A TORELLI THEOREM FOR THE MODULI SPACE OF HIGGS BUNDLES ON A CURVE

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Abstract. Let $X$ be a smooth projective curve over $\mathbb{C}$, and let $M^{n,\xi}_X$ be the moduli space of stable Higgs bundles on $X$ (with genus $g > 1$), with rank $n$ and fixed determinant $\xi$, with $n$ and $\deg(\xi)$ coprime. Let $X'$ and $\xi'$ be another such curve and line bundle. We prove that if $M^{n,\xi}_X$ and $M^{n,\xi'}_{X'}$ are isomorphic as algebraic varieties, then $X$ and $X'$ are isomorphic.

1. Introduction

The classical Torelli theorem says that the Jacobian $J(X)$ of a curve, together with the polarization given by the theta divisor, determines the curve $X$, i.e., if $J(X)$ and $J(X')$ are isomorphic as polarized varieties, then $X$ is isomorphic to $X'$.

A similar result holds for $SU^{n,\xi}_X$, the moduli space of stable vector bundles on $X$ with rank $n$ and fixed determinant $\xi$, with $x$ and $\deg(\xi)$ coprime (assuming $g > 1$). Namely, the isomorphism class of $SU^{n,\xi}_X$ determines the isomorphism class of $X$ (note that this moduli space has a unique generator of polarization). It was proved by Mumford and Newstead [MN] for $n = 2$, and later extended to any rank by Narasimhan and Ramanan ([NR] and [NR2]). They consider the intermediate Jacobian associated to the third cohomology $H^3(SU^{n,\xi}_X)$. It has a canonical polarization defined by the positive generator of Pic($SU^{n,\xi}_X$). They show that this canonically polarized intermediate Jacobian is isomorphic (as a polarized variety) to the Jacobian $J(X)$ of the curve with the polarization given by the theta divisor, and then the result follows from the classical Torelli theorem.

In this paper we consider the same question for the moduli space of Higgs bundles, with fixed determinant of degree coprime to the rank, and traceless Higgs field.

Let $X$ be a connected smooth projective curve over $\mathbb{C}$ (we will assume that its genus $g > 1$). A Higgs bundle on $X$ is a pair $(E, \varphi)$, where $E$ is a vector bundle on $X$ and

$$\varphi : E \rightarrow E \otimes K_X$$

is a morphism, called the Higgs field. A subsheaf $F$ of $E$ is called $\varphi$-invariant if $\varphi$ maps $F$ to $F \otimes K_X \subset E \otimes K_X$. We say that a Higgs bundle is stable (respectively, semistable) if for all $\varphi$-invariant proper subsheaves $F$ of $E$,

$$\frac{\deg(F)}{\rk(F)} < \frac{\deg(E)}{\rk(E)} \quad \text{respectively, } \leq.$$

Denote by $\mathcal{M}^{n,d}_X$ the moduli space of semistable Higgs sheaves with $\rk(E) = n$ and $\deg(E) = d$. Note that the trace of $\varphi$ is a section of $K_X$.

We denote by $\mathcal{M}^{n,\xi}_X$ the moduli space of semistable Higgs bundles with $\tr(\varphi) = 0$, $\rk(E) = r$, and $\det(E) \cong \xi$, where $\xi$ is a fixed line bundle on $X$. Both $\mathcal{M}^{n,d}_X$ and

Date: 13 July 2001.

1991 Mathematics Subject Classification. 14D20, 14C34.
\(M^{n,\xi}\) are irreducible and if \(r\) and \(d\) (respectively, \(\deg(\xi)\)) are coprime, then \(M^{n,d}\) (respectively, \(M^{n,\xi}\)) is smooth.

The main result of this paper is the following theorem.

**Theorem 1.1.** Let \(n > 0\) and \(d\) be two coprime integers. Let \(X\) and \(X'\) be two smooth projective curves with \(g \geq 2\), and let \(\xi\) and \(\xi'\) be line bundles on \(X\) and \(X'\), with \(\deg(\xi) = \deg(\xi') = d\). If there is an isomorphism of algebraic varieties

\[M^X_{\xi} \cong M^{X'}_{\xi'},\]

then \(X\) and \(X'\) are isomorphic.

**Outline of the proof.** Recall that there is a surjective morphism, called the Hitchin map, from the moduli space of Higgs bundles to a vector space of dimension \((n^2 - 1)(g - 1)\), called the Hitchin space. The fiber over the origin is called the nilpotent cone. It has several irreducible components, and one of them is isomorphic to \(SU_n^{\xi}\). In fact, it is the only irreducible component of the nilpotent cone that does not admit a nontrivial \(\mathbb{C}^*\) action (cf. Proposition 3.1). Hence, if we were given \(M^X_{\xi}\) together with the Hitchin map, we would recover \(SU_X^{\xi}\), and using the Torelli theorem for vector bundles, we would also recover \(X\).

The problem, of course, is that we are given only the isomorphism class of \(M^X_{\xi}\), but not the Hitchin map.

We start with an algebraic variety \(Y\), isomorphic to \(M^X_{\xi}\). Look at the natural map

\[m : Y \longrightarrow \text{Spec } \Gamma(Y),\]

where \(\Gamma(Y)\) is the ring of global functions. Since the fibers of the Hitchin map are complete, it turns out that this map is isomorphic to the Hitchin map. More precisely, by Lemma 2.2, \(\text{Spec } \Gamma(Y) \cong \mathbb{A}^{(n^2 - 1)(g - 1)}\), and there is a commutative diagram like (5.2).

This gives the Hitchin map, but only up to automorphism (as an algebraic variety) of \(\mathbb{A}^{(n^2 - 1)(g - 1)}\). More precisely, we have recovered the Hitchin fibration, but not the Hitchin map. We have to find which point of \(\mathbb{A}^{(n^2 - 1)(g - 1)}\) corresponds to the origin of the Hitchin space.

Recall that the moduli space of Higgs bundles has a \(\mathbb{C}^*\) action given by sending \((E, \varphi)\) to \((E, t \varphi)\). This action descends to an action on the Hitchin space, and the origin has the property that it is the only fixed point of this descended action. In Proposition 5.1 we prove that this property characterizes the origin. More precisely, if \(g\) is a \(\mathbb{C}^*\) action on the Hitchin space, having exactly one fixed point, and admitting a lift to \(M^X_{\xi}\), then this fixed point is the origin. Hence, if \(g\) is such an action on \(\mathbb{A}^{(n^2 - 1)(g - 1)}\), its fixed point is precisely the point corresponding to the origin of the Hitchin space, the fiber of \(m\) over this point is isomorphic to the nilpotent cone, and by the previous observations, the theorem is proved.

Now we will give an outline of the proof of Proposition 5.1. Let \(s\) be a point in the Hitchin space such that the corresponding spectral curve \(X_s\) is smooth. Then we have a Kodaira-Spencer map \(u_0\) from the tangent space at \(s\) to the space of deformations of \(X_s\). The kernel of this map is described in Proposition 4.2. In the case of \(M^X_{\xi}\), this kernel has dimension one. The direction defined by this kernel is precisely the direction given by the standard \(\mathbb{C}^*\) action. The fiber over \(s\) is isomorphic to the Prym variety \(P_s\) (cf. (2.7)), hence we also have a Kodaira-Spencer map \(v_0\) from the tangent space at \(s\) to the space of deformations of \(P_s\). It can be shown that a nontrivial deformation of the spectral curve produces a nontrivial
deformation of $P_s$, and hence the kernel of the homomorphism $v_0$ coincides with the kernel of $u_0$.

If $g$ is a $\mathbb{C}^*$ action on $\mathbb{A}^{(n^2-1)(g-1)}$ that lifts to $\mathcal{M}_X^n$, then the tangent vector at $s$ defined by the action $g$ is in the kernel of the Kodaira-Spencer map $u_0$. This implies that the orbit $g(\mathbb{C}^*, s)$ is contained in the orbit of the standard action, and then the origin of the Hitchin space is in the closure of $g(\mathbb{C}^*, s)$. Now, a point in the closure of an orbit is a fixed point, hence if $g$ has exactly one fixed point, this fixed point has to be the origin. This proves Proposition 5.1.

2. Preliminaries

In this section we will recall some general facts about Higgs bundles (see, for instance, [HI] and [BNR]). Define the Hitchin space

$$\mathcal{H} = H^0(K_X) \oplus H^0(K_X^2) \oplus \cdots \oplus H^0(K_X^n).$$

Its dimension is $n^2(g-1) + 1$.

Let $S$ be the total space of the line bundle $K_X$, and let $p$ be the projection

$$p : S = \text{Spec}(\text{Sym}^n K_X) \longrightarrow X.$$

Given $(s) = (s_1, \ldots, s_n) \in \mathcal{H}$, we define the spectral curve $X_s$ in $S$ as the zero scheme of the following section of $p^n K_X$:

$$f = x^n + s_1 x^{n-1} + s_2 x^{n-2} + \cdots + s_n,$$

where $s_i = p^* s_i$, and $x \in H^0(S, p^* K_X)$ is the tautological section. In other words,

$$X_s = \text{Spec}(\text{Sym}^n (K_X^{-1})/I)$$

where $I$ is the ideal sheaf generated by the image of the homomorphism

$$K_X^{-n} \longrightarrow \text{Sym}^n (K_X^{-1})$$

$$\alpha \longmapsto \alpha \sum_{i=0}^n s_i$$

where we take $s_0 = 1$. Note that the projection $\pi : X_s \to X$, which is the restriction of $p$, has degree $n$. From (2.1) we obtain the following isomorphism

$$\pi_* \mathcal{O}_{X_s} = \mathcal{O}_X \oplus K_X^{-1} \oplus K_X^{-2} \oplus \cdots \oplus K_X^{-(n-1)}$$

There is a dense open set $U$ in $\mathcal{H}$ (respectively, $U_0$ in $\mathcal{H}_0$) such that the spectral curve $X_s$ is smooth for $s \in U$ (respectively, $U_0$). Let $X_s$ be such a curve, and consider the short exact sequence

$$0 \longrightarrow T_{X_s} \longrightarrow T_S|_{X_s} \longrightarrow N_{X_s/S} \longrightarrow 0.$$  

Since $S$ is the cotangent space of $C$, it is a holomorphic symplectic variety, and then $\det(T_S) \cong \mathcal{O}_S$. Hence, we have

$$N_{X_s/S} \cong K_{X_s}.$$  

On the other hand,

$$N_{X_s/S} \cong \mathcal{O}(X_s)|_{X_s} = p^* K_X^n|_{X_s} = \pi^* K_X^n,$$

and then the ramification line bundle of the projection $\pi : X_s \to X$ is

$$\mathcal{O}(R) = K_{X_s} \otimes \pi^* K_X^{-1} = \pi^* K_X^{n-1}.$$  

The section

$$\frac{\partial f}{\partial x} = nx^{n-1} + (n-1)s_1 x^{n-2} + \cdots + s_{n-1} \in H^0(S, p^* K_X^{-1}),$$

for $s \in U$. The spectral curve $X_s$ is smooth for $s \in U$, and the spectral curve $X_s$ is a smooth curve for $s \in U$. Now, a point in the closure of an orbit is a fixed point, hence if $g$ has exactly one fixed point, this fixed point has to be the origin. This proves Proposition 5.1.
when restricted to $X_s$, gives a section of $\mathcal{O}(R)$, and its scheme of zeroes is exactly the ramification divisor $R$.

Given a Higgs bundle $(E, \varphi)$, the formula
\[
\det(x \cdot \text{id} - \varphi) = x^n + s_1 x^{n-1} + s_2 x^{n-2} + \cdots + s_n
\]
defines sections $s_i \in H^0(K_X^i)$, and this defines the Hitchin map
\[
h : \mathcal{M}^{n,d} \to \mathcal{H}.
\]
The dimension of $\mathcal{M}^{n,d}$ is $2n^2(g-1) + 2$, and the fibers of $h$ are equidimensional projective schemes of dimension $n^2(g-1) + 1$. If $X_s$ is smooth, then the fiber $h^{-1}(s)$ is isomorphic to the Jacobian $J(X_s)$ of the spectral curve [1], [BNR].

We can restrict this map to the subscheme $\mathcal{M}^{n,\xi} \subset \mathcal{M}^{n,d}$ corresponding to fixed determinant and traceless $\varphi$. Since $s_1 = \text{tr}(\varphi)$, the image is actually in the traceless Hitchin space
\[
(2.6) \quad h_0 : \mathcal{M}^{n,\xi} \to \mathcal{H}_0 = \bigoplus_{i=2}^{n} H^0(K_X^i).
\]
We have
\[
\text{dim}(\mathcal{M}^{n,\xi}) = 2(n^2 - 1)(g-1),
\]
\[
\text{dim}(\mathcal{H}_0) = (n^2 - 1)(g-1),
\]
and the fibers of $h_0$ are equidimensional projective schemes of dimension $(n^2 - 1)(g-1)$. If the spectral curve $X_s$ is smooth, then the fiber $h_0^{-1}(s)$ is the Prym variety
\[
(2.7) \quad P_s = \{ L \in \text{Pic}(X_s) : \det(\pi_*L) \equiv \xi \},
\]
where $\pi$ is the obvious projection of $X_s$ to $X$.

**Lemma 2.1.** Let $s \in \mathcal{H}_0$ such that $X_s$ is smooth. Then the morphism
\[
\alpha : P_s \times J(X) \to J(X_s)
\]
\[(L_1, L_2) \mapsto L_1 \otimes \pi^*L_2
\]
is an unramified covering of degree $n^{2g}$.

**Proof.** First note that $\text{dim}(J(X) \times P_s) = \text{dim}(J(X_s))$. If $L_1 \otimes \pi^*L_2 \cong L'_1 \otimes \pi^*L'_2$, then
\[
(\pi_*L_1) \otimes L_2 \cong (\pi_*L'_1) \otimes L'_2,
\]
\[
\xi \otimes L_2^n \cong \xi \otimes L'_2^n,
\]
and hence $L'_2 \cong L_2 \otimes \zeta$, for some $\zeta$ in $J(X)[n]$ (the $n$-torsion subgroup of $J(X)$), and $L_1 \cong L'_1 \otimes \pi^*(\zeta)$. Hence, the fiber of $\alpha$ is isomorphic to $J(X)[n]$, and the lemma follows. \(\square\)

**Lemma 2.2.** The Hitchin map induces an isomorphism between the rings of global functions
\[
\Gamma(\mathcal{M}^{n,\xi}) \cong \Gamma(\mathcal{H}_0) \cong \mathbb{C}[y_1, y_2, \ldots, y_{(n^2-1)(g-1)}].
\]
The last isomorphism follows from $\mathcal{H}_0 \cong \mathbb{A}^{(n^2-1)(g-1)}$.

**Proof.** Since the Hitchin map is surjective, it gives an inclusion
\[
\Gamma(\mathcal{M}^{n,\xi}) \supset \Gamma(\mathcal{H}_0).
\]
Since the fibers of $h_0$ are projective, any function on $\mathcal{M}^{n,\xi}$ is constant on these fibers, hence it factors through the Hitchin space, and hence we have an equality. \(\square\)
3. The nilpotent cone

The fiber of the Hitchin map $h_0$ over the origin $s = 0$ is called the nilpotent cone. It is a reducible scheme of dimension $(n^2 - 1)(g - 1)$. Note that $(E, \varphi)$ is in the nilpotent cone if and only if $\varphi$ is a nilpotent endomorphism.

**Proposition 3.1.** The nilpotent cone has a unique irreducible component that does not admit a nontrivial $\mathbb{C}^*$ action, and this is isomorphic to $SU^n_{\xi}$, the moduli space of semistable vector bundles on $X$ of rank $n$ and fixed determinant $\xi$.

Before proving this proposition, we need the following lemma (recall that we are assuming that $r$ and $d$ are coprime, and hence semistable implies stable).

**Lemma 3.2** (Lemma 11.9 in [SII]). Let $(E, \varphi)$ be a Higgs bundle in the nilpotent cone, with $\varphi \neq 0$. Consider the standard $\mathbb{C}^*$ action sending $(E, \varphi)$ to $(E, t\varphi)$. Assume that $(E, \varphi)$ is a fixed point, i.e., for every $t$ there is an isomorphism with $(E, t\varphi)$. Then there is another Higgs bundle $(F, \psi)$ in the nilpotent cone, not isomorphic to $(E, \varphi)$, such that $\lim_{t \to \infty} (F, t\psi) = (E, \varphi)$.

**Proof.** This lemma is stated in [SII] with the extra assumption that $\deg(E) = 0$ (because Simpson is interested in representations of the fundamental group), but the proof actually works for Higgs bundles of any degree. For convenience of the reader, we will give the necessary details.

Since $(E, \varphi)$ is a fixed point, by [S, Lemma 4.1] we know that it is of the form

$$E = \bigoplus_{i=0}^{m} E_p$$

with $\varphi$ sending $E_i$ to $E_{i-1} \otimes K_X$. Since $\varphi \neq 0$, we have $m > 0$ and $E_0$ and $E_m$ are nontrivial. Since $r$ and $d$ are coprime, $(E, \varphi)$ is stable. Then, since $E_0$ is $\varphi$-invariant, the stability condition implies

$$\frac{\deg(E_0)}{\text{rk}(E_0)} < \frac{d}{r}$$

The subbundle $\bigoplus_{i=1}^{m} E_p$ is also $\varphi$-invariant, and the stability condition implies

$$\frac{d}{r} < \frac{\deg(E_m)}{\text{rk}(E_m)}.$$

Combining both inequalities, we obtain $\deg \text{Hom}(E_m, E_0) < 0$, and Riemann-Roch implies that $\text{Ext}^1(E_m, E_0)$ is nontrivial. Furthermore, if $A \subset E_m$ is the $\beta$-subbundle of $E_m$ (i.e., the first term in the Harder-Narasimhan filtration), then the slope of $A$ is bigger or equal than the slope of $E_m$, so we still have

$$\text{Ext}^1(E_m, E_0) \to \text{Ext}^1(A, E_0) \neq 0.$$

Let $\eta$ be a nonzero element of $\text{Ext}^1(E_m, E_0)$ that maps to a nonzero element in $\text{Ext}^1(A, E_0)$. For each $t \in \mathbb{C}^*$, let $M_t$ be the extension given by $t^m \eta$

$$0 \to E_0 \to M_t \to E_m \to 0.$$

The Higgs bundle $(F_t, \psi_t)$ is defined taking

$$F_t = M_t \oplus \bigoplus_{0<p<m} E_p$$
and the Higgs field $\psi$ is given by

$$
E_i \xrightarrow{\varphi} E_{i-1} \otimes K_X, \quad 1 < i < m \\
E_1 \xrightarrow{\varphi} E_0 \otimes K_X \rightarrow M_t \otimes K_X \\
M_t \rightarrow E_m \xrightarrow{\varphi} E_{m-1} \otimes K_X
$$

By construction, we see that $\psi_t$ is nilpotent. We define $(F, \psi) = (F_t, \psi_t)$. The rest of the proof is identical to the proof given in [SiII], so we only give a sketch. First one checks that $(F, t^{-1} \psi)$ is isomorphic to $(F_t, \psi_t)$, and that these Higgs bundles are stable, hence they are in the nilpotent cone. Furthermore,

$$
\lim_{t \to \infty} (F, t\psi) = (E, \varphi).
$$

Finally Simpson checks that $(F, \psi)$ is not isomorphic to $(E, \varphi)$ (here is where we use the fact that $\eta$ has nonzero image in $\text{Ext}^1(A, E_0)$).

Now we can prove the proposition.

**Proof of Proposition 3.4.** The map that sends a vector bundle $E$ to the Higgs bundle $(E, 0)$ defines an inclusion of $SU^{n, \xi}$ in the nilpotent cone. Since they have the same dimension, this gives one component of the nilpotent cone. It does not have a nontrivial $\mathbb{C}^*$ action, because if it had, then it would produce a nontrivial section of the tangent bundle of $SU^{n, \xi}$, but this is known to have no nontrivial sections ([NR] and [NR2]).

In the rest of the components we have the $\mathbb{C}^*$ action given by $(E, \varphi) \mapsto (E, t\varphi)$. To show that this action is nontrivial, we use the previous lemma: Let $(E, \varphi)$ be a fixed point (with $\varphi \neq 0$), and let $(F, \psi)$ be the Higgs bundle given by Lemma 3.2. Since the limit of the action moves $(F, \psi)$ to $(E, \varphi)$, they are in the same irreducible component of the nilpotent cone, and since both bundles are stable and nonisomorphic, they correspond to different points in the moduli space, and hence the $\mathbb{C}^*$ action is nontrivial in this component.

**4. Kodaira-Spencer map**

The projection $\pi : X_s \rightarrow X$ has degree $n$. The spectral curve construction gives a bijection between points in $\mathcal{H}$ and projective curves on $S$ such that the restriction of the projection has degree $n$.

Let $s \in \mathcal{H}$ be a point such that the corresponding spectral curve $X_s$ is smooth. There is a morphism from the tangent space $T_s\mathcal{H} \cong \mathcal{H}$ to the space $H^1(X_s, T_{X_s})$ of all infinitesimal deformations of $X_s$, the Kodaira-Spencer map $u$.

We can also define the Kodaira-Spencer map $u_0$ for the traceless Hitchin space $\mathcal{H}_0$ (cf. (2.6)). It is obtained by restricting $u$ to $T_s\mathcal{H}_0 \subset T_s\mathcal{H}$.

**Lemma 4.1.** If $X_s$ is smooth, then there are natural isomorphisms

$$
H^0(X_s, N_{X_s/S}) \cong H^0(X_s, \pi^* K_X^n) \cong T_s\mathcal{H}.
$$

**Proof.** Using the isomorphisms (2.4) and (2.2) and the projection formula, we have

$$
H^0(X_s, N_{X_s/S}) = H^0(X_s, \pi^* K_X^n) = H^0(X, K_X^n \otimes \pi_* \mathcal{O}_{X_s}) = H^0(X, \oplus_{i=1}^n K_X^n) = \mathcal{H} \cong T_s\mathcal{H}
$$

\[\square\]
The objective of this section is to calculate the kernels of $u$ and $u_0$. There are some elements in $H^0(X_s, N_{X_s}/S)$ that are clearly in the kernel. For instance, let $\lambda \in \mathbb{C} \cong H^0(X, \mathcal{O}_X)$, and denote a point in $X_s \subset S$ by $(\omega, x)$, where $\omega$ is a point in $X$ and $x$ is a coordinate in the fiber of $S$ over $\omega$. Then the deformation sending $(\omega, x)$ to $(\omega, e^{\lambda}x)$ clearly does not change the isomorphism class of $X_s$. In fact, this is the deformation produced by the standard $\mathbb{C}^*$ action, and it is clearly in the kernel of the Kodaira-Spencer map $u_0$.

Furthermore, for any $\alpha \in H^0(X, K_X)$, sending $(\omega, x)$ to $(\omega, x + \alpha(\omega))$ also preserves the isomorphism class of $X_s$. The deformations defined in this way do not preserve the condition $0 = \text{tr}(\varphi) (= s_1)$, and hence they are in the kernel of $u$, but not in the domain of $u_0$. The following proposition says that these two constructions describe the kernels.

**Proposition 4.2.** There is an exact sequence

$$0 \longrightarrow H^0(X, K_X \oplus \mathcal{O}_X) \longrightarrow T_s \mathcal{H} \overset{u}{\longrightarrow} H^1(X_s, T_{X_s}),$$

hence the kernel of the Kodaira-Spencer map $u$ has dimension $g + 1$. If we fix the determinant, we have an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow T_s \mathcal{H}_0 \overset{u_0}{\longrightarrow} H^1(X_s, T_{X_s}),$$

and hence the kernel of the restricted Kodaira-Spencer map $u_0$ has dimension $1$.

**Proof.** Consider the following diagram, constructed using (2.3) and (2.4):

\[
\begin{array}{cccccc}
0 & \longrightarrow & T_{X_s} & \longrightarrow & T_S|X_s & \longrightarrow & N_{X_s}/S & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & T_p|X_s \cong \pi^* K_X & \longrightarrow & \pi^* K_X^n \cong K_{X_s} & & \\
& & \otimes \frac{\partial f}{\partial x}|_{X_s} & & & & \\
\end{array}
\]

where $T_p$ denotes the relative tangent bundle for the projection $p$. Note that the diagram is well defined, since $\frac{\partial f}{\partial x}|_{X_s}$ is a section of $\pi^* K_X^{n-1} \cong \mathcal{O}(R)$ (cf. (2.5)). The diagram is commutative because the zero scheme of the two morphisms between the line bundles $T_p|X_s$ and $N_{X_s}/S$ are the same, namely the ramification divisor, hence the maps differ by a scalar, but this scalar is absorbed in the isomorphism between $K_{X_s}$ and $N_{X_s}/S$.

Since the tangent line bundle $T_X$ has negative degree, $H^0(X_s, p^* T_X|X_s) = 0$. Therefore, the previous diagram gives

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(T_S|X_s) & \overset{I}{\longrightarrow} & H^0(N_{X_s}/S) & \overset{\beta}{\longrightarrow} & H^1(T_{X_s}) \\
& & \cong & & \cong & & \\
0 & \longrightarrow & H^0(X_s, \pi^* K_X) & \overset{I'}{\longrightarrow} & H^0(X_s, \pi^* K_X^n) & \overset{\alpha}{\cong} & H^1(T_{X_s}) \\
& & \cong & & \cong & & \\
0 & \longrightarrow & H^0(X, K_X \oplus \mathcal{O}_X) & \overset{I''}{\longrightarrow} & T_s \mathcal{H} & \overset{u}{\longrightarrow} & H^1(T_{X_s}) \\
\end{array}
\]
where $\alpha$ and $\beta$ are the isomorphism of Lemma 4.1, the isomorphism $\gamma$ is given by

$$H^0(X_s, \pi^* K_X) = H^0(X, K_X \otimes \pi_* O_{X_s}) = H^0(X, \oplus_{i=1}^n K_X^i) = H^0(X, K_X \oplus O_X),$$

$I' = H^0(\partial f/\partial x \mid X_s)$, $I''$ is defined by composition, and $u$ is the Kodaira-Spencer map.

Now we will restrict the Kodaira-Spencer map $u$ to $T_s \mathcal{H}_0$. To do this, we need an explicit description of the morphism $I'$.

Using the isomorphism $H^0(X_s, \pi^* K_X) \cong H^0(X, \oplus_{i=1}^n K_X^i)$ (cf. proof of Lemma 4.1), an element of this group is written as $\tilde{a}_0 x^n + \tilde{a}_1 x^{n-1} + \cdots + \tilde{a}_n$ with $a_i \in H^0(X, K_X^i)$ and $\tilde{a}_i = \pi^* a_i$.

On the other hand, using the isomorphism $\gamma$, an element of this group will be written as $\tilde{b}_1 + \tilde{b}_0 x$ with $b_i \in H^0(X, K_X^i)$ and $\tilde{b}_i = \pi^* b_i$.

Since $I'$ comes from multiplication with $\partial f/\partial x \mid X_s$, a short calculation using $f|X_s = 0$ gives

$$I'(\tilde{b}_1 + \tilde{b}_0 x) = \sum_{i=1}^n ((n - i + 1)\tilde{s}_{i-1}\tilde{b}_1 - i\tilde{s}_i\tilde{b}_0)x^{n-i}$$

(4.1)

The subspace $\mathcal{H}_0 \subset \mathcal{H}$ is the zero locus of the trace map sending $(s_1, \ldots, s_n)$ to $s_1$. Then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & T_s \mathcal{H}_0 & \rightarrow & T_s \mathcal{H} & \rightarrow & H^0(X, K_X) & \rightarrow & 0 \\
\cong & & & \cong & & & & \\
0 & \rightarrow & H^0(\oplus_{i=2}^n K_X^i) & \rightarrow & H^0(\oplus_{i=1}^n K_X^i) & \rightarrow & H^0(X, K_X) & \rightarrow & 0
\end{array}
$$

where $p_1$ is projection to the first summand.

Now, if $s \in \mathcal{H}_0$, then $s_1 = 0$, and using the explicit formula (4.1) for $I'$, we obtain

$$(d(tr) \circ I'')(\tilde{b}_1 + \tilde{b}_0 x) = nb_1,$$

and hence the following diagram is commutative

$$
\begin{array}{cccccc}
0 & \rightarrow & H^0(O_X) & \rightarrow & T_s \mathcal{H}_0 & \rightarrow & H^1(X_s, T_{X_s}) & \rightarrow & 0 \\
\rightarrow & & \downarrow & & \uparrow & & \rightarrow & & \\
0 & \rightarrow & H^0(K_X \oplus O_X) & \rightarrow & T_s \mathcal{H} & \rightarrow & H^1(X_s, T_{X_s}) & \rightarrow & 0 \\
\rightarrow & & \downarrow q & & \uparrow u & & \rightarrow & & \\
0 & \rightarrow & H^0(K_X) & \rightarrow & 0 & \rightarrow & H^0(K_X) & \rightarrow & 0
\end{array}
$$
where \( q \) is projection to the first summand followed by multiplication by \( n \). The top row is the second exact sequence in the statement of the proposition.

5. Torelli theorem

Recall that there is a \( \mathbb{C}^* \) action on \( \mathcal{M}^{n, \xi} \), sending \((E, \varphi)\) to \((E, t\varphi)\). This action descends to give an action on \( \mathcal{H}_0 \), whose only fixed point is the origin \( s_i = 0 \).

**Proposition 5.1.** Let \( g : \mathbb{C}^* \times \mathcal{H}_0 \to \mathcal{H}_0 \) be an action, having exactly one fixed, and admitting a lift to \( \mathcal{M}^{n, \xi} \). Then this fixed point is the origin \( s_i = 0 \).

**Proof.** As we have seen, the standard action has this property. Now let \( s \in \mathcal{H}_0 \) be a point giving a smooth spectral curve \( X_s \). The tangent vector at \( s \) defined by the standard action is contained in the kernel of the Kodaira-Spencer map \( u_0 \), because the standard action does not change the isomorphism class of the spectral curve. Now we will show that the tangent vector defined by any action that lifts to \( \mathcal{M}^{n, \xi} \) is also in the kernel of the Kodaira-Spencer map.

Denote \( J = J(X) \) and \( J_s = J(X_s) \). In general, if \( M_1 \to M_2 \) is a covering map between differentiable manifolds, a holomorphic structure on \( M_2 \) induces a holomorphic structure on \( M_1 \). Hence, using the morphism \( \alpha \) of Lemma [2.1], a deformation of \( J_s \) gives a deformation of \( J \times P_s \). The map \( \epsilon \) between the deformation spaces is the composition

\[
\epsilon : H^1(J_s, T_{J_s}) \to H^1(J_s, \mathcal{O}_{J_s} \otimes \mathcal{O}_s) \to H^1(J \times P_s, T_{J \times P_s})
\]

where the inclusion is induced by the adjunction map \( \mathcal{O}_{J_s} \to \alpha_\ast \mathcal{O}_{J \times P_s} \) and the isomorphism is given by the projection formula and the isomorphism \( \alpha^\ast T_{J_s} \cong T_{J \times P_s} \).

We have several varieties associated to a point \( s \in \mathcal{H}_0 \), namely the spectral curve \( \pi : X_s \to X \), the Jacobian \( J_s \) and the Prym variety \( P_s \). Hence the \( g \) action produces deformations of all these objects

\[
\eta_1 \in H^1(X_s, T_{X_s}) \\
\eta_2 \in H^1(J_s, T_{J_s}) \\
\eta_3 \in H^1(P_s, T_{P_s})
\]

We will study the relationship between \( \eta_2 \) and \( \eta_3 \). Using the projection formula and the Leray spectral sequence for the projections \( q_1 \) and \( q_2 \) from \( J \times P_s \) to \( J \) and \( P_s \), we obtain that the space of deformations of \( J \times P_s \) is

\[
H^1(J \times P_s, T_{J \times P_s}) = H^1(J \times P_s, q_1^\ast T_J) \oplus H^1(J \times P_s, q_2^\ast T_P) =
\]

\[
= H^1(T_J) \oplus \left( H^0(T_J) \otimes H^1(\mathcal{O}_P) \right) \oplus \left( H^0(T_P) \otimes H^1(\mathcal{O}_J) \right) \oplus H^1(T_{P_s})
\]

The first and last terms correspond to deformations of \( J \) and \( P_s \). To understand the second term, view \( J \times P_s \to P_s \) as a trivial fiber bundle. The deformations of this as a fiber bundle (i.e., keeping the fiber and the base fixed) are parametrized by

\[
H^1(P, \mathcal{O}_P \otimes \text{Lie}(\text{Aut}(J))) = H^1(P, \mathcal{O}_P \otimes H^0(J, T_J)) = H^1(P, \mathcal{O}_P) \otimes H^0(J, T_J),
\]

i.e., the second term. Analogously, the third term corresponds to deformations of \( J \times P_s \to J \) as a fiber bundle.

The class \( \eta_3 \) lies in the fourth summand \( H^1(P_s, T_{P_s}) \). Indeed, if we move \( s \) to a nearby point \( s' \), the fiber \( J \times P_s \) is deformed to \( J \times P_{s'} \), hence the components of \( \eta_3 \) in the first three summands of (5.1) have to be zero. Hence \( \epsilon(\eta_2) \) lies in \( H^1(P_s, T_{P_s}) \), and is equal to \( \eta_3 \).

\[
\epsilon(\eta_2) = \eta_3 \in H^1(P_s, T_{P_s}) \subset H^1(J \times P_s, T_{J \times P_s})
\]
A deformation of a curve produces a deformation of its Jacobian, hence there is a natural map

\[ H^1(X_s, T_{X_s}) \rightarrow H^1(J_s, T_{J_s}), \]

and the infinitesimal version of the classical Torelli theorem says that this map is injective. Clearly, in our situation \( \eta_2 \) is the image of \( \eta_1 \) under this map. Combining the injectivity of this map with the injectivity of \( \epsilon \), we obtain that if \( \eta_3 = 0 \), then \( \eta_1 = 0 \).

Since the action \( g \) lifts, the fiber \( P_s \) of the Hitchin map \( h_0 \) at \( s \) is isomorphic to the fiber \( P_{g(t, s)} \) at \( g(t, s) \) for all \( t \in \mathbb{C}^* \). This implies that \( \eta_3 = 0 \), hence \( \eta_1 = 0 \).

This means that the tangent vector at \( s \) defined by the action \( g \) is in the kernel of the Kodaira-Spencer map \( u_0 \), and since by Proposition 4.2 this has dimension 1, we obtain that the orbit \( g(\mathbb{C}^*, s) \) of \( g \) through \( s \) is included in the orbit of the standard action through \( s \). In particular, the origin is a limiting point of the orbit \( g(\mathbb{C}^*, s) \), i.e., it is in the closure of the orbit, but not in the orbit (note that the origin is not in the orbit, because the fiber over the origin is not isomorphic to the fiber over \( s \)). The limiting points of an orbit are fixed points of the action. Then the origin is a fixed point of the action, and by hypothesis is the only fixed point.

Finally we prove the Torelli theorem for Higgs bundles.

**Proof of Theorem 1.1.** Let \( Y \) be an algebraic variety isomorphic to the moduli space \( \mathcal{M}^{n,\xi}_X \). By Lemma 2.4, choosing a set of generators of the ring of global functions on \( Y \) we obtain an isomorphism \( \Gamma(Y) \cong \mathbb{C}[y_1, y_2, \ldots, y_{(n^2-1)(g-1)}] \), and then the natural morphism \( Y \rightarrow \text{Spec} \Gamma(Y) \) gives a morphism

\[ m : Y \rightarrow \mathbb{A}^{(n^2-1)(g-1)}. \]

Note that \( m \) depends on the set of generators chosen: a different choice will give a morphism that differs by an automorphism (as an algebraic variety) of \( \mathbb{A}^{(n^2-1)(g-1)} \). Up to isomorphism, this is the Hitchin map. More precisely, if \( \alpha : Y \rightarrow \mathcal{M}^{n,\xi}_X \) is an isomorphism, then there is an isomorphism (as algebraic varieties) \( \beta \) such that the following diagram commutes

\[
\begin{array}{ccc}
Y & \xrightarrow{\beta} & \mathcal{M}^{n,\xi}_X \\
\downarrow m & & \downarrow h_0 \\
\mathbb{A}^{(n^2-1)(g-1)} & \underset{\alpha}{\cong} & \mathcal{H}_0
\end{array}
\]

Let \( g : \mathbb{C}^* \times \mathbb{A}^{(n^2-1)(g-1)} \rightarrow \mathbb{A}^{(n^2-1)(g-1)} \) be a \( \mathbb{C}^* \) action with exactly one fixed point \( y \), and such that it admits a lift to \( Y \).

We know that such an action exists (using the standard \( \mathbb{C}^* \) action on \( \mathcal{H}_0 \) and the isomorphism \( \beta \)), and by Proposition 5.1, \( \beta(y) \) is the origin \( s_i = 0 \) of \( \mathcal{H}_0 \). Indeed, if \( \beta(y) \) were a different point, using \( g \) and \( \beta \) we would have an action contradicting Proposition 5.1.

Then the fiber \( m^{-1}(y) \) over \( y \) is isomorphic to the nilpotent cone. This has several irreducible components, but by Proposition 5.1, only one of them (call it \( Z \)) does not admit nontrivial \( \mathbb{C}^* \) actions, and furthermore \( Z \) is isomorphic to \( SU^{n,\xi}_X \), the moduli space of semistable vector bundles on \( X \) of rank \( n \) and fixed determinant \( \xi \).

Now, if \( \mathcal{M}^{n,\xi}_X \) is isomorphic to \( \mathcal{M}^{n,\xi'}_X \), then \( Z \) is isomorphic to \( Z' \), hence \( SU^{n,\xi}_X \) is isomorphic to \( SU^{n,\xi'}_X \), and by the Torelli theorem for the moduli space of vector bundles, \( X \) is isomorphic to \( X' \).
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