Existence, uniqueness and approximation for $L^p$ solutions of reflected BSDEs under weaker assumptions

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Abstract

We put forward and prove several existence and uniqueness results for $L^p$ ($p > 1$) solutions of reflected BSDEs with continuous barriers and generators satisfying a one-sided Osgood condition together with a general growth condition in $y$ and a uniform continuity condition or a linear growth condition in $z$. A necessary and sufficient condition with respect to the growth of barrier is also explored to ensure the existence of a solution. And, we show that the solutions may be approximated by the penalization method and by some sequences of solutions of reflected BSDEs. Our results improve considerably some known works.

Keywords: Reflected backward stochastic differential equation, $L^p$ solutions, Comparison theorem, One-sided Osgood condition, Uniform continuity condition

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space carrying a standard $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$, and $(\mathcal{F}_t)_{t \geq 0}$ the completed $\sigma$-algebra filtration generated by $(B_t)_{t \geq 0}$. Assume that $T > 0$ is a real number and $\mathcal{F} = \mathcal{F}_T$. In this paper we are given an $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable continuous process $(L_t)_{t \in [0,T]}$, an $\mathcal{F}_T$-measurable random variable $\xi$ such that $\xi \geq L_T$, a random function

$$g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

such that $g(\cdot, y, z)$ is $(\mathcal{F}_t)$-progressively measurable for each $(y, z)$, and an $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable continuous process $(V_t)_{t \in [0,T]}$ with finite variation. By a solution of the reflected backward stochastic differential equation (reflected BSDE or directly RBSDE for short) with terminal time $T$, terminal value $\xi$, generator $g + dV$ and barrier $L$, we understand a triple $(Y_t, Z_t, K_t)_{t \in [0,T]}$ of $(\mathcal{F}_t)_{t \geq 0}$-progressively

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measurable processes such that \( t \mapsto |g(t, Y_t, Z_t)| \) belongs to \( L^1(0, T) \), \( t \mapsto |Z_t| \) belongs to \( L^2(0, T) \) and

\[
\begin{cases}
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s dB_s, & t \in [0, T], \\
Y_t \geq L_t, & t \in [0, T], \\
K \text{ is nondecreasing, continuous, } K_0 = 0, & \int_0^T (Y_t - L_t) dK_t = 0.
\end{cases}
\]  

(1)

This equation is usually denoted by RBSDE \((\xi, g + dV, L)\). The second condition in (1) says that the first component \( Y \) of the solution is forced to stay above \( L \). The role of \( K \) is to push \( Y \) upwards in order to keep it above \( L \) in a minimal way, which means that the third condition in (1) is satisfied. Note that the usual BSDEs may be considered as a special case of RBSDEs with \( L \equiv -\infty \) (and \( K \equiv 0 \)).

Nonlinear BSDEs were initially put forward in 1990 by Pardoux and Peng [33], which proved an existence and uniqueness result for square-integrable solutions of BSDEs with generators satisfying the Lipschitz condition in \((y, z)\) where the data \( \xi, g(\cdot, 0, 0) \) are square-integrable and \( V \equiv 0 \). In El Karoui, Peng and Quenez [8] and Peng [34], the authors further investigated BSDEs with \( V \) being not zero. As a generation of the notion of nonlinear BSDEs, El Karoui, Kapoudjian, Pardoux, Peng and Quenez [6] first introduced nonlinear RBSDEs in 1997 and proved the existence and uniqueness for square-integrable solutions of RBSDEs with generators satisfying the Lipschitz condition in \((y, z)\) where the data \( \xi, g(\cdot, 0, 0) \) and \( \sup_{t \in [0, T]} |L_t| \) are all square-integrable and \( V \equiv 0 \). Recently, Klimsiak [25] further considered RBSDEs with \( V \) being not zero and with discontinuous barriers. BSDEs and RBSDEs have attracted more and more interests, and due to the closely connections with many problems, they have gradually become a very useful and efficient tool in different mathematical fields including mathematical finance, game theory, optimal switching problem, partial differential equations and others (see, e.g., [2, 3, 6–8, 13, 14, 16, 17, 22, 32, 34–36]).

The assumptions on the data in [6, 33] are sometimes too strong for many interesting applications. Therefore many attempts have been made to prove the existence and uniqueness of solutions of BSDEs or RBSDEs under less restrictive assumptions. For example, many papers were devoted to relaxing the continuity assumption for the barrier \( L \) of RBSDEs, see [3, 13, 25, 28, 35]; many papers aimed to solving BSDEs or RBSDEs with data that are not square-integrable but only in \( L^p \) \((p > 1)\) or \( L^1 \), see [1, 3–5, 8, 9, 15, 20, 24, 25, 29, 37]; and more papers were interested in weakening the linear growth and Lipschitz-continuity of the generator \( g \) with respect to \((y, z)\), see [4, 5, 8–12, 18, 20–22, 26, 29, 30, 32] for BSDEs and [1–3, 16, 19, 23–25, 27, 31, 37, 38] for RBSDEs.

In the present paper we focus attention on solving RBSDE (1) with \( L^p \) \((p > 1)\) data and continuous barriers under some weaker assumptions of the generator \( g \) with respect to \((y, z)\). RBSDEs with \( L^1 \) data and discontinuous barriers under
weaker assumptions will be our object of research in the near future. Here, we would like to mention some known results related closely to our work. Firstly, in Briand, Delyon, Hu, Pardoux and Stoica [4], the authors established the existence and uniqueness of $L^p$ solutions of BSDEs with $L^p$ ($p \geq 1$) data under assumptions that the generator $g$ satisfies a monotonicity condition together with a general growth condition in $y$ and the Lipschitz condition in $z$ (see respectively (H1s), (H3) and (H2s) in Section 2 for details). Recently, this result was extended to RBSDE (1) with $V \equiv 0$ by Lepeltier, Matoussi and Xu [27] and Rozkosz and Słomiński [37], where some additional assumptions relating the growth of generator $g$ with that of barrier $L$. were put forward. Under the same assumptions with respect to the generator $g$ as in [4, 37], Klimsiak [24] further explored a necessary and sufficient condition with respect to the growth of barrier $L$. (see (H6) in Section 2 for details) to ensure the existence and uniqueness of $L^p$ solutions of RBSDE (1) with $L^p$ ($p \geq 1$) data, continuous barrier $L$ and $V \equiv 0$. And, by establishing and applying a general monotonic limit theorem of BSDEs, Klimsiak [25] investigated RBSDEs with two irregular reflecting barriers.

Furthermore, in Matoussi [31] and Xu [38] the authors obtained respectively an existence result of square-integrable solutions for RBSDE (1) with $L^2$ data and $V \equiv 0$ where the generator $g$ satisfies a linear growth condition in $z$ instead of the Lipschitz condition. Jia and Xu [23] further proved a uniqueness result where the generator $g$ satisfies a uniform continuity condition in $z$ (see (H2) in Section 2 for details). With respect to this condition, we also refer to [10, 11, 21, 22].

On the other hand, Fan and Jiang [11] put forward a kind of one-sided Osgood condition in $y$ of the generator $g$ (see (H1) in Section 2 for details), which is weaker than not only Mao’s condition used in Mao [30] and the Osgood condition put forward in Fan, Jiang and Davison [12] but also the monotonicity condition (see (H1s) in Section 2) applied in [3, 4, 24, 25, 27, 37, 38]. Under (H1) and (H2), a comparison theorem of square-integral solutions of BSDEs was proved in [11], whose a direct corollary is the uniqueness of solutions. Furthermore, the existence and uniqueness result for $L^p$ solutions of BSDEs with $L^p$ ($p > 1$) data obtained in [4] has been extended by Fan [9], where the generator $g$ satisfies (H1), (H3) and (H2) or (H2w). Then, the following questions are naturally asked:

- Can we establish a comparison theorem for $L^p$ solutions of RBSDEs with $L^p$ ($p > 1$) data under assumptions (H1) and (H2)?

- Can we establish an existence result for $L^p$ solutions of RBSDE (1) with $L^p$ ($p > 1$) data under some appropriate conditions with respect to the growth of barrier $L$. if the generator $g$ only satisfies (H1), (H3) and (H2) or (H2w), a linear growth condition in $z$ (see Section 2 for details)?

- Can we give a necessary and sufficient condition with respect to the growth of barrier $L$. to ensure the existence of $L^p$ solutions of RBSDE (1) with $L^p$ ($p > 1$) data under (H1), (H3) and (H2) or (H2w)?
Does the sequence of $L^p$ solutions of usual penalization equation for RBSDE (1) with $L^p$ ($p > 1$) data still converge under (H1), (H3) and (H2) or (H2w)?

The present paper gives positive answers for all these questions. It should be mentioned that our results improve considerably some works mentioned before and that many technical results in our work, including some a priori estimates of BSDEs, the convergence of sequence of $L^p$ solutions for penalization and approximation equations of RBSDE (1) with $L^p$ ($p > 1$) data and the comparison theorem for $L^p$ solutions of RBSDEs, are all respectively established under some very general and elementary conditions, for example, the assumptions (H1), (H2), (HH) and (A) in Section 2 and the assumptions (B) and (C) in Section 3.

The remainder of this paper is organized as follows. Section 2 contains some notation and hypotheses which will be used later. Section 3 is devoted to establishing several a priori estimates on $L^p$ solutions of BSDEs with $L^p$ ($p > 0$) data as well as on some sequences of $L^p$ solutions of BSDEs with $L^p$ ($p > 1$) data, which will play important roles in the proof of our main results. The convergence of sequence of $L^p$ solutions for penalization and approximation equations of RBSDE (1) with $L^p$ ($p > 1$) data under assumption (HH) and some very elementary conditions, and a comparison theorem for $L^p$ solutions of RBSDEs with $L^p$ ($p > 1$) data under assumptions (H1) and (H2) are put forward and proved in Section 4. Based on these results, Section 5 focus on establishing some existence, uniqueness and approximation results on $L^p$ solutions of BSDEs with $L^p$ ($p > 1$) data (Section 5.1) and RBSDEs with $L^p$ ($p > 1$) data (Section 5.2) under weaker assumptions, which answers those questions put forward before.

We note that the work of this paper (even for Theorems 1-2 on non-reflected BSDEs) improves considerably some corresponding known results including those obtained in Briand, Lepeltier and San Martin [5], El Karoui, Peng and Quenez [8], Fan [9], Fan and Jiang [11], Hamadène and Popier [15], Klimsiak [24], Lepeltier, Matoussi and Xu [27], Rozkosz and Slomiński [37] and Xu [38] (see, for example, Remark 5 in Section 4 and Remarks 6 and 8 in Section 5.2 for more details).

2. Notation and hypotheses

In the whole paper we fix a real number $T > 0$ and a positive integer $d$, and let $\mathbb{R}_+ := [0, +\infty)$, $a^+ := \max\{a, 0\}$ and $a^- := (-a)^+$ for any real number $a$. Let $1_A$ represent the indicator function of a set $A$, and $\text{sgn}(x)$ the sign of a real number $x$. The Euclidean norm of a vector $z \in \mathbb{R}^d$ is denoted by $|z|$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space carrying a standard $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$, and $(\mathcal{F}_t)_{t \geq 0}$ the completed $\sigma$-algebra filtration generated by $(B_t)_{t \geq 0}$ and assume that $\mathcal{F} = \mathcal{F}_T$. In the whole paper all equalities and inequalities between random elements are understood to hold $\mathbb{P} - a.s.$.

For $p > 0$, denote by $L^p(\mathcal{F}_T)$ the set of all $\mathcal{F}_T$-measurable random variables $\xi$ such that

$$
\|\xi\|_{L^p} := (\mathbb{E}[|\xi|^p])^{1/p} < +\infty,
$$
and define the following spaces of processes or functions:

$\mathcal{S}$ — the set of all continuous $(\mathcal{F}_t)$-progressively measurable processes;

$\mathcal{S}^p$ — the set of all processes $Y \in \mathcal{S}$ such that

$$
\|Y\|_{\mathcal{S}^p} := \left( \mathbb{E}\left[ \sup_{t \in [0,T]} |Y_t|^p \right] \right)^{1 \wedge 1/p} < +\infty;
$$

$\mathcal{M}$ — the set of all $(\mathcal{F}_t)$-progressively measurable processes $Z$ such that

$$
\mathbb{P}\left( \int_0^T |Z_t|^2 dt < +\infty \right) = 1;
$$

$\mathcal{M}^p$ — the set of all processes $Z \in \mathcal{M}$ such that

$$
\|Z\|_{\mathcal{M}^p} := \left\{ \mathbb{E}\left[ \left( \int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{1 \wedge 1/p} < +\infty;
$$

$\mathcal{H}$ — the set of all $(\mathcal{F}_t)$-progressively measurable processes $X$ such that

$$
\mathbb{P}\left( \int_0^T |X_t| dt < +\infty \right) = 1;
$$

$\mathcal{H}^p$ — the set of all processes $X \in \mathcal{H}$ such that

$$
\|X\|_{\mathcal{H}^p} := \left\{ \mathbb{E}\left[ \left( \int_0^T |X_t| dt \right)^p \right] \right\}^{1 \wedge 1/p} < +\infty;
$$

$\mathcal{M}$ — the set of all continuous local $(\mathcal{F}_t)$-martingales.

$\mathcal{M}^p$ — the set of all martingales $M \in \mathcal{M}$ such that $\mathbb{E}\left[ (\langle M \rangle_T)^{p/2} \right] < +\infty$;

$\mathcal{V}$ — the set of all continuous $(\mathcal{F}_t)$-progressively measurable processes of finite variation;

$\mathcal{V}^p$ — the set of all processes $V \in \mathcal{V}$ such that $\mathbb{E}\left[ [V]^p_T \right] < +\infty$;

$\mathcal{V}^+$ — the set of all continuous $(\mathcal{F}_t)$-progressively measurable increasing processes;

$\mathcal{V}^{+p}$ — the set of all processes $V \in \mathcal{V}^+$ such that $\mathbb{E}\left[ [V]^p_T \right] < +\infty$;

$\mathcal{S}$ — the set of nonnegative functions $\psi_t(\omega, r) : \Omega \times [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following two conditions:

- $d\mathbb{P} \times dt$ a.e., the function $r \mapsto \psi_t(\omega, r)$ is increasing and $\psi_t(\omega, 0) = 0$;
− for each $r \geq 0$, $\psi(r) \in \mathcal{H}$.

For $p > 0$, we introduce the following hypotheses.

\textbf{(H1)} $g$ satisfies the one-sided Osgood condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_0^+ \frac{du}{\rho(u)} = +\infty$ such that $d\mathbb{P} \times dt - a.e., \ \forall \ y_1, y_2 \in \mathbb{R}, z \in \mathbb{R}^d,$

$$(g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \text{sgn}(y_1 - y_2) \leq \rho(|y_1 - y_2|).$$

\textbf{(H1s)} $g$ satisfies the monotonicity condition in $y$, i.e., there exists a constant $\mu \in \mathbb{R}$ such that $d\mathbb{P} \times dt - a.e., \ \forall \ y_1, y_2 \in \mathbb{R}, z \in \mathbb{R}^d,$

$$(g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \text{sgn}(y_1 - y_2) \leq \mu|y_1 - y_2|.$$

\textbf{(H2)} $g$ satisfies the uniform continuity condition in $z$, i.e., there exists a nondecreasing and continuous function $\phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\phi(0) = 0$ such that $d\mathbb{P} \times dt - a.e., \ \forall \ y \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d,$

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \phi(|z_1 - z_2|).$$

\textbf{(H2s)} $g$ satisfies the Lipschitz condition in $z$, i.e., there exists a nonnegative constant $\lambda$ such that $d\mathbb{P} \times dt - a.e., \ \forall \ y \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d,$

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \lambda|z_1 - z_2|.$$

\textbf{(H2w)} $g$ has a stronger linear growth in $z$, i.e., there exists two constants $\mu, \lambda \geq 0$ and a nonnegative process $f. \in \mathcal{H}^p$ such that $d\mathbb{P} \times dt - a.e., \ \forall \ y \in \mathbb{R}, z \in \mathbb{R}^d,$

$$|g(\omega, t, y, z) - g(\omega, t, y, 0)| \leq f_t(\omega) + \mu|y| + \lambda|z|.$$

\textbf{(H3)} $g$ has a general growth in $y$, i.e. $\forall r > 0$, $\varphi(r) := \sup_{|y| \leq r} |g(\cdot, y, 0) - g(\cdot, 0, 0)|$ belongs to the space $\mathcal{H}$. And, $g(\cdot, 0, 0) \in \mathcal{H}^p$.

\textbf{(H3s)} $g$ has a linear growth in $y$, i.e., there exists a constant $\mu \geq 0$ and a nonnegative process $f. \in \mathcal{H}^p$ such that $d\mathbb{P} \times dt - a.e., \ \forall \ y \in \mathbb{R}, |g(\omega, t, y, 0)| \leq f_t(\omega) + \mu|y|.$

\textbf{(H4)} $g$ is continuous in $(y, z)$, i.e., $d\mathbb{P} \times dt - a.e., g(\omega, t, \cdot, \cdot)$ is continuous.

\textbf{(H4s)} $g$ is stronger continuous in $(y, z)$, i.e., $d\mathbb{P} \times dt - a.e., \ \forall \ y \in \mathbb{R}$, $g(\omega, t, y, \cdot)$ is continuous, and $g(\omega, t, z, \cdot)$ is continuous uniformly with respect to $z$.

\textbf{(H4w)} $d\mathbb{P} \times dt - a.e., \ \forall \ z \in \mathbb{R}^d$, $g(\omega, t, \cdot, z)$ is continuous.

\textbf{(H5)} $\xi \in \mathbb{L}^p(\mathcal{F}_T)$, $L \in \mathcal{S}$ and $L_T \leq \xi$. 

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(H6) There exists a semi-martingale $X \in M^p + \mathcal{V}^p$ such that $g(\cdot, X, 0) \in \mathcal{H}^p$ and for each $t \in [0, T]$, $X_t \geq L_t$.

(HH) $g$ has a certain general growth in $(y, z)$, i.e., there exists a constant $\lambda \geq 0$, a nonnegative process $f. \in \mathcal{H}^p$ and a nonnegative function $\psi. (r) \in S$ such that
\[
|g(\omega, t, y, z)| \leq f_t(\omega) + \psi_t(\omega, |y|) + \lambda|z|.
\]

(A) There exists two nonnegative constants $\bar{\mu}$ and $\bar{\lambda}$ such that $\mathbb{P} \times dt - a.e.$,
\[
\forall y \in \mathbb{R}, z \in \mathbb{R}^d,
\]
\[
g(\omega, t, y, z)\text{sgn}(y) \leq \bar{f}_t(\omega) + \bar{\mu}|y| + \bar{\lambda}|z|.
\]

Remark 1 Without loss of generality, we will always assume that the functions $\rho(\cdot)$ and $\phi(\cdot)$ defined respectively in (H1) and (H2) are of linear growth, i.e., there exists a constant $A > 0$ such that
\[
\forall x \in \mathbb{R}_+, \quad \rho(x) \leq A(x + 1) \quad \text{and} \quad \phi(x) \leq A(x + 1).
\]

Remark 2 It is not very hard to verify the following statements hold:

(i) (H1s) $\Rightarrow$ (H1); (H2s) $\Rightarrow$ (H2) $\Rightarrow$ (H2w); (H3s) $\Rightarrow$ (H3); (H4s) $\Rightarrow$ (H4) $\Rightarrow$ (H4w);

(ii) (H2w) $+$ (H3) $\Rightarrow$ (HH) $\Rightarrow$ (H3); If (H2) holds, then (H4w) $\Leftrightarrow$ (H4);

(iii) (H6) $\Rightarrow$ $L^+_t \in S^p$; If $L^+_t \in S^p$ and
\[
\left( g(t, \sup_{s \in [0,t]} L^+_s, 0) \right)_{t \in [0,T]} \in \mathcal{H}^p,
\]
then (H6) holds; If (H3s) holds, then (H6) $\Leftrightarrow$ $L^+_t \in S^p$

(iv) (H1) $+$ (H2w) $+$ $g(\cdot, 0, 0) \in \mathcal{H}$ $\Rightarrow$ (A); (H1) $+$ (HH) $\Rightarrow$ (A).

We only show (iv). In fact, if $g$ satisfies (H1) and (H2w) with $g(\cdot, 0, 0) \in \mathcal{H}$, then in view of Remark 1, it follows that $\mathbb{P} \times dt - a.e.$, for each $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,
\[
g(\cdot, y, z)\text{sgn}(y) \leq |(g(\cdot, y, z) - g(\cdot, y, 0))\text{sgn}(y)|
\]
\[
+ (g(\cdot, y, 0) - g(\cdot, 0, 0))\text{sgn}(y) + |g(\cdot, 0, 0)|
\]
\[
\leq f + \mu|y| + \lambda|z| + \rho(|y|) + |g(\cdot, 0, 0)|
\]
\[
\leq f + |g(\cdot, 0, 0)| + A + (\mu + A)|y| + \lambda|z|.
\]
Hence, \( g \) satisfies the assumption (A) with
\[
\tilde{f} = f + |g(\cdot, 0, 0)| + A, \quad \tilde{\mu} = \mu + A \quad \text{and} \quad \tilde{\lambda} = \lambda.
\]
Furthermore, if \( g \) satisfies assumptions (H1) and (HH), then in view of Remark 1, it follows that \( d\mathbb{P} \times dt - a.e. \), for each \( y \in \mathbb{R}, z \in \mathbb{R}^d \),
\[
g(\cdot, y, z) \text{sgn}(y) \leq (g(\cdot, y, z) - g(\cdot, 0, z)) \text{sgn}(y) + |g(\cdot, 0, z)|
\leq \rho(|y|) + f + \lambda|z|
\leq f + A + A|y| + \lambda|z|.
\]
Hence, \( g \) satisfies the assumption (A) with
\[
\tilde{f} = f + A, \quad \tilde{\mu} = A \quad \text{and} \quad \tilde{\lambda} = \lambda.
\]

3. A priori estimates

By virtue of Itô’s formula, the Burkholder-Davis-Gundy (BDG for short) inequality and Hölder’s inequality as well as the stopping time technique and Fatou’s Lemma, using a similar argument as that in the proof of Proposition 2.4 of Izumi [20] we can prove the following lemma 1. The proof is omitted here.

**Lemma 1** Let \((\tilde{Y}, \tilde{Z}, \tilde{V}) \in \mathcal{S} \times M \times \mathcal{V}\) satisfy the following equation:
\[
\tilde{Y}_t = \tilde{Y}_T + \int_t^T \tilde{V}_s d\tilde{V}_s - \int_t^T \tilde{Z}_s dB_s, \quad t \in [0, T]. \tag{2}
\]
We have

(i) If \( \tilde{Y} \in \mathcal{S}^p \) for some \( p > 0 \), then there exists a constant \( C_1 > 0 \) depending only on \( p \) such that for each \( t \in [0, T] \) and \((\mathcal{F}_t)\)-stopping time \( \tau \) valued in \([0, T]\),
\[
\mathbb{E} \left[ \left( \int_{t \wedge \tau}^{T \wedge \tau} |\tilde{Z}_s|^2 ds \right)^{\frac{p}{2}} \mid \mathcal{F}_t \right] \leq C_1^p \mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}_{s \wedge \tau}|^p + \sup_{s \in [t, T]} \left( \int_{s \wedge \tau}^{T \wedge \tau} |\tilde{Y}_r| d\tilde{V}_r \right)^+ \right] \mid \mathcal{F}_t \right].
\]

(ii) If \( \tilde{Y} \in \mathcal{S}^p \) for some \( p > 1 \), then there exists a constant \( C_2 > 0 \) depending only on \( p \) such that for each \( t \in [0, T] \) and \((\mathcal{F}_t)\)-stopping time \( \tau \) valued in \([0, T]\),
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}_{s \wedge \tau}|^p + \int_{t \wedge \tau}^{T \wedge \tau} |\tilde{Y}_s|^{p-2} \mathbb{1}_{\{|\tilde{Y}_s| \neq 0\}} |\tilde{Z}_s|^2 ds \mid \mathcal{F}_t \right] \leq C_2 \mathbb{E} \left[ |\tilde{Y}_T|^p + \int_{s \wedge \tau}^{T \wedge \tau} |\tilde{Y}_r|^{p-1} \text{sgn}(\tilde{Y}_r) d\tilde{V}_r \right] \mid \mathcal{F}_t \right].
\]
By virtue of Lemma 1 we can prove the following Lemma 2. The proof is classical, see, for example, the proof of Lemma 3.1 and Proposition 3.2 in Briand, Delyon, Hu, Pardoux and Stoica [4], we omit it here.

**Lemma 2** Assume that the assumption (A) is satisfied for the generator $g$. Let $(Y, Z, V) \in S \times M \times V$ satisfy the following equation:

$$
Y_t = Y_T + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T dV_s - \int_t^T Z_s dB_s, \quad t \in [0, T],
$$

If $Y \in S^p$ and $(V, \tilde{f}) \in V^p \times H^p$ for some $p > 1$, then $Z \in M^p$, and there exists a constant $C > 0$ depending only on $p, \bar{\mu}, \bar{\lambda}, T$ such that for each $t \in [0, T]$,

$$
\mathbb{E} \left[ \sup_{s \in [t,T]} |Y_s|^p + \left( \int_t^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right| \mathcal{F}_t \leq C \mathbb{E} \left[ |Y_T|^p + |V|^p_T + \left( \int_t^T \tilde{f}_s \, ds \right)^p \right| \mathcal{F}_t, \right.
$$

**Remark 3** Note that in case of $t = 0$, Lemma 2 has been obtained in Proposition 3.5 of Klimsiak [25].

By Lemma 1 we can also deduce the following important a priori estimate.

**Lemma 3** Let $(Y, Z, V, K) \in S \times M \times V \times V^+$ satisfy the following equation:

$$
Y_t = Y_T + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0, T],
$$

and let $p > 0$. We have

(i) Assume that the assumption (A) is satisfied for the generator $g$. If $Y \in S^p$, then there exists a nonnegative constant $C$ depending only on $p, \bar{\mu}, \bar{\lambda}, T$ such that for each $t \in [0, T]$,

$$
\mathbb{E} \left[ \left( \int_t^T |Z_s|^3 ds \right)^{\frac{p}{2}} \right| \mathcal{F}_t \leq C \mathbb{E} \left[ \sup_{s \in [t,T]} |Y_s|^p + |V|^p_T + \left( \int_t^T \tilde{f}_s \, ds \right)^p + \left( \int_t^T |Y_s| dK_s \right)^{\frac{p}{2}} \right| \mathcal{F}_t \right]
$$

and

$$
\mathbb{E} \left[ \left( \int_t^T |g(s, Y_s, Z_s)| ds \right)^p \right| \mathcal{F}_t \leq C \mathbb{E} \left[ \sup_{s \in [t,T]} |Y_s|^p + |V|^p_T + \left( \int_t^T \tilde{f}_s \, ds \right)^p + |K_T - K_t|^p \right. \right.

$$

$$
+ \left( \int_t^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right| \mathcal{F}_t \].
$$
(ii) Assume that the following assumption (B) holds:

(B) There exists two nonnegative constants $\tilde{\mu}$ and $\tilde{\lambda}$ such that $d\mathbb{P} \times dt - a.e.,$

$$g(t, Y_t, Z_t) \geq - \left( \tilde{f}_t + \tilde{\mu}|Y_t| + \tilde{\lambda}|Z_t| \right),$$

where $\tilde{f}_t$ is a nonnegative process belonging to $\mathcal{H}.$

Then there exists a nonnegative constant $\tilde{C}$ depending only on $p, \tilde{\mu}, \tilde{\lambda}, T$ such that for each $t \in [0, T],$

$$\mathbb{E} \left[ |K_T - K_t|^p \big| \mathcal{F}_t \right] \leq \tilde{C} \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p + |V|^p_T + \left( \int_t^T \tilde{f}_s \, ds \right)^p + \left( \int_t^T |Z_s|^2 \, ds \right)^{\frac{p}{2}} \big| \mathcal{F}_t \right]. \quad (5)$$

Proof. (i) Observe that

$$(Y, \tilde{Z}, \tilde{V}) := \left( Y, Z, \int_0^T g(s, Y_s, Z_s) \, ds + V + K \right)$$

satisfies equation (2). It follows from (i) of Lemma 1 that if $\tilde{Y} \in \mathcal{S}^p,$ then there exists a constant $C_1 > 0$ depending only on $p$ such that for each $t \in [0, T],$

$$\mathbb{E} \left[ \left( \int_t^T |Z_s|^2 \, ds \right)^{\frac{p}{2}} \big| \mathcal{F}_t \right] \leq C_1 \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p + \sup_{s \in [t, T]} \left( \int_s^T Y_r \, d\tilde{V}_r \right)^+ \big| \mathcal{F}_t \right]. \quad (8)$$

Furthermore, it follows from (A) and Hölder inequality that for each $0 \leq t \leq s \leq T,$

$$\left( \int_s^T Y_r \, d\tilde{V}_r \right)^+ \leq \tilde{\mu} \sup_{r \in [t, T]} |Y_r|^2 + \sup_{r \in [t, T]} |Y_r| \left[ \int_t^T \tilde{f}_r \, dr + \tilde{\lambda}T \left( \int_t^T |Z_r|^2 \, dr \right)^{\frac{1}{2}} \right] + \sup_{r \in [t, T]} |Y_r||V_T| + \int_t^T |Y_r| \, dK_r,$$
and then by virtue of the basic inequalities
\[ \frac{ab}{2} \leq (a^2 + b^2)/2 \quad \text{and} \quad (|a| + |b|)^q \leq 2^q(|a|^q + |b|^q), \quad q > 0, \]
we can get the existence of a constant \( C_2 > 0 \) depending only on \( p, \bar{\mu}, \bar{\lambda}, T \) such that for each \( t \in [0, T] \),
\[
C_1 \sup_{s \in [t, T]} \left[ \left( \int_s^T Y_r d\bar{V}_r \right)^+ \right] ^{\frac{q}{2}} \leq \frac{1}{2} \left( \int_t^T |Z_s|^2 ds \right) ^{\frac{q}{2}} + C_2 \sup_{s \in [t, T]} |Y_s|^p + C_2 |V|^p_T + C_2 \left( \int_t^T |Y_s|^q ds \right) ^{\frac{q}{2}}.
\]

Thus, if \( Z \in M_p \), then (3) follows from (8) and (9). Otherwise, for each positive integer \( k \geq 1 \), define the following \((\mathcal{F}_t\)-stopping time:
\[
\tau_k := \inf \{ t \in [0, T] : \int_0^t |Z_s|^2 ds \geq k \} \wedge T. \tag{10}
\]
Note that \( \tau_k \to T \) as \( k \to +\infty \) due to the fact that \( Z \in M \). In the above argument beginning from (8) till (9), replacing respectively
\[
\int_t^T, \quad \int_0^T, \quad \sup_{s \in [t, T]} |Y_s|^p, \quad \sup_{r \in [t, T]} |Y_r|^2, \quad \sup_{r \in [t, T]} |Y_r|, \quad |V|_T
\]
with
\[
\int_{t \wedge \tau_k}^{T \wedge \tau_k}, \quad \int_{s \wedge \tau_k}^{T \wedge \tau_k}, \quad \sup_{s \in [t, T]} |Y_s\wedge \tau_k|^p, \quad \sup_{r \in [t, T]} |Y_r\wedge \tau_k|^2, \quad \sup_{r \in [t, T]} |Y_r\wedge \tau_k|, \quad |V|_{T\wedge \tau_k}
\]
yields that for each \( k \geq 1 \) and each \( t \in [0, T] \),
\[
\mathbb{E} \left[ \left( \int_{t \wedge \tau_k}^{T \wedge \tau_k} |Z_s|^2 ds \right) ^{\frac{q}{2}} \mid \mathcal{F}_t \right] \leq C \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s\wedge \tau_k|^p + |V|_{T\wedge \tau_k}^p + \left( \int_{t \wedge \tau_k}^{T \wedge \tau_k} \tilde{f}_s ds \right) ^p + \left( \int_{t \wedge \tau_k}^{T \wedge \tau_k} |Y_s|dK_s \right) ^{\frac{q}{2}} \mid \mathcal{F}_t \right],
\]
where the constant \( C > 0 \) depends only on \( p, \bar{\mu}, \bar{\lambda}, T \). Thus, letting \( k \to +\infty \) in the above inequality and using Fatou’s lemma we get (3).

In the sequel, it follows from (A) that \( d\mathbb{P} \times dt \) a.e.,
\[
|g(\cdot, Y, Z)| = \left| - \text{sgn}(Y)g(\cdot, Y, Z) \right| \leq |f + \bar{\mu}|Y| + \bar{\lambda}|Z| - \text{sgn}(Y)g(\cdot, Y, Z) + f + \bar{\mu}|Y| + \bar{\lambda}|Z| \tag{11}
\]
\[
= 2 \left( f + \bar{\mu}|Y| + \bar{\lambda}|Z| \right) - \text{sgn}(Y)g(\cdot, Y, Z).
\]
On the other hand, by Itô-Tanaka’s formula we know that for each $t \in [0, T]$,

$$
- \int_t^T \text{sgn}(Y_s)g(s, Y_s, Z_s)ds \\
\leq |Y_T| - |Y_t| + \int_t^T \text{sgn}(Y_s)dV_s + \int_t^T \text{sgn}(Y_s)dK_s - \int_t^T \text{sgn}(Y_s)Z_sdB_s
$$

(12)

Thus, (4) follows by combining (11) and (12) and using H{"o}lder’s inequality as well as the BDG inequality.

(ii) It follows from (B) and H{"o}lder’s inequality that for each $t \in [0, T]$,

$$
|K_T - K_t| = Y_t - Y_T - \int_t^T g(s, Y_s, Z_s)ds - \int_t^T dV_s + \int_t^T Z_sdB_s
\leq Y_t - Y_T + \int_t^T (\tilde{f}_s + \mu|Y_s| + \tilde{\lambda}|Z_s|)ds + |V|_T + \left| \int_t^T Z_sdB_s \right|
\leq (2 + \mu T) \sup_{s \in [t, T]} |Y_s| + |V|_T + \int_t^T \tilde{f}_s ds
\quad + \lambda T \left( \int_t^T |Z_s|^2 ds \right)^{\frac{1}{2}} + \sup_{r \in [t, T]} \left| \int_r^T Z_sdB_s \right|,
$$

from which (5) follows immediately by using the basic inequality

$$
(\sum_{k=1}^n |a_k|)^p \leq n^p \sum_{k=1}^n |a_k|^p
$$

and the BDG inequality.

(iii) Since both (A) and (B) are satisfied and $Y \in S^p$, then it follows from (i) and (ii) that both (3), (4) and (5) hold true with constants $C$ and $\tilde{C}$ respectively. It follows from the basic inequality $2ab \leq \epsilon a^2 + b^2/\epsilon$, $\epsilon > 0$ with $\epsilon = \tilde{C}$ that

$$
C \left( \int_t^T |Y_s|dK_s \right)^{\frac{2}{p}} \leq C \sup_{s \in [0, T]} |Y_s|^{\frac{2}{p}} |K_T - K_t|^{\frac{2}{p}}
\leq \frac{\tilde{C}}{2} C^2 \sup_{s \in [0, T]} |Y_s|^p + \frac{1}{2C} |K_T - K_t|^p.
$$

Combining (3), (5) and the above inequality yields the existence of a constant $\tilde{C} > 0$ depending only on $p, \mu, \tilde{\lambda}, \tilde{\mu}, \tilde{\lambda}, T$ such that for each $t \in [0, T]$,

$$
E \left[ \left( \int_t^T |Z_s|^2 ds \right)^{\frac{2}{p}} \bigg| \mathcal{F}_t \right] \\
\leq \tilde{C} E \left[ \sup_{s \in [t, T]} |Y_s|^p + |V|_T^p + \left( \int_t^T \tilde{f}_s ds \right)^p + \left( \int_t^T \tilde{f}_s^p ds \right) \bigg| \mathcal{F}_t \right],
$$

(13)
provided that \( Z \in \mathbb{M}^p \). Using the stopping time technique and Fatou’s lemma, similar to the argument beginning from (10) to the end of that paragraph, we can deduce that (13) holds still true for \( Z \) which only belongs to \( M \). Thus, (7) follows from (5), (4) and (13), and in view of (6), \((Z, K_i) \in \mathbb{M}^p \times \mathbb{V}^{+p}\). \( \square \)

By virtue of Lemmas 2 and 3 we can prove the following Propositions 1 and 2, which will play important roles later.

**Proposition 1** Assume that \( p > 1, \xi \in L^p(\mathcal{F}_T) \) and \( V \in \mathbb{V}^p \). For each \( n \geq 1 \), suppose that generators \( g_n \) satisfy (H1) and (HH) with the same \( \rho(\cdot), f, \psi(r) \) and \( \lambda \), and let \((Y^n, Z^n) \in \mathcal{S}^p \times \mathbb{M}^p \) satisfy

\[
Y^n_t = \xi + \int_t^T g_n(s, Y^n_s, Z^n_s)ds + \int_t^T dB_s, \quad t \in [0, T].
\]

Then, there exists a nonnegative constant \( C \) depending only on \( p, A, \lambda, T \) such that for each \( t \in [0, T] \) and \( n \geq 1 \),

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y^n_s|^p + \left( \int_t^T |Z^n_s|^2 ds \right)^{\frac{p}{2}} + \left( \int_t^T |g_n(s, Y^n_s, Z^n_s)| ds \right)^p \right] F_t \\
\leq C \mathbb{E} \left[ |\xi|^p + |V|^p_T + \left( \int_t^T f_s ds \right)^p + 1 \right] F_t.
\]

**Proof.** It follows from (iv) of Remark 2 that all \( g_n \) satisfy (A) with the same

\[
\tilde{f} = f + A, \quad \tilde{\mu} = A \quad \text{and} \quad \tilde{\lambda} = \lambda.
\]

Then by Lemma 2 we know that there exists a constant \( C_2 > 0 \) depending only on \( p, A, \lambda, T \) such that for each \( t \in [0, T] \) and \( n \geq 1 \),

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y^n_s|^p + \left( \int_t^T |Z^n_s|^2 ds \right)^{\frac{p}{2}} \right] F_t \\
\leq C_2 \mathbb{E} \left[ |\xi|^p + |V|^p_T + \left( \int_t^T f_s ds \right)^p + 1 \right] F_t,
\]

form which together with (4) in Lemma 3 the conclusion follows. \( \square \)

**Proposition 2** Assume that \( p > 1, \xi \in L^p(\mathcal{F}_T) \) and \( V \in \mathbb{V}^p \). For \( n \geq 1 \), suppose that generators \( g_n \) and \( g \) satisfy (H1) with the same \( \rho(\cdot), g(\cdot, 0, 0) \in \mathcal{H}^p \) and that there exist two nonnegative constants \( \mu, \lambda \) and a nonnegative process \( f \in \mathcal{H}^p \) such that \( d\mathbb{P} \times dt - a.e., \forall y \in \mathbb{R}, z \in \mathbb{R}^d, \)

\[
|g_n(\omega, t, y, z) - g(\omega, t, y, 0)| \leq f_t(\omega) + \mu|y| + \lambda|z|. \quad (14)
\]

For each \( n \geq 1 \), let \((Y^n, Z^n, K^n) \in \mathcal{S}^p \times \mathbb{M}^p \times \mathbb{V}^{+p} \) satisfy

\[
Y^n_t = \xi + \int_t^T g_n(s, Y^n_s, Z^n_s)ds + \int_t^T dB_s + \int_t^T dK^n_s - \int_t^T Z^n_s dB_s, \quad t \in [0, T].
\]

If the following assumption (C) holds:
(C) There exists a process \( \bar{X} \in \mathcal{S}^p \) such that \( g(\cdot, \bar{X}, 0) \in \mathcal{H}^p \) and \( \bar{X}_t \geq Y^n_t \) for each \( t \in [0, T] \) and \( n \geq 1 \),

then there exists a nonnegative constant \( C \) depending only on \( p, \mu, \lambda, A, T \) such that for each \( t \in [0, T] \) and \( n \geq 1 \),

\[
E \left[ \left( \int_t^T \left| Z^n_s \right|^2 ds \right)^{\frac{p}{2}} + |K^n_T - K^n_t| + \left( \int_t^T |g_n(s, Y^n_s, Z^n_s)| ds \right)^p \bigg| \mathcal{F}_t \right] 
\leq C E \left[ \sup_{s \in [t,T]} |Y^n_s|^p + |V|^p + \sup_{s \in [t,T]} |\bar{X}_s|^p + \left( \int_t^T f_s ds \right)^p + 1 
+ \left( \int_t^T |g(s, \bar{X}_s, 0)| ds \right)^p + \left( \int_t^T |g(s, 0, 0)| ds \right)^p \bigg| \mathcal{F}_t \right].
\]

\[ (15) \]

**Proof.** In view of Remark 1, it follows from \((14)\) together with \((H1)\) for \( g \) that \( d\mathbb{P} \times dt = a.e., \) for each \( y \in \mathbb{R}, \ z \in \mathbb{R}^d \) and \( n \geq 1 \),

\[
g_n(\cdot, y, z) \text{sgn}(y) \leq \left| (g_n(\cdot, y, z) - g(\cdot, y, 0)) \text{sgn}(y) \right| 
+ (g(\cdot, y, 0) - g(\cdot, 0, 0)) \text{sgn}(y) + |g(\cdot, 0, 0)| 
\leq f + \mu |y| + \lambda |z| + \rho |y| + |g(\cdot, 0, 0)| 
\leq f + |g(\cdot, 0, 0)| + A + (\mu + A)|y| + \lambda |z|.
\]

Hence, for each \( n \geq 1 \), \( g_n \) satisfies the assumption \((A)\) with the same \( \bar{f} = f + |g(\cdot, 0, 0)| + A, \ \bar{\mu} = \mu + A \) and \( \bar{\lambda} = \lambda \).

Furthermore, note by \((C)\) that \( \bar{X}_t \geq Y^n_t \) for each \( t \in [0, T] \) and \( n \geq 1 \). Once again, in view of Remark 1, it follows from \((H1)\) for \( g_n \) together with \((14)\) that \( d\mathbb{P} \times dt = a.e., \) for each \( n \geq 1 \),

\[
-g_n(\cdot, Y^n_s, Z^n_s) \leq \left( g_n(\cdot, Y^n_s, Z^n_s) - g_n(\cdot, Y^n_s, Z^n_s) \right) 
+ |g(\cdot, \bar{X}_s, 0) - g_n(\cdot, \bar{X}_s, Z^n_s)| + |g(\cdot, \bar{X}_s, 0)| 
\leq \rho(|\bar{X} - Y^n_t|) + f + \mu |\bar{X}_s| + \lambda |Z^n_s| + |g(\cdot, \bar{X}_s, 0)| 
\leq |g(\cdot, \bar{X}_s, 0)| + (\mu + A)|\bar{X}_s| + f + A + A|Y^n_t| + \lambda |Z^n_t|.
\]

Hence, the assumption \((B)\) holds also true for each \( g_n \) with the same \( \bar{f} = |g(\cdot, \bar{X}_s, 0)| + (\mu + A)|\bar{X}_s| + f + A, \ \bar{\mu} = \mu + A \) and \( \bar{\lambda} = \lambda \).

Thus, \((15)\) follows from \((iii)\) of Lemma 3. \( \square \)

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4. Penalization, approximation and comparison theorem

In this section, we will put forward and prove the convergence of sequence of $L^p$ solutions for penalization and approximation equations of RBSDE (1) with $L^p$ ($p > 1$) data under assumption (HH) and some very elementary conditions, and a comparison theorem for $L^p$ solutions of RBSDEs with $L^p$ ($p > 1$) data under assumptions (H1) and (H2).

**Proposition 3** (Penalization) Assume that the generator $g$ satisfies (H4) and (HH) with $f_\cdot, \psi_\cdot, r_\cdot$ and $\lambda_\cdot$. Let $p > 1$, $V_\cdot \in \mathcal{V}^p$ and (H5) be satisfied for $\xi_\cdot$ and $L_\cdot$. For $n \geq 1$, assume that $(Y^n_\cdot, Z^n_\cdot) \in \mathcal{S}^p \times \mathcal{M}^p$ satisfies the following penalization BSDE:

$$Y^n_t = \xi + \int_t^T g(s, Y^n_s, Z^n_s)ds + \int_t^T dV_s + \int_t^T dK^n_s - \int_t^T Z^n_s dB_s, \quad t \in [0, T]$$  \hfill (16)

with

$$K^n_t := n \int_0^t (Y^n_s - L_s)^- ds, \quad t \in [0, T].$$  \hfill (17)

If $Y^n$ increases in $n$ and there exists a random variable $\eta \in \mathbb{L}^1(\mathcal{F}_T)$ such that for each $n \geq 1$ and $t \in [0, T]$,

$$\mathbb{E} \left[ \sup_{s \in [0, T]} |Y^n_s|^p + \left( \int_t^T |Z^n_s|^2 ds \right)^{\frac{p}{2}} + |K^n_T - K^n_t|^p + \left( \int_t^T |g(s, Y^n_s, Z^n_s)| ds \right)^{p} \right]_{\mathcal{F}_t} \leq \mathbb{E} [\eta | \mathcal{F}_t] \leq \mathbb{E} [\eta | \mathcal{F}_t]$$  \hfill (18)

then there exists a triple $(Y, Z, K, \cdot) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{V}^p$ which solves RBSDE (1),

$$\lim_{n \to \infty} (\|Y^n - Y\|_{\mathcal{S}^p} + \|Z^n - Z\|_{\mathcal{M}^p}) = 0,$$

and there exists a subsequence $\{K^{n_j}\}$ of $\{K^n\}$ such that

$$\lim_{j \to \infty} \sup_{t \in [0, T]} |K^{n_j}_t - K_t| = 0.$$

**Proof.** Since $Y^n$ increases in $n$, there exists a process $Y$ such that $\mathbb{P} - a.s., \ Y^n_t \uparrow Y_t$ for each $t \in [0, T]$. By Fatou’s lemma and (18) we can deduce that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^p \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} \liminf_{n \to \infty} |Y^n_t|^p \right] \leq \mathbb{E} \left[ \liminf_{n \to \infty} \sup_{t \in [0, T]} |Y^n_t|^p \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |Y^n_t|^p \right] \leq \mathbb{E} [\eta] < +\infty.$$  \hfill (19)

Furthermore, by (18) we can also get that

$$\sup_{n \geq 1} |Y^n_t| \leq (\mathbb{E} [\eta | \mathcal{F}_t])^{\frac{1}{p}}, \quad t \in [0, T]$$  \hfill (20)
and
\[
\sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T |Z^n_t|^2 dt \right)^{\frac{p}{2}} + |K^n_T|^p + \left( \int_0^T |g(t, Y^n_t, Z^n_t)| dt \right)^p \right] \leq \mathbb{E} |\eta| < +\infty. \tag{21}
\]

The rest proof is divided into 6 steps.

**Step 1.** We show that \( Y \) is a càdlàg process. For each integer \( l, q \geq 1 \), introduce the following two \((\mathcal{F}_t)\)-stopping times:

\[
\tau_l := \inf \left\{ t \geq 0 : \left( \mathbb{E} |\eta| \mathcal{F}_t \right)^{\frac{1}{p}} + \int_0^t \psi_s(l) ds + L_t \geq l \right\} \land T;
\]

\[
\sigma_{l,q} := \inf \left\{ t \geq 0 : \int_0^t \psi_s(l) ds \geq q \right\} \land \tau_l.
\]

Then we have, \( \tau_l \to T \) as \( l \to \infty \), \( \sigma_{l,q} \to \tau_l \) as \( q \to \infty \) for each \( l \geq 1 \),

\[
\mathbb{P} (\{ \omega : \exists l_0(\omega) \geq 1, \forall l \geq l_0(\omega), \tau_l(\omega) = T \}) = 1
\]

and

\[
\mathbb{P} (\{ \omega : \exists l_0(\omega), q_0(\omega) \geq 1, \forall l \geq l_0(\omega), \forall q \geq q_0(\omega), \sigma_{l,q}(\omega) = T \}) = 1. \tag{22}
\]

Now, let us arbitrarily fix a pair of \( l, q \geq 1 \). Since \( g \) satisfies (HH) with \( f_\cdot, \psi_\cdot \) and \( \lambda \), and (20) is satisfied, in view of the definitions of \( \tau \) and \( \sigma_{l,q} \), we know that
d\( \mathbb{P} \times dt \) - a.e., for each \( n \geq 1 \),

\[
|h_{n,l,q}^n| \leq 1_{\leq \tau_l} f_\cdot + 1_{\leq \sigma_{l,q}} \psi_\cdot(l) + \lambda |Z^n|
\]

with \( h_{n,l,q}^n := 1_{\leq \sigma_{l,q}} g_\cdot, Y^n, Z^n \),

\[
\mathbb{E} \left[ \int_0^T 1_{t \leq \tau_l} f_t dt \right] \leq l \quad \text{and} \quad \mathbb{E} \left[ \int_0^T 1_{t \leq \sigma_{l,q}} \psi_t(l) dt \right] \leq q, \tag{24}
\]

from which together with (21), we can deduce that there exists a subsequence \( \{h_{n,l,q}^j\}_{j=1} \) of the sequence \( \{h_{n,l,q}^n\}_{n=1} \) which converges weakly to a process \( h_{l,q}^\tau \) in \( \mathcal{H}^1 \). Now, take any bounded linear functional \( \Phi(\cdot) \) defined on \( \mathbb{L}^1 (\mathcal{F}_T) \). Then there exists a constant \( b > 0 \) such that for each \( \mathbb{F} \in \mathcal{H}^1 \) and every \((\mathcal{F}_t)\)-stopping time \( \bar{\tau} \) valued in \([0, T] \), we have

\[
|\Phi (\int_0^{\bar{\tau}} \mathbb{F}_s ds) | \leq b \| \int_0^{\bar{\tau}} \mathbb{F}_s ds \|_{\mathbb{L}^1} \leq b \| \mathbb{F} \|_{\mathcal{H}^1}.
\]

Hence, for each \((\mathcal{F}_t)\)-stopping time \( \bar{\tau} \) valued in \([0, T] \), \( \Phi(\int_0^{\bar{\tau}} \cdot ds) \) is a bounded linear functional defined on \( \mathcal{H}^1 \), which means that

\[
\lim_{j \to \infty} \Phi \left( \int_0^{\bar{\tau}} h_{n,l,q}^j ds \right) = \Phi \left( \int_0^{\bar{\tau}} h_{l,q}^\tau ds \right).
\]

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As a result, for every \((\mathcal{F}_t)\)-stopping time \(\tau\) with \(0 \leq \tau \leq \sigma_{t,q}\), as \(j \to \infty\),
\[
\int_0^\tau g(s, Y_s^{n_j}, Z_s^{n_j}) ds = \int_0^\tau h_s^{n_j;l,q} ds \to \int_0^\tau h_s^{l,q} ds \quad \text{weakly in } L^1(\mathcal{F}_T). 
\] (25)
Furthermore, it follows from (21) and Lemma 4.4 of Klimsiak [24] that there exists a process \(Z \in \mathbb{M}^p\) and a subsequence of the sequence \(\{n_j\}_{j=1}^\infty\), still denoted by itself, such that for every \((\mathcal{F}_t)\)-stopping time \(\tau\) valued in \([0, T]\), as \(j \to \infty\),
\[
\int_0^\tau Z_s^{n_j} dB_s \to \int_0^\tau Z_s dB_s \quad \text{weakly in } L^p(\mathcal{F}_T) \text{ and then in } L^1(\mathcal{F}_T). 
\] (26)

In the sequel, we define

\[
K_t^{l,q} := Y_0 - Y_t - \int_0^t h_s^{l,q} ds - \int_0^t dV_s + \int_0^t Z_s dB_s, \quad t \in [0, T].
\]

Then, in view of (25), (26) and the fact that for each \((\mathcal{F}_t)\)-stopping time \(\tau\) valued in \([0, T]\), \(Y_\tau^n \uparrow Y_\tau\) in \(L^1(\mathcal{F}_T)\), we can get that for every \((\mathcal{F}_t)\)-stopping time \(\tau\) such that \(0 \leq \tau \leq \sigma_{t,q}\), the sequence of random variables

\[
K_{\tau,j}^n = Y_0^j - Y_{\tau,j} - \int_0^\tau g(s, Y_s^{n,j}, Z_s^{n,j}) ds - \int_0^\tau dV_s + \int_0^\tau Z_s^{n,j} dB_s
\]
converges weakly to \(K_t^{l,q}\) in \(L^1(\mathcal{F}_T)\) as \(j \to \infty\). Consequently, since \(K^n \in \mathcal{V}^+\) for each \(n \geq 1\), we know that

\[
K_{\tau_1 \wedge \sigma_{t,q}}^{l,q} \leq K_{\tau_2 \wedge \sigma_{t,q}}^{l,q}
\]
for any \((\mathcal{F}_t)\)-stopping times \(\tau_1 \leq \tau_2\) valued in \([0, T]\), and in view of the definition of \(K_t^{l,q}\) and the facts that \(Y^n \uparrow Y\) and \(Y^n \in \mathcal{S}^p\) for each \(n \geq 1\), it is not hard to check that \(K_t^{l,q}\) is a \((\mathcal{F}_t)\)-optional process with \(\mathbb{P} - a.s.\) upper semi-continuous paths. Thus, Lemma A.3 in Bayraktar and Yao [3] yields that \(K_{\tau_1 \wedge \sigma_{t,q}}^{l,q}\) is a nondecreasing process, and then it has \(\mathbb{P} - a.s.\) right lower semi-continuous paths. Hence, \(K_{\tau_1 \wedge \sigma_{t,q}}^{l,q}\) is càdlàg and so is \(Y_{\tau \wedge \sigma_{t,q}}\) from the definition of \(K_t^{l,q}\). Finally, it follows from (22) that \(Y_t\) is also a càdlàg process.

**Step 2.** We show that \(Y_t \geq L_t\) for each \(t \in [0, T]\) and as \(n \to \infty\),
\[
\sup_{t \in [0, T]} (Y_t^n - L_t)^- \to 0. 
\] (27)
In fact, it follows from (21) and the definition of \(K^n\) that for each \(n \geq 1\),
\[
\mathbb{E} \left[ \left( \int_0^T (Y_t^n - L_t)^- dt \right)^p \right] \leq \frac{\mathbb{E}[\eta]}{n^p}.
\]
Hence, by Fatou’s lemma and Hölder’s inequality,
\[
\mathbb{E} \left[ \int_0^T (Y_t - L_t)^- dt \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \int_0^T (Y_t^n - L_t)^- dt \right] \leq \lim_{n \to \infty} \frac{(\mathbb{E}[\eta])^{\frac{1}{p}}}{n} = 0,
\]
which implies that
\[ \mathbb{E} \left[ \int_0^T (Y_t - L_t)^- \, dt \right] = 0. \]

Since \( Y - L \) is a càdlàg process, \((Y_t - L_t)^- = 0\) and hence \( Y_t \geq L_t \) for each \( t \in [0, T) \). Moreover, \( Y_T = Y_T^m = \xi \geq L_T \). Hence
\[ (Y_t^m - L_t)^- \downarrow 0 \]
for each \( t \in [0, T] \) and by Dini’s theorem, (27) follows.

**Step 3.** We show the convergence of the sequence \( \{Y^n\} \). Let \( \tau_n \) and \( \sigma_{t,q} \) be the sequences of \((\mathcal{F}_t)\)-stopping times defined in Step 1. For each \( n, m \geq 1 \), observe that
\[
(Y^n, \bar{Z}, \bar{V}) := (Y^n - Y^m, Z^n - Z^m, \int_0^\tau (g(s, Y^n_s, Z^n_s) - g(s, Y^m_s, Z^m_s)) \, ds + (K^n - K^m))
\]
satisfies equation (2). It then follows from (ii) of Lemma 1 with \( p = 2, t = 0 \) and \( \tau = \sigma_{t,q} \) that there exists a constant \( C > 0 \) such that for each \( n, m, l, q \geq 1 \),
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_{t \wedge \sigma_{t,q}} - Y^m_{t \wedge \sigma_{t,q}}|^2 \right] \leq C \mathbb{E} \left[ \left( \int_{t \wedge \sigma_{t,q}}^{\sigma_{t,q}} (Y^n - Y^m) (dK^n - dK^m) \right)^+ \right.
\]
\[ + \left. \int_0^{\sigma_{t,q}} |Y^n_t - Y^m_t| \, |g(t, Y^n_t, Z^n_t) - g(t, Y^m_t, Z^m_t)| \, dt \right]. \]

Furthermore, by virtue of the definition of \( K^n \) we know that for each \( t \in [0, T] \),
\[
\int_{t \wedge \sigma_{t,q}}^{\sigma_{t,q}} (Y^n_s - Y^m_s) \, (dK^n_s - dK^m_s)
\]
\[ = \int_{t \wedge \sigma_{t,q}}^{\sigma_{t,q}} [(Y^n_s - L_s) - (Y^m_s - L_s)] \, dK^n_s - \int_{t \wedge \sigma_{t,q}}^{\sigma_{t,q}} [(Y^n_s - L_s) - (Y^m_s - L_s)] \, dK^m_s \]
\[ \leq \int_{t \wedge \sigma_{t,q}}^{\sigma_{t,q}} (Y^n_s - L_s)^- \, dK^n_s + \int_{t \wedge \sigma_{t,q}}^{\sigma_{t,q}} (Y^m_s - L_s)^- \, dK^m_s \]
\[ \leq \sup_{t \in [0,T]} (Y^n_{t \wedge \sigma_{t,q}} - L_{t \wedge \sigma_{t,q}})^- \, |K^n_T| + \sup_{t \in [0,T]} (Y^m_{t \wedge \sigma_{t,q}} - L_{t \wedge \sigma_{t,q}})^- \, |K^m_T|. \]
Combining (23), (29) and (30) together with Hölder’s inequality yields that

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |Y^n_{t \land \sigma_{l,q}} - Y^m_{t \land \sigma_{l,q}}|^2 \right] \\
\leq \left( C \mathbb{E}\left[ |Y^n_{\sigma_{l,q}} - Y^m_{\sigma_{l,q}}|^2 \right] + 2 \int_0^T |Y^n_t - Y^m_t| \left( \mathbb{1}_{t \leq \tau_l} f_t + \mathbb{1}_{t \leq \sigma_{l,q}} \psi_t(l) \right) \, dt \right)^{\frac{1}{2}} \\
+ C \left( \mathbb{E}\left[ \sup_{t \in [0,T]} \left( |Y^n_{t \land \sigma_{l,q}} - L_{t \land \sigma_{l,q}}| \right)^\frac{p}{p-1} \right] \left( \mathbb{E}[K^n_T]^\frac{1}{p} \right) \right)^{\frac{1}{2}} \\
+ C \left( \mathbb{E}\left[ \sup_{t \in [0,T]} \left( |Y^n_{t \land \sigma_{l,q}} - L_{t \land \sigma_{l,q}} - Y^m_{t \land \sigma_{l,q}}| \right)^\frac{p}{p-1} \right] \left( \mathbb{E}[K^m_T]^\frac{1}{p} \right) \right)^{\frac{1}{2}} \\
+ 2C\lambda \left( \mathbb{E}\left[ \left( \int_0^{\sigma_{l,q}} |Y^n_t - Y^m_t|^2 \, dt \right)^\frac{p}{2(p-1)} \right] \left( \mathbb{E}\left[ \int_0^T (|Z^n_t| + |Z^m_t|)^2 \, dt \right] \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\]

(31)

Thus, note that \(Y^n_t \uparrow Y_t\) for each \(t \in [0,T]\). In view of the definitions of \(\tau_l\) and \(\sigma_{l,q}\), (20), (21), (24) and (27), by (31) and Lebesgue’s dominated convergence theorem we can deduce that for each \(l,q \geq 1\), as \(n,m \to \infty\),

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |Y^n_{t \land \sigma_{l,q}} - Y^m_{t \land \sigma_{l,q}}|^2 \right] \to 0,
\]

which implies that for each \(l,q \geq 1\), as \(n,m \to \infty\),

\[
\sup_{t \in [0,T]} |Y^n_{t \land \sigma_{l,q}} - Y^m_{t \land \sigma_{l,q}}| \to 0 \text{ in probability } \mathbb{P}.
\]

And, by (22) and the fact that \(Y^n\) increases in \(n\) we know that \(\mathbb{P} - a.s.\),

\[
\sup_{t \in [0,T]} |Y^n_t - Y_t| \to 0, \quad \text{as } n \to \infty.
\]

(32)

So, \(Y\) is a continuous process. Finally, note that \(|Y^n| \leq |Y^1| + |Y|\) for each \(n \geq 1\) and that (19) is satisfied. From (32) and Lebesgue’s dominated convergence theorem it follows that

\[
\lim_{n \to \infty} \|Y^n - Y\|_{L^p}^p = \lim_{n \to \infty} \mathbb{E}\left[ \sup_{t \in [0,T]} |Y^n_t - Y_t|^p \right] = 0.
\]

(33)

**Step 4.** We show the convergence of the sequence \(\{Z^n\}\). Note that (28) solves (2). It follows from (i) of Lemma 1 with \(t = 0\) and \(\tau = T\) that there exists a
constant $C' > 0$ such that for each $m, n \geq 1$,

\[
\mathbb{E} \left[ \left( \int_0^T |Z^n_t - Z^m_t|^2 dt \right)^{\frac{p}{2}} \right] \\
\leq C' \mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_t - Y^m_t|^p \right] + C' \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_t - Y^m_t|^p \right] \right)^{\frac{1}{2}} \left( \mathbb{E} [ |K^n_T|^p ] \right)^{\frac{1}{2}} \\
+ (\mathbb{E} [ |K^m_T|^p ])^{\frac{1}{2}} \left( \mathbb{E} \left[ \left( \int_0^T |g(t, Y^n_t, Z^n_t) - g(t, Y^m_t, Z^m_t)| dt \right)^p \right] \right)^{\frac{1}{2}},
\]

from which together with (21), (33) and (26) yields that

\[
\lim_{n \to \infty} \| Z^n - Z \|_{\mathbb{M}^p} = \lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^T |Z^n_t - Z^m_t|^2 dt \right)^{\frac{p}{2}} \right] = 0. \tag{34}
\]

**Step 5.** We show the convergence of the sequence $\{K^n\}$. Let $\tau_l$ and $\sigma_{l,q}$ be the sequences of $(\mathcal{F}_t)$-stopping times defined in Step 1. Since $g$ satisfies (H4), by (23), (24), (21), (32) and (34) we can deduce that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that for each $l, q \geq 1$,

\[
\lim_{j \to \infty} \int_0^{\sigma_{l,q}} |g(t, Y^n_{t^{n_j}}, Z^n_{t^{n_j}}) - g(t, Y_t, Z_t)| dt = 0.
\]

Then, in view of (22), we have

\[
\lim_{j \to \infty} \sup_{t \in [0,T]} \left| \int_0^t g(t, Y^n_{t^{n_j}}, Z^n_{t^{n_j}}) dt - \int_0^t g(t, Y_t, Z_t) dt \right| = 0. \tag{35}
\]

Combining (32), (34) and (35) yields that $\mathbb{P} - a.s.,$ for each $t \in [0,T],$

\[
K^{n_j}_t = Y^{n_j}_0 - Y^{n_j}_t - \int_0^t g(s, Y^{n_j}_s, Z^{n_j}_s) ds - \int_0^t dV_s + \int_0^t Z^{n_j}_s dB_s
\]

tends to

\[
K_t := Y_0 - Y_t - \int_0^t g(s, Y_s, Z_s) ds - \int_0^t dV_s + \int_0^t Z_s dB_s
\]
as \( j \to \infty \) and that
\[
\lim_{j \to \infty} \sup_{t \in [0, T]} |K_t^{n_j} - K_t| = 0. \tag{36}
\]

Hence, \( K \) is a continuous process.

**Step 6.** We show that \( K \in \mathcal{V}^{+p} \) and \((Y, Z, K) \in \mathcal{S}^p \times M^p \times \mathcal{V}^{+p}\) is a solution of RBSDE (1). In fact, by Fatou’s lemma with (36) and (21) we get that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |K_t| \right] = \mathbb{E} \left[ \lim_{j \to \infty} \sup_{t \in [0, T]} |K_t^{n_j}| \right] \leq \liminf_{j \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |K_t^{n_j}| \right] \\
\leq \sup_{j \geq 1} \mathbb{E} \left[ |K_T| \right] \leq \mathbb{E} [\eta] < +\infty.
\]

Hence, \( K \in \mathcal{V}^{+p} \) and \((Y, Z, K) \in \mathcal{S}^p \times M^p \times \mathcal{V}^{+p}\) solves
\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0, T].
\]

By Step 2 we know that \( Y_t \geq L_t \) for each \( t \in [0, T] \), and then
\[
\int_0^T (Y_t - L_t) dK_t \geq 0.
\]

On the other hand, in view of (32) and (36), it follows from the definition of \( K_t^{n_j} \) that
\[
\int_0^T (Y_t - L_t) dK_t = \lim_{j \to \infty} \int_0^T (Y_t^{n_j} - L_t) dK_t^{n_j} \leq 0.
\]

Consequently, we have
\[
\int_0^T (Y_t - L_t) dK_t = 0.
\]

Thus, \((Y, Z, K) \in \mathcal{S}^p \times M^p \times \mathcal{V}^{+p}\) solves (1.1). Proposition 3 is then proved. \( \square \)

**Proposition 4** (Approximation) Assume that for each \( n \geq 1 \), the generator \( g_n \) satisfies (HH) with the same \( f, \psi, r \) and \( \lambda \). Let \( p > 1, V \in \mathcal{V}^p \) and (H5) be satisfied for \( \xi \) and \( L \). For \( n \geq 1 \), assume that \((Y^n, Z^n, K^n) \in \mathcal{S}^p \times M^p \times \mathcal{V}^{+p}\) is a solution of RBSDE \((\xi, g_n + dV, L)\). If \( Y^n \) increases or decreases in \( n \), \( g_n \) tends locally uniformly in \((y, z)\) to the generator \( g \) as \( n \to \infty \) and there exists a random variable \( \eta \in \mathbb{L}^1(\mathcal{F}_T) \) such that for each \( n \geq 1 \) and \( t \in [0, T] \),
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s^n|^p + \left( \int_t^T |Z_s^n|^2 ds \right)^{\frac{p}{2}} + |K_T^n - K_T|^p + \left( \int_t^T |g_n(s, Y_s^n, Z_s^n)| ds \right)^p \bigg\vert \mathcal{F}_t \right] \\
\leq \mathbb{E} [\eta \bigg\vert \mathcal{F}_t], \tag{37}
\]

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then there exists a triple \((Y, Z, K) \in S^p \times M^p \times Y^{+p}\) which solves RBSDE (1),
\[
\lim_{n \to \infty} (\|Y^n - Y\|_{S^p} + \|Z^n - Z\|_{M^p}) = 0,
\]
and there exists a subsequence \(\{K^{n_j}\}\) of \(\{K^n\}\) such that
\[
\lim_{j \to \infty} \sup_{t \in [0, T]} |K^{n_j}_t| = 0.
\]
Furthermore, if \(K^n\) increases or decreases in \(n\), then we have
\[
\lim_{n \to \infty} \|K^n - K\|_{S^p} = 0.
\]

**Proof.** Since \(Y^n\) increases or decreases in \(n\), there exists a process \(Y\) such that \(\mathbb{P}-a.s., Y^n_t \to Y_t\) for each \(t \in [0, T]\). In the same way as in the proof of Proposition 3, by Fatou’s lemma together with (37) we can deduce that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^p \right] < +\infty, \tag{38}
\]
and
\[
\sup_{n \geq 1} |Y^n_t| \leq (\mathbb{E} [\eta | \mathcal{F}_t])^{\frac{1}{p}}, \quad t \in [0, T] \tag{39}
\]
and
\[
\sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T \|Z^n_t\|^2 dt \right)^{\frac{p}{2}} + |K^n_T|^p + \left( \int_0^T |g_n(t, Y^n_t, Z^n_t)| dt \right)^p \right] \leq \mathbb{E} [\eta] < +\infty. \tag{40}
\]
For each positive integer \(l, q \geq 1\), as in the proof of Proposition 3, we introduce the following two \((\mathcal{F}_t)\)-stopping times:
\[
\tau_l := \inf \left\{ t \geq 0 : (\mathbb{E} [\eta | \mathcal{F}_t])^{\frac{1}{p}} + \int_0^t f_s ds \geq l \right\} \wedge T;
\]
\[
\sigma_{l,q} := \inf \left\{ t \geq 0 : \int_0^t \psi_l(s) ds \geq q \right\} \wedge \tau_l.
\]
Then we have
\[
\mathbb{P} \left( \{ \omega : \exists l_0(\omega), q_0(\omega) \geq 1, \forall l \geq l_0(\omega), \forall q \geq q_0(\omega), \sigma_{l,q}(\omega) = T \} \right) = 1. \tag{41}
\]
Furthermore, since all \(g_n\) satisfy (HH) with the same \(f, \psi(r)\) and \(\lambda\), and (39) is satisfied, in view of the definitions of \(\tau_l\) and \(\sigma_{l,q}\), we know that \(d\mathbb{P} \times dt - a.e.,\) for each \(l, q, n \geq 1,
\[
1_{t \leq \sigma_{l,q}} |g_n(t, Y^n_t, Z^n_t)| \leq 1_{t \leq \tau_l} f_t + 1_{t \leq \sigma_{l,q}} \psi_l(t) + \lambda |Z^n_t| \tag{42}
\]
with
\[
\mathbb{E} \left[ \int_0^T 1_{t \leq \tau_l} f_t dt \right] \leq l \quad \text{and} \quad \mathbb{E} \left[ \int_0^T 1_{t \leq \sigma_{l,q}} \psi_l(t) dt \right] \leq q. \tag{43}
\]
The rest proof is divided into 4 steps.

**Step 1.** We show the convergence of the sequence \( \{ Y^m \} \). For each \( n, m \geq 1 \), observe that

\[
(\bar{Y}, \bar{Z}, \bar{V}) := (Y^n - Y^m, Z^n - Z^m, \quad (44)
\]

satisfies equation (2). It then follows from (ii) of Lemma 1 with \( p = 2 \), \( t = 0 \) and \( \tau = \sigma_{l,q} \) that there exists a constant \( C > 0 \) such that for each \( n, m, l, q \geq 1 \),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_{t \wedge \sigma_{l,q}} - Y^m_{t \wedge \sigma_{l,q}}|^2 \right]
\leq C \mathbb{E} \left[ |Y^n_{\sigma_{l,q}} - Y^m_{\sigma_{l,q}}|^2 + \sup_{t \in [0,T]} \left( \int_{t \wedge \sigma_{l,q}}^{\sigma_{l,q}} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \right)^+ \right. \\
+ \left. \int_0^{\sigma_{l,q}} |Y^n_t - Y^m_t| |g_n(t, Y^n_t, Z^n_t) - g_m(t, Y^m_t, Z^m_t)| dt \right].
\]

Furthermore, note that \( Y^n_t \geq L_t \) for each \( t \in [0, T] \) and \( n \geq 1 \) and that \( \int_0^T (Y^n_t - L_t) dK^n_t = 0 \) for each \( n \geq 1 \). It follows that for each \( t \in [0, T] \) and \( l, q, m, n \geq 1 \),

\[
\int_{t \wedge \sigma_{l,q}}^{\sigma_{l,q}} (Y^n_s - Y^m_s) (dK^n_s - dK^m_s) \\
= \int_{t \wedge \sigma_{l,q}}^{\sigma_{l,q}} [(Y^n_s - L_s) - (Y^m_s - L_s)] dK^n_s - \int_{t \wedge \sigma_{l,q}}^{\sigma_{l,q}} [(Y^n_s - L_s) - (Y^m_s - L_s)] dK^m_s \\
\leq \int_{t \wedge \sigma_{l,q}}^{\sigma_{l,q}} (Y^n_s - L_s) dK^n_s + \int_{t \wedge \sigma_{l,q}}^{\sigma_{l,q}} (Y^m_s - L_s) dK^m_s = 0.
\]

Combining (42), (45) and (46) together with Hölder’s inequality yields that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_{t \wedge \sigma_{l,q}} - Y^m_{t \wedge \sigma_{l,q}}|^2 \right]
\leq C \mathbb{E} \left[ |Y^n_{\sigma_{l,q}} - Y^m_{\sigma_{l,q}}|^2 + 2 \int_0^T |Y^n_t - Y^m_t| \left( \mathbb{1}_{t \leq \tau} f_t + \mathbb{1}_{t \leq \sigma_{l,q}} \psi_t(l) \right) dt \right] \\
+ 2C \lambda \left( \mathbb{E} \left[ \left( \int_0^{\sigma_{l,q}} |Y^n_t - Y^m_t|^2 dt \right)^{p/(2p-1)} \right] \right)^{2(p-1)\over p} \\
\times \left( \mathbb{E} \left[ \left( \int_0^T (|Z^n_t|^2 + |Z^m_t|^2) dt \right)^{p\over 2} \right] \right)^{1\over p}.
\]

Thus, note that \( Y^n_t \to Y_t \) for each \( t \in [0, T] \). In view of the definitions of \( \tau_l \) and \( \sigma_{l,q} \), (39), (40) and (43), by (47) and Lebesgue’s dominated convergence theorem.
we can deduce that for each $l, q \geq 1$, as $n, m \to \infty$,

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_{l \wedge \sigma_{l,q}} - Y^m_{l \wedge \sigma_{l,q}}|^2 \right] \to 0,$$

which implies that for each $l, q \geq 1$, as $n, m \to \infty$,

$$\sup_{t \in [0,T]} |Y^n_{l \wedge \sigma_{l,q}} - Y^m_{l \wedge \sigma_{l,q}}| \to 0 \text{ in probability } \mathbb{P}.$$ 

And, by (41) and the monotonicity of $Y^\cdot$ with respect to $n$ we know that $\mathbb{P} - a.s.$,

$$\sup_{t \in [0,T]} |Y^n_t - Y^m_t| \to 0, \text{ as } n \to \infty.$$ (48)

So, $Y^\cdot$ is a continuous process. Finally, note that $|Y^n| \leq |Y^1| + |Y|$ for each $n \geq 1$ and that (38) is satisfied. From (48) and Lebesgue’s dominated convergence theorem it follows that

$$\lim_{n \to \infty} \|Y^n - Y\|_{\mathcal{S}^p} = \lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_t - Y^m_t|^p \right] = 0.$$ (49)

**Step 2.** We show the convergence of the sequence $\{Z^n\}$. Note that (44) solves (2). It follows from (i) of Lemma 1 with $t = 0$ and $\tau = T$ that there exists a constant $C' > 0$ such that for each $m, n \geq 1$,

$$\mathbb{E} \left[ \left( \int_0^T |Z^n_t - Z^m_t|^2 dt \right)^{\frac{p}{2}} \right] \leq C' \mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_t - Y^m_t|^p \right] + C' \left( \mathbb{E} \left[ \left( \int_0^T (Y^n_t - Y^m_t)(dK^n_t - dK^m_t) \right)^p \right] \right)^{\frac{1}{p}}$$

$$+ C' \left( \int_0^T |Y^n_t - Y^m_t| |g_n(t, Y^n_t, Z^n_t) - g_m(t, Y^m_t, Z^m_t)| dt \right)^{\frac{p}{2}}.$$ 

Then, in view of (46), it follows from Hölder’s inequality that for each $m, n \geq 1$,

$$\mathbb{E} \left[ \left( \int_0^T |Z^n_t - Z^m_t|^2 dt \right)^{\frac{p}{2}} \right] \leq C'' \mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n_t - Y^m_t|^p \right] + C'' \left( \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |Y^n_t - Y^m_t|^p \right) \right]^{\frac{1}{p}} \right)^{\frac{1}{p}}$$

$$\cdot \left( \mathbb{E} \left[ \left( \int_0^T (|g_n(t, Y^n_t, Z^n_t) - g_m(t, Y^m_t, Z^m_t)| dt \right)^p \right] \right)^{\frac{1}{p}},$$

from which together with (49) and (40) yields that there exists a process $Z^\cdot \in \mathcal{M}^p$ such that

$$\lim_{n \to \infty} \|Z^n - Z\|_{\mathcal{M}^p} = \lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^T |Z^n_t - Z^m_t|^2 dt \right)^{\frac{p}{2}} \right] = 0.$$ (50)
Step 3. We show the convergence of the sequence \( \{K^n\} \). Since \( g_n \) tends locally uniformly in \((y, z)\) to the generator \( g \) as \( n \to +\infty \), by (48), (50), (40), (42) and (43) we can deduce that there exists a subsequence \( \{n_j\} \) of \( \{n\} \) such that for each \( l, q \geq 1, \)

\[
\lim_{j \to \infty} \int_0^{t_{l,q}} |g_{n_j}(t, Y^n_{t_j}, Z^n_{t_j}) - g(t, Y_t, Z_t)|dt = 0.
\]

Then, in view of (41), we have

\[
\lim \sup_{j \to \infty} \sup_{t \in [0, T]} \left| \int_0^t g_{n_j}(t, Y^n_{t_j}, Z^n_{t_j})dt - \int_0^t g(t, Y_t, Z_t)dt \right| = 0. \tag{51}
\]

Combining (48), (50) and (51) yields that for each \( t \in [0, T], \)

\[
K^n_t = Y^n_0 - Y^n_t - \int_0^t g_{n_j}(s, Y^n_{s_j}, Z^n_{s_j})ds - \int_0^t dV_s + \int_0^t Z^n_{s_j}dB_s
\]

tends to

\[
K_t := Y_0 - Y_t - \int_0^t g(s, Y_s, Z_s)ds - \int_0^t dV_s + \int_0^t Z_sdB_s
\]
as \( j \to \infty \) and that

\[
\lim \sup_{j \to \infty} \sup_{t \in [0, T]} |K^n_t - K_t| = 0. \tag{52}
\]

Hence, \( K_t \) is a continuous process.

Step 4. We show that \( K_t \in \mathcal{V}^{+p} \) and \((Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{V}^{+p}\) is a solution of RBSDE (1). In fact, by Fatou’s lemma with (52) and (40) we get that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |K_t|^p \right] = \mathbb{E} \left[ \lim_{j \to \infty} \sup_{t \in [0, T]} |K^n_t|^p \right] \leq \liminf_{j \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |K^n_t|^p \right] \leq \sup_{j \geq 1} \mathbb{E} \left[ |K^n_T|^p \right] \leq \mathbb{E}[\eta] < +\infty.
\]

Hence, \( K_t \in \mathcal{V}^{+p} \) and \((Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{V}^{+p}\) solves

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_sdB_s, \quad t \in [0, T].
\]

Since \( Y^n_t \geq L_t, \ n \geq 1 \) and \( Y^n_t \to Y_t \) for each \( t \in [0, T], \) we have \( Y_t \geq L_t \) for each \( t \in [0, T]. \) Furthermore, in view of (48) and (52), it follows that

\[
\int_0^T (Y_t - L_t)dK_t = \lim_{j \to \infty} \int_0^T (Y^n_{t_j} - L_t)dK^n_{t_j} = 0.
\]

So, \((Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{V}^{+p}\) solves RBSDE (1).
Finally, if $K^n$ increases or decrease in $n$, then $\mathbb{P} - a.s.$, for each $t \in [0, T]$, $K^n_t \to K_t$ as $n \to \infty$ and $|K^n_t| \leq |K^1_t| + |K_t|$. Thus, it follows from Dini’s theorem and Lebesgue’s dominated convergence theorem that
\[
\lim_{n \to \infty} \|K^n - K\|_{\mathbb{P}} = 0.
\]
Proposition 4 is then proved.

**Remark 4** In the case when $L = -\infty$ and $K^n = 0$ for each $n \geq 1$, by Proposition 4 we can get the approximation result for $L^p$ solutions of non-reflected BSDEs.

**Proposition 5** (Comparison theorem) Let $p > 1$, $V^i \in \mathcal{V}^p$, $g^i$ be a generator, $\xi^i$ and $L^i$ satisfy (H5), and $(Y^i, Z^i, K^i) \in \mathbb{S}^p \times \mathcal{M}^p \times \mathcal{V}^{+p}$ be a solution of RBSDE $(\xi^i, g^i + dV^i, L^i)$ for each $i = 1, 2$. If $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$, $L^1 \leq L^2$, and either
\[
\begin{align*}
&\left\{ \begin{array}{ll} 
&g^1 \text{ satisfies (H1) and (H2);} \\
&d\mathbb{P} \times dt - a.e., \quad \mathbbm{1}_{\{Y^1_t > Y^2_t\}}(g^1(t, Y^1_t, Z^1_t) - g^2(t, Y^2_t, Z^2_t)) \leq 0
\end{array} \right.
\end{align*}
\]
or
\[
\begin{align*}
&\left\{ \begin{array}{ll} 
&g^2 \text{ satisfies (H1) and (H2);} \\
&d\mathbb{P} \times dt - a.e., \quad \mathbbm{1}_{\{Y^1_t > Y^2_t\}}(g^1(t, Y^1_t, Z^1_t) - g^2(t, Y^1_t, Z^1_t)) \leq 0
\end{array} \right.
\end{align*}
\]
is satisfied, then $\mathbb{P} - a.s., Y^1_t \leq Y^2_t$ for each $t \in [0, T]$.

**Proof.** By Itô-Tanaka’s formula we know that for each $t \in [0, T],$
\[
(Y^1_t - Y^2_t)^+ \leq (\xi^1 - \xi^2)^+ + \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+)(dV^1_s - dV^2_s)
\]
\[
+ \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+)(g^1(s, Y^1_s, Z^1_s) - g^2(s, Y^2_s, Z^2_s)) \, ds
\]
\[
+ \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+)(dK^1_s - dK^2_s)
\]
\[
+ \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+)(Z^1_s - Z^2_s) \, dB_s.
\]
Since $L^1_t \leq L^2_t \leq Y^2_t, L^1_t \leq Y^1_t, t \in [0, T]$ and $\int_0^T (Y^1_s - L^1_s) \, dK^1_s = 0$, we have
\[
\int_t^T \text{sgn}((Y^1_s - Y^2_s)^+)(dK^1_s - dK^2_s)
\]
\[
\leq \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+) \, dK^1_s \leq \int_t^T \text{sgn}((Y^1_s - L^1_s)^+) \, dK^1_s
\]
\[
= \int_t^T \mathbbm{1}_{\{Y^1_t > L^1_t\}}|Y^1_s - L^1_s|^{-1}(Y^1_s - L^1_s) \, dK^1_s = 0.
\]
Thus, noticing that $\xi^1 \leq \xi^2$ and $dV^1_t \leq dV^2_t$ for each $t \in [0, T]$, we can get that

$$(Y^1_t - Y^2_t)^+ \leq \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+) \left( g^1(s, Y^1_s, Z^1_s) - g^2(s, Y^2_s, Z^2_s) \right) ds$$

$$+ \int_t^T \text{sgn}((Y^1_s - Y^2_s)^+) (Z^1_s - Z^2_s) dB_s, \quad t \in [0, T].$$

Now, in view of the assumptions of $g^1$ and $g^2$, the rest proof runs as the proof of Theorem 1 in Fan and Jiang [11]. The only difference lies in that in order to deal with the $L^p$ solution we need to use

$$\mathbb{E}[|XY|] \leq \left( \mathbb{E}[|X|^p] \right)^{\frac{1}{p}} \left( \mathbb{E}\left[|Y|^\frac{p}{p-1}\right]\right)^{\frac{p-1}{p}}$$

instead of the inequality

$$\mathbb{E}[|XY|] \leq \left( \mathbb{E}[|X|^2] \right)^{\frac{1}{2}} \left( \mathbb{E}[|Y|^2] \right)^{\frac{1}{2}}$$

for any $\mathcal{F}_T$-measurable random variables $X$ and $Y$. \hfill \Box

From Proposition 5, the following corollary is immediate.

**Corollary 1** Let $p > 1$, $V^i \in \mathcal{V}^p$, $g^i$ be a generator, $\xi^i$ and $L^i$ satisfy (H5), and $(Y^i, Z^i, K^i) \in \mathcal{S}^p \times M^p \times \mathcal{V}^{+p}$ be a solution of RBSDE $(\xi^i, g^i + dV^i, L^i)$ for each $i = 1, 2$. If $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$, $L^1 \leq L^2$, $g^1$ or $g^2$ satisfies (H1) and (H2), and

$$d\mathbb{P} \times dt - a.e., \quad g^1(t, y, z) \leq g^2(t, y, z)$$

for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, then $\mathbb{P}$ - a.s., $Y^1_t \leq Y^2_t$ for each $t \in [0, T]$.

**Remark 5** In the case when $L^1 = L^2 = -\infty$ and $K^1 = K^2 = 0$, by Proposition 5 and Corollary 1 we can get the comparison result for $L^p$ solutions of non-reflected BSDEs. In addition, it should be noted that Proposition 5 and Corollary 1 improves greatly the corresponding results obtained in El Karoui, Peng and Quenez [8], Fan and Jiang [11], Hamadène and Popier [15], Klìmsìak [24], Lepeltier, Matoussi and Xu [27], Rozkosz and Słomiński [37] and etc.

## 5. Existence, uniqueness and approximation results

In this section, based on the results obtained in previous sections, we will establish some existence, uniqueness and approximation results on $L^p$ solutions of BSDEs and RBSDEs with $L^p$ ($p > 1$) data under weaker assumptions, which answers those questions put forward in the Introduction.

### 5.1. Non-reflected BSDEs

Let us start with the following existence and uniqueness result—Proposition 6. It improves Corollary 2 of Fan [9] in the one-dimensional case, where $V \equiv 0$ and the $\varphi(r)$ in (H3) is assumed to be in $\mathcal{H}^1$. 

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Proposition 6 Let \( p > 1 \), \( V \in \mathcal{V}^p \) and let \( g \) satisfy assumptions (H1), (H2s), (H3) and (H4w). Then for each \( \xi \in \mathcal{L}^p(\mathcal{F}_T) \), the following BSDE, denoted by BSDE \((\xi, g + dV)\) here and hereafter,

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T dV_s - \int_t^T Z_s dB_s, \quad t \in [0, T]
\]

admits a unique solution in \( \mathcal{S}^p \times M^p \).

Proof. Note that (H2s)\( \Rightarrow \) (H2). The uniqueness part follows immediately from Proposition 5 or Corollary 1 in view of Remark 5. In the sequel, we prove the existence part. Let \( p > 1, \xi \in \mathcal{L}^p(\mathcal{F}_T), V \in \mathcal{V}^p \) and let the generator \( g \) satisfy (H1) with \( \rho(\cdot), (H2s) \) with \( \lambda, (H3) \) with \( \varphi(r) \) and (H4w).

We first assume that \( g \) is bounded. It then follows from Corollary 2 in Fan [9] that the following BSDE

\[
\tilde{Y}_s = \xi + V_T + \int_t^T g(s, \tilde{Y}_s - V_s, \tilde{Z}_s)ds - \int_t^T \tilde{Z}_s dB_s, \quad t \in [0, T]
\]

admits a unique solution \((\tilde{Y}, \tilde{Z}) \in \mathcal{S}^p \times M^p\). Then the pair \((Y, Z) := (\tilde{Y} - V, \tilde{Z})\) is the unique solution of BSDE \((\xi, g + dV)\) in \( \mathcal{S}^p \times M^p \).

Now suppose that \( g \) is bounded from below. Write \( g_n = g \land n \). Then \( g_n \) is bounded, nondecreasing in \( n \) and tends locally uniformly to \( g \) as \( n \to \infty \), and it is not difficult to check that all \( g_n \) satisfy (H1), (H2s) and (H3) with the same \( \rho(\cdot), \lambda \) and \( \varphi(r) \) as well as (H4w). Then by the first step of the proof there exists a unique solution \((Y^n, Z^n)\) of BSDE \((\xi, g_n + dV)\). By Corollary 1 together with Remark 5, \( Y^n \) increases in \( n \). Furthermore, \( d\mathbb{P} \times dt \) a.e., for each \( n \geq 1 \) and \((y, z) \in \mathbb{R} \times \mathbb{R}^d\),

\[
|g_n(\cdot, y, z)| \leq |g_n(\cdot, y, z) - g_n(\cdot, y, 0)| + |g_n(\cdot, y, 0) - g_n(\cdot, 0, 0)| + |g_n(\cdot, 0, 0)|
\]

\[
\leq \lambda|z| + \varphi(|y|) + |g(\cdot, 0, 0)|,
\]

which means that all \( g_n \) satisfy (HH) with the same

\[
f := |g(\cdot, 0, 0)|, \quad \psi(r) := \varphi(r) \quad \text{and} \quad \lambda.
\]

It then follows from Proposition 1 that (37) in Proposition 4 holds with

\[
\eta := C \left[ |\xi|^p + |V|^p_T + \left( \int_0^T |g(t, 0, 0)|dt \right)^p + 1 \right],
\]

where \( K^n \equiv 0 \) and \( C > 0 \) is a constant depending only \( p, \lambda, A, T \). Thus, we have checked all the conditions in Proposition 4, and then in view of Remark 4, it follows from Proposition 4 that BSDE \((\xi, g + dV)\) admits a solution in \( \mathcal{S}^p \times M^p \).

Finally, in the general case, we can approximate \( g \) by the sequence \( g_n \), where \( g_n := g \lor (-n), \quad n \geq 1 \). By the previous step there exists a unique solution
$(Y^n, Z^n) \in \mathcal{S}^p \times M^p$ of BSDE $(\xi, g_n + dV)$ for each $n \geq 1$. Repeating arguments in the proof of the previous step yields that $(Y^n, Z^n)$ converges in $\mathcal{S}^p \times M^p$ to the unique solution of BSDE $(\xi, g + dV)$.

By Propositions 1 and 4–6, we can prove the following Theorems 1 and 2, which further extend Proposition 6 to the case of BSDEs with generators satisfying the weaker assumptions (H2) or (H2w) than (H2s). They generalize respectively Theorems 2–3 in Fan and Jiang [11] and Theorem 4.1 in Briand, Lepeltier and San Martin [5], where some stronger assumptions are assumed to be satisfied.

**Theorem 1** Let $p > 1$, $V \in \mathcal{V}^p$ and let $g$ satisfy (H1), (H2), (H3) and (H4w). Then for each $\xi \in L^p(\mathcal{F}_T)$, BSDE $(\xi, g + dV)$ admits a unique solution in $\mathcal{S}^p \times M^p$.

**Proof.** The uniqueness part is a direct consequence of Proposition 5 in view of Remark 5.

Now, we prove the existence part. Firstly, let $p > 1$, $\xi \in L^p(\mathcal{F}_T)$, $V \in \mathcal{V}^p$ and let the generator $g$ satisfy assumptions (H1) with $\rho(\cdot)$, (H2) with $\phi(\cdot)$, (H3) with $\varphi(r)$ and (H4w). By a similar argument to that in the proof of the existence part of Theorem 1 of Ma, Fan and Song [29] we can prove that for each $n \geq 1$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the following function

$$g_n(\omega, t, y, z) := \inf_{u \in \mathbb{R}^d} [g(\omega, t, y, u) + (n + 2A)|u - z|]$$

is well defined and $(\mathcal{F}_t)$-progressively measurable, $d\mathbb{P} \times dt - a.e.$, $g_n$ increases in $n$ and converges uniformly in $(y, z)$ to the generator $g$,

$$\sup_{n \geq 1} |g_n(\cdot, 0, 0)| \leq |g(\cdot, 0, 0)| + \phi(2A)$$

and all $g_n$ satisfy (H1) with the same $\rho(\cdot)$, (H2) with the same $\phi(\cdot)$, (H3) with the same $\varphi(r) + 2\phi(2A)$, (H4) and (H2s) with $\lambda := n + 2A$. It then follows from Proposition 6 that there exists a unique solution $(Y^n, Z^n) \in \mathcal{S}^p \times M^p$ of BSDE $(\xi, g_n + dV)$ for each $n \geq 1$. By Corollary 1 together with Remark 5, $Y^n$ increases in $n$. Furthermore, it follows from Remark 1 that $d\mathbb{P} \times dt - a.e.$, for each $n \geq 1$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|g_n(\cdot, y, z)| \leq |g_n(\cdot, y, z) - g_n(\cdot, y, 0)| + |g_n(\cdot, y, 0) - g_n(\cdot, 0, 0)| + |g_n(\cdot, 0, 0)|$$

$$\leq A|z| + A + \varphi(|y|) + 2\phi(2A) + |g(\cdot, 0, 0)| + \phi(2A),$$

which means that all $g_n$ satisfy (HH) with the same

$$f := |g(\cdot, 0, 0)| + 3\phi(2A) + A, \quad \psi(r) := \varphi(r) \quad \text{and} \quad \lambda := A.$$

It then follows Proposition 1 that (37) in Proposition 4 holds with

$$\eta := C \left[ |\xi|^p + |V|^p_T + \left( \int_0^T |g(t, 0, 0)|^p \, dt \right)^{\frac{p}{p}} + 1 \right],$$
where $K^n \equiv 0$ and $C > 0$ is a constant depending only $p, A, T$. Thus, we have checked all the conditions in Proposition 4, and then in view of Remark 4, the conclusion of Theorem 1 follows from Proposition 4 immediately.

\textbf{Theorem 2} Let $p > 1$, $V, \in \mathcal{V}^p$ and let $g$ satisfy (H1), (HH) and (H4s). Then for each $\xi \in \mathbb{L}^p(\mathcal{F}_T)$, BSDE $\langle \xi, g + dV \rangle$ admits a maximal (resp. minimal) solution $(Y, Z)$ in $\mathcal{S}^p \times \mathcal{M}^p$, i.e., if $(Y'_t, Z'_t)$ is also a solution of BSDE $\langle \xi, g + dV \rangle$ in $\mathcal{S}^p \times \mathcal{M}^p$, then $\mathbb{P} - a.s., Y_t \geq Y'_t$ (resp. $Y_t \leq Y'_t$) for each $t \in [0, T]$.

\textbf{Proof.} We only prove the case of the maximal solution. In the same way, we can prove another case.

Assume that $p > 1$, $\xi \in \mathbb{L}^p(\mathcal{F}_T)$, $V, \in \mathcal{V}^p$ and let the generator $g$ satisfy assumptions (H1) with $\rho(\cdot)$, (HH) with $f, \varphi(r)$ and $\lambda$ as well as (H4s). It is not very hard to prove that for each $n \geq 1$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the following function

$$g_n(\omega, t, y, z) := \sup_{u \in \mathbb{R}^d} [g(\omega, t, y, u) - (n + 2\lambda)|u - z|]$$

is well defined and $(\mathcal{F}_t)$-progressively measurable, $d\mathbb{P} \times dt - a.e., g_n$ decreases in $n$ and converges locally uniformly in $(y, z)$ to the generator $g$ as $n \to \infty$, and all $g_n$ satisfy (H1) with the same $\rho(\cdot)$, (HH) with the same $f, \varphi(r)$ and $\lambda$, (H4) and (H2s) with $\lambda := n + 2\lambda$. Note by (i) and (ii) of Remark 2 that (HH)$\Rightarrow$(H3) and (H4)$\Rightarrow$(H4w). It then follows from Proposition 6 that there exists a unique solution $(Y^n, Z^n) \in \mathcal{S}^p \times \mathcal{M}^p$ of BSDE $\langle \xi, g_n + dV \rangle$ for each $n \geq 1$. By Corollary 1 together with Remark 5, $Y^n$ decreases in $n$. Furthermore, it follows from Proposition 1 that (37) in Proposition 4 holds true with

$$\eta := C \left[ |\xi|^p + |V|^p_T + \left( \int_0^T f_t dt \right)^p + 1 \right],$$

where $K^n \equiv 0$ and $C > 0$ is a constant depending only $p, A, T$. Thus, we have checked all the conditions in Proposition 4, and then in view of Remark 4, it follows from Proposition 4 that BSDE $\langle \xi, g + dV \rangle$ admits a solution $(Y, Z)$ in $\mathcal{S}^p \times \mathcal{M}^p$ such that

$$\lim_{n \to \infty} (\|Y^n - Y\|_{\mathcal{S}^p} + \|Z^n - Z\|_{\mathcal{M}^p}) = 0. \quad (54)$$

Finally, we show that $(Y, Z)$ is just the maximal solution of BSDE $\langle \xi, g + dV \rangle$ in $\mathcal{S}^p \times \mathcal{M}^p$. In fact, if $(Y'_t, Z'_t)$ is also a solution of BSDE $\langle \xi, g + dV \rangle$ in $\mathcal{S}^p \times \mathcal{M}^p$, then noticing that for each $n \geq 1$, $g_n \geq g$ and $g_n$ satisfies (H1) and (H2) due to (H2s)$\Rightarrow$(H2), it follows from Corollary 1 together with Remark 5 that $Y^n_t \geq Y'_t$ for each $t \in [0, T]$ and $n \geq 1$. Thus, by (54) we know that $\mathbb{P} - a.s., Y_t \geq Y'_t$ for each $t \in [0, T]$. Theorem 2 is then proved.

In view of (ii) of Remark 2, the following corollary follows from Theorem 2 immediately.

\textbf{Corollary 2} Let $p > 1$, $V, \in \mathcal{V}^p$ and let $g$ satisfy (H1), (H2w), (H3) and (H4s). Then for each $\xi \in \mathbb{L}^p(\mathcal{F}_T)$, BSDE $\langle \xi, g + dV \rangle$ admits a maximal (resp. minimal) solution $(Y, Z)$ in $\mathcal{S}^p \times \mathcal{M}^p$. 

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By Corollary 1 together with Remark 5 and the proof of Theorem 2 it is easy to verify that under assumptions (H1), (HH) and (H4s), the comparison theorem for the maximal (resp. minimal) solutions of BSDEs holds. More precisely, we have

**Corollary 3** Let \( p > 1 \) and for \( i = 1, 2 \), assume that \( \xi_i \in L^p(\mathcal{F}_T), V_i \in \mathcal{V}^p, g^i \) satisfies (H1), (HH) and (H4s), and that \((Y_1^i, Z_1^i) \in S^p \times M^p\) is the maximal (resp. minimal) solution of BSDE \((\xi_i, g_i^* + dV_i)\). If \( \xi_1 \leq \xi_2, dV_1 \leq dV_2 \), and \( dP \times dt - a.e., g^1(t, y, z) \leq g^2(t, y, z) \) for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), then \( P - a.s., Y_1^t \leq Y_2^t \) for each \( t \in [0, T] \).

5.2. Reflected BSDEs

The following theorem shows that under conditions of (H1), (H2w) and (H5) with \( g(\cdot, 0, 0) \in \mathcal{H}^p \), (H6) is necessary to ensure the existence of \( L^p \) solutions for RBSDEs with \( L^p (p > 1) \) data, which is one of our main results.

**Theorem 3** Assume that \( p > 1, V \in \mathcal{V}^p \), the generator \( g \) satisfies (H1) and (H2w) with \( g(\cdot, 0, 0) \in \mathcal{H}^p \), and that (H5) holds for \( \xi \) and \( L \). If RBSDE (1) admits a solution \((Y, Z, K) \in S^p \times M^p \times \mathcal{V}^+\), then \( g(\cdot, Y, 0) \in \mathcal{H}^p \). So (H6) holds.

**Proof.** By (iv) of Remark 2 we know that \( g \) satisfies (A) with \( \bar{f} := |g(\cdot, 0, 0)| + f + A, \bar{\mu} := \mu + A \) and \( \bar{\lambda} := \lambda \). It then follows from (4) in Lemma 3 that

\[
E \left[ \left( \int_0^T |g(t, Y_t, Z_t)| dt \right)^p \right] < +\infty.
\]

Then, by (H2w) together with Hölder’s inequality we can deduce that

\[
E \left[ \left( \int_0^T |g(t, Y_t, 0)| dt \right)^p \right] \\
\leq 4^p E \left[ \left( \int_0^T |g(t, Y_t, Z_t)| dt \right)^p \right] + 4^p E \left[ \left( \int_0^T f_t dt \right)^p \right] \\
+(4\mu T)^p E \left[ \sup_{t \in [0,T]} |Y_t|^p \right] + (4\lambda)^p E \left[ \left( \int_0^T |Z_t|^2 dt \right)^{\frac{p}{2}} \right] \\
< +\infty.
\]

Thus, Theorem 3 is proved. \( \square \)

The following Proposition 7 establishes an a priori estimate on the \( L^p \) solution of the penalization equation of RBSDE \((\xi, g + dV, L)\) with \( L^p (p > 1) \) data under the assumptions of (H1), (H2) or (H2w), (H3), (H5) and (H6), which will play an important role in the proofs of the following Theorems 4 and 5. The proof is based on Propositions 1-2 and 5, Theorems 1-2, and Corollaries 1-3.
Proposition 7 Let \( p > 1, V \in \mathcal{V}^p \) and let \( g \) satisfy (H1), (H2w) (resp. (H2)), (H3) and (H4s) (resp. (H4w)). Assume that (H5) and (H6) hold for \( \xi, L \) and some \( X \). For each \( n \geq 1 \), let \( (Y^n, Z^n) \in S^p \times M^p \) be the maximal or minimal (resp. unique) solution of the penalization equation (16) with (17) (Recall Corollary 2 (resp. Theorem 1)). Then, (18) appearing in Proposition 3 holds true.

Proof. We only prove the case that (H2w) and (H4s) are satisfied and \( (Y^n, Z^n) \) is the maximal solution. In the same way, we can prove other cases.

We first show that (C) appearing in Proposition 2 holds true for \( X^n \) and \( g \). In fact, since \( X \in M^p + \mathcal{V}^p \) and the Brownian filtration has the representation property, there exist \( H \in M^p \) and \( C \in \mathcal{V}^p \) such that

\[
X_t = X_T - \int_t^T dC_s - \int_t^T H_s dB_s, \quad t \in [0, T].
\]

(55)

It follows from (H2w) that \( d\mathbb{P} \times dt - a.e., \)

\[
|g(\cdot, X, H)| \leq |g(\cdot, X, 0)| + f + \mu|X| + \lambda|H|,
\]

from which together with (H6) we know that \( g(\cdot, X, H) \in \mathcal{H}^p \). Then, the equation (55) can be rewritten in the form

\[
X_t = X_T + \int_t^T g(s, X_s, H_s)ds + \int_t^T dV_s - \int_t^T (g^+(s, X_s, H_s)ds + dC^+_s + dV^+_s)
\]

\[
+ \int_t^T (g^-(s, X_s, H_s)ds + dC^-_s + dV^-_s) - \int_t^T H_s dB_s, \quad t \in [0, T].
\]

On the other hand, it follows from Corollary 2 that there exists a maximal solution \( (\bar{X}, \bar{Z}) \in S^p \times M^p \) of the BSDE

\[
\bar{X}_t = X_T \vee \xi + \int_t^T g(s, \bar{X}_s, \bar{Z}_s)ds + \int_t^T dV_s
\]

\[
+ \int_t^T (g^-(s, X_s, H_s)ds + dC^-_s + dV^-_s) - \int_t^T \bar{Z}_s dB_s, \quad t \in [0, T].
\]

Note that \( g \) satisfies (HH) by (ii) of Remark 2. It then follows from Proposition 1 with \( g_n \equiv g \) that

\[
g(\cdot, \bar{X}, \bar{Z}) \in \mathcal{H}^p,
\]

and by (H2w) and Remark 1 together with Hölder’s inequality,

\[
g(\cdot, \bar{X}, 0) \in \mathcal{H}^p.
\]

Furthermore, it follows from Corollary 3 together with (H6) that \( L_t \leq X_t \leq \bar{X}_t \) for each \( t \in [0, T] \). Therefore, we have

\[
\bar{X}_t = X_T \vee \xi + \int_t^T g(s, \bar{X}_s, \bar{Z}_s)ds + \int_t^T dV_s + \int_t^T (\bar{X}_s - L_s)^- ds
\]

\[
+ \int_t^T (g^-(s, X_s, H_s)ds + dC^-_s + dV^-_s) - \int_t^T \bar{Z}_s dB_s, \quad t \in [0, T].
\]
Thus, by Corollary 3 again we know that $Y^n_t \leq \bar{X}_t$ for each $t \in [0, T]$ and $n \geq 1$. That is to say, (C) holds true for $Y^n$ and $g$.

Thus, we have verified that all conditions in Proposition 2 are satisfied with $g_n \equiv g$. It then follows from Proposition 2 that (15) holds true. Furthermore, it follows from Corollary 3 that $Y^n$ increases in $n$. Then we have

$$|Y^n| \leq |Y^1| + |\bar{X}| \in S^p. \tag{56}$$

By combining (15) and (56) we can deduce that (18) holds true with

$$\eta := C \left[ \sup_{t \in [0,T]} |Y^1_t|^p + |V|^p + \sup_{t \in [0,T]} |\bar{X}_t|^p + \left( \int_0^T f_t dt \right)^p + 1 \\
+ \left( \int_0^T |g(t, \bar{X}_t, 0)| dt \right)^p + \left( \int_0^T |g(t, 0, 0)| dt \right)^p \right],$$

where $C$ is a nonnegative constant depending on $p, \mu, \lambda, A, T$. \hfill $\square$

Making use of Theorems 1-2, Propositions 3, 5 and 7 together with Corollaries 2-3, we can prove the following existence and uniqueness results (see Theorems 4-5) for $L^p$ solutions of RBSDE (1) with $L^p$ ($p > 1$) data under the assumptions of (H1), (H2) (resp. (H2w)), (H3), (H4w) (resp. (H4s)), (H5) and (H6), which also shows that the solutions can be approximated by the penalization method.

**Theorem 4** Let $p > 1$, $V \in \mathcal{V}^p$ and let $g$ satisfy (H1), (H2), (H3) and (H4w). Assume that (H5) and (H6) hold for $\xi, L$ and some $X$. For each $n \geq 1$, let $(Y^n, Z^n) \in S^p \times M^p$ be the unique solution of the penalization equation (16) with (17) (Recall Theorem 1). Then, there exists a triple $(Y, Z, K) \in S^p \times M^p \times \mathcal{V}^{+, p}$ such that

$$\lim_{n \to \infty} (\|Y^n - Y\|_{S^p} + \|Z^n - Z\|_{M^p} + \|K^n - K\|_{S^p}) = 0. \tag{57}$$

And, $(Y, Z, K)$ is the unique solution of RBSDE $(\xi, g + dV, L)$ in $S^p \times M^p \times \mathcal{V}^{+, p}$.

**Proof.** It follows from Proposition 5 or Corollary 1 that $Y^n$ increases in $n$. By (i) and (ii) of Remark 2, we know that (H4) holds true, and (H2) and (H3) can imply (HH). At the same time, it follows from Proposition 7 that (18) holds. Thus, we have checked all conditions in Proposition 3. It then follows from Proposition 3 that there exists a solution $(Y, Z, K) \in S^p \times M^p \times \mathcal{V}^{+, p}$ of RBSDE $(\xi, g + dV, L)$ such that

$$\lim_{n \to \infty} (\|Y^n - Y\|_{S^p} + \|Z^n - Z\|_{M^p}) = 0, \tag{58}$$

and there exists a subsequence $\{K^{n_j}\}$ of $\{K^n\}$ such that $\lim_{j \to \infty} \sup_{t \in [0,T]} |K^{n_j}_t - K_t| = 0$.

In the sequel, we prove that

$$\lim_{n \to \infty} \left\| \int_0^t g(s, Y^n_s, Z^n_s) ds - \int_0^t g(s, Y_s, Z_s) ds \right\|_{S^p} = 0. \tag{59}$$
In fact, it follows from (H2) that $d\mathbb{P} \times dt - a.e.$, for each $n \geq 1$,

$$\left| g(\cdot, Y^n, Z^n) - g(\cdot, Y, Z) \right|$$

$$\leq \left| g(\cdot, Y^n, Z^n) - g(\cdot, Y^n, Z) \right| + \left| g(\cdot, Y^n, Z) - g(\cdot, Y, Z) \right|$$

$$\leq \left| g(\cdot, Y^n, Z) - g(\cdot, Y, Z) \right| + \phi(\|Z^n - Z\|).$$

Thus, making use of the following basic inequality (see Fan and Jiang [10] for details)

$$\phi(x) \leq (m + 2A)x + \phi \left( \frac{2A}{m + 2A} \right), \quad \forall \, x \geq 0, \quad \forall m \geq 1$$

together with Hölder’s inequality, we get that for each $n, m \geq 1$,

$$\left\| \int_0^t g(s, Y^n_s, Z^n_s)ds - \int_0^t g(s, Y_s, Z_s)ds \right\|_{\mathcal{H}^p}$$

$$\leq \| g(\cdot, Y^n, Z^n) - g(\cdot, Y, Z) \|_{\mathcal{H}^p} \leq \| g(\cdot, Y^n, Z) - g(\cdot, Y, Z) \|_{\mathcal{H}^p}$$

$$+(m + 2A)T^{\frac{1}{p}}\|Z^n - Z\|_{\mathcal{H}^p} + \phi \left( \frac{2A}{m + 2A} \right)T.$$

Furthermore, note that $Y^1_t \leq Y^n_t \leq Y_t$ for each $n \geq 1$ and $t \in [0,T]$. It follows from (H1) and (H2) together with Remark 1 that $d\mathbb{P} \times dt - a.e.$, for each $n \geq 1$, in view of (18), (iv) of Remark 2, and (4) in Lemma 3,

$$\left| g(\cdot, Y^n, Z) \right| \leq \left| g(\cdot, Y^1, Z) \right| + \left| g(\cdot, Y, Z) \right| + 2A|Y - Y^1| + 2A$$

$$\leq \left| g(\cdot, Y^1, Z^1) \right| + \left| g(\cdot, Y, Z) \right| + 2A|Y - Y^1| + A|Z - Z^1| + 3A \in \mathcal{H}^p.$$

Then, Lebesgue’s dominated convergence theorem together with (H4w) and the fact of $d\mathbb{P} \times dt - a.e., Y^n \uparrow Y$, yields that

$$\lim_{n \to \infty} \| g(\cdot, Y^n, Z) - g(\cdot, Y, Z) \|_{\mathcal{H}^p} = 0.$$  \hspace{1cm} (61)

Thus, letting first $n \to \infty$, and then $m \to \infty$ in (60), in view of (61), (58) and the fact that $\phi(\cdot)$ is continuous and $\phi(0) = 0$, we get (59).

Finally, (57) follows from (58) and (59). And, the uniqueness part is a direct corollary of Proposition 5 or Corollary 1. The proof of Theorem 4 is completed. \square

**Corollary 4** Let $p > 1$, $V^1, V^2 \in \mathcal{V}^p$ and both $g^1$ and $g^2$ satisfy assumptions (H1), (H2), (H3) and (H4w). For $i = 1,2$, assume that (H5) and (H6) hold for $\xi^i$, $L^i$ and $X^i$ associated with $g^i$, and that $(Y^i, Z^i, K^i) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{V}^{1,p}$ is the unique solution of RBSDE $\xi^i, g^i + dV^i, L^i$. If $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$, $L^1 = L^2$, and

$$d\mathbb{P} \times dt - a.e., \quad g^1(t, y, z) \leq g^2(t, y, z)$$

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for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), then \(\mathbb{P} - a.s.,\) \(dK^1_t \geq dK^2_t\) for each \(t \in [0, T]\).

**Proof.** For \(n \geq 1\) and \(i = 1, 2\), by Theorem 1 let \((Y^{i,n}, Z^{i,n}) \in \mathcal{S}^p \times \mathcal{M}^p\) be the unique solution of the following penalization BSDE:

\[
Y^{i,n}_t = \xi^i + \int_t^T g^i(s, Y^{i,n}_s, Z^{i,n}_s)ds + \int_t^T dV^i_s + \int_t^T dK^{i,n}_s - \int_t^T Z^{i,n}_s dB_s, \quad t \in [0, T]
\]

with

\[
K^{i,n}_t := n \int_0^t (Y^{i,n}_s - L^i_s)^- ds, \quad t \in [0, T).
\]

In view of the assumptions of Corollary 4, it follows from Proposition 5 that for each \(n \geq 1\), \(Y^{1,n} \leq Y^{2,n}\), and then

\[
K^{1,n}_{t_2} - K^{1,n}_{t_1} = n \int_{t_1}^{t_2} (Y^{1,n}_s - L^1_s)^- ds \geq n \int_{t_1}^{t_2} (Y^{2,n}_s - L^2_s)^- ds = K^{2,n}_{t_2} - K^{2,n}_{t_1}
\]

for every \(n \geq 1\) and \(0 \leq t_1 \leq t_2 \leq T\). Since

\[
\|K^{1,n} - K^1\|_{\mathcal{S}^p} \to 0 \quad \text{and} \quad \|K^{2,n} - K^2\|_{\mathcal{S}^p} \to 0
\]

as \(n \to \infty\) by Theorem 4, it follows that \(\mathbb{P} - a.s.,\)

\[
K^{1}_{t_2} - K^{1}_{t_1} \geq K^{2}_{t_2} - K^{2}_{t_1}
\]

for every \(0 \leq t_1 \leq t_2 \leq T\), which proves the desired result. \(\square\)

**Remark 6** By (i) and (iii) of Remark 2, it is clear that Theorems 3-4 together with Corollary 4 strengthen the corresponding results for RBSDE (1) established in Klmiśak [24], Lepeltier, Matoussi and Xu [27] and Rozkosz and Slomiński [37], which the stronger assumptions (H1s) and (H2s) than (H1) and (H2) are satisfied.

**Theorem 5** Let \(p > 1, V, \in \mathcal{V}^p\) and let \(g\) satisfy (H1), (H2w), (H3) and (H4s). Assume that (H5) and (H6) hold for some \(\xi, L, \) and \(X\). For each \(n \geq 1\), let \((Y^n, Z^n) \in \mathcal{S}^p \times \mathcal{M}^p\) be the maximal (resp. minimal) solution of the penalization BSDE (16) with (17) (Recall Corollary 2). Then, there exists a solution \((Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{Y}^+\) of RBSDE \((\xi, g + dV, L)\) such that

\[
\lim_{n \to \infty} (\|Y^n - Y\|_{\mathcal{S}^p} + \|Z^n - Z\|_{\mathcal{M}^p}) = 0,
\]

and there exists a subsequence \(\{K^{n_j}\}\) of \(\{K^n\}\) such that

\[
\lim_{j \to \infty} \sup_{t \in [0, T]} |K^{n_j}_t - K_t| = 0.
\]

**Proof.** It follows from Corollary 3 that \(Y^n\) increases in \(n\). By (i) and (ii) of Remark 2 we know that (H2w) and (H3) imply (HH), and (H4) holds true. And, it follows from Proposition 7 that (18) holds true. Thus, we have checked all conditions in Proposition 3. Then the conclusion follows from Proposition 3. \(\square\)
Remark 7 Let us remark that it is not clear whether \((Y, Z, \cdot)\) obtained in Theorem 5 is the maximal (resp. minimal) solution of RBSDE \((\xi, g + dV, L)\) or not.

The following Theorem 6 further proves that under the conditions of Theorem 5, RBSDE \((\xi, g + dV, L)\) admits both a minimal and a maximal solution in \(S^p \times M^p \times V^{+,p}\), which also shows that the solution can be approximated by some sequence of solutions of RBSDEs. The proof is based on Theorem 4, Corollaries 1 and 4, Propositions 1, 2, 4 and 5.

Theorem 6 Let \(p > 1\), \(V \in \mathcal{V}^p\) and let \(g\) satisfy (H1), (H2w), (H3) and (H4s). Assume that (H5) and (H6) hold for some \(\xi, L\) and \(X\). Then, RBSDE \((\xi, g + dV, L)\) admits a minimal (resp. maximal) solution \((Y, Z, K)\) in \(S^p \times M^p \times V^{+,p}\), i.e., if \((Y', Z', K')\) is also a solution of RBSDE \((\xi, g + dV, L)\) in the space \(S^p \times M^p \times V^{+,p}\), then \(\mathbb{P} - a.s., Y_t \leq Y'_t\) (resp. \(Y_t \geq Y'_t\)) for each \(t \in [0, T]\).

Proof. Assume that \(p > 1\), \(V \in \mathcal{V}^p\) and the generator \(g\) satisfies (H1) with \(\rho(\cdot)\), (H2w) with \(f, \mu\) and \(\lambda\), (H3) with \(\varphi(r)\) and (H4s). Assume further that (H5) and (H6) hold for some \(\xi, L\) and \(X\).

We first show the existence of the minimal solution. In view of the assumptions of \(g\), it is not very hard to prove that for each \(n \geq 1\) and \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), the following function

\[
g_n(\omega, t, y, z) := \inf_{u \in \mathbb{R}^d} [g(\omega, t, y, u) + (n + 2\lambda)|u - z|]
\]  

(62)

is well defined and \((\mathcal{F}_t)\)-progressively measurable, \(d\mathbb{P} \times dt - a.e., g_n\) increases in \(n\) and converges locally uniformly in \((y, z)\) to the generator \(g\) as \(n \to \infty\), and all \(g_n\) satisfy (H1) with the same \(\rho(\cdot)\), (H2s) with \(n + 2\lambda\), (H3) with the same \(\varphi(r) + \mu r + 2f\) and (H4). In addition, we can also prove that for each \(n \geq 1\), \(g_n\) and \(g\) satisfy (14) (appearing in Proposition 2) with the same \(f, \mu\) and \(\lambda\), and then \(d\mathbb{P} \times dt - a.e.,\) for each \(n \geq 1\) and each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\),

\[|g_n(\cdot, y, z)| \leq |g(\cdot, y, 0)| + f. + \mu |y| + \lambda |z| \leq |g(\cdot, 0, 0)| + f. + \varphi(|y|) + \mu |y| + \lambda |z|.
\]

That is to say, all \(g_n\) satisfy (HH) with the same parameters.

Note that (H2s) implies (H2), and (H4) implies (H4w) by (i) of Remark 2. It then follows from Theorem 4 that there exists a unique solution \((Y^n, Z^n, K^n)\) in \(S^p \times M^p \times V^{+,p}\) of RBSDE \((\xi, g_n + dV, L)\) for each \(n \geq 1\). By Corollary 1, \(Y^n\) increases in \(n\). Furthermore, it follows from (14) and (H6) that for each \(n \geq 1\),

\[|g_n(\cdot, X, 0)| \leq |g(\cdot, X, 0)| + f. + \mu |X| \in \mathcal{H}^p.
\]

Then, by Corollary 4, \(K^n\) decreases in \(n\).

In the sequel, we show that (C) appearing in Proposition 2 holds true for \(Y^n\) and \(g\). In fact, let

\[\underline{g}(\cdot, y, z) := g(\cdot, y, 0) - f. - \mu |y| - \lambda |z|
\]

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Then both \( g \) and \( \bar{g} \) satisfy (H1), (H2s), (H3) and (H4s),
\[
g(\cdot, X, 0) = g(\cdot, X, 0) - f - \mu|X| \in \mathcal{H}^p,
\]
\[
\bar{g}(\cdot, X, 0) = g(\cdot, X, 0) + f + \mu|X| \in \mathcal{H}^p,
\]
and by (14) for each \( n \geq 1 \),
\[
g \leq g_n \leq \bar{g}.
\]
Thus, it follows from Theorem 4 that RBSDE \((\xi, g + dV, L)\) and RBSDE \((\xi, \bar{g} + dV, L)\) admit respectively a unique solution \((Y, \bar{Z}, K)\) in \( S^p \times M^p \times V^{+p}\) and \((\bar{Y}, \bar{Z}, \bar{K})\) in \( S^p \times M^p \times V^{+p}\), and by Corollary 1, we know that \( \mathbb{P} \)-a.s.,
\[
Y_t^n \leq Y^n_t \leq \bar{Y}_t
\]
for each \( t \in [0, T] \) and \( n \geq 1 \). In addition, in view of (i) and (ii) of Remark 2, by Proposition 1 with \( g_n \equiv \bar{g} \) we know that \( \bar{g}(\cdot, \bar{Y}, \bar{Z}) \in \mathcal{H}^p \), and then
\[
g(\cdot, \bar{Y}, 0) = \bar{g}(\cdot, \bar{Y}, \bar{Z}) - f - \mu|\bar{Y}| - \lambda|\bar{Z}| \in \mathcal{H}^p.
\]
By (63) and (64) we know that (C) is true for \( Y^n \) and \( g \).

Now we have checked that all conditions in Proposition 2 are satisfied. It then follows from Proposition 2 that (15) holds true. Furthermore, in view of (63), we can deduce that (37) appearing in Proposition 4 holds true with
\[
\eta := C \left[ \sup_{t \in [0, T]} |Y_t|^p + |V_t|^p + \sup_{t \in [0, T]} |\bar{Y}_t|^p + \left( \int_0^T f_t dt \right)^p + 1 + \left( \int_0^T |g(t, \bar{Y}_t, 0)| dt \right)^p + \left( \int_0^T |g(t, 0, 0)| dt \right)^p \right],
\]
where \( C \) is a nonnegative constant depending on \( p, \mu, \lambda, A, T \). Hence, all conditions in Proposition 4 are satisfied, and then it follows from Proposition 4 that RBSDE \((\xi, g + dV, L)\) admits a solution \((Y', Z', K')\) in \( S^p \times M^p \times \mathcal{H}^p\) such that
\[
\lim_{n \to \infty} (\|Y^n - Y\|_{S^p} + \|Z^n - Z\|_{M^p} + \|K^n - K\|_{\mathcal{H}^p}) = 0.
\]

Finally, let us show that \((Y', Z', K')\) is just the minimal solution of RBSDE \((\xi, g + dV, L)\) in \( S^p \times M^p \times \mathcal{H}^p\). In fact, if \((Y'', Z'', K'')\) is also a solution of RBSDE \((\xi, g + dV, L)\) in \( S^p \times M^p \times \mathcal{H}^p\), then noticing that for each \( n \geq 1 \), \( g_n \leq g \) and \( g_n \) satisfies (H1) and (H2) due to (H2s) \( \Rightarrow \) (H2), it follows from Proposition 5 that \( \mathbb{P} \)-a.s., \( Y^n_t \leq Y''_t \) for each \( t \in [0, T] \) and \( n \geq 1 \). Thus, by (65), \( \mathbb{P} \)-a.s., \( Y_t \leq Y'_t \) for each \( t \in [0, T] \).

As for the case of the maximal solution, we only need to replace (62) with (53). And, by a similar argument as above we can obtain the desired result. The proof of Theorem 6 is then completed. \( \square \)
Remark 8 It follows from (i) and (iii) of Remark 2 that Theorem 6 improves Theorem 5.1 in Xu [38], where the stronger assumption (H1s) than (H1) is satisfied, the barrier \( L \) is assumed to be bounded and only \( L^2 \) solution is considered.

By Corollaries 1, 4 and the proof of Theorem 6 it is not hard to verify that under assumptions (H1), (H2w), (H3), (H4s), (H5) and (H6), the comparison theorem for the maximal (resp. minimal) \( L^p \) solutions of RBSDEs with \( L^p \) (\( p > 1 \)) data is true. More precisely, we have

Proposition 8 Let \( p > 1 \) and for \( i = 1, 2 \), assume that \( V^i \in \mathcal{V}^p \), \( g^i \) satisfies (H1), (H2w), (H3) and (H4s), \( \xi^i, L^i \) and \( X^i \) satisfy (H5) and (H6) associated with \( g^i \), and \( (Y^i, Z^i, K^i) \in \mathcal{S}^p \times \mathbb{M}^p \times \mathcal{V}^{p+} \) is the maximal (resp. minimal) solution of RBSDE \( (\xi^i, g^i + dV^i, L^i) \) (Recall Theorem 6). If \( \xi^1 \leq \xi^2 \), \( dV^1 \leq dV^2 \), \( L^1 \leq L^2 \), and

\[
d\mathbb{P} \times dt - a.e., \quad g^1(t, y, z) \leq g^2(t, y, z)
\]

for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\), then \( \mathbb{P} - a.s., Y^1_t \leq Y^2_t \) for each \( t \in [0, T] \). Furthermore, if \( L^1 = L^2 \), then \( \mathbb{P} - a.s., dK^1_t \geq dK^2_t \) for each \( t \in [0, T] \).

Remark 9 In a subsequent work we will further discuss the problem on the existence and uniqueness for \( L^p \) solutions of RBSDE (1) with \( L^p \) (\( p > 1 \)) or \( L^1 \) data, irregular barriers and generators satisfying (H1), (H2) or (H2w), and (H3).

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