On linear coupling of acoustic and cyclotron waves in plasma flows

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It is found that in magnetized electrostatic plasma flows the velocity shear couples ion-acoustic waves with ion-cyclotron waves and leads, under favorable conditions, to their efficient reciprocal transformations. It is shown that in a two-dimensional setup this coupling has a remarkable feature: it is governed by equations that are exactly similar to the ones describing coupling of sound waves with internal gravity waves [Rogava & Mahajan: Phys. Rev. E 55, 1185 (1997)] in neutral fluid flows. Using another noteworthy quantum mechanical analogy we calculate transformation coefficients and give fully analytic, quantitative description of the coupling efficiency for flows with low shearing rates.

II. MAIN CONSIDERATION

Let us consider the simplest possible setup: the equilibrium magnetic field \( B_0 \) is homogeneous and directed along the \( X \) axis \( B_0 \equiv (B_0, 0, 0) \); the mean velocity \( u_0 \equiv (AY, 0, 0) \) is also directed along the \( X \) axis and has a linear shear along the \( Y \) axis. The equilibrium densities of both electrons \( N_e \) and ions \( N_i \) are homogeneous. The temperature of electrons \( T_e \) is constant, while ions are supposed to be cold \( T_i = 0 \).

It is well known that ISW and ICW with phase velocities considerably less than Alfvén speed \( \omega/k_x \ll V_A \equiv B_0/\sqrt{4\pi\epsilon_0 M N_i} \) (\( M \) is the ion mass) do not perturb the background magnetic field, and a good approximation for the electric field \[ E \] is: \( E = -\nabla \phi \).

Another common assumption for low-frequency electrostatic waves is that the electrons “thermalize along the field lines” and the electron number density can be described by the Boltzmann distribution \[ N_e = N_0 \exp(e\phi/T_e) \approx N_0 [1 + e\phi/T_e] \];

(1)
while dynamics of ions is governed by fluid equations of continuity and motion:

\[ \partial_t N_i + \nabla \cdot (N_i V_i) = 0, \quad (2) \]

\[ [\partial_t + (V_i \cdot \nabla)] V_i = \frac{e}{M} \left( -\nabla \varphi + \frac{1}{c} (V_i \times B) \right). \quad (3) \]

The basic system of linearized equations for ions, describing the evolution of small-scale, 3D perturbations in this flow, takes the form:

\[ D_t n_i + N_0 (\partial_x u_x + \partial_y u_y + \partial_z u_z) = 0, \quad (4) \]

\[ D_t u_x + A v_y = -(e/M) \partial_x \varphi, \quad (5a) \]

\[ D_t u_y = -(e/M) \partial_y \varphi + \Omega_c u_z, \quad (5b) \]

\[ D_t u_z = -(e/M) \partial_z \varphi - \Omega_c v_y, \quad (5c) \]

while the Poisson equation relates the perturbation of the electric potential \( \varphi \) with the number density perturbations for electrons \( n_e \) and ions \( n_i \), respectively:

\[ [\partial_x^2 + \partial_y^2 + \partial_z^2] \varphi = 4\pi e(n_e - n_i). \quad (6) \]

Here \( \Omega_c \equiv eB_0/Mc \) is the ion-cyclotron frequency and \( D_t \equiv \partial_t + A v_y \partial_x \) is the convective derivative operator.

Employing for all perturbational variables appearing in the above equations the ansatz \( F(x, y, z; t) = F(t) \exp[i(k_x x + k_y y + k_z z)] \) with \( k_y(t) \equiv k_y(0) - At k_x \). This ansatz guarantees that \( D_t F = \exp[i(k_x x + k_y(0)t + k_z z)]\partial_t \Phi \) and we can reduce the initial set of partial differential equations to the set of ordinary differential equations for the amplitudes \( F \) (see for details, e.g., [1]). It is convenient to write these equations in dimensionless notation. We define \( \omega_c \equiv \Omega_c/C_s k_x \) as the normalized ion-cyclotron frequency and \( \xi \equiv \lambda_D k_x \), using for normalization conventional definitions of the ion-sound speed \( C_s \equiv (T_e/M)^{1/2} \) and electron Debye length \( \lambda_D \equiv (T_e/eN_0 e^2)^{1/2} \). Other notation are: \( R \equiv A/C_s k_x \), \( \tau \equiv C_s k_x \), \( \beta_0 \equiv k_y/k_x \), \( \beta(\tau) \equiv \beta_0 - R \tau \), \( \gamma_0 \equiv k_z/k_x \), \( \bar{v}_{x,y,z} \equiv \bar{u}_{x,y,z}/C_s \), \( D \equiv i(n_i/N_0) \), \( \Phi \equiv i e \varphi/MC_s^2 \). In the language of these terms the set of first order, ordinary differential equations, derived from (4) and (5) takes the following form:

\[ D^{(1)} = v_x + \beta(\tau) v_y + \gamma v_z, \quad (7) \]

\[ v_x^{(1)} = -\Phi - R v_y, \quad (8a) \]

\[ v_y^{(1)} = -\beta(\tau) \Phi + \omega_c v_z, \quad (8b) \]

\[ v_z^{(1)} = -\gamma \Phi - \omega_c v_y. \quad (8c) \]

Additionally we have the algebraic relation between \( D \) and \( \Phi \), which follows from the Poisson equation (6):

\[ D = [1 + \xi^2(1 + \beta^2(\tau) + \gamma^2)] F, \quad (9) \]

Note that \( \xi \omega_c = \Omega_c/\Omega_p = V_A/c \ [\Omega_p = (4\pi N_0 e^2/M)^{1/2} \) is the ion plasma frequency]. When the Debye length is sufficiently small (\( \xi < 1 \)) the oscillations can be considered quasi-neutral \( (D=F) \). In the forthcoming consideration we shall employ this approximation. However, we should bear in mind that due to the “k(t)-drift”, induced by the existence of the non-uniform motion, \( \beta^2(\tau) \) sooner or later becomes large enough and violates the quasineutrality condition.

As regards \( \omega_c \), its value may be as less as greater than unity, depending on the relative smallness of \( \xi \) and \( \gamma \). The simplest sort of waves, which may be considered in this system [19] are ones propagating in the XOY plane \( (\gamma = 0) \). In this case if we introduce an auxiliary notation \( Y \equiv -\omega_c D - \beta(\tau) v_z \), the system (7-8) reduces to the following pair of coupled second-order differential equations:

\[ Y^{(2)} + \omega_2^2 Y + C_{12}(\tau) v_z = 0, \quad (10a) \]

\[ v_z^{(2)} + \omega_2^2(\tau) v_z + C_{21}(\tau) Y = 0, \quad (10b) \]

describing coupled oscillations with two degrees of freedom, with: \( C_{12}(\tau) = C_{21}(\tau) \equiv \beta(\tau) \) as the coupling coefficients and with \( \omega_1 \equiv 1 \) and \( \omega_2(\tau) \equiv \omega_2^2 + \beta^2(\tau) \) as the two respective eigenfrequencies. The presence of shear in the flow \( (R \neq 0) \) makes coefficients variable and opens the door for mutual ICW–ISW transformations.

In the “shearless” \( (R = 0) \) limit (10) describe two independent oscillations with frequencies:

\[ \Omega_{12}^2 = \frac{1}{2} \left[ \omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4C_{12}^2} \right] = \frac{1}{2} \left[ \omega_c^2 + 1 + \beta^2 \pm \sqrt{(\beta^2 + \omega_c^2 + 1)^2 - 4\omega_c^2} \right], \quad (11) \]

that can be readily identified as ISW and ICW frequencies respectively. Corresponding eigenfunctions (sometimes called normal variables) are:

\[ \Psi_1 = \frac{(\Omega_1^2 - \omega_2^2) Y + C_{12} v_z}{\sqrt{(\Omega_1^2 - \omega_2^2)^2 + C_{12}^2}}, \quad (12a) \]

\[ \Psi_2 = \frac{(\Omega_1^2 - \omega_2^2) v_z - C_{12} Y}{\sqrt{(\Omega_1^2 - \omega_2^2)^2 + C_{12}^2}}. \quad (12b) \]
A. Transformation coefficient

The coupled oscillator systems similar to (10) with slowly varying coefficients are well known in different branches of physics. Mathematical methods for their analysis were first developed for quantum mechanical problems: non-elastic atomic collisions [20] and non-adiabatic transitions in two level quantum systems [21]. Later, similar asymptotic methods were successfully applied to various other problems [22].

In [18] these efficient mathematical tools were used, for the first time, for the study of the velocity shear induced coupling and transformation of MHD waves. The problem which we are studying now, also allows thorough probing by means of this asymptotic method. We consider (most interesting for practical applications) case $R \ll 1$, when coefficient in (10) are slowly varying functions of $\tau$ and, therefore, WKB approximation is valid everywhere except nearby the turning points $(\Omega_i(\tau_i) = 0)$ and resonant points $(\Omega_i(\tau_r) = \Omega_2(\tau_r))$. Using (11) one can check that the condition

$$\Omega_i^{(1)} \ll \Omega_i^2, \quad (13)$$

is satisfied for both wave modes at any moment of time, or equivalently, none of the turning points are located near the real $\tau$-axis. From physical point of view this means, that there are no (over) reflection phenomena [18] and only the resonant coupling between different waves modes with the same sign of phase velocity can occur.

From (11) we also learn that there are two pairs of complex conjugated resonant points of the first order$^1$:

$$\beta(\tau^+) = \pm i(\omega_c - 1), \quad \beta(\tau^+) = \pm i(\omega_c + 1). \quad (14)$$

Therefore all the resonant points are located on the axis $\Re[\beta(\tau)] = 0$ in the complex $\tau$-plane and consequently, the resonant coupling can take place only in a vicinity of the point $\tau_r$ where $\beta(\tau_r) = 0$. Generally, the time scale of resonant coupling $\Delta \tau$ is of the order of $\Delta \tau \sim R^{-1/2}$ [18] and the evolution of the waves is adiabatic when

$$|\beta(\tau)| \gg R^{1/2}. \quad (15)$$

If this condition is met the temporal evolution of the waves is described by the standard WKB solutions:

$$\Psi^\pm_i = \frac{D_i^\pm}{\sqrt{\Omega_i(\tau)}} e^{\pm i \int \Omega_i(\tau) d\tau}, \quad (16)$$

where $D_i^\pm$ are WKB amplitudes of the wave modes with positive and negative phase velocity along the $x$-axis, respectively. All the physical quantities can be easily found by combining (12). One can check that the energies of the involved wave modes satisfy the standard adiabatic evolution condition:

$$E_i = \Omega_i(\tau)(|D_i^+|^2 + |D_i^-|^2). \quad (17)$$

Let us assume that initially $\beta(0) \gg R^{1/2}$ and, therefore, evolution of the waves is originally adiabatic. Due to the linear drift in the $k$-space, $\beta(\tau)$ decreases and when the condition (15) fails to work, the mode dynamics becomes non-adiabatic due to the resonant coupling between the modes. The duration of the non-adiabatic evolution is given by $\Delta \tau \sim R^{-1/2}$. Afterwards, when $\beta(\tau) \ll -R^{-1/2}$, the evolution becomes adiabatic again. Denoting WKB amplitudes of the wave modes before and after the coupling region (i.e., for the $\tau < \tau_r - \Delta \tau/2$ and $\tau > \tau_r + \Delta \tau/2$) by $D_{i,B}^\pm$ and $D_{i,A}^\pm$ respectively and employing the formal analogy with the $S$-matrix of the scattering theory [23] and the transition matrix from the theory of multi-level quantum systems [24], one can connect $D_{i,A}^\pm$ with $D_{i,B}^\pm$ via the $4 \times 4$ transition matrix:

$$\begin{pmatrix} D_A^+ \\ D_A^- \end{pmatrix} = \begin{pmatrix} T & T^\pm \\ T^\mp & T^* \end{pmatrix} \begin{pmatrix} D_B^+ \\ D_B^- \end{pmatrix}, \quad (18)$$

where $D_L^\pm$ and $D_R^\pm$, while $1 \times 2$ matrices and $T$, its Hermitian conjugated matrix $T^*$, $T^\pm$, and $T^\mp$ are $2 \times 2$ matrices.

None of the turning points are located near the real $\tau$-axis and, therefore, only wave modes with the same sign of the phase velocity along the $x$-axis can effectively interact. It is well known [24] that in this case components of $T^\pm$ and $T^\mp$ are exponentially small with respect to the large parameter $1/R$ and can be neglected. Consequently, (18) decomposes and reduces to:

$$D_A^+ = TD_B^+, \quad (19a)$$

$$D_A^- = T^*D_B^- - . \quad (19b)$$

Since all coefficients in the (12) are real and $C_{12} = C_{21}$, the matrix $T$ is unitary [25], and

$$\sum_j |T_{ij}|^2 = 1. \quad (20)$$

Generally, this equation represents conservation of the wave action. When $R \ll 1$ it transcribes into the energy conservation throughout the resonant coupling of wave modes [18]. The components of the transition matrix in (18) are complex, i.e., the coupling of different wave modes changes not only the absolute values of $D_i^\pm$, but also their phases. The value of the quantity $|T_{12}|^2$ represents a part of the energy transformed during the resonant coupling of the modes. The absolute values of the transition matrix components $|T_{12}|$ and $|T_{21}|$ are called the transformation coefficients of corresponding

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$^1$The resonant point is said to be of the order $n$ if $(\Omega_i^2 - \Omega_j^2) \sim (\tau - \tau_r)^{n/2}$ in the neighborhood of the $\tau_r$.  

wave modes. Unitarity of the $T$ provides an important symmetry property of the transition matrix:

$$|T_{ij}| = |T_{ji}|,$$

i.e., transformation coefficients are reciprocally equal to each other.

It is well known, that if in the neighborhood of the real $\tau$-axis only a pair of complex conjugated first order resonant points $\tau_1^c$, $\tau_2^c$ exists, the transformation coefficient is [18]:

$$|T_{12}| = \exp \left( -\left| \operatorname{Im} \int_{0}^{\tau} (\Omega_1 - \Omega_2) dr \right| \right) \left[ 1 + O(R^{1/2}) \right].$$

(22)

As it was mentioned earlier, coupling between ICW and ISW can be effective if $|\omega_c - 1| \ll 1$. Otherwise transformation coefficient is exponentially small with respect to the large parameter $R^{-1}$ [24,25]. If this condition is satisfied the resonant points $\tau_1^c$ [see (14)] tend to the real $\tau$-axis and, hence, the effective coupling is possible. Noting, that according to (13), in the neighborhood of the resonant points:

$$\Omega_1 - \Omega_2 \approx \sqrt{\beta^2(\tau) + (\omega_c - 1)^2},$$

and then from (22) one can readily obtain:

$$|T_{12}| \approx \exp \left[ -\pi(\omega_c - 1)^2 / 4R \right].$$

(24)

From the (14) we also see that if $\omega_c \to 1$, the resonant points tend to the real $\tau$-axis. Then from (24) it follows that $|T_{12}| \to 1$, i.e., one wave mode is totally transformed into the another. On the other hand, if $|\omega_c - 1| \gg R^{1/2}$, transformation is negligible.

In the case of moderate or high shearing rates, similarly to the case $R \ll 1$, one can show that the WKB approximation is valid only when $\beta(\tau) \gg 1, R$. It means that the asymptotic problem can still be formulated. However, when the shearing rate is not small, non-adiabatic evolution of the modes consists of both transformation and reflection phenomena. From mathematical point of view it means that, in general, all the components of the transition matrix significantly differ from zero. Conservation of the wave action remains valid and provides following important relation [18]:

$$\sum_j |T_{ij}|^2 - \sum_j |T_{ij}^+|^2 = 1.$$

(25)

However, it should be also stressed that for high shearing rates no analytical expressions for the components of the transition matrix can be written explicitly.

### B. Hydrodynamical analogy

The remarkable feature of our governing equations (10) is that they are almost identical with the pair of equations from [11] (numbered there as equations (16) and (17)):

$$e^{(2)} + \psi + \beta(\tau)e = 0,$$

(26a)

$$e^{(2)} + [W^2 + \beta^2(\tau)]e + \beta(\tau)e = 0.$$

(26b)

These equations also describe coupling of two wave modes. But this is totally different physical system: shear flow of a gravitationally stratified neutral fluid, which sustains sound waves and internal gravity waves. In [11] it was shown that these modes are coupled through the agency of the shear and may effectively transform into each other, providing the condition $W \approx 1$ is met.

The presence of this analogy implies that all the details of the transformation coefficient asymptotic analysis, which were given above, can also be applied to the named hydrodynamic example of the shear induced wave couplings. The only factual difference is that we have to replace the dimensionless frequency of electrostatic ion-cyclotron waves $\omega_c$ by the characteristic dimensionless frequency $W$ of the internal gravity waves.

This analogy allows also to predict that within the electrostatic problem we should have a new kind of electrostatic beat waves [just as they exist in the hydrodynamical problem! Beat waves are excited when an initial perturbation propagates almost along the flow axis ($\beta_0 \ll 1$ and when, additionally, $\tau_*$ is sufficiently large in comparison with the beat period $\Omega_b \equiv \Omega_1 - \Omega_2$.

### III. CONCLUSION

Summarizing main properties of the resonant coupling of ICW and ISW for small shearing rates ($R \ll 1$) we can state that: (a) Only the wave modes with the same sign of the phase velocity can effectively interact - there are no reflection phenomena; (b) The duration of the effective coupling of the modes is of the order of $\Delta \tau \sim R^{-1/2}$, i.e., resonant coupling is slow compared to the wave period $\tau_1 \sim 1$ but fast enough compared to the adiabatic change of the system parameters $\tau_2 \sim R^{-1}$; (c) The total energy of the modes is conserved during the resonant coupling - the transformed wave is generated at the expense of the energy of the initial wave mode; (d) The mode coupling process is symmetric - transformation coefficient of one mode into another one equals the coefficient of the inverse process. (e) The transformation coefficients are given by (22) and (24).

None of these features remain valid for moderate and high shearing rates. If $R \ll 1$ is not satisfied, there are no ‘long’ and ’short’ timescales in the problem and all the processes have approximately the same characteristic timescale. Hence, the coupling of the modes represents some mixture of transformation and reflection processes, that are accompanied by the energy exchange between the waves and the background flow.

The discovered ICW–ISW transformations are likely to be important in a number of applications. One possible example is the problem of the ICW observed by low altitude satellites and ground based magnetometers [26]. Observational surveys indicate that these waves are correlated with the ICW observed in the equatorial magneto-
tosphere. However, theoretical studies of the ICW propagation from the magnetosphere to the ground suggest that these waves can not penetrate through the Buchsbaum resonance and can not reach ionospheric layers of the atmosphere. Thus, one could expect that the magnetospheric ICW shouldn’t be correlated with the ionospheric ICW, while observational evidence shows the correlation. Recently Johnson and Cheng [26] reconsidered this problem and found that strong mode coupling occurs near the He$^+$ and O$^+$ resonance locations. They argued that this coupling may help the equatorial ICW to penetrate to ionospheric altitudes.

It seems reasonable to admit that the velocity-shear-induced ICW-ISW coupling may provide yet another mode transformation mechanism, which in conjunction with the one, found by Johnson and Cheng, may account for the penetration of the ICW through the Buchsbaum resonance. This may work in a quite similar way to the scenario given in the [27] for the penetration of the fast magnetosonic waves (FMW) from the chromosphere to the ground suggest that these waves can not penetrate through the Buchsbaum resonance and to reach the low ionospheric altitudes.

Finally, the remarkable exact analogy of the ICW-ISW coupling with the coupling of internal gravity waves and sound waves in hydrodynamic flows [11] points out, once again, at the universal character of the velocity shear induced phenomena in the physics of fluids and plasmas.

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