TYPES OF MIXINGS AND TRANSITIVITIES IN TOPOLOGICAL DYNAMICS

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Abstract. Using the combinatorial properties of subsets of integers, a classification of metric dynamical systems was given in [V. Bergelson and T. Downarowicz, Large sets of integers and hierarchy of mixing properties of measure-preserving systems, Colloquium Mathematicum 110(1), (2008), 117-150]. As a result, some new families emerged. Here, their counterparts in topological dynamics has been considered. The differences will be discussed and new classes of systems in topological dynamics will be introduced.

Introduction

The parallel concepts for measure-theoretical dynamical systems (MDS) and topological dynamical systems (TDS) such as ergodicity vs minimality, measure-theoretical entropy vs topological entropy, measure-theoretical mixing vs topological mixing, etc. has been always a subject for investigation [11, 13, 14]. In this respect, we first consider the different notions of mixing in TDS, that is, we partition the family of weak mixing TDS into different sub-classes and will compare them to the similar ones considered for MDS in [8]; however, we extend this task to transitive TDS as well. That is, we will also introduce different sub-classeses in transitive topological systems. The main tool in [8] was combinatorial number theory which is one of ours and we also use spacing shifts to construct our examples. Spacing shifts, a class of subshifts, was first introduced by Lau and Zame in 1973 for constructing an example of a weak mixing system which is not strong mixing [21]. The fact that this can be done is due to the peculiarity of the spacing shifts that allows one to investigate dynamical properties of a subshift via combinatorial number theory. For instance, Lau and Zame noticed that if $P$ is thick but not cofinite, then the associated spacing shift denoted by $\Sigma_P$ is weak mixing but not strong mixing. Our goal is more or less the same. We use families of subsets of integers, say from Hindman’s table presented in [12] and try to establish different examples of mixings and transitives in systems. An objective is to sort out if different families of integers do introduce different dynamics on the respective spacing shifts as well. The families we are considering appear in diagram (2.6).

Our arrangement in this note is as follows. The Hindman’s table is introduced in Section 1. The properties of a family $\mathcal{F}$ when an $\mathcal{F}$-mixing system occurs is given in section 2 and via examples by spacing shifts we show that they are really different mixings. Section 3 is devoted to different transitive families using families of difference sets like $\mathcal{F} - \mathcal{F}$ and we show that how these difference sets can be placed in Hindman’s table. The application of our earlier results to minimal systems

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are considered in Section 4. There we show that for minimals, the results very much resemble the corresponding results in MDS.

1. Preliminaries

A TDS is a pair \((X, T)\) such that \(X\) is a compact metric space and \(T\) is a homeomorphism. In a TDS, the return times set is defined to be \(N(U, V) = \{n \in \mathbb{Z} : T^n(U) \cap V \neq \emptyset\}\) where \(U\) and \(V\) are open (non-empty and open) sets. A TDS is transitive if for any two open sets \(U\) and \(V\), \(N(U, V) \neq \emptyset\); and it is totally transitive if \((X, T^n)\) is transitive for any \(n\). A TDS is weak mixing if the product system \((X \times X, T \times T)\) is transitive and it is (strong) mixing if \(N(U, V)\) is cofinite for any open sets \(U, V\).

Let \(\mathcal{F}\) be a family of nonempty subsets of \(\mathbb{Z}\). This means \(\mathcal{F}\) is hereditary upward: if \(F_1 \in \mathcal{F}\) and \(F_1 \subseteq F_2\), then \(F_2 \in \mathcal{F}\). The dual of \(\mathcal{F}\), denoted by \(\mathcal{F}^*\), is defined to be all subsets of \(\mathbb{Z}\) meeting all sets in \(\mathcal{F}\):

\[\mathcal{F}^* = \{G \subseteq \mathbb{Z} : G \cap F \neq \emptyset, \forall F \in \mathcal{F}\}\.

A family \(\mathcal{F}\) is called partition regular if \(F \in \mathcal{F}\) is partitioned into finite sets \(F = F_1 \cup \cdots \cup F_k\), then there is \(i\) such that \(F_i \in \mathcal{F}\). A non-empty family closed under finite intersections is called a filter. It is known that if \(\mathcal{F}\) is partition regular, then \(\mathcal{F}^*\) is a filter. A filter which is partition regular is called an ultrafilter. The collection of all ultrafilters is denoted by \(\beta\mathbb{Z}\) and when endowed with appropriate topology, becomes the Stone–Čech compactification of \(\mathbb{Z}\). There is a natural semigroup structure in \(\beta\mathbb{Z}\) extending the addition operation of \(\mathbb{Z}\) [7]. An ultrafilter \(p \in \beta\mathbb{Z}\) is called idempotent if \(p + p = p\).

For a family \(\mathcal{F}\) and \(k \in \mathbb{Z}\), the shifted family is defined as \(\mathcal{F} + k = \{F + k : F \in \mathcal{F}\}\) where \(F + k = \{n + k : n \in F\}\). An example of a shift invariant is \(\mathbb{Z}\), the family of all cofinite sets (\(I = \{F \subseteq \mathbb{Z} : |F| = \infty\}\)). Also, starting with a family \(\mathcal{F}\), families \(\mathcal{F}_+\) and \(\mathcal{F}_\ast\) defined as

\[\mathcal{F}_+ := \bigcup_{k \in \mathbb{Z}} (\mathcal{F} + k), \quad \mathcal{F}_\ast := \bigcap_{k \in \mathbb{Z}} (\mathcal{F} + k)\]

are shift invariant families [8].

We have \(\mathcal{F}_\ast \subseteq \mathcal{F} \subseteq \mathcal{F}_+\) and \(\mathcal{F}^* = (\mathcal{F}_+)^*\) [8]. Also, if \(\mathcal{F} \subseteq \mathcal{F}'\) then \(\mathcal{F}_+ \subseteq \mathcal{F}'_+\) and \(\mathcal{F}^* \subseteq \mathcal{F}'^*\) which implies that \(\mathcal{F}^*_+ \subseteq \mathcal{F}_+^*\). If \(\mathcal{F}\) is a filter, so is any shift of \(\mathcal{F}\) and since the intersection of filters is again a filter, \(\mathcal{F}_\ast\) is a filter.

A set \(F \subseteq \mathbb{Z}\) is called an IP-set if it contains the set of finite sums of some sequence of nonzero integers \(A = \{a_n\}_{n \geq 1}\) and is denoted by \(FS(A)\); thus, \(FS(A) = \{a_i + a_{i_2} + \cdots + a_{i_n} : i_j < i_{j+1}\} \subseteq F\). Equivalently, a subset \(F \subseteq \mathbb{Z}\) is an IP-set if and only if there is an idempotent \(p \in \beta\mathbb{Z}\) such that \(F \in p\) [7, Theorem 1.5]. Denote by \(IP\) the family of all IP-sets.

Set \(d^+(A) := \limsup_{M-N \to -\infty} \frac{|A\cap [N,M]|}{M-N+1}\) and call it the upper Banach density of a set \(A \subseteq \mathbb{Z}\). We denote the family of all positive upper Banach density by \(\mathcal{F}_{\text{pubd}}\). Also, the lower Banach density of a set \(A \subseteq \mathbb{Z}\) is defined similarly as \(d^-(A) = \liminf_{M-N \to -\infty} \frac{|A\cap [N,M]|}{M-N+1}\).

A subset \(E \subseteq \mathbb{Z}\) is called

- Delta-set if there exists a sequence of natural numbers \(S = (s_n)_{n \in \mathbb{N}}\) such that the difference set \(\Delta(S) = \{s_i - s_j : i > j\} \subseteq E\). Let \(\Delta\) be the family of all Delta-sets.
Let $\mathcal{P}(\mathbb{Z})$ be the power set of $\mathbb{Z}$. For any family $\mathcal{F}$
\[ \tau \mathcal{F} = \{ F \in \mathcal{P}(\mathbb{Z}) : (F - k_1) \cap \cdots \cap (F - k_n) \in \mathcal{F}, \{k_1, \ldots, k_n\} \subseteq \mathbb{Z} \} \]
is defined to be the \textit{thick family} contained in $\mathcal{F}$. A family $\mathcal{F}$ is \textit{thick family} if and only if $\tau \mathcal{F} = \mathcal{F}$ [18].

\textbf{Theorem 2.1.} $\tau \mathcal{F} \subset \mathcal{F}_\bullet$ and if $\mathcal{F}$ is a filter, then $\tau \mathcal{F} = \mathcal{F}_\bullet$.

\textit{Proof.} $\tau \mathcal{F} \subset \mathcal{F}_\bullet$ follows trivially from definition. For the second part, since $\mathcal{F}$ is a filter, $\mathcal{F}_\bullet$ is a filter as well. Let $F \in \mathcal{F}_\bullet$ and pick $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Since $\mathcal{F}_\bullet$ is a shift invariant family and it is closed under finite intersections, by the filter property, we have $(F - k_1) \cap \cdots \cap (F - k_n) \in \mathcal{F}_\bullet$. Hence $F \in \tau \mathcal{F}$ and $\tau \mathcal{F} = \mathcal{F}_\bullet$. \hfill \Box

The classical different mixing concepts for a TDS are (strong) mixing, mild mixing and weak mixing. They can be characterized in terms of combinatorial properties of $\mathcal{N}$, the family which is generated by return times set of $(X, T)$. A TDS is mixing, mild mixing or weak mixing if and only if $\mathcal{N}$ is cofinite, $(\mathcal{I} \mathcal{P} - \mathcal{I} \mathcal{P})^*$ or thick respectively [14].

A TDS $(X, T)$ is called $\mathcal{F}$-\textit{transitive} if for any two open sets $U, V \subset X$, $N(U, V) \in \mathcal{F}$ and it is called $\mathcal{F}$-\textit{mixing} if the product system $(X \times X, T \times T)$ is $\mathcal{F}$-transitive [18]. It is easy to see that $(X, T)$ is $\mathcal{F}$-mixing if and only if it is weak mixing and $\mathcal{F}$-transitive if and only if $N(U, V) \cap N(U, U) \in \mathcal{F}$ for any two open sets $U, V$.

Now we may explicitly define other mixings by considering the combinatorial structure of $\mathcal{N}$. A motivation for this is the following hierarchy in the combinatorial number theory:

\begin{equation}
\mathcal{I}^* \subset \mathcal{D}^* \subset \mathcal{P}^* \subset \mathcal{C}^*.
\end{equation}

As an application, we will define $\mathcal{D}^*$, $\mathcal{P}^*$, $\mathcal{D}^*$ and $\mathcal{C}^*$-mixings and by constructing examples, the fact that they are different families will be shown. First a theorem:

\textbf{Theorem 2.2.} \hspace{1cm} (1) A TDS $(X, T)$ is $\mathcal{F}$-\textit{transitive} if and only if it is $\mathcal{F}_\bullet$-\textit{transitive}.

(2) If $\mathcal{F}$ is a filter, then $(X, T)$ is $\mathcal{F}$-\textit{mixing} if and only if it is $\mathcal{F}_\bullet$-\textit{mixing}.

(3) $(X, T)$ is $\mathcal{F}$-\textit{mixing} if and only if it is $\tau \mathcal{F}$-\textit{mixing}.
Lemma 2.5. Suppose $P$. See [4] for a detailed study of $\Sigma$. 

Definition 2.3. Let $P$ be a subset of integers with $P = \overline{P}$. Define an invertible spacing shifts to be the subshift

$$\Sigma_P^Z = \{s \in \{0, 1\}^Z : s_i = s_j = 1 \Rightarrow i - j \in P \cup \{0\}\}.$$ (2.2)

It follows from the definition that $P = N([1], [1])$.

Remark 2.4. Definition 2.3 is an invertible version of the classical spacing shifts which are defined for $P \subset \mathbb{N}$. We will denote them by $\Sigma_P^N$ and then (2.2) changes to

$$\Sigma_P^N = \{s \in \{0, 1\}^N : s_i = s_j = 1 \Rightarrow |i - j| \in P \cup \{0\}\}.$$ 

See [4] for a detailed study of $\Sigma_P^N$. Note that if $P^+ := \{p \in P : p \geq 0\}$ and $P^- := -P^+$ then $P = P^+ \cup P^-$ and $\mathcal{L}(\Sigma_P^Z) = \mathcal{L}(\Sigma_P^{N+})$.

Lemma 2.5. Suppose $U, V \in \mathcal{L}(\Sigma_P^Z)$. Then there exist $k_1, k_2, \ldots, k_n \in \mathbb{Z}$ such that $N(U, V) \supseteq \bigcap_{i=1}^n (P - k_i)$.

Proof. If either $U$ or $V$ is 0 then $N(U, V) = \mathbb{Z}$. Otherwise, we may assume

$$U = 0^{a_1}10^{p_1-1}10^{p_2-1}11\cdots10^{p_{r-1}}10^{b_1}$$

(2.3)

$$V = 0^{c_1}10^{q_1-1}10^{q_2-1}11\cdots10^{q_{r-1}}10^{d_1}.$$
where \( p_i, q_j \in P^+ \) for \( 1 \leq i \leq r, 1 \leq j \leq t \) and \( a_1, b_1, c_1, d_1 \in \mathbb{N} \cup \{0\} \) and there are \( A, B \) and \( C \) such that \( N(U, V) = A \cup B \cup C \) where

\[
A = \{|W| : VWU \in \mathcal{L}(\Sigma^Z_P)\} + |V|, \\
B = \{-|W| : UWV \in \mathcal{L}(\Sigma^Z_P)\} - |U|, \\
|C| < \infty \ (|C| \leq |V| + |U|).
\]

In particular, \( A \cup B \subseteq N(U, V) \).

Let \( W \in \mathcal{L}(\Sigma^Z_P) \). If we replace some of the 1’s in \( W \) with zero’s and calling the new word \( W' \), then \( W' \in \mathcal{L}(\Sigma^Z_P) \). Therefore, we may let

\[
A = \{n \in \mathbb{N} : V0^nU \in \mathcal{L}(\Sigma^Z_P)\} + |V|
\]

and the same for \( B \). Suppose \( n \in A \). Then by the way \( V \) and \( U \) are presented in \((2.3)\), we must have \( d_1 + n + a_1 \in P^+ \), that is, \( n \in P^+ - (d_1 + n_1) \). Note that \( d_1 + n + a_1 \) is the length of space between the last 1 in \( V \) and the first 1 in \( U \). In fact, \( d_i + n + a_j \in P^+ \) where \( d_i \) is the position of \( i \)’th 1 from right on \( V \) and \( a_j \) is the position of the \( j \)’th 1 from left in \( U \). Hence, \( n \in P^+ - (d_i + a_j) \) where \( 1 \leq i \leq t, 1 \leq j \leq r \). Let \( \{l_1, \cdots, l_m\} = \{d_i + a_j + |V| : 1 \leq i \leq t, 1 \leq j \leq t\} \) for appropriate \( m \). This means that \( A = \cap_{i=1}^m (P^+ - l_i) \). Similarly, \( B = \cap_{k=1}^m (P^+ - k_i') \) and the proof is complete by setting \( \{k_1, \cdots, k_n\} = \{l_i : 1 \leq i \leq m\} \cup \{l_i' : 1 \leq i' \leq m'\} \).

**Corollary 2.6.** If \( P \in \tau F \) then \( \Sigma^Z_P \) is \( \mathcal{F} \)-transitive.

Combinatorial properties of \( P \) could lead to dynamical properties for \( \Sigma^N_P \) or \( \Sigma^Z_P \). For instance, \( \Sigma^Z_P \) is mixing if and only if \( P \) is cofinite and it is weak mixing if and only if \( P \) is thick \([4]\). The same is true for \( \Sigma^Z_P \).

**Theorem 2.7.** \( \Sigma^Z_P \) is \( \mathcal{F} \)-mixing if and only if \( P \in \tau F \).

**Proof.** The necessity is a consequence of Theorem 2.2. For the converse, \( P \in \tau F \) if and only if any intersection of finite shifts of \( P \) is in \( \mathcal{F} \). Applying Lemma 2.5, \( N(U, V) \supseteq \bigcap_{k=1}^m (P-k_i) \) and \( N(U, U) \supseteq \bigcap_{k=1}^m (P-k_i') \) for some \( k_1, k_2 \). So \( N(U, V) \cap N(U, U) \) is in \( \mathcal{F} \) which means \( \Sigma^Z_P \) is \( \mathcal{F} \)-mixing. \( \square \)

Therefore, if \( P \in \{\mathcal{I}^*, \Delta^*, \mathcal{I}P^*, \mathcal{D}^*, \mathcal{C}^*\} \), then \( \Sigma^Z_P \) is \( \mathcal{F} \)-mixing for the respective \( \mathcal{F} \in \{\mathcal{I}^*, \Delta^*, \mathcal{I}P^*, \mathcal{D}^*, \mathcal{C}^*\} \). The classical mixing is the \( \mathcal{I}^* \)-mixing, so we just say mixing instead of \( \mathcal{I}^* \)-mixing.

In \([8]\), it has been shown that not all the mixings defined for the families in \( \{\mathcal{I}^*, \Delta^*, \mathcal{I}P^*, \mathcal{D}^*, \mathcal{C}^*\} \) are different in the measure theoretic case. See also section 4 for the equivalencies of some of these mixings for certain classes in TDS. However, as one may expect, they are different families in a general TDS as the next result shows.

**Theorem 2.8.** The following hierarchy is proper.

\[
\text{mixing} \subset \Delta^*-\text{mixing} \subset \mathcal{I}P^*\text{-mixing} \subset \mathcal{D}^*\text{-mixing} \subset \mathcal{C}^*\text{-mixing} \subset \text{weak mixing}
\]

**Proof.** We choose our examples from spacing shifts. First, we show that there is a \( \Delta^* \)-mixing spacing shifts which is not mixing. Let \( E = \{n^2 : n \in \mathbb{N}\} \) and note that \( E \) is not a \( \Delta \)-set and set \( P^+ = \mathbb{N} \setminus E \). Then \( P \) is \( \Delta^*_+ \) and as a result \( \Sigma^Z_P \) is \( \Delta^* \)-mixing. However, it cannot be mixing because \( P \) is not cofinite.
For the remaining cases, recall first that there are examples of subsets of integers which are $C^*$ but not $D^*$, $D^*$ but not $IP^*$ and $IP^*$ but not $\Delta^*$ [8]. This implies that the associated spacing shifts are different.

To show that there exists weak mixing system which is not $C^*$-mixing notice that \( \mathcal{PS} = (\mathcal{C}_+)^* \) [8], thus
\[
(2.5) \quad C^*_* = (\mathcal{C}_+)^* = (\mathcal{PS})^* = \mathcal{T}\mathcal{S}.
\]
In particular, $\mathcal{T}\mathcal{S}$ is a filter. Now if we pick $E \subset \mathbb{Z}$ so that both $E$ and $E^c$ are thick sets, then $E$ is not $C^*$ and therefore, $\Sigma^*_E$ is a weak mixing which is not $C^*$-mixing.

Let us recall that in MDS mixing and $\Delta^*$-mixing in one side and $D^*$-mixing and weak mixing (and so $C^*$-mixing as well) in the other side were the same [8].

By definition a TDS is mild mixing if it is $(IP - IP)^*$-transitive. This motivates us to consider the following diagram, an extension of (2.4), for achieving a possible classification in TDS via these mixings and transitivities concepts.

\[
\begin{align*}
\mathcal{T}^* & \subset \Delta^* \subset IP^* \subset D^* \subset C^* \\
\Delta^* = (\mathcal{I} - \mathcal{I})^* & \subset (\Delta - \Delta)^* \subset (IP - IP)^* \subset (D - D)^* \subset (C - C)^*.
\end{align*}
\]

We showed in Theorem 2.7 that how different mixings can be defined for the first row. The families in the second row are not necessarily filters and we may only define transitivity for them. The above list in (2.6) is rather complete with respect to Hindman’s table [12]; for as we will point out later, $(\mathcal{S} - \mathcal{S})^* = (\mathcal{C} - \mathcal{C})^* = (\mathcal{PS} - \mathcal{PS})^*$ and $(\mathcal{F}_{pubd} - \mathcal{F}_{pubd})^* = (D - D)^*$. Now the first problem to be set is if they actually represent different families in TDS; though from set theoretical point of view, they are all different families. Recall that whether there is a $(C - C)^*$-transitive system which is not $(D - D)^*$-transitive is yet an open problem first raised in [16]. Except this case, we will show in sequel that all other families in (2.6) represent different transitive TDS. One case is already known: there exists $(IP - IP)^*$-transitive which is not $IP^*$-mixing [18, Example C]. Other cases are treated in Theorem 3.3 and Lemma 3.7.

We also look for the thick families inside the families in the second row in (2.6). By Theorem 2.2, these thick families define some other mixings. For instance, any thick families in the $(IP - IP)^*$-transitive family will be a mild mixing system. Then we investigate how these new mixing families are different from those on the first row.

3. Characterization of $(\mathcal{F} - \mathcal{F})^*$-transitive and $\mathcal{F}^*$-mixing

An increasing set \( \{a_n\}_{n \in \mathbb{N}} \) is said to have progressive gaps if it contains a subsequence \( \{a_{n_k}\} \) (call each finite subset \( \{a_{n_k+1}, a_{n_k+2}, \ldots, a_{n_{k+1}}\} \) a chunk) such that for \( n_k + 1 < i \leq n_{k+1} \) one has \( a_i - a_{i-1} > a_{n_{k+1}} - a_i \) (inside each chunk every gap is larger than the distance to the right end of the chunk) and \( a_{n_{k+1}} - a_{n_k} \rightarrow \infty \) (the gaps between the chunks tend to infinity) [8].

Lemma 3.1. There exists a set $B \subset \mathbb{N}$ such that $\Delta(B)$ is not a $(\Delta - \Delta)$-set and $(\Delta(B) - \Delta(B))$ is not an $IP$-set.

Proof. Suppose $B = \{b_n\}_{n \in \mathbb{N}}$ is a rapidly growing sequence of natural numbers, for instance, let $b_n > 4 \sum_{i=1}^{n-1} b_i$ ($b_n > 4 \sum_{i=1}^{n-1} (b_i + i)$) was considered in [18, Example
C]). Let $A = \Delta(B)$ and observe that $A$ has progressive gaps. Also, for a certain distance $d$, $A_d = \{a_{ij} \in A : a_{ij+1} - a_{ij} = d\}$ is either finite or $\lim_{j \to \infty} (a_{ij+1} - a_{ij}) = \infty$. This is implied by the fact that the distance $d$ between any two elements must eventually occur only inside the chunks of $A$ and then only once in every chunk.

To show that $\Delta(B)$ is not a $(\Delta - \Delta)$-set, it suffices to show that for any arbitrary increasing sequence $S \subset \mathbb{N}$, $L = (\Delta(S) - \Delta(S))$ cannot have progressive gaps.

First, since $s_j - s_i = (s_j - s_k) - (s_k - s_i) \in L$, $\Delta(S) \subset L$. Now pick any $r, t, j$; $r > t > j$ and note that for any $n > r$ we have $(s_n - s_r), (s_n - s_t), (s_n - s_j)$ and $(s_n - s_j) - (s_r - s_t)$ in $L$ and

$$d = (s_n - s_j) - (s_n - s_t) = [(s_n - s_j) - (s_r - s_t)] - (s_n - s_r).$$

This means that for any $n$, $\lim_{n \to \infty} \{(s_n - s_j) - (s_r - s_t)\} = (s_t - s_r) \neq \infty$ and this in turn shows that $L$ does not have progressive gaps.

For the second part assume that $(\Delta(B) - \Delta(B))$ contains an IP-set $F \subset \mathbb{N}$. Since $\Delta(B) \subset (\Delta(B) - \Delta(B))$ and $\Delta(B)$ has progressive gaps, infinitely many members of $F$ are in $(\Delta(B) - \Delta(B)) \setminus \Delta(B)$. This enables us to choose $f_1, f_2 \in F \cap ((\Delta(B) - \Delta(B)) \setminus \Delta(B))$ so that if $f_1 = (b_{t_1} - b_{r_1}) - (b_{r_1} - b_{r_1}), f_2 = (b_{t_2} - b_{r_2}) - (b_{r_2} - b_{r_2})$, then $b_{r_1} < b_{r_i}, 1 \leq i \leq 7$. Let $f = f_1 + f_2 = (b_{t_1} - b_{r_1}) - (b_{r_1} - b_{r_2}) \in F$. Then by the way $B$ has been chosen, $r_1 = t_1, \{r_2, r_3\} = \{t_2, t_3\}$ and $r_4 = t_4$. This means $f_2$ must be zero which is absurd. □

Using the concept of chunks, Bergelson–Downarowicz prove that there exists an IP*-set which is not $\Delta^*_c$ [8, Theorem 2.11]. To prove this fact, they show that any IP*-set, likewise any $(\Delta - \Delta)$-set, cannot have progressive gaps. Then they consider a set $E$ as a union over all integers $k$ of shifted by $r_k$ of $\Delta$-sets $E_k$ such that $E$ has progressive gaps, and as a result it cannot contain any shifted IP-set. This argument shows that $E$ does not have any $(\Delta - \Delta)$-set either. Now, the complement of such $E$ is $(\Delta - \Delta)^*_c$ and not $\Delta^*_c$. See [8, proof of Theorem 2.11] for the construction of $E$. So we have:

**Lemma 3.2.** There exists a $(\Delta - \Delta)^*_c$-set which is not $\Delta^*_c$-set.

**Theorem 3.3.**

1. There exists a $(\Delta - \Delta)^*$-transitive which is not $\Delta^*$-mixing.
2. There exists an $(\mathcal{LP} - \mathcal{LP})^*$-transitive which is not $(\Delta - \Delta)^*$-transitive.

**Proof.** (1) We show that $(\Delta - \Delta)^*_c$ contains a thick family $F$ different from $\Delta^*_c$. Then by applying Corollary 2.6, $\Sigma^*_p$ will be the required space.

Let $E$ be the set given in Lemma 3.2 and set $P^+ := E^c \subset \mathbb{N}$, $P := P^+ \cup P^-$. Also let

$$F := \{(P + k_1) \cap (P + k_2) \cap \cdots \cap (P + k_n) : \{k_1, k_2, \ldots, k_n\} \subset \mathbb{Z}\}.$$ 

One needs to show that any element of $F$ is a $(\Delta - \Delta)^*_c$-set and to have this we prove that the complement of an element of $F$ does not contain a $(\Delta - \Delta)$-set. Since $(P + k_1) \cap \cdots \cap (P + k_n) = ((P^+ + k_1) \cap \cdots \cap (P^+ + k_n)) \cup ((P^- + k_1) \cap \cdots \cap (P^- + k_n))$, $F$ will be a thick family in $(\Delta - \Delta)^*$ if we show that $(P^+ + k_1) \cap \cdots \cap (P^+ + k_n)$ is a $(\Delta - \Delta)^*$-set or equivalently $E + k_1 \cup \cdots \cup (E + k_n)$ does not contain any $(\Delta - \Delta)$-set.

We show this fact for $E \cup (E + k)$, general case follows similarly. Assume the contrary and let $L$ be a $(\Delta - \Delta)$-sets such that $L \subset E \cup (E + k)$. For any $i \in \mathbb{N}$, choose $\{l_i^1, l_i^2, l_i^3, l_i^4\} \subset \mathbb{N}$ satisfying (3.1), i.e. $l_i^2 - l_i^1 = l_i^4 - l_i^3 = d$ and observe that for all $i$, $l_i^3 - l_i^1 = l_i^4 - l_i^2$. We may assume that $l_i^3 - l_i^1 > k$. Clearly $l_i^1, l_i^2, l_i^3, l_i^4$ cannot
be in $E$ or $E+k$. Therefore, suppose for some $i$, $l_1^i \in (E+k)$ and $l_2^i \in E$, $2 \leq j \leq 4$ (similar argument applies for other cases). Then $l_1^i - k$ is in the same chunk of $E$ containing $l_2^j$’s. But one has $l_2^j - (l_1^i - k) = k + d < l_4^j - l_2^j$. This is absurd since $E$ has progressive gaps.

(2) Choose $B$ satisfying Lemma 3.1 and set $P^+ := (\Delta(B) - \Delta(B))^c \subseteq \mathbb{N}$; then $P = P^+ \cup P^-$ is $\mathcal{IP}_*^*$ which is not $(\Delta - \Delta)^*$. So the associated $\Sigma_p^\mathbb{N}$ is $\mathcal{IP}^*$-mixing but not $(\Delta - \Delta)^*$-transitive. Now since any $\mathcal{IP}^*$-mixing is $(\mathcal{IP} - \mathcal{IP})^*$-transitive, $\Sigma_p^\mathbb{N}$ must be $(\mathcal{IP} - \mathcal{IP})^*$-transitive. □

For any $\epsilon > 0$ there exists $A \subseteq \mathbb{N}$ such that $d(A) > 1 - \epsilon$ and there is not any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathbb{N}$ and $t \in \mathbb{Z}$ such that $t + FS(\{x_n\}_{n \in \mathbb{N}}) \subseteq A$ [6, Theorem 2.20]. This means that there is a set with positive density which is not a $D$-set and so $D$ and $\mathcal{F}_\mathcal{P}_d$ are actually different families. However, in the next result, we will show that $\Delta(D) = \Delta(\mathcal{F}_\mathcal{P}_d)$.

Before that we need some more terminologies.

Let $\mathcal{F}$ be a family. The block family of $\mathcal{F}$, denoted by $b\mathcal{F}$, is the family consisting of sets $F \subseteq \mathcal{F}$ for which there exists some $F' \in \mathcal{F}$ such that for every finite subset $W$ of $F'$, $m + W \subseteq F$ for some $m \in \mathbb{Z}$ [20]. We have $b\mathcal{S} = \mathcal{P}\mathcal{S}$, $b\mathcal{F}_\mathcal{P}_d = \mathcal{F}_\mathcal{P}_d$ and $b\mathcal{I}^* = \mathcal{I}$. Also, by [20, Lemma 2.1], $\Delta(\mathcal{F}) = \Delta(b\mathcal{F})$.

For $A \subseteq \mathbb{Z}$, let $1_A = (x_n)_{n \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ with $x_n = 1$ whenever $n \in A$. If $d^\omega(A) > 0$, then there exists a transitive system $(Y, \sigma)$, $Y \subset \overline{O}(1_A)$ with an invariant Borel probability measure $\mu$ with full support (see comments prior to [18, Theorem 2.4]). Any TDS with such an invariant measure is called an $E$-system.

**Theorem 3.4.** $b\mathcal{D} = \mathcal{F}_\mathcal{P}_d$. In particular, $(D - D) = (\mathcal{F}_\mathcal{P}_d - \mathcal{F}_\mathcal{P}_d)$ and any set in $(\mathcal{F}_\mathcal{P}_d - \mathcal{F}_\mathcal{P}_d)$ is an $(\mathcal{IP} - \mathcal{IP})$-set.

**Proof.** The proof will be similar to proof in [13, Example 2.4]. We have $\mathcal{D} \subseteq \mathcal{F}_\mathcal{P}_d$ and hence $b\mathcal{D} \subseteq b\mathcal{F}_\mathcal{P}_d = \mathcal{F}_\mathcal{P}_d$. Let $A \in \mathcal{F}_\mathcal{P}_d$ and set $X = \overline{O}(1_A)$. Then $A = N(1_A, [1])$. Let $(Y, \sigma)$ be the $E$-system where $Y \subseteq X$, $Y \neq \{0^\infty\}$ and let $y$ be a transitive point in $Y$. The fact that $Y$ is an $E$-system, $\overline{O}(y) = Y$ is measure saturated and so $y$ is essentially recurrent [8, Theorem 2.6]. Apply [8, Theorem 2.8] for $x = y$ and $U(y, y) = [1] \times [1]$ to see that

$$F = \{n \in \mathbb{Z} : (\sigma^n y, \sigma^n y) \in [1] \times [1]\} = N(y, [1])$$

is a $D$-set. Since $y \in \overline{O}(1_A)$, for any finite subset $W$ of $F$ one has $m + W \subseteq A$ for some $m \in \mathbb{Z}$. This means that $A \in b\mathcal{D}$ and $b\mathcal{D} = \mathcal{F}_\mathcal{P}_d$. □

### 3.1. Scattering systems

Theorems 2.8 and 3.3 show different transitive systems arising from (2.6). We continue for the remaining cases in (2.6) using the results and routines from scattering.

Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $X$. We call this cover non-trivial if $U_i$ is not dense in $X$ for $1 \leq i \leq n$. Let $N(\mathcal{U})$ be the least cardinality of a sub-cover of $\mathcal{U}$. For this $\mathcal{U}$ and an infinite sequence $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ the complexity function of $\mathcal{U}$ along a sequence $A$ is defined to be

$$C_A(\mathcal{U}) = \lim_{n \to \infty} N(\bigvee_{i=1}^n T^{-a_i} \mathcal{U}).$$

This complexity function is a tool which is used to classify some transitive systems [16, 18].

A TDS $(X, T)$ is called full scattering, strong scattering, scattering if for any finite non-trivial cover $\mathcal{U}$, $C_A(\mathcal{U}) = \infty$ for $A \in \mathcal{I}$, $A \in \mathcal{F}_\mathcal{P}_d$ and $A \in \mathcal{P}\mathcal{S}$ respectively.
There are other characterization for scatterings. For instance, \((X, T)\) is strong scattering if and only if \(N(U, V) \in (F_{pubd} - F_{pubd})^*\) and it is scattering if and only if \(N(U, V) \in (S - S)^*\) where \(S\) is the family of syndetic sets \([16]\) and in \([17]\) it is shown that \((X, T)\) is weak scattering if and only if it is Bohr*-transitive.

It is known that \([18]\)
mixing \(\subseteq\) full scattering \(\subseteq\) \(\Delta^*\) - mixing \(\subseteq\) mild mixing \(\subseteq\) weak mixing \(\subseteq\) totally transitive.

There are two sorts of inclusions appearing above, \(\subseteq\) and \(\subseteq\). For \(\subseteq\), we actually do not know whether it can be replaced with equality or a proper inclusion. In fact, the equivalency of full scattering and \(\Delta^*\) - mixing is an open problem first raised in \([18]\) and the same problem has been questioned between weak scattering and strong scattering in \([10, 16]\).

Now we give a sufficient condition for a certain situation that strong scattering and weak scattering are equivalent. Another case will be dealt in Theorem 3.8.

**Theorem 3.5.** Let \((X, T)\) be a weak scattering TDS. If for any opene sets \(U, V \subset X\) and Bohr set \(B\), \(N(U, V) \cap B\) has positive upper Banach density, then \((X, T)\) is strongly scattering.

**Proof.** By the definition of strong scattering, it is sufficient to show that \(N(U, V) \cap (A - A) \neq \emptyset\) when \(d^*(A) > 0\). For any such set \(A\), there exists a Bohr set \(B\) and a subset of integers \(N\) with \(d^*(N) = 0\) such that \(B \subset (A - A) \cup N\) \([5, Corollary 5.3]\).

Set \(C := N(U, V) \cap B\) and note that by the assumption \(d^*(C) > 0\) and so \(C \not\subset N\). This means \((A - A) \cap C \neq \emptyset\) and as a result, \(N(U, V) \cap (A - A) \neq \emptyset\).

By \([16, Proof\ of\ Theorem 3.1]\), there is a strong scattering \((X, T)\) such that \(N(U, V)\) has zero upper Banach density for some opene sets \(U, V\). So the necessity does not hold in the above theorem.

Huang and Ye’s example in \([16, Proof\ of\ Theorem 3.1]\) implies that \((TP - TP)^*\)-transitive and \((D - D)^* = (F_{pubd} - F_{pubd})^*\)-transitives are different families. Since when \(d^*(N(U, V)) = 0\), then \(N(U, V)\) is not thick and therefore, \((X, T)\) cannot be mild mixing.

**Lemma 3.6.** Let \((X, T)\) be a TDS. Then the following are equivalent.

1. \((X, T)\) is scattering, that is, \((S - S)^*\)-transitive.
2. \((X, T)\) is \((PS - PS)^*\)-transitive.
3. \((X, T)\) is \((C - C)^*\)-transitive.

**Proof.** From \([20, Lemma 2.1]\), \((S - S)^* = (PS - PS)^*\). So (1) \(\iff\) (2). Also, a set is \(PS\) if and only if it is \(C_+\) \([8]\). Hence \((C - C) = (PS - PS)\); or, \((C - C)^* = (PS - PS)^*\). Thus (2) \(\iff\) (3).

Next theorem shows that \(D^*\)-mixing, \(C^*\)-mixing and \((C - C)^*\)-transitive are different.

**Theorem 3.7.** \(C^*\)-mixing is a proper subfamily of \((D - D)^*\)-transitive.

**Proof.** By Theorem 3.4, \((X, T)\) is strong scattering if and only if it is \((D - D)^*\)-transitive. Also, there is a strong scattering TDS which is not weakly mixing and hence it is not \(C^*\)-mixing \([16]\).

□
Trivially, the above theorem shows that $\mathcal{D}^*$-mixing and $(\mathcal{D} - \mathcal{D})^*$-transitive are different families. In fact, in a TDS we have the following inclusions (see (2.6)).

\[
\mathcal{D}^*\text{-mixing} \subseteq \mathcal{C}^*\text{-mixing} \subseteq \text{weak mixing} \subseteq (\mathcal{D} - \mathcal{D})^*\text{-transitive} \subseteq (\mathcal{C} - \mathcal{C})^*\text{-transitive}.
\]

**Theorem 3.8.** An scattering TDS with full support measure is weak mixing.

**Proof.** In [19], it has been shown that an MDS $(X, \mathcal{B}, \mu, T)$ is weak mixing if and only if \( \{ n : \mu(A \cap T^{-n}B) > 0 \} \) is a recurrence set (that is $(\mathcal{C} - \mathcal{C})^*$-set) for any $A, B \in \mathcal{B}$.

Now let $(X, T)$ be an scattering TDS with $\mathcal{B}$ its Borel sigma algebra. By Lemma 3.6, $(X, T)$ is $(\mathcal{C} - \mathcal{C})^*$-transitive and with its full support measure $\mu$, $(X, \mathcal{B}, \mu, T)$ is a weakly mixing MDS and thus a weakly mixing TDS. □

## 4. Measure dynamical systems and TDS with full support

For an MDS $(X, \mathcal{B}, \mu, T)$, let

\[ B^+ = \{ B \in \mathcal{B} : \mu(B) > 0 \} \]

and define

\[ N_\mu(A, B) = \{ n \in \mathbb{Z} : \mu(A \cap T^{-n}B) > 0 \} \]

and call

\[ R^{\epsilon}_{A, B} = \{ n \in \mathbb{Z} : \mu(A \cap T^{-n}B) > \mu(A)\mu(B) - \epsilon \} \]

a *fat intersection* [8].

An MDS is called $\mathcal{F}$-ergodic if for any $A, B \in B^+$, $N_\mu(A, B) \in \mathcal{F}$ and it is called $\mathcal{F}$-mixing if and only if $R^{\epsilon}_{A, B} \in \mathcal{F}$. In [8], it has been shown that in spite of different combinatorial families appearing in (2.1), the associated MDS families may be the same; namely $\mathcal{D}^*$-mixing and $\mathcal{C}^*$-mixing are equal to weak mixing and $\Delta^*$-mixing is strong mixing. Now we aim to extend this investigation for families in (2.6) for MDS.

In [19], Kuang and Ye show that if $\mathcal{F} \neq \Delta$, $(\mathcal{F} - \mathcal{F})^*$-ergodic implies $\mathcal{F}^*$-mixing in MDS. For instance $(IP - IP)^*$-ergodic is measure-theoretical mild mixing or equivalently $IP^*$-mixing. Also by Theorem 3.8, $(\mathcal{C} - \mathcal{C})^*$-ergodic is $\mathcal{C}^*$-mixing or equivalently weak mixing. But for $\mathcal{F} = \Delta$, $\Delta^*$-ergodic is not anymore $\Delta^*$-mixing; an example is provided in [19]. Thus $(\Delta - \Delta)^*$-ergodic is not $\Delta^*$-mixing unlike any other $(F - F)^*$-mixing. Let us summarize our above discussion in Figure 1. These are in fact the same inclusions as in (2.6), with three boxes. First let $\mathcal{F}$ be a family in the middle or the right box, in these situations, $\mathcal{F}$-mixing and $\mathcal{F}$-ergodic are equivalent. In fact in the middle, they are mild mixing while in the right they are weak mixing. Now for $\mathcal{F}$ in the left box, $\mathcal{F}$-mixings are equivalent and all are strong mixing and whether $(\Delta - \Delta)^*$-ergodic is $\Delta^*$-ergodic (which is as above said different from $\Delta^*$-mixing) is an open problem [19]. This problem is solved for some important classes of topological dynamics. For instance, in [1] it has been shown that totally transitive coded systems are strong mixing and since all weak mixing are totally transitive so in coded systems all the families are the same.

### 4.1. TDS with full support measure and minimal systems

Systems with full support measure share a few mixing and ergodic properties with their counterparts in MDS with this reservation that ergodic in MDS will be replaced with transitive in TDS.
Thus by the above discussion, in a full support TDS, if \( F \neq \Delta \), then \( (F - F)^* \)-transitive and \( F^* \)-mixing are equivalent. Here we briefly examine our above results for systems with full support and recall that an important class of systems with full support is the class of minimal system.

Moreover, the equivalency of two right boxes in Figure 1 transfer to this case. Since \( \Delta^* \)-ergodic is a different property from strong mixing in MDS, \( \Delta^* \)-transitive cannot imply strong mixing. So we have Figure 2 for TDS with full support measure from (2.6).

In minimal systems, we consider the following diagram. Chacon is a minimal mild mixing which is not mixing, so at least one of the following inclusion must be proper in minimal case.

\[
\text{mixing} \subseteq \Delta^*\text{-mixing} \subseteq \Delta^*\text{-ergodic} \subseteq (\Delta - \Delta)^*\text{-ergodic} \subseteq \mathcal{IP}^*\text{-mixing}
\]

In fact, the equality for the first inclusion was asked in [18]; and if one can give an example of a minimal \( \mathcal{IP}^* \)-mixing rigid (mild mixing), then by [2, Corollary 2.13] \( (\Delta - \Delta)^*\)-ergodic \( \not\subseteq \mathcal{IP}^*\)-mixing.

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