CAUSAL HYDRODYNAMICS FROM KINETIC THEORY BY DOUBLET SCHEME IN RENORMALIZATION-GROUP METHOD

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Abstract. We develop a general framework in the renormalization-group (RG) method for extracting a mesoscopic dynamics from an evolution equation by incorporating some excited (fast) modes as additional components to the invariant manifold spanned by zero modes. We call this framework the doublet scheme. We apply the doublet scheme to construct causal hydrodynamics as a mesoscopic dynamics of kinetic theory, i.e., the Boltzmann equation, in a systematic manner with no ad-hoc assumption. It is found that our equation has the same form as Grad’s thirteen-moment causal hydrodynamic equation, but the microscopic formulae of the transport coefficients and relaxation times are different. In fact, in contrast to the Grad equation, our equation leads to the same expressions for the transport coefficients as given by the Chapman-Enskog expansion method and suggests novel formulae of the relaxation times expressed in terms of relaxation functions which allow a natural physical interpretation of the relaxation times. Furthermore, our theory nicely gives the explicit forms of the distribution function and the thirteen hydrodynamic variables in terms of the linearized collision operator, which in turn clearly suggest the proper ansatz forms of them to be adopted in the method of moments.

Key words. Reduction theory of dynamics, Renormalization-group method, Boltzmann equation, Causal hydrodynamics

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1. Introduction . Dissipative hydrodynamic equation is a powerful means for describing the long-wavelength and low-frequency dynamics of many-body systems, which are close to equilibrium state. A typical equation is the Navier-Stokes equation, whose dynamical variables are five fields consisting of temperature, density, and fluid velocity.

One of the problems inherent in the Navier-Stokes equation is instantaneous propagation of information, i.e., the lack of causality, which is attributed to parabolicity of the equation [1,2,3,4]. Here, the parabolicity is a character of diffusion equations containing first-order (second-order) temporal-derivative (spatial-derivative) terms of dynamical variables. This character plagues a relativistically covariant extension of the Navier-Stokes equation.

In 1949, Grad [5] showed that the lack of causality could be circumvented within the framework of kinetic theory, i.e., the Boltzmann equation by employing a method of moments, where an ad-hoc but seemingly plausible ansatz for the functional forms of the distribution function and the moments leads to a closed system of differential equations as the hydrodynamic equations. In particular, for the thirteen-moment approximation to the functional forms, the resultant equation is similar to the Navier-Stokes equation but respects the causality, because the character of the equation is hyperbolic instead of parabolic, with finite propagation speeds. This thirteen-moment causal equation is called the Grad equation, whose dynamical variables are thirteen fields, i.e., temperature, density, fluid velocity, viscous pressure, and heat flux. Since the advent of the Grad equation, a large amount of extensions to relativistic systems have been proposed [6,7,8,9,10,11,12,13,14].

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In 1996, Jou and his collaborators \[15, 16\] called the description by the Grad equation *mesoscopic* since it occupies an intermediate level between the descriptions by the Navier-Stokes equation and the Boltzmann equation. In fact, the Grad equation has been applied to various kinetic problems, e.g., in plasma and in photon transport \[17\], whose dynamics often cannot be described by the Navier-Stokes equation since the systems are not close enough to the equilibrium state.

However, it has recently turned out that the dynamics described by the Grad equation is not consistent with the Boltzmann equation in the mesoscopic scales of space and time. In fact, Karlin and his collaborators \[18, 19\] showed that some additional terms to the Grad equation should be necessary to ensure the consistency with the mesoscopic dynamics of the linearized Boltzmann equation, by employing their method of invariant manifold. Moreover, they demonstrated that these additional terms contain second-order spatial-derivative terms of viscous pressure and heat flux, which cause a significant modification in the dispersion relation of hydrodynamic modes, and hence the causality is lost in the resultant equations. Struchtrup and Torrilhon \[20\] also applied a similar approach to the nonlinear Boltzmann equation for the Maxwell molecules to derive an equation where the causality is similarly lost although the phase velocity given by their equation compares well with experiments. These facts mean that it is still a challenge to construct the equation that respects both of the causality and the consistency with the Boltzmann equation in the mesoscopic regime. Indeed, although an extension of the Grad’s thirteen-moment approximation that ensures the causality has been recently proposed by Levermore \[21\], Torrilhon \[22\], and Öttinger \[23\], with respective different approaches, the consistency between the resultant equations and the mesoscopic dynamics of the Boltzmann equation remains totally unclear.

The purpose of this paper is to extract the mesoscopic dynamics from the Boltzmann equation without recourse to any ansatz for the functional forms of the distribution function and the moments, and investigate whether or not the resultant equation respects the causality. For this purpose, the method of invariant manifold employed by Karlin et al. \[18, 19\] is a promising method since it is a reduction theory of the dynamics \[23\]. Thus, we shall analyze critically the pioneering work by Karlin et al. \[18, 19\]: The lack of causality may be traced back to an ad-hoc assumption used in their approach based on the method of invariant manifold. Indeed, to derive the equation consistent with the Boltzmann equation in the mesoscopic regime, they assumed that the mesoscopic dynamics of thirteen variables exists and is invariant with respect to the Boltzmann equation, and the Grad equation is a suitable first approximation to this mesoscopic dynamics. However, one can find that the number of dynamical variables and the first approximation they adopted might be plausible but just a kind of ansatz, and the form of the resultant equation might not be based on a sound foundation. Therefore, their way of derivation does not necessarily deny the possible existence of the causal equation consistent with the Boltzmann equation in the mesoscopic regime.

To extract the mesoscopic dynamics from the Boltzmann equation in a more systematic way with no ad-hoc assumption, we take the “renormalization-group (RG) method” \[24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39\] as an alternative reduction theory: The RG method as formulated in Refs \[33, 34, 35, 36, 37, 38, 39\] is a powerful tool for reducing evolution equations based on the notion of attractive manifold or invariant manifold \[10\], which the dynamical variables approach to and after some time are eventually confined in. In fact, the RG method \[33, 34, 35, 36, 37\]
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[38, 39] has been applied to reduce kinetic equations to a slower dynamics with fewer degrees of freedom, which is realized on the invariant manifold asymptotically.

Hatta and the authors [38, 39] used the RG method to derive the Navier-Stokes equation from the Boltzmann equation. An essential point in the derivation of the Navier-Stokes equation was to utilize five zero modes of the linearized collision operator, which form the invariant manifold on which hydrodynamics is defined; the would-be constant five zero modes, corresponding to temperature, density, and fluid velocity, acquire the time dependence on the manifold by the RG equation.

Thus, a basic observation presented in the extraction of the mesoscopic dynamics from the Boltzmann equation is to include some excited (fast) modes as additional components of the invariant/attractive manifold, because the mesoscopic dynamics is faster than hydrodynamics that is defined on the invariant manifold spanned by the five zero modes. Which excited modes should we adopt? In this paper, we try to determine the excited modes based on the following consistency condition and basic principle in the reduction theory of the dynamics [24]: (A) the resultant dynamics should be consistent with the reduced dynamics obtained by employing only the zero modes in the asymptotic regime; (B) the resultant dynamics should be as simple as possible because we are interested to reduce the dynamics to a simpler one. Here, we note that the principle (B) is the very spirit of the reduction theory of the dynamics [24], and here the term “simple” is used to express that the resultant dynamics is described with a fewer number of dynamical variables and is given by an equation composed of a fewer number of terms.

We demonstrate that these condition and principle lead us to the doublet scheme in the RG method, which uniquely determines the number and form of the excited modes that should be included in the invariant/attractive manifold on which the mesoscopic dynamics of the Boltzmann equation is defined: The doublet scheme can be applied to a wide class of evolution equations. We also demonstrate that the equation obtained by the RG equation contains thirteen dynamical variables and respects the causality. We show that the form of the resultant equation is the same as that of the Grad equation, but the microscopic formulae of the coefficients, e.g., the transport coefficients and relaxation times, are different, and our theory leads to the same expressions for the transport coefficients as given by the Chapman-Enskog method [41] and also novel formulae of the relaxation times in terms of relaxation functions, which allow a natural physical interpretation of the relaxation times. Moreover, the distribution function and the moments which are explicitly constructed in our theory provide a proper new ansatz for the functional forms of the distribution function and the moments in the method of moments proposed by Grad.

We here remark that some results in the present paper have been announced in the proceedings [42] by the present authors. In the present paper, we shall make a detailed and complete account on the derivations of the causal hydrodynamic equations together with those of the microscopic expressions of the transport coefficients and relaxation times.

This paper is organized as follows: In Sec. 2 we summarize a brief but self-contained account of the Boltzmann equation, Grad’s thirteen-moment approximation in the method of moments, and the thirteen-moment approximation that Karlin et al. have obtained. In Sec. 3 we present the causal hydrodynamic equation and the microscopic representations of the transport coefficients and relaxation times that are obtained with the doublet scheme in the RG method. In Appendix A we describe the detailed development of the doublet scheme in the RG method. Furthermore, in
2. Preliminaries. In this section, we give the basic notions and a brief summary of the previous works as preliminaries so that the significance of the present work will become clear. First, a brief account is given on the basic properties of the Boltzmann equation with a focus on those of the collision operator. Next, we introduce Grad’s moment method and its thirteen-moment approximation [5] for the functional forms of the distribution function and the moments, and present the explicit form of the Grad equation. Finally, we give an account of the method proposed by Karlin et al. [18, 19] for constructing the functional forms of the distribution function and the moments with some comments.

2.1. Basic properties of Boltzmann equation. The Boltzmann equation that we consider in the present work reads

\[ \frac{\partial}{\partial t} f_{\mathbf{v}}(t, \mathbf{x}) + \mathbf{v} \cdot \nabla f_{\mathbf{v}}(t, \mathbf{x}) = C[f|_{\mathbf{v}}(t, \mathbf{x}), \mathbf{x}] \]

Here, \( f_{\mathbf{v}}(t, \mathbf{x}) \) denotes the distribution function defined in phase space \((\mathbf{x}, \mathbf{v})\) with \( t \) and \( \mathbf{x} = (x^1, x^2, x^3) \) being the space-time coordinate and \( \mathbf{v} = (v^1, v^2, v^3) \) the velocity of the one-shell particle whose mass, momentum, and energy are given as \( m, m \mathbf{v}, \) and \( m |\mathbf{v}|^2/2, \) respectively. The right-hand side of Eq. (2.1) is the collision integral,

\[ C[f|_{\mathbf{v}}(t, \mathbf{x})] \equiv \frac{1}{2!} \sum_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \times \left( f_{\mathbf{v}_2}(t, \mathbf{x}) f_{\mathbf{v}_3}(t, \mathbf{x}) - f_{\mathbf{v}}(t, \mathbf{x}) f_{\mathbf{v}_1}(t, \mathbf{x}) \right), \]

where \( \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \) denotes the transition probability due to the microscopic two particle interaction. We note that \( \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \) contains the delta function representing the energy-momentum conservation,

\[ \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \propto \delta^3(m \mathbf{v} + m \mathbf{v}_1 - m \mathbf{v}_2 - m \mathbf{v}_3) \times \delta(m |\mathbf{v}|^2/2 + m |\mathbf{v}_1|^2/2 - m |\mathbf{v}_2|^2/2 - m |\mathbf{v}_3|^2/2), \]

and also has the symmetric properties due to the indistinguishability of the particles and the time reversal invariance of the microscopic transition probability,

\[ \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) = \omega(\mathbf{v}_2, \mathbf{v}_3 | \mathbf{v}, \mathbf{v}_1) = \omega(\mathbf{v}_1, \mathbf{v} | \mathbf{v}_3, \mathbf{v}_2) = \omega(\mathbf{v}_3, \mathbf{v}_2 | \mathbf{v}_1, \mathbf{v}). \]

It should be stressed here that we have confined ourselves to the case in which the particle number is conserved in the collision process. In the following, we suppress the arguments \((t, \mathbf{x})\) when no misunderstanding is expected.

To make explicit the correspondence to the general formulation of the RG method given in Ref. [37], we treat the velocity as a discrete variable, but the summation with respect to the velocity may be interpreted as the integration: \( \sum_{\mathbf{v}} \equiv \int d^3 \mathbf{v}. \)

The property of the transition probability shown in Eq. (2.4) leads to the following identity satisfied for an arbitrary vector \( \varphi_{\mathbf{v}}, \)

\[ \sum_{\mathbf{v}} \varphi_{\mathbf{v}} C[f|_{\mathbf{v}}] = \frac{1}{2!} \frac{1}{4} \sum_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \times (\varphi_{\mathbf{v}_2} + \varphi_{\mathbf{v}_1} - \varphi_{\mathbf{v}_2} - \varphi_{\mathbf{v}_3}) (f_{\mathbf{v}_2} f_{\mathbf{v}_3} - f_{\mathbf{v}} f_{\mathbf{v}_1}). \]
A function $\varphi_v$ is called a collision invariant when it satisfies

$$\sum_v \varphi_v C[f]_v = 0. \quad (2.6)$$

As is easily confirmed using the identity (2.5) and the property (2.3), $\varphi_v = 1$, $m_v$, and $m_v |v|^2/2$ are collision invariants;

$$\sum_v (1, m_v, m_v |v|^2/2) C[f]_v = 0, \quad (2.7)$$

which represent the conservation of the particle number, momentum, and energy by the collision process, respectively. We see that the linear combination of these collision invariants given by $\varphi_v = a + b \cdot m_v + c m_v |v|^2/2$ is also a collision invariant with $a$, $b$, and $c$ being arbitrary functions of $t$ and $x$.

Using the Boltzmann equation (2.1) together with the collision invariants $(1, m_v, m_v |v|^2/2)$, we have the following balance equations,

$$\sum_v (1, m_v, m_v |v|^2/2) \left[ \frac{\partial}{\partial t} + v \cdot \nabla \right] f_v = 0, \quad (2.8)$$

which are reduced to the following forms expressed with the macroscopic variables

$$\frac{\partial}{\partial t} \rho = -\nabla \cdot (\rho \mathbf{V}), \quad (2.9)$$
$$m \rho \frac{\partial}{\partial t} V^i = -m \rho \mathbf{V} \cdot \nabla V^i - \nabla P^{ij}, \quad (2.10)$$
$$\rho \frac{\partial}{\partial t} e = -\rho \mathbf{V} \cdot \nabla e - P^{ij} \nabla V^j - \nabla \cdot \mathbf{Q}, \quad (2.11)$$

respectively. Here, the particle density $\rho$, the fluid velocity $V^i$, the internal energy density $e$, the pressure tensor $P^{ij}$, and the heat current $Q^i$ have the following microscopic expressions, respectively;

$$\rho \equiv \sum_v f_v, \quad V^i \equiv \frac{1}{\rho} \sum_v v^i f_v, \quad e \equiv \frac{1}{\rho} \sum_v \frac{m}{2} |v - V|^2 f_v, \quad (2.12)$$
$$P^{ij} \equiv \sum_v m (v^i - V^i) (v^j - V^j) f_v, \quad Q^i \equiv \sum_v \frac{m}{2} |v - V|^2 (v^i - V^i) f_v. \quad (2.13)$$

It is noted that while these equations have the same forms as the hydrodynamic equation, nothing about the dynamical properties is contained in these equations before the evolution of the distribution function $f_v$ is obtained by solving Eq. (2.1).

In this kinetic theory, the entropy density $s$ and current $J_s$ may be defined by

$$s \times J_s \equiv -\sum_v (1, v) f_v \left( \ln \left[ (2 \pi)^3 f_v \right] - 1 \right). \quad (2.14)$$

Using Eq. (2.14), we have

$$\frac{\partial}{\partial t} s + \nabla \cdot J_s = -\sum_v \left( \ln \left[ (2 \pi)^3 f_v \right] \right) C[f]_v. \quad (2.15)$$
The above equation tells us that the entropy \( S(t) \equiv \int d^3x \, s(t, x) \) is conserved only if \( \ln[(2\pi)^3 f_v] \) is a collision invariant, or a linear combination of the basic collision invariants \((1, v, m|v|^2/2)\). In other words, the entropy-conserving distribution function is parametrized as

\[
 f_v = n \left[ \frac{m}{2\pi T} \right]^{\frac{3}{2}} \exp \left[ -\frac{m|v-u|^2}{2T} \right] \equiv f_v^{eq},
\]

which is identified with the Maxwellian, i.e., the local equilibrium distribution function. The quantities \( T = T(t, x), n = n(t, x), \) and \( u = u(t, x) \) in Eq. (2.16) are the temperature, density, and flow velocity with space- and time-dependence, respectively.

We note that for \( f_v = f_v^{eq} \) the collision integral identically vanishes,

\[
 C[f^{eq}] v = 0,
\]

because of the identity derived from Eq. (2.18):

\[
 \omega(v, v_1|v_2, v_3) (f_v^{eq} f_v^{eq} - f_v^{eq} f_v^{eq}) = 0.
\]

Substituting \( f_v = f_v^{eq} \) into the balance equations (2.19)-(2.21), we have

\[
 \frac{\partial}{\partial t} n = -\nabla \cdot (n u),
\]

\[
 m n \frac{\partial}{\partial t} u^i = -m n u \cdot \nabla u^i - \nabla (n T),
\]

\[
 n \frac{\partial}{\partial t} (3T/2) = -n u \cdot \nabla (3T/2) - n T \nabla \cdot u,
\]

where we have used the fact that Eqs. (2.12) and (2.13) are reduced to \( \rho = n, V^i = u^i, e = 3T/2, P^{ij} = \delta^{ij} n T, \) and \( Q^i = 0, \) respectively. We remark that Eqs. (2.19)-(2.21) are identical with the Euler equation, which describes the fluid dynamics with no dissipative effects, and \( e \) and \( P^{ij} \) are the equations of state of an isotropic dilute gas. Since the entropy-conserving distribution function \( f_v^{eq} \) reproduces the Euler equation, we see that the dissipative effect is attributable to a deviation of \( f_v \) from \( f_v^{eq} \).

2.2. Grad’s thirteen-moment approximation and Grad equation . In Grad’s theory [5], the dissipative distribution function \( f_v \) is first expanded around the local equilibrium one \( f_v^{eq} \) as

\[
 f_v = f_v^{eq} (1 + \Phi_v),
\]

where the deviation \( \Phi_v \) is supposed to be small. Neglecting the second and higher orders of \( \Phi_v \), the Boltzmann equation (2.21) can be linearized as

\[
 (f_v^{eq})^{-1} \left[ \frac{\partial}{\partial t} + v \cdot \nabla \right] f_v^{eq} (1 + \Phi_v) = \sum_k L_{\nu k} \Phi_k,
\]

where the following expansion has been used:

\[
 C[f] = C[f^{eq}] v + \sum_k \frac{\partial}{\partial f_k} C[f] v \bigg|_{f = f^{eq}} f_k^{eq} \Phi_k = f_v^{eq} \sum_k L_{\nu k} \Phi_k.
\]
Here, \( L_{vk} \) denotes the linearized collision operator whose explicit form reads

\[
(2.25) \quad L_{vk} = -\frac{1}{2\tau} \sum_{v_1v_2v_3} \omega(v, v_1v_2, v_3) f^{eq}(\delta v_k + \delta v_1k - \delta v_2k - \delta v_3k).
\]

To express \( \Phi_v \) in terms of hydrodynamic variables, let us introduce \( v \)-dependent quantities \( \hat{\pi}^{ij}_v \) and \( \hat{J}^i_v \) defined by

\[
(2.26) \quad \hat{\pi}^{ij}_v = m \Delta^{ijkl} \delta v^i \delta v^j,
\]

and

\[
(2.27) \quad \hat{J}^i_v = \left( \frac{m}{2} |\delta v|^2 - \frac{5}{2} T \right) \delta v^i,
\]

with the peculiar velocity \( \delta v = v - u \) and the projection matrix

\[
(2.28) \quad \Delta^{ijkl} = \frac{1}{2} \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \frac{2}{3} \delta^{ij} \delta^{kl} \right).
\]

Here, \( \hat{\pi}^{ij}_v \) and \( \hat{J}^i_v \) are identified as the microscopic representations of the viscous pressure and heat flux, respectively. Thanks to the symmetry property of \( \Delta^{ijkl} \), \( \hat{\pi}^{ij}_v \) is symmetric and traceless:

\[
(2.29) \quad \hat{\pi}^{ij}_v = \hat{\pi}^{ji}_v, \quad \delta^{ij} \hat{\pi}^{ij}_v = 0.
\]

It is to be noted that \( \hat{\pi}^{ij}_v \) and \( \hat{J}^i_v \) are orthogonal to the collision invariants as

\[
(2.30) \quad \langle \varphi, \hat{\pi}^{ij}_v \rangle_{eq} = \langle \varphi, \hat{J}^i_v \rangle_{eq} = 0, \quad \varphi_v = 1, m v, m |v|^2 / 2,
\]

where the inner product for two arbitrary functions \( \psi_v \) and \( \chi_v \) is defined by

\[
(2.31) \quad \langle \psi, \chi \rangle_{eq} \equiv \sum_v f^{eq}_v \psi_v \chi_v.
\]

With the use of the vector fields \( \hat{\pi}^{ij}_v \) and \( \hat{J}^i_v \), the conventional ansatz for the form of \( \Phi_v \) is expressed as

\[
(2.32) \quad \Phi_v = -\frac{\hat{\pi}^{ij}_v \pi^{ij}}{5} \langle \hat{\pi}^{kl}_v, \hat{\pi}^{kl}_v \rangle_{eq} - \frac{\hat{J}^i_v J^i}{3} \langle \hat{J}^k, \hat{J}^k \rangle_{eq} \equiv \Phi_v^G.
\]

Here, \( \pi^{ij} \) and \( J^i \) are expansion coefficients which have a dependence on \( (t, x) \) as well as \( T, n, \) and \( u^i \); \( \pi^{ij} = \pi^{ij}(t, x) \) and \( J^i = J^i(t, x) \). The coefficients \( \pi^{ij} \) and \( J^i \) should be interpreted as the viscous pressure and heat flux, respectively. It is noted that the total number of independent components of \( T, n, u^i, \pi^{ij} \), and \( J^i \) is thirteen because without loss of generality we can suppose that

\[
(2.33) \quad \pi^{ij} = \pi^{ji}, \quad \delta^{ij} \pi^{ij} = 0,
\]

owing to the symmetric and traceless properties of \( \hat{\pi}^{ij}_v \) in Eq. (2.29). The factors \( 5/\langle \hat{\pi}^{kl}_v, \hat{\pi}^{kl}_v \rangle_{eq} \) and \( 3/\langle \hat{J}^k, \hat{J}^k \rangle_{eq} \) in \( \Phi_v^G \) have been introduced so that the followings are satisfied:

\[
(2.34) \quad \pi^{ij} = -\langle \hat{\pi}^{ij} \rangle_{eq}, \quad J^i = -\langle \hat{J}^i \rangle_{eq}.
\]
where we have used the relations
\begin{equation}
\langle \tilde{v}^{ij}, \tilde{v}^{kl} \rangle = \frac{1}{5} \Delta^{ijkl} \langle \hat{v}^{ab}, \hat{v}^{ab} \rangle_{\text{eq}}, \langle \tilde{j}^{i}, \tilde{j}^{j} \rangle = \frac{1}{3} \delta^{ij} \langle j^{a}, J_{a} \rangle_{\text{eq}},
\end{equation}
\begin{equation}
\langle \tilde{v}^{ij}, \tilde{j}^{k} \rangle_{\text{eq}} = \langle \tilde{j}^{i}, \tilde{v}^{kl} \rangle_{\text{eq}} = 0.
\end{equation}

To determine the \((t, \mathbf{x})\)-dependence of the thirteen coefficients \(T, n, u^{i}, \pi^{ij}, \text{and} \ J^{i}\), one may utilize the linearized Boltzmann equation (2.23), the inner product of which with any independent thirteen variables dependent on \(v\) would give a closed system of equations for the thirteen coefficients. Let us denote such a set of the thirteen variables by \(\phi_{13v}\). In the Grad’s thirteen-moment approximation, the five collision invariants \((1, m \mathbf{v}, m |v|^{2}/2)\) and the eight quantities \(\hat{v}^{ij}_{v}\) and \(\hat{J}^{i}_{v}\) are adopted as \(\phi_{13v}\):
\begin{equation}
\phi_{13v} = \left\{ 1, m \mathbf{v}, \frac{m}{2} |v|^{2}, \hat{v}^{ij}_{v}, \hat{J}^{i}_{v} \right\} = \phi^{G}_{13v}.
\end{equation}
Here, it should be emphasized that this is merely a possible choice without any foundation. The resultant closed equations consist of the five balance equations
\begin{equation}
\sum_{\mathbf{v}} (1, m \mathbf{v}, m |v|^{2}/2) \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f_{\mathbf{v}}^{\text{eq}} (1 + \phi^{G}_{v}) = 0,
\end{equation}
and the eight relaxation equations
\begin{equation}
\sum_{\mathbf{v}} (\hat{v}^{ij}_{v}, \hat{J}^{i}_{v}) \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f_{\mathbf{v}}^{\text{eq}} (1 + \phi^{G}_{v}) = \langle (\hat{v}^{ij}, \hat{J}^{i}), L \phi^{G} \rangle_{\text{eq}},
\end{equation}
where the following representations are used: \([L \phi^{G}]_{v} = \sum_{k} L_{v k} \phi^{G}_{k}\).

By carrying out the integration with respect to \(v\), Eqs. (2.38) and (2.39) are reduced to
\begin{align}
&\frac{\partial}{\partial t} n = -\mathbf{v} \cdot (n \mathbf{u}), \\
&mn \frac{\partial}{\partial t} u^{i} = -mn \mathbf{u} \cdot \nabla u^{i} - \nabla (nT \delta^{ij} - \pi^{ij}), \\
&n \frac{\partial}{\partial t} (3T/2) = -n \mathbf{u} \cdot \nabla (3T/2) - (nT \delta^{ij} - \pi^{ij}) \nabla u^{j} + \nabla J^{i}, \\
&\pi^{ij} + \tau^{G}_{\pi} \frac{\partial}{\partial t} \pi^{ij} = 2\eta^{G} \Delta^{ijkl} \nabla^{k} u^{l} + \text{terms involving relaxation lengths}, \\
&J^{i} + \tau^{G}_{J} \frac{\partial}{\partial t} J^{i} = \lambda^{G} \nabla T + \text{terms involving relaxation lengths},
\end{align}
which govern the dynamics of \(T, n, u^{i}, \pi^{ij}, \text{and} \ J^{i}\). Here, \(\eta^{G}\) and \(\tau^{G}_{\pi} \lambda^{G} \text{and} \ \tau^{G}_{J}\) denote the transport coefficient and relaxation time associated with the viscous pressure \(\pi^{ij}\) (the heat flux \(J^{i}\)), respectively, whose microscopic representations are given by
\begin{align}
\eta^{G} &\equiv -\frac{1}{10T} \left[ \langle \hat{v}^{ij}, \hat{v}^{ij} \rangle_{\text{eq}} \right]^{2}, \\
\lambda^{G} &\equiv -\frac{1}{3T^{2}} \left[ \langle \hat{j}^{i}, \hat{j}^{i} \rangle_{\text{eq}} \right]^{2}, \\
\tau^{G}_{\pi} &\equiv -\frac{\langle \hat{v}^{ij}, \hat{v}^{ij} \rangle_{\text{eq}}}{\langle \hat{v}^{kl}, L \hat{v}^{kl} \rangle_{\text{eq}}}, \\
\tau^{G}_{J} &\equiv -\frac{\langle \hat{j}^{i}, \hat{j}^{i} \rangle_{\text{eq}}}{\langle \hat{j}^{k}, L \hat{j}^{k} \rangle_{\text{eq}}},
\end{align}
The transport coefficients $\eta^G$ and $\lambda^G$ are to be identified with the shear viscosity and thermal conductivity, respectively. It is well known, however, that the shear viscosity and thermal conductivity in the Grad equation are not in accord with those by the Chapman-Enskog method \[41\]. In fact, the microscopic representations by Chapman and Enskog read

\begin{equation}
\eta_{CE} \equiv -\frac{1}{10T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq}, \quad \lambda_{CE} \equiv -\frac{1}{3T^2} \langle \hat{j}^{i}, L^{-1} \hat{j}^{i} \rangle_{eq},
\end{equation}

and one easily sees that

\begin{equation}
\eta^G \neq \eta_{CE}, \quad \lambda^G \neq \lambda_{CE}.
\end{equation}

Here, $L^{-1}_{vk}$ denotes the inverse matrix of $L_{vk}$.

2.3. Distribution function and moments derived by Karlin et al. . The setting of $\Phi_{v}$ and $\phi_{13v}$ in the Grad’s thirteen-moment approximation is just ansatz, and it is not a unique choice for leading to a closed system of equations that describes the fluid dynamics with the dissipative effect. In fact, Karlin et al. \[18, 19\] have derived the following setting on the basis of the method of invariant manifold:

\begin{equation}
\Phi_{v} = -\frac{1}{10} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq} - \frac{1}{3} \langle \hat{j}^{i}, L^{-1} \hat{j}^{i} \rangle_{eq} + \Delta_{v} = \Phi_{v}^{KGDN},
\end{equation}

\begin{equation}
\phi_{13v} = \{1, m, \frac{m}{2} | v |^2, \hat{\pi}^{ij}, \hat{j}^{i} \} \equiv \phi_{13v}^{KGDN}.
\end{equation}

We note that

\begin{equation}
\phi_{13v}^{KGDN} = \phi_{13v}^{G}.
\end{equation}

Here, $\Delta_{v}$ consists of spatial derivatives of $\pi^{ij}$ and $J^{i}$, with which the causality of the equations is lost \[18, 19\]. If $\Delta_{v}$ is neglected, the resultant equation has the same form as the Grad equation given by Eqs. (2.40)-(2.44) but the microscopic structure of the transport coefficients are different. We write the coefficients as $\eta^{KGDN}$, $\lambda^{KGDN}$, $\tau^{KGDN}_{\pi}$, and $\tau^{KGDN}_{J}$ instead of $\eta^G$, $\lambda^G$, $\tau^G_{\pi}$, and $\tau^G_{J}$ in Eqs. (2.43) and (2.44), respectively. Then their microscopic representations read

\begin{equation}
\eta^{KGDN} \equiv -\frac{1}{10T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq}, \quad \lambda^{KGDN} \equiv -\frac{1}{3T^2} \langle \hat{j}^{i}, L^{-1} \hat{j}^{i} \rangle_{eq},
\end{equation}

\begin{equation}
\tau^{KGDN}_{\pi} \equiv -\frac{\langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq}}{\langle \hat{\pi}^{kl}, \hat{\pi}^{kl} \rangle_{eq}}, \quad \tau^{KGDN}_{J} \equiv -\frac{\langle \hat{j}^{i}, L^{-1} \hat{j}^{i} \rangle_{eq}}{\langle \hat{J}^{k}, \hat{J}^{k} \rangle_{eq}}.
\end{equation}

We note that in contrast to $\Phi_{v}^{G}$ and $\phi_{13v}^{G}$ by Grad, their setting gives the same representations of the transport coefficients as those by Chapman and Enskog:

\begin{equation}
\eta^{KGDN} = \eta_{CE}^{G}, \quad \lambda^{KGDN} = \lambda_{CE}^{G}.
\end{equation}

3. Reduction of Boltzmann equation to mesoscopic dynamics with doublet scheme in RG method . In this section, we first give a brief summary of a newly developed general reduction scheme called the doublet scheme for extracting the mesoscopic dynamics from a given microscopic evolution equation on the basis of the RG method, with the detailed account of the doublet scheme being presented
in Appendix A. Next, we show how a natural application of the doublet scheme to the Boltzmann equation leads to the causal hydrodynamic equation in the mesoscopic scale, with the straightforward but somewhat involved derivation of it being presented in Appendix B. On the basis of the explicit form of the resultant hydrodynamic equation, we clarify the desirable properties of it.

3.1. Mesoscopic dynamics derived with doublet scheme. Since we are interested in the mesoscopic solution whose space-time scales are coarse-grained from those in the kinetic regime, we solve the Boltzmann equation (2.1) in the mesoscopic regime where the space-time variation of \( f_{\mathbf{v}}(t, \mathbf{x}) \) is small. To make a coarse graining in a systematic manner, we convert Eq. (2.1) into

\[
\frac{\partial}{\partial t} f_{\mathbf{v}} = C[f]_{\mathbf{v}} - \epsilon \mathbf{v} \cdot \nabla f_{\mathbf{v}},
\]

where a parameter \( \epsilon \) has been introduced to express that the space derivatives are small for the system that we are interested in. Here, \( \epsilon \) is identified with the ratio of the average particle distance over the mean free path, i.e., the Knudsen number.

To make the generic structure of Eq. (3.1) clear, let us write as

\[
\mathbf{X} = \left\{ f_{\mathbf{v}} \right\}_{\mathbf{v}}, \quad G(\mathbf{X}) = \left\{ C[f]_{\mathbf{v}} \right\}_{\mathbf{v}}, \quad F(\mathbf{X}) = \left\{ -\mathbf{v} \cdot \nabla f_{\mathbf{v}} \right\}_{\mathbf{v}}.
\]

It is noted that the velocity \( \mathbf{v} \) is interpreted as an index of the vector, while we treat the space coordinate \( \mathbf{x} \) solely as a parameter, in accordance with the past works by Kuramoto [24] and Hatta and Kunihiro [38]. From now on, let us omit \( \left\{ \cdot \right\}_{\mathbf{v}} \). Then, Eq. (3.1) is converted to the following generic form of evolution equation

\[
\frac{\partial}{\partial t} \mathbf{X} = G(\mathbf{X}) + \epsilon F(\mathbf{X}).
\]

From now on, we shall explain how the mesoscopic dynamics can be extracted by the doublet scheme from the generic equation (3.3) without recourse to the special identification given in Eq. (3.2). \( \mathbf{X} \) now denotes dynamical variables, \( G(\mathbf{X}) \) and \( F(\mathbf{X}) \) non-linear functions with respect to \( \mathbf{X} \), and \( \epsilon \) an infinitesimal expansion parameter.

We assume that the unperturbative equation \( \partial \mathbf{X} / \partial t = G(\mathbf{X}) \) has an \( M_0 \) dimensional invariant manifold \( \mathbf{X} = \mathbf{X}_{\text{st}} \) as an asymptotic solution, the invariant/attractive manifold of Eq. (3.3) will be expressed as

\[
\mathbf{X} = \mathbf{X}_{\text{st}}(C(1), \cdots, C(M_0)) + \epsilon \sum_{\mu=1}^{M_1} A^{-1} \varphi^{(\mu)}_1 \delta X^{(\mu)} + O(\epsilon^2)
\]

in the perturbation theory. Here, \( C(\alpha) \) with \( \alpha = 1, \cdots, M_0 \) are variables that parameterize the unperturbative manifold \( \mathbf{X}_{\text{st}} \), and \( A^{-1} \) is the inverse matrix of a linearized evolution operator \( A \) given by the first derivative of \( G(\mathbf{X}) \) on \( \mathbf{X} = \mathbf{X}_{\text{st}} \), \( \varphi^{(\mu)}_1 \) with \( \mu = 1, \cdots, M_1 \) denote vectors orthogonal to the manifold \( \mathbf{X}_{\text{st}} \) which can be derived from \( F(\mathbf{X}_{\text{st}}) \), and \( \delta X^{(\mu)} \) denotes an amplitude corresponding to \( \varphi^{(\mu)}_1 \).

The dynamics of \( \mathbf{X} \) is given through the \( M_0 + M_1 \) variables \( C(\alpha) \) with \( \alpha = 1, \cdots, M_0 \) and \( \delta X^{(\mu)} \) with \( \mu = 1, \cdots, M_1 \) whose dynamics is governed by the following \( M_0 + M_1 \) equations:

\[
\sum_{\beta=1}^{M_0} \left\langle \varphi_0^{(\alpha)}, \varphi_0^{(\beta)} \right\rangle \frac{\partial}{\partial t} C^{(\beta)} + \epsilon \sum_{\mu=1}^{M_1} \left( \varphi^{(\mu)}_0, \varphi^{(\mu)}_1 \right) \delta X^{(\mu)}
\]
obtained by substituting the exact solution to Eqs. (3.5) and (3.6) into

\[ (3.3) \]

The global solution to the evolution equation (3.3) in the mesoscopic regime can be

In this subsection, we shall show that the general scheme developed in the previous

\[ (3.7) \]

microscopic representations of transport coefficients and relaxation times

First, we construct \( A, B, F_0, F_1, \delta X(\mu), C(\alpha), \gamma_{X(\mu)}, \varphi_0(\alpha), \) and \( \varphi_1(\mu) \) based on the correspondences in Eq. (3.2). Then, by substituting these into Eqs. (3.4)-(3.6), we derive the hydrodynamic equation as the mesoscopic dynamics of the Boltzmann equation (3.1). Details of the construction and the derivation are presented in Appendix B.

The resultant invariant/attractive manifold is parameterized by the thirteen variables \( T, n, u_i, \pi^{ij}, \) and \( J^i \) as

\[ (3.7) \]

and the equations that govern the dynamics of these variables are given by

\[ (3.8) \]

\[ (3.9) \]

\[ (3.10) \]

\[ (3.11) \]
(3.12) \[ \epsilon \tau_j^{TK} \frac{\partial}{\partial t} f^i = -\epsilon \left[ f^i - \lambda^{TK} \nabla^i T \right] + O(\epsilon^2), \]

where \( \eta^{TK}, \lambda^{TK}, \tau_\pi^{TK}, \) and \( \tau_j^{TK} \) are the transport coefficients and relaxation times. In Eqs. (3.4), (3.10), and (3.11), we have presented the invariant/attractive manifold and the relaxation equations valid up to \( O(\epsilon) \) for the sake of simplicity. Full expressions valid up to \( O(\epsilon^2) \) are given in Appendix B.3.

Setting \( \epsilon = 1 \), we find that the form of Eqs. (3.8)-(3.12) is the same as that of the Grad equation given by Eqs. (2.40)-(2.44), and hence our equation has the hyperbolic character with the causality being hold, as the Grad equation does. We stress that Eqs. (3.8)-(3.12) are identical to the thirteen-moment causal equation consistent with the Boltzmann equation in the mesoscopic regime, which has been long sought for.

Furthermore, the relaxation equations valid up to \( \tau \)

In Eq. (3.7), (3.11), and (3.11), we have presented the invariant/attractive manifold and identities satisfied for \( \hat{\psi} \).

The microscopic representation of the coefficients in our causal hydrodynamic equations read

(3.13) \[ \eta^{TK} = \frac{1}{10 T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{eq}, \quad \lambda^{TK} = \frac{1}{3 T} \langle \hat{J}^i, L^{-1} \hat{J}^i \rangle_{eq}, \]

(3.14) \[ \tau_\pi^{TK} = -\frac{1}{\lambda^{TK}} \langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle_{eq}, \quad \tau_j^{TK} = -\frac{1}{\lambda^{TK}} \langle \hat{J}^i, L^{-2} \hat{J}^i \rangle_{eq}. \]

The transport coefficients in Eq. (3.13) perfectly agree with those derived in the Chapman-Enskog method or the method of invariant manifold by Karlin et al., but not in Grad’s thirteen-moment approximation:

(3.15) \[ \eta^{TK} = \eta^{CE} = \eta^{KGDN} \neq \eta^{G}, \quad \lambda^{TK} = \lambda^{CE} = \lambda^{KGDN} \neq \lambda^{G}. \]

Furthermore, the relaxation times in Eq. (3.14) differ from those in the previous works and have novel representations, that is,

(3.16) \[ \tau_\pi^{TK} \neq \tau_\pi^{KGDN}, \quad \tau_j^{TK} \neq \tau_j^{KGDN}. \]

To make the physical meaning of the microscopic representations of \( \tau_\pi^{TK} \) and \( \tau_j^{TK} \) clearer, we convert the definitions in Eq. (3.14) into the forms as the Green-Kubo formula in the linear response theory. For this purpose, we utilize the following identities satisfied for \( \hat{\psi}_\nu = (\hat{\pi}_\nu, \hat{J}_\nu) \):

(3.18) \[ [L^{-1} \hat{\psi}]_\nu = -\int_0^\infty ds \left[ e^{sL} \hat{\psi} \right]_\nu = -\int_0^\infty ds \hat{\psi}_\nu(s), \]

(3.19) \[ [L^{-2} \hat{\psi}]_\nu = \int_0^\infty ds_1 \int_0^\infty ds_2 \left[ e^{(s_1+s_2)L} \hat{\psi} \right]_\nu = \int_0^\infty ds \hat{\psi}_\nu(s), \]

where we have defined \( \hat{\psi}_\nu(s) \equiv \left[ e^{sL} \hat{\psi} \right]_\nu \). It is noted that \( \hat{\psi}_\nu(s) \) could be interpreted as a “time-evolved” vector of \( \hat{\psi}_\nu \) by \( L_{\nu \kappa} \). With the use of the above identities, we can obtain the compact forms for the transport coefficients and relaxation times as follows:

(3.20) \[ \eta^{TK} = \int_0^\infty ds R_\pi(s), \quad \lambda^{TK} = \int_0^\infty ds R_J(s), \]

(3.21) \[ \tau_\pi^{TK} = \frac{\int_0^\infty ds \hat{s} R_\pi(s)}{\int_0^\infty ds R_\pi(s)}, \quad \tau_j^{TK} = \frac{\int_0^\infty ds \hat{s} R_J(s)}{\int_0^\infty ds R_J(s)}. \]
where \( R_\pi(s) \) and \( R_J(s) \) are defined by

\[
R_\pi(s) = \frac{1}{10T} \left\langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s) \right\rangle_{eq}, \quad R_J(s) = \frac{1}{3T^2} \left\langle \hat{J}^i(0), \hat{J}^i(s) \right\rangle_{eq}.
\]

It is noted that \( R_\pi(s) \) and \( R_J(s) \) denote the relaxation functions introduced in the linear response theory. We remark that the formulae of \( \tau_\pi^{TK} \) and \( \tau_J^{TK} \) allow the natural interpretation of them as the correlation times of \( R_\pi(s) \) and \( R_J(s) \), respectively.

Here, we discuss the reason why the novel and natural microscopic expressions of the relaxation times \( \tau_\pi^{TK} \) and \( \tau_J^{TK} \) have been obtained, together with those of the transport coefficients in agreement with the Chapman-Enskog formulae. First of all, it should be noted that our method is based on a faithful solution of the Boltzmann equation in the perturbation theory as the Chapman-Enskog theory is, with the secular terms being resummed by the RG method or multiple-scale method, although the latter method fails in deriving the causal hydrodynamic equation. Indeed, the analytical forms of our excited modes are derived by solving the Boltzmann equation in a faithful manner based on the perturbation theory with the secular terms resummed by the RG method. Indeed, the analytical forms of our excited modes are constructed so as to solve the Boltzmann equation and represent the relaxation process to the local equilibrium distribution function: The doublet scheme in the RG method developed in Appendix A provides us with the powerful scheme for describing the relaxation dynamics in the mesoscopic scale, and hence deriving the natural microscopic representations of the relaxation times \( \tau_\pi^{TK} \) and \( \tau_J^{TK} \). Thus, we are confident that we have arrived at the correct formulae of the relaxation times for the first time.

3.3. Functional forms of distribution function and moments to reproduce causal hydrodynamics by RG method. We can read off the form of the derivation \( \Phi_v \) and the thirteen quantities that reproduce the causal hydrodynamic equations (3.8)-(3.12) as

\[
\Phi_v = -\frac{1}{2} \left\langle \hat{\pi}^{kl}, L^{-1} \hat{\pi}^{kl} \right\rangle_{eq} - \frac{1}{2} \left\langle \hat{J}^i, L^{-1} \hat{J}^i \right\rangle_{eq} \equiv \Phi_v^{TK},
\]

\[
\phi_{13v} = \left\{ 1, m, \frac{m}{2} |v|^2, [L^{-1} \hat{\pi}^{ij}]_v, [L^{-1} \hat{J}^i]_v \right\} \equiv \phi_{13v}^{TK}.
\]

It is obvious that in general

\[
\Phi_v^{TK} = \Phi_v^{KGDN} \neq \Phi_v^G,
\]

\[
\phi_{13v}^{TK} \neq \phi_{13v}^{KGDN} = \phi_{13v}^G,
\]

if the term \( \Delta_v \) in \( \Phi_v^{KGDN} \) is neglected. We stress that the set of \( \Phi_v^{TK} \) and \( \phi_{13v}^{TK} \) provides the correct functional forms of the distribution function and the moments to be used in the method of moments, which thereby should lead to the causal hydrodynamic equation compatible with the Boltzmann equation in the mesoscopic regime.

Finally, we point out that since the linearized collision operator \( L_{vk} \) is specified by the microscopic transition probability \( \omega(v, v_1 | v_2, v_3) \), our \( \Phi_v^{TK} \) and \( \phi_{13v}^{TK} \) may happen to coincide with those given in the previous works by other authors when
the transition probability has some peculiar properties. Indeed, such a coincidence is possible when both the equality $\Phi_{TK}^v = \Phi_G^v$ and $\phi_{13v}^{TK} = \phi_{13v}^{KGDN}$ are satisfied. We find that such a case is realized only when both $\tilde{\alpha}_{ij}^v$ and $J_{ij}^v$ are eigenvectors of $L_{\nu k}$. It is well known that the linearized collision operator for the Maxwell molecules has such a property \[18, 19\]. Thus, we conjecture that the method of moments with the use of our $\Phi_{TK}^v$ and $\phi_{13v}^{TK}$ can provide the causal hydrodynamic equation for generic systems with the particle interaction not restricted to that of the Maxwell molecules type, whereas those of Grad or Karlin et al. may be at most compatible with the Boltzmann equation only for the Maxwell molecules. It is left as a future work to show that the conjecture is true, which may imply that the present theory makes the correct and general method for constructing mesoscopic dynamics for a given microscopic dynamics.

4. Summary and concluding remarks. In this paper, we have derived the mesoscopic dynamics from the Boltzmann equation on the basis of the renormalization group (RG) method in a systematic manner with no ad-hoc assumption: A basic observation presented in the extraction of the mesoscopic dynamics from the Boltzmann equation is to include some excited (fast) modes of the linearized collision operator as additional components for the invariant manifold spanned by the zero modes, where the hydrodynamics is defined. We have demonstrated that the number and form of the excited modes that should be included in the invariant/attractive manifold is uniquely determined by the doublet scheme in the RG method, which is developed based on the following consistency condition and basic principle in the reduction theory of the dynamics: (A) the resultant dynamics should be consistent with the reduced dynamics obtained by employing only the zero modes in the asymptotic regime; (B) the resultant dynamics should be as simple as possible because we are interested to reduce the dynamics to a simpler one. We have also demonstrated that the mesoscopic dynamics of the Boltzmann equation obtained by the doublet scheme in the RG method respects the hyperbolic character, i.e., the causality, where the number of the dynamical variables is thirteen. We have shown that the form of the resultant equation is the same as that of the Grad equation \[5\], but the microscopic formulae of the coefficients, e.g., the transport coefficients and relaxation times, are different. It has turned out that our theory leads to the same expressions for the transport coefficients as given by the Chapman-Enskog method \[11\] as the method of invariant manifold of Karlin et al. \[18, 19\] does. We have found that our microscopic representations of the relaxation times differ from those of the previous works, and can be converted into formulae written in terms of relaxation functions, which allow a natural physical interpretation of the relaxation times. We have shown that the distribution function and the moments which are explicitly constructed in our theory provides a new ansatz for the functional forms of the distribution function and the moments in the method of moments proposed by Grad. Furthermore, we have conjectured that the functional forms of the distribution function and the moments in the previous works of Grad or Karlin et al. are valid only for a specific interacting systems such as the Maxwell molecules, while our functional forms can be applied to generic interacting systems not restricted to the Maxwell molecules.

It is interesting and important to present numerical simulations to elucidate that a solution of the causal hydrodynamic equation obtained in this work is actually consistent with that of the Boltzmann equation in the mesoscopic regime, even if the systems of interest are generic interacting systems instead of the Maxwell molecules.

It is also interesting to apply the doublet scheme in the RG method to extract
the mesoscopic dynamics of the relativistic Boltzmann equation. This is because the fourteen-moment approximation for the distribution function of the relativistic Boltzmann equation proposed by Israel and Stewart [6, 7], i.e., the most famous relativistically covariant extension of Grad’s thirteen-moment approximation, has encountered the same difficulty as in the non-relativistic case: Numerical simulations by several groups [8, 9] show that the dynamics described by Israel-Stewart’s fourteen-moment equation is not consistent with that of the relativistic Boltzmann equation in the mesoscopic regime. Indeed, we can show [42, 43, 44] that the equation obtained by the RG method, which ensures the consistency with the mesoscopic dynamics of the relativistic Boltzmann equation, respects the causality, and the number of the dynamical variables are fourteen. We can also show that the form of our fourteen-moment causal equation is the same as the Israel-Stewart’s fourteen-moment equation, but the formulae of the coefficients, i.e., the transport coefficients and relaxation times, include in the equation are different and the microscopic representations of the coefficients are given as natural forms in terms of the relaxation functions as in the non-relativistic case shown in this work.

Finally, we note that the doublet scheme in the RG method itself has a universal nature and can be applied to derive a mesoscopic dynamics from kinetic equations other than the simple Boltzmann equation, e.g., Kadanoff-Baym equation [45]. Furthermore, we also note that an extension of the doublet scheme to a method applicable to a more generic case where the evolution operator $A$ is not self-adjoint or has Jordan cell will be studied elsewhere.

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Appendix A. Derivation of mesoscopic dynamics from generic evolution equation with doublet scheme in RG method. In this section, we develop a method on the basis of the RG method to extract the mesoscopic dynamics from a generic evolution equation by constructing the invariant/attractive manifold incorporating some appropriate excited modes as well as the zero modes of its linearized evolution operator based on the following consistency condition and general principle of the reduction theory of the dynamics [24]: (A) the resultant dynamics should be consistent with the slow dynamics obtained by employing only the zero modes in the asymptotic regime; (B) the resultant dynamics should be as simple as possible because we are interested in reducing the dynamics to a simpler one. As mentioned in Appendix A we use the principle (B) to derive an equation describing the mesoscopic dynamics, where the number of dynamical variables and terms are reduced as few as possible. We will see that these condition and principle uniquely determine the number and form of the excited modes that should be included in the invariant/attractive manifold on which the mesoscopic dynamics of the evolution equation is defined.

A.1. Evolution equation. As a generic evolution equation, we shall treat a system of differential equations with two non-linear terms, which represent the relaxation to a static solution and weak perturbation, respectively. The equation reads

$$\frac{\partial}{\partial t}X_i = G_i(X_1, \cdots, X_N) + \epsilon F_i(X_1, \cdots, X_N), \quad i = 1, \cdots, N,$$  

(A.1)
which is also rewritten in a more convenient vector form

\[ \frac{\partial}{\partial t} \mathbf{X} = G(\mathbf{X}) + \epsilon F(\mathbf{X}). \]  

(A.2)

In Eq. (A.2), the dynamical variables are represented by \( N \)-component vector \( \mathbf{X} \) \((1 < N \leq \infty)\), whereas \( G(\mathbf{X}) \) and \( F(\mathbf{X}) \) are non-linear functions of \( \mathbf{X} \), and \( \epsilon \) is introduced as an indicator of the smallness of \( F(\mathbf{X}) \) that is finally set equal to 1; the vector \( \mathbf{X}(t) \) governed by Eq. (A.2) without \( F(\mathbf{X}) \) relaxes to the static solution \( \mathbf{X}_{st} \) under time evolution as

\[ \mathbf{X}(t \to \infty) \to \mathbf{X}_{st}, \]  

(A.3)

which is given as a solution to

\[ G(\mathbf{X}_{st}) = 0. \]  

(A.4)

Here, we suppose that the static solution \( \mathbf{X}_{st} \) forms a well-defined \( M_0 \)-dimensional invariant manifold with \( M_0 \) being smaller than or equal to \( N \). This means that \( \mathbf{X}_{st} \) is parametrized by \( M_0 \) integral constants \( C_{(\alpha)} \) with \( \alpha = 1, \cdots, M_0 \):

\[ \mathbf{X}_{st} = \mathbf{X}_{st}(C_{(1)}, \cdots, C_{(M_0)}). \]  

(A.5)

We first define the linearized evolution operator \( A \) by

\[ A_{ij} = \left. \frac{\partial}{\partial X_j} G_i(X_1, \cdots, X_N) \right|_{X=\mathbf{X}_{st}}. \]  

(A.6)

We note that in accordance with Eq. (A.5), \( A \) has \( M_0 \) eigenvectors belonging to the zero eigenvalue, i.e., zero modes, and the dimension of the kernel of \( A \) is \( M_0 \); i.e., \( \dim \ker A = M_0 \). In fact, by differentiating Eq. (A.4) with respect to the \( M_0 \) integral constants \( C_{(\alpha)} \), we have

\[ A \frac{\partial \mathbf{X}_{st}}{\partial C_{(\alpha)}} = 0, \]  

(A.7)

which means that \( \varphi_0^{(\alpha)} \) defined by

\[ \varphi_0^{(\alpha)} = \frac{\partial \mathbf{X}_{st}}{\partial C_{(\alpha)}}, \]  

(A.8)

are the \( M_0 \) zero modes. The invariant manifold is spanned by \( \varphi_0^{(\alpha)} \) with \( \alpha = 1, \cdots, M_0 \).

We shall define the projection operator \( P_0 \) onto the kernel of \( A \), which is called the \( P_0 \) space, and the projection operator \( Q_0 \) onto the \( Q_0 \) space as the complement to the \( P_0 \) space: With the use of an inner product which satisfies the positive definiteness of the norm as \( \langle \psi, \psi \rangle > 0 \) with \( \psi \neq 0 \), we define

\[ P_0 \psi = \sum_{\alpha, \beta=1}^{M_0} \varphi_0^{(\alpha)} \eta_{0(\alpha)(\beta)}^{-1} \langle \varphi_0^{(\beta)}, \psi \rangle, \quad Q_0 \equiv 1 - P_0, \]  

(A.9)

where \( \eta_{0(\alpha)(\beta)}^{-1} \) is the inverse matrix of the \( P_0 \)-space metric matrix \( \eta_{0(\alpha)(\beta)} \) defined by

\[ \eta_{0(\alpha)(\beta)} \equiv \langle \varphi_0^{(\alpha)}, \varphi_0^{(\beta)} \rangle. \]  

(A.10)
We assume that the other eigenvalues of $A$ are real negative and negative eigenvalues closest to zero are discrete. Accordingly we suppose that with the inner product $A$ is self-adjoint,

$$\langle A \psi, \chi \rangle = \langle \psi, A \chi \rangle,$$

where $\psi$ and $\chi$ are arbitrary vectors. We will see that this self-adjoint nature of $A$ plays an essential role in making the form of the resultant equation simpler.

Furthermore, we shall focus on the case that $A$ has no Jordan cell.

**A.2. Approximate solution around arbitrary initial time**. In accordance with the general formulation of the RG method [33, 34, 36, 37], we first try to obtain the perturbative solution $\tilde{X}$ to Eq. (A.2) around an arbitrary initial time $t = t_0$ with the initial value $X(t_0)$,

$$\tilde{X}(t = t_0 ; t_0) = X(t_0),$$

where we have made explicit that the solution has a $t_0$ dependence. It is noted that in the RG method the initial value $X(t_0)$ and the RG equation applied to the perturbative solution $\tilde{X}(t ; t_0)$ provide the invariant/attractive manifold and the reduced dynamics defined on it, respectively. We suppose that the initial value is on a yet unknown exact solution. The initial value as well as the perturbative solution is expanded with respect to $\epsilon$ as follows:

$$\tilde{X}(t ; t_0) = \tilde{X}_0(t ; t_0) + \epsilon \tilde{X}_1(t ; t_0) + \epsilon^2 \tilde{X}_2(t ; t_0) + \cdots,$$

$$X(t_0) = X_0(t_0) + \epsilon X_1(t_0) + \epsilon^2 X_2(t_0) + \cdots,$$

The respective initial conditions at $t = t_0$ are set up as

$$\tilde{X}_i(t = t_0 ; t_0) = X_i(t_0), \quad i = 0, 1, 2, \cdots.$$  

In the expansion, the zeroth-order initial value $\tilde{X}_0(t_0 ; t_0) = X_0(t_0)$ is supposed to be as close as possible to an exact solution.

Substituting the above expansions into Eq. (A.2), we obtain the series of the perturbative equations with respect to $\epsilon$. Here, we shall carry out the perturbative analysis up to the second order.

The zeroth-order equation reads

$$\frac{\partial}{\partial t} \tilde{X}_0(t ; t_0) = G(\tilde{X}_0(t ; t_0)).$$

As mentioned above, we shall construct the invariant/attractive manifold by incorporating appropriate excited modes, as additional components to the invariant manifold spanned by the $M_0$ zero modes. To achieve the construction in a systematic manner, we suppose that the initial value $X_0(t_0)$ can be expanded around the stationary solution $X_{\text{st}}(t_0)$ as

$$X_0(t_0) = X_{\text{st}}(t_0) + \phi(t_0),$$

where $\phi(t_0)$ denotes additional components due to the excited modes. Without loss of generality, we can impose the boundary condition

$$\tilde{X}_0(t \to \infty ; t_0) \rightarrow X_{\text{st}}(t_0).$$
whose geometrical image is shown in Fig. A.1.

In most cases, it is difficult to obtain an exact solution to Eq. (A.16) with the initial value \( X_0(t_0) \) as given in Eq. (A.17) due to the non-linearity of \( G(X) \). Here, we shall treat \( \phi(t_0) \) as a small quantity, whose amplitude of \( \phi(t_0) \) is the same as that of the perturbation term \( F(X) \) in the evolution equation (A.2), i.e.,

\[
\phi(t_0) \sim O(\epsilon).
\]

In view of the consistency of the perturbative analysis with respect to \( \epsilon \), \( \phi(t_0) \) may be identified as the first-order initial value \( X_1(t_0) \):

\[
X_1(t_0) = \tilde{X}_1(t = t_0; t_0) = \phi(t_0),
\]

which is further constrained so as to satisfy the boundary condition Eq. (A.18).

Thus, we can write the zeroth-order solution and initial value as

\[
\tilde{X}_0(t; t_0) = X_{st}(t_0), \quad X_0(t_0) = \tilde{X}_0(t = t_0; t_0) = X_{st}(t_0),
\]

respectively. We note that \( X_{st}(t_0) \) depends on \( t_0 \) through the would-be \( M_0 \) integral constants \( C(\alpha)(t_0) \) with \( \alpha = 1, \ldots, M_0 \) defined in Eq. (A.4):

\[
X_{st}(t_0) = X_{st}(C(1)(t_0), \ldots, C(M_0)(t_0)).
\]

In the following, we shall suppress the initial time \( t_0 \) when no misunderstanding is expected.

**A.3. Doublet scheme.** As announced, we determine the first-order initial value \( \phi \) on the basis of the consistency condition (A) and general principle of the reduction theory of the dynamics (B). Here, we present an explicit form of \( \phi \). For that, let us utilize the doublet scheme, which will be explained below.

First, we introduce new vectors \( \tilde{\phi}^{(\mu)} \) with \( \mu = 1, \ldots, M_1 \), assuming that the perturbation term \( F(X) \) at \( X = X_{st} \) is expressed as

\[
F_0 \equiv F(X_{st}) = \sum_{\mu=1}^{M_1} \tilde{\phi}^{(\mu)} \delta \tilde{X}(\mu).
\]
Here, we suppose that $\delta \bar{X}_{(\mu)}$ with $\mu = 1, \cdots, M_1$ are functions of $C_{(\alpha)}$ with $\alpha = 1, \cdots, M_0$ which satisfy the linear independence condition
\begin{equation}
\sum_{\mu=1}^{M_1} d^{(\mu)} \delta \bar{X}_{(\mu)} = 0 \implies d^{(1)} = \cdots = d^{(M_1)} = 0,
\end{equation}
for arbitrary $C_{(1)}, \cdots, C_{(M_0)}$. We emphasize that the linear independence of $\delta \bar{X}_{(\mu)}$ determines the number and form of $\bar{\varphi}_1^{(\mu)}$ without mathematical ambiguity.

Here, we present a systematic procedure to obtain an explicit form of $\bar{\varphi}_1^{(\mu)}$. First, with the use of a complete set of the vector space spanned by the dynamical variables $\bar{X}$, i.e., $\varphi^{(i)}$ with $i = 1, \cdots, N$, we expand $F_0$ as
\begin{equation}
F_0 = \sum_{i=1}^{N} f^{(i)} \varphi^{(i)}.
\end{equation}

When $\varphi^{(i)}$ are orthogonal and normalized vectors, i.e., $\langle \varphi^{(i)}, \varphi^{(j)} \rangle = \delta^{ij}$, the amplitudes $f^{(i)}$ with $i = 1, \cdots, N$ are given by
\begin{equation}
f^{(i)} = \langle \varphi^{(i)}, F_0 \rangle.
\end{equation}

Next, we examine the linear independence of $f^{(i)}$. If the number of the independent functions in $f^{(i)}$ is $M_1$, there exist $M_1$ linear independent functions $g_{(\mu)}$ with $\mu = 1, \cdots, M_1$, and $f^{(i)}$ are expressed as a linear combination of $g_{(\mu)}$,
\begin{equation}
f^{(i)} = \sum_{\mu=1}^{M_1} D^{(\mu)}_{(i)} g_{(\mu)}, \quad i = 1, \cdots, N,
\end{equation}
with $D^{(\mu)}_{(i)}$ being a matrix independent of $C_{(\alpha)}$. Substituting Eq. (A.28) into Eq. (A.26) and comparing with Eq. (A.24), we identify $\bar{\varphi}_1^{(\mu)}$ and $\delta \bar{X}_{(\mu)}$ as
\begin{equation}
\bar{\varphi}_1^{(\mu)} = \sum_{i=1}^{N} \varphi^{(i)} D^{(\mu)}_{(i)} g_{(\mu)}, \quad \delta \bar{X}_{(\mu)} = g_{(\mu)},
\end{equation}
respectively. It is noted that $\bar{\varphi}_1^{(\mu)}$ cannot be the eigenvector of $A$ in the generic case.

Then, with the use of $\bar{\varphi}_1^{(n, \mu)}$, we shall define the $P_1$ space that $\varphi$ belongs to, and introduce the $Q_1$ space which is the complement to the $P_0$ and $P_1$ spaces. We introduce the vectors spanning the $P_1$ space as
\begin{equation}
\Phi_1^{(n, \mu)} \equiv A^{-n} \bar{\varphi}_1^{(\mu)}, \quad n = 0, 1,
\end{equation}
with
\begin{equation}
\varphi_1^{(\mu)} = Q_0 \bar{\varphi}_1^{(\mu)}.
\end{equation}
We note that the existence of $Q_0$ in front of $\bar{\varphi}_1^{(\mu)}$ ensures that the zero modes $\varphi_0^{(\alpha)}$ are not included in $\varphi_1^{(\mu)}$, and hence $A^{-1}$ can be applied to $\varphi_1^{(\mu)}$ as shown in Eq. (A.30).
without mathematical ambiguity. Since \( \Phi_1^{(1,\mu)} = A^{-1} \varphi_1^{(\mu)} \) do not contain the zero modes either, the following identity is satisfied:

\[
Q_0 \Phi_1^{(n,\mu)} = \Phi_1^{(n,\mu)}.
\]

We call \( \Phi_1^{(n,\mu)} \) with \( n = 0, 1 \) the doublet modes. It is noted that the doublet modes cannot also be the eigenvector of \( A \), and is generically written as a linear combination of the eigenvectors of different finite eigenvalues of \( A \). The projection operators onto the \( P_1 \) and \( Q_1 \) spaces are given by

\[
P_1 \psi \equiv \sum_{n,m=0,1} \sum_{\mu,\nu=1}^{M_1} \Phi_1^{(n,\mu)} \eta_{1(n,\mu)(m,\nu)}^{-1} \langle \Phi_1^{(m,\nu)}, \psi \rangle, \quad Q_1 \equiv Q_0 - P_1,
\]

respectively, where \( \eta_{1(n,\mu)(m,\nu)}^{-1} \) has been introduced as the inverse matrix of the \( P_1 \)-space metric matrix given by

\[
\eta_{1(n,\mu)(m,\nu)} \equiv \langle \Phi_1^{(n,\mu)}, \Phi_1^{(m,\nu)} \rangle.
\]

We note that \( P_0, P_1, \) and \( Q_1 \) satisfy the following properties:

\[
1 = P_0 + P_1 + Q_1, \quad P_0^2 = P_0, \quad P_1^2 = P_1, \quad Q_1^2 = Q_1,
\]

\[
P_0 P_1 = P_1 P_0 = P_0 Q_1 = Q_1 P_0 = P_1 Q_1 = Q_1 P_1 = 0.
\]

It is noted that we utilize the projection operators \( P_0, P_1, \) and \( Q_1 \) to construct the first- and second-order solutions, where only the motion caused by the \( P_0 \) and \( P_1 \) spaces appears.

Finally, with the use of one of the doublet modes, i.e., \( \Phi_1^{(1,\mu)} \), we define \( \phi \) as

\[
\phi = \sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta X_\mu,
\]

where \( \Phi_1^{(1,\mu)} \) with \( \mu = 1, \ldots, M_1 \) are the excited modes that should be considered as the additional components to the invariant manifold spanned by the zero modes \( \varphi_0^{(\alpha)} \) with \( \alpha = 1, \ldots, M_0 \), and \( \delta X_\mu \) with \( \mu = 1, \ldots, M_1 \) denote the would-be integral constants that acquire the time dependence by the RG equation as well as \( C_\alpha \) with \( \alpha = 1, \ldots, M_0 \) in \( X_m \).

We call the scheme of the construction of the mesoscopic dynamics based on the choice of the \( P_1 \)-space vectors in Eq. (A.30) and the form of \( \phi \) in Eq. (A.37) the doublet scheme in the RG method. We will see that the doublet scheme makes the resultant equation governing the dynamics of \( C_\alpha \) and \( \delta X_\mu \) satisfy the consistency condition (A) and general principle of the reduction theory of the dynamics (B) by the mechanism shown in Table A.1.

**A.4. First-order solution.** The first-order equation reads

\[
\frac{\partial}{\partial t} \tilde{X}_1(t) = A \tilde{X}_1(t) + F_0.
\]

By solving Eq. (A.38) with the initial value \( \tilde{X}_1(t_0) = \phi \) shown in Eq. (A.20), we have the first-order solution given by

\[
\tilde{X}_1(t; t_0) = e^{A(t-t_0)} \tilde{X}_1(t_0) + \int_{t_0}^{t} ds e^{A(t-s)} F_0.
\]
The doublet scheme in the RG method and the mechanism to produce a simple reduced dynamics which can reproduce the slow dynamics as described with only the zero modes in the asymptotic regime are summarized. Thanks to Eqs. (A.40), (A.56), and (A.57) derived from the explicit form of the modes belonging to the P_0 and P_1 spaces and the would-be integral constants correspondent to the modes, we can obtain Eqs. (A.44) and (A.45) as an equation governing the dynamics of C_[(α)] and δX(μ), which is consistent with the evolution equation (A.3) in the mesoscopic regime. We have defined P_A = P_1 (A - ∂/∂τ) P_1 (A - ∂/∂τ)^(-1) Q_0 and R(τ) = F_0 or K(τ).

| Subspace | Mode | Integration constant |
|----------|------|---------------------|
| P_0      | φ_0^[[α]] | C_[(α)] |
| P_1      | A^(-1) φ_1^[[μ]] | δX(μ) | \( Q_1 A^{-1} Q_0 F_0 = 0 \) in Eq. (A.40) |
| Q_0      | φ_1^[[μ]] | \( \langle \Phi_1^{(1,μ)}, P_A R(τ) \rangle = \langle \Phi_1^{(1,μ)}, R(τ) \rangle \) in Eqs. (A.56) and (A.57) |
| Q_1      |      |                    |

\[
A^t φ + (t - t_0) P_0 F_0 + (A^t - 1) A^-1 Q_0 F_0 \\
= e^{A(t-t_0)} [φ + Q_1 A^{-1} Q_0 F_0] \\
+ (t - t_0) P_0 F_0 + (e^{A(t-t_0)} - 1) P_1 A^{-1} Q_0 F_0 - Q_1 A^{-1} Q_0 F_0 \\
(39) \quad = e^{A(t-t_0)} \phi + (t - t_0) P_0 F_0 + (e^{A(t-t_0)} - 1) P_1 A^{-1} Q_0 F_0.
\]

We should stress that in the first-order solution the motion coming from the P_0 and P_1 spaces appears and the fast motion caused by the Q_1 space is absent due to

\[
(40) \quad Q_1 A^{-1} Q_0 F_0 = Q_1 \sum_{μ=1}^{M_1} Φ_1^{(1,μ)} δX(μ) = 0,
\]

because Φ_1^{(1,μ)} belongs to the P_1 space as shown in Eq. (A.39).

We note the appearance of the secular term proportional to \( t - t_0 \) in Eq. (A.39), which apparently invalidate the perturbative solution when \( |t - t_0| \) becomes large.

### A.5. Second-order solution

The second-order equation is written as

\[
(41) \quad \frac{∂}{∂τ} \tilde{X}_2(t) = A \tilde{X}_2(t) + K(t - t_0),
\]

with the time-dependent inhomogeneous term given by

\[
(42) \quad K(τ) = \frac{1}{2} B \left[ e^{Aτ} φ + τ P_0 F_0 + (e^{Aτ} - 1) P_1 A^{-1} Q_0 F_0 \right]^2 \\
+ F_1 \left[ e^{Aτ} φ + τ P_0 F_0 + (e^{Aτ} - 1) P_1 A^{-1} Q_0 F_0 \right].
\]

Here, we have introduced B and F_1 whose components are given by

\[
(43) \quad B_{ijk} = \left. \frac{∂^2}{∂X_j∂X_k} G_i(X) \right|_{X=X_α}, \quad F_{1ij} = \left. \frac{∂}{∂X_i} F_i(X) \right|_{X=X_α}.
\]
To obtain appropriate initial values and solutions with the motion coming from the \( P_0 \) and \( P_1 \) spaces to Eq. (A.41), we utilize the formulae (A.74) and (A.75) given in Appendix A.10. By setting \( \mathbf{R}(t - t_0) = \mathbf{K}(t - t_0) \) in Eqs. (A.74) and (A.75), we find that the initial value and solution to Eq. (A.41) read

\[
\mathbf{X}_2(t_0) = -Q_1 (A - \partial / \partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \bigg|_{\tau = 0},
\]

and

\[
\mathbf{X}_2(t; t_0) = (1 - e^{(t-t_0)\partial / \partial \tau}) (-\partial / \partial \tau)^{-1} P_0 \mathbf{K}(\tau) \bigg|_{\tau = 0} \\
+ (e^{A(t-t_0)} - e^{(t-t_0)\partial / \partial \tau}) P_1 (A - \partial / \partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \bigg|_{\tau = 0} \\
- e^{(t-t_0)\partial / \partial \tau} Q_1 (A - \partial / \partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \bigg|_{\tau = 0},
\]

respectively. We notice again the appearance of secular terms in Eq. (A.45).

Summing up the perturbative solutions up to the second order with respect to \( \epsilon \), we have the full expression of the initial value and the approximate solution around \( t \sim t_0 \) to the second order:

\[
\mathbf{X}(t_0) = \mathbf{X}_{st} + \epsilon \phi - \epsilon^2 Q_1 (A - \partial / \partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \bigg|_{\tau = 0} + O(\epsilon^3),
\]

and

\[
\mathbf{X}(t; t_0) = \mathbf{X}_{st} + \epsilon \left[ e^{A(t-t_0)} \phi + (t - t_0) P_0 \mathbf{F}_0 + (e^{A(t-t_0)} - 1) P_1 A^{-1} Q_0 \mathbf{F}_0 \right] \\
+ \epsilon^2 \left[ (1 - e^{(t-t_0)\partial / \partial \tau}) (-\partial / \partial \tau)^{-1} P_0 \mathbf{K}(\tau) \bigg|_{\tau = 0} \\
+ (e^{A(t-t_0)} - e^{(t-t_0)\partial / \partial \tau}) P_1 (A - \partial / \partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \bigg|_{\tau = 0} \\
- e^{(t-t_0)\partial / \partial \tau} Q_1 (A - \partial / \partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \bigg|_{\tau = 0} \right] + O(\epsilon^3).
\]

We note that in Eq. (A.47) the fast motion caused by the \( Q_1 \) space has been eliminated by adopting the appropriate initial value (A.46).

A.6. RG improvement of perturbative expansion. We emphasize that the solution (A.47) contains the secular terms that apparently invalidate the perturbative expansion for \( t \) away from the initial time \( t_0 \). The point of the RG method lies in the fact that we can utilize the secular terms to obtain a solution valid in a global domain as discussed in Refs. [33, 34, 35, 36, 37, 38, 39]. By applying the RG equation to the local solution (A.47), we can convert the set of the locally valid approximate solutions to the solution valid in a global domain:

\[
\frac{\partial}{\partial t} \mathbf{X}(t; t_0) \bigg|_{t_0=t} = 0,
\]

which is reduced to

\[
\frac{\partial}{\partial t} \mathbf{X}_{st} + \epsilon \left[ -A \phi + \frac{\partial}{\partial t} \phi - P_0 \mathbf{F}_0 - A P_1 A^{-1} Q_0 \mathbf{F}_0 \right]
\]
It is noted that the RG equation (A.49) gives the equation of motion governing the dynamics of the would-be integral constant \( C \) in \( X_{\text{st}} \) and \( \delta X_{(\mu)} \) in \( \phi \). The global solution can be obtained as the initial value (A.46)

\[
X_F(t) \equiv X(t_0 = t)
\]

(A.50) \[= X_{\text{st}} + \epsilon \phi - \epsilon^2 Q_1 (A - \partial / \partial \tau)^{-1} Q_0 K(\tau) \bigg|_{\tau=0} + O(\epsilon^3), \]

where the exact solution to Eq. (A.49) is inserted. It is noteworthy that we have derived the mesoscopic dynamics of Eq. (A.2) in the form of the pair of Eqs. (A.49) and (A.50): Equation (A.50) is nothing but the invariant/attractive manifold of Eq. (A.2), and Eq. (A.49) describes the mesoscopic dynamics defined on it.

### A.7. Reduction of RG equation to simpler form
Here, we shall convert Eq. (A.49) into a more convenient form with the use of \( \phi \) and \( P_1 \) defined in Eqs. (A.47) and (A.33), respectively.

By applying \( P_0 \) and \( P_1 \) from the left-hand side of Eq. (A.49), we have

\[
P_0 \frac{\partial}{\partial t} X_{\text{st}} + \epsilon \left[ P_0 \frac{\partial}{\partial t} \phi - P_0 F_0 \right] - \epsilon^2 P_0 K(0) + O(\epsilon^3) = 0,
\]

\[
P_1 \frac{\partial}{\partial t} X_{\text{st}} + \epsilon \left[ - P_1 A \phi + P_1 \frac{\partial}{\partial t} \phi - P_1 A P_1 A^{-1} Q_0 F_0 \right] - \epsilon^2 P_1 (A - \partial / \partial \tau)^{-1} Q_0 K(\tau) \bigg|_{\tau=0} + O(\epsilon^3) = 0,
\]

(A.52) respectively.

Then, we take the inner product with the zero modes \( \varphi^{(0)}_0 \) and the excited modes \( \Phi_1^{(1,\mu)} \) used in the definition of \( \phi \), respectively. In this procedure, we note that the direct use of the definitions (A.47), (A.48), (A.33), and (A.34) leads to the followings:

\[
\langle \varphi^{(0)}_0, P_0 \psi \rangle = \langle \varphi^{(0)}_0, \psi \rangle, \quad \langle \Phi_1^{(1,\mu)}, P_1 \psi \rangle = \langle \Phi_1^{(1,\mu)}, \psi \rangle,
\]

with \( \psi \) being an arbitrary vector. Thus, we arrive at

\[
\langle \varphi^{(0)}_0, \frac{\partial}{\partial t} (X_{\text{st}} + \epsilon \phi) \rangle - \epsilon \langle \varphi^{(0)}_0, F_0 + \epsilon F_1 \phi \rangle = \epsilon^2 \frac{1}{2} \langle \varphi^{(0)}_0, B \phi^2 \rangle + O(\epsilon^3),
\]

(A.53) \[= \epsilon^2 \frac{1}{2} \langle \varphi^{(0)}_0, B \phi^2 \rangle + O(\epsilon^3), \]

\[
\langle \Phi_1^{(1,\mu)}, \frac{\partial}{\partial t} (X_{\text{st}} + \epsilon \phi) \rangle - \epsilon \langle \Phi_1^{(1,\mu)}, F_0 + \epsilon F_1 \phi \rangle = \epsilon \langle \Phi_1^{(1,\mu)}, A \phi \rangle + \epsilon^2 \frac{1}{2} \langle \Phi_1^{(1,\mu)}, B \phi^2 \rangle + O(\epsilon^3).
\]
In the derivation of Eqs. (A.54) and (A.55), we have used the following identities:

\[
\langle \Phi_1^{(1,\mu)}, A P_1 A^{-1} Q_0 F_0 \rangle = \langle \Phi_1^{(0,\mu)}, P_1 A^{-1} Q_0 F_0 \rangle = \langle \Phi_1^{(0,\mu)}, A^{-1} Q_0 F_0 \rangle = \langle \Phi_1^{(1,\mu)}, F_0 \rangle,
\]

and

\[
\langle \Phi_1^{(1,\mu)}, (A - \partial/\partial \tau) P_1 (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \rangle_{\tau=0} = \langle \Phi_1^{(0,\mu)}, P_1 (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \rangle_{\tau=0} = \langle \Phi_0^{(0,\mu)} - \Phi_1^{(1,\mu)}, P_1 (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \rangle_{\tau=0} = \langle \Phi_1^{(1,\mu)}, (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \rangle_{\tau=0} = \langle \Phi_1^{(1,\mu)}, K(0) \rangle = \frac{1}{2} \langle \Phi_1^{(1,\mu)}, B \phi^2 \rangle + \langle \Phi_1^{(1,\mu)}, F_1 \phi \rangle,
\]

where we have used the self-adjoint nature of \( A \) shown in Eq. (A.11), the identities given by (A.53), and the relation derived from Eq. (A.42): \( K(0) = \frac{1}{2} B \phi^2 + F_1 \phi \). We note that the pair of Eqs. (A.54) and (A.55) is also the equation of motion governing \( C(\alpha) \) in \( X_{\text{st}} \) and \( \delta X_{(\mu)} \) in \( \phi \), which is much simpler than Eq. (A.49).

With the use of the explicit forms of \( \phi \), \( F_0 \), and \( \Phi_1^{(1,\mu)} \) in Eqs. (A.37), (A.24), and (A.30), respectively, we obtain an alternative form of Eqs. (A.54) and (A.55):

\[
\langle \phi_0^{(\alpha)}, \frac{\partial}{\partial t} X_{\text{st}} \rangle + \epsilon \sum_{\mu=1}^{M_1} \langle \phi_0^{(\alpha)}, \bar{\phi}_1^{(\mu)} \rangle \delta X_{(\mu)} = -\epsilon \sum_{\mu=1}^{M_1} \langle \phi_0^{(\alpha)}, \left[ \frac{\partial}{\partial t} - \epsilon F_1 \right] A^{-1} \phi_1^{(\mu)} \rangle \delta X_{(\mu)} \]

\[
+ \epsilon^2 \frac{1}{2} \sum_{\mu,\nu=1}^{M_1} \langle \phi_0^{(\alpha)}, B \left[ A^{-1} \phi_1^{(\mu)} \right] \left[ A^{-1} \phi_1^{(\nu)} \right] \rangle \delta X_{(\mu)} \delta X_{(\nu)} + O(\epsilon^3),
\]

\[
\}

\[
+ \epsilon \sum_{\nu=1}^{M_1} \langle A^{-1} \phi_1^{(\mu)}, \left[ \frac{\partial}{\partial t} - \epsilon F_1 \right] A^{-1} \phi_1^{(\nu)} \rangle \delta X_{(\nu)} \]

\[
= \epsilon \sum_{\nu=1}^{M_1} \langle \phi_1^{(\mu)}, A^{-1} \phi_1^{(\nu)} \rangle (\delta X_{(\nu)} + \delta \bar{X}_{(\nu)})
\]

\[
+ \epsilon^2 \frac{1}{2} \sum_{\mu,\nu=1}^{M_1} \langle A^{-1} \phi_1^{(\mu)}, B \left[ A^{-1} \phi_1^{(\nu)} \right] \left[ A^{-1} \phi_1^{(\nu)} \right] \rangle \delta X_{(\nu)} \delta X_{(\mu)} + O(\epsilon^3).
\]

Here, we have used the fact that \( \partial X_{\text{st}} / \partial t \) is the zero modes, which can be shown by the following identity: \( \frac{\partial}{\partial t} X_{\text{st}} = \sum_{\alpha=1}^{M_0} \frac{\partial}{\partial \phi^{(\alpha)}} X_{\text{st}} \epsilon \frac{\partial}{\partial \phi^{(\alpha)}} C(\alpha) = \sum_{\alpha=1}^{M_0} \phi_0^{(\alpha)} \frac{\partial}{\partial \phi^{(\alpha)}} C(\alpha) \), where we have used Eqs. (A.8) and (A.23). Then, the initial value (A.50) is written as

\[
X_E(t) = X_{\text{st}} + \epsilon \sum_{\mu=1}^{M_1} A^{-1} \phi_1^{(\mu)} \delta X_{(\mu)}
\]
\( -\epsilon Q_1 (A - \partial/\partial \tau)^{-1} Q_0 (B \phi^2(\tau)/2 + F_1 \phi(\tau)) \bigg|_{\tau=0} + O(\epsilon^3), \)

where

\[
\phi(\tau) = \sum_{\mu=1}^{M_1} \left[ e^{A\tau} A^{-1} \phi_1^{(\mu)} \delta X_{(\mu)} + \left( \tau P_0 \phi_1^{(\mu)} + (e^{A\tau} - 1) A^{-1} \phi_1^{(\mu)} \right) \delta \bar{X}_{(\mu)} \right].
\]

We note that the set of Eqs. \((A.54)/(A.58), (A.55)/(A.59), \) and \((A.60)\) is the main result in this section, and the global solution to Eq. \((A.2)\) in the mesoscopic regime can be obtained by substituting the exact solution to Eqs. \((A.54) \) and \((A.55)\) (Eqs. \((A.58) \) and \((A.59)\)) into Eq. \((A.60)\). We also note that Eqs. \((A.58), (A.59), \) and \((A.60)\) are identical to Eqs. \((3.5), (3.6), \) and \((3.4)\).

**A.8. Naturalness of mesoscopic dynamics: Consistency with slow dynamics as described with only zero modes in asymptotic regime.**

We will show that the form of the mesoscopic dynamics obtained above approaches asymptotically the slow dynamics obtained by employing only the zero modes from the outset, which is given by the following equation:

\[
\langle \varphi_0^{(\alpha)}, \delta X_{st} - \epsilon A^{-1} Q_0 F_0 \rangle - \epsilon \langle \varphi_0^{(\alpha)}, F_0 - \epsilon F_1 A^{-1} Q_0 F_0 \rangle = \epsilon^2 \frac{1}{2} \langle \varphi_0^{(\alpha)}, B \left[ A^{-1} Q_0 F_0 \right]^2 \rangle + O(\epsilon^3),
\]

of which derivation is shown in Appendix A.11.

It is important to note that Eq. \((A.54)\) can be reduced to Eq. \((A.62)\) when the equality

\[
\phi = -A^{-1} Q_0 F_0 + O(\epsilon),
\]

is satisfied in the asymptotic regime. In fact, \(\phi\) and \(-A^{-1} Q_0 F_0\) are composed of the same vectors, i.e., \(\Phi_1^{(1,\mu)}\), because

\[
A^{-1} Q_0 F_0 = \sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta \bar{X}_{(\mu)},
\]

which can be derived from Eqs. \((A.24) \) and \((A.31)\). We emphasize that the definition of \(\phi\) based on \(\Phi_1^{(1,\mu)}\) adopted in the doublet scheme is just a necessary condition for the consistency with the slow dynamics, and our task is to show that the amplitudes of \(\Phi_1^{(1,\mu)}\) in \(\phi\) agree with those in \(-A^{-1} Q_0 F_0\) in the asymptotic regime.

Here, we should notice the time-scale separation between the fast motion of \(\delta X_{(\mu)}\) caused by the \(P_1\) space and the slow motion of \(C_{(\alpha)}\) by the \(P_0\) space. Due to this separation which becomes significant in the asymptotic regime, we can obtain the closed equations with respect to \(C_{(\alpha)}\) by eliminating \(\delta X_{(\mu)}\) adiabatically from Eqs. \((A.54)/(A.58), (A.55)/(A.59)\). First, we obtain

\[
\delta X_{(\mu)} = -\delta \bar{X}_{(\mu)} + O(\epsilon),
\]

\(\text{Causal hydrodynamics in RG method}\)
which has been derived from Eq. (A.59) governing the fast motion of $\delta X_{(\mu)}$ as a stationary solution that is realized asymptotically with $C_{(\alpha)}$ being treated as a constant. We note that Eq. (A.65) is equivalent to

$$
\phi = -\sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta X_{(\mu)} + O(\epsilon).
$$

(A.66)

Then, combining Eq. (A.66) with (A.64), we obtain Eq. (A.63), and find that the slow dynamics of $C_{(\alpha)}$ is the same as Eq. (A.62). Thus, we complete the proof that Eqs. (A.54)/(A.58) and (A.55)/(A.59) can reproduce Eq. (A.62) asymptotically.

**A.9. Discussion.** We shall discuss whether or not there exist other schemes which can produce a mesoscopic dynamics that is simpler than that of the doublet scheme shown in Table A.1 and respects the consistency with the slow dynamics given in terms of the zero modes in the asymptotic regime. Here, we should notice that the identities (A.40), (A.56), and (A.57) have played an essential role in the derivation of the simple form shown in Eqs. (A.54) and (A.55).

First, let us see the significance of Eq. (A.40): If Eq. (A.40) is not satisfied, that is, $\Phi_1^{(1,\mu)}$ does not belong to the $P_1$ space, $\Phi_1^{(1,\mu)}$ cannot be utilized in the construction of $\phi$, and hence an asymptotic form of the mesoscopic dynamics is not in accord with the slow dynamics as described with only the zero modes as we discussed in Appendix A.8. In fact, the equality (A.63) is not realized in the asymptotic regime because $\phi$ and $-A^{-1} Q_0 F_0$ are no longer composed of the same vectors in general. We conclude that Eq. (A.40) is necessary to obtain a reduced dynamics which can reproduce the slow dynamics given in terms of the zero modes in the asymptotic regime and $\Phi_1^{(1,\mu)}$ should be incorporated into the vectors that are used to define $\phi$.

Next, we shall focus Eqs. (A.56) and (A.57). To make the explanation clearer, we consider the case where $\phi$ is constructed with the use of not only $\Phi_1^{(1,\mu)}$ but also $\Phi_0^{(0,\mu)}$ as follows:

$$
\phi = \sum_{\mu=1}^{M_1} \Phi_1^{(0,\mu)} \delta X_{(0,\mu)} + \sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta X_{(1,\mu)},
$$

(A.67)

where $\delta X_{(0,\mu)}$ and $\delta X_{(1,\mu)}$ denote the dynamical variables. In this case, it becomes necessary to take an inner product with not only $\Phi_0^{(0,\alpha)}$ and $\Phi_1^{(1,\mu)}$ but also $\Phi_0^{(0,\mu)}$ in Eq. (A.52). In contrast to the inner product with $\Phi_1^{(1,\mu)}$, however, the inner product with $\Phi_0^{(0,\mu)}$ does not have a simple form:

$$
\langle \Phi_1^{(0,\mu)} , (A - \partial/\partial \tau) P_1 (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \bigg|_{\tau=0} \rangle = \langle (A - \partial/\partial \tau) \Phi_1^{(0,\mu)} , P_1 (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \bigg|_{\tau=0} \rangle = \langle \Phi_1^{(0,\mu)} , K(0) \rangle,
$$

(A.68)

because $A \Phi_1^{(0,\mu)} = A \phi_1^{(\mu)}$ does not belong to the $P_1$ space. Owing to the inequality shown in the final line of Eq. (A.68), the reduced RG equation corresponding to the choice of $\phi$ defined in Eq. (A.67) is given by the set of Eqs. (A.54), (A.55), and

$$
\langle \Phi_1^{(0,\mu)} , \frac{\partial}{\partial \epsilon} (X_{st} + \epsilon \phi) \rangle - \epsilon \langle \Phi_1^{(0,\mu)} , A P_1 A^{-1} Q_0 (F_0 + \epsilon F_1 \phi) \rangle.
$$
\[ \epsilon \langle \Phi_1^{(0,\mu)} , A\phi \rangle + \epsilon^2 \frac{1}{2} \langle \Phi_1^{(0,\mu)} , A P_1 A^{-1} Q_0 B \phi^2 \rangle \]

(A.69) \[ + \epsilon^2 \sum_{k=1}^{\infty} \langle \Phi_1^{(0,\mu)} , (A P_1 A^{-1} - P_1) A^{-k} Q_0 \frac{\partial^k}{\partial \tau^k} K(\tau) \rvert_{\tau=0} \rangle + O(\epsilon^3), \]

where we have used the identity

\[ (A - \partial/\partial \tau) P_1 (A - \partial/\partial \tau)^{-1} Q_0 K(\tau) \rvert_{\tau=0} \]

(A.70) \[ = A P_1 A^{-1} K(0) + (A P_1 A^{-1} - P_1) \sum_{k=1}^{\infty} A^{-k} Q_0 \frac{\partial^k}{\partial \tau^k} K(\tau) \rvert_{\tau=0}. \]

Noticing that \( \partial^k K(\tau)/\partial \tau^k \rvert_{\tau=0} \) does not vanish for any \( k \) since \( K(\tau) \) has an exponential dependence as shown in Eq. (A.61), we find that the last term in Eq. (A.69) gives an infinite number of terms. Although we can avoid this difficulty by making the \( P_1 \) space spanned by \( \Phi_1^{(n,\mu)} \) with \( n = -1, 0, 1 \) instead of \( \Phi_1^{(n,\mu)} \) with \( n = 0, 1 \) shown in Eq. (A.30), the resultant equation becomes an equation describing the dynamics of \( C_{(a)} \), \( \delta X_{(0,\mu)} \), and \( \delta X_{(1,\mu)} \), which is more complicated than Eqs. (A.54) and (A.55), because the number of the variables \( C_{(a)} \), \( \delta X_{(0,\mu)} \), and \( \delta X_{(1,\mu)} \) is obviously more than that of \( C_{(a)} \) and \( \delta X_{(\mu)} \) governed by Eqs. (A.54) and (A.55). Thus, the \( P_1 \) space should be spanned by \( \Phi_1^{(0,\mu)} \) together with \( \Phi_1^{(1,\mu)} \) that is utilized to define \( \phi \).

From the above observations, it is found that any modification to the doublet scheme does not make the reduced dynamics simpler than the original one, when the consistency with the slow dynamics as described with only the zero modes in the asymptotic regime is required. Thus, we suggest that the doublet scheme shown in Table A.1 is one of the most adequate schemes to produce the simplest equation describing the mesoscopic dynamics whose asymptotic form is in accord with that of the slow dynamics.

A remark is in order here: The doublet scheme in the RG method itself has a universal nature and can be applied to derive a mesoscopic dynamics from a wide class of evolution equations, as far as the equation can be written as Eq. (A.2) and the linearization evolution operator \( A \) is self-adjoint as shown in Eq. (A.11) and has no Jordan cell. As will be seen in Appendix B.1, the Boltzmann equation satisfies the above conditions, and its mesoscopic dynamics can be extracted by the doublet scheme. An extension of the doublet scheme to a method applicable to a more generic case where \( A \) is not self-adjoint or has Jordan cell will be studied elsewhere.

**A.10. Solution to linear differential equation with time dependent inhomogeneous term.** We present formulae for obtaining an appropriate solution of linear differential equations with a time-dependent inhomogeneous term. Let us consider the solution of the equation given by

\[ \frac{\partial}{\partial t} Y(t) = A Y(t) + R(t - t_0). \]

The solution reads

\[ Y(t) = e^{A(t-t_0)} Y(t_0) + \int_{t_0}^{t} ds P_0 R(s - t_0) + \int_{t_0}^{t} ds e^{A(t-s)} Q_0 R(s - t_0), \]

where we have inserted \( 1 = P_0 + Q_0 \) in front of \( R(s - t_0) \). Substituting the following Taylor expansion, \( R(s - t_0) = e^{(s-t_0)\partial/\partial \tau} R(\tau) \rvert_{\tau=0} \), into Eq. (A.72) and carrying out
integration with respect to \( s \), we have
\[
Y(t) = e^{A(t-t_0)}Y(t_0) + (1 - e^{(t-t_0)\partial/\partial \tau})(-\partial/\partial \tau)^{-1} P_0 R(\tau) \bigg|_{\tau=0} \\
+ (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial \tau}) (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0} \\
e^{A(t-t_0)} \left[ Y(t_0) + Q_1 (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0} \right] \\
+ (1 - e^{(t-t_0)\partial/\partial \tau})(-\partial/\partial \tau)^{-1} P_0 R(\tau) \bigg|_{\tau=0} \\
+ (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial \tau}) P_1 (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0} \\
- e^{(t-t_0)\partial/\partial \tau} Q_1 (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0},
\]
(A.73)
where \( 1 = P_0 + P_1 + Q_1 \) has been inserted in front of \((A - \partial/\partial \tau)^{-1} Q_0 R(\tau)\) in the second line of Eq. (A.73). We note that the contributions from the inhomogeneous term \( R(t - t_0) \) are decomposed into two parts, whose time dependencies are given by \( e^{A(t-t_0)} \) and \( e^{(t-t_0)\partial/\partial \tau} \), respectively. The former shows a fast motion according to the vector spaces operated by itself, while the time dependence of the latter is universal due to the absence of \( A \). Since we are interested in the motion coming from the \( P_0 \) and \( P_1 \) spaces, we eliminate the former associated with the \( Q_1 \) space with a choice of the initial value \( Y(t_0) \) that has not yet specified as follows:
\[
Y(t_0) = -Q_1 (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0},
\]
which reduces Eq. (A.73) to
\[
Y(t) = (1 - e^{(t-t_0)\partial/\partial \tau})(-\partial/\partial \tau)^{-1} P_0 R(\tau) \bigg|_{\tau=0} \\
+ (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial \tau}) P_1 (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0} \\
- e^{(t-t_0)\partial/\partial \tau} Q_1 (A - \partial/\partial \tau)^{-1} Q_0 R(\tau) \bigg|_{\tau=0}.
\]
(A.75)
Equations (A.74) and (A.75) are nothing but the formulae we wanted.

**A.11. Slow dynamics as described with would-be zero modes.** We derive the slow dynamics obtained by employing only the zero modes from the generic evolution equation (A.2) with the RG method for completeness, although a detailed derivation can be seen in Ref. [37].

As mentioned in Appendix A, we first try to obtain the perturbative solution \( \tilde{X} \) to Eq. (A.2) around an arbitrary initial time \( t = t_0 \) with the initial value \( X(t_0) \); \( \tilde{X}(t; t_0) = X(t_0) \). We expand the initial value as well as the solution with respect to \( \epsilon \) as shown in Eqs. (A.13) and (A.14), and obtain the series of the perturbative equations with respect to \( \epsilon \).

The zeroth-order equation is the same as Eq. (A.10). Since we are interested in the slow motion realized asymptotically for \( t \to \infty \), we adopt the stationary solution \( X_{st} \) as the zeroth-order solution: \( \tilde{X}_0(t; t_0) = X_{st} \), which means that the zeroth-order initial value reads \( X_0(t_0) = \tilde{X}_0(t_0; t_0) = X_{st} \).

The first-order equation is \( \partial_t \tilde{X}_1(t) = A \tilde{X}_1(t) + F_0 \), where \( A \) and \( F_0 \) have been defined in Eqs. (A.6) and (A.24), respectively. A solution to the first-order equation reads
\[
\tilde{X}_1(t) = e^{A(t-t_0)} \left[ \tilde{X}_1(t_0) + A^{-1} Q_0 F_0 \right] + (t-t_0) P_0 F_0 - A^{-1} Q_0 F_0.
\]
(A.76)
Here, $P_0$ denotes the projection operator onto the $P_0$ space spanned by the zero modes $\varphi^{(\alpha)}$, i.e., the kernel space of $A$, and $Q_0$ the projection operator onto the $Q_0$ space as the complement to the $P_0$ space.

Since we are interested in the slow motion caused by the $P_0$ space, we eliminate the fast motion coming from the $Q_0$ space with the use of the initial value $\tilde{X}_1(t_0)$ that has not yet specified as follows: $X_1(t_0) = \tilde{X}_1(t_0; t_0) = -A^{-1}Q_0F_0$, which reduces Eq. (A.76) to $\tilde{X}_1(t; t_0) = (t-t_0)P_0F_0 - A^{-1}Q_0F_0$.

The second-order equation is $\frac{\partial}{\partial \tau} \tilde{X}_2(t) = A\tilde{X}_2(t) + U(t-t_0)$, with

$$U(\tau) = \frac{1}{2}B\left[\tau P_0F_0 - A^{-1}Q_0F_0\right]^2 + F_1\left[\tau P_0F_0 - A^{-1}Q_0F_0\right],$$

where $B$ and $F_1$ have been defined in Eq. (A.43). With the use of the method developed in Appendix A.10, we have a solution to the second-order equation as

$$\tilde{X}_2(t) = e^{A(t-t_0)}\left[\tilde{X}_2(t_0) + (A - \partial/\partial \tau)^{-1}Q_0U(\tau)_{\tau=0}\right] + (1 - e^{A(t-t_0)}(A - \partial/\partial \tau)^{-1}P_0U(\tau)_{\tau=0}) - e^{A(t-t_0)}(A - \partial/\partial \tau)^{-1}Q_0U(\tau)_{\tau=0}. (A.78)$$

As in the case of the first order, we shall eliminate the fast motion caused by the $Q_0$ space using the initial value $\tilde{X}_2(t_0)$ as follows: $X_2(t_0) = \tilde{X}_2(t_0; t_0) = -Q_0U(\tau)_{\tau=0}$, which leads to $\tilde{X}_2(t; t_0) = (1 - e^{A(t-t_0)}(A - \partial/\partial \tau)^{-1}P_0U(\tau)_{\tau=0}) - e^{A(t-t_0)}(A - \partial/\partial \tau)^{-1}Q_0U(\tau)_{\tau=0}$.

Summing up the solutions and initial values constructed in the perturbative analysis up to $O(\epsilon^2)$, we have

$$X(t_0) = X_{st} - \epsilon A^{-1}Q_0F_0 - \epsilon^2 (A - \partial/\partial \tau)^{-1}Q_0U(\tau)_{\tau=0} + O(\epsilon^3),$$

$$\tilde{X}(t; t_0) = X_{st} + \epsilon\left[(t-t_0)P_0F_0 - A^{-1}Q_0F_0\right] + \epsilon^2\left[(1 - e^{A(t-t_0)}(A - \partial/\partial \tau)^{-1}P_0U(\tau)_{\tau=0}) - e^{A(t-t_0)}(A - \partial/\partial \tau)^{-1}Q_0U(\tau)_{\tau=0}\right] + O(\epsilon^3). (A.80)$$

We note the appearance of the secular term proportional to $t - t_0$, which invalidates the perturbative solution when $|t-t_0|$ becomes large. For obtaining the global solution from this local perturbative solution, we apply the RG equation $\frac{\partial}{\partial \tau} \tilde{X}_1(t; t_0) / \partial t|_{t_0=\tau} = 0$ to Eq. (A.80). The RG equation reads

$$\frac{\partial}{\partial \tau}(X_{st} - \epsilon A^{-1}Q_0F_0) - \epsilon P_0F_0$$

$$+ \epsilon^2\left[P_0U(0) + (A - \partial/\partial \tau)^{-1}Q_0U(\tau)_{\tau=0}\right] + O(\epsilon^3) = 0, (A.81)$$

which is the equation governing the slow motion of $C^{(\alpha)}_0$ in $X_{st}$. By taking the inner product with the zero modes $\varphi^{(\alpha)}_0$, we can convert Eq. (A.81) into a simpler
equation, which is found to be identically Eq. (A.62). Here, we have used $U(0) = \frac{1}{2} B \left[ -A^{-1} Q_{0} F_{0} \right]^{2} + F_{1} \left[ -A^{-1} Q_{0} F_{0} \right]$, which can be derived from Eq. (A.77).

Appendix B. Detailed derivation of explicit form of mesoscopic dynamics of Boltzmann equation. In this section, we present a detailed derivation of the mesoscopic dynamics of the Boltzmann equation (3.1) based on the doublet scheme in the RG method developed in Appendix A.

B.1. Set up suited for doublet scheme. We build $A$, $B$, $F_{0}$, $F_{1}$, $\delta X(\mu)$, $C(\alpha)$, $\delta X_{(\mu)}$, $\varphi_{0}^{(\alpha)}$, and $\varphi_{1}^{(\mu)}$ in the case of the Boltzmann equation (3.1) which are required for the doublet scheme.

With the use of Eq. (A.4), we find that the static solution reads

$$X_{\mu}(t_{0}) = f_{\mu}^{eq}(x; t_{0}) = n(x; t_{0}) \left[ \frac{m}{2 \pi T(x; t_{0})} \right]^{\frac{3}{2}} \exp \left[ -\frac{m|v - u(x; t_{0})|^{2}}{2 T(x; t_{0})} \right],$$

(B.1)

which is nothing but the Maxwellian (2.16) and satisfies $C[f^eq]v = 0$ as discussed in Eq. (2.17). We note that the five would-be integral constants $n(x; t_{0})$, $T(x; t_{0})$, and $u(x; t_{0})$ corresponding to $C(\alpha)(t_{0})$ in Appendix A are lifted to the dynamical variables by applying the RG equation. In the following, we shall suppress $(x; t_{0})$ when no misunderstanding is expected.

Using Eq. (A.6), we have the linearized evolution operator $A$ as

$$A = \frac{\partial}{\partial f_{k}} C[f]_{v} \bigg|_{f = f^{eq}} = f^{eq}_{v} L_{v k} \left( f^{eq}_{v} \right)^{-1}$$

(B.2)

$$= \frac{1}{2!} \sum_{v_{1}v_{2}v_{3}} \omega(v, v_{1}|v_{2}, v_{3}) (\delta_{v_{2}k} f_{v_{3}}^{eq} + f_{v_{2}}^{eq} \delta_{v_{3}k} - \delta_{v_{k}} f_{v_{1}}^{eq} - f_{v}^{eq} \delta_{v_{1}k}).$$

Here, let us examine the properties of $A$. We define the inner product by

$$\langle \psi, \chi \rangle = \sum_{v} (f^{eq}_{v})^{-1} \psi_{v} \chi_{v} = \langle f^{eq}_{v} \psi, f^{eq}_{v} \chi \rangle_{eq},$$

(B.3)

with $\psi_{v}$ and $\chi_{v}$ being arbitrary vectors and the diagonal matrix $f^{eq}_{v k} \equiv \delta_{v k} f^{eq}_{v}$. We note that the norm through this inner product is positive definite,

$$\langle \psi, \psi \rangle = \sum_{v} (f^{eq}_{v})^{-1} (\psi_{v})^{2} > 0, \quad \psi \neq 0.$$

(B.4)

It is noteworthy that $A$ is self-adjoint with respect this inner product as

$$\langle \psi, A \chi \rangle = -\frac{1}{2!} \frac{1}{4} \sum_{uv_{1}v_{2}v_{3}} \omega(v, v_{1}|v_{2}, v_{3}) (\psi_{v} + \psi_{v_{1}} - \psi_{v_{2}} - \psi_{v_{3}})$$

$$\times (\chi_{v} + \chi_{v_{1}} - \chi_{v_{2}} - \chi_{v_{3}})$$

(B.5)

$$= \langle A \psi, \chi \rangle,$$

and real semi-negative definite;

$$\langle \psi, A \psi \rangle = -\frac{1}{2!} \frac{1}{4} \sum_{uv_{1}v_{2}v_{3}} \omega(v, v_{1}|v_{2}, v_{3}) (\psi_{v} + \psi_{v_{1}} - \psi_{v_{2}} - \psi_{v_{3}})^{2} \leq 0.$$
Thanks to these properties of $A$, we can apply the doublet scheme presented in Appendix A to extract the mesoscopic dynamics from Eq. (3.1).

The zero modes of $A$ are found to be

$$
\Phi^{(\alpha)}_0 = \mathcal{F}_\alpha v_0 = [\mathcal{F}_\alpha v_0^\alpha], \quad \alpha = 0, 1, 2, 3, 4.
$$

In Eq. (B.7), we have introduced the following five vectors:

$$
\Phi^0_{0v} = \frac{1}{\sqrt{n}}; \quad \Phi^i_{0v} = \frac{1}{\sqrt{n}} \sqrt{\frac{m}{T}} \delta^i v, \quad \Phi^3_{0v} = \frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \left( \frac{m}{2T} |\delta v|^2 - \frac{3}{2} \right),
$$

with the peculiar velocity $\delta v = v - u$. It is noted that $\Phi^0_{0v}$ with $\alpha = 0, \ldots, 4$ coincide with the collision invariants shown in Eq. (2.7), and the dimension of the kernel space of $A$ is five, i.e., $M_0 = 5$. In fact, using the particle-number and energy-momentum conservation laws presented in Eq. (2.3), we can show that

$$
\left[ \mathcal{A}^\alpha \Phi^0_{0v} \right]_v = -\frac{1}{2T} \sum_{v_1 v_2 v_3} \omega(v, v_1|v_2, v_3) f^{eq}_{v_2} f^{eq}_{v_3} \left( \Phi^0_{0v} + \Phi^0_{0v1} - \Phi^0_{0v2} - \Phi^0_{0v3} \right)
$$

(B.9)

$$
= 0.
$$

With the use of $\Phi^0_{0v}$ and the inner product defined in Eqs. (B.7) and (B.8), respectively, we have the $P_0$-space metric matrix as follows,

$$
\eta^{(\alpha)(\beta)}_0 = \{ \mathcal{F}_\alpha v_0^\alpha, \mathcal{F}_\beta v_0^\beta \} = \sum_{v} f^{eq}_{v} \Phi^\alpha_{0v} \Phi^\beta_{0v} = \delta^{\alpha\beta}.
$$

(B.10)

Thus, we have the projection operators $P_0$ and $Q_0$ given as

$$
[P_0 \psi]_v = \sum_{\alpha=0}^{4} f^{eq}_{v} \Phi^\alpha_{0v} \{ \mathcal{F}_\alpha v_0^\alpha, \psi \},
$$

(B.11)

and $Q_0 = 1 - P_0$.

The perturbative term defined in Eq. (A.24) now takes the form

$$
\mathbf{F}_0 = -v^i f^{eq}_{v} \left[ \frac{1}{n} \nabla^i n + \left( \frac{m}{2T} |\delta v|^2 - \frac{3}{2} \right) \frac{1}{T} \nabla^j T + \frac{m}{T} \delta v^i \nabla_j T \right].
$$

(B.12)

By comparing Eq. (B.12) with Eq. (A.24), we can read $\tilde{\Phi}^{(\mu)}_1$ and $\delta \tilde{X}^{(\mu)}$ as

$$
\tilde{\Phi}^{(\mu)}_1 = f^{eq}_{v} \tilde{\Phi}^{(\mu)}_{1v}, \quad \delta \tilde{X}^{(\mu)} = \delta \tilde{X}^{(\mu)} n
$$

(B.13)

respectively, where

$$
\tilde{\Phi}^{(\mu)}_{1v} \equiv v^i \Phi^0_{0v} = (\delta v^i + u^i) \Phi^0_{0v},
$$

(B.14)

and

$$
\delta \tilde{X}^{i0} \equiv -\frac{1}{\sqrt{n}} \nabla^i n, \quad \delta \tilde{X}^{ij} \equiv -\sqrt{n} \left\{ \frac{m}{T} \nabla^i u^j \right\}, \quad \delta \tilde{X}^{i4} \equiv -\sqrt{\frac{3}{2T}} \frac{1}{T} \nabla^i T,
$$

(B.15)

We can check that $\mathbf{F}_0$ in Eq. (B.12) is expressed in terms of $\delta \tilde{X}^{i\alpha}$ as

$$
\mathbf{F}_0 = \sum_{\alpha=0}^{4} f^{eq}_{v} \tilde{\Phi}^{(\mu)}_{1v} \delta \tilde{X}^{i\alpha}.
$$

(B.16)
It is obvious that $\delta X^{i\alpha}$ defined in Eq. (B.13) are linear independent functions of the hydrodynamic variables $n$, $T$, and $u^i$ with $i = 1$, 2, 3.

Through the straightforward calculation shown in Appendix B.3, we have the vectors belonging to the $P_1$ space as

$$(B.17) \quad \varphi_{i\alpha}^{(1)} = f_{v^i} \varphi_{i\alpha}^{(eq)},$$

with

$$(B.18) \quad \varphi_{0\alpha}^{(1)} = 0, \quad \varphi_{i\alpha}^{(1)} = \frac{1}{\sqrt{n}} \sqrt{\frac{m}{T}} \hat{\pi}_{ij}^{\alpha}, \quad \varphi_{i\alpha}^{(4)} = \frac{1}{\sqrt{n}} \sqrt{2 \frac{1}{3}} \hat{J}_i^{\alpha}.$$

We remark that $\hat{J}_i^{\alpha}$ and $\hat{\pi}_{ij}^{\alpha}$ are identical to the vector fields introduced in the method of moments in Eqs. (2.26) and (2.27), respectively. Thus, the number of independent components of $\varphi_{i\alpha}^{(1)}$ is eight, i.e., $M_1 = 8$.

Using the $P_1$-space vectors $\Phi_{(n,\mu)}^{(n,\mu)} = [A^{-n} f_{v^1} \varphi_{1\alpha}^{(1)}]_v, \quad n = 0, 1,$ we have the $P_1$-space metric matrix

$$(B.19) \quad \eta_{\alpha\beta}^{(n,\mu)}(m,\nu) = \langle A^{-n} f_{v^1} \varphi_{1\alpha}^{(1)} , A^{-m} f_{v^1} \varphi_{1\beta}^{(1)} \rangle \equiv \eta_{\alpha\beta}^{(n,\mu)}(m,\nu),$$

which leads us to the projection operators $P_1$ and $Q_1$ given as

$$(B.20) \quad [P_1 \psi]_v = \sum_{n,m=0,1} \sum_{\alpha,\beta=0}^4 [A^{-n} f_{v^1} \varphi_{1\alpha}^{(1)}]_v \eta_{\alpha\beta}^{-1} \langle A^{-m} f_{v^1} \varphi_{1\beta}^{(1)} , \psi \rangle,$$ and $Q_1 = Q_0 - P_1$.

We introduce the integral constants that represents the deviation from $f_{v^1}^{eq}$ as

$$(B.22) \quad \delta X_{(\mu)}(t_0) = \delta X^{i\alpha}(x; t_0).$$

As shown in Appendix A, $\delta X^{i\alpha}$ appears in the form of $\phi = \sum_{\alpha=1}^{M_1} A^{-1} \varphi_{(1)}^{(1)} \delta X_{(\mu)}$ in the resultant equation. In this case, $\phi$ reads

$$(B.23) \quad \phi = \sum_k A_{v^k}^{-1} \sum_{\alpha=0}^4 f_{v^k} \varphi_{1\alpha}^{(1)} \delta X^{i\alpha} = \frac{1}{T} \sum_k A_{v^k}^{-1} f_{v^k} \langle \hat{\pi}_{ij}^{\alpha} \delta X^{ij} + \hat{J}_i^{\alpha} J_i \rangle,$$

where

$$(B.24) \quad \hat{\pi}_{ij}^{\alpha} = \frac{1}{\sqrt{n}} \sqrt{\frac{T}{m}} \Delta_{ijkl} \delta X^{kl}, \quad J_i^{\alpha} = \frac{1}{\sqrt{n}} \sqrt{2 \frac{1}{3}} \delta X^{i4}.$$ Instead of $\delta X^{i\alpha}$, we use $\pi_{ij}^{\alpha}$ and $J_i$, i.e., the viscous pressure and heat flux, as the fundamental quantities from now on. We note that the definition of $\pi_{ij}^{\alpha}$ ensures that

$$(B.25) \quad \pi_{ij} = \pi_{ji}^{\alpha}, \quad \delta^{ij} \pi_{ij}^{\alpha} = 0.$$

Due to the properties shown in Eq. (B.25), the number of independent components of $\pi_{ij}^{\alpha}$ is five, and the total number of the would-be integral constants $T$, $n$, $u^i$, $\pi_{ij}^{\alpha}$ and $J_i$ are thirteen. Although this number is the same as that of the dynamical variables
introduced in the thirteen-moment approximation proposed by Grad, we mention that this number and form of the dynamical variables have been automatically determined from the Boltzmann equation by the doublet scheme in the RG method developed in Appendix A, which does not demand any ansatz at all in contrast to the traditional approaches. This agreement strongly suggests the reliability of the doublet scheme in the RG method, and encourage us to proceed to the higher orders.

For later convenience, let us introduce the following quantities:

\[
\bar{X}^{ij}_\pi = -\frac{1}{\sqrt{n}} \sqrt{\frac{T}{m}} \delta^{ijkl} \delta \bar{X}^{kl},
\]

\[
\bar{X}^i_j = -\frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \delta \bar{X}^{i4} = \frac{1}{T} \nabla^i T.
\]

With the use of \(\bar{X}^{ij}_\pi\) and \(\bar{X}^i_j\), we have

\[
\sum_{\mu=1}^{M_1} \varphi^{(\mu)}_1 \delta \bar{X}_{(\mu)} = \sum_{\alpha=0}^4 f^{eq}_\alpha \varphi^{(i\alpha)}_1 \delta \bar{X}^{i\alpha} = -\frac{1}{T} f^{eq}_{\bar{v}} (\bar{X}^{ij}_\pi + \bar{J}_\bar{v} \bar{X}^i_j).
\]

The definitions presented in Eq. (A.38) lead to

\[
B = \left. \frac{\partial^2}{\partial f_k \partial f_l} C[f] v \right|_{f = f^{eq}} = \frac{1}{2!} \sum_{v_1 v_2 v_3} \omega(v, v_1|v_2, v_3) \times (\delta v_2 k \delta v_3 l + \delta v_2 \delta v_3 k - \delta v_k \delta v_1 l - \delta v l \delta v_1 k),
\]

\[
F_1 = -v \cdot \nabla \delta v k.
\]

B.2. Mesoscopic dynamics of Boltzmann equation with doublet scheme

Substituting \(A, B, F_0, F_1, \bar{X}_{(\mu)}, C_{(\alpha)}, \delta \bar{X}_{(\mu)}, \varphi^{(\alpha)}_1\), and \(\varphi^{(i\alpha)}_1\) constructed in Appendix B.1 into Eqs. (A.58), (A.59), and (A.60), we obtain the mesoscopic dynamics of the Boltzmann equation.

First, we start with Eqs. (A.58) and (A.59):

\[
\langle f^{eq} \varphi^0_{\alpha} \frac{\partial}{\partial t} f^{eq} \rangle / e^\alpha + \epsilon \sum_{\beta=0}^4 \langle f^{eq} \varphi^0_{\alpha} , f^{eq} \varphi^{i\beta}_1 \rangle \delta \bar{X}^{i\beta} / e^\alpha = -\epsilon \sum_{\beta=0}^4 \langle f^{eq} \varphi^0_{\alpha} , K^\mu \partial_\mu (A^{-1} f^{eq} \varphi^{i\beta}_1 \delta X^{j\beta}) \rangle / e^\alpha + O(\epsilon^3),
\]

and

\[
\epsilon \sum_{\beta=0}^4 \langle A^{-1} f^{eq} \varphi^{i\alpha}_1 , K^\mu \partial_\mu (A^{-1} f^{eq} \varphi^{j\beta}_1 \delta X^{i\beta}) \rangle / e^\alpha
\]

\[
= \epsilon \sum_{\beta=0}^4 \langle f^{eq} \varphi^{i\alpha}_1 , A^{-1} f^{eq} \varphi^{j\beta}_1 \rangle \langle \delta X^{i\beta} + \delta \bar{X}^{i\beta} \rangle / e^\alpha
\]
\[ + \epsilon^2 \sum_{\beta=0}^4 \sum_{\gamma=0}^4 \langle A^{-1} f_{\alpha} \varphi_{i}^{\alpha}, B \left[ A^{-1} f_{\gamma} \varphi_{j}^{\beta} \right] \left[ A^{-1} f_{\beta} \varphi_{k}^{\gamma} \right] \rangle \frac{\delta X^i \delta X^j}{c^\alpha} \]

\[ + O(\epsilon^3). \quad (B.32) \]

Here, we have used the identity

\[ \frac{1}{2} \sum_{\beta=0}^4 \sum_{\gamma=0}^4 \langle f^{\alpha} \varphi_0^{\beta}, B \left[ A^{-1} f_{\gamma} \varphi_{i}^{\beta} \right] \left[ A^{-1} f_{\beta} \varphi_{k}^{\gamma} \right] \rangle \frac{\delta X^i \delta X^j}{c^\alpha} \]

\[ = \sum_{\nu} \varphi_0^{\nu} C \left[ \sum_{\beta=0}^4 A^{-1} f_{\gamma} \varphi_{i}^{\beta} \delta X^j \right] v = 0, \quad (B.33) \]

where we have used the fact that \( \varphi_0^{\nu} \) are collision invariants shown in Eq. (2.7).

Furthermore, we have divided the obtained equations by \( c^\alpha \) given by

\[ c^0 \equiv \frac{1}{\sqrt{n}}, \quad c^{i=1,2,3} \equiv \frac{1}{\sqrt{n}} \sqrt{\frac{m}{T}}, \quad c^4 \equiv \frac{1}{\sqrt{n}} \sqrt{\frac{2}{3} T}, \quad (B.34) \]

and have used the relation

\[ \left[ \frac{\partial}{\partial t} - \epsilon F_1 \right] \delta v_k = \left[ \frac{\partial}{\partial t} + \epsilon v \cdot \nabla \right] \delta v_k = K_{\mu \kappa}^\mu \partial_{\kappa}, \quad (B.35) \]

with the definitions

\[ (\partial_0, \partial_1, \partial_2, \partial_3) \equiv (\partial/\partial t, \epsilon \nabla^1, \epsilon \nabla^2, \epsilon \nabla^3), \quad (B.36) \]

\[ (K^0_{\nu k}, K^1_{\nu k}, K^2_{\nu k}, K^3_{\nu k}) \equiv (1, v^1, v^2, v^3) \delta_{\nu k}. \quad (B.37) \]

Now we shall show the explicit form of each terms in Eqs. (B.31) and (B.32) one by one: The first and second terms in the left-hand side of Eq. (B.31) read

\[ \langle f^{\alpha} \varphi_0^0, \frac{\partial}{\partial t} f^{\alpha} \rangle / c^\alpha = \begin{cases} \frac{\partial}{\partial t} n, & \alpha = 0, \\ m n \frac{\partial}{\partial t} u^i, & \alpha = i, \\ n \frac{\partial}{\partial t} (3T/2), & \alpha = 4, \end{cases} \quad (B.38) \]

\[ \epsilon \sum_{\beta=0}^4 \langle f^{\alpha} \varphi_0^\beta, f^{\alpha} \varphi_1^{i, j} \rangle \frac{\delta X^i \delta X^j}{c^\alpha} = \begin{cases} -\epsilon \nabla \cdot (n u), & \alpha = 0, \\ -\epsilon m n u \cdot \nabla u^i - \epsilon \nabla^i (n T), & \alpha = i, \\ -\epsilon n u \cdot \nabla (3T/2) - \epsilon n T \nabla \cdot u, & \alpha = 4, \end{cases} \quad (B.39) \]

respectively. The first term in the right-hand side of Eq. (B.32) reads

\[ \epsilon \sum_{\beta=0}^4 \langle f^{\alpha} \varphi_1^\beta, A^{-1} f^{\alpha} \varphi_1^{i, j} \rangle \frac{\delta X^i \delta X^j + \delta \tilde{X}^{i, j}}{c^\alpha} = \begin{cases} 0, & \alpha = 0, \\ \epsilon 2 \eta (-\pi^{ij} + \tilde{X}_n^{ij}), & \alpha = j, \\ \epsilon T \lambda (-J^i + \tilde{X}^i), & \alpha = 4, \end{cases} \quad (B.40) \]
where we have defined
\[(B.41) \quad \eta \equiv -\frac{1}{10 T} \langle f^{eq} \hat{\pi}^{ij}, A^{-1} f^{eq} \hat{\pi}^{ij} \rangle, \quad \lambda \equiv -\frac{1}{3 T^2} \langle f^{eq} \hat{j}^i, A^{-1} f^{eq} \hat{j}^i \rangle,\]
and have utilized the following identities:
\[(B.42) \quad \langle f^{eq} \hat{\pi}^{ij}, A^{-1} f^{eq} \hat{\pi}^{kl} \rangle = \frac{1}{5} \Delta^{ijkl} \langle f^{eq} \hat{\pi}^{ab}, A^{-1} f^{eq} \hat{\pi}^{ab} \rangle,\]
\[(B.43) \quad \langle f^{eq} \hat{j}^i, A^{-1} f^{eq} \hat{j}^j \rangle = \frac{1}{3} \delta^{ij} \langle f^{eq} \hat{j}^a, A^{-1} f^{eq} \hat{j}^a \rangle.\]

We note that the transport coefficients, i.e., $\eta$ and $\lambda$, given by Eq. (B.41) accord with $\eta^{TK}$ and $\lambda^{TK}$ in Eq. (B.39), on account of the inner product defined in Eq. (B.31) and $A^{-1} = f^{eq} L^{-1} (f^{eq})^{-1}$.

The term in the right-hand side of Eq. (B.31) is more complicated than the other terms. First, we expand this term as
\[
\begin{align*}
\epsilon \sum_{\beta=0}^{4} \langle f^{eq} \varphi^\alpha_0, K^\mu \partial_\mu (A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta}) \rangle / c^\alpha \\
= \epsilon \sum_{\beta=0}^{4} \left[ \partial_\mu \langle f^{eq} K^\mu (\varphi^\alpha_0 / c^\alpha), A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta} \rangle \right] \\
- \langle f^{eq} K^\mu \partial_\mu (\varphi^\alpha_0 / c^\alpha), A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta} \rangle \\
= \epsilon \sum_{\beta=0}^{4} \left[ \partial_\mu \langle Q_0 f^{eq} K^\mu (\varphi^\alpha_0 / c^\alpha), A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta} \rangle \right] \\
- \langle Q_0 f^{eq} K^\mu \partial_\mu (\varphi^\alpha_0 / c^\alpha), A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta} \rangle.
\end{align*}
\]
\[(B.44) \quad \epsilon \sum_{\beta=0}^{4} \langle f^{eq} \varphi^\alpha_0, K^\mu \partial_\mu (A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta}) \rangle / c^\alpha.
\]

We note that the projection operator $Q_0$ can be inserted in the final line in Eq. (B.44) without a change in the values of the inner product because $A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta}$ belongs to the $Q_0$ space.

Then, with the direct manipulation based on the definitions (B.38), (B.24), (B.34), (B.36), and (B.37), we can show the following identities:
\[(B.45) \quad [Q_0 f^{eq} K^\mu (\varphi^\alpha_0 / c^\alpha)]_{\mathbf{v}} = f^{eq}_{\mathbf{v}} (0, \varphi^{2\alpha}_1 \mathbf{v}, \varphi^{3\alpha}_1 \mathbf{v}) / c^\alpha,\]
\[(B.46) \quad [Q_0 f^{eq} K^\mu \partial_\mu (\varphi^\alpha_0 / c^\alpha)]_{\mathbf{v}} = \begin{cases} 0, & \alpha = 0, \\
0, & \alpha = i, \\
-\epsilon f^{eq} \hat{j}^k_{\mathbf{v}} X^i_{jk}, & \alpha = 4.\end{cases}\]

Substituting these into Eq. (B.44), we have
\[
\begin{align*}
\epsilon \sum_{\beta=0}^{4} \langle f^{eq} \varphi^\alpha_0, K^\mu \partial_\mu (A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta}) \rangle / c^\alpha \\
= \begin{cases} 0, & \alpha = 0, \\
-\epsilon^2 \nabla^j (2 \eta \pi^{ij}), & \alpha = i, \\
-\epsilon^2 \nabla^j (T \lambda J^j) + \epsilon^2 2 \eta \pi^{jk} X^i_{jk}, & \alpha = 4.\end{cases}
\end{align*}
\[(B.47) \quad \epsilon \sum_{\beta=0}^{4} \langle f^{eq} \varphi^\alpha_0, K^\mu \partial_\mu (A^{-1} f^{eq} \varphi^j_1 \delta X^{j\beta}) \rangle / c^\alpha.
\]
Substituting Eqs. (B.48), (B.49), and (B.51) into Eq. (B.51), we have the balance equations as

\[ \frac{\partial}{\partial t} n = -\epsilon \nabla \cdot (n \mathbf{u}) + O(\epsilon^3), \]

\[ m n \frac{\partial}{\partial t} u^i = \epsilon m n \mathbf{u} \cdot \nabla u^i - \epsilon \nabla^j (nT \delta^{ij} - \epsilon^2 \eta \pi^{ij}) + O(\epsilon^3), \]

\[ n \frac{\partial}{\partial t} (3T/2) = -\epsilon n \mathbf{u} \cdot \nabla (3T/2) - \epsilon n T \nabla \cdot \mathbf{u} + \epsilon^2 \nabla^j (T \lambda J^j) + O(\epsilon^3). \]

We emphasize that Eqs. (B.48) - (B.50) describe the slow motion of \( n \), \( u^i \), and \( T \), because the time derivative of them is the first order of \( \epsilon \), as is manifest.

The term in the left-hand side of Eq. (B.52) reads

\[ \epsilon \sum_{\beta=0}^4 \left( \frac{A^{-1} f^{eq} \phi^{i\alpha}}{c^\alpha} , K^{\mu} \partial_\mu (A^{-1} f^{eq} \phi^{j\beta} \delta X^{j\beta}) \right) / c^\alpha \]

\[ \begin{cases} 
0, & \alpha = 0, \\
\epsilon T \left( \tilde{\pi}^{ij}, K^{\mu} \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \right), & \alpha = j, \\
\epsilon T \left( \tilde{J}^i, K^{\mu} \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \right), & \alpha = 4, 
\end{cases} \]

with \( \tilde{\pi}^{ij} \equiv \frac{1}{T} [A^{-1} f^{eq} \tilde{\pi}^{ij}] \mathbf{u} \) and \( \tilde{J}^i \equiv \frac{1}{T} [A^{-1} f^{eq} \tilde{J}^i] \mathbf{u} \). Using the expansions

\[ T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle = T \langle \tilde{\pi}^{ij}, K^{\mu} \pi^{kl} \partial_\mu \pi^{kl} + T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_\mu \tilde{\pi}^{kl} \rangle \pi^{kl} + T \langle \tilde{\pi}^{ij}, K^{\mu} \tilde{J}^k \partial_\mu J^k \rangle \rangle \]

\[ T \langle \tilde{J}^i, K^{\mu} \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle = T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle + T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_\mu \tilde{J}^k \partial_\mu \pi^{kl} \rangle \pi^{kl} \]

we proceed to the further analysis of Eq. (B.51). First, the first and third terms in the right-hand side of Eqs. (B.52) and (B.53) read

\[ T \langle \tilde{\pi}^{ij}, K^{\mu} \pi^{kl} \partial_\mu \pi^{kl} \rangle = 2 \eta \tau_\pi \left( \frac{\partial}{\partial t} + \epsilon \mathbf{u} \cdot \nabla \right) \pi^{ij}, \]

\[ T \langle \tilde{\pi}^{ij}, K^{\mu} \tilde{J}^k \partial_\mu J^k \rangle = 2 \eta \ell_{\pi J} \Delta^{lmk} \nabla^m J^k, \]

\[ T \langle \tilde{\pi}^{ij}, K^{\mu} \tilde{J}^k \partial_\mu \pi^{kl} \rangle = 2 \eta \ell_{\pi J} \Delta^{lmk} \nabla^m \pi^{kl}, \]

\[ T \langle \tilde{J}^i, K^{\mu} \tilde{J}^k \partial_\mu J^k \rangle = 2 \eta \ell_{\pi J} \Delta^{lmk} \nabla^m \pi^{kl}, \]

respectively. In Eqs. (B.54) - (B.57), we have used the definitions given by

\[ \tau_\pi \equiv \frac{T}{10 \eta} \langle \tilde{\pi}^{ij}, \tilde{\pi}^{ij} \rangle = \frac{1}{10 T} \langle f^{eq} \tilde{\pi}^{ij}, A^{-2} f^{eq} \tilde{\pi}^{ij} \rangle, \]

\[ \tau_J \equiv \frac{1}{3 \lambda} \langle \tilde{J}^i, \tilde{J}^i \rangle = \frac{1}{3 T^2 \lambda} \langle f^{eq} \tilde{J}^i, A^{-2} f^{eq} \tilde{J}^i \rangle, \]

\[ \ell_{\pi J} \equiv \frac{T}{10 \eta} \langle \tilde{\pi}^{ij}, \delta K^i \tilde{J}^j \rangle = \frac{1}{10 T} \langle A^{-1} f^{eq} \tilde{\pi}^{ij}, \delta K^i A^{-1} f^{eq} \tilde{J}^j \rangle, \]

\[ \ell_{J \pi} \equiv \frac{1}{5 \lambda} \langle \tilde{J}^i, \delta K^j \tilde{\pi}^{ij} \rangle = \frac{1}{5 T^2 \lambda} \langle A^{-1} f^{eq} \tilde{J}^i, \delta K^j A^{-1} f^{eq} \tilde{\pi}^{ij} \rangle, \]
and the following identities:

\begin{align*}
\langle \tilde{\pi}^{ij}, \tilde{\pi}^{kl} \rangle &= \frac{1}{5} \Delta^{ijkl} \langle \tilde{\pi}^{ab}, \tilde{\pi}^{ab} \rangle, \quad \langle \tilde{J}^{i}, \tilde{J}^{k} \rangle = \frac{1}{3} \delta^{ik} \langle \tilde{J}^{a}, \tilde{J}^{a} \rangle, \\
\langle \tilde{\pi}^{ij}, \delta K^{m} \tilde{J}^{k} \rangle &= \frac{1}{5} \Delta^{imnk} \langle \tilde{\pi}^{ab}, \delta K^{a} \tilde{J}^{k} \rangle, \\
\langle \tilde{J}^{i}, \delta K^{m} \tilde{\pi}^{kl} \rangle &= \frac{1}{5} \Delta^{imkl} \langle \tilde{J}^{a}, \delta K^{b} \tilde{\pi}^{ab} \rangle, \\
\langle \tilde{\pi}^{ij}, \tilde{J}^{k} \rangle &= \langle \tilde{\pi}^{ij}, \delta K^{m} \tilde{\pi}^{kl} \rangle = \langle \tilde{J}^{i}, \delta K^{m} \tilde{J}^{k} \rangle = 0,
\end{align*}

with $\delta K^{i} \equiv \delta v^{i} \delta v_{k}$. We note that the relaxation times, i.e., $\tau_{\pi}$ and $\tau_{J}$, given by Eqs. (B.58) and (B.59) agree with $\tau_{\pi K}$ and $\tau_{J K}$ in Eq. (3.14), respectively, using the inner product (2.31) and $A^{-1} = f_{eq} L^{-1} (f_{eq})^{-1}$, as well as the transport coefficients given by Eq. (B.41). Furthermore, we note that $\ell_{\pi, J}$ and $\ell_{J, \pi}$ defined in Eqs. (B.60) and (B.61) denote the relaxation lengths.

Next, we consider the second and fourth terms in the right-hand side of Eqs. (B.52) and (B.53). We notice the following power counting with respect to $\epsilon$:

\begin{align*}
T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_{\mu} \tilde{\pi}^{kl} \rangle &\sim O(\epsilon), \quad T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_{\mu} \tilde{J}^{k} \rangle \sim O(\epsilon), \\
T \langle \tilde{J}^{i}, K^{\mu} \partial_{\mu} \tilde{\pi}^{kl} \rangle &\sim O(\epsilon), \quad T \langle \tilde{J}^{i}, K^{\mu} \partial_{\mu} \tilde{J}^{k} \rangle \sim O(\epsilon).
\end{align*}

This order counting can be derived from the fact that the above terms contain the temporal and spatial first-order derivatives of $n$, $T$, and $u^{j}$. The temporal derivatives can be converted into the spatial derivatives with the use of the balance equations (B.48) - (B.50), and there exists $\epsilon$ in front of the spatial derivatives. Accordingly, we can represent the above terms as the quantities of $O(\epsilon)$.

\begin{align*}
T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_{\mu} \tilde{\pi}^{kl} \rangle &\equiv -2 \eta \bar{X}_{\pi \pi}^{ijkl}, \quad T \langle \tilde{\pi}^{ij}, K^{\mu} \partial_{\mu} \tilde{J}^{k} \rangle \equiv -2 \eta \bar{X}_{\pi J}^{ij}, \\
T \langle \tilde{J}^{i}, K^{\mu} \partial_{\mu} \tilde{\pi}^{kl} \rangle &\equiv -\epsilon T \lambda \bar{X}_{\pi \pi}^{ijkl}, \quad T \langle \tilde{J}^{i}, K^{\mu} \partial_{\mu} \tilde{J}^{k} \rangle \equiv -\epsilon T \lambda \bar{X}_{\pi J}^{ij}.
\end{align*}

It is noteworthy that $\bar{X}_{\pi \pi}^{ijkl}$, $\bar{X}_{\pi J}^{ijkl}$, and $\bar{X}_{\pi J}^{ij}$ might be also called the thermodynamic forces as $\bar{X}_{\pi \pi}^{ijkl} = \Delta^{ijkl} \nabla^{k} u^{l}$ and $\bar{X}_{\pi J}^{ij} = \nabla^{i} u^{j}$ are. Their explicit forms are given by

\begin{align*}
\bar{X}_{\pi \pi}^{ijkl} &= \Delta^{ijkl} C_{\pi \pi}^{(0)} \nabla \cdot u + \Delta^{ijac} \Delta^{ckl} (C_{\pi \pi}^{(2)} \delta^{ab} \nabla d_{u} c + C_{\pi \pi}^{(3)} \omega^{ab}), \\
\bar{X}_{\pi J}^{ijkl} &= \Delta^{ijkl} C_{\pi J}^{(0)} \nabla a T + C_{\pi J}^{(2)} \nabla a n, \\
\bar{X}_{\pi J}^{ijkl} &= \Delta^{akl} C_{\pi J}^{(1)} \nabla a T + C_{\pi J}^{(2)} \nabla a n, \\
\bar{X}_{\pi J}^{ik} &= \delta^{ik} C_{\pi J}^{(1)} \nabla \cdot u + C_{\pi J}^{(2)} \Delta^{ikl} \nabla u b + C_{\pi J}^{(3)} \omega^{ik},
\end{align*}

with the vorticity term $\omega^{ij} = (\nabla^{i} u^{j} - \nabla^{j} u^{i})/2$. Here, the coefficients $C_{\pi \pi}^{(0)}$, $C_{\pi \pi}^{(2)}$, $C_{\pi \pi}^{(3)}$, $C_{\pi J}^{(0)}$, $C_{\pi J}^{(2)}$, $C_{\pi J}^{(3)}$, $C_{\pi J}^{(4)}$, $C_{\pi J}^{(5)}$, and $C_{\pi J}^{(6)}$ are defined by

\begin{align*}
C_{\pi \pi}^{(1)} &= \frac{\Delta^{ijkl}}{-10 T} \langle \tilde{\pi}^{ij}, ((\delta^{ab} / 3) \delta K^{a} \partial / \partial u^{b} - (2 T / 3) \partial / \partial T - n \partial / \partial n) \tilde{\pi}^{kl} \rangle, \\
C_{\pi \pi}^{(2)} &= \frac{\Delta^{ijac} \Delta^{ckl} + \Delta^{ijbc} \Delta^{akl} - 2 \Delta^{ijkl} \delta^{ab}}{-15 T} \langle \tilde{\pi}^{ij}, \delta K^{a} \partial / \partial u^{b} \tilde{\pi}^{kl} \rangle, \\
C_{\pi \pi}^{(3)} &= \frac{\Delta^{ijac} \Delta^{ckl} - \Delta^{ijbc} \Delta^{akl}}{-15 T} \langle \tilde{\pi}^{ij}, \delta K^{a} \partial / \partial u^{b} \tilde{\pi}^{kl} \rangle,
\end{align*}

\begin{align*}
C_{\pi J}^{(1)} &= \frac{\Delta^{ijkl}}{-10 T} \langle \tilde{\pi}^{ij}, ((\delta^{ab} / 3) \delta K^{a} \partial / \partial u^{b} - (2 T / 3) \partial / \partial T - n \partial / \partial n) \tilde{\pi}^{kl} \rangle, \\
C_{\pi J}^{(2)} &= \frac{\Delta^{ijac} \Delta^{ckl} + \Delta^{ijbc} \Delta^{akl} - 2 \Delta^{ijkl} \delta^{ab}}{-15 T} \langle \tilde{\pi}^{ij}, \delta K^{a} \partial / \partial u^{b} \tilde{\pi}^{kl} \rangle, \\
C_{\pi J}^{(3)} &= \frac{\Delta^{ijac} \Delta^{ckl} - \Delta^{ijbc} \Delta^{akl}}{-15 T} \langle \tilde{\pi}^{ij}, \delta K^{a} \partial / \partial u^{b} \tilde{\pi}^{kl} \rangle.
\end{align*}
We note that \( J_i \), whose definitions are given by

\[
\varphi_i = \frac{\pi J_i}{\partial \tau} = \frac{\pi J_i}{\partial n} - (T/m \partial \partial u_a \bar{\pi}^k),
\]

Finally, we examine the second term in the right-hand side of Eq. (B.32), which reads

\[
e^2 \frac{1}{2} \sum_{\beta=0}^{4} \sum_{\gamma=0}^{4} \langle A^{-1} f^{eq} \varphi_i^{\alpha} \rangle, B \left[ A^{-1} f^{eq} \varphi_1^{\beta} \right] \left[ A^{-1} f^{eq} \varphi_1^{\gamma} \right] \rangle \delta X^{\alpha} \delta X^{\beta} / e^\alpha
\]

\[
= \begin{cases} 
0, & \alpha = 0, \\
\epsilon^2 T \lambda b_{\pi J J} \pi^{\alpha} J^k, & \alpha = j, \end{cases}
\]

\[
\epsilon = \frac{\pi J_i}{\partial \tau} \frac{\pi J_i}{\partial n} - (T/m \partial \partial u_a \bar{\pi}^k).
\]

We present an explicit form of the invariant/attractive manifold: Equation (A.60) reads

\[
f_{\mathbf{v}} = f_{\mathbf{v}}^{eq} + \epsilon \sum_{\alpha=1}^{4} \left[ A^{-1} f^{eq} \varphi_i^{\alpha} \right] \mathbf{v} \delta X^{\alpha}
\]

\[
- \epsilon^2 \left[ Q_1 (A - \partial/\partial \tau)^{-1} Q_0 (B \phi^2(\tau)/2 + F_1 \phi(\tau)) \right]_{\tau=0} \mathbf{v} + O(\epsilon^3),
\]
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\[ \phi(\tau) = \sum_{\alpha=0}^{4} \left[ e^{A_{\tau}} A^{-1} f_{\text{eq}} \phi_1^{\alpha} \delta X^{\alpha} + (\tau P_0 f_{\text{eq}} \bar{\phi}_1^{\alpha} + (e^{A_{\tau}} - 1) A^{-1} f_{\text{eq}} \phi_1^{\alpha}) \delta \bar{X}^{\alpha} \right]. \]

(B.90)

Here, we consider the form valid up to \( O(\epsilon) \):

\[ f_{\text{EV}} = f_{\text{v}}^{\text{eq}} + \epsilon \frac{1}{T} [A^{-1} f_{\text{eq}} \pi^{ij}] v \pi^{ij} + \epsilon \frac{1}{T} [A^{-1} f_{\text{eq}} \bar{J}^{i}] v J^{i} + O(\epsilon^2). \]

(B.91)

By applying the redefinitions of \( \pi^{ij} \) and \( J^{i} \) as

\[ 2 \eta \pi^{ij} \rightarrow \pi^{ij}, \quad T \lambda J^{i} \rightarrow J^{i}, \]

(B.92)

to the balance equations \((B.48)-(B.50)\), relaxation equations \((B.87)\) and \((B.88)\), and invariant/attractive manifold \((B.91)\), we arrive at Eqs. \((3.8)-(3.12)\), and \((3.7)\), respectively.

B.3. Detailed derivation of explicit form of excited modes . We show a detailed derivation of \( \phi_1^{\alpha} \) as shown in Eq. \((B.18)\). The definitions in Eqs. \((B.13)\) and \((B.17)\) lead to

\[ f_{\text{v}}^{\text{eq}} \phi_1^{\alpha} = [Q_0 f_{\text{eq}} \phi_1^{\alpha}] v = f_{\text{v}}^{\text{eq}} \phi_1^{\alpha} - \sum_{\beta=0}^{4} f_{\text{v}}^{\text{eq}} \phi_0^{\beta} \langle f_{\text{eq}} \phi_0^{\beta}, f_{\text{eq}} \phi_1^{\alpha} \rangle. \]

(B.93)

Then, we can convert Eq. \((B.93)\) into

\[ \phi_1^{\alpha} v = \delta v^{\alpha} \phi_0^{\alpha} - \sum_{\beta=0}^{4} \phi_0^{\alpha} M^{\beta \alpha}, \]

(B.94)

where

\[ M^{\alpha \beta} \equiv \langle f_{\text{eq}} \phi_0^{\beta}, f_{\text{eq}} \phi_1^{\alpha} \rangle - u^{i} \delta^{\alpha \beta} = \sum_{v} f_{\text{v}}^{\text{eq}} \phi_0^{\alpha} \delta v^{i} \phi_0^{\beta}. \]

(B.95)

We note that \( M^{\alpha \beta} \) read

\[ M^{00} = M^{04} = M^{4k} = M^{44} = 0, \]

(B.96)

\[ M^{0i} = M^{i0} = T^{i} \delta^{ij}, \quad M^{iji} = M^{iij} = \sqrt{\frac{2}{3}} T^{ij}, \]

(B.97)

which have been derived straightforwardly with the use of

\[ \sum_{v} f_{\text{v}}^{\text{eq}} (1, \delta v^{i}, \delta v^{i} \delta v^{j}, \delta v^{i} \delta v^{j} \delta v^{k}, \delta v^{i} \delta v^{j} \delta v^{k} \delta v^{l}) \]

(B.98)

Substituting \( M^{\alpha \beta} \) into Eq. \((B.94)\), we obtain the explicit from of \( \phi_1^{\alpha} \) shown in Eq. \((B.18)\).
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