PSEUDO-CONFORMAL FIELD THEORY

C. N. Ragiadakos*

ABSTRACT
The fundamental structure of the 4-dimensional spacetime is assumed to be the lorentzian CR-structure (LCR-structure), which contains two correlated 3-dimensional CR-structures. It is defined by explicit Frobenius integrable relations characterized by ”left” and ”right” CP(3) points. This LCR-structure is invariant under a very restrictive tetrad-Weyl symmetry, which permits a unique special second order partial differential equation applied to a Yang-Mills field, identified with the gluon field. A class of metrics with the corresponding self-dual forms are defined. After partially fixing the tetrad-Weyl symmetry, the electroweak connection is also defined, which is directly related with the class of LCR-tetrads of the structure. The ”free electron” LCR-structure is identified, which has gravitational and electroweak potentials (dressings) with fermionic gyromagnetic ratio $g=2$. The corresponding massless ”neutrino” LCR-structure is also found. These two solitonic configurations constitute the first leptonic generation identified with the Petrov type D LCR-structure. The muon and tau leptonic generations are identified with the Petrov type II and I respectively. Using the electron LCR-structure, I compute the corresponding quark having an additional gluonic potential and providing the explication of the lepton-quark correspondence. The standard model is implied via the causal Bogoliubov, Epstein-Glaser and Scharf procedure viewed as a targeted harmonic analysis in the rigged Hilbert-Fock space of precise Poincare representations of the computed geometric structures.
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1 INTRODUCTION

The standard model (SM) actually provides an experimentally well established description of elementary particles, except the graviton. The discovery of the Higgs particle made the SM renormalizable and hence well defined in the context of quantum field theory (QFT). But the recent experimental results of ATLAS and CMS from the LHC of CERN show that the minimal supersymmetric SM does not describe nature. No supersymmetric particles have been observed. On the other hand highly sensitive experiments did not find any sign of weakly interacting massive particles. Hence string theory has to be abandoned, and elementary particle physics has to look for other quantum models, which do not need supersymmetry to describe nature. The present work provides such an alternative by simply replacing Einstein’s metric with a special totally real (Cauchy-Riemann) CR-structure in the tangent space of a differential manifold. The first investigators (E. Cartan, Tanaka, Severi etc) of CR-structures used the term ”pseudo-conformal transformations” [6], therefore the present model is called pseudo-conformal field theory (PCFT) and its fundamental CR-structure is called lorentzian Cauchy-Riemann (LCR) structure. In fact it generalizes the metric independence of the 2-dimensional Polyakov action into a four dimensional highly symmetric model which derives all the particles of the SM. The present work systematically describes the emergence of the leptonic and hadronic sector with the amazing lepton-quark correspondence. These Poincaré representations (free fields) are the basis of the Bogoliubov [3]-Epstein-Glaser [10] perturbative causal approach [28] as reformulated by Scharf [29] into a self-consistent (BEGS) procedure, which is briefly reviewed in the following subsection.

The metric independence is usually related with topological field theories, which are essentially dynamically ”empty”. The 2-dimensional generally covariant Polyakov action

\[ I_S = \frac{1}{2} \int d^2\xi \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \]  

(1.1)
is also metric independent, but it is very rich, because it is not topological. In the light-cone coordinates \((\xi_-, \xi_+)\), the action takes the metric independent form

\[ I_S = \int d^2z \partial_- X^\mu \partial_+ X^\nu \eta_{\mu\nu} \]  

(1.2)
The light-cone coordinates \((\xi_-, \xi_+)\) may be viewed as the real part of the structure coordinates \((z^0, \tilde{z}^0)\) of the following trivial totally real CR-manifold of \(\mathbb{C}^2\)

\[ \rho_{11}(z^0, \tilde{z}^0) = 0 \quad , \quad \rho_{22}(z^0, \tilde{z}^0) = 0 \]  

(1.3)
because its corresponding normal form [2] is

\[ \text{Im} z^0 = 0 \quad , \quad \text{Im} \tilde{z}^0 = 0 \]  

(1.4)
The CR-structures [18] are metric independent. In general relativity, the geodetic and shear free condition of a null congruence coincides [31] with
the CR-structure of conventional mathematics. In order to make the present work self-consistent, a very brief review of the 3-dimensional CR-structure will be presented in section II, where the 4-dimensional LCR-structure will be defined\[26\] without using the notion of the metric. It essentially consists of two 3-dimensional CR-substructures with a common complex tangent sub-space. The emergence of Einstein’s gravity is described in section III and the emergence of electroweak gauge field is described in section V. The solitonic LCR-structures of the free electron and its neutrino are explicitly computed in section IV.

1.1 A brief review of BEGS procedure

After the Wightman axioms it became clear that the fundamental structures of QFT are the Poincaré group and the Schwartz distributions\[30\]. The later are functionals applied over vector spaces of functions with infinite number of successive finer norms (nuclear spaces) called test functions. The useful kinds of test functions on $\mathbb{R}^4$ are: 1) Smooth functions $\phi(x)$ with compact support and infinite number of derivatives ($\in C\infty$) denoted with $\mathcal{D}$. The corresponding functionals $f(\phi) \in \mathbb{C}$ are denoted $\mathcal{D}'$. 2) Smooth functions $\phi(x)$ decreasing faster than any polynomial of $x$ denoted with $\mathcal{S}$ usually called Schwartz space. The corresponding functionals $f(\phi) \in \mathbb{C}$ are denoted $\mathcal{S}'$. 3) Real analytic functions $\phi(x)$ denoted with $\mathcal{A}$. The corresponding functionals $f(\phi) \in \mathbb{C}$ are denoted $\mathcal{A}'$ usually called Sato’s distributions. 4) General smooth functions $\phi(x) \in C\infty$ denoted with $\mathcal{E}$. The corresponding functionals $f(\phi) \in \mathbb{C}$ are denoted $\mathcal{E}'$ usually called distributions with compact support. Apparently the following set relations

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{A} \subset \mathcal{E}$$

$$\mathcal{D}' \subset \mathcal{A}' \subset \mathcal{S}' \subset \mathcal{E}'$$

are valid. The Schwartz distributions permit the definition of rigged Hilbert space (Gelfand triple)\[3\] as the completion of the test-function normed nuclear spaces. The Gelfand-Maurin theorem permits the proper definition of the self-adjoint and unitary operators with their combined discrete and continuous eigenvalues. The conventional QFT is based on the rigged Hilbert space

$$\mathcal{S}(\mathbb{R}^4) \subset L^2(\mathbb{R}^4) \subset \mathcal{S}'(\mathbb{R}^4)$$

where the space $\mathcal{S}(\mathbb{R}^4)$ is dense in $L^2(\mathbb{R}^4)$ of square integrable functions. In section VI we will find that the quark gluonic dressing has line singularities implying that the physical hadrons are colorless distributions with compact support i.e. they belong to $\mathcal{E}'(\mathbb{R}^4)$, which is interpreted as confinement. The Schwartz test functions $\mathcal{S}$ and the corresponding distributions $\mathcal{S}'$ decay as

$$|\varphi(x)| \leq C_N |x|^{-N} \quad \text{as} \quad |x| \to \infty$$

$$|f(x)| \leq C_N |x|^N \quad \text{as} \quad |x| \to \infty$$

(1.7)
This decaying difference essentially distinguishes the wavefront singularities\textsuperscript{10} which characterize the representatives of the (proper) distributions. The above conditions may be replaced with the condition that the distributions are represented with functions which are derivatives of locally integrable "potentials" with singularities. The typical example is the potential \((\frac{1}{|x|})\) of a point charge, which is locally integrable but its grad, the corresponding field strength \((\frac{1}{|x|}^2)\), is not locally integrable.

The free fields are operator valued distributions \(S'\) which belong to precise representations of the Poincaré group. The Bogoliubov expansion of the S-matrix into the coupling \(c(x)\) constant considered as a test function is

\[
S = 1 + \sum_{n \geq 1} \frac{1}{n!} \int S_n(x_1, x_2, ..., x_n)c(x_1)c(x_2)...c(x_n)[dx]
\]

\[
S_n(x_1, x_2, ..., x_n) = T\{L_I(x_1)L_I(x_2)...L_I(x_n)\}
\]

where \(T\) denotes the time-ordering of the interaction lagrangian \(L_I(x)\), which depends on the free field operators. It has the normal product form of operator valued distributions. In conventional QFT time-ordering is imposed using the step distribution, which causes mathematical problems, because its multiplication with the distribution \(L_I(x)\) is not permitted, implying\textsuperscript{28} the appearance of the well known infinities. Epstein and Glaser\textsuperscript{10} found a procedure to bypass these infinities. They multiply with a \(C^\infty\) regulator and after separate the advanced from the retarded parts. After, they find the degree of the divergence at \(x = 0\). These divergencies are subtracted. Grangé et.al.\textsuperscript{13} use a more effective method introducing test functions as partitions of unity.

Up to now the procedure is equivalent to the ordinary renormalization procedure. The final essential step was the use by Scharf and his collaborators\textsuperscript{29} a variation of the BRS transformation procedure to establish order by order the complete interaction lagrangian without the unphysical modes of the free fields. Considering the free fields as operator valued distributions, they make the infinitesimal gauge transformation

\[
A'^\mu(x) = A^\mu(x) + \lambda \partial^\mu u(x) + O(\lambda^2)
\]

as an operator transformation

\[
A'^\mu(x) = e^{-i\lambda Q} A^\mu(x) e^{i\lambda Q} \simeq A^\mu(x) - i\lambda [Q, A^\mu(x)] + O(\lambda^2)
\]

\[
[Q, A^\mu(x)] = i\partial^\mu u(x)
\]

implying the commutator of the charge \(Q\) with the field operator \(A^\mu(x)\). It determines \(Q\) up to a complex number

\[
Q = \int_{x^0 = c} d^3 x (\partial_\mu A^\mu \partial_0 u - (\partial_0 \partial_\mu A^\mu) u)
\]

The gauge charge is nilpotent \(Q^2 = \{Q, Q\} = 0\) and the field \(u(x)\) is a ghost massless "fermionic" field. Its partner is denoted \(\tilde{u}(x)\). The purpose of these
ghost fields is to eliminate the unphysical longitudinal and scalar components determined by the gauge condition \( \partial_\mu A^\mu = 0 \). That is, a state \( \Phi \) of the Fock space is physical if
\[
\{ Q^\dagger, Q \} \Phi = 0 \tag{1.12}
\]
This procedure was generalized for the non-abelian groups and applied to the derivation of the SM of elementary particles. It demystified the spontaneous symmetry breaking and provided an understanding of the "naturalness problems" and the neutrino and quark mixing problems. The application of the algorithm to the coordinate invariance of the spin-2 field \( h^{\mu\nu} \) implies the proper Einstein-Hilbert action with a cosmological constant, which we should expect, because it is well known that higher order derivatives generate negative norm (unphysical) states. Hence the entire BEGS procedure provides a well defined ("renormalizable") quantum general relativity.

Concluding this brief review I point out that BEGS procedure essentially separates QFT into two parts. The first part where the experimentally observed particles and symmetries (gravity, electroweak and gluon gauge fields) are assumed and the second part the mathematical harmonic analysis properly done in the rigged Hilbert-Fock space of the Poincaré representations. The assumed symmetries and particles of the first part are implied by PCFT. The second pure mathematical part is implied by the proper treatment of the Schwartz distributions and the rigged Hilbert space (Gelfand triple) described by the BEGS procedure as an operational harmonic analysis in the asymptotic Poincaré representations found in PCFT. This interplay of PCFT and the BEGS procedure provides the known SM of elementary particles as described in section VI.

2 THE LORENTZIAN CR-STRUCTURE

In the complex plain \( \mathbb{C} \), the Riemann mapping theorem states that two lines (real hypersurfaces) are holomorphically equivalent. Poincaré has showed that it is not valid in higher dimensions. That is, the real surfaces \( \rho(\overline{z^\alpha}, z^\alpha) = 0 \) (for any real function \( \rho \)) on the complex plane \( \mathbb{C}^n, n > 1 \) cannot be transformed to each other with an holomorphic transformation. These different structures of the real surfaces are called Cauchy-Riemann structures (CR-structures) and the manifolds endowed with such structures are called CR-manifolds. Surfaces of \( \mathbb{C}^n \) determined by one real function are called CR-manifolds of the hypersurface type and those determined by \( n \) independent real functions are called totally real CR-manifolds. In the following subsections I review the 3-dimensional CR-manifolds of the hypersurface type and a special form of 4-dimensional totally real CR-manifolds, which I call lorentzian CR-manifolds (LCR-manifolds). These two kinds of CR-structures are sufficient for the reader to understand the fundamental geometric structure of pseudo-conformal field theory (PCFT).

Throughout this mathematical review, the reader should notice that the notion of the CR-structure does not need the notion of the metric to be defined. CR-structure is not a notion of the riemannian geometry, while in general rela-
activity the CR-structure emerged through the notion of a geodetic and shear free null congruence.

2.1 Three-dimensional CR-structures

The simple three-dimensional CR-manifold $M$ is determined by the annihilation of a real function $\rho(z^\alpha, z^\alpha) = 0$ in $\mathbb{C}^2$. The condition $d\rho(z^\alpha, z^\alpha)|_M = 0$ implies that this surface admits the following cotangent real 1-form

$$\omega_0 = 2i(\partial \rho)|_M = i((\partial - \bar{\partial})\rho)|_M \quad (2.1)$$

If we use the regular coordinates $(u, \zeta, \zeta)$, which provide the following graph form for the surface

$$z^0 = w = u + iU \quad , \quad z^1 = \zeta$$

$$U = z^0 - \bar{z}^0 = \phi(u, \zeta, \zeta) \quad , \quad \phi(0) = 0 \quad , \quad d\phi(0) = 0$$

and the real surface takes the simple form

$$\rho(z^\alpha, z^\alpha) = z^0 - \bar{z}^0 = \phi(z^0, z^1, \bar{z}^1) \quad (2.3)$$

We find the following basis of the cotangent space

$$\omega_0 = du - i(\frac{\partial \phi}{\partial z^1})dz^1 + i(\frac{\partial \phi}{\partial \zeta})d\bar{z}^1$$

$$\omega_1 = dz^1 \quad , \quad \omega_1 := \bar{\omega}_1 = d\bar{z}^1$$

which admit arbitrary multiplication factors. The dual basis of the tangent space is

$$k_0 = \frac{\partial}{\partial u}$$

$$k_1 = \frac{\partial}{\partial z^1} + 2i\frac{\partial U}{\partial U} \frac{\partial}{\partial u}$$

$$\bar{k}_1 := \bar{k}_1$$

which is normalized by the (interior product $\cdot$) relations $k_0, \omega_0 = 1$ and $k_1, \omega_1 = 1$ and the other interior products vanish. Do not confuse the interior product with the inner product defined by a metric tensor!

If the defining function $\rho(z^\alpha, z^\alpha)$ is real analytic, the surface is called real analytic. This surface is \textit{diffeomorphically} equivalent to the "flat" hyperplane $U = 0$. But there is not always a \textit{holomorphic} transformation which performs this transformation.

In its abstract definition a 3-dimensional CR-manifold is a (real) differentiable manifold which admits a complex field $k_1 = k_1^\beta \partial_\beta$ in the tangent
space $T^\ast(M)$, which is linearly independent to its complex conjugate. The CR-structure is invariant under the transformation

$$k'_1 = bk_1$$

(2.6)

with $b \neq 0$, $\infty$ complex function

The CR-structure may be equivalently defined by a real covector field $\omega_0 = \omega_{0\alpha}d\xi^\alpha$ and a complex covector $\omega_1 = \omega_{1\alpha}d\zeta^\alpha$ of the cotangent space $T^\ast(M)$ such that

$$k_1,\omega_0 = 0 \quad , \quad k_1,\omega_1 = 1 \quad , \quad k_1,\omega_1 = 0$$

(2.7)

\[ \omega_0 \wedge \omega_1 \neq 0 \]

The corresponding equivalence transformation in the cotangent space is

$$\omega'_0 = a\omega_0 \quad , \quad \omega'_1 = \omega_1 + c\omega_0$$

(2.8)

$\quad a \neq 0$ real, and $b \neq 0$, $\infty$ and $c$ complex functions

The CR-structures have some local invariants (called relative invariants), which take the discrete values 0 or 1. The first relative invariant emerges from the observation that under a general CR-structure preserving transformation (2.8) the relation

$$d\omega_0 = iA\omega_1 \wedge \overline{\omega_1} + B\omega_0 \wedge \omega_1 + \overline{B}\omega_0 \wedge \overline{\omega_1}$$

(2.9)

takes the form

$$d\omega'_0 = iAab\omega'_1 \wedge \overline{\omega'_1} \mod[\omega'_0]$$

(2.10)

Hence, if at a point $y$ the function does not vanish $A(y) \neq 0$, it will not vanish for any other set of representative 1-forms of the CR-structure, which is then called non-degenerate at $y$. On the other hand if the CR-structure at the point $y$ has $A(y) = 0$, it will vanish for any other set of representative 1-forms of the CR-structure and it is called degenerate at $y$. Notice that the non-vanishing condition of the CR-structure defining 1-forms $[\omega_0 , \omega_1 , \omega^\ast_1]$ is related to its degeneracy. Vanishing points of $A(x)$ may be interpreted as a different CR-manifold, because the transformation

$$\omega'_0 = \frac{1}{A}\omega_0 \quad , \quad \omega'^1 = \omega^1$$

(2.11)

at any point of a neighborhood of the zeros of $A(x)$ removes the zeros, but it makes $\omega'_0$ not well defined at $x = y$. This is the reason that the transformation (2.9) must respect the condition $A(x)$ related to a coordinate atlas of the CR-manifold. We can generally define non-degenerate CR-manifolds with non-vanishing coefficient $A(x)$, $\forall x \in M$. A degenerate CR-manifold has $A(x) = 0$, $\forall x \in M$. Hence if in the form (2.9) $\frac{\partial^2 U}{\partial \zeta^2}|_{u=0} = 0$, $\forall \zeta$, the CR-manifold is degenerate. A degenerate CR-structure is equivalent to the trivial one

$$\omega_0 = du \quad , \quad \omega^1 = d\zeta \quad , \quad \overline{\omega}^1 = d\overline{\zeta}$$

(2.12)
A non-degenerate CR-structure (also called pseudoconvex) on a smooth manifold can always take the form $d\omega_0 = i\omega_1 \wedge \overline{\omega_1} (\mod \omega_0)$, which is invariant under the transformation

$$\omega_0' = |\lambda|^2 \omega_0 , \quad \omega_1' = \lambda (\omega_0 + \mu \omega_1)$$

(2.13)

with $\lambda(x) \neq 0$ and $\mu(x)$ arbitrary complex functions.

Moser used the holomorphic transformations to restrict the real function $\phi(u, \zeta, \overline{\zeta})$ to the following form up to special linear transformations.

$$\phi = \frac{1}{2} \zeta \overline{\zeta} + \sum_{k \geq 2, j \geq 2} N_{jk}(u) \zeta^j \overline{\zeta}$$

(2.14)

$$N_{22} = N_{32} = N_{33} = 0$$

The functions $N_{jk}(u)$ characterize the CR-structure. By their construction these functions belong into representations of the isotropy subgroup of $SU(1, 2)$ symmetry group of the hyperquadric.

The classical domains are usually described as regions of projective spaces. The $SU(1, 2)$ symmetric classical domain is the region of $CP^2$ determined by the relation

$$\overline{Z} C_{mn} Z^m > 0$$

(2.15)

where $Z^m$ are the homogeneous coordinates of $CP^2$ and $C_{mn}$ are $SU(1, 2)$ symmetric matrices. The matrix

$$C_B = \begin{pmatrix} 1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \end{pmatrix}$$

(2.16)

gives the bounded realization of the classical domain and the matrix

$$C_S = \begin{pmatrix} 0 & 0 & -i \\
0 & -1 & 0 \\
i & 0 & 0 \end{pmatrix}$$

(2.17)

gives the unbounded realization, which is also called Siegel domain. The boundary of the classical domain is a $SU(1, 2)$ symmetric real submanifold of $CP^2$. In the bounded realization it takes the form of $S^3$ and in the unbounded realization it takes the form of the hyperquadric. The unitary transformation of the hermitian matrices and $C_S = U^\dagger C_B U$ with

$$U = \begin{pmatrix} 1 / \sqrt{2} & 0 & \overline{1} / \sqrt{2} \\
0 & -1 & 0 \\
\overline{1} / \sqrt{2} & 0 & \overline{1} / \sqrt{2} \end{pmatrix}$$

(2.18)

implies the holomorphic transformation between the $S^3$ and the hyperquadric CR-structure coordinates.
The boundary of the classical domain is invariant under the $SU(1, 2)$. The action of the group on the boundary is transitive, because for any two points of the boundary there is a group element which transforms the one point to the other. But the group action is not effective (faithful), because there are many group elements, which transform one point to the other. The boundary is the coset space $SU(1, 2)/P$, where $P$ is the isotropy group (subgroup of $SU(1, 2)$) which preserves a point of the boundary. Hence the hyperquadric and $S^3$ may be viewed as a base manifold of a 8-dimensional bundle with the background Cartan connection of the group $SU(1, 2)$. If we define the connection with $\omega = g^{-1}dg$, where $g \in SU(1, 2)$, we find that its curvature $\Omega = d\omega + \omega \wedge \omega = 0$ vanishes.

Now it is trivial to see that the Moser form (2.14) is a deformation of the projective form

$$X^*C_5X = (1 \ z \ x)^T \begin{pmatrix} 0 & 0 & -i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z^1 \\ z^0 \end{pmatrix}$$

i.e. of the boundary of the hyperquadric, the unbounded realization of the $SU(1, 2)$ classical domain. The unbounded realization is needed because of the local translation is needed to formulate the expansion of the holomorphic transformation. This point permits the application of the Cartan method to classify the non-degenerate 3-dimensional CR-structures.

Cartan generalized the Klein geometry by osculating a general manifold with a coset space. The forms of a general non-degenerate CR-manifold are extended to the following relations\[18\]

\[
\begin{align*}
  d\omega_0 &= i\omega_1 \wedge \overline{\omega_1} - \omega_0 \wedge (\omega_2 + \overline{\omega_2}) \\
  d\omega_1 &= -\omega_1 \wedge \omega_2 - \omega_0 \wedge \omega_3 \\
  d\omega_2 &= 2i\omega_1 \wedge \overline{\omega_3} + \overline{\omega_1} \wedge \omega_3 - \omega_0 \wedge \omega_4 - R\omega_0 \wedge \overline{\omega_1} \\
  d\omega_3 &= i\omega_3 \wedge \overline{\omega_3} + \omega_4 \wedge (\omega_2 + \overline{\omega_2}) - S\omega_0 \wedge \omega_1 - \overline{S}\omega_0 \wedge \overline{\omega_1}
\end{align*}
\]

where $R$ and $S$ are the components of the curvature because for $R = S = 0$ we find the $SU(1, 2)$ structure equations. If we assume the following normalization and notation

\[
\begin{align*}
  d\omega_0 &= i\omega_1 \wedge \overline{\omega_1} + b\omega_0 \wedge \omega_1 + \overline{b}\omega_0 \wedge \overline{\omega_1} \\
  \omega_1 &= d\zeta \\
  \overline{\omega}_1 &= d\overline{\zeta}
\end{align*}
\]

we find\[18\]

\[
\begin{align*}
  R &= \frac{k(x)}{6\lambda^3} \\
  k(x) &= \overline{e} - 2b\overline{c} \\
  e &= c_1 - bc - 2ib_0 \\
  c &= b
\end{align*}
\]

where $k(x)$ depends on the coordinates of the CR-manifold. If $R = 0$ we find $S = 0$ and the CR-manifold is holomorphically equivalent to the hyperquadric. From the above form of $R \neq 0$, we see that we can always find a function $\lambda(x)$
such that $R = 1$. Hence $R$ is a relative invariant of the CR-structure, which may take the values $R = 0$ or $R = 1$. In the latter case the CR-structure may be characterized by the sections $\omega_2$, $\omega_3$, $\omega_4$ where $\omega_2$ is imaginary.

I will now apply the Cartan extension to the following CR-manifold

$$U = -2a \frac{\zeta}{1 + \zeta} \ , \ a \neq 0$$

(2.23)

Then

$$\omega_0 = du + \frac{2ia}{(1+\zeta\overline{\zeta})^2}(\zeta \, d\zeta - \zeta \, d\overline{\zeta}) \ , \ \omega_1 = d\zeta$$

(2.24)

and

$$d\omega_0 = -\frac{4ia(1-\zeta^2)}{(1 + \zeta \overline{\zeta})^3} \omega_1 \wedge \overline{\omega_1} \ , \ \omega_1 = d\zeta$$

(2.25)

We see that at the points $\zeta \overline{\zeta} = 1$ the CR-structure is degenerate. The normalized (real) 1-form $\omega_0$ is

$$\omega_0 = -\frac{(1+\zeta\overline{\zeta})^3}{4a(1-\zeta^2)} [du + \frac{2ia}{(1+\zeta\overline{\zeta})^2}(\zeta \, d\zeta - \zeta \, d\overline{\zeta})]$$

(2.26)

$$\omega_1 = d\zeta$$

which is not defined at $\zeta \overline{\zeta} = 1$. After some calculations I find $R \neq 0$. Hence I conclude that this CR-manifold is not holomorphically equivalent neither to the degenerate one nor to the hyperquadric.

Concluding the present introductory subsection I point out that the important notion is the existence of the partial complex structure in the tangent (or equivalently the cotangent) space which is inherited from the embedding in an ambient complex manifold. The differences between these two definitions are the dubbed "realizability problems", which will not concern us. But the reader should realize that the notion of the metric is not needed to define the CR-structure. On the other hand a related metric may also be defined. In the present 3-dimensional CR-manifold we can define the following class of Kähler metrics and symplectic forms

$$ds^2 = 2 \frac{\partial z^\alpha}{\partial z^\beta} \frac{\partial \overline{z}^\beta}{\partial \overline{z}^\alpha} \, dz^\alpha d\overline{z}^\beta \ , \ \omega = 2i \frac{\partial^2 \rho^2}{\partial z^\alpha \partial \overline{z}^\beta} \, dz^\alpha \wedge d\overline{z}^\beta$$

(2.27)

It is essentially a class of metrics, because the CR-structure does not uniquely determine $\rho$. Notice that $\rho' = f \rho$, with $f \neq 0$ determines the same CR-structure but it gives different ambient metrics.

### 2.2 Definition of the lorentzian CR-structure

The lorentzian CR-manifold (LCR-manifold) is defined as a 4-dimensional totally real submanifold of $\mathbb{C}^4$ determined by three special functions, with $z^b := (z^\alpha, \overline{z}^\alpha)$, $\alpha = 0, 1$.

$$\rho_{11}(\overline{z}^\alpha, z^\alpha) = 0 \ , \ \rho_{12}(\overline{z}^\alpha, z^\alpha) = 0 \ , \ \rho_{22}(\overline{z}^\alpha, z^\alpha) = 0$$

(2.28)
where \( \rho_{11}, \rho_{22} \) are real functions and \( \rho_{12} \) is a complex function. Notice the special dependence of the defining functions on the structure coordinates.

Because of \( d\rho_{ij}|M = 0 \) and the special dependence of each function on the structure coordinates \((z^\alpha, \tilde{z}^{\tilde{\alpha}})\), we find the following real 1-forms in the cotangent space of the manifold

\[
\ell := 2i\partial\rho_{11}|M = 2i\partial'\rho_{11}|M = i(\partial' - \overline{\partial'})\rho_{11}|M = -2i\overline{\partial'\rho_{11}}|M
\]

\[
n := 2i\partial\rho_{22}|M = 2i\partial'\rho_{22}|M = i(\partial'' - \overline{\partial''})\rho_{22}|M = -2i\overline{\partial''\rho_{22}}|M
\]

\[
m_1 := 2i\partial\frac{\rho_{12}}{2z_2}\rho_{22}|M = i(\partial' + \partial'' - \overline{\partial'} - \overline{\partial''})\frac{\rho_{12}}{2z_2}|M
\]

\[
m_2 := 2i\partial'\frac{\rho_{12}}{2z_2}|M = i(\partial' + \partial'' - \overline{\partial'} - \overline{\partial''})\frac{\rho_{12}}{2z_2}|M
\]

where the accented symbols of the partial exterior derivatives are defined as follows

\[
d = \partial + \overline{\partial} = (\partial' + \partial'') + (\overline{\partial'} + \overline{\partial''})
\]

\[
\partial' f := \frac{\partial f}{\partial z^\alpha} dz^\alpha , \quad \partial'' f := \frac{\partial f}{\partial \tilde{z}^{\tilde{\alpha}}} d\tilde{z}^{\tilde{\alpha}}
\]

\[
A_\mu dx^\mu := A'_\alpha dz^\alpha + A''_\tilde{\alpha} d\tilde{z}^{\tilde{\alpha}}
\]

In order to familiarize the reader with this new formalism we make the transcription \( A \rightarrow A' + A'' \) in details

\[
A_\mu dx^\mu = A_\mu \delta_\nu^\sigma dx_\nu = A_\mu (\ell^\nu n_\nu + n^\nu \ell_\nu - \overline{m^\nu} m_\nu - m^\nu \overline{m}_\nu) dx_\nu =
\]

\[
= [(n^\mu A_\mu) \ell_\alpha - (\overline{m^\mu} A_\mu) m_\alpha] dz^\alpha + [(\ell^\mu A_\mu) n_{\tilde{\alpha}} - (m^\mu A_\mu) \overline{m}_{\tilde{\alpha}}] d\tilde{z}^{\tilde{\alpha}} =
\]

\[
A'_\alpha dz^\alpha + A''_{\tilde{\alpha}} d\tilde{z}^{\tilde{\alpha}}
\]

where the induced symbols are neglected.

The 1-forms are real, because we consider them restricted on the defined submanifold. The relations become simpler, if we use the complex form

\[
m = m_1 + im_2 = 2i\partial'\rho_{12} = -2i\overline{\partial''\rho_{12}} = i(\partial' - \overline{\partial''})\rho_{12}
\]

We see that a proper LCR-manifold is characterized by a pair of 3-dimensional CR-submanifolds with a common complex tangent (and cotangent) vector \( m \).

The corresponding tangent basis with the real vectors \( \ell'^\mu \partial_\mu, n'^\nu \partial_\nu \) and the complex one \( m'^\mu \partial_\mu \) is defined via the contractions

\[
(\ell'^\mu \partial_\mu)_+(n_\nu dx_\nu) = 1 , \quad (n'^\nu \partial_\nu)_+(\ell_\nu dx_\nu) = 1
\]

\[
(m'^\mu \partial_\mu)_+ (\overline{m}_\nu dx_\nu) = -1 , \quad (\overline{m'^\mu} \partial_\mu)_+(m_\nu dx_\nu) = 1
\]

\textit{all the other vanish}
An abstract LCR-manifold is defined by two real \( \ell, n \), and a complex \( m \) 1-forms, such that \( \ell \wedge n \wedge m \wedge \overline{m} \neq 0 \) and

\[
d\ell = Z_1 \wedge \ell + i\Phi_1 m \wedge \overline{m}
\]
\[
dn = Z_2 \wedge n + i\Phi_2 m \wedge \overline{m}
\]
\[
dm = Z_3 \wedge m + \Phi_3 \ell \wedge n
\]

where the vector fields \( Z_1, Z_2 \) are real, the vector field \( Z_3 \) is complex, the scalar fields \( \Phi_1, \Phi_2 \) are real and the scalar field \( \Phi_3 \) is complex. Using the relations (2.33) one can prove that the above definition of the LCR-structure in the cotangent space is equivalent to the following commutation relations

\[
[(\ell'^\mu \partial_\mu), (m'^\mu \partial_\mu)] = f_0(\ell'^\mu \partial_\mu) + f_1(m'^\mu \partial_\mu)
\]
\[
[(n'^\mu \partial_\mu), (\overline{m}'^\mu \partial_\mu)] = h_0(n'^\mu \partial_\mu) + h_1(\overline{m}'^\mu \partial_\mu)
\]

of the corresponding tangent directional derivatives. That is, the LCR-structure conditions constitute an integrable system where the application of the (holomorphic) Frobenius theorem in the cotangent and the tangent space implies the embedding relations (2.28), which will be explicitly proved below.

The above conditions are invariant under the transformation

\[
\ell'_\mu = \Lambda \ell_\mu, \quad \ell'^\mu = \frac{1}{N} \ell'^\mu
\]
\[
n'_\mu = NN_\mu, \quad n'^\mu = \frac{1}{N} n'^\mu
\]
\[
m'_\mu = MM_\mu, \quad m'^\mu = \frac{1}{M} m'^\mu
\]

with non-vanishing \( \Lambda, N, M \) functions and which imply the following transformations of the vector and scalar fields

\[
Z'_1 = Z_1 + \partial_\mu \ln \Lambda, \quad Z'_2 = Z_2 + \partial_\mu \ln N
\]
\[
Z'_3 = Z_3 + \partial_\mu \ln M
\]

Hence \( \Phi_1, \Phi_2, \Phi_3 \) are LCR-structure relative invariants and the differential forms

\[
F_1 = dZ_1, \quad F_2 = dZ_2, \quad F_3 = dZ_3
\]

are LCR-invariants. These relative invariants act as topological invariants characterizing the different sectors of the LCR-structure solitons. Hence a LCR-structure defines a class of LCR-tetrads, which will imply a class of Einstein metrics.
Notice that this definition permits us to apply the holomorphic Frobenius theorem for \((\ell, m)\) and \((n, \overline{m})\), which is always possible, if the tetrad 1-forms are real analytic functions. For that, we have to complexify the coordinates \(x^\mu\), considering the basis covectors real analytic. This theorem implies that there are two sets of generally complex coordinates \((z^\alpha(x), z^\tilde{\alpha}(x))\), \(\alpha = 0, 1\) such that

\[
dz^\alpha = f_\alpha \ell_\mu dx^\mu + h_\alpha m_\mu dx^\mu,
\]

\[
dz^0 \wedge dz^1 \wedge d\overline{z}^0 \wedge d\overline{z}^1 \neq 0
\]

(2.39)

\[
\ell = \ell_\alpha dz^\alpha, \quad m = m_\alpha dz^\alpha, \quad n = n_\tilde{\alpha} d\overline{z}^\tilde{\alpha}, \quad \overline{m} = \overline{m}_\tilde{\alpha} d\overline{z}^\tilde{\alpha}
\]

After their computation we make \(x^\mu\) real again, but real analyticity condition assures that the matrices \(\partial_\mu z^b(x)\) and its inverse \(\partial^b x_\mu(z^a)\) do not vanish on the LCR-manifold. The reality conditions of the tetrad imply that these structure coordinates satisfy the relations

\[
dz^0 \wedge dz^1 \wedge d\overline{z}^0 \wedge d\overline{z}^1 = 0
\]

\[
dz^0 \wedge dz^\tilde{1} \wedge d\overline{z}^0 \wedge d\overline{z}^1 = 0
\]

(2.40)

\[
dz^0 \wedge dz^\tilde{1} \wedge d\overline{z}^0 \wedge d\overline{z}^\tilde{1} = 0
\]

that is, there are two real functions \(\rho_{11}\), \(\rho_{22}\) and a complex one \(\rho_{12}\), such that the abstract LCR-structure is realized in \(\mathbb{C}^4\) via the totally real surface (2.28). Hence in the case of real analytic differentiable manifolds, the abstract LCR-structure is embeddable in \(\mathbb{C}^4\) through real functions of special form and the two definitions of LCR-structure coincide.

### 2.3 A \(G(4,2)\) embedding of the LCR-structures

The permitted (restricted) holomorphic transformations \(z'^\alpha = f^\alpha(z^\beta)\), \(z'^{\tilde{\alpha}} = f^{\tilde{\alpha}}(z^\tilde{\beta})\) may be used [2] to find structure coordinates (called regular coordinates) such that (2.28) take the forms

\[
\rho_{11}(z, z') = \text{Im} \ z^0 - \phi_{11}(z^1, z^\tilde{1}, \text{Re} \ z^0)
\]

\[
\rho_{12}(z, z') = z^\tilde{1} - \overline{z}^1 - \phi_{12}(z^1, z^\tilde{1}, \text{Re} \ z^0)
\]

\[
\rho_{22}(z, z') = \text{Im} \ z^\tilde{0} - \phi_{22}(z^1, z^\tilde{1}, \text{Re} \ z^0)
\]

\[
\phi_{ij}(0) = 0, \quad d\phi_{ij}(0) = 0
\]

(2.41)

where \(z^1, z^\tilde{1}\), are the complex coordinates of \(CP^1\), because this regular form of the LCR-structure continues to permit the following \(SL(2, \mathbb{C})\) transformation

\[
z'^{\alpha} = \frac{z^\alpha + z^\tilde{\alpha}}{\alpha + \overline{\beta}}^2 \quad , \quad z'^{\tilde{\alpha}} = \frac{z^\mu + z^\tilde{\alpha}}{\alpha + \overline{\beta}}^2
\]

\[
ad - bc = 1
\]

(2.42)
That is, the corresponding spinors transform relative to the conjugate representations of $SL(2, \mathbb{C})$

\[
\begin{pmatrix}
\lambda'
\lambda' z'^1
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\lambda
\lambda z^1
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\frac{\lambda' z'^i}{\lambda'}
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1}
\begin{pmatrix}
-\frac{\lambda z^i}{\lambda}
\end{pmatrix}
\]

\[ad - bc = 1\] (2.43)

We saw that the non-degenerate realizable 3-dimensional CR-structures can be osculated with the boundary of the $SU(1, 2)$ classical domain. In the present case of the LCR-structure the convenient 4-dimensional symmetric space is the boundary of the $SU(2, 2)$ classical domain. For that we first projectivize (2.28) to the form

\[
\rho_{11}(Z^{m1}, Z^{n1}) = 0 \quad , \quad \rho_{12}(Z^{m1}, Z^{n2}) = 0 \quad , \quad \rho_{22}(Z^{m2}, Z^{n2}) = 0
\]

\[K(Z^{m1}) = 0 = K(Z^{m2})\] (2.44)

where $Z^{ni} \in CP^3$ and $K(Z^n)$ is a homogeneous function in $\mathbb{C}^4$. We will call it Kerr function because it is related to the Kerr theorem in Minkowski space.

The grassmannian projective manifold $G(4, 2)$ is the set of the $4 \times 2$ complex matrices

\[
X = \begin{pmatrix}
X^{01} & X^{02} \\
X^{11} & X^{12} \\
X^{21} & X^{22} \\
X^{31} & X^{32}
\end{pmatrix}
\] (2.45)

of rank-2 with the equivalence relation $X \sim X'$ if there exists a $2 \times 2$ regular ($\det S \neq 0$) matrix $S$ such that

\[X' = XS\] (2.46)

Its typical coordinates are the projective coordinates, which are defined in every coordinate chart determined by every $2 \times 2$ submatrix with non-vanishing determinant. In the coordinate chart with

\[
\det \begin{pmatrix}
X^{01} & X^{02} \\
X^{11} & X^{12}
\end{pmatrix} \neq 0
\] (2.47)

the projective coordinates $\hat{z}$ are defined by the relation

\[
X = \begin{pmatrix}
X_1 \\
\hat{z}X_1
\end{pmatrix}
\] (2.48)
The projective form (2.44) is projectively invariant under the general linear $SL(4, \mathbb{C})$ transformations. But in order to apply the Cartan expansion we need the precise form of the $SU(2, 2)$ classical domain.

The $SU(2, 2)$ symmetric classical domain is the following region of $G(4, 2)$

$$X^\dagger E X > 0 \quad (2.49)$$

which means that the $2 \times 2$ matrix is positive definite. $E$ is a $SU(2, 2)$ symmetric $4 \times 4$ matrix. The bounded realization of the classical domain is achieved with the matrix

$$E_B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (2.50)$$

and it is

$$\begin{pmatrix} Y_1^\dagger & Y_2^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} > 0 \iff I - \hat{w}^\dagger \hat{w} > 0 \quad (2.51)$$

where $\hat{w} \equiv w_{ij}$ is the symbol we will use for the projective coordinates in the bounded realization. It is invariant under the $SU(2, 2)$ transformations which take the following explicit form

$$\begin{pmatrix} Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$\hat{w}' = (A_{21} + A_{22} \hat{w}) (A_{11} + A_{12} \hat{w})^{-1} \quad (2.52)$$

$$A_{11} A_{11} - A_{21} A_{21} = I \quad , \quad A_{11} A_{12} - A_{21} A_{22} = 0$$

$$A_{12} A_{22} - A_{12} A_{12} = I$$

The characteristic (Shilov) boundary of this domain is the $S^1 \times S^3 [\approx U(2)]$ manifold with $\hat{w}^\dagger \hat{w} = I$.

The unbounded (Siegel) realization of the classical domain is determined with the matrix

$$E_U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (2.53)$$

and it has the form

$$\begin{pmatrix} X_1^\dagger & X_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} > 0 \iff \frac{-1}{2} (\hat{r} - \hat{r}^\dagger) = \hat{g} > 0 \quad (2.54)$$

$$\hat{r} = \hat{x} + i \hat{y} = i X_2 X_1^{-1}$$

where the projective coordinates in the Siegel realization $\hat{r} \equiv r_{A'A}$ are defined with an additional factor $i$ for convenience. The fractional transformations,
which preserve the unbounded domain, are

\[
\begin{pmatrix}
X'_1 \\
X'_2
\end{pmatrix} = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix} \begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
\]

\[\hat{r}' = (B_{22} \hat{r} + iB_{21})(B_{11} - iB_{12})^{-1}\] (2.55)

\[B_{11}^\dagger B_{22} + B_{21}^\dagger B_{12} = I, \quad B_{11}^\dagger B_{21} + B_{12}^\dagger B_{11} = 0\]

\[B_{22}^\dagger B_{12} + B_{12}^\dagger B_{22} = 0\]

Notice that the linear part of these transformations \((B_{12} = 0)\), which preserves the infinity of the Siegel domain, are the Poincaré×Dilation transformations.

The unitary transformation

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
I & I \\
I & -I
\end{pmatrix} \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
I & I \\
I & -I
\end{pmatrix}
\] (2.56)

implies the following Cayley transformation of the projective coordinates

\[\hat{r} = i(I - \hat{w})(I + \hat{w})^{-1} = i(I + \hat{w})^{-1}(I - \hat{w})\]

\[\hat{w} = (iI - \hat{r})(iI + \hat{r})^{-1} = (iI + \hat{r})^{-1}(iI - \hat{r})\] (2.57)

Restricted on the boundary it becomes \(U(2) \to \mathbb{R}^4\), which is not bijective. The \(SU(2)\) group manifold implied by exponentiation of its Lie algebra is

\[SU(2) \ni U_0 = e^{i\psi} \begin{pmatrix}
\frac{\psi}{2} \\
\frac{\psi}{2}
\end{pmatrix} = \cos \frac{\psi}{2} + i\hat{\psi}_j \sigma^j \sin \frac{\psi}{2}, \quad \hat{\psi}_j =: \psi \hat{w}_j, \quad \psi = \sqrt{\sum_{j=1}^{3} (\psi_j)^2}\]

\[U_0(\hat{\psi}_j, \psi + 4\pi) = U_0(\hat{\psi}_j, \psi) = -U_0(\hat{\psi}_j, \psi + 2\pi)\] (2.58)

where \(\hat{\psi}_j\) is the unit vector. Its most convenient image is the unit ball with radius \(2\pi\), center at \(U_0 = I\) and its surface identified with the point \(U_0 = -I\). When \(\psi =: 2\rho\) is in this domain, the cartesian coordinates of the \(w = I\) chart

\[\hat{x}_+ = i(I - \hat{w})(I + \hat{w})^{-1} = i(I + \hat{w})^{-1}(I - \hat{w})\]

\[\hat{w}^\dagger = \hat{w}^{-1}\] (2.59)

is found assuming

\[\hat{\psi}_j = (-\sin \sigma \cos \chi, -\sin \sigma \sin \chi, \cos \sigma)\]

\[\hat{w} = e^{i\tau} \begin{pmatrix}
\cos \rho + i \sin \rho \cos \sigma & -i \sin \rho \sin \sigma e^{-i\chi} \\
-i \sin \rho \sin \sigma e^{i\chi} & \cos \rho - i \sin \rho \cos \sigma
\end{pmatrix}\]

\[\tau \in (-\pi, \pi), \quad \rho \in (0, \pi), \quad \sigma \in [0, \pi), \quad \chi \in (0, 2\pi)\] (2.60)
It has the form

\[
\begin{align*}
  x_0^+ &= \frac{\sin \tau}{\cos \tau + \cos \rho} \\
  x_1^+ + i x_2^+ &= \frac{\sin \rho}{\cos \tau + \cos \rho} \sin \sigma e^{i\chi} \\
  x_3^+ &= \frac{\sin \rho}{\cos \tau + \cos \rho} \cos \sigma \\
  \tau &\in (-\pi, \pi), \quad \rho \in [0, \pi), \quad \sigma \in [0, \pi), \quad \chi \in (0, 2\pi)
\end{align*}
\]

(2.61)

which describes the one \(\mathbb{R}^4\)-chart around the point \(w = I\).

One may view the rest of \(U(2)\) as the Cayley transformation centered at \(w = -I\). But I find more convenient the extension of the above parameter \(s\) to negative values to correspond to the rest of \(U(2)\), the region \(\sin \rho \cos \tau + \cos \rho < 0\) to be the other sheet of \(\mathbb{R}^4\) with

\[
\begin{align*}
  x_0^+ &= \frac{\sin \tau}{\cos \tau + \cos \rho} \\
  x_1^+ + i x_2^+ &= -\frac{\sin \rho}{\cos \tau + \cos \rho} \sin \sigma e^{i\chi} \\
  x_3^+ &= -\frac{\sin \rho}{\cos \tau + \cos \rho} \cos \sigma \\
  \tau &\in (-\pi, \pi), \quad \rho \in [0, \pi), \quad \sigma \in [0, \pi), \quad \chi \in (0, 2\pi)
\end{align*}
\]

(2.62)

\[s := \frac{\sin \rho}{\cos \tau + \cos \rho} < 0 \iff \cos \tau + \cos \rho < 0\]

Hence we conclude that the unbounded realization shows only the one sheet of the universe. These two cartesian sheets do not overlap, therefore they do not constitute an atlas of the universe \(U(2)\). Recall that for the \(SU(1,1)\) classical domain, the correspondence between the circle of the disc and the real line is bijective

\[w = e^{i\varphi}, \quad \varphi \in (-\pi, \pi) \leftrightarrow x = \tan \frac{\varphi}{2} \in \mathbb{R}\]

(2.63)

All the circle (boundary) is covered by one real line \(\mathbb{R}\).

The Shilov boundary of the \(SU(2,2)\) symmetric classical domain is

\[
\rho_{ij}(X^m, X^n) = X^m E_{mn}^U X^n = 0
\]

\[K(X^m) = 0, \quad E_{mn}^U := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\]

(2.64)

in the unbounded realization. Hence, for these LCR-structures, which we call "algebraically flat", we have real projective coordinates \(\widehat{r} = \widehat{x} = \widehat{z}^1\), and the LCR-structure is determined only by the homogeneous holomorphic function \(K(X^m)\). That is, the points of the Shilov boundary take the representation

\[
X^m = \begin{pmatrix} \lambda \\ -i \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \lambda^A_j \\ -iz_{A'B}^B \lambda^B_j \end{pmatrix}
\]

\[
\widehat{x} = \eta_{\mu\nu} x^\mu \sigma^\nu = \begin{pmatrix} x^0 - x^3 & -(x^1 + ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} = x_{A'B}
\]

(2.65)
in homogeneous coordinates. The Poincaré×dilation transformations preserve
the above infinity
\[
\begin{pmatrix}
\lambda' \\
-i\hat{x}'\lambda'
\end{pmatrix}
= \begin{pmatrix}
B & 0 \\
-iTB & (B^\dagger)^{-1}
\end{pmatrix}
\begin{pmatrix}
\lambda \\
-i\hat{x}\lambda
\end{pmatrix}
\]
(2.66)
\[
\lambda' = B\lambda , \quad \hat{x}' = (B^{-1})^\dagger \hat{x}B^{-1} + T
\]
\[
\det B = \det B , \quad T^\dagger = T
\]
It is the spinorial Poincaré×dilation group. Its Poincaré subgroup is identified
with the Poincaré symmetry observed in nature, but there is an essential difference
between the present symmetry and that implied by the metric of general
relativity. The present group is the proper orthochronous group of special rel-
avity (Minkowski space). The spatial and temporal reflections are external automorphisms
of its Lie algebra, i.e. they do not exponentiate into elements
of the group (2.66). Therefore they may be broken in PCFT as it has been
observed in nature.

In order to osculate the general LCR-structure relations with the "alge-
braically flat" LCR-structure conditions, I write
\[
\rho_{ij}(\overline{X}^{mi}, X^{nj}) = \overline{X}^{mi}X^{nj}E_{mn} - G_{ij}(\overline{X}^{mi}, X^{nj}) = 0
\]
\[
K(X^{mj}) = 0
\]
(2.67)
where \(G_{ij}(\overline{X}^{mi}, X^{nj})\) is viewed as the non-flat part of the LCR-structure. Notice
that the Kerr homogeneous function persists even in the deformed boundary.

### 2.4 LCR-structures determined with complex trajectories

Newman introduced\[20\] complex trajectories to describe geodetic and shear free
null congruences in Minkowski space. Besides, it is well known that the Lienard-
Wiechert radiating electromagnetic potentials are related to accelerating traject-
ories. In the present formalism and in the case of embeddable LCR-structures
(2.64) the ruled surfaces of \(CP(3)\) are characterized by complex trajectories
\(\xi(\tau)\) in the corresponding grassmannian \(G(4,2)\). The ruled surfaces have a
special explicit parameterization, which reveals their internal property to be
made up of lines, which determine a generally complex trajectory in the grass-
mannian space. That is, they have the form
\[
Z^m(\tau, s) = (1 - s)Z^{m1}(\tau) + sZ^{m2}(\tau) = Z^{m1}(\tau) + sT^m(\tau)
\]
(2.68)
\[
T^m(\tau) := Z^{m2}(\tau) - Z^{m1}(\tau)
\]
where \(T^m(\tau)\) indicates the direction of the generating line which meets \(Z^{m1}(\tau)\)
(the generatrix) at \(\tau\). In this definition the Kerr function is replaced with its
proper parametrization.
The generating lines correspond to complex points of the grassmannian manifold \( G(4, 2) \), with projective coordinates
\[
\hat{\xi}(\tau) =: iX_2X_1^{-1} =: \begin{pmatrix}
\xi^0 - \xi^3 & -\xi^1 - i\xi^2 \\
-(\xi^1 + i\xi^2) & \xi^0 + \xi^3
\end{pmatrix}
\]
\[
X_1 =: \begin{pmatrix}
X^{01} & X^{02} \\
X^{11} & X^{12}
\end{pmatrix} = \begin{pmatrix}
Z^0(\tau, 0) & Z^0(\tau, 1) \\
Z^0(\tau, 0) & Z^1(\tau, 1)
\end{pmatrix}
\]
\[
X_2 =: \begin{pmatrix}
X^{21} & X^{22} \\
X^{31} & X^{32}
\end{pmatrix} = \begin{pmatrix}
Z^2(\tau, 0) & Z^2(\tau, 1) \\
Z^3(\tau, 0) & Z^3(\tau, 1)
\end{pmatrix}
\]

Using homogeneous coordinates, this curve of \( G(4, 2) \) is spanned by the two points \( Z^{ai}(\tau) \); \( i = 1, 2 \) of \( \mathbb{C}^4 \). The curve is called non-degenerate if the following determinant does not identically vanish
\[
\det[Z^{a1}, Z^{a2}, \frac{dZ^{a1}}{d\tau}, \frac{dZ^{a2}}{d\tau}] = \det \begin{pmatrix}
X_1 & \dot{X}_1 \\
-\dot{\hat{\xi}}X_1 & -i(\dot{\hat{\xi}}X_1 + \dot{\hat{\xi}}X_1)
\end{pmatrix} = \det(\xi)(\det X_1)^2
\]

This happens if and only if \( \xi A (\xi, \eta_{ab} \neq 0) \), because \( \det X_1 \neq 0 \) in this projective coordinate chart. This condition will differentiate the massive from the massless partner (neutrino) of a leptonic generation. The complex trajectory is related to the ordinary classical trajectory of the particle viewed as a soliton. If they are real, they are identified with the well known trajectories of the Lienard-Wiechert potential. If the curve is degenerate \( (\xi A (\eta_{ab} = 0)) \), the gaussian curvature of the ruled surface vanishes and the ruled surface is called developable. The developable surfaces of \( CP(3) \) are cones, cylinders and tangent developables with \( T^a(\tau) = \frac{dZ^a(\tau)}{d\tau} \).

In the context of special relativity, Newman showed that a complex trajectory in complex Minkowski spacetime defines a geodetic and shear free null congruence. A quite general Poincaré covariant explicit parameterization of a ruled hypersurface of \( CP(3) \) and its corresponding grassmannian patch, is
\[
X^{ai} = \begin{pmatrix}
Z^0(\tau, s) & Z^0(\tilde{\tau}, \tilde{s}) \\
Z^1(\tau, s) & Z^1(\tilde{\tau}, \tilde{s}) \\
Z^2(\tau, s) & Z^2(\tilde{\tau}, \tilde{s}) \\
Z^3(\tau, s) & Z^3(\tilde{\tau}, \tilde{s})
\end{pmatrix} = \begin{pmatrix}
\lambda^A_i \\
-\dot{\eta}_{B'i} \lambda^{B_i}
\end{pmatrix}
\]

where the \( r_{B'B} = r_0 \sigma_{B'B}^b \) are the projective coordinates, generally outside the \( \xi_b(\tau) \) trajectory of the ruled surface. Because simply not all the pairs of points of a ruled surface belong to rulings. If \( r_b \in \xi_b(\tau) \) the projective line of \( CP(3) \) coincides with a ruling line of the ruled surface.

The reparametrization \( (\tau) \) ambiguity may be fixed with either the condition \( \xi^0(\tau) = \tau \) or the more restrictive one \( \xi^0(\tau) = \tau \). In the coordinate chart
$Z^0 = 1$ of $CP(3)$, a general point of the ruled surface determined by a trajectory $\xi^b(\tau)$ has the form

$$Z^n(\tau, \lambda) = \begin{pmatrix} 1 \\ 0 \\ -i(\xi^0 - \xi^3) \\ i(\xi^1 + i\xi^2) \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ i(\xi^1 - i\xi^2) \\ -i(\xi^0 + \xi^3) \end{pmatrix}$$

(2.72)

$$\lambda = \frac{(1-s)\lambda^{11}(\tau) + s\lambda^{12}(\tau)}{(1-s)\lambda^{00}(\tau) + s\lambda^{02}(\tau)}$$

The first term is the directrix curve of the ruled surface and the second is the generating line (ruling) of the surface. We will see that the linear trajectory $\xi^b(\tau) = v^b\tau + d^b$ with $v^a v^b \eta_{ab} = 1$ corresponds to the "free" electron and with $v^a v^b \eta_{ab} = 0$ corresponds to its neutrino.

A general line ($\hat{r} \in G(4, 2)$) of $CP(3)$, generally intersects $d$ times the ruled surface and $d$ coincides with the algebraic degree of the ruled surface. Two of these intersection points $(X^{n1}(\tau_1, s_1), X^{n2}(\tau_2, s_2))$ determine the line and subsequently

$$X^{ni} = \begin{pmatrix} \lambda^{A1}(\tau_1, s_1) \\ -i\xi_{B' B}(\tau_1) \lambda^{B1} \\ \lambda^{A2}(\tau_2, s_2) \\ -i\xi_{B' B}(\tau_2) \lambda^{B2} \end{pmatrix} = \begin{pmatrix} \lambda^{Ai} \\ -ir_{A'B}\lambda^{Bi} \end{pmatrix}$$

(2.73)

$$\lambda^{Ai}(\tau_i, s_i) = \begin{pmatrix} \lambda^{o i}(\tau_i, s_i) \\ \lambda^{i i}(\tau_i, s_i) \end{pmatrix}, \quad i = 1, 2$$

Using spinorial coordinates, the above relation takes the form

$$(r_{A'B} - \xi_{A'B}(\tau_j))\lambda^{Bj} = 0$$

(2.74)

which are two homogeneous linear equations for every $j = 1, 2$. They admit a (projectively) non-vanishing solution $\lambda^{Bj}$ for every $j$, if

$$\det(r_{A'B} - \xi_{A'B}(\tau)) = \det(\hat{r} - \xi(\tau)) = (r^a - \xi^a)(v^b - \xi^b)\eta_{ab} = 0$$

(2.75)

Every generally complex solution $\tau(r_{A'B})$ of this equation is replaced back into (2.73) and find the corresponding spinor $\lambda^A$. For every column of $X^{ni}$ (point of $CP(3)$) we get a pair of generally complex functions

$$z^0(r) = \tau_1(r), \quad z^j(r) = \frac{\lambda^{A_j}(\tau_j(r))}{\lambda^{00}(\tau_j(r))}$$

(2.76)

$$z^0(r) = \tau_2(r), \quad z^j(r) = -\frac{\lambda^{A_j}(\tau_j(r))}{\lambda^{00}(\tau_2(r))}$$

which may be assumed as the structure coordinates in the ambient complex manifold of the LCR-structure. Notice that $z^b(r^a)$ are generally holomorphic functions. After the projection to the real LCR-submanifold, they become proper structure coordinates.

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The reader must not confuse the set of "straight" lines of $CP(3)$, which are all the points $r_{A'A}$ of the Grassmannian manifold $G(4,2)$, with the rulings of the ruled surface, i.e. the "straight" lines which belong (as sets of points) to the ruled hypersurface of $CP(3)$, and which are just the points of the complex trajectory $\xi_{\lambda'\lambda}(\tau)$ in $G(4,2)$.

A geometric visualization of the above mathematical procedure is the following. A point of the Grassmannian manifold with projective coordinates $\hat{r}$ determines a line of $CP(3)$. This line intersects the hypersurface of $CP(3)$ at a number of points (equal to the polynomial degree of the surface), which belong to different sheets of the surface of $CP(3)$. Every pair of intersection points with homogeneous coordinates $X^{a1}$ may be taken as the corresponding homogeneous coordinates of the Grassmannian point $\hat{r}$. Hence every point $\hat{r} \in G(4,2)$, determines (and is determined by) two points $\xi^b(\tau_1)$ and $\xi^b(\tau_2)$ of the complex trajectory $\xi^b(\tau)$ with two corresponding spinors $\lambda^{A1}(\tau_1, s_1)$ and $\lambda^{A2}(\tau_2, s_2)$. Notice that the trajectory $\xi^b(\tau)$ determines the algebraic subcurve of $CP(3)$, where the two sheets intersect, $\lambda^A(\tau_1) = \lambda^A(\tau_2)$. This is valid for any point $\hat{r}$ of the ambient complex manifold. If the point belongs in the real LCR-submanifold, i.e. if $r^a = x^a + iy^a(x)$, the "observer" at the point $x^a$, has the local null system

$$L^a = \frac{1}{\sqrt{2}}\sigma^a_{AA'}\bar{\lambda}^{A'}\lambda^A, \quad N^a = \frac{1}{\sqrt{2}}\sigma^a_{AA'}\bar{\lambda}^{A'}\lambda^{A2}, \quad M^a = \frac{1}{\sqrt{2}}\sigma^a_{AA'}\bar{\lambda}^{A'}\lambda^{A1}$$

$$\epsilon_{AB}\lambda^{A1}\lambda^{B2} = 1$$

and conditions (2.67) are formally "solved" by

$$y^a = \frac{1}{2\sqrt{2}}[G_{22}N^a + G_{11}L^a - G_{12}M^a - \bar{G}_{12}\bar{M}^a] \quad (2.78)$$

The computation of $\lambda^A$ as functions of $r^a$ using the Kerr condition $K(X^{m1})$, permits us to perturbatively compute $y^a$ as functions of the real part of $r^a$. This procedure gives the canonical form $y^a = h^a(x)$ of the (totally real) Lorentzian CR submanifold expressed in the projective coordinates of $G(4,2)$.

The explicit form of $h^a(x)$ is perturbatively implied by the precise dependence of $G_{ij}(X^{m1}, X^{m2})$ from $X^{m2}$. If we choose the $\xi^0(\tau) = \tau$ normalization, we find

$$z^0(x) = r^0(x) - \sqrt{(r^1(x) - \xi^1(z^0))^2} \quad , \quad z^1(x) = \frac{z^0 + \sqrt{r^1(x) - r^1(z^0)} - \xi^1(z^0)}{\sqrt{r^0 - r^0(z^0) - \xi^0(z^0)}}$$

$$\bar{z}^0(x) = r^0(x) + \sqrt{(r^1(x) - \xi^1(z^0))^2} \quad , \quad \bar{z}^1(x) = \frac{\bar{z}^0 + \sqrt{r^1(x) - r^1(z^0)} + \xi^1(z^0)}{\sqrt{r^0 - r^0(z^0) + \xi^0(z^0)}} \quad (2.79)$$

Hence, the left column of $X^{a1}$ provides the retarded coordinates $z^a(x)$ and the right column $X^{a2}$ provides the advanced coordinates $\bar{z}^a(x)$. A (curved) LCR-tetrad is found as usual by simply taking the differential forms of the structure coordinates and using their reality conditions.
One can easily see that in the zero gravity approximation \( y^a(x) = 0 \), the structure coordinates (and the null tetrad) are completely determined by the generally complex trajectory as we should expect from Kerr’s theorem (in Minkowski spacetime). In the first \( \frac{1}{c} \) approximation, the LCR-structure coordinates take the form

\[
\begin{align*}
z_0(x) &\simeq x^0 - \frac{1}{c} \sqrt{(x^i - \xi^i(x^0))^2}, & z^1(x) &\simeq \frac{x^1 + ix^2 - \xi^1(x^0) - i\xi^2(x^0)}{x^{02} + x^1 - \xi^0(x^0) - \xi^1(x^0)}, \\
z_\tilde{0}(x) &\simeq x^0 + \frac{1}{c} \sqrt{(x^i - \xi^i(x^0))^2}, & z^\tilde{1}(x) &\simeq \frac{x^1 - ix^2 - \xi^1(x^0) + i\xi^2(x^0)}{x^{02} - x^1 - \xi^0(x^0) + \xi^1(x^0)} 
\end{align*}
\]

(2.80)

where the (dimensional) light velocity factor is made apparent in order to reveal the newtonian approximation.

The points of the trajectory \( \xi^i(x^0) \) are the singularities of the structure coordinates. If the trajectory is real, the singularity is just a curve in LCR-manifold. But if the trajectory is complex \( \xi^j(x^0) = \xi^j_R(x^0) + i\xi^j_I(x^0) \), the singularity is the surface

\[
\sum_j (x^j - \xi^j_R(x^0))^2 = \sum_j [(x^j - \xi^j_R(x^0))^2 - (\xi^j_I(x^0))^2] = 0
\]

(2.81)

This is the well-known ring-like singularity of Kerr-type metrics in general relativity. The imaginary part of the trajectory is related to the spin of the LCR-structure. It is exactly this imaginary part that generates the fermionic gyromagnetic ratio of the Kerr-Newman spacetime. Hence the complex trajectory \( \xi^a(\tau) \) is the singular curve where two sheets of the ruled surface of \( CP(3) \) intersect.

### 3 EMERGENCE OF EINSTEIN’S METRICS

The LCR-tetrad \( (\ell, m; n, \overline{m}) \) may be identified with a geodetic and shear-free null tetrad\([12]\) of general relativity, which is defined with the condition \( \kappa = 0 = \lambda = \nu \) in the Newman-Penrose formalism. But the LCR-tetrad is not uniquely determined by the LCR-structure. The tetrad-Weyl symmetry \([2.36]\) implies that we may only define a class of metrics and the corresponding self-dual 2-forms

\[
\begin{align*}
[g_{\mu\nu}] &= \ell_\mu n_\nu + \ell_\nu n_\mu - m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu \\
[V_1] &= \ell \wedge m, & [V_2] &= n \wedge \overline{m} \\
[V_3] &= \ell \wedge n - m \wedge \overline{m}
\end{align*}
\]

(3.1)

Notice that only metrics compatible with two geodetic and shear-free null congruences can be defined in PCFT. The other metrics cannot have any physical
meaning. The observed in nature Kerr type metrics admit two geodetic and shear-free null congruences and are defined through \( \mathbf{m} \). I want to point out that the additional Einstein’s equations will be derived by the BEGS procedure, which constitutes the QFT emergence in the context of PCFT and will be described in section VII.

Recall that the Newman-Penrose formalism is essentially the Cartan moving frame formalism adapted to a null tetrad, not necessarily geodetic and shear-free. Its fundamental connection symbols are

\[
\begin{align*}
\alpha &= \left( (\ell n) + (\ell m) - (n m) - 2(\ell m) \right) \\
\beta &= \left( (\ell m) + (n m) - (n m) + 2(\ell m) \right) \\
\gamma &= \left( (\ell m) - (\ell m) - (n m) + 2(\ell n) \right) \\
\varepsilon &= \left( (\ell n) - (\ell m) + (n n) + 2(\ell n) \right) \\
\mu &= \left( (\ell m) + (\ell m) + (n n) + 2(\ell n) \right) \\
\pi &= \left( (\ell m) - (n m) - (\ell m) \right) \\
\rho &= \left( (\ell m) - (n m) - (\ell m) \right) \\
\tau &= \left( (\ell m) + (\ell m) + (n n) + 2(\ell n) \right) \\
\kappa &= (\ell m), \quad \sigma = (\ell m) \\
\nu &= -(n m) , \quad \lambda = -(n m) \\
\end{align*}
\]

(3.2)

where the symbols \( (\ldots) \) are constructed according to the rule of the following example \((\ell n) = (\ell n) \), without using a metric or connection. In general relativity, where the metric is the fundamental quantity, the symmetry is not the tetrad-Weyl transformations but the well known internal tetrad-Lorentz transformations, which generally do not conserve the LCR-structure conditions \( \kappa = \sigma = 0 = \lambda = \nu \).

Hence the Newman-Penrose formalism is easily incorporated in PCFT, by simply imposing the conditions \( \kappa = \sigma = 0 = \lambda = \nu \). The general Cartan relation for the independent covectors of a basis in the cotangent vector basis are

\[
\begin{align*}
d\ell &= -(+\ell \wedge n + (\alpha + \beta - \gamma) \ell \wedge m + (\beta + \gamma) \ell \wedge m - \\
&\quad -\gamma n \wedge m + \gamma n \wedge m + (\mu + n) m \wedge m \\
dn &= -\gamma n \wedge m + \gamma n \wedge m + (\pi - \alpha + \beta) n \wedge m + \\
&\quad + (\sigma - \rho) n \wedge m + (\mu + m) m \wedge m \\
dm &= -\gamma n \wedge m + \gamma n \wedge m + (\pi - \alpha + \beta) n \wedge m + \\
&\quad + (\sigma - \rho) n \wedge m + (\mu + m) m \wedge m \\
\end{align*}
\]

(3.3)

without any reference to the metric. If we apply the \( \kappa = \sigma = 0 = \lambda = \nu \) conditions it takes the form of the LCR-structure condition \( (3.3) \). That is \( \ell, dx^\mu \) and \( n, dx^\mu \) are not qualified as null vectors. They are just real vectors, which combine with the complex vector \( m, dx^\mu \) to form a basis. The corresponding relations in the tangent space is

\[
\begin{align*}
[\ell^\mu \partial_\mu, \ m^\nu \partial_\nu] &= (\pi - \alpha - \beta) \ell^\rho \partial_\rho + (\gamma - \alpha - \beta) m^\rho \partial_\rho + (\sigma n + m) n^\rho \partial_\rho \\
[\ell^\mu \partial_\mu, \ m^\nu \partial_\nu] &= n^\rho \partial_\rho + (\sigma - \beta) m^\rho \partial_\rho \end{align*}
\]

(3.4)
which becomes the LCR-condition \((2.35)\) if \(\kappa = \sigma = 0 = \lambda = \nu\) is imposed. Hence we may cast the LCR-tetrad in the form of Newman-Penrose formalism and use it freely, but we can apply the tetrad-Weyl transformation only for the particular LCR-integrable tetrad. Notice that the real quantity

\[ i(\rho - \overline{\rho})(\mu - \overline{\mu})(\tau + \overline{\tau})(\tau + \pi)\ell \wedge m \wedge n \wedge \overline{m} \quad (3.5) \]

is an LCR-structure invariant, but it is not invariant under a tetrad-Lorentz transformation. Another interesting property is that the complex scalars of the conformal tensor \((\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4)\) which correspond to a LCR-tetrad satisfy the condition \(\Psi_0 = 0 = \Psi_4\). Hence the LCR-tetrad is determined\([8]\) by two solutions of the quartic polynomial

\[ \Psi'_0 + 4b\Psi'_1 + 6b^2\Psi'_2 + 4b^3\Psi'_3 + b^4\Psi'_4 = 0 \quad (3.6) \]

where \(\Psi'_i\) are the conformal tensor components relative to any other null tetrad of the metric. That is the class of LCR-tetrads is determined from the principal null directions of the conformal tensor. On the other hand the above relation poses a maximum number of LCR-tetrads of the solitonic gravitating LCR-structures which will be identified below with the leptonic generations (it will be described in section IV).

The LCR-structure condition \((2.28)\) determines a class of Kaehler metrics and symplectic forms

\[ ds^2 = 2i\overline{\rho_j} \frac{\partial^2 \det(\rho_{ij})}{\partial z^a \partial \overline{z}^b} dz^a d\overline{z}^b, \quad \omega = 2i\overline{\rho_j} \frac{\partial^2 \det(\rho_{ij})}{\partial z^a \partial \overline{z}^b} dz^a \wedge d\overline{z}^b \quad (3.7) \]

in the ambient complex manifold. Like in the 3-dimensional CR-structure it is not unique nor the induced Einstein metric which is analogous to \((3.1)\). Because a given LCR-structure may be implied by any \(\rho'_ij = f_{ij}\rho_{ij}\) without summation but the same dependence on the \(z^a\) and \(\overline{z}^3\) structure variables. In fact the symplectic form vanishes implying that the LCR-manifold is a lagrangian submanifold. We will consider below the two symmetric "algebraically flat" LCR-structures and their corresponding symmetric metrics.

Recall that the "algebraically flat" LCR-structures \((2.64)\) are essentially determined by the Kerr function. In the unbounded realization we consider the reducible quadric polynomial \(X^0X^1 = 0\). Then the homogeneous coordinates are

\[ X^{ni} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -iz^0 & iz^1 \\ iz^1 & -iz^0 \end{pmatrix} = : \begin{pmatrix} I \\ -i\overline{x} \end{pmatrix} \quad (3.8) \]

where we have introduced the convenient structure coordinates so that the struc-
ture conditions are

\[ \rho_{ij} = \overline{X^m E_{nm}} X^{mj} = 0 \]

\[ \begin{align*}
\rho_{11} &= -i(z^1 - \bar{z}^1) = 0 \\
\rho_{12} &= i(z^1 - \bar{z}^1) = 0 \\
\rho_{22} &= i(z^0 - \bar{z}^0) = 0
\end{align*} \tag{3.9} \]

and the dependence of the structure coordinates with the cartesian coordinates is

\[ \begin{align*}
z^0 &= i \frac{X^{21}}{X^{01}} = x^0 - x^3 \\
z^1 &= -i \frac{X^{31}}{X^{01}} = x^1 + ix^2 \\
z^0^\prime &= i \frac{X^{32}}{X^{12}} = x^0 + x^3 \\
z^1^\prime &= -i \frac{X^{22}}{X^{12}} = x^1 - ix^2
\end{align*} \tag{3.10} \]

which we call "light-cone" LCR-structure. The general LCR-tetrad is

\[ \begin{align*}
\ell_\mu dx^\mu &= \Lambda dz^0 |_M \\
m_\mu dx^\mu &= M dz^1 |_M \\
n_\mu dx^\mu &= N dz^0^\prime |_M \\
\overline{m}_\mu dx^\mu &= \overline{M} dz^1^\prime |_M
\end{align*} \tag{3.11} \]

with \( \Lambda, N \) and \( M \) the arbitrary non vanishing factors of the tetrad-Weyl transformation. Notice that this LCR-structure is degenerate with vanishing all its relative invariants, \( \Phi_j = 0 \).

After a straightforward computation we find for the ambient Kaehler manifold

\[ ds^2 = 2(dz^0 \overline{dz^0} + dz^0 d\overline{z^0} - dz^1 \overline{dz^1} - dz^1^\prime \overline{dz^1^\prime}) \tag{3.12} \]

\[ \omega = 2i(dz^0 \wedge \overline{dz^0} + dz^0 \wedge \overline{dz^0} - dz^1 \wedge \overline{dz^1} - dz^1^\prime \wedge \overline{dz^1^\prime}) \]

and the induced manifold has \( \omega |_M = 0 \) as expected and the induced metric

\[ ds^2 |_M = (dz^0)^2 - (dz^1)^2 - (dz^2)^2 - (dz^3)^2 \tag{3.13} \]

properly normalized. It is the symmetric Minkowski spacetime with vanishing curvature.

Apparently the present "light-cone" LCR-structure is regular \( \forall x^\mu \in \mathbb{R}^4 \). But singularities may be hidden at the "infinites". Therefore this problem should be treated in the largest LCR-manifold \( \mathbb{R} \times SU(2) \) being the boundary of the bounded realization of the \( SU(2,2) \) classical domain. This is formally done by transcribing the Kerr polynomial condition \( X^0 X^1 = 0 \) into the bounded homogeneous coordinates

\[ \begin{align*}
X^{11} &= \frac{1}{\sqrt{2}} (Y^{11} + Y^{31}) = 0 \\
X^{02} &= \frac{1}{\sqrt{2}} (Y^{02} + Y^{22}) = 0
\end{align*} \tag{3.14} \]
In the normalization $Y^{01} = 1 = Y^{12}$ we have

$$
\hat{w} = \begin{pmatrix}
Y^{21} & -Y^{02} \\
-Y^{01} & Y^{02}
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
Y^{11} & 1
\end{pmatrix}^{-1} =
\begin{pmatrix}
\cos \rho + i \sin \rho \cos \sigma & -i \sin \rho \sin \sigma e^{-i\chi} \\
-i \sin \rho \sin \sigma e^{i\chi} & \cos \rho - i \sin \rho \cos \sigma
\end{pmatrix}
\tag{3.15}
$$

which implies that the singularities occur when

$$
\det\left(\begin{array}{cc}
w_{11} + 1 & w_{12} \\
w_{21} & w_{22} + 1
\end{array}\right) = 0
\tag{3.16}
$$

$$
\cos \tau + \cos \rho = 0
$$

which are the future and past celestial spheres. In fact the singularities happen when the two roots of the quadratic Kerr polynomial $(Y^0 + Y^2)(Y^1 + Y^3)$ vanish. Apparently the ”light-cone” LCR-structure has singularities (hidden) at the conformal infinities of $\mathbb{R}^4$.

In the unbounded realization we consider the reducible quadric polynomial $Y^0 Y^1 = 0$. Then the homogeneous coordinates are

$$
Y^{nj} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
w^0 & -iw^1 \\
-iw^1 & \bar{w}^0
\end{pmatrix} =: \left(\begin{array}{c}
I \\
\hat{w}
\end{array}\right)
\tag{3.17}
$$

where the proper structure coordinates are fixed. Then the LCR-structure embedding conditions are

$$
\rho_{ij} = \overline{Y^{mi}} E^B_{jm} Y^{mj} = 0
$$

$$
\rho_{11} = w^0 \bar{w}^0 + w^1 \bar{w}^1 - 1 = 0
$$

$$
\rho_{12} = w^0 \bar{w}^1 - w^1 \bar{w}^0 = 0
$$

$$
\rho_{22} = w^0 \bar{w}^0 + w^1 \bar{w}^1 - 1 = 0
\tag{3.18}
$$

This LCR-structure will be called ”natural U(2)” structure. One may check that the Taub-NUT spacetime$^{15,12}$ admits a LCR-structure equivalent to the present one. From (3.17) we find the LCR-structure coordinates

$$
w^0 = (\cos \rho + i \sin \rho \cos \sigma)e^{i\tau} \quad , \quad w^1 = \sin \rho \sin \sigma e^{i\chi} e^{i\tau}
$$

$$
\bar{w}^0 = (\cos \rho - i \sin \rho \cos \sigma)e^{i\tau} \quad , \quad \bar{w}^1 = \sin \rho \sin \sigma e^{-i\chi} e^{i\tau}
\tag{3.19}
$$

and the LCR-tetrad

$$
\ell = d\tau + \cos \sigma dp - \sin \rho \cos \rho \sin \sigma d\sigma + \sin^2 \rho \sin^2 \sigma d\chi
$$

$$
n = d\tau - \cos \sigma dp + \sin \rho \cos \rho \sin \sigma d\theta - \sin^2 \rho \sin^2 \sigma d\chi
$$

$$
m = -e^{i\chi}[\sin \sigma dp + \sin \rho (\cos \rho \cos \sigma + i \sin \rho) d\sigma +
+(\cos \rho + i \sin \rho \cos \sigma) \sin \rho \sin \sigma d\chi]
\tag{3.20}
$$
which satisfies the conditions
\[
\begin{pmatrix} \ell & m \\ m & n \end{pmatrix} = \hat{\ell}, \quad d\hat{\ell} - \hat{i} \epsilon \wedge \hat{\ell} = 0
\] (3.21)

\[d\ell = im \wedge \overline{m}, \quad dn = -im \wedge \overline{m}, \quad dm = i(\ell - n) \wedge m\]

Notice that this LCR-structure has relative invariants \(\Phi_1 = i, \Phi_2 = -i\) and \(\Phi_3 = 0\). Hence the "natural \(U(2)\)" structure is not equivalent to the degenerate "light-cone" LCR-structure. Besides, the possibility to cast the LCR-tetrad (with \(\Phi_1 \neq 0, \Phi_2 \neq 0\)) into the above hermitian matrix form with vanishing Cartan \(U(2)\) curvature provides the possibility to interpret it as the electroweak \(U(2)\) potential with vanishing field strength. This will be verified in section V by simply computing this field strength (Cartan curvature) in the electron LCR-structure solitonic sector, where its electromagnetic potential is derived.

We will now see how the "natural \(U(2)\)" looks like in a projective chart of the unbounded realization. The \(Y_0Y^1 = 0\) Kerr polynomial takes the form \((X^0 + X^2)(X^1 + X^3) = 0\) and the LCR-conditions

\[X^{nj} \equiv \begin{pmatrix} 1 & -z^\dagger \\ -iz^0 + i & 1 \\ z^\dagger & -z^1 \end{pmatrix} = \begin{pmatrix} \lambda & -i\tilde{x}\lambda \\ i\tilde{x}\lambda & -\lambda \end{pmatrix} \] (3.22)

are satisfied with \(\tilde{x}^\dagger = \tilde{x}\) and the explicit forms of the structure coordinates are

\[

z^1 = x^1 + i\frac{\sigma}{\sqrt{1 + \sigma^2}}, \quad z^0 = x^0 - \frac{(x^1)^2 + (x^2)^2}{1 + \sigma^2 + x^2},
\]

\[

z^\dagger = x^\dagger + i\frac{\sigma}{\sqrt{1 + \sigma^2}}, \quad z^\dagger = x^0 + x^3 - i - \frac{(x^1)^2 + (x^2)^2}{1 + \sigma^2 + x^2}.
\]

\[z^0 - z^\dagger + 2i(1 - z^1 z^\dagger) = 0, \quad z^\dagger - z^0 + 2i(1 - z^1 z^\dagger) = 0, \quad z^\dagger z^0 + z^0 z^\dagger = 0
\] (3.23)

There is no singularity in \(\mathbb{R}^4\), because \(\det \lambda \neq 0\).

After a straightforward computation we find the metric and the symplectic form for the ambient Kaehler manifold. The induced metric is

\[ds^2|_M = (dt)^2 - (d\rho)^2 - \sin^2 \rho (d\theta)^2 - \sin^2 \rho \sin^2 \theta (d\varphi)^2\] (3.24)

properly normalized. Recall that \(\rho_{ij}\) is determined up to conformal factors. Hence the above metric is equivalent to the symmetric de Sitter metric

\[ds^2_S = (dt)^2 - T_0^2 \cosh^2 \frac{\tau}{T_0} [(d\rho)^2 + \sin^2 \rho (d\sigma)^2 + \sin^2 \rho \sin^2 \sigma (d\chi)^2] = T_0^2 \cosh^2 \frac{\tau}{T_0} [ds^2|_M]\]

\[\tau = 2 \arctan(e^{\frac{\tau}{T_0}}), \quad T_0 := \sqrt{\frac{T}{T_0}}\]

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with \( \rho \in [0, 2\pi) \). It is fixed being the general symmetric space which covers the entire covering spacetime \( R \times SU(2) \). We will use this result (or rather the inverse argument) to show the existence of dark energy. The existence of dark energy suggests us to assume that the empty (symmetric) universe is the de Sitter metric and not its conformally equivalent Minkowski metric.

4 LEPTONIC LCR-STRUCTURES

Identifying the Poincaré subgroup of the \( SU(2, 2) \) group of the classical domain with the observed Poincaré symmetry in nature opens up the possibility to look for automorphic LCR-structures relative to time translation and z-rotation, i.e. static and axially symmetric LCR-structures. Trying to impose the additional dilation we find a singular LCR-structure. We will essentially work in two steps. First we will look for automorphic algebraically flat LCR-structures and after we will find a curved deformation using the Kerr-Schild ansatz adapted to the LCR-tetrad.

4.1 Free electron (and positron) LCR-structure

Recall that an algebraically flat LCR-structure is completely determined by the roots of the Kerr function. We need two roots to fix the flat null tetrad (2.77). On the other hand higher than 4-degree polynomials are excluded by the condition (3.6) on the non-vanishing conformal tensor. Therefore we will study quadric, cubic and quartic polynomials.

The infinitesimal z-rotation, time translation and dilation are

\[
\delta_z X^{0i} = -i^{12} X^{0i}, \quad \delta_z X^{1i} = i^{12} X^{1i}, \quad \delta_z X^{2i} = -i^{12} X^{2i}, \quad \delta_z X^{3i} = i^{12} X^{3i}
\]

\[
\delta_t X^{0i} = 0, \quad \delta_t X^{1i} = 0, \quad \delta_t X^{2i} = -i^{0} X^{0i}, \quad \delta_t X^{3i} = i^{0} X^{1i}
\]

\[
\delta_d X^{0i} = -\frac{\xi}{2} X^{0i}, \quad \delta_d X^{1i} = -\frac{\zeta}{2} X^{1i}, \quad \delta_d X^{2i} = \frac{\xi}{2} X^{2i}, \quad \delta_d X^{3i} = \frac{\zeta}{2} X^{3i}
\]

The automorphic quadratic Kerr polynomial \( K(X^m) = \sum_{m,n} A_{mn} X^m X^n \) relative to the z-rotation has the form

\[
K_z = A_{01} X^0 X^1 + A_{03} X^0 X^3 + A_{12} X^1 X^2 + A_{23} X^2 X^3
\]

The two solutions of this quadratic polynomial are generally time dependent. The time translation automorphism restricts it into the form

\[
K_{zt} = A_{01} X^0 X^1 + A_{12} (X^1 X^2 - X^0 X^3)
\]

A singular point \( X^n \in CP(3) \) is solution of

\[
\frac{\partial K}{\partial X^n} = (A_{01} X^1 - A_{12} X^3, A_{01} X^0 + A_{12} X^2, A_{12} X^1, -A_{12} X^0) = 0
\]
If $A_{01} \neq 0 \neq A_{12}$ we find $X^n = 0, \forall n$. But this solution of the homogeneous coordinates does not represent a point of $CP(3)$. Hence this is a regular surface of $CP(3)$ and gives the algebraically regular "free electron" LCR-structure. If $A_{01} \neq 0 = A_{12}$ we find the "light-cone" LCR-structure. If $A_{01} = 0 \neq A_{12}$ we find an algebraically regular LCR-structure, which is also automorphic relative to the dilation transformation implying that the parameter $A_{01}$ (which will become the rotating parameter) essentially breaks the dilation transformation. But the corresponding dilation invariant LCR-manifold is singular, because the $\ell^a$ and $n^a$ congruences have infinite concentrations into finite points. We will investigate this effect below in comparison with the "free electron" LCR-structure.

The non-existence of the automorphic LCR-structure relative to all the three automorphisms is directly related to the "spontaneous" breaking of the $SU(2,2)$ group to its Poincaré subgroup as we will see in the section VII of the SM derivation.

If we make a general translation and after a boost transformation, the automorphic Kerr polynomial (4.3) gives the general flatprint form of the "free electron" LCR-structure. But I find more impressive to start from the linear trajectory $\xi^a = v^a \tau + c^a$ with $v^a$ a general real velocity and $c^a$ generally complex. After eliminating the complex parameter $\tau$ using (2.72), the Kerr polynomial takes the form

$$K(X^n) = iX^0X^0((v^0 - v^3)(c^1 + ic^2) - (v^1 + iv^2)(c^0 - c^3)) +$$

$$+ iX^0X^1((v^0 + v^3)(c^0 - c^3) - (v^0 - v^3)(c^0 + c^3)) +$$

$$+ (v^1 + iv^2)(c^1 - ic^2) - (v^1 - iv^2)(c^1 + ic^2) -$$

$$- X^0X^2(v^1 + iv^2) - X^0X^3(v^0 - v^3) +$$

$$+ iX^1X^1((v^1 - iv^2)(c^0 + c^3) - (v^0 + v^3)(c^1 - ic^2)) +$$

$$+ X^1X^2(v^0 + v^3) + X^1X^3(v^1 - iv^2)$$

This is the most general quadratic Kerr polynomial which incorporates all the parameters of the Poincaré representation. The singular points of this quadratic surface satisfy the relations $\partial_n K(X^n) = 0$ and $X^n \neq \overline{0}$. We finally find that there are the following two cases

1st: $\quad If \quad v^a v^b \eta_{ab} \neq 0 \quad the \ surface \ is \ irreducible$

2nd: $\quad If \quad v^a v^b \eta_{ab} = 0 \quad the \ surface \ is \ reducible$

The first case gives the electron and positron LCR-solitons and the second reducible surface gives the left-handed chiral part of the neutrino. The electron LCR-structure is determined with an irreducible quadratic polynomial and the neutrino LCR-structure is determined with the corresponding reducible quadratic polynomial.

Returning back to the automorphic Kerr polynomial $K_{zt}$, we will first find
the algebraically flat LCR-structure with

\[
X^{\alpha i} = \begin{pmatrix}
1 & -z^1 \\
2(z^0 - i\alpha) & 1 \\
-i(z^0 + i\alpha)z^1 & -i(z^0 + i\alpha)
\end{pmatrix}
\] (4.8)

where \((z^\alpha; z^\tilde{\alpha})\) are now the structure coordinates, which imply simple formula. The LCR-structure coordinates are

\[
\begin{align*}
z^0 &= t - r + i\alpha \cos \theta, & z^1 &= e^{i\varphi} \tan \frac{\theta}{2} \\
\tilde{z}^0 &= t + r - i\alpha \cos \theta, & \tilde{z}^1 &= \frac{r + i\alpha}{r - i\alpha} e^{-i\varphi} \tan \frac{\theta}{2}
\end{align*}
\] (4.9)

from which we find the LCR-tetrad

\[
\begin{align*}
\ell_\mu dx^\mu &= \Lambda [dt - dr - a \sin^2 \theta d\varphi] \\
n_\mu dx^\mu &= N [dt + \frac{r^2 + a^2 + a^2 \cos^2 \theta - a^2}{r^2 + a^2} dr - a \sin^2 \theta d\varphi] \\
m_\mu dx^\mu &= M [-i a \sin \theta (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i \sin \theta (r^2 + a^2) d\varphi]
\end{align*}
\] (4.10)

where the tetrad-Weyl factors are not determined, as expected. They are determined by simply imposing that the tetrad gives the symmetric Minkowski metric. But for that, we have to find first the relation of the cartesian coordinates \(x^\mu\) with the present convenient coordinates \((t, r, \theta, \varphi)\).

The general relation between the projective coordinates and the homogeneous coordinates of \(G(4,2)\) is found by simply inverting their definition formula. We finally find

\[
\begin{align*}
\rho^0 &= \frac{(X^{01}X^{32} - X^{31}X^{02}) + (X^{21}X^{12} - X^{11}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} \\
\rho^1 &= \frac{(X^{11}X^{32} - X^{31}X^{12}) + (X^{21}X^{12} - X^{11}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} \\
\rho^2 &= \frac{(X^{11}X^{32} - X^{31}X^{12}) - (X^{21}X^{12} - X^{11}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})} \\
\rho^3 &= \frac{(X^{01}X^{32} - X^{31}X^{02}) - (X^{21}X^{12} - X^{11}X^{22})}{2(X^{01}X^{12} - X^{11}X^{02})}
\end{align*}
\] (4.11)

We already know that the imaginary part of \(r^b = x^b + iy^b\) determines the gravitational "dressing", because the "flatness" condition implies \(y^b = 0\). The Minkowski coordinates \(x^\mu\) are related with \((t, r, \theta, \varphi)\) via the relation

\[
\begin{align*}
x^0 &= t \\
x^1 + ix^2 &= (r - i\alpha) \sin \theta e^{i\varphi} = \sqrt{r^2 + a^2} e^{-i \arctan \frac{\alpha}{r}} \sin \theta e^{i\varphi} \\
x^3 &= r \cos \theta \\
\frac{r^4 - [(x^1)^2 + (x^2)^2] + \sqrt{((x^1)^2 + (x^2)^2)^2 - (a^2)^2}}{r^4 + a^2} \\
\cos \theta &= \frac{x^3}{r}, & \sin \theta &= \sqrt{\frac{((x^1)^2 + (x^2)^2)^2 - (a^2)^2}{r^4 + a^2}, & \frac{(x^1)^2 + (x^2)^2}{r^4 + a^2} + (x^3)^2} &= 1
\end{align*}
\] (4.12)
We finally find the geodetic and shear free null tetrad

\[
L_\mu dx^\mu = [dt - dr - a \sin^2 \theta d\phi] \\
N_\mu dx^\mu = \frac{1}{2(r^2 + a^2 \cos^2 \theta)} [dt + (r^2 + 2a^2 \cos^2 \theta - a^2)dr - a \sin^2 \theta d\phi] \\
M_\mu dx^\mu = \frac{1}{\sqrt{2(r^2 + a^2 \cos^2 \theta)}} \left[ -ia \sin \theta (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i \sin \theta (r^2 + a^2) d\phi \right]
\]

of the Minkowski metric.

A static axisymmetric gravitating tetrad is found with the "Kerr-Schild" ansatz adapted to the LCR-structure formalism

\[
\ell_\mu = L_\mu, \quad m_\mu = M_\mu, \quad n_\mu = N_\mu + \frac{h(r)}{2(r^2 + a^2 \cos^2 \theta)} L_\mu
\]

I want to point out that we find the same static LCR-structure looking for a general LCR-structure admitting time translation and axisymmetric automorphisms applied on the regular coordinates.

With the above definition of the coordinates \((t, r, \theta, \phi)\), the structure coordinates have the form

\[
z^0 = t - r + i a \cos \theta \quad , \quad z^1 = e^{i \varphi} \tan \frac{\theta}{2} \\
z^0 = t + r - i a \cos \theta - 2 f_1 \quad , \quad z^1 = \frac{r + i a}{r - i a} e^{2i a f_2} e^{-i \varphi} \tan \frac{\theta}{2}
\]

where the two new functions are

\[
f_1(r) = \int \frac{h}{r^2 + a^2 + h} \, dr \quad , \quad f_2(r) = \int \frac{h}{(r^2 + a^2 + h)(r^2 + a^2)} \, dr
\]

The Newman-Penrose spin coefficients are found to be

\[
\begin{array}{c|c|c|c|c|c}
\alpha &= \frac{i a (1 + \sin^2 \theta) - r \cos \theta}{2 \sqrt{2} \sin \theta \left( r - i a \cos \theta \right)^2} & \beta = \frac{\cos \theta}{2 \sqrt{2} \sin \theta \left( r + i a \cos \theta \right)} \\
\gamma &= -\frac{r^2 - 2 a^2}{2 \mu} \frac{r - i a \cos \theta}{r + i a \cos \theta} + \frac{\mu}{r^2 + a^2} & \delta = 0 \\
\mu &= -\frac{1}{2} \frac{r^2 + a^2 + h}{(r - i a \cos \theta)} & \nu = \frac{i a \sin \theta}{\sqrt{2} \rho} \\
\rho &= \frac{1}{\sqrt{2} \sin \theta} & \kappa = 0 \quad , \quad \sigma = 0 \quad , \quad \tau = -\frac{i a \sin \theta}{\sqrt{2} \mu} \quad , \quad \lambda = 0
\end{array}
\]

which will be useful for our computations. Recall that the Kerr-Newman spacetime has \(h(r) = -2Mr + e^2\). In this case the integrals are

\[
f_1(r) = \int \frac{-2Mr + e^2}{r^2 + a^2 - 2Mr + e^2} \, dr = -M \ln \left[ \frac{\Delta}{r^2} + \frac{2M^2 - e^2}{er} \right] \arctan \frac{2r}{r - M} \\
f_2(r) = \int \frac{-2Mr + e^2}{(r^2 + a^2 - 2Mr + e^2)(r^2 + a^2)} \, dr = \frac{1}{2ia} \ln \left[ r^2 \frac{r - ia}{r + ia} \frac{(r - M + i \Theta)^2}{(r - M - i \Theta)^2} \right]
\]

\[
\Delta := r^2 + a^2 - 2Mr + e^2 \quad , \quad \Theta := \sqrt{a^2 + e^2 - M^2}
\]

and the structure coordinates of the "free electron" LCR-manifold are

32
\[ z^0 = t - r + ia \cos \theta , \quad z^1 = e^{i\varphi} \tan \frac{\theta}{2} \]
\[ z^\tilde{0} = t + r - ia \cos \theta + 2M \ln \left| \frac{\Delta}{r_1} \right| + \frac{2(e^2 - 2M^2)}{e^2} \arctan \frac{\Theta}{r - M} \]
\[ z^\tilde{1} = r_2 \left( \frac{r - M + i\Theta}{r - M - i\Theta} \right)^{\frac{1}{2}} e^{-i\varphi} \tan \frac{\theta}{2} \]

in the coordinates \((t, r, \theta, \varphi)\). The constants \(r_1\) and \(r_2\) are normalization constants. Notice the singularities in the ambient complex manifold occur at the two complex values of \(r = M \pm i\Theta\). It is well known to general relativists that this choice of tetrad-Weyl factors preserve the electromagnetic current, and the energy-momentum and angular momentum currents. Identifying the “electron” LCR-manifold with the above Kerr-Newman manifold we fix the electromagnetic and gravitational “dressings” of the electron to be

\[ A = \frac{i_\pi (r^2 + a^2 \cos^2 \theta)}{(r^2 + a^2 \cos^2 \theta)} (dt - dr - a \sin^2 \theta d\varphi) \]
\[ h_{\mu\nu} = \frac{-2Mr + e^2}{(r^2 + a^2 \cos^2 \theta)} L_{\mu} N_{\nu} \] (4.20)

The general form (2.28) of the embedding of the LCR-manifold in the ambient complex manifold may be viewed as a deformation of the 3-dimensional CR-manifold \(\rho_{11}(\overline{z^\alpha}, z^\beta) = 0\) through a formal anti-meromorphic transformation

\[ z^\tilde{\beta} = f^\beta(z^\alpha; s) \] (4.21)

which generalizes the trivial transformation of the degenerate LCR-structure. In the present electron LCR-structure this deformation takes the form

\[ z^\tilde{0} = z^0 + 2(r - f_1) \]
\[ z^\tilde{1} = r_2 z^1 \left( \frac{r - M + i\Theta}{r - M - i\Theta} \right)^{\frac{1}{2}} \] (4.22)

where the deformation parameter is the real variable \(r\).

The static axially symmetric LCR-structure (identified with the electron) is stable, because all its relative invariants

\[ \Phi_1 = \frac{\rho - \overline{\rho}}{r} = \frac{-2a \cos \theta}{r^2 + a^2 \cos^2 \theta} \]
\[ \Phi_2 = \frac{\mu - \overline{\mu}}{r} = \frac{(r^2 + a^2 + h)a \cos \theta}{(r^2 + a^2 \cos^2 \theta)^2} \] (4.23)
\[ \Phi_3 = -(\tau + \overline{\tau}) = \frac{2ia \sin \theta}{\sqrt{2} (r + ia \cos \theta)^2 (r - ia \cos \theta)} \]

do not vanish.
The positron LCR-structure is the conjugate of the electron one

\[ z^{0} = z^{0} = t - r - ia \cos \theta , \quad z^{1} = z^{1} = e^{-i \varphi} \tan \frac{\theta}{2} \]

\[ z^{0} = z^{0} = t + r + ia \cos \theta + 2M \ln \left| \frac{\Delta}{r_{1}} \right| + \frac{2(e^{2} - 2M^{2})}{e^{2}} \arctan \frac{e^{2}}{r-M} \]  

(4.24)

\[ z^{1} = z^{1} = r_{2}(\frac{r-M-2M}{r-M+2})^{1/2} e^{i \varphi} \tan \frac{\theta}{2} \]

which has the LCR-tetrad

\[ \ell'_{\mu} = \ell_{\mu} , \quad m'_{\mu} = m_{\mu} , \quad n'_{\mu} = n_{\mu} , \quad \bar{m}'_{\mu} = m_{\mu} \]  

(4.25)

which has the same gravitational dressing but opposite charge electromagnetic dressing.

### 4.2 Solving the electron naked singularity "problem"

We saw that the LCR-structure implies Einstein’s general relativity and the "free electron" LCR-structure is related to the Kerr-Newman metric. This metric admits two geodetic and shear free null congruences, which are related with the LCR-tetrad. It also admits two commuting killing vectors, which are identified with the time-translation and \( z \)-rotation generators of the Poincaré group.

Carter’s discovery that the gyromagnetic ratio of the Kerr-Newman manifold is fermionic, which implies that it has gyromagnetic ratio \( g = 2 \) [22] shocked the community of general relativists. Many tried to identify the Kerr-Newman spacetime with the electron without success, because the electron constants imply the existence of a naked singularity in the Kerr-Newman spacetime.

The electron mass \( M_{e} \), charge \( e^{2} \) and spin parameter \( a \) have the values

\[ M = \frac{M_{e}}{M_{P}} = 4.18 \times 10^{-23} \]

\[ e^{2} = \frac{e^2}{4 \pi \varepsilon_{0} \hbar c} = \frac{1}{137} \]

\[ a = \frac{a}{M_{e}} = 2.09 \times 10^{23} \]  

(4.26)

Hence \( a^{2} + e^{2} - M^{2} > 0 \), and the electron metric has an essential naked singularity, which is not permitted in riemannian geometry. This is a problem for general relativity, because its fundamental quantity, the metric, does not "see" the algebraic structure. It is known (and well described in many books of general relativity) that its analytic extension has two sheets \( x^{0} \) and \( x^{9} \) which are determined by the two roots

\[ r = \pm \left\{ \frac{(x^{1})^{2} + (x^{2})^{2} + (x^{3})^{2} - a^{2}}{2} + \sqrt{\frac{[(x^{1})^{2} + (x^{2})^{2} + (x^{3})^{2} - a^{2}]^{2}}{4} + a^{2}(x^{3})^{2}} \right\}^{1/2} \]  

(4.27)

These two surfaces constitute the boundary \( U(2) \) of the bounded realization of the \( SU(2,2) \) classical domain and their correspondence is the well known
Cayley transformation, which has been described through the formula (2.58-2.62). The spinorial electron naked singularity in \( U(2) \) universe can be properly incorporated in PCFT, while it is rejected as "unphysical" by the riemannian formalism. In the context of the unbounded realization this may be studied using the ray tracing in the flatprint "free electron" LCR-structure (4.12) and the cartesian coordinates (4.12). Then \( L^\mu \partial_\mu z^\alpha = 0 \) implies that the outgoing \( L^\mu \) null integral curves (rays) are determined by the surfaces

\[
s_1 := t - r \quad , \quad s_2 := \theta \quad , \quad s_3 := \varphi
\]

(4.28)

Assuming the coordinates \((r, s_1, s_2, s_3)\), which have the property \((0, s_1, \frac{\pi}{2}, s_3)\) to be on the caustic. In this caustic coordinate system the LCR-rays are traced by the relation

\[
x^0_L(r) = s_1 + r
\]
\[
x^1_L(r) = (r \cos \varphi + a \sin \varphi) \sin \theta
\]
\[
x^2_L(r) = (r \sin \varphi - a \cos \varphi) \sin \theta
\]
\[
x^3_L(r) = r \cos \theta
\]

(4.29)

**Jacobian** \( = [r^2 + a^2 \cos^2 \theta] \sin \theta \)

The source of the LCR-rays are at \( r = 0 \), i.e.

\[
x^0_L(0) = s_1
\]
\[
x^1_L(0) = a \sin \varphi \sin \theta
\]
\[
x^2_L(0) = -a \cos \varphi \sin \theta
\]
\[
x^3_L(0) = 0
\]

(4.30)

which is the disc found above. Notice that the rays with \( s_2 := \theta \neq \frac{\pi}{2} \) pass through the disc.

The \( N^\mu \partial_\mu z^\alpha = 0 \) implies that its incoming \( N^\mu \) rays are determined by the surfaces

\[
s'_1 := t + r \quad , \quad s'_2 := \theta \quad , \quad s'_3 := \varphi + \arctan \frac{2ar}{a^2 - r^2}
\]

(4.31)

Then we find the congruence

\[
x^0_N(r) = s'_1 - r
\]
\[
x^1_N(r) = (r \cos s'_1 - a \sin s'_3) \sin \theta
\]
\[
x^2_N(r) = (r \sin s'_1 + a \cos s'_3) \sin \theta
\]
\[
x^3_N(r) = r \cos \theta
\]

(4.32)

**Jacobian** \( = [r^2 + a^2 \cos^2 \theta] \sin \theta \)

We will now show that the origin of the essential singularity of the Kerr manifold is the intersection of the two sheets of the static electron regular quadric (in the unbounded Siegel realization) \( 4.13 \) of \( CP^3 \). In the flatprint case and after the projection to the real surface \( \mathbb{R}^4 \) we have

\[
X^0 = 1 \quad , \quad X^1 = \lambda \quad , \quad X^2 = -i[(x^0 - x^3) - (x^1 - ix^2) \lambda]
\]
\[
X^3 = -i[ -(x^1 + ix^2) + (x^0 + x^3) \lambda]
\]

(4.33)

\[
K_{24}(X^n) = X^1 X^2 - X^0 X^3 + 2a X^0 X^1 = 0
\]
The two solutions (sheets) of the above quadric are
\[ \lambda_{1(2)} = \frac{-x^3 + ia \pm \sqrt{\Delta}}{x^2 + ix} \quad , \quad \Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 - a^2 - 2ia x^3 \] (4.34)

The intersection of the two sheets (which occur for \( \lambda_1 = \lambda_2 \)) is the singularity ring of the "free electron" structure. Notice that the quadratic surface is algebraically regular and the intersection of the two branches is implied by the projection. The points of the algebraic intersection curve (the branch curve) of the (regular) quadric of \( CP^3 \) are regular points like any other point of the quadric.

The bounded realization of a flat LCR-manifold is \( U(2) \), which is covered by two \( \mathbb{R}^4 \) sheets through the Cayley \( 2 \to 1 \) transformation with the \( x'^\mu \) (2.61) written in the present context

For \( s := R_0 \frac{\sin \rho}{\cos \tau + \cos \rho} > 0 \)
\[ x^0 = T_0 \frac{\sin \tau}{\sin \rho} \frac{\sin \rho}{\cos \tau + \cos \rho}, \]
\[ x^1 + ix^2 = R_0 \frac{\sin \rho}{\cos \tau + \cos \rho} \sin \sigma e^{ix} \]
\[ x^3 = R_0 \frac{\sin \rho}{\cos \tau + \cos \rho} \cos \sigma \] (4.35)

and \( x'^\mu \) (2.62) the second \( \mathbb{R}^4 \) is identified with \( s < 0 \),

For \( s := R_0 \frac{\sin \rho}{\cos \tau + \cos \rho} < 0 \)
\[ x^0 = T_0 \frac{\sin \tau}{\sin \rho} \frac{\sin \rho}{\cos \tau + \cos \rho}, \]
\[ x^1 + ix^2 = -R_0 \frac{\sin \rho}{\cos \tau + \cos \rho} \sin \sigma e^{ix} \]
\[ x^3 = -R_0 \frac{\sin \rho}{\cos \tau + \cos \rho} \cos \sigma \] (4.36)

The constants \( T_0 \) and \( R_0 \) are related to the time and space sizes. Notice that this is the Penrose artificial compactification of the Minkowski spacetime, but in the context of PCFT, this is implied by the formalism itself. In the case of the Penrose artificial compactification these two sheets \( s \geq 0 \) communicate through the \( \text{scri}^+ \) and \( \text{scri}^- \) infinities. In the case of the electron flatprint LCR-structure, these two sheets communicate through the glued two discs \((x^1)^2 + (x^2)^2 < a^2\) too, because we may assume

\[ r = + \left\{ \frac{x^2 - a^2}{2} + \sqrt{\left(\frac{x^2 - a^2}{2}\right)^2 + a^2 x^3} \right\}^{\frac{1}{2}} \quad \text{for } s > 0 \]
\[ r = - \left\{ \frac{x^2 - a^2}{2} + \sqrt{\left(\frac{x^2 - a^2}{2}\right)^2 + a^2 x^3} \right\}^{\frac{1}{2}} \quad \text{for } s < 0 \] (4.37)

Notice that in the identified region (the disc for both sheets) \( r = 0 \) in both sheets. That is, \( r = 0 \) occurs at \( x^3 = 0 \) and \( s^2 \leq a^2 \) for both sheets \( s \geq 0 \).

The two LCR-congruences \( L^\mu = \frac{dx^\mu}{dt} \) and \( N^\mu = \frac{dx'^\mu}{dt} \) of the flatprint electron LCR-manifold can be easily implied from the calculations of the previous
section. The starting idea is that the structure coordinates \( z^a(x) \) provide the three invariants \((s_1, s_2, s_3)\) along the ray, which label the \( L \)-ray \( x^a_L(r) \), and the structure coordinates \( z^a(x) \) provide the invariants \((s'_1, s'_2, s'_3)\), which label the \( N \)-ray \( x^a_N(r) \). Hence we simply have the same forms, but we let \( r \in (\infty, +\infty) \) and at \( r = 0 \) we pass from the one \( \mathbb{R}^4 \) sheet to the other.

A complete visualization of the rays \( w_{L,N}(r; s_1, s_2, s_3) \in U(2) \) taking \( r \in (-\infty, +\infty) \) can be done in the bounded realization of the flatprint electron (as the \( U(2) \) boundary of the \( SU(2, 2) \) classical domain). From (2.56) we find the relation

\[
Y^0 = \frac{1}{\sqrt{2}}(X^0 + X^2), \quad Y^1 = \frac{1}{\sqrt{2}}(X^1 + X^3) \quad (4.38)
\]

between the bounded \( Y^a \) and unbounded \( X^a \) homogeneous coordinates and from (4.8) we find

\[
Y^{a_i} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 - i(z^0 - ia) & (-1 + i(z^0 - ia))z^1 \\
(1 - i(z^0 + ia))z^1 & 1 - i(z^0 + ia) \\
1 + i(z^0 - ia) & (1 + i(z^0 - ia))z^1 \\
(1 + i(z^0 + ia))z^1 & 1 + i(z^0 + ia)
\end{pmatrix} \quad (4.39)
\]

The relation between the projective coordinates and the homogeneous coordinates of \( G(4, 2) \) is found by simply inverting their definition formula in the bounded realization. We finally find the relation

\[
w_{11} = \frac{Y^{21}Y^{12} - Y^{11}Y^{22}}{Y^{21}Y^{12} + Y^{11}Y^{22}}, \quad w_{12} = \frac{Y^{01}Y^{22} - Y^{21}Y^{02}}{Y^{21}Y^{12} + Y^{11}Y^{22}} \quad (4.40)
\]

between the bounded projective \( \tilde{w} \in U(2) \) and homogeneous \( Y^{a_i} \) coordinates. In principle we may compute the rays \( \tilde{w}_{L,N}(r; s_1, s_2, s_3) \in U(2) \) in the complete bounded universe \( U(2) \), but it seems to be complicated. But the intersection (touching) of the two \( \mathbb{R}^4 \) sheets in \( U(2) \) coordinates can be computed by simply making the Cayley transformation of the cartesian form of the ring singularity. Then we find that in \((\tau, \rho, \sigma, \chi)\) coordinates the ring singularity (the caustic of the congruence) is

\[
\sigma = \frac{\pi}{2}, \quad R_0^2 \frac{\sin^2 \rho}{(\cos \tau + \cos \rho)^2} \leq a^2 \\
-\pi < \rho < \pi, \quad -\pi < \tau < \pi \quad (4.41)
\]

We have already identified \( \ell^\mu \) and \( n^\mu \) the positron with the conjugate LCR-structure of the electron. In order to geometrically distinguish the positron from the electron we have to consider the (unbounded) two \( \mathbb{R}^4 \)-sheets realization, where the \( \ell^a \) and \( n^a \) congruences have opposite directions passing through the ring singularity. In a given \( \mathbb{R}^4 \)-sheet, assuming that the electron is the LCR-manifold with the \( \ell^a \) congruence outgoing and the \( n^a \) congruence ingoing, the
positron is the LCR-manifold with the $\ell'^\mu$ congruence outgoing too but with opposite twist $a$ and the $n'^\mu$ congruence ingoing too but with opposite twist too. Hence the outgoing (retarded) and ingoing (advanced) character of their LCR-rays is preserved.

In order to make clear the importance of the spin parameter $a$ to the regularity of "free electron" LCR-structure, we will describe in details the simple quadratic surface (4.5) of $CP(3)$. It is the quadratic Kerr polynomial which is symmetric relative to $z$-rotations, time translations and dilations. Apparently the quadric of $CP(3)$ is algebraically regular as we have already showed in (4.4). But, we will see that its reduction to the real LCR-manifold is going to generate a non-permitting singularity. The (4.33) quadratic polynomial now becomes

$$K_{std}(X^n) = X^1X^2 - X^0X^3 = 0$$

$$X^0 = 1 \quad X^1 = \lambda \quad X^2 = -i[(x^0 - x^3) - (x^1 - i x^2)\lambda]$$

$$X^3 = -i[-(x^1 + ix^2) + (x^0 + x^3)i\lambda]$$

The Kerr polynomial and its two solutions are

$$(x^1 - i x^2)\lambda^2 + 2x^3\lambda - (x^1 + i x^2) = 0$$

$$\lambda_{1(2)} = \frac{-x^3 \pm \sqrt{\Delta}}{x^1 + i x^2} \quad \Delta = (x^1)^2 + (x^2)^2 + (x^3)^2$$

where $\lambda_{1(2)}$ are the two values of $\lambda$ on the two sheets of the quadric. The intersection of the two sheets of $CP(3)$ becomes

$$\Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 = 0$$

a point.

The preceding calculations are described as follows in the algebraic picture. The two points

$$X^{n1} = \begin{pmatrix} 1 \\ \lambda_1(x) \\ -i[x^0 - x^3 - (x^1 - i x^2)\lambda] \\ -i[-(x^1 - i x^2) + (x^0 + x^3)i\lambda] \end{pmatrix}$$

$$X^{n2} = \begin{pmatrix} 1 \\ \lambda_2(x) \\ -i[x^0 - x^3 - (x^1 - i x^2)\lambda] \\ -i[-(x^1 - i x^2) + (x^0 + x^3)i\lambda] \end{pmatrix}$$

of the above quadric belong to different sheets created by the considered projection and they correspond to a point $x^a$ of the characteristic boundary $\mathbb{R}^4$ of the "upper half-plane" domain of $G(4,2)$. If det($\lambda_{4i}$) = $\lambda_2 - \lambda_1 = 0$, the two points coincide, that is, the projection line is tangent to the quadric.

In the present case the branch curve is reduced to a point $\overrightarrow{a} = -\overrightarrow{0}$. Therefore the branch cut should be reduced to a line joining $\overrightarrow{0}$ and $\infty$. The structure
coordinates are
\[ z^0 = iX^{21} = x^0 - |\mathbf{x}| , \quad z^1 = \lambda_1 = \frac{|\mathbf{x}| - x^3}{x^1 - ix^2} = \frac{x^1 + ix^2}{\sqrt{x^1 + x^2}} \]
\[ \tilde{z}^0 = iX^{22} = x^0 + |\mathbf{x}| , \quad \tilde{z}^1 = \frac{1}{\lambda_2} = \frac{x^1 - ix^2}{|\mathbf{x}| + x^3} \]  
\[
(4.46)
\]
and the derived tetrad is
\[
\ell_\mu dx^\mu = \Lambda[|\mathbf{x}| dx^0 - \mathbf{x} \cdot d\mathbf{x}] \\
m_{\mu} dx^\mu = M[|\mathbf{x}| (x^3 + |\mathbf{x}|) - (x^1 + ix^2)x^1]dx^1 + \\
(i|\mathbf{x}| (x^3 + |\mathbf{x}|) - (x^1 + ix^2)x^2)dx^2 - (x^1 + ix^2)(x^3 + |\mathbf{x}|)dx^3 \\
n_{\mu} dx^\mu = N[|\mathbf{x}| dx^0 + \mathbf{x} \cdot d\mathbf{x}] 
\]
\[ \ell \land m \land n \land \mathbf{m} = -4i|\mathbf{x}|^4(x^3 + |\mathbf{x}|)^2 dx^0 \land dx^3 \land dx^2 \land dx^3 \neq 0, \quad \forall x^\mu \in \mathbb{R}^4 - \{\mathbb{R}\} \]
\[
(4.47)
\]
where the tetrad-Weyl factors are arbitrary as expected. The tetrad is singular (because it cannot be a basis of the tangent space) in the negative z-axis, where the branch cut in the algebraic quadric is reduced.

We can make the same calculations in the compact realization of complete spacetime. In this coordinate patch, \( Y^n \) is given by the linear transformation \((4.38)\). Then the Kerr polynomial has the same form \( K_B^{2d}(Y^n) = Y^1Y^2 - Y^0Y^3 \) \((4.48)\) as in the unbounded realization. This quadratic LCR-structure has the same form in the bounded and unbounded realizations. In the bounded realization the homogeneous coordinates of \( G(4, 2) \) have the form
\[
Y^{ni} = \begin{pmatrix} Y^{01} & Y^{02} \\ Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \\ Y^{31} & Y^{32} \end{pmatrix} = \begin{pmatrix} k \\ \tilde{w}k \end{pmatrix} \]
\[
(4.49)
\]
where the elements of the \( 2 \times 2 \) matrix \( \tilde{w} \) are the projective coordinates. Hence we will substitute
\[
Y^n = \begin{pmatrix} 1 \\ k \\ w_{00} + w_{01}k \\ w_{10} + w_{11}k \end{pmatrix} \]
\[
(4.50)
\]
in the new (bounded) form of the Kerr quadric. Then it takes the form
\[
k^2w_{01} + k(w_{00} - w_{11}) - w_{10} = 0 \]
\[
(4.51)
\]
with singularities at the points
\[
w = e^{i\tau}I \quad \text{and} \quad w = -e^{i\tau}I \\
\rho = 0 \quad \text{and} \quad \rho = \pi \]
\[
(4.52)
\]
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The $\ell^\mu$ and $n^\mu$ rays, which pass from these two points are determined by 

$$s_1 := \frac{\sin \tau \pm \sin \rho}{\cos \tau \pm \cos \rho},$$

$$s_2 := \sigma$$

and

$$s_3 := \chi$$

respectively. Notice that the concentration of rays at the above points of $SU(2)$ stop at two points of the regular manifold $U(2)$. Hence this LCR-structure should be rejected, because it is not defined in the entire $U(2)$ universe. I want to point out that all the spherically symmetric metrics of general relativity are compatible with this singular LCR-structure. Hence the fact, that the $K_{ztd}$ symmetric soliton does not exist for PCFT, its physical space is going to restrict the $SU(2, 2)$ rigged Hilbert space down to the rigged Hilbert space of the Poincaré representations as it will be described in section VII.

4.3 Free neutrino LCR-structure

We know that the neutrino is a massless particle therefore we will search for the corresponding automorphisms. The form of the infinitesimal $z$-rotation is the same. But the infinitesimal translation has to be taken for $\delta (x^0 - x^3)$, which is

$$\begin{align*}
\delta X^0 &= 0, \quad \delta X^1 = 0 \\
\delta X^2 &= -i \varepsilon z^0 X^0, \quad \delta X^3 = 0
\end{align*}$$

(4.53)

A linear trajectory $\xi^a = v^a \tau + c^a$ of a free massless particle implies a reducible quadratic Kerr polynomial (1.6), which has an analogous form. The two linear polynomials are arranged such that

$$X^{11} + a X^{31} = 0, \quad X^{02} = 0$$

(4.54)

for the left and right homogeneous coordinates. Then in $CP(3)$, the intersections of the line $\hat{x}$ with the first and second planes are

$$\begin{align*}
\hat{x} &:= i X_2 X_1^{-1} \\
-a(x^1 + ix^2) X^{01} + [a(x^0 + x^3) + i] X^{11} &= 0 \\
X^{02} &= 0
\end{align*}$$

(4.55)

Notice that this LCR-structure is regular in the present affine chart ($\det(X_1) \neq 0$), because $[a(x^0 + x^3) + i] \neq 0$.

The convenient structure coordinates are

$$X^{n \bar{i}} = \begin{pmatrix}
1 & 0 \\
\alpha z^1 & 1 \\
-i z^0 & z^1 (1 + i \alpha z^0) \\
-z^1 & -i z^0
\end{pmatrix}$$

(4.56)

The LCR-structure conditions ($X^j E_{ij} X = 0$) are
The flat LCR-tetrad and its differential conditions are found following the general computational rules

\[ L = du - iaz^1 dz^1 + iaz^1 d\overline{z}^1, \quad M = dz^1, \quad N = dv \]

\[ u := \frac{z^0 + x^3}{2}, \quad v := \frac{z^0 - x^3}{2} \]

\[ dL = 2iaM \wedge \overline{M}, \quad dM = 0, \quad dN = 0 \]

(4.59)

The integral curves \( x^\mu_L(s) \) of the causal vector \( L^\mu \partial_\mu \) are found using the definition of the projective coordinates and the fact that \( z^0 \) and \( z^1 \) are constant along the curves, because \( L^\mu \partial_\mu z^0 = 0 = L^\mu \partial_\mu z^1 \). We assume the ray (affine) parameter \( s := z^0 = x^0 + x^3 \) and the ray labels \( s_1 = \text{Re}(z^0) \), \( s_2 = \text{Re}(z^1) \), and \( s_3 = \text{Im}(z^1) \). Then the jacobian is

\[ ds \wedge ds_1 \wedge ds_2 \wedge ds_3 = \frac{8}{a(x^0 + x^3)^2 + 1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \]

(4.60)

Hence the caustic of the causal rays is at infinity as we algebraically found it above.

The coordinate singularity is hidden at infinity, like in the "light-cone" LCR-structure. In order to "see" it, we have to work in the patch det \( X^2 \neq 0 \), where

\[ \tilde{x}' := -iX_1(X_2)^{-1} = \tilde{x}^{-1} \]

\[ -(x'^1 + ix'^2)X^{21} + [(x'^0 + x'^3) - ia]X^{31} = 0 \]

(4.61)

The singularity occurs at

\[ \det X_2 = 0 \]

\[ x'^0 - x'^3 = 0, \quad x'^1 = 0 = x'^2 \]

(4.62)

which describes an object moving with the velocity of light.

Using the transformation (1.38), we may have a global view of the LCR-manifold in the "bounded realization" coordinate chart \( Y^{ni} \) where the two planes take the form
\[(a + 1)Y^{11} + (1 - a)Y^{31} = 0 \quad , \quad Y^{02} + Y^{22} = 0 \]
\[(1 - a)w_{21}Y^{01} + [(a + 1) + (1 - a)w_{22}]Y^{11} = 0 \]
\[(1 + w_{11})Y^{02} + w_{12}Y^{12} = 0 \]

and the singular surface \((\det Y_1 = 0)\) is
\[
[w_{11}w_{22} - w_{12}w_{21} + 1 + w_{11} + w_{22}] + a[-w_{11}w_{22} + w_{12}w_{21} + 1 + w_{11} - w_{22}] = 0
\]
\[
\cos \tau + \cos \rho = 0 \quad , \quad \sin \tau + \sin \rho \cos \sigma = 0
\]
\[
\tau \in (0, 2\pi) \quad , \quad \rho \in [0, 2\pi) \quad , \quad \sigma \in [0, \pi]
\]

Notice the difference between the "light-cone" LCR-manifold and the present "neutrino" LCR-manifold. The former is singular in the entire null infinity \((\text{scri}^+ \text{ and scri}^-)\) \((3.16)\), while the present is singular at a part of it. This subset must satisfy

\[
(sin \rho)(sin \sigma) = 0 \quad ; \quad \rho \in [0, 2\pi), \quad \sigma \in [0, \pi]
\]

For \(\sigma = 0\), we find the \(\tau = 2\pi - \rho\) curve and for \(\sigma = \pi\), we find the \(\tau = \rho\) curve.

The \((4.59)\) integrability conditions of the flat "neutrino" LCR-tetrad implies that its relative invariants are \(\Phi_1 = 2a\) and \(\Phi_2 = 0 = \Phi_3\). But the LCR-tetrad implied by the Kerr-Schild ansatz is

\[
\ell = L \quad , \quad m = M \quad , \quad n = N + f(v)L
\]

where \(f(v)\) is an arbitrary function of \(v\). The LCR-structure relations are

\[
d\ell = 2iam \wedge m \quad , \quad dm = 0 \quad , \quad dn = -f'\ell \wedge n + 2iafm \wedge m
\]

Notice that the curved LCR-structure changes category with the relative invariants being now \(\Phi_1 = 2a\), \(\Phi_2 = 2af\) and \(\Phi_3 = 0\). This is the category of the "natural \(U(2)\)" LCR-structure \((3.21)\). We see that the right part of the massless soliton is no longer trivial, like in the flatprint case. That is the curved neutrino is not a smooth deformation of the flat one. Does it mean that the neutrino small mass is due to its gravitational dressing?

### 4.4 The muon and tau generations

In the spinorial formulation \([24]\) of general relativity the conformal tensor is a spinorial tensor \(\Psi_{A B C D}\). A geodetic and shear free null congruence \(\ell^\mu = \lambda^A A^B \sigma^b_{A B} \rho^\mu_b\) satisfies the condition

\[
\Psi_{A B C D} \lambda^A \lambda^B \lambda^C \lambda^D = 0
\]

This is the spinorial form of the relation \((3.10)\). Formally expanding this relation relative to the gravitational constant at a point, we find a general quartic polynomial plus higher order terms. We may generally arrange the tetrad \(e^\mu_b\) such that the first
term of the expansion is identified with the Kerr function of the formal flatprint of $\ell_\mu$, and subsequently $X^4$ is projectively related with $(1, b)$. But the two real vectors $\ell_\mu$ and $n_\mu$ of the LCR-tetrad are geodetic and shear-free null congruences relative to the defined class of metrics (5.1). Hence locally the number of gravitationally permitted solutions are restricted to be roots of quadratic, cubic and quartic Kerr polynomials. We claim that the limited number of three particle generations observed in nature is related to the above limitation on the degree of the Kerr polynomial.

In the previous subsections we have found that an irreducible and a reducible quadratic Kerr polynomial provides the static "electron" and the stationary "neutrino" LCR-structures respectively. Proceeding into the case of cubic and quartic polynomials I have not found regular irreducible LCR-structures. We should expect it, because we know that muon and tau leptons are not static particles. But reducible symmetric cubic and quartic polynomials may be found. The z-rotation invariant cubic polynomial is

$$K_z = A_{001}(X^0)^2X^1 + A_{003}(X^0)^2X^3 + A_{012}X^0X^1X^2 + A_{023}X^0X^2X^3 + A_{122}X^1(X^2)^2 + A_{223}(X^3)^2X^3$$

(4.68)

The static axially symmetric cubic polynomial is reducible

$$K_{zt} = [A_{001}X^0X^1 + A_{003}(X^0X^3 - X^1X^2)]X^0$$

(4.69)

which has the form of quadratic electron up to a factor $X^0$. On the other hand the massless automorphism restricts (4.68) to

$$\delta X^0 = 0 \quad , \quad \delta X^1 = 0$$

$$\delta X^2 = -i\epsilon^{03}X^0 \quad , \quad \delta X^3 = 0$$

(4.70)

$$K_{zt'} = [A_{001}X^1 + A_{003}X^3](X^0)^2$$

which is analogous to electron neutrino two planes of $CP(3)$, up to a factor $(X^0)^2$.

The z-rotation quartic invariant polynomial is

$$K_z = A_{0011}(X^0)^2(X^1)^2 + A_{0013}(X^0)^2(X^1)(X^3) + A_{0033}(X^0)^2(X^3)^2 + A_{0112}(X^0)(X^1)^2(X^2) + A_{0123}(X^0)(X^1)(X^2)(X^3) + A_{0233}(X^0)(X^2)(X^3)^2 + A_{1122}(X^1)(X^2)^2(X^3) + A_{2233}(X^2)(X^3)^2(X^3)$$

(4.71)

The static axially symmetric quartic polynomial has the form

$$K_{zt} = A(X^1X^2 - X^0X^3)^2 + BX^0X^1(X^1X^2 - X^0X^3) + C(X^0X^1)^2$$

(4.72)

which is essentially the product of two independent quadrics (4.3). The massless automorphism restricts (4.71) to

$$K_{zt'} = [A_{0011}(X^1)^2 + A_{0013}X^1X^3 + A_{0033}(X^3)^2](X^0)^2$$

(4.73)
which is essentially the product of two independent reducible quadrics of CP(3).

The general meaning of the LCR-structures locally described with reducible surfaces of CP(3) may be realized considering the case of the product of ruled surfaces with different trajectories \( \xi_a(\tau) \) and \( \tilde{\xi}_a(\tau) \). Then we may have the left and right columns from different trajectories

\[
X^{n\beta} = \begin{pmatrix}
\lambda \\
-i\xi_a(z^0)\sigma^a\lambda \\
-i\tilde{\xi}_a(z^0)\sigma^a\tilde{\lambda}
\end{pmatrix}
\]

\[
\lambda = \begin{pmatrix}
z^1 \\
1
\end{pmatrix}, \quad \tilde{\lambda} = \begin{pmatrix}
-z^1 \\
1
\end{pmatrix}
\]

(4.74)

In the zero algebraic gravity approximation we have the structure conditions

\[
\rho_{ij} = X^\dagger \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix} X = \begin{pmatrix}
i(\xi_a - \xi_a)\lambda^\dagger\sigma^a\lambda & i(\xi_a - \tilde{\xi}_a)\lambda^\dagger\sigma^a\tilde{\lambda} \\
-i(\xi_a - \tilde{\xi}_a)\tilde{\lambda}\sigma^a\lambda & i(\xi_a - \tilde{\xi}_a)\tilde{\lambda}^\dagger\sigma^a\tilde{\lambda}
\end{pmatrix} = 0
\]

(4.75)

where \( z^0, z^1 \) and \( \tilde{z}^0, \tilde{z}^0 \) are the structure coordinates of the two independent trajectories. The null integral curves (LCR-rays) of the retarded \( \ell^\mu \) congruence are labeled by the three independent functions \( \text{Re}(z^0), \text{Re}(z^1) \) and \( \text{Im}(z^1) \) of the structure coordinates \( z^\beta \). The first is the wavefront surface. Hence, the algebraic trajectory \( \xi^\alpha(z^0) \) is the movement of the source of retarded rays in the grassmannian space \( G(4, 2) \). In the case of the advanced \( n^\mu \) rays the \( z^\beta \) structure coordinates are related to the second trajectory.

5 EMERGENCE OF ELECTROWEAK FIELD

In conventional field theory the electroweak interactions are imposed as \( U(2) \) connections. In the Cartan formalism the group \( U(2) \) gauge field \( B = B_{I\mu}dx^\mu t_I \) with \( t_I \) being the generators of the Lie algebra of \( U(2) \) have generally non-vanishing curvature \( F = dB - iB \wedge B \). We precisely have

\[
B = B_{I\mu}dx^\mu t_I = \begin{pmatrix}
A \\
W \\
Z
\end{pmatrix}, \quad t_0 = I, \quad t_j = \frac{1}{2}\sigma^j
\]

\[
F = dB - iB \wedge B \rightarrow DF := dF + iB \wedge F - iF \wedge B = 0
\]

(5.1)

where \( A_\mu, Z_\mu \) and \( W_\mu \) are the electromagnetic, neutral and charged fields respectively. In the computation of the electron and its neutrino distributional LCR-structures, these fields appear as gravitational and electromagnetic dressing distributions with precise compact singular support. The purpose of the present section is to provide the algorithmic derivation of the electroweak connection observed in the SM formulation. The surprising result is that the electroweak
connection is directly related with a precise LCR-tetrad. This essentially generalizes the surprising identification of the electron electromagnetic dressing $A_{\mu}(x)$ with the vector $\ell_{\mu}$ of the LCR-tetrad (and the induced gravitational dressing).

The first observation is that the LCR-structure conditions $\rho_{ij}$ (2.28) form a $2 \times 2$ hermitian matrix, implying that the LCR-tetrad $i(\partial - \bar{\partial})\rho_{ij}$ may be cast as a hermitian 1-form. But recall that the LCR-tetrad is not uniquely determined because of the tetrad-Weyl symmetry (2.36-2.37). Hence we may make a tetrad-Weyl transformation (2.37) fixing the relative invariants $\Phi_1 = 1 = -\Phi_2$. Then this special hermitian LCR-tetrad

$$\hat{e}' := \left( \ell' \ \ m' \ \ n' \right) = i(\partial - \bar{\partial}) \begin{pmatrix} \rho_{11}' & \rho_{12}' \\ \rho_{12}' & \rho_{22}' \end{pmatrix}$$

(5.2)

considered as a $U(2)$ connection has the generally non-vanishing Cartan curvature (3.21)

$$d\hat{e}' - i\hat{e}' \wedge \hat{e}' = \Omega \neq 0$$

(5.3)

Then identifying the Cartan $U(2)$ connection $\hat{e}'$ (5.2) with the (5.1) gauge potential

$$\hat{e}' := \left( \ell' \ \ m' \ \ n' \right) = \begin{pmatrix} A \\ W \\ Z \end{pmatrix}$$

(5.4)

we find the direct relation between the electroweak potentials with a member of the LCR-tetrad class of the solitonic LCR-structure. We precisely have

$$B_{0\mu} + \frac{1}{2}B_{3\mu} = \ell'_\mu \ , \ B_{0\mu} - \frac{1}{2}B_{3\mu} = n'_\mu \ , \ \frac{1}{2}(B_{1\mu} + iB_{2\mu}) = m'_\mu$$

(5.5)

$$F_{0\mu \nu} = \partial_\mu B_{0\nu} - \partial_\nu B_{0\mu} \ , \ F_{i\mu \nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu} - \epsilon_{ijk} B_{j\mu} B_{k\nu}$$

In order to clarify the procedure we will give two precise examples. The first is the simple observation that the "natural $U(2)$" LCR-structure (3.18) has vanishing electroweak field strength. The second example is the "electron" LCR-structure. The relative invariant (4.23) are tetrad-Weyl transformed to reach the conditions $\Phi'_1 = 1 = -\Phi'_2$. We find

$$N = -\frac{2(r^2 + a^2 \cos^2 \theta)}{r^2 + a^2 + h} \Lambda$$

$$M'M = -\frac{2a \cos \theta}{r^2 + a^2 \cos^2 \theta} \Lambda$$

(5.6)

The electromagnetic dressing (4.20) is found with $\Lambda = \frac{\alpha}{4\pi(r^2 + a^2 \cos^2 \theta)}$. Then the
electroweak connection $B$ (5.3) is found with

$$
\Lambda = \frac{qr}{4\pi(r^2 + a^2 \cos^2 \theta)}
$$

$$
N = -\frac{qr}{2\pi(r^2 + a^2 + h)}
$$

$$
M\bar{M} = -\frac{qra \cos \theta}{2\pi(r^2 + a^2 \cos^2 \theta)^2}
$$

up to an $M$ phase tetrad-Weyl transformation. That is, we find the following electroweak potentials (dressings) of the electron

$$
A = \frac{qr}{4\pi(r^2 + a^2 \cos^2 \theta)} (dt - dr - a \sin^2 \theta d\varphi)
$$

$$
Z = \frac{-qr}{4\pi(r^2 + a^2 \cos^2 \theta)} (dt + \frac{r^2 + 2a^2 \cos^2 \theta - a^2 - h}{r^2 + a^2 + h} dr - a \sin^2 \theta d\varphi)
$$

$$
W = \frac{-M}{\sqrt{2(r^2 + ia \cos \theta)}} [-ia \sin \theta (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i a \sin \theta (r^2 + a^2) d\varphi]
$$

(5.8)

where the $h = -2mr + q^2$ and the tetrad-Weyl factor $M$ will be computed below through the third scalar (relative invariant) dressing.

The third (complex) relative invariant $\Phi'_3$ is not completely fixed. Its phase is absorbed by $W$ and the remaining scalar real field

$$
|\Phi'_3|^2 = \frac{(4\pi a)^3 \sin^2 \theta \cos \theta (r^2 + a^2 + h)^2}{q^2 r (r^2 + a^2 \cos^2 \theta)^2}
$$

(5.9)

will be finally related with the Higgs field of the SM. The $U(2)$ gauge field is directly related to the LCR-tetrad and the breaking of the tetrad-Weyl symmetry through $\Phi'_1 = 1 = -\Phi'_2$, while the value of the the third relative invariant $|\Phi'_3|$ indicates the difference of the compact deSitter vacuum (with dark Energy) and the matter sector. The electroweak field strength and $\Phi'_3$ exist even in the case of vanishing algebraic gravity, that is electroweak field strength (de Sitter space curvature) and $\Phi'_3$ may not vanish even without algebraic gravity. The above form (5.9) of $|\Phi'_3|$ indicates that at the large universe scale we have the regions of matter concentrations with non-vanishing algebraic gravity, and the asymptotic empty space being a de Sitter space with vanishing algebraic gravity, electroweak field strength and $|\Phi'_3| \simeq 0$. But there may be an intermediate region where algebraic gravity vanishes while the third scalar relative invariant does not vanish.

The tetrad-Weyl transformation is the local symmetry of the fundamental geometric LCR-structure. This symmetry is broken by the local Lorentz $SO(1,3)$ transformation of the tetrad, which preserves the Einstein metric. It is also broken by the electroweak $U(2)$ transformation. That is the $SO(1,3)$ and $U(2)$ transformations are transversal to the tetrad-Weyl transformations.
6 GLUONIC FIELD AND QUARKS

The hadronic sector of the elementary particles is (about) a copy of the leptonic sector relative to the electroweak interactions. Quarks simply have the additional gluonic interaction, which should provide a confining mechanism. The SM does not explain the lepton-quark correspondence, while the artificial add-on of the SU(3) gauge group gives some answers to some phenomena and the annihilation of the anomalies, but it fails to imply (in the continuum) confinement, which is the characteristic property of strong interactions.

The tetrad-Weyl symmetry (2.36) of PCFT is highly constrained. The only permitted field equations have the following two dual forms

\[
\frac{1}{\sqrt{-g}} (D_\mu)_{ij} \{ \sqrt{-g} (\Gamma^{\mu\rho\sigma} + \Gamma^\rho_{\mu\sigma}) F_{j\rho\sigma} \} = 0
\]

\[
\Gamma^{\mu\rho\sigma} := \frac{1}{2} [(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho m^\sigma - n^\sigma m^\rho) + (n^\mu m^\nu - n^\nu m^\mu)(\ell^\rho m^\sigma - \ell^\sigma m^\rho)]
\]

\[
F_{j\mu\nu} := \partial_j A_{\mu\nu} - \partial_{\mu} A_{j\nu} - \gamma f_{ijkl} A_{\mu l} A_{k\nu}
\]

\[
(D_\mu)_{ij} := \delta_{ij}\partial_\mu - \gamma f_{ijkl} A_{k\mu}
\]

\[
\sqrt{-g} = \frac{1}{4} \epsilon^{\mu\rho\sigma\tau} (\ell_\mu n_\rho m_\sigma m_\tau)
\]

(6.1)

No scalar or spinorial sources are permitted because simply they would break tetrad-Weyl symmetry. Besides only the tetrad-Weyl covariant self-dual forms \([V_1]\) and \([V_2]\) (3.1) appear in the differential operator. Essentially the two \(\mp\) differential equations are the imaginary and real part of the sum of \([V_1]\) and \([V_2]\) self-dual forms. Notice that the one equation is the dual of the other equation.

In the first subsection we define the quarks as the permitted real source of the sum of \([V_1]\) and \([V_2]\) self-dual forms. The immediate consequence is the observed in nature lepton-quark correspondence, based on the same LCR-structure between the leptons and the corresponding quarks. I explicitly solve the differential equations in the "electron" LCR-structure and find the gluonic dressing of the corresponding quark which apparently continues to have the electroweak dressings. The magneto-gluonic field has line singularities not permitted for free field configurations. In the BEGS procedure this must be described with colorless distributions of compact support. In the second subsection I find convex bag-like picture for the hadronic structures. In the third subsection I explicitly find that the "natural U(2)" LCR-structure does not have gluonic sources. Recall that this LCR-structure is the vacuum of the electroweak interactions and the origin of the de Sitter metric (3.22), which describes dark energy.

6.1 Emergence of quarks

The quarks emerge in PCFT as the permitted distributional sources

\[
( - ) \rightarrow \frac{1}{\sqrt{-g}} (D_\mu)_{ij} \{ \sqrt{-g} [(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho m^\sigma - n^\sigma m^\rho)] \} = -i k^\rho_i
\]

\[
( + ) \rightarrow \frac{1}{\sqrt{-g}} (D_\mu)_{ij} \{ \sqrt{-g} [(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho m^\sigma - n^\sigma m^\rho)] \} = -i k^\rho_i
\]

(6.2)
where \( k^i_\nu(x) \) is a real vector field. That is, taking into account that the sum of the two terms is self-dual we find

\[
\frac{1}{2}(G^\mu_\nu - i \ast G^\mu_\nu) := \Gamma^\mu_\nu_\rho_\sigma F_{\rho_\sigma} = \frac{1}{2}\Gamma^\mu_\nu_\rho_\sigma(F_{\rho_\sigma} - i \ast F_{\rho_\sigma})
\]

\[
\Gamma^\mu_\nu_\rho_\sigma = \frac{1}{2}[(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\rho m^\sigma - n^\sigma m^\rho) + (n^\mu m^\nu - n^\nu m^\mu)(\ell^\rho m^\sigma - \ell^\sigma m^\rho)]
\]

\[
(6.3)
\]

In (-) PDE \( k^i_\nu \) is a source of the real part \( G^\mu_\nu \), while in (+) \( k^i_\nu \) is a source of its dual part imaginary part. The PDEs look like the equations of a gauge field with a color-electric and color-magnetic source respectively. Notice the essential difference of the present equations (6.1) and the conventional gluonic equations. The covariant gauge field derivative \( (D_\mu)_{ij} \) applies only on the tetrad-Weyl invariant part of the gauge field and not over the entire gauge field. Notice that the LCR-structure defining equations completely decouple from the gauge field equations. The LCR-structure is first fixed and after we proceed to the solution of the field equations, which involve the gluonic gauge field. This property of PCFT is essentially behind the physical observation of the lepton-quark correspondence! That is, a quark has the same LCR-structure with the corresponding lepton and the implied electroweak gauge field (5.2). But the quark has in addition a stable non-vanishing distributional gauge field configuration "dressing" (from which it gets its color), while the lepton has vanishing gluonic "dressing".

Like in the case of the static axisymmetric LCR-structure (static electron) we looked for a solution in the abelian Cartan subgroup of the Poincaré group, we now look for a solution with only \( A_3^\mu \) and \( A_8^\mu \) non-vanishing, the abelian subgroup of \( SU(3) \). That is we will look for solutions in the abelian PDE (6.2), which admit compact distributional sources. Then Stoke’s theorem will be applied on their distributional singularities. We will first look for a solution of the electro-gluonic form (-) and finally we will show that \( \epsilon^{\mu\nu_\rho_\sigma}\partial_\nu F_{\rho_\sigma} = 0 \) does not permit magneto-gluonic solutions of (+).

It is more convenient to make calculations using the differential forms. In the vanishing gravity case (4.13), the (-) PDE takes the form

\[
d\{(N^\rho M^\sigma F^+_{\rho_\sigma})L \wedge M + (L^\rho M^\sigma F^-_{\rho_\sigma})N \wedge M\} = i \ast k_j
\]

(6.4)

We find the non-vanishing closed 2-forms (with real compact sources) in the case of the flatprint massive LCR-tetrad

\[
d\left\{\frac{C_j^i}{\sin \theta (r - ia \cos \theta)} L \wedge M + \frac{C_j^i (r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} N \wedge \overline{M}\right\} = i \ast k_j
\]

(6.5)

\[
G_j^+ := \frac{1}{2}(G_j - i \ast G_j) = \frac{C_j^i}{\sin \theta (r - ia \cos \theta)} L \wedge M + \frac{C_j^i (r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} N \wedge \overline{M}
\]

where \( C_j^i \) and \( C_j^i \) with \( j = 3, 8 \) are arbitrary complex constants, which are fixed using Stokes’ theorem and the reality conditions for gluonic sources. That is, we have
\[(L^\mu M^\nu F_{j\rho\sigma}) = \frac{C''(r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} \quad , \quad (N^\nu M^\nu F_{j\rho\sigma}) = \frac{C''}{\sin \theta (r - ia \cos \theta)}\]

\[(L^\rho M^\sigma F_{j\rho\sigma}^+) = \frac{C''(r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} \quad , \quad (N^\rho M^\nu F_{j\rho\sigma}^+) = \frac{C''}{\sin \theta (r - ia \cos \theta)}\]  \hspace{1cm} (6.6)

\[F^+_{j\rho\sigma} := \frac{1}{3}(F_{j\rho\sigma} - i * F_{j\rho\sigma}) \quad \text{and} \quad F_{j\rho\sigma} := \partial_\rho A_{j\sigma} - \partial_\sigma A_{j\rho} \quad , \quad j = 3, 8\]

We apparently have

\[[(L^\mu N^\nu - M^\mu M^\nu) G_{j\mu\nu}^+] = 0 \hspace{1cm} (6.7)\]

\[F^+ = -G^+ - f(L \wedge N - M \wedge M)\]

In order to avoid any ambiguity, we assume \(f = 0\). Hence

\[F_{j\rho\sigma}^+ = -\frac{C''}{\sin \theta (r - ia \cos \theta)}(L^\rho M^\sigma - L^\sigma M^\rho) - \frac{C''(r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} (N^\rho M^\nu - N^\nu M^\rho)\] \hspace{1cm} (6.8)

For the static flatprint LCR-tetrad the solutions have the explicit forms

\[F_j - i * F_j := -\frac{2C''}{\sin \theta (r - ia \cos \theta)} L \wedge M - \frac{2C''(r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} N \wedge M = \]

\[= \frac{2C''}{\sqrt{2}} \left[-\frac{2a \sin \theta dt \wedge dr + \frac{1}{a^2} \sin \theta dt \wedge d\theta + \frac{a^2 \sin^2 \theta dr \wedge d\theta}{(r^2 + a^2) \sin \theta} \right] + \frac{2C'' - C'''}{\sqrt{2}} \left[\frac{a \sin \theta dt \wedge d\phi + \frac{1}{a^2} \sin \theta dt \wedge d\phi - \frac{a^2 \cos^2 \theta}{(r^2 + a^2) \sin \theta} dr \wedge d\theta\right]\] \hspace{1cm} (6.9)

After a straightforward calculation I find

\[\int_{t, r = \text{const.}} \left[-\frac{2C''}{\sin \theta (r - ia \cos \theta)} L \wedge M - \frac{2C''(r - ia \cos \theta)}{(r^2 + a^2) \sin \theta} N \wedge M\right] = \frac{(2C'' + C''')4\pi a}{\sqrt{2}} =: -i\gamma_j\] \hspace{1cm} (6.10)

which implies that the (dimensionless) constants \(\gamma_j\) and \(\gamma_8\) must be real for the sources to be real and the original field equations to be satisfied.

From \(\Box M\) we see that we actually have the sum of a retarded and an advanced null 2-form. Assuming \(2C'' - C''' = 0\) (because it is not determined by the gluonic charge), the arbitrary constants are completely fixed and the solutions are

\[F_j = \frac{-\gamma_j}{4\pi a} \left[\frac{a}{\tan^{-1} \frac{a}{r} (\tan^{-1} \frac{a}{r} dt - rd\phi)}\right]
\]

\[*F_j = \frac{\gamma_j}{4\pi} \left[\frac{1}{a \sin \theta} dt \wedge d\theta + \frac{a \sin \theta}{r^2 + a^2} dr \wedge d\theta + \sin \theta d\theta \wedge d\varphi\right]\] \hspace{1cm} (6.11)
where $j = 3, 8$ for the $su(3)$ color algebra and the corresponding potentials being apparent.

It is interesting to compare the electromagnetic dressing potential of the "electron" LCR-structure

$$F^{EM} = \frac{q}{4\pi(r^2 + a^2 \cos^2 \theta)} [(r^2 - a^2 \cos^2 \theta) dt \wedge dr - 2a^2 r \cos \theta \sin \theta dt \wedge d\theta +$$

$$+ 2a^2 r \cos \theta \sin \theta d\theta \wedge d\theta + a(r^2 - a^2 \cos^2 \theta) \sin^2 \theta dr \wedge d\varphi -$$

$$- 2ar(r^2 + a^2) \cos \theta \sin \theta d\theta \wedge d\varphi =$$

$$= d[\frac{q r}{4\pi(r^2 + a^2 \cos^2 \theta)} (dt - dr - a \sin^2 \theta d\varphi)]$$

$$r^4 - [(r^1)^2 + (r^2)^2 + (r^3)^2 - a^2]r^2 - a^2(r^3)^2 = 0$$

(6.12)

with the electromagnetic potential in cartesian coordinates

$$A^{EM} = \frac{q r}{4\pi(r^2 + a^2 \cos^2 \theta)} (dt - dr - a \sin^2 \theta d\varphi) =$$

$$= \frac{q r^3}{4\pi(r^2 + a^2)(r^3)} (dx^0 - \frac{x^1}{r^2 + a^2} dx^1 - \frac{x^2 - a x^3}{r^2 + a^2} dx^2 - \frac{x^3}{r} dx^3)$$

(6.13)

The most important differences are the singularities relative to the rotation parameter $a$, and the cylindrical variable $\rho$ of the magneto-gluonic field (and potential) as we will explicitly show below.

The gluonic dressing potential of the static quark LCR-structure in cartesian coordinates

$$x^0 = t$$
$$x^1 = (r \cos \varphi + a \sin \varphi) \sin \theta$$
$$x^2 = (r \sin \varphi - a \cos \varphi) \sin \theta$$
$$x^3 = r \cos \theta$$

(6.14)

inverted into

$$dt = dx^0$$
$$dr = \frac{r}{r^2 + a^2(x^3)^2} [r^2 x^1 dx^1 + r^2 x^2 dx^2 + (r^2 + a^2)x^3 dx^3]$$
$$d\theta = \frac{\rho}{r^2 + a^2(x^3)^2} [(r^2 + a^2)x^3(x^1 dx^1 + x^2 dx^2) - r^2 \rho^2 dx^3]$$
$$d\varphi = -\frac{x^2}{r^2 dx^1 + \frac{x^1}{\rho^2} dx^2 - \frac{a}{r^2 + a^2} dr$$

$$\rho^2 := (x^1)^2 + (x^2)^2 , \quad \frac{x^2}{r^2 + a^2} + \frac{(x^3)^2}{r^2} = 1$$

(6.15)

I find the following gauge potential (up to a gauge transformation)

$$A^g = \frac{\gamma_i}{4\pi a} (\tan^{-1} \frac{r}{a} dt - rd\varphi) =$$

$$= \frac{\gamma_i}{4\pi a} [\tan^{-1} \frac{1}{a} dx^0 + \frac{r x^2}{(x^1)^2 + (x^2)^2} dx^1 - \frac{r x^1}{(x^1)^2 + (x^2)^2} dx^2]$$

$$= \frac{\gamma_i}{4\pi a} [\tan^{-1} \frac{1}{a} dx^0 - rd \arctan \frac{x^2}{x^1}$$

$$d \arctan \frac{x^2}{x^1} = \frac{x^2}{(x^1)^2 + (x^2)^2} dx^1 + \frac{x^1}{(x^1)^2 + (x^2)^2} dx^2$$

(6.16)
where I considered the gauge transformation $A^{(g)}_j \rightarrow A^{(g)}_j - \frac{\gamma_j r}{4\pi a^2 (r^2 + x^2)^2} dr$. The gauge field strength is

$$F^{(g)}_j = \frac{-\gamma_j r}{4\pi a^2 (r^2 + x^2)^2} dt \wedge dr + dr \wedge d\varphi =$$

$$= \frac{1}{4\pi (r^2 + x^2)(x^2 + (x^3)^2)} [(\gamma_j r)^2 x^1 \wedge dx^1 + r^2 x^2 \wedge dx^2 + (r^2 + a^2)x^3 \wedge dx^3] +$$

$$+ \frac{1}{4\pi (r^2 + x^2)(x^2 + (x^3)^2)} [(2x^2 \wedge dx^3 - x^1 \wedge dx^3)$$

\[ (6.17) \]

Notice that the electromagnetic potential is real analytic relative to the rotation parameter $a$, while the magneto-gluonic potential and field strength are singular. Both also has a line singularity along the z-axis. Recall that the Dirac magnetic monopole field strength has the ordinary Coulomb singularity while line singularity appears in its magnetic potential. The above field strength line singularity \{$x^1 = 0 = x^2, x^3 \neq 0$\} cannot be removed. But the components of the gauge potential $A^{(g)}_j$ are locally integrable functions while the components of the field $F^{(g)}_j$ are not, as it usually happens in a proper Schwartz distribution.

Using the notation

$$F^{(g)}_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

\[ (6.18) \]

we find

$$\bar{E}^{(g)}_j = \frac{\gamma_j r}{4\pi a^2 (r^2 + x^2)^2} [(x^2 + (x^3)^2) x^1, x^2, (r^2 + a^2)x^3] = -\nabla A^{(g)}_j$$

$$\bar{B}^{(g)}_j = \frac{\gamma_j r}{4\pi a^2 (r^2 + x^2)^2} [(r^2 + a^2)x^1 x^3, (r^2 + a^2)x^2 x^3, -\rho^2 r^2] = \nabla \times \bar{A}^{(g)}_j$$

\[ (6.19) \]

r

\[ (6.19) \]

where the electro-gluonic $\bar{E}^{(g)}_j$ and magneto-gluonic $\bar{B}^{(g)}_j$ fields are singular at the disc \{$x^3 = 0, \rho \leq a$\} and besides $A^{(g)}_j$ is singular at the line $\rho = 0$. The corresponding potentials are

$$A^{(g)}_0 = \frac{\gamma_j r}{4\pi a^2} (\arctan \frac{x^3}{a})$$

$$A^{(g)}_j = \frac{\gamma_j r}{4\pi a^2} \left[ \frac{x^3}{(x^1)^2 + (x^3)^2}, \frac{x^1}{(x^1)^2 + (x^3)^2}, 0 \right]$$

\[ (6.20) \]

\[ \nabla \cdot A^{(g)}_j = 0 \]
where the vector potential $\vec{A}^{(g)}_j$ has a line singularity at the z-axis and a singularity along the negative part of the $x^1$ axis, which is the characteristic singularity implied by the gap $\varphi + 2\pi$ chosen to be the negative $x^1$ axis. Notice that both the gluonic potential and the field strength are singular while they do not have magnetic charge. They are not Dirac gluonic monopoles. These singularities are compatible with the Schwartz distribution, because

$$\lim_{\varepsilon \to 0} \frac{\Lambda}{\varepsilon} \int \vec{A}^{(g)}_j dx^1 dx^2 dx^3 = \lim_{\varepsilon \to 0} \frac{\Lambda}{\varepsilon} \int \frac{\gamma_j}{4\pi a} (\arctan \frac{r}{a}) \rho d\rho d\psi dz = \text{finite}$$

(6.21)

where $\Lambda$ is an arbitrary upper bound of the integration region. Hence the gauge potential is a distribution, but analogous calculations imply that the field strength components are not locally integrable. As usual, the potential is the base locally integrable function, on which the "ladder" of the corresponding Schwartz distributions is built.

The energy density is

$$u = \frac{1}{2} \sum_{3+8} (\vec{E}^\perp_j \cdot \vec{E}^\parallel_j + \vec{B}^\perp_j \cdot \vec{B}^\parallel_j) = \sum_{3+8} \frac{\gamma_j^2 r^2}{32 \pi^2 (r^2 + a^2)^2} \left[ \frac{1}{r^2 + a^2} + \frac{r^2 + a^2}{\sigma^2 \rho^2} \right]$$

$$\vec{E}^\perp_j, \vec{E}^\parallel_j = \frac{\gamma_j^2 r^2}{16 \pi^2 a \rho^2 (r^2 + a^2)(x^3)^2}$$

$$\vec{B}^\perp_j, \vec{B}^\parallel_j = \frac{\gamma_j^2 r^2 (r^2 + a^2)}{16 \pi^2 a \rho^2 (r^2 + a^2)(x^3)^2}$$

(6.22)

and the momentum density is

$$\vec{s} = \sum_{3+8} \vec{E}^\perp_j \times \vec{B}^\parallel_j = \sum_{3+8} \frac{\gamma_j^2 r^2}{16 \pi^2 a \rho^2 (r^2 + a^2)(x^3)^2} [x^2 \vec{e}^\perp_j - x^1 \vec{e}^\parallel_j]$$

(6.23)

The singularities of the magneto-gluonic dressing of quark is apparent in the corresponding energy densities and the circularly rotating Poynting vector.

### 6.2 Confinement problem

The well-known confinement mechanism\[11\] based on the surface singularity of a locally defined potential of the Dirac magnetic formalism does not apply here, because the quark defining PDE (6.2) (case (-)) assures that the magneto-gluonic charge vanishes.

From (6.11) we see that in the oblate coordinates the electro-gluonic part $E_j^{(g)} = \frac{\gamma_j}{4\pi (r^2 + a^2)} dr$ of the gluonic field is rather conventional but its magneto-gluonic part $B_j^{(g)} = \frac{\gamma_j}{4\pi a} dr \wedge d\varphi$ is a constant 2-form in the 3-dimensional space. The oblate coordinates are singular, therefore it is convenient to use the vector
fields (6.16) and the corresponding electro-gluonic and magneto-gluonic potentials (6.18) where \( A^{(g)}_j \) is defined up to a gauge transformation. Notice that this line singularity is present in the magneto-gluonic field too. It is not a singularity only on a non-globally defined magnetic potential like in the Dirac magnetic monopole.

Such a singularity appears in the magneto-gluonic flux through a surface \( S \) bounded by the closed loop \( \Gamma \) which has the form

\[
\phi_j = \int_S \vec{B}^{(g)}_j \cdot \vec{n} \, dS = \int_{S} (\vec{\nabla} \times \vec{A}^{(g)}_j) \cdot \vec{n} \, dS = \oint_{\Gamma} \vec{A}^{(g)}_j \cdot d\vec{r} \tag{6.24}
\]

In the cylindrical coordinates the above magneto-gluonic flux can be computed at a circle of radius \( \rho = \sqrt{(x_1^2 + x_2^2)} \) at a point \( x_3 = z \). We find

\[
x^1 = \rho \cos \psi, \quad x^2 = \rho \sin \psi, \quad dx^1 = -\rho \sin \psi \, d\psi, \quad dx^2 = \rho \cos \psi \, d\psi
\]

\[
\phi_j = \frac{\gamma r_0}{2a} \left\{ \frac{a^2 + z^2 - a^2}{2} + \sqrt{\left[ \frac{a^2 + z^2 - a^2}{2} \right]^2 + a^2 z^2} \right\}^{\frac{1}{2}}
\]

Hence we precisely find

\[
\begin{align*}
at \{ z = 0, \rho \leq a \} & \rightarrow \phi_j = 0 \\
at \{ z = 0, \rho > a \} & \rightarrow \phi_j = \frac{\gamma r_0}{2a} \sqrt{(\rho^2 - a^2)} \tag{6.26} \\
at \{ z \neq 0 \} & \rightarrow \phi_j = \frac{\gamma r_0}{2a}
\end{align*}
\]

But at a closed surface the magneto-gluonic flux must vanish, because \( \vec{\nabla} \cdot \vec{B}^{(g)} = 0 \) i.e. no compact (distributional) magneto-gluonic source exists. We interpret it that the line singularity enters and leaves the closed surface without internal source or sink. There is an additional singularity implied by the jump of the angle \( \psi = \arctan \frac{x_2}{x_1} \) at the ends of the \([ -\pi, \pi ]\) interval. This means that the singularity is the entire \( x - z \) plane, without magneto-gluonic charge.

In (6.24) we checked that the potentials define proper Schwartz distributions. Hence the line and surface singularities are "compatible" with the Schwartz distributions in \( \mathbb{R}^3 \). They do not affect it because the singular surfaces in \( \mathbb{R}^3 \), have dimension less than three. But they cannot be physically acceptable. Infinite singular straight lines and surfaces cannot traverse the universe, but curved line singularities (vortices) in a bounded domain may exist [11].

In the present case the extended singularities are removed by simply considering that the hadronic Schwartz distributions have compact support. Apparently this implies confinement (a MIT bag-like picture), because the gluonic fields and their sources are restricted into convex 3-dimensional bounded subsets of \( \mathbb{R}^3 \). Because of (1.5), it is a subset of tempered distributions, opening up the possibility to study hadrons in the context of BEGS procedure.
In the case of the "quantum" approach (the BEGS procedure) the computations seem to be feasible. We start by including to the asymptotic rigged Hilbert space the MIT bag bound states in dynamical spheroids. The relation (Lemma 1.4.3 in Hormander’s book\[16\])

\[ B \subset K \subset B\sqrt{3} \]

(6.27)

of the compact support \( K \) between two spheroids provides a good approximation for the hadron. On the other hand from a Payley-Wienner theorem (7.3.1 in Hormander’s book\[16\]) we know that the Fourier-Laplace transform of a distribution with compact support is an entire holomorphic function of the exponential type, which permits the application of the dispersion relations. The study of hydrogen-like atoms in the rigged Hilbert space of the tempered distributions is a very effective method\[4\]. Hence the above two ways of the incorporation of the compactness of the hadronic Schwartz distributions into the BEGS procedure seem to provide the feasible computations in the multihadron interactions.

6.3 The gluonic dressing of the ”natural U(2)” LCR-structure

We consider the ”natural U(2)” LCR-manifold as the compact ”vacuum” universe, because it is compatible with the Minkowski metric and the corresponding U(2)-Cartan curvature (the electroweak field strength) vanishes. In this subsection I will show that the gluonic field on the ”natural U(2)” LCR-manifold does not have any source, which I interpret that it does not correspond to any hadronic particle. These calculations are more convenient to be done in the following local coordinates \((z^\alpha, \tilde{z}^\beta)\) of ”natural U(2)” LCR-structure which has the structure coordinates \((w^\alpha, \tilde{w}^\beta)\) satisfying the structure conditions (3.18). The precise relations are

\[ w^0 = e^{i\frac{\theta}{2}} e^{i\frac{\phi}{2}}, \quad w^1 = e^{i\phi} \tan \frac{\theta}{2} \]

\[ \tilde{w}^0 = e^{-i\frac{\theta}{2}} e^{i\frac{\phi}{2}}, \quad \tilde{w}^1 = e^{-i\phi} \tan \frac{\theta}{2} \]

(6.28)

which imply

\[ z^0 = t - r' - 4il \ln(\cos \frac{\theta}{2}) \quad , \quad z^1 = e^{i\phi} \tan \frac{\theta}{2} \]

\[ \tilde{z}^0 = t + r' - 4il \ln(\cos \frac{\theta}{2}) \quad , \quad \tilde{z}^1 = e^{-i\phi} \tan \frac{\theta}{2} \]

(6.29)

The variable \( \frac{dr}{d\theta} \) is the \( U(1) \) parameter \( \tau \) and \( \frac{\phi}{2}, \theta, \phi \) are related with the Euler angles of \( SU(2) \). We finally find the following LCR-tetrad

\[ \ell_{\mu} dx^\mu = \Lambda [dt - dr' + 4l \sin^2 \frac{\theta}{2} d\phi] \]

\[ n_{\mu} dx^\mu = N [dt + dr' + 4l \sin^2 \frac{\theta}{2} d\phi] \]

\[ m_{\mu} dx^\mu = M [d\theta + i \sin \theta d\phi] \]

(6.30)
up to a tetrad-Weyl transformation. This tetrad is compatible with the Taub metric, therefore I call \((z^\alpha, \tilde{z}^\beta)\) "Taub" structure coordinates.

The corresponding closed self-dual null 2-forms, which are invariant under \(t\) and \(\varphi\) translations, have the form

\[
G' + \tilde{G}' = \frac{C'}{\sin \theta} \ell \wedge m \tag{6.31}
\]

and the static self-dual 2-form is

\[
F_j - i \ast F_j := -\frac{2C'}{\sin \theta} \ell \wedge m - \frac{2C''}{\sin \theta} n \wedge \overline{m} = -\frac{2(C'_j + C''_j)}{\sin \theta}(dt \wedge d\theta - i \sin \theta dr' \wedge d\varphi + 2l \cos \theta d\theta \wedge d\varphi) - \frac{2(C'_j - C''_j)}{\sin \theta}(i \sin \theta dt \wedge d\varphi - dr' \wedge d\theta) \tag{6.32}
\]

where the tetrad-Weyl factors are ignored. The source could come from the \(d\theta \wedge d\varphi\) term. Therefore we assume \(C'_j - C''_j = 0\). But even then, if we integrate over the surface \(t = \text{const}\) and \(r' = \text{const}\), we find zero too, because

\[
\int \int (F_j - i \ast F_j) = -4l(C'_j + C''_j) \int \int \frac{\cos \theta}{\sin \theta} d\theta \wedge d\varphi = -8\pi l(C'_j + C''_j) \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi - \varepsilon} d\ln |\sin \theta| = 0 \tag{6.33}
\]

Hence the "natural \(U(2)\)" LCR-manifold does not have any distributional compact source, and no further restriction can be found on \((C'_j + C''_j)\). This means that the electroweak vacuum does not have a gluonic source.

## 7 STANDARD MODEL DERIVATION

All the previous sections were devoted to the geometric consequences of the fundamental LCR-structure on the tangent space of the spacetime. In the present section I will describe how conventional "quantum field theory" emerges as a mathematical targeted harmonic analysis through the BEGS procedure. Electron and its neutrino are stable solitonic LCR-manifolds which must be the time asymptotic free structures of any other LCR-surface of \(C^4\). This background geometric solitonic decay phenomenon is essentially described by the Dyson expansion of the S-matrix into asymptotic free fields identified with the elementary particles viewed as Poincaré representations. In the context of BEGS formalism the Poincaré group and Bogoliubov causality are assumed and applied to the expansion of the S-matrix into representations of the Poincaré group. This procedure is usually called "causal approach" which is essentially equivalent to the ordinary QFT. The great advantage of the BEGS formalism is that it fully "respects" the mathematical properties of the Schwartz distributions and the
order by order built up of the S-matrix expansion starting from the symmetry properties of the free fields and using a BRS procedure for the elimination of their unphysical modes as briefly described in the introduction. But the Poincaré group and the elementary particles are axiomatically assumed like in ordinary QFT. PCFT geometrically determines the Poincaré group and all the "elementary particles" which appear in the SM, making the BEGS procedure a kind of targeted harmonic analysis in the Hilbert-Fock space of Poincaré group representations (precise free fields).

The convenient way to relate PCFT with the BEGS procedure is to consider the representations of the background $SU(2,2)$ group in the context of the generalized functions, which are necessary to be used in order to pass from the basis of its maximal compact $SU(2) \times SU(2)$ subgroup to its affine (non-compact) Poincaré subgroup. This is the subject of the first subsection. We first start with the simple example of the analogous $SU(1,1)$ case and continue with the relation between the $SU(2) \times SU(2)$ Lie algebra and the Poincaré Lie algebra. In the second subsection I will describe how the results of order by order expansion of the "quantum" BEGS procedure reconcile with the "classical" potentials (dressings) of the "electron" LCR-structure. Besides I will point out how the BEGS procedure provides the Einstein equations with a cosmological constant implying the origin of dark energy already appeared through (3.24) in the geometric level of PCFT. In the third subsection we reconcile the disc singularity of the dressings of the "free electron" LCR-structure with the point singularity of "classical electron" potential implied by the BEGS procedure.

The assumption of the LCR-structure as the fundamental dynamical geometric principle, contains all the observed particles as solitonic configurations. It is clear that the identification criterion should be the distributional singularities. Let me describe the case of every particle.

**Electron**: The free electron is the unique static LCR-structure with the disc singularity, moving with a constant velocity. It is represented with a massive Dirac free field, because of its fermionic gyromagnetic ratio. Notice that the free field representation of the corresponding rigged Hilbert space does not contain all the surface properties. Even its linear complex trajectory (recall that the free electron LCR-structure is a ruled surface) and its disc singularity do not appear in the corresponding free field rigged Hilbert space, on which QFT is built. Its distributional singularity is the wavefront singularity implied by its free field representation, like all the other waves.

**Neutrino**: The neutrino is the developable surface (consisting by two $CP(3)$ planes) corresponding to the electron ruled surface. The one plane has a complex trajectory while the other coincides with the trivial one of the light-cone LCR-structure. It is represented as the left-hand part of a massless Dirac field. Like in the electron case the simple Poincaré representation (free field) does not contain the geometric properties. I mention these points in order to point out that QFT is only a restricted view of the geometry.

**Scaling is broken**: The massive static "electron" LCR-structure is automorphic relative to the z-rotation and time-translation. The stationary "neutrino" LCR-structure is automorphic relative to the z-rotation and the light-cone
translational. But no regular solitonic LCR-structure exists which is automorphic relative to an additional dilation transformation. The Cartan subalgebra of the conformal SU(2, 2) group is not realized in the solitonic sectors.

**Graviton:** The general LCR-structure defines a class of Einstein metrics and the corresponding self-dual 2-forms. The Minkowski metric is compatible to the "flat" LCR-structures of the Shilov boundary of the SU(2, 2) classical domain. Linearized deformations of this classical domain define a symmetric tensor $h_{\mu\nu}$. In the case of the electron and the neutrino LCR-structures it satisfies the wave equation with source their singularities and the additional wavefront singularity of the free wave equation. Recall that the distributional solutions inherit[10] the singularities of their source and the principal symbol of the PDE. This last distributional singularity is identified with the spin-2 representation of the (free) graviton.

**Electroweak gauge fields:** After fixing the tetrad-Weyl symmetry by imposing the conditions $\Phi_1 = 1 = -\Phi_2$ and $\Phi = |\Phi_3|$ of the relative invariants we showed that the hermitian matrix (5.4) coincides with a tetrad of the corresponding class of LCR-tetrads. It is the SM gauge field with $\ell'$ the electromagnetic potential which coincides with the electromagnetic field of the Kerr-Newman manifold. The potentials describe 1-spin free fields (representations of the Poincaré group) identified with the corresponding wavefront singularities.

**Higgs particle:** The remaining relative invariant component $\Phi = |\Phi_3|$ is the remnant potential which plays the role of the Higgs field.

**Gluons and quarks:** The gluonic field appears in PCFT through the unique special PDE which is invariant under the tetrad-Weyl symmetry of the fundamental geometric LCR-structure. No sources are permitted to be included in the initial special PDE. For every leptonic generation a corresponding quark generation emerges as sources in explicit distributional solitonic solutions of the special symmetric PDE.

You may have noticed that we did not say anything yet for the field equations. We have not introduced neither the Einstein equations nor the $U(2)$ gauge field equations. Only the existing free fields were introduced. The field equations are essentially implied[29] by the BRS algorithm applied in the causal approach.

### 7.1 From SU(2,2) down to Poincaré rigged Hilbert space

Lindblad and Nagel[19] studied the semisimple group SU(1, 1) and showed that its compact subgroup $U(1)$ with the discrete basis determines the space of test functions of the SU(1, 1) rigged Hilbert space and they computed the eigenfunctions (Schwartz distributions) of its non-compact generators. In its bounded realization

$$E_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} Y^0 \\ Y^1 \end{pmatrix} = : \begin{pmatrix} Y^0 \\ wY^0 \end{pmatrix}$$

$$Y^\dagger E_B Y = \overline{Y^0}Y^0(1 - \overline{w}w) = 0 \quad (7.1)$$
$SU(1, 1)$ has the generators

$$J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ (7.2)

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0, \quad J_\pm := J_1 \pm iJ_2$$

$C_2 = J_0^2 - J_1^2 - J_2^2$

where $C_2$ is the Casimir invariant. The eigenstates of $J_0$ and $C_2$ define the following standard basis

$$<j, m|j, m'> = \delta_{mm'}$$

$$C_2|j, m> = j(j + 1)|j, m>, \quad J_0|j, m> = m|j, m>$$ (7.3)

In the unbounded realization

$$E_U = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} X^0 \\ -irX^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Y^0 \\ wY^0 \end{pmatrix}$$ (7.4)

$$X^\dagger E_U X = 2X^0 X^0 (r - r)^{-1} = 0$$

we find the real line, which is the boundary of the upper half-plane and the Cayley transformation between the projective coordinates is

$$r = i(I - w)(I + w)^{-1} \iff w = (iI - r)(iI + r)^{-1}$$ (7.5)

In this unbounded realization the $SU(1, 1)$ group is decomposed into the translation, dilation and conformal subgroups with the corresponding generators

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ (7.6)

$$[D, P] = -iP, \quad [D, K] = iK, \quad [K, P] = -2iD$$

which are related with the $J_i$ generators with the relations

$$J_0 = \frac{1}{2}(P' + K'), \quad J_1 = \frac{1}{2}(P' - K'), \quad J_2 = -D'$$

$$P' = J_0 + J_1, \quad K' = J_0 - J_1, \quad D' = -J_2$$ (7.7)

where the accents indicate that the non-compact generators are first transferred to the compact coordinates. The parabolic (energy) generator $P' = J_0 + J_1$ has a continuous spectrum. Its remarkable feature is that the discrete principal series $D'_{\pm}$ corresponds to the positive real line spectrum while $D'_{\pm}$ corresponds to the negative real line spectrum, because of the existence of the external automorphism
\((J_0', J_1', J_2') = (-J_0, -J_1, J_2)\)

\((P'', K'', D'') = (-P', -K', D')\)

which is the “temporal reflection” in \(su(1, 1)\) algebra and apparently commutes with dilation. Recall that the \(D_j^+\) is the basis of functions on the boundary which analytically extend in the interior (exterior) of the disc.

The existence of a maximal compact subgroup \(e^{-iJ_0\phi}\) defines the countable basis \(|j, m\rangle\) such that \(J_0|j, m\rangle = |j, m\rangle\), which determines the Hilbert space

\[H = \{x = \sum_m a_m|j, m\rangle ; \quad ||x|| = \sqrt{\sum_m |a_m|^2} < \infty\}\]

(7.9)

which is the central part of the Gelfand triplet \(S \subset H \subset S'\). \(S\) is the space of ”rapidly decreasing sequences”

\[S = \{a = \sum_m a_m|j, m\rangle ; \quad \lim_{|m| \to \infty} m^n a_m = 0 , \forall n\}\]

(7.10)

where the second line defines the infinite set of norms, which provide the strong topology of \(S\). Notice that \(p_0(a)\) is the norm of the Hilbert space. The distributions \(S'\) is the space of ”slowly increasing sequences”

\[S' = \{a' = \sum_m a'_m|j, m\rangle ; \quad \exists N \in \mathbb{N} : \lim_{|m| \to \infty} m^{-N} a_m = 0\}\]

(7.11)

Starting from the \(SU(1, 1)\) classical domain, the above abstract method becomes precise with the Celeghini et. al.\([5]\) method. The boundary \(w^{\infty} = 1\) of bounded realization of the \(SU(1, 1)\) classical domain provides the discrete Fourier transform with \(|j, m\rangle = \frac{1}{\sqrt{2\pi}} e^{im\phi}\). In the unbounded realization, the parabolic energy generator \(P\) is diagonalized. Its eigenstates belong in the distribution space \(S'\) and their expansion in the countable basis has been computed \([19]\).

Celeghini et. al.\([5]\) defined the rigged Hilbert space using the Laguerre functions

\[M_n(y) := e^{-y/2}L_n(y) , \quad n = 0, 1, 2, \ldots\]

(7.12)

where \(L_n(y)\) are the Laguerre polynomials. The fundamental operators of the Laguerre basis are

\[YM_n(y) := yM_n(y) , \quad (YD_y)M_n(y) := y\frac{d}{dy}M_n(y) , \quad NM_n(y) := nM_n(y)\]

(7.13)
with
\[ K_\pm = \pm(YD_y) + N + I - \frac{Y}{2} \quad , \quad K_3 = N + \frac{1}{2}I \]

\[ K_\pm M_n(y) = \sqrt{(n + \frac{1}{2} ± \frac{1}{2})(n + \frac{1}{2} ± \frac{1}{2})M_{n+1}(y)} \]
\[ K_3M_n(y) = (n + \frac{1}{2})M_n(y) \]  \hspace{1cm} (7.14)

\[ [K_3, K_\pm] = ±K_\pm \quad , \quad [K_+, K_-] = -2K_3 \]
\[ C_2 = K_3^2 - \frac{1}{2}(K_+, K_-) = -\frac{1}{4}I \]

and we find the unbounded operators
\[ Y = -(K_+ + K_-) + 2K_3 \]
\[ (YD_y) = \frac{1}{2}(K_+ - K_-) \]  \hspace{1cm} (7.15)

The relation of the compact support test functions with the precise continuous basis index \( y \) is
\[ |n> = \int_0^\infty dy M_n(y)|y> \quad ⇔ \quad |y> = \sum_{n=0}^\infty M_n(y)|n> \]  \hspace{1cm} (7.16)

The eigenstate state \( |y> \) is a distribution because it is a "slowly increasing sequence"
\[ |y> = \sum_{n=0}^\infty M_n(y)|n> ; \quad \exists N \in \mathbb{N} : \lim_{n \to \infty} n^{-N}M_n(y) = 0 \]  \hspace{1cm} (7.17)

In the present case of \( SU(2,2) \) group, its maximal compact subgroup is \( S(U(2) \times U(2)) \). The 15 \( SU(2,2) \) generators are
\[ H_0 = \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad H_j = \frac{1}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad H_{3+j} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^j \end{pmatrix} \]
\[ H_7 = \frac{1}{2} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad , \quad H_8 = \frac{1}{2} \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix} \]
\[ H_{8+j} = \frac{1}{2} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \quad , \quad H_{11+j} = \frac{1}{2} \begin{pmatrix} 0 & i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \]  \hspace{1cm} (7.18)

in the bounded realization. \( \sigma^j \) are the Pauli matrices.

In the unbounded realization the conventional generators are
\[ P^\mu = \begin{pmatrix} 0 & 0 \\ \sigma^\mu & 0 \end{pmatrix} \quad , \quad K^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix} \]
\[ S^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \quad , \quad B^j = i \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \]
\[ D = \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]  \hspace{1cm} (7.19)
where \( S^i = \frac{1}{4} \epsilon_{ijk} M_{jk} \) and \( B^j = M_{0j} \). The relations between the generators of \( SU(2, 2) \) in their bounded and unbounded realization are

\[
P^{00} = H_0 + H_7 \quad , \quad P^{ij} = H_j - H_{3+j} + H_{8+j} \\
K^{00} = H_0 - H_7 \quad , \quad K^{ij} = H_j - H_{3+j} - H_{8+j} \quad (7.20)
\]

The generators of the maximal compact subgroup \( S(U(2) \times U(2)) \) are \( H_0 \), \( H_j \) and \( H_{3+j} \), which determine the test functions for the rigged Hilbert space of \( SU(2, 2) \). But the non-existence of the automorphic LCR-structure relative to \( z \)-rotation, time translation and dilation implies that the \( SU(2, 2) \) symmetry is broken down to the Poincaré symmetry. Hence the harmonic analysis must be restricted to the rigged Fock space of the representations of the Poincaré group (wavefront singularities) which appear to the well defined LCR-manifolds of the above list.

Using the Celeghini et al.\[5\] method, the orthogonal functions are based on the Hermite polynomials

\[
\Psi_n(x) := \frac{e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) \quad , \quad n = 0, 1, 2, ... \quad , \quad x \in (-\infty, +\infty)
\]

\[
\int_{-\infty}^{+\infty} \Psi_n(x) \Psi_m(x) dx = \delta_{nm} \quad , \quad \sum_{n=0}^{\infty} \Psi_n(x) \Psi_n(x') = \delta(x - x') \quad (7.21)
\]

We define the operators

\[
X \Psi_n(x) := x \Psi_n(x) \quad , \quad D_x \Psi_n(x) := \frac{d}{dx} \Psi_n(x) \quad , \quad N \Psi_n(x) := n \Psi_n(x)
\]

\[
a := \frac{1}{\sqrt{2}} (X + D_x) \quad , \quad a^\dagger := \frac{1}{\sqrt{2}} (X - D_x) \quad , \quad \Psi_n(x) = \sqrt{n + 1} \Psi_{n+1}(x) \quad (7.22)
\]

It is well known that

\[
[a, a^\dagger] = I \quad , \quad [N, a^\dagger] = a^\dagger \quad , \quad [N, a] = -a
\]

\[
a \Psi_n(x) = \sqrt{n} \Psi_{n-1}(x) \quad , \quad a^\dagger \Psi_n(x) = \sqrt{n + 1} \Psi_{n+1}(x) \quad (7.23)
\]

is the Heisenberg algebra. The discrete \(|n\rangle \) and the continuous basis \(|x\rangle \) are related with the relation

\[
|n\rangle = \int_{-\infty}^{\infty} dx \Psi_n(x) |x\rangle \quad \Leftrightarrow \quad |x\rangle = \sum_{n=0}^{\infty} \Psi_n(x) |n\rangle \quad (7.24)
\]

In our case of the 3-dimensional space we have to start with the 3-dimensional Hermite function being the product of the three Hermite functions \((7.21)\). They span the vector space \( S \) of rapidly decreasing functions in \( \mathbb{R}^3 \). It is dense in the Hilbert space of square integrable functions \( L^2(\mathbb{C}) \). The functionals on \( S \) determine the generalized functions \( S' \) in the basis \( |\vec{x}\rangle \). For the following details I refer the reader to the chapter 4 of the classical book of Bogoliubov, Logunov and Todorov\[3\].
7.2 BEGS formulation of the standard model

My claim that the geometry prevails and that the "quantum" SM S-matrix is a targeted harmonic expansion into precise representations of the Poincaré group has to clarify how the dressing potentials of an LCR-structure become the radiation fields used in the BEGS procedure. We will recall the formulation of the Lienard-Wiechert radiating potentials by an accelerating charge, studying the LCR-structure based on a ruled surface with a general (accelerating) complex trajectory (2.73). Then the homogeneous coordinates are

\[
X^{\alpha j} = \begin{pmatrix}
\lambda^A_1 & \lambda^A_2 \\
-i\xi_a(z^0)\sigma^a_A \lambda^A_1 & -i\xi_a(z^0)\sigma^a_A \lambda^B_2
\end{pmatrix}
\]

(7.25)

where \(z^\alpha, \tilde{z}^\alpha\) are two solutions of the following equations

\[
\det[(r_a - \xi_a(\tau))\sigma^a_A] = 0, \quad \tau_{1,2} = \xi^0 = r^0 \pm \sqrt{(r^i - \xi^i(\tau_1,2(\tau^b)))^2}
\]

\[
(r_a - \xi_a(z^0))\sigma^a_A \lambda^A_1 = 0 = (r_a - \xi_a(\tilde{z}^0))\sigma^a_A \lambda^A_2
\]

(7.26)

Notice that if the two roots \(\tau_{1,2}\) are different, the corresponding \(\lambda^{Aj}\) are also different and they provide the natural "retarded" and "advanced" intersection points of the line \(r^a\) with two sheets of the algebraic surface of \(CP(3)\). This becomes more clear in the case of zero algebraic gravity (\(\text{Im} r^a = 0\)) and \(\xi_a(\tau) = \xi^a_R(\tau) + i\xi^a_I(\tau)\) with the vectors \(\xi^a_R(\tau), \xi^a_I(\tau)\) real analytic. Then up to 1st order approximation relative to \(\frac{1}{c}\) implies

\[
\begin{align*}
\tilde{z}^0 &\simeq t - \frac{1}{z^1} \sqrt{(x^i - \xi^i_R(t) - i\xi^i_I(t))^2} \\
z^0 &\simeq t + \frac{1}{z^1} \sqrt{(x^i - \xi^i_R(t) - i\xi^i_I(t))^2}
\end{align*}
\]

(7.27)

Apparently the trajectory (source) singularity of the LCR-structure occurs at the space curve

\[
(x^i - \xi^i_R(t))^2 = (\xi^i_I(t))^2
\]

\[
\sum_{i=1}^{3} (x^i - \xi^i_R(t))\xi^i_I(t) = 0
\]

(7.28)

which is a moving disc with radius \(\xi^i_I(t)\) around the real trajectory \(\xi^i_R(t)\), which may be viewed as an extension of ring singularity of the "free electron" LCR-structure. I think it is clear that these geometric calculations indicate the emergence of the radiation fields but they turn out to be very difficult to be computed.

Detailed calculations for the derivation of the SM are presented in the second book [29] of Scharf, using the nilpotent charge \(Q\) (1.11) as described in the
introduction. This procedure is valid if no anomalies appear, which is the case in the $S(3) \times SU(2) \times U(1)$ SM. Recall that the choice of the lagrangian of a field theory respected the rule not to include higher order derivatives. Einstein proposed his field equations for the metric respecting this rule taking up to second order derivatives including the cosmological constant term. After the Pais-Uhlenbeck work\cite{23} it became clear that higher order derivatives may generate negative norm (unphysical) modes. The BRS algorithm of Scharf and coworkers essentially replaces the above ad hoc assumption following the opposite way. It starts from the free wave field and it eliminates order by order all the interaction terms which could create unphysical modes. In the case of the spin-2 field $h^{\mu\nu}$ considered as an operator valued distribution describing the spin-2 Poincaré group representation through

$$[h^{\alpha\beta}(x), h^{\mu\nu}(y)] = -i\delta^{\alpha\beta\mu\nu} D(x - y)$$

(7.29)

where $D(x)$ is the mass-zero Jordan-Pauli distribution. The nilpotent gauge charge is

$$Q := \int_{x^a = t} d^3 x (\partial_\beta h^{\alpha\beta}) \partial_0 u_\alpha$$

(7.30)

and the ghosts fields $u^\mu$ and $\bar{u}^\nu$ satisfy the anticommutation relation

$$\{u^\mu(x), \bar{u}^\nu(y)\} = i\eta^{\mu\nu} D(x - y)$$

(7.31)

The variations of the fundamental fields are

$$d_Q h^{\mu\nu} = -\frac{i}{2}(\partial^\mu u^\nu + \partial^\nu u^\mu - \eta^{\mu\nu} \partial_\alpha u_\alpha)$$

$$d_Q u^\mu = 0 \quad d_Q \bar{u}^\nu(y) = i\partial_\nu h^{\mu\nu}$$

(7.32)

Applying the invariance (up to a total derivative) under the above variations to the possible self-interactions of the spin-2 field and after very long calculations the group of Scharf found that up to third order the terms of the S-matrix coincide with those of the Hilbert-Einstein action including a cosmological constant. The interested reader must study the original works of the group or the books of Scharf for more details. The first important point for the present case is that the compact U(2) universe, indicated by the background geometry 3.24, is also found from the BEGS harmonic analysis. The second important point is the cancellation of the anomalies for the precise Weinberg-Salam model implied by PCFT. An analogous BRS algorithm was used to prove that in the 2-dimensional PCFT (Polyakov action)\cite{32} that the conformal anomaly vanishes for the well known 26-dimension of the vector space of $X^\mu$. Notice that in the present 4-dimensional PCFT the gluonic gauge field 3.1 corresponds to the $X^\mu$ field of the 2-dimensional PCFT. Hence the physically observed $SU(3)$ unitary gluonic group may be considered as a consequence of the cancellation of the axial anomaly.
I have already pointed out that the Poincaré group of PCFT is the proper orthochronous group of special relativity (Minkowski space). The spatial and temporal reflections are external automorphisms of its Lie algebra, i.e. they do not exponentiate into elements of the group like \((7.8)\) in the \(SU(1,1)\) example. In PCFT there are two additional discrete external transformations. The asymmetry between the left \(X^a_1\) and the right \(X^a_1\) homogeneous coordinates in the case of reducible Kerr polynomials (related to the chirality) and the \((z^a, z^\tilde{b}) \leftrightarrow (\tilde{z}^a, z^\tilde{b})\) transformation related to the particle-antiparticle transformation. All these four discrete transformations are also permitted in the BEGS harmonic analysis. In the following subsection I discuss the possible manifestation of the electron ring singularity in the BEGS procedure.

### 7.3 On self-consistency conditions

The perturbative approach permits the definition of general dynamical variables through the generating functional introduced considering the formal existence of a "classical" current \(J(x)\) for every field \(\phi(x)\) of the action. The generating functional \(Z_0(J)\) and the connected generating functional \(Z_c(J)\) are

\[
Z_0(J) = \langle 0 | T \{ \exp \left[ i \int (L_I(x) + \phi(x) J(x)) d^4x \right] \} | 0 \rangle
\]

\[
Z_c(J) = -i \ln [Z_0(J)]
\]

The general and the connected Green functions are defined taking (formal) functional derivatives of the generating functionals. Through this formal procedure, the symmetries of the action become Ward identities for the Green functions. The anomalies appear as disagreements between the formal and the exact (quantum) computations.

Any field \(\phi(x)\) defines a generating field \(\Phi(x; J)\) and the Legendre transformation

\[
\Phi(x; J) = \frac{\delta Z_c(J)}{\delta J(x)}
\]

\[
Z_c(J) \rightarrow W(\Phi) = Z_c(J) - \int \Phi(x; J) J(x) d^4x
\]

In the context of the Bogoliubov-Shirkov notation \([11]\), Chap. VII]

\[
\Phi(x; g) = -\frac{\delta H(x; g)}{\delta J(x)} = -i \frac{\delta S}{\delta J(x)} S|_{J=0}
\]

\[
H(x; g) := i \left( \frac{\delta S(g)}{\delta J(x)} S(g) \right)
\]

where \(H(x; g)\) is the "quantum" hamiltonian of the system. Hence the causal approach formalism permits the computation of a classical potential with the following formula.
\[ \phi(x; 1) = -\frac{\delta J(x)}{\delta \phi(x)} \bigg|_{J=0} = \frac{\delta S}{\delta J(x)} \Phi_1 \bigg|_{J=0} \]  

(7.36)

where \( \Phi_1 \) is the one-electron state. Notice that the elementary particle has the same initial and final energies and their creation and annihilation operators are outside the time ordering. The physical intuition is that we use the classical current \( J(x) \) as a sensor of the potential generated by a particle.

Hence the BEGS procedure permits the computation of the electromagnetic dressing of the electron with the following first order term

\[ A_{1\mu}(x; 1) \approx -\frac{1}{2} \Phi_1 \frac{\delta \hat{S}_2(J)}{\delta J(x)} |_{J=0} \]  

(7.37)

\[ \hat{S}_2(J) = \int T(L_I(x_1) + A_{\nu}(x_1) J^\nu(x_1))(L_I(x_2) + A_{\nu}(x_2) J^\nu(x_2))[dx] \]

which becomes

\[ A_{1\mu}^\nu(x) \approx -e \int D_0(x - y) \Phi_1 \gamma^\mu \psi_e(y) : \Phi_1 \psi_e(y) : \Phi_1 |_{J=0} \]  

(7.38)

\[ \Phi_1 = (2\pi)^{\frac{3}{2}} a_0^+ \left( \frac{k}{\hbar} \right) \Phi_0 \]

This potential is singular at the point \( \vec{x} = 0 \), but the electromagnetic dressing of the “electron” LCR-structure (1.20) in cartesian coordinates

\[ A = \frac{q^2}{4\pi(e + a(\tau + a)^2)}(dx^0 - \frac{r x^1 - a x^2}{\tau^2 + a^2} dx^1 - \frac{r x^2 + a x^1}{\tau^2 + a^2} dx^2 - \frac{e^3}{r} dx^3) \]  

(7.39)

\[ dF = 0, \quad d * F = -* j_e \]

is singular at the entire disc with radius \( a \), which is a non-local interaction. This incompatibility may be solved from the observation that the Kerr-Newman manifold attributed to the “electron” LCR-structure has spin \( ma \). Notice that the expansion of the electromagnetic dressing relative to \( a \), has first term (7.38).

But because of the relation \( a = \frac{1}{\sqrt{m}} \), the higher order terms of (7.39) should emerge from the loop diagrams. It would be interesting to check it for the electrogravity potentials (4.20).

8 DISCUSSION

The very interesting result of the present work is that by simply replacing the Einstein metric \( g_{\mu\nu} \) on the tangent space of spacetime with the Frobenius integrable LCR-structure (about) all current phenomenology in elementary particle physics emerges. The solitonic static quadratic LCR-structure is the static electron with electroweak, “Higgs” and gravitational potentials which generate
wavefront singularities in the accelerating case. Its corresponding neutrino is also computed. Quantum field theory emerges by simply applying the BEGS procedure to the rigged Hilbert-Fock space of the Poincaré representations of the above wavefront singularities. Besides it seems that in PCFT the only static (non-baryonic) particle is the electron LCR-structure. Hence there is no place for the existence of weakly interacting massive particles (WIMPs), which is compatible with current observations. It is apparent that geometry prevails as expected by Einstein, implying that all the assumptions of the standard model are provided by the geometry.

In the context of astrophysical observations the formulation of the mathematical problem changes. Recall that algebraic gravity of matter emerges from a deformation of the SU(2,2) classical domain with a Kerr-Schild ansatz. In the present case the asymptotic space must be the deSitter space. Recall that the tetrd-Weyl transformations and the corresponding $\rho'_{ij} = f_{ij} \rho_{ij}$ transformation show that the LCR-structure does not uniquely determine neither the metric of spacetime nor the metric of the ambient Kaehler manifold. But in the case of the vacuum LCR-structure it is natural to assume the symmetric ambient metric with curvature

$$\tilde{R}_{\alpha\beta\gamma\delta} = R_0(a_{\alpha\gamma} a_{\beta\delta} - a_{\alpha\delta} a_{\beta\gamma})$$

which implies that the Gauss equation

$$R_{ijkl} \approx \sum_{\sigma=5}^8 e_\sigma (\Omega_{\sigma|ik} \Omega_{\sigma|jl} - \Omega_{\sigma|il} \Omega_{\sigma|jk}) + \tilde{R}_0 g_{ik} g_{jl} - g_{il} g_{jk}$$

(8.3)

where $g_{ik}$ is the first fundamental form (metric), $\Omega_{\sigma|ik}$ are the four second fundamental forms and $\xi_\sigma^\alpha$ are the four normal vectors to the spacetime being formally embedded in the ambient complex manifold. The notation of the Eisenhart book[9] is used. The emergence of the deSitter spacetime is apparent, but there is an additional term implied by the second fundamental forms of the flatprint LCR-manifold of matter (stars, galaxies, etc). Explicit use of this formula into the astrophysical computations with the typical flatprint embedding

$$z^0 = t - r + ia \cos \theta , \quad z^1 = e^{i\varphi} \tan \frac{\theta}{\pi}$$

$$z^\tilde{0} = t + r - ia \cos \theta , \quad z^\tilde{1} = \frac{r + ia}{r - ia} e^{-i\varphi} \tan \frac{\theta}{\pi}$$

(8.4)

of a rotating star or galaxy could be a test of PCFT.

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