Two types of branching programs with bounded repetition that cannot efficiently compute monotone 3-CNFs

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Abstract

It is known that there are classes of 2-CNFs requiring exponential size non-deterministic read-once branching programs to compute them. However, to the best of our knowledge, there are no superpolynomial lower bounds for branching programs of a higher repetition computing a class of 2-CNFs. This work is an attempt to make a progress in this direction.

We consider a class of monotone 3-CNFs that are almost 2-CNFs in the sense that in each clause there is a literal occurring in this clause only. We prove exponential lower bounds for two classes of non-deterministic branching programs. The first class significantly generalizes monotone read-\(k\)-times \( \text{NBP}\)s and the second class generalizes oblivious read \(k\) times branching programs. The lower bounds remain exponential for \(k \leq \log n/a\) where \(a\) is a sufficiently large constant.

1 Introduction

Statement of Results. It is known that there are classes of 2-CNFs that require Nondeterministic Read-Once Branching Programs (NROBPs) of exponential size for their representation. For example, in [8], we have essentially shown that a NROBP computing a monotone 2-CNF with a bounded number of occurrences of each variable is of size exponential in the pathwidth of the primal graph of the 2-CNF. An exponential lower bound is thus easy to obtain by taking a 2-CNF corresponding to an expander graph.

A natural direction for further research is to understand the complexity of branching programs with higher repetition on monotone 2-CNFs. Indeed, to the best of our knowledge, for 2-CNFs, there are no superpolynomial lower bounds for non-deterministic branching programs with a repetition higher than one, even for the restricted cases where the branching programs are allowed to be monotone or oblivious and deterministic.
This work is an attempt to make a progress in this direction. Instead of monotone 2-CNFs we consider padded monotone 3-CNFs that are ‘almost’ 2-CNFs. In particular each clause of a padded monotone 3-CNF contains a padding literal, that is a literal appearing in this clause only, the number of occurrences of the rest two literals is larger than 1. It is not hard to see that there is a one-to-one correspondence between padded monotone 3-CNFs and graphs without isolated vertices or vertices of degree 1. In particular, the CNF corresponding to such a graph \( G \) has \( V(G) \cup E(G) \) as the set of variables and the clauses correspond to the edges. In particular, literals of the clause corresponding to \( e \in E(G) \) are the variable corresponding to \( e \) and the variables corresponding to the end vertices of \( e \). We denote the CNF corresponding to \( G \) by \( CNF(G) \).

In this paper we consider the class of CNFs \( CNF(K_n) \) where \( K_n \) is the complete graph of \( n \) vertices. We establish exponential lower bounds for non-deterministic read-\( k \)-times monotone branching programs and non-determinisitic read-\( k \)-times oblivious branching programs where \( k \leq \log m/c \) where \( m \) is the number of variables of the considered CNF and \( c \) is a sufficiently large constant.

In fact, in order to ‘push’ as far as possible towards exponential lower bounds for non-deterministic read-\( k \)-times branching programs, we establish the exponential lower bounds for generalizations of non-deterministic monotone and oblivious read \( k \)-times branching programs that are specified below.

Let us partition the variables of \( CNF(G) \) into vertex variables, i.e. those that correspond to the vertices of \( G \) and edge variables )those that correspond to the edges of \( G \). The generalization of non-deterministic read-\( k \)-times monotone branching programs is called \( k \)-Vertex Edge Monotone Branching Programs (\( k \)-VEMBP). In this model vertex variables are allowed to appear at most \( k \) times (monotonicity is not required), while edge variables are required to be monotone, though the number of their occurrences is not limited.

The generalization of non-deterministic oblivious read-\( k \) times branching programs requires vertex variables to be oblivious and to occur at most \( k \) on each consistent path (that is, this restriction is semantic, while for \( k \)-VEMBP, the restriction is syntactic because it applies to each path). The edge variables are not constrained at all. We refer to this model \( k \)-vertex oblivious branching programs (\( k \)-VOBP).

**Overview of the proof.** As mentioned above, the considered class of CNFs is associated with (undirected simple) graphs as follows. Given a graph \( G \), the CNF \( CNF(G) \) [9] has variables \( \{X_u | u \in V(G)\} \) (the vertex variables) and \( \{X_{u,v} | \{u,v\} \in E(G)\} \) (the edges variables) and the clauses \( \{X_u \lor X_{u,v} \lor X_v\} \) for each \( \{u,v\} \in E(G) \). The above class \( K \) consists of \( CNF(K_n) \) for even \( n \) where \( K_n \) is a complete graph of \( n \) vertices.

The lower bounds are stated in Theorem 1. In order to prove the theorem, we introduce a new graph parameter, which we call \( (k,c) \)-fold matching width. This is a generalization of matching width which we used to prove lower bounds in [9] [8] [7], in particular, the matching width can be seen as \( (1,2) \)-fold matching width. Putting \( c = 8k + 10 \) and using Theorem 1.1. from [2], we show (Theorem 2) that the \( (k,c) \)-fold matching width of clique is at least \( n/2^c \). Then we show
(resp. Theorems 3 and 4) that for a graph $G$ with $(k, c)$-fold matching width at least $t$, both $k$-VEMBP and $k$-VOBP need an exponential size in $t/(c - 1)$ to compute $CNF(G)$. Theorem 1 follows from combination of Theorems 2, 3 and 4.

In order to prove Theorem 3, we associate a matching $M$ of $G$ with an (essentially) disjunction of cnf which we call a decision tree w.r.t. $M$. The main part of the proof is Theorem 4, stating that if $A$ is a set of decision trees w.r.t. matching of size $t$ such that every satisfying assignment of $CNF(G)$ satisfies at least one element of $A$ then $|A|$ must be exponential in $t$. We prove this by a probabilistic argument, defining a probability space over satisfying assignments of $CNF(G)$ and showing that the probability of satisfying an individual decision tree w.r.t. a matching of size at least $t$ is exponentially small in $t$. Then Theorem 4 immediately follows by the union bound. Theorem 3 is then proved by showing that it is possible to define a function from a subset of length $c - 1$ sequences of nodes of a $k$-VEMBP $Z$ computing $CNF(G)$ to decision trees w.r.t matchings of size at least $t$ so that every satisfying assignment of $CNF(G)$ would satisfy a decision tree in the range of this function. It will immediately follow from Theorem 4 that size of the range is exponential in $t$ and hence so is the size of the domain (upper-bounded by the number of $c - 1$-sequences of nodes of $Z$) of the function. We will conclude that the number of nodes of $Z$ is exponential in $t/(c - 1)$.

To prove Theorem 4, we specify a set $S$ of $2^t$ satisfying assignments of $CNF(G)$. For each $S \in S$, we specify a consistent root-leaf path $P^S$ of a $k$-VOBP $Z$ computing $CNF(G)$ whose labelling set of literals is a subset of $S$. Then, using ‘superposition’ of paths, a standard ‘fooling’ technique for proving lower bounds for oblivious branching programs, we identify on each $P^S_1$ and $P^S_2$ with $S_1 \neq S_2$. Then we apply the counting argument similar to the one mentioned in the end of the previous paragraph.

Related work. Exponential lower bounds for branching programs with bounded repetition are well known both when occurrences of each individual variable are restricted [3] and the total length of root-leaf paths is restricted [1].

The author considered monotone padded 3-CNFs in [10] for non-deterministic semantic $k$-OBDD, a special case on nondeterministic semantic oblivious read $k$-times branching programs where the sequence of variables on each computational path is a subsequence of $k$ copies of a fixed permutation. We are not aware of other results separating branching programs of bounded repetition from monotone 2-CNFs or monotone padded 3-CNFs.

The only other results we are aware of that separate branching programs with repetition higher than 1 and CNFs with constant length clauses are [5] (for nondeterministic read-$k$-times branching programs with $k \leq c \log / \log \log n$ for some constant $c$) and [11] (for randomized read-$k$-times branching programs with repetition $k \leq c \log n$ for some constant $c$ and probability of error exponentially small in $2^k$). Both these results consider CNFs with all literals being negative, so the respective lower bounds clearly hold for monotone CNFs resulting from ‘switching sign’ of literals.
Definition of monotone NBPs can be found in [4] or [6], we are not aware of previous results considering monotone NBPs with bounded repetition.

2 Preliminaries

In this paper by a set of literals we mean one that does not contain both an occurrence of a variable and its negation. For a set $S$ of literals we denote by $\text{Var}(S)$ the set of variables whose literals occur in $S$ (the $\text{Var}$ notation naturally generalizes to CNFs and Boolean functions). A set $S$ of literals represents the truth assignment to $\text{Var}(S)$ where variables occurring positively in $S$ (i.e. whose literals in $S$ are positive) are assigned with true and the variables occurring negatively are assigned with false. For example, the assignment $\{x_1 \leftarrow \text{true}, x_2 \leftarrow \text{true}, x_3 \leftarrow \text{false}\}$ to variables $x_1, x_2, x_3$ is represented as $\{x_1, x_2, \neg x_3\}$.

Let $CC$ be a CNF. A set $S$ of literals satisfies a clause $C$ of $CC$ if at least one literal of $C$ belongs to $S$. If all clauses of $CC$ are satisfies by $S$ then $S$ satisfies $CC$. If, in addition, $\text{Var}(CC) = \text{Var}(S)$ then we say that $S$ is a satisfying assignment of $CC$. The notion of a satisfying assignment naturally extends to Boolean functions $F$ meaning a truth assignment to $\text{Var}(F)$ on which $F$ is true.

We are now going to present a general terminology related to Non-deterministic branching programs (NBPs). An NBP $Z$ computing a Boolean function $F$ is a directed acyclic graph (DAG) with one root and one leaf with some edges labelled with literals of variables of $\text{Var}(F)$. Multiple edges are allowed. A (directed) path $P$ of $Z$ is consistent if opposite literals of the same variable do not occur on it (as labels of its edges). For a consistent path $P$, we denote by $A(P)$ the set of literals labelling its edges. A consistent root-leaf path of $Z$ is called a computational path of $Z$. The connection between $Z$ and $F$ is the following. A set $S$ of literals with $\text{Var}(S) = \text{Var}(F)$ is a satisfying assignment of $F$ if and only if there is a computational path $P$ of $Z$ such that $A(P) \subseteq S$. The size of $Z$ denoted by $|Z|$ is the number of its vertices. A well known way to obtain special classes of NBPs is by imposing constraints on its root-leaf paths (e.g. the number occurrences of each variable). If such a restriction is imposed on computational paths only, this restriction is called semantic. Otherwise, if the restriction is imposed on all root-leaf paths, it is called syntactic.

Remark. Note that if $Z$ computes a CNF $CC$ then, for every computational path $P$, $A(P)$ satisfies $CC$. Indeed, otherwise, $A(P)$ can be extended to an assignment to all the variables of $\text{Var}(CC)$ that falsifies one of its clauses.

3 The lower bounds

Let $G$ be a graph. Then by $CNF(G)$ we denote the monotone 3-CNF having the set of variables $V \text{Var}(G) \cup E \text{Var}(G)$ where $V \text{Var}(G) = \{X_u | u \in V(G)\}$ and $E \text{Var}(G) = \{X_{u,v} | \{u, v\} \in E(G)\}$ and the set of clauses $\{(X_u \lor X_{u,v} \lor X_v) | \{u, v\} \in E(G)\}$. We call variables of $V \text{Var}(G)$ and $E \text{Var}(G)$ the vertex
variables and the edge variables of $\text{CNF}(G)$, respectively.

Now we define two types of NBPs computing $\text{CNF}(G)$. The first type has the following constraints: (i) on each root-leaf path, each vertex variable occurs at most $k$ times; (ii) all the occurrences of edge variables on each root-leaf path are positive (the branching program is monotone on edge variables) but their repetition is unbounded. Note that the above constraints are syntactic because they apply to each root-leaf path of the considered branching program. We call a branching program obeying the above constraints a $k$-Vertex Edge Monotone Branching Program ($k$-VEMBP). Note that $k$-VEMBPs generalize monotone read-$k$-times NBPs.

The second considered class of NBPs obeys the following constraints: (i) on each computational path, each vertex variable occurs at most $k$ times; (ii) there is a sequence $XSV$ of $V\text{Var}(G)$ where each variable occurs exactly $k$ times such that the sequence of vertex variables along each computational path is a subsequence of $XSV$. Note that the above constraints are semantic because they apply to computational paths only. Moreover, there are no constraints on edge variables even on computational paths. We call an NBP obeying the above constraints a $k$-Vertex-Oblivious Branching Program ($k$-VOBP). Note that a $k$-VOBP generalizes semantic oblivious read-$k$-times NBPs.

Now we are ready to state the lower bounds for the above branching programs. For that purpose, recall that $K_n$ is a clique on $n$ variables.

**Theorem 1** Let $s = 8k + 10$. Then the size of $k$-VEMBP computing $\text{CNF}(K_n)$ is at least $(8/7)^{\Omega(n/(2^s(s-1)))}$ and the size of $k$-VOBP computing is at least $2^{n/(2^s(s-1))}$.

Note that the number $m$ of variables of $\text{CNF}(K_n)$ is $n(n+1)/2$. That is, for sufficiently large $n$, $m^{1/2} < n < 2m^{1/2}$ and hence, in terms of the number of variables, the above lower bounds can be restated as $(8/7)^{\Omega(m^{1/2}/(2^s(s-1)))}$ and $2^{n^{1/2}/(2^s(s-1))}$, respectively, for a sufficiently large $n$. Clearly the lower bound remains exponential for $k \leq \log n/a$ where $a$ is a sufficiently large constant.

In order to prove Theorem 1 we introduce a graph parameter and show this parameter is large for cliques and that both $k$-VEMBP and $k$-VOBP are of exponential size on $\text{CNF}(G)$ when this parameter for $G$ is large. Theorem 1 will immediately follow from the combination of these two statements. The parameter is a generalization of matching width that we used in [8] to prove a lower bound for read-once branching programs.

An interval of a sequence $S = s_1, \ldots, s_q$ is a subsequence $s_i, \ldots, s_j$ ($1 \leq i \leq j \leq q$).

**Definition 1** $c$-separation. Let $S$ be a sequence of elements of a universe set $U$. Let $S_1, \ldots, S_c$ be a partition of $S$ into intervals. Suppose $X, Y \subseteq U$ are disjoint sets such that, for each $1 \leq i \leq c$, the following is true: (i) $S_i \cap (X \cap Y) \neq \emptyset$; (ii) if $i$ is odd then $S_i \cap (X \cap Y) \subseteq X$; (iii) if $i$ is even then $S_i \cap (X \cap Y) \subseteq Y$. Then $S$ has a $c$-separation w.r.t. $(X, Y)$ and $S_1, \ldots, S_c$ are witnessing intervals of this separation.
Example 1 Let $U = \{1, 2, 3, 4, 5\}$ and let $S = (1, 2, 3, 1, 4, 5, 1, 1, 3, 2)$. Then the intervals $S_1 = (1, 2, 3), S_2 = (1, 4, 5, 1), S_3 = (1, 3, 2)$ witness a 3-separation of $S$ w.r.t. $\langle \{2, 3\}, \{4, 5\} \rangle$.

Definition 2 A graph $G$ has $c_1, c_2$-fold matching width at least $t$ if for each sequence $SV$ of elements of $V(G)$ where each element appears exactly $c_1$ times, there is a matching $M = \{\{u_1, v_1\}, \ldots, \{u_t, v_t\}\}$ of $G$ such that $SV$ has a $c \leq c_2$-separation w.r.t. $\langle U = \{u_1, \ldots, u_t\}, V = \{v_1, \ldots, v_t\} \rangle$.

An equivalent but, possibly, more intuitive definition is that for each sequence $SV$ of elements of $V(G)$ where each element occurs at most $c_1$ times there is a partition of $SV$ into at most $c_2$ intervals and a matching $M$ as above so that there is no interval containing elements of both $U$ and $V$.

Theorem 2 Let $s = 8k + 10$. Then for each even $n$, the $(k, s)$-fold matching width of $K_n$ is at least $\lfloor n/2^s \rfloor$.

Proof. Let $SV$ be a sequence of vertices of $K_n$ where each vertex appears exactly $k$ times. Let $U, V$ be two disjoint subsets of $V(G)$. An interval $I$ of $SV$ is a link between $U$ and $V$ if intermediate elements of $I$ do not belong to $U \cup V$ and either (i) the first element of $I$ belong to $U$ and the last element of $I$ belongs to $V$ or (ii) the first element of $I$ belong to $V$ and the last element of $I$ belongs to $U$.

Claim 1 There are disjoint sets $U, V$ of size at least $\lfloor n/2^s \rfloor$ such that there are at most $c \leq s - 1$ links between them.

Proof. Assume the claim is not true. Put $\ell = \lfloor n/2^s \rfloor$. Let $\{1, \ldots, n\}$ be elements of $V(G)$ being arbitrarily enumerated and let $U_1 = \{1, \ldots, n/2\}$ and $U_2 = \{n/2 + 1, \ldots, n\}$. By our assumption, for each $U \subseteq U_1, V \subseteq U_2$ with $|U| = |V| = \ell$ there are at least $s$ links between $U$ and $V$. By Theorem 1.1. of Alon and Maass [2], $|SV| \geq n(s - 9)/8 = n(8k + 10 - 9)/8 = nk + n/8 > nk$ in contradiction to the definition of $SV$. □

Let $U = \{u_1, \ldots, u_\ell\}$ and let $V = \{v_1, \ldots, v_\ell\}$ be as stated in the claim and let $I_1, \ldots, I_c$ be the links between them in $SV$. Note that $M = \{\{u_1, v_1\}, \ldots, \{u_\ell, v_\ell\}\}$ is a matching of $K_n$. It remains to show that there is $c + 1$ separation either w.r.t. $(V, U)$ or w.r.t. $(U, V)$. Let $x_1, \ldots, x_c$ be the respective first elements of $I_1, \ldots, I_c$. It is not hard to see that they are all distinct elements of $SV$. Assume w.l.o.g. that they occur on $SV$ in the order they are listed. Let $SV_1, \ldots, SV_{c+1}$ be partitioning of $SV$ into intervals defined as follows. $SV_1$ is the prefix of $SV$ whose last element is $x_1$. For $1 < i \leq c, SV_i$ is an interval whose first element is the immediate successor of $x_{i-1}$ and the last element is $x_i$. Finally, $SV_{c+1}$ is the suffix of $SV$ starting at the immediate successor of $x_c$. Observe that $SV_1, \ldots, SV_{c+1}$ is well defined. In particular, notice that $SV_{c+1}$ is not empty because it contains the last element of $I_c$.

Notice that by construction, each $SV_i$ contains an end vertex of a link between $U$ and $V$ and hence intersects with either $U$ or $V$. On the other, no $SV_i$ intersects with both $U$ and $V$ because no $SV_i$ contains a whole $U - V$-link.
follows that each \( SV_i \) is either ‘U-only’ or ‘V-only. Next, by construction, any \( SV_j + SV_{j+1} \) contains \( I_j \) and hence intersects with both \( U \) and \( V \). It follows that if \( SV_j \) is U-only then \( SV_{j+1} \) is V-only and vice versa. We conclude that either all \( SV_j \) with odd \( j \) are U only and all \( SV_j \) with even \( j \) are V-only or, all \( SV_j \) with odd \( j \) are V-only and all \( SV_j \) with even \( j \) are U only. It follows that \( SV_1, \ldots, SV_{c+1} \) are witnessing intervals of \( c+1 \leq s \) separation of \( SV \) w.r.t. either \((U,V)\) or \((V,U)\).

**Theorem 3** Let \( G \) be a graph of \( n \) vertices. Let \( k, c, t \) be integers such that the \( k,c \)-fold matching of \( G \) is at least \( t \). Let \( Z \) be a \( k \)-VEMBP implementing \( CNF(G) \). Then \( Z \) has at least \( (8/7)^{(t/(c-1))} \) nodes.

**Theorem 4** Let \( G, n, k, c, t \) be as in Theorem 3 and let \( Z \) be a \( k \)-VOBP implementing \( CNF(G) \). Then \( Z \) has at least \( 2^{t/(c+1)} \) nodes.

Theorems 3 and 4 are proved in Sections 4 and 5, respectively. Now, we already to prove Theorem 1.

**Proof of Theorem 1**. Put \( c = 8k + 10 \) and \( t = n/2^c \). Then the hypotheses of both Theorem 3 and 4 are satisfied by Theorem 2. Theorem 1 now immediately follows from Theorems 3 and 4 by substitution \( t = n/2^c \).

## 4 Proof of Theorem 3

A \( k \)-VEMBP is uniform if on each root-leaf path each \emph{vertex} variable occurs exactly \( k \) times. The following lemma is proved in Section A of the Appendix.

**Lemma 1** Let \( Z \) be a \( k \)-VEMBP computing \( CNF(G) \) for a graph \( G \). Then there is a uniform \( k \)-VEMBP of size \( O(n^mZ^{c/k}) \) computing \( CNF(G) \), where \( m \) is the number of edges of \( G \).

We prove Theorem 3 for a uniform \( k \)-VEMBP as it follows from Lemma 1 that this assumption does not restrict generality.

**Lemma 2** Let \( Z \) be a uniform \( k \)-VEMBP implementing \( CNF(G) \). Let \( P_1 \) and \( P_2 \) be two paths having the same initial and final vertices. Then \( Var(P_1) \cap VVar(G) = Var(P_2) \cap VVar(G) \). Put it differently, a vertex variable \( X_u \) occurs on \( P_1 \) if and only if \( X_u \) occurs on \( P_2 \).

**Proof.** Let \( u \) and \( v \) be the starting and ending vertices of \( P_1 \) and \( P_2 \). Denote by \( rt \) and \( lf \) the root and leaf vertices of \( Z \). Let \( P_0 \) be a \( rt - u \) path of \( Z \) and \( P_3 \) be a \( v - lf \) path of \( Z \). Then, due to the acyclicity both \( Q_1 = P_0 + P_1 + P_3 \) (the concatenation of \( P_0, P_1, P_3 \)) and \( Q_2 = P_0 + P_2 + P_3 \) are root-leaf paths of \( Z \). Suppose that \( V_1 = Var(P_1) \cap VVar(G) \neq V_2 = Var(P_2) \cap VVar(G) \) and that, say, \( V_1 \backslash V_2 \neq \emptyset \). Let \( X_u \in V_1 \backslash V_2 \). Since \( X_u \) does not occur in \( P_2 \), it occurs \( k \) times in \( P_1 \cup P_3 \) due to the uniformity of \( Z \), in particular, requiring \( k \) occurrences of \( X_u \) on \( Q_2 \). It follows that in \( Q_1 \) \( X_u \) occurs at least \( k + 1 \) times in contradiction to the definition of a \( k \)-VEMBP.
Definition 3 MCNF\((M)\) and XIR\((M)\). Let \(G\) be a graph and let \(M = \{\{u_1,v_1\}, \ldots, \{u_t,v_t\}\}\) be a matching of \(G\). A matching \(\text{cnf}\) w.r.t. \(M\) consists of \(t\) clauses \(C_1, \ldots, C_t\) such that \(C_i\) is either \((X_{u_i} \lor X_{v_i})\) or \((X_{u_i} \land X_{v_i})\). We denote the set of all matching \(\text{cnfs}\) w.r.t. \(M\) by \(\text{MCNF}(M)\). We denote the set \(\{X_u \mid u \in V(G) \setminus V(M)\}\) by \(\text{XIR}(M)\) (‘IR’ stands for ‘irrelevant’).

Definition 4 Decision tree w.r.t. \(M\). Let \(G\) and \(M\) be as in Definition 3. A decision tree \(T\) w.r.t. \(M\) is a directed tree whose edges are oriented from the root towards the leaf. Each leaf node \(a\) is associated with an element of \(\text{MCNF}(M)\) denoted by \(\varphi(a)\). Each non-leaf node \(a\) is associated with a variable of \(\text{XIR}(M)\) denoted by \(\text{Var}(a)\) so that on any root-leaf path each variable of \(\text{XIR}(M)\) occurs at most once as a label of a non-leaf node. Each non-leaf node \(a\) has two outgoing edges called the positive and negative edges which are associated with the respective literals of \(\text{Var}(a)\) (i.e. \(\text{Var}(a)\) and \(\neg\text{Var}(a)\), respectively). For a root-leaf path \(P\) of \(T\) we denote by \(L(P)\) the set of literals associated with the edges of \(P\). A set \(S\) of literals satisfies \(T\) if there is a root-leaf path \(P\) of \(T\) such that \(L(P) \subseteq S\) and \(S\) satisfies \(\varphi(a)\) where \(a\) is the last vertex of \(P\).

Example 2 Let \(G\) be a graph on vertices \(\{u_1, \ldots, u_6\}\) and suppose that \(M = \{\{u_3,u_4\}, \{u_5,u_6\}\}\) is a matching of \(G\). Then \(C_1 = (X_{u_3} \lor X_{u_3,u_4}) \land (X_{u_5} \lor X_{u_5,u_6}), C_2 = (X_{u_3} \lor X_{u_3,u_4}) \land (X_{u_5} \lor X_{u_5,u_6}), \) and \(C_3 = (X_{u_4} \lor X_{u_3,u_4}) \land (X_{u_6} \lor X_{u_5,u_6})\) are all matching \(\text{cnfs}\) w.r.t. \(M\). Consider a decision tree \(T\) shown in Figure 1. Labels \(u_i, \neg u_i\) denote, respectively the positive and negative edges with a tail \(a\) such that \(\text{Var}(a) = X_{u_i}\). A leaf \(b\) labelled with \(C_i\) means that \(\varphi(b) = C_i\). Observe that the set \(S_1 = \{X_{u_1}, \neg X_{u_2}, X_{u_3}, X_{u_6}\}\) satisfies \(T\) because it contains the literals labelling the root-leaf path to \(C_2\) and also satisfies \(C_2\). On the other hand, the set \(S_2 = \{X_{u_1}, X_{u_2}, \neg X_{u_3}, \neg X_{u_5,u_4}\}\) does not satisfy \(T\) because the only possible root-leaf path whose literals are contained in \(S_2\) is the one leading to \(C_1\); however \(S_2\) falsifies \(C_1\).

Figure 1: Illustration of a decision tree

For a directed path \(P\), we say that \(P_1, \ldots, P_c\) is a partition of \(P\) into subpaths if \(P_1, \ldots, P_c\) are subpaths of \(P\) such that \(P_1\) is a prefix of \(P\), \(P_c\) is a suffix of \(P\) and...
for any $P_i, P_{i+1}$, the last vertex of $P_i$ is the first vertex of $P_{i+1}$. If $x_1, \ldots, x_{c-1}$ are last vertices of $P_1, \ldots, P_{c-1}$, respectively, then we say that $P_1, \ldots, P_c$ is a partition of $P$ into subpaths w.r.t. $x_1, \ldots, x_{c-1}$.

For a path $P$ of a $k$-VEMBP $Z$ computing $CNF(G)$, we denote by $SV(P)$ the sequence of vertices of $G$ listed in the order the respective vertex variables occur along $P$. For example if the sequence of literals occurring on $P$ is $(X_{u_1}, X_{u_2}, X_{u_2}, X_{u_2}, u_3, X_{u_1}, X_{u_1}, X_{u_2})$ then $SV(P) = (u_1, u_2, u_2, u_3)$ (simply remove the edge variables and replace each $X_{u_i}$ of the resulting sequence by $u_i$). The two next lemmas are proved in the following two respective subsections of this section.

**Lemma 3** Let $Z$ be a uniform $k$-VEMBP computing $CNF(G)$ and let $P$ be a computational path of $Z$ (recall that a computational path is a consistent root-leaf path). Let $M = \{(u_1, v_1), \ldots, (u_t, v_t)\}$ be a matching of $G$. Suppose that there is a partition $P_1, \ldots, P_c$ of $P$ into subpaths such that $SV(P_1), \ldots, SV(P_c)$ are intervals witnessing a c-separation of $SV(P)$ w.r.t. $(U = \{u_1, \ldots, u_t\}, V = \{v_1, \ldots, v_t\})$. Let $x_1, \ldots, x_{c-1}$ be the respective end vertices of $P_1, \ldots, P_{c-1}$. Then there is a decision tree $D$ w.r.t. $M$ such that for any computational path $Q$ passing through $x_1, \ldots, x_{c-1}$, $A(Q)$ satisfies $D$.

**Lemma 4** Let $A$ be a family of decision trees w.r.t. matchings of size at least $t$ such that any satisfying assignment $S$ of $CNF(G)$ satisfies at least one element of $A$. Then $|A| \geq (8/7)^t$.

**Proof of Theorem 3** Let $P$ be a computational path of $Z$. Let $(x_1, \ldots, x_{c'})$ be a sequence of distinct intermediate vertices of $P$, occurring on $P$ in that order and let $P_1, \ldots, P_{c'+1}$ be the partition of $P$ into subpaths w.r.t. $x_1, \ldots, x_{c'}$. Suppose there are $U = \{u_1, \ldots, u_t\}, V = \{v_1, \ldots, v_t\}$ and a matching $M = \{(u_1, v_1), \ldots, (u_t, v_t)\}$ of $G$ such that $SV(P_1), \ldots, SV(P_{c'+1})$ are intervals witnessing separation of $SV(P)$ w.r.t. $(U, V)$. Then we say that $(x_1, \ldots, x_{c'})$ is a t-separation vector of $P$.

Let $XV$ be the set of all sequences of vertices of $Z$ of length at most $c - 1$ such that each $XV \in XV$ is a $t$-separation vector of some computational path of $Z$. By Lemma 3 for each $XV \in XV$, we can associate a decision tree $DT(XV)$ w.r.t. a matching of size $t$ such that for any computational path $Q$ of $Z$ passing through $XV$, $A(Q)$ satisfies $DT(XV)$.

Let $A$ be the set of all $DT(XV)$. Observe that each satisfying assignment $S$ of $CNF(G)$ satisfies an element of $A$. Indeed, let $P$ be a computational path of $Z$ such that $A(P) \subseteq S$. By definition of $k, c$-fold matching width, there is a matching $M = \{(u_1, v_1), \ldots, (u_t, v_t)\}$ such that there is a $c' \leq c$-separation of $SV(P)$ w.r.t. $(U = \{u_1, \ldots, u_t\}, V = \{v_1, \ldots, v_t\})$. Let $SV_1, \ldots, SV_{c'}$ be a partition of $SV(P)$ into intervals witnessing the $c$-separation. It is not hard to see that there is a partition $P_1, \ldots, P_{c'}$ of $P$ into subpaths such that $SV(P_i) = SV_i$ for $1 \leq i \leq c'$. It follows that the sequence $XV = (x_1, \ldots, x_{c'-1})$ of respective end vertices of $P_1, \ldots, P_{c'-1}$ is a $t$-separation vector of $P$. Due to the bound on the length, $XV \in XV$ and hence, by the previous paragraph,
A(P) satisfies \( DT(XV) \) and hence \( S \) satisfies \( DT(XV) \) as well. It follows from Lemma 4 that \( |A| \geq (8/7)^4 \).

On the other hand, \( |A| \leq |XV| \) and \( |XV| \) is at most as the number of sequences of (not necessarily distinct) vertices of \( Z \) of length \( c - 1 \). That is, \( |A| \leq |XV| \leq |Z|^{c-1} \). Combining this with the previous paragraph, we obtain \( |Z|^{c-1} \geq (8/7)^4 \) and the theorem follows.

### 4.1 Proof of Lemma 3

Let \( Q^1 \) and \( Q^2 \) be two computational paths of \( Z \) passing through \( x_1, \ldots, x_{c-1} \).

Let \( Q^1_1, \ldots, Q^1_c \) and \( Q^2_1, \ldots, Q^2_c \) be respective partitions of \( Q^1 \) and \( Q^2 \) into subpaths w.r.t. \( x_1, \ldots, x_{c-1} \). Let \( Q^* \) be the root-leaf path passing through \( x_1, \ldots, x_{c-1} \) whose respective partition \( Q^*_1, \ldots, Q^*_c \) into subpaths w.r.t. \( x_1, \ldots, x_{c-1} \) is as follows. For each \( 1 \leq i \leq c \), \( Q^*_i = Q^1_i \) whenever \( i \) is odd and \( Q^*_i = Q^2_i \) whenever \( i \) is even. We denote \( Q^* \) by \( Mix(Q^1, Q^2) \).

Using ‘superposition’ of two paths such as \( Mix(Q^1, Q^2) \) is a standard ‘fooling’ technique in proving lower bounds for branching programs. In case of 3VEMBP such a technique is not applicable directly because \( Mix(Q^1, Q^2) \) may be labelled by opposite literals of the same variable and thus is not necessarily a computational path. However, as shown in the next lemma, such an approach is possible if one more condition is imposed on \( Q^1 \) and \( Q^2 \).

**Lemma 5** Suppose \( XIR(M) \subseteq Var(A(Q^1) \cap A(Q^2)) \) (that is, all variables of \( XIR(M) \) are assigned by the same values by both \( Q^1 \) and \( Q^2 \)). Then \( Q^* = Mix(Q^1, Q^2) \) is a computational path.

**Proof.** Assume, by contradiction, that there is a variable \( X \) of \( CNF(G) \) such that both \( X \) and \( \neg X \) occur as labels of \( Q^* \). Then \( X \) is not an edge variable because, due to their monotonicity, edge variables do not occur negatively on \( Z \) and also \( X \not\in XIR(M) \) by assumption of the lemma. It remains to assume that \( X \in XV \cup XV \) where \( XV = \{X_u | u \in U \} \) and \( XV = \{X_v | v \in V \} \). Assume that \( X \in XV \), that is \( X = X_u \) for some \( u \in U \). Then the opposite occurrences happen one of \( Q^1_j \) and another on \( Q^2_j \) such that \( i \) is odd and \( j \) is even. Indeed, if \( i \) and \( j \) are of the same parity then they both subpaths of either \( Q^1 \) or \( Q^2 \) in contradiction to being \( Q^1 \) and \( Q^2 \) computational paths. It follows that \( X_u \) occurs on \( Q^1_j \) and \( X_v \) occurs on \( P_j \) and hence \( u \) occurs on \( SV(P_j) \). However, this is a contradiction to our assumption that \( SV(P_1), \ldots, SV(P_c) \) witness a \( c \) separation of \( SV(P) \) w.r.t. \( U, V \) (recall that elements of \( U \) cannot occur on the ‘even’ intervals of the witness). The reasoning for the case \( X \in XV \) is symmetric.

Using Lemma 5, we can prove a restricted version of Lemma 4 that then will be used for the induction basis.

**Lemma 6** Let \( S \) be a set of literals of variables of \( XIR(M) \). Then there is a matching CNF CC w.r.t. \( M \) such that any computation path \( Q \) passing through \( x_1, \ldots, x_{c-1} \) with \( S \subseteq A(Q) \) satisfies CC.
\textbf{Proof.} We are going to show that for each each \(\{u_i, v_i\}\) of \(M\) there is a clause \(C(u_i, v_i) \in \{(X_{u_i} \lor X_{u_i, v_i}), (X_{u_i, v_i} \lor X_{v_i})\}\) such that for each path \(Q\) as in the statement of the lemma \(A(Q)\) satisfies \(C(u_i, v_i)\). This will immediately imply that each \(A(Q)\) satisfies the matching \text{cnf} consisting of clauses \(C(u_1, v_1) \ldots C(u_t, v_t)\).

Assume by contradiction that there are computational paths \(Q^1\) and \(Q^2\) such that \(Q^1\) does not satisfy \((X_{u_i} \lor X_{u_i, v_i})\) and \(Q^2\) does not satisfy \((X_{u_i, v_i} \lor X_{v_i})\). Let 
\[Q^* = \text{Mix}(Q^1, Q^2).\]
By Lemma 5, \(Q^*\) is a computational path and hence \(A(Q^*)\) satisfies \((X_{u_i} \lor X_{u_i, v_i} \lor X_{v_i})\) by definition of \(Z\). We derive a contradiction by showing that none of these three literals occurs in \(A(Q^*)\). Indeed, by definition of \(Q^1\) and \(Q^2\), \((X_{u_i, v_i} \notin A(Q_1)\) and \((X_{u_i, v_i} \notin A(Q_2)\) and hence, clearly, \((X_{u_i, v_i} \notin A(Q^*)\). By the same reasoning as in the proof of Lemma 5, a literal of \(X_u\) can only occur in \(Q^*_i\) for an odd \(i\). However, such a \(Q^*_i\) is a subpath of \(Q^1\) where \(X_u\) does not occur positively. The reasoning regarding \(X_{v_i}\) is symmetric. ■

\textbf{Proof of Lemma 3} We prove the following more general statement. Let \(S\) be a set of literals with \(\text{Var}(S) \subseteq XIR(M)\). Let \(Q(S)\) be the set of computational paths of \(Z\) going through \(x_1, \ldots, x_{c-1}\) such that \(S \subseteq A(Q)\). Then there is a decision tree \(DT\) w.r.t. \(M\) where \(\text{Var}(S)\) do not occur as labels and such that for each \(Q \in Q(S)\), \(A(Q)\) satisfies \(DT\). The lemma will follow as a special case with \(S = \emptyset\).

The proof is by induction on \(|XIR(M) \setminus \text{Var}(S)|\). Assume that \(|XIR(M) \setminus \text{Var}(S)| = 0\), that is \(\text{Var}(S) = XIR(M)\). Let \(CC\) be a matching \text{cnf} w.r.t. \(M\) satisfied by \(A(Q)\) for all \(Q \in Q(S)\) according to Lemma 6. Then all these \(A(Q)\) satisfy a decision tree with a single node \(a\) such that \(\varphi(a) = CC\).

Assume now that \(\text{Var}(S) \subseteq XIR(M)\). If there is \(S' \supseteq S\) with \(\text{Var}(S') = XIR(M)\) such that \(Q(S) \subseteq Q(S')\) then the previous paragraph applies. Otherwise, there is a variable \(X \in XIR(M) \setminus \text{Var}(S)\) such that \(Q(S) = Q(S_1) \cup Q(S_2)\) where \(S_1 = S \cup \{X\}\) and \(S_2 = S \cup \{\neg X\}\) (recall that, due to the uniformity, vertex variables and, in particular, variables of \(XIR(M)\), occur on all computational paths of \(Z\)).

By the induction assumption, there are decision trees \(DT_1\) and \(DT_2\) that are, respectively related to \(Q(S_1)\) and \(Q(S_2)\) as specified in the first paragraph of this proof. Let \(rt_1, rt_2\) be the respective roots of \(DT_1\) and \(DT_2\). Let \(DT\) be a decision tree obtained from \(DT_1\) and \(DT_2\) by introduction of a new vertex \(rt\) with \(\text{Var}(rt) = X\) as the root of \(DT\) with \(rt_1, rt_2\) being the children of \(rt\) and \((rt, rt_1), (rt, rt_2)\) being, respectively, positive and negative edges of \(rt\). It remains to observe that \(DT\) is satisfied by \(A(Q)\) for all \(Q \in Q(S)\). Indeed, assume that \(Q \in Q(S_1)\) and let \(Q'\) be the path of \(DT_1\) witnessing that \(A(Q)\) satisfies \(DT_1\) (that is \(L(Q') \subseteq A(Q)\) and \(A(Q)\) satisfies \(\varphi(a)\)). Let \(Q''\) be the path of \(DT\) obtained by appending \(Q'\) to \((rt, rt_1)\). As \(X \in S_1 \subseteq A(Q)\), \(L(Q'') \subseteq A(Q)\) and hence \(Q''\) witnesses that \(A(Q)\) satisfies \(DT\). If \(Q \in Q(S_2)\) then the reasoning is symmetric. ■
4.2 Proof of Lemma 4

We define a probability space over the set SAT of satisfying assignments of CNF and show that the probability that a decision tree w.r.t. a matching of size at least \( t \) is satisfied by an element of SAT is exponentially small in \( t \). It will follow then that the number of decision trees w.r.t. such matchings must be large to ensure that each element of SAT satisfies one of them.

We first define a probability space over all possible assignments of \( \text{Var}(G) \) and then observe that the assignments having non-zero probabilities are precisely those that belong to SAT and hence this space is, in fact, over SAT. A random assignment \( S \) in this space is chosen by the following procedure. For each vertex variable \( X \), choose either \( X \) or \( \neg X \) with probability 1/2. Then for each edge variable \( X_{u,v} \), choose \( X_{u,v} \) with probability 1 if both \( \neg X_u \) and \( \neg X_v \) have been chosen. Otherwise, choose either \( X_{u,v} \) or \( \neg X_{u,v} \) with probability 1/2. The probability of the chosen assignment is the product of probabilities of assignments of individual variables. Due to the way we choose assignments to edge variables it is not hard to see that indeed, \( \Pr(S) > 0 \) if and only if \( S \in \text{SAT}(G) \). In particular (although, this is not relevant for the further reasoning), for each \( S \in \text{SAT}(G) \), \( \Pr(S) = (1/2)^{m - \text{ne}(S)} \), where \( m \) is the number of variables of CNF and \( \text{ne}(S) \) is the number of edges \( \{u, v\} \) such that \( \{\neg X_u, \neg X_v\} \subseteq S \).

An event in the probability space we have just defined is a subset of SAT(G). We say that a subset \( V \) of variables of \( \text{CNF}(G) \) is a prefix set of variables if either \( V \subseteq \text{Var}(G) \) or \( \text{Var}(G) \subseteq V \). For example, if \( G = K_4 \) with the set of vertices \{u_1, u_2, u_3, u_4\} then the sets \{X_{u_3, u_4}\} and \{X_{u_1}, \ldots, X_{u_4}, X_{u_2, u_3}\} are both prefix sets, while the set \{X_{u_2}, X_{u_3}, X_{u_1, u_2}\} is not a prefix set. Let \( S \) be a set of literals over a prefix set \( V \) of variables. We denote by EC(S) the event containing all the assignments \( S' \) such that \( S \subseteq S' \). We first need to prove two basic facts related to the EC events. The proofs of these facts are provided in Section B of the Appendix.

**Observation 1** Let \( S \) be a set of literals such that \( \text{Var}(S) \) is a prefix set. Let \( X \notin \text{Var}(S) \) be a variable such that \( \text{Var}(S) \cup \{X\} \) is still a prefix set. Let \( S_1 = S \cup \{X\} \) and \( S_2 = S \cup \{\neg X\} \). Suppose that \( X = X_{u,v} \) and both \( X_u \) and \( X_v \) occur negatively in \( S \). Then \( \Pr(\text{EC}(S_1) | \text{EC}(S)) = 1 \) and \( \Pr(\text{EC}(S_2) | \text{EC}(S)) = 0 \). Otherwise, both \( \Pr(\text{EC}(S_1) | \text{EC}(S)) = 0.5 \) and \( \Pr(\text{EC}(S_2) | \text{EC}(S)) = 0.5 \).

**Lemma 7** Let \( X, S, S_1, S_2 \) be as in Observation 1. Let \( X \) be an arbitrary event of SAT(G). Suppose that \( X = X_{u,v} \) and both \( X_u \) and \( X_v \) occur negatively in \( S \). Then \( \Pr(X | \text{EC}(S)) = \Pr(X | \text{EC}(S_1)) \). Otherwise, \( \Pr(X | \text{EC}(S)) = \Pr(X | \text{EC}(S_1)) * 0.5 + \Pr(X | \text{EC}(S_2)) * 0.5 \).

In the rest of the proof, we, essentially, first show that the probability of satisfying a matching CNF is exponentially small and then, using induction, extend this statement to decision trees. However, for the induction to take hold, we need to define a generalization of a matching CNF, which we call a matching CNF with a tail. This is a CNF CC whose set of clauses can be partitioned into
two subsets $MT(CC)$ and $TL(CC)$, respectively referred to as the matching clauses and the tail clauses. $MT(CC)$ is a matching CNF w.r.t. a matching $M$ and the clauses of $TL(CC)$ are singletons each contains a literal of a variable of $XIR(M)$. For such a $CC$, we denote by $ES(CC)$ the event consisting of all the elements of $SAT(G)$ satisfying $CC$. We are going to bound from above the probability of $ES(CC) \cap EC(S)$ where $S$ is an assignment to a prefix set of variables. Let us first extend the notation.

A clause $C$ of $CC$ is satisfied by $S$ if $S$ contains a literal of $C$ and falsified if $S$ contains negations of all the literals of $S$. Let us first extend the notation.

Let $ES(clauses)$ and the clauses and retained non resolved matching clauses of all the elements of $XIR$ of $TL$ and the clauses of $S$.

Example 3 Let $G$ be a graph on vertices $u_1, \ldots, u_{15}$ and suppose that $M = \{\{u_4, u_5\}, \{u_6, u_7\}, \{u_8, u_9\}, \{u_{10}, u_{11}\}\}$. Consider the following CNF.

$$CC = (X_{u_1}) \land (\neg X_{u_2}) \land (\neg X_{u_3}) \land (X_{u_4} \lor X_{u_5} \lor X_{u_8}) \land (X_{u_6} \lor X_{u_9} \lor X_{u_{10}}) \land (X_{u_{11}} \lor X_{u_{10}} \lor X_{u_{11}})$$

Clearly, it is a matching CNF with a tail. In particular, $MT(CC) = (X_{u_4} \lor X_{u_5} \lor X_{u_6} \lor X_{u_7}) \land (X_{u_9} \lor X_{u_8} \lor X_{u_{10}}) \land (X_{u_{11}} \lor X_{u_{10}} \lor X_{u_{11}})$ is a matching CNF w.r.t. $M$ and $TL(CC) = (X_{u_4}) \land (\neg X_{u_5}) \land (\neg X_{u_6}) \land (\neg X_{u_7}) \land (\neg X_{u_8}) \land (\neg X_{u_9}) \land (\neg X_{u_{10}}) \land (\neg X_{u_{11}})$. Let $S = \{X_{u_1}, X_{u_2}, \neg X_{u_3}, \neg X_{u_5}, \neg X_{u_6}, X_{u_7}, X_{u_8}\}$. Then clauses $(X_{u_1})$ and $(\neg X_{u_2})$ are, respectively, satisfied and falsified, clause $(X_{u_4} \lor X_{u_5} \lor X_{u_6} \lor X_{u_7})$ is resolved and $DET(CC, S) = \{\neg X_{u_3}\}, (X_{u_4} \lor X_{u_5} \lor X_{u_6} \lor X_{u_7}, (X_{u_9} \lor X_{u_8} \lor X_{u_{10}}), (X_{u_{11}} \lor X_{u_{10}} \lor X_{u_{11}})\}$. Finally, $R_{tl}(CC, S) = \{\neg X_{u_3}\}, R_{bc}(CC, S) = \{X_{u_4} \lor X_{u_5} \lor X_{u_6} \lor X_{u_7}\}, R_{hf}(CC, S) = \{X_{u_9} \lor X_{u_8} \lor X_{u_{10}}\}, R_{bc}(CC, S) = \{X_{u_{11}} \lor X_{u_{10}} \lor X_{u_{11}}\}$, and $R_{hf}(CC, S) = \emptyset$

We define $weight(CC, S)$ as follows.

$$weight(CC, S) = (7/8)^{|R_{bc}(CC, S)|} (1/4)^{|R_{hf}(CC, S)|} (3/4)^{|R_{tf}(CC, S)|}$$

(2)

Lemma 8 Let $S$ be an assignment to the prefix set $VR$ of variables. Then $Pr[ES(CC) \cap EC(S)] \leq weight(CC, S)$.

Proof. We use induction on the number of variables $Var(CNF(G)) \setminus VR$, i.e. the number of variables not assigned by $S$. Suppose the number of such variables is 0. Then none of the clauses of $CC$ is retained w.r.t. $S$. That is, $weight(CC, S) = 1$. If all the clauses of $CC$ are satisfied by $S$ then $Pr[ES(CC) \cap EC(S)] = 1$, otherwise it is 0, hence the lemma holds for the considered case.
Assume now that the number of variables not assigned by $S$ is greater than 0 and let $X \notin VR$ be a variable such that $VR \cup \{X\}$ is a prefix set (such a variable clearly exists). Let $S_1 = S \cup \{X\}$ and $S_2 = S \cup \{\neg X\}$.

We say that $X$ is relevant to a clause $C$ of $CC$ if one of the following is true:
(i) $C$ is a tail clause and $X$ occurs in $C$ or (ii) $C = (X_u \lor X_{u,v})$ (that is it is a clause of $MT(CC)$) and $X$ is one of $X_u, X_{u,v}, X_v$. Clearly $X$ can be relevant to at most one clause of $CC$. Below we use case analysis to demonstrate that the lemma holds for all types of relevance of $X$ to $CC$. The cases are organized in a three-level hierarchy. To make it easier for the reader to go through the rest of the proof we enumerate the cases in a way that clearly shows that one case is a subcase of another. For example, case 1.2 is a subcase of case 1.

**Case 1.** Assume first that $X$ is not relevant to any determining clause of $CC$ w.r.t. $S$ Then $weight(CC, S) = weight(CC, S_1) = weight(CC, S_2)$ because $R_j(CC, S) = R_j(CC, S_1) = R_j(CC, S_2)$ for all $j \in \{tl, bc, bf, hc, hf\}$. By the induction assumption and Lemma 7 $\Pr(ES(CC)|EC(S)) \leq weight(CC, S_1)$ if $X = X_u, \neg X_u \subseteq S$ and $\Pr(ES(CC)|EC(S)) \leq weight(CC, S_1) \ast 0.5 + weight(CC, S_2) \ast 0.5$. Clearly, the right part of both inequalities does not exceed $weight(CC, S)$.

**Case 2.** Assume now that $X$ is relevant to a clause $C \in R_{tl}(CC, S)$. Then $R_{tl}(CC, S_1) = R_{tl}(CC, S_2) = R_{tl}(CC, S) \setminus \{C\}$ and $R_{tl}(CC, S) = R_{tl}(CC, S_1) = R_{tl}(CC, S_2)$ for all $j \in \{bc, bf, hc, hf\}$. It follows that $weight(CC, S_1) = weight(CC, S_2) = 2 \ast weight(CC, S)$.

Assume that $C = (X)$. Then $S_2$ falsifies $C$ and hence $\Pr(ES(CC)|EC(S_2)) = 0$. Hence, by Lemma 7 and the induction assumption, $\Pr(ES(CC)|EC(S)) \leq weight(CC, S_1) \ast 0.5 = weight(CC, S)$. If $C = (\neg X)$, the reasoning is symmetric.

**Case 3.** It remains to assume that $X$ is relevant to a clause $C = (X_u \lor X_{u,v})$ of $MT(CC)$.

**Case 3.1.** Assume first that $X = X_{u,v}$. As $XV \cup \{X\}$ is a prefix set, both $X_u$ and $X_v$ occur in $S$. Moreover, as $C$ is retained and non-resolved, $\{\neg X_u, X_v\} \subseteq S$. It follows that $C \in R_{bf}(CC, S)$. Clearly, $R_{bf}(CC, S_1) = R_{bf}(CC, S_2) = R_{bf}(CC, S) \setminus \{C\}$ and $R_{bf}(CC, S) = R_{bf}(CC, S_1) = R_{bf}(CC, S_2)$ for all $j \in \{bc, bf, hc, hf\}$. It follows that $weight(CC, S_1) = weight(CC, S_2) = 2 \ast weight(CC, S)$. As $S_2$ falsifies $C$, $\Pr(ES(CC)|EC(S_2)) = 0$. Hence, by Lemma 7 and the induction assumption, $\Pr(ES(CC)|EC(S)) \leq weight(CC, S_1) \ast 0.5 = weight(CC)$ thus confirming the lemma for the considered case.

**Case 3.2.** Assume now that $X = X_u$. That is, $X_u \notin VR$ hence $\neg X_u \notin S$ and hence, in turn $C$ is a binary clause, that is $C \in R_{bc}(CC, S) \cup R_{bf}(CC, S)$. (Note that $C$ is not falsified by $S_2$ because $X_{u,v}$ does not occur in $VR$ (indeed, as $VR$ is a prefix set, $X_{u,v} \in VR$ implies $X_u \in VR$ in contradiction to our assumption that $X \notin VR$). It follows that $C \in R_{bc}(CC, S_2)$ and $R_{bc}(CC, S_2) = R_{bc}(CC, S) \setminus \{C\}$ and $R_{bc}(CC, S_2) = R_{bc}(CC, S) \cup \{C\}$ and $R_{bf}(CC, S_2) = R_{bf}(CC, S) \cup \{C\}$ and $R_{bf}(CC, S_2) = R_{bf}(CC, S) \cup \{C\}$ for all $j \in \{tl, bc, bf, hc, hf\}$.)
of reasoning as in the previous paragraph, we first observe that \( R_{bf}(CC, S_1) = R_{bf}(CC, S) \setminus \{C\} \) and \( R_j(C, S) = R_j(C, S_1) \) for all \( j \in \{tl, bc, hc, hf\} \). It follows that \( \text{weight}(CC, S_1) = 4/3 \times \text{weight}(CC, S) \). Then we observe that \( R_{bc}(CC, S_2) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_2) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_1) = R_j(C, S_2) \) for all \( j \in \{tl, bc, hc\} \). Consequently, \( \text{weight}(CC, S_2) = \text{weight}(CC, S) \times 4/3 \times 1/2 = \text{weight}(CC, S) \times 2/3 \). Combining Lemma \( \text{Pr}(ES)(CC)) \text{EC}(S)) \leq \text{weight}(CC, S_1) \times 0.5 + \text{weight}(CC, S_2) \times 0.5 = \text{weight}(CC, S) \times 6/7 \times 1/2 + \text{weight}(CC, S) \times 8/7 \times 1/2 = \text{weight}(CC, S) \) as required.

Case 3.2.2. Assume now that \( C \in R_{bc}(CC, S) \). Applying the same line of reasoning as in the previous paragraph, we first observe that \( R_{bc}(CC, S_1) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_1) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_1) = R_j(C, S_2) \) for all \( j \in \{tl, bc, hc, hf\} \). Hence \( \text{weight}(CC, S_1) = \text{weight}(CC, S) \times 6/7 \). Next, \( C \) is resolved w.r.t. \( S_2 \). That is, \( R_{bc}(CC, S_2) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_2) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_2) = R_j(C, S) \) for all \( j \in \{tl, bc, hc, hf\} \). Thus \( \text{weight}(CC, S_2) = \text{weight}(CC, S) \times 8/7 \). Combining Lemma \( \text{Pr}(ES)(CC)) \text{EC}(S)) \leq \text{weight}(CC, S_1) \times 0.5 + \text{weight}(CC, S_2) \times 0.5 = \text{weight}(CC, S) \times 6/7 \times 1/2 + \text{weight}(CC, S) \times 8/7 \times 1/2 = \text{weight}(CC, S) \) as required.

Case 3.3. Assume now that \( C \in R_{bc}(CC, S) \). It remains to consider the case where \( X = X_v \). As \( X_v \notin VR, C \) is constrained w.r.t. \( S \). The same line of reasoning as in the previous paragraph, we first observe that \( R_{bc}(CC, S_1) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_1) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_1) = R_j(C, S) \) for all \( j \in \{tl, bc, hf\} \). Thus \( \text{weight}(CC, S_1) = \text{weight}(CC, S) \times 6/7 \). Next, \( C \) is resolved w.r.t. \( S_2 \). That is, \( R_{bc}(CC, S_2) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_2) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_2) = R_j(C, S) \) for all \( j \in \{tl, bc, hf\} \). Thus \( \text{weight}(CC, S_2) = \text{weight}(CC, S) \times 8/7 \). Combining Lemma \( \text{Pr}(ES)(CC)) \text{EC}(S)) \leq \text{weight}(CC, S_1) \times 0.5 + \text{weight}(CC, S_2) \times 0.5 = \text{weight}(CC, S) \times 6/7 \times 1/2 + \text{weight}(CC, S) \times 8/7 \times 1/2 = \text{weight}(CC, S) \) as required.

Case 3.4. Finally, assume that \( C \in R_{bc}(CC, S) \). Applying the same line of reasoning as in the previous paragraph, we first observe that \( R_{bc}(CC, S_1) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_1) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_1) = R_j(C, S) \) for all \( j \in \{tl, bc, hf\} \). Hence \( \text{weight}(CC, S_1) = \text{weight}(CC, S) \times 6/7 \). Next, \( C \) is resolved w.r.t. \( S_2 \). That is, \( R_{bc}(CC, S_2) = R_{bc}(CC, S) \setminus \{C\} \) and \( R_{bf}(CC, S_2) = R_{bf}(CC, S) \cup \{C\} \) and \( R_j(C, S_2) = R_j(C, S) \) for all \( j \in \{tl, bc, hf\} \). Thus \( \text{weight}(CC, S_2) = \text{weight}(CC, S) \times 8/7 \). Combining Lemma \( \text{Pr}(ES)(CC)) \text{EC}(S)) \leq \text{weight}(CC, S_1) \times 0.5 + \text{weight}(CC, S_2) \times 0.5 = \text{weight}(CC, S) \times 6/7 \times 1/2 + \text{weight}(CC, S) \times 8/7 \times 1/2 = \text{weight}(CC, S) \) as required.

Observe that the case analysis is complete at this point. Indeed, a variable \( X \) can only be irrelevant to any clause of \( DET(CC, S) \) or relevant to a tail clause of \( DET(CC, S) \) or relevant to a matching clause of \( DET(CC, S) \). For each of these cases, we have provided exhaustive classification of subcases and shown that the lemma holds for each of them. Hence the proof is complete.
by $\text{ES}(DT, TL)$ the event consisting of assignment satisfying both $DT$ and $TL$.

**Lemma 9**  $Pr(\text{ES}(DT, TL)) \leq (1/2)^{|TL|} \ast (7/8)^{|M|}$.  

**Proof.** By induction on the number of nodes of $DT$. Assume that $DT$ has only one node. Then it is associated with $CC$, a matching $\text{CNF}$ w.r.t. $M$. Then $CC' = CC$ $\cup$ $TL$ is matching $\text{CNF}$ with a tail w.r.t. $M$. Clearly, $Pr(\text{ES}(DT, TL)) = Pr(\text{ES}(CC') \ast \text{EC}(\emptyset))$. Also, it is not hard to see that $R_u(CC', \emptyset) = TL$, $R_v(CC, \emptyset) = CC$ (here $TL$ and $CC$ are treated as sets of clauses) and $R_j(CC, \emptyset) = \emptyset$ for all other roles $j$. Then the statement of this lemma follows by Lemma 8 and, taking into account that $|CC| = |M|$. 

Assume now that $DT$ has more than one node. Let $u$ be the root of $DT$, let $v_1, v_2$ be, respectively, positive and negative children of $u$, and let $DT_1, DT_2$ be the respective subtrees of $DT$ rooted by $v_1$ and $v_2$. Let $X$ be the variable associated with $u$. Let $TL_1 = TL \cup \{(X)\}$ and $TL_2 = TL \cup \{\neg(X)\}$. By definition of $DT$, $X$ does not occur in $DT_1$, nor in $DT_2$.

Therefore, $(DT_1, TL_1)$ and $(DT_2, TL_2)$ are decision trees with tails w.r.t. $M$. Note that $\text{ES}(DT, TL) \subseteq \text{ES}(DT, TL_1) \cup \text{ES}(DT, TL_2)$. Indeed, let $S$ be a satisfying assignment of $\text{CNF}(G)$ satisfying $DT$ and $TL$. Assume w.l.o.g. that $X$ is assigned positively. Then $(u,v_1)$ is the first edge of the path $P$ witnessing that $S$ satisfies $DT$. Hence, the prefix of $P$ starting at $v_1$ witnesses that $S$ satisfies $DT_1$. To see that $S$ satisfies $TL_1$, notice that $TL \subseteq S$ by definition and $X \in S$ because $X$ is the label of $(u,v_1)$. Applying the union bound and then the induction assumption and taking into account that $|TL_1| = |TL_2| = |TL| + 1$, we get $Pr(\text{ES}(DT, TL)) \leq Pr(DT, TL_1) + Pr(DT, TL_2) \leq 2 \ast (1/2)^{|TL|+1} \ast (7/8)^{|M|} = (1/2)^{|TL|} \ast (7/8)^{|M|}$ as required. 

**Proof of Lemma 4**. Let $DT$ be a decision tree w.r.t. a matching $M$ of size at least $t$ and let $\text{ES}(DT)$ be the event consisting of the satisfying assignments of $\text{CNF}(G)$ satisfying $DT$. Clearly, $\text{ES}(DT) = \text{ES}(DT, \emptyset)$ and hence, by Lemma 9 $Pr(\text{ES}(DT)) \leq (7/8)^t$.

Let $\text{ES}(A)$ be the event consisting of satisfying assignments of $\text{CNF}(G)$ satisfying at least one element of $A$. Then $\text{ES}(A) \leq \bigcup_{DT \subseteq A} \text{ES}(DT)$. By the union bound and lemma 9 $Pr(\text{ES}(A)) \leq \sum_{DT \subseteq A} \text{ES}(DT) \leq |A| \ast (7/8)^t$.

Suppose $|A| < (8/7)^t$. Then $Pr(\text{ES}(A)) < 1$. That is, there is a satisfying assignment of $\text{CNF}(G)$ that does not satisfy any element of $A$ in contradiction to the definition of $A$. 

**5 Proof of Theorem 4**

By definition of $k$-VOBP, there is a sequence $XSV$ of vertex variables of $\text{CNF}(G)$ where each variable occurs exactly $k$ times and such that the sequence of vertex variables occurring on each computational path of $Z$ is a subsequence of $XSV$. Let $SV$ be the sequence of variables of $G$ obtained from $XSV$ by replacement of each $X_u$ with $u$. By definition of $k, c$-fold matching width, there is a matching $M = \{(u_1,v_1),\ldots,(u_t,v_t)\}$ of $G$ such that $SV$ has a $c' \leq c$-separation w.r.t.
Let $S$ be the set of all assignments $S$ to the variables of $CNF(G)$ satisfying the following conditions: (i) $\neg X_{u_i,v_i} \in S$ for each $1 \leq i \leq t$; (ii) for each $1 \leq i \leq t$, the occurrences of $X_{u_i}$ and $X_{v_i}$ have distinct signs (that is, the former occurs positively if and only if the latter occurs negatively); (iii) the variables besides $\bigcup_{i=1}^{t} \{X_{u_i}, X_{u_i,v_i}, X_{v_i}\}$ are assigned positively. It is not hard to see that each $S \in S$ is a satisfying assignment of $CNF(G)$. Indeed, for each clause $(X_\cup X_{u_i,v_i} \lor X_{v_i})$, either $X_{u_i,v_i} \in S$ (if $\{u, v\} \neq \{u_i, v_i\}$) for all $1 \leq i \leq t$ or one of $X_{u_i}, X_{v_i}$ belongs to $S$ otherwise (due them being assigned oppositely).

It follows that for each $S \in S$, we can pick a computational path $P^S$ such that $A(P^S) \subseteq S$. It is not hard to see that there is a partition $P^S_1, \ldots, P^S_c$ into subpaths such that each $SV(P^S_i)$ is a subsequence of $SV_i$ (note that we cannot put equality here because a $k$-vobp does not have to obey the uniformity condition). Let $XV^S = (x_1, \ldots, x_{c_1})$ be the sequence of respective ends of $P^S_1, \ldots, P^S_{c-1}$.

**Lemma 10** Let $S_1, S_2$ be two distinct elements of $S$. Then $XV^{S_1} \neq XV^{S_2}$.

**Proof.** Assume the opposite and let $S_1, S_2$ be two distinct elements of $S$ such that $XV^{S_1} = XV^{S_2} = (x_1, \ldots, x_{c_1})$. Note that for some $u_i, S_1$ and $S_2$ have opposite occurrences of $X_{u_i}$. Indeed, otherwise, the occurrences of all $X_{v_i}$ (determined by $X_{u_i}$) are the same and hence $S_1 = S_2$. We assume w.l.o.g. that there is $u_i$ such that $X_{u_i}$ occurs negatively in $S_1$ and positively in $S_2$.

Let $P^{S_1}_1, \ldots, P^{S_1}_{c_1}$ and $P^{S_2}_1, \ldots, P^{S_2}_{c_2}$ be respective partitions of $P^{S_1}$ and $P^{S_2}$ w.r.t. $x_1, \ldots, x_{c_1}$. Let $P^*$ be a root-leaf path of $Z$ passing through $x_1, \ldots, x_{c_1}$ with partition $P^{S_1}_1, \ldots, P^{S_1}_{c_1}$ into subpaths w.r.t. $x_1, \ldots, x_{c_1}$ such that $P^*_i = P^{S_i}_i$ whenever $i$ is odd and $P^*_i = P^{S_2}_i$ whenever $i$ is even.

Observe that $P^*$ is a computational path. Indeed, all the variables outside $XU = \{X_u | u \in U\}$ and $XV = \{X_v | v \in V\}$ have the same occurrence in both $P^{S_1}$ and $P^{S_2}$. If we assume that $X_u \in XU$ has two opposite occurrences on $P^*$ then such occurrences must happen on some $P^*_i$ and $P^*_j$ such that $i$ is odd and $j$ is even (otherwise, both these occurrences happen either on $P^{S_1}$ or on $P^{S_2}$ in contradiction to their consistency). Notice however that $SV(P^*_j) = SV(P^{S_2}_j)$ is a subsequence of $SV_j$ where $X_u$ does not occur by definition and hence $X_u$ cannot occur on $P^*_j$, a contradiction. The reasoning regarding $X_v \in XV$ is symmetric.

It follows that $P^*$ is a computational path and hence $A(P^*)$ satisfies all the clauses of $CNF(G)$. We derive a contradiction by showing that $A(P^*)$ does not satisfy $(X_{u_i} \lor X_{u_i,v_i} \lor X_{v_i})$. Indeed, as both $S_1$ and $S_2$ contain $\neg X_{u_i,v_i}$ neither $A(P^{S_1})$ nor $A(P^{S_2})$ contain $X_{u_i,v_i}$ and hence, clearly, $X_{u_i,v_i} \notin A(P^*)$. Furthermore, $X_{u_i}$ does not belong to $A(P^*_i)$ for an odd $i$ because in this case $A(P^*_i) \subseteq A(P^{S_1}_i) \subseteq S_1$ and $S_1$ contains $\neg X_{u_i}$. If $i$ is even $X_{u_i} \notin A(P^*_i)$ simply because, as verified in the previous paragraph, $X_{u_i}$ does not occur at all on $P^*_i$ for an even $i$. Thus we have shown that $X_{u_i} \notin A(P^*)$. It can be verified symmetrically that $X_{v_i} \notin A(P^*)$. ■
Proof of Theorem 4
Let $\mathbf{X}^S = \{X^S \mid S \in \mathbf{S}\}$. It follows from Lemma 10 that $|\mathbf{X}^S| \geq |S|$. Observe that $|S| \geq 2^t$. Indeed, there are $2^t$ distinct assignments to variables $X_{u_1}, \ldots, X_{u_t}$. It is not hard to see that each of these assignments can be extended to an assignment $S \in \mathbf{S}$ and that any two assignments obtained this way are distinct, just because their restrictions to $\{X_{u_1}, \ldots, X_{u_t}\}$ are distinct.

It follows that $|\mathbf{X}^S| \geq 2^t$. The rest of the reasoning is analogous to the last paragraph of the proof of Theorem 3.

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A Proof of Lemma 1

Let $Z$ be a $k$-vembp. The in-degree $d^+(v)$ of a node $v$ is a number of in-neighbours (this is essential point because of the possibility of multiple edges). Let us call $Z$ clean if the following conditions are true.

- All the incoming edges of nodes $v$ with $d^+(v) > 1$ are unlabelled.
- For each pair of nodes $u, v$, all the $(u, v)$-edges are either unlabelled or labelled with literals of the same variable.

We assume that $Z$ is clean. This assumption does not restrict generality because a $k$-vembp can be transformed into a clean one having at most twice more edges than the original $k$-vembp and computing the same function. Indeed, let $v$ be a node with in-degree greater than 1 and let $(u, v)$ be an edge labelled with a literal $x$. Subdivide $(u, v)$ and let $u, w, v$ be the path that replaced $(u, v)$. Then label $(u, w)$ with $x$. Clearly, as a result we get a $k$-vembp implementing the same function as the original one. Notice that $w$ has in-degree 1, that is the number of edges violating the assumption has decreased by 1. Thus, one can inductively argue that in case of $q$ ‘violating’ edges, there is a transformation to a $k$-vembp satisfying the above assumption that creates at most $q$ additional edges. As for multiple edges, suppose there are multiple edges $(u, v)$ violating the second condition of the cleanness. Then subdivide as above all the $(u, v)$ edges labelled with a literal. As a result we get a clean $k$-vembp in which each edge has been subdivided at most once.

Suppose $Z$ computes $CNF(G)$ and let $X$ be a vertex variable of $CNF(G)$. For a node $v$ of $Z$, denote by $fr_Z(v, X)$ the largest number of occurrences of $X$ on a path from the root to $v$. We call the edges $(u, v)$ of $Z$ such that $d^+(v) > 1$ relevant. Denote the set of relevant edges of $Z$ by $RL_Z$. A relevant edge is irregular w.r.t. $X$ if $fr_Z(v, X) - fr_Z(u, X) > 0$.

Let $(u, v)$ be an edge such that $fr_Z(v, X) - fr_Z(u, X) = q > 0$. Then transform $Z$ as follows.

1. Remove the edge $(u, v)$.
2. Introduce new vertices $u_1, \ldots, u_q$; we will refer to $u$ as $u_0$ for the sake of convenience.
3. For each $1 \leq i \leq q$, introduce two edges $(u_{i-1}, u_i)$ and label them them $X$ and $\neg X$, respectively.
4. Introduce an unlabelled edge $(u_q, v)$.

Let $Z'$ be the graph obtained as a result of the above transformation. The following properties can be observed by a direct inspection.

Observation 2

1. $fr_Z(v, X) = fr_{Z'}(X)$.
2. The edge $(u_q, v)$ is regular w.r.t. $X$ in $Z'$. 


3. $Z'$ is clean.

4. $Z$ and $Z'$ have the same set of nodes with in-degree greater than one.

5. For each vertex variable $Y \neq X$, and node $w$ of $Z$, $fr_Z(w, Y) = fr_{Z'}(w, Y)$. Moreover, $fr_Z(u, Y) = fr_{Z'}(u, Y)$.

6. $RL_{Z'} = RL_Z \setminus \{(u, v)\} \cup \{(u_q, v)\}$

**Lemma 11** $Z'$ is a $k$-VEMBP computing the same function as $Z$.

**Proof.** Clearly, the transformation from $Z$ to $Z'$ preserves monotonicity of edge variables of $CNF(G)$. In light of statement 4 of Observation 2 it is sufficient to establish that there are no more than $k$ occurrences of $X$ on each root-leaf path $P$ of $Z'$ that is not a path of $Z$. By construction, such a path $P$ includes $u$ and $v$ and the subpath $P_{u,v}$ starting at $u$ and ending at $v$ goes through $u_1, \ldots, u_q$ as defined above. Let $P_u$ be the prefix of $P$ ending at $u$, and $P_v$ be the suffix of $P$ beginning at $v$. By definition of the $fr$ function, the number of occurrences of $X$ on $P_u$ is at most $fr_Z(v) - q$. Furthermore, the number of occurrences of $X$ on $P_v$ is at most $k - fr_Z(v)$. Indeed, otherwise, let $P'$ be a root-$v$ path witnessing $fr_Z(v)$. Then, appending $P_v$ to the end of $P'$ we obtain a root-leaf path of $Z$ with more than $k$ occurrences of $X$, a contradiction.

Thus, the number of occurrences of $X$ on $P$ is at most $fr_Z(v, X) - q$ on $P_u$ plus $q$ occurrences of $P_{u,v}$ plus at most $k - fr_Z(v, X)$ occurrences on $P_v$. Clearly, on $P$, there are at most $k$ occurrences of $X$ in total.

Let $S$ be a satisfying assignment of the function computed by $Z$ and let $P$ be a computational path of $Z$ with $A(P) \subseteq S$. If $P$ does not include $(u, v)$ then $P$ is a computational path of $Z'$. Otherwise, let $P_u$ and $P_v$ be as in the previous paragraph and let $P'$ be a $u - v$ path with $u_1, \ldots, u_q$ being the intermediate vertices and the in-edge for each $u_i$ is the one labelled with the literal of $X$ that belongs to $S$ (by construction, such a selection is possible) and, as a result $A(P') \subseteq S$. Taking into account that $A(P_u) \cup A(P_v) \subseteq A(P) \subseteq S$, we conclude that $A(P_u + P' + P_v) \subseteq S$. That is, in any case there is a computational path of $Z$ whose set of literals is a subset of $S$ and hence $S$ is a satisfying assignment of the function computed by $Z'$.

Conversely, let $S$ be a satisfying assignment of the function computed by $Z'$. Let $P$ be a computational path of $Z'$ such that $A(P) \subseteq S$. If $P$ is not a path of $Z$ then, by construction, $P$ includes both $u$ and $v$ and a path of $Z$ can be obtained by replacement of the subpath of $P$ between $u$ and $v$ by an edge $(u, v)$. Clearly, the set of literals of this resulting path is a subset of $A(P)$, hence $S$ is a satisfying assignment of the function computed by $Z$. □

Denote by $RG_Z(Y)$, $IR_Z(Y)$ the respective sets of regular and irregular edges of $Z$ w.r.t. a vertex variable $Y$.

**Lemma 12** $|IR_{Z'}(X)| < |IR_Z(X)|$ and for each $Y \neq X$, $|IR_{Z'}(Y)| \leq |IR_Z(Y)|$.

**Proof.** Statement 6 of Observation 2 lets us define a bijection from the relevant edges of $Z$ to the relevant edges of $Z'$ so that $(u, v)$ corresponds to
fr\(u_q, v\) and each other edge corresponds to itself. Observe that for each \(Y \neq X\) each regular edge of \(Z\) w.r.t. \(Y\) corresponds to a regular edge of \(Z'\) w.r.t. \(Y\). Indeed, assume that \((u, v)\) is regular w.r.t. \(Y\). Then, by the statement 5 of Observation 2 \(fr_{Z'}(v, Y) - fr_{Z'}(u_q, Y) = fr_{Z}(v, Y) - fr_{Z}(u, Y) = 0\). That is, the corresponding edge \((u_q, v)\) of \(Z'\) is also regular w.r.t. \(Y\). For other regular edges, the argumentation is similar. It follows that \(|RG_{Z'}(Y)| \geq |RG_{Z}(Y)|\). Then, since \(|RL_Z| = |RL_{Z'}|\), it follows that \(|IR_{Z'}(Y)| = |RL_{Z'} \setminus RG_{Z}(Y)| \leq |RL_{Z} \setminus RG_{Z}(Y)| = |IR_{Z}(Y)|\), proving the second statement.

For the first statement, we need an additional claim.

**Claim 2** For each node \(w\) of \(Z\), \(fr_{Z}(w, X) = fr_{Z'}(w, X)\).

**Proof.** By the first statement of Observation 2 the claim holds for \(v\). For \(w \neq v\), it is not hard to see that \(fr_{Z}(w, X) \leq fr_{Z'}(w, X)\). We are going to establish that \(fr_{Z}(w, X) \geq fr_{Z'}(w, X)\). Let \(P\) be a root-\(w\) path of \(Z\) with the largest number of occurrences of \(X\). If \(P\) is also a path of \(Z\) then \(fr_{Z}(w, X) \geq fr_{Z'}(w, X)\) follows immediately. Otherwise, \(P\) passes through \(v\). Let \(P_v\) be the prefix of \(P\) ending at \(v\). Let \(P_v'\) be a root-\(v\) path of \(Z\) witnessing \(fr_{Z}(v, X)\). As \(fr_{Z}(v, X) = fr_{Z'}(v, X)\), the number of occurrences of \(X\) on \(P_v'\) is not smaller than on \(P_v\). Due to the irregularity of \((u, v)\), it is not an edge of \(P_v'\) and hence \(P_v'\) is also a path of \(Z\). It follows that, transforming \(P\) by replacing \(P_v\) with \(P_v'\), we obtain a new root-\(w\) path \(P'\) of \(Z\) with a number of occurrences of \(X\) at least \(fr_{Z'}(w, X)\). As \(P'\) is also a path of \(Z\), it follows that \(fr_{Z}(w, X) \geq fr_{Z'}(w, X)\). \(\square\)

It follows that the above bijection from \(RL_Z\) to \(RL_{Z'}\), maps each regular edge of \(Z\) w.r.t. \(X\) to a regular edge of \(Z'\) w.r.t. \(X\). Indeed, let \((u', v')\) be a regular edge of \(Z\) w.r.t. \(X\). By assumption about \((u, v), (u', v') \neq (u, v)\) and hence the bijection maps \((u', v')\) to itself. It follows from the claim that \(fr_{Z'}(v', X) = fr_{Z'}(u', X) = fr_{Z}(v', Y) - fr_{Z}(u', Y) = 0\) and hence \((u', v')\) is regular w.r.t. \(X\) in \(Z'\). In addition, by Observation 2 \((u, v)\), an irregular edge w.r.t. \(X\) in \(Z\) corresponds to a regular edge \((u_q, v)\) w.r.t. \(X\) in \(Z'\). It follows that \(|RG_{Z}(X)| < |RG_{Z'}(X)|\) and hence the first statement follows by a calculation similar to the one we used for the second statement. ■

**Proof of Lemma 1.** Suppose that \(Z\) is clean. Denote \(\sum_{X \in \text{Var}(G)} |IR_{Z}(X)|\) by \(ir_{Z}\). We prove, by induction on \(ir_{Z}\), that by adding at most \(ir_{Z} + k\) new vertices, \(Z\) can be transformed into a \(k\)-\textsc{vembp} \(Z^*\) where all the relevant edges are regular w.r.t. all the variables.

If \(ir_{Z} = 0\) then no transformation is needed. Otherwise, pick an edge \((u, v)\) irregular w.r.t. \(X\) and apply transformation as above. Let \(Z'\) be the resulting graph. Then, by Lemma 1 \(ir_{Z'} < ir_{Z}\). By Observation 2 \(Z'\) is clean and by Lemma 1 \(Z'\) is a \(k\)-\textsc{vembp} computing \(CNF(G)\). By the induction assumption, \(Z'\) can be transformed to \(Z^*\) as above by adding at most \(ir_{Z'} + k\) vertices. Hence, the transformation from \(Z\) requires at most \((ir_{Z'} + 1)k \leq ir_{Z} + k\) new vertices as required.

Observe that for each node \(v\) of \(Z^*\) and each vertex variable \(X\), all root-\(v\) paths of \(Z^*\) carry the \(fr_{Z'}(v, X)\) occurrences of \(X\). Indeed, assume that this is not so and let \(P\) be the shortest path violating this statement w.r.t. a variable...
Let $w$ be the last node of $P$. Then the number of occurrences of $X$ on $P$ is smaller than $fr_{Z^*}(w, X)$.

Assume first that $w$ has in-degree 1 and let $u$ be the only in-neighbour. Then, it follows from the second condition of a clean $k$-vembp, the number of occurrences of $X$ on $P_u$, the prefix of $P$ ending at $u$ is smaller than $fr_{Z^*}(u, X)$. Assume now that the in-degree of $w$ is larger than 1 and let $P'$ be a root-$w$ path witnessing $fr_{Z^*}(w, X)$. If $P$ and $P'$ have the same penultimate vertex then we get the same contradiction with the minimality of $P$ as in the previous case. Otherwise, we conclude that the last edge of $P$ is irregular w.r.t. $X$, in contradiction to $ir_{Z^*} = 0$.

Let $lf$ be the leaf of $Z^*$. If for each variable $X$, $fr_{Z^*}(lf, X) = k$ then we are done. Otherwise, subdividing in-coming edges of $lf$ as described in the above transformation, we can add the required number of occurrences of each variable $X$ with $fr_{Z^*}(lf, X) < k$.

Clearly, during the above transformation, each edge of $Z$ is subdivided at most $O(nk)$ times. Hence, the total number of nodes in the resulting $k$-vembp is $O(mnk) + |Z|$ where $m$ is the number of edges of $Z$. Now, what is an upper bound on $m$ in terms of $n$ and $Z$? The number of multiple edges between any two particular vertices $u$ and $v$ of $Z$ can be assumed $O(n)$ because if there are two multiple edges unlabelled or two labelled with the same literal then one of them can be safely removed. Then $m = O(nZ^2)$. It follows that for any $k$-vembp there is a uniform $k$-vembp with $O(n^2|Z|^2k)$ nodes as required. ■

B Proofs omitted from Section 4.2.

Proof of Observation 1 First of all, we need an auxiliary claim.

Claim 3 Let $S$ be a set of literals such that $Var(S)$ is a prefix set. Let $Y \in S$ be a literal of a variable $X$. Let us define the individual probability $pr_S(Y)$ of $Y$ (w.r.t. $S$) as follows.

- If $X$ is a vertex variable then $pr(Y) = 0.5$.
- If $X = X_{u,v}$ and either $X_u \in S$ or $X_v \in S$ then $pr(Y) = 0.5$.
- If $X = X_{u,v}$ and neither of $X_u$ and $X_v$ belongs to $S$ (that is, both of the negations belong to $S$, by definition of a prefix set) then $pr(Y) = 1$ if $Y = X$ and $pr(Y) = 0$ if $Y = \neg X$.

Denote $\prod_{Y \in S} pr_S(Y)$ by $pr(S)$. Then $Pr(EC(S)) = pr(S)$.

Proof. By induction on the number of variables not assigned by $S$. If the number of such variables is 0 then $S$ assigns all the variables of $CNF(G)$. In this case the claim follows by definition of the probability space. Assume now that $S$ does not assign all the variables of $CNF(G)$ and let $X \in Var(CNF(G)) \setminus Var(S)$ such that $Var(S) \cup \{X\}$ is a prefix set (it is not hard to see that such a variable always exists). Then, clearly $EC(S)$ is the disjoint union of
**EC(S ∪ {X})** and **EC(S ∪ {¬X})**. That is, \( Pr(\text{EC}(S)) = Pr(\text{EC}(S ∪ \{X\})) + Pr(\text{EC}(S ∪ \{¬X\})) \). By the induction assumption, \( Pr(\text{EC}(S ∪ \{X\})) = pr(S ∪ \{X\}) \) and \( Pr(\text{EC}(S ∪ \{¬X\})) = pr(S ∪ \{¬X\}) \).

By definition, \( pr(S ∪ \{X\}) = (\prod_{Y ∊ S} pr_{S \cup \{X\}}(Y)) \times pr_{S \cup \{X\}}(X) \). Notice that for each \( Y ∊ S \), \( pr_{S \cup \{X\}}(Y) = pr_S(Y) \). This is certainly true if \( Y \) is a literal of a vertex variable: the individual probability of \( Y \) is always 0.5. If \( Y \) is a literal of an edge variable \( X_{u,v} \), then, due to being \( Var(S) \) a prefix set, literal of \( X_u \) and \( X_v \) belong to \( S \) and hence the very same literals belong to \( S ∪ X \), confirming the observation. Thus, we can write, \( Pr(\text{EC}(S ∪ \{X\})) = (\prod_{Y ∊ S} pr_S(Y) \times pr_{S ∪ \{X\}}(X) \)

Hence, adding up \( Pr(\text{EC}(S ∪ \{X\})) \) and \( Pr(\text{EC}(S ∪ \{¬X\})) \) results in \( pr(S) \), as required. □

Observe that since \( \text{EC}(S) ⊆ \text{EC}(S_1) \) and \( \text{EC}(S) ⊆ \text{EC}(S_2) \), \( Pr(\text{EC}(S_1) ∪ \text{EC}(S)) = Pr(\text{EC}(S_1)) \) and \( Pr(\text{EC}(S_2) ∪ \text{EC}(S)) = Pr(\text{EC}(S_2)) \). By definition of conditional probability and the above claim, \( Pr(\text{EC}(S_1) | \text{EC}(S)) = \frac{Pr(\text{EC}(S_1))}{Pr(\text{EC}(S))} = pr(S_1) / pr(S) \) and, analogously, \( Pr(\text{EC}(S_2) | \text{EC}(S)) = pr(S_2) / pr(S) \). The observation now immediately follows from definition of individual probability in the above claim. ■

**Proof of Lemma 7**. It is not hard to see that \( \text{EC}(S) \) is the disjoint union of \( \text{EC}(S_1) \) and \( \text{EC}(S_2) \). Therefore, by the law of full probability, \( Pr(X \cap \text{EC}(S)) = Pr(X \cap \text{EC}(S_1)) + Pr(X \cap \text{EC}(S_2)) \). Or, in terms of conditional probabilities

\[
Pr(X | \text{EC}(S)) \times \text{EC}(S) = Pr(X | \text{EC}(S_1)) \times \text{EC}(S_1) + Pr(X | \text{EC}(S_2)) \times \text{EC}(S_2)
\]

(3)

Notice that since \( \text{EC}(S_1) ⊆ \text{EC}(S) \), we can write

\[
Pr(\text{EC}(S_1)) = Pr(\text{EC}(S_1) \cap \text{EC}(S)) = Pr(\text{EC}(S_1) | \text{EC}(S)) \times Pr(\text{EC}(S))
\]

(4)

Likewise,

\[
Pr(\text{EC}(S_2)) = Pr(\text{EC}(S_2) | \text{EC}(S)) \times Pr(\text{EC}(S))
\]

(5)

Substituting (4) and (5) into the right hand side of (3) and dividing both sides by \( Pr(\text{EC}(S)) \) gives us

\[
Pr(X | \text{EC}(S)) = Pr(X | \text{EC}(S_1)) \times Pr(\text{EC}(S_1) | \text{EC}(S)) + Pr(X | \text{EC}(S_2)) \times Pr(\text{EC}(S_2) | \text{EC}(S))
\]

(6)

The Lemma now follows from Observation 1 by choosing the appropriate \( Pr(\text{EC}(S_1) | \text{EC}(S)) \) and \( Pr(\text{EC}(S_2) | \text{EC}(S)) \) for each considered case. ■