Wavelet-based estimation in a semiparametric regression model

Emmanuel de Dieu NKOU and Guy Martial NKIET

Université des Sciences et Techniques de Masuku
BP 943 Franceville, Gabon
E-mail: emmanueldiedieunkou@gmail.com, gnikiet@hotmail.com.

Abstract. In this paper, we introduce a wavelet-based method for estimating the EDR space in Li’s semiparametric regression model for achieving dimension reduction. This method is obtained by using linear wavelet estimators of the density and regression functions that are involved in the covariance matrix of conditional expectation whose spectral analysis gives the EDR directions. Then, consistency of the proposed estimators is proved. A simulation study that allow one to evaluate the performance of the proposal with comparison to existing methods is presented.

AMS 1991 subject classifications: 62F05, 62G05, 62J02.
Key words: dimension reduction; wavelet-based estimation; semiparametric regression, sliced inverse regression.

1 Introduction

Modeling by regression models allowing to establish links between a response variable and several explanatory variables is an approach that is both old and important in statistical analysis. In this perspective, many kinds of regression models have been introduced in the statistical literature, and the estimation problems related to these models have been intensively studied. For achieving both estimation and dimension reduction, a semiparametric regression model having the form

$$Y = F(\beta_1^T X, ..., \beta_N^T X, \varepsilon),$$

have been introduced in [13]. In this model, $Y$ is a real response random variable, $X$ is a $d$-dimensional random vector containing the explanatory variables, $N$ is an integer of $\mathbb{N}^*$ such that $N < d$, $\beta_1, \ldots, \beta_N$ are unknown vectors in $\mathbb{R}^d$, $\varepsilon$ is a real random variable that is independent of $X$, and $F$ is an arbitrary unknown function. It expresses the fact that the projection
of $X$ onto the subspace of $\mathbb{R}^d$ spanned by $\beta_1, \ldots, \beta_N$, named the effective dimension reduction (EDR) space, contains all information about $Y$. Estimating $N$ and the EDR space is then a crucial issue that have been tackled in several works ([1], [2], [4], [5], [11], [13], [15], [16], [19], [20], [23], [25]). For estimating the EDR space, it is enough to estimate the $\beta_j$’s which are characterized, under some conditions, as eigenvectors of the covariance matrix $\Lambda$ of $E(X|Y)$. For doing that, Li [13] proposed a method, called sliced inverse regression (SIR), based on slicing the range of $Y$. Although there exist alternative methods, this method stills the most popular for dimension reduction. Based on the fact that the aforementioned matrix is expressed as a function of the density of $Y$ and regression functions, Zhu and Fang [25] proposed a nonparametric estimation procedure by using kernel estimates of the preceding density and regression functions. But, as it is well known, there exist alternative nonparametric estimators for these functions. Among them, the wavelet-based estimators are known to have interesting properties and have been successfully used in many fields of Statistics (see, e.g. [24]). However, they never have been used for estimation in the model (1). That is why, we propose in this paper an estimation method for the EDR space related to this model, based on wavelet-based estimates of the density and regression functions involved in $\Lambda$. The rest of the paper is organized as follows. In Section 2 we construct an estimator of $\Lambda$ based on wavelet method. Consistency of the resulting estimators are then given in Section 3. Section 4 is devoted to a simulation study made in order to evaluate the performance of the proposal with comparison to existing estimation methods. The proofs of the main results are postponed in Section 5.

2 Wavelet-based estimation

Putting $X = (X_1, \ldots, X_d)^T$ and letting $f$ be the density of $Y$, we suppose that for all $y \in \mathbb{R}$, we have $f(y) > 0$; then, for any $j = 1, \ldots, d$, we consider

$$R_j(y) = \mathbb{E}(X_j|Y = y) = \frac{g_j(y)}{f(y)} \text{ where } g_j(y) = \int_{\mathbb{R}} x f_{(X_j,Y)}(x,y)dx,$$

$$R(Y) = (R_1(Y), \ldots, R_d(Y))^T = (\mathbb{E}(X_1|Y), \ldots, \mathbb{E}(X_d|Y))^T = \mathbb{E}(X|Y),$$

where $f_{(X_j,Y)}$ denotes the bivariate density of the pair $(X_j, Y)$. The covariance matrix $\Lambda = Cov \left(\mathbb{E}(X|Y)\right)$ is of a great importance since the EDR
space, which is to be estimated, is obtained from its spectral analysis\cite{13}. More precisely, $\beta_j$ is taken as an eigenvector of $\Lambda$ associated with the $j$-th largest eigenvalue $\lambda_j$. For estimating this matrix Li\cite{13} used an approach based on slicing the range of $Y$ whereas Zhu and Fang\cite{25} introduced an estimator based on kernel estimates of the involved density and regression functions. Here, we propose an estimator of $\Lambda$ obtained from wavelets-based estimates of $f$ and $g_j$, $j = 1, \ldots, d$. We assume that these functions belong to the space $L^2(\mathbb{R})$ of square integrable functions from $\mathbb{R}$ to itself. Let $\varphi \in L^2(\mathbb{R})$ be a father wavelet, and $\psi$ the associated mother wavelet, so that $
{\varphi_k(\cdot) = \varphi(\cdot - k), \psi_{\ell k}(\cdot) = 2^{\ell/2} \psi(2^\ell \cdot - k) : k \in \mathbb{Z}, \ell \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$ (ee, e.g., \cite{3}, \cite{9}, \cite{14}). Considering an i.i.d. sample $\{(X^{(i)}, Y^{(i)})\}_{i=1,\ldots,n}$ of $(X,Y)$, and putting $X^{(i)} = (X_{i1}, \ldots, X_{id})^T$, we make use of the following estimators of $f$ and $g_j$:

\[
\hat{f}_n(y) = \sum_{k \in \mathbb{Z}} \hat{\alpha}_k^{(n)} \varphi_k(y) + \sum_{\ell=0}^{j_n} \sum_{k \in \mathbb{Z}} \hat{\gamma}_{\ell k}^{(n)} \psi_{\ell k}(y),
\]

\[
\hat{g}_{j,n}(y) = \sum_{k \in \mathbb{Z}} \hat{\delta}_{j,k}^{(n)} \varphi_k(y) + \sum_{\ell=0}^{j_n} \sum_{k \in \mathbb{Z}} \hat{\eta}_{j,\ell k}^{(n)} \psi_{\ell k}(y),
\]

where

\[
\hat{\alpha}_k^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \varphi_k(Y_i), \quad \hat{\gamma}_{ijk}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(Y_i),
\]

\[
\hat{\delta}_{j,k}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} X_{ij} \varphi_k(Y_i), \quad \hat{\eta}_{j,\ell k}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} X_{ij} \psi_{\ell k}(Y_i),
\]

and $(j_n)_{n \in \mathbb{N}}$ is an increasing sequence of integers such that $j_n \nearrow +\infty$ as $n \to +\infty$. As it was already done\cite{25}, in order to avoid small values in the denominator, we consider

\[
f_{b_n}(y) = \max\left(f(y), b_n\right) \quad \text{and} \quad \hat{f}_{b_n}(y) = \max\left(\hat{f}_n(y), b_n\right),
\]

where $(b_n)_{n \in \mathbb{N}^\star}$ is a sequence of positive real numbers such that $\lim_{n \to +\infty} (b_n) = 0$, and we estimate the ratio $R_{b_n,j}(y) = g_j(y)/f_{b_n}(y)$ by $\hat{R}_{b_n,j}(y) = \hat{g}_{j,n}(y)/\hat{f}_{b_n}(y)$. Then, putting $\hat{R}_{b_n}(y) = \left(\hat{R}_{b_n,1}(y), \ldots, \hat{R}_{b_n,d}(y)\right)^T$, we take as estimator of $\Lambda$
the random matrix:

\[ \hat{\Lambda}_n = \frac{1}{n} \sum_{i=1}^{n} \hat{R}_b(Y_i)\hat{R}_b(Y_i)^T. \]

An estimate of the EDR space is obtained from the spectral analysis of this matrix. Indeed, if \( \hat{\beta}_j \) is an eigenvector of \( \hat{\Lambda}_n \) associated with the \( j \)-th largest eigenvalue \( \hat{\lambda}_j(n) \), we estimate the EDR space by the subspace of \( \mathbb{R}^d \) spanned by \( \hat{\beta}_1, \ldots, \hat{\beta}_N \).

**Remark 1** It is well known that, under some conditions, the preceding wavelets estimators have linear forms given by

\[ \hat{f}_n(y) = \frac{2^{jn}}{n} \sum_{i=1}^{n} K(2^{jn}y, 2^{jn}Y_i) \quad \text{and} \quad \hat{g}_{j,n}(y) = \frac{1}{n} \sum_{i=1}^{n} X_{ij}2^{jn}K(2^{jn}Y_i, 2^{jn}y), \]

where

\[ K(x, y) = \sum_{k \in \mathbb{Z}} \varphi(x - k)\varphi(y - k). \] (2)

This is the case when compactly supported wavelets, such as the Haar and Daubechies wavelets (see, e.g., [24]) for example, are used. In this case, the sum given in (2) reduces to a finite one (see [9]).

### 3 Assumptions and main results

In this section, we give the used assumptions, then the main results which give consistency of \( \hat{\Lambda}_n \) and the \( \hat{\beta}_j \)’s are stated.

**Assumption 1** The random variable \( X \) is bounded, i.e. there exists \( G > 0 \) such that \( \|X\|_d \leq G \), where \( \|\cdot\|_d \) is the usual euclidean norm of \( \mathbb{R}^d \).

**Assumption 2** The \( g_j \)’s and \( f \) are 3-times differentiable and their third derivatives satisfy the following condition: there exists a neighborhood of the origin, say \( U \), and a constant \( c > 0 \) such that, for any \( u \in U \),

\[ |f^{(3)}(y + u) - f^{(3)}(u)| \leq c|u| \quad \text{and} \quad |g^{(3)}_j(y + u) - g^{(3)}_j(u)| \leq c|u|, \]

for \( j = 1, \ldots, d. \)
Assumption 3 For any $j \in \{1, \cdots, d\}$ and any $u \in U$, $|R_j(y + u) - R_j(y)| \leq c|u|$.

Assumption 4 The father wavelet $\varphi$ and the mother wavelet $\psi$ are bounded and compactly supported.

Assumption 5 The kernel $K$ given in (2) satisfies the following properties:

1. $|K(x, y)| \leq \Phi(x - y)$, where $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ is a bounded, compactly supported and symmetric function satisfying:

$$\int u^2 \Phi^2(u) du < +\infty$$

and

$$\int |u|^k \Phi(u) dv < +\infty$$

for $k \in \{0, 1, 4\}$.

2. $\forall x \in \mathbb{R}, \forall k \in \{1, 2, 3\}, \int K(x, y)(y - x)^k dy = 0$.

Assumption 6 When $n$ is large enough $2^{-jn} \sim n^{-c_1}$ and $b_n \sim n^{-c_2}$ where $c_1$ and $c_2$ are the positive numbers satisfying: $c_1 > 0$, $0 < c_2 < 1/10$ and $1/8 + c_2/4 < c_1 < 1/4 - c_2$.

Assumption 7 For all $j \in \{1, \cdots, d\}$, $E[R_j^2(Y)] < +\infty$.

2. $\sqrt{n} E \left[ |R_k(Y)R_l(Y)| 1_{\{f(Y) \leq a_n\}} \right] = o(1)$ for $1 \leq k, l \leq d$ and any sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_b \sim b_n$ as $n \rightarrow +\infty$.

Assumption 8 The eigenvalues $\lambda_1, \cdots, \lambda_d$ of $\Lambda$ satisfy: $\lambda_1 > \cdots > \lambda_d > 0$.

Remark 2 Assumptions 2 and 5 are of a type which is classical in nonparametric statistics literature. They were used in particular in some papers on nonparametric estimation of the model which is tackled in this paper (see [16], [25]). Several wavelet functions satisfy Assumption 4 (see [3], [9]). That is the case, for instance, for the Daubechies wavelets and the Haar wavelets. The first condition in Assumption 5(1) just is the condition $H$ introduced in Härdle et al. [24], the third condition is the condition $H(k)$ and Assumption 5(2) is a part of the condition $M(k)$. Note that if Assumption 4 holds, then for any $x \in \mathbb{R}$, $\int K(x, y) dy = 1$, and there exists a non-zero constant $D$ independent of $x$ such that the inequality $\int K^2(x, y) dy \leq D^2$ holds [3]. In
Assumption 6, one can take $b_n = \min(a, n^{-c_2})$, where $a$ is a fixed strictly positive number and sufficiently small. This yields a more accurate estimation of $f$; see more details in Remark 3.1 of Nkou and Nkiet [16]. Assumption 8-(2) is of a kind that has already been used in the literature (see [25]).

We now present the main results.

**Theorem 1** Under the assumptions 1 to 7, as $n \to +\infty$, we have

$$
\sqrt{n} \left( \hat{\Lambda}_n - \Lambda \right) \overset{d}{\to} \mathcal{H},
$$

where $\overset{d}{\to}$ denotes convergence in distribution, $\mathcal{H}$ is a random variable having a normal distribution, in the space $\mathcal{M}_d(\mathbb{R})$ of $d \times d$ matrices, such that, for any $A = (a_{k\ell}) \in \mathcal{M}_d(\mathbb{R})$, $A \neq 0$, one has $\text{tr} \left( A^T \mathcal{H} \right) \sim N(0, \sigma_A^2)$ with:

$$
\sigma_A^2 = \text{Var} \left( \sum_{k=1}^d \sum_{\ell=1}^d \frac{a_{k\ell}}{2} \left( X_\ell R_k (Y) + X_k R_\ell (Y) \right) \right). \quad (3)
$$

From Theorem 1, we can derive the asymptotic normality of the eigenvectors. For $(j, k) \in \{1, \cdots, d\}^2$, we put $\beta_j = (\beta_{j1}, \cdots, \beta_{jd})^T$ and we consider the random variable

$$
W_{jk} = \left( \sum_{r=1}^d \frac{\beta_{rk}}{\lambda_j - \lambda_r} \right) \sum_{p=1}^d \sum_{q=1}^d \frac{\beta_{jp} \beta_{jq}}{2} \left( X_q R_p (Y) + X_p R_q (Y) \right). \quad (4)
$$

Then, we have:

**Theorem 2** Under the assumptions 1 to 8, we have

$$
\sqrt{n} \left( \hat{\beta}_j - \beta_j \right) \overset{d}{\to} N(0, \Sigma_j),
$$

as $n \to +\infty$, where $\Sigma_j$ is the $d \times d$ covariance matrix of the random vector $W_j = (W_{j1}, \cdots, W_{jd})^T$.  

6
4 Simulation results

In this section, we present results of simulations that was made in order to evaluate the performance of the introduced wavelet-based method and to compare it with some existing methods. We estimate the EDR directions corresponding to the following models with dimension \( d = 5 \):

**Model 1:** \( Y = X_1 + X_2 + X_3 + X_4 + \varepsilon \);

**Model 2:** \( Y = X_1 (X_1 + X_2 + 1) + \varepsilon \);

**Model 3:** \( Y = X_1 \left( 0.5 + (X_2 + 1.5)^2 \right)^{-1} + \varepsilon \).

Model 1 corresponds to \( N = 1 \) and \( \beta_1 = (1,1,1,0)^T \) whereas for Model 2 and Model 3 we have \( N = 2 \), and the EDR directions are, respectively, \( \beta_1 = (1,0,0,0)^T \), \( \beta_2 = (1,1,0,0)^T \) and \( \beta_1 = (1,0,0,0)^T \), \( \beta_2 = (0,1,0,0)^T \). Each data set was obtained as follows: \( X = (X_1, X_2, X_3, X_4, X_5)^T \) is generated from a multivariate normal distribution \( \mathcal{N}(0, I_5) \), where \( I_5 \) is the \( 5 \times 5 \) identity matrix, \( \varepsilon \) is generated from a standard normal distribution and \( Y \) is computed according to the above models. We simulated 100 independent replications of samples of size \( n = 500 \) generated as indicated above, over which means and standard deviations of the \( \hat{\beta}_j \)'s were computed, together with the means of squared cosines between \( \hat{\beta}_j \) and \( \beta_j \), \( j = 1, 2 \), given by [13]:

\[
R^2 \left( \hat{\beta}_j \right) = \frac{\left( \hat{\beta}_j^T \beta_j \right)^2}{\hat{\beta}_j^T \hat{\beta}_j \cdot \beta_j^T \beta_j}.
\]

Four methods were used for estimating the EDR directions: sliced inverse regression (denoted by SIR) with number of slices equal to \( H = 5 \), the kernel method (denoted by Kernel) with quadratic kernel

\[
K(x) = 0.9375 \left( 1 - x^2 \right)^2 1_{[-1,1]}(x)
\]

and bandwith \( h_n = n^{-0.2} \approx 0.2885 \), the wavelet-based method with Haar wavelets (denoted by Wavelet (H)) and the wavelet-based method with Daubechies wavelets of order 2 (denoted by Wavelet (D)). The wavelet-based methods were taken with resolution level \( j_n = 0 \) and \( b_n = 0.01 \). Table 1 presents the obtained means and standard deviations (in parentheses) of

\[
\hat{\beta}_1 = (\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{13}, \hat{\beta}_{14}, \hat{\beta}_{15})^T
\]
for Model 1, whereas the obtained means and standard deviation of $R^2(\hat{\beta}_j)$, $j = 1, 2$, for all three models, are given in Table 2. Boxplots showing these later are also given in Figures 1 to 3. As it is seen in Table 1, the four methods have good behaviours, but the wavelet-based method with Daubechies wavelets of order 2 seems to yield more accurate estimates. This fact is confirmed in Table 2 and Figures 1 to 3. Indeed, it is seen in Table 3 that wavelet (D) correspond to a standard deviation relatively reduced compared with that of the other methods. Figures show that its estimate of $\beta_1$ is very good like that of the other methods, but it is very powerful in estimating $\beta_2$. 

| Method     | $\hat{\beta}_{11}$ | $\hat{\beta}_{12}$ | $\hat{\beta}_{13}$ | $\hat{\beta}_{14}$ | $\hat{\beta}_{15}$ |
|------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| SIR        | 0.9806              | 0.9940              | 0.9947              | 0.9893              | 0.0076              |
|            | (0.0508)            | (0.0528)            | (0.0431)            | (0.0471)            | (0.0512)            |
| Kernel     | 0.9737              | 0.9866              | 0.9993              | 0.9992              | 0.0032              |
|            | (0.0455)            | (0.0466)            | (0.0486)            | (0.0459)            | (0.0514)            |
| Wavelet (H)| 0.9815              | 0.9836              | 0.9914              | 1.0029              | -0.0017             |
|            | (0.0567)            | (0.0465)            | (0.0423)            | (0.0472)            | (0.0544)            |
| Wavelet (D)| 0.9916              | 0.9893              | 0.9988              | 0.9973              | -0.0031             |
|            | (0.0339)            | (0.0318)            | (0.0336)            | (0.0384)            | (0.0383)            |

Table 1: Means and standard deviations of $\hat{\beta}_1$ over 100 replications for Model 1 with $n = 500$. 


| Method      | Model 1     | Model 2     | Model 3     |
|-------------|-------------|-------------|-------------|
|             | $R^2(\hat{\beta}_1)$ | $R^2(\hat{\beta}_1)$ | $R^2(\hat{\beta}_2)$ | $R^2(\hat{\beta}_1)$ | $R^2(\hat{\beta}_2)$ |
| SIR         | 0.9986      | 0.9357      | 0.7658      | 0.9512      | 0.7624      |
|             | (0.0010)    | (0.0852)    | (0.1555)    | (0.0649)    | (0.2041)    |
| Kernel      | 0.9987      | 0.9726      | 0.8914      | 0.9756      | 0.8722      |
|             | (0.0010)    | (0.0190)    | (0.1261)    | (0.0202)    | (0.1441)    |
| Wavelet (H) | 0.9986      | 0.9572      | 0.8679      | 0.9716      | 0.9023      |
|             | (0.0009)    | (0.0413)    | (0.0979)    | (0.0298)    | (0.0943)    |
| Wavelet (D) | 0.9994      | 0.9740      | 0.9604      | 0.9775      | 0.9501      |
|             | (0.0006)    | (0.0237)    | (0.0362)    | (0.0228)    | (0.0447)    |

Table 2: Means and standard deviations of $R^2(\hat{\beta}_j)$, $j = 1, 2$, over 100 replications with $n = 500$. 
Figure 1: Boxplots showing $R^2(\hat{\beta}_j)$ for Model 1, $n = 500$.

Figure 2: Boxplots showing $R^2(\hat{\beta}_j)$, $j = 1, 2$, for Model 2 with $n = 500$. 
5 Proofs

5.1 Asymptotic properties of $\hat{f}_n$ and $\hat{g}_{j,n}$

In this section, we give some results on asymptotic properties of $\hat{f}_n$ and $\hat{g}_{j,n}$ that are useful for proving the main result of the paper.

Proposition 1 Under the assumptions \(4\) and \(5\), we have

$$
\sup_{y \in \mathbb{R}} \left| \hat{f}_n(y) - \mathbb{E} \left( \hat{f}_n(y) \right) \right| = O \left( \left( \log n \right)^\beta \times 2^{jn} n^{-1/2} \right) \quad \text{a.s.}
$$

with \(\beta > 1/2\).

Proof. Consider the class \(\mathcal{F}_n = \{ \theta_y : u \mapsto \theta_y(u) = \frac{2jn}{n} K \left( 2^{jn} y, 2^{jn} u \right), y \in \mathbb{R} \}\) of functions; each \(\theta_y\) is measurable since it is a finite sum of measurable functions (Assumption \(4\)). By using the same arguments than in the proof
of Lemma 2 of Giné and Nickl\[8\], it is easy to check that $F_n$ is a VC-class of functions with respect to an envelope $\theta$ such that $|\theta| \leq \frac{2^j n}{n} \|\Phi\|_\infty$. Moreover,

$$\mathbb{E} (|\theta_y(\cdot)|) \leq \frac{2^j}{n} \|\Phi\|_\infty =: \mu_n \quad \text{and} \quad \mathbb{E} (|\theta_y^2(\cdot)|) \leq \left( \frac{2^j}{n} \|\Phi\|_\infty \right)^2 = \sigma_n^2.$$

Then, applying Talagrand's inequality (see Proposition 2.2 in Giné and Guillou\[7\]; here $\mu_n = \sigma_n$), there exist positive constants $K_1$ and $K_2$ such, that for all $t \geq K_1 \left( \mu_n \log(A) + \sqrt{n} \sigma_n \sqrt{\log(A)} \right)$,

$$P\left\{ \sup_{y \in \mathbb{R}} \left| \hat{f}_n(y) - \mathbb{E} \left( \hat{f}_n(y) \right) \right| > t \right\} \leq K_2 \exp \left\{ - \frac{1}{K_2} \frac{t \mu_n}{\log \left( \mu_n + K_2 \left( \sqrt{n} \sigma_n + \sqrt{n} \log(A) \right) \right)^2} \right\} =: v_n(t),$$

where $A > 0$. Taking $t := t_n = \frac{2^j n}{\sqrt{n}} (\log n)^{\beta}$, with $\beta > 1/2$, we have $\mu_n^{-1} t_n \to +\infty$ and $\sqrt{n} \mu_n t_n^{-1} \to 0$ as $n \to +\infty$. Thus, for $n$ large enough, $\mu_n^{-1} t_n - \sqrt{n} K_1 \sqrt{\log(A)} \geq K_1 \log(A)$, that is $t_n \geq K_1 \mu_n \left( \log(A) + \sqrt{n} \log(A) \right)$. Therefore, the preceding inequality holds for $n$ large enough and, using $\log(1+u) \sim u$ and $(1+u)^2 \sim 1$ as $u \to 0$, we get

$$v_n(t_n) \sim K_2 \exp \left\{ - \frac{(\log n)^{2\beta}}{K_2^2 \|\Phi\|_\infty^2} \right\},$$

from which we deduce that $\sum_{n=1}^{+\infty} v_n(t_n) < +\infty$. Hence,

$$\sum_{n=1}^{+\infty} P \left\{ \sup_{y \in \mathbb{R}} \left| \hat{f}_n(y) - \mathbb{E} \left( \hat{f}_n(y) \right) \right| > t_n \right\} < +\infty,$$

and the required result is obtained by using Borel Cantelli’s lemma.

\[ \Box \]

**Proposition 2** Under the assumptions \[2\], \[4\] and \[8\], we have

$$\sup_{y \in \mathbb{R}} \left| \hat{f}_n(y) - f(y) \right| = O \left( (\log n)^{2\beta} \times 2^{j_n} n^{-1/2} + 2^{-4j_n} \right) \text{ a.s.}$$
Proof. We can see that
\[
E\left( \hat{f}_n(y) \right) - f(y) = \int K \left( 2^{jn} y, 2^{jn} y + u \right) \left( f(y + u2^{-jn}) - f(y) \right) du,
\]
and by a Taylor expansion, we have:
\[
f(y + u2^{-jn}) - f(y) = \sum_{k=1}^{2} \frac{2^{-kj_n} f^{(k)}(y)}{k!} u^k + \frac{2^{-3j_n} u^3}{2} \int_0^1 (1-v)^2 f^{(3)} \left( y + vu2^{-jn} \right) dv.
\]
Then, using Assumption [5]-(2) we get
\[
E\left( \hat{f}_n(y) \right) - f(y) = \frac{2^{-3j_n}}{2} \int_0^1 \int_0^1 u^3 (1-v)^2 K \left( 2^{jn} y, 2^{jn} y + u \right) \left( f^{(3)} \left( y + vu2^{-jn} \right) - f^{(3)}(y) \right) dv du,
\]
and, therefore,
\[
\left| E\left( \hat{f}_n(y) \right) - f(y) \right| \leq \frac{2^{-3j_n}}{2} \int_0^1 \int_0^1 |u|^3 (1-v)^2 |K \left( 2^{jn} y, 2^{jn} y + u \right)| \left| f^{(3)} \left( y + vu2^{-jn} \right) - f^{(3)}(y) \right| dv du
\]
\[
\leq \frac{2^{-3j_n} c}{2} \int_0^1 \int_0^1 |u|^4 \Phi(u) v2^{-jn} (1-v)^2 dv du
\]
\[
\leq \frac{2^{-4j_n} c}{2} \int_0^1 |u|^4 \Phi(u) du.
\]
We deduce that \( \sup_{y \in \mathbb{R}} \left| E\left( \hat{f}_n(y) \right) - f(y) \right| = O \left( 2^{-4jn} \right) \). Combining this result to that of Proposition \([2]\) yields the required result.

Proposition 3 Under the assumptions \([7, 4]\) and \([5]\), we have for \( \beta > 1/2 \)
\[
\sup_{y \in \mathbb{R}} \left| \hat{g}_{j,n}(y) - g_j(y) \right| = O \left( 2^{j_n} n^{-1/2} (\log n) \beta + 2^{-4jn} \right) a.s.
\]
Proof. A similar reasoning than that of Proposition \([2]\) based on the class \( \mathcal{G}_n = \{ \theta_y : [-G; G] \times \mathbb{R} \rightarrow \mathbb{R} / \theta_y(u,v) = 2^{jn} K_{\varphi} \left( 2^{jn} y, 2^{jn} v \right), y \in \mathbb{R} \} \), where \( G \) is the constant introduced in Assumption \([4]\) gives
\[
\sup_y \left| \hat{g}_{j,n}(y) - \mathbb{E} \left( \hat{g}_{j,n}(y) \right) \right| = O \left( 2^{j_n} n^{-1/2} (\log n) \beta \right) a.s.
\]
A similar reasoning than that of Proposition \([2]\) allows to obtain \( \sup_{y \in \mathbb{R}} \left| \mathbb{E} \left( \hat{g}_{j,n}(y) \right) - g_j(y) \right| = O \left( 2^{-4jn} \right) \). □
5.2 Preliminary lemmas

Lemma 5.1 Under assumptions 2, 4 and 5 we have
\[
\left| \int K(2^{jn}x,2^{jn}y) (g_\ell(x) - g_\ell(y)) \, dx \right| \leq C_1 2^{-5jn},
\]
where \(C_1 > 0\).

Proof. From Taylor expansion of \(g_\ell\) and Assumption 5-(2) we get
\[
\int K(2^{jn}x,2^{jn}y) (g_\ell(x) - g_\ell(y)) \, dx
= \frac{1}{2} \int_0^1 (x-y)^3 K(2^{jn}x,2^{jn}y) (1-v)^2 \left( g_\ell^{(3)}(y + v(x-y)) - g_\ell^{(3)}(y) \right) \, dv \, dx
\]
and, therefore,
\[
\left| \int K(2^{jn}x,2^{jn}y) (g_\ell(x) - g_\ell(y)) \, dx \right| \leq \frac{c}{2} \int |x-y|^4 \Phi(2^{jn}(x-y)) \, dx
= \frac{2^{-5jn}c}{2} \int |u|^4 \Phi(u) \, du.
\]
\[\square\]

Define, for \(1 \leq k, \ell \leq d\),
\[
U_{n,k,\ell}^{(1)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ g_k(Y_i) (\hat{g}_{\ell,n}(Y_i) - g_\ell(Y_i)) + g_\ell(Y_i) (\hat{g}_{k,n}(Y_i) - g_k(Y_i)) \right\} \frac{\hat{f}^2_{bn}(Y_i) - f^2_{bn}(Y_i)}{f^2_{bn}(Y_i)},
\]
\[
U_{n,k,\ell}^{(2)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(\hat{g}_{k,n}(Y_i) - g_k(Y_i)) (\hat{g}_{\ell,n}(Y_i) - g_\ell(Y_i))}{f^2_{bn}(Y_i)},
\]
\[
U_{n,k,\ell}^{(3)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_{bn,k}(Y_i) R_{bn,\ell}(Y_i) \left( \frac{\hat{f}^2_{bn}(Y_i) - f^2_{bn}(Y_i)}{f^2_{bn}(Y_i)} \right)^2,
\]
\[
U_{n,k,\ell}^{(4)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \hat{f}_{bn}(Y_i) - f_{bn}(Y_i) \right)^2 \frac{R_{bn,k}(Y_i) R_{bn,\ell}(Y_i)}{f^2_{bn}(Y_i)}.
\]

Lemma 5.2 Under the assumptions 1, 2, 4, 5 and 6, we have \(\left| U_n^{(q)} \right| = o_p(1)\) for any \(q \in \{1, \cdots, 4\}\).
Proof. Using Proposition 3 and Assumption 6, we get

\[ |g_k(Y_i) (\hat{g}_{\ell,n}(Y_i) - g_\ell(Y_i)) + g_\ell(Y_i) (\hat{g}_{k,n}(Y_i) - g_k(Y_i))| \leq A_1 \left( |g_k(Y_i)| + |g_\ell(Y_i)| \right) \tau_n, \]

where \( A_1 > 0 \) and \( \tau_n = n^{\epsilon_1 - 1/2} (\log n)^\beta + 2^{-4n} \). Then, using the inequalities \( |g_k(Y_i)| + |g_\ell(Y_i)| \leq f_{b_n}(Y_i) (|R_k(Y_i)| + |R_\ell(Y_i)|) \), \( \left| f_{b_n}(Y) - f_{b_n}(Y_i) \right| \leq \left| \hat{f}_n(Y) - f(Y) \right| \), \( f_{b_n}(Y_i) \geq b_n, f_{b_n}(Y_i) \geq b_n \), together with Proposition 2, we obtain

\[ \left| U_{n,k,\ell}^{(1)} \right| \leq A_1 \tau_n \frac{\sqrt{n}}{n} \sum_{i=1}^{n} \left( |R_k(Y_i)| + |R_\ell(Y_i)| \right) \left[ \frac{\left( f_{b_n}(Y_i) - f_{b_n}(Y_i) \right)^2}{f_{b_n}^2(Y_i) f_{b_n}(Y_i)} + 2 \frac{\hat{f}_{b_n}(Y_i) - f_{b_n}(Y_i)}{f_{b_n}^2(Y_i)} \right] \]

\[ \leq A_1 A_2 \tau_n^2 \sqrt{n} b_n^{-2} \left( A_2 b_n^{-1} \tau_n + 2 \right) \frac{1}{n} \sum_{i=1}^{n} \left( |R_k(Y_i)| + |R_\ell(Y_i)| \right) , \]

where \( A_2 > 0 \). This implies that \( \left| U_{n,k,\ell}^{(1)} \right| = o_p(1) \), due to the law of large numbers and the fact that: \( \lim_{n \to +\infty} \frac{1}{n} \sqrt{n} b_n^{-2} (b_n^{-1} \tau_n + 2) = 0 \). By a similar reasoning, we get

\[ \left| U_{n,k,\ell}^{(2)} \right| \leq n^{-1/2} b_n^{-2} \sum_{i=1}^{n} |\hat{g}_{k,n}(Y_i) - g_k(Y_i)| |\hat{g}_{\ell,n}(Y_i) - g_\ell(Y_i)| \leq A_1^2 n^{1/2} b_n^{-2} \tau_n^2 , \]

since \( \lim_{n \to +\infty} \left( n^{1/2} b_n^{-2} \tau_n^2 \right) = 0 \), it follows \( \left| U_{n,k,\ell}^{(2)} \right| = o_p(1) \). In the same way, using in addition the inequality \( |R_{b_n,k}(Y_i)| \leq |R_k(Y_i)| \), we get

\[ \left| U_{n,k,\ell}^{(3)} \right| \leq A_2^2 \tau_n^2 \sqrt{n} b_n^{-2} \left( A_2^2 b_n^{-2} \tau_n^2 + 4 A_2 b_n^{-1} \tau_n + 4 \right) \frac{1}{n} \sum_{i=1}^{n} |R_k(Y_i) R_\ell(Y_i)| , \]

\[ \left| U_{n,k,\ell}^{(4)} \right| \leq A_2^2 \left( n^{1/2} b_n^{-2} \tau_n^2 \right) \frac{1}{n} \sum_{i=1}^{n} |R_k(Y_i) R_\ell(Y_i)| , \]

and we deduce that \( \left| U_{n,k,\ell}^{(3)} \right| = o_p(1) \) and \( \left| U_{n,k,\ell}^{(4)} \right| = o_p(1) \).
In the following, we define the functions:

\[ A_{k\ell}^{(1)}(y) = \frac{g_k(y)g_\ell(y)}{f_{b_n}(y)} = R_{b_n,k}(y)R_{b_n,\ell}(y), \]

\[ A_{k\ell}^{(2)}(y) = \frac{g_k(y)\hat{g}_{\ell,n}(y) + g_\ell(y)\hat{g}_{k,n}(y)}{f_{b_n}(y)} = \frac{R_{b_n,k}(y)\hat{g}_{\ell,n}(y)}{f_{b_n}(y)} + \frac{R_{b_n,\ell}(y)\hat{g}_{k,n}(y)}{f_{b_n}(y)}, \]

\[ A_{k\ell}^{(3)}(y) = 2R_{b_n,k}(y)R_{b_n,\ell}(y)\frac{\hat{f}_{b_n}(y)}{f_{b_n}(y)}. \]

Then, we first have:

**Lemma 5.3** Under the assumptions 6 and 7, we have:

\[ \sqrt{n} \left| \mathbb{E} \left( A_{k\ell}^{(1)}(Y) \right) - \mathbb{E} \left( R_k(Y)R_\ell(Y) \right) \right| = o(1). \]

**Proof.** The proof is identical to (4.17) in Zhu and Fang [25]. \( \square \)

**Lemma 5.4** Under the assumptions 2, 4, 5, 6 and 7, we have:

\[ \sqrt{n} \mathbb{E} \left[ A_{k\ell}^{(2)}(Y) \right] = 2\sqrt{n} \mathbb{E} \left[ R_\ell(Y)R_k(Y) \right] + o(1). \]

**Proof.** With the assumptions 6 and 7 we have

\[ \sqrt{n} \mathbb{E} \left[ \frac{R_{b_n,k}(Y)\hat{g}_{\ell,n}(Y)}{f_{b_n}(Y)} \right] = \sqrt{n} \mathbb{E} \left[ \frac{R_{b_n,k}(Y)}{f_{b_n}(Y)} 2^{jn}X_1\ell K \left( 2^{jn}Y, 2^{jn}Y_1 \right) \right]. \]

Since \((X_1, Y_1)\) and \(Y\) are independent, it follows

\[ \sqrt{n} \mathbb{E} \left[ \frac{R_{b_n,k}(Y)\hat{g}_{\ell,n}(Y)}{f_{b_n}(Y)} \right] = 2^{jn} \sqrt{n} \int \int \int \frac{R_{b_n,k}(y)}{f_{b_n}(y)} xK \left( 2^{jn}z, 2^{jn}y \right) f_{(Y_1, X_1\ell)}(z, x) f(y) \, dx \, dz \, dy \]

\[ = \sqrt{n} \int \int K \left( 2^{jn}y - u, 2^{jn}y \right) R_\ell \left( y - 2^{-jn}u \right) f \left( y - 2^{-jn}u \right) \frac{R_{b_n,k}(y)f(y)}{f_{b_n}(y)} \, du \, dy \]

\[ = \mathcal{I}_n + \mathcal{J}_n, \]

16
where

\[ I_n = \sqrt{n} \int \int K(2^{jn}y, 2^{jn}y - u) (R_\ell(y - 2^{-jn}u) f(y - 2^{-jn}u) - R_\ell(y)f(y)) \]

\[ \times \frac{R_{bn,k}(y)f(y)}{f_{bn}(y)} \ du \ dy \]

and \( J_n = \sqrt{n} \mathbb{E}\left(R_\ell(Y)R_k(Y) \frac{f^2(Y)}{f^2_{bn}(Y)}\right) \). Furthermore, by Assumption 7, we have

\[ \sqrt{n} \left| \mathbb{E}\left[R_\ell(Y)R_k(Y) \frac{f^2(Y)}{f^2_{bn}(Y)}\right] - \mathbb{E}[R_\ell(Y)R_k(Y)] \right| \leq \sqrt{n} \left| \mathbb{E}\left[R_\ell(Y)R_k(Y) 1_{\{f(Y) < b_n\}}\right] \right| = o(1). \]

Then, when \( n \) is large enough, \( J_n = \sqrt{n} \mathbb{E}[R_\ell(Y)R_k(Y)] + o(1) \). For having the required result it is enough to show that \( I_n = o(1) \). First,

\[ |I_n| \leq b_n^{-1} \sqrt{n} \left| \int \left( \int K(2^{jn}y, 2^{jn}y + u) \left( g_\ell(y + 2^{-jn}u) - g_\ell(y) \right) \ du \right) R_{bn,k}(y)f(y)dy \right| \]

\[ = b_n^{-1} \sqrt{n} \left| \int \left( \int K(2^{jn}x, 2^{jn}x) \left( g_\ell(x) - g_\ell(y) \right) \ dx \right) R_{bn,k}(y)f(y)dy \right|. \]

Using Lemma 5.1 and the inequality \( |R_{bn,k}(y)| \leq |R_k(y)| \), we get

\[ |I_n| \leq C b_n^{-1} \sqrt{n} 2^{-5jn} \int |R_k(y)| f(y)dy = O( b_n^{-1} \sqrt{n} 2^{-5jn} ) \].

From Assumption 6 \( b_n^{-1} \sqrt{n} 2^{-5jn} \sim n^{c_2+1/2-4c_1} \leq n^{-(5c_1+c_2+1/2)} \leq n^{-(1/8+c_2/4)} \); thus \( I_n = o(1) \).

\[ \square \]

**Lemma 5.5** Under the assumptions 2, 4, 5, 6 and 7, we have:

\[ \sqrt{n} \mathbb{E}\left[A_{k\ell}^{(3)}(Y)\right] = \sqrt{n} \mathbb{E}[R_\ell(Y)R_k(Y)] + o(1). \]

**Proof.** Clearly,

\[ \widehat{f}_{bn}(Y) = V_n^{(1)}(Y) + V_n^{(2)}(Y) + V_n^{(3)}(Y) + V_n^{(4)}(Y) + V_n^{(5)}(Y), \]
where
\[ V_n^{(1)}(Y) = \hat{f}_n(Y)1_{(f(Y) > b_n + C_2\rho_n)}, \quad V_n^{(2)}(Y) = f(Y) \left\{ 1_{(\hat{f}_n(Y) > b_n)} - 1_{(f(Y) > b_n + C_2\rho_n)} \right\}, \]
\[ V_n^{(3)}(Y) = \left( \hat{f}_n(Y) - f(Y) \right) \left\{ 1_{(\hat{f}_n(Y) > b_n)} - 1_{(f(Y) > b_n + C_2\rho_n)} \right\}, \]
\[ V_n^{(4)}(Y) = b_n 1_{(f(Y) < b_n - C_2\rho_n)}, \quad V_n^{(5)}(Y) = b_n \left\{ 1_{(\hat{f}_n(Y) < b_n)} - 1_{(f(Y) < b_n - C_2\rho_n)} \right\}, \]
\[ C_2 > 0, \text{ and } \rho_n = (\log n)^\beta \times 2^{j_n} n^{-1/2} + 2^{-4j_n}. \]
It is then enough to show that
\[ \sqrt{n} \mathbb{E} \left[ \frac{R_{b_n,k}(Y)R_{b_n,\ell}(Y)V_n^{(1)}(Y)}{f_{b_n}(Y)} \right] = \sqrt{n} \mathbb{E} [R_k(Y)R_{\ell}(Y)] + o(1), \tag{6} \]
and that for any \( t \in \{2, \ldots, 5\}, \)
\[ \sqrt{n} \mathbb{E} \left[ \frac{R_{b_n,k}(Y)R_{b_n,\ell}(Y)V_n^{(t)}(Y)}{f_{b_n}(Y)} \right] = o(1). \tag{7} \]
The proof of (7) is similar to that of (4.23) in Zhu and Fang. It remains to prove (6). We have:
\[
\begin{align*}
\sqrt{n} \mathbb{E} & \left[ \frac{R_{b_n,k}(Y)R_{b_n,\ell}(Y)V_n^{(1)}(Y)}{f_{b_n}(Y)} \right] \\
= & \sqrt{n} \mathbb{E} \left[ \frac{R_k(Y)R_{\ell}(Y)\hat{f}_n(Y)}{f(Y)} 1_{(f(Y) > b_n + C_2\rho_n)} \right] \\
= & \sqrt{n} \mathbb{E} \left[ \frac{R_k(Y)R_{\ell}(Y)2^{j_n} K \left( 2^{j_n} y_1, 2^{j_n} y \right) 1_{(f(Y) > b_n + C_2\rho_n)} }{f(Y)} \right] \\
= & \sqrt{n} \int \int_{\{f(y) > b_n + C_2\rho_n\}} \frac{R_k(y)R_{\ell}(y)2^{j_n} K \left( 2^{j_n} y_1, 2^{j_n} y \right) f(y_1)f(y)}{f(y)} \ dy_1 \ dy \\
= & \sqrt{n} \int \int_{\{f(y) > b_n + C_2\rho_n\}} R_k(y)R_{\ell}(y)K \left( 2^{j_n} y + u, 2^{j_n} y \right) f(y) du \ dy \\
& + \sqrt{n} \int \int_{\{f(y) > b_n + C_2\rho_n\}} R_k(y)R_{\ell}(y)K \left( 2^{j_n} y + u, 2^{j_n} y \right) \left( f(y + u2^{-j_n}) - f(y) \right) du \ dy \\
= & \sqrt{n} \int_{\{f(y) > b_n + C_2\rho_n\}} R_k(y)R_{\ell}(y) f(y) dy \\
& + \sqrt{n} \int \int_{\{f(y) > b_n + C_2\rho_n\}} R_k(y)R_{\ell}(y)K \left( 2^{j_n} y + u, 2^{j_n} y \right) \left( f(y + u2^{-j_n}) - f(y) \right) du \ dy. \tag{8}
\end{align*}
\]
Further, using Assumption 7-(2) we get

$$\sqrt{n} \int_{\{f(y) > b_n + C_2 \rho_n\}} R_k(y) R_\ell(y) f(y) \, dy = \sqrt{n} \mathbb{E}[R_k(Y) R_\ell(Y)]$$

$$- \sqrt{n} \mathbb{E}[R_k(Y) R_\ell(Y) 1_{\{f(Y) \leq b_n + C_2 \rho_n\}}]$$

$$= \sqrt{n} \mathbb{E}[R_k(Y) R_\ell(Y)] + o(1). \quad (9)$$

On the other hand, by a Taylor expansion of $f$, assumptions 2 and 5-(2), we obtain

$$\sqrt{n} \int \int_{\{f(y) > b_n + C_2 \rho_n\}} R_k(y) R_\ell(y) K (2^{j_n} y + u, 2^{j_n} y) (f(y + u 2^{-j_n}) - f(y)) \, du \, dy$$

$$= \sqrt{n} \int \int_{\{f(y) > b_n + C_2 \rho_n\}} R_k(y) R_\ell(y) K (2^{j_n} y + u, 2^{j_n} y)$$

$$\times \left(\frac{2^{-3j_n}}{2} \int_0^1 u^3 (1 - v)^2 f^{(3)} (y + vu 2^{-j_n}) \, dv\right) \, du \, dy$$

$$= \sqrt{n} \frac{2^{-3j_n}}{2} \int \int_{\{f(y) > b_n + C_2 \rho_n\}} R_k(y) R_\ell(y) K (2^{j_n} y + u, 2^{j_n} y)$$

$$\times \int_0^1 u^3 (1 - v)^2 \left(f^{(3)} (y + vu 2^{-j_n}) - f^{(3)} (y)\right) \, dv \, du \, dy.$$

Hence

$$\left|\sqrt{n} \int \int_{\{f(y) > b_n + C_2 \rho_n\}} R_k(y) R_\ell(y) K (2^{j_n} y + u, 2^{j_n} y) (f(y + u 2^{-j_n}) - f(y)) \, du \, dy\right|$$

$$\leq \sqrt{n} \frac{c 2^{-3j_n}}{2} \left(\int \Phi(u) \, |u|^4 \, du\right) \int_{\{f(y) > b_n + C_2 \rho_n\}} \frac{|R_k(y) R_\ell(y)|}{f(y)} \, f(y) \, dy.$$
1/8 + c_2/4 < c_1 by Assumption 6, we deduce that \( \lim_{n \to +\infty} (b_n^{-1} \rho_n) = 0 \).

Finally, we get:

\[
\sqrt{n} \int \int_{\{f(Y) > b_n + c_2 \rho_n\}} R_k(y) R_\ell(y) K \left( 2^{j_n} y, 2^{j_n} y + u \right) \left( f(y + u 2^{-j_n}) - f(y) \right) \, du \, dy = o(1);
\]

this equality and the equations (8), (9) allow to conclude that (6) holds. \( \square \)

In all what follows, we consider for \( j \in \{1, 2, 3\} \):

\[
E^{(j)}_{k \ell} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ A^{(j)}_{k \ell} (Y_i) - E \left( A^{(j)}_{k \ell} (Y) \right) \right\}.
\]

We first have:

**Lemma 5.6** Under assumptions 1, 2, 4, 5, 6, and 7, we have:

\[
E^{(1)}_{k \ell} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_k(Y_i) R_\ell(Y_i) - E \left( R_k(Y) R_\ell(Y) \right) \right\} + o_p(1).
\]

*Proof.* The proof is identical to that of step 2 of the proof of Theorem 2.1 in Zhu and Fang[25]. \( \square \)

**Lemma 5.7** Under assumptions 1, 2, 3, 4, 5, 6, and 7 we have

\[
E^{(2)}_{k \ell} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) R_{b_n,\ell}(Y_i) + \frac{1}{2} X_{\ell i} R_{b_n,k}(Y_i) \frac{f(Y_i)}{f_{b_n}(Y_i)} + \frac{1}{2} X_{k i} R_{b_n,\ell}(Y_i) \frac{f(Y_i)}{f_{b_n}(Y_i)} \right\}
\]

\[
- E \left\{ R_{b_n,k}(Y) R_{b_n,\ell}(Y) + \frac{1}{2} X_{\ell i} R_{b_n,k}(Y) \frac{f(Y)}{f_{b_n}(Y)} + \frac{1}{2} X_{k i} R_{b_n,\ell}(Y) \frac{f(Y)}{f_{b_n}(Y)} \right\} + o_p(1)
\]

*Proof.* Applying the similar argument used in several works in order to approximate a sum by a U-statistic ([9], [10], [17], [18], [21]), we obtain

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) \frac{\hat{g}_{\ell,n}(Y_i)}{f_{b_n}(Y_i)} - E \left[ R_{b_n,k}(Y) \frac{\hat{g}_{\ell,n}(Y)}{f_{b_n}(Y)} \right] \right\}
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{n,k,i} - E(\nu_{n,k,i}) + o_p(1), \quad (10)
\]
where

$$\mathcal{V}_{n,k,i} = \frac{1}{2} \int \int 2^{i_n} K (2^{i_n} y, 2^{i_n} Y_i) \left( \frac{x R_{b_n,k}(Y_i)}{f_{b_n}(Y_i)} + \frac{X_{\ell_i} R_{b_n,k}(y)}{f_{b_n}(y)} \right) f_{(X_{\ell_i},Y)}(x,y) dx dy.$$ 

From the equalities

$$\int x \frac{f_{(X_{\ell_i},Y)}(x,y)}{f(y)} dx = R_{\ell}(y), \quad \int f_{(X_{\ell_i},Y)}(x,y) dx = f(y), \quad R_{b_n,k}(Y_i) = R_k(Y_i) \varepsilon_{b_n}(Y_i),$$

where \( \varepsilon_{b_n}(y) = f(y)/f_{b_n}(y) \), and the property \( \int K(x,y) dy = 1 \), it follows

$$\mathcal{V}_{n,k,i} = \frac{R_{b_n,k}(Y_i)}{2f_{b_n}(Y_i)} \int 2^{i_n} K (2^{i_n} y, 2^{i_n} Y_i) R_{\ell}(y) f(y) dy$$

$$\quad + \frac{X_{\ell_i}}{2} \int 2^{i_n} K (2^{i_n} y, 2^{i_n} Y_i) \frac{R_{b_n,k}(y)}{f_{b_n}(y)} f(y) dy$$

$$= \frac{1}{2} R_k(Y_i) R_{\ell}(Y_i) \varepsilon_{b_n}^2(Y_i) + \frac{1}{2} X_{\ell_i} R_k(Y_i) \varepsilon_{b_n}^2(Y_i) + U_n^{(5)}(Y_i) + U_n^{(6)}(Y_i),$$

where

$$U_n^{(5)}(Y_i) = \frac{R_{b_n,k}(Y_i)}{2f_{b_n}(Y_i)} \int 2^{i_n} K (2^{i_n} y, 2^{i_n} Y_i) \left( R_{\ell}(y) f(y) - R_{\ell}(Y_i) f(Y_i) \right) dy,$$

$$U_n^{(6)}(Y_i) = \frac{X_{\ell_i}}{2} \int 2^{i_n} K (2^{i_n} y, 2^{i_n} Y_i) \left( \frac{R_{b_n,k}(y) f(y)}{f_{b_n}(y)} - \frac{R_{b_n,k}(Y_i) f(Y_i)}{f_{b_n}(Y_i)} \right) dy.$$

However, from Lemma 5.1

$$\mathbb{E} \left( \left( U_n^{(5)}(Y) \right)^2 \right)$$

$$= \left( \frac{2^{i_n}}{2} \right)^2 \int \left( \frac{R_{b_n,k}(y)}{f_{b_n}(y)} \right)^2 \left( \int K (2^{i_n} x, 2^{i_n} y) \left( g_{\ell}(x) - g_{\ell}(y) \right) dx \right)^2 f(y) dy$$

$$\leq \frac{C_1^2}{4} 2^{-8j_n} b_n^{-2} \int R_k(y)^2 f(y) dy = \frac{C_1^2}{4} 2^{-8j_n} b_n^{-2} \mathbb{E} \left( R_k(Y)^2 \right).$$

From Assumption 7(1), \( \mathbb{E} \left( R_k(Y)^2 \right) < +\infty \), and from Assumption 6, \( 2^{-8j_n} b_n^{-2} \sim n^{-2(4c_1-c_2)} \). Since \( 4c_1 - c_2 > 0 \) because \( c_1 > 1/8 + c_2/4 > c_2/4 \), it follows that

$$\mathbb{E} \left( \left( U_n^{(5)}(Y) \right)^2 \right) = o(1).$$

Since \( \text{Var} \left( \sum_{i=1}^{n} U_n^{(5)}(Y_i) \right) = \text{Var} \left( U_n^{(5)}(Y) \right) \leq$$
\( \mathbb{E}\left( \left( U_n^{(5)}(Y) \right)^2 \right) \), it follows that \( n^{-1/2} \sum_{i=1}^{n} \left( U_n^{(5)}(Y_i) - \mathbb{E}(U_n^{(5)}(Y_i)) \right) = o_p(1) \).

On the other hand,

\[
\mathbb{E}\left[ \left( U_n^{(6)}(Y) \right)^2 \right] = \mathbb{E}\left[ \left( \frac{X_{\ell}}{2} \int 2^{jn}K(2^{jn}x, 2^{jn}Y) \left( \frac{R_{bn,k}(x)f(x)}{f_{bn}(x)} - \frac{R_{bn,k}(Y)f(Y)}{f_{bn}(Y)} \right) dx \right)^2 \right] \\
= \frac{1}{4} \int \int \left( t \int 2^{jn}K(2^{jn}x, 2^{jn}y) \left( \frac{R_{bn,k}(x)f(x)}{f_{bn}(x)} - \frac{R_{bn,k}(y)f(y)}{f_{bn}(y)} \right) dx \right)^2 f_{(X, Y)}(t, y) dt dy \\
= \frac{1}{4} \int \left( \int t^2 f_{(X, Y)}(t, y) dt \right) \\
\times \left( \int 2^{jn}K(2^{jn}x, 2^{jn}y) \left( \frac{R_{bn,k}(x)f(x)}{f_{bn}(x)} - \frac{R_{bn,k}(y)f(y)}{f_{bn}(y)} \right) dx \right)^2 f(y) dy \\
= \frac{1}{4} \mathbb{E}\left[ \mathbb{E}\left( X_{\ell}^2 \mid Y \right) \left( \int 2^{jn}K(2^{jn}x, 2^{jn}Y + u) \left\{ \frac{R_{bn,k}(x)f(x)}{f_{bn}(x)} - \frac{R_{bn,k}(Y)f(Y)}{f_{bn}(Y)} \right\} dx \right)^2 \right] \\
= \frac{1}{4} \mathbb{E}\left[ \mathbb{E}\left( X_{\ell}^2 \mid Y \right) h_n^2(Y) \right],
\]

where \( h_n(Y) = \int w_n(Y, u) du \) with \( w_n(Y, u) = K(2^{jn}Y, 2^{jn}Y + u) m_n(Y, u) \) and

\[
m_n(Y, u) = \frac{R_{bn,k}(Y + 2^{-jn}u)f(Y + 2^{-jn}u)}{f_{bn}(Y + 2^{-jn}u)} - \frac{R_{bn,k}(Y)f(Y)}{f_{bn}(Y)}. \]

From continuity of \( f \) and \( g_k \) and the fact that the sequence \( (K(2^{jn}Y, 2^{jn}Y + u))_{n \in \mathbb{N}} \) is bounded (see Assumption 5(1)), we obtain \( \lim_{n \to +\infty} (w_n(Y, u)) = 0 \). Fur-
ther, since $\varepsilon_{b_n} = f/f_{b_n}$ ≤ 1 and $R_{b_n,k} = R_k \varepsilon_{b_n}$, it follows

$$|m_n(Y,u)| \leq \left| \left( R_k(Y + 2^{-jn}u) - R_k(Y) \right) \varepsilon_{b_n}^2(Y + 2^{-jn}u) + R_k(Y) \left( \varepsilon_{b_n}^2(Y + 2^{-jn}u) - \varepsilon_{b_n}^2(Y) \right) \right|$$

$$\leq \left| R_k(Y + 2^{-jn}u) - R_k(Y) \right| + 2|R_k(Y)|$$

$$\leq c2^{-jn}|u| + 2|R_k(Y)|.$$  

The sequence $(2^{-jn})_{n \in \mathbb{N}}$ being bounded, there exists $M > 0$ such that $|m_n(Y,u)| \leq M|u| + 2|R_k(Y)|$. Hence

$$\int |w_n(Y,u)| du \leq M \int |u|\Phi(u) du + 2|R_k(Y)| \int \Phi(u) du < +\infty;$$

and using the dominated convergence theorem we get: $\lim_{n \to +\infty} (h_n(Y)) = 0$.

On the other hand,

$$h_n^2(Y) \leq \int w_n^2(Y,u) du = \int K^2(2^{jn}Y,2^{jn}Y + u) m_n^2(Y,u) du$$

$$\leq \int K^2(2^{jn}Y,2^{jn}Y + u) (M|u| + 2|R_k(Y)|)^2 du$$

$$\leq 2M^2 \int K^2(2^{jn}Y,2^{jn}Y + u) u^2 du + 8R_k^2(Y) \int K^2(2^{jn}Y,2^{jn}Y + u) du.$$  

Then, using Assumption 3.5-(1) and the property $\int K^2(x,y) dy \leq D^2$ (see Remark 2), we get $h_n^2(Y) \leq M_1 + M_2R_k^2(Y)$, where $M_1 = 2M^2 \int u^2\Phi^2(u) du$ and $M_2 = 8D^2$. Therefore, $|E(X_k^2|Y) h_n^2(Y)| \leq \phi(Y)$, where $\phi(Y) = M_1E(X_k^2|Y) + M_2E(X_k^2R_k^2(Y)||Y)$. This random variable is integrable since

$$E(|\phi(Y)|) = M_1E(X_k^2) + M_2E(X_k^2R_k^2(Y)) \leq M_1G^2 + M_2G^2E(R_k(Y)) < +\infty.$$  

Applying the dominated convergence theorem, we obtain

$$\lim_{n \to +\infty} E \left[ (U_n^{(6)}(Y))^2 \right] = \frac{1}{4} E \left[ E(X_k^2|Y) \lim_{n \to +\infty} (h_n^2(Y)) \right] = 0$$
and we deduce, since \( \text{Var} \left( n^{-1/2} \sum_{i=1}^{n} U_n^{(6)}(Y_i) \right) = \text{Var} \left( U_n^{(6)}(Y) \right) \leq \mathbb{E} \left( \left( U_n^{(6)}(Y) \right)^2 \right) \),
that \( n^{-1/2} \sum_{i=1}^{n} \left( U_n^{(6)}(Y_i) - \mathbb{E}(U_n^{(6)}(Y_i)) \right) = o_p(1) \). These results allow to conclude that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_{n,k,i} - \mathbb{E} \left( V_{n,k,i} \right)
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{2} R_k(Y_i) R_\ell(Y_i) \varepsilon_{b_n}(Y_i) + \frac{1}{2} X_{\ell,i} R_k(Y_i) \varepsilon_{b_n}(Y_i) \right. \\
- \mathbb{E} \left( \frac{1}{2} R_k(Y_i) R_\ell(Y_i) \varepsilon_{b_n}(Y) + \frac{1}{2} X_{\ell,i} R_k(Y) \varepsilon_{b_n}(Y) \right) \left. \right\} + o_p(1),
\]

and from Eq.(10), we obtain

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) \frac{\hat{g}_{\ell,n}(Y_i)}{f_{b_n}(Y_i)} - \mathbb{E} \left[ R_{b_n,k}(Y) \frac{\hat{g}_{\ell,n}(Y)}{f_{b_n}(Y)} \right] \right\}
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{2} R_k(Y_i) R_\ell(Y_i) \varepsilon_{b_n}(Y_i) + \frac{1}{2} X_{\ell,i} R_k(Y_i) \varepsilon_{b_n}(Y_i) \right. \\
- \mathbb{E} \left( \frac{1}{2} R_k(Y) R_\ell(Y) \varepsilon_{b_n}(Y) + \frac{1}{2} X_{\ell,i} R_k(Y) \varepsilon_{b_n}(Y) \right) \left. \right\} + o_p(1)
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{2} R_{b_n,k}(Y_i) R_{b_n,\ell}(Y_i) + \frac{1}{2} X_{\ell,i} R_{b_n,k}(Y_i) \right. \\
- \mathbb{E} \left( \frac{1}{2} R_{b_n,k}(Y) R_{b_n,\ell}(Y) + \frac{1}{2} X_{\ell,i} R_{b_n,k}(Y) \right) \left. \right\} + o_p(1). \tag{11}
\]

By replacing \( k \) with \( \ell \) and adding the results, we obtain the required result.

**Lemma 5.8** Under assumptions 1, 2, 4, 5, 6 and 7, we have

\[
\mathcal{E}_{k,\ell}^{(3)} = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) R_{b_n,\ell}(Y_i) \frac{f(Y_i)}{f_{b_n}(Y_i)} - \mathbb{E} \left[ R_{b_n,k}(Y) R_{b_n,\ell}(Y) \frac{f(Y)}{f_{b_n}(Y)} \right] \right\} + o_p(1).
\]

**Proof.** The proof similar to that of step 2 of the proof of Theorem 2.1 in Zhu and Fang[25].
Lemma 5.9  Under assumptions [1, 2, 4, 5, 6, and 7] we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) R_{b_n,\ell}(Y_i) + \frac{X_{\ell_i} R_{b_n,k}(Y_i) f(Y_i)}{2 f_{b_n}(Y_i)} + \frac{X_{k_i} R_{b_n,\ell}(Y_i) f(Y_i)}{2 f_{b_n}(Y_i)} - \mathbb{E} \left[ R_{b_n,k}(Y) R_{b_n,\ell}(Y) + \frac{X_{\ell_i} R_{b_n,k}(Y) f(Y)}{2 f_{b_n}(Y)} + \frac{X_{k_i} R_{b_n,\ell}(Y) f(Y)}{2 f_{b_n}(Y)} \right] \right\}
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{k_i}(Y_i) R_{\ell_i}(Y_i) + \frac{X_{\ell_i} R_{k_i}(Y_i) f(Y_i)}{2 f_{b_n}(Y_i)} + \frac{X_{k_i} R_{\ell_i}(Y_i) f(Y_i)}{2 f_{b_n}(Y_i)} - 2 \mathbb{E} \left[ R_{k_i}(Y) R_{\ell_i}(Y) \right] \right\} + o_p(1)
\]

Proof.  Putting
\[
Z_{k,\ell,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{X_{\ell_i} R_{b_n,k}(Y_i) f(Y_i)}{2 f_{b_n}(Y_i)} - \frac{1}{2} X_{\ell_i} R_{k_i}(Y_i) - \mathbb{E} \left[ \frac{X_{\ell_i} R_{b_n,k}(Y) f(Y)}{2 f_{b_n}(Y)} - \frac{1}{2} X_{\ell_i} R_{k_i}(Y) \right] \right\},
\]
and using Assumption [1] we get
\[
\mathbb{E} \left( Z_{k,\ell,n}^2 \right) \leq \frac{G^2}{4} \mathbb{E} \left( \frac{R_{b_n,k}(Y) f(Y)}{f_{b_n}(Y)} - R_k(Y) \right)^2
\]
\[
\leq \frac{G^2}{4} \int_{\{f(y)<b_n\}} \left( \frac{R_{b_n,k}(y) f(y)}{f_{b_n}(y)} + |R_k(y)| \right)^2 f(y) dy
\]
\[
\leq \frac{G^2}{4} \int_{\{f(y)<b_n\}} (|R_{b_n,k}(y)| + |R_k(y)|)^2 f(y) dy
\]
\[
\leq G^2 \mathbb{E} \left[ (R_k(Y))^2 \mathbf{1}_{\{f(y)<b_n\}} \right],
\]
from Assumption [7] we deduce that \( \mathbb{E} \left( Z_{k,\ell,n}^2 \right) \to 0 \) as \( n \to +\infty \). Thus
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{X_{\ell_i} R_{b_n,k}(Y_i) f(Y_i)}{2 f_{b_n}(Y_i)} - \frac{1}{2} X_{\ell_i} R_{k_i}(Y_i) - \mathbb{E} \left[ \frac{X_{\ell_i} R_{b_n,k}(Y) f(Y)}{2 f_{b_n}(Y)} - \frac{1}{2} X_{\ell_i} R_{k_i}(Y) \right] \right\} = o_p(1).
\]

(12)
By inverting $k$ and $\ell$, and adding the result to the previous one, we get
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{X_{ki} R_{b_n,\ell}(Y_i) f(Y_i)}{2f_{b_n}(Y_i)} + \frac{X_{\ell i} R_{b_n,k}(Y_i) f(Y_i)}{2f_{b_n}(Y_i)} \right\} - \mathbb{E} \left[ \frac{R_{\ell}(Y) R_{b_n,k}(Y) f(Y)}{2f_{b_n}(Y)} + \frac{R_{k}(Y) R_{b_n,\ell}(Y) f(Y)}{2f_{b_n}(Y)} \right]
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{2} X_{\ell i} R_{k}(Y_i) + \frac{1}{2} X_{ki} R_{\ell}(Y_i) - \mathbb{E} [R_k(Y) R_{\ell}(Y)] \right\} + o_p(1),
\]
and adding this result to the one of Lemma 5.9 yields the required result. $\square$

Lemma 5.10 Under assumptions 1, 2, 4, 5, 6 and 7, we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) R_{b_n,\ell}(Y_i) \frac{f(Y_i)}{f_{b_n}(Y_i)} - \mathbb{E} \left[ R_{b_n,k}(Y) R_{b_n,\ell}(Y) \frac{f(Y)}{f_{b_n}(Y)} \right] \right\}
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{k}(Y_i) R_{\ell}(Y_i) - \mathbb{E} [R_k(Y) R_{\ell}(Y)] \right\} + o_p(1).
\]

Proof. The proof is obtained by using similar arguments than in the proof of Lemma 5.9 from
\[
\mathcal{Y}_{k,\ell,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ R_{b_n,k}(Y_i) R_{b_n,\ell}(Y_i) \frac{f(Y_i)}{f_{b_n}(Y_i)} - R_{k}(Y_i) R_{\ell}(Y_i) \right. \nonumber \\
- \mathbb{E} \left[ R_{b_n,k}(Y) R_{b_n,\ell}(Y) \frac{f(Y)}{f_{b_n}(Y)} - R_{k}(Y) R_{\ell}(Y) \right] \left. \right\}.
\]

$\square$

5.3 Proof of Theorem 1

Let us denote by $\hat{\lambda}_{k,\ell}^{(n)}$ the $(k,\ell)$-th entry of the $d \times d$ matrix $\hat{\Lambda}_n$. It is easily seen that
\[
\sqrt{n} \hat{\lambda}_{k,\ell}^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{g}_{k,n}(Y_i) \hat{g}_{\ell,n}(Y_i)}{f_{b_n}(Y_i)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ A_{k,\ell}^{(1)}(Y_i) + A_{k,\ell}^{(2)}(Y_i) - A_{k,\ell}^{(3)}(Y_i) \right\}
\]
\[
- U_{n,k,\ell}^{(1)} + U_{n,k,\ell}^{(2)} + U_{n,k,\ell}^{(3)} - U_{n,k,\ell}^{(4)}.
\]
and from Lemma 5.2 we get

\[ \sqrt{n} \hat{\lambda}_{k,\ell}^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ A_{k\ell}^{(1)}(Y_i) + A_{k\ell}^{(2)}(Y_i) - A_{k\ell}^{(3)}(Y_i) \right\} + o_p(1). \]  

(13)

Therefore, putting \( \nu_{k,\ell} = \mathbb{E} \left( A_{k\ell}^{(1)}(Y) + A_{k\ell}^{(2)}(Y) - A_{k\ell}^{(3)}(Y) \right) \), we have \( \sqrt{n} \left( \hat{\lambda}_{k,\ell}^{(n)} - \nu_{k,\ell} \right) = \mathcal{E}_{k\ell}^{(1)} + \mathcal{E}_{k\ell}^{(2)} - \mathcal{E}_{k\ell}^{(3)} + o_p(1) \). Then, from Lemmas 5.6, 5.7, 5.9, 5.8 and 5.10 it follows

\[ \sqrt{n} \left( \hat{\lambda}_{k,\ell}^{(n)} - \nu_{k,\ell} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{2} \left( X_{\ell i} R_k(Y_i) + X_{ki} R_\ell(Y_i) \right) - \mathbb{E} \left( R_k(Y) R_\ell(Y) \right) \right\} + o_p(1), \]

and using Lemmas 5.3, 5.4 and 5.5, we get \( \sqrt{n} \nu_{k,\ell} = \sqrt{n} \lambda_{k,\ell} + o(1) \), where \( \lambda_{k,\ell} = \mathbb{E} \left( R_k(Y) R_\ell(Y) \right) \), that is the \((k, \ell)\)-th entry of the \(d \times d\) matrix \( \Lambda \). Hence,

\[ \sqrt{n} \left( \hat{\lambda}_{k,\ell}^{(n)} - \lambda_{k,\ell} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{2} \left( X_{\ell i} R_k(Y_i) + X_{ki} R_\ell(Y_i) \right) - \mathbb{E} \left( R_k(Y) R_\ell(Y) \right) \right\} + o_p(1), \]

Clearly,

\[ \mathbb{E} \left( \frac{1}{2} \left( X_{\ell i} R_k(Y) + X_{ki} R_\ell(Y) \right) \right) = \mathbb{E} \left( X_{\ell i} R_k(Y) \right) = \mathbb{E} \left( R_\ell(Y) R_k(Y) \right), \]

and, putting \( \mathcal{H}_n = \sqrt{n} \left( \hat{\Lambda}_n - \Lambda \right) \) and \( \mathcal{H}_{k\ell}^{(n)} = \sqrt{n} \left( \hat{\lambda}_{k,\ell}^{(n)} - \lambda_{k,\ell} \right) \), we have

\[ tr \left( A^T \mathcal{H}_n \right) = \sum_{k=1}^{d} \sum_{\ell=1}^{d} a_{k\ell} \mathcal{H}_{k\ell}^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( U_i - \mathbb{E} \left( U_i \right) \right) + o_p(1), \]

where

\[ U_i = \sum_{k=1}^{d} \sum_{\ell=1}^{d} \frac{a_{k\ell}}{2} \left( X_{\ell i} R_k(Y_i) + X_{ki} R_\ell(Y_i) \right). \]

From the central limit theorem and Slutsky’s theorem we deduce that \( tr \left( A^T \mathcal{H}_n \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\mathcal{A}^2) \), as \( n \to +\infty \), where \( \sigma_\mathcal{A}^2 \) is given in [3]. Then, using Levy’s theorem, we conclude that \( \mathcal{H}_n \xrightarrow{\mathcal{D}} \mathcal{H} \), as \( n \to +\infty \), where \( \mathcal{H} \) has a normal distribution in \( \mathcal{M}_d(\mathbb{R}) \) with \( tr \left( A^T \mathcal{H} \right) \sim \mathcal{N}(0, \sigma_\mathcal{A}^2) \). \( \square \)
5.4 Proof of Theorem 2

It is known from Theorem 2.2 in Zhu and Fang [25] that
\[ \sqrt{n} \left( \hat{\beta}_j - \beta_j \right) \xrightarrow{D} \mathcal{G}_j = \sum_{r=1, r\neq j}^d (\lambda_j - \lambda_r)^{-1} \beta_r \beta_j^T \mathcal{H} \beta_j. \]
Since \( \mathcal{G}_j \) is a linear function of \( \mathcal{H} \), it has a normal distribution \( \mathcal{N}(0, \Sigma_j) \), where \( \Sigma_j \) is a covariance matrix that will now be specified. Clearly, for any \( u = (u_1, \cdots, u_d) \in \mathbb{R}^d \), we have \( u^T \mathcal{G}_j = tr(A \mathcal{H}) \), where \( A = \beta_j u^T \xi_j \beta_j^T \) with \( \xi_j = \sum_{r=1, r\neq j}^d (\lambda_j - \lambda_r)^{-1} \beta_r \). Then, from Theorem 1 we get \( u^T \mathcal{G}_j \sim \mathcal{N}(0, \sigma_A^2) \), where \( \sigma_A^2 = V ar \left( \sum_{p=1}^d \sum_{q=1}^d 2^{-1} a_{pq} (X_q R_p(Y) + X_p R_q(Y)) \right) \), and \( a_{pq} \) is the \((p,q)\)-th entry of \( A \). However, \( a_{pq} = \sum_{k=1}^d \beta_{jp} u_k \xi_{jk} \beta_{jq} \), and, therefore,
\[
\sum_{p=1}^d \sum_{q=1}^d a_{pq} (X_q R_p(Y) + X_p R_q(Y)) = \sum_{k=1}^d u_k \xi_{jk} \sum_{p=1}^d \sum_{q=1}^d \beta_{jp} \beta_{jq} (X_q R_p(Y) + X_p R_q(Y)).
\]
Since \( \xi_{jk} = \sum_{r=1, r\neq j}^d (\lambda_j - \lambda_r)^{-1} \beta_{rk} \), it follows \( \sigma_A^2 = V ar \left( \sum_{k=1}^d u_k \mathcal{W}_{jk} \right) = u^T \Theta_j u \), where \( \Theta_j \) is the covariance matrix of \( \mathcal{W}_j = (\mathcal{W}_{j1}, \cdots, \mathcal{W}_{jd})^T \). Since this later equality holds for all \( u \) in \( \mathbb{R}^d \), we deduce that \( \Sigma_j = \Theta_j \). \( \square \)

References

[1] Y. Aragon and J. Sarraco, Sliced inverse regression (SIR) : an appraisal of small sample alternatives to slicing, *Comput. Statist.* 12 (1997) 109–130.

[2] E. Bura and D. Cook, Extending SIR: the weighted chi-square test, *J. Amer. Statist. Assoc.* 96 (2001) 996–1003.

[3] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics 61, (SIAM, 1992).

[4] N. Duan and K.C. Li, Slicing regression: a link-free regression method, *Ann. Statist.* 19 (1991) 505–530.

[5] L. Ferré, Determining the dimensionality in sliced inverse regression and related methods, *J. Amer. Statist. Assoc.* 93 (1998) 132–140.
[6] E. Giné and A. Guillou, Law of iterated logarithm for censored data, *Ann. Probab.* **27** (1999) 2042–2067.

[7] E. Giné and A. Guillou, On consistency of kernel density estimators for randomly censored data: rates holding uniformly over adaptive intervals, *Ann. Inst. Henri Poincaré* **37** (2001) 503–522.

[8] E. Giné and R. Nickl R, Uniform limit theorems for wavelets density estimators, *Ann. Probab.* **37** (2009) 1605–1646.

[9] W. Härdle, G. Kerkyacharian, D. Picard and A. Tsybakov, *Wavelets, Approximation, and Statistical Applications* (Springer, 1998).

[10] W. Härdle and T.M. Stoker, Investigating smooth multiple regression by the method of average derivatives, *J. Amer. Statist. Assoc.* **84** (1989) 986–995.

[11] T. Hsing and R.J. Carroll, An asymptotic theory for sliced inverse regression, *Ann. Statist.* **20** (1992) 1040–1061.

[12] G. Kerkyacharian and D. Picard, (1992). Density estimation in Besov spaces, *Statist. Probab. Lett.* **13** (1992) 15–24.

[13] K.C. Li, (1991) Sliced inverse regression for dimension reduction, *J. Amer. Statist. Assoc.* **86** (1991) 316–327.

[14] Y. Meyer, *Wavelets and Operators* ( Cambridge Univ. Press, 1992).

[15] G.M. Nkiet, Consistent estimation of the dimensionality in sliced inverse regression, *Ann. Inst. Statist. Math.* **60** (1998) 257–271.

[16] E.D.D. Nkou and G.M. Nkiet, Strong consistency of kernel estimator in a semiparametric regression model, *Statistics* **53** (2019) 1289–1305.

[17] D. Nolan and D. Pollard, U-processes: rate of convergence, *Ann. Statist.* **15** (1987) 780–799.

[18] J.L. Powell, J.H. Stock and T.M. Stoker, (1989) . Semiparametric estimation of index coefficients, *Econometrica* **57** (1989) 1403–1430.

[19] J. Sarraco, An asymptotic theory for sliced inverse regression, *Comm. Stat.- Theory Meth.* **26** (1997) 2141–2171.

29
[20] J.R. Schott, Determining the dimensionality in sliced inverse regression and related methods, *J. Amer. Statist. Assoc.* **89** (1998) 141–148.

[21] C.J. Stone, Consistent nonparametric regression, *Ann. Statist.*, **54** (1977) 595–645.

[22] M. Talagrand, Sharper bounds for Gaussian and empirical processes, *Ann. Probab.* **22** (1994) 28–76.

[23] S. Velilla, Assessing the number of linear components in a general regression problem, *J. Amer. Statist. Assoc.* **93** (1998) 1088–1098.

[24] B. Vidakovic, *Statistical modeling by Wavelets* (Wiley, 1999).

[25] L.X. Zhu and K.T. Fang, Asymptotics for kernel estimate of sliced inverse regression, *Ann. Statist.* **24** (3) (1996) 1053–1068.