The non-GRS properties for the twisted generalized Reed-Solomon code and its extended code

Canze Zhu, Quying Liao

Abstract—In 2017, Beelen et al. firstly introduced twisted generalized Reed-Solomon (in short, TGRS) codes, and constructed a large subclass of MDS TGRS codes. Later, they proved that TGRS code is non-GRS when the code rate is less than one half. In this letter, basing on the dual code of the TGRS code or the extended TGRS code, by using the Schur product, we prove that almost all of TGRS codes and extended TGRS codes are non-GRS when the code rate more than one half.

Index Terms—Twisted generalized Reed-Solomon codes, Extended twisted generalized Reed-Solomon codes, Generalized Reed-Solomon codes, Non-GRS properties

I. INTRODUCTION

Throughout this paper, let \( F_q \) be the finite field with \( q \) elements, where \( q \) is the power of a prime. An \([n, k, d]\) linear code \( C \) over \( F_q \) is a \( k \)-dimensional subspace of \( F_q^n \) with minimum (Hamming) distance \( d \) and length \( n \). If the parameters reach the Singleton bound, namely, \( d = n - k + 1 \), then \( C \) is maximum distance separable (in short, MDS). Especially, generalized Reed-Solomon (in short, GRS) codes are a well known class of MDS codes, it is very important in coding theory and applications \([12], [8], [18], [20], [25], [6], [17], [24], [26], [10], [1], [7], [13], [5]\). Other known MDS codes have been constructed from \( n \)-arcs in projective geometry \([9]\), circulant matrices \([22]\), Hankel matrices \([22]\), or twisted Reed-Solomon (in short, TGRS) codes \([2], [3]\).

In 2017, inspired by the construction for twisted Gabidulin codes \([23]\), Beelen et al. firstly introduced TGRS codes, which is a generalization for GRS codes, they also showed that TGRS codes could be well decoded. Different from GRS codes, they showed that a TGRS code is not necessarily MDS and presented a sufficient and necessary condition for a TGRS code to be MDS \([2], [4], [3]\). Especially, the authors showed that TGRS codes are not GRS when the code rate is less than one half \([3]\). Later, by TGRS codes, Lavauzelle et al. presented an efficient key-recovery attack used in the McEliece cryptosystem \([19]\). TGRS codes are also used to construct LCD MDS codes by their applications in cryptography \([15], [21]\). Recently, the authors gave the parity check matrix for the TGRS code and obtained some self-dual TGRS codes with small Singleton defect \([14], [27]\). More relative results about self-orthogonal TGRS codes can be seen in \([28], [29], [11]\).

In this letter, we focus on the non-GRS properties for the TGRS code and its extended code. Let \( C \) be the TGRS code or extended TGRS (in short, ETGRS) code with code rate more than one half, by calculating the dimension and the minimum Hamming distance of the Schur square for \( C^\perp \) under some assumptions, we show that \( C^\perp \) is non-GRS, and then \( C \) is non-GRS. The rest of this letter is organized as follows. In section 2, some basic notations and results about linear codes are given. In section 3, we show that almost all of the TGRS code and ETGRS code are non-GRS. In section 4, we conclude the whole paper.

II. PRELIMINARIES

In this section, we review some basic knowledge.

A. The dual codes, the Schur product and equivalence for linear codes

The notion of the dual code is given in the following.

For \( \mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in F_q^n \), the inner product is defined as \( \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i b_i \). And then the dual code of \( C \) is defined as

\[ C^\perp = \{ \mathbf{c}^\prime \in F_q^n \mid \langle \mathbf{c}^\prime, \mathbf{c} \rangle = 0 \text{ for any } \mathbf{c} \in C \}. \]

The Schur product is defined as follows.

Definition 1 \((\[15]\)):

For \( \mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in F_q^n \), the Schur product between \( \mathbf{x} \) and \( \mathbf{y} \) is defined as \( \mathbf{x} \ast \mathbf{y} := (x_1 y_1, \ldots, x_n y_n) \). The Schur product of two \( q \)-ary codes \( C_1 \) and \( C_2 \) with length \( n \) is defined as

\[ C_1 \ast C_2 = \langle c_1 \ast c_2 \mid c_1 \in C_1, c_2 \in C_2 \rangle. \]

Especially, for a code \( C \), we call \( C^2 := C \ast C \) the Schur square of \( C \).

Remark 1: For any linear code \( C = \langle v_1, \ldots, v_k \rangle \) with \( v_i \in F_q^n \), \( i = 1, \ldots, k \), we have

\[ C^2 = \langle v_i \ast v_j \mid i, j \in \{1, \ldots, k\} \rangle. \]

The definition of the equivalence for linear codes is given in the following.

Definition 2: \((\[16]\)) Let \( C_1 \) and \( C_2 \) be \( q \)-ary linear codes with length \( n \). We say that \( C_1 \) and \( C_2 \) are equivalent if there is a permutation \( \pi \) in the permutation group with order \( n \) and \( \mathbf{v} = (v_1, \ldots, v_n) \in (F_q^n)^n \) such that \( C_2 = \Phi_{\pi, \mathbf{v}}(C_1) \), where

\[ \Phi_{\pi, \mathbf{v}} : F_q^n \to F_q^n, \quad (c_1, \ldots, c_n) \mapsto (v_1 c_{\pi(1)}, \ldots, v_n c_{\pi(n)}). \]

Remark 2: By Definition \([2]\) if \( C_1 \) and \( C_2 \) are equivalent, then 

- \( C_1^2 \) and \( C_2^2 \) are equivalent;
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B. The GRS, TGRS and ETGRS code

The definition of the GRS code is given in the following.

Definition 3 ([16]): Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n \) with \( \alpha_i \neq \alpha_j \) (i \( \neq \) j) and \( \nu = (v_1, \ldots, v_n) \in (\mathbb{F}_q^n)^n \). Then the GRS code is defined as

\[
GRS_{k,n}(\alpha, \nu) = \{(v_1 f(\alpha_1), \ldots, v_n f(\alpha_n)) | f(x) \in \mathbb{F}_q[x], \deg f(x) \leq k-1 \}.
\]

The dual code of the GRS code is given in the following.

Lemma 1 ([13]): Let \( u = (u_1, \ldots, u_n) \) with \( u_j = \prod_{i=1, i \neq j}^{n} (\alpha_j - \alpha_i)^{-1} \) (j = 1, \ldots, n), then

\[
(\overline{GRS}_{k,n}(\alpha, 1))^\perp = GRS_{n-k,n}(\alpha, u).
\]

By the definition of the GRS code, Lemma [1] and Remark [1] we have

Proposition 1: For \( \frac{n}{2} \leq k < n \), let \( u = (u_1, \ldots, u_n) \) with \( u_j = -\prod_{i=1, i \neq j}^{n} (\alpha_j - \alpha_i) \) (j = 1, \ldots, n), then

\[
(\overline{GRS}_{k,n}(\alpha, 1))^2 = GRS_{2(n-k)-2,n}(\alpha, u^2)
\]

with dimension \( 2(n-k) - 1 \) and minimum Hamming distance \( 2k + 2 \).

The definition of the twisted polynomials linear space \( \mathcal{V}_{k,t,h,\eta} \) is given in the following.

Definition 4 ([2]): Let \( \eta \in \mathbb{F}_q^* \) and \( t, h, k, \eta \in \mathbb{N} \) with \( 0 \leq h < k \leq \eta q \). Then the set of \((k, t, h, \eta)\)-twisted polynomials is defined as

\[
\mathcal{V}_{k,t,h,\eta} = \{ f(x) = \sum_{i=0}^{k-1} a_i x^i + \eta a_h x^{k-1+t} | a_i \in \mathbb{F}_q \} \quad (i = 0, \ldots, k-1),
\]

which is a \( k \)-dimensional \( \mathbb{F}_q \)-linear subspace. \( k, t \) are the hook and the twist, respectively.

From the twisted polynomials linear space \( \mathcal{V}_{k,t,h,\eta} \), the definitions of the TGRS code and the ETGRS code are given in the following, respectively.

Definition 5 ([2]): For any \( t, h, k, \eta, n \in \mathbb{N} \) with \( 0 \leq h < k < k+1 \leq t < n \leq \eta q \) and \( \eta \neq \eta_q \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q^n \) with \( \alpha_i \neq \alpha_j \) (i \( \neq \) j) and \( \nu = (v_1, \ldots, v_n) \in (\mathbb{F}_q^n)^n \). Then the TGRS code is defined as

\[
C_{t,h,k,n}(\alpha, \nu, \eta) = \{(v_1 f(\alpha_1), \ldots, v_n f(\alpha_n)) | f(x) \in \mathcal{V}_{k,t,h,\eta}\}.
\]

For \( h > 0 \), the ETGRS code is defined as

\[
C_{t,h,k,n}(\alpha, \nu, \eta, \infty) = \{(v_1 f(\alpha_1), \ldots, v_n f(\alpha_n), f_h) | f(x) \in \mathcal{V}_{k,t,h,\eta}\},
\]

where \( f_h \) is the coefficient of \( x^h \) in \( f(x) \). Especially, \( C_{t,h,k,n}(\alpha, 1, \eta) \) and \( C_{t,h,k,n}(\alpha, 1, \eta, \infty) \) are called the TRS code and the ETRS code, respectively.

Remark 3: • Note that \( f(0) = f_0 \), thus it is not necessary to define the ETGRS code when \( h = 0 \).

• In [2], \( C_{t,h,k,n}(\alpha, \nu, \eta) \) and \( C_{t,h,k,n}(\alpha, \nu, \eta, \infty) \) are called TGRS codes without difference. But it is not different from Definition [5].
Theorem 1: Let $t, h, k, n \in \mathbb{N}$ with $0 \leq h \leq k - 1$ and $k + t \leq n \leq q$. If $\frac{n}{2} \leq k \leq n - 3$ and one of the following conditions (1.1)-(1.6) holds, then $C_{t,h,k,n}(\alpha, \eta)$ is non-GRS.

(1.1) $n = k + t$, $2k \geq n + 2$, and $3 \leq h \leq k - 3$; 
(1.2) $n = k + t + 1$, $2k \geq n + 2$, and $2 \leq h \leq k - 3$; 
(1.3) $n \geq k + t + 2$, $2k \geq n$, $t = 1$ and $2 \leq h \leq k - 2$; 
(1.4) $n \geq k + t + 2$, $2k \geq n + 1$, $t = 2$ and $1 \leq h \leq k - 3$; 
(1.5) $n \geq k + t + 2$, $2k \geq n$, $t \geq 3$ and $h \in \{0, \ldots, k - 1\}\backslash\{k - t\}$; 
(1.6) $n \geq k + t + 2$, $2k \geq n + 1$, $t \geq 3$ and $h = k - t$.

Proof. We show that $\dim \left( \left( C_{t,h,k,n}(\alpha, 1, \eta) \right)^2 \right) \geq 2(n - k)$ as follows.

By Remark 4, we have
$$C_{t,h,k,n}(\alpha, 1, \eta) = \left\{ (u_1 f(\alpha_1), \ldots, u_n f(\alpha_n)) \mid f(x) \in V_{k,t,h,n}^\perp \right\},$$
where
$$V_{k,t,h,n}^\perp = \left\{ f(x) = \sum_{i=0}^{n-(k+t+1)} a_i x^i + \sum_{j=1}^{t-1} a_{n-k-t+j} x^{n-k-t+j} \right\} \quad a_i \in \mathbb{F}_q \ (i = 0, \ldots, n - k - 1)$$
with
$$h_{n-k-t+i}(x) = \begin{cases} x^{n-(h+1)} + \tilde{L}_k x^{n-k-t} - \sum_{m=1}^{k-h+1} L_m x^{n-(h+1)-m}, & \text{if } i = 0; \\ x^{n-k-t+i} - L_i x^{n-k-1}, & \text{if } 1 \leq i \leq t. \end{cases}$$
Thus
$$\left( C_{t,h,k,n}(\alpha, 1, \eta) \right)^2 = \left\{ (g(\alpha_1), \ldots, g(\alpha_n)) \mid g(x) \in \left( V_{k,t,h,n}^\perp \right)^2 \right\}$$
with
$$\left( V_{k,t,h,n}^\perp \right)^2 = \langle g(x) = f_1(x) f_2(x) \mid f_1(x), f_2(x) \in V_{k,t,h,n}^\perp \rangle.$$ 

Now we show that if one of the conditions (1.1)-(1.6) holds, then there exists $g_i(x) \in \left( V_{k,t,h,n}^\perp \right)^2$ ($i = 0, \ldots, 2(n - k) - 1$) such that
$$\deg g_i(x) \leq n - 1 \text{ and } \deg g_i(x) \neq \deg g_j(x) \ (i \neq j). \quad (3)$$

(1.1) For $n = k + t$, $2k \geq n + 2$, and $3 \leq h \leq k - 3$, we have
$$\left( V_{k,t,h,n}^\perp \right)^2 = \langle h_i(x) h_s(x) \rangle, \ h_o(x) h_i(x), \ h^2_i(x) \ (l, s, i \in \{1, \ldots, n - k - 1\}).$$

By (2), one has
$$\deg \ (h_l(x) h_s(x)) = l + s \leq 2(n - k) - 2 \leq n - 1 \quad (l, s \in \{1, \ldots, n - k - 1\}). \quad (4)$$
Furthermore, for $h_0(x) h_i(x)$, we have the following two cases.
If $3 \leq h \leq 2k - n + 1$, then $2(n - k) - 2 \leq n - (h + 1) \leq n - 4$. Thus
$$2(n - k) - 1 \leq \deg (h_0(x) h_i(x)) = 2(n - k) - 2 + i \leq n - 1 \ (i \in \{1, 2, 3\}). \quad (5)$$
If $2k - n + 2 \leq h \leq k - 3$, then $1 \leq n - 2k + h < n - 2k + h + 1 < n - 2k + h + 2 \leq n - k - 1$. Now by $2k \geq n + 2$, one has
$$2(n - k) - 1 \leq \deg \ (h_0(x) h_{n-2k+h+1+i}(x)) = 2(n - k) - 2 + i \leq n - 1 \ (i \in \{1, 2, 3\}). \quad (6)$$
So far, by (1)-(6), (3) holds.

(1.2) For $n = k + t + 1$, $2k \geq n + 2$, and $2 \leq h \leq k - 3$, we have
$$(V_{k,t,h,n}^\perp)^2 = \langle h_0(x), h_m(x), h_i(x) h_s(x), h_o(x) h_i(x), h^2_i(x) \rangle \quad (m, l, s, i \in \{2, \ldots, n - k - 1\}).$$
By (3), one has
$$\deg h_m(x) = m \ (m = 2, 3), \quad 4 \leq \deg (h_1(x) h_s(x)) = l + s \leq n - 1 \ (l, s \in \{2, \ldots, n - k - 1\}). \quad (7)$$
Furthermore, for $h_0(x) h_i(x)$, we have the following two cases.
If $3 \leq h \leq 2k - n + 1$, then $2(n - k) - 2 \leq n - (h + 1) \leq n - 4$. Thus
$$2(n - k) - 1 \leq \deg (h_0(x) h_i(x)) = n - (h + 1) + i \leq n - 1 \ (i \in \{2, 3\}). \quad (8)$$
If $2k - n + 2 \leq h \leq k - 3$, then $2 \leq n - 2k + h + 1 < n - 2k + h + 2 \leq n - k - 1$. Thus
$$2(n - k) - 1 \leq \deg (h_0(x) h_{n-2k+h+1+i}(x)) = 2(n - k) - 2 + i \leq n - 1 \ (i \in \{1, 2\}). \quad (9)$$
Now by $1 \leq (V_{k,t,h,n}^\perp)^2$ and (7)-(9), (3) holds.

(1.3) For $n = k + t + 2$, $2k \geq n$, $t = 1$ and $2 \leq h \leq k - 2$, we have
$$\left( V_{k,t,h,n}^\perp \right)^2 = \langle x^{l+s}, \ x^i h_0(x), \ h^2_o(x) \rangle \ (l, s, i \in \{0, 1, \ldots, n - k - 2\}).$$
Obviously,
$$\deg (x^{l+s}) = l + s \leq n - 1 \ (l, s \in \{0, 1, \ldots, n - k - 2\}). \quad (10)$$
Furthermore, for $x^i h_i(x)$, we have the following two cases.
If $2 \leq h \leq 2k - n + 2$, then $2(n - k) - 3 \leq n - (h + 1) \leq n - 3$. Thus
$$2(n - k) - 3 \leq \deg (x^i h_0(x)) = n - (h + 1) + i \leq n - 1 \ (i \in \{0, 1, 2\}). \quad (11)$$
If $2k - n + 3 \leq h \leq k - 2$, then $h + 1 \leq n - 2k + h - 2 < n - 2k + h + 1 < n - 2k + h - 3 \leq n - k - 2$, thus
$$2(n - k) - 4 \leq \deg (x^{n-2k+h-2+i} h_0(x)) = 2(n - k) - 3 + i \leq n - 1 \ (i \in \{0, 1, 2\}). \quad (12)$$
Now by (10)-(12), (3) holds.
(1.4) For $n \geq k + t + 2$, $2k \geq n + 1$, $t = 2$ and $1 \leq k \leq k - 3$, we have
\[
(V^t_{k,t,h,n})^2 = \langle x^{s_1 + s_2}, x^t h_0(x), x^t h_1(x), h_0(x) h_1(x), h_0^2(x) \rangle
\]
\[
\langle \{s, s_1, s_2 \in \{0, \ldots, n - k - t - 1\} \rangle, \{l, l_1, l_2 \in \{0, \ldots, t - 1\}\} \rangle.
\]
By (2), one has
\[
\text{deg}(x^{s_1 + s_2}) = l + s \leq 2n - 2k - 6 \leq n - 1
\]
\[
\text{deg}(x^{n - k - t - 2} h_0(x)) = 2(n - k) - 2l - t - 2 + l = 2(n - k) - 3,
\]
\[
\text{deg}(h_0^2(x) = 2(n - k) - 2. \quad (15)
\]
Furthermore, for $x^l h_0(x)$ and $h_1(x) h_0(x)$, we have the following three cases.
If $0 \leq h \leq 2k - n$, then $2(n - k) - 1 \leq n - (h + 1) \leq n - 1$, thus
\[
\text{deg}(h_0(x)) = n - (h + 1) \leq n - 1. \quad (17)
\]
If $2k - n + 1 \leq h \leq k - t - 1$, then $h + 1 \leq n - 2k + h \leq n - k - t + 1$, thus
\[
\text{deg}(x^{n - 2k + h} h_0(x)) = 2(n - k) - 1 \leq n - 1. \quad (18)
\]
If $k - t + 1 \leq h \leq k - 1$, then $n - k - t + 1 \leq n - 2k + h \leq n - k - 1$, thus
\[
\text{deg}(h_{n - 2k + h}(x) h_0(x)) = 2(n - k) - 1 \leq n - 1. \quad (19)
\]
Now by (10) (19), (3) holds.

(1.6) For $n \geq k + t + 2$, $2k \geq n + 1$ and $h = k - t$, in the similar proof as that for (1.5), we can get (3).

Let $V = \{ \sum_{i=0}^{2(n-k)-1} a_i g_i(x) | a_i \in F_q \}$, where $g_i(x)$ $(i = 0, \ldots, 2(n - k) - 1)$ is given in (3), then $\dim(V) = 2(n - k)$ and $V \subseteq (V^t_{k,t,h,n})^2$. Thus
\[
C_V = \{(g(a_1), \ldots, g(a_n)) | g(x) \in V \} \subseteq \langle C^t_{k,t,h,n}(\alpha, \eta, \eta) \rangle. \quad (20)
\]
Now by deg($g(x)$) $\leq n - 1$ ( $\forall g(x) \in V$ ), we have
\[
(g(a_1), \ldots, g(a_n)) \neq 0 \quad (\forall g(x) \in V \{0\}).
\]
It implies that
\[
\dim(\langle C^t_{k,t,h,n}(\alpha, \eta, \eta) \rangle) \geq \dim(C_V) = \dim(V) = 2(n - k).
\]
By Proposition 1 for any $[n, k]$ GRS code $C$, $(C^t_{k,t,h,n})^2 = 2(n - k) - 1$. Thus if one of conditions (1.1)-(1.6) holds, then $C^t_{k,t,h,n}(\alpha, \eta, \eta)$ is non-GRS, and so $C_{k,h,k,n}(\alpha, \eta, \eta)$ is non-GRS.

The following lemma is necessary to determine some code-words in $C^t_{k,t,h,n}(\alpha, \eta, \eta, \eta, \infty)$.

Lemma 5: For any $m \in \mathbb{N}$ and $A \subseteq F_q$ with $|A| > 2$, let $L_A(m) = \sum_{\alpha \in A} \prod_{\beta \in F_q \setminus A} (\alpha - \beta)$, then
\[
L_A(m) = \begin{cases} 0, & \text{if } m \leq |A| - 2; \\ -1, & \text{if } m = |A| - 1; \\ \sum_{\alpha \in A} \alpha, & \text{if } m = |A|. \end{cases} \quad (21)
\]

Proof. For any $l \in \mathbb{Z}^+$, it is well-known that
\[
\sum_{\beta \in F_q} \beta^l = \begin{cases} 1, & \text{if } q - 1 | l; \\ 0, & \text{otherwise.} \end{cases} 
\]

Note that
\[
\prod_{\beta \in F_q \setminus A} (\alpha - \beta) = \alpha^{q - |A|} - \sum_{\beta \in F_q \setminus A} \beta \alpha^{q - |A| - 1} + \cdots + (-1)^{q - |A|} \prod_{\beta \in F_q \setminus A} \beta
\]
\[
= \alpha^{q - |A|} + \sum_{\gamma \in A} \gamma \alpha^{q - |A| - 1} + \cdots + (-1)^{q - |A|} \prod_{\beta \in F_q \setminus A} \beta,
\]
thus we have
\[
L_A(m) = \sum_{\alpha \in A} \alpha^m \prod_{\beta \in F_q \setminus A} (\alpha - \beta)
\]
\[
= \sum_{\alpha \in F_q} (\alpha^{q - |A| + m} + \sum_{\gamma \in A} \gamma \alpha^{q - |A| - 1 + m} + \cdots + (-1)^{q - |A| + m} \prod_{\beta \in F_q \setminus A} \beta^m). \quad (22)
\]

Now by (21), (22), we obtain (20).

Theorem 2: For $t \geq 2$, $3 \leq k \leq n - 2$ and $n \geq k + t + 1$, $C_{k,h,k,n}(\alpha, \eta, \eta, \infty)$ is non-GRS.

Proof. By Lemma 3 if $3 \leq k < \frac{n + 1}{t + 1}$, then $C_{k,h,k,n}(\alpha, \eta, \eta, \infty)$ is non-GRS. Thus it is enough to prove that the theorem is true for $2k \geq n + 1$. 


In the following, we prove that $c_i \in C_{t,h,k,n}^{(1)}(\alpha, \nu, \eta, \infty)$ ($i = 1, 2, 3$), where
\[
\begin{align*}
c_1 &= \left( \frac{u_1}{v_1} \alpha_{n-k-t-1}, \ldots, \frac{u_n}{v_n} \alpha_{n-k-t-1}, 0 \right), \\
c_2 &= \left( \frac{u_1}{v_1} \alpha_{n-k-t}, \ldots, \frac{u_n}{v_n} \alpha_{n-k-t}, -\eta \right), \\
c_3 &= \left( \frac{u_1}{v_1} \alpha_{n-k-t+1}, \ldots, \frac{u_n}{v_n} \alpha_{n-k-t+1}, -\eta \sum_{i=1}^{n} \alpha_i \right).
\end{align*}
\]

Denote $A_\alpha = \{\alpha_1, \ldots, \alpha_n\}$, for $s \in \{n - k - t - 1, n - k - t, n - k - t + 1\}$ and $l \in \{0, \ldots, k - 1, k - 1 + t\}$, by the assumption $t \geq 2$ and Lemma 5 we have
\[
\sum_{i=0}^{n} u_i \alpha_i^s \alpha_i^l = - \sum_{\alpha \in A_\alpha} \alpha^{s+l} \prod_{\beta \in F_q^* \setminus A_\alpha} (\alpha - \beta) =
\begin{cases}
1, & \text{if } s = n - k - t \text{ and } l = k - 1 + t; \\
\sum_{i=1}^{n} \alpha_i, & \text{if } s = n - k - t + 1 \text{ and } l = k - 1 + t; \\
0, & \text{otherwise}.
\end{cases}
\]

By above equation, we can verify that
\[
c_i \in C_{t,h,k,n}^{(1)}(\alpha, \nu, \eta, \infty) \quad (i = 1, 2, 3)
\]
directly. Thus
\[
\mathcal{C} = c_1 \times c_3 - c_2 \times c_2
\]
\[
= \left(0, 0, \ldots, 0, \eta^2 \right) \in (C_{k,n}^{(2)}(\alpha, \nu, \eta, \infty))^2.
\]

By Proposition 1 for an $[n + 1, k] GRS$ code $\mathcal{C}$, $(\mathcal{C}^\perp)^2$ is with minimum Hamming distance
\[
d = 2k - (n + 1) + 2 \geq 2.
\]
Thus $\mathcal{C} \notin (\mathcal{C}^\perp)^2$, and then $C_{t,h,k,n}^{(1)}(\alpha, \nu, \eta, \infty)$ is non-GRS, it implies that $C_{t,h,k,n}(\alpha, \nu, \eta, \infty)$ is non-GRS. \hfill \Box

Remark 5: \textbullet \text{ Note that any punctured code of the GRS code is a GRS code and $C_{t,h,k,n}(\alpha, \nu, \eta, \infty)$ is a punctured code of $C_{t,h,k,n}(\alpha, \nu, \eta, \infty)$. Thus, if one of conditions (1.1)-(1.6) in Theorem 1 holds, then $C_{t,h,k,n}(\alpha, \nu, \eta, \infty)$ is non-GRS.}

\textbullet \text{ The assumption in Theorem 2 is weaker than that in Theorem 1 when } t \geq 2 \text{ and } n \geq k + t + 1.

IV. Conclusion

In this letter, by using the Schur product, we prove that almost all of TGRS codes and ETGRS codes are non-GRS when the code rate more than one half. However, if $n, k, t, h$ do not satisfy one of the conditions in Theorem 1 we can not obtain the non-GRS property for the TGRS code by the method given in the proof for Theorem 1. The main reason is that we can not obtain $\dim \left( \left[ C_{t,h,k,n}(\alpha, \nu, \eta) \right]^2 \right) \geq 2k$ without the conditions in Theorem 1.

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