Coefficient Bounds Problem For Functions associated with Universally Prestarlike Functions

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Abstract. Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathbb{C} \setminus [1, \infty)$ have been introduced by S. Ruscheweyh. This notion generalizes the corresponding one for functions in the unit disk $\Delta$ (and other circular domains in $\mathbb{C}$). In this paper, we obtain coefficient bounds for certain class of analytic functions associated with universally prestarlike function.

1. Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain $\Omega$. For domain $\Omega$ containing the origin $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when $\Omega$ is the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, we use the abbreviation $H, H_0$ and $H_1$ respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order $\alpha$ with $0 \leq \alpha < 1$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \Delta)$$

and the set of all such functions is denoted by $S_\alpha$. The convolution or Hadamard Product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in H_1$ is called prestarlike of order $\alpha$ if

$$\frac{z}{(1-z)^{2-2\alpha}} \ast f(z) \in S_\alpha$$

The set of all such functions is denoted by $R_\alpha$. The notion of prestarlike functions has been extended from the unit disk to other disk and half planes containing the origin by Ruscheweyh.
and Salinas[1]. Let Ω be one such disk or half plane. Then there are two unique parameters \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \rho \in [0,1] \) such that
\[
\Omega_{\gamma,\rho} = \{ w_{\gamma,\rho}(z) : z \in \Delta \}
\]
(3)
where, \( w_{\gamma,\rho}(z) = \frac{\gamma z}{1 - \rho z} \). Note that \( \frac{1}{z} \in \Omega_{\gamma,\rho} \) iff \( |\gamma + \rho| \leq 1 \).

**Definition 1:**[1] Let \( \alpha \leq 1 \), and \( \Omega = \Omega_{\gamma,\rho} \) for some admissible pair \((\gamma, \rho)\). A function \( f \in H_1(\Omega_{\gamma,\rho}) \) is called prestarlike of order \( \alpha \) in \( \Omega_{\gamma,\rho} \) if
\[
f_{\gamma,\rho}(z) = \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha.
\]
(4)
The set of all such functions \( f \) is denoted by \( \mathcal{R}_\alpha(\Omega) \).

Let \( \Lambda \) be the slit domain \( \mathbb{C} \setminus [1, \infty) \) (the slit being along the positive real axis).

**Definition 2:**[1] Let \( \alpha \leq 1 \). A function \( f \in H_1(\Lambda) \) is called universally prestarlike of order \( \alpha \) if and only if \( f \) is prestarlike of order \( \alpha \) in all sets \( \Omega_{\gamma,\rho} \) with \( |\gamma + \rho| \leq 1 \). The set of all such functions is denoted by \( \mathcal{R}^u_\alpha \).

For a univalent function \( f(z) \) of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
(5)
the \( k^{th} \) root transform is defined as
\[
F(z) = [f(z^k)]^{\frac{1}{k}} := z + \sum_{n=1}^{\infty} d_{kn+1} z^{kn+1}
\]
(6)
k \( \in \mathbb{N} = \{1, 2, \ldots \} \).

**Definition 3:**[2],[3],[4],[5] Let \( \phi(z) \) be an analytic function with positive real part on \( \Delta \), which satisfies \( \phi(0) = 1 \), \( \phi'(0) > 0 \) and which maps the unit disc \( \Delta \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class \( \mathcal{R}^u_\alpha(\phi) \) consists of all analytic function \( f \in H_1(\Lambda) \) satisfying
\[
\frac{D^{3-2\alpha} f(z)}{D^{2-2\alpha} f(z)} < \phi(z).
\]
(7)
where \( \prec \) denotes the subordination, where \( (D^\beta f)(z) = \frac{z}{(1-z)^\beta} * f \), for \( \beta \geq 0 \). In particular, for \( \beta = n \in \mathbb{N} \), we have \( D^{n+1} f = \frac{z}{(1-z)^n} (z^n f)^{(n)} \).

**Note:** Let \( F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz} \) where \( a_k = \int_0^1 t^k d\mu(t) \), \( \mu(t) \) is a probability measure on \([0,1] \). Let \( T \) denote the set of all such functions \( F \). They are analytic in the slit domain \( \Lambda \).

To prove our result we need the following theorems.
Theorem 1:[2],[3],[4],[5] Let $0 \leq \alpha \leq 1$ and $f \in H_1(\Lambda)$. Then $f \in R_\alpha^u$ if and only if
\[
\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in T.
\] (8)

This admits an explicit representation of the function in $R_\alpha^u$. If $f \in H_0$ has all its taylor coefficients at the origin different from zero we write $f^{(-1)}$ for the (possibly formal but) unique solution of $f \ast f^{(-1)} = \frac{1}{1-z}$.

Theorem 2:[2] Let $f$ be an universally prestarlike function of order $\alpha \leq 1$, then the function $f(z)$ has a representation of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]
for $n = 2, 3, \ldots$ where,
\[
a_n = \left\{ \frac{n-1}{C'(\alpha, n) - C(\alpha, n)} \right\}
\]
\[
C(\alpha, n) = \prod_{k=2}^{n} (k - 2\alpha) \frac{1}{(n-1)!}, \quad C(\alpha, k) = \prod_{m=2}^{k} (m - 2\alpha) \frac{1}{(k-1)!}, \quad C(\alpha, 1)a_1 = 1
\]
\[
C'(\alpha, n) = \prod_{k=2}^{n} (k + 1 - 2\alpha) \frac{1}{(n-1)!}, \quad b_n = \int_0^1 t^n d\mu(t) \text{ and } \mu(t) \text{ a probability measure on } [0,1].
\]

Let $\Omega_1$ be the class of analytic functions $\omega$, normalized by $\omega_1(0) = 0$, satisfying the condition $|\omega_1(z)| < 1$. The following two lemmas regarding the coefficients of functions in $\Omega_1$ are needed to prove our main results.

The following lemma is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda[6].

Lemma 1:[7] If $\omega \in \Omega_1$ and
\[
\omega(z) = \omega_1 z + \omega_2 z^2 + \ldots, (z \in \Delta)
\] (10)
then,
\[
|\omega_2 - t\omega_1^2| \leq \left\{ \begin{array}{ll}
-t, & t \leq -1 \\
1, & -1 \leq t \leq 1 \\
t, & t \geq 1.
\end{array} \right.
\]

For $t < -1$, or $t > 1$, the equality holds if and only if $\omega(z) = z$ or one of its rotations. For $-1 < t < 1$, then the equality holds if and only if $\omega(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $\omega(z) = \frac{\lambda + z}{1 + \lambda z} (0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $\omega(z) = -z \frac{\lambda + z}{1 + \lambda z} (0 \leq \lambda \leq 1)$ or one of its rotations.
Lemma 2:[8] If $\omega \in \Omega_1$, then $|\omega^2 - t\omega^2| \leq \max\{1; |t|\}$, for any complex number $t$. The result is sharp for the function $\omega(z) = z^2$ or $z$.

Lemma 3:[6] If $P_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in $\Delta$, then
\[
|c_2 - vc_1^2| \leq \begin{cases} 
-4v + 2, & v \leq 0 \\
2, & 0 \leq v \leq 1 \\
4v + 2, & v \geq 1 
\end{cases}
\]

when $v < 0$, or $v > 1$, the equality holds if and only if $P_1(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. when $0 < v < 1$, then the equality holds if and only if $P_1(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if $P_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}$ for any complex number $t$. The reciprocal of one of the function for which the equality holds in the case of $v = 0$. Also the above upper bound can be improved as follows when $0 < v < 1$
\[
|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \quad \left(0 < v \leq \frac{1}{2}\right).
\]
\[
|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2, \quad \left(\frac{1}{2} < v \leq 1\right).
\]

We now estimate the sharp bound for the coefficient functional $|d_{2k+1} - \mu d_{2k+1}^2|$ corresponding to the $k^{th}$ root transformation of universally prestarlike functions of order $\alpha$ with respect to $\phi$.

2. Coefficient bounds for the $k^{th}$ root transformation

Theorem 3: Let $\phi(z) = 1 + \frac{B_1}{2} z + \frac{B_2}{2} z^2 + \ldots$, and
\[
\sigma_1 = \frac{-2k}{(3-2\alpha)B_1} + \frac{2kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2}
\]
\[
\sigma_2 = \frac{2k}{(3-2\alpha)B_1} + \frac{2kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2}
\]
\[
t = -\frac{B_2}{B_1} - \frac{(2-2\alpha)B_1}{2} + \frac{(3-2\alpha)B_1}{2} \left[\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right]
\]
If $f \in \mathcal{R}_{\alpha}^n(\phi)$ and $F$ is the $k^{th}$ root transformation of $f$ given by (5), then,
\[
|d_{2k+1} - \mu d_{2k+1}^2| \leq \begin{cases} 
-\frac{B_1}{(3-2\alpha)2k} t, & \mu \leq \sigma_1 \\
\frac{B_1}{(3-2\alpha)2k}, & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{B_1}{(3-2\alpha)2k} t, & \mu \geq \sigma_2,
\end{cases}
\]
and where $\mu$ complex
Now, by using (15) in (6) and equating the coefficients of $z$ and $d$, we get,

$$d_{k+1} = \frac{a_2}{k}; \quad d_{2k+1} = \frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2}$$

Next, for a function $f$, a computation shows that

$$\frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = \phi(\omega(z)).$$

(11)

We know that, $D^{3-2\alpha}f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ where $b_n = \int_0^1 t^n d\mu(t)$ and $\mu(t)$ is a probability measure on $[0, 1]$. and

$$\phi(\omega(z)) = 1 + \frac{B_1}{2} \omega_1 z + (\frac{B_1}{2} \omega_2 + B_2 \omega_3^2) z^2 + \ldots$$

Therefore,

$$1 + b_1 z + b_2 z^2 + \ldots = 1 + \frac{B_1}{2} \omega_1 z + (\frac{B_1}{2} \omega_2 + B_2 \omega_3^2) z^2 + \ldots$$

Now, equating the coefficients of $z$ and $z^2$ we get

$$b_1 = \frac{B_1}{2} \omega_1, \quad b_2 = \frac{B_1}{2} \omega_2 + \frac{B_2}{2} \omega_3^2$$

(12)

Now, $D^{3-2\alpha}f(z) = 1 + [C'(\alpha, 2)a_2 - C(\alpha, 2)a_2] z +$

$$[C'(\alpha, 3)a_3 - C(\alpha, 2)C'(\alpha, 2)^2 - C(\alpha, 3) + (C(\alpha, 2)a_2]^2] z^2 + \ldots$$

$$= 1 + \frac{B_1}{2} z + b_2 z^2 + \ldots$$

where, $C(\alpha, n) = \prod_{k=2}^{n} (k-2\alpha)/(n-1)!$, $C'(\alpha, n) = \prod_{k=2}^{n} (k+1-2\alpha)/(n-1)!$, $b_n = \int_0^1 t^n d\mu(t)$ for $n = 2, 3, \ldots$ and $\mu(t)$ a probability measure on $[0, 1]$.

Equating the coefficients of $z$ and $z^2$ respectively and simplifying we get,

$$a_2 = b_1; \quad a_3 = \frac{b_2 + (2 - 2\alpha)b_1^2}{(3 - 2\alpha)}$$

(13)

Now, using (12) in (13) we get,

$$a_2 = \frac{B_1}{2} \omega_1; \quad a_3 = \frac{2B_1 \omega_2 + (2B_2 + (2 - 2\alpha)B_1^2) \omega_3^2}{4(3 - 2\alpha)}$$

(14)

Now, for a function $f$, a computation shows that

$$\frac{1}{f(z^k)^k} = z + \frac{a_2}{k} z^{k+1} + \left(\frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2}\right) z^{2k+1} + \ldots$$

(15)

Now, by using (15) in (6) and equating the coefficients of $z$ and $z^2$ we get,

$$d_{k+1} = \frac{a_2}{k}; \quad d_{2k+1} = \frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2}$$

(16)
Now, using (14) in (16) we get,
\[ d_{k+1} = \frac{B_1 \omega_1}{2k} \]
and
\[ d_{2k+1} = \frac{1}{k} \left[ \frac{2B_1 \omega_2 + (2B_2 + (2 - 2\alpha)B_1^2) \omega_1^2}{4(3 - 2\alpha)} - \frac{B_1^2 \omega_1^2}{4} + \frac{B_1^2 \omega_1^2}{4k} \right] \]

Now,
\[ d_{2k+1} - \mu d_{k+1} = \frac{1}{k} \left[ \frac{2B_1 \omega_2 + (2B_2 + (2 - 2\alpha)B_1^2) \omega_1^2}{4(3 - 2\alpha)} - \frac{B_1^2 \omega_1^2}{4} + \frac{B_1^2 \omega_1^2}{4k} \right] \]
\[ - \mu B_1^2 \omega_1^2 \frac{1}{4k^2} \]
and hence
\[ d_{2k+1} - \mu d_{k+1} = \frac{B_1}{(3 - 2\alpha)2k} [\omega_2 - \omega_1^2 t] \]

The first result is established by an application of Lemma 1

If \( t \leq -1 \), then,
\[ \mu \leq \frac{-2k}{(3 - 2\alpha)B_1} + \frac{2kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - k \frac{1}{2} + \frac{1}{2} \quad (\mu \leq \sigma_1), \]
and Lemma 1 gives:
\[ |d_{2k+1} - \mu d_{k+1}| \leq \frac{B_1}{(3 - 2\alpha)2k} t. \]

For \(-1 \leq t \leq 1\), we have \( \sigma_1 \leq \mu \leq \sigma_2 \), where
\[ \sigma_1 = \frac{-2k}{(3 - 2\alpha)B_1} + \frac{2kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - k \frac{1}{2} + \frac{1}{2} \]
\[ \sigma_2 = \frac{2k}{(3 - 2\alpha)B_1} + \frac{2kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - k \frac{1}{2} + \frac{1}{2} \]
\[ t = \frac{-B_2}{B_1} - \frac{(2 - 2\alpha)B_1}{2} + \frac{(3 - 2\alpha)B_1}{2} \left[ \frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k} \right] \]
and Lemma 1 yields:
\[ |d_{2k+1} - \mu d_{k+1}| \leq \frac{B_1}{(3 - 2\alpha)2k}. \]

For \( t \geq 1 \), we have,
\[ \mu \geq \frac{2k}{(3 - 2\alpha)B_1} + \frac{2kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - k \frac{1}{2} + \frac{1}{2} \quad (\mu \geq \sigma_2), \]
and it follows from Lemma 1 that
\[ |d_{2k+1} - \mu d_{k+1}| \leq \frac{B_1}{(3 - 2\alpha)2k} t. \]

For the sharpness of the results in the above theorem we have the following:
(i) If $\mu = \sigma_1$, then the equality holds in the Lemma 1 if and only if 
\[ \omega(z) = z \frac{\lambda + z}{1 + \lambda z} (0 \leq \lambda \leq 1) \text{ or one of its rotations.} \]
(ii) If $\mu = \sigma_2$, then $\omega(z) = -z \frac{\lambda + z}{1 + \lambda z} (0 \leq \lambda \leq 1) \text{ or one of its rotations.}$
(iii) If $\sigma_1 < \mu < \sigma_2$, then $\omega(z) = z^2$.

The second result follows by an application of lemma 2

For $k = 1$, the $k^{th}$ root transformation of $f$ reduces to the given function $f$ itself. Thus the estimate given in the above theorem is an extension of the corresponding result for the Fekete-Szego functional corresponding to universally prestarlike functions of order $\alpha$ with respect to $\phi$.

**Theorem 4:** Let $\phi(z) = 1 + B_1 \frac{z}{2} + B_2 \frac{z^2}{2} + \ldots$, where $B_n'$s are real with $B_1 > 0, B_2 \geq 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a universally prestarlike function of order $\alpha$ then

\[
|a_3 - \mu a_2^2| \leq \begin{cases}
\frac{2B_2 + (2 - 2\alpha)B_1^2 - (3 - 2\alpha)B_1^2 \mu}{4(3 - 2\alpha)}, & \mu \leq \sigma_1 \\
\frac{B_1}{2(3 - 2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{-2B_2 - (2 - 2\alpha)B_1^2 + (3 - 2\alpha)B_1^2 \mu}{4(3 - 2\alpha)}, & \mu \geq \sigma_2,
\end{cases}
\]

where

\[
\sigma_1 = \frac{(2B_2 - 2B_1) + (2 - 2\alpha)B_1^2}{(3 - 2\alpha)B_1^2},
\]
\[
\sigma_2 = \frac{(2B_2 + 2B_1) + (2 - 2\alpha)B_1^2}{(3 - 2\alpha)B_1^2}
\]

the result is sharp.

For the choice $\alpha = 0$, theorem 4 coincides with the following result obtained for the class $C(\phi)$ by Ma and Minda [6]

**Corollary 1:** Let $0 \leq \mu \leq 1$, Further Let $\phi(z) = 1 + \frac{B_1}{2} z + \frac{B_2}{2} z^2 + \ldots$, where $B_n'$s are real with $B_1 > 0, B_2 \geq 0$ and

\[
\sigma_1 = \frac{2(B_2 - B_1) + B_1^2}{3B_1^2},
\]
\[
\sigma_2 = \frac{2(B_2 + B_1) + B_1^2}{3B_1^2}
\]

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\phi)$, then
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6} \left[ B_2 + B_1^2 - \frac{3B_1^2 \mu}{2} \right], & \mu \leq \sigma_1 \\ \frac{B_1}{6}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{1}{6} \left[ -B_2 - B_1^2 + \frac{3B_1^2 \mu}{2} \right], & \mu \geq \sigma_2, \end{cases} \]

If \( \sigma_1 \leq \mu \leq \sigma_2 \) then in view of lemma 3 theorem 4 can be improved.

**Corollary 2:** Let \( \sigma_3 \) be given by

\[ \sigma_3 = \frac{2B_2 + (2 - 2\alpha)B_1^2}{(3 - 2\alpha)B_1^2} \]

If \( \sigma_1 \leq \mu \leq \sigma_3 \), then,

\[ |a_3 - \mu a_2^2| + \left( \frac{(3 - 2\alpha)\mu B_1^2 - [(2B_2 - 2B_1) + (2 - 2\alpha)B_1^2]}{(3 - 2\alpha)B_1^2} \right) |a_2^2| \leq \frac{B_1}{2(3 - 2\alpha)} \]

If \( \sigma_2 \leq \mu \leq \sigma_3 \), then,

\[ |a_3 - \mu a_2^2| + \left( \frac{-3(3 - 2\alpha)\mu B_1^2 + 2B_2 + 2B_1 + (2 - 2\alpha)B_1^2}{(3 - 2\alpha)B_1^2} \right) |a_2^2| \leq \frac{B_1}{2(3 - 2\alpha)} \]

For the choice \( \alpha = 0 \), Corollary 2 coincides with the following result obtained for the class \( C(\phi) \) by Ma and Minda[6].

**Corollary 3:** If \( \sigma_1 \leq \mu \leq \sigma_2 \) then in view of Lemma 3 Corollary 2 can be improved. Let \( \sigma_3 \) be given by

\[ \sigma_3 = \frac{2(B_2 + B_1^2)}{3B_1^2} \]

If \( \sigma_1 \leq \mu \leq \sigma_3 \), then,

\[ |a_3 - \mu a_2^2| + \left( \frac{3\mu B_1^2 - [2(B_2 - B_1 + B_1^2)]}{3B_1^2} \right) |a_2^2| \leq \frac{B_1}{6} \]

If \( \sigma_2 \leq \mu \leq \sigma_3 \), then,

\[ |a_3 - \mu a_2^2| + \left( \frac{-3\mu B_1^2 + 2(B_2 + B_1 + B_1^2)}{3B_1^2} \right) |a_2^2| \leq \frac{B_1}{6} \]

**3. Fractional Derivative**

We begin by recalling the following definitions of operators of fractional calculus (that is fractional derivatives and fractional integrals) which was used by Owa and Srivastava.
Definition 4:[9] Let \( f \) be analytic in a simply connected region of the \( z \)-plane containing the origin. The fractional derivative of \( f \) of order \( \lambda \) is defined by

\[
D_\zeta^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 < \lambda < 1)
\]

where the multiplicity of \((z-\zeta)^\lambda\) is removed by requiring that \( \log(z-\zeta) \) is real for \( z - \zeta > 0 \).

Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava(1987) introduced the operator \( \Omega^\lambda \) : \( \mathcal{A} \rightarrow \mathcal{A} \) for \( \lambda \) any positive real number \( \neq 2, 3, 4, \ldots \) defined by

\[
(\Omega^\lambda f)(z) = \Gamma(2-\lambda)z^\lambda D_\zeta^\lambda f(z)
\]

and \( \mathcal{A} = H_1(\Delta) \).

The class \((\mathcal{R}_\alpha^u)^\lambda(\phi)\) consists of function \( f \in \mathcal{A} \) for which \( \Omega^\lambda f \in (\mathcal{R}_\alpha^u)(\phi) \).

Note that \((\mathcal{R}_\alpha^u)^\lambda(\phi)\) is the special case of the class \((\mathcal{R}_\alpha^u)^\sigma(\phi)\) when

\[
g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n
\]

Let \( g(z) = z + \sum_{n=2}^{\infty} g_n z^n \) \((g_n > 0)\), \( g \) be analytic in \( \Delta \) and \( f \ast g \neq 0 \). Since \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^\sigma(\phi) \) if and only if

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in (\mathcal{R}_\alpha^u)(\phi),
\]

we obtain the coefficient estimate for functions in the class \((\mathcal{R}_\alpha^u)^\sigma(\phi)\), from the corresponding estimate for functions in the class \((\mathcal{R}_\alpha^u)(\phi)\).

Theorem 5: Let \( \phi(z) = 1 + \frac{B_1}{2} z + \frac{B_2}{2} z^2 + \ldots \). If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^\sigma(\phi) \) then

\[
-a_3 - \mu a_2^2 \leq \left\{ \begin{array}{ll}
\frac{1}{4g_3(3-2\alpha)} \left( 2B_2 + B_1^2(2-2\alpha) - \nu \right), & \mu \leq \sigma_1 \\
\frac{B_1}{2g_3(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{1}{4g_3(3-2\alpha)} \left( -2B_2 - B_1^2(2-2\alpha) + \nu \right), & \mu \geq \sigma_2,
\end{array} \right.
\]

where

\[
\nu = \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2}
\]

\[
\sigma_1 = \frac{g_2}{g_3} \left[ \frac{(2B_2 - 2B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right],
\]

\[
\sigma_2 = \frac{g_2}{g_3} \left[ \frac{(2B_2 + 2B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right].
\]
the result is sharp.

**Note:** Since
\[ g(z) = (\Omega^zf)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n, \]  
(17)

We have
\[ g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \]
(18)

and
\[ g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \]
(19)

For \( g_2 \) and \( g_3 \) mentioned above substitute in theorem 5 reduces to the following

**Theorem 6:** Let \( \phi(z) = 1 + \frac{B_1}{2} z + \frac{B_2}{2} z^2 + \ldots \), If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (R_n^\alpha)^g(\phi) \) then

\[-a_3 - \mu a_2^2 \leq \begin{cases} 
\frac{(2-\lambda)(3-\lambda)}{24(3-2\alpha)} \left(2B_2 + B_1^2(2-2\alpha) - \nu_1\right), & \mu \leq \sigma_1 \\
\frac{(2-\lambda)(3-\lambda)B_1}{12(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{(2-\lambda)(3-\lambda)}{24(3-2\alpha)} \left(-2B_2 - B_1^2(2-2\alpha) + \nu_1\right), & \mu \geq \sigma_2,
\end{cases}\]

where
\[ \nu_1 = \frac{3(3-2\alpha)\mu(2-\lambda)}{2(3-\lambda)} \]
\[ \sigma_1 = \frac{2(3-\lambda)}{3(2-\lambda)} \left[\frac{2B_2 - B_1 + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2}\right], \]
\[ \sigma_2 = \frac{2(3-\lambda)}{3(2-\lambda)} \left[\frac{2B_2 + B_1 + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2}\right]. \]

the result is sharp.

For \( \alpha = 0 \) in theorem 5, we get the following result by Ma and Minda (1994) [6] for the class \( C(\phi) \)

**Corollary 4:** Let the function \( \phi \) given by \( \phi(z) = 1 + \frac{B_1}{2} z + \frac{B_2}{2} z^2 + \ldots \),

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\phi) \]

\[-a_3 - \mu a_2^2 \leq \begin{cases} 
\frac{1}{6g_3} \left[B_2 + B_1^2 \left(1 - \frac{3g_3\mu}{2g_2}\right)\right], & \mu \leq \sigma_1 \\
\frac{B_1}{6g_3}, & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{-1}{6g_3} \left[B_2 + B_1^2 \left(1 - \frac{3g_3\mu}{2g_2}\right)\right], & \mu \geq \sigma_2,
\end{cases}\]

where
\[ \sigma_1 = \frac{2g_2^2}{3g_3B_1^2} \left[B_2 - B_1 + B_1^2\right]. \]
\[ \sigma_2 = \frac{2g_2^2}{3g_3B_1^2} \left[ B_2 + B_1 + B_1^2 \right] \]

the result is sharp.

For \( \alpha = 0 \) in corollary 4, we get the following result by Ma and Minda (1994)[6] for the class \( C(\phi) \).

**Corollary 5:** Let the function \( \phi \) given by \( \phi(z) = 1 + \frac{B_1}{2} z + \frac{B_2}{2} z^2 + \ldots \),

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\phi) \]

then,

\[
-a_3 - \mu a_2^2 \leq
\begin{cases}
\frac{(2 - \lambda)(3 - \lambda)}{36} \left[ B_2 + B_1^2 \left( 1 - \frac{9(2 - \lambda)\mu}{4(3 - \lambda)} \right) \right], & \mu \leq \sigma_1 \\
\frac{B_1(2 - \lambda)(3 - \lambda)}{36}, & \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{-(2 - \lambda)(3 - \lambda)}{36} \left[ B_2 + B_1^2 \left( 1 - \frac{9(2 - \lambda)\mu}{4(3 - \lambda)} \right) \right], & \mu \geq \sigma_2,
\end{cases}
\]

where

\[ \sigma_1 = \frac{4(3 - \lambda)}{9B_1^2(2 - \lambda)} \left[ B_2 - B_1 + B_1^2 \right], \]

\[ \sigma_2 = \frac{4(3 - \lambda)}{9B_1^2(2 - \lambda)} \left[ B_2 + B_1 + B_1^2 \right] \]

the result is sharp.

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