Abstract

Given a configuration of pebbles on the vertices of a connected graph $G$, a pebbling move removes two pebbles from some vertex and places one pebble on an adjacent vertex. The pebbling number of a graph $G$ is the smallest integer $k$ such that for each vertex $v$ and each configuration of $k$ pebbles on $G$ there is a sequence of pebbling moves that places at least one pebble on $v$.

First, we improve on results of Hurlbert, who introduced a linear optimization technique for graph pebbling. In particular, we use a different set of weight functions, based on graphs more general than trees. We apply this new idea to some graphs from Hurlbert’s paper to give improved bounds on their pebbling numbers.

Second, we investigate the structure of Class 0 graphs with few edges. We show that every $n$-vertex Class 0 graph has at least $\frac{5}{3}n - \frac{11}{3}$ edges. This disproves a conjecture of Blasiak et al. For diameter 2 graphs, we strengthen this lower bound to $2n - 5$, which is best possible. Further, we characterize the graphs where the bound holds with equality and extend the argument to obtain an identical bound for diameter 2 graphs with no cut-vertex.

1 Introduction

Graph pebbling was introduced by Chung in 1989. Following a suggestion of Lagarias and Saks, she computed the pebbling number of Cartesian products of paths to give a combinatorial proof of the following number-theoretic result of Kleitman and Lemke.
Theorem 1. [3, 13] Let $\mathbb{Z}_n$ be the cyclic group on $n$ elements and let $|g|$ denote the order of a group element $g \in \mathbb{Z}_n$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{Z}_n$, there exists a zero-sum subsequence $(g_k)_{k \in K}$, such that $\sum_{k \in K} \frac{1}{|g_k|} \leq 1$. Here $K$ is the set of indices of the elements in the subsequence.

Chung developed the pebbling game to give a more natural proof of this theorem. Results of this type are important in this area of number theory, as they generalize zero-sum theorems such as the Erdős-Ginzburg-Ziv theorem [6]. Over the past two decades, pebbling has developed into its own subfield [11, 12], with over 80 papers.

We consider a connected graph $G$ with $\mathbb{N}$ elements and let $|g|$ denote the order of a group element $g \in \mathbb{N}$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{N}$, there exists a zero-sum subsequence $(g_k)_{k \in K}$, such that $\sum_{k \in K} \frac{1}{|g_k|} \leq 1$. Here $K$ is the set of indices of the elements in the subsequence.

Postle, Streib, and Yerger [17] strengthened Bukh’s result, proving the exact bound $f(n, d) = 2^d$ for diameter $2$ graphs with few edges. We show that every $n$-vertex graph $G$ with $\pi(G) = |V(G)|$ has at least $\frac{5}{3}n - \frac{11}{3}$ edges. This disproves a conjecture of Blasiak et al [1]. For diameter $2$ graphs, we strengthen this bound to $2n - 5$, which is best possible. We characterize the graphs where it holds with equality and extend the argument to obtain an identical bound for diameter $2$ graphs with no cut-vertex.


2 Linear Programming Preliminaries

Computing a graph’s pebbling number is hard. Watson [18] and Clark and Milans [4] studied the complexity of graph pebbling and some of its variants, including optimal pebbling and cover pebbling. Watson showed that it is NP-complete to determine whether a given configuration is solvable for a given rooted graph \((G, r)\). Clark and Milans refined this result, showing that deciding whether \(\pi(G) \leq k\) is \(\Pi^p_2\)-complete; this means that pebbling is in the class of problems computable in polynomial time by a co-NP machine equipped with an oracle for an NP-Complete language.

Hurlbert [10] introduced a new linear programming technique, in hopes of more efficiently computing bounds on pebbling numbers. Before we describe our improvements on it, we briefly explain his method. Let \(G\) be a graph and let \(T\) be a subtree of \(G\) rooted at \(r\). For each \(v \in V(T) - r\), let \(v^+\) be the parent of \(v\), the neighbor of \(v\) in \(T\) that is closer to \(r\). A tree strategy is a tree \(T\) and an associated nonnegative weight function \(w_T\) (or \(w\) if the context is clear) where \(w(r) = 0\) and \(w(v^+) = 2w(v)\) for every vertex not adjacent to \(r\). Further, \(w(v) = 0\) if \(v \not\in V(T)\). Let \(w_G\) be the weight vector on \(V(G)\) in which every entry is 1.

Hurlbert [10] proposed a general method for defining such a weight function through tree strategies. He proved the following result (here \(\cdot\) denotes dot product).

Lemma 1. Let \(T\) be a tree strategy of \(G\) rooted at \(r\), with associated nonnegative weight function \(w\). If \(p\) is an \(r\)-unsolvable configuration of pebbles on \(V(G)\), then \(w \cdot p \leq w \cdot w_G\).

The proof idea is easy. Suppose that \(p\) is a configuration with \(w \cdot p > w \cdot w_G\). This implies that some vertex \(v\) in \(T\) has at least two pebbles. Now we make a pebbling move from \(v\) toward the root, i.e., from \(v\) to \(v^+\), to get a new configuration \(p'\). Since \(w(v^+) = 2w(v)\), we have \(w \cdot p' = w \cdot p > w \cdot w_G\). By repeating this process, we can eventually move a pebble to the root, \(r\).

Since every \(r\)-unsolvable pebbling configuration \(p\) satisfies \(w \cdot p \leq w \cdot w_G\), it follows that \(\pi(G, r)\) is bounded above by one plus the number of pebbles in the largest configuration \(p\) such that \(w \cdot p \leq w \cdot w_G\). Let \(T_r\) be the set of all tree strategies in \(G\) associated with root vertex \(r\). By applying Lemma 1 to all of \(T_r\) simultaneously, we arrive at the following integer linear program:

\[
\begin{align*}
\text{max} & \quad \sum_{v \neq r} p(v) \\
\text{s.t.} & \quad w \cdot p \leq w \cdot w_G \\
& \quad \text{for all } T \in T_r.
\end{align*}
\]

Let \(z_{G,r}\) be the optimal value of this integer linear program and let \(\hat{z}_{G,r}\) be the optimum of the linear relaxation, so that configurations can be rational. Since \(z_{G,r} \leq \lfloor \hat{z}_{G,r} \rfloor\), we get the bound \(\pi(G, r) \leq z_{G,r} + 1 \leq \lfloor \hat{z}_{G,r} \rfloor + 1\). Let \(w_1, \ldots, w_k\) be weight functions of tree strategies for trees (possibly different) rooted at \(r\), and let \(w'\) be a convex combination of \(w_1, \ldots, w_k\). If \(p\) is an \(r\)-unsolvable configuration, then \(w' \cdot p \leq w' \cdot w_G\) (otherwise \(w_i \cdot p > w_i \cdot w_G\), for some \(i\), a contradiction). Further, if \(w'(v) \geq 1\) for all \(v\), then \(|p| \leq \sum_{v \neq r} |w'(v)|p(v) \leq \lfloor w' \rfloor \cdot w_G\). For ease of application, we state this observation in a slightly more general form. We call this the Covering Lemma.
Lemma 2 (Covering Lemma). For a graph $G$ and a root $r \in V(G)$, let $w'$ be a convex combination of tree strategies for $r$, and let $C$ and $M$ be positive constants. If $w'(v) \geq C$ for all $v \in V(G) \setminus \{r\}$ and $\sum_{v \in V(G) \setminus \{r\}} w'(v) < M$, then $\pi(G, r) \leq \left\lfloor \frac{M}{C} \right\rfloor + 1$. In particular, if $\sum_{v \in V(G) \setminus \{r\}} w'(v) < C|V(G)|$, then $\pi(G, r) = |V(G)|$.

For any bound on $\pi(G)$ arising from such a $w'$, a certificate of the bound consists of the strategies $w_i$ and their coefficients in the convex combination forming $w'$.

Hurlbert applies this linear programming method more broadly by considering strategies on trees where $w(v^+) \geq 2w(v)$, called nonbasic strategies. Since nonbasic strategies are conic combinations of basic strategies \cite[Lemma 5]{10}, this extension does not strengthen the method. However, it often yields simpler certificates.

3 More General Weight Functions

Here we generalize the notion of weight function from the previous section to allow weight functions for graphs $G$ that are not trees. A weight function is a map $w : V(G) \to \mathbb{R}^+ \cup \{0\}$. A weight function for a graph $G$ and root $r$ is valid if $w(r) = 0$ and every $r$-unsolvable configuration $p$ satisfies $w \cdot p \leq w \cdot 1_G$. Although it is harder to show that one of these more general weight functions is valid, when we can, this often leads to improved pebbling bounds for a variety of graph families. Given a graph $G$ and a root $r$, it is straightforward to check that the theory developed in the previous section extends to any weight function $w$ such that $w \cdot p \leq w \cdot 1_G$ for every configuration $p$ that is not $r$-solvable. Our next result establishes a new family of such weight functions. A $k$-vertex is a vertex of degree $k$.

Lemma 3. Form $G$ from an even cycle $C_{2t}$ by identifying one vertex with the endpoint of a path of length $s - t$. Let $x_t$ be the resulting 3-vertex and $x_0$ be the 2-vertex farthest from $x_t$; now $x_0$ and $x_t$ split the even cycle into two paths, $P_1$ and $P_2$. Label the internal vertices of $P_1$ as $x'_1, x'_2, \ldots, x'_{t-2}, x'_{t-1}$ and the internal vertices of $P_2$ as $x''_1, x''_2, \ldots, x''_{t-2}, x''_{t-1}$. Call the 1-vertex $r$, and let $P_3$ be the path from $x_t$ to $r$. Label the internal vertices of $P_3$ as $x_{t+1}, x_{t+2}, \ldots, x_{s-1}, x_s$. For each $i \neq 0$, give weight $2^t$ to vertex $x_i$ or vertices $x'_i$ and $x''_i$. Let $\alpha = \frac{2^t + 2^t - 2}{2^t - 1}$ and give weight $\alpha$ to $x_0$. Fix some order on the vertices, and let $w$ be the vector of length $|V(G)|$ where entry $i$ is the weight of vertex $i$. If $p$ is an $r$-unsolvable configuration, then $w \cdot p \leq w \cdot 1_G$.

Proof. Figures 1 and 0 both show examples of this lemma, which we apply later.

Let $p$ be an $r$-unsolvable configuration. We will show that $w \cdot p \leq w \cdot 1_G$. Let $M = \alpha + 2^{s+1} + 2^t - 4$. Note that $w \cdot 1_G = M$. Let $W_0 = \alpha w(x_0)$, $W_L = \sum_{v \in P_1 \setminus \{x_0, x_t\}} w(v)p(v)$, $W_R = \sum_{v \in P_2 \setminus \{x_0, x_t\}} w(v)p(v)$, and $W_C = \sum_{v \in P_3 \setminus \{r\}} w(v)p(v)$. (Here $L$, $R$, and $C$ stand for left, right, and center.) We will show that $W_0 + W_L + W_R + W_C \leq M$.

Claim 1. If $W_L = 0$ or $W_R = 0$, then the lemma is true.

By symmetry, assume that $W_L = 0$. Now Lemma 1 implies that $\alpha^{-1}W_0 + W_R + W_C \leq 2^{s+1} - 1$. Multiplying by $\alpha$ gives $W_0 + W_L + W_R + W_C \leq W_0 + W_L + \alpha(W_R + W_C) \leq \frac{2^{s+1}}{2^t - 1} - 1$. Thus, $W_0 + W_L + W_R + W_C \leq M$. Therefore, the lemma holds in this case.
\( \alpha(2^{s+1} - 1) = \alpha + \alpha(2^{s+1} - 2) = \alpha + 2(2^s + 2^{t-1} - 2) = \alpha + 2^{s+1} + 2^t - 4 = M. \) This proves the claim.

**Claim 2.** If \( W_L + \alpha^{-1}W_0 > 2^t - 1 \), then the lemma is true.

Suppose that \( W_L + \alpha^{-1}W_0 > 2^t - 1 \). We can assume that \( W_L < 2^t \); otherwise we can move weight down to \( x_t \), without changing the sum \( W_L + W_C \). Now we move some pebbles toward the root and reduce to the case in Claim 1. Specifically, we remove \( 2^t - W_L \) pebbles from \( x_0 \) and place half that many on \( x'_t \). Call the new configuration \( p' \) and define \( W'_0, W'_L, W'_R, \) and \( W'_C \), analogously. Note that \( \alpha^{-1}W'_0 + W'_L + W'_R + W'_C = \alpha^{-1}W_0 + W_L + W_R + W_C \). Since \( W'_L = 2^t \), we can move all weight from internal vertices of \( P_1 \) to \( x_t \). This gives a new configuration \( p'' \) with \( W''_L = 0 \). Again \( \alpha^{-1}W''_0 + W''_L + W''_R + W''_C = \alpha^{-1}W_0 + W_L + W_R + W_C \). So the claim holds by Claim 1.

By Claim 2, we now assume that \( W_L + \alpha^{-1}W_0 \leq 2^t - 1 \). By symmetry, we assume that \( W_R + \alpha^{-1}W_0 \leq 2^t - 1 \). By Lemma 2, we can also assume that \( W_C \leq 2^{s+1} - 2^t \).

Adding these inequalities yields
\[
W_C + W_L + W_R + 2\alpha^{-1}W_0 \leq 2(2^t - 1) + 2^{s+1} - 2^t = 2^{s+1} + s^t - 2.
\]

We can assume that \( W_0 > 0 \), since otherwise the lemma holds by Lemma 2. Thus, we have \( W_0 \geq \alpha \), so \( (2\alpha^{-1} - 1)W_0 \geq 2 - \alpha \). Subtracting this inequality from (1) gives the desired result.

The following observation extends our class of valid weight functions a bit further.

**Observation 1.** Let \( G \) be a graph and \( r \) a root; let \( w \) be a weight function on \( G \) such that \( w \cdot p \leq w \cdot 1_G \) for every \( r \)-unsolvable configuration \( p \). Form \( G' \) from \( G \) by adding a new vertex \( u \) adjacent to some vertex \( u^+ \) of \( G \) (with \( u^+ \neq r \)), and form \( w' \) from \( w \), where \( w'(u) = \frac{1}{2}w(u^+) \) and \( w'(v) = w(v) \) for every \( v \in V(G) \). For every \( r \)-unsolvable configuration \( p' \) in \( G' \), we have \( w' \cdot p' \leq w' \cdot 1_G \). Further, we can allow, more generally, that \( w'(u) \leq \frac{1}{2}w(u^+) \). We can also attach trees, rather than single vertices.

**Proof.** If the new vertex \( u \) has more than one pebble, we move as much weight as possible from \( u \) to \( u^+ \), which does not decrease the total weight on \( G \). This proves the first statement. The second statement follows from taking convex combinations of \( w \) and \( w' \). The final statement follows by induction on the size of the tree \( T \) that we attach (we just proved the induction step, and the base case, \( |T| = 0 \), is trivial).

### 3.1 The Cube and the Lemke Graph

To illustrate the usefulness of Lemma 3 and Observation 1, we give two easy applications of this method. We show that \( \pi(Q_3) = 8 \) and \( \pi(L) = 8 \), where \( Q_3 \) is the 3-dimensional cube and \( L \) is the Lemke graph, shown in Figure 3. When using tree strategies alone, Hurlbert’s method cannot handle these graphs.
Proposition 1. If $Q_3$ is the 3-cube, then $\pi(Q_3) = 8$.

Proof. Every graph $G$ satisfies $\pi(G) \geq |V(G)|$, so $\pi(Q_3) \geq 8$. Thus, we focus on proving that $\pi(Q_3) \leq 8$.

To show that $\pi(Q_3) \leq 8$, we first note that the weight function in Figure 1 is valid. Since a valid weight function remains valid when multiplied by a positive constant (in this case 3), this statement follows from Lemma 3, with $t = 2$ and $s = 0$.

The convex combination of the three strategies shown in Figure 2 (each taken with weight 1) yields $w'$ such that $w'(v) = 12$ for all $v \neq r$. Thus, the Covering Lemma shows that $\pi(Q_3) \leq 8$, so $Q_3$ is Class 0. These three strategies in Figure 2 also serve as a certificate that $\pi(Q_3) \leq 8$, and they yield an efficient algorithm for getting a pebble to $r$, starting from any configuration $p$ with $|p| \geq 8$.

The most famous long-standing pebbling problem is Graham’s conjecture: for all graphs $G_1$ and $G_2$, $\pi(G_1 \square G_2) \leq \pi(G_1)\pi(G_2)$; here $\square$ denotes the Cartesian product. This conjecture has been verified only for a few classes of graphs. Specifically, it holds when $G_1$ and $G_2$ are both cycles [9], both trees [14], both complete bipartite graphs [7], or a fan and a wheel [8].

When considering Graham’s conjecture, we are interested in the Lemke Graph, denoted $L$ and shown on the left in Figure 3. This graph is of interest because it is the
smallest graph without the 2-pebbling property. The exact definition is unimportant for us here; what matters, is that if $G$ has this property, then $\pi(G \Box H) \leq \pi(G)\pi(H)$ for every graph $H$. This makes $L \Box L$ a natural candidate for disproving Graham’s conjecture.

Hurlbert asserted that it is impossible, using tree strategies alone, to obtain the pebbling number of the Lemke graph via the linear programming technique. However, by using this method with more general weight functions, we prove that $\pi(L) = 8$.

![Figure 3: The Lemke graph.](image)

Theorem 2. If $L$ is the Lemke graph, then $\pi(L) = 8$.

Proof. Note that $\pi(L) \geq |V(L)| = 8$, so we focus on proving the upper bound. Hurlbert [10] showed that $\pi(L,v) = 8$ for all vertices $v \in L$ except for $r$, as shown on the left in Figure 3. So we only need to show that $\pi(L,r) = 8$. Now we need the weight function in Figure 4.

Claim 1. The weight function in Figure 4 is valid.

The proof of this claim is very similar to the proof of Observation 1, so we just sketch the ideas. If any vertex weighted 6 has no pebbles, then we invoke the weight
function in Figure 1, and multiply the resulting inequality by $\frac{5}{4}$ to get one that implies what we want; so we assume that each vertex weighted 6 has a pebble. If the vertex weighted 12 has a pebble, then the vertex weighted 5 has at most one pebble, so we are done. Otherwise, the vertex weighted 5 has at most 3 pebbles; again, we are done. This proves the claim.

The proof that $\pi(L, r) \leq 8$ uses the two strategies in Figure 3. The rightmost is a nonbasic tree strategy. The center strategy is derived from the weight function in Figure 4 by adding a vertex with weight 3 adjacent to some vertex with weight 6. This weight function is valid, by Observation 1. When we sum the weights of the two strategies, each vertex has weight at least 7 and the total weight is 55. Now the Covering Lemma implies that $\pi(L, r) \leq \lceil \frac{55}{7} \rceil + 1 = 8$. □

3.2 Larger Graphs

In this section we determine the pebbling number of the Bruhat graph of order 4. The (weak) Bruhat graph of order $m$ has as its vertices the permutations of $\{1, \ldots, m\}$; two vertices are adjacent if the corresponding permutations differ by an adjacent transposition. Since this graph is vertex-transitive, we can choose the root vertex arbitrarily. Using the linear programming method, Hurlbert proved that $\pi(B_4) \leq 72$. By using more general weight functions, we calculate the pebbling number of this graph exactly.

![Figure 5: The Bruhat graph of order 4 and a set of strategies proving $\pi(B_4) = 64$.](image)
Theorem 3. If $B_4$ is the Bruhat graph of order 4, then $\pi(B_4) = 64$.

Proof. The diameter of $B_4$ is 6, so $\pi(B_4) \geq 2^6 = 64$. We need to show that $\pi(B_4) \leq 64$. Note that the rightmost graph in Figure 6 describes two strategies, as we explain below. We combine these four strategies, as shown in Figure 5 (weighted with multiplicities $\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}$) to get a weight function $w'$, such that $w'(v) \geq 1$ for all $v$ and $w' \cdot 1_{B_4} = 63$. This proves the desired upper bound $\pi(B_4) \leq 63 + 1$. Thus, we only need to show that the four strategies in Figure 6 are valid.

The weight function on the right denotes two different weight functions on $G$; the first includes the vertex labeled $(8,0)$ but not the one labeled $(0,8)$, and the second vice versa. By Lemma 3, the leftmost and rightmost strategies are valid (the former with $t = 4$ and $s = 1$; the latter with $t = 5$ and $s = 0$).

The proof that the middle strategy is valid is similar to the proof of Lemma 3, so we just sketch the ideas. Note that weights 30, 15, 10, and 5 (with the other vertices unweighted) are consistent with Lemma 3 (multiplied by $\frac{15}{2}$), when $t = 2$ and $s = 0$, and adding a vertex by Observation 4. Thus, we know that if $p$ is $r$-unsolvable, then these five vertices have weight at most 75. The key observation is that these five vertices can play the role of $P_3$ in the proof of Lemma 3. Let $x_0$, $x_1'$, and $x_1''$ denote the vertex labeled $\frac{40}{31}$ and its two neighbors, respectively. We first consider the case where
$x'_1$ or $x''_1$, say $x'_1$, has no pebbles. In this case we move as much weight as possible to the vertex labeled 5 from $x_0$ and $x''_1$. We also consider the case where both $x'_1$ and $x''_1$ have pebbles. Either we can reduce to the previous case, or else we get two inequalities. We add these two to the inequality for the bottom 5 vertices, which gives the desired inequality.

4 Class 0 Graphs

4.1 Preliminaries

In this section, we study Class 0 graphs. We focus on graphs with diameter at least 2 since those with diameter 0, a single vertex, and diameter 1, a complete graph, are well understood. A graph $G$ is Class 0 if its pebbling number is equal to its number of vertices, i.e., $\pi(G) = |V(G)|$. Recall that always $\pi(G) \geq |V(G)|$, so Class 0 graphs are those where this trivial lower bound holds with equality. For each vertex $v$, we write $N(v)$ for the set of vertices adjacent to $v$, and we write $N[v]$ to denote $N(v) \cup \{v\}$. For a graph $G$, let $e(G)$ denote the number of edges in $G$. In this section, we prove lower bounds on $e(G)$ for all Class 0 graphs.

Blasiak et al. [1] showed that every $n$-vertex Class 0 graph $G$ has $e(G) \geq \left\lceil \frac{3n}{2} \right\rceil$. They also conjectured (see [10, p. 19]) that for some constant $C$ and for all sufficiently large $n$ there exist $n$-vertex Class 0 graphs with $e(G) \leq \left\lceil \frac{3n}{2} \right\rceil + C$. In particular, they defined a family of “generalized Petersen graphs” of arbitrary size and diameter with one vertex of some fixed degree $m$ and all other vertices of degree 3. They conjectured that these graphs are all Class 0. We disprove this conjecture in a very strong sense. Shortly, we prove that for fixed $m$, all sufficiently large graphs of this form are not Class 0. (Figure 7 shows $P_{8,2}$, one of these generalized Petersen graphs that is not Class 0.) Later in this section, we extend this idea to show that every $n$-vertex Class 0 graph $G$ has $e(G) \geq \frac{5}{2}n - \frac{11}{2}$. To conclude the section, for all diameter 2 graphs $G$ we strengthen this lower bound to $e(G) \geq 2n - 5$. Further, we characterize the graphs where this bound holds with equality (which include two infinite families).

Figure 7: The generalized Petersen graph, $P_{8,2}$, is not Class 0.
Our main tool for proving bounds on \( e(G) \) is the following lemma.

**Lemma 4 (Small Neighborhood Lemma).** Let \( G \) be a Class 0 graph. If \( u, v \in V(G) \), \( d(u) = 2 \), and \( u \) and \( v \) are distance at least 3 apart, then \( d(v) \geq 4 \). Similarly, if \( u, v \in V(G) \), \( d(u) = 3 \), and \( u \) and \( v \) are distance at least 4 apart, and each neighbor of \( v \) is a 3-vertex, then \( d(v) \geq 4 \).

**Proof.** The proofs for both statements are similar. In each case, we assume the statement is false and construct a configuration with \(|V(G)|\) vertices that is \( u \)-unsolvable. Consider the first statement first. Suppose, to the contrary, that \( u \) and \( v \) are as required, but \( d(v) \leq 3 \). Form configuration \( p \) by putting 7 pebbles on \( v \), 0 pebbles on each vertex of \( N[u] \cup N(v) \), and 1 pebble on each other vertex. Since \(|N[u] \cup N[v]| \leq 7\), this configuration has at least \(|V(G)|\) pebbles. Now no pebble can reach \( u \), since at most one pebble can leave \( N[v] \). This contradicts that \( G \) is Class 0, so \( d(v) \geq 4 \).

Now consider the second statement. Suppose, to the contrary, that \( u \) and \( v \) are as required, but \( d(v) \leq 3 \). Form configuration \( p \) by putting 15 pebbles on \( v \), 0 pebbles on each vertex of \( N[u] \cup (N[N[v]] \setminus \{v\}) \), and 1 pebble on each other vertex. Since \(|N[u] \cup N[N[v]]| \leq 15\), the configuration has at least \(|V(G)|\) pebbles, but no pebble can reach \( v \), since at most one pebble can leave \( N[N[v]] \). This contradicts that \( G \) is Class 0. Thus, \( d(v) \geq 4 \). \( \square \)

**Corollary 1.** For each integer \( C \), there exists an integer \( n_0 \), such that if \( G \) is any \( n \)-vertex graph with \( \delta(G) = 3 \), \( n \geq n_0 \), and \( e(G) \leq \frac{3}{2}n + C \), then \( G \) is not Class 0.

**Proof.** We can choose \( n_0 \) sufficiently large so that there exists some pair of vertices \( u, v \) violating the second statement of the Small Neighborhood Lemma. Specifically, it suffices to find a 3-vertex \( v \) such that every vertex within distance four of \( v \) is a 3-vertex. To guarantee such a vertex \( v \), we can take, for example, \( n_0 = 2C \cdot 3^5 \). \( \square \)

### 4.2 Diameter at least 3

Now we use the Small Neighborhood Lemma to prove, in Theorem 4, that every \( n \)-vertex Class 0 graph \( G \) with diameter at least 3 has \( e(G) \geq \frac{5}{3}n - \frac{11}{3} \). The case \( \delta(G) = 2 \) is complicated, so we handle it separately, in Lemma 6. For the case \( \delta(G) \leq 1 \), we use the following easy lemma from [5].

**Lemma 5 ([5]).** Every Class 0 graph \( G \) has no cut-vertices. Specifically, \( \delta(G) \geq 2 \).

**Proof.** Let \( G \) be a graph with a cut-vertex \( u \) and neighbors \( v_1 \) and \( v_2 \) that are in different components of \( G - u \). Consider the distribution \( p \) with 3 pebbles on \( v_1 \), 0 pebbles on each of \( u \) and \( v_2 \), and 1 pebble on each other vertex. Distribution \( p \) has \(|V(G)|\) pebbles, but no pebble can reach \( v_2 \), which we now show. If a pebble ever moves to \( u \), then at that point each vertex has at most one pebble, and \( v_2 \) has no pebbles. Otherwise, every pebbling move is within the component of \( G - u \) containing \( v_1 \), so no pebble reaches \( v_2 \). Thus no pebble can reach \( v_2 \), so \( G \) is not Class 0. \( \square \)
**Lemma 6.** If an n-vertex Class 0 graph $G$ has diameter at least 3 and $\delta(G) = 2$, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.

**Proof.** Let $G$ be an n-vertex Class 0 graph with diameter at least 3 and $\delta(G) = 2$. We assign each vertex $v$ a charge $ch(v)$, where $ch(v) = d(v)$. Now we redistribute these charges, without changing their sum, so that all but a few vertices finish with charge at least $\frac{10}{3}$; the charge of each vertex $v$ after redistributing is $ch^*(v)$. If at most $k$ vertices finish with charge less than $\frac{10}{3}$ (but all charges are nonnegative), then

$$
e(G) = \frac{1}{2} \sum_{v \in V} ch(v) = \frac{1}{2} \sum_{v \in V} ch^*(v) \geq \frac{1}{2} \left( \frac{10}{3} (n - k) \right) = \frac{5}{3}n - \frac{5}{3}k.$$  

Choose $r \in V(G)$ such that $d(r) = 2$. For each positive integer $i$, let $N_i$ denote the set of vertices at distance $i$ from $r$. Also, let $N_{3+} = \bigcup_{i \geq 3} N_i$. By the Small Neighborhood Lemma, with $u = r$, if $v \in N_{3+}$, then $d(v) \geq 4$.

We redistribute charge according to the following two discharging rules.

1. Each vertex $v \in N_2$ takes charge 1 from some neighbor in $N_1$. If $d(v) = 2$, then $v$ also takes charge $\frac{1}{3}$ from its other neighbor.

2. Each vertex $v \in N_{3+}$ with $d(v) = 4$ takes charge $\frac{1}{3}$ from each neighbor $u$ with $d(u) \geq 3$.

We show that nearly all vertices finish with charge at least $\frac{10}{3}$. Consider a vertex $v \in V(G) \setminus N[r]$. If $d(v) \geq 5$, then $ch^*(v) \geq d(v) - \frac{1}{3}d(v) = \frac{2}{3}d(v) \geq \frac{10}{3}$. Now suppose $v \in N_2$ and $d(v) \geq 3$. In this case, $ch^*(v) \geq d(v) + 1 - \frac{1}{3}(d(v) - 1) = \frac{2}{3}d(v) + \frac{4}{3} \geq \frac{10}{3}$.

Suppose instead that $v \in N_2$, $d(v) = 2$, and either $v$ has both neighbors in $N_1$ or the neighbor of $v$ outside of $N_1$ has degree at least 3. Now $ch^*(v) = d(v) + \frac{4}{3} = \frac{10}{3}$.

We show that $G$ has at most two 2-vertices in $N_2$ with 2-neighbors in $N_2$. Suppose, to the contrary, that $u_1$, $u_2$, and $u_3$ are 2-vertices in $N_2$, each with a 2-neighbor in $N_2$; by symmetry, assume $u_1u_2 \in E(G)$. By Lemma 5, $u_1$ and $u_2$ cannot have a common neighbor $v \in N_1$, since then $v$ would be a cut-vertex. Thus, $u_1$ and $u_2$ have distinct neighbors in $N_1$. However, now $u_3$ is distance three from either $u_1$ or $u_2$; by symmetry, say $u_1$. Now $u_1$ and $u_3$ contradict the Small Neighborhood Lemma. So indeed $N_2$ has at most two 2-vertices with 2-neighbors in $N_2$.

Now we consider 4-vertices in $N_{3+}$. Rather than compute the charges of these 4-vertices individually, we group them together as follows. Let $H$ be the subgraph induced by 4-vertices in $N_{3+}$, and let $H_1$ be a component of $H$ with $k$ vertices. If $H_1$ contains a cycle, then $H_1$ contains at least $k$ edges, so vertices of $H_1$ give charge to at most $4k - 2(k) = 2k$ vertices outside $H_1$. Thus, $ch^*(H_1) \geq ch(H_1) - 2k(\frac{1}{3}) = 4k - \frac{2k}{3} = \frac{10}{3}k$. Similarly, if $H_1$ has some adjacent vertex that is not a 2-vertex, then $ch^*(H_1) \geq ch(H_1) - (2k + 1)(\frac{1}{3}) + \frac{1}{3} = \frac{10}{3}k$. Instead, assume that $H_1$ is a tree and every vertex adjacent to $H_1$ is a 2-vertex. Recall that each such 2-vertex is in $N_2$.

If every 2-neighbor of $H$ is adjacent to the same vertex of $N_1$, call it $v$, then $v$ is a cut-vertex. Thus, $H_1$ has 2-neighbors that are adjacent to both vertices of $N_1$; call these 2-neighbors $u_1$ and $u_2$. By the Small Neighborhood Lemma, every pair of 2-vertices in $N_2$ are adjacent or have a common neighbor. Since $u_1$ and $u_2$ are both adjacent to $H_1$, they can’t be adjacent to each other; thus, they must have a common
neighbor, \( u_3 \). Further, every 2-vertex in \( N_2 \) must be adjacent to \( u_3 \). Since \( u_3 \in V(H_1) \), \( u_3 \) is a 4-vertex, so \( N_2 \) has at most four 2-vertices. Thus, \( H_1 \) is the only component of \( H \) with final charge less than \( \frac{10}{3} \) times its size. Furthermore, \( H_1 \) has only a single vertex, and \( \text{ch}^*(H_1) = 4 - 4 \left( \frac{1}{3} \right) = \frac{8}{3} \).

Now we compute the total final charge of \( V(G) \). For each vertex \( v \) not in \( H \), the final excess of \( v \) is \( \text{ch}^*(v) \) at most \( \frac{10}{3} \). For each component \( H_i \) of \( H \) with order \( k \), the final excess is \( \text{ch}^*(H_i) - \frac{10}{3}k \). We now show that the sum of all final excesses is greater than or equal to \(-\frac{22}{3}\), which proves the lemma.

If \( v \in N_{3^+} \) and \( d(v) \geq 5 \), then \( \text{ch}^*(v) \geq \frac{10}{3} \), so \( v \) has nonnegative excess. Each component of \( H \), other than (possibly) \( H_1 \), has nonnegative excess. Further, \( H_1 \) has excess greater than or equal to \(-\frac{2}{3}\). Each \( v \in N_2 \) with \( d(v) \geq 3 \) has nonnegative excess. Also, each \( v \in N_2 \) with \( d(v) = 2 \) has excess 0, except for at most two adjacent 2-vertices, which each have excess \(-\frac{1}{3}\). Finally, the sum of the final charges on \( N[v] \) is at least 4 (since \( r \) takes no charge from \( N(r) \)). Thus, the sum of excesses of \( N[r] \) is at least \( 4 - 3 \left( \frac{10}{3} \right) = -6 \). So the sum of excesses over all vertices is at least \( 2 (-\frac{1}{3}) + (-\frac{2}{3}) + (-6) = -\frac{22}{3} \). Thus \( \sum_{v \in V(G)} d(v) \geq \frac{10}{3}n - \frac{22}{3} \), so \( e(G) \geq \frac{5}{3}n - \frac{11}{3} \).

Now we prove our main theorem of this section.

**Theorem 4.** If \( G \) is an \( n \)-vertex Class 0 graph with diameter at least 3, then \( e(G) \geq \frac{5}{3}n - \frac{11}{3} \).

**Proof.** Let \( G \) be Class 0 with diameter at least 3. By Lemma 5 \( \delta(G) \geq 2 \). Lemma 6 proves the bound when \( \delta(G) = 2 \). If \( \delta(G) \geq 4 \), then \( e(G) \geq \frac{\delta(G)n}{2} \geq 2n \). Thus, we assume that \( \delta(G) = 3 \).

The proof is similar to that of Lemma 6 but easier. Recall that a \( k \)-vertex is a vertex of degree \( k \). Similarly, a \( k^+ \)-vertex has degree at least \( k \) and a \( k \)-neighbor of a vertex \( v \) is a \( k \)-vertex adjacent to \( v \). Choose \( r \) to be a 3-vertex with as few vertices at distance 2 as possible. For each integer \( i \), let \( N_i \) denote the set of vertices at distance \( i \) from \( r \). Also, let \( N_{4^+} = \bigcup_{i \geq 4} N_i \). We first handle the case \(|N_2| \geq 8\), which is short.

**Claim 1.** If \(|N_2| \geq 8\), then \( e(G) \geq \frac{5}{3}n \).

Since \( r \) was chosen among all 3-vertices to minimize \( N_2 \), each 3-vertex has at least three 4-neighbors or at least two 4-neighbors. Thus, we let \( \text{ch}(v) = d(v) \) and use the following discharging rule.

1. Each 3-vertex takes \( \frac{1}{6} \) from each 4-neighbor and \( \frac{1}{3} \) from each \( 5^+ \)-neighbor.

   If \( d(v) \geq 5 \), then \( \text{ch}^*(v) \geq d(v) - \frac{1}{6}d(v) = \frac{5}{6}d(v) \geq \frac{10}{3} \). If \( d(v) = 4 \), then \( \text{ch}^*(v) \geq d(v) - \frac{1}{6}d(v) = 4 - \frac{4}{6} = \frac{10}{3} \). If \( d(v) = 3 \), then \( \text{ch}^*(v) \geq 3 + \frac{1}{3} = \frac{10}{3} \) or \( \text{ch}^*(v) \geq 3 + \frac{2}{6} = \frac{10}{3} \). Hence, \( e(G) = \frac{1}{2} \sum_{v \in V(G)} \text{ch}(v) = \frac{1}{2} \sum_{v \in V(G)} \text{ch}^*(v) \geq \frac{5}{3}n \). This proves the claim.

Hereafter, we assume that \(|N_2| \leq 7 \). Now a variation on the Small Neighborhood Lemma implies that \( d(v) \geq 4 \) for each vertex \( v \in N_{4^+} \). Suppose instead that \( d(v) = 3 \) for some vertex \( v \in N_{4^+} \). Let \( p \) be the configuration with 15 pebbles on \( r \), 0
pebbles on each vertex in \( N_1 \cup N_2 \cup N[v] \), and 1 pebble on each other vertex. Since \( |\{r\} \cup N_1 \cup N_2 \cup N[v]| \leq 15 \), the configuration has at least \( n \) pebbles, but no pebble can reach \( v \), since at most one pebble can leave \( N[r] \cup N_2 \). This contradicts that \( G \) is Class 0. Thus, \( d(v) \geq 4 \) for each \( v \in N_{4^+} \).

Now we again redistribute charge. We let \( ch(v) = d(v) \) and we use the following two discharging rules.

1. Each vertex in \( N_2 \) takes charge 1 from its neighbor in \( N_1 \).
2. Each vertex in \( N_3 \) takes charge \( \frac{1}{3} \) from its neighbor in \( N_2 \).

We show that each vertex in \( V(G) \setminus N[r] \) finishes with charge at least \( \frac{10}{3} \). If \( v \in N_{4^+} \), then \( ch^*(v) = ch(v) = d(v) \geq 4 \). If \( v \in N_3 \), then \( ch^*(v) \geq d(v) + \frac{1}{3} \geq \frac{10}{3} \).

We now prove that every \( 3 \)-vertex diameter 2 Class 0 graph \( G \) has at least \( 2n - 5 \) edges. This bound is best possible. Before proving this result, we describe some graphs where equality holds. In what follows, we show that these are the only graphs where equality holds. To begin, we need the following lemma.

**Lemma 7.** Given a graph \( G \) and a vertex \( v \in V(G) \), form \( G' \) from \( G \) by adding a new vertex, \( v' \), with \( N(v') = N(v) \). If \( G \) is Class 0, then \( G' \) is also Class 0.

**Proof.** Let \( G \) be Class 0, and form \( G' \) from \( G \) as in the lemma. We show that \( G' \) is Class 0. Let \( p' \) be a configuration of size \( |V(G')| \) on \( G' \) and \( r \) be a target vertex in \( G' \).

First suppose that \( r \notin \{v,v'\} \). We form configuration \( p \) for \( G \) as follows. Let \( p(w) = p'(w) \) for all \( w \in V(G) \setminus \{v\} \), and let \( p(v) = \max(p'(v) + p'(v') - 1,0) \). Now \( |p| \geq |V(G)| \), so \( r \) is reachable from \( p \) in \( G \); let \( \sigma \) be a pebbling sequence that reaches \( r \) in \( G \). If \( \sigma \) reaches \( r \) from \( p' \) in \( G' \), then we are done. Otherwise, \( v \) must make more moves in \( \sigma \) in \( G' \) from \( p \) than are possible in \( G' \) from \( p' \). Now all of these “extra” moves from \( v \) can be made instead from \( v' \) (precisely because \( p(v) = p'(v) + p'(v') - 1 \)). Thus \( r \) is reachable in \( G' \) from \( p' \).

Suppose instead that \( r \in \{v,v'\} \); by symmetry, assume that \( r = v \). We may assume that \( p'(v) = 0 \) and \( p'(v') < 4 \). If \( p'(v') \leq 1 \), then we can proceed as in the previous paragraph. So assume that \( p'(v') \in \{2,3\} \). Since \( G \) is Class 0, Lemma 5 implies that \( d(v) \geq 2 \). Choose \( u_1, u_2 \in N(v) \). Since \( p(v') \in \{2,3\} \), we can assume that \( p(u_1) = p(u_2) = 0 \). We form \( p \) for \( G \) as follows. Let \( p(w) = p'(w) \) for all \( w \in V(G) \setminus \{u_1, u_2\} \) and \( p(u_1) = p(u_2) = 1 \). Now \( |p| \geq |V(G)| \), so \( v \) is reachable from \( p \) in \( G \); let \( \sigma \) be a pebbling sequence that reaches \( v \) in \( G \). If \( \sigma \) makes no moves from \( u_1 \) or \( u_2 \), then \( \sigma \) also reaches \( v \) from \( p' \) in \( G' \). So assume that \( \sigma \) makes a move from \( u_1 \) or \( u_2 \). Form \( \sigma' \) from \( \sigma \) by truncating \( \sigma \) just before the first time that it moves from
Figure 8: The tree strategies for $C_5$ and for $K_4$ with the edges of a $K_{1,3}$ subdivided.

$u_1$ or $u_2$, say $u_1$, and then appending a move from $v'$ to $u_1$ and a move from $u_1$ to $v$. Now $\sigma'$ reaches $v$ from $p'$ in $G$. Thus, $G'$ is Class 0.

Now we use Lemma 7 to show that two infinite families of graphs are all Class 0.

**Example 1.** The following are two infinite families of Class 0 graphs. Each $n$-vertex
graph has exactly $2n - 5$ edges. To form an instance of $F_{p,q}$, begin with $K_3$ and replace the two edges incident to some vertex $v$ with $p$ parallel edges and $q$ parallel edges (where $p$ and $q$ are positive); finally, subdivide each of these $p + q$ new edges. To form an instance of $G_{p,q,r}$, begin with $K_4$ and replace the three edges incident to some vertex $v$ with $p$ parallel edges, $q$ parallel edges, and $r$ parallel edges (where $p$, $q$, and $r$ are positive); finally, subdivide each of these $p + q + r$ new edges.

It is easy to see that each $n$-vertex graph in $F_{p,q}$ has $2n - 5$ edges, since the 2-vertices induce an independent set (when $p \geq 2$ and $q \geq 2$), and the three high-degree vertices have among them a single edge. Similarly, $G_{p,q,r}$ has $2n - 5$ edges, since the 2-vertices induce an independent set and the four high-degree vertices have among them 3 edges.

We prove that all of $F_{p,q}$ is Class 0, by induction on $p + q$; the induction step follows immediately from Lemma 7. The base case is $F_{1,1}$, which is the 5-cycle. To show that it is Class 0, we use the tree strategies shown in the first row of Figure 8. Since $C_5$ is vertex-transitive, we can pick the root arbitrarily. Let $w$ be the sum of the weights in the two tree strategies for $C_5$. Note that $w(v) \geq 3$ for every vertex $v \in V(G) \setminus \{r\}$ and $\sum_{v \in V(G) \setminus \{r\}} w(v) = 14 < 3(4 + 1)$. Thus, by the Covering Lemma, $C_5$ is Class 0.

We prove that all of $G_{p,q,r}$ is Class 0, by induction on $p + q + r$; the induction step follows immediately from Lemma 7. The base case is $G_{1,1,1}$. To show that $G_{1,1,1}$ is Class 0, we use the tree strategies shown in Figure 8. Up to symmetry, $G_{1,1,1}$ has three types of vertices: a degree 2 vertex, the center degree 3 vertex, and a peripheral degree 3 vertex. The tree strategies for these cases are given in the first, second, and third row below the strategies for $C_5$.

Let $r$ be a degree 2 vertex, and let $w(v)$ be the sum of the two weight functions in the second row of Figure 8. Note that $w(v) \geq 3$ for all $v \in V(G) \setminus \{r\}$. Further, $\sum_{v \in V(G) \setminus \{r\}} w(v) = 20 < 3(6 + 1)$. Thus, the Covering Lemma implies that $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$. Now let $r$ be the center vertex, and let $w(v)$ be the sum of the three weight functions in the third row of Figure 8. Note that $w(v) = 4$ for all $v \in V(G) \setminus \{r\}$. Thus, $\sum_{v \in V(G) \setminus \{r\}} w(v) = 24 < 4(6 + 1)$. Thus, the Covering Lemma implies that $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$. Finally, let $r$ be a peripheral vertex, and let $w(v)$ be the sum of the three weight functions in the fourth row of Figure 8. Note that $w(v) \geq 7$ for all $v \in V(G) \setminus \{r\}$. Further, $\sum_{v \in V(G) \setminus \{r\}} w(v) = 46 < 7(6 + 1)$. Thus, the Covering Lemma implies that $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$. Since $\pi(G_{1,1,1}, r) \leq |V(G_{1,1,1})|$ for each root $r$, we conclude that $\pi(G_{1,1,1}) \leq |V(G_{1,1,1})|$. So, $G_{1,1,1}$ is Class 0.

Now we show that every diameter 2 Class 0 graph $G$ has $e(G) \geq 2n - 5$ and characterize when equality holds. Clarke et al. [5, Theorem 2.4] characterized diameter 2 graphs that are not Class 0. It seems likely that we could derive our result from theirs. However, we prefer the proof below, since it seems simpler and more straightforward. Further, the proof below generalizes to diameter 2 graphs with no cut-vertices.

**Theorem 5.** Let $G$ be an $n$-vertex graph with diameter 2. If $G$ has no cut-vertex (in particular, if $G$ is Class 0) then $e(G) \geq 2n - 5$. Further, equality holds if and only if $G$ is the Petersen graph or one of the graphs in Example 1.
Proof. If \( \delta(G) \geq 4 \), then \( e(G) \geq \frac{4n}{2} = 2n \) and the theorem is true. So we assume \( \delta(G) \leq 3 \). Lemma 3 implies that \( \delta(G) \geq 2 \), so \( e(G) \geq \frac{n\delta(G)}{2} \geq n \). If \( n \leq 5 \), then \( e(G) \geq n \geq 2n - 5 \), so the theorem is true. Thus, we assume \( n \geq 6 \). We consider two cases: (i) \( \delta(G) = 3 \) and (ii) \( \delta(G) = 2 \).

Case 1: \( \delta(G) = 3 \). Choose \( r \in V(G) \) with \( d(r) = 3 \), and let \( S = N(r) \). Each vertex \( v \in V(G) \setminus S \) has a neighbor in \( S \) and \( r \) has 3 neighbors in \( S \), so \( \sum_{v \in S} d(v) \geq (n - 4) + 3 = n - 1 \). Also, \( \sum_{v \in V(G) \setminus S} d(v) \geq \sum_{v \in V(G) \setminus S} 3 = 3(n - 3) \). So \( e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \geq \frac{1}{2}((n - 1) + 3(n - 3)) = \frac{1}{2}(4n - 10) = 2n - 5 \).

If equality holds in \( e(G) \geq 2n - 5 \), then each vertex in \( V \setminus S \) has degree 3 and each vertex in \( V \setminus (S \cup \{r\}) \) has exactly one neighbor in \( S \). Let \( \{v_1, v_2, v_3\} = S \), let \( S_i = N(v_i) - r \) for each \( i \in \{1, 2, 3\} \), and let \( H = G[S_1 \cup S_2 \cup S_3] \). Note that \( H \) is a disjoint union of cycles, since each vertex has degree 3 and has exactly one neighbor in \( S \). Also \( |S_i| \geq 2 \) for each \( i \), since \( \delta(G) = 3 \). Suppose that \( |S_1| \geq 3 \), and choose \( v \in S_2 \). Now \( |S_1 \cup S_2| \geq 3 + 2 = 5 \), so \( v_3 \) is distance at least 3 from some vertex of \( S_1 \cup S_2 \) (precisely because \( H \) is a disjoint union of cycles). Hence \( |S_1| = 2 \) and, by symmetry, \( |S_2| = |S_3| = 2 \). Similarly, if \( H \) consists of two 3-cycles, then some pair of its vertices is distance at least 3 apart. Hence, \( H \) is a 6-cycle. Further, each pair of vertices in the same \( S_i \) are distance 3 apart in \( H \). Thus, if \( e(G) = 2n - 5 \), then \( G \) is the Petersen graph.

Case 2: \( \delta(G) = 2 \). Choose \( r \in V(G) \) with \( d(r) = 2 \), and let \( \{v_1, v_2\} = N(r) \). We partition \( V(G) \setminus N(r) \) into three sets, \( S_1 \), \( S_2 \), and \( S_{1,2} \). (Note that \( r \in S_{1,2} \).) Let \( S_1 \) consist of all vertices adjacent only to \( v_1 \), \( S_2 \) of all vertices adjacent only to \( v_2 \), and \( S_{1,2} \) of all vertices adjacent to both \( v_1 \) and \( v_2 \). Let \( H \) be the subgraph induced by \( S_1 \cup S_2 \), and let \( H_1, \ldots, H_t \) be the components of \( H \). We first show that \( e(G) \geq 2n - 4 \) if every \( H_i \) either contains a cycle or has a vertex adjacent to some vertex of \( S_{1,2} \).

We assign each edge to one of its endpoints as follows, so that each vertex other than \( v_1 \) and \( v_2 \) has at least 2 assigned edges. Each edge with exactly one endpoint in \( \{v_1, v_2\} \) is assigned to its other endpoint. Thus, each vertex of \( S_{1,2} \) has 2 assigned edges and each vertex of \( S_1 \cup S_2 \) has 1 assigned edge. Suppose that \( T \) is some tree component of \( H \) and \( t \) is a vertex of \( T \) with a neighbor in \( S_{1,2} \). We can direct the edges of \( T \) so that \( t \) has outdegree 0 and each other vertex has outdegree 1. Now we assign to \( t \) its edge to \( S_{1,2} \) and assign to each other vertex of \( T \) its out-edge. When \( H_i \) is a component of \( H \) with a cycle, the process is similar. We choose some spanning tree \( T \) of \( H_i \) and choose \( t \) to be some vertex incident to an edge of \( H_i \) not in \( T \).

So assume that some component \( H_1 \) is a tree and has no neighbor in \( S_{1,2} \). If \( V(H_1) \subseteq S_1 \), then \( v_1 \) is a cut-vertex, which is forbidden. Hence, \( V(H_1) \not\subseteq S_1 \); similarly, \( V(H_1) \not\subseteq S_2 \). Suppose that \( H \) has another component, with some vertex \( w \). By symmetry, assume that \( w \in S_1 \). Since \( H_1 \) has vertices in both \( S_1 \) and \( S_2 \), \( w \) is distance at least 3 from some vertex of \( H_1 \) in \( S_2 \), a contradiction. Thus, \( H_1 \) is the only component of \( H \); so from now on, we say \( H \) for \( H_1 \). Choose \( t \) arbitrarily in \( H \), and direct \( E(H) \) and assign edges as above. Now \( t \) has 1 assigned edge and each other vertex has 2 assigned edges, so \( e(G) \geq 2n - 5 \). If \( v_1 v_2 \in E(G) \), then \( e(G) \geq 2n - 4 \), so we assume \( v_1 v_2 \notin E(G) \). Similarly, if any edge has both endpoints in \( S_{1,2} \), then
$e(G) \geq 2n - 4$, so we assume that $S_{1,2}$ induces an independent set. In what follows, we characterize when equality holds in $e(G) \geq 2n - 5$.

First suppose that $H$ has leaves in both $S_1$ and $S_2$; call these $u_1$ and $u_2$, respectively. If $u_1$ and $u_2$ are adjacent, then $H$ is a single edge, which is possible; this is $F_{1,q}$, where $q = |S_{1,2}|$. Now suppose that $u_1$ and $u_2$ are nonadjacent. Since $G$ is diameter 2, $u_1$ and $u_2$ have some common neighbor, $u_3$. By symmetry, assume that $u_3 \in S_1$. Now $u_1$ and $v_2$ must have a common neighbor, so $v_1v_2$ is an edge. However, now $e(G) \geq 2n - 4$.

So assume instead that $H$ has leaves only in one of $S_1$ and $S_2$; by symmetry, say $S_2$. Let $S_1'$ denote the vertices of $S_1$ adjacent to a leaf. Now $S_1'$ induces a graph with diameter at most 1 (otherwise some leaf is distance at least 3 from some vertex of $S_1'$). Since $H$ is acyclic, $|S_1'| \leq 2$.

First suppose that $|S_1'| = 1$. Let $\{u_1\} = S_1'$. Now all vertices in $S_2$ are leaves of $H$, since $H$ is acyclic. Further, $u_1$ is the only vertex in $S_1$. Thus, $H$ is a star centered at $u_1$. This is possible; $G = F_{p,q}$, where $q = |S_{1,2}|$ and $p$ is the number of leaves of $H$.

Suppose instead that $|S_1'| = 2$, and let $\{u_1, u_2\} = S_1'$. Now $u_1$ and $u_2$ are adjacent, since $G$ has diameter 2; otherwise some leaf in $S_2$ is distance at least 3 from $u_1$ or $u_2$. Again, each vertex $u_3 \in S_2$ must be a leaf, since $G$ is acyclic. Finally, $S_1 = \{u_1, u_2\}$, again since $G$ has diameter 2. Thus, $H$ is a double star, centered at $u_1$ and $u_2$, with all leaves in $S_2$. This is also possible; $G = G_{p,q,r}$, where $p = |S_{1,2}|$ and $q$ and $r$ are (respectively) the numbers of leaves of $H$ adjacent to $u_1$ and $u_2$.

Hence, $e(G) = 2n - 5$ implies that $H$ is (i) a single edge, which is $F_{1,q}$; (b) a star with its center in $S_1$ (by symmetry) and all of its leaves in $S_2$, which is $F_{p,q}$; or (c) a double star with both of its centers in $S_1$ and all of its leaves in $S_2$, which is $G_{p,q,r}$.

This finishes the characterization of when $e(G) = 2n - 5$. 

\[ \square \]

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