REMARKS ON AFFINE COMPLETE DISTRIBUTIVE LATTICES

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Abstract. We characterise the Priestley spaces corresponding to affine complete bounded distributive lattices. Moreover we prove that the class of affine complete bounded distributive lattices is closed under products and free products. We show that every (not necessarily bounded) distributive lattice can be embedded in an affine complete one and that $\mathbb{Q} \cap [0,1]$ is initial in the class of affine complete lattices.\footnote{The author is grateful for financial support from the Swiss National Science Foundation.}

1. Affine complete lattices

A $k$-ary function $f$ on a bounded distributive lattice $L$ is called compatible if for any congruence $\theta$ on $L$ and $(a_i, b_i) \in \theta$, $(i = 1, \ldots, k)$ we always have $(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)) \in \theta$. It is easy to see that the projections $p_i : L^k \to L$ are compatible. With induction on polynomial complexity one shows that every polynomial function is compatible (see [4]). A lattice $L$ is called affine complete, if conversely every compatible function on $L$ is a polynomial.

G. Grätzer [2] gave an intrinsic characterization of bounded distributive lattices that are affine complete:

Theorem 1.1. ([2]) A bounded distributive lattice is affine complete if and only if it does not contain a proper interval that is a Boolean lattice in the induced order.

Note that in particular, no finite bounded distributive lattice $L$ is affine complete: Let $x \in L$ be an element distinct from 1. Then $x$ has an upper neighbor, i.e., there exists $y \in L$ such that $[x, y] = \{x, y\}$ which is isomorphic to the 2-element Boolean lattice.

Example 1.2. The bounded distributive lattices $[0, 1]$ and $[0, 1] \times [0, 1]$ are affine complete.

Proof. First, take any $x < y$ in $[0, 1]$. Then the element $a = \frac{x+y}{2} \in [x, y]$ has no complement $a'$ in $[x, y]$: Otherwise we would have $a \land a' = x$ which would imply $a' = x$, but then $a \lor a' = a \neq y$. So $[x, y]$ is not Boolean, whence $[0, 1]$ has no proper Boolean interval.

Secondly, let $(x_1, x_2) < (y_1, y_2) \in [0, 1] \times [0, 1]$. With a similar argument as before, the element

$\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right) \in [(x_1, x_2), (y_1, y_2)]$

does not have a complement in $[(x_1, x_2), (y_1, y_2)]$. Thus, $[0, 1] \times [0, 1]$ has no proper Boolean interval and is therefore affine complete. \hfill \Box
2. Priestley duality

In [5], Priestley proved that the category $D$ of bounded distributive lattices with $(0,1)$-preserving lattice homomorphisms and the category $P$ of compact totally order-disconnected spaces (henceforth referred to as Priestley spaces) with order-preserving continuous maps are dually equivalent. (A compact totally order-disconnected space $(X; \tau, \leq)$ is a poset $(X; \leq)$ endowed with a compact topology $\tau$ such that, for $x, y \in X$, whenever $x \nless y$, then there exists a clopen decreasing set $U$ such that $x \in U$ and $y \notin U$.) The functor $D : D \to P$ assigns to each object $L$ of $D$ a Priestley space $(D(L); \tau(L), \subseteq)$, where $D(L)$ is the set of all prime ideals of $L$ and $\tau(L)$ is a suitably defined topology (the details of which will not be required here). The functor $E : P \to D$ assigns to each Priestley space $X$ the lattice $(E(X); \cup, \cap, \emptyset, X)$, where $E(X)$ is the set of all clopen decreasing sets of $X$.

Priestley duality therefore provides us with a “dictionary” between the world of bounded distributive lattices and a certain category of ordered topological spaces. This is interesting in particular because free products of lattices are “translated” into products of Priestley spaces. We will use this fact for showing that the class of affine complete bounded distributive lattices is closed under free products.

3. Affine complete Priestley spaces

The aim of this section is to characterize the Priestley spaces corresponding to affine complete distributive (0,1)-lattices. Such spaces will be called affine complete Priestley spaces. In other words, a Priestley space $X$ is affine complete iff $E(X)$ is affine complete.

The following theorem provides a rather straightforward translation of the algebraic concept of affine completeness in order-topological terms.

**Theorem 3.1.** Let $X$ be a Priestley space. Then the following statements are equivalent:

1. $E(X)$ is affine complete.
2. If $U \subseteq V$ are clopen down-sets and $U \neq V$, then the subposet $V \setminus U$ of $X$ is not an antichain, i.e. $V \setminus U$ contains a pair of distinct comparable elements.

**Proof.** (1) $\implies$ (2). Suppose $V \setminus U$ is an antichain. Let $C \in [U, V] \subseteq E(X)$. Take $C' = U \cup (V \setminus C)$.

**Claim:** $C'$ is a clopen down-set of $X$.

It is clear that $C'$ is a clopen subset of $X$ since $V \setminus C = V \cap (X \setminus C)$. Now, let $c \in C'$ and assume $x < c$. Then if $c \in U$, we are done, since $U$ is a down-set. Assume $c \in V \setminus U$. Since $V$ is a down-set, we get $x \in V$, and the fact that $V \setminus U$ is an antichain tells us that $x$ cannot be a member of $V \setminus U$. Therefore $x \in U \subseteq C'$ which proves that $C'$ is indeed a (clopen) down-set.

Moreover, $C'$ is the complement of $C$ in $[U, V]$, i.e. $C \cap C' = U$ and $C \cup C' = V$. Because $C$ was arbitrary, we see that $[U, V]$ is a proper Boolean interval of $E(X)$, whence $E(X)$ is not affine complete.

(2) $\implies$ (1). Suppose $U \subseteq V$ are distinct clopen down-sets. By assumption, there are elements $x, y \in V \setminus U$ such that $x < y$. There is a clopen down-set $A$ with $x \in A$ and $y \notin A$. Consider $B = (A \cap V) \cup U$. So $B \in [U, V]$ and $y \notin B$. Now we show that $B$ has no complement...
in $[U,V]$: Take any $C \in [U,V]$ with $C \cup B = V$. Then $y \in C$, but since $C$ is a down-set, we have $x \in C$, thus $x \in (B \cap C) \setminus U$ and $B \cap C \neq U$. So whatever $C$ we pick, $C$ is no complement for $B$, i.e. $B$ is not complemented, and consequently $[U,V]$ is not Boolean. It follows that no proper interval of $E(X)$ is Boolean. \hfill \Box

We can formulate the above result in a more concise way:

**Corollary 3.2.** A Priestley space $X$ is affine complete if and only if each nonempty open set contains two distinct comparable points.

**Proof.** It follows directly from theorem 3.1 that if each nonempty open set contains two distinct points that are comparable, then $X$ is affine complete.

Conversely, suppose that $U$ is a nonempty open set which is an antichain, then there exist open down-sets $C_1, C_2$ such that $\emptyset \neq C_1 \cap (X \setminus C_2) \subseteq U$. Then $[C_1 \cap C_2, C_1]$ is a proper interval such that $C_1 \setminus (C_1 \cap C_2) = C_1 \cap (X \setminus C_2)$ is an antichain (as a subset of the antichain $U$). Thus theorem 3.1 implies that $X$ is not affine complete. \hfill \Box

Note that the proof works exactly the same way if each occurrence of “open” is replaced by “clopen” (basically because each Priestley space is zero-dimensional). So we can state as well:

A Priestley space $X$ is affine complete if and only if each nonempty clopen set contains two distinct comparable points.

4. **Products of affine complete lattices**

We prove in this section that arbitrary products of affine complete lattices are affine complete. We don’t need Priestley duality to do this. Priestley duals of affine complete lattices, i.e. affine complete Priestley spaces, will come into play when we consider coproducts of affine complete lattices.

**Theorem 4.1.** If $(L_i)_{i \in I}$ is a family of (bounded) affine complete lattices, then $\Pi_{i \in I}L_i$ is affine complete.

**Proof.** We prove the contrapositive of the theorem. Suppose that $\Pi_{i \in I}L_i$ is not affine complete. Then it contains a proper interval $[\xi, \eta]$ that is Boolean. There exists some $k \in K$ such that $\xi(k) < \eta(k)$. We claim that $[\xi(k), \eta(k)] \subseteq L_k$ is a Boolean interval. Set $x = \xi(k), y = \eta(k)$. Suppose $l \in [x,y]$ and define $\lambda \in \Pi_{i \in I}L_i$ by

$$\lambda(i) = \begin{cases} l & \text{if } i = k \\ \xi(i) & \text{if } i \neq k \end{cases}$$

Because $[\xi, \eta]$ is Boolean, there exists $\lambda' \in \Pi_{i \in I}L_i$ such that $\lambda \land \lambda' = \xi$ and $\lambda \lor \lambda' = \eta$. Thus it is easy to see that $l' := \lambda'(k)$ is the complement of $l \in [x,y]$. Therefore, $[x,y]$ is a proper Boolean interval of $L_k$ and whence $L_k$ is not affine complete. \hfill \Box

**Example 4.2.** Theorem 4.1 implies that $[0,1]^\mathbb{N}$ is affine complete.
5. Free products of affine complete lattices

Now we turn our attention to free products of affine complete bounded distributive lattices; we prove they are complete. A convenient way to obtain this result is to dualise the problem into the category of Priestley spaces. Free products (that is, coproducts) in $\mathbf{D}$ correspond to products in $\mathbf{P}$ and vice versa; this is stated in the following proposition in a more general way.

**Proposition 5.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and assume that $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are contravariant functors that form a dual equivalence. Then:

1. If $\mathcal{A}$ is a product of a family of objects $(A_i)_{i \in I}$ of $\mathcal{A}$, then $F(\mathcal{A})$ is a coproduct of $(F(A_i))_{i \in I}$.

2. If $\mathcal{A}$ is a coproduct of a family of objects $(A_i)_{i \in I}$ of $\mathcal{A}$, then $F(\mathcal{A})$ is a product of $(F(A_i))_{i \in I}$.

Moreover we have shown that affine complete lattices correspond to affine complete spaces under the Priestley duality.

**Theorem 5.2.** If $(X_i)_{i \in I}$ is a family of affine complete Priestley spaces, then $\Pi_{i \in I} X_i$ is affine complete.

**Proof.** Suppose that $X_i$ is affine complete for every $i \in I$. It suffices to show that every nonempty subset $V$ of $\Pi_{i \in I} X_i$ of the form

$$V = \pi_{i_1}^{-1}(U_1) \cup \ldots \cup \pi_{i_r}^{-1}(U_r)$$

contains two distinct comparable elements (where $U_k \subseteq X_{i_k}$ open, nonempty). Take $U_1$. It contains elements $a < b$, because $X_{i_1}$ is affine complete. Now pick $\xi \in V$. Define $\xi_1, \xi_2 \in V$ by

$$\xi_1(i) = \begin{cases} \xi(i) & \text{if } i \neq i_1 \\ a & \text{if } i = i_1 \end{cases}$$

and

$$\xi_2(i) = \begin{cases} \xi(i) & \text{if } i \neq i_1 \\ b & \text{if } i = i_1 \end{cases}$$

Clearly, $\xi_1, \xi_2$ are distinct comparable elements of $V$. □

Applying the Priestley duality now yields:

**Corollary 5.3.** The class of (bounded) affine complete lattices is closed under free products.

6. Embedding lattices in affine complete lattices

First we will stay away from affine completeness in the worst possible way: we will embed each $L$ into a powerset of some set, which, being Boolean, is as affine incomplete as it gets.

**Lemma 6.1.** Let $L$ be a distributive lattice ($L$ need not be bounded). There is a set $X$ and a lattice embedding

$$j : L \hookrightarrow \mathcal{P}(X)$$

where $\mathcal{P}(X)$ is the powerset of the set $X$. 
Proof. First, endow $L$ with a smallest element and a greatest element. Call this new bounded distributive lattice $L_{01}$. By Priestley duality, there is a Priestley space $(X, \tau, \leq)$ such that the lattice $E(X)$ of clopen down-sets is isomorphic to $L_{01}$. Since $E(X)$ is a sublattice of $P(X)$, we are done.

Next, we will embed that powerset in an affine complete lattice.

**Lemma 6.2.** Let $X$ be a set and let $Q = \{ q \in Q ; 0 \leq q \leq 1 \}$. Then there is a lattice embedding

$$j : P(X) \hookrightarrow Q^X.$$  

Moreover, $Q$ is affine complete.

**Proof.** Set $j : S \mapsto \chi_S \in Q^X$ for every $S \subseteq X$, where $\chi_S$ is defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

It is easy to see that $j$ is a lattice embedding. Next, we claim that $Q$ is affine complete. Take any $x < y$ in $Q$. Then the element $a = \frac{x+y}{2} \in [x, y]$ has no complement $a'$ in $[x, y]$: Otherwise we would have $a \land a' = x$ which would imply $a' = x$, but then $a \lor a' = a \neq y$. So $[x, y]$ is not Boolean, whence $Q$ has no proper Boolean interval. Therefore, $Q$ is affine complete.

Moreover, by **6.2** $Q^X$ is affine complete which concludes the proof.

Lemmas **6.1** and **6.2** now imply:

**Corollary 6.3.** Every distributive lattice (not necessarily bounded) can be embedded in a bounded affine complete lattice.

Admittedly, the construction provided by **6.1** and **6.2** is highly non-unique and has no minimality properties.

7. $Q_{01}$ as Initial Object in the Category of Affine Complete Lattices

The aim of this section is to show that the lattice $Q_{01} = Q \cap [0, 1]$ can be embedded into each affine complete lattice, which amounts to saying that $Q_{01}$ is an initial object of the category of affine complete lattices (with $(0,1)$-homomorphisms, i.e. a full subcategory of the category bounded distributive lattices). The key will be the notion of a dense chain.

**Definition 7.1.** A chain $(X, \leq)$ is called dense if for all $x < y \in X$ there is $z \in X$ with $x < z < y$.

The first tool we need here is a well known result of model theory. It states that the theory of dense linear orders is complete and has $(Q, \leq)$ as prime model. We will state this result in a more primitive way and prove it.

**Proposition 7.2.** If $(X, \leq)$ is a bounded dense chain, there is a $(0, 1)$-embedding

$$\varphi : Q_{01} \hookrightarrow X.$$
Proof. Let $a : \omega \to \mathbb{Q}_{01}\setminus\{0,1\}$ be a bijection. We will write $a_k$ instead of $a(k)$ to simplify notation and will inductively build a subset

$$f \subseteq (\mathbb{Q}_{01}\setminus\{0,1\}) \times (X\setminus\{0_X,1_X\})$$

that’s an injective function from $\mathbb{Q}_{01}\setminus\{0,1\}$ to $X\setminus\{0_X,1_X\}$ which is even order-preserving.

$n = 0$: Choose $b_0 \in X\setminus\{0_X,1_X\}$ and set $f_0 := \{(a_0, b_0)\}$.

$n \to n + 1$: Assume that $f_n$ has been defined in a way that for all $k, l \in \{0, ..., n\}$ the relation $a_k \leq a_l$ implies $f_n(a_k) \leq f_n(a_l)$ and that $f_n$ is an injective function from $\{a_0, ..., a_n\}$ to $X\setminus\{0_X,1_X\}$. Now consider the element $a_{n+1} \in \mathbb{Q}_{01}\setminus\{0,1\}$.

Case 1: $a_{n+1} \geq a_i$ for all $i \in \{0, ..., n\}$. Then, since $X$ is dense, there is $b_{n+1} \in X$ such that $1_X > b_{n+1} \geq f_n(a_i)$ for all $i \in \{0, ..., n\}$. So,

$$f_{n+1} := f_n \cup \{(a_{n+1}, b_{n+1})\}$$

is an injective order-preserving function that continues $f_n$.

Case 2: $a_{n+1} \leq a_i$ for all $i \in \{0, ..., n\}$. Proceed similarly as in Case 1.

Case 3: There are $k, l \in \{0, ..., n\}$ such that $a_k < a_{n+1} < a_l$. We may assume that there is no $k' \in \{0, ..., n\}$ with $a_k < a_{k'} < a_{n+1}$ and likewise that there is no $l' \in \{0, ..., n\}$ with $a_{n+1} < a_{l'} < a_l$. Consider $b_k = f_n(a_k)$ and $b_l = f_n(a_l)$. Since $f_n$ is order-preserving and injective by assumption, we get $b_k < b_l$. Because $X$ is dense, there is an element $b_{n+1}$ such that $b_k < b_{n+1} < b_l$. Then

$$f_{n+1} := f_n \cup \{(a_{n+1}, b_{n+1})\}$$

is easily seen to be an injective order-preserving map that continues $f_n$.

Now, it is easy to see that

$$f := \bigcup_{n \in \omega} f_n$$

is an injective order-preserving function from $\mathbb{Q}_{01}\setminus\{0,1\}$ to $X\setminus\{0_X,1_X\}$ which is even order-preserving. So

$$\varphi := f \cup \{(0,0_X),(1,1_X)\}$$

is an order embedding from $\mathbb{Q}_{01}$ to $X$. \qed

Proposition 7.3. Let $L$ be a bounded affine complete distributive lattice. Then

a) There is a maximal chain $C \subseteq L$, i.e., a chain that is not properly contained in another chain in $L$.

b) If $C$ is a maximal chain of $L$ then $C$ is dense.

Proof. a) is a standard application of Zorn’s Lemma: If $\mathcal{K}$ is a set of chains of $L$ such that for any $C_1, C_2 \in \mathcal{K}$ we either have $C_1 \subseteq C_2$ or $C_1 \supseteq C_2$, then $\bigcup \mathcal{K}$ is easily checked to be a chain in $L$: Let $x, y \in \bigcup \mathcal{K}$, then there are members $C, D$ containing $x, y$ respectively; now since $\mathcal{K}$ is a chain with respect to $\subseteq$, at least one of the statements $x, y \in C$ or $x, y \in D$ holds. Since $C, D$ are chains in $L$, either statement leads us to $x \leq_L y$ or $x \geq_L y$. So $\mathcal{K}$ is bounded in the poset of all chains of $L$, thus Zorn’s Lemma implies that there is a maximal chain.

As for b), assume that $C$ is a maximal chain such that $x < y \in C$ but there is no $z \in C$ with $x < z < y$. Now if there were no $z$ in the whole lattice $L$ such that $x < z < y$, then $[x, y] = \{x, y\}$ is a proper Boolean interval of $L$ which implies that $L$ is not affine complete,
leading to a contradiction. Thus there is such a \( z \), whence \( C \cup \{ z \} \) is a chain of \( L \) that properly contains \( C \), contradicting the maximality of \( C \).

Now the propositions 7.2 and 7.3 directly imply the following.

**Theorem 7.4.** If \( L \) is an affine complete lattice, then there exists a \((0,1)\)-embedding \( \varphi : \mathbb{Q}_{01} \hookrightarrow L \).

*Proof.* Pick any maximal chain \( C \) in \( L \). Note that by maximality of \( C \) we have \( 0, 1 \in C \) since \( C \cup \{0, 1\} \) is a chain. So the inclusion map \( \iota : C \hookrightarrow L \) is a \((0,1)\)-embedding as well as the embedding from \( \mathbb{Q}_{01} \) to \( C \) provided by proposition 7.3. Composing these two, we get a \((0,1)\)-embedding from \( \mathbb{Q}_{01} \) to \( L \). \( \square \)

**8. Open questions**

In chapter 6 we showed that every bounded distributive lattice can be extended to an affine complete lattice. This was achieved by making use of \( \mathbb{Q}_{01} \) which happens to be embeddable in any affine complete lattice, i.e., the “smallest” affine complete lattice. Now the question is: Is the construction carried out in chapter 6 in some way canonical? For an arbitrary lattice \( L \), does its ’affine hull’ have any interesting universal properties?

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