I. INTRODUCTION

Since the first proposal of the adiabatic geometrical pumping by Thouless [1, 2], it has been recognized that a current can flow without any averaged bias if there is a Berry-phase-like variable or the Berry-Sinitsyn-Nemenman (BSN) curvature in the space of the modulation parameters [1–6]. This phenomenon, known as Thouless pumping or geometrical pumping, has been observed experimentally in various processes such as charge transport [7–14] and spin pumping [15]. There are various theoretical papers on geometrical pumping processes based on scattering theories [16–23], classical master equations [24–33] and quantum master equations [34–39]. The extended fluctuation theorem for geometrical pumping processes has also been studied [40–42].

The phenomenon of geometrical pumping can be used in nanoscale thermodynamics. The geometrical framework has already been used in finite-time thermodynamics [43], in which the thermodynamic length plays a key role. The thermodynamic length is originally introduced for macroscopic systems [44–48], and it has been used in a wide range of thermodynamic systems such as a classical nanoscale system [49], a closed quantum system [50] and an open quantum system [51]. These concepts are important, in particular, for quantum thermodynamics or thermodynamics for nanoscale machines [52–54]. The current status of the subject is presented in a recent review [55]. Thus, the geometrical concept is of fundamental importance in understanding and controlling nanoscale machines.

There are several attempts to formulate the geometrical thermodynamics for a microscopic heat engine in the adiabatic regime [56–62]. In particular, it is remarkable that Ref. [58] has formulated the geometrical theory of the heat engine induced by the BSN curvature without average temperature difference between the two reservoirs. Nevertheless, Ref. [58] has, at least, two shortcomings. First, its analysis is only applicable to systems obeying a classical master equation, which means that the quantum effects of the heat engine cannot be discussed. Second, the implementation of the microscopic heat engine may not be realistic, because the periodic control of the temperatures of the two reservoirs is difficult in practice. If we replace the temperatures with chemical potentials, such a system would be easy to realize experimentally. Thus, in this paper, we control the chemical potentials of reservoirs under the isothermal condition. We apply the formulation developed in Ref. [58] to the Anderson model, a single quantum dot system described by a quantum master equation within the wide-band approximation [63].

The organization of this paper is as follows. In Sec. II, we explain the setup and geometrical formulation for describing the heat engine under an adiabatic pumping process. In Sec. III, we apply our formulation to the Anderson model for a quantum dot coupled to two reservoirs within the wide-band approximation. In Sec. IV we discuss our results including (i) whether a perfectly periodic engine is possible, (ii) what happens if we control the temperature of a reservoir, and (iii) Carmér-Rao bound for this engine [64–66]. Finally, we summarize our results and perspectives in Sec. V. In Appendix A, we prove the positivity of the quantum Kullback-Leibler divergence. In Appendix B, we describe some general properties of the quantum master equation such as the perturbation method with a slowly modulated parameter and a math-
II. GENERAL FRAMEWORK

A. Setup

In this paper, we consider a system S of a quantum dot coupled to two reservoirs L and R under periodic modulation of some system parameters, with the period \( \tau_p \). Each reservoir \( \alpha = L \) or \( R \) is characterized by the chemical potential \( \mu^\alpha \) and temperature \( T \) (or the inverse temperature \( \beta := 1/T \)). Although we will specify the system Hamiltonian to describe a quantum dot attached to two reservoirs later, we assume that we can control some parameters in the Hamiltonian through a control parameter \( \lambda \). We also assume that the heat generation under the operation is negligible, i.e., that we can impose the isothermal condition with constant temperature \( T \). Throughout the paper, we set \( \hbar = 1 \).

We illustrate the setup by a schematic picture in Fig. 1 where we modulate \( \mu^L(\theta) \) and \( \mu^R(\theta) \) in the reservoirs with different phases. The system Hamiltonian \( \hat{H}(\lambda(\theta)) \) is also modulated by an external agent.

In this paper, we control the set of parameters

\[
\Lambda := \left( \lambda, \frac{\mu^L}{\mu^L}, \frac{\mu^R}{\mu^R} \right),
\]

where we have introduced

\[
\overline{\mu}^\alpha := \frac{1}{\tau_p} \int_0^{\tau_p} dt \mu^\alpha(t)
\]

for \( \alpha = L \) and \( R \). When we consider the case without any difference in time-averaged chemical potentials, i.e. \( \overline{\mu}^L = \overline{\mu}^R \), we can replace \( \overline{\mu}^\alpha \) in Eq. (1) by \( \overline{\mu} \) which is independent of \( \alpha \).

We assume that the dynamics of S is described by the quantum master equation

\[
\frac{d}{dt} |\hat{\rho}(\theta)\rangle = e^{-1} \hat{K}(\Lambda(\theta)) |\hat{\rho}(\theta)\rangle,
\]

where \( |\hat{\rho}(\theta)\rangle \) is the density matrix of the system S. Here, we have introduced the dimensionless time (which is the phase of the modulation) \( \theta := 2\pi(t - t_0)/\tau_p \) and the dimensionless operation speed \( \epsilon := 1/(\tau_p \Gamma) \), where \( t_0 \) is the time after which the system reaches a periodic state and \( \Gamma \) is the coupling strength or the characteristic transition rate between the system and the reservoirs. The dynamics of physical relevant quantum master equations such as the Lindblad equation are completely positive and trace-preserving (CPTP) [67–69], but the modulation process may not be CPTP. Therefore, we do not assume that the dynamics including the modulation is CPTP. Moreover, we assume that Eq. (3) is a Markovian type master equation, though the modulation is included in \( \hat{K}(\Lambda(\theta)) \) through \( \Lambda(\theta) \) and thus non-Markovianity might be important [70]. If the diagonal elements of the density matrix \( |\rho\rangle \) are decoupled from the off-diagonal elements, the system can be regarded as classical, but if the diagonal elements are mixed with off-diagonal elements, the system is regarded as quantum.

In Eq. (3) we have also introduced the vector consisting of the elements of the density matrix

\[
|\hat{\rho}(\theta)\rangle := \begin{pmatrix} \rho_{11}(\theta) \\ \rho_{12}(\theta) \\ \vdots \\ \rho_{nn}(\theta) \end{pmatrix},
\]

where \( \rho_{ij}(\theta) \) is the \((ij)-element of \( |\hat{\rho}(\theta)\rangle \), which is assumed to be a \( n \times n \) matrix. The density matrix \( |\hat{\rho}(\theta)\rangle \) satisfies the condition of probability conservation \( Tr|\hat{\rho}(\theta)\rangle = 1 \). Thus, the transition matrix \( \hat{K}(\Lambda(\theta)) \) is a superoperator acting on \( |\hat{\rho}(\theta)\rangle \), which is expressed as a \( n^2 \times n^2 \) matrix.

We assume that the master equation (3) has a unique steady state \( |\hat{\rho}^\text{ss}(\Lambda(\theta))\rangle \) which satisfies

\[
\hat{K}(\Lambda(\theta))|\hat{\rho}^\text{ss}(\Lambda(\theta))\rangle = 0.
\]

Equation (5) means that the steady density matrix is equivalent to the right zero eigenstate of \( \hat{K}(\Lambda(\theta)) \). Since the system is coupled to two reservoirs having different chemical potentials, \( |\hat{\rho}^\text{ss}(\Lambda(\theta))\rangle \) is a nonequilibrium steady state.

The matrix \( \hat{K}(\Lambda(\theta)) \) consists of the transition matrices between the reservoir \( \alpha \) to the system as \( \hat{K}(\Lambda(\theta)) = \sum_{\alpha=L,R} \hat{K}_\alpha(\Lambda(\theta)) \). From probability conservation, the element \( k_{ij}^\alpha(\Lambda(\theta)) \) of \( \hat{K}_\alpha(\Lambda(\theta)) \) satisfies \( \sum_{\alpha=L,R} \sum_i k_{ij}^\alpha(\Lambda(\theta)) = 0 \). where \( k_{ij}^\alpha(\Lambda(\theta)) \) expresses the transition rate of \( j \) to \( i \) due to interaction with the reservoir \( \alpha \) at \( \theta \). We assume that the \( \theta \)-dependence of
The excess entropy \( S^\alpha \) of Eq. (9) is the von Neumann entropy where the first term on the right hand side (RHS) of Eq. (9) contains the excess entropy \( S^{ex}(\theta) \) defined as

\[
S^{ex}(\theta) := -\text{Tr}[(\hat{\rho} \ln \hat{\rho} - \hat{\rho}^a)],
\]

where \( \hat{\rho}^a \) is the \( i \)-th component of \( |\hat{\rho}^a\rangle \).

**B. Thermodynamic Quantities**

The second law of thermodynamics governs the performance of an engine. It is well-known that the quantum relative entropy or quantum Kullback-Leibler (KL) divergence

\[
S^{KL}(\hat{\rho}, \hat{\rho}^a) := \text{Tr}[\hat{\rho} \ln \hat{\rho} - \hat{\rho} \ln \hat{\rho}^a],
\]

with density matrices \( \hat{\rho} \) and \( \hat{\rho}^a \) satisfying \( \text{Tr}[\hat{\rho} - \hat{\rho}^a] = 0 \) is positive semidefinite (see Appendix A) [69, 71–74]:

\[
S^{KL}(\hat{\rho}|\hat{\rho}^a) \geq 0.
\]

We can rewrite the KL divergence as

\[
S^{KL}(\hat{\rho}|\hat{\rho}^a) := -S^{VN}(\theta) + S^{ex}(\theta),
\]

where the first term on the right hand side (RHS) of Eq. (9) is the von Neumann entropy \( S^{VN} := -\text{Tr}[\hat{\rho} \ln \hat{\rho}] \) of the system S. Equation (9) contains the excess entropy \( S^{ex}(\theta) \) defined as

\[
S^{ex}(\theta) := -\text{Tr}[(\hat{\rho} \ln \hat{\rho}^a)].
\]

To discuss the performance of the engine, we introduce the one-cycle averaged entropy production defined as:

\[
\sigma := \frac{1}{2\pi} \int_0^{2\pi} S^{KL}(\theta) d\theta.
\]

According to the KL inequality \( S^{KL}(\theta) \geq 0 \), the entropy production is always positive semidefinite, i.e. \( \sigma \geq 0 \). This \( \sigma \) is related to the dissipative availability \( A := T \sigma \) [56, 58]. In other words, nonzero \( \sigma \) means that the system is driven by a geometrical variable such as the BSN curvature. Thus, \( \sigma \) plays a key role in nonequilibrium thermodynamics. Introducing

\[
W := \frac{1}{2\pi} \int_0^{2\pi} \text{Tr} \left[ (\hat{\rho} \partial \hat{H}(\lambda(\theta)) \partial \lambda(\theta)) \right] \hat{\lambda}(\theta) d\theta
\]

with \( \hat{\lambda} := \frac{d}{d\theta} \lambda(\theta) \), we also have a scalar function

\[
Q := W + T \sigma,
\]

which satisfies the inequality

\[
W = Q - T \sigma \leq Q,
\]

because \( T \sigma \geq 0 \). The scalar quantity \( Q \) is the available energy corresponding to the absorbed heat of this engine. Although \( W \) can be negative in the definition of Eq. (12), it must be positive semidefinite if we restrict our interest to the case of \( \dot{\lambda} \geq 0 \) and \( \text{Tr}[\hat{\rho} \partial \hat{H}(\lambda(\theta)) \partial \lambda] \geq 0 \). If these conditions are satisfied, \( W, Q \) and Eq (14) correspond to the work done on the system through the modulation of the system Hamiltonian \( \hat{H}(\lambda(\theta)) \), the absorbed heat and the second law of thermodynamics, respectively. However, the system cannot be perfectly periodic if the modulation satisfies the condition \( \dot{\lambda} \geq 0 \). Indeed, the eigenvalue of \( \hat{H}(\lambda(\theta)) \) should increase after a one-cycle operation by an external agent, while a completely periodic operation must satisfy \( \hat{H}(\lambda(\theta + 2\pi)) = \hat{H}(\lambda(\theta)) \). The possibility of the implementation of a perfectly periodic engine under the condition \( \hat{H}(\lambda(\theta + 2\pi)) = \hat{H}(\lambda(\theta)) \) will be discussed in Sec. IV.

Let us introduce the ratio of the work \( W \) to the available energy \( Q \) as

\[
\eta^{eff} := \frac{W}{Q} = \frac{W}{W + T \sigma}.
\]

From Eq. (14), \( \eta^{eff} \) satisfies \( 0 \leq \eta^{eff} \leq 1 \) if \( \dot{\lambda}(\theta) \geq 0 \) is satisfied for any \( \theta \). Thus, we call \( \eta^{eff} \) the effective efficiency because it is an indicator of the performance of the heat engine. In the quasi-static limit (\( \epsilon \to 0 \)), \( \eta^{eff} \) converges to 1. The scaled power defined as

\[
P := \epsilon W
\]

converges to zero in this limit. Note that \( P \) does not have the dimension of power because we set the time scale by the dimensionless parameter \( \epsilon \) under fixed \( \Gamma \). To obtain larger power, we need a higher speed of operation \( \epsilon \), which leads to the effective efficiency \( \eta^{eff} \) becoming smaller. Although \( \eta^{eff} \) differs from the conventional thermodynamic efficiency, \( \eta^{eff} \) is related to the entropy production during a cyclic operation of the engine, and can thus be regarded as a new efficiency for a thermodynamic engine as indicated by Refs. [56, 58].

Before we proceed with the argument, we mention the possibility of the replacement of Eq. (11) by

\[
\Delta S := -S^{KL}(\hat{\rho}(2\pi)||\hat{\rho}^a(2\pi)) + S^{KL}(\hat{\rho}(0)||\hat{\rho}^a(0))
\]

\[
= -\int_0^{2\pi} d\theta \hat{S}^{KL}(\hat{\rho}(\theta)||\hat{\rho}^a(\theta)),
\]

because \( \Delta S \) is also positive semidefinite if the dynamics satisfies the CPTP properties [69, 71–74]. We will discuss the proper choice of the entropy production for the dynamics in Sec. IV.

\[1\] If we are interested in the work done by the engine to the external environment, we need to focus on the situation \( \text{Tr}[\hat{\rho} \partial \hat{H} \partial \lambda] \leq 0 \) and \( \lambda \geq 0 \) in which the restored energy in \( \hat{H} \) is released to perform the work to the environment. In this case the argument in the main text is almost unchanged except for the sign of \( W \) as \( W \leq 0 \). Therefore, we only discuss the case of positive semidefinite \( W \).
C. Linear response regime

In this subsection, we consider the thermodynamics of the engine introduced in the previous subsection in the linear response regime for small $\epsilon$. When we assume that $\hat{\rho}^{ss}$ is diagonalizable, there exists $(\hat{\rho}^{ss})^{-1}$. In this case, we can expand

$$\hat{\rho}(\theta) = \hat{\rho}^{ss}(\theta) + \epsilon \hat{\rho}^{(1)}(\theta) + O(\epsilon^2). \quad (18)$$

Substituting this into the expression for $\sigma$ in Eq. (11), with the help of Eqs. (9) and (10), we can expand the KL entropy as

$$S_{KL}(\hat{\rho}||\hat{\rho}^{ss}) = \frac{1}{2} \text{Tr} \left[ \hat{\rho}^{(1)}(\hat{\rho}^{ss})^{-1} \hat{\rho}^{(1)} - \text{Tr}[\hat{\rho}^{ss}] \right] + O(\epsilon^3), \quad (19)$$

where we have used $\text{Tr}\hat{\rho}^{(1)} = 0$. Thus, we can rewrite $\sigma$ as

$$\sigma = \epsilon^2 \sigma^{(2)} + O(\epsilon^3), \quad (20)$$

where $\sigma^{(2)}$ is expressed as

$$\sigma^{(2)} = \frac{1}{4\pi} \int_0^{2\pi} d\theta \text{Tr}[\hat{\rho}^{(1)}(\hat{\rho}^{ss})^{-1} \hat{\rho}^{(1)}]. \quad (21)$$

As will be shown, $\text{Tr}[\hat{\rho}^{(1)}(\hat{\rho}^{ss})^{-1} \hat{\rho}^{(1)}]$ should be positive semidefinite.

The remaining problem is to evaluate $\hat{\rho}^{(1)}$ to obtain $\sigma^{(2)}$. For this purpose we rewrite Eq. (18) as

$$|\hat{\rho}(\theta)| \approx |\hat{\rho}^{ss}(A(\theta)) + \epsilon |\hat{\rho}^{(1)}(A(\theta))| + O(\epsilon^2). \quad (22)$$

With the aid of Appendix B, the second term on the RHS of Eq. (22) can be written as

$$|\hat{\rho}^{(1)}(A(\theta))| = |\partial_{\nu} \hat{\rho}^{ss}(A(\theta)) A_{\nu}(\theta), \quad (23)$$

where we have introduced

$$|\partial_{\nu} \hat{\rho}^{ss}(A(\theta))| := \hat{K}^+(A(\theta)) \frac{\partial}{\partial A_{\nu}(\theta)} |\hat{\rho}^{ss}(A(\theta))|. \quad (24)$$

using the pseudo-inverse $\hat{K}^+(A(\theta))$ \cite{77} of the transition matrix $\hat{K}(A(\theta))$ and Eq. (B5) for $n = 1$. Here we define $\hat{A}_\mu = \frac{\partial}{\partial A_{\mu}} A_{\mu}$. The definition of $\hat{K}^+(A(\theta))$ is given as follows. If $\hat{K}(A)$ is diagonalizable, we can use the spectral decomposition as

$$\hat{K}(A) = \sum_m \varepsilon_m(A) |r_m(A)\rangle \langle \ell_m(A)|, \quad (25)$$

where $\varepsilon_m(A)$ is the eigenvalue. $|r_m(A)\rangle$ and $|\ell_m(A)\rangle$ are the corresponding right and left eigenvectors of $\hat{K}(A)$, respectively. Then, $\hat{K}^+(A)$ is defined as

$$\hat{K}^+(A) = \sum_{m \neq 0} \varepsilon_m(A)^{-1} |r_m(A)\rangle \langle \ell_m(A)|. \quad (26)$$

Thus, we can express $\hat{\rho}^{(1)}$ as

$$\hat{\rho}^{(1)} := \partial_{\nu} \hat{\rho}^{ss} A_{\nu}, \quad (27)$$

where $\partial_{\nu} \hat{\rho}^{ss}$ is the $n \times n$ matrix defined as

$$\partial_{\nu} \hat{\rho}^{ss} := \left( \begin{array}{ccc} \partial_{\nu} \rho_{11}^{ss} & \cdots & \partial_{\nu} \rho_{1n}^{ss} \\ \vdots & \ddots & \vdots \\ \partial_{\nu} \rho_{n1}^{ss} & \cdots & \partial_{\nu} \rho_{nn}^{ss} \end{array} \right), \quad (28)$$

corresponding to $|\partial_{\nu} \hat{\rho}^{ss}|$ which has $n^2$ components. Substituting Eq. (27) into Eq. (21) we obtain

$$\sigma^{(2)} = \frac{1}{2} \int_0^{2\pi} d\theta g_{\mu\nu}(A(\theta)) A_{\mu}(\theta) A_{\nu}(\theta), \quad (29)$$

where the metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu}(A) := \frac{1}{2} \text{Tr} [\partial_{\mu} \hat{\rho}^{ss}(A) (\hat{\rho}^{ss}(A))^{-1} \partial_{\nu} \hat{\rho}^{ss}(A)]. \quad (30)$$

There are two important properties of the metric tensor $g_{\mu\nu}$. First, $g_{\mu\nu}$ is related to the Fisher information matrix \cite{75} and the Hessian matrix, since Eq. (30) can be rewritten as

$$g_{\mu\nu} = \frac{1}{2} \text{Tr} [\partial_{\mu} \hat{\rho}^{ss}(A) \ln \hat{\rho}^{ss}(A) \partial_{\nu} \hat{\rho}^{ss}(A)], \quad (31)$$

where we have used Eq. (B16). The expression on the first line of Eq. (31) is the Fisher information of $\hat{\rho}^{ss}$ and the expression on the second line is the negative of the Hessian matrix of $\ln \hat{\rho}^{ss}$. Thus, the metric tensor can be expressed in terms of the Fisher information and Hessian matrix \cite{75}. Second, $g_{\mu\nu} A_{\mu} A_{\nu}$ is positive semidefinite. Indeed, it is straightforward to show that

$$\hat{A}_\mu g_{\mu\nu} A_\nu = \frac{1}{2} \text{Tr} \left[ \hat{\rho}^{ss}(A) \left( \hat{A}_\mu \hat{K}^+ \frac{\partial}{\partial A_\mu} \ln \hat{\rho}^{ss} \right)^2 \right] \geq 0. \quad (32)$$

Using the Cauchy-Schwartz inequality, the entropy production is bounded as

$$\sigma^{(2)} \geq \mathcal{L}^2, \quad (33)$$

where

$$\mathcal{L} := \int_{\partial \Omega} \sqrt{g_{\mu\nu}(A) A_{\mu} A_{\nu}} \quad (34)$$

is the thermodynamic length corresponding to the length along the closed trajectory $\partial \Omega$ surrounding the domain $\Omega$ of the Riemannian manifold with the metric $g_{\mu\nu}(A)$. The inequality (33) has an identical form to that of the system coupled only to one reservoir \cite{56}. The equality
in Eq. (33) holds when $g_{\mu\nu}(\Lambda(\theta)) \Lambda_{\mu}(\theta) \Lambda_{\nu}(\theta)$ is a non-negative $\theta$-independent constant. This equality cannot be achieved if the BSN curvature is meaningful, because $g_{\mu\nu}(\Lambda(\theta)) \Lambda_{\mu}(\theta) \Lambda_{\nu}(\theta)$ should be a $\theta$-dependent variable when the trajectory in the parameter space makes a closed loop to generate BSN curvature.

Let us consider the leading-order contribution to the work $W$, written as $W = W + O(\epsilon)$, where

$$W := \oint_{\partial \Omega} A_{\mu}(\Lambda) d\Lambda_{\mu}$$

(35)

is the adiabatic work expressed as the line integral of the thermodynamic vector potential

$$A_{\mu}(\Lambda) := \delta_{\mu,0} \text{Tr} \left[ \frac{\partial \hat{H}}{\partial \Lambda} \rho_{\text{ss}}(\Lambda) \right]$$

(36)

along the trajectory $\partial \Omega$ of parameter control [56]. Note that $A_{\mu}(\Lambda)$ corresponds to the BSN vector in adiabatic pumping processes [5, 6]. We also note that the adiabatic work $W$ is independent of $\epsilon$. We stress that the adiabatic work in Eq. (35) is expressed as the geometrical vector potential in Eq. (36) as expected below Eq. (12). Therefore, if $W$ in Eq. (35) is finite, we extract the work from the quantum chemical engine under the control of external chemical potentials $\mu^R$ and $\mu^L$.

The adiabatic work is related to the flux penetrating the area $\Omega$ surrounded by $\partial \Omega$. By using the Stokes theorem, one can rewrite the adiabatic work as

$$W = \oint_{\partial \Omega} A = \int_{\Omega} dA.$$  

(37)

Here we define the 1-form $A = A_{\mu} d\Lambda_{\mu}$. Introducing the thermodynamic curvature $F_{\mu\nu}$

$$F_{\mu\nu} := \frac{\partial}{\partial \Lambda_{\mu}} A_{\nu} - \frac{\partial}{\partial \Lambda_{\nu}} A_{\mu},$$

(38)

we can rewrite the 2-form $dA$ as

$$dA = \frac{1}{2} F_{\mu\nu} d\Lambda^\mu \wedge d\Lambda^\nu.$$  

(39)

This $dA$ is directly related to the thermodynamic axial field $\vec{B} = (B_\mu)$ and the thermodynamic flux $\Phi_{\text{TD}}$ as

$$B_\mu := \epsilon_{\mu\nu\rho} F_{\nu\rho},$$

$$\Phi_{\text{TD}} := \int_{\Omega} \vec{B} \cdot d\vec{S},$$

(40)

(41)

where $dS_\mu := \frac{1}{2} \epsilon_{\mu\nu\rho} d\Lambda_{\nu} \wedge d\Lambda_{\rho}$ with the Levi-Civita symbol $\epsilon_{\mu\nu\rho}$, i.e. $\epsilon_{\mu\nu\rho} = 1$ for an even permutation of $(\mu, \nu, \rho)$, $\epsilon_{\mu\nu\rho} = -1$ for an odd permutation of $(\mu, \nu, \rho)$ and $\epsilon_{\mu\nu\rho} = 0$ if two of three suffices are equal. From Eqs. (36), (38) and (40), $B_0 = 0$, and thus, $\vec{B}$ can be regarded as a two-dimensional vector which has only the components $B_1$ and $B_2$.

From the definition (38), $B_\mu$ satisfies the Gaussian law as

$$\partial_\mu B_\mu = 0.$$  

(42)

With the aid of Eqs. (37), (38), (39) and (40), Eq. (41) can be rewritten as

$$\Phi_{\text{TD}} = \oint_{\partial \Omega} A_{\mu} d\Lambda_{\mu} = W.$$  

(43)

Thus, the thermodynamic flux $\Phi_{\text{TD}}$ reduces to the line integral along the closed path $\partial \Omega$, which is the adiabatic work $W$.

The average power can be expressed as

$$P = \epsilon W + O(\epsilon^2)$$

(44)

for small $\epsilon$.

By using the equality (20), the effective efficiency $\eta^{\text{eff}}$ introduced in Eq. (15) is rewritten as

$$\eta^{\text{eff}} = \frac{W}{W + \epsilon^2 T \sigma^{(2)}} = 1 - \epsilon^2 T \sigma^{(2)} + O(\epsilon^2).$$

(45)

Using Eq. (33), the relation (45) can be rewritten as

$$1 - \eta^{\text{eff}} \geq \epsilon^2 \frac{L^2}{W} = \epsilon^2 \frac{L^2}{P},$$

(46)

where we have used Eq. (16) for the final expression. This relation tells us that the decrement of the effective efficiency is bounded by the thermodynamic length $L$ and $W$ or the power $P$, which becomes smaller if $W$ or $P$ is larger.

It is obvious that $\eta^{\text{eff}}$ in Eq. (45) becomes 1 in the adiabatic limit ($\epsilon \rightarrow 0$). This result corresponds to the Carnot efficiency in conventional thermodynamics. Note that the conventional thermodynamic efficiency does not exist in our system because there is no heat current between two reservoirs. Therefore, the bound of the efficiency of this engine is not the Carnot efficiency, even if $\eta^{\text{eff}}$ reaches the maximum value. Nevertheless, we do not consider any cost of controlling the system Hamiltonian, chemical potentials, and temperature, which is most important for the realization of this engine.

### III. APPLICATION TO A QUANTUM DOT SYSTEM

In this section, we apply the general framework developed in the previous section to the Anderson model for a quantum dot (QD) in which a single dot is coupled to two electron reservoirs (see Fig. 2 for a schematic of our system).
A. Anderson model for a quantum dot

The total system consists of the single-dot system and baths (reservoirs). Thus, the total Hamiltonian $\hat{H}^{\text{tot}}$ is written as

$$\hat{H}^{\text{tot}} := \hat{H} + \hat{H}^r + \hat{H}^{\text{int}},$$

where the system Hamiltonian $\hat{H}$, reservoir Hamiltonian $\hat{H}^r$ and interaction Hamiltonian $\hat{H}^{\text{int}}$ are, respectively, given by

$$\hat{H} = \sum_{\sigma} \epsilon_\sigma \hat{a}_\alpha \hat{d}_\sigma + U(\theta)\hat{n}_\uparrow \hat{n}_\downarrow.$$

$$\hat{H}^r = \sum_{\sigma, \alpha, k, \sigma} \epsilon_{k, \alpha, \sigma} \hat{a}^\dagger_{\alpha, k, \sigma} \hat{a}_{\alpha, k, \sigma}$$

$$\hat{H}^{\text{int}} = \sum_{\alpha, k, \sigma} V_\alpha \hat{d}_\alpha^{\dagger} \hat{a}_{\alpha, k, \sigma} + \text{h.c.},$$

where $\hat{a}^\dagger_{\alpha, k, \sigma}$ and $\hat{a}_{\alpha, k, \sigma}$ are, respectively, the creation and annihilation operators for the electron in the reservoirs $\alpha = (L \text{ or } R)$ with the wave number $k$, energy $\epsilon_k$, and spin $\sigma = (\uparrow \text{ or } \downarrow)$. Moreover, $\hat{d}_\sigma$ and $\hat{d}^\dagger_\sigma$ are those in the QD, and $\hat{n}_\sigma = \hat{d}_\sigma \hat{d}^\dagger_\sigma$. $U(\theta)$ and $V_\alpha$ are, respectively, the time-dependent electron-electron interaction in the QD and the transfer energy between the QD and the reservoir $\alpha$. We adopt a model in the wide-band limit for the reservoirs. In this paper, the line width is given by $\Gamma = \pi n V^2$, where $V^2 = V_L^2 + V_R^2$ and $n$ is the density of states in the reservoirs.

In this paper, we consider geometrical pumping caused by an adiabatic modulation of the parameters. As stated, we adiabatically control $\mu^L$ and $\mu^R$ with the condition

$$\overline{\theta} := \overline{\mu^L} \text{ and } U(\theta), \text{ and fix the other parameters.}$$

The modulation of $U(\theta)$ is described as

$$U(\theta) = U_0 \lambda(\theta), \quad \lambda(\theta) = \theta + r_\lambda \cos \theta,$$

where we have assumed $|r_\lambda| \leq 1$.

The Anderson model for a quantum dot should have the following four states (corresponding to $n = 4$ in the previous section): doubly occupied, singly occupied with an up-spin, singly occupied with a down-spin and empty. Therefore, the density matrix should be expressed as a $4 \times 4$ matrix. As is shown in Ref. [63], however, the density matrix of the quantum master equation of the Anderson model within the wide-band approximation is reduced to a four-component matrix

$$\hat{\rho} = \begin{pmatrix} \rho_d & 0 & 0 & 0 \\ 0 & \rho_\uparrow & 0 & 0 \\ 0 & 0 & \rho_\downarrow & 0 \\ 0 & 0 & 0 & \rho_e \end{pmatrix},$$

where $\rho_d, \rho_\uparrow, \rho_\downarrow,$ and $\rho_e$ correspond to probabilities of the doubly occupied state, singly occupied state with up-spin, singly occupied state with down-spin, and empty state, respectively. This means that the model is not fully quantum-mechanical but quasi-classical.

Since Eq. (52) is diagonal, $|\hat{\rho}|$ also has only four components and the transition matrix $\hat{K}(\Lambda(\theta))$ in Eq. (3) in the wide band approximation is given by the $4 \times 4$ matrix (see Appendix C and Ref. [76] for the derivation)

$$\hat{K}(\Lambda(\theta)) = \begin{pmatrix} -2f^{(1)}_d & f^{(1)}_+ & f^{(1)}_- & 0 \\ f^{(1)}_- & -f^{(0)}_+ & f^{(1)}_+ & f^{(1)}_+ \\ -f^{(0)}_- & 0 & f^{(0)}_+ & 0 \\ f^{(0)}_+ & 0 & -f^{(0)}_+ & 2f^{(0)}_+ \end{pmatrix},$$

where we have introduced

$$f^{(j)}_d := f^{(j)}_L(\mu^L, U) + f^{(j)}_R(\mu^L, U)$$

$$f^{(j)}_0 := 2 - \{f^{(j)}_L(\mu^L, U) + f^{(j)}_R(\mu^L, U)\}$$

with the Fermi distribution

$$f^{(j)}_a(\mu^a(\theta), U(\theta)) := \frac{1}{1 + e^{2(\mu^a(\theta) - \mu^a(\theta))}},$$

in the lead $\alpha (= L \text{ or } R)$. Note that Eqs. (54) and (55) satisfy the relation

$$f^{(j)}_+ + f^{(j)}_- = 2$$

for $j = 0$ and 1.

It is straightforward to obtain the eigenvalues of $K(\Lambda(\theta))$ in Eq. (53) as

$$\varepsilon_0 = 0,$$

$$\varepsilon_1 = -(f^{(0)}_+ + f^{(1)}_-),$$

$$\varepsilon_2 = -(f^{(0)}_+ + f^{(1)}_+),$$

$$\varepsilon_3 = -4.$$
The left and right eigenfunctions corresponding to $\varepsilon_0 = 0$ in Eq. (58) are given by

$$\langle \ell_0 \rangle = (1, 1, 1, 1),$$

and

$$|r_0\rangle = \frac{1}{2(f^0_+ + f^{(1)}_+)} \begin{pmatrix} f^{(0)}_+ f^{(1)}_+ \\ f^{(0)}_+ f^{(1)}_+ \\ f^{(0)}_+ f^{(1)}_+ \\ f^{(0)}_+ f^{(1)}_+ \end{pmatrix},$$

respectively. Because of Eq. (5) there is the trivial relation $|r_0\rangle = |\hat{\rho}^{\text{ss}}\rangle$ for the diagonal element of the density matrix. Note that $|r_0\rangle$ satisfies $\langle \ell_0 | r_0 \rangle = \text{Tr} \hat{\rho}^{\text{ss}} = 1$ for Eq. (52). The left and right eigenfunctions corresponding to $\varepsilon_1$ in Eq. (59) are given by

$$\langle \ell_1 \rangle = 2 \left( f^{(1)}_-, \frac{-f^{(0)}_+ + f^{(1)}_+}{2}, \frac{-f^{(0)}_+ + f^{(1)}_+}{2}, \frac{-f^{(0)}_+ + f^{(1)}_+}{2} \right),$$

and

$$|r_1\rangle = \frac{1}{(f^{(0)}_+ + f^{(1)}_+)(f^{(0)}_+ + f^{(1)}_+)} \begin{pmatrix} f^{(1)}_+ \\ -f^{(0)}_+ + f^{(1)}_+ \\ -f^{(0)}_+ + f^{(1)}_+ \\ -f^{(0)}_+ + f^{(1)}_+ \end{pmatrix}.$$  

The left and right eigenfunctions corresponding to $\varepsilon_2$ in Eq. (60) are

$$\langle \ell_2 \rangle = 2(0, 1, -1, 0),$$

and

$$|r_2\rangle = \frac{1}{4} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

respectively.

The left and right eigenfunctions corresponding to $\varepsilon_3$ in Eq. (61) are

$$\langle \ell_3 \rangle = \left( f^{(0)}_- f^{(1)}_-, -f^{(0)}_+ f^{(1)}_+, -f^{(0)}_+ f^{(1)}_+, f^{(0)}_+ f^{(1)}_+ \right),$$

and

$$|r_3\rangle = \frac{1}{2(f^{(0)}_+ + f^{(1)}_+)} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$
where $\delta$ is the phase difference between the chemical potentials in the left and right reservoirs.

If we take $\delta \neq 0$, the chemical potential difference between two reservoirs remains finite. For the explicit calculation, we can set

$$r := r_L = r_R = r_T$$

for simplicity.

For the Anderson model, all the matrices involved are diagonalizable $4 \times 4$ matrices. Therefore, we can calculate the one-cycle average entropy production $\sigma^{(2)}$ and the square of the thermodynamic length $\mathcal{L}^2$ defined in Eq. (34) for the protocol in Eq. (73). We note that above the units have been non-dimensionalised and we take $\beta \epsilon_0 = \beta \mu = \beta U_0 = 0.1$. We plot $\sigma^{(2)}$ against $\delta$ for various $r$ in Fig. 3. As we can see, $\sigma^{(2)}$ increases as $r$ increases. We plot $\mathcal{L}^2$ and $\sigma^{(2)}$ as functions of the phase difference $\delta$ for $r = 0.5$ and $r = 0.9$ (see Fig. 4). Figure 4 is consistent with the inequality Eq. (33).

IV. DISCUSSION

So far, we have assumed the conditions $\lambda \geq 0$ and $\text{Tr}[\hat{\rho}(\theta)\partial \tilde{H}(\lambda(\theta))/\partial \lambda] \geq 0$ to ensure the positive semidefinite property of $\mathcal{W}$. This means that the system Hamiltonian is not perfectly periodic, i.e. $\tilde{H}(\lambda(\theta + 2\pi)) \neq \tilde{H}(\lambda(\theta))$ in our analysis. In this section, we mainly discuss what happens if we remove these requirements.

First, we discuss whether $\Delta S$ introduced in Eq. (17) can be used. Indeed, $\Delta S$ is directly connected to the entropy production in a one-cycle operation. If the conditions $\lambda \geq 0$ and $\text{Tr}[\hat{\rho}\partial \tilde{H}/\partial \lambda] \geq 0$ are satisfied, $\Delta S$ is given by

$$\Delta S = r^2 \left[ g_{\mu\nu}(0)\hat{A}_\mu(0)\hat{A}_\nu(0) - g_{\mu\nu}(2\pi)\hat{A}_\mu(2\pi)\hat{A}_\nu(2\pi) \right]$$

(77)

in the linear response regime is expected to be positive semidefinite, because $S^{KL}_{\mu\nu} \leq 0$ is always satisfied under any CPTP process [69, 71–74]. It is, however, sur-

The adiabatic work $\mathcal{W}$ defined in Eq. (35) can also be calculated as in Appendix D. The plot of $\beta \mathcal{W}$ versus $\delta$ for various $r$ is presented in Fig. 5. It is noteworthy that the adiabatic work takes a maximum at $\delta = \pi$, where the geometrical contribution from the BSN curvature does not exist. On the other hand, $\mathcal{W}$ take minimum at $\delta = 2n\pi$ with an integer $n$. The functional form $\beta \mathcal{W}$ can be fitted by a sinusoidal function of $\delta$.

The effective efficiency $\eta^{\text{eff}}$ defined in Eq. (45) as a function of $\delta$ for fixed $\epsilon = 0.1$ is presented in Fig. 6. Although $\eta^{\text{eff}}$ is close to unity for all $\delta$, the maximum value of $\eta^{\text{eff}}$ at $\delta = \pi$ remains less than unity for finite $\epsilon$. The functional form of $1 - \eta^{\text{eff}}$ can be also fitted by a sinusoidal function.

We also plot the two-dimensionalessential thermodynamic axial field $\vec{B}$ in the plane of $\mu^L/\mu$ and $\mu^R/\mu$ (see Fig. 7). For both cases, $\vec{B}$ flows in the direction of $\mu^L$ increases and $\mu^R$ decreases.
situation, it is an interesting one to consider.

Let us consider the case where we control one of the temperatures of the reservoirs, and use if \( \dot{S}^{\text{KL}} \) is negative semidefinite, as in Refs. [56, 58], we can use the dissipative availability \( A \) defined by

\[
A := - \int_0^{2\pi} d\theta \Theta(\theta) \dot{S}^{\text{KL}}(\theta),
\]

where \( \Theta(\theta) \) is the modulated temperature of one of the reservoirs satisfying \( \Theta = T \), as the basis of thermodynamic engine. However, as is shown in Fig. 8 with Eq. (77), we cannot ensure the positive semidefinite property of \( A \). Therefore, the problem of modulating the temperature is non-trivial. Analyzing this setup is one of our future tasks.

Finally, let us discuss the Cramér-Rao bound [64–66] which is associated with Fisher information. Actually, it is straightforward to derive the Cramér-Rao bound as

\[
\text{Tr}
\left[
\delta \Lambda \mu
\delta \Lambda \nu
\hat{\rho}^{\text{ss}}(\theta)
\right] \geq g_{\mu \nu}^{-1}
\]

with \( \delta \Lambda \mu := \Lambda \mu(\theta) - \Lambda \mu(\theta) \), but its physical meaning is unclear. To clarify the meaning of the bound will be a topic of our future research.

V. CONCLUDING REMARKS

In this paper, we successfully extended the geometrical thermodynamics formulated in Refs. [56–58] to a quantum system coupled to two reservoirs whose chemical potentials are slowly modulated. While our system is an example of adiabatic quantum (Thouless) pumping, the explicit calculation in Sec. III is still quasi-classical. In the adiabatic regime, the work is expressed as the line integral of the vector potential in the parameter space. On the other hand, the lower bound of the one-cycle entropy production can be written as the square of the thermodynamic length along the path. These results are unchanged from those for the geometrical thermodynamic engine obtained in Ref. [58]. We applied the formulation to the Anderson model within the wide-band approximation to obtain the explicit values of one-cycle averaged entropy production, thermodynamic length, adiabatic work, and effective efficiency. In contrast to Ref. [58], we have analyzed a pumping system under the isothermal condition. Therefore, our engine in which the geometrical contribution plays a dominant role is not a heat engine but a chemical engine.

Our future tasks are as follows: (i) To calculate the thermodynamic metric tensor or the vector potential, we need to know the explicit form of the steady-state of the quantum master equation. In other words, we cannot use our method in systems for which a steady solution is not explicitly known. Since we cannot obtain non-equilibrium steady states analytically in most cases, we need to extend our formulation to cases where the steady state is not explicitly known. (ii) When we consider a perfectly periodic case, the effective efficiency \( \eta^{\text{eff}} \) can
reach unity if we choose parameters satisfying $W \geq 0$. Of course, this setup is not a thermodynamic situation, but it might be possible to achieve this if we know the details of the properties of the engine a priori. The perfectly periodic situation will need to be analyzed in detail. Indeed, it is known that the energy fluctuation under unitary operations is subject to certain bounds [78], even though we ignore the energy fluctuations in our non-unitary dynamics. Including the control costs in our formulation and clarifying the role of such fluctuations will be a subject of our future work. As an alternative interpretation, an engine with efficiency equal to unity can be achieved in some setups such as the sensor-gate model [79] and the autonomous Maxwell's demon [80]. Therefore, we need to consider the implementation of real chemical engines reaching $\eta_{\text{eff}} \to 1$ in the future. (iii) Because the present method, at least, for the argument after Sec. II C, is restricted to the adiabatic case $\epsilon \to 0$, we will need to extend the analysis to the non-adiabatic regime of finite $\epsilon$. Reference [81] obtained the non-adiabatic solution of a classical master equation and a geometrical representation of the non-adiabatic current in a two-level system. We expect to apply these methods to investigate the non-adiabatic effect in heat engines. (iv) Although we have analyzed a quantum system, our treatment in Sec. III is still quasi-classical. Thus, we have so far been unable to clarify the role of quantum coherence. Reference [56] showed that quantum coherence reduces the performance of slowly driven heat engines. On the other hand, it was shown that coherence can enhance the performance of heat engines in Ref. [82]. Therefore, we will need to resolve the current confusing situation in which quantum coherence leads to the enhancement or reduction of efficiency by using a fully quantum mechanical model. (v) We assumed that the master equation (3) is still Markovian even though parameter modulation is present. However, this assumption seems to not be valid in general as indicated in Ref. [70]. This is important, because the modulation process becomes non-CPTP if we use the Markovian dynamics described by Eq. (3). Therefore, the effect of non-Markovianity arising from the modulation will need to be clarified.

ACKNOWLEDGEMENTS

The authors thank Yuki Hino for fruitful discussions. We also appreciate useful comments by Hiroyasu Tajima, Naoto Shiraishi and Kiyoshi Kanazawa. RY appreciates useful comments by Asahi Yamaguchi. This work is partially supported by a Grant-in-Aid of MEXT for Scien-
tific Research (Grant Nos. 16H04025 and 21H01006). The work of RY is supported by JSPS Grant-in-Aid for Scientific Research (KAKENHI Grant No. 19K14616 and 20H01838).

Appendix A: Positive semidefinite property of quantum Kullback-Leibler divergence

In this appendix, we prove the positive semidefinite property of the Kullback-Leibler (KL) divergence

\[ S^{KL}(\hat{\rho}|\hat{\sigma}) \geq 0. \tag{A1} \]

based on the description in Ref. [69]. See also Refs. [71–74].

Now, let us prove Eq. (A1). For this purpose, we introduce the Hilbert-Schmidt inner product defined as

\[ \langle \hat{Y}, \hat{X} \rangle_{HS} := \text{Tr}[\hat{Y}^\dagger \hat{X}]. \tag{A2} \]

We also introduce the left and right multiplications of \( \hat{\rho} \) as

\[ \mathcal{L}_{\hat{\rho}}(\hat{X}) := \hat{\rho} \hat{X}; \quad \mathcal{R}_{\hat{\rho}}(\hat{X}) := \hat{X} \hat{\rho}. \tag{A3} \]

Here, we assume that \( \mathcal{L}_{\hat{\rho}} \) and \( \mathcal{R}_{\hat{\rho}} \) are commutable and Hermitians with respect to the Hilbert-Schmidt inner product. Then, we introduce the modular operator

\[ \mathcal{D}_{\hat{\rho}, \hat{\sigma}}(\hat{X}) := \mathcal{L}_{\hat{\rho}} \mathcal{R}_{\hat{\sigma}^{-1}}(\hat{X}) = \hat{\rho} \hat{X} \hat{\sigma}^{-1}. \tag{A4} \]

Let \( \hat{\rho} = \sum_i p_i \hat{P}_i \) and \( \hat{\sigma} = \sum_j q_j \hat{Q}_j \) be the spectral decompositions, where \( \hat{P}_i \) and \( \hat{Q}_j \) are the projections onto the eigenspace with the assumption \( p_i \neq p_j \) and \( q_i \neq q_j \) for any pair of \( i \neq j \). Then, the spectral decomposition of \( \mathcal{D}_{\hat{\rho}, \hat{\sigma}} \) is expressed as

\[ \mathcal{D}_{\hat{\rho}, \hat{\sigma}} = \sum_{i,j} p_i \mathcal{P}_{ij}, \tag{A5} \]

where \( \mathcal{P}_{ij} \) is defined as \( \mathcal{P}_{ij} := \hat{P}_i \hat{Q}_j \).

Now, let us introduce a divergence-like quantity or Petz’s quasi-entropy [71, 72, 74]:

\[ D_f(\hat{\rho}||\hat{\sigma}) := \langle \hat{\sigma}^{1/2}, f(\mathcal{D}_{\hat{\rho}, \hat{\sigma}}) \hat{\sigma}^{1/2} \rangle_{HS} = \sum_{i,j} q_j f\left(\frac{p_i}{q_j}\right) \text{Tr}[\hat{P}_i \hat{Q}_j]. \tag{A6} \]

If we use \( f(x) = x \ln x \), with the aid of \( f(\mathcal{D}_{\hat{\rho}, \hat{\sigma}}) = \mathcal{L}_{\hat{\rho}} \mathcal{R}_{\hat{\sigma}^{-1}}(\ln \mathcal{L}_{\hat{\rho}} + \ln \mathcal{R}_{\hat{\sigma}^{-1}}) \) and the commutability of \( \mathcal{L}_{\hat{\rho}} \) and \( \mathcal{R}_{\hat{\sigma}^{-1}} \), we obtain

\[ f(\mathcal{D}_{\hat{\rho}, \hat{\sigma}})(\hat{X}) = (\hat{\rho} \ln \hat{\rho}) \hat{X} \hat{\sigma}^{-1} - \hat{\rho} \hat{X} (\hat{\sigma}^{-1} \ln \hat{\sigma}). \tag{A7} \]

This leads to

\[ D_f(\hat{\rho}||\hat{\sigma}) = \text{Tr}[\hat{\sigma}^{1/2} (\hat{\rho} \ln \hat{\rho}) \hat{\sigma}^{1/2} \hat{\sigma}^{-1}] - \text{Tr}[\hat{\sigma}^{1/2} \hat{\rho} \hat{\sigma}^{1/2} (\hat{\sigma}^{-1} \ln \hat{\sigma})] = S^{KL}(\hat{\rho}|\hat{\sigma}). \tag{A8} \]

Thus, KL divergence can be mapped onto the divergence-like quantity Eq. (A6).

Then, to prove the non-negativity of Eq. (A1) is sufficient to prove [69, 71, 72, 74]

\[ D_f(\hat{\rho}||\hat{\sigma}) \geq 0. \tag{A9} \]

The proof of Eq. (A9) is simple as follows. \( T_{ij} := \text{Tr}[\hat{P}_i \hat{Q}_j] \) introduced in Eq. (A6) is a doubly stochastic matrix, i.e., the stochastic matrix \( T_{ij} \) satisfies \( \sum_i T_{ij} = 1 \). Suppose \( h(x) \) is a convex function, \( T_{ij} \) and \( h(x) \) satisfy

\[ \sum_i h\left(\frac{p_i}{q_j}\right) T_{ij} \geq h\left(\frac{p'_i}{q'_j}\right), \tag{A10} \]

where \( p'_j := \sum_i p_i T_{ij} \). Thus, we obtain

\[ D_f(\hat{\rho}||\hat{\sigma}) \geq D^h(\rho' || q) := \sum_{i,j} q_i h\left(\frac{p'_i}{q_i}\right). \tag{A11} \]

Using the convexity of \( h(x) \), \( D_f(\rho' || q) \) satisfies the relation \( D_f(\rho' || q) \geq h(\sum_i q_i (p_i(q_i))) = h(1) \). Equality, i.e., \( D_f(\hat{\rho}||\hat{\sigma}) = h(1) \), is held if and only if \( \hat{\sigma} = \hat{\rho} \). If we choose \( h(x) = x \ln x \), which is one of convex functions with \( h(1) = 0 \), we reach Eq. (A1).

Appendix B: Some detailed properties of general framework

In this appendix, we explain some general properties of the quantum master equation such as the outline of the perturbation method with a slowly modulated parameter in Appendix B 1 and the mathematical description of pseudo-inverse of transition matrix in Appendix B 2.

1. Slow-driving perturbation

In this subsection, we explain the outline of the perturbation theory of the quantum master equation with a slowly modulated parameter \( \epsilon \) [8]. First, we expand the solution of Eq. (3) in terms of \( \epsilon \) as

\[ |\hat{\rho}(\theta)\rangle = \sum_{n=0}^{\infty} \epsilon^n |\hat{\rho}^{(n)}(\Lambda(\theta))\rangle \tag{B1} \]

with \( |\hat{\rho}^{(0)}\rangle = |\hat{\rho}^{ss}\rangle \) as in Eq. (22). Since the normalization condition \( \text{Tr}[\hat{\rho}(\theta)] = 1 \) holds for any \( \epsilon \), \( |\hat{\rho}^{(n)}(\Lambda(\theta))\rangle \) satisfies

\[ \text{Tr}[\hat{\rho}^{ss}(\Lambda(\theta))] = 1, \tag{B2} \]

\[ \text{Tr}[\hat{\rho}^{(n)}(\Lambda(\theta))] = 0 \tag{B3} \]

for \( n \geq 1 \). Substituting these into Eq. (3), we obtain Eq. (5) and

\[ \hat{K}(\Lambda(\theta))|\hat{\rho}^{(n)}(\Lambda(\theta))\rangle = \frac{d}{d\theta} |\hat{\rho}^{(n-1)}(\Lambda(\theta))\rangle \tag{B4} \]

where
for $n \geq 1$, By using the pseudo-inverse $\hat{K}^+(\Lambda(\theta))$ of $\hat{K}(\Lambda(\theta))$, Eq. (B4) can be written as

$$|\hat{\rho}^{(n)}(\Lambda(\theta))\rangle = \hat{K}^+(\Lambda(\theta))\frac{d}{d\theta}|\hat{\rho}^{(n-1)}(\Lambda(\theta))\rangle = \left(\hat{K}^+(\Lambda(\theta))\frac{d}{d\theta}\right)^n|\hat{\rho}^{ss}(\Lambda(\theta))\rangle.$$  \hspace{1cm} (B5)

Ignoring terms of $O(\epsilon^2)$ and higher in Eq. (B1), we obtain Eq. (23) of the main text.

2. Pseudo-inverse of the transition matrix

In this subsection, we introduce the pseudo-inverse $\hat{K}^+(\Lambda)$ of $\hat{K}(\Lambda)$, which satisfies following conditions [77, 84]

$$\hat{K}^+(\Lambda)\hat{K}(\Lambda) = \hat{K}(\Lambda)\hat{K}^+(\Lambda) = 1 - |\hat{\rho}^{ss}(\Lambda)\rangle\langle 1|,$$ \hspace{1cm} (B6)

$$\hat{K}^+(\Lambda)|\hat{\rho}^{ss}(\Lambda)\rangle = 0,$$ \hspace{1cm} (B7)

$$\langle 1|\hat{K}^+(\Lambda) = 0.$$ \hspace{1cm} (B8)

Here, the eigenvalue equations are given by

$$\langle \ell_m(\Lambda)|\hat{K}(\Lambda) = \varepsilon_m(\Lambda)\langle \ell_m|$$ \hspace{1cm} (B9)

$$\hat{K}(\Lambda)|r_m(\Lambda)\rangle = \varepsilon_m(\Lambda)|r_m(\Lambda)\rangle.$$ \hspace{1cm} (B10)

We note that $\varepsilon_0(\Lambda) = 0$, then $|r_0(\Lambda)\rangle = |\hat{\rho}^{ss}(\Lambda)\rangle$ and $\langle \ell_0(\Lambda)| = \langle 1|$. For simplicity, we assume that these eigenstates do not degenerate throughout this paper. Thus, we need to solve the eigenvalue problem (B9) or Eq. (B10) to express the pseudo-inverse operator.

The definition of $\hat{K}^+$ in Eq. (26) satisfies the requirements of Eqs. (B6)-(B8). Indeed, using Eqs. (25) and (26) we obtain

$$\hat{K}^+ = \sum_{m \neq 0} \sum_n \frac{\varepsilon_n}{\varepsilon_m}|r_m\rangle\langle \ell_m|r_n\rangle\langle \ell_n|$$

$$= \sum_{m \neq 0} \sum_n \frac{\varepsilon_n}{\varepsilon_m}|r_m\rangle\langle \ell_n|\delta_{mn}$$

$$= \sum_n |r_n\rangle\langle \ell_n| + |r_0\rangle\langle \ell_0| - |\hat{\rho}^{ss}\rangle\langle 1|$$

$$= 1 - |\hat{\rho}^{ss}\rangle\langle 1|,$$ \hspace{1cm} (B11)

where we have used $\varepsilon_0 = 0$ in the second line, $|r_0\rangle = |\hat{\rho}^{ss}\rangle$ and $\langle \ell_0| = \langle 1|$ in the third line, and $\sum_n |r_n\rangle\langle \ell_n| = 1$ in the last expression. The second expression of Eq. (B6) can be obtained by the parallel calculation to Eq. (B11). Equation (B7) is the definition of the right zero eigenvector $|r_0\rangle = |\hat{\rho}^{ss}\rangle$. The proof of Eq. (B8) is also straightforward. Indeed, substituting Eq. (26) into the left hand side of Eq. (B8) we can write

$$\langle 1|\hat{K}^+ = \sum_{m \neq 0} \frac{1}{\varepsilon_m}(1|\hat{r}_m\langle \ell_m| = 0,$$ \hspace{1cm} (B12)

where we have used the orthogonal relation $\langle \ell_0|\hat{r}_m\rangle := (1|\hat{r}_m) = 0$ for $m \neq 0$. Thus, Eq. (26) satisfies all requirements of the pseudo-inverse.

With the aid of Eq. (B8) it is straightforward to obtain

$$\text{Tr}[\hat{K}^+\hat{A}] = \langle 1|\hat{K}^+\hat{A}|1\rangle = 0$$ \hspace{1cm} (B13)

for an arbitrary matrix $\hat{A}$. Thus, we have the relation

$$\text{Tr}[\partial_{\alpha}\partial_{\beta}\hat{\rho}^{ss}] = \text{Tr}\left[\hat{K}^+\frac{\partial}{\partial A_{\mu}} \left(\hat{K}^+\frac{\partial}{\partial A_{\nu}}\hat{\rho}^{ss}\right)\right] = 0.$$ \hspace{1cm} (B14)

Multiplying $\hat{\rho}^{ss}$ on the both side of the relation

$$\partial_{\mu}\partial_{\nu}\ln\hat{\rho}^{ss} = (\hat{\rho}^{ss})^{-1}\partial_{\mu}\partial_{\nu}\hat{\rho}^{ss} - \partial_{\mu}\ln\hat{\rho}^{ss} \cdot \partial_{\nu}\ln\hat{\rho}^{ss}$$ \hspace{1cm} (B15)

and take the trace with the aid of Eq. (B14) we obtain

$$\text{Tr}[\hat{\rho}^{ss}\partial_{\mu}\partial_{\nu}\ln\hat{\rho}^{ss}] = -\text{Tr}[\hat{\rho}^{ss}\partial_{\mu}\ln\hat{\rho}^{ss} \cdot \partial_{\nu}\ln\hat{\rho}^{ss}].$$ \hspace{1cm} (B16)

Substituting Eqs. (59), (60), (61), (64), (65), (66), (67), (68), (69) into Eq. (26) we obtain

$$\hat{K}^+ = C\begin{pmatrix}
  f_{a_1}^{(1)} & f_{a_2}^{(1)} & f_{a_2}^{(1)} & f_{a_2}^{(0)} f_{a_3}^{(1)} \\
  f_{a_1}^{(1)} & a_4 & a_5 & f_{a_6}^{(0)} \\
  f_{a_2}^{(1)} & a_5 & a_4 & f_{a_6}^{(0)} \\
  f_{a_3}^{(1)} & f_{a_6}^{(0)} & f_{a_6}^{(0)} & f_{a_1}^{(0)}
\end{pmatrix},$$ \hspace{1cm} (B17)
where

\[
\mathcal{C} = \frac{1}{8(f_+^{(0)} + f_-^{(1)})^2(f_-^{(0)} + f_+^{(1)})},
\]

\[
a_1 = -f_-^{(0)}(f_+^{(0)} + f_-^{(1)})^2 - 16f_+^{(1)},
\]

\[
a_2 = f_-^{(0)}(f_+^{(0)} + f_-^{(1)})^2 + 8(f_-^{(0)} - f_-^{(1)}),
\]

\[
a_3 = 16 - (f_-^{(0)} + f_+^{(1)})^2,
\]

\[
a_4 = 8[(f_-^{(0)})^2 + (f_-^{(1)})^2] - f_-^{(0)}f_+^{(1)}(f_+^{(0)} + f_-^{(1)})^2,
\]

\[
a_5 = 16f_+^{(1)}f_-^{(0)} - f_-^{(0)}f_+^{(1)}(f_-^{(1)} + f_+^{(0)}),
\]

\[
a_6 = 8(f_-^{(1)} - f_+^{(0)}) + f_+^{(1)}(f_-^{(1)} + f_+^{(0)})^2.
\]  

We can easily check that the form of the pseudo-inverse in Eq. (26) satisfies the above conditions (B6) - (B8).

### Appendix C: The derivation of the transition matrix

In this appendix, we present the details of the derivation of the transition matrix which appears in Eq. (3). This appendix consists of two parts. In the first part, we explain the outline of the derivation of the quantum master equation. In the second part, we derive the transition matrix in Eq. (3).

#### 1. Quantum master equation for the Anderson model

In this subsection we explain the outline of the derivation of the quantum master equation for the Anderson model. In this part, we use the time \( t \) instead of using the phase \( \theta \) because the system may not relax to a steady state if the time is not long enough. Before moving to the detail, we mention that the double bracket notation is used for the super-vector. The super-vectors appearing in this part corresponds to those in the main text as follows.

\[
|e, e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow, \uparrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
\]

\[
|\downarrow, \downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |d, d\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]  

As is assumed, the density matrix of the total system \( \hat{\rho}_{\text{tot}}^{\text{t}} \) is decomposed into the matrix of system \( \hat{\rho} \) and the matrix of the bath in thermal equilibrium \( \hat{\rho}^\circ \) at initial time \( t_0 \),

\[
\hat{\rho}_{\text{tot}}^{\text{t}}(t_0) = \hat{\rho}(t_0) \otimes \hat{\rho}^\circ.
\]  

The time evolution of \( \hat{\rho}_{\text{tot}}^{\text{t}}(t) \) is described by Liouvillian as

\[
\frac{d}{dt} \hat{\rho}_{\text{tot}}^{\text{t}}(t) = \hat{\mathcal{K}}^{\text{tot}} \hat{\rho}_{\text{tot}}^{\text{t}}(t).
\]  

As is the case of Hamiltonian (48), the Liouvillian \( \hat{\mathcal{K}}^{\text{tot}} \) for the total system can be decomposed into \( \hat{\mathcal{K}}, \hat{\mathcal{K}}^\circ \), and \( \hat{\mathcal{K}}^\text{int} \). As can be seen from the form of time evolution equation, Dyson’s equation in quantum system can be used. By the Laplace transformation of \( \hat{\rho}_{\text{tot}}^{\text{t}}(t) \)

\[
\hat{\rho}_{\text{tot}}^{\text{t}}(z) = \int_{t_0}^{\infty} dt e^{-z(t-t_0)} \hat{\rho}_{\text{tot}}^{\text{t}}(t),
\]  

Eq. (C3) becomes

\[
\hat{\rho}_{\text{tot}}^{\text{t}}(z) = \frac{1}{z - \hat{\mathcal{K}}^{\text{tot}}(t_0)}
\]

\[
= (G^0(z) + G^0(z)\hat{\mathcal{K}}^{\text{int}}G^0(z) + \cdots)\hat{\rho}_{\text{tot}}^{\text{t}}(t_0),
\]

where \( G^0(z) = (z - \hat{\mathcal{K}} - \hat{\mathcal{K}}^\circ)^{-1} \). From Eqs. (C2) and (C5), the reduced density matrix which is obtained by tracing out the bath degrees of freedom becomes

\[
\hat{\rho}(z) = \text{Tr}\left[ (G^0(z) + G^0(z)\hat{\mathcal{K}}^{\text{int}}G^0(z)\hat{\mathcal{K}}^{\text{int}}G^0(z)) \hat{\rho}(t_0) \otimes \hat{\rho}^\circ \right],
\]

in the second order of \( \hat{\mathcal{K}}^{\text{int}} \). It can be shown that the term linear to \( \hat{\mathcal{K}}^{\text{int}} \) vanishes. By using \( \hat{\mathcal{K}}^\circ \hat{\rho}^\circ = 0 \), the first term of RHS of Eq. (C6) can be rewritten as

\[
\text{Tr}\left[ (G^0(z)\hat{\rho}(t_0) \otimes \hat{\rho}^\circ) \right] = G^0_z(z)\hat{\rho}(t_0),
\]

where \( G^0_z(z) = (z - \hat{\mathcal{K}})^{-1} \). Now we define the effective Liouvillian \( \hat{\mathcal{K}}^{\text{eff}} \), which describes the time evolution of \( \hat{\rho} \) as

\[
\hat{\rho}(z) = \frac{1}{z - \hat{\mathcal{K}}^{\text{eff}}(t_0)}\hat{\rho}(t_0).
\]  

This is equivalent to the following time evolution equation

\[
\frac{d}{dt} \hat{\rho}(t) = \int_{t_0}^{t} d\tau \hat{\mathcal{K}}^{\text{eff}}(t - \tau)\hat{\rho}(\tau).
\]
Here we can see the non-Markovian memory effect. By decomposing $\hat{K}_{\text{eff}}(z)$ into the free part $\hat{K}$ and the “self energy part” $\Sigma$ as $\hat{K}_{\text{eff}} = \hat{K} + \hat{\Sigma}(z)$, it becomes more clear that the memory effect is induced by the interaction. By expanding (C8) in $\hat{K}_{\text{eff}}(z)$, we obtain

$$\hat{\rho}(z) = \left( G_0^a(z) + G_0^b(z) \hat{\Sigma}(z) G_0^a(z) + \cdots \right) \hat{\rho}(t_0). \tag{C10}$$

From Eqs. (C6), (C7), and (C10), we can easily see that the state $\hat{\Sigma}$ is decomposed as

$$\hat{\Sigma}(z) = \sum_{c, c', \xi, \sigma = \pm} cc' J_{\xi, \sigma}^c |aa'\rangle \langle aa'| J_{\xi, \sigma}^c I(\xi, c, a, a'),$$

where $|a, b\rangle = |a\rangle \langle b|$ is the two state vector and $J_{\xi, \sigma}^c$ is the ladder operators (for the case of $\downarrow$, the definition is the same) defined as

$$J_{\uparrow, \downarrow}^c = \sum_{a = \uparrow, \downarrow, d} \langle \uparrow, a| \langle \uparrow, a| \rightarrow \downarrow, a \rangle \langle \downarrow, a|, \tag{C12}$$

$$J_{\downarrow, \uparrow}^c = \sum_{a = \uparrow, \downarrow, d} \langle \downarrow, a| \langle \downarrow, a| \rightarrow \uparrow, a \rangle \langle \uparrow, a|, \tag{C13}$$

$$J_{\uparrow, \uparrow}^c = \sum_{a = \uparrow, \downarrow, d} \langle \uparrow, a| \langle \uparrow, a| \rightarrow \downarrow, a \rangle \langle \downarrow, a|, \tag{C14}$$

$$J_{\downarrow, \downarrow}^c = \sum_{a = \uparrow, \downarrow, d} \langle \downarrow, a| \langle \downarrow, a| \rightarrow \uparrow, a \rangle \langle \uparrow, a|, \tag{C15}$$

and $I(\xi, c, a, a')$ is given by

$$I = V^2 \sum_k f_{-\xi \epsilon(\omega)}(\omega_k) \frac{1}{z + i\omega + i\Delta_{a, a'}}. \tag{C16}$$

where the line width $\Gamma$ is defined as $\Gamma = \pi n V^2$ with the density of states in reservoirs $n$.

Next we make an assumption which corresponds to neglect the memory effect in Eq. (C9). This is valid when the time scale of the dynamics of the system is much larger than that of bath. By taking the long-time limit $z \rightarrow +0$, we can use the Sokhotski-Plemelj relation

$$\lim_{\eta \rightarrow +0} (\omega + i\eta)^{-1} = -i\pi \delta(\omega) + P\omega^{-1}. \tag{C17}$$

Assuming the wide band limit $D \rightarrow \infty$, the imaginary part of $\Sigma$ can be negligible and thus, we obtain

$$\hat{\Sigma}(+0) = -\frac{1}{2} \sum_{c, c', \xi, \sigma} cc' J_{\xi, \sigma}^c |aa'\rangle \langle aa'| J_{\xi, \sigma}^c f_a^{+\xi}(-\xi \Delta_{aa'}). \tag{C18}$$

By replacing $\hat{\Sigma}(z) \rightarrow \hat{\Sigma}(+0)$ (long-time limit), Eq. (C9) can be rewritten as

$$\frac{d}{dt} \hat{\rho}(t) = \hat{K}_{\text{eff}}(z = +0) \hat{\rho}(t). \tag{C19}$$

By using this method, we arrive at Eq. (3) by setting $\hat{K}_{\text{eff}}(z = +0) = 2\pi \Gamma \hat{K}$ and $\theta = 2\pi (t - t_0)/\tau_p$ where $t_0$ and $\tau_p$ are the time to reach the periodic state and the period, respectively.

2. The derivation of the transition matrix $\hat{K}$ in Eq. (3)

As is explained in the previous subsection, the transition matrix $\hat{K} = \hat{K}_{\text{eff}}$ is given by

$$\hat{K} = -\frac{1}{2} \sum_{c, c', \xi, \sigma} cc' J_{\xi, \sigma}^c |aa'\rangle \langle aa'| J_{\xi, \sigma}^c f_a^{+\xi}(-\xi \Delta_{aa'}). \tag{C20}$$

where $\alpha = \text{L or R}$, $a$ and $a'$ can take the state of $e$, $\uparrow, \downarrow$ or $d$, $e = \pm 1$, $\xi = \pm 1$, and $\sigma$ is the spin taking the values $\sigma = \uparrow$ and $\downarrow$. All the subscriptions and superscriptions are summed up. Here $J_{\xi, \sigma}$ is the ladder operator given by Eqs. (C12)-(C15).

2.a. Calculation for $\langle i, i| \hat{K}| j, j \rangle$

In this part, we write the result of the operation $\hat{K}$ onto the super-vector $|e, e\rangle$, $|\uparrow, \uparrow\rangle$, $|\downarrow, \downarrow\rangle$, and $|d, d\rangle$ as

$$\hat{K}|e, e\rangle = -2G f_+(e_0)|e, e\rangle + G f_+(e_0)|\uparrow, \uparrow\rangle + G f_+(e_0)|\downarrow, \downarrow\rangle, \tag{C21}$$

$$\hat{K}|\uparrow, \uparrow\rangle = G f_-(e_0)|e, e\rangle + G f_+(e_0 + U)|d, d\rangle - G f_+(e_0) + f_+(e_0 + U)|\uparrow, \uparrow\rangle, \tag{C22}$$

$$\hat{K}|\downarrow, \downarrow\rangle = G f_-(e_0)|e, e\rangle + G f_+(e_0 + U)|d, d\rangle - G f_+(e_0) + f_+(e_0 + U)|\downarrow, \downarrow\rangle, \tag{C23}$$

$$\hat{K}|d, d\rangle = -2G f_-(e_0 + U)|d, d\rangle + G f_-(e_0 + U)|\uparrow, \uparrow\rangle + G f_-(e_0 + U)|\downarrow, \downarrow\rangle. \tag{C24}$$
where we use $\Delta_{\alpha,\alpha'} = -\Delta_{\alpha',\alpha} = \varepsilon_0$. These results yields the each components of $K$-matrix as:

$$
\begin{pmatrix}
\langle \langle d, d | \mathbf{K} | d, d \rangle \rangle & \langle \langle d, d | \mathbf{K} | \uparrow, \uparrow \rangle \rangle & \langle \langle d, d | \mathbf{K} | \downarrow, \downarrow \rangle \rangle & \langle \langle d, d | \mathbf{K} | e, e \rangle \rangle \\
\langle \langle \uparrow, \uparrow | \mathbf{K} | d, d \rangle \rangle & \langle \langle \uparrow, \uparrow | \mathbf{K} | \uparrow, \uparrow \rangle \rangle & \langle \langle \uparrow, \uparrow | \mathbf{K} | \downarrow, \downarrow \rangle \rangle & \langle \langle \uparrow, \uparrow | \mathbf{K} | e, e \rangle \rangle \\
\langle \langle \downarrow, \downarrow | \mathbf{K} | d, d \rangle \rangle & \langle \langle \downarrow, \downarrow | \mathbf{K} | \uparrow, \uparrow \rangle \rangle & \langle \langle \downarrow, \downarrow | \mathbf{K} | \downarrow, \downarrow \rangle \rangle & \langle \langle \downarrow, \downarrow | \mathbf{K} | e, e \rangle \rangle \\
\langle \langle e, e | \mathbf{K} | d, d \rangle \rangle & \langle \langle e, e | \mathbf{K} | \uparrow, \uparrow \rangle \rangle & \langle \langle e, e | \mathbf{K} | \downarrow, \downarrow \rangle \rangle & \langle \langle e, e | \mathbf{K} | e, e \rangle \rangle
\end{pmatrix}
= \Gamma \begin{pmatrix}
-2f_-(\varepsilon_0 + U) & f_+(\varepsilon_0 + U) & 0 & f_+(\varepsilon_0) \\
F_-(\varepsilon_0 + U) & -f_-(\varepsilon_0) - f_+(\varepsilon_0 + U) & 0 & f_+(\varepsilon_0) \\
0 & f_-(\varepsilon_0) & 0 & f_+(\varepsilon_0 + U) \\
0 & f_-(\varepsilon_0) & 0 & -2f_+(\varepsilon_0)
\end{pmatrix}.
$$

(C25)

This is identical to Eq. (53).

**Appendix D: Some detailed properties for the Anderson model**

The system Hamiltonian is expressed as the matrix form:

$$
\hat{H} = \begin{pmatrix}
U(\theta) & 0 & 0 & 0 \\
0 & \varepsilon_0 & 0 & 0 \\
0 & 0 & \varepsilon_0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

(D1)

With the aid of Eqs. (51), (52) and (D1) we obtain

$$
\mathcal{F} := \text{Tr} \left[ \frac{\partial \hat{H}}{\partial \lambda} \rho_{\text{eq}} \right] = U_0 E_4.
$$

(D2)

Substituting the first component of (63) into Eq. (D2) we can rewrite it as

$$
\mathcal{F} = \frac{U_0 f_+(\varepsilon_0)}{2[f_+^2(\varepsilon_0) + f_-^2(\varepsilon_0)]}.
$$

(D3)

Note that $\mathcal{F}$ does not depend on $f^{(1)}$ explicitly. Therefore we can write each component of the vector potential $A_\mu$ in Eq. (36) as

$$
A_0 = \lambda \left( \frac{\partial \mathcal{F}}{\partial f_+^{(1)}} - \frac{\partial \mathcal{F}}{\partial f_-^{(1)}} \right) \frac{\partial f_+^{(1)}}{\partial \lambda},
$$

$$
A_1 = \lambda \left( \frac{\partial \mathcal{F}}{\partial f_+^{(1)}} - \frac{\partial \mathcal{F}}{\partial f_-^{(1)}} \right) \frac{\partial f_+^{(1)}}{\partial \mu_L} + \frac{\partial \mathcal{F}}{\partial f_-^{(1)}} \frac{\partial f_-^{(1)}}{\partial \mu_L},
$$

$$
A_2 = \lambda \left( \frac{\partial \mathcal{F}}{\partial f_+^{(1)}} - \frac{\partial \mathcal{F}}{\partial f_-^{(1)}} \right) \frac{\partial f_+^{(1)}}{\partial \mu_R} + \frac{\partial \mathcal{F}}{\partial f_-^{(1)}} \frac{\partial f_-^{(1)}}{\partial \mu_R},
$$

where we have used

$$
\frac{\partial f_+^{(1)}}{\partial \lambda} = -\beta U_0 e^{\beta(\varepsilon_0 + U_0 \lambda(\theta))} \left[ \frac{e^{-\beta \mu_L}}{\{1 + e^{\beta(\varepsilon_0 + U_0 \lambda(\theta) - \mu_L(\theta))}\}^2} + \frac{e^{-\beta \mu_R}}{\{1 + e^{\beta(\varepsilon_0 - \mu_R(\theta))}\}^2} \right],
$$

(D7)
\[
\frac{\partial f_+^{(0)}}{\partial \mu^{\alpha}} = \frac{\beta \mu^{\alpha} (e^{\mu^\alpha} - \mu^\alpha)}{1 + e^{\beta (\mu^\alpha - \mu^\alpha)}}, \quad (D8)
\]
\[
\frac{\partial f_+^{(1)}}{\partial \mu^{\alpha}} = \frac{\beta \mu^{\alpha} (e^{\mu^\alpha} + U_0 \lambda (\mu^\alpha - \mu^\alpha))}{1 + e^{\beta (\mu^\alpha + U_0 \lambda (\mu^\alpha - \mu^\alpha))}}, \quad (D9)
\]
\[
\frac{\partial \mathcal{F}}{\partial f_+^{(0)}} = \frac{(2-f_+^0)U_0 f_+^{(0)}}{2(2+f_+^0-f_+^1)^2}, \quad (D10)
\]
\[
\frac{\partial \mathcal{F}}{\partial f_+^{(1)}} = \frac{U_0 f_+^{(0)}}{2} \frac{2+f_+^1}{(2+f_+^0-f_+^1)^2}. \quad (D11)
\]

Substituting Eqs. (D4), (D5) and (D6) into Eq. (35) we obtain the expression of the work.

**Appendix E: Perturbation calculation of Anderson model in high temperature and weak barrier \(\beta U_0 \ll 1\) limit**

1. **Framework of the perturbation method**

Now, let us solve eigenvalue problem given by Eqs. (B9) or (B10) in the high temperature (or classical) limit. In this case, \(\hat{K}(\Lambda(\theta))\) in Eq. (53) can be expanded in terms of \(\beta := \beta U\) as

\[
\hat{K} = \hat{K}^{(0)} + \beta \hat{K}^{(1)} + O(\beta^2),
\]

(E1)

where \(\hat{K}^{(0)} := \hat{K}_{\beta=0}\) is given by

\[
\hat{K}^{(0)} = \begin{pmatrix}
-2g_+ & g_+ & g_+ & 0 \\
-g_+ & -g_+ & g_+ & 0 \\
g_+ & 0 & -g_+ & -g_+ \\
0 & g_+ & 0 & -2g_+
\end{pmatrix}
\]

(E2)

where \(g_+ := f_+^{(0)} = \lim_{\beta \to 0} f_+^{(1)}\). \(\hat{K}^{(1)}\) in Eq. (E1) is given by

\[
\hat{K}^{(1)} := \left. \frac{\partial \hat{K}}{\partial \beta} \right|_{\beta=0} = \begin{pmatrix}
-2g_+ & \hat{g}_+ & \hat{g}_+ & 0 \\
\hat{g}_+ & -g_+ & \hat{g}_+ & 0 \\
\hat{g}_+ & 0 & -g_+ & -\hat{g}_+ \\
0 & \hat{g}_+ & 0 & 0
\end{pmatrix},
\]

(E3)

where \(\hat{g}_+ := (\partial f_+^{(1)}/\partial \beta)_{\beta=0}\).

Then, we can adopt the perturbation method to solve the eigenvalue problem Eqs. (B9) or (B10). The eigenequation Eqs. (B9) and (B10) can be rewritten as

\[
\langle \ell_m | \hat{K} = \left( \langle \ell_m^{(0)} | + \beta \langle \ell_m^{(1)} | \right) (\hat{K}^{(0)} + \beta \hat{K}^{(1)}),
\]

(E4)

\[
\hat{K} | r_m \rangle = \left( \hat{K}^{(0)} + \beta \hat{K}^{(1)} \right) \left( | r_m^{(0)} \rangle + \beta | r_m^{(1)} \rangle \right),
\]

(E5)

\[
\varepsilon_m | r_m \rangle = \left( \varepsilon_m^{(0)} + \beta \varepsilon_m^{(1)} \right) \left( | r_m^{(0)} \rangle + \beta | r_m^{(1)} \rangle \right),
\]

(E6)

2. **Properties of the unperturbed state**

Now, let us develop the perturbation of the eigenvalue equations Eqs. (B9) and (B10) for \(\beta \ll 1\).

Since \(\hat{K}^{(0)}\) can be decomposed into the spin \(\uparrow\) space and spin \(\downarrow\) space

\[
\hat{K}^{(0)} = \hat{K}_\uparrow^{(0)} + \hat{K}_\downarrow^{(0)}
\]

(E7)

with

\[
\hat{K}_\uparrow^{(0)} := \begin{pmatrix}
-g_+ & 0 & g_+ & 0 \\
0 & -g_+ & 0 & g_+ \\
g_+ & 0 & -g_+ & 0 \\
0 & g_+ & 0 & -g_+
\end{pmatrix}
\]

(E8)

and

\[
\hat{K}_\downarrow^{(0)} := \begin{pmatrix}
-g_+ & g_+ & 0 & 0 \\
g_+ & -g_+ & 0 & 0 \\
0 & 0 & -g_+ & g_+ \\
0 & 0 & g_+ & -g_+
\end{pmatrix}
\]

(E9)

where we have introduced

\[
\hat{K}_0 := \begin{pmatrix}
-g_+ & g_+ \\
g_+ & -g_+
\end{pmatrix},
\]

(E10)

the properties of unperturbed state are determined by the eigenvalue problem of \(K_0\).

Their corresponding eigenvalues are, respectively, given by

\[
\varepsilon_0 = 0, \quad \varepsilon_1 = -2,
\]

(E11)

It should be noted that the left and right eigenvectors \((\ell_0)\) and \(| r_0 \rangle\) corresponding to \(\varepsilon_0 = 0\) are, respectively, given by

\[
| 0 \rangle = (1,1), \quad | 0 \rangle = \frac{1}{2} \begin{pmatrix}
g_+ \\
g_-
\end{pmatrix}.
\]

(E12)

The left and right eigenvectors corresponding to \(\varepsilon_1 = -2\) are given by

\[
| 1 \rangle = (g_-, g_+), \quad | 1 \rangle = \frac{1}{2} \begin{pmatrix}
1 \\
-1
\end{pmatrix},
\]

(E13)

respectively. It is easy to check the orthonormal relation \(\langle i | j \rangle = \delta_{ij}\) for \(i, j = 0\) and 1. It is also easy to check the completeness:

\[
| 0 \rangle \langle 0 | + | 1 \rangle \langle 1 | = \frac{1}{2} \begin{pmatrix}
g_+ & g_- \\
g_- & g_+
\end{pmatrix} (1,1) + \frac{1}{2} \begin{pmatrix}
1 \\
-1
\end{pmatrix} (g_-, g_+)
\]

(E14)
where we have used Eq. (57).

The full eigenstates of $\hat{K}^{(0)}$ can be constructed by the Kronecker products of eigenstates for the subspace. The left eigenstates are

$$
\langle \ell_0^{(0)} | = \langle e | := | 0 \rangle \otimes | 0 \rangle,
\langle \ell_1^{(0)} | = \langle \uparrow | := | 1 \rangle \otimes | 0 \rangle,
\langle \ell_2^{(0)} | = \langle \downarrow | := | 0 \rangle \otimes | 1 \rangle,
\langle \ell_3^{(0)} | = \langle d | := | 1 \rangle \otimes | 1 \rangle,
$$

(E15)

and the right eigenstates are

$$
|r_0^{(0)} \rangle = | e \rangle := | 0 \rangle \otimes | 0 \rangle,
|r_1^{(0)} \rangle = | \uparrow \rangle := | 1 \rangle \otimes | 0 \rangle,
|r_2^{(0)} \rangle = | \downarrow \rangle := | 0 \rangle \otimes | 1 \rangle,
|r_3^{(0)} \rangle = | d \rangle := | 1 \rangle \otimes | 1 \rangle.
$$

(E16)

These eigenstates satisfy orthonormal property, i.e.,

$$
\langle r_m^{(0)} | r_n^{(0)} \rangle = \delta_{mn}.
$$

The eigenvalues corresponding to $\langle \ell_0^{(0)} |$ and $| r_m^{(0)} \rangle$ are, respectively, given by

$$
\varepsilon_0^{(0)} = 2\varepsilon_0 = 0,
\varepsilon_1^{(0)} = \varepsilon_0 + \varepsilon_1 = -2,
\varepsilon_2^{(0)} = \varepsilon_1 + \varepsilon_0 = -2
$$

(E17)

$$
\varepsilon_3^{(0)} = 2\varepsilon_1 = -4.
$$

(E18)

These eigenstates correspond to the empty ($m = 0$), single occupied by the upspin ($m = 1$), single occupied by the downspin ($m = 2$) and double occupied ($m = 3$) states, respectively.

3. Perturbed states

a. $K^{(1)}$ and basis for the degenerated system

Similarly, $K^{(1)}$ in Eq. (3) can be rewritten as

$$
\hat{K}^{(1)} = \hat{g}_- \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \hat{g}_- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},
$$

(E21)

where we have used $\hat{g}_+ = -\hat{g}_- \text{ derived from } \partial (g^{(1)}_+ + g^{(1)}_-)/\partial \beta = 0$ because of Eq. (57).

Before moving the detailed calculation, we rearrange the eigenbasis for the unperturbed states, because $| \uparrow \rangle$ and $| \downarrow \rangle$ are degenerated. With the aid of Eqs. (E12), (E13) and (E16), one can check that $K^{(1)}$ satisfies

$$
\hat{K}^{(1)}| r_0^{(0)} \rangle = \frac{\hat{g}_-}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} g_- \end{pmatrix},
$$

(E22)

$$
\hat{K}^{(1)}| \uparrow \rangle = \frac{\hat{g}_-}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix},
$$

(E23)

$$
\hat{K}^{(1)}| \downarrow \rangle = \frac{\hat{g}_-}{2} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \end{pmatrix},
$$

(E24)

$$
\hat{K}^{(1)}| r_3^{(0)} \rangle = \frac{\hat{g}_-}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix}.
$$

(E25)

Thus, we obtain

$$
\langle \downarrow | \hat{K}^{(1)} | \uparrow \rangle = -\hat{g}_-, \quad \langle \uparrow | \hat{K}^{(1)} | \uparrow \rangle = 0, \quad \langle \downarrow | \hat{K}^{(1)} | \downarrow \rangle = 0, \quad \langle \uparrow | \hat{K}^{(1)} | \downarrow \rangle = -\hat{g}_-.
$$

(E26)

Therefore, we choose the following basis:

$$
\langle \ell_0^{(0)} | = \langle \ell_1^{(0)} | = \begin{pmatrix} 1 \rangle + \langle -1 | \rangle, \quad | r_1^{(0)} \rangle = \frac{1}{2} (| \uparrow \rangle + | \downarrow \rangle),
$$

(E30)

$$
\langle \ell_2^{(0)} | = \langle \ell_3^{(0)} | = \begin{pmatrix} 1 \rangle - \langle -1 | \rangle, \quad | r_2^{(0)} \rangle = \frac{1}{2} (| \uparrow \rangle - | \downarrow \rangle),
$$

(E31)

$$
\langle \ell_3^{(0)} | = \langle \ell_3^{(0)} | = \begin{pmatrix} 1 \rangle + \langle -1 | \rangle.
$$

(E32)

This set of basis also satisfies the orthonormal relation $\langle \ell_m^{(0)} | r_n^{(0)} \rangle = \delta_{mn}$. The eigenvalues corresponding to $\langle \ell_m |$ and $| r_m \rangle$ are unchanged as

$$
\varepsilon_0^{(0)} = 0, \quad \varepsilon_1^{(0)} = -2, \quad \varepsilon_2^{(0)} = -2, \quad \varepsilon_3^{(0)} = -4.
$$

(E33)

(E34)

It is easy to verify the completeness as

$$
\sum_{i=0}^{3} \langle \ell_i^{(0)} | \ell_i^{(0)} \rangle = | 0 \rangle \otimes | 0 \rangle + | 1 \rangle \otimes | 0 \rangle + | 1 \rangle \otimes | 1 \rangle + | 1 \rangle \otimes | 1 \rangle.
$$

(E35)

b. Perturbation

Now, let us obtain the perturbed eigenstates. As the usual perturbation method, we expand the eigenstates and eigenvalues with the aid of $\langle \ell_m |$ and $| r_m \rangle$ as

$$
\langle \ell_m | = \langle \ell_m^{(0)} | + \beta \sum_{n=0}^{3} a_{mn} \langle \ell_n^{(0)} |,
$$

(E36)

$$
| r_m \rangle = | r_m^{(0)} \rangle + \beta \sum_{n=0}^{3} b_{mn} | r_n^{(0)} \rangle,
$$

(E37)

$$
\varepsilon_m = \varepsilon_m^{(0)} + \beta \varepsilon_m^{(1)}.
$$

(E38)
Substituting Eqs. (E36), (E37) and (E38) into Eqs. (B9) or (B10) multiplying $|\tilde{r}_k^{(0)}\rangle$ or $\langle \tilde{r}_k^{(0)} |$, we obtain

$$\varepsilon_m^{(1)} = \langle \tilde{r}_k^{(0)} | \hat{\Lambda}^{(1)} | \tilde{r}_m^{(0)} \rangle.$$  \hspace{1cm} (E39)

Substituting Eqs. (E22) and (E25) into Eq. (E39) we immediately obtain

$$\varepsilon_0^{(1)} = \varepsilon_3^{(1)} = 0,$$  \hspace{1cm} (E40)

due to $\langle \tilde{r}_0^{(0)} | \hat{\Lambda}^{(1)} | \tilde{r}_0^{(0)} \rangle = \langle \tilde{r}_3^{(0)} | \hat{\Lambda}^{(1)} | \tilde{r}_3^{(0)} \rangle = 0$. For $\varepsilon_1^{(1)}$ and $\varepsilon_2^{(1)}$, with the aid of Eqs. (E26,E27,E28,E29) and (E31,E32) we obtain

$$\varepsilon_1^{(1)} = \frac{1}{2}(\langle \uparrow | + \langle \downarrow | \hat{\Lambda}^{(1)} (| \uparrow \rangle + | \downarrow \rangle) = -\tilde{g}_-, \hspace{1cm} (E41)$$

$$\varepsilon_2^{(1)} = \frac{1}{2}(\langle \uparrow | - \langle \downarrow | \hat{\Lambda}^{(1)} (| \uparrow \rangle - | \downarrow \rangle) = \tilde{g}_-. \hspace{1cm} (E42)$$

Similarly, using Eqs. (E36), (E37), (B9) and (B10) we obtain the relations at $O(\tilde{\beta})$ for $m \neq n$ as

$$a_{mn} = \frac{\langle \tilde{r}_m^{(0)} | \hat{\Lambda}^{(1)} | \tilde{r}_n^{(0)} \rangle}{\varepsilon_m^{(0)} - \varepsilon_n^{(0)}} = -b_{nm}, \hspace{1cm} (E43)$$

$$b_{mn} = \frac{\langle \tilde{r}_n^{(0)} | \hat{\Lambda}^{(1)} | \tilde{r}_m^{(0)} \rangle}{\varepsilon_m^{(0)} - \varepsilon_n^{(0)}}. \hspace{1cm} (E44)$$

Then, we obtain

$$b_{01} = \frac{1}{2}\tilde{g}_-, \hspace{1cm} (E45)$$

$$b_{02} = b_{10} = b_{20} = b_{23} = b_{30} = b_{31} = b_{32} = 0, \hspace{1cm} (E46)$$

$$b_{03} = -\tilde{g}_-, \hspace{1cm} (E47)$$

$$b_{13} = -\frac{1}{2}\tilde{g}_-. \hspace{1cm} (E48)$$

We also have the relations

$$a_{10} = \frac{1}{2}\tilde{g}_+, \hspace{1cm} (E49)$$

$$a_{20} = a_{01} = a_{02} = a_{32} = a_{03} = a_{13} = a_{23} = 0, \hspace{1cm} (E50)$$

$$a_{31} = \frac{1}{2}\tilde{g}_-. \hspace{1cm} (E51)$$

$$a_{31} = \frac{1}{2}\tilde{g}_-. \hspace{1cm} (E52)$$

Note that Eq. (E44) cannot be used for $b_{12}$ and $b_{21}$ (as well as $a_{12}$ and $a_{21}$) because of the degeneracy of the ground state energy, i.e. zero denominator in Eq. (E44). In this case, we may replace the denominator with $\varepsilon_1^{(1)} - \varepsilon_2^{(1)}$ for $b_{12}$. If we accept this approximation, we estimate $b_{12}, b_{21}, a_{12}$ and $a_{21}$ as

$$b_{12} = b_{21} = a_{12} = a_{21} = 0. \hspace{1cm} (E53)$$

Substituting these relations into Eq. (E37) we obtain

$$\langle \ell_0 \rangle = \langle \epsilon \rangle, \hspace{1cm} (E54)$$

$$| r_0 \rangle = | \epsilon \rangle - \tilde{\alpha} \left[ \frac{1}{2}g_+ | r_1^{(0)} \rangle + g_- | d \rangle \right], \hspace{1cm} (E55)$$

$$\langle \ell_1 \rangle = \langle \tilde{r}_1^{(0)} \rangle + \frac{1}{2} \tilde{\alpha} g_+| \epsilon \rangle, \hspace{1cm} (E56)$$

$$| r_1 \rangle = | r_1^{(0)} \rangle - \frac{1}{2} \tilde{\alpha} g_- | d \rangle, \hspace{1cm} (E57)$$

$$\langle \ell_2 \rangle = \langle \tilde{r}_2^{(0)} \rangle, \hspace{1cm} (E58)$$

$$| r_2 \rangle = | r_2^{(0)} \rangle, \hspace{1cm} (E59)$$

$$\langle \ell_3 \rangle = \langle \tilde{r}_3^{(0)} \rangle - \frac{1}{2} \left[ 2g_+ | \epsilon \rangle + g_- | \tilde{r}_1^{(0)} \rangle \right], \hspace{1cm} (E60)$$

$$| r_3 \rangle = | d \rangle, \hspace{1cm} (E61)$$

where we have introduced $\tilde{\alpha} := \tilde{\beta}\tilde{g}_-.$

It is easy to verify that Eqs. (E54)-(E61) satisfy the completeness relation as

$$\sum_{m=0}^{3} | r_m \rangle \langle \ell_m | = \sum_{m=0}^{3} \beta \langle \tilde{r}_m^{(0)} \rangle + \tilde{\beta} \sum_{n=0}^{3} b_{mn} | r_n^{(0)} \rangle \{ \langle \tilde{r}_m^{(0)} \rangle + \tilde{\beta} \sum_{n=0}^{3} a_{mn} \langle \tilde{r}_n^{(0)} \rangle \}$$

$$= \sum_{m=0}^{3} | r_m^{(0)} \rangle \langle \tilde{r}_m^{(0)} | + \tilde{\beta} \sum_{m,n} (b_{mn} + a_{mn}) | r_n^{(0)} \rangle \langle \tilde{r}_m^{(0)} | + O(\tilde{\beta}^2)$$

$$= \sum_{m=0}^{3} | r_m^{(0)} \rangle \langle \tilde{r}_m^{(0)} | = 1, \hspace{1cm} (E62)$$

where we have used Eq. (E43).

4. Determination of $\hat{\rho}^{(1)}$

In this subsection, let us determine $\hat{\rho}^{(1)}$ in the super-vector notation based on

$$\langle \hat{\rho}^{(1)}(\Lambda(\theta)) \rangle = \hat{\Lambda}_\nu(\theta)\hat{K}_+^{+}(\Lambda(\theta)) \partial_{\Lambda_\nu(\theta)} \hat{\rho}^{\infty}(\Lambda(\theta))), \hspace{1cm} (E63)$$
Thus, once we obtain the eigenfuctions and eigenvalues of $\hat{\rho}$ or the left eigenstates we obtain
\[
|\hat{\rho}(\Lambda(\theta))\rangle = \hat{\Lambda}_\nu(\theta) \sum_{m \neq 0} \frac{1}{\varepsilon_m(\Lambda(\theta))} |r_m(\Lambda(\theta))\rangle \langle \ell_m(\Lambda(\theta))| \frac{\partial}{\partial \Lambda_\nu(\theta)} |r_0(\Lambda(\theta))\rangle
\]
\[
= \hat{\Lambda}_\nu(\theta) \sum_{m \neq 0} \frac{1}{\varepsilon_m(\Lambda(\theta))} |r_m(\Lambda(\theta))\rangle \left( \frac{\partial}{\partial \Lambda_\nu(\theta)} \langle \ell_m(\Lambda(\theta)) | r_0(\Lambda(\theta)) \rangle \right) |r_0(\Lambda(\theta))\rangle,
\]
\[
= -\hat{\Lambda}_\nu(\theta) \sum_{m \neq 0} \frac{1}{\varepsilon_m(\Lambda(\theta))} |r_m(\Lambda(\theta))\rangle \left( \frac{\partial}{\partial \Lambda_\nu(\theta)} \langle \ell_m(\Lambda(\theta)) | r_0(\Lambda(\theta)) \rangle \right) |r_0(\Lambda(\theta))\rangle.
\]
(E64)

Since the left eigenvector $|\ell_2\rangle$ does not depend on $\Lambda$ in the present case (see Eq. (66)), it can be further simplified as
\[
|\hat{\rho}(\Lambda(\theta))\rangle = -\hat{\Lambda}_\nu(\theta) \sum_{m=1,3} \frac{1}{\varepsilon_m(\Lambda(\theta))} |r_m(\Lambda(\theta))\rangle \left( \frac{\partial}{\partial \Lambda_\nu(\theta)} \langle \ell_m(\Lambda(\theta)) | r_0(\Lambda(\theta)) \rangle \right) |r_0(\Lambda(\theta))\rangle.
\]
(E65)

Thus, once we obtain the eigenfunctions and eigenvalues of $\hat{K}$, we can determine $\hat{\rho}(^{(1)})$.

5. Thermodynamic length

By using eigenvalues and eigenstates obtained in the previous subsection, we calculate the thermodynamic length in this subsection. First we rewrite the eigenstates for the unperturbed part. The right eigenstates written as the Kronecker product of spin-up and spin-down subspace can be rewritten as follows
\[
|e\rangle = |0\rangle \otimes |0\rangle = \frac{1}{2} \left( |g+\rangle \otimes |g-\rangle \right) = \frac{1}{4} \left( \begin{pmatrix} g+ \\ g- \\ g- \\ g- \end{pmatrix} \right) = \frac{1}{4} \left( \begin{pmatrix} g^2+ \\ g+g- \\ g- \\ g^2- \end{pmatrix} \right).
\]
(E66)

The same procedures yield
\[
|\hat{\rho}_1^{(0)}\rangle = \frac{1}{2} \left( |\uparrow\rangle + |\downarrow\rangle \right) = \frac{1}{2} \left[ \frac{1}{4} \left( \begin{pmatrix} g+ \\ g- \\ -g+ \\ -g- \end{pmatrix} \right) \right] + \frac{1}{4} \left( \begin{pmatrix} g+ \\ -g+ \\ -g- \\ g- \end{pmatrix} \right) = \frac{1}{4} \left( \begin{pmatrix} g+ \\ 1-g+ \\ 1-g+ \\ -g- \end{pmatrix} \right),
\]
(E67)
\[
|\hat{\rho}_2^{(0)}\rangle = \frac{1}{2} \left( |\uparrow\rangle - |\downarrow\rangle \right) = \frac{1}{2} \left[ \frac{1}{4} \left( \begin{pmatrix} g+ \\ g- \\ -g+ \\ g- \end{pmatrix} \right) \right] - \frac{1}{4} \left( \begin{pmatrix} g+ \\ g- \\ -g+ \\ g- \end{pmatrix} \right) = \frac{1}{4} \left( \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right),
\]
(E68)
\[
|d\rangle = \frac{1}{4} \left( \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right).
\]
(E69)

For the left eigenstates we obtain
\[
|e\rangle = (1,1,1,1),
\]
\[
|\hat{\rho}_1^{(0)}\rangle = (g+g-,g-,g-,g-) + (g-,g+,g-,g+) = 2(g-,1-g+,1-g+),
\]
\[
|\hat{\rho}_2^{(0)}\rangle = (g-,g-,g-,g+) - (g-,g-,g-,g+) = 2(0,1,-1,0),
\]
\[
|d\rangle = (g^2-,g+g-,g-,g^2-).
\]

Thus we can rewrite the perturbed eigenstate \( |r_0 \rangle = |\tilde{\rho}^m \rangle \) in Eq. (63) as

\[
|r_0 \rangle = \frac{1}{4} \begin{pmatrix} g_+^2 \\ 2g_+g_- - g_+g_- \\ g_+g_- - g_-g_+ \\ g_-^2 \end{pmatrix} - \frac{1}{4} \tilde{\alpha} \begin{pmatrix} 1 \\ g_+ \\ 1 - g_+ \\ -g_- \end{pmatrix} - \frac{1}{8} \tilde{\alpha}g_+g_- \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}
\]

= \frac{1}{8} \begin{pmatrix} 2g_+^2 - 2\tilde{\alpha}g_+ - \tilde{\alpha}g_+g_- \\ 2g_+ - 2\tilde{\alpha}(1 - g_+) + \tilde{\alpha}g_+g_- \\ 2g_+ - 2\tilde{\alpha}(1 - g_+) + \tilde{\alpha}g_+g_- \\ 2g_-^2 - 2\tilde{\alpha}g_- - \tilde{\alpha}g_+g_- \end{pmatrix}.
\]

(E74)

Similarly, \( |r_1 \rangle \) obeys

\[
|r_1 \rangle = \frac{1}{4} \begin{pmatrix} g_+ \\ 1 - g_+ \\ -g_- \end{pmatrix} - \frac{1}{8} \tilde{\alpha}g_- \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 2g_+ - \tilde{\alpha}g_- \\ 2 - 2g_+ + \tilde{\alpha}g_- \\ 2 - 2g_+ + \tilde{\alpha}g_- \\ -2g_- - \tilde{\alpha}g_- \end{pmatrix}.
\]

(E75)

A parallel procedure for the left super-eigenvectors leads to

\[
\langle \ell_1 | = \frac{1}{2} (4g_+ + \tilde{\alpha}g_+, 4 - 4g_+ + \tilde{\alpha}g_+, 4 - 4g_+ + \tilde{\alpha}g_+, -4g_+ + \tilde{\alpha}g_+),
\]

(E76)

\[
\langle \ell_3 | = (g_+^2 - 2\tilde{\alpha}g_+, -g_+g_-, -\tilde{\alpha}g_-, -g_+g_-, g_+^2),
\]

(E77)

With the aid of the relation \( g_+ + g_- = 2 \), one can show

\[
\frac{\partial}{\partial \Lambda_\mu} (\ell_1 | = \frac{1}{2} \frac{\partial (-4 + \tilde{\alpha})g_+}{\partial \Lambda_\mu} (1, 1, 1, 1).
\]

(E78)

Thus, we obtain

\[
\left( \frac{\partial}{\partial \Lambda_\mu} (\ell_1 | \right) |r_0 \rangle = \frac{1}{2} \frac{\partial (-4 + \tilde{\alpha})g_+}{\partial \Lambda_\mu} = -2 \frac{\partial g_+}{\partial \Lambda_\mu} + O(\tilde{\beta}).
\]

(E79)

Similarly, from the relation

\[
\frac{\partial}{\partial \Lambda_\mu} \langle \ell_3 | = \frac{\partial}{\partial \Lambda_\mu} (g_+^2, -g_+g_-, -g_+g_-, g_+^2) - \frac{\partial (\tilde{\alpha}g_-)}{\partial \Lambda_\mu} (2, 1, 1, 0),
\]

(E80)

we obtain

\[
\left( \frac{\partial}{\partial \Lambda_\mu} (\ell_3 | \right) |r_0 \rangle = \left[ \frac{1}{4} \frac{\partial g_+^2}{\partial \Lambda_\mu} (g_+^2, -g_+g_-, -g_+g_-, g_+^2) \right] \begin{pmatrix} g_+^2 \\ g_+g_- \\ g_+g_- \\ g_-^2 \end{pmatrix} + O(\tilde{\beta})
\]

\[
= 1 \left[ \frac{1}{4} \frac{\partial g_+^2}{\partial \Lambda_\mu} g_+^2 + \frac{\partial g_+g_-}{\partial \Lambda_\mu} g_+ + \frac{\partial g_+^2}{\partial \Lambda_\mu} g_-^2 \right] + O(\tilde{\beta}) = O(\tilde{\beta}).
\]

(E81)

Substituting Eqs. (E79) and (E81) into Eq. (E65), the leading order of \(|\tilde{\rho}^{(1)}| \) in \( \tilde{\beta} \) expansion becomes

\[
|\tilde{\rho}^{(1)}(\Lambda(\theta)) \rangle = 2\Lambda_\nu(\theta) \frac{\partial g_+}{\partial \Lambda_\nu} \frac{1}{\epsilon_1(\Lambda(\theta))} |r_1(\Lambda(\theta)) \rangle + O(\tilde{\beta})
\]

\[
= 2\Lambda_\nu(\theta) \frac{\partial g_+}{\partial \Lambda_\nu} \frac{1}{\epsilon_1(\Lambda(\theta))} |r_1(\Lambda(\theta)) \rangle + O(\tilde{\beta})
\]

\[
= - \frac{d g_+}{d \theta} |r_1(\Lambda(\theta)) \rangle + O(\tilde{\beta}).
\]

(E82)

Thus, we can rewrite Eq. (E83) in the matrix form as

\[
\tilde{\rho}^{(1)}(\Lambda(\theta)) = - \frac{1}{4} \frac{d g_+}{d \theta} \text{diag}(g_+, 1 - g_+, 1 - g_+, -g_-) + O(\tilde{\beta}).
\]

(E83)
Finally, we get
\[ \sigma^{(2)} = \frac{1}{4\pi} \int_0^{2\pi} d\theta \, \frac{1}{4} \left( \frac{dg_+}{d\theta} \right)^2 \text{Tr} \{ \text{diag} [1, (1 - g_+)^2 / g_+, (1 - g_+)^2 / g_-] \} + O(\beta) \]
\[ = \frac{1}{4\pi} \int_0^{2\pi} d\theta \, \frac{1}{2} \left( \frac{dg_+}{d\theta} \right)^2 \frac{g_+g_- + (1 - g_+)^2}{g_+g_-} + O(\beta) \]
\[ = \frac{1}{8\pi} \int_0^{2\pi} d\theta \, \left( \frac{dg_+}{d\theta} \right)^2 \frac{1}{g_+g_-} + O(\beta). \]  

(E84)

The integrant of the leading term is positive definite as expected. Similar calculation yields

\[ \mathcal{L} = \frac{1}{4\pi} \int_0^{2\pi} d\theta \, \left( \frac{dg_+}{d\theta} \right)^2 \frac{1}{g_+g_-} + O(\beta) = \frac{1}{4\pi} \int_0^{2\pi} d\theta \, \left( \frac{dg_+}{d\theta} \right) \frac{1}{\sqrt{g_+g_-}} + O(\beta). \]  

(E85)

6. Adiabatic work

In this subsection, we present the details of the adiabatic work \( \mathcal{W} \).

By using the curvature introduced in Eq. (38), the 2-form \( dA \) can be written as \( dA = \frac{1}{2} F_{\mu\nu} d\Lambda_{\mu} \wedge d\Lambda_{\nu} \). If both of \( \Lambda_{\mu} \) and \( \Lambda_{\nu} \) are not \( \lambda \), \( A_{\nu} \) for \( \partial H / \partial \lambda = 0 \) at \( \theta = 2\pi \) satisfies the relation

\[ \frac{\partial}{\partial \Lambda_{\mu}} A_{\nu} = -\frac{\partial}{\partial \Lambda_{\mu}} \left[ \lambda \frac{\partial}{\partial \Lambda_{\nu}} \text{Tr} \left( \frac{\partial H}{\partial \Lambda_{\lambda}} \hat{\rho}^{ss} \right) \right] = -\lambda \frac{\partial}{\partial \Lambda_{\nu}} \frac{\partial}{\partial \Lambda_{\nu}} \text{Tr} \left( \frac{\partial H}{\partial \Lambda_{\lambda}} \hat{\rho}^{ss} \right) = \frac{\partial}{\partial \Lambda_{\nu}} A_{\mu}. \]  

(E86)

Thus, \( F_{\mu\nu} \) can be nonzero only if \( \Lambda_{\mu} \) or \( \Lambda_{\nu} \) is chosen to be \( \lambda \). For instance, if we choose \( \mu = 3, \nu = 1 \) with \( \Lambda_3 = \lambda \) and \( \Lambda_1 = \mu^{\perp}/\mu^{\perp} \), we obtain \( F_{31} \) as

\[ F_{31} = -\frac{\mu}{2U_0} \frac{\partial}{\partial \mu^L} \text{Tr} \left( \frac{\partial H}{\partial \Lambda} \hat{\rho}^{ss} \right). \]  

(E87)

If the Hamiltonian \( \hat{H} \) depends on \( \lambda \) only through \( U = U_0 \lambda(\theta) \) as we consider in this paper, one can further calculate the thermodynamic curvature (E87) as

\[ F_{31} = -\frac{\mu}{2U_0} \frac{\partial}{\partial \mu^L} \text{Tr} \left( U_0 \hat{n}_+ \hat{n}_- \hat{\rho}^{ss} \right) = -\frac{\mu}{2U_0} \frac{\partial}{\partial \mu^L} \left( \rho^{ss}_1 (A) \right) \]
\[ = -\frac{1}{8} \frac{\partial}{\partial \mu^L} (2g_+^2 - 2\hat{\alpha}_- \hat{g}_+ - \hat{\alpha}_+ \hat{g}_-). \]  

(E88)

where we have used the matrix representation of the double occupied state \( (\hat{n}_+ \hat{n}_-)_{ij} = \delta_{i,1} \delta_{j,1} \). In order to visualize the curvature, we introduce the thermodynamic axial field \( \hat{B} \) as in Eq. (40).

In the present case, the thermodynamic axial field \( \hat{B} \) is given by

\[ \hat{B} = -\frac{1}{8} \frac{\mu}{U_0} \left( \begin{array}{c} \frac{\partial}{\partial \mu^L} (2g_+^2 - 2\hat{\alpha}_- \hat{g}_+ - \hat{\alpha}_+ \hat{g}_-) \\ -\frac{\partial}{\partial \mu^L} (2g_+^2 - 2\hat{\alpha}_- \hat{g}_+ - \hat{\alpha}_+ \hat{g}_-) \end{array} \right), \]  

(E89)

The axial field \( \hat{B} \) can be expanded as \( \hat{B} = \hat{\beta} \hat{B}^{(1)} \) for small \( \hat{\beta} = \hat{\beta}^{\perp} U_0 \) and \( \hat{\alpha} = \hat{\beta} \hat{J}_- \), where \( \hat{B}^{(1)} \) is given by

\[ \hat{B}^{(1)} = -\frac{1}{2\mu} \left( \begin{array}{c} g_+ + \frac{\partial g_+}{\partial \mu^L} \\ -g_+ + \frac{\partial g_+}{\partial \mu^L} \end{array} \right). \]  

(E90)

Thus, the thermodynamic axial field \( \hat{B}_\mu \) is independent of \( \lambda(\theta) \). Therefore, the path in the parameter space \( \Lambda_\mu \) is embedded in the two-dimensional space spanned by \( \mu^L \) and \( \mu^{\perp} \).
For a divergence free variable $\mathcal{B}_\mu$ satisfying Eq. (42), we can introduce the stream function $\Psi(\mu^L, \mu^R)$ satisfying

$$\mathcal{B} = \begin{pmatrix} -\frac{\partial}{\partial \mu^L} \Psi(\mu^L, \mu^R) \\ \frac{\partial}{\partial \mu^R} \Psi(\mu^L, \mu^R) \end{pmatrix}. \quad (E91)$$

Note that $\Psi(\mu^L, \mu^R)$ is symmetric satisfying $\Psi(\mu^L, \mu^R) = \Psi(\mu^R, \mu^L)$. Since the operational path enclosed by $\partial \Omega$ embedded in the two-dimensional space, the normal vector perpendicular to the enclosed path is proportional to $(1, 1, 0)^T$ where $\tau$ expresses the transverse for the case of $\delta = \pi$. Thus, the thermodynamic flux penetrating the path $\partial \Omega$ at $\mu^L = a$ and $\mu^R = b$ is expressed as

$$\Phi_{TD}(a, b) \propto \mathcal{B} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{\partial}{\partial \mu^L} \Psi(\mu^L, \mu^R)|_{\mu^L=a, \mu^R=b} - \frac{\partial}{\partial \mu^R} \Psi(\mu^L, \mu^R)|_{\mu^L=a, \mu^R=b}. \quad (E92)$$

Since $\Psi(\mu^L, \mu^R) = \Psi(\mu^R, \mu^L)$, the thermodynamic flux satisfies

$$\Phi_{TD}(b, a) \propto \frac{\partial}{\partial \mu^L} \Psi(\mu^L, \mu^R)|_{\mu^L=b, \mu^R=a} - \frac{\partial}{\partial \mu^R} \Psi(\mu^L, \mu^R)|_{\mu^L=b, \mu^R=a}$$

$$= \frac{\partial}{\partial \mu^L} \Psi(\mu^R, \mu^L)|_{\mu^L=b, \mu^R=a} - \frac{\partial}{\partial \mu^R} \Psi(\mu^R, \mu^L)|_{\mu^L=a, \mu^R=b}$$

$$= - \frac{\partial}{\partial \mu^R} \Psi(\mu^R, \mu^L)|_{\mu^L=a, \mu^R=b} + \frac{\partial}{\partial \mu^L} \Psi(\mu^R, \mu^L)|_{\mu^L=a, \mu^R=b}. \quad (E93)$$

Thus, one can find the relation $\Phi_{TD}(b, a) = -\Phi_{TD}(a, b)$. This means that the thermodynamic flux penetrating the area becomes zero if the shape of the area is symmetric under the exchange of $\mu^L$ and $\mu^R$. Thus, the adiabatic work $W$ becomes zero at $\delta = \pi$, if the modulation of the parameter is cyclic.

Now, let us write the vector potential explicitly as

$$A_\mu = \frac{1}{8} U_0 \lambda \frac{\partial}{\partial \Lambda_\mu} (2g_+^2 - 2\tilde{\alpha} g_+ - \tilde{\alpha} g_-)$$

$$= \frac{1}{8} U_0 \lambda \frac{\partial}{\partial \Lambda_\mu} (2g_+^2 - 2\beta U_0 \tilde{g}_- g_+ - \beta U_0 \tilde{g}_- g_-). \quad (E94)$$

One can rewrite $A_\mu$ furthermore

$$A_\mu = -\frac{1}{8} \frac{\partial}{\partial \Lambda_\mu} \left[ \lambda(2g_+^2 - 2\beta U_0 \tilde{g}_- g_+ - \beta U_0 \tilde{g}_- g_-) \right] + \frac{1}{8} \frac{\partial \lambda}{\partial \Lambda_\mu} (2g_+^2 - 2\beta U_0 \tilde{g}_- g_+ - \beta U_0 \tilde{g}_- g_-)$$

$$= -\frac{1}{8} \frac{\partial}{\partial \Lambda_\mu} \left[ \lambda(2g_+^2 - 2\beta U_0 \tilde{g}_- g_+ - \beta U_0 \tilde{g}_- g_-) \right] + \frac{1}{8} U_0 \delta_{\mu, \beta} (2g_+^2 - 2\beta U_0 \tilde{g}_- g_+ - \beta U_0 \tilde{g}_- g_-). \quad (E95)$$

If the modulation of the parameters is carried out in $(\mu^L, \mu^R)$ plane, with keeping $\lambda =$ const., the adiabatic work vanishes as

$$W(\mu^L, \mu^R) = -\frac{1}{8\pi} U_0 \lambda \left\{ \oint d\mu^L \frac{\partial g_+^2}{\partial \mu^L} + \oint d\mu^R \frac{\partial g_+^2}{\partial \mu^R} \right\} = 0. \quad (E96)$$

Thus, with the aid of Eq. (E95), the modulation with changing $\lambda$ leads to the adiabatic work as

$$W = \frac{1}{2\pi} \oint d\Lambda_\mu A_\mu = \frac{1}{16\pi} U_0 \oint d\lambda (2g_+^2 - 2\beta U_0 \tilde{g}_- g_+ - \beta U_0 \tilde{g}_- g_-) \approx \frac{1}{8\pi} U_0 \oint d\lambda g_+^2. \quad (E97)$$

where we have used the assumption of the perturbation i.e. $\beta U_0 \ll 1$ to obtain the final expression.

[1] D. J. Thouless, Quantization of particle transport, Phys. Rev. B 27, 6083 (1983).

[2] Q. Niu and D. J. Thouless, Quantised adiabatic charge transport, Phys. Rev. B 27, 6083 (1983).
transport in the presence of substrate disorder and many-body interaction, J. Phys. A : Math. Gen. 17, 2453 (1984).

3. M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. London Ser. A 392, 45 (1984).

4. D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1959 (2010).

5. N. A. Sinitsyn and I. Nemenman, The Berry phase and the pump flux in stochastic chemical kinetics, Europhys. Lett. 77, 58001 (2007).

6. N. A. Sinitsyn and I. Nemenman, Universal Geometric Theory of Mesoscopic Stochastic Pumps and Reversible Ratchets, Phys. Rev. Lett. 99, 220408 (2007).

7. L. P. Kouwenhoven, A. T. Johnson, N. C. van der Vaart, C. J. P. M. Harmans, and C. T. Foxon, Quantized current in a quantum-dot turnstile using oscillating tunnel barriers, Phys. Rev. Lett. 67, 1626 (1991).

8. H. Pothier, P. Lafarge, C. Urbina, D. Esteve, and M. H. Devoret, Single-Electron Pump Based on Charging Effects, Europhys. Lett. 17, 249 (1992).

9. M. Switkes, C. M. Marcus, K. Campman, and A. C. Gossard, An Adiabatic Quantum Electron Pump, Science 283, 1905 (1999).

10. A. Fuhrer, C. Fasth, and L. Samuelson, Single electron pumping in InAs nanowire double quantum dots, Appl. Phys. Lett. 91, 052109 (2007).

11. B. Kaestner, V. Kashcheyevs, G. Hein, K. Pierz, U. Siegner, and H. W. Schumacher, Robust single-parameter quantized charge pumping, Appl. Phys. Lett. 92, 192106 (2008).

12. S. J. Chorley, J. Frake, C. G. Smith, G. A. C. Jones, and M. R. Buiterlaar, Quantized charge pumping through a carbon nanotube double quantum dot, Appl. Phys. Lett. 100, 143104 (2012).

13. S. Nakajima, T. Tomita, S. Taie, T. Ichinose, H. Ozawa, L. Wang, M. Troyer, and Y. Takahashi, Topological Thouless pumping of ultracold fermions, Nature Physics 12, 296 (2016).

14. M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelsburger and I. Bloch, A Thouless quantum pump with ultracold bosonic atoms in an optical superlattice, Nature Physics 12, 350 (2016).

15. S. K. Watson, R. M. Potok, C. M. Marcus, and V. Umansky, Experimental Realization of a Quantum Spin Pump, Phys. Rev. Lett. 91, 258301 (2003).

16. P. W. Brouwer, Scattering approach to parametric pumping, Phys. Rev. B 58, R10135 (1998).

17. J. E. Avron, A. Elgart, G. M. Graf, and L. Sadun, Geometry, statistics, and asymtotics of quantum pumps, Phys. Rev. B 62, R10618 (2000).

18. M. Moskalets and M. Büttiker, Effect of inelastic scattering on parametric pumping, Phys. Rev. B 64, 201305(R) (2001).

19. J. N. H. J. Cremers and P. W. Brouwer, Dephasing in a quantum pump, Phys. Rev. B 65, 115333 (2002).

20. A. Andreev and A. Kamenev, Counting Statistics of an Adiabatic Pump, Phys. Rev. Lett. 85, 1294 (2000).

21. Y. Makhlin and A. D. Mirlin, Counting Statistics for Arbitrary Cycles in Quantum Pumps, Phys. Rev. Lett. 87, 276803 (2001).

22. I. L. Aleiner and A. V. Andreev, Adiabatic Charge Pumping in Almost Open Dots, Phys. Rev. Lett. 81, 1286 (1998).

23. E. R. Mucciolo, C. Chamon, and C. M. Marcus, Adiabatic Quantum Pump of Spin-Polarized Current, Phys. Rev. Lett. 89, 146802 (2002).

24. J. M. R. Parrondo, Reversible ratchets as Brownian particles in an adiabatically changing periodic potential, Phys. Rev. E 57, 7297 (1998).

25. O. Usmani, E. Lutz, and M. Büttiker, Noise-assisted classical adiabatic pumping in a symmetric periodic potential, Phys. Rev. E 66, 021111 (2002).

26. R. D. Astumian, Adiabatic Pumping Mechanism for Ion Motive ATPases, Phys. Rev. Lett. 91, 118102 (2003).

27. R. D. Astumian, Adiabatic operation of a molecular machine, Proc. Natl. Acad. Sci. USA 104, 19715 (2007).

28. S. Rahav, J. Horowitz, and C. Jarzynski, Directed Flow in Nonadiabatic Stochastic Pumps, Phys. Rev. Lett. 101, 140602 (2008).

29. V. Y. Chernyak, J. R. Klein, and N. A. Sinitsyn, Quantization and fractional quantization of currents in periodically driven stochastic systems. I. Average currents, J. Chem. Phys. 136, 154107 (2012).

30. V. Y. Chernyak, J. R. Klein, and N. A. Sinitsyn, Quantization and fractional quantization of currents in periodically driven stochastic systems. II. Full counting statistics, J. Chem. Phys. 136, 154108 (2012).

31. J. Ren, P. Hänggi, and B. Li, Berry-Phase-Induced Heat Pumping and Its Impact on the Fluctuation Theorem, Phys. Rev. Lett. 104, 170601 (2010).

32. T. Sagawa and H. Hayakawa, Geometrical expression of excess entropy production, Phys. Rev. E 84, 051110 (2011).

33. V. M. Puasonen and H. Hayakawa, Pumping current in a non-Markovian N-state model Phys. Rev. Research 3, 023238 (2021).

34. F. Renzoni and T. Brandes, Charge transport through quantum dots via time-varying tunnel coupling, Phys. Rev. B 64, 245301 (2001).

35. T. Brandes and T. Vorrath, Adiabatic transfer of electrons in coupled quantum dots, Phys. Rev. B 66, 075341 (2002).

36. E. Cota, R. Aguado, and G. Platero, ac-Driven Double Quantum Dots as Spin Pumps and Spin Filters, Phys. Rev. Lett. 94, 107202 (2005).

37. J. Spletstosser, M. Governale, J. König, and R. Fazio, Adiabatic pumping through a quantum dot with coulomb interactions: A perturbation expansion in the tunnel coupling, Phys. Rev. B 74, 085305 (2006).

38. T. Yuge, T. Sagawa, A. Sugita, and H. Hayakawa, Geometrical pumping in quantum transport: Quantum master equation approach, Phys. Rev. B 86, 235308 (2012).

39. T. Yuge, T. Sagawa, A. Sugita, and H. Hayakawa, Geometrical Excess Entropy Production in Nonequilibrium Quantum Systems, J. Stat. Phys. 153, 412 (2013).

40. K. L. Watanabe and H. Hayakawa, Geometric fluctuation theorem for a spin-boson system, Phys. Rev. E 96, 022118 (2017).

41. Y.Hino and H. Hayakawa, Fluctuation relations for adiabatic pumping, Phys. Rev. E 120, 012115 (2020).

42. K. Takahashi, Y. Hino, K. Fujii and H. Hayakawa, Full Counting Statistics and Fluctuation?Dissipation Relation for Periodically Driven Two-State Systems, J. Stat. Phys. 181, 2206 (2020).

43. B. Andresen, Finite-time thermodynamics and thermodynamic length, Rev. Gén. Therm. 35, 647 (1996).

44. F. Weinhold, Metric geometry of equilibrium thermody-
