Semiclassical limits of distorted plane waves in chaotic scattering without a pressure condition

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Abstract

In this paper, we study the semi-classical behavior of distorted plane waves, on manifolds that are Euclidean near infinity or hyperbolic near infinity, and of non-positive curvature. Assuming that there is a strip without resonances below the real axis, we show that distorted plane waves are bounded in $L^2_{\text{loc}}$ independently of $h$, that they admit a unique semiclassical measure, and we prove bounds on their $L^p_{\text{loc}}$ norms.

1 Introduction

Consider a Riemannian manifold $(X, g)$ of dimension $d \geq 2$ which is Euclidean near infinity, that is to say, such that there exists $X_0 \subset X$ and $R_0 > 0$ such that $(X \setminus X_0, g)$ and $(\mathbb{R}^d \setminus B(0, R_0), g_{\text{eucl}})$ are isometric. The distorted plane waves on $X$ are a family of functions $E_h(x; \xi)$ with parameters $\xi \in S^{d-1}$ (the direction of propagation of the incoming wave) and $h$ (a semiclassical parameter corresponding to the inverse of the square root of the energy) such that

$$(-h^2 \Delta - 1)E_h(x; \xi) = 0,$$

and which can be put in the form

$$E_h(x; \xi) = (1 - \chi_0)e^{ix \cdot \xi / h} + E_{\text{out}}.$$  

(1.1)

Here, $\chi_0 \in C_0^\infty$ is such that $\chi_0 \equiv 1$ on $X_0$, and $E_{\text{out}}(x; \xi, h)$ is outgoing in the sense that it satisfies the Sommerfeld radiation condition, were $|x|$ is the distance to any fixed point in $X$:

$$\lim_{|x| \to \infty} |x|^{(d-1)/2}(\frac{\partial}{\partial |x|} - \frac{i}{h})E_{\text{out}} = 0.$$  

(1.3)

It can be shown (cf. [Mel95, §2] or [DZ, §4]) that there is only one function $E_h(\cdot; \omega)$ such that (1.1) is satisfied and which can be put in the form (1.2).

Actually, the term $E_{\text{out}}$ can be given an explicit expression in terms of the outgoing resolvent, that is to say, in terms of the family of operators $R_+(z; h) = (-h^2 \Delta - z)^{-1}$, which is well defined for $\Im(z) > 0$ as an operator from $L^2(X)$ to itself.

It is well-known (see [DZ, §4 and §5]) that, if $\chi \in C_0^\infty(X)$, then for any $h > 0$, $z \mapsto \chi R_+(z; h)\chi$ can be extended to $\mathbb{C}\setminus(-\infty, 0]$ as a meromorphic function. Its poles, which are independent of the
choice of $\chi$, are called the resonances of $-h^2\Delta$. Since there are no resonances on $[0, \infty)$, $R_+(1; h)$ is well defined as an operator $L^2_{comp} \to L^2_{loc}$.

Let us write, for $x \in X \setminus X_0$,

$$E_h^0(x; \xi) = e^{\pm x \cdot \xi}.$$

With $\chi_0$ as in (1.2), we set

$$F_h(\cdot; \xi) = [h^2\Delta, \chi_0]E_h^0(\cdot; \xi),$$

which is compactly supported, and satisfies $\|F_h\|_{L^2} = O(h)$.

We then have

$$E_{out}(\cdot; \xi, h) := R_+(1; h)F_h(\cdot; \xi), \quad (1.4)$$

In this paper, we will be interested in the behavior of $E_h(\cdot; \xi)$ in the semiclassical limit $h \to 0$. The first question we would like to address is whether $E_h$ is bounded in $L^2_{loc}$ uniformly with respect to $h$. More generally, we will be interested in the semiclassical limits of $E_h$, and in the behavior of the $L^2_{loc}$ norms of $E_h(\cdot; \xi)$ as $h \to 0$.

In [Ing17a] and [Ing17b], the author answered these questions under some assumptions on the dynamics of the geodesic flow. Let us denote by $p$ the classical Hamiltonian $p : T^*X \ni (x, \xi) \mapsto \|\xi\|_2^2 \in \mathbb{R}$.

For each $t \in \mathbb{R}$, we denote by $\Phi^t : T^*X \to T^*X$ the geodesic flow generated by $p$ at time $t$. We will write by the same letter its restriction $\Phi^t : S^*X \to S^*X$ to the energy layer $p(x, \xi) = 1$.

The trapped set is defined as

$$K := \{(x, \xi) \in S^*X; \Phi^t(x, \xi) \text{ remains in a bounded set for all } t \in \mathbb{R}\}. \quad (1.5)$$

One of the main result of [Ing17a] was the following. Suppose that the trapped set is hyperbolic, and that the topological pressure associated to half the unstable jacobian is negative: $P(1/2) < 0$. (see section 2.2 for the definition of a hyperbolic set, and of the topological pressure). Then $E_h$ is uniformly bounded in $L^2_{loc}(\cdot; \xi)$, and it has a unique semiclassical measure. In [Ing17b], under the additional assumption that $(X, g)$ has non-positive curvature, it was shown that $E_h(\cdot; \xi)$ is uniformly bounded in $L^2_{loc}$.

The aim of this paper is to extend some of the results of [Ing17a] and [Ing17b] in the case where no assumption is made on the topological pressure associated to half the unstable jacobian. Instead, we will make the weaker assumption that there is a resonance-free strip below the real axis.

**Resonance-free strip** In the sequel, we will suppose that there exists $\varepsilon_0, h_0, C_0 > 0$, such that for all $0 < h < h_0$, $-h^2\Delta$ has no resonances in

$$D_h := \left\{ z \in \mathbb{C}; \Re z \in [1 - \varepsilon_0, 1 + \varepsilon_0] \text{ and } \Im z \geq -C_0 h \right\}. \quad (1.6)$$

Furthermore, we suppose that there exists $\alpha > 0$ such that the following holds. For any $\chi \in C^\infty_c(X)$, there exists $C_\chi > 0$ such that for all $z \in D_h$,

$$\|\chi R_+(z; h)\chi\|_{L^2 \to L^2} \leq C_\chi h^{-\alpha}. \quad (1.7)$$

It was shown in [NZ09a], [NZ09b] that (1.7) holds when the topological pressure $P(1/2)$ is strictly negative. In [BD16], (1.7) was shown to hold on all convex co-compact surfaces, even when

\footnote{Actually, in [Ing17a], we also make a transversality assumption on the direction $\xi$, which is always satisfied if $(X, g)$ has non-positive curvature.}
the condition $P(1/2) < 0$ is not satisfied. By gluing resolvent estimates thanks to the methods of [DV12], we can modify a convex co-compact surface near infinity, by replacing the hyperbolic funnels by Euclidean ends, so that (1.7) still holds. Hence, there are some examples of Euclidean near infinity manifolds such that (1.7) holds, but $P(1/2) \geq 0$. Actually, it was conjectured in [Zwo17, Conjecture 3, §3.2] that (1.7) holds on any Euclidean near infinity manifold with a compact hyperbolic trapped set.

**Theorem 1.1.** Let $(X, g)$ be a Riemannian manifold which is Euclidean near infinity. We suppose that $(X, g)$ has nonpositive sectional curvature, that the trapped set is hyperbolic (Hypothesis 2.1), and that (1.7) is satisfied. Let $\xi \in \mathbb{S}^{d-1}$ and $\chi \in C_c^\infty(X)$. Then there exists $C_{\xi, \chi} > 0$ such that, for any $h > 0$, we have

$$\|\chi E_h(\cdot, \xi)\|_{L^2} \leq C_{\xi, \chi}. $$

We will actually give more precise results about the $L^p_{loc}$ norms and the semi-classical measure of $E_h$ in section 2.3.

**More general framework** In [Ing17a], a general framework was introduced for the study of distorted plane waves in the presence of a hyperbolic trapped set. Though we only describe the simplest case of a Euclidean near infinity manifold in this section, Theorem 1.1 still holds with the same proof in the framework of [Ing17a], replacing the assumption that $P(1/2)$ by the weaker assumption (1.7). In particular the analogue of Theorem 1.1 holds for Eisenstein series on all convex co-compact hyperbolic surfaces, thanks to the results of [BD16].

Our results would also hold if (1.6) and (1.7) were replaced by the weaker assumption

$$D'_h := \left\{ z \in \mathbb{C}; \Re z \in [1 - \varepsilon_0, 1 + \varepsilon_0] \text{ and } \Im z \geq -C_0 h |\log h|^\beta \right\}$$

for some $\beta \geq 0$. However, since we know no example of such a resonance free strip with $\beta > 0$, so we will only work with the assumptions (1.6) and (1.7).

**Relation to other works** The study of the high frequency behaviour of eigenfunctions of the Laplacian, and of their semiclassical measures, in the case where the associated classical dynamics has a chaotic behaviour, has a long story. It goes back to the classical works [Shn74], [Zel87] and [CDV85] dealing with Quantum Ergodicity on compact manifolds.

Analogous results on manifolds of infinite volume are much more recent. Although distorted plane waves are a natural family of eigenfunctions, they may not be uniformly bounded in $L^2_{loc}$, so that it may not be possible to define their semiclassical measure.

In [DG14], the authors studied the semiclassical measures associated to distorted plane waves in a very general framework, with very mild assumptions on the classical dynamics. The counterpart of this generality is that the authors have to average on directions $\xi$ and on an energy interval of size $h$ to be able to define the semiclassical measure of distorted plane waves. Their result can be seen as a form of Quantum Ergodicity result on non-compact manifolds, although no “ergodicity” assumption is made.

In [GN14], the authors considered the case where $X = \Gamma \backslash \mathbb{H}^d$ is a manifold of infinite volume, with sectional curvature constant equal to $-1$ (convex co-compact hyperbolic manifold), and with the assumption that the Hausdorff dimension of the limit set of $\Gamma$ is smaller than $(d-1)/2$. In this setting, distorted plane waves are often called *Eisenstein series*. The authors prove that there is a unique semiclassical measure for the Eisenstein series with a given incoming direction, and they
give a very explicit formula for it. This result can hence be seen as a Quantum Unique Ergodicity result in infinite volume.

The results of [GN14] were extended to the case of variable curvature in [Ing17a, Ing17b], under the assumption that the topological pressure of half the unstable Jacobian is negative: \( \mathcal{P}(1/2) < 0 \), which naturally generalizes the assumption that the Hausdorff dimension of the limit set of \( \Gamma \) is smaller than \((d - 1)/2\).

Showing resonance gaps without the assumption \( \mathcal{P}(1/2) < 0 \) is a very delicate issue (see [Non11] and [Zwo17] for a review of the known results and conjectures). Actually, it is not even known that for a general hyperbolic trapping, the resolvent is polynomially bounded on the real axis. The main examples where a resonance-free strip with polynomial bounds on the resolvent is known are convex co-compact hyperbolic surfaces ([BD16]) as well as some families of convex co-compact hyperbolic manifolds of higher dimension ([DZ16]).

In this paper, we will also study the behavior of the \( L^p_{\text{loc}} \) norms of \( E_h \) as \( h \) goes to zero. To this end, we will use a method introduced in [HR14], which consists in showing \( L^2 \) bounds on \( E_h \) restricted to balls whose radius depend on \( h \).

### Structure of the paper

In section 2, we will recall the definition of hyperbolicity and topological pressure, and state our results on distorted plane waves. In particular, we will describe their semiclassical measure, and show bounds on their \( L^p_{\text{loc}} \) norms. In section 3, we shall recall a few facts of classical dynamics which were proven in [Ing17a] and [Ing17b]. We will give the proof of our results in section 4. Finally, we shall recall a few facts of semiclassical analysis in appendix A.

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### 2 Assumptions and statement of the results

Before recalling the definitions of hyperbolicity and of topological pressure, let us recall how distorted plane waves can be constructed on manifolds that are hyperbolic near infinity.

#### 2.1 The case of convex co-compact hyperbolic manifolds

Our results do not apply only in the case of Euclidean near infinity manifolds, but also in the case of hyperbolic near infinity manifolds. We shall recall here the definition of distorted plane waves on hyperbolic near infinity manifolds. In the framework of convex co-compact hyperbolic manifolds, distorted plane waves are often referred to as \textit{Eisenstein series}.

**Definition 2.1.** We say that \( X \) is hyperbolic near infinity if it fulfills the following assumptions

1. There exists a compactification \( \overline{X} \) of \( X \), that is, a compact manifold with boundaries \( \overline{X} \) such that \( X \) is diffeomorphic to the interior of \( \overline{X} \). The boundary \( \partial \overline{X} \) is called the boundary at infinity.

2. There exists a boundary defining function \( b \) on \( X \), that is, a smooth function \( b : \overline{X} \to [0, \infty) \) such that \( b > 0 \) on \( X \), and \( b \) vanishes to first order on \( \partial \overline{X} \).
3. There exists a constant \( \epsilon_0 > 0 \) such that for any point \((x, \xi) \in S^*X, \)

\[
\text{if } b(x, \xi) \leq \epsilon_0 \text{ and } \dot{b}(x, \xi) = 0 \text{ then } \ddot{b}(x, \xi) < 0.
\]

4. In a collar neighborhood of \( \partial X \), the metric \( g \) has sectional curvature \(-1\) and can be put in the form

\[
g = \frac{db^2 + h(b)}{b^2},
\]

where \( h(b) \) is a smooth 1-parameter family of metrics on \( \partial X \) for \( b \in [0, \epsilon) \).

**Construction of \( E^0_h \)** Let us fix a \( \xi \in \partial X \). Since \( X \) is hyperbolic near infinity, there exists a neighborhood \( V_{\xi} \) of \( \xi \) in \( X \) and an isometric diffeomorphism \( \psi_{\xi} \) from \( V_{\xi} \cap X \) into a neighborhood \( V_{q_0, \delta} \) of the north pole \( q_0 \) in the unit ball \( B := \{ q \in \mathbb{R}^d ; |q| < 1 \} \) equipped with the metric \( g_0 \):

\[
V_{q_0, \delta} := \{ q \in \mathbb{B} ; |q - q_0| < \delta \}, \quad g_0 = \frac{4dq^2}{(1 - |q|^2)^2},
\]

where \( \psi_{\xi}(\xi) = q_0 \), and \(| \cdot |\) denotes the Euclidean length. We shall choose the boundary defining function on the ball \( \mathbb{B} \) to be

\[
b_0 = \frac{2 - |q|}{1 + |q|}, \tag{2.1}
\]

and the induced metric \( b_0^2g_0|_{S^d} \) on \( S^d = \partial \mathbb{B} \) is the usual one with curvature \(+1\). The function \( b_{\xi} := b_0 \circ \psi_{\xi}^{-1} \) can be viewed locally as a boundary defining function on \( X \).

For each \( p \in S^d \), we define the Busemann function on \( \mathbb{B} \)

\[
\phi_p^\mathbb{B}(q) = \log \left( \frac{1 - |q|^2}{|q - p|^2} \right).
\]

There exists an \( \epsilon > 0 \) such that the set

\[
U_{\xi} := \{ x \in \mathbb{X} ; d_{\mathbb{X}, \xi}(x) < \epsilon \}
\]

lies inside \( V_{\xi} \), where \( g = b_{\xi}^2g \) is the compactified metric. We define the function

\[
\phi_\xi(x) := \phi_{q_0}^\mathbb{B}(\psi_{\xi}(x)), \quad \text{for } x \in U_{\xi}, \quad 0 \text{ otherwise.}
\]

Let \( \chi_0 : \mathbb{X} \rightarrow [0, 1] \) be a smooth function which vanishes outside of \( U_{\xi} \), which is equal to one in a neighborhood of \( \xi \).

The incoming wave is then defined as

\[
E^0_h(x; \xi) := \chi_0(x)e^{(i(d-1)/2 + i/h)\phi_\xi(x)} \quad \text{if } x \in U_{\xi}, \quad 0 \text{ otherwise.}
\]

\( E^0_h \) is then a Lagrangian state associated to the Lagrangian manifold

\[
\Lambda_{\xi} = \{(x, \partial_x \phi_\xi(x)) , x \in U_{\xi} \}.
\]
Construction of $E_h$  

We set $E_{\text{out}} := -RhF_h$, where $R_h$ is the outgoing resolvent

$$(-\hbar^2 \Delta - \frac{(d-1)^2}{4} \hbar^2 - 1 - i0)^{-1},$$

and $F_h := [\hbar^2 \Delta, \bar{\chi}] e^{((d-1)/2+i/\hbar)\phi(x)}$.

We then define

$$E_h := E_0^h + E_{\text{out}}.$$

We refer the reader to [DG14, §7] for other equivalent definitions of distorted plane waves in this context, showing that our definition is intrinsic.

2.2 Assumptions on the classical dynamics

Let $(X, g)$ be a manifold which is Euclidean near infinity, or hyperbolic near infinity.

In the sequel, we will always assume that $(X, g)$ has non-positive sectional curvature. Since the curvature vanishes outside of a compact set, we may define

$$-b_0$$

is the minimal value taken by the sectional curvature on $X$. (2.2)

Let us describe more precisely the hyperbolicity assumption we make.

**Hyperbolicity**  

For $\rho \in S^*X$, we will say that $\rho \in \Gamma^{\pm}$ if $\{\Phi^t(\rho), \pm t \leq 0\}$ is a bounded subset of $S^*X$; that is to say, $\rho$ does not “go to infinity”, respectively in the past or in the future. The sets $\Gamma^{\pm}$ are called respectively the outgoing and incoming tails.

The trapped set is defined as

$$K := \Gamma^+ \cap \Gamma^-.$$ 

It is a flow invariant set, and it is compact by the geodesic convexity assumption.

**Hypothesis 2.1**  

(Hyperbolicity of the trapped set). We assume that $K$ is non-empty, and is a hyperbolic set for the flow $\Phi^t$. That is to say, there exists an adapted metric $g_{\text{ad}}$ on a neighborhood of $K$ included in $S^*X$, and $\lambda > 0$, such that the following holds. For each $\rho \in K$, there is a decomposition

$$T_\rho(S^*X) = \mathbb{R} \frac{\partial (\Phi^t(\rho))}{\partial t} \oplus E^+_{\rho} \oplus E^-_{\rho}$$

such that

$$\|d\Phi^t(\rho)\|_{g_{\text{ad}}} \leq e^{-\lambda|t|}\|v\|_{g_{\text{ad}}} \text{ for all } v \in E^\pm_{\rho}, \pm t \geq 0.$$ 

The spaces $E^\pm_{\rho}$ are respectively called the unstable and stable spaces at $\rho$.

We may extend $g_{\text{ad}}$ to a metric on $S^*X$, so that outside of the interaction region, it coincides with the restriction of the metric on $T^*X$ induced from the Riemannian metric on $X$. From now on, we will denote by

$$d_{\text{ad}}$$

the Riemannian distance associated to the metric $g_{\text{ad}}$ on $S^*X$. 

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**Topological pressure** We shall say that a set $S \subset K$ is $(\epsilon, t)$-separated if for $\rho_1, \rho_2 \in S$, $\rho_1 \neq \rho_2$, we have $d_{ad}(\Phi^t(\rho_1), \Phi^t(\rho_2)) > \epsilon$ for some $0 \leq t \leq t'$. (Such a set is necessarily finite.)

The metric $g_{ad}$ induces a volume form $\Omega$ on any $d$-dimensional subspace of $T(T^*\mathbb{R}^d)$. Using this volume form, we will define the unstable Jacobian on $K$. For any $\rho \in K$, the determinant map $\Lambda_n\Phi_t^\rho : \Lambda^n E^+_\rho \to \Lambda^n E^+_{\Phi_t^\rho}(\rho)$ can be identified with the real number

$$\det (d\Phi^t(\rho)|_{E^+_{\rho}}) := \frac{\Omega_{\Phi_t^\rho}(v_1 \wedge v_2 \wedge \ldots \wedge v_n)}{\Omega_{\rho}(v_1 \wedge v_2 \wedge \ldots \wedge v_n)},$$

where $(v_1, \ldots, v_n)$ can be any basis of $E^+_{\rho}$. This number defines the unstable Jacobian:

$$\exp \lambda_1^+(\rho) := \det (d\Phi^t(\rho)|_{E^+_{\rho}}). \quad (2.3)$$

From there, we take

$$Z_t(\epsilon, s) := \sup_S \sum_{\rho \in S} \exp(-s\lambda_1^+(\rho)), \quad (2.4)$$

where the supremum is taken over all $(\epsilon, t)$-separated sets. The pressure is then defined as

$$P(s) := \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log Z_t(\epsilon, s). \quad (2.5)$$

This quantity is actually independent of the volume form $\Omega$ and of the metric chosen: after taking logarithms, a change in $\Omega$ or in the metric will produce a term $O(1)/t$, which is not relevant in the $t \to \infty$ limit.

One of the main assumptions made in [Ing17a] and [Ing17b] was that $P(1/2) < 0$. Here, we will use instead the weaker fact, proven in [BR75, Theorem 5.6] that

$$P(1) < 0. \quad (2.5)$$

To take advantage of this fact, we will introduce in section 3.1 another definition of the topological pressure, which was introduced in [NZ09a]. We refer the reader to this paper for the proof that the two definitions are equivalent.

### 2.3 Statement of the results

Our main result is the following.

**Theorem 2.1.** Let $(X, g)$ be a Riemannian manifold which is Euclidean or hyperbolic near infinity. We suppose that $(X, g)$ has nonpositive sectional curvature, that the trapped set is hyperbolic (Hypothesis 2.1), and that (1.7) is satisfied.

Let $\chi \in C^\infty_0(X)$ have a small enough support.

There exists an infinite set $\tilde{B}^\times$ and a function $\tilde{n} : \tilde{B}^\times \to \mathbb{N}$ such that the number of elements in $\{\tilde{\beta} \in \tilde{B}^\times ; \tilde{n}(\tilde{\beta}) \leq N\}$ grows at most exponentially with $N$, and such that the following holds.

For any $\epsilon > 0$, there exists $M^\epsilon > 0$ such that
\[ \chi E_h(x) = \sum_{\beta \in \mathcal{B}^x \atop \tilde{n}(\beta) \leq M^* \log h} e^{i\varphi_\beta(x)/h} a_{\beta, \chi}(x; h) + O_{L^2} \left( h^{2\epsilon_0 - \epsilon} \right), \]  

(2.6)

where \( a_{\beta, \chi} \in S^{\text{comp}}(X) \) is a classical symbol in the sense of Definition A.1 and each \( \varphi_\beta \) is a smooth function defined in a neighborhood of the support of \( a_\beta \).

For any \( \epsilon' > 0 \), there exists \( C_{\epsilon'} > 0 \) such that for any \( n \in \mathbb{N} \)

\[ \sum_{\beta \in \mathcal{B}^x \atop \tilde{n}(\beta) = n} \| a_{\beta, \chi} \|_{L^2}^2 \leq C_{\epsilon'} e^{n(P(1) + \epsilon')}, \]  

(2.7)

and there exists a constant \( C_\chi > 0 \) such that for all \( \tilde{\beta} \neq \tilde{\beta}' \in \tilde{\mathcal{B}}^x \), we have

\[ |\partial \varphi_\beta(x) - \partial \varphi_{\beta'}(x)| \geq C_\chi e^{-\sqrt{h} \max(\tilde{n}(\beta), \tilde{n}(\beta'))}, \]  

(2.8)

Actually, our proof shows that we may obtain a smaller remainder, of the order of \( h^N \) for any \( N \), by taking more terms into account in the sum (i.e. replacing \( M^\epsilon \) by some larger constant \( M(N) \)), but the new terms would not satisfy (2.8) anymore.

**Proof that Theorem 2.1 implies Theorem 1.1** Note that it suffices to prove the statement for \( \chi \) with a small enough support so that Theorem 2.1 applies. For such a \( \chi \), we may use Theorem 2.1 and denote by \( R_h \) the remainder in (2.6). We obtain

\[ \| \chi(E_h - R_h) \|_{L^2}^2 = \left\langle \sum_{\beta \in \mathcal{B}^x \atop \tilde{n}(\beta) \leq M^* \log h} e^{i\varphi_\beta(x)/h} a_{\beta, \chi}(x; h), \sum_{\beta \in \mathcal{B}^x \atop \tilde{n}(\beta) \leq M^* \log h} e^{i\varphi_\beta(x)/h} a_{\beta, \chi}(x; h) \right\rangle, \]

\[ = \sum_{\beta \in \mathcal{B}^x \atop \tilde{n}(\beta) \leq M^* \log h} \| a_{\beta, \chi}(x; h) \|^2 + \sum_{\beta, \beta' \in \tilde{\mathcal{B}}^x \atop \tilde{n}(\beta), \tilde{n}(\beta') \leq M^* \log h} \left\langle e^{i\varphi_\beta(x)/h} a_{\beta, \chi}(x; h), e^{i\varphi_{\beta'}(x)/h} a_{\beta, \chi}(x; h) \right\rangle. \]

Now, by (2.8) and Proposition A.1 each term in the second sum is a \( O(h^\infty) \), and since the number of terms is bounded by some power of \( h \), the second term is a \( O(h^\infty) \). As to the first term, equation (2.7) implies that it is bounded independently of \( h \). Therefore, \( \chi(E_h - R_h) \) is bounded independently of \( h \). Since \( \| R_h \|_{L^2} = O \left( h^{2\epsilon_0 - \epsilon} \right) \), this concludes the proof. \( \square \)

Theorem 2.1 also allows us to characterize the semiclassical measure of \( E_h \). The proof of the following corollary is exactly the same as that of Theorem 1.1.

**Corollary 2.1.** We make the same hypotheses as in Theorem 1.1 Let \( \chi \in C_0^\infty(X) \) and let \( \epsilon > 0 \). Then there exists a finite measure \( \mu_\chi \) on \( S^*X \) such that we have for any \( \psi \in S^{\text{comp}}(S^*X) \)

\[ \langle Op_h(\psi) \chi E_h, \chi E_h \rangle = \int_{T^*X} \psi(x, \xi) d\mu_\chi(x, \xi) + O \left( h^{\min \left( \frac{1}{2\epsilon_0}, \frac{1}{\sqrt{\epsilon_0}} \right) - \epsilon} \right), \]

Furthermore, if \( \chi \) has a small enough support, the measure \( \mu_\chi \) is then given by

\[ d\mu_\chi(x, \xi) = \sum_{\beta \in \mathcal{B}^x} |a_{\beta, \chi}^0|^2(x) \delta_{\xi = \partial \varphi_\beta(x)} dx. \]
where $a_\beta$ is as in (2.6), and $a^0_\beta$ is its principal symbol as defined in Definition A.1.

Finally, Theorem 2.1 allows us to obtain some bounds on the $L^p$ norms of $E_h$.

**Corollary 2.2.** We make the same assumptions as in the previous theorem. Let us write

$$\lambda_0 := \frac{|\mathcal{P}(1)|}{2d\sqrt{b_0}}$$  \hspace{1cm} (2.9)

For any small $\varepsilon > 0$, set

$$r_{h,\varepsilon} := h^{\lambda_0 - \varepsilon}.$$  

For any $\varepsilon > 0$, there exists $C_{\chi,\varepsilon}$ such that for all $h$ small enough, we have

$$\|\chi E_h\|_{L^\infty} \leq C_{\chi,\varepsilon} \left( \frac{h}{r_{h,\varepsilon}} \right)^{(d-1)/2}. \hspace{1cm} (2.10)$$

Let

$$p_d := \frac{2(d+1)}{d-1}.$$  

There exists $C'_{\chi,\varepsilon} > 0$ such that

$$\|\chi E_h\|_{L^{p_d}} \leq C \left( \frac{h}{r_{h,\varepsilon}} \right)^{-1/p_d}. \hspace{1cm} (2.11)$$

**Remark 2.1.** One could also obtain $L^p$ bounds on $\chi E_h$ for any $p > 2$, by interpolation. The special value $p_d$ corresponds to the critical exponent in the Sogge inequalities (see [Zwo12 §10.4] and the references therein). Actually, our estimates are analogous to the Sogge estimates for eigenvalues on compact manifolds, except that in the right hand side, the negative power of $h$ is replaced by the same power of $h/r_{h,\varepsilon}$. We hence have an improvement of some power of $h$.

### 3 Facts from classical dynamics

In this section, we shall recall a few constructions of classical dynamics which shall be useful in the proof of Theorem 2.1.

Let us start with the alternative definition of topological pressure given in [NZ09a].

#### 3.1 A useful definition of topological pressure

For any $\delta > 0$ small enough, we define

$$\mathcal{E}^\delta := \{(x, \xi) \in T^*X; |\xi|^2 \in (1 - \delta, 1 + \delta)\},$$

$$K^\delta := \{(x, \xi) \in \mathcal{E}^\delta; \Phi^t(x, \xi) \text{ remains in a bounded set for all } t \in \mathbb{R}\}.$$  

Let $\mathcal{W} = (W_\alpha)_{\alpha \in A_1}$ be a finite open cover of $K^{\delta/2}$, such that the $W_\alpha$ are all strictly included in $\mathcal{E}^\delta$. For any $T \in \mathbb{N}^*$, define $W(T) := (W_\alpha)_{\alpha \in A_1^T}$ by

$$W_\alpha := \bigcap_{k=0}^{T-1} \Phi^{-k}(W_{a_k}),$$
where $\alpha = a_0, \ldots, a_{T-1}$. Let $\mathcal{A}_T'$ be the set of $\alpha \in A_T^1$ such that $W_\alpha \cap K^\delta \neq \emptyset$. If $V \subset \mathcal{E}^\delta$, $V \cap K^\delta/2 \neq \emptyset$, define

$$S_T(V) := - \inf_{\rho \in V \cap K^\delta/2} \lambda_T^\rho, \quad \text{with} \ \lambda_T^\rho \ \text{as in (2.3).}$$

$$Z_T(W, s) := \inf \left\{ \sum_{\alpha \in A_T} \exp \{ sS_T(W_\alpha) \} : A_T' \subset K^{\delta/2} \subset \bigcup_{\alpha \in A_T} W_\alpha \right\}$$

$$\mathcal{P}_\delta(s) := \lim_{diam W \to 0} \lim_{T \to \infty} \frac{1}{T} \log Z_T(W, s).$$

The topological pressure is then:

$$\mathcal{P}(s) = \lim_{\delta \to 0} \mathcal{P}_\delta(s). \quad (3.1)$$

Let us fix $\epsilon_0 > 0$ so that $\mathcal{P}(1) + 2\epsilon_0 < 0$. Then there exists $t_0 > 0$, and $\hat{W}$ an open cover of $K^\delta$ with $diam(\hat{W}) < \epsilon_0$ such that

$$\left| \frac{1}{t_0} \log Z_{t_0}(\hat{W}, s) - \mathcal{P}_\delta(s) \right| \leq \epsilon_0. \quad (3.2)$$

We can find $A_{t_0}$ so that $\{W_\alpha, \alpha \in A_{t_0}\}$ is an open cover of $K^\delta$ in $\mathcal{E}^\delta$ and such that

$$\sum_{\alpha \in A_{t_0}} \exp \{ sS_{t_0}(W_\alpha) \} \leq \exp \{ t_0(\mathcal{P}_\delta(s) + \epsilon_0) \}.$$

Therefore, if we take $\delta$ small enough, and if we rename $\{W_\alpha, \alpha \in A_{t_0}\}$ as $\{V_b, b \in B_1\}$, we have:

$$\sum_{b \in B_1} \exp \{ S_{t_0}(V_b) \} \leq \exp \{ t_0(\mathcal{P}(1) + 2\epsilon_0) \}. \quad (3.3)$$

### 3.2 Truncated propagation of Lagrangian manifolds

Let us recall the results given in [Ing17a, §2] and [Ing17b, §4] concerning the propagation of $\Lambda_\xi$ by the geodesic flow. To do this, we have to introduce a nice open covering of $S^*X$.

We may find a finite number of open sets $(V_b)_{b \in B_2 \cup \{0\}}$ with $V_0 = T^*(X \setminus X_0) \cap \mathcal{E}^\delta$, with $V_0 \cap K = \emptyset$ and $V_b$ is bounded if $b \in B_2$, such that $(V_b)_{b \in B}$ is an open cover of $S^*X$ included in $\mathcal{E}^\delta$, with $B = B_1 \cup B_2 \cup \{0\}$, and such that the following holds.

**Truncated Lagrangians** Let $N \in \mathbb{N}$, and let $\beta = \beta_0, \beta_1, \ldots, \beta_{N-1} \in B^N$. Let $\Lambda$ be a Lagrangian manifold in $T^*X$. We define the sequence of (possibly empty) Lagrangian manifolds $(\Phi^k_\beta(\Lambda))_{0 \leq k \leq N-1}$ by recurrence by:

$$\Phi^0_\beta(\Lambda) = \Lambda \cap V_{\beta_0}, \quad \Phi^{k+1}_\beta(\Lambda) = V_{\beta_{k+1}} \cap \Phi^k_{\beta}(\Phi^k_{\beta}(\Lambda)). \quad (3.4)$$

If $\beta \in B^N$, we will often write

$$\Phi_\beta(\Lambda) := \Phi^{N-1}_{\beta}(\Lambda).$$

For any $\beta \in B^N$ such that $\beta_{N-1} \neq 0$, we will define

$$\tau(\beta) := \max \{ 1 \leq i \leq N-1; \beta_i = 0 \}. \quad (3.5)$$

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if there exists $1 \leq i \leq N - 1$ with $\beta_i = 0$, and $\tau(\beta) = 0$ otherwise.

In the sequel, we will be interested in the truncated propagation of the manifolds $\Lambda_\xi$, defined as follows for $\xi \in \mathbb{S}^{d-1}$:

$$\Lambda_\xi := \{(x, \xi) ; x \in X \setminus X_0 \} \subset S^* X.$$  

By possibly taking the set $X_0$ bigger, we can assume that

$$\forall \xi \in \mathbb{S}^{d-1}, \quad \Phi_0, \ldots, 0(\Lambda_\xi) = \Lambda_\xi.$$  

(3.6)

The main results of Ing17b concerning the truncated propagation of $\Lambda_\xi$ can be summed up as follows.

**Theorem 3.1.** Let $(X, g)$ be a Euclidean near infinity manifold of non-positive sectional curvature, whose trapped set is hyperbolic, and let $\xi \in \mathbb{S}^{d-1}$.

Let $O \subset X$ be an open set which is small enough so that we may define local coordinates on it. Then the manifolds $\Phi_\beta^N(\Lambda_\xi) \cap T^* O$ satisfy the following properties.

1. (Finite time away from the trapped set) Let $O'$ be a bounded open set. There exists $N_{O,O'}$, $N'_{O,O'} \in \mathbb{N}$ such that, for all $N \in \mathbb{N}$ and $\beta \in B^N$, if $\Phi_\beta^N(\Lambda_\xi \cap O') \cap O \neq \emptyset$, then we have

$$\forall i \in \{N_{O,O'}, \ldots, N - N_{O,O'}\}, \beta \in B_1 \cup B_2$$

$$\forall i \in \{N'_{O,O'}, \ldots, N - N'_{O,O'}\}, \beta \in B_1.$$

2. (Smooth projection) Then for any $N \in \mathbb{N}$ and any $\beta \in B^N$, $\Phi_\beta(\Lambda_\xi) \cap (S^* O)$ is a Lagrangian manifold which may be written, in local coordinates, in the form

$$\Phi_\beta^N(\Lambda_\xi) \cap T^* O \equiv \{(x, \partial_x \varphi_{\beta,O}(x)) ; x \in O^\beta\},$$

where $O^\beta$ is an open subset of $O$.

Furthermore, for any $\ell \in \mathbb{N}$, there exists a $C_{\ell,O} > 0$ such that for any $N \in \mathbb{N}$, $\beta \in B^N$, we have

$$\|\partial_x \varphi_{\beta,O}\|_{C^\ell} \leq C_{\ell,O}.$$  

(3.7)

3. (Expansion) If $x \in O^\beta$, let us write

$$\Phi^{-N}(x, \partial_x \varphi_{\beta,O}(x)) = (g_{\beta,O}(x), \xi)$$  

(3.8)

Set

$$J_{N,\beta,O} := \sup_{x \in O^\beta} \det g_{\beta,O}^{1/2}.$$  

(3.9)

If $N \in \mathbb{N}$ and $\beta \in B^N$, we shall write $\beta^1$ for the sequence obtained by keeping only the elements of $\beta$ which belong to $B_1$. We have $\beta^1 \in (B_1)^{n_1(\beta)}$ for some $n_1(\beta) \leq N$.

There exists $C > 1$ and $N_0 > 0$ such that for all $N \in \mathbb{N}$ and all $\beta \in B^N$, we have

$$J_{N,\beta,O} \leq C \exp \left[ \sum_{i=0}^{n_1(\beta)} \frac{S_{1b}(V_{1b})}{2} \right].$$  

(3.10)

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4. (Distance between the Lagrangian manifolds) Finally, there exists a constant $C_\mathcal{O} > 0$ such that for any $n, n' \in \mathbb{N}$, for any $\beta, \beta' \in \mathbb{R}$, and for any $x \in \mathcal{O}$, such that $x \in \pi_X(\Phi_{\beta, \mathcal{O}}(\lambda_l)) \cap \pi_X(\Phi_{\beta', \mathcal{O}}(\lambda_l))$, we have either $\partial_{\Phi_{\beta, \mathcal{O}}}(x) = \partial_{\Phi_{\beta', \mathcal{O}}}(x)$ or
\[
|\partial_{\Phi_{\beta, \mathcal{O}}}(x) - \partial_{\Phi_{\beta', \mathcal{O}}}(x)| \geq C_1 e^{-\sqrt{N} \max(n - \tau(\beta), n' - \tau(\beta'))},
\]
with $\tau(\beta)$ defined as in (3.10), and where

$- b_0$ is the minimum of the sectional curvature on $X$.

4. Proof of Theorem 2.1

The starting point of the proof will be the following decomposition, which is proven in [Ing17a §5.1]. For any $\chi \in C^\infty_c(\mathbb{R}; [0, 1])$, there exists $\chi_{t_0} \in C^\infty_c(\mathbb{R})$ such that the following holds for any $N \leq M|\log h|$ for some $M > 0$:

\[
\chi_{E_h} = (\chi \tilde{U}(t_0))^N \chi_{t_0} E_h + \sum_{k=1}^N (\chi \tilde{U}(t_0))^k (1 - \chi) \chi_{t_0} E_h^0 + O_{L^2}(h^\infty),
\]

where

$\tilde{U}(t_0) := e^{i \Phi_{\beta}(1 - h^2 \Delta)}$.

The proof of (4.1) in [Ing17a], which is based on [DG14, Lemma 3.10] did not use the hypothesis on the topological pressure, but merely the hypothesis that the resolvent is bounded polynomially on the real axis.

Let us now show that thanks to the assumptions (1.6) and (1.7) we made on the resolvent, the term $(\chi \tilde{U}(t_0))^N \chi_{E_h}$ is negligible provided we take $N = M|\log h|$ for $M$ large enough.

Lemma 4.1. Let $r > 0$. We may find a constant $M_r > 0$ such that for any $M > M_r$, for any $M_r|\log h| \leq N \leq M|\log h|$, we have:

\[
\|(\chi \tilde{U}(t_0))^N \chi_{t_0} E_h\|_{L^2} = O(h^r).
\]

Proof. First of all, note that by (1.4) combined with (1.7), we have that $\|\chi_{t_0} E_h\|_{L^2} = O(h^{-\alpha})$ for some $\alpha > 0$.

Let us fix a function $\psi \in C^\infty_c(1 - \varepsilon_0, 1 + \varepsilon_0)$ such that $\psi(x) = 1$ for $x \in (1 - \varepsilon_0/2, 1 + \varepsilon_0/2)$. By the proof of [Zwo12, Theorem 6.4], we have that

$$(1 - \psi(-h^2 \Delta)) E_h = O(h^\infty).$$

Therefore, the proof follows from the following lemma, which comes from [Ing17c], and whose proof we recall.

Let us denote the Schrödinger propagator by

$U_h(t) := e^{-ith \Delta}$.

Lemma 4.2. For any $r' > 0$, there exists $M_{r'} > 0$ and $C_{r'} > 0$ such that

\[
\|\chi_{U_h(M_{r'})|\log h|\psi(-h^2 \Delta)} \chi\|_{L^2 \rightarrow L^2} \leq C_{r'} h^{r'}.
\]
Proof. Let us consider the incoming resolvent \( R_-(z; h) := (-h^2 \Delta - z)^{-1} \), which is analytic for \(-\Re z > 0\). Using Stone’s formula, we obtain that for any \( t > 0 \), we have
\[
\chi U_h(t) \psi(-h^2 \Delta) \chi = \frac{1}{2i\pi} \int_\mathbb{R} e^{-itz/h} \chi(R_-(z; h) - R_+(z; h)) \psi(z) \, dz.
\]

Let \( \tilde{\psi} \) be an almost analytic extension of \( \psi \), that is to say, a function \( \tilde{\psi} \in C_c^\infty(\mathbb{C}) \) such that
\[
\partial_z \psi(z) = O((\Im z)^\infty)
\] (4.2)
and such that \( \tilde{\psi}(z) = \psi(z) \) for \( z \in \mathbb{R} \). We may furthermore assume that
\[
spt \tilde{\psi} \subset \{ z; \Re z \in \text{spt} \psi \}.
\]

We refer the reader to [Mar02, §2] for the construction of such a function.

Using Green’s formula, we obtain that
\[
\chi U_h(t) \psi(-h^2 \Delta) \chi = \frac{1}{2i\pi} \int_{\Im z = -\infty} e^{-itz/h} \chi(R_-(z; h) - R_+(z; h)) \tilde{\psi}(z) \chi \, dz + \frac{1}{2i\pi} \int_{-\infty}^{\infty} e^{-itz/h} \chi(R_-(z; h) - R_+(z; h)) \partial_z \psi(z) \chi \, dz.
\]

Thanks to (1.7) and to (4.2), the second term is \( O(h^\infty) \), independently of \( t \). On the other hand, by (1.7), the first term is bounded by \( C e^{-C_0 t} h^{-\alpha} \). Therefore, taking \( t = M|\log h| \) with \( M \) large enough proves the lemma.

Therefore, we have
\[
\chi E_h = \sum_{k=1}^{M_r|\log h|} (\chi \tilde{U}(t_0))^k (1 - \chi) \chi_{t_0} E_h^0 + O_{L^2}(h^\gamma).
\] (4.3)

We now have to decompose the propagators in order to take advantage of Theorem 3.1.

4.1 Microlocal partition

We take a partition of unity \( \sum_{b \in B} \pi_b \) such that :
\[
\sum_{b \in B} \pi_b(x) \equiv 1 \text{ for all } x \in \mathcal{E}^\delta,
\]
and \( \text{supp} (\pi_b) \subset V_b \subset \mathcal{E}^\delta \) for all \( b \in B \).

For \( b \in B_1 \cup B_2 \), we set \( \Pi_b := Op_h(\pi_b) \). We have
\[
WF_h(\Pi_b) \subset V_b \cap \mathcal{E}^\delta, \quad \text{and } \Pi_b = \Pi_b^r.
\]

We then set
\[
\Pi_0 := Id - \sum_{b \in B_1 \cup B_2} \Pi_b.
\]
We can decompose the propagator at time $t_0$ into:

$$\hat{U}(t_0) = \sum_{b \in B} \hat{U}_b, \text{ where } \hat{U}_b := \chi \Pi_b e^{it_0/h} \hat{U}(t_0).$$

The propagator at time $t_0$ may then be decomposed as follows:

$$(\chi \hat{U}(t_0))^N = \sum_{\beta \in B^N} \hat{U}_\beta,$$

where $\hat{U}_\beta := \hat{U}_{\beta_{N-1}} \circ \ldots \circ \hat{U}_{\beta_0}$.

### 4.2 Truncated propagation of Lagrangian states

From (4.3) and the construction of the previous subsection, we have

$$\chi E_h = \sum_{k=1}^{M_{\log h}} \sum_{\beta \in B_k} \hat{U}_\beta(1 - \chi) \chi t_0 E_h^0 + O_{L^2}(h^r),$$

(4.5)

Let us fix an open set $O \subset X$ small enough so that we may define local coordinates on it. Let us fix $\chi \in C^\infty_c(O)$. Thanks to Theorem 3.1 we may apply Proposition A.2 to describe $\hat{U}_\beta(1 - \chi) \chi t_0 E_h^0$.

We obtain that

$$\hat{U}_\beta(1 - \chi) \chi t_0 E_h^0 = a_{\beta, \chi} e^{\hat{z}_r \varphi_{\beta, O}},$$

(4.6)

with $a_{\beta, \chi} \in S^{comp}(O)$ satisfying

$$\|a_{\beta, \chi}\|_{C^0} \leq 2J_{N, \beta, O} \leq 2C \exp \left[ \sum_{i=0}^{n_1(\beta)} S_{t_0}(V_{\beta_i}) \right].$$

(4.7)

In particular, we have

$$\|a_{\beta, \chi}\|_{L^2}^2 \leq C' \exp \left[ \sum_{i=0}^{n_1(\beta)} S_{t_0}(V_{\beta_i}) \right].$$

(4.8)

Thanks to the first point in Theorem 3.1 we have that if $\hat{U}_\beta(1 - \chi) \chi t_0 E_h^0 \neq O(h^{\infty})$, then we must have $n_1(\beta) \geq N - N_O$ for some $N_O > 0$.

Therefore, for any $k \geq N_O$, we have

$$\sum_{\beta \in B_k} \|a_{\beta, \chi}\|_{L^2}^2 \leq C' \exp \left[ (k - N_O) \sum_{b \in B_1} S_{t_0}(V_b) \right] \leq C' \exp \left[ (k - N_O) t_0 (P(1) + 2\varepsilon_0) \right].$$

(4.9)

In particular, thanks to (2.5), we see that for any $M > 0$, $\sum_{k=1}^{M_{\log h}} \sum_{\beta \in B_k} \|\hat{U}_\beta(1 - \chi) \chi t_0 E_h^0\|_{L^2}^2$ is bounded independently of $h$. 

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4.3 Regrouping the Lagrangian states

From now on, we fix a compact set \( K \subset X \), and we take \( O \subset K \).

In \cite{Ing17b}, it is shown that there exists \( \varepsilon_K > 0 \) such that, if \( O \subset K \) has a diameter smaller than \( \varepsilon_K \), it is possible to build an equivalence relation \( \sim_O \) is built on the set \( \bigcup_{k \in \mathbb{N}} B^k \) with the following properties:

1. If \( \beta \sim_O \beta' \), then for all \( x \in O \) belonging to the domain of definition of \( \varphi_{\beta,O} \) and \( \varphi_{\beta',O} \), we have \( \partial \varphi_{\beta,O} = \partial \varphi_{\beta',O} \).

   Therefore, it is possible for each \( \tilde{\beta} \in \tilde{B}^O \) to build a phase function \( \varphi_{\tilde{\beta}} : O \to \mathbb{R} \) such that for every \( \beta \in \tilde{\beta} \), we have \( \partial \varphi_{\beta}(x) = \partial \varphi_{\beta,O}(x) \) for every \( x \in \text{supp}(\varphi_{\beta,O}) \).

2. In particular, there exists a constant \( C_O > 0 \) such that for all \( \tilde{\beta} \neq \tilde{\beta}' \in \tilde{B}^O \), we have

\[
|\partial \varphi_{\beta}(x) - \partial \varphi_{\beta'}(x)| \geq C_O e^{-\sqrt{b_0} \max(n(\tilde{\beta}), n(\tilde{\beta}'))}. \tag{4.10}
\]

3. Let us write \( \tilde{B}_O = (\bigcup_{k \in \mathbb{N}} B^k) \sim_O \). Each equivalence class \( \tilde{\beta} \in \tilde{B}_O \) of \( \sim_O \) is finite.

   We shall define for each \( \tilde{\beta} \in \tilde{B}_O \):

\[
n(\tilde{\beta}) := \min \{ n \in \mathbb{N} ; \exists \beta \in B^k \text{ such that } \beta \in \beta' \}. \tag{4.11}
\]

Let \( O \subset K \) have diameter smaller than \( \varepsilon_K \), and let \( \chi \in C_c^\infty(\mathcal{O}) \). For every \( \tilde{\beta} \in \tilde{B}^O \), we set

\[
a_{\tilde{\beta},\chi} := \sum_{\beta \in \tilde{\beta}} a_{\beta,\chi} e^{i\varphi_{\beta} / \hbar}. \]

Defined this way, \( a_{\tilde{\beta},\chi} \in S^{\text{comp}}(X) \).

We then have

\[
\chi E_h = \sum_{\tilde{\beta} \in \tilde{B}^O \atop n(\tilde{\beta}) \leq M^* |\log h|} a_{\tilde{\beta},\chi} e^{i\varphi_{\tilde{\beta}}} + O_{L^2}(h^r). \tag{4.12}
\]

Furthermore, by \eqref{4.13} and \eqref{4.14}, there exists a \( C_{\chi,r} > 0 \) such that

\[
\sum_{\tilde{\beta} \in \tilde{B}^O \atop n(\tilde{\beta}) = n} \|a_{\tilde{\beta},\chi}\|_{L^2} \leq C_{\chi,r} e^{n(P(1)+\alpha)}. \tag{4.13}
\]

4.4 End of the proof of Theorem 2.1

Let \( \varepsilon > 0 \), and set \( M^* := \frac{1}{2\sqrt{b_0}} - \varepsilon \).

We have

\[
\chi E_h = \sum_{\tilde{\beta} \in \tilde{B}^O \atop n(\tilde{\beta}) \leq M^* |\log h|} a_{\tilde{\beta},\chi} e^{i\varphi_{\tilde{\beta}}} + \sum_{\tilde{\beta} \in \tilde{B}^O \atop M^* |\log h| < n(\tilde{\beta}) \leq M^* |\log h|} a_{\tilde{\beta},\chi} e^{i\varphi_{\tilde{\beta}}} + O_{L^2}(h^r). \tag{4.14}
\]
Now, thanks to (4.10), we have for all $\tilde{\beta}, \tilde{\beta}'$ such that $\tilde{n}(\tilde{\beta}), \tilde{n}(\tilde{\beta}') \leq M^\varepsilon |\log h|$ that

$$|\partial \varphi_{\tilde{\beta}}(x) - \partial \varphi_{\tilde{\beta}'}(x)| \geq Ch^{1/2-\varepsilon}.$$ 

Therefore, by Proposition A.1, we have that for all $\tilde{\beta} \neq \tilde{\beta}'$ such that $\tilde{n}(\tilde{\beta}), \tilde{n}(\tilde{\beta}') \leq M^\varepsilon |\log h|$, 

$$(a_{\tilde{\beta}, \chi} e^{i \varphi_{\tilde{\beta}}}, a_{\tilde{\beta}', \chi} e^{i \varphi_{\tilde{\beta}'}})_{L^2} = O(h^\infty).$$

Hence, we have for any $N \leq M^\varepsilon |\log h|$

$$\left\| \sum_{\tilde{\beta} \in B^{\varepsilon}} a_{\tilde{\beta}, \chi} e^{i \varphi_{\tilde{\beta}}} \right\|_{L^2}^2 = \sum_{\tilde{\beta} \in B^{\varepsilon}} \left\| a_{\tilde{\beta}, \chi} e^{i \varphi_{\tilde{\beta}}} \right\|_{L^2}^2 + O(h^\infty)$$

$$\leq C e^{N(P(1)+\varepsilon_0)} + O(h^\infty).$$

We may deduce from this that, for any $N \leq M^\varepsilon |\log h|$, we have

$$\| (\chi \tilde{U}(t_0))^N (1 - \chi) \chi_{t_0} E^0_h \|_{L^2} \leq C e^{N(P(1)+\varepsilon_0)} + O(h^\infty)$$

In particular, we have

$$\| (\chi \tilde{U}(t_0))^{[M^\varepsilon |\log h|]} (1 - \chi) \chi_{t_0} E^0_h \|_{L^2} = O\left(h^{\sqrt{b_0}} - \varepsilon'\right)$$

for some $\varepsilon'$ which can be made arbitrarily small.

Now, since $\| \chi \tilde{U}(t_0) \|_{L^2 \rightarrow L^2} \leq 1$, we also have for any $N \geq M^\varepsilon |\log h|$ that

$$\| (\chi \tilde{U}(t_0))^N (1 - \chi) \chi_{t_0} E^0_h \|_{L^2} = O\left(h^{\sqrt{b_0}} - \varepsilon'\right).$$

Putting these estimates together, we obtain that there exists $C_\chi > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$, we have

$$\| \chi E_h \|_{L^2} \leq C_\chi.$$

Since this is true for any $\chi$ with a small enough diameter, we deduce that the same estimate holds true for any $\chi \in C^\infty_c(X)$, which concludes the proof of Theorem 1.1.

From these considerations, (4.14) gives us

$$\chi E_h = \sum_{\tilde{\beta} \in B^{\varepsilon}} a_{\tilde{\beta}, \chi} e^{i \varphi_{\tilde{\beta}}} + O\left(h^{\sqrt{b_0}} - \varepsilon'\right),$$

which is precisely (2.6), and (2.7) follows from (4.13).

### 4.5 Proof of Corollary 2.2

Corollary 2.2 will follow from the following small-scale $L^2$ bound, which has interest on its own. If $x \in X$ and $r > 0$, we shall write $B(x, r)$ for the geodesic ball of center $x$ and of radius $r$. 
Proposition 4.1. Then for any \( x_0 \in X \), there exists \( C > 0 \) such that

\[
\int_{B(x_0, r_h)} |E_h|^2 \leq C r_h^d.
\]  

(4.16)

Here, the constant \( C \), can be taken independent of \( x_0 \) if \( x_0 \) varies in a compact set.

Remark 4.1. Using the methods of [Ing17b, §6], it is possible to show that \( \int_{B(x_0, r_h)} |E_h|^2 \) is also bounded from below by some constant times \( r_h^d \). Since we won’t need the lower bound for our \( L^p \) estimates, we will not give a proof here.

Proof. Let \( x_0 \in X \). By Theorem 2.1 in a neighborhood of \( x_0 \), we may write \( E_h \) as

\[ E_h = S_h + R_h, \]

with \( \|R_h\|_{L^2(X)} = o(r_h^d) \) and \( S_h \) is a sum of Lagrangian states of the form

\[ S_h = \sum_{\hat{\beta} \in \hat{B}} e^{i\varphi_{\hat{\beta}}(x) / h} a_{\hat{\beta}}(x; h), \]

with

\[
\sum_{\hat{\beta} \in \hat{B}} \|a_{\hat{\beta}}\|_{L^2}^2 \leq C e^{n(P(1)+\epsilon')}. \tag{4.17}
\]

and there exists a constant \( C > 0 \) such that for all \( \hat{\beta} \neq \hat{\beta}' \in \hat{B} \), we have

\[
|\partial \varphi_{\hat{\beta}}(x) - \partial \varphi_{\hat{\beta}'}(x)| \geq C e^{-\sqrt{\log(n(\hat{\beta}), n(\hat{\beta}'))}}. \tag{4.18}
\]

We have

\[
\int_{B(x_0, r_h)} |E_h|^2 = \int_{B(x_0, r_h)} |S_h|^2 + \int_{B(x_0, r_h)} \left( R_h^2 + 2S_hR_h \right).
\]

Therefore, if we can show that \( \int_{B(x_0, r_h)} |S_h|^2 = O(r_h^d) \), then we will have that \( \int_{B(x_0, r_h)} |E_h|^2 = O(r_h^d) \).

Let \( \chi_h \in C_c^\infty(X; [0, 1]) \) be supported in \( B(x_0, 2r_h) \), and equal to one in \( B(x_0, r_h) \), so that \( \int_{B(x_0, r_h)} |S_h|^2 \leq \int_X \chi_h |S_h|^2 \). We have

\[
\int_X \chi_h |S_h|^2 = \sum_{\hat{\beta} \in \hat{B}} \int_X \chi_h |a_{\hat{\beta}}(x; h)|^2
\]

\[
+ \sum_{\hat{\beta} \neq \hat{\beta}' \in \hat{B}} \int_X \chi_h e^{i\varphi_{\hat{\beta}}(x) / h} a_{\hat{\beta}, \hat{\beta}'}(x; h), \tag{4.19}
\]

where \( \varphi_{\hat{\beta}, \hat{\beta}'} = \varphi_{\hat{\beta}} - \varphi_{\hat{\beta}'} \) and \( a_{\hat{\beta}, \hat{\beta}'} = a_{\hat{\beta}} a_{\hat{\beta}'} \).

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From (4.17), we see that
\[
\int_X \chi_h |a_{\beta,\chi}|^2 \leq C \kappa_h^{d} \exp \left[ \sum_{i=0}^{n_1(\beta)} S_{\ell_0}(V_{\beta_1}) \right],
\]
so that by (2.5), we see that the first sum in (4.19) is a $O(\kappa_h)$.

Let us consider the terms of the other sum in (4.19). By a change of variables, they can be put in the form
\[
\int_X \chi_h a_{\beta,\beta'}(x; h) e^{i \phi_{\beta,\beta'}(x) / h} dx = r_h \int_{B(0,1)} \chi(x) a_{\beta,\beta'}(r_h x + x_0; h) e^{i \phi_{\beta,\beta'}(r_h x + x_0) / h} dx,
\]
with $\chi$ independent of $h$, supported in a ball of radius 2. We may apply Proposition A.1 to it, to conclude that this integral is $O(\kappa_h^{\infty})$. Since the number of terms in the second sum in (4.19) is bounded by a power of $h$, we conclude that the second term in (4.19) is a $O(\kappa_h^{\infty})$.

This proves the claim.

Let us now prove Corollary 2.2.

Proof of Corollary 2.2. Let $x_0 \in X$, and $r_h = h^{\lambda_0 - \varepsilon}$ for some small $\varepsilon > 0$. We shall denote by $\exp_{x_0} : T_{x_0} X \simeq \mathbb{R}^d \to X$ the exponential map centered at $x_0$. Fix $\psi \in C_c^\infty(\mathbb{R}^+; [0,1])$ be equal to one on $[0,1/2]$ and vanish on $[1,\infty)$. For $y \in \mathbb{R}^d$, we set
\[
\tilde{E}_h(y; x_0) = \psi(|y|_{x_0}) E_h(\exp_{x_0}(r_h y)).
\]

By Proposition 4.1, we have
\[
\|\tilde{E}_h(\cdot; x_0)\|_{L^2(\mathbb{R})} = O_{h \to 0}(1).
\]

We define the operator $Q_h$ on $T_{x_0}^* X \simeq \mathbb{R}^d$.
\[
Q_h := -\left(\frac{h}{r_h}\right)^2 \psi(|y|_{x_0}/10) \left( \sum_{i,j} g^{ij}(r_h y) \frac{\partial^2}{\partial y_i \partial y_j} + \frac{1}{D_g(r_h y)} \frac{\partial}{\partial y_i}((D_g g^{ij})(r_h y)) \frac{\partial}{\partial y_j} \right),
\]
where $D_g := \sqrt{\det(g_{ij})}$, and where $g^{ij}$ are the coefficients of the metric in the coordinates $y = \exp_{x_0}^{-1}(x)$.

As shown in [HR14 §3.4], we have
\[
(Q_h - 1)\tilde{E}_h = O\left(\frac{h}{r_h}\right)\|\tilde{E}_h\|_{L^2}.
\]

Therefore, we can apply [Zwo12 Theorem 7.12] to obtain
\[
\|\tilde{E}_h\|_{L^\infty} \leq C\left(\frac{h}{r_h}\right)^{-\frac{(d-1)}{2}}
\]
and (2.10) follows.
Let
\[ p_d := \frac{2(d + 1)}{d - 1}. \]
We may also apply [Zwo12, Theorem 10.10] to obtain
\[ \|	ilde{E}_h\|_{L^{p_d}} \leq C \left( \frac{h}{r_h} \right)^{-1/p_d}. \]

Now, by a change of variables, there exists \( C > 0 \) such that
\[ \int_{B(x_0, r_h)} |E_h|^{p_d} \leq C r_h^d \int_{B(0,10)} |\tilde{E}_h|^{p_d}(y)dy. \]

We may then cover the support of \( \chi \) by geodesic balls of radius \( r_h \). Since the number of such balls is \( O(r_h^{-d}) \), we obtain
\[ \int_X \chi |E_h|^{p_d} \leq C \frac{h}{r_h}, \]
which gives us the result. \( \square \)

Appendix A  Reminder of semiclassical analysis

A.1 Pseudodifferential calculus

Let \( Y \) be a Riemannian manifold. We will say that a function \( a(x, \xi; h) \in C^\infty(T^*Y \times (0,1]) \) is in the class \( S^{comp}(T^*Y) \) if it can be written as
\[ a(x, \xi; h) = \tilde{a}_h(x, \xi) + O\left(\frac{h}{|\xi|}\right)^\infty, \]
where the functions \( \tilde{a}_h \in C^\infty(T^*Y) \) have all their semi-norms and supports bounded independently of \( h \). If \( U \) is an open set of \( T^*Y \), we will sometimes write \( S^{comp}(U) \) for the set of functions \( a \) in \( S^{comp}(T^*Y) \) such that for any \( h \in [0,1] \), \( \tilde{a}_h \) has its support in \( U \).

Definition A.1. Let \( a \in S^{comp}(T^*Y) \). We will say that \( a \) is a classical symbol if there exists a sequence of symbols \( a_k \in S^{comp}(T^*Y) \) such that for any \( n \in \mathbb{N} \),
\[ a - \sum_{k=0}^{n} h^k a_k \in h^{n+1} S^{comp}(T^*Y). \]
We will then write \( a^0(x, \xi) := \lim_{h \to 0} a(x, \xi; h) \) for the principal symbol of \( a \).

We will sometimes write that \( a \in S^{comp}(Y) \) if it can be written as
\[ a(x; h) = \tilde{a}_h(x) + O(h^\infty), \]
where the functions \( \tilde{a}_h \in C^\infty_c(Y) \) have all their semi-norms and supports bounded independently of \( h \).
We associate to $S^\text{comp}(T^*X)$ the class of pseudodifferential operators $\Psi^\text{comp}_h(X)$, through a surjective quantization map

$$Op_h : S^\text{comp}(T^*X) \to \Psi^\text{comp}_h(X).$$

This quantization map is defined using coordinate charts, and the standard Weyl quantization on $\mathbb{R}^d$. It is therefore not intrinsic. However, the principal symbol map

$$\sigma_h : \Psi^\text{comp}_h(X) \to S^\text{comp}(T^*X)/hS^\text{comp}(T^*X)$$

is intrinsic, and we have

$$\sigma_h(A \circ B) = \sigma_h(A)\sigma_h(B)$$

and

$$\sigma_h \circ Op : S^\text{comp}(T^*X) \to S^\text{comp}(T^*X)/hS^\text{comp}(T^*X)$$

is the natural projection map.

For more details on all these maps and their construction, we refer the reader to [Zwo12 Chapter 14].

For $a \in S^\text{comp}(T^*X)$, we say that its essential support is equal to a given compact $K \subset T^*X$, if and only if, for all $\chi \in S(T^*X)$,

$$\text{supp} \chi \subset (T^*X \setminus K) \Rightarrow \chi a \in h^\infty S(T^*X).$$

For $A \in \Psi^\text{comp}_h(X)$, $A = Op_h(a)$, we define the wave front set of $A$ as:

$$WF_h(A) = \text{ess supp}_h a,$$

noting that this definition does not depend on the choice of the quantization. When $K$ is a compact subset of $T^*X$ and $WF_h(A) \subset K$, we will sometimes say that $A$ is microsupported inside $K$.

Let us now state a lemma which is a consequence of Egorov theorem [Zwo12 Theorem 11.1]. Recall that $U(t)$ is the Schrödinger propagator $U(t) = e^{itP_h}/h$.

**Lemma A.1.** Let $A, B \in \Psi^\text{comp}_h(X)$, and suppose that $\Phi^t(WF_h(A)) \cap WF_h(B) = \emptyset$. Then we have

$$A U(t) B = O_{L^2 \to L^2}(h^\infty).$$

If $U, V$ are bounded open subsets of $T^*X$, and if $T, T' : L^2(X) \to L^2(X)$ are bounded operators, we shall say that $T \equiv T'$ microlocally near $U \times V$ if there exist bounded open sets $\tilde{U} \supset U$ and $\tilde{V} \supset V$ such that for any $A, B \in \Psi^\text{comp}_h(X)$ with $WF(A) \subset \tilde{U}$ and $WF(B) \subset \tilde{V}$, we have

$$A(T - T')B = O_{L^2 \to L^2}(h^\infty).$$

**Tempered distributions** Let $u = (u(h))$ be an $h$-dependent family of distributions in $\mathcal{D}'(X)$. We say it is $h$-tempered if for any bounded open set $U \subset X$, there exists $C > 0$ and $N \in \mathbb{N}$ such that

$$\|u(h)\|_{H^{-N}_h(U)} \leq Ch^{-N},$$

where $\|\cdot\|_{H^{-N}_h(U)}$ is the semiclassical Sobolev norm.

For a tempered distribution $u = (u(h))$, we say that a point $\rho \in T^*X$ does not lie in the wave front set $WF(u)$ if there exists a neighbourhood $V$ of $\rho$ in $T^*X$ such that for any $A \in \Psi^\text{comp}_h(X)$ with $WF(a) \subset V$, we have $Au = O(h^\infty)$. 20
Stationary phase  Let $a, \phi \in S^{\text{comp}}(X)$, with $\text{supp}(a) \subset \text{supp}(\phi)$. We consider the oscillatory integral:

$$I_h(a, \phi) := \int_X a(x)e^{i\phi(x)/h}\,dx.$$ 

The following result is classical, and its proof similar to that of [Zwo12, Lemma 3.12].

**Proposition A.1** (Non stationary phase). Let $\epsilon > 0$. Suppose that there exists $C > 0$ such that, $\forall x \in \text{supp}(a), \forall 0 < h < h_0$, $|\partial \phi(x, h)| \geq Ch^{1/2-\epsilon}$. Then

$$I_h(a, \phi) = O(h^\infty).$$

A.2  Lagrangian distributions and Fourier Integral Operators

**Phase functions** Let $\phi(x, \theta)$ be a smooth real-valued function on some open subset $U_\phi$ of $X \times \mathbb{R}^L$, for some $L \in \mathbb{N}$. We call $x$ the base variables and $\theta$ the oscillatory variables. We say that $\phi$ is a nondegenerate phase function if the differentials $d(\partial_{\theta_1}\phi)\ldots d(\partial_{\theta_L}\phi)$ are linearly independent on the critical set

$$C_\phi := \{(x, \theta); \partial \theta \phi = 0\} \subset U_\phi.$$ 

In this case

$$\Lambda_\phi := \{(x, \partial_x \phi(x, \theta)); (x, \theta) \in C_\phi\} \subset T^*X$$

is an immersed Lagrangian manifold. By shrinking the domain of $\phi$, we can make it an embedded Lagrangian manifold. We say that $\phi$ generates $\Lambda_\phi$.

**Lagrangian distributions** Given a phase function $\phi$ and a symbol $a \in S^{\text{comp}}(U_\phi)$, consider the $h$-dependent family of functions

$$u(x; h) = h^{-L/2} \int_{\mathbb{R}^L} e^{i\phi(x, \theta)/h} a(x, \theta; h) d\theta.$$  

(A.1)

We call $u = (u(h))$ a Lagrangian distribution, (or a Lagrangian state) generated by $\phi$. By the method of non-stationary phase, if $\text{supp}(a)$ is contained in some $h$-independent compact set $K \subset U_\phi$, then

$$WF_h(u) \subset \{(x, \partial_x \phi(x, \theta)); (x, \theta) \in C_\phi \cap K\} \subset \Lambda_\phi.$$ 

**Definition A.2.** Let $\Lambda \subset T^*X$ be an embedded Lagrangian submanifold. We say that an $h$-dependent family of functions $u(x; h) \in C^\infty(X)$ is a (compactly supported and compactly microlocalized) Lagrangian distribution associated to $\Lambda$, if it can be written as a sum of finitely many functions of the form (A.1), for different phase functions $\phi$ parametrizing open subsets of $\Lambda$, plus an $O(h^\infty)$ remainder. We will denote by $I^{\text{comp}}(\Lambda)$ the space of all such functions.

**Fourier integral operators** Let $X, X'$ be two manifolds of the same dimension $d$, and let $\kappa$ be a symplectomorphism from an open subset of $T^*X$ to an open subset of $T^*X'$. Consider the Lagrangian

$$\Lambda_\kappa = \{(x, \nu; x', -\nu'); \kappa(x, \nu) = (x', \nu')\} \subset T^*X \times T^*X' = T^*(X \times X').$$

A compactly supported operator $U : \mathcal{D}'(X) \rightarrow C^\infty_c(X')$ is called a (semiclassical) Fourier integral operator associated to $\kappa$ if its Schwartz kernel $K_U(x, x')$ lies in $h^{-d/2}I^{\text{comp}}(\Lambda_\kappa)$. We write $U \in$
$I^{\text{comp}}(\kappa)$. The $h^{-d/2}$ factor is explained as follows: the normalization for Lagrangian distributions is chosen so that $\|u\|_{L^2} \sim 1$, while the normalization for Fourier integral operators is chosen so that $\|U\|_{L^2(X) \to L^2(X')} \sim 1$.

Note that if $\kappa \circ \kappa'$ is well defined, and if $U \in I^{\text{comp}}(\kappa)$ and $U' \in I^{\text{comp}}(\kappa')$, then $U \circ U' \in I^{\text{comp}}(\kappa \circ \kappa')$.

If $U \in I^{\text{comp}}(\kappa)$ and $O \subset T^*X$ is an open bounded set, we shall say that $U$ is microlocally unitary near $O$ if $U^*U \equiv I_{L^2(X) \to L^2(X)}$ microlocally near $O \times \kappa(O)$.

### A.3 Local properties of Fourier integral operators

In this section we shall see that, if we work locally, we may describe many Fourier integral operators without the help of oscillatory coordinates. In particular, following [NZ09a, 4.1], we will recall the effect of a Fourier integral operator on a Lagrangian distribution which has no caustics.

Let $\kappa : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ be a local symplectic diffeomorphism. By performing phase-space translations, we may assume that $\kappa$ is defined in a neighbourhood of $(0,0)$ and that $\kappa(0,0) = (0,0)$. We furthermore assume that $\kappa$ is such that the projection from the graph of $\kappa$

$$T^*\mathbb{R}^d \times T^*\mathbb{R}^d \ni (x^1, \xi^1, x^0, \xi^0) \mapsto (x^1, \xi^0) \in \mathbb{R}^d \times \mathbb{R}^d, \quad (x^1, \xi^1) = \kappa(x^0, \xi^0),$$

(A.2)

is a diffeomorphism near the origin. Note that this is equivalent to asking that

the $n \times n$ block $(\partial x^1/\partial x^0)$ in the tangent map $dx(0,0)$ is invertible. (A.3)

It then follows that there exists a unique function $\psi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for $(x^1, \xi^0)$ near $(0,0), \quad \kappa(\partial_\xi \psi(x^1, \xi^0), \xi^0) = (x^1, \partial_\xi \psi(x^1, \xi^0)), \quad \det \partial^2_{x, \xi} \psi \neq 0$ and $\psi(0,0) = 0$.

The function $\psi$ is said to generate the transformation $\kappa$ near $(0,0)$.

Thanks to assumption [A.2], a Fourier integral operator $T \in I^{\text{comp}}(\kappa)$ may then be written in the form

$$Tu(x^1) := \frac{1}{(2\pi h)^d} \int \int_{\mathbb{R}^{2n}} e^{i\psi(x^1, \xi^0) - (x^0, \xi^0)/h} \alpha(x^1, \xi^0, h)u(x^0)dx^0dx^1d\xi^0, \quad (A.4)$$

with $\alpha \in S^{\text{comp}}(\mathbb{R}^{2d})$.

Now, let us state a lemma which was proven in [NZ09a, Lemma 4.1], and which describes the effect of a Fourier integral operator of the form [A.3] on a Lagrangian distribution which projects on the base manifold without caustics.

**Lemma A.2.** Consider a Lagrangian $\Lambda_0 = \{(x_0, \phi_0(x_0)) : x \in \Omega_0\}, \phi_0 \in C^\infty(\Omega_0)$, contained in a small neighbourhood $V \subset T^*\mathbb{R}^d$ such that $\kappa$ is generated by $\psi$ near $V$. We assume that

$$\kappa(\Lambda_0) = \Lambda_1 = \{(x, \phi_1(x)) : x \in \Omega_1\}, \quad \phi_1 \in C^\infty_b(\Omega_1).$$

(A.5)

Then, for any symbol $a \in S^{\text{comp}}(\Omega_0)$, the application of a Fourier integral operator $T$ of the form [A.4] to the Lagrangian state

$$a(x)e^{i\phi_0(x)/h}$$

associated with $\Lambda_0$ can be expanded, for any $L > 0$, into

$$T(ae^{i\phi_0/h})(x) = e^{i\phi_1(x)/h} \left( \sum_{j=0}^{L-1} b_j(x)h^j + h^L r_L(x, h) \right),$$

where $r_L(x, h)$ tends to zero as $h \to 0$.
where \( b_j \in S^{\text{comp}} \), and for any \( \ell \in \mathbb{N} \), we have
\[
\|b_j\|_{C^\ell(\Omega_1)} \leq C_{\ell,j}\|a\|_{C^{\ell+1}(\Omega_0)}, \quad 0 \leq j \leq L - 1,
\]
\[
\|r_L(\cdot, h)\|_{C^\ell(\Omega_1)} \leq C_{\ell,L}\|a\|_{C^{\ell+L+n}(\Omega_0)}.
\]

The constants \( C_{\ell,j} \) depend only on \( \kappa, \alpha \) and \( \sup_{t_0} |\partial^\beta \phi_0| \) for \( 0 < |\beta| \leq 2\ell + j \). Furthermore, if we write \( g : \Omega_1 \ni x \mapsto g(x) := \pi \circ \kappa^{-1}(x, \phi_i(x)) \in \Omega_0 \), the principal symbol \( b_0 \) satisfies
\[
b_0(x^1) = e^{i\theta/h} \frac{\alpha_0(x^1, \xi^0)}{|\det \psi_{x,\xi}(x^1, \xi^0)|^{1/2}} |\det dg(x^1)|^{1/2} a \circ g(x^1), \quad (A.6)
\]
where \( \xi^0 = \phi_0 \circ g(x^1) \) and where \( \theta \in \mathbb{R} \).

### A.4 Iterations of Fourier integral operators

We recall here the main results from [NZ09a §4] concerning the iterations of semiclassical Fourier integral operators in \( T^*\mathbb{R}^d \).

Let \( V \subset T^*\mathbb{R}^d \) be an open neighbourhood of 0, and take a sequence of symplectomorphisms \((\kappa_i)_{i=1,\ldots,N}\) from \( V \) to \( T^*\mathbb{R}^d \) such that \( \forall i \in \{1,\ldots,N\} \), we have \( \kappa_i(0) \in V \), and the following projection:
\[
(x_1, \xi_1; x_0, \xi_0) \mapsto (x_1, \xi_0) \text{ where } (x_1, \xi_1) = \kappa(x_0, \xi_0)
\]
is a diffeomorphism close to the origin. We consider Fourier integral operators \((T_i)_i\) which quantise \( \kappa_i \) and which are microlocally unitary near an open set \( U \times \tilde{U} \), where \( U \subset V \) which contains the origin. Let \( \Omega \subset \mathbb{R}^d \) be an open set such that \( U \subset T^*\Omega \), and, for all \( i, \kappa_i(U) \subset T^*\Omega \). For each \( i \), we take a smooth cut-off function \( \chi_i \in C_0^\infty(U; [0,1]) \), and let
\[
S_i := Op_{\chi_i}(T_i) \quad (A.7)
\]
Let us consider a family of Lagrangian manifolds \( \Lambda_k = \{(x, \phi_k(x)); x \in \Omega \} \subset T^*\mathbb{R}^d, \ k = 0,\ldots,N \) such that:
\[
|\partial^\alpha \phi_k| \leq C_\alpha, \quad 0 \leq k \leq N \quad \alpha \in \mathbb{N}^d. \quad (A.8)
\]
We assume that there exists a sequence of integers \((i_k \in \{1,\ldots,J\})_{k=1,\ldots,N}\) such that
\[
\kappa_{i_k+1}(\Lambda_k \cap U) \subset \Lambda_{k+1}, \quad k = 0,\ldots,N - 1.
\]

We define \( g_k \) by
\[
g_k(x) = \pi \circ \kappa^{-1}_{i_k}(x, \phi'_k(x)).
\]
That is to say, \( \kappa^{-1}_{i_k}(x, \phi'_k(x)) = (g_k(x), \phi'_{k-1}(g_k(x))) \).

We will say that a point \( x \in \Omega \) is \( N \)-admissible if we can define recursively a sequence by \( x^N = x \), and, for \( k = N,\ldots,1, \ x^{k+1} = g_k(x^k) \). This procedure is possible if, for any \( k \), \( x^k \) is in the domain of definition of \( g_k \).

Let us assume that, for any admissible sequence \((x^N,\ldots,x^0)\), the Jacobian matrices are uniformly bounded from above:
\[
\left\| \frac{\partial x^k}{\partial x^l} \right\| = \left\| \frac{\partial (g_{k+1} \circ g_{k+2} \circ \cdots \circ g_1)}{\partial x^l}(x^1) \right\| \leq C_D, \quad 0 \leq k < l \leq N,
\]

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where $C_D$ is independent on $N$. This assumption roughly says that the maps $g_k$ are (weakly) contracting.

We will also use the notation

$$D_k := \sup_{x \in \Omega} |\det dg_k(x)|^{1/2}, \quad J_k := \prod_{k' = 1}^k D_{k'},$$

and assume that the $D_k$'s are uniformly bounded: $1/C_D \leq D_k \leq C_D$.

The following result can be found in [NZ09a, Proposition 4.1].

**Proposition A.2.** We use the above definitions and assumptions, and take $N$ arbitrarily large, possibly varying with $h$. Take any $a \in S^{\text{comp}}$ and consider the Lagrangian state $u = ae^{i\phi_0/h}$ associated with the Lagrangian $\Lambda_0$. Then we may write:

$$(S_i \circ ... \circ S_1)(ae^{i\phi_0/h})(x) = e^{i\phi_N(x)/h} \left( \sum_{j=0}^{L-1} h^j a_j^N(x) + h^L R_L^N(x,h) \right),$$

where each $a_j^N \in C^\infty(\Omega)$ depends on $h$ only through $N$, and $R_L^N \in C^\infty((0,1)_h, S(\mathbb{R}^d))$. If $x_N \in \Omega$ is $N$-admissible, and defines a sequence $(x,k) = N, ..., 1$, then

$$|a_0^N(x_N)| = \left( \prod_{k=1}^N \chi_k(x_k, \phi_k'(x_k)) |\det dg_k(x_k)|^{1/2} \right) |a(x_0)|,$$

otherwise $a_j^N(x_N) = 0, \quad j = 0, ..., L - 1$. We also have the bounds

$$\|a_j^N\|_{C^\ell(\Omega)} \leq C_{j,\ell} J_N (N + 1)^{\ell+3} \|a\|_{C^{\ell + 2j}(\Omega)}, \quad j = 0, ..., L - 1, \ell \in \mathbb{N}, \quad (A.9)$$

$$\|R_L^N\|_{L^2(\mathbb{R}^d)} \leq C_L \|a\|_{C^{2L+4}(\Omega)} (1 + C_0 h)^N \sum_{k=1}^N J_k k^{3L+d}, \quad (A.10)$$

$$\|R_L^N\|_{C^\ell(\mathbb{R}^d)} \leq C_{L,h} h^{-d/2-\ell} \|a\|_{C^{2L+4}(\Omega)} (1 + C_0 h)^N \sum_{k=1}^N J_k k^{3L+d}. \quad (A.11)$$

The constants $C_{j,\ell}, C_0$ and $C_L$ depend on the constants in (A.8) and on the operators $\{S_j\}_{j=1}^J$.

We shall mainly be using this proposition in the case where for all $k$, we have $D_k \leq \nu < 1$. In this case, the estimates (A.9), (A.10) and (A.11) imply that for any $\ell \in \mathbb{N}$, there exists $C_\ell$ independent of $N$ such that for any $N \in \mathbb{N}$, we have

$$\|a^N\|_{C^\ell} \leq \|a_0^N\|_{C^\ell} (1 + C_{\ell} h). \quad (A.12)$$

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