METHOD FOR CONSTRUCTING RHEOLOGICAL MODELS OF INCOMPRESSIBLE MEDIA UNDER FINITE DEFORMATIONS

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Introduction. The viscoelastic behavior and viscous-flow properties of incompressible materials undergoing finite deformations can be simulated in terms of models of integral and differential type. The integral models describe the history of medium deformation using integral equations. However, these equations are difficult to generalize to complicated problems such as, for instance, simulation of mass exchange processes in dissipative media. Attempts to present damage accumulation and thixotropy of elastomeric materials in terms of integral models add complexity to these models and hamper their practical application.

The differential models describe the rheological properties of materials in terms of tensor internal variables and, as a rule, assign them the physical meaning of stresses [1, 2, 3, 4] or strains [5, 6, 7, 8]. These models can be conveniently represented with the aid of symbolic diagrams illustrating the mechanical behavior of the medium. In the paper presented here a fairly simple method of obtaining constitutive equations is described. The method has been constructed using symbolic diagrams. The main advantage of the method is rather simple physical meaning of mathematical expressions. The method has its origin in the ideas reviewed in works [9, 10, 11, 12, 13, 14].

Singular points in the symbolic diagram. This work presents the method allowing one to construct the mathematical model for the medium whose mechanical behavior is illustrated by a diagram consisting of blocks of interrelated elastic and viscous elements. Consider this method using the diagram shown in Fig. 1 as an example. To each junction of elements from the symbolic diagram illustrating the material behavior, a vector in nine-dimensional vector space corresponds. The space forms a set of deformation gradient tensors.

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Fig. 1: Schematic representation of the model for viscoelastic material. Arrows show linear mappings of the infinitesimal domain of the material. Corresponding deformation gradients present information about the mappings which couple reference and examined points in the diagram and are used for introducing internal variables (point coordinates of the diagram in nine-dimensional vector space).

Between the points there are elastic and viscous interrelations. Special role is assigned to three points: left, right and reference.

The left point in the scheme is denoted by symbol $A$ (Fig. 1) to which the deformation gradient of the material corresponds. That is,

$$F_A = F = \frac{\partial x}{\partial x_0},$$

where $x = x(t, x_0)$ is the radius-vector of location of medium points at current time $t$; $x_0$ is the radius-vector of location of medium points at reference time. It is evident that the rate of deformation tensor $D$ and the spin $W$, employed in the theory of nonlinear media, are connected with point $A$.

The right point in the scheme is denoted by symbol $E$ in Fig. 1. Its peculiarity is that the strain gradient $F_E$, corresponding to point $E$, is the rotation tensor $R_{\text{gr}}$, which determines the rotation of the infinitesimal bulk of the medium as an absolutely solid body and is independent of the material deformation

$$F_E = R_{\text{gr}}.$$
Fig. 2: Diagram of the $i$th symbolic element. In constructing of constitutive equations, deformation gradients $F_A$, $F_B$ and tensor measures of the rate of deformation of $D_A$, $D_B$ are assigned to points $A$ and $B$.

Hence, the rate of deformation tensor, calculated for the right points of the system, is equal to zero

$$D_\varepsilon = \frac{1}{2} \left( \dot{F}_\varepsilon F_\varepsilon^{-1} + F_\varepsilon^{-T} \dot{F}_\varepsilon T \right) = 0.$$  

*The reference point* is denoted by symbol $F$ in Fig. 1. In nine-dimensional space, the unit tensor $I$ corresponds to this point

$$F_F = I.$$  

**Tensors used for establishing the strained state of the elements in the symbolic diagram and the rate of change of this state.** Let us examine some peculiarities of the mathematical description of the strained state of the arbitrarily selected element of the system (Fig. 2). Assume that the element is numbered by $i$, and its strained state is determined by comparing two deformation gradients and two tensors of rate of deformation.

The element $i$ is connected with other elements at points $A$ and $B$. Deformation gradients $F_A$ and $F_B$ correspond to points $A$ and $B$. Using these gradients, the rate of deformation tensors $D_A$, $D_B$ and spins $W_A$, $W_B$ are determined as

$$D_A = \frac{1}{2} \left( \dot{F}_A F_A^{-1} + F_A^{-T} \dot{F}_A T \right),$$

$$D_B = \frac{1}{2} \left( \dot{F}_B F_B^{-1} + F_B^{-T} \dot{F}_B T \right),$$

$$W_A = \frac{1}{2} \left( \dot{F}_A F_A^{-1} - F_A^{-T} \dot{F}_A T \right),$$

$$W_B = \frac{1}{2} \left( \dot{F}_B F_B^{-1} - F_B^{-T} \dot{F}_B T \right).$$
To interpret tensor values from physical standpoint, it is reasonable to elaborate the appropriate mathematical model of the medium starting from the following premises. The rate of deformation tensor of the \( i \)th element \( D_i \) can be calculated for viscous and elastic elements from the formula

\[
D_i = D_A - D_B. \tag{1}
\]

Furthermore, in the elastic elements of the scheme the relation between deformation gradients, corresponding to points \( A \) and \( B \), can be defined by the law of multiplicative decomposition

\[
F_A = V_i F_B, \tag{2}
\]

where the stretch tensor of the \( i \)th element \( V_i \) is found by three orthonormalized eigenvectors \( n^i_1, n^i_2, n^i_k \) (defining the direction of principal tensor axes) and by stretch ratios \( \lambda^i_1, \lambda^i_2, \lambda^i_3 \)

\[
V_i = \sum_{k=1}^{3} \lambda^i_k n^i_k \otimes n^i_k.
\]

It is easy to check that condition (2) determines the relation between tensors \( D_A, W_A \) and \( D_B, W_B \) in elastic elements

\[
D_B = \frac{1}{2} V_i^{-1} \left( B_i (D_A + W_A^T) + (D_A + W_A) B_i - \dot{B}_i \right) V_i^{-1}, \tag{3}
\]

\[
W_B = \frac{1}{2} V_i^{-1} \left( V_i \dot{V}_i - \dot{V}_i V_i + (D_A + W_A) B_i - -B_i (D_A + W_A^T) \right) V_i^{-1}, \tag{4}
\]

and the rate of time variation of the stretch ratio

\[
\dot{\lambda}_k^i = \lambda_k^i n_k^i \otimes n_k^i \cdot D_i, \tag{5}
\]

where the tensor \( B_i \) is the left Cauchy-Green tensor of the elastic element

\[
B_i = V_i^2.
\]

**Coupling of elastic elements in the diagram.** In the proposed method of constructing of constitutive equations there is a condition which should be
fulfilled when coupling the elements. The elastic elements must be connected by their left ends with other elastic elements or with the left point in the symbolic diagram. This restriction is caused by the necessity to calculate by formula (3) the time derivatives of tensors $B_i$ for all elastic elements using for this tensors $D_i$, $D$ and $W$. Calculation starts with elastic elements connected by their left ends with the left-hand point in the diagram. At this step we find not only the time derivatives of tensors $B_i$, but also the rate of deformation and spin tensors of the right junction with the adjacent elements (tensors $D_B$ and $W_B$ in Fig. 2). Calculations are performed using formulas (1), (3), (4). Then we carry out calculations for elastic elements connected on the right with the elements just examined, and so on.

**Simulation of mechanical properties of medium elements.** In isothermal case, as an expression for mass free-energy density $f$, we use the function of stretch ratios $\lambda^i_k$ of the elastic elements of the medium

$$f = f(\lambda^i_1, \lambda^i_2, \lambda^i_3, \ldots, \lambda^i_1, \lambda^i_2, \lambda^i_3),$$

(6)

where $n$ is the number of elastic elements. It is assumed that each elastic element satisfies the volume invariance condition when subjected to deformation

$$\lambda^i_1 \lambda^i_2 \lambda^i_3 = 1.$$

(7)

Therefore, the Cauchy stress tensor is calculated by the formula from the nonlinear elasticity theory for incompressible materials

$$T_i = p_i I + \rho \sum_{k=1}^{3} \lambda^i_k \frac{\partial f}{\partial \lambda^i_k} n^i_k \otimes n^i_k,$$

(8)

where $\rho$ is the medium density, $p_i$ is an indefinite parameter, and $i$ is the elastic element number. The characteristic features of viscous elements can be simulated by ordinary state equations for viscous liquids

$$T_j = p_j I + 2 \eta_j D_j$$

(9)

and their incompressibility condition

$$I \cdot D_j = 0,$$

(10)

where $T_j$ is the Cauchy tensor for viscous elements, $\eta_j$ is the coefficient of shear viscosity, and $j$ is the viscous element number.

**Interrelations between stresses for coupled elements.** For determination of the closed system of constitutive equations, it is necessary to establish
the interrelations for stresses at diagram junctions. They must be formulated in such a way that the sum of all forces acting on the arbitrarily selected junction would be equal to zero. Since we study the points in nine-dimensional vector space (space generates a set of possible values of deformation gradient tensors), then we can use the Cauchy stress tensors of the diagram elements as the nine-dimensional force vectors.

Now let us formulate the rule. The sum of Cauchy stress tensors of the elements coupled on the left is equal to the sum of Cauchy stress tensors of the elements coupled on the right at any diagram junction. It is assumed in this case that the forces, determined in nine-dimensional vector space by the Cauchy stress tensor, act from left on the left point of the symbolic diagram and from right on its right point. It means that for the diagram given in Fig. 1 the following equalities are valid:

\[
\begin{align*}
T &= T_1 + T_2 + T_3 + T_4 \quad \text{(at point } A), \\
T_1 &= T_5 \quad \text{(at point } B), \\
T_2 + T_5 &= T_6 \quad \text{(at point } C), \\
T_3 + T_6 &= T_7 \quad \text{(at point } D). 
\end{align*}
\]

The right point allows us to formulate the linearly dependent equation

\[
T_4 + T_7 = T \quad \text{(at point } E) 
\]

which gives no additional information.

**Dissipation inequality.** In the isothermal case, the thermodynamic constraint (second law of thermodynamics) imposed on processes in the medium

\[
T \cdot D - \rho \dot{f} \geq 0
\]

takes the form

\[
\sum_{i=1}^{n} T_i \cdot D_i + \sum_{j=n+1}^{n+m} T_j \cdot D_j - \rho \dot{f} \geq 0,
\]

where elastic and viscous elements are numbered by \( i \) and \( j \), respectively, \( n \) is the number of elastic elements in the system, and \( m \) is the number of viscous elements in the system. Finally, using the type of free energy \( (6) \) and equalities \( (5), (7) - (10) \), the dissipation inequality can be written as

\[
\sum_{j=n+1}^{n+m} 2\eta_j D_j \cdot D_j \geq 0.
\]
This inequality is satisfied automatically provided that the shear viscosity coefficients can not be negative.

**Example.** Consider the material whose behavior is illustrated by a diagram in Fig. 3. Dots denote the blocks of elements. We will not describe here the way of coupling elements into these blocks.

In modeling the viscoelastic properties of polymers, a proper consideration of hysteresis losses is of particular importance. Usually the most challenging task is to provide the accurate description of large losses in the region of large values of stretch ratios and small losses in the region of small deformations of the material. In the considered example it is possible to describe such material behavior using the potential of free energy

\[ \rho_f = -\sum_{i=1}^{2} \frac{c_i J_i^m}{6} \ln \left( 1 - \frac{I_1(B_i)}{J_i^m} - 3 \right) + c_3 \left( I_1(B_1) - 3 \right) \left( I_1(B_3) - 3 \right), \]

where \( c_1, c_2, c_3, J_1^m, J_2^m \) are material constants, and \( I_1(B_i) \) is the first invariant of tensor \( B_i \). The first two terms in the potential are known in the literature as Gent potentials formulated for corresponding elastic elements [18].

Two peculiarities of the material behavior defined by elements under consideration should be noted. During the first loading cycle there are hysteresis significant losses and during the second and other cycles the losses are essentially less. However, such behavior of the material is viscoelastic by nature, and not a manifestation of Mullins softening effect.
Another distinguishing feature of the model is that the free energy potential may involve the cross terms interrelating the behavior of different elastic elements. In the example presented, this is the last term. It means that the deformation of the third element changes the elastic behavior of the first, and the deformation of the first element specifies the properties of the third. The physical meaning of symbolic elements remains unchanged.

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