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\textbf{Q-FACTORIAL LAURENT RINGS}

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\textbf{Abstract.} Dolgachev proves that the ring naturally associated to a generic Laurent polynomial in \(d\) variables, \(d \geq 4\), is factorial [4, 5] (for any field \(k\)). We prove a sufficient condition for the ring associated to a very general complex Laurent polynomial in \(d = 3\) variables to be \(\mathbb{Q}\)-factorial.

1. \textbf{Introduction}

In [4] and Dolgachev [5] proves that the ring \(A_F\) naturally associated to generic Laurent polynomial \(F\) in \(d\) variables, \(d \geq 4\), with coefficients in any field \(k\), is factorial. The basic ingredient in Dolgachev’s proof is Grothendieck’s Lefschetz-type theorem ([5], Prop. 3.12) which, among other things, shows that under suitable conditions, the natural restriction map \(\text{Pic}(X) \to \text{Pic}(Y)\), where \(X\) is a scheme and \(Y\) is subvariety corresponding to an ideal sheaf in \(\mathcal{O}_X\), is an isomorphism. This result can be applied only when \(d \geq 4\).

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In this paper we consider the case $d = 3$, assuming that $k = \mathbb{C}$, and prove a sufficient condition for the ring $A_F$ to be $\mathbb{Q}$-factorial (Theorem 3.1). The proof of this fact follows the lines of Dolgachev’s proof, with Grothendieck’s result replaced by a Noether-Lefschetz theorem for hypersurfaces in toric 3-folds (Theorem 2.5) that we proved in [2].

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2. Preliminaries

We follow the notation in [1] and [2]. Let $M$ be a $d$-dimensional lattice, $N = \text{Hom}(M, \mathbb{Z})$ and $T_N = N \otimes \mathbb{C}^*$ the associated algebraic torus. Let $\Sigma \subset N_{\mathbb{R}}$ be a complete simplicial fan, and denote by $X_\Sigma$ the corresponding complete toric variety. The torus $T_N$ naturally acts on $X_\Sigma$; $T_\tau \subset X_\Sigma$ denotes the orbit of a subset of $X_\Sigma$ corresponding to a face $\tau$ of $\Sigma$ under this action; the open dense orbit is denoted by $T_0$.

Definition 2.1. [1, Def. 4.13] A hypersurface $X$ in $X_\Sigma$ is nondegenerate if $X \cap T_\tau$ is a smooth 1-codimensional subvariety of $T_\tau$ for all faces $\tau$ in $\Sigma$.

$X_\Sigma$ has only abelian quotient singularities, and is therefore an orbifold.

Proposition 2.2. [1, Prop. 3.5, 4.15] Let $L$ be an ample line bundle on $X_\Sigma$. The hypersurface $X \subset X_\Sigma$ given by the zero locus of a generic section of $L$ is nondegenerate. Moreover, $X$ is an orbifold.

Since $X$ is an orbifold, its complex cohomology has a pure Hodge structure [9]. This is an essential point in the proof of our Theorem 2.5.

Definition 2.3 (The Cox Ring [3]). Consider a variable $z_i$ for each 1-dimensional cone $\varsigma_i$, $i = 1, \ldots, n$ in $\Sigma$, and let $S(\Sigma)$ be the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$.

The Cox ring has a natural gradation given by its class group $\text{Cl}(\Sigma)$ of $X_\Sigma$.

Let $L$ be an ample line bundle on $X_\Sigma$, and let $f \in H^0(X_\Sigma, L) \simeq S(\Sigma)_\beta$, where $\beta = \text{deg}(L)$.

Definition 2.4. The Jacobian ring of $f$ is the quotient $R(f) = S(\Sigma)/J(f)$, where $J(f)$ is the ideal in $S(\Sigma)$ generated by the derivatives of $f$. 

The Jacobian ring $R(f)$ inherits a natural gradation from $S(\Sigma)$.

The next theorem was proved in [2], and will be key to proving our result about Laurent rings. We assume $d = 3$. We recall that the Picard number is defined as the rank of the class group.

**Theorem 2.5.** [2] Let $X_\Sigma$ a complete simplicial toric variety, and $X \subset X_\Sigma$ a very general hypersurface cut by a section $f$ of an ample line bundle $L$ such that the multiplication morphism

$$R(f)_\beta \otimes R(f)_{\beta - \beta_0} \to R(f)_{2\beta - \beta_0}$$

is surjective (here $\beta = \deg(L)$ and $\beta_0 = -\deg(K_{X_\Sigma})$, where $K_{X_\Sigma}$ is the canonical sheaf of $X_\Sigma$). Then $X$ has the same Picard number as $X_\Sigma$.

Recall that a property is very general if it holds in the complement of countably many proper subvarieties.

If $X$ is a quartic surface in $\mathbb{P}^3$, or more generally a $K3$ surface defined by a section of the anticanonical divisor in a simplicial toric variety, then the above map is surjective [2]. It is a classical result that the map is not surjective if $X$ is a cubic in $\mathbb{P}^3$.

### 3. $\mathbb{Q}$-factorial Laurent rings

The ring $\mathbb{C}[M]$ may be identified with the ring of regular functions on the torus $T_N \simeq T_0 \subset X_\Sigma$. An element $F \in \mathbb{C}[M]$ is called a *Laurent polynomial*; $F$ may be regarded as a section of the ample line bundle $L$, and it defines a hypersurface $X_F$ in $X_\Sigma$.

Let $\Delta \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ be the polytope uniquely determined by the fan $\Sigma$ and $L$ (see [3], Lemma 2.14). To each Laurent polynomial $F$ on can associate a polytope $\Delta_F$, called the *Newton polytope* of $F$. This is most easily described by choosing an isomorphism $M \simeq \mathbb{Z}^d$, writing

$$F = \sum_{i_1, \ldots, i_d \in \mathbb{Z}^d} a_{i_1, \ldots, i_d} t_1^{i_1} \cdots t_d^{i_d}$$

and defining

$$\text{supp}(F) = \{i_1, \ldots, i_d \in \mathbb{Z}^d \mid a_{i_1, \ldots, i_d} \neq 0\}.$$  

$\Delta_F$ is then defined to be the convex hull of $\text{supp}(F)$ and $\Gamma(\Delta)$ the set of all Laurent polynomials such that $\Delta_F \subset \Delta$. $\Gamma(\Delta)$ is a finite dimensional vector space over $\mathbb{C}$.

By results given in [7] (see also [3], Chapter 2) a Laurent polynomial $F$ extends to a meromorphic function on $X_\Sigma$, which is a section of an ample line bundle $L_F$. Thus, $F$
may be regarded as an element in $S(\Sigma)_\beta$, where $\beta = \deg(L_F)$. Denote by $A_F$ the ring $\mathbb{C}[M]/(F)$.

**Theorem 3.1.** Let $d = 3$, and let $F$ be a very general Laurent polynomial in $\Gamma(\Delta)$; set $\beta = \deg(L_F)$ and $\beta_0 = -\deg(K_{X_\Sigma})$. If the multiplication morphism
\[
R(F)_\beta \otimes R(F)_{\beta - \beta_0} \to R(F)_{2\beta - \beta_0}
\]
is surjective, the ring $A_F$ is $\mathbb{Q}$-factorial.

The proof that $A_F$ is $\mathbb{Q}$-factorial follows closely the proof of Theorem 1.1 in [4]. The basic idea is to formulate the problem in a geometric way:

**Proof.** Let $X_F \subset X_\Sigma$ be the hypersurface cut by $F$ (as a section of $L_F$). By Proposition 2.2 the hypersurface $X_F$ is nondegenerate, and is an orbifold.

Note that the ring $A_F$ may be identified with the ring of regular functions on the affine part $U_F = X_F \cap T_0$ of $X_F$. Since the Picard group of $T_0$ is trivial, every Cartier divisor in $X_\Sigma$ is linearly equivalent to a divisor supported in $X_\Sigma - T_0$. By Theorem 2.5 $X_F$ has the same Picard number as $X_\Sigma$, i.e., $\rho(X_F) = \rho(X_\Sigma)$. Then any Cartier divisor in $X_F$ is linearly equivalent modulo torsion to a divisor supported in $X_F - U_F$, so that $\text{Pic}(U_F) \otimes \mathbb{Q} = 0$. Since $U_F$ is normal (actually smooth), then $\text{Cl}(U_F) \otimes \mathbb{Q} = 0$. As $U_F \simeq \text{Spec}(A_F)$, we have $\text{Cl}(A_F) \otimes \mathbb{Q} = 0$. □

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