ON MANIFOLDS WITH TRIVIAL LOGARITHMIC TANGENT BUNDLE: THE NON-KÄHLER CASE.

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Abstract. We study non-Kähler manifolds with trivial logarithmic tangent bundle. We show that each such manifold arises as a fiber bundle with a compact complex parallelizable manifold as basis and a toric variety as fiber.

1. Introduction

By a classical result of Wang [11] a connected compact complex manifold $X$ has holomorphically trivial tangent bundle if and only if there is a connected complex Lie group $G$ and a discrete subgroup $\Gamma$ such that $X$ is biholomorphic to the quotient manifold $G/\Gamma$. In particular $X$ is homogeneous. If $X$ is Kähler, $G$ must be commutative and the quotient manifold $G/\Gamma$ is a compact complex torus.

Given a divisor $D$ on a compact complex manifold $\bar{X}$, we can define the sheaf of logarithmic differential forms. This is a coherent sheaf. If $D$ is locally s.n.c., this sheaf is locally free and we may ask whether it is globally free, i.e. isomorphic to $\mathcal{O}^{\dim \bar{X}}$.

In [13] we investigated this question in the case where the manifold is Kähler (or at least in class $C$) and showed in particular that under these conditions there is a complex semi-torus $T$ acting on $X$ with $X \setminus D$ as open orbit and trivial isotropy at points in $X \setminus D$.

In this article we investigate the general case, i.e., we do not require the complex manifold $X$ to be Kähler or to be in class $C$.

There are two obvious classes of examples: First, if $G$ is a connected complex Lie group with a discrete cocompact subgroup $\Gamma$, then $X = G/\Gamma$ is such an example with $D$ being the empty divisor. Second, if $T$ is a semi-torus and $T \hookrightarrow \bar{T}$ is a smooth equivariant compactification such that all the isotropy groups are again semi-tori, then $X = \bar{T}$, $D = \bar{T} \setminus T$ yields such examples ([13]).

Both classes contain many examples. E.g., every semisimple Lie group admits a discrete cocompact subgroup ([2]). On the other hand,
every semi-torus admits a smooth equivariant compactification such that all isotropy groups are semi-tori. If the semi-torus under consideration is algebraic, i.e. a semi-abelian variety, then this condition on the isotropy groups is fulfilled for every smooth algebraic equivariant compactification.

We will show that in general every compact complex manifold \( \bar{X} \) with a divisor \( D \) such that \( \Omega^1(\log D) \) is trivial must be constructed out of these two special classes. First we show that there is a fibration \( \bar{X} \to Y \) where the fiber is an equivariant compactification \( \bar{A} \) of a semi-torus \( A \) and the base \( Y \) is biholomorphic to a quotient of a complex Lie group \( G \) by a discrete cocompact subgroup \( \Gamma \). From this one can deduce that any such manifold \( X \) is a compactification of a \((\mathbb{C}^*)^l\)-principal bundle over a compact complex parallelizable manifold, just like semi-tori are extension of compact complex tori by some \((\mathbb{C}^*)^l\).

2. The main results

**Theorem 1.** Let \( \bar{X} \) be a compact complex manifold and let \( D \) be a locally s.n.c. divisor such that \( \Omega^1(\log D) \) is globally trivial (i.e. isomorphic to \( \mathcal{O}_{\bar{X}}^{\dim \bar{X}} \)).

Let \( G \) denote the connected component of
\[
\text{Aut}(\bar{X}, D) = \{ g \in \text{Aut}(\bar{X}) : g(D) = D \},
\]
and let \( x \in X = \bar{X} \setminus D \).

Then \( G \) acts transitively on \( X = \bar{X} \setminus D \) with discrete isotropy group \( \Gamma = \{ g \in G : g \cdot x = x \} \).

Furthermore there exists a central subgroup \( C \simeq (\mathbb{C}^*)^l \) of \( G \), a smooth equivariant compactification \( \bar{C} \to \bar{C} \), a compact complex parallelizable manifold \( Y \) and a holomorphic fibration \( \pi : \bar{X} \to Y \) such that
- \( \pi \) is a locally holomorphically trivial fiber bundle with \( \bar{C} \) as fiber and \( C \) as structure group.
- The \( \pi \)-fibers are closures of orbits of \( C \),
- The projection map \( \pi \) is \( G \)-equivariant and admits a \( G \)-invariant flat holomorphic connections.
- There is a connected complex Lie subgroup \( H \subset G \) acting transitively on \( Y \) with discrete isotropy group \( \Lambda \).
- \( \text{Lie}(G) = \text{Lie}(H) \oplus \text{Lie}(C) \) and \( \Gamma = \{ \gamma \cdot \rho(\gamma) : \gamma \in \Lambda \} \) for some group homomorphism \( \rho : \Lambda \to C \).

Conversely, any such fiber bundle yields a log-parallelizable manifold:

**Theorem 2.** Let \( Y \) be a compact complex parallelizable manifold, \( C \simeq (\mathbb{C}^*)^l \), \( \bar{C} \) a smooth equivariant compactification of \( C \). Assume that all the isotropy groups of the \( C \)-action on \( \bar{C} \) are reductive. Let \( E_1, \ldots, E_l \)
be $\mathbb{C}^*$-principal bundles over $Y$, each endowed with a holomorphic connection.

Define $\tilde{X}$ as the total space of the $\tilde{C}$-bundle associated to the $C$-principal bundle $\Pi E_i$ and let $D$ be the divisor on $\tilde{X}$ induced by the divisor $C \setminus C \subset \tilde{C}$.

Then $D$ is a locally s.n.c. divisor and $\Omega^1(\tilde{X}; \log D)$ is globally trivial.

This can also be expressed in group-theoretic terms:

**Theorem 3.** Let $H$ be a complex connected Lie group, $\Lambda$ a discrete cocompact subgroup, $l \in \mathbb{N}$, $C = (\mathbb{C}^*)^l$, $i : C \hookrightarrow \tilde{C}$ a smooth equivariant compactification and $\rho : \Lambda \rightarrow C$ a group homomorphism. Assume that all the isotropy groups of the $C$-action on $\tilde{C}$ are reductive.

Define $\tilde{X} = (H \times \tilde{C}) / \sim$ and $X = (H \times C) / \sim$ where $(h, x) \sim (h', x')$ iff there is an element $\lambda \in \Lambda$ such that $(h', x') = (h\lambda, \rho(\lambda^{-1})x)$. Define $D = \tilde{X} \setminus X$.

Then $D$ is a locally s.n.c. divisor and $\Omega^1(\tilde{X}; \log D)$ is globally trivial.

### 3. Logarithmic Forms

Let $X$ be a complex manifold and $D$ a divisor. We say that $D$ is “locally s.n.c.” (where s.n.c. stands for “simple normal crossings”) if if for every point $x \in X$ there exists local coordinates $z_1, \ldots, z_n$ and a number $d \in \{0, \ldots, n\}$ such that in a neighbourhood of $x$ the divisor $D$ equals the zero divisor of the holomorphic function $\Pi_{i=1}^d z_i$.

It is called a “divisor with only simple normal crossings as singularities” or “s.n.c. divisor” if in addition every irreducible component of $D$ is smooth.

Let $\tilde{X}$ be a compact complex manifold with a locally s.n.c. divisor $D$. There is a stratification as follows: $X_0 = X = \tilde{X} \setminus D$, $X_1 = D \setminus \text{Sing}(D)$ and for $k > 1$ the stratum $X_k$ is the non-singular part of $\text{Sing}(\tilde{X}_{k-1})$.

If in local coordinates $D$ can be written as $\{z : \Pi_{i=1}^d z_i = 0\}$, then $z = (z_1, \ldots, z_n) \in X_k$ iff $\# \{i : 1 \leq i \leq d, z_i = 0\} = k$.

Let $D$ be an effective divisor on a complex manifold $X$. The sheaf $\Omega^k(\log D)$ of logarithmic $k$-forms with respect to $D$ is defined as the sheaf of all meromorphic $k$-forms $\omega$ for which both $f\omega$ and $fd\omega$ are holomorphic for all $f \in I_D$. If $D$ is locally s.n.c., we may also define the algebra (with respect to exterior product) of logarithmic $k$-forms as the $\mathcal{O}_X$-algebra generated by 1 and all $df/f$ where $f$ is a section $\mathcal{O}_X \cap \mathcal{O}_X^*$. Note: A logarithmic 0-form is simply a holomorphic 0-form, i.e., a holomorphic function.

For every natural number $k$ the sheaf of logarithmic $k$-forms is a coherent $\mathcal{O}$-module sheaf. It is locally free if $D$ is a locally s.n.c. divisor.
Especially, if $D = \{ z_1 \cdot \ldots \cdot z_d = 0 \}$, then $\Omega^1(\log D)$ is locally the free $\mathcal{O}_\tilde{X}$-module over $dz_1/z_1, \ldots, dz_d/z_d, dz_{d+1}, \ldots, dz_n$.

For a locally s.n.c. divisor $D$ on a complex manifold we define the logarithmic tangent bundle $T(-\log D)$ as the dual bundle of $\Omega^1(\log D)$.

Then $T(-\log D)$ can be identified with the sheaf of those holomorphic vector fields $V$ on $\tilde{X}$ which fulfill the following property:

$V_x$ is tangent to $X_k$ at $x$ for every $k$ and every $x \in X_k$.

In local coordinates: If $D = \{ z : \prod_{i=1}^d z_i = 0 \}$, then $T(-\log D)$ is the locally free sheaf generated by the vector fields $z_i \frac{\partial}{\partial z_i}$ ($1 \leq i \leq d$) and $\frac{\partial}{\partial z_i}$ ($d < i \leq n$).

This implies in particular (compare also \cite{12}):

**Proposition 1.** Let $X$ be a compact complex manifold and $D$ a locally s.n.c. divisor such that $\Omega^1(\log D)$ is globally trivial.

Let $G$ denote the connected component of the group $\text{Aut}(X, D)$ of all holomorphic automorphisms of $X$ which stabilize $D$.

Then the Lie algebra of $G$ can be identified with $\Gamma(\tilde{X}, \Omega^1(\log D))^*$ and the $G$-orbits in $\tilde{X}$ coincide with the connected components of the strata $X_k$.

For every $x \in X_0 = \tilde{X} \setminus D$ the isotropy group $G_x = \{ g \in G : g(x) = x \}$ is discrete.

The isotropy representation of $G_x$ on $T_x$ is almost faithful (i.e. the kernel is discrete) for every $x \in \tilde{X}$.

4. Residues

Let $\tilde{X}$ be a complex manifold and $D$ a locally s.n.c. divisor. In the preceding section we introduced the notion of logarithmic forms. Now we introduce “residues”.

Let $\tau : \tilde{D} \rightarrow D$ denote a normalization of $D$. Note that $\tilde{D}$ is a (not necessarily connected) compact manifold. It is smooth, because $D$ is locally s.n.c. Furthermore $D$ being locally s.n.c. implies that $D' = \tau^* \text{Sing}(D)$ defines a divisor on $\tilde{D}$.

We are going to define

$$\text{res} : \Omega^k(X; \log D) \longrightarrow \Omega^{k-1}(\tilde{D}, \log D').$$

Let $\hat{x} \in \tilde{D}$ and $x = \tau(\hat{x}) \in X$. $x$ admits an open neighbourhood $W$ in $X$ with local coordinates $z_1, \ldots, z_n$ such that

1. $z_1(x) = \ldots = z_n(x) = 0$.
2. $\tau$ induces an isomorphism between an open neighbourhood $\hat{W}$ of $\hat{x}$ in $\tilde{D}$ with $\{ p \in W : z_1(p) = 0 \}$. 
(3) There is a number $1 \leq k \leq n$ such that $D|_W$ is the zero divisor of the function $z_1 \cdots z_k$.

(4) The restriction of $D'$ to $\hat{W}$ is the pull-back of the zero divisor of the function $z_2 \cdot z_3 \cdots z_k$ on $W$.

Now let $\omega \in \Gamma(W, \Omega^k(\log D))$. Then

$$\omega = \eta + \frac{dz_1}{z_1} \wedge \mu$$

where $\eta$ and $\mu$ have poles only along the zero set of $z_2 \cdot \cdots \cdot z_k$. We define the restriction of $\text{res}(\omega)$ to $\hat{W}$ as

$$\text{res}(\omega)|_{\hat{W}} = \tau^* \mu$$

It is easily verified that $\text{res}(\omega)$ is independent of the choice of local coordinates and the decomposition $\omega = \eta + \frac{dz_1}{z_1} \wedge \mu$ and that furthermore the only poles of $\text{res}(\omega)$ are logarithmic poles along $D'$.

Moreover:

**Lemma 1.** Let $X$ be a complex manifold, $D$ a locally s.n.c. divisor and $\omega$ a logarithmic $k$-form ($k \in \mathbb{N} \cup \{0\}$).

Then $\omega$ is holomorphic iff $\text{res}(\omega) = 0$. Furthermore

$$\text{res}(d\omega) = -d\text{res}(\omega)$$

**Proof.** This is easily verified by calculations in local coordinates. \qed

As a consequence we obtain the following result which we will need later on.

**Lemma 2.** Let $\bar{X}$ be a compact complex manifold, $D$ a locally s.n.c. divisor and $\omega$ a logarithmic $1$-form.

Then $d\omega$ is holomorphic.

**Proof.** The residue $\text{res}(\omega)$ is a logarithmic 0-form on the normalization $\hat{D}$ of $D$. But logarithmic 0-forms are simply holomorphic functions, and every holomorphic function on a $\hat{D}$ is locally constant by our compactness assumption. Therefore $d\text{res}(\omega) = 0$. Consequently $\text{res}(d\omega) = 0$, which is equivalent to the condition that $d\omega$ has no poles. \qed

**Remark 1.** Residues can also be introduced for divisors which are not locally s.n.c., see (Saito?).

5. **Group theoretic preparations**

**Definition.** A (not necessarily closed) subgroup $H$ of a topological group $G$ is called cocompact iff there exists a compact subset $B \subset G$ such that $B \cdot H = \{bh : b \in B, h \in H\} = G$. 
If $H$ is a closed subgroup of a Lie group $G$, then $H$ is cocompact if and only if the quotient space $G/H$ is compact.

**Proposition 2.** Let $G$ be a connected complex Lie group, $Z$ its center and $\Gamma$ a subgroup. Let $C = Z_G(\Gamma)^0$ denote the connected component of the centralizer $Z_G(\Gamma) = \{ g \in G; g\gamma = \gamma g \ \forall \gamma \in \Gamma \}$.

Assume that $Z\Gamma$ is cocompact. Then $C = Z^0$.

**Proof.** Let $B$ be a compact subset of $G$ such that $BZ\Gamma = G$.

For each $c \in C$ we define a holomorphic map from $G$ to $G$ by taking the commutator with $c$:

$$\zeta_c : g \mapsto cgc^{-1}g^{-1}$$

Since $c$ commutes with every element of $Z\Gamma$, we have $\zeta_c(G) = \zeta_c(B)$. Thus compactness of $B$ implies that the image $\zeta_c(G) = \zeta_c(B)$ is compact. Considering the adjoint representation

$$Ad : G \to GL(\text{Lie } G) \subset \mathbb{C}^{n \times n}$$

we deduce that for every holomorphic function $f$ on $GL(\text{Lie } G)$ the composed map $f \circ Ad \circ \zeta_c : G \to \mathbb{C}$ is a bounded holomorphic function on $G$ and therefore constant. It follows that the image $\zeta_c(G)$ is contained in $\ker Ad = Z$.

Thus $\zeta_c(h)$ is central in $G$ for all $c \in C, h \in G$. This implies

$$\zeta_c(g)\zeta_c(h^{-1}) = cgc^{-1}g^{-1}(ch^{-1}c^{-1}h) = cgc^{-1}(ch^{-1}c^{-1}h)g^{-1}$$

$$= cgh^{-1}c^{-1}hg^{-1} = \zeta_c(gh^{-1})$$

for all $c \in C$ and $g, h \in G$. Hence $\zeta_c : G \to G$ is a group homomorphism for all $c \in C$.

It follows that $\zeta_c(G)$ is a connected compact complex Lie subgroup of $Z$. In particular, $\zeta_c(G)$ is a compact complex torus and the commutator group $G'$ is contained in $\ker \zeta_c$. Let $A$ denote the smallest closed complex Lie subgroup of $G$ containing $G'Z\Gamma$. Then $A$ is normal, because it is a subgroup containing $G'$, and $G/A$ is compact, since $Z\Gamma$ is cocompact. Therefore $G/A$ is a connected compact complex Lie group, i.e., a compact complex torus. Thus for every $c$ the map $\zeta_c : G \to G$ fibers through $G/A$ and is induced by some holomorphic Lie group homomorphism from $G/A$ to $T$ where $T$ denotes the maximal compact complex subgroup of $Z$. Note that both $G/A$ and $T$ are compact tori. This implies that $\text{Hom}(G/A, T)$ is discrete. Now $c \mapsto \zeta_c$ defines a map from the connected complex Lie group $C$ to $\text{Hom}(G/A, T)$. Since $\text{Hom}(G/A, T)$ is discrete, this map is constant. Furthermore $\zeta_c \equiv e$. Hence $\zeta_c \equiv e$ for all $c \in C$. This implies $C \subset Z$. On the other hand the
inclusion $Z^0 \subset C$ is obvious. This completes the proof of the equality $C = Z^0$. □

**Proposition 3.** Let $G$ be a connected complex Lie group, $Z$ its center and $\Gamma$ a discrete subgroup. Assume that $Z\Gamma$ is cocompact.

Then $Z^0\Gamma$ is closed in $G$.

*Proof.* Let $C = C_G(\Gamma)^0$ be the connected component of the centralizer. Then $C = Z^0$ be the preceding proposition. On the other hand $G\Gamma$ is closed by an argument of Raghunathan, see [12], lemma 3.2.1. □

**Corollary 1.** Let $G$ be a connected complex Lie group, $Z$ its center and $\Gamma$ a discrete subgroup. Assume that $Z\Gamma$ is cocompact.

Then there exists a discrete cocompact subgroup $\Gamma_1$ in $G$ with $\Gamma \subset \Gamma_1$.

Moreover the commutator groups $\Gamma'$, $\Gamma'_1$ can be required to be equal.

*Proof.* Note that $Z^0$ is a connected commutative Lie group. It is easy to see that there is a discrete subgroup $\Lambda \subset Z^0$ which is cocompact in $Z^0$ and contains $\Gamma \cap Z^0$. Define $\Gamma_1 = \Gamma \cdot \Lambda$. This is a subgroup, because $\Lambda$ is central in $G$. By prop. 3 both $Z^0\Gamma_1$ and $Z^0\Gamma$ are closed in $G$. Hence we may consider the fibration

$$X = G/\Gamma_1 \rightarrow G/Z^0\Gamma_1 = G/Z^0\Gamma = Y.$$  

Now $Y$ and $Z^0/\Lambda \simeq Z^0\Gamma_1/\Gamma_1$ are both compact, hence $X$ is likewise compact.

Finally we note that the equality $\Gamma' = \Gamma'_1$ follows from the fact that $\Lambda$ is central. □

**Proposition 4.** Let $G$ be a connected complex Lie group, $Z$ its center and $\Gamma$ a discrete subgroup. Assume that $Z\Gamma$ is cocompact.

Then $\Gamma' \cap Z$ is cocompact in $G' \cap Z$.

*Proof.* By cor. 1 there is a discrete cocompact subgroup $\Gamma \subset \Gamma_1 \subset G'$ with $\Gamma' = \Gamma'_1$.

By passing to the universal covering there is no loss in generality in assuming that $G$ is simply-connected.

Now $(G' \cap R)/(\Gamma_1 \cap G' \cap R)$ and $Z/(Z \cap \Gamma_1)$ are both compact, hence $G'/Z \cap G' \cap \Gamma$ is compact, too. Recall that $\Gamma_1' = \Gamma' \subset \Gamma$. By [12], prop. 3.11.2 $R \cap \Gamma'_1$ is of finite index in $G' \cap R \cap \Gamma_1$. Using $Z \subset R$, it follows that $Z \cap G' \cap \Gamma$ is of finite index in $Z \cap G' \cap \Gamma_1$. Hence $Z \cap G' \cap \Gamma$ is cocompact in $Z \cap G'$. □

**Corollary 2.** Let $G$ be a connected complex Lie group, $Z$ its center and $\Gamma$ a discrete subgroup. Assume that $Z\Gamma$ is cocompact and assume further that $G$ acts effectively on $G/\Gamma$.

Then $G' \cap Z$ is compact.
Proof. The subgroup $\Gamma \cap Z$ acts trivially on $G/\Gamma$, hence $\Gamma' \cap Z = \{e\}$ if $G$ acts effectively on $G/\Gamma$. However, the trivial subgroup $\{e\}$ is cocompact in $G' \cap Z$ if and only if the latter is compact. \hfill \Box

6. The commutative case

Proposition 5. Let $G$ be a commutative connected complex Lie group and let $G \hookrightarrow \bar{X}$ be a smooth equivariant compactification, $D = \bar{X} \setminus G$.

Then $\Omega^1(\log D)$ is a holomorphically trivial vector bundle on $\bar{X}$ if and only if the following two conditions are fulfilled:

1. $G$ is a semi-torus, and
2. every isotropy group for the $G$-action on $\bar{X}$ is a semi-torus.

Proof. This is (1) $\Rightarrow$ (2) of the Main Theorem in [13]. In [13] we assumed that $\bar{X}$ is Kähler or in class $C$. However, this assumption is used only in order to deduce that $G$ is commutative. Hence the line of arguments in [13] still applies if instead of requiring $\bar{X}$ to be Kähler we merely assume that $G$ is commutative. \hfill \Box

7. The isotropy groups

Proposition 6. Let $\bar{X}$ be a compact complex manifold and $D$ a locally s.n.c. divisor such that $\Omega^1(\bar{X}; \log D)$ is spanned by global sections. Let $G$ be a connected complex Lie group acting holomorphically on $\bar{X}$ such that $D$ is stabilized and $p \in \bar{X}$.

Then the connected component of the isotropy group $G_p = \{g \in G : g(p) = p\}$ is central in $G$.

Proof. Let $w \in \text{Lie}(G)$ and $v \in \text{Lie}(G_p)$. We have to show that $[w, v] = 0$ for any such $w, v$. By abuse of notation we identify the elements $w, v$ of $\text{Lie}(G)$ with the corresponding fundamental vector fields on $\bar{X}$. Now let us assume that $[w, v] \neq 0$. Then there exists a logarithmic one-form $\omega \in \Omega^1(\bar{X}; \log D)$ such that $(\omega, [w, v])$ is not identically zero. Since $G$ stabilizes $D$, the vector fields $w, v$ are tangent to $D$ and thus sections in $T(-\log D)$ which is dual to $\Omega^1(\log D)$. Therefore $(\omega, w)$ and $(\omega, v)$ are global holomorphic functions on $\bar{X}$. Because $\bar{X}$ is compact, they are constant. Hence $w(\omega, v)$ and $v(\omega, w)$ vanish both. Therefore

$$\omega([w, v]) = d\omega(w, v).$$

Now $d\omega$ has no poles (lemma 2) and $v$ vanishes at one point, namely $p$. Thus $\omega([w, v])$ is a global holomorphic function on a compact connected manifold which vanishes at one point, i.e., $\omega([w, v]) \equiv 0$ contrary to our assumption on $\omega$. We have thus deduced that the assumption $[w, v] \neq 0$ leads to a contradiction. Hence $[w, v] = 0$ for all $w \in \text{Lie}(G)$ and all $v \in \text{Lie}(G_p)$. It follows that $G^0_p$ is central in $G$. \hfill \Box
8. Cocompactness of $Z\Gamma$

We fix some notation to be used throughout this section:

- $\tilde{X}$ is a compact complex manifold,
- $D$ a loc. s.n.c. divisor on $\tilde{X}$ and $X = \tilde{X} \setminus |D|$, 
- We assume that $\Omega^1(\tilde{X}, \log D)$ is a globally trivial $\mathcal{O}_{\tilde{X}}$-module,
- $\text{Aut}(\tilde{X}, D) = \{g \in \text{Aut}(\tilde{X}) : g(D) = D\}$,
- $G$ denotes the connected component of $\text{Aut}(\tilde{X}, D)$ which contains the identity map,
- $x$ denotes a fixed point in $X = \tilde{X} \setminus |D|$, 
- $\Gamma$ denotes the isotropy group at $x$, i.e., $\Gamma = \{g \in G : g \cdot x = x\}$
- $Z$ denotes the center of $G$.

**Proposition 7.** $Z\Gamma$ is cocompact in $G$.

**Proof.** Equip $\tilde{X}$ with some Riemannian metric. Let $p \in |D|$. Assume that $p \in D_1 \cap \ldots \cap D_j$ and $p \notin D_i$ for $i > j$. Then there are logarithmic vector fields $v_i$ and local holomorphic coordinates $z_i$ near $p$ such that $v_i = z_i \frac{\partial}{\partial z_i}$ for $1 \leq i \leq j$. As a consequence, for every point $p \in \tilde{X}$ there exists an open neighborhood $W_p$ and a positive number $\epsilon_p > 0$ such that the following assertion holds: For every $y \in W_p \setminus |D|$ there exists an element $g \in G^0_p$ such that $d(g \cdot w, |D|) > \epsilon_p$. Since $\tilde{X}$ is compact, it can be covered by finitely many of the open sets $W_p$. We choose such a finite collection and define $\epsilon$ as the minimum of the $\epsilon_p$. Let $H$ be the subgroup of $G$ generated by all the subgroups $G^0_p$.

Then we have: For every $y \in X \setminus |D|$ there exists an element $g \in H$ such that $d(g \cdot y, |D|) > \epsilon$.

Compactness of $X$ implies that $C = \{z \in X : d(z, |D|) \geq \epsilon\}$ is compact. We have $H \cdot C = X \setminus |D|$. We can choose a compact subset $K \subset G$ such that the natural projection from $G$ onto $G/\Gamma \simeq X \setminus |D|$ maps $K$ onto $C$. Then $G = H \cdot K \cdot \Gamma$. Observe that $H$ is central in $G$ due to prop 6. Hence

$$G = H \cdot K \cdot \Gamma = K \cdot H \cdot \Gamma$$

and $H \subset Z$. It follows that $Z\Gamma$ is cocompact in $G$. \qed

**Corollary 3.** $Z^0\Gamma$ is closed and cocompact in $G$.

**Proof.** Imply by prop. 3 and 7. \qed

**Proposition 8.** $G/Z^0\Gamma$ is a compact parallelizable manifold and the natural projection $G/\Gamma \to G/Z^0\Gamma$ extends to a holomorphic map from $\tilde{X}$ onto $G/Z^0\Gamma$. 

Proof. $G/Z^0\Gamma$ can be described as the quotient of the complex Lie group $G/Z$ by the discrete subgroup $\Gamma/(\Gamma \cap Z^0)$ and is therefore a compact parallelizable manifold.

Now let $p$ be a point in $D \setminus \text{Sing}(D)$. Choose local cooordinates near $p$ such that $D = \{ z_1 = 0 \}$. Since $\Omega^1(\log D)$ is trivial, there is a global vector field $v$ such that 

$$\left( \frac{1}{z_1} v \right)_p = \frac{\partial}{\partial z_1} .$$

Now $v$ is central in $\text{Lie}(G)$ and $G$ acts locally transitively on $D \setminus \text{Sing}(D)$. Therefore $v$ vanishes as holomorphic vector field on $D$. It follows that $w = \frac{1}{z_1} v$ is a holomorphic vector field near $p$. Moreover $w_p = \frac{\partial}{\partial z_1}$. Now, as any vector field, is locally integrable. Therefore there is a local coordinate system near $p$ in which $w$ equals $\frac{\partial}{\partial z_1}$. Since $v$ is central as element in $\text{Lie}(G)$, $v$, as a holomorphic vector field on $X \setminus D$, is tangent to the fibers of $\tau : G/\Gamma \to G/Z^0\Gamma$. Thus in this local coordinate system near $p$, the map $\tau$ depends only on the variables $z_2, \ldots, z_n$. But this implies that $\tau$, which is defined on $\{ z_1 \neq 0 \}$, extends to a map defined on the whole neighbourhood of $p \in D$. In this way we say that $\tau$, originally defined on $X \setminus D$, extends to a holomorphic map defined on $X \setminus \text{Sing}(D)$.

Next we observe that the image space $G/Z^0\Gamma$ is complex parallizable, and therefore has a Stein universal covering. It follows that we can use the classical Hartogs theorem to extend $\tau$ through the set $\text{Sing}(D)$ which has codimension at least two in $X$. □

Proposition 9. Let $Y$ be a complex manifold on which $G$ acts transitively, and $\pi : X \to Y$ an equivariant holomorphic map.

Then each fiber $F$ of $\pi$ is smooth, $D \cap F$ is a s.n.c. divisor in $F$ and $\Omega^1(F, \log(D \cap F))$ is trivial.

Proof. A generic fiber is smooth. Since $\pi$ is equivariant and $G$ acts transitively on $Y$, it follows that every fiber is smooth. Let $S = \{ y \in Y : \pi^{-1}(y) \subset D \}$. Then $S$ is $G$-invariant. Hence either $\pi(S) = Y$ or $S = \emptyset$. But $\pi(S) = Y$ would imply $D = X$ contrary to $D$ being a divisor. Hence $S = \emptyset$, i.e. $D$ does not contain any fiber of $\pi : X \to Y$.

Now we consider the usual stratification of $D$ (as described in §3). At each point $p \in D_k$ the divisor $D$ can locally be defined as $D = \{ f_1 \ldots f_k = 0 \}$ where $df_1, \ldots, df_k$ are linearly independent in $T^*_p X$. Since $D$ can not be a pull-back of a divisor on $Y$, for a generic choice of $p$ there will be local functions $f_1, \ldots, f_k$ near $p$ such that $D = \{ f_1 \ldots f_k = 0 \}$ and in addition such that the $df_i$ are linearly independent as elements in $T_p F$ (with $F = \pi^{-1}(\pi(p))$). In the same spirit as in the
definition of $S$ above we can now define the set of points in $D$ where
this fails. Observing that these sets are invariant, and recalling that
$G$ acts transitively on the connected components of the strata $D_k$, we
deduce that they must empty. Thus $D \cap F$ is always a s.n.c. divisor.

By similar arguments we see that $\Omega^1(F, \log(D \cap F))$ is trivial. □

9. Proofs for the theorems

Proof of theorem 1. Let $Z$ denote the center of $G$. By definition $G$ is a
subgroup of the automorphism group of $\bar{X}$ and therefore acts effectively
on $\bar{X}$. By cor. 3 $Z^0\Gamma$ is closed in $G$ and due to prop. 8 there is a
holomorphic fiber bundle $\pi: \bar{X} \rightarrow Y_0 = G/Z^0\Gamma$ which extends the
natural projection $G/\Gamma \rightarrow Y_0 = G/Z^0\Gamma$. From prop 9 we deduce that
a typical fiber $F$ of $\bar{X}$ is log-parallelizable with respect to the divisor
$F \cap D$. Since $Z$ is commutative, this implies (due to prop. 5) that $Z^0$
is a semi-torus. Let $C$ be a maximal connected linear subgroup of $Z^0$.
Then $C \simeq (\mathbb{C}^*)^l$ for some $l \in \mathbb{N}$ and $Z^0/C$ is a compact complex torus.
The fibration $Z^0 \rightarrow Z^0/C$ induces a tower of fibrations
$$G/\Gamma \rightarrow G/CT \rightarrow G/Z^0\Gamma$$
which (using prop. 8) extends to fibrations
$$\bar{X} \rightarrow Y = G/CT \rightarrow G/Z^0\Gamma$$
Now $Y$ is parallelizable because $C$ is normal in $G$ and compact because
both $G/Z^0\Gamma$ and $Z^0/C$ are compact.

From cor. 2 we deduce that $G' \cap Z$ is compact. Since $C \subset Z$ and
$C \simeq (\mathbb{C}^*)^l$, it follows that $G' \cap C$ is discrete. Thus we can choose
a complex vector subspace $V \subset \text{Lie}(G)$ such that $\text{Lie}(G') \subset V$ and
$\text{Lie}(G) = V \oplus \text{Lie}(C)$. The condition $\text{Lie}(G') \subset V$ implies that $V$ is an
ideal in $\text{Lie}(G)$. Hence $V$ is the Lie algebra of a normal Lie subgroup
$H$ of $G$. Furthermore $H \cdot C = G$ due to $\text{Lie}(G) = V \oplus \text{Lie}(C)$. The
condition $H \cdot C = G$ implies that $H$ acts transitively on $Y = G/CT$. For
dimension reasons the isotropy group $\Lambda = H \cap (CT)$ is discrete.

Because $G$ acts effectively on $X$, the intersection $Z \cap \Gamma$ must be
trivial. Hence $C \cap \Gamma = \{e\}$. It follows that the projection map $\tau: G \rightarrow
G/C \simeq H/(H \cap C)$ maps $\Gamma$ injectively onto $\tau(\Gamma) = CT/C$.

We claim that $\tau(\Gamma) = \tau(\Lambda)$. Indeed, $\Lambda = H \cap (CT) \subset CT$ implies
$\tau(\Lambda) \subset \tau(\Gamma)$. On the other hand, if $\gamma \in \Gamma$, then there exists an element
$c \in C$ such that $\gamma c \in H$, because $H \cdot C = G$. Then $\tau(\gamma) = \tau(\gamma c)$ and
$\gamma c = c \gamma \in H \cap (CT) = \Lambda$. Hence $\tau(\Gamma) \subset \tau(\Lambda)$.

It follows that there exists a map $\rho: \Lambda \rightarrow C$ such that for each $\lambda \in \Lambda$
the product $\lambda \rho(\lambda)$ is the unique element $\gamma$ of $\Gamma$ with $\tau(\lambda) = \tau(\gamma)$. 
One verifies easily that $\rho$ is a group homomorphism, using the facts that $\tau$ is a group homomorphism and that $C$ is central.

To obtain the $G$-invariant flat connection, we observe that the fundamental vector fields of $\text{Lie}(H) = V$ induce a decomposition $T_x\tilde{X} = V \oplus \ker(d\pi)$ in each point $x \in \tilde{X}$. This connection is obviously $G$-invariant. Moreover it is flat, because $V = \text{Lie}(H)$ is a Lie subalgebra of $\text{Lie}(G)$.

This completes the proof. □

Proof of theorem 2. Due to prop. 5 we know that $\tilde{C}$ is log-parallelizable, i.e., that $\Omega^1(\tilde{C}, \log(\tilde{C} \setminus C))$ is isomorphic to $\mathcal{O}_{\tilde{C}}$. Moreover, this isomorphism is given by meromorphic 1-forms which are dual to a basis of the $C$-fundamental vector fields. Let $V_1, \ldots, V_l$ be such a basis of $C$-fundamental vector fields and $\eta_1, \ldots, \eta_l$ a dual basis for the logarithmic one-forms on $\tilde{C}$. We may regard $\tilde{C}$ as one fiber of the projection map from $\tilde{X}$ onto $Y$. Using the $C$-principal right action of $C$ we extend $V_1, \ldots, V_l$ to holomorphic $C$-fundamental vector fields on all of $\tilde{X}$. Let $H \subset T\tilde{X}$ be the horizontal subbundle defined by the connection. Then we can extend the meromorphic one-forms $\eta_i$ as follows: For each $i$ we require that $\eta_i(Y_j) = \delta_{ij}$ and that $\eta_i$ vanishes on $H$.

Let $\omega_1, \omega_r$ be a family of holomorphic 1-forms on $Y$ which gives a trivialization of the tangent bundle $TY$. Define $\mu_i = \pi^*\omega_i \in \Omega^1(\tilde{X})$.

We claim that the family $(\mu_i)_i$ together with the family $\eta_i$ gives a trivialization of the sheaf of logarithmic one-forms on $\tilde{X}$.

Evidently the $\mu_i$ are holomorphic. Since the restriction of $\eta_i$ to a fiber is logarithmic and $\eta_i|_H \equiv 0$, it is clear that $f\eta_i$ is holomorphic for any locally given function $f$ vanishing on $D$. It remains to show that $f d\eta_i$ is holomorphic as well. To see this, we calculate $f d\eta_i(Y, Z)$ for holomorphic vector fields $Y, Z$. Recall that

$$d\eta_i(Y, Z) = Y \eta_i Z - Z \eta_i Y - \eta_i ([Y, Z])$$

It suffices to verify holomorphicity for a base of vector fields. Thus we may assume that each of the vector fields $Y$ and $Z$ is horizontal or vertical (with respect to the connection). If both are vertical, there is no problem since $\eta_i$ restricted to a fiber is logarithmic. If both are horizontal, then $\eta_i Y = \eta_i Z = 0$. Furthermore $[Y, Z]$ is horizontal, because the connection is flat. Hence $\eta_i ([Y, Z]) = 0$ and consequently $d\eta_i(Y, Z) = 0$. Finally let us discuss the case where $Y$ is horizontal and $Z$ is vertical. Then $\eta_i Y = 0$ and moreover $[Y, Z] = 0$ because $H$ is defined by a connection for the $C$-bundle and therefore $C$-invariant. Thus $f d\eta_i(Y, Z) = f Y \eta_i Z$. Since the fiber $\tilde{C}$ is log-parallelizable, it
suffices to consider the case where, up to multiplication by a meromorphic function, $Z$ agrees with a $C$-fundamental vector field $V$. Let $\phi$ be a defining function for the zero locus of $V$. We may then assume that $Z = \frac{1}{\phi} V$. Then
\[
fd\eta_i(Y, Z) = fY d\eta_i \frac{1}{\phi} V = -f \frac{Y \phi}{\phi^2} (\eta_i V) + f \frac{\eta V}{\phi} (\eta_i V)
\]
Since $\eta_i$ is dual to $V_i$, the function $\eta_i V$ is constant and the second term vanishes. Therefore
\[
fd\eta_i(Y, Z) = - \left( \frac{f}{\phi} \right) \left( \frac{Y \phi}{\phi} \right) (\eta_i V).
\]
We claim that all three factors are holomorphic. Indeed, the zero locus of $\phi$ is contained in $D$ (with multiplicity one) and $f$ vanishes on $D$. By construction the horizontal vector field $V$ is tangent to $D$, hence $V(\phi)$ vanishes along the zero locus of $\phi$ (which is contained in $D$) and therefore $V\phi/\phi$ is holomorphic. Finally $\eta V$ is evidently holomorphic, since it is a constant function.\qed

Proof of theorem 3. The obvious connection on the trivial bundle $H \times C \to H$ induces a flat connection on $H \times (\bar{C}/\sim) \to Y = H/\Lambda$.

Hence the statement follows from the preceding theorem (thm. 2).\qed

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