A projection and an effect in a synaptic algebra

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Abstract

We study a pair \( p, e \) consisting of a projection \( p \) (an idempotent) and an effect \( e \) (an element between 0 and 1) in a synaptic algebra (a generalization of the self-adjoint part of a von Neumann algebra). We show that some of Halmos’s theory of two projections (or two subspaces), including a version of his CS-decomposition theorem, applies in this setting, and we introduce and study two candidates for a commutator projection for \( p \) and \( e \).

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1 Introduction

In [16], P. Halmos studied two projection operators \( P \) and \( Q \) on a Hilbert space and proved a basic theorem, now called the \textit{CS-decomposition theorem}, that expresses \( Q \) in terms of \( P \) and positive contraction operators \( C \) and \( S \), called the \textit{cosine} and the \textit{sine} operators, respectively, for \( Q \) with respect to \( P \). For a lucid and extended exposition of Halmos’s theory of two projections,

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see [2]. In [12], we proved a generalization of the CS-decomposition theorem in the setting of a so-called synaptic algebra [12, Theorem 5.6].

In what follows, \( A \) is a synaptic algebra with enveloping algebra \( R \supseteq A \), \( P \) is the orthomodular lattice \([1, 18]\) of projections in \( A \), and \( E \) is the convex effect algebra \([4, 14]\) of all effects in \( A \). To help fix ideas, we note that the self-adjoint part of a von Neumann algebra, and more generally of an AW\(^*\)-algebra, forms a synaptic algebra. Numerous additional examples are given in the literature cited above.

In this article we generalize the CS-decomposition theorem for two projections \( p, q \in P \subseteq A \) to the case of a projection \( p \in P \) and an effect \( e \in E \) (Theorem 3.9 below), and we investigate two candidates for the commutator projection for the pair \( p \) and \( e \) (Section 5 below).

In our generalization of the CS-decomposition theorem, which we call the CBS-decomposition theorem, the cosine and sine effects \( c \) and \( s \) introduced in [12, Definition 4.2] are generalized (Definition 3.1 below) and supplemented by a third effect \( b \) (Definition 3.6 below).

Part of our motivation for the work in this article derives from our interest in the infimum problem as applied to the synaptic algebra \( A \), i.e., the problem of determining just when two effects \( e, f \in E \) have an infimum \( e \wedge f \) in \( E \), and if possible, finding a perspicuous formula for \( e \wedge f \) when it does exist. That this problem is non-trivial is indicated by a remark of P. Lahti and M. Mączynski in [19, p. 1674] that the partial order structure of \( E \) is “rather wild.” The development in [15] and [20] suggests that it might be possible to make progress on the infimum problem for \( A \) if the problem can be solved for the pair \( p, e \) with \( p \in P \) and \( e \in E \). We hope that our results in this article will cast some light on the latter problem. In Section 6 below, we illustrate the utility of the CBS-decomposition theorem by applying it to generalize a result of T. Moreland and S. Gudder concerning the infimum problem [20] to the setting of a synaptic algebra.

### 2 Some basic definitions, notation, and facts

In this section we briefly outline some notions that we shall need below. For the definition of a synaptic algebra and more details, see the literature cited above, especially [3] and [10]. In what follows, the notation := means ‘equals by definition,’ the ordered field of real numbers and its subfield of rational numbers are denoted by \( \mathbb{R} \) and \( \mathbb{Q} \), and ‘iff’ abbreviates ‘if and only if.’
The enveloping algebra $R$ of $A$ is a real linear associative algebra and if $a, b \in A$, it is understood that the product $ab$, which may or may not belong to $A$, is calculated in $R$. However, if $a$ commutes with $b$, in symbols $aCb$, then $ab = ba \in A$. The commutant and bicommutant of $a$ are defined and denoted by

$$C(a) := \{b \in A : aCb\} \text{ and } CC(a) := \{c \in A : c \in C(b) \text{ for all } b \in C(a)\},$$

respectively. There is a unity element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

As a subset of $R$, the synaptic algebra $A$ forms a real linear space which is partially ordered by $\leq$ and for which $1$ is a (strong) order unit. If $a, b \in A$ and $a \leq b$, we say that $b$ dominates $a$, or equivalently, that $a$ is a subelement of $b$.

If $a, b, c \in A$, then $ab + ba$, $abc + cba \in A$. Also $aba \in A$ and the quadratic mapping $b \mapsto aba$ is linear and order preserving on $A$.

If $0 \leq a \in A$, there exists a unique square root, denoted $a^{1/2} \in A$ such that $0 \leq a^{1/2}$ and $(a^{1/2})^2 = a$; moreover $a^{1/2} \in CC(a)$. Thus, if $0 \leq a$, then $C(a) = C(a^2) = C(a^{1/2})$. If $a \in A$, then $0 \leq a^2$, and the absolute value of $a$ is denoted and defined by $|a| := (a^2)^{1/2}$. We note that $|a| \in CC(a)$.

The positive part of $a$ is denoted and defined by $a^+ := \frac{1}{2}(|a| + a)$. Clearly, $a^+ \in CC(a)$.

Partially ordered by the restriction of $\leq$, the set $P := \{p \in A : p = p^2\}$ of projections in $A$ forms an orthomodular lattice (OML) \cite{1 \cite{18}, \cite{3}, \cite{5}] with $p \mapsto p^+ := 1 - p$ as the orthocomplementation. The meet (greatest lower bound) and join (least upper bound) of projections $p, q \in P$ are denoted by $p \land q$ and $p \lor q$, respectively. The projections $p, q \in P$ are orthogonal, in symbols $p \perp q$, iff $p \leq q^+$, and it turns out that $p \perp q \Rightarrow p + q = p \lor q$. A minimal nonzero projection in $P$ is called an atom. If $p, q \in P$ and $p$ is an atom, then either $p \land q = p$ (i.e., $p \leq q$) or else $p \land q = 0$.

Calculations in the OML $P$ are facilitated by the following theorem \cite{18} Theorem 5, p. 25].

**2.1 Theorem.** For $p, q, r \in P$, if any two of the relations $pCq$, $pCr$, or $qCr$ hold, then $p \land (q \lor r) = (p \land q) \lor (p \land r)$ and $p \lor (q \land r) = (p \lor q) \land (p \lor r)$.

To each element $a \in A$ is associated a unique projection $a^o \in P$ called the carrier of $a$ such that, for all $b \in A$, $ab = 0 \iff a^ob = 0 \iff ba^o = 0 \iff ba = 0$. It turns out that $aa^o = a^oa = a$, $a^o \in CC(a)$, $(a^2)^o = a^o$, $|a|^o = a^o$, and if
\( p \in P \) and \( e \in E \), then \( p^o = p \) and \( e \leq e^o \). Also, \( 0 \leq a \leq b \Rightarrow a^o \leq b^o \). Moreover, if \( p, q \in P \), then \((pq)^o = p \land (p^\perp \lor q)\) \([3\text{, Theorem 5.6}].\)

We shall have use for the next two lemmas which follow from [10\text{, Lemma 4.1}] and [11\text{, Theorem 5.5}].

2.2 Lemma. If \( 0 \leq a_1, a_2, \ldots, a_n \in A \), then \((\sum_{i=1}^n a_i)^o = \bigvee_{i=1}^n (a_i)^o\).

2.3 Lemma. If \( a, b, ab \in A \), then \((ab)^o = a^o b^o = b^o a^o = a^o \land b^o\).

The set \( E := \{ e \in A : 0 \leq e \leq 1 \} \) of effect elements (or for short, simply \textit{effects}) in \( A \) forms a convex effect algebra \([4\text{, 14}].\) If \( e \in E \), then the orthosupplement of \( e \) is denoted and defined by \( e^\perp := 1 - e \in E \). Two effects \( e \) and \( f \) are disjoint iff the only effect \( g \in E \) with \( g \leq e, f \) is \( g = 0 \). Every projection is an effect, i.e., \( P \subseteq E \); in fact, \( P \) is the extreme boundary of the convex set \( E \).

2.4 Lemma. Let \( p, q \in P \). Then: (i) The infimum \( p \land q \) of \( p \) and \( q \) in \( P \) is also the infimum of \( p \) and \( q \) in \( E \). (ii) The supremum \( p \lor q \) of \( p \) and \( q \) in \( P \) is also the supremum of \( p \) and \( q \) in \( E \).

Proof. (i) Of course \( p \land q \leq p, q \), and it remains to prove that if \( e \in E \) with \( e \leq p, q \), then \( e \leq p \land q \). But, if \( e \leq p, q \), then \( e^o \leq p, q \), whence \( e \leq e^o \leq p \land q \).

(ii) Of course \( p, q \leq p \lor q \), and it remains to prove that if \( e \in E \) with \( p, q \leq e \), then \( p \lor q \leq e \). So assume that \( p, q \leq e \), and therefore that \( e^\perp \leq p^\perp, q^\perp \). It follows that \((e^\perp)^o \leq p^\perp, q^\perp \), whence \( p, q \leq ((e^\perp)^o)^\perp \in P \). Consequently, \( p \lor q \leq ((e^\perp)^o)^\perp \). But \( e^\perp \leq (e^\perp)^o \), so \(((e^\perp)^o)^\perp \leq e^\perp = e \), and we have \( p \lor q \leq e \).

In view of Lemma 2.3, no confusion will result if an existing infimum (respectively, supremum) in \( E \) of effects \( e, f \in E \) is denoted by \( e \land f \) (respectively, by \( e \lor f \)).

By [3\text{, Theorem 2.6 (v)}], an effect \( e \in E \) is a projection iff \( e \) is sharp, i.e., iff \( e \) is disjoint from its own orthosupplement \( e^\perp \) iff \( e \land e^\perp = 0 \). Moreover, the carrier \( e^o \) of an effect \( e \in E \) is the smallest projection that dominates \( e \), so \( E \) is a \textit{sharply dominating} effect algebra \([13]\).

The next theorem and its corollary provide useful ways to stipulate that a projection \( p \) either dominates or is dominated by an effect \( e \).

2.5 Theorem ([3\text{, Theorem 2.4}]). Let \( p \in P \) and \( e \in E \). Then the following conditions are mutually equivalent: (i) \( e \leq p \). (ii) \( e = ep = pe \). (iii) \( e = pep \). (iv) \( e = ep \). (v) \( e = pe \).
2.6 Corollary. If \( p \in P \) and \( e \in E \), then the following conditions are mutually equivalent: (i) \( p \leq e \). (ii) \( p = ep = pe \). (iii) \( p = ep \). (iv) \( p = pe \).

Proof. In Theorem 2.5, replace \( e \) by \( e^\perp = 1 - e \) and \( p \) by \( p^\perp = 1 - p \). Then \( p \leq e \iff e^\perp \leq p^\perp, 1 - e = (1 - e)(1 - p) \iff p = ep \), and \( 1 - e = (1 - p)(1 - e) \iff p = pe \). □

As a consequence of Theorem 2.5 and its corollary, if a projection \( p \) and an effect \( e \) are comparable (i.e., \( e \leq p \) or \( p \leq e \)), then \( pCe \). One of the reasons that the order structure of \( E \) is so “wild” is that the same does not hold for two effects.

2.7 Lemma. Suppose that \( e, f \in E \) and \( p \in P \). Then: (i) If \( eCf \), then \( ef \in E \) and \( ef \leq e, f \). (ii) If \( pCf \), then \( pf = fp = pf = p \land f \), the infimum of \( p \) and \( f \) in \( E \).

Proof. (i) Assume that \( e, f \in E \) and \( ef = fe \). By [3, Lemma 1.5], \( 0 \leq ef \). Likewise, \( 0 \leq e, 1 - f \) and \( eCf(1 - f) \), so \( 0 \leq e(1 - f) = e - ef \), whence \( ef \leq e \leq 1 \), so \( ef \in E \). By symmetry, \( ef \leq f \).

(ii) Suppose that \( pCf \) and let \( g \in E \) with \( g \leq p, f \). By (i), \( pf \leq p, f \). Also, by Theorem 2.5, \( g = pgp \), and as \( g \leq f \), we have \( g = pgp \leq pf = p^2f = pf \), whence \( pf = p \land f \). □

In part (i) of Lemma 2.7, we note that although \( ef = fe \in E \), it is not necessarily the infimum of \( e \) and \( f \) in \( E \). In fact, P.J. Lahti and M.J. Mączynski [19] page 1675] give an example of an effect operator \( e \) on a two-dimensional Hilbert space such that the infimum of the commuting effects \( e \) and \( e^\perp = 1 - e \) does not exist in \( E \).

2.8 Lemma. Suppose that \( e \in A \) with \( 0 \leq e \). Then: (i) \( e \in E \Rightarrow 0 \leq e^2 \leq e \leq 1 \Rightarrow e^2 \in E \). (ii) \( e^2 \leq 1 \Leftrightarrow e \in E \). (iii) \( e \in E \Rightarrow e - e^2 = ee^\perp \in E \).

Proof. (i) If \( e \in E \), then \( e^2 \leq e \) by Lemma 2.7 (i).

(ii) Suppose that \( e^2 \leq 1 \). Then \( 0 \leq (1 - e)^2 + (1 - e^2) = 2(1 - e) \), so \( e \leq 1 \), whence \( e \in E \). Conversely, if \( e \in E \), then by (i), \( e^2 \leq e \leq 1 \).

(iii) If \( e \in E \), then \( 0 \leq e - e^2 = e(1 - e) = ee^\perp \) by (i) and \( e - e^2 \leq e \leq 1 \), so \( e - e^2 \in E \). □

Each element \( a \in A \) determines and is determined by a one-parameter family of projections \( (p_{a,\lambda})_{\lambda \in \mathbb{R}} \) called its spectral resolution and defined by \( p_{a,\lambda} := 1 - ((a - \lambda I)^+)^\circ \) for all \( \lambda \in \mathbb{R} \) [3, Definition 8.2]. See [3, §8], especially
Theorem 4.10] for the basic properties of the spectral resolution. We note that by [3, Theorem 8.10], if \(a, b \in A\), then \(bCa\) iff \(bCp_{a,\lambda}\) for all \(\lambda \in \mathbb{R}\).

By [3, Theorem 8.4 (vii)], the spectral resolution \((p_{a,\lambda})_{\lambda \in \mathbb{R}}\) is uniquely determined by the corresponding rational spectral resolution \((p_{a,\mu})_{\mu \in \mathbb{Q}}\) according to the formula

\[
p_{a,\lambda} = \bigwedge \{p_{a,\mu} : \lambda \leq \mu \in \mathbb{Q}\} \quad \text{for each } \lambda \in \mathbb{R}.
\]

2.9 Remark. If \(a \in A\) and \(q \in P\), then since commutativity of projections is preserved under the formation of arbitrary existing infima, the formula above implies that \(qCa\) iff \(qCp_{a,\mu}\) for all \(\mu \in \mathbb{Q}\).

Let \(q \in P\). Then with the partial order and operations inherited from \(A\), the subset

\[
qAq := \{qaq : a \in A\} = \{a \in A : a = qa\} = \{a \in A : a = qa = aq\} \subseteq A
\]
is a synaptic algebra in its own right with unity element \(q\) and with \(qRq\) as its enveloping algebra [3, Theorem 4.10]. The OML of projections in \(qAq\) is \(P[0, q] := \{v \in P : v \leq q\}\) with the orthocomplementation \(v \mapsto v^\perp := v^\perp \land q\).

Likewise, the set of all effects in \(qAq\) is \(E[0, q] := \{f \in E : f \leq q\}\) with the orthosupplementation \(f \mapsto f^\perp := q - f = (1 - f)q = f^\perp q = qf^\perp = f^\perp \land q\) (Lemma 2.7 (ii)). Let \(a \in qAq\). Then \(|a|, a^+, a^o\), and if \(0 \leq a, a^{1/2}\), belong to \(qAq\) and coincide with the absolute value, the positive part, the carrier, and the square root of \(a\), respectively, as calculated in \(qAq\).

2.10 Lemma. Let \(a \in A\), \(f \in E\), and \(q \in P\). Then: (i) If \(qCa\), then the spectral resolution of \(qa = aq \in qAq\) as calculated in \(qAq\) is given by \((qp_{a,\lambda})_{\lambda \in \mathbb{R}} = (p_{a,\lambda} \land q)_{\lambda \in \mathbb{R}}\). (ii) If \(qCf\), then the spectral resolution of \(qf =fq = f \land q \in qAq\), as calculated in \(qAq\), is given by \((qp_{f,\lambda})_{\lambda \in \mathbb{R}} = (p_{f,\lambda} \land q)_{\lambda \in \mathbb{R}}\).

Proof. Part (i) is proved by a direct calculation using [3, Definition 8.2 and Theorem 4.10] and the fact that \(qCa\) implies \(qCp_{a,\lambda}\), whence \(p_{a,\lambda} \land q = qp_{a,\lambda}\) for all \(\lambda \in \mathbb{R}\). Part (ii) follows from (i) and Lemma 2.7 (ii).

2.11 Lemma. Suppose that \(p\) is an atom in \(P\). Then: (i) \(pAp = \{\lambda p : \lambda \in \mathbb{R}\}\). (ii) If \(a \in A\), there exists a unique \(\lambda \in \mathbb{R}\) such that \(pap = \lambda p\). (iii) If \(f \in E\) and \(pfp = \lambda p\), then \(0 \leq \lambda \leq 1\).

Proof. (i) Since \(p\) is an atom, it follows that \(0\) and \(p \neq 0\) are the only projections in the synaptic algebra \(pAp\), from which, using spectral theory in \(pAp\), (i) follows. Part (ii) follows from the fact that \(p \neq 0\), and (iii) is a consequence of \(0 \leq f \leq 1 \Rightarrow 0 \leq pfp \leq p1p = p^2 = p \leq 1\).
An element \( u \in A \) is said to be a symmetry \([10]\) iff \( u^2 = 1 \), and a partial symmetry is an element \( t \in A \) such that \( t^2 \in P \). As a consequence of the uniqueness theorem for square roots, a projection is the same thing as a partial symmetry \( p \) such that \( 0 \leq p \). If \( t \in A \) is a partial symmetry, then \( u := t + (t^2)^{\perp} \) is a symmetry called the canonical extension of \( t \).

If \( a \in A \) there is a uniquely determined partial symmetry \( t \in A \), called the signum of \( a \), such that \( t^2 = a^o \) and \( a = |a|t \). Moreover, \( t \in CC(a) \), \( t^o = a^o \), and if \( u = t + (t^2)^{\perp} \) is the canonical extension of \( t \) to a symmetry, then \( u \in CC(a) \) and \( a = |a|u = u|a| \). The latter formula is called the polar decomposition of \( a \). It turns out that the symmetry \( u \) in the polar decomposition of \( a \) is uniquely determined.

If \( a, b \in A \) and \( u \in A \) is a symmetry, it is not difficult to verify that \( a \leq b \Leftrightarrow uau \leq ubu \) and that \( ua^o u = (uau)^o \).

Two projections \( p, q \in P \) are exchanged by a symmetry \( u \in A \) iff \( upu = q \) (whence, automatically, \( uqu = p \) and they are exchanged by a partial symmetry \( t \in A \) iff \( tpt = q \) and \( tqt = p \). If \( p \) and \( q \) are exchanged by a partial symmetry \( t \), then they are exchanged by the canonical extension \( u := t + (t^2)^{\perp} \) of \( t \) to a symmetry.

If \( p \in P \) and \( a \in A \), then by direct calculation using the fact that \( p^\perp = 1 - p \), one obtains the well-known Peirce decomposition of \( a \) with respect to \( p \), namely

\[
a = pap + pap^\perp + p^\perp ap + p^\perp ap^\perp.
\]

We refer to \( pap + p^\perp ap^\perp \) as the diagonal part of \( a \) with respect to \( p \) and to \( pap^\perp + p^\perp ap \) as the off-diagonal part of \( a \) with respect to \( p \). We note that \( pap, p^\perp ap^\perp \), and the diagonal part \( pap + p^\perp ap^\perp \) of \( a \) belong to \( A \). Also, although \( pap^\perp \) and \( p^\perp ap \) belong to the enveloping algebra \( R \), but not necessarily to \( A \), the off-diagonal part \( pap^\perp + p^\perp ap \) belongs to \( A \).

**2.12 Lemma** ([12] Theorem 2.12). If \( 0 \leq a \in A \) and \( p \in P \), then \( a = 0 \) iff the diagonal part of \( a \) with respect to \( p \) is zero.

**2.13 Lemma.** Let \( a \in A \) and \( p \in P \). Then the following conditions are mutually equivalent: (i) \( pCa \). (ii) The off-diagonal part of \( a \) with respect to \( p \) is zero. (iii) \( pa \in A \). (iv) \( ap \in A \). (v) \( pap^\perp = 0 \). (vi) \( p^\perp ap = 0 \).

*Proof.* The equivalence (i) \( \Leftrightarrow \) (ii) follows from [12] Theorem 2.12. If \( pa \in A \), then since \( pa + ap \in A \), we have \( ap = (pa + ap) - pa \in A \); similarly, \( ap \in A \Rightarrow pa \in A \), and we have (iii) \( \Leftrightarrow \) (iv). To prove that (i) \( \Leftrightarrow \) (iii), note
that \( pCa \Rightarrow pa = ap \in A \). Conversely, suppose that \( pa \in A \). Then, since (iii) \( \Leftrightarrow \) (iv), \( ap \in A \). Also, \((1 - p)pa = 0\), so \( pa(1 - p) = 0\), and we have \( pa = pap \). Similarly, \( ap(1 - p) = 0\), so \( (1 - p)ap = 0\), i.e., \( ap = pap\), whence \( pa = pap = ap \). This proves that (i) \( \Leftrightarrow \) (iii), and it follows that conditions (i)–(iv) are mutually equivalent.

If (i) holds, then \( pap \perp = app \perp = 0\), so (i) \( \Rightarrow \) (v). Conversely, if (v) holds, then \( 0 = pap \perp = pa(1 - p) = pa - pap\), so \( pa = pap \in A\), and we have (v) \( \Rightarrow \) (iii). Similarly, (i) \( \Rightarrow \) (vi) \( \Rightarrow \) (iv).

\[ \square \]

3 A projection and an effect

3.1 Standing Assumption. For the remainder of this article we assume that \( p \in P\), and \( e \in E\).

In this section we associate with the pair \( p, e\) four special effects, \( c, s, j, \) and \( b\) (Definitions 3.2, 3.4, and 3.6) and a symmetry \( k\) (Definition 3.8). Using \( c, s, j, b, \) and \( k\), we rewrite the Peirce decomposition of \( e\) with respect to \( p\), thus obtaining the CBS-decomposition of \( e\) with respect to \( p\) (Theorem 3.9).

In the next definition we generalize to the present case the definitions of the cosine and sine effects for a projection \( q\) with respect to the projection \( p\) [12, Definition 4.2].

3.2 Definition. Since \( 0 \leq e, e^\perp\), we have \( 0 \leq pep + p^\perp e^\perp p^\perp\) and \( 0 \leq pe^\perp p + p^\perp ep^\perp\). Thus, we define the cosine effect \( c\) and the sine effect \( s\) for \( e\) with respect to the projection \( p\) as follows:

\[
(1) \quad c := (pep + p^\perp e^\perp p^\perp)^{1/2}.
\]

\[
(2) \quad s := (pe^\perp p + p^\perp ep^\perp)^{1/2}.
\]

3.3 Lemma. (i) \( c^2 = 1 - p + pe + ep - e\). (ii) \( s^2 = p - pe - ep + e\). (iii) \( c^2 + s^2 = 1\). (iv) \( c^2 p = pc^2 = pep\) and \( s^2 p^\perp = p^\perp s^2 = p^\perp ep^\perp\). (v) \( c, s \in C(p)\) and \( cCs\). (vi) \( c, s, cs, c^2, s^2, c^2 s^2 \in E\), \( c^2 \leq c\), and \( s^2 \leq s\).

Proof. Parts (i) and (ii) follow from straightforward calculations using the facts that \( p^\perp = 1 - p\) and \( e^\perp = 1 - e\). Obviously, (iii) follows from (i) and (ii).

By (i) we have \( c^2 p = p - p + pe + ep - ep = pep\) and \( pc^2 = p - p + pe + pep - pe = pep\). Using (ii), a similar calculation yields \( s^2 p^\perp = p^\perp s^2 = p^\perp ep^\perp\), and we have (iv).
As $0 \leq c, s$, it follows that $C(c) = C(c^2)$ and $C(s) = C(s^2)$. By (iv), $p \in C(c^2)$ and $p \in C(s^2)$, whence $pCc$ and $pCs$, and (v) is proved.

We have $0 \leq c, s$ and since $c^2, s^2 \leq c^2 + s^2 = 1$, we have $c^2, s^2 \leq 1$, whence by Lemma 2.8, $c^2 \leq c \in E$, and $s^2 \leq s \in E$. Thus, since $c, s \in E$ and $cCs$, Lemma 2.7 (i) implies that $cs \in E$, and (vi) is proved.

As $e \in E$, we have $e^2 \in E$ with $e - e^2 = ee^\perp \in E$ (Lemma 2.8 (iii)), whence $p(e - e^2)p + p^\perp(e - e^2)p^\perp \geq 0$.

3.4 Definition. We define $j \in A$ by

$$j := (p(e - e^2)p + p^\perp(e - e^2)p^\perp)^{1/2},$$

i.e., $0 \leq j$ and $j^2$ is the diagonal part of $e - e^2 = ee^\perp$ with respect to $p$.

In the next lemma we obtain an important relation between $c^2 s^2$, the diagonal part $j^2$ of $e - e^2$ with respect to $p$, and the square of the off-diagonal part $p e p^\perp + p^\perp e p$ of $e$ with respect to $p$.

3.5 Lemma. $c^2 s^2 = (cs)^2 = j^2 + (p e p^\perp + p^\perp e p)^2$.

Proof. By parts (i) and (ii) of Lemma 3.3,

$$c^2 s^2 = (1 - p + pe + ep - e)(p - pe - ep + e) = (p - pe - ep + e) - (p - pe - ep + e)^2$$

$$= p - pe - ep + e - p + pe + pep - pep - pepe - pe^2 p + pe^2$$

$$+ ep - epe - epep + epe - ep + epe + e^2 p - e^2$$

$$= e - e^2 + 2p(e - e^2)p - (e - e^2)p - p(e - e^2) + pe^2 p + epe - epep - pepe. \quad (1)$$

Also,

$$(p e p^\perp + p^\perp e p)^2 = p e p^\perp e p + p^\perp e p e p^\perp = pe^2 p + epe - epep - pepe \quad (2)$$

and

$$j^2 = p(e - e^2)p + (1 - p)(e - e^2)(1 - p)$$

$$= e - e^2 + 2p(e - e^2)p - (e - e^2)p - p(e - e^2). \quad (3)$$

Combining Equations (1), (2), and (3), we obtain the desired result. \qed
3.6 Definition. By Lemma 3.5, \(0 \leq c^2s^2 - j^2\), which enables us to define
\[b := (c^2s^2 - j^2)^{1/2}\.\]
We refer to \(b\) as the commutator effect for the pair \(p, e\) (see Lemma 3.11 below).

3.7 Theorem. (i) \(pCj\) and \(pCb\). (ii) \(b \in E\). (iii) \(b = |pep^\perp + p^\perp ep|\).

Proof. (i) Since
\[p(p(e - e^2)p + p^\perp(e - e^2)p^\perp) = p(e - e^2)p + p^\perp(e - e^2)p^\perp,\]
we have \(pC(p(e - e^2)p + p^\perp(e - e^2)p^\perp)\), and since
\[j = (p(e - e^2)p + p^\perp(e - e^2)p^\perp)^{1/2}\]
it follows that \(pCj\). Also, by Lemma 3.3 (v), \(pC(c^2s^2)\), and therefore \(pC(c^2s^2 - j^2)\). As \(b = (c^2s^2 - j^2)^{1/2}\), it follows that \(pCb\).

(ii) Evidently, \(0 \leq b\). Also by Lemma 3.3 (vi), \(b^2 \leq c^2s^2 \leq 1\), and it follows from Lemma 2.8 (ii) that \(b \in E\).

Part (iii) follows immediately from Lemma 3.5 and Definition 3.6.

3.8 Definition. As per Theorem 3.7 (iii), we define the symmetry \(k\) by polar decomposition of \(pep^\perp + p^\perp ep\), so that
\[pep^\perp + p^\perp ep = |pep^\perp + p^\perp ep|k = bk = kb\]
where \(k \in CC(pep^\perp + p^\perp ep)\).

3.9 Theorem (CBS-decomposition).
\[e = c^2p + bk + s^2p^\perp, \text{ where}\]
(i) \(pep = c^2p = pc^2\) and \(p^\perp ep^\perp = s^2p^\perp = p^\perp s^2\).
(ii) \(b = |pep^\perp + p^\perp ep| = (c^2s^2 - j^2)^{1/2} \in E\).
(iii) \(k\) is a symmetry and \(pep^\perp + p^\perp ep = bk = kb\).
(iv) \(cCp, sCp, cCs, bCp,\) and \(k \in CC(pep^\perp + p^\perp ep)\).
(v) \(pbk = bpk = bkp^\perp = pep^\perp, \text{ whence } b(pk - kp^\perp) = b^\perp(pk - kp^\perp) = 0\).
(vi) \( p^+bk = bp^+k = bkp = p^+ep. \)

**Proof.** Parts (i), (ii), (iii), and the formula \( e = c^2p + bk + s^2p^+ \) follow from Lemma 3.3 (iv), Lemma 3.7 (iii), Definition 3.8, and the Pierce decomposition of \( e \) with respect to \( p \). Part (iv) is a consequence of Lemma 3.3 (v), Lemma 3.7 (i), and Definition 3.8.

By (iii) and the fact that \( bCp \), we have \( bpk = pbk = p(pkp + p^+ep) = pep = (pep + p^+ep)p^+ = bkp^+, \) whence \( b(pk - kp^+) = pep - pep^+ = 0, \) proving (v). Part (vi) follows immediately from (v). \( \square \)

As a consequence of the next lemma, in case \( e \) is a projection, then the CBS-decomposition theorem reduces to the generalized CS-decomposition theorem \([12, \text{Theorem 5.6}]\).  

**3.10 Lemma.** The following conditions are mutually equivalent: (i) \( e \) is a projection. (ii) \( j = 0 \). (iii) \( b = cs. \)

**Proof.** By Lemma 2.12 (i), \( e - e^2 = 0 \) iff \( j = 0 \), whence (i) \( \iff \) (ii). That (ii) \( \iff \) (iii) is an immediate consequence of Definition 3.6. \( \square \)

**3.11 Lemma.** The following conditions are mutually equivalent: (i) \( pCe. \) (ii) \( b = 0 \). (iii) \( b^o = 0 \). (iv) \( cs = j. \) (v) \( e = c^2p + s^2p^+. \)

**Proof.** The equivalence (i) \( \iff \) (ii) follows from Lemma 2.13 and Theorem 3.7 (iii), and the equivalence (ii) \( \iff \) (iii) is obvious. The equivalence (ii) \( \iff \) (iv) is a consequence of Definition 3.6 so (i)–(iv) are mutually equivalent. That (ii) \( \Rightarrow \) (v) follows from Theorem 3.9 and since \( p \) commutes with both \( c^2 \) and \( s^2 \), it is clear that (v) \( \Rightarrow \) (i). \( \square \)

**3.12 Definition.** If \( a \in A, q \in P, \) and \( aCq \), then the component of \( a \) in the synaptic algebra \( qAq \) is denoted and defined by \( a_q := aq = qa = qaq \in qAq. \)

If \( a \in A, q \in P, \) and \( aCq \), it is easy to see that \( a = a_q + a_{q^+} \) is the unique decomposition of \( a \) as a sum of an element in \( qAq \) and an element in \( q^+Aq^+. \) This decomposition can be useful in deducing properties of \( a \) from properties of its components \( a_q \in qAq \) and \( a_{q^+} \in q^+Aq^+. \)

**3.13 Lemma.** Let \( f \in E, q \in P, \) and suppose that \( fCq. \) Then: (i) The component \( f_q = fq = qf \) is an effect in \( qAq. \) (ii) The orthosupplement of \( f_q \) in \( E[0, q] \) is the component of \( f^\perp \) in \( qAq, i.e., f^\perp_q = qf^\perp = f^\perp q = f^\perp \land q = (f^\perp)_q. \)
4.1 Definition. If we derive some information about the carriers of the effects
The assumptions and notation of Section 3 remain in force. In this section

4 Carriers and projection-free effects

Proof. By Lemma 2.7 (ii), \( qf = fq = f \wedge q \in E[0, q] \), proving (i). Also,
\( f_q^{-1} = (fq)^{-1} = (1 - fq)q = q - fq = (1 - f)q = f^\perp q = f^\perp \wedge q = (f^\perp)_q \),
proving (ii).

3.14 Theorem. For \( p \in P \) and \( e \in E \), suppose that \( q \in P \) with \( qCp \) and \( qCe \). Then: (i) \( q \) commutes with \( c, s, b, \) and \( k \). (ii) The cosine, sine, and
commutator effects for \( e_q \) with respect to \( p_q = pq = qp = p \wedge q \) as calculated
in \( qAq \) are \( c_q = cq = qc = c\wedge q, s_q = sq = qs = s\wedge q, \) and \( b_q = bq = qb = b\wedge q, \)
respectively. (iii) The CBS-decomposition of \( e_q \) with respect to \( p_q \) in \( qAq \) is
\( e_q = c_q^2p_q + b_qk_q + s_q^2p_q^{-1}q = q(c^2p + bk + s^2p^\perp) = (c^2p + bk + s^2p^\perp)q. \)

Proof. (i) As \( c = (pep + p^\perp e^\perp p^\perp)^{1/2} \in CC(pep + p^\perp e^\perp p^\perp), \) we have \( qCc \) and
similarly \( qCs \). Likewise, \( qCb \) follows from \( b = |pep + p^\perp ep| \) (Theorem 3.7
(iii)), and \( qCk \) follows from \( k \in CC(pep^\perp + p^\perp ep). \)

(ii) Obviously, \( p_qe_qp_q = qpep = pepq. \) Also, as \( pCq, \) we have \( p_q^{-1} = p^\perp \wedge q = p^\perp q = qp^\perp. \) Moreover, \( e_q^{-1} = ge^\perp = e^\perp q = ge^\perp q. \) Therefore the
cosine effect for \( e_q \) with respect to \( p_q \) in \( qAq \) is
\[ (p_qe_qp_q + p_q^{-1}e_q^{-1}p_q^{-1})^{1/2} = (pepq + p^\perp e^\perp p^\perp q)^{1/2} = cq = c_q. \]

Similar computations take care of \( s_q \) and \( b_q. \) Part (iii) follows from (ii).

4 Carriers and projection-free effects

The assumptions and notation of Section 3 remain in force. In this section
we derive some information about the carriers of the effects \( e, c, s, j, \) and
\( b. \) Also, we introduce two special projections, \( z \) and \( t, \) associated with the
effect \( e \) (Definition 4.3 below).

If \( f \in E, q \in P, \) and \( q \leq f, \) we say that \( q \) is a subprojection of \( f; \) likewise,
if \( g \in E \) and \( g \leq f, \) we say that \( g \) is a subeffect of \( f. \)

4.1 Definition. If \( f \in E \) and the only subprojection of \( f \) is 0, we say that
\( f \) is projection free.

Obviously, every subeffect of a projection-free effect is projection free.

4.2 Lemma. (i) If \( f \in E, \) then \( ((f^\perp)^o)^\perp \) is the largest subprojection of \( f. \)
(ii) \( f \) is projection free iff \( (f^\perp)^o = 1. \) (iii) \( f^\perp \) is projection free iff \( f^o = 1. \)
(iv) \( f - ((f^\perp)^o)^\perp \) and \( f^\perp - (f^o)^\perp \) are projection-free effects.
Proof. Part (i) follows from the fact that \((f^\perp)^o\) is the smallest projection that dominates \(f^\perp\) [3 Theorem 2.10 (iv)], and parts (ii) and (iii) are immediate consequences of (i).

(iv) By (i), \(((f^\perp)^o)^\perp\) is a subprojection of \(f\), so \(g := f - ((f^\perp)^o)^\perp\) is an effect. We have \(g^\perp = 1 - f + ((f^\perp)^o)^\perp = f^\perp + ((f^\perp)^o)^\perp\), whence by Lemma 4.2, \((g^\perp)^o = (f^\perp)^o \lor ((f^\perp)^o)^\perp = 1\), so \(g\) is projection free by (ii). Similarly, \(f^\perp - (f^o)^\perp\) is a projection-free effect. \(\square\)

4.3 Definition. In what follows, \(z := ((e^\perp)^o)^\perp\) is the largest subprojection of \(e\) and \(t := ((e^\perp)^o)^{\perp\perp} = (e^\perp)^o\) is the largest subprojection of \(e^\perp\).

We note that \((e^\perp)^o = z^\perp\) and \(e^o = t^\perp\). Evidently, \(e \in P \Leftrightarrow e = z = t^\perp\).

4.4 Theorem.

(i) \(z, t \in P \cap CC(e), z \leq e \leq e^o, t \leq e^\perp \leq (e^\perp)^o, \) and \(e - z, e^\perp - t \in E\).

(ii) \(e\) is projection free iff \(z = 0\) iff \((e^\perp)^o = 1\) and \(e^\perp\) is projection free iff \(t = 0\) iff \(e^o = 1\).

(iii) \(z \perp t\), i.e., \((e^o)^\perp \leq (e^\perp)^o\).

(iv) \(e - z\) and \(e^\perp - t\) are projection-free effects.

(v) \((e^o - z)^o = e^o - z = e^o \land z^\perp = t^\perp \land z^\perp = (t \lor z)^\perp = (t + z)^\perp\).

(vi) \((e^\perp - t)^o = (e - z)^o = (t + z)^\perp\).

Proof. (i) By [3 Theorem 2.10 (vi)], \(z^\perp = ((e^\perp)^o)^\perp \in CC(e^\perp)\), from which \(z \in P \cap CC(e)\) follows; similarly, \(t \in P \cap CC(e)\).

(ii) Part (ii) follows immediately from Lemma 4.2 (ii).

(iii) Since \(e \leq e^o\), it follows that \((e^o)^\perp \leq e^\perp\), and therefore \(t = (e^o)^\perp = ((e^o)^\perp)^o \leq (e^\perp)^o = z^\perp\).

(iv) Part (iv) follows immediately from Lemma 4.2 (iv).

(v) We have \(e = z + (e - z)\), where \(z, e - z \in E\), whence by Lemma 2.2, \(e^o = z^o \lor (e - z)^o = z \lor (e - z)^o\). Also, \(e - z \leq 1 - z = z^\perp\), whence \((e - z)^o \leq z^\perp\), and it follows that \(e^o = z \lor (e - z)^o = z + (e - z)^o\), so \((e - z)^o = e^o - z\). Also, since \(z \leq e^o\), we have \(e^o - z = e^o \land z^\perp = t^\perp \land z^\perp\), and the remaining equalities follow from De Morgan and the fact that \(z \perp t\).

(vi) Proceeding as in the proof of (v), we have \((e^\perp - t)^o = (e^\perp)^o - t = (e^\perp)^o \land t^\perp = z^\perp \land t^\perp = t^\perp \land z^\perp = (e - z)^o = (t + z)^\perp\). \(\square\)
4.5 Corollary. (i) \( e - e^2 = (e - z) - (e - z)^2 \leq e - z \). (ii) \( e - e^2 \) is projection free. (iii) \((e - e^2)^o = t^\perp \wedge z^\perp = (e - z)^o = e^o - z\).

Proof. (i) Since \( z \leq e \) and \( z \in P \), we have \( ze = ez = z \), whence \( (e - z) - (e - z)^2 = e - z - (e^2 - ez - ze + z) = e - e^2 \) and \( e - e^2 \leq e - z \).

(ii) By Theorem 4.4 (iv), \( e - z \) is projection free; by part (i), \( e - e^2 \) is a subeffect of \( e - z \); therefore \( e - e^2 \) is projection free.

(iii) By Lemma 2.2 (ii) \( (e - e^2)^o = (e^o(e^o))^o = (e^o)^o = t^\perp z^\perp = t^\perp \wedge z^\perp \).

4.6 Theorem.

(i) \( c^o = (p \lor z^\perp) \land (p^\perp \lor t^\perp) \) and \( s^o = (p \lor t^\perp) \land (p^\perp \lor z^\perp) \).

(ii) \( (cs)^o = c^o s^o = s^o c^o = c^o \land s^o \).

(iii) \( (c^2 s^2)^o = (cs)^o = (p \lor z^\perp) \land (p \lor t^\perp) \land (p^\perp \lor z^\perp) \land (p^\perp \lor t^\perp) \).

(iv) \( j^o = (p \lor (t^\perp \lor z^\perp)) \land (p^\perp \lor (t^\perp \lor z^\perp)) \).

(v) \( s^o \leq c^2 \leq c \) and \( c^o \leq s^2 \leq s \).

(vi) \( (s^o)^o = e(s^o)^o = (s^o)^o \land e = (s^o)^o p = p(s^o)^o = (s^o)^o \land p \).

(vii) \( (c^o)^o = e(c^o)^o = (c^o)^o \land e = (c^o)^o p^\perp = p^\perp (c^o)^o = (c^o)^o \land p^\perp \).

Proof. (i) Since \( 0 \leq pep, p^\perp q^\perp p^\perp \), we infer from [3, Theorem 4.9 (v)] and Lemma 2.2 that

\[
c^o = [(pep + p^\perp e^o p^\perp)^{1/2}]^o = (pep + p^\perp e^o p^\perp)^o = (pep)^o \lor (p^\perp e^o p^\perp)^o = (pep)^o \lor (p^\perp e^o p^\perp)^o = (pep)^o \lor (p^\perp e^o p^\perp)^o = [p \land (p^\perp \lor t^\perp)] \lor [p \land (p^\perp \lor z^\perp)] = [p \land (p^\perp \lor t^\perp)] \lor w, \tag{1}
\]

where \( w := p^\perp \land (p \lor z^\perp) \). Now \( pC(p^\perp \lor t^\perp) \) and \( pCw \), whence

\[
[p \land (p^\perp \lor t^\perp)] \lor w = (p \lor w) \land (p^\perp \lor t^\perp \lor w). \tag{2}
\]

But \( pCp^\perp \) and \( pC(p \lor z^\perp) \), whence

\[
p \lor w = p \lor [p^\perp \land (p \lor z^\perp)] = (p \lor p^\perp) \land (p \lor p \lor z^\perp) = p \lor z^\perp. \tag{3}
\]

Furthermore, since \( w \leq p^\perp \),

\[
p^\perp \lor t^\perp \lor w = p^\perp \lor t^\perp. \tag{4}
\]

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By Equations (3) and (4),

\[(p \lor w) \land (p^\perp \lor t^\perp \lor w) = (p \lor z^\perp) \land (p^\perp \lor t^\perp),\]

whence by Equations (2) and (1), \(c^o = (p \lor z^\perp) \land (p^\perp \lor t^\perp)\). By a similar calculation, \(s^0 = (p \lor t^\perp) \land (p^\perp \lor z^\perp)\).

Part (ii) follows from Lemma 2.3 and (iii) follows from (i) and (ii).

To prove (iv), put \(q := (e - e^2)^o\), noting that by Corollary 4.5 (iii), \(q = t^\perp \land z^\perp\). By Definition 3.3 \(j^2 = p(e - e^2)p + p^\perp(e - e^2)p^\perp\), and again it follows from 3 Theorem 4.9 (v)] and Lemma 2.2 that

\[j^o = [p \land (p^\perp \lor q)] \lor [p^\perp \land (p \lor q)] = [p \land (p^\perp \lor q)] \lor v,\]

where \(v := p^\perp \land (p \lor q)\). Now \(pC(p^\perp \lor q)\) and \(pCv\), whence

\[[p \land (p^\perp \lor q)] \lor v = (p \lor v) \land (p^\perp \lor q \lor v).\]

But \(pCp^\perp\) and \(pC(p \lor q)\), so

\[p \lor v = (p \lor p^\perp) \land (p \lor p \lor q) = p \lor q.\]

Furthermore, since \(v \leq p^\perp\),

\[p^\perp \lor q \lor v = p^\perp \lor q.\]

Combining Equations (5)–(8) and the fact that \(q = e^o \land z^\perp\), we obtain (iv).

(v) Since \(s^o = s^0 = s^\perp = s^0 = s^\perp = 0 = s^0 = s^\perp\), we have \(s^0 \leq c^2 \leq c\).

Similarly, \(c^\perp s^2 = c^\perp (1 - c^2) = c^\perp - 0 = c^\perp\), whence \(c^\perp \leq s^2 \leq s\).

(vi) Since \(sCp\), we have \((s^o)^\perp Cp\). Moreover, \((s^o)^\perp c^2 = (s^o)^\perp (1 - s^2) = (s^o)^\perp\); by (v), \(b^o \leq s^o\), so \((s^o)^\perp b = 0\); and \((s^o)^\perp s^2 p^\perp = 0\); whence \((s^o)^\perp e = (s^o)^\perp (c^2 p + bk + s^2 p^\perp) = (s^o)^\perp p = (s^o)^\perp \land p\). Similarly, \(e(s^o)^\perp = (pe^2 + bk + p^\perp s^2)(s^o)^\perp = p(s^o)^\perp = p \land (s^o)^\perp\), so \((s^o)^\perp C e\) and \((s^o)^\perp e = (s^o)^\perp \land e\) by Lemma 2.7(ii). The proof of (vii) is similar.

4.7 Corollary. If both \(e\) and \(e^\perp\) are projection free, then \(c^o = s^0 = d^o = 1\).

A reasonable formula for \(b^o\) seems to be elusive; however, we do have partial results as per the following lemma. (Also, see Theorem 5.19 below.)

4.8 Lemma. Let \(v := kpk\). Then: (i) \(v\) is a projection, the symmetry \(k\) exchanges \(p\) and \(v\), \(bCv\), and \(b^o \leq (p \land v^\perp) \lor (p^\perp \land v) = (p \land v^\perp) + (p^\perp \land v)\).

(ii) If \(p\) is an atom and \(pe \neq ep\), then \(p \perp v\) and \(b^o = p \lor v = p + v\). (iii) If \(p\) is an atom and \(pe \neq ep\), then there exists \(\beta \in \mathbb{R}\) with \(b = \beta b^o\), \(0 < \beta \leq 1\).
Proof. (i) Obviously, \( v \) is a projection and \( k \) exchanges \( p \) and \( v \). By parts (iii) and (iv) of Theorem 3.9 \( bCk \) and \( bCp \), so \( bCv \). Moreover, by Theorem 3.9 (v), \( bpkp = bp^\perp p = 0 \), whence, since \( v = kpk \in P \),

\[
b^o \leq ((pkp)^o)^\perp = (((pkp)^2)^o)^\perp = ((p(kpk)p)^o)^\perp = ((pvp)^o)^\perp = (p \land (p^\perp \lor v)) \land (p \lor (p^\perp \land v^\perp)). \tag{1}
\]

Starting with the observation that \( bp^\perp kp^\perp = bkp^\perp = 0 \), and arguing as above, we deduce that

\[
b^o \leq p \lor (p^\perp \land v). \tag{2}
\]

By (1) and (2),

\[
b^o \leq [p^\perp \lor (p \land v^\perp)] \lor [p \lor (p^\perp \land v)],
\]

and using Theorem 2.1 to simplify the right side of the latter inequality, we obtain (i).

(ii) Suppose that \( p \) is an atom and \( pe \neq ep \). Since \( k \) exchanges \( p \) and \( v \), it follows that \( v \) is also an atom. By Lemma 3.11 \( b^o \neq 0 \), whence by (i), at least one of the conditions \( p \land v^\perp \neq 0 \) or \( p^\perp \land v \neq 0 \) must hold. Since \( p \) and \( v \) are atoms, we have \( p \perp v \) in either case, whence \( p \land v^\perp = p, p^\perp \land v = v \), so \( p \perp v \) and by (i),

\[
0 \neq b^o \leq p \lor v = p + v. \tag{3}
\]

We claim that \( p \leq b^o \). Suppose not. Then, since \( p \) is an atom, \( b^o \land p = 0 \). Thus, as \( bCp \), we have \( \overline{b^o}Cp \), whence \( \overline{b^o}p = b^o \land p = 0 \) and it follows that \( bp = pb = 0 \). Consequently, by Theorem 3.9 (vi), \( 0 = bkp = pep^\perp \), and it follows from Lemma 2.13 that \( pCe \), contradicting \( pe \neq ep \). Therefore, \( p \leq b^o \).

We claim that \( v \leq b^o \). Suppose not. Then since \( v \) is an atom, \( b^o \land v = 0 \). Thus, as \( bCv \), we have \( \overline{b^o}Cv \), whence \( \overline{b^o}v = b^o \land v = 0 \), and it follows that \( bv = vb = 0 \). By Theorem 3.9 (vi), \( bkp = bp^\perp k \), and we have

\[
0 = bv = bkp^\perp k^2 = bp^\perp = b(1 - p) = b - bp, \text{ so } b = bp. \tag{4}
\]

By Theorem 3.9 (vi) again, \( pep^\perp = bkp \) and \( p^\perp ep = bkp \), whence by (4),

\[
pep^\perp = bkp = bk \text{ and therefore } p^\perp ep = bkp = (pep^\perp)p = 0,
\]

and again it follows from Lemma 2.13 that \( pCe \), contradicting \( pe \neq ep \). Therefore, \( v \leq b^o \).

Now we have \( p, v \leq b^o \), whereupon \( p + v = p \lor v \leq b^o \), which together with (3) yields \( b^o = p \lor v = p + v \).
(iii) Assume the hypotheses of (iii). By Theorem 3.7 (i), \( bp = pb = pbp \) and by Lemma 2.11 (ii), (iii), \( pb = bp = pbp = \beta p \) with \( 0 \leq \beta \leq 1 \). Moreover, \( bk = kb \) by Theorem 3.9 (iii), and by Theorem 3.9 (v), \( pbk = bpk = bkp^\perp \). Multiplying both sides of \( b = bp + bp^\perp \) by \( k \), we obtain \( kb = kbp + kbp^\perp = \beta kp + \beta pk = \beta (kp + pk) \). Multiplying by \( k \) again, we get \( b = \beta (p + kp) = \beta (p + v) = \beta b^o \) by (ii). Finally, since \( pe \neq ep \), we have \( b \neq 0 \) by Lemma 3.11 whence \( 0 < \beta \).

5 Two commutators

The assumptions and notation set forth above remain in force. In this section we study two candidates for a commutator projection for the pair \( p \in P, e \in E \). Recall that in [12, Definition 2.3] the Marsden commutator of two projections \( p, q \in P \) is denoted and defined by

\[
[p, q] := (p \lor q) \land (p \lor q^\perp) \land (p^\perp \lor q) \land (p^\perp \lor q^\perp)
\]

and has the property that \( pCq \Leftrightarrow [p, q] = 0 \). With this in mind, for a projection \( w \in P \) to be regarded as a commutator for the pair \( p, e \), we shall require—at least—that \( pCe \Leftrightarrow w = 0 \). (Observe that the commutators defined in [22, §5.1] satisfy the dual condition that commutativity obtains iff the commutator equals 1.)

The simplest candidate for a commutator projection for \( p \) and \( e \) is the carrier projection \( b^o \) of the commutator effect \( b \). By Lemma 3.11 \( b^o \) satisfies our basic condition \( pCe \Leftrightarrow b^o = 0 \).

5.1 Remark. If it happens that \( e \in P \), then \( z = e, t = e^\perp \), and \( b = cs \), whence by Theorem 4.4 (iii),

\[
b^o = (cs)^o = (p \lor e) \land (p \lor e^\perp) \land (p^\perp \lor e) \land (p^\perp \lor e^\perp)
\]

is the Marsden commutator \([p, e]\) of the pair of projections \( p \) and \( e \).

Two projections are in so-called generic position [12, Definition 2.1] iff their Marsden commutator is 1; hence, by analogy, we say that the projection \( p \) and the effect \( e \) are in generic position iff \( b^o = 1 \).

5.2 Theorem. Suppose that \( p \) and \( e \) are in generic position. Then:

(i) \( (cs)^o = c^o = s^o = 1 \).
(ii) \( p \land z = p \land t = p^\perp \land z = p^\perp \land t = 0 \).

(iii) The symmetry \( k \) in the CBS-decomposition of \( e \) with respect to \( p \) exchanges the projections \( p \) and \( p^\perp \).

Proof. Assume that \( p \) and \( e \) are in generic position, i.e., \( b^o = 1 \). Since \( b^2 = c^2 s^2 - d^2 \leq c^2 s^2 \), it follows that \( 1 = b^o = (b^2)^o \leq (c^2 s^2)^o = (c s)^o = c^o s^o \), proving (i). Part (ii) follows from (i), Theorem 4.6 (iii), and De Morgan. By Theorem 3.9 (v), \( pk = kp^\perp \), whence \( k pk = p^\perp \), proving (ii). \( \square \)

There are two possible shortcomings of \( b^o \) as a commutator projection for the pair \( p \) and \( e \): First, although \( p \) commutes with \( b^o \), in general, \( e \) fails to commute with \( b^o \) (see Example 5.23 below). Second, as we mentioned earlier, obtaining a perspicuous formula for \( b^o \) in terms of \( e^o, (e^\perp)^o, p, c^o, s^o, z, t, \) and \( k \) seems to offer a challenge.

In the following definition, we shall extend the Marsden commutator for two projections to a commutator \([F]\) for a finite set \( F \subseteq P \) of projections. We note that this definition is dual to [22 Definition 5.1.4], i.e., suprema and infima have been interchanged.

5.3 Definition. Suppose that \( F = \{w_1, w_2, \ldots, w_n\} \subseteq P \) is a finite set of projections. For any \( w \in P \), let us write \( w^\perp := w \) and \( w^{-1} := w^\perp \). Further, let \( D := \{1, -1\} \). Then, as \( d = (d_1, d_2, \ldots, d_n) \) runs through \( D^n \), the commutator of the set \( F \) is denoted and defined by

\[
[F] := \bigwedge_{d \in D^n} (w_1^{d_1} \lor w_2^{d_2} \lor \cdots \lor w_n^{d_n}) \in P.
\]

Also, we define \([\emptyset] := 0\).

Clearly, \([\{w_1\}] = 0\). Also, if \( F = \{w_1, w_2\} \), then

\[
[F] = (w_1 \lor w_2) \land (w_1^\perp \lor w_2) \land (w_1 \lor w_2^\perp) \land (w_1^\perp \lor w_2^\perp)
\]

is the Marsden commutator of \( w_1 \) and \( w_2 \). We note that if the special projections \( 0 \) or \( 1 \) are present in \( F \), then \([F \setminus \{0, 1\}] = [F]\).

5.4 Remark. Suppose that \( F \) is a finite subset of \( P \), \( q \in P \), and \( qCw \) for every \( w \in F \). Then since commutativity is preserved under formation of orthocomplements, finite suprema, and finite infima, it follows that \( qC[F] \).
5.5 Remark. If $F$ is a finite subset of $P$, it is obvious that $[F]$ is unchanged if one of the projections in $F$ is replaced by its orthocomplement. As a consequence, if both $w \in F$ and $w^\perp \in F$, then $w^\perp$ can be omitted from $F$ without affecting the value of $[F]$.

By dualizing [22, Theorem 5.1.5 and Prop. 5.1.8], we obtain the following characterization of $[F]$.

5.6 Lemma. Let $F \subseteq P$ be a finite set of projections and put $r := [F]$. Then:

(i) $w \in F \Rightarrow rCw$.

(ii) The projections in the set $\{w \wedge r^\perp : w \in F\}$ commute pairwise.

(iii) $r$ is the smallest projection that satisfies (i) and (ii).

(iv) $r = 0$ iff the projections in the set $F$ commute pairwise.

Now, by dualizing [22, Def. 5.1.6], we shall extend Definition [5.3] to arbitrary countable subsets $W$ of $P$. (By countable, we mean finite or countably infinite.) However our definition will require that the OML $P$ is $\sigma$-complete, i.e., that every countable subset of $P$ has a supremum (whence also an infimum) in $P$. It is known that $P$ is $\sigma$-complete iff it is $\sigma$-orthocomplete, i.e., iff every countable and pairwise orthogonal subset of $P$ has a supremum in $P$ [17, Corollary 3.4]. According to the discussion in [3, §6], every generalized Hermitian algebra [5, 6, 8] is a synaptic algebra with a $\sigma$-complete projection lattice. For instance, the self-adjoint part of a von Neumann algebra has a $\sigma$-complete (and in fact, a complete) projection lattice. Thus we make the following assumption.

5.7 Standing Assumption. Henceforth in this section, we assume that the OML $P$ is $\sigma$-orthocomplete; hence $\sigma$-complete.

5.8 Remarks. Since there are only countably many finite subsets of a countable set, the supremum in the following definition exists. Also, if $W \subseteq P$ is a finite set, then (as is easily seen) $[W] = \bigvee \{ [F] : F \subseteq W \}$. Therefore, the following definition provides a true generalization of $[F]$ for a finite set $F \subseteq P$. 

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5.9 Definition. For an arbitrary countable subset $W \subseteq P$, the commutator of $W$ is denoted and defined by

$$[W] = \bigvee \{ [F] : F \subseteq W \text{ and } F \text{ is finite} \}.$$ 

5.10 Remark. Suppose that $W$ is a countable subset of $P$, $q \in P$, and $qCw$ for every $w \in W$. Then since commutativity is preserved under formation of arbitrary existing suprema, it follows from Remark 5.4 that $qC[W]$.

5.11 Remarks. If $W$ is a countable subset of $P$, then as a consequence of Remark 5.5, $[W]$ is unchanged if one of the projections in $W$ is replaced by its orthocomplement. As a consequence, if both $w \in W$ and $w^\perp \in W$, then $w^\perp$ can be omitted from $W$ without affecting the value of $[W]$.

By dualizing [22] Theorem 5.1.7 and Prop. 5.1.8, we obtain the following characterization of $[W]$.

5.12 Theorem. If $W \subseteq P$, $W$ is countable, and $r := [W]$, then:

(i) $w \in W \Rightarrow rCw$.

(ii) The projections in the set $\{w \wedge r^\perp : w \in W\}$ commute pairwise.

(iii) $r$ is the smallest projection with properties (i) and (ii).

(iv) $r = 0$ iff the projections in the set $W$ commute pairwise.

Using Assumption 5.7, Definition 5.9, and the notion of a rational spectral resolution, we are now in a position to define an alternative $[p, e]$ to $b^0$ as a commutator for the pair $p, e$.

5.13 Definition. For $p \in P$ and $e \in E$, the commutator of the pair $p, e$ is denoted and defined by

$$[p, e] := \{ \{p\} \cup \{pe, \mu : \mu \in \mathbb{Q}\} \}.$$ 

As we shall see in Corollary 5.21 (ii) below, no notational conflict with the Marsden commutator of two projections in [12] will result from the use of the notation $[p, e]$ in Definition 5.13.

We note that, in Definition 5.13, only the set of projections in the rational spectral resolution of $e$ is involved—the labeling of these projections by rational numbers plays no role in the computation of $[p, e]$.

In the following theorem, which characterizes $[p, e]$, recall that by Lemma 2.7 (ii), if $q \in P$ and $qCe$, then $q^\perp e = eq^\perp = e \wedge q^\perp$, the infimum of $e$ and $q^\perp$ in $E$. 

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5.14 Theorem. If \( p \in P \) and \( e \in E \), then \([p, e]\) is the smallest projection \( q \in P \) such that \( qCP \), \( qCe \), and \( (p \land q^\perp)C(e \land q^\perp) \).

Proof. Put \( W := \{p\} \cup \{p_{e, \mu} : \mu \in \mathbb{Q}\} \) and \( r := [p, e] = [W] \). By Theorem 5.12 we have: (i) \( w \in W \Rightarrow rCw \). (ii) The projections in the set \( \{w \land r^\perp : w \in W\} \) commute pairwise. (iii) \( r \) is the smallest projection with properties (i) and (ii).

We claim that (iv) \( rCp \), (v) \( rCe \), and (vi) \( (p \land r^\perp)C(e \land r^\perp) \). Indeed, since \( p \in W \), (i) implies that \( rCp \). Also by (i), for every \( \mu \in \mathbb{Q} \), \( rCp_{e, \mu} \), whence by Remark 2.9 \( rCe \). Moreover, for every \( \mu \in \mathbb{Q} \), we have both \( p \in W \) and \( p_{e, \mu} \in W \), whence \( (p \land r^\perp)C(p_{e, \mu} \land r^\perp) \) by (ii). But by Lemma 2.10 \((p_{e, \lambda} \land r^\perp)_{\lambda \in \mathbb{R}} \) is the spectral resolution of \( e \land r^\perp \) as calculated in \( r^\perp Ar^\perp \); hence by Remark 2.9 again, \( p \land r^\perp \) commutes with \( e \land r^\perp \) in \( r^\perp Ar^\perp \), and therefore also in \( A \). Thus we have (iv), (v), and (vi).

Now assume that \( v \in P \), \( vCp \), \( vCe \), and \( (p \land v^\perp)C(e \land v^\perp) \). We have to prove that \( r \leq v \). By (iii) it will be sufficient to show that (i') \( w \in W \Rightarrow vCw \) and (ii') the projections in the set \( \{w \land v^\perp : w \in W\} \) commute pairwise. To prove (i'), suppose \( w \in W \). If \( w = p \), we have \( vCw \), so we can assume that \( w = p_{e, \mu} \) for some \( \mu \in \mathbb{Q} \). But since \( vCe \), it follows that \( vCp_{e, \mu} \), and we have (i').

To prove (ii'), suppose that \( w, q \in W \). First we consider the case \( w = p \) and \( q = p_{e, \nu} \) with \( \nu \in \mathbb{Q} \). Since \( vCe \), we have \( eCv^\perp \), whence by Lemma 2.10 (ii), \((p_{e, \lambda} \land v^\perp)_{\lambda \in \mathbb{R}} \) is the spectral resolution of \( e \land v^\perp \) as calculated in \( v^\perp Av^\perp \). By hypothesis, \((p \land v^\perp)C(e \land v^\perp) \), and it follows that \((p \land v^\perp)C(p_{e, \nu} \land v^\perp) \). This reduces our argument to the case \( w = p_{e, \mu} \) and \( q = p_{e, \nu} \) with \( \mu, \nu \in \mathbb{Q} \). But, the projections in a spectral resolution commute pairwise, whence \((p_{e, \mu} \land v^\perp)C(p_{e, \nu} \land v^\perp) \), proving (ii').

By the following corollary to Theorem 5.14 \([p, e]\) qualifies as a commutator of \( p \) and \( e \).

5.15 Corollary. If \( p \in P \) and \( e \in E \), then \( pCe \Leftrightarrow [p, e] = 0 \).

Proof. If \( pCe \), then \( 0CP \), \( 0Ce \), and \( (p \land 0^\perp)C(e \land 0^\perp) \), whence \([p, e] \leq 0 \), i.e., \([p, e] = 0 \). Conversely, if \([p, e] = 0 \), then \((p \land 0^\perp)C(e \land 0^\perp) \), i.e., \( pCe \).

5.16 Lemma. Let \( r := [p, e] \). Then: (i) In the CBS-decomposition of \( e \) with respect to \( p \), we have \( rCP \), \( rCe \), \( rCp \), \( rCs \), \( rCj \), \( rCb \) and \( rCk \). (ii) If \( q \in P \), \( qCP \), and \( qCe \), then \( qCr \).
Proof. (i) By Theorem 5.14, \( rCp \) and \( rCe \). Since \( c = (pqp + p^+ e^+ p^+)^{1/2} \in CC(pqp + p^+ e^+ p^+) \), it follows that \( rCe \), and similarly, \( rCs \). Also, \( rC(e - e^2) \), and because \( j = (p(e - e^2)p + p^+ (e - e^2)p)^{1/2} \), it follows that \( rCj \). Therefore, as \( b = (c^2 s^2 - j^2)^{1/2} \), we have \( rCb \). Finally, \( rCk \) follows from \( k \in CC(pqp^+ + p^+ e)p \).

(ii) Suppose \( q \in P, qCp \), and \( qCe \). Then \( qCp_{e,\mu} \) for all \( \mu \in \mathbb{Q} \), whence \( qCr \) by Definition 5.13 and Remark 5.10.

5.17 Theorem. Let \( q \in P \), suppose that \( qCp \) and \( qCe \), let \( r := [p, e] \) and let \( v \in P[0, q] \) be the commutator \([p_q, e_q]_{qAq}\) of \( p_q = pq \) and \( e_q = eq \) as calculated in \( qAq \). Then \( qCr, pCr, eCr, qCv, pCv, eCv \), and \( v = r_q = r_q = q = q \wedge r \).

Proof. Since \( qCp \) and \( qCe \), we have \( qCr \) by Lemma 5.16 (ii). Also, \( pCr \) and \( eCr \) by Lemma 5.16 (i). As \( v \in P[0, q] \), we have \( v = vq = vq \) and \( v^{-1} = vq^{-1} = vq^{-1} \). Thus, by Theorem 5.14 applied to \( p_q \) and \( e_q \) in the synaptic algebra \( qAq \), we infer that \( v \) is the smallest projection in \( P[0, q] \) such that

(i) \( vC(pq) \), (ii) \( vC(eq) \), and (iii) \((pq) \wedge (v^{-1}q)\)C((eq) \wedge (v^{-1}q)).

Since \( vC(pq) \) and \( qCp \), it follows that \( pv = p(vq) = (pq)v = v(pq) = v(p) = v(q)p = vp \), whence \( pCv \). Likewise, since \( vC(eq) \) and \( qCe \), it follows that \( ev = e(vq) = (eq)v = v(eq) = v(qe) = (vq)e = ve \), whence \( eCv \). Thus the three elements \( p, q, \) and \( v^{-1} \) commute in pairs, and so do the three elements \( e, q, \) and \( v^{-1} \). Consequently, \( p \wedge (v \vee q^{-1}) = p \wedge v^{-1} \wedge q = pvq^{-1} = (pq)q^{-1} = (pq) \wedge (v^{-1}q) \), similarly \( e \wedge (v \vee q^{-1}) = (eq) \wedge (v^{-1}q) \), and we can rewrite (iii) as \((p \wedge (v \vee q^{-1}))C(e \wedge (v \vee q^{-1})) \). Furthermore, \((v \vee q^{-1})Cp \) and \((v \vee q^{-1})Ce \), and it follows from Theorem 5.14 that \( r = [p, e] \leq v \vee q^{-1} \). Therefore, \( r_q = r_q = r \wedge q \leq (v \vee q^{-1}) \wedge q = v \wedge q = v \).

To complete the proof, we have to show that \( v \leq r_q \), i.e., that \( v \leq r_q \). Since \( rCp, rCq, \) and \( qCp, \) we have \((rq)C(pq) \). Likewise, since \( rCe, rCq, \) and \( qCe \), we have \((rq)C(eq) \). Thus, with \( v \) replaced by \( rq \), conditions (i) and (ii) hold; hence, to prove that \( v \leq r_q \), it will be sufficient to prove that condition (iii) holds with \( v \) replaced by \( rq \), i.e., that \((pq) \wedge ((pq)^{-1}q)\)C((eq) \wedge ((eq)^{-1}q)) \). Since \( rCq \), we have \((rq)^{-1}q = (r \wedge q)^{-1} \wedge q = (r^{-1} \vee q^{-1}) \wedge q = r^{-1} \wedge q = qr^{-1} \). Thus, as \( p, q, \) and \( r \) commute pairwise, we have \((pq) \wedge ((pq)^{-1}q) = (pq) \wedge (qr^{-1}) = pqr^{-1} \). Likewise, as \( e, q, \) and \( r \) commute pairwise, we deduce that \((eq)^{-1} \wedge (eq)^{-1}q = eqr^{-1} \). Thus, it will be sufficient to show that \((pq) \wedge (eq)(erp^{-1})) \). By Theorem 5.14 \((p(p)^{-1})C(erp^{-1}) \); hence, as \( qCp, qCr^{-1} \), and \( qCe \), we have

\[
(pqr^{-1})(eqr^{-1}) = q(p)^{-1}(er^{-1}q) = q(p)^{-1}(er^{-1})q
\]
\[ = q(er^\perp)(pr^\perp)q = (eqr^\perp)(pqr^\perp), \]

so \((pqr^\perp)C(eqr^\perp)\).

5.18 Theorem. Let \(r := [p,e]\). Then: (i) \(p_r^eC_{r^e}\). (ii) \(b_{r^e} = 0\) and \(e_{r^e} = c_r^e p_{r^e} + s_r^e (p_{r^e})^{\perp r^e}\).

Proof. (i) By Theorem 5.14 \((p \wedge r^\perp)C(e \wedge r^\perp)\), proving (i).

(ii) By Theorem 3.14 (iii) with \(q := r^\perp\), the CBS-decomposition of \(e_{r^e}\) with respect to \(p_{r^e}\) in \(p^eAp^e\) is \(e_{r^e} = c_r^e p_{r^e} + b_{r^e} k_{r^e} + s_r^e (p_{r^e})^{\perp r^e}\). But by (i) and Lemma 3.11 \(b_{r^e} = 0\).

5.19 Theorem. \(b \leq b^0 \leq [p,e] \leq c^0 \wedge s^0 = (cs)^0 = c^0 s^0\).

Proof. Put \(r := [p,e]\). By Theorem 5.18 (ii), \(b r^\perp = b_{r^e} = 0\), so \(b \leq r\), and therefore \(b \leq b^0 \leq r = [p,e]\).

Put \(q := s^0\). Then by Theorem 4.6 (vi), \(q Cp, q Ce, and p \wedge q^\perp = e \wedge q^\perp\), so \((p \wedge q^\perp)C(e \wedge q^\perp)\). Therefore, by Theorem 5.14 \([p,e] \leq q = s^0\). A similar argument using Theorem 4.6 (vii) shows that \([p,e] \leq c^0\), and we have \([p,e] \leq c^0 \wedge s^0 = (cs)^0 = c^0 s^0\) (Theorem 4.6 (ii)).

Using the fact that \(b^0 \leq [p,e]\), we obtain the following alternative characterization of \([p,e]\).

5.20 Theorem. \([p,e]\) is the smallest projection \(v\) such that \(vCp, vCe, and b^0 \leq v\).

Proof. Put \(r := [p,e]\). By Lemma 5.16 (i), \(r Cp\) and \(r Ce\) and by Theorem 5.19 \(b^0 \leq r\). Suppose that \(v \in P, vCp, vCe\), and \(b^0 \leq v\). We have to prove that \(r \leq v\). We have \(b \leq b^0 \leq v\), whence \(bv^\perp = v^\perp b = 0\). Moreover, as \(vCp, vCe\), and \(c^2 = pep + p^e e^\perp p^\perp\), it follows that \(vCc^2\). Likewise, \(vCs^2\), whence \(v^\perp\) commutes with both \(e\) and \(p\), whereas both \(v^\perp\) and \(p\) commute with \(c^2, p, s^2\), and \(p^\perp\). Therefore, by the CBS-decomposition of \(e\) with respect to \(p\),

\[ ev^\perp = v^\perp e = v^\perp e^2 p + v^\perp b k + v^\perp s^2 p^\perp = v^\perp c^2 p + v^\perp s^2 p^\perp, \]

and since \(pv^\perp\) commutes with both \(v^\perp c^2 p\) and \(v^\perp s^2 p^\perp\), it follows that \(pv^\perp\) commutes with \(ev^\perp\), i.e., \((p \wedge v^\perp)C(e \wedge v^\perp)\). Consequently, by Theorem 5.14 \(r \leq v\).

5.21 Corollary. (i) \(b^0 = [p,e]\) iff \(eC b^0\). (ii) If \(e \in P\), then \(b^0 = [p,e]\) is the Marsden commutator of the two projections \(p\) and \(e\).
5.22 Lemma. Suppose that \( v \in \mathbb{C} \) can omit the projections 0 and 1, whence \( b^o = [p, e] \) by Theorem 5.20. Conversely, by Theorem 5.20 again, if \( b^o = [p, e] \), then \( eCb^o \).

(ii) Suppose that \( e \in P \). Temporarily denoting the Marsden commutator of \( p \) and \( e \) by \( [p, e]_M \), we infer from Remark 5.1 that \( b^o = (cs)^o = [p, e]_M \). By [12, Theorem 3.8 (vi)] with \( q := e \), we have \( eC[p, e]_M \), whence \( eCb^o \). Therefore by (i), \( [p, e]_M = b^o = [p, e] \).

The projection \( p \) and the effect \( e \) are said to be totally noncompatible iff \( [p, e] = 1 \). If \( p \) and \( e \) are in generic position, then \( 1 = b^o \leq [p, e] \), and it follows that \( p \) and \( e \) are totally noncompatible.

5.22 Lemma. Suppose that \( p \) and \( e \) are totally noncompatible. Then: (i) If \( v \in P, vCp, vCe, (p \wedge v)C(e \wedge v) \), then \( v = 0 \). (ii) \( c^o = s^o = (cs)^o = 1 \). (iii) \( p \wedge z = p \wedge t = p^\perp \wedge z = p^\perp \wedge t = 0 \).

Proof. By hypothesis, \( [p, e] = 1 \). Part (i) follows from Theorem 5.14, part (ii) follows from Theorem 5.19, and (iii) is a consequence of (ii), parts (ii) and (iii) of Theorem 4.6 and De Morgan.

The following example shows that it is possible to have \( p \) and \( e \) totally noncompatible (i.e., \([p, e] = 1 \)), where \( p \) and \( e \) are not in generic position (i.e., \( b^o \leq [p, e] = 1 \)).

5.23 Example. Let \( \mathbb{R}^3 \) be organized as usual into a 3-dimensional real Hilbert space and let \( A \) be the synaptic algebra of all self-adjoint linear operators on \( \mathbb{R}^3 \). Let \( p_1, p_2, p_3 \), and \( p \) be the (orthogonal) projections onto the one-dimensional subspaces \( \{(\alpha, 0, 0) : \alpha \in \mathbb{R}\} \), \( \{(0, \beta, 0) : \beta \in \mathbb{R}\} \), \( \{(0, 0, \gamma) : \gamma \in \mathbb{R}\} \), and \( \{((\xi, \xi, \xi) : \xi \in \mathbb{R}\} \), respectively. Put \( e := \frac{1}{4}p_1 + \frac{1}{2}p_2 + \frac{3}{4}p_3 \), noting that \( e \) is an effect in \( A \) and that the set of projections in the spectral resolution of \( e \) is \( \{0, p_1, p_1 + p_2, 1\} \). As observed above, in forming \([p, e]\), we can omit the projections 0 and 1, whence \([p, e] = \{p, p_1, p_1 + p_2\} \). Also, \( p_1 + p_2 = p_3^\perp \), so by Remark 5.5 \([p, e] = \{p, p_1, p_3\} \).

We observe that each of the atoms \( p_1, p_3 \), and \( p_1^\perp \wedge p_3^\perp = p_2 \) is disjoint from both \( p \) and \( p_1^\perp \), whence with the notation of Definition 5.3, \( p^d_1 \wedge p_1^d_2 \wedge p_3^d_3 = 0 \) for all \( d \in D^3 \). Therefore, by De Morgan,

\[
[p, e] = \bigwedge_{d \in D^3} (p^d_1 \vee p_1^d_2 \vee p_3^d_3) = 1,
\]

i.e. \( p \) and \( e \) are totally noncompatible. In particular \( pe \neq ep \). As \( p \) is an atom in \( P \), so is \( v := kpk \). Thus by Lemma 4.8 (ii), \( p \perp v \) and \( b^o = p \vee v = p + v \) so
\( b^o \) is a two-dimensional (i.e., rank 2) projection, and therefore \( b^o \neq 1 = [p, e] \). As a consequence (Corollary 5.21), \( b^o \) does not commute with \( e \).

5.24 Theorem. Let \( r := [p, e] \). Then the projection \( p_r = pr = rp = p \wedge r \) and the effect \( e_r = er = re = e \wedge r \) are totally noncompatible in \( rAr \).

Proof. By Lemma 5.16 (i), \( rCp \) and \( rCe \), whence, putting \( q := r \) in Theorem 5.17, we find that the commutator \( [p_r, e_r]_{rAr} \) as calculated in \( rAr \) is given by \( [p_r, e_r]_{rAr} = r \wedge r = r \). But \( r \) is the unit element in \( rAr \), proving the theorem.

By Theorem 5.18 (i) and Theorem 5.24, the projection \( p = p_r + p_r^\perp \) and the effect \( e = e_r + e_r^\perp \) are decomposed into components \( p_r, e_r \) that are totally noncompatible in \( rAr \) and components \( p_r^\perp, e_r^\perp \) that commute in \( r^\perp Ar^\perp \).

6 An application of CBS-decomposition

If \( A \) is the synaptic algebra of all self-adjoint operators on a complex Hilbert space, then (transcribed to our current notation), T. Morland and S. Gudder prove that, if \( e \in E \) and \( p \) is an atom in \( P \), then \( e \wedge p^\perp \) exists in \( E \) [20, Lemma 3.8]. Morland and Gudder’s proof uses the Hilbert-space inner product and the Schwarz inequality, and thus is not available for our more general synaptic algebra. However, using the CBS-decomposition we generalize [20, Lemma 3.8] to our present setting in Theorem 6.6 below.

6.1 Standing Assumptions. In this section the notation and assumptions of Sections 2–5 remain in force. In addition, we assume that (i) \( p \) is an atom in \( P \) and (ii) as per Lemma 2.11, \( pep = \alpha p \) with \( \alpha \in \mathbb{R} \), \( 0 \leq \alpha \leq 1 \).

6.2 Definition. If \( \alpha > 0 \), we define: (1) \( a := \alpha^{-1}bk = \alpha^{-1}(pep^\perp + p^\perp ep) \in A \). (2) \( y := p^\perp(1 - a) \in R \). (3) \( y^* := (1 - a)p^\perp \in R \).

Provided that \( \alpha > 0 \), the mapping \( f \mapsto yfy^* \) for \( f \in A \) is the composition of the quadratic mappings \( f \mapsto g := (1 - a)f(1 - a) \) and \( g \mapsto p^\perp gp^\perp \), whence it is a linear and order-preserving mapping on \( A \).

We omit the straightforward computational proofs of the next three lemmas.

6.3 Lemma. Suppose that \( \alpha > 0 \). Then: (i) \( ap = p^\perp a = p^\perp ap = \alpha^{-1}p^\perp ep \), \( ap^\perp = pa = pap^\perp = \alpha^{-1}pep^\perp \), and \( ap + pa = a \). (ii) \( (ap)^2 = 0 \). (iii) \( (pa)^2 = 0 \). (iv) \( y = p^\perp - ap \) and \( y^* = p^\perp - pa \).
6.4 Lemma. Suppose that \( \alpha > 0 \). Then: (i) The CBS-decomposition of \( e \) with respect to the atom \( p \) is \( e = \alpha p + \alpha a + s^2 p^\perp \). (ii) \( \alpha^2 a^2 = b^2 \). (iii) \( epe = \alpha^2 p + \alpha^2 a + b^2 p^\perp \). (iv) \( e - \alpha^{-1} epe = (s^2 - \alpha^{-1} b^2) p^\perp = p^\perp (s^2 - \alpha^{-1} b^2) \).

6.5 Lemma. Suppose that \( f \in A \) and \( \alpha > 0 \). Then: (i) \( 0 \leq f \leq p^\perp \Rightarrow yfy^* = f \). (ii) \( 0 \leq yey^* = (s^2 - \alpha^{-1} b^2) p^\perp = e - \alpha^{-1} epe \).

6.6 Theorem. The infimum \( e \land p^\perp \) exists in \( E \). In fact, if \( \alpha = 0 \), then \( e \land p^\perp = e \), and if \( \alpha > 0 \), then \( e \land p^\perp = (s^2 - \alpha^{-1} b^2) p^\perp = e - \alpha^{-1} epe \).

Proof. If \( \alpha = 0 \), then \( pep = 0 \), and as \( 0 \leq e \) it follows that \( pe = ep = 0 \) ([3, Axiom SA4]), whence \( e \leq p^\perp \), so \( e = e \land p^\perp \).

Now suppose that \( \alpha > 0 \). By Lemma 6.5 (ii), \( 0 \leq (s^2 - \alpha^{-1} b^2) p^\perp = e - \alpha^{-1} epe \). Since \( 0 \leq epe \), we also have \( e - \alpha^{-1} epe \leq e \leq 1 \), so \( e - \alpha^{-1} epe \in E \). Moreover, \( e - \alpha^{-1} epe = (s^2 - \alpha^{-1} b^2) p^\perp \leq p^\perp \). Suppose that \( f \in E \) with \( f \leq e, p^\perp \). Then by Lemma 6.5 (i), \( 0 \leq y(e - f)y^* = yey^* - yfy^* = (s^2 - \alpha^{-1} b^2) p^\perp - f \), whence \( f \leq (s^2 - \alpha^{-1} b^2) p^\perp \), and it follows that \( e \land p^\perp = (s^2 - \alpha^{-1} b^2) p^\perp = e - \alpha^{-1} epe \). \( \square \)

6.7 Corollary (Cf. [20] Corollaries 3.9 and 3.10).

(i) If \( p_1, p_2, \ldots, p_n \) is a finite sequence of mutually orthogonal atoms in \( P \), then \( e \land (p_1 \lor p_2 \lor \cdots \lor p_n)^\perp \) exists in \( E \).

(ii) Suppose that every nonzero projection in \( P \) is a supremum of a finite sequence of mutually orthogonal atoms in \( P \). Then, for all \( q \in P \), the infimum \( e \land q \) exists in \( E \).

Proof. (i) The infimum \( e \land p_1^\perp \) exists by Theorem 6.6. Similarly, as \( e \land p_1^\perp \in E \), the infimum \( (e \land p_1^\perp) \land p_2^\perp = e \land (p_1 \lor p_2)^\perp \) exists in \( E \). Continuing in this way by induction, we obtain (i).

(ii) Obviously, \( e \land 1 = e \), so we can assume that \( q \neq 1 \), whence \( q^\perp \neq 0 \). Therefore by hypothesis, there is a finite sequence \( p_1, p_2, \ldots, p_n \) of mutually orthogonal atoms in \( P \) such that \( q^\perp = p_1 \lor p_2 \lor \cdots \lor p_n \), and it follows from (i) that \( e \land q \) exists in \( E \). \( \square \)

The synaptic algebra \( A \) is said to be of rank \( r \), \( r = 1, 2, 3, \ldots \) iff there are \( r \), but not \( r + 1 \) mutually orthogonal nonzero projections in \( P \). Clearly, a synaptic algebra of rank \( r \) satisfies the hypothesis of Corollary 6.7 (ii). By [6] and [7] Corollary 4.4, a positive-definite spin factor of dimension 2 or more is the same thing as a synaptic algebra of rank 2. Therefore:
6.8 Corollary. If $A$ is a positive-definite spin factor of dimension 2 or more, $e \in E$, and $q \in P$, then $e \wedge q$ exists in $E$.

We note that there are infinite-dimensional positive-definite spin factors.

References

[1] L. Beran, Orthomodular Lattices, An Algebraic Approach, Mathematics and its Applications, Vol. 18, D. Reidel Publishing Company, Dordrecht, 1985.

[2] A. Böttcher, I.M. Spitkovsky, A gentle guide to the basics of two projections theory, Linear Algebra Appl. 432 (2010) 1412–1459.

[3] D.J. Foulis, Synaptic algebras, Math. Slovaca 60 (2010) 631–654.

[4] D.J. Foulis, M.K. Bennett, Effect algebras and unsharp quantum logics, Found. Phys. 24 (1994) 1331–1352.

[5] D.J. Foulis, S. Pulmannová, Generalized Hermitian algebras, Internat. J. Theoret. Phys. 48 (2009) 1320–1333.

[6] D.J. Foulis, S. Pulmannová, Spin factors as generalized Hermitian algebras, Found. Phys. 39 (2009) 237–255.

[7] D.J. Foulis, S. Pulmannová, Projections in synaptic algebras, Order 27 (2010) 235–257.

[8] D.J. Foulis, S. Pulmannová, Regular elements in generalized Hermitian algebras, Math. Slovaca 61 (2011) 155–172.

[9] D.J. Foulis, S. Pulmannová, Type-decomposition of a synaptic algebra, Found. Phys. 43 (2013) 948–968.

[10] D.J. Foulis, S. Pulmannová, Symmetries in synaptic algebras, Math. Slovaca 64 (2014) 751–776.

[11] D.J. Foulis, S. Pulmannová, Commutativity in a synaptic algebra. To appear in Math. Slovaca.
[12] D.J. Foulis, A. Jenčová, S. Pulmannová, Two projections in a synaptic algebra, Linear Algebra Appl. 478 (2015) 189–204.

[13] Stanley P. Gudder, Sharply dominating effect algebras, Quantum structures, II (Liptovský Ján, 1998), Tatra Mt. Math. Publ. 15 (1998) 23–30.

[14] S. Gudder, S. Pulmannová, S. Bugajski, E. Beltrametti, Convex and linear effect algebras, Rep. Math. Phys. 44 (1999) 359–379.

[15] A. Gheondea, S. Gudder, P. Jonas, On the infimum of quantum effects, J. Math. Phys. 46, 062102 (2005) 11 pp.

[16] P. Halmos, Two subspaces, Trans. Amer. Math. Soc. 144 (1969) 381–389.

[17] G. Jenča, A Cantor-Bernstein type theorem for effect algebras, Algebra Univers. 48 (2002) 399–411.

[18] G. Kalmbach, Orthomodular Lattices, Academic Press, Inc., London/New York, 1983.

[19] P.J. Lahti, M.J. Mączynski, Partial order of quantum effects, J. Math. Phys. 36 (1995) 1673–1680.

[20] T. Morland, S. Gudder, Infima of Hilbert space effects, Linear Alg. Appl. 286 (1999) 1–17.

[21] S. Pulmannová, A note on ideals in synaptic algebras, Math. Slovaca 62 (2012) 1091–1104.

[22] P. Pták, S. Pulmannová, Orthomodular Structures as Quantum Logics, Veda SAV, Bratislava-Kluwer, Dordrecht, 1991.