CONNECTIVITY OF SINGLE-ELEMENT COEXTENSIONS OF A BINARY MATROID

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Abstract. Given an $n$-connected binary matroid, we obtain a necessary and sufficient condition for its single-element coextensions to be $n$-connected.

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1. Introduction

For undefined terminologies, we refer to Oxley [6]. The point-splitting operation is a fundamental operation in respect of connectivity of graphs. It is used to characterize 3-connected graphs in the classical Tutte’s Wheel Theorem [9] and also to characterize 4-connected graphs by Slater [8]. This operation is defined as follows.

Definition 1.1 ([8]). Let $G$ be a graph with a vertex $v$ of degree at least $2n - 2$ and let $T = \{v_1, v_2, \ldots, v_{n-1}\}$ be a set of $n - 1$ edges of $G$ incident to $v$. Let $G'_T$ be the graph obtained from $G$ by replacing $v$ by two adjacent vertices $u$ and $w$ such that $u$ is adjacent to $v_1$, $v_2$, $\ldots$, $v_{n-1}$, and $w$ is adjacent to the vertices which are adjacent to $v$ except $v_1$, $v_2$, $\ldots$, $v_{n-1}$. We say $G'_T$ arises from $G$ by $n$-point splitting (see the following figure).

Slater [8] obtained the following result to characterize 4-connected graphs.

Theorem 1.2 ([8]). Let $G$ be an $n$-connected graph and let $T$ be a set of $n - 1$ edges incident to a vertex of degree at least $2n - 2$. Then the graph $G'_T$ is $n$-connected.

In this paper, we extend the above theorem to binary matroids.

Azadi [1] extended the $n$-point splitting operation on graphs to binary matroids as follows.

Definition 1.3 ([1]). Let $M$ be a binary matroid with standard matrix representation $A$ over the field $GF(2)$ and let $T$ be a subset of the ground set $E(M)$ of $M$. Let $A'_T$ be the matrix obtained from $A$ by adjoining one extra row to matrix $A$ whose entries are 1 in the columns labeled by the elements of $T$ and 0 otherwise and also having one extra column labeled by $a$ with 1 in the last row and 0 elsewhere. Denote the vector matroid of $A'_T$ by $M'_T$. We say $M'_T$ is obtained from $M$ by element splitting with respect to the set $T$.

For example, the following matrices $A$ and $A'_T$ represent the Fano matroid $F_7$ and its element splitting matroid with respect to the set $T = \{1, 2, 3\} \subset E(F_7)$.

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}, \quad A'_T = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & a \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Given a graph $H$, let $M(H)$ denote the circuit matroid of $H$. A matroid $N$ is a single-element coextension of a matroid $M$ if $N/e = M$ for some element $e$ of $N$. 

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Definition \[ \text{(3)} \] is an extension of Definition \[ \text{(1)} \] as \( M(G'_T) = M(G'_T) \) for a set \( T \) of edges incident to a vertex of a graph \( G \). Note that if \( M \) is a binary matroid, then the element splitting matroid \( M'_T \) is also binary and it is a coextension of \( M \) by the element \( a \) as \( M'_T/a = M \). In fact, we prove in Lemma \[ \text{2.1} \] that every coextension of a binary matroid \( M \) by a non-loop and non-coloop element is the element splitting matroid \( M'_T \) for some \( T \subset E(M) \).

Dalvi et al. \[ \text{[4, 5]} \] characterized the graphic (cographic) matroids \( M \) whose single-element coextensions \( M'_T \) are again graphic (cographic). Let \( M \) be an \( n \)-connected binary matroid. Borse and Mundhe \[ \text{[3]} \] obtained sufficient conditions for the matroid \( M'_T \setminus a \) to be \( n \)-connected. In this paper, we obtain a necessary and sufficient condition for \( M'_T \) to be \( n \)-connected. The following is the main theorem of the paper.

**Main Theorem 1.4.** Let \( n \geq 2 \) be an integer and \( M \) be an \( n \)-connected binary matroid with \( |E(M)| \geq 2n-2 \). Suppose \( T \subset E(M) \) with \( |T| = n-1 \). Then \( M'_T \) is \( n \)-connected if and only if \( |Q| \geq 2|Q \cap T| \) for every cocircuit \( Q \) of \( M \) intersecting \( T \).

We also prove that Theorem \[ \text{1.2} \] follows from Main Theorem \[ \text{1.4} \] under a mild restriction.

Azadi \[ \text{[1]} \] obtained the following result for \( M'_p \) to be \( n \)-connected, in terms of the circuits of \( M \) containing an odd number of elements of \( T \).

**Theorem 1.5 (\[ \text{1} \]).** Let \( n \geq 2 \) be an integer and \( M \) be an \( n \)-connected binary matroid with \( |E(M)| \geq 2n-2 \). Suppose \( T \subset E(M) \) with \( |T| = n-1 \). Then \( M'_T \) is \( n \)-connected if and only if for any set \( A \subset E(M) \) with \( |A| = n-2 \), there exists a circuit \( C \subset T \) containing an odd number of elements of \( T \) and is contained in \( E(M) \setminus A \).

We provide an alternate shorter proof of Theorem \[ \text{1.5} \] in the third section.

In Section 2, we provide some properties of \( M'_T \). Main Theorem \[ \text{1.4} \] is proved in Section 3. In the last section, we discuss consequences of Main Theorem \[ \text{1.4} \] to the graphs.

## 2. Preliminaries

We prove below that the single-element coextension of a binary matroid \( M \) by a non-loop and non-coloop element is nothing but an element splitting matroid \( M'_T \) for some \( T \subset E(M) \).

**Lemma 2.1.** Let \( M \) and \( N \) be binary matroids. Then \( N \) is a coextension of \( M \) by a non-loop and non-coloop element if and only if \( N = M'_T \) for some \( T \subset E(M) \).

**Proof.** Suppose \( N = M'_T \) for some \( T \subset E(M) \). Then the ground set of \( N \) is \( E(M) \cup \{a\} \) and \( N/a = M \). Hence \( N \) is a coextension of \( M \) by the element \( a \). Let \( A \) be the standard matrix representation of \( M \) over \( GF(2) \). By Definition \[ \text{(3)} \] in the matrix \( A'_T \) of \( M'_T \), the column labeled by \( a \) has 1 in the last row and 0 elsewhere, and the columns labeled by the elements of \( T \) have 1 in the last row. This shows that \( a \) is neither a loop nor a coloop of \( N \).

Conversely, suppose \( N \) is a coextension of \( M \) by a non-loop and non-coloop element \( a \). Let \( T_1 \) be a cocircuit of \( N \) containing \( a \) and let \( T = T_1 \setminus \{a\} \). Then \( T \) is a non-empty subset of \( E(M) \). We can write the standard matrix representation \( B \) of \( N \) such that the column of \( B \) labeled by \( a \) has entry 1 in the last row and 0 elsewhere. Since \( T_1 \) is a cocircuit of \( N \), the last row of \( B \) contains 1 in the columns corresponding to \( T_1 \) and 0 elsewhere. Let \( C \) be the matrix obtained from \( B \) by deleting the last row and the column corresponding to \( a \). Then \( M[C] = N/a = M \). Thus \( B \) can be obtained from \( C \) by adding one extra row which has entries 1 below the elements corresponding to \( T \) and then adding a column labeled by \( a \) which has entry 1 in the last row and 0 elsewhere. Therefore, by Definition \[ \text{(3)} \], \( B = C'_T \). Hence \( N = M[B] = M[C'_T] = M'_T \). \( \Box \)

Henceforth, we use the notation \( M'_T \) for a single-element coextension of a binary matroid \( M \).

We need the following results.

**Lemma 2.2 (\[ \text{1} \]).** Let \( M \) be a binary matroid and \( T \subset E(M) \). If \( \mathcal{C} \) is the collection of circuits of \( M \), then every circuit of \( M'_T \) belongs to one of the following type.

(i). \( \mathcal{C}_1 = \{C \in \mathcal{C}: |C \cap T| \text{ is even} \} \)

(ii). \( \mathcal{C}_2 = \{C \cup \{a\} : C \in \mathcal{C} \text{ and contains an odd number of elements of } T \} \)

(iii). \( \mathcal{C}_3 = \text{ set of minimal members of } \{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset \text{ and } C_1 \text{ and } C_2 \text{ each contains an odd number of elements of } T \text{ such that } C_1 \cup C_2 \text{ does not contain any member of } \mathcal{C}_1 \} \)
Lemma 2.3 ([4]). Let $M$ be a binary matroid. Suppose $r$ and $r'$ are the rank functions of $M$ and $M'_T$, respectively. If $A \subset E(M) \cup \{a\}$, then rank of $A$ is given by

(i). $r'(A) = r(A - \{a\}) + 1$ if $a \in A$.
(ii). $r'(A) = r(A) + 1$ if $a \notin A$ and $A$ contains a circuit $C$ of $M$ with $|C \cap T|$ odd.
(iii). $r'(A) = r(A)$ if $a \notin A$ and $A$ does not contain any circuit $C$ of $M$ with $|C \cap T|$ odd.

Corollary 2.4. Let $M$ be a binary matroid and $T \subseteq E(M)$. Then $r'(M'_T) = r(M) + 1$.

Lemma 2.5 ([7]). Let $M$ be a binary matroid and $\mathcal{C}$ be the collection of cocircuits of $M$. Suppose $T \subseteq E(M)$ does not contain a cocircuit of $M$. Then every cocircuit of $M'_T$ belongs to one of the following type.

(i). $\mathcal{Q}_1 = \{(C^* - T) \cup \{a\}: C^* \in \mathcal{C}^* \text{ and } T \text{ is a proper subset of } C^*\}$,
(ii). $\mathcal{Q}_2 = \{C^* : C^* \in \mathcal{C}^*\}$,
(iii). $\mathcal{Q}_3 = \{(C^* \Delta T) \cup \{a\}: C^* \in \mathcal{C}^*, 1 \leq |C^* \cap T| < |T| \text{ and } C^* \text{ does not contain } D^* - T \text{ for any } D^* \in \mathcal{C}^* \text{ and } T \subset D^*\}$,
(iv). $\mathcal{Q}_4 = \{(C^*_1 \cup C^*_2 \cup \ldots \cup C^*_k - T) \cup \{a\}: k \geq 2, C^*_i \in \mathcal{C}^*, C^*_i \cap T \neq \emptyset, C^*_i \text{ are mutually disjoint and } (C^*_1 \cup C^*_2 \cup \ldots \cup C^*_k - T) \text{ does not contain } D^* - T \text{ for any } D^* \in \mathcal{C}^* \text{ and } T \subset D^*\}$,
(v). $\mathcal{Q}_5 = \{T \cup \{a\}\}$.

3. Proofs

In this section, we prove Main Theorem [14] and also provide an alternate shorter proof of Theorem [15].

We need the following result.

Lemma 3.1 ([4], pp 296). If $n \geq 2$ and $M$ is an $n$-connected matroid with $|E(M)| \geq 2(n - 1)$, then all circuits and all cocircuits of $M$ have at least $n$ elements.

Suppose $M$ is an $n$-connected binary matroid with $|E(M)| \geq 2(n - 1)$ and $T \subset E(M)$. By Definition [13], there is a cocircuit of $M'_T$ contained in $T \cup \{a\}$. Therefore, if $|T| < n - 1$, then $M'_T$ contains a cocircuit of size less than $n$ by Lemma [2.5] and hence $M'_T$ is not $n$-connected by Lemma [3.1]. Hence we assume that $|T| \geq n - 1$.

We obtain below an obvious necessary condition for $M'_T$ to be $n$-connected.

Lemma 3.2. Let $n \geq 2$ be an integer and $M$ be an $n$-connected binary matroid with $|E(M)| \geq 2n - 2$. Suppose $T \subset E(M)$ with $|T| = n - 1$. If $M'_T$ is $n$-connected, then $|Q| \geq 2|Q \cap T|$ for every cocircuit $Q$ of $M$ intersecting $T$.

Proof. Suppose $M'_T$ is $n$-connected. Assume that there is a cocircuit $Q$ of $M$ intersecting $T$ such that $|Q| < 2|Q \cap T|$. By Lemma [2.5] (iii), $Q \Delta T \cup \{a\}$ contains a cocircuit, say $X$, of $M'_T$. Then $|X| \leq |Q \Delta T \cup \{a\}| = |Q| + |T| - 2|Q \cap T| + 1 < |T| + 1 = n$, a contradiction by Lemma 3.1. □

We now prove that the obvious necessary condition for $M'_T$ to be $n$-connected stated in the above lemma is sufficient also.

Proposition 3.3. Let $n \geq 2$ be an integer and $M$ be an $n$-connected binary matroid with $|E(M)| \geq 2n - 2$. Suppose $T \subset E(M)$ with $|T| = n - 1$. If $|Q| \geq 2|Q \cap T|$ for every cocircuit $Q$ of $M$ intersecting $T$, then $M'_T$ is $n$-connected.

Proof. Assume that $|Q| \geq 2|Q \cap T|$ for every cocircuit $Q$ of $M$ intersecting $T$. We proceed by contradiction. Suppose $M'_T$ is not $n$-connected. Then there exists an $(n - 1)$-separation $(A, B)$ of $M'_T$. Therefore

$$\min\{|A|, |B|\} \geq n - 1 \text{ and } r'(A) + r'(B) - r'(M'_T) \leq n - 2. \ldots (*)$$

Suppose $|A| \geq n$ and $|B| \geq n$. Without loss of generality, we may assume that $a \in B$. By Lemma [2.3] and by (*),

$$r(A) + r(B - \{a\}) - r(M) \leq r'(A) + r'(B) - 1 - (r'(M'_T) - 1) \leq n - 2.$$ 

Therefore $(A, B - \{a\})$ forms an $(n - 1)$-separation of $M$, a contradiction.
Therefore $|A| = n - 1$ or $|B| = n - 1$. We may assume that $|A| = n - 1$. Then $A$ is independent in $M$ by Lemma 3.1. Hence, by Lemma 2.2 $A$ is independent in $M'_T$ also.

Claim: $A$ is a coindependent in $M'_T$.

Assume that $A$ is not coindependent in $M'_T$. Then $A$ contains some cocircuit $Q$ of $M'_T$. Therefore $|Q| \leq |A| = n - 1$. By Lemma 3.1, $Q$ is not a cocircuit of $M$. Further, by Lemma 2.5, $Q$ does not belong to $\mathcal{Q}_3$. Hence $Q$ belongs to one of the four classes $\mathcal{Q}_1^*$, $\mathcal{Q}_2^*$, $\mathcal{Q}_3^*$ and $\mathcal{Q}_4^*$.

$(1)$. Suppose $Q \in \mathcal{Q}_1^*$. Then $Q = (C^* - T) \cup \{a\}$, where $C^*$ is a cocircuit of $M$ containing $T$. Then, by hypothesis, $|C^*| \geq 2|C^* \cap T| = 2|T| = 2n - 2$. Therefore

$$n - 1 \geq |Q| = |C^*| - |T| + 1 \geq (2n - 2) - (n - 1) + 1 = n,$$

a contradiction.

$(2)$. Suppose $Q \in \mathcal{Q}_2^*$. Then $Q = ((C_1^* \cup C_2^* \cup \cdots \cup C_k^*) - T) \cup \{a\}$, where $k \geq 2$ and $C_i^*$ are mutually disjoint cocircuits of $M$ and each of them contains at least one element of $T$. Since $M$ is $n$-connected, $|C_i^*| \geq n$ for each $i$ by Lemma 3.1. Hence, we have

$$|Q| \geq |(C_1^* \cup C_2^*) - T| + 1 \geq |C_1^*| + |C_2^*| - |T| + 1 \geq 2n - (n - 1) + 1 = n + 2 > n - 1 \geq |Q|,$$

again a contradiction.

$(3)$. Suppose $Q \in \mathcal{Q}_3^*$. Then $Q = (C^* \Delta T) \cup \{a\}$, where $C^*$ is a cocircuit of $M$ intersecting $T$. Hence

$$|Q| = |C^* \Delta T| + 1 = |C^*| + |T| - 2|C^* \cap T| + 1 \geq |T| + 1 = n > n - 1 \geq |Q|,$$

a contradiction.

$(4)$. Suppose $Q \in \mathcal{Q}_4^*$. So $Q = T \cup \{a\}$. This gives $|Q| = n$, a contradiction.

Thus in all the four cases, we get a contradiction. This proves the claim.

Therefore $A$ is independent and coindependent in the matroid $M'_T$. Hence $r'(A) = |A|$ and $r'(B) = r'(M'_T)$. This gives $n - 1 = |A| = r'(A) = r'(A) + r'(B) - r'(M'_T) \leq n - 2$, a contradiction. Thus we get a contradiction in each case. Therefore $M'_T$ is $n$-connected.

□

Main Theorem 1.4 follows obviously from Lemma 3.2 and Proposition 3.3.

For $2 \leq n \leq 4$, we get the following weaker sufficient conditions for $M'_T$ to be $n$-connected.

Corollary 3.4. Let $n \in \{2, 3, 4\}$ and let $M$ be $n$-connected binary matroid. Suppose $T \subseteq E(M)$ with $|T| = n - 1$. If $|Q| \geq 2n - 2$ for every cocircuit $Q$ containing $T$, then $M'_T$ is $n$-connected.

Proof. Let $Q$ be a cocircuit of $M$ intersecting $T$. By Proposition 3.3, it is sufficient to prove that $|Q| \geq 2|Q \cap T|$. If $T \subseteq Q$, then $|Q| \geq 2n - 2 = 2|T| = 2|Q \cap T|$. Suppose $T \not\subseteq Q$. Then $|Q \cap T| < |T| = n - 1$ and hence $|Q \cap T| \leq n - 2$. Since $2 \leq n \leq 4$, we have $2|Q \cap T| \leq 2(n - 2) = 2n - 4 \leq n$. By Lemma 3.1, $|Q| \geq n$ and so $|Q| \geq 2|Q \cap T|$.

We combine Main Theorem 1.4 and Theorem 1.5 and provide a shorter proof of Theorem 1.5.

Theorem 3.5. Let $n \geq 2$ be an integer and $M$ be an $n$-connected binary matroid with $|E(M)| \geq 2n - 2$. Suppose $T \subset E(M)$ with $|T| = n - 1$. Then the following statements are equivalent.

(i). $M'_T$ is $n$-connected.

(ii). $|Q| \geq 2|Q \cap T|$ for every cocircuit $Q$ of $M$ intersecting $T$.

(iii). For any subset $A \subseteq E(M)$ with $|A| = n - 2$, there exists a circuit $C$ of $M$ containing an odd number of elements of $T$ and is contained in $E(M) - A$.

Proof. (i) $\implies$ (ii) follows from Lemma 3.2 and (ii) $\implies$ (i) follows from Proposition 3.3.

(i) $\implies$ (iii). Suppose (i) holds but (iii) does not hold. Then there is a subset $A \subseteq E(M)$ with $|A| = n - 2$ such that no circuit of $M$ containing an odd number of elements of $T$ is contained in $E(M) - A$. Let $A' = A \cup \{a\}$ and $B = E(M) - A$. Then $|A'| = n - 1$ and $|B| \geq n - 1$. Let $r$ and $r'$ be the rank function of $M$ and $M'_T$, respectively. By Lemma 3.1, $A$ contains neither a cocircuit of $M$ nor a cocircuit of $M'_T$. Hence $r(B) = r(M)$ and $r'(B) = r'(M'_T)$. Also, by Lemma 2.3 (iii), $r(B) = r'(B)$. This gives $r(M) = r'(M'_T)$, a contradiction by Corollary 2.4. Hence (ii) implies (iii).

(iii) $\implies$ (i). Suppose (iii) holds but (i) does not hold. Then $M'_T$ has an $(n - 1)$-separation $(A, B)$. Therefore

$$\min\{|A|, |B|\} \geq n - 1$$

and $r'(A) = r'(B) = r'(M'_T) \leq n - 2$. . . . . . (∗)
Without loss of generality, assume that $a \in A$. By Lemma 2.3(i), $r'(A) = r(A - \{a\}) + 1$. If $|A| \geq n$, then, by (*),

$$r(A - \{a\}) + r(B) - r(M) \leq r'(A) - 1 + r'(B) - (r'(M_T) - 1) \leq n - 2.$$ 

Therefore $(A - \{a\}, B)$ is an $(n - 1)$-separation of $M$, a contradiction. Hence $|A| = n - 1$. Then $|A - \{a\}| = n - 2$. By (iii) and Lemma 2.3(ii), $r'(B) = r(B) + 1$. Therefore

$$r(A - \{a\}) + r(B) - r(M) \leq r'(A) - 1 + r'(B) - 1 - (r'(M_T) - 1) = n - 3.$$ 

This shows that $(A, B)$ is an $(n - 2)$-separation of $M$, a contradiction. Thus (iii) implies (i). □

4. Consequences to Graphs

In this section, we prove that Proposition 3.3 is a matroid extension of Theorem 1.2. We need the following result.

**Theorem 4.1** ([6], pp. 328). For $n \geq 2$, let $G$ be a graph without isolated vertices and with at least $n + 1$ vertices. Then the circuit matroid $M(G)$ is $n$-connected if and only if $G$ is $n$-connected and has no cycle with fewer than $n$ edges.

By Theorem 4.1, the circuit matroid $M(G)$ of an $n$-connected graph $G$ is not $n$-connected if $G$ contains a cycle of length less than $n$. Therefore we derive Theorem 1.2 from Proposition 3.3 by assuming that $G$ has girth at least $n$.

**Theorem 4.2.** Suppose $G$ is an $n$-connected graph of girth at least $n$, where $n \geq 2$. Let $T$ be a set of $n - 1$ edges incident to a vertex of degree at least $2n - 2$ in $G$. Then the $n$-point splitting graph $G'_T$ is $n$-connected.

**Proof.** Let $M = M(G)$. Then $M'_T = M(G'_T)$. We prove that $M'_T$ is $n$-connected. By Theorem 4.1, $M$ is $n$-connected. Let $Q$ be a cocircuit of $M$ intersecting $T$. By Proposition 3.3, it is sufficient to prove that $|Q| \geq 2|Q \cap T|$. On the contrary, assume that $|Q| < 2|Q \cap T|$. As $Q = (Q - T) \cup (Q \cap T)$,

$$|Q| = \frac{|Q|}{2} + \frac{|Q|}{2} = |Q - T| + |Q \cap T|$$

and hence $|Q - T| < \frac{|Q|}{2} < |Q \cap T|$. Let $u$ be the vertex of $G$ of degree at least $2n - 2$ such that the edges of $G$ belonging to $T$ are incident to $u$. Since $Q$ is a cocircuit of $M(G)$, the graph $G - Q$ is disconnected and it has two components, say $C_1$ and $C_2$. We may assume that $C_2$ contains the vertex $u$. Let $Q \cap T = \{w_1u_1, w_2u_2, \ldots, w_ku_k\}$. Then $v_1, v_2, \ldots, v_k$ are vertices of $C_1$. Let $v_1, v_2, \ldots, v_r$ be the end vertices of the edges belonging to $Q - T$ in $C_1$. Then $r \leq |Q - T| < |Q \cap T|$. Since $|Q| < 2|Q \cap T| \leq 2|T| = 2n - 2$ and degree of $u$ is at least $2n - 2$, there is at least one edge $uw$ incident to $u$ in $G - Q$. Then the edge $uw$ is in $C_2$. Let $A = \{v_1, v_2, \ldots, v_r, u\}$. Then $G - A$ is a disconnected component, leaving $u_i$ for some $i \in \{1, 2, \ldots, k\}$ in one component and the vertex $w$ is in another component. However, $|A| = r + 1 \leq |Q \cap T| \leq |T| = n - 1$, a contradiction to the fact that $G$ is $n$-connected. Thus $M'_T$ is $n$-connected. By Theorem 4.1, $G'_T$ is $n$-connected. □

**Corollary 4.3.** Let $G$ be a 3-connected simple graph and $T$ be a set of two edges incident to a vertex of $G$ of degree at least four. Then the graph $G'_T$ is 3-connected.

We now prove that one can obtain a 3-regular, 3-connected graph from the given 3-connected simple graph by repeated applications of 3-point splitting operation.

**Corollary 4.4.** A 3-regular, 3-connected simple graph can be obtained from the given 3-connected simple graph by a finite sequence of the 3-point splitting operation.

**Proof.** Let $G$ be a 3-connected simple graph. Then degree of every vertex of $G$ is at least three. Suppose $G$ contains a vertex $v$ of degree $k > 3$. Let $T = \{x, y\}$ be a set of two edges incident at $v$. By Corollary 4.3, $G'_T$ is 3-connected. The vertex of $v$ of $G$ is replaced by two vertices $v'$ and $v''$ with degrees 3 and $k - 1$, respectively in $G'_T$. Thus one application of 3-point splitting on a vertex of degree $t > 3$ results into a 3-connected graph with one additional vertex of degree less than $t$. By a finite sequence of 3-point splitting operation we can get a 3-connected graph with no vertex of degree greater than three. Clearly, this graph will be 3-regular. □
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