Dynamical fluctuations in classical adiabatic processes: General description and their implications

Qi Zhang\textsuperscript{a,b}, Jiangbin Gong\textsuperscript{c,d,*}, C.H. Oh\textsuperscript{b,e}

\textsuperscript{a}Center for Clean Energy and Quantum Structures and School of Physics and Engineering, Zhengzhou University, Zhengzhou, 450052, China
\textsuperscript{b}Centre for Quantum Technologies and Department of Physics, National University of Singapore, 117543, Singapore
\textsuperscript{c}Department of Physics and Centre for Computational Science and Engineering, National University of Singapore, 117542, Singapore
\textsuperscript{d}NUS Graduate School for Integrative Sciences and Engineering, Singapore 117597, Republic of Singapore
\textsuperscript{e}Institute of Advanced Studies, Nanyang Technological University, Singapore 639798, Republic of Singapore

Abstract

Dynamical fluctuations in classical adiabatic processes are not considered by the conventional classical adiabatic theorem. In this work a general result is derived to describe the intrinsic dynamical fluctuations in classical adiabatic processes. Interesting implications of our general result are discussed via two subtopics, namely, an intriguing adiabatic geometric phase in a dynamical model with an adiabatically moving fixed-point solution, and the possible “pollution” to Hannay’s angle or to other adiabatic phase objects for adiabatic processes involving non-fixed-point solutions.

Keywords: Dynamical fluctuations, classical adiabatic theorem, adiabatic geometric phase, “pollution” to Hannay’s angle

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\textsuperscript{*}Email address: phygj@nus.edu.sg (Jiangbin Gong)

\textsuperscript{1}Department of Physics, 2 Science Drive 3, Singapore, 117542, phone 65-6516-1154, Fax 65-6777-6126
1. Introduction

The adiabatic theorem is important in both classical and quantum mechanics [1]. It predicts a system’s dynamical behavior subject to slowly varying system parameters. Although a general and mathematically rigorous proof of the adiabatic theorem is not obvious in both classical mechanics and quantum mechanics, the adiabatic theorem has been widely used. Indeed, it is always highly useful so long as there exist two drastically different time scales. The adiabatic theorem has also led to the discoveries of Berry phase [2] and the classical counterpart, i.e., Hannay’s angle [3].

We focus on the classical adiabatic theorem (CAT), but as shown below, some of our results can be applied to quantum systems as well. Our interest here is not in a rigorous proof of the CAT, but in dynamical fluctuations around what is predicted by CAT. As discussed below, the possible consequences of the fluctuations neglected by the conventional CAT can be far reaching. The motivation of considering the fluctuations is based on a simple observation. That is, CAT, whose proof is based on an average over fast-varying variables, only reflects a mean dynamical behavior. As such fluctuations on top of a mean dynamical behavior should exist in classical adiabatic processes. Though fluctuations should be intuitively smaller in a slower adiabatic process, their effects are accumulated over a longer time scale and hence might not vanish even in the adiabatic limit. For instance, in a few early studies [4, 5, 6], including the study of “Hannay’s angle of the world” [6, 7], the actual total change in canonical variables may depend on the smoothness of the evolving adiabatic parameters. This abnormal behavior was shown to be connected with subtle fluctuations in the action variables from their average behavior predicted by CAT. Clearly then, a general description of the dynamical fluctuations in adiabatically evolving and classically integrable systems should be of importance.

We shall present in this work a general result that describes the dynamical fluctuations inherent to classical adiabatic processes. Roughly speaking, it establishes an interesting connection between the actual rate of change of slowly varying system parameters and the actual classical orbits deformed from that predicted by CAT. To illustrate the usefulness of our general result, we design a simple dynamical model with an adiabatically moving fixed-point solution, from which an intriguing classical geometric phase can emerge. We then exploit our general result to discuss the “pollution” to Hannay’s angle in classical adiabatic processes. A mean-field model that describes a two-
mode Bose-Einstein condensate (BEC) is also proposed to study fluctuation-induced “pollution” to adiabatic quantum evolution.

To tackle with dynamical fluctuations, one may quickly think of an equation describing the time dependence of the fluctuations around ideal adiabatic orbits. But this approach may not be fruitful because in principle, the time dependence of any canonical variables is already fully captured by classical canonical equations of motion. Instead, we are concerned with how fluctuations distort trajectories as compared with that predicted by CAT. In this sense, our approach is somewhat in a similar spirit as an early “multiple-time-scale-expansion” approach to corrections to classical adiabatic invariants in chaotic systems [8]. However, we focus on fluctuations associated with individual orbits in integrable systems, rather than fluctuations associated with an ensemble of chaotic trajectories in an energy shell.

This paper is organized as follows. In Sec. II we derive a differential equation describing the dynamical fluctuations in classical adiabatic processes. Some related details are also provided in Appendix. As an application, in Sec. III we study the case of an adiabatically moving fixed-point solution and show how an intriguing geometric angle may emerge in a simple toy model. Based on our general result, Sec. IV discusses why the “pollution” to Hannay’s angle may exist and then proposes a physical system to study analogous fluctuation-induced pollution. We finally give a brief summary in Sec. V.

2. General Description of Dynamical Fluctuations in Classical Adiabatic Processes

Consider a classical integrable system with $N$ degrees of freedom. Its Hamiltonian is given by $H(p, q, R)$, where canonical variables $p = (p_1, p_2, \ldots, p_N)$ and $q = (q_1, q_2, \ldots, q_N)$ represent canonical momenta and coordinates, and $R$ represents a collection of system parameters. Let $F(I, q, R)$ be the generating function that induces the $R$-dependent canonical transformation from $(p, q)$ to the action-angle variables $(I, \Theta)$, where

$$I_i = \frac{1}{2\pi} \oint p_i dq_i$$

and

$$\Theta = (\theta_1, \theta_2, \ldots, \theta_N).$$

To clearly present our derivation of a differential equation that describes dynamical fluctuations in classical adiabatic processes, this section is divided into four subsections representing the four steps in our derivation. First, after expressing classical equations of motion in the action-angle variables $(I, \Theta)$, we define dynamical fluctuations on top of the idealized solution given by
CAT. Second, the time dependence of the canonical variables \((p, q)\) is expressed in terms of the dynamical fluctuations we define. Third, directly using the canonical equations of motion and the canonical transformation between the action-angle variables and the canonical variables, we reexpress the time dependence of the canonical variables in terms of the dynamical fluctuations as well as the action-angle variables along idealized classical orbits. Finally, by comparing results in the second and third steps a differential equation describing the dynamical fluctuations around idealized adiabatic orbits is obtained.

2.1. Dynamical fluctuations

In the \((I, \Theta)\) representation an integrable Hamiltonian becomes \(H(I, R)\), which is independent of the angle variables \(\Theta\). For time-varying \(R = R(t)\), the equations of motion for \((I, \Theta)\) are given by

\[
\frac{dI_i}{dt} = -\frac{\partial W}{\partial \theta_i} \cdot \frac{dR}{dt},
\]

\[
\frac{d\theta_i}{dt} = \omega_i(I; R) + \frac{\partial W}{\partial I_i} \cdot \frac{dR}{dt},
\]

where \(\omega_i(I, R) = \partial H / \partial I_i\) is the angular frequency, and \(W\) is defined by

\[
W \equiv \nabla_R F[I, q(I, \Theta, R), R] - p \cdot \nabla_R q(I, \Theta, R).
\]

Note that \(\nabla_R\) refers to the gradient in the parameter space under fixed \((I, \Theta)\). If

\[
\epsilon = \left| \frac{dR}{dt} \right|
\]

is much smaller than \(\omega_i|R|\), one can take the average of Eq. (1) over the rapidly oscillating angle variables, yielding \(\frac{dI_i}{dt} \approx 0\) (\(W\) is a periodic function of \(\Theta\)). CAT hence identifies the action variables as adiabatic invariants, i.e., in adiabatic processes their values are fixed at \(\bar{I} \equiv (\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_N)\). For clarity, angle variables associated with this idealized solution are defined as \(\bar{\Theta} \equiv (\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)\). We also use \(\bar{p} \equiv p(\bar{I}, \bar{\Theta}, R)\) and \(\bar{q} \equiv q(\bar{I}, \bar{\Theta}, R)\) to describe the idealized solution in terms of the (old) set of canonical variables. With the action variables fixed at \(\bar{I}\), one may then solve Eq. (2) for a cyclic process from \(t = 0\) to \(t = T\) in a straightforward manner. One may further
take the average of the idealized solution over all possible initial angle values to obtain Hannay’s angle, which is the total mean angle change minus a dynamical angle.

The above discussion does not represent a complete description of classical adiabatic processes. Clearly, Eq. (1) tells us that \( \frac{dI}{dt} \) is not mathematically zero: it may possess fluctuations of the order \( O(\epsilon) \) (i.e., to the first order of \( \epsilon \)). As such, in performing an averaging procedure as is done in CAT one neglects the dynamical correlation between \( \Theta \) and \( I \). It is hence necessary to reconsider Eq. (1) in order to consider any possible real-orbit fluctuations on top of CAT. On a real orbit we assume we have \( I_i = I_i + \delta I_i \), where we have used \( \delta \) to represent fluctuations from the behavior predicted by CAT. Equivalent to that, one can describe the same fluctuations from the idealized orbit in terms of \( \delta q_j \) and \( \delta p_j \).

There are now both idealized adiabatic orbits without considering fluctuations and true orbits with fluctuations: the geometry of an idealized orbit can be characterized by \( I = \bar{I} \) and \( \Theta \in [0, 2\pi) \); and that of a true orbit with fluctuations is slightly deformed to

\[
\begin{align*}
I &= \bar{I} + \delta I, \\
\Theta &= \bar{\Theta} + \delta \Theta,
\end{align*}
\]

where \( \delta I \) and \( \delta \Theta \) are assumed to be at most of the order \( O(\epsilon) \). By our definitions above, we have

\[
\begin{pmatrix}
\delta I \\
\delta \Theta
\end{pmatrix} = \begin{pmatrix}
\frac{\partial I}{\partial p_i} \delta p_j + \frac{\partial I}{\partial q_i} \delta q_j \\
\frac{\partial \Theta}{\partial p_j} \delta p_j + \frac{\partial \Theta}{\partial q_j} \delta q_j
\end{pmatrix} \equiv \begin{pmatrix}
K \\
M
\end{pmatrix} \begin{pmatrix}
\delta p \\
\delta q
\end{pmatrix}.
\]

Here and in the following the summation convention for repeated indices is adopted. Equation (7) also defines two \( N \times 2N \) matrices \( K \) and \( M \), corresponding to the upper and lower halves of a Jacobi matrix. Note that throughout we use \( \frac{\partial f}{\partial x} \) to indicate \( \frac{\partial f}{\partial x} \) evaluated at \( x = \bar{x} \).

As will be seen below, it suffices to consider fluctuations of the first order of \( \epsilon \) because higher-order effects cannot be accumulated with time. We stress that the fluctuations are intrinsic: they are nonzero so long as \( \epsilon \) is not identically zero. In other words, fluctuations considered here exist in any classical adiabatic process and should not be thought of an effect arising from a too-large \( \epsilon \). It should be also noted that in principle, all the dynamical information is contained in Eqs. (1) and (2). However, we are interested in
developing a framework to describe how fluctuations might behave along an idealized classical orbit.

2.2. Canonical equations of motion in terms of fluctuations

In terms of the fluctuations $\delta q_j$ and $\delta p_j$, we next expand $H(p, q, R)$ around $\bar{H} \equiv H(\bar{p}, \bar{q}, \bar{R})$ to the order $O(\epsilon)$, yielding the following canonical equations of motion for $(q, p)$:

$$
\frac{dp_i}{dt} = -\frac{\partial \bar{H}}{\partial q_i} - \frac{\partial^2 \bar{H}}{\partial q_i \partial p_j} \delta p_j - \frac{\partial^2 \bar{H}}{\partial q_i \partial q_j} \delta q_j
$$

$$
\frac{dq_i}{dt} = \frac{\partial \bar{H}}{\partial p_i} + \frac{\partial^2 \bar{H}}{\partial p_i \partial p_j} \delta p_j + \frac{\partial^2 \bar{H}}{\partial p_i \partial q_j} \delta q_j
$$

where we have used the following two canonical relations

$$
\frac{\partial I_j}{\partial q_i} = -\frac{\partial \bar{p}_i}{\partial \theta_j};
$$

$$
\frac{\partial I_j}{\partial p_i} = \frac{\partial \bar{q}_i}{\partial \theta_j}.
$$

Through Eq. (8) it is seen that the time dependence of the canonical variables $(p, q)$ is connected to the dynamical fluctuations $\delta q_j$ and $\delta p_j$, to the first order of $\epsilon$.

2.3. Time-dependence of canonical variables from action-angle variables

The time evolution of the canonical variables $(p, q)$ may be also directly obtained from the canonical transformation from the action-angle variables to $(p, q)$ and from the equations of motion given by Eqs. (1) and (2). In particular, using

$$
\frac{dp_i}{dt} = \frac{\partial p_i}{\partial R} \frac{dR}{dt} + \frac{\partial p_i}{\partial I_j} \frac{dI_j}{dt} + \frac{\partial p_i}{\partial \theta_j} \frac{d\theta_j}{dt}
$$
and the analogous expression for $\frac{dq_i}{dt}$, rewriting the derivatives in Eqs. (1) and (2) at $(I, \Theta)$ in terms of those at $(\bar{I}, \bar{\Theta})$, and neglecting all terms that are at least $O(\epsilon^2)$, one arrives at (see Appendix for details)

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial R} \frac{dR}{dt} - \frac{\partial p_i}{\partial I_j} \frac{dW}{\partial \theta_j} \cdot dR + \frac{\partial dp_i}{\partial \theta_j} \omega_j(\bar{I}, R)$$

$$\frac{dq_i}{dt} = \frac{\partial q_i}{\partial R} \frac{dR}{dt} - \frac{\partial q_i}{\partial I_j} \frac{dW}{\partial \theta_j} \cdot dR + \frac{\partial dq_i}{\partial \theta_j} \omega_j(\bar{I}, R)$$

(11)

Interestingly, due to the direct connection between $(p, q)$ and $(I, \Theta)$, the full time dependence of $(p, q)$ is connected with dynamical fluctuations in a highly nontrivial manner. In particular, the terms $\frac{\partial p_i}{\partial \theta_j}$ and $\frac{\partial q_i}{\partial \theta_j}$ in Eq. (11) indicate that it is important to account for how dynamical fluctuations change with $\bar{\theta}_j$. This is a crucial piece of information regarding the overall feature of the dynamical fluctuations.

2.4. A differential equation describing dynamical fluctuations

Both Eq (8) and Eq. (11) deal with the same time dependence of $(p, q)$ and hence they should be consistent with each other. Comparing these two equations term by term, we arrive at the following equation,

$$\Gamma \left( \frac{\delta p}{\delta q} \right) = \Sigma \cdot \frac{dR}{dt} + \Pi \left( \begin{array}{c} \frac{\delta I_1}{\delta q} \\ \frac{\delta I_2}{\delta q} \\ \vdots \\ \frac{\delta I_N}{\delta q} \end{array} \right)$$

(12)

where

$$\Gamma = \begin{pmatrix} \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial q \partial \theta} \\ \frac{\partial^2 H}{\partial p \partial p} & \frac{\partial^2 H}{\partial p \partial \theta} \end{pmatrix}$$

(13)

is a $2N \times 2N$ matrix;

$$\Sigma = \begin{pmatrix} -\frac{\partial W}{\partial I_j} \frac{\partial q_i}{\partial \theta_j} + \frac{\partial W}{\partial I_j} \frac{\partial q_i}{\partial \theta_j} + \frac{\partial W}{\partial I_j} \frac{\partial q_i}{\partial \theta_j} \\ -\frac{\partial W}{\partial I_j} \frac{\partial q_i}{\partial \theta_j} + \frac{\partial W}{\partial I_j} \frac{\partial q_i}{\partial \theta_j} + \frac{\partial W}{\partial I_j} \frac{\partial q_i}{\partial \theta_j} \end{pmatrix}$$

(14)
is a $2N \times 1$ vector along each direction of $\mathbf{R}$; and
\[
\Pi = \begin{pmatrix}
\frac{\partial p}{\partial \theta_j}, \frac{\partial q}{\partial \omega_j}, & \frac{\partial p}{\partial \theta_1}, \frac{\partial q}{\partial \omega_1}, & \cdots
\end{pmatrix}
\tag{15}
\]
is a $2N \times N$ matrix. Substituting Eq. (7) into Eq. (12), one finally obtains an equation for $\delta p$ and $\delta q$ only:
\[
\left(\begin{array}{c}
\frac{\partial \delta p}{\partial \theta_j} \omega_j \\
\frac{\partial \delta q}{\partial \theta_j} \omega_j
\end{array}\right) + (\Pi K - \Gamma) \left(\begin{array}{c}
\delta p \\
\delta q
\end{array}\right) + \sum \cdot \frac{d\mathbf{R}}{dt} = 0.
\tag{16}
\]

For a given integrable Hamiltonian, except for those related to $\delta p$ and $\delta q$ and their derivatives, all the matrices contained in Eq. (16) are evaluated at an idealized orbit and hence can be explicitly obtained.

Some remarks are in order. First, Eq. (16) is not about evolving the fluctuations $(\delta q, \delta p)$ at one moment to the next moment. Instead, it describes, when the system parameters reach the current configuration $\mathbf{R}$ with a small but nonzero rate $\frac{d\mathbf{R}}{dt}$, the deviation of the overall shape of one true orbit from the idealized orbit without dynamical fluctuations, i.e., the overall deformed orbit in phase space. To our knowledge, this result is obtained for the first time here. This detailed description of the dynamical fluctuations can be very useful for both quantitative and qualitative considerations. The derivation here is somewhat lengthy because of the physical meaning of $(\mathbf{I}, \mathbf{\Theta})$ in terms of $(\mathbf{q}, \mathbf{p})$ and hence the idealized orbit itself is changing as $\mathbf{R}$ varies. Second, consistent with our treatment to the first order of $\epsilon$, $(\delta p, \delta q)$ is seen to depend on $\frac{d\mathbf{R}}{dt}$. If $\frac{d\mathbf{R}}{dt}$ were identically zero, then $\delta p = \delta q = 0$ is one possible solution. If $\delta p \neq 0$ and $\delta q \neq 0$ is still the solution for $\frac{d\mathbf{R}}{dt} = 0$, then this solution describes the relationship between two infinitely close orbits. Third, in the absence of the detailed information of $(\delta p, \delta q)$ for at least one phase space location, Eq. (16) alone does not suffice to predict $(\delta p, \delta q)$ because of its differential form. As will be discussed later, this implies that in general, detailed information of the time-dependence of $\mathbf{R}$, e.g., its smoothness, can be important for determining the dynamical fluctuations. Finally, because the linear Schrödinger equation and nonlinear Gross-Pitaevskii (GP) equation have an exact canonical structure of Hamiltonian dynamics, our results here can be also relevant to quantum adiabatic processes.

If we now consider the mean behavior of $(\delta p, \delta q)$ along an ideal orbit (denoted by $\langle \cdot \rangle$), then using the fact that $(\delta p, \delta q)$ are periodic functions of
Because the matrices $\Pi$, $K$, $\Gamma$ vary along the orbit, one may infer from Eq. (17) the statistical correlations $\langle (\Pi K - \Gamma) \delta p \rangle$ and $\langle (\Pi K - \Gamma) \delta q \rangle$, but the mean fluctuations $\langle \delta p \rangle$, $\langle \delta q \rangle$, or $\langle \delta I \rangle$ remain unknown.

3. Emergence of a geometric angle from an adiabatically moving fixed-point solution

As a direct application of our central result in Eq. (16), here we focus on a rather simple case, where the solution to Hamilton’s equation of motion is a fixed point in phase space if the system parameters are not changing. We denote the fixed-point solution as $(p, q)$, which are of course functions of $R$.

Consider now an adiabatic process in which $R$ is changing slowly. Then the idealized orbit according to CAT is just one adiabatically moving fixed point. In addition, at this fixed point all functions of $(p, q)$ are independent of $\Theta$ (otherwise they would be time-dependent), thus forcing their derivatives with respect to $\Theta$ to vanish and making an averaging over $\Theta$ [e.g., in Eq. (17)] unnecessary. We therefore obtain

\begin{align*}
\Pi &= 0, \\
\Sigma &= \left( \frac{\partial p}{\partial R}, \frac{\partial q}{\partial R} \right)^T.
\end{align*}

Using these results we have the following relation from Eq. (16):

\begin{equation}
\begin{pmatrix}
\delta p \\
\delta q
\end{pmatrix} = \Gamma^{-1} \begin{pmatrix}
\frac{\partial p}{\partial R} \\
\frac{\partial q}{\partial R}
\end{pmatrix} \cdot \frac{dR}{dt}.
\end{equation}

Note that the values of $\theta_i$ at a fixed point can be taken as arbitrary. Hence the fluctuations obtained in Eq. (20) do not have any interesting consequence for the evolution of $\theta_i$. Furthermore, since the $K$ matrix vanishes at fixed points (where the action reaches its minimum), one would also arrive at $\delta I = 0$ to the first order of $\epsilon$ even though $\delta q \neq 0$ and $\delta p \neq 0$.

Consider then the coupling of this system with another degree of freedom, whose canonical coordinates are denoted by $(J, \phi)$. The total Hamiltonian is
assumed to be independent of $\phi$, denoted $H^{\text{tot}}(p, q, J)$. Because $J$ is a strict constant of motion and can be regarded as a fixed system parameter for the motion of $(p, q)$, the expression for $\delta p$ and $\delta q$ in Eq. (20) still applies to fixed points in the phase space of $(p, q)$. To seek how fluctuations predicted by Eq. (20) may affect the motion in $\phi$, let us now examine the angular frequency associated with $\phi$, i.e.,

$$\omega_J(p, q, J) \equiv \frac{\partial H^{\text{tot}}}{\partial J}. \tag{21}$$

Clearly, the fluctuations $\delta p$ and $\delta q$ will lead to

$$\delta \omega_J(p, q, J) = \frac{\partial \omega_J(p, q, J)}{\partial p} \cdot \delta p + \frac{\partial \omega_J(p, q, J)}{\partial q} \cdot \delta q. \tag{22}$$

This fluctuation in $\omega_J(p, q, J)$ induces an correction to the evolution of $\phi$. Using Eq. (20), one finds an explicit expression for this correction as follows,

$$\phi^{\text{corr}} = \int_0^T \left[ \frac{\partial \omega_J}{\partial p} \cdot \delta p + \frac{\partial \omega_J}{\partial q} \cdot \delta q \right] dt = \oint \left( \frac{\partial \omega_J}{\partial p}, \frac{\partial \omega_J}{\partial q} \right) \Gamma^{-1} \left( \frac{\partial \omega_J}{\partial p}, \frac{\partial \omega_J}{\partial q} \right) \cdot dR. \tag{23}$$

As seen from Eq. (23), $\phi^{\text{corr}}$ obtained above no longer depends on $T$ (so it will not vanish even in the $\epsilon \to 0$ or $T \to +\infty$ limit). Rather, it depends on the geometry in the parameter space only. $\phi^{\text{corr}}$ is hence identified as a geometric angle that arises from the fluctuations in a classical adiabatic process. This is particularly interesting because here $\delta I = 0$, i.e., even when the fluctuations in the original action variables are vanishing, there can still be a physical effect on another degree of freedom due to the dynamical fluctuations.

To illustrate the result in Eq. (23) we have designed a simple toy model with two degrees of freedom in total. Specifically, the total Hamiltonian is given by

$$H^{\text{tot}}(p_1, q_1; J) = \alpha J + \frac{1}{2} \left[ \left( \frac{p_1^2}{X^2} - J \right)^2 + \left( \frac{q_1^2}{Y^2} - J \right)^2 \right], \tag{24}$$

with $R = (X > 0, Y > 0)$, $\phi$ being a cyclic angular coordinate that forms a canonical pair with $J$, and $\alpha$ being a free parameter. For the $(p_1, q_1)$ degree
of freedom, this system has a \( \mathbf{R} \)-dependent fixed point

\[
\overline{q}_1 = \sqrt{\bar{J}Y}; \\
\overline{p}_1 = \sqrt{\bar{J}X},
\]

where \( \bar{J} \) represents a conserved value of the variable \( J \).

To calculate the fluctuation-induced geometric angle seen in the evolution of \( \phi \), note first

\[
\omega_J = \frac{\partial H^\text{tot}}{\partial J} = \alpha + 2J - \frac{p_1^2}{X^2} - \frac{q_1^2}{Y^2},
\]

and

\[
\frac{\partial \omega_J}{\partial \overline{p}_1} = \frac{2 \overline{p}_1}{X^2}, \quad \frac{\partial \omega_J}{\partial \overline{q}_1} = \frac{2 \overline{q}_1}{Y^2}.
\]

One may also easily obtain that the matrix \( \Gamma \) here is just a \( 2 \times 2 \) matrix, i.e.,

\[
\Gamma_{2 \times 2} = \begin{pmatrix}
-\frac{\partial^2 H}{\partial \overline{q}_1 \partial \overline{p}_1} & -\frac{\partial^2 H}{\partial \overline{q}_1 \partial \overline{q}_1} \\
\frac{\partial^2 H}{\partial \overline{p}_1 \partial \overline{p}_1} & \frac{\partial^2 H}{\partial \overline{p}_1 \partial \overline{q}_1}
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{4 \overline{J}}{X^2} \\
\frac{4 \overline{J}}{X^2} & 0
\end{pmatrix};
\]

and

\[
\left( \frac{\partial \overline{p}_1}{\partial \overline{R}} \right) = \begin{pmatrix}
\sqrt{\bar{J}} \\
0
\end{pmatrix} \hat{X} + \begin{pmatrix}
0 \\
\sqrt{\bar{J}}
\end{pmatrix} \hat{Y},
\]

where \( \hat{X} \) and \( \hat{Y} \) are unit vectors along the \( X \) and \( Y \) coordinates. Finally, substituting these intermediate results into Eq. (23), one finds the fluctuation-induced geometric angle

\[
\phi^\text{corr} = \oint_C \left( \frac{\partial \omega_J}{\partial \overline{p}_1} \right) \Gamma^{-1} \left( \frac{\partial \overline{p}_1}{\partial \overline{R}} \right) \cdot d\mathbf{R}
\]

\[
= \oint_C \left( \frac{\partial \omega_J}{\partial \overline{p}_1} \frac{X^2}{4J} \sqrt{\bar{J}} \hat{Y}, -\frac{\partial \omega_J}{\partial \overline{q}_1} \frac{Y^2}{4J} \sqrt{\bar{J}} \hat{X} \right) \cdot d\mathbf{R}
\]

\[
= \frac{1}{2} \oint_C \left( YdX - XdY \right) = -\int_C dS.
\]

As seen from the above result, here the geometric angle induced by the fluctuations in the first degree of freedom may be interpreted as the flux of an effective “magnetic charge” uniformly distributed on the \((X,Y)\) plane. The emergence of such a new classical geometric angle from our simple calculations is hence intriguing. It should be emphasized that in obtaining \( \phi^\text{corr} \)}
in Eq. (30), we did not seek new action-angle variables \((\tilde{I}_1, \tilde{\theta}_1)\) and \((\tilde{I}_2, \tilde{\theta}_2)\) such that \(H^{\text{tot}}\) becomes a function of \(\tilde{I}_1\) and \(\tilde{I}_2\) only. Indeed it can be highly complicated in general to find such a new representation due to the coupling between the two degrees of freedom. This indicates that \(\phi^{\text{corr}}\) here has a different meaning than Hannay’s angle, because it represents a geometrical correction to the \(\phi\) evolution, not to the evolution of the yet-to-be-found new angle variables \(\tilde{\theta}_1\) or \(\tilde{\theta}_2\).

We also note that our result here is consistent with one of the found terms in the previous study of the so-called “nonlinear Berry phase” based on GP equation [12]. In particular, the GP equation considered in Ref. [12] can be mapped to that of a classical Hamiltonian with two degrees of freedom, with the nonlinear eigenstates mapped to classical fixed points (see also Sec. IV-B). Adopting our perspective here, the geometric phase contributed by deviations from nonlinear eigenstates as analyzed in Ref. [12] may be understood as a classical geometric angle due to intrinsic fluctuations in classical adiabatic processes. Indeed, we have checked that if we apply Eq. (20) to the model considered in Ref. [12], then we can obtain a fluctuation-induced geometric phase term that is identical with a Berry-phase correction term discovered in Ref. [12]. Note however, the focus of our perspective is on a general description of the important dynamical fluctuations in a broad class of classical adiabatic processes. In our fully classical considerations here, a totally classical geometry angle is shown to arise in a second degree of freedom that is coupled with the first degree of freedom (with one adiabatically moving fixed point solution); whereas in Ref. [12], the emphasis was placed on a quantum adiabatic evolution context and the main concern is with the sum of one familiar Berry phase and a fluctuation-induced geometric phase as a correction.

4. Discussion

4.1. Pollution to Hannay’s angle

As mentioned above, in some early studies about Hannay’s angle in some Hamiltonian systems [1, 6, 7], it was numerically found that during an adiabatic process the total angle change minus the dynamical angle may not be Hannay’s angle. This subtle behavior was connected with dynamical fluctuations in classical adiabatic processes. Here we exploit our general result of Eq. (16) to shed more light on possible pollution to Hannay’s angle.
According to Eq. (2) and CAT, the total change in angle variables in a cyclic adiabatic process is given by

\[ \Delta \theta_i^\text{ideal}(T) = \int_0^T \omega_i(I, R) \, dt - \frac{\partial}{\partial I_i} \oint (\nabla_R \mathbf{q}) \cdot d\mathbf{R}. \]  

(31)

On the right hand side of Eq. (31), the first term is often called the dynamical angle, and the second term gives Hannay’s angle (upon an average over initial angle variables). We have also used the notation \( \theta_i^\text{ideal} \) to emphasize that it is for idealized cases without considering any dynamical fluctuations. Indeed, the angular frequency \( \omega_i \) in Eq. (31) is naively assumed to be the one determined by the idealized and constant action \( I \).

However, as suggested by Eq. (2), fluctuations in the action variables \( \delta I \) can then correct the angular frequency from \( \omega_i(I, R) \) to \( \omega_i(I, R) + \frac{\partial \omega_i(I, R)}{\partial I} \cdot \delta I \). In terms of the canonical variables \((p, q)\), fluctuations in \( p \) and \( q \) will lead to fluctuations in the angular frequency

\[ \delta \omega(I, R) = \frac{\partial \omega(I, R)}{\partial p} \cdot \delta p + \frac{\partial \omega(I, R)}{\partial q} \cdot \delta q. \]  

(32)

For this reason, the dynamical angle obtained by a time-integral of the idealized frequency \( \omega_i(I, R) \), [see Eq. (31)] should be re-examined with care. In terms of \( \delta p \) and \( \delta q \), the real change in the angular variables should be given by

\[ \Delta \theta_i^\text{real}(T) = \Delta \theta_i^\text{ideal}(T) + \int_0^T \delta \omega_i(I, R) \, dt \\
= \Delta \theta_i^\text{ideal}(T) + \int_0^T \frac{\partial \omega(I, R)}{\partial p} \cdot \delta p \, dt \\
+ \int_0^T \frac{\partial \omega(I, R)}{\partial q} \cdot \delta q \, dt. \]  

(33)

Because \( \delta p, \delta q \) and hence \( \delta \omega \) are of the same order with \( \epsilon = |dR/dt| \), just like the above fixed-point solution case, the term \( \int_0^T \delta \omega_i dt \) may not be negligible as it accumulates the fluctuations \( \delta \omega_i(I, R) \) over an entire adiabatic process. So the term \( \int_0^T \delta \omega_i dt \) should not be neglected without a clear understanding of the dynamics. At this point it is also clearer why we only consider \( \delta p \) and \( \delta q \) to the first order of \( \epsilon \) including higher-order terms are unnecessary because they will vanish in the \( \epsilon \to 0 \) limit.
The correction term $\int_0^T \delta \omega_i \, dt$ can hence give the difference between two objects: the standard Hannay’s angle, and a numerical calculation of a geometric angle based on the expression of $(\Delta \theta_i^{\text{real}} - \int_0^T \omega_i(I, R) \, dt)$. Unfortunately, unless for special fixed-point solution cases analyzed above, we in general cannot determine the fluctuations $\delta p$ and $\delta q$ from the differential equation in Eq. (16). In particular, $\delta p$ and $\delta q$ can only be determined if we have information about them for at least one given $\Theta$ (as the input). Therefore, without some detailed information of an adiabatic process, e.g., the detailed dependence of adiabatic parameter $R(t)$ on time, information about $\delta p$ and $\delta q$ is not available in general.

To see more clearly, let us discretize the adiabatic process by dividing one adiabatic process into many time intervals $t_1, t_2, \cdots$, during each of which $R = R_j$, followed by a jump onto the next value $R_{j+1}$ after the temporal interval $t_j$ (different time intervals and different choices for $R_j$ define different adiabatic processes with different details). Note that even for a continuous adiabatic process, this discretized version is rather typical in numerical simulations (as the discretized time steps decrease, the simulated dynamics approaches a continuous process). Now for each point $R_j$, we may use Eq. (16) to describe the dynamical fluctuations, but Eq. (16) is dependent on $R_j$. For a particular segment where $R = R_j$, the angle variable $\Theta$ changes rapidly. Obviously, different timing for the next jump will result in different initial values of $\Theta$ for next segment $R = R_{j+1}$, leading to another initial condition for the differential equation (16) associated with $R = R_{j+1}$. This process then continues. According to Eq. (32), $\delta \omega$ and thus the correction term $\int_0^T \delta \omega_i \, dt$ will then depend on great details of a particular adiabatic process. It is for this reason that the correction term $\int_0^T \delta \omega_i \, dt$ is identified as “pollution” to Hannay’s angle, with the latter independent of how an adiabatic process is implemented. Analysis here also makes it clearer that the fixed-point solution case in Sec. III is special because a definite prediction about fluctuations can be made therein.

It is also worth noting that, according to Eq. (32), the pollution vanishes if the angular frequency $\omega_i$ does not depend on the action $I$. This is the case in a linear system such as a harmonic oscillator.

4.2. “Pollution” to adiabatic phase evolution in a two-mode BEC model

Finally, we propose to use a two-mode GP equation to study pollution to a geometric phase associated with quantum adiabatic cycles, thus making
a connection between our theoretical considerations here and a reachable experimental context. In particular, there are a number of possibilities to experimentally realize a two-mode BEC. For example, one may consider a BEC in a double-well potential, or a BEC in an optical lattice occupying two bands [13]. On the mean-field level, a two-mode BEC can be described by the following GP equation \((\hbar = 1)\)

\[
i \frac{d}{dt} \begin{pmatrix} a \\ b \end{pmatrix} = H_{\text{GP}} \begin{pmatrix} a \\ b \end{pmatrix},
\]

\[
= \frac{1}{2} \left( \begin{array}{cc}
\gamma + c(|b|^{2} - |a|^{2}) & \Delta \\
\Delta & -\gamma - c(|b|^{2} - |a|^{2}) \end{array} \right) \begin{pmatrix} a \\ b \end{pmatrix},
\]

(34)

where \(\gamma\) denotes an energy bias between the two modes, \(|a|^{2}\) and \(|b|^{2}\) (with \(|a|^{2} + |b|^{2} = 1\)) represent occupation probabilities of the two modes, \(c\) gives the self-interaction strength, and \(\Delta\) denotes the coupling between the two modes. We can consider, for example, the two parameters \(\gamma\) and \(\Delta\) to implement an adiabatic cyclic process.

The dynamics described by the above GP equation can be translated into Hamiltonian dynamics. In particular, let \(p = \phi_{a} - \phi_{b}, \quad q = |a|^{2}, \quad a = |a|e^{i\phi_{a}}, \quad b = |b|e^{i\phi_{b}}\), then apart from an overall phase parameter \(\phi_{b}\), Eq. (34) leads to

\[
\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}.
\]

(35)

where

\[
H = \Delta \sqrt{q(1-q)} + \frac{\gamma}{2}(2q - 1) - \frac{c}{4}(2q - 1)^{2}.
\]

(36)

It is also straightforward to find that the evolution of \(\phi_{b}\) obeys

\[
\frac{d\phi_{b}}{dt} = i(\sqrt{q}e^{-ip}, \sqrt{1-q}) \frac{d}{dt} \left( \sqrt{q}e^{ip} \right) - H - \Lambda,
\]

(37)

where

\[
\Lambda = -\frac{c}{4}(2q - 1)^{2}.
\]

(37)

It is seen that the evolution of the overall phase \(\phi_{b}\) is determined by, but will not have a back action on, the classical trajectories determined by \(H\)
in Eq. (36). In this sense, the $\phi_b$ parameter plays a similar role as the $\phi$ parameter in Sec. III.

It is now clear that our general result of dynamical fluctuations in classical adiabatic processes can be directly relevant to understanding the adiabatic evolution of a two-mode BEC system. If the adiabatic process starts from a stationary state of the GP equation, then the dynamics is just about an adiabatically evolving fixed-point solution of the Hamiltonian in Eq. (36). As shown earlier (see also Ref. [12]), in this case a definite prediction can be made about how accumulation of dynamical fluctuations can eventually lead to a geometry-like correction to $\phi_b$. Consider now a superposition state of two stationary states of the above two-mode GP equation as the initial state of an adiabatic process. This case then corresponds to a classical adiabatic process with non-fixed-point solutions. As indicated by Eq. (36), dynamical fluctuations can now affect the evolution of the adiabatically evolving phase $\phi_b$, in an unpredictable way if we do not know the details of the adiabatic process. Pollution to the quantum phase $\phi_b$ hence emerges. Interestingly, in the same context, how $\phi_b$ may develop an adiabatic geometric phase for general superpositions of stationary states was already considered in Ref. [14] without considering dynamical fluctuations. It is hence of interest to numerically or even experimentally examine the actual pollution due to the accumulation of dynamical fluctuation effects in such type of quantum adiabatic processes.

5. Summary

To summarize, we have obtained a general description of the intrinsic dynamical fluctuations in classical adiabatic processes associated with integrable systems. These fluctuations are typically neglected by the conventional classical adiabatic theorem. The dynamical fluctuations are described in this work in terms of deviations from idealized adiabatic trajectories. As an application, we have shown how a new kind of classical geometric phase may emerge using an explicit example with an adiabatically evolving fixed-point solution. We then discussed the origin of the pollution to Hannay’s angle and proposed to use a two-mode BEC system to further study possible fluctuation-induced pollution to one type of quantum adiabatic evolution described on a mean-field level.

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Appendix A. On the derivation of Eq. (11)

Here we present some necessary details in deriving Eq. (11). The summation convention by using repeated indices is also adopted here. First, by definition we have 
\[ p_i = \bar{p}_i + \delta p_i, \]
and hence
\[ \frac{\partial p_i}{\partial R} = \frac{\partial \bar{p}_i}{\partial R} + \frac{\partial \delta p_i}{\partial R}. \]  
(A.1)

As a second step, let us expand the expressions \( \frac{\partial p_i}{\partial I_j} \) and \( \frac{\partial p_i}{\partial \theta_j} \) around \( \frac{\partial \bar{p}_i}{\partial I_j} \) and \( \frac{\partial \bar{p}_i}{\partial \theta_j} \), to the first order of \( \delta I \) and \( \delta \Theta \), leading to
\[ \frac{\partial p_i}{\partial I_j} = \frac{\partial \bar{p}_i}{\partial I_j} + \frac{\partial^2 \bar{p}_i}{\partial I_j \partial I_k} \delta I_k + \frac{\partial^2 \bar{p}_i}{\partial I_j \partial \theta_k} \delta \theta_k \]  
(A.2)
and
\[ \frac{\partial p_i}{\partial \theta_j} = \frac{\partial \bar{p}_i}{\partial \theta_j} + \frac{\partial^2 \bar{p}_i}{\partial \theta_j \partial I_k} \delta I_k + \frac{\partial^2 \bar{p}_i}{\partial \theta_j \partial \theta_k} \delta \theta_k. \]  
(A.3)

To proceed further we shall use the obvious two relations
\[ \frac{dp_i}{dt} = \frac{\partial p_i}{\partial R} \frac{dR}{dt} + \frac{\partial p_i}{\partial I_j} \frac{dI_j}{dt} + \frac{\partial p_i}{\partial \theta_j} \frac{d\theta_j}{dt} \]  
(A.4)
and
\[ \omega_j(I, R) = \omega_j(\bar{I}, \bar{R}) + \frac{\partial \omega_j}{\partial I_k} \delta I_k. \]  
(A.5)

Substituting Eqs. (1), (2), (A1), (A2), and (A3) into Eq. (A4), we find
\[ \frac{dp_i}{dt} = \frac{\partial \bar{p}_i}{\partial R} \frac{dR}{dt} + \frac{\partial \bar{p}_i}{\partial I_j} \frac{dI_j}{dt} + \frac{\partial \bar{p}_i}{\partial \theta_j} \frac{d\theta_j}{dt} \]
\[ + \frac{\partial \bar{p}_i}{\partial I_k} \left[ \frac{\partial W}{\partial I_j} \frac{dR}{dt} + \omega_j(\bar{I}, \bar{R}) + \frac{\partial \omega_j}{\partial I_k} \right] \]
\[ + \left( \frac{\partial^2 \bar{p}_i}{\partial \theta_j \partial I_k} \delta I_k + \frac{\partial^2 \bar{p}_i}{\partial \theta_j \partial \theta_k} \delta \theta_k \right) \omega_j(\bar{I}, \bar{R}), \]  
(A.6)

and
\[ \omega_j(\bar{I}, \bar{R}) \]  
(A.7)
with all the terms of the second order of \( \frac{dR}{dt} \) or higher neglected. Note that consistent with our final result, we have assumed that \((\delta p, \delta q)\) or \((\delta I, \delta \Theta)\) are of the first order of \( \epsilon \equiv \frac{dR}{dt} \).

As the last step we use the relation

\[
\frac{\partial^2 p_i}{\partial \theta_j \partial I_k} \delta I_k + \frac{\partial^2 p_i}{\partial \theta_j \partial \theta_k} \delta \theta_k = \frac{\partial}{\partial \theta_j} \left( \frac{\partial p_i}{\partial I_k} \delta I_k + \frac{\partial p_i}{\partial \theta_k} \delta \theta_k \right) = \frac{\partial \delta p_i}{\partial \theta_j}. \tag{A.8}
\]

Plugging this simple relation into Eq. (A6), we obtain the first equality in Eq. (11). The second equality in Eq. (11) can be obtained in the same manner.
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