A 2-element antichain that is not contained in any finite retract

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Abstract. We give an example of an ordered set $P$ which contains a 2-element antichain that is not contained in any finite retract of $P$.

1. Introduction

The question in [2, p. 259, Remark 8] asks if every finite subset of an (infinite) ordered set is actually contained in a finite retract. The question was motivated by the product problem for the fixed point property, but, as a possible structural property, it is interesting in its own right.

Moreover, by [1, Theorem 2], every isometric spanning fence is a retract, which means that in a chain-complete ordered set, every 2-antichain (that is, 2-element antichain) consisting of minimal or maximal elements is contained in a finite retract. The construction can be generalized to show that in an arbitrary ordered set, every 2-antichain in which one of the two elements is maximal or minimal is contained in a finite retract: Let $\{m, a\}$ be an antichain and without loss of generality let $m$ be minimal. Let $m = f_0 < f_1 > f_2 < \cdots f_n = a$ be a shortest possible fence from $m$ to $a$. Mapping the elements whose distance to $m$ is $j < n$ to $f_j$ and mapping the elements whose distance to $m$ is $\geq n$ to $f_n = a$ is a retraction.

Given the generality and simplicity of the above construction, it is all the more surprising that there is an ordered set of height 2 with a 2-antichain that is not contained in any finite retract.

2. The construction

Lemma 1. Let $P$ be an ordered set, let $\{a, b\} \subseteq P$ be a 2-antichain, and let $F = \{a = f_0 < f_1 > \cdots f_n = b\}$ be a shortest possible fence from $f_0$ to $f_n$. If there is no other fence from $a$ to $b$ that is of length $n$ or if any other fence $F'$ from $a$ to $b$ that is of length $n$ and has the property that $a = f'_0 < f'_1$, then any retract of $P$ that contains $\{a, b\}$ must contain $F$.

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Proof. Let \( r: P \to P \) be a retraction such that \( r(a) = a \) and \( r(b) = b \). Now,
\[
a = f_0 < f_1 > \cdots f_n = b
\]
implies
\[
a = r(a) = r(f_0) \leq r(f_1) \geq \cdots r(f_n) = r(b) = b.
\]
Because the distance from \( a \) to \( b \) is \( n \), the \( r(f_j) \) must form a fence of length \( n \)
from \( a \) to \( b \). In particular, we have
\[
a = r(a) = r(f_0) < r(f_1) > \cdots r(f_n) = r(b) = b.
\]
By the conditions on \( P \), the image of \( F \) under \( r \) must be \( F \) itself. \( \square \)

To construct the ordered set \( P \) for our example, let \( a \) and \( b \) be two points. Let
\[
U := \{a = u_0 < u_1 > u_2 < u_3 > \cdots > u_{90} = b\}
\]
and
\[
L := \{a = l_0 > l_1 < l_2 > l_3 < \cdots < l_{90} = b\}
\]
be two fences that are disjoint, except for the endpoints. Let
\[
G := \{u_{32} = g_0 < g_1 > g_2 < \cdots > g_{10} = l_{33}\},
\]
\[
H := \{u_{34} = h_0 < h_1 > h_2 < \cdots > h_{10} = l_{35}\},
\]
\[
I := \{u_{36} = i_0 < i_1 > i_2 < \cdots > i_{10} = l_{37}\},
\]
\[
J := \{u_{38} = j_0 < j_1 > j_2 < \cdots > j_{10} = l_{39}\},
\]
\[
K := \{u_{40} = k_0 < k_1 > k_2 < \cdots > k_{10} = l_{41}\},
\]
be pairwise disjoint fences that only intersect \( U \) at their starting points and
that only intersect \( L \) at their endpoints. (The numbers 10 and 90 are in no
way optimal. They were chosen to make it obvious that the construction works
as desired.) Let \( Q_0 := \{g_4, h_4, i_4, j_4, k_4\} \). For \( n \geq 1 \), let \( Q_n \) be the set of two
element subsets of \( Q_{n-1} \), considered as an antichain. Let
\[
P := U \cup L \cup G \cup H \cup I \cup J \cup K \cup \bigcup_{n=1}^{\infty} Q_n.
\]
The order on \( P \) is the union of the orders on \( U, L, G, H, I, J, \) and \( K \) together
with the element relation \( \leq : = \in \) on \( Q_{2k} \cup Q_{2k+1} \) and the reverse element
relation \( \leq := \supset \) on \( Q_{2k+1} \cup Q_{2k+2} \). The resulting ordered set has height 2.

Now let \( r: P \to P \) be a retraction such that \( \{a, b\} \subseteq r[P] \). Then, by
Lemma 1, \( U \subseteq r[P] \) and \( L \subseteq r[P] \). Similarly, because \( G, H, I, J, \) and \( K \)
are the shortest fences between their endpoints, we must have \( G \subseteq r[P], H \subseteq r[P], \)
\( I \subseteq r[P], J \subseteq r[P], \) and \( K \subseteq r[P] \). Thus, in particular, \( r \) is the identity on \( Q_0 \).
Once more by Lemma 1, we must have that \( r \) is the identity on \( Q_1 \), because
for every \( \{x, y\} \in Q_1 \), the fence \( x \in \{x, y\} \supset y \) is the shortest fence from \( x \)
to \( y \).

We now prove inductively that \( r \) is the identity on \( \bigcup_{n=2}^{\infty} Q_n \). Let \( x \in Q_m \),
assume that \( r \) is the identity on \( \bigcup_{j=0}^{m-1} Q_j \), and, without loss of generality,
assume that \( m \) is even. Then \( x = \{g, h\} \) for two distinct elements \( g, h \in Q_{m-1} \).
and \(x < g, h\). By definition of \(Q_{m-1}\), as the set of two-element subsets of \(Q_{m-2}\), there is at most one \(v \in Q_{m-2}\) such that \(v < g\) and \(v < h\). If there is no such \(v \in Q_{m-2}\), then \(r(x) = x\), because \(g\) and \(h\) are fixed by \(r\) and \(x\) is their only common lower cover. In this case there is such a \(v \in Q_{m-2}\), suppose for a contradiction that \(r(x) = v\). Let \(s, t, u, v,\) and \(w\) be 5 distinct elements of \(Q_{m-2}\). We may assume that \(g = \{u, v\}\) and \(h = \{v, w\}\). Let \(g_* := \{u, w\} \in Q_{m-1}, h_* := \{s, t\} \in Q_{m-1}\), and \(x_* := \{g_*, h_*\} \in Q_m\). Then, because \(x_*\) is the only common lower cover of \(g_*\) and \(h_*\), which are both fixed by \(r\), we have \(r(x_*) = x_*\). Now, \(x, x_* \in Q_{m+1}\), so \(r(x), r(x_*) < r(\{x, x_*\})\). But there is no element greater than both \(v = r(x)\) and \(x_* = r(x_*)\), a contradiction. Thus, \(r(x) \neq v\). Because \(x\) and \(v\) are the only common lower covers of the elements \(g\) and \(h\), which are fixed by \(r\), we must thus have \(r(x) = x\) in this case, too. This proves that \(r\) is the identity on \(Q_m\). Hence, \(r\) is the identity on \(\bigcup_{n=0}^{\infty} Q_n\), and thus it is the identity on \(P\). So in fact, the only retract of \(P\) that contains \(\{a, b\}\) is \(P\) itself.

If an example without infinite fences is desired, the union \(\bigcup_{n=1}^{\infty} Q_n\) could be replaced with a disjoint union of sets \(Z_n := \bigcup_{j=1}^{n} Q_j\) attached in the same fashion to \(Q_0\). In this set, once more \(\{a, b\}\) is not contained in any finite retract, but there are retracts other than \(P\) that contain \(\{a, b\}\): First of all, because the induction above also used an element of \(Q_{m+1}\), a retraction that fixes the elements of a union, \(\bigcup_{j=1}^{m} Q_j\) in \(Z_n\), could still not fix some elements of \(Q_n\). Second, a retraction could map a set \(Z_n\) to a set \(Z_{n+j}\). But because no retraction can map a set \(Z_{n+j}\) to a set \(Z_n\), any retraction that fixes \(a\) and \(b\) must, for infinitely many \(n\), fix the first \(n-1\) stages of the set \(Z_n\), which means that the retract must be infinite.

### 3. Concluding remarks

As we have noted, our set \(P\) has height 2. By [1, Theorem 2], every antichain consisting of 2 elements in a poset of height 1 is contained in a finite retract. This is no longer true for 3-element antichains. To see this, modify our construction by replacing \(L\) with

\[ U' = \{a = u'_0 < u'_1 > u'_2 < \cdots > u'_{g_0} < u'_{g_1} > u'_{g_2} = b\}, \]

and put \(g_{10} = u'_{32}, h_{10} = u'_{34}, i_{10} = u'_{36}, j_{10} = u'_{38}, k_{10} = u'_{40}\). It is easy to see that the antichain \(\{a, b, u'_{45}\}\) is not contained in any finite retract.

The original motivation for the question in [2] came from fixed point theory. Neither the set \(P\) nor any of its modifications considered in this note do have the fixed point property. Therefore, the question remains open whether every finite subset of a poset with the fixed point property is contained in a finite retract of this poset. Also, an answer for posets that do not contain infinite antichains could be interesting.
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