ANALYTIC RESULTS FOR THE LINEAR STABILITY OF THE EQUILIBRIUM POINT IN ROBE’S RESTRICTED ELLIPTIC THREE-BODY PROBLEM

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(Communicated by Chongchun Zeng)

Abstract. We study the Robe’s restricted three-body problem. Such a motion was firstly studied by A. G. Robe in [13], which is used to model small oscillations of the earth’s inner core taking into account the moon’s attraction. Earlier results for the linear stability of the elliptic equilibrium point in Robe’s restricted problem depend on a lot of numerical computations, while we give an analytic approach to it. The linearized Hamiltonian system near the elliptic equilibrium point in our problem coincides with the linearized system near the Euler elliptic relative equilibria in the classical three-body problem except for the range of the mass parameter. We first establish some relations of the linear stability problem to the properties of some symplectic paths and some corresponding linear operators. Then using the Maslov-type $\omega$-index theory of symplectic paths and the theory of linear operators, we compute $\omega$-indices and obtain certain properties of the linear stability of the elliptic equilibrium point of Robe’s restricted three-body problem.

1. Introduction and main results. A new kind of restricted three-body problem that incorporates the effect of buoyancy forces was introduced by Robe in 1977. In [13], he regarded one of the primaries as a rigid spherical shell $m_1$ filled with a homogenous incompressible fluid of density $\rho_1$. The second primary is a mass point $m_2$ outside the shell and the third body $m_3$ is a small solid sphere of density $\rho_3$, inside the shell, with the assumption that the mass and radius of $m_3$ are infinitesimal. He has shown the existence of the equilibrium point with $m_3$ at the center of the shell, where $m_2$ describes a Keplerian orbit around it, see Figure [1].

Furthermore, he discussed two cases of the linear stability of the equilibrium point of such a restricted three-body problem. In the first case, the orbit of $m_2$ around $m_1$ is circular and in the second case, the orbit is elliptic, but the shell is empty (that is no fluid inside it) or the densities of $m_1$ and $m_3$ are equal. In the second case, we use “elliptic equilibrium point” to call the equilibrium point.

2010 Mathematics Subject Classification. Primary: 70F07, 70H14; Secondary: 34C25.

Key words and phrases. Restricted three-body problem, equilibrium point, linear stability, Maslov-type $\omega$-index.

The first author is supported by NSFC (No. 11501330, No. 11425105) and CPSF (Grant No. 2015M582071). The second author is supported by the Fundamental Research Funds for the Central Universities (Grant No. N142303010).
Figure 1. The Robe’s restricted three-body problem considered: $m_1$ is a spherical shell filled with a fluid of density $\rho_1$; $m_2$ a mass point outside the shell and $m_3$ a small solid sphere of density $\rho_3$ inside the shell.

In each case, the domain of stability has been investigated for the whole range of parameters occurring in the problem.

Later on, A. R. Plastino and A. Plastino ([12]) studied the linear stability of the equilibrium point and the connection between the effect of the buoyancy forces and a perturbation of a Coriolis force. In 2001, P. P. Hallen and N. Rana ([3]) found other new equilibrium points of the restricted problem and discussed their linear stabilities. K. T. Singh, B. S. Kushvah and B. Ishwar ([15]) examined the stability of triangular equilibrium point in Robe’s generalized restricted three body problem where the problem is generalized in the sense that a more massive primary has been taken as an oblate spheroid.

However, for the elliptic case, the studies of the linear stability of equilibrium point are much more complicated than that of the circular case, thus in [13], the bifurcation diagram of linear stability was obtained just by numerical methods. In [12, 3, 14, 15], the authors studied the stability of equilibrium points in a modified problem, but their studies did not contain the elliptic case.

On the other hand, in [6, 7] of 2009–2010, X. Hu and S. Sun found a new way to relate the stability problem to the iterated Morse indices. Recently, by observing new phenomena and discovering new properties of elliptic Lagrangian solutions, in the joint paper [4] of X. Hu, Y. Long and S. Sun, the linear stability of elliptic Lagrangian solutions is completely solved analytically by index theory (cf. [8]) and the new results are related directly to $(\beta, e)$ in the full parameter rectangle. Here $\beta$ is the mass parameter which was given by (1.4) of [4], and $e$ is the common eccentricity of the elliptic orbit of each body. Inspired by the analytic method, Q. Zhou and Y. Long in [16] studied the linear stability of elliptic triangle solutions of a charged three-body problem.

Recently, in [17, 18], Q. Zhou and Y. Long studied the linear stability of elliptic Euler-Moulton solutions of $n$-body problem for $n = 3$ and for general $n \geq 4$, respectively. Also, the linear stability of Euler collision solutions of 3-body problem was studied by X. Hu and Y. Ou in [5].
In the current paper, we study an analytical approach to the linear stability of equilibrium point of the Robe’s restricted three-body problem. We related their linear stabilities to the Maslov-type and Morse indices of them. For such elliptic equilibrium point, we use index theory to compute the Maslov-type indices of the corresponding symplectic paths and determine their stability properties.

Following Robe’s notation in [13], various forces acting on \( m_3 \) are:

1. The attraction of \( m_2 \),
2. The gravitational force of attraction
   
   \[ F_A = -\left(\frac{4}{3}\pi\right)G\rho_1 m_3 \overrightarrow{M_1 M_3} \]  
   
   exerted by the fluid of density \( \rho_1 \), where \( M_1 \) is the center of the spherical shell \( m_1 \), \( M_3 \) is the center of \( m_3 \), and \( \overrightarrow{M_1 M_3} \) is the vector from \( M_1 \) to \( M_3 \),
3. The buoyancy force
   
   \[ F_B = \left(\frac{4}{3}\pi\right)G\rho_2 m_3 \overrightarrow{M_1 M_3}/\rho_3 \]  
   
   of fluid density \( \rho_1 \).

Let the orbital plane of \( m_2 \) around \( m_1^* \) (that is the shell with its fluid) be taken as the \( xy \)-plane and let the origin of the coordinate system be at the center of the mass, \( O \), of the two primaries. The equation of motion of \( m_3 \) is

\[ \ddot{R}_3 = Gm_2 \frac{R_{32}}{R_{32}^3} \left(1 - \frac{\rho_1}{\rho_3}\right)\dot{R}_{13}, \]

where \( R_3 = \overrightarrow{OM_3} \) and \( R_{ij} = \overrightarrow{M_i M_j} \). Assume that \( m_2 \) describes around \( m_1^* \) an elliptical orbit of eccentricity \( e \) and long semi-axis \( a \), the distance \( l \) between the two bodies is given by

\[ l = \frac{a(1 - e^2)}{1 + e \cos \theta}, \]

where \( \theta \) is the true anomaly.

The equations of motion in the \( xy \)-plane obtained by Robe are:

\[ \ddot{x} - 2\dot{y} = \left(1 + e \cos \theta\right)^{-1}V_x, \]  

\[ \ddot{y} + 2\dot{x} = \left(1 + e \cos \theta\right)^{-1}V_y, \]

where \( V \) is given by

\[ V = \frac{1}{2}(x^2 + y^2 + z^2) + \frac{\mu}{(x-x_1)^2 + y^2 + z^2} - \frac{K}{2}\frac{1 - e^2}{1 + e \cos \theta} \left[(x-x_1)^2 + y^2 + z^2\right] \]

with

\[ \mu = \frac{m_2}{m_1^* + m_2}, 0 < \mu < 1; \quad K = \frac{4}{3}\pi \frac{\rho_1 a^3}{m_1^* + m_2}(1 - \frac{\rho_1}{\rho_3}), \]

\[ x_1 \text{ and } x_2 \text{ being the } x \text{ coordinates of } M_1 \text{ and } M_2: \]

\[ x_1 = \mu, \quad x_2 = 1 - \mu. \]  

Here Robe used a non-uniformly rotating and pulsating coordinate system.

In [13], H. Robe studied the equilibrium point at the center of \( m_1 \) in the circular case and in the elliptic case. He also studied the linear stability of the above two cases of such equilibrium point. But for the elliptic case, only numerical results was obtained. Later on, in [3], P. P. Hallen and N. Rana studied the existence of all the equilibrium points in the Robe’s restricted three-body problem. They found that, in the case of equilibrium points with circular, there are several different situations.
depending on \( K \), and the linear stability of such equilibrium points were carefully studied. More details can be seen in \([3]\).

We focus on the case when there is no fluid inside the shell or when the densities of \( m_1 \) and \( m_3 \) are equal \((\rho_1 = \rho_3)\), i.e., \( K = 0 \). Here \( K \) reflect the joint effect of the gravitational force \( \mathbf{F}_A \) and the buoyancy force \( \mathbf{F}_B \). \( K = 0 \) implies that the total effect of the fluid is disappeared. By (17)-(19) in \([13]\), the linearized equations of motion around the equilibrium point are:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \frac{1 + 2\mu}{1 + e \cos \theta} \dot{x}, \\
\ddot{y} + 2\dot{x} &= \frac{-1 - \mu}{1 + e \cos \theta} \dot{y},
\end{align*}
\]  

(9)

which is a set of linear homogeneous equations with periodic coefficients of periodic \( 2\pi \).

Now we study equations \((9)-(10)\) by another form. Let \((W_1, W_2, w_1, w_2)^T = (\dot{x} - y, \dot{y} + x, x, y)^T \) and \( t = \theta \), then we have

\[
\frac{d}{dt} \begin{pmatrix}
W_1 \\
W_2 \\
w_1 \\
w_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 & -1 + \frac{1 + 2\mu}{1 + e \cos t} & 0 \\
-1 & 0 & 0 & -1 + \frac{-1 - \mu}{1 + e \cos t} \\
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{pmatrix} \begin{pmatrix}
W_1 \\
W_2 \\
w_1 \\
w_2
\end{pmatrix}.
\]

(11)

Let \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) be the standard symplectic matrix for any \( n \in \mathbb{N} \), and

\[
B(t) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 - \frac{1 + 2\mu}{1 + e \cos t} & 0 \\
1 & 0 & 0 & 1 - \frac{-1 - \mu}{1 + e \cos t}
\end{pmatrix},
\]

(12)

then (11) can be written as

\[
\dot{w} = JB(t)w,
\]

(13)

where \( w = (W_1, W_2, w_1, w_2)^T \). When \( \mu = \beta + 1 \), \( B(t) \) of (12) coincides with \( B(t) \) of (2.35) in \([17]\). Thus a lot of results which were developed in \([17]\) can be applied to this paper.

Following \([9]\) and \([11]\), denote by \( \text{Sp}(2n) \) the symplectic group of real \( 2n \times 2n \) matrices. For any \( \omega \in U = \{ z \in \mathbb{C} \mid |z| = 1 \} \) we can define a real function \( D_\omega(M) = (-1)^{n-1} \Im \det(M - \omega I_{2n}) \) for any \( M \in \text{Sp}(2n) \). Then we define \( \text{Sp}(2n)\omega = \{ M \in \text{Sp}(2n) \mid D_\omega(M) = 0 \} \) and \( \text{Sp}(2n)\omega_\in = \text{Sp}(2n) \setminus \text{Sp}(2n)\omega \). The orientation of \( \text{Sp}(2n)\omega_\in \) at any of its point \( M \) is defined to be the positive direction \( \frac{d}{dt}Me^{tJ}|_{t=0} \) of the path \( Me^{tJ} \) with \( t > 0 \) small enough. Let \( \nu_\omega(M) = \dim \ker_c(M - \omega I_{2n}) \). Let \( P_{2\pi}(2n) = \{ \gamma \in C([0, 2\pi], \text{Sp}(2n)) \mid \gamma(0) = I \} \) and \( \xi(t) = \text{diag}(2 - \frac{t}{2\pi}, (2 - \frac{t}{2\pi})^{-1}) \) for \( 0 \leq t \leq 2\pi \).

Given any two \( 2m_k \times 2m_k \) matrices of square block form \( M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \) with \( k = 1, 2 \), the symplectic sum of \( M_1 \) and \( M_2 \) is defined (cf. \([9]\) and \([11]\)) by the following \( 2(m_1 + m_2) \times 2(m_1 + m_2) \) matrix \( M_1 \odot M_2 \):

\[
M_1 \odot M_2 = \begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix},
\]
and $M^\otimes k$ denotes the $k$ copy $\diamond$-sum of $M$. For any two paths $\gamma_j \in \mathcal{P}_r(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \circ \gamma_1(t) = \gamma_0(t) \circ \gamma_1(t)$ for all $t \in [0, \tau]$.

As in [11], for $\lambda \in \mathbb{R} \setminus \{0\}$, $a \in \mathbb{R}$, $\theta \in (0, \pi) \cup (\pi, 2\pi)$, $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbb{R}$ for $i = 1, \ldots, 4$, and $c_j \in \mathbb{R}$ for $j = 1, 2$, we denote respectively some normal forms by

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

$$N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad N_2(e^{\sqrt{-1} \theta}, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix},$$

$$M_2(\lambda, c) = \begin{pmatrix} \lambda & 1 & c_1 & 0 \\ 0 & \lambda & c_2 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & -\lambda^{-2} & \lambda^{-1} \end{pmatrix}.$$

Here $N_2(e^{\sqrt{-1} \theta}, b)$ is trivial if $(b_2 - b_3) \sin \theta > 0$, or non-trivial if $(b_2 - b_3) \sin \theta < 0$, in the sense of Definition 1.8.11 on p.41 of [11].

For every $M \in \text{Sp}(2n)$ and $\omega \in \mathbb{U}$, as in Definition 1.8.5 on p.38 of [11], we define the $\omega$-homotopy set $\Omega_{\omega}(M)$ of $M$ in $\text{Sp}(2n)$ by

$$\Omega_{\omega}(M) = \{ N \in \text{Sp}(2n) \mid \nu_{\omega}(N) = \nu_{\omega}(M) \},$$

and the homotopy set $\Omega(M)$ of $M$ in $\text{Sp}(2n)$ by

$$\Omega(M) = \{ N \in \text{Sp}(2n) \mid \sigma(N) \cap \mathbb{U} = \sigma(M) \cap \mathbb{U}, \text{ and } \nu_{\lambda}(N) = \nu_{\lambda}(M) \quad \forall \lambda \in \sigma(M) \cap \mathbb{U} \}.$$

We denote by $\Omega^{0}(M)$ (or $\Omega^{0}_{\omega}(M)$) the path connected component of $\Omega(M)$ ($\Omega_{\omega}(M)$) which contains $M$, and call it the homotopy component (or $\omega$-homotopy component) of $M$ in $\text{Sp}(2n)$. Following Definition 5.0.1 on p.111 of [11], for $\omega \in \mathbb{U}$ and $\gamma_i \in \mathcal{P}_r(2n)$ with $i = 0, 1$, we write $\gamma_0 \sim \gamma_1$ if $\gamma_0$ is homotopic to $\gamma_1$ via a homotopy map $h \in C([0, 1] \times [0, \tau], \text{Sp}(2n))$ such that $h(0) = \gamma_0$, $h(1) = \gamma_1$, $h(s)(0) = I$, and $h(s)(\tau) \in \Omega^{0}_{\omega}(\gamma_0(\tau))$ for all $s \in [0, 1]$. We write also $\gamma_0 \sim \gamma_1$ if $h(s)(\tau) \in \Omega^{0}(\gamma_0(\tau))$ for all $s \in [0, 1]$ is further satisfied. We write $M \approx N$, if $N \in \Omega^{0}(M)$.

For any $\gamma \in \mathcal{P}_{2\pi}(2n)$ we define $\nu_{\omega}(\gamma) = \nu_{\omega}(\gamma(2\pi))$ and

$$i_{\omega}(\gamma) = [\text{Sp}(2n)_{\omega} : \gamma * \xi^n], \quad \text{ if } \gamma(2\pi) \not\in \text{Sp}(2n)_{\omega}^{0},$$

i.e., the usual homotopy intersection number, and the orientation of the joint path $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points. When $\gamma(2\pi) \in \text{Sp}(2n)_{\omega}^{0}$, we define $i_{\omega}(\gamma)$ to be the index of the left rotation perturbation path $\gamma_-, \epsilon > 0$ small enough (cf. Def. 5.4.2 on p.129 of [11]). The pair $(i_{\omega}(\gamma), \nu_{\omega}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}$ is called the index function of $\gamma$ at $\omega$. When $\nu_{\omega}(\gamma) = 0$ ($\nu_{\omega}(\gamma) > 0$), the path $\gamma$ is called $\omega$-non-degenerate ($\omega$-degenerate). For more details we refer to the Appendix or [11].

Based on the above notation, we will give some statements on the stability and instability of the periodic solutions of the Hamiltonian systems via indices of the orbits. Recall that, for $M \in \text{Sp}(2n)$, it is linearly stable if $||M^j|| \leq C$ for some constant $C$ and all $j \in \mathbb{N}$. Note that this implies $M$ is diagonalizable and the eigenvalues of $M$ are all on the unit circle $\mathbb{U}$ of the complex plane. We call $M$ to be spectrally stable if all its eigenvalues are on the unit circle.
Definition 1.1. Given a $T$-periodic solution $z(t)$ to a first order Hamiltonian system with fundamental solution $\gamma(t)$, we say $z$ is spectrally stable (linearly stable) if $\gamma(T)$ is spectrally stable (linearly stable, respectively).

In the literature there are many papers concerning the stability of the periodic solutions of the Hamiltonian system using the Maslov-type index [2, 9, 11]. The complete iteration formula developed by Y. Long and his collaborators is a very effective tool for this purpose.

Precisely, if $M \in \text{Sp}(2n)$ is linearly stable as defined, then there exists $P \in \text{Sp}(2n)$, such that [1] (p. 223, Remark(c))

$$P^{-1}MP = I_2^J \circ M_{j+1} \circ \ldots \circ M_n,$$

where $M_i = \left(\begin{array}{cc} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{array}\right), \theta_i \in (0, 2\pi)$. Moreover, $\det(M_i - I_2) > 0$, which means that $\det(e^{-2i}P^{-1}MP - I_{2n}) > 0$ with real $\epsilon > 0$ small enough. Additionally, if $M$ is $1$-nondegenerate, $j = 0$ must hold; if $M$ is $\omega$-nondegenerate for some $\omega = e^{\sqrt{-1} \theta_0} \in \mathbb{U} \setminus \{1\}$, then $\theta_i \neq \pm \theta_0(\text{mod } 2\pi)$ must hold for any $1 \leq i \leq n$.

The following two theorems describe the main results proved in this paper.

Theorem 1.2. In the Robe’s restricted three-body problem, we denote by $\gamma_{\mu,e} : [0, 2\pi] \to \text{Sp}(4)$ the fundamental solution of the linearized Hamiltonian system [15] near the equilibrium point. Then $\gamma_{\mu,e}(2\pi)$ is non-degenerate for all $(\mu, e) \in (0, 1) \times [0, 1)$; and when $\mu = 0$ or $1$, it is degenerate. Moreover, the following results on the Maslov-type indices of $\gamma_{\mu,e}$ hold.

(i) For $e \in [0, 1)$, we have

$$i_1(\gamma_{\mu,e}) = 0 \quad \forall 0 \leq \mu \leq 1,$$

$$\nu_1(\gamma_{\mu,e}) = \begin{cases} 2, & \text{if } \mu = 0, \\ 0, & \text{if } 0 < \mu < 1, \\ 3, & \text{if } \mu = 1. \end{cases}$$

(ii) Let

$$\mu^* = \frac{5 + \sqrt{97}}{16}.$$ 

We have

$$i_{-1}(\gamma_{\mu,0}) = \begin{cases} 0, & \text{if } \mu \in [0, \mu^*], \\ 2, & \text{if } \mu \in (\mu^*, 1], \end{cases}$$

$$\nu_{-1}(\gamma_{\mu,0}) = \begin{cases} 2, & \text{if } \mu = \mu^*, \\ 0, & \text{if } \mu \in [0, 1] \setminus \{\mu^*}\end{cases}.$$ 

(iii) For fixed $e \in [0, 1)$ and $\omega \in \mathbb{U} \setminus \{1\}$, $i_\omega(\gamma_{\mu,e})$ is non-decreasing and tends from 0 to 2 when $\mu$ increases from 0 to 1. In particular, for every $e \in [0, 1)$, the $-1$-index $i_{-1}(\gamma_{\mu,e})$ is strictly increasing only on the two values $\mu = \mu_1(e), \mu_2(e) \in (0, 1)$. Here the parameter $-1$ in the functions $\mu_1$ and $\mu_2$ reflect the $-1$-indices.

Remark 1. (i) Here we are specially interested in indices with eigenvalues 1 and $-1$. The reason is that the major changes of the linear stability of the elliptic equilibrium point happen near the eigenvalues 1 and $-1$, and such information is used in the next theorem to get the separation curves of the linear stability domain $[0, 1] \times [0, 1)$ of the mass and eccentricity parameter $(\mu, e)$.
Theorem 1.3. Using notation in Theorem 1.2, define
\[ \mu_m(e) = \min\{\mu_1(e, -1), \mu_2(e, -1)\}, \quad \mu_r(e) = \max\{\mu_1(e, -1), \mu_2(e, -1)\}, \]
and
\[ \mu_l(e) = \inf\{\mu' \in [0, 1] | \sigma(\gamma_{\mu,e}(2\pi)) \cap U \neq \emptyset, \forall \mu \in (0,\mu']\}, \]
for \( e \in [0, 1) \). Let
\[ \Gamma_j = \{(\mu_j(e), e) \in [0, 1] \times [0, 1)\}, \]
for \( j = l, m, r \), i.e., the curves \( \Gamma_l, \Gamma_m \) and \( \Gamma_r \) are the diagrams of the functions \( \mu_l, \mu_m \) and \( \mu_r \) with respect to \( e \in [0, 1) \), respectively. These three curves separated the parameter rectangle \( \Theta \) into four regions, and we denote them by I, II, III and IV (see Figure 2), respectively. Then we have the following:

(i) \( 0 < \mu_l(e, -1) < 1, i = 1, 2 \). Moreover, \( \mu_1(0, -1) = \mu_2(0, -1) = \mu^* \) with \( \mu^* \) given by (17), and \( \lim_{e \to 1} \mu_1(e, -1) = \lim_{e \to 1} \mu_2(e, -1) = 1 \);

(ii) The two functions \( \mu_1 \) and \( \mu_2 \) are real analytic in \( e \), and with derivatives \( -\frac{291+15\sqrt{37}}{3104}, \frac{291+15\sqrt{37}}{3104} \) at \( e = 0 \) with respect to \( e \) respectively, thus they are different and the intersection points of their diagrams must be isolated if there exist when \( e \in (0, 1) \). Consequently, \( \Gamma_m \) and \( \Gamma_r \) are different piecewise real analytic curves;

(iii) We have
\[ \gamma_{\mu,e}(2\pi) \]
and \( \Gamma_m \) and \( \Gamma_r \) are precisely the \( -1 \)-degenerate curves of the path \( \gamma_{\mu,e} \) in the \( (\mu,e) \) rectangle \( \Theta = [0, 1] \times [0, 1) \);

(iv) Every matrix \( \gamma_{\mu,e}(2\pi) \) is hyperbolic when \( \mu \in [0,\mu_l(e)), e \in (0,1) \), and there holds
\[ \mu_l(e) = \sup\{\mu \in [0, 1] | \sigma(\gamma_{\mu,e}(2\pi)) \cap U = \emptyset, \forall e \in [0, 1)\}. \]

Consequently, \( \Gamma_l \) is the boundary curve of the hyperbolic region of \( \gamma_{\mu,e}(2\pi) \) in \( \Theta \);

(v) \( \Gamma_l \) is continuous in \( e \in (0, 1) \), and \( \lim_{e \to 1} \mu_l(e) = 1 \);

(vi) \( \Gamma_l \) is different from the curve \( \Gamma_m \) at least when \( e \in (0, e) \) for some \( e \in (0, 1) \);

(vii) In Region I, i.e., when \( 0 < \mu < \mu_l(e) \), we have \( \gamma_{\mu,e}(2\pi) \) is hyperbolic, and thus it is linearly unstable;

(viii) In Region II, i.e., when \( \mu_l(e) < \mu < \mu_m(e) \), we have \( \gamma_{\mu,e}(2\pi) \approx R(\theta_1) \circ R(\theta_2) \) for some \( \theta_1 \in (\pi, 2\pi) \) and \( \theta_2 \in (0, \pi) \), and thus it is strongly linear stable;

(ix) In Region III, i.e., when \( \mu_m(e) < \mu < \mu_r(e) \), we have \( \gamma_{\mu,e}(2\pi) \approx R(\theta) \circ D(-2) \) for some \( \theta \in (\pi, 2\pi) \), and thus it is linearly unstable;

(x) In Region IV, i.e., when \( \mu_r(e) < \mu < 1 \), we have \( \gamma_{\mu,e}(2\pi) \approx R(\theta_1) \circ R(\theta_2) \) for some \( \theta_1, \theta_2 \in (\pi, 2\pi) \), and thus it is strongly linear stable.

Remark 2. When \( (\mu,e) \) is in Region II or IV, i.e., the shaded regions in Figure 2 then \( \gamma_{\mu,e}(2\pi) \), and hence the elliptic equilibrium point is linear stable. Comparing Figure 3 of [13] with our Figure 2, I is the point \( \mu = 8/9 \), \( F \) is the point \( \mu = \mu^* \) in the \( \mu \)-axis and the shaded regions represent the same parameter regions. In Robe's figure, the boundary point \( G \) of the right shaded region is inside \( \Theta \). But in our theorem, we can strictly obtain that \( G \) is just the point \( (\mu,e) = (1,1) \).

Moreover, for \( (\mu,e) \) located on these three special curves \( \Gamma_l, \Gamma_m \) and \( \Gamma_r \), we have the following:
Figure 2. Stability bifurcation diagram of elliptic equilibrium point of the Robe’s restricted three-body problem in the $(\mu, e)$ rectangle $[0, 1] \times [0, 1]$.

(i) If $\mu_l(e) < \mu_m(e) \leq \mu_r(e)$, we have $\gamma_{\mu_l(e), e}(2\pi) \approx N_2(e^{\sqrt{-1}\theta}, b)$ for some $\theta \in (0, \pi)$ and $b = (b_1, b_2, b_3, b_4)$ satisfying $(b_2 - b_3)\sin \theta > 0$. Consequently, the matrix $\gamma_{\mu_l(e), e}(2\pi)$ is spectrally stable and linearly unstable;

(ii) If $\mu_l(e) = \mu_m(e) < \mu_r(e)$, we have $\gamma_{\mu_l(e), e}(2\pi) \approx M_2(-1, c)$ with $c_1, c_2 \in \mathbb{R}, c_2 \neq 0$, and it is spectrally stable and linearly unstable;

(iii) If $\mu_l(e) = \mu_m(e) = \mu_r(e)$, we have $\gamma_{\mu_l(e), e}(2\pi) \approx N_1(-1, 1) \odot N_1(-1, -1)$ and it is spectrally stable and linearly unstable;

(iv) If $\mu_l(e) < \mu_m(e) < \mu_r(e)$, we have $\gamma_{\mu_l(e), e}(2\pi) \approx N_1(-1, -1) \odot R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and thus is spectrally stable and linearly unstable;

(v) If $\mu_l(e) < \mu_m(e) = \mu_r(e)$, we have $\gamma_{\mu_l(e), e}(2\pi) \approx -I_2 \odot R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and thus is linearly stable but not strongly linearly stable;

(vi) If $\mu_l(e) < \mu_r(e)$, we have $\gamma_{\mu_l(e), e}(2\pi) \approx N_1(-1, -1) \odot R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and thus is spectrally stable and linearly unstable.

The paper is organized as follows. In Section 2 we associate $\gamma_{\mu, e}(t)$, the fundamental solution of the system (13), with a corresponding second order self-adjoint operator $A(\mu, e)$. Some connections between $\gamma_{\mu, e}(t)$ and $A(\mu, e)$ are given there. In Section 3 we compute the $\omega$-indices along the three boundary segments of $(\mu, e)$ rectangle $[0, 1] \times [0, 1)$. In Section 4 the non-decreasing property of $\omega$-index is proved in Lemma 4.1 and Corollary 1. Also Theorem 1.2 and Theorem 1.3 are proved there. For Theorem 1.2 the index properties in (i) and (ii) are established in Section 3 the non-decreasing property (iii) is proved in Theorem 4.2. Theorem 1.3 parts (i)-(iii) are proved in Section 4.3 and the remaining part is proved in Section 4.4.
2. **Associate** \(\gamma_{\mu,e}(t)\) with a second order self-adjoint operator \(A(\mu, e)\). In the Appendix, we give a brief review on the Maslov-type \(\omega\)-index theory for \(\omega\) in the unit circle of the complex plane following [11]. In the following, we use notation introduced there.

Let

\[
J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q_{\mu,e}(t) = \begin{pmatrix} \frac{1+2\mu}{1+\varepsilon \cos t} & 0 \\ 0 & \frac{1-\mu}{1+\varepsilon \cos t} \end{pmatrix},
\]

and set

\[
L(t,x,\dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + J_2 x(t) \cdot \dot{x}(t) + \frac{1}{2} Q_{\mu,e}(t) x(t) \cdot x(t), \quad \forall \, x \in W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^2),
\]

where \(a \cdot b\) denotes the inner product in \(\mathbb{R}^2\). By Legendrian transformation, the corresponding Hamiltonian function to system (13) is

\[
H(t,w) = \frac{1}{2} B(t) w \cdot w, \quad \forall \, w \in \mathbb{R}^1.
\]

Now let \(\gamma = \gamma_{\mu,e}(t)\) be the fundamental solution of the (13) satisfies:

\[
\begin{align*}
\dot{\gamma}(t) &= J B(t) \gamma(t), \\
\gamma(0) &= I_4.
\end{align*}
\]

In order to transform the Lagrangian system (26) to a simpler linear operator corresponding to a second order Hamiltonian system with the same linear stability as \(\gamma_{\mu,e}(2\pi)\), using \(R(t)\) and \(R_4(t) = \text{diag}(R(t), R(t))\) as in Section 2.4 of [4], we let

\[
\xi_{\mu,e}(t) = R_4(t) \gamma_{\mu,e}(t), \quad \forall \, t \in [0, 2\pi], (\mu, e) \in [0, 1] \times [0, 1).
\]

One can show by direct computation that

\[
\frac{d}{dt} \xi_{\mu,e}(t) = J \begin{pmatrix} I_4 & 0 \\ 0 & R(t)(I_2 - Q_{\mu,e}(t))R(t)^T \end{pmatrix} \xi_{\mu,e}(t).
\]

Note that \(R_4(0) = R_4(2\pi) = I_4\), so \(\gamma_{\mu,e}(2\pi) = \xi_{\mu,e}(2\pi)\) holds and the linear stabilities of the systems (27-28) and (30) are precisely the same.

By (29) the symplectic paths \(\gamma_{\mu,e}\) and \(\xi_{\mu,e}\) are homotopic to each other via the homotopy \(h(s,t) = R_4(st) \gamma_{\mu,e}(t)\) for \((s,t) \in [0,1] \times [0,2\pi]\). Because \(R_4(s) \gamma_{\mu,e}(2\pi)\) for \(s \in [0,1]\) is a loop in \(Sp(4)\) which is homotopic to the constant loop \(\gamma_{\mu,e}(2\pi)\), we have \(\gamma_{\mu,e} \sim_1 \xi_{\mu,e}\) by the homotopy \(h\). Then by Lemma 5.2.2 on p.117 of [11], the homotopy between \(\gamma_{\mu,e}\) and \(\xi_{\mu,e}\) can be realized by a homotopy which fixes the end point \(\gamma_{\mu,e}(2\pi)\) all the time. Therefore by the homotopy invariance of the Maslov-type index (cf. (i) of Theorem 6.2.7 on p.147 of [11]) we obtain

\[
i_\omega(\xi_{\mu,e}) = i_\omega(\gamma_{\mu,e}), \quad \nu_\omega(\xi_{\mu,e}) = \nu_\omega(\gamma_{\mu,e}), \quad \forall \, \omega \in U, (\mu, e) \in [0,1] \times [0,1).
\]

Note that the first order linear Hamiltonian system (30) corresponds to the following second order linear Hamiltonian system

\[
\ddot{x}(t) = -x(t) + R(t)Q_{\mu,e}(t)R(t)^T x(t).
\]

For \((\mu, e) \in [0,1] \times [0,1]\), the second order differential operator corresponding to (32) is given by

\[
A(\mu, e) = -\frac{d^2}{dt^2} I_2 - I_2 + R(t)Q_{\mu,e}(t)R(t)^T T
\]

\[
= -\frac{d^2}{dt^2} I_2 - I_2 + \frac{1}{2(1+\varepsilon \cos t)} [(2+\mu)I_2 + 3\mu S(t)],
\]
where $S(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}$, defined on the domain $D(\omega, 2\pi)$ in (132). Then it is self-adjoint and depends on the parameters $\mu$ and $e$. By Lemma 5.9 we have for any $\mu$ and $e$, the Morse index $\phi_\omega(A(\mu, e))$ and nullity $\nu_\omega(A(\mu, e))$ of the operator $A(\mu, e)$ on the domain $D(\omega, 2\pi)$ satisfy

$$\phi_\omega(A(\mu, e)) = i_\omega(\xi_{\mu,e}), \quad \nu_\omega(A(\mu, e)) = \nu_\omega(\xi_{\mu,e}), \quad \forall \omega \in U. \quad (34)$$

In the rest of this paper, we shall use both of the paths $\gamma_{\mu,e}$ and $\xi_{\mu,e}$ to study the linear stability of $\gamma_{\mu,e}(2\pi) = \xi_{\mu,e}(2\pi)$. Because of (31), in many cases and proofs below, we shall not distinguish these two paths.

3. Computation of the $\omega$-indices on the boundary of the bounded rectangle $[0, 1] \times [0, 1]$. We first know the full range of $(\mu, e)$ is $(0, 1) \times [0, 1)$. For convenience in the mathematical study, we extend the range of $(\mu, e)$ to $[0, 1] \times [0, 1)$.

Furthermore, we need more precise information on indices of $\gamma_{\mu,e}$ at the boundary of the $(\mu, e)$ rectangle $[0, 1] \times [0, 1)$. In this section, we will compute the $\omega$-indices $(i_\omega, \nu_\omega)$ through the Morse indices $(\phi_\omega, \nu_\omega)$ by (34).

3.1. $\omega$-indices on the boundary segments $\{0\} \times [0, 1)$ and $\{1\} \times [0, 1)$. When $\mu = 0$, from (33), we have

$$A(0, e) = -\frac{d^2}{dt^2} I_2 - I_2 + \frac{1}{1 + e \cos t} I_2, \quad (35)$$

this is just the same case which has been discussed in Section 4.1 of [17]. Using Lemma 4.1 of [17], $A(0, e)$ is non-negative definite for the $\omega = 1$ boundary condition, and $A(0, e)$ is positive definite for the $\omega \in U \setminus \{1\}$ boundary condition. Hence we have

$$i_\omega(\gamma_{0,e}) = i_\omega(\xi_{0,e}) = \begin{cases} 0, & \text{if } \omega = 1, \\ 0, & \text{if } \omega \in U \setminus \{1\}, \end{cases} \quad (36)$$

$$\nu_\omega(\gamma_{0,e}) = \nu_\omega(\xi_{0,e}) = \begin{cases} 2, & \text{if } \omega = 1, \\ 0, & \text{if } \omega \in U \setminus \{1\}. \end{cases} \quad (37)$$

When $\mu = 1$, from (33), we have

$$A(1, e) = -\frac{d^2}{dt^2} I_2 - I_2 + \frac{3}{2(1 + e \cos t)} (I_2 + 3S(t)). \quad (38)$$

This is just the case which has been discussed in Section 3.1 of [4]. We just cite the results here:

$$i_\omega(\gamma_{1,e}) = i_\omega(\xi_{1,e}) = \begin{cases} 0, & \text{if } \omega = 1, \\ 2, & \text{if } \omega \in U \setminus \{1\}, \end{cases} \quad (39)$$

$$\nu_\omega(\gamma_{1,e}) = \nu_\omega(\xi_{1,e}) = \begin{cases} 3, & \text{if } \omega = 1, \\ 0, & \text{if } \omega \in U \setminus \{1\}. \end{cases} \quad (40)$$

3.2. $\omega$-indices on the boundary $[0, 1] \times \{0\}$. In this case $e = 0$. We shall first recall the properties of eigenvalues of $\gamma_{\mu,0}(2\pi)$. Then we carry out the computations of normal forms of $\gamma_{\mu,0}(2\pi)$, and $\pm 1$ indices $i\pm 1(\gamma_{\mu,0})$ of the path $\gamma_{\mu,0}$ for all $\mu \in [0, 1]$, which are new.
In this case, the system \(13\) becomes an ODE system with:

\[
B = B(t) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & -1 & -2\mu & 0 \\
1 & 0 & 0 & \mu
\end{pmatrix}.
\] (41)

The characteristic polynomial \(\det(JB - \lambda I)\) of \(JB\) is given by

\[
\lambda^4 + (2 - \mu)\lambda^2 + (1 - \mu)(1 + 2\mu) = 0.
\] (42)

Letting \(\alpha = \lambda^2\), the two roots of the quadratic polynomial \(\alpha^2 + (2 - \mu)\alpha + (1 - \mu)(1 + 2\mu)\) are given by \(\alpha_1 = \frac{\mu - 2 + \sqrt{9\mu^2 - 8\mu}}{2}\) and \(\alpha_2 = \frac{\mu - 2 - \sqrt{9\mu^2 - 8\mu}}{2}\). Therefore the four roots of the polynomial (42) are given by

\[
\alpha_{1,\pm} = \pm \sqrt{\alpha_1} = \pm \sqrt{\frac{\mu - 2 + \sqrt{9\mu^2 - 8\mu}}{2}},
\] (43)

\[
\alpha_{2,\pm} = \pm \sqrt{\alpha_2} = \pm \sqrt{\frac{\mu - 2 - \sqrt{9\mu^2 - 8\mu}}{2}}.
\] (44)

(A) Eigenvalues of \(\gamma_{\mu,0}(2\pi)\) for \(\mu \in [0, 1]\).

When \(0 < \mu < \frac{8}{9}\), from (43) and (44), by direct computation the four characteristic multipliers of the matrix \(\gamma_{\mu,0}(2\pi)\) are given by

\[
\rho_{1,\pm}(\mu) = e^{2\pi \alpha_{1,\pm}} \in \mathbb{C}\backslash(U \cap \mathbb{R}),
\]

\(\rho_{2,\pm}(\mu) = e^{2\pi \alpha_{2,\pm}} \in \mathbb{C}\backslash(U \cap \mathbb{R}).\n\) (45) (46)

When \(\frac{8}{9} \leq \mu \leq 1\), by (43) and (44), we get four characteristic multipliers of \(\gamma_{\mu,0}(2\pi)\)

\[
\rho_{i,\pm}(\mu) = e^{2\pi \alpha_{i,\pm}} = e^{\pm 2\pi \sqrt{-\theta_i(\mu)}}, \quad i = 1, 2,
\] (47)

where

\[
\theta_1(\mu) = \sqrt{\frac{2 - \mu - \sqrt{9\mu^2 - 8\mu}}{2}}, \quad \theta_2(\mu) = \sqrt{\frac{2 - \mu + \sqrt{9\mu^2 - 8\mu}}{2}}.
\] (48)

Moreover, when \(\frac{8}{9} \leq \mu \leq 1\), we have

\[
\frac{d\theta_1(\mu)}{d\mu} = -\frac{1 + \frac{9\mu - 4}{2\sqrt{9\mu^2 - 8\mu}}}{4 \sqrt{2 - \mu - \sqrt{9\mu^2 - 8\mu}}} < 0,
\] (49)

\[
\frac{d\theta_2(\mu)}{d\mu} = \frac{-1 + \frac{9\mu - 4}{2\sqrt{9\mu^2 - 8\mu}}}{4 \sqrt{2 - \mu + \sqrt{9\mu^2 - 8\mu}}} > 0.
\] (50)

Thus \(\theta_1(\mu)\) and \(\theta_2(\mu)\) are monotonic with respect to \(\mu\) in this case.

From (48), \(\theta_1\left(\frac{8}{9}\right) = \theta_2\left(\frac{8}{9}\right) = \frac{\sqrt{2}}{16}\) and \(\theta_1(1) = 0, \theta_2(1) = 1\). Letting \(\mu^*\) be the \(\mu\) such that \(\theta_1(\mu^*) = \frac{1}{2}\), then we have

\[
\mu^* = \frac{5 + \sqrt{97}}{16}.
\] (51)

It is obvious that \(\mu^* > \frac{8}{9}\).

Specially, we obtain the following results:

When \(\mu = 0\), we have \(\sigma(\gamma_{0,0}(2\pi)) = \{1, 1, 1, 1\}\).
Thus we obtain
\[ \theta_1 = \begin{cases} 1 & \text{if } 0 < \mu < \frac{8}{9}, \\ \frac{1}{2} & \text{if } \frac{8}{9} < \mu < \frac{5 + \sqrt{37}}{16}, \\ \theta_2 & \text{if } \frac{5 + \sqrt{37}}{16} < \mu < 1. \end{cases} \]
and
\[ \phi = \begin{cases} 1 & \text{if } 0 < \mu < \frac{8}{9}, \\ \frac{1}{2} & \text{if } \frac{8}{9} < \mu < \frac{5 + \sqrt{37}}{16} < \mu < 1, \\ \phi_3 & \text{if } \frac{5 + \sqrt{37}}{16} < \mu < 1. \end{cases} \]
Indeed, we have
\[ \left( A(\mu, 0) \begin{array}{c} f_{0,1} \\ f_{0,2} \end{array} \right) = \left( \begin{array}{cc} 1 + 2\mu & 0 \\ 0 & 1 - \mu \end{array} \right) \left( \begin{array}{c} f_{0,1} \\ f_{0,2} \end{array} \right), \]
\[ \left( A(\mu, 0) \begin{array}{c} f_{n,1} \\ f_{n,2} \end{array} \right) = \left( \begin{array}{cc} n^2 + 1 + 2\mu & 2n \\ 2n & n^2 + 1 + \mu \end{array} \right) \left( \begin{array}{c} f_{n,1} \\ f_{n,2} \end{array} \right), \]
\[ \left( A(\mu, 0) \begin{array}{c} f_{n,3} \\ f_{n,4} \end{array} \right) = \left( \begin{array}{cc} n^2 + 1 + 2\mu & -2n \\ -2n & n^2 + 1 + \mu \end{array} \right) \left( \begin{array}{c} f_{n,3} \\ f_{n,4} \end{array} \right), \]
for \( n \in \mathbb{N} \). Then \( f_{0,1}, f_{0,2} \) and \( f_{n,1}, f_{n,2}, f_{n,3}, f_{n,4} \) form an orthogonal basis of \( D(1, 2\pi) \). By (43) and \( \frac{dt(t)}{dt} = JR(t) \), computing \( A(\mu, 0) f_{n,1} \) yields
\[
A(\mu, 0) f_{n,1} = \left[ -\frac{d^2}{dt^2} + R(t) K_{\mu, 0}(t) R(t)^T \right] R(t) \begin{pmatrix} \cos nt \\ 0 \end{pmatrix}
= R(t) \begin{pmatrix} (n^2 + 1 + 2\mu) \cos nt \\ 2n \sin nt \end{pmatrix}
= (n^2 + 1 + 2\mu) f_{n,1} + 2n f_{n,4}.
\]
for \( n \in \mathbb{N} \). Define
\[
B_0 = \begin{pmatrix} 1 + 2\mu & 0 \\ 0 & 1 - \mu \end{pmatrix},
\]
\[
B_n = \begin{pmatrix} n^2 + 1 + 2\mu & 2n \\ 2n & n^2 + 1 - \mu \end{pmatrix},
\]
\[
\tilde{B}_n = \begin{pmatrix} n^2 + 1 + 2\mu & -2n \\ -2n & n^2 + 1 - \mu \end{pmatrix}.
\]
Denoting the characteristic polynomial of \( B_n \) and \( \tilde{B}_n \) by \( p_n(\lambda) \) and \( \tilde{p}_n(\lambda) \) respectively, then we have
\[
p_n(\lambda) = \tilde{p}_n(\lambda) = \lambda^2 - (2n^2 + 2 + \mu)\lambda - [2\mu^2 - (n^2 + 1)\mu - (n^2 - 1)^2].
\]

If \( \mu = 0 \), then \( B_0 > 0 \) and \( p_n(\lambda) = [\lambda - (n + 1)^2][\lambda - (n - 1)^2] \), and hence both \( B_1 \) and \( \tilde{B}_1 \) have the zero eigenvalue, and all other eigenvalues of \( B_n \) and \( \tilde{B}_n(n \geq 1) \) are positive. Then we have \( \phi_1(A(0,0)) = 0 \), \( \nu_1(A(0,0)) = 2 \), and hence \( i_1(\gamma_{0,0}) = 0 \), \( \nu_1(\gamma_{0,0}) = 2 \) by Lemma 5.6 in the Appendix.

If \( 0 < \mu < 1 \), then \( \tilde{B}_0 > 0 \) and both \( B_1 \) and \( \tilde{B}_1 \) have two positive eigenvalues. Moreover, we have \( B_n > \begin{pmatrix} n^2 & 2n \\ 2n & n^2 \end{pmatrix} \), and hence when \( n \geq 2 \), \( B_n \) has two positive eigenvalues. Similarly, when \( n \geq 2 \), \( \tilde{B}_n \) has two positive eigenvalues. Then we have \( \phi_1(A(\mu,0)) = 0 \), \( \nu_1(A(\mu,0)) = 0 \), and hence \( i_1(\gamma_{\mu,0}) = 0 \), \( \nu_1(\gamma_{\mu,0}) = 0 \) by Lemma 5.6 in the Appendix.

The case \( \mu = 1 \) was given by (39) and (40).

Therefore, we have
\[
i_1(\gamma_{\mu,0}) = 0;
\]
\[
\nu_1(\gamma_{\mu,0}) = \begin{cases} 
2, & \text{if } \mu = 0, \\
0, & \text{if } 0 < \mu < 1, \\
3, & \text{if } \mu = 1.
\end{cases}
\]

(C) Indices \( i_{-1}(\gamma_{\mu,0}) \) and \( \nu_{-1}(\gamma_{\mu,0}) \) for \( \mu \in [0,1] \).

Using the notation in (A), we have
\[
\nu_{-1}(\gamma_{\mu,0}) = \begin{cases} 
2, & \text{if } \mu = \frac{5 + \sqrt{37}}{16} (= \mu^*), \\
0, & \text{if } \mu \neq \frac{5 + \sqrt{37}}{16}.
\end{cases}
\]

Note that \( D(-1,2\pi) \) has the following orthogonal basis
\[
\tilde{f}_{n,1} = R(t) \begin{pmatrix} \cos(n + \frac{1}{2})t \\ 0 \end{pmatrix}, \quad \tilde{f}_{n,2} = R(t) \begin{pmatrix} 0 \\ \cos(n + \frac{1}{2})t \end{pmatrix}, \quad \tilde{f}_{n,3} = R(t) \begin{pmatrix} \sin(n + \frac{1}{2})t \\ 0 \end{pmatrix}, \quad \tilde{f}_{n,4} = R(t) \begin{pmatrix} 0 \\ \sin(n + \frac{1}{2})t \end{pmatrix}, \quad n \in \mathbb{N} \cup \{0\}.
\]

Then we have
\[
\begin{pmatrix}
A(\mu,0) & O \\
O & A(\mu,0)
\end{pmatrix}
\begin{pmatrix}
\tilde{f}_{n,1} \\
\tilde{f}_{n,4}
\end{pmatrix}
= \begin{pmatrix}
(n + \frac{1}{2})^2 + 1 + 2\mu & 2(n + \frac{1}{2}) \\
2(n + \frac{1}{2}) & (n + \frac{1}{2})^2 + 1 - \mu
\end{pmatrix}
\begin{pmatrix}
\tilde{f}_{n,1} \\
\tilde{f}_{n,4}
\end{pmatrix},
\]
\[
\begin{pmatrix}
A(\mu,0) & O \\
O & A(\mu,0)
\end{pmatrix}
\begin{pmatrix}
\tilde{f}_{n,3} \\
\tilde{f}_{n,2}
\end{pmatrix}
for any \( n \in \mathbb{N} \cup \{0\} \).

Hence by a similar argument to (B), we have

\[
i_{-1}(\gamma_{\mu, 0}) = \begin{cases} 
0, & \text{if } 0 \leq \mu \leq \mu^*, \\
2, & \text{if } \mu^* < \mu \leq 1.
\end{cases}
\]  (66)

4. The separation curves of the different linear stability patterns of the elliptic equilibrium point via \((\mu, e)\) parameters.

4.1. The increasing of \(\omega\)-indices of \(\gamma_{\mu, e}\). For \((\mu, e) \in (0, 1) \times [0, 1)\), we can rewrite \(A(\mu, e)\) as follows

\[
A(\mu, e) = -\frac{d^2}{dt^2}I_2 - I_2 + \frac{I_2}{1 + e \cos t} + \frac{\mu}{2(1 + e \cos t)}(I_2 + 3S(t)) \quad \text{and consequently,}
\]

\[
\bar{A}(\mu, e) = -\frac{d^2}{dt^2}I_2 - I_2 + \frac{I_2}{1 + e \cos t} + \frac{\mu}{2(1 + e \cos t)}(I_2 + 3S(t)) + \frac{A(0, e)}{\mu} + \frac{I_2 + 3S(t)}{2(1 + e \cos t)}. \]  (68)

Note that \(A(\mu, e)\) is the same as \(A(\beta, e)\) when \(\beta = \mu - 1\) in [17] with different parameter ranges, then by Lemma 4.2 in [17] and modifying its proof to the different range of parameters, we get the following important lemma:

**Lemma 4.1.** (i) For each fixed \(e \in [0, 1)\), the operator \(\bar{A}(\mu, e)\) is non-increasing with respect to \(\mu \in (0, 1)\) for any fixed \(\omega \in \mathbb{U}\). Specially

\[
\frac{\partial}{\partial \mu} \bar{A}(\mu, e)|_{\mu=\mu_0} = -\frac{1}{\mu_0^2}A(0, e),
\]  (69)

is a non-negative definite operator for \(\mu_0 \in (0, 1)\).

(ii) For every eigenvalue \(\lambda_{\mu_0} = 0\) of \(\bar{A}(\mu_0, e_0)\) with \(\omega \in \mathbb{U}\) for some \((\mu_0, e_0) \in (0, 1) \times [0, 1)\), there holds

\[
\frac{d}{d\mu} \lambda_{\mu}|_{\mu=\mu_0} < 0. \]  (70)

(iii) For every \((\mu, e) \in (0, 1) \times [0, 1)\) and \(\omega \in \mathbb{U}\), there exists \(e_0 = e_0(\mu, e) > 0\) small enough such that for all \(e \in (0, e_0)\) there holds

\[
i_{\omega}(\gamma_{\mu+e, e}) - i_{\omega}(\gamma_{\mu, e}) = \nu_{\omega}(\gamma_{\mu, e}). \]  (71)

Consequently we arrive at

**Corollary 1.** For every fixed \(e \in [0, 1)\) and \(\omega \in \mathbb{U}\), the index function \(\phi_{\omega}(A(\mu, e))\), and consequently \(i_{\omega}(\gamma_{\mu, e})\), is non-decreasing as \(\mu\) increases from 0 to 1. When \(\omega = 1\), these index functions are constantly equal to 0, and when \(\omega \in \mathbb{U} \setminus \{1\}\), they are increasing and tends from 0 to 2.

**Proof.** For \(0 < \mu_1 < \mu_2 \leq 1\) and fixed \(e \in [0, 1)\), when \(\mu\) increases from \(\mu_1\) to \(\mu_2\), it is possible that positive eigenvalues of \(\bar{A}(\mu_1, e)\) pass through 0 and become negative ones of \(\bar{A}(\mu_2, e)\), but it is impossible that negative eigenvalues of \(\bar{A}(\mu_2, e)\) pass through 0 and become positive by (ii) of Lemma 4.1. \(\square\)
4.2. The $\omega$-degenerate curves of $\gamma_{\mu,e}$. By a similar analysis to the proof of Proposition 6.1 in [4], for every $e \in (0, 1)$ and $\omega \in \mathbb{U}\{1\}$, the total multiplicity of $\omega$-degeneracy of $\gamma_{\mu,e}(2\pi)$ for $\mu \in [0, 1]$ is always precisely 2, i.e.,
\[ \sum_{\mu \in [0, 1]} \nu_\omega(\gamma_{\mu,e}(2\pi)) = 2, \quad \forall \omega \in \mathbb{U}\{1\}. \]  

Consequently, together with the definiteness of $A(0,e)$ for the $\omega \in \mathbb{U}\{1\}$ boundary condition, we have

**Theorem 4.2.** For any $\omega \in \mathbb{U}\{1\}$, there exist two analytic $\omega$-degenerate curves $(\mu_i(e,\omega), e)$ in $e \in (0, 1)$ with $i = 1, 2$. Specially, each $\mu_i(e,\omega)$ is a real analytic function in $e \in [0, 1]$, and $0 < \mu_i(e,\omega) < 1$ and $\gamma_{\mu_i(e,\omega),e}(2\pi)$ is $\omega$-degenerate for $\omega \in \mathbb{U}\{1\}$ and $i = 1, 2$.

**Proof.** We prove first that $i_\omega(\gamma_{\mu,e}) = 0$ when $\mu$ is near 0. By Lemma 4.1(ii) in [17], $A(0,e)$ is positive definite on $\mathcal{D}(\omega,2\pi)$. Therefore, there exists an $\epsilon > 0$ small enough such that $A(\mu, e)$ is also positive definite on $\mathcal{D}(\omega,2\pi)$ when $0 < \mu < \epsilon$. Hence $\nu_\omega(\gamma_{\mu,e}) = \nu_\omega(A(\mu,e)) = 0$ when $0 < \mu < \epsilon$. Thus we have proved our claim.

Then under similar steps to those of Lemma 6.2 and Theorem 6.3 in [4], we can prove the theorem. \hfill \square

4.3. The $\omega = -1$ degenerate curves of $\gamma_{\mu,e}$. Specially, for $\omega = -1$, $e \in [0, 1)$, we define
\[ \mu_m(e) = \min\{\mu_1(e, -1), \mu_2(e, -1)\}, \quad \mu_r(e) = \max\{\mu_1(e, -1), \mu_2(e, -1)\}, \]  

where $\mu_i(e, -1)$ are the two $-1$-degenerate curves as in Theorem 4.2.

By [62], $-1$ is a double eigenvalue of the matrix $\gamma_{\mu^*,e}(2\pi)$, then the two curves bifurcate out from $(\mu^*,0)$ when $e > 0$ is small enough.

Recall that $A(\mu^*, 0)$ is $-1$-degenerate. Then, by [62], we find that $\dim \ker A(\mu^*, 0) = \nu_{-1}(\gamma_{\mu^*,0}) = 2$. By the definition of (132), we have $R(t) \left( \tilde{a}_n \sin(n + \frac{1}{2})t \cos(n + \frac{1}{2})t \right) \in \mathcal{D}(-1,2\pi)$ for some constant $\tilde{a}_n$.

Moreover, $A(\mu,0)R(t) \left( \frac{\tilde{a}_n \sin(n + \frac{1}{2})t}{\cos(n + \frac{1}{2})t} \right) = 0$ reads
\[ \left\{ \begin{array}{l} (n + \frac{1}{2})^2 \tilde{a}_n - 2(n + \frac{1}{2}) + (1 + 2\mu)\tilde{a}_n = 0, \\ (n + \frac{1}{2})^2 - 2(n + \frac{1}{2})\tilde{a}_n + 1 - \mu = 0. \end{array} \right. \]  

Then $2\mu^2 - ((n + \frac{1}{2})^2 + 1)\mu - [(n + \frac{1}{2})^2 - 1]^2 = 0$ holds only when $n = 0$ and $\mu = \frac{5 + \sqrt{97}}{16} = \mu^*$, and hence
\[ \tilde{a}_0 = \frac{1}{4} + 1 - \mu^* = \frac{15 - \sqrt{97}}{16}. \]  

Then we have $R(t) \left( \frac{\tilde{a}_0 \sin \frac{1}{2}}{\cos \frac{1}{2}} \right) \in \ker A(\mu^*, 0)$. Similarly, we have $R(t) \left( \frac{\tilde{a}_0 \cos \frac{1}{2}}{\sin \frac{1}{2}} \right) \in \ker A(\mu^*, 0)$, and hence
\[ \ker A(\mu^*, 0) = \text{span} \left\{ R(t) \left( \frac{\tilde{a}_0 \sin \frac{1}{2}}{\cos \frac{1}{2}} \right), \quad R(t) \left( \frac{\tilde{a}_0 \cos \frac{1}{2}}{\sin \frac{1}{2}} \right) \right\}. \]  

Indeed, we have the following theorem:
Theorem 4.3. The tangent directions of the two curves $\Gamma_m$ and $\Gamma_r$ at the same bifurcation point $(\mu^*, 0)$ are given by
\[
\mu'_m(e)|_{e=0} = -\frac{291 + 15\sqrt{97}}{3104}, \quad \mu'_r(e)|_{e=0} = \frac{291 + 15\sqrt{97}}{3104}.
\] (77)

Proof. Now let $(\mu(e), e)$ be one of such curves (i.e., one of $(\mu_i(-1, e), e), i = 1, 2,$) which starts from $\mu^*$ with $e \in [0, \epsilon)$ for some small $\epsilon > 0$ and $x_e \in D(-1, 2\pi)$ being the corresponding eigenvector, that is
\[
A(\mu(e), e)x_e = 0.
\] (78)

Without loss of generality, by (76), we suppose
\[
z = (\tilde{a}_0 \sin \frac{t}{2}, \cos \frac{t}{2})^T
\]
and
\[
x_0 = R(t)z = R(t)(\tilde{a}_0 \sin \frac{t}{2}, \cos \frac{t}{2})^T.
\] (79)

There holds
\[
\langle A(\mu(e), e)x_e, x_e \rangle = 0.
\] (80)

Differentiating both side of (80) with respect to $e$ yields
\[
\mu'(e)\langle \frac{\partial}{\partial \mu} A(\mu(e), e)x_e, x_e \rangle + \langle \frac{\partial}{\partial e} A(\mu(e), e)x_e, x'_e \rangle = 0,
\]
where $\mu'(e)$ and $x'_e$ denote the derivatives with respect to $e$. Then evaluating both sides at $e = 0$ yields
\[
\mu'(0)\langle \frac{\partial}{\partial \mu} A(\mu^*, 0)x_0, x_0 \rangle + \langle \frac{\partial}{\partial e} A(\mu^*, 0)x_0, x_0 \rangle = 0.
\] (81)

Then by the definition (33) of $A(\mu, e)$ we have
\[
\frac{\partial}{\partial \mu} A(\mu, e) \bigg|_{(\mu, e) = (\mu^*, 0)} = R(t) \frac{\partial}{\partial \mu} K_{\mu, e}(t) \bigg|_{(\mu, e) = (\mu^*, 0)} R(t)^T,
\] (82)
\[
\frac{\partial}{\partial e} A(\mu, e) \bigg|_{(\mu, e) = (\mu^*, 0)} = R(t) \frac{\partial}{\partial e} K_{\mu, e}(t) \bigg|_{(\mu, e) = (\mu^*, 0)} R(t)^T.
\] (83)

By direct computations from the definition of $K_{\mu, e}(t)$ in (25), we obtain
\[
\frac{\partial}{\partial \mu} K_{\mu, e}(t) \bigg|_{(\mu, e) = (\mu^*, 0)} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},
\] (84)
\[
\frac{\partial}{\partial e} K_{\mu, e}(t) \bigg|_{(\mu, e) = (\mu^*, 0)} = -\cos t \begin{pmatrix} 1 + 2\mu^* & 0 \\ 0 & 1 - \mu^* \end{pmatrix}.
\] (85)

Therefore from (79) and (82)-(85) we have
\[
\langle \frac{\partial}{\partial \mu} A(\mu^*, 0)x_0, x_0 \rangle = \langle \frac{\partial}{\partial \mu} K_{\mu^*, 0}z, z \rangle = \int_0^{2\pi} [2\tilde{a}_0^2 \sin^2 \frac{t}{2} - \cos^2 \frac{t}{2}] dt
\]
\[
= \pi(2\tilde{a}_0^2 - 1)
\]
\[
= \pi \frac{97 - 15\sqrt{97}}{64}
\] (86)
and
\[
\langle \frac{\partial}{\partial e} A(\mu, 0)x_0, x_0 \rangle = \langle \frac{\partial}{\partial e} K_{\mu, 0} z, z \rangle = -\int_0^{2\pi} [(1 + 2\mu^*)\tilde{a}_0^2 \cos t \sin^2 \frac{t}{2} + (1 - \mu^*) \cos t \cos^2 \frac{t}{2}] dt = \pi \frac{(1 + 2\mu^*)\tilde{a}_0^2 - (1 - \mu^*)}{2} = \pi \frac{-33 + 15\sqrt{97}}{1024}.
\]

Therefore by (81) and (86)-(87), we obtain
\[
\mu'(0) = \frac{291 + 15\sqrt{97}}{3104}.
\] (88)

The other tangent can be computed similarly. Thus the theorem is proved. \(\square\)

Now we can give

*The proofs of Theorem 1.3 parts (i)-(iii).*

(i) By the computations in Section 3.2, the only \(-1\)-degenerate point in the segment \([0, 1] \times \{0\}\) is \((\mu, e) = (\mu^*, 0)\), which is a two-fold \(-1\)-degenerate point. That \(\mu_i(e, -1) \rightarrow 1\) as \(e \rightarrow 1\) for \(i = 1, 2\) follows by the similar arguments in the Section 5 of [4].

(ii) By Theorem 4.2, for \(\omega = -1\), \(\mu_i(e, -1)\) is real analytic on \(e \in [0, 1)\) for \(i = 1, 2\). Because these two curves bifurcate out from \((\mu^*, 0)\) in different angles with tangents \(\pm \frac{291 + 15\sqrt{97}}{3104}\) respectively when \(e > 0\) is small by Theorem 4.3, they are different from each other at least near \((\mu^*, 0)\). Because of analyticity, the intersection points of these two curves can only be isolated.

(iii) It follows from the computations in Section 3.2, Lemma 4.1 and Theorem 4.2. \(\square\)

4.4. The region division and the symplectic normal forms of \(\gamma_{\mu, e}(2\pi)\). For every \(e \in [0, 1)\), we recall
\[
\mu_I(e) = \inf\{\mu' \in [0, 1]|\sigma(\gamma_{\mu, e}(2\pi)) \cap U \neq \emptyset, \forall \mu \in [0, \mu']\},
\]
and
\[
\Gamma_I = \{(\mu_I(e), e) \in [0, 1] \times [0, 1]\}.
\]

By similar arguments of Lemma 9.1 and Corollary 9.2 in [4], we have

*Lemma 4.4.* (i) If \(0 \leq \mu_1 < \mu_2 \leq 1\) and \(\gamma_{\mu_2, e}(2\pi)\) is hyperbolic, so does \(\gamma_{\mu_1, e}(2\pi)\). Consequently, the hyperbolic region of \(\gamma_{\mu, e}(2\pi)\) in \([0, 1] \times [0, 1]\) is connected.

(ii) For any fixed \(e \in [0, 1]\), every matrix \(\gamma_{\mu, e}(2\pi)\) is hyperbolic if \(0 \leq \mu < \mu_1(e)\) for \(\mu_1(e)\) defined by (21).

(iii) We have
\[
\sum_{\mu \in [\mu_1(e), 1]} \nu_{\omega}(\gamma_{\mu, e}(2\pi)) = 2, \quad \forall \omega \in U \backslash \{1\}.
\] (89)

(iv) For every \(e \in [0, 1]\), we have
\[
\sum_{\mu \in [0, \mu_m(e)]} \nu_{-1}(\gamma_{\mu, e}(2\pi)) = 0, \quad \sum_{\mu \in [\mu_m(e), 1]} \nu_{-1}(\gamma_{\mu, e}(2\pi)) = 2.
\] (90)
Now we can give

\textit{The Proofs of Theorem 1.3 parts (iv)-(x).} Some arguments below are use the methods in the proof of Theorem 1.2 in \cite{4}. \\
\textbf{(v)} In fact, if the function $\mu_t(e)$ is not continuous in $e \in [0,1]$, then there exist some $\hat{e} \in [0,1]$, a sequence $\{e_i|i \in \mathbb{N}\}\backslash \{\hat{e}\}$ and $\mu_0 \in [0,1]$ such that $\mu_t(e_i) \to \mu_0 \neq \mu_t(\hat{e})$ and $e_i \to \hat{e}$ as $i \to +\infty$. 

We continue in two cases according to the sign of the difference $\mu_0 - \mu_t(\hat{e})$. 

On the one hand, by the definition of $\mu_t(e_i)$ we have $\sigma(\gamma_{\mu_t(e_i),e_i}(2\pi)) \cap U \neq \emptyset$ for every $e_i$. By the continuity of eigenvalues of $\gamma_{\mu_t(e_i),e_i}(2\pi)$ in $i$ and \cite{11}, we obtain $\sigma(\gamma_{\mu_0,\hat{e}}(2\pi)) \cap U \neq \emptyset$.

Thus by Lemma \ref{4.4}, this would yield a contradiction if $\mu_0 < \mu_t(\hat{e})$.

On the other hand, we suppose $\mu_0 > \mu_t(\hat{e})$. By Lemma \ref{4.4} for all $i \geq 1$, we have $\sigma(\gamma_{\mu_t(e_i),e_i}(2\pi)) \cap U = \emptyset, \forall \mu \in (0,\mu_t(e_i))$. 

Then by the continuity of $\mu_m(e)$ in $e$, \cite{91} and \cite{92}, we obtain

$$\mu_t(\hat{e}) < \mu_0 \leq \mu_m(\hat{e}).$$

Let $\omega_0 \in \sigma(\gamma_{\mu_t(\hat{e}),\hat{e}}(2\pi)) \cap U$, which exists by the definition of $\mu_t(\hat{e})$.

Moreover, let $L = \{(\mu,\hat{e})|\mu \in (0,\mu_t(\hat{e}))\}$ and $L_i = \{(\mu,e_i)|\mu \in (0,\mu_t(e_i))\}$ for all $i \geq 1$. By \cite{36}, Lemma \ref{4.3}iii) and Lemma \ref{4.4} we obtain

$$i_{\omega_0}(\gamma_{\mu,e}) = \nu_{\omega_0}(\mu,e) = \sum_{i \geq 1} L_i.$$

Specially, we have

$$i_{\omega_0}(\gamma_{\mu_t(\hat{e}),\hat{e}}) = 0, \quad \nu_{\omega_0}(\gamma_{\mu_t(\hat{e}),\hat{e}}) \geq 1.$$ 

Therefore by Lemma \ref{4.1}iii) and the definition of $\omega_0$, there exists $\hat{\mu} \in (\mu_t(\hat{e}),\mu_0)$ sufficiently close to $\mu_t(\hat{e})$ such that

$$i_{\omega_0}(\gamma_{\hat{\mu},\hat{e}}) = i_{\omega_0}(\gamma_{\mu_t(\hat{e}),\hat{e}}) + \nu_{\omega_0}(\gamma_{\mu_t(\hat{e}),\hat{e}}) \geq 1.$$ 

This estimate \cite{96} in facts holds for all $\mu \in (\mu_t(\hat{e}),\hat{\mu}]$ too. Note that $(\hat{\mu},\hat{e})$ is an accumulation point of $\cup_{i \geq 1} L_i$. Consequently for each $i \geq 1$, there exists $(\mu,e_i) \in L_i$ such that $\gamma_{\mu,e_i} \in \mathcal{P}_{2\pi}(4)$ is $\omega_0$ non-degenerate, $\mu_i \to \hat{\mu}$ in $\mathbb{R}$, and $\gamma_{\mu,e_i} \to \gamma_{\hat{\mu},\hat{e}}$ in $\mathcal{P}_{2\pi}(4)$ as $i \to \infty$. Therefore by \cite{94}, \cite{96}, the Definition 5.4.2 of the $\omega_0$-

index of $\omega_0$-degenerate path $\gamma_{\hat{\mu},\hat{e}}$ on p.129 and Theorem 6.1.8 on p.142 of \cite{11}, we obtain the following contradiction

$$1 \leq i_{\omega_0}(\gamma_{\hat{\mu},\hat{e}}) \leq i_{\omega_0}(\gamma_{\mu,e}) = 0,$$

for $i \geq 1$ large enough. Thus the continuity of $\mu_t(e)$ in $e \in [0,1)$ is proved.

Now we prove the claim $\lim_{e \to 1} \mu_t(e) = 1$. We argue by contradiction, and suppose there exist $e_i \to 1$ as $i \to +\infty$ such that $\lim_{e \to 1} \mu_t(e) = \mu_0$ for some $0 \leq \mu_0 < 1$. Then at least one of the following two cases must occur: (A) There exists a subsequence $\hat{e}_i$ of $e_i$ such that $\mu_t(\hat{e}_{i+1}) \leq \mu_t(\hat{e}_i)$ for all $i \in \mathbb{N}$; (B) There exists a subsequence $\hat{e}_i$ of $e_i$ such that $\mu_t(\hat{e}_{i+1}) \geq \mu_t(\hat{e}_i)$ for all $i \in \mathbb{N}$.

If Case (A) happens, for this $\mu_0$, by a similar argument of Theorem 1.7 in \cite{11}, there exists $\varepsilon_0 > 0$ sufficiently close to 1 such that $\gamma_{\mu,e}(2\pi)$ is hyperbolic for all $(\mu,e)$ in the region $(0,\mu_t(\hat{e}_i)] \times [\varepsilon_0,1)$. Then by the monotonicity of Case (A) we obtain

$$\mu_0 \leq \mu_t(\varepsilon_{i+m}) \leq \mu_t(\varepsilon_i), \quad \forall m \in \mathbb{N}.$$
Therefore \( \mu(\epsilon_{i+m}, \epsilon_{i+m}) \) will get into this region for sufficiently large \( m \geq 1 \), which contradicts the definition of \( \mu(\epsilon_{i+m}) \).

If Case (B) happens, the proof is similar. Thus (v) holds.

(vi) By our study in Section 3.3, we have \( (\frac{\epsilon}{2}, 0) \in \Gamma_i \setminus \Gamma_m \). Thus there exists an \( \epsilon \in (0, 1] \) such that \( \mu(e) < \mu_m(e) \) for all \( e \in [0, \epsilon) \). Therefore, \( \Gamma_i \) and \( \Gamma_m \) are different curves.

(vii) It follows from (21).

(viii) If \( \mu(e) < \mu < \mu_m(e) \), then by the definitions of the degenerate curves and Lemma 4.1 (iii), we have

\[
i_1(\gamma_{\mu,e}) = 0, \quad \mu_1(\gamma_{\mu,e}) = 0, \quad (99)
\]

and

\[
i_{-1}(\gamma_{\mu,e}) = 0, \quad \mu_{-1}(\gamma_{\mu,e}) = 0. \quad (100)
\]

Assume \( \gamma_{\mu,e}(2\pi) = N_2(e^{\sqrt{-1}\theta}, b) \) for some \( \theta \in (0, \pi) \cup (\pi, 2\pi) \). Without loss of generality, we suppose \( \theta \in (0, \pi) \). Let \( \omega_0 = e^{\sqrt{-1}\theta} \), we have \( \omega_0(\gamma_{\mu,e}(2\pi)) \geq 1 \). Then for any \( \omega \in U, \omega \neq \omega_0 \), we have

\[
i_\omega(\gamma_{\mu,e}) = i_1(\gamma_{\mu,e}) = 0 \quad (101)
\]

or

\[
i_\omega(\gamma_{\mu,e}) = i_1(\gamma_{\mu,e}) - S^-_{N_2(e^{\sqrt{-1}\theta}, b)}(e^{\sqrt{-1}\theta}) + S^+_{N_2(e^{\sqrt{-1}\theta}, b)}(e^{\sqrt{-1}\theta}) = 0. \quad (102)
\]

Then by the sub-continuous of \( i_\omega(\gamma_{\mu,e}) \) with respect to \( \omega \), we have \( i_\omega(\gamma_{\mu,e}) = 0 \), \( \forall \omega \in U \). Moreover, by Corollary 1, we have

\[
i_\omega(\gamma_{\mu,e}) = 0, \quad \forall \omega \in U, \mu \in (0, \mu). \quad (103)
\]

Therefore, by the definition of \( \mu_1(e) \) of (21), we have \( \mu_1(e) \geq \mu \). It contradicts \( \mu_1(e) < \mu < \mu_m(e) \).

Then we can suppose \( \gamma_{\mu,e}(2\pi) = M_1 \circ M_2 \) where \( M_1 \) and \( M_2 \) are two basic normal forms in \( \text{Sp}(2) \). By Lemma 3.1 in [17], there exist two paths \( \gamma_1, \gamma_2 \in P_{2\pi}(2) \) such that \( \gamma_1(2\pi) = M_1, \gamma_2(2\pi) = M_2 \) and \( \gamma_{\mu,e} \sim \gamma_1 \circ \gamma_2 \). Then

\[
i_1(\gamma_{\mu,e}) = i_1(\gamma_1) + i_1(\gamma_2). \quad (104)
\]

By the definition of \( \mu_1, M_1 \) and \( M_2 \) cannot be both hyperbolic, and without loss of generality, we suppose \( M_1 = R(\theta_1) \). Then \( i_1(\gamma_1) \) is odd, and hence \( i_1(\gamma_2) \) is also odd. By Theorem 4 to Theorem 7 of Chapter 8 on pp.179-183 in [17] and using notation there, we must have \( M_2 = D(-2) \) or \( M_2 = R(\theta_2) \) for some \( \theta_2 \in (0, \pi) \cup (\pi, 2\pi) \).

If \( M_2 = D(-2) \), then we have \( i_{-1}(\gamma_1) - i_1(\gamma_1) = \pm 1 \) and \( i_{-1}(\gamma_2) - i_1(\gamma_2) = 0 \). Therefore \( i_{-1}(\gamma_{\mu,e}(2\pi)) = i_{-1}(\gamma_1) + i_{-1}(\gamma_2) \) and \( i_1(\gamma_{\mu,e}(2\pi)) = i_1(\gamma_1) + i_1(\gamma_2) \) has the different oddity, which contradicts (99) and (100). Then we have \( M_2 = R(\theta_2) \).

Moreover, if \( \theta_1 \in (\pi, 2\pi) \), we must have \( \theta_2 \in (0, \pi) \), otherwise \( i_{-1}(\gamma_1) - i_1(\gamma_1) = 1 \) and \( i_{-1}(\gamma_2) - i_1(\gamma_2) = 1 \) and hence

\[
i_{-1}(\gamma_{\mu,e}) = i_{-1}(\gamma_1) + i_{-1}(\gamma_2) = i_1(\gamma_1) + i_1(\gamma_2) = 2, \quad (105)
\]

which contradicts (100). Similarly, if \( \theta_1 \in (0, \pi) \), we must have \( \theta_2 \in (\pi, 2\pi) \).

(ix) If \( \mu_m(e) < \mu < \mu_{\mu}(e) \), then by the definitions of the degenerate curves and Lemma 4.1 (iii), we have

\[
i_1(\gamma_{\mu,e}) = 0, \quad \mu_1(\gamma_{\mu,e}) = 0, \quad (106)
\]

and

\[
i_{-1}(\gamma_{\mu,e}) = 1, \quad \mu_{-1}(\gamma_{\mu,e}) = 0. \quad (107)
\]
which contradicts (106) and (107).

Firstly, if $\gamma_{\mu,e}(2\pi) \approx N_2(e^{-\sqrt{-1}\theta},b)$ for some $\theta \in (0, \pi) \cup (\pi, 2\pi)$, we have

$$i_0(\gamma_{\mu,e}) = i_1(\gamma_{\mu,e}) = -S_{N_2(e^{-\sqrt{-1}\theta},b)}(e^{\sqrt{-1}\theta}) + S_{N_2(e^{-\sqrt{-1}\theta},b)}(e^{\sqrt{-1}\theta}) = i_1(\gamma_{\mu,e})$$

or

$$i_0(\gamma_{\mu,e}) = i_1(\gamma_{\mu,e}) - S_{N_2(e^{-\sqrt{-1}\theta},b)}(e^{\sqrt{-1}(2\pi-\theta)}) + S_{N_2(e^{-\sqrt{-1}\theta},b)}(e^{\sqrt{-1}(2\pi-\theta)}) = i_1(\gamma_{\mu,e}),$$

which contradicts (106) and (107).

By Lemma 3.1 in [17], there exist two paths $\gamma_1, \gamma_2 \in \mathcal{P}_2(2)$ such that $\gamma_1(2\pi) = M_1, \gamma_2(2\pi) = M_2$ and $\gamma_{\mu,e} \sim \gamma_1 \circ \gamma_2$. Then

$$0 = i_1(\gamma_{\mu,e}) = i_1(\gamma_1) + i_1(\gamma_2).$$

Remark 3. Remark 2 can be obtained by similar arguments to Theorem 1.3 (viii)-(x).

5. Appendix: $\omega$-Maslov-type indices and $\omega$-Morse indices.

5.1. A brief review on index theory for symplectic paths. Let $(\mathbb{R}^{2n}, \Omega)$ be the standard symplectic vector space with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the symplectic form $\Omega = \sum_{i=1}^n dx_i \wedge dy_i$. Let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ be the standard symplectic matrix, where $I_n$ is the identity matrix on $\mathbb{R}^n$.

As usual, the symplectic group $\text{Sp}(2n)$ is defined by

$$\text{Sp}(2n) = \{ M \in \text{GL}(2n, \mathbb{R}) \mid M^TJM = J \},$$

whose topology is induced from that of $\mathbb{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $\text{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \},$$

which is equipped with the topology induced from that of $\text{Sp}(2n)$. For any $\omega \in \mathbb{U}$ and $M \in \text{Sp}(2n)$, the following real function was introduced in [9]:

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}).$$

Thus for any $\omega \in \mathbb{U}$ the following codimension 1 hypersurface in $\text{Sp}(2n)$ is defined [9]:

$$\text{Sp}(2n)_\omega^0 = \{ M \in \text{Sp}(2n) \mid D_\omega(M) = 0 \}.$$
For any $M \in \text{Sp}(2n)^0$, we define a co-orientation of $\text{Sp}(2n)^0$ at $M$ by the positive direction $\frac{d}{dt} M e^{\lambda t}|_{t=0}$ of the path $M e^{\lambda t}$ with $0 \leq t \leq \varepsilon$ and $\varepsilon$ being a small enough positive number. Let

$$\text{Sp}(2n)^0 = \text{Sp}(2n) \setminus \text{Sp}(2n)^0,$$

$$\mathcal{P}^*_\tau,\omega(2n) = \{ \gamma \in \mathcal{P}_\tau(2n) | \gamma(\tau) \in \text{Sp}(2n)^0 \},$$

$$\mathcal{P}^0_{\tau,\omega}(2n) = \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_\tau,\omega(2n).$$

As in [11], for $\lambda \in \mathbb{R} \setminus \{0\}$, $\tau \in (0, \pi) \cup (\pi, 2\pi)$, $b = (b_1 b_2 b_3 b_4)$ with $b_i \in \mathbb{R}$ for $i = 1, \ldots, 4$, and $c_j \in \mathbb{R}$ for $j = 1, 2$, we denote respectively some normal forms by

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

$$N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad N_2(e^{\sqrt{-1}\theta}, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix},$$

$$M_2(\lambda, c) = \begin{pmatrix} \lambda & 1 & c_1 & 0 \\ 0 & \lambda & c_2 \cos \theta & 0 \\ 0 & 0 & \lambda^{-1} \left(-\lambda \right)_{\sqrt{-1}} & 0 \\ 0 & 0 & -\lambda^{-2} \lambda^{-1} & \lambda^{-1} \end{pmatrix}.$$

Here $N_2(e^{\sqrt{-1}\theta}, b)$ is trivial if $(b_2 - b_3) \sin \theta > 0$, or non-trivial if $(b_2 - b_3) \sin \theta < 0$, in the sense of Definition 1.8.11 on p.41 of [11]. Note that by Theorem 1.5.1 on pp.24-25 and (1.4.7)-(1.4.8) on p.18 of [11], when $\lambda = -1$ there hold

$$c_2 \neq 0 \quad \text{if and only if} \quad \dim \ker(M_2(-1, c) + I) = 1,$$

$$c_2 = 0 \quad \text{if and only if} \quad \dim \ker(M_2(-1, c) + I) = 2.$$  

Note that we have $N_1(\lambda, a) \approx N_1(\lambda, a/|a|)$ for $a \in \mathbb{R} \setminus \{0\}$ by symplectic coordinate change, because

$$\begin{pmatrix} 1/\sqrt{|a|} & 0 \\ 0 & \sqrt{|a|} \end{pmatrix} \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \sqrt{|a|} & 0 \\ 0 & 1/\sqrt{|a|} \end{pmatrix} = \begin{pmatrix} \lambda & a/|a| \\ 0 & \lambda \end{pmatrix}.$$

Following Definition 1.8.9 on p.41 of [11], we call the above matrices $D(\lambda)$, $R(\theta)$, $N_1(\lambda, a)$ and $N_2(\omega, b)$ basic normal forms of symplectic matrices. As proved in [9] and [11] (cf. Theorem 1.9.3 on p.46 of [11]), every $M \in \text{Sp}(2n)$ has its basic normal form decomposition in $\Omega^0(M)$ as a $\alpha$-sum of these basic normal forms.

For any two continuous paths $\xi$ and $\eta : [0, \tau] \to \text{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, we define their concatenation by

$$\eta \ast \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau. \end{cases}$$

Moreover, we define a special continuous symplectic path $\xi_n \subset \text{Sp}(2n)$ by

$$\xi_n(t) = \left( 2 - \frac{t}{\tau} \right)^n \begin{pmatrix} 0 & 0 \\ 2 - \frac{t}{\tau} & 1 \end{pmatrix} \quad \text{for } 0 \leq t \leq \tau. \quad (112)$$

**Definition 5.1.** ([9], [11]) For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).$$

(113)

If $\gamma \in \mathcal{P}^*_\tau,\omega(2n)$, define

$$i_\omega(\gamma) = [\text{Sp}(2n)^0 : \gamma \ast \xi_n].$$

(114)
where the right hand side of (114) is the usual homotopy intersection number, and the orientation of \( \gamma \ast \xi_n \) is its positive time direction under homotopy with fixed end points.

If \( \gamma \in \mathcal{P}_{r,\omega}^n(2n) \), we let \( \mathcal{F}(\gamma) \) be the set of all open neighborhoods of \( \gamma \) in \( \mathcal{P}_r(2n) \), and define

\[
i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{ i_\omega(\beta) \mid \beta \in U \cap \mathcal{P}_{r,\omega}^n(2n) \}.
\]

Then

\[
(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},
\]

is called the index function of \( \gamma \) at \( \omega \).

The \( \omega \)-index function have the following properties:

Lemma 5.2. (Y. Long, [11], pp 147-148) For \( \omega \in \mathcal{U} \) and \( \tau > 0 \), the \( \omega \)-index part of the index function defined on paths in \( \mathcal{P}_r(2n) \) starting from the identity is uniquely determined by the following five axioms:

1. **(Homotopy invariant)** For \( \gamma_0 \) and \( \gamma_1 \in \mathcal{P}_r(2n) \), if \( \gamma_0 \sim_\omega \gamma_1 \) on \([0, \tau]\), then

\[
i_\omega(\gamma_0) = i_\omega(\gamma_1).
\]

2. **(Symplectic additivity)** For any \( \gamma_j \in \mathcal{P}_r(2n_j) \) with \( j = 0, 1 \),

\[
i_\omega(\gamma_0 \circ \gamma_1) = i_\omega(\gamma_0) + i_\omega(\gamma_1).
\]

3. **(Clockwise continuity)** For any \( \gamma \in \mathcal{P}_r(2) \) satisfying \( \gamma(\tau) = N_1(\omega, b) \) with \( b = \pm 1 \) or \( 0 \) if \( \omega = \pm 1 \), and \( b = 0 \) if \( \omega \in \mathcal{U} \setminus \mathbb{R} \), there exists a \( \theta_0 > 0 \) such that

\[
i_\omega([\gamma(\tau) \phi_{\tau,-\theta}] \ast \gamma) = i_\omega(\gamma), \quad \forall 0 < \theta \leq \theta_0,
\]

where \( \phi_{\tau,\theta} \) is defined by

\[
\phi_{\tau,\theta} = R(\theta \left\lfloor \frac{t}{\tau} \right\rfloor), \quad \forall t \in [0, \tau], \theta \in \mathbb{R}.
\]

4. **(Counterclockwise jumping)** For any \( \gamma \in \mathcal{P}_r(2) \) satisfying \( \gamma(\tau) = N_1(\omega, b) \) with \( b = \pm 1 \) or \( 0 \) if \( \omega = \pm 1 \), and \( b = 0 \) if \( \omega \in \mathcal{U} \setminus \mathbb{R} \), there exists a \( \theta_0 > 0 \) such that

\[
i_\omega([\gamma(\tau) \phi_{\tau,\theta}] \ast \gamma) = i_\omega(\gamma) + 1, \quad \forall 0 < \theta \leq \theta_0.
\]

5. **(Normality)** For \( \hat{\alpha}_0(t) = D(1 + t/\tau) \),

\[
i_\omega(\hat{\alpha}_0) = 0.
\]

Splitting numbers are very useful in the computation on the \( \omega \)-indices. Now we give their definition:

Definition 5.3. ([9], [11]) For any \( M \in \text{Sp}(2n) \) and \( \omega \in \mathcal{U} \), choosing \( \tau > 0 \) and \( \gamma \in \mathcal{P}_r(2n) \) with \( \gamma(\tau) = M \), we define

\[
S_N^\pm_M(\omega) = \lim_{\varepsilon \to 0^+} i_{\exp(\varepsilon \sqrt{-\omega})}(\gamma) - i_\omega(\gamma).
\]

They are called the splitting numbers of \( M \) at \( \omega \).

Splitting numbers have the following properties:

Lemma 5.4. (Y. Long, [11], pp. 191) Splitting numbers \( S_N^\pm_M(\omega) \) are well defined, i.e., they are independent of the choice of the path \( \gamma \in \mathcal{P}_r(2n) \) satisfying \( \gamma(T) = M \). For \( \omega \in \mathcal{U} \) and \( M \in \text{Sp}(2n) \), splitting numbers \( S_N^\pm_M(\omega) \) are constant for all \( N = P^{-1}MP \), with \( P \in \text{Sp}(2n) \).
Lemma 5.5. (Y. Long, [11], pp. 198–199) For $M \in \text{Sp}(2n)$ and $\omega \in U$, $\theta \in (0, \pi)$, there hold

\begin{align*}
S_M^+(\omega) &= 0, \quad \text{if } \omega \not\in \sigma(M), \quad (117) \\
S_M^- (\omega) &= S_M^-(\omega), \quad (118) \\
0 \leq S_M^+(\omega) \leq \dim \ker(M - \omega I), \quad (119) \\
S_M^+(\omega) + S_M^-(\omega) \leq \dim \ker(M - \omega I)^{2n}, \quad \omega \in \sigma(M), \quad (120)
\end{align*}

\begin{align*}
(S_{N_1(1,b)}^+(1), S_{N_1(1,b)}^-(1)) &= \begin{cases} (1, 1), & \text{if } b = 0, 1, \\
(0, 0), & \text{if } b = -1, \end{cases} \quad (121)
\end{align*}

\begin{align*}
(S_{N_1(-1,a)}^+(1), S_{N_1(-1,a)}^-(1)) &= \begin{cases} (1, 1), & \text{if } a = -1, 0, \\
(0, 0), & \text{if } a = 1, \end{cases} \quad (122)
\end{align*}

\begin{align*}
(S_{R(\theta)}^+(e^{\sqrt{-1} \theta}), S_{R(\theta)}^-(e^{\sqrt{-1} \theta})) &= (0, 1), \quad (123)
\end{align*}

\begin{align*}
(S_{R(2\pi - \theta)}^+(e^{\sqrt{-1} \theta}), S_{R(2\pi - \theta)}^-(e^{\sqrt{-1} \theta})) &= (1, 0), \quad (124)
\end{align*}

\begin{align*}
(S_{N_2(\omega,b)}^+(\omega), S_{N_2(\omega,b)}^-(\omega)) &= (1, 1) \text{ for } N_2(\omega,b) \text{ being non-trivial}, \quad (125)
\end{align*}

\begin{align*}
(S_{N_2(\omega,b)}^+(\omega), S_{N_2(\omega,b)}^-(\omega)) &= (0, 0) \text{ for } N_2(\omega,b) \text{ being trivial}. \quad (126)
\end{align*}

Moreover, for any $M_i \in \text{Sp}(2n_i)$ with $i = 1$ and 2, there holds

\begin{align*}
S_{M_1 \circ M_2}^+(\omega) = S_{M_1}^+(\omega) + S_{M_2}^+(\omega). \quad (127)
\end{align*}

From the definition and property of splitting numbers, for any $\gamma \in P_\tau(2n)$ with $\gamma(T) = M$, we have

\begin{align*}
i_{\omega_0}(\gamma) = i_1(\gamma) + S_M^+(1) + \sum_{\omega} (S_M^+(\omega) - S_M^-(\omega)) - S_M^-(\omega_0), \quad (128)
\end{align*}

where the sum runs over all the eigenvalues $\omega$ of $M$ belonging to the part of $U^+ = \{\text{Re} z \geq 0 | z \in U\}$ or $U^- = \{\text{Re} z \leq 0 | z \in U\}$ strictly located between 1 and $\omega_0$.

We refer to [11] for more details on this index theory of symplectic matrix paths and periodic solutions of Hamiltonian system.

5.2. Morse index of Lagrangian system. Our purpose here is to give the relation for the Morse index and the Maslov-type index which covers the applications to our problem.

For $T > 0$, suppose $x$ is a critical point of the functional

\begin{align*}
F(x) = \int_0^T L(t, x, \dot{x}) dt, \quad \forall x \in W^{1,2}(\mathbb{R}/T \mathbb{Z}, \mathbb{R}^n),
\end{align*}

where $L \in C^2(\mathbb{R}/T \mathbb{Z}) \times \mathbb{R}^{2n}, \mathbb{R})$ and satisfies the Legendrian convexity condition $L_{p,p}(t, x, p) > 0$. It is well known that $x$ satisfies the corresponding Euler-Lagrangian equation:

\begin{align*}
\frac{d}{dt} L_p(t, x, \dot{x}) - L_x(t, x, \dot{x}) = 0, \quad (129)
x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \quad (130)
\end{align*}
For such an extremal loop, define
\[
P(t) = L_{p,p}(t, x(t), \dot{x}(t)),
Q(t) = L_{x,p}(t, x(t), \dot{x}(t)),
R(t) = L_{x,x}(t, x(t), \dot{x}(t)).
\]

For \( \omega \in U \), set
\[
D(\omega, T) = \{ y \in W^{1,2}([0, T], C^n) \mid y(T) = \omega y(0) \}. \tag{131}
\]
and
\[
\overline{D}(\omega, T) = \{ y \in W^{2,2}([0, T], C^n) \mid y(T) = \omega y(0), \dot{y}(T) = \omega \dot{y}(0) \}. \tag{132}
\]
Suppose \( x \) is an extreme of \( F \) in \( \overline{D}(\omega, T) \). The index form of \( x \) is given by
\[
I(y_1, y_2) = \int_0^T \{ (P \dot{y}_1 + Q y_2) \cdot \dot{y}_2 + Q^T \dot{y}_1 \cdot y_2 + R y_1 \cdot y_2 \}, \quad y_1, y_2 \in D(\omega, T). \tag{133}
\]
The Hessian of \( F \) at \( x \) is given by
\[
I(y_1, y_2) = \langle F''(x)y_1, y_2 \rangle, \quad y_1, y_2 \in D(\omega, T), \tag{134}
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2 \). Linearization of \( (129) \) at \( x \) is given by
\[
- \frac{d}{dt}(P(t)\dot{y} + Q(t)y) + Q^T \dot{y} + R(t)y = 0, \tag{135}
\]
and \( y \) is solution of \( (135) \) if and only if \( y \in \ker(I) \).

We define the \( \omega \)-Morse index \( \phi_\omega(x) \) of \( x \) to be the dimension of the largest negative definite subspace of the index form \( I \) which was defined on \( D(\omega, T) \times D(\omega, T) \). Moreover, \( F''(x) \) is a self-adjoint operator on \( L^2([0, T], R^n) \) with domain \( \overline{D}(\omega, T) \). We also define
\[
\nu_\omega(x) = \dim \ker(F''(x)).
\]

In general, for a self-adjoint operator \( A \) on the Hilbert space \( H \), we set \( \nu(A) = \dim \ker(A) \) and denote by \( \phi(A) \) its Morse index which is the maximum dimension of the negative definite subspace of the symmetric form \( \langle A \cdot, \cdot \rangle \). Note that the Morse index of \( A \) is equal to the total multiplicity of the negative eigenvalues of \( A \).

On the other hand, \( \tilde{x}(t) = (\partial L/\partial \dot{x}(t), x(t))^T \) is the solution of the corresponding Hamiltonian system of \( (129)-(130) \), and its fundamental solution \( \gamma(t) \) is given by
\[
\gamma(t) = JB(t)\gamma(t), \quad \gamma(0) = I_{2n}, \tag{136}
\]
with
\[
B(t) = \begin{pmatrix}
P^{-1}(t) & -P^{-1}(t)Q(t) \\
-Q(t)^TP^{-1}(t) & Q(t)^TP^{-1}(t)Q(t) - R(t)
\end{pmatrix}. \tag{138}
\]

**Lemma 5.6.** (Y. Long, [11], p.172) For the \( \omega \)-Morse index \( \phi_\omega(x) \) and nullity \( \nu_\omega(x) \) of the solution \( x = x(t) \) and the \( \omega \)-Maslov-type index \( i_\omega(\gamma) \) and nullity \( \nu_\omega(\gamma) \) of the symplectic path \( \gamma \) corresponding to \( \tilde{x} \), for any \( \omega \in U \) we have
\[
\phi_\omega(x) = i_\omega(\gamma), \quad \nu_\omega(x) = \nu_\omega(\gamma). \tag{139}
\]

A generalization of the above lemma to arbitrary boundary conditions is given in [9]. For more information on these topics, we refer to [11].
Acknowledgments. The authors thank sincerely Professor Yiming Long for his precious help and useful discussions. They also thank sincerely the anonymous referees on their careful reading and helpful comments on the manuscript of this paper.

REFERENCES

[1] W. Ballmann, G. Thorbergsson and W. Ziller, Closed geodesics on positively curved manifolds. Ann. of Math., 116 (1982), 213–247.
[2] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer-Verlag, Berlin, 1990.
[3] P. P. Hallen and D. N. Rana, The Existence and stability of equilibrium points in the Robe’s restricted three-body problem. Celest. Mech. Dyn. Astr., 79 (2001), 145–155.
[4] X. Hu, Y. Long and S. Sun, Linear stability of elliptic Lagrangian solutions of the classical planar three-body problem via index theory Arch. Rational. Mech. Anal., 213 (2014), 993–1045.
[5] X. Hu and Y. Ou, Collision index and stability of elliptic relative equilibria in planar n-body problem. Commun. Math. Phys., 348 (2016), 803–845.
[6] X. Hu and S. Sun, Morse index and stability of symmetric periodic orbits in Hamiltonian systems with its application to figure-eight orbit. Commun. Math. Phys., 290 (2009), 737–777.
[7] X. Hu and S. Sun, Index and stability of elliptic Lagrangian solutions in the planar three-body problem. Adv. Math., 223 (2010), 98–119.
[8] Y. Long, The structure of the singular symplectic matrix set, Sci. China. Ser. A. (English Ed.), 34 (1991), 897–907.
[9] Y. Long, Bott formula of the Maslov-type index theory. Pacific J. Math., 187 (1999), 113–149.
[10] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. Adv. Math., 154 (2000), 76–131.
[11] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Math. 207, Birkhäuser, Basel, 2002.
[12] A. R. Plastino and A. Plastino, Robe’s restricted three-body problem revisited. Celest. Mech. Dyn. Astr., 61 (1995), 197–206.
[13] H. A. G. Robe, A new kind of three-body problem, Celest. Mech., 16 (1977), 343–351.
[14] A. K. Shrivastava and D. Garain, Effect of perturbation on the location of liberation point in the Robe restricted problem of three bodies. Celest. Mech., 51 (1991), 67–73.
[15] K. T. Singh, B. S. Kushvah and B. Ishwar, Stability of triangular equilibrium points in Robe’s generalised restricted three body problem, in Celestial Mechanics: Recent Trends (eds. B.Ishwar), Narosa Publishing House Pvt. Ltd., New Delhi, India, (2006), 65–70.
[16] Q. Zhou and Y. Long, Equivalence of linear stabilities of elliptic triangle solutions of the planar charged and classical three-body problem J. Diff. Equa., 258 (2015), 3851–3879.
[17] Q. Zhou and Y. Long, Maslov-type indices and linear stability of elliptic Euler solutions of the three-body problem, preprint. arXiv:1510.06622
[18] Q. Zhou and Y. Long, The reduction on the linear stability of elliptic Euler-Moulton solutions of the n-body problem to those of 3-body problems arXiv:1511.00070 Celest. Mech. Dyn. Astr., (2016), 1–32.

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