishes.

**Stability analysis** Figure 3 shows the vector fields that characterize the continuous-time approximation: its stationary points correspond to the steady states of the original system. The figures highlight the presence of two stationary points when $g$ is large. The steady state on the boundary disappears for lower values of $g$. We can characterize these points analytically:

**Proposition 2.** There always exists a steady state $q_{eq}^D \in \omega_{D,D}$. Moreover, if $\varepsilon < 1 - \sqrt{\frac{2-g}{g}}$ there exists another steady state $q_{eq}^C \in \overline{\omega}_{D,D} \cap \overline{\omega}_{C,C}$, given by

$$q_{eq}^C(C) = q_{eq}^C(D) = \frac{1 + g + \sqrt{(g-1)(g-\varepsilon g + \frac{\varepsilon^2 g}{2})}}{(1-\gamma)}.$$

We will refer to the point $q_{eq}^C$ as the *cooperative equilibrium*. Its existence hinges on the relationship between the value of cooperation and the exploration rate.

Figure 4 shows the minimum exploration $\varepsilon$ that guarantees a cooperative equilibrium and how it varies with the value of cooperation. Intuitively, as $g$ increases, the relative

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Formally, the boundary can be divided in three regions. A *crossing region* occurs where the normal components have the same sign, so that there is no need to define a field on the boundary. A *repulsive region* occurs where the normal components point away from the boundary, which will then never be reached. Finally a *sliding region* occurs when both normal components point towards the boundary, thus forcing a trajectory to slide along it, and therefore the vector field is defined as explained above. For more details on how the field is calculated we refer the reader to the proof of Proposition 2 in Appendix A.
Figure 3: Stationary points are marked with a red dot. The domain of attraction of the cooperative outcome is green-shaded, the one for the non-cooperative outcome is blue-shaded.
benefit of defecting decreases (and vanishes completely when $g = 2$), so more and more exploration is needed to realize that $D$ is a dominant action. For example, if $g = 1.8$ the exploration rate required to guarantee convergence on $\{D, D\}$ is about 70%, which is considerably larger than the standard employed in practice.\(^9\)

**Sustaining cooperation** Note that the system of ODE, when simulated with small but discrete time steps, closely mimics the path of play of the discrete Q-learning. This behavior is clearly observed in Figure 5a and can be compared with Figure 2b, shedding some light on how cooperation can be sustained. Suppose $C$ is the preferred action of both players: Alice and Bob cooperate, but with probability $\frac{1}{2}$ one defects and realizes its benefit. Over time, $Q(D)$ rises above $Q(C)$. However, once Alice defects, Bob will also defect almost immediately after, because cooperating when Alice defects makes Bob considerably worse off. Joint defection decreases the value of $Q(D)$ for Alice and Bob and reinforces the value of joint cooperation. Thus, when the exploration rate is too small, the algorithms play a symmetric profile of actions too often. We call this effect the *coordination bias*: the estimated value of action $a$ is correct *conditional on the opponent playing (almost always) the same action*. When Alice begins defecting, her cooperative experiments are too infrequent, and her estimate of the value of cooperation remains biased. Effectively, Alice is too slow to realize the downside from cooperating when Bob defects,

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\(^9\)The literature on Q-learning in games usually employs $\varepsilon = 0.1$ or smaller, either fixed or decreasing over time. E.g., see Gomes and Kowalczyk (2009).
and the benefit of defecting when Bob cooperates.

(a) Cycles of play around the cooperative equilibrium in the discretized ODE system.

(b) Proportion of time spent playing the cooperative outcome in $q^\text{eq}_C$, analytically and in the simulations.

Figure 5

The steady state on the boundary is the continuous-time counterpart of these cooperative cycles in discrete system. Agents spend some “local time” playing cooperation: we interpret the weights assigned to the fields on the boundary as the fraction of time dedicated to cooperation and defection, respectively.\footnote{This intuition can be made formal using the idea of hysteresis loop around the boundary; see di Bernardo et al. (2008)} The local time is such that the infinitesimal incentives around the stationary point are balanced.

**Corollary 1.** In the collusive equilibrium, agents spend $\tau_C$ fraction of their time cooperating, where

$$
\tau_C = \frac{\frac{\epsilon^2 \gamma}{2} + \epsilon - 2 - q^\text{eq}_C (\gamma - 1)(1 - \epsilon)}{2(\epsilon - 1)(1 + g + (\gamma - 1)q^\text{eq}_C)} \in \left[ \frac{1}{2}, 1 \right].
$$

Figure 5b plots this quantity and shows that the proportion of time spent playing the cooperative profile at equilibrium varies with $g$. We notice that the local time spent cooperating obtained analytically approximates well the estimates from the simulations. Moreover, $\tau_C$ is always greater than $\frac{1}{2}$, which squares well with the results from Figure 2a: if the cooperative equilibrium exists, the agents collude more than 50% of the time, which lead to conclude that the learned outcome is $\{C,C\}$. 

\footnote{This intuition can be made formal using the idea of hysteresis loop around the boundary; see di Bernardo et al. (2008)}
4 Main Results

After building some intuition in Section 3, we are ready to formally understand collusion by algorithms in general dominant-strategy games. We map the coordination bias identified in the Prisoner’s Dilemma in a characteristic of the algorithms: their relative learning speed for different actions. We show that market design can guarantee convergence to dominant-strategy equilibria by constraining the relative speeds of learning of each algorithm.

4.1 Speed of Learning

Suppose Alice’s preferred action is cooperation: at rate $1 - \frac{\varepsilon}{2}$ she learns about the payoffs of cooperating, while she learns about the payoffs of defection more slowly, at rate $\frac{\varepsilon}{2}$. Different rates imply that Alice’s estimates enjoy rather different persistence, thus impairing the algorithm’s ability to wash away existing bias. Low exploration rate $\varepsilon$ makes discovering the value of the dominant strategy difficult.

Let us consider a stylized model of competition. We call a collusion game a normal-form game $G = (N_i, (A_i)_{i \in N}, (u_i)_{i \in N})$ such that each agent has a (strictly) dominant action but there are Pareto-improving outcomes in dominated strategies. We focus on continuous-time algorithms, with the understanding that these can be construed as limits of discrete-time algorithms. In this section and the next, we will analyze procedures that take the following simplified form:

**Definition 3.** A reinforcer is an adaptive algorithm such that its update for each action $a_i \in A_i$ when the opponents’ actions are $a_{-i} \in A_{-i}$ is of the form

$$\dot{\theta}^{a_i}(t) = \alpha^{a_i} U(\theta^{a_i}(t), r_t(a_i, a_{-i})), $$

and $U$ is Lipschitz, increasing in $r_t(a_i, a_{-i})$ and decreasing in $\theta^{a_i}(t)$.

Essentially, a reinforcer always increases the value of $\theta$ after good news, but the larger the value of $\theta$ the smaller the update is. We will assume throughout that the algorithm $\theta$ is bounded. Note that, while the update function $U$ is common to all actions, the statistic for each action carries a personalized rate $\alpha^{a_i}$. We continue to consider algorithms which trade off exploration and exploitation: for this reason, each reinforcer will have identical updates $U$ but possibly different rates $\alpha$ depending on which action is currently preferred. The magnitude of the $\alpha$’s does not affect the stability analysis, which is why we focus on the relative rates, or speeds.
**Definition 4.** The relative learning speed of action $a_i$ is the ratio

$$RLS(a_i) = \frac{\alpha_{a_i}}{\sum_{a \in A_i} \alpha^a}.$$

The relative learning speed can capture differences in the exploration policy or the frequency of action selection. As mentioned before, the coordination bias arises from differences in learning speeds. The following proposition shows that this is the case for generic reinforcers, not just the Q-learning example of Section 3.

**Proposition 3.** In any collusion game, let each agent learn through a greedy reinforcer. Then, there exist $RLS(a_i)$ for all $a_i \in A_i$ and $i \in N$ such that a dominated strategy is played in equilibrium.

The proof of this proposition is quite simple: all it takes is for the relative speeds of learning to be zero for all actions but for a dominated strategy. In the appendix the argument is made formal, and we also show that even non-zero speeds of learning yield dominated strategies in equilibrium. The proof in full detail can be found in Appendix A.

**Proposition 3** seems to indicate that there is no hope for reinforcers to converge on dominant strategies. Instead, when the relative learning speeds are identical across all actions, we can show that the algorithms can only converge on the (strict) dominant strategy, regardless of the opponents’ actions. We will need the following technical assumption:

**Assumption A3 (Thickness).** Let $G^n_{-i}(a)$ be the distribution over actions of all players but $i$ in period $n$. There exists a $\chi > 0$ such that $G^n_{-i}(a) \geq \chi$ for all $a \in A_{-i}$ for all $i$.

Thickness ensures that sufficient exploration is carried out by all players in the limit. Thickness is easily satisfied by any algorithmic system which adopts a $\varepsilon$-greedy policy, for example.

**Theorem 2.** Under Assumption A3, in any game with a dominant strategy equilibrium a $\varepsilon$-greedy reinforcer with $RLS(a_i) = RLS(\tilde{a}_i)$ for all $a_i, \tilde{a}_i \in A_i$ converges on the dominant strategy.

This result is surprisingly powerful. In particular, we make no assumption about the opponent’s play: as long as all actions are played with some positive probability even in the limit, the greedy reinforcer will learn to play its dominant strategy. Convergence is guaranteed for any number of opponents, as well as opponents who adopt different learning algorithms, or the same learning algorithm with different parameters. All these asymmetries can be accommodated by Theorem 2.
The assumption that relative speeds of learning be identical across actions may appear stringent: it might for example require restricting the exploration of the algorithm to try each action uniformly at random. However, suppose that the algorithms were able to compute counterfactuals. That is, suppose that, after choosing an action $a_i$ in period $t$, the algorithmic agent was able to compute $r_t(\tilde{a}_i, a_{-i})$ for all $\tilde{a}_i \neq a_i$. Then, the statistics of all actions could be updated simultaneously, using the reward that each action would have procured had it been played in that period. Simultaneous updates are sometimes referred to as synchronous learning:\textsuperscript{11} in this case learning happens at the same rate for all actions. The following corollary provides a powerful result.

**Corollary 2.** Under the same assumptions of Theorem 2, a greedy reinforcer who can compute counterfactuals always converges to the dominant strategy equilibrium.

It is not surprising that counterfactuals help to learn to play equilibria. In fact, the theory of Nash equilibria is based on the assumption that agents can compute the payoff that would have obtained if they had played a different action, taking the opponents’ strategies fixed, which in turn allows them to evaluate incentives to deviate. Corollary 2 establishes that reinforcer algorithms successfully rule out dominated strategies, provided they have access to a method to compute counterfactuals, which then implies that they converge to the (unique) equilibrium.

Elimination of dominated actions is one of the most basic procedures that players in games are supposed to be able to perform, and it is understood even by level-one rational agents, who have the lowest level of strategic sophistication.\textsuperscript{12} For games with dominant strategies, level-one rationality is enough to guarantee that the equilibrium will be played. However, dominant strategies do not exist in many settings of strategic interaction, which instead require higher levels of rationality for players to play an equilibrium.\textsuperscript{13} For example, it is well known that players need level-$k$ rationality for dominance-solvable games such that Iterated Elimination of Strictly Dominated Strategies (IESDS) terminates in $k$ rounds.

Our next result shows that reinforcers actually achieve a special form of level-$k$ rationality. We consider here IESDS where a strategy can be eliminated only when it is strictly dominated by another pure strategy. Under this assumption, the order of elimination is important in determining which profile(s) of actions survive the procedure, as shown in

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\textsuperscript{11}The term synchronous appears in Asker et al. (2022), but the idea of agents learning from counterfactuals is introduced by Tumer and Khani (2009).
\textsuperscript{12}See the seminal paper on level-$k$ rationality by Stahl II and Wilson (1994).
\textsuperscript{13}E.g., consider first-price auctions and Bertrand pricing: in both cases, if the players can choose more than two actions, there is no dominant strategy.
a simple example by Gilboa et al. (1990). The restriction to domination by pure strategies is necessary because the adaptive algorithms we defined do not deal well with mixed strategies. Fix a finite normal-form game \( G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \).

**Definition 5.** We say that an action \( a_i \in A_i \) is pure-rationalizable if \( a_i \) survives some IESDS procedure.

In general, for a certain order of elimination of dominated strategies, action \( a_i \) might get eliminated. However, as long as there is an order such that \( a_i \) survives the IESDS process, we consider \( a_i \) pure-rationalizable. Our next theorem shows that reinforcers with access to counterfactuals only play pure-rationalizable strategies.

**Theorem 3.** Let all players in game \( G \) learn through a greedy reinforcer with \( RLS(a_i) = RLS(\tilde{a}_i) \) for all \( a_i, \tilde{a}_i \in A_i \) for all \( i \in N \). Then, there exists a \( T \) such that for all \( t \geq T \) all actions played are pure-rationalizable by the same IESDS.

In the next section we will leverage these result to provide a mechanism for algorithmic markets that implements the social optimum. Furthermore, Section 5 shows how Theorems 2 and 3 can be applied to study analytically two environments that received considerable attention in the experimental literature.

### 4.2 Dominant Strategy Implementation

Parkes (2004) highlighted the role that mechanism design might play in shaping algorithmic systems, focusing mainly on how to improve learning speed and to select the designer’s preferred equilibrium when players in the mechanism are algorithmic. When agents are not fully rational, learnable mechanism design proposes mechanisms that are explicitly designed to maximize their performance considering the agent’s lack of rationality. As the author suggests, “a useful learnable mechanism would provide information, for example via price signals, to maximize the effectiveness with which individual agents can learn equilibrium strategies”. In this section we develop this intuition, and we show that mechanism design can improve competition in markets dominated by algorithms. We do so leveraging our results, which guarantee that competition is restored provided counterfactuals can be computed. Supplying enough information so that algorithms can evaluate counterfactual returns falls onto the mechanism designer, and we embrace this perspective. Concretely, we define price-VCG, a class of mechanisms based upon the traditional

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14**Definition 2** makes clear that it is impossible for adaptive algorithms to learn the value of randomizing across actions.
VCG mechanism. In addition to the VCG price and allocation, price-VCG communicates personalized prices for all outcomes to each participant. This information is sufficient to compute counterfactuals for alternative reports. We show that these mechanisms can implement the surplus-maximizing allocation, whereas traditional VCG can instead suffer from the coordination bias.

In this setting, let $\mathcal{X}$ be a set of possible outcomes, and let there be $n$ agents with values $v_i(x)$ for all $x \in \mathcal{X}$ and quasilinear utility in payments. The goal of the designer is to select the allocation $x^*$ such that

$$x^* = \arg\max_{x \in \mathcal{X}} \sum_{i=1}^{n} v_i(x).$$

Adopting the usual argument, we will focus on truthful reporting mechanisms. It is well-known that the designer can choose a VCG mechanism to implement such a utilitarian choice function. In particular, the mechanism with the Clarke pivot rule selects the optimal allocation for the given reports and assigns payments for agent $i$ equal to

$$p_i = \sum_{j \neq i} v_j(x^*) - \max_{x \in \mathcal{X}} \sum_{j \neq i} v_j(x).$$

Given these payments, rational agents are incentivized to report truthfully: reporting their true values is a dominant strategy. However, we showed in Section 4.1 that algorithmic agents need not converge to the dominant strategy equilibrium, and can play dominated strategies even in the limit. In general, traditional VCG implementations will fail.

Nevertheless, we can design a mechanism that implements the socially optimal outcome. Intuitively, we leverage Corollary 2: if the algorithms could compute counterfactuals in a VCG mechanism, the mechanism would achieve its goal. The feedback from the traditional VCG is rather limited: with only the knowledge of the price for the implemented outcome, counterfactual returns for other actions are impossible to compute. In what follows we define a new mechanism, the Price-VCG, and we show that it implements the socially optimal outcome.

**Definition 6.** A Price-VCG mechanism is a pair of functions:

- An outcome function, which takes as input the reported vectors of valuations $(v_i(x))_{x \in \mathcal{X}}$ of the players and returns the outcome that maximizes the social surplus,

- A payment function, which takes as input the reported vectors of valuations $(v_i(x))_{x \in \mathcal{X}}$
of the bidders and returns a vector of prices \( (p_i(x))_{x \in X} \) for each player, such that

\[
p_i(x) = \sum_{j \neq i} v_j(x) - \max_{x' \in X} \sum_{j \neq i} v_j(x').
\]

Intuitively, a price-VCG mechanism provides each algorithmic player with personalized prices for all possible outcomes that could have been implemented. These prices aggregate the information about other player’s valuations and allow to compute counterfactuals. Note that a player’s own prices depend on its reported valuations only through the outcome selection. Therefore, each \( p_i(x) \) is independent of the vector of reported \( (v_i(x))_{x \in X} \).

**Theorem 4.** The price-VCG mechanism implements the social optimum even when players bid through greedy adaptive agents.

**Proof.** It suffices to show that each algorithm can compute its counterfactual return from reporting any vector \( (\hat{v}_i(x))_{x \in X} \). For any report \( (\hat{v}_i(x))_{x \in X} \), the player can compute which outcome \( \hat{x} \) would have been implemented:

\[
\hat{x} = \arg\max_{x \in X} \hat{v}_i(x) + p_i(x)
\]

Fixed the other player’s reports, \( p_i(x) \) represents the sum of other players’ valuations for outcome \( x \), up to a constant. It is then easy to compute the counterfactual return for reporting \( (\hat{v}_i(x))_{x \in X} \): player \( i \) would obtain \( v_i(\hat{x}) + p_i(\hat{x}) \). Now, because VCG is a strategy-proof mechanism, Corollary 2 applies and adaptive algorithms learn to play truthfully, thus implementing the social optimum.

In this general setting, price-VCG is also the mechanism with the smallest feedback to enable counterfactual evaluation. For any mechanism with less feedback in fact one can produce a valuation for which the counterfactual is not uniquely determined.

Notice how in a second-price auction for a single item, the price-VCG mechanism reduces to providing the lowest-bid-to-win to all the losers. Anecdotally, this practice is becoming more common in online auctions, even in games that do not have a dominant strategy (but can be solved by iterated elimination), as pointed out in Banchio and Skrzypacz (2022). When the auctioneer is selling multiple goods and each bidder has unit demand, price-VCG has a price-theoretic interpretation. The designer is offering personalized prices to each consumer, essentially offering them to purchase any goods at those prices. Consumers have a budget constraint, given by their unit demand and their valuations for the items, and will choose the item whose price maximizes their utility while...
satisfying their budget constraint. In this sense, the feedback of price-VCG is an aggre-
gator of market information, which helps agents evaluating the true value of truthful reporting.

5 Application

In this section we better position our results vis-a-vis the literature on algorithmic collu-
sion by applying the arguments of Section 4 to two recent papers by Asker et al. (2022) and by Banchio and Skrzypacz (2022).

5.1 Bertrand Competition

Consider the setting of Asker et al. (2022). The authors simulate algorithmic competition in a Bertrand oligopoly, and find that collusion depends critically on the synchronicity of the algorithm. We show that their results can be derived analytically, and how compe-
tition is enforced only when players have access to counterfactuals with respect to their pricing decisions.

There are two firms, Alice Inc. and Bob Ltd., which face a common demand for their product. For a simple illustrative model, we assume that the market demand is \( D(p_A,p_B) = 3 - \min\{p_A,p_B\} \), and if the two firms charge the same price they split demand equally. Suppose that each firm has 0 marginal cost, and for simplicity let the firms choose only between two prices: \( p \in \{0.5, 2\} \). Profits equal price times individual de-
mand. This Bertrand game has only one static Nash equilibrium: the profile \( \{0.5, 0.5\} \).

Asker et al. (2022) consider two variations of the Q-learning algorithm, both of which are greedy, i.e., the action taken is always the one with the highest estimated value.

(i) Asynchronous Greedy Q-learning: the algorithm updates only the Q-value of the action taken in each period;

(ii) Synchronous Greedy Q-learning: the algorithm updates all Q-values in each period, with the return that he could have obtained had he played the other action instead, but holding the opponent’s action fixed.

We plot both systems in Figure 6 for \( \gamma = 0.15 \). The figures depict the vector fields govern-
ing the dynamics of the continuous-time analogous of each Q-learning procedure, with

\[ \text{The choice of } \gamma = 0 \text{ reflects the specification of Asker et al. (2022), but we provide the vector fields for general values of } \gamma, \text{ and notice that the same intuitions apply.} \]
the value of the competitive price on the vertical axis and that of the collusive price on the horizontal axis.

Figure 6: The green-shaded area denotes the domain of attraction of the competitive outcome, while the blue-shaded area is the domain of attraction of the collusive outcome. The red dot and red lines are the equilibria of the systems.

Asynchronous Learning. The fields when both firms play asynchronous Q-learning and both choose 0.5 and 2, respectively, are:

\[
\begin{aligned}
\frac{dQ_{0.5}}{dt} &= \alpha \left( \frac{5}{8} + (\gamma - 1)Q_{0.5} \right) & & \text{on } \omega_{0.5,0.5} \\
\frac{dQ_2}{dt} &= 0 & & \text{on } \omega_{2,2}
\end{aligned}
\]

Figure 6a plots the fields within their domains. As shown in the picture, there are two equilibrium regions. For values of \( Q_2 \leq Q_{0.5} = \frac{5}{8} \) the algorithms converge on the competitive outcome. However, for \( Q_{0.5} \leq Q_2 = 1 \) the algorithms collude. Notice also that the domains of attraction of these equilibria are profoundly unbalanced: in order to converge on competition, Q-learning needs to be initialized with a strong bias towards competition. Otherwise, the collusive outcome is an attractor. In particular, the optimistic initialization used in Asker et al. (2022) and common in the literature leads to collusive outcomes.

These results are robust to an \( \varepsilon \)-greedy specification: the \textit{coordination bias} introduced
in Section 3 again sustains collusion. Because most observations of the returns from a supra-competitive price are obtained when colluding, the estimates of returns from charging a price of 2 are biased during a competitive phase. The persistence of this bias amounts to insufficient exploration of alternative strategies.

**Synchronous Learning.** Instead, if both firms adopt Synchronous Q-learning, the system is described by the following fields:

\[
\begin{align*}
\frac{dQ_{0.5}}{dt} &= \alpha \left( \frac{3}{8} + (\gamma - 1)Q_{0.5} \right) \quad \text{on } \omega_{0.5,0.5} \\
\frac{dQ_2}{dt} &= \alpha \left( \gamma Q_{0.5} - Q_2 \right) \\
\frac{dQ_2}{dt} &= \alpha \left( 1 + (\gamma - 1)Q_2 \right) \quad \text{on } \omega_{2,2}
\end{align*}
\]

There is only one equilibrium, at \( Q_{0.5} = \frac{5}{8}, Q_2 = 0 \). The plot of Figure 6b shows a clear pattern: the two firms can only converge on competition. There is no sliding along the boundary between the competitive and collusive pricing: everywhere the trajectories move from colluding to competing. When the two firms are colluding, all arrows point upward: the intuition is that they overestimate the value of undercutting the opponent, because they do not internalize the effect that defecting from a collusive outcome will have on returns in the future. The counterfactual rewards do not account for the losses that will stem from a change in equilibrium. Once the firms start competing it is then impossible to revert back to collusion: the counterfactual return of a deviation is zero. Observe the contrast with the asynchronous case, where once agents begin to compete, they almost immediately revert back to collusion. The bias present in the asynchronous algorithm disappears in the synchronous version: actions are updated according to counterfactual returns, therefore the value of joint collusion is short-lived after competition begins. These result are not surprising, as Theorem 2 proves that collusion is unsustainable in a dominant-strategy game.

**General Bertrand.** The simple model above reduces the Bertrand game to a dominant-strategy game. It is a convenient simplification for the purposes of inspecting and plotting the dynamical systems, but the theory developed in Section 4 allows us to deal with much more general models.

Take for example the full model from Asker et al. (2022). Alice Inc. and Bob Ltd. have now constant marginal costs \( c_A = c_B = 2 \). They sell homogeneous goods and compete by setting prices. The set of feasible prices is composed of 100 equally spaced numbers between 0.01 and 10, inclusive. The set of prices is denoted by \( P = \{p^1, \ldots, p^{100}\} \). Consumers
buy from the firm with the lowest price, and demand is parametrized as

\[ d_i(p_i, p_{-i}) = \begin{cases} 
1 & \text{if } p_i < p_{-i} \text{ and } p_i \leq 10 \\
1 & \text{if } p_i = p_{-i} \text{ and } p_i \leq 10 \\
0 & \text{otherwise}
\end{cases} \]

As the authors note, there are two Nash equilibria of this game, one \((E_1)\) with \(p_A = p_B = 2.0282\) and one \((E_2)\) with \(p_A = p_B = 2.1291\). The multiplicity is a consequence of the discretization of the space in equally spaced prices.

**Proposition 4.** In a Bertrand oligopoly, if Alice Inc. and Bob Ltd. adopt any greedy reinforcers such that the relative speed of learning is the same across all prices, they either converge on \(E_1\) or \(E_2\).

**Proof.** The proof of this proposition is a simple consequence of Theorem 3. By applying iterated elimination of strictly dominant strategies, only one of two pairs of prices can survive. Those pairs are exactly the equilibrium pairs. \(\square\)

In particular, this result shows that the parameters of the algorithms are irrelevant for the convergence result. How fast the equilibria are reached will depend on the different discounting or learning rates, but convergence on an equilibrium is guaranteed.

### 5.2 First-Price Auction

Consider now the setting of Banchio and Skrzypacz (2022). The authors simulate algorithmic competition in an auction environment, and they show that revenues in first- and second-price auction are substantially different when Q-learning algorithms choose bids. While the second-price auction appears to be competitive, the first-price auction is prone to collusion. However, they show that providing counterfactuals in the first-price auction restores competition.

Suppose Alice and Bob are bidding in a first-price auction. Both have value of 1 for the item being auctioned, and they have access to a set of 19 equally-spaced bids \(b_i \in \{0.05, 0.1, \ldots, 0.95\}\). This game has two equilibria: one \((E_1)\) sees both Alice and Bob bidding \(b_A = b_B = 0.95\), and one \((E_2)\) sees them bidding \(b_A = b_B = 0.9\). As in the Bertrand oligopoly, this multiplicity is given by the equally-spaced discretization of the bid space. Again, we can show:

**Proposition 5.** In a first-price auction, if Alice and Bob adopt any greedy reinforcers with same relative speed of learning across bids, they converge on either \(E_1\) or \(E_2\).
The proof is identical to that of Proposition 4 and is therefore omitted. In fact, first-price auctions and homogeneous Bertrand oligopolies are strategically equivalent. Banchio and Skrzypacz (2022) argue that Google’s minimum-bid-to-win feedback in their first-price auction acts in the interest of Google, by simultaneously improving convergence and guaranteeing more competitive outcomes. The latter can be seen clearly in our framework, both in Proposition 5 as well as in the implementation problem we presented in Section 4.2: feedback restores the level-k reasoning which in turn guarantees successful implementation.

6 Conclusion

This paper analyses collusion in games played by online learning algorithms: we take a theoretical perspective and, moving from burgeoning empirical and numerical evidence, we identify the drivers of collusive behaviour in a widely studied class of games. In particular, we first address the issue of the analytical intractability of strategic interaction among algorithms by showing they can be approximated with a system of ODEs. Then we apply this framework to dominant-strategy games, and prove that (ε-)greedy algorithms can learn to collude. We identify the mechanism sustaining collusion in the coordination bias: when algorithms realize the benefit of the dominant action too slowly, joint collusion appears more attractive. We demonstrate this intuition in a Prisoner’s Dilemma with Q-learning agents. We expect the techniques developed to analyze the simple Prisoner’s Dilemma to yield insight in games with more complex strategic structure.

We show that convergence on the dominant action is instead guaranteed for greedy algorithms if learning occurs at the same speed for all actions. Algorithms that evaluate counterfactual rewards from actions not taken learn simultaneously for all actions, thus converge on dominant strategies. Following this intuition, we design a mechanism, price-VCG, that implements the social optimum in dominant strategies. Price-VCG enhances traditional VCG mechanisms with additional feedback. Personalized prices for all outcomes aggregate the necessary information efficiently and allow counterfactual evaluations for any report. We suspect that these design ideas could be successfully applied to settings with regret-minimizing agents, as additional feedback simplifies regret evaluations. Finally, we validate our theoretical results by analyzing the Bertrand oligopoly introduced in Asker et al. (2022), where our framework delivers an analysis of the driving forces while recovering their experimental results, and the first-price auction of Banchio and Skrzypacz (2022).

We view our paper as contributing to the growing literature studying strategic in-
teraction of algorithmic agents. Algorithms shape the dimensions of rationality of these decision makers, and allow us to carry out a disciplined analysis of equilibria and market design for such boundedly-rational agents. Our work provides a theoretical analysis of collusion in games played by online learning algorithms. In doing so we prove a continuous-time approximation technique that can be applied more generally to the study of systems otherwise deemed intractable. We hope that this work will stimulate further analysis of the strategic interaction of learning algorithms. We focused on dominant strategy games, which intrinsically make collusion the hardest to sustain: the outcomes of games where the separation between competition and collusion is less stark remains unclear, and worthwhile to pursue. Our algorithms interact with the environment and adapt according to the feedback they receive, but many deployed market algorithms are instead trained offline. Our analysis points to coordination as a key driver of collusion, thus suggesting that offline algorithms may be less prone to collusive behavior. Finally, our approximation allows for the analysis of stochastic games. We hope that by adopting this framework future research will be able to shed light on the kind of policies reached by these systems in such settings.

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Appendix A

Proof of Theorem 1. The existence of a solution for the Cauchy problem is guaranteed by assumption Assumption A2.

Consider now an auxiliary process $w_k$. The first $d \times f$ components of $w_k$ are equal to the column vector $[\theta_k^1, \ldots, \theta_k^f]$. Then, for each state $s$ of the Markov Chain $X_k$ we add a component to the process $w_k$, projecting the (controlled) Markov chain into a trivial embedding on a $f$-dimensional simplex.\(^{16}\) Finally, we add other $f$ components to keep track of the number of iterations $k$ spent in each state $s$. This yields a Markov process in discrete time, $w_k \in \mathbb{R}^{d \times 2f}$, that updates according to $w_k = w_{k-1} + \hat{T}(w_{k-1}, Y_k)$. The function $\hat{T}$ remains Lipschitz in every component. To produce the sequence of processes $\theta^n$, let us consider the “Poissonization” of the process $w_k$.

Let $N^0(t)$ be a Poisson process of rate $\lambda^0 = 1$. Consider the sequence of (stochastic) arrival times $\{0 < \tau_1 < \tau_2 < \tau_3 \ldots\}$. We define $w(t)$ as

$$w(t) = w_k \quad \text{if} \quad \tau_{k+1} > t \geq \tau_k$$

for all times $t \geq 0$. The process $w(t)$ is now a compound Poisson process, particularly cadlag and Markov.

We consider a sequence $(w^n(t))_n$ of Continuous Time Markov Chains indexed by $n \in \mathbb{N}$ as follows:

- The jump rate $\lambda^n(w^n(t))$ depends only on the current state $s$, and it is defined as

$$\lambda^n(s) = \sum_{j=1}^{n} P_{ss}^{(j)}$$

Intuitively, the jump rate measures the expected number of jumps from a given state in a unit of time, and we increase the frequency at which updates occur.

- At each jump, the update in the first $d \times f$ components is

$$w^n(t) - w^n(t^-) = \frac{1}{n} \hat{T}(w^n(t^-), Y)$$

- At each jump, the update in the coordinates from $d \times f + 1$ to $d \times f + f$ is

$$w^n(t) - w^n(t^-) = \frac{1}{n^2} \hat{T}(w^n(t^-), Y)$$

\(^{16}\)I.e., states are represented as a vector with all zeros but for the $s$th entry, which is 1.
• At each jump, the update in the last $f$ coordinates is

$$w^n(t) - w^n(t^-) = \frac{1}{n} \hat{T}(w^n(t^-), Y)$$

Intuitively, the updates of the original process $\theta$ are scaled down by a factor $n$, we keep track of the states in a $\frac{1}{n^2}$-simplex and the last $f$ coordinates keep track of how many updates have occurred in each state $s$.

Fix now a generic vector $x \in \mathbb{R}^{d \times 3f}$. Consider the measure $\mu(x, dz)$ of updates at $x$ with

$$\mu(x, dz) = \mathbb{P}\{w^n(\tau) \in dz | w^n(0) = x\}$$

where $\tau$ is the first exit time of $w^n$ from $x$. The expected update of $w(t)$ over a unit of time is summarized in the function defined component-wise:

$$F^n(x)_m = \lambda^n(x) \int (z_m - x_m) \mu^n(x, dz)$$

where $\lambda^n(x) = \lambda^n(s)$ if $x_{d \times f + s} = \frac{1}{n^2}$. For the last $f$ components, this amounts to $F^n(x)_m = \frac{1}{n} \sum_{k=0}^{n} P_{ss}^{(k)} \leq 1$ if $x_{d + s} = 1$ and $0$ otherwise. Similarly, for the Markov states the expected update is bounded above by $\frac{1}{n} \sum_{k=0}^{n} P_{ss}^{(k)} \leq 1$ for $x_{d + s} = 1$ and $0$ everywhere else. For the first $d \times f$ components instead, the expected update over one unit of time takes the form:

$$F^n(x)_m = \sum_{k=0}^{n} P_{ii}^{(k)} \int \alpha \sum_{s \in S} P_{ss} T(x, i, i, s, Y, t) \nu(Y)$$

The first term is bounded above by $1$, while the second term is bounded over any compact domain because it is a continuous function of $x$. We prove the following auxiliary lemma:

**Lemma 1.** Let $D$ be a compact ball in $\mathbb{R}^{d \times 2f}$. There exists a sequence $\{\varepsilon^n\}_n > 0$ with $\lim_{n \to \infty} \varepsilon^n = 0$ such that

$$\limsup_{n \to \infty} \sup_{x \in D} \lambda^n(x) \int_{|z - x| > \varepsilon^n} |z - x| \mu^n(x, dz) = 0$$

Moreover,

$$\sup \sup_{n} \sup_{x \in D} \lambda^n(x) \int_{D} |z - x| \mu^n(x, dz) < \infty$$

**Proof.** We showed that the instantaneous updates are all bounded above by a constant $\frac{M}{n}$ for all $n$ over a compact set $D$. Let $\varepsilon^n = \frac{M + 1}{n}$: the sequence satisfies the assumptions of the
lemma and the first equality is satisfied. The second claim follows from the observation that the increment is absolutely bounded by $\frac{M}{n}$ for all $x$. Since $\frac{\sum_{k=0}^{n} P_{ik}(k)}{n}$ is bounded above by $n$, jointly the integral is bounded by $M < \infty$.

Define $F(x) = \lim_{n \to \infty} F^n(x)$. It follows from the elementary renewal theorem (Karlin and Taylor (1975)) that

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n} P_{ss}(k)}{n} = \pi_s$$

where $\pi_s$ is the unique stationary distribution of the irreducible Markov Chain. Therefore, the last $f$ components of $F(x)$ equal the stationary distribution $\pi$. The components keeping track of the state instead all equal 0: in the limit we collapse the markov chain transitions in their expected values. Finally, the first $d \times f$ components equal the expected update of the process $\theta$ weighted by the stationary probability of currently belonging to their own state: $F(x) = [\pi_1 \alpha E[T^1(x,X_t,Y_t,t)],...\pi_f \alpha E[T^f(x,X_t,Y_t,t)]]$. The function $F$ is clearly Lipschitz. Together with the conditions of Lemma Lemma 1, this allow us to invoke the following result from Kurtz (1970):

**Theorem 2.11.** Suppose there exists $E \subset \mathbb{R}^k$, a function $F: E \to \mathbb{R}^k$ and a constant $M$ such that $|F(x) - F(y)| \leq M|x - y|$ for all $x,y \in E$ and

$$\lim_{n \to \infty} \sup_{x \in E_n \cup E} |F_n(x) - F(x)| = 0$$

Let $w(t,x_0), 0 \leq t \leq T, x_0 \in E$ satisfy

$$w(0,x_0) = x_0, \quad w(t,x_0) = F(w(t,x_0))$$

Suppose additionally that the sequence $F^n$ satisfies the conditions of Lemma 1, then for every $\eta > 0$

$$\lim_{n \to \infty} P\{\sup_{t \leq T} |w^n(t) - w(t,x_0)| \geq \eta\} = 0$$

This concludes the proof.

**Proof of Proposition 1.** This result relies on the following theorem from Filippov (1988):

**Theorem 1, Chapter 2, Section 7.** Let $D$ be a compact domain. Let $G(t,x)$ be a nonempty, bounded, closed, convex set-valued function that is upper semicontinuous in $t,x$ for all $(t,x) \in D$
D. Then for any point \((t_0, x_0) \in D\) there exists a solution of the problem

\[ \dot{x} \in G(t, x), \quad x(t_0) = x_0 \]

and if the domain \(D\) contains a cylinder \(Z(t_0 \leq t \leq t_0 + a, |x - x_0| \leq b)\), the solution exists at least on the interval

\[ t_0 \leq t \leq t_0 + d, \quad d = \min \left\{ a; \frac{b}{m} \right\} \quad m = \sup_Z |G(t, x)| \]

Let \(G(t, x)\) be defined as

\[ \text{co}\{F_j(x) \mid \forall j \} \quad \text{over } \omega_j \quad \forall j \]
\[ \text{co}\{F_k(x) \mid \forall k \in K \subset j \} \quad \text{over } \bigcap_{k \in K} \omega_k \quad \forall K \subset J \]

and notice that \(G\) is time-invariant. Let \(D\) be a compact ball in \(\mathbb{R} \times \mathbb{R}^{I \times S \times A}\). We show that \(G\) satisfies all the conditions:

- it is nonempty over \(\mathbb{R} \times \mathbb{R}^{I \times S \times A}\),
- it is bounded everywhere, since each individual \(F_j\) is bounded over \(\omega_j \cup D\),
- it is closed and convex, as it is clear from the definition above,
- it is upper semicontinuous: to see this, select a convergent sequence in the domain. If the sequence is entirely contained in a \(\omega_j\), then continuity is clear. Instead, suppose that the sequence lies in an \(\omega_j\) but its limit lies on \(\partial \omega_j\). Upper semicontinuity is guaranteed by the definition of \(G\) as a convex combination of \(F\) over the overlapping boundaries. Finally, suppose that the sequence lies within a \(\bigcap_{k \in K} \omega_k\) for some \(K \subset J\), but the limit lies in a different intersection \(\bigcap_{h \in H} \omega_h\). Since the sequence is convergent, it has to be the case that \(K \subset H\), therefore upper semicontinuity is satisfied.

Additionally, note that the domain is any compact ball, therefore we can find a \(D\) that contains any cylinder \(Z\) and a solution to this differential inclusion is global within any compact subset of \(\mathbb{R}^{I \times S \times A}\).

Proof of Proposition 2  The existence of the equilibrium \(q_{eq}^D\) follows directly from setting the field over \(\omega_{D,D}\) to zero.
We prove existence of $q_{eq}^C$ and its related property for one agent; by symmetry, the other agent’s Q-values enjoy the same properties. The boundary is defined as $\Sigma = \{q \in \mathbb{R}^2 : c \cdot q = 0\}$ where $c = (1, -1)$ and $\cdot$ denotes the usual dot product. Using the Filippov convention, we can further divide $\Sigma$ in three regions:

- a crossing region, $\Sigma^c = \{q : (c \cdot (A_Cq + b_C))(c \cdot (A_Dq + b_D)) > 0\}$
- a repulsive region, $\Sigma^r = \{q : c \cdot (A_Cq + b_C) > 0, c \cdot (A_Dq + b_D) < 0\}$
- a sliding region, $\Sigma^s = \{q : c \cdot (A_Cq + b_C) < 0, c \cdot (A_Dq + b_D) > 0\}$

where we have

$$A_C = \begin{bmatrix} a(1 - \xi)(\gamma - 1) & 0 \\ a\gamma^\xi & -a\xi \end{bmatrix}, \quad A_D = \begin{bmatrix} -a\xi & a\gamma^\xi \\ 0 & a(1 - \xi)(\gamma - 1) \end{bmatrix},$$

and

$$b_C = \begin{bmatrix} a(1 - \xi)(2 - \xi)^g \\ a\xi(2 + g - g^\xi) \end{bmatrix}, \quad b_D = \begin{bmatrix} a(1 + \xi)^g \xi \\ a(1 - \xi)(2 + \xi g) \end{bmatrix}.$$

We can define the sliding solution as the field $\frac{dQ}{dt} = F^s(Q)$ over the sliding region where

$$F^s(Q) = \frac{c \cdot (A_DQ + b_D)(A_CQ + b_C) - c \cdot (A_CQ + b_C)(A_DQ + b_D)}{c \cdot (A_DQ + b_D) - c \cdot (A_CQ + b_C)}.$$

The relative time spent on $\omega_{C,C}$ at point $Q$ is defined as

$$\tau_C = \frac{c \cdot (A_DQ + b_D)}{c \cdot (A_DQ + b_D) - c \cdot (A_CQ + b_C)}.$$

The sliding vector field becomes

$$\frac{dQ_j}{dt} = \frac{a\left(\frac{1}{2} \varepsilon^2 g(2 - \varepsilon)(g - 1) + (2g + (\gamma - 1)Q_j)(2 + (\gamma - 1)Q_j)\right)}{2(1 + g + (\gamma - 1)Q_j)}$$

for every direction $j$. By setting the field equal to zero and solving for $Q_j$, we find that there is an equilibrium at

$$q_{eq}^{C,j} = \frac{1 + g - \sqrt{(g - 1)(g - 1 - \varepsilon g + \frac{\varepsilon^2 g}{2})}}{\gamma - 1}.$$
for all j. This equation has a feasible solution for all \( \varepsilon < 1 - \sqrt{\frac{2-g}{g}} \).

**Proof of Corollary 1.** The result follows immediately from the proof of Proposition 2. In particular, it is sufficient to compute

\[
\tau_C = \frac{c \cdot (A_D Q + b_D)}{c \cdot (A_D Q + b_D) - c \cdot (A_C Q + b_C)}
\]

at \( Q = q_C^{eq} \).

**Proof of Proposition 3.** Let \( RLS(a_i) \) be positive only when \( a_i \) is the action such that \( \theta^{a_i} > \theta^a \) for all other \( a \in A_i \). Clearly, there are a multiplicity of equilibria where a dominated action is played in equilibrium, because values of \( \theta^a \) can be completely meaningless when action \( a \) is not played.

More interesting is a proof when \( RLS(a) \) is positive for more than one action. For this case we assume that \( RLS(a) \) is zero for all but two actions, such that the remaining game is a Prisoner’s Dilemma (which always exists in a collusion game). We suspect that a similar proof approach would carry over to more general positive \( RLS(a) \). First off, note that there is a given ordering of rewards in a Prisoner’s Dilemma:

\[
r(D, C) > r(C, C) > r(D, D) > r(C, D)
\]

We will prove existence of an equilibrium of the following system:

\[
\begin{align*}
\dot{\theta}^C &= \alpha_m U(\theta^C, r(C, C)) \\
\dot{\theta}^D &= (1 - \alpha_m) U(\theta^D, r(D, C))
\end{align*}
\]

in the half-plane where \( C \) is the preferred action, and

\[
\begin{align*}
\dot{\theta}^C &= (1 - \alpha_m) U(\theta^C, r(C, D)) \\
\dot{\theta}^D &= \alpha_m U(\theta^D, r(D, D))
\end{align*}
\]

in the half-plane where \( D \) is the preferred action. Note that we are assuming WLOG that the learning speeds sum to 1, since all it matters is the relative speed of learning.
We want to show that there exist an $\alpha_m$ and a $\theta^*$ such that:

$$\begin{align*}
\alpha_m U(\theta, r(C, C)) + (1 - \alpha_m) U(\theta, r(C, D)) &= 0 \\
(1 - \alpha_m) U(\theta, r(C, D)) + \alpha_m U(\theta, r(D, D)) &= 0
\end{align*}$$

For a given $\theta$, we can write this problem as a convex combination of two vectors:

$$\alpha_m \vec{x} + (1 - \alpha_m) \vec{y}$$

where

$$\vec{x}(\theta) = \begin{bmatrix} U(\theta, r(C, C)) \\ U(\theta, r(D, D)) \end{bmatrix} \quad \vec{y}(\theta) = \begin{bmatrix} U(\theta, r(C, D)) \\ U(\theta, r(D, C)) \end{bmatrix}$$

Let $\theta^1$ be the value of $\theta$ such that $U(\theta^1, r(C, C)) = 0$. Then, using the monotonicity of $U$ in rewards, the two vectors $\vec{x}, \vec{y}$ computed in $\theta^1$ will be positioned as in Figure 7. Let $\theta^2$ instead be the value of $\theta$ such that $U(\theta^2, r(C, D)) = 0$. Again, the two vectors $\vec{x}, \vec{y}$ are positioned as in Figure 7. Notice that by monotonicity of $U$ in its first component, $\theta^1 < \theta^2$. 
Define \( f : [0,1] \rightarrow \mathbb{R}^2 \) as

\[
\begin{align*}
\begin{cases}
(1 - 4t)\tilde{x}(\theta^1) + 4t \tilde{y}(\theta^1) & t \in [0, \frac{1}{4}] \\
\tilde{y}(\theta^1 + (\theta^2 - \theta^1)(4t - 1)) & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\
(3 - 4t)\tilde{y}(\theta^2) + (4t - 2)\tilde{x}(\theta^2) & t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\
\tilde{x}(\theta^1 + 4(\theta^2 - \theta^1)(1-t)) & t \in \left[\frac{3}{4}, 1\right]
\end{cases}
\end{align*}
\]

Note that, by continuity of \( U \), \( f(t) \) is a loop based in \( \tilde{x}(\theta^1) \). We can show that \( f \) is null-homotopic by providing the following simple homotopy \( H : [0,1] \times [0,1] \rightarrow \mathbb{R}^2 \).

\[
\begin{align*}
\begin{cases}
(1 - 4ts)\tilde{x}(\theta^1) + 4ts \tilde{y}(\theta^1) & t \in [0, \frac{1}{4}] \\
(1-s)\tilde{x}(\theta^1 + s(4t - 1)(\theta^2 - \theta^1)) + s \tilde{y}(\theta^1 + s(4t - 1)(\theta^2 - \theta^1)) & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\
(1-s(3 - 4t))\tilde{x}(1-s)\theta^1 + s\theta^2) + s(3 - 4t)\tilde{y}(1-s)\theta^1 + s\theta^2) & t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\
\tilde{x}(\theta^1 + 4s(\theta^2 - \theta^1)(1-t)) & t \in \left[\frac{3}{4}, 1\right]
\end{cases}
\end{align*}
\]

We verify that \( H \) is a homotopy between \( f(t) \) and the constant path at \( \tilde{x}(\theta^1) \).

- \( H(0,s) = H(1,s) = \tilde{x}(\theta^1) \)
- \( H(t,0) = \tilde{x}(\theta^1) \)
- \( H(t,1) = f(t) \)

Suppose now that there is no pair \( t,s \) such that \( H(t,s) = (0,0) \). Then, by continuity it must be the case that there is an open neighborhood \( V \ni (0,0) \) such that for all points \( z \in V, z \in \text{Im}(H) \). Note that we can restrict the loop \( f \) to a the domain \( \mathbb{R}^2 \setminus V \), and because \( V \in \text{Im}(H) \) we can restrict the homotopy to a homotopy \( H : [0,1]^2 \rightarrow \mathbb{R}^2 \setminus V \). Thus we proved that a loop around the open set \( V \) is null-homotopic, which is equivalent to proving that \( \mathbb{R}^2 \setminus V \) is simply connected, a contradiction.

Therefore, there exists a pair \( \bar{t}, \bar{s} \) such that \( H(\bar{t}, \bar{s}) = (0,0) \). There is a bijective transformation between \( t,s \) and \( \theta, \alpha \) over their respective domains, which provides an equilibrium exhibiting coordination bias.

**Proof of Theorem 2.** Regardless of the opponent’s policy, the reinforcer in every step observes a return in hindsight for every action. Let action \( n \) be the dominant-strategy action: we denote by \( r_n(t) \) the return from playing action \( n \) in period \( t \), whatever the opponents’ actions are. We consider the evolution of the weights pairwise: \( \theta^n \) and \( \theta^i \) for
all \(i\). We assume that for any sequence of actions taken by the opponent(s), \(r_n(t) > r_i(t)\).\(^{17}\) Thus, the derivative \(\dot{\theta^n}\) strictly dominates \(\dot{\theta^i}\) in each point \((x,t)\): \(U(x, r_n(t)) > U(x,., r_i(t))\).

In particular, there exists an \(\epsilon\) such that \(U(x, r_n(t)) > U(x,., r_i(t)) + \epsilon\). Suppose that for some \(t \geq 0\), \(\theta^n(t) \geq \theta^i(t)\): then it can never be the case that \(\theta^i(T) > \theta^n(T)\) for some \(T > t\). This is because of continuity and because if at any time \(t' > t\) we had \(\theta^i(t') = \theta^n(t') - \delta\) for any small \(\delta > 0\), the derivative of \(\theta^n(t')\) is larger than the derivative of \(\theta^i(t')\) and the trajectories pull farther apart again:

\[
\dot{\theta^n}(t' + dt) - \dot{\theta^i}(t' + dt) \sim \dot{\theta^n}(t') + dt(U(\theta^n(t'), r_n(t')) - w_i(t')) - dtU(\theta^i(t'), r_i(t')) \\
> \dot{\theta^n}(t') + dt(U(\theta^i(t'), r_i(t')) + \epsilon) - \dot{\theta^i}(t') - dtU(\theta^i(t'), r_i(t')) \\
= \delta + dt\epsilon > \delta
\]

Instead, suppose now that \(\theta^n(t) \leq \theta^i(t)\) for some \(t\). The derivative \(\dot{\theta^n}(t)\) always dominates the derivative \(\dot{\theta^i}(t)\). Since we assumed that the processes \(w\) are bounded, there has to be a time where the trajectories cross: \(\theta^n(T) = \theta^i(T)\). This brings us back to the case where \(\theta^n(t) \geq \theta^i(t)\), which we analyzed before, and the trajectories will never cross again. If the algorithm is greedy, the action taken after time \(t\) will always be the dominant action, which proves our claim.

\[\square\]

**Proof of Theorem 3.** The proof is largely based upon Theorem 2. We will show that there is always a \(T\) such that the actions taken after \(T\) survive an IESDS procedure.

Let \(a_i\) be a strategy for player \(i\) which is dominated by \(b_i\) in the game \(G\). With the same argument of the proof of Theorem 2 we can show that it must be the case that, in the limit, \(\theta^{b_i} > \theta^{a_i}\). Equivalently, there exists a \(T_i\) such that for all \(t \geq T_i\) we have \(\theta^{b_i}(t) > \theta^{a_i}(t)\). Therefore, after time \(T_i\) it is as if the agents were playing in a reduced game \(G^1 = (N, (A^1_i)_i, (u_i)_i)\), where \(A^1_i = A_i \setminus \{a_i\}\) and \(A^1_j = A_j\) for all \(j \neq i\).\(^{18}\) We can now apply the same idea to this new reduced game \(G^1\) and eliminate a strictly dominated strategy in this reduced game. While IESDS eliminates strategies “in place”, the algorithms abandon dominated strategies over time, reducing the effective game played. Of course, the components of \(\theta\) that correspond to dominated actions keep getting updated, but note that if an action is dominated given a larger subset of opponent’s strategies, it

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\(^{17}\)This is not necessarily the case. If the dominant action has the same reward across different strategies of the opponents, we can prove that \(\theta^n\) will be the largest statistic, but we can’t prove it will be unique. In that case, we need additional information about the strategies of the opponents. It is here that Assumption A3 (thickness) saves the day. If all opponent’s actions are taken with positive probability, then \(\theta^n\) will be the unique maximum.

\(^{18}\)Of course, Assumption A3 guarantees that even action \(a_i\) will be played with some positive probability, but as long as such probability is vanishingly small the argument made next follows immediately.
is also dominated given a smaller subset. Therefore, the $\theta$ corresponding to a dominated action will always remain below that of actions surviving IESDS. We then define $T$ as the largest among the $T_k$ that correspond to an action being eliminated, and this concludes the proof.