Viscous damping of deep-water waves: dispersion relations and non-linear corrections

Andrea Armaroli,1 Debbie Eeltink,1 Maura Brunetti,1 and Jérôme Kasparian1
1 GAP-Nonlinearity&Climate, Institute for Environmental Sciences,
Université de Genève, Boulevard Carl-Vogt 66, 1211 Genève 4, Switzerland

We discuss the impact of viscosity on non-linear propagation of deep-water surface waves. After a survey of the available approximations of their dispersion relation, we propose to modify the hydrodynamic boundary conditions to model both short and long waves. From them we derive a non-linear Schrödinger equation where both linear and non-linear parts are modified by dissipation and show that the former plays the main role in both gravity and capillary-gravity waves, while, in most situations, the latter represents only small corrections. This provides a justification of the conventional approaches to damped propagation found in the literature.

I. INTRODUCTION

Deep-water surface waves are among the most studied examples of non-linear physical system. The non-linearity stems from kinematic and dynamic boundary conditions at the air-fluid interface. Triplet [1] and quadruplet [2] interactions can be included and a general integro-differential equation derived. A suitable expansion of the integral kernel [3, 4] was shown to connect it to simple propagation models, like the ubiquitous non-linear Schrödinger equation (NLS) [5] and its generalizations (modified NLS—MNLS), the best known among which is the Dysthe equation [6]. The universal NLS possesses many remarkable properties and solutions, such as the modulation instability (MI, also known as Benjamin-Feir instability—BFI) [7, 8], solitons and breathers, the clearest examples of the balance of dispersion and non-linearity.

In the propagation of deep-water waves, gravity effects dominate for long waves (small wave-numbers) and surface tension for short waves. Moreover, viscosity is the ubiquitous damping mechanism [10, 11] (other routes to dissipation include impurities, obstacles, wave-breaking...). It induces a vorticity in the fluid and requires therefore to solve the full Navier-Stokes equations. For small viscosity, the vorticity is significant only in a small boundary layer close to the surface [10, 12].

For small amplitude waves, i.e. when non-linearities are negligible, the velocity potential formulation valid for an inviscid flow can be corrected to include the effects of viscosity [13, 14]. Different loss rates and corrections to the group velocity can be found in Ref. [17]. The availability of a quasi-potential formulation greatly simplifies the numerical solution of the hydrodynamic problem [18, 19]. Moreover, it is always assumed, without justification, that the linear dissipation is so small that no non-linear correction is required for the formulation of a modified NLS including damping [20, 21].

In this work we justify this assumption by quantifying the non-linear corrections to propagation equations stemming from kinematic viscosity. Our approach is based on the dissipation method approximation detailed in [17], but can be generalized to any expression for the dispersion relations and damping mechanisms. After recalling some simple considerations about dispersion relations and justify the choice of a specific form (Section III), we propose a modified set of hydrodynamic boundary conditions consistent with our approach (section III). These are straightforwardly adapted in Section IV to generalize the 1D MNLS [3]

\[
\frac{\partial B}{\partial t} + i (\omega(k_0 + \kappa) - \omega_0) \hat{B} + i \frac{\omega k_0^2}{2} j_B \hat{B}^2 = 0, \quad (1)
\]

with \( \hat{B}(k, t) \) the Fourier transform of the envelope \( B(x, t) \) of the surface elevation centered around the carrier wavenumber \( k_0; \omega(k) \) the real-valued dispersion relation \( (\omega_0 \equiv \omega(k_0)) \). More details are provided below. We quantify the non-linear corrections to the recurrence period and spectral-mean downshift to be less than one percent. We also verify that the full dispersion relation plays a key role in the non-linear evolution of the MI, e.g. in explaining the frequency downshift first observed in [22]. Finally (section V), we show that the considered dispersion relation can be inverted so that not only the space-like, but also the time-like formulation of the NLS naturally generalizes to the dissipative case. The same approach can be adapted to high-order NLS, such as the Dysthe [6] or the recently proposed compact [23] and super-compact [24] equations. Conclusions and perspectives complete our manuscript.

II. VISCOUS DISPERSION RELATIONS

We first briefly review how to express the dispersion relation for deep-water capillary-gravity waves at the surface of an incompressible viscous fluid. The solution of the linearized hydrodynamic equations for 1D propagation in the x-direction is a plane wave \( \exp(i k x - i \omega(k) t) \). We decompose \( \omega(k) = \omega_R(k) + i \omega_I(k) \), where \( \omega_R > 0 \) (resp. \( \omega_R < 0 \)) represents forward- (resp. backward-) propagating waves and \( \omega_I < 0 \) is the damping rate.

The dispersion relation can be shown to be the solution...
\( \omega(k) \) of the implicit equation [10–11]

\[
\left(2 - i \frac{\omega}{\nu k^2}\right) + \frac{|k|(g + sk^2)}{\nu^2 k^4} = 4 \left(1 - i \frac{\omega}{\nu k^2}\right)^{\frac{1}{2}}
\]  

(2)

where \( g \) is the standard acceleration due to gravity, \( s \equiv T/\rho_t \) with \( T \) the surface tension (in \( \text{Nm}^{-1} \)) of the fluid-air interface and \( \rho_t \) is the density of the fluid. The gravity and capillary contributions dominate, respectively, for small and large \( k \), i.e. for large and small wavelength. Finally, \( \nu \) denotes the kinematic viscosity of the fluid (in \( \text{m}^2/\text{s} \)) and is the physical origin of \( \omega \).

The detailed derivation of Eq. (2) from the linearized Euler equations for an incompressible fluid, along with their kinematic and boundary conditions, consists in including a vorticity field and assuming that the mass transport occurs only in a boundary layer close to the air-fluid interface [10–12, 14, 21, 25, 26]. Eq. (2) being quartic, two forward-traveling wave branches exist: one is unphysical (it corresponds to a velocity potential diverging for infinite depth \( z \to -\infty \), see also in section III). The physical one is shown in Fig. 1 for a gravity-capillary wave propagating at the interface of air and water (\( \omega \geq 0 \)). The left (right) axis refers to \( \text{Re}\omega \) (\( \text{Im}\omega \)). The vertical dotted line refers to the cut-off \( k_c \) beyond which \( \omega_R = 0 \) and \( \omega_I \) bifurcates in two standing wave branches (red solid and green dashed lines), yielding two different damping rates.

It was shown that simpler explicit expressions can be obtained, by expressing the vorticity in terms of the velocity potential and surface elevation, relying on physical arguments, by assuming linear propagation and small viscosity [14–17, 27]. Nevertheless, a simple algebraic manipulation of Eq. (2) allows us to re-derive them easily. Let us define \( \tilde{\theta} \equiv -i \frac{\omega}{\nu k^2} \) and \( \theta = -i \frac{\omega}{\nu k^2} \) (related to the Reynolds number defined for water waves), with \( \tilde{\omega}(k) \equiv \sqrt{|k|(g + sk^2)} \) (i.e. the inviscid dispersion relation). Eq. (2) is written compactly as

\[
\left(\frac{2 + \theta}{\theta}\right)^2 - 1 = 4 \left(\frac{1 + \theta}{\theta}\right)^{\frac{1}{2}}
\]

(3)

Notice that for \( \tilde{\theta} \gg 1 \), it is trivial to understand that the RHS can be considered small. The standard dispersion relation is indeed obtained by neglecting the RHS of Eq. (3): we write \( \theta = \pm \theta - 2, \) i.e.

\[
\omega = \pm \tilde{\omega} - 2i\nu k^2,
\]

(4)

which we refer to as (small-\( k \)) Lamb approximation [10]. It was shown that its physical justification can be traced back to the smallness of the vortical contribution to pressure and the fact that the surface deformation and boundary layer (where the vorticity is non-negligible) are small [14, 21]. However, the viscosity enters only in the imaginary part, so no cut-off for traveling waves can possibly appear. Lamb derived from Eq. (3) also the two branches of the damping rate in the opposite case of small \( \tilde{\theta} \), which read respectively \( \omega_1 = -\frac{\tilde{\omega}^2}{\nu k^2} \) and \( \omega_1 = -0.91\nu k^2, \) the former being the most physically important. The cut-off is estimated from Eq. (3), by looking for a real solution \( \theta \) of double multiplicity: it is easy to obtain \( \tilde{\omega}^2 = 0.58(\nu k^2)^2, \) where \( 0.58 = 4/\alpha - \alpha^2, \) \( \alpha \equiv \tilde{\theta} + 2, \) with \( \tilde{\theta} \) the real solution of \( \tilde{\theta}^3 + 5\tilde{\theta}^2 + 8\tilde{\theta} + 3 = 0. \)

In order to have a single expression for small and large \( k \), different approximations of the RHS of Eq. (2) are shown behave better than the Lamb approach. Indeed, if we let \( 1 + \theta \approx 1 \) on the RHS of Eq. (3), we re-obtain the result of the dissipation method (DM), [17, 28], i.e.

\[
\theta = -2 \pm \sqrt{\tilde{\theta}^2 + 4}.
\]

(5)

In contrast to Eq. (4), this relation exhibits a cut-off in the real part of \( \omega \), for \( \tilde{\theta} = -2i \). Beyond, the two signs represent the two branches of dissipation of the standing mode.

Further, we notice that the alternative approach proposed in [15] (viscous potential flow—VPF) modifies only the dynamic boundary condition (Bernoulli equation), analogously to [20], by introducing viscosity as an external pressure perturbation. It can be reproduced by the expansion \( (1 + \theta)^{\frac{1}{2}} \approx 1 + \theta/2 \) and its solution is written, in our notation

\[
\theta = -1 \pm \sqrt{\tilde{\theta}^2 + 1}.
\]

(6)
Finally, we may ask ourselves what is the result of expanding the RHS of Eq. (8) to the second order: we write a third approximation,

$$\theta = -\frac{2}{3} + \frac{2}{3} \sqrt{\frac{3}{2} \tilde{\omega}^2 + 1},$$

which we will refer to as modified VPF (MVPF).

Among these three last dispersion relations, the DM is the sole to provide the correct approximation (compared to Lamb) of damping at small $k$, while the cut-off is not accurate. The real part of the VPF is a better approximation at the cut-off and matches very well with the conventional dispersion relation, but the damping is half of the expected one at small $k$, while standing waves ($k > k_c$) of lesser damping have the correct asymptotics (compared to the estimation made by Lamb, see above). Finally the MVPF behaves better at the cut-off, but does not reproduce the behavior of either $\omega_0$ or $\omega_1$ for small values of $k$, so it is of little practical use in the most accessible oceanic regimes. In [29], it was shown that irrotational theories fail to provide a good approximation around $k_c$. The MVPF shows instead that a good approximation of the cut-off is incompatible to a satisfactory asymptotic behavior at both small and large $k$. The results are summarized in Table I.

The numerical values of cut-off for the fluids considered in [17] are reported in Table I where we include also the MVPF results, for the sake of completeness. In order to visually confirm the formulas of Table I we compare the different approximations in Fig. 2 for a surface between glycerin and air ($\nu = 6.21 \times 10^{-4}$ m$^2$s$^{-1}$, $s = 5.03 \times 10^{-5}$ m$^3$s$^{-2}$), the larger viscosity allows us to have a cut-off for relatively small $k$ and observe both short and long wave ranges in linear scale. The same behavior applies to other fluids.

We include also a further simplification. Suppose we choose the DM relation of Eq. (5), which provides the best approximation of damping and group velocity for standing waves ($k > k_c$) of lesser damping have the correct asymptotics (compared to Lamb) of damping at small $k$, while the cut-off is not accurate. The real part of the VPF is a better approximation at the cut-off and matches very well with the conventional dispersion relation, but the damping is half of the expected one at small $k$, while standing waves ($k > k_c$) of lesser damping have the correct asymptotics (compared to the estimation made by Lamb, see above). Finally the MVPF behaves better at the cut-off, but does not reproduce the behavior of either $\omega_0$ or $\omega_1$ for small values of $k$, so it is of little practical use in the most accessible oceanic regimes. In [29], it was shown that irrotational theories fail to provide a good approximation around $k_c$. The MVPF shows instead that a good approximation of the cut-off is incompatible to a satisfactory asymptotic behavior at both small and large $k$. The results are summarized in Table I.

The numerical values of cut-off for the fluids considered in [17] are reported in Table I where we include also the MVPF results, for the sake of completeness. In order to visually confirm the formulas of Table I we compare the different approximations in Fig. 2 for a surface between glycerin and air ($\nu = 6.21 \times 10^{-4}$ m$^2$s$^{-1}$, $s = 5.03 \times 10^{-5}$ m$^3$s$^{-2}$), the larger viscosity allows us to have a cut-off for relatively small $k$ and observe both short and long wave ranges in linear scale. The same behavior applies to other fluids.

We include also a further simplification. Suppose we choose the DM relation of Eq. (5), which provides the best approximation of damping and group velocity for standing waves ($k > k_c$) of lesser damping have the correct asymptotics (compared to Lamb) of damping at small $k$, while the cut-off is not accurate. The real part of the VPF is a better approximation at the cut-off and matches very well with the conventional dispersion relation, but the damping is half of the expected one at small $k$, while standing waves ($k > k_c$) of lesser damping have the correct asymptotics (compared to the estimation made by Lamb, see above). Finally the MVPF behaves better at the cut-off, but does not reproduce the behavior of either $\omega_0$ or $\omega_1$ for small values of $k$, so it is of little practical use in the most accessible oceanic regimes. In [29], it was shown that irrotational theories fail to provide a good approximation around $k_c$. The MVPF shows instead that a good approximation of the cut-off is incompatible to a satisfactory asymptotic behavior at both small and large $k$. The results are summarized in Table I.

| $\omega_R(k \ll 1)$ | $\omega_I(k \ll 1)$ | cut-off | $\omega_I(k \gg 1)$ |
|---------------------|---------------------|---------|---------------------|
| Exact [3] $\tilde{\omega}$ | $-2\nu k^2$ | $\tilde{\omega}^2 = 0.58(\nu k^2)^2 - \frac{\nu^2}{6} + \frac{\nu}{6}$ | $\tilde{\omega}$ |
| DM [3] $\tilde{\omega}$ | $-2\nu k^2$ | $\tilde{\omega}^2 = 4(\nu k^2)^2 - \frac{\nu^2}{4} + \frac{\nu}{2}$ | $\tilde{\omega}$ |
| SDM [10] $\tilde{\omega}$ | $-2\nu k^2$ | n/a | $\tilde{\omega}$ |
| VPF [9] $\tilde{\omega}$ | $-\nu k^2$ | $\tilde{\omega}^2 = (\nu k^2)^2 - \frac{\nu^2}{2} + \frac{\nu}{4}$ | $\tilde{\omega}$ |
| MVPF [7] $\sqrt{3} \tilde{\omega}$ | $-2\nu k^2$ | $\tilde{\omega}^2 = \frac{4}{3}(\nu k^2)^2 - \frac{\nu^2}{4} + \frac{\nu}{4}$ | $\tilde{\omega}$ |

TABLE I. Comparison of the asymptotic behavior of the exact and approximated dispersion relations. The classical estimations of Lamb on the implicit formula, the DM, VPF, and MVPF are simply derived in the text from the full dispersion relation, SDM is the simplified DM found by neglecting the viscous terms under the square-root in Eq. (9). The first two are also discussed in [17].

Finally, we may ask ourselves what is the result of expanding the RHS of Eq. (8) to the second order: we write a third approximation,

$$\theta = -\frac{2}{3} + \frac{2}{3} \sqrt{\frac{3}{2} \tilde{\omega}^2 + 1}, \quad (7)$$

which we will refer to as modified VPF (MVPF).

FIG. 2. Dispersion relation for gravity-capillary waves propagating at the interface of air and glycerin. (a) real part; (b) imaginary part. Exact solutions of Eq. (2) for forward propagating waves (\(\omega \geq 0\)) are shown as a solid dark yellow line. The conventional approximation is shown by a green dotted line. The different approximations are represented by dashed lines (black—DM, red—VPF, pink—MVPF). Finally, the dashed-dotted lines correspond to the SDM: it does not exhibit a clear cut-off, but provides a good approximation of damping at both $k$ limits.

For small $\nu k^2$, we can neglect the term proportional to $\nu^2$ under the square root in Eq. (9) and obtain the simplified DM (SDM)

$$\omega \approx \frac{\tilde{\omega} \pm \frac{2\nu k^2}{\omega}}{\pm 1 + \frac{2\nu k^2}{\omega}} \equiv \tilde{D}(k).$$

TABLE II. Cut-off values (in m$^{-1}$) for the three examples of Ref. [17] and their comparison to the MVPF result.

| Type   | Water | Glycerin | SO10000 |
|--------|-------|----------|----------|
| DM [5] | $1.82 \times 10^7$ | 196.81 | 28.50 |
| VPF [9] | $7.28 \times 10^7$ | 344.64 | 45.29 |
| MVPF [7] | $4.09 \times 10^8$ | 416.10 | 51.86 |
| Exact [3] | $1.25 \times 10^8$ | 445.18 | 54.30 |

It is trivial to verify—see also table [4]—that Eq. (10) provides a better asymptotics than Eq. (5) for damping at large $k$, at the expense of a smooth transition around the cut-off, as shown in Fig. 2 as a cyan dash-dotted line. That is, the series expansion of $\tilde{\omega}/\omega$ provides a more robust approximation than the conventional direct Taylor expansion.
expansion of \( \omega \) (i.e. the low-\( k \) Lamb approach, green dotted lines in Fig. 2), which predicts ever increasing phase velocity \( v_p \equiv \omega_R/k \) and damping. This is similar to the fit of a dispersion relation by means of a Padé approximant, well known in optics 30.

III. HYDRODYNAMIC EQUATIONS

Viscosity induces vorticity, thus making the solution of the non-linear hydrodynamic problem (Navier-Stokes equations) extremely complicated. The most practical workaround is to extend the use of a velocity potential \( \phi(x, z, t) \), where \( z \) is the depth coordinate and \( x \) is the longitudinal propagation direction, to the viscous case.

By denoting the free surface elevation \( \eta(x, t) \), the system of hydrodynamic equations in an inviscid and infinitely deep fluid reads as

\[
\begin{align*}
\phi_{xx} + \phi_{zz} &= 0 \quad \text{for } -\infty < z < \eta \\
\nabla \phi &= 0 \quad \text{for } z \rightarrow -\infty \\
\eta_t + \phi_x \eta_x - \phi_z &= 0 \quad \text{for } z = \eta \\
\phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + g \eta &= -s \frac{\eta_{xx}}{(1 + \eta_z^2)^2} = 0 \quad \text{for } z = \eta
\end{align*}
\]

These are respectively the Laplace equation in the fluid, the rigid bottom condition, the kinematic and the dynamic boundary conditions at the free surface. In the linear limit, the solutions of Eq. (11) are plane waves, \( \eta = \eta_0 \exp(ikx - i\omega t) \) and \( \phi = \phi_0 \exp(ikx - i\omega t + |k|z) \). The inviscid dispersion relation \( \omega(k) \) is the compatibility condition of the homogeneous system

\[
\begin{bmatrix}
\omega \\
-(g + sk^2) \omega
\end{bmatrix}
\frac{\eta_0}{\phi_0} = 0.
\]

Eq. (11) is obtained by the substitution \( \omega \rightarrow \omega + 2i\nu k^2 \) in Eq. (12). In the wavenumber domain, we obtain a correction \(-2i\nu k^2 \eta_0 \) to the kinematic boundary condition and \(-2i\nu k^2 \phi_0 \) to the dynamic one. To transform them back to the spatial domain, we notice that \( \eta \) does not depend on \( z \) and \( \phi \) is the solution of the Laplace equation. The operator correspondence \( ik \leftrightarrow \frac{D}{Dz} \) (established by the plane-wave definition) is correct for both terms and allows us to obtain the well-known weakly viscous hydrodynamic system 14, 21

\[
\begin{align*}
\phi_{xx} + \phi_{zz} &= 0 \quad \text{for } -\infty < z < \eta \\
\nabla \phi &= 0 \quad \text{for } z \rightarrow -\infty \\
\eta_t + \phi_x \eta_x - \phi_z &= 2\nu \eta_{xx} \quad \text{for } z = \eta \\
\phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + g \eta &= -s \frac{\eta_{xx}}{(1 + \eta_z^2)^2} = 2\nu \phi_{xx} = -2\nu \phi_{zz} \quad \text{for } z = \eta
\end{align*}
\]

Its physical motivation is that the velocity field can be decomposed in a potential and vorticity contribution, \((u, w) = (\phi_x - \Omega_y \eta, \phi_z + \Omega_x)\). The vorticity pseudo-vector has only the \( y \)-component, \( \Omega \equiv (0, \Omega_x, 0) \), and is expressed as a function of \( \phi \) and \( \eta \) by using the linearized boundary conditions and by assuming \( \Omega_x \approx 0 \) 14, 21.

The fully non-linear Navier-Stokes equations couple different velocity components and is hard to write in as simple terms as Eqs. (11) and (13).

We propose a solution to this difficulty based on the approximation presented in the previous section. The principle behind our choice is that, in Eq. (11) as well as the unidirectional models derived from it (e.g. the NLS or the Dysthe equation [4]), the energy is the sum of a kinetic part, which depends on dispersion, and a potential part, which is ascribed to non-linear interaction. A periodic exchange between the two parts characterizes, e.g., the nonlinear stage of ML. We find thus sound that both are damped by kinematic viscosity. Since the Hamiltonian density which encompasses the different energy terms is associated to the evolution in time, the associated operator must be modified as a whole. This allows us to better quantify the role of dissipation in the non-linear propagation of waves. Consider Eq. (3): it is the solution of

\[
\begin{bmatrix}
\omega \hat{D}(k) \\
-(g + sk^2) \omega \hat{D}(k)
\end{bmatrix} = 0.
\]

Thus, in Eq. (11), we make the substitution \( \frac{\partial}{\partial z} \rightarrow \partial_t \equiv D(-i \frac{\partial}{\partial z}) \frac{\partial}{\partial t} \), where \( D(-i \frac{\partial}{\partial z}) \) is the operator in the physical space associated in the Fourier space to \( \hat{D}(k) \) in Eq. (9).

This allows us to formally obtain an alternative form for the hydrodynamic equations

\[
\begin{align*}
\phi_{xx} + \phi_{zz} &= 0 \quad \text{for } -\infty < z < \eta \\
\nabla \phi &= 0 \quad \text{for } z \rightarrow -\infty \\
\partial_t \eta + \phi_x \eta_x - \phi_z &= 0 \quad \text{for } z = \eta \\
\partial_t \phi + \frac{1}{2} (\phi_x^2 + \phi_z^2) + g \eta &= -s \frac{\eta_{xx}}{(1 + \eta_z^2)^2} = 0 \quad \text{for } z = \eta
\end{align*}
\]

The proposed system reduces to Eq. (13) for \( \nu k^2 \ll \omega \), by replacing \(-1/\omega \rightarrow \int dt \) and \(-k^2 \rightarrow \frac{\partial^2}{\partial z^2} \).

Eq. (11) provides however an alternative set of equations to be employed in full hydrodynamic solvers, like the high-order spectral method (HOSM) 18, 19, 31, [32].

In the next section we will discuss how to derive a non-linear NLS-like propagation equation and show that small albeit measurable differences between the linear and non-linear dissipation terms exist.
IV. A SPACE-LIKE PROPAGATION EQUATION

In order to generalize the MNLS of Eq. (1) to the dissipative case, we can follow the approach of [3], based on the Zakharov’s method [2], to provide the justification for an NLS in which the full dispersion relation and not a truncation thereof is included. The free-surface elevation is reconstructed from the envelope $B(x,t)$ as
\[ \eta = \frac{1}{2} [b(x,t) + \text{c.c.}], \]
where $b \equiv Be^{-i(k_0x-\omega_0t)}$ is the free-wave component and c.c. denotes the complex conjugate. Notice that the plane wave factors are defined to be real: $\omega_0 \equiv \omega(k_0)$.

Eq. (1), where the nonlinearity is assumed small and viscosity is neglected ($\omega_H = \tilde{\omega}$), is written for $b(k,t) \equiv \mathcal{F}_k[b(x,t)]$ as
\[ \dot{b}_i + i\tilde{\omega}(k)b_i + i\frac{\omega_0 k^2}{2} \mathcal{F}_k[|b|^2b] = 0 \]
from which, by means of the substitution $\partial_t \rightarrow \bar{\partial}_t$, we derive the equation of motion for
\[ \frac{\partial \bar{b}}{\partial \bar{t}} + i\tilde{\omega}(k)\bar{b} + i\frac{\omega_0 k^2}{2D(k)} \mathcal{F}_k [|b|^2b] = 0. \]
Back to the slowly-varying variable $B$, we can write, for $B(\kappa,t) = \mathcal{F}_\kappa[B(x,t)]$
\[ \frac{\partial \bar{B}}{\partial \bar{t}} + i\tilde{\omega}(k+\kappa)\bar{B} - \omega_0 \bar{B} + i\frac{\omega_0 k^2}{2D(k+\kappa)} \mathcal{F}_\kappa [|B|^2B] = 0. \]
This represents our generalization of Eq. (1) in the dissipative case.

Notice that the linear part of Eq. (17) represents the complete dispersion relation including damping. If we Taylor-expand it to fourth order in $\kappa$ and write the resulting terms in the physical space by replacing powers of $\kappa$ by derivatives in $x$, we obtain the linear terms of Ref. [33]. This approach may prove convenient also for generalizing a forced MNLS model [34].

Below, in Figs. 3-4, we show that the dispersive contribution explains alone most of the frequency downshift observed in experiments. Our approach provides also a non-linear viscous damping (the imaginary part of the non-linearity), valid for small $k$ where the bound modes (small corrections to $\eta$ oscillating at integer multiples of $k_0$ and enslaved, for pure gravity waves, to the free-modes $B$) are not resonantly excited. It also introduces a wavenumber-dependent correction to non-linearity (proportional to $\nu^2$ under the square root in $\bar{D}$, see Eq. [33]). We expand the non-linear damping coefficients as
\[ \frac{1}{2} \alpha^2 \omega_0 k_0^2 \approx \nu k_0^2 \left[ k_0 + \left( 2 - \frac{\nu}{k_0^2} \right) \kappa \right], \]
where $\nu_0 \equiv \tilde{\omega}(k_0)/k_0$ and $\nu_0 \equiv \tilde{\omega}'(k_0)$ are the phase and group velocities at $k_0$, respectively, neglecting viscosity. The first contribution is a homogeneous non-linear damping (which can be obtained independently by the method of multiple scales [33]), while the second is a derivative damping term, i.e. the dissipative counterpart of the self-steepening in non-linear optics [34]. Its effect is small, because energy dissipation caused by linear attenuation limits the bandwidth of $B$.

As an example, we show in Fig. 3 the solution for a harmonically perturbed Stokes’ wave propagating at the water-air interface ($B(x,0) = B_0 \sqrt{1 - \alpha + \sqrt{2\alpha \cos k_0 x}}$), with initial steepness $\varepsilon \equiv \frac{k_0B_0}{\sqrt{2}} = 0.1$, $k_0 = 10$, $\alpha = 1 \times 10^{-2}$ and neglecting surface tension. The perturbation wave-number is $k_0 = 2\varepsilon k_0 \sqrt{2B_0}$, i.e. the maximally unstable mode predicted by the NLS. We notice that the linear and non-linear dissipation scale as $2\nu k_0^3$ and $2\nu k_0^3|B|^2 \approx 2\nu k_0^3\varepsilon^2$, respectively. We thus expect that only at extreme steepness and breather peaks is the non-linear damping non-negligible with respect to the linear one.

In Fig. 3 we compare the dynamics of Eq. (17) (full model), with the results of neglecting $\bar{D}(k)$ in either the linear or non-linear part (we refer to them as non-linear and linear damping, respectively). The comparison shows both the energy attenuation and downshift of the spectral mean ($\kappa_m \equiv P/N$, with $P = \text{Im} \int dx B_2 B^*$ the momentum and $N = \int dx |B|^2$ the norm of the field) depend mainly on the linear damping. As shown in [37], the slightest amount of dissipation causes the recurrence period to stretch and a period-1 orbit to be attracted to a period-2 one.

This behavior is found in results of the full model (blue solid lines) and in the linearly damped (dashed red lines) simulations. The discrepancy in recurrence periods is just a contraction of about 1% per period. In panel (a), we notice also that the non-linear damping alone (black
dotted line) leads instead to a behavior more similar to the undamped result. As far as the downshift of $k_m$ is concerned [panel (b)], the full model exhibits about 1% more shift than the linearly damped one. Notice that the difference of the two nearly equals the pure non-linear contribution (black dotted line).

The blue solid lines exhibit the same behaviour if Eq. (10) is used instead of Eq. (8) (not shown). The discrepancies between the solid and dashed lines, observed in Fig. 3 are mainly explained by the non-linear damping $\nu k_0 B^2$: viscous corrections to group velocity have an even smaller impact on non-linear coefficients.

This behavior does not qualitatively change if we include higher-order non-linear terms in Eq. (17), see e. g. [4], the downshift is associated to a transient upshift at each recurrence cycle [37].

A different dynamics is observed for shorter wavelengths even though stronger damping limits the impact of non-linear phenomena. We thus simulated the same initial conditions as above, with $k_0 = 75$, i. e. near the transition from gravity to capillary waves. We show in Fig. 4 the results of the purely linear and purely non-linear (dashed lines) damping results. Parameters are $k_0 = 75$, $\varepsilon = 0.1$, $\nu = 1 \times 10^{-6}$, $s = 7.28 \times 10^{-5}$, $\alpha = 0.01$. (a) Central mode $\kappa = 0$; (b) main unstable modes $\kappa = \pm k_0$. We notice that the downshifted mode at $-k_0$ soon acquires most of the energy. While the linear damping irreversibly stops recurrence, the non-linear damping is not sufficient.

We finally remark that the present non-linear viscous damping does not exclude the existence of other non-linear loss mechanisms. We would like to mention the Landau damping [40], which plays a role at large $k$, where the bound modes are resonantly excited and dissipated by viscous damping. Its even symmetry allows however neither a peak nor mean spectral downshift. Alternatively, a loss mechanism like the $\beta$-term introduced in [14, 12] explains the frequency downshift observed in the non-linear stage of MI, by virtue of its odd symmetry but has no clear physical origin: it may represent a model of wave-breaking, not included here. Nevertheless, we have shown that the spectral mean shifts permanently simply due to the variation of damping with frequency.

Both examples, Figs. 3 and 4 show that an MNLS where the full dispersion (with dissipation) is included in the linear part is sufficient to explain at least qualitatively the partial recurrence and spectral downshift in the non-linear behavior of MI. Thus the implicit assumption of [21] that a non-linear correction due to viscosity plays a minor role in deep-water wave propagation is confirmed in its physical soundness.

V. A TIME-LIKE PROPAGATION EQUATION

Laboratory conditions usually imply the measurement of the temporal profiles at different positions along a wave-tank by means of wave-gauges. The time-like formulation of the NLS and its generalizations are thus more practical than their space-like counterparts for describing and interpreting measurements.

In order to obtain such a formulation, we need to invert the dispersion relation and derive an explicit expression for $k(\omega)$. The method we used above to derive the DM approximation provides a straightforward solution, at least for $s = 0$, that reads

$$k(\omega) = \frac{\tilde{k}}{D_k(\omega)}$$

with $\tilde{k}(\omega) \equiv \frac{\omega^2}{g}$, the conventional dispersion of deep-water gravity waves, and $D_k(\omega) \equiv \sqrt{1 + 16i\frac{\tilde{k}(\omega)}{g^2}}$ the correction due to viscosity, which contributes to both the real and imaginary parts of $k(\omega)$. Eq. (18) can be simplified as $k(\omega) \approx \frac{\omega^2}{g} \left[1 - 4i\frac{\tilde{k}(\omega)}{g^2}\right]^{-1}$.

As shown in Sec. IV the non-linear damping is negligible in most cases. We write the modified time-like NLS in the frequency domain—with respect to the detuning.
\[ \Omega \equiv \omega - \omega_0 \text{ (here } \hat{B}(x, \Omega) \equiv \mathcal{F}_\Omega[B(x, t)]\text{)—as} \]
\[
\frac{\partial \hat{B}}{\partial x} - i \left[ \frac{\hat{k} (\omega_0 + \Omega)}{D_k (\omega_0 + \Omega)} - k_0 \right] \hat{B} - i \frac{k_0^3 B}{2} \mathcal{F}_\Omega [\vert B \vert^2 B] = 0. \tag{19}
\]

This model can be applied to assess the effects of the dispersive damping in a wave-tank experiment.

**VI. CONCLUSIONS**

After having recalled the different forms of dispersion relation for deep-water gravity-capillary waves in the presence of viscosity and unified their derivation, we discussed their physical validity all over the wavenumber/frequency range. We exploited these results to reformulate the hydrodynamic equations and quantify the impact of kinematic viscosity on non-linear damping. We showed that the simplest NLS model with full dispersion (in both the real and imaginary part) provides most of the justification for the downshift of the spectral mean during the non-linear stage of evolution of the MI: corrections in the non-linear behavior have only a small effect. This provides an \textit{a posteriori} justification of the choice of using dispersion relations only in the linear part of a non-linear propagation equation \cite{3, 21}. This does not forbid however to consider other damping mechanisms, e.g, in order to model wave-breaking. Our approach allows one to include them in the form of a rational dispersion relation, to be used in fully non-linear hydrodynamic simulations.

**ACKNOWLEDGMENTS**

We acknowledge the financial support from the Swiss National Science Foundation (Projects Nos. 200021-155970 and 200020-1756797). We would like to thank John D. Carter for fruitful discussions.

\[\begin{align*}
[1] & \quad L. F. Mcgoldrick, “Resonant interactions among capillary-gravity waves,” J. Fluid Mech. 21, 305 (1965).
[2] & \quad V. E. Zakharov, “Stability of periodic waves of finite amplitude on the surface of a deep fluid,” J. Applied Mech. Tech. Phys. 9, 190–194 (1968).
[3] & \quad K. Trulsen, Igor Kliakhandler, K. B. Dysthe, and M. G. Velarde, “On weakly nonlinear modulation of waves on deep water,” Phys. Fluids 12, 2432 (2000).
[4] & \quad M. Stiassnie, “Note on the modified nonlinear Schrödinger equation for deep water waves,” Wave Motion 6, 431–433 (1984).
[5] & \quad C. Sulem and P.-L. Sulem, The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse (Applied Mathematical Sciences), 1st ed. (Springer, 1999) p. 350.
[6] & \quad K. B. Dysthe, “Note on a Modification to the Nonlinear Schrodinger Equation for Application to Deep Water Waves,” Proc. R. Soc. A 369, 105–114 (1979).
[7] & \quad T. B. Benjamin and J. E. Feir, “The disintegration of wave trains on deep water Part 1. Theory,” J. Fluid Mech. 27, 417 (1967).
[8] & \quad E. Lo and C. C. Mei, “A numerical study of water-wave modulation based on a higher-order nonlinear Schrodinger equation,” J. Fluid Mech. 150, 395–416 (1985).
[9] & \quad V. E. Zakharov and L. A. Ostrovsky, “Modulation instability: The beginning,” Physica D 238, 540–548 (2009).
[10] & \quad H. Lamb, Hydrodynamics (Dover publications, New York, 1945).
[11] & \quad L. D. Landau and E. M. Lifshitz, Fluid Mechanics: Landau and Lifshitz: Course of Theoretical Physics, Volume 6 (Pergamon, 2013).
[12] & \quad M. S. Longuet-Higgins, “Mass Transport in Water Waves,” Phil. Trans. R. Soc. A 245, 535–581 (1953).
[13] & \quad M. S. Longuet-Higgins, “Mass transport in the boundary layer at a free oscillating surface,” J. Fluid Mech. 8, 293 (1960).
[14] & \quad K. D. Ruvinsky, F. I. Feldstein, and G. I. Freidman, “Numerical simulations of the quasi-stationary stage of ripple excitation by steep gravity-capillary waves,” J. Fluid Mech. 230, 339 (1991).
[15] & \quad D. D. Joseph and J. Wang, “The dissipation approximation and viscous potential flow,” J. Fluid Mech. 505, 365–377 (2004).
[16] & \quad J. Wang and D. D. Joseph, “Purely irrotational theories of the effect of the viscosity on the decay of free gravity waves,” J. Fluid Mech. 559, 461 (2006).
[17] & \quad J. C. Padrino and D. D. Joseph, “Correction of Lamb’s dissipation calculation for the effects of viscosity on capillary-gravity waves,” Phys. Fluids 19, 082105 (2007).
[18] & \quad B. J. West, K. A. Braeckner, R. S. Janda, D. M. Milller, and R. L. Milton, “A new numerical method for surface hydrodynamics,” J. Geophys. Res. 92, 11803 (1987).
[19] & \quad D. G. Dommermuth and D. K. P. Yue, “A high-order spectral method for the study of nonlinear gravity waves,” J. Fluid Mech. 184, 267 (1987).
[20] & \quad G. Wu, Y. Liu, and D. K. P. Yue, “A note on stabilizing the BenjaminFeir instability,” J. Fluid Mech. 556, 45 (2006).
[21] & \quad F. Dias, A. I. Dyachenko, and Vladimir E. Zakharov, “Theory of weakly damped free-surface flows: A new formulation based on potential flow solutions,” Phys. Lett. A 372, 1297–1302 (2008).
[22] & \quad B. M. Lake, H. C. Yuen, H. Rungaldier, and W. E. Ferguson, “Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train,” J. Fluid Mech 83, 49–74 (1977).
[23] & \quad A. I. Dyachenko and V. E. Zakharov, “Compact equation for gravity waves on deep water,” JETP Lett. 93, 701–705 (2011).
[24] & \quad A. I. Dyachenko, D. I. Kachulin, and V. E. Zakharov, “Super compact equation for water waves,” J. Fluid Mech. 828, 661–679 (2017).
\end{align*}\]
[25] M. S. Longuet-Higgins, “Mass transport in the boundary layer at a free oscillating surface,” J. Fluid Mech. 8, 293–306 (1960).
[26] M. S. Longuet-Higgins, “Theory of weakly damped Stokes waves: a new formulation and its physical interpretation,” J. Fluid Mech. 235, 319–324 (1992).
[27] D. D. Joseph, “Viscous potential flow,” J. Fluid Mech. 479, 191–197 (2003).
[28] A. Prosperetti, “Viscous effects on small-amplitude surface waves,” Phys. Fluids 19, 195 (1976).
[29] J. Wang and D. D. Joseph, “Purely irrotational theories of the effect of the viscosity on the decay of free gravity waves,” J. Fluid Mech. 559, 461 (2006).
[30] S. Amiranashvili, U. Bandelow, and A. Mielke, “Padé approximant for refractive index and nonlocal envelope equations,” Opt. Comm. 283, 480–485 (2010).
[31] J. Touboul and C. Kharif, “Nonlinear evolution of the modulational instability under weak forcing and damping,” Nat. Hazards Earth Syst. Sci. 10, 2589–2597 (2010).
[32] C. Kharif and J. Touboul, “Under which conditions the Benjamin-Feir instability may spawn an extreme wave event: A fully nonlinear approach,” Eur. Phys. J. Spec. Top. 185, 159–168 (2010).
[33] J. D. Carter and A. Govan, “Frequency downshift in a viscous fluid,” Eur. J. Mech. B 59, 177–185 (2016).
[34] D. Eeltink, A. Lemoine, H. Branger, O. Kimmoun, C. Kharif, J. D. Carter, A. Chabchoub, M. Brunetti, and J. Kasparian, “Spectral up- and downshifting of Akhmediev breathers under wind forcing,” Phys. Fluids 29, 107103 (2017).
[35] D. Eeltink, “Private communication,” (2018).
[36] G. P. Agrawal, Nonlinear Fiber Optics, fifth edit ed. (Academic Press, Oxford, 2012) p. 648.
[37] A. Armaroli, D. Eeltink, M. Brunetti, and J. Kasparian, “Nonlinear stage of Benjamin-Feir instability in forced/damped deep-water waves,” Phys. Fluids 30, 017102 (2018).
[38] C. Skandrani, C. Kharif, and J. Poitevin, “Nonlinear evolution of water surface waves: the frequency downshift phenomenon,” in Mathematical problems in the theory of water waves (Luminy, 1995), Contemp. Math., Vol. 200 (Amer. Math. Soc., Providence, RI, 1996) pp. 157–171.
[39] F. Dias and C. Kharif, “Nonlinear gravity and capillary-gravity waves,” Annual Rev. Fluid Mech. 31, 301–346 (1999).
[40] A. L. Fabrikant, “On nonlinear water waves under a light wind and Landau type equations near the stability threshold,” Wave Motion 2, 355–360 (1980).
[41] Y. Kato and M. Oikawa, “Wave Number Downshift in Modulated Wavetrain through a Nonlinear Damping Effect,” J. Phys. Soc. Japan 64, 4660–4669 (1995).
[42] C. M. Schober and M. Strawn, “The effects of wind and nonlinear damping on rogue waves and permanent downshift,” Physica D 313, 81–98 (2015).