Critical graph of a polynomial quadratic differential related to a Schrödinger equation with quartic potential

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Abstract
In this paper, we study the weak asymptotic in the \( \mathbb{C} \)-plane of some wave functions resulting from the WKB-techniques applied to a Schrödinger equation with quartic oscillator and having some boundary condition. As a first step, we make transformations of our problem to obtain a Heun equation satisfied by the polynomial part of the WKB wave functions. Especially, we investigate the properties of the Cauchy transform of the root counting measure of re-scaled solutions of the Schrödinger equation, to obtain a quadratic algebraic equation of the form
\[
C(z)^2 + r(z)C(z) + s(z) = 0,
\]
where \( r, s \) are also polynomials. As a second step, we discuss the existence of solutions (in the form of Cauchy transform of a signed measure) of this algebraic equation. It remains to describe the critical graph of a related quadratic differential \( -p(z)dz^2 \) where \( p(z) \) is a quartic polynomial. In particular, we discuss the existence (and their number) of finite critical trajectories of this quadratic differential.

Keywords Quantum mechanics · WKB-analysis · Quadratic differentials · Cauchy transform

Mathematics Subject Classification 30C15 · 31A35 · 34E05

1 Introduction

Quadratic differentials appear in different mathematical and mathematical-physical domains, such as, orthogonal polynomials, potential theory, ordinary differential equations, quantum mechanics, moduli spaces of curves, etc.
Recently, quadratic differentials have provided an important tool in the asymptotic study of some solutions of differential equations.

In quantum mechanics, trajectories of some quadratic differentials play crucial role in the WKB-analysis. More precisely, consider the time-independent Schrödinger equation on the complex plane $\mathbb{C}$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(z, \hbar)}{dz^2} + V(z, \hbar)\psi(x, \hbar) = E\psi(z, \hbar),$$  \hspace{1cm} (1)

where $\hbar$ denotes the Planck constant, $z$ is the local coordinate, $\psi$ is the wave function, $V(z, \hbar)$ is the potential, and $E$ is the energy of a system of mass $m$. The determination of an explicit solution of Eq. (1) is difficult in general, but a general solution can be written as a linear combination of particular solutions. The series expansion in $\hbar^{-1}$ of $\hbar^{-\frac{2}{2}}V(z, \hbar)$ provides a principal part $V_0$ which defines a meromorphic quadratic differential $\phi$ in the complex plane. The critical graph of $\phi$ (i.e. the closure of the union of critical trajectories of $\phi$), called also Stokes graph, is important for the WKB-analysis which gives an important method to determine particular solutions of the Schrödinger equation. General study can be found in [15] and [23], where the authors considered the “classical” case $\hbar^{-2}V(z, \hbar) = V_0(z)$. The classical Schrödinger equation is

$$H\psi(z) = E\psi(z),$$

where

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z)$$

is the Hamiltonian. By definition, $\psi$ is an eigenfunction of the operator $H$. In quantum mechanics, $H$ is generally supposed to be Hermitian; in fact this condition guarantees a crucial property that the energy levels are real. Clearly these energy levels are the eigenvalues of $H$. It is known that the Hermitian property condition of $H$ is sufficient but not necessary; see [4]. In fact, under the PT-symmetry condition (space-time reflection) Bender has shown in [3] that the hamiltonian $H = -\frac{d^2}{dx^2} + ix^3$ (which is not Hermitian) also has a real spectrum. PT-symmetry condition allows to study many new Hamiltonians that were discarded before.

The PT-symmetry condition has its origin in the numerical and asymptotic studies of the spectrum of operators $H$, when $V$ is polynomial. Bender and Boettcher conjectured in [5] that when $V(z) = (iz)^m + \alpha z^2$ ($\alpha \in \mathbb{R}$), the eigenvalues of $H$ are all real and positive. Many works supported this conjecture, see [6,12].

The first rigorous proof of reality and positivity of the eigenvalues of some non-self-adjoint hamiltonian $H$ was given by Dorey et al. [11]. However, there are some PT-symmetric hamiltonians with polynomial potentials producing non-real eigenvalues, see [10]. This PT-symmetric condition of the hamiltonian with polynomial potentials can be expressed by

$$V(-\overline{z}) = V(z), \ z \in \mathbb{C}. \hspace{1cm} (2)$$
In the present paper, we focus on the case of a quartic (i.e. \( V \) is a polynomial of degree 4) PT-symmetric hamiltonian and we study the weak asymptotic of its spectrum. Starting from a Schrödinger differential equation and using the Cauchy transform defined in (14) we get algebraic equation (7), which gives rise to a particular quadratic differential. In other words,

\[
\text{Schrödinger equation} \Rightarrow \text{Algebraic equation} \Rightarrow \text{Quadratic differential}.
\]

More precisely, we consider the eigenvalue problem

\[
\begin{cases}
-y'' + (t^4 + 2bm^2 t^2 + 2mit)y = \lambda y, \\
y \left( te^{-i\left(\frac{7\pi}{3}\right)} \right) \rightarrow 0, t \rightarrow \infty,
\end{cases}
\]

where \( b \) is real, \( m \) is integer and we choose \( \hbar = 1 \); see [4,13,19]. This problem is quasi-exactly solvable, i.e. for each \( b \), there are \( m \) eigenvalues \( \lambda_{m,k} \), with elementary eigenfunctions

\[
y_{m,k}(t) = p_{m,k}(t) e^{-i\frac{3\pi}{3} - ibm^2 t},
\]

and \( p_{m,k} \) are polynomials of degree \( m - 1 \); \( k = 1, \ldots, m \).

Under condition (2), the potential

\[
V(t) = t^4 + 2bm^2 t^2 + 2mit
\]

is PT-symmetric. Using \( z = it \), we obtain the following system

\[
\begin{cases}
-y'' + (z^4 - 2bm^2 z^2 + 2mz)y = \lambda y, \\
y \left( te^{\pm i\frac{\pi}{3}} \right) \rightarrow 0, t \rightarrow \infty,
\end{cases}
\]

with \( y \in \left\{ pe^{\pm z^3 - bm^2 z}; \deg p \leq m - 1 \right\} \).

To simplify the notation, we set \( n = m - 1 \) and denote \( y_{m,m-1} \) by \( y_n \) and \( p_{m,m-1} \) by \( p_n \). Substituting in (4), we obtain

\[
p_n'' - 2 \left[ z^2 - b(n+1)^{\frac{2}{3}} \right] p_n' + \left[ 2n z - (\lambda_n + b^2(n+1)^{\frac{4}{3}}) \right] p_n = 0.
\]

It’s clear that \( p_n \) are also the eigenfunctions of the operator

\[
T_n(y) = y'' - 2 \left[ z^2 - b(n+1)^{\frac{2}{3}} \right] y' + 2nzy,
\]

associated to the eigenvalues \( \beta_n = \lambda_n + b^2(n+1)^{\frac{4}{3}} \).
Let us denote by $C_{\nu_n}$ the Cauchy transform of the normalized root-counting measure $\nu_n$ of $p_n$. It is straightforward from [20, Theorem 29.1] that the eigenvalue problem (3) has infinitely many eigenvalues $\lambda_n$ tending to infinity, with the asymptotic expansion

$$\lambda_n = n^{4/3} [c + o(1)], \ n \to \infty,$$

for some positive real $c$. In order to study the asymptotic of the sequence $\{\nu_n\}$, we shall consider $q_n(z) = p_n(n^\varepsilon z)$ a re-scaled polynomial of $p_n$ and

$$\rho_n(z) = \frac{q'_n(z)}{n q_n(z)} = n^\varepsilon C_{\nu_n}(z).$$

Substituting in (5), we get that

$$C_{\nu_n}^2(z) - 2 \left( \frac{z^2 - b(n+1)^{2/3}}{n} \right) C_{\nu_n}(z) + \left( \frac{2z}{n} - \frac{\beta_n}{n^2} \right) + \frac{C'_{\nu_n}(z)}{n} = 0,$$

and

$$\rho_n^2(z) - 2 \left( z^2 n^{\varepsilon-1} - b(n+1)^{2/3} n^{\varepsilon-1} \right) \rho_n(z) + (2zn^{\varepsilon-1} - \beta_n n^{2\varepsilon-2}) + \frac{\rho'_n(z)}{n} = 0.$$

In particular, for $\varepsilon = \frac{1}{3}$, we get

$$\rho_n^2(z) - 2 \left( z^2 - b(n+1)^{2/3} n^{\frac{2}{3}} \right) \rho_n(z) + \left( 2z - \beta n^{\frac{4}{3}} \right) + \frac{\rho'_n(z)}{n} = 0.$$

It follows from (6), that the sequence $\{\beta_n/n^{4/3}\}$ is bounded, (see also [19]). By the Helly selection theorem, we can choose a subsequence such that

$$\lim_{n \to \infty} \beta_n/n^{4/3} = \beta, \ \beta \in \mathbb{C}.$$ 

Then, there exists a compactly-supported positive measure $\nu$ such that

$$\lim_{n \to \infty} \nu_n = \nu, \ \lim_{n \to \infty} \rho_n = C.$$

Finally, we obtain the algebraic equation:

$$C^2 - 2(z^2 - b)C + (2z - \beta) = 0. \quad (7)$$

Notice that if we start with the eigenvalue problem

$$\begin{cases}
    -y'' + (t^4 + 2bm t^2 + 2mit)y = \lambda y \\
    y(t e^{-i(\frac{\pi}{6} \pm \frac{\pi}{3})}) \to 0, \ t \to \infty,
\end{cases}$$

then...
then we analyse the case when \( \lim_{m \to \infty} \frac{b_m}{m^{3/2}} = b \neq 0 \). The case \( \lim_{m \to \infty} \frac{b_m}{m^{3/2}} = 0 \) is studied in [19].

We are looking for solutions of (7) as Cauchy transforms of some compactly supported Borel positive measure in the complex plane \( \mathbb{C} \). Any connected curve in the support of such a measure (if exists) coincides with a finite critical trajectory of a quadratic differential on the Riemann sphere \( \overline{\mathbb{C}} \) of the form \(- p(t) \, dt^2\), where \( p \) is polynomial of degree 4. A necessary and sufficient condition for the existence of these solutions is that the above quadratic differential has at least two short trajectories, (Sect. 2.2). Possible configurations of the critical graph of this kind of quadratic differential are known, see [23], but the main goal of this work (besides the precise description of the critical graph) is the investigation of the number of finite critical trajectories when \( p(z) = (z^2 - 1)(z - a)(z - \overline{a}), \) for some complex number \( a \). More precisely, we show that this number equals 3 when \( a \) belongs to a certain curve \( \Gamma \setminus [-1, 1] \) (that will be described), while it takes one of the values 1 or 2 on each connected component of \( \mathbb{C} \setminus \Gamma \), (Sect. 2.1).

### 2 Quadratic differentials

Below, we describe the critical graphs of the family of quadratic differentials on the Riemann sphere \( \overline{\mathbb{C}} \)

\[
\varpi_p(z) = -p(z) \, dz^2,
\]

where \( p \) is a monic real polynomial of degree 4.

We begin discussions with immediate observations from the theory of quadratic differentials. For more details, we refer the reader to [16, 21].

Recall that finite critical points of the polynomial quadratic differential \( \varpi_p \) are its zeros and simple poles; poles of order 2 or greater then 2 are called infinite critical points. All other points of \( \overline{\mathbb{C}} \) are called regular points of \( \varpi_p \).

With the parametrization \( u = 1/z \), we get

\[
\varpi_p(u) = \left( -\frac{1}{u^8} + \mathcal{O}\left( \frac{1}{u^7}\right) \right) \, du^2, \quad u \to 0;
\]

thus, infinity is an infinite critical point of \( \varpi_p \), as a pole of order 8. Horizontal trajectories (or just trajectories) of the quadratic differential \( \varpi_p \) are the zero loci of the equation

\[
\Re \int z \sqrt{p(t)} \, dt = \text{const}, \quad (8)
\]

or, equivalently,

\[
\varpi_p = -p(z) \, dz^2 > 0.
\]
If \( z(t), t \in \mathbb{R} \) is a horizontal trajectory, then the function
\[
 t \mapsto \Im \int \sqrt{p(t)} z'(u) \, du
\]
is monotone.

The vertical (or, orthogonal) trajectories are obtained by replacing \( \Im \) by \( \Re \) in Eq. (8). The horizontal and vertical trajectories of the quadratic differential \( \sigma_p \) produce two pairwise orthogonal foliations of the Riemann sphere \( \mathbb{C} \).

A trajectory passing through a critical point of \( \sigma_p \) is called a critical trajectory. In particular, if it starts and ends at a finite critical point, it is called finite critical trajectory or short trajectory, otherwise, we call it an infinite critical trajectory. A short trajectory is called unbroken if it does not pass through any finite critical points except its two endpoints. The closure of the set of finite and infinite critical trajectories is called the critical graph of the quadratic differential.

The local structure of the trajectories is as follows:

- At any regular point, horizontal (resp. vertical) trajectories look locally as simple analytic arcs passing through this point, and through every regular point of \( \sigma_p \) passes a uniquely determined horizontal (resp. vertical) trajectory of \( \sigma_p \); these horizontal and vertical trajectories are locally orthogonal at this point.
- From each zero with multiplicity \( r \) of \( \sigma_p \), there emanate \( r + 2 \) critical trajectories spacing under equal angles \( 2\pi/(r + 2) \). See Fig. 1.
- Since \( \infty \) is a pole of order 8, there are 6 asymptotic directions (called critical directions) spacing under equal angle \( \pi/3 \), and a neighborhood \( \mathcal{U} \) of this pole, such that each trajectory entering \( \mathcal{U} \) stays in \( \mathcal{U} \) and tends to \( \infty \) following one of the critical directions. These critical directions are:
\[
 D_k = \left\{ z \in \mathbb{C} : \arg(z) = \frac{\pi}{2} + \frac{k\pi}{3} \right\}; k = 0, \ldots, 5.
\]

See Fig. 1. The same thing happens to the orthogonal trajectories at \( \infty \), but their critical directions are:
\[
 D_k^\perp = \left\{ z \in \mathbb{C} : \arg(z) = \frac{k\pi}{3} \right\}; k = 0, \ldots, 5.
\]

Observe that if two trajectories diverge to \( \infty \) in a same direction \( D_k \), then there exists a neighborhood \( \mathcal{V} \) of \( \infty \), such that any orthogonal trajectory traversing \( D_k \) in \( \mathcal{V} \), must traverse these two trajectories.

We have the following observations:

- If \( \gamma \) is a horizontal trajectory of \( \sigma_p \), then \( \Im \int z \sqrt{p(t)} \, dt \) is monotone on \( \gamma \).
- If two different trajectories are not disjoint, then their intersection must be a zero of the quadratic differential.
A necessary condition for the existence of a short trajectory connecting finite critical points of \( \varpi_p \) is the existence of a Jordan arc \( \gamma \) connecting them, such that

\[
\Re \int_\gamma \sqrt{p(t)} dt = 0.
\]  

(9)

However, we will see that this condition is not sufficient.

- Since the quadratic differential \( \varpi_p \) has only one pole, Jenkins Three-pole Theorem (see [21, Theorem 15.2]) asserts that the situation of the so-called recurrent trajectory (whose closure might be dense in some domain in \( \mathbb{C} \)) cannot happen.
- Any critical trajectory which is not finite diverges to \( \infty \) following one of the directions \( D_k \) described above.

One of the most common problems in the study of a given quadratic differential is the existence and the number of its short trajectories, which is the main motivation of this Section. Here we apply the technique used in many previous papers such as [1,2,9,17,22].

A very helpful tool for our investigation is the Teichmüller lemma (see [21, Theorem 14.1]).

**Definition 1** A domain in \( \mathbb{C} \) bounded only by segments of horizontal and/or vertical trajectories of \( \varpi_p \) (and their endpoints) is called \( \varpi_p \)-polygon.

**Lemma 2** (Teichmüller) Let \( \Omega \) be a \( \varpi_p \)-polygon, and let \( z_j \) be the critical points on the boundary \( \partial \Omega \) of \( \Omega \), and let \( t_j \) be the corresponding interior angles with vertices at \( z_j \), respectively. Then

\[
\sum \left( 1 - \frac{(n_j + 2) t_j}{2\pi} \right) = 2 + m,
\]

(10)

where \( n_j \) are the multiplicities of \( z_j = 1 \), and \( m \) is the number of zeros of \( \varpi_p \) inside \( \Omega \).
2.1 Critical graph of $\varpi_a$

We focus on the case when $p$ is a real monic polynomial having two simple real zeros and two conjugate zeros. By a linear change of variables, we may assume that

$$p(z) = (z^2 - 1)(z - a)(z - a^*),$$

and the associated quadratic differential

$$\varpi_a (z) = -(z^2 - 1)(z - a)(z - a^*)\, dz^2,$$

(11)

where $a \in \mathbb{C}_+ = \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$.

**Lemma 3** For any $a \in \mathbb{C}_+$, condition (9) is satisfied for the pairs of zeros $(-1, 1)$ and $(a, a^*)$.

**Proposition 4**

(i) The segment $[-1, 1]$ is always a short trajectory connecting $\pm 1$.

(ii) Two critical trajectories emanating from the same zero of $p(z)$ cannot diverge to $\infty$ in the same direction.

(iii) No critical trajectory emanating from $z = 1$ diverges to $\infty$ in the direction $D_2$.

(iv) Exactly one trajectory emanating from $z = -1$ diverges to $\infty$ in the upper half plane, and it follows the direction $D_2$.

(v) There are at most three short trajectories.

Let us consider the set

$$\Gamma = \left\{ z \in \mathbb{C} : \Re \left( \int_0^z \sqrt{(t - z)(t - \bar{z})(t^2 - 1)} \, dt = 0 \right) \right\}. \tag{12}$$

Then we have the following observations:

- the choice of the square root does not play any role in the integral defining (12);
- $[-1, 1] \subset \Gamma$;
- since $p$ is a real polynomial, it can be easily shown that the set $\Gamma$ is symmetric with respect to the real and imaginary axis;
- direct calculations show that $\pm 1 + 2 \exp \left( \pm \frac{i\pi}{3} \right) \in \Gamma$.

**Lemma 5** Let $x$ be the unique real number defined by:

$$\sinh \left( \frac{2}{3 \cot x \sin^3 x} - \frac{\cot x}{\sin x} \right) = \cot x, \quad x \in \left[ 0, \frac{\pi}{2} \right]. \tag{13}$$

Then, we have

$$\lim_{z \to \infty, z \in \mathbb{C}_+} \frac{z}{|z|} = \exp(ix);$$

$$\lim_{z \to 1, z \in \mathbb{C}_+} \frac{z - 1}{|z - 1|} = \exp \left( i \frac{\pi}{3} \right),$$
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where we denote by $\mathbb{C}^+_+ = \{ z \in \mathbb{C} \mid \Re(z), \Im(z) > 0 \}$.

**Remark 6** Numeric methods give the approximate value $x \simeq 0.898 \in \left( \frac{\pi}{6}, \frac{\pi}{2} \right)$.

**Lemma 7** The set $\Gamma$ is formed by 5 Jordan arcs:

- the segment $[-1, 1]$;
- two curves $\Gamma^+_1$ emerging from $z = 1$, and diverging respectively to infinity in $\mathbb{C}_\pm$;
- two curves $\Gamma^-_1$ emerging from $z = -1$, and diverging respectively to infinity in $\mathbb{C}_\pm$.

By Lemma 7, $\Gamma$ splits $\mathbb{C}$ into 4 connected domains:

- $\Omega_1$ limited by $\Gamma^+_1$ and containing $z = 2$;
- $\Omega_2$ limited by $\Gamma^-_1$ and containing $z = -2$;
- $\Omega_+ \subset \mathbb{C}_+$, limited by $[-1, 1]$ and $\Gamma^+_1$;
- $\Omega_- \subset \mathbb{C}_-$, limited by $[-1, 1]$ and $\Gamma^-_1$. See Fig. 2.

Then main result of this paper is as follows:

**Proposition 8** For any complex number $a \in \mathbb{C}^+$, the segment $[-1, 1]$ is a short trajectory of the quadratic differential $\sigma_a$. Moreover,

(i) For $a \in \Omega_1 \cup \Omega_2$, $\sigma_a$ has an unbroken short trajectory connecting $a$ and $\overline{a}$. The critical graph $\Gamma_a$ split $\mathbb{C}$ into six half-plane domains and one strip domain; see Fig. 3.

(ii) For $a \in \Gamma^+_\pm$, $\sigma_a$ has two short trajectories connecting $\pm 1$ to $a$ and to $\overline{a}$. The critical graph $\Gamma_a$ split $\mathbb{C}$ into six half-plane domains; see Fig. 4.

(iii) For $a \in \Omega_+ \cup \Omega_-$, there is no short trajectory connecting $a$ and $\overline{a}$. The critical graph $\Gamma_a$ split $\mathbb{C}$ into six half-plane domains and two strip domains; see Fig. 5.

**Remark 9** If $p \in \mathbb{C}[X]$ and $\sigma_a$ has no short trajectory, then $\Gamma_a$ split $\mathbb{C}$ into six half-plane domains and three strip domains. If $p \in \mathbb{R}[X]$, then the critical graph $\Gamma_p$ is symmetric with respect to the real axis. It is obvious that when $p = \prod_{i=1}^4 (z - a_i)$ with simple real zeros $a_1 < a_2 < a_3 < a_4$, then the segments $[a_1, a_2]$ and $[a_3, a_4]$ are two short trajectories of $\sigma_a$. See Fig. 6.
Fig. 3 Critical graph for $a \in \Omega_1$, where $a = 2 + i$ (left), and $a = 2 + 1.5i$ (right).

Fig. 4 Critical graph for $a \in \Gamma$, here $a = 2 + 1.752i$ (left), and $a = 100 + 125.4i$ (right).

Fig. 5 Critical graph for $a \in \Omega_+$, where $a = 2 + 1.9i$ (left), and $a = 2 + 2.4i$ (right).
2.2 The algebraic equation

The Cauchy transform $C_\nu$ of a compactly-supported Borel complex measure $\nu$ is the analytic function defined in $\mathbb{C} \setminus \text{supp}(\nu)$ by:

$$C_\nu(z) = \int_{\mathbb{C}} \frac{d\nu(t)}{z - t}. \quad (14)$$

It satisfies the condition:

$$C_\nu(z) = \frac{\nu(C)}{z} + \mathcal{O}(z^{-2}), \quad z \to \infty, \quad (15)$$

and the inversion formula (which should be understood in the distributional sense):

$$\nu = \frac{1}{\pi} \frac{\partial C_\nu}{\partial z}. $$

In particular, the normalized root-counting measure $\nu_n = \nu(P_n)$ of a complex polynomial $P_n$ of degree $n$ is defined for any compact set $K$ in $\mathbb{C}$ by:

$$\int_K d\nu_n = \frac{\text{number of zeros of } P_n \text{ in } K}{n};$$

the Cauchy transform of $\nu_n$ is given by:

$$C_{\nu_n}(z) = \int_{\mathbb{C}} \frac{d\nu_n(t)}{z - t} = \frac{P_n'(z)}{nP_n(z)}; \quad P_n(z) \neq 0.$$

Let us come back to the algebraic equation (7). The following Lemma gives a sufficient condition for a solution of (7) (as a quadratic equation) to be the Cauchy transform of some compactly supported measure in $\mathbb{C}$:
Lemma 10 ([14, comp. Th. 1.2, Ch. II,]) Suppose that $f \in L^1_{\text{loc}}(\mathbb{C})$ and that $f(z) \to 0$ as $z \to \infty$ and let $\mu$ be a compactly-supported measure in $\mathbb{C}$ such that

$$\mu = \frac{1}{\pi} \frac{\partial f}{\partial z}$$

in the sense of distributions. Then $f(z) = C \mu(z)$ almost everywhere in $\mathbb{C}$.

However, for a general algebraic equation

$$C^2(z) + r(z) C(z) + s(z) = 0, \quad (16)$$

where $r$ and $s$ are two polynomials with degrees 2 and at most 1:

$$r(z) = a z^2 + b z + c; \quad a \in \mathbb{C} \setminus \{0\}, \ b, c \in \mathbb{C}$$

$$s(z) = e z + f; \quad e, f \in \mathbb{C},$$

the following Proposition gives a necessary condition on the polynomials $r$ and $s$ to insure the existence of a compactly-supported Borel signed measure $\nu$, such that its Cauchy transform $C_\nu$ satisfies (16) almost everywhere in $\mathbb{C}$.

**Proposition 11** If Eq. (16) admits a solution almost everywhere in $\mathbb{C}$ as the Cauchy transform of a compactly-supported signed measure $\nu$, then:

- $\frac{e}{a} = -1$;
- any connected curve in the support of $\nu$ coincides with a short trajectory of the quadratic differential

$$\sigma = -Q(z) \, dz^2, \quad (17)$$

where $Q(z) = r^2(z) - 4s(z)$ is the discriminant of the quadratic equation (16),

- the quadratic differential (17) has two short trajectories.

**3 Proofs**

**Proof of Lemma 3** Since $p$ is a real polynomial, then $\sqrt{p(t)} = \sqrt{p(\bar{t})}$. After the change of variable $u = \bar{t}$ in the second integral, we get

$$\Re\left( \int_{\bar{z}}^z \sqrt{p(t)} \, dt \right) = \Re\left( \int_1^z \sqrt{p(t)} \, dt - \int_{1}^\bar{z} \sqrt{p(t)} \, dt \right)$$

$$= \Re\left( \int_1^z \sqrt{p(t)} \, dt - \int_{1}^z \sqrt{p(t)} \, dt \right)$$

$$= \Re\left( 2i \Im\left( \int_1^z \sqrt{p(t)} \, dt \right) \right)$$

$$= 0.$$
Let us now give a necessary condition to obtain two short trajectories joining two different pairs of zeros of \( p \) in the general case. We write
\[
p(z) = z^4 + \alpha z^3 + \beta z^2 + \gamma z + \delta \in \mathbb{C}[X].
\]
and consider two disjoint oriented Jordan arcs \( \gamma_1 \) and \( \gamma_2 \) connecting two different pairs zeros. We define the single-valued function \( \sqrt{p(z)} \) in \( \mathbb{C} \setminus (\gamma_1 \cup \gamma_2) \) by condition
\[
\sqrt{p(z)} \sim z^2, \quad z \to \infty.
\]
From the Laurent expansion of \( \sqrt{p(z)} \) at \( \infty \), we have
\[
\sqrt{p(z)} = z^2 + \frac{\alpha}{2} z - \frac{\alpha^2 - 4 \beta}{8} + \frac{\alpha^3 - 4 \alpha \beta + 8 \gamma}{16} z^{-1} + O(z^{-2}), \quad z \to \infty,
\]
which gives
\[
-\text{res}_{\infty} \left( \sqrt{p(z)} \right) = \frac{1}{16} \left( \alpha^3 - 4 \alpha \beta + 8 \gamma \right).
\]
For \( s \in \gamma_1 \cup \gamma_2 \), we denote by \( (\sqrt{p(s)})_+ \) and \( (\sqrt{p(s)})_- \) the limits from the + and − sides, respectively. (As usual, the + side of an oriented curve lies to the left and the − side lies to the right, if one traverses the curve according to its orientation.) Let
\[
I = \int_{\gamma_1} (\sqrt{p(s)})_+ \, ds + \int_{\gamma_2} (\sqrt{p(s)})_+ \, ds.
\]
Since
\[
(\sqrt{p(s)})_+ = - (\sqrt{p(s)})_-, \quad s \in \gamma_1 \cup \gamma_2,
\]
we have
\[
2I = \int_{\gamma_1 \cup \gamma_2} \left[ (\sqrt{p(s)})_+ - (\sqrt{p(s)})_- \right] \, ds = \oint_{\Gamma} \sqrt{p(z)} \, dz,
\]
where \( \Gamma \) is a closed contour encircling \( \gamma_1 \) and \( \gamma_2 \). After the deformation of the contour we pick up residue at \( z = \infty \). For any choice of the square roots we get
\[
I = \frac{1}{2} \oint_{\Gamma} \sqrt{p(z)} \, dz = \pm i \pi \text{res}_{\infty} \left( \sqrt{p(z)} \right)
\]
\[
= \pm \frac{\pi i}{16} \left( \alpha^3 - 4 \alpha \beta + 8 \gamma \right).
\]
Thus, we obtain the necessary condition of the existence of two short trajectories:
\[
\Im \left( \alpha^3 - 4 \alpha \beta + 8 \gamma \right) = 0.
\]
Proof of Lemma 4  (i) For \( s \in [-1, 1] \), it is straightforward that \( \Re \int_{-1}^{s} \sqrt{p(t)} \, dt = 0 \).

(ii) Suppose that \( \gamma_1 \) and \( \gamma_2 \) are two such trajectories emanating from the same zero \( z_j \), spacing with angle \( t \in \{2\pi/3, 4\pi/3\} \). Consider the \( \sigma \)-polygon with edges \( \gamma_1 \) and \( \gamma_2 \), and vertices \( z_j \), and infinity. The right-hand side of (10) can only take the values 0, or \(-1 \) while the left-hand side is \( \geq 2 \); a contradiction.

(iii) If a critical trajectory emanating from \( z = 1 \) diverges to \( \infty \) in the direction \( D_2 \), then, also, a critical trajectory emanating from \( z = -1 \) must diverge to \( \infty \) in the direction \( D_2 \). Together with the segment \([-1, 1]\) and \( \infty \), these two critical trajectories will form an \( \sigma \)-polygon and, again, this violates Lemma 2.

(iv) The result follows by combining (ii) and (iii), and using Lemma 2.

(v) The case of four short trajectories can be discarded immediately by Lemma 2. It is an immediate consequence of Lemma 2 that two short trajectories cannot connect the same two zeros of a holomorphic quadratic differential on the Riemann sphere. \( \square \)

Proof of Lemma 7  The first observation is that:

\[
\Gamma \cap \{ z \in \mathbb{C} \mid \Re (z) = 1, \Im (z) \geq 0 \} = \{1\}. \tag{18}
\]

Indeed, if for some \( y > 0 \), \( z = 1 + iy \in \Gamma \), then

\[
0 = \Re \left( \int_{1}^{1+iy} \sqrt{(u - (1 + iy))(u - (1 - iy))(u^2 - 1)} \, du \right) \\
= y^3 \int_{1}^{y} \sqrt{ty(1-t^2)} \Re \left( \sqrt{ty-2i} \right) \, dt, \quad u = 1 + ity, \quad 0 \leq t \leq 1.
\]

However, if we choose the argument in \([0, 2\pi]\), then, for any \( t \in ]0, 1[ \), we have

\[
\frac{3\pi}{2} < \arg (ty - 2i) < 2\pi \implies \frac{3\pi}{4} < \arg \sqrt{ty - 2i} < \pi \\
\implies \sqrt{ty(1-t^2)} \Re \left( \sqrt{ty-2i} \right) < 0 \\
\implies \int_{0}^{1} \sqrt{ty(1-t^2)} \Re \left( \sqrt{ty-2i} \right) \, dt < 0,
\]

which gives a contradiction.

In order to prove that \( \Gamma \) is a curve, since we know that \([-1, 1] \subset \Gamma \), by the observation (18), it is sufficient to consider the set

\[
\Pi = \{(x, y) \mid x > 1, y > 0 \},
\]
and the real functions $F$ and $G$ defined for $(x, y)$ in $\Pi$ by:

$$F(x, y) = \Re \left( \int_0^x \sqrt{(u - (x + iy))(u - (x - iy))(u^2 - 1)} \, du \right)$$

$$= \int_1^x \sqrt{(u - x)^2 + y^2} \, (u^2 - 1) \, du,$$

$$G(x, y) = \Re \int_x^{x+iy} \sqrt{(u - (x + iy))(u - (x - iy))(u^2 - 1)} \, du.$$

In other words, the set $\Gamma$ is defined by:

$$\{(x, y \in \Pi \mid (F + G)(x, y) = 0\}.$$

Observe that $F + G$ is differentiable in $\Pi$, and $z = 1$ is the only singular point of $F + G$.

We have

$$\frac{\partial F}{\partial x}(x, y) = \sqrt{y^2(x^2 - 1)} + \Re \left( \int_1^x \frac{(x - t)(t^2 - 1)}{\sqrt{(t - x)^2 + y^2}(t^2 - 1)} \, dt \right) > 0.$$

On the other hand, with the change of variable $u = x + ity$, $0 \leq t \leq 1$, we get:

$$\frac{\partial G}{\partial x}(x, y) = \frac{\partial}{\partial x} \left[ \Re \int_0^1 i y^2 \sqrt{1 - t^2} ((x + ity)^2 - 1) \, dt \right]$$

$$= \Re \int_0^1 i y^2 (x + ity) \frac{1 - t^2}{\sqrt{(x + ity)^2 - 1}(1 - t^2)} \, dt$$

$$= -y^2 \int_0^1 \sqrt{1 - t^2} \Re \left( \frac{z_t}{\sqrt{z_t^2 - 1}} \right) \, dt$$

where $z_t = x + ity$.

Thus, for any $t \in [0, 1]$:

$$0 \leq \arg(z_t) < \frac{\pi}{2} \implies 0 \leq \arg(z_t) \leq \arg \left( \sqrt{z_t^2 - 1} \right) \leq \frac{\pi}{2}$$

$$\implies -\frac{\pi}{2} \leq \arg \left( \frac{z_t}{\sqrt{z_t^2 - 1}} \right) = \arg(z_t) - \arg \left( \sqrt{z_t^2 - 1} \right) \leq 0$$

$$\implies \sqrt{1 - t^2} \Re \left( \frac{z_t}{\sqrt{z_t^2 - 1}} \right) \leq 0 \text{ (non identically vanishing).}$$
It follows that

\[ \frac{\partial G}{\partial x}(x, y) > 0. \]

We conclude that the set \( \Gamma \) is a regular curve in \( \mathbb{C} \) by applying the Implicit Function Theorem to the function \( F + G \).

\[ \square \]

**Proof of Lemma 5** From the power series expansion

\[ \sqrt{p(t)} = \sqrt{(z - 1)(2z - 2)}\sqrt{t - 1} + O(t - 1)^{\frac{3}{2}}, t \to 1, \]

we get for \( z \to 1, z \in \Gamma \cap \mathbb{C}_+^+ \)

\[
0 = \Re \int_1^z \frac{(t - z)(t - \bar{z})(t^2 - 1)}{3} \, dt \\
= \Re \left( \frac{2\sqrt{2}}{3} |z - 1|(z - 1)^{\frac{3}{2}} \right) \\
= \frac{2\sqrt{2}}{3} |z - 1| \Re (z - 1)^{\frac{3}{2}).
\]

Thus,

\[ \arg (z - 1)^{\frac{3}{2}} \equiv \pi \mod (2\pi), z \to 1, z \in \Gamma \cap \mathbb{C}_+^+, \]

which gives the local structure of \( \Gamma \) near \( z = 1 \).

Substituting \( z = re^{ix} \in \Gamma \cap \mathbb{C}_+^+ \) with \( r > 0, x \in (2, \pi) \), in the integral defining \( \Gamma \), we get:

\[ \Re \int_0^1 e^{ix} \sqrt{(1 - t)(1 - te^{2ix})(e^{2ix}t^2 - \frac{1}{r^2})} = 0 \quad (19) \]

by the change of variable \( s = tr e^{ix}, t \in [0, 1] \). Taking the limit when \( r \to \infty \), equality \( (19) \) becomes

\[ \Re \left( e^{3ix} \int_0^1 t \sqrt{(t - 1)(t - e^{-2ix})} \, dt \right) = 0. \]

With the change of variable \( t = \alpha u + \beta, \) where

\[ \alpha = \frac{1 - e^{-2ix}}{2}, \beta = \frac{1 + e^{-2ix}}{2}, \]
we get

\[ 0 = \Re \int_{i \cot x}^{1} \alpha^2 e^{3ix} (\alpha u + \beta) \sqrt{(u - 1 - \beta)(u - e^{-2ix} - \beta)} du \]

\[ = \Re \left( \alpha^3 e^{3ix} \int_{i \cot x}^{1} (u - i \cot x) \sqrt{u^2 - 1} du \right) \]

\[ = 3 \left( \int_{i \cot x}^{1} (t - i \cot x) \sqrt{t^2 - 1} \right) \sin^3 x = 0. \]

It follows that

\[ \Im \left( \int_{i \cot x}^{0} (t - i \cot x) \sqrt{t^2 - 1} \right) + \Im \left( \int_{0}^{1} (t - i \cot x) \sqrt{t^2 - 1} \right) = 0, \]

and then

\[ 0 = \int_{0}^{\cot x} t \sqrt{t^2 + 1} \cot x \int_{0}^{\cot x} \sqrt{t^2 + 1} + \int_{0}^{1} t \sqrt{1 - t^2} \]

\[ = \frac{1}{3 \sin^3 x} - \frac{1}{2} \cot x \left( \frac{\cot x}{\sin x} + \text{arg sinh} (\cot x) \right), \]

which gives Eq. (13).

To prove the existence and uniqueness of solution of Eq. (13) in \((0, \frac{\pi}{2})\), we need to study the function

\[ t \mapsto f \sinh (v (t)) - \cot t; t \in \left(0, \frac{\pi}{2}\right). \]

where

\[ v (t) = \frac{2}{3} \frac{1}{\cot t \sin^3 t} - \frac{\cot t}{\sin t}; t \in \left(0, \frac{\pi}{2}\right); \]

Since

\[ v' (t) = \frac{8 - 3 (\sin t + \sin 3t)}{12 \cos^2 t \sin^2 t} > 0; t \in \left(0, \frac{\pi}{2}\right). \]

Thus, \( f \) increases from \(-\infty\) to \(+\infty\) for \( t \in \left(0, \frac{\pi}{2}\right) \), which gives the existence and uniqueness of \( x \) in \( \left(0, \frac{\pi}{2}\right) \). \( \square \)

**Proof of Proposition 8** From (9), it is clear that if \( a \notin \Gamma \), then there is no short trajectory connecting 1 and \( a \). Suppose now that there is no short trajectory connecting 1 and \( a \) for some \( a \in \Gamma \cap C^+_1 \), and denote by \( \gamma_{1,a} \) the only critical trajectory emanating from \( z = 1 \) and diverging to \( \infty \) in the upper half-plane, then there emanates a critical trajectory \( \gamma_a \) from \( a \) that diverges to \( \infty \) in the same direction of \( \gamma_{1,a} \). From the behaviour of
orthogonal trajectories at $\infty$, we can take an orthogonal trajectory $\sigma$ that hits $\gamma_{1,a}$ and $\gamma_{a}$ respectively in two points $b$ and $c$ (there are infinitely many such orthogonal trajectories $\sigma$). We consider a path $\gamma$ connecting $z = 1$ and $a$ formed by the part of $\gamma_{1,a}$ from $z = 1$ to $b$, the part of $\sigma$ from $b$ to $c$, and the part of $\gamma_{a}$ from $c$ to $a$. Then

$$\Re \int_{\gamma} \sqrt{p(t)} dt = \Re \int_{1}^{b} \sqrt{p(t)} dt + \Re \int_{b}^{c} \sqrt{p(t)} dt + \Re \int_{c}^{a} \sqrt{p(t)} dt$$

$$= \Re \int_{b}^{c} \sqrt{p(t)} dt \neq 0,$$

which provides a contradiction, and proves (i).

Let us continue our proof by the observation from Lemma 4, that, for any $a \in \mathbb{C}^{+}$, $\gamma_{1,a}$ diverges to $\infty$ in the direction $D_{0}$, if and only if there is no short trajectory connecting $a$ and $\overline{a}$; $\gamma_{1,a}$ diverges to $\infty$ in the direction $D_{1}$, if and only if there exists an unbroken short trajectory connecting $a$ and $\overline{a}$.

By its definition, $\Omega_{1}$ is an open subset of $\mathbb{C}$. Suppose that there exists a ball included in $\Omega_{1} \cap \mathbb{C}^{+}$ containing two points $a$ and $a'$, such that $a$ and $\overline{a}$ are connected by a short trajectory, while $a'$ and $\overline{a'}$ are not. Since $\gamma_{1,a}$ and $\gamma_{1,a'}$ diverge to $\infty$ respectively in the directions $D_{1}$ and $D_{0}$, then, by continuity of the trajectories in the Hausdorff metric, there exists $a''$ in the same ball such that $\gamma_{1,a''}$ hits one of the trajectories emanating from $a''$, which means that $\gamma_{1,a''}$ is a short trajectory connecting $1$ and $a''$; in other word $a'' \in \Gamma$. A contradiction. We deduce that, in each component defined by $\Gamma$, the critical graph $\Gamma_{a}$ has the same configuration. It suffices then to see a particular case in each of these components.

For $a = i \in \Omega_{+}$, it is straightforward that $\{xi; x \geq 1\} \subset \Gamma_{i}$, and $\Gamma_{i}$ is symmetric with respect to the real and imaginary axis (see Fig. 7), which proves (ii).

Suppose that, for some $a_{\epsilon} = x + i\epsilon \in \Omega_{1} \cap \mathbb{C}^{+}$ ($x > 1, \epsilon > 0$ small enough), there is no short trajectory connecting $a_{\epsilon}$ and $\overline{a_{\epsilon}}$. Then $\gamma_{1,a_{\epsilon}}$ still diverges to $\infty$ in the direction $D_{0}$ when $\epsilon \to 0$.

$$\lim_{\epsilon \to 0} \left( \inf_{z \in \gamma_{1,a_{\epsilon}}} \{|x - z|\} \right) = \lim_{\epsilon \to 0} \left( \inf_{z \in \gamma_{1,a_{\epsilon}}} \{|a_{\epsilon} - z|\} \right) = 0.$$
Fig. 8  Critical graphs of $\sigma_\rho$ when $b = 1 + 1.9i$ (left), $b = 1 + 2.04i$ (center), and $b = 1 + 2.2i$ (right).

Then the segment $[1, x]$ will be a short trajectory of $\sigma_x$ which cannot hold, and proves (iii). \qed

**Proof of Proposition 11**  Solutions of the quadratic equation (16) are given by

$$C^\pm (z) = \frac{-r(z) \pm \sqrt{r^2(z) - 4s(z)}}{2},$$

for any choice of the square root. If the Cauchy transform $C_\nu$ of some signed measure $\nu$ is a solution of (16), then, the choice of the square root in the above equation must be done so that condition (15) is fulfilled for $C_\nu$. In this case, using the Laurent expansion of $\sqrt{r^2(z) - 4s(z)}$ at $\infty$, we get

$$C_\nu(z) = \frac{-r(z) + \sqrt{r^2(z) - 4s(z)}}{2} = \frac{-\epsilon}{z} + O\left(z^{-2}\right), \; z \to \infty.$$

Hence

$$1 = \nu(\mathbb{C}) = -\frac{\epsilon}{a}.$$

The fact that the measure $\nu$ lives on the horizontal trajectories of the quadratic differential (17) follows from the so-called Plemelj-Sokhotsky’s formula. (For more details, we refer the reader to [7,8,18]).

Since the Cauchy transform of $\nu$ is a single-valued function in $\mathbb{C}\setminus\text{supp}(\nu)$, therefore $\text{supp}(\nu)$ must include all branching points (finite critical points) of the quadratic differential (17). Then, the quadratic differential (17) must have two short trajectories connecting two different pairs of zeros.

The measure $\nu$ is absolutely continuous with respect to the linear Lebesgue measure and it is given on its support (with an adequate orientation) by the expression

$$d\nu(t) = \frac{1}{2\pi i} \left(\sqrt{Q(t)}\right)_+ \; dt.$$

Indeed, it is straightforward to check (like in the Proof of Lemma 3) that the Cauchy transform of this measure $\nu$ satisfies (a.e.) in $\mathbb{C}$ Eq. (7). \qed
Finally, notice that this analysis can be done in the case where $p$ is a real polynomial without real zeros: Suppose that

$$p(z) = (z - a) (z - b) (z - \overline{a}) (z - \overline{b}),$$

with $a \neq b \in \mathbb{C}$, $\Im(a), \Im(a) > 0$. Then the quadratic differential has at list one short trajectory. Indeed, if no one of the trajectories emanating from $a$ is short, then, they must diverge to $\infty$ in three different asymptotic directions in $\mathbb{C}_+$. It follows from Lemma 2 that at most two trajectories emanating from $b$ can diverge to $\infty$ in $\mathbb{C}_+$, and then, using symmetry, the remaining one is a short trajectory that connects $b$ and $\overline{b}$. See Fig. 8, where we use $a = i$.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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