On the existence of polynomial-time algorithms to the subset sum problem

Jorma Jormakka

Department of Military Technology
National Defence University
Karhekuja 4, 01660 Vantaa, Finland
jorma.o.jormakka@kolumbus.fi
Abstract

This paper proves that there does not exist a polynomial-time algorithm to the subset sum problem. As this problem is in $NP$, the result implies that the class $P$ of problems admitting polynomial-time algorithms does not equal the class $NP$ of problems admitting nondeterministic polynomial-time algorithms.

Keywords: computational complexity, polynomial-time, algorithm, knapsack problem
1 Introduction

The paper shows that there does not exist polynomial-time algorithms for the subset sum problem, called the knapsack problem in the Merkle-Hellman cryptosystem. We start by defining the formulations used in this paper. Let $\mathbb{N}$ and $\mathbb{R}$ indicate natural and real numbers respectively.

**Definition 1.** A knapsack is a pair of the form $(j, (d_1, \ldots, d_n))$ where $j, n \in \mathbb{N}$, $j, n > 0$ and $d_k \in \mathbb{N}$, $d_k > 0$ for $1 \leq k \leq n$.

The knapsack problem means the following: given a knapsack $(j, (d_1, \ldots, d_n))$ determine if there exist binary numbers $c_k \in \{0, 1\}$, $1 \leq k \leq n$, such that

$$j = \sum_{k=1}^{n} c_k d_k.$$

Let $B, \alpha \in \mathbb{R}$, $B \geq 1$, $\alpha \geq 0$ be fixed numbers. An algorithm $A$ is called polynomial-time algorithm to the knapsack problem if there exist numbers $C, \beta \in \mathbb{R}$ that depend on $B$ and $\alpha$ but not on $n$ such that the following condition is true: For any sequence of knapsacks of the form

$$(j_n, (d_{1,n}, \ldots, d_{n,n}))_{n \geq 1}$$

satisfying

$$\log_2 j_n < Bn^\alpha, \quad \log_2 d_{k,n} < Bn^\alpha, \quad (1 \leq k \leq n), (n \geq 1) \quad (1.1)$$

the number $N_n$ of elementary operations that the algorithm $A$ needs to produce an answer yes or no to the question if there exists binary numbers $c_{k,n} \in \{0, 1\}$, $1 \leq k \leq n$, such that

$$j_n = \sum_{k=1}^{n} c_{k,n} d_{k,n} \quad (1.2)$$

satisfies $N_n < Cn^\beta$ for all $n \geq 1$. 

3
Remark 1. In the definition of a polynomial-time algorithm for the knapsack problem we have included an upper bound on \( j_n \) and on each \( d_{k,n}, 1 \leq k \leq n \). Such bounds are necessary for the following reason. The number \( m \) of bits in the binary representation of \( j_n \) satisfies \( m \leq \log_2 j_n < m + 1 \). Thus, if \( \log_2 j_n \) grows faster than any polynomial as a function of \( n \) then so does the length of \( j_n \) in the binary representation. It is necessary to verify that (1.2) is satisfied. It requires making some operations (like compare, copy, read, add, subtract, multiply, divide, modulus) that act on a representation of \( j_n \) on some base number. We may assume that the number base is 2 as changing a number base does not change the character of the algorithm from polynomial-time to non-polynomial-time. Any operations that require all bits of \( j_n \) must require more than a polynomial number of elementary operations from any algorithm \( A \) if the number of bits in \( j_n \) grows faster than any polynomial. Similar comments apply to \( d_{k,n} \).

Remark 2. The problem that has been described is used in the Merkle-Hellman knapsack cryptosystem and today it is commonly known as the knapsack problem. The name Subset sum problem is used for it in [2] p. 301, while the name Knapsack problem is reserved for a more general problem involving selecting objects with weights and profits. The name knapsack is more convenient than subset sum and it is often used in this paper.

2 The method of the proof

In this article we give a proof that the subset sum problem cannot be solved in polynomial time. The result settles the \( P \) versus \( NP \) problem [1]. It is understandable to be sceptical of proofs purporting to solve well-known problems but sometimes the proofs are correct. The Poincare conjecture was proven some years ago and the Statement D of the Clay Mathematics Institute Navier-Stokes problem was against expectations proven by the author and the proof was recently published in a peer-reviewed journal [3]. Thus,
oversceptism is not always good either. There are rumored to be thousands of so called crank proofs of the $P$ versus $NP$ problem. This is very much an overstatement. The list $P$ versus $NP$ page kept by G. J. Woeginger at www.win.tur.nl lists 59 attempts by 3. August 2010. In September 2008, the time the present proof was put to arxiv, there were 44 attempts. It would not be a major effort for the mathematical community to check all of these 59 proofs because many are not serious and most have been or can be easily shown incorrect. The presented proof was over 15 months in a peer-reviewed journal and in four referee statements only a few very minor mistakes of the type of misprints were found. However, the editor did not see the possibilities of finding a referee who would read the manuscript to the end. This was probably because the manuscript was not well structured and the proof was not broken into small lemmas. In this new version the logical structure of the proof is improved while no new arguments have been added. As no errors were found in the reviews, there are no corrections in the proof. Much discussion is added to address the issues where the previous referee thought there was some unclarity. These discussion parts and remarks make the article longer but they should not be removed before the method of the proof is correctly understood. The main points of unclarity that a previous referee stated were two. There was a question if Section 3 makes assumptions on the way the algorithm works and the other one was if a lower bound proof of Section 4 can be made without specifying a generic model of computation. The added discussion hopefully shows that these issues are correctly treated in the proof.

Before going to the proof let us look at a simple algorithm that demonstrates that the problem in finding a polynomial time algorithm to the subset sum problem is caused by the sum $j_n$ in (1.2) having a bound that grows faster than any polynomial, as indicated in (1.1). If the upper bound of $j$ grows polynomially there exists a polynomial time algorithm to the subset sum problem. The algorithm in Lemma 2 is very slow but it runs in polynomial
time. It calculates an exponentially growing number of combinations of $c_k$ in the same polynomial time run. The algorithm in Lemma 2 is not a practical competitor to existing algorithms for solving the subset sum problem. Effective algorithms exist for many, or maybe even for almost all, cases of the subset sum problem. Effective algorithms usually do not compute an exponential number of combinations of values $c_k$ at the same run. They limit the search to some subtrees.

**Lemma 1.** Let $B \geq 1$, $\alpha \geq 0$ and $\gamma \geq 0$ be selected. Let $r_n > 0$ and $j_n$ be integers satisfying

$$r_n < n^\gamma, \quad \log_2 j_n < B n^\alpha \quad (n \geq 1).$$

There exist numbers $C, \beta \in \mathbb{R}, C \geq 1$, $\beta \geq 0$ and an algorithm that given any sequence of knapsacks

$$\left( (j_n, (d_{1,n}, \ldots, d_{n,n})) \right)_{n \geq 1}$$

can determine for each $n$ if there exist binary numbers $c_{k,n}$, $1 \leq k \leq n$, such that

$$j_n \equiv \sum_{k=1}^{n} c_{k,n} d_{k,n} \pmod{r_n}. \quad (2.1)$$

The number $N_n$ of elementary operations needed by the algorithm satisfies $N_n < C n^\beta$ for every $n > 1$.

**Proof.** The bound on the logarithm of $j_n$ guarantees that modular arithmetic operations on $d_{k,n}$ can be made in polynomial time since we can assume that $d_{k,n} \leq j_n$. We can find the numbers $c_{k,n}$ by computing numbers $s_{k,j,n}$ from the recursion equations for $k$

$$s_{k,j,n} = s_{k-1,j,n} + s_{k-1,(j-d_{k,n}) \pmod{r_n}, n} \quad (2.2)$$

where the index $j$ ranges from 0 to $r_n - 1$ and is calculated modulo $r_n$. The index $n$ is fixed and only indicates that the numbers are for the $n^{th}$ knapsack. Here $\delta_x$ is an indicator function: $\delta_x = 1$ if
the statement $x$ ( i.e., $j$ equals 0 in (2.2) ) is true and $\delta_x = 0$ if $x$ is false. Let

$$G_{k,n}(x) = \sum_{j=0}^{r_n-1} s_{k,j,n} x^j,$$

where $|x| < 1$. From (2.2) follows

$$\sum_{j=0}^{r_n-1} s_{k,j,n} x^j = \sum_{j=0}^{r_n-1} s_{k-1,j,n} x^j + \sum_{j=0}^{r_n-1} s_{k-1,(j-d_{k,n})(\text{mod } r_n),n} x^j.$$

Changing summation to $j' = j - d_{k,n}$ yields

$$G_{k,n}(x) = G_{k-1,n}(x) + \sum_{j'=-d_{k,n}}^{r_n-1-d_{k,n}} s_{k-1,j' \text{(mod } r_n),n} x^{j'+d_{k,n}}.$$

Changing the order of summation of $j'$ shows that

$$G_{k,n}(x) = G_{k-1,n}(x) + x^{d_{k,n}} \sum_{j'=0}^{r_n-1} s_{k-1,j',n} x^{j'}.$$ (2.3)

Simplifying (2.3) gives

$$G_{k,n}(x) = G_{k-1,n}(x) + x^{d_{k,n}} G_{k-1,n}(x).$$

As $G_{0,n}(x) = s_{0,0,n} = 1$, we get

$$G_{n,n}(x) = \prod_{k=1}^{n} (1 + x^{d_{k,n}}).$$

Expanding the product shows that $s_{k,j,n} \neq 0$ if and only if there exist binary numbers $c_m$, $c_m \in \{0, 1\}$, $1 \leq m \leq n$, satisfying

$$j \equiv \sum_{m=1}^{n} c_m d_{m,n} \pmod{r_n}.$$ (mod $r_n$).

For $j = j_n$ and $k = n$ we get the knapsack problem. This means that we can solve the knapsack problem by computing all $s_{k,j,n}$ form (2.2). We do not actually need the numbers $s_{k,j,n}$ but only the information if $s_{k,j,n} \neq 0$. Therefore we will not compute the terms $s_{k,j,n}$ directly but calculate binary numbers $b_{j,k} \in \{0, 1\}$ by Algorithm A0 below. The number $b_{k,j}$ calculated by A0 is zero if and only if the number $s_{k,j,n} = 0$ is zero.
Algorithm A0:

Loop from $k = 0$ to $k = n$ with the step $k := k + 1$ do {
  Loop from $j = 0$ to $j = r_n - 1$ with the step $j := j + 1$ do
  $b_{j,k} := 0$
}

$b_{0,0} := 1$

Loop from $k = 1$ to $k = n$ with the step $k := k + 1$ do {
  $M := \min\{r_n - 1, \sum_{m=1}^{k} d_{m,n}\}$
  Loop from $j = 0$ to $j = M$ with the step $j := j + 1$ do {
    If $(b_{k-1,j} = 0$ and $b_{k-1,(j-d_{k,n})(\text{mod } r_n)} = 0)$ do $b_{j,k} := 0$
    else do $b_{j,k} := 1$
  }
}

If $b_{n,j_n} = 1$ do $\text{result} := \text{TRUE}$ else do $\text{result} := \text{FALSE}$

Algorithm A0 loops from $k = 0$ to $k = n$ and from $j = 0$ to $j = r_n - 1 < n$. Thus A0 needs a polynomial number of elementary operations as a function of $n$ in order to give the result TRUE or FALSE to the existence of a solution to (2.1).

**Lemma 2.** Let $B, \alpha \in \mathbb{R}, B \geq 1, \alpha \geq 0$ be fixed. There exist numbers $C, \beta \in \mathbb{R}, C \geq 1, \beta \geq 0$ and an algorithm that for any sequence

$((j_n, (d_{1,n}, \ldots, d_{n,n})))_{n \geq 1}$

of knapsacks satisfying

$j_n \leq Bn^\alpha, \quad d_{k,n} \leq j_n \quad (1 \leq k \leq n), \quad (1 \leq n \leq n).$
can determine if there exist binary numbers $c_{k,n}$, $1 \leq k \leq n$, such that

$$j_n = \sum_{k=1}^{n} c_{k,n} d_{k,n}.$$

The number $N_n$ of elementary operations needed by the algorithm satisfies $N_n < C n^\beta$ for every $n > 1$.

**Proof:** The result follows directly from Lemma 1 by selecting $r_n = \sum_{k=1}^{n} d_{k,n} \leq n j_n$.

**Remark 3.** Lemma 2 solves all possible values of $j_n < B n^\alpha$ with the same polynomial time run of Algorithm A0 because $j_n$ is not used in A0 before checking the final result $b_{n,k}$. Let us consider the case when $j_n$ is not limited from above by a polynomial of $n$. Lemma 1 runs in polynomial time even if the upper bound for $j_n$ grows faster than a polynomial of $n$ but it does not produce results that can tell if there exists a solution for a particular value $j_n$. A polynomial-time test, such as taking a modulus in (2.1), maps the superpolynomial set of possible values of $j = \sum_{k=1}^{n} c_{k,n} d_{k,n}$ into a polynomial number of classes. In (2.1) the classes are all sums $j$ with the same moduli by $r_n$. At least one such a class corresponds to an superpolynomial number of values $j$. In order to check if any value $j$ in the class equals $j_n$, the algorithm should in some way check all of the values $j$ in the class but if the algorithm at the same run checks all values of $j$ then it apparently should loop over a superpolynomial set which is not possible for a polynomial time algorithm. In general, we can say that a single polynomial time run of an algorithm cannot solve all values of $j_n$ that are below a superpolynomial upper bound because the algorithm can only produce a polynomial number of results and there exist a superpolynomial number of possible values $j_n$. A polynomial time algorithm that solves the subset sum problem for any value $j_n$ below a superpolynomial upper bound must limit search and there must be values $j_n$ that are solved with different runs of the algorithm.
Remark 4. An algorithm is a finite set of rules that at every step tell what to do next. We can implement an algorithm as a computer program in a second generation language on a von Neumann machine and a polynomial time algorithm can be implemented in this way so that it requires time and memory that grow polynomially with respect to the problem dimension. In the case when the smallest upper bound of \( j_n \) in Remark 3 grows exponentially a program in a second generation computer language implementing a polynomial time algorithm needs to limit search by branching instructions. Thus, we can find values of \( j_n \) such that the algorithm uses different branches in solving the subset sum problem.

Remark 5. It does not seem possible to select a sequence of specific subset sum problems and to show that no algorithm can solve this specific sequence of problems in polynomial time. This is so because it is seems plausible that we can create an algorithm that treats these specific problems in a particular way and can solve that sequence of problem in a fast way. Instead, we must first select the algorithm and pose that selected algorithm a sequence of subset sum problems that are particularly hard for that specific algorithm. As the algorithm can be any possible algorithm, the sequence of problems can only be defined by using some suitable definition of a difficult problem to the selected algorithm and we cannot give any numerical values for all of the numbers \( c_{k,n} \) in (1.2). We will do the selection by using the following definition of the computation time of a subset sum problem.

For convenience, let us select \( n \) to be of the form \( n = 2^{i+2} \) for some \( i > 0 \). This simplifies expressions since it is not necessary to truncate numbers to integers.

Definition 2.

We define a function \( f(n) \) that describes (in a certain sense) the
worst computation time for a selected algorithm.

Let the worst in the median \( n \)-tuple as be as follows. Let

\[ h(d_{1,n}, \ldots, d_{n,n}, j_n) \]

be the computation time for deciding if the knapsack

\[ (j_n, (d_{1,n}, \ldots, d_{n,n})) \]

has a solution or not. Let

\[ \text{Median}_{j_n} h(d_{1,n}, \ldots, d_{n,n}, j_n) \] (2.4)

be the median computation time where \( j_n \) ranges over numbers

\[ j_n \in \{C + 1, \ldots, 2^{n+1} - 1\} \] (2.5)

satisfying the two conditions

\[ j_{n,t} = j_n - C \left\lfloor \frac{j_n}{C} \right\rfloor > 2^{\frac{n}{2}+2} \] (2.6)

where \( C = 2^{\frac{n}{2}+1} \), and that there is no solution to the knapsack \((j_n, (d_{1,n}, \ldots, d_{n,n}))\). The values of \( j_n \) are computed separately in calculation of the median, i.e., no partial results from previously computed values of \( j_n \) are used.

Let \((d_{1,n}, \ldots, d_{n,n})\) range over all knapsack sequences with

\[ \lceil \log_2 \sum_{k=1}^{n} d_{k,n} \rceil = n \]

and \( d_{k,n} \leq 2^{\frac{n}{2}-1} \). Because of this requirement at most every second value of \( j_n \) in (2.5) is a solution to the knapsack, i.e., there are \( 2^n \) combinations of \((c_{1,n}, \ldots, c_{n,n})\) mapped to numbers from zero to \( 2^{n+1} - 1 \). The worst in the median tuple for \( n \) is an \( n \)-tuple \((d_{1,n}, \ldots, d_{n,n})\) (possibly not unique) that maximizes the median computation time (2.4).

Let this maximal median computation time be denoted by \( f(n) \).
Thus

$$f(n) = \max_{d_1,n,\ldots,d_{n,n}} \text{Median}_{j_n} h(d_1,n,\ldots,d_{n,n},j_n).$$

(2.7)

**Lemma 3.** Let $m$ be fixed and $n$ be a power of $m$. If $f(n)$ satisfies the inequality

$$\frac{n}{m} f\left(\frac{n}{m}\right) < f(n)$$

(2.8)

then $f(n)$ does not grow polynomially with $n$.

**Proof:** Iterating we get

$$\frac{n}{m} f\left(\frac{n}{m^2}\right) < f(n)$$

and iterating up to $k$ yields

$$\frac{n^k}{m^{\sum_{i=1}^k i}} f\left(\frac{n}{m^k}\right) < f(n)$$

i.e.,

$$e^{k \ln n - \frac{1}{2} k^2 \ln m - \frac{k}{2} \ln m} f\left(\frac{n}{m^k}\right) < f(n).$$

Setting $k = \frac{\ln n}{\ln m}$ gives

$$\left(n^{\ln n}\right)^{\frac{1}{2 \ln m}} n^{-\frac{1}{2}} f(1) < f(n).$$

If $m$ is any fixed number we see that $f(n)$ satisfying (2.8) is not bounded by a polynomial function of $n$.

**Lemma 4.** Let $n$ be a power of 2. If $f(n) = f_1(n) + f_2(n)$ where $f_1(n)$ is a polynomial function of $n$ and $f_2(n)$ satisfies the inequality

$$\frac{n}{2} f_2\left(\frac{n}{2}\right) < f_2(n)$$

(2.9)

then $f(n)$ does not grow polynomially with $n$.

**Proof:** If $f(n)$ is a polynomial function of $n$ and since $f_1(n)$ is a polynomial function of $n$ by assumption, it follows that $f_2(n)$
must also be a polynomial function of \( n \). By Lemma 3, \( f_2(n) \) is not a polynomial function of \( n \), thus neither is \( f(n) \).

Remark 6. We use the median in Definition 2 instead of the worst case or the worst in the average case because we need \( \frac{n}{2} \) almost as long computations as the worst in (2.9). In the worst and in the worst in the average, a very slow computation of one value \( j_n \) can be the reason for the long computation time. We include only unsuccessful cases of \( j_n \) in the computation of the median because then there is no need to argue that the median over unsuccessful \( j_n \) is at least as high as the median over all \( j_n \).

3 Construction of a special subset sum problem

We will use the denotation \( n_1 = \frac{n}{2} \) throughout this article for brevity. In this section we will define a special subset sum problem \( K_{1,j_n} \) in Definition 3 and show that it can only be solved by solving \( n_1 \) subknapsacks \( (j_i', (d_{1,n}, \ldots, d_{n_1,n})) \) with different values of \( j_i' \).

**Definition 3. Construction of \( K_{1,j_n} \).** We first make a knapsack where the only solutions must satisfy the condition that exactly one \( c_k \) must be 1 and the others must be zero for \( k = n_1 + 1 \) to \( k = n \). Let us construct the values \( d_{k,n}, k = n_1 + 1, \ldots, n \) of \( K_{1,j_n} \) for a given \( j_n \). Let \( C = 2^{\frac{n}{2}+1} \) and

\[
\begin{align*}
    j_{n,h} &= C \left\lfloor \frac{j_n}{C} \right\rfloor, \quad j_{n,l} = j_n - j_{n,h} \\
\end{align*}
\]

be the high and low bit parts of \( j_n \). Because of (2.5), \( j_{n,h} \neq 0 \). Let

\[
    d_{n_1+k,n} = j_{n,h} + a_k
\]

where \( 0 < a_k < \min\{j_{n,l}, \frac{2^{n_1}-1}{n_1}\} \) are distinct integers and there exists no solution to the knapsack problem for the knapsack

\[
    (j_i', (d_{1,n}, \ldots, d_{n_1,n}))
\]

where

\[
    j_i' = j_{n,l} - a_i.
\]
Let us also require that the computation time for $j'_i$ is at least as long as the median computation time $f(n_1)$ for $(j, (d_{1,n}, \ldots, d_{n_1,n}))$. We can select $j'_i$ filling this condition because half of the values $j$ are above the median. Notice that we compute the median only over values $j$ that do not give a solution to the knapsack. We will also assume that the $j'_i$ are in the set corresponding to (2.5)-(2.6) for $f(n_1)$, i.e.,

$$j'_i \in \{C' + 1, \ldots, 2^{n_1+1} - 1\} \quad (3.4)$$

satisfying the condition

$$j'_i - C' \left\lfloor \frac{j'_i}{C'} \right\rfloor > 2^{n_2+2} \quad (3.5)$$

where $C' = 2^{n_1+1}$. We may assume so because there are enough values from which to choose $j'_i$.

**Remark 7.** On (3.2) we select the numbers $a_k$ in such a way that the $d_{n_1+k,n}$ satisfy the size condition $d_{n_1+k,n} \leq \frac{2^{n_1+1}}{n}$. Because of the bound (2.6) we have an exponential number of choices for $a_i$. It is possible to find numbers $j'_i$ such that there is no solution since only for about half of the values of $j$ there exists a solution for $(j, (d_{1,n}, \ldots, d_{n_1,n}))$. If $j_{n,l}$ is too small and we cannot find values $j'_i$, we take a carry from $j_{n,h}$ in (3.3) and reselect $a_k$. Because of the lower bound on $j$ in (2.5), $j_{n,h}$ is not zero and we can take the carry. Then $j_{n,h}$ is decreased by the carry.

**Remark 8.** Exactly one $c_k$ must be 1 and the others must be zero for $k = n_1 + 1$ to $k = n$. There cannot be more values $c_k = 1$ for $k > n_1$ because then the higher bits of $j_n$ are not matched. The unknown algorithm can try also other combinations but these are the only possible combinations and the algorithm must also try them (i.e., check these cases in some way unknown to us). The sum of the numbers $d_{k,n}$, $k \leq \frac{n}{2}$ is less than $2^{n_2+1} - 1$. Adding one $c_k$ can give a carry and there may not be a solution to the knapsack because the high bits of $j_n$ do not match but this is not an issue since we do not want solutions. We select the $n$-tuple so that there are no solutions to the knapsack already because the
lower bits do not match.

**Lemma 5.** The algorithm cannot stop to finding a solution because for every $j_n$ none of the $\frac{n}{2}$ values of $j'_i$ solve the knapsack problem. Every value $j'_i$ gives at least as long computation as the median computation time $f(n_1)$.

**Proof:** We have selected $K_{1,j_n}$ such that $(j'_i, (d_{1,n}, \ldots, d_{n_1,n}))$ has no solution for any $j'_i$. Thus the algorithm cannot stop because it finds a solution. By construction the values $j'_i$ give at least as long computation time as the median for the tuple at $k = 1, \ldots, n_1$. Since that tuple is the worst in the median tuple for $n_1$, the computation time for each $j'_i$ is at least $f(n_1)$.

**Lemma 6.** There is no way to discard any values $j'_i$ without checking if they solve the subknapsack from $k = 1$ to $k = n_1$. Any case of using the values of $d_{k,n}$ in order to get the result is considered checking.

**Proof:** We can select any $a_k$ in such a way that there either exists a solution or does not exist. Knowledge from other $c_{i,n}$ ($i \neq n_1+k$) cannot give any information on how this $a_k$ was selected. Thus, the existence of a solution must be checked using the value $d_{n_1+k,n}$.

**Lemma 7.** Several values of $j'_i$ cannot be evaluated on the same run. The median computation time of $K_{1,j_n}$ is at least

$$f_1(n_1) + n_1f_2(n_1)$$

where $f(n) = f_1(n) + f_2(n)$ and $f_1(n)$ is a polynomial function of $n$.

**Proof:** As explained in Remark 3, a polynomial time algorithm cannot solve all values of $j'_i$ at the same run because it would require an exponential amount of memory. As explained in Remark 4, we can assume that the algorithm is implemented in a
second generation computer language on a von Neumann machine and its code has branching instructions. These branching instructions define a branching tree describing the execution of the algorithm for any input data. The tree is fixed when the algorithm is selected. At each branching point the input data is divided into a finite number of classes. Because this division is fixed, we can always find two values \( j'_i \) which are not executed by the same polynomial time run. After finding two, we can continue to find three values \( j'_i \) which all are executed by different polynomial time runs of the algorithm. This can be extended to \( \frac{n}{2} \) values \( j'_i \). Especially, it is not possible for such an implementation of an algorithm to have the property that for any selection of \( \frac{n}{2} \) values \( j'_i \) there always exists a polynomial time run such that the subset sum problem for every value \( j'_i \) is computed in the same run. Instead, we can select \( j'_i \) in such a way that no two values \( j'_i \) are computed in the same run. The runs in Lemma 6 do not need to be completely separate but they can have parts that are shared, as long as the shared parts are computed in polynomial time. This is necessarily the case, the runs must share at least the beginning of the code before branch instructions are reached. The shared part can be described by a polynomial function \( f_1(n) \) and the computation time to solve all these problems is as in in Lemma 4.

**Discussion of the method of Section 3**

We explain why the method of Section 3 does not make assumptions on the way the algorithm solves the subset sum problem. The method is based on the following logical division:

C1) There can be a case that a subtree of possible solutions can be discarded without checking the subproblems in the subtree.

C2) If a subtree cannot be discarded, all its subproblems must be checked, but several subproblems may be checked by the same run of the algorithm.

C3) If a subtree cannot be discarded, all its subproblems must
be checked. If several cannot be checked by the same run of the algorithm then they must be checked separately.

There is an analogy between checking a proof by a proof checking algorithm and checking a subset sum problem by a subset sum problem checking algorithm. In both cases there is an outcome that stops the search (finding an error, finding a solution) and an outcome that does not stop the search (not finding an error, not finding a solution). Let a proof have \( n_1 \) lemmas and the main theorem be that all lemmas are correct. If the proof checking algorithm concludes that the main theorem does not have an error, it must have checked all lemmas. If it finds an error, it does not need to check all lemmas as it stops to an error.

The proof checking algorithm can discard a set of lemmas without checking if it can reason that this set is impossible. This corresponds to case (C1). We assume that the main theorem is correct, i.e., no lemmas have errors. As there are no errors the theorem checking algorithm cannot discard lemmas and it must check them, i.e., verify if they have an error or not. In the presented proof of the subset sum we construct subset sum problems that do not have any solutions, corresponding to the case of a theorem with no errors.

This checking does not need to be done by reading each lemma in any particular order, nor do we assume any such order. Several lemmas may be checked at the same time by some argument, as in case (C2). The proof that the first lemma is correct can e.g. be valid for all lemmas satisfying some properties and in this case the proof can verify a set of lemmas. Let us assume that we know the finite code of the proof checking algorithm and select the lemmas depending on the algorithm. In the case of lemmas that can be over anything that is possible to imagine it is easy to believe that we indeed can select a set of \( n_1 \) unrelated lemmas that must be checked with a different run each by this special previously selected algorithm. In the case of the subset sum problem we show that this is the case in Lemma 7. Assuming that we got
$n_1$ lemmas such that each must be solved by a different run by the selected algorithm, the running times for the lemmas are essentially additive - there may be some common parts in the runs but the running times add as in Lemma 4, which contains an additive term given by a polynomial function $f_1(n_1)$ to account for the shared parts of the runs.

This method does not make any assumptions of the algorithm. It needs the following properties of an algorithm: the algorithm solves the problem correctly and it has a finite set of instructions. The algorithm runs in polynomial time. It needs to be deterministic because with a subroutine capable of guessing correctly the algorithm could check all lemmas on the same run. A non-deterministic algorithm could also check explicitly more than a polynomial number of lemmas in the same run by guessing which need to be checked. Additionally, it is needed that the space of possible lemmas has a superpolynomial size, so that we can find the $n_1$ lemmas. This is a demand on the problem: not all problems have this property. The method does not restrict the proof checking algorithm to read the lemmas as they are written. It can work any way. This logical division has nothing to do with any special way of computing subset sums. The logical division can be applied to any similar problem and it does not make restrictive assumptions of the algorithm.

Let us look at some concrete examples explaining why it is possible to construct a subset sum problem containing parts that must be all checked.

Let $j_n = 21 + 2^{10}$ and let the $n$-tuple be

$$(d_{1,n}, \ldots, d_{n_1,n}, 9 + 2^{10}, 7 + 2^{10}, \ldots).$$

We only look at the two subproblems: $j'_1 = 21 - a_1 = 21 - 9 = 12$, $j'_2 = 21 - a_2 = 21 - 7 = 14$. We assume that the carry from the sum

$$
\sum_{k=1}^{n_1} d_{k,n} + \sum_{k=n_1+1}^{n} a_k
$$
cannot reach $2^{10}$ and therefore exactly one $c_{k,n} = 1$ when $k > n_1$. The other $c_{k,n} = 0$ for $k > n_1$.

**Example 1.** We ask do we need to solve the problems

$$12 = \sum_{k=1}^{n_1} c_{k,n} d_{k,n}$$

$$14 = \sum_{k=1}^{n_1} c_{k,n} d_{k,n}$$

before concluding that

$$20 + 2^{10} = \sum_{k=1}^{n} c_{k,n} d_{k,n}$$

does not have a solution.

Let us assume that all $d_{k,n}$, $k = 1, \ldots, n_1$, are even. We can use a divisibility condition and conclude that 20 cannot be made from a subset sum of even $d_{k,n}$ and one odd number. We do not need to check if 12 and 14 can be made as a subset sum of the smaller knapsack up to $n_1$, while they probably can. This is case (C1) in the logical division: we can discard a whole subtree without checking the subproblems. We do not have this possibility in Lemma 6: $a_i$ is the lower bit part of $d_{n_1+i,n}$. This is why we cannot discard the subproblems without checking. They are all possible. So, we conclude that the subproblems in Lemma 6 must be checked. They may be checked at the same time by the same run of the algorithm and we must go to (C2).

**Example 2.** Let us take another example. Let $j'_1 = 21 - a_1 = 21 - 8 = 13$, $j'_2 = 21 - a_2 = 21 - 6 = 15$. Let us assume that all $d_{k,n}$, $k = 1, \ldots, n_1$, are even. There are two subproblems:

$$13 = \sum_{k=1}^{n_1} c_{k,n} d_{k,n}$$

$$15 = \sum_{k=1}^{n_1} c_{k,n} d_{k,n}.$$
We ask do we have to check these subproblems before concluding that
\[ 21 + 2^{10} = \sum_{k=1}^{n} c_{k,n}d_{k,n} \]
does not have a solution.

We must check both of these subproblems since they are possible, but we can check both of them at the same time by computing \( j \mod 2 = 1 \) and noticing that there cannot be a solution to
\[ 21 + 2^{10} = \sum_{k=1}^{n} c_{k,n}d_{k,n} \]
This calculation checks both of the subproblems. That is, we know that
\[ 13 = \sum_{k=1}^{n_1} c_{k,n}d_{k,n} \]
\[ 15 = \sum_{k=1}^{n_1} c_{k,n}d_{k,n} \]
are also impossible.

In the case (C2), when the subproblems are solved several on the same run, the algorithm does not look at each subproblem separately and compute subset sums for them. Implicitly it solves all subproblems, i.e., the algorithm has proven that no subproblem has a solution unlike in Example 1 where the algorithm did not solve the subproblems at all.

We are checking the subproblems by noticing that \( j_n \) belongs to an infinite set of odd numbers. The set of odd numbers that are smaller than an exponential upper bound is exponential. Thus, it is possible to check an exponential number of problems at the same time. The argument in the proof is not that the set of \( j'_i \) has an exponential size, but that it has an exponential range: it is going through all numbers up to some exponential upper bound. In this range we have also the even numbers. The numbers on a certain range do not have common properties, such as divisability by some number.
Let us see how Lemma 7 shows that (C2) cannot happen. It is because we can select $a_i$ freely from an exponential set ranging over all numbers with some exponential upper bound. If for each choice of the $n_1$ values $a_i$ the algorithm can check the subset sums on the same run, then the algorithm checks an exponential number of values $j'_i$ on the same run, where $j'_i$ can range over all numbers with an exponential upper bound and some lower bound. This is not possible for a polynomial-time algorithm.

**Example 3.** Let $j'_1 = 21 - a_1 = 21 - 9 = 12$, $j'_2 = 21 - a_2 = 21 - 6 = 15$. In this case all $d_{k,n}$, $k = 1, \ldots, n$, cannot be even. There are two subproblems:

\[
12 = \sum_{k=1}^{n_1} c_{k,n}d_{k,n}
\]

\[
15 = \sum_{k=1}^{n_1} c_{k,n}d_{k,n}.
\]

Again the algorithm must check both subproblems before concluding that

\[
21 + 2^{10} = \sum_{k=1}^{n} c_{k,n}d_{k,n}
\]

does not have a solution. The second subproblem can be checked by divisibility considerations, while the first must be checked in some other way, probably on another run of the algorithm. This the case (C3). As we can select the $a_i$, we can select such subproblems that they cannot be checked on the same run. It depends on the algorithm what problems must be solved in different runs and therefore we must look at the algorithm before selecting the $a_k$.

### 4 On the inequality (2.9)

In this section we try to establish the inequality (2.9) for any algorithm solving the knapsack problem. Let the algorithm be chosen. We selected a tuple $K_{1,j_n}$ for every $j_n$ and showed in
Lemma 7 that the median computation time for the set of $K_{1,j_n}$ is at least as high as the left hand side of (2.9). However, the set of $K_{1,j_n}$ depends on $j_n$ and we should get a constant $n$-tuple that can be compared to the worst in the median knapsack with the computation time $f(n)$.

Let this constant $n$-tuple be called $K_2$. $K_2$ has at most as long median computation time as the worst in the median tuple. What we have to show is that the set of $K_{1,j_n}$ is not harder to solve for the algorithm than $K_2$. In order to show this, we first define another set of $n$-tuples $K_{3,j_n}$ and show that $K_{1,j_n}$ is not harder than $K_{3,j_n}$, and then try to show that $K_{3,j_n}$ is not harder than $K_2$.

**Definition 4. Construction of $K_{3,j_n}$.** Let $j_n$ be given and let us define a $n$-tuple $K_{3,j_n}$ by specifying the elements

\[ d_{2,k} = d_{k,n} \quad (k = 1, \ldots, \frac{n}{2}) \]

\[ d_{2,k} = e_1 \quad (k = \frac{n}{2} + 1, \ldots, \frac{3n}{4}) \] \hspace{1cm} (4.1)

\[ d_{2,k} = e_2 \quad (k = \frac{3n}{4} + 1, \ldots, n - 1) \]

\[ d_{2,n} = j_{n,h}. \]

We select two nonnegative integers $e_i \leq \frac{2^{n_1-1}}{n_1}, i = 1, 2$. The selected $e_1$ and $e_2$ are so small that if $c_n = 0$ the higher bits of $j_n$ are not matched because there is no carry.

**Remark 9.** This $n$-tuple has a simple upper half tuple. The sum of the numbers $d_{k,n}$, $k \leq \frac{n}{2}$ is less than $2^{rac{n}{2}+1} - 1$. It is always necessary to set $c_n = 1$ and this satisfies the upper half bits of $j_n$.

**Definition 5. Construction of $K_2$.** Let us define $K_2$ as an $n$-tuple with elements $(d_{0,1}, \ldots, d_{0,n})$ where each $d_{0,k} \leq \frac{2^{n_1-1}}{n_1}$ and let $n$-tuple $(d_{1,n}, \ldots, d_{2,n})$ be the worst in the median tuple for $\frac{n}{2}$. We define

\[ d'_k = C d_{0,k} + d_{k,n} \] \hspace{1cm} (4.2)
as the values of the elements of $K_2$ for $k = 1, \ldots, n_1$. The numbers $e_1$ and $e_2$ are as in $K_{3,j_n}$ and we define the elements of $K_2$ for $k = n_1 + 1$ to $k = n$ as

$$d'_k = Cd_{0,k} + e_1 \quad (k = \frac{n}{2} + 1, \ldots, \frac{3n}{4})$$

$$d'_k = Cd_{0,k} + e_2 \quad (k = \frac{3n}{4} + 1, \ldots, n - 1) \quad (4.3)$$

$$d'_n = Cd_{0,n}.$$ 

Thus, $K_2$ has the same lower half tuple elements as $K_{3,j_n}$.

**Remark 10.** In $K_{3,j_n}$ our chosen algorithm fast finds a solution and stops. The tuple $K_2$ can be split into two $n$-tuples: the lower half tuple with elements smaller than $C$ and the upper half tuple that has the higher bit parts. In $K_2$ the algorithm usually does not stop to a solution of the lower half tuple since the upper half tuple is usually not satisfied by $c_k$ that satisfy the lower half knapsack. We construct another algorithm $A2$ that is similar to the chosen algorithm $A1$ but such that it does not stop when it finds a solution. Otherwise it is similar to the chosen algorithm $A1$. Thus, for $j_n$ that yields no solution to $K_2$ the new algorithm $A2$ works in the same way as the chosen algorithm. The cases when $K_2$ has a solution are not computed to the median time of deciding if $K_2$ has a solution since we count the median over unsuccessful cases of $A1$ only. We compute the median time for $A2$ solving $K_{3,j_n}$ so that the median is taken over values of $j_n$ for which $K_2$ does not have solution.

**Lemma 8.** *It is at most as fast to compute $K_{1,j_n}$ than to compute $K_{3,j_n}$."

*Proof:* We compare the algorithm $A2$ solving the set of $K_{3,j_n}$ and the algorithm $A1$ solving the set of $K_{1,j_n}$. In $K_{3,j_n}$ the indices $k > n_1$ yield $\frac{(n+4)n}{16}$ values of $j$ that could be made as subset sums from the the worst in the median $n_1$-tuple in the indices $k \leq n$. As we can select $e_1$ and $e_2$ from an exponential set of numbers, we may assume that the numbers $j$ are sufficiently well randomly
distributed over the possible range of the numbers \( j \). Therefore over half of the values \( j \) are likely to be on the range \((3.4)\) and about half of the values of \( j \) that do not give a solution in \( K2 \) are likely to yield a longer computation time than \( f(n_1) \). We can select the values \( j_i' \) in \( K_{1,j_n} \) to be the values that give the smallest possible computation time larger or equal to \( f(n_1) \). The time to compute the sums of about \( \frac{1}{8} \frac{(n+4)n}{16} \) values of \( j \) for \( K_{3,j_n} \) that are above \( f(n_1) \) is larger than the time to compute \( \frac{n}{2} \) values of \( j_i' \) for \( K_{1,j_n} \) that are slightly above \( f(n_1) \). Most of the numbers \( j \) in \( K_{3,j_n} \) will also require different runs of the algorithm as one run can only give an answer to a polynomial number of values \( j \) and the values \( j \) in \( K_{3,j_n} \) take values sufficiently randomly over an exponential range.

**Remark 11.** The median computation time in (2.4) is calculated over the no instances only. Thus, yes instances are ignored. It is sufficient that there are at least some no instances so that (2.4) can be calculated. We give an argument that estimates the number of solutions to the knapsack problem \( (j_n, K_2) \). The argument makes use of averages but it is quite sufficient for showing that there are some no instances for computation of (2.4) if the upper bits of \( K_2 \) are selected in a suitable way, indeed a random selection of these bits is likely to yield many no instances. Let us mention that Lemmas 9-13 are not used in the proof of \( P \neq NP \) and the probabilistic nature of the proofs of these lemmas has no relevance to Theorems 1 and 2.

**Lemma 9.** There are in average \( 2^{\frac{n}{2}} \) solutions possible choises of \((c_1, \ldots, c_n)\) that give the same sum \( \sum_{k=1}^{n} c_k d_{o,k} \).

**Proof:** The number of combinations of \( c_k \) is \( 2^n \) and the sum \( \sum_{k=1}^{n} d_{o,k} \) is at most \( 2^{\frac{n}{2}} \). There are fewer combinations that yield very small or large sums and most sums are in the middle ranges.

**Lemma 11.** We can select the numbers \( d_{o,k} \) in such a way that there are in average about \( 2^{\frac{n}{2}} \) solutions possible choises of
that give the same sum $\sum_{k=1}^{n_1} c_k d_{o,k}$.

**Proof.** Most random selections of the numbers $d_{o,k}$ give this result. There are fewer combinations that yield very small or large sums and most sums are in the middle ranges.

**Lemma 12.** The lower half tuple in the indices $k = n_1 + 1, \ldots, n$ has only $\frac{n+4}{4}$ possible values $j$.

**Proof.** These numbers are

$$j = \sum_{k=n_1+1}^{n} c_k (d'_k - Cd_{0,k}) = k_1 e_1 + k_2 e_2 \quad (4.4)$$

where $0 \leq k_1 \leq \frac{n}{4}$ and $0 \leq k_2 \leq \frac{n}{4} - 1$.

**Remark 12.** The elements in the worst in the median tuple for $n_1$ satisfy $d_{k,n} \leq \frac{2^{n_1}-1}{n_1}$ because we only maximize over such elements. Also $e_i \leq \frac{2^{n_1}-1}{n_1}$. Thus, there is no carry from the lower half tuple to the upper half tuple.

**Lemma 13.** It is possible to compute the median (2.4) for $K_2$.

**Proof.** Let us assume that the values $c_k$ are fixed for the indices $k > n_1 + 1$. This fixes some value $j$ that must be obtained from the knapsack in the indices $k = 1, \ldots, n_1$ as the subset sum. By Lemma 12 there are only $\frac{n+4}{4}$ possible values $j$. The upper half tuple yields about $2^\frac{n}{4}$ possible solutions for a given $j$ in the indices $k = 1, \ldots, n_1$ by Lemma 11. The worst in the median tuple in the lower half tuple has $\frac{n}{2}$ elements, thus $2^\frac{n}{2}$ possible numbers can be constructed as sums $\sum_{k=1}^{n_1} c_k d'_k$ in the lower half tuple. The set of the about $2^\frac{n}{4}$ possible solutions of the upper half tuple for a randomly selected $j$ is a small subset of all possible combinations of $c_k$ in the lower half tuple in the indices $k = 1, \ldots, n_1$. The probability that any of the possible solutions from the upper half tuple is a solution of the lower half tuple is only on the range of $\frac{(n+4)n}{16} 2^{-\frac{n}{4}}$. The events of selecting the upper half tuple, the
lower half tuple, and the value $j$ can all be considered independent events. There are only a polynomial number of sums (4.4), thus when $j_n$ is selected, there are only a polynomial number of possible values for the lower half of $j$ in $(j, (d'_1, \ldots, d'_{n_1}))$. For a randomly selected $j_n$ there are then only a polynomial number of $c_k, k \leq n_1$, that satisfy the lower half bits of $j_n$. The choice of $c_k, k \leq n_1$, fixes the upper half of $j$. We are left with an upper half knapsack problem for the indices $k = n_1 + 1, \ldots, n$. In this knapsack problem the elements have the size about $2^{n_1}$ and there are $n_1$ elements. Thus, for a randomly selected $j_n$ we expect about one solution. The solution is constrained by the demand that the lower half bits give $j$, i.e., not all combinations are possible. We conclude that we get at least some no instances for computation of (2.4) for some choice of $(d_{0,1}, \ldots, d_{0,n})$.

**Remark 13.** In order to solve the subset sum problem for $K_2$ the algorithm should find a common solution to two knapsacks, i.e., both the upper bits and the lower bits knapsacks in $K_2$ must be solved with the same numbers $(c_1, \ldots, c_n)$. As we may choose any difficult knapsack $(d_{0,1}, \ldots, d_{0,n})$ to the upper bits of $K_2$, it seems to be a much harder problem to solve $K_2$ than to solve $K_{3,j_n}$ where the upper bits are trivial to satisfy. However, it might theoretically be faster to solve $K_2$ since the algorithm does not need to check that there are no solutions to the lower half tuple, only that none of the solutions to the upper half tuple are solutions to the lower half tuple. This is as far as we get in trying to prove the inequality (2.9) directly. We cannot show that there exists such upper half tuples $(d_{0,1}, \ldots, d_{0,n})$ that guarantee that $K_2$ is slower to solve than the set of $K_{3,j_n}$ even though it seems obvious that this claim is true. In Section 5 we make a different argument by considering another algorithm that is faster than the original algorithm.

**Discussion of the method of Section 4.**

Figure 1 shows the main idea of the proof. The set $K_{1,j_n}$ has the worst in the median $n_1$-tuple in the left side and the right side
has numbers from which it is necessary to select exactly one in order to satisfy the high bits of $j_n$. This yields $n_1$ separate subset sum problems and we get the computation time corresponding to the left side of (2.9). The set $K_{3,j_n}$ has only one element which has high order bits and it must always be selected in order to satisfy the high bits of $j_n$. There is the same worst in the median $n_1$-tuple and the remaining $n_1 - 1$ elements can be assigned in any way yielding of the order $n^2$ knapsack problems. Therefore, it is easy to believe that $K_{1,j_n}$ is easier to solve than $K_{3,j_n}$ for almost any $j_n$. The $n$-tuple $K_2$ has some difficult upper half knapsack problem which has to be satisfied with the same values $c_k$ as the lower half knapsack. It certainly seems that $K_2$ is more difficult to solve than $K_{3,j_n}$. Finally, the inequality from $K_2$ to the worst in the median $n$-tuple is obtained directly by the definition of what the worst means.

Fig. 1. The idea of the proof.

We have not specified a generic computational model for the algorithm and are presenting a proof using a lower bound. It is sometimes stated that specifying what computation model is used is
important for a lower bound proof. This may be the case for good lower bounds but a weak lower bound does not require knowing the algorithm. Lower bounds of functions are taken in several branches of mathematics and no computation model in the sense of the theory of computational complexity is usually specified. Thus, a computation model is not necessary for proofs of lower bounds. The presented proof is a simple case of approximating functions as is done in many fields of mathematics. As an example showing that there is no need to have a computational model for the algorithm for deriving a weak lower bound let us consider a hypothetical algorithm that has the property that it always wins a game of solitaire provided that there is a way to win the game. There is a lower bound to the running time of such an algorithm derived directly from the problem it solves. In order to win a game of solitaire, all cards must be turned face up in the end. We know how many cards are face down in the beginning. Thus, the time to flip the cards that are face down is a lower bound to the running time of any algorithm with the stated property. No computational model needs to be specified for this lower bound. The lower bound is the property determined by the problem posed to the algorithm. In the presented proof the situation is similar. We present an unknown algorithm with a carefully selected set of subset sum problems. Any algorithm solving the selected problems must do certain things determined by the problem, and the weak lower bound is a result of these necessary things. A computational model, like the Turing model, describes how an algorithm solves the problem. The proof presented in this article only looks at what the algorithm necessarily must do in order to solve the presented problems, not how it does it. Tracking how an unknown algorithm might do things is difficult and therefore no computational model is used in the proof.
We will construct another algorithm $A_3$ that is at least as fast as the original algorithm $A_1$ and for which the inequality (2.9) can be established. It follows that $A_3$ is not a polynomial-time algorithm. Thus, $A_1$ is not a polynomial-time algorithm either.

We use the following notations. Let $K_a = (d_1, \ldots, d_k)$ and $K_b = (d'_1, \ldots, d'_k)$ be $k$-tuples. Then we write

$$(K_a|K_b) = (d_1, \ldots, d_k, d'_1, \ldots, d'_k)$$

and

$$(K_a + 2^n K_b) = (d_1 + 2^n d'_1, \ldots, d_k + 2^n d'_k).$$

Let $W_n$ be the worst in the median tuple for $n$ (thus, it is an $n$-tuple). Let

$$A_n = (2^n A_n^2 + A_n^2 | W_n^2)$$

define recursively $A_n$ starting from some large $n$ and $A_n$ that the algorithm $A_1$ finds difficult for subset sum problems. Here we let $n$ be divisible by 4. Each number in $W_n$ has bit length less than $n$ and the numbers in $A_n$ have bit lengths less than $n_1 = \frac{n}{2}$.

**Definition 6. The algorithm $A_3$.** Let us construct the new algorithm $A_3$. For each $n$ let us select a fixed $n$-tuple $(d_{0,1}, \ldots, d_{0,n})$ using the recursive definition (5.1) and let $A_3$ work as follows. It replaces $j_n$ by $j_n + 2^n C^{-1} j_{n,h}$ and it replaces each $d_{k,n}$ by $d_{k,n} + 2^n d_{0,k}$. Then $A_3$ either solves the extended knapsack with $A_1$, or solves the extended knapsack by ignoring the bits above $2^{n-1}$ (i.e., solves the original knapsack) with $A_1$, whichever is faster. This means that $A_3$ makes a non-deterministic decision, but it does not need to be a deterministic algorithm. We only want it to be at least as fast as $A_1$. The time taken by extending knapsacks and deciding which way to solve the knapsack is not counted into the median computation time for $A_3$.

**Remark 14.** Because of the Definition 6 the algorithm $A_3$ is never slower than $A_1$. If there is a solution to the original knapsack...
sack, there may not be a solution to the extended knapsack since no solution to the original knapsack needs to satisfy the new high bits. This means that A3 may make errors if there exists a solution for the knapsack problem \((j_n, K_2)\). This error is not important because of the following reason. The algorithm computing the median does not check if there is a solution from the (possibly wrong) answers given by the tested algorithm but it makes an exhaustive search and it does know if there is a solution to the knapsack. The algorithm computing the median is anyway an exponential-time algorithm and there is no need to worry about its slowness. Therefore only such values \(j_n\) that do not have a solution in the original knapsack are counted in the median. The erroneously solved cases are not included. The difference between the original algorithm A1 solving extended knapsacks and the new algorithm A3 solving the original knapsacks is in the search space for \(j_n\). When we compute the median for A3, the number \(j_n\) loops over values where A1 cannot find a solution for the original knapsacks. A3 can search through a smaller space of possible combination candidates by using the extended knapsack and therefore it can be faster than A1 in solving the original knapsack problems.

**Remark 15.** We have defined that A3 solves problems by first expanding the knapsack problems, thus we know something of A3. Lemma 14 is given only for showing how A3 solves knapsacks by extending them. It is not used in the proof of \(P \neq NP\). The conclusion in Lemma 14 is that for A3 it is more difficult to solve \(K_{1,j_n}\) that to solve a set of knapsack problems \((j'_i, W_{n_1})\). However, the sense how this is more difficult is only that it is more difficult to solve two knapsack problems than to solve one. This does not say anything of the time needed. A3 may have some clever way to solve two problems faster than one. Our main interest is in Lemma 15. A similar calculation as in Lemma 14 shows in Lemma 15 that for A3 it is at least as slow to compute the median for \(K_2\) as for the set \(K_{3,j_n}\).
Lemma 14. Let the algorithm be A3. It is more difficult to conclude that $K_{1,j_n}$ has no solutions than it is to conclude that $n_1$ values of $j'_i$ do not solve $(j_i', W_{n_1})$.

Proof: The algorithm A3 solves $(j_{n_1}, W_{n_1})$ by expanding it into

\[(j_{n_1} + 2^n C' j_{n_1,h}, W_{n_1} + 2^n A_{n_1}) \quad (5.2)\]

where $j_{n_1,h} = C'[j_{n_1} C' - 1]$, $C'$ is as in (3.5). Let $B_1$ be defined by $K_{1,j_n} = (W_{n_1} | B_1).$ The algorithm A3 solves $(j_n, K_{1,j_n})$ by expanding it as

\[(j_n + 2^n C^{-1} j_{n,h} + 2^n A_n) = (j_n + 2^n C^{-1} j_{n,h}, (W_{n_1} + 2^n A_{n_1} + 2^{n+\frac{n_1}{2}} A_{n_1} | B_1 + 2^n W_{n_1})).\]

This implies solving

\[(j'_i + 2^n j + 2^{n+\frac{n_1}{2}} j, W_{n_1} + 2^n A_{n_1} + 2^{n+\frac{n_1}{2}} A_{n_1}) \quad (5.3)\]

for some $j'_i$ and $j$. It is also necessary to solve

\[(j_n + 2^n C^{-1} j_{n,h} - j'_i - 2^n j - 2^{n+\frac{n_1}{2}} j, B_1 + 2^n W_{n_1}). \quad (5.4)\]

Solving (5.3) is the same as solving

\[(j'_i + 2^n j, W_{n_1} + 2^n A_{n_1}). \quad (5.5)\]

Solving (5.4) is the same as solving the following two knapsack problems

\[(C^{-1} j_{n,h} - j - 2^n j, W_{n_1}), \quad (j_n - j'_i, B_1). \quad (5.6)\]

$B_1$ allows $m = \frac{n}{2}$ solutions $j'_i$. Let us compare (5.2) and (5.5). In order to show that there are no solutions to (5.2) it is enough to check one value $j = C'^{-1} j_{n_1,h}$. In order to show that there are no solutions to (5.5) it is necessary to check all values $j$ that satisfy the left side knapsack in (5.6). It follows that it is more difficult to conclude that $K_{1,j_n}$ has no solutions than it is to conclude that $n_1$ values of $j'_i$ do not solve $(j'_i, W_{n_1})$.

Lemma 15. Let the algorithm be A3. It is at least as slow to compute $K_2$ than it is to compute $K_{3,j_n}$. 

\[\text{31}\]
Proof: An easy verbal explanation of the following proof is that $K_{3,j_n}$ is solved by A3 by first extending it with $(d_{0,1}, \ldots, d_{0,n})$. We define $K_2$ in such a way that the upper half tuple $(d_{0,1}, \ldots, d_{0,n})$ in (4.2) is the same $(d_{0,1}, \ldots, d_{0,n})$ in Definition 6. When $K_2$ is solved by extending it with $(d_{0,1}, \ldots, d_{0,n})$ there are two identical extensions in $K_2$. The complexity of solving two identical extensions is the same as solving one extension, especially for A3 that selects the faster alternative of either first extending and solving with A1, or not extending and solving directly with A1. Thus, $K_2$ is essentially as difficult to solve for the new algorithm as $K_{3,j_n}$. We give this argument more precisely in what follows. Let us define $B_2$ by $K_{3,j_n} = (W_{n,1}, B_2)$. By the proof of Lemma 8 the tuple $K_{3,j_n}$ yields about $m = \frac{1}{8} \cdot \frac{n^2}{16}$ values of $j$ that may give solutions to $(j, W_{n,1})$. The median computation time for $K_2$ is computed from knapsacks $(j_n, K_2)$ that are expanded by A3 as

$$(j_n + 2^nC^{-1}j_{n,h}, K_2 + 2^nA_n)$$

(5.6)

$$= (j_n + 2^nC^{-1}j_{n,h}, K_{3,j_n} - j_{n,h}(0, \ldots, 1) + CA_n + 2^nA_n)$$

$$= (j_n + j_{n,h} + 2^nC^{-1}j_{n,h}, K_{3,j_n} + CA_n + 2^nA_n).$$

Here we have inserted the definition of $K_2$ from (4.2). The median over the set $K_{3,j_n}$ is computed from

$$(j_n + 2^nC^{-1}j_{n,h}, K_{3,j_n} + 2^nA_n)$$

(5.7)

and there is the additional condition that $c_n = 1$. We notice that in $K_2$ there are identical repeated rows but otherwise the problems (5.6) and (5.7) are the same. Thus, the median over $K_2$ is at least as slow to compute as the median over the set $K_{3,j_n}$. It may be a bit easier to solve $K_{3,j_n}$ because the condition $c_n = 1$ restricts possibilities.

**Lemma 16.** Let the algorithm be A3. It is possible to compute the median (2.4) for $K_2$.

**Proof:** Let us approximate the number of $j_n$ that do not give solutions to $K_2$. Let us expand $A_n$ in (5.6) by using the definition
(5.1). The knapsack in indices $1, \ldots, n_1$ has the expression

$$j'_i + 2^n j + 2^{n+\frac{3}{2}} j, W_{n_1} + 2^n A_{n_1} + 2^{n+\frac{3}{2}} A_{n_1}).$$

At the indices $n_1 + 1, \ldots, n$ the solution must solve $(B_2, j_n - j_i)$ in the low bits. Let there be $m'$ values of $j'_i$ that solve $(B_2, j_n - j_i)$. $B_2$ has only two different values of $e_i$ in (4.1). Thus, many combinations of $c_{n_1+1}, \ldots, c_n$ yield the same value $j'_i$. Let us divide the combinations of $c_{n_1+1}, \ldots, c_n$ into $m$ sets in such a way that each combination in the set yields the same value $j'_i$. Each combination $c_{n_1+1}, \ldots, c_n$ determines a value $j_0$ that the upper bits knapsack $(W_{n_1}, j_0)$ in the indices $n_1 + 1, \ldots, n$ should satisfy. About half of the values $j_0$ yield a solution to $(j_0, W_{n_1})$. At the indices $1, \ldots, n_1$ there is the knapsack $(j'_i, W_{n_1})$. It has a solution for about half of the values $j'_i$. Each solution determines a combination $c_1, \ldots, c_{n_1}$ which determines a value $j$ to the upper bits in indices $1, \ldots, n_1$. The numbers $j_0$ and $j$ must satisfy the equality

$$j + 2^{\frac{n}{2}} j + j_0 = (1 + 2^{\frac{n}{2}})C^{-1} j_{n,h}. \quad (5.8)$$

Let us select $j'_i$. In about half of the cases there is no such $j$ since $(j'_i, K_{n_1})$ does not have a solution. (If $j$ exists, it is usually unique because the $2^\frac{n}{2}$ combinations of $c_k$ in $W_{n_1}$ are mapped to the range $2^\frac{n}{2} + 1$ and the most probable cases are that there is no solution or one solution.) When $j$ is obtained, the equation (5.8) determines the value $j_0$. This $j_0$ must belong the the set of possible values for $j_0$ to this $j'_i$. There are $m$ sets. Though the sets do not have the same sizes, we can in the average approximate that the probability of the $j_0$ belonging to the correct set is about $\frac{1}{m}$. If it belongs to the correct set, it solves $(j_0, W_{n_1})$ with probability 0.5. Then it is a solution to $K_2$. Thus, for selected $j'_i$ there is a solution with the probability that $j$ exists (about 0.5) times $j_0$ is in the set (about $\frac{1}{m}$) times $j_0$ solves $(j_0, W_{n_1})$ (about 0.5). There are $m$ values $j'_i$. This yields the approximation to finding a solution to $K_2$ as $m\frac{1}{2} \frac{1}{m} = 0.25$. This rough approximation shows that at least half of the values $j_n$ do not give a solution. In (2.4) the exact proportion of successful and unsuccessful values of $j_n$ is not important as long as there are at least some values to
compute the median. The argument shows that the proportion of unsuccessful values of $j_n$ does not approach zero.

**Lemma 17.** Let the algorithm be A3. The algorithm cannot stop because of finding a solution because for every $j_n$ none of the $\frac{n}{2}$ values of $j'_i$ solve the knapsack problem. Every value $j'_i$ gives at least as long computation as the median computation time $f(n_1)$.

*Proof:* The proof for Lemma 5 does not need changes: there are no solutions for $(j'_i, W_n)$ for any of the $\frac{n}{2}$ values $j'_i$ from $B_1$ and the selection of $j'_i$ can be made to give at least as long computation as the median computation time $f(n_1)$.

**Lemma 18.** Let the algorithm be A3. There is no way to discard any values $j'_i$ without checking if they solve the subknapsack from $k = 1$ to $k = n_1$. Any case of using the values of $d_{k,n}$ in order to get the result is considered checking.

*Proof:* The proof of Lemma 6 uses the property that $K_{1,j_n}$ has a simple upper half tuple. It may appear that this property has changed since the algorithm extends the knapsacks by adding a nontrivial knapsack to high bits. However, the knapsack $(j_n, K_{1,j_n})$ has not changed. The proof of Lemma 6 is true for any (arbitrary) algorithm solving knapsack problems. The algorithm can solve in any way, e.g. by adding an extension. What is important is that the algorithm needs to check if the value $j'_i$ yields a solution. If there exists at least one solution to the upper knapsack then the new algorithm cannot decide that there is no solution without checking the value of $j'_i$ in some way. This condition is valid since there are many possible solutions to the upper half knapsack problem $(j, A_{n_1})$ for a selected $j_n$ in almost all cases: $2^n$ combinations of $c_k$ are mapped to $2^2$ values of $j$. If the upper half of $j_n$ is very large or very small there are only few possible combinations to the upper half knapsack and A3 may conclude fast that there are no solutions because the upper half bits do not match. There cases are so rare (necessarily a polynomial size set) that they do
not affect the median. Thus, the proof of Lemma 6 also proves Lemma 18.

**Lemma 19.** Let the algorithm be A3. Several values of $j'_i$ cannot be evaluated on the same run. The median computation time of $K_{1,j_n}$ is at least

$$f_1(n_1) + n_1 f_2(n_1)$$

where $f(n) = f_1(n) + f_2(n)$ and $f_1(n)$ is a polynomial function of $n$.

**Proof:** If the upper half knapsack can be solved by polynomially many possible combinations for $c_1, \ldots, c_{n_1}$ only, then the lower half knapsack $(j,W_{n_1})$ can be solved by checking these combinations. Then there is no need to solve the knapsack $(j'_i,W_{n_1})$ in the hard way. As long as the search space of possible combinations of $c_1, \ldots, c_{n_1}$ gives an exponential number of values $j$ that are possible in the upper bits but do not solve the lower bit knapsack in indices $1, \ldots, n_1$, the argument in the proof of Lemma 7 holds. For the choise (5.1) there exists an exponential number of combinations $c_1, \ldots, c_n$ that solve the upper half knapsack $(j, A_{n_1})$ for any fixed $j$. It can be assumed that a large proportion of these choises give $j$ that does not yield a solution to $(j, W_{n_1})$, Thus, the proof of Lemma 7 also proves Lemma 19.

**Lemma 20.** Let the algorithm be A3. It is at most as fast to compute $K_{1,j_n}$ than to compute $K_{3,j_n}$.

**Proof:** The proof of Lemma 8 also proves Lemma 20.

**Lemma 21.** Let the algorithm be A3. The set of selected tuples $K_{1,j_n}$ gives a longer median computation time than the worst in the median tuple for $n$.

**Proof:** By Lemma 20 the set $K_{3,j_n}$ is slower to solve than the set $K_{1,j_n}$ when the median is taken over the set of $j_n$. By Lemma 15
the tuple \( K_2 \) is not faster to solve than the set of \( K_{3,j_n} \) when the median is computed over the set \( j_n \) such that there is no solution for \( K_2 \). As \( K_2 \) is a fixed \( n \)-tuple and does not depend on \( j_n \) it follows from the definition of the worst in the median tuple that \( K_2 \) has at most as long median computation time as the worst in the median tuple for \( n \), i.e., \( f(n) \) for A3. We conclude that by selecting \( e_i \) and \( a_i \) in a suitable way, the median time of deciding if \( K_{1,j_n} \) has a solution is smaller than \( f(n) \), where the median is taken over the set of values of \( j_n \) as in (2.5)-(2.6).

**Lemma 22.** Algorithm A3 is not a polynomial time algorithm

*Proof:* We have constructed in Sections 3 and 4 a sequence of \( j_n \) growing as \( 2^n \) since it is sufficient to find one case of a knapsack sequence that cannot be solved in polynomial time. By Lemmas 19 and 21 the inequality of Lemma 4 holds. Consequently, \( f(n) \) is not bounded by a polynomial function of \( n \) and thus the algorithm A3 is not a polynomial-time algorithm.

**Discussion of the method of Section 5**

Figure 2 demonstrates the recursive definition (5.1) and extension of knapsacks in Definition 6.

| Fixed extension \( A_n \) | \( A_n \) structure |
|--------------------------|--------------------|
| Original knapsack        | \( A_{n/2} \)      |
|                          | \( A_{n/2} \)      |
|                          | Worst \( n/2 \)     |

Fig. 2. Extending knapsacks in A3.
The problem in Section 4 was comparing the median computation times of $K_2$ and the set $K_{3,n}$. It seems natural that $K_2$ is more difficult to solve of these two as it has a difficult upper half knapsack problem that must be solved with the same set of $c_k$ as the lower half knapsack problem, while the set $K_{3,n}$ only has the same lower half knapsack problem and a trivial upper half knapsack problem. However, this we could not show since the unknown algorithm might in some way use the possible solutions to the upper half knapsack problem and in such a way check fewer cases. This is why we modified the original algorithm A1 into A3, which always can take advantage of the upper half knapsack solutions. This is done by always extending the knapsack problem before solving it with A1 as one alternative, or solving it directly with A1, whichever is the fastest. Extending the upper half knapsack with a difficult knapsack only reduces the possible solution space. Instead of having all possible combinations of $c_k$, the algorithm A3 might only check those combinations of $c_k$ which solve the upper half knapsack. For the inequality (2.9) this only means that the possible cases are an exponential size sample of the original set of the original possible cases. As the set has an exponential size and it can be freely selected, we still have a large enough space of possible values $j_n$ for the lemmas of Section 3. The difficult part to prove in Section 4 becomes simple as both $K_{3,n}$ and $K_2$ have the same upper half knapsack in the case when A3 extends knapsacks.

6 A proof that P does not equal NP

**Theorem 1.** Let an algorithm for the knapsack problem be selected. There exist numbers $B, \alpha \in \mathbb{R}$, $B \geq 1$, $\alpha \geq 0$ and a sequence

$$( (j_n, (d_{1,n}, \ldots, d_{n,n})) )_{n \geq 1}$$

of knapsacks satisfying

$$\log_2 j_n < Bn^\alpha, \quad \log_2 d_{k,n} < Bn^\alpha, (1 \leq k \leq n), \quad (n \geq 1)$$
such that the algorithm cannot determine in polynomial time if there exist binary numbers $c_{k,n}$, $1 \leq k \leq n$, satisfying

$$j_n = \sum_{k=1}^{n} c_{k,n}d_{k,n}.$$

**Proof.** The idea in this proof is to compare the computation time of the worst (in some sense) knapsack of size $n$ to the computation time of (in the same sense) worst knapsack of $\frac{n}{2}$. This computation time was defined in 2.7 and denoted by $f(n)$. In Sections 3 and 4 we showed several parts of inequality (2.9) but did not manage to show the last part for (2.9). However, in Section 5 we created another algorithm A3 and showed in Lemma 22 that is not a polynomial time algorithm. As A3 is never slower than the original selected algorithm, the original algorithm is not a polynomial time algorithm.

**Theorem 2.** P does not equal NP.

**Proof.** The knapsack problem is well known to be in NP.

**References**

[1] S. Cook, The P versus NP problem. *available on-line at* www.claymath.org.

[2] D. L. Kreher and D. R. Stinson, Combinatorial algorithms, generation, enumeration, and search, CRC Press, Boca Raton, 1999.

[3] J. Jormakka, Solutions to three-dimensional Navier-Stokes equations for incompressible fluids, EJDE Vol 2010(2010), No. 93, pp. 1-14. *available on-line at* ejde.math.txstate.org.