Cuts, flows and gradient conditions on harmonic functions.

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Abstract
Reduced cohomology motivates to look at harmonic functions which satisfy certain gradient conditions. If $G$ is a direct product of two infinite groups or a (FC-central)-by-cyclic group, then there are no harmonic functions with gradient in $c_0$ on its Cayley graphs. From this, it follows that a metabelian group $G$ has no harmonic functions with gradient in $\ell^p$. Furthermore, under a radial isoperimetric condition, groups whose isoperimetric profile is bounded by power of logarithms also have no harmonic functions with gradient in $\ell^p$.

1. Introduction
The subject matter of this paper is to investigate which graphs and Cayley graph of groups possess harmonic functions whose gradient (the difference of the values of the function at the end of the edges) belongs to $\ell^p$ or $c_0$. The motivation comes mainly from groups: for example when $p = \infty$, one gets the class of Lipschitz harmonic functions which is of known importance, e.g. see Shalom & Tao [32].

Furthermore, if a Hilbertian representation of the group has non trivial reduced cohomology in degree 1, then there exist a non-constant harmonic function on the Cayley graph. The gradient of this harmonic function is related to the mixing property of this representation. For example, for a strongly mixing representation this yields (in any Cayley graph) a harmonic functions with gradient in $c_0$. Hence if a group has no harmonic function with gradient in $c_0$ in some Cayley graph, then the reduced cohomology in degree 1 of any strongly mixing representation is trivial; see [15, §2] or [12, §3] for details and references.

Lastly, let us mention the reduced $\ell^p$-cohomology in degree 1 (of a group or graph), an useful quasi-isometry invariant. Under some assumptions on the isoperimetry, the non-vanishing of this cohomology is equivalent to the presence of harmonic function with gradient in $\ell^p$; see [14] for details and references. An underlying question due to Gromov [16, §8.A1,A2, p.226] is whether any amenable group has harmonic function with gradient in $\ell^p$.

For Cayley graphs of groups, the main results are:

Theorem 1.1 (see Theorem 4.17 and Corollary 4.18). Let $G$ be a finitely generated group which is a cyclic extension of a group with an infinite FC-centraliser. Then (on any Cayley graph of $G$) any harmonic function with gradient in $c_0$ is constant.
Consequently, on any Cayley graph of a metabelian group there are no harmonic function with gradient in $\ell^p$.

Note that any virtually nilpotent group has an infinite FC-centraliser (the FC-centraliser is the subgroup of elements with a finite conjugacy class; more see §4.4). The previous result is akin to a result of Brieussel & Zheng [4, Theorem 1.1].

The current technique sometime only apply to a restricted generating set:

**Theorem 1.2** (see Theorem 4.16). Let $G$ be a group and $H$ an infinite subgroup so that $G$ is generated by $H$ and its FC-centraliser $Z := Z^F_C(H)$. Then there is a Cayley graph of $G$ which has no non-constant harmonic function with gradient in $c_0$.

Note that if $Z \cap H$ is infinite, then the FC-centraliser of $G$ itself is infinite. In that case [15, Lemma 2.7] could already yield the conclusion; see also Corollary 4.15.

Direct products of two infinite groups satisfy the hypothesis of Theorem 1.2. For such groups, the absence of harmonic functions with gradient in $c_0$ can be established in any Cayley graph; see Corollary 4.14. The results mention so far rely on the concepts of almost-malnormal subgroups and quasi-normalisers; in that sense, they build up on some techniques from [12].

Using some analysis of the $p$-spectral gap of finite graphs as well as estimates on the separation profile from [6], an appreciable family of groups may be covered, modulo an hypothesis, see 3.14. This hypothesis that has been dubbed “radial isoperimetric inequality” is a strengthening of a known inequality conjectured by Sikorav and proved by Żuk [34].

**Theorem 1.3** (see Corollary 4.8). Let $G$ be a group and consider some Cayley graph of $G$. Assume that the isoperimetric ratio function is bounded above and below by powers of logarithms and that a radial isoperimetric inequality (3.14) holds. Then there are no non-constant harmonic functions with gradient in $\ell^p$.

See §3.3 for the definitions of isoperimetric ratio function and radial isoperimetric inequality. The hypothesis on the isoperimetric ratio function is fairly weak (there are many groups which will satisfy them, see e.g. [6, §4.2] or Pittet & Saloff-Coste [29, Theorem 7.2.1] and references therein).

Some further results are also presented. §4.5 gives a quick estimate of how slowly the gradient of a non-constant harmonic may decay by using the idea of divergence, see Ol’shanskii, Osin and Sapir [28]. §3.4 reformulate this kind of isoperimetry directly as an inequality between entropy and the gradient of probability measures. This is then used to get an estimate on the decay of the gradient of the $n^{th}$-distribution of a random walk. These might be amusing in view of [32, §6]. §3.5 uses Rényi entropy to capture three usual large scale properties of a group (and its generating set): the return probability of a random walk, the growth of balls and the entropy.

The article is organised as follows. All of the questions are gathered in §5. §2 introduces the space of cuts and flows which are used to study the space of harmonic function with gradient conditions. One of the aim of this section is to give a dual characterisation of having no non-constant harmonic function with gradient in $\ell^p$ (or $c_0$). This dual characterisation will allow to reduce the non-existence of harmonic functions to the existence of transport.
plans to deal with other cases. §3 develops tools which are necessary for Theorem 1.3 (or Corollary 4.8): §3.1 bridges estimate on the Cheeger constant in finite graphs with the \( p \)-spectral gap and §3.3 focuses on the existence of useful finite sets inside groups. These finite set as well as the bounds on the spectral gap will be used to construct transport plans. §3.2 introduces the idea of exit distributions which also come in the construction of some transport plans. §4 contains the bulk of the proofs. §4.1 defines transport plans and use them to give alternate proofs of results from [10], namely that graph without bounded harmonic functions also have no [potentially unbounded] harmonic function with gradient in \( \ell^p \), provided some isoperimetric inequality holds; this uses the exit distributions from §3.2. §4.2 contains the proof of Theorem 1.3; it relies on the well-chosen sets from §3.3 and the bounds of §3.1 to do so. §4.3 reproves §15, Lemma 2.7 using slightly different techniques. §4.4 hinges on the idea of q-normalisers to show that the constancy of a harmonic function with gradient in \( c_0 \) propagates to malnormal hulls. This eventually lead to the proof of Theorems 1.1 and 1.2.

Acknowledgments: The author learned about Theorem 3.10 through an answer V. Befara gave on MathOverflow [2] ans was given the correct reference thanks to V. Kaimanovich.

2 Harmonic functions and reduced \( \ell^p \)-cohomology

2.1 Cuts and flows

The aim of this section is to define the subspaces of \( \ell^p E \) and \( c_0 E \) which are of interest here (cuts and flows) and relate them to harmonic functions.

For convenience, the edges will be seen as a subset of \( V \times V \). However to avoid dealing with “alternate” functions\(^2\), only one of the pair \((x, y)\) or \((y, x)\) will be allowed. So \( x, y \in V \) are neighbours if either \((x, y) \in E\) or \((y, x) \in E\); but \((x, y)\) and \((y, x)\) may not both belong to \( E \).

That said, given a function on the vertices \( f : V \rightarrow \mathbb{R} \), \( \nabla f \) is defined as \( \nabla f(x, y) = f(y) - f(x) \) (whenever \((x, y) \in E\)). On any countable set \( X \), one can define the inner product of finitely supported functions \( f, g : X \rightarrow \mathbb{R} \) by

\[
\langle f \mid g \rangle = \sum_{x \in X} f(x)g(x).
\]

This extends to larger spaces, e.g. \( f \in \ell^p X \) and \( g \in \ell^p X \). The adjoint of \( \nabla \) for this product is \( \nabla^* \). It is defined (as usual) by \( \langle \nabla^* f \mid g \rangle = \langle f \mid \nabla g \rangle \) where \( f : E \rightarrow \mathbb{R} \) and \( g : V \rightarrow \mathbb{R} \). This gives

\[
\nabla^* f(x) = -\sum_{(x, y) \in E} f(x, y) + \sum_{(y, x) \in E} f(y, x).
\]

A function \( f : V \rightarrow \mathbb{R} \) is harmonic if \( \nabla^* \nabla f = 0 \).

\(^2\)When both \((x, y)\) and \((y, x)\) are allowed as edges, the gradient of a function is alternated in the sense it satisfies \( f(x, y) = -f(y, x) \). Since alternated functions in \( \ell^p E \) are complemented, this is not a serious difficulty, simply a convention.
The standing hypothesis that the graph is connected is important in order that the only function with trivial gradient are constant functions. The [other] standing hypothesis that the graph has bounded valency is crucial in order for the gradient to be a bounded operator (from $\ell^p V \to \ell^p E$). Note that the identity $\langle \nabla^* f \mid g \rangle = \langle f \mid \nabla g \rangle$ holds if $f \in \ell^p E$ and $g \in \ell^p V$ (where $p'$ is the H"older conjugate of $p$ and $\ell^\infty$ can be replaced by $c_0$).

The notation $\ell^0 X$ will be used to speak of the finitely supported functions on $X$. The space $c_0 X$ is the completion of $\ell^0 X$ with respect to the $\ell^\infty$-norm. A function $f$ belongs to $c_0 X$ if there is an increasing sequence of finite sets $X_n$ such that $\cup X_n = X$ and $\|f\|_{\ell^\infty(X_n)} \to 0$. In pagan words, “it decreases to 0 at infinity”.

One of the spaces that will be used are the spaces of cuts. Roughly said, these are the image of the gradient. Intuitively, this is because the gradient of the characteristic function of $f$ is the set $\partial f = \{x \in \partial A \mid f(x) \neq 0\}$. So this might be seen as a third possibility for a space of cycles. See Question 5.2. 

Remark 2.1. Sometimes there is a good basis for $C$. Namely, there is a collection of cycles $C'$ so that every cycle can be decomposed as a sum of cycles in $C'$ and there is a uniform bound on the length of the cycles in $C'$. This property implies that a given edge belongs to finitely many element of $C'$. When looking at the Cayley graph of a group, this property coincides with the fact that the group is finitely presented.

In those cases, $\nabla_2$ is a bounded operator from $\ell^p C'$ to $\ell^p E$. It is then relatively straightforward to check that $k_p = \ker \nabla_2 \subseteq \ell^p E$ (and likewise for $c_0$). There is also a natural candidate for $l_\infty$, namely $\nabla_2^{\ell^\infty} C^{\ell^\infty} E$. 

On the other hand, $\nabla_2 \mathbb{R} C' \cap \ell^p E \neq \nabla_p$ (and likewise for $c_0$). The simplest case of a graph where this is an inequality happens when there are no cycles (i.e. a tree, e.g. the Cayley graphs of a free group with respect to a free generating set). So this might be seen as a third possibility for a space of cycles. See Question 5.2.
Recall that the space of harmonic functions with gradient in $c_0$ can be identified, modulo the constant functions, with $F_c \cap K_c$ (by looking at the gradient of these functions). Indeed, since the graph is connected, one can always “integrate” a gradient back to a function, up to a constant function. Hence, an element of $K_c$ is (as a function) harmonic if and only if it lies also in $F_c$. This “proves” (one can essentially take this as a definition): 

**Lemma 2.2.** There are no non-constant harmonic functions with gradient in $c_0$ if and only if $F_c \cap K_c = \{0\}$.

Likewise, the space of Lipschitz harmonic function is $F_\infty \cap K_\infty$. In a group, it is never trivial by Shalom & Tao [32].

A graph $G$ is called Liouville if there are no non-constant bounded harmonic functions, i.e. $(\nabla \ell_\infty V) \cap F_\infty = \{0\}$.

**Remark 2.3.** It is not clear that there is an equivalence between “$G$ is Liouville” and “$F_\infty \cap K_\infty = \{0\}$”. Indeed, as will be shortly shown (see Proposition 2.5), $K_\infty$ contains much more than $\nabla \ell_\infty V$. See Questions 5.3 and 5.4. 

### 2.2 Some reductions via reduced cohomology in degree 1

The difference between $K_1$ and $k_1$ is called the reduced $\ell_1$-cohomology in degree 1 (likewise for $c_0$). It is a well-known fact that the dimension of the quotient vector space $K_1/k_1$ is the number of ends of the graph minus 1 (see Proposition A.2 in [10]). This statement is given a number here for further uses:

**Corollary 2.4.** Assume $G$ is infinite. Then $G$ has one end if and only if $K_1 = k_1$.

Equalities in the dual and pre-dual are easier to check:

**Proposition 2.5.** $k_c = K_c$.

Before moving to the proof, let us first recall the key lemma (see [10, Lemma 2.1]) for the convenience of the reader. This Lemma was inspired to the author by Lemma 4.4 from Holopainen & Soardi [18], which itself builds on a classical truncation lemma on Dirichlet harmonic functions.

**Lemma 2.6.** Let $g \in K_c$ be such that $g \notin k_c$. Consider $g$ as a function $g : V \to \mathbb{R}$. For $t \in \mathbb{R}_{>0}$, let $g_t$ be defined as

$$g_t(x) = \begin{cases} g(x) & \text{if } |g(x)| < t, \\ t \frac{g(x)}{|g(x)|} & \text{if } |g(x)| \geq t. \end{cases}$$

Then there exists $t_0$ such that $g_t \notin k_c$, for any $t > t_0$. In particular, $K_c = k_c$ if and only if $K_c \cap \nabla \ell_\infty V = k_c \cap \nabla \ell_\infty V$.
Proof. Although $g$ and $g_t$ are seen as functions, their norms is still given by their gradient. Assume, without loss of generality that $g(\emptyset) = 0$ for some preferred vertex (i.e. root) $\emptyset \in V$. Given $v \in V$ and $P$ a path from $\emptyset$ to $v$,
\[
|g(v)| = |g(v) - g(\emptyset)| = \sum_{e \in P: \emptyset \rightarrow v} \nabla g(e) \cdot d(\emptyset, v)||\nabla g||_{L^\infty E}.
\]
In particular, $g_t$ is identical to $g$ on $B_{t/K}$ where $K = ||\nabla g||_{L^\infty E}$ and $B_t$ is the ball of radius $t$ around $\emptyset$ (in the combinatorial graph distance). Hence $||\nabla g - \nabla g_t||_{L^\infty E} \leq ||\nabla g||_{L^\infty(B_{t/K})}$, where $L^\infty(B_{t/K})$ denotes the $L^\infty$-norm restricted to edges which are not inside $B_{t/K}$. Because $\nabla g \in \ell^0 E$, $||\nabla g||_{L^\infty(B_{t/K})}$ tends to $0$, as $t$ tends to $\infty$.

Now if there is a infinite sequence $t_n$ such that $g_{t_n}$ are in $k_c$ and $t_n \rightarrow \infty$, then $g_{t_n}$ is a sequence of functions in $k_c$ which tends (in $\ell^0 E$-norm) to $g$. This implies $g \in k_c$, a contradiction. Hence, for some $t_0$, $g_t \notin k_c$ given that $t > t_0$. □

Proof of Proposition 2.5. Given $f : V \rightarrow \mathbb{R}$ with $\nabla f \in \ell^0 E$, the aim is to show that $\nabla f \in k_c$, i.e. that the gradient of $f$ can be approximated by the gradient of finitely supported functions. By Lemma 2.6 (see also [10, Lemma 2.1]), one can assume that $f \in \ell^0 V$ without loss of generality.

So let $f \in \ell^0 V$ with $\nabla f \in \ell^0 E$. Fix some “root” vertex $\emptyset$ again and let $B_{t_n}$ be the ball of radius $n$ (for the combinatorial distance on the graph) around $\emptyset$. Let $n_\varepsilon$ be so that, for any $n > n_\varepsilon$, $||\nabla f||_{L^\infty(B_n^c)} \leq \varepsilon$ where $B_n^c$ is the complement of $B_n$. Let $g_\varepsilon$ be identical to $f$ on $B_{n_\varepsilon}$. Let $S_n := B_{n+1} \backslash B_n$. Then, using $\delta = \varepsilon/||f||_{L^0 V}$, let
\[
g_\varepsilon(x) = \begin{cases} f(x) & \text{if } x \in B_{n_\varepsilon} \\ (1 - (n - n_\varepsilon)\delta) f(x) & \text{if } x \in S_n \text{ and } (n - n_\varepsilon)\delta \in ]0, 1[ \\ 0 & \text{if } x \in S_n \text{ and } (n - n_\varepsilon)\delta \geq 1 \end{cases}
\]
Note that $g_\varepsilon$ is finitely supported hence in $L^0 V$. We claim that $\nabla g_\varepsilon \rightarrow \nabla f$ in $L^\infty$. This follows by checking that $||\nabla g_\varepsilon - \nabla f||_{L^\infty} \leq 3\varepsilon$. To do so, separate in cases depending on where the edge $(x, y)$ (or $(y, x)$, the arguments are symmetric) lies:
- $g_\varepsilon - f \equiv 0$ inside $B_{n_\varepsilon}$, so $\nabla g_\varepsilon - \nabla f = \nabla(g_\varepsilon - f) \equiv 0$ too;
- Let $n_0 = \lceil \delta^{-1} \rceil + n_\varepsilon - 1$. On $B_{n_0}^c$, $g_\varepsilon \equiv 0$ and $||\nabla f||_{L^\infty(B_{n_0}^c)} \leq \varepsilon$ so $||\nabla g_\varepsilon - \nabla f||_{L^\infty(B_{n_0}^c)} \leq \varepsilon$;
- when $x, y \in S_n$ and $n \in [n_\varepsilon, \delta^{-1} + n_\varepsilon[$, $|\nabla f(x, y)| < \varepsilon$ and $|\nabla g_\varepsilon(x, y)| \leq |\nabla f(x, y)|$ (the scaling is the same inside a given sphere) hence $|\nabla g_\varepsilon(x, y) - \nabla f(x, y)| \leq 2\varepsilon$;
- when $x \in S_n$, $y \in S_{n+1}$ and $n \in [n_\varepsilon, \delta^{-1} + n_\varepsilon[$. Then $|\nabla f(x, y) - \varepsilon$ while $|g_\varepsilon(y) - g_\varepsilon(x)| \leq |\nabla f(x, y)| + \delta f(x)$. Hence
\[
|\nabla g_\varepsilon(x, y) - \nabla f(x, y)| \leq 2|\nabla f(x, y)| + \delta f(x) < 3\varepsilon,
\]
by the choice of $\delta$.

To sum up, for any $\varepsilon > 0$, there is a finitely supported function $g_\varepsilon$ so that $||\nabla g - \nabla f||_{L^\infty} \leq 3\varepsilon$. This implies $f \in \ell^0 V^c_{L^\infty} = k_c$ and proves the claim. □

In case the reader is curious, the above equality is always false in $L^\infty$ (unless $G$ is finite): $k_{\infty} \neq k_{L^\infty}$. See [10, Proposition A.3] for details. However
Proposition 2.7. $\mathbb{K}_\infty^* = \mathbb{K}_\infty$

Before moving on to the proof, let us quickly recall a basic fact about weak*-convergence:

Lemma 2.8. Let $p \in [1, \infty]$ and $X$ be a countable set. The sequence $\{y_n\}_{n \in \mathbb{N}}$ is weak* convergent in $\ell^p X$ if and only if it is bounded in $\ell^p X$ and point-wise convergent.

Proof of Proposition 2.7. It is easy to see from either Lemma 2.8 or Lemma 2.9 that $\mathbb{K}_\infty$ is weak*-closed, hence $\overline{\mathbb{K}_{\infty}} \subseteq \mathbb{K}_\infty$.

For the other direction, the proof essentially goes as in the proof of Proposition 2.5. Note that by Lemma 2.8, it suffices to show that any $f \in \mathbb{K}_\infty$ is a point-wise limit of a bounded sequence $g_n$. To do so, it is (again) more convenient to think of these as functions (and the norm is taken on their gradient). Consider $f_t$ as in Lemma 2.6 by truncating $f$. As in said lemma, $f_t \equiv f$ on $B_{t/K}$ where $K = \|\nabla f\|_{L^\infty}$, so $\nabla f_t \to \nabla f$ point-wise. On the other hand, $\|\nabla f_t\|_{L^\infty} \leq \|\nabla f\|_{L^\infty}$ and $f_t \in \mathbb{K}_\infty$. Hence $f_t$ is a bounded point-wise convergent sequence of elements of $\mathbb{K}_\infty$. Thus, $f \in \overline{\mathbb{K}_{\infty}}$.

2.3 Annihilators and duals

Some facts about annihilators will be required. The reader unfamiliar with these is invited to read [31, §4.8]. Our first step here is to identify the annihilators of cuts and flows.

Lemma 2.9.

| $\mathbb{K}_1$ | $\mathbb{F}_c^\perp$ | $\mathbb{F}_1^\perp$ | $\mathbb{K}_1^\perp$ | $\mathbb{F}_1^\perp$ |
| $\mathbb{K}_\infty^*$ | $\mathbb{F}_1$ | $\mathbb{K}_1^\perp$ | $\mathbb{F}_1^\perp$ |

Proof. First, one shows the equalities $\mathbb{K}_1^\perp = \mathbb{F}_1$, $\mathbb{K}_1^\perp = \mathbb{F}_1$, $\mathbb{K}_1^\perp = \mathbb{F}_1$ and $\mathbb{K}_\infty^* = \mathbb{F}_1$. These are all instances of the identity $(\text{Im } L)^\perp = \ker L^*$ (where $L$ is a bounded operator). For example,

\[ \mathbb{K}_c^\perp = (\nabla \ell^0 V)^\perp = \{ y \in \ell^1 E \mid \forall x \in \nabla \ell^0 V \langle y, x \rangle = 0 \} = \{ y \in \ell^1 E \mid \forall x' \in \ell^0 V \langle y, \nabla x' \rangle = 0 \} = \{ y \in \ell^1 E \mid \forall x' \in \ell^0 V, \langle \nabla^* y, x' \rangle = 0 \} = \ker \nabla^* \subseteq \ell^1 E = \mathbb{F}_1. \]

From those four equalities, one gets: $\mathbb{K}_c = \mathbb{F}_1^\perp$, $\mathbb{K}_1 = \mathbb{F}_1^\infty$, $\mathbb{K}_1^\perp = \mathbb{F}_1^\perp$ and $\mathbb{K}_\infty^* = \mathbb{F}_1^\perp$.

To show $\mathbb{F}_c^\perp = \mathbb{K}_1$, one needs to argue slightly differently\(^4\). First, if $y \in \mathbb{K}_1$ then it sums to zero on any oriented cycle, hence any finite combination of such, hence a dense subspace of $\mathbb{F}_c$ and so, by continuity, $y$ is in the annihilator of $\mathbb{F}_c$: $\mathbb{K}_1 \subseteq \mathbb{F}_c^\perp$. Second, if $x \in \ell^1 E$ is in $\mathbb{F}_c^\perp$, then it sums to zero on any oriented cycle. This is exactly the condition that $x$ is the gradient of some function, hence $\mathbb{F}_c^\perp \subseteq \mathbb{K}_1$.

\(^4\)If $\nabla^2$ is available as a bounded operator, then the proof is absolutely identical.
The equalities $f^+_1 = K_\infty$ and $f^+_1 = K_c$ are obtained by the same arguments. By taking [weak]-annihilators, one gets: $f_c = K^+_1$, $f_1 = K^*_\infty$ and $f^*_1 = K^*_c$.

Inclusions are presented in the statement so as to be obvious.

Combining Lemma 2.9 with Corollary 2.4, Proposition 2.5 and Proposition 2.7:

$$k_c = K_c, \quad k^+_1 \subseteq K_1, \quad k_\infty \subseteq K^*_\infty = K_\infty$$

where $f_\infty$ is only defined if $\nabla^*_2$ can be defined as a bounded operator, and $\alpha, \epsilon, ?$ means the inclusion is strict if and only if the graph has $> 1$ ends.

Note that there are probably more direct proofs of some of the equalities above. For example, it is easy to check that $\text{Im} \nabla^*_2 \cap \ell^1 E = \varnothing_1$. From there, it is not too difficult to conclude $f_1 = \varnothing_1$.

The “dual” criterion for the absence of non-constant harmonic functions with gradient in $c_0$ can now be proven. In order to alleviate notations, the shorthand “has $\mathcal{A}$” will be used in order to say “has no non-constant harmonic functions with gradient in $c_0$”.

**Lemma 2.10.** $G$ has $\mathcal{A}$ if and only if $\ell^1 E = \varnothing^+_1$.

If $G$ has one end, then $G$ has $\mathcal{A}$ if and only if $\ell^1 E = \varnothing^+_1$.

**Proof.** Lemma 2.2 shows that $G$ has $\mathcal{A}$ exactly when $\varnothing_c \cap K_c = \{0\}$. Using the basic properties of annihilators $(A \cap B)^\perp = A^\perp + B^\perp$ and that $\ell^1 E$ is the annihilator of $\{0\} \subset c_0 E$, one gets

$$\mathcal{A} \Rightarrow \ell^1 E = \varnothing^+_1 = \varnothing^+_c + \varnothing^*_c$$

By proposition 2.5, $K_c = K_c$. Lemma 2.9 then implies $K^+_1 = \varnothing^+_1$ and $\varnothing^*_c = \varnothing^*_c$. Hence

$$\mathcal{A} \iff \ell^1 E = \varnothing^+_1$$

By Corollary 2.4 (see also directly [10, Proposition A.2]), if $G$ has one-end then $K_1 = K_1$, so one-end implies $\mathcal{A} \iff \ell^1 E = \varnothing^+_1$.

**Remark 2.11.**

- Since all infinite groups have a non-trivial Lipschitz harmonic function (see Shalom & Tao [32]), $K_\infty \cap \varnothing_\infty \neq \{0\}$ and by the same arguments as in the proof of Lemma 2.10, $\varnothing^+_1 = \varnothing^+_1 \subseteq \ell^1 E$. If further $G$ is one-ended, then $\varnothing^+_1 \subseteq \ell^1 E$.

- Since there are one-ended groups (e.g., $\mathbb{Z}^2$) with $\mathcal{A}$, there are groups where $\varnothing^+_1 \subseteq \varnothing^+_1 \subseteq \ell^1 E$.

- Since any two-ended group (e.g., $\mathbb{Z}$) has $\mathcal{A}$, there are groups where $\varnothing^+_1 \subseteq \varnothing^+_1 \subseteq \ell^1 E$. In fact, in the usual Cayley graph of $\mathbb{Z}$ (the line), it is easy to see that $\varnothing^+_1 \subseteq \ell^1 E$ while $\varnothing^+_1 = \emptyset$. Hence there are groups where $\varnothing^+_1 \subseteq \varnothing^+_1 \subseteq \varnothing^+_1 = \varnothing^+_1 \subseteq \ell^1 E$.

- In a the Cayley graph of group with infinitely many ends (e.g., a regular tree), one can check that $K_1 \subseteq K_1$. To do so, consider a half-tree $H$ (a connected component after removing an edge) and look at $\nabla_1 H \in K_1$. The fact the half-tree has a strong isoperimetric ratio function (IS, see §3.3) implies that $\nabla_1 H \notin K^*_1$.  

\[ \diamond \]
The proof of Lemma 2.9 can be mimicked without problem in the reflexive case. The output is cleaner, since closure and weak \(^*\)-closure coincide.

**Lemma 2.12.** Let \( p' = \frac{p}{p-1} \) be the Hölder conjugate of \( p \in [1, \infty[ \), then

\[
\mathcal{K}_p = \mathcal{F}^\perp_{p'} \supseteq \mathcal{F}^\perp_p = \mathcal{K}_p \quad \mathcal{F}_p = \mathcal{K}^\perp_{p'} \supseteq \mathcal{K}^\perp_p = \mathcal{F}_p
\]

Lemma 2.10 also has an analogue for \( \ell^p \) (proved using Lemma 2.12). Here “has zero \( \Theta\mathcal{D}^p \)” means “has no non-constant harmonic functions with gradient in \( \ell^p \).

**Lemma 2.13.** \( G \) has zero \( \Theta\mathcal{D}^p \) if and only if \( \ell^{p'}_0 E = \mathcal{K}_p + \mathcal{F}_p \). 

**Remark 2.14.** If a group has a presentation with generators \( s_1, \ldots, s_n \) none of which are of order 2 and relations \( t_1^{n_1} t_2^{n_2} \cdots t_k^{n_k} \) so that \( t_i \in S \) and (for each relator) \( \sum n_i = 0 \), then \( k_1 \) and \( f_1 \) both lie in a weak \(^*\) closed subspace of \( \ell^1 E \). Indeed, orient the edges so that \( s_i \) is the positive direction (so \( s_i^{-1} \) is the negative direction). Then \( \nabla \delta_k \) has zero sum. Also, for any cycle \( c \), \( \nabla^+_2 c \) has exactly the same number of positive and negative edges. Hence \( \nabla^+_2 c \) also has zero sum. This implies that \( k_1 + f_1 \subset \ell^1_0 E \subseteq \ell^1 E \).

Note that it is obvious that such groups have non-constant Lipschitz harmonic functions. Indeed, they have an infinite Abelianisation: a surjection \( \Gamma \rightarrow \mathbb{Z} \) is given by sending each generator to 1 ∈ \( \mathbb{Z} \).

\( \diamond \)

## 3 Isoperimetry, Laplacian in \( \ell^p V \) and entropy

### 3.1 Inverting the Laplacian in \( \ell^p \)

In this subsection only \( d \)-regular finite graphs are considered. Here \( \Delta \) is the Laplacian with spectrum in \([0, 2] \). Just to fix notations, let \( P \) be the simple random walk operator (i.e. \( (Pf)(x) = \frac{1}{d} \sum_{y \sim x} f(y) \)) where \( \sim \) denotes the neighbour relation) on the graph and \( \Delta = I - P \).

**Definition 3.1.** Assume the graph \( G = (V, E) \) is finite. The \( p \)-spectral gap of \( \Delta \) is the largest constant \( \lambda_p \) in

\[
\sum_{x \in V} f(x) = 0 \implies \|\Delta^{-1} f\|_{\ell^p V} \leq \lambda_p^{-1} \|f\|_{\ell^p V}
\]

The \( p \)-conductance constant is the largest constant \( \kappa_p \) in

\[
\sum_{x \in V} f(x) = 0 \implies \|\nabla f\|_{\ell^p E} \geq \kappa_p \|f\|_{\ell^p V}
\]

The aim of this subsection is to show the various inequalities between these constants.

If \( p = 2 \), \( \lambda_2 \) would be the first non-zero eigenvalue of \( \Delta \). By decomposing a function into its level set, one can prove that \( \kappa_1 \) is the usual isoperimetric (or conductance or Cheeger) constant. Also

\( (3.2) \quad \kappa_2^2 = d \lambda_2 \)
and the classical result relating isoperimetry to the 2-spectral gap is

\begin{equation}
\frac{\kappa^2}{2d^2} \leq \lambda_2 \leq \frac{2\kappa_1}{d}.
\end{equation}

See (among many possibilities) Mohar [26].

**Lemma 3.4.** Let \( \frac{1}{p'} = 1 - \frac{1}{p} \) and \( \bar{p} = \max\{p', p\} \). Then \( \lambda_p \geq \frac{2}{\bar{p}} \lambda_2 \).

**Proof.** The easy bound on \( \lambda_p \) follows by observing that (for functions \( f \) with \( \sum_{x \in V} f(x) = 0 \)) \( \Delta^{-1} = \sum_{n \geq 0} P^n \). Then, one has (again restricting to the space of functions with zero mean)

\[ \|P\|_{\ell^2 \to \ell^2} = 1 - \lambda_2 < 1 \text{ while } \|P\|_{\ell^1 \to \ell^1} \leq 1 \text{ and } \|P\|_{\ell^\infty \to \ell^\infty} \leq 1 \]

So that, by Riesz-Thorin interpolation, \( \|P\|_{\ell^p \to \ell^p} < 1 \) for any \( p \in (1, \infty) \). This suffices to see that the above series converges. More precisely, this gives

\[ \frac{1}{\lambda_p} \leq \frac{1}{1 - (1 - \lambda_2)^2/\bar{p}} \leq \frac{\bar{p}}{2\lambda_2}, \]

where the last inequality follows by Taylor-Lagrange.

**Lemma 3.5.** \( \kappa_p \geq \frac{d^{1/p}}{2} \lambda_p \)

**Proof.** The implication “\( \sum_{x \in V} f(x) = 0 \) \( \implies \|\Delta^{-1} f\|_{\ell^p V} \leq \lambda_p^{-1} \|f\|_{\ell^p V} \)” is equivalent (letting \( f = \Delta g \)) to “\( \sum_{x \in V} g(x) = 0 \) \( \implies \lambda_p \|g\|_{\ell^p V} \leq \|\Delta g\|_{\ell^p V} \)” Since \( \Delta = \frac{\bar{p}}{2} \nabla^* \nabla \) and \( \|\nabla^*\|_{\ell^p V \to \ell^p V} = \|\nabla\|_{\ell^p V \to \ell^{p'}} \leq 2d^{1/p'} \), one gets

\[ \sum_x g(x) = 0 \implies \lambda_p \|g\|_{\ell^p} \leq \frac{2}{d^{1/p}} \|\nabla g\|_{\ell^p} \]

The next inequality is probably one of the easiest.

**Lemma 3.6.** \( 2^{p-1} \kappa_1 \geq \kappa_p \)

Combined with (3.2) it also gives a proof of a part of (3.3).

**Proof.** Let \( F \subset V \) with \( |F| \leq |V|/2 \) and take \( f = |F^c| \mathbb{1}_F - |F| \mathbb{1}_{F^c} \) where \( \mathbb{1}_A \) is the characteristic function of a set \( A \). Note that \( f \) has zero sum, \( \nabla f = |V| \mathbb{1}_{\partial F} \) and \( \|f\|_{\ell^p V} = \left( |F^c|^p |F| + |F|^p |F^c| \right)^{1/p} \). Applying this to the inequality for \( \kappa_p \) yields

\[ \|\nabla\|_{\ell^p V \to \ell^p V} \]

or \( \frac{\|\nabla\|_{\ell^p V \to \ell^p V}}{\|\| \} \leq \kappa_p \left( |F^c|^p |F| + |F|^p |F^c| \right)^{1/p}. \) Since \( |F| \leq |F^c| \), the right-hand side can bounded by finding the minimum of \( x \mapsto \frac{x^{p-1} + 1}{(x+1)^p} \) for \( x \in [0, 1] \). One then gets that \( \frac{\|\nabla\|_{\ell^p V \to \ell^p V}}{\|\| \}} \geq \kappa_p^{p-1} |F|^{p-1} \]. By optimising on \( F \), one gets \( \kappa_1 \geq \kappa_p |F|^{p-1} \).

Lastly, the upcoming inequality goes back to Matoušek [25, Proposition 3]. The upcoming lemma uses the same proof (with the obvious extension to \( p \leq 2 \) and checking the constants).
Lemma 3.7. Assume \( \sum_{x \in V} f(x) = 0 \). Then \( \|\nabla f\|_p \geq (\frac{2}{7})^{1/p'} \min(\frac{1}{2}, \frac{1}{p}) \kappa_1 \|f\|_p \). In particular, \( \kappa_p \geq (\frac{2}{7})^{(p-1)/p} \max(\frac{1}{4}, p) \kappa_1 \).

Proof. By applying another variant of the inequality for \( \kappa_1 \) (see Matoušek [25, Lemma 2] or Lovász [23, Ex.11.30 on p.83, p.144 and p.472]) to \( f^p \), one gets

\[
\kappa_1 \|f\|^p_p \leq \|\nabla (f^p)\|_1.
\]

Because \( |x^p - y^p| \leq \max(1, \frac{p}{2}) |x - y| |x^{p-1} + y^{p-1}| \leq \max(1, \frac{p}{2}) |x - y| (|x|^{p-1} + |y|^{p-1}) \), one has

\[
\kappa_1 \|f\|^p_p \leq \max(1, \frac{p}{2}) (\nabla f \mid \nabla^+ (|f|^{p-1}))
\]

where \( \nabla^+ : \ell^2 V \rightarrow \ell^2 E \) is defined by \( \nabla^+ \phi(x, y) := \phi(x) + \phi(y) \). By Hölder's inequality, this is \( \leq \max(1, \frac{p}{2}) \|\nabla f\|_p \|\nabla^+ (|f|^{p-1})\|_{p'} \). But now

\[
(|x|^{p-1} + |y|^{p-1})^{p'} \leq 2^{p-1} (|x|^{p'-(p-1)} + |y|^{p'-(p-1)}) = 2^{p-1} (|x|^{p} + |y|^p)
\]

so

\[
\|\nabla^+ (|f|^{p-1})\|_{p'} \leq 2^{1/p} d^{1/p'} \|f\|_p \|f\|_{p'} = 2^{1/p} d^{1/p'} \|f\|_p^{p'}.
\]

Then

\[
\kappa_1 \|f\|_p^p \leq 2^{1/p} d^{1/p'} \max(1, \frac{p}{2}) \|\nabla f\|_p = (\frac{2}{7})^{1/p'} \max(2, p) \|\nabla f\|_p.
\]

As a summary:

**Theorem 3.8.**

1) \( 2^{p-1} \kappa_1 \geq \kappa_p^p \)

2) \( \kappa_2 = d \lambda_2 \)

3) \( \max \{2, p\} d^{\frac{p-1}{p}} \kappa_p \geq 2^{(p-1)/p} \kappa_1 \)

4) \( \kappa_p \geq d^{1/p} \lambda_p \)

5) \( \bar{p} \lambda_p \geq 2 \lambda_2 \)

6) \( 4d \kappa_1 \geq 2d^2 \lambda_2 \geq \kappa_1^2 \)

In particular, the \( p \)-spectral gap is bounded (above and below) by functions of \( \lambda_2 \) and \( \kappa_1 \):

\[
\frac{\kappa_1^2}{d^2 \bar{p}} \leq \frac{2 \lambda_2}{\bar{p}} \leq \lambda_p \leq \frac{2 \kappa_1}{d^{1/p}} \leq \frac{2^{(2p-1)/p} \kappa_1^{1/p}}{d^{1/p}} \leq \frac{2^{(4p-1)/2p} \lambda_2^{1/2p}}{}.
\]

### 3.2 The Laplacian in \( \ell^1 \)

This subsection only deals with connected infinite graphs of bounded degree. Kesten [20] showed that a [finitely generated] group has \( \text{IS}_\omega \) (i.e. is non-amenable) if and only if the Laplacian \( \Delta : \ell^2 V \rightarrow \ell^2 V \) is invertible (in any Cayley graph). This also holds for graphs (see e.g. Woess [33, 10.3 Theorem]). Using the same interpolation trick as in Lemma 3.4, one can then show that the same is true for \( \Delta : \ell^p V \rightarrow \ell^p V \) as long as \( p \in ]1, \infty] \).

Let \( c_0 V \) (the closure in \( \ell^\infty \)-norm of the finitely supported function). Let us start by recalling the image and kernel of the Laplacian. Sometimes \( \text{Im} \chi \Delta \) (likewise for ker) will be used to denote the image of \( \Delta : X \rightarrow X \).

**Proposition 3.9.** Assume \( G \) is an infinite connected graph of bounded degree.

1. In \( \ell^p V \) (for \( 1 \leq p < \infty \)) and \( c_0 V \), ker \( \Delta = \{0\} \).
2. In $\ell^\infty$, $\ker \Delta \supset \{ r \mathbb{1}_V \mid r \in \mathbb{R} \}$ where $\mathbb{1}_V$ is the constant function.
3. $\Im \rho_1 \Delta \subset \ell_0^1 V = \{ f \in \ell^1 V \mid \sum_{v \in V} f(v) = 0 \}$.
4. $\Im \rho_1 \Delta = \ell_0^1 V$ if and only if there are no non-constant bounded harmonic functions.
5. In $\ell^0$ and $\ell^\infty$, $\Im \Delta$ is weak* dense.
6. In $\ell^p V$ (for $1 < p < \infty$) and $\ell_0^1 V$, $\Im \Delta$ is dense.
7. Let $X = \ell_0^1 V, \ell^\infty V$ or $\ell_0^0 V$. There are sequences $f_n \in X$, so that $\|\Delta f_n\|_X \to 0$.
8. In $\ell^1, \ell^\infty$ and $\ell_0^0$, $\Delta$ has no bounded inverse and the image is not closed.

Proof. The first point is a consequence of the maximum principle. Harmonic functions (i.e. elements of $\ker \Delta$) in $\ell_0^1 V$ tend to 0 at infinity. By the maximum principle, they are 0 everywhere. The second is trivial: constant functions are in $\ell^\infty V$ so the Laplacian has a kernel.

A consequence of 2 and of $\Delta^* = \Delta$ is 3: since $\Im \rho_1 \Delta = (\ker \Delta)^\perp$ (this is proven as in Lemma 2.9) and $\ell_0^1 V$ is the annihilator of the constant function, one gets the conclusion. The same considerations yield also 4: the existence of other elements in $\ker \Delta$ makes the annihilator of $\Delta \ell_0^1 V$ larger.

The fifth follows from $\Im \rho_1 \Delta^* = (\ker_{\ell_0^1} \Delta)^\perp, \Im \ell_0^1 \Delta^* = (\ker_{\ell_0^1} \Delta)^\perp$ and the first point. Likewise for 6.

For 7 with $X = \ell_0^1 V$, consider some root $\phi \in V$ and the balls $B_n$ centred at $\phi$. Let $f_n = \sum_{i=0}^n \frac{1}{n+1} \mathbb{1}_{B_i}$. Then $f_n$ is finitely supported, $\|f_n\|_{\ell^\infty} = f_n(\phi) = 1$ and $\|\nabla f_n\|_{\ell^\infty} \leq \frac{1}{n+1}$ (so, in particular $\Delta f_n \overset{\ell^\infty}{\to} 0$).

When $X = \ell^\infty V$, the same sequence works for the proof 7. One do need to be careful with the kernel of $\Delta$. Namely, the norm of $g_n$ in the quotient space $\ell^\infty V / \ker \Delta$ should not tend to 0. Elements of $\ker \Delta$ are never 0 at infinity. Indeed, if the value of such an element is non-zero at $\phi$, then it must take this value at infinity. Since the $g_n$ are 0 at infinity (they are in $\ell_0^1 V$), one gets that $\|g_n\|_{\ell^\infty V / \ker \Delta} \geq 1/2$.

The case $\ell_0^1$ in 7 is done here by a sequence which is reminiscent of Green’s kernel. Let $f_n = \frac{1}{n+1} \sum_{i=0}^n P^n \delta_x$ where $P$ is the random walk operator. Since $P^n \delta_x$ are the [positive] probability distributions of the random walk starting at $x$, $\|f_n\|_{\ell_0^1} = 1$. Now $\Delta f_n = (I - P) f_n = \frac{1}{n+1} (I - P^{n+1}) \delta_x = \frac{1}{n+1} (\delta_x - P^{n+1} \delta_x)$. By linearity of the norm and the triangle inequality, $\|\Delta f_n\|_{\ell_0^1} \leq \frac{2}{n+1}$. This gives the claim.

The last point 8 is a direct consequence of 1 and 7. 

It is well established that $\Im \Delta$ is closed in $\ell^p$ (for $1 < p < \infty$) if and only if the graph is non-amenable (i.e. has $\mathbb{S}_\omega$; see for example Lohoue [22]).

A much more precise statement of Proposition 3.9.4 (due to V. Kaimanovich) will be interesting for our upcoming purposes. In order to shorten the notation, let $D_1 = \Delta \ell_0^1 V$ be the closure of the image of the Laplacian. Also, recall that a graph is called Liouville if it has no non-constant harmonic functions.

The formulation which will be interesting here is a characterisation of the Liouville property in terms of exit distributions. For $A \subset V$, denote by $\delta A$ the set of vertices in $A^c$ which are neighbours to some vertex of $A$. Given a finite connected subset $A$ and a vertex $v \in A$, define the exit probability $\text{ex}_v^A : \delta A \to [0, 1]$ by $\text{ex}_v^A(x)$ is the probability that a simple random walk starting at $v$ lands in $x$ the first time it exits the set $A$. 

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It turns out there is another way of obtaining this probability measure. Take $P_A$ to be the random walk operator “stopping outside $A$” (it’s a Markovian operator). It is defined on the Dirac mass basis of $\ell^1 V$ as follows: $P_A \delta_x = \frac{1}{|A|} \mathbb{1}_{\{y \mid y \sim x\}}$ if $x \in A$ and $P_A \delta_x = \delta_x$ if $x \notin A$. In other words, the random walker only walks as long as it is in $A$ and stays where he is once he has left $A$.

The author learned about the following result on Math Overflow, thanks to V. Beffara (see [2]). This theorem is a particular case of a result of V. Kaimanovich (see [19, Theorem 2.6])

**Theorem 3.10.** $G$ is Liouville if and only if the following holds. For any increasing sequence of finite connected sets $A_n$ so that $\cup A_n = V$ and any vertices $v, w \in V$,

$$
\lim_{n \to \infty} \|\exp_{A_n} - \exp_{A_n}^{A}\|_{\ell^1} = 0.
$$

Note that the first few terms of the sequence might not be defined, but there is a $N$ so that, for $n > N$, they all are defined. Also one could replace the $\lim$ by a $\liminf$ without changing anything.

### 3.3 Separation, relative isoperimetry and geometry of optimal sets

This section is a short review of following question: given an infinite graph with a rather large isoperimetry, does the same holds for the subgraphs induced on its subsets? As shown in [6], given reasonable upper and lower bounds on the isoperimetric ratio function one can get lower bounds on the Cheeger constant on some finite subsets.

The **isoperimetric ratio function** is the function $F : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ defined to be the largest function so that $F(|F|) \leq |\partial F|$, i.e. $F(x) = \inf\{|\partial F| \mid |F| = x\}$. The **isoperimetric ratio function** is the function $G : \mathbb{Z}_{>0} \rightarrow \mathbb{R}_{>0}$ defined by $G(x) := x^{-1}F(x) = \inf\{|\partial F| \mid |F| = x\}$. Be aware that some texts use “isoperimetric ratio function” in place of “isoperimetric ratio function” The (decreasing) function $G^{-1}(x) := \inf\{|\partial F| \mid |F| \leq x\}$ derived from $G$ will also come in handy.

Looking at the decreasing function $G^{-1}$ instead of $G$ is not much of thing. Recall that $F$ is subadditive: $F(a + b) \leq F(a) + F(b)$. Hence, up to a minor change (taking the concave hull of a subadditive function changes it by at most a factor 2), $F$ is concave. If $f$ is concave, then $x^{-1}f(x)$ is decreasing.

A graph has a...

- strong isoperimetric ratio ($\text{IS}_\omega$) if $\exists K > 0$ so that $G(x) \geq K$.
- $d$-dimensional isoperimetric ratio ($\text{IS}_d$) if $\exists K > 0$ so that $G(x) \geq \frac{K}{x^{d-1}}$.
- $\nu$-intermediate isoperimetric ratio ($\text{IS}_\nu$) if $\exists K > 0$ so that $G(x) \geq \frac{K}{(1 + \ln x)^{1/\nu}}$.

$\text{IS}_\omega$ is equivalent to non-amenability. Typical graphs with $\text{IS}_d$ are Cayley graph of $\mathbb{Z}^d$. For Cayley graphs of groups, only those of virtually nilpotent groups do not have $\text{IS}_{d+1}$ for some $d$ (this $d$ is related to the growth of balls, but not to the dimension of the continuous counterpart of the group). There are no current example of a group which has none of the above three properties, i.e. all known groups which are neither amenable nor virtually nilpotent satisfy some $\text{IS}_\nu$. 

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On a finite graph $(X, E)$, the functions $\mathcal{F}$ and $\mathcal{G}$ will only be considered to be defined on the range $0 \leq x \leq |X|/2$. For example, the isoperimetric constant of a finite graph (also called Cheeger constant) is $\kappa_1 := \min \{ \mathcal{G}(x) \mid 0 < x \leq |X|/2 \}$.

Note that, when speaking of isoperimetry, some authors consider rather: the (increasing) function $\mathcal{F}'(x) = \inf \{ |\partial F| \mid |F| \geq x \}$, the (increasing) function $J'(x) = \mathcal{G}'(x)^{-1}$ or $J = \mathcal{G}^{-1}$.

In [6, §3 and §4] the authors develop a machinery to obtain lower bounds on the Cheeger constant of some sets. More precisely, if lower and upper bounds on $\mathcal{G}$ are available, then there are infinitely many finite sets $F$ for which $\kappa_1$ is bounded below. The methods cover quite a wide range of groups, so let us focus on the strictly necessary set-up and on a particular family.

The sets $F$ for which a lower bound on $\kappa_1(F)$ is given are [a subset of] **optimal sets**. A set is said to be optimal (w.r.t. to isoperimetry) if $\mathcal{G}(|F|) = |\partial F|/|F|$. In other words, for any other set $F'$ with $|F'| \leq |F|$ one has $|\partial F'|/|F'| \geq |\partial F|/|F|$. Although optimal sets have essentially not been explicitly given for most graphs (see [6, Question 6.1]), there are many of them in Cayley graphs.

Indeed, if $n$ is an integer so that there is an optimal set with cardinality $n$, then the next such integer is at most $2n$ (since taking two disjoint copies of an optimal set yield a set with the same isoperimetric ratio). As $n = 1$ is an integer with an optimal set, one gets that there are many optimal sets.

The results of [6, §3 and §4] will be used mostly to show that, when the isoperimetric ratio function of a [Cayley graph of a] group satisfies $K_1(\log n)^{-b} \leq \mathcal{G}(n) \leq K_2(\log n)^{-a}$ (where $a < b \in \mathbb{R}_{>0}$ and $K_1, K_2 \in \mathbb{R}_{>0}$), then for infinitely many optimal sets $F$, $\kappa_1(F) \geq K_3\mathcal{G}(|F|)/\log(|F|)$ (for some $K_3 \in \mathbb{R}_{>0}$).

Such groups are nice, in the sense that both $\kappa_1(F)$ and $\mathcal{G}(|F|)$ have similar decay (for some optimal sets). There are however groups (like $\mathbb{C}_2 \wr (\mathbb{C}_2 \wr \mathbb{Z})$) where $\kappa_1(F)$ decays much faster than $\mathcal{G}(|F|)$.

Let us also discuss two further quantities which will be relevant in this context. Given a subset $F$ of a graph $G$, define its inradius to be $\text{inrad}(F) = \max \{ r \mid \exists x \in F \text{ s.t. } B_x(r) \subset F \}$ (here $B_x(r)$ is the ball of radius $r$ around $x$). The diameter $\delta(F)$ of a connected set $F$ is the maximal combinatorial distance of the graph induced on $F$.

For the following lemma the minimal and maximal volume growths $(f_v(n) := \inf_{x \in G} |B_n(x)|$ and $f_V(n) := \sup_{x \in G} |B_n(x)|$ respectively) will come in handy. Note that both functions are increasing.

**Lemma 3.11.** Assume $F$ is a connected subset of $G$. Then $\delta(F) \geq f_v^{-1}(|F|)$ and $\text{inrad}(F) \leq f_v^{-1}(|F|)$.

In particular, if $e^{n^{1/3}} \leq f_v(n) \leq e^{n}$ then $\delta(F) \geq (\log |F|)^{1/3}$ and $\text{inrad}(F) \leq (\log |F|)^{1/2}$.

**Proof.** Note that if the graph induced on the set $F$ is connected, $r = \text{inrad}(F)$ is its inradius and $\delta$ is its diameter, then $|B_r(x)| \leq |F| \leq |B_\delta(x)|$ for some $x \in F$. Hence $f_v(r) \leq |F| \leq f_V(\delta)$ and the conclusion follows. \qed
Since any infinite connected graph contains a geodesic ray, there is no better generic bound than \( \delta(F) \leq |F| \) for a connected set \( F \). For optimal sets, one can do better:

**Lemma 3.12.** Assume \( G \) is the Cayley graph of a group and \( K_1(\log n)^{-b} \leq \mathcal{G}(n) \leq K_2(\log n)^{-a} \) (where \( a \leq b \in \mathbb{R}_{>0} \) and \( K_1, K_2 \in \mathbb{R}_{>0} \). Then there are infinitely optimal sets \( F \) such that \( \delta(F) \leq K \frac{(\log |F|)^2}{\mathcal{V}(|F|)} \leq K' (\log |F|)^{2+b} \) where \( K \) and \( K' \) are constants depending only on the isoperimetric ratio function and the degree of \( G \).

*Proof.* Using arguments from [6, Proposition 5.5], one has that for a connected set \( F \) (whose diameter is \( \geq 3 \)) the diameter of \( F \) is bounded by \( \delta \leq \frac{3k \log |F|}{\kappa_1(F) \log 2} \) (where \( k \) is the maximal degree of the graph induced on \( F \)).

Using [6, §3 and §4] one gets when the isoperimetric ratio function of a [Cayley graph of a group satisfies \( K_1(\log n)^{-b} \leq \mathcal{G}(n) \leq K_2(\log n)^{-a} \) (where \( a < b \in \mathbb{R}_{>0} \) and \( K_1, K_2 \in \mathbb{R}_{>0} \), then for infinitely many optimal sets \( F \), \( \kappa_1(F) \geq \mathcal{G}(|F|)/\log(|F|) \).

Putting these together yield the conclusion. \( \square \)

Note that if the Cayley graph has a lower bound on the size of balls of the form \( |B_r(x)| \geq K_3 e^{K_4 r^a} \), then \( b \) can be taken to be \( \frac{1}{2} \).

Also if the separation profile is known, one gets a similar lower bounds on the sets which are optimal with respect to the separation profile. There is however no reason to believe that these sets are the optimal sets from the isoperimetric perspective.

There are also similar bounds for the inradius.

**Lemma 3.13.** Assume \( F \) is a finite optimal set, then \( \text{inrad}(F) \geq f_V^{-1}\left(\frac{1}{\mathcal{G}(|F|)}\right) \).

*Proof.* Let \( r \) be the inradius of \( F \), then a ball of radius \( r \) has at most \( f_V(r) \) elements. Hence \( |F| \leq f_V(r) |\partial F| \). The conclusion follows for optimal sets since \( \frac{|F|}{|\partial F|} = \mathcal{G}(|F|)^{-1} \). \( \square \)

In particular, when the growth is exponential (the “worse case” for a graph of bounded degree) \( \text{inrad}(F) \gg -\log \mathcal{G}(|F|) \).

Note that the lower bound \( \text{inrad}(F) \gg -\log \mathcal{G}(|F|) \) is sharp for some graphs. An example is to consider the graph given by attaching the leaves of a binary tree to \( \mathbb{Z}_{\geq 0} \).

This graph has an isoperimetric ratio function which satisfies \( \mathcal{G}(2^n - 1) = 1/(2^n - 1) \) and the corresponding optimal sets are easy to find. It can be turned into a regular graph (by taking three copies of it and gluing the leaves together).

The author expects a much better lower bound for vertex-transitive graphs. Indeed, for groups of polynomial growth one has \( \mathcal{G}(n) < n^{-1/d} \) and \( f_V(n) = n^d \); so the above lemma yields only \( n^{1/d^2} \) (whereas \( n^{1/d} \) is expected).

Since a better bounds might be possible in groups, a property, from which such a bound would follow, will be introduced. Say a graph satisfies a radial isoperimetric inequality if, there exists constants \( K \geq 1 \) and \( k \geq 1 \) so that, for any finite set \( A \) whose complement has no finite connected components,

\[
(3.14) \quad K |\partial A| (1 + \text{inrad}(A))^k \geq |A|
\]

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Note this inequality is weaker than a strong isoperimetric inequality (since \( k = 0 \) would suffice in that case). Also graphs of polynomial growth satisfy this inequality trivially (with \( K \) and \( k \) such that \(|B(n)| \leq Kn^k\)).

For Cayley graphs, the inequality might be true with \( k = 1 \) independently of the group. Note that if one replaces the inradius by the diameter, then the inequality holds for \( K = k = 1 \) (see Žuk [34]).

**Lemma 3.15.** Assume \( G \) is a one-ended Cayley graph where (3.14) holds for some \( k, K \geq 1 \). Then for any optimal set \( F \), \( \text{inrad}(F) \geq \frac{1}{K||f||_{nu}} - 1 \).

**Proof.** The conclusion follows for optimal sets since \( \frac{|F|}{|\nu(F)|} = \frac{1}{K||f||_{nu}} \).

Lastly, let us give a bound on the average distance to the boundary: given \( F \) a finite connected set, let \( \bar{r}(F) = \frac{1}{|F|} \sum_{x \in F} d(x, \partial F) \).

**Lemma 3.16.** Assume \( G \) satisfies IIS_\nu (with constant \( k \)), then \( \bar{r}(F) \leq k(\ln |F|)^{1/\nu} \).

**Proof.** Consider the function \( f(x) = d(x, \partial F) \) and let \( f_0 = f/\|f\|_1 \). Note that \( \|\nabla f\|_1 \leq k|F| \) (since the value of \( f \) can change by at most 1 along any edge). Hence \( \|\nabla f_0\|_1 \leq k/\bar{r}(F) \). Then using Proposition 3.17, one gets \( H(f_0)^{-1/\nu} \leq \|\nabla f_0\|_1 \leq k/\bar{r}(F) \). This implies \( \bar{r}(F) \leq kH(f_0)^{1/\nu} \). Using the naive bound \( H(f_0) \leq \ln |F| \) yields the claim.

A similar argument with the \( d \)-dimensional isoperimetric ratio function IS_\nu yields \( \bar{r}(F) \leq k|F|^{1/d} \)

### 3.4 Isoperimetry and entropy

In this subsection, only infinite graphs are considered. More precisely, the focus will be on graphs which satisfy IIS_\nu (see the beginning of §3.3). The reason to consider such a profile is that groups where balls grow at least \( |B_n| \geq K_1\exp(K_2n^\nu) \) will satisfy such a profile (here \( \nu \in [0, 1] \)), see Coulhon & Saloff-Coste [5, Théorème 1].

The aim of this section is to express this isoperimetric ratio function directly in term of the entropy. Recall if \( X \) is a countable set and \( f : X \rightarrow [0, 1] \) is such that \( \|f\|_{1_X} = 1 \), then the entropy of \( f \) is \( H(f) = -\sum_{x \in X} f(x) \ln f(x) \).

**Proposition 3.17.** Assume \( G = (V, E) \) has IIS_\nu with constant \( K \). Then, for any \( f : V \rightarrow [0, 1] \) is finitely supported with \( \|f\|_{1_V} = 1 \) and \( \|f\|_{\infty} \leq \exp(-1) \),

\[
\|\nabla f\|_1 \geq \frac{K\nu}{\nu + 1}H(f)^{-1/\nu}.
\]

Note that except for the constant \( \kappa \) which became \( \frac{1}{\nu + 1} \), this is equivalent to the original isoperimetric ratio function (up to a change in constant). Indeed, taking \( f = 1_F/|F| \) (and under the mild assumption that \( |F| \geq 3 \)) this reads

\[
\frac{|\partial F|}{|F|} \geq K'(\ln F)^{-1/\nu}.
\]
The following proof can be seen as an adaptation of the Sobolev inequality obtained from IS (see e.g. Woess [33, 4.3 Proposition]). In Coulhon & Saloff-Coste [5], it is mentioned that IS leads to some inequality on the Orlicz norm. The motivation for the current formulation is to get insight on the growth of entropy; see Question 5.6.

Proof. Let \( \{f > t\} = \{x \mid f(x) > t\} \) and by abuse of notation its cardinality too (likewise for \( \partial f > t \)). By assumption, \( f \) is not a Dirac mass. Note that
\[
\|\nabla f\|_{L^1 V} = \sum_x \sum_{y \sim x, f(y) > f(x)} f(y) - f(x) \\
= \sum_x \sum_{y \sim x, f(y) > f(x)} \int_0^{\infty} \mathbb{1}_{[f(x), f(y)]}(t) dt \\
= \int_0^{\infty} \sum_x \sum_{y \sim x, f(y) > f(x)} 1 dt \\
= \int_0^{\infty} \partial f > t dt \\
\geq \int_0^{\infty} \frac{[f > t] / (\ln[f > t])^{1/v}}{dt} \\
\geq \int_0^{1} \frac{[f > t] / (\ln t)^{1/v}}{dt} \\
\text{where IS is the use of the isoperimetric ratio function and MI is (Markov's inequality)} \\
[f > t] \leq \|f\|_1/t = 1/t \text{ (so } \ln[f > t] \leq -\ln t \text{ and } (\ln[f > t])^{-1/v} \geq (-\ln t)^{-1/v}. \)

Since \( \|f\|_\infty \leq e^{-1} \), for \( t \leq e^{-1} \), one has \( -\ln t \geq 1 \) so \( \frac{1+1/v}{(-\ln t)^{1/v}} \geq \frac{1}{(-\ln t)^{1/v}} + \frac{1/v}{(-\ln t)^{1+1/v}} \).

Hence
\[
\|\nabla f\|_1 \geq \int_0^{1} \frac{[f > t] / (\ln t)^{1/v}}{dt} \geq \frac{1}{v+1} \int_0^{1} \frac{1}{(-\ln t)^{1/v}} + \frac{1/v}{(-\ln t)^{1+1/v}} [f > t] \\
\text{Or}
\[
\|\nabla f\|_1 \geq \frac{v}{v+1} \int_0^{1} H(t)[f > t] \\
\text{where } H(t) = t / (\ln t)^{1/v}.
\]

But notice that, if \( t_i \) are the values of \( f \)
\[
\sum_x f(x) / (-\ln f(x))^{1/v} = \sum_{i=1}^{n} \frac{t_i}{(-\ln t_i)^{1/v}} ([f > t_{i-1}] - [f > t_i]) \\
= \sum_{i=0}^{n-1} \frac{t_{i+1} - t_i}{(-\ln t_{i+1})^{1/v} - (-\ln t_i)^{1/v}} [f > t_i] \\
= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} H(t)[f > t_i] dt \\
= \int_0^{1} H(t)[f > t] dt
\]

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So
\[
\int_0^1 \mathcal{H}(t) \left[ f > t \right] = \sum_x f(x) \left( - \ln f(x) \right)^{-1/v} \geq \left( \sum_x f(x)(-\ln f(x)) \right)^{-1/v} = H(f)^{-1/v}
\]
where J stands for Jensen’s inequality. Indeed, as \( \phi(x) = x^{-1/v} \) is convex, \( \sum p_x \phi(a_x) \geq \phi(\sum p_x a_x) \). Here, \( p_x = f(x) \) and \( a_x = -\ln f(x) \) were used. Putting these inequalities together gives
\[
\|\nabla f\|_1 \geq \frac{v}{v+1} KH(f)^{-1/v}
\]
as desired. \( \square \)

**Remark 3.18.** Let \( P^n \) be the time \( n \) distribution of a random walk starting at the identity in a Cayley graph and assume \( P^1(e) \geq 1/2 \). One of the motivations for Proposition 3.17 was to obtain a good lower bound on the entropy in terms of volume growth via isoperimetric ratio functions (see Question 5.6). Indeed, a typical trick is to go as follows
\[
H(P^{n+1}) - H(P^n) \geq \langle \nabla P^n, \nabla \ln P^n \rangle_{LS} \geq \|\nabla (|P^n|^{1/2})\|^2_{L^2} \geq \|\nabla P^n\|^2_{L^1}/\|P^n\|^2_{L^1}
\]
where CS stands for the Cauchy-Schwartz inequality and LS for a “log-Sobolev” trick (this inequality is a commonplace when one wishes to prove log-Sobolev estimates). Since \( P^n \) is a measure, this reads
\[
H(P^{n+1}) - H(P^n) \geq \|\nabla P^n\|^2_{L^1} \geq H(P^n)^{-2/v}
\]
(The first inequality can be made rigorous, see Erschler & Karlsson [8]) However, this only gives \( H(P^n) \geq n^{v/(2+v)} \). This estimate can be directly deduced from the fact that \( H(P^0) \geq -\ln \|P^n\|^2_{L^1} \) and estimates on the probability of return. \( \square \)

Proposition 3.17 still allows to prove interesting lower bounds on \( \|\nabla P^n\|_{L^1} \). Let us start by the \( IS_d \) case.

**Corollary 3.19.** Assume \( G \) is a Cayley graph of a group. If \( G \) has \( IS_d \) (i.e. the group has polynomial growth of degree \( d \)), then \( \|\nabla P^n\|_{L^1} \geq K^n n^{-1/2} \) when \( n \) is even.

**Proof.** For \( d \)-dimensional isoperimetric ratio function and \( d \geq 2 \), one has \( \|\nabla P^n\|_1 \geq \|P^n\|_{d/(d-1)} =: N_d(n) \) (see [33, 4.3 Proposition]). By interpolation, \( \|P^n\|_2 \leq \|P^n\|_{d/(d-1)}^{1-t} \|P^n\|_{\infty}^{t} \)
where \( t = \frac{d-1}{d} t + \frac{1}{\infty} (1-t) \), i.e. \( t = \frac{d}{2(d-1)} \) and, consequently, \( 1-t = \frac{d-2}{2(d-1)} \). Since \( \|P^n\|_2 = P^{2n}(e) = K_1 n^{-d/2} = \|P^{2n}\|_{\infty} \), this means that (for \( n \) even)
\[
2^{-(d-2)/d} K_1^{d-2-2/d} n^{-1/2} \leq N_d(n)
\]
Hence, for \( n \) even, \( \|\nabla P^n\|_1 \geq K_2 n^{-1/2} \). \( \square \)

Recall that \( \mathbb{E}|P^n| = \sum_{x \in V} d(x, o) P^n(x) \) is the speed of the random walk (\( d \) is the combinatorial distance in the Cayley graph and \( o \) is the neutral element). Let \( \overline{\beta} := \mathbb{E}|P^n| = \limsup_{n \to \infty} \frac{\ln \mathbb{E}|P^n|}{\ln n} \).
Corollary 3.20. Assume $G$ is a Cayley graph of a group. If $G$ has IIS$_\nu$ (e.g. $|B_n| \geq K_1 \exp(K_2 n^\nu)$) and $|B_n| \leq K_3 \exp(K_4 n^\nu)$, then $\|\nabla P^n\|_1 \geq K n^{-\frac{\nu}{\nu+1}}$.

In particular, if the group has exponential growth, then $\|\nabla P^n\|_1 \geq K n^{-\frac{\nu}{\nu+1}}$.

Proof. If $G$ has IIS$_\nu$ then $\|\nabla P^n\|_1 \geq H(P^n)^{-1/\nu}$ by Proposition 3.17. Furthermore, $H(P^n) \leq K_5 f_t(\mathbb{E}[P^n])$ where $f$ is $n \rightarrow \ln |B_n|$ extended in a piecewise linear fashion (see [13, Lemma 4.4], or also Erschler [9, Lemma 5.1]). So $H(P^n) \leq K_6 n^{\frac{\nu}{\nu+1}}$ which in turn implies the claim.

For groups of exponential growth, the bound in Corollary 3.20 is somehow counter intuitive. Indeed, when the speed is linear (which implies $\overline{\nu} = 1$), the group is not Liouville and one can use other methods to show that $\|\nabla P^n\|_{\ell_1}$ is bounded below by an absolute constant.

In view of Shalom & Tao [32, §6] it might be good to have bounds on $\|\nabla g_n\|_1$ where $g_n = \frac{1}{n} \sum_{i=0}^{n-1} P_i$ (it is a partial sum of Green’s kernel renormalised to be a probability measure).

Corollary 3.21. Assume $G$ is a Cayley graph of a group and let $g_n$ be as above. If $G$ has IIS$_\nu$ (e.g. $|B_n| \geq K_1 \exp(K_2 n^\nu)$), then $\|\nabla g_n\|_1 \geq K H(P^n)^{-1/\nu}$.

Proof. As in the previous corollary, the main point is to get an upper bound on $H(g_n)$ (since $g_n \|_1 = 1$). Using the formula for convex combinations of measure: $H(g_n) = \sum_i \frac{1}{n} H(P_i) + \sum_i \frac{1}{n} D(P_i || P^{n-1})$ where $D$ is the divergence. Bound the first sum using $H(P_i) \leq H(P^n)$ and the second with $D(P_i || P^{n-1}) \leq D(P_i || P^n) = - \log P^{n-1}(e)$ together with $- \log P^{n-1}(e) \leq H(P^n)$ and the fact that $- \log P^{2k}(e)$ is increasing (see e.g. [13, §4]). This implies $H(g_n) \leq H(P^n)$. In combination with Proposition 3.20 this gives the desired result.

Note that the entropy grows slower than the volume when the speed of the random walk is not linear (see [13, §4] for the explicit estimates). For example, if a group has exponential growth and speed $n^{\frac{\nu}{\nu+1}}$, then $\|\nabla g_n\|_1 \geq K / n^{\frac{\nu}{\nu+1}}$. This indicates that the “optimistic” bound $\|\nabla g_n\|_1 \leq 1/n$ probably seldom holds (so that the different cases in [32, §6] are necessary).

Note that this estimate is also an important contrast to the fact that $\|\Delta g_n\|_1 \leq K / n$ (for some constant $K$ which depend on the way the Laplacian $\Delta$ is defined).

3.5 Rényi entropy and groups

The main aim of this section is to show that 3 invariants for groups (growth, return probability and entropy) are essentially instance of Rényi entropy.

For any $\mu$ probability measure the entropy is defined as

$$H(\mu) = \sum_x \mu(x)(- \ln \mu(x))$$

Denote for $q \in \mathbb{R} \setminus \{0\}$:

$$\|\mu\|_q = \left( \sum_{x: \mu(x) \neq 0} \mu(x)^q \right)^{1/q}$$

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For the limit, apply l’Hôpital’s rule:

\[ \lim_{p \to 1^\pm} -\frac{\ln \|\mu\|_p^p}{p-1} = \lim_{p \to 1^\pm} -\frac{\ln \|\mu\|_p}{p-1} \]

Another possible way of writing this is \( -\frac{\ln \|\mu\|_p^p}{p-1} = -\frac{\ln \|\mu\|_{p'}}{p'} \) where \( p' \) is the Hölder conjugate of \( p \).

The following lemma (whose proof could be left as an exercise) recalls why this has anything to do with entropy.

**Lemma 3.22.** Let \( q \in \mathbb{R} \setminus \{0, 1\} \) and \( p \in [1, \infty[ \) then

\[ (*) \quad H_q(\mu) \geq H(\mu) \geq H_p(\mu) \]

Also

\[ \lim_{p \to 1^\pm} H_p(\mu) = H(\mu), \quad \lim_{q \to 0^\pm} H_q(\mu) = \ln \|\mu\|_0, \]

\[ \lim_{q \to +\infty} \frac{\ln \|\mu\|_q}{q-1} = \ln \|\mu\|_{\infty} \quad \text{and} \quad \lim_{q \to -\infty} \frac{\ln \|\mu\|_q}{q-1} = \ln \|\mu\|_{-\infty} \]

where \( \|\mu\|_0 = |\text{supp} \mu| = \{|x \mid \mu(x) \neq 0\} \), \( \|\mu\|_{\infty} = \sup_x |\mu(x)| \) and \( \|\mu\|_{-\infty} = \left( \inf_x \{|\mu(x)| \mid \mu(x) \neq 0\} \right)^{-1} \).

**Proof.** Let \( f \) be defined by \( f = \mu^q \) for \( q \neq 0, 1 \). Then

\[ H(\mu) = \sum_x \mu(x)(-\ln f(x)^{1/q}) = \frac{1/q}{1-1/q} \sum_x \mu(x)(-\ln f(x)^{1-1/q}) \]

If \( \frac{1}{q-1} > 0 \) (i.e. \( q > 1 \)), then use convexity of \( x \to -\ln x \) to get

\[ H(\mu) = \frac{-1}{q-1} \sum_x \mu(x) \ln f(x)^{1-1/q} \geq \frac{-1}{q-1} \ln \left( \sum_x \mu(x) f(x)^{1-1/q} \right) = \frac{-1}{q-1} \ln \left( \sum_x \mu(x)^q \right) \]

Hence

\[ H(\mu) \geq H_q(\mu) \]

As desired. If \( \frac{1}{q-1} < 0 \) (i.e. \( q < 1 \) and \( q \neq 0 \)), then use concavity of \( x \to \ln x \) to get

\[ H(\mu) = \frac{-1}{q-1} \sum_x \mu(x) \ln f(x)^{1-1/q} \leq \frac{-1}{q-1} \ln \left( \sum_x \mu(x) f(x)^{1-1/q} \right) = \frac{-1}{q-1} \ln \left( \sum_x \mu(x)^q \right) \]

Which is

\[ H(\mu) \leq H_q(\mu) \]

For the limit, apply l’Hôpital’s rule:

\[ \lim_{p \to 1^\pm} \frac{-\ln \sum_x \mu(x)^p}{p-1} = \lim_{p \to 1^\pm} -\frac{\sum_x \mu(x)^p \ln \mu(x)}{\|\mu\|_{p'}^p} = \sum_x \mu(x)(-\ln \mu(x)) = H(\mu). \]

The other limits are also straightforward computations. \( \square \)
Note that (*) is stable under using the interpolation of \( p \)-norm from the 1-norm and a \( r \)-norm (with \( r > p > 1 \)). This is a convoluted way of showing that the limits as \( p \to 1^\pm \) are monotone. More precisely:

**Lemma 3.23.** If \( p < q \)

\[
H_q(f) \leq H_p(f)
\]

If further \( 1 < p < q \),

\[
H_p(f) \leq \frac{p(q-1)}{(p-1)q} H_q(f).
\]

**Proof.** Recall the Lyapounov’s interpolation inequality. First, if \( q, r \in ]-\infty, \infty[ \) and \( p = \lambda r + (1 - \lambda)q \) (with \( \lambda \in ]0, 1[ \)), using Hölder with exponent \( 1/\lambda > 1 \), one gets

\[
\int |f|^p = \int |f|^\lambda |f|^{(1-\lambda)q} \leq \left( \int |f|^r \right) ^{\lambda} \left( \int |f|^q \right) ^{1-\lambda} = \|f\|_r^\lambda \|f\|_q^{(1-\lambda)},
\]

First, let’s show the ratio is monotone on \( ]1, \infty[ \). Assume \( p = \theta 1 + (1-\theta)q \), i.e. \( \theta = \frac{q-p}{q-1} \), then

\[
\|f\|_p^\theta \leq \|f\|_q^{1-\theta} \cdot \|f\|_q^{(1-\theta)} = \|f\|_q^{1-\theta}
\]

using \( \|f\|_1 = 1 \). Since \( 1 - \theta = \frac{1-p}{q-1} \), one has

\[
-\ln \|f\|_p^\theta \geq -\ln \|f\|_q^{1-\theta} = -\ln \|f\|_q^{\frac{1-p}{q-1}} = -\ln \|f\|_q^{\frac{q-p}{q-1}}
\]

Next, let us show they are equivalent up to a constant on \( ]1, \infty[ \). For \( p < q \), \( \|f\|_p \geq \|f\|_q \) so \( -\ln \|f\|_p^\theta \geq \frac{q}{q-1} - \ln \|f\|_q^\theta \) and so, if \( p > 1 \)

\[
-\ln \|f\|_p^\theta \leq \frac{p(q-1)}{(p-1)q} - \ln \|f\|_q^\theta
\]

It turns out using this inequality for \( p < 1 \) is not interesting. A better way to finish on the interval \( ]-\infty, 1[ \), is to note that Lyapounov’s inequality still holds: \( \ln \|f\|_p^\theta \leq \frac{1-p}{1-q} - \ln \|f\|_q^\theta \) if \( p > q \). This time \( q < p < 1 \), so

\[
\frac{\ln \|f\|_p^\theta}{1-p} \leq \frac{\ln \|f\|_q^\theta}{1-q}.
\]

Random walks are a very useful tool to study countable groups. Let \( \mu = P \) be a probability measure with support on a generating set of \( \Gamma \). Let us assume that the measure is symmetric (that is \( P(s) = P(s^{-1}) \)). Assume further that \( P(\sigma) = \frac{1}{2} \) where \( \sigma \) is the identity element of \( \Gamma \) (this makes the random walk “lazy” and simplifies many of the following statements).

The distribution at time \( n \) of the random walk is \( P^n = \mu^n \) (\( \mu \) convoluted \( n \) times with itself). From this distribution, one commonly studies the two following function: \( n \mapsto H(P^n) \) (entropy at time \( n \)) and \( n \mapsto P^n(\sigma) \) (return probability at time \( n \)). It turns out that \( P^n(\sigma) = \|P^n\|_{\ell^\infty} \) and \( P^{2n}(\sigma) = \|P^n\|_{\ell^2} \).

Another important function in geometric group theory is the growth function of the group: \( n \mapsto |B_n| \) (the size of the ball of radius \( n \); so called growth function).
Remark 3.24. It is interesting to notice that all these functions are (up to a logarithm and a sign), a certain Rényi entropy, namely:

- $H_0(P^n)$ is the logarithm the growth function.
- $H_1(P^n)$ is the entropy.
- $H_2(P^n)$ is minus the logarithm of the return probability (at even times).
- $H_\infty(P^n)$ is minus the logarithm of the return probability.

This is noteworthy in view of question 5.7. The logarithms may look artificial, but the logarithm of the growth function (resp. minus the logarithm of the return probability) are subadditive functions.

4 Applications to $\mathcal{TD}^p$

4.1 Liouville graphs and direct products

Recall that Liouville is “$\mathcal{F}_\infty \cap \nabla \ell_\infty V = \{0\}$” and $\mathcal{TD}^p$ is, by definition (or an analogue of Lemma 2.2), equivalent to “$k_p \cap \{0\}$” (i.e. no non-constant harmonic functions with gradient in $\ell^p$). As $k_p \subseteq \nabla \ell^\infty V$, it is not trivial that Liouville implies $\mathcal{TD}^p$ for all $p \in [1, \infty]$. Theorem 4.5 below was already proven in [10, Theorem 1.2 or Corollary 3.14], but the following proof is more direct (and avoids $\ell^p$-cohomology; see also [14] for a streamlined proof using $\ell^p$-cohomology). An ingredient which will come in the proof is that of a transport pattern from $\xi$ to $\phi$ (where $\xi$ and $\phi$ are measures). This is a finitely supported function $\tau : E \to \mathbb{R}$ so that $\nabla^* \tau = \phi - \xi$.

Note that for any $h \in \ell^1 E$, $\nabla^* h =: \pi$ can be decomposed into two positive measures $\pi_\pm \in \ell^1 V$ so that $\pi = \pi_+ - \pi_-$. For any such $\pi = \pi_+ - \pi_-$ of finite support, let $TP(\pi)$ be the set of all (finitely supported) transport patterns with $\nabla^* \pi = \pi$.

Lemma 4.1. Let $h$ be a finitely supported function on the edges. The norm $h$ in the quotient $\ell^p E/\ell_p^+$ is

\[
\inf_{\tau \in TP(\nabla^* h)} \|\tau\|_{\ell^p} = \frac{\inf_{\tau \in TP(\nabla^* h)} \|\tau\|_{\ell^p}}{\pi_+ - \pi_-}.
\]

Proof. By definition, the closure (in $\ell^p E$) of the set $TP(\nabla^* h)$ equals $h + \ell_p$. The infimal norm of such an element is the quotient norm. \qed

In the particular case $p = 1$, the quotient norm $\ell^1 E/\ell_1^+$ is the Wasserstein distance between $p_+$ and $p_-$. Transport patterns will be a combination of [the characteristic function of oriented] geodesic paths between the support of $p_+$ and the support of $p_-$. Still for $p = 1$, the optimal transport patterns are automatically finitely supported if $p$ is.

For $p > 1$, the quotient norm $\ell^p E/\ell_p$ has nothing to do with the $p$-Wasserstein distance. For example, when $p = 2$, it expresses the energy of a current flowing between sinks and sources. The infimum is probably never realised by a finitely supported transport pattern.

Remark 4.2. It is quite important to point out that one can essentially always assume that $\|\tau\|_{\ell^\infty} \leq \|\pi_\pm\|_{\ell^1} = \frac{1}{2}\|\pi\|_{\ell^1}$. Indeed, if any transport pattern $\tau$ (from $\pi_-$ to $\pi_+$) is such that $\|\tau\|_{\ell^\infty} > \|\pi_\pm\|_{\ell^1}$ then it means some mass is transported twice along some edge. This implies that the said mass is transported along a cycle (something which is definitely not very efficient). Hence there is another transport pattern $\tau'$ where this [useless from the view of transport] cycle is avoided and $|\tau'| \leq |\tau|$ (point-wise). \qed
Lemma 4.3. Let $G$ be $D$-regular graph. If for any neighbours $v$ and $w$ there is an increasing and exhausting sequence of sets $A_n$ and a sequence of transport patterns $\tau_n$ from $\text{ex}_{v}^{A_n}$ to $\text{ex}_{w}^{A_n}$ so that $\tau_n$ tends weak* to 0 in $\ell^p$-E, then $G$ has $\Theta D^p$.

Proof. The aim is to show that $\tau_n$ tends weak* to 0 in $\ell^p$-E. By density of the Dirac masses, it is then equal to $\ell^p$-E.

To do so consider $f_n = \sum_{i\geq 0} P^n_{A_i} \nabla^* \delta_{v,w}$, $f_n$ is a function with bounded support, so $\nabla f_n \in k_p$. Note that $\nabla^* (\nabla f_n - \delta_{v,w}) = \text{ex}_{v}^{A_n} - \text{ex}_{w}^{A_n}$. By the property of the transport patterns $\tau_n$, $\nabla f_n - \delta_{v,w}$ belongs to the weak* closure of $k_p$.

Lemma 4.4. Let $G$ be $D$-regular graph with $IS_d$ for some $d > 2$ and let $p < d/2$. Then for any neighbours $v$ and $w$ and any increasing and exhausting sequence of sets $A_n$ there is a sequence of transport patterns $\tau_n$ from $\text{ex}_{v}^{A_n}$ to $\text{ex}_{w}^{A_n}$ so that $\|\tau_n\|_{\ell^p(E)}$ is bounded.

Proof. The transport pattern $\tau_n$ presented here are extremely inefficient. Construct $\tau_n$ as a concatenation of many transport patterns. The first series of transport pattern bring the mass from $\text{ex}_{v}^{A_n}$ to $\delta_v$. Note that the transport between $P^n_{A_n} \delta_v$ and $P^n_{A_n} \delta_y$ can be achieved by just making a random step (for the mass lying within $A_n$, the rest does not move): at every given vertex, split and move the mass evenly to all the neighbours. The $\ell^p$-norm of this “random step” transport is $\leq \|P^n_{A_n} \delta_y\|_{\ell^p}$.

So transport from $\delta_v$ to $\text{ex}_{w}^{A_n}$ by this method. This transport moves less mass than the transport from $P^n_{A_n} \delta_v$ to $P^{n+1}_{A_n} \delta_y$ (because one only moves what lies inside $A_n$). Hence, the $\ell^p$-norm (for the transport from $\delta_v$ to $\text{ex}_{w}^{A_n}$) is bounded by $\sum_{n\geq 0} \|P^n_{A_n} \delta_y\|_{\ell^p}$.

Using $IS_d$, one has that $\|P^n_{A_n} \delta_y\|_{\ell^p} \leq Kn^{-d/2}$ (for some $K > 0$). A simple interpolation (one also has $\|P^n_{A_n} \delta_y\|_{\ell^1} = 1$, and for any $f \in \ell^1\mathbb{N}$, $\|f\|_q \leq \|f\|_1 \|f\|_q^{-1}$) gives an upper bound on the $\ell^p$-norm. So the $\ell^p$-norm of this [very inefficient] transport pattern from $\text{ex}_{v}^{A_n}$ to $\delta_v$ is bounded by $\sum_{n=0}^{\infty} \|P^n_{A_n} \delta_y\|_{\ell^p}$. It turns out that the series converges when $d > 2p$. The value of the series only depends on the constant in the profile $IS_d$.

A similar transport pattern bring the mass from $\text{ex}_{w}^{A_n}$ to $\delta_w$. A last (and very obvious) transport pattern can bring the mass from $\delta_v$ to $\delta_w$. By combining these three transport patterns, one gets $\tau_n$ and the $\ell^p$-norm of $\tau_n$ is bounded (independently of $n$, $A_n$, $v$ and $w$).

Theorem 4.5. Assume $G$ is $D$-regular Liouville graph with $IS_d$ for some $d > 2$ and let $p < d/2$. Then $G$ has $\Theta D^p$.

Proof. By the Lemma 4.4, there is a plan $\tau_n$ which is bounded. However, this plan might not necessarily tend weak* to 0. By Remark 4.2, one can check that there is an alternative plan $\tau'_n$ which is point-wise smaller than $\tau_n$ and so that $\|\tau'_n\|_{\ell^p} \leq \|\text{ex}_{v}^{A_n} - \text{ex}_{w}^{A_n}\|_{\ell^p}$. The latter implies $\tau'_n$ is bounded in norm (since $\|\tau'_n\|_{\ell^p} \leq \|\tau_n\|_{\ell^p}$ while the former together with a simpler form of Kaimanovich’s 0-2 law (Theorem 3.10) implies it tends to 0 point-wise. Boundedness together with point-wise convergence to 0 is equivalent to weak* convergence (see Lemma 2.8). The conclusion follows by Lemma 4.3.
In the proof of Theorem 4.5, it is unfortunately not possible to prove that $\tau'_n \to 0$ in the weak* topology on $\ell^1 E$ (if so, it would imply the absence of non-constant harmonic functions with gradient in $c_0$).

Note that the previous result may also be found in [10, Theorem 1.2] or [14, Corollary 4.15]. By [12, Theorem 5.12], if this result can be applied to (the Cayley graph of) a subgroup which is not almost-malnormal, then it extends to the whole group.

Recall that if $G_1 = (X_1, E_2)$ and $G_2 = (X_2, E_2)$ are graphs, their direct product is the graph $G$ with vertex set $X = X_1 \times X_2$ and the pair $(x_1, x_2)$ is a neighbour of $(y_1, y_2)$ if either $x_1 \sim x_2$ or $y_1 \sim y_2$ but not both. It would be possible to use the techniques of this section to prove that direct products of graphs have $\text{HLD}_p$ (a variant of [11, Proposition 2]). However work of Amir, Gerasimova and Kozma [1] and Corollary 4.14 contain stronger results.

**Remark 4.6.** As soon as a graph satisfies $\text{IS}_d$ for $d > 2p'$, one has that $\| k_p + \ell^*_p \| = \ell^1 E$. Indeed, under this hypothesis $\mathbf{\delta}_x \in \ell^1 V \ast \{ \} = \delta_x$ (here $\mathbf{\delta}_x = \sum_{i \in X} P^i$ is Green’s kernel) so that taking $H = \mathbf{\delta}_x = \delta_x - \delta_w$ (for any neighbours $v \sim w$), one has

$$\nabla H \in \| k_p \|, \quad \nabla H - \delta_{v,w} \in \ker \nabla^* = \ell^1 E.$$

so $\delta_{v,w} \in \| k_p \|$. Taking closure one gets $\ell^1 E$. ◊

### 4.2 Transport using relative isoperimetry

The aim of this section is to show that one can (using the estimates from §3.3) also deduce $\text{HLD}_p$ if the isoperimetry in some sets is well-behaved (as in §3.3).

The idea is very similar to Theorem 4.5. Instead of constructing a transport from Dirac masses to the exit distributions, the idea is to construct a transport pattern from Dirac masses to $\frac{1}{|\rho|} \sum_\rho F$. For reasons to come, this will work better if $F$ is a very particular set: $F$ will be optimal in the sense that, for any finite set $F'$ with $|F'| \leq |F|$, $\frac{\| F' \|}{|F'|} \geq \frac{\| F \|}{|F|}$

which have a bounded $\ell^p'$-norm (the bound being independent of $F$). See §3.3 or [6] for details. In order to keep the constants legible, only the case $p \in ]1, 2[$ will be taken into account.

**Theorem 4.7.** Let $G$ be the Cayley graph of a group. Let $F_n$ be a sequence of finite sets. Let $r(F_n) = \max \{ k \mid \text{a ball of radius } k + 2 \text{ is contained in } F_n \}$ (this is, up to the constant 2, the inner radius of $F_n$). Let $\rho_k = \| P^{k+1} \delta_w \|_{\ell^\infty}$ be the return probability of the random walk after $n$ steps. If $k_1(F_n)^{-2} \rho_k^{1/q}$ tends to 0, then $G$ has $\text{HLD}_q$.

**Proof.** Let $p = q'$ be the Hölder conjugate of $q$. As in Theorem 4.5, the goal is to show that for any pair of neighbouring vertices $v$ and $w$, $\delta_{v,w} \in \| k_p + \ell^*_p \|$. Let $v$ and $w$ be two neighbouring vertices. Since $G$ is assumed to be a Cayley graph, one can assume the $F_n$ are chosen so that a ball of radius $r(F_n)$ around $v$ or $w$ is contained in $F_n$.

Let $f = \sum_{i=0}^{r(F_n)} P^i (\delta_w - \delta_v)$. Then $\nabla^* (\delta_{v,w} - \nabla f) = P^{r(F_n)+1} \delta_w - P^{r(F_n)+1} \delta_v$. So if one can construct a transport pattern between these measures
Let $\Delta_n$ be the Laplacian restricted to graph induced on $F_n$ and $g = P_r(F_n)^{+1}\delta_w - P_r(F_n)^{+1}\delta_v$. Note that $g_n$ has zero sum, hence lies in the image of $\Delta_n$. Let $h_n = \Delta_n^{-1}g$. Let $\tau_n$ be the function (on the edges) which is identically equal to $\nabla_n h_n$ inside $F_n$ (where $\nabla_n$ is the gradient on the graph induced on $F_n$) and 0 everywhere else. This is a transport pattern since $\nabla^* \tau_n$ only is non-zero on vertices in $F_n$. Furthermore, on such vertices, it is equal to $\nabla^* \nabla_n h_n = \Delta_n h_n = g_n$.

It remains to check that the norm of $\tau_n$ tends to 0. For this note that $\|g_n\|_{\ell^p} \leq \|P_{r}(F_n)^{+1}\delta_v\|_{\ell^p} + \|P_{r}(F_n)^{+1}\delta_w\|_{\ell^p}$. Furthermore,

$$\|P_{r}(F_n)^{+1}\delta_w\|^p_{\ell^p} \leq \|P_{r}(F_n)^{+1}\delta_w\|_{\ell^\infty}^p \|P_{r}(F_n)^{+1}\delta_w\|_{\ell^1} \leq \|P_{r}(F_n)^{+1}\delta_w\|_{\ell^\infty}^{p-1}.$$  

So $\|g_n\|_{\ell^p} \leq 2\rho_{r}(F_n)^{1/q}$ (recall $q$ is the Hölder conjugate of $p$).

Next

$$\|\tau_n\|_{\ell^p_E} = \|\nabla h_n\|_{\ell^p_E} \leq d\|h_n\|_{\ell^p_V} \leq d\|h_n\|_{\ell^p_V} \leq d\|h_n\|_{\ell^p_V} \leq 2qd^3\kappa_1(F_n)^{-2}\rho_{r}(F_n)^{1/q},$$

where the last inequality follows from Theorem 3.8. So, if $\kappa_1(F_n)^{-2}\rho_{r}(F_n)^{1/q} \to 0$, then the transport patterns $\tau_n$ tend in norm to 0. This implies that $\delta_{c,v} \in \mathbb{K}_p + \mathbb{F}_p$. By Lemma 2.13, the groups has $\theta\mathbb{D}^p$. 

Theorem 4.7 is fairly hard to apply to examples. Examples which seem feasible are groups of polynomial growth, polycylic groups and wreath products. In all those examples, the property $\theta\mathbb{D}^p$ can be deduced using a different method.

The major hindrance is that the optimal sets are not known in any groups except $\mathbb{Z}^d$. This is not in itself that bad (one could be happy with their existence alone). The real problem comes in when one tries to find bounds on the inner radius of such sets.

Here is an application of Theorem 4.7.

**Corollary 4.8.** Let $G$ be a group and consider some Cayley graph of $G$. Assume that, for some constants $a, b, K_1$ and $K_2 \in ]0, \infty[$, one has $\frac{K_1}{(\ln x)^a} \leq G(x) \leq \frac{K_2}{(\ln x)^b}$. Assume further that

- either a radial isoperimetric inequality (3.14) holds
- or there are $c$ and $K_3 \in ]0, \infty[$ so that, for any optimal set $F$, the inner radius $r(F)$ is bounded below: $r(F) \geq K_3(\ln |F|)^c$.

Then $G$ has $\theta\mathbb{D}^p$.

**Proof.** Note that the radial isoperimetric inequality implies the bound on the inradius by Lemma 3.15.

As a consequence of [6] there are infinitely many optimal sets such that $\kappa_1(F) \geq K_4/(\ln |F|)^t$ for some other constants $K_4$ and $t$. On the other hand, the isoperimetric ratio function implies that there are constants $K_5$ and $\gamma$ so that $\rho_{t} \leq \exp(-K_5\kappa^\gamma)$. Hence $\kappa_1(F_n)^{-2}\rho_{r}(F_n)^{1/q} = K'(\ln |F_n|)^{2d} \cdot \exp(-K''(\ln |F_n|)^{\gamma}).$ Using L'Hôpital's rule (\(\frac{2d}{\ln n}\) times) one sees that this tends to 0.

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The conclusion follows by Theorem 4.7.

The hypothesis of the previous corollary are still very hard to check, but hopefully gives the feeling it covers a large class of groups. The main obstacle is the estimate on the inner radius (or the radial isoperimetric inequality).

For comparison, in the usual Følner sequences [which may not be optimal] used for groups of intermediate growth, polycyclic groups (which are not nilpotent), wreath products \( A \wr N \) (where \( A \) is finite and \( N \) has polynomial growth of degree \( d \)) and \( \mathbb{Z} \wr \mathbb{Z} \), the corresponding bound on the inner radius \( r(F) \) are respectively \((\ln |F|)^{c} \) with \( c > 1 \), \( \ln |F| \), \((\ln |F|)^{c} \) with \( c = \frac{1}{2} \) and \( \ln |F| / \ln \ln |F| \).

In fact, a lower bound of the form \((\ln |F|)^{c} \) with \( c = \frac{1}{2} - \varepsilon \) exists for \( N' \wr N \) (where \( N' \) and \( N \) are of polynomial growth and \( d \) is the degree of the growth of \( N \)). This suggests that Corollary 4.8 applies to free metabelian groups (of any rank \( r \)) since they embed in \( \mathbb{Z}' \wr \mathbb{Z}' \).

There are also groups where it seems very unlikely that Theorem 4.7 applies. A possible candidate could be as simple as \( A \wr (A/\mathbb{Z}) \) where \( A \) is some finite group. There are two reasons for this. First, [6] show that (even for optimal sets) \( \kappa_{1}(F) \leq (\ln |F|)^{c} \) (which is much worse than the isoperimetric ratio function). Second, the inner radius for the usual Følner sets is probably fairly small, e.g. one would expect something like \( r(F) \approx \ln \ln |F| \). This indicates that the methods probably breaks for groups whose isoperimetry ratio function decays more slowly than a power of \( \ln \).

### 4.3 \( \mathcal{D}^{c} \) and commuting elements

The main method of this section is to interpret the norm and the equivalence classes of elements from \( \ell^{p}E/\ell_{p}^{*}f \) and deduce the non-existence of harmonic functions with gradient conditions.

Recall that “has \( \mathcal{D}^{c} \)” is a shorthand for “has no non-constant harmonic functions with gradient in \( c_{0} \)” (like \( \mathcal{D}^{p} \) but with \( c_{0} \) in place of \( \ell^{p} \)).

Given a function \( f \in \ell^{p}E \), recall that the norm of \( f \in \ell^{p}E/\ell_{p}^{*}f \) is the norm of a transport plan of \( \nabla^{*}f \). So it remains to interpret the contribution of \( \ell_{p}^{*}f \). To this end consider the Dirac mass \( \delta_{v} \in \ell^{p}V \). As an element of \( \ell_{p}^{*}f \), it is sent to \( \nabla \delta_{v} \). Hence its effect on the above norm is to change \( \nabla^{*}f \) to \( \nabla^{*}f + \nabla^{*} \nabla \delta_{v} \).

Seeing the function \( \nabla^{*}f \) as pile of chips on each vertex, this can be interpreted as “firing up" a vertex: [part of] the value of \( \nabla^{*}f \) can be removed from a vertex and redistributed (equally) on all the neighbours. Note that this is possible even \( \nabla^{*}f \) has value 0 at a vertex. The random walk can be seen as a special case where one “fires up" (the full value at) all the vertices.

When looking at \( \ell^{1}E/\ell_{1} \), note that it further suffices to show that there are bounded transport plans that tend point-wise to 0 (see Lemma 2.8 or the proof of Theorem 4.5).

**Lemma 4.9.** Assume that for an exhausting and increasing sequence \( A_{n} \) and for some vertices \( x \) and \( y \), the \( \ell^{1} \)-transport distance of the \( n^{th} \)-step random walk distribution \( P^{n}\delta_{x} \) and \( P^{n}\delta_{y} \) is bounded independently of \( n \). Then any path from \( x \) to \( y \) belongs to \( \ell_{1} \).
Proof. Let $p_{xy} \in \ell^1 E$ be such a path, then $\nabla^* p_{xy} = \delta_x - \delta_y$. Modulo $k_1$ this belongs to the same class as $P^n \delta_x - P^n \delta_y$. The norm of this class in $\ell^1 E/\ell_1$ is bounded (by hypothesis). Furthermore, since the $P^n$ tend point-wise to 0, the (bounded) transport pattern must tend point-wise to 0.

Indeed, assume there some edge $e$ for which $\tau(e) > c$ independently of $n$. Let $k_n$ be the largest integer, such that a ball of radius $k_n$ around $e$ contains a mass of at most $c/2$ i.e. $P^n \delta_x(B_{k_n}(e)) \leq c/2$. Since $P^n \delta_x$ tends point-wise to 0, $k_n \to \infty$. Then in order to transport the (at least] remaining $c/2$ mass, the transport plan must transport this mass over at least a distance of $k_n$. Since $k_n, c/2$ is not bounded, this contradicts the hypothesis.

This means that $p_{xy} \in \ell^1 E/k_1 + \ell_1^1$.

In particular, the previous lemma implies that any harmonic function with $c_0$-gradient will take the same values at the vertices $x$ and $y$. It is also possible to prove the previous lemma with the exit distributions (w.r.t. to some sequence of sets $A_n$), but the author could not apply the upcoming results to this distribution.

Although the required property is hard to ensure in a generic graph, note that if there is a central element in a group, then one gets many such pairs.

**Lemma 4.10.** Let $\Gamma$ be a group and assume $z \in Z(\Gamma)$. Then in any Cayley graph of $\Gamma$ and for any $x \in \Gamma$, the distributions $P^n \delta_x$ and $P^n \delta_{zx}$ stay at bounded transport distance.

**Proof.** Since $z$ belongs to the centraliser, applying a step of the random walk, and then moving along the path [in the Cayley graph] labelled by writing $z$ as a word in the generators is the same as first moving along $z$ and applying the random walk: since $P = \sum_s \delta_s$ and $\delta_s \ast \delta_z = \delta_z \ast \delta_s$, then $P \ast \delta_z = \delta_z \ast P$.

This means that the transport plan from $P^n \delta_x$ to $P^n \delta_{zx}$ simply consists in following the path labelled by $z$. This plan is bounded by the word length of $z$. \qed

**Theorem 4.11.** Assume $\Gamma$ has an infinite centre. Then (in any Cayley graph) $\Theta D^C$ holds.

**Proof.** The aim is to use Lemma 2.10, namely to show that $\ell^1 E = k_1 + \ell_1^1$. Let $p_{xy}$ be a path between $x$ and $y$. Take $c_n$ to be a sequence of central elements (which tend to infinity, i.e. leaves any fixed ball). Let $x_n = x c_{n} = c_n x$ and $y_n = y c_n = c_n y$. Since $p_{xy} = p_{x_n y_n} + p_{x_n y_n} + p_{y_n y}$. By Lemmas 4.9 and 4.10, this means that $p_{xy}$ and $p_{x_n y_n}$ belong to the same equivalence class. Note that $\| p_{x_n y_n} \|_{\ell^1 E} = \| p_{xy} \|_{\ell^1 E}$, since the distance from $x_n$ to $y_n$ is the same as the distance from $x$ to $y$. Lastly, since $p_{x_n y_n}$ is bounded and tends point-wise to 0, one get that $p_{xy}$ is in the same class as $0$ in $\ell^1 E/k_1 + \ell_1^1$.

Note that this result can be improved to include the case where there are infinitely many elements with a finite conjugacy class. To do this one just needs to adapt the proofs of [15, Proposition 1.5 and Lemma 2.7].

The reader is directed to the work of Amir, Gerasimova and Kozma [1] for a proof that direct product of graphs have $\Theta D^C$.

In the case of two groups $G_1$ and $G_2$, note that the direct product $G = G_1 \times G_2$ of the groups admits many generating sets for which the resulting graph is not a direct product of the Cayley graphs (in the sense of graph products). See Corollary 4.14 for a result about direct product of groups.
4.4 Centralisers and malnormal subgroups

This section deals only with Cayley graph of finitely generated groups. Although the main focus is for harmonic function in the usual sense, it will be useful to consider functions which are harmonic with respect to other measures. It will be assumed that the measure is finitely supported and symmetric (i.e. \( \mu(s) = \mu(s^{-1}) \)); see Remark 4.19 for some variations.

Recall that the centraliser of \( H \) in \( G \) is \( Z_G(H) := \{ g \in G \mid \forall h \in H, gh = hg \} \). If \([g, H]\) denotes the set \( \{ [g, h] \mid h \in H \} \), note that this can be rewritten as \( Z_G(H) := \{ g \in G \mid [g, H] \) has one element \}. This second definition is closer to the definition of the FC-centraliser of \( H \) in \( G \): \( Z^\text{FC}_G(H) := \{ g \in G \mid [g, H] \) is finite \}.

**Lemma 4.12.** Let \( H < G \) be an infinite subgroup of \( G \). Let \( f \) be a function on \( G \) with gradient in \( c_0 \). Assume \( f \) is harmonic with respect to the [finitely supported symmetric] measure \( \mu \) whose support generate an infinite subgroup of \( H \). Then \( f \) is constant on the right cosets of \( Z = Z_G(H) \) or \( Z^\text{FC}_G(H) \) (i.e. \( f \) is constant on the sets \( gZ \) for any \( g \in G \)).

**Proof.** For readability, the case \( Z = Z_G(H) \) will be done first. Consider \( g \in G \) and \( z \in Z_G(H) \). Then let \( d = h(g) - h(gz) = \langle h | \delta_g - \delta_{gz} \rangle \). Since \( h \) is harmonic with respect to \( \mu_H \), \( d = \langle h | \delta_g * \mu^*_H - \delta_{gz} * \mu^*_H \rangle \). However, since \( z \) lies in the centraliser, \( \delta_{gz} * \mu^*_H = \delta_g * \delta_z * \mu^*_H = \delta_g * \mu^*_H \). This means there is a transport plan from \( \delta_g * \mu^*_H \) to \( \delta_{gz} * \mu^*_H \), which is bounded (by the word length of \( z \)) and tends to 0 (since \( H \) is infinite and the support of \( \mu_H \) generates \( H \)). Consequently, \( d = 0 \) which means that \( h \) is constant on the right cosets of \( Z_G(H) \).

To get the conclusion for \( Z = Z^\text{FC}_G(H) \), note that \( \delta_z * \mu_H = \delta_z * \sum_{h \in H} \mu_H(h) \delta_h = \sum_{h \in H} \mu_H(h) \delta_z * \delta_h = \sum_{h \in H} \mu_H(h) \delta_z * \delta_{c(h)} \) where \( c(h) = |z^{-1}, h^{-1}| \) depends on \( h \), but takes only finitely many values. So the transport plan is still bounded (by \( \max_{h \in H} |zc(h)h| \)) and the conclusion also holds. \( \square \)

There are some further definitions and properties which should be introduced before moving on to the next lemma. A subgroup \( K < G \) is almost-malnormal if \( \forall g \in G \setminus K, K \cap gKg^{-1} \) is finite. (The whole group \( G \) is an almost-malnormal subgroup of itself.) Recall that the q-normaliser of \( H \) in \( G \) is \( N^q_G(H) = \{ g \in G \mid H \cap gHg^{-1} \) is infinite \} (this subgroup is not always finitely generated).

Note that, if \( Z = Z_G(H) \) or \( Z^\text{FC}_G(H) \), then \( H \subset N^q(Z) \) and \( Z \subset N^q(H) \). Also if one starts at a subgroup \( K \) and considers the [transfinite] sequence of iterated q-normalisers of \( K \), then this sequence stabalises at the almost-malnormal hull of \( K \) (the smallest almost-malnormal subgroup containing \( K \)). Furthermore, the almost-malnormal hull of a finite subgroup is itself. Lastly, if \( L < K \) is an infinite subgroup and \( L < H < G \) then \( N^q(K) \) (and hence the almost-malnormal hull) contains \( H \).

**Lemma 4.13.** Let \( G \) be an infinite groups, \( S \) a finite generating set and consider the associated Cayley graph. Assume \( f \) is so that \( \nabla f \in c_0E \) and \( f \) is constant on the right-cosets of some subgroup \( K < G \). Then \( f \) is constant on the almost-malnormal hull of \( K \).

**Proof.** Let \( N = N^q_G(K) \) and take \( g \in N \) to be one of the generators of \( N \), i.e. \( K \cap gKg^{-1} \) is infinite. Then \( gK \cap Kg \) is infinite, so let \( g_n \) be an infinite sequence of (distinct) elements
in this intersection. Note that $\gamma_n \in gK$, hence $d = f(\gamma_n)$ is a constant. On the other hand all the $\gamma_n$ are at distance at most $|g|_S$ from $K$, because $\gamma_n \in Kg$. Since $f$ is constant on $K$, it follows that $f(\gamma_n) = f(K) + \sum_{k \in \pi_{K,\gamma_n}} \nabla f(k)$ where $\pi_{K,\gamma_n}$ is some path of length at most $|g|_S$ from $K$ to $\gamma_n$. But $f \in c_0E$ and consequently, the sum tends to $0$ as $n \to \infty$. Hence $f(K) = f(\gamma_n) = f(gK)$.

By induction this can be extended to any element of $g$. Indeed, assume that $f$ is constant on all cosets of the form $g_1 \cdots g_k K$ (where the $g_j$ are generators of $N$ and $0 \leq k \leq n$). Let $h_k = g_1 \cdots g_k$ and consider $h_kg_{k+1} \in N$. Then (since $g_{k+1}$ is a generator of $N$) $h_kg_{k+1}Kg_k^{-1}h_k^{-1}$ has an infinite intersection with $h_kKh_k^{-1}$. Hence $h_kg_{k+1}K \cap h_kKg_{k+1}$ is also infinite. Consider again a sequence $\gamma_n$ of elements of this intersection. Then $f(\gamma_n)$ is constant (since $h_kg_{k+1}K$ is a right-coset of $K$) and the $\gamma_n$ are at distance at most $|g_{k+1}|_S$ from $h_kK$. But by induction, $f(h_kK) = f(K) = d$. Hence $f$ takes the same constant value on $h_kg_{k+1}K$. So induction shows that the statement hold for any element of $N$.

This shows that $f$ is constant on $N$. To show that $f$ is constant on any right-coset of $N$, replace $f$ by a translate: let $\lambda_x f(g) := f(x^{-1}g)$. Then $\lambda_x f$ is also constant on the right-cosets of $K$ and $\nabla \lambda_x f \in c_0E$. As a consequence $\lambda_x f$ is constant on $N$, which means that $f$ is constant on $xN$.

Since $f$ is constant on the right-cosets of $N = N_1 = N^q(K)$, then it is also constant on the right-cosets of $N_2 = N^q(N)$. This argument can be repeated to any successive ordinal of the sequence $N_\lambda := N^q(N_\lambda)$. If $\lambda$ is a limit ordinal, then the conclusion holds as $N_\lambda = \cup_{\nu<\lambda} N_\nu$. Thus transfinite induction may be applied to show that $f$ is constant on the malnormal hull of $Z$.

The previous two lemmas could (and in part will) be used as follows. First, one needs to show that a function which is harmonic and has gradient in $c_0$ is also harmonic with respect to some other measure. Then conclude that it is constant on the right cosets of some subgroup (here using Lemma 4.12; but there could be other possibilities). Finally, use Lemma 4.13 to show it’s constant on the whole group. The upcoming results are some possibilities for this strategy.

**Corollary 4.14.** Let $G = G_1 \times G_2$ be a direct product of two infinite (finitely generated) groups. Then (in any Cayley graph of $G$) if $f$ is harmonic and $\nabla f \in c_0E$, then $f$ is constant.

**Proof.** Indeed, since $G_1 \subset Z(G_2)$ then by Lemma 4.12, $f$ is constant on the $G_1$-cosets. Since one also has $G_2 \subset Z(G_1)$, $f$ is also constant on the $G_2$-cosets. This means that $f$ is constant.

**Corollary 4.15.** Let $G$ be a finitely generated group and let $H < G$ be an infinite subgroup. Let $Z < Z^F_G(C)$. Assume $H \cap Z$ is infinite and $G = \langle H, Z \rangle$. Then (in any Cayley graph of $G$) if $f$ is harmonic and $\nabla f \in c_0E$, then $f$ is constant.

**Proof.** If $C := Z \cap H$ is infinite, then actually $Z^F_C(G)$ is infinite. Hence, for any generating set $S$, Lemma 4.12 shows that $f$ (which is harmonic on $G$) is constant on $C$. Since $N^q(C) \supset H$ and $N^q(H) \supset Z$, the almost-malnormal hull of $C$ is $G$ and Lemma 4.13 can then be applied to conclude that $f$ is constant on $G$.  

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Upon restricting to a specific Cayley graph, the condition that $H \cap Z$ is infinite can be relaxed.

**Theorem 4.16.** Let $G$ be a finitely generated group and let $H < G$ be an infinite subgroup. Assume $Z < Z^F_C(H)$ is infinite and $G = \langle H, Z \rangle$. Then there is a [finite symmetric] generating set $S$, so that, in the associated Cayley graph, if $f$ is harmonic and $\nabla f \in c_0E$, then $f$ is constant.

**Proof.** Since Corollary 4.15 deals with the case $H \cap Z$ is infinite, the proof reduces to the case where $C := Z \cap H$ is finite.

The generating set $S$ which will come into the proof, is any generating can be written as $S = (\bigcup_i C_i) \cup (\bigcup_{j,k} D_j h_k)$ with $C_i, D_j \subset Z$ being $H$-conjugacy classes and $h_k \in H$.

To see that such a generating set always exists, note that $Z < G$. Hence generating set $S = \{g_i\}_{1 \leq i \leq n}$ can be written as $g_i = z_i h_i$ with $z_i \in Z$ and $h_i \in H$. As such $G$ is generated by $C_i$ and $h_i$ where $C_i$ is the $H$-conjugacy class of $z_i$ (in that case $D_j$ is reduced to the identity element).

For such a set, the random walk on the Cayley graph is given by convolution with $P = \sum_i p_i \delta_z + \left(\sum_j q_j \delta_{z_j}\right)\left(\sum_k r_k \delta_{h_k}\right)$ where the $p_i, q_j$ and $r_k$ are some positive real numbers (for the simple random walk $p_i = q_i^2 = r_i^2$, but here, it only matters that conjugating the elements of $Z$ under $H$ does not affect the coefficients). Look at the associated random walk on $H$, given by convolution with $\mu = p \delta_e + q \sum_n r_n \delta_{h_n}$ where $p = \sum_i p_i$ and $q = \sum_j q_j$.

Then

$$\mu \ast P = pP + q\left(\sum_n r_n \delta_{h_n}\right)\left(\sum_i p_i \delta_{z_i}\right) + q\left(\sum_n r_n \delta_{h_n}\right)\left(\sum_j q_j \delta_{z_j}\right)\left(\sum_k r_k \delta_{h_k}\right)$$

$$= \mu P + \left(\sum_i p_i \delta_{z_i}\right)q\left(\sum_n r_n \delta_{h_n}\right) + \left(\sum_j q_j \delta_{z_j}\right)\left(\sum_n r_n \delta_{h_n}\right)\left(\sum_k r_k \delta_{h_k}\right)$$

$$= \mu P + \left(\sum_i p_i \delta_{z_i}\right)q\left(\sum_n r_n \delta_{h_n}\right) + \left(\sum_j q_j \delta_{z_j}\right)\left(\sum_k r_k \delta_{h_k}\right)$$

where the interchange in the second and third terms happen only because the whole conjugacy class of $z_i$ is in the sum. In the second to the third line only a change of indices occur.

From there, one can then consider a harmonic function $f$ on the Cayley graph of $G$ (with generating set $S$). Then, by harmonicity of $f$ and then the fact that $P \ast \mu = \mu \ast P$, $d = \langle f | \delta_g - \delta_g \ast \mu \rangle = \langle f | \delta_g \ast P^n - \delta_g \ast \mu \ast P^n \rangle = \langle f | \delta_g \ast P^n - \delta_g \ast P^n \ast \mu \rangle$. There is an obviously bounded transport plan $\tau_n$ from $P^n$ to $P^n \ast \mu$ (just do a random step in $H$).

Since $G$ is infinite, $P^n$ (and hence $\tau_n$) tend point-wise to 0. Consequently, $d = 0$ which means that $f(g) = \sum_h f(gh)\mu(h)$.

This implies that $f$ is harmonic with respect to $\mu$, and $\mu$ generate $H$. By Lemma 4.12, $f$ is constant on $Z$. By Lemma 4.13, $f$ is constant on the malnormal hull of $Z$, which is $G$.

The following result applies, among others, to Abelian-by-cyclic groups such as the soluble Baumslag-Solitar groups.

**Theorem 4.17.** Let $G$ be a [finitely generated] group which is an cyclic extension $1 \to H \to G \xrightarrow{\pi} Z \to 1$. If $Z^F_C(H)$ is infinite, then any harmonic function on a Cayley graph of $G$ with $c_0$-gradient is constant.
Proof. Let $\mu$ denote the random walk on the Cayley graph and $\pi : G \to \mathbb{Z}$ be the quotient map. Define $\mu_n$ to be the measure defined by firing up $\mu$ at all vertices $g$ such that $\pi(g) \neq 0$ or $|\pi(g)| > n$. In other words, let a random walker start at the identity, and he stops if he hits the set $\pi^{-1}\{-n,0,n\}$. Then $\mu_n$ is the probability distribution of where he stopped.

Write $\mu_n = \beta_n + \zeta_n$ where $\zeta_n$ is the part of $\mu_n$ supported on $\pi^{-1}(0) = H$. By the gambler’s ruin, one has that $\zeta_n$ has $1 - 1/n$ of the mass.

Next let $f$ be a harmonic function. Consider $g \in G$ and, for simplicity, assume $c \in Z_H(H)$ (instead of $Z_H^{FC}(H)$). Then, since $f$ is also harmonic with respect to $\mu_n$,

$$f(g) - f(gc) = \langle f|\delta_g * (1-\delta_c)\rangle = \langle f|\delta_g * (\zeta_n - \delta_c * \zeta_n + \beta_n - \delta_c * \beta_n)\rangle.$$ 

where the fact that $c$ commutes with $H$ is used (to commute $\delta_c$ with $\zeta_n$, which has support in $H$). The transport plan from $\beta_n$ to $\delta_c * \beta_n$ can use paths of length at most $2n + |c|$ and since the mass of $\beta_n$ is at most $1/2$, the $\ell^1$-norm of the transport remains bounded (and it tends to 0 point-wise). Hence $f(g) - f(gc) = \langle f|\delta_g * (\zeta_n - \zeta_c)\rangle$ where $\zeta$ is the limit measure.

Repeating this argument $k$ times again yield $f(g) - f(gc) = \langle f|\delta_g * (\zeta_n - \zeta_c)\rangle$. The transport plan from $\zeta_n$ to $\zeta_c$ is bounded by the word length of $c$ and tends point-wise to 0. It follows that $f(g) - f(gc) = 0$.

Consequently, $f$ is constant on the cosets of $Z_H(H)$. Since $N^q(Z_H(H)) \supset H$ and $N^q(H) = G$, Lemma 4.13 shows that $f$ is constant on $G$.

The proof can be adapted to $Z_H^{FC}(H)$ just as in Lemma 4.13.

The previous result is similar to a result of Brieussel & Zheng [4, Theorem 1.1]. In their result $G$ is a [locally normally finite]-by-cyclic group and they show that $G$ has trivial reduced cohomology in degree one for any weakly mixing representations (Shalom’s property $H_{FD}$). Such a group $G$ necessarily fulfills the hypothesis of Theorem 4.17, since a locally finitely normal group is FC-central. On the other hand, Theorem 4.17 only implies the triviality of the reduced cohomology in degree one for strongly mixing representations. As such it seems that both results are not comparable.

Using Theorem 4.17, as well as [12, Theorem 5.12], one gets a result on the absence of non-constant harmonic function with gradient in $\ell^p$:

**Corollary 4.18.** Any [finitely generated] metabelian group has $\Theta \mathcal{D}^p$ for any $1 < p < \infty$.

*Proof.* If $G$ is virtually nilpotent, then the result can be seen as a consequence of Theorem 4.11 (but there are many possible earlier proofs).

Otherwise if $G$ is not virtually nilpotent, then $G$ contains an Abelian-by-cyclic subgroup $G_0$ of exponential growth (see, for example, Groves [17] or Breuillard [3, Proposition 4.1]). $G_0$ has exponential growth, hence it has $\mathcal{S}_d$ for any $d$ (see §3.3). Furthermore, by Theorem 4.17, $G_0$ has $\Theta \mathcal{D}^c$, in particular is also has $\Theta \mathcal{D}^p$ for any $p \in [1,\infty]$.

Furthermore, if $1 \to A_1 \to G \to A_2 \to 1$ is decomposition of $G$ as an extension of $A_1$ by $A_2$, then note that $N^q(G_0) \supset \langle G_0, A_1 \rangle =: G_1$ (this follows from the fact that the “Abelian”
subgroup of $G_0$ is infinite and commutes with $A_1$). It follows that $N^q(G_1) \supset N^q(A_1) = G$. So $G$ is the almost-malnormal hull of $G_0$.

By [12, Theorem 5.12] (and the correspondence between $\ell^p$-cohomology and harmonic functions from [10] or [14]) reduced $\ell^p$-cohomology of $G$ is trivial for all $p \geq 1$, which in turn implies that $\Theta\mathcal{D}^p$ holds for all $p \geq 1$. \hfill \Box

[12, Question 6.2] asks whether $\Theta\mathcal{D}^p$ holds for any finitely generated soluble groups. The previous corollary gives a positive answer for case where the soluble rank is 2.

**Remark 4.19.** Let $G$ be a group which is generated by the finite set $S$. Let $\mu$ be a measure on $G$. Given $\alpha > 0$, a measure has a finite $\alpha$-moment if $\sum_{g \in G} \mu(g)|g|_S^{\alpha} < +\infty$ where $|g|_S$ denotes the word length of $g$ (i.e. the distance between $g$ and the identity in the Cayley graph with respect to $S$).

Let $f$ be a function with $\ell^\infty$-gradient (i.e. $f$ is Lipschitz). If $\mu$ has finite 1-moment, then $f * \mu(g) := \sum_{h \in G} f(gh)\mu(h)$ will be finite for any $g$. Indeed, if $\pi_{g,gh}$ denotes a path from $g$ to $gh$ (of length $|h|_S$), one has

$$\sum_{h \in G} f(gh)\mu(h) \leq \sum_{h \in G} (f(g) + \sum_{k \in \pi_{g,gh}} \nabla f(k))\mu(h) \leq \sum_{h \in G} (f(g) + |h|_S\|\nabla f\|_{\ell^\infty})\mu(h) \leq f(g) + \|\nabla f\|_{\ell^\infty} \sum_{h \in G} |h|_S\mu(h).$$

Hence, for functions $f$ with $\ell^\infty$-gradient, one may consider them harmonic with respect to $\mu$ if $f * \mu = f$.

Note that it is also possible to speak of transport plans between measures which are not finitely supported. Indeed, given $f$ with gradient in $\ell^\infty$ as well as two measures with finite first moment $\mu$ and $\nu$, If there is is a $\tau \in \ell^1 E$ so that $\nabla^\tau \nu = \mu - \nu$, then the equality $\langle \nabla f|\tau \rangle = \langle f|\nabla^* \tau \rangle$ holds (since $f$ is Lipschitz, see above).

This might be useful to prove $\Theta\mathcal{D}^c$ for groups. Indeed, when a group $\Gamma$ has a measure $P$ whose support generate $\Gamma$ and another measure $\mu$ which commutes with $P$ and generates a smaller group, then the methods of this current subsection have allowed us to show that $\Gamma$ has $\Theta\mathcal{D}^c$. By considering measure which have finite 1-moment, instead of finitely supported, one could be able to prove this property for more groups.

Similar considerations can be made for functions with $\ell^p$-gradient. Indeed, as before

$$\sum_{h \in G} f(gh)\mu(h) \leq \sum_{h \in G} (f(g) + \sum_{k \in \pi_{g,gh}} \nabla f(k))\mu(h)$$

Since $\nabla f \in \ell^p E$, one has that $\|\sum_{k \in \pi_{g,gh}} \nabla f(k)\|_{\ell^\infty} \leq \|\nabla f\|_{\ell^p}|h|_S^{1/p'}$. Hence the above sum makes sense for measures $\mu$ with finite $\frac{1}{p'}$-moment. \hfill $\diamondsuit$

By the upcoming work of Amir, Gerasimova & Kozma [1], it is however not possible to show $\Theta\mathcal{D}^c$ for any metabelian group. Indeed, they show that the lamplighter on $\mathbb{Z}^5$ contain such functions.

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4.5 Divergence and decay of the gradient

The goal of this section is to mention a connection between the divergence (see e.g. Ol’shanskii, Osin and Sapir [28, Definition 1.5]) and the decay of the gradient. For our purpose, let the divergence be defined as follows.

Given some vertex $\ell$ and an integer $K > 1$, consider $S_{\ell,K}(n)$ to be vertices which are in the complement of the ball of radius $n$ around $\ell$ but not in the infinite components of the complement of the ball of radius $Kn$. Let $d_S$ be the combinatorial distance on the finite graph induced on $S_{\ell,K}(n)$ and $S^{out}$ to be the set of vertices which in $S_{\ell,K}(n)$ and are neighbours to a vertex at distance $Kn + 1$ von $\ell$. Then the divergence is $D_{\ell,K}(n) = \max\{d_S(x,y) \mid x, y \in S^{out}(n)\}$.

There are many groups where this divergence is a linear function (for any $K$ large enough) such as soluble groups, uniformly amenable groups and groups with a non-trivial law (see Ol’shanskii, Osin and Sapir [28, §1.3]).

Given a function $f$ on the vertices of the graph $G$, let $\text{gd}_f(n) = \|\nabla f\|_{\ell^\infty(E \cap B_0(n)^{c,\infty})}$ be the supremum on edges of the gradient which are in an infinite component of the complement of the ball of radius $n$.

**Corollary 4.20.** If $h$ is a non-constant harmonic function, then $\lim_{n \to \infty} D_{\ell,K}(n) \cdot \text{gd}_h(n) \neq 0$.

In particular, if $h$ is a non-constant harmonic function on a soluble group then the gradient of $h$ has to decay as slow as $1/n$.

**Proof.** Consider two vertices $x$ and $y$ so that $h(x) \neq h(y)$. Then, if $n$ is large enough so that $x, y \in B_0(Kn)$, $h(x) - h(y) = \langle \text{ex}^{S^{out}(n)}_x - \text{ex}^{S^{out}(n)}_y \mid h \rangle$. Now let $\tau$ be a transport plan from $\text{ex}^{S^{out}(n)}_x$ to $\text{ex}^{S^{out}(n)}_y$ which avoids the ball of radius $n$. For this transport plan, one has an upper bound $\|\tau\|_{\ell^1 E} \leq D_{\ell,K}(n) \cdot \|\text{ex}^{S^{out}(n)}_x - \text{ex}^{S^{out}(n)}_y\|_{\ell^1 X} \leq 2D_{\ell,K}(n)$. From $h(x) - h(y) = \langle \nabla^* \tau \mid h \rangle = \langle \tau \mid \nabla h \rangle$ and the fact that $\tau$ is supported in $S_{\ell,K}(n)$ one gets that $h(x) - h(y) \leq \|\tau\|_{\ell^1 E} \|\nabla h\|_{\ell^\infty(E \cap B_0(n)^{c,\infty})} \leq 2D_{\ell,K}(n) \cdot \text{gd}_h(n)$. Consequently the right-hand side is bounded away from 0.

Note that if the graph is Liouville, then one gets a small improvement since the $\ell^1$-norm of the difference of exit distributions tends to 0. It would be interesting to apply this result to prove the absence of bounded harmonic functions with gradient in $\ell^p$.

5 Questions

Regarding the first question. Here is an example of a graph with infinitely many ends where $K_1 = \frac{1}{K_1}$. Consider the half line and attach to every vertex another half-line (the vertices of these attached half-line will be labelled by $\mathbb{N}$). It’s not too difficult to show that the Dirac mass (in $\ell^1 E$) of any edge belongs to $K_1$. Indeed, if the edge $(x - 1, x)$ belongs to an attached half-line then consider $\nabla f_n$ where $f_n$ is the function supported on $(x, x + n)$. This $f_n$ tends weak* to the desired Dirac mass. Dirac masses which are on the original half-line can then easily be obtained.
Here is an example where \( f_c \neq F_c \). Consider an infinite tree without vertices of degree \( \leq 2 \). Recall that in a tree \( f_c \) is trivial. Hence it suffices to construct an element of \( F_c \). Let \( \sigma_\pm \) be two extremities of some edge. One removes this edge and gets two rooted trees \( T_\pm \) (with roots \( \sigma_\pm \)). Draw those trees in a graded fashion (say with the root \( \sigma_+ \) at the top of its tree and the root \( \sigma_- \) at the bottom of its tree). Define the value of the function \( f \) (on the edges) by setting \( f(\vec{e}) = 1/\prod(d_i - 1) \) where \( d_i \) are the degrees of the vertices between \( \vec{e} \) and the root. The sign of \( f \) should be positive when going down. Now put back the edge between \( \sigma_+ \) and \( \sigma_- \) and set the value of \( f \) on this edge to be \( 1 \). It is fairly obvious to see that this function is a flow and tends to \( 0 \) at infinity (i.e. belongs to \( F_c \)).

Assume \( f \in \ell^1 \iota E \setminus K_1 + F_1 \). Look at elements \( g \) of minimal norm inside \( f + K_1 + F_1 \). Each \( g \) can be used as a weight on the edges and to create a length-metric space \((G, m_g)\) where \( m_g(x, y) = \inf \sum_{\vec{e} \in \pi} |g(\vec{e})| \) where the infimum runs over all paths from \( x \) to \( y \) and the sum over all edges in the path. Let \( \overline{\mathcal{G}}^g \) be the completion of \( G \) with respect to the metric \( m_g \).

**Question 5.1.** Are any of the \( \overline{\mathcal{G}}^g \setminus G \) Floyd boundaries?

Given a finite presentation of a group, there is a natural space of cycles as well as an operator \( \nabla_2 : \ell^p C \to \ell^p E \). The fact that \( \nabla_2 \) has closed image for \( p = 1 \) seems to be related to hyperbolicity. A more general question would be:

**Question 5.2.** Does the fact that \( \nabla_2 \) has closed image in \( \ell^2 \) is related to a group-theoretic property?

For Cayley graphs, the property \( \Theta \mathcal{D}^p \) (absence of non-constant harmonic functions with gradient in \( \ell^p \)) is very close to be invariant under quasi-isometry. Indeed, in [10], it is shown that if a graph has \( \Theta \mathcal{D}^p \) then its reduced \( \ell^p \)-cohomology is trivial and that this in turn implies \( \Theta \mathcal{D}^q \) for any \( q < p \). Since the triviality of the reduced \( \ell^p \)-cohomology is an invariant of quasi-isometry, one gets that, for any fixed \( p \), “having \( \Theta \mathcal{D}^q \) for any \( q < p \)” is also an invariant of quasi-isometry. However it remains open to check that:

**Question 5.3.** Is \( \Theta \mathcal{D}^p \) an invariant of quasi-isometry (for Cayley graphs) ?

In the case of generic graphs, some trouble could come up in the class of graphs which do not have IS\( d \) for some \( d \). The question could also be asked for graphs which have IS\( d \) for all \( d \) (instead of Cayley graphs). The quasi-isometry invariance of \( \Theta \mathcal{D}^c \) (absence of non-constant harmonic functions with gradient in \( c_0 \)) is also open. In this case the following is also unclear:

**Question 5.4.** Is \( \Theta \mathcal{D}^c \) equivalent to the absence of non-constant bounded harmonic function with gradient in \( c_0 \)?

In some sense, the quasi-isometry invariance could be as hard as the quasi-isometry invariance of the Liouville property. However, note that, contrary to the space of bounded harmonic functions, the space of harmonic functions with gradient in \( \ell^p \) (resp. \( c_0 \)) is closed in \( \ell^p E \) (resp. \( c_0 E \)).
In [7], Elder & Rogers investigate how to recover the amenable radical using some properties on random walks. Let $H \subset G$ be the set of all elements $h$ so that
\[ \|e_h^A - e_h^n\|_{\ell^1} \to 0 \]
The triangle inequality combined with group invariance shows this is actually a subgroup of $G$. This may not be the amenable radical simply because it will not equal $G$ in any group which is not Liouville. For example, on the lamplighter on $\mathbb{Z}^3$, $G = C_2 \wr \mathbb{Z}^3$, $H \leq G$ (it seems that $H = \{e\}$). Also, if $G = G_1 \times G_2$ where $G_1$ is Liouville and $G_2$ is not, then $H = G_1$.

**Question 5.5.** What are the properties of this subgroup $H < G$?

On the topic of entropy the author often heard the following

**Conjecture [folklore]:** If $G$ has exponential growth, then $H(n) \geq Kn^{1/2}$.

This motivates two possible behaviours for groups of growth $\geq \exp(n^v)$ (for $v \in [0, 1]$):

**Question 5.6.** Assume $|B_n| \geq K_1\exp(K_2n^v)$, does

1. $H(P^n) \geq Kn^{v/2}$?
2. $H(P^n) \geq Kn^{v/(1+v)}$?

However in view of the fact that II$S_{\nu}$ implies a volume growth of the type $|B_n| \geq \exp(n^{v/(1+v)})$, one naturally asks whether this would extend to entropy (which is sometimes described as a form of volume growth).

Note that a positive answer to Question 5.6.2 would imply a weak form of the gap conjecture. Indeed, by Erschler [9, Lemma 5.1] or [13, Lemma 4.4], if $K_1\exp(K_2n^v) \leq |B_n| \leq K_3\exp(K_4n^V)$ then $H(n) \leq K' n^{V/(2-v)}$. Combining this with an hypothetical positive answer to Question 5.6.2, one gets $V \geq v^{2-v/(1+v)}$. Under the (very strong) assumption that $V = v$, this would again in turn forbid that $V = v \leq 1/2$. It does not make impossible arbitrarily small growth (e.g. with only $v = 1/5$ one gets $V \geq 3/10$), it simply forces oscillations in the growth function.

An estimate of the form $\|\nabla P^n\|_{\ell^1} \leq n^{-a}$ could also be useful to get a lower bound on entropy *via* Corollary 3.20.

It is established in §3.5 that three of the characteristic function used to study infinite countable groups (volume growth, entropy growth and return probability) can be formulated in terms of Rényi entropy ($H_0$, $H_1$ and $H_\infty$ respectively). Note that $H_0(P^n)$, $H_1(P^n)$, $H_2(P^n)$ and $H_\infty(P^{2n})$ are all sub-additive. Furthermore $H_1(P^n)$, $H_2(P^n)$ and $H_\infty(P^{2n})$ are concave.

**Question 5.7.** Assume $\Gamma$ is a group of exponential growth. Is the function $n \mapsto H_0(P^{2n})$ concave? In other words, is the function $n \mapsto \ln |B_{2n}|$ (where $B_n$ are the balls of radius $n$) concave?

It seems likely that the answer is no (the function $n \mapsto \ln |B_n|$ is not always concave, see Machi [24]). However, a positive answer would mean that numerical estimates on the
[exponential] growth factor of groups are much more reliable. (This is because the ratio test converges much more quickly than the root test, and concavity is what is needed for the ratio test to be a monotone sequence.)

In general, these functions are only considered up to linear factors (i.e. says that $f(x)$ and $Af(ax + b) + B$, where $A, B, a$ and $b \in \mathbb{R}_{>0}$, belong to the same equivalence class). Otherwise, they are not an invariant of the group, but depend on the choice of $P$. Note it is still unknown whether or not [the equivalence class of] $H_1(P^n)$ depends on the choice of $P$.

Lemma 3.23 shows that the functions $H_p(P^n)$ give, for $1 < p \leq \infty$, the same invariant.

**Question 5.8.** How many different equivalence classes of functions are there in the sequence $H_p(P^n)$ where $-\infty \leq p < 1$?

Note that, for $-\infty \leq p < 0$, $H_p(P^n)$ is probably some exponential function (they all belong to the same equivalence class) independently of the group considered (as long as it is infinite). As such the interesting range is probably $0 < p < 1$. If there happened to be many equivalence classes inside that range, there might be many interesting related questions: what are the critical values of $p$ where the equivalence class changes? does this value depend on $P$?

In a Cayley graph, a special case [restricting to degree 1] of a question (dating back at least to Gromov [16, §8.A1, A2, p.226]) can be formulated (via the results of [10] and Lemma 2.13) as

**Question 5.9.** Is it true that $\ell_p + \mathbb{K}_p$ is dense in $\ell^p E$ for the Cayley graph of any amenable group and all $p \in [1, 2]$?

The main objective in proof of Theorem 4.5, is that one notes immediately that only part in the proof where we use the Liouville condition is to show $\tau_{p-w} \Rightarrow 0$. To develop this further, say that a transport pattern which only uses geodesic will be called optimal. A transport pattern is $K$-optimal (for some $K \geq 1$) if the mass is transported by at most $K$ times the minimal length (the distance it would travel in the optimal transport). Note that if $\tau''$ is $K$-optimal and $\tau'$ is optimal, then $\|\tau''\|_{\ell^1} \leq K\|\tau'\|_{\ell^1}$.

Let $\mu_k = P^k \delta_o - P^k \delta_s$. Take a sequence $\tau_k$ such $\nabla^* \tau_k = \mu_k$ so that $\tau_k \Rightarrow 0$. Let $K_k$ be the smallest real number such that $\tau_k$ is $K$-optimal. For each sequence the function $k \mapsto K_k$ gives some kind of divergence as measured by the random walk.

**Question 5.10.** Are there groups for which the divergence is linear, but there is a sequence $\tau_k$ as above so that $k \mapsto K_k$ is bounded?

Although the existence of such a sequence is sufficient for $\Theta_\mathcal{D}_p$, it might not be necessary. Still, this gives a good hint at which amenable groups might not have $\Theta_\mathcal{D}_p$.

**Question 5.11.** Is there an amenable lacunary hyperbolic group with $\Theta_\mathcal{D}_p$?

More precisely, does the group introduced in Ol’shanskii, Osin & Sapir [28, §3.5] has $\Theta_\mathcal{D}_p$? Note that it is not known whether this group has the Liouville property (no non-constant bounded functions).
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