A PSH HOPF LEMMA FOR DOMAINS WITH CUSP CONDITIONS

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ABSTRACT. We obtain a psh Hopf lemma for domains satisfying certain cusp conditions by using a sharp estimate for the Green function of a planar cusp along the axis. As an application, we obtain a negative psh exhaustion function with certain global growth estimate on a pseudoconvex domain with Hölder boundary.

1. INTRODUCTION

Let \( \Omega \subset \mathbb{R}^n \) be a domain satisfying the inner ball condition (e.g., a domain with \( C^2 \) boundary). The classical Hopf lemma asserts that if \( u \) is a negative harmonic function on \( \Omega \) with \( u(x_0) = 0 \) for some \( x_0 \in \partial \Omega \), then \( \partial u(x_0)/\partial \nu < 0 \), where \( \nu \) is the inner normal vector field of \( \partial \Omega \). In particular, we have \( u \lesssim -\delta_\Omega \), where \( \delta_\Omega \) is the boundary distance. Sometimes, such an inequality is also called a Hopf lemma in literature. Indeed, we only need \( u \lesssim -\delta_\Omega^{1/\alpha} \) \((0 < \alpha < 1)\) for some applications (cf. [6]), which is available for domains satisfying cone conditions (cf. [10], [13]).

In this paper, we obtain a Hopf lemma for plurisubharmonic (psh) functions on domains satisfying cusp conditions. A domain of the form

\[
\Gamma = \Gamma(C, \alpha) = \{ z \in \mathbb{C}^n; \text{Re } z_n > C(|z'|^2 + (\text{Im } z_n)^2)^{\alpha/2}\}
\]

is called a \((C, \alpha)\)-cusp along the \text{Im} \( z_n \)-axis and with vertex at the origin. By a finite \((C, \alpha, r)\)-cusp we mean an intersection of \( \Gamma \) with a ball centered at the vertex with radius \( r \). Roughly speaking, a domain is said to satisfy the \((C, \alpha, r)\)-cusp condition if for any \( p \in \partial \Omega \), one can transplant a finite \((C, \alpha, r)\)-cusp inside \( \Omega \) with vertex \( p \) (through translations and rotations). We refer to Definition 3.1 for more details.

The main result of the paper is the following

**Theorem 1.1** (Hopf lemma). Let \( \Omega \subset \mathbb{C}^n \) be a domain satisfying the \((C, \alpha, r)\)-cusp condition. Then

\[
\rho \lesssim -\exp \left( -\frac{A}{\delta_\Omega^{1/\alpha-1}} \right)
\]

for every \( \rho \in PSH^{-}(\Omega) \), where \( PSH^{-}(\Omega) \) denotes the set of negative psh functions on \( \Omega \), and

\[
A := \frac{\pi C^{1/\alpha}}{2(1/\alpha - 1)}.
\]
Usually, Hopf lemmas for subharmonic functions are proved by constructing barrier functions for the complement of the closure of the domain (see, e.g., [11], [12], [13]). This breaks down in our case. Indeed, one can apply Wiener’s criterion to show that a closed cusp $\Gamma$ is thin at the vertex for $0 < \alpha < 1$ and $n \geq 2$ (see Appendix). That is, there does not exist a barrier function at the vertex of $\Gamma$ for $C^1 \setminus \overline{\Gamma}$, which clearly satisfies certain cusp condition. On the other hand, a planar cusp is not thin at the vertex. The proof of Theorem 1.1 is based on the following optimal estimate of the Green function on a planar cusp along the axis, which is of independent interest.

**Theorem 1.2.** Let

$$\Gamma := \{z = x + iy; x > C|y|^\alpha\},$$

where $C > 0$ and $0 < \alpha < 1$, and $\Delta_R$ be the disk centered at 0 with radius $R$. For $a := R/2$ and $0 < t \ll 1$, the (negative) Green function of $\Gamma \cap \Delta_R$ satisfies

$$g_{\Gamma \cap \Delta_R}(t, a) \approx -\exp\left(-\frac{A}{t^{1/\alpha - 1}}\right).$$

The key ingredient of the proof of Theorem 1.2 is an explicit construction of a comparison function of a planar cusp at the vertex by using conformal mappings and a result of Li-Nirenberg [9].

This paper is motivated by a recent work [2] of the first author, where the local order of hyperconvexity is obtained for bounded pseudoconvex domains with H"older boundaries (written as H"older domains for short). This together with a result of Kerzman and Rosay [7] imply the hyperconvexity of such domains. It remains a question whether there exists a negative psh exhaustion function with certain global estimate on bounded pseudoconvex H"older domains.

We shall give a partial answer to this question by using Theorem 1.1. Let $\Omega$ be a bounded pseudoconvex H"older domain, which naturally satisfies certain $(C, \alpha, r)$-cusp condition (see Proposition 3.2). Theorem 1.1 and the main result in [2] yield

$$\psi(-\delta_\Omega(z)) \leq \rho_j(z) \leq \varphi(-\delta_\Omega(z))$$

where $\psi(t) := -C_1(-\log(-t))^{-\beta}$ and $\varphi(t) := -C_2 \exp(-A/(-t)^{1/\alpha - 1})$ for certain positive constants $\alpha, \beta, C_1, C_2, M$. Let $\{\alpha_\nu\}$ be an increasing sequence with $\alpha_\nu < 0$, $\alpha_1$ sufficiently close to 0, and

$$\alpha_{\nu+1} = \psi^{-1}(\varphi(\alpha_\nu)/2).$$

We define

$$\lambda(t) := \max\{\nu \in \mathbb{Z}^+; \alpha_\nu \leq -t\}, \quad t > 0.$$ 

Since $\lim_{\nu \to \infty} \alpha_\nu = 0$, it follows that $\lambda(t) \to +\infty$ as $t \to 0+$. It is easy to see that $\lambda$ is not an elementary function. The method of Coltoiu-Mihalche [4] together with Lemma 2.1 in [2] yield the following

**Theorem 1.3.** Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex H"older domain and $\overline{B}$ be a closed ball in $\Omega$. Then there exists a constant $\varepsilon_0 > 0$ depending only on $\Omega$ and $\overline{B}$ such that

$$\theta_{\Omega, \overline{B}} \gtrsim -e^{-\varepsilon_0 \lambda(\delta_\Omega)}.$$
Lemma 2.1. There exists a constant $R > 0$ depending only on $C$ and $\alpha$ such that the holomorphic function $F$, given in (2.2), maps $\Gamma \cap \Delta_R$ conformally to a domain $D \subset \mathbb{C}$.
Proof. It suffices to show that $F$ is injective on $\Gamma \cap \Delta_R$ for some sufficiently small $R$. We first verify the injectivity of $G(z) := -A/z^{1/\alpha-1}$. Given $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, $G(z_1) = G(z_2)$ if and only if $r_1 = r_2$ and $(1/\alpha - 1)(\theta_1 - \theta_2) = 2k\pi$ for some integer $k$. Since

$$|\theta(s)| \leq |c|s^{1/\alpha-1} \leq C^{1/\alpha}s^{1/\alpha-1}$$

in view of (2.1), we have $|\theta_1 - \theta_2| < 2\pi/(1/\alpha - 1)$ for $z_1, z_2 \in \Gamma \cap \Delta_R$, provided that

$$C^{1/\alpha}R^{1/\alpha-1} < \frac{\pi}{(1/\alpha - 1)}.$$ 

Thus $G$ is injective on $\Gamma \cap \Delta_R$. Moreover, for $z = \gamma_c(s) \in \Gamma$, we have $\mathrm{Im} G(z) = \overline{\theta}(s) \in (-\pi/2, \pi/2)$ in view of (2.3). Since the exponential mapping $w \mapsto e^w$ is injective on the strip $\{\mathrm{Im} w < \pi/2\}$, we conclude that $F = e^G$ is injective on $\Gamma \cap \Delta_R$. \hfill $\Box$

The domain $D$ is also symmetric about the $\overline{x}$-axis and $\partial D$ is more regular than $\partial \Gamma$. In a neighbourhood of $0 \in \partial D$, the curve $\partial D$ is precisely $F \circ \gamma_c$ with $c = \pm C^{-1/\alpha}$. Since $\overline{\theta}(s) \to \pm \pi/2$ as $s \to 0$ and $c = \pm C^{-1/\alpha}$, $\partial D$ is tangent to the $y$-axis at $0$. To obtain an estimate for the Green function of $D$ near $0 \in \partial D$, we need more information about the regularity of $\partial D$ near $0$. Choose a neighbourhood $U \ni 0$ such that

$$D \cap U = \{z = x + iy \in U; \ x > h(|y|), \ |y| < \varepsilon_0\},$$

where $h$ is a real function on $[0, \varepsilon_0)$ and $0 < \varepsilon_0 \ll 1$ is a constant depending only on $C$ and $\alpha$. Note that $h$ is smooth on $(0, \varepsilon_0)$ and differentiable at $0$ with $h(0) = h'(0) = 0$. Now we fix $c = C^{-1/\alpha}$ and set

$$\overline{x}(s) = \overline{r}(s) \cos \overline{\theta}(s), \quad \overline{y}(s) = \overline{r}(s) \sin \overline{\theta}(s).$$

The function $h$ is given by the equation

$$\overline{x}(s) = h(\overline{y}(s)), \quad 0 < s \ll 1.$$ 

Thus

$$(2.6) \quad h(\overline{y}(s)) = \frac{\overline{y}(s)}{\tan \overline{\theta}(s)} \sim \overline{y}(s) \cdot \left(\frac{\pi}{2} - \overline{\theta}(s)\right), \quad s \to 0.$$ 

Moreover, (2.4) together with (2.1) imply

$$\overline{\theta}(s) = \frac{A \sin \left((1/\alpha - 1)\arctan(C^{-1/\alpha}s^{1/\alpha-1})\right)}{s^{1/\alpha-1}(1 + s^2/\alpha - 1)(1/\alpha - 1)/2}$$

$$(2.7) \quad = \frac{A \sin \left((1/\alpha - 1)(C^{1/\alpha}s^{1/\alpha-1} - \frac{1}{2}C^{-3/\alpha}s^{3/\alpha-3} + o(s^{3/\alpha-3}))\right)}{s^{1/\alpha-1} + \frac{1}{2}(1/\alpha - 1)s^{3/\alpha-3} + o(s^{3/\alpha-3})}$$

$$= \frac{\pi}{2} - A_1s^{2/\alpha-2} + o(s^{2/\alpha-2}), \quad s \to 0.$$
for some constant $A_1 = A_1(C, \alpha) > 0$. It follows from (2.1), (2.3) and (2.7) that

$$
\log \frac{1}{|y(s)|} = \log \frac{1}{r(s)} - \log \sin \bar{\theta}(s) = \frac{A \cos \left( \frac{(1/\alpha - 1)\theta(s)}{r(s)^{1/\alpha - 1}} \right) - \log \sin \bar{\theta}(s)}{s^{1/\alpha - 1}} \quad \text{as} \quad s \to 0.
$$

By (2.6), (2.7) and (2.8), we finally obtain

$$
h(|y|) \sim A_2 |y| \left( \log \frac{1}{|y|} \right)^{-2}, \quad y \to 0,
$$

where $A_2 = A_1 A^2$.

Recall that the (negative) Green function of a planar domain $\Omega$ can be defined as

$$
g_\Omega(z, a) := \sup \left\{ \varphi \in SH^-(\Omega); \varphi(z) - \log \frac{1}{|z - a|} = O(1) \right\},
$$

where $SH^-(\Omega)$ denotes the set of negative subharmonic functions on $\Omega$. It is well-known that $g_\Omega(\cdot, a)$ is a subharmonic function on $\Omega$ and is harmonic on $\Omega \setminus \{a\}$.

**Proof of Theorem 7.2** By Lemma 2.1 we have

$$
g_{\Omega \cap \Delta_r}(t, a) = g_D(F(t), F(a)).
$$

It suffices to estimate $g_D(x, F(a))$ when $x \to 0^+$ and $x \in \mathbb{R}$.

From (2.9), there is a constant $B > 0$ such that

$$
h(u) < Bu \left( \log \frac{1}{u} \right)^{-2} =: h_B(u), \quad 0 < u < u_0 < 1.
$$

Here, the constant $u_0$ depends only on $C$ and $\alpha$. Let

$$
D_B := \{ z = x + iy; \ x > h_B(|y|), \ -u_0 < y < u_0 \}.
$$

It follows from (2.11) that $D_B \cap \Delta_r \subset D$ for some $0 < r \ll 1$. We may choose $r = r(C, \alpha)$ so small that $F(a) \notin \Delta_r$. Thus $g_D(\cdot, F(a))$ is a harmonic function on $D \cap \Delta_r$.

The boundary of $D_B \cap \Delta_r$ can be divided into the following two parts:

$$
E_1 := \partial D_B \cap \{ |z| = r \}, \quad E_2 := \{ z = x + iy; \ x = h_B(|y|), \ |z| < r \}.
$$

By (2.11), $E_1$ is a compact subset in $D$, while $E_2$ satisfies the following conditions:

1. $h_B$ is smooth on $(0, u_0)$ and $C^1$ on $[0, u_0]$;
2. $\int_0^x h_B(u)/u^2du < \infty$ for $0 < \varepsilon < \min\{u_0, 1\}$;
3. $h_B(0) = h'_B(0) = 0$ and $h''_B(u) \geq 0$;
4. $h''_B(u) + h_B(u)/u$ is nonincreasing.

It follows from the proof of Theorem 1 of [9] that

$$
\varphi(z) := x + 2h_B(x) + 2x \int_0^x \frac{h_B(u)}{u^2}du - 2h_B(|z|), \quad (z = x + iy)
$$
is a subharmonic function on $D_B \cap \Delta_r$ with $\varphi \leq 0$ on $E_2$ and $\varphi(x) > 0$ for $x > 0$. The constant
\[
M := \frac{\sup_{E_1} \varphi}{\inf_{E_1} (-g_D)}
\]
is positive and depends only on $C$ and $\alpha$. Thus
\[
(2.12) \quad -g_D(z, F(a)) \geq \frac{\varphi(z)}{M}, \quad z \in E_1.
\]
The same inequality clearly holds for $z \in E_2$ since $g_D \leq 0$. Thus (2.12) holds for all $z \in D_B \cap \Delta_r$ in view of the maximum principle. In particular,
\[
g_D(x, F(a)) \lesssim -\varphi(x) \leq -x, \quad 0 < x \ll 1.
\]
On the other hand, by comparing $g_D(\cdot, F(a))$ with the negative harmonic function $z = x + iy \mapsto -x$ on $D$, we have
\[
g_D(x, F(a)) \gtrsim -x, \quad 0 < x \ll 1.
\]
The conclusion follows immediately from (2.10). \qed

3. THE CUSP CONDITION

For a unit vector $v \in \mathbb{C}^n$, we consider the half space
\[
H_v := \{z \in \mathbb{C}^n; \ \text{Re} \langle z, v \rangle > 0\}.
\]
Let $\pi_v$ be the orthogonal projection $H_v \to \partial H_v$. A $(C, \alpha)$-cusp with axis $v$ and vertex $p$ is defined to be
\[
\Gamma(p, v, C, \alpha) := \{z \in H_v; \ \text{Re} \langle z - p, v \rangle > C|\pi_v(z - p)|^\alpha\}.
\]

Definition 3.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $B_r(p)$ be the ball with center $p$ and radius $r$. We say that $\Omega$ satisfies the $(C, \alpha, r)$-cusp condition if there is some $r > 0$, such that every $z \in \Omega$ sufficiently close to $\partial\Omega$ lies on the axis of a $(C, \alpha)$-cusp $\Gamma(p, v, C, \alpha)$ with
\begin{enumerate}
\item $p \in \partial\Omega$;
\item $\Gamma(p, v, C, \alpha) \cap B_r(p) \subset \Omega$;
\item $|z - p| < r/2$.
\end{enumerate}

The condition (3) implies that $z$ is not very close to $\partial B_r(p)$ so that $|z - p| < \delta_{\partial B_r(p)}(z)$. Hence
\[
\delta_{\Gamma \cap B_r(p)}(z) = \delta_{\Gamma}(z),
\]
where $\Gamma := \Gamma(p, v, C, \alpha)$.

Lemma 3.1. For $z := p + tv \in \Gamma(p, v, C, \alpha)$ $(0 < t \ll 1)$, we have
\[
\delta_{\Gamma}(z) \approx t^{1/\alpha}.
\]

Proof. We may assume that $p = 0$ and $v = (0, \cdots, 0, 1)$. Set $z = (0, \cdots, 0, \text{Re} z_n)$ (i.e., $t = \text{Re} z_n$). For any $z^* \in \partial \Gamma(p, v, C, \alpha)$, we define
\[
w^* = (z_1^*, \cdots, z_{n-1}^*, \text{Im} z_n^*) \in \mathbb{C}^{n-1} \times \mathbb{R}
\]
so that Re $z_n^* = C |w^*|^\alpha$. In particular, if we take $z_0^* = (C^{-1/\alpha} t^{1/\alpha}, 0, \ldots, 0, t) \in \partial \Gamma(p, v, C, \alpha)$, then
\[
\delta_\Gamma(z) \leq |z - z_0^*| = C^{-1/\alpha} |z - p|^{1/\alpha}.
\]
On the other hand, we have
\[
|z - z^*|^2 = |w^*|^2 + |\text{Re} (z_n - z_n^*)|^2.
\]
We shall divide the argument into the following two cases:

(1) If $\text{Re} z_n^* \leq \frac{1}{2} \text{Re} z_n$, then $\text{Re} (z_n - z_n^*) \geq \frac{1}{2} \text{Re} z_n > 0$, so that
\[
|z - z^*| \geq |\text{Re} (z_n - z_n^*)| \geq \frac{1}{2} \text{Re} z_n = \frac{1}{2} t \geq \frac{1}{2} t^{1/\alpha};
\]

(2) If $\text{Re} z_n^* > \frac{1}{2} \text{Re} z_n$, then we have
\[
|z - z^*| \geq |w^*| = \left( \frac{1}{C} \right)^{1/\alpha} (\text{Re} z_n^*)^{1/\alpha} > \left( \frac{1}{2C} \right)^{1/\alpha} (\text{Re} z_n)^{1/\alpha}
\]
\[
= \left( \frac{1}{2C} \right)^{1/\alpha} t^{1/\alpha}.
\]
Hence $\delta_\Gamma(z) \geq B t^{1/\alpha}$ with $B := \min \{ 1/2, (2C)^{-1/\alpha} \}$. \hfill \Box

**Remark.** Let $\Omega$ be a domain satisfying the $(C, \alpha, r)$-cusp condition and $z \in \Omega$ satisfying the conditions in Definition\[3.7\] Lemma\[3.7\] together with (3.1) imply that
\[
|z - p| \lesssim \delta_\Gamma(z)^\alpha = \delta_{\Gamma \cap B_r(p)}(z)^\alpha \leq \delta_\Omega(z)^\alpha.
\]
That is, if $z$ is sufficiently close to the boundary, then $|z - p|$ is also sufficiently small.

Recall that $\Omega$ is a bounded Hölder domain if $\partial \Omega$ is locally the graph of a Hölder continuous function. More precisely, there exist

1. a finite open covering $\{ V_j \}$ of $\partial \Omega$;
2. $p_j \in V_j$;
3. a unit vector $v_j$;
4. a neighbourhood $V_j'$ of 0 in $\partial H_{v_j}$ with $\pi_{v_j} (w - p_j) \in V_j'$ for all $w \in V_j$;
5. a Hölder continuous function $h_j$ of order $\alpha_j$ on $V_j'$ with $h_j(0) = 0$, such that
\[
\Omega \cap V_j = \{ w \in V_j; \text{Re} (w - p_j, v_j) > h_j(\pi_{v_j}(w - p_j)) \}.
\]

**Proposition 3.2.** If $\Omega \subset \mathbb{C}^n$ is a bounded Hölder domain, then there exist constants $C, \alpha$ and $r$ such that $\Omega$ satisfies the $(C, \alpha, r)$-cusp condition. Moreover precisely, $\alpha = \min \{ \alpha_j \}$, where $\alpha_j$ is given as above.

**Proof.** Let $V_j, p_j, v_j, V_j', h_j$ be as above. Suppose that
\[
|h_j(x_1) - h_j(x_2)| \leq C_j |x_1 - x_2|^\alpha_j, \quad x_1, x_2 \in V_j'
\]
for some $C_j > 0$. For any $w \in V_j$, let $w^* \in \partial \Omega$ be the point with
\[
\pi_{v_j} (w - p_j) = \pi_{v_j} (w^* - p_j).
\]
That is, $w = w^* + tv_j$ for some $t$. We may take another covering $\{U_j\}$ of $\partial \Omega$ with $p_j \in U_j \subset \subset V_j$, such that $w^* \in U_j \cap \partial \Omega$ and $|w - w^*| < r/2$ for some $0 < r \ll 1$ whenever $w \in U_j$. Moreover, we may take $r$ so small that $B_r(x) \subset V_j$ for all $x \in U_j \cap \partial \Omega$. Set

$$U_j^+ := U_j \cap \Omega.$$ 

Then any $z \in \Omega$ sufficiently close to $\partial \Omega$ lies in some $U_j^+$. For $p := z^*$, it follows from (3.2) and the definition of $U_j$ that $z \in \Gamma(p, v_j, C_j, \alpha_j) \cap B_r(p)$ and $|z - p| < r/2$. Moreover, for any $w \in \Gamma(p, v_j, C_j, \alpha_j) \cap B_r(p)$, we have $w \in V_j$ and

$$\text{Re} \langle w - p, v_j \rangle > C_j |\pi_{v_j}(w - p)|^\alpha = C_j |\pi_{v_j}(w - p_j) - \pi_{v_j}(p - p_j)|^\alpha \\
\geq h(\pi_{v_j}(w - p_j)) - h(\pi_{v_j}(p - p_j)).$$

Since $p \in \partial \Omega$, we have

$$\text{Re} \langle p - p_j, v_j \rangle = h(\pi_{v_j}(p - p_j)).$$

Hence $\text{Re} \langle w - p_j, v_j \rangle > h(\pi_{v_j}(w - p_j))$, i.e.,

$$\Gamma(p, v_j, C_j, \alpha_j) \cap B_r(p) \subset \Omega \cap V_j \subset \Omega.$$ 

It suffices to take $\alpha = \min\{\alpha_j\}$ and $C = \max\{C_j\}$. 

4. PROOF OF THEOREM 1.1

The following lemma is essentially known, but we shall provide a proof for the sake of completeness.

**Lemma 4.1.** Let $\Omega \subset \mathbb{C}$ be a bounded domain and $\rho \in SH^{-} (\Omega)$. If $\Delta_{R_1} (a) \subset \subset \Omega \subset \subset \Delta_{R_2} (a)$, then

$$\rho(z) \leq \frac{\inf_{\partial \Delta_{R_1} (a)} (-\rho)}{\log (R_2/R_1)} \cdot g_{\Omega}(z, a)$$

in a neighbourhood of $\partial \Omega$.

**Proof.** Let $\{\Omega_m\}$ be a sequence of domains with smooth boundaries such that $\Omega_m \subset \subset \Omega_{m+1}$ and $\bigcup \Omega_m = \Omega$. Then $g_{\Omega_m} \searrow g_{\Omega}$ when $m \to \infty$. We may assume that $\Delta_{R_1} (a) \subset \subset \Omega_m \subset \subset \Delta_{R_2} (a)$. Thus

$$g_{\Omega_m}(z, a) \geq \log \frac{|z - a|}{R_2}, \quad z \in \Omega_m.$$ 

In particular,

$$g_{\Omega_m}(z, a) \geq -\log \frac{R_2}{R_1}, \quad z \in \partial \Delta_{R_1} (a).$$

For $C_0 := \inf_{\partial \Delta_{R_1} (a)} (-\rho) = -\sup_{\partial \Delta_{R_1} (a)} \rho$, we have

$$\frac{\log (R_2/R_1)}{C_0} \rho \leq -\log \frac{R_2}{R_1}, \quad z \in \partial \Delta_{R_1} (a).$$

Note that $g_{\Omega_m}(\cdot, a) = 0$ on $\partial \Omega_m$. It follows from the maximum principle that

$$\rho(z) \leq \frac{C_0}{\log (R_2/R_1)} g_{\Omega_m}(z, a).$$
on $\Omega_m \setminus \Delta_{R_1}(a)$. Letting $m \to \infty$, we complete the proof.

**Proof of Theorem 1.1** Let $R$ be the constant in Lemma 2.1 for planar $(C,\alpha)$-cusps. We may assume that $0 < R < r$. Given $z \in \Omega$ sufficiently close to $\partial \Omega$ (i.e., $\delta_\Omega(z) \ll 1$), there exists a $(C,\alpha)$-cusp $\Gamma(p, v, C, \alpha)$ satisfying the conditions (1)-(3) in Definition 3.1 with $z$ lying on the axis. By the remark after Lemma 3.1 we have $|z - p| \ll 1$. We may identify

$$D_p := \{p + tv; t \in \mathbb{C}\} \cap \Gamma(p, v, C, \alpha) \cap B_R(p)$$

with a domain in $\mathbb{C}$. Thus $\rho|_{D_p}$ is a subharmonic function on $D_p$. Set $a = p + Rv/2$. By Lemma 4.1 and Theorem 1.2 we have

$$\rho(z) \lesssim g_{D_p}(z, a) \leq g_{D_p \cap B_R(p)}(z, a) \lesssim -\exp \left(-\frac{A}{|z - p|^{1/\alpha - 1}}\right).$$

Since $|z - p| \geq \delta_\Omega(z)$, we conclude the proof.

### 5. Proof of Theorem 1.3

Recall that $\{U_j\}$ is a finite covering of $\partial \Omega$, $\rho_j$ is a negative psh function on $U_j$ with $\psi(-\delta_\Omega) \leq \rho_j \leq \varphi(-\delta_\Omega)$, and $\alpha_{\nu+1} = \psi^{-1}(\varphi(\alpha_{\nu})/2)$ with $\alpha_1$ sufficiently close to 0. We set

$$a_\nu = \begin{cases} \varphi(\alpha_{\nu}), & \text{if } \nu \text{ is odd;} \\ \psi(\alpha_{\nu}), & \text{if } \nu \text{ is even.} \end{cases}$$

From (1.5) and the fact that $\varphi \geq \psi$, we obtain

$$a_{2\nu} = a_{2\nu-1}/2, \quad a_{2\nu+1} = \varphi \circ \psi^{-1}(\varphi(\psi^{-1}(a_{2\nu})/2)) \geq a_{2\nu}/2.$$ 

In particular, $\{a_\nu\}$ is an increasing sequence with $a_\nu \to 0$ as $\nu \to \infty$. We consider the following convex increasing function $\tau : (-\infty, 0) \to [0, +\infty)$ introduced in [4]:

$$\tau(x) = \begin{cases} 0, & x \leq a_1, \\ \nu - \sum_{k=1}^{\nu-1} a_{k+1}/a_k - x/a_\nu, & a_\nu \leq x \leq a_{\nu+1}, \end{cases}$$

which satisfies

$$\tau(a_{\nu+1}) - \tau(a_\nu) = 1 - \frac{a_{\nu+1}/a_\nu}{a_\nu} < 1, \quad \forall \nu \in \mathbb{Z}^+.$$ 

On the other hand, we infer from (5.2) that $\tau(a_{\nu+1}) - \tau(a_\nu) \geq 1/2$. Hence

$$\tau(a_\nu) \geq \frac{\nu}{2} - c_0$$

for some constant $c_0 > 0$. If $z \in \Omega \cap U_j \cap U_k$ and $a_{2\nu} \leq -\delta_\Omega(z) \leq a_{2\nu+2}$, then it follows from (1.4) and (5.1) that

$$\min\{\rho_j(z), \rho_k(z)\} \geq \psi(\alpha_{2\nu}) = a_{2\nu}$$

and

$$\max\{\rho_j(z), \rho_k(z)\} \leq \varphi(\alpha_{2\nu+2}) \leq \varphi(\alpha_{2\nu+3}) = a_{2\nu+3}.$$ 

Thus $|\tau(\rho_j) - \tau(\rho_k)| < 3$ on $\Omega \cap U_j \cap U_k$. Since $\tau$ is convex, we have

$$|\tau(\rho_j - \varepsilon) - \tau(\rho_k - \varepsilon)| < 3$$
on $\Omega \cap U_j \cap U_k$ for any $\varepsilon > 0$. Moreover,
\begin{equation}
\tau(\rho_j - \varepsilon) \leq \tau(a_{2\nu+3} - \varepsilon) \leq \tau(a_{2\nu} - \varepsilon) + 3
\end{equation}
and
\begin{equation}
\tau(\rho_j - \varepsilon) \geq \tau(a_{2\nu} - \varepsilon) \geq \tau(a_{2\nu+2} - \varepsilon) - 2
\end{equation}
Next we use the standard Richberg technique (compare [5]) to patch these $\rho_j$ together. Choose $U'_j \subset U'_j \subset U_j$ and $U_0 \subset \Omega$ such that $\overline{\Omega_j \setminus U_0} \subset U'_j$. Take $\chi_j \in C_0^\infty(U'_j)$ with $\chi_j \equiv 1$ in a neighbourhood of $\overline{U'_j}$. There are constant $M, N$ satisfying $|z|^2 - M < 0$ and
\begin{equation}
3\chi_j + N(|z|^2 - M) \in PSH(\mathbb{C}^n).
\end{equation}
Set
$$u_{j,\varepsilon}(z) := \tau(\rho_j(z) - \varepsilon) + 3\chi_j(z) - 3 + N(|z|^2 - M).$$
Then $u_{j,\varepsilon}$ is a plurisubharmonic function on $U_j$, and it follows from (5.4) that $u_{j,\varepsilon} < u_{k,\varepsilon}$ in a neighbourhood of $\Omega \cap \overline{U'_k} \cap \partial U'_j$. Choose $a$ so that $\sup_{U_0 \cup U_j} \rho_j < a < 0$ for all $j \geq 1$ and fix $N \gg 1$ such that (5.7) holds and $\tau(a) + N(|z|^2 - M) < 0$ on $\overline{\Omega}$. Hence
$$u_\varepsilon(z) := \max\{u_{j,\varepsilon}(z), \tau(a) + N(|z|^2 - M)\} \in PSH(\Omega)$$
when $\varepsilon \ll 1$. Moreover, it follows from (5.5) and (5.6) that
\begin{equation}
\tau(\rho_j - \varepsilon) - c_1 \leq u_\varepsilon \leq \tau(\rho_j - \varepsilon) + c_2
\end{equation}
for some constants $c_1, c_2 > 0$.
Now we shall derive a global estimate by the method in [2]. Fix $l \in \mathbb{Z}^+$ for a moment. We set
$$w_\nu := \frac{u_{-a_{2\nu+2l}}}{\tau(a_{2\nu+2l})}.$$  
For $\Omega_\nu := \{\delta_\Omega > -a_{2\nu}\}$, we infer from (5.8) that
\begin{equation}
\frac{\tau(a_{2\nu} + a_{2\nu+2l}) - c_1}{\tau(a_{2\nu+2l})} \leq \sup_{\partial \Omega_\nu} w_\nu \leq \frac{\tau(a_{2\nu} + a_{2\nu+2l}) + c_2}{\tau(a_{2\nu+2l})}
\end{equation}
and
\begin{equation}
\inf_{\Omega_\nu \cap \Omega_{\nu+1}} w_\nu \geq \frac{\tau(2a_{2\nu+2l}) - c_1}{\tau(a_{2\nu+2l})} = \frac{\tau(a_{2\nu+2l-1}) - c_1}{\tau(a_{2\nu+2l})}.
\end{equation}
If we choose $l \gg 1$ so that $(2l - 1)/2 > c_1 + c_2$, then

$\kappa_\nu := \frac{\inf_{\Omega \setminus \Omega_{\nu+l}} w_\nu - \sup_{\partial \Omega_\nu} w_\nu}{1 - \sup_{\partial \Omega_\nu} w_\nu} \geq \frac{\tau(a_{2\nu+2l-1}) - \tau(a_{2\nu} + a_{2\nu+2l}) - c_1 - c_2}{\tau(a_{2\nu+2l}) - \tau(a_{2\nu} + a_{2\nu+2l}) + c_1} \geq \frac{\tau(a_{2\nu+2l-1}) - \tau(a_{2\nu}) - c_1 - c_2}{\tau(a_{2\nu+2l}) - \tau(a_{2\nu-1}) + c_1} \geq \frac{(2l - 1)/2 - c_1 - c_2}{2l + 1 + c_1} \geq c_3$

for some constant $c_3 > 0$.

The rest part of the proof is essentially parallel to [2], which we still include here for the sake of completeness. Set $M_\nu := \sup_{\Omega \setminus \Omega_\nu} (-\varrho_{\Omega, \overline{B}})$. For $z \in \partial \Omega_\nu$, we have

$(5.10) \quad (1 - w_\nu(z)) M_\nu \geq \left[1 - \sup_{\partial \Omega_\nu} w_\nu\right](-\varrho_{\Omega, \overline{B}}(z)).$

Since $\varrho_{\Omega, \overline{B}}(z) \to 0$ when $z \to \partial \Omega$, (5.10) also holds on $\partial \Omega_{\nu+k}$ for $k \gg 1$. An equivalent statement of (5.10) is

$(5.11) \quad \varrho_{\Omega, \overline{B}}(z) \geq \frac{M_\nu}{1 - \sup_{\partial \Omega_\nu} w_\nu} (w_\nu(z) - 1),$

which actually holds for all $z \in \Omega_{\nu+k} \setminus \Omega_\nu$ by the maximal property of $\varrho_{\Omega, \overline{B}}$. Finally, letting $k \to \infty$, we conclude that (5.11) remains valid for $z \in \Omega \setminus \Omega_{\nu+1}$, i.e.,

$-\varrho_{\Omega, \overline{B}}(z) \leq \frac{1 - \inf_{\Omega \setminus \Omega_{\nu+l}} w_\nu}{1 - \sup_{\partial \Omega_\nu} w_\nu} M_\nu = (1 - \kappa_\nu) M_\nu, \quad z \in \Omega \setminus \Omega_{\nu+1}.$

This combined with (5.9) gives

$M_{\nu+l} \leq (1 - c_3) M_\nu.$

Thus

$-\varrho_{\Omega, \overline{B}}(z) \gtrsim (1 - c_3)^{\nu/l} = \exp \left(-\nu l \log \frac{1}{1 - c_3}\right)$

for $z \in \Omega \setminus \Omega_\nu$. The assertion follows immediately from the definition of $\lambda(t)$. \hfill \Box

6. APPENDIX: THINNESS AT THE VERTEX OF A CLOSED CUSP

Recall that a boundary point $x_0$ of a domain $\Omega \subset \mathbb{R}^m$ is regular if and only if $\Omega$ admits a barrier at $x_0$. This is also equivalent to the thinness of $\mathbb{R}^m \setminus \Omega$ at $x_0$ (cf. [8], Theorem 4.8 and 5.10). Thinness can be characterized by using Wiener’s criterion. For a compact set $K \subset \mathbb{R}^m$ ($m \geq 3$), we define the capacity of $K$ by

$\text{Cap}(K) := \inf \left\{ \int_{\mathbb{R}^m \setminus K} |\nabla \varphi|^2; \varphi \in C_0^1(\mathbb{R}^m), \varphi|_K \geq 1 \right\}.$
Indeed, $\text{Cap} (\cdot)$ is precisely the Newtonian capacity up to a constant multiplier (cf. [3], Chapter V, 25). Wiener’s criterion asserts that a closed set $E \subset \mathbb{R}^m (m \geq 3)$ is thin at some $x_0 \in \partial E$ if and only if
\[
\sum_{k=1}^{\infty} 2^{k(m-2)} \text{Cap} (E_k) < \infty,
\]
where $E_k := E \cap \{x; 2^{-k-1} \leq |x-x_0| \leq 2^{-k}\}$ (cf. [8] Theorem 5.2).

Consider the closed cusp $\Gamma := \{z \in \mathbb{C}^n; \text{Im} z \geq C(|z'|^2 + (\text{Re} z)^2)^{\alpha/2}\} \subset \mathbb{C}^n = \mathbb{R}^{2n}$.

Clearly, $\Gamma$ is not thin at the vertex when $n = 1$ since $\mathbb{C} \setminus \Gamma$ is simply connected (cf. [14], Theorem 4.2.1). Cusps can be also defined in real Euclidean spaces and every closed cusp in $\mathbb{R}^3$ is not thin at the vertex (cf. [8], Chapter V, §1, No.3). In contrast, the following conclusion holds

**Proposition 6.1.** Every closed cusp in $\mathbb{C}^n = \mathbb{R}^{2n}$ is thin at the vertex when $n \geq 2$.

This result might be known. However, we still provide a proof since we cannot find the result in literature explicitly.

**Proof of Proposition 6.1.** Take $m = 2n$ and $E = \overline{\Gamma}$. It follows that
\[
E_k \subset \{z \in \mathbb{C}^n; |z'|^2 + (\text{Re} z)^2 \leq (2^{-k}/C)^{2/\alpha}, 2^{-k-2} \leq \text{Im} z \leq 2^{-k+1}\} = \overline{B(2^{-k}/C)^{1/\alpha}(0)} \times [2^{-k-2}, 2^{-k+1}] \subset \mathbb{R}^{2n-1} \times \mathbb{R}.
\]

Set $F_k := \overline{B(2^{-k}/C)^{1/\alpha}(0)} \times [1/4, 2]$. The affine mapping
\[
T : \mathbb{C}^n \to \mathbb{C}^n, \quad z \mapsto ((2^{-k}/C)^{1/\alpha} z', (2^{-k}/C)^{1/\alpha} \text{Re} z, 2^{-k} \text{Im} z)
\]
maps the set $F := \overline{B}(0) \times [1/4, 2]$ onto $F_k$. For any $\varphi \in C_0^1(\mathbb{C}^n)$ with $\varphi|_{F_k} \geq 1$, we have
\[
\int_{F_k} |\nabla_x \varphi(x)|^2 dx = \int_F |(\nabla_x \varphi)(Ty)|^2 |\det T'(y)| dy \leq 2^{2k/\alpha} \times 2^{-(2n-1)k/\alpha} \times 2^{-k} \int_F |\nabla_y (\varphi \circ T)|^2
\]
\[
= 2^{-(2n-3)/\alpha + 1} k \int_F |\nabla_y (\varphi \circ T)|^2.
\]

Therefore,
\[
\text{Cap} (F_k) \lesssim 2^{-(2n-3)/\alpha + 1} k
\]
and
\[
\sum_{k=1}^{\infty} 2^{2k(2n-2)} \text{Cap} (E_k) \leq \sum_{k=1}^{\infty} 2^{k(2n-2)} \text{Cap} (F_k) \lesssim \sum_{k=1}^{\infty} 2^{-(2n-3)(1/\alpha - 1)k} < \infty,
\]
i.e., $\Gamma$ is thin at the vertex when $n \geq 2$. \qed
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