EQUIVALENCE OF KRYLOW SUBSPACE METHODS
FOR SKEW-SYMMETRIC LINEAR SYSTEMS

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Abstract. In recent years two Krylov subspace methods have been proposed for solving skew symmetric linear systems, one based on the minimum residual condition, the other on the Galerkin condition. We give new, algorithm-independent proofs that in exact arithmetic the iterates for these methods are identical to the iterates for the conjugate gradient method applied to the normal equations and the classic Craig’s method, respectively, both of which select iterates from a Krylov subspace of lower dimension. More generally, we show that projecting an approximate solution from the original subspace to the lower-dimensional one cannot increase the norm of the error or residual.

Key words. Krylov methods, skew-symmetric systems

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1. Introduction. Consider the system of linear equations

\[ Ax = b \]

where the \( n \times n \) coefficient matrix \( A \) is real, skew symmetric (i.e., \( A^t = -A \)), and nonsingular (so that \( n \) is even). Krylov subspace methods for solving \( Ax = b \) compute a sequence \( \{x_m\} \) of approximate solutions where

\[ x_m \in \text{span}\{b, Ab, \ldots, A^{m-1}b\} \equiv \mathcal{K}_m(A, b). \]

The iterate \( x_m \) is often the unique vector that satisfies either the Galerkin condition

\[ p^t(b - Ax_m^G) = 0, \quad \text{for any } p \in \mathcal{K}_m(A, b), \]

or the minimum residual condition

\[ x_m^M = \arg\min_{z \in \mathcal{K}_m(A, b)} \|b - Az\|, \]

where \( \| \cdot \| \) denotes the Euclidean norm. The latter is easily seen to be equivalent to

\[ (Ap)^t(b - Ax_m^M) = 0, \quad \text{for any } p \in \mathcal{K}_m(A, b). \]

A classic approach to solving \( Ax = b \) is the conjugate gradient method (itself a Krylov subspace method based on the Galerkin condition) applied to the normal equations, either \( AA^ty = b \) or \( A^tAx = A^tb \).

CGNE [5, 2, p. 105] (also known as Craig’s method [1]; see Figure 1) uses CG to solve \( AA^ty = b \) and sets \( x = A^ty \). Thus the iterate \( x_q^E \) is the unique vector satisfying

\[ x_q^E = A^ty_q^E \in A^t\mathcal{K}_q(AA^t, b) = \mathcal{K}_q(A^tA, A^tb) \]

and

\[ p^t(b - Ax_q^E) = p^t(b - AA^ty_q^E) = 0, \quad \text{for any } p \in \mathcal{K}_q(AA^t, b). \]

For simplicity we take \( x_0 = 0 \).

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1
\[ r_0 = b; \quad p_0 = A^t r_0 \]

**for** \( q = 1 \) **step 1** **until** convergence

\[
\alpha_q = \frac{\|y_{q-1}\|^2/\|y_{q-1}\|^2}{\|A^t y_{q-1}\|^2/\|A^t y_{q-1}\|^2} \quad (E) \\
\beta_q = \frac{\|y_{q-1}\|^2/\|y_{q-1}\|^2}{\|A^t y_{q-1}\|^2/\|A^t y_{q-1}\|^2} \quad (R) \\
x_q = x_{q-1} + \alpha_q p_q - 1 \\
r_q = r_{q-1} - \alpha_q A p_q - 1 \\
p_q = A^t x_q + \beta_q p_q - 1
\]

**for** \( q = 1 \) **step 1** **until** convergence

\[
\alpha_q = \frac{\|y_{q-1}\|^2/\|y_{q-1}\|^2}{\|A^t y_{q-1}\|^2/\|A^t y_{q-1}\|^2} \quad (E) \\
\beta_q = \frac{\|y_{q-1}\|^2/\|y_{q-1}\|^2}{\|A^t y_{q-1}\|^2/\|A^t y_{q-1}\|^2} \quad (R) \\
x_q = x_{q-1} + \alpha_q p_q - 1 \\
r_q = r_{q-1} - \alpha_q A p_q - 1 \\
p_q = -A r_q + \beta_q p_q - 1
\]

**FIG. 1.** CGNE \((E)\) and CGNR \((R)\) for general \((\text{left})\) and skew symmetric \((\text{right})\) systems.

Since \( A \) is skew symmetric, this can be written as \( x^E_q \in K_q(A^2, Ab) \) and

\[(4) \quad p^t(b - Ax^E_q) = 0, \quad \text{for any } p \in K_q(A^2, b).\]

Moreover, it follows that [5,2] p. 106]

\[ x^E_q = \arg \min_{z \in K_q(A^2, Ab)} \|z - x\|. \]

CGNR [6,2] p. 105] (also known as CGLS [10]; see Figure 1) uses CG to solve \( A^t Ax = A^t b \). Thus the iterate \( x^R_q \) is the unique vector satisfying \( x^R_q \in K_q(A^t A, A^t b) \) and

\[ (Ap)^t(b - Ax^R_q) = p^t(Ab - A^t Ax^R_q) = 0, \quad \text{for any } p \in K_q(A^t A, A^t b). \]

Since \( A \) is skew symmetric, this can be written as \( x^R_q \in K_q(A^2, Ab) \) and

\[(5) \quad (Ap)^t(b - Ax^R_q) = 0, \quad \text{for any } p \in K_q(A^2, Ab).\]

Moreover, it follows that [6,2] pp. 105–6]

\[ x^R_q = \arg \min_{z \in K_q(A^2, Ab)} \|b - Az\|. \]

CGNE and CGNR are often disparaged since they square the condition number (which may slow convergence) and may be more susceptible to round-off error (which is why the algorithms in Figure 1 avoid multiplication by \( AA^t \) and \( A^t A \), respectively).

Thus in recent years several authors have derived Krylov subspace methods that solve \( A^t Ax = A^t b \) directly. Gu and Qian [4] and Greif and Varah [3] impose the Galerkin condition [2] on the subspace \( K_m(A, b) \); while Jiang [9], Idema and Vuik [5] and Greif and Varah [3] impose the minimum residual condition [3]. Greif and Varah [3] show that the odd iterates \( x^G_{2q+1} \) do not exist, that their algorithm for the even iterates \( x^G_{2q} \) is equivalent to CGNE, and that \( x^M_{2q+1} = x^M_{2q} \).

In this paper we give new, algorithm-independent proofs that \( x^G_{2q} = x^E_q \) and that \( x^M_{2q+1} = x^M_{2q} = x^R_q \). More generally, we show that any approximate solution \( z \) that belongs to \( K_m(A, b) \) but not to \( K_{m/2}(A^2, Ab) \) has a larger error \( \|z - x\| \) and residual \( \|b - Az\| \) than its projection onto the lower-dimensional subspace. Thus there does not seem to be any advantage to seeking an approximate solution in \( K_m(A, b) \).

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2 Greenbaum [2] p. 106] rebuts this view.

3 Gu and Qian [4] claim incorrectly that they are imposing the minimum residual condition.

4 That \( x^G_{2q} = x^R_q \) also follows from the observation (see [5] [2.4]) that the Huang, Wathen, and Li [2] algorithm, which computes only the even iterates \( x^M_{2q} \) is equivalent to CGNR.
2. Main results. We begin with a simple consequence of skew symmetry.

**Lemma 1.** If $A$ is skew symmetric, the subspaces $K_s(A^2, Ab)$ and $K_t(A^2, b)$ are orthogonal and the solution $x$ of $Ax = b$ is orthogonal to $K_s(A^2, b)$, for any $s, t \geq 0$.

**Proof.** Without loss of generality it suffices to show that both $(A^2)^k Ab$ and $x$ are orthogonal to $(A^2)^s b$ for any $0 \leq k < s$ and $0 \leq \ell < t$. But

\[ ((A^2)^k Ab)^t ((A^2)^t b) = (A^{2k+1} b)^t (A^{2t} b) = (-1)^{k+t+1} (A^{k+t} b)^t A (A^{k+t} b) = 0 \]

and

\[ x^t ((A^2)^t b) = x^t (A^{2t} Ax) = (-1)^t (A^t x)^t A (A^t x) = 0 \]

since $z^t A z = 0$ for any $z$.

By grouping even and odd powers of $A$, any $p \in K_m(A, b)$ can be written as $p = p_e + p_o$ for some $p_e \in K_{qe}(A^2, b)$ and $p_o \in K_{qo}(A^2, Ab)$, where $q_e = \lfloor m/2 \rfloor$ and $q_o = \lceil m/2 \rceil$. By Lemma 1, we have that $p_e$ is orthogonal to $p_o$.

**Theorem 2.** If $A$ is skew symmetric, the Galerkin iterates \( \{x^G_m\} \) and the CGNE iterates \( \{x^E_m\} \) satisfy \( x^G_2 = x^E_2 \); and the minimum residual iterates \( \{x^M_m\} \) and the CGNR iterates \( \{x^R_m\} \) satisfy \( x^R_{2q+1} = x^M_2 = x^R_q \).

**Proof.** \( x^G_2 = x^E_2 \): Since \( x^E_2 \in K_q(A^2, Ab) \subseteq K_{2q}(A, b) \), by the Galerkin condition it suffices to prove that \( x^E_2 \) satisfies

\[ p^t (b - Ax^E_2) = 0, \quad \text{for any} \ p \in K_{2q}(A, b). \]

Any $p \in K_{2q}(A, b)$ can be written as $p = p_e + p_o$ as above. Since $p_e \in K_q(A^2, b)$,

\[ p^t (b - Ax^E_2) = p^t_e (b - Ax^E_2) + p^t_o (b - Ax^E_2) = p^t_o (b - Ax^E_2) \]

by (1). But since $p_o \in K_q(A^2, Ab)$ and

\[ b - Ax^E_2 \in b + AK_q(A^2, Ab) \subseteq K_{q+1}(A^2, b), \]

we have that $p_o$ is orthogonal to $b - Ax^E_2$ by Lemma 1 and so $p^t_o (b - Ax^E_2) = 0$.

\[ x^M_{2q+1} = x^M_2 = x^E_2. \quad \text{Note that} \ K_q(A^2, Ab) \subseteq K_{2q}(A, b) \subseteq K_{2q+1}(A, b). \quad \text{Thus} \ x^R_2 \in K_{2q+1}(A, b) \text{ and} \ x^R_2 \in K_{2q}(A, b);\ \text{and by the minimum residual condition it suffices to prove that} \ x^E_2 \]

satisfies

\[ (Ap)^t (b - Ax^R_2) = 0, \quad \text{for any} \ p \in K_{2q+1}(A, b), \]

for then

\[ (Ap)^t (b - Ax^R_2) = 0, \quad \text{for any} \ p \in K_{2q}(A, b) \]

as well. Any $p \in K_{2q+1}(A, b)$ can be written as $p = p_e + p_o$ as above. Since $p_o \in K_q(A^2, Ab)$,

\[ (Ap)^t (b - Ax^R_2) = (Ap_e)^t (b - Ax^R_2) + (Ap_o)^t (b - Ax^R_2) = -p^t_e (A(b - Ax^R_2)) \]

by (3). But since $p_e \in K_{q+1}(A^2, b)$ and

\[ A(b - Ax^R_2) \in Ab + A^2K_q(A^2, Ab) \subseteq K_{q+1}(A^2, Ab), \]

for any $s, t \geq 0$. \hfill $\square$
we have that $p_e$ is orthogonal to $A(b - Ax^R)$ and so $(Ap)^t(b - Ax^R) = 0. \square$

Finally we show that the extra dimensions in $K_m(A, b)$ versus $K_{\lfloor m/2 \rfloor}(A^2, Ab)$ can not decrease the norm of the error or the residual.

**Theorem 3.** Let $z \in K_m(A, b)$ and write $z = z_e + z_o$ for some $z_e \in K_qe(A^2, b)$ and $z_o \in K_qo(A^2, Ab)$, where $q_e = \lceil m/2 \rceil$ and $q_o = \lfloor m/2 \rfloor$. If $A$ is skew symmetric, the solution $x$ of $Ax = b$ satisfies

$$\|z - x\|^2 = \|z_o - x\|^2 + \|z_e\|^2 \quad \text{and} \quad \|b - Az\|^2 = \|b - Az_o\|^2 + \|Az_e\|^2.$$ 

**Proof.** Since $z_o \in K_qo(A^2, Ab)$, we have $z_o$ and $x$ orthogonal to $z_e \in K_qe(A^2, b)$ by Lemma 1. Similarly, since $b - Az_o \in b + AK_qo(A^2, Ab) \subseteq K_{q_o+1}(A^2, b)$ and $Az_e \in AK_qe(A^2, b) = K_qe(A^2, Ab)$, we have $b - Az_o$ orthogonal to $Az_e$. Now apply the Pythagorean Theorem. \square

3. Conclusions. **Theorem 3** shows that there is no advantage to using all of $K_m(A, b)$, and **Theorem 2** shows that CGNE and CGNR compute the Galerkin and minimum residual iterates, at least in exact arithmetic. Thus a Krylov subspace method based on $K_m(A, b)$ would have to be at least as efficient and/or accurate to warrant consideration.

Normally a Krylov subspace method is applied to a preconditioned system

$$\tilde{A}\tilde{x} \equiv (M_L^{-1}AM_R^{-1})(M_Rx) = (M_L^{-1}b) \equiv \tilde{b}. \quad (6)$$

Greif and Varah [3] derive a preconditioner (i.e., an $M_L$ and an $M_R$) for which $\tilde{A}$ is skew symmetric, but many preconditioners do not have this property and CGNE and CGNR applied to $\tilde{A}$ do not require it. Thus a preconditioner for $A$ that does preserve skew symmetry in $\tilde{A}$ would have to be at least as efficient and/or accurate as the best general preconditioner used with CGNE or CGNR / LSQR to warrant consideration.

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**References.**

[1] J. Craig, *The n-step iteration procedure*, J. Math. and Phys., 34 (1955), pp. 64–73.
[2] A. Greenbaum, *Iterative Methods for Solving Linear Systems*, no. 17 in Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1997.
[3] C. Greif and J. M. Varah, *Iterative solution of skew-symmetric linear systems*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 584–601.
[4] C. Gu and H. Qian, *Skew-symmetric methods for nonsymmetric linear systems with multiple right-hand sides*, J. Comput. Appl. Math., 223 (2009), pp. 567–577.
[5] M. R. Hestenes, *The conjugate gradient method for solving linear systems*, in Numerical Analysis, vol. VI of Proceedings of the Symposium on Applied Mathematics, Providence, RI, 1956, American Mathematical Society, pp. 83–102. The Symposium was held at Santa Monica City College, August 26–28, 1953.

Paige and Saunders’ LSQR is a more stable equivalent to CGNR if round-off error is an issue [10].
[6] M. R. HESTENES and E. STIEFEL, *Methods of conjugate gradients for solving linear systems*, J. Res. Nat. Bur. Standards, 49 (1952), pp. 409–436.

[7] Y. HUANG, A. J. WATHEN, and L. LI, *An iterative method for skew-symmetric systems*, Information, 2 (1999), pp. 147–153.

[8] R. IDEMA and C. VUIK, *A minimum residual method for shifted skew-symmetric systems*, Report 07-09, Department of Applied Mathematical Analysis, Delft University of Technology, Delft, The Netherlands, 2007.

[9] E. JIANG, *Algorithm for solving shifted skew-symmetric linear system*, Front. Math. China, 2 (2007), pp. 227–242.

[10] C. C. PAIGE and M. A. SAUNDERS, *LSQR: An algorithm for sparse linear equations and sparse least squares*, ACM Trans. Math. Software, 8 (1982), pp. 43–71.