Towards a CPT Invariant Quantum Field Theory
on Elliptic de Sitter Space

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Abstract

Consequences of Schrödinger's antipodal identification on quantum field theory in de Sitter space are investigated. The elliptic $\mathbb{Z}_2$ identification provides observers with complete information. We show that a suitable confinement on dimension of the elliptic de Sitter space guarantees the existence of globally defined spinors and orientable $dS/\mathbb{Z}_2$ manifold. In Beltrami coordinates, we give exact solutions of scalar and spinor fields. The CPT invariance of quantum field theory on the elliptic de Sitter space is presented explicitly.

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1 Introduction

Quantum field theory in curved space is of fundamental interest in the understanding of a conceptual unification of general relativity and quantum mechanics. Quantum field theory on de Sitter space is of great interest for three reasons. First, it provides a promising “inflationary” model of the very early universe [1]. The second, it is a highly symmetric curved space, in which one can quantize fields and obtain simple exact solutions of fields. Finally, the astronomical observations on supernova and the cosmic microwave radiation suggest that the real universe resembles de Sitter space [2].

The de Sitter space contains the past and future null infinity \( \mathcal{I}^+ \) and \( \mathcal{I}^- \). They are the surfaces where all null geodesics originated and terminated. This fact rises absurdities in the study of quantum field theory on de Sitter space. The first one is that a single immortal observer in de Sitter space can see at most half of the space. The second problem, as pointed in [3], is that if enough matter is presented, the asymptotic de Sitter space may collapse at \( \mathcal{I}^- \) in finite time. And there will be no future asymptotic de Sitter region \( \mathcal{I}^+ \) at all. Even if the collapse does not occur, presence of matter alters the causal structure also [4]. The third and most fatal problem arises in trying to define the analogue of an S matrix. In quantum field theory, asymptotic incoming and outgoing states are properly defined only in the asymptotic regions of spacetime. However, an observer can not denote the states continuously from past infinity to future infinity. Consequently, the matrix elements of S-like matrix are not measurable. When one considers quantum gravity in asymptotically de Sitter space, the situation becomes even more serious. As has been pointed out by Witten [5], the only available pairing between incoming states and outgoing states, CPT, should be used to obtain an inner product in the Hilbert space. The conventional formulation of quantum field theory and string theory is based on the existence of an S matrix.

In order to construct a consistent quantum field theory in de Sitter space, one should deal with the problems shown above carefully. The elliptic interpretation pioneered by Schrödinger [6] has been gaining interests again. In the elliptic de Sitter space, each point is identified with its antipode. This simple \( \mathbb{Z}_2 \) identification makes all observers able of gaining complete information about all events. For black holes, the identification destroys the thermal features and Fock space construction [7, 8].

Quantum field theory on the elliptic de Sitter space has been discussed extensively [9–11]. Unfortunately, a general \( d \)-dimensional elliptic de Sitter space \( dS_d/\mathbb{Z}_2 \) is non-orientable. This is a fatal problem for constructing a quantum field theory. In this paper, we have calculated the first Stiefel-Whitney class of the elliptic de Sitter space and showed that the manifold \( dS_d/\mathbb{Z}_2 \) is orientable for odd \( d \). Furthermore, we have noted that the second Stiefel-Whitney class \( \omega_2(dS_d/\mathbb{Z}_2) \) vanished while the dimension of elliptic de Sitter space equals to 3 mod 4. Thus, for \( (3 + 4n) \)-dimensional elliptic de Sitter space, there exists globally defined spinor fields. In Beltrami coordinates, we have obtain exact solutions of scalar field and spinor field. The CPT invariance of the quantum field theory on the elliptic de Sitter space is shown explicitly.

This paper is organized as follows. In Sec.2, we briefly review properties of de Sitter space and present arguments on why we should introduce the elliptic identification. In Sec.3, we investigate geometrical properties of the elliptic de Sitter space and present explicitly a confinement on dimension of the elliptic de Sitter space. This confinement guarantees the manifold \( dS_d/\mathbb{Z}_2 \) orientable and existence of globally defined spinor field. Sec.4 is devoted to explicit construction of scalar field and spinor field in Beltrami coordinates on the elliptic de Sitter space. In Sec.5, field quantization in the elliptic de Sitter space is discussed in a general way. The CPT invariance of the scalar quantum field theory is shown. In the final, we give conclusions and remarks in Sec.6.
2 Elliptic identification

De Sitter space is a maximally symmetric space\[12, 13\]. In the \((d + 1)\)-dimensional Minkowski space with metric \(\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)\), the de Sitter space \(dS_d\) is a hyperboloid with embedding equation

\[
\eta_{ab} X^a Y^b = a^2. \tag{1}
\]

Here the constant \(a\) is a parameter with units of length called the de Sitter radius. In the paper, we use the lower case letter \(x\) denotes a \(d\)-dimensional coordinate on \(dS_d\) and the upper case letter \(X\) denotes the corresponding \((d + 1)\)-dimensional coordinate in the embedding space.

The de Sitter space has a constant positive curvature \(R = d(d-1)a^{-2}\). Its symmetry group is \(O(1,d)\), which consists of four disconnected components \(G_I, G_T, G_S\) and \(G_{ST}\). Here \(I, T\) and \(S\) denote the identity, time reversal and space reflecting, respectively, \(I \equiv \text{diag}(1, 1, \ldots, 1)\), \(T \equiv \text{diag}(-1, 1, \ldots, 1)\), \(S \equiv \text{diag}(1, -1, \ldots, -1)\).

\[
G_I \text{ denotes the component which contains the identity. } G_T \text{ is obtained by acting } T \text{ on } G_I, \tag{3}
\]

In a similar way, we can get \(G_S\) and \(G_{ST}\) components. Define an antipodal transformation \(A\), which is an element of \(G_{ST}\), as

\[
A \equiv \text{diag}(-1, -1, \ldots, -1). \tag{4}
\]

The antipodal transformation sends a point \(p\) with coordinate \(X^a(p)\) to its antipode \(\bar{p}\) with coordinate \(X^a(\bar{p}) = -X^a(p)\). Since \(A\) is an element of \(G_{ST}\), it reverses direction of time.

It is convenient to introduce a quadratic real form

\[
Z(x, y) = a^{-2} \eta_{ab} X^a(x) Y^b(y). \tag{5}
\]

Then, the geodesic distance between points \(p(x)\) and \(q(y)\) can be written into the form

\[
d(x, y) = a \arccos Z(x, y). \tag{6}
\]

From the definition of the quadratic real form \(Z(x, y)\), we know that it changes sign when one point \(p(x)\) is sent to its antipode \(\bar{p}(\bar{x})\),

\[
Z(x, y) = -Z(\bar{x}, y). \tag{7}
\]

To investigate properties of the elliptic de Sitter space, one may begin with discussing on relations between point and its antipode in ordinary de Sitter space\[9\]. First, the point \(p\) and its antipode \(\bar{p}\) is always space-like separated, since \(Z(p, \bar{p}) = -a^2 X^2(p) < 0\). Moreover, the interiors of the light cones of the point \(p\) and its antipode \(\bar{p}\) do not intersect. If such intersection exist and a point \(q(Y)\) is belong to the intersection, one would have\[10\]

\[
(Y + X)^2 = (Y - X)^2 = 0 \Rightarrow Y^2 = -a^2. \tag{8}
\]

So the point \(q(Y)\) does not lie on the de Sitter hypersurface. This shows that the light cones of two antipodal points within de Sitter space do not intersect. All the events observed by a single immortal
Theorem 1: The de Sitter space.

Theorem 2: The Whitney sum \( \omega \) de Sitter space to have a vanishing second Stiefel-Whitney class \([\omega_2(E) = 0]\).

For constructing a quantum spinor field theory in elliptic de Sitter space globally, we need the elliptic identification maps \( I^+ \) and \( I^- \) to each other. This will be useful for constructing an \( S \) matrix.

However, the quotient space \( dS/\mathbb{Z}_2 \) is non-simply-connected and non-time-orientable for arbitrary dimension. The operator \( A \) reverses direction of time. Non-time-orientable elliptic de Sitter space may rise causality problems\([10]\).

In the rest of the paper, we set \( a = 1 \).

3 Geometrical properties

The general elliptic manifold \( dS_d/\mathbb{Z}_2 \) is non-orientable. For the non-orientable manifold, we can not use the “Stokes formula”. Fortunately, we will show that the manifold \( dS_d/\mathbb{Z}_2 \) is orientable while \( d \) is odd.

A real vector bundle \( E \) over \( M \) is called orientable if the structure group of \( E \) can be reduced to \( GL^+(\mathbb{R}, n) \). A manifold is called orientable if its tangent bundle is orientable. There is a theorem, which indicates that the bundle \( E \) with its first Stiefel-Whitney class \( \omega_1(E) = 0 \) must be oriented\([14]\).

Manifolds with a non-vanishing second Stiefel-Whitney class do not admit globally defined spinors\([15]\).

For constructing a quantum spinor field theory in elliptic de Sitter space globally, we need the elliptic de Sitter space to have a vanishing second Stiefel-Whitney class \([\omega_2(E) = 0]\).

In order to calculate the first and second Stiefel-Whitney class of the elliptic de Sitter space, we introduce a line bundle over \( \mathbb{R}P^n - E(\gamma^n) \),

\[
E(\gamma^n) = \{ (\{\pm x\}, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} | v = \lambda x, \lambda \in \mathbb{R} \}, \\
\pi : E(\gamma^n) \rightarrow \mathbb{R}P^n : (\{\pm x\}, v) \mapsto \{\pm x\}.
\]

There are two useful theorems on \( \gamma^n \)\([16]\) for calculating the Stiefel-Whitney classes of the elliptic de Sitter space.

Theorem 1: \( \omega_1(\gamma^n) \) generates \( H^1(\mathbb{R}P^n, \mathbb{Z}_2) \), thus \( \gamma^n \) is non-orientable.

Theorem 2: The Whitney sum \( T(\mathbb{R}P^n) \oplus \varepsilon^1 \) is isomorphic to the \((n+1)\)-fold Whitney sum \( \gamma^n \oplus \gamma^n \cdot \cdot \cdot \oplus \gamma^n \)

Here \( T(\mathbb{R}P^n) \) is the tangent bundle of \( \mathbb{R}P^n \) and \( \varepsilon^1 \) is a trivial line bundle over \( \mathbb{R}P^n \).

From the mapping \( i : S^n/\mathbb{Z}_2 \rightarrow (\mathbb{R}^1 \times S^n)/\mathbb{Z}_2 \) (a homotopy equivalence), we get the induced map

\[
i^*T(\mathbb{R}^1 \times S^n)/\mathbb{Z}_2 = (\mathbb{R}^1 \times S^n \oplus T(S^n))/\mathbb{Z}_2 \\
= \gamma^n \oplus T(\mathbb{R}P^n).
\]

By making use of relation \([11]\) and axioms of Stiefel-Whitney class, we get\([17]\)

\[
\omega(T(\mathbb{R}^1 \times S^{d-1} \oplus T(S^{d-1}))/\mathbb{Z}_2) = \omega((\mathbb{R}^1 \times S^{d-1} \oplus T(S^{d-1}))/\mathbb{Z}_2) \\
= \omega(\gamma^{d-1}) \circ \omega(T(\mathbb{R}P^{d-1})) \\
= \omega(\gamma^{d-1}) \circ \omega(T(\mathbb{R}P^{d-1}) + \varepsilon^1) \\
= \omega(\gamma^{d-1}) \circ \omega(d\gamma^{d-1}) \\
= (\omega(\gamma^{d-1}))^{d+1} = (1 + \omega_1(\gamma^{d-1}))^{d+1} \\
= 1 + (d + 1)\omega_1(\gamma^n) + \frac{(d+1)d}{2} \omega_1(\gamma^n)^2 + \cdots.
\]
From the second and third part of the above equations, we obtain
\[
\begin{align*}
\omega_1(i^* T(\mathbb{R}^1 \times S^{d-1})/\mathbb{Z}_2)) &= (d+1)\omega_1(\gamma^n) = 0, \quad \text{for } d = 1 \mod 2 \\
\omega_2(i^* T(\mathbb{R}^1 \times S^{d-1})/\mathbb{Z}_2)) &= \frac{(d+1)d}{2}\omega_2(\gamma^n) = 0, \quad \text{for } d = 3 \mod 4. \quad (12)
\end{align*}
\]
Since \(i\) is a homotopy equivalence, one has
\[
\omega_j(i^* TM) = 0 \iff \omega_j(TM) = 0. \quad (13)
\]
Thus, we can conclude that the \((3+4n)\)-dimensional elliptic de Sitter space is orientable and there exists globally defined spinors on it.

4 Beltrami coordinates and Fock space

4.1 Manifold structure of \(\mathbb{RP}^d\) and Beltrami coordinates of elliptic de Sitter space

Before discussing quantum field theory on elliptic de Sitter space in Beltrami coordinates, we recall the nature manifold structure of the projective spaces \(\mathbb{RP}^d\). Consider the set of all straight lines in \(\mathbb{R}^{d+1}\) passing through the origin. Since such a straight line is completely determined by any direction vector, and since any non-zero scalar multiple of any particular direction vector serves equally well, we may take these straight lines as the points of \(\mathbb{RP}^d\). Now each of these straight lines intersects the sphere \(S^d\) (with equation \((X^0)^2 + \cdots + (X^d)^2 = 1\)) at exactly two (diametrically opposite) points. Thus the points of \(\mathbb{RP}^d\) are in one-to-one correspondence with the pairs of diametrically opposite points of \(d\)-sphere. We may therefore think of projective space \(\mathbb{RP}^d\) as obtain from \(S^d\) by “glueing”, as they say, that is by identifying, diametrically opposite points. \(\mathbb{RP}^d\) is topologically equivalent to \(S^d/\mathbb{Z}_2\). This will be useful for us to discuss elliptic de Sitter space.

To give a manifold structure on the projective spaces \(\mathbb{RP}^d\), let us return to original characterization of \(\mathbb{RP}^d\) as consisting of equivalence classes of non-zero vectors in the space \(\mathbb{R}^{d+1}\) with coordinates \(X^0, \ldots, X^d\). Let \(U_q\) \((q = 0, 1, \ldots, d)\) denote the set of equivalence classes of vectors \((X^0, \ldots, X^d)\) with \(X^q \neq 0\). On each such region \(U_q\) of \(\mathbb{RP}^d\) we introduce the local coordinates
\[
\begin{align*}
x_1^q &= \frac{X^0}{X^q}, \ldots, x_q^q = X_q^{q-1} X_q^{q+1}, x_{q+1}^q = \frac{X^0}{X^q}, \ldots, x_d^q = \frac{X^d}{X^q}. \quad (14)
\end{align*}
\]
Clearly the regions \(U_q\) \((q = 0, 1, \ldots, d)\) cover the whole of projective \(d\)-dimensional space. The general formulae for the transition functions on \(U_j \cap U_k\) can be obtained from those for \(U_0 \cap U_1\) by the appropriate replacement of indices. In the region \(U_0 \cap U_1\), where both \(X^0, X^1 \neq 0\), the transition function from \((x_0)\) to \((x_1)\) is obviously
\[
\begin{align*}
x_1^1 &= \frac{1}{x_0^1}, x_2^2 = \frac{x_0^2}{x_0^1}, x_3^3 = \frac{x_0^3}{x_0^1}, \ldots, x_1^d = \frac{x_0^d}{x_0^1}. \quad (15)
\end{align*}
\]

The Beltrami coordinates in elliptic de Sitter space can be defined in the same way as in \(\mathbb{RP}^d\),
\[
x^a = \frac{X^a}{X^d}, \quad (a = 0, \ldots, d-1), \quad X^d \neq 0. \quad (16)
\]
Here we simply choose the \(U_d\) patch of the elliptic de Sitter space to discuss. On the other patches, same discuss can be given. For the sake of convenience, we define
\[
\sigma(x^a, x^b) \equiv 1 + \sum_{a,b=0}^d g_{ab} x^a x^b, \quad (17)
\]

5
where \( g = \text{diag}(-1, 1, \cdots, 1) \). The metric of elliptic de Sitter space in the Beltrami coordinates is given as

\[
ds^2 = -\frac{d\mathbf{x}g(I - x^Txg)^{-1}dx^T}{1 - xg\mathbf{x}^T}.
\]

(18)

In order to compare with the global time which defined in the global coordinates \((t, \Omega)\), we write the metric (18) in terms of coordinates \((X^0, x^1, \cdots, x^{d-1})\). The relation between \(X^0\) and the global time \(\tau\) is \(X^0 = \sinh \tau\). By making use of the relations between the coordinates \((x^a)\) and \((X^0, x^1, \cdots, x^{d-1})\)

\[
\sigma(x^a, x^b) = \frac{1 + \bar{x}x^T}{1 + X^0X^0},
\]

(19)

\[
x^0x^0 = \frac{X^0X^0}{1 + X^0X^0(1 + \bar{x}x^T)},
\]

(20)

we deform the metric (18) as

\[
ds^2 = -\frac{dX^0dX^0}{1 + X^0X^0} + (1 + X^0X^0)\frac{d\bar{x}(I + \bar{x}x^T)^{-1}d\bar{x}^T}{1 + \bar{x}x^T},
\]

(21)

where \(\bar{x}\) means just take the spatial part. Writing the spatial section of the metric (21) in spherical coordinates, we can get

\[
ds^2 = -d\tau^2 + \cosh^2 \tau(1 + \rho^2)^{-2}d\rho^2 + (1 + \rho^2)^{-1}\rho^2d\Omega_{d-2}^2.
\]

(22)

Rewrite metric (22) in terms of Beltrami time \(\tau\) as

\[
ds^2 = -d\tau^2 + \cosh^2 \tau(1 + \rho^2)^{-2}d\rho^2 + (1 + \rho^2)^{-1}\rho^2d\Omega_{d-2}^2.
\]

(23)

### 4.2 Scalar field

Let us consider a scalar field in \(dS_d/\mathbb{Z}_2\) with action

\[
S = -\frac{1}{2} \int d^dx\sqrt{g} \left[(\nabla \phi)^2 + m^2\phi^2\right].
\]

(24)

By writing the scalar field \(\phi\) into variable-separating form

\[
\phi(\tau, \rho, \Omega) = T(\tau)U(\rho)Y_{lm}(\Omega),
\]

(25)

we can transform the equations of motion of the scalar field in \(dS_d/\mathbb{Z}_2\) as

\[
cosh^2 \tau\dot{T}(\tau) + (d - 1)\sinh \tau \cosh \tau \dot{T}(\tau) + (m^2\cosh^2 \tau - \epsilon)T(\tau) = 0,
\]

(26)

\[
U''(\rho) + \frac{d - 2 + 2\rho^2}{\rho(1 + \rho^2)}U'(\rho) - \left[\frac{\epsilon}{(1 + \rho^2)^2} + \frac{\mu(l + d - 3)}{\rho(1 + \rho^2)}\right]U(\rho) = 0,
\]

(27)

\[
[\Delta_{S^{d-2}} + l(l + d - 3)]Y_{lm}(\Omega) = 0,
\]

(28)

where \(\epsilon\) is an arbitrary real constant. Here the \(Y_{lm}\) is not in standard form. In terms of usual spherical harmonics \(U_{lm}\), the \(Y_{lm}\) is expressed as

\[
Y_{lm} = \frac{1}{\sqrt{2}}[e^{i\pi/4}U_{lm} + e^{-i\pi/4}U^*_{lm}].
\]

(29)
The spherical harmonic functions $Y_{lm}$ are also orthonormal and form a complete set. In particular, we have

$$Y_{lm}(\Omega_A) = (-1)^l Y_{lm}^*(\Omega).$$  \hfill (30)

To solve the equation (26), first we change its variable as $X^0$. Then, the equation (26) transforms as

$$(1 + X^{02})^2 \frac{d^2 T(X^0)}{dX^0} d^2 + 2 \cdot X^0 \frac{dT(X^0)}{dX^0} dX^0 + \left[ m^2 (1 + X^{02}) - \epsilon \right] T(X^0) = 0.$$  \hfill (31)

By substituting

$$(1 + X^{02}) \frac{d^2 f(X^0)}{dX^{02}} + 2 X^0 \frac{df(X^0)}{dX^0} + \left[ m^2 - \frac{d(d - 2)}{4} + \frac{(d - 2)^2 - 4\epsilon}{4(1 + X^{02})} \right] f(X^0) = 0.$$  \hfill (32)

The solution of Eq. (33) is

$$f(-iX^0) = \begin{cases} P_\nu^{\mu}(-iX^0) \\ Q_\nu^{\mu}(-iX^0) \end{cases},$$  \hfill (34)

where

$$\nu (\nu + 1) = \frac{d}{2} \left( \frac{d}{2} - 1 \right) - m^2, \quad \mu = \sqrt{\left( \frac{d}{2} - 1 \right)^2 - \epsilon},$$  \hfill (35)

and $P_\nu^{\mu}(-iX^0)$, $Q_\nu^{\mu}(-iX^0)$ are the first and second kind of associate Legendre function respectively.

By substituting

$$U(\rho) = \rho^l (1 + \rho^2)^{\frac{d}{2}} F(\rho),$$  \hfill (36)

where

$$\kappa^2 - \kappa (d - 2) = -\epsilon,$$  \hfill (37)

the equation (27) transforms as

$$(1 + \rho^2) \frac{d^2 F(\rho)}{d\rho^2} + [(2l + 2\kappa + 2)\rho + (2l + d - 2)\rho^{-1}] \frac{dF(\rho)}{d\rho} + [l(l + 1) + \kappa (2l + 1) + \kappa^2] F(\rho) = 0.$$  \hfill (38)

Thus, we have

$$F(\rho) = \binom{\frac{l + \kappa + 1}{2}}{\frac{l + \kappa}{2}, \frac{l - 1}{2}} \frac{\Gamma(l + d - 1)}{\Gamma(l + \kappa + d - 1)}.$$  \hfill (39)

Consequently, we get solution for Eq. (26), (27) respectively,

$$T(\tau) = (\cosh \tau)^{-\frac{1}{2}} \left( \frac{d}{2} - 1 \right), \begin{cases} P_\nu^{\mu}(-i \sinh \tau) \\ Q_\nu^{\mu}(-i \sinh \tau) \end{cases},$$  \hfill (40)

$$U(\rho) = \rho^l (1 + \rho^2)^{\frac{d}{2}} \binom{\frac{l + \kappa + 1}{2}}{\frac{l + \kappa}{2}, \frac{l - 1}{2}} \frac{\Gamma(l + d - 1)}{\Gamma(l + \kappa + d - 1)}.$$  \hfill (41)
A quantum field theory should be \( CPT \) invariant. To the end, we set
\[
\mu = 0 \quad \text{and} \quad \nu \in \mathbb{R}.
\] (42)

It means that
\[
\frac{(d - 2)^2}{4} = \epsilon \quad \text{and} \quad \kappa = \frac{d - 2}{2}.
\] (43)

By making use of the relation between the associate Legendre function and hypergeometric function
\[
P_{\nu}^{\mu}(z) = \frac{2^\nu \Gamma(\nu + \frac{1}{2}) z^{\nu + \mu} (z^2 - 1)^{-\mu/2}}{\Gamma(\frac{1}{2}) \Gamma(1 + \nu - \mu)} \, _2F_1 \left( \frac{1 - \nu - \mu}{2}, -\frac{\nu + \mu}{2}, \frac{1}{2} - \nu; z^{-2} \right)
\]
\[
+ \frac{2^{-\nu-1} \Gamma(-\nu - \frac{1}{2}) z^{-\nu+\mu-1} (z^2 - 1)^{-\mu/2}}{\Gamma(\frac{1}{2}) \Gamma(-\nu - \mu)} \, _2F_1 \left( \frac{2 + \nu - \mu}{2}, \frac{1 + \nu - \mu}{2}, \frac{3}{2} + \nu; z^{-2} \right),
\] (44)

we can get
\[
T(-\tau) = T^*(\tau),
\] (46)
\[
U(-\rho) = (-1)^l U^*(\rho).
\] (47)

From the relations (30), (46) and (47) we get
\[
\phi(-x) = \phi^*(x).
\] (48)

In Sec. 5, we will show that property (48) guarantees that the scalar field theory is \( CPT \) invariant.

4.3 Spinor field

The Dirac equation in curved spacetime has been discussed extensively by Brill and Wheeler [21], and Kibble [22]. With the help of the so-called tetrad or vierbein formalism [23], one can convert general tensors into local, Lorentz-transforming tensors and shift the additional spacetime dependence into vierbeins. The metric of curved space \( g_{ij} \) is related to the metric of Minkowski space \( \eta_{ab} \) by
\[
g_{ij} = e^a_i e^b_j \eta_{ab},
\] (49)
where \( e^a_i \) is the so called vierbein. Converting the derivative \( \partial_i \) into the covariant derivative \( \nabla_i \), one yields the covariant Dirac equation in curved space
\[
(i \gamma^i \nabla_i + im)\Psi = 0,
\] (50)
where
\[
\nabla_i = \partial_i + \Gamma_i,
\] (51)
\[
\gamma^i = e^i_a \gamma^a
\] are the curved space counterparts of the Dirac \( \gamma \) matrices which satisfy
\[
\{\gamma^i, \gamma^j\} = 2g^{ij},
\] (52)
and the connection is given as
\[ \Gamma_i = \frac{1}{2} \Sigma^{ab} e^a_i (\partial_i e^b_j + \Gamma^k_{ij} e^b_k). \] (53)

Here \( \Sigma^{ab} \) is the generator of the Lorentz group given as
\[ \Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b], \] (54)
and \( \Gamma^k_{ij} \) is the Christoffel symbol defined as
\[ \Gamma^k_{ij} = -\frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \] (55)

The Dirac matrices is defined as follows (remind that the spacetime dimension is even).

From the representations of Clifford algebra, we can give an iterative expression of Dirac matrices starting from 2-dimensional space, where
\[ \gamma^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \] (56)

Then, in \( d = 2n + 2 \), we have
\[ \gamma'^a = \gamma^a \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a = 0, \ldots, d-3, \] (57)
\[ \gamma'^{d-2} = I \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma'^{d-1} = I \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \] (58)

with \( \gamma^a \) the \( 2^n \times 2^n \) Dirac matrices in \( (d - 2) \) dimension and \( I \) the \( 2^n \times 2^n \) identity.

In the Beltrami coordinates with metric (22), we can write the Dirac equation (55) as
\[ \left\{ i \cosh \tau \gamma_0 \left( \partial_\tau + \frac{d-1}{2} \tan \tau \right) - i \left[ \gamma_1 \left( 1 + \rho^2 \right) \partial_\rho + \frac{d-2}{2} \rho^{-1} \right] + \sqrt{(1 + \rho^2)^{-1}} \left[ \gamma_2 \left( \partial_{\theta_1} + \frac{d-3}{2} \cot \theta_1 \right) + \sin^{-1} \theta_1 \gamma_3 \left( \partial_{\theta_2} + \frac{d-4}{2} \cot \theta_2 \right) + \cdots + \sin^{-1} \theta_1 \cdots \sin^{-1} \theta_{d-3} \gamma_{d-1} \partial_{\theta_{d-2}} \right] \right\} \Psi(\tau, \rho, \theta_1, \cdots, \theta_{d-2}) = 0. \] (59)

By rewriting \( \Psi \) as
\[ \Psi(\tau, \rho, \theta_1, \cdots, \theta_{d-2}) = \cosh \frac{d-1}{2} \tau \sin \frac{d-2}{2} \theta_1 \cdots \sin \frac{d}{2} \theta_{d-3} \psi(\tau, \rho, \theta_1, \cdots, \theta_{d-2}), \] (60)
we transform the equation (59) as
\[ \cosh \tau (\partial_\tau + m \gamma_0) \psi = \left[ \gamma_0 \gamma_1 \left( 1 + \rho^2 \right) \partial_\rho + \frac{d-2}{2} \rho^{-1} \right] - i \sqrt{(1 + \rho^2)^{-1}} \gamma_1 \hat{K}(\theta_1, \cdots, \theta_{d-2}) \psi, \] (61)

where the Hermitian operator \( \hat{K} \) is introduced as
\[ \hat{K} = i \gamma_0 \gamma_1 (\gamma_2 \partial_{\theta_1} + \sin^{-1} \theta_1 \gamma_3 \partial_{\theta_2} + \cdots + \sin^{-1} \theta_1 \cdots \sin^{-1} \theta_{d-3} \gamma_{d-1} \partial_{\theta_{d-2}}). \] (62)
Analogous to the discussing in paper\[19\][24], we can conclude that \( \hat{K} \) commutes with \( \hat{h} \)

\[
\hat{h} = -\gamma_0 \gamma_1 (1 + \rho^2) \partial_\rho + \frac{d - 2}{2} \rho^{-1} - i \sqrt{(1 + \rho^2) \rho^{-1}} \gamma_1 \hat{K},
\]

(63)

where \( \hat{h} \) is related to the total angular momentum. Since \( \hat{K} \) commutes with \( \hat{h} \), they have common eigenfunction \( \omega \)

\[
\hat{K} \omega = k \omega, \quad k = 0, \pm 1, \pm 2, \ldots;
\]

(64)

\[
-\hat{h} \omega = E \omega, \quad E \in \mathbb{R}.
\]

(65)

Then equation (61) can be transformed as

\[
[cosh \tau (\partial_\tau + i m) + E] \varphi = 0,
\]

(66)

\[
-\gamma_0 \gamma_1 (1 + \rho^2) \partial_\rho + \frac{d - 2}{2} \rho^{-1} - i \sqrt{(1 + \rho^2) \rho^{-1}} \gamma_1 k + E \omega = 0.
\]

(67)

By making use of similitude transformation of Dirac matrices, one can always transform \( \gamma_0 \) and \( \gamma_1 \) as

\[
\gamma_0 = i \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix},
\]

(68)

where \( I \) is the \( 2^{d-2} \times 2^{d-2} \) identity and \( B \) the \( 2^{d-2} \times 2^{d-2} \) matrix satisfying \( B^2 = I \). The transformation make the solving of equation (61) easier.

For the sake of convenience, we write the eigenfunction \( \omega \) into two-component spinor function form

\[
\omega = D_k(\theta_1, \cdots, \theta_{d-2}) \begin{bmatrix} \varphi \\ \chi \end{bmatrix},
\]

(69)

where \( D_k(\theta_1, \cdots, \theta_{d-2}) \) is the eigenfunction of operator \( \hat{K} \). In the two-component spinor form, the equation of field is

\[
[cosh \tau (\partial_\tau + i m) + E] \varphi = 0,
\]

(70)

\[
\left[(1 + \rho^2) \partial_\rho + (1 + k \sqrt{1 + \rho^2}) \right] \left[(1 + \rho^2) \partial_\rho + (1 - k \sqrt{1 + \rho^2}) \right] \varphi + E^2 \varphi = 0.
\]

(71)

\[
[cosh \tau (\partial_\tau - i m) + E] \chi = 0,
\]

(72)

\[
\left[(1 + \rho^2) \partial_\rho + (1 - k \sqrt{1 + \rho^2}) \right] \left[(1 + \rho^2) \partial_\rho + (1 + k \sqrt{1 + \rho^2}) \right] \chi + E^2 \chi = 0.
\]

(73)

The difference between the equations (70), (71) and the equations (72), (73) is just changing \( m \) to \(-m\) and \( k \) to \(-k\). So that, we just need solve equations (70) and (71).

The solution of equation (70) is

\[
\varphi(\tau) = e^{-im\tau} (\tanh \tau + \sech \tau)^{-Ea}.
\]

(74)

The solution of equation (71) is more complicate. To solve it, we make a transformation,

\[
y = \frac{1 - \sqrt{1 + \rho^2}}{1 + \sqrt{1 + \rho^2}}.
\]

(75)
Then, the equation (71) is transformed into the form
\[
y(1-y) \frac{d^2 \varphi(y)}{dy^2} + \frac{(d-5)y + (d-1) d \varphi(y)}{2} + \frac{1}{4y(1-y)}
\times \left[ k(1-y^2) - k^2(1-y)^2 + 2(d-2)y - 4E^2y + \frac{(d-2)(d-4)}{4}(1+y)^2 \right] \varphi(y) = 0.
\] (76)

By choosing
\[
\varphi(y) = y^{\nu}(1-y)^{\mu} F(y), \quad \nu = \pm \left( \frac{k}{2} \frac{1}{4} \right) - \frac{d-3}{4}, \quad \mu = \pm E + \frac{d-2}{2},
\] (77)
we get
\[
y(1-y) \frac{d^2 F(y)}{dy^2} + \left[ \left( 2 \nu + \frac{d-1}{2} \right) - \left( 2 \nu + 2 \mu - \frac{d-5}{2} \right) \right] \frac{dF(y)}{dy}
\times \left[ \left( \nu - \frac{d-3}{2} \right) + 2 \nu \mu + \mu \left( \mu - \frac{d-3}{2} \right) - \frac{k(k+1)}{2} + \frac{(d-2)(d-4)}{16} \right] F(y) = 0.
\] (78)

The solution of the above equation is of the form
\[
F(y) = \begin{cases} 
2F_1(k + \frac{1}{2} \pm E, \pm E, k + \frac{1}{2}; y) \\
2F_1(1 \pm E, \frac{1}{2} - k \pm E, \frac{3}{2} - k; y)
\end{cases}
\] (79)

Then, the solution \(\varphi(\tau, \rho)\) is
\[
\varphi(\tau, \rho) = \varphi(\tau) \left( \frac{1 - \sqrt{1 + \rho^2}}{1 + \sqrt{1 + \rho^2}} \right)^{\nu} \left( \frac{2 \sqrt{1 + \rho^2}}{1 + \sqrt{1 + \rho^2}} \right)^{\mu} F \left( \frac{1 - \sqrt{1 + \rho^2}}{1 + \sqrt{1 + \rho^2}} \right).
\] (80)

In the final, we get the solution of the Dirac spinor function \(\Psi\) in Beltrami coordinates
\[
\Psi = \cosh \frac{d-1}{2} \tau \sinh \frac{d-3}{2} \theta_1 \cdots \sinh \frac{d-3}{2} \theta_{d-3} D_2(\theta_1, \cdots, \theta_{d-2}) \varphi(\tau, \rho),
\] (81)

5 **Field quantization and CPT invariance**

Consider the symmetric two-point function
\[
G^{(1)}_{\lambda}(x, y) = \langle \lambda | \Phi(x) \Phi(y) + \Phi(y) \Phi(x) | \lambda \rangle
\] (82)
in a de Sitter-invariant state \(\lambda\). We assume \(\lambda\) is invariant under the full disconnected group \(O(1, 4)\), so that \(G^{(1)}_{\lambda}(x, y)\) can only depend upon the geodesic distance \(d(x, y)\). Because the geodesic distance \(d(x, y)\) is a function of \(Z(x, y)\), the two-point function obeys the scalar field equation
\[
(\Box - m^2)G^{(1)}(x, y) = 0,
\] (83)
which can be written in terms of \(Z\), as
\[
\left[ (Z^2 - 1) \frac{d^2}{dZ^2} + d \cdot Z \frac{d}{dZ} + m^2 \right] G^{(1)}(Z) = 0.
\] (84)

This wave equation is invariant under \(Z \rightarrow -Z\). Thus the solution of (84) is of the form
\[
G^{(1)}(Z) = c_1 f(Z) + c_2 f(Z),
\] (85)
\[
f(Z) = 2F_1 \left( h_+, h_-, \frac{d}{2}, \frac{1 + Z}{2} \right),
\] (86)
\[
h_+ h_- = m^2.
\] (87)
where \( {}_2F_1(h_+, h_-, \frac{z}{2}, \frac{z}{2}) \) is the hypergeometric function, \( c_1 \) and \( c_2 \) are constants.

The general solution (55) has two poles at \( Z = 1 \) and \( Z = -1 \). If \( x \) is on the light cone of \( y \), one has \( Z(x, y) = 1 \). This is analogous to the short-distance singularity along the light cone of Minkowski space. Similarly, if \( x \) is on the light cone of \( \bar{y} \), then \( Z = -1 \). We see that, singularity in de Sitter space comes in pairs.

A real scalar field \( \Phi_s \) on the elliptic de Sitter space, is expressed in terms of a scalar field on ordinary de Sitter space, as

\[
\Phi_s(x) = \frac{1}{\sqrt{2}} [\Phi(x) + \Phi(\bar{x})].
\]  

(88)

It is symmetric under action of the antipodal operator \( A \).

Thus, the Green function on the elliptic de Sitter space is given as

\[
G_{s\lambda}(x, y) = \langle \lambda | \Phi_s(x) \Phi_s(y) + \Phi_s(y) \Phi_s(x) | \lambda \rangle.
\]  

(89)

In terms of \( G_{\lambda}^{(1)}(x, y) \), \( G_{s\lambda}^{(1)}(x, y) \) can be expressed as

\[
G_{s\lambda}^{(1)}(x, y) = \frac{1}{2} [G_{\lambda}^{(1)}(x, y) + G_{\lambda}^{(1)}(x, \bar{y}) + G_{\lambda}^{(1)}(\bar{x}, y) + G_{\lambda}^{(1)}(\bar{x}, \bar{y})].
\]  

(90)

Noticing that in de Sitter space \( G_{\lambda}^{(1)}(x, y) \) is a function of \( Z(x, y) \), and making use of Equation(51), we obtain the following relations,

\[
G_{\lambda}^{(1)}(x, y) = G_{\lambda}^{(1)}(\bar{x}, \bar{y}),
\]  

(91)

\[
G_{\lambda}^{(1)}(\bar{x}, y) = G_{\lambda}^{(1)}(x, \bar{y}).
\]  

(92)

Therefore, we have

\[
G_{s\lambda}^{(1)}(x, y) = G_{\lambda}^{(1)}(x, y) + G_{\lambda}^{(1)}(x, \bar{y}).
\]  

(93)

Before considering quantization of fields on the elliptic de Sitter space, let us recall briefly the canonical quantization of a scalar field in de Sitter space.

Normally, one expands the scalar field in terms of its modes as

\[
\Phi(x) = \sum_n [a_n \phi_n(x) + a_n^\dagger \phi_n^*(x)].
\]  

(94)

The modes \( \phi_n(x) \) satisfy the wave equation

\[
(\Box - m^2) \phi_n(x) = 0.
\]  

(95)

They are orthonormal each other with the Klein-Gordon inner product

\[
(\phi_n, \phi_m) = -i \int_\Sigma d\Sigma^\mu (\phi_n \partial_\mu \phi_m^* - \phi_m \partial_\mu \phi_n^*),
\]  

(96)

where \( \Sigma \) is a Cauchy surface. The operators \( a_n \) and \( a_n^\dagger \) satisfy commutation relations

\[
[a_n, a_m^\dagger] = \delta_{nm}.
\]  

(97)

And the vacuum state \( |\Omega\rangle \) is defined uniquely as

\[
a_n |\Omega\rangle = 0.
\]  

(98)
The Wightman function is defined by

\[ G_\Omega(x, y) \equiv \langle \Omega | \Phi(x) \Phi(y) | \Omega \rangle = \sum_n \phi_n(x) \phi^*_n(y). \] (99)

There is a unique state, the “Euclidean” vacuum \(|E\rangle\). The short distance singularities in \(G_E^{(1)}\) have the Hadamard form, the coefficient of the \(\sigma^{-1} = 2/d(x, y)^{-2}\) is the same as in flat space. The Wightman function in state \(|E\rangle\) is

\[ G_E(x, y) = \langle E | \phi(x) \phi(y) | E \rangle = \frac{\Gamma(h_+) \Gamma(h_-)}{(4\pi)^d / 2 \Gamma(\frac{d}{2})} 2 F_1 \left( h_+, h_-; \frac{1 + Z(x, y)}{2} \right). \] (100)

In de Sitter space, there is a family of de Sitter invariant Green functions parametrized by \(\alpha\) [20–30]. The “Euclidean” vacuum corresponds to \(\alpha = 0\). The Green function for these vacuum can be expressed as

\[ G^{(1)}_\alpha(x, y) = \cosh 2\alpha G^{(1)}_0(x, y) + \sinh 2\alpha G^{(1)}_0(x, \bar{y}). \] (101)

In the elliptic de Sitter space, by making use of Eq.(93), we have

\[ G^{(1)}_{s\alpha} = e^{2\alpha} [G^{(1)}_E(x, y) + G^{(1)}_E(x, \bar{y})]. \] (102)

Unlike the case of de Sitter space, in the elliptic de Sitter space even for \(\alpha = 0\), \(G^{(1)}_{s\alpha}\) contains singularities both for \(Z(x, y) = 1\) and \(Z(x, y) = -1\) and with the same strength. However, we regard the \(\alpha\) vacua for \(\alpha \neq 0\) as unphysical. Since their Green functions do not have the same short distance singularities as in flat space. As discussed in [31], in standard renormalization theory the subtractions include nonlocal contributions to the effective action for \(\alpha \neq 0\).

Going around from a point in de Sitter space to its antipodal point has the effect of acting on tangent space by \(PT\) and the \(\mathbb{Z}_2\) map also requires charge conjugation \(C\). The cumulative effect is to relate a point of its antipodal point by \(CPT\) [10]. After the \(\mathbb{Z}_2\) identification, the effect of \(CPT\) is just relate \(\Phi(x)\) to itself in the elliptic de Sitter space. The mode expansion of complex scalar field \(\Phi\) is

\[ \Phi(x) = \sum_n [a_n \phi_n(x) + b^*_n \phi^*_n(x)]. \] (103)

We define the action of \(PT\) by

\[ PT \Phi(x) T^{-1} \mathcal{P}^{-1} = \Phi(-x), \] (104)

and the action of \(C\) by

\[ C \Phi(x) C^{-1} = \Phi^\dagger(x). \] (105)

Then, we get

\[ PT a_n T^{-1} \mathcal{P}^{-1} = b^\dagger_n, \quad \text{and} \quad PT b^\dagger_n T^{-1} \mathcal{P}^{-1} = a_n, \] (106)

\[ Ca_n C^{-1} = b_n, \quad \text{and} \quad Cb_n C^{-1} = a_n. \] (107)

This is in agreement with analytic proof of the CPT theorem on quantum field theory in de Sitter space [22–25]. In fact, one can check without difficulty that the quantum field theory in the Beltrami coordinate of elliptic de Sitter space satisfies the so called assumptions of covariance, weak spectral condition and locality.

In the limit of the Hubble constant \(\frac{1}{a} \to 0\), the de Sitter space simply reduces to two copies of Minkowski space. The second copy is the CPT conjugate of the first. And the elliptic de Sitter space goes to Minkowski space.
6 Conclusions and remarks

In this paper, we have discussed consequences of Schrödinger’s antipodal identification of de Sitter space on quantum field theory. The elliptic $\mathbb{Z}_2$ identification provides observers with complete information. Unfortunately, a general $d$-dimensional elliptic de Sitter space $dS_d/\mathbb{Z}_2$ is non-orientable. This is a fatal problem for constructing a quantum field theory. We calculated the first Stiefel-Whitney class of the elliptic de Sitter space and showed that the manifold $dS_d/\mathbb{Z}_2$ is orientable for odd $d$. Furthermore, we have noted that the second Stiefel-Whitney class $\omega_2(dS_d/\mathbb{Z}_2)$ vanished while the dimension of the elliptic de Sitter space equal to 3 mod 4. Thus, for 3 mod 4-dimensional elliptic de Sitter space, there exists globally defined spinors. In Beltrami coordinates, we have given exact solutions of scalar field and spinor field. The CPT invariance of the quantum field theory on the elliptic de Sitter space was shown explicitly.

Though the asymptotic geometry of elliptic de Sitter space consists of only a single boundary $S^{d-1}$, the $\mathbb{Z}_2$ identification makes the boundary of elliptic de Sitter space much more complicated. The discussing about $S$ matrix, and interaction between scalar field and spinor field will be published separately.

The striking elliptic interpretation given by Schrödinger is the implementation of observer complementarity. However, Schrödinger did not tell us how this can be done in detail. Here we suggest that the implementation of observer complementarity is relate to the entanglement entropy. According to the big bang model of universe observers are close enough. We simply considered the total quantum system as a pure state. Next, we divide the total system into two subsystems and they represent the observer and his antipodal observer respectively. The universe will eventually become an asymptotic de Sitter Space. Then, knowing the entanglement entropy of one system means knowing the entanglement entropy of antipodal system. This can be regard as the implementation of observer complementarity.

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A Vierbein and Christoffel symbol

In this appendix, we present the detail vierbeins and Christoffel symbol used in the Sec. 4.3.

Vierbein:

\[ e^a_i = \text{diag}(1, \cosh \tau(1 + \rho^2)^{-1}, \cosh \tau \cdot \rho(1 + \rho^2)^{-\frac{1}{2}}(1, \sin \theta_1, \cdots, \sin \theta_1 \cdots \sin \theta_{d-2})). \quad (108) \]

By making use of the substitution

\[ R = \cosh t \text{ and } \lambda = 1 + \rho^2, \quad (109) \]
we may express the Chirstoffel symbol as:

\[
\begin{align*}
\Gamma_\tau^\rho &= -R \frac{dR}{dr} \lambda^{-2}, \quad \Gamma_\rho^\rho = 2\lambda^{-1} \rho, \quad \Gamma_\tau^\rho = -R^{-1} \frac{dR}{dr}, \\
\Gamma_{\theta_1,\theta_1} &= -R \frac{dR}{dr} \lambda^{-1} \rho^2, \quad \Gamma_{\theta_n,\theta_n} = \sin^2 \theta_{n-1} \Gamma_{\theta_{n-1},\theta_{n-1}} (n = 2, \cdots, d-2), \\
\Gamma_{\theta_1,\theta_1} &= \rho, \quad \Gamma_{\theta_n,\theta_n} = \sin^2 \theta_{n-1} \Gamma_{\theta_{n-1},\theta_{n-1}} (n = 2, \cdots, d-2), \\
\Gamma_{\tau_{n-1},\tau_{n-1}} &= -R^{-1} \frac{dR}{dr} (n = 2, \cdots, d-2), \\
\Gamma_{\rho_{n-1},\rho_{n-1}} &= -\rho^{-1} \lambda^{-1} (n = 2, \cdots, d-2), \\
\Gamma_{\theta_n,\theta_n} &= \sin \theta_{n-1} \cos \theta_{n-1} (n = 2, \cdots, d-2), \\
\Gamma_{\theta_n,\theta_n} &= -\mathrm{ctg} \theta_m \ [(n = 2, \cdots, d-2) \text{ and } (m = 1, \cdots, d-3) \ m < n], \\
\Gamma_{\theta_{l+1},\theta_{l+1}} &= \sin^2 \theta_l \Gamma_{\theta_l,\theta_l} \ [(l = 2, \cdots, d-3) \text{ and } (m = 1, \cdots, d-3)].
\end{align*}
\] (110)

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