Opposite product system for the multiparameter CAR flows

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Abstract

We consider the multiparameter CAR flows and describe its opposite. We also characterize the symmetricity of CAR flows in terms of associated isometric representations.

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1 Introduction

Let $P$ be a closed convex cone in $\mathbb{R}^d$. We assume that $P - P = \mathbb{R}^d$ and $P \cap -P = \{0\}$. Let $V$ be a pure isometric representation of $P$ and let $\alpha$ be the CCR flow associated to the isometric representation $V$. The author in [4] have shown that the CCR flow is not cocycle conjugate to the CAR flow when the isometric representation $V$ is proper. The product system associated with the CAR flow is not decomposable in general; see [1]. It was shown in [5] that $\alpha$ is cocycle conjugate to $\alpha^{\text{op}}$ if and only if $V$ is unitary equivalent to its opposite $V^{\text{op}}$. This result uses the characterization of decomposable product system which admits a unit; see [6]. It is natural to ask whether the analogous result holds true for the multiparameter CAR flows. In this article we answer this question affirmatively; see Theorem 3.5. We will achieve this by identifying the opposite of the product system for a CAR flow with the product system for an appropriate CAR flow. Also we will also use this to study the symmetricity of the CAR flows.
2 Preliminaries

Let $H$ be a Hilbert space and let $\Gamma_a(H)$ be the antisymmetric Fock space over $H$. For $f \in H$, define a bounded operator $a(f)^*$ on $\Gamma_a(H)$ as

$$a(f)^*(\Omega) = f$$

and

$$a(f)^*(h_1 \wedge h_2 \wedge \ldots \wedge h_n) = f \wedge h_1 \wedge h_2 \wedge \ldots \wedge h_n$$

where $\Omega$ is the vacuum vector of $\Gamma_a(H)$ and $h_1 \wedge h_2 \wedge \ldots \wedge h_n$ is an arbitrary antisymmetric elementary tensor element with $h_1, h_2, \ldots, h_n \in H$ and $n \geq 1$. Let $a(f)$ be the adjoint of $a(f)^*$. The operators $a(f)^*$ and $a(f)$ are called the creation and the annihilation operator associated to a vector $f$.

By an isometric representation of $P$ on a Hilbert space $H$, we mean a strongly continuous map $V : P \rightarrow B(H)$ such that each $V_x$ is an isometry and $V_x V_y = V_{x+y}$ for each $x, y \in P$. For a given isometric representation $V : P \rightarrow B(H)$, there exists a unique $E_0$-semigroup, denoted by $\beta^V$, on $\Gamma_a(H)$ satisfying

$$\beta^V_x(a(f)) = a(V_x f)$$

for each $f \in H$.

This $E_0$-semigroup $\beta^V$ is called the CAR flow associated to the isometric representation $V$; see [4].

Let $H$ and $K$ be Hilbert spaces. For an isometry $W : H \rightarrow K$, there exists a unique bounded operator $\Gamma_a(W)$, called the second quantization of $W$, from $\Gamma_a(H)$ to $\Gamma_a(K)$, satisfying

$$\Gamma_a(W)(\Omega) = \Omega,$$

$$\Gamma_a(W)(f_1 \wedge f_2 \wedge \ldots \wedge f_n) = W f_1 \wedge W f_2 \wedge \ldots \wedge W f_n,$$

where $\Omega$ is the vacuum vector in the appropriate antisymmetric Fock space and $f_1 \wedge f_2 \wedge \ldots \wedge f_n$ is any antisymmetric elementary tensor element with $f_1, f_2, \ldots, f_n \in H$ and $n \geq 1$.

3 Opposite product system for a CAR flow

Let $V$ be a pure isometric representation of $P$ on a Hilbert space $H$. Let $\beta^V$ be the CAR flow associated to the isometric representation $V$ and denote its concrete product system by $\mathcal{E}_{\beta^V}$. Set $E^V(x) = \Gamma_a(\text{Ker}(V_x^*))$. Consider the set $E^V$ as

$$E^V = \{(x, f) : x \in \Omega \text{ and } f \in E^V(x)\}.$$

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Since $E^V$ is a Borel subset of $\Omega \times \Gamma_a(H)$, $E^V$ is a standard Borel space. Define a multiplication on $E^V$ as

$$(x, f) \cdot (y, g) := (x + y, f \otimes \Gamma_a(V_x)g)$$

for every $(x, f), (y, g) \in E^V$. $E^V$ equipped with the above multiplication defines a product system structure over $\Omega$. We define another multiplication $\circ$ on $E^V$ as

$$(x, f) \circ (y, g) := (x + y, g \otimes \Gamma_a(V_y)f).$$

Then the pair $(E^V, \circ)$ also has a structure of product system over $\Omega$, called the opposite product system for $(E^V, \cdot)$, denoted by $(E^V)^{\text{op}}$.

Let $x \in \Omega$ and let $f \in E^V(x)$ be given. Define a bounded operator $T_f$ on $\Gamma_a(H)$ as

$$T_f \eta = f \otimes \Gamma_a(V_x)\eta,$$

for every $\eta \in \Gamma_a(H)$.

Then we have the following lemma.

**Lemma 3.1** The map $\theta : E^V \ni (x, f) \mapsto (x, T_f) \in \mathcal{E}_{\beta^V}$ is an isomorphism as product systems.

**Proof:** Let $(x, f), (y, g) \in E^V$ be given. Since $T_fT_g = T_f \otimes \Gamma_a(V_x)g$, it follows that $\theta(x, f)\theta(y, g) = \theta((x, f)(y, g))$. For each $x \in \Omega$, the restriction of $\theta$ to $E^V(x)$, $\theta|_{E^V(x)} : E^V(x) \to \mathcal{E}_{\beta^V}(x)$ is a unitary. For let $f, g \in E^V(x)$ be given. Note that $T_g^*T_f = \langle f, g \rangle 1_{E^V(x)}$ and $T_f \in \mathcal{E}_{\beta^V}(x)$. This implies that the map $E^V(x) \ni f \mapsto T_f \in \mathcal{E}_{\beta^V}(x)$ is an isometry. To prove that the map is a unitary it suffices to show that whenever $T \in \mathcal{E}_{\beta^V}(x)$ such that $\langle T_f, T \rangle = 0$ for all $f$, then $T = 0$. Since the linear span of the set $\{f \otimes \Gamma_a(V_x)\eta : f \in E^V(x) \text{ and } \eta \in \Gamma_a(H)\}$ is dense in $\Gamma_a(H)$, we see that $T = 0$.

Since $E^V$ and $\mathcal{E}_{\beta^V}$ are standard Borel spaces and the restriction of $\theta$ to each fibre is a unitary, it follows that the map $\theta$ is a Borel isomorphism and hence it is an isomorphism as product systems by [2].

Let us recall the opposite isometric representation $V^{\text{op}}$ for the given isometric representation $V$ considered in [6]. Let $U$ be a minimal unitary dilation of $V$. More precisely, there exists a Hilbert space $\tilde{H}$ containing $H$ as a subspace and a unitary representation $U$ of $\mathbb{R}^d$ on a Hilbert $\tilde{H}$ such that the following conditions hold.

1. For $x \in P$, $U_x \xi = V_x \xi$.

2. The set $\cup_{x \in P}U_x^*H$ is dense in $\tilde{H}$. 

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Note that for $x \in P$, $K = H^\perp$ is invariant under $U_x$. For $x \in P$, define $V_x^{\text{op}}$ on $K$ to be the restriction of $U_{-x}$ to $K$ i.e. $V_x^{\text{op}} := U_{-x}|_K$. Then $V^{\text{op}} := \{V_x^{\text{op}}\}_{x \in P}$ is an isometric representation of $P$, called the opposite isometric representation for $V$. This isometric representation $V^{\text{op}}$ is pure [5, Proposition 3.2].

**Proposition 3.2** The map $\phi : (E^V)^{\text{op}} \ni (x, f) \mapsto (x, \Gamma_a(U_{-x})f) \in E^{V^{\text{op}}}$ is an isomorphism as product systems.

**Proof:** For each $x \in \Omega$, the map $\text{Ker}(V_x^*)h \mapsto U_{-x}h \in \text{Ker}((V_x^{\text{op}})^*)$ is a unitary; see the proof of [5, Proposition 3.2]. Then it follows that the map $\phi : (E^V)^{\text{op}} \ni (x, f) \mapsto (x, \Gamma_a(U_{-x})f) \in E^{V^{\text{op}}}$ is a continuous bijection and its inverse is given by $E^{V^{\text{op}}} \ni (x, \xi) \mapsto (x, \Gamma_a(U_x)\xi) \in (E^V)^{\text{op}}$. Hence it is a Borel isomorphism by [2]. Now it remains to show that $\phi$ follows product system structure. Let $(x, f), (y, g) \in E^V$ be given. Then we have

$$
\phi((x, f)(y, g)) = \phi(x + y, f \otimes \Gamma_a(V_x)g) \\
= (x + y, \Gamma_a(U_{-(x+y)})(f \otimes \Gamma_a(V_x)g)) \\
= (x + y, \Gamma_a(U_{-(x+y)})\Gamma_a(V_x)g \otimes \Gamma_a(U_{-(x+y)})f) \\
= (x + y, \Gamma_a(U_{-y})g \otimes \Gamma_a(U_{-y})\Gamma_a(U_{-x})f) \\
= (y, \Gamma_a(U_{-y})g)(x, \Gamma_a(U_{-x})f) \\
= \phi(y, g)\phi(x, f).
$$

Hence the map $\phi$ is an isomorphism as product systems. \qed

Let $\mathcal{E}_{\beta^V}$ be the concrete product system for $\beta^V$ and let $\mathcal{E}_{\beta^V}^{\text{op}}$ be its opposite product system. By [3, Theorem 3.14], there exists an $E_0$-semigroup denoted by $(\beta^V)^{\text{op}}$ such that $\mathcal{E}_{\beta^V}^{\text{op}}$ is isomorphic to $\mathcal{E}_{(\beta^V)^{\text{op}}}$.

**Corollary 3.3** An $E_0$-semigroup $(\beta^V)^{\text{op}}$ is cocycle conjugate to $\beta^{V^{\text{op}}}$.

**Proof:** By Proposition 3.2 and Lemma 3.1, we conclude that $(E^V)^{\text{op}}$ is isomorphic to $\mathcal{E}_{\beta^{V^{\text{op}}}}$. This implies that the product system $\mathcal{E}_{(\beta^{V^{\text{op}}})}$ is isomorphic to $\mathcal{E}_{\beta^{V^{\text{op}}}}$ by Lemma 3.1. Then by [3, Theorem 2.9], we have $(\beta^V)^{\text{op}}$ is cocycle conjugate to $\beta^{V^{\text{op}}}$. \qed

**Remark 3.4** The above corollary implies that the opposite of a CAR flow over $P$ is again a CAR flow over $P$.

**Theorem 3.5** Let $\beta^V$ be the CAR flow associated to an isometric representation $V$. Then the following are equivalent.

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1. The CAR flow $\beta^V$ is cocycle conjugate to its opposite $(\beta^V)^{op}$

2. The isometric representation $V$ is unitary equivalent to its opposite $V^{op}$.

Proof: Proof follows from [4, Proposition 4.7] and Corollary 3.3. \qedsymbol

4 Examples for symmetric and asymmetric CAR flows

By a $P$-module we mean a non-empty closed subset $A$ of $\mathbb{R}^d$ such that $A + P \subseteq A$. Let $A$ be a $P$-module. For $x \in P$, define an operator $V_x^A$ on $L^2(A)$ as

$$(V_x^A f)(y) = \begin{cases} f(y - x) & \text{if } y - x \in A, \\ 0 & \text{if } y - x \notin A. \end{cases}$$

for each $f \in L^2(A)$. Then the family $\{V_x^A\}_{x \in P}$ is an isometric representation of $P$.

**Proposition 4.1** (See [5, Proposition 3.4]) We have the following.

1. The isometric representation $(V^A)^{op}$ is unitary equivalent to $V^A$.

2. There exists an element $z \in \mathbb{R}^d$ such that $A = -(\text{int}(A)^c) + z$.

Here $\text{int}(A)$ is the interior of $A$ and $\text{int}(A)^c$ is the complement of $\text{int}(A)$ in $\mathbb{R}^d$.

Let $\beta^A$ be the CAR flow associated to the isometric representation $V^A$. It follows from Theorem 3.5 and Proposition 4.1 that the CAR flow $\beta^A$ is cocycle conjugate to its opposite $(\beta^A)^{op}$ if and only if $A = -(\text{int}(A)^c) + z$ for some $z \in \mathbb{R}^d$.

**Remark 4.2** By considering the existence of such $P$-modules, we can see that there are uncountably many symmetric CAR flows as well as asymmetric CAR flows over $P$.

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