EXTENDED BLOCH GROUP AND THE CHERN-SIMONS CLASS
(INCOMPLETE WORKING VERSION)

WALTER D. NEUMANN

Abstract. We define an extended Bloch group and show it is isomorphic to $H_3(\text{PSL}(2, \mathbb{C}), \mathbb{Z})$. Using the Rogers dilogarithm function this leads to an exact simplicial formula for the universal Cheeger-Simons class on this homology group. It also leads to an independent proof of the analytic relationship between volume and Chern-Simons invariant of hyperbolic manifolds conjectured in [14] and proved in [17], as well as an effective formula for the Chern-Simons invariant of a hyperbolic manifold.

1. Introduction

There are several variations of the definition of the Bloch group in the literature; by [7] they differ at most by torsion and they agree with each other for algebraically closed fields. In this paper we shall use the following.

Definition 1.1. Let $k$ be a field. The pre-Bloch group $P(k)$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(k - \{0, 1\})$ by all instances of the following relation:

\[(1) \; [x] - [y] + \frac{y}{x} - \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \left[ \frac{1 - x}{1 - y} \right] = 0,\]

This relation is usually called the five term relation. The Bloch group $B(k)$ is the kernel of the map $P(k) \to k^* \otimes_{\mathbb{Z}} k^*$, $[z] \mapsto 2(z \otimes (1 - z))$.

(In [13] the additional relations $[x] = [1 - 1/x] = [1 - x^{-1}] = -1/x = [x - 1] = -[1 - x]$ were used. These follow from the five term relation when $k$ is algebraically closed, as shown by Dupont and Sah [7]. Dupont and Sah use a different five term relation but it is conjugate to the one used here by $z \mapsto \frac{1}{z}$.)

There is an exact sequence due to Bloch and Wigner:

$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(\text{PSL}(2, \mathbb{C}), \mathbb{Z}) \to B(\mathbb{C}) \to 0$.

The superscript $\delta$ means “with discrete topology.” We will omit it from now on.

$B(\mathbb{C})$ is known to be uniquely divisible, so it has canonically the structure of a $\mathbb{Q}$-vector space (Suslin [10]). It’s $\mathbb{Q}$-dimension is infinite and conjectured to be countable (the “Rigidity Conjecture,” equivalent to the conjecture that $B(\mathbb{C}) = B(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers). In particular, the $\mathbb{Q}/\mathbb{Z}$ in the Bloch-Wigner exact sequence is precisely the torsion of $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$, so any finite torsion subgroup is cyclic.
In the present paper we define an extended Bloch group $\mathcal{EB}(\mathbb{C})$ by replacing $\mathbb{C} - \{0, 1\}$ in the definition of $\mathcal{B}(\mathbb{C})$ by its universal abelian cover $(\mathbb{C} - \{0, 1\})^{ab}$ and appropriately lifting the five term relation (1). Our main results are that we can lift the Bloch-Wigner map $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathcal{B}(\mathbb{C})$ to an isomorphism

$$\lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathcal{EB}(\mathbb{C})$$

Moreover, the “Roger’s dilogarithm function” (see below) gives a natural map

$$R: \mathcal{EB}(\mathbb{C}) \to \mathbb{C}/2\pi^2\mathbb{Z}.$$ 

We show that the composition

$$R \circ \lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathbb{C}/2\pi^2\mathbb{Z}$$

is the Cheeger-Simons class (cf [4]), so it can also be described as $i(\text{vol} + i \text{cs})$, where $cs$ is the universal Chern-Simons class. Dupont in [3] gave an answer modulo $\pi^2\mathbb{Q}$ and our computation is a natural lift of his.

Another consequence of our result is that any complete hyperbolic 3-manifold $M$ of finite volume has a natural “fundamental class” in $H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})/C_2$, where $C_2$ is the (unique) order 2 subgroup. For compact $M$ the existence of this class, even without the $C_2$ ambiguity, is easy and well known: $M = \mathbb{H}^3/\Gamma$ is a $K(\Gamma, 1)$-space, so the inclusion $\Gamma \to \text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$ induces $H_3(M) = H_3(\Gamma) \to H_3(\text{PSL}(2, \mathbb{C}))$, and the class in question is the image of the fundamental class $[M] \in H_3(M)$. For non-compact $M$ the existence of such a class is somewhat surprising, although it was already strongly suggested by earlier results.

We can describe this fundamental class nicely in terms of an ideal triangulation of $M$. However, this ideal triangulation has to be a “true” ideal triangulation rather than the less restrictive “degree 1” ideal triangulations used in [3]. The ideal triangulations resulting from Dehn filling that are used by the programs Snappea and Snap [2] are not true. Nevertheless, we can describe the fundamental class in terms of these “Dehn filling triangulations.” This leads also to an exact simplicial formula for the Chern-Simons invariant of a hyperbolic 3-manifold, refining the formula of [9].

We work initially with a different version of the extended Bloch group, based on a disconnected $\mathbb{Z} \times \mathbb{Z}$ cover of $\mathbb{C} - \{0, 1\}$. This group, which we call $\hat{\mathcal{B}}(\mathbb{C})$ is a quotient of $\mathcal{EB}(\mathbb{C})$ by a subgroup of order 2.

**Acknowledgements.** The definition of the extended Bloch group was suggested by an idea of Jun Yang, to whom I am grateful also for many useful conversations. In particular, he informs me that this work can be interpreted as giving a motivic complex for $K_3(\mathbb{C})$. The main results of this paper were announced in [10]. This research is supported by the Australian Research Council.

## 2. The preliminary version of extended Bloch group

We shall need a $\mathbb{Z} \times \mathbb{Z}$ cover $\hat{\mathbb{C}}$ of $\mathbb{C} - \{0, 1\}$ which can be constructed as follows. Let $P$ be $\mathbb{C} - \{0, 1\}$ split along the rays $(-\infty, 0)$ and $(1, \infty)$. Thus each real number $r$ outside the interval $[0, 1]$ occurs twice in $P$, once in the upper half plane of $\mathbb{C}$ and once in the lower half plane of $\mathbb{C}$. We denote these two occurrences of $r$ by $r + 0i$...
and \( r - 0i \). We construct \( \hat{C} \) as an identification space from \( P \times \mathbb{Z} \times \mathbb{Z} \) by identifying
\[
(x + 0i, p, q) \sim (x - 0i, p + 2, q) \quad \text{for each } x \in (-\infty, 0)
\]
\[
(x + 0i, p, q) \sim (x - 0i, p + 2, q) \quad \text{for each } x \in (1, \infty).
\]
We will denote the equivalence class of \((z, p, q)\) by \((\hat{z}; p, q)\). \( \hat{C} \) has four components:
\[
\hat{C} = X_{00} \cup X_{01} \cup X_{10} \cup X_{11}
\]
where \( X_{\epsilon_0, \epsilon_1} \) is the set of \((z; p, q) \in \hat{C} \) with \( p \equiv \epsilon_0 \) and \( q \equiv \epsilon_1 \) (mod 2).

We may think of \( X_{00} \) as the riemann surface for the function \( C - \{0, 1\} \rightarrow \mathbb{C}^2 \) defined by \( z \mapsto (\log z, -\log(1 - z)) \). Taking the branch \((\log z + 2p\pi i, -\log(1 - z) + 2q\pi i)\) of this function on the portion \( P \times \{(2p, 2q)\} \) of \( X_{00} \) for each \( p, q \in \mathbb{Z} \) defines an analytic function from \( X_{00} \) to \( \mathbb{C}^2 \). In the same way, we may think of \( \hat{C} \) as the riemann surface for the collection of all branches of the functions \((\log z + p\pi i, -\log(1 - z) + q\pi i)\) on \( C - \{0, 1\} \).

Consider the set
\[
\text{FT} := \left\{(x, y, y' \frac{1-x^{-1}}{1-y^{-1}}, x, 1-y) : x \neq y, x, y \in C - \{0, 1\}\right\} \subseteq (C - \{0, 1\})^5
\]
of 5-tuples involved in the five term relation (4). An elementary computation shows:

**Lemma 2.1.** The subset \( \text{FT}^+ \) of \((x_0, \ldots, x_4) \in \text{FT} \) with each \( x_i \) in the upper half plane of \( C \) is the set of elements of \( \text{FT} \) for which \( y \) is in the upper half plane of \( C \) and \( x \) is inside the triangle with vertices \( 0, 1, y \). Thus \( \text{FT}^+ \) is connected (even contractible).

**Definition 2.2.** Let \( V \subset (\mathbb{Z} \times \mathbb{Z})^5 \) be the subspace
\[
V := \{(p_0, q_0, (p_1, q_1), (p_1 - p_0, q_2), (p_1 - p_0 + q_1 - q_0, q_2 - q_1), (q_1 - q_0, q_2 - q_1 - p_0)) : p_0, p_1, q_0, q_1, q_2 \in \mathbb{Z}\}.
\]
Let \( \hat{\text{FT}}_0 \) denote the unique component of the inverse image of \( \text{FT} \) in \( \hat{C}^5 \) which includes the points \((x_0; 0, 0), \ldots, (x_4; 0, 0)\) with \((x_0, \ldots, x_4) \in \text{FT}^+ \), and define
\[
\hat{\text{FT}} := \hat{\text{FT}}_0 + V = \{x + v : x \in \hat{\text{FT}}_0 \text{ and } v \in V\},
\]
where we are using addition to denote the action of \((\mathbb{Z} \times \mathbb{Z})^5 \) by covering transformations on \( \hat{C}^5 \). (Although we do not need it, one can show that the action of \( 2V \) takes \( \hat{\text{FT}}_0 \) to itself, so \( \hat{\text{FT}} \) has \( 2^6 \) components, determined by the parities of \( p_0, p_1, q_0, q_1, q_2 \).)

Define \( \hat{\mathcal{P}}(C) \) as the free \( \mathbb{Z} \)-module on \( \hat{C} \) factored by all instances of the relations:
\[
(2) \quad \sum_{i=0}^{4} (-1)^i (x_i; p_i, q_i) = 0 \quad \text{with } ((x_0; p_0, q_0), \ldots, (x_4; p_4, q_4)) \in \hat{\text{FT}}
\]
and
\[
(3) \quad (x; p, q) + (x; p', q') = (x; p, q') + (x; p', q) \quad \text{with } p, q, p', q' \in \mathbb{Z}
\]
We shall denote the class of \((z; p, q)\) in \( \hat{\mathcal{P}}(C) \) by \([z, p, q]\).
We call relation (4) the lifted five term relation. We shall see that its precise form arises naturally in several contexts. In particular, we give it a geometric interpretation in Sect. 3.

We call relation (3) the transfer relation. It is almost a consequence of the lifted five term relation, since we shall see that the effect of omitting it would be to replace \( \hat{\mathcal{P}}(\mathbb{C}) \) by \( \hat{\mathcal{P}}(\mathbb{C}) \otimes \mathbb{Z}/2 \), with \( \mathbb{Z}/2 \) generated by an element \( \kappa := [x, 1, 1] + [x, 0, 0] - [x, 1, 0] - [x, 0, 1] \) which is independent of \( x \).

**Lemma 2.3.** There is a well-defined homomorphism \( \nu: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C} \wedge \mathbb{C} \) defined on generators by \([z, p, q] \mapsto (\log z + p\pi i) \wedge (-\log(1 - z) + q\pi i)\).

**Proof.** We must verify that \( \nu \) vanishes on the relations that define \( \hat{\mathcal{P}}(\mathbb{C}) \). This is trivial for the transfer relation (3). We shall show that the lifted five term relation is the most general lift of the five term relation (4) for which \( \nu \) vanishes. If one applies \( \nu \) to an element \( \sum_{i=0}^{4} (-1)^i [x_i, p_i, q_i] \) with \((x_0, \ldots, x_4) = (x, y, \ldots) \in \text{FT}^+\) one obtains after simplification:

\[
((q_0 - p_2 - q_2 + p_3 + q_3) \log x + (p_0 - q_3 + q_4) \log(1 - x) + (-q_1 + q_2 - q_3) \log y + (-p_1 + p_3 + q_3 - p_4 - q_4) \log(1 - y) + (p_2 - p_3 + p_4) \log(x - y)) \wedge \pi i.
\]

An elementary linear algebra computation shows that this vanishes identically if and only if \( p_2 = p_1 - p_0, p_3 = p_1 - p_0 + q_1 - q_3, q_3 = q_2 - q_1, q_4 = q_1 - q_0, \) and \( q_4 = q_2 - q_1 - p_0 \), as in the lifted five term relation. The vanishing of \( \nu \) for the general lifted five term relation now follows by analytic continuation. \( \square \)

**Definition 2.4.** Define \( \hat{\mathcal{B}}(\mathbb{C}) \) as the kernel of \( \nu: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C} \wedge \mathbb{C} \).

Define

\[
R(z; p, q) = \mathcal{R}(z) + \frac{\pi i}{2} (p \log(1 - z) + q \log z) - \frac{\pi^2}{6}
\]

where \( \mathcal{R} \) is the Rogers dilogarithm function

\[
\mathcal{R}(z) = \frac{1}{2} \log(z) \log(1 - z) - \int_0^z \frac{\log(1 - t)}{t} dt.
\]

Then

**Proposition 2.5.** \( R \) gives a well defined map \( R: \hat{\mathcal{C}} \to \mathbb{C}/\pi^2 \mathbb{Z} \). The relations which define \( \hat{\mathcal{P}}(\mathbb{C}) \) are functional equations for \( R \) modulo \( \pi^2 \) (the lifted five term relation is in fact the most general lift of the five term relation (4) with this property). Thus \( R \) also gives a homomorphism \( R: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C}/\pi^2 \mathbb{Z} \).

**Proof.** If one follows a closed path from \( z \) that goes anti-clockwise around the origin it is easily verified that \( R(z; p, q) \) is replaced by \( R(z; p, q) + \pi i \log(1 - z) - q \pi^2 = R(z, p + 2, q) - q \pi^2 \). Similarly, following a closed path clockwise around 1 replaces \( R(z; p, q) \) by \( R(z, p + q, q + 2) + p \pi^2 \). Thus \( R \) modulo \( \pi^2 \) is well defined on \( \hat{\mathcal{C}} \) (in fact \( R \) itself is well defined on a \( \mathbb{Z} \) cover of \( \hat{\mathcal{C}} \) which is a nilpotent cover of \( \mathbb{C} \setminus \{0, 1\} \)).

It is well known that \( L(z) := \mathcal{R}(z) - \frac{\pi^2}{6} \) satisfies the functional equation

\[
L(x) - L(y) + L\left(\frac{y}{x}\right) - L\left(\frac{1 - x^{-1}}{1 - y^{-1}}\right) + L\left(\frac{1 - x}{1 - y}\right) = 0
\]
for $0 < y < x < 1$. Since the 5-tuples involved in this equation are on the boundary of $\mathcal{F}T^+$, the functional equation
\[ \sum (-1)^i R(x_i; 0, 0) = 0 \]
is valid by analytic continuation on the whole of $\mathcal{F}T^+$. Now
\[ \sum (-1)^i R(x_i; p_i, q_i) \]
differs from this by
\[ \frac{\pi i}{2} \sum (-1)^i (p_i \log(1 - x_i) + q_i \log x_i) \]
and it is an elementary calculation to verify that this vanishes identically for $(x_0, \ldots, x_4) \in \mathcal{F}T^+$ if and only if the $p_i$ and $q_i$ are as in the lifted five term relation. Thus the lifted five-term relation gives a functional equation for $R$ when $(x_0, \ldots, x_4) \in \mathcal{F}T^+$. By analytic continuation, it is a functional equation for $R \mod \pi^2$ in general. The transfer relation is trivially a functional equation for $R$. □

The first version of our main result is

**Theorem 2.6.** There exists an epimorphism $\lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \hat{\mathcal{B}}(\mathbb{C})$ with kernel of order 2 such that the composition $\lambda \circ R: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}$ is the characteristic class given by $i(\text{vol} + i\text{cs})$.

We shall later modify the definition of $\hat{\mathcal{B}}(\mathbb{C})$ to eliminate the $\mathbb{Z}/2$ kernel. To describe the map $\lambda$ we must give a geometric interpretation of $\hat{\mathcal{C}}$.

### 3. Parameters for Ideal Hyperbolic Simplices

In this section we shall interpret $\hat{\mathcal{C}}$ as a space of parameters for what we call “combinatorial flattenings” of ideal hyperbolic simplices. We need this to define the above map $\lambda$. It also gives a geometric interpretation of the lifted five term relation.

We shall denote the standard compactification of $\mathbb{H}^3$ by $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{C}P^1$. An ideal simplex $\Delta$ with vertices $z_1, z_2, z_3, z_4 \in \mathbb{C}P^1$ is determined up to congruence by the cross ratio
\[ z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}. \]
Permuting the vertices by an even (i.e., orientation preserving) permutation replaces $z$ by one of
\[ z, \quad z' = \frac{1}{1 - z}, \quad \text{or} \quad z'' = 1 - \frac{1}{z}. \]
The parameter $z$ lies in the upper half plane of $\mathbb{C}$ if the orientation induced by the given ordering of the vertices agrees with the orientation of $\mathbb{H}^3$. But we allow simplices whose vertex ordering does not agree with their orientation. We also allow degenerate ideal simplices whose vertices lie in one plane, so the parameter $z$ is real. However, we always require that the vertices are distinct. Thus the parameter $z$ of the simplex lies in $\mathbb{C} - \{0, 1\}$ and every such $z$ corresponds to an ideal simplex.

There is another way of describing the cross-ratio parameter $z = [z_1 : z_2 : z_3 : z_4]$ of a simplex. The group of orientation preserving isometries of $\mathbb{H}^3$ fixing the points $z_1$ and $z_2$ is isomorphic to $\mathbb{C}^*$ and the element of this $\mathbb{C}^*$ that takes $z_4$ to $z_3$ is $z$. Thus the cross-ratio parameter $z$ is associated with the edge $z_1z_2$ of the simplex.
The parameter associated in this way with the other two edges $z_1z_4$ and $z_1z_3$ out of $z_1$ are $z'$ and $z''$ respectively, while the edges $z_3z_4$, $z_2z_3$, and $z_2z_4$ have the same parameters $z$, $z'$, and $z''$ as their opposite edges. See fig. 1.

Note that $zz'z'' = -1$, so the sum
\[
\log z + \log z' + \log z''
\]
is an odd multiple of $\pi i$, depending on the branches of log used. In fact, if we use standard branch of log then this sum is $\pi i$ or $-\pi i$ depending on whether $z$ is in the upper or lower half plane.

**Definition 3.1.** We shall call any triple of the form
\[
w = (w_0, w_1, w_2) = (\log z + p\pi i, \log z' + q\pi i, \log z'' + r\pi i)
\]
with
\[
p, q, r \in \mathbb{Z} \quad \text{and} \quad w_0 + w_1 + w_2 = 0
\]
a combinatorial flattening for our simplex.

Each edge $E$ of $\Delta$ is assigned one of the components $w_i$ of $w$, with opposite edges being assigned the same component. We call $w_i$ the log-parameter for the edge $E$ and denote it $l_E(\Delta, w)$.

This combinatorial flattening can be written
\[
\ell(z; p, q) := (\log z + p\pi i, -\log(1 - z) + q\pi i, \log(1 - z) - \log z - (p + q)\pi i),
\]
and $\ell$ is then a map of $\hat{\mathbb{C}}$ to the set of combinatorial flattenings of simplices.

**Lemma 3.2.** This map $\ell$ is a bijection, so $\hat{\mathbb{C}}$ may be identified with the set of all combinatorial flattenings of ideal tetrahedra.

**Proof.** We must show that we can recover $(z; p, q)$ from $(w_0, w_1, w_2) = \ell(z; p, q)$. It clearly suffices to recover $z$. But $z = \pm e^{w_0}$ and $1 - z = \pm e^{-w_1}$, and the knowledge of both $z$ and $1 - z$ up to sign determines $z$. \[\square\]

We can give a geometric interpretation of the choice of parameters in the five term relation (2). If $z_0, \ldots, z_4$ are five distinct points of $\partial \mathbb{H}^3$, then each choice of four of five points $z_0, \ldots, z_4$ gives an ideal simplex. We denote the simplex which
omits vertex $z_i$ by $\Delta_i$. The cross ratio parameters $x_i = [z_0 : \ldots : \hat{z}_i : \ldots : z_4]$ of these simplices can be expressed in terms of $x := x_0$ and $y := x_1$ as

\[
\begin{align*}
x_0 &= [z_1 : z_2 : z_3 : z_4] =: x \\
x_1 &= [z_0 : z_2 : z_3 : z_4] =: y \\
x_2 &= [z_0 : z_1 : z_3 : z_4] = \frac{y}{x} \\
x_3 &= [z_0 : z_1 : z_2 : z_4] = \frac{1 - x^{-1}}{1 - y^{-1}} \\
x_4 &= [z_0 : z_1 : z_2 : z_3] = \frac{1 - x}{1 - y}
\end{align*}
\]

The lifted five term relation has the form

\[
\sum_{i=0}^{4} (-1)^i (x_i; p_i, q_i) = 0
\]

with certain relations on the $p_i$ and $q_i$. We will give a geometric interpretation of these relations.

Using the map of Lemma 3.2, each summand in this relation \([4]\) represents a choice $\ell(x_i; p_i, q_i)$ of combinatorial flattening for one of the five ideal simplices. For each edge $E$ connecting two of the points $z_i$ we get a corresponding linear combination

\[
\sum_{i=0}^{4} (-1)^i l_E(\Delta_i, \ell(x_i; p_i, q_i))
\]

of log-parameters (Definition 3.1), where we put $l_E(\Delta_i, \ell(x_i; p_i, q_i)) = 0$ if the line $E$ is not an edge of $\Delta_i$. This linear combination has just three non-zero terms corresponding to the three simplices that meet at the edge $E$. One easily checks that the real part is zero and the imaginary part can be interpreted (with care about orientations) as the sum of the “adjusted angles” of the three flattened simplices meeting at $E$.

**Definition 3.3.** We say that the $(x_i; p_i, q_i)$ satisfy the flattening condition if each of the above linear combinations \([4]\) of log-parameters is equal to zero. That is, the adjusted angle sum of the three simplices meeting at each edge is zero.

**Lemma 3.4.** Relation \([4]\) is an instance of the lifted five term relation \([2]\) if and only if the $(x_i; p_i, q_i)$ satisfy the flattening condition.

**Proof.** We first consider the case that $(x_0, \ldots, x_4) \in \text{FT}^+$. Recall this means that each $x_i$ is in $\mathbb{H}$. Geometrically, this implies that each of the above five tetrahedra is positively oriented by the ordering of its vertices. This implies the configuration of fig. 2 with $z_1$ and $z_3$ on opposite sides of the plane of the triangle $z_0 z_2 z_4$ and the line from $z_1$ to $z_3$ passing through the interior of this triangle. Denote the combinatorial flattening of the $i^{th}$ simplex by $\ell(x_i; p_i, q_i)$. If we consider the log-parameters at the edge $z_3 z_4$ for example, they are $\log x + p_0 \pi i$, $\log y + p_1 \pi i$, and $\log (y/x) + p_2 \pi i$ and the condition is that $(\log x + p_0 \pi i) - (\log y + p_1 \pi i) + (\log (y/x) + p_2 \pi i) = 0$. This implies $p_2 = p_1 - p_0$. Similarly the other edges lead to other relations among the $p_i$ and $q_i$, namely:
Elementary linear algebra verifies that these relations are equivalent to the equations
\[ p_2 = p_1 - p_0, \quad p_3 = p_1 - p_0 + q_1 - q_0, \quad q_3 = q_2 - q_1, \quad p_4 = q_1 - q_0, \quad \text{and} \quad q_4 = q_2 - q_1 - p_0, \]
as in the lifted five term relation (2). The lemma thus follows for \((x_0, \ldots, x_4) \in \FT^+\). It is then true in general by analytic continuation. \(\square\)

4. Definition of \(\lambda\)

We can now describe the map \(\lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \hat{\mathcal{P}}(\mathbb{C})\).

We shall first recall a standard chain complex for homology of \(G = \text{PSL}(2, \mathbb{C})\), the chain complex of “homogeneous simplices for \(G\).” We will, however, diverge from the standard by using only non-degenerate simplices, i.e., simplices with distinct vertices — we may do this because \(G\) is infinite.

Let \(C_n(G)\) denote the free \(\mathbb{Z}\)-module on all ordered \((n+1)\)-tuples \((g_0, \ldots, g_n)\) of distinct elements of \(G\). Define \(\delta: C_n \to C_{n-1}\) by
\[
\delta(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \langle g_0, \ldots, \hat{g}_i, \ldots, g_n \rangle.
\]
Then each \(C_n\) is a free \(\mathbb{Z}G\)-module under left-multiplication by \(G\). Since \(G\) is infinite the sequence
\[
\cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0
\]
is exact, so it is a \(\mathbb{Z}G\)-free resolution of \(\mathbb{Z}\). Thus the chain complex
\[
\cdots \to C_2 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_0 \otimes_{\mathbb{Z}G} \mathbb{Z} \to 0
\]
computes the homology of $G$. Note that $C_n \otimes_{\mathbb{Z}_2} \mathbb{Z}$ is the free $\mathbb{Z}$-module on symbols $\langle g_0 : \ldots : g_n \rangle$, where the $g_i$ are distinct elements of $G$ and $\langle g_0 : \ldots : g_n \rangle = \langle g_0' : \ldots : g_n' \rangle$ if and only if there is a $g \in G$ with $gg_i = g'_i$ for $i = 0, \ldots, n$.

Thus an element of $\alpha \in H_3(G; \mathbb{Z})$ is represented by a sum

$$\sum \epsilon_i(g_0^{(i)} : \ldots : g_3^{(i)})$$

of homogeneous 3-simplices for $G$ and their negatives (here each $\epsilon_i$ is $\pm 1$). The fact that this is a cycle means that the 2-faces of these homogeneous simplices cancel in pairs. We choose some specific way of pairing cancelling faces and form a geometric quasi-simplicial complex $K$ by taking a 3-simplex $\Delta_i$ for each homogeneous 3-simplex of the above sum and gluing together 2-faces of these $\Delta_i$ that correspond to 2-faces of the homogeneous simplices that have been paired with each other.

We call a closed path $\gamma$ in $K$ a normal path if it meets no 0- or 1-simplices of $K$ and crosses all 2-faces that it meets transversally. When such a path passes through a 3-simplex $\Delta_i$, entering and departing at different faces, there is a unique edge $E$ of the 3-simplex between these faces. We say the path passes this edge $E$.

Consider a choice of combinatorial flattening $w_i$ for each simplex $\Delta_i$. Then for each edge $E$ of a simplex $\Delta_i$ of $K$ we have a log-parameter $l_E = l_E(\Delta_i, w_i)$ assigned. Recall that this log-parameter has the form $\log z + s\pi i$ where $z$ is the cross-ratio parameter associated to the edge $E$ of simplex $\Delta_i$ and $s$ is some integer. We call $(s \mod 2)$ the parity parameter at the edge $E$ of $\Delta_i$ and denote it $\delta_E = \delta_E(\Delta_i, w_i)$.

**Definition 4.1.** Suppose $\gamma$ is a normal path in $K$. The parity of $\gamma$ is the sum $(\sum_E \delta_E)$ modulo 2 of the parity parameters of all the edges $E$ that $\gamma$ passes. Moreover, if $\gamma$ runs in a neighbourhood of some fixed vertex $V$ of $K$, then the log-parameter for the path is the sum $\sum_E \pm \epsilon_i(E) l_E$, summed over all edges $E$ that $\gamma$ passes, where:

- $i(E)$ is the index $i$ of the simplex $\Delta_i$ that the edge $E$ belongs to and $\epsilon_i(E)$ is the corresponding coefficient $\pm 1$ from equation (6);
- the extra sign $\pm$ is $+$ or $-$ according as the edge $E$ is passed in a counterclockwise or clockwise fashion as viewed from the vertex.

**Theorem 4.2.** Choose $z \in \mathbb{H}^3$ such that $g_0^{(i)} z, g_1^{(i)} z, g_2^{(i)} z, g_3^{(i)} z$ are distinct points for each $i$. This defines an ideal hyperbolic simplex shape for each simplex $\Delta_i$ of $K$ and an associated cross ratio $x_i = [g_0^{(i)} z : g_1^{(i)} z : g_2^{(i)} z : g_3^{(i)} z]$. There is a way of assigning combinatorial flattenings $w_i = \ell(x_i; p_i, q_i)$ to the simplices of $K$ such that the parity of any normal path in $K$ is zero and the log-parameter of any normal path in any vertex neighbourhood of $K$ is zero.

For such an assignment the element $\sum_i \epsilon_i[x_i, p_i, q_i] \in \hat{\mathcal{P}}(\mathbb{C})$ is independent of choices and only depends on the original homology class $\alpha$. We denote it $\lambda(\alpha)$. Moreover, $\lambda(\alpha) \in \hat{\mathcal{B}}(\mathbb{C})$ and $\lambda: H_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \to \hat{\mathcal{B}}(\mathbb{C})$ is a homomorphism.

To prove this theorem we will need a general relation (Lemma 4.1 below) in $\hat{\mathcal{P}}(\mathbb{C})$ that follows from the lifted five term relation.

5. Consequences of the Lifted Five Term Relation

Let $K$ be a simplicial complex obtained by gluing 3-simplices $\Delta_1, \ldots, \Delta_n$ together in sequence around a common edge $E$. Thus, for each index $j$ modulo $n$, $\Delta_j$ is glued to each of $\Delta_{j-1}$ and $\Delta_{j+1}$ along one of the two faces of $\Delta_j$ incident to $E$. Suppose,
moreover, that the vertices of each $\Delta_j$ are ordered such that orderings agree on the common 2-faces of adjacent 3-simplices.

There is then a sequence $\epsilon_1 = \pm 1, \ldots, \epsilon_n = \pm 1$ such that the 2-faces used for gluing all cancel in the boundary of the 3-chain $\sum_{j=1}^n \epsilon_j \Delta_j$. (Proof: choose $\epsilon_1 = 1$ and then for $i = 2, \ldots, n$ choose $\epsilon_i$ so the common face of $\Delta_{i-1}$ and $\Delta_i$ cancels. The common face of $\Delta_n$ and $\Delta_1$ must then cancel since otherwise that face occurs with coefficient $\pm 2$ in $\partial \sum_{j=1}^n \epsilon_j \Delta_j$, and $E$ occurs with coefficient $\pm 2$ in $\partial \partial \sum_{j=1}^n \epsilon_j \Delta_j$.)

Suppose now further that a combinatorial flattening $w_j$ has been chosen for each $\Delta_j$ such that the “signed sum” of log parameters around the edge $E$ vanishes and the same for parity parameters:

$$
\sum_{j=1}^n \epsilon_j l_E(\Delta_j, w_j) = 0, \quad \sum_{j=1}^n \epsilon_j d_E(\Delta_j, w_j) = 0.
$$

We think of the edge $E$ as being vertical, so that we can label the two edges other than $E$ of the common triangle of $\Delta_j$ and $\Delta_{j+1}$ as $T_j$ and $B_j$ (for “top” and “bottom”). Let $w_j'$ be the flattening obtained from $w_j$ by adding $\epsilon_j \pi i$ to the log parameter at $T_j$ and its opposite edge and subtracting $\epsilon_j \pi i$ from the log parameter at $B_j$ and its opposite edge. If we do this for each $j$ then the total log parameter and parity parameter at any edge of the complex $K$ is not changed (we sum log-parameters with the appropriate sign $\epsilon_j$): — at $E$ no log-parameter has changed while at every other edge $\pi i$ has been added at one of the two simplices at the edge and subtracted at the other.

**Lemma 5.1.** With the above notation,

$$
\sum_{j=1}^n \epsilon_j [w_j] = \sum_{j=1}^n \epsilon_j [w_j'] \in \hat{\mathcal{P}}(\mathbb{C}),
$$

where we are using $[w]$ as a shorthand for $[\ell^{-1} w]$ (i.e., $[w]$ means $[z, p, q]$ where $\ell(z; p, q) = w$; see Lemma 3.2).

**Proof.** Each of the simplices $\Delta_i$ has an associated ideal hyperbolic structure compatible with the combinatorial flattenings $w_j$. This ideal hyperbolic structure is also compatible with the flattening $w_j'$. Choose a realization of $\Delta_1$ as an ideal simplex in $\mathbb{H}^3$. We think of this as a mapping of $\Delta_1$ to $\mathbb{H}^3$. We can extend this to a mapping of $K$ to $\mathbb{H}^3$ which maps each $\Delta_j$ to an ideal simplex with shape appropriate to its combinatorial flattening. Adjacent simplices will map to the same side of their common face in $\mathbb{H}^3$ if their orientations or the signs $\epsilon_j$ do not match and will be on opposite sides otherwise. The fact that the signed sums of log and parity parameters at edge $E$ are zero guarantees that the identifications match up as we go once around the edge $E$ of $K$.

Note that $K$ has $n + 2$ vertices. We first consider the special case that $n = 3$ and there is an ordering $v_0, \ldots, v_4$ of the five vertices of $K$ that restricts to the given vertex ordering for each simplex. We also assume the five vertices of $K$ map to distinct points $z_0, \ldots, z_4$ of $\partial \mathbb{H}^3$.

Each simplex $\Delta_j$ for $j = 1, 2, 3$ has vertices obtained by omitting one the five vertices $v_0, \ldots, v_4$. Denote by $\Delta_4$ and $\Delta_5$ the simplices obtained by omitting each of the other two vertices. The fact that the common 2-faces of the $\Delta_j$ cancel when taking
boundary of the chain $\epsilon_1 \Delta_1 + \epsilon_2 \Delta_2 + \epsilon_3 \Delta_3$ means that, up to sign this sum corresponds to three summands of the chain $\partial \langle v_0, \ldots, v_4 \rangle = \sum (-1)^j \langle v_0, \ldots, \hat{v}_i, \ldots, v_4 \rangle$.

Choose $\epsilon_4$ and $\epsilon_5$ so that $\sum_{j=1}^5 \epsilon_j \Delta_j$ is $\pm \partial \langle v_0, \ldots, v_4 \rangle$.

We now claim that we can choose unique combinatorial flattenings $w_4$ and $w_5$ of $\Delta_4$ and $\Delta_5$ so that the signed sum of log parameters and parity parameters at any edge of $K \cup \Delta_4 \cup \Delta_5$ is zero. Indeed, this claim does not depend on the order of the vertices, so by permuting the vertices we can assume the five vertices are ordered so that $\Delta_1$, $\Delta_2$ and $\Delta_3$ are the first three simplices occurring in the five term relation. Then the common edge $E$ is $v_3 v_4$ and the fact that the simplices fit together around this edge is the condition that their cross-ratio parameters $x_0$, $x_1$, and $x_2$ satisfy $x_0 x_1^{-1} x_2 = 1$. Writing the flattenings as elements of $\tilde{C}$ as $(x_0; p_0, q_0)$, $(x_1; p_1, q_1)$, and $(x_2; p_2, q_2)$, the equation saying signed sum of log parameters at this edge is zero is $(\log x_0 + p_0 \pi i) - (\log x_1 + p_1 \pi i) + (\log x_2 + p_2 \pi i) = 0$. If $x_0$, $x_1$, and $x_2$ are in the complex upper half plane this implies the equation $p_2 = p_1 - p_0$ of the lifted five term relation, while otherwise it implies the appropriate analytic continuations in $\tilde{C}$ of this. The desired choice of flattenings of $\Delta_4$ and $\Delta_5$ is thus determined as in the lifted five term relation by the choice of $p_0$, $p_1$, $q_0$, $q_1$, and $q_2$ (namely $p_3 = p_1 - p_0 + q_1 - q_0$, $q_3 = q_2 - q_1$, $p_4 = q_1 - q_0$, and $q_4 = q_2 - q_1 - p_0$ if $x_0$, $x_1$, and $x_2$ are in the upper half plane and otherwise the appropriate analytic continuation).

Note that $w_4$ and $w_5$ do not change if we replace $w_1, \ldots, w_3$ by $w'_1, \ldots, w'_3$ (with the above reordering of vertices this just subtracts 1 from each of $q_0$, $q_1$, and $q_2$ in the lifted five term relation, so it does not alter $p_3, q_3, p_4, q_4$). By Lemma 3.4 we then have

$$\epsilon_1 w_1 + \epsilon_2 w_2 + \epsilon_3 w_3 = -(\epsilon_4 w_4 + \epsilon_5 w_5)$$

$$\epsilon_1 w'_1 + \epsilon_2 w'_2 + \epsilon_3 w'_3 = -(\epsilon_4 w'_4 + \epsilon_5 w'_5),$$

proving this case.

We next consider the case that for some index $j$ modulo $n$ the images of $\Delta_j$ and $\Delta_{j+1}$ in $\mathbb{H}^3$ do not coincide, so their union has five distinct vertices. By cycling our indices we may assume $j = 1$. Since the orderings of the vertices of $\Delta_1$ and of $\Delta_2$ agree on the three vertices they have in common, there is an ordering of all five vertices compatible with both $\Delta_1$ and $\Delta_2$. Let $\Delta_0$ be the simplex determined by the common edge $E$ and the two vertices that $\Delta_1$ and $\Delta_2$ do not have in common. Then there is an $\epsilon_0 = \pm 1$ such that the common faces of $\Delta_0$, $\Delta_1$, and $\Delta_2$ cancel in the boundary of the chain $\epsilon_0 \Delta_0 + \epsilon_1 \Delta_1 + \epsilon_2 \Delta_2$. Choose a flattening $w_0$ of $\Delta_0$ such that $\epsilon_0 I_E(\Delta_0, w_0) + \epsilon_1 I_E(\Delta_1, w_1) + \epsilon_2 I_E(\Delta_2, w_2) = 0$. Then the relation of the lemma has already been proved for $w_0$, $w_1$, and $w_2$, and by subtracting this relation from the relation to be proved for $w_1, \ldots, w_n$ we obtain a case of the lemma with one less simplices. Thus, if we assume the lemma proved for $n - 1$ simplices then this case is also proved.

The above induction argument fails only for the case that there are $2m$ simplices that alternately “fold back on each other” so that their images in $\mathbb{H}^3$ all have the same four vertices. The above induction eventually reduces us to this case (usually with $m = 1$). We must therefore deal with this situation to complete the proof. We first consider the case that $m = 1$ so $n = 2$. We then have four vertices $z_0, \ldots, z_3$ in $\partial \mathbb{H}^3$. We assume the edge $E$ is $z_0 z_1$. Then the ordering of the vertices of the faces $z_0 z_3 z_2$ and $z_0 z_1 z_3$ is the same in each of $\Delta_1$ and $\Delta_2$. Choose a new point $z_4$ in $\partial \mathbb{H}^3$ distinct from $z_0, \ldots, z_3$ and consider the ordered simplex with vertices
\(z_0, z_1, z_2\) ordered as above followed by \(z_4\). Call this \(\Delta_3\). Similarly make \(\Delta_4\) using \(z_0, z_1, z_3, z_4\) ordered as above followed by \(z_4\). Choose flattenings of \(\Delta_3\) and \(\Delta_4\) so that the signed sum of log parameters for \(\Delta_1, \Delta_3, \Delta_4\) around \(E\) is zero. Then we obtain a three simplex relation of the type already proved for \(\Delta_1\) using \(\Delta_2, \Delta_3, \Delta_4\) and another for \(\Delta_2, \Delta_3, \Delta_4\), and the difference of these two relations gives the desired two-simplex relation.

More generally, if we are in the above “folded” case with \(m > 1\) we can use an instance of the three-simplex relation to replace one of the \(2^m\) simplices by two. We then use the induction step to replace one of these new simplices together with an adjacent old simplex by one simplex and then repeat for the other new simplex. In this way we reduce to a relation involving \(2m - 1\) simplices, completing the proof.

Before we continue, we note a consequence of the first case we considered in the above proof that we will need later. If vertices have been reordered as in that proof then the relation we proved can be written (with the appropriate relationship among \(p_0, p_1, p_2\)):

\[
[x, p_0, q_0] - [y, p_1, q_1] + [y/x, p_2, q_2] =
\]

\[
[x, p_0, q_0 - 1] - [y, p_1, q_1 - 1] + [y/x, p_2, q_2 - 1].
\]

This is true for any choice of \(q_0, q_1, q_2\) so long as \(p_0, p_1, p_2\) satisfy the appropriate relation. Thus if we just change \(q_0\) and subtract the resulting equation from the above we get

\[
[x, p_0, q_0] - [x, p_0, q_0'] = [x, p_0, q_0 - 1] - [x, p_0, q_0' - 1].
\]

From the versions of the above three-simplex case with different orderings of the vertices we can derive three versions of this relation:

\[
[x, p, q] - [x, p, q'] = [x, p, q - 1] - [x, p, q' - 1]
\]

\[
[x, p, q] - [x, p', q] = [x, p - 1, q] - [x, p' - 1, q]
\]

\[
[x, p, q] - [x, p + s, q - s] = [x, p + 1, q - 1] - [x, p + s + 1, q - s - 1]
\]

From these we obtain:

**Lemma 5.2.** \([x, p, q] = pq[x, 1, 1] - (pq - p)[x, 1, 0] - (pq - q)[x, 0, 1] + (pq - p - q + 1)[x, 0, 0]\).

*Proof.* The first of the relations (8) implies \([x, p, q] = [x, p, q - 1] + [x, p, 1] - [x, p, 0]\) and applying this repeatedly shows

\[
[x, p, q] = q[x, p, 1] - (q - 1)[x, p, 0].
\]

The second equation of (8) implies similarly that \([x, p, q] = p[x, 1, q] - (p - 1)[x, 0, q]\) and using this to expand each of the terms on the right of (10) gives the desired equation.

Up to this point we have not used the transfer relation (8). We digress briefly to show that the transfer relation almost follows from the five term relation.

**Proposition 5.3.** If \(\hat{\mathcal{P}}'(\mathbb{C})\) and \(\hat{\mathcal{B}}'(\mathbb{C})\) are defined like \(\hat{\mathcal{P}}(\mathbb{C})\) and \(\hat{\mathcal{B}}(\mathbb{C})\) but without the transfer relation, then in \(\hat{\mathcal{P}}'(\mathbb{C})\) the element \(\kappa := [x, 1, 1] + [x, 0, 0] - [x, 1, 0] - [x, 0, 1]\) is independent of \(x\) and has order 2. Moreover, \(\hat{\mathcal{P}}'(\mathbb{C}) = \hat{\mathcal{P}}(\mathbb{C}) \times C_2\) and \(\hat{\mathcal{B}}'(\mathbb{C}) = \hat{\mathcal{B}}(\mathbb{C}) \times C_2\), where \(C_2\) is the cyclic group of order 2 generated by \(\kappa\).
**Proof.** If we subtract equation (8) with \( p_0 = p_1 = q_0 = q_1 = q_2 = 1 \) from the same equation with \( p_0 = p_1 = 0, q_0 = q_1 = q_2 = 1 \) we obtain \([x, 1, 1] - [y, 1, 1] - [x, 0, 1] + [y, 0, 1] = [x, 1, 0] - [y, 1, 0] - [x, 0, 0] + [y, 0, 0] \), which rearranges to show that \( \kappa \) is independent of \( x \). The last of the equations (8) with \( p = 0 \) and \( s = -1 \) gives 2\([x, 0, 0] = [x, 1, -1] + [x, -1, 1] \) and expanding the right side of this using (ii) gives 2\([x, 0, 0] = -2[x, 1, 1] + 2[x, 1, 0] + 2[x, 0, 1] \), showing that \( \kappa \) has order dividing 2.

To show \( \kappa \) has order exactly 2 we note that there is a homomorphism \( \epsilon : \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{Z}/2 \) defined on generators by \([z, p, q] \to (pq \mod 2) \). Indeed, it is easy to check that this vanishes on the lifted five-term relation, and is thus well defined on \( \hat{\mathcal{P}}(\mathbb{C}) \).

Since \( \epsilon(\kappa) = 1 \) we see \( \kappa \) is non-trivial. Finally, Lemma 5.2 implies that the effect of the transfer relation is simply to kill the element \( \kappa \), so the final sentence of the proposition follows.

**Lemma 5.4.** For any \([x, p, q] \in \hat{\mathcal{P}}(\mathbb{C}) \) one has \([x, p, q] + [1 - x, -q, -p] = 2[1/2, 0, 0] \).

**Proof.** Assume first that \( 0 < y < x < 1 \). Then, as remarked in the proof of Proposition 2.3,

\[
[x, p_0, q_0] - [y, p_1, q_1] + \left[\frac{y}{x}, p_1 - p_0, q_2\right] - \left[\frac{1 - x}{1 - y}, p_1 - p_0 + q_1 - q_0, q_2 - q_1\right] + \left[\frac{1 - x}{1 - y}, q_1 - q_0, q_2 - q_1 - p_0\right] = 0
\]

is an instance of the lifted five term relation. Replacing \( y \) by \( 1 - x \), \( x \) by \( 1 - y \), \( p_0 \) by \( -q_1, q_1 \) by \( -q_0, q_0 \) by \( -p_1, q_1 \) by \( -p_0 \), and \( q_2 \) by \( q_2 - q_1 - p_0 \) replaces this relation by exactly the same relation except that the first two terms are replaced by \([1 - y, -q_1, -p_1] - [1 - x, -q_0, -p_0]\). Thus subtracting the two relations gives:

\[
[x, p_0, q_0] - [y, p_1, q_1] + [1 - y, -q_1, -p_1] + [1 - x, -q_0, -p_0] = 0.
\]

Putting \([y, p_1, q_1] = [1/2, 0, 0]\) now proves the lemma for \(1/2 < x < 1\). But since we have shown this as a consequence of the lifted five term relation, we can analytically continue it over the whole of \( \hat{\mathbb{C}} \).

**Proposition 5.5.** The following sequence is exact:

\[
0 \to \mathbb{C}^* \xrightarrow{\chi} \hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}) \to 0
\]

where \( \hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}) \) is the natural map and \( \chi(z) := [z, 0, 1] - [z, 0, 0] \) for \( z \in \mathbb{C}^* \).

**Proof.** Denote \( \{z, p\} := [z, p, q] - [z, p, q - 1] \) which is independent of \( q \) by the first equation of (9). By Lemma 5.2 we have \([z, p, q] - [z, p - 1, q] = -\{1 - z, -q\}\). It follows that elements of the form \( \{z, p\} \) generate \( \text{Ker}(\hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \). Computing \( \{z, p\} \) using Lemma 5.2 and the transfer relation, one finds \( \{z, 0\} = \{z, 0\} \) which only depends on \( z \). Thus the elements \( \{z, 0\} = \chi(z) \) generate \( \text{Ker}(\hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \).

If we take equation (9) with even \( p_i \) and subtract the same equation with the \( q_i \) reduced by 1 we get an equation that says that \( \chi : \mathbb{C}^* \to \text{Ker}(\mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \) is a homomorphism. We have just shown that it is surjective, and it is injective because \( R \circ \chi \) is the map \( \mathbb{C}^* \to \mathbb{C}/\pi^2 \) defined by \( z \mapsto \frac{\pi}{\pi} \log z \).

We can now describe the relationship of our extended groups with the “classical” ones.
Theorem 5.6. There is a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
\downarrow & & & & \\
0 & \mu^* & \mathbb{C}^* & \mathbb{C}^*/\mu^* & 0 \\
\chi|\mu^* & \chi & \beta & & \\
\Downarrow & \Downarrow & \Downarrow & & \\
\hat{B}(\mathbb{C}) & \hat{P}(\mathbb{C}) & \nu & \mathbb{C} \wedge \mathbb{C} & K_2(\mathbb{C}) & 0 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & & \\
\hat{B}(\mathbb{C}) & \hat{P}(\mathbb{C}) & \nu' & \mathbb{C}^* \wedge \mathbb{C}^* & K_2(\mathbb{C}) & 0 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & & \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Here \(\mu^*\) is the group of roots of unity and the labelled maps defined as follows:

\[
\begin{align*}
\chi(z) &= [z, 0, 1] - [z, 0, 0] \in \hat{P}(\mathbb{C}); \\
\nu[z, p, q] &= (\log z + p\pi i) \wedge (-\log(1 - z) + q\pi i); \\
\nu'[z] &= 2(z \wedge (1 - z)); \\
\beta[z] &= \log z \wedge \pi i; \\
\epsilon(w_1 \wedge w_2) &= -2(e^{w_1} \wedge e^{w_2});
\end{align*}
\]

and the unlabelled maps are the obvious ones.

Proof. The top horizontal sequence is trivially exact while the other two are exact at their first two non-trivial groups by definition of \(\hat{B}\) and \(B\). The bottom row is exact also at its other two places by Milnor’s definition of \(K_2\). The exactness of the third vertical sequence is elementary and the second one has just been proved. The commutativity of all but the top left square is elementary. A diagram chase confirms that \(\chi\) maps \(\mu^*\) to \(\hat{B}(\mathbb{C})\) and that the left vertical sequence is also exact. Another confirms exactness of the middle row. \(\square\)

6. Proof of Theorem 4.2

We must first recall some notation and results from \[9\].

The complex \(K\) of Theorem 4.2 is what is called an “oriented 3-cycle” in \[8\]. That is, it is a finite quasi-simplicial 3-complex such that the complement \(K - K^{(0)}\) of the vertices is an oriented 3-manifold. (“Quasi-simplicial” means that \(K\) is a CW-complex obtained by gluing together simplices along their faces in such a way that the interior of each face of each simplex injects into the resulting complex.) Each simplex \(\Delta_i\) of \(K\) has an orientation compatible with the ordering of its vertices, and this orientation is compatible with or opposite to the orientation of \(K\) according as \(\epsilon_i\) is \(+1\) or \(-1\).

To an oriented 3-simplex \(\Delta\) of \(K\) we associate a 2-dimensional bilinear space \(J_\Delta\) over \(\mathbb{Z}\) as follows. As a \(\mathbb{Z}\)-module \(J_\Delta\) is generated by the six edges \(e_0, \ldots, e_5\) of \(\Delta\)
(see Fig. 3) with the relations:

\[ e_i - e_{i+3} = 0 \quad \text{for } i = 0, 1, 2. \]
\[ e_0 + e_1 + e_2 = 0. \]

Thus, opposite edges of \( \Delta \) represent the same element of \( J_\Delta \), so \( J_\Delta \) has three “geometric” generators, and the sum of these three generators is zero. The bilinear form on \( J_\Delta \) is the non-singular skew-symmetric form given by

\[ \langle e_0, e_1 \rangle = \langle e_1, e_2 \rangle = \langle e_2, e_0 \rangle = -\langle e_1, e_0 \rangle = -\langle e_3, e_1 \rangle = -\langle e_0, e_2 \rangle = 1. \]

Let \( J \) be the direct sum \( \coprod J_\Delta \), summed over the oriented 3-simplices of \( K \). For \( i = 0, 1 \) let \( C_i \) be the free \( \mathbb{Z} \)-module on the unoriented \( i \)-simplices of \( K \). Define homomorphisms

\[ \alpha: C_0 \rightarrow C_1 \quad \text{and} \quad \beta: C_1 \rightarrow J \]

as follows. \( \alpha \) takes a vertex to the sum of the incident edges (with an edge counted twice if both endpoints are at the given vertex). The \( J_\Delta \) component of \( \beta \) takes an edge \( E \) of \( K \) to the sum of those edges \( e_i \) in the edge set \( \{ e_0, e_1, \ldots, e_5 \} \) of \( \Delta \) which are identified with \( E \) in \( K \).

The natural basis of \( C_i \) gives an identification of \( C_i \) with its dual space and the bilinear form on \( J \) gives an identification of \( J \) with its dual space. With respect to these identifications, the dual map

\[ \alpha^*: C_1 \rightarrow C_0 \]

is easily seen to map an edge \( E \) of \( K \) to the sum of its endpoints, and the dual map

\[ \beta^*: J \rightarrow C_1 \]

can be described as follows. To each 3-simplex \( \Delta \) of \( K \) we have a map \( j = j_\Delta \) of the edge set \( \{ e_0, e_1, \ldots, e_5 \} \) of \( \Delta \) to the set of edges of \( K \); put \( j(e_i) \) equal to the edge that \( e_i \) is identified with in \( K \). For \( e_i \) in \( J_\Delta \) we have

\[ \beta^*(e_i) = j(e_{i+1}) - j(e_{i+2}) + j(e_{i+4}) - j(e_{i+5}) \quad (\text{indices mod } 6). \]

Let \( K_0 \) be the result of removing a small open cone neighborhood of each vertex \( V \) of \( K \), so \( \partial K_0 \) is the disjoint union of the links \( L_V \) of the vertices of \( K \).
Theorem 6.1 ([3], Theorem 4.2). The sequence
\[ J : 0 \to C_0 \xrightarrow{\alpha} C_1 \xrightarrow{\beta} J \xrightarrow{\beta^*} C_1 \xrightarrow{\alpha^*} C_0 \to 0 \]
is a chain complex. Its homology groups \( H_i(J) \) (indexing the non-zero groups of \( J \) from left to right with indices 5, 4, 3, 2, 1) are

\[
\begin{align*}
H_5(J) &= 0, & H_4(J) &= \mathbb{Z}/2, & H_1(J) &= \mathbb{Z}/2, \\
H_3(J) &= \mathcal{H} \oplus H^1(K; \mathbb{Z}/2), & H_2(J) &= H_1(K; \mathbb{Z}/2),
\end{align*}
\]

where \( \mathcal{H} = \text{Ker}(H_1(\partial K_0; \mathbb{Z}) \to H_2(K_0; \mathbb{Z}/2)) \). Moreover, the isomorphism \( H_2(J) \to H_1(K; \mathbb{Z}/2) \) results by interpreting an element of \( \text{Ker}(\alpha^*) \subset C_1 \) as an unoriented 1-cycle in \( K \). □

If \( \Delta \) is an ideal simplex and \( \mathbf{w} = (w_0, w_1, w_2) \) is a flattening of it, then denote
\[ \xi(\mathbf{w}) := w_1e_0 - w_0e_1 \in J_\Delta \otimes \mathbb{C}. \]
This definition is only apparently unsymmetric since \( w_1e_0 - w_0e_1 = w_2e_1 - w_1e_2 = w_0e_2 - w_2e_0 \).

Give each simplex \( \Delta_i \) of the complex \( K \) of Theorem [4.2] the flattening \( \mathbf{w}_i^{(0)} := \ell(x_i; 0, 0) \). Denote by \( \omega \) the element of \( J \otimes \mathbb{C} \) whose \( \Delta_i \)-component is \( \xi(\mathbf{w}_i^{(0)}) \) for each \( i \). That is, the \( \Delta_i \)-component of \( \omega \) is \( - (\log(1 - x_i)e_0 + \log(x_i)e_1) \).

Lemma 6.2. \( \frac{1}{\pi i} \beta^*(\omega) \) is an integer class in the kernel of \( \alpha^* \), so it represents an element of the homology group \( H_2(J) \). Moreover this element in \( H_2(J) \) vanishes, so \( \frac{1}{\pi i} \beta^*(\omega) = \beta^*(x) \) for some \( x \in J \).

Proof. Let \( J_\Delta \) be defined like \( J_\Delta \) but without the relation \( e_0 + e_1 + e_2 = 0 \), so it is generated by the six edges \( e_0, \ldots, e_5 \) of \( \Delta \) with relations \( e_i = e_{i+3} \) for \( i = 0, 1, 2 \). Let \( \overline{J} \) be the direct sum \( \bigsqcup J_\Delta \) over 3-simplices \( \Delta \) of \( K \).

The map \( \beta^*: J \to C_2 \) factors as
\[ \beta^*: J \xrightarrow{\beta_1} \overline{J} \xrightarrow{\beta_2} C_2, \]
with \( \beta_1 \) and \( \beta_2 \) defined on each component by:
\[
\begin{align*}
\beta_1(e_i) &= e_{i+1} - e_{i+2} \\
\beta_2(e_i) &= j(e_i) + j(e_{i+3})
\end{align*}
\]
for \( i = 0, 1, 2 \).

Note that \( \beta_1(\xi(\mathbf{w})) = w_1e_0 + w_1e_1 + w_2e_2 \in \overline{J}_\Delta \otimes \mathbb{C} \). Thus if \( E \) is an edge of \( K \) then the \( E \)-component of \( \beta^*(\omega) = \beta_2\beta_1(\omega) \) is the sum of the log-parameters for \( E \) in the ideal simplices of \( K \) around \( E \) and is hence a multiple of \( \pi i \). In fact, it would be a multiple of \( 2\pi i \) except for the adjustments by multiples of \( \pi i \) that are involved in forming the log-parameters from the logarithms of cross-ratio parameters of the simplices. We claim that these adjustments add to an even multiple of \( \pi i \) around each edge, so in fact
\[
\frac{1}{\pi i} \beta^*(\omega) \in 2C_2,
\]
Once this is proved the lemma follows, since the isomorphism \( H_2(J) \to H_1(K; \mathbb{Z}/2) \) is the map which interprets an element of \( \text{Ker}(\alpha^*) \) as an unoriented 1-cycle in \( K \), and equation (11) says this 1-cycle is zero modulo 2.

In the terminology of Definition [3,1] our claim can be restated that the parity of the normal path that circles the edge \( E \) is zero for each edge \( E \). In fact, the parity
of any normal path in $K$ is zero. To see this, note that as we follow a normal path
the contribution to the parity as we pass an edge of a simplex is 0 if the edge is the
01, 03, 12, or 23 edge of the simplex and the contribution is ±1 if it is the 02 or
13 edge. Consider the orientations of the triangular faces we cross as we follow the
path, where the orientation is the one induced by the ordering of its vertices. As
we pass a 02 or 13 edge this orientation changes while for the other edges it does
not. Since $K$ is oriented, we must have an even number of orientation changes as
we traverse the normal path, proving the claim. (This argument, which simplifies
my original one, is due to Brian Bowditch.)

Let $\omega := \omega - \pi ix \in J \otimes \mathbb{C}$ with $x$ as in the lemma, so $\beta^*(\omega') = 0$. The $\Delta_i$-
component of $\omega'$ is $\xi(w_i)$, where $w_i = l(x_i; p_i, q_i)$ for suitable integers $p_i, q_i$. These
integers $p_i$ and $q_i$ are the coefficients occurring in the element $x \in J$, which is only
determined by the lemma up to elements of $\text{Ker}(\beta^*)$. We claim that for suitable
choice of $x$, the $w_i$ satisfy the parity and log-parameter conditions of first para-
graph of Theorem 4.2. To see this we need to review a computation of $H_3(J)$ from [9].

We define a map $\gamma': H_3(J) \to H^1(\partial K_0; \mathbb{Z}) = \text{Hom}(H_1(\partial K_0), \mathbb{Z})$ as follows.
Given elements $a \in H_3(J)$ and $c \in H_1(\partial K_0)$ we wish to define $\gamma'(a)(c)$. It is
easy to do this for a class $c$ which is represented by a normal path $C$ in the
link of some vertex of $K$. Represent $a$ by an element $A \in J$ with $\beta^*(A) = 0$ and
consider the element $\beta_1(A) \in \overline{J}$. This element has a coefficient for each edge of each
simplex of $K$. To define $\gamma'(a)(c)$ we consider the coefficients of $\beta_1(A)$ corre-
sponding to edges of simplices that $C$ passes and sum these using the orientation conventions
of Definition 4.1. It is easy to see that the result only depends on the homology
class of $C$.

We can similarly define a map $\gamma_2': H_3(J) \to H^1(K_0; \mathbb{Z}/2) = \text{Hom}(H_1(K_0), \mathbb{Z}/2)$
by using normal paths in $K_0$ and taking modulo 2 sum of coefficients of $\beta_1(A)$.

**Lemma 6.3 ([9], Theorem 5.1).** The sequence

$$0 \to H_3(J) \xrightarrow{(\gamma' \gamma_2')} H^1(\partial K_0; \mathbb{Z}) \oplus H^1(K_0; \mathbb{Z}/2) \xrightarrow{i - i^*} H^1(\partial K_0; \mathbb{Z}/2) \to 0$$

is exact, where $r: H^1(\partial K_0; \mathbb{Z}) \to H^1(\partial K_0; \mathbb{Z}/2)$ is the coefficient map and the map
$i^*: H^1(K_0; \mathbb{Z}/2) \to H^1(\partial K_0; \mathbb{Z}/2)$ is induced by the inclusion $\partial K_0 \to K_0$.

Returning to the choice of $x$ above, assume we have made a choice so that the
resulting flattenings $w_i$ do not lead to zero log-parameters and parities for normal
paths. Taking $\frac{1}{\gamma_1}$ times the log-parameters of normal paths leads as above to an
element $c \in H^1(\partial K_0; \mathbb{Z})$. Similarly, parities of normal paths leads to an element
of $c_2 \in H^1(K_0; \mathbb{Z}/2)$. These elements satisfy $r(c) = i^*(c_2)$. The lemma thus gives
an element of $H_3(J)$ that maps to $(c, c_2)$, and subtracting a representative for this
element from $x$ gives the desired correction of $x$ so the log-parameters and parities
of normal paths with respect to the corresponding changed $w_i$’s are zero. This
completes the proof of the first paragraph of Theorem 4.2.

Suppose now that we have a different choice of flattenings $w_i$ satisfying the parity
and log-parameter conditions of the Theorem. Then the above lemma implies that
the difference between the corresponding elements $x$ represents $0$ in $H_3(J)$ and is
thus in the image of $\beta$. For $E \in C_2$ the effect of changing $x$ by $\beta(E)$ is to change
the element $\sum \epsilon_i[x_i, p_i, q_i] \in \mathcal{P}($ by the corresponding relation of Lemma 5.1.
Since this is a consequence of the lifted five term relations, the element in $\mathcal{P}($ is unchanged.
Finally we need to show that none of the other choices we made in defining the element \( \lambda(\alpha) \) in Theorem 4.3 have an effect. These were:

- the representative of the homology class \( \alpha \);
- the choice of pairing of faces of simplices of \( \alpha \) to form the complex \( K \);
- the choice of the point \( z \in \partial K \) in Theorem 4.2.

To prove this we will use a bordism theory based on \( n \)-cycles which we describe next.

An ordered \( n \)-cycle with boundary is defined by the following:

- a finite collection of ordered \( n \)-simplices \( \Delta_i \) is given;
- each simplex has a sign \( \epsilon_i = \pm 1 \);
- the \((n-1)\)-faces of the simplices \( \Delta_i \) are given signs as follows: the sign of the \((n-1)\)-face that omits the \( j \)-th vertex of \( \Delta_i \) is \((-1)^j \epsilon_i \);
- a subset of the \( D \) of the \((n-1)\)-faces of the collection of simplices is given;
- the \((n-1)\) faces which are not in \( D \) are paired in such a way the the two simplices in each pair have opposite sign.

We can form a geometric \( n \)-complex \( K \) by realizing the \( \Delta_i \) by geometric \( n \)-simplices and gluing (by affine isomorphisms that respect the vertex orderings) faces that are paired. The result is a space that is an oriented \( n \)-manifold with boundary in the complement of its \((n-2)\)-skeleton. We shall use the same letter \( K \) to denote the underlying combinatorial object and the geometric realization.

If \( K \) is an ordered \( n \)-cycle with boundary as above, we say it is closed if \( D \) is empty. If \( K \) is not closed, then \( D \) inherits the structure of a closed ordered \((n-1)\)-cycle, and we denote this structure by \( \partial K \). The pairing on the faces of the simplices of \( D \) can be described as follows. Consider the set of pairs \((\sigma, \tau)\) consisting of an \((n-2)\)-face \( \tau \) of an \((n-1)\)-face \( \sigma \) of a simplex of \( K \). Construct a graph with this set as vertex set and with edges of two types: \((\sigma, \tau)\) is connected to \((\sigma', \tau')\) by an edge if either \( \sigma \) and \( \sigma' \) are adjacent faces of an \( n \)-simplex of \( K \) and \( \tau = \tau' \) is their common \((n-2)\) face, or if \( \sigma \) and \( \sigma' \) are paired \((n-1)\)-simplices and \( \tau \) and \( \tau' \) are corresponding faces under an order-preserving matching of \( \sigma \) with \( \sigma' \). Each vertex \((\sigma, \tau)\) of this graph has valency 2 except when \( \sigma \) is in \( D \), in which case the valency is 1. Thus the graph consists of a collection of arcs and cycles. The endpoints of each arc gives two \((n-2)\)-faces of elements of \( D \) that are to be paired.

### 7. The true extended Bloch group

Recall that \( \widehat{C} \) consists of four components \( X_{00}, X_{01}, X_{10}, \) and \( X_{11} \), of which \( X_{00} \) is naturally the universal abelian cover of \( C - \{0, 1\} \).

Let \( \widehat{FT}_{00} \) be \( \widehat{FT} \cap (X_{00})^3 \), so \( \widehat{FT}_{00} = \widehat{FT}_0 + 2V \) in the notation of Definition 2.2.

As mentioned earlier, \( \widehat{FT}_0 \) is, in fact, equal to \( \widehat{FT}_0 \), but we do not need this.

Define \( \mathcal{E}\mathcal{P}(C) \) to be the free \( \mathbb{Z} \)-module on \( X_{00} \) factored by all instances of the relation

\[
(12) \quad \sum_{i=0}^{4} (-1)^i (x_i; 2p_i, 2q_i) = 0 \quad \text{with} \quad ((x_0; 2p_0, 2q_0), \ldots, (x_4; 2p_4, 2q_4)) \in \widehat{FT}_{00}.
\]

As before, we have a well-defined map

\[
\nu: \mathcal{E}\mathcal{P}(C) \rightarrow C \wedge C,
\]
given by \( \nu[z; 2p, 2q] = (\log z + 2\pi i) \wedge (\log(1 - z) + 2q\pi i) \), and we define
\[
(13) \qquad \mathcal{EB}(\mathbb{C}) := \text{Ker} \nu.
\]

The proof of Proposition 2.5 shows:

**Proposition 7.1.** The function \( R(z; 2p, 2q) := \mathcal{R}(z) + \pi i (p \log(1 - z) + q \log z) - \frac{\pi^2}{6} \)
gives a well defined map \( X_{00} \to \mathbb{C}/2\pi^2 \mathbb{Z} \) and induces a homomorphism \( \mathcal{EP}(\mathbb{C}) \to \mathbb{C}/2\pi^2 \mathbb{Z} \).

We can repeat the computations in Sect. 5 word-for-word, replacing anything of the form \([x, p, q]\) by \([x, 2p, 2q]\), to show that \( \text{Ker}(\mathcal{EP}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})) \) is generated by elements of the form \( \hat{\chi}(z) := [z, 0, 2] - [z, 0, 0] \). We thus get:

**Proposition 7.2.** The following sequence is exact:
\[
0 \to \mathbb{C}^* \xrightarrow{\hat{x}} \mathcal{EP}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}) \to 0
\]
where \( \hat{\mathcal{P}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}) \) is the natural map and \( \hat{\chi}(z) := [z, 0, 2] - [z, 0, 0] \) for \( z \in \mathbb{C}^* \).

**Proof.** The only thing to prove is the injectivity of \( \hat{x} \) which follows by noting that \( R(\hat{\chi}(z)) = \pi i \log z \in \mathbb{C}/2\pi^2 \).

**Corollary 7.3.** We have a commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & & & & & & \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
0 & \longrightarrow & \mathbb{C}^* & \xrightarrow{\hat{x}} & \mathcal{EP}(\mathbb{C}) & \longrightarrow & \mathcal{P}(\mathbb{C}) & \longrightarrow & 0 \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
& & \longrightarrow & & & & \longrightarrow & \longrightarrow & \\
0 & \longrightarrow & \mathbb{C}^* & \xrightarrow{x} & \hat{\mathcal{P}}(\mathbb{C}) & \longrightarrow & \mathcal{P}(\mathbb{C}) & \longrightarrow & 0 \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
& & \longrightarrow & & & & & & \\
0 & 0 & & & & & & & \\
\end{array}
\]

and analogously for the Bloch group:

\[
\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & & & & & & \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
0 & \longrightarrow & \mu^* & \xrightarrow{\hat{x}} & \mathcal{EB}(\mathbb{C}) & \longrightarrow & \mathcal{B}(\mathbb{C}) & \longrightarrow & 0 \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
& & \longrightarrow & & & & \longrightarrow & \longrightarrow & \\
0 & \longrightarrow & \mu^* & \xrightarrow{x} & \hat{\mathcal{B}}(\mathbb{C}) & \longrightarrow & \mathcal{B}(\mathbb{C}) & \longrightarrow & 0 \\
& & & & & & & & \\
& & \downarrow & & & & & & \\
& & \longrightarrow & & & & & & \\
0 & 0 & & & & & & & \\
\end{array}
\]
References

[1] S. Bloch, Higher regulators, algebraic K-theory, and zeta functions of elliptic curves, Lecture notes U.C. Irvine (1978).
[2] D. Coulson, O.A. Goodman, C.D. Hodgson W.D. Neumann, Computing arithmetic invariants of 3-manifolds, to appear.
[3] S. Chern, J. Simons, Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, Proc. Nat. Acad. Sci. U.S.A. 68 (1971), 791-794.
[4] J. Cheeger and J. Simons, Differential characters and geometric invariants, Springer Lect. Notes in Math. 1167 (1985), 50–80.
[5] J. Dupont, The dilogarithm as a characteristic class for flat bundles, J. Pure and App. Algebra 44 (1987), 137–164.
[6] J. Dupont, Algebra of polytopes and homology of flag complexes, Osaka J. Math. 19 (1982), 599–641.
[7] J. Dupont, H. Sah, Scissors congruences II, J. Pure and App. Algebra 25 (1982), 159–195.
[8] R. Meyerhoff and W. Neumann, An asymptotic formula for the \( \eta \)-invariant of hyperbolic 3-manifolds, Comment. Math. Helvetici 67 (1992), 28–46.
[9] W. D. Neumann, Combinatorics of triangulations and the Chern Simons invariant for hyperbolic 3-manifolds, in Topology 90, Proceedings of the Research Semester in Low Dimensional Topology at Ohio State (Walter de Gruyter Verlag, Berlin - New York 1992), 243–272.
[10] W. D. Neumann, Hilbert’s 3rd problem and invariants of 3-manifolds, to appear in the Epstein 60th Birthday volume of Geometry and Topology.
[11] W. D. Neumann, J. Yang, Problems for K-theory and Chern-Simons Invariants of Hyperbolic 3-Manifolds, L’Enseignement Mathématique 41 (1995), 281–296.
[12] W. D. Neumann, J. Yang, Invariants from triangulation for hyperbolic 3-manifolds, Electronic Research Announcements of the Amer. Math. Soc. 1 (2) (1995), 72–79.
[13] W. D. Neumann, J. Yang, Bloch invariants of hyperbolic 3-manifolds, preprint available from http://www.maths.mu.oz.au/~neumann
[14] W. D. Neumann, D. Zagier, Volumes of hyperbolic 3-manifolds, Topology 24 (1985), 307-332.
[15] C. S. Sah, Scissors congruences, I, Gauss-Bonnet map, Math. Scand. 49 (1982) 181–210.
[16] A. A. Suslin, Algebraic K-theory of fields, Proc. Int. Cong. Math. Berkeley 86, vol. 1 (1987), 222–244.
[17] T. Yoshida, The \( \eta \)-invariant of hyperbolic 3-manifolds, Invent. Math. 81 (1985), 473–514
[18] D. Zagier, The Bloch-Wigner-Ramakrishnan polylogarithm function, Math. Annalen 286 (1990), 613–624.

Department of Mathematics, The University of Melbourne, Carlton, Vic 3052, Australia

E-mail address: neumann@maths.mu.oz.au