ON QUANTUM HUYGENS PRINCIPLE AND RAYLEIGH SCATTERING

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ABSTRACT. We prove several minimal photon velocity estimates below the ionization threshold for a particle system coupled to the quantized electromagnetic or phonon field. Using some of these results, we prove the asymptotic completeness (for the Rayleigh scattering) on the states for which the expectation of the photon number is uniformly bounded.

1. Introduction

In this paper we study the long-time dynamics of a non-relativistic particle system coupled to the quantized electromagnetic or phonon field. For energies below the ionization threshold, we prove several lower bounds on the growth of the distance of the escaping photons to the particle system. (Here and in what follows we use the term photon for both photon and phonon.) Using some of these results, we prove the asymptotic completeness (for the Rayleigh scattering) on the states for which the expectation of the photon number is uniformly bounded.

Model. The state space for our model is given by $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$ and the dynamics is generated by the Hamiltonian

$$H = H_p + H_f + I(g),$$

acting on it. Here $\mathcal{H}_p$ is the particle state space, $\mathcal{F}$ is the bosonic Fock space, $\mathcal{F} \equiv \Gamma(h) := \bigoplus_0^\infty \otimes_n h$, based on the one-photon space $h := L^2(\mathbb{R}^3)$, $H_p$ is a self-adjoint particle system Hamiltonian, acting on $\mathcal{H}_p$, and $H_f := d\Gamma(\omega)$ is the photon Hamiltonian, acting on $\mathcal{F}$, where $\omega = \omega(k)$ is the photon dispersion law ($k$ is the photon wave vector) and $d\Gamma(b)$ denotes the lifting of a one-photon operator $b$ to the photon Fock space.

$$d\Gamma(b)|_{\otimes^n h} = \sum_{j=1}^n \underbrace{1 \otimes \cdots \otimes 1}_{j-1} b \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}.$$  

Here $\otimes^n h$ stands for the symmetrized tensor product of $n$ factors (for $n = 0$, $h$ is replaced by $\mathbb{C}$ and $d\Gamma(b)|_\mathbb{C} = 0$). The operator $I(g)$ acts on $\mathcal{H}$ and represents an interaction energy, labeled by a coupling family $g(k)$ of operators acting on the particle space $\mathcal{H}_p$.

For photons $\omega(k) = |k|$, for acoustic phonons, $\omega(k) \propto |k|$ for small $|k|$ and $c \leq \omega(k) \leq c^{-1}$, for some $c > 0$, away from 0, while for optical phonons, $c \leq \omega(k) \leq c^{-1}$, for some $c > 0$, for all $k$. To fix ideas we consider below only the most difficult case of $\omega(k) = |k|$. (For photons, to accommodate their polarizations, the one-boson space $L^2(\mathbb{R}^3)$ should be replaced by $L^2(\mathbb{R}^3; \mathbb{C}^2)$, but the resulting modifications are trivial, see e.g. [29] [33].) In the simplest case of linear coupling (the dipole approximation in QED or the phonon models), $I(g)$ is given by

$$I(g) := \int (g^*(k) \otimes a(k) + g(k) \otimes a^*(k))dk,$$

with $a^*(k)$ and $a(k)$, the creation and annihilation operators, acting on $\mathcal{F}$ (see Supplement II for definitions).

A primary model for the particle system to have in mind is an electron in a vacuum or in a solid in an external potential $V$. In this case, $H_p := \epsilon(p) + V(x)$, $p = -i\nabla_x$, with $\epsilon(p)$ being the standard non-relativistic kinetic energy, $\epsilon(p) = |p|^2 \equiv -\Delta_x$ (the Nelson model), or the electron dispersion
law in a crystal lattice (a standard model in solid state physics), acting on $\mathcal{H}_p := L^2(\mathbb{R}^3)$, and the coupling family is given by

$$g(k) = |k|^\mu \xi(k) e^{ikx},$$

where $\xi(k)$ is the ultraviolet cut-off. For phonons, $\mu = 1/2$. To have a self-adjoint operator $H_p$ we assume that $V$ is a Kato potential. A key fact here is that there is a spectral point $\Sigma \in \sigma(H)$, called the ionization threshold, s.t. below $\Sigma$, the particle system is well localized:

$$\| e^{\delta|x|} f(H) \| \lesssim 1,$$

for any $0 \leq \delta < \text{dist}(\text{supp } f, \Sigma)$ and any $f \in C^\infty_0((-, \Sigma))$, i.e. states decay exponentially in the particle coordinates $x$ ([20, 5, 6]). This can be easily upgraded to an $N$–body system (e.g. an atom or a molecule, see e.g. [29, 31]). Another example – the spin-boson model – will be defined below.

Finally, the above can be extended to the standard model of non-relativistic quantum electrodynamics in which particles are minimally coupled to the quantized electromagnetic field, which leads to $I(g)$ being quadratic in the creation and annihilation operators $a^\#(k)$.

**Problem.** In all above cases, the Hamiltonian $H$ is self-adjoint and generates the dynamics through the Schrödinger equation,

$$i\partial_t \psi_t = H \psi_t.$$  

As initial conditions, $\psi_0$, we consider states below the ionization threshold, $\Sigma$, defined in [11, 12], i.e. $\psi_0$ in the range of the spectral projection $E_\Delta(H)$, $\Delta := (-\Sigma, \Sigma)$. In other words, we are interested in processes, like emission and absorption of radiation, or scattering of photons on an electron bound by an external potential (created e.g. by an infinitely heavy nucleus or impurity of a crystal lattice), in which the particle system (say, an atom or a molecule) is not being ionized.

Denote by $\Phi_j$ and $E_j$ the eigenfunctions and the corresponding eigenvalues of the Hamiltonian $H$, below $\Sigma$, i.e. $E_j < \Sigma$. The following are the key characteristics of evolution of a physical system, in progressive order the refined information they provide and in our context:

- **Local decay** stating that some photons are bound to the particle system while others (if any) escape to infinity, i.e. the probability that they occupy any bounded region of the physical space tends to zero, as $t \to \infty$.
- **Minimal photon velocity bound** with speed $c$ stating that, as $t \to \infty$, with probability $\to 1$, the photons are either bound to the particle system or depart from it with the distance $\geq c't$, for any $c' < c$.
- **Asymptotic completeness** on the interval $(-\Sigma, \Sigma)$ stating that, for any $\psi_0 \in \text{Ran } \chi_{(-\infty, \Sigma)}(H)$, and any $\epsilon > 0$ there are photon wave functions $f_{j\epsilon} \in F$, with a finite number of photons, s.t. the solution, $\psi_t = e^{-iHt} \psi_0$, of the Schrödinger equation, (1.6), satisfies

$$\limsup_{t \to \infty} \| e^{-iHt} \psi_0 - \sum_j e^{-iE_{j\epsilon} t} \Phi_j \otimes_s f_{j\epsilon} \| \leq \epsilon.$$  

(It will be shown in the text that $\Phi_j \otimes_s f_{j\epsilon}$ is well-defined, at least for the ground state $(j = 0)$.) In other words, for any $\epsilon > 0$ and with the probability $\geq 1 - \epsilon$, the Schrödinger evolution $\psi_t$ approaches asymptotically a superposition of states in which the particle system with a photon cloud bound to it is in one of its bound states $\Phi_j$, with additional photons (or possibly none) escaping to infinity with the velocity of light.

The reason for $\epsilon > 0$ in (1.7) is that for the state $\Phi_j \otimes_s f$ to be well defined, as one would expect, one would have to have a very tight control on the number of photons in $f$, i.e. the number of photons escaping the particle system. (See the remark at the end of Subsection 5.3 for a more
technical explanation.) For massive bosons $\epsilon > 0$ can be dropped (set to zero), as the number of photons can be bound by the energy cut-off.

We describe the photon position by the operator $y := i\nabla_k$ on $L^2(\mathbb{R}^3)$, canonically conjugate to the photon momentum $k$ (see [9] for a discussion of the notion of the photon position in our context). We say that the system obeys the quantum Huygens principle if the Schrödinger evolution, $\psi_t = e^{-itH}\psi_0$, obeys the estimates

$$\int_1^{\infty} dt \ t^{-\sigma} \|d\Gamma(\chi_{|y|\geq 1})^{1\over 2}\psi_t\|^2 \lesssim \|\psi_0\|_0^2,$$

(1.8)

for some norm $\|\psi_0\|_0$, some $0 < \sigma' \leq 1$, and for any $\alpha > 0$ and $c > 0$ such that either $\alpha < 1$ or $\alpha = 1$ and $c < 1$. In other words there are no photons which either diffuse or propagate with speed $< 1$. Here $\chi_\Omega$ denotes a smoothed out characteristic function of the set $\Omega$, which is defined at the end of the introduction. The maximal velocity estimate, as proven in [9], states that, for $\mu > 0$, any $\tilde{c} > 1$, and $\gamma < \mu \over 2 \min(3\tilde{c} - 1, 2\mu)$,

$$\|d\Gamma(\chi_{|y|\geq 1})^{1\over 2}\psi_t\| \lesssim t^{-\gamma} \| (d\Gamma(y)) + 1 \|^{1\over 2}\psi_0\|.$$  

(1.9)

Considerable progress has been made in understanding the asymptotic dynamics of non-relativistic particle systems coupled to quantized electromagnetic or phonon field. The local decay property was proven in [6, 7, 22, 23, 20, 21, 8, 10], by positive commutator techniques and the combination of the renormalization group and positive commutator methods. The maximal velocity estimate was proven in [9].

An important breakthrough was achieved recently in [11], where the authors proved relaxation to the ground state and uniform bounds on the number of emitted massless bosons in the spin-boson model.

In scattering theory, asymptotic completeness was proven for (a small perturbation of) a solvable model involving a harmonic oscillator (see [2, 39]), and for models involving massive boson fields ([14, 17, 18, 19]). Moreover, [24] obtained some important results for massless bosons. Motivated by the many-body quantum scattering, [14, 24, 17, 18, 19] defined main notions of the scattering theory on Fock spaces, such as wave operators, asymptotic completeness and propagation estimates.

**Results.** Now we formulate our results. For notational simplicity we consider (1.1), with the linear coupling (1.3). The coupling operators $g(k)$ are assumed to satisfy

$$\|\eta^{1\alpha} \partial^\alpha g(k)\|_{H_p} \lesssim |k|^{\mu - 1\alpha} \xi(k), \quad |\alpha| \leq 2,$$

(1.10)

where $\xi(k)$ is the ultra-violet cut-off (a smooth function decaying sufficiently rapidly at infinity) and $\eta$ is an estimating operator on the particle space $H_p$ (a bounded, positive operator with unbounded inverse), satisfying

$$\|\eta^{-n} f(H)\| \lesssim 1,$$

(1.11)

for any $n = 1, 2$ and $f \in C^\infty_0((-\infty, \Sigma))$. For the particle model discussed in the paragraph containing (1.4), (1.10) holds with $\eta = \langle x \rangle^{-1}$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$, and the ionization threshold, $\Sigma$, for which (1.11) is true, is given by

$$\Sigma := \lim_{R \to \infty} \inf_{\phi \in D_R} \langle \phi, H \phi \rangle,$$

(1.12)

where the infimum is taken over $D_R = \{ \phi \in \mathcal{D}(H) \mid \phi(x) = 0 \text{ if } |x| > R, \|\phi\| = 1 \}$ (see [26]; $\Sigma$ is close to $\inf \sigma_{ess}(H_p)$). For the spin-boson model defined below, $\eta = 1$.

Below, we assume $\mu > -1/2$ or $\mu > 0$. To apply our techniques to minimally coupled particle systems, where $\mu = -1/2$, one would have to perform first the generalized Pauli-Fierz transform of [33], as it is done in [9] (see also [23, 34]), which brings it to $\mu = 1/2$.

It is known (see [6, 27]) that the operator $H$ has the unique ground state (denoted here as $\Phi_{gs}$) and that generally (e.g. under the Fermi Golden Rule condition) it has no eigenvalues in the
interval \((E_{gs},\Sigma)\), where \(E_{gs}\) is the ground state energy (see [7]). We assume that this is exactly the case:

\begin{equation}
\text{Fermi's Golden Rule} \quad (1.13)
\end{equation}

Treatment of the (exceptional) situation when such eigenvalues do occur requires, within our approach, proving a delicate estimate \(\|P_f H\| \lesssim \langle g \rangle\), where \(P_f\) denotes the projection onto \(\mathcal{H}_p \otimes \Omega\) (\(\Omega := 1 \oplus 0 \oplus \ldots\) is the vacuum in \(\mathcal{F}\)) and \(f \in C_0^\infty((E_{gs}\Sigma) \setminus \sigma_p(H))\), uniformly in dist(supp \(f\), \(\sigma_p(H)\)).

In what follows we let \(\psi_t\) denote the Schrödinger evolution, \(\psi_t = e^{-itH}\psi_0\), i.e. the solution of Schrödinger equation \((1.4)\), with an initial condition \(\psi_0\), satisfying \(\psi_0 = f(H)\psi_0\), with \(f \in C_0^\infty((-\infty, \Sigma))\).

For \(A \geq -C\), we denote \(\|\psi_t\|_A := (\|\psi_0\|^2 + \|(A + C)\frac{1}{2}\psi_0\|_2^2)^{1/2}\). We define \(\nu(\rho) \geq 0\) by the inequality

\begin{equation}
\langle \psi_t, d\Gamma(\omega^\rho)\psi_t \rangle \lesssim t^{\nu(\rho)}\|\psi_0\|_0^2, \quad (1.14)
\end{equation}

where \(\|\psi\|_0^2 = \|\psi\|^2_H + \|\psi\|^2_D(\omega^\rho)\). It was shown in [9] (see (A.1) of Appendix A) that, for any \(-1 \leq \rho \leq 1\), the inequality \((1.14)\) holds for the the exponent \(\nu(\rho) = \frac{\rho^2\nu}{2 + \mu}\) (this generalizes an earlier bound due to [24]). Also, the bound

\begin{equation}
\|\psi_t\|_{H^\rho} \lesssim \|\psi_0\|_H, \quad (1.15)
\end{equation}

shows that \((1.14)\) holds for \(\rho = 1\) with \(\nu(1) = 0\). With \(\nu(\delta)\) defined by \((1.14)\), we prove the following two results.

**Theorem 1.1** (Quantum Huygens principle). Assume \((1.10)\) with \(\mu > -1/2\) and \((1.11)\). Let either \(\beta < 1\), or \(\beta = 1\) and \(c < 1\). Assume

\begin{equation}
\beta > \max\left(\frac{5}{6} + \frac{\nu(-1) - \nu(0)}{6}, \frac{1}{2} + \frac{1}{2(\frac{3}{2} + \mu)}\right), \quad (1.16)
\end{equation}

Then for any initial condition \(\psi_0 \in f(H)D(d\Gamma(\omega^{-1})^{1/2})\), for some \(f \in C_0^\infty((-\infty, \Sigma))\), the Schrödinger evolution, \(\psi_t\), satisfies, for any \(a > 1\), the following estimate

\begin{equation}
\int_1^\infty dt\ t^{-\beta - a\nu(0)}\|d\Gamma(\chi_{|y| \leq ct^\beta})\frac{1}{2}\psi_t\|^2_2 \lesssim \|\psi_0\|^2_2. \quad (1.17)
\end{equation}

To formulate our next result we let \(\Gamma(\chi)\) be the lifting of a one-photon operator \(\chi\) (e.g. a smoothed out characteristic function of \(y\)) to the photon Fock space, defined by

\begin{equation}
\Gamma(\chi) = \oplus_{n=0}^\infty (\otimes^n \chi), \quad (1.18)
\end{equation}

(so that \(\Gamma(e^b) = e^{d\Gamma(b)}\), and then to the space of the total system. We have

**Theorem 1.2** (Weak minimal photon escape velocity estimate). Assume \((1.10)\) with \(\mu > -1/2\), \((1.11)\) and \((1.13)\). Let the norm \(\langle g \rangle := \sum_{|\alpha| \leq 2} \|\eta^{\alpha}|\partial x^\alpha g\|_{L^2(\mathbb{R}^2, \mathcal{H})}\) of the coupling function \(g\) be sufficiently small and \(\nu(-1) < \alpha < 1 - \nu(0)\). Then for any initial condition \(\psi_0 \in f(H)D(d\Gamma(\langle y \rangle))\), for some \(f \in C_0^\infty((E_{gs}\Sigma))\), the Schrödinger evolution, \(\psi_t\), satisfies the estimate

\begin{equation}
\|\Gamma(\chi_{|y| \leq ct^\alpha})\psi_t\|_2 \lesssim t^{-\gamma}\|\psi_0\|_{d\Gamma(y)^2}, \quad (1.19)
\end{equation}

where \(\gamma < \frac{1}{2} \min(1 - \alpha - \nu(0), \frac{1}{2}(\alpha - \nu(0) - \nu(-1)))\).

**Remarks.**

1) The estimate \((1.17)\) is sharp if \(\nu(0) = 0\). Assuming this and taking \(\nu(-1) = (3/2 + \mu)^{-1}\) (see (A.3)), the condition \((1.16)\) on \(\beta\) in Theorem 1.1 becomes \(\beta > \frac{5}{6} + \frac{1}{6(3/2 + \mu)}\), and the condition on \(\alpha\) in Theorem 1.2 \((3/2 + \mu)^{-1} < \alpha < 1\).

2) The estimate \((1.19)\) states that, as \(t \to \infty\), with probability \(\to 1\), either all photons are attached to the particle system in the combined ground state, or at least one photon departs the
We formalize this method in the next section. (1.19) for \( \mu \geq 1/2 \), some \( \alpha > 0 \) and \( \psi_0 \in E_2(H) \), with \( \Delta \subset (E_{\text{gs}}, e_1 - O(\langle g \rangle)) \) and \( e_1 \) the first excited eigenvalue of \( H_p \), can be derived directly from [8][9].

3) With some more work, one can remove Assumption (1.13) and relax the condition on \( \psi_0 \) in Theorem 1.2 to the natural one: \( \psi_0 \in P_2 D(d\Gamma(\langle g \rangle)) \), where \( P_2 \) is the spectral projection onto the orthogonal complement of the eigenfunctions of \( H \) with the eigenvalues in the interval \( (-\infty, \Sigma) \).

Let \( N := d\Gamma(1) \) be the photon (or phonon) number operator. Our next result is

**Theorem 1.3** (Asymptotic Completeness). Assume (1.10) with \( \mu > 0 \), (1.11) and (1.13). Let the norm \( \langle g \rangle := \sum_{|\sigma| \leq 2} \| \eta^{(\sigma)} g \|_{L^2(\mathbb{R}^3, H_p)} \) of the coupling function \( g \) be sufficiently small. Suppose that

\[
\| N^\frac{1}{2} \psi_t \| \leq \| N^\frac{1}{2} \psi_0 \| + \| \psi_0 \|, \tag{1.20}
\]

uniformly in \( t \in [0, \infty) \), for any \( \psi_0 \in D(N^{1/2}) \). Then the asymptotic completeness holds on \( \text{Ran} E(-\infty, \Sigma)(H) \).

As we see from the results above, the uniform bound, (1.20), on the number of photons (or phonons) emerges as the remaining stumbling block to proving the asymptotic completeness without qualifications.

For massive bosons (e.g. optical phonons), the inequality (1.20) (as well as (1.11), with \( \nu(0) = 0 \)) is easily proven and the proof below simplifies considerably as well. In this case, the result is unconditional. It was first proven in [14] for the models with confined particles, and in [17] for the Rayleigh scattering.

The difficulty in proving this bound for massless particles is due to the same infrared problem which pervades this field and which was successfully tackled in other central issues, such as the theory of ground states and resonances (see [4][34] for reviews), the local decay and the maximal velocity bound. As was mentioned above, for the spin-boson model (see below), a uniform bound, \( \langle \psi_t, e^{\delta N} \psi_t \rangle \leq C(\psi_0) < \infty \), \( \delta > 0 \), on the number of photons, on a dense set of \( \psi_0 \)'s, was recently proven in the remarkable paper [11], which gives substance to our conjecture that the bound (1.20) holds for a dense set of states.

**Spin-boson model.** Another example of the particle system, and the simplest one, is the spin-boson model, describing an idealized two-level atom, with state space \( \mathcal{H}_p = \mathbb{C}^2 \), the Hamiltonian \( H_p = \varepsilon \sigma^3 \), where \( \sigma^1, \sigma^2, \sigma^3 \) are the usual 2 \( \times \) 2 Pauli matrices, and \( \varepsilon > 0 \) is an atomic energy. The coupling family is given by \( g(k) = \omega^\mu \kappa(k) \sigma^+, \sigma^\pm = \frac{1}{2} (\sigma^1 \mp i \sigma^2) \). In this case, \( g \) satisfies (1.10) with \( \eta = 1 \). For the spin-boson model, we can take \( \Sigma = \infty \).

**Approach and organization of the paper.** In this paper, as in earlier works, we use the method of propagation observables, originating in the many body scattering theory ([36][37][32][25][41][42], see [13][31] for a textbook exposition and a more recent review), and extended to the non-relativistic quantum electrodynamics in [14][24][16][17][18][19] and to the \( P(\varphi)_2 \) quantum field theory, in [15]. We formalize this method in the next section.

After that we prove key propagation estimates in Sections 3 and 4. Instead of \( |y| \), these estimates involve the operator \( b_\epsilon \) defined as \( b_\epsilon := \frac{1}{2} (v(k) \cdot y + y \cdot v(k)) \), where \( v(k) := \frac{k}{w + \epsilon} \), for \( \epsilon = t^{-\kappa} \), with some \( \kappa > 0 \). Since the vector field \( v(k) \) is Lipschitz continuous and therefore generates a global flow, the operator \( b_\epsilon \) is self-adjoint. We show in Section 5 that these propagation estimates give the estimates (1.17) and (1.19). (The operator \( b_\epsilon \) was considered in [I.M. Sigal and A. Soffer, Unpublished, 2004], as a regularization of the non-self-adjoint operator \( b_0 \) used in [24]. We could have also used the operators \( b_\epsilon \), with \( 0 < \epsilon < \gamma_0 := \text{dist}(\Delta, \sigma^0(H_{el})) \) constant, \( b := \frac{1}{2} (\frac{k}{w} \cdot y + \frac{y}{w} \cdot k) \), or \( \tilde{b} := \frac{1}{2} (k \cdot y + k \cdot y) \). Using \( b_\epsilon \) avoids some (trivial) technicalities, as compared to the other two operators. At the expense of slightly lengthier computations but gaining simpler technicalities, one can also modify \( b_\epsilon \) to make it bounded, by multiplying it with the cut-off function \( \chi_{|y| \leq \epsilon t} \), with \( \epsilon > 1 \).
such that the maximal velocity estimate (1.9) holds, or use the smooth vector field \( v(k) := \frac{k}{\sqrt{\omega^2 + \epsilon^2}} \), instead of \( v(k) := k(\sqrt{\omega^2 + \epsilon^2}) \).

Theorem 1.3 is proven in Section 5. As it is standard in the scattering theory, to prove the asymptotic completeness, we establish the existence of the Deift-Simon wave operator \( W_+ \), mapping solutions of the Schrödinger equation into the scattering data (see [14, 17, 24] and [35, 25, 41, 12] for earlier works). We prove the existence of \( W_+ \) in Subsection 5.2 and then deduce from it Theorem 1.3 in Subsection 5.4. A low momentum bound of [9] and some standard technical statements are given in Appendices A, B and C.

The paper is essentially self-contained. In order to make it more accessible to non-experts, we included Supplement I giving standard definitions, proof of the existence and properties of the wave operators, and Supplement II defining and discussing the creation and annihilation operators.

Notations. For functions \( A \) and \( B \), we will use the notation \( A \lesssim B \) signifying that \( A \leq CB \) for some absolute (numerical) constant \( 0 < C < \infty \). The symbol \( E_{\Delta} \) stands for the characteristic function of a set \( \Delta \), while \( \chi_{\cdot} \leq 1 \) denotes a smoothed out characteristic function of the interval \( (-\infty, 1] \), that is it is in \( C^\infty(\mathbb{R}) \), is non-decreasing, and \( 1 \) if \( x \leq 1/2 \) and \( 0 \) if \( x \geq 1 \). Moreover, \( \chi_{\cdot}/2 \leq 1 \) and \( \chi_{\cdot} = 1 \) stands for the derivative of \( \chi_{\cdot}/2 \). Given a self-adjoint operator \( a \) and a real number \( \alpha \), we write \( \chi_{a} \leq \alpha := \chi_{a} \leq 1 \), and likewise for \( \chi_{a} \geq \alpha \). Finally, \( D(A) \) denotes the domain of an operator \( A \).

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2. Method of propagation observables

Many steps of our proof use the method of propagation observables which we formalize in what follows. In this section we consider the Hamiltonian (1.1) and assume (1.10) and (1.11). Let \( \psi_t = e^{-itH}\psi_0 \). The method reduces propagation estimates for our system say of the form

\[
\int_0^\infty dt \| G_t^{1/2} \psi_t \|^2 \lesssim \| \psi_0 \|^2
\]

for some norm \( \| \cdot \| \geq \| \cdot \| \), to differential inequalities for certain families \( \phi_t \) of positive, one-photon operators on the one-photon space \( L^2(\mathbb{R}^3) \). Let

\[
d\phi_t = \partial_t \phi_t + i[\omega, \phi_t],
\]

and let \( \nu(\rho) \geq 0 \) be determined by the estimate (1.4). We isolate the following useful class of families of positive, one-photon operators:

Definition 2.1. A family of positive operators \( \phi_t \) on \( L^2(\mathbb{R}^3) \) will be called a one-photon weak propagation observable, if it has the following properties

- there are \( \delta \geq 0 \) and a family \( p_t \) of non-negative operators, such that

\[
\| \omega^{-\delta/2} \phi_t \omega^{-\delta/2} \| \lesssim t^{-\nu(\delta)} \quad \text{and} \quad d\phi_t \geq p_t + \sum_{\text{finite}} \text{rem}_i,
\]

where \( \text{rem}_i \) are one-photon operators satisfying

\[
\| \omega^{-\rho_i/2} \text{rem}_i \omega^{-\rho_i/2} \| \lesssim t^{-\lambda_i},
\]

for some \( \rho_i \) and \( \lambda_i \), s.t. \( \lambda_i > 1 + \nu(\rho_i) \),
• for some $\lambda' > 1 + \nu(\delta)$ and with $\eta$ satisfying (1.11),
\[
\left( \int \|\eta\phi_i g\|^2_{\mathcal{L}_p} \omega^{-\delta} d^3 k \right)^{\frac{1}{2}} \lesssim t^{-\lambda'}.
\] (2.4)

(Here $\phi_i$ acts on $g$ as a function of $k$.)

Similarly, a family of operators $\phi_i$ on $L^2(\mathbb{R}^3)$ will be called a one-photon strong propagation observable, if
\[
d\phi_i \leq -p_i + \sum_{\text{finite}} \text{rem}_i,
\] with $p_i \geq 0$, $\text{rem}_i$ are one-photon operators satisfying (2.3) for some $\lambda_1 > 1 + \nu(\rho_i)$, and (2.4) holds for some $\lambda' > 1 + \nu(\delta)$.

The following proposition reduces proving inequalities of the type of (2.1) to showing that $\phi_i$ is a one-photon weak or strong propagation observable, i.e. to one-photon estimates of $d\phi_i$ and $\phi_i g$.

**Proposition 2.2.** If $\phi_i$ is a one-photon weak (resp. strong) propagation observable, then we have either the weak estimate, (2.1), or the strong propagation estimate,
\[
\langle \psi_t, \Phi_t \psi_t \rangle + \int_0^\infty dt \|G_t^{1/2} \psi_t\|^2 \lesssim \|\psi_0\|_\Delta^2 + \|\psi_0\|_\Omega^2,
\] with the norm $\|\psi_0\|_\Delta^2 := \|\psi_0\|_\Delta^2 + \|\psi_0\|_\Omega^2$, where $\Phi_t := d\Gamma(\phi_t)$ and $G_t := d\Gamma(p_t)$, on the subspace $\mathcal{H}_f \subset \mathcal{H}$, with $f \in C_0^\infty((-\infty, \Sigma))$. Here $\|\psi_0\|_\Delta := \|\psi_0\|_\Delta$ and $\|\psi_0\|_\Omega := \sum \|\psi_0\|_{\rho_i}$.

Before proceeding to the proof we present some useful definitions. Consider families $\Phi_t$ of operators on $\mathcal{H}$ and introduce the Heisenberg derivative
\[
D\Phi_t := \partial_t \Phi_t + i[H, \Phi_t],
\] with the property
\[
\partial_t \langle \psi_t, \Phi_t \psi_t \rangle = \langle \psi_t, D\Phi_t \psi_t \rangle.
\] (2.7)

**Definition 2.3.** A family of operators $\Phi_t$ on a subspace $\mathcal{H}_1 \subset \mathcal{H}$ will be called a (second quantized) weak propagation observable, if for all $\psi_0 \in \mathcal{H}_1$, it has the following properties

- $\sup_t \langle \psi_t, \Phi_t \psi_t \rangle \lesssim \|\psi_0\|^2$;
- $D\Phi_t \geq G_t + \text{Rem}$, where $G_t \geq 0$ and $\int_0^\infty dt \langle \psi_t, \text{Rem} \psi_t \rangle \lesssim \|\psi_0\|^2_\Delta$,

for some norms $\|\psi_0\|_\Delta, \|\cdot\|_\Delta \geq \|\cdot\|$. Similarly, a family of operators $\Phi_t$ will be called a strong propagation observable, if it has the following properties

- $\Phi_t$ is a family of non-negative operators;
- $D\Phi_t \leq -G_t + \text{Rem}$, where $G_t \geq 0$ and $\int_0^\infty dt \langle \psi_t, \text{Rem} \psi_t \rangle \lesssim \|\psi_0\|^2_\Omega$,

for some norm $\|\cdot\|_\Omega \geq \|\cdot\|$.

If $\Phi_t$ is a weak propagation observable, then integrating the corresponding differential inequality sandwiched by $\psi_t$’s and using the estimate on $\langle \psi_t, \Phi_t \psi_t \rangle$ and on the remainder Rem, we obtain the (weak propagation) estimate (2.1), with $\|\psi_0\|^2_\Delta := \|\psi_0\|^2_\Delta + \|\psi_0\|^2_\Omega$. If $\Phi_t$ is a strong propagation observable, then the same procedure leads to the (strong propagation) estimate
\[
\langle \psi_t, \Phi_t \psi_t \rangle + \int_0^\infty dt \|G_t^{1/2} \psi_t\|^2 \lesssim \|\psi_0\|^2_\Delta + \|\psi_0\|^2_\Omega.
\] (2.8)

**Proof of Proposition 2.2.** Let $\Phi_t := d\Gamma(\phi_t)$. To prove the above statement we use the relations
\[
D_0 d\Gamma(\phi_t) = d\Gamma(d\phi_t), \quad i[I(g), d\Gamma(\phi_t)] = -I(i\phi_t g),
\] where $D_0$ is the free Heisenberg derivative,
\[
D_0 \Phi_t := \partial_t \Phi_t + i[H_0, \Phi_t],
\]

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valid for any family of one-particle operators $\phi_t$, to compute

$$D\Phi_t = d\Gamma(d\phi_t) - I(i\phi_t g). \quad (2.10)$$

Denote $\langle A \rangle_\psi := \langle \psi, A\psi \rangle$. Applying the Cauchy-Schwarz inequality, we find the following version of a standard estimate

$$|\langle I(g) \rangle_\psi| \leq \left( \int \|g\|^2_d\nu \omega^{-\delta} d^3k \right)^{\frac{1}{2}} \eta^{-1}\|\psi\| \|\psi\|_{d\Gamma(\omega^\delta)}. \quad (2.11)$$

Using that $\psi_t = f_1(H)\psi_t$, with $f_1 \in C_0^\infty((-\infty, \Sigma))$, $f_1 f = f$, and using (1.11), we find $\|\eta^{-1}\psi_t\| \lesssim \|\psi_t\|$. Taking this into account, we see that the equations (2.11), (2.4) and (1.15) yield

$$|\langle I(i\phi_t g) \rangle_\psi| \lesssim t^{-X_++\nu(\delta)}\|\psi_0\|^2_d. \quad (2.12)$$

Next, using (2.3), we find $\text{rem}_i \leq \|\omega^{-\rho_i/2}\text{rem}_i \omega^{-\rho_i/2}\|\omega^{\rho_i} \lesssim t^{-\lambda_i} \omega^{\rho_i}$. This gives $d\Gamma(\text{rem}_i) \lesssim t^{-\lambda_i} d\Gamma(\omega^{\rho_i})$, which, due to the bound (1.11), leads to the estimate

$$\langle \langle d\Gamma(\text{rem}_i) \rangle_\psi \rangle \lesssim t^{-\lambda_i + \nu(\rho_i)}\|\psi_0\|^2_d. \quad (2.13)$$

In the strong case, (2.5) and (2.10) imply

$$D\Phi_t \leq -d\Gamma(p_t) + \sum_{\text{finite}} d\Gamma(\text{rem}_i) - I(i\phi_t g), \quad (2.14)$$

which together with (2.12) and (2.13) implies that $\Phi_t$ is a strong propagation observable.

In the weak case, (2.2) and (2.10) imply

$$D\Phi_t \geq d\Gamma(p_t) + \sum_{\text{finite}} d\Gamma(\text{rem}_i) - I(i\phi_t g). \quad (2.15)$$

Next, since $\phi_t \leq \|\omega^{-\delta/2}\phi_t \omega^{-\delta/2}\|\omega^\delta \lesssim t^{-\nu(\delta)}\omega^\delta$, we have $d\Gamma(\phi_t) \lesssim t^{-\nu(\delta)} d\Gamma(\omega^\delta)$. Using this estimate and using again the bound (1.14), we obtain

$$\langle \langle d\Gamma(\omega^\delta) \rangle_\psi \rangle \lesssim \|\psi_0\|^2_d. \quad (2.16)$$

Hence $\Phi_t$ is a weak propagation observable. \hfill \square

**Proposition 2.4.** Let $\phi_t$ be a one-photon family satisfying

- either, for some $\delta \geq 0$,

  $$\|\omega^{-\delta/2}\phi_t \omega^{-\delta/2}\|\omega^\delta \lesssim t^{-\nu(\delta)} \quad \text{and} \quad d\phi_t \geq p_t - d\tilde{\phi}_t + \text{rem}, \quad (2.17)$$

  or

  $$d\phi_t \leq -p_t + d\tilde{\phi}_t + \sum_{\text{finite}} \text{rem}_i, \quad (2.18)$$

  where $p_t \geq 0$, rem$_i$ are one-photon operators satisfying (2.3), and $\tilde{\phi}_t$ is a weak propagation observable,

- (2.4) holds.

Then, depending on whether (2.17) or (2.18) is satisfied, $\Phi_t := d\Gamma(\phi_t)$ is a weak, or strong, propagation observable, with the norm $\|\psi_0\|_\# = \|\psi_0\|_\rho$, on the subspace $f(H)\mathcal{H} \subset \mathcal{H}$, with $f \in C_0^\infty((-\infty, \Sigma))$, and therefore we have either the weak or strong propagation estimates, (2.1) or (2.8), on this subspace.

**Proof.** Given Proposition 2.4 and its proof, the only term we have to control is $d\Gamma(d\tilde{\phi}_t)$. Using that $\tilde{\phi}_t$ is a weak propagation observable and using (2.7), (2.10) and (2.12) for $\Phi_t := d\Gamma(\tilde{\phi}_t)$, we obtain

$$|\int_0^\infty dt \langle \Phi_t \rangle_\psi | \lesssim \|\psi_0\|_\#^2, \quad (2.19)$$

with $\|\psi_0\|_\#^2 := \|\psi_0\|_\#^2 + \|\psi_0\|_\rho^2 \|\psi_0\|_\rho$ and $\|\psi_0\|_\#, \|\psi_0\|_\rho$ might be different now), which leads to the desired estimates. \hfill \square
Remarks.

1) Proposition 2.2 reduces a proof of propagation estimates for the dynamics (1.6) to estimates involving the one-photon datum \((\omega, g)\) (an ‘effective one-photon system’), parameterizing the hamiltonian (1.1). (The remaining datum \(H_p\) does not enter our analysis explicitly, but through the bound states of \(H_p\) which lead to the localization in the particle variables, (1.5)).

2) The condition on the remainder in (2.2) can be weakened to \(\text{rem} = \text{rem}' + \text{rem}''\), with \(\text{rem}'\) and \(\text{rem}''\) satisfying (2.3) and

\[
|\text{rem}''| \lesssim \chi_{|\nu| \geq c},
\]

for \(c\) as in (1.9), respectively. Moreover, (2.3) can be further weakened to

\[
\int_0^\infty |\langle \psi_t, d\Gamma(\text{rem}_i)\psi_t \rangle| < \infty.
\]

3) An iterated form of Proposition 2.3 is used to prove Theorem 1.1

3. The First Propagation Estimate

Let \(\nu(\delta) \geq 0\) be the same as in (1.14) and recall the operator \(b_\epsilon\) defined in the introduction. We write it as

\[
b_\epsilon := \frac{1}{2}(\theta_\epsilon \nabla \omega \cdot y + y \cdot \nabla \omega \theta_\epsilon), \quad \text{where} \quad \theta_\epsilon := \frac{\omega}{\omega_\epsilon}, \quad \omega_\epsilon := \omega + \epsilon, \quad \epsilon = t^{-\kappa}.
\]

**Theorem 3.1.** Assume (1.10) with \(\mu > -1/2\) and (1.11). Let \(\nu(-1) - \nu(0) < \kappa < 1\).

If either \(\beta < 1\), or \(\beta = 1\) and \(c < 1\), and

\[
\beta > \max((3/2 + \mu)^{-1}, (1 + \kappa)/2, 1 - \kappa + \nu(-1) - \nu(0)),
\]

then for any initial condition \(\psi_0 \in f(H)D(d\Gamma(\omega^{-1})^{1/2})\), for some \(f \in C_0^\infty((- \infty, \Sigma))\), the Schrödinger evolution, \(\psi_t\), satisfies, for any \(a > 1\), the following estimates

\[
\int_0^\infty dt \ t^{-\beta - a
\nu(0)} \|d\Gamma(\chi_{\epsilon t^\beta = 1})^{1/2} \psi_t \|^2 \lesssim \|\psi_0\|^2_1.
\]

If \(\nu(0) = 0\), \(\mu > 0\), and \(\beta\) satisfies (3.2) and \(\beta < \frac{1}{c}\), with \(c > 1\), then, with the notation \(\chi \equiv \chi(\frac{\omega}{\omega_\epsilon})^2 \leq 1\),

\[
\int_0^\infty dt \ t^{-\beta} \|d\Gamma(\theta_\epsilon^{1/2} \chi \chi_{\epsilon t^\beta = 1})^{1/2} \psi_t \|^2 \lesssim \|\psi_0\|^2_6.
\]

**Proof.** We will use the method of propagation observables outlined in Section 2. We consider the one-parameter family of one-photon operators

\[
\phi_t := t^{-a\nu(0)} \chi_{v \geq 1}, \quad v := \frac{b_\epsilon}{ct\beta},
\]

where \(a > 1\). To show that \(\phi_t\) is a weak one-photon propagation observable, we obtain differential inequalities for \(\phi_t\). We use the notation \(\chi_{\beta} \equiv \chi_{v \geq 1}\). To compute \(d\phi_t\), we use the expansion

\[
d\phi_t = t^{-a\nu(0)}(dv)\chi_{\beta} + \sum_{i=1}^2 \text{rem}_i, \quad \text{rem}_1 := t^{-a\nu(0)}[d\chi_{\beta} - (dv)\chi_{\beta}'], \quad \text{rem}_2 := -a\nu(0)t^{-1} \phi_t.
\]

Using the definitions in (3.1), we compute

\[
dv = \frac{\theta_\epsilon}{ct\beta} - \frac{\beta b_\epsilon}{ct\beta + 1} + \frac{1}{ct\beta} \partial_t b_\epsilon.
\]
Next, we have \( \partial_t b_\epsilon = \frac{\kappa}{t^{1+\kappa}}(\omega_\epsilon^{-1} \theta_t \nabla \cdot y + \text{h.c.}) \) on \( D(b_\epsilon) \), which, due to the relation \( \frac{1}{2}(\omega_\epsilon^{-1} \theta_t \nabla \cdot y + \text{h.c.}) = \omega_\epsilon^{-1/2} b_\epsilon \omega_\epsilon^{-1/2} \), becomes

\[
\partial_t b_\epsilon = \frac{\kappa}{t^{1+\kappa}} \omega_\epsilon^{-1/2} b_\epsilon \omega_\epsilon^{-1/2}. \tag{3.8}
\]

Using that (see Lemma B.1 of Appendix B)

\[
\omega_\epsilon^{-1/2} b_\epsilon \omega_\epsilon^{-1/2} \chi'_\beta = \omega_\epsilon^{-1/2} b_\epsilon \chi'_\beta \omega_\epsilon^{-1/2} + \mathcal{O}(t^\kappa),
\]

and that \( b_\epsilon \geq 0 \) on \( \text{supp} \chi'_\beta \), we obtain

\[
\frac{1}{ct^\beta} \partial_t b_\epsilon \chi'_\beta \geq - \frac{\text{const}}{t^{1+\beta-\kappa}}. \tag{3.9}
\]

The relations (3.5)-(3.9), together with \( \frac{b_\epsilon}{t^\beta} \chi'_\beta \leq \chi'_\beta \), imply

\[
d\phi_t \geq \left( \frac{\theta_t}{ct^\beta} - \frac{\beta}{t} \right) \chi'_\beta + \sum_{i=1}^3 \text{rem}_i, \tag{3.10}
\]

where \( \text{rem}_1 \) and \( \text{rem}_2 \) are given in (3.6) and

\[
\text{rem}_3 = \mathcal{O}(t^{-1-\beta+\kappa-av(0)}). \tag{3.11}
\]

This, together with \( \theta_t = 1 - \frac{r^\kappa}{\omega_\epsilon} \) and \( \omega_\epsilon^{-1/2} \chi'_\beta = \omega_\epsilon^{-1/2} \chi'_\beta \omega_\epsilon^{-1/2} + \mathcal{O}(t^{-\beta+\kappa}) \) (see again Lemma B.1 of Appendix B), implies

\[
d\phi_t \geq \left( \frac{1}{ct^\beta} - \frac{\beta}{t} \right) \chi'_\beta + \sum_{i=1}^4 \text{rem}_i, \quad \text{rem}_4 := \frac{1}{ct^\beta + \kappa + av(0)} \omega_\epsilon^{-1/2} \chi'_\beta \omega_\epsilon^{-1/2}. \tag{3.12}
\]

We have \( \|\phi_t\| \leq t^{-av(0)} \) and therefore, due to (1.14), the first estimate in (2.2) holds. If either \( \beta < 1 \) (and \( t \) sufficiently large), or \( \beta = 1 \) and \( c < 1 \), then \( \rho_t := (\frac{1}{ct^\beta} - \frac{\beta}{t}) \) is non-negative, which implies the second estimate in (2.2). Thus (2.2) holds. By the definition (3.13) and Corollary B.3 of Appendix B, for \( i = 1 \), and by an explicit form for \( i = 2, 3, 4 \), we have the estimates

\[
\|\omega^{-\rho_t/2} \text{rem}_i \omega^{-\rho_t/2}\| \lesssim t^{-\lambda_i}, \tag{3.13}
\]

\( i = 1, 2, 3, 4 \), with \( \rho_1 = \rho_2 = \rho_3 = 0, \rho_4 = -1, \lambda_1 = 2\beta - \kappa + av(0), \lambda_2 = 1 + av(0), \lambda_3 = 1 + \beta - \kappa + av(0), \) and \( \lambda_4 = \beta + \kappa + av(0) \). We remark here that the \( i = 2 \) term is absent if \( v(0) = 0 \).

The relation (3.13) together with the assumption \( \kappa \leq 1 \) implies (2.3) with \( p = \rho_t \) and \( \lambda = \lambda_t \), for rem = rem_\kappa, provided \( \lambda_t > 1 + \nu(\rho_t) \).

Finally, (2.4) with \( \lambda' < av(0) + (\frac{3}{2} + \mu) \beta \), holds, by [9 Lemma 3.1], with \( b_\epsilon \) instead of \( |y| \) (See Lemma B.6 in Appendix B of the present paper.). Hence \( \phi_t \) is a weak one-photon propagation observable, provided \( 2\beta > 1 + \kappa + av(0) - av(0), \beta - \kappa > v(0) - av(0), \beta + \kappa > 1 + v(-1) - av(0), \) and \( (\frac{3}{2} + \mu) \beta > 1 \). Therefore, by Proposition 2.2 and under the conditions on the parameters above, we have

\[
\int_0^\infty dt \ t^{-\beta-av(0)} \|d\Gamma(\chi'_\beta)^\sharp \psi_t\|^2 \lesssim \|\psi_0\|^2_{-1}. \tag{3.14}
\]

This, due to the definition of \( \chi'_\beta \), implies the estimate (3.3).

We now prove (3.4). We use again the notation \( \chi_\beta \equiv \chi_{\nu \geq 1} \), where \( v := \frac{b_\epsilon}{ct^\beta} \), and we denote \( w := (\frac{w}{ct^\beta})^2 \). We consider the one-parameter family of one-photon operators

\[
\phi_t := \chi_\beta \psi_t, \tag{3.15}
\]
and show that $\phi_t$ is a weak one-photon propagation observable. We have $\|\phi_t\| \leq 1$ and therefore, due to (1.14) and the assumption $\nu(0) = 0$, the first estimate in (2.2) holds. Now, we show the second estimate in (2.2). To compute $d\phi_t$, we use the expansion

$$d\phi_t = \chi(d\nu)\chi_\beta' + \chi'(d\nu) \chi_\beta' \chi + \chi \chi_\beta (d\nu) \chi' + \sum_{i=1,2} \text{rem}_i,$$

where

$$\text{rem}_1 := \chi(d\chi - (d\nu)\chi_\beta') \chi, \quad \text{rem}_2 := (d\chi - (d\nu)\chi') \chi_\beta \chi + \text{h.c.}. \quad (3.16)$$

As in (3.7)–(3.9), we have

$$\chi(d\nu)\chi_\beta' \geq \chi(\frac{\theta c}{ct^\beta} - \frac{\beta b c}{ct^{\beta+1}}) \chi_\beta' \chi + \text{rem}_3,$$

where $\text{rem}_3 = O(t^{-1-\beta+\kappa})$. We consider the term $-\frac{\beta b c}{ct^{\beta+1}}$ in (3.18). Since $b_c = \theta_{\epsilon_{\chi}}^{1/2} b t_{\epsilon_{\chi}}^{1/2}$, we obtain, using in particular Lemma B.1 of Appendix [3] that

$$\chi b_c \chi_\beta' \chi = \chi(\chi_\beta')^{1/2} \theta_{\epsilon_{\chi}}^{1/2} b t_{\epsilon_{\chi}}^{1/2} \chi_\beta' \chi_\beta^{1/2} \chi = \theta_{\epsilon_{\chi}}^{1/2} (\chi_\beta')^{1/2} b t_{\epsilon_{\chi}}^{1/2} \chi_\beta^{1/2} + O(t^{\nu_{\epsilon_{\chi}}}),$$

and the maximal velocity cut-off gives $\beta b \chi \leq \bar{c} t$. Thus, commuting again $\chi$ through $\theta_{\epsilon_{\chi}}^{1/2}$ and $(\chi_\beta')^{1/2}$, we obtain

$$-\chi \beta b c t^{-1+1} \chi_\beta' \chi \geq -\bar{c} \chi_\beta' \chi_\beta^{1/2} \chi + O(\frac{1}{t^{1+\beta-\kappa}}).$$

(3.19)

Proceeding in the same way for the term $\frac{\theta c}{ct^\beta}$ in (3.18) gives

$$\chi(\frac{\theta c}{ct^\beta} - \frac{\beta b c}{ct^{\beta+1}}) \chi_\beta' \chi \geq \frac{1 - \beta c}{ct^\beta} \chi_\beta' \chi_\beta^{1/2} \chi + O(\frac{1}{t^{1+\beta-\kappa}}).$$

(3.20)

Next, we compute $dw = 2(\frac{b}{ct^\beta} - (\frac{b}{ct^\beta})^{1/2})$, where, recall, $b = \frac{1}{2} (\nabla \omega \cdot y + \text{h.c.})$. By Lemma B.1 of Appendix [3] we have

$$\chi'(d\nu) \chi_\beta' + \chi \chi_\beta (d\nu) \chi' = -2(\chi_\beta)^{1/2}(-\chi')^{1/2} (d\nu)(-\chi')^{1/2} (\chi_\beta)^{1/2} + O(\frac{1}{t^{1+\beta-\kappa}}).$$

(3.21)

Using that $dw \leq (\frac{1}{c} - 1) \frac{1}{t}$ on the support of $\chi'$ and that $\chi' \leq 0$ and $\bar{c} > 1$, we obtain

$$(-\chi')^{1/2} (d\nu)(-\chi')^{1/2} \geq (1 - \frac{1}{c}) \frac{1}{t} (-\chi').$$

(3.22)

The relations (3.16), (3.18), (3.21) and (3.22) imply

$$d\phi_t \geq p_t + \bar{p}_t - \sum_{i=1,2,3,4} \text{rem}_i,$$

where $\text{rem}_4 = O(\frac{1}{t^{2\beta-\kappa}})$ and

$$p_t := \frac{1 - \beta c}{ct^\beta} \theta_{\epsilon_{\chi}}^{1/2} \chi_\beta' \chi_\beta^{1/2}, \quad (3.24)$$

$$\bar{p}_t := (1 - \frac{1}{c}) \frac{1}{t} \chi_\beta' (-\chi') \chi_\beta^{1/2}. \quad (3.25)$$

The terms $p_t$ and $\bar{p}_t$ are non-negative, provided $\beta < 1/c$ and $\bar{c} > 1$. Together with the assumption $\nu(0)$, this implies (2.2). Next, we claim the estimates

$$\|\text{rem}_i\| \lesssim t^{-\lambda},$$

(3.26)
i = 1, 2, 3, 4, with \( \lambda = 2\beta - \kappa \). Indeed, the definition (3.17) and Corollary B.3 of Appendix B imply (3.26) for \( i = 1 \). The estimate for \( i = 3, 4 \) are obvious. To estimate \( \text{rem}_2 \), we write

\[
(d\chi - (dw)\chi')\chi\chi = (d\chi - (dw)\chi')\frac{b_i}{ct^2}\tilde{\chi}_b\chi,
\]

where \( \tilde{\chi}_b = \frac{b_i}{ct^2} - 1 \chi_b \), and \( b_\varepsilon = \theta_b + i\varepsilon c^2 \). Using that, by Lemma B.4 of Appendix B

\[
\|d\chi - (dw)\chi'\| \lesssim t^{-1},
\]

and commuting \( b \) through \( \tilde{\chi}_b \) gives

\[
(d\chi - (dw)\chi')\chi\chi = \frac{1}{ct^2}(d\chi - (dw)\chi')\theta_b\tilde{\chi}_b\chi + \mathcal{O}(\frac{1}{t^{1+\beta-\kappa}}).
\] (3.27)

By Lemma B.4 we also have

\[
\| (d\chi - (dw)\chi')\omega \| \lesssim t^{-2}.
\]

Combining this with (3.27) and the estimates \( \omega_\varepsilon^{-1} = \mathcal{O}(t^\nu) \) and \( b\chi = \mathcal{O}(t) \), we obtain

\[
(d\chi - (dw)\chi')\chi\chi = \mathcal{O}(\frac{1}{t^{1+\beta-\kappa}}),
\] (3.28)

and hence the estimate for \( i = 2 \) follows.

The relation (3.26) implies (2.3) with \( \lambda = 2\beta - \kappa \), for \( \text{rem} = \text{rem}_1 \), provided \( 2\beta - \kappa > 1 \). Finally, as above, (2.3) holds with \( \lambda' < \alpha(0) + (\frac{2}{3} + \mu)\beta \) by Lemma B.6 of Appendix B. This yields (3.4). \( \square \)

4. The second propagation estimate

We introduce the norm \( \langle g \rangle := \sum_{|\alpha| \leq 2} \| \eta^{\alpha} \partial^\alpha g \|_{L^2(\mathbb{R}^3, \mathcal{H}_p)} \), for the coupling function \( g \).

**Theorem 4.1.** Assume (1.10) with \( \mu > -1/2 \), (1.11) and (1.13). Let \( \langle g \rangle \) be sufficiently small, \( \nu(-1) < \kappa < 1 \), and \( 0 < \alpha < 1 \). Let \( f \in C_0^\infty((E_{gs}, \Sigma)) \) and \( \psi_0 \in \mathcal{D} := f(H)D(d\Gamma(\langle y \rangle)) \). Then the Schrödinger evolution, \( \psi_t \), satisfies the estimate

\[
\| \Gamma(\chi_{b_\varepsilon \leq c't^\alpha}) \frac{1}{t} \psi_t \| \lesssim t^{-\delta} \| \psi_0 \|_{d\Gamma(\langle y \rangle)^2},
\] (4.1)

for \( 0 \leq \delta < \frac{1}{2} \min(\kappa - \nu(-1), 1 - \kappa, 1 - \alpha - \nu(0)) \) and any \( c' > 0 \).

We define \( B_\varepsilon := d\Gamma(b_\varepsilon) \). As is [1] Proposition B.3 and Remark B.4, one verifies that \( \mathcal{D} \subset D(d\Gamma(\langle y \rangle)) \subset D(B_\varepsilon) \). The proof of Theorem 4.1 is based on the following result (cf. [36], [32]).

**Proposition 4.2.** Under the conditions of Theorem 4.1, the evolution \( \psi_t = e^{-itH}\psi_0 \) obeys

\[
\| \chi_{B_\varepsilon \leq c't} \psi_t \| \lesssim t^{-\delta'} \| \psi_0 \|_{d\Gamma(\langle y \rangle)^2},
\] (4.2)

where \( \delta' := \frac{1}{2} \min(\frac{1-C(\langle q \rangle)}{\varepsilon} - 1 - \kappa, 1 - \kappa, \kappa - \nu(-1)) \).

**Remark.** The constant \( C \) is independent of \( \gamma_0 := \text{dist}(E_{gs}, \text{supp } f) \) (but the implicit constant appearing in the right hand side of (4.2) does depend on \( \gamma_0 \)).

**Proof.** Let \( \varepsilon > 0 \) be a constant. Let \( \rho < \min(\frac{1-C(\langle q \rangle)}{\varepsilon} - 1, 1) \) where \( C > 0 \) is a positive constant defined below. Consider the propagation observable

\[
\Phi_t := -t^\rho \varphi \left( \frac{B_\varepsilon}{ct} \right),
\]

where \( \varphi \left( \frac{B_\varepsilon}{ct} \right) := (\frac{B_\varepsilon}{ct} - 2) \chi_{B_\varepsilon \leq c't} \). Note that \( \varphi \leq 0 \), but \( \varphi' \geq 0 \). Let \( \varphi' = \varphi_1^2 \). The relations below are understood in the sense of quadratic forms on \( \mathcal{D} \). The IMS formula gives

\[
D\Phi = M + R,
\] (4.3)
where $M := -it^\varphi_1 D((ct)^{-1} B_\epsilon) \varphi_1 - \rho t^{-1} \varphi$ and
\begin{equation}
R := \frac{1}{ct^{1-p}} [[B_1, \varphi_1], \varphi_1] + t^\varphi ([H, \varphi] - \frac{1}{2ct}(\varphi'B_1 + B_1 \varphi')), \tag{4.4}
\end{equation}
where $B_1 := i[H, B_\epsilon]$. First, we compute the main term, $M$, in \cite{4.3}. We leave out a standard proof of $f(H) \in C^1(B_\epsilon)$ (see e.g. \cite{20} Theorem 8) and standard domain questions such as that $D \subset D(B_\epsilon)$. We have
\begin{equation}
D\left(\frac{B_\epsilon}{ct}\right) = \frac{1}{ct} DB_\epsilon - \frac{1}{ct} \frac{B_\epsilon}{t}, \tag{4.5}
\end{equation}
The computations below are understood in the sense of quadratic forms on $D$. Since $DB_\epsilon = i[H_f, B_\epsilon] = N_\epsilon$, where $N_\epsilon := d\Gamma(\theta_\epsilon)$, we have
\begin{equation}
DB_\epsilon = N_\epsilon + \tilde{I}, \tag{4.6}
\end{equation}
where $\tilde{I} := i[I(g), B_\epsilon]$. To estimate the operator $N_\epsilon$ from below, we use that $\theta_\epsilon = 1 - \frac{\xi}{\omega_\epsilon}$ to obtain
\begin{equation}
N_\epsilon \geq N - c\Im(\omega_\epsilon^{-1}). \tag{4.7}
\end{equation}
Next, we estimate the term $\varphi_1 d\Gamma(\omega_\epsilon^{-1}) \varphi_1$. Using
\begin{equation*}
[d\Gamma(\omega_\epsilon^{-1}), i\left(\frac{B_\epsilon}{ct} - z\right)^{-1}] = -(ct)^{-1}(\frac{B_\epsilon}{ct} - z)^{-1} d\Gamma(\theta_\epsilon \omega_\epsilon^{-2})(\frac{B_\epsilon}{ct} - z)^{-1},
\end{equation*}
we obtain that
\begin{equation*}
\|d\Gamma(\omega_\epsilon^{-1}, \varphi_1)(N + 1)^{-1}\| \lesssim t^{-1} \epsilon^{-2} |\Im z|^{-2},
\end{equation*}
and hence, since $B_\epsilon$ commutes with $N$, the Helffer-Sjöstrand formula shows that
\begin{equation*}
\|d\Gamma(\omega_\epsilon^{-1}, \varphi_1)(N + 1)^{-1}\| \lesssim t^{-1} \epsilon^{-2}.
\end{equation*}
Since, in addition, $\|d\Gamma(\omega_\epsilon^{-1}) u\| \leq \|d\Gamma(\omega_\epsilon^{-1}) u\|$, we deduce that
\begin{equation*}
\|d\Gamma(\omega_\epsilon^{-1}) \varphi_1 d\Gamma(\omega_\epsilon^{-1}) + t^{-1} \epsilon^{-2}(N + 1)^{-1}\| \lesssim 1,
\end{equation*}
and therefore, by interpolation and \cite{1.14}, we arrive at
\begin{equation*}
\langle \varphi_1 d\Gamma(\omega_\epsilon^{-1}) \varphi_1 \rangle_\psi \leq t^{\nu(-1)} \|\psi_0\|^2_1 + t^{-1+\nu(0)} \epsilon^{-2} \|\psi_0\|^2_2. \tag{4.8}
\end{equation*}
By the condition $\mu > -1/2$ and \cite{2.11} (with $\delta = 0$), we have $\tilde{I} \geq -C(g)(N + \eta^{-2} + 1)$. Combining this with the definition of $M$, \cite{1.11}, \cite{4.5}, \cite{4.6}, \cite{1.7} and \cite{4.8}, we obtain
\begin{equation*}
\langle M \rangle_\psi \leq -\frac{1}{ct^{1-p}} \langle \varphi_1 [(1 - C(g)) N - t^{-1} B_\epsilon - C(g)] \varphi_1 + cp \varphi \rangle_\psi \tag{4.9}
\end{equation*}
\begin{equation*}
+ \frac{C}{t^{1-p}} (ct^{\nu(-1)} \|\psi_0\|^2_1 + t^{-1+\nu(0)} \epsilon^{-1} \|\psi_0\|^2_2).
\end{equation*}
Let $\Omega := 1 \oplus 0 \oplus \ldots$ be the vacuum in $F$ and $P_{\Omega}$, the orthogonal projection on the subspace $H_{p} \otimes \Omega$, $P_\Omega \Psi := \langle \Omega, \Psi \rangle_F \Omega \otimes \Omega$. We now use the following
\begin{lemma} \textbf{Lemma 4.3.} Assume \cite{1.10} with $\mu > -1/2$, \cite{1.11} and \cite{1.13}. Let $\langle g \rangle$ be sufficiently small and $f \in C^\infty_0 ((E_g, \Sigma))$. Then
\begin{equation*}
\|P_{\Omega} e^{-itH} f(H) u\| \lesssim t^{-s} \|f(B) u\|, \quad s < 1/2,
\end{equation*}
where $\tilde{B} = d\Gamma(\tilde{b})$ with, recall, $\tilde{b} = \frac{1}{2}(k \cdot y + y \cdot k)$. \end{lemma}
Proof. We use the local decay properties established in [21] and [7]. Let \( c_j := (e_j + e_{j+1})/2 \) and \( \delta_j := e_{j+1} - e_j \). We decompose the support of \( f \) into different regions, writing

\[
 f(H) = f(H)\chi_{H \leq c_0} + \sum_{\text{finite}} f(H)\chi_j(H),
\]

where \( \chi_j(H) \) denotes a smoothed out characteristic function of the interval \([e_j - \delta_j/4, e_j + \delta_{j+1}/4]\). Using \( P_\Omega = P_\Omega(\tilde{B}) \), and [21], we obtain

\[
 \|P_\Omega e^{-itH} f(H)\chi_{H \leq c_0}u\| = \|\tilde{B}^{-1} e^{-itH} f(H)\chi_{H \leq c_0}u\| \lesssim t^{-s}\|\tilde{B}u\|,
\]

for \( s < 1/2 \).

To estimate \( \|P_\Omega e^{-itH} f(H)\chi_j(H)u\| \), we let \( \tilde{\chi}_j(H) := f(H)\chi_j(H) \). In [7], assuming (1.13), a conjugate operator \( \tilde{B}_j \) is constructed in such a way that the commutators \([\tilde{\chi}_j(H), \tilde{B}_j] \) and \([\tilde{\chi}_j(H), \tilde{B}_j], \tilde{B}_j \] are bounded. Moreover, the Mourre estimate

\[
 \tilde{\chi}_j(H)[H, iB_j]\tilde{\chi}_j(H) \geq m_0 \tilde{\chi}_j(H)^2,
\]

holds for some positive constant \( m_0 \). By an abstract result of [32], this implies

\[
 \|\tilde{B}_j\|^{-s} e^{-itH} \tilde{\chi}_j(H)\tilde{B}_j^{-s} \| \lesssim t^{-s},
\]

for \( s < 1 \). Since the operator \( \tilde{B}_j \) is of the form \( \tilde{B}_j = \tilde{B} + M_j \), where \( M_j \) is a bounded operator, it then follows that

\[
 \|\tilde{B}\|^{-s} e^{-itH} \tilde{\chi}_j(H)\tilde{B}^{-s} \| \lesssim t^{-s},
\]

and hence, using again that \( P_\Omega(\tilde{B}) = P_\Omega \), we obtain

\[
 \|P_\Omega e^{-itH} \tilde{\chi}_j(H)u\| = \|\tilde{B}^{-1} e^{-itH} \tilde{\chi}_j(H)u\| \lesssim t^{-s}\|\tilde{B}u\|.
\]

Equations (4.11), (4.12) and (4.13) give (4.10). \( \square \)

Together with \( \varphi P_\Omega = P_\Omega \), the estimate (4.10) gives

\[
 \langle \varphi P_\Omega \varphi_1 \rangle_{\psi_t} = \langle P_\Omega \rangle_{\psi_t} \lesssim t^{-2s}\|\tilde{B}\|\psi_0\|_B^2 \lesssim t^{-2s}\|\psi_0\|_{B^2}^2.
\]

Combining this with \( N \geq 1 - P_\Omega \) and (4.9), we obtain

\[
 \langle M \rangle_{\psi_t} \leq -\frac{1}{ct^{1-\rho}} \langle \varphi_1 [1 - t^{-1}B_e - C\langle g \rangle] \varphi_1 + c\varphi_2 \rangle_{\psi_t} + \frac{C}{t^{1-\rho}}(ct^{\nu(-1)}\|\psi_0\|^2_2 + t^{-1+\nu(0)}\|\psi_0\|^2_0 + t^{-2s}\|\psi_0\|^2_{B^2}).
\]

Now, using the definition \( \varphi(B_e ct) := (B_e ct - 2)\chi_{B_e \leq ct} \), we compute

\[
 \frac{B_e}{ct} \varphi' + \rho(-\varphi) = \frac{B_e}{ct}(\chi + (B_e ct - 2)^c - \rho(B_e ct - 2)) \chi = ((1 - \rho)\frac{B_e}{ct} + 2\rho)\chi + \frac{B_e}{ct}(B_e ct - 2)\chi'.
\]

Next, using that \( \frac{B_e}{ct} \chi \leq \chi, \frac{B_e}{ct}(\frac{B_e}{ct} - 2)\chi' \leq (\frac{B_e}{ct} - 2)\chi' \), we find furthermore

\[
 \frac{B_e}{ct} \varphi' + \rho(-\varphi) \leq (1 + \rho)\chi + (\frac{B_e}{ct} - 2)\chi' = \rho\chi + \varphi' \leq (1 + \rho)\varphi'.
\]

This, together with (4.15), with \( \varphi_1^2 = \varphi' \), gives

\[
 \langle M \rangle_{\psi_t} \leq -\left[\frac{\sigma}{c} - 1 - \rho \right] \frac{1}{t^{1-\rho}} \langle \varphi' \rangle_{\psi_t} + \frac{C}{t^{1-\rho}}(ct^{\nu(-1)}\|\psi_0\|^2_2 + t^{-1+\nu(0)}\|\psi_0\|^2_0 + t^{-2s}\|\psi_0\|^2_{\Gamma(y)^2}),
\]

where \( \sigma := 1 - C\langle g \rangle \).
Next, we show that the remainder, $R$, in (4.3) is bounded as
\begin{equation}
\|(1 + \eta^{-2} + N)^{-1/2}R(1 + \eta^{-2} + N)^{-1/2}\| \lesssim t^{-2}e^{-1}.
\end{equation}
Indeed, proceeding as in the proof of Lemma 4.2 using the Helffer-Sjöstrand formula, one verifies that
\begin{equation}
\|(1 + \eta^{-2} + N)^{-1/2}R(1 + \eta^{-2} + N)^{-1/2}\|
\lesssim t^{-2}\|(1 + \eta^{-2} + N)^{-1/2}B_2(1 + \eta^{-2} + N)^{-1/2}\|,
\end{equation}
where $B_2 := [B_1, B_2]$. Now, an elementary computation (see (2.9)) gives $B_2 = d\Gamma(\epsilon\theta, \omega^{-2}) + I(b^2g)$. Using $\epsilon\theta, \omega^{-2} \leq \epsilon^{-1}$ and $\|I(\eta b^2g)(1 + N)^{-1/2}\| \lesssim \|\eta b^2g\| \leq \epsilon^{-1}$ since $\epsilon > -1/2$, we obtain
\begin{equation}
\|(1 + \eta^{-2} + N)^{-1/2}B_2(1 + \eta^{-2} + N)^{-1/2}\| \lesssim t^{-2}e^{-1},
\end{equation}
which together with (4.20) implies (4.19). Together with Equations (4.3) and (4.18) and the fact that $\|\eta^{-2}f(H)\| \leq 1$, this implies
\begin{equation}
\langle D\Phi_t \rangle_{\psi_0} \leq -\frac{\sigma}{c} - \rho^{1+\rho}\|\psi_0\|^2_1 + t^{-2+\rho(0)+\rho}\|\psi_0\|^2_0 + t^{-1+\rho}\|\psi_0\|^2_{B_2}. \tag{4.22}
\end{equation}

Thus, choosing $s$ such that $2s - \rho > 0$, together with the observation $\Phi_t \geq t^\rho \chi_{B_1, \leq t}$, the conditions $2s - \rho - \rho > 0$, $\rho < 1 \leq 2 - \rho(0)$, the trivial inequalities $\|\psi_0\|^2_0 \leq \|\psi_0\|^2_{d\Gamma(\psi)}$, $\|\psi_0\|^2_{B_2} \lesssim \|\psi_0\|^2_{d\Gamma(\psi)}^2$, and Hardy’s inequality $\|\psi_0\|^2_{\leq 1} \lesssim \|\psi_0\|^2_{d\Gamma(\psi)}^2$ implies that
\begin{equation}
\|\Phi_t \|_{\psi_0} \leq \langle \Phi_t \rangle_{\psi_0} = \langle \Phi_t \rangle_{\psi_0}|_{t=0} + \int_0^t \langle D\Phi_s \rangle_{\psi_0} ds
\end{equation}
\begin{equation}
\leq \langle -B_\epsilon \chi_{B_1, \leq 0} \rangle_{\psi_0} + C(\epsilon^{-1} + \epsilon t^{\rho+\nu(-1)} + 1)\|\psi_0\|^2_{d\Gamma(\psi)}^2.
\end{equation}
Using $\langle -B_\epsilon \chi_{B_1, \leq 0} \rangle_{\psi_0} \lesssim \|\psi_0\|^2_{d\Gamma(\psi)}$, and choosing $\epsilon = t^{-\kappa}$, we find
\begin{equation}
\langle \chi \rangle_{\psi_0} \leq C(1 + \epsilon^{-1+\kappa} + t^{-\rho})\|\psi_0\|^2_{d\Gamma(\psi)}^2;
\end{equation}
which in turn gives (4.12).

**Proof of Theorem 4.1.** Since $N := d\Gamma(1)$ and $B_\epsilon := d\Gamma(b_\epsilon)$, commute we have
\begin{equation}
\Gamma(\chi_{B_\epsilon, \leq c\theta^\mu}) \leq \chi_{B_\epsilon, \leq c\theta^\mu} = \chi_{B_\epsilon, \leq c\theta^\mu} \chi_{N \leq c\theta^\mu} \chi_{N \geq c\theta^\mu}
\leq \chi_{B_\epsilon, \leq c\theta^\mu} \chi_{N \geq c\theta^\mu},
\end{equation}
where $\nu := \alpha + \gamma$ and $c := c_\epsilon\epsilon^\nu$. We choose $\epsilon^\nu \ll 1/c$, so that $0 < c \ll 1$. Next, we have
\begin{equation}
\|\chi_{N \geq c\theta^\mu}\| \leq (\epsilon^\nu)^{-\frac{\gamma}{\nu-\frac{\gamma}{2}}} \|\chi_{N \geq c\theta^\mu}\| \lesssim (\epsilon^\nu)^{-\frac{\gamma}{\nu-\frac{\gamma}{2}}} \|\chi_{N \geq c\theta^\mu}\|,
\end{equation}
which, together with (1.14) (with $\rho = 0$), implies
\begin{equation}
\|\chi_{N \geq c\theta^\mu}\| \lesssim t^{-\frac{\gamma}{\nu-\frac{\gamma}{2}}} \|\psi_0\|_0.
\end{equation}
The inequality (1.23) with $\nu = 1$, Proposition 1.2 and the inequality (4.24) (with $\gamma = 1 - \alpha$) imply the estimate (1.1).
5. Proof of Theorem 1.3

5.1. Partition of unity. We begin with a construction of a partition of unity which separates photons close to the particle system from those departing it. Following [14, 17] (cf. the many-body scattering construction), it is defined by first constructing a partition of unity \( \tilde{\Gamma}(j) \) on the one-photon space \( \mathfrak{h} = L^2(\mathbb{R}^3) \), with \( j_0 \) localizing a photon to a region near the particle system (the origin) and \( j_\infty \) near infinity, and then associating with it the map \( j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \), given by \( j : h \mapsto j_0 h + j_\infty h \). After that we lift the map \( j \) to the Fock space \( \mathcal{F} := \Gamma(\mathfrak{h}) \) by using \( \Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \) (defined in (1.18)). Next, we consider the adjoint map \( j^* : h_0 \oplus h_\infty \to j_0^* h_0 + j_\infty^* h_\infty \), which we also lift to the Fock space \( \mathcal{F} := \Gamma(\mathfrak{h}) \) by using \( \Gamma(j^*) : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h}) \). By definition, the operator \( \Gamma(j) \) has the following properties

\[
\Gamma(j)^* = \Gamma(j^*), \quad \Gamma(\tilde{j}) \Gamma(j) = \Gamma(\tilde{j}j).
\]  

(5.1)

Since \( j^* j = j_0^2 + j_\infty^2 = 1 \), this implies the relation \( \Gamma(j)^* \Gamma(j) = 1 \), which is what we mean by a partition of unity of the Fock space \( \mathcal{F} := \Gamma(\mathfrak{h}) \).

We refine this construction further by defining the unitary map \( U : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \), through the relations

\[
U \Omega = \Omega \otimes \Omega, \quad U a^*(h) = [a^*(h_1) \otimes 1 + 1 \otimes a^*(h_2)]U,
\]  

for any \( h = (h_1, h_2) \in \mathfrak{h} \oplus \mathfrak{h} \), and introducing the operators

\[
\tilde{\Gamma}(j) := U \Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}).
\]  

(5.3)

We lift \( \Gamma(j) \), as well as \( \tilde{\Gamma}(j) \), from the Fock space \( \mathcal{F} := \Gamma(\mathfrak{h}) \) to the full state space \( \mathcal{H} := \mathcal{H}_f \otimes \mathcal{F} \), so that e.g., \( \tilde{\Gamma}(j) : \mathcal{H} \to \mathcal{H} \otimes \Gamma(\mathfrak{h}) \). Now, the partition of unity relation on \( \mathcal{H} \) becomes \( \tilde{\Gamma}(j)^* \tilde{\Gamma}(j) = 1 \) (in particular, \( \tilde{\Gamma}(j) \) is an isometry).

Finally, we specify \( j_0 \) to be the operator \( \chi_{b_\epsilon \leq \epsilon^\alpha} \), with \( b_\epsilon \) defined in the introduction, and \( j_\infty \) is defined by \( j_0^2 + j_\infty^2 = 1 \) and is of the form \( \chi_{b_\epsilon \geq \epsilon^\alpha} \), where \( \epsilon := t^{-\kappa} \), and \( \alpha \) and \( \kappa \) satisfy

\[ 1 - \mu/(6 + 3\mu) < \alpha < 1 \quad \text{and} \quad 1 + \nu(-1) - \alpha < \kappa < \frac{1}{2}(5\alpha - 3). \]

5.2. Deift-Simon wave operators. We define the auxiliary space \( \hat{\mathcal{H}} := \mathcal{H} \otimes \mathcal{F} \), which will serve as our repository of asymptotic dynamics, which is governed by the hamiltonian \( \hat{H} := H \otimes 1 + 1 \otimes H_f \) on \( \hat{\mathcal{H}} \). With the partition of unity \( \tilde{\Gamma}(j) \), we associate the Deift-Simon wave operators,

\[
W_\pm := \text{s-lim}_{t \to \infty} W(t), \quad \text{where} \quad W(t) := e^{it\hat{H}} \tilde{\Gamma}(j) e^{-it\hat{H}},
\]  

(5.4)

which map the original dynamics, \( e^{-it\hat{H}} \), into auxiliary one, \( e^{-it\hat{H}} \) (to be further refined later). Our goal is to prove

**Theorem 5.1.** Assume (1.10) with \( \mu > 0 \), (1.11) and (1.20). Then the Deift-Simon wave operators exist on \( \text{Ran} \, E_{(-\infty, \Sigma)}(\hat{H}) \) and satisfy

\[
W_+ P_{gs} = P_{gs},
\]  

(5.5)

and, for any smooth, bounded function \( f \),

\[
W_+ f(\hat{H}) = f(\hat{H}) W_+.
\]  

(5.6)

**Proof.** We want to show that the family \( W(t) := e^{it\hat{H}} \tilde{\Gamma}(j) e^{-it\hat{H}} \) form a strong Cauchy sequence as \( t \to \infty \). To this end, we define \( \chi_m := \chi_{\hat{N} \leq m} \) and \( \bar{\chi}_m := \chi_{\hat{N} \geq m} \), where \( \hat{N} := N \otimes 1 + 1 \otimes N \), the number operator on \( \hat{\mathcal{H}} \), so that \( \chi_m + \bar{\chi}_m = 1 \). First, we show that, for any \( \psi_0 \in D(N^\frac{1}{2}) \),

\[
\sup_t \| \bar{\chi}_m W(t) \psi_0 \| \lesssim m^{-\frac{1}{2}} \| \psi_0 \|_N.
\]  

(5.7)
Indeed, by the assumption (1.20),
\[ \| \hat{N} \frac{1}{2} e^{iHt} \hat{\Gamma}(j) e^{-iHs} \psi_0 \| \lesssim \| \hat{N} \frac{1}{2} \hat{\Gamma}(j) e^{-iHs} \psi_0 \| + \| \hat{\Gamma}(j) e^{-iHs} \psi_0 \|. \tag{5.8} \]

The boundedness of \( \hat{\Gamma}(j) \) implies \( \| \hat{\Gamma}(j) e^{-iHt} \psi_0 \| \leq \| \psi_0 \| \leq \| \psi_0 \|_N \). Moreover, we claim that
\[ \hat{\Gamma}(j) N = \hat{N} \hat{\Gamma}(j), \tag{5.9} \]
Indeed, a straightforward computation gives \( \Gamma(j) d\Gamma(c) = d\Gamma(c) \Gamma(j) + d\Gamma(j, j c - c j) \), where \( c = \text{diag}(c, c) : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \) and
\[ d\Gamma(a, c)|_{\otimes c \mathfrak{h}} = \sum_{j=1}^n a \otimes \cdots \otimes a \otimes c \otimes a \otimes \cdots \otimes a. \tag{5.10} \]

It follows from this relation and the equalities \( U d\Gamma(c) = (d\Gamma(c) \otimes 1 + 1 \otimes d\Gamma(c))U \) that (14, 17)
\[ \hat{\Gamma}(j) d\Gamma(c) = (d\Gamma(c) \otimes 1 + 1 \otimes d\Gamma(c)) \hat{\Gamma}(j) + d\Gamma(j, j c - c j), \tag{5.11} \]
where and \( \hat{\Gamma}(j, a, c) := U d\Gamma(a, c) \). For \( c = 1 \), the latter relation gives (5.9). Equation (5.9) implies \( \hat{N} \frac{1}{2} \hat{\Gamma}(j) = \hat{\Gamma}(j) N \frac{1}{2} \), and this relation, boundedness of \( \hat{\Gamma}(j) \) and the assumption (1.20) give
\[ \| \hat{N} \frac{1}{2} \hat{\Gamma}(j) e^{-iHs} \psi_0 \| = \| \hat{\Gamma}(j) N \frac{1}{2} e^{-iHs} \psi_0 \| \leq \| \psi_0 \|_N, \]
and therefore, by (5.8), \( \| \hat{N} \frac{1}{2} \hat{\Gamma}(j) e^{-iHs} \psi_0 \| \lesssim \| \psi_0 \|_N \). Since this is true uniformly in \( t, s \), it implies \( \| \hat{N} \frac{1}{2} W(t) \psi_0 \| \lesssim \| \psi_0 \|_N \), which yields (5.12). Equation (5.12) implies that
\[ \sup_{t, t'} \| \chi_m(W(t') - W(t)) \psi_0 \| \lesssim m^{-\frac{1}{2}}. \tag{5.12} \]

Now we show that, for any \( m > 0 \) and for any \( \psi_0 \in D(N \frac{1}{2}) \cap \text{Ran } E_{(-\infty, \Sigma)}(H) \),
\[ \| \chi_m (W(t') - W(t)) \psi_0 \| \to 0, \tag{5.13} \]
as \( t, t' \to \infty \). This together with (5.12) implies that \( W(t) \) form a strong Cauchy sequence. Lemma 5.2, proven below, implies that, in order to show (5.13), it suffices to prove
\[ \| \chi_m f(\hat{H})(W(t') - W(t)) \psi_0 \| \to 0, \tag{5.14} \]
to which we now proceed. We write
\[ (W(t') - W(t)) \psi_0 = \int_t^{t'} ds \partial_s W(s) \psi_0 \tag{5.15} \]
and compute \( \partial_t W(t) = e^{iHt} G e^{-iHt} \), where \( G := i(\hat{H} \Gamma(j) - \hat{\Gamma}(j) H) + \partial_t \hat{\Gamma}(j) \). We write \( G = G_0 + G_1 \), where
\[ G_0 := i(\hat{H} \Gamma(j) - \hat{\Gamma}(j) H) \Gamma(j) \]
and
\[ G_1 := i(I(g) \otimes 1) \hat{\Gamma}(j) - \hat{\Gamma}(j) I(g). \tag{5.16} \]

We consider \( G_0 \). Using \( (H_p \otimes 1 \otimes 1)(1 \otimes \hat{\Gamma}(j)) = (1 \otimes \hat{\Gamma}(j))(H_p \otimes 1) \) and using the notation \( \delta j := i(w(j - j) + \partial_t j) \), with \( w = \text{diag}(\omega, \omega) \), and (5.11), we compute readily
\[ G_0 = U d\Gamma(j \delta j) = d\Gamma(j \delta j). \tag{5.17} \]

Write \( j' = (j_0', j_\infty') \), where \( j_0', j_\infty' \) are the derivatives of \( j_0, j_\infty \) as functions of \( v = \frac{\partial}{\partial \omega} \). We first find a convenient decomposition of \( \delta j \). Using \( \delta j f = (d_0 j f, d_\infty j f) \), with \( d_0 c = i[\omega, c_\omega] + \partial_t c_\omega \), (5.7) and Corollary B.3 we compute
\[ d j = (j_0', j_\infty')(\frac{\partial_t}{c_\alpha} - \frac{\alpha b_\omega}{c_\alpha + 1}) + O(t^{-2\alpha + \kappa}) \tag{5.18} \]
We insert the maximal velocity partition of unity \( \chi_{(\frac{b}{c})^2 \leq 1} + \chi_{(\frac{b}{c})^2 \geq 1} = 1 \), with \( \bar{c} > 1 \), into this formula and use the notation \( \chi \equiv \chi_{(\frac{b}{c})^2 \leq 1} \) and the relation \( \frac{b}{c} j_\# = O(1) j_\# \), valid due to the localization of \( j_\# \), to obtain

\[
d_{j} = \frac{1}{ct^{\alpha}} \theta_{1/2} \chi(j_0, j_\theta') \theta_{1/2} + \text{rem}_t, \tag{5.19}
\]

\[
\text{rem}_t = O(t^{-1}) \chi(j_0, j_\theta') + O(t^{-2\alpha-\kappa}) + O(t^{-\alpha}) \chi_{(\frac{b}{c})^2 \geq 1}. \tag{5.20}
\]

These relations give

\[
G_0 = G_0' + \text{Rem}_t, \tag{5.21}
\]

where \( G_0' := \frac{1}{ct^\alpha} Ud\Gamma(j, \omega) \), with \( \omega = (c_0, c_\infty) := (\theta_{1/2} \chi j_0 \theta_{1/2}, \theta_{1/2} \chi j_\theta' \theta_{1/2}) \), and

\[
\text{Rem}_t := G_0 - G_0' = Ud\Gamma(j, \text{rem}_t).
\]

Next, we write \( A := \sup_{\|c_0\|=1} \int_t^{t'} ds \langle \hat{\phi}_s, G_0 \psi_s \rangle \), where \( \hat{\phi}_s := e^{-iH_t f(H) \chi_m \hat{\phi}_0} \). By (C.1) of Appendix C, \( G_0' \) satisfies

\[
|\langle \hat{\phi}, G_0' \psi \rangle| \leq \frac{1}{ct^\alpha} (\|d\Gamma(|c_0|)^{1/2} \otimes 1 \hat{\phi} \| \|d\Gamma(|c_0|)^{1/2} \psi \|
+ \|1 \otimes d\Gamma(|c_\infty|)^{1/2} \hat{\phi} \| \|d\Gamma(|c_\infty|)^{1/2} \psi \|). \tag{5.22}
\]

By the Cauchy-Schwarz inequality, (5.22) implies

\[
\int_{t}^{t'} ds |\langle \hat{\phi}_s, G_0' \psi_s \rangle| \leq \left( \int_{t}^{t'} ds \|d\Gamma(|c_0|)^{1/2} \otimes 1 \hat{\phi}_s \|^2 \right)^{1/2} \left( \int_{t}^{t'} ds \|d\Gamma(|c_0|)^{1/2} \psi_s \|^2 \right)^{1/2}
+ \left( \int_{t}^{t'} ds \|1 \otimes d\Gamma(|c_\infty|)^{1/2} \hat{\phi}_s \|^2 \right)^{1/2} \left( \int_{t}^{t'} ds \|d\Gamma(|c_\infty|)^{1/2} \psi_s \|^2 \right)^{1/2}.
\]

Since \( |c_0|, |c_\infty| \) are of the form \( \theta_{1/2} \chi \chi_{b_0 = ct^\alpha} \theta_{1/2} \), the minimal velocity estimate (5.33) implies

\[
\int_{1}^{\infty} ds s^{-\alpha} \|d\Gamma_\#(|c|)^{1/2} \hat{\phi}_s \|^2 \lesssim \|\chi \chi_{m \hat{\phi}_0} \|^2 \lesssim m \hat{\phi}_0 \|^2,
\]

where \( d\Gamma_\#(|c|)^{1/2} \) stands for \( d\Gamma(|c_0|)^{1/2} \otimes 1 \) or \( 1 \otimes d\Gamma(|c_\infty|)^{1/2} \), and

\[
\int_{1}^{\infty} ds s^{-\alpha} \|d\Gamma_\#(|c|)^{1/2} \psi_s \|^2 \lesssim \|\psi_0 \|^2
\]

with \( d\Gamma_\#(|c|)^{1/2} = d\Gamma(|c_0|)^{1/2} \) or \( d\Gamma(|c_\infty|)^{1/2} \). The last three relations give

\[
\sup_{\|\hat{\phi}_0\|=1} |\int_{t}^{t'} ds \langle \hat{\phi}_s, G_0' \psi_s \rangle| \to 0, \quad t, t' \to \infty. \tag{5.23}
\]

Likewise, applying (C.2) of Appendix C, first with \( c_1 = c_2 = 1 \), next with \( c_1 = 1 \) and \( c_2 = \chi_{(\frac{b}{c})^2 \geq 1} \), and then applying (C.1) with \( c_0 = \chi j_0 \chi \) and \( c_\infty = \chi j_\infty \chi \), we see that \( \text{Rem}_t \) satisfies

\[
|\langle \hat{\phi}, \text{Rem}_t \psi \rangle| \lesssim N^{1/2} \hat{\phi} (t^{-2\alpha+\kappa} \|N^{1/2} \psi \| + t^{-1} \|d\Gamma(\chi j_\infty \chi)^{1/2} \psi \| + t^{-\alpha} \|d\Gamma(\chi_{(\frac{b}{c})^2 \geq 1})^{1/2} \psi \|).
\]

Now, using (5.24) and the Cauchy-Schwarz inequality, we obtain

\[
|\int_{t}^{t'} ds \langle \hat{\phi}_s, \text{Rem}_s \psi_s \rangle| \leq \left( \int_{t}^{t'} ds s^{-\tau} \|N^{1/2} \hat{\phi}_s \|^2 \right)^{1/2} \left( \int_{t}^{t'} ds s^{-2(2\alpha-\kappa)+\tau} \|N^{1/2} \psi_s \|^2 \right)^{1/2}
+ \left( \int_{t}^{t'} ds s^{-2\alpha+\tau} \|d\Gamma(\chi j_\infty \chi)^{1/2} \psi_s \|^2 \right)^{1/2} \left( \int_{t}^{t'} ds s^{-2\alpha+\tau} \|d\Gamma(\chi_{(\frac{b}{c})^2 \geq 1})^{1/2} \psi_s \|^2 \right)^{1/2}.
\]
Let $\tau > 1$ and $\alpha = 2 - \tau$. Then by the estimate (3.3),
\[
\int_1^\infty ds \, s^{-2+\tau} \|d\Gamma(\chi j_\infty\lambda)\frac{\partial}{\partial s}\| \psi_s \| \leq \|\psi_0\|^{-1},
\]
provided $\alpha < \frac{1}{c}$, and by the maximal velocity estimate (1.9),
\[
\int_1^\infty ds \, s^{-2\alpha+\tau} \|d\Gamma(\chi^2(\frac{\partial}{\partial s})^2 \geq 1)\frac{\partial}{\partial s}\| \psi_s \| \leq \|\psi_0\|^{-1},
\]
provided that $\alpha > 1 - 2\gamma/3$, where, recall, $\gamma < \frac{3}{2} \min(\frac{c-1}{\sqrt{c-1}}, \frac{1}{2\sqrt{\alpha}})$. One verifies that $c > 1$ can be chosen such that the two conditions above are satisfied. Moreover, by Assumption (1.20),
\[
\int_1^\infty ds \, s^{-2(2\alpha-\kappa)+\tau} \|N^{\frac{\alpha}{2}} \psi_s \| \leq \|\psi_0\|^{-1},
\]
provided that $5\alpha > 3 + 2\kappa$. This and the fact that, by Assumption (1.20), the first integral on the r.h.s. of (5.25) converge yield
\[
\sup_{\|\psi_0\|=1} \int_t^{t'} ds (\hat{\phi}_s, \text{Rem}_s \psi_s) \to 0, \quad t, t' \to \infty. \tag{5.26}
\]
Equations (5.23) and (5.26) imply
\[
A = \| \int_t^{t'} ds \chi_m f(\hat{H}) \psi G_0 \| \to 0, \quad t, t' \to \infty. \tag{5.27}
\]

Now we turn to $G_1$. We use the definition $\tilde{\Gamma}(j) := U \Gamma(j)$ to obtain $\tilde{\Gamma}(j) a\#(h) = U a\#(jh) \Gamma(j)$, then (5.2), and then $j_\infty j_0 + j_\infty j_\infty = 1$, to derive
\[
\tilde{\Gamma}(j) a\#(h) = (a\#(j_0 h) \otimes 1 + 1 \otimes a\#(j_\infty h)) \tilde{\Gamma}(j), \tag{5.28}
\]
where $a\#$ stands for $a$ or $a^*$, which implies
\[
\tilde{\Gamma}(j) I(g) = (I(j_0 g) \otimes 1 + 1 \otimes I(j_\infty g)) \tilde{\Gamma}(j) \otimes 1. \tag{5.29}
\]
The equation (5.29) gives
\[
G_1 = (I((1 - j_0) g) \otimes 1 - 1 \otimes I(j_\infty g)) \tilde{\Gamma}(j). \tag{5.30}
\]
Due to [9] Lemma 3.1 (see Appendix [13] Lemma [13.6], we have $\|j_\infty g\| \leq t^{-\lambda}$, $\|(1 - j_0) g\| \leq t^{-\lambda}$ with $\lambda < (\mu + \frac{3}{2})\alpha$. This, (2.11) (with $\delta = 0$), and $N^\frac{\alpha}{2} \tilde{\Gamma}(j) = \tilde{\Gamma}(j) N^\frac{\alpha}{2}$ imply that
\[
\|f(\hat{H}) G_1 (N + 1)^{-\frac{\alpha}{4}} \| \leq t^{-\alpha - \frac{\lambda}{2}} \tag{5.31}
\]
This together with Assumption (1.20) implies that $\|f(\hat{H}) G_1 \psi_t\| \leq t^{-\alpha - \frac{\lambda}{2}} \|\psi_0\|$, and hence
\[
\| \int_t^{t'} ds f(\hat{H}) e^{iHs} G_1 \psi_s \| \to 0, \quad t, t' \to \infty,
\]
provided that $\alpha > (\mu + \frac{3}{2})^{-1}.$ This together with (5.27) gives (5.14), and therefore (5.13), which, as was mentioned above, together with (5.12) shows that $W(t)$ is a Cauchy sequence as $t \to \infty$. This implies the existence of $W_+$. 

Finally, the proofs of (5.5) and (5.6) are standard. We present the second one. By (5.4), we have $W_+ e^{iHs} = s \cdot \lim e^{iH(t-s)} \tilde{\Gamma}(j) e^{-iH(t-s)} = s \cdot \lim e^{iH(t-s)} \hat{\phi}_s = e^{iHs} W_+$, which implies (5.6). \(\square\)

Now we establish the following lemma used in the proof of Theorem 5.1.
Lemma 5.2. Under the conditions of Theorem 5.1, for any \( f \in C^\infty_0(\Delta) \), \( \Delta \subset (E_g, \Sigma) \), and \( \psi_0 \in \text{Ran } E_\Delta(H) \cap D(N^{1/2}) \),

\[
\|(N + 1)^{-\frac{1}{2}}(\tilde{\Gamma}(j)f(H) - f(\tilde{H})\tilde{\Gamma}(j))\psi_t\| \lesssim t^{-\alpha}\|\psi_0\|.
\] (5.32)

Proof. We compute, using the Helffer-Sjöstrand formula, \( \tilde{\Gamma}(j)f(H)\psi_t - f(\tilde{H})\tilde{\Gamma}(j)\psi_t = R \), where

\[
R := \frac{1}{\pi} \int \partial_z \tilde{f}(z)(\tilde{H} - z)^{-1}(\tilde{H}\tilde{\Gamma}(j) - \tilde{\Gamma}(j)\tilde{H})(\tilde{H} - z)^{-1}\psi_t \, d\text{Re } z \, d\text{Im } z.
\] (5.33)

We have \( \tilde{H}\tilde{\Gamma}(j) - \tilde{\Gamma}(j)\tilde{H} = \tilde{G}_0 - iG_1 \), where \( \tilde{G}_0 := U\partial\Gamma(j, \omega j - j\omega) \) and \( G_1 := (I(g) \otimes 1)\tilde{\Gamma}(j) - \tilde{\Gamma}(j)I(g) \) was defined in (5.10).

We consider \( \tilde{G}_0 \). As in the proof of Theorem 5.1, we have \( \omega j - j\omega = ([\omega, j_0], [\omega, j_\infty]) \), and, by Corollary B.3

\[
[\omega, j\#] = \frac{\theta_g}{\pi} j_\# + r,
\] (5.34)

where \( j_\# \) stands for \( j_0 \) or \( j_\infty \), \( j'_\# \) is the derivative of \( j_\# \) as a function of \( \frac{\theta_g}{\pi} \), and \( r \) satisfies \( \|r\| \lesssim t^{-2\alpha + \kappa} \). Since \( \theta_k \leq 1 \) and since \( \kappa < \alpha \), we deduce that \( [\omega, j\#] = O(t^{-\alpha}) \). By (C.2) of Appendix C, we then obtain that

\[
\|(N + 1)^{-\frac{1}{2}}\tilde{G}_0(N + 1)^{-\frac{1}{2}}\| \lesssim t^{-\alpha}.
\]

Since \( H \in C^1(\mathbb{N}) \), we have \( \|(N + 1)^{-\frac{1}{2}}(H - z)^{-1}(N + 1)^{-\frac{1}{2}}\| \lesssim |\text{Im } z|^{-2} \), and likewise \( \|(N + 1)^{-\frac{1}{2}}(\tilde{H} - z)^{-1}(N + 1)^{-\frac{1}{2}}\| \lesssim |\text{Im } z|^{-2} \). Moreover, by Assumption (1.20), \( \|(N + 1)^{\frac{1}{2}}e^{-i\tilde{H}t}(N + 1)^{-\frac{1}{2}}\| \lesssim 1 \), and \( \|(N + 1)^{-\frac{1}{2}}e^{i\tilde{H}t}(\tilde{N} + 1)^{-\frac{1}{2}}\| \lesssim 1 \). The previous estimates imply

\[
\|(N + 1)^{-\frac{1}{2}}e^{i\tilde{H}t}(\tilde{H} - z)^{-1}\tilde{G}_0(H - z)^{-1}\psi_t\| \lesssim t^{-\alpha}|\text{Im } z|^{-4}\|\psi_0\|_N.
\] (5.35)

As in (5.30)-(5.31), we have in addition

\[
\|(N + 1)^{-\frac{1}{2}}G_1E_\Delta(H)\| \lesssim t^{-(\mu + \frac{3}{2})\alpha},
\]

and hence

\[
\|(N + 1)^{-\frac{1}{2}}e^{i\tilde{H}t}(\tilde{H} - z)^{-1}G_1(H - z)^{-1}\psi_t\| \lesssim t^{-(\mu + \frac{3}{2})\alpha}|\text{Im } z|^{-3}\|\psi_0\|.
\] (5.36)

From (5.33), (5.35), (5.36) and the properties of the almost analytic extension \( \tilde{f} \), we conclude that (5.32) holds.

5.3. Scattering map. We define the space \( \mathcal{H}_{\text{fin}} := \mathcal{H}_p \otimes \mathcal{F}_{\text{fin}} \otimes \mathcal{F}_{\text{fin}} \), where \( \mathcal{F}_{\text{fin}} \equiv \mathcal{F}_{\text{fin}}(\mathfrak{h}) \) is the subspace of \( \mathcal{F} \) consisting of vectors \( \Psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F} \) such that \( \psi_n = 0 \), for all but finitely many \( n \), and the (scattering) map \( I : \mathcal{H}_{\text{fin}} \to \mathcal{H} \) as the extension by linearity of the map (see [30, 14, 17])

\[
I : \Phi \otimes \prod_{1}^{n} a^*(h_i)\Omega \to \prod_{1}^{n} a^*(h_i)\Phi,
\] (5.37)

for any \( \Phi \in \mathcal{H}_p \otimes \mathcal{F}_{\text{fin}} \) and for any \( h_1, \ldots h_n \in \mathfrak{h} \). (Another useful representation of \( I \) is \( I : \Phi \otimes f \to \left( \begin{array}{c} p + q \\ p \end{array} \right)^{1/2} \Phi \otimes_s f \), for any \( \Phi \in \mathcal{H}_p \otimes (\otimes^q_s \mathfrak{h}) \) and \( f \in (\otimes^q_s \mathfrak{h}) \). As already clear from (5.37), the operator \( I \) is unbounded. Let

\[
\mathfrak{h}_0 := \{ h \in L^2(\mathbb{R}^3), \int dk(1 + \omega^{-1})|h(k)|^2 < \infty \}.
\] (5.38)

Properties of the operator \( I \) used below are recorded in the following
Lemma 5.3 ([14] [17] [24]). For any operator \( j : h \to j_0h \oplus j_\infty h \) and \( n \in \mathbb{N} \), the following relations hold

\[
\Gamma(j)^* = \Pi(j_0^*) \otimes \Gamma(j_\infty^*),
\]

\[
D((H + i)^{-n/2}) \otimes (\otimes_i^n \mathfrak{h}_0) \subset D(I).
\]

**Proof.** Since the operators involved act only on the photonic degrees of freedom, we ignore the particle one. For \( g, h \in \mathfrak{h} \), we define embeddings \( i_0g := (g, 0) \in \mathfrak{h} \oplus h \) and \( i_\infty h := (0, h) \in \mathfrak{h} \oplus h \). By the definition of \( U \) (see (5.22)), we have the relations \( U^*a^*(g) \otimes 1 = a^*(i_0g)U^* \), and \( U^*1 \otimes a^*(h) = a^*(i_\infty h)U^* \). Hence, using in addition \( U^*\Omega \otimes \Omega = \Omega \), we obtain

\[
U^* \prod_i a^*(g_i)\Omega \otimes \prod_i a^*(h_i)\Omega = \prod_i a^*(i_0g_i) \prod_i a^*(i_\infty h_i)\Omega.
\]

By the definition of \( \Gamma(j) \) and the relations \( j^*i_0g = j_0^*g \) and \( j^*i_\infty h = j_\infty^*h \), this gives

\[
\Gamma(j)^*U^* \prod_i a^*(g_i)\Omega \otimes \prod_i a^*(h_i)\Omega = \prod_i a^*(j_0^*g_i) \prod_i a^*(j_\infty^*h_i)\Omega.
\]

Now, by the definition of \( \tilde{\Gamma}(j) \) (see (5.23)), we have \( \tilde{\Gamma}(j)^* = \Gamma(j)^*U^* \). On the other hand by (5.37), the r.h.s. is \( \tilde{\Gamma}(j_0^*) \otimes \Gamma(j_\infty^*) \prod_i^\infty a^*(g_i)\Omega \otimes \prod_i^\infty a^*(h_i)\Omega \). This proves (5.39).

To prove (5.40), we use the following elementary properties (17 [24]):

The operator \( H_f^n(H + i)^{-n} \) is bounded \( \forall n \in \mathbb{N} \), and, for any \( h_1, \ldots, h_n \in \mathfrak{h}_0 \), where \( \mathfrak{h}_0 \) is defined in (5.38),

\[
\|a^*(h_1) \cdots a^*(h_n)(H_f + 1)^{-n/2}\| \leq C_n\|h_1\|_\omega \cdots \|h_n\|_\omega,
\]

where \( h_\omega := \int dk(k + 1)^{-1}|h(k)|^2 \). The previous two estimates and the representation (5.37) imply that for any \( \Phi \in D((H + i)^{-n/2}) \) and \( h_1, \ldots, h_n \in \mathfrak{h}_0 \), we have \( \|\Phi \otimes \prod_i^\infty a^*(h_i)\Omega\| \leq C_n\|h_1\|_\omega \cdots \|h_n\|_\omega \|\Phi\|(H + i)^{n/2}\Phi\| < \infty \). This gives the second statement of the lemma. \( \Box \)

5.4. **Asymptotic completeness.** Recall that \( P_{gs} \) denotes the orthogonal projection onto the ground state subspace of \( H \). Below, the symbol \( C(\epsilon')\omega(1) \) stands for a positive function of \( \epsilon \) and \( t \) such that \( \|C(\epsilon')\omega(1)\| \to 0 \) as \( t \to \infty \) and we denote by \( \chi_\Omega(\lambda) \) a smoothed out characteristic function of a set \( \Omega \). In this section we prove the following result.

**Theorem 5.4.** Assume the conditions of Theorem 1.3 and let \( a < \Sigma \), \( \Delta = [E_{gs}, a] \subset \mathbb{R} \). Then, for every \( \epsilon' > 0 \) there is \( \phi_{\epsilon'} \), s.t.

\[
\limsup_{t \to \infty} \|\psi_t - I(e^{-iE_{gs}t}P_{gs} \otimes e^{-iH_f t}\chi_{[0,a-E_{gs}]}(H_f))\phi_{\epsilon'}\| = O(\epsilon'),
\]

which implies (1.7).

**Proof.** Let \( \alpha, \beta, \kappa \) be fixed such that the conditions of Theorems 5.1 [4] and 5.4 hold, with \( \alpha = \beta \). Let \( (j_0, j_\infty) := (\chi_{b_0 < x < \alpha}, \chi_{b_\infty > \alpha}) \) be the partition of unity defined in Subsection 5.1. Since \( j_0^2 + j_\infty^2 = 1 \), the operator \( \tilde{\Gamma}(j) \) is, as mentioned above, an isometry. Using the relation \( \Gamma(j)^*\Gamma(j) = 1 \), the boundedness of \( \Gamma(j)^* \), and the existence of \( W_j^* \), we obtain

\[
\psi_t = \tilde{\Gamma}(j)^*e^{-i\hat{H}t}e^{i\hat{H}t}\tilde{\Gamma}(j)e^{-i\hat{H}t}\psi_0 = \tilde{\Gamma}(j)^*e^{-i\hat{H}t}\phi_0 + \omega(1),
\]

where \( \phi_0 := W_+\psi_0 \). Next, using the property \( W_+\chi_\Delta(H) = \chi_\Delta(\hat{H})W_+ \), which gives \( W_+\psi_0 = \chi_\Delta(\hat{H})W_+\psi_0 \), and \( \chi_\Delta(\hat{H}) = (\chi_{[E_{gs}, a]}(H) \otimes \chi_{[0,a-E_{gs}]}(H_f))\chi_\Delta(H) \), and again using \( \chi_\Delta(\hat{H})W_+\psi_0 = W_+\psi_0 = \phi_0 \), we obtain

\[
\phi_0 = (\chi_{[E_{gs}, a]}(H) \otimes \chi_{[0,a-E_{gs}]}(H_f))\phi_0.
\]
For all $\epsilon' > 0$, there is $\delta' = \delta'(\epsilon') > 0$, such that

$$\| (\chi_{[E_{gs}, a]}(H) \otimes 1) \phi_0 - (\chi_{\Delta_e}(H) \otimes 1) \phi_0 - (P_{gs} \otimes 1) \phi_0 \| \leq \epsilon',$$

with $\Delta_e = [E_{gs} + \delta', a]$. The last two relations give

$$\phi_0 = ( (\chi_{\Delta_e}(H) + P_{gs}) \otimes \chi_{[0, a - E_{gs}]}(H_f)) \phi_0 + \mathcal{O}(\epsilon').$$

(5.48)

Now, let $\phi_{0\epsilon'} \in \mathcal{F}_{\text{fin}}(D(d\Gamma'(y))) \otimes \mathcal{F}_{\text{fin}}(\phi_0)$ be such that $\| \phi_0 - \phi_{0\epsilon'} \| \leq \epsilon'$. (We require that the ‘first components’ of $\phi_{0\epsilon'}$ are in $D(d\Gamma'(y))$ in order to apply the minimal velocity estimate below, and that the ‘second components’ are in $\mathcal{F}_{\text{fin}}(\phi_0)$ in order that $(P_{gs} \otimes 1) \phi_{0\epsilon'}$ is in $D(I)$).

This together with (5.45) and (5.48) gives

$$\psi_t = \tilde{\Gamma}(j)^* e^{-iHt}((\chi_{\Delta_e}(H) + P_{gs}) \otimes \chi_{[0, a - E_{gs}]}(H_f)) \phi_{0\epsilon'} + \mathcal{O}(\epsilon') + o_t(1).$$

(5.49)

Furthermore, let $(\tilde{j}_0, \tilde{j}_\infty)$ be of the form $\tilde{j}_0 = \tilde{\chi}_{b_0 \leq \epsilon t^\alpha}$, $\tilde{j}_\infty = \tilde{\chi}_{b_\infty \geq \epsilon t^\alpha}$ where $\tilde{\chi}$, has the same properties as $\chi$, and satisfy $j_0 \tilde{j}_0 = j_0$, $j_\infty \tilde{j}_\infty = j_\infty$. Then, by (5.39), the adjoint $\tilde{\Gamma}(j)^*$ to the operator $\tilde{\Gamma}(j)$ can be represented as

$$\tilde{\Gamma}(j)^* = \tilde{\Gamma}(j)^* (\Gamma(\tilde{j}_0) \otimes \Gamma(\tilde{j}_\infty)).$$

(5.50)

Using this equation in (5.49), together with the relations $e^{-iHt} = e^{-iHt} \otimes e^{-iH_f t}$ and $e^{-iHt} P_{gs} = e^{-iE_{gs} t} P_{gs}$, gives

$$\psi_t = \tilde{\Gamma}(j)^* \psi_{t\epsilon'} + A + B + C + \mathcal{O}(\epsilon') + o_t(1),$$

(5.51)

where

$$\psi_{t\epsilon'} := (e^{-iE_{gs} t} P_{gs} \otimes e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) \phi_{0\epsilon'},$$

(5.52)

$$A := \tilde{\Gamma}(j)^* (\Gamma(\tilde{j}_0) e^{-iHt} \chi_{\Delta_e}(H) \otimes \Gamma(\tilde{j}_\infty) e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) \phi_{0\epsilon'},$$

(5.53)

$$B := \tilde{\Gamma}(j)^* ((\Gamma(\tilde{j}_0) - 1) e^{-iE_{gs} t} P_{gs} \otimes \Gamma(\tilde{j}_\infty) e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) \phi_{0\epsilon'},$$

(5.54)

$$C := \tilde{\Gamma}(j)^* (e^{-iE_{gs} t} P_{gs} \otimes (\Gamma(\tilde{j}_\infty) - 1) e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) \phi_{0\epsilon'},$$

(5.55)

Since $\Gamma(j)^*$ is bounded, the minimal velocity estimate, (4.11), gives (here we use that the first components of $\phi_{0\epsilon'}$ are in $D(d\Gamma'(y)))$

$$\|A\| \leq \| (\Gamma(\tilde{j}_0) e^{-iHt} \chi_{\Delta_e}(H) \otimes 1) \phi_{0\epsilon'} \| = C(\epsilon') o_t(1).$$

Now we consider the term given by $B$. We begin with

$$\|B\| \leq \| (\Gamma(\tilde{j}_0) - 1) P_{gs} \|.$$

(5.56)

Since $0 \leq \tilde{j}_0 \leq 1$, we have that $0 \leq 1 - \Gamma(\tilde{j}_0) \leq 1$. Using this, the relations $1 - \Gamma(\tilde{j}_0) \leq d\Gamma(\tilde{\chi}_{b_\infty \geq \epsilon t^\alpha})$ and $d\Gamma(\tilde{\chi}_{b_\infty \geq \epsilon t^\alpha}) \leq t^{-2\alpha} d\Gamma(b_t^2)$, we obtain the bound

$$\| (\Gamma(\tilde{j}_0) - 1) u \|^2 \leq \| (1 - \Gamma(\tilde{j}_0)) {\bar{\partial}}u \|^2 \leq \| d\Gamma(\tilde{\chi}_{b_\infty \geq \epsilon t^\alpha}) {\bar{\partial}}u \|^2 \leq t^{-2\alpha} \| d\Gamma(b_t^2) {\bar{\partial}}u \|^2.$$

(5.57)

Using the pull-through formula, one verifies that $d\Gamma(b_t^2) {\bar{\partial}}P_{gs}$ is bounded and that $\| d\Gamma(b_t^2) {\bar{\partial}}P_{gs} \| = \mathcal{O}(t^\kappa)$ (see Appendix C, Lemma C.3). Hence, since $\kappa < \alpha$, the above estimates yield

$$\|B\| = o_t(1).$$

(5.58)

Next, using $\Gamma(\tilde{j}_\infty) e^{-iH_f t} = e^{-iH_f t} \Gamma(e^{i\omega t} \tilde{j}_\infty e^{-i\omega t})$ and $e^{i\omega t} b_\epsilon e^{-i\omega t} = b_\epsilon + \theta_\epsilon t$, it is not difficult to verify (see Appendix C, Lemma C.3) that

$$\|C\| \leq \| 1 \otimes (\Gamma(e^{i\omega t} \tilde{j}_\infty e^{-i\omega t}) - 1) \phi_{0\epsilon'} \| \to 0,$$
Remark. Proof of Theorem 1.1. Inserting the previous estimates into (5.51) shows that
\[ \psi_t = \hat{\Gamma}(j)^* \psi_{t \epsilon} + O(\epsilon') + C(\epsilon')o_t(1). \] (5.59)

Next, we want to pass from \( \hat{\Gamma}(j)^* \) to \( I \) using the formula (5.39). To this end we use estimates of the type (5.58) and (5.59) in order to remove the term \( \Gamma(j_0) \otimes \Gamma(j_\infty) \). Hence, we need to bound \( I \), for instance by introducing a cutoff in \( N \). Let \( \chi_m := \chi_{N \leq m} \) and \( \check{\chi}_m := 1 - \chi_m \) and write \( \hat{\Gamma}(j)^* \psi_{t \epsilon} = \chi_m \hat{\Gamma}(j)^* \psi_{t \epsilon} + \check{\chi}_m \hat{\Gamma}(j)^* \psi_{t \epsilon} \). Using that \( N^{1/2} \hat{\Gamma}(j)^* = \hat{\Gamma}(j)^* \hat{N}^{1/2} \), and that \( \psi_{t \epsilon} \in D(\hat{N}^{1/2}) \) (see Appendix C, Lemma C.4), we estimate
\[ \| \check{\chi}_m \hat{\Gamma}(j)^* \psi_{t \epsilon} \| \lesssim m^{-\frac{1}{2}} \| \hat{N}^{1/2} \psi_{t \epsilon} \| = m^{-\frac{1}{2}} C(\epsilon'). \]

Now, we can use (5.39) to obtain
\[ \psi_t = \chi_m I (\Gamma(j_0) \otimes \Gamma(j_\infty)) \psi_{t \epsilon} + O(\epsilon') + C(\epsilon')o_t(1) + C(\epsilon')o_m(1). \] (5.60)

Using \( \| \chi_m I \| \leq 2^{m/2} \) together with estimates of the type (5.58) and (5.59), we find (here we need the cutoff \( \chi_m \))
\[ \psi_t = \chi_m I \psi_{t \epsilon} + O(\epsilon') + C(\epsilon', m) o_t(1) + C(\epsilon')o_m(1). \] (5.61)

Since \( \phi_{0 \epsilon} \in \mathcal{H} \otimes \mathcal{F}_{\text{fin}}(h_0) \), we can write \( \psi_{t \epsilon} \) as \( \psi_{t \epsilon} = \Phi_{gs} \otimes f_{t \epsilon} \), with \( f_{t \epsilon} \in \mathcal{F}_{\text{fin}}(h_0) \), and therefore \( \psi_{t \epsilon} \in D(I) \) (here we need that \( f_{t \epsilon} \) is in \( \mathcal{F}_{\text{fin}}(h_0) \)). Hence \( \chi_m I \psi_{t \epsilon} = I \psi_{t \epsilon} + C(\epsilon')o_m(1) \). Combining this with (5.62) and remembering (5.52), we obtain
\[ \psi_t = I (e^{-iE_{\epsilon} t} P_{gs} \otimes e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) \phi_{0 \epsilon} + O(\epsilon') + C(\epsilon', m) o_t(1) + C(\epsilon')o_m(1). \] (5.63)

Letting \( t \to \infty \), next \( m \to \infty \), the equation (5.44) follows. \( \square \)

Remark. The reason for \( \epsilon' \) in the statement of the theorem is we do not know whether \( \text{Ran}(P_{gs} \otimes 1)W_+ \psi_0 \in D(I) \). Indeed, if the latter were true, then the relations (5.63) and \( \| \phi_0 - \phi_{0 \epsilon} \| \leq \epsilon' \), where \( \phi_0 := W_+ \psi_0 \), would give
\[ \psi_t = I (e^{-iE_{\epsilon} t} P_{gs} \otimes e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) \phi_0 + O(\epsilon') + C(\epsilon', m) o_t(1) + C(\epsilon')o_m(1), \] (5.64)

which, after letting \( t \to \infty \), next \( m \to \infty \) and then \( \epsilon' \to 0 \), gives
\[ \lim_{t \to \infty} \| \psi_t - I (e^{-iE_{\epsilon} t} P_{gs} \otimes e^{-iH_f t} \chi_{[0, a - E_{gs}]}(H_f)) W_+ \psi_0 \| = 0. \] (5.65)

6. Proof of minimal velocity estimates

In this section we use Theorems 3.1 and 4.1 to prove the minimal velocity estimates of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. To prove (1.17), we use several iterations of Proposition 2.4. We consider the one-parameter family of one-photon operators
\[ \phi_t := t^{-\nu(0)} \chi_{w \geq 1}, \]

with \( w := \left( \frac{|b|}{c't^{\beta}} \right)^2, a > 1 \), and \( \nu(\delta) \geq 0 \), the same as in (1.14). We use the expansion (3.6). We compute
\[ dw = \frac{2b}{(c't^{\beta})^2} - \frac{2\beta w}{t}, \] (6.1)
where, recall, \( b := \frac{1}{2}(\nabla \omega \cdot y + \text{h.c.}) \). We use the notation \( \tilde{\chi}_{\beta} \equiv \chi_{\nu \geq 1} \). We write \( b = b_{\epsilon} + \epsilon \left( \frac{1}{\omega_{\epsilon}} \nabla \omega \cdot y + \text{h.c.} \right) \), where, recall, \( \omega_{\epsilon} := \omega + \epsilon \), \( \epsilon = t^{-\kappa} \). We choose \( \kappa > 0 \) satisfying
\[
4\beta - 3 > \kappa > 2 - 2\beta + \nu(-1) - \nu(0).
\]
(6.2)

Using the notation \( v := \frac{b_{\epsilon}}{\omega_{\epsilon}} \) and the partition of unity \( \chi_{v \geq 1} + \chi_{v = 1} = 1 \), we find \( b_{\epsilon} \geq c t^{\beta} + (b_{\epsilon} - c t^{\beta}) \chi_{v \leq 1} \). Commutator estimates of the type considered in Appendix [B] (see Lemma [B.5]) give \( \chi_{v \leq -1}(\tilde{\chi}_{\beta})^{1/2} = O(t^{-\beta + \kappa}) \) for \( \tilde{c} > c/2 \), which, together with \( b_{\epsilon}(\tilde{\chi}_{\beta})^{1/2} = O(t^{\beta}) \), yields
\[
(\tilde{\chi}_{\beta})^{1/2} b_{\epsilon} \chi_{v \leq 1}(\tilde{\chi}_{\beta})^{1/2} \geq -\tilde{c} t^{\beta}(\tilde{\chi}_{\beta})^{1/2} \chi_{v \leq 1}(\tilde{\chi}_{\beta})^{1/2} - C t^{\beta} \tilde{\chi}_{\beta}.
\]
The last two estimates, together with \( v \leq 1 \) on \( \text{supp} \tilde{\chi}_{\beta} \geq 1 \), give \( d_{\epsilon} \phi t \geq p_{t} - \tilde{p}_{t} + \text{rem} \), where
\[
p_{t} := \frac{2}{t^{\alpha}(0)} \left( \frac{c}{(c')^{2}t^{\beta} - \frac{\beta}{t}} \right) \tilde{\chi}_{\beta},
\]
\[
\tilde{p}_{t} := \frac{2(\tilde{c} + c)}{c^{2}t^{\beta + \alpha}(0)}(\tilde{\chi}_{\beta})^{1/2} \chi_{v \leq 1}(\tilde{\chi}_{\beta})^{1/2},
\]
and \( \text{rem} = \sum_{i=1}^{4} \text{rem}_{i} \), with \( \text{rem}_{1} \) given by (3.6) with \( \chi_{\beta} \) replaced by \( \tilde{\chi}_{\beta} \),
\[
\text{rem}_{2} := \frac{c}{(c't^{\beta})^{2}(t^{\alpha} + \alpha(0))}(\frac{1}{\omega_{\epsilon}} \nabla \omega \cdot y + \text{h.c.}) \tilde{\chi}_{\beta},
\]
\[
\text{rem}_{3} = O(t^{-2\beta + \kappa - \alpha\nu(0)}), \text{ and } \text{rem}_{4} := -\alpha \nu(0) t^{-1} \phi t. \text{ If } \beta = 1, \text{ then we choose } c > (c')^{2} \text{ so that } p_{t} \geq 0.
\]

As in the proof of Theorem [3.1] we deduce that the remainders \( \text{rem}_{i} \), \( i = 1, 2, 3, 4 \), satisfy the estimates \( (3.13) \), \( i = 1, 2, 3, 4 \), with \( p_{1} = p_{2} = -1, p_{3} = p_{4} = 0, \lambda_{1} = 2\beta + \alpha \nu(0), \lambda_{2} = 2\beta + \kappa + \alpha \nu(0), \lambda_{3} = 2\beta - \kappa + \alpha \nu(0) \) and \( \lambda_{4} = 1 + \alpha \nu(0) \). In particular, the estimate for \( i = 1 \) follows from Lemma [B.4]. Since \( 2\beta > 1 + \nu(-1) - \alpha \nu(0) \) and \( 2\beta - \kappa > 1 \), the remainder \( \text{rem} = \sum_{i=1}^{4} \text{rem}_{i} \) gives an integrable term. (Note \( \text{rem}_{2} = 0 \), if \( \nu(0) = 0 \).)

Now, we estimate the contribution of \( \tilde{p}_{t} \). Let \( \gamma = 2\beta - 1 \leq \beta \) and decompose \( \tilde{p}_{t} = p_{t1} + p_{t2} \), where
\[
p_{t1} := \frac{\text{const}}{t^{\beta + \alpha}(0)}(\tilde{\chi}_{\beta})^{1/2} \chi_{c_{1} t^{\gamma} \leq b_{\epsilon} \leq c t^{\beta}}(\tilde{\chi}_{\beta})^{1/2},
\]
\[
p_{t2} := \frac{\text{const}}{t^{\beta + \alpha}(0)}(\tilde{\chi}_{\beta})^{1/2} \chi_{c_{1} t^{\gamma} \leq b_{\epsilon} \leq c t^{\beta}}(\tilde{\chi}_{\beta})^{1/2},
\]
with \( c_{1} < 1 \), if \( \gamma = 1 \), and \( c_{1} < (c')^{2} \) if \( \gamma < 1 \), and const = \( \frac{\epsilon}{c} \). First, we estimate the contribution of \( p_{t1} \). Since \( [(\tilde{\chi}_{\beta})^{1/2}, (\chi_{c_{1} t^{\gamma} \leq b_{\epsilon} \leq c t^{\beta}})]^{1/2} = O(t^{-\beta + \kappa}) \) (see Lemma [B.1] of Appendix [B]) and since \( 2\beta - \kappa > 1 \), it suffices to estimate the contribution of \( \frac{\text{const}}{t^{\beta + \alpha}(0)}(\tilde{\chi}_{\beta})^{1/2} \chi_{c_{1} t^{\gamma} \leq b_{\epsilon} \leq c t^{\beta}} \). To this end, we use the propagation observable
\[
\phi_{t1} := t^{-\alpha \nu(0)} h_{\beta} \chi_{\gamma},
\]
(6.3)

where \( h_{\beta} \equiv h\left(\frac{b_{\epsilon}}{ct^{\beta}}\right) \), \( h(\lambda) := \int_{\lambda}^{\infty} ds \chi_{s \leq 1} \), and \( \chi_{\gamma} \equiv \chi_{\frac{b_{\epsilon}}{c t^{\gamma}} \geq 1} \). As in (3.9), we have
\[
\frac{1}{ct^{\gamma}} h_{\beta} \partial_{\gamma} b_{\epsilon} \chi_{\gamma} \leq \frac{\text{const}}{t^{1 + \gamma - \kappa}}, \quad \frac{1}{ct^{\gamma}} h_{\beta} \partial_{\gamma} b_{\epsilon} \chi_{\gamma} \geq -\frac{\text{const}}{t^{1 + \beta - \kappa}}.
\]
Using this together with (3.7), we compute
\[
d\phi_{t1} \leq -\frac{\theta_{\epsilon}}{ct^{\beta + \alpha}(0)} - \frac{\beta b_{\epsilon}}{ct^{\beta + 1 + \alpha}(0)} h_{\beta} \chi_{\gamma} + h_{\beta} \chi_{\gamma} \left(\frac{\theta_{\epsilon}}{c_{1} t^{\gamma} \alpha \nu(0)} - \frac{\gamma b_{\epsilon}}{c_{1} t^{\gamma + 1 + \alpha}(0)}\right) + \sum_{i=1}^{3} \text{rem}_{i}^{\prime},
\]
where \( \text{rem}_{i}^{\prime} \) is a sum of two terms of the form of \( \text{rem}_{1} \) given in (3.6), with \( \chi_{\beta} \) replaced by \( h_{\beta} \), in one, and by \( \chi_{\gamma} \), in the other, \( \text{rem}_{2}^{\prime} := O(t^{-1 + \kappa - \alpha \nu(0)}) \), and \( \text{rem}_{3}^{\prime} := -\alpha \nu(0) t^{-1} \phi_{t1} \). We estimate
Since \( \theta - \frac{2h}{c^2} \geq 1 - \frac{1}{\omega c^2} - \frac{2\epsilon}{c^2} \) on supp \( h' \) and \( \theta - \frac{2h}{c^2} \leq 1 - \frac{1}{\omega c^2} - \frac{2\epsilon}{c^2} \) on supp \( \chi' \). Using \( h' \leq 0 \), \( \chi' \geq 0 \), \( h \leq 1 - \frac{h}{c^2} \) and \( \frac{h}{c^2} \) is a strong 1-photon propagation observable and therefore we have the estimate
\[
|\int_1^\infty dt ||d\Gamma(p_{t_1})|^{1/2} \psi_t|^{2} \lesssim |\int_1^\infty dt ||d\Gamma(p'_{t_1})|^{1/2} \psi_t|^{2} \lesssim ||\psi_0||_{\Lambda}^{-1}. \tag{6.5}
\]
(In fact, by multiplying the observable \( |d\Gamma| \) by \( t^\delta \) for an appropriate \( \delta > 0 \), we can obtain a stronger estimate.)

Now, we consider \( p_{t_2} \). Let \( f_{\beta} \equiv f(w) \), where \( f(\lambda) := \chi_{\lambda \geq 1} \) and, recall, \( w = \left( \frac{1}{c^2} \right)^2 \), and \( h_{\gamma} \equiv h(v_\gamma) \), with \( h(\lambda) := \int_{C} d\lambda \chi_{\lambda \leq 1} \) and \( v_\gamma := \frac{h}{c^2} \). We use the propagation observable \( \phi_{t_2} := t^{-\alpha(0)}(f_{\beta} h_{\gamma} + h_{\gamma} f_{\beta}) \).

Using \( \gamma \geq 1 \), \( \beta \geq 2 \), \( \frac{h}{c^2} \leq c_1 t^\gamma \) on supp \( \chi_{v_\gamma \leq 1} \), \( \gamma = 2\beta - 1 \) and \( \int(f_{\beta}')^{1/2}, h_{\gamma} = O(t^{-\gamma+\kappa}) \) (see Lemma 3.1 of Appendix B), we compute
\[
d\phi_{t_2} \leq t^{-\alpha(0)} \left[ \left( \frac{c_1}{(c^2)^2} - \beta \right) \frac{2h}{t} (f_{\beta}')^{1/2} h_{\gamma} (f_{\beta}')^{1/2} + f_{\beta} h_{\gamma}'(dv_{\gamma}) + (dv_{\gamma}) h_{\gamma} f_{\beta} \right] + \sum_{i=1}^{4} rem''_i,
\]
where \( dv_{\gamma} = \frac{\theta}{c^2} - \frac{\gamma h}{c^2} \), \( rem''_i \) is a term of the form of \( rem_1 \) given in (3.6), with \( \chi_{\beta} \) replaced by \( f_{\beta} \), likewise, \( rem''_2 \) is the form of \( rem_1 \) given in (3.6), with \( \chi_{\beta} \) replaced by \( h_{\gamma} \), and \( rem''_3 = O(t^{-1/2} - a(0)) \) and \( rem''_4 := -av(0) t^{-1} \phi_{t_2} \). To estimate \( dv_{\gamma} = \frac{\theta}{c^2} - \frac{\gamma h}{c^2} \), we use that \( f_{\beta}' \geq 0, h_{\gamma}' \leq 0, \theta \leq 1 - t^{-\kappa} \omega_{c^2}, v_{\gamma} h_{\gamma}' \leq h_{\gamma}', \) and \( f_{\beta} h_{\gamma}'(dv_{\gamma}) + (dv_{\gamma}) h_{\gamma} f_{\beta} = -f_{\beta}' h_{\gamma}'(dv_{\gamma}) - h_{\gamma}'^{1/2} (dv_{\gamma}) - h_{\gamma}'^{1/2} f_{\beta} + O(t^{-\gamma+\kappa}) \) (see again Lemma 3.1 of Appendix B), to obtain
\[
d\phi_{t_2} \leq -t_{0}^{-1} + rem'_{a},
\]
with \( rem'_{a} := \sum_{i=1}^{6} rem'_{i}, rem''_{a} = O(t^{-2\gamma+\kappa - a(0)}), rem''_{b} = O(t^{-\gamma+\kappa - a(0)}), \omega^{-1/2} \) and (at least for \( t \) sufficiently large)
\[
p_{t_2} := t^{-\alpha(0)} \left[ \left( \frac{2c_1}{(c^2)^2} - \beta \right) \frac{1}{t} (f_{\beta}')^{1/2} h_{\gamma} (f_{\beta}')^{1/2} + (1 - \frac{\gamma c_1}{t^2}) \frac{1}{t} (f_{\beta}')^{1/2} h_{\gamma}'^{1/2} f_{\beta}^{1/2} \right].
\]
Since \( \frac{c_1}{(c^2)^2} < \beta \) and either \( \gamma < 1 \) or \( \gamma = 1 \) and \( c_1 < 1 \), and \( f_{\beta}' \geq 0 \) and \( h_{\gamma}' \leq 0 \), both terms in the square braces on the r.h.s. are non-positive. We deduce as above that the remainders \( rem''_{i}, i = 1, \ldots, 6, \) satisfy the estimates (3.13), with \( \rho_1 = \rho_6 = -1, \rho_2 = \rho_3 = \rho_4 = \rho_5 = 0, \lambda_1 = 2\beta + a(0), \lambda_2 = \lambda_5 = 2\gamma - \kappa + a(\delta), \lambda_3 = 1 + \gamma - \kappa + a(0), \lambda_4 = 1 + a(0), \lambda_6 = \gamma + \kappa + a(0). \) Since \( 2\beta > \gamma + \kappa > 1 + \nu(-1) - a(0), 2\gamma - \kappa > 1 \) and \( \gamma > \kappa \), the condition (2.3) is satisfied.
Moreover, (2.4) with $\lambda' < \alpha \nu(0) + (\frac{3}{2} + \mu)\beta$ holds by [9, Lemma 3.1]. Therefore $\phi_{t^2}$ is a strong one-photon propagation observable and we have the estimate

$$\int_1^\infty dt ||d\Gamma(p_{t^2})||^{1/2} \lesssim \int_1^\infty dt ||d\Gamma(p_{t^2})||^{1/2} \lesssim ||\psi_0||^2_{-1}. \quad (6.7)$$

(In fact, by multiplying the observable (6.6) by $\delta$ for an appropriate $\delta > 0$, we can obtain a stronger estimate.)

Since $\tilde{p}_t = p_{t^1} + p_{t^2}$, estimates (6.5) and (6.7) imply the estimate

$$\int_1^\infty dt ||d\Gamma(p_{t})||^{1/2} \lesssim ||\psi_0||^2_{-1}, \quad (6.8)$$

which due to $\chi'_\beta \approx \chi_{\nu=1}$, implies the estimate (1.17).

\[\square\]

**Proof of Theorem 7.2.** To prove (1.19), we begin with the following estimate, proven in the localization lemma [13,5] of Appendix [3]

$$\chi_{b_c \geq c't^\alpha \chi_{|y| \leq ct^\alpha}} = \mathcal{O}(t^{-(\alpha - \kappa)}), \quad (6.9)$$

for $\epsilon = t^{-\kappa}, \kappa < \alpha$, and $c < c'/2$. Now, let $\chi_{b_c \leq c't^\alpha}^2 + \chi_{b_c \leq c't^\alpha} = 1$ and write

$$\chi_{b_c \leq c't^\alpha}^2 = \chi_{b_c \leq c't^\alpha} \chi_{|y| \leq ct^\alpha} \chi_{b_c \leq c't^\alpha} + R \leq \chi_{b_c \leq c't^\alpha} + R, \quad (6.10)$$

where $R := \chi_{b_c \leq c't^\alpha} \chi_{|y| \leq ct^\alpha} \chi_{b_c \leq c't^\alpha} + \chi_{b_c \leq c't^\alpha} \chi_{|y| \leq ct^\alpha} \chi_{b_c \leq c't^\alpha}$. The estimates (6.9) and (6.10) give

$$\chi_{b_c \leq c't^\alpha}^2 \leq \chi_{b_c \leq c't^\alpha} + \mathcal{O}(t^{-(\alpha - \kappa)}), \quad (6.11)$$

which in turn implies

$$||\Gamma(\chi_{|y| \leq ct^\alpha})||^{1/2} \lesssim ||\Gamma(\chi_{b_c \leq c't^\alpha})||^{1/2} + Ct^{-(\alpha - \kappa)/2}((N + 1)^{1/2} \psi). \quad (6.12)$$

This, together with (4.1), yields (1.19).

\[\square\]

**APPENDIX A. PHOTON # AND LOW MOMENTUM ESTIMATE**

Recall the notation $\langle A \rangle := \langle \psi, A \psi \rangle$. The idea of the proof of the following estimate follows [24] and [9].

**Proposition A.1.** Assume (1.10) with $\mu > -1/2$. Let $\psi_0 \in D(d\Gamma(\omega^\rho)^{1/2})$. Then for any $\rho \in [-1, 1],

$$\langle d\Gamma(\omega^\rho) \rangle_{\psi_0} \lesssim \nu(\rho) ||\psi_0||^2_H + \langle d\Gamma(\omega^\rho) \rangle_{\psi_0}, \quad \nu(\rho) = \frac{1-\rho}{2 + \mu}. \quad (A.1)$$

**Proof.** Decompose $d\Gamma(\omega^\rho) = K_1 + K_2$, where

$$K_1 := d\Gamma(\omega^\rho | t^{\alpha}, \omega \leq 1) \quad \text{and} \quad K_2 := d\Gamma(\omega^\rho | t^{\alpha}, \omega > 1).$$

Then by (1.15),

$$\langle K_2 \rangle_\psi \lesssim \langle d\Gamma(t^{\alpha(1-\rho)} \chi_{t^{\alpha}, \omega \leq 1}) \rangle_{\psi_0} \lesssim t^{\alpha(1-\rho)} \langle H_f \rangle_\psi \lesssim t^{\alpha(1-\rho)} ||\psi_0||^2_H. \quad (A.2)$$

On the other hand, we have by (2.10),

$$DK_1 = d\Gamma(\alpha \omega^{-1} t^{\alpha-1} \chi_{t^{\alpha}, \omega \leq 1}) - I(i \omega^\rho \chi_{t^{\alpha}, \omega \leq 1} g). \quad (A.3)$$

Since $||g(k)||_{H_\rho} \lesssim |k|^\mu \xi(k)$ (see (1.10)), we obtain

$$\int \omega^{2\rho} \chi_{t^{\alpha}, \omega \leq 1} ||g(k)||^2_{H_\rho} (\omega^{-1} + 1) d^3k \lesssim t^{-2(1+\mu+\rho)\alpha}. \quad (A.4)$$
This together with (2.11) and (1.15) gives
\[ |\langle I(i\omega^\rho \chi_{t^\rho \omega \leq 1} \rangle\psi_1| \lesssim t^{-(1+\mu+\rho)\alpha} \| \psi_0 \|_H^2. \] (A.5)
Hence, by (A.3), since \( \partial_t \langle K_1 \rangle \psi_2 = \langle DK_1 \rangle \psi_1, \chi_{t^\rho \omega \leq 1} \leq 0 \), we obtain
\[ \partial_t \langle K_1 \rangle \psi_2 \lesssim t^{-(1+\mu+\rho)\alpha} \| \psi_0 \|_H^2 \]
and therefore
\[ \langle K_1 \rangle \psi_2 \leq C t^{\nu} \| \psi_0 \|_H^2 + (d\Gamma(\omega^\rho))_\psi_0, \] (A.6)
where \( \nu' = 1 - (1 + \mu + \rho)\alpha \), if \( (1 + \mu + \rho)\alpha < 1 \) and \( \nu' = 0 \), if \( (1 + \mu + \rho)\alpha > 1 \). Estimates (A.6) and (A.2) with \( \alpha = \frac{1}{2+r_\mu} \), if \( \rho < 1 \), give (A.1). The case \( \rho = 1 \) follows from (1.15). \( \square \)

**Corollary A.2.** Assume (1.10) with \( \mu > -1/2 \), let \( \psi_0 \in D(d\Gamma(\omega^\rho)^{1/2}) \), and denote \( K_\rho := d\Gamma(\omega^\rho) \). Then for any \( \gamma \geq 0 \) and any \( c > 0 \),
\[ \| \chi_{K_\rho \geq ct^n} \psi_t \| \lesssim t^{-\frac{\gamma}{2} + \frac{1}{2+2\mu}} \| \psi_0 \|_H^2 + t^{-\frac{\gamma}{2}} \langle K_\rho \rangle \psi_0. \] (A.7)

**Proof.** We have
\[ \| \chi_{K_\rho \geq ct^n} \psi_t \| \leq c^{-\frac{\gamma}{2}} t^\frac{\gamma}{2} \| \chi_{K_\rho \geq ct^n} K_\rho^{1/2} \psi_t \| \leq c^{-\frac{\gamma}{2}} t^\frac{\gamma}{2} \| K_\rho^{1/2} \psi_t \| \]
Now applying (A.1) we arrive at (A.7). \( \square \)

**Remark.** A minor modification of the proof above give the following bound for \( \rho < 0 \) and \( \nu_1(\rho) := \frac{-\rho}{2+\mu} \),
\[ (d\Gamma(\omega^\rho))_\psi_t \lesssim t^{\nu_1(\rho)} (\| \psi_t \|_H^2 + \| \psi_0 \|_H^2) + (d\Gamma(\omega^\rho))_\psi_0. \] (A.8)

**APPENDIX B. Commutator estimates**

In this appendix, we estimate some localization terms and commutators appearing in Section 3. Recall that \( b_\epsilon := \frac{1}{2}(\theta_\epsilon \nabla \omega \cdot y + \text{h.c.}) \), where \( \theta_\epsilon = \frac{\omega}{\omega_\epsilon} \), \( \omega_\epsilon := \omega + \epsilon \), \( \epsilon = t^{-\kappa} \), with \( \kappa \geq 0 \). The following lemma is a straightforward consequence of the Helffer–Sjöstrand formula. We do not detail the proof.

**Lemma B.1.** Let \( \tilde{h}, h \) be smooth function satisfying the estimates \( |\partial_t^n h(s)| \leq C_n \langle s \rangle^{-n} \) for \( n \geq 0 \) and likewise for \( \tilde{h} \). Let \( w_a = |y|/(c_1 t^\alpha) \), \( v_\beta = b_\epsilon/(c_2 t^\beta) \), with \( 0 < \alpha, \beta \leq 1 \). The following estimates hold
\[ [h(w_a), \omega] = O(t^{-\alpha}), \quad [\tilde{h}(v_\beta), \omega] = O(t^{-\beta}), \quad [h(w_a), b_\epsilon] = O(t^\kappa), \]
\[ [h(w_a), \tilde{h}(v_\beta)] = O(t^{-\beta+\kappa}), \quad b_\epsilon [h(w_a), \tilde{h}(v_\beta)] = O(t^\kappa). \]

Now we prove the following abstract result.

**Lemma B.2.** Let \( h \) be a smooth function satisfying the estimates \( |\partial_t^n h(s)| \leq C_n \langle s \rangle^{-n} \) for \( n \geq 0 \). Assume that the commutators \( [v, \omega] \) and \( [v, [v, \omega]] \) are bounded, and for some \( z \in \mathbb{C} \setminus \mathbb{R} \), \( (v - z)^{-1} \) preserves \( D(\omega) \). Then the operator \( r := [h(v), v] - [v, [v, \omega]]h'(v) \) is bounded as
\[ \| r \| \lesssim \| [v, [v, \omega]] \|. \] (B.1)

**Proof.** We would like to use the Helffer–Sjöstrand formula for \( h \). Since \( h \) might not decay at infinity, we cannot directly express \( h(v) \) by this formula. Therefore, we approximate \( h(v) \) as follows. Consider \( \varphi \in C^\infty_0(\mathbb{R}; [0,1]) \) equal to 1 near 0 and \( \varphi_R(\cdot) = \varphi(\cdot/R) \) for \( R > 0 \). Let \( \tilde{h} \) be an almost analytic extensions of \( h \) such that \( \tilde{h}|_{\mathbb{R}} = h \),
\[ \text{supp} \tilde{h} \subset \{ z \in \mathbb{C}; |\text{Im } z| \leq C(\text{Re } z) \}, \] (B.2)
\[ \left| \partial_z \tilde{h}(z) \right| \leq C_n(\operatorname{Re} z)^{\rho-1-n} | \operatorname{Im} z|^n. \]  

(B.3)

Similarly let \( \tilde{\varphi} \in C_0^\infty(\mathbb{C}) \) be an almost analytic extension of \( \varphi \) satisfying these estimates. As a quadratic form on \( D(\omega) \), we have

\[ \left[ (\varphi R h)(v), \omega \right] = \frac{1}{\pi} \int \partial_z(\tilde{\varphi} \tilde{h})(z)[(v-z)^{-1}, \omega] \, d\operatorname{Re} z \, d\operatorname{Im} z \]

(B.4)

Since \( (v-z)^{-1} \) preserves \( D(\omega) \) for some \( z \) in the resolvent set of \( v \) (and hence for any such \( z \), see [1] Lemma 6.2.1), we can compute, using the Helffer–Sjöstrand representation for \( (\varphi R h)(v) \),

\[ \left[ (\varphi R h)(v), \omega \right] = \frac{1}{\pi} \int \partial_z(\tilde{\varphi} \tilde{h})(z)[(v-z)^{-1}, \omega] \, d\operatorname{Re} z \, d\operatorname{Im} z \]

(B.5)

as a quadratic form on \( D(\omega) \), where

\[ r_R = -\frac{1}{\pi} \int \partial_z(\tilde{\varphi} \tilde{h})(z)[(v-z)^{-1}, [v,\omega]](v-z)^{-1} d\operatorname{Re} z \, d\operatorname{Im} z \]

(B.6)

Now, using \( (v-z)^{-1} = O(|\operatorname{Im} z|^{-1}) \), we obtain that

\[ \| (v-z)^{-1} [v,[v,\omega]](v-z)^{-2} \| \lesssim |\operatorname{Im} z|^{-3} \| [v,[v,\omega]] \| . \]  

(B.7)

Besides, for all \( n \in \mathbb{N} \),

\[ |\partial_z(\tilde{\varphi} \tilde{h})(z)| \leq C_n(\operatorname{Re} z)^{\rho-1-n} | \operatorname{Im} z|^n, \]  

(B.8)

where \( C_n > 0 \) is independent of \( R \geq 1 \). Using (B.6) together with (B.7), we see that there exists \( C > 0 \) such that \( \| r_R \| \leq C \| [v,[v,\omega]] \| \), for all \( R \geq 1 \). Finally, since \( (\varphi R h)'(v) \) converges strongly to \( h'(v) \), the lemma follows from (B.5) and the previous estimate.

\[ \square \]

We want apply the lemma above to the time-dependent self-adjoint operator \( v := \frac{h'}{c^{\epsilon}} \).

**Corollary B.3.** Let \( h \) be a smooth function satisfying the estimates \( |\partial_s^n h(s)| \leq C_n(s)^{-n} \) for \( n \geq 0 \) and let \( v := \frac{h'}{c^{\epsilon}} \), where \( c > 0, \epsilon = t^{-\kappa} \), with \( 0 \leq \kappa \leq \beta \leq 1 \). Then the operator \( r := dh(v)-(dv)h'(v) \) is bounded as

\[ \| r \| \lesssim t^{-\lambda}, \quad \lambda := 2\beta - \kappa. \]  

(B.9)

**Proof.** Observe that

\[ dh(v)-(dv)h'(v) = [h(v),i\omega] - [v,i\omega]h'(v) + \partial_t h(v) - (\partial_t v)h'(v). \]

It is not difficult to verify that \( (v-z)^{-1} \) preserves \( D(\omega) \) for any \( z \in \mathbb{C} \setminus \mathbb{R} \). Hence it follows from the computations

\[ [v,i\omega] = t^{-\beta} \theta_{\epsilon}, \quad [v,[v,i\omega]] = t^{-2\beta} \theta_{\epsilon} \omega_{\epsilon}^{-2} \epsilon, \]  

(B.10)

that we can apply Lemma B.2. The estimate

\[ [v,[v,\omega]] = O(\omega_{\epsilon}^{-1} t^{-2\beta}) = O(t^{-2\beta + \kappa}) \]  

(B.11)

then gives

\[ \| [h(v),i\omega] - [v,i\omega]h'(v) \| \lesssim t^{-2\beta + \kappa}. \]
It remains to estimate \( \| \partial_t h(v) - (\partial_v v) h'(v) \|. \) It is not difficult to verify that \( D(b_c) \) is independent of \( t \). Using the notations of the proof of Lemma B.2 and the fact that \( \partial_t h(v) = s\text{-}\lim_{R \to \infty} \partial_t (\varphi_R h)(v) \), we compute

\[
\partial_t (\varphi_R h)(v) = \frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) \partial_t (v - z)^{-1} d\text{Re} \ z \ d\text{Im} \ z
\]

\[
= -\frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) (v - z)^{-1} \partial_t v (v - z)^{-1} d\text{Re} \ z \ d\text{Im} \ z
\]

\[
= (\partial_t v)(\varphi_R h)'(v) + r'_R,
\]

where

\[
r'_R = -\frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) [(v - z)^{-1}, \partial_t v](v - z)^{-1} d\text{Re} \ z \ d\text{Im} \ z
\]

\[
= \frac{1}{\pi} \int \partial_z (\varphi_R \tilde{h})(z) (v - z)^{-1} [v, \partial_t v] (v - z)^{-2} d\text{Re} \ z \ d\text{Im} \ z. \quad (B.12)
\]

Now using \( \partial_t v = -\beta b_c + \frac{1}{ct^\alpha} \partial_t b_c \) together with (3.23), we estimate

\[
[v, \partial_t v] = \mathcal{O}(t^{-1-2\beta+\kappa}) b_c + \mathcal{O}(t^{-1-2\beta+2\kappa}).
\]

From this, the properties of \( \varphi, \tilde{h}, \) and \( \kappa \leq \beta \), we deduce that \( \| r'_R \| \lesssim t^{-1-\beta+\kappa} \lesssim t^{-\beta+\kappa} \) uniformly in \( R \geq 1 \). This concludes the proof of the corollary. \( \square \)

The following lemma is taken from [9]. Its proof is similar to the proof of Lemma B.2

**Lemma B.4.** Let \( h \) be a smooth function satisfying the estimates \( |\partial_x^\alpha h(s)| \leq C_n |s|^{-n} \) for \( n \geq 0 \) and \( 0 \leq \delta \leq 1 \). Let \( w = y^2/(ct^\alpha)^2 \) with \( 0 < \alpha < 1 \). We have

\[
[h(w), i\omega] = \frac{1}{ct^\alpha} h'(w) \left( \frac{y}{ct^\alpha} \cdot \nabla \omega + \nabla \omega \cdot \frac{y}{ct^\alpha} \right) + \text{rem},
\]

with

\[
||\omega^\frac{1}{2} \text{rem} \omega^\frac{1}{2}|| \lesssim t^{-\alpha(1+\delta)}.
\]

Now we prove a localization lemma.

**Lemma B.5.** Let \( \kappa < \alpha \). We have, for \( c < c'/2 \),

\[
\chi_{b_c \geq ct^\alpha} \chi_{|y| \leq ct^\alpha} = \mathcal{O}(t^{-(\alpha-\kappa)}). \quad (B.13)
\]

**Proof.** Observe that by the definition of \( \chi \) (see Introduction) and the condition \( c < c'/2 \), we have \( \chi_{|y| \geq ct^\alpha} \phi_{|y| \leq ct^\alpha} = 0 \). Let \( c < c' < c'/2 \) and let \( \tilde{\chi}_{|y| \leq ct} \) be such that \( \chi_{|y| \leq ct} \tilde{\chi}_{|y| \leq ct} = \chi_{|y| \leq ct} \) and \( \chi_{|y| \geq ct} \tilde{\chi}_{|y| \leq ct} = 0 \). Define \( b_c = \tilde{\chi}_{|y| \leq ct} b_c \tilde{\chi}_{|y| \leq ct} \). It follows from the expression of \( b_c \) that \( |\langle u, b_c u \rangle| \leq \|u\|\|y|u\| \), and hence we deduce that \( |\langle u, b_c u \rangle| \leq ct^\alpha \|u\|^2 \). This gives \( \chi_{b_c \geq ct^\alpha} = 0 \).

Using this, we write

\[
\chi_{b_c \geq ct^\alpha} \chi_{|y| \leq ct^\alpha} = (\chi_{b_c \geq ct^\alpha} - \chi_{b_c \geq ct^\alpha}) \chi_{|y| \leq ct^\alpha}. \quad (B.14)
\]

Let \( a := \frac{b_c}{ct^\alpha} \) and \( \bar{a} := \frac{b_c}{ct^\alpha} \). Denote \( g(a) := \chi_{b_c \geq ct^\alpha} \) and \( g(\bar{a}) := \chi_{b_c \geq ct^\alpha} \). We will use the construction and notations of the proof of Lemma B.2 Using the Helffer-Sjöstrand formula for \( (\varphi_R g)(e) \), we write

\[
(\varphi_R g)(a) - (\varphi_R g)(\bar{a}) = \frac{1}{\pi} \int \partial_z (\varphi_R \tilde{g})(z) [(a - z)^{-1} - (\bar{a} - z)^{-1}] d\text{Re} \ z \ d\text{Im} \ z
\]

\[
= -\frac{1}{\pi} \int \partial_z (\varphi_R \tilde{g})(z) (a - z)^{-1} (\bar{a} - z)^{-1} d\text{Re} \ z \ d\text{Im} \ z. \quad (B.15)
\]
Now we show that \((a - \bar{a})(\bar{a} - z)^{-1} \chi_{|y| \leq ct^\alpha} = O(t^{-(\alpha - \kappa)}|\text{Im} z|^{-2})\). We have
\[
a - \bar{a} = (1 - \bar{\chi}_{|y| \leq ct^\alpha}) \frac{b_c}{ct^\alpha} + \bar{\chi}_{|y| \leq ct^\alpha} \frac{b_c}{ct^\alpha}(1 - \bar{\chi}_{|y| \leq ct^\alpha}),
\]
and we observe that, by Lemma \[\text{B.1}\]
\[
[(1 - \bar{\chi}_{|y| \leq ct^\alpha}), b_c] = O(t^\kappa).
\]
Thus
\[
a - \bar{a} = (1 + \bar{\chi}_{|y| \leq ct^\alpha}) \frac{b_c}{ct^\alpha}(1 - \bar{\chi}_{|y| \leq ct^\alpha}) + O(t^{-(\alpha - \kappa)}),
\]
Moreover, we can write
\[
(1 - \bar{\chi}_{|y| \leq ct^\alpha})(\bar{a} - z)^{-1} \chi_{|y| \leq ct^\alpha} = [(1 - \bar{\chi}_{|y| \leq ct^\alpha}), (\bar{a} - z)^{-1}] \chi_{|y| \leq ct^\alpha}
\]
\[
= - (\bar{a} - z)^{-1} [(1 - \bar{\chi}_{|y| \leq ct^\alpha}), b_c/ct^\alpha] (\bar{a} - z)^{-1} \chi_{|y| \leq ct^\alpha}
\]
\[
= O(t^{-(\alpha - \kappa)}|\text{Im} z|^{-2}),
\]
where we used \[\text{B.16}\] to obtain the last estimate. This implies the statement of the lemma.

**Remark.** The estimate \[\text{B.13}\] can be improved to \(\chi_{b_c \geq ct^\alpha} \chi_{|y| \leq ct^\alpha} = O(t^{-m(\alpha - \kappa)})\), for any \(m > 0\), if we replace \(\omega_\epsilon := \omega + \epsilon\) in the definition of \(b_\epsilon\) by the smooth function \(\omega_\epsilon := \sqrt{\omega^2 + \epsilon^2}\).

In conclusion of this appendix we reproduce a statement corresponding to \[9\] Lemma 3.1] with \(b_\epsilon\) instead of \(|y|\). The proof is the same.

**Lemma B.6.** Assume Hypothesis \[1.10\] on the coupling function \(g\) is satisfied for some \(-\frac{1}{2} \leq \mu \leq \frac{1}{2}\). Then
\[
\|\eta \chi_{b_\epsilon \geq ct^\alpha} g(k)\|_{L^2(\mathbb{R}^3; \mathcal{H}_\mu)} \lesssim t^{-\tau}, \quad \tau < \left(\frac{3}{2} + \mu\right)\alpha.
\]

**Appendix C. Technicalities**

In this appendix we prove technical statements used in the main text. Most of the results we present here are close to known ones. We begin with the following standard result, which was used implicitly at several places.

**Lemma C.1.** Let \(a, b\) be two self-adjoint operators on \(\mathfrak{h}\) with \(b \geq 0\), \(D(b) \subset D(a)\) and \(\|a\varphi\| \leq \|b\varphi\|\) for all \(\varphi \in D(b)\). Then \(D(d\Gamma(b)) \subset D(d\Gamma(a))\) and \(\|d\Gamma(a)\Phi\| \leq \|d\Gamma(b)\Phi\|\) for all \(\Phi \in D(d\Gamma(b))\).

We recall that, given two operators \(a, c\) on \(\mathfrak{h}\), the operator \(d\Gamma(a, c)\) was defined in \[5.10\], and \(d\Gamma(a, c) := U_d\Gamma(a, c)\).

**Lemma C.2.** Let \(j = (j_0, j_\infty)\) and \(c = \text{diag}(c_0, c_\infty)\), where \(j_0, j_\infty, c_0, c_\infty, c_1, c_2\) are operators on \(\mathfrak{h}\). Furthermore, assume that \(j_0^2 + j_\infty^2 \leq 1\). Then we have the relations
\[
|\langle \hat{\phi}, d\Gamma(j, c)\psi \rangle| \leq |d\Gamma(c_0)\hat{\phi}||d\Gamma(c_0)\frac{1}{2}\psi| + |d\Gamma(c_\infty)\frac{1}{2}\hat{\phi}||d\Gamma(c_\infty)\frac{1}{2}\psi|, \quad (C.1)
\]
\[
|\langle u, d\Gamma(j, c_1 c_2) v \rangle| \leq |d\Gamma(c_1)\frac{1}{2}u||d\Gamma(c_2)\frac{1}{2}v|. \quad (C.2)
\]
Proof. Let $\tilde{\phi} = U^*\phi$ and for an operator $b$ on $\mathfrak{h}$ define operators $i_0 b := \text{diag}(b, 0)$ and $i_\infty b := \text{diag}(0, b)$ on $\mathfrak{h} \oplus \mathfrak{h}$. Since $U^* d\Gamma(|c_0|)^{\frac{1}{2}} \otimes 1_U = d\Gamma(i_0 |c_0|)^{\frac{1}{2}}$ and $U^* 1 \otimes d\Gamma(|c_\infty|)^{\frac{1}{2}} U = d\Gamma(i_\infty |c_\infty|)^{\frac{1}{2}}$, the statement of the lemma is equivalent to

$$
|\langle \tilde{\phi}, d\Gamma(j, c)\psi \rangle| \leq \|d\Gamma(i_0 |c_0|)^{\frac{1}{2}}\tilde{\phi}\| \|d\Gamma(|c_0|)^{\frac{1}{2}}\psi\| + \|d\Gamma(i_\infty |c_\infty|)^{\frac{1}{2}}\tilde{\phi}\| \|d\Gamma(|c_\infty|)^{\frac{1}{2}}\psi\| .
$$

(C.3)

We decompose $d\Gamma(j, c) = d\Gamma(j, i_0 c_0) + d\Gamma(j, i_\infty c_\infty)$ and estimate each term separately. We have, using that $\|j\| \leq 1$,

$$
|\langle \tilde{\phi}, d\Gamma(j, i_0 c_0)\psi \rangle| \leq \sum_{l=1}^n \frac{|\langle i_0 c_0 |l^2 \tilde{\phi}, |i_0 c_0 |l^2 \psi \rangle|}{\|i_0 c_0 |l^2 \tilde{\phi}\| \|i_0 c_0 |l^2 \psi\|} \leq \left( \sum_{l=1}^n \|i_0 c_0 |l^2 \tilde{\phi}\|^2 \right)^{\frac{1}{2}} \left( \sum_{l=1}^n \|i_0 c_0 |l^2 \psi\|^2 \right)^{\frac{1}{2}} \leq \|d\Gamma(i_0 |c_0|)^{\frac{1}{2}}\tilde{\phi}\| \|d\Gamma(|c_0|)^{\frac{1}{2}}\psi\| .
$$

Since $\|d\Gamma(i_0 |c_0|)^{\frac{1}{2}}\tilde{\phi}\|_{F(\mathfrak{h} \oplus \mathfrak{b})} = \|d\Gamma(|c_0|)^{\frac{1}{2}}\psi\|_{F(\mathfrak{h})}$, we obtain the first term in the r.h.s. of (C.3). The second one is obtained exactly in the same way. (C.2) can be proven in a similar manner.

In the following lemma, as in the main text, the operator $j_\infty$ on $L^2(\mathbb{R}^3)$ is of the form $j_\infty = \chi_{b_\epsilon \geq ct^\alpha}$, where, recall, $b_\epsilon = \frac{1}{2}(v_\epsilon(k) \cdot y + \text{h.c.})$, where $v_\epsilon(k) := \theta \nabla \omega$, $\theta = \frac{\omega}{\omega + \epsilon}$, and $\epsilon = t^{-\kappa}$, $\kappa > 0$.

Lemma C.3. Assume $\alpha + \kappa > 1$. Let $u \in F$. Then $\|(j_\infty - 1)e^{-iH_j t}u\| \to 0$, as $t \to \infty$.

Proof. Assume that $u \in D(d\Gamma(|y|))$. Using unitarity of $e^{-iH_j t}$ and the fact that $e^{-iH_j t} = \Gamma(e^{-i\omega t})$, we obtain

$$
\|(j_\infty - 1)e^{-iH_j t}u\| = \|(\Gamma(e^{i\omega t}j_\infty e^{-i\omega t}) - 1)u\| \leq \|d\Gamma(e^{i\omega t}j_\infty e^{-i\omega t})u\| ,
$$

(C.4)

where $j_\infty = 1 - j_\infty$. Using the identity $e^{i\omega t}b_\epsilon e^{-i\omega t} = b_\epsilon + \theta_\epsilon t$ and the Helffer-Sjöstrand formula show that

$$
e^{i\omega t} \chi_{\frac{b_\epsilon}{ct^\alpha} \leq 1} e^{-i\omega t} = \chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1} .
$$

Since $\alpha + \kappa > 1$, we have $\chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1} = 1 + O(t^{-(\alpha + \kappa - 1)})$. Due to $\frac{-2b_\epsilon}{\epsilon} \geq 1$ on supp $\chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1}$ for $t$ sufficiently large, we have

$$
\|\chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1} \phi\| \leq \|\frac{-2b_\epsilon}{\epsilon} \chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1} \phi\| \leq \|\frac{2(y)}{\epsilon} \phi\| ,
$$

and therefore

$$
\|\chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1} u\| \leq \|\frac{2(y)}{\epsilon} d\Gamma(|y|) u\| .
$$

Together with (C.4), this shows that $\|(j_\infty - 1)e^{-iH_j t}u\| \to 0$, for $u \in D(d\Gamma(|y|))$. Since $D(d\Gamma(|y|))$ is dense in $F$, this concludes the proof.

Lemma C.4. Assume (1.10) with $\mu > -1/2$ and (1.11). Then $\text{Ran}(P_{gs}) \subset D(N^{\frac{1}{2}}) \cap D(d\Gamma(b_\epsilon^{\frac{1}{2}}))$, in other words, the operators $N^{\frac{1}{2}} P_{gs}$ and $d\Gamma(b_\epsilon^{\frac{1}{2}}) P_{gs}$ are bounded. Moreover, we have $\|d\Gamma(b_\epsilon^{\frac{1}{2}}) P_{gs}\| = O(t^\kappa)$. 

Since $\|a(k)\Phi\|$, $k \mapsto \|b(\alpha)\Phi\| \in L^2(\mathbb{R}^3)$, and (1.10)–(1.11), one easily deduces that $\|a(k)\Phi\| \in L^2(\mathbb{R}^3)$ for any $\mu > -1/2$. Likewise, using in addition that $\|b(\alpha)\Phi\| \in L^2(\mathbb{R}^3)$ for any $\Psi \in H$, we have $\Omega_{\pm} \in H$ as $\alpha \to \pm\infty$. Using the relation $\|\hat{n}\|^2 |h|^{-1} \lesssim |h|^{-1}$, one easily deduces that $\|b(\alpha)\Phi\| \in L^2(\mathbb{R}^3)$ for any $\mu > -1/2$.

Supplement I. The wave operators

In this supplement we briefly review the definition and properties of the wave operator $\Omega_{\pm}$, and establish its relation with $W_+$ in Theorem I.2 below. Let $\mathcal{H}_b \equiv \mathcal{H}_b \cap \mathbf{1}_{(-\infty, \Sigma)}(H)$ be the space spanned by the eigenfunctions of $H$ with the eigenvalues in the interval $(-\infty, \Sigma)$. Define $\mathcal{H}_b \cap \mathcal{H}_b = \{h \in L^2(\mathbb{R}^3), \int |h|^2 (|k|^{-1} + |k|^2)d^3k < \infty\}$. The wave operator $\Omega_{\pm}$ on the space $\mathcal{H}_b \cap \mathcal{H}_b$, is defined by the formula

\[ \Omega_{\pm} := \lim_{t \to \infty} e^{itH} I(e^{-itH} \otimes e^{-itH}) \] (I.1)

As in [14, 16, 17, 28], it is easy to show

**Theorem I.1.** Assume (1.10) with $\mu \geq -1/2$ and (1.11). The wave operator $\Omega_{\pm}$ exists on $\mathcal{H}_b \cap \mathcal{H}_b$ and extends to an isometric map, $\Omega_{\pm} : \mathcal{H}_{\text{as}} \to \mathcal{H}$, on the space of asymptotic states, $\mathcal{H}_{\text{as}} := \mathcal{H}_b$.

**Proof.** Let $h_t(k) := e^{-it|k|}h(k)$. For $h \in D(\omega^{-1/2})$, s. t. $\partial_a h \in D(\omega^{1/2})$, $|a| \leq 2$, we define the asymptotic creation and annihilation operators by (see [14, 16, 17, 24, 28])

\[ a^\pm_a(h) := \lim_{t \to \pm \infty} e^{itH} a^\pm(h_t) e^{-itH} \] for any $h \in D(\omega^{-1/2}) \cap \text{Ran} E_{(-\infty, \Sigma)}(H)$. Here $a^\pm_a$ stands for $a$ or $a^*$. To show that $a^\pm_a(h)$ exists (see [16, 28]), we define $a^\pm_a(h) := \lim_{t \to \infty} e^{itH} a^\pm(h_t) e^{-itH}$ and compute $a^\pm_a(h) = \int d^3k \delta(\omega) a^\pm_a(k)$ and $\partial_a a^\pm_a(h) = i e^{itH} G e^{-itH} t^{-1}$, where $G := [H, a^\pm_a(h)] = a^\pm_a(\omega h_t) - a^\pm_a(\omega h_t) e^{itH} = \sum_{n \geq 0} (h_t, g^{n}) L^2(dk)$ for $a^\pm_a = a^\pm_a$. Thus the proof of existence reduces to showing that one-photon terms of the form $\langle g, h_t \rangle$ are integrable in $t$. By (1.10), we have $\|\langle g, h_t \rangle L^2(dk)\| \lesssim (1 + t)^{-1 - \varepsilon}$, with $0 < \varepsilon < \mu + 1$, which is integrable. Moreover, as in [16, 28] one can show that $a^\pm_a(h)$ satisfy the canonical commutation relations and relations $a^\pm_a(h) = 0$, and

\[ \lim_{t \to \infty} e^{itH} a^\pm_a(h_{1,t}) \cdots a^\pm_a(h_{n,t}) e^{-itH} \Phi = a^\pm_a(h_1) \cdots a^\pm_a(h_n) \Phi, \] (I.2)

for any $\Psi \in \mathcal{H}_b$, $h, h_1, \cdots, h_n \in \mathcal{H}_b$, and any $\Phi \in \mathbf{1}_{(-\infty, \Sigma)}(H)$. We define the wave operator $\Omega_{\pm}$ on $\mathcal{H}_b$ by

\[ \Omega_{\pm} (\Phi \otimes a^\pm_a(h_1) \cdots a^\pm_a(h_n)) := a^\pm_a(h_1) \cdots a^\pm_a(h_n) \Phi. \] (I.3)

Using the canonical commutation relations, one sees that $\Omega_{\pm}$ extends to an isometric map $\Omega_{\pm} : \mathcal{H}_{\text{as}} \to \mathcal{H}$. Using the relation $e^{itH} (\Phi \otimes a^\pm_a(h_1) \cdots a^\pm_a(h_n)) = (e^{itH} \Phi_{\text{gs}}) \otimes (a^\pm_a(h_{1,t}) \cdots a^\pm_a(h_{n,t}))$, the definition of $I$ and (1.2), we identify the definition (I.3) with (1.1).
Recall that $P_{gs}$ denotes the orthogonal projection onto the ground state subspace of $H$. Let $P_{gs} := 1 - P_{gs}$ and $P_{\Omega} := 1 - P_{\Omega}$, where, recall, $P_{\Omega}$ is the projection onto the vacuum sector in $F$. Theorem [5.4] and its proof imply the following result.

**Theorem I.2.** Under the conditions of Theorem [5.4] we have on $\text{Ran} \chi_\Delta(H)$

$$\Omega_+(P_{gs} \otimes P_{\Omega})W_+P_{gs} + P_{gs} = 1. \tag{1.4}$$

**Proof.** Let $\psi_0 \in \text{Ran} \chi_\Delta(H)$. For every $\epsilon'' > 0$ there is $\delta'' = \delta(\epsilon'') > 0$, s.t.

$$\|\psi_0 - \psi_{0,\epsilon''} - P_{gs}\psi_0\| \leq \epsilon'' \tag{1.5}$$

where $\psi_{0,\epsilon''} = \chi_{\Delta_{\epsilon''}}(H)\psi_0$, with $\Delta_{\epsilon''} = [E_{gs} + \delta, a]$. Proceeding as in the proof of Theorem [5.4] with $\psi_{0,\epsilon''}$ instead of $\psi_0$, we arrive at (see (5.63))

$$\psi_{0,\epsilon''} = e^{-iHt}(e^{-iE_{gs}t}P_{gs} \otimes e^{-iHt}\chi_{(0, a - E_{gs})}(H_f))\phi_{0,\epsilon''} + \mathcal{O}(\epsilon') + C(\epsilon', m_0(1) + C(\epsilon')o\epsilon(1), \tag{1.6}$$

where we choose $\phi_{0,\epsilon''}$ such that $\phi_{0,\epsilon''} \in D(\Gamma((y))) \otimes F_{\text{fin}}(\tilde{h}_0)$ and $\|W_+\psi_{0,\epsilon''} - \phi_{0,\epsilon''}\| \leq \epsilon'$. Now using Theorem I.1, we let $t \to \infty$, next $n \to \infty$ to obtain

$$\psi_{0,\epsilon''} = \Omega_+(P_{gs} \otimes \chi_{(0, a - E_{gs})}(H_f))\phi_{0,\epsilon''} + \mathcal{O}(\epsilon'). \tag{1.7}$$

Since $\Omega_+$ is isometric, hence bounded, we can let $\epsilon' \to 0$, which gives

$$\psi_{0,\epsilon''} = \Omega_+(P_{gs} \otimes \chi_{(0, a - E_{gs})}(H_f))W_+\psi_{0,\epsilon''} = \Omega_+(P_{gs} \otimes P_{\Omega})W_+P_{gs}\psi_{0,\epsilon''}. \tag{1.8}$$

Here we used that $\chi_{(0, a - E_{gs})}(H_f) = P_{\Omega}\chi_{(0, a - E_{gs})}(H_f)$, together with $\chi_{(0, a - E_{gs})}(H_f)W_+\psi_{0,\epsilon''} = W_+\psi_{0,\epsilon''}$ and $\psi_{0,\epsilon''} = P_{gs}\psi_{0,\epsilon''}$. Introducing (1.8) into (1.5) and letting $\epsilon'' \to 0$, we obtain

$$\psi_0 = \Omega_+(P_{gs} \otimes P_{\Omega})W_+P_{gs}\psi_0 + P_{gs}\psi_0,$$

which gives (1.4). $\square$

**Supplement II. Creation and annihilation operators on Fock spaces**

With each function $f \in \mathfrak{h}$, one associates creation and annihilation operators $a(f)$ and $a^\ast(f)$ defined, for $u \in \otimes^n_0 \mathfrak{h}$, as

$$a^\ast(f) : u \to \sqrt{n+1}f \otimes_s u \quad \text{and} \quad a(f) : u \to \sqrt{n}\langle f, u \rangle_\mathfrak{h},$$

with $(f, u)_\mathfrak{h} := \int f(k)u(k, k_1, \ldots, k_{n-1})dk$. They are unbounded, densely defined operators of $\Gamma(\mathfrak{h})$, adjoint of each other (with respect to the natural scalar product in $F$) and satisfy the canonical commutation relations (CCR):

$$[a^\#(f), a^\#(g)] = 0, \quad [a(f), a^\ast(g)] = \langle f, g \rangle,$$

where $a^\# = a$ or $a^\ast$. Since $a(f)$ is anti-linear and $a^\ast(f)$ is linear in $\varphi$, we write formally

$$a(f) = \int f(k)a(k)dk, \quad a^\ast(f) = \int f(k)a^\ast(k)dk,$$

where $a(k)$ and $a^\ast(k)$ obey (again formally) the canonical commutation relations

$$[a^\#(k), a^\#(k')] = 0, \quad [a(k), a^\ast(k')] = \delta(k - k'),$$

Finally, given an operator $b$ acting on the one-photon space, the operator $d\Gamma(b)$ defined on the Fock space $F$ by (1.2) can be written (formally) as $d\Gamma(b) := \int a^\ast(k)ba(k)dk$, where $b$ acts on the variable $k$.

The following bounds on $a(f)$ and $a^\ast(f)$ are standard (see e.g. [29]).
Lemma II.1. Recall the notation \( \|h\|_\omega := \int dk (1 + \omega^{-1})|h(k)|^2 \). Let \( f \in L^2(\mathbb{R}^3) \). The operators \( a(f)(N + 1)^{-1/2} \) and \( a^*(f)(N + 1)^{-1/2} \) extend to bounded operators on \( \mathcal{H} \) satisfying
\[
\|a(f)(N + 1)^{-\frac{1}{2}}\| \leq \|f\|, \quad \|a^*(f)(N + 1)^{-\frac{1}{2}}\| \leq \sqrt{2}\|f\|.
\]
If, in addition, \( f \) satisfy \( \omega^{-1/2}f \in L^2(\mathbb{R}^3) \), then the operators \( a(f)(H_f + 1)^{-1/2} \) and \( a^*(f)(H_f + 1)^{-1/2} \) extend to bounded operators on \( \mathcal{H} \) satisfying
\[
\|a(f)(H_f + 1)^{-\frac{1}{2}}\| \leq \|\omega^{-\frac{1}{2}}f\|, \quad \|a^*(f)(H_f + 1)^{-\frac{1}{2}}\| \leq \|f\|_\omega.
\]

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