Abstract

In this article, we study the following question raised by Mel Hochster: let $(R, m, K)$ be a local ring and $S$ be a flat extension with regular closed fiber. Is $\mathcal{V}(mS) \cap \text{Ass}_SH_i^I(S)$ finite for every ideal $I \subset S$ and $i \in \mathbb{N}$? We prove that the answer is positive when $S$ is either a polynomial or a power series ring over $R$ and $\dim(R/I \cap R) \leq 1$. In addition, we analyze when this question can be reduced to the case where $S$ is a power series ring over $R$. An important tool for our proof is the use of $\Sigma$-finite $D$-modules, which are not necessarily finitely generated as $D$-modules, but whose associated primes are finite. We give examples of this class of $D$-modules and applications to local cohomology.

1 Introduction

Throughout this article $A, R$ and $S$ will always denote commutative Noetherian rings with unit. If $M$ is an $S$-module and $I \subset S$ is an ideal, we denote the $i$-th local cohomology of $M$ with support in $I$ by $H_i^I(M)$. The structure of these modules has been widely studied by several authors [5, 9, 11, 12, 13, 16, 17]. Among the results obtained is that the set of associated primes of $H_i^I(R)$ is finite for certain regular rings. Huneke and Sharp proved this for characteristic $p > 0$ [6]. Lyubeznik showed this finiteness property for regular local rings of equal characteristic zero and finitely generated regular algebras over a field of characteristic zero [8]. We point out that this property does not necessarily hold for ring that are not regular [7 [19]. Motivated by these finiteness results, Mel Hochster raised the following related questions:
**Question 1.1.** Let \((R, m, K)\) be a local ring and \(S\) be a flat extension with regular closed fiber. Is

\[
\text{Ass}_SH^0_{mS}(H^i_I(S)) = \mathcal{V}(mS) \cap H^i_I(S)
\]

finite for every ideal \(I \subset S\) and \(i \in \mathbb{N}\)?

**Question 1.2.** Let \((R, m, K)\) be a local ring and \(S\) denote either \(R[x_1, \ldots, x_n]\) or \(R[[x_1, \ldots, x_n]]\). Is

\[
\text{Ass}_SH^0_{mS}(H^i_I(S)) = \mathcal{V}(mS) \cap H^i_I(S)
\]

finite for every ideal \(I \subset S\) and \(i \in \mathbb{N}\)?

It is clear that Question 1.2 is a particular case of Question 1.1. In Proposition 6.2, we show that under minor additional hypothesis these questions are equivalent. Question 1.2 has a positive answer when \(R\) is a ring of dimension 0 or 1 of any characteristic \([15]\). In her thesis \([18]\), Robbins answered Question 1.2 positively for certain algebras of dimension smaller than or equal to 3 in characteristic 0. In addition, several of her results can be obtained in characteristic \(p > 0\), by working in the category \(C(S, R)\) (see the discussion after Remark 2.1). In addition, a positive answer for Question 1.1 would help to that the associated primes of local cohomology modules, \(H^i_I(R)\), over certain regular local rings of mixed characteristic, \(R\). For example,

\[
V[[x, y, z_1, \ldots, z_n]]/(\pi - xy)V[[x, y, z_1, \ldots, z_n]] = \left(\frac{V[[x, y]]}{\pi - xyV[[x, y]]}\right)[[z_1, \ldots, z_n]],
\]

where \((V, \pi V, K)\) is a complete DVR of mixed characteristic. This is, to the best of our knowledge, the simplest example of a regular local ring of ramified mixed characteristic that the finiteness of \(\text{Ass}_R H^j_I(R)\) is unknown.

In this manuscript, we give a partial positive answer for Question 1.1 and 1.2. Namely:

**Theorem 1.3.** Let \((R, m, K)\) be any local ring. Let \(S\) denote either \(R[x_1, \ldots, x_n]\) or \(R[[x_1, \ldots, x_n]]\). Then, \(\text{Ass}_SH^0_{mS}(H^i_I(S))\) is finite for every ideal \(I \subset S\) such that \(\dim R/I \cap R \leq 1\) and every \(i \in \mathbb{N}\). Moreover, if \(mS \subset \sqrt{I}\),

\[
\text{Ass}_SH^j_{J_1} \cdots H^j_{J_t} H^i_I(S)
\]

is finite for all ideals \(J_1, \ldots, J_t \subset S\) and integers \(j_1, \ldots, j_t \in \mathbb{N}\).

**Theorem 1.4.** Let \((R, m, K) \to (S, \eta, L)\) be a flat extension of local rings with regular closed fiber such that \(R\) contains a field. Let \(I \subset S\) be an ideal such that \(\dim R/I \cap R \leq 1\). Suppose that the morphism induced in the completions \(\hat{R} \to \hat{S}\) maps a coefficient field of \(R\) into a coefficient field of \(S\). Then,

\[
\text{Ass}_SH^0_{mS}(H^i_I(S))
\]
is finite for every \( i \in \mathbb{N} \).

In Theorem 1.4 the hypothesis that \( \hat{\varphi} \) maps a coefficient field of \( \hat{R} \) to a coefficient field of \( \hat{S} \) is not very restrictive. For instance, it is satisfied when \( L \) is a separable extension of \( K \) \cite[Theorem 28.3]{SR}. In particular, this holds when \( K \) is a field of characteristic 0 or a perfect field of characteristic \( p > 0 \).

A key part of the proof of Theorem 1.3 is the use of \( \Sigma \)-finite \( D \)-modules, which are directed unions of finite length \( D \)-modules that satisfy certain condition (see Definition 3.2). One of the main properties that a \( \Sigma \)-finite \( D \)-module satisfies is that its set of associated primes is finite. In addition, the local cohomology of a \( \Sigma \)-finite \( D \)-module is again \( \Sigma \)-finite. Proving that the local cohomology modules supported on \( H^i_{mS}H^j_I(S) \) would answer Question 1.1.

This manuscript is organized as follows. In section 2 we recall some definitions and properties of local cohomology and \( D \)-modules. Later, in Section 3 we define \( \Sigma \)-finite \( D \)-modules and give their first properties. In Section 4 we prove that certain local cohomology modules are \( \Sigma \)-finite; as a consequence, we give a proof of Theorem 1.3. Later, in Section 5 we give several examples of \( \Sigma \)-finite \( D \)-modules. Finally, in Section 6 we prove that under the certain hypothesis Question 1.1 and 1.2 are equivalent. In addition, we prove Theorem 1.4.

\section{Preliminaries}

\subsection{Local cohomology}

Let \( R \) be a ring, \( I \subset R \) an ideal, and \( M \) an \( R \)-module. If \( I \) is generated by \( f_1, \ldots, f_s \in R \), the local cohomology group, \( H^i_I(M) \), can be computed using the \( \check{\text{C}} \)ech complex, \( \check{\mathcal{C}}(f; M) \),

\[ 0 \to M \to \bigoplus_j M_{f_j} \to \cdots \to M_{f_1 \cdots f_s} \to 0. \]

Let \( \mathcal{K}(f_1, \ldots, f_s; M) \) denote the Koszul complex associated to the sequence \( \underline{f} = f_1, \ldots, f_s \). In Figure 2.1 there is a direct limit involving \( \mathcal{K}(f^i; M) \), whose limit is \( \check{\mathcal{C}}(f; M) \).

\begin{figure}[h]
\begin{center}
\begin{tikzpicture}
\node (M) at (0,0) {$M$};
\node (Mf1) at (1,0) {$M_{f_1}$};
\node (Mf2) at (2,0) {$M_{f_2}$};
\node (Mf3) at (3,0) {$M_{f_3}$};
\node (Mf4) at (4,0) {$M_{f_4}$};
\node (Mfn) at (5,0) {$M_{f_n}$};
\node (Mf) at (0,-1) {$M_{\underline{f}}$};
\node (Mf1) at (1,-1) {$M_{f_1}$};
\node (Mf2) at (2,-1) {$M_{f_2}$};
\node (Mf3) at (3,-1) {$M_{f_3}$};
\node (Mf4) at (4,-1) {$M_{f_4}$};
\node (Mfn) at (5,-1) {$M_{f_n}$};
\node (Mf) at (0,-2) {$M_{\underline{f}}$};

\draw[->] (M) -- (Mf1);
\draw[->] (Mf1) -- (Mf2);
\draw[->] (Mf2) -- (Mf3);
\draw[->] (Mf3) -- (Mf4);
\draw[->] (Mf4) -- (Mfn);
\draw[->] (Mfn) -- (Mf);
\draw[->] (M) -- (Mf);
\draw[->] (Mf1) -- (Mf1);
\draw[->] (Mf2) -- (Mf2);
\draw[->] (Mf3) -- (Mf3);
\draw[->] (Mf4) -- (Mf4);
\draw[->] (Mfn) -- (Mfn);
\draw[->] (Mf) -- (Mf);
\end{tikzpicture}
\end{center}
\caption{Direct limit of Koszul complexes}
\end{figure}

Let \( \underline{f}^i \) denote the sequence \( f_1^i, \ldots, f_s^i \). Since

\[ \mathcal{K}(\underline{f}^i; M) = \mathcal{K}(f_1; M) \otimes_R \cdots \otimes_R \mathcal{K}(f_s; M), \]
we have that
\[
\bar{\mathcal{C}}(f; M) = \bar{\mathcal{C}}(f_1; M) \otimes_S \ldots \otimes_S \bar{\mathcal{C}}(f_s; M)
= \lim_{\to t} \mathcal{K}(f_1^t; M) \otimes_S \ldots \otimes_S \lim_{\to t} \mathcal{K}(f_s^t; M)
= \lim_{\to t} \mathcal{K}(f_1^t; M) \otimes_S \ldots \otimes_S \mathcal{K}(f_s^t; M).
\]

We define the **cohomological dimension of** $I$ by

\[
\text{cd}_RI = \text{Max}\{i \mid H^i_R \neq 0\}.
\]

### 2.2 D-modules

Given two commutative rings $R$ and $S$ such that $R \subset S$, we define the **ring of R-linear differential operators of** $S$, $D(S, R)$, as the subring of $\text{Hom}_R(S, S)$ obtained inductively as follows. The differential operators of order zero are morphisms induced by multiplication by elements in $S$ ($\text{Hom}_S(S, S) = S$). An element $\theta \in \text{Hom}_R(S, S)$ is a differential operator of order less than or equal to $k+1$ if $\theta \cdot r - r \cdot \theta$ is a differential operator of order less than or equal to $k$ for every $r \in S$.

We recall that if $M$ is a $D(S, R)$-module, then $M_f$ has the structure of a $D(S, R)$-module such that, for every $f \in S$, the natural morphism $M \to M_f$ is a morphism of $D(S, R)$-modules. As a consequence, $H^0_{\bar{\mathcal{C}}} \cdots H^n_{\bar{\mathcal{C}}}(S)$ is also a $D(S, R)$-module [Examples 2.1].

**Remark 2.1.** If $(R, m, K)$ is a local ring and $S$ is either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$, then

\[
D(S, R) = R \left[ \frac{1}{t!} \frac{\partial^t}{\partial x_i^t} \mid t \in \mathbb{N}, 1 \leq i \leq n \right] \subset \text{Hom}_R(S, S)
\]

[3] Theorem 16.12.1]. Then, there is a natural surjection $\rho : D(S, R) \to D(S/1S, R/1R)$ for every ideal $I \subset R$. Moreover,

(i) If $M$ is a $D(S, R)$-module, then $IM$ is a $D(S, R)$-submodule and the structure of $M/IM$ as a $D(S, R)$-module is given by $\rho$, i.e.,

$\delta \cdot v = \rho(\delta) \cdot v$ for all $\delta \in D(S, R)$ and $v \in M/IM$.

(ii) If $R$ contains the rational numbers $D(S, R)$ is a Noetherian ring.

Let $\Gamma_i = \{\delta \in D(S, R) \mid \text{ord}(\delta) \leq i\}$. We have that $\text{gr}^f D = S[y_1, \ldots, y_n]$, which is Noetherian and then so $D$ is.

We recall a subcategory of $D(S, R)$-modules introduced by Lyubeznik [10]. We denote by $C(S, R)$ the smallest subcategory of $D(S, R)$-modules that contains $S_f$ for all $f \in S$ and that is closed under subobjects, extensions and quotients. In particular, the kernel, image and
cokernel of a morphism of $D(S, R)$-modules that belongs to $C(S, R)$ are also objects in $C(S, R)$. We note that if $M$ is an object in $C(S, R)$, then $H^1_{I_1} \cdots H^\ell_{I_\ell}(M)$ is also an object in this subcategory; in particular, $H^1_{I_1} \cdots H^\ell_{I_\ell}(S)$ belongs to $C(S, R)$ [10, Lemma 5].

A $D(S, R)$-module, $M$, is simple if its only $D(S, R)$-submodules are 0 and $M$. We say that a $D(S, R)$-module, $M$, has finite length if there is a strictly ascending chain of $D(S, R)$-modules, $0 \subset M_0 \subset M_1 \subset \ldots \subset M_h = M$, called a composition series, such that $M_{i+1}/M_i$ is a nonzero simple $D(S, R)$-module for every $i = 0, \ldots, h$. In this case, $h$ is independent of the filtration and it is called the length of $M$. Moreover, the composition factors, $M_{i+1}/M_i$, are the same, up to permutation and isomorphism, for every filtration.

**Notation 2.2.** If $M$ is a $D(S, R)$-module of finite length, we denote the set of its composition factors by $\mathcal{C}(M)$.

**Remark 2.3.**

(i) If $M$ is a nonzero simple $D(S, R)$-module, then $M$ has only one associated prime. This is because $H^0_P(M)$ is a $D(S, R)$-submodule of $M$ for every prime ideal $P \subset S$. As a consequence, if $M$ is a $D(S, R)$-module of finite length, then $\text{Ass}_S M \subseteq \bigcup_{N \in \mathcal{C}(M)} \text{Ass}_S N$, which is finite.

(ii) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $D(S, R)$-modules of finite length, then $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M'')$.

3 \quad $\Sigma$-finite $D$-modules

**Notation 3.1.** Thorough this section $(R, m, K)$ denotes a local ring and $S$ denotes either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. In addition, $D$ denotes $D(S, R)$.

**Definition 3.2.** Let $M$ be a $D$-module supported at $mS$ and $\mathcal{M}$ be the set of all $D$-submodules of $M$ that have finite length. We say that $M$ is $\Sigma$-finite if:

(i) $\bigcup_{N \in \mathcal{M}} N = M$,

(ii) $\bigcup_{N \in \mathcal{M}} \mathcal{C}(N)$ is finite, and

(iii) For every $N \in \mathcal{C}(M)$ and $L \in \mathcal{C}(N)$, $L \in C(S/mS, R/mR)$.

We denote the set of composition factors of $M$, $\bigcup_{N \in \mathcal{M}} \mathcal{C}(N)$, by $\mathcal{C}(M)$.

**Remark 3.3.** We have that

$$\text{Ass}_S M \subseteq \bigcup_{N \in \mathcal{C}(M)} \text{Ass}_S M$$

for every $\Sigma$-finite $D$-module, $M$. In particular, $\text{Ass}_S M$ is finite.
Lemma 3.4. Let $M$ be a $\Sigma$-finite $D$-module and $N$ be a $D$-submodule of $S$. Then, $N$ has finite length as $D$-module if and only if $N$ is a finitely generated as $D$-module.

Proof. Suppose that $N$ is finitely generated. Let $v_1, \ldots, v_\ell$ be a set the generators of $N$. Since $\bigcup_{N \in M} N = M$, there exists a finite length module $N_i$ that contains $v_i$. Then, $N \subset N_1 + \ldots + N_\ell$ and it has finite length. It is clear that if $N$ has finite length then it is finitely generated.

Proposition 3.5. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $D$-modules. If $M$ is $\Sigma$-finite, then $M'$ and $M''$ are $\Sigma$-finite. Moreover, $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M'')$.

Proof. We first assume that $M$ is $\Sigma$-finite. We have that $M' = \bigcup_{N \in M} N \cap M' = \bigcup_{N' \in M} N'$ and then $M$ is $\Sigma$-finite by Remark 2.3. Let $\rho$ denote the morphism $M \to M''$ and $N'' \in M''$ and $\ell = \text{length}_D N''$. There are $v_1, \ldots, v_\ell \in N''$ such that $N'' = D \cdot v_1 + \ldots + D \cdot v_\ell$. Let $w_j$ be the preimage of $v_j$ and $N$ be the $D$-module generated by $w_1, \ldots, w_\ell$. We have that $N \to N''$ is a surjection, and that $N$ has finite length by Lemma 3.4. Therefore, $M'' = \bigcup_{N \in M} \rho(N) = \bigcup_{N'' \in M''} N''$ and the result follows by Remark 2.3.

Proposition 3.6. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $D$-modules. Suppose that $R$ contains the rational numbers. Then, $M$ is $\Sigma$-finite if and only if $M'$ and $M''$ are $\Sigma$-finite. Moreover, $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M'')$.

Proof. We first assume that $M'$ and $M''$ are $\Sigma$-finite. Let $v \in M$. We have a short exact sequence

$$0 \to M' \cap D \cdot v \to D \cdot v \to D \cdot \bar{v} \to 0.$$  

$M' \cap D \cdot v$ is finitely generated because $D$ is Noetherian by Remark 2.1. Then, $M' \cap D \cdot v$ has finite length by Lemma 3.4 and so $D \cdot \bar{v}$ has finite length. Therefore, $M = \bigcup_{N \in M} N$. Let $N \in M$. Then, $N \cap M' \in M'$ and $\rho(N) \in M''$. We have a short exact sequence

$$0 \to N \cap M' \to N \to \rho(N) \to 0$$

of finite length $D$-modules, and then result follows by Remark 2.3.

The other direction follows from Proposition 3.5.
Proposition 3.7. Let $M$ be a $\Sigma$-finite $D$-module. Then, $M_f$ is $\Sigma$-finite for every $f \in S$.

Proof. Let $N \subset M_f$ be a module of finite length. We have that $N$ is a finitely generated $D$-module. Then there exists a finitely generated $D$-submodule $N'$ of $M$ such that $N \subset N'$. We have that $N'$ has finite length and $\mathcal{C}(N') = \bigcup_{V \in \mathcal{C}(N)} \mathcal{C}(V)$ because $V_f$ is in $C(S/mS, R/m)$ \[10\]. Then, $M_f = \bigcup_{N \subset M_f} N = \bigcup_{N \subset M} N_f = M_f$ and the result follows.

Lemma 3.8. Let $M$ and $M'$ be $\Sigma$-finite $D$-modules. Then, $M \oplus M'$ is also $\Sigma$-finite.

Proof. It is clear that $M \oplus M'$ is supported on $mS$. For every $(v, v') \in M \oplus M'$, there exist $N$ and $N'$, $D$-modules of finite length, such that $v \in N$ and $v' \in N'$. Then, $N \oplus N' \subset M \oplus M'$ has finite length and $(v, v') \in N \oplus N'$. Therefore, $\bigcup_{N \subset M, N' \subset M'} N \oplus N' = M \oplus M'$, and the $M \oplus M'$ is union of its $D$-modules of finite length. The rest follows from Remark 2.3.

Corollary 3.9. Let $M$ be a $\Sigma$-finite $D$-module. Then, $H^i_I(M)$ is $\Sigma$-finite for every ideal $I \subset S$ and $i \in \mathbb{N}$.

Proof. Let $f_1, \ldots, f_\ell$ be generators for $I$. We have that $\mathcal{C}(f_i; M)$ is $\Sigma$-finite by Lemma 3.8. Then $H^i_I(M)$ is also $\Sigma$-finite by Proposition 3.5.

Proposition 3.10. Let $M_t$ be an inductive direct system of $\Sigma$-finite $D$-modules. If $\bigcup_t \mathcal{C}(M_t)$ is finite, then $\lim_{\rightarrow t} M_t$ is $\Sigma$-finite and $\mathcal{C}(M) \subset \bigcup_t \mathcal{C}(M_t)$.

Proof. Let $M = \lim_{\rightarrow t} M_t$ and $\phi_t : M_t \to M$ the morphism induced by the limit. We have that $\phi_t(M_t)$ is a $\Sigma$-finite $D$-module by Proposition 3.5. We may replace $M_t$ by $\phi_t(M_t)$ by Remark 2.3 and assume that $M = \bigcup M_t$ and $M_t \subset M_{t+1}$. If $N \subset M$ has finite length as $D$-module, then it is finitely generated and there exists a $t$ such that $N \subset M_t$. Therefore, $M = \bigcup_t M_t = \bigcup_t \bigcup_{N \in \mathcal{M}_t} N$ and the result follows.
4 Associated Primes

Notation 4.1. Throughout this section \((R, m, K)\) denotes a local ring and \(S\) denotes either \(R[x_1, \ldots, x_n]\) or \(R[[x_1, \ldots, x_n]]\). In addition, \(D\) denotes \(D(S, R)\).

Lemma 4.2. Let \(J \subset S\) be an ideal and \(M\) be an \(R\)-module of finite length. Then, \(H^1_J(M \otimes_R S)\) is a \(D(S, R)\)-module of finite length. Moreover, \(\mathcal{C}(H^1_J(M \otimes_R S)) \subset \bigcup_j \mathcal{C}(H^1_J(S/mS))\).

Proof. Our proof will be by induction on \(h = \text{length}_R(M)\). If \(h = 1\), we have that \(H^1_J(R/m \otimes_R S) = H^1_J(S/mS)\), which has finite length as a \(D(S, R)\)-module [10, Theorem 2, Corollary 3] and by Remark 2.3. In addition, \(\mathcal{C}(H^1_J(M \otimes_R S)) \subset \mathcal{C}(H^1_J(S/mS))\).

Clearly,

\[
\mathcal{C}(H^1_J(M \otimes_K A)) = \mathcal{C}(H^1_J(R/m \otimes_K A)) = \bigcup_j \mathcal{C}(H^1_J(S/mS))
\]

in this case. Suppose that the statement is true for \(h\) and \(\text{length}_R(M) = h + 1\). We have a short exact sequence of \(R\)-modules, \(0 \to K \to M \to M' \to 0\), where \(h = \text{length}_R(M')\). Since \(S\) is flat over \(R\), we have that \(0 \to K \otimes_R S \to M \otimes_R S \to M' \otimes_R S \to 0\) is also exact. Then, we have a long exact sequence

\[
\cdots \to H^1_J(K \otimes_R S) \to H^1_J(M \otimes_R S) \to H^1_J(M' \otimes_R S) \to \cdots
\]

Then \(H^1_J(M \otimes_R S)\) has finite length by the induction hypothesis and Remark 2.3. In addition,

\[
\mathcal{C}(H^1_J(M \otimes_R S)) \subset \mathcal{C}(H^1_J(M' \otimes_R S)) \bigcup \mathcal{C}(H^1_J(K \otimes_R S)) \subset \bigcup_j \mathcal{C}(H^1_J(S/mS)).
\]

and the result follows by the induction hypothesis and Remark 2.3.

Proposition 4.3. Let \(I \subset S\) be an ideal containing \(mS\). Then \(H^1_I(S)\) is \(\Sigma\)-finite for every \(i \in \mathbb{N}\).

Proof. Let \(f_1, \ldots, f_d\) be a system of parameters for \(R\) and \(g_1, \ldots, g_{\ell}\) be a set of generators for \(I\). Let \(f^I_t\) denote the sequence \(f^I_1, \ldots, f^I_t\). Let \(T_i = \{T^{p,q}_i\}\) be the double complex of \(D(S, R)\)-modules given by the tensor product \(\mathcal{K}(f^I_t ; S) \otimes_R \mathcal{C}(g^I_s)\). The direct limit \(\mathcal{K}(f^I_t ; R)\) introduced in Figure 2.1 induces a direct limit of double complexes \(\text{Tot}(T_i) \to \text{Tot}(T_{i+1})\). Since \(\lim_{\to i} \mathcal{K}(f^I_t ; R) = \mathcal{C}(f^I_t ; R)\), we have that \(\lim_{\to i} \text{Tot}(T_i) = \mathcal{C}(f^I_t ; S)\). Let \(E^{p,q}_{r,t}\) be the spectral sequence associated to \(T_i\). We have that

\[
E^{p,q}_{2,r,t} = H^{p,q}_t(\mathcal{K}(f^I_t ; S)) \Rightarrow E^{p,q}_{\infty, t} = H^{p+q} \text{Tot}(T_i).
\]
We note that $H^g(K(f^I; S)) = H^g(K(f^I; R)) \otimes_R S$, because $S$ is $R$-flat. Since $H^g(K(f^I; R))$ has finite length as an $R$-module, we have that $E_{t,t}^{p,q}$ is a $D(S, R)$-module of finite length for all $p, q \in \mathbb{N}$ and that $\mathcal{E}(E_{t,t}^{p,q}) = \bigcup_j \mathcal{E}(H^j(S/mS))$ by Lemma 3.10. Moreover, $E_{r,t}^{p,q}$ is a $D(S, R)$-module of finite length, and

$$
\mathcal{E}(E_{r,t}^{p,q}) \subset \bigcup_{p,q} \mathcal{E}(E_{2t}^{p,q}) = \bigcup_j \mathcal{E}(H^j(S/mS))
$$

for $r > 2$. Then, $\mathcal{E}(H^i(T_{t,j})) \subset \bigcup_j \mathcal{E}(H^j(S/mS))$ for every $j, t \in \mathbb{N}$ by Remark 2.3 in particular, $\bigcup_j \mathcal{E}(H^i(T_{t,j}))$ is finite and every element there belongs to $C(S/mS, R/mR)$. Therefore, $E_{r,t}^{p,q}$ is a $\Sigma$-finite $D(S, R)$-module. Moreover,

$$
H^j_i(S) = H^j_i(\mathcal{C}(f, g; S)) = H^j_i(\lim_{\to t} \text{Tot}(T_t)) = \lim_{\to t} H^i_j(\text{Tot}(T_t))
$$

because the direct limit is exact. Hence, $H^j_i(S)$ is $\Sigma$-finite by Proposition 3.10. \(\square\)

**Corollary 4.4.** Let $I \subset S$ be an ideal containing $mS$ and $J_1, \ldots, J_t \subset S$ be any ideals. Then $H^j_{i_1} \cdots H^j_{i_t} H^i_j(S)$ is $\Sigma$-finite.

**Proof.** This is a consequence of Proposition 4.3 and Corollary 3.3. \(\square\)

**Proof of Theorem 4.4.** This is a consequence of Remark 3.3 and Corollary 4.4. \(\square\)

**Proposition 4.5.** Let $(R, m, K)$ be any local ring. Let $S$ denote either $R[[x_1, \ldots, x_n]]$ or $R[[x_1, \ldots, x_n]]$. Let $I \subset S$ be an ideal, such that $\dim R/I \cap R \leq 1$. Then,

$$
\text{Ass}_S H^0_m H^i_j(S)
$$

is finite for every $i \in \mathbb{N}$.

**Proof.** Since $\dim R/(I \cap R) \leq 1$, there exists $f \in R$ such that $mS \subset \sqrt{I + fS}$. We have the exact sequence

$$
\cdots \to H^i_{(I, f)} S(S) \xrightarrow{\alpha} H^i_j(S) \xrightarrow{\beta} H^i_j(S_f) \to \cdots.
$$

Then,

$$
\text{Ass}_S H^i_j(S) \cap V(mS) \subset (\text{Ass}_S \text{Im}(\alpha) \cap V(mS)) \bigcup (\text{Ass}_S \text{Im}(\beta) \cap V(mS))
$$

Since $H^i_{(I, f)} S(S)$ is a $\Sigma$-finite $D(S, R)$-module by Proposition 4.3, we have that $\text{Im}(\alpha)$ is also $\Sigma$-finite by Proposition 3.3, and so $\text{Ass}_S \text{Im}(\alpha)$ is finite. Since $\text{Im}(\beta) \subset H^i_j(S_f)$, $\text{Ass}_S \text{Im}(\beta) \cap V(mS) = \emptyset$. Therefore,

$$
\text{Ass}_S H^i_j(S) \cap V(mS) = \text{Ass}_S H^0_m H^i_j(S)
$$

is finite. \(\square\)
Proposition 4.6. Suppose that $R$ is a ring of characteristic 0 and that $\dim R/(I \cap R) \leq 1$. Then $H^j_{mS}H^i_I(S)$ is $\Sigma$-finite for every $i, j \in \mathbb{N}$.

Proof. Since $\dim R/(I \cap R) \leq 1$, there exists $g \in R$, such that $mS \subset \sqrt{(I, g)S}$. We have the long exact sequence

$$\ldots \to H^i_{(I, g)S}(S) \to H^i_S(S) \to H^i_I(S_g) \to \ldots$$

Let $M_i = \ker(H^i_{(I, g)S}(S) \to H^i_S(S))$, $N_i = \im(H^i_{(I, g)S}(S) \to H^i_S(S))$ and $W_i = \im(H^i_S(S) \to H^i_I(S_g))$. We have the following short exact sequences:

$$0 \to M_i \to H^i_{(I, g)S}(S) \to N_i \to 0,$$

$$0 \to N_i \to H^i_I(S) \to W_i \to 0$$

and

$$0 \to W_i \to H^i_I(S_g) \to M_{i+1} \to 0.$$ 

Since $mS \subset \sqrt{(I, g)S}$, $H^i_{(I, g)S}(S)$ is $\Sigma$-finite by Proposition 1.3. Then, $M_i$ and $N_i$ is $\Sigma$-finite for every $i \in \mathbb{N}$ by Proposition 3.3. By the long exact sequences

$$\ldots \to H^j_{mS}(M_i) \to H^j_{mS}H^i_{(I, g)S}(S) \to H^j_{mS}(N_i) \to \ldots,$$

$$\ldots \to H^j_{mS}(N_i) \to H^j_{mS}H^i_I(S) \to H^j_{mS}(W_i) \to \ldots$$

and

$$\ldots \to H^j_{mS}(W_i) \to H^j_{mS}H^i_I(S_g) \to H^j_{mS}(M_{i+1}) \to \ldots,$$

$H^j_{mS}(M_i)$, $H^j_{mS}H^i_{(I, g)S}(S)$ and $H^j_{mS}(M_i)$ are $\Sigma$-finite for every $i, j \in \mathbb{N}$. Since $H^j_{mS}H^i_I(S_g) = 0$, $H^j_{mS}(W_i) = H^j_{mS}(M_{i+1})$. Then, $H^j_{mS}(W_i)$ is $\Sigma$-finite, and so $H^j_{mS}H^i_I(S)$ is $\Sigma$-finite by 3.6.

5 More examples of $\Sigma$-finite $D$-modules

In the previous section we gave a positive answer for specific cases for Question 1.2. Our method consisted in proving that $H^j_{mS}H^i_I(S)$ is $\Sigma$-finite and then applying Remark 3.3. This motivates the following question:

Question 5.1. Is $H^j_{mS}H^i_I(S)$ $\Sigma$-finite for every ideal $I \subset S$ and $i, j \in \mathbb{N}$?

In this section, we provide positive examples for Question 5.1.

Proposition 5.2. Let $(R, m, K)$ be any local ring. Let $S$ denote either $R[x_1, \ldots, x_n]$ or $R[[x_1, \ldots, x_n]]$. Let $I \subset S$ be an ideal such that $\text{depth}_S I = \text{cd}_S I$. Then, $H^j_{mS}H^i_I(S)$ is $\Sigma$-finite for every $i \in \mathbb{N}$.
Proof. We have that the spectral sequence
\[ E_2^{i,j} = H_i^pH_j^q(S) \Rightarrow E_\infty^{i,j} = H_{(i,m)S}^{p+q}(S), \]
converges at the second spot, because depth$_S I = \text{cd}_S I$. Hence,
\[ H_i^pH_j^q(S) = H_{(i,m)S}^{p+q}(S) \]
and the result follows by Proposition 4.3.

**Proposition 5.3.** Let \((R, m, K)\) be any local ring. Let \(S\) denote either \(R[x_1, \ldots, x_n]\) or \(R[[x_1, \ldots, x_n]]\). Let \(I \subset S\) be an ideal such that \(\text{Ext}^i_S(S/mS, H_j^1(S))\) is a \(D\)-module in \(C(R, S)\) for every \(i \in \mathbb{N}\). Then, \(H_i^mH_j^1(S)\) is a \(\Sigma\)-finite \((D, R, m, K)\)-module for every \(i, j \in \mathbb{N}\).

**Proof.** We claim that \(\text{Ext}^i_S(N \otimes_R S, H_j^1(S))\) is a \(D(mS, S)\)-module in \(C(S/mS, K)\) for every \(i \in \mathbb{N}\) and every finite length \(R\)-module \(N\). Moreover, \(\bigcup_i \text{Ext}^i_S(N \otimes_R S, H_j^1(S))\) is a \(\Sigma\)-finite \((D, R, S, K)\)-module. The proof of our claim is analogous to Lemma 4.2.

The direct system \(\text{Ext}^i(S/m^tS, H_j^1(S)) \to \text{Ext}^i(S/m^{t+1}S, H_j^1(S))\) satisfies the hypotheses of Proposition 5.10. Hence,
\[ H_i^mH_j^1(S) = \lim_{t \to \infty} \text{Ext}^i(S/m^tS, H_j^1(S)) \]
is a \(\Sigma\)-finite \((D, S, R)\)-module.

**Remark 5.4.** The condition that \(\text{Ext}^i_S(S/mS, H_j^1(S))\) be a \(D(S/mS, K)\)-module in \(C(S/mS, K)\) for every \(i \in \mathbb{N}\) is not necessary.

Let \(R = K[[s, t, u, w]]/(us + vt)\), where \(K\) is a field. This is the ring given by Hartshorne’s example 41. Let \(I = (s, t)A\). Hartshorne showed that \(\dim_K \text{Hom}_A(K, H_j^1(A))\) is not finite. Let \(S\) be either \(R[x_1, \ldots, x_n]\) or \(R[[x_1, \ldots, x_n]]\). Therefore,
\[ \text{Ext}^0_S(S/mS, H_j^1(S)) = \text{Hom}_R(S/mS, H_j^1(S)) = \text{Hom}_R(K, H_j^1(R)) \otimes_R S = S/mS, \]
where the direct sum is infinite. Then, \(\text{Ext}^0_S(S/mS, H_j^1(S))\) does not belong to \(C(S, R)\).

On the other hand, \(H_i^mH_j^1(S)\) is a direct limit of finite direct sums of \(S/mS\). This direct limit satisfies the hypotheses of Proposition 5.10. Therefore, \(H_i^mH_j^1(S)\) is a \(\Sigma\)-finite \((D, S, R)\)-module.

**Proposition 5.5.** Let \((R, m, K)\) be any local ring and let \(S\) denote \(R[x_1, \ldots, x_n]\). Let \(I \subset S\) be an ideal. Then, \(H_i^mH_j^1(S)\) is \(\Sigma\)-finite for every \(i \in \mathbb{N}\). In addition, if \(\text{cd}_S I \leq 1\), then \(H_i^iH_j^1(S)\) is \(\Sigma\)-finite for every \(i, j \in \mathbb{N}\).
Proof. We claim that there exists an ideal $J \subset R$ such that $H^0_I(S) = JS$. We have that $H^0_I(S)$ is a $D(S, R)$-module. For every $f = \sum c_\alpha x^\alpha \in H^0_I(S)$ and $\theta \in D(S, R)$, $\theta f \in H^0_I(S)$. Therefore, $c_\alpha \in H^0_I(S)$, and $H^0_I(S) = JS$, where $J = \{ c_\alpha \mid \sum c_\alpha x^\alpha \in H^0_I(S) \}$.

We have that
\[
\text{Ext}^S_0(S/mS, H^0_I(S)) = \text{Ext}^i_S(R/mR \otimes_R S, J \otimes_R S) = \text{Ext}^i_R(K, J) \otimes_S S = \oplus^\mu S/mS, \quad \text{where } \mu = \dim_K \text{Ext}^i_R(K, J),
\]
and it is a $D(S, R)$-module in $C(S, R)$ for every $i \in \mathbb{N}$. The first claim follows from Proposition [5.3].

We have that $H^1_I(S) = \text{Ext}^1_{I(S/J)}(S/J)$ [11 Corollary 2.1.7]. In addition, $S/JS = (R/J) [x_1, \ldots, x_n]$ and
\[
\text{depth}_{I(S/J)} = \text{cd}_{I(S/JS)}(S/JS) = 1.
\]
The second claim follows from Proposition [5.2].

6 Reduction to power series rings

Discussion 6.1. Suppose that $(R, m, K)$ and $(S, \eta, L)$ are complete local rings and that $\varphi : R \to S$ is a flat extension of local rings with regular closed fiber. Assume that $\varphi$ maps a coefficient field of $R$ to a coefficient field of $S$. We pick such coefficient fields, and then $\varphi(K) \subset L$. Thus, $R = K[[x_1, \ldots, x_n]]/I$ for some ideal $I \subset K[[x_1, \ldots, x_n]]$. Let $A = L \otimes_K R = L[[x_1, \ldots, x_n]]/IL[[x_1, \ldots, x_n]]$. We note that $A$ is a flat local extension of $R$, such that $mA$ is the maximal ideal of $A$. Let $\theta : A \to S$ be the morphism induced by $\varphi$ and our choice of coefficient fields.

We claim that $S$ is a flat $A$-algebra. Let $F_\ast$ be a free resolution of $R/mR$. Then, $A \otimes_R F_\ast$ is a free resolution for $A/mA$. We have that
\[
\text{Tor}_A^1(S, A/mA) = H_1(S \otimes_A A \otimes_R F_\ast) = H_1(S \otimes_R F_\ast) = \text{Tor}_A^0(S, R/mR) = 0
\]
because $S$ is a flat extension. Since $mA$ is the maximal ideal of $A$, we have that $S$ is a flat $A$-algebra by the local criterion of flatness [2 Theorem 6.8].

Let $d = \dim(S/mS)$ and $z_1, \ldots, z_d \in S$ be preimages of a regular system of parameters for $S/mS$. Let $\varphi : A[[y_1, \ldots, y_d]] \to S$ be the morphism given by sending $A$ to $S$ via $\theta$ and $y_i$ to $z_i$. Since
\[
(mA + (z_1, \ldots, z_d))S = \eta
\]
and the morphism induced by $\varphi$ in the quotient fields of $A$ and $S$ is an isomorphism. Hence, $\varphi$ is an isomorphism.
Proposition 6.2. Questions 1.1 and 1.2 are equivalent when we restrict them to a local extensions, such that the induced morphism in the completions maps a coefficient field of the domain to a coefficient field of the target.

Proof. Let \( \varphi : (R, m, K) \to (S, \eta, L) \) be a flat extension of local rings with regular closed fiber. Suppose that \( \widehat{\varphi} : \widehat{R} \to \widehat{S} \), the induced morphism in the completions, maps a coefficient field of the \( \widehat{R} \) to a coefficient field of \( \widehat{S} \). We have that \( \text{Ass}_R H^0_{mR} H^i_I(S) \) is finite if and only if \( \text{Ass}_{\widehat{R}} H^0_{m\widehat{R}} H^i_I(\widehat{S}) \) is finite. Let \( A \) be as in the previous discussion and \( d = \dim(S/mS) \). The result follows, because \( \widehat{S} = A[[y_1, \ldots, y_d]] \) and \( mS = (mA)S \).

Remark 6.3. In the previous proposition, the hypothesis that \( \widehat{\varphi} \) maps a coefficient field of \( \widehat{R} \) to a coefficient field of \( \widehat{S} \) is satisfied when \( L \) is a separable extension of \( K \) [14, Theorem 28.3].

Proof of Theorem 1.4. By Discussion 6.1 we may assume that \( R \) is complete and \( S \) is a power series ring over \( R \). The rest is a consequence of Proposition 4.5.

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