A Detailed Fluctuation Theorem for Arbitrary Measurement and Feedback Schemes

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Fluctuation theorems are powerful equalities that hold far from equilibrium. However, the standard approach to include measurement and feedback schemes may become inapplicable in certain situations, including continuous measurements, precise measurements of continuous variables, and feedback induced irreversibility. Here we overcome these shortcomings by providing a recipe for producing detailed fluctuation theorems. Based on this recipe, we derive a fluctuation theorem which holds for arbitrary measurement and feedback protocols. The key insight is that fluctuations inferable from the measurement outcomes may be suppressed by post-selection. Our detailed fluctuation theorem results in a stringent and experimentally accessible inequality on the extractable work, which is saturated when the full entropy production is inferable from the data.

Introduction.— Most devices that simplify our daily lives are far from equilibrium, consuming and dissipating energy. A thorough understanding of non-equilibrium physics is therefore of pivotal importance for the development of novel technologies. However, systems that are far from equilibrium are notoriously difficult to describe. This holds especially true for small systems, where fluctuations cannot be neglected. During the last 25 years, a number of powerful thermodynamic equalities that hold far from equilibrium have been developed (for recent reviews, see Ref. [1–6]). The most prominent of these are the Jarzynski relation [7, 8] and the Crooks fluctuation theorem [9, 13]. These equalities involve the probability distributions of work or entropy production along trajectories through phase space and constitute important results in the field of stochastic thermodynamics.

Recent experimental advances in observing and controlling small systems opened up the possibility of using information to optimize the process at hand by feedback control [14]. Promising platforms for such experiments include electronic systems [15–19], DNA molecules [20, 21], photons [22], Brownian particles [23], and superconducting circuits in the quantum regime [24–26]. These experiments probe the thermodynamics of information [27–30], a field which goes back to the thought experiments of Maxwell and Szilard [31, 32], where microscopic information is used to seemingly violate the second law and to produce useful work. Under measurement and feedback schemes, fluctuation theorems and second-law-like inequalities can still be derived by including a term that represents the obtained information [31, 52]. The resulting apparent violation of the second law vanishes when taking into account the physical apparatus that performs the measurement [45]. For the Jarzynski relation, the most prominent generalizations read [36, 40]

\[
\langle e^{-\sigma - I} \rangle = 1 \Rightarrow \langle \sigma \rangle \geq -\langle I \rangle, \quad (1)
\]

\[
\langle e^{-\gamma} \rangle = \gamma \Rightarrow \langle \sigma \rangle \geq -\ln \gamma, \quad (2)
\]

where \( I \) denotes the transfer entropy (the average of which reduces to the mutual information if only a single measurement is performed), \( \gamma \) the efficacy parameter, and \( \sigma \) the entropy production.

While existing fluctuation theorems constitute powerful results, they are unfortunately not always applicable and a detailed fluctuation theorem for arbitrary measurement and feedback scenarios is still lacking. The problems that can arise can be exemplified with the help of Eqs. (1) and (2), where we identified three key shortcomings: (i) The quantities \( I \), \( I \), and \( \gamma \) can diverge, rendering Eqs. (1) and (2) inapplicable or the corresponding inequalities uninformative. In particular, \( I \) diverges when the feedback introduces absolute irreversibility. A naive evaluation of the Jarzynski relation in Eq. (1) then results in the wrong result [37, 53]. The average of the transfer entropy \( \langle I \rangle \) can diverge, e.g., for continuous measurements, when the amount of information extracted from the system diverges [47]. Moreover, the efficacy parameter \( \gamma \) can diverge for feedback schemes that include a large number of protocols to choose from (see below). (ii) The transfer entropy \( I \) is not directly measurable as it contains information on the correlations between system and measurement apparatus [11, 52]. This limits the practical relevance of Eq. (1). (iii) For Eq. (2), there is no corresponding detailed fluctuation theorem which relates probabilities in a forward experiment to probabilities in a backward experiment. Given these shortcomings, it is highly desirable to obtain refined detailed fluctuation theorems which hold for any measurement and feedback scheme. For error-free measurements, an effort in this direction has been made in Ref. [46].

In this letter, we overcome the shortcomings of fluctuation theorems in the presence of measurement and feedback with two interrelated contributions. First, we provide a novel recipe for obtaining fluctuation theorems. Upon defining a backward experiment our recipe provides the associated fluctuation theorem, including the corresponding information terms. This allows one to tailor useful fluctuation theorems, Jarzynski relations, and second-law-like inequalities for the problem at hand. Second, we use this recipe to find a detailed fluctuation theorem that circumvents the problems (i)–(iii) listed above. In the case of error-free measurements, our fluctuation
where $\Delta P$ to be infinitesimally small. Similarly, $Y$ denotes a trajectory of the system state (detector output) and $\Lambda$ an experimental protocol. A detailed fluctuation theorem for the full experiment can be obtained, where the total entropy production $\sigma$ is reduced by the inferable entropy production $\sigma_Y$. Probability distributions are defined in the text.

A recipe for fluctuation theorems.— Our starting point is the detailed fluctuation theorem for a fixed protocol [40,41,40,54,58]. In the notation of Ref. [40], largely followed throughout the paper, we have

$$P[X,\Lambda] = e^{-\sigma_{X,\Lambda}}. \tag{3}$$

Here the vector $X = (x_1, \ldots, x_N)$ denotes a system trajectory through phase space, where time is discretized and $x_j$ denotes a point in phase space the system occupies at time $t_j$. The time-step $t_{j+1} - t_j = \delta t$ is assumed to be infinitesimally small. Similarly, $Y = (y_1, \ldots, y_N)$ denotes a trajectory of measurement outcomes, assigning an outcome to each moment in time. Discrete measurements can be obtained by taking most of the $y_j$ to be independent of the system trajectory $X$. Moreover, $\Lambda = (\lambda_1, \ldots, \lambda_N)$ denotes a protocol which specifies all settings for a single run of the experiment. To include feedback, we assume that the protocol depends on the previous measurement outcomes $\lambda_j(\{y_{k<j}\})$. The daggeder quantities denote the time-reverse of the undagged ones, e.g., $X^\dagger = (x_{N}^*, \ldots, x_1^*)$, where $x_j^*$ is the time-reverse of $x_j$. Equation (3) can then be understood as follows: $P[X,\Lambda]$ denotes the probability that the system takes trajectory $X$ when an experiment with protocol $\Lambda$ is performed (for ease of notation, we omit the $Y$-dependence of $\Lambda$ whenever there is no explicit $Y$-dependence). The probability $P[X^\dagger,\Lambda]^\dagger$ of realizing the time-reversed trajectory when applying the time-reversed protocol is related to $P[X,\Lambda]$ by the exponentiated entropy production [58]. For experiments that start in thermal equilibrium and systems coupled to a single bath at temperature $T$, the entropy production can be written as

$$k_B T \sigma_{X,\Lambda} = \Delta F[\Lambda] - W[X,\Lambda], \tag{4}$$

where $\Delta F[\Lambda]$ corresponds to the free energy difference of the equilibrium states at the beginning and at the end of the protocol and $W[X,\Lambda]$ to the work extracted from the system. If not specifically stated otherwise, our results only require Eq. (3) to hold and do not depend on the specifics of the entropy production. We note that for non-equilibrium initial states, there are cases when Eq. (5) becomes inapplicable [49,53]. This problem can be circumvented by including the preparation of the initial state.

In the presence of measurement and feedback, the forward experiment is described by a joint probability distribution for system trajectory $X$ and measurement outcome $Y$. This joint probability distribution is related to the conditional probability distribution given in Eq. (3) as [40]

$$P[X, Y] = P_m[Y|X]P[X,\Lambda(Y)], \tag{5}$$

where $P_m[Y|X]$ (playing an important role below) denotes the probability that a fixed trajectory $X$ results in the measurement outcomes $Y$. If each measurement only depends on the instantaneous system state, we have $P_m[Y|X] = \prod_j p_j(y_j|x_j)$, with $p_j(y_j|x_j)$ the probability of measuring $y_j$ at time $t_j$ given the system is in the state $x_j$. Rewriting Eq. (5), we arrive at our first main contribution, a general detailed fluctuation theorem for joint probabilities

$$P[X^\dagger, Y^\dagger] = e^{-\sigma_{X,\Lambda}(Y)^\dagger - I[X,Y^\dagger]}, \tag{6}$$

where $P[X^\dagger, Y^\dagger]$ denotes the probability distribution for the backward experiment; unspecified thus far. Here we introduced the transfer entropy in the forward experiment

$$I[X : Y] = \ln \frac{P[X, Y]}{P[X,\Lambda(Y)]P[Y]} = \ln \frac{P_m[Y|X]}{P[Y]}, \tag{7}$$

and in the backward experiment

$$I[X^\dagger : Y^\dagger] = \ln \frac{P[X^\dagger, Y^\dagger]}{P[X^\dagger,\Lambda(Y)^\dagger]P[Y]}, \tag{8}$$

and $P[Y] = \int dX P[X, Y]$. To illustrate the usefulness of Eq. (6) as a recipe for fluctuation theorems, we consider the following scenario: An experiment using measurement and feedback has been designed and it is desired to investigate the physics of the experiment with fluctuation theorems. While the forward experiment is fixed by the designed experiment, there is a freedom in choosing the backward experiment. For any chosen backward experiment (which might include feedback or not), Eq. (6) provides a fluctuation theorem and allows for identifying the corresponding information terms.

It is instructive to see how previous results [cf. Eq. (1)] can be recovered from Eq. (6). To this end, we consider a backward experiment where no feedback is performed. Instead, the fixed protocol $\Lambda^\dagger$ is performed with the same
probability as $\Lambda$ is applied in the forward experiment (where it arises from feedback). This corresponds to the backward probability $P^\dagger[X^\dagger,Y^\dagger] = P[X^\dagger|\Lambda(Y)^\dagger]P[Y]$ (i.e., $I^\dagger[X^\dagger : Y^\dagger] = 0$). Equation (6) then results in the fluctuation theorem associated to Eq. (1) [36, 40].

The backward experiment can however be chosen differently. We note that in the absence of feedback, the freedom of defining the backward experiment has led to the unification of various fluctuation theorems [4, 59–62], considerably enhancing our understanding of nonequilibrium processes. Equation (6) can be seen as a generalization of this approach to processes including measurement and feedback schemes. More generally, one can demand conditions on the backward experiment and/or the information terms in Eq. (6) to find novel fluctuation theorems. Generalized Jarzynski relations and second-law-like inequalities can then be derived in a straightforward manner.

A versatile fluctuation theorem.— We now apply our recipe to find a fluctuation theorem which circumvents the shortcomings (i)-(iii) listed in the introduction. To this end, we impose two conditions:

I The quantity $\Delta I[Y] \equiv I[X : Y] - I^\dagger[X^\dagger : Y^\dagger]$ is fully determined by the measurement outcomes.

II The marginals of the forward and backward probabilities for the measurement outcomes are the same

$$\int dXP^\dagger[X^\dagger,Y^\dagger] = P[Y]. \tag{9}$$

The first condition ensures that the information term $\Delta I$ is experimentally accessible, overcoming shortcoming (ii). This allows for obtaining a detailed fluctuation theorem for the entropy production by integrating Eq. (6) over all $X$ which result in the same $\sigma$. We note that this is not generally possible for previous fluctuation theorems in the literature. The second condition demands that a given set of measurement outcomes $Y$ is equally likely in the forward and in the backward experiment.

These two conditions uniquely fix the backward probability distribution as well as the information term in Eq. (6) and result in our second main contribution, a detailed fluctuation theorem applicable for arbitrary measurement and feedback scenarios

$$\frac{P^\dagger[-\sigma,Y^\dagger]}{P[\sigma,Y]} = e^{-(\sigma - \sigma_{cg}[Y])}. \tag{10}$$

Here probability distributions for the entropy production are obtained through $P^{\dagger}[\sigma,Y] = \int dX\delta[\sigma[X,\Lambda(Y)] - \sigma]P^{\dagger}[X,Y]$ and we introduced the coarse-grained entropy production [40, 63]

$$e^{-\sigma_{cg}[Y]} \equiv \int dXe^{-\sigma[X,\Lambda(Y)]}P[X|Y], \tag{11}$$

where $P[X|Y] = P[X,Y]/P[Y]$. We note that as long as the total entropy production remains finite, $\sigma_{cg}$ remains finite as well, preventing the divergences related to shortcoming (i). From Eq. (10), we find the generalized Jarzynski relation

$$\langle e^{-(\sigma - \sigma_{cg}[Y])}\rangle = 1 \Rightarrow \langle \sigma \rangle \geq \langle \sigma_{cg}[Y] \rangle, \tag{12}$$

where $\langle \cdots \rangle$ denotes an average over the forward probability distribution and the second-law-like inequality follows from Jensen’s inequality.

We now consider scenarios which fulfill the measurement time-reversal symmetry

$$P_m[Y|X] = P_m[Y^\dagger|X^\dagger]. \tag{13}$$

This condition leads to a particularly illuminating physical interpretation of the fluctuation theorem in Eq. (10) and ensures that the backward probability distribution has an operational meaning. We also note that this condition underlies Eq. (2) (see below). In this case, it can be shown that a detailed fluctuation theorem for the detector output holds [40]

$$e^{-\sigma_{cg}[Y]} = e^{-\sigma_Y} = \frac{P[Y^\dagger|\Lambda(Y)^\dagger]}{P[Y|\Lambda(Y)]}, \tag{14}$$

where $P[Y|\Lambda] = \int dXP_m[Y|X]P[X|\Lambda]$ denotes the probability of obtaining the outcomes $Y$ given the protocol $\Lambda$. From Eq. (5), we thus find $P[Y|\Lambda(Y)] = P[Y]$. Comparing Eq. (14) with the detailed fluctuation theorem in Eq. (3), we conclude that $\sigma_Y$ is the entropy production that we infer from observing only the measurement outcomes (see also Fig. 1). We thus call it the inferable entropy production. We note that the coarse-grained entropy production is only equal to the inferable entropy production when Eq. (13) holds. In the following, we thus identify $\sigma_Y = \sigma_{cg}$, deferring a discussion on scenarios where this is not the case to the supplemental material.

The efficacy parameter introduced in Eq. (2) is defined as $\gamma = \int dYP[Y^\dagger|\Lambda(Y)^\dagger]$. In the presence of feedback, $\gamma$ can differ from unity. Using $\langle \exp(-\gamma) \rangle = \gamma$, we recover Eq. (10) from Eq. (10). From Jensen’s inequality we further find $\langle \gamma \rangle \geq -\ln \gamma$. The inequality in Eq. (12) is thus strictly more stringent than the inequality based on the efficacy parameter given in Eq. (2).

The backward probability obtained from our conditions reads

$$P^\dagger[X^\dagger,Y^\dagger] = \frac{P[X^\dagger|\Lambda(Y)^\dagger]}{P[Y^\dagger|\Lambda(Y)^\dagger]}P_m[Y^\dagger|X^\dagger]P[Y]. \tag{15}$$

This distribution has an operational meaning [overcoming shortcoming (iii)] and can be obtained as follows: In a backward experiment, the protocol $\Lambda(Y)^\dagger$ is applied with probability $P[Y]$ and measurements are performed.
The data is then post-selected on measurement outcomes $Y^\dagger$, corresponding to the time-reverse of the outcomes that result in $\Lambda(Y)$ in a forward experiment. Under this post-selection, $P^\dagger[X^\dagger,Y^\dagger]$ is the joint probability of the system and the measurement outcomes realizing the respective trajectories $X^\dagger$ and $Y^\dagger$. It is this post-selection which results in the reduction of the entropy production in Eq. (10) by the inferable entropy production $\sigma_Y$. Intuitively, having access to the measurement outcomes, their fluctuations can be suppressed. This is illustrated in Fig. 2. In case the full entropy production is inferable from the measurement outcomes, i.e. $\sigma_Y = \sigma$, the fluctuation theorem in Eq. (10) reduces to the trivial equality $1 = 1$ reflecting the fact that we have all information on the entropy production at our disposal. Finding deviations from this trivial identity in experiments then reflects the fact that not all entropy producing degrees of freedom are perfectly measured. To verify this, an experimenter must be able to measure the entropy production independently from $Y$.

For an entropy production given by Eq. (4), we can further derive a fluctuation theorem for the extracted work $W$

$$P[W,Y] \frac{P([-W,Y])}{P([W,Y])} = e^{-\beta(W-\Delta F(\Lambda(Y)))-\sigma_Y}. \quad (16)$$

This results in the second-law-like inequality

$$\langle W \rangle \leq \langle \Delta F(\Lambda(Y)) \rangle - k_B T \langle \sigma_Y \rangle. \quad (17)$$

We note that in the absence of feedback, the probability distributions factorize and Eq. (16) reduces to a simple product between the Crooks fluctuation theorem and Eq. (14). To illustrate our results, we consider two well-studied examples, the Szilard engine and a Brownian particle in a harmonic trap. We note that Eq. (13) holds for both examples.

**The Szilard engine.**— We consider a particle in a box of volume $v = 1$. A separation in the middle of the box is introduced and the particle will be found to the left $x = L$ or to the right $x = R$ of the separation with equal probabilities. Subsequently, the location of the particle is measured with an error $\varepsilon$ resulting in a measurement outcome $y = l/r$. The separation is then slowly moved with the aim of increasing the volume available to the particle to $v_\epsilon$, depending on the outcome of the measurement. Finally the separation is removed and the system returns to its initial state.

Detailed calculations are given in the supplemental material (see below), where we verify the detailed fluctuation theorem given in Eq. (10). In Fig. 2(a), we show the extracted work and compare it to the bounds given in Eqs. (11), (12), and (17). We find that the inequality involving the inferable entropy production gives a tighter bound than the established inequalities for a range of parameters.

**Brownian particle in a harmonic trap.**— Our second example consists of a Brownian particle in a harmonic trap potential with spring constant $k$. After a position measurement is performed, the trap potential is shifted, such that the new minimum coincides with the measurement outcome. As long as the thermal spread, $k_B T/k$, is larger than the measurement error, denoted by $\Sigma^2$, a positive amount of work is extracted from the particle.
on average. As for the Szilard engine, detailed calculations are given in the supplemental material (see below) where Eq. [10] is explicitly verified. In Fig. 2(b), the extracted work is compared to the transfer entropy and the inferable entropy production. The efficacy parameter diverges in this scenario since the position measurement has infinitely many outcomes, resulting in infinitely many protocols. The transfer entropy diverges as the measurement error goes to zero. The inferable entropy production provides a useful bound for all parameters. We note that Ref. [10] discussed the same example in the limit Σ → 0, where the bound provided by the inferable entropy becomes tight.

As an additional example published elsewhere, our results are applied to continuous measurements in single molecule force spectroscopy experiments [61].

Conclusions. — We provided a recipe for obtaining fluctuation theorems in the presence of measurement and feedback. This recipe relies on the freedom of choosing a backward experiment and can be employed to develop useful and experimentally relevant fluctuation theorems. The usefulness of the recipe is illustrated with a detailed fluctuation theorem which overcomes the shortcomings identified in previous works. This fluctuation theorem entails an entropy production that is reduced by the amount inferred from the measurement outcomes and it provides a stringent upper bound on the extracted work in situations where previous fluctuation theorems break down.

The freedom of choosing a backward experiment indicates that there is no single fluctuation theorem which is universally optimal, but that each class of problems might be best described by a tailor-made fluctuation theorem. The general validity of our recipe allows for the construction of relevant fluctuation theorems for any given problem including measurement and feedback. The approach outlined here has thus great potential for obtaining a better understanding of non-equilibrium processes and will likely result in additional practically useful equalities and inequalities.

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Supplemental information: A Detailed Fluctuation Theorem for Arbitrary Measurement and Feedback Schemes

This supplemental information provides a discussion on measurements without time-reversal symmetry as well as detailed derivations for the examples in the main text. Equation and Figure numbers not preceded by an ‘S’ refer to the main text.

A. MEASUREMENTS WITHOUT TIME-REVERSAL SYMMETRY

Here we consider the case where \( P_m[Y^\dagger|X^\dagger] \neq P_m[Y|X] \) in some detail. In this case, the two conditions given in the main text result in the following detailed fluctuation theorem

\[
\frac{P[X^\dagger,Y^\dagger]}{P[X,Y]} = e^{-\langle \sigma[X,\Lambda(Y)] - \sigma_{cg}[Y] \rangle}, \tag{S1}
\]

where the coarse grained entropy production is given in Eq. (11) in the main text and the backward probability distribution reads

\[
P[X^\dagger,Y^\dagger] = \frac{P_m[Y|X]P[X^\dagger|\Lambda(Y)^\dagger]}{\int dXP_m[Y|X]P[X^\dagger|\Lambda(Y)^\dagger]} P[Y]. \tag{S2}
\]

While the last equation denotes a normalized probability distribution, it has no clear operational meaning, i.e., it does not correspond to the measured distribution of an implementable experiment. We stress that the efficacy parameter also loses its operational meaning for measurements without time-reversal symmetry. We note that by integrating Eq. (S1) over all trajectories \( X \) which, for a given \( Y \), result in the same entropy production, we recover Eq. (10) in the main text. For completeness, we reprint here the generalized Jarzynski relation following from Eq. (S1)

\[
\langle e^{-\langle \sigma[X,\Lambda(Y)] - \sigma_{cg}[Y] \rangle} \rangle = 1, \tag{S3}
\]

which implies the second law-like inequality

\[
\langle \sigma[X,\Lambda(Y)] \rangle \geq \langle \sigma_{cg}[Y] \rangle. \tag{S4}
\]

For measurements without time-reversal symmetry, we thus find that our conditions only remedy the shortcomings (i) and (ii) but not (iii). The generalized Jarzynski relation in Eq. (S3) has thus the same shortcoming as Eq. (2) in the main text but it results in a strictly more stringent second-law-like inequality.

Alternatively, we can define the backward experiment through the operational meaning of the backward probability distribution in the case where \( P_m[Y^\dagger|X^\dagger] = P_m[Y|X] \). This results in Eq. (15) in the main text which is reprinted here for convenience

\[
P[X^\dagger,Y^\dagger] = \frac{P[X^\dagger|\Lambda(Y)^\dagger]}{P[Y|\Lambda(Y)^\dagger]} P_m[Y^\dagger|X^\dagger] P[Y]. \tag{S5}
\]

As discussed in the main text, this distribution describes an experiment, overcoming shortcoming (iii). We can now relax the condition \( P_m[Y^\dagger|X^\dagger] = P_m[Y|X] \) but still keep the backward probability distribution in Eq. (S5). This results in the detailed fluctuation theorem

\[
\frac{P[X^\dagger,Y^\dagger]}{P[X,Y]} = e^{-\langle \sigma[X,\Lambda(Y)] - \sigma_Y - \sigma_m[X,Y] \rangle}, \tag{S6}
\]

where \( \sigma_Y \) denotes the inferable entropy production defined in Eq. (14) and we introduced

\[
e^{-\sigma_m[X,Y]} \equiv \frac{P_m[Y^\dagger|X^\dagger]}{P_m[Y|X]}. \tag{S7}
\]

Equation (S6) results in the generalized Jarzynski relation

\[
\langle e^{-\langle \sigma[X,\Lambda(Y)] - \sigma_Y - \sigma_m[X,Y] \rangle} \rangle = 1, \tag{S8}
\]
which implies the second law-like inequality
\[ \langle \sigma | X, \Lambda(Y) \rangle \geq \langle \sigma \rangle + \langle \sigma_m | X, Y \rangle. \] (S9)

We note that the price to pay in order to keep the operational meaning of the backward experiment is that the information term is no longer only dependent on the measurement outcome \( Y \). We thus find that shortcomings (ii) and (iii) are overcome by two separate fluctuation theorems for measurements without time-reversal symmetry.

B. THE SZILARD ENGINE

We consider a particle in a box of volume \( v = 1 \). Starting in thermal equilibrium, the particle is equally likely to be found in the left and in the right half of the box. A partition (wall) is then inserted in the middle of the box and a measurement of the position of the particle is performed. We denote the location of the particle by \( x = L, R \) and the measurement outcome by \( y = l, r \). We assume that a measurement error happens with probability \( \varepsilon \), i.e.,
\[ P_m[l|L] = P_m[r|R] = 1 - \varepsilon, \quad P_m[l|R] = P_m[r|L] = \varepsilon. \] (S10)

Since the particle is equally likely to be in the left and in the right half of the box, the joint probability for \( x \) and \( y \) reads
\[ P[x, y] = \delta_{x,y}(1 - \varepsilon)/2 + \delta_{x,\bar{y}}\varepsilon/2, \] (S11)
where the Kronecker delta is defined as \( \delta_{L,l} = \delta_{R,r} = 1 \) and zero otherwise. We further introduced \( \bar{y} \neq y \). Having measured \( y \), the partition is then moved away from where the particle is assumed to be, extending the volume it presumably occupies to \( v_y \leq 1 \). This process results in the extracted work
\[ \beta W[x, y] = \delta_{x,y} \ln(2v_y) + \delta_{x,\bar{y}} \ln(2 - 2v_y), \] (S12)
where \( \beta = 1/(k_B T) \) denotes the inverse temperature. The protocol is then completed by removing the partition, such that the particle returns to its initial state. We note that there are two protocols, \( \Lambda(y) \), which differ by the direction in which the partition is moved upon insertion. In this scenario, the entropy production is determined completely by the work, i.e., \( \sigma = -\beta W \). We note that the work cost diverges if the measurement outcome is erroneous and if \( v_y = 1 \) because in this case the particle is squeezed into a vanishingly small volume. For a finite \( \varepsilon \) and \( v_y = 1 \), there are thus trajectories for which the entropy production diverges.

We note that because the two protocols are the same up to the measurement, the protocol does not influence the value of \( x \) (which is given by the actual particle location when the measurement happens). We therefore find
\[ P[x|\Lambda(y)] = P[x] = \frac{1}{2}. \] (S13)

It is then straightforward to verify Eq. (3) in the main text. From Eq. (S11), we further find that obtaining each measurement outcome is equally likely, i.e., \( P[y] = 1/2 \). The transfer entropy in the forward experiment then reduces to the mutual information
\[ I[x : y] = \delta_{x,y} \ln(2 - 2\varepsilon) + \delta_{x,\bar{y}} \ln(2\varepsilon). \] (S14)

Note that in the limit \( \varepsilon \to 0 \), the mutual information diverges when a measurement error occurs because this becomes infinitely unlikely. As a consequence, the standard detailed fluctuation theorem involving the mutual information is no longer applicable (see below). Also note that the mutual information does not contain any information on \( v_y \). It can thus not take into account any limitation by the protocol we apply. This can be seen most drastically by taking \( v_y = 1/2 \), i.e., the protocol corresponding to doing nothing. Clearly no work can be extracted in this case. The second-law-like inequality involving the mutual information alone does not take this into account [cf. Eq. (1)]. The mean mutual information reads
\[ \langle I[x : y] \rangle = \ln(2) + (1 - \varepsilon) \ln(1 - \varepsilon) + \varepsilon \ln(\varepsilon), \] (S15)
and is shown in Fig. 2(a). Just as Eq. (S14), it does not take into account the feedback protocol. Note that the mean mutual information remains finite in the limit of error-free measurements. As noted in Ref. [40], the extracted work for a given measurement error is maximized for \( v_y = 1 - \varepsilon \) where \( \beta(W) = \langle I \rangle \).
The backward experiment discussed in the main text is obtained as follows. First, protocol \( \Lambda(y) \) is applied with probability \( P[y] \). The partition is thus inserted such that the box is divided into parts of volume \( v_y \) and \( 1 - v_y \). The partition is then moved to the middle of the box and a measurement of the particle location is performed. The backward experiments are then postselected on the measurement outcomes \( y \) which correspond to the applied protocol (note that in this case \( y = y' = y \) and \( x' = x \)). For the backward experiment, the two protocols are different even before the measurement happens. We thus find

\[
P[x|\Lambda(y)] = \delta_{x,y}v_y + \delta_{x,\bar{y}}(1 - v_y), \tag{S16}
\]

and

\[
P[y|\Lambda(y)] = \sum_{x = L, R} P_m[y|x]P[x|\Lambda(y)] = v_y(1 - \varepsilon) + \varepsilon(1 - v_y). \tag{S17}
\]

For the joint backward probability distribution we then get from Eq. (15)

\[
P[y, x] = \frac{1}{2} \delta_{y, y'}v_y(1 - \varepsilon) + \delta_{y, \bar{y}}v_y(1 - \varepsilon) + \delta_{x, y}v_y(1 - \varepsilon) + \delta_{x, \bar{y}}v_y(1 - \varepsilon) + \varepsilon(1 - v_y), \tag{S18}
\]

and we can easily verify that \( \sum_x P[y, x] = P[y] = 1/2 \).

From Eq. (14), we find

\[
e^{-\sigma_y} = 2v_y(1 - \varepsilon) + 2\varepsilon(1 - v_y), \tag{S19}
\]

and we can verify the detailed fluctuation theorem

\[
\frac{P[y, x]}{P[y, x]} = e^{\beta W[x, y] + \sigma_y} = \frac{\delta_{y, y'}v_y(1 - \varepsilon) + \delta_{y, \bar{y}}v_y(1 - \varepsilon) + \varepsilon(1 - v_y)}{v_y(1 - \varepsilon) + \varepsilon(1 - v_y)}. \tag{S20}
\]

We note that for error-free measurements, we obtain \( P'[y, x] = P[y, x] \) and \( \beta W = -\sigma_y = \ln(2v_y) \), reflecting the fact that the full entropy production (or extracted work) can be inferred from the measurement outcome \( y \). The average of the inferable entropy production is given by

\[
\langle \sigma_y \rangle = -\ln(2) - \frac{1}{2} \ln(v_y(1 - \varepsilon) + \varepsilon(1 - v_y)). \tag{S21}
\]

Finally, the efficacy parameter is given by

\[
\gamma = \sum_{y = l, r} P[y|\Lambda(y)] = \langle e^{-\sigma_y} \rangle = \sum_{y = l, r} [v_y(1 - \varepsilon) + \varepsilon(1 - v_y)]. \tag{S22}
\]

For \( v_l = v_r \), we thus find \( \ln(\gamma) = -\langle \sigma_y \rangle \). Otherwise, \( \langle \sigma_y \rangle \) gives us a strictly stronger bound on the extracted work. The different bounds on the work obtained by the mutual information, the efficacy parameter, and the inferable entropy are shown in Fig. 2(a).

We close this section with a brief discussion on the conventional definition of the backward probability including feedback

\[
\hat{P}[y, x] = P[x|\Lambda(y)]P[y] = \delta_{x,y}v_y/2 + \delta_{x,\bar{y}}(1 - v_y)/2. \tag{S23}
\]

This results in the detailed fluctuation theorem

\[
\frac{\hat{P}[y, x]}{P[y, x]} = e^{\beta W[x, y] - I[x, y]} = \delta_{x,y} v_y \frac{1}{1 - \varepsilon} + \delta_{x,\bar{y}} \frac{1 - v_y}{\varepsilon}, \tag{S24}
\]

which diverges for \( \varepsilon \to 0 \) because \( P[x, y] \) is equal to zero for measurement outcomes that do not correspond to \( x \) whereas \( \hat{P}[x, y] \) remains finite as it is independent of \( \varepsilon \).
C. BROWNIAN PARTICLE IN A HARMONIC TRAP

We consider a Brownian particle in a harmonic trap. Based on the outcome of a position measurement, the minimum of the trap is moved in order to extract work. The particle is initially in thermal equilibrium and the trap potential is centered around \( x = 0 \)

\[
V_0(x) = \frac{k}{2}x^2, \quad P[x] = P[x|\Lambda(y)] = \frac{e^{-\beta V_0(x)}}{\sqrt{2\pi k_B T/k}}. \tag{S25}
\]

As for the Szilard engine, the initial position of the particle, \( x \) is independent of the protocol. A measurement of position is then performed. We assume the measurement outcome to have a Gaussian distribution

\[
P_m[y|x] = \frac{e^{-\frac{(y-x)^2}{2\Sigma^2}}}{\sqrt{2\pi \Sigma}}, \tag{S26}
\]

where \( \Sigma \to 0 \) corresponds to an error-free measurement. The trapping potential is then shifted such that the minimum coincides with the measurement outcome

\[
V_y(x) = \frac{k}{2}(x-y)^2. \tag{S27}
\]

Finally, the system equilibrates in the new trap potential.

The work extracted by this process can be written as

\[
W[x,y] = ky(x-y/2), \quad \langle W[x,y] \rangle = \frac{k_B T}{2} - \frac{k\Sigma^2}{2}, \tag{S28}
\]

where we used \( P[x,y] = P_m[y|x]P[x] \) to evaluate the average. The mutual information (transfer entropy) is given by

\[
I[x:y] = \frac{1}{2} \ln \left( \frac{k_B T}{k\Sigma^2} + 1 \right) - \frac{y^2k_B T + x(2y + x)(k_B T + k\Sigma^2)}{\Sigma^2(k_B T + k\Sigma^2)}, \tag{S29}
\]

with an average value of

\[
\langle I[x:y] \rangle = \frac{1}{2} \ln \left( \frac{k_B T}{k\Sigma^2} + 1 \right) \geq \beta \langle W[x,y] \rangle, \tag{S30}
\]

where the last inequality can easily be proven. We note that the average mutual information diverges in the error-free measurement limit where \( \Sigma \to 0 \). The reason for this is that a perfect position measurement gives an infinite amount of information.

For the backward experiment, the system starts in thermal equilibrium with the external potential \( V_y(x) \) chosen with probability

\[
P[y] = \int dx P_m[y|x]P[x] = \sqrt{\frac{k}{2\pi(k_B T + k\Sigma^2)}} e^{-\frac{k_B T}{2(k_B T + k\Sigma^2)}}, \tag{S31}
\]

The external potential is then shifted to \( V_0(x) \) and the particle location is measured immediately. Finally, the particle thermalizes to recover the initial state. We note that since all variables are position variables, we have \( x^\dagger = x \) and \( y^\dagger = y \).

The probability that the particle is located at position \( x \), given the initial trapping potential \( V_y(x) \), reads

\[
P[x|\Lambda(y)\dagger] = \frac{e^{-\beta V_y(x)}}{\sqrt{2\pi k_B T/k}}. \tag{S32}
\]

The probability that the particle is measured to be in the minimum of the applied trapping potential reads

\[
P[y|\Lambda(y)\dagger] = \int dx P_m[y|x]P[x|\Lambda(y)\dagger] = \sqrt{\frac{k}{2\pi(k_B T + k\Sigma^2)}}, \tag{S33}
\]
From $\gamma = \int dy P[y|\Lambda(y)^\dagger]$, we find that the efficacy parameter diverges. The reason for this is that there are infinitely many protocols since there are infinitely many measurement outcomes for a position measurement. The inferable entropy production however remains finite. From Eq. (14) in the main text, we find

$$\sigma_y = -\frac{k y^2}{2(k_B T + k \Sigma^2)}; \quad \langle \sigma_y \rangle = -\frac{1}{2}. \quad (S34)$$

While the efficacy parameter does not provide an inequality, and the mutual information provides an irrelevant inequality as the measurement error becomes small, the inferable entropy production always provides a reasonable bound on the extracted work. The tightness of this bound gives insight into how sharply the measurement resolves the position of the particle.

Finally, from Eq. (15) in the main text, we find

$$P^\dagger[x, y] = \frac{\sqrt{k}}{2\pi \sqrt{k_B T k \Sigma^2}} \exp \left[ -\frac{(x - y)^2 k_B T + k \Sigma^2}{2 k_B T k \Sigma} - \frac{k y^2}{2(k_B T + k \Sigma^2)} \right], \quad (S35)$$

and it is straightforward to verify the detailed fluctuation theorem

$$\frac{P^\dagger[x, y]}{P[x, y]} = e^{\sigma_y + \beta W[x, y]}. \quad (S36)$$