EXISTENCE OF GLOBAL CLASSICAL SOLUTIONS
FOR THE SAINT-VENANT EQUATIONS

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Abstract. Nowadays, investigations of the existence of global classical solutions for non linear evolution equations is a topic of active mathematical research. In this article, we are concerned with a classical system of shallow water equations which describes long surface waves in a fluid of variable depth. This system was proposed in 1871 by Adhémar Jean-Claude Barré de Saint-Venant. Namely, we investigate an initial value problem for the one dimensional Saint-Venant equations. We are especially interested in question of what sufficient conditions the initial data and the topography of the bottom must verify in order that the considered system has global classical solutions. In order to prove our main results we use a new topological approach based on the fixed point abstract theory of the sum of two operators in Banach spaces. This basic and new idea yields global existence theorems for many of the interesting equations of mathematical physics.

Key words: Saint-Venant equations, classical solution, fixed point, initial value problem.

AMS Subject Classification: 35Q35, 35A09, 35E15.

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1. Introduction

One of the most important works in the study of nonlinear science is the investigation of the existence of global classical solutions. In this paper, we investigate the Cauchy problem for a classical system of shallow water equations which describes long surface waves in a fluid of variable depth. This system was proposed in 1871 by Adhémar Jean-Claude Barré de Saint-Venant [1]. Namely, we consider the following initial value problem for the Saint-Venant equations:

$$\begin{align*}
\partial_t u + \partial_x (uv) &= 0, \quad t \in (0, \infty), \quad x \in \mathbb{R}, \\
\partial_t (uv) + \partial_x \left( uv^2 + \frac{1}{2} ku^2 \right) + ku \partial_x f(t, x) &= 0, \quad t \in (0, \infty), \quad x \in \mathbb{R},
\end{align*}$$

(1.1)

$$\begin{align*}
u(0, x) &= u_0(x), \quad x \in \mathbb{R}, \\
v(0, x) &= v_0(x), \quad x \in \mathbb{R},
\end{align*}$$

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where \( k \in \mathbb{R}_+ \) represents the gravitational constant, the initial conditions \( u_0, v_0 \) and the topography of the bottom \( f \) are given functions. Here the unknowns \( u = u(t, x) \) and \( v = v(t, x) \) denote respectively, the depth and the average horizontal velocity of the fluid.

There are many papers devoted to the study of shallow water flows especially by numerical methods, we can cite [2–7] and the references therein. Let us also mention some works related to the mathematical analysis of Saint-Venant equations. The correct formulation of the boundary conditions for the Saint-Venant system for simulating closed and open basins is investigated in [8]. Local and global temporal existence of classical solutions for the dissipative shallow water equations is considered in [9] and [10], respectively. In [11], the uniqueness of the classical solution of the system of 1-D Saint-Venant equations is proved. In [12], an initial-boundary value problem for a system of Saint-Venant equations is solved by two methods. In [13], Li et al. proved the local-in-time well-posedness of classical solutions with finite total mass and/or energy, allowing the surface height goes to zero in the far field. In [14] and [15], Cheng et al. proved long-time and global existence and asymptotic behavior of classical solutions for two-dimensional rotating shallow water system. In [16] classical solutions for the Cauchy problem of the rotating shallow water equations with physical viscosity are obtained. In [17], local-time well-posedness and breakdown for solutions of regularized Saint-Venant equations are established. In [18], global solutions to Saint-Venant equations are investigated by a method of an additional argument. In [19], under regular initial data with small energy but possibly large oscillations, the global well-posedness of classical solution for Cauchy problem of two-dimensional chemotaxis-shallow water system is studied. In the note [20], a mathematical analysis, based on the theory of semigroups of operators on Hilbert space, of a linearized problem involving the Saint-Venant equations is given.

In this paper, we are especially interested in question of what conditions the initial data \( u_0, v_0 \) and the topography of the bottom \( f \) must verify in order that Problem (1.1) has classical solutions. Here, by a classical solution to the Saint-Venant equations we mean a solution which is along with its derivatives that appear in the equations of class \( C([0, \infty) \times \mathbb{R}) \). In other words, \((u, v)\) belongs to the space \( C^1([0, \infty) \times \mathbb{R}) \times C^1([0, \infty) \times \mathbb{R}) \) of continuously differentiable functions on \([0, \infty) \times \mathbb{R}\). The main assumptions on the functions \( u_0, v_0 \) and \( f \) are

\[
(H1) \quad u_0, v_0 \in C^1(\mathbb{R}), \quad 0 < u_0 \leq B, \quad 0 \leq v_0 \leq B \text{ on } \mathbb{R} \text{ for some positive constant } B.
\]

\[
(H2) \quad f \in C([0, \infty), C^1(\mathbb{R})), \quad 0 \leq |\partial_x f| \leq B \text{ on } [0, \infty) \times \mathbb{R}.
\]

Our approach is based on the use of the fixed point theory for the sum of operators in Banach spaces.

The paper is organized as follows. In the next section, we describe a new topological approach which uses fixed point abstract theory of the sum of two operators. In Section 3 we give some properties of solutions of Problem (1.1). These properties concern integral representation of the solutions and some energy estimates to be made more precise later on. In Section 4 we prove existence and multiplicity of solutions for the Saint-Venant system (1.1). Finally, in Section 5 an example illustrating our main results will be given.

2. On Fixed Points for the Sum of Two Operators

In this section, some definitions and results related to fixed point theory will be given. We will recall two approaches for the existence of fixed points for the sum of two operators.

2.1. First approach.

Theorem 2.1. Let \( E \) be a Banach space and

\[ E_1 = \{ x \in E : \|x\| \leq R \}, \]

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with \( R > 0 \). Consider two operators \( T \) and \( S \), where

\[
Tx = -\epsilon x, \quad x \in E_1,
\]

with \( \epsilon > 0 \) and \( S : E_1 \to E \) be continuous and such that

(i) \( (I - S)(E_1) \) resides in a compact subset of \( E \) and

(ii) \( \{ x \in E : x = \lambda(I - S)x, \|x\| = R \} = \emptyset \), for any \( \lambda \in (0, \frac{1}{\epsilon}) \). Then there exists \( x^* \in E_1 \) such that

\[
Tx^* + Sx^* = x^*.
\]

Theorem 2.1 will be used to prove Theorem 4.1 and its proof can be found in [21]. Let us recall the proof of Theorem 2.1 for convenience.

Define

\[
r\left( -\frac{1}{\epsilon} x \right) = \begin{cases} -\frac{1}{\epsilon} x, & \text{if } \|x\| \leq R\epsilon, \\ \frac{R\epsilon}{\|x\|}, & \text{if } \|x\| > R\epsilon. \end{cases}
\]

Then \( r\left( -\frac{1}{\epsilon}(I - S)x^* \right) : E_1 \to E_1 \) is continuous and compact. Hence and the Schauder fixed point theorem, it follows that there exists \( x^* \in E_1 \) so that

\[
r\left( -\frac{1}{\epsilon}(I - S)x^* \right) = x^*.
\]

Assume that \( -\frac{1}{\epsilon}(I - S)x^* \notin E_1 \). Then

\[
\|(I - S)x^*\| > R\epsilon, \quad \frac{R}{\|(I - S)x^*\|} < \frac{1}{\epsilon}
\]

and

\[
x^* = \frac{R}{\|(I - S)x^*\|}(I - S)x^* = r\left( -\frac{1}{\epsilon}(I - S)x^* \right)
\]

and hence, \( \|x^*\| = R \). This contradicts the condition (ii). Therefore \( -\frac{1}{\epsilon}(I - S)x^* \in E_1 \) and

\[
x^* = r\left( -\frac{1}{\epsilon}(I - S)x^* \right) = -\frac{1}{\epsilon}(I - S)x^*
\]

or

\[-\epsilon x^* + Sx^* = x^*, \]

or

\[Tx^* + Sx^* = x^*.
\]

This completes the proof. ▷

2.2. Second approach. Let \( E \) be a real Banach space.

Definition 2.1. A closed, convex set \( \mathcal{P} \) in \( E \) is said to be cone if

1) \( \alpha x \in \mathcal{P} \) for any \( \alpha \geq 0 \) and for any \( x \in \mathcal{P} \),

2) \( -x \in \mathcal{P} \) implies \( x = 0 \).

Definition 2.2. A mapping \( K : E \to E \) is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. Let \( X \) and \( Y \) be real Banach spaces. A mapping \( K : X \to Y \) is said to be expansive if there exists a constant \( h > 1 \) such that

\[
\|Kx - Ky\|_Y \geq h\|x - y\|_X
\]

for any \( x, y \in X \).
The following result (its proof relies on [22, Proposition 2.16]) will be used to prove Theorem 4.2.

**Theorem 2.2.** Let $\mathcal{P}$ be a cone of a Banach space $E$; $\Omega$ a subset of $\mathcal{P}$ and $U_1, U_2$ and $U_3$ three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P} \setminus \{0\}$ such that

(i) $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$;
(ii) there exists $\varepsilon > 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda > 1 + \varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$;
(iii) $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

\[ x_1 \in \partial U_2 \cap \Omega \quad \text{and} \quad x_2 \in (U_3 \setminus \overline{U}_2) \cap \Omega \]

or

\[ x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \text{and} \quad x_2 \in (U_3 \setminus \overline{U}_2) \cap \Omega. \]

3. Integral Representation and a Priori Estimates for Solutions of Problem (1.1)

Let $X = X^1 \times X^1$, where $X^1 = C^1([0, \infty) \times \mathbb{R})$. For $(u, v) \in X$, define the operators $S_1^1$, $S_1^2$ and $S_1$ as follows.

\[
S_1^1(u, v)(t, x) = \int_0^x (u(t, x_1) - u_0(x_1)) \, dx_1 + \int_0^t u(t_1, x)v(t_1, x) \, dt_1, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

\[
S_1^2(u, v)(t, x) = \int_0^x (u(t, x_1)v(t_1, x_1) - u_0(x_1)v_0(x_1)) \, dx_1
\]

\[ + \int_0^t \left( u(t_1, x)(v(t_1, x))^2 + \frac{1}{2}k(u(t_1, x))^2 \right) \, dt_1
\]

\[ + k \int_0^t \int_0^x u(t_1, x_1) \partial_x f(t_1, x_1) \, dx_1 \, dt_1, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

\[
S_1(u, v)(t, x) = (S_1^1(u, v)(t, x), S_1^2(u, v)(t, x)), \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]

**Lemma 3.1.** Suppose that (H1) and (H2) are satisfied. If $(u, v) \in X$ satisfies the equation

\[
S_1(u, v)(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

then $(u, v)$ is a solution of the IVP (1.1).

Let $(u, v) \in X$ be a solution to the equation (3.1). Then

\[
S_1^1(u, v)(t, x) = 0, \quad S_1^2(u, v)(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
We differentiate the first equation of (3.2) with respect to \( t \) and \( x \) and we find
\[
\partial_t u(t, x) + \partial_x (uv)(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
We put \( t = 0 \) in the first equation of (3.2) and we arrive at
\[
\int_0^x (u(0, x_1) - u_0(x_1)) \, dx_1 = 0, \quad x \in \mathbb{R},
\]
which we differentiate with respect to \( x \) and we find
\[
u(0, x) = u_0(x), \quad x \in \mathbb{R}.
\]
Now, we differentiate the second equation of (3.2) with respect to \( t \) and \( x \) and we find
\[
\partial_t (uv)(t, x) + \partial_x \left( uv^2 + \frac{1}{2} ku^2 \right)(t, x) + ku(t, x) \partial_x f(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
We put \( t = 0 \) in the second equation of (3.2) and we get
\[
\int_0^x (u(0, x_1)v(0, x_1) - u_0(x_1)v_0(x_1)) \, dx_1 = 0, \quad x \in \mathbb{R},
\]
which we differentiate with respect to \( x \) and we obtain
\[
u(0, x)v(0, x) - u_0(x)v_0(x) = 0, \quad x \in \mathbb{R},
\]
whereupon
\[
v(0, x) = v_0(x), \quad x \in \mathbb{R}.
\]
Thus, \((u, v)\) is a solution to the IVP (1.1). This completes the proof.

\[\blacksquare\]

**Lemma 3.2.** Suppose that \((H1)\) and \((H2)\) are satisfied. Let \( h \in C([0, \infty) \times \mathbb{R}) \) be a positive function almost everywhere on \([0, \infty) \times \mathbb{R}\). If \((u, v) \in X\) satisfies the following integral equations:
\[
\int_0^t \int_0^x (t - t_1)^2(x - x_1)^2 h(t_1, x_1) S_1^1(u, v)(t_1, x_1) \, dx_1 \, dt_1 = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
and
\[
\int_0^t \int_0^x (t - t_1)^2(x - x_1)^2 h(t_1, x_1) S_1^2(u, v)(t_1, x_1) \, dx_1 \, dt_1 = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
then \((u, v)\) is a solution to the IVP (1.1).

\[\leftarrow\]
We differentiate three times with respect to \( t \) and three times with respect to \( x \) the integral equations of Lemma 3.2 and we find
\[
h(t, x) S_1(u, v)(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
whereupon
\[
S_1(u, v)(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
Hence and Lemma 3.1, we conclude that \((u, v)\) is a solution to the IVP (1.1). This completes the proof. \[\blacksquare\]
Now, let us prove some \textit{a priori} estimates related to solutions of Problem (1.1). In the sequel, \( X = X^1 \times X^1 \), where \( X^1 = C^1([0, \infty) \times \mathbb{R}) \) will be endowed with the norm
\[
\|(u,v)\| = \max\{\|u\|_{X^1}, \|v\|_{X^1}\}, \quad (u,v) \in X,
\]
with
\[
\|u\|_{X^1} = \max\left\{ \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u(t,x)|, \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u_t(t,x)|, \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u_x(t,x)| \right\},
\]
provided it exists. Let
\[
B_1 = \max\left\{ 2B, B^2 \left( 1 + \frac{1}{2}k \right), kB^2, 2B^2 \right\}.
\]

\textbf{Lemma 3.3.} Under hypothesis (H1) and (H2) and for \((u,v) \in X\) with \( \|(u,v)\| \leq B \), the following estimates hold:
\[
|S^1_1(u,v)(t,x)| \leq B_1(1 + t)(1 + |x|), \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
and
\[
|S^2_1(u,v)(t,x)| \leq B_1(1 + t)(1 + |x|), \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
\(<\) Suppose that (H1) and (H2) are satisfied and let \((u,v) \in X\) with \( \|(u,v)\| \leq B \).

(i) Estimation of \(||S^1_1(u,v)(t,x)||\), \((t, x) \in [0, \infty) \times \mathbb{R} :\)
\[
|S^1_1(u,v)(t,x)| = \left| \int_0^x (u(t,x_1) - u_0(x_1)) \, dx_1 + \int_0^t u(t_1,x)v(t_1,x) \, dt_1 \right|
\]
\[
\leq \left| \int_0^x (|u(t,x_1)| + u_0(x_1)) \, dx_1 + \int_0^t |u(t_1,x)||v(t_1,x)| \, dt_1 \right|
\]
\[
\leq 2B|x| + B^2t \leq B_1(t + |x|) \leq B_1(1 + t)(1 + |x|),
\]

(ii) Estimation of \(||S^2_1(u,v)(t,x)||\), \((t, x) \in [0, \infty) \times \mathbb{R} :\)
\[
|S^2_1(u,v)(t,x)| = \left| \int_0^x (u(t,x_1)v(t,x_1) - u_0(x_1)v_0(x_1)) \, dx_1 \right.
\]
\[
+ \left. \int_0^t \left( u(t_1,x)(v(t_1,x))^2 + \frac{1}{2}k(u(t_1,x))^2 \right) \, dt_1 + k \int_0^t \left| u(t_1,x)\partial_x f(t_1,x) \right| \, dx_1 \right|
\]
\[
\leq \left| \int_0^x (|u(t,x_1)||v(t,x_1)| + u_0(x_1)v_0(x_1)) \, dx_1 \right| + \int_0^t \left( |u(t_1,x)||v(t_1,x)|^2 + \frac{1}{2}k(u(t_1,x))^2 \right) \, dt_1
\]
\[
+ k \int_0^t \left| \int_0^x |u(t_1,x_1)|\partial_x f(t_1,x_1) \, dx_1 \right| \, dt_1 \leq 2B^2|x| + \left( B^2 + \frac{1}{2}kB^2 \right)t + kB^2t|x|
\]
\[
\leq B_1(t + |x| + t|x|) \leq B_1(1 + |x|)(1 + t).
\]

This completes the proof. \(\triangleright\)
Suppose

\((H3)\) there exist a positive constant \(A\) and a nonnegative function \(g \in C([0, \infty) \times \mathbb{R})\) such that

\[
8(1 + t) (1 + t + t^2) (1 + |x|) (1 + |x| + x^2) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \leq A, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]

In the last section, we will give an example for a function \(g\) that satisfies \((H3)\). For \((u, v) \in X\), define the operators \(S_1^2, S_2^2\) and \(S_2\) as follows.

\[
S_2^1(u, v)(t, x) = \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) S_1^1(u, v)(t_1, x_1) \, dx_1 \, dt_1, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

\[
S_2^2(u, v)(t, x) = \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) S_2(u, v)(t_1, x_1) \, dx_1 \, dt_1, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

and

\[
S_2(u, v)(t, x) = (S_1^2(u, v)(t, x), S_2^2(u, v)(t, x)), \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]

**Lemma 3.4.** Under hypothesis \((H1)\), \((H2)\) and \((H3)\) and for \((u, v) \in X\), with \(
\|g(u, v)\| \leq B,
\)

the following estimate holds:

\[
\|S_2(u, v)\| \leq AB_1.
\]

<sup>\(\triangleright\)</sup> Suppose that \((H1)\), \((H2)\) and \((H3)\) are satisfied. Let \((u, v) \in X\), with \(
\|g(u, v)\| \leq B.
\)

(i) Estimation of \(
|S_2^1(u, v)(t, x)|, \quad (t, x) \in [0, \infty) \times \mathbb{R} :
\)

\[
|S_2^1(u, v)(t, x)| = \left| \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) S_1^1(u, v)(t_1, x_1) \, dx_1 \, dt_1 \right|
\]

\[
\leq \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) S_1^1(u, v)(t_1, x_1) \, dx_1 \, dt_1
\]

\[
\leq B_1 \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1)(1 + t_1)(1 + |x_1|) \, dx_1 \, dt_1
\]

\[
\leq 4B_1(1 + t)t^2(1 + |x|)x^2 \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1
\]

\[
\leq 8B_1(1 + t)(1 + t^2)(1 + |x|)(1 + |x| + x^2) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \leq AB_1.
\]
(ii) Estimation of $\left| \frac{\partial}{\partial t} S_2^1(u, v)(t, x) \right|$, $(t, x) \in [0, \infty) \times \mathbb{R}$:

$$\left| \frac{\partial}{\partial t} S_2^1(u, v)(t, x) \right| = 2 \left| \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1) S_1^1(u, v)(t_1, x_1) \, dx_1 \, dt_1 \right|$$

$$\leq 2 \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1) \left| S_1^1(u, v)(t_1, x_1) \right| \, dx_1 \, dt_1$$

$$\leq 2B_1 \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1)(1 + t_1)(1 + |x_1|) \, dx_1 \, dt_1$$

$$\leq 8B_1(1 + t)(1 + t^2)(1 + |x|)(1 + |x| + x^2) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \leq AB_1.$$ 

(iii) Estimation of $\left| \frac{\partial}{\partial x} S_2^1(u, v)(t, x) \right|$, $(t, x) \in [0, \infty) \times \mathbb{R}$:

$$\left| \frac{\partial}{\partial x} S_2^1(u, v)(t, x) \right| = 2 \left| \int_0^t \int_0^x (t - t_1)^2(x - x_1) g(t_1, x_1) S_1^1(u, v)(t_1, x_1) \, dx_1 \, dt_1 \right|$$

$$\leq 2 \int_0^t \int_0^x (t - t_1)^2|x - x_1| g(t_1, x_1) \left| S_1^1(u, v)(t_1, x_1) \right| \, dx_1 \, dt_1$$

$$\leq 2B_1 \int_0^t \int_0^x (t - t_1)^2|x - x_1| g(t_1, x_1)(1 + t_1)(1 + |x_1|) \, dx_1 \, dt_1$$

$$\leq 4B_1(1 + t)t^2(1 + |x|)|x| \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1$$

$$\leq 8B_1(1 + t)(1 + t^2)(1 + |x|)(1 + |x| + x^2) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \leq AB_1.$$ 

As above,

$$\left| S_2^1(u, v)(t, x) \right|, \quad \left| \frac{\partial}{\partial t} S_2^1(u, v)(t, x) \right|, \quad \left| \frac{\partial}{\partial x} S_2^1(u, v)(t, x) \right| \leq AB_1, \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$ 

Therefore

$$\| S_2(u, v) \| \leq AB_1.$$

This completes the proof. $\lhd$
4. Existence and Multiplicity of Nonnegative Solutions

In the sequel, suppose that the constants \( B \) and \( A \) which appear in the conditions (H1) and (H3), respectively, satisfy the following inequality:

\[(H4)\] \( AB_1 < B, \) where \( B_1 = \max \{ 2B, B^2(1 + \frac{1}{B^2}),kB^2,2B^2 \} \).

Our first main result for existence of classical solutions of the IVP (1.1) is as follows.

**Theorem 4.1.** Under hypotheses (H1), (H2), (H3) and (H4), the IVP (1.1) has at least one nonnegative solution \((u, v)\) for some \( \lambda \). Existence of Global Classical Solutions for the Saint-Venant Equations

Choose \( \iota \in (0,1) \), such that \( \iota B_1(1+A) < B \). For \((u,v)\in X=\mathcal{C}^1([0,\infty)\times \mathbb{R})\times \mathcal{C}^1([0,\infty)\times \mathbb{R})\), we will write

\[
(u,v) \geq 0 \quad \text{if} \quad u(t,x) \geq 0 \quad \text{and} \quad v(t,x) \geq 0 \quad \text{for any} \quad (t,x) \in [0,\infty) \times \mathbb{R}.
\]

Let \( \tilde{Y} \) denote the set of all equi-continuous families in \( X \) with respect to the norm \( \| \cdot \| \), \( \tilde{Y} = \bar{Y} \) be the closure of \( \tilde{Y} \), \( \bar{Y} = Y \cup \{(u_0,v_0)\} \) and

\[
Y = \left\{ (u,v) \in \bar{Y} : (u,v) \geq 0, \| (u,v) \| \leq B \right\}.
\]

Note that \( Y \) is a compact set in \( X \). For \((u,v)\in X\), define the operators

\[
T(u,v)(t,x) = -\iota(u,v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R},
\]

\[
S(u,v)(t,x) = (u,v)(t,x) + \iota(u,v)(t,x) + \iota S_2(u,v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.
\]

For \((u,v)\in Y\) and by using Lemma 3.4, it follows that

\[
\| (I-S)(u,v) \| = \| \iota(u,v) - \iota S_2(u,v) \| \leq \iota \| (u,v) \| + \iota \| S_2(u,v) \| \leq \iota B_1 + \iota AB_1 = \iota B_1(1+A) < B.
\]

Thus, \( S : Y \to X \) is continuous and \( (I-S)(Y) \) resides in a compact subset of \( X \). Now, suppose that there is a \((u,v)\in X\) so that \( \| (u,v) \| = B \) and

\[
(u,v) = \lambda (I-S)(u,v)
\]

or

\[
\frac{1}{\lambda} (u,v) = (I-S)(u,v) = -\iota(u,v) - \iota S_2(u,v),
\]

or

\[
\left( \frac{1}{\lambda} + \iota \right)(u,v) = -\iota S_2(u,v)
\]

for some \( \lambda \in (0,\frac{1}{2}) \). Hence, \( \| S_2(u,v) \| \leq AB_1 < B \),

\[
\iota B < \left( \frac{1}{\lambda} + \iota \right) B = \left( \frac{1}{\lambda} + \iota \right) \| (u,v) \| = \iota \| S_2(u,v) \| < \iota B,
\]

which is a contradiction. Hence and Theorem 2.1, it follows that the operator \( T + S \) has a fixed point \((u^*,v^*)\in Y\). Therefore

\[
(u^*,v^*)(t,x) = T(u^*,v^*)(t,x) + S(u^*,v^*)(t,x) = -\iota(u^*,v^*)(t,x) + (u^*,v^*)(t,x) + \iota S_2(u^*,v^*)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R},
\]
whereupon
\[ 0 = S_2(u^*, v^*)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}. \]

Lemma 3.2 yields that \((u^*, v^*)\) is a solution to the IVP (1.1). This completes the proof. \(\Box\)

In the sequel, suppose that the constants \(B\) and \(A\) which appear in the conditions (H1) and (H3), respectively, satisfy the following inequality:

**H5** \(AB_1 < \frac{L}{10}\), where \(B_1 = \max\{2B, B^2(1 + \frac{1}{2}k), kB^2, 2B^2\}\) and \(L\) is a positive constant that satisfies the following conditions:

\[ r < L < R_1 \leq B, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \]

with \(r\) and \(R_1\) are positive constants and \(m > 0\) is large enough.

Our second main result for existence and multiplicity of classical solutions of the IVP (1.1) is as follows.

**Theorem 4.2.** Under Hypotheses (H1), (H2), (H3) and (H5), the IVP (1.1) has at least two nonnegative solutions \((u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}) \times \mathcal{C}^1([0, \infty) \times \mathbb{R})\).

Set \(X = \mathcal{C}^1([0, \infty) \times \mathbb{R}) \times \mathcal{C}^1([0, \infty) \times \mathbb{R})\) and let

\[ \tilde{P} = \{(u, v) \in X : (u, v) \geq 0 \text{ on } [0, \infty) \times \mathbb{R}\}. \]

With \(\mathcal{P}\) we will denote the set of all equi-continuous families in \(\tilde{P}\). For \((u, v) \in X\), define the operators \(T_1\) and \(S_3\) as follows:

\[ T_1(u, v)(t, x) = (1 + m\epsilon)(u, v)(t, x) - \left(\frac{L}{10}, \frac{L}{10}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}, \]

\[ S_3(u, v)(t, x) = -\epsilon S_2(u, v)(t, x) - m\epsilon(u, v)(t, x) - \left(\epsilon \frac{L}{10}, \epsilon \frac{L}{10}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}, \]

where \(\epsilon\) is a positive constant, \(m > 0\) is large enough and the operator \(S_2\) is given by formula (3.3). Note that any fixed point \((u, v) \in X\) of the operator \(T_1 + S_3\) is a solution to the IVP (1.1). Now, let us define

\[ U_1 = \mathcal{P}_r = \{(u, v) \in \mathcal{P} : \|(u, v)\| < r\}, \]

\[ U_2 = \mathcal{P}_L = \{(u, v) \in \mathcal{P} : \|(u, v)\| < L\}, \]

\[ U_3 = \mathcal{P}_{R_1} = \{(u, v) \in \mathcal{P} : \|(u, v)\| < R_1\}, \]

\[ \Omega = \overline{\mathcal{P}_{R_2}} = \{(u, v) \in \mathcal{P} : \|(u, v)\| \leq R_2\}, \quad R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m}. \]

1) For \((u_1, v_1), (u_2, v_2) \in \Omega\), we have

\[ \|T_1(u_1, v_1) - T_1(u_2, v_2)\| = (1 + m\epsilon)\|(u_1, v_1) - (u_2, v_2)\|, \]

whereupon \(T_1 : \Omega \to X\) is an expansive operator with a constant \(h = 1 + m\epsilon > 1\).

2) Let \((u, v) \in \mathcal{P}_{R_1}\), then Lemma 3.4 yields

\[ \|S_3(u, v)\| \leq \epsilon\|S_2(u, v)\| + m\epsilon\|(u, v)\| + \epsilon \left(\frac{L}{10}\right) \leq \epsilon \left(AB_1 + mR_1 + \frac{L}{10}\right). \]

Therefore \(S_3(\mathcal{P}_{R_1})\) is uniformly bounded. Since \(S_3 : \mathcal{P}_{R_1} \to X\) is continuous, we have that \(S_3(\mathcal{P}_{R_1})\) is equi-continuous. Consequently \(S_3 : \mathcal{P}_{R_1} \to X\) is completely continuous.
3) Let \((u_1, v_1) \in \overline{P}_{R_1}\). Set
\[
(u_2, v_2) = (u_1, v_1) + \frac{1}{m} S_2(u_1, v_1) + \left( L \frac{1}{5m}, L \frac{1}{5m} \right).
\]
Note that \(S_2(u_1, v_1) + \frac{L}{5} \geq 0\), \(S_2(u_1, v_1) + \frac{L}{5} \geq 0\) on \([0, \infty) \times \mathbb{R}\). We have \(u_2, v_2 \geq 0\) on \([0, \infty) \times \mathbb{R}\) and
\[
\|u_2, v_2\| \leq \|u_1, v_1\| + \frac{1}{m} \|S_2(u_1, v_1)\| + \frac{L}{5m} \leq R_1 + \frac{A}{m} B_1 + \frac{L}{5m} = R_2.
\]
Therefore \((u_2, v_2) \in \Omega\) and
\[
-em(u_2, v_2) = -em(u_1, v_1) - \epsilon S_2(u_1, v_1) - \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right) - \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right)
\]
or
\[
(I - T_1)(u_2, v_2) = -em(u_2, v_2) + \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right) = S_3(u_1, v_1).
\]
Consequently, \(S_3(\overline{P}_{R_1}) \subset (I - T_1)(\Omega)\).

4) Assume that for any \((w_0, z_0) \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}\) there exist \(\lambda \geq 0\) and \((u, v) \in \partial \mathcal{P}_r \cap (\Omega + \lambda(w_0, z_0))\) or \((u, v) \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda(w_0, z_0))\) such that
\[
S_3(u, v) = (I - T_1)((u, v) - \lambda(w_0, z_0)).
\]
Then
\[
-em(u, v) - \epsilon S_2(u, v) - \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right) = -em((u, v) - \lambda(w_0, z_0)) + \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right)
\]
or
\[
-S_2(u, v) = \lambda m(w_0, z_0) + \left( L \frac{1}{5}, L \frac{1}{5} \right).
\]
Hence,
\[
\|S_2(u, v)\| = \left\| \lambda m(w_0, z_0) + \left( L \frac{1}{5}, L \frac{1}{5} \right) \right\| > L \frac{1}{5}.
\]
This is a contradiction.

5) Let \(\varepsilon_1 = \frac{2}{5m}\). Assume that there exist \((u_1, v_1) \in \partial \mathcal{P}_L\) and \(\lambda_1 \geq 1 + \varepsilon_1\) such that
\[
\lambda_1(u_1, v_1) \in \overline{\mathcal{P}}_{R_1}
\]
and
\[
S_3(u_1, v_1) = (I - T_1)(\lambda_1(u_1, v_1)).
\]
(4.1)
Since \((u_1, v_1) \in \partial \mathcal{P}_L\) and \(\lambda_1(u_1, v_1) \in \overline{\mathcal{P}}_{R_1}\), it follows that
\[
\left( \frac{2}{5m} + 1 \right) L < \lambda_1 L = \lambda_1 \|u_1, v_1\| \leq R_1.
\]
Moreover,
\[
-em(u_1, v_1) - \epsilon S_2(u_1, v_1) - \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right) = -\lambda_1 m(u_1, v_1) + \epsilon \left( L \frac{1}{10}, L \frac{1}{10} \right)
\]
or
\[
S_2(u_1, v_1) + \left( L \frac{1}{5}, L \frac{1}{5} \right) = (\lambda_1 - 1)m(u_1, v_1).
\]
From here,
\[
2 \frac{L}{5} \geq \left\| S_2(u_1, v_1) + \left( \frac{L}{5}, \frac{L}{5} \right) \right\| = (\lambda_1 - 1)m\|(u_1, v_1)\| = (\lambda_1 - 1)mL
\]
and
\[
\frac{2}{5m} + 1 \geq \lambda_1,
\]
which is a contradiction.

Therefore all conditions of Theorem 4.2 hold. Hence, the IVP (1.1) has at least two solutions \((u_1, v_1)\) and \((u_2, v_2)\) so that
\[
\|(u_1, v_1)\| = L < \|(u_2, v_2)\| \leq R_1
\]

or
\[
r \leq \|(u_1, v_1)\| < L < \|(u_2, v_2)\| \leq R_1.
\]

5. An Example

Below, we will illustrate our main results. Let
\[
h(s) = \log \frac{1 + s^{11} \sqrt{2} + s^{22}}{1 - s^{11} \sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11} \sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.
\]
Then
\[
h'(s) = \frac{22 \sqrt{2} s^{10} (1 - s^{22})}{(1 - s^{11} \sqrt{2} + s^{22})(1 + s^{11} \sqrt{2} + s^{22})}, \quad l'(s) = \frac{11 \sqrt{2} s^{10} (1 + s^{22})}{1 + s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.
\]
Therefore
\[
-\infty < \lim_{s \to \pm \infty} (1 + s + s^2 + s^3 + s^4 + s^5 + s^6) h(s) < \infty,
\]
\[
-\infty < \lim_{s \to \pm \infty} (1 + s + s^2 + s^3 + s^4 + s^5 + s^6) l(s) < \infty.
\]
Hence, there exists a positive constant \(C_1\) so that
\[
(1 + s + s^2 + s^3 + s^4 + s^5 + s^6) \left( \frac{1}{44 \sqrt{2}} \log \frac{1 + s^{11} \sqrt{2} + s^{22}}{1 - s^{11} \sqrt{2} + s^{22}} + \frac{1}{22 \sqrt{2}} \arctan \frac{s^{11} \sqrt{2}}{1 - s^{22}} \right) \leq C_1,
\]
\(s \in \mathbb{R}\). Note that \(\lim_{s \to \pm 1} l(s) = \frac{\pi}{2}\) and by [23, p. 707, Integral 79], we have
\[
\int \frac{dz}{1 + z^4} = \frac{1}{4 \sqrt{2}} \log \frac{1 + z \sqrt{2} + z^2}{1 - z \sqrt{2} + z^2} + \frac{1}{2 \sqrt{2}} \arctan \frac{z \sqrt{2}}{1 - z^2}.
\]
Let
\[
Q(s) = \frac{s^{10}}{(1 + s^{14})(1 + s + s^2)^2}, \quad s \in \mathbb{R},
\]
and
\[
g_1(t, x) = Q(t)Q(x), \quad t \in [0, \infty), \quad x \in \mathbb{R}.
\]
Then there exists a constant $C_2 > 0$ such that
\[
8(1 + t)(1 + t^2)(1 + |x|)(1 + |x| + |x|^2) \int_0^t \int_0^x g_1(t_1, x_1) \, dx_1 \, dt_1 \leq C_2, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
Let
\[
g(t, x) = \frac{A}{C_2} g_1(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]
Then
\[
8(1 + t)(1 + t^2)(1 + |x|)(1 + |x| + |x|^2) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \leq A, \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
and (H3) holds. Now, consider the initial value problem
\[
\begin{align*}
p_t u + p_x(uv) &= 0, \quad t > 0, \quad x \in \mathbb{R},
p_t(uv) + p_x(uv^2 + u^2) + \frac{4vu}{1 + x^2} &= 0, \quad t > 0, \quad x \in \mathbb{R},
u(0, x) = v(0, x) = \frac{3}{1 + x^{16}}, \quad x \in \mathbb{R},
\end{align*}
\]
so that (H1) and (H2) hold, with $B = 10$, for example. Take $B = 10$ and $A = \frac{1}{10^4}$. Then
\[
B_1 = \max \{20, 2 \cdot 10^2\} = 2 \cdot 10^2
\]
and
\[
AB_1 = \frac{1}{10^4} \cdot 2 \cdot 10^2 < B.
\]
So, Condition (H4) is fulfilled. Thus, the conditions (H1), (H2), (H3) and (H4) are satisfied. Hence, by Theorem 4.1, it follows that Problem 6 has at least one nonnegative solution $(u, v) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}) \times \mathcal{C}^1([0, \infty) \times \mathbb{R})$.

In the sequel, take
\[
R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.
\]
Clearly,
\[
r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L, \quad AB_1 < \frac{L}{5},
\]
i.e., (H5) holds. Thus, the conditions (H1), (H2), (H3) and (H5) are satisfied. Hence, by Theorem 4.2, it follows that the initial value Problem 6 has at least two nonnegative solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}) \times \mathcal{C}^1([0, \infty) \times \mathbb{R})$.

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Аннотация. В настоящее время исследования существования глобальных классических решений нелинейных эволюционных уравнений являются предметом активных математических исследований. В этой статье нас интересует классическая система уравнений мелкой воды, описывающая длинные поверхностные волны в жидкости переменной глубины. Эта система была предложена в 1871 г. Адмари Жан-Клодом Барром де Сен-Венаном. А именно, мы исследуем начальную задачу для одномерных уравнений Сен-Венана. Нас особенно интересует вопрос, при каких достаточных...
условиях должны верифицироваться начальные данные и топография dna, чтобы рассматриваемая система имела глобальные классические решения. Для доказательства наших основных результатов мы используем новый топологический подход, основанный на абстрактной теории суммы двух операторов в банаховых пространствах с фиксированной точкой. Эта основная и новая идея приводит к глобальным теоремам существования для многих интересных уравнений математической физики.

Ключевые слова: уравнения Сен-Венана, классическое решение, неподвижная точка, начальная задача.

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