TWO-SIDED BOUNDARY FUNCTIONALS FOR KOU PROCESS

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In this paper the moment generating function of two-sided boundary functionals for double exponential jump diffusion processes are treated.

Using the results of [1, 2] we investigate the so called two-sided boundary functionals for a Kou process, which is a Lévy process with bounded variation positive and negative jumps that have exponential distribution. To find the representation of moment generating functions for the functionals we apply the factorization method (see for instance [2]), by which the solution of the corresponding integral equation can be determined in terms of the distributions of killed extrema. For another methods we refer the reader to [3, 4] (for the method of successive iterations) and [5] (for the resolvent method).

1 Kou process and its approximating process

Consider the stochastic process
\[ \xi(t) = at + \sigma W(t) + S(t), \quad \xi(0) = 0, \quad t \geq 0, \]
where \( a \) is a finite constant, \( \sigma > 0 \), \( W(t) \) is a standard Wiener process, \( S(t) \) is a Poisson process with intensity \( \lambda > 0 \) and dual exponential jumps with density
\[ f(x) = pce^{-cx}I_{\{x\geq0\}} + qbe^{bx}I_{\{x<0\}}, \quad (c, b, p, q > 0, p + q = 1). \]

We call process (1) a Kou process.

Let \( \theta_s \) be an exponential random variable with parameter \( s > 0 \) independent of \( \xi(t) \), then the moment generating function (m.g.f.) of a Kou process killed at rate \( s \) is as follows
\[ E[e^{r\xi(\theta_s)}] = \int_0^\infty se^{-st}E[e^{r\xi(t)}]dt = \frac{s}{s - k(r)}, \quad \text{Re}[r] = 0, \]
where \( k(r) \) is the cumulant function of the form
\[ k(r) = ar + r^2\frac{\sigma^2}{2} + \lambda r \left( \frac{p}{c - r} - \frac{q}{b + r} \right). \]

To find the distribution of a functional for a Lévy process important role plays the approximating method (see [4] and references therein). By this method, we construct a prelimit process and deduce an integro-differential equation for the m.g.f. of the corresponding functional. Then solving the equation and passing to the limit yields the solution for the initial process.

The prelimit Kou process is given by
\[ \xi_n(t) = a_n t + S_n(t), \]
where \( a_n = a + 3n\sigma^2/2 \), \( S_n(t) \) is a compound Poisson process with intensity \( \lambda_n = \lambda p + 3n^2\sigma^2 + \lambda qe^{-b/n} \) and the density of jumps
\[ f_n(x) = \begin{cases} \frac{p_1(n)}{n}ce^{-cx}, & x \geq 0; \\ \frac{p_2(n)}{n}, & -\frac{1}{n} \leq x < 0; \\ \frac{p_3(n)}{n}be^{b(x+1/n)}, & x \leq -\frac{1}{n}. \end{cases} \]
where $p_1(n) = \lambda p/\lambda_n, p_2(n) = 3n^2\sigma^2/\lambda_n, p_3(n) = e^{-b/n}\lambda q/\lambda_n$. That is, the prelimit process is a Poisson process with positive drift (possibly for large enough $n$), exponentially distributed positive jumps and negative jumps that have a mixture of shifted exponential and uniform distributions.

Denote the event $A_n(T) = \{\omega : \sup_{t \leq T} |\xi(t) - \xi_n(t)| > 1/\sqrt{n}\}$, then by Kolmogorov's inequality: $P(A_n(T)) \leq Tn^{2e^{-b/n}}(e^{b/n} - 1 - b/n - 2/n^2) \sim 1/n$ (for more general case see [4]). Therefore, for any $T > 0$ the series $\sum_{n \geq 1} P(A_n(T))$ converges and using the Borel–Cantelli lemma we obtain $P\left\{\bigcup_{n \geq 1} \bigcap_{k \geq n} A_k(T)\right\} = 1$. Thus

$$P\left\{\bigcap_{T > 0} \lim_{n \to \infty} \sup_{0 \leq t \leq T} |\xi(t) - \xi_n(t)| = 0 \right\} = 1.$$ 

For the cumulant function of prelimit process we have

$$k_n(r) = a_n r + \int_{-\infty}^{\infty} \left(e^{rx} - 1\right) \lambda_n f_n(x) \, dx = ar + \lambda q e^{-b/n} \left(\frac{b}{b + r} e^{r/n} - 1\right) + \lambda p \left(\frac{c}{c - r} - 1\right) - \frac{3n^3\sigma^2}{r} \left(e^{-r/n} - 1 + r/n - \frac{1}{2} (r/n)^2\right) \to k(r).$$

By [6], finite-dimensional distributions of the prelimit process tend to ones of the Kou process and for some constant $C$, which possibly depends on $r > 0$

$$\lim_{n \to \infty} \sup_{|t_1 - t_2| \leq h} P\{|\xi_n(t_1) - \xi_n(t_2)| > \epsilon\} \leq Ch.$$ 

Moreover, let $f_T(x(t))$ be a functional determined on the Skorokhod space $D_{[0,T]}(R)$. If $f_T(x(t))$ almost everywhere continuous in the Skorokhod topology, then the distribution of $f_T(\xi_n(t))$ tends to the distribution of $f_T(\xi(t))$. By [4], the class of such functionals comprises supremum, infimum, overshoots and two-sided boundary functionals.

The basic functionals are the extrema of the process:

$$\xi^+(t) = \sup_{u \leq t} \xi(u), \quad \xi^-(t) = \inf_{u \leq t} \xi(u),$$

for which the factorization identity (the Spitzer-Rogozin identity) is hold

$$E e^{r\xi(\theta_s)} = E e^{\xi^+(\theta_s)} E e^{\xi^-(\theta_s)} e^{r|\theta_s|}, \quad \text{Re}[r] = 0,$$

where factors $E e^{r\xi(\theta_s)}$ have the analytic continuation into the half-plane $\pm \text{Re}[r] \geq 0$, respectively. The distribution of the other mentioned functionals can be represented in terms of the distribution of extrema.

Following [1], the cumulant equation $k(r) = s$ for a Kou process $\xi(t)$ has exactly two positive roots $\rho_{1,2}(s) : \rho_1(s) < c < \rho_2(s)$ and two negative roots $-r_{1,2}(s) : r_1(s) < b < r_2(s)$, which define the destiny functions of killed extrema and the m.g.f. of overshoot functionals.

**Lemma 1.1.** [1] For a Kou process the density function of killed supremum and infimum can be represented as the sum of exponents:

$$P^+_\xi(s,x) = \frac{\partial}{\partial x} P\{\xi^+(\theta_s) < x\} = A^+_1 e^{-\rho_1(s)x} + A^+_2 e^{-\rho_2(s)x}, x > 0,$$

$$P^-\xi(s,x) = \frac{\partial}{\partial x} P\{\xi^-(\theta_s) < x\} = A^-_1 e^{r_1(s)x} + A^-_2 e^{r_2(s)x}, x < 0,$$

where $A^+_1 = (-1)^{i-1} \frac{c-\rho_1(s)}{c} \frac{\rho_1(s)\rho_2(s)}{\rho_2(s)-\rho_1(s)}$ and $A^-_i = (-1)^{i-1} \frac{b-r_1(s)}{b} \frac{r_1(s)r_2(s)}{r_2(s)-r_1(s)}$, $i = 1, 2$. 

2
The first passage time, \( \tau^+ (x) = \inf \{ t \geq 0 : \xi (t) > x \} , x \geq 0 \), and the overshoot, \( \gamma^+ (x) = \xi (\tau^+ (x)) - x \), are conditionally independent and the overshoot is conditionally memoryless given that \( \gamma^+ (x) > 0 \):

\[
\mathbb{E} \left[ e^{-s \tau^+(x)}, \gamma^+ (x) = 0, \tau^+ (x) < \infty \right] = \frac{c - \rho_1 (s)}{\rho_2 (s) - \rho_1 (s)} e^{-\rho_1 (s)x} + \frac{\rho_2 (s) - c}{\rho_2 (s) - \rho_1 (s)} e^{-\rho_2 (s)x}, \quad (6)
\]

\[
\mathbb{E} \left[ e^{-s \tau^+(x)-u \gamma^+(x)}, \gamma^+ (x) > 0, \tau^+ (x) < \infty \right] = \left( \frac{c - \rho_1 (s)}{\rho_2 (s) - \rho_1 (s)} \right) \left( e^{-\rho_1 (s)x} - e^{-\rho_2 (s)x} \right) \frac{c}{c + u}. \quad (7)
\]

Using formulas \((4) - (7)\) we can deduce the relations for the m.g.f. of functionals connected with the first exit time from a fixed interval.

## 2 Two-sided boundary functionals

Define the first exit time from the interval \((x - T, x)\) \((0 < x < T)\) by the Kou process as

\[
\tau (x,T) = \inf \{ t \geq 0 : \xi (t) \notin (x - T, x) \},
\]

and assume that \( \tau (x,T) = 0 \) for \( x \notin (0, T) \). Introduce the events of the exit from the interval through the upper and lower bounds

\[
A_+ (x) = \{ \omega : \xi (\tau (x,T)) \geq x \}, \quad A_- (x) = \{ \omega : \xi (\tau (x,T)) \leq x - T \},
\]

and the m.g.f. of the exit time through the corresponding bound as

\[
Q_T (s, x) = \mathbb{E} \left[ e^{-s \tau(x,T)}, A_+ (x) \right], \quad Q_T (s, x) = \mathbb{E} \left[ e^{-s \tau(x,T)}, A_- (x) \right].
\]

The m.g.f. of exit time for the Kou process we derive as the limit of the corresponding m.g.f. for the prelimit process. For simplicity, we suppress the explicit dependence on \( \eta \) in the notation of parameters of the prelimit process.

### 2.1 M.g.f. for the moment of exit from an interval

Consider the next stochastic relation for \( \tau (x,T) \) on \( A_+ (x) \). Let \( \zeta, \eta \) denote the moment and value of the first jump of \( \xi_\eta (t) \), respectively (see Fig. 1), then

\[
\tau (x,T) = \begin{cases} 
\frac{x/a}{\zeta}, & a\zeta > x; \\
\zeta + \tau (x - a\zeta - \eta, T), & a\zeta < x, x - T < a\zeta + \eta < x; \\
\zeta, & a\zeta < x, a\zeta + \eta \geq x.
\end{cases}
\]

Using this stochastic relation and strong Markov property of \( \xi_\eta (t) \) we deduce

\[
Q_T (s, x) = \mathbb{E} \left[ e^{-sx/a}, a\zeta > x \right] + \mathbb{E} \left[ e^{s(z+\tau(x-a\zeta-\eta,T))}, a\zeta < x, x - T < a\zeta + \eta < x \right] + \mathbb{E} \left[ e^{-s\zeta}, a\zeta < x, a\zeta + \eta \geq x \right].
\]

Taking into account that \( \zeta \) have exponential distribution with parameter \( \lambda \), and \( \eta \) have density function \( f (x) \) we obtain

\[
Q_T (s, x) = e^{-(s+\lambda)\frac{x}{a}} + \int_0^{x/a} \lambda e^{-(s+\lambda)y} \int_{x-ay}^{x-ay-T} f(z) \, dz \, dy + \int_0^{x/a} \lambda e^{-(s+\lambda)y} \int_{x-ay}^{\infty} f(z) \, dz \, dy.
\]
Combining this equation with boundary conditions

\[ Q_T(s, x) = \begin{cases} 1, & x \leq 0, \\ 0, & x \geq T, \end{cases} \]

yields

\[ Q_T(s, x) = e^{-(s+\lambda)\xi} + \frac{\lambda}{\alpha} \int_0^x e^{-(s+\lambda)\xi} \int_{-\infty}^{\infty} Q_T(s, y - z) f(z) dy. \]

Differentiating the last equation with respect to \( x \) \((0 < x < T)\) gives

\[ a \frac{\partial}{\partial x} Q_T(s, x) = -(s + \lambda) Q_T(s, x) + \lambda \int_{-\infty}^\infty Q_T(s, x - z) f(z) dz. \]

Using the boundary conditions we can extend the last equation for \( x \geq T \) as

\[ a \frac{\partial}{\partial x} Q_T(s, x) = -(s + \lambda) Q_T(s, x) + \lambda \int_{-\infty}^\infty Q_T(s, x - z) f(z) dz - C_T(x), \]

where \( C_T(x) = \lambda \int_{-\infty}^\infty Q_T(s, x - z) f(z) dz I_{(x \geq T)} \) and \( I_{(A)} \) is the indicator of the event \( A \).

Using the factorization identity and projection operation \([\cdot], J \subset (-\infty, \infty)\), which is defined for an absolutely integrable function \( g(x) \) as

\[
\left[ \int_{-\infty}^\infty e^{rx} g(x) dx \right]_f = \int_f e^{rx} g(x) dx, \left[ \int_{-\infty}^\infty e^{rx} g(x) dx + C \right]_0^\pm = \mp \int_0^\pm e^{rx} g(x) dx + C,
\]

from the integro-differential equation we can deduce the integral transform of m.g.f. of the exit time through upper bound.

**Lemma 2.1.** The integral transform of m.g.f. \( Q_T(s, x) \) for the prelimit Kou process has the representation

\[
\int_0^\infty e^{rx} Q_T(s, x) dx = s^{-1} E e^{\xi^+(\theta_s)} \left[ E e^{\xi^-(-\theta_s)} \left( 1 - e^{rT} Q_T(s, T - 0) \right) \right]_+^0 + \\
+ s^{-1} E e^{\xi^+(\theta_s)} \left[ E e^{\xi^-(-\theta_s)} \int_0^\infty e^{rx} \left( \int_0^s Q_T(s, z) \lambda f(z - x) dz - C_T(x) \right) dx \right]_+^0. \tag{8}
\]
Proof. Since function $Q^T(s, x)$ has jump at point $T$: $Q^T(s, T - 0) \neq Q^T(s, T) = 0$, we see that

$$-ar \int_0^\infty e^{rx} Q^T(s, x) \, dx = a \left( 1 - e^{rT} Q^T(s, T - 0) \right) + \int_0^\infty e^{rx} \frac{\partial}{\partial x} Q^T(s, x) \, dx,$$

$$\int_0^\infty e^{rx} \frac{\partial}{\partial x} Q^T(s, x) \, dx = -(s + \lambda) \int_0^\infty e^{rx} Q^T(s, x) \, dx +$$

$$+ \int_0^\infty e^{rx} \int_{-\infty}^s Q^T(s, x) \lambda f(x - z) \, dz \, dx - \int_0^\infty e^{rx} C_T(x) \, dx.$$

Whence

$$(s - k(r)) \int_0^\infty e^{rx} Q^T(s, x) \, dx = a \left( 1 - e^{rT} Q^T(s, T - 0) \right) - \int_0^\infty e^{rx} C_T(x) \, dx +$$

$$+ \int_0^\infty e^{rx} \int_{-\infty}^0 Q^T(s, z) \lambda f(x - z) \, dz \, dx - \int_0^\infty e^{rx} \int_0^\infty Q^T(s, z) \lambda f(x - z) \, dz \, dx,$$

Utilizing now the factorization identity and the projection operation yields the equality

$$s \int_0^\infty e^{rx} Q^T(s, x) \, dx = E_{e^{\xi^+ \theta_s}} \left[ (1 - e^{rT} Q^T(s, T - 0)) \right] +$$

$$- E_{e^{\xi^+ \theta_s}} \left[ E_{e^{\xi^- \theta_s}} \int_{-\infty}^0 e^{rx} \int_0^\infty Q^T(s, z) \lambda f(x - z) \, dz \, dx \right] +$$

$$+ E_{e^{\xi^+ \theta_s}} \left[ E_{e^{\xi^- \theta_s}} \int_0^\infty e^{rx} \left( \int_{-\infty}^0 Q^T(s, z) \lambda f(x - z) \, dz - C_T(x) \right) \right] \right]_0^0.$$

The second term is zero and we obtain (8).

Inverting (8) with respect to $r$ gives

$$sQ^T(s, x) = sP'_-(s, x) C_+(s) - aQ^T(s, T - 0) \times$$

$$\times \left( \int_{\min\{x-T,0\}} P'_+(s, x - y - T) P'_-(s, y) \, dy + p_-(s) P'_+(s, x - T) I_{(x \geq T)} \right) +$$

$$+ \int_0^x \int_{-\infty}^0 \int_{-\infty}^s Q^T(s, u) \lambda f(x - y - z - u) \, du \, dz \, dr -$$

$$- \int_0^x \int_{-\infty}^0 C_T(x - y - z) \, dr \, dz \, dr.$$

Observe, that for $u \leq 0 : Q^T(s, u) = 1$, and for $u > 0 : \lambda f(u) = \lambda pe^{-cu}$, hence

$$\int_0^x Q^T(s, u) \lambda f(x - y - z - u) \, du = \int_{-\infty}^0 \lambda f(x - y - z - u) \, du = \lambda pe^{c(x-y-z)},$$

$$C_T(x - y - z) = \lambda pe^{-c(x-y-z)} \left( \int_0^T Q^T(s, u) ce^{cu} \, du + 1 \right) I_{(x-y-z \geq T)}.$$
Formula (10) determines the m.g.f. of exit time through the upper bound for the prelimit Kou process.

**Theorem 2.1.** For a Kou process the m.g.f. of exit time from the interval \((x-T, x), 0 \leq x \leq T\) through upper bound has the representation

\[
Q^T (s, x) = P \{ \xi^+ (\theta_s) \geq x \} - C_1 (T) \int_{-T}^{x-T} P_+ (s, x - y - T) P_- (s, y) \, dy - C_0 (T) \int_{-\infty}^{x} \int_{-\infty}^{x-y-T} s^{-1} \lambda pe^{-c(x-y-z)} P'_+ (s, z) P'_- (s, y) \, dzdy. \tag{11}
\]

where densities \(P'_\pm (s, x)\) are defined by (4) – (5), \(C_0 (T)\) and \(C_1 (T)\) satisfy equations \(C_0 (T) = \int_0^T Q^T (s, u) \, c e^{cu} du + 1\) and \(Q^T (s, T) = 0\).

For the m.g.f. of exit time through the lower bound the next formula holds

\[
Q_T (s, x) = P \{ \xi^- (\theta_s) \leq x - T \} - C_1 (T) \int_{x-T}^{\infty} P_+ (s, x - y) P'_- (s, y) \, dy - C_0 (T) \int_{x-y}^{\infty} \int_{-\infty}^{x-y-T} s^{-1} \lambda pe^{-b(x-y-z)} P'_+ (s, z) P'_- (s, y) \, dzdy. \tag{12}
\]

where \(C_0 (T), C_1 (T)\) obey equations \(C_0 (T) = \int_0^T Q_T (s, u) \, b e^{-b(u-T)} du + 1\) and \(Q_T (s, 0) = 0\).

**Proof.** Taking into account that as \(n \to \infty\) the distribution of killed extrema and the distribution of exit time from the interval for the prelimit process tend to the distributions of the corresponding functionals of Kou process: \(P_\pm^n (s, x) \to P_\pm (s, x), Q_n^T (s, x) \to Q^T (s, x)\), we have that \(C_0^n (T) \to C_0 (T)\) and \(C_1^n (T) \to C_1 (T)\). Moreover, \(P \{ \xi^n_\pm (\theta_s) = 0 \} \to 0\) as \(n \to \infty\) gives (11) when combined with (10).

To derive the m.g.f. for the exit time through the lower bound we use the fact that \(Q_T (s, x) = Q_T^1 (s, T-x)\), where \(Q_T^1 (s, x)\) is the m.g.f. of exit time through the upper bound for the dual process \(\xi_1 (t) = -\xi (t)\).

Denote the integrals in (11) by

\[
J_1 (s, x, T) = \int_{-T}^{x-T} P'_+ (s, x - y - T) P'_- (s, y) \, dy,
\]

\[
J_2 (s, x, T) = \int_{-\infty}^{x} \int_{-\infty}^{x-y-T} s^{-1} \lambda pe^{-c(x-y-z)} P'_+ (s, z) P'_- (s, y) \, dzdy,
\]

then

\[
Q^T (s, x) = P \{ \xi^+ (\theta_s) \geq x \} - C_1 (T) J_1 (s, x, T) - C_0 (T) J_2 (s, x, T).
\]

To find \(C_0 (T)\) and \(C_1 (T)\) use the boundary conditions for \(Q^T (s, x)\): \(Q^T (s, T) = 0\) and \(\int_0^T Q^T (s, u) \, c e^{cu} du + 1 = C_0 (T)\).

Then

\[
C_0 (T) = \frac{1 + \tilde{J}_0}{\tilde{J}_1 (1 + \tilde{J}_2) - \tilde{J}_2 \tilde{J}_1} = \frac{1 + \tilde{J}_2}{\tilde{J}_1 (1 + \tilde{J}_2) - \tilde{J}_2 \tilde{J}_1},
\]

where \(J_0 = P \{ \xi^+ (\theta_s) \geq T\}, J_1 = J_1 (s, T, T), J_2 = J_2 (s, T, T), \tilde{J}_0 = J_0^T P_+ (s, u) \, c e^{cu} du, \tilde{J}_1,2 = \int_0^T J_1,2 (s, u, T) \, c e^{cu} du.\)
3 The joint distribution of two-boundary functionals

We include into consideration the value of overshoot through a boundary at the exit moment from the interval by a Kou process:

\[
\gamma_T (x) = (\xi (\tau (x, T)) - x) I_{A_+ (x)} + (x - T - \xi (\tau (x, T))) I_{A_- (x)}.
\]

Applying similar arguments as in Section 2.1 we can establish the following integral equation for the joint m.g.f. of \( \{ \tau (x, T), \gamma_T (x) \} \) (\( \text{Im}(\alpha) = 0 \))

\[
E \left[ e^{-s \tau (x, T) + i \alpha \gamma_T (x)}, A_+ (x) \right] = E \left[ e^{-s \tau^+ (x) + i \alpha \gamma^+ (x)}, \tau^+ (x) < \infty \right] - \nonumber \\
-C_1 (T, \alpha) J_1 (s, x, T) - C_0 (T, \alpha) J_2 (s, x, T), \quad (13)
\]

where \( C_0 (T, \alpha) \) and \( C_1 (T, \alpha) \) obey equations \( E \left[ e^{-s \tau (T, T) + i \alpha \gamma_T (T)}, A_+ (T) \right] = 0 \) and \( C_0 (T, \alpha) = \int_0^T E \left[ e^{-s \tau (u, T) + i \alpha \gamma_T (u)}, A_+ (u) \right] \text{ce}^{nu} du + c (c - i \alpha)^{-1} \).

Denote \( V_0 = E \left[ e^{-s \tau^+ (T)}, \gamma^+ (T) = 0, \tau^+ (T) < \infty \right] \), \( V_0 = E \left[ e^{-s \tau^+ (T)}, \gamma^+ (T) > 0, \tau^+ (T) < \infty \right] \),

\[
\tilde{V}_0 = \int_0^T E \left[ e^{-s \tau^+ (u)}, \gamma^+ (u) = 0, \tau^+ (u) < \infty \right] \text{ce}^{nu} du,
\]

\[
\tilde{V}_{>0} = \int_0^T E \left[ e^{-s \tau^+ (u)}, \gamma^+ (u) > 0, \tau^+ (u) < \infty \right] \text{ce}^{nu} du,
\]

then from (13) the matrix form for the joint m.g.f. follows

\[
E \left[ e^{-s \tau (x, T) + i \alpha \gamma_T (x)}, A_+ (x) \right] = E \left[ e^{-s \tau^+ (x)}, \gamma^+ (x) = 0, \tau^+ (x) < \infty \right] + \nonumber \\
+ E \left[ e^{-s \tau^+ (x)}, \gamma^+ (x) > 0, \tau^+ (x) < \infty \right] \frac{c}{c - i \alpha} - \nonumber \\
- (J_1 (s, x, T); J_2 (s, x, T)) \left( \frac{J_1}{J_1 1 + J_2} \right)^{-1} \left( \frac{V_0 + \frac{c}{c - i \alpha} V_{>0}}{V_0} \right) \right). \nonumber
\]

We thus get the next assertion.

**Theorem 3.1.** For the joint m.g.f. of the first exit time from \( (x - T, x) \), \( 0 \leq x \leq T \) by a Kou process through the upper bound and the corresponding overshoot the following equalities are hold

\[
E \left[ e^{-s \tau (x, T)}, \gamma_T (x) = 0, A_+ (x) \right] = E \left[ e^{-s \tau^+ (x)}, \gamma^+ (x) = 0, \tau^+ (x) < \infty \right] - \nonumber \\
- (J_1 (s, x, T); J_2 (s, x, T)) \left( \frac{J_1}{J_1 1 + J_2} \right)^{-1} \left( \frac{V_0}{V_0} \right) \right) \quad (14)
\]

and

\[
E \left[ e^{-s \tau (x, T) + i \alpha \gamma_T (x)}, \gamma_T (x) > 0, A_+ (x) \right] = \frac{c}{c - i \alpha} \left( E \left[ e^{-s \tau^+ (x)}, \gamma^+ (x) > 0, \tau^+ (x) < \infty \right] - \nonumber \\
- (J_1 (s, x, T); J_2 (s, x, T)) \left( \frac{J_1}{J_1 1 + J_2} \right)^{-1} \left( \frac{V_{>0}}{V_{>0}} \right) \right) \right). \quad (15)
\]

Formulas (14) – (15) imply that the exit time and the corresponding overshoot are conditionally independent and the overshoot is conditionally memoryless given that \( \{ \gamma_T (x) > 0 \} \).
3.1 Density function of the process before exit from the interval

To find the m.g.f. for the process before the exit from the interval use the Pecherskii identity (see [2, Th. 4.3]):

\[
E \left[ e^{i\alpha \xi(T_\tau)} \right]_{\tau (x, T) > \theta_s} = \\
= E e^{i\alpha \xi(T_\tau)} E \left[ e^{\frac{1}{\alpha} \iota\xi(T_\tau)} (1 - e^{i\alpha \xi}) \left( e^{-sT_\tau(x, T) + i\alpha \gamma T(x)} A_+(x) \right) \right]_{x, T, \infty} = \\
= E e^{i\alpha \xi(T_\tau)} \left[ E e^{i\alpha \xi} \right]_{x, T, \infty} - \\
- E e^{i\alpha \xi(T_\tau)} \left[ E e^{i\alpha \xi} \right]_{x, T, \infty} - \\
- E e^{i\alpha \xi(T_\tau)} e^{i\alpha \xi} \left[ e^{-sT_\tau(x, T) + i\alpha \gamma T(x)} \gamma T(x) = 0, A_+(x) \right]_{x, T, \infty} - \\
- E e^{i\alpha \xi(T_\tau)} e^{i\alpha \xi} \left[ e^{-sT_\tau(x, T) + i\alpha \gamma T(x)} \gamma T(x) > 0, A_+(x) \right]_{x, T, \infty}. \\
\tag{16}
\]

For the first term in (16) we have

\[
E e^{i\alpha \xi(T_\tau)} \left[ E e^{i\alpha \xi} \right]_{x, T, \infty} = \int_{x-T}^{\infty} e^{i\alpha z} \int_{x-T}^{\min(0,z)} P'_+ (s, z - y) P'_- (s, y) dy dz,
\]

for the second

\[
E e^{i\alpha \xi(T_\tau)} e^{i\alpha \xi} \left[ e^{-sT_\tau(x, T)} \gamma T(x) = 0, A_+(x) \right]_{x, T, \infty} = \\
\int_{x-T}^{\infty} e^{i\alpha z} \int_{x-T}^{\min(z,x)} P'_+ (s, z - y) P'_- (s, y) dy \left[ e^{-sT_\tau(x, T)} \gamma T(x) = 0, A_+(x) \right],
\]

and for the third we find that

\[
E e^{i\alpha \xi(T_\tau)} e^{i\alpha \xi} \left[ e^{-sT_\tau(x, T) + i\alpha \gamma T(x)} \gamma T(x) > 0, A_+(x) \right]_{x, T, \infty} = \\
\int_{x-T}^{\infty} e^{i\alpha z} \int_{x-T}^{z} P'_+ (s, z - v) \int_{-\infty}^{\min(x,v)} e^{-c(v-y)} P'_- (s, y) dy dv dz \times \\
\times \left[ e^{-sT_\tau(x, T)} \gamma T(x) > 0, A_+(x) \right].
\]

Hence, inverting (16) with respect to \(\alpha\) we can deduce the next statement.

**Theorem 3.2.** The density function of killed Kou process until the exit time from \((x - T, x)\) has the representation

\[
h_s (T, x, z) = \frac{\partial}{\partial z} P \left\{ \xi (\theta_s) < z, \tau (x, T) > \theta_s \right\} = \int_{x-T}^{\min(0,z)} P'_+ (s, z - y) P'_- (s, y) dy - \\
- \int_{x-T}^{\min(z,x)} P'_+ (s, z - y) P'_- (s, y) dy \left[ e^{-sT_\tau(x, T)} \gamma T(x) = 0, A_+(x) \right] - \\
- \int_{x-T}^{z} P'_+ (s, z - v) \int_{-\infty}^{\min(x,v)} e^{-c(v-y)} P'_- (s, y) dy dv E \left[ e^{-sT_\tau(x, T)} \gamma T(x) > 0, A_+(x) \right].
\tag{17}
\]

Density \(h_s (T, x, y)\) determines the joint distribution of \(\{\xi^- (\theta_s), \xi (\theta_s), \xi^+ (\theta_s)\}\) (for details see [3]).

**Example.** We consider the two sets of parameters (see Tabl. 3.1). The main difference is that \(m = E\xi(1)\) have different signs: for the first case \(m < 0\), and for the second \(m > 0\). The sign of the expectation determines the asymptotic behavior of the roots of cumulant equation.
From the cumulant equation

\[ ar + r^2 \frac{\sigma^2}{2} + \lambda r \left( \frac{p}{c - r} - \frac{q}{b + r} \right) = s \]

we find the approximated values of roots, then using (4)–(5) we obtain the densities of killed extrema (see Tabl. 3.1).

### Table 1: Density functions of killed extrema

| Case | Parameters | Roots | Densities |
|------|------------|-------|-----------|
| 1    | \( a = -3.05, p = 0.75, \lambda = 4, c = 2, b = 8, \sigma = 0.5, s = 4/45 \) | \( r_2 \approx 8.2084, r_1 \approx 0.0515, \rho_1 \approx 1.0000, \rho_2 \approx 13.4599 \) | \( P_-(s, x) \approx 0.0515 e^{0.0515x} + 0.0014 e^{8.2084x}, x < 0 \) \( P_+(s, x) \approx 6.1898 e^{-13.4599x} + 0.5401 e^{-1.0000x}, x > 0 \) |
| 2    | \( a = 1, p = 0.2, \lambda = 6, c = 2, b = 8, \sigma = 2, s = 1 \) | \( r_2 \approx 8.6470, r_1 \approx 1.3654, \rho_1 \approx 0.5504, \rho_2 \approx 2.4621 \) | \( P_-(s, x) = 1.3447 e^{1.3654x} + 0.1311 e^{8.6470x}, x < 0 \) \( P_+(s, x) = 0.1638 e^{-2.4621x} + 0.5138 e^{-0.5504x}, x > 0 \) |

Substitution these densities in (11) – (12) yields the representations for \( Q_T(s, x) \), \( Q^T(s, x) \) and their sum \( Q(s, x, T) = \mathbb{E}e^{-s\tau(x,T)} \) (see Fig. 2). Due to complexity we omit the corresponding expressions.

![Figure 2: M.g.f. for the exit time from \((x - T, x)\)](image-url)

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