A note on graphs with disjoint cliques and a link with evasiveness

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Abstract

In this note, we prove that any non complete finite vertex-transitive graph doesn’t have a clique which intersects all other cliques. This gives a positive answer in the case of vertex-transitive graphs to a question raised by Berge and Payan. It also gives a positive answer to a special case of the evasiveness conjecture.

1 Introduction

Independently, Berge [2] and Payan [7] made the following conjecture: any non complete finite regular graph contains two disjoint cliques. Although this conjecture has been proved in several cases of graphs with relatively small degrees [2, 8, 9], Payan also proved [9, 10] that it is false in this full generality. Let us recall the counter-example $P$ that he gave in [9]:

![Graph P](image)

Figure 1: The graph $P$ where each $C_i$ and $C'_i$ is isomorphic to a complete graph with 35 vertices and the 70 vertices $\{s_{1234}, s_{1235}, \ldots, s_{5678}\}$ induce a complete graph $K_{70}$ with 70 vertices. There is no edges between the $C_i$'s and no edges between the $C'_i$'s, so the sets $\{C_1, C_2, \ldots, C_7, C_8\}$ and $\{C'_1, C'_2, \ldots, C'_7, C'_8\}$ each induce a subgraph isomorphic to the lexicographic product $S_8[K_{35}]$ where $S_8$ is the independent set with 8 vertices. On the contrary, any vertex of a $C_i$ is linked to all the vertices of the 8 subgraphs $C'_j$. Finally, a vertex $s_{ijkl}$ is linked to all the vertices of the of completes $C_\alpha$ and $C'_\alpha$, with $\alpha \notin \{i, j, k, l\}$.

It is easy to see that the graph $P$ is regular of degree 350 and that there is no pairwise disjoint cliques, contradicting the initial claim of Berge and Payan.

However, the neighborhood of $s_{1234}$ is not isomorphic to that of any vertex of $C_1$ and so $P$ is not vertex-transitive. In fact, in this note, we prove that if we restrict to vertex-transitive graphs, we have the following stronger conclusion than the initial one claimed by Berge and Payan:

**Theorem:** For any clique $A$ of a non complete finite vertex-transitive graph, there exists a clique $B$ disjoint from $A$.

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Actually, this result was settled as a conjecture in [11] where the authors, following a remark due to Lovász, mentioned that it was consequence of the evasiveness conjecture. In the last section, we will recall this link between the initial question about disjoint cliques and the question of evasiveness and will present the above Theorem as a positive answer to the evasiveness conjecture for graphs, a weak version of the general evasiveness conjecture.

In section 2, we give some definitions and preliminary results, in section 3, we prove the above Theorem for Cayley graphs and, in section 4, we extend the result to vertex-transitive graphs thanks to a theorem of Sabidussi [12] which allows to build a Cayley graph from a vertex-transitive graph.

2 Notations and definitions

In this paper, all graphs are finite, without parallel edges; loops are allowed. The vertex set and the edge set of a graph $X$ will be denoted by $V(X)$ and $E(X)$. The automorphism group of $X$ is denoted $\text{Aut}(X)$. For any vertex $x$ of $X$, $N_X(x) = \{y, x \sim y\}$ is the set of all vertices of $X$ which are neighbors of $x$ and $N_X[x] = \{y, x \sim y\} \cup \{x\}$ is the closed neighborhood of $x$; so, $N_X(x) = N_X[x]$ if, and only if, there is a loop $x \sim x$.

We call $C(X)$ (resp. $K(X)$) the set of complete subgraphs (resp. cliques - i.e. maximal complete subgraphs) of $X$. For $A \subseteq V(X)$, to lighten the writing, we will say that $A$ is a complete or a clique rather than $A$ induces a complete subgraph or induces a clique and write $A \in C(X)$ or $A \in K(X)$.

Finally, we write $X \cong Y$ when $X$ and $Y$ are isomorphic graphs.

**Definition 1** A graph $X$ is vertex-transitive if $\text{Aut}(X)$ acts transitively on $V(X)$: $\forall v, w \in V(X), \exists g \in \text{Aut}(X), g.v = w$.

Let us recall some general facts concerning Cayley graph. For any discrete group $G$ and any subset $S$ of $G$, one defines $X = \text{Cay}(G, S)$ by

$$V(X) = G \quad \text{and} \quad E(X) = \{g \sim h, g^{-1}h \in S\}$$

In other words, $g \sim h$ if, and only if, there is some $s$ in $S$ such that $h = gs$ and, for all $g \in G$, $N_{\text{Cay}(G, S)}(g) = gS$. In order to define a graph (and not only a digraph), we suppose that $S = S^{-1}$. We also suppose that $\langle S \rangle = G$ (i.e. $G$ is generated by $S$) which is a necessary and sufficient condition in order to have $X$ connected. It is easy to prove that a Cayley graph is vertex-transitive. The reverse is false and the Petersen graph is a counter-example. However, we know from [12] that every vertex-transitive graph admits some multiple which is isomorphic to a Cayley graph (see section 4).

Usually, it’s supposed that $1 \notin S$ but we don’t make this assumption and we allow reflexive graphs. For every subset $A$ of $G$ we denote $A^+ = A \cup \{1\}$. So, $A^+ = A$ if and only if $1 \in A$.

**Definition 2** Let $X$ a graph.

1. A subgraph $A$ of $X$ is called a BP-complete if $A \in C(X)$ and $A \cap K \neq \emptyset$ for all clique $K \in K(X)$; we denote $BP(X)$ the set of BP-completes of $X$.

2. $X$ is called a BP-graph if $BP(X) \neq \emptyset$.

So, a BP-complete of $X$ is a complete of $X$ which is also a clique transversal. As an example, the graph $P$ in Figure 1 is a BP-graph: the « central » clique $K_{70}$ is a BP-complete (even more, one can verify that every clique of $P$ is a BP-complete).

The following lemmas help to precise the structure of $BP(X)$.

**Lemma 1** Let $X$ a BP-complete graph. Then for all BP-complete $\Sigma$ and all automorphism $\varphi$ of $X$, $\varphi(\Sigma) \in BP(X)$.

**Proof:** Let $\Sigma \in BP(X), K = \{x_1, \ldots, x_n\} \subseteq K(X)$ and $\varphi \in \text{Aut}(X)$. As $\varphi^{-1}(K) = \{\varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_n)\}$ is also a clique, there is a vertex $a \in \Sigma$ such that $a \in \varphi^{-1}(K)$. So, we have $\varphi(a) \in K$ and this proves that $\varphi(\Sigma) \in BP(X)$.

The following result is an evidence:

**Lemma 2** Let $X$ a BP-complete graph and $\Sigma \in BP(X)$. Then every clique which contains $\Sigma$ is also a BP-complete of $X$.


3 The case of Cayley graphs

For any discrete group $G$ and any $S \subseteq G$, let us define two sets which will be the basis of helpful characterizations of the complete and cliques in Cayley graphs:

$$\mathcal{F}_S = \{A, A \subseteq S, A^{-1}A \subseteq S^+\} \quad \text{and} \quad \mathcal{F}_S^{\text{max}} = \{A, A \in \mathcal{F}_S, |A| \text{ maximal}\}$$

Hence we have:

**Lemma 3** For any Cayley graph $X = \text{Cay}(G, S)$:

i) $\mathcal{C}(X) = \{xA^+, x \in G, A \in \mathcal{F}_S\}$

ii) $\mathcal{K}(X) = \{xA^+, x \in G, A \in \mathcal{F}_S^{\text{max}}\}$

**Proof:** i) Let $C$ be a set of elements of $G$ which induces a complete subgraph of $X$ and let $x \in C$. As $C$ is a complete, $C \subseteq N_X[x] = xS = \{x, xs, s \in S\}$; so, $C = xA^+$ with $A \subseteq S$. Let $a, b \in A$ and consider $\alpha = xa$ and $\beta = xb$ in $C$. Since $C$ is a complete graph, by adjacency of $\alpha$ and $\beta$, we get $a^{-1}b = (xa)^{-1}(xb) = a^{-1}b \in S^+$; so, $A^{-1}A \subseteq S^+$ and every complete $C$ of $X$ can be written $xA^+$ with $A \in \mathcal{F}_S$.

Conversely, let $A \in \mathcal{F}_S$, $x \in G$ and $u, v \in xA^+$ with $u \neq v$. There is $a, b \in A^+$ with $a \neq b$ such that $u = xa$, $v = xb$ and $u^{-1}v = (xa)^{-1}(xb) = a^{-1}b$ is in $S^+$ because $A \in \mathcal{F}_S$. This shows that $uv$ is an edge of $X$ and that $xA^+$ induces a complete subgraph of $X$.

ii) Follows from the fact that a clique is a complete of maximal cardinality.

The following lemma point a useful relation between $G$ and $\text{Cay}(G, S)$.

**Lemma 4** Let $A \subseteq G$ and $x, y$ two distinct vertices of $X = \text{Cay}(G, S)$ such that $xA^+ \cap yA^+ \neq \emptyset$. Then the two following assertions are equivalent:

1. $\exists a, b \in A^+$ s.t. $ab^{-1} \in S$

2. $x \sim y$

**Proof:** The equivalence follows directly from:

$$xA^+ \cap yA^+ \neq \emptyset \iff \exists a, b \in A^+ \text{ s.t. } xa = yb \iff \exists a, b \in A^+ \text{ s.t. } x^{-1}y = ab^{-1}$$

We can now prove the Berge-Payan conjecture for Cayley graphs:

**Theorem 1** Let $X$ a Cayley graph. If $X$ is a BP-graph, then $X$ is a complete.

**Proof:** Let $X = \text{Cay}(G, S)$ and $\Sigma \subseteq BP(X)$. By Lemma 2, we can suppose that $1 \in \Sigma$ and, by Lemma 4, that $\Sigma$ is a clique; in other terms, we can suppose that $\Sigma = 1A^+ = A^+$ for some $A \in \mathcal{F}_S^{\text{max}}$.

If $|\Sigma| = 1$, every vertex of $X$ is a neighbor of the unique element of $\Sigma$ and, by vertex-transitivity, $X$ is complete. If $|\Sigma| \geq 2$, we note that $\Sigma = A^+$ possesses two vertices $a$ and $b$ verifying $ab^{-1} \in S$. Indeed, let $s \in S$, then $sA^+ \in \mathcal{K}(X)$ by Lemma 3. So, $sA^+$ intersects the BP-complete $A^+$, which gives the existence of $a, b \in A^+$ such that $sb = a$ or $ab^{-1} = s \in S$.

Now, let $x$ in $X$. Then $xA^+ \in \mathcal{K}(X)$ by Lemma 3 and intersects the BP-complete $A^+$. By Lemma 4, we get that $1 \sim x$. This proves that $N_X[1] = X$ and, as $X$ is vertex-transitive, implies that $X$ is a complete graph.

4 The case of vertex-transitive graphs

Sabidussi [4] [12] proved that any vertex-transitive graph is isomorphic to a quotient of some Cayley graphs and gave a characterization of such a Cayley graph. In order to use these results, we need to remind two definitions.

Firstly, given any graphs $X$ and $Z$ (looped or not), the lexicographic product $X[Z]$ is the graph with vertex set $V(X) \times V(Z)$ and there is an edge between $(x_1, z_1)$ and $(x_2, z_2)$ if and only if either $x_1 \sim x_2$ or $\{x_1 = z_2 \text{ and } z_1 \sim z_2\}$.

Secondly, the stabilizer $\text{Aut}(X)_x$ of $x \in V(X)$ is $\{g \in \text{Aut}(X), g \cdot x = x\}$ and the cardinal of $\text{Aut}(X)_x$ will be denoted $n$ (by vertex transitivity, it does not depend on the choice of the vertex $x$).
**Theorem 2** Let $X$ be a vertex-transitive graph with no loops and $x \in V(X)$. Then $\text{Cay}(\text{Aut}(X), S_x)$ is isomorphic to $X[K_n]$ with $S_x = \{g, g \in \text{Aut}(X) \text{ and } g.x \sim x\}$ and $n = |\text{Aut}(X)|$.

Let us recall an explicit characterization of this isomorphism for a chosen $x \in V(X)$. We suppose that $X$ is of order $N$ and put $V(X) = \{x_i, i \in [N]\}$ with $x_1 = x$ and the notation $[N]$ for the set of integers $\{1, 2, \ldots, N\}$. By vertex-transitivity, we choose $g_1 = \text{Id}_X, g_2, \ldots, g_N$ in $\text{Aut}(X)$ such that $g_i.x = x_i$ for $i \in [N]$. We also put $\text{Aut}(X)_x = \{h_j, j \in [n]\}$. The family $\{g_i\text{Aut}(X)_x, i \in [N]\}$ of left cosets of $\text{Aut}(X)_x$ is a partition of $\text{Aut}(X)$. For $f$ fixed in $[n]$, the set $\{g_ih_j, i \in [N]\}$ is in bijection with the set $\{g.x, i \in [N]\} = V(X)$ and, by considering the last set as $V(X) \times \{f\}$, the expression of adjacencies in $\text{Cay}(\text{Aut}(X), S_x)$:

$$g_ih_j \sim g_ih’_j \iff g_i.x \sim g_i.x \iff x_v \sim x_i$$

proves the isomorphism of Theorem 2

Now, it is clear that $X$ is reflexive if, and only if, for all vertices $x$ of $X$, $1 \in S_x$ and we will consider the following reflexive counterpart of Theorem 2 which will be more suitable to our case:

**Proposition 1** Let $X$ be a vertex-transitive reflexive graph and $x \in V(X)$. Then $\text{Cay}(\text{Aut}(X), S_x)$ is isomorphic to $X[K_n]$ with $S_x = \{g, g \in \text{Aut}(X) \text{ and } g.x \sim x\}$ and $n = |\text{Aut}(X)|$.

**Definition 3** ([12]) Let $X$ a graph, $n \in \mathbb{N}$ and $[n] = \{1, 2, \ldots, n\}$. The $n$-multiple of $X$ is the graph $nX$ defined by

$$V(nX) = V(X) \times [n] \quad \text{and} \quad (x, i) \sim (x’, i’) \iff x \sim x’$$

We note that if $X$ is reflexive, then $nX \cong X[K_n]$ and if $X$ is without any loop, $nX \cong X[\overline{K_n}]$. So, Theorem 2 and Proposition 1 are unified by saying that for every vertex-transitive graph and every vertex $x$ of $X$, $\text{Cay}(\text{Aut}(X), S_x)$ $\cong$ $nX$ where $n$ is the order of the stabilizer of $x$.

The difference between the two cases (reflexive or not) is that in the reflexive case, the cliques of $X$ are in a one to one correspondence with those of the lexicographic product, which is not the case otherwise.

**Lemma 5** Let $X$ a graph, $nX$ a multiple of $X$ and $p : nX \to X$ the canonical projection.

1. If $K \in \mathcal{K}(nX)$, then $p(K) \in \mathcal{K}(X)$

2. If $K \in \mathcal{K}(X)$ and $X$ is reflexive, then $\mathcal{K}(nX) = p^{-1}(\mathcal{K}(X))$.

**Proof:**

1. In $nX$, the completes are given by sets

$$\{(x_i, m_j), x_i \in V(X), m_j \in [n], i \in I \text{ s.t. } \forall i_1, i_2 \in I, x_{i_1} \sim x_{i_2}\}$$

So, it is clear that the projection of a complete of $nX$ is a complete of $X$. Now, let $K \in \mathcal{K}(nX)$ and let us suppose that $p(K) \notin \mathcal{K}(X)$. As $p(K) \in \mathcal{C}(X)$, it means that there is some vertex $x \notin p(K)$ such that $p(K) \cup \{x\} \in \mathcal{C}(X)$. But this would imply that $K \cup \{(x, 1)\}$ is a complete which contains strictly $K$.

2. For all $A \subset X$, by definition of the $n$-multiple, $p^{-1}(A) \cong nA$. Moreover, as $X$ is reflexive, all the edges $(x, i) \sim (x', i')$ exist and $p^{-1}(K)$ is a clique of $nX$ whenever $K$ is a clique of $X$; so, $\mathcal{K}(nX) \supset p^{-1}(\mathcal{K}(X))$. For the reverse inclusion, let $K \in \mathcal{K}(nX)$. By 1., $p(K) \in \mathcal{C}(X)$ and we have seen that $p^{-1}(p(K)) \in \mathcal{K}(nX)$. By maximality, we have necessarily $K = p^{-1}(p(K))$ and $\mathcal{K}(nX) \subset p^{-1}(\mathcal{K}(X))$. \hfill $\Box$

**Lemma 6** Let $X$ be a reflexive BP-graph. Then, for all integer $n \geq 1$, $nX$ is a BP-graph.

**Proof:** If $A \in B(P(X), p^{-1}(A) \cong nA$ is also a BP-complete of $nX$. Indeed, by Lemma 5, the cliques of $nX$ are of the form $p^{-1}(K)$ with $K \in \mathcal{K}(X)$ and $p^{-1}(A) \cap p^{-1}(K) = p^{-1}(A \cap K) \neq \emptyset$ because $A \cap K \neq \emptyset$ as $A \in B(P(X)$. Therefore, $p^{-1}(A)$ intersects all the cliques of $nX$. \hfill $\Box$

A direct consequence of Proposition 1 and Lemma 6 is:

**Corollary 1** Let $X$ a reflexive vertex-transitive graph and $x \in V(X)$. If $X$ is a BP-graph, then $\text{Cay}(\text{Aut}(X), S_x)$ is a BP-graph.

Now, from corollary 1 we get the result announced in the introduction:
Theorem 3. Every finite vertex-transitive BP-graph is a complete graph.

Proof: Let $X$ a vertex-transitive BP-graph of order $N$. Without loss of generality, we can suppose that $X$ is reflexive. By Corollary 1 we know that $Cay(Aut(X), S_x)$ is also a BP-graph. By Theorem 1 we conclude that $Cay(Aut(X), S_x)$ is a complete graph. We also know by Proposition 1 that $Cay(Aut(X), S_x) \cong X[K_n]$. So, $X[K_n]$ is a complete graph, which means that $X$ is a complete graph (actually, we have $X \cong X[K_n] \cong X[K_n]$).

From Theorem 3, the only vertex-transitive finite BP-graphs are the complete graphs. In particular, in a vertex-transitive graph, the existence of a BP-complete implies (trivially, as the graph is a complete) that all cliques are BP-complete. In a regular and not vertex-transitive graph, this implication is false. As an illustration and contrary to the graph $P$ of Figure 1 where all cliques are BP-completes, the variation $Q$ from $P$ presented in Figure 2 is a non complete regular finite BP-graph in which one can find cliques which are not BP-complete (for example, $C_1 \cup C_2' \cup C_3'' \cup \{s_{1,2,3}\}$ and $C_2 \cup C_1' \cup C_4'' \cup \{s_{1,2,4}\}$ are disjoint cliques).

Let us note also that $Q$ is not vertex-transitive; one can check it directly but, actually, it is a consequence of Theorem 3. Indeed, as a corollary of Theorem 3 we get the following criterion for identifying non vertex-transitive regular graphs:

Proposition 2. Let $X$ a non complete finite regular graph. If $X$ is a BP-graph, then $X$ is not vertex-transitive.

![Figure 2: In this variation $Q$ from the graph $P$, each graph $C_i$, $C_j'$ and $C_k''$ (with $i, j, k \in \{1, 2, 3, 4, 5\}$) is isomorphic to $K_2$ and there is a central complete graph of order 10; so, the graph is of order 40. Next, any $s_{i,j,k}$ is also adjacent to all vertices of $C_i \cup C_j \cup C_k \cup C_i' \cup C_j' \cup C_k' \cup C_i'' \cup C_j'' \cup C_k''$. The other edges are $c \sim c'$, $c \sim c''$ and $c' \sim c''$ with $c \in C_i$, $c' \in C_j'$ and $c'' \in C_k''$, for all mutually distinct $i, j, k$. This graph is 28-regular and is a BP-graph (the central clique $K_{10}$ is a BP-complete).](image)

5. Connection with evasiveness

Given a boolean function $F : \{0, 1\}^n \to \{0, 1\}$, the question of evasiveness refers to the following situation: a person who knows $F$ has to guess $F(\epsilon)$ for any (unknown) element $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{0, 1\}^n$ by asking question of the form «is $\epsilon_i = 0$?» for any $1 \leq i \leq n$ (assuming that this person always get
the correct answer, YES or NO). Of course, after \( n \) questions, this person will know all coordinates of \( \epsilon \) and, consequently, will also know \( F(\epsilon) \). The function \( F \) is said non-evasive if, whatever is the element \( \epsilon \), the person can know the value of \( F(\epsilon) \) in at most \( (n - 1) \) questions \([1, 6]\). The function is said evasive (or exhaustive in \([11]\)) when it is not non-evasive. This formulation is a generalization of the original question on evasiveness of graph properties, a graph property on (non oriented) graphs of order \( k \) being nothing but a particular boolean function on \( \{0, 1\}^n \) with \( n = \binom{k}{2} \).

A boolean function \( F \) is said transitive if it is invariant under the transitive action on \( \{1, \ldots, n\} \) of a subgroup \( \Gamma \) of the permutations group \( S_n \) (i.e., for all permutations \( \sigma \in \Gamma \) of \( \{1, 2, \ldots, n\} \), \( F(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = F(\sigma(\epsilon_1), \sigma(\epsilon_2), \ldots, \sigma(\epsilon_n)) \)) and \( F \) is said monotone if \( \epsilon \preceq \epsilon' \) implies \( F(\epsilon) \preceq F(\epsilon') \) where the poset-structure of \( \{0, 1\}^n \) is given by \((\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \preceq (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n)\) if, and only if, \( \epsilon_i \leq \epsilon'_i \) for all \( 1 \leq i \leq n \). The generalized Aanderaa-Karp-Rosenberg conjecture (or GAKR-conjecture) claims that any non constant, monotone and transitive boolean function must be evasive \([6]\).

Equivalently, one can reformulate these questions in terms of set systems. For a set \( V \) of cardinality \( n \), a set system \( F \subseteq 2^V \) (as \( F^{-1}(0) \) or \( F^{-1}(1) \) in the preceding description) is called non-evasive if, whatever is \( A \subseteq V \), a person who knows \( F \) can know if \( A \in F \) in at most \( (n - 1) \) questions of the form \( \{ \text{is } v_i \text{ in } A \? \} \) for any \( 1 \leq i \leq n \) and \( V = \{v_1, \ldots, v_n\} \). Now, a monotone set system is a family of sets closed for inclusion, i.e. an abstract simplicial complex. So we get the notion of evasive simplicial complex. A simplicial complex is vertex-homogeneous if its automorphism group of simplicial automorphisms acts transitively on its vertex set and the evasiveness conjecture for simplicial complexes \([5]\), which claims that a non-evasive, vertex-homogeneous and non empty simplicial complex is a simplex, is equivalent to the GAKR-conjecture.

We recall that a flag complex is nothing but the clique complex of its 1-skeleton considered as a graph, so, by restricting to flag complexes, we have the following weaker claim:

**Evasiveness conjecture for graphs:** A non-evasive, vertex-transitive and non empty finite graph is a complete graph.

We must emphasize that this formulation is different from the original question of evasiveness conjecture for graph properties, which is a special case obtained by considering a graph with \( \binom{k}{2} \) vertices for some integer \( k \). Now, for a known graph \( X \), the tested set system is the family of sets of vertices which induce of complete subgraph of \( X \) and \( X \) is non-evasive if, for any \( A \subseteq V(X) \), one can guess if \( A \) induces a complete subgraph of \( X \) in at most \( n - 1 \) questions of the form \( \{ \text{is } x_i \text{ in } A \? \} \) where \( V(X) = \{x_1, x_2, \ldots, x_n\} \) is the set of vertices of \( X \).

In \([11]\), following a remark due to Lovász, the authors pointed out that a positive answer to the evasiveness conjecture implies that a finite vertex-transitive graph with a clique which intersects all other cliques (i.e., a vertex-transitive BP-graph) is a complete.

Indeed, a BP-graph is non-evasive; let us recall the proof of this fact. Let \( X \) a BP-graph and let \( \Sigma \in BP(X) \). Now, let \( A \) an (unknown) set of vertices of \( X \). We must verify that we can know if \( A \) is a complete of \( X \) in at most \( n - 1 \) questions (with \( X \) of order \( n \)). First, we ask \( \{ \text{is } x \text{ in } A \? \} \) for every \( x \in V(X) \) \( \setminus \Sigma \). After these questions, either there exist two nonadjacent vertices in \( A \) and we know that \( A \) is not a complete, or \( A' = A \cap (V(X) \setminus \Sigma) \) is a complete. In this second case, as \( \Sigma \) intersects all cliques of \( X \), there is at least one \( x \in \Sigma \) such that \( A' \cup \{x\} \) is a complete. Actually, one can complete \( A' \) in a clique by adding vertices of \( K = \{x, x \in \Sigma, A' \cup \{x\} \text{ is a complete}\} \). The clique \( A' \cup K \) is the unique clique containing \( A' \) and it contains \( A' \) strictly. Now, in order to know if \( A \) is a complete, it is sufficient to ask \( \{ \text{is } x \text{ in } A \? \} \) for every \( x \in V(X) \) \( \setminus (A' \cup K) \) (and \( A \) will be a complete if, and only if, the answer is NO for all these \( x \)). It means that \( X \) is non-evasive.

So, in other terms, Theorem \( 3 \) of section 4 gives a positive answer to the particular case of the evasiveness conjecture for graphs obtained by restricting to vertex-transitive BP-graphs.

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