Hypergeometric viable models in $f(R)$ gravity

Roger Hurtado ⓒ and Robel Arenas
Observatorio Astronómico Nacional, Universidad Nacional de Colombia, Av. Carrera 30 45-03 Edif. 476, Bogotá, Colombia
E-mail: rarhurtadom@unal.edu.co and jrarenass@unal.edu.co

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Abstract

A cosmologically viable hypergeometric model within the framework of the modified gravity theory $f(R)$ has been derived based on the requirements of asymptotic behavior towards $\Lambda$CDM, the presence of an inflection point in the $f(R)$ curve, and the viability conditions dictated by the phase space curves $(m, r)$, where $m$ and $r$ denote characteristic functions of the model. To examine the constraints associated with these viability criteria, the models were expressed in terms of a dimensionless variable, namely $R \rightarrow x$ and $f(R) \rightarrow y(x) = x + h(x) + \lambda$, where $h(x)$ represents the deviation of the model from General Relativity. By employing the geometric properties imposed by the inflection point, differential equations were formulated to establish the relationship between $h'(x)$ and $h''(x)$. The resulting solutions yielded models of the Starobinsky (2007) and Hu-Sawicki types. However, it was subsequently discovered that these differential equations correspond to specific cases of a hypergeometric differential equation, indicating that these models can be derived from a more general hypergeometric model. The parameter domains of this model were thoroughly analyzed to ensure its viability.

1. Introduction

General Relativity (GR) stands as the widely accepted theory of gravity, accurately predicting phenomena such as the expansion of the Universe and the associated redshifts of galaxies as consequences of its evolution from the Big Bang. Furthermore, it successfully explains remarkable occurrences like the gravitational lensing effect [1, 2], black holes, and recently detected gravitational waves [3, 4]. However, the observations of Supernovae type Ia (SN Ia) [5, 6] have revealed that the Universe is undergoing an accelerated expansion phase. This fact lacks a clear interpretation within the framework of GR, necessitating the introduction of an unknown form of negative pressure force called Dark Energy (DE). It is postulated that DE dominates gravitational attraction on large scales [6–8]. This model is referred to as Lambda Cold Dark Matter (ΛCDM), which also incorporates Dark Matter (DM), a new and peculiar type of gravitating matter that does not interact with radiation [9, 10]. The presence of DM rectifies the discrepancy between theory and the observed flat rotation curves of spiral galaxies [11, 12].

ΛCDM aligns well with a wide range of cosmological observations [2, 13–16]. However, the nature of DM and DE remains unknown. According to the observations made by the ESA’s Planck satellite in 2013 [17], within the theoretical framework of ΛCDM, the Universe comprises 27% DM and 68% DE, meaning that only 5% of the energy density is attributed to known constituents. This realization raises more questions than answers.

Undoubtedly, ΛCDM stands as a paradigm, supplemented by additional concepts throughout the last century. For instance, the theory of cosmological inflation explains the homogeneity and isotropy of the Universe at large scales by postulating an accelerated expansion during its early stages [14]. This theory resolves issues such as the flatness problem [18] and the magnetic monopole problem [19]. However, a universally accepted model for inflation is yet to be established. Likewise, the standard model of cosmology faces certain challenges (refer to [20] for a concise summary) that necessitate a reconsideration of our comprehension of GR on cosmological scales. One alternative, motivated primarily by the search for a geometrical explanation for late-time acceleration, is the $f(R)$ theory [21, 22]. The dynamics of this theory are derived from an action formulated in terms of a general function of the scalar curvature, denoted as $R$. Numerous $f(R)$ models have been proposed.
to address issues related to DM [23–25], DE [25–27], and even the inflationary phase. The pioneering model in the context of \( f(R) \) theory, introduced by Starobinsky in 1980, incorporates a quadratic term for the curvature scale in the Einstein-Hilbert (EH) action, yielding \( f(R) = R + \alpha R^2 \), where \( \alpha \) is a constant. Extensive studies have confirmed the viability of this model, which aligns well with recent data obtained from the Planck satellite [28].

Finding a viable \( f(R) \) model that reproduces inflation, a radiation-dominated stage followed by a matter-dominated phase, and late-time accelerated expansion, while successfully passing Solar system tests, is a formidable task. However, [27] establishes the general conditions for a model to be cosmologically acceptable, providing examples of viable models such as Starobinsky [29], Hu-Sawicki [30], Tsujikawa [31], and exponential models [32]. Notably, the first two models have been rigorously tested using redshift data from SN Ia, and their cosmological and free parameters have been calculated through Markov chain Monte Carlo simulations, demonstrating their exceptional fit to the data [33]. One characteristic of these models is the presence of an inflection point, which will be the focus of this study, elucidating the conditions that \( f(R) \) models must satisfy to be deemed cosmologically valid.

This work is structured as follows: In section 2, we review the theoretical framework of \( f(R) \) theory, presenting an outline of the field equations. Sections 3 and 4 delve into the analysis of the viability conditions and the existence of an inflection point in the function, respectively. In sections 5 and 6, a differential equation is derived from the geometric properties imposed by the aforementioned conditions. The solutions are presented and generalized as a hypergeometric model in section 7. Finally, the conclusions are summarized in section 8.

2. Field equations in \( f(R) \) theory

The \( f(R) \) theory is formulated by modifying the Einstein-Hilbert (E-H) action, where the Lagrangian density is an arbitrary function of \( R \), defined over a hypertemplate \( \Sigma \) [21].

\[
I = \frac{1}{2\kappa} \left( \int_\Sigma d^4x \sqrt{-g} f(R) + I_{GYH} \right) + I_M,
\]

where \( I_{GYH} \) is the Gibbons-York-Hawking boundary (\( \partial \Sigma \)) term [34], \( I_M \) is the contribution of matter, and \( \kappa = 8\pi G \). In the metric formalism, the modified field equations are obtained by varying the action with respect to the metric \( g_{\mu\nu} \).

\[
FR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} F - g_{\mu\nu} F^{\alpha}_{\alpha} = \kappa T_{\mu\nu},
\]

where \( f = f(R), \ F = F(R) = f'(R), \) the D’Alembertian is defined by \( F^{\alpha}_{\alpha} \), the covariant derivatives \( \nabla_{\mu} \nabla_{\nu} F = F_{\mu\nu} \), and the definition of the energy-momentum tensor

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta I_M,\]

The trace of the field equations (2) is obtained by multiplying them by the metric tensor

\[
FR - 2f + 3F_{\mu\nu} = \kappa T,
\]

where \( T \) is defined as \( T = g^{\mu\nu} T_{\mu\nu} \). Although equation (4) is a differential equation, in the context of \( f(R) \) theory, it is often treated as an algebraic equation to establish a relationship between \( R, f(R), \) and \( F(R) \). In General Relativity (GR), \( T = 0 \) implies \( R = 0 \), but this does not hold in \( f(R) \) theory. When there are coupled Maxwell fields with a traceless stress-energy tensor, the scalar curvature becomes constant, i.e., \( R = R_0 \). However, in the presence of non-linear electromagnetic fields, such as those of the Born-Infeld type, the solutions yield \( R = R(r) \), where \( r \) is the radius vector [35]. By utilizing the trace equation, the field equations can be reduced to

\[
FR_{\mu\nu} - \kappa T_{\mu\nu} - F_{\mu\nu} - \frac{1}{4} (FR - \kappa T - F^{\alpha}_{\alpha}) g_{\mu\nu} = 0.
\]

This equation depends on the second covariant derivatives of the scalar function \( f(R) \), which are combinations of partial derivatives of the metric. One of the main advantages of \( f(R) \) theories of gravity is the flexibility they offer in returning to General Relativity (GR) easily by setting \( f(R) \) equal to \( R \).

3. Model constraints and inflection point

The general form of the function \( f(R) \) can be expressed explicitly as the sum of the linear term \( R \) which reproduces GR plus a perturbation,

\[
f(R) = R + \tilde{f}(R) + \lambda R_0,
\]
where \( \bar{f}(R) \) represents the deviation of the model from GR, and the \( \Lambda \)CDM model can be obtained as a special case with \( \bar{f}(R) = 0 \), where \( \lambda = -2 \Lambda / R_0 \), and \( \Lambda \) is the cosmological constant. Therefore, by defining the dimensionless coordinate \( x \) as \( x = R / R_0 \),

\[
y(x) = x + h(x) + \lambda,
\]

where \( y(x) = f(R_0 x) / R_0 \), \( \bar{f}(R_0 x) = R_0 h(x) \), and with the definition of the characteristic functions \[27\]

\[
m = \frac{Rf''(R)}{f'(R)} = \frac{x h''(x)}{1 + h'(x)},
\]

and

\[
r = -\frac{Rf''(R)}{f'(R)} = -\frac{x [1 + h'(x)]}{x + h(x) + \lambda}.
\]

Let us consider the scenario where \( h(x) \) is a continuous function with continuous derivatives in a domain \( I \). Suppose \( x_i \in I \) represents an inflection point of \( h(x) \) that is neither stationary nor of infinite slope. Specifically, we have \( h''(x_i) = 0 \), indicating that \( h'(x_i) \) corresponds to a local maximum. The existence of an inflection point, along with the following conditions of asymptotic behavior towards \( \Lambda \)CDM \[30\], (where \( k \) is a constant that depends on the model)

\[
\lim_{x \to 0} h(x) = -\lambda,
\]

and

\[
\lim_{x \to \infty} h(x) = \frac{k}{R_0} - \lambda,
\]

impose restrictions on the form of the function \( h(x) \), allowing for two possibilities: decreasing concave (increasing convex) for \( 0 < x < x_i \), and decreasing convex (increasing concave) for \( x_i < x \). These constraints are motivated by the successful outcomes of the Starobinsky model \[33\], which utilizes a quadratic term to reproduce the accelerated expansion of the Universe without the need for Dark Matter. Additionally, in order for \( h(x) \) to contain only quadratic terms when expanded in a Maclaurin series, it must be an even function. As a result, \( x = 0 \) becomes a maximal point, leading to the condition

\[
\lim_{x \to 0} h'(x) = 0,
\]

and at the other hand at infinity the curve is flattened according to equation \[11\], thus

\[
\lim_{x \to \infty} h'(x) = 0,
\]

and in turn

\[
\lim_{x \to 0} h''(x) = c,
\]

where \( c \) is a constant, and

\[
\lim_{x \to \infty} h''(x) = 0,
\]

The two possible behaviors of the function affect the sign of its derivatives, as discussed in table 1.

| \( h(x) \) | Decreasing | Increasing | Domain |
|-----------|------------|------------|--------|
| \( h'(x) \) | -          | +          | \( x > 0 \) |
| \( h''(x) \) | -          | +          | \( 0 < x < x_i \) |
| \( h''(x) \) | +          | -          | \( x > x_i \) |
| \( h'(x) - h'(x_i) \) | + | - | \( x > 0 \) |
| \( h''(x) \) | + | - | \( x > 0 \) |

Table 1. Sign of the first derivatives of the function \( h(x) \) according to its monotonicity in the domain given in the last column.

Only models with a characteristic function, \( (8) \), \( m \geq 0 \) and close to \( \Lambda \)CDM, can be considered cosmologically viable \[27\]. Therefore, there are two options for \( h(x) \) that satisfy this condition: either \( h''(x) \geq 0 \) and \( h'(x) > -1 \), or \( h''(x) \leq 0 \) and \( h'(x) < -1 \). If \( h(x) \) is a decreasing function for \( x \geq x_i \),
or for $0 \leq x \leq x_i$,

$$h'(x) < -1.$$  \hfill (17)

From the second option, when $h(x)$ is an increasing function, it is only possible to choose

$$h'(x) > 0,$$  \hfill (18)

for $0 < x < x_m$. Due to the condition (12), option (17) is discarded. Additionally, in order to avoid singularities in the characteristic functions $m(x)$ and $r(x)$, option (18) is also ruled out. The presence of the inflection point $x_i$ gives rise to a minimum (maximum) in $h'(x)$ when $h(x)$ is a decreasing (increasing) function. Moreover, according to equation (15), there exists an inflection point $x_m$ on the curve of $h'(x)$. Consequently, $h''(x_m)$ will be a maximum for $x > x_m$, and $h''(x)$ will exhibit a decreasing (increasing) behavior for $x > x_m$.

The function $h''(x)$ is integrable throughout its domain, as indicated by limits (12) and (13), yielding

$$\int_0^\infty h''(x)\,dx = 0.$$  \hfill (19)

Therefore, it is reasonable to assume

$$0 < x + h(x) + \lambda.$$  \hfill (22)

Conditions (21) and (22) are valuable for determining the limits of the function $r(\varphi)$, as demonstrated below.

In order for the model to smoothly tend towards $\Lambda$CDM and for $h(x)$ to be decreasing in the entire domain, let us assume that $0 < x_i < 1$. Since $\kappa$ and $\lambda$ are constants, as $x_i \to 0$, $h'(x)$ will have a steeper slope as $x \to 0$. Therefore, the absolute value of $c$ must increase. At the same time, as $x_m$ decreases, $h''(x_m)$ increases (since the integral of $h''(x)$ is zero in the entire domain). This implies that

$$\int_0^{x_i} h''(x)\,dx = -c.$$  \hfill (23)

On the other hand, as $x_i \to 1$, the shape of $h(x)$ becomes flatter, tending towards $-\lambda$, and all its derivatives become identically zero, so $c \to 0$. With these considerations, it is evident that $c$ is inversely dependent on $x_i$, and therefore it must have the following form

$$h''(0) \propto \left(1 - \frac{1}{x_i^2}\right).$$  \hfill (24)

and from equation (23), as $c$ increases, the maximum of $h''(x)$ increases, and consequently $h^{(4)}(0)$ increases. Therefore, it is reasonable to assume

$$h^{(4)}(0) \propto -\left(1 - \frac{1}{x_i^4}\right).$$  \hfill (25)

These assumptions will be used later on. In figure 1, the presence of an inflection point in the function $h(x)$ can be observed, along with the limits it must satisfy, as well as its derivatives.

### 4. Characteristic functions

A model that reproduces a matter-dominated era with a corresponding transition to accelerated expansion must satisfy [27]

$$m(r) \approx +0, \quad \text{and} \quad m'(r) > -1 \quad \text{at} \quad r \approx -1,$$  \hfill (26)

where

$$\lim_{r \to x} \frac{dm}{dr} = \frac{x m'(x)}{r(x)[1 + m(x) + r(x)].}$$  \hfill (27)
There are three points $x$ for which $r(x) \approx -1$, these are $x_1 \to 0$,
\[
\lim_{x \to 0} \frac{x[1 + h'(x)]}{x + h(x) + \lambda} = 1,
\]
where we have used equation (10) and (12);
\[
x_2 = \frac{h(x_2) + \lambda}{h'(x_2)};
\]
and $x_3 \to \infty$, since
\[
\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \frac{1 + h'(x) + xh''(x)}{1 + h'(x)} = 1,
\]
where L'Hôpital’s rule and condition (21) were used. Now, since $m(x_1) = -0$, and
\[
m'(r)_{x=0} = -2,
\]
(see appendix), we discard $x_1$, and observe that $m(x)$ is directly proportional to $h''(x)$. As $x_3$ is also a root of $m(x)$, denoted as $x_2 \to x_3$, we require $x_2$ to tend to $x_3$ from the right, denoted as $x_{r2}$, satisfying $h(x_{r2}) = x_2 + h'(x_{r2}) - \lambda$.

However, the last term of equation (27) diverges as $x$ approaches $x_3$, and $h''(x_{r2}) > 0$ because $h'(x_{r2})$ is a minimum. Consequently, point $x_3$ is discarded. On the other hand, $x_2$ is a valid point that provides viability to the model. According to equation (13) and (21), we have
\[
\lim_{x \to \infty} m = +0,
\]
and
\[
m'(r) \to 0.
\]
Since $0 < 1 + h'(x)$, for $0 < x < x_2$, $h''(x) < 0$, and $m(x) < 0$, and simultaneously for $x > x_2$, $m(x) > 0$, therefore equation (32) expresses that $m(x)$ should be flattened towards zero at infinity. On the other hand, $r(x) > -1$ for $0 < x < x_2$, then it will have a minimum and will tend to $-1$ at infinity, figure 2.

The behavior of $m'(r)$ in the phase space can now be drawn as shown in figure 3, where $m'(r)$ is also observed. It should be noted that $r'(x) = r(1 + m + r)/x$, so that the maximal points of $r$ are found when $m = -1 - r$, that is, the points where $m'(r)_{x=0}$ diverges. Since $r$ takes the value of $-1$ at $x_1$, $x_2$, and $x_3$, it exhibits two maximum points, namely $x_{r1}$ and $x_{r2}$ as shown in figure 2. At these points, the derivative becomes infinite, as illustrated in figure 3, and is represented by the coordinates $(r_{a1}, m_{a})$ and $(r_{a2}, m_{b})$.

In the upcoming section, a differential equation for $h(x)$ will be derived based on its geometric properties. This will involve considering the parities of the functions $h'(x)$ and $h''(x)$ and multiplying them by suitable functions to ensure that their roots coincide.

### 5. Starobinsky type models

Considering that $h(x)$ and its derivatives are continuous functions, we can suppose that
\[
h'(x) = e^{c \int x^{-1} K(x) dx},
\]
where $K(x)$ is a function that satisfies $\int K(x) x^{-1} dx < 0$ according to equation (16). The constant $c$ is determined by expanding in a series to satisfy equation (18). It is worth noting that
\[
\lim_{x \to 0, \infty} \int x K(x) dx = -\infty.
\]

The function $h''(x)$ expressed in exponential form, equation (34), obeys the differential equation
\[
xh''(x) = K(x) h'(x),
\]
where the left-hand side allows us to express $h''(x)$ as
\[
h''(x) = \frac{m(x)}{K(x) - m(x)},
\]
or equivalently
\[
x \frac{m'(x)}{m(x)} + m(x) = \frac{K'(x)}{K(x)} + K(x),
\]
and based on the form of equation (36), it is concluded that $K(x)$ has only one root, which corresponds to the inflection point,
At the same time, since $h(x)$ is an even function, $h'(x)$ and $h''(x)$ are odd and even functions, respectively, as shown in figure 1. Therefore, it can be deduced that the function $xh''(x)$ will be odd and have roots at $x = 0$ and $x = x_i$. Similarly, by multiplying $h'(x)$ by the factor $(x_i^2 - x^2)$, the same intervals of increase and decrease of $xh''(x)$ are obtained, along with the same roots, for both negative and positive $x$. Therefore, $K(x)$ is proposed to be an even rational function

$$K(x) = \frac{x_i^2 - x^2}{q(1 + p(x))},$$

where $q$ is a constant and $p(x)$ is a function linearly independent of $h(x)$. Note that $p(x)$ must be an even function to ensure that the left-hand side of equation (36) remains odd. The choice of the denominator of $K(x)$ as $1 + p(x)$ will be explained in the following lines. In this way, it is possible to express equation (36) as follows:

$$q(1 + p(x))xh''(x) = (x_i^2 - x^2)h'(x).$$

In this manner, it is possible to observe that $p(x)$ satisfies the following limits:

$$\lim_{x \to 0} p(x) = 0,$$

and

$$\lim_{x \to \infty} \frac{1}{1 + p(x)} = \lim_{x \to \infty} \frac{q}{x^2 - x_i^2} = 0.$$ (43)

since $q$ is a constant, it is possible to evaluate it at some limit, for example

$$q = \lim_{x \to 0} \frac{(x_i^2 - x^2)h'(x)}{(1 + p(x))xh''(x)} = \lim_{x \to 0} \frac{x_i^2}{1 + p(x)} = x_i^2,$$

thus equation (41) can be rewritten as

$$(1 + p(x))\frac{1}{x}h''(x) + \left(\frac{1}{x_i^2} - \frac{1}{x^2}\right)h'(x) = 0,$$

and integrating it

$$\frac{h(x)}{x_i^2} + \frac{h'(x)}{x} + \int x p(x)h''(x)dx = 0,$$

where it can be seen why $1 + p(x)$ was chosen in equation (41) rather than $p(x)$. By performing integration by parts twice and taking into account the integration limits, we obtain

$$-\frac{\lambda p''(t)}{2} + \frac{\lambda}{x_i^2} - c + (1 + p(x))h'(x) \frac{1}{x} + \left(\frac{1}{x_i^2} + \frac{p(x)}{x^2} - \frac{p'(x)}{x}\right)h(x)$$

Figure 1. Behavior of the function $h(x)$ and its derivatives with respect to $x$, taking into account the necessary conditions for the model to be considered viable. The conditions include the asymptotic behaviors specified by equations (10) to (15), along with the presence of an inflection point denoted by $x_i$. The diagram also highlights the existence of the maximum of $h''(x)$ at $x_m$. The chosen value for the parameter $R_0$ is 1.
when evaluating at $x_i \to 1$,

$$\lambda \left( p(1) - p'(1) + \frac{p''(0)}{2} \right) - \int_0^{x_i} \frac{h(t)}{t^3} (t^2 p''(t) - 2tp'(t) + 2p(t)) dt = 0,$$

(49)

where we have used the limits of $h(x)$ and $h'(x)$ at $x_i \to 1$. And by analyzing the behavior of the limits of $h(x)$ and its derivatives as $x_i \to 1$, it is possible to determine that

$$p(1) - p'(1) = -1.$$

(50)

Furthermore, by performing a series expansion of equation (41), it is found that

$$p''(0) = -2 \lim_{x \to 0} \left( \frac{h''(x)}{x^2} - \frac{h'(x)}{x^3} \right) = -2 \frac{2}{x_i^2} = \frac{2}{c} h''(0) = 2.$$

(51)

Thus, the factor accompanying $-\lambda$ disappears in equation (49), leaving the integral as the only option to be zero. Since $h(x) \neq 0$ when $x_i \to 1$, and the upper limit of the integral varies between zero and one, the only option for the integral to be zero is for the integrand to be zero, i.e.,

$$x^2 p''(x) - 2xp'(x) + 2p(x) = 0,$$

(52)

from which we can directly obtain the value of the function $p(x)$, that is

$$p(x) = x(\alpha + \beta x),$$

(53)

with $\alpha$ and $\beta$ constants, but for $p(x)$ to be an even function, $\alpha = 0$, so the equation is reduced to

$$\left( \frac{1}{x} + \beta x \right) h'(x) + \left( \frac{1}{x_i^2} - \beta \right) h(x) + c_1 = 0,$$

(54)

whose solution is

$$h(x) = \frac{k}{R_0} \left[ 1 - (1 + \beta x_i^2)^{\frac{1}{1 + \beta x_i^2}} \right] - \lambda,$$

(55)

where the integration constants were found by equation (10) and (11) for $\beta > 0$ and $\beta x_i^2 < 1$, which, in turn, requires that the power must be negative,

$$\frac{1}{2} \left( 1 - \frac{1}{\beta x_i^2} \right) = -m,$$

(56)

with $m > 0$, or analogously, when $x_i = (2m - 1)^{-1/2}$ and $\beta = 1$,

$$h(x) = \frac{k}{R_0} \left[ 1 - (1 + x^2)^{-m} \right] - \lambda,$$

(57)

so without loss of generality, it can be concluded that equation (55) actually represents Starobinsky’s model [29]. In this way it is easy to find the value of the constant $c$, equation (14),

$$c = \frac{2km}{R_0},$$

(58)

where $c \neq 0$ of course, and therefore

$$y(x) = x + \frac{k}{R_0} \left[ 1 - (1 + x^2)^{-m} \right],$$

(59)

where $m, k, R_0$ are parameters. In the next section, a generalized Hu-Sawicki model will be established through a similar procedure.

6. Hu-Sawicki type models

In the previous section, we established relationships among the derivatives of $h(x)$ by using geometric considerations and imposing the condition dictated by the existence of the inflection point, which is determined by the root of $h''(x)$ on the left-hand side of equation (41), and explicitly expressed as the difference $x_i^2 - x^2$ on the right-hand side. One characteristic of the Starobinsky model is that the inflection point is smaller than one. However, in this section, we will employ an alternative condition by assuming that
\( x_i > 1 \), and we will reformulate equation (41) using the difference \( x'_i - x' \), where \( r \) is a parameter of the model, that is

\[
t(1 + s(x))xh''(x) = (x'_i - x')h'(x),
\]  
where \( t \) is constant, \( r > 0 \) is an even number and \( s(x) \) is a continuous even function that, as in the previous section, will be requested linearly independent of \( h(x) \), satisfying

\[
\lim_{s \to x_i} s(x) = \frac{r x'_i - 2h'(x_i)}{h''(x_i)},
\]  
and

\[
\lim_{x \to \infty} \frac{1}{s(x)} = 0.
\]  

Note that equation (60) can be integrated as

\[
\frac{1}{1 + r}x[x' - (1 + r)x'_i]h'(x) + \int \left[ts(x) - \frac{x'}{1 + r} + x'_i\right]xh''(x)dx + c_1 = 0,
\]  
and integrating by parts twice and taking into account the limits of integration

\[
\frac{1}{1 + r}x[x' - (1 + r)x'_i]h'(x) + \left[ts(x) - \frac{x'}{1 + r} + x'_i\right]xh'(x) + \{x' - x'_i -
\]

\[
t\{s(x) + x^s(x')\} h(x) - x'_i \lambda - \int [r x'^{-1} - t(2s'(x) + x^s(x))]h(x)dx + c_1 = 0,
\]  
By evaluating the expression when \( x_i \to 1 \) and taking into account that in this limit \( h(x) = -\lambda \) and \( h'(x) = 0 \), we obtain

\[
\lambda [t(s'(1) + s(1)) - 1] - \int_{x_i}^{x} [r x'^{-1} - t(2s'(x) + x^s(x))]h(x)dx = 0,
\]  
due to the fact that \( x' - 1 > 0 \), we have \( s(1)h''(1) < 0 \), but since \( h''(1) < 0 \), it follows that \( s(1) > 0 \). Similarly, \( s'(1) > 0 \). Therefore, motivated by the results of the previous section, we can assume that \( t(s(1) + s'(1)) = 1 \) for some value of the constant \( t \), in order for the integrand to be equal to zero, thus

\[
s(x) = \frac{x'}{t(1 + r)} - \frac{\alpha}{x} + \beta,
\]  
with \( \alpha = 0 \) to allow the function to be even, thus equation (60) can be written as

\[
\left(\frac{x'}{1 + r} + \frac{\beta}{t}\right)xh'(x) - (x'_i + \beta)h(x) + c_1 = 0,
\]  
whose solution is

\[
h(x) = \frac{k}{R_0}x^{1 + \frac{\beta}{t}}[x' + (1 + r)t]^{\frac{\beta}{t} - 1} - \lambda,
\]  
where \( \beta \) was absorbed in \( t \), the constants were established from conditions (10) and (11), and by (16), \( k < 0 \). However, for this function to be even, the ratio \( x'_i / t \) must be an odd number, and since the parameter \( r \) is even, they can be related by means of

\[
x'_i = (nr - 1)t,
\]  
where \( n \) is a natural number, and consequently, the function can be expressed as

\[
h(x) = \frac{k}{R_0}[1 + (1 + r)tx^{-n}]^{-\frac{n}{n-1}} - \lambda,
\]  
with which for \( nr > 2 \) and \( t > 0 \), \( c = 0 \). In this scenario equation (7) is now expressed as

\[
y'(x) = x + \frac{k}{R_0}[1 + (1 + r)tx^{-n}]^{-n}.
\]  
As a particular case, when \( n = 1 \) and defining

\[
c_1 = -\frac{c_1 k}{R_0}, \quad \text{and} \quad c_2 = \frac{1}{(1 + r)t},
\]  
the inflection point is obtained at

\[
x'_i = \frac{r - 1}{c_2(r + 1)},
\]  
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and

\[ h(x) = -\lambda - \frac{c_1 x'}{1 + c_2 x'}, \quad (74) \]

so

\[ y(x) = x - \frac{c_1 x'}{1 + c_2 x'}, \quad (75) \]

which is the Hu-Sawicki model \[30\], where \( c = 0 \). Nevertheless, because \( r > 0 \), it is not possible to obtain Starobinsky’s model from equation (70) and therefore equation (55) and (70) represent different models, however, these models are part of a more general class of models, as will be seen in the next section.

7. Hypergeometric models

The similarities of models given by equations (55) and (70) can be found in the form of their differential equations, as well as in the possible values of their respective parameters. To see this, equation (54) is rewritten as

\[ \frac{1 + \beta x^2}{(1 + 2m - \beta x)} h'(x) + h(x) + \lambda - \frac{k}{R_0} = 0, \quad (76) \]

and equation (41)

\[ (1 + \beta x^2) x h''(x) + [(1 + 2m)x^2 - 1] h'(x) = 0, \quad (77) \]

multiplying equation (76) for \( nr \) and making \( \beta = 1, \nu = -1 \) and \( r = -2 \), these equations can be combined as

\[ (\nu - x')x^2h''(x) + [\nu(1 - (m + n)r) + (1 + nr)x']xh'(x) + mn^2r = 0. \quad (78) \]

In the the same manner, it is possible to express equation (67) and (60), respectively, as

\[ \left( \frac{x'}{t(1 + r)} + 1 \right) xh''(x) - nrh(x) - nr\lambda = 0, \quad (79) \]

and

\[ \left( \frac{x'}{t(1 + r)} + 1 \right) xh''(x) + \left( \frac{x'}{t} + 1 - nr \right) h'(x) = 0, \quad (80) \]

or combined as

\[ (\nu - x')x^2h''(x) + [\nu(1 - (m + n)r) + (m - 1)r - 1)x']xh'(x) + mn^2r = 0, \quad (81) \]

where \( \nu = -t/2(1 + r) \) and \( \lambda = 0 \). Therefore, a generalization of equation (78) and (81) can be made as follows,

\[ (\nu - x')x^2h''(x) + [\nu(1 - (m + n)r) + (m - 1)r - 1)x']xh'(x) + mn^2r = 0, \quad (82) \]

where \( \nu \) is a parameter that can be adjusted according to the type of model, when \( u = n \) and \( r = -2 \), the Starobinsky type model is obtained, equation (57), and when \( u = m \), the Hu-Sawicki one, equation (70), is found. Now with the variable change

\[ z = vx^{-r}, \quad (83) \]

it is realized that equation (82) is in effect the hypergeometric equation

\[ (1 - z) zg''(z) + (u - (1 + m + n)z) g'(z) - mg(z) = 0, \quad (84) \]

where

\[ g(z) = h(z) + \lambda - \frac{c}{2m}, \quad (85) \]

by choosing the constants appropriately, according to equations (57) and (70), the solution can be written as (for \( m, n, u, v, r \neq 0 \))

\[ h(z) = \left( \frac{k}{R_0} - \frac{c}{m} \right) F_2(m, n, u, z) + \frac{c}{2m} - \lambda, \quad (86) \]
together with the condition of existence of the inflection point, given by the algebraic equation
\[
\frac{\binom{m + 1, n + 1; u + 1}{z}}{\binom{m + 2, n + 2; u + 2}{z}} = \frac{(m + 1)(n + 1)r}{(r + 1)(u + 1)z}
\]  
(87)

If we want to ensure the existence of a single point of inflection, we need to examine the parameter space in which \( h(x) \) is viable. This involves studying the behavior of the derivatives of \( h(x) \) and imposing certain conditions on their signs and values. By carefully examining these conditions, we can determine the range of parameters for which the desired properties of \( h(x) \), such as a single point of inflection, are satisfied. When \( r > 0 \), the model naturally satisfies limits equation (13) and equation (15), however to satisfy the limits (10) and (11), \( c = 0 \), and at the same time, by the series expansion of \( h'(x) \) and \( h''(x) \), it is observed that \( m > 2/r \), \( n > 2/r \), \( v = 0 \), \( m - n \notin \mathbb{Z} \) and \( u = w \) with \( w \notin \mathbb{Z} \) and \( w \leq 0 \), for the model to meet limits (12) and (14). In addition, Euler’s integral representation of the hypergeometric function allows to further restrict the parameters of the model according to equation (16), since for \( x > 0 \), \(-1 < u < n + v < 0\) (with \( R_0 = 1 \))
\[
h'(x) = -\frac{kmrvx^{m−1}Γ(u)}{Γ(n)Γ(u−n)}\int_0^1 \frac{(1−x)^{−n−1}x^u}{(x'−v)^{m+1}}dx',
\]  
(88)
where \( Γ(x) \) is the Gamma function. Due to the positive integrand and the integral definition interval, the integral is positive, and because
\[
\frac{Γ(u)}{Γ(n)Γ(u−n)} > 0,
\]  
(89)
for \( n < u \), then for \( h'(x) < 0 \), \( k < 0 \). Simultaneously, in order that \( h'(x) > −1 \),
\[
\frac{u(r + 1)(u + 1)x_i^{2r+1}}{|k|^2v^2mn(m + 1)(n + 1)} > \frac{\binom{m + 2, n + 2; u + 2}{z}}{\binom{m + 1, n + 1; u + 1}{z}}.
\]  
(90)
where \( x_i \) is the inflection point, obtained from equation (87). Numerically it is found that for \( 0 < r < 2 \), \( m \geq 1 \), \( n \geq 1 \), \( n < u < n + 1 \), \( v < −1 \),
\[
x_i^{r+1} > \binom{m + 1, n + 1; u + 1}{z} x_i^{r+1},
\]  
(91)
so that if \( u > kmrv \), then \( h'(x) > −1 \).

Alternatively, a sufficient condition for \( h'(x) > −1 \) is
\[
\left(1 + \frac{1}{r}\right)_{w−1} (u) w > \frac{\binom{m + w, n + w; u + w + \frac{v}{x_i}}{z}}{|k|^w}(m)(w)(n)_w x_i^{r+1},
\]  
(92)
where \( (a)_w \) is the Pochhammer symbol.

On the other hand, when \( r < 0 \), for limit equation (10) to be satisfied, \( c = 2km/R_0 \), whereas for limits (11), (13) and (15), \( m > 0 \), \( n > 0 \) and, as in the previous case, \( u = w \). Likewise, when \( r \leq −2 \), limit equation (12) is satisfied\(^1\), however when \( r = −2 \),
\[
\lim_{x \to 0} h''(x) = -\frac{2kmv}{uR_0},
\]  
(93)
so that \( u = −nv \), which in turn implies that \( v < −1 \). When \( r = −2 \), limit given by equation (14) is fulfilled. Note that although the value of the constant \( c \) depends on the sign of \( r \), the equation (88) is still valid in this case, \( r \leq −2 \), so the restrictions on the parameters that were made previously, i.e. \( n < u \) and \( k < 0 \), remain valid.

Finally, the hypergeometric model can be expressed using equation (7),
\[
y(x) = x + \left(\frac{k}{R_0} - \frac{c}{m}\right) F_1\left(m, n, u; \frac{v}{x^3}\right) + \frac{c}{2m},
\]  
(94)
and contains the generalized Starobinsky type (59) and Hu–Sawicki (71) type models, since when \( u = n, r = −2 \), \( v = −1 \) and \( c = 2km/R_0 \), the first one is obtained, and when \( u = m, v = −(1 + r)t \) and \( c = 0 \), the second one is obtained, therefore equation (94) can be considered as a generalization of these models. An interesting property that can be observed when expressing the limits in equation (10) and (12) is that viable models in the hypergeometric form equation (94), admit constant solutions for the scalar curvature. In other words, the equation (4) is immediately satisfied when \( x \to 0 \).
\[
\frac{x(h''(x) - 1)}{2} = 2h(x) + 2λ.
\]  
(95)
This behavior can be observed when plotting the hypergeometric function for certain parameter values, as shown in figure 4, where the curve, \( f(x) = x \), can be appreciated. Specifically, we consider General Relativity, the

\(^{1}\) When \( r \in (0, 2) \), \( r = −1 \), both limits equation (12) and equation (14) are indeterminate, and for \( r = −1 \), \( \lim_{x \to 0} h''(x) = -\frac{kmv}{uR_0} \neq 0 \).
A cosmologically viable hypergeometric model within the $f(R)$ gravity theory has been developed based on the assumption of the existence of an inflection point in the $f(R)$ curve. The model satisfies viability conditions in the $(m, r)$ parameter plane and reproduces the $\Lambda$CDM model in a certain limit. This limit is expressed in terms of the dimensionless variable $x$, where $y(x) = x + h(x) + \lambda$, with $h(x)$ representing the deviation of the model from General Relativity (GR).

From a geometric perspective, the existence of the inflection point $x_i$, along with the monotonically decreasing behavior of $h(x)$, guarantees the satisfaction of the limits (10) and (11). This allows us to consider the model as a perturbation around the $\Lambda$CDM model. Moreover, as $x$ approaches infinity, the conditions $r = -1$, $m = +0$, and $m'(−1) = 0$ are fulfilled, as shown in figures 2 and 3. This enables $y(x)$ to exhibit a matter-dominated epoch.

The significance of the inflection point $x_i$ lies in allowing the model to approach asymptotically the behavior of the $\Lambda$CDM model.

The geometric constraints imposed by $x_i$ on the function $h(x)$ and its derivatives were utilized to formulate a differential equation. This equation was designed such that the roots of $xh''(x)$, modulated by a function $p(x)$, align with a term $h'(x)$ multiplied by the factor $(x^2 - x_f^2)$. Integrating this differential equation, the function $p(x)$ was chosen to facilitate the expression of the integrand as an exact differential. The resulting solution, equation (59), corresponds to Starobinsky’s model from 2007 [29].

Similarly, employing a comparable procedure, but this time expressing the factor as $(x^2 - x_f^2)$, with $r$ as a parameter of the model, a differential equation was constructed. The solution of this equation, equation (71), corresponded to a generalization of the Hu-Sawicki model [30].

Upon closer examination, it was discovered that the differential equations governing each individual model belonged to specific cases of the hypergeometric differential equation. This realization led to the establishment of the hypergeometric model, as described in equation (94). The model is characterized by five parameters: $m$, $n$, $r$, $u$, and $v$. However, the equation for the inflection point (87) imposes a constraint on the model as it serves as a necessary condition for its viability. Consequently, the number of independent parameters is reduced to four.

Furthermore, the constant $k$ is restricted to negative values, and the value of $c$ can be determined in a specific manner based on the classification into two sub-models: Hu-Sawicki type ($r > 0$), where $c = 0$; and Starobinsky type ($r < 0$), where $c = 2km/R_0$.

8. Concluding remarks

Starobinsky model, and the Hu-Sawicki model. At the same time, figure 5 illustrates an extended domain of the same data presented in figure 4. It provides a broader perspective on the behavior of the viable models and their incorporation of the cosmological constant. By extending the domain, we can observe how the models continue to exhibit parallel trends and convergence towards the cosmological constant, reinforcing their viability and consistency with observational data. Figures 4 and 5 demonstrate that the viable models do indeed incorporate the cosmological constant.

Figure 2. Characteristic functions $m$ and $r$ versus $x$. The inflection point, $x_i$, the $x_2$ point, in which $r = −1$, and the maximum points of $r(x), x_A$ and $x_B$, are appreciated.
When \( r > 0 \) and \( x > 0 \), in order for the model to satisfy limits (10) to (15), as well as condition equation (16), the parameters must satisfy certain conditions: \( m > 2/r, u > n > 2/r, v < 0 \), and \( m - n \not\in \mathbb{Z} \), in addition to equation (90). For instance, when \( 0 < r < 2, m \geq 1, n \geq 1, n < u < n + 1, v < -1 \), and \( u > kmnr \), the hypergeometric model is cosmologically viable.

On the other hand, only when \( r < 2, m > 0, \) and \( u > n > 0 \), the hypergeometric model is considered viable. Specifically, when \( r = -2 \), it is observed that \( u = -nv \) and \( v < -1 \).

The key feature of the hypergeometric model is its ability to encompass a family of functions that exhibit an inflection point while simultaneously mimicking the behavior of the \( \Lambda \text{CDM} \) model. Prominent examples of such models are the well-known Starobinsky and Hu-Sawicki models. The proposed hypergeometric model relies on four independent parameters, providing increased flexibility for adjusting to observational constraints at both cosmological and local scales.

To achieve this objective, appropriate computational tools, as mentioned in [33], are required. Nevertheless, the outlook for attaining this goal is promising. Future advancements in observational techniques, particularly in high-curvature scenarios like black holes or neutron stars, could offer valuable opportunities to test modified gravity theories. In these extreme environments, the effects of \( f(R) \) gravity on the dynamics of spacetime may become more pronounced and observable, leading to further insights into the viability of the hypergeometric model and related theories.
Data availability statement

No new data were created or analysed in this study.

Appendix A. Limit of $m'(r)$

Note that the derivative of $m(r)$ can be expressed as, where $h = h(x)$ and the same notation is used for its derivatives,

$$m'|_x = -\frac{(\lambda + x + h')^2}{\lambda + h} \frac{(1 + h' - xh'' + (1 + h')xh''')}{(1 + h')(1 + h')\left(1 - \frac{xh'}{\lambda + h}\right) + xh'' + \frac{x^2h'''}{\lambda + h}},$$

(A1)

such that, by equations (10), (12) and (14),

$$\lim_{x \to 0} \frac{(\lambda + x + h')^2}{\lambda + h} = \lim_{x \to 0} \frac{2(1 + h')^2 + 2(\lambda + x + h)h''}{h''} = \frac{2}{c},$$

(A2)

$$\lim_{x \to 0} \frac{xh'}{\lambda + h} = \lim_{x \to 0} \frac{2h'' + xh'''}{h''} = 2,$$

(A3)

and

$$\lim_{x \to 0} \frac{x^2h'''}{\lambda + h} = \lim_{x \to 0} \frac{2h'' + 4xh'' + x^3h^{(4)}}{h''} = 2,$$

(A4)

therefore

$$\lim_{x \to 0} m'(r)|_x = -2.$$  

(A5)

ORCID iDs

Roger Hurtado  @  https://orcid.org/0000-0002-3677-0969

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Figure 5. Starobinsky and Hu-Sawicki models have achieved remarkable success and have convincingly passed all observational tests, because they are small perturbations of the $\Lambda$CDM model. This behavior is equally observed here for viable hypergeometric models (same parameters as figure 4), it can be seen that all models tend to be parallel as the value of $x$ increases.
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