Recent theoretical progress on an information geometrodynamical approach to chaos

Carlo Cafaro

Department of Physics, University at Albany-SUNY, 1400 Washington Avenue, Albany, NY 12222, USA

In this paper, we report our latest research on a novel theoretical information-geometric framework suitable to characterize chaotic dynamical behavior of arbitrary complex systems on curved statistical manifolds. Specifically, an information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth and an information-geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain in a tilted (transverse) external magnetic field are presented.

PACS numbers: 02.50.Tt-Inference methods; 02.40.Ky- information geometry; 05.45.-a- chaos.

I. INTRODUCTION

Research on complexity [1, 2] has created a new set of ideas on how very simple systems may give rise to very complex behaviors. Moreover, in many cases, the "laws of complexity" have been found to hold universally, caring not at all for the details of the system’s constituents. Chaotic behavior is a particular case of complex behavior and it will be the object of the present work.

In this paper we make use of the so-called Entropic Dynamics (ED) [3]. ED is a theoretical framework that arises from the combination of inductive inference (Maximum Entropy Methods (ME), [4, 5]) and Information Geometry (IG) [6]. The most intriguing question being pursued in ED stems from the possibility of deriving dynamics from purely entropic arguments. This is clearly valuable in circumstances where microscopic dynamics may be too far removed from the phenomena of interest, such as in complex biological or ecological systems, or where it may just be unknown or perhaps even nonexistent, as in economics. It has already been shown that entropic arguments do account for a substantial part of the formalism of quantum mechanics, a theory that is presumably fundamental [7]. Laws of physics may just be consistent, objective ways to manipulate information. Following this line of thought, we extend the applicability of ED to temporally-complex (chaotic) dynamical systems on curved statistical manifolds and identify relevant measures of chaoticity of such an information geometrodynamical approach to chaos (IGAC).

The layout of the paper is as follows. In the section II, we give an introduction to the main features of our IGAC. In section III, we apply our theoretical construct to three complex systems. We emphasize that we have omitted technical details some of which can be found in our previous articles [8, 9, 10, 11, 12, 13, 14, 15]. Finally, in section IV we present our final remarks and future research directions.

II. GENERAL FORMALISM OF THE IGAC

The IGAC is an application of ED to complex systems of arbitrary nature. ED is a form of information-constrained dynamics built on curved statistical manifolds $\mathcal{M}_S$ where elements of the manifold are probability distributions $\{P(X|\Theta)\}$ that are in a one-to-one relation with a suitable set of macroscopic statistical variables $\{\Theta\}$ that provide a convenient parametrization of points on $\mathcal{M}_S$. The set $\{\Theta\}$ is called the parameter space $\mathcal{D}_\Theta$ of the system.

In what follows, we schematically outline the main points underlying the construction of an arbitrary form of entropic dynamics. First, the microstates of the system under investigation must be defined. For the sake of simplicity, we assume our system is characterized by an $l$-dimensional microspace with microstates $\{x_i\}$ where $i = 1, \ldots, l$. The main goal of an ED model is that of inferring "macroscopic predictions" in the absence of detailed knowledge of the microscopic nature of arbitrary complex systems. Once the microstates have been defined, we have to select the relevant information about such microstates. In other words, we have to select the macrospace of the system. For the sake of the argument, we assume that our microstates are Gaussian-distributed. They are defined by $2l$-information constraints, for example their expectation values $\mu_i$ and variances $\sigma_i$. In addition to information constraints, each Gaussian distribution $p_k(x_k|\mu_k, \sigma_k)$ of each microstate $x_k$ must satisfy the usual normalization conditions. Once the

*Electronic address: carlocafar2000@yahoo.it
microstates have been defined and the relevant information constraints selected, we are left with a set of probability distributions \( p(X|Θ) = \prod_{k=1}^{l} p_k(x_k|μ_k, σ_k) \) encoding the relevant available information about the system where \( X \) is the \( l \)-dimensional microscopic vector with components \( (x_1, ..., x_l) \) and \( Θ \) is the 2\( l \)-dimensional macroscopic vector with coordinates \( (μ_1, ..., μ_l; σ_1, ..., σ_l) \). The set \{\( Θ \)\} define the 2\( l \)-dimensional space of macrostates of the system, the statistical manifold \( M_S \). A measure of distinguishability among macrostates is obtained by assigning a probability distribution \( P(X|Θ) \equiv M_S \) to each macrostate \( Θ \). Assignment of a probability distribution to each state endows \( M_S \) with a metric structure. Specifically, the Fisher-Rao information metric \( g_{\mu\nu}(Θ) \) defines a measure of distinguishability among macrostates on \( M_S \), with \( M_S \) being defined as the set of probabilities \{\( p(X|Θ) \)\} described above where \( X \in \mathbb{R}^{3N}, Θ \in D_Θ = [I_μ × I_σ]^{3N} \). The parameter space \( D_Θ \) (homeomorphic to \( M_S \)) is the direct product of the parameter subspaces \( I_μ \) and \( I_σ \), where \( I_μ = (−∞, +∞)_μ \) and \( I_σ = (0, +∞)_σ \) in the conventional Gaussian case. Once \( M_S \) and \( D_Θ \) are defined, the ED formalism provides the tools to explore dynamics driven on \( M_S \) by entropic arguments. Specifically, given a known initial macrostate \( Θ^{(\text{initial})} \) (probability distribution), and that the system evolves to a final known macrostate \( Θ^{(\text{final})} \), the possible trajectories of the system are examined in the ED approach using ME methods.

The geodesic equations for the macrovariables of the Gaussian ED model are given by nonlinear second order coupled ordinary differential equations. They describe a reversible dynamics whose solution is the trajectory between an initial \( Θ^{(\text{initial})} \) and a final macrostate \( Θ^{(\text{final})} \). Given the Fisher-Rao information metric, we can apply standard methods of Riemannian differential geometry to study the information-geometric structure of the manifold \( M_S \) underlying the entropic dynamics. Connection coefficients \( \Gamma^ρ_{\mu\nu} \), Ricci tensor \( R_{\mu\nu} \), Riemannian curvature tensor \( R_{\mu\nu\rho\sigma} \), sectional curvatures \( K_{M_S} \), scalar curvature \( S_{M_S} \), scalar curvature \( R_{M_S} \), Weyl anisotropy tensor \( W_{\mu\nu\rho\sigma} \), Killing fields \( ξ^μ \) and Jacobi fields \( J^μ \) can be calculated in the usual way.

To characterize the chaotic behavior of complex entropic dynamical systems, we are mainly concerned with the signs of the scalar and sectional curvatures of \( M_S \), the asymptotic behavior of Jacobi fields \( J^μ \) on \( M_S \), the existence of Killing vectors \( ξ^μ \) and the asymptotic behavior of the information-geometrodynamical entropy (IGE) \( S_{M_S} \) (see 1). It is crucial to observe that true chaos is identified by the occurrence of two features: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., stretching of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., folding of dynamical trajectories. The negativity of the Ricci scalar \( R_{M_S} \) implies the existence of expanding directions in the configuration space manifold \( M_s \). Indeed, since \( R_{M_S} \) is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements \( \{e_ρ = \partial_Θ^ρ\} \), the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability. A powerful mathematical tool we use to investigate the stability or instability of a geodesic flow is the Jacobi-Levi-Civita equation (JLC-equation) for geodesic spread. Finally, the asymptotic regime of diffusive evolution describing the possible exponential increase of average volume elements on \( M_s \) provides another useful indicator of dynamical chaoticity. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifold. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the von Neumann entropy increases linearly at a rate determined by the Lyapunov exponents. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [16, 17]. In our information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing ED models is the information geometrodynamical entropy (IGE) defined as 3,

\[
S_{M_s}(τ) \overset{\text{def}}{=} \lim_{τ \to +∞} \log V_{M_s} \text{ with } V_{M_s}(τ) = \frac{1}{τ} \int_0^τ \int_{M_s(τ')} \sqrt{g} d^2θ
\]

and \( g = |\det (g_{\mu\nu})| \). IGE is the asymptotic limit of the natural logarithm of the statistical weight defined on \( M_s \) and represents a measure of temporal complexity of chaotic dynamical systems whose dynamics is underlined by a curved statistical manifold.

### III. APPLICATIONS OF THE IGAC

As a first example, we apply our IGAC to study the dynamics of a system with \( l \) degrees of freedom, each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statisti-
The line element \( ds^2 = g_{\mu \nu} (\Theta) d\Theta^\mu d\Theta^\nu \) on \( M_s \) with \( \mu, \nu = 1, \ldots, 2l \) is defined by \[ (2) \]

\[
\sum_{k=1}^{l} \left( \frac{1}{\sigma_k^2} d\mu_k^2 + \frac{2}{\sigma_k^2} d\sigma_k^2 \right).
\]

This leads to consider an ED model on a non-maximally symmetric 2l-dimensional statistical manifold \( M_s \). It is shown that \( M_s \) possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, \( R_{M_s} = -l \). It is shown that the system explores statistical volume elements on \( M_s \) at an exponential rate. The information geometrodynamical entropy \( S_{M_s} \) increases linearly in time (statistical evolution parameter) and is moreover, proportional to the number of degrees of freedom of the system, \( S_{M_s} \sim l \Lambda \). The parameter \( \lambda \) characterizes the family of probability distributions on \( M_s \). The asymptotic linear information-geometrodynamical entropy growth may be considered the information-geometric analogue of the von Neumann entropy growth introduced by Zurek-Paz, a quantum feature of chaos. The geodesics on \( M_s \) are hyperbolic trajectories. Using the JLC-equation, we show that the Jacobi vector field intensity \( J_{M_s} \) diverges exponentially and is proportional to the number of degrees of freedom of the system, \( J_{M_s} \sim \exp (\lambda \tau) \). The exponential divergence of the Jacobi vector field intensity \( J_{M_s} \) is a classical feature of chaos. Therefore, we conclude that \( R_{M_s} = -l, J_{M_s} \sim \exp (\lambda \tau), S_{M_s} \sim \Lambda \). Thus, \( R_{M_s}, S_{M_s}, J_{M_s} \) and \( M_s \) behave as proper indicators of chaoticity.

In our second example, we employ ED and “Newtonian Entropic Dynamics” (NED) \[ (4) \]. In our special application, we consider a manifold with a line element \( ds^2 = g_{\mu \nu} (\Theta) d\Theta^\mu d\Theta^\nu \) with \( \mu, \nu = 1, \ldots, l \) given by \[ (3) \]

\[
[1 - \Phi (\Theta)] \delta_{\mu \nu} d\Theta^\mu d\Theta^\nu, \Phi (\Theta) = \sum_{k=1}^{l} u_k (\theta_k)
\]

where \( u_k (\theta_k) = -\frac{1}{2} \omega_k^2 \theta_k^2 \theta_k^k \) and \( \theta_k = \theta_k (s) \). The geodesic equations for the macrovariables \( \theta_k (s) \) are strongly nonlinear and their integration is not trivial. However, upon a suitable change of the affine parameter \( s \) used in the geodesic equations, we may simplify the differential equations for the macroscopic variables parametrizing points on the manifold \( M_s \), with metric tensor \( g_{\mu \nu} \). Recalling that the notion of chaos is observer-dependent and upon changing the affine parameter from \( s \) to \( \tau \) in such a way that \( ds^2 = 2 (1 - \Phi) \delta_{\mu \nu} d\Theta^\mu d\Theta^\nu, \Phi (\Theta) = \sum_{k=1}^{l} u_k (\theta_k) \), we obtain new geodesic equations describing a set of macroscopic inverted harmonic oscillators (IHOs). In order to ensure the compactification of the parameter space of the system, we choose a Gaussian distributed frequency spectrum for the IHOs. Thus, with this choice, the folding mechanism required for true chaos is restored in a statistical (averaging over \( \omega \) and \( \tau \)) sense. Upon integrating these differential equations, we obtain the expression for the asymptotic behavior of the IGE \( S_{M_s} \), namely \( S_{M_s} (\tau) \sim \Lambda \tau \) with \( \Lambda = \sum_{i=1}^{l} \omega_i \). This result may be considered the information-geometric analogue of the Zurek-Paz model used to investigate the implications of decoherence for quantum chaos. They considered a chaotic system, a single unstable harmonic oscillator characterized by a potential \( V (x) = -\frac{\Omega x^2}{2} (\Omega \) is the Lyapunov exponent), coupled to an external environment. In the reversible classical limit \[ (5) \], the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent, \( S_{\text{chaotic}} (\tau) \sim \Lambda \Omega \tau \).

In our final example, we use our IGAC to study the entropic dynamics on curved statistical manifolds induced by classical probability distributions of common use in the study of regular and chaotic quantum energy level statistics. It is known \[ (6) \] that integrable and chaotic quantum antiferromagnetic Ising chains are characterized by asymptotic logarithmic and linear growths of their operator space entanglement entropies, respectively. In this last example, we consider the information-geometrodynamics of a Poisson distribution coupled to an Exponential bath (spin chain in a transverse magnetic field, regular case) and that of a Wigner-Dyson distribution coupled to a Gaussian bath (spin chain in a tilted magnetic field, chaotic case). Remarkably, we show that in the former case the IGE exhibits asymptotic logarithmic growth while in the latter case the IGE exhibits asymptotic linear growth. In the regular case, the line element \( ds^2_{\text{integrable}} \) on the statistical manifold \( M_s \) is given by \[ (7) \]

\[
\frac{1}{\mu_A} d\mu_A^2 + \frac{1}{\mu_B} d\mu_B^2
\]

where the macrovariable \( \mu_A \) is the average spacing of the energy levels and \( \mu_B \) is the average intensity of the magnetic energy arising from the interaction of the transverse magnetic field with the spin \( \frac{1}{2} \) particle magnetic moment. In such a case, we show that the asymptotic behavior of \( S_{M_s}^{\text{integrable}} \) is sub-linear in \( \tau \) (logarithmic IGE growth), \( S_{M_s}^{\text{integrable}} (\tau) \sim \log \tau \). Finally, in the chaotic case, the line element \( ds^2_{\text{chaotic}} \) on the statistical manifold \( M_s \) is
given by \[13, 15\],
\[
d s^2_{\text{chaotic}} = \frac{4}{\mu_A^2} d\mu_A^2 + \frac{1}{\sigma_B^2} d\mu_B^2 + \frac{2}{\sigma_B^2} d\sigma_B^2
\] (5)

where the (nonvanishing) macrovariable $\mu_A'$ is the average spacing of the energy levels, $\mu_B'$ and $\sigma_B'$ are the average intensity and variance, respectively of the magnetic energy arising from the interaction of the tilted magnetic field with the spin $\frac{1}{2}$ particle magnetic moment. In this case, we show that asymptotic behavior of $S_{M_s}^{(\text{chaotic})}$ is linear in $\tau$ (linear IGE growth), $S_{M_s}^{(\text{chaotic})}(\tau) \sim \tau$. The equations for $S_{M_s}^{(\text{integrable})}$ and $S_{M_s}^{(\text{chaotic})}$ may be considered as the information-geometric analogues of the entanglement entropies defined in standard quantum information theory in the regular and chaotic cases, respectively.

IV. CONCLUSION

In this paper we presented a novel theoretical information-geometric framework suitable to characterize chaotic dynamical behavior of arbitrary complex systems on curved statistical manifolds. Specifically, an information-geometric analogue of the Zurek-Paz quantum chaos criterion of linear entropy growth and an information-geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain in a tilted (transverse) external magnetic field were presented.

The descriptions of a classical chaotic system of arbitrary interacting degrees of freedom, deviations from Gaussianity and chaoticity arising from fluctuations of positively curved statistical manifolds are currently under investigation. Furthermore, we are investigating the possibility to extend the IGAC to quantum Hilbert spaces constructed from classical curved statistical manifolds and we are considering the information-geometric macroscopic versions of the Henon-Heiles and Fermi-Pasta-Ulam $\beta$-models to study chaotic geodesic flows on statistical manifolds. Finally, the information geometry of a chaotic spring pendulum and a periodically perturbed spherical pendulum are currently under investigation. Soft chaos regimes arising in chemical physics are being considered as well. Our objective is to study transitions from order to chaos in a floppy molecule using inference methods and information geometry.

At this stage of its development, IGAC remains an ambitious unifying information-geometric theoretical construct for the study of chaotic dynamics with several limitations and unsolved problems \[12, 13\]. However, based on our recent findings, we believe it could provide an interesting, innovative and potentially powerful way to study and understand the very important and challenging problems of classical and quantum chaos. Therefore, we believe our research program deserves further investigation and developments.

Acknowledgments

The author is grateful to S. A. Ali, Ariel Caticha and Adom Giffin for very useful comments. Thanks are extended to all MaxEnt 2008 participants in Sao Paulo- Brazil, especially to Rafael M. Gutierrez for his sincere interest and very important comments on our works.

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