A topological definition of the Maslov bundle

Colette Anné
anne@math.univ-nantes.fr
Laboratoire de Mathématiques Jean Leray
Université de Nantes, BP 92208
44322 Nantes-Cedex 03, France

26th September 2018

Abstract

We give a definition of the Maslov fibre bundle for a lagrangian submanifold of the cotangent bundle of a smooth manifold. This definition generalizes the definition given, in homotopic terms, by Arnol’d for lagrangian submanifolds of $T^*\mathbb{R}^n$. We show that our definition coincides with the one of Hörmander in his works about Fourier Integral Operators.

Key words: fourier integral operators, Maslov bundle, Hörmander’s index.

Contents

1 Introduction 1

1.1 Arnol’d’s definition of the Maslov index ........................................ 1
1.2 Hörmander’s definition of the Maslov bundle ................................. 2
1.3 Definition of the Maslov index and results .................................. 2

2 Study of the index $\mu_0$ 3

2.1 The index $\mu_0$ on $L(n)$ is also an intersection number .............. 3
2.2 Proof of the proposition 1.2 ......................................................... 3

3 Links with the definition of Hörmander 4

3.1 Maslov’s index in term of signature ......................................... 5
3.2 Hörmander’s index ................................................................. 5
3.3 Proof of theorem 1.1 .............................................................. 7

4 Topological comments 9

5 References 10
1 Introduction

The Maslov index appears as the phase term when one tries to define the symbol of a Fourier Integral Operator (FIO). This symbol is then defined as a section of the Maslov bundle contructed on a lagrangian submanifold of $T^*X$. In his historical paper [1], Hörmander proposes a construction of this bundle in terms of cocycles and tries to make the links with the strictly topological presentation (representation of the fundamental group) proposed by Arnol’d [3], originally in an appendix of the book of Maslov [12]. This link is established only for the lagrangian submanifolds of $T^*\mathbb{R}^n$. I propose in this work a new construction [14] for the lagrangian submanifolds of $T^*X$, $X$ a smooth manifold, based on a definition of the Maslov index (1.1) which generalize the one of Arnol’d, and satisfies the cocycles conditions of Hörmander. These correspondances are established in the sections 2 and 3.

acknowledges. — To François Laudenbach and his interest in this work.

1.1 Arnol’d’s definition of the Maslov index

Recall first the construction of Arnol’d [3]. The space $T^*\mathbb{R}^n$ has a symplectic structure by the standard symplectic form

$$\omega = \sum_{j=1}^{n} \, d\xi_j \wedge dx_j.$$

Let $L(n)$ be the Grassmanian manifold of the Lagrangian subspaces of $T^*\mathbb{R}^n$; we identify $L(n) = U(n)/O(n)$. The map $Det^2$ is well defined on $L(n)$, it is showed in [3] that every path $\gamma : S^1 \to L(n)$ such that $Det^2 \circ \gamma : S^1 \to S^1$ is a generator of $\Pi_1(S^1)$, gives a generator of $\Pi_1(L(n))$. It follows that $\Pi_1(L(n)) \simeq \mathbb{Z}$ and that the cocycle $\mu_0$ defined by

$$\forall \gamma \in \Pi_1(L(n)) \quad \mu_0(\gamma) = \text{Degree } (Det^2 \circ \gamma)$$

is a generator of the group $H^1(L(n)) \simeq \mathbb{Z}$. It is then possible to define a Maslov bundle $M(n)$ on $L(n)$ by the representation $\exp(i\frac{x}{2} \mu_0) = i^{\mu_0}$ of $\Pi_1(L(n))$. It is a flat bundle with torsion because $M(n)^{\otimes 4}$ is trivial.

Now the Maslov bundle of a submanifold $\mathcal{L}$ of $T^*\mathbb{R}^n$ is the pullback of $M(n)$ by the natural map

$$\varphi_n : \mathcal{L} \to L(n) \quad \nu \mapsto T_\nu \mathcal{L}.$$

Arnol’d precisely shows that $\mu = \varphi_n^* \mu_0$ is the Maslov index of $\mathcal{L}$. One can write

$$\mu : \Pi_1(\mathcal{L}) \to \mathbb{Z} \quad \gamma \mapsto <\mu_0, \varphi_n \circ \gamma> = \text{Degree } (Det^2 \circ \varphi_n \circ \gamma). \quad (1.1.1)$$

We have to take care of the structural group of this bundle. As a $U(1)$—bundle it is always trivial. But it is considered as a $\mathbb{Z}_4 = \{1, i, -1, -i\}$—bundle. In fact one can see, using the expression of the Maslov cocycle $\sigma_{jk}$ given by [3] (3.2.15) that the Chern classes of this bundle are null but $\sigma_{jk}$ can not be written in general as the coboundary of a constant cochain.

We recall now the theorem of symplectic reduction as it is presented in [6] Proposition 3.2. p.132.

**Proposition 1.1 (Guillemin, Sternberg)**. — Let $\Delta$ be an isotropic subspace of dimension $m$ in $T^*\mathbb{R}^{(n+m)}$. Define $S_\Delta = \{\lambda \in L(n+m) / \lambda \supset \Delta\}$. Then $S_\Delta$ is a submanifold of $L(n+m)$ of codimension $(n+m)$, if we define $\rho$ to be the map

$$\begin{array}{ccc}
L(n+m) & \xrightarrow{\rho} & L(n) \\
\lambda & \mapsto & \lambda \cap \Delta^\omega / \lambda \cap \Delta
\end{array}$$

1
1.2 Hörmander’s definition of the Maslov bundle

Let $X$ be a smooth manifold, then $T^*X \rightarrow X$ is endowed with a canonical symplectic structure by $\omega = dt \wedge dx$. Let $L$ be a lagrangian (homogeneous) submanifold of $T^*X$. Hörmander, in [7] p.155, defines the Maslov bundle of $L$ by its sections.

A Lagrangian manifold owns an atlas such that the cards $(C_\phi, D_\phi)$ are defined by non degenerated phase functions $\phi$ defined on $U \times \mathbb{R}^N$ $U$ open in a domain diffeomorphic to a ball of a card of $X$ and

$$C_\phi = \{(x, \theta); \phi'(x, \theta) = 0\} \xrightarrow{D_\phi} L_\phi \subset L$$

$$(x, \theta) \mapsto (x, \phi'(x, \theta)).$$

For the function $\phi$, to be non degenerate means that $\phi'$ is a submersion and thus $C_\phi$ is a submanifold and $D_\phi$ an immersion.

A section is then given by a family of functions $z_\phi : C_\phi \rightarrow C$ satisfying the change of cards formulae:

$$z_\phi = \exp i \pi 4 (\text{sgn}\phi''_\theta - \text{sgn}\tilde{\phi}''_\theta) z_\phi. \tag{1.2.2}$$

In fact $(\text{sgn}\phi''_\theta - \text{sgn}\tilde{\phi}''_\theta)$ is even (see below, proposition [8]) and we have indeed constructed by this way a $\mathbb{Z}_4$–bundle.

1.3 Definition of the Maslov index and results

In the same situation as before, we can construct on any lagrangian submanifold $L$ of $T^*X$ (and in fact on all $T^*X$) the following fibre bundle

$$L(n) \rightarrow L(L) \xrightarrow{\pi} L$$

of the lagrangian subspaces of $T_\nu(T^*X), \nu \in L$.

This bundle has two natural sections:

$$\lambda(\nu) = T_\nu(L), \text{ and } \lambda_0(\nu) = \text{vert}(T_\nu(T^*X))$$

defined by the tangent to $L$ and the tangent to the vertical $T^{*\nu}_{T_\nu(L)}X$.

To a fibre bundle is associated a long exact sequence of homotopy groups, here:

$$\cdots \rightarrow \Pi_2(L) \rightarrow \Pi_1(L(n)) \rightarrow \Pi_1(L) \rightarrow \Pi_0(L(n)) \rightarrow 0.$$

But our fibre bundle possesses a section (two in fact), as a consequence the maps $\Pi_k(L(L)) \rightarrow \Pi_k(L)$ are onto and the maps $\Pi_{k+1}(L) \rightarrow \Pi_k(L(n))$ are null; this gives a split exact sequence

$$0 \rightarrow \Pi_1(L(n)) \rightarrow \Pi_1(L(L)) \rightarrow \Pi_1(L) \rightarrow 0.$$

Take a base point $\nu_0 \in L$ and fix a path $\sigma$ from $\lambda(\nu_0)$ to $\lambda_0(\nu_0)$ lying in the fibre $L(L)_{\nu_0}$. For $\gamma \in \Pi_1(L)$ we denote $\lambda_0^\sigma.*(\gamma)$ the composition of $\sigma$, $\lambda_0$, $\gamma$ and finally $\sigma^{-1}$ (we use here the conventions of writing of $[10]$).

Then $\forall \gamma \in \Pi_1(L), \pi_* \left(\lambda_0^\sigma.*(\lambda_0^\sigma.*(\gamma^{-1})\right) = 0$ and $\lambda_0^\sigma.*(\lambda_0^\sigma.*(\gamma^{-1}))$ is in $\Pi_1(L(n))$. Let us take the
Definition 1.1. — The Maslov index of $\mathcal{L}$ is the map $\mu$:
\[ \forall \gamma \in \Pi_1(\mathcal{L}), \quad \mu(\gamma) = \mu_0\left(\chi_\gamma * \lambda_0 \sigma_0(\gamma^{-1})\right). \]

Proposition 1.2. — This definition does not depend on the path $\sigma$ that we have chosen to joint $\lambda(\nu_0)$ to $\lambda_0(\nu_0)$: moreover $\mu$ is a morphism of group, that is $\mu \in H^1(\mathcal{L}, \mathbb{Z})$.

First remark: in the case where $X = \mathbb{R}^n$ the fibre bundle $L(\mathcal{L})$ can be trivialized in such a way that the section $\lambda_0$ is constant. In this case our definition coincide with the one of $\mathcal{M}$. A natural consequence of the proposition is the following definition:

Definition 1.2. — The Maslov bundle $\mathcal{M}(\mathcal{L})$ over $\mathcal{L}$ is defined as in section 1.1 by the representation $\exp(i\pi \mu) = i^{\mu}$ of $\Pi_1(\mathcal{L})$ in $\mathbb{C}$.

This means that the sections of the bundle are identified with functions $f$ on the universal cover of $\mathcal{L}$ with complex values and satisfying the relation:
\[ \forall \gamma \in \Pi_1(\mathcal{L}), \quad f(x, \gamma) = i^{-\mu(\gamma)}f(x), \quad (1.3.3) \]
like in $\mathcal{M}$ formula (2.19).

Theorem 1.1. — The sections of the Maslov bundle of a Lagrangian (homogeneous) submanifold as defined by the definition 1.2 satisfy the gluing conditions of Hörmander, it means that our definition coincides with the one of Hörmander.

2 Study of the index $\mu$.

2.1 The index $\mu_0$ on $\mathbb{L}(n)$ is also an intersection number.

For $\alpha \in L(n)$ et $k \in \mathbb{N}$ one defines $\mathbb{L}^k(n)(\alpha) = \{\beta \in \mathbb{L}(n); \dim \alpha \cap \beta = k\}$. Since $\mathcal{M}$ we know that $\mathbb{L}^k(n)(\alpha)$ is an open submanifold of codimension $\frac{k(n+1)}{2}$, in particular $\mathbb{L}^1(n)(\alpha)$ is an oriented cycle of codimension 1 and his intersection number coincides with $\mu_0$.

2.2 Proof of the proposition 1.2.

It is a consequence of the two following lemmas. Provide $\mathbb{L}(\mathcal{L})$ with a connection of $U(n)$-bundle. Indeed any symplectic manifold $(M, \omega)$, like $T^*X$, can be provided with an almost complex structure $J$ which is compatible with the symplectic structure (see $\mathcal{M}$ p.102), it means such that $g(X,Y) = \omega(JX,Y)$ is a riemannian metric. By this way the tangent bundle of $M$ is provided with an hermitian form $g_C = g + i\omega$, and its structural group restricts to $U(n)$ it is also the case for the grassmannian of Lagrangians or its restriction to a submanifold.

We will denote by $\tau(\gamma)_{x \to y}$ the parallel transport for this connection from $\mathbb{L}(\mathcal{L})_x$ to $\mathbb{L}(\mathcal{L})_y$ along the path $\gamma$ joining $x$ to $y$ in $\mathcal{L}$.

Let’s now $\gamma : S^1 \to \mathcal{L}$ be a closed path such that $\gamma(0) = \nu_0$, we define $\lambda(t) = \lambda_0(\gamma)(t)$ and in the same way $\lambda_0^{-1}(t) = \lambda_0(\gamma^{-1})(t)$.

If, as before, $\sigma$ is a path from $\lambda(0)$ to $\lambda_0(0)$ in the fibre $L(\mathcal{L})_{\gamma(0)}$; then the path of $L(\mathcal{L}) : \lambda * \sigma * \lambda_0^{-1} \sigma^{-1}$ is homotopic to a path in the fibre, we have to calculate the Maslov index $\mu_0$ of this last one. For this we use the parallel transport along $\gamma$ to deform $\lambda * \sigma * \lambda_0^{-1}$.

Definition 2.1. — For $t \in [0,1]$ let’s $\sigma_t$ denote the path included in the fibre $L(\mathcal{L})_{\gamma(t)}$ joining $\lambda(t)$ to $\lambda_0(t)$ and obtained by the parallel transport of $\lambda_{[t,1]} \sigma (\lambda_0_{[t,1]})^{-1}$.
This path has three distinct parts: first $\hat{\lambda}(t, s) = \tau(\gamma^{-1})\gamma(s)\rightarrow\gamma(t)\lambda(s)$ then $\tilde{\sigma}(t, s) = \tau(\gamma^{-1})\gamma(1)\rightarrow\gamma(t)\sigma(s)$ and finally $\tilde{\lambda}_0^{-1}(t, s) = \tau(\gamma^{-1})\gamma(s)\rightarrow\gamma(0^{-1}(t))$.

By the definition \[\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1})\].

**Lemma 2.1**. — This definition does not depend on the path $\sigma$ chosen to link $\lambda(0)$ to $\lambda_0(0)$ staying in the fibre above $\gamma(0)$.

The index $\mu_0$ is defined on the free homotopy group so

$$
\mu_0(\sigma_0 * \sigma^{-1}) = \mu_0(\sigma^{-1} * \sigma_0) = \mu_0(\sigma^{-1} * \hat{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1})
$$

if, here, $\tilde{\lambda}(s) = \hat{\lambda}(0, s)$ and the same notations for $\lambda_0$ and $\sigma$.

If $\sigma'$ is an other path from $\lambda(0)$ to $\lambda_0(0)$, then by the preceding remark and the fact that $\mu_0$ is a morphism of group, one has:

$$
\begin{align*}
\mu_0(\sigma' * \sigma'^{-1}) - \mu_0(\sigma_0 * \sigma^{-1}) &= \mu_0(\sigma'^{-1} * \sigma'_0) - \mu_0(\sigma_0^{-1} * \sigma_0) = \\
\mu_0(\sigma'^{-1} * \sigma'_0) + \mu_0(\sigma_0^{-1} * \sigma) &= \mu_0(\sigma_0^{-1} * \sigma_0^{-1} * \sigma) = \\
\mu_0(\sigma * \sigma'^{-1} * \hat{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1} * \tilde{\lambda}_1^{-1}) &= \mu_0((\sigma * \sigma'^{-1}) * \hat{\lambda} * (\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1} * \tilde{\lambda}_1^{-1}) = \\
\mu_0(\sigma * \sigma'^{-1}) + \mu_0(\hat{\lambda} * (\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1} * \tilde{\lambda}_1^{-1}) &= \\
\mu_0(\sigma * \sigma'^{-1}) + \mu_0(\tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1}) &= \\
\mu_0(\sigma * \sigma'^{-1}) + \mu_0((\tilde{\sigma} * \tilde{\sigma}'^{-1})^{-1}) &= \mu_0(\sigma * \sigma'^{-1}) - \mu_0(\tilde{\sigma} * \tilde{\sigma}'^{-1}) = 0
\end{align*}
$$

because $\tilde{\sigma} * \tilde{\sigma}'^{-1}$ is the image of $\sigma * \sigma'^{-1}$ by the parallel transport $\tau(\gamma)$ along $\gamma$; but $\tau(\gamma) \in U(n)$ preserves the Maslov index $\mu_0$.

**Lemma 2.2**. — $\mu$ is a morphism of groups.

Indeed, if $\alpha$ and $\beta$ are two elements of $\Pi_1(L)$ it is sufficient to calculate $\mu(\alpha) + \mu(\beta)$ beginning the first circle at $\hat{\sigma}^{-1}(1) = \tau(\alpha)\sigma(0)$ and applying $\tau(\alpha)$ to the second circle which was chosen to begin at $\sigma(0)$.

### 3 Links with the definition of Hörmander

To make the link of this definition with signature terms of the formula in \textit{[7]} we follow the calculation from \textit{[4]}.
3.1 Maslov’s index in term of signature.

Let \( \gamma \in \mathbb{L}^k(n)(\alpha) \) and \( \beta \in \mathbb{L}^0(n)(\alpha) \cap \mathbb{L}^0(n)(\gamma) \). Then \( \alpha \) and \( \beta \) are transversal and \( \gamma \) can be presented as a graph : there exists a unique linear map \( C : \alpha \to \beta \) such that \( \gamma = \{(x, Cx), x \in \alpha\} \).

p. 181, defines a quadratic form in \( \alpha \) by :

\[
Q(\alpha, \beta; \gamma) = \omega(C, \gamma)_{\in} Q(\alpha).
\]

One sees easily that \( \ker Q(\alpha, \beta; \gamma) = \ker C = \alpha \cap \gamma \). and if we choose a basis on \( \alpha \) such that \( Q(\alpha, \beta; \gamma) \) has the form \[
\begin{pmatrix}
B_0 & 0 \\
0 & 0
\end{pmatrix},
\]
the null part corresponds to \( \alpha \cap \gamma \).

Let now \( \gamma(t) \) be a path in \( \mathbb{L}^0(n)(\beta) \) such that \( \gamma(0) = \gamma \). The goal of the following calculations is to control the jump of the signature of the quadratic form \( Q(\alpha, \beta; \gamma(t)) \) in the neighbourhood of \( t = 0 \).

**Proposition 3.1.** — Let \( \gamma(t) \) be a path in \( \mathbb{L}^0(n)(\beta) \) such that \( \gamma(0) = \gamma \). If

\[
Q(\alpha, \beta; \gamma(t)) = \begin{pmatrix}
B(t) & C(t) \\
C^t(t) & D(t)
\end{pmatrix}
\]

with \( D(t) \) in \( \alpha \cap \gamma \). Then, if \( D'(t) \) is invertible in the neighbourhood of 0, there exists \( \varepsilon > 0 \) such that

\[
\forall t, 0 < t < \varepsilon \ sgn Q(\alpha, \beta; \gamma(t)) - sgn Q(\alpha, \beta; \gamma(-t)) = 2 sgn D'(0).
\]

**Proof.** — We know that \( B(t) \) is invertible and \( C(t), D(t) \) are small. The identity

\[
\begin{pmatrix}
B & C \\
C^t & D
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & B^{-1}C \\
0 & 1
\end{pmatrix}
\]

(3.1.5)
gives \( sgn Q(\alpha, \beta; \gamma(t)) = sgn(B(t)) + sgn(D(t) - C(t)B(t)^{-1}C(t)) \). When \( t \) is small \( sgn B(t) = sgn Q(\alpha, \beta; \gamma) \) and \( sgn \{D(t) - C(t)B(t)^{-1}C(t)\} = sgn(t) sgn(D'(0)) \) by the mean value theorem.

Now if \( \gamma \) is a path which cross transversally \( \mathbb{L}^1(n)(\alpha) \) at \( \gamma(0) \) then the assumption on \( D' \) is satisfied.

**Theorem 3.1.** — Let \( \alpha \in \mathbb{L}(n) \) and \( \gamma \) a closed path in \( \mathbb{L}(n) \) which cross \( \mathbb{L}^1(n)(\alpha) \) transversally, then for all \( \beta \in \mathbb{L}(n) \) transversal to \( \alpha \) and to \( \gamma(t) \) one has

\[
\mu_0(\gamma) = \frac{1}{2} \sum_{t, \gamma(t) \in \mathbb{L}^1(n)(\alpha)} \left( sgn Q(\alpha, \beta; \gamma(t^+)) - sgn Q(\alpha, \beta; \gamma(t^-)) \right).
\]

Indeed, in this case \( T_\gamma \mathbb{L}(n)/T_\gamma \mathbb{L}^1(n)(\alpha) \sim S^2(\alpha \cap \gamma) \) which is oriented by the positive-definite quadratic forms and \( sgn D'(0) = \pm 1 \), we use then the previous formula.

**Remark 3.1.** — This formula allows to define index of path not necessarily closed, see [13].

3.2 Hörmander’s index.

Let \( \alpha, \beta, \beta' \) be three elements of \( \mathbb{L}(n) \) such that \( \beta, \beta' \in \mathbb{L}^0(n)(\alpha) \). For any path \( \sigma \) joining \( \beta \) to \( \beta' \) one defines

\[
|\sigma, \alpha| = \mu_0(\hat{\sigma})
\]

where \( \hat{\sigma} \) is the closed path obtained from \( \sigma \) by linking its endpoints staying in \( \mathbb{L}^0(n)(\alpha) \):

\[
\hat{\sigma} = \sigma \ast \sigma_\alpha \text{ and } \sigma_\alpha \subset \mathbb{L}^0(n)(\alpha).
\]

The theorem [331] shows that \( |\sigma, \alpha| \) does not depend on the way \( \sigma \) is closed staying in \( \mathbb{L}^0(n)(\alpha) \). Let now \( \alpha' \) be a point in \( \mathbb{L}^0(n)(\beta) \cap \mathbb{L}^0(n)(\beta') \). The index of Hörmander is the number

\[
s(\alpha, \alpha'; \beta, \beta') = |\sigma, \alpha'| - |\sigma, \alpha| = \mu_0(\sigma \ast \sigma_\alpha \ast (\sigma \ast \sigma_\alpha)^{-1}) = \mu_0(\sigma_\alpha \ast \sigma_\alpha^{-1})
\]

because the calculation of \( \mu_0 \) does not depend on the base point in \( S^1 \).

This index depends only on the four points in \( \mathbb{L}(n) \) and not on the paths :
Proposition 3.2. — Let \( \beta, \beta' \in \mathbb{L}^0(n) / (\alpha) \) then
\[
s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2} \left( \text{sgn } Q(\alpha, \beta'; \alpha') - \text{sgn } Q(\alpha, \beta; \alpha') \right).
\]

Indeed, first suppose that \( \alpha \) and \( \alpha' \) are transversal; the theorem (3.1) can be applied and also the proposition (3.1); this gives
\[
s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2} \left( \text{sgn } Q(\alpha, \alpha'; \beta) - \text{sgn } Q(\alpha, \alpha'; \beta') \right).
\]

On the other hand \( \beta \in \mathbb{L}^0(n) / (\alpha) \) can be written as the graph of \( C \in \text{End}(\alpha, \alpha') \) and so \( Q(\alpha, \alpha'; \beta) = \omega(C, \cdot) \). But also \( \alpha' \) is the graph of \( D \in \text{End}(\alpha, \beta) \) with \( \forall x \in \alpha, \quad D(x) = -(x + C(x)) \), then \( Q(\alpha, \beta; \alpha') = \omega(D, \cdot) = -\omega(C, \cdot) = Q(\alpha, \alpha'; \beta) \). As a consequence
\[
s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2} \left( \text{sgn } Q(\alpha, \beta'; \alpha') - \text{sgn } Q(\alpha, \beta; \alpha') \right).
\]

This formula can be generalized by the symplectic reduction (3.1). \( \blacksquare \)

Let us recall finally the

Proposition 3.3. — Let \( \alpha, \alpha', \beta, \beta' \) be four points in \( \mathbb{L}(n) \) such that \( \beta \) and \( \beta' \) are in \( \mathbb{L}^0(n) / (\alpha) \cap \mathbb{L}^0(n) / (\alpha') \) then
\[
s(\alpha, \alpha'; \beta, \beta') = -s(\alpha', \alpha; \beta, \beta') = -s(\alpha, \alpha'; \beta', \beta) = -s(\beta, \beta'; \alpha, \alpha').
\]

Only the third equality is not obvious. It can be shown by the formula of proposition (3.2). Choose symplectic coordinates \((x, \xi)\) such that \( \alpha = \{x = 0\} \) and \( \beta = \{\xi = 0\} \). By the transversality hypothesis there exist homomorphisms \( A \) and \( B \) such that
\[
\alpha' = \{x = A\xi\} \quad \beta' = \{\xi = Bx\}.
\]

If \( \alpha' \) is the graph of \( A' \in \text{Hom}(\alpha, \alpha') \), then for all \( \xi \in \alpha \) we must find \( \xi' \in \alpha \) and \( x \in \beta \) with
\[
A'\xi = (x, Bx) \quad \text{and} \quad (A'\xi', \xi') = (x, Bx + \xi).
\]

This gives \( x = A\xi' \) and \( \xi' = Bx + \xi = BA\xi' + \xi \) so \( \xi' = (1 - BA)^{-1}\xi \) and
\[
A'\xi = (A(1 - BA)^{-1}\xi, (1 - BA)^{-1}\xi - \xi).
\]

We remark that \( (1 - BA) \) is indeed invertible: if \( \xi \in \ker(1 - BA) \) then \( (A\xi, \xi) = (A\xi, BA\xi) \in \alpha' \cap \beta' = \{0\} \) so \( \xi = 0 \).

Therefore by the proposition (3.2)
\[
2s(\alpha, \alpha'; \beta, \beta') = \text{sgn } \omega(A(1 - BA)^{-1}, \cdot) - \text{sgn } \omega(A, \cdot) = \text{sgn } \begin{vmatrix} 0 & 0 \\ A & -A(1 - BA)^{-1} \end{vmatrix}.
\]

Suppose now that \( A \) is invertible then, because a symmetric matrix and its inverse have same signature:
\[
\text{sgn } \begin{vmatrix} A & 0 \\ 0 & -A(1 - BA)^{-1} \end{vmatrix} = \text{sgn } \begin{vmatrix} A & 0 \\ 0 & -(1 - BA)A^{-1} \end{vmatrix} = \text{sgn } \begin{vmatrix} A & 0 \\ 0 & B - A^{-1} \end{vmatrix} = \text{sgn } \begin{vmatrix} A & 0 \\ 0 & B - A^{-1} \end{vmatrix}.
\]

by formula (3.1.12). By the same calculus, and because \( \omega \) is skewsymmetric, one has:
\[
2s(\beta, \beta'; \alpha, \alpha') = \text{sgn } Q(\beta, \alpha'; \beta') - \text{sgn } Q(\beta, \alpha; \beta')) = -\text{sgn } \begin{vmatrix} B & 0 \\ 1 & A \end{vmatrix}.
\]

\( \blacksquare \)
3.3 Proof of theorem 1.1

Following [7], we denote by \( T(\mathcal{L}) \subset \mathbb{L}(\mathcal{L}) \) the set of the \( \alpha \in \mathbb{L}(\mathcal{L}) \) transversal to \( \lambda(\pi(\alpha)) \) and to \( \lambda_0(\pi(\alpha)) \). If \( p : T(\mathcal{L}) \to \mathcal{L} \) is the associated projection, then for all \( \nu \in \mathcal{L} \)

\[
p^{-1}(\nu) = \mathbb{L}^0(n)(\lambda(\nu)) \cap \mathbb{L}^0(n)(\lambda_0(\nu)).
\]

n.b. On the neighbourhood of points where the two Lagrangian are not transversal this map is not a fibration.

**Lemma 3.1**. — Let \( \alpha : S^1 \to T(\mathcal{L}) \) satisfying \( p \circ \alpha = \gamma \) and \( \sigma \) be a path as before. The index \([\sigma, \alpha(t)]\) is constant in \( t \).

Indeed the index is a continuous map : let \( t_0 \in [0, 1] \) and \( \beta \) a path in the fibre over the point \( \gamma(t_0) \) and linking \( \lambda_0(t_0) \) to \( \lambda(t_0) \) staying transversal to \( \alpha(t_0) \); by definition \([\sigma_{t_0}, \alpha(t_0)] = \mu_0(\sigma_{t_0} \ast \beta)\) but the property of transversality is open : if we denote \( \beta_t \) the path in the fibre over the point \( \gamma(t) \) resulting of the parallel transport of \( \lambda_0\left|_{[t,t_0]} \ast \beta \ast \lambda_{t_0}^{-1} \right|_{[t,t_0]} \); then there exists \( \varepsilon > 0 \) such that for all \( |t - t_0| < \varepsilon \) one has \( \beta_t \) is transversal to \( \alpha(t) \). This parallel transport realizes an homotopy, so for all \( |t - t_0| < \varepsilon \) one has \( \mu_0(\sigma_{t_0} \ast \beta) = \mu_0(\sigma_{t} \ast \hat{\beta}_t) \).

**Corollary 3.1**. — The induced fibre bundle \( p^*\mathcal{M}(\mathcal{L}) \) is trivial.

**Proof.** — We have to show that for all path \( \alpha : S^1 \to T(\mathcal{L}) \) continuous, if we define \( \gamma = p \circ \alpha \), then \( \mu(\gamma) = 0 \). To this goal take \( \sigma \) as before, a path in the fibre over \( \gamma(0) \) linking \( \lambda(0) \) to \( \lambda_0(0) \). Choose \( \sigma \) transversal to \( \alpha(1) \) and do the same constrution as before, then

\[
[\sigma, \alpha(1)] = [\sigma_0, \alpha(0)] = 0
\]

by the definition of \([\sigma, \alpha(1)]\) and lemma 3.1. But \( \alpha(0) = \alpha(1) \) so

\[
\mu(\gamma) = \mu_0(\sigma_0 \ast \sigma^{-1}) = [\sigma_0, \alpha(1)] = 0.
\]

**Corollary 3.2**. — Let \( s \) be a section of the Maslov bundle over \( \mathcal{L} \), and \( \gamma : S^1 \to \mathcal{L} \) a closed path such that \( \gamma(0) = \nu_0 = \pi(\lambda_0) \). Let \( \alpha : [0, 1] \to T(\mathcal{L}) \) be a continuous path satisfying \( \gamma = p \circ \alpha \). Then

\[
p^*(\alpha(1)) = i^s(\lambda_0, \lambda(\alpha(1); \alpha(0))) \cdot p^*(\alpha(0)).
\]

**Proof.** — Let \( \sigma \) be a path linking \( \lambda(0) \) to \( \lambda_0 \) staying transversal to \( \alpha(1) \). By lemma 3.1, \([\sigma_0, \alpha(0)] = [\sigma, \alpha(1)] = 0 \) and

\[
\mu(\gamma) = \mu_0(\sigma_0 \ast \sigma^{-1}) = [\sigma_0, \alpha(1)] = [\sigma_0, \alpha(0)] = s(\alpha(0), \alpha(1); \lambda(0), \lambda_0(0))
\]

and \( s(\alpha(0), \alpha(1); \lambda, \lambda_0) = -s(\lambda, \lambda; \alpha(1), \alpha(0)) \) by the proposition 3.3. Therefore

\[
-\mu(\gamma) = s(\lambda_0, \lambda(0); \alpha(1), \alpha(0)).
\]

This gives the result by the equivalent relation 3.3.3.

From these two corollaries one obtains

**Corollary 3.3**. — The sections of \( \mathcal{M}(\mathcal{L}) \) are identified with functions \( f \) on \( T(\mathcal{L}) \) satisfying the relation : \( \forall \alpha, \tilde{\alpha} \in T(\mathcal{L}) \)

\[
p(\alpha) = p(\tilde{\alpha}) \Rightarrow f(\tilde{\alpha}) = i^s(\lambda_0, \lambda(\tilde{\alpha}; \alpha)) f(\alpha).
\]

This result gives the gluing condition of Hörmander, in view of the theorem 3.3.3, 7 and finish the proof of the theorem. For completeness we recall this last step.
Proposition 3.4. — The functions $f$ on $\mathcal{T}(\mathcal{L})$ which satisfy:

$$p(\alpha) = p(\tilde{\alpha}) \Rightarrow f(\tilde{\alpha}) = i^{s(\lambda_0, \lambda\tilde{\alpha}; \alpha)}f(\alpha).$$

are the sections defined by the gluing conditions of the section 12.

Proof. — Let $\phi$ be a non degenerated phase function as in section 12. and $\nu_0 = (x_0, \xi_0) = (x_0, \phi'_0(x_0, \theta_0))$ a point in $\mathcal{L}_\phi$. For each $\alpha \in \mathcal{T}(\mathcal{L})$ such that $p(\alpha) = \nu_0$, there exists a function $\psi$ defined on an open set $U$ such that the graph $L_\psi = \{(x, d\psi(x)), x \in U\}$ of the differential $d\psi$ intersect transversally $\mathcal{L}_\phi$ at $\nu_0$, one has $L_{\nu_0} \mathcal{L}_\phi = \alpha$.

Or equivalently one can say: the following quadratic form defined on $\mathbb{R}^{n+N}$ by the matrix

$$Q_\psi \left| \begin{array}{ccc} \phi''_{xx} & -\psi''_{xx} & \phi''_{x\theta} \\ -\psi''_{xx} & \phi''_{xx} & \phi''_{\theta\theta} \\ \phi''_{x\theta} & \phi''_{\theta\theta} & \phi''_{\theta\theta} \end{array} \right|$$

(3.3.6)

is non degenerated.

The restriction of this quadratic form to the tangent $W$ of $\mathcal{L}_\phi$ at $\nu_0$ only depends on $\mathcal{L}$ and $\psi$ (and not on $\phi$). Indeed $\phi$ defines a card in which

$$\lambda(\nu_0) = T_{\nu_0}(\mathcal{L}) = \{(X, \phi''_{xx}X + \phi''_{x\theta}A); (X, A) \in \mathbb{R}^{n+N}, \phi''_{xx}X + \phi''_{x\theta}A = 0\};$$

if now $(X, A), (X', A')$ define two tangent vectors $V$ and $V' \in \lambda(\nu_0)$

$$Q_\psi \left( \begin{array}{c} (X, A) \\ (X', A') \end{array} \right) = \begin{array}{c} <X, (\phi''_{xx} - \psi''_{xx})X' + \phi''_{x\theta}A' > \\ <-\psi''_{xx} X, X' > - <X, \phi''_{xx}X' + \phi''_{x\theta}A' > = Q_\psi \left( \begin{array}{c} \lambda(\nu_0), \alpha; \lambda_0(\nu_0) \end{array} \right)(V, V') \end{array}$$

by definition (3.1.4). More precisely $\alpha$ is transverse to the two lagrangians $\lambda(\nu_0)$ and $\lambda_0(\nu_0)$ so the vertical $\lambda_0(\nu_0)$ is the graph of an homomorphism $A_\psi$ from $\lambda(\nu_0)$ to $\alpha = T_{\nu_0} L_\psi$:

$$\forall (0, \Xi) \in \lambda(\nu_0), \exists (X, A)\text{unique such that } \Xi = \phi''_{xx}X + \phi''_{x\theta}A \text{ et } \phi''_{xx}X + \phi''_{x\theta}A = 0$$

because $Q_\psi$ is non degenerated, and one can write

$$(0, \Xi) = (X, \phi''_{xx}X + \phi''_{x\theta}A) - (X, \psi''_{xx}X),$$

it means that $A_\psi(X, \phi''_{xx}X + \phi''_{x\theta}A) = (-X, -\psi''_{xx}X)$.

We see now that the orthogonal $WQ_\psi$ of $W$ with respect to $Q_\psi$ is $\mathbb{R}^N = \{(0, A)\}$ and that $Q_{\psi W}Q_\psi = \phi''_{x\theta}$. But the lemma below gives $\text{sgn } Q_\psi = \text{sgn } Q_{\psi | W} + \text{sgn } Q_{\psi | W}Q_\psi$, so :

$$\text{sgn } Q_\psi = \text{sgn } Q(\lambda(\nu_0), \alpha; \lambda_0(\nu_0)) + \text{sgn } \phi''_{x\theta}.$$  (3.3.7)

Let now $\nu_0$ be a section in the sens of H"{o}rmander. For any $\alpha \in \mathcal{T}(\mathcal{L}), p(\alpha) = \nu_0$, if $\phi$ and $\tilde{\phi}$ are two phase functions defining $\mathcal{L}$ in a neighbourhood of $\nu_0$ and if $\psi$ is a function on $X$ satisfying $\alpha = T_{\nu_0} L_\psi$, we denote by $Q_\psi$ and $\tilde{Q}_\psi$ the respective quadratic forms defined by (3.3.6). Put

$$f(\alpha) = \exp(i\pi \alpha \text{sgn } Q_\psi)z_\phi(\nu_0).$$

By the relation (3.3.7) one has $\text{sgn } \phi''_{x\theta} - \text{sgn } \phi''_{x\theta} = \text{sgn } Q_\psi - \text{sgn } \tilde{Q}_\psi$; the compatibility condition 12.12.2 gives then

$$\exp(i\pi \alpha \text{sgn } Q_\psi)z_\phi(\nu_0) = \exp(i\pi \alpha \text{sgn } \tilde{Q}_\psi)z_\tilde{\phi}(\nu_0)$$

and the function $f$ is well defined on $\mathcal{T}(\mathcal{L})$. On the other hand if $\tilde{\alpha}$ is an other point in $\mathcal{T}(\mathcal{L})$ such that $p(\tilde{\alpha}) = \nu_0$ and if $\tilde{\psi}$ is an adapted function, then

$$f(\tilde{\alpha}) = \exp(i\pi \alpha \text{sgn } \tilde{Q}_\psi - \text{sgn } Q_\psi)\tilde{f}(\alpha)$$

$$= \exp(\left(i\pi \left(\alpha \text{sgn } Q(\lambda(\nu_0), \tilde{\alpha}; \lambda_0(\nu_0)) - \text{sgn } Q(\lambda(\nu_0), \alpha; \lambda_0(\nu_0))\right)\right)\tilde{f}(\alpha)$$

$$= \exp\left(i\pi \alpha s(\lambda_0(\nu_0), \lambda(\nu_0); \alpha, \tilde{\alpha})\right)\tilde{f}(\alpha)$$

$$= \exp\left(i\pi \alpha s(\lambda_0(\nu_0), \lambda(\nu_0); \tilde{\alpha}, \alpha)\right)\tilde{f}(\alpha)$$

and the relation (3.3.7) becomes valid.
As a consequence of the works of Arnol’d recalled above, a generator of $\Pi\lambda$ by $\text{Det} \lambda$ is trivial and $\sigma(1) = 1$. — Let’s construct the following homotopy of $\tilde{\gamma}$. — Let $\rho_* : \Pi_1(\mathbb{L}(n)) \rightarrow \Pi_1(\mathbb{L}(L))$ be the restriction of the parallel transport along $\gamma$. Then $\rho_* = \rho_\lambda$. — As was seen in paragraph 2, the parallel transport along $\gamma$ defines an homotopy of $\lambda(\gamma) \cdot i_* (\sigma) \cdot (\lambda(\gamma))^{-1}$ to a path which can be written $\tilde{\lambda}(\gamma) \cdot \tilde{\sigma} \cdot (\tilde{\lambda}(\gamma))^{-1}$ where $\tilde{\sigma}$ is a path in restriction to $\mathbb{R}^q$. — This lemma can be showed using an induction on dim $V$. — This lemma can be showed using an induction on dim $V$.

4 Topological comments

Let’s have a look to the exact sequence : $0 \rightarrow \Pi_1(\mathbb{L}(n)) \rightarrow \Pi_1(\mathbb{L}(L)) \rightarrow \Pi_1(L) \rightarrow 0$. The group $\Pi_1(\mathbb{L}(L))$ is the semidirect product of $\Pi_1(\mathbb{L}(n))$ and $\Pi_1(L)$. It means that $\Pi_1(L)$ acts on $\Pi_1(\mathbb{L}(n))$ by conjugation. More precisely for all $\gamma \in \Pi_1(L)$ let’s define

$$\rho_\gamma : \Pi_1(\mathbb{L}(n)) \rightarrow \Pi_1(\mathbb{L}(n)) \quad \sigma \mapsto \lambda(\gamma) \cdot i_* (\sigma) \cdot (\lambda(\gamma))^{-1}$$

Lemma 4.1 This representation is trivial and $\Pi_1(\mathbb{L}(L))$ is in fact the direct product of $\Pi_1(\mathbb{L}(n))$ and $\Pi_1(L)$.

Proof. — As was seen in paragraph 2, the parallel transport along $\gamma$ defines an homotopy of $\lambda(\gamma) \cdot i_* (\sigma) \cdot (\lambda(\gamma))^{-1}$ to a path which can be written $\tilde{\lambda}(\gamma) \cdot \tilde{\sigma} \cdot (\tilde{\lambda}(\gamma))^{-1}$ where $\tilde{\sigma}$ is the image of $\sigma$ by $\tau(\gamma)$. But

$$\mu_0(\tilde{\lambda}(\gamma) \cdot \tilde{\sigma} \cdot (\tilde{\lambda}(\gamma))^{-1}) = \mu_0((\tilde{\lambda}(\gamma))^{-1} \cdot \tilde{\lambda}(\gamma) \cdot \tilde{\sigma}) = \mu_0(\tilde{\sigma}) = \mu_0(\sigma).$$

As a consequence of the works of Arnol’d recalled above, a generator of $\Pi_1(\mathbb{L}(n))$ is characterized by $\mu_0(\sigma) = 1$.

Theorem 4.1. — Let $\mathbb{L}^1(L)$ be the set of the points $l \in L$ which are not transversal to $\lambda(\pi(l))$. It is an oriented cycle of $L$ of codimension 1 ; if $m$ is its Poincaré dual form, then

$$\mu = \lambda^* m.$$

Proof. — We keep the notations of paragraph 2. By choosing the starting point one can suppose that the two lagrangians $\lambda_0 = \lambda(0)$ and $\lambda_0(0)$ are transversal. We will use a deformation of the path $\tilde{\lambda} \cdot \tilde{\sigma} \cdot \tilde{\lambda}_0^{-1}$ joining $\lambda(0)$ to $\lambda(0)$. Recall that $\tilde{\sigma}(t) = \tau(\gamma)(\sigma(t))$.

There exists a (continuous) path $u(t) \in U(n)$ such that $u(0) = I$ and

$$\forall t \in [0, 1] \quad \tilde{\lambda}_0(t) = u(t)\lambda(0).$$

But $\tilde{\lambda}_0(1) = \tau(\gamma)(\lambda(0))$, so $\tau(\gamma)$ and $u(1)$ differ by an element of $O(n)$:

$$\exists a \in O(n) : \tau(\gamma) = u(1) \circ a.$$

Let’s construct the following homotopy of $\tilde{\lambda} \cdot \tilde{\sigma} \cdot \tilde{\lambda}_0^{-1}$ by the concatenation of $u(st)^{-1} \tilde{\lambda}(t)$, next $u(s)^{-1} \tilde{\sigma}$ and finally the inverse of $u(st)^{-1} \tilde{\lambda}_0(t)$. The end of this homotopy is a path, result of the concatenation of $\tilde{\lambda}(t) = u(t)^{-1} \tilde{\lambda}(t)$ and $u(1)^{-1} \tilde{\sigma} = a \sigma$ because $u(t)^{-1} \tilde{\lambda}_0(t) = \lambda_0$ is a constant path.

We have now to calculate $\mu_0(\sigma^{-1} \cdot \tilde{\lambda} \cdot a \sigma)$. Because $a \in O(n)$

$$\text{Det}^2(\sigma(t)) = \text{Det}^2(a \sigma(t));$$

$\text{Det}^2 \circ \tilde{\lambda}$ is a closed path even if $\tilde{\lambda}$ is not, so $\mu(\gamma) = \text{Degree}(\text{Det}^2 \circ \tilde{\lambda})$.

Considering the results of section 2.1, we have obtained
Proposition 4.1 $\mu(\gamma)$ is the intersecting number of the submanifold $\mathbb{L}^1(n)(\lambda_0)$ and the cycle obtained from $\bar{\lambda}$, by closing it with a path staying transversal to $\lambda_0$.

Remark that $\bar{\lambda}(0) = \lambda(0)$ and $\bar{\lambda}(1) = a\lambda(0)$ are both transversal to $\lambda_0$. Let’s now

$$\mathbb{L}^1(\mathcal{L}) = \left\{ l \in \mathcal{L} ; \lambda_0(\pi(l)) \cap l \neq \emptyset \right\}.$$ 

It is a fibration above $\mathcal{L}$ with fibre $\mathbb{L}^1(n)(\lambda_0)$, so it is an oriented cycle of codimension 1 in $\mathcal{L}$. If $\lambda \circ \gamma$ cuts $\mathbb{L}^1(\mathcal{L})$ transversally at $\lambda \circ \gamma(t)$ then $\bar{\lambda}$ cuts transversally $\mathbb{L}^1(n)(\lambda_0)$ at $\bar{\lambda}(t)$ and conversely. Moreover the transformations which permit to pass from $\lambda \circ \gamma$ to $\bar{\lambda}$ realise a continuous deformation of $\mathbb{L}^1(\mathcal{L})$ to $\mathbb{L}^1(n)(\lambda_0)$ above $\gamma$. This argument finishes the proof of the theorem.

5 References

1. Aebischer B, Borer M & al. — Symplectic Geometry, Birhäuser, Basel, 1992.
2. Anné C., Charbonnel A-M. — Bohr Sommerfeld conditions for several commuting Hamiltonians, preprint (Juillet 2002), ArXiv math-ph/0210026.
3. Arnol’d V.I. — Characteristic Class entering in Quantization Conditions. Funct. Anal. and its Appl. 1 (1967), 1-14.
4. Duistermaat J. — Morse Index in Variational Calculus. Adv. in Maths 21 (1976), 173–195.
5. Duistermaat J., Guillemin V. — The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. 29 (1975), 39-79.
6. Guillemin V., Sternberg S. — Geometric Asymptotics Math. Surveys and Monograph n° 14, AMS (1990).
7. Hörmander L. — Fourier Integral operators. Acta Math. 127 (1971), 79–183.
8. Hörmander L. — The Weyl calculus of pseudodifferential operators. Comm. Pure Appl. Math. 32 (1979), 359-443.
9. Hörmander L. — The Analysis of Linear Partial Differential Operators III., Springer, Berlin - Heidelberg - New York, 1985.
10. Hörmander L. — The Analysis of Linear Partial Differential Operators IV., Springer, Berlin - Heidelberg - New York, 1985.
11. Husemoller D. — Fibre Bundles, Springer, Berlin - Heidelberg - New York, 1975.
12. Maslov V.P. — Théorie des perturbations et méthodes asymptotiques, Dunod, Gauthier-Villars, Paris 1972.
13. Robin J., Salamon D. — The Maslov Index for Path. Topology 32 (1993), 827–844.