Geometric Inequalities and Trapped Surfaces in Higher Dimensional Spacetimes

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Abstract
Geometric inequalities of classical differential geometry are used to extend to higher dimensional spacetimes the Penrose-Gibbons isoperimetric inequality and the hoop conjecture of general relativity.

1 Introduction

Higher dimensional spacetimes is now a common ingredient to most theories trying to unify gravity with the other forces of nature. While in the earlier works the extra-dimensions were compact with a size comparable with the Planckian scale, some recent models consider large [1] or even infinite [2] extra dimensions. In the so-called brane-world models the standard fields are confined to a four-dimensional timelike hypersurface (the brane) embedded in the higher dimensional spacetime (the bulk) where only gravity can propagate. Black holes being the gravitational solitons can be either attached to the brane or move in the bulk space. The higher dimensional generalization of the vacuum black hole solutions were obtained long time ago by Tangherlini [3] for a non-rotating black hole and by Myers and Perry [4] for a rotating one. With the development of those models black holes in higher dimensional spacetimes have come to play a fundamental role and have received much attention, see e.g. [5]–[9] and references therein. Another interesting consequence of large extra-dimensions is the possibility of production of mini black holes in high energy experiments such as collision of ultrarelativistic particles in future colliders or in cosmic rays, see e.g. [10] [11] [12] and references therein. The purpose of this paper...
is to use geometric inequalities of classical differential geometry to derive conditions for
the formation of trapped surfaces and horizons in spacetimes with dimensions $D \geq 4$.

In ordinary general relativity where $D = 4$ two inequalities play an important role in
the formation of horizons during a gravitational collapse. The first one is the so-called
Gibbons-Penrose “isoperimetric” inequality \[13], \[14]
\[ \mathcal{A} \leq 4\pi (2GM)^2. \] (1.1)
It gives a relation between the area $\mathcal{A}$ of a trapped surface formed during the collapse and
the mass $M$ of the resulting black hole. The second inequality arises in the formulation
of the hoop conjecture \[15\] which states that black hole with horizons form when and only
when a mass $M$ gets compacted into a region whose circumference $C$ in every direction is
bounded by
\[ C \leq 2\pi (2GM). \] (1.2)
The formulation of the hoop conjecture is rather vague as the type of horizons is not
specified and various interpretations can be given to the mass and the circumference.

In an attempt to produce counter-examples to the cosmic censorship conjecture, Pen-
rose \[13\] considered a convex thin shell collapsing at the speed of light from infinity, in
an initially flat spacetime, and he pointed that consistency with conventional theory de-
manded the validity of (1.1). Gibbons \[14\] noticed the connection between (1.1) and the
geometric Minkowski inequality \[16\]
\[ 4\pi \mathcal{A} \leq Q^2, \] (1.3)
which holds for any convex domain of $\mathbb{R}^3$, $\mathcal{A}$ being the area of the boundary of this convex
and $Q$ its total mean curvature -see \[17\] for extension to the case of non-convex domains.
In the Penrose construction the mean curvature is shown to be equal to $Q = 8\pi GM$ at
the moment when the surface of the collapsing shell gets trapped. Once this value of $Q$ is
introduced into (1.3) the inequality (1.1) immediately follows. In this approach $M$ is the
Bondi advanced mass (also equal to the ADM mass and to the Hawking quasilocal mass)
and is conserved during the collapse.

The Penrose construction has also been used to give a more precise formulation to the
hoop conjecture. Calling $l$ the maximum circumference of plane projections of the convex
shell, and $L$ the maximum circumference $L$ of plane curves drawn on the surface of the
shell, the following inequalities
\[ \pi L \leq 16\pi GM \leq 4l, \] (1.4)
were derived in \[18\]. Tod \[19\] found the slightly different result
\[ \pi l \leq 16\pi GM \leq 4l. \] (1.5)
which provides a stronger condition for the lower bound as $L \leq l$ for any convex. Accor-
ding to say, (1.4) the necessary condition for an apparent horizon to form is that the mass
$M$ gets compacted into a region such that $L \leq 16GM$ Some examples satisfying this
inequality were described in \[21\], and a numerical analysis of axisymmetric distributions
by Chiba et al \[22\] led to the condition $C \leq 15.8GM$ (their definition of $C$ does not
exactly coincide with $L$). Equality to the upper bound in (1.4) occurs for a sphere, and
the lower bound is approached when a cylinder ended by two hemispherical caps collapses
toward its axis of symmetry to form a spindle singularity.
2 Gibbons-Penrose inequality in higher dimensional spacetimes

Before extending the area inequality (1.1) and the hoop conjecture inequalities (1.4) to higher dimensional spacetimes let us briefly describe the Penrose-construction in a $D$-dimensional spacetime - for a detailed derivation in the 4-dimensional case see [23]. We consider a $(D-2)$-dimensional convex thin shell which collapses at the speed of light from infinity in an initially flat $D$-dimensional spacetime$^1$. The history of the shell is a null hypersurface whose interior geometry remains flat as long as the shell stays convex. As the shell implodes outgoing light rays emerging from the interior of the shell get more and more focussed when they cross the shell surface. A trapped surface momentarily coincident with the shell forms when the expansion rate of the outgoing light rays vanishes after crossing the shell. Integrating the Raychaudhuri equation for the outgoing light rays, accross the null hypersurface at the moment of formation of the trapped surface, one obtains the following relation

$$K = 16\pi G_D \sigma,$$

(2.1)

between the the extrinsic curvature of the $(D-2)$-surface of the shell (calculated in the interior Euclidean $(D-1)$-space), and the surface energy-density $\sigma$ of the shell. Here $G_D$ is the gravitational constant of the $D$-dimensional spacetime. It is worth mentioning that the main advantage of the Penrose construction is that it allows to derive the relation (2.1) without knowing the spacetime geometry to the future of the imploding null shell. The total mean curvature of the convex surface of the shell which is defined as

$$Q = \frac{1}{D-2} \int K \, dS,$$

(2.2)

becomes using (2.1)

$$Q = \frac{16\pi}{D-2} G_D M,$$

(2.3)

where $M$ is the Bondi advanced mass of the shell which is conserved during the collapse and is also equal to the $D$-dimensional analogues of the ADM mass and the Hawking quasi-local mass.

The generalization of the Gibbons-Penrose inequality (1.1) to $D$-dimensional spacetimes follows from the generalized Minkowski inequality (see [16] p.212) which states that for any closed convex $m$-dimensional surface immersed in $\mathbb{R}^n$, with $2 \leq m < n$, its area $A_m$ satisfies

$$s_m (A_m)^{m-1} \leq Q^m,$$

(2.4)

where $Q$ is the total mean curvature of the surface, and $s_m$ is the area of the unit $m$-sphere

$$s_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

(2.5)

$^1$It should be emphasized that in the higher dimensional case there is no analogue of the uniqueness theorem. In particular, one cannot exclude that the topology of a higher dimensional black hole differs from the topology of the sphere. A black ring solution [5] in 5-dimensional case with the topology of the horizon $S^2 \times S^1$ is one of the examples. In the general case the existence of such solutions and their stability is an open question [6]. In our approach we consider only the black holes having spherical topology for the horizon.
Γ(x) being the Euler function. If one applies (2.4) to the imploding convex shell of the Penrose-construction, then \( m = D - 2 \), and using the expression (2.3) for \( Q \) one obtains

\[
 s_{D-2} (A_{D-2})^{D-3} \leq \left( \frac{16\pi G_D M}{D - 2} \right)^{D-2}.
\] (2.6)

This inequality puts an upper bound on the area \( A_{D-2} \) of any trapped surface (this includes of course the apparent horizon) in terms of the mass \( M \) in a spacetime with \( D \geq 3 \) dimensions. When \( D = 4 \) one recovers the Gibbons-Penrose inequality of general relativity, \( A_2 \leq 4\pi (2GM)^2 \), and for say, \( D = 5 \) one gets

\[
 A_3 \leq \frac{32}{3} \left( \frac{2\pi}{3} \right)^{1/2} (G_5 M)^{3/2}.
\] (2.7)

The dimension of the trapped surface increases with the dimension of spacetime, recall that \( (A_{D-2})^{D-3} \sim (G_D M)^{D-2} \sim (\text{length})^{(D-2)(D-3)} \). Inequalities involving the area of the boundary of plane sections of the trapped surface with lower dimensions cannot be obtained from the method followed here. One could introduce the maximum area \( \Sigma_{\text{max}} \) of section of a convex of \( \mathbb{R}^n \) with an hyperplane by using the inequality \( A_{n-1} \geq (b_n/b_{n-1}) \Sigma_{\text{max}} \), where \( b_n \) is the volume of the unit \( n \)-ball (see [16] p.152). However this would not lower the exponents appearing in (2.6) as \( \Sigma_{\text{max}} \) has the same dimension as \( A_{n-1} \), which corresponds here to \( A_{D-2} \) as \( n = D - 1 \).

The Schwarzschild radius \( r_H \) of a spherically symmetric \( D \)-dimensional black hole with mass \( M \) is equal to [4]

\[
 r_H = \left[ \frac{16\pi G_D M}{(D - 2)s_{D-2}} \right]^{1/2}.
\] (2.8)

Using this relation the inequality (2.6) takes the familiar form

\[
 A_{D-2} \leq s_{D-2} r_H^{D-2},
\] (2.9)

from which one immediately sees that equality occurs in the spherical case. Another consequence of (2.6) is that it can be used to obtain an upper bound for the energy \( E \) emitted as gravitational radiation during the collapse of a mass \( M \). If cosmic censorship holds and if a black hole is formed, then the following quantity

\[
 E_{\text{max}} = M - \frac{(D - 2)(s_{D-2})^{D-2}}{16\pi G_D} (A_{D-2})^{D-3},
\] (2.10)

is always positive and yields the upper bound for \( E \). In the case of brane-world models it is known that the gravitational radiation will be emitted in the bulk as only gravitons can propagate in the bulk.

3 Hoop inequalities in higher dimensional spacetimes

Let us now consider the generalization to \( D \)-dimensional spacetimes of the inequalities (1.4) associated with the hoop-conjecture. The following proposition will be used:

\textit{Let} \( D \) \textit{be a convex domain of} \( \mathbb{R}^n \) \textit{and} \( Q \) \textit{the total mean curvature of the boundary} \( \partial D \) \textit{of} \( D \). \textit{Let} \( \omega_{n-2} \) \textit{be the maximum area of the boundary of its orthogonal hyperplane projections}
and $\Omega_{n-2}$ the maximum area of $(n-2)$-dimensional sections of $\partial D$ by hyperplanes. Then the total mean curvature satisfies the following inequalities

$$\frac{s_n}{2s_{n-2}} \Omega_{n-2} \leq Q \leq \frac{s_{n-1}}{s_{n-2}} \omega_{n-2}.$$  \hspace{1cm} (3.1)

**Proof:** Let $K$ be a convex domain embedded in $\mathbb{R}^n$, and $\partial K$ its boundary. Let $V_n(K)$ be the volume of $K$ and $A_{n-1}(\partial K)$ the area $\partial K$. We define the $t$-expanded domain $K_t$ associated with $K$ as $K_t = \{ x \in \mathbb{R}^n \mid d(x, K) \leq t \}$ where $d$ is the Euclidean distance. The total mean curvature $Q$ of $\partial K$ is such that

$$Q = \frac{1}{n-1} \frac{dA_{n-1}(\partial K_t)}{dt} \bigg|_{t=0}. \hspace{1cm} (3.2)$$

The Cauchy formula, [16] p. 142, provides a relation between the area $A_{n-1}(\partial K)$ of the boundary $\partial K$, and the mean volume $V_{n-1}(p_\xi(K))$ of the orthogonal plane projections of $K$ in arbitrary directions

$$A_{n-1}(\partial K) = \frac{1}{b_{n-1}} \int_{\xi \in S_{n-1}} V_{n-1}(p_\xi(K)) \, d\xi, \hspace{1cm} (3.3)$$

where $b_n$ is the volume of the unit $n$-ball, $\xi$ is a unit vector, $S_{n-1}$ is the unit-sphere and $p_\xi$ indicates the projection in the direction of $\xi$ onto a hyperplane orthogonal to $\xi$. The integral in (3.3) is taken over the unit-sphere $S_{n-1}$.

One now applies the Cauchy formula to the $t$-expanded convex domain $K_t$ and notices that $p_\xi(K_t) = (p_\xi(K))_t$. Then, using the following property

$$\frac{dV_{n-1}(p_\xi(K))}{dt} \bigg|_{t=0} = A_{n-2}(\partial p_\xi(K)), \hspace{1cm} (3.4)$$

where the r.h.s. of this equation represents the area of the boundary of the projection of $K$, one gets

$$Q = \frac{1}{s_{n-2}} \int_{\xi \in S_{n-1}} A_{n-2}(\partial p_\xi(K)) \, d\xi. \hspace{1cm} (3.5)$$

The relation $s_{n-1} = n b_n$ between the area $s_{n-1}$ of the unit $(n-1)$-sphere and the volume $b_n$ of the unit $n$-ball has been used.

In order to derive the upper bound of (3.1) one denotes $\omega_{n-2} \equiv \sup_\xi [A_{n-2}(\partial p_\xi(K))]$, the maximum area of the boundary of the hyperplane projections of $K$. Then the integration in (3.5) yields

$$Q \leq \frac{s_{n-1}}{s_{n-2}} \omega_{n-2}. \hspace{1cm} (3.6)$$

For the lower bound of (3.1) one considers the intersection of $K$ with an arbitrary hyperplane $\Pi$. Let $\Sigma_\Pi$ be the closed plane $(n-2)$-dimensional domain resulting from the intersection of $\Pi$ with the boundary $\partial K$ of $K$. The following property [24] applies to any compact $q$-dimensional manifold $V$ embedded in $\mathbb{R}^{p+q}$ with $p \geq 2$

$$A_q(V) = \frac{s_p}{s_{p-1}s_{p+q}} \int_{\xi \in S_{p+q-1}} A_q(p_\xi(V)) \, d\xi. \hspace{1cm} (3.7)$$
where, as above $p_\xi$ indicates the projection in the direction of the unit vector $\xi$ onto a hyperplane orthogonal to $\xi$. Applying this property to $V = \Sigma_\Pi$ with $q = n - 2$ and $p + q = n$, one gets

$$A_{n-2}(\Sigma_\Pi) = \frac{2}{s_{n-2}} \int_{\xi \in S_{n-1}} A_{n-2}(p_\xi(\Sigma_\Pi)) d\xi. \quad (3.8)$$

As all the projections of $\Sigma_\Pi$ are boundaries of convex domains which are contained within the projection of $\partial K$, one has $A_{n-2}(p_\xi(\Sigma_\Pi)) \leq A_{n-2}(p_\xi(\partial K))$. Introducing this into (3.8) and using (3.5) one gets

$$\frac{s_n}{2} A_{n-2}(\Sigma_\Pi) \leq s_{n-2} Q. \quad (3.9)$$

This relation holds for any hyperplane $\Pi$ intersecting the convex domain $K$. Calling $\Omega_{n-2} \equiv \sup_\Pi [A_{n-2}(\Sigma_\Pi)]$, the maximum area of plane $(n-2)$-dimensional sections of $\partial K$ by hyperplanes, one obtains

$$\frac{s_n}{2 s_{n-2}} \Omega_{n-2} \leq Q. \quad (3.10)$$

The two results (3.6) and (3.10) provides the two geometric inequalities appearing (3.1) Q.E.D.

The following relations derived from (2.5)

$$\frac{s_n}{s_{n-2}} = \frac{2\pi}{n-1}; \quad \frac{s_{n-1}}{s_{n-2}} = \sqrt{\pi} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \quad (3.11)$$

could be used to present the proved inequalities in the form

$$\frac{\pi}{n-1} \Omega_{n-2} \leq Q \leq \sqrt{\pi} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \omega_{n-2}. \quad (3.12)$$

### 3.1 Hoop of dimension $D - 3$

Let us apply the inequalities (3.1) to an imploding null shell of the Penrose construction in a $D$-dimensional spacetime. For this case $n = D - 1$ and $Q$ is given by (2.3). It immediately follows that

$$\frac{s_{D-1}}{2 s_{D-3}} \Omega_{D-3} \leq \frac{16\pi}{D - 2} G_D M \leq \frac{s_{D-2}}{s_{D-3}} \omega_{D-3}, \quad (3.13)$$

where $\Omega_{D-3}$ and $\omega_{D-3}$ are defined above. The left inequality in (3.13) implies that a necessary condition for the formation of a trapped surface is that a convex body with mass $M$ gets compacted in a region such that the largest area $\Omega_{D-3}$ of a plane closed $D - 3$-surface drawn on its boundary satisfies, using (3.11)

$$\Omega_{D-3} \leq 16 G_D M, \quad (3.14)$$

which has the same form as the corresponding condition $L \leq 16 GM$ of general relativity. A similar result was proposed by [20], see their equation (53), and examples in 5-dimensional spacetimes were considered.
The right inequality in (3.13) yields a sufficient condition and it states that a trapped surface forms before the mass $M$ gets compacted into a region whose orthogonal plane projections have a maximum area satisfying

$$\omega_{D-3} \leq \frac{16\pi s_{D-3}}{(D-2)s_{D-2}} G_D M, \quad (3.15)$$

In terms of the Schwarzschild radius $r_H$ defined in (2.8) the inequalities (3.13) becomes

$$\frac{\pi}{D-2} \Omega_{D-3} \leq s_{D-3} r_H^{D-3} \leq \omega_{D-3}, \quad (3.16)$$

which shows, as in general relativity, that equality for the upper bound occurs in the case of a sphere.

The inequalities (3.1) and (3.13) generalize to spacetimes with $D$ dimensions the relations (2) and (11) of [18] which were valid in general relativity. Let us mention that Tod’s approach [19] does not seem to be straightforwardly generalizable to $\mathbb{R}^n$ with $n \geq 4$, i.e. to spacetimes with dimension $D \geq 5$. As $G_D M \sim \Omega_{D-3} \sim \omega_{D-3} \sim (\text{length})^{D-3}$, what was referred to as a hoop in general relativity becomes in fact a closed $(D-3)$-dimensional strip in $D$-dimensional spacetimes. In the brane-world models only one of the $D-3$ dimensions of this strip belongs to the brane and the $D-4$ remaining ones correspond to the bulk.

### 3.2 Hoop of dimension $D - 4$

Introducing the circumference $C$ of a curve and writing an inequality of the form $C \leq 2\pi r_H$ giving a necessary and sufficient condition to form horizons, as proposed in some works [23], is not possible within our approach. One can however obtain for the upper bound of (3.1) an expression involving the area of a $(D - 4)$-dimensional surface by repeating the projection procedure used in the derivation of (3.4), and by using the isoperimetric inequality. Once the orthogonal projection $p_\xi$ of convex domain $K$ is performed, the convex domain $p_\xi(p_\xi(K))$ is all over orthogonally projected in the direction of a new vector $\chi$ onto an hyperplane lying inside the hyperplane used in the $p_{xi}$ projection. The domain $p_\chi(p_\xi(K))$ obtained after these projections is a $(n-2)$-dimensional domain which is also convex. The Cauchy formula (3.3) then gives

$$A_{n-2}(\partial p_\xi(K)) = \frac{1}{b_{n-2}} \int_{\chi \in S_{n-2}} V_{n-2}(p_\chi(p_\xi(K))) d\chi, \quad (3.17)$$

and introducing this result into (3.5) one obtains for the total mean curvature

$$Q = \frac{n-2}{s_{n-2}s_{n-3}} \int_{\xi \in S_{n-1}} \int_{\chi \in S_{n-2}} V_{n-2}(p_\chi(p_\xi(K))) d\chi d\xi. \quad (3.18)$$

The next step is to apply to the domain $p_\chi(p_\xi(K))$ the isoperimetric inequality [16] which states that between the volume $V_n$ and the area $A_{n-1}$ of any convex of $\mathbb{R}^n$ one has

$$n^n b_n V_n^{n-1} \leq A_{n-1}^n. \quad (3.19)$$

One then gets a new upper bound for the total mean curvature $Q$

$$Q \leq \frac{1}{s_{n-2}(s_{n-3})^{\frac{n-2}{n-3}}} \int_{\chi \in S_{n-2}} \int_{\xi \in S_{n-1}} A_{n-3}(\partial(p_\chi(p_\xi(K))))^{\frac{n-2}{n-3}} d\chi d\xi. \quad (3.20)$$
If one denotes by $\sigma_{n-3} \equiv \sup_{\chi,\xi} A_{n-3}(\partial(p_\chi(p_\xi(K))))$, the maximum area of the boundary of two successive hyperplane projections of $K$, then the integration in (3.20) yields

$$Q \leq s_{n-1} \left( \frac{\sigma_{n-3}}{s_{n-3}} \right)^{\frac{n-3}{n-1}}.$$  \hspace{1cm} (3.21)

Using the expression (2.3) for $Q$ and making $n = D - 1$ one obtains another expression for the upper bound of (3.1) which now involves the area of a $(D - 4)$-dimensional surface. It is worth noticing that this procedure only works for the upper bound and cannot be translated to rewrite the lower bound of (3.1) in a similar manner. Also it can be easily noticed that the procedure of successive projections cannot be iterated more than twice as the the Cauchy formula can no longer be used after two projections. In terms of the Schwarzschild radius $r_H$ the upper bound can be rewritten into the simple form

$$s_{D-4} (r_H)^{D-4} \leq \sigma_{D-4},$$  \hspace{1cm} (3.22)

which can be easily compared with the right inequality in (3.16).

As an example let us consider the case where $D = 5$. In that case the relation (3.13) gives

$$\Omega_2 \leq 16 G_5 M \leq \frac{3}{2} \omega_2,$$  \hspace{1cm} (3.23)

as $s_1 = 2\pi$, $s_2 = 4\pi$, $s_3 = 2\pi^2$, and $s_4 = 8\pi^2/3$. Suppose now that this situation applies to a brane-world model, and that the 2-surface with area $\Omega_2$ has the form of an ellipsoid. Then one can write, omitting a factor of order unity, $\Omega_2 \simeq L l_5$, where $L$ corresponds to the size of the part of the ellipsoid lying on the brane and $l_5$ in the bulk. The necessary condition to form an apparent horizon is that the product $L l_5$ be small enough, i.e. the larger (smaller) $l_5$ will be the smaller (larger) $L$ will have to be in order to satisfy the inequality $\Omega_2 \leq 16 G_5 M$. Also when $D = 5$ two successive projections, as used in (3.21), yields a two dimensional domain whose perimeter has a maximum value $\sigma_1$ which satisfies the inequality $2\pi r_H \leq \sigma_1$. This provides as mentioned earlier a sufficient condition to form a trapped surface. The condition takes a form which is apparently similar to the inequality used in the hoop conjecture of general relativity, except that we have now for the Schwarzschild radius, $r_H^2 = 8 G_5 M / 3\pi$ instead of $r_H = 2 G M$.

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