Numerical analysis of the SIMP model for the topology optimization of minimizing compliance in linear elasticity

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Abstract

We study the finite element approximation of the solid isotropic material with penalization (SIMP) model for the topology optimization of the compliance of a linearly elastic structure. To ensure the existence of a minimizer to the infinite-dimensional problem, we consider two popular restriction methods: $W^{1,p}$-type regularization and density filtering. Previous results prove weak(-*) convergence in the solution space of the material distribution to an unspecified minimizer of the infinite-dimensional problem. In this work, we show that, for every isolated local or global minimizer, there exists a sequence of finite element minimizers that strongly converges to the minimizer in the solution space. As a by-product, this ensures that there exists a sequence of unfiltered discretized material distributions that does not exhibit checkerboarding.

1 Introduction

Topology optimization is an important mathematical technique extensively used in the initial stages of design. The goal is to find the optimal design of a structure or device that minimizes an objective functional. In this work, we focus on the numerical analysis of the topology optimization of the compliance of linearly elastic materials. This is perhaps the most studied topology optimization problem, with roots that go back to the original works by Bendsøe and Kikuchi [6]. The topology optimization of elastic structures and its extensions have been used in numerous applications, for instance airplane wing design [1], cantilevers, and beams [5].

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We consider the density formulation via the solid isotropic material with penalization model (SIMP). Here the optimal design of elastic material is encoded by a function, $\rho$, known as the material distribution. Regions where $\rho = 1$ a.e. indicate the presence of elastic material, whereas where $\rho = 0$ a.e. are areas where material is absent, as modelled by a material with negligible stiffness. The density approach results in an infinite-dimensional and nonconvex optimization problem with PDE, box, and inequality constraints. In order to obtain a well-posed problem, the infinite-dimensional model must be coupled with a restriction method [5]. Irrespective of the choice of restriction method, even with fixed model parameters, there may exist multiple (local) minimizers to the same density formulated problem, e.g. [27, Sec. 4.5 & 4.6].

Due to the nonlinear nature of the problem, closed-form solutions are difficult to find. Hence, these models are often discretized and the finite element method (FEM) is a common choice. In this context, the FEM has been highly successful. However, the guarantee that the finite element method converges to the true solutions, in a suitably strong sense, is often missing in the literature. Partly this is due to the diverse number of restriction methods that are combined with the SIMP model. The other main difficulty is caused by the nonconvexity of the problem (in the general case). Literature that proves convergence shows that a subsequence of the finite element minimizers of the discretized problem converge to an infinite-dimensional (possibly local) minimizer of the original problem. However, these arguments do not guarantee that every minimizer can be arbitrarily closely approximated by a sequence of finite element minimizers. In particular, they do not guarantee that the global minimizer of the problem is necessarily well approximated by the finite element method.

In the context of the topology optimization of Stokes flow [10], the issue of approximating multiple minimizers with the finite element method was recently resolved [26–29]. It was shown that, for every isolated local or global minimizer of the problem, there exists a sequence of finite element solutions to the first-order optimality conditions that strongly converges to the infinite-dimensional minimizer as the mesh size tends to zero. We endeavour to prove a similar result here for any conforming finite element spaces with an approximation property.

We consider two restriction methods in this work: $W^{1,p}$-type regularization and density filtering (both described in the next section). Recall that $\rho$ denotes the material distribution and let $u$ denote the displacement of the elastic material. Our goal is to show that for every isolated local or global infinite-dimensional minimizer of the SIMP model, there exists a sequence of finite element minimizers that converges to the infinite-dimensional minimizer. In particular $u_h \to u$ strongly in $H^1(\Omega)^d$ and $\rho_h \rightharpoonup \rho$ weakly-* in $L^\infty(\Omega)$. Moreover, if a $W^{1,p}$-type regularization is used, then $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$. If a linear density filter is used, then we show that $\rho_h \to \rho$ strongly in $L^p(\Omega)$ for any $p \in [1, \infty)$. In addition, for a general class of filters, we show that $\tilde{\rho}_h \to \tilde{\rho}$ strongly
in $W^{1,p}(\Omega)$, where $p \in (1, \infty)$ depends on the discretization and $\tilde{\rho}$ and $\tilde{\rho}_h$ denote the filtered density and the discretized filtered density, respectively. The strong convergence results in the filtered, unfiltered, and regularized material distribution are novel, even for an arbitrary infinite-dimensional minimizer. As a direct consequence, there exists a sequence of strongly converging unfiltered material distributions that does not exhibit checkerboarding.

The SIMP model is introduced in Section 2 as well as the two restriction methods we consider. We provide a literature review of previous finite element results in Section 2.1. In Section 2.2 we state several properties of the topology optimization problem and its minimizers. In particular, we prove a first-order optimality condition which provides the bedrock for the proofs of the strong convergence results. We establish the finite element discretization in Section 3 where we discuss the assumptions of the discretization. In Section 3.1 we give useful auxiliary results known in the literature. Their proofs are provided in Appendix A for completeness. For each isolated local or global minimizer, we prove the existence of a sequence such that the displacement strongly converges in $H^1(\Omega)$ and the material distribution weak-* converges in $L^\infty(\Omega)$ in Section 3.2. In Section 3.3, we focus on $W^{1,p}$-type regularization and strengthen the convergence of the material distribution to strong convergence in $W^{1,p}(\Omega)$. We consider density filtering in Section 3.4 and prove that, for a linear filter, the unfiltered material distribution strongly converges in $L^p(\Omega)$ for any $p \in [1, \infty)$. Moreover, for a more general class of filters, we show that if the unfiltered material distribution weakly-* converges in $L^\infty(\Omega)$, then the filtered material distribution converges strongly in $W^{1,p}(\Omega)$ for some $p \in (1, \infty)$. Finally, in Section 4, we give our conclusions.

2 The SIMP model

Consider a bounded and open Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. Let $W^{s,p}(\Omega)$, $s \in (0, \infty)$, $p \in [1, \infty]$, and $L^q(\Omega)$, $q \in [1, \infty]$, denote the standard Sobolev and Lebesgue spaces, respectively [2]. Define $H^s(\Omega) := W^{s,2}(\Omega)$. Let $\Gamma \subseteq \partial \Omega$ be a subset of the boundary with nonzero Hausdorff measure $\mathcal{H}^{d-1}(\Gamma) > 0$. Then we define:

$$H^1_\Gamma(\Omega)^d := \{ v \in H^1(\Omega)^d : v|_\Gamma = 0 \}, \quad (2.1)$$

where $|_\Gamma$ is the standard boundary trace operator $|_\Gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p}, d}(\partial \Omega)$ [20].

The topology optimization problem is to find $u : \mathbb{R}^d \to \mathbb{R}^d$, $u \in H^1(\Omega)^d$, that minimizes

$$J(u, \rho) := l(u) + R(\rho) \quad (2.2)$$
where
\[
    l(u) := \int_{\Omega} f \cdot u \, dx + \int_{\Gamma_N} g' u \cdot \hat{t} \, ds + \int_{\Gamma_N} g^n u \cdot \hat{n} \, ds,
\]
subject to the linear elasticity PDE constraint
\[
\begin{align*}
    \text{div}(S) &= f \quad \text{in } \Omega, \\
    S &= k(F(\rho)) [2\mu D(u) + \lambda \text{div}(u) I] \quad \text{in } \Omega, \\
    u \cdot \hat{t} &= 0 \quad \text{on } \Gamma_D, \\
    u \cdot \hat{n} &= 0 \quad \text{on } \Gamma_D, \\
    \hat{t} \cdot S \hat{n} &= g^t \quad \text{on } \Gamma_N, \\
    \hat{n} \cdot S \hat{n} &= g^n \quad \text{on } \Gamma_N,
\end{align*}
\]
as well as the following box and inequality constraints on \( \rho \in L^\infty(\Omega) \):
\[
0 \leq \rho \leq 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} F(\rho) \, dx \leq \gamma |\Omega|.
\]

The state \( u \) denotes the displacement of the structure and \( S \) denotes the stress tensor. The body force \( f \in L^2(\Omega)^d \) and traction forces \( g^t \in L^2(\Gamma_D^t) \) and \( g^n \in L^2(\Gamma_D^n) \) are known. \( \Gamma_D^n, \Gamma_D^t \subset \partial \Omega \) (respectively \( \Gamma_D^t, \Gamma_D^n \subset \partial \Omega \)) are known disjoint boundaries on \( \partial \Omega \) such that \( \Gamma_D^n \cup \Gamma_D^t = \partial \Omega \) (respectively \( \Gamma_D^t \cup \Gamma_D^n = \partial \Omega \)). \( \mu \) and \( \lambda \) are the Lamé coefficients, \( \text{tr}(\cdot) \) is the matrix-trace operator, \( I \) is the \( d \times d \) identity matrix, \( \hat{t} \) and \( \hat{n} \) are the unit tangent and outward normal, respectively, and
\[
    D(u) = \frac{1}{2} (\nabla u + \nabla u^\top), \quad k(\rho) = \epsilon_{\text{SIMP}} + (1 - \epsilon_{\text{SIMP}}) \rho^p_s,
\]
where \( 0 < \epsilon_{\text{SIMP}} \ll 1 \) and \( p_s \geq 1 \). A typical “magic” choice resulting in crisp solutions is \( p_s = 3 \) [35, Sec. 3]. The constant \( \gamma \in (0, 1) \) is the volume fraction of the total volume of the domain that the material distribution can occupy. Here, \( |\cdot| \) denotes the Lebesgue measure of a set. Let \( H_0^1(\Omega)^d := \{ v \in H^1(\Omega)^d : v \cdot \hat{t}|_{\Gamma_D^t} = 0, v \cdot \hat{n}|_{\Gamma_D^n} = 0 \} \) and denote the space of admissible rigid body motions by \( \text{RM}_D^f \) [4, Sec. 2.2]:
\[
    \text{RM}_D^f := \{ Rx + a : a \in \mathbb{R}^d, R \in \text{skew}_d \land a \},
\]
where \( \text{skew}_d \) denotes the vector space of constant skew-symmetric \( d \times d \) matrices and \( \land \) denotes the wedge product. For instance, \( \text{RM}_D^2 = \text{span}\{[1,0]^\top, [0,1]^\top, [-y,x]^\top\} \). Define the space of displacements by \( V_D \) where
\[
    V_D := \{ v \in H_0^1(\Omega)^d : (\nabla v, R)_{L^2(\Omega)} = 0 \text{ for all } R \in \nabla(\text{RM}_D^f \cap H_0^1(\Omega)^d) \}. \tag{2.7}
\]

For a physical interpretation of the SIMP model we refer to Bendsøe and Sigmund [5, Ch. 1]. When \( \rho \approx 1, k(\rho) \approx 1 \), indicating the presence of material, whereas where
\( \rho \approx 0, k(\rho) \approx c_{\text{SIMP}}, \) indicating void. By raising \( \rho \) to the power of \( p_s \), values of \( \rho \) between 0 and 1 are penalized.

The optimization problem (2.2)–(2.5) is normally stated with \( R(\rho) = 0 \) and \( F(\rho) = \rho \). With these choices (2.2)–(2.5) is ill-posed in general, i.e. it does not have minimizers in the continuous setting. Naïve attempts at finding minimizers often yield checkerboard patterns of \( \rho \); a phenomenon where the discretized material distribution wildly oscillates in value between neighbouring elements. Checkerboarding can be avoided, without modifications to the discretized optimization problem induced by (2.2)–(2.5), by particular choices of finite element spaces for \( u \) and \( \rho \). However, the solutions will still be mesh dependent i.e. as the mesh is refined, the beams of the solutions will become ever thinner, leading to nonphysical solutions in the limit. There are several schemes employed by the topology optimization community to obtain physically reasonable solutions for \( \rho \) known as restriction methods \([5]\). An excellent review is given by Sigmund and Petersson \([36]\) as well as Borrvall \([8]\). We consider two restriction methods in this work: \( W^{1,p} \)-type regularization and density filtering.

In order to guarantee the existence of a minimizer of (2.2)–(2.5), Borrvall notes one requires both weak(*) compactness in the solution space and weak(*) lower semicontinuity of the objective functional on the same function space \([8]\). Although the material distribution solution space is weakly-* compact with respect to the \( L^\infty \)-norm, the objective functional is not weakly-* lower semicontinuous on \( L^\infty(\Omega) \). Thus, any attempt at a proof of existence fails. Consider the material distribution solution space

\[
\mathcal{H}_\gamma := \{ \eta \in L^\infty(\Omega) : 0 \leq \rho \leq 1 \text{ a.e., } \int_\Omega F(\rho) \, dx \leq \gamma |\Omega| \}.
\]

(2.8)

Restriction methods are typically seen in one of two ways:

(M1) We seek a solution \( \rho \in C_\gamma \subset \mathcal{H}_\gamma \) such that \( C_\gamma \) is weakly compact in the correct space;

(M2) We modify the objective functional so that it is weakly-* lower semicontinuous on \( L^\infty(\Omega) \).

We incorporate two of the most common types of restriction methods via \( R(\rho) \) and \( F(\rho) \). We use the functional \( R \) to model \( W^{1,p} \)-type regularization of \( \rho \). This forces one to seek a solution \( \rho \in W^{1,p}(\Omega) \cap \mathcal{H}_\gamma \subset \mathcal{H}_\gamma \), i.e. this is an (M1) type method. Examples of choices for \( R \) include, for \( \epsilon > 0, \beta > 0 \),

\[
R(\rho) = \frac{\epsilon}{p} \| \nabla \rho \|_{L^p(\Omega)}^p, \quad p \in (1, \infty), \quad \text{(\( W^{1,p} \)-regularization), (2.9)}
\]

\[
R(\rho) = \epsilon \| \nabla \rho \|_{L^1(\Omega)} \quad \text{(Total variation regularization), (2.10)}
\]

\[
R(\rho) = \epsilon \| \nabla \rho \|_{L^\infty(\Omega)} \quad \text{(Lipschitz continuity), (2.11)}
\]

\[
R(\rho) = \frac{\beta \epsilon}{2} \| \nabla \rho \|_{L^2(\Omega)}^2 + \frac{\beta}{2\epsilon} \int_\Omega \rho(1 - \rho) \, dx \quad \text{(Ginzburg–Landau regularization). (2.12)}
\]
Here we only consider $W^{1,p}$-type regularization where $p \in (1, \infty)$. Hence, total variation regularization and Lipschitz continuity regularization are beyond the scope of this work.

**Definition 2.1** ($W^{1,p}$-type regularization). We say that the restriction method is a $W^{1,p}$-type regularization restriction method if $R(\rho) = \frac{1}{p} \|\nabla \rho\|_{L^p(\Omega)}^p + m(\rho)$, where $m(\rho)$ is continuously Fréchet differentiable and models lower order terms not involving $\nabla \rho$. For instance $m(\rho) = \rho(1 - \rho)$.

Two common assumptions for $R(\cdot)$ are the following:

(R1) $R(\cdot)$ is weakly lower semicontinuous on $W^{1,p}(\Omega)$;

(R2) $R(\eta)$ is Fréchet differentiable at $\eta$ for all $\eta \in W^{1,p}(\Omega)$.

Often the $W^{1,p}(\Omega)$-type regularization is added as an additional constraint rather than at the level of the objective function. For instance, $R(\rho) = 0$ but one adds the constraint $\|\nabla \rho\|_{L^p(\Omega)} \leq M < \infty$ for a user-chosen $M > 0$. Although the implementation differs, the analysis is similar and we expect many of the results discussed in this work to continue to hold.

The function $F(\cdot)$ models a density filtering of the problem [11, 22, 38]. Filtering averages $\rho$ in small neighbourhoods at every point. The first application in the context of topology optimization of compliance is attributed to Bruns and Tortorelli [14]; existence and FEM convergence was first shown by Bourdin [11]. We make the following assumptions on the filters:

(F1) For any $\eta \in \mathcal{H}_\gamma$, then $\|F(\eta)\|_{W^{1,\infty}(\Omega)} \leq C < \infty$ with $C$ independent of $\eta$;

(F2) For any $\eta \in \mathcal{H}_\gamma$, we have $0 \leq F(\eta) \leq 1$ a.e. in $\Omega$;

(F3) $F$ is Fréchet differentiable with respect to its argument;

(F4) Suppose there exists an $L^p$-bounded sequence, $p \in [1, \infty]$, such that $\eta_n \rightharpoonup \eta$ weakly in $L^p(\Omega)$ ($\eta_n \rightharpoonup^* \eta$ weakly-* in $L^\infty(\Omega)$ if $p = \infty$). Then there exists a subsequence (not indicated) such that $F(\eta_n) \to F(\eta)$ strongly in $L^p(\Omega)$.

To prove convergence of the filtered material distribution in $W^{1,p}(\Omega)$, $p \in (1, \infty)$, we require the following assumption:

(F5) The space of filtered material distributions, $\tilde{C}_\gamma$, is a norm-closed and convex subset of $W^{1,p}(\Omega)$ for any $p \in (1, \infty)$.

Filtering is often described via the framework of (M2). The following proposition is a well known result and we give a proof in Appendix A for completeness.
Proposition 2.1. Consider a filter of the form
\[ F(\rho)(x) = \int_{\Omega} f(x-y)\rho(y)\,dy, \] (2.13)
where \( f \in W^{1,\infty}(\mathbb{R}^d), \ f \geq 0 \ a.e. \ and \ \|f\|_{L^1(\mathbb{R}^d)} = 1. \) Then, (F1)–(F5) are satisfied.

We do not discuss sensitivity filtering in this work [34]. The general consensus is that, although sensitivity filtering is cheap and effective, it is heuristic, and not theoretically thoroughly understood. In particular there is a lack of existence proofs. We refer to the works of Lazarov and Sigmund as well as Evgrafov and Bellido for some studies [17, 24].

Although the restriction methods guarantee existence, they do not guarantee uniqueness. Indeed, the optimization problem is nonconvex and there may exist multiple local minima to the same optimization problem. In [27, Sec. 4.5 & 4.6], there are examples of cantilevers and MBB beams (see Example 2.1), coupled with a Ginzburg–Landau restriction method, where two local minima are found for each example. The nonconvexity complicates any numerical analysis. Given a bounded sequence of finite element minimizers, whereby the mesh size tends to zero, it may be possible to take one subsequence that converges to one solution and a different subsequence that converges to a different solution.

A key assumption for the convergence results in Theorems 3.1–3.3 is that the local and global minimizers are isolated.

Definition 2.2 (Isolated minimizer). Let \( Z \) be a Banach space and suppose that \( z_0 \in Z \) is a local or global minimizer of the functional \( J : Z \to \mathbb{R} \). We say that \( z_0 \) is isolated if there exists an open neighborhood \( E \subset Z \) of \( z_0 \) such that there are no other minimizers contained in \( E \).

An isolation assumption is reasonable in this context. All multiple local minima found in the literature have visibly different structures and topologies.

We denote the restricted solution space for the material distribution as \( C_\gamma \),
\[
C_\gamma := \begin{cases} 
\mathcal{H}_\gamma \cap W^{1,p}(\Omega) & \text{if a } W^{1,p}\text{-type regularization is used,} \\
\mathcal{H}_\gamma & \text{otherwise.}
\end{cases}
\] (2.14)

Example 2.1 (MBB beam). To illustrate the model, we provide an example of a Messerschmitt–Bölkow–Blohm (MBB) beam. Consider the domain \( \Omega = (0, 3) \times (0, 1) \). Consider the boundaries:
\[
\Gamma_D^0 = \{x = 0\}, \ \Gamma_D^y = \{y = 0, \ 2.9 \leq x \leq 3\}, \ \Gamma_N^y = \{y = 1, \ 0 \leq x \leq 0.1\}. \] (2.15)
The Dirichlet and Neumann boundaries are
\[
\Gamma_D^n = \Gamma_D^0 \cup \Gamma_D^y, \ \Gamma_D^l = \emptyset, \ \Gamma_N^n = \partial \Omega \setminus \Gamma_D^n, \ \Gamma_N^l = \partial \Omega. \] (2.16)
The body and traction forces are $f = 0$, $g' = 0$, and

$$g''(x, y) = \begin{cases} -10 & \text{if } (x, y) \in \Gamma^N_{N_1}, \\ 0 & \text{otherwise.} \end{cases}$$

(2.17)

These conditions describe a half-beam that is fixed horizontally on the $y$-axis and fixed vertically at its bottom right corner on the $x$-axis. There is a boundary force pushing vertically downwards at the top left corner, which represents the middle of the beam when the half-beam is mirrored. This is visualised in Fig. 1. With these boundary conditions, a check reveals that $RM^D \cap H^1_D(\Omega)^d = \{0\}$. Thus $V_D = H^1_D(\Omega)^d$.

Consider the volume fraction $\gamma = 0.535$, the Lamé coefficients $\mu = 75.38$ and $\lambda = 64.62$, the SIMP parameters, $\epsilon_{\text{SIMP}} = 10^{-5}$ and $p_s = 3$. If one uses a Ginzburg–Landau regularization, it is possible to find two local minima as depicted in Fig. 2.

Figure 1: Setup of the MBB beam. The remaining unlabeled boundary conditions are $(\mathbf{S\hat{n}}) \cdot \mathbf{i} = 0$ on $\Gamma^N_{D_1} \cup \Gamma^N_{D_2}$ and $\mathbf{S\hat{n}} = (0, 0)^T$ on $\partial \Omega \setminus (\Gamma^N_{D_1} \cup \Gamma^N_{D_2} \cup \Gamma^N_{N_1})$.

Figure 2: The material distribution of two (local) minima of the MBB beam problem [27, Sec. 4.6]. In white regions $\rho = 1$ whereas in black regions $\rho = 0$. The parameters are $\epsilon = 1.90 \times 10^{-2}$, $\beta = 9 \times 10^{-3}$, $\gamma = 0.535$, $\epsilon_{\text{SIMP}} = 10^{-5}$, $p_s = 3$, and the Lamé coefficients are $\mu = 75.38$ and $\lambda = 64.62$.

For a finite element discretization of the linear elasticity PDE constraint, we must recast (2.4) in variational form. The most common choice, in the topology optimization literature, is the primal formulation due to its simplicity and comparatively computational inexpense due to fewer degrees of freedom. Despite the issue of locking (discussed
later), all known FEM convergence results for the SIMP model deal solely with the primal formulation. Hence, in this work, we study the primal formulation and provide results for some open problems in the literature.

By examining (2.4) one can eliminate the stress tensor $\mathbf{S}$. In the variational form, the primal formulation of the optimization problem reduces to, for any $\mathbf{v} \in \mathbf{V}_D$,

$$\min_{(\mathbf{u}, \rho) \in \mathbf{V}_D \times \mathbf{C}_\gamma} J(\mathbf{u}, \rho) \text{ subject to } a(\mathbf{u}, \mathbf{v}; \rho) = l(\mathbf{v}),$$

(SIMP)

where

$$a(\mathbf{u}, \mathbf{v}; \rho) := \int_{\Omega} k(F(\rho)) \mathbf{E}\mathbf{u} : \mathbf{E}\mathbf{v} \, dx,$$

$$\mathbf{E}\mathbf{u} : \mathbf{E}\mathbf{v} := 2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \lambda \text{div}(\mathbf{u}) \text{div}(\mathbf{v}).$$

(2.18)

(2.19)

An alternative view of filtering is that instead of searching for a function $\rho \in \mathbf{C}_\gamma$ that minimizes (2.2)–(2.5), we seek a function $\tilde{\rho}

\tilde{\rho} \in \tilde{\mathbf{C}}_\gamma := \{ \tilde{\eta} \in W^{1,\infty}(\Omega) : \exists \eta \in \mathbf{C}_\gamma \text{ s.t. } \tilde{\eta} = F(\eta) \}.

(2.20)

Note that $\rho$ only appears in the optimization problem (2.2)–(2.5) in the box and volume constraints as well as the regularization term $\mathcal{R}$. However, by assumption (F2), $0 \leq \rho \leq 1$ a.e. implies that $0 \leq F(\rho) \leq 1$. If $\mathcal{R}(\rho) = 0$, (SIMP) is equivalent to

$$\min_{(\mathbf{u}, \tilde{\rho}) \in \mathbf{V}_D \times \tilde{\mathbf{C}}_\gamma} \tilde{J}(\mathbf{u}, \tilde{\rho}) \text{ subject to } \tilde{a}(\mathbf{u}, \mathbf{v}; \tilde{\rho}) = l(\mathbf{v}),$$

(F-SIMP)

where $\tilde{J}(\mathbf{u}, \tilde{\rho}) = J(\mathbf{u}, \rho)$ and $\tilde{a}(\mathbf{u}, \mathbf{v}; \tilde{\rho}) = a(\mathbf{u}, \mathbf{v}; \rho)$.

2.1 Literature review

Below we give a (perhaps non-exhaustive) list of the known FEM convergence results. Let $h$ denote the finite element discretization on a mesh with mesh size $h$. The discretized filtered material distribution $F_h(\rho_h)$ is defined in Section 3.

- One of the first FEM convergence results for the topology optimization of a linearly elastic material was shown by Petersson and Haslinger [32]. They showed that a sequence of FEM solutions, $(\mathbf{u}_h, \rho_h)$, of a problem similar to the discretization of (SIMP), converge to an unspecified solution $(\mathbf{u}, \rho)$. In particular $\mathbf{u}_h \to \mathbf{u}$ strongly in $H^1(\Omega)^d$ and $\rho_h \rightharpoonup \rho$ weakly-* in $L^\infty(\Omega)$.

- For the case where the SIMP parameter $p_s = 1$, the problem reduces to the problem of the optimal variable thickness of sheets. Petersson analyzed such problems [30] and showed that, under a regularity and biaxiality assumption on the optimal stress field, the minimizer $\rho$ is unique. Moreover, for any conforming discretization
of \( u \) and a \( \text{DG}_0 \) (piecewise constant) discretization for \( \rho \), the minimizing sequence of FEM solutions converges to the unique minimizer \((u, \rho)\) such that \( u_h \to u \) strongly in \( H^1(\Omega)^2 \) and \( \rho_h \to \rho \) strongly in \( L^p(\Omega) \) for any \( p \in [1, \infty) \).

- Petersson and Sigmund analyzed the primal formulation in the context of slope constraints [33], which have a similar flavour to \( W^{1, \infty} \)-regularization [8, Sec. 6.3]. Here the admissible set of material distributions is reduced to Lipschitz continuous functions and a pointwise bound is placed on \( \partial x_i \rho \) for \( i = 1, 2 \). They showed that a \((\text{CG}_1)^2\) (continuous piecewise linear) discretization for \( u \) and a \( \text{DG}_0 \) discretization for \( \rho \) converges to an unspecified minimum of the problem. In particular \( u_h \to u \) strongly in \( H^1(\Omega)^2 \) and \( \rho_h \to \rho \) uniformly in \( \Omega \) and, therefore, \( \rho_h \to \rho \) strongly in \( L^\infty(\Omega) \).

- Petersson [31] investigated a regularization in the flavour of a \( W^{1, 1}(\Omega) \)-regularization together with penalization terms such as \( \epsilon^{-1} \int_\Omega \rho(1-\rho) \, dx \). He used a \((\text{CG}_1)^2\) discretization for \( u \) and a \( \text{DG}_0 \) discretization for \( \rho \). He showed that a sequence of FEM minimizers of the discretized (SIMP), \((u_h, \rho_h)\), converges to an unspecified minimizer \((u, \rho)\) such that \( u_h \to u \) strongly in \( H^1(\Omega)^d \) and \( \rho_h \to \rho \) strongly in \( L^p(\Omega) \) for any \( p \in [1, \infty) \).

- Talischi and Paulino studied a \( W^{1, 2}\)-regularization [37]. They used a \((\text{CG}_k)^2\) and \((\text{CG}_k)\), \( k \geq 1 \), for the discretization of \( u \) and \( \rho \), respectively. However, they allow for \( \rho \) to be projected into the space \( \text{DG}_0 \) in the PDE constraint. They showed that there exists a sequence of FEM solutions, \((u_h, \rho_h)\) converging to an unspecified minimizer \((u, \rho)\) such that \( u_h \to u \) strongly in \( H^1(\Omega)^2 \) and \( \rho_h \to \rho \) weakly in \( L^s(\Omega) \) for any \( s \in [1, \infty) \).

- In the context of linear filtering, Bourdin showed that, for a \((\text{CG}_1)^2\) discretization for \( u \) and a \( \text{DG}_0 \) discretization for \( \rho \), the sequence of minimizers \((u_h, \rho_h)\) of the discretized (SIMP) converges to an unspecified minimizer \((u, \rho)\) such that \( u_h \to u \) strongly in \( H^1(\Omega)^d \), \( F_h(\rho_h) \to F(\rho) \) uniformly on \( \Omega \), and \( \rho_h \to \rho \) weakly in \( L^s(\Omega) \) for any \( s \in [1, \infty) \) [11].

- Rather than applying the filtering at the level of the PDE constraint, Borrvall and Petersson [9] instead added the kernel at the level of the objective functional, a regularization they called the regularized intermediate density control. They showed that for any conforming discretization for \( u \) and \( \text{DG}_0 \) discretization for \( \rho \), there exists a sequence of FEM solutions \( \rho_h \) such that \( \rho_h \xrightarrow{\ast} \rho \) weakly-* in \( L^\infty(\Omega) \) where \( \rho \) is an unspecified minimizer of the problem. This is subsequently strengthened to \( \rho_h \to \rho \) strongly in \( L^s(\Omega_b) \), for any \( s \in [1, \infty) \), where \( \Omega_b = \{ \rho = 0 \text{ or } 1 \text{ a.e. in } \Omega \} \), if such a set \( \Omega_b \) exists and is measurable.

- Greifenstein and Stingl [21] consider a material distribution that is partially filtered and partially regularized with slope constraints. Let the pair \((\rho, \theta)\) denote the
filtered and slope constrained designs in that order. They show that there exists a sequence of FEM solutions that converge to an unspecified minimizer \((u, \rho, \theta)\) such that \(u_h \to u\) strongly in \(H^1(\Omega)^d\), \(\rho_h \rightharpoonup^* \rho\) weakly-* in \(L^\infty(\Omega)\), \(F_h(\rho_h) \to F(\rho)\) uniformly and \(\theta_h \to \theta\) uniformly.

- Berggren and Kasolis [7] derived error bounds for a \((\text{CG}_k)^2, k \geq 1\), discretization for \(u\) in a model that incorporates the weak material approximation (i.e. the fact that \(\epsilon_{\text{SIMP}} > 0\)). However, the (0-1) material distribution \(\rho\) is fixed a priori, and, therefore, the minimizer is unique.

Although the above list might not be exhaustive, all works follow very similar arguments as Petersson and Sigmund [33] or Bourdin [11] for regularization and filtering techniques, respectively.

Our goal is to answer the following open problems in the numerical analysis:

(P1) Apart from the specific cases where there is a unique solution, it is not clear which minimizer the sequence is converging to as the nonconvexity of the problem provides multiple candidates for the limits;

(P2) (Regularization). Although some results prove strong convergence of \(\rho\) in the \(L^s\)-norm, for any \(s \in [1, \infty)\), one expects convergence in the \(W^{1,p}\)-norm in a \(W^{1,p}\)-type regularization;

(P3) (Filtering). It is well known that the discretized filtered material distribution \(F(\rho)\) uniformly converges and the unfiltered material distribution \(\rho\) weakly(-*) converges. However, it is an open problem if the unfiltered material distribution strongly converges in \(L^p(\Omega)\), \(p \in (1, \infty)\);

(P4) (Filtering). Given that the infinite-dimensional filtered distribution \(F(\rho) \in W^{1,\infty}(\Omega)\), it is an open problem whether the discretized filtered material distribution converges strongly in \(W^{1,p}(\Omega)\) for some \(p \in [1, \infty]\).

In both Theorems 3.1 and 3.2 we tackle the issue of multiple minimizers (P1). In Theorem 3.1 we resolve the open problem (P2). If one chooses a linear density filter as a restriction method in the sense of Bourdin [11], then, in Theorem 3.2, we consider (P3). Finally, in Theorem 3.3, we address (P4).

2.2 Properties of (SIMP)

In this subsection we provide existence theorems for (SIMP) and prove a first-order optimality condition with respect to taking variations in the material distribution.

Proposition 2.2 (Existence & uniqueness with \(\rho\) fixed). Suppose that \(\rho \in C_\gamma\) is fixed, \(l(v) = 0\) for all \(v \in \text{RM}_D\), and (F2) holds. Then, there exists a unique \(u \in V_D\) that satisfies the PDE constraint in (SIMP).
Proof. The result is standard and follows as a direct consequence of Korn’s inequality with tangential or normal boundary conditions [4, Th. 11] and the Lax–Milgram Theorem [16, Ch. 6.2, Th. 1]. See also [13, Ch. 11].

An essential result utilized in this work is the existence of a minimizer with either a regularization or a density filter.

**Theorem 2.1** (Existence of a minimizer with regularization: Ch. 5.2.2 in [5]). Suppose that a $W^{1,p}$-type regularization is used and $(R1)$ holds. Then, there exists a minimizer $(u, \rho) \in V_D \times (H_\gamma \cap W^{1,p}(\Omega))$ of (SIMP).

**Theorem 2.2** (Existence of a minimizer with density filtering: Sec. 3 in [11]). Suppose that a density filtering is used and $(F1)$–$(F4)$ hold. Then, there exists a minimizer $(u, \rho) \in V_D \times H_\gamma$ of (SIMP).

A useful property of the primal formulation of (SIMP) is its reformulation as a saddle-point problem and the subsequent first-order optimality conditions. In the following two propositions, we assume that $(u, \rho)$ is a minimizer of (SIMP). The result of the following proposition is standard [23, Ch. 1.4].

**Proposition 2.3** (Saddle point reformulation). There exists a Lagrange multiplier $u_a \in V_D$ such that the tuple $(u, \rho, u_a)$ is a saddle point of the Lagrangian

$$
\mathcal{L}(u, \rho, u_a) := l(u) + \int_\Omega k(F(\rho)) |E u| d\Omega \cdot u_a - l(u_a) + R(\rho),
$$

i.e. for any $(v, \eta, v_a) \in V_D \times C_\gamma \times V_D$, within a neighborhood of $(u, \rho, u_a)$,

$$
\mathcal{L}(u, \rho, v_a) \leq \mathcal{L}(u, \rho, u_a) \leq \mathcal{L}(v, \eta, u_a).
$$

**Proposition 2.4** (First-order optimality condition). Suppose that $(R1)$–$(R2)$ and $(F1)$–$(F4)$ hold. Then, $(u, \rho)$ satisfies the first-order optimality condition, for any $\eta \in C_\gamma$,

$$
- \int_\Omega k'(F(\rho)) |E u|^2 \langle F'(\rho), \eta - \rho \rangle d\Omega + \langle R'(\rho), \eta - \rho \rangle \geq 0,
$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairings between $F'(\rho)$ and $\eta - \rho$ as well as $R'$ and $\eta - \rho$.

Proof. By considering the variation in $u$ of the Lagrangian in (2.21), a quick calculation reveals that $u_a = -u$ a.e. Hence, $(u, \rho)$ is a stationary point of

$$
L(u, \rho) = 2l(u) - \int_\Omega k(F(\rho)) |E u|^2 d\Omega + R(\rho)
$$

(2.24)
such that $L(u, \rho) \leq L(v, \eta)$ for all $(v, \eta) \in V_D \times C_\gamma$ within a neighbourhood of $(u, \rho)$. $C_\gamma$ is a convex set. Hence, for any $\eta, \zeta \in C_\gamma$, then $\eta + t(\eta - \zeta) \in C_\gamma$ for any $t \in [0, 1]$. Thus, for any $\eta = \rho + t(\rho - \zeta)$,

$$L(u, \rho + t(\rho - \zeta)) - L(u, \rho) \geq 0. \quad (2.25)$$

By dividing by $t$, sending $t \to 0^+$, utilizing assumptions (R2) and (F3) as well as an application of the chain rule, one achieves the result.

## 3 Finite element approximation

Consider a family of quasi-uniform and non-degenerate meshes $(T_h)$ as $h \to 0$ [13, Def. 4.4.13]. We choose conforming finite element discretizations $V_h \subset H^1(\Omega)^d$ and $C_{\gamma,h} = P_k \cap C_\gamma$, $k \geq 0$, where $P_k$ denotes piecewise polynomials of degree $k$ that are continuous if a $W^{1,p}$-type regularization is used. Except conformity, the only property these finite element spaces require is that any function in the infinite-dimensional solution space can be approximated arbitrarily well as the mesh size tends to zero.

In general, the filtered discretized material distribution $F(\rho_h) \notin P_k$. Hence, we are required to project $F(\rho_h)$ into the space $P_k$. We denote the projection of $F(\rho_h)$ in $P_k$ as $F_h(\rho_h) = \Pi_h F(\rho_h) \in P_k$. We assume that:

(F6) For any $\eta_h \in C_{\gamma,h}$, then $\|F_h(\eta_h)\|_{W^{1,\infty}(\Omega)} \leq C < \infty$ with $C$ independent of $\eta_h$ and $h$;

(F7) The projection is a linear operator, i.e. for any $u, v \in L^p(\Omega)$, $p \in [1, \infty]$, $a, b \in \mathbb{R}$,

$$\Pi_h(au + bv) = a\Pi_h u + b\Pi_h v;$$

(F8) For any $\eta \in C_\gamma$, $\|F(\eta) - F_h(\eta)\|_{L^p(\Omega)} \leq C h^s \|F(\eta)\|_{W^{1,p}(\Omega)}$, $p \in [1, \infty]$, for some $s > 0$, where $C$ only depends on the degeneracy of the mesh.

The interpolant would satisfy the assumptions (F6)–(F8), see [13, Th. 4.4.20] and [13, Prop. 3.3.4].

We denote the restriction of $V_h$ to the boundary conditions as:

$$V_{D,h} := V_h \cap V_D. \quad (3.1)$$

Recall the definition of the $\tilde{C}_\gamma$ in (2.20). If filtering is used, we define the space $\tilde{C}_{\gamma,h}$ as

$$\tilde{C}_{\gamma,h} := \{ \tilde{\eta}_h \in C_{\gamma,h} : \exists \eta_h \in C_{\gamma,h} \text{ such that } \tilde{\eta}_h = F_h(\eta_h) \}. \quad (3.2)$$

In general, it will not be possible to represent the body and traction forces $f$, $g^n$, $g^t$ exactly in the displacement finite element space. Hence, for each $h$, we instead consider $f_h$, $g^n_h$, $g^t_h$ (which can be represented) and assume that
(A1) \( f_h \rightarrow f \) strongly in \( L^2(\Omega)^d \), \( g^n_h \rightarrow g^n \) strongly in \( L^2(\Gamma_N^n) \), and \( g^t_h \rightarrow g^t \) strongly in \( L^2(\Gamma_N^t) \).

Moreover,

(A2) The finite element spaces are dense in their respective function spaces, i.e., for any \((v, \eta, \tilde{\eta}) \in H^1(\Omega)^d \times C_\gamma \times \tilde{C}_\gamma\),

\[
\lim_{h \rightarrow 0} \inf_{w_h \in V_h} \|v - w_h\|_{H^1(\Omega)} = \lim_{h \rightarrow 0} \inf_{\zeta_h \in C_{\gamma,h}} \|\eta - \zeta_h\|_Y = \lim_{h \rightarrow 0} \inf_{\tilde{\zeta}_h \in \tilde{C}_{\gamma,h}} \|\tilde{\eta} - \tilde{\zeta}_h\|_Y = 0,
\]

where \( Y = W^{1,p}(\Omega) \) if a \( W^{1,p} \)-type regularization is used and \( Y = L^2(\Omega) \) otherwise.

The discretized compliance is denoted \( J_h(u_h, \rho_h) := l_h(u_h) + \mathcal{R}(\rho_h) \),

(3.3)

where

\[
l_h(u_h) := \int_{\Omega} f_h \cdot u_h \, dx + \int_{\Gamma_N^d} g^d_h u_h \cdot \hat{t} \, ds + \int_{\Gamma_N^t} g^t_h u_h \cdot \hat{n} \, ds.
\]

(3.4)

We define the bilinear form

\[
a_h(u_h, v_h; \rho_h) := \int_{\Omega} k(F_h(\rho_h)) \mathbf{E} u_h : \mathbf{E} v_h \, dx.
\]

(3.5)

The following three theorems are the core aims of this work. In all three theorems, we assume that \( \Omega \subset \mathbb{R}^d \) is a polygonal domain in two dimensions or a polyhedral Lipschitz domain in three dimensions, (A1)–(A2) hold, and \( l(v) = l_h(v) = 0 \) for all \( v \in \mathbb{R}M_D^d \).

Moreover, we assume there exists a (local or global) minimizer of (SIMP) and fix any local or global isolated minimizer \((u, \rho) \in V_D \times C_\gamma\) of (SIMP) in all three theorems.

In both Theorems 3.1 and 3.2 we tackle the open issue of multiple minimizers (P1).

We prove that every isolated minimizer is arbitrarily closely approximated by the finite element method. If a \( W^{1,p} \)-type regularization is used, in Theorem 3.1, we show that there exists a sequence of FEM minimizers satisfying \( u_h \rightarrow u \) strongly in \( H^1(\Omega)^d \) and \( \rho_h \rightarrow \rho \) strongly in \( W^{1,p}(\Omega) \). This resolves the open problem (P2). If one chooses a linear density filter as a restriction method, in the sense of Bourdin [11], then, in Theorem 3.2, we show that \( \rho_h \rightarrow \rho \) strongly in \( L^p(\Omega) \) for any \( p \in [1, \infty) \). Hence, treating the open problem (P3). Finally, suppose that there exists a sequence of FEM minimizers such that \( u_h \rightarrow u \) weakly in \( H^1(\Omega)^d \) and \( \rho_h \rightharpoonup^* \rho \) weakly-* in \( L^\infty(\Omega) \) and \( C_{\gamma,h} \subset W^{1,p}(\Omega) \), for some \( p \in [1, \infty) \). Then, for any density filter satisfying (F1)–(F8), we have that \( u_h \rightarrow u \) strongly in \( H^1(\Omega)^d \) and \( F_h(\rho_h) \rightarrow F(\rho) \) strongly in \( W^{1,p}(\Omega) \) as a result of Theorem 3.3. This observation resolves (P4).
**Theorem 3.1** (Main theorem for $W^{1,p}$-type regularization). Suppose that a $W^{1,p}$-type regularization is used for some $p \in (1, \infty)$. Consider the finite element spaces $V_h \subset H^1(\Omega)^d$ and $C_{\gamma,h} \subset C_{\gamma} = \mathcal{H}_{\gamma} \cap W^{1,p}(\Omega)$. Suppose that the regularization satisfies (R1) and (R2).

Then, there exists an $\bar{h} > 0$ such that, for $h \leq \bar{h}$, $h \to 0$, there exists a sequence of solutions $(u_h, \rho_h) \in V_{D,h} \times C_{\gamma,h}$ that satisfies, for any $v_h \in V_{D,h}$,

$$\min_{(u_h, \rho_h) \in V_{D,h} \times C_{\gamma,h}} J_h(u_h, \rho_h) \quad \text{subject to} \quad a_h(u_h, v_h; \rho_h) = l_h(v_h),$$

(SIMP$_h$)

where $J_h$, $l_h$, and $a_h(\cdot, \cdot, \cdot)$ are defined in (3.3), (3.4), and (3.5), respectively. Moreover, $u_h \to u$ strongly in $H^1(\Omega)^d$ and $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$.

**Theorem 3.2** (First main theorem for density filtering). Suppose that a linear density filter is used, in the sense of Bourdin [11], that satisfies (F1)–(F8). Consider the finite element spaces $V_h \subset H^1(\Omega)^d$ and $C_{\gamma,h} \subset C_{\gamma} = \mathcal{H}_{\gamma}$.

Then, there exists an $\bar{h} > 0$ such that, for $h \leq \bar{h}$, $h \to 0$, there exists a sequence of solutions $(u_h, \rho_h) \in V_{D,h} \times C_{\gamma,h}$ that satisfies (SIMPh). Moreover, $u_h \to u$ strongly in $H^1(\Omega)^d$ and $\rho_h \to \rho$ strongly in $L^p(\Omega)$ for any $p \in [1, \infty)$.

Recall that in the case of pure filtering (SIMP) is equivalent to (F-SIMP). Hence, if $(u, \tilde{\rho})$ is the minimizer of (F-SIMP), then $(u, \tilde{\rho}) = (u, F(\rho))$ where $(u, \rho)$ is the equivalent minimizer of (SIMP). We assume that the minimizer $(u, \tilde{\rho})$ defined via $(u, \rho)$ is isolated.

**Theorem 3.3** (Second main theorem for density filtering). Suppose that a density filter is used that satisfies (F1)–(F8). Consider the finite element spaces $V_h \subset H^1(\Omega)^d$ and $C_{\gamma,h} \subset \mathcal{H}_{\gamma} \cap W^{1,p}(\Omega)$ for some $p \in (1, \infty)$.

Moreover, assume that the minimizer $(u, \tilde{\rho}) = (u, F(\rho))$ is an isolated minimizer of (F-SIMP). Let $(u_h, \rho_h)$ be a sequence of minimizers of (SIMPh) such that $u_h \to u$ weakly in $H^1(\Omega)^d$ and $\rho_h \rightharpoonup^{\ast} \rho$ weakly-* in $L^\infty(\Omega)$.

Then, there exists a subsequence (not indicated) such that

$$u_h \to u \quad \text{strongly in} \quad H^1(\Omega)^d, \quad (3.6)$$

$$F_h(\rho_h) \to F(\rho) \quad \text{strongly in} \quad W^{1,p}(\Omega). \quad (3.7)$$

**Remark 3.1.** A common criticism of the primal formulation of classical linear elasticity (without topology optimization) is that as the material becomes incompressible ($\lambda \to \infty$), the operator norm becomes unbounded. This can result in locking, the phenomenon where the approximation has sub-optimal convergence, or even appear to diverge, for large ranges of the mesh size $h$. In classical linear elasticity, locking can be alleviated by using a high-order element for $u$, e.g. quartics or higher [3] or alternative formulations such as the pressure and symmetric stress formulations [12, Ch. VI.3]. However, these results do not (immediately) translate to the topology optimization case. Hence, any proof of non-locking formulations is beyond the scope of this work.
3.1 Auxiliary results

The following lemmas are useful results and are used later in the convergence analysis. Their proofs are provided in Appendix A for completeness.

Lemma 3.1. Consider a sequence $(\eta_h) \subset L^\infty(\Omega)$, $0 \leq \eta_h \leq 1$ a.e. such that $\eta_h \rightharpoonup \eta$ strongly in $L^p(\Omega)$ for some $p \in [1, \infty)$. Then $\eta_h \rightharpoonup \eta$ strongly in $L^s(\Omega)$ for any $s \in [1, \infty)$.

Lemma 3.2 (Bourdin [11]). Suppose that $(F1)$–$(F8)$ hold. Consider any sequence $(\eta_h) \subset C_{\gamma,h}$ such that $\eta_h \rightharpoonup \eta \in C_\gamma$ weakly in $L^p(\Omega)$ $(\eta_h \rightharpoonup^* \eta$ weakly* in $L^\infty(\Omega)$ if $p = \infty)$ for $p \in [1, \infty]$. Then $F_h(\eta_h) \rightharpoonup F(\eta)$ strongly in $L^p(\Omega)$.

Lemma 3.3. Suppose that $(A1)$–$(A2)$ hold and, if filtering is used, then $(F1)$–$(F8)$ are also satisfied. Consider any sequence $(\hat{u}_h, \hat{\rho}_h) \in V_{D,h} \times C_{\gamma,h}$ satisfying the PDE constraint in (SIMP) such that $\hat{u}_h \rightharpoonup \hat{u} \in V_D$ weakly in $H^1(\Omega)^d$. Moreover, suppose that either $\hat{\rho}_h \rightharpoonup \hat{\rho} \in C_\gamma$ strongly in $L^p(\Omega)$ for some $p \in [2, \infty)$ or filtering is used and $\hat{\rho}_h \rightharpoonup \hat{\rho}$ weakly in $L^p(\Omega)$ for some $p \in [2, \infty)$ $(\hat{\rho}_h \rightharpoonup^* \hat{\rho}$ weakly* in $L^\infty(\Omega)$ if $p = \infty)$. Then the limit $(\hat{u}, \hat{\rho})$ satisfies the PDE constraint in (SIMP).

Results similar to Lemma 3.3 are found in the literature e.g. [31, Lem. 2.1].

Lemma 3.4 (Existence of a strongly converging sequence $\hat{u}_h$ in $H^1(\Omega)^d$). Consider any sequence such that $F_h(\hat{\rho}_h) \rightharpoonup F(\rho)$ or $\hat{\rho}_h \rightharpoonup \rho$ strongly in $L^p(\Omega)$, for some $p \in [2, \infty]$, and the corresponding (unique) sequence of displacements such that $(\hat{u}_h, \hat{\rho}_h)$ satisfies the PDE constraint in (SIMP). Then, $\hat{u}_h \rightharpoonup u$ strongly in $H^1(\Omega)^d$.

3.2 FEM convergence results

In this subsection, we assume that $\Omega \subset \mathbb{R}^d$ is a polygonal domain in two dimensions or a polyhedral Lipschitz domain in three dimensions and (R1)–(R2), (F1)–(F8), and (A1)–(A2) hold and $l(v) = l_h(v) = 0$ for all $v \in \mathbb{R}^d_M$. Moreover, we fix a local or global isolated minimizer $(u, \rho) \in V_D \times C_\gamma$ of (SIMP).

As we make an isolation assumption in all the following results, we define closed balls around functions and tuples of functions. Consider a function $u \in U$ where $U$ is a Banach space. We denote the closed ball of radius $r$ by $B_{r,U}(u)$ as defined by

$$B_{r,U}(u) := \{v \in U : \|u - v\|_U \leq r\}. \tag{3.8}$$

This is extended to tuples of functions as follows: consider $u_i \in U_i$, for $i = 1, \ldots, n$, where $U_i$ are Banach spaces. We define $B_{r,U_1 \times \cdots \times U_n}(u_1, \ldots, u_n)$ as

$$B_{r,U_1 \times \cdots \times U_n}(u_1, \ldots, u_n) := \{(v_1, \ldots, v_n) \in U_1 \times \cdots \times U_n : \sum_{i=1}^n \|v_i - u_i\|_{U_i} \leq r\}. \tag{3.9}$$
We define the radius of the basin of attraction as the largest value \( r \) such that \((u, \rho)\) is the unique local minimizer in \( B_{r_1,2,2}V(Y) × Y(u, \rho) \cap (V_D × C_1)\). Recall that \( Y = W^{1,p}(Ω) \) if a \( W^{1,p}\)-type regularization is used and \( Y = L^2(Ω) \) otherwise. We note that
\[
(V_D × B_{r_1,2,2}V(Y)(u)) × (C_2 × B_{r_1,2,2}V(Y)(\rho))
\subset B_{r_1,2,2}V(Y)(u) × (V_D × C_2)
\]
and hence \((u, \rho)\) is also the unique minimizer in \((V_D × B_{r_1,2,2}V(Y)(u)) × (C_2 × B_{r_1,2,2}V(Y)(\rho))\).

**Remark 3.2.** We make the assumption that \( \rho \in C_2 \) is isolated with respect to \( V(Y) \)-norm (as opposed to the \( L^\infty \)-norm). This is a stronger isolation assumption as discussed in [28, Rem. 7]. As far as we are aware, the \( V(Y) \)-isolation assumption is valid for all practical problems found in the literature, e.g. in Example 2.1. We make this stronger isolation assumption as simple and continuous functions are not dense in \( L^\infty(Ω) \), but are dense in \( Y \). This has implications in the assumption \( A_2 \).

Consider the finite-dimensional optimization problem: find \((u_h, \rho_h)\) that minimizes
\[
\min_{(v_h, \eta_h)∈(V_D,h) × (C_2,ρ_h)} J_h(v_h, \eta_h),
\]
subject to
\[
a_h(u_h, v_h; \rho_h) = l_h(v_h) \text{ for any } v_h ∈ V_D,h.
\]

In Proposition 3.1, we prove the existence of finite-dimensional minimizers to the discretized problem \((I-SIMP_h)\). Then, in Proposition 3.2, we prove weak(-*) convergence of the discretized minimizers to the isolated infinite-dimensional minimizer as \( h → 0 \). We conclude this subsection with direct consequences of the weak(-*) convergence in Corollaries 3.1, 3.2, and 3.3.

**Proposition 3.1.** A minimizer \((u_h, \rho_h)\) of \((I-SIMP_h)\) exists.

**Proof.** The set \((V_D,h) × (C_2,ρ_h)\) is a finite-dimensional, closed and bounded set. Moreover, for sufficiently small \( h \) it is non-empty. Therefore, it is sequentially compact by the Heine–Borel theorem [18, Th. 11.18]. We say that \((v, \eta)\) satisfies the PDE constraint in \((I-SIMP_h)\). By Korn’s inequality [4, Th. 11], and the assumption that \( \epsilon_{\text{min}} > 0 \), then the bilinear form \( a_h(\cdot, \cdot; \eta_h) \) is coercive for any \( \eta_h ∈ C_2 \). Moreover, the bilinear form is bounded, \( l_h(v) = 0 \) for all \( v ∈ RM_D \), and therefore, by the Lax–Milgram theorem [16, Ch. 6.2, Th. 1], for every \( \eta_h ∈ C_2 \), there exists a unique \( v_h ∈ V_D,h \) such that \((v_h, \eta_h) ∈ S\). Moreover, \( v_h \) is bounded as \( f_h, g_h^i \) and \( g_h^j \) are bounded. Hence, \( S × (V_D,h × C_2) \) is finite-dimensional, closed, and bounded and thus sequentially compact by the Heine–Borel theorem. As the spaces are Hausdorff, the intersection of two compact sets is compact and, hence, \( S × ((V_D,h) × (C_2,h) × B_{r_1,2,2}V(Y)(ρ))\) is sequentially compact.
By assumption the functional $J_h$ is continuous. Hence $J_h$ obtains its infimum in $S \times ((V_{D,h} \cap B_{r/2,H^1(\Omega)}(u)) \times (C_{\gamma,h} \cap B_{r/2,Y}(\rho)))$ and therefore, a minimizer $(u_h, \rho_h)$ of $(I-\text{SIMP}_h)$ exists.

The following proposition is the first step in tackling the open problem (P1). The result does not rely on a particular restriction method for $(\text{SIMP})$. The only key assumption is that a minimizer exists and the (local or global) minimizer is isolated.

**Proposition 3.2** (Weak×weak-* convergence of $(u_h, \rho_h)$ in $H^1(\Omega)^d \times L^\infty(\Omega)$). There exist subsequences (up to relabeling) of finite element minimizers of $(I-\text{SIMP}_h)$, that satisfy

\begin{align}
  u_h &\rightharpoonup u \text{ weakly in } H^1(\Omega)^d, \\
  \rho_h &\rightharpoonup \rho \text{ weakly-* in } L^\infty(\Omega).
\end{align}

**Proof.** By a corollary of Kakutani’s Theorem [19, Th. A.65], if a Banach space is reflexive then every norm-closed, bounded and convex subset of the Banach space is weakly compact and thus, by the Eberlein–Šmulian theorem [19, Th. A.62], sequentially weakly compact. It can be checked that $V_{D,h} \cap B_{r/2,H^1(\Omega)}(u)$ and $C_{\gamma,h} \cap B_{r/2,Y}(\rho)$ are norm-closed, bounded and convex subsets of the reflexive Banach spaces $H^1(\Omega)^d$ and $Y$, respectively. Therefore, $V_{D,h} \cap B_{r/2,H^1(\Omega)}(u)$ is weakly sequentially compact in $H^1(\Omega)^d$ and $C_{\gamma,h} \cap B_{r/2,Y}(\rho)$ is weakly sequentially compact in $Y$.

Hence we extract subsequences (up to relabeling), $(u_h)$ and $(\rho_h)$ of the sequence generated by the global minimizers of $(I-\text{SIMP}_h)$ such that

\begin{align}
  u_h &\to u_0 \in V_{D,h} \cap B_{r/2,H^1(\Omega)}(u) \text{ weakly in } H^1(\Omega)^d, \\
  \rho_h &\to \rho_0 \in C_{\gamma,h} \cap B_{r/2,Y}(\rho) \text{ weakly in } Y.
\end{align}

In the remainder of the proof, we show that the weak limit $(u_0, \rho_0)$ is in fact the isolated minimizer $(u, \rho)$. If a regularization is used then by the Rellich–Kondrachov theorem we have that $\rho_h \to \rho_0$ strongly in $L^1(\Omega)$ and thus $\rho_h \to \rho_0$ strongly in $L^p(\Omega)$ for any $p \in [1, \infty)$ by Lemma 3.1. Hence, for either a filter or a regularization, we satisfy the requirement of Lemma 3.3 and deduce that $(u_0, \rho_0)$ satisfies the PDE constraint in $(\text{SIMP})$.

The next step is to prove that $(u_0, \rho_0)$ is a minimizing pair. By assumption (A2), there exists a sequence of finite element functions $\hat{\rho}_h \in C_{\gamma,h}$ that strongly converges to $\rho$ in $Y$. Moreover, by Lemma 3.4, there exists a sequence $(\hat{u}_h)$ such that $(\hat{u}_h, \hat{\rho}_h)$ satisfies the PDE constraint in $(I-\text{SIMP}_h)$ and $\hat{u}_h \to u$ strongly in $H^1(\Omega)^d$. Thus

$$|J_h(u_h, \rho_h) - J(u, \rho)| \leq |l_h(u_h) - l(u)| + |R(\rho) - R(\hat{\rho}_h)|.$$  

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By (A1), we have strong convergence of \( f_h \) in \( L^2(\Omega) \), \( g_{h_n}^0 \) in \( L^2(\Gamma_n^0) \), and \( g_n^h \) in \( L^2(\Gamma_n^1) \). Thus together with the strong convergence of \( \hat{\rho}_h \) and \( \hat{u}_h \) we see that
\[
J_h(\hat{u}_h, \hat{\rho}_h) \to J(u, \rho) \quad \text{as} \quad h \to 0.
\]

Furthermore, for sufficiently small \( h > 0 \),
\[
(\hat{u}_h, \hat{\rho}_h) \in (\mathcal{V}_{D,h} \cap B_{r/2, H^1(\Omega)}(u)) \times (C_{\gamma,h} \cap B_{r/2,Y}(\rho)).
\]

Therefore,
\[
J_h(u_h, \rho_h) \leq J_h(\hat{u}_h, \hat{\rho}_h).
\] (3.15)

By taking the limit as \( h \to 0 \) and utilizing the strong convergence of \( \hat{u}_h \) and \( \hat{\rho}_h \) to \( u \) and \( \rho \), respectively, we see that
\[
\lim_{h \to 0} J_h(u_h, \rho_h) \leq J(u, \rho).
\] (3.16)

By linearity of the first term and (R1), \( J \) is weakly lower semicontinuous on \( H^1(\Omega)^d \times Y \). Therefore,
\[
J(u_0, \rho_0) \leq \liminf_{h \to 0} J_h(u_h, \rho_h).
\] (3.17)

Since \( (u, \rho) \) is the unique minimizer of \( (\mathcal{V}_{D} \cap B_{r/2, H^1(\Omega)}(u)) \times (C_{\gamma} \cap B_{r/2,Y}(\rho)) \). Thus \( J(u, \rho) \leq J(u_0, \rho_0) \). Hence, from (3.16) and (3.17), it follows that
\[
J(u_0, \rho_0) = J(u, \rho).
\] (3.18)

Corollary 3.1. If a filter is used, then \( F_h(\rho_h) \to F(\rho) \) strongly in \( L^\infty(\Omega) \).

Proof. In Proposition 3.2 we showed that \( \rho_h \rightharpoonup^* \rho \) weakly-* in \( L^\infty(\Omega) \). The result is a direct consequence of Lemma 3.2.
Corollary 3.2. If a $W^{1,p}$-type regularization is used, then there exists a subsequence of FEM minimizers $(u_h, \rho_h)$ of (I-SIMP) not relabeled such that $\rho_h \to \rho$ strongly in $L^s(\Omega)$ for any $s \in [1, \infty)$.

Proof. By Proposition 3.2, $\rho_h \rightharpoonup \rho$ weakly in $W^{1,p}(\Omega)$. Thus, for $p \geq 1$, the Rellich–Kondrachov theorem implies that there exists a subsequence such that $\rho_h \to \rho$ strongly in $L^1(\Omega)$. The result follows from Lemma 3.1. \hfill \Box

Corollary 3.3 (Strong convergence of $u_h$ in $H^1(\Omega)^d$). There exists a subsequence of the FEM minimizers $(u_h, \rho_h)$ of (I-SIMP) such that $u_h \to u$ strongly in $H^1(\Omega)^d$.

Proof. By Corollaries 3.1 and 3.2, we have that $F_h(\rho_h) \to F(\rho)$ strongly in $L^\infty(\Omega)$ or $\rho_h \to \rho$ strongly in $L^2(\Omega)$ for a filtering or regularization, respectively. Hence, the result follows as a direct consequence of Lemma 3.4. \hfill \Box

3.3 FEM analysis of $W^{1,p}$-type regularization

In the following subsection we focus on proving the stronger results of Theorem 3.1 and assume that no filtering is used, i.e. $F(\rho) = \rho$, $F_h(\rho_h) = \rho_h$. We show that if one uses a $W^{1,p}$-type regularization of (SIMP), then $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$.

Proposition 3.3 (Strong convergence of $\rho_h$ in $W^{1,p}(\Omega)$, $p \in (1, \infty)$). Suppose that a $W^{1,p}$-type regularization is used, satisfying the requirements of Theorem 3.1, with no filtering, i.e. $F(\rho) = \rho$. Then, there exists a subsequence (up to relabeling) of finite element minimizers of (I-SIMP), that satisfy

$$\rho_h \to \rho \text{ strongly in } W^{1,p}(\Omega).$$

To simplify the proof, we choose $R(\rho) = \frac{\epsilon}{p} \|\nabla \rho\|_{L^p(\Omega)}^p$. We omit a proof for regularization with lower-order terms (e.g. in the Ginzburg–Landau regularization). By assumption, these lower-order terms are continuously Fréchet differentiable and the result still holds with small changes to the proof.

Proof. $C_{\gamma,h} \cap B_{r/2,W^{1,p}(\Omega)}(\rho)$ is a convex set, and hence for any $\eta_h \in C_{\gamma,h} \cap B_{r/2,W^{1,p}(\Omega)}(\rho)$, $t \in [0,1]$, we have that $\rho_h + t(\eta_h - \rho_h) \in C_{\gamma,h} \cap B_{r/2,W^{1,p}(\Omega)}(\rho)$. Since $(u_h, \rho_h)$ is a minimizer of (I-SIMP), by the arguments used in Propositions 2.3 and 2.4, we deduce that, for all $\eta_h \in C_{\gamma,h} \cap B_{r/2,W^{1,p}(\Omega)}(\rho)$,

$$-\int_{\Omega} k'(\rho_h)|\nabla u_h|^2(\eta_h - \rho_h)dx + \epsilon \int_{\Omega} |\nabla \rho_h|^{p-2} \nabla \rho_h \cdot \nabla (\eta_h - \rho_h) \, dx \geq 0.$$

(3.20)
Thus (3.24) and (3.25) imply that, if
\[ p \in C_\gamma \text{ and } \eta_h \in C_{\gamma,h} \cap B_{r/2,W^{1,p}(\Omega)}(\rho), \]
\[
- \int_{\Omega} k'(\rho)|\mathbf{E}u|^2\rho \, dx + \epsilon \int_{\Omega} |\nabla \rho|^{p-2}\nabla \rho \cdot \nabla d x \\
\leq - \int_{\Omega} k'(\rho)|\mathbf{E}u|^2 \eta \, dx + \epsilon \int_{\Omega} |\nabla \rho|^{p-2}\nabla \rho \cdot \nabla \eta \, dx,
\]
\[
- \int_{\Omega} k'(\rho_h)|\mathbf{E}u_h|^2\rho_h \, dx + \epsilon \int_{\Omega} |\nabla \rho_h|^{p-2}\nabla \rho_h \cdot \nabla \rho_h \, dx \\
\leq - \int_{\Omega} k'(\rho_h)|\mathbf{E}u_h|^2 \eta_h \, dx + \epsilon \int_{\Omega} |\nabla \rho_h|^{p-2}\nabla \rho_h \cdot \nabla \eta_h \, dx.
\tag{3.22}
\]
By subtracting \(- \int_{\Omega} k'(\rho)|\mathbf{E}u|^2\rho_h \, dx + \epsilon \int_{\Omega} |\nabla \rho|^{p-2}\nabla \rho \cdot \nabla \rho_h \, dx\) from (3.21),
\[- \int_{\Omega} k'(\rho_h)|\mathbf{E}u_h|^2\rho_h \, dx + \epsilon \int_{\Omega} |\nabla \rho_h|^{p-2}\nabla \rho_h \cdot \nabla \rho_h \, dx\) from (3.22), summing the result, and rearranging the left-hand side, we see that
\[
- \int_{\Omega} (k'(\rho) - k'(\rho_h))|\mathbf{E}u|^2(\rho - \rho_h) \, dx - \int_{\Omega} k'(\rho_h)(|\mathbf{E}u|^2 - |\mathbf{E}u_h|^2)(\rho - \rho_h) \, dx \\
+ \epsilon \int_{\Omega} (|\nabla \rho|^{p-2}\nabla \rho - |\nabla \rho_h|^{p-2}\nabla \rho_h) \cdot \nabla (\rho - \rho_h) \, dx \\
\leq - \int_{\Omega} k'(\rho)|\mathbf{E}u|^2(\eta - \rho_h) \, dx - \int_{\Omega} k'(\rho_h)|\mathbf{E}u_h|^2(\eta_h - \rho) \, dx \\
+ \epsilon \int_{\Omega} |\nabla \rho|^{p-2}\nabla \rho \cdot \nabla (\eta - \rho_h) \, dx + \epsilon \int_{\Omega} |\nabla \rho_h|^{p-2}\nabla \rho_h \cdot \nabla (\eta_h - \rho) \, dx.
\tag{3.23}
\]
We note that \((u_h, \rho_h)\) is a bounded sequence, and as, by construction, \(k(\cdot)\) is continuously Fréchet differentiable, we see that \(k(\rho)\) is bounded. By assumption (A2), there exists a sequence \(\eta_h \in C_{\gamma,h}\) such that \(\eta_h \rightarrow \rho\) strongly in \(W^{1,p}(\Omega)\). Moreover, for sufficiently small \(h\), we have that \(\eta_h \in B_{r/2,W^{1,p}(\Omega)}(\rho)\) due to the strong convergence. Set \(\eta = \rho_h\). Corollary 3.3 implies that there is a subsequence such that \(u_h \rightarrow u\) strongly in \(H^1(\Omega)^d\). Moreover, by Corollary 3.1, \(\rho_h \rightarrow \rho\) weakly in \(W^{1,p}(\Omega)\), and therefore, by the Rellich–Kondrachov compactness theorem and Lemma 3.1, there exists a subsequence (not relabeled) such that \(\rho_h \rightarrow \rho\) strongly in \(L^2(\Omega)\). Now by moving the first two terms of (3.23) to the right-hand side, we find that there exists a subsequence of FEM minimizers such that
\[
\epsilon \int_{\Omega} (|\nabla \rho|^{p-2}\nabla \rho - |\nabla \rho_h|^{p-2}\nabla \rho_h) \cdot \nabla (\rho - \rho_h) \, dx \leq C(h),
\tag{3.24}
\]
where \(C(h) \rightarrow 0\) as \(h \rightarrow 0\).

(Case 1: \(p \in [2, \infty)\)). We note that [15, Th. 2.2], for any vectors \(a, b \in \mathbb{R}^n, p \in [2, \infty)\),
\[
|b|^{p-2} - |a|^{p-2}a \cdot (b - a) \geq 2^{2-p}|b - a|^p.
\tag{3.25}
\]
Thus (3.24) and (3.25) imply that, if \(p \in [2, \infty)\),
\[
2^{2-p} \epsilon \|\nabla (\rho - \rho_h)\|_{L^p(\Omega)}^p \leq C(h).
\tag{3.26}
\]
Therefore, $\nabla \rho_h \to \nabla \rho$ strongly in $L^p(\Omega)$ if $p \in [2, \infty)$.

(Case 2: $p \in (1, 2)$). We note that [25, Ch. 12, (VII)], for any vectors $a, b \in \mathbb{R}^n$, $p \in (1, 2)$,

$$
(|b|^{p-2}b - |a|^{p-2}a) \cdot (b - a) \geq (p - 1)(1 + |a|^2 + |b|^2)^{\frac{p-2}{2}}|b - a|^2.
$$

(3.27)

Thus (3.24) and (3.27) imply that, if $p \in (1, 2)$,

$$
(p - 1)c \int_\Omega \frac{(|\nabla (\rho - \rho_h)|^2)}{(1 + |\nabla \rho|^2 + |\nabla \rho_h|^2)^{\frac{2-p}{2}}} \, dx \leq C(h).
$$

(3.28)

An application of Hölder’s inequality reveals that

$$
\|\nabla (\rho - \rho_h)\|_{L^p(\Omega)} \leq \left\| \frac{\nabla (\rho - \rho_h)}{(1 + |\nabla \rho|^2 + |\nabla \rho_h|^2)^{\frac{2-p}{2}}} \right\|_{L^2(\Omega)} \left\| (1 + |\nabla \rho|^2 + |\nabla \rho_h|^2)^{\frac{2-p}{2}} \right\|_{L^{\frac{2-p}{2}}(\Omega)}.
$$

(3.29)

The second term on the right-hand side of (3.29) is bounded since $\nabla \rho, \nabla \rho_h \in L^p(\Omega)$.

Moreover, the first term on the right-hand side of (3.29) tends to zero as $h \to 0$ thanks to the inequality in (3.28). Hence, $\nabla \rho_h \to \nabla \rho$ strongly in $L^p(\Omega)$.

Since $\rho_h \to \rho$ strongly in $L^2(\Omega)$ then Lemma 3.1 implies that $\rho_h \to \rho$ strongly in $L^p(\Omega)$, $p \in [1, \infty)$. Hence $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$ if $p \in (1, \infty)$. □

**Corollary 3.4.** If a $W^{1,p}$-type regularization is used, then for sufficiently small $h$, there exists a subsequence of minimizers $(u_h, \rho_h)$ of (I-SIMP$_h$) that are also minimizers of (SIMP$_h$).

**Proof.** Corollary 3.3 and Proposition 3.3 imply that there exists a subsequence $(u_h, \rho_h)$ such that $u_h \to u$ strongly in $H^1(\Omega)^d$ and $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$. By the definition of strong convergence, there exists an $\tilde{h}$ such that for all $h \leq \tilde{h}$, $\|u - u_h\|_{H^1(\Omega)} + \|\rho - \rho_h\|_{W^{1,p}(\Omega)} \leq \eta / 4$. Thus the basin of attraction constraint is not active and the subsequence of minimizers of (I-SIMP$_h$) are also minimizers of (SIMP$_h$). □

We now have the sufficient ingredients to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 3.1, there exists a sequence $(u_h, \rho_h)$ minimizing (I-SIMP$_h$) that satisfies $u_h \rightharpoonup u$ weakly in $H^1(\Omega)^d$ and $\rho_h \rightharpoonup^* \rho$ weakly-* in $L^\infty(\Omega)$.

In Corollary 3.3, we show that there exists a subsequence such that $u_h \to u$ strongly in $H^1(\Omega)^d$. Then, in Proposition 3.3, we demonstrate that there exists a subsequence such that $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$. Finally, in Corollary 3.4, we conclude that a subsequence of the minimizers are also minimizers of (SIMP$_h$). □
3.4 FEM analysis of density filtering

In this subsection we will focus on proving the stronger results of Theorem 3.2 and Theorem 3.3 and assume that no regularization is used, i.e. $R(\rho) = 0$. In particular, we tackle the open problems (P3) and (P4).

We first prove the following two lemmas.

**Lemma 3.5.** A minimizer exists to the following $\epsilon$-perturbed optimization problem:

$$
\min_{(v,\eta) \in (V_D \cap B_{r/2,H^1(\Omega)}(u)) \times (C_\gamma \cap B_{r/2,L^2(\Omega)}(\rho))} J_\epsilon(v, \eta) := J(v, \eta) + \frac{\epsilon}{2} \|\eta\|_{L^2(\Omega)}^2,
$$

subject to

$$
a(u_\epsilon, v; \rho_\epsilon) = l(v) \text{ for all } v \in V_D.
$$

**Proof.** Consider a minimizing sequence $(u_n, \rho_n) \in (V_D \cap B_{r/2,H^1(\Omega)}(u)) \times (C_\gamma \cap B_{r/2,L^2(\Omega)}(\rho))$. Since $V_D \cap B_{r/2,H^1(\Omega)}(u)$ and $C_\gamma \cap B_{r/2,L^2(\Omega)}(\rho)$ are norm-closed, bounded, and convex subsets of $H^1(\Omega)^d$ and $L^2(\Omega)$, respectively, then there exists a limit $(u_\epsilon, \rho_\epsilon) \in (V_D \cap B_{r/2,H^1(\Omega)}(u)) \times (C_\gamma \cap B_{r/2,L^2(\Omega)}(\rho))$ such that

$$
u_n \to u_\epsilon \text{ weakly in } H^1(\Omega)^d, \quad (3.30)
$$

$$\rho_n \to \rho_\epsilon \text{ weakly in } L^2(\Omega). \quad (3.31)
$$

By Lemma 3.3, $(u_\epsilon, \rho_\epsilon)$ satisfies the PDE constraint in (I-SIMP$\epsilon$). Moreover, $J(u, \rho)$ is bounded and weak-to-weak lower semicontinuous on $H^1(\Omega)^d \times L^2(\Omega)$. Thus

$$
J_\epsilon(u_\epsilon, \rho_\epsilon) \leq \liminf_{n \to \infty} J_\epsilon(u_n, \rho_n).
$$

Hence, a minimizer exists. $\Box$

**Lemma 3.6.** Consider a sequence of minimizers $(u_\epsilon, \rho_\epsilon)$ of (I-SIMP$\epsilon$). Then, there exists a subsequence (up to relabelling) such that $u_\epsilon \to u$ weakly in $H^1(\Omega)^d$ and $\rho_\epsilon \to \rho$ weakly in $L^2(\Omega)$ as $\epsilon \to 0$.

**Proof.** Since $V_D \cap B_{r/2,H^1(\Omega)}(u)$ and $C_\gamma \cap B_{r/2,L^2(\Omega)}(\rho)$ are norm-closed, bounded, and convex subsets of $H^1(\Omega)^d$ and $L^2(\Omega)$, respectively, then there exists a limit $(u_0, \rho_0) \in (V_D \cap B_{r/2,H^1(\Omega)}(u)) \times (C_\gamma \cap B_{r/2,L^2(\Omega)}(\rho))$ such that

$$
u_\epsilon \to u_0 \text{ weakly in } H^1(\Omega)^d, \quad (3.32)
$$

$$\rho_\epsilon \to \rho_0 \text{ weakly in } L^2(\Omega). \quad (3.33)
$$

By Lemma 3.3, $(u_0, \rho_0)$ satisfies the PDE constraint in (SIMP). Consider any sequence of functions such that $(u_n, \rho_n) \in V_D \times C_\gamma$ such that $u_n \to u$ strongly in $H^1(\Omega)^d$ and $\rho_n \to \rho$ strongly in $L^2(\Omega)$. Then, for any $\epsilon > 0$ and sufficiently large $n \in \mathbb{N}$,

$$
J_\epsilon(u_\epsilon, \rho_\epsilon) \leq J_\epsilon(u_n, \rho_n). \quad (3.34)
$$
$J_\epsilon$ is weak $\times$ weak lower semicontinuous on $H^1(\Omega)^d \times L^2(\Omega)$. Thus by taking the limits $\epsilon \to 0$ and $n \to \infty$ and using the weak convergence of $(u_\epsilon, \rho_\epsilon)$ and the strong convergence of $(u_n, \rho_n)$ we see that

$$J_\epsilon(u_0, \rho_0) \leq J_\epsilon(u, \rho).$$

(3.35)

Since $(u, \rho)$ is the unique minimizer of (SIMP) in $(V_D \cap B_{r/2, H^1(\Omega)}(u)) \times (C_\gamma \cap B_{r/2, L^2(\Omega)}(\rho))$, we identify $u_0 = u$ and $\rho_0 = \rho$ a.e. in $\Omega$.

The following proposition is concerned with the open problem (P3). We show that if a linear filter is used in the sense of Bourdin [11], then there exists a sequence of the unfiltered discretized material distributions that converges strongly in $L^p(\Omega)$ for any $p \in [1, \infty)$.

**Proposition 3.4** (Strong convergence of $\rho_h$ in $L^p(\Omega)$, $p \in [1, \infty)$). Suppose that a linear filter is used, with no regularization, i.e. $R(\rho) = 0$ and for $f \in W^{1,\infty}(\mathbb{R}^d)$, $f \geq 0$ a.e., $\|f\|_{L^1(\mathbb{R}^d)} = 1$,

$$F(\rho)(x) = \int_{\Omega} f(x-y)\rho(y)\,dy.$$  

(3.36)

Then, there exists a subsequence (up to relabeling) of finite element minimizers of (I-SIMP$_h$), that satisfy, for any $p \in [1, \infty)$,

$$\rho_h \to \rho \text{ strongly in } L^p(\Omega).$$

(3.37)

In Fig. 3, we provide the structure of the proof. The first step is to prove the two downward limits, as $\epsilon \to 0$, of minimizers of the $\epsilon$-perturbed problem (I-SIMP$_\epsilon$) to minimizers of the original problem (SIMP). This is achieved via weak convergence, the convergence of the supremum value of the norms, and the Radon–Riesz property [19, Th. A.70]. Next, the top arrow, as $h \to 0$, in the $\epsilon$-perturbed problem is proven using the first-order optimality conditions. Then, since $\rho_{\epsilon,h}$, $\rho_\epsilon$, $\rho_h$, and $\rho$ are all bounded in $L^p(\Omega)$ uniformly in $h$ and $\epsilon$, we invoke the Osgood theorem [39, Ch. 4, Sec. 11, Th. 2'] for exchanging limits together with the Radon–Riesz property to conclude the result.

**Proof of Proposition 3.4.** Consider the global minimizer $(u_\epsilon, \rho_\epsilon)$ of (I-SIMP$_\epsilon$). If there is more than one global minimizer, then fix one. After discretizig (I-SIMP$_\epsilon$), then using the same arguments as in the proof of Proposition 3.2, there exists a sequence as $h \to 0$ such that $u_{\epsilon,h} \to u_\epsilon$ strongly in $H^1(\Omega)^d$ and $\rho_{\epsilon,h} \rightharpoonup \rho_\epsilon$ weakly-* in $L^\infty(\Omega)$ as $h \to 0$.

By Lemma 3.6, there exists a sequence (up to relabeling) $(u_\epsilon, \rho_\epsilon)$ such that $\rho_\epsilon \to \rho$ weakly in $L^2(\Omega)$ and $u_\epsilon \rightharpoonup u$ weakly in $H^1(\Omega)^d$ as $\epsilon \to 0$. We now strengthen the convergence of material distribution to strong convergence in $L^2(\Omega)$ as $\epsilon \to 0$. Suppose that, for a contradiction, for any $\epsilon > 0$,

$$\|\rho\|_{L^2(\Omega)} < \|\rho_\epsilon\|_{L^2(\Omega)}.$$  

(3.38)
This would imply that
\[ J(u, \rho) + \frac{\epsilon}{2} \|\rho\|_{L^2(\Omega)}^2 < J(u_\epsilon, \rho_\epsilon) + \frac{\epsilon}{2} \|\rho_\epsilon\|_{L^2(\Omega)}^2. \] (3.39)

However, this would be a contradiction by definition of \((u_\epsilon, \rho_\epsilon)\). Thus
\[ \limsup_{\epsilon \to 0} \|\rho_\epsilon\|_{L^2(\Omega)} \leq \|\rho\|_{L^2(\Omega)}. \] (3.40)

Since \(L^2(\Omega)\) is a uniformly convex Banach space, then by a generalized result of the Radon–Riesz property [19, Th. A.70], we have that \(\rho_\epsilon \rightharpoonup \rho\) strongly in \(L^2(\Omega)\). A similar argument gives \(\rho_{\epsilon, h} \rightharpoonup \rho_h\) strongly in \(L^2(\Omega)\).

The next step is to show that there exists a subsequence such that, for any \(\epsilon > 0\), \(\rho_{\epsilon, h} \to \rho_\epsilon\) strongly in \(L^2(\Omega)\) as \(h \to 0\). Since the filter is linear, for any \(\eta \in C_\gamma\), \(\langle F'(\rho), \eta - \rho \rangle = F(\eta - \rho)\). Moreover, by assumption \(F7\), the projection is a linear operator. Hence for any \(\eta_h \in C_{\gamma, h}\), \(F'_h(\rho_h), \eta_h - \rho_h = F_h(\eta_h - \rho_h)\). By following the derivation of the first-order optimality condition in Proposition 2.4, we find that, for all \(\eta \in C_\gamma \cap B_{r/2, L^2(\Omega)}(\rho)\),
\[ -\int_{\Omega} k'(F_h(\rho_{\epsilon, h}))|E u_{\epsilon, h}|^2 F_h(\eta_h - \rho_{\epsilon, h}) \, dx + \epsilon \int_{\Omega} \rho_{\epsilon, h}(\eta_h - \rho_{\epsilon, h}) \, dx \geq 0. \] (3.41)

Similarly, for any \(\eta \in C_\gamma \cap B_{r/2, L^2(\Omega)}(\rho)\),
\[ -\int_{\Omega} k'(F(\rho_\epsilon))|E u_\epsilon|^2 F(\eta - \rho_\epsilon) \, dx + \epsilon \int_{\Omega} \rho_\epsilon(\eta - \rho_\epsilon) \, dx \geq 0. \] (3.42)

Hence, \((3.41)\) and \((3.42)\) imply that for all \(\eta \in C_{\gamma \cap B_r/L^2(\Omega)}(\rho)\) and \(\eta_h \in C_{\gamma, h \cap B_{r/2, L^2(\Omega)}(\rho)}\) we have that
\[
-\int_{\Omega} k'(F(\rho_\epsilon))|E u_\epsilon|^2 \rho_\epsilon + \epsilon \rho_\epsilon^2 \, dx \leq -\int_{\Omega} k'(F(\rho_\epsilon))|E u_\epsilon|^2 F(\eta) + \epsilon \rho_\epsilon \eta \, dx, 
\] (3.43)

\[
-\int_{\Omega} k'(F_h(\rho_{\epsilon, h}))|E u_{\epsilon, h}|^2 F_h(\rho_{\epsilon, h}) + \epsilon \rho_{\epsilon, h}^2 \, dx 
\leq -\int_{\Omega} k'(F_h(\rho_{\epsilon, h}))|E u_{\epsilon, h}|^2 F_h(\eta_{\epsilon, h}) + \epsilon \rho_{\epsilon, h} \eta_h \, dx. 
\] (3.44)
By subtracting \(- \int_{\Omega} k'(F(\rho_\varepsilon)) |E u_{\varepsilon}|^2 F_h(\rho_{\varepsilon,h}) + \epsilon \rho_\varepsilon \rho_{\varepsilon,h} \, dx\) from (3.43),
\(- \int_{\Omega} k'(F_h(\rho_{\varepsilon,h})) |E u_{\varepsilon,h}|^2 F(\rho_\varepsilon) + \epsilon \rho_\varepsilon \rho_{\varepsilon,h} \, dx\) from (3.44), summing the result, and rearranging, we see that
\[
\int_{\Omega} (\rho_\varepsilon - \rho_{\varepsilon,h})^2 \, dx \\
\leq \int_{\Omega} (k'(F(\rho_\varepsilon)) - k'(F_h(\rho_{\varepsilon,h}))) |E u_{\varepsilon}|^2 (F(\rho_\varepsilon) - F_h(\rho_{\varepsilon,h})) \, dx \\
+ \int_{\Omega} k'(F_h(\rho_{\varepsilon,h})) (|E u_{\varepsilon}|^2 - |E u_{\varepsilon,h}|^2) (F(\rho_\varepsilon) - F_h(\rho_{\varepsilon,h})) \, dx \\
- \int_{\Omega} k'(F(\rho_\varepsilon)) |E u_{\varepsilon}|^2 (F(\eta) - F_h(\rho_{\varepsilon,h})) \, dx \\
- \int_{\Omega} k'(F_h(\rho_{\varepsilon,h})) |E u_{\varepsilon,h}|^2 (F_h(\eta_h) - F(\rho_\varepsilon)) \, dx \\
+ \epsilon \int_{\Omega} \rho_\varepsilon (\eta - \rho_{\varepsilon,h}) + \rho_{\varepsilon,h} (\eta_h - \rho_\varepsilon) \, dx.
\] (3.45)

We must now show that each term on the right-hand side of (3.45) tends to zero as \(h \to 0\). We label the five terms on the right-hand side with the Roman numerals (I)-(V). Since \(\rho_{\varepsilon,h} \rightharpoonup \rho_\varepsilon\) in \(L^\infty(\Omega)\), then by Lemma 3.2, \(F_h(\rho_{\varepsilon,h}) \to F(\rho_\varepsilon)\) strongly in \(L^\infty(\Omega)\). Hence
\[
(\text{I}) \leq \|k'(F(\rho_\varepsilon)) - k'(F_h(\rho_{\varepsilon,h}))\|_{L^\infty(\Omega)} \|E u_{\varepsilon}\|_{L^2(\Omega)}^2 \|F(\rho_\varepsilon) - F_h(\rho_{\varepsilon,h})\|_{L^\infty(\Omega)}. \\
(3.46)
\]
The first two terms are bounded and the last term tends to zero. Similarly (II) also tends to zero due to the strong convergence of the filter and the strong convergence of \(E u_{\varepsilon,h}\) in \(L^2(\Omega)^{d \times d}\) due to Corollary 3.3. For the third term (III), we choose \(\eta = \rho_{\varepsilon,h}\), \(F(\rho_{\varepsilon,h}) \to F(\rho_\varepsilon)\) strongly in \(L^\infty(\Omega)\) thanks to the weak-* convergence of \(\rho_{\varepsilon,h}\) in \(L^\infty(\Omega)\). Similarly \(F_h(\rho_{\varepsilon,h}) \to F(\rho_\varepsilon)\) strongly in \(L^\infty(\Omega)\) thanks to Lemma 3.2. Hence (III) \(\to 0\).

In (IV), we choose \(\eta_h\) such that \(\eta_h \to \rho_\varepsilon\) strongly in \(L^2(\Omega)\). Such a sequence is assumed to exist in (A2). Since we have strong convergence, then for sufficiently small \(h\), we have that \(\eta_h \in B_{r/2,L^2(\Omega)}(\rho)\). Thus this choice is admissible. By utilizing the density of smooth compactly supported functions in \(L^1(\Omega)\), one can show that for \(\eta_h, \rho_\varepsilon \in L^\infty(\Omega)\) then \(\eta_h \to \rho_\varepsilon\) strongly in \(L^2(\Omega)\) implies that \(\eta_h \rightharpoonup \rho_\varepsilon\) weakly-* in \(L^\infty(\Omega)\). Hence, by Lemma 3.2, \(F_h(\eta_h) \to F(\rho_\varepsilon)\) strongly in \(L^\infty(\Omega)\). Therefore, (IV) \(\to 0\). Finally, since \(\eta = \rho_{\varepsilon,h}\) and \(\eta_h \to \rho_\varepsilon\) strongly in \(L^2(\Omega)\), we have that (V) \(\to 0\). Hence, we conclude that \(\rho_{\varepsilon,h} \to \rho_\varepsilon\) strongly in \(L^2(\Omega)\) for any \(\varepsilon > 0\).

Since the sequence of norms \(\|\rho_{\varepsilon,h}\|_{L^2(\Omega)}\) is a uniformly bounded sequence in \(\varepsilon\) and \(h\), we may extract a converging subsequence (not indicated) such that
\[
\lim_{\varepsilon,h \to 0} \|\rho_{\varepsilon,h}\|_{L^2(\Omega)} = L. \quad \text{We have already shown that} \quad \lim_{\varepsilon \to 0} \|\rho_{\varepsilon,h}\|_{L^2(\Omega)} = \|\rho_h\|_{L^2(\Omega)} \quad \text{and} \quad \lim_{h \to 0} \|\rho_{\varepsilon,h}\|_{L^2(\Omega)} = \|\rho_\varepsilon\|_{L^2(\Omega)}.
\]
An application of the Osgood Theorem [39, Ch. 4, Sec. 11, Th. 2] for exchanging limits reveals that the limits \(\lim_{h \to 0} \lim_{\varepsilon \to 0} \|\rho_{\varepsilon,h}\|_{L^2(\Omega)}\)
and \( \lim_{\epsilon \to 0} \lim_{h \to 0} \| \rho_{\epsilon,h} \|_{L^2(\Omega)} \) exist and
\[
L = \lim_{\epsilon \to 0} \lim_{h \to 0} \| \rho_{\epsilon,h} \|_{L^2(\Omega)} = \lim_{h \to 0} \lim_{\epsilon \to 0} \| \rho_{\epsilon,h} \|_{L^2(\Omega)}.
\] (3.47)

By first taking the inner limit and then the outer limit in the second term and only the inner limit in the third term of (3.47), we find that
\[
\| \rho \|_{L^2(\Omega)} = \lim_{h \to 0} \lim_{\epsilon \to 0} \| \rho_h \|_{L^2(\Omega)}.
\] (3.48)

Since \( L^2(\Omega) \) is a uniformly convex Banach space, \( \rho_h \rightharpoonup \rho \) weakly in \( L^2(\Omega) \) and (3.48) holds, then by the Radon–Riesz property [19, Th. A.70], we have that \( \rho_h \to \rho \) strongly in \( L^2(\Omega) \). Lemma 3.1 implies that \( \rho_h \to \rho \) strongly in \( L^p(\Omega) \) for any \( p \in [1, \infty) \).

Remark 3.3. The result in Proposition 3.4 still holds for a nonlinear filter provided there exists a linear operator \( G(\rho; \cdot) \) such that
\[
\langle F'(\rho), \rho - \eta \rangle = G(\rho; \rho - \eta) = G(\rho; \rho) - G(\rho; \eta),
\] (3.49)
and \( G(\rho; \cdot) \) satisfies (F1)–(F8).

Corollary 3.5. Suppose a filter in the sense of Remark 3.3 is used. Then, for sufficiently small \( h \), there exists a subsequence of minimizers \( (u_h, \rho_h) \) of \( (I-\text{SIMP}_h) \) that are also minimizers of \( (\text{SIMP}_h) \).

Proof. Corollaries 3.3 and 3.4 imply that there exists a subsequence \( (u_h, \rho_h) \) such that \( u_h \to u \) strongly in \( H^1(\Omega)^d \) and \( \rho_h \to \rho \) strongly in \( L^2(\Omega) \). By the definition of strong convergence, there exists an \( \bar{h} \) such that for all \( h \leq \bar{h} \), \( \| u - u_h \|_{H^1(\Omega)^d} + \| \rho - \rho_h \|_{L^2(\Omega)} \leq r/4 \). Thus the basin of attraction constraint is not active and the sequence of minimizers of \( (I-\text{SIMP}_h) \) are also minimizers of \( (\text{SIMP}_h) \).

We now have the sufficient ingredients to prove Theorem 3.2.

Proof of Theorem 3.2. By Proposition 3.1, there exists a sequence of FEM solutions \( (u_h, \rho_h) \) minimizing \( (\text{SIMP}_h) \) that satisfy \( u_h \rightharpoonup u \) weakly in \( H^1(\Omega)^d \) and \( \rho_h \rightharpoonup^* \rho \) weakly-* in \( L^\infty(\Omega) \). In Corollary 3.3, we show that there exists a subsequence such that \( u_h \to u \) strongly in \( H^1(\Omega) \). Then, in Proposition 3.3, we show that \( \rho_h \to \rho \) strongly in \( L^p(\Omega) \). Finally, in Corollary 3.5, we show that a subsequence of the minimizers are also minimizers of \( (\text{SIMP}_h) \).

The final result we consider is the proof of Theorem 3.3 concerning the strong convergence of the filtered material distributions in \( W^{1,p}(\Omega) \). The proof of this result follows a similar pattern to the proof of Proposition 3.4. Essentially, the goal is to prove strong convergence of an \( \epsilon \)-perturbed problem, then by proving convergence in the value of the norms, weak convergence, and utilizing the Radon–Riesz property, we deduce strong convergence. A summary is given in Fig. 4.

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Proof of Theorem 3.3. Let $\tilde{\rho} := F(\rho)$ and $\tilde{\rho}_h := F_h(\rho_h)$. We note that as $h \to 0$, the sequence of $\tilde{\rho}_h$ is a bounded sequence in $W^{1,p}(\Omega)$ thanks to (F1) and (F6). Hence, as $W^{1,p}(\Omega)$ is a reflexive Banach space, we may extract a subsequence (not indicated) such that $\tilde{\rho}_h \rightharpoonup \tilde{\rho}_0$ weakly in $W^{1,p}(\Omega)$ for some $\tilde{\rho}_0 \in W^{1,p}(\Omega)$. By assumption, $\rho_h \overset{\ast}{\rightharpoonup} \rho$ weakly-* in $L^\infty(\Omega)$. Hence, by Lemma 3.2, $\tilde{\rho}_h \to F(\rho)$ strongly in $L^\infty(\Omega)$. Thus, by the uniqueness of limits, $\tilde{\rho}_0 = F(\rho) = \tilde{\rho}$. Moreover, by Lemma 3.4, there exists a subsequence (not indicated) such that $u_h \to u$ strongly in $H^1(\Omega)^d$. By assumption $(u, \tilde{\rho})$ is isolated. Let $\tilde{r}$ denote the radius of the basin of attraction.

Denote the minimizers of the following optimization problem by $(u_\epsilon, \tilde{\rho}_\epsilon)$:

$$\min_{(v, \tilde{\eta}) \in (V_\Omega \cap B_{\tilde{r}/2, W^{1,p}(\Omega)(\tilde{\rho})}) \times (\tilde{C}_\gamma \cap B_{\tilde{r}/2, W^{1,p}(\tilde{\rho})})} \tilde{J}(v, \tilde{\eta}) + \frac{\epsilon}{p} \|
abla \tilde{\eta}\|_{L^p(\Omega)}^p,$$

subject to

$$\tilde{a}(u_\epsilon, v; \tilde{\rho}_\epsilon) = l(v) \text{ for all } v \in V_\Omega.$$  

(3.54) + Proposition 3.3

Radon–Riesz+ (3.53)

$\frac{\partial \tilde{\rho}_{\epsilon, h}}{\partial \epsilon} \rightarrow \frac{\partial \tilde{\rho}_{\epsilon}}{\partial \epsilon}$ (I-F-SIMP)$_\epsilon$

Radon–Riesz+ (3.53)

$\frac{\partial F_h(\rho_h)}{\partial h} \rightarrow \frac{\partial F(\rho)}{\partial h}$ (I-SIMP)$_h$

Osgood+Radon–Riesz

$h \to 0$

Figure 4: A summary of the proof of Theorem 3.3.
However, this would be a contradiction by definition of \((u_\epsilon, \tilde{\rho}_\epsilon)\). Thus

\[
\limsup_{\epsilon \to 0} \| \nabla \tilde{\rho}_\epsilon \|_{L^p(\Omega)} \leq \| \nabla \tilde{\rho} \|_{L^p(\Omega)}. \tag{3.53}
\]

Since \(L^p(\Omega)\) is a uniformly convex Banach space for any \(p \in (1, \infty)\), then by a generalized result of the Radon–Riesz property [19, Th. A.70], we have that \(\nabla \tilde{\rho}_\epsilon \to \nabla \tilde{\rho}\) strongly in \(L^p(\Omega)\). By an application of the Rellich–Kondrachov theorem, (F2), and Lemma 3.1 we deduce that \(\tilde{\rho}_\epsilon \to \tilde{\rho}\) strongly in \(W^{1,p}(\Omega)\). A similar argument gives \(\tilde{\rho}_{\epsilon, h} \to \tilde{\rho}_h\) strongly in \(W^{1,p}(\Omega)\) as \(h \to 0\).

By instead considering variations in \(\tilde{\rho}\) rather than \(\rho\) in Proposition 2.4, we find that, for any \(\tilde{\eta} \in \tilde{C}_\gamma\),

\[
- \int_{\Omega} k'(\tilde{\rho}_\epsilon)|\mathbf{E}u_\epsilon|^2(\tilde{\eta} - \tilde{\rho}_\epsilon)\,dx + \epsilon \int_{\Omega} |\nabla \tilde{\rho}_\epsilon|^{p-2} \nabla \tilde{\rho}_\epsilon \cdot \nabla (\tilde{\eta} - \tilde{\rho}_\epsilon)\,dx \geq 0. \tag{3.54}
\]

With only small changes to the proof of Proposition 3.3 we see that, for any \(\epsilon > 0\), we have that \(\tilde{\rho}_{\epsilon, h} \to \tilde{\rho}_\epsilon\) strongly in \(W^{1,p}(\Omega)\) as \(h \to 0\).

Since the sequence of norms \(\| \tilde{\rho}_{\epsilon, h} \|_{W^{1,p}(\Omega)}\) is a uniformly bounded sequence in \(\epsilon\) and \(h\), we may extract a converging subsequence (not indicated) such that

\[
\lim_{\epsilon, h \to 0} \| \tilde{\rho}_{\epsilon, h} \|_{W^{1,p}(\Omega)} = L. \tag{3.55}
\]

Thus by first taking the inner limit and then the outer limit in the second term and only the inner limit in the third term of (3.55), we find that

\[
\| \tilde{\rho} \|_{W^{1,p}(\Omega)} = \lim_{h \to 0} \| \tilde{\rho}_h \|_{W^{1,p}(\Omega)}. \tag{3.56}
\]

Since \(W^{1,p}(\Omega)\) is a uniformly convex Banach space for any \(p \in (1, \infty)\), \(\tilde{\rho}_h \to \tilde{\rho}\) weakly in \(W^{1,p}(\Omega)\) and (3.56) holds, then by the Radon–Riesz property [19, Th. A.70], we have that \(\tilde{\rho}_h \to \tilde{\rho}\) strongly in \(W^{1,p}(\Omega)\). Thus \(F_h(\rho_h) \to F(\rho)\) strongly in \(W^{1,p}(\Omega)\).

4 Conclusions

In this work we studied the convergence of a conforming finite element discretization of the SIMP model for the linear elasticity compliance topology optimization problem. To ensure existence, we considered two types of restriction methods: \(W^{1,p}\)-type regularization and density filtering. The nonconvexity of the optimization problem was
handled by fixing any isolated local or global minimizer and introducing a modified optimization problem with the chosen isolated minimizer as its unique solution. We then showed that there exists a sequence of discretized FEM minimizers that converges to the infinite-dimensional minimizer in the appropriate norms. The displacement and material distribution converge strongly in $H^1(\Omega)$ and $L^p(\Omega)$, for any $p \in [1, \infty)$, respectively, for either a $W^{1,p}$-type regularization or a linear density filter. Even for an arbitrary minimizer (rather than any isolated minimizer) this is a novel result for a linear density filter. If a $W^{1,p}$-type regularization is used, we further deduced that there exists a sequence such that $\rho_h \to \rho$ strongly in $W^{1,p}(\Omega)$. In contrast, if a general class of density filters is used then the filtered material distribution converges strongly in $W^{1,p}(\Omega)$ provided the discretization for the filtered material distribution is $W^{1,p}(\Omega)$-conforming. Thanks to the strong convergence, a subsequence of the FEM minimizers of the modified optimization problem are also minimizers of the original optimization problem.

We expect many of these results to generalize to other topology optimization problems such e.g. minimizing the maximum displacement and maximizing the minimum eigenfrequency of free vibrations. Future work could include adapting these results for such problems as well as the consideration of nonlinear elasticity and FEM discretizations that avoid locking.

A Supplementary proofs

Proof of Proposition 2.1. We first note that for any $\rho \in H_\gamma$

$$\|F(\rho)\|_{L^\infty(\Omega)} = \left\| \int_\Omega f(x-y)\rho(y) \, dy \right\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\mathbb{R}^d)} \leq C < \infty. \quad (A.1)$$

It can be shown that $h(x,y) := f(x-y)\rho(y)$ satisfies the condition for the Leibniz integral rule and, therefore,

$$\|\nabla F(\rho)\|_{L^\infty(\Omega)} = \left\| \int_\Omega \nabla_x[f(x-y)]\rho(y) \, dy \right\|_{L^\infty(\Omega)} \leq \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \leq C < \infty. \quad (A.2)$$

This implies that $F(\rho) \in W^{1,\infty}(\Omega)$. Hence, (F1) is satisfied.

Since $f, \rho \geq 0$ a.e. then $F(\rho)(x) \geq 0$ for all $x \in \Omega$. Moreover,

$$|F(\rho)(x)| = \left| \int_\Omega f(x-y)\rho(y) \, dy \right| \leq \|f\|_{L^1(\mathbb{R}^d)} \|\rho\|_{L^\infty(\Omega)} \leq 1. \quad (A.3)$$

Thus (F2) is satisfied.

Consider a weakly-* converging sequence $\rho_n \xrightarrow{\ast} \rho \in C_\gamma$ weakly-* in $L^\infty(\Omega)$. This defines a bounded sequence of Lipschitz functions $F(\rho_n)$. Thus, by the Arzelà–Ascoli theorem [2, Th. 1.33], there exists a limit $F_0 \in L^\infty(\Omega)$ such that a subsequence (not
indicated) satisfies $F(\rho_n) \to F_0$ uniformly in $\Omega$. Hence, $F(\rho_n) \to F_0$ strongly in $L^\infty(\Omega)$. Moreover,

$$F(\rho_n)(x) = \int_\Omega f(x-y)\rho_n(y) \, dy \to \int_\Omega f(x-y)\rho(y) \, dy = F(\rho)(x). \quad (A.4)$$

Hence $F(\rho_n) \to F(\rho)$ pointwise. Thus, by the uniqueness of limits, $F_0 = F(\rho)$ and $F(\rho_n) \to F(\rho)$ strongly in $L^\infty(\Omega)$. A similar argument reveals that if $\rho_n \rightharpoonup \rho$ weakly in $L^p(\Omega)$, $p \in [1, \infty]$, then $F(\rho_n) \to F(\rho)$ strongly in $L^p(\Omega)$. Thus (F4) is satisfied.

Note that $F(\rho) + t(F(\rho) - F(\eta)) = F(\rho + t(\eta - \rho)) \in \tilde{C}_\gamma$ since $\rho + t(\eta - \rho) \in C_\gamma$. Hence $\tilde{C}_\gamma$ is a convex space. Consider any sequence such that $F(\rho_n) \to F_0$ strongly in $W^{1,p}(\Omega)$ for any $p \in [1, \infty]$. This defines a bounded sequence $(\rho_n)$. Hence, since $C_\gamma$ is a norm-closed, bounded, and convex subset of $L^\infty(\Omega)$, then by the Banach–Alaoglu theorem, there exists an $\rho_0 \in C_\gamma$ such that a subsequence (not indicated) satisfies $\rho_n \rightharpoonup \rho_0$ weakly-* in $L^\infty(\Omega)$. As previously shown, this implies that $F(\rho_n) \to F(\rho_0)$ strongly in $L^\infty(\Omega)$. Hence, $F_0 = F(\rho_0)$ and, therefore, $F_0 \in \tilde{C}_\gamma$. Hence, $\tilde{C}_\gamma$ is a norm-closed, bounded, and convex subset of $W^{1,\infty}(\Omega)$. A similar argument reveals that $\tilde{C}_\gamma$ is a norm-closed, bounded, and convex subset of $W^{1,p}(\Omega)$ for any $p \in [1, \infty]$. Therefore, (F5) is satisfied.

**Proof of Lemma 3.1.** By Hölder’s inequality $\|\eta - \eta_n\|_{L^1(\Omega)} \leq |\Omega|^{1/p}\|\eta - \eta_n\|_{L^p(\Omega)}$. Thus $\eta_n \to \eta$ strongly in $L^1(\Omega)$. Therefore, for any $s \in (1, \infty)$,

$$\int_\Omega |\eta - \eta_n|^s \, dx = \int_\Omega |\eta - \eta_n|^s \, dx \leq 1^{s-1}\|\eta - \eta_n\|_{L^1(\Omega)}, \quad (A.5)$$

which implies that $\eta_n \to \eta$ strongly in $L^s(\Omega)$ for any $s \in [1, \infty)$. □

**Proof of Lemma 3.2.** For any $\eta \in L^p(\Omega)$,

$$\|F(\eta) - F_h(\eta_h)\|_{L^p(\Omega)} \leq \|F(\eta) - F(\eta_h)\|_{L^p(\Omega)} + \|F(\eta_h) - F_h(\eta_h)\|_{L^p(\Omega)}. \quad (A.6)$$

The first term on the right-hand side of (A.6) tends to zero thanks to assumption (F4). For the second term we note that, by assumptions (F1) and (F8),

$$\|F(\eta_h) - F_h(\eta_h)\|_{L^p(\Omega)} \leq C h^s \|F(\eta_h)\|_{W^{1,p}(\Omega)} \leq C' h^s, \quad (A.7)$$

for some $s > 0$ and a constant $C' < \infty$ independent of $\eta_h$. Hence $\eta_h \to \eta$ weakly in $L^p(\Omega)$ ($\eta_h \rightharpoonup \eta$ weakly-* in $L^\infty(\Omega)$ if $p = \infty$) implies that $F_h(\eta_h) \to F(\eta)$ strongly in $L^p(\Omega)$.

**Proof of Lemma 3.3.** For any $v \in V_D$ and $v_h \in V_{D,h}$, consider

$$a_h(\tilde{u}_h, v; \tilde{\rho}_h) = a_h(u_h, v; \rho_h) + a_h(u_h, v - v_h; \rho_h)$$

$$= l_h(v_h) + a_h(u_h, v - v_h; \rho_h). \quad (A.8)$$
By (A2) we may choose \(v_h\) such that \(v_h \to v\) strongly in \(H^1(\Omega)^d\). Thus
\[
l_h(v_h) \to l(v).
\]
(A.9)
due to the linearity of \(l(\cdot)\), (A1), and (A2). Moreover,
\[
|a_h(\hat{u}_h, v - v_h; \hat{\rho}_h)| \leq \|k(F_h(\hat{\rho}_h))\|_{L^\infty(\Omega)} \|\mathbf{E}\hat{u}_h\|_{L^2(\Omega)} \|\mathbf{E}(v - v_h)\|_{L^2(\Omega)} 
\to 0,
\]
(A.10)
thanks to the boundedness of \(k(F_h(\hat{\rho}_h)), \mathbf{E}\hat{u}_h\) and (A2). Hence,
\[
a_h(\hat{u}_h, v; \hat{\rho}_h) \to l(v).
\]
(A.11)

We note that
\[
|a_h(\hat{u}_h, v; \hat{\rho}_h) - a(\hat{u}, v; \hat{\rho})|
\leq |a_h(\hat{u}_h - \hat{u}, v; \hat{\rho})| + |a_h(\hat{u}_h, v; \hat{\rho}_h) - a(\hat{u}_h, v; \hat{\rho})|.
\]
(A.12)
The first term on the right-hand side tends to zero thanks to weak convergence of \(\hat{u}_h\) to \(\hat{u}\). For the second term we see that
\[
|a_h(\hat{u}_h, v; \hat{\rho}_h) - a(\hat{u}_h, v; \hat{\rho})| \leq \|(k(F_h(\hat{\rho}_h)) - k(F(\hat{\rho})))\mathbf{E}v\|_{L^2(\Omega)} \|\mathbf{E}\hat{u}_h\|_{L^2(\Omega)}.
\]
(A.13)

By assumption either \(\hat{\rho}_h \rightharpoonup \hat{\rho}\) in \(W^{1,p}(\Omega)\) for some \(p \in [2, \infty)\) and, by the Rellich–Kondrachov theorem and Lemma 3.1, \(\hat{\rho}_h \to \hat{\rho}\) strongly in \(L^2(\Omega)\) or \(\hat{\rho}_h \rightharpoonup \hat{\rho}\) weakly in \(L^p(\Omega)\) but (F1)–(F8) hold and thus, by Lemma 3.2, \(F_h(\hat{\rho}_h) \to F(\hat{\rho})\) strongly in \(L^p(\Omega)\). Thus by utilizing Lemma 3.1, for any smooth test function \(v \in C^\infty_c(\Omega)^d\),
\[
a_h(\hat{u}_h, v; \hat{\rho}_h) \to a(\hat{u}, v; \hat{\rho}).
\]
(A.14)

(A.14) follows for any \(v \in \mathbf{V}_D\) by the density of smooth and compactly supported functions in \(\mathbf{V}_D\). Hence, (A.11) and (A.14) imply that, for any \(v \in \mathbf{V}_D\),
\[
a(\hat{u}, v; \hat{\rho}) = l(v).
\]
(A.15)

Proof of Lemma 3.4. The first step is to show that \(\hat{u}_h \rightharpoonup u\) in \(H^1(\Omega)^d\). The sequence generated by \(\hat{\rho}_h\) defines a bounded sequence of \(\hat{u}_h\) as \(h \to 0\). Since \(\mathbf{V}_D\) is a reflexive space, then there exists a limit \(\hat{u} \in \mathbf{V}_D\) and a subsequence (not indicated) such that \(\hat{u}_h \rightharpoonup \hat{u} \in \mathbf{V}_D\). We must now identify \(\hat{u}\) with \(u\).

The requirements of Lemma 3.3 are satisfied and thus, for any \(v \in \mathbf{V}_D\), \(a(\hat{u}, v; \rho) = l(v)\). Due to the uniqueness of limits and the uniqueness of solutions of the linear elasticity problem with a fixed \(\rho\) (Proposition 2.2), we deduce that \(\hat{u} = u\) a.e. and \(\hat{u}_h \to u\) in \(H^1(\Omega)^d\).
We now strengthen the convergence to strong convergence. By Korn’s inequality [4, Th. 11], and the assumption that $\varepsilon_{\text{SIMP}} > 0$, there exists a $c > 0$ such that
\[
\|u - \hat{u}_h\|_{H^1(\Omega)} \leq a_h(u - \hat{u}_h, u - \hat{u}_h; \hat{\rho}_h)
\]
\[
= a_h(u - \hat{u}_h, u; \hat{\rho}_h) + a_h(\hat{u}_h, \hat{u}_h; \hat{\rho}_h) - a_h(u, \hat{u}_h; \hat{\rho}_h)
\]
\[
= a_h(u - \hat{u}_h, u; \hat{\rho}_h) + l(u_h) - a_h(u, \hat{u}_h; \hat{\rho}_h).
\]
(A.16)
The first term on the rightmost-hand side of (A.16) tends to zero as
\[
a_h(u - \hat{u}_h, u; \hat{\rho}_h)
\]
\[
\leq a(u - u_h, u; \rho) + \|k(F(\rho)) - k(F_h(\rho_h))\|_{L^2(\Omega)}\|E(u - \hat{u}_h)\|_{L^2(\Omega)}\|E u\|_{L^2(\Omega)}.
\]
(A.17)
The first term tends to zero due to the weak convergence in $H^1(\Omega)^d$. The second term converges to zero since either $\rho_h \to \rho$ or $F_h(\rho_h) \to F(\rho)$ strongly in $L^p(\Omega)$ for some $p \in [2, \infty]$ (and thus for any $p \in [1, \infty]$ by Lemma 3.1 and (F2)).

The second term on the rightmost-hand side of (A.16) tends to $l(v)$ thanks to (A1) and the weak convergence of $\hat{u}_h$. By the same arguments as (A.8)–(A.10), we note that $a_h(u, \hat{u}_h; \hat{\rho}_h) \to l(v)$. Hence, we conclude that $\hat{u}_h \to u$ strongly in $H^1(\Omega)^d$. $\square$

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