KMS-weights on $C^*$-algebras

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Abstract
In this paper, we build a solid framework for the use of KMS-weights on $C^*$-algebras. We will use another definition than the one introduced by Combes in [4], but we will prove that they are equivalent. However, the subject of KMS-weights is approached from a somewhat different angle. We introduce a construction procedure for KMS-weights, prove the most important properties of them, construct the tensor product of KMS-weights and construct weights which are absolutely continuous to a given weight.

Introduction
In [3], Combes studied lower semi-continuous weights on $C^*$-algebras and proved the major result that a lower semi-continuous weight can be approximated by the positive linear functionals which are majorated by it.
At the moment, these lower semi-continuous weights seem not to behave well enough to be used in an efficient way. So it is natural to search for extra conditions conditions on lower semi-continuous weights which makes them manageable. An attempt in this direction was undertaken by Jan Verding in [22] where he introduced the so-called regular weights. These are lower semi-continuous weights which have a well-behaved truncating net.
It turns out that this regularity condition is also very useful in the theory of $C^*$-valued weights (see [10]).
There is however a class of lower semi-continuous weights which behave very well and were introduced by Combes in definition 4.1 of [3], the so-called KMS-weights:

Consider a $C^*$-algebra $A$ and a densely defined lower semi-continuous weight $\varphi$ on $A$ such that there exists a norm continuous one-parameter group $\sigma$ on $A$ satisfying:

1. We have for every $t \in \mathbb{R}$ that $\varphi(\sigma_t) = \varphi$.
2. For every $x, y \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, there exists a bounded continuous function $f$ from $S(i)$ into $\mathbb{C}$ which is analytic on $S(i)^0$ and such that:
   - We have for every $t \in \mathbb{R}$ that $f(t) = \varphi(\sigma_t(x)y)$.
   - We have for every $t \in \mathbb{R}$ that $f(t+i) = \varphi(y\sigma_t(x))$.

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Then $\varphi$ is called a KMS-weight on $A$, $\sigma$ is called a modular group for $\varphi$.

Here, $S(i)$ denotes the horizontal strip in the complex plane between 0 and $i$.

By a proof of Combes in [5], we know that such a KMS-weight can be extended to normal KMS-weight in the GNS-representation of $\varphi$. In this way, we can make use of the very rich theory of normal weights on von Neumann algebras and use for instance the Radon Nikodym theorem for normal weights (see [14]).

In short, we get a class of weights which which posses the same rich structure as the Borel measures on locally compact spaces.

The last years, there is a lot of interest for lower-semicontinuous weights in general (and KMS-weights in particular) from people working in the field of C$^*$-algebraic quantum groups. This because of the role of the left Haar weight in C$^*$-algebraic quantum group theory.

At the moment, it is not yet clear whether the left Haar weight should be a KMS-weight. There are however some indications that this will be the case:

- Masuda, Nakagami & Woronowicz are momentarily working on a definition for reduced C$^*$-algebraic quantum group and their left Haar weight will be a KMS-weight.

- In all the known examples, the left Haar weight turns out to satisfy the KMS-condition.

- A. Van Daele proves in [21] that the left Haar functional on an algebraic quantum group satisfies automatically some sort of weak KMS-condition.

- It is proven in [8] that this weak KMS-condition implies that the left Haar weight on the reduced C$^*$-algebraic group arising from the algebraic one, is a KMS-weight.

- In [11], we prove that this KMS-condition on this reduced C$^*$-algebraic quantum group implies that the left Haar weight on the universal C$^*$-algebraic quantum group is also KMS. It appears that the same principle can be used to prove such a result in the general case.

Especially the universal case necessitates to investigate KMS-weights in a C$^*$-algebraic framework.

In this paper, we will work with another definition of a KMS-weight than the one mentioned above but we will show that both are equivalent. One of the reasons to use the other definition is the fact that Masuda, Nakagami & Woronowicz seemed to be going to use this other definition in their approach to quantum groups.

The main aim of this paper is to build a solid framework for KMS-weights on C$^*$-algebras. We believe that it gives a technically useful overview of KMS-weights in the C$^*$-algebra picture.

Most of the results will look familiar to the ones known in normal weight theory but will be proven within the C$^*$-algebra framework.

In the first section, we fix notations and give an overview of the results about one-parameter representations and their analytic continuations.

We give the definition of a KMS-weight in the second section. At the same time, we prove some useful properties concerning lower semi-continuous weights.

In section 3, we have gathered some results from [22] about the construction of a weight starting from a GNS-construction.

The fourth section concerns the behaviour of one-parameter representations which are relatively invariant with respect to some closed mapping.

In the fifth section, we prove that a closed linear mapping from within a C$^*$-algebra into a Hilbert space and satisfying certain KMS-characteristics gives rise to a KMS-weight on this C$^*$-algebra.
The most important technical properties concerning KMS-weights are collected in the sixth section. In section 7, we construct the tensor product of KMS-weights. Given a KMS-weight \( \varphi \) and a strictly positive element \( \delta \) affiliated to the C*-algebra which is relatively invariant under a modular group of \( \varphi \), we construct the weight \( \varphi(\delta^{\frac{1}{2}}, \delta^{\frac{1}{2}}) \) in section 8. We have to use another method than the one used by Pedersen & Takesaki because of the relative invariance.

Let us now fix some terminology and conventions.

Consider a complex number \( z \), then \( S(z) \) will denote the horizontal strip \( \{ c \in \mathbb{C} \mid \text{Im } c \in [0, \text{Im } z] \} \). The interior of \( S(z) \) will be denoted by \( S(z)^0 \).

The domain of a mapping \( f \) will be denoted by \( D(f) \), its image by \( \text{Ran } f \).

For any Banach space \( E \), we denote the set of bounded operators on \( E \) by \( B(E) \) and the topological dual by \( E^* \).

Let \( A \) be a C*-algebra. We denote the multiplier algebra of \( A \) by \( M(A) \).

Any element \( \omega \in A^* \) has a unique strictly continuous linear extension to \( M(A) \) and we denote this extension by \( \overline{\omega} \). We put \( \omega(x) = \overline{\omega}(x) \) for every \( x \in M(A) \).

For a good introduction into elements affiliated with \( A \), we refer to [12] and [24]. Whenever we speak of a positive element affiliated element with \( A \), we mean a (possibly unbounded) positive and selfadjoint element affiliated with \( A \). The same remark applies to positive operators in Hilbert spaces. A positive element affiliated to \( A \) is called strictly positive if it has dense range. As usual, we can raise a strictly positive affiliated element to any complex power.

Consider a Hilbert space \( H \). Let \( \pi \) a non-degenerate *-homomorphism from \( A \) into \( B(H) \). Then \( \pi \) has a unique extension to a *-representation \( \overline{\pi} \). We put \( \pi(x) = \overline{\pi}(x) \) for every \( x \in M(A) \).

If \( T \) is an element affiliated with \( A \), then \( \pi(T) \) denotes the closed densely defined operator in \( H \) such that \( \pi(D(T)) \) is a core for \( \pi(T) \) and \( \pi(T)(\pi(a)v) = \pi(T(a))v \) for every \( a \in D(T) \) and \( v \in H \).

In the following, \( \circ \) will denote the composition of mappings and elements of \( M(A) \) will be considered as linear mappings from \( A \) into \( A \).

Consider an element \( T \) affiliated with \( A \) and \( a \in M(A) \). Then

- We call \( a \) a left multiplier of \( T \) if \( a \circ T \) is bounded as a mapping from \( A \) into \( A \) and \( a \overline{\circ} T \) belongs to \( M(A) \). In this case, we put \( aT = a \overline{\circ} T \).

- We call \( a \) a right multiplier of \( T \) if \( D(T \circ a) = A \). You can prove in this case that \( T \circ a \) belongs to \( M(A) \) and we put \( Ta = T \circ a \).

It is not very difficult to prove that \( a \) is a left multiplier of \( T \) if and only if \( a \) is a right multiplier of \( T^* \). If this is the case, we have that \( (aT)^* = T^*a^* \).

We will freely use calculation rules involving this left and right multipliers, e.g. Let \( a, x \in M(A) \) and \( T \) an element affiliated with \( A \). If \( a \) is a left multiplier of \( T \), then \( xa \) is a left multiplier of \( T \) and \( (xa)T = x(aT) \).

Consider elements \( S, T \) affiliated with \( A \) and \( a \in M(A) \). Then we call \( a \) a middle multiplier of \( S, T \) if \( D(S \circ a \circ T) = D(T) \), \( S \circ a \circ T \) is bounded and \( S \circ a \circ T \) belongs to \( M(A) \). In this case, we put \( SaT = S \circ a \circ T \).
1 One-parameter representations

In this first section, we recall the definition of one-parameter representations and an overview of the most important results. Most of the proofs can be found in [9]. A standard reference for one-parameter representation is [25].

Whenever we use the notion of integrability of a function with values in a Banach space, we mean the strong form of integrability (e.g. Analysis II, S. Lang):

Consider a measure space \((X, \mathcal{M}, \mu)\) and a Banach space \(E\).

- It is obvious how to define integrability for step functions from \(X\) into \(E\).
- Let \(f\) be a function from \(X\) into \(E\). Then \(f\) is \(\mu\)-integrable if and only if there exists a sequence of integrable step functions \((f_n)_{n=1}^\infty\) from \(X\) into \(E\) such that:
  1. We have for almost every \(x \in X\) that \((f_n(x))_{n=1}^\infty\) converges to \(f(x)\) in the norm topology.
  2. The sequence \((f_n)_{n=1}^\infty\) is convergent in the \(L_1\)-norm.

In this case, the sequence \((\int f_n \, d\mu)_{n=1}^\infty\) is convergent and the integral of \(\int f \, d\mu\) is defined to be the limit of this sequence (Of course, one has to prove that this limit is independent of the choice of the sequence \((f_n)_{n=1}^\infty\)).

- It is possible to define a form of \(\mu\)-measurability (see Lang) in such a way that a function \(f\) is integrable if and only if the function \(f\) is measurable and \(\|f\|\) is integrable.

We start this section with a basic but very useful lemma.

**Lemma 1.1** Consider Banach spaces \(E, F\), \(\Lambda\) a closed linear mapping from within \(E\) into \(F\). Let \(f\) be a function from \(\mathbb{R}\) into \(D(\Lambda)\) such that

- \(f\) is integrable.
- The function \(\mathbb{R} \to F : t \mapsto \Lambda(f(t))\) is integrable.

Then \(\int f(t) \, dt\) belongs to \(D(\Lambda)\) and \(\Lambda(\int f(t) \, dt) = \int \Lambda(f(t)) \, dt\).

**Proof:** Define \(G\) as the graph of the mapping \(\Lambda\). By assumption, we have that \(G\) is a closed subspace of \(E \oplus F\). Next, we define the mapping \(g\) from \(\mathbb{R}\) into \(G\) such that \(g(t) = (f(t), \Lambda(f(t)))\) for every \(t \in \mathbb{R}\). It follows that \(g\) is integrable and

\[\int g(t) \, dt = \left(\int f(t) \, dt, \int \Lambda(f(t)) \, dt\right) .\]

Because \(G\) is a closed subspace of \(E \oplus F\), we have that \(\int g(t) \, dt\) belongs to \(G\). This implies that \(\left(\int f(t) \, dt, \int \Lambda(f(t)) \, dt\right)\) belongs to \(G\).

When we speak of analyticity, we will always mean norm analyticity. But we have the following well known result (which follows from the Uniform Boundedness principle).

**Result 1.2** Consider a Banach space \(E\) and a subspace \(F\) of \(E^*\) such that \(\|x\| = \sup\{\|\omega(x)\| | \omega \in F \text{ with } \|\omega\| \leq 1\}\) for every \(x \in E\). Let \(O\) be an open subset of \(C\) and \(f\) a function from \(O\) into \(E\).

Then \(f\) is analytic \(\iff\) We have for every \(\omega \in F\) that \(\omega \circ f\) is analytic.

Notice that the result is true for \(F = E^*\).
1.1 One-parameter representations on Banach spaces

We introduce the notion of strongly continuous one-parameter groups and discuss the analytic continuations of them. The standard reference for the material in this section is [25].

**Terminology 1.3** Consider a Banach space $E$. By a one-parameter representation on $E$, we will always mean a mapping $\alpha$ from $\mathbb{R}$ into $B(E)$ such that :

- We have for every $s,t \in \mathbb{R}$ that $\alpha_{s+t} = \alpha_s \alpha_t$.
- $\alpha_0 = \iota$
- We have for every $t \in \mathbb{R}$ that $\|\alpha_t\| \leq 1$.

We call $\alpha$ strongly continuous $\iff$ We have for every $a \in E$ that the mapping $\mathbb{R} \to E : t \mapsto \alpha_t(a)$ is continuous.

This definition implies for every $t \in \mathbb{R}$ that $\alpha_t$ is invertible in $B(E)$, that $(\alpha_t)^{-1} = \alpha_{-t}$ and that $\|\alpha_t\| = 1$.

**Remark 1.4** We would like to mention the following special cases :

- If $H$ is a Hilbert space and $u$ is a strongly continuous one-parameter representation on $H$ such that $u_t$ is unitary for every $t \in \mathbb{R}$, we call $u$ a strongly continuous unitary one-parameter group on $H$.
- If $A$ is a C*-algebra and $\alpha$ is a strongly continuous one-parameter representation on $A$ such that $\alpha_t$ is a C*-automorphism on $A$ for every $t \in \mathbb{R}$, we call $\alpha$ a norm continuous one-parameter group on $A$.
- If $A$ is a C*-algebra and $u$ is a strongly continuous one-parameter representation on $A$ such that $u_t$ is a unitary element in $M(A)$ for every $t \in \mathbb{R}$, we call $u$ a strictly continuous unitary one-parameter group on $A$.

Now we will summarize the theory of analytic continuations of such norm continuous one-parameter representations. For the most part of this subsection, we will fix a Banach space $E$ and a strongly continuous one-parameter representation $\alpha$ on $E$.

**Definition 1.5** Consider $z \in \mathbb{C}$. We define the mapping $\alpha_z$ from within $E$ into $E$ such that :

- The domain of $\alpha_z$ is by definition the set $\{ a \in E \mid$ There exists a function $f$ from $S(z)$ into $E$ such that
  1. $f$ is continuous on $S(z)$
  2. $f$ is analytic on $S(z)^0$
  3. We have that $\alpha_t(a) = f(t)$ for every $t \in \mathbb{R} \}$. 

- Choose $a$ in the domain of $\alpha_z$ and $f$ the function from $S(z)$ into $E$ such that
  1. $f$ is continuous on $S(z)$
  2. $f$ is analytic on $S(z)^0$
  3. We have that $\alpha_t(a) = f(t)$ for every $t \in \mathbb{R}$

Then we have by definition that $\alpha_z(a) = f(z)$.

In the case that $z$ belongs to $\mathbb{R}$, this definition of $\alpha_z$ corresponds with $\alpha_z$ which we started from. Therefore, the notation is justified.
Remark 1.6  

- Consider \( z \in \mathbb{C} \) and \( y \in S(z) \). Then it is clear that \( D(\alpha_z) \) is a subset of \( D(\alpha_y) \).
- It is also clear that \( D(\alpha_y) = D(\alpha_z) \) for \( y, z \in \mathbb{C} \) with \( \text{Im } y = \text{Im } z \).
- Let \( z \in \mathbb{C} \) and \( a \in D(\alpha_z) \). Then the function \( S(z) \to E : u \mapsto \alpha_u(a) \) is continuous on \( S(z) \) and analytic on \( S(z)^0 \) (because this function must be equal to the function \( f \) from the definition).
- Consider an element \( a \) in \( E \). We say that \( a \) is analytic with respect to \( \alpha \) if \( a \) belongs to \( D(\alpha_z) \) for every \( z \in \mathbb{C} \). If \( a \) is analytic with respect to \( \alpha \), then the function \( \mathbb{C} \to E : u \mapsto \alpha_u(a) \) is analytic.

Now we give a list of the most important properties of these analytic continuations. The most important tool for proving the results is the Phragmen-Lindelof theorem.

Proposition 1.7  *The mapping \( \alpha_z \) is a closed linear operator in \( E \) which is densely defined and has dense range.*

We would like to mention that the proof of the closedness is substantially easier than in the case of strongly continuous one-parameter groups on von Neumann algebras (see [25]).

Result 1.8  *Consider \( z \in \mathbb{C} \) and \( a \in D(\alpha_z) \). Then the function \( S(z) \to E : u \mapsto \alpha_u(a) \) is bounded.*

Proposition 1.9  *Consider \( z \in \mathbb{C} \) and \( t \in \mathbb{R} \). Then \( \alpha_z \alpha_t = \alpha_t \alpha_z = \alpha_{z+t} \). Furthermore, the following equalities hold :

- We have that \( \alpha_t(D(\alpha_z)) = D(\alpha_z) \) and \( \alpha_t(\text{Ran } \alpha_z) = \text{Ran } \alpha_z \).
- We have that \( D(\alpha_{z+t}) = D(\alpha_z) \) and \( \text{Ran } \alpha_{z+t} = \text{Ran } \alpha_z \).

Proposition 1.10  *Consider \( z \in \mathbb{C} \). Then \( \alpha_z \) is injective and \( (\alpha_z)^{-1} = \alpha_{-z} \). Therefore \( \text{Ran } \alpha_z = D(\alpha_{-z}) \) and \( D(\alpha_z) = \text{Ran } \alpha_{-z} \).*

Proposition 1.11  *Consider \( y, z \in \mathbb{C} \). Then

1. \( \alpha_y \alpha_z \subseteq \alpha_{y+z} \).
2. If \( y \) and \( z \) lie at the same side of the real axis, we have that \( \alpha_y \alpha_z = \alpha_{y+z} \).*

As usual, we smear elements to construct elements which behave well with respect to \( \alpha \).

Notation 1.12  *Consider \( a \in E \), \( n > 0 \) and \( z \in \mathbb{C} \). Then we define the element \( a(n, z) \) in \( E \) such that

\[
a(n, z) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t-z)^2) \alpha_t(a) \, dt .
\]

Then we have that \( \|a(n, z)\| \leq \|a\| \exp(n^2(Im \, z)^2) \).

For \( n > 0 \) and \( z \in \mathbb{C} \), we will use the notation

\[
a(n) = a(n, 0) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \alpha_t(a) \, dt .
\]
**Proposition 1.13** Let $a \in E$, $n > 0$ and $z \in \mathbb{C}$. Then $a(n, z)$ is analytic with respect to $\alpha$ and $\alpha_y(a(n, z)) = a(n, z + y)$ for every $y \in \mathbb{C}$.

**Proposition 1.14** Consider $n > 0$, $y, z \in \mathbb{C}$ and $a \in D(\alpha_y)$. Then $\alpha_y(a(n, z)) = \alpha_y(a)(n, z)$.

A proof of the next result can be found in [23].

**Proposition 1.15** Let $z$ be a complex number. Consider a dense subset $K$ of $E$. Then the set $\langle a(n) \mid a \in K, n \in \mathbb{N} \rangle$ is a core for $\alpha_z$.

We end this subsection with some special cases:

The first one is a familiar one and the proof can be found in corollary 9.21 of [18].

**Proposition 1.16** Consider a Hilbert space $H$ and an injective positive operator $M$ in $H$. Define the strongly continuous unitary one-parameter group $u$ on $H$ such that $u_t = M^{it}$ for every $t \in \mathbb{R}$. Then $u_z = M^{iz}$ for every $z \in \mathbb{C}$.

The next result is a $C^*$-version of the previous one.

**Proposition 1.17** Consider a $C^*$-algebra $A$ and a strictly positive element $\delta$ affiliated with $A$. Define the strictly continuous unitary one-parameter group $u$ on $A$ such that we have for every $t \in \mathbb{R}$ that $u_t = \delta^{it}$ considered as a left multiplier. Then $u_z = \delta^{iz}$ for every $z \in \mathbb{C}$.

The previous result implies immediately the following result.

**Proposition 1.18** Consider a $C^*$-algebra $A$ and a strictly positive element $\delta$ affiliated with $A$. Define the strictly continuous unitary one-parameter group $u$ on $A$ such that we have for every $t \in \mathbb{R}$ that $u_t = \delta^{it}$ considered as a right multiplier. Let $z$ be a complex number and $a$ an element in $A$, then:

- The element $a$ belongs to $D(u_z) \iff a$ is a left multiplier of $\delta^{iz}$ and $a \delta^{iz}$ belongs to $A$.
- If $a$ belongs to $D(u_z)$, then $u_z(a) = a \delta^{iz}$.

The following proposition is also a $C^*$-version of a known result in Hilbert space theory (see proposition 9.24 of [18]).

**Proposition 1.19** Consider a $C^*$-algebra $A$ and a strictly positive element $\delta$ affiliated with $A$. Define the norm continuous one-parameter group $\alpha$ on $A$ such that $\alpha_t(a) = \delta^{-it}a \delta^{it}$ for every $t \in \mathbb{R}$ and $a \in A$. Let $z$ be a complex number and $a$ an element in $A$, then:

- The element $a$ belongs to $D(\alpha_z) \iff a$ is a middle multiplier of $\delta^{-iz}$, $\delta^{iz}$ and $\delta^{-iz}a \delta^{iz}$ belongs to $A$.
- If $a$ belongs to $D(\alpha_z)$, then $\alpha_z(a) = \delta^{-iz}a \delta^{iz}$.

A proof of the last 3 propositions can be found in [3].
1.2 One-parameter groups on C*-algebras

For this section, we will fix a norm-continuous one-parameter group $\alpha$ on a C*-algebra $A$. We will investigate the properties of the analytic continuations in this case a little bit further. Proofs of the results in this section can be found in [9].

By the remarks of the previous section, we have for any $z \in \mathbb{C}$ a closed linear operator $\alpha_z$ from within $A$ into $A$ which is generally unbounded.

Because $\alpha_t$ is a $\ast$-homomorphism for every $t \in \mathbb{R}$, we have the two following algebraic properties.

**Proposition 1.20** The mapping $\alpha_z$ is a multiplicative linear operator in $A$.

**Proposition 1.21** Consider $z \in \mathbb{C}$. Then $D(\alpha_z) = D(\alpha_z)^*$ and $\text{Ran } \alpha_z = (\text{Ran } \alpha_z)^*$. We have for every $a \in D(\alpha_z)$ that $\alpha_z(a)^* = \alpha_z(a^*)$.

This implies for all $z \in \mathbb{R}$ and $a \in D(\alpha_z)$ that $\alpha_z(a)^*$ belongs to $D(\alpha_z)$ and $\alpha_z(\alpha_z(a)^*)^* = a$.

For the rest of this section, we want to concentrate on strictly analytic continuations of $\alpha$.

Concerning analyticity, we have the following result. The proof of this fact uses result 1.2 and the fact that every continuous linear functional on $A$ is of the form $a \omega$ with $\omega \in A^*$ and $a \in A$.

**Result 1.22** Consider an open subset $O$ of the complex plane, a function from $O$ into $M(A)$. Then $f$ is analytic $\iff$ We have for every $\omega \in A^*$ that the function $\omega \circ f$ is analytic $\iff$ We have for every $a \in A$ that the function $O \to A : z \mapsto f(z)a$ is analytic.

It is possible to give the following definition.

**Definition 1.23** Consider a complex number $z$. Then the mapping $\alpha_z$ is closable for the strict topology on $M(A)$ and we denote the strict closure by $\overline{\alpha_z}$.

In the case where $z$ belongs to $\mathbb{R}$, the previous definition implies that $\overline{\alpha_z}$ is the unique $\ast$-homomorphism from $M(A)$ into $M(A)$ which extends $\alpha_z$. It is also clear that $\alpha_z \subseteq \overline{\alpha_z}$ for every $z \in \mathbb{C}$.

The two previous remarks justify the following notation. For every $z \in \mathbb{C}$ and $a \in D(\overline{\alpha_z})$, we put $\alpha_z(a) = \overline{\alpha_z}(a)$.

**Result 1.24** Let $a$ be an element in $M(A)$. Then the function $\mathbb{R} \to M(A) : t \mapsto \alpha_t(a)$ is strictly continuous.

The following theorem is one of the most important results of [9].

**Theorem 1.25** Consider $z \in \mathbb{C}$. Then we have the following properties

- Let $a$ be an element in $M(A)$.

  Then $a$ belongs to $D(\overline{\alpha_z}) \iff$ there exists a function $f$ from $S(z)$ into $M(A)$ such that

  1. $f$ is strictly continuous on $S(z)$
  2. $f$ is analytic on $S(z)^0$
  3. We have that $\alpha_t(a) = f(t)$ for every $t \in \mathbb{R}$
• Let $a$ be an element in $D(\pi_z)$ and $f$ the function from $S(z)$ into $M(A)$ such that
  1. $f$ is strictly continuous on $S(z)$
  2. $f$ is analytic on $S(z)^0$
  3. We have that $\alpha_z(a) = f(t)$ for every $t \in \mathbb{R}$

Then we have that $\alpha_z(a) = f(z)$.

It is also possible to give another characterization of $\pi_z$. First look at the following result.

**Proposition 1.26** Consider $z \in \mathbb{C}$ and $a \in D(\pi_z)$, $b \in D(\alpha_z)$. Then $ab$ and $ba$ belong to $D(\alpha_z)$ and $\alpha_z(ab) = \alpha_z(a)\alpha_z(b)$ and $\alpha_z(ba) = \alpha_z(b)\alpha_z(a)$.

There is also a converse of this:

**Proposition 1.27** Consider elements $a, b \in M(A)$ and $z \in \mathbb{C}$. Then

1. If we have for every $c \in D(\alpha_z)$ that $ac$ belongs to $D(\alpha_z)$ and $\alpha_z(ac) = b\alpha_z(c)$, then $a$ belongs to $D(\pi_z)$ and $\alpha_z(a) = b$.

2. If we have for every $c \in D(\alpha_z)$ that $ca$ belongs to $D(\alpha_z)$ and $\alpha_z(ca) = \alpha_z(c)b$, then $a$ belongs to $D(\pi_z)$ and $\alpha_z(a) = b$.

These strictly analytic extensions satisfy the same kind of properties as the norm analytic extensions. Now we give a list of these properties.

**Remark 1.28**

• Consider $z \in \mathbb{C}$ and $y \in S(z)$. Then it is clear that $D(\pi_z)$ is a subset of $D(\pi_y)$.

• It is also clear that $D(\pi_y) = D(\pi_z)$ for $y, z \in \mathbb{C}$ with $\text{Im } y = \text{Im } z$.

• Let $z \in \mathbb{C}$ and $a \in D(\pi_z)$. Then the function $S(z) \to M(A) : u \mapsto \alpha_u(a)$ is strictly continuous on $S(z)$ and analytic on $S(z)^0$.

• Consider an element $a$ in $M(A)$. We say that $a$ is strictly analytic with respect to $\alpha$ if $a$ belongs to $D(\pi_z)$ for every $z \in \mathbb{C}$. If $a$ is strictly analytic with respect to $\alpha$, then the function $\mathbb{C} \to M(A) : u \mapsto \alpha_u(a)$ is analytic.

**Proposition 1.29** The mapping $\pi_z$ is a strictly closed multiplicative linear operator in $M(A)$.

**Proposition 1.30** Consider $z \in \mathbb{C}$. Then $D(\pi_z) = D(\pi_z)^*$ and $\text{Ran } \pi_z = (\text{Ran } \pi_z)^*$. We have for every $a \in D(\alpha_z)$ that $\alpha_z(a)^* = \alpha_{\pi_z}(a^*)$.

This implies that for all $z \in \mathbb{R}i$ and $a \in D(\pi_z)$ that $\alpha_z(a)^*$ belongs to $D(\pi_z)$ and $\alpha_z(\alpha_z(a)^*)^* = a$.

**Result 1.31** Consider $z \in \mathbb{C}$ and $a \in D(\pi_z)$. Then the function $S(z) \to M(A) : u \mapsto \alpha_u(a)$ is bounded.

**Proposition 1.32** Consider $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Then $\pi_{z+t} = \pi_z \pi_t = \pi_{z+t}$. Furthermore the following equalities hold:
We have that $\alpha_t(D(\alpha z)) = D(\alpha z)$ and $\alpha_t(\text{Ran} \alpha z) = \text{Ran} \alpha z$.

We have that $D(\alpha_{z+t}) = D(\alpha z)$ and $\text{Ran} \alpha_{z+t} = \text{Ran} \alpha z$.

**Proposition 1.33** Consider $z \in \mathbb{C}$. Then $\alpha z$ is injective and $(\alpha z)^{-1} = \overline{\alpha_{-z}}$. Therefore $\text{Ran} \alpha z = D(\alpha z)$ and $D(\alpha z) = \text{Ran} \alpha_{-z}$.

**Proposition 1.34** Consider $y, z \in \mathbb{C}$. Then

1. $\overline{\alpha y \alpha z} \subseteq \overline{\alpha y z}$.
2. If $y$ and $z$ lie at the same side of the real axis, we have that $\overline{\alpha y \alpha z} = \overline{\alpha y z}$.

As before, we smear elements to construct elements which behave well with respect to $\alpha$.

**Notation 1.35** Consider $a \in M(A)$, $n > 0$ and $z \in \mathbb{C}$. Then we define the element $a(n, z)$ in $M(A)$ such that

$$a(n, z) b = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t - z)^2) \alpha_t(a) b \, dt$$

for every $b \in A$. Then we have that $\|a(n, z)\| \leq \|a\| \exp(n^2(\text{Im} z)^2)$.

For $n > 0$ and $z \in \mathbb{C}$, we will use the notation $a(n) = a(n, 0)$, so

$$a(n) b = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \alpha_t(a) b \, dt$$

for every $b \in A$.

**Proposition 1.36** Let $a \in M(A)$, $n > 0$ and $z \in \mathbb{C}$. Then $a(n, z)$ is strictly analytic with respect to $\alpha$ and $\alpha_y(a(n, z)) = a(n, z + y)$ for every $y \in \mathbb{C}$.

**Proposition 1.37** Consider $n > 0$, $y, z \in \mathbb{C}$ and $a \in D(\alpha y)$. Then $\alpha_y(a(n, z)) = \alpha_y(a)(n, z)$.

Another useful property of smearing is contained in the following result.

**Proposition 1.38** Let $z \in \mathbb{C}$ and $n > 0$. Consider $a \in M(A)$ and a bounded net $(a_i)_{i \in I}$ in $M(A)$ such that $(a_i)_{i \in I}$ converges strictly to $a$. Then $(a_i(n, z))_{i \in I}$ is also bounded and converges strictly to $a(n, z)$.
2 The definition of KMS-weights

In this section, we introduce the definition of a KMS-weight. We will also introduce the definition of a so-called regular weight. In a later section, it is shown that a KMS-weight is automatically regular. We will also prove a result about such a regular weight which will be very useful in a later section. Also some results about lower semi-continuous weights will be included. The standard reference for lower semi-continuous weights is [3].

Let us first introduce notations and conventions concerning weights. Consider a $C^*$-algebra $A$ and a densely defined weight $\varphi$ on $A$. We use the following notations:

- $M^+_{\varphi} = \{ a \in A^+ \mid \varphi(a) < \infty \}$
- $N_{\varphi} = \{ a \in A \mid \varphi(a^*a) < \infty \}$
- $M_{\varphi} = \text{span} M^+_{\varphi} N_{\varphi}^*$

A GNS-construction of $\varphi$ is by definition a triple $(H, \pi, \Lambda)$ such that

- $H$ is a Hilbert space
- $\Lambda$ is a linear map from $N_{\varphi}$ into $H$ such that
  1. $\Lambda(N_{\varphi})$ is dense in $H$
  2. We have that $\langle \Lambda(a), \Lambda(b) \rangle = \varphi(b^*a)$ for every $a, b \in N_{\varphi}$.
- $\pi$ is a representation of $A$ on $H$ such that $\pi(a) \Lambda(b) = \Lambda(ab)$ for every $a \in A$ and $b \in N_{\varphi}$.

The following concepts play a central role in the theory of lower semi-continuous weights:

- We define the set $F_{\varphi} = \{ \omega \in A_+^* \mid \omega \leq \varphi \}$.
- Put $G_{\varphi} = \{ \alpha \omega \mid \omega \in F_{\varphi}, \alpha \in [0,1] \}$. Then $G_{\varphi}$ is a directed subset of $F_{\varphi}$.

A proof of the last result can be found in proposition [3,3].

The major result about lower semi-continuous weights was proven by Combes (proposition 1.7 of [3]):

**Theorem 2.1** Suppose that $\varphi$ is lower semi-continuous, then we have the following results:

- We have for every $x \in A^+$ that $\varphi(x) = \sup \{ \omega(x) \mid \omega \in F_{\varphi} \}$
- Consider $x \in A^+$, then the net $(\omega(x))_{\omega \in G_{\varphi}}$ converges to $\varphi(x)$
- Consider $x \in M_{\varphi}$, then the net $(\omega(x))_{\omega \in G_{\varphi}}$ converges to $\varphi(x)$

By proposition 2.4 of [3], we have the following result.

**Result 2.2** Consider $\omega \in F_{\varphi}$. Then there exists a unique element $T_\omega \in B(H)$ with $0 \leq T_\omega \leq 1$ such that $\langle T_\omega \Lambda(a), \Lambda(b) \rangle$ for every $a, b \in N_{\varphi}$.

The theorem above implies that $(T_\omega)_{\omega \in G_{\varphi}}$ converges strongly to 1 if $\varphi$ is lower semi-continuous.

**Result 2.3** Suppose that $\varphi$ is lower semi-continuous. Then we have the following properties.
1. The mapping $\Lambda$ is closed.

2. The $^*$-representation $\pi$ is non-degenerate and $\pi(a)\Lambda(b) = \Lambda(ab)$ for every $a \in M(A)$ and $b \in N_\varphi$.

Proof:

1. Choose a sequence $(a_n)^\infty_{n=1} \in N_\varphi$, $a \in A$ and $v \in H$ such that $(a_n)^\infty_{n=1}$ converges to $a$ and $(\Lambda(a_n))^\infty_{n=1}$ converges to $v$.

Choose $\omega \in F_\varphi$. We have for every $n \in \mathbb{N}$ that $\langle T_\omega \Lambda(a_n), \Lambda(a_n) \rangle = \omega(a_n^*a_n)$, which implies that $\langle T_\omega v, v \rangle = \omega(a^*a)$. So we see that $\omega(a^*a) \leq \|v\|^2$.

Hence, theorem 2.1 implies that $a$ belongs to $N_\varphi$.

Choose $\omega \in F_\varphi$.

Take $b \in N_\varphi$. We have for every $n \in N_\varphi$ that $\langle T_\omega \Lambda(a_n), \Lambda(b) \rangle = \omega(b^*a_n)$, which implies that $\langle T_\omega v, \Lambda(b) \rangle = \omega(b^*a) = \langle T_\omega \Lambda(a), \Lambda(b) \rangle$.

This implies that $T_\omega v = T_\omega \Lambda(a)$.

The lower semi-continuity of $\varphi$ implies that $(T_\omega)_{\omega \in F_\varphi}$ converges to 1. Hence $v = \Lambda(a)$.

2. From [3], we know already that $\pi$ is non-degenerate. Choose $a \in M(A)$ and $b \in N_\varphi$.

Take $e \in A$. Then $e\ a$ belongs to $A$ which implies that $e\ b$ belongs to $N_\varphi$ and $\Lambda(e\ a) = \Lambda(e\ b) = \pi(e)\pi(a)\Lambda(b)$.

It is clear that $N_\varphi$ is a left ideal in $M(A)$ so $a\ b$ belongs to $N_\varphi$. This implies that $e\ a$ belongs to $N_\varphi$ and $\Lambda(e\ a\ b) = \pi(e)\Lambda(ab)$.

So we see that $\pi(e)\pi(a)\Lambda(b) = \pi(e)\Lambda(ab)$.

Therefore, the non-degeneracy of $\pi$ implies that $\Lambda(ab) = \pi(a)\Lambda(b)$.

\[ \square \]

If $\varphi$ is lower semi-continuous, the weight $\varphi$ has a natural extension to a weight $\overline{\varphi}$ on $M(A)$ by putting

$$\overline{\varphi}(x) = \sup \{ \omega(x) \mid \omega \in F_\varphi \}$$

for every $x \in M(A)^+.$

Then $\overline{\varphi}$ is the unique strictly lower semi-continuous weight on $M(A)$ which extends $\varphi$. The unicity follows from the fact that any element in $M(A)^+$ can be strictly approximated by an increasing net in $A^+$.

We define $\overline{M}_\varphi = M_\overline{\varphi}$ and $\overline{N}_\varphi = N_\overline{\varphi}$. For any $x \in \overline{M}_\varphi$, we put $\varphi(x) = \overline{\varphi}(x)$.

Theorem 2.1 implies that $\overline{\varphi}$ is an extension of $\varphi$. So we have that $M_\varphi^+ = \overline{M}_\varphi^+ \cap A$, $N_\varphi = \overline{N}_\varphi \cap A$ and $M_\varphi \subseteq \overline{M}_\varphi \cap A$.

Then we have that

- Consider $x \in M(A)^+$, then the net $(\omega(x))_{\omega \in G_\varphi}$ converges to $\varphi(x)$
- Consider $x \in \overline{M}_\varphi$, then the net $(\omega(x))_{\omega \in G_\varphi}$ converges to $\varphi(x)$
Lemma 2.4 Suppose that \( \varphi \) is lower semi-continuous. Then the mapping \( \Lambda : \mathcal{N}_\varphi \mapsto H \) is strictly closable.

Proof: Choose a net \((a_i)_{i \in I}\) in \( \mathcal{N}_\varphi \) and \( v \in H \) such that \((a_i)_{i \in I}\) converges strictly to 0 and such that \((\Lambda(a_i))_{i \in I}\) converges to \( v \).

Take \( e \in A \). Then we have that \((ea_i)_{i \in I}\) converges to 0. We have also for every \( i \in I \) that \( ea_i \) belongs to \( \mathcal{N}_\varphi \) and \( \Lambda(ea_i) = \pi(e)\Lambda(a_i) \). Hence, \((\Lambda(ea_i))_{i \in I}\) converges to \( \pi(e)v \). So the closedness of \( \Lambda \) implies that \( \pi(e)v = 0 \).

Therefore, the non-degeneracy of \( \pi \) implies that \( v = 0 \). \( \blacksquare \)

Definition 2.5 Suppose that \( \varphi \) is lower semi-continuous. Then we define the mapping \( \overline{\Lambda} \) from within \( M(A) \) into \( H \) such that \( \mathcal{N}_\varphi \) is a strict core for \( \overline{\Lambda} \) and \( \overline{\Lambda} \) extends \( \Lambda \). We put \( \Lambda(a) = \overline{\Lambda}(a) \) for every \( a \in D(\overline{\Lambda}) \).

Using result 2.3.2, it is not very difficult to prove for every \( x \in M(A) \) and every \( a \in D(\overline{\Lambda}) \) that \( xa \) belongs to \( D(\overline{\Lambda}) \) and \( \Lambda(xa) = \pi(a)\Lambda(x) \).

Proposition 2.6 Suppose that \( \varphi \) is lower semi-continuous. Then \((H, \overline{\Lambda}, \overline{\pi})\) is a GNS-construction for \( \overline{\varphi} \).

Proof:

- Choose \( a \in D(\overline{\Lambda}) \). By definition, there exists a net \((a_i)_{i \in I}\) in \( \mathcal{N}_\varphi \) such that \((a_i)_{i \in I}\) converges strictly to \( a \) and \((\Lambda(a_i))_{i \in I}\) converges to \( \Lambda(a) \).

Take \( \omega \in F_\varphi \). Then we have for every \( i, j \in I \) that \( \langle T_\omega \Lambda(a_i), \Lambda(a_j) \rangle = \omega(a_j^*a_i) \).

This implies that \( \langle T_\omega \Lambda(a), \Lambda(a) \rangle = \omega(a^*a) \).

By the definition of \( \overline{\varphi} \) and because \( \langle T_\omega, \omega \in G_\varphi \) converges strongly to 1, we get that \( a \) belongs to \( \overline{\mathcal{N}}_\varphi \) and that \( \varphi(a^*a) = \langle \Lambda(a), \Lambda(a) \rangle \).

- Choose \( a \in \overline{\mathcal{N}}_\varphi \). Take an approximate unit \((e_i)_{i \in I}\) of \( A \). Then \((e_i a)_{i \in I}\) converges strictly to \( a \).

We have for every \( i \in I \) that \( e_i a \in \overline{\mathcal{N}}_\varphi \cap A = \mathcal{N}_\varphi \).

Because \((a^*e_i a)_{i \in I}\) is an increasing net in \( \overline{\mathcal{N}}_\varphi \), which converges strictly to \( a^*a \), the strict lower semi-continuity of \( \overline{\varphi} \) implies that \((\varphi(a^*e_i a))_{i \in I}\) converges to \( \varphi(a^*a) \).

We have moreover for every \( j, k \in I \) with \( j \leq k \) that \( 0 \leq e_k - e_j \leq 1 \), so

\[
\|\Lambda(e_j a) - \Lambda(e_j a)\| = \varphi(a^*(e_k - e_j)^2 a) \leq \varphi(a^*(e_k - e_j) a) = \varphi(a^*e_k a) - \varphi(a^*e_j a)
\]

So we get that \((\Lambda(e_i a))_{i \in I}\) is cauchy and hence convergent in \( H \).

Hence, we get by the definition of \( \overline{\Lambda} \) that \( a \) belongs to \( D(\overline{\Lambda}) \). \( \blacksquare \)

Remark 2.7 By definition, we have that \( \mathcal{N}_\varphi \) is a strict core for \( \overline{\Lambda} \). But we have even more:
Consider \( a \in \overline{\mathcal{N}}_\varphi \). Then there exists a net \((a_k)_{k \in K}\) in \( \mathcal{N}_\varphi \) such that

- We have for every \( k \in K \) that \( \|a_k\| \leq \|a\| \) and \( \|\Lambda(a_k)\| \leq \|\Lambda(a)\| \).
• The net \((a_k)_{k \in K}\) converges strictly to \(a\) and the net \((\Lambda(a_k))_{k \in K}\) converges strongly* to \(\Lambda(a)\).

This follows immediately by multiplying \(a\) to the left by an approximate unit of \(A\).

We will not use this extension to the multiplier in this paper. However, most of the results in this paper concerning \(\varphi, \mathcal{M}_\varphi, \mathcal{M}_\varphi^+, \mathcal{N}_\varphi, \Lambda\) have obvious variants concerning \(\varphi, \mathcal{M}_\varphi^+, \mathcal{M}_\varphi, \mathcal{N}_\varphi, \Lambda\).

They can be proven using the results in this paper (using strict approximation arguments) or by similar proofs.

Now we give the definition of a KMS-weight.

**Definition 2.8** Consider a \(C^*\)-algebra \(A\) and a densely defined lower semi-continuous weight \(\varphi\) on \(A\) such that there exist a norm-continuous one parameter group \(\sigma\) on \(A\) such that

1. We have that \(\varphi \sigma_t = \varphi\) for every \(t \in \mathbb{R}\).
2. We have that \(\varphi(a^*a) = \varphi(\sigma_{\frac{t}{2}}(a)\sigma_{\frac{t}{2}}(a)^*)\) for every \(a \in D(\sigma_{\frac{t}{2}})\).

Then \(\varphi\) is called a KMS-weight and \(\sigma\) is called a modular group for \(\varphi\).

Later we will prove that the second condition on \(\sigma\) can be weakened.

This definition of a KMS-weight is different from the usual one (see definition 4.1 of [4]) but we will prove in theorem 5.36 that this definition is equivalent with the usual one.

In [10], Jan Verding introduced the notion of so-called regular weights. We will formulate this definition and prove a useful property concerning regular weights.

**Definition 2.9** Consider a \(C^*\)-algebra \(A\). Let \(\varphi\) be a weight on \(A\) with GNS-construction \((H, \Lambda, \pi)\). We call \(\varphi\) regular \(\iff\) \(\varphi\) is densely defined, lower semi-continuous and there exists a net \((u_i)_{i \in I}\) in \(\mathcal{N}_\varphi\) such that

- We have for every \(i \in I\) that:
  1. There exists \(S_i \in B(H)\) such that \(S_i \Lambda(a) = \Lambda(au_i)\) for every \(a \in \mathcal{N}_\varphi\).
  2. \(\|u_i\| \leq 1\) and \(\|S_i\| \leq 1\).

- Furthermore, we assume that
  1. \((u_i)_{i \in I}\) converges strictly to 1.
  2. \((S_i)_{i \in I}\) converges strongly to 1.

We call the net \((u_i)_{i \in I}\) a truncating net for \(\varphi\).

It is not difficult to see that the definition of regularity is independent of the choice of GNS-construction.

**Proposition 2.10** Consider a regular weight \(\varphi\) on a \(C^*\)-algebra \(A\) with GNS-construction \((H, \Lambda, \pi)\). Let \(B\) be a \(\sigma^*\)-algebra of \(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*\) such that \(B\) is a core for \(\Lambda\) and let \(a\) be an element in \(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*\). Then there exists a sequence \((a_n)_{n=1}^\infty\) in \(B\) such that

1. \((a_n)_{n=1}^\infty \to a\)
2. \((\Lambda(a_n))_{n=1}^\infty \to \Lambda(a)\)
3. \((\Lambda(a_n^*))_{n=1}^{\infty} \to \Lambda(a^*)\).

**Proof:** Choose a truncating net \((u_i)_{i \in I}\) for \(\varphi\). Define for every \(i \in I\) the operator \(S_i \in B(H)\) such that \(S_i \Lambda(a) = \Lambda(au_i)\) for every \(a \in \mathcal{N}_\varphi\).

Choose \(n \in \mathbb{N}\). Because \((u_i)_{i \in I}\) is a bounded net which converges strictly to 1 and \((S_j)_{i \in I}\) is a bounded net which converges strongly to 1, there exist an element \(j \in I\) such that

- \(\|\Lambda(a) - \pi(u_j^*) S_j \Lambda(a)\| \leq \frac{1}{n}\)
- \(\|\Lambda(a^*) - \pi(u_j^*) S_j \Lambda(a^*)\| \leq \frac{1}{n}\)
- \(\|a - u_j^* au_j\| \leq \frac{1}{n}\)

By the definition of \(S_j\), we get that

- \(\|\Lambda(a) - \pi(u_j^*) \pi(a) \Lambda(u_j)\| \leq \frac{1}{n}\)
- \(\|\Lambda(a^*) - \pi(u_j^*) \pi(a^*) \Lambda(u_j)\| \leq \frac{1}{n}\).

Choose \(n \in \mathbb{N}\). Because \(B\) is a core for \(\Lambda\), there exists an element \(b_n \in B\) such that

- \(\|u_j - b_n\| \leq \min\{\frac{1}{2n}, \frac{1}{n}\}\)
- \(\|\Lambda(u_j) - \Lambda(b_n)\| \leq \frac{1}{n}\)
- \(\|b_n\| \leq 2\)

Furthermore, there exists an element \(d_n \in B\) such that

\[
\|a - d_n\| \leq \frac{1}{n}\ \|b_n\|^2 + 1, \quad \frac{1}{n}\ (\|b_n\| + 1)(\|\Lambda(b_n)\| + 1)\]

We define \(a_n = b_n^* d_n b_n\). Then \(a_n\) belongs to \(B\) and \(a_n^* = b_n^* d_n^* b_n\).

Moreover,

1. Using the above estimations, we have that
   \[
   \|\Lambda(a_n) - \Lambda(a)\| = \|\pi(b_n^*) \pi(d_n) \Lambda(b_n) - \Lambda(a)\| \\
   \leq \|\pi(b_n^*) \pi(d_n) \Lambda(b_n) - \pi(b_n^*) \pi(a) \Lambda(u_j)\| + \|\pi(b_n^*) \pi(a) \Lambda(b_n) - \pi(b_n^*) \pi(a) \Lambda(u_j)\| \\
   + \|\pi(b_n^*) \pi(a) \Lambda(u_j) - \pi(u_j^*) \pi(a) \Lambda(u_j)\| + \|\pi(u_j^*) \pi(a) \Lambda(u_j) - \Lambda(a)\| \\
   \leq \|b_n\| \|\Lambda(b_n)\| \|d_n - a\| + \|b_n\| \|a\| \|\Lambda(b_n) - \Lambda(u_j)\| + \|b_n - u_j\| \|a\| \|\Lambda(u_j)\| + \frac{1}{n} \\
   \leq \frac{1}{n} + 2 \|a\| \|\Lambda(b_n) - \Lambda(u_j)\| + \frac{1}{n} + \frac{1}{n} \leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{4}{n}
   \]

2. Similarly, one can prove \(\|\Lambda(a_n^*) - \Lambda(a^*)\| \leq \frac{4}{n}\).

3. Using the above estimations once again, we get that
   \[
   \|a_n - a\| = \|b_n^* d_n b_n - a\| \\
   \leq \|b_n^* d_n b_n - b_n^* ab_n\| + \|b_n^* ab_n - b_n^* au_j\| + \|b_n^* au_j - u_j^* au_j\| + \|u_j^* au_j - a\| \\
   \leq \|b_n\| \|d_n - a\| + \|b_n\| \|a\| \|b_n - u_j\| + \|b_n - u_j\| \|a\| \|u_j\| + \|u_j^* au_j - a\| \\
   \leq \frac{1}{n} + 2 \|a\| \|b_n - u_j\| + \|b_n - u_j\| \|a\| + \frac{1}{n} \leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{4}{n}.
   \]
Looking at 1, 2 and 3, we see that \((a_n)_{n=1}^\infty \to a\), \((\Lambda(a_n))_{n=1}^\infty \to \Lambda(a)\) and \((\Lambda(a_n^*))_{n=1}^\infty \to \Lambda(a^*)\).

**Corollary 2.11** Consider a regular weight \(\varphi\) on a C\(^\ast\)-algebra \(A\) with GNS-construction \((H, \Lambda, \pi)\). Let \(a\) be an element in \(N_\varphi \cap N_\varphi^*\). Then there exists a sequence \((a_n)_{n=1}^\infty\) in \(M_\varphi\) such that

1. \((a_n)_{n=1}^\infty \to a\)
2. \((\Lambda(a_n))_{n=1}^\infty \to \Lambda(a)\)
3. \((\Lambda(a_n^*))_{n=1}^\infty \to \Lambda(a^*)\).

We need the following result from [14] (lemma 3.1). The proof remains valid for lower semi-continuous weights on C\(^\ast\)-algebras.

**Lemma 2.12** Consider a C\(^\ast\)-algebra \(A\) and a densely defined lower semi-continuous weight \(\varphi\) on \(A\). Let \(f\) be a function from \(\mathbb{R}\) into \(M_\varphi^+\) which is continuous and integrable. Then the function \(\mathbb{R} \to \mathbb{R}^+: t \mapsto \varphi(f(t))\) is lower semi-continuous and

\[
\varphi\left(\int f(t)\, dt\right) = \int \varphi(f(t))\, dt.
\]

**Lemma 2.13** Consider a C\(^\ast\)-algebra \(A\) and a densely defined lower semi-continuous weight \(\varphi\) on \(A\), \(\alpha\) a norm continuous one-parameter group on \(A\) such that there exists a strictly positive number \(\lambda\) such that \(\varphi(\alpha_t) = \lambda^t \varphi\) for every \(t \in \mathbb{R}\). Let \(z \in \mathbb{C}\), \(n \in \mathbb{N}\) and \(a \in M_\varphi\).

Then we have that the element \(\int \exp(-(t-z)^2) \alpha_t(a)\, dt\) belongs to \(M_\varphi\) and

\[
\varphi(\int \exp(-(t-z)^2) \alpha_t(a)\, dt) = \lambda^z \sqrt{\pi} \exp\left(\frac{(\ln \lambda)^2}{4}\right) \varphi(a).
\]

**Proof** Choose \(b \in M_\varphi^+\). Define the function \(f : \mathbb{R} \to \mathbb{C} : t \mapsto \exp(-(t-z)^2)\).

There exist continuous positive functions \(f_1, \ldots, f_4\) such that \(f_1, \ldots, f_4 \leq |f|\) and \(f = f_1 - f_2 + if_3 - if_4\).

Fix \(j \in \{1, \ldots, 4\}\). It is clear that the function \(\mathbb{R} \to M_\varphi^+ : t \mapsto f_j(t)\alpha_t(b)\) is continuous. Because we also have that \(\|f_j(t)\alpha_t(b)\| = f_j(t)||b|| \leq |f(t)||b||\) for every \(t \in \mathbb{R}\), we see that the function \(\mathbb{R} \to M_\varphi^+ : t \mapsto f_j(t)\alpha_t(b)\) is integrable. So, the previous lemma implies that the function \(\mathbb{R} \to \mathbb{R}^+ : t \mapsto \varphi(f_j(t)\alpha_t(b))\) is lower semi-continuous and

\[
\varphi(\int f_j(t)\alpha_t(b)\, dt) = \int \varphi(f_j(t)\alpha_t(b))\, dt = (\int f_j(t)\lambda^t\, dt) \varphi(b).
\]

Because \(f_j \leq |f|\), the function \(\mathbb{R} \to \mathbb{R}^+ : t \mapsto f_j(t)\lambda^t\) is integrable. Therefore we see that \(\int f_j(t)\alpha_t(b)\, dt\) belongs to \(M_\varphi\).

Using the fact that \(f = f_1 - f_2 + if_3 - if_4\), we get that \(\int f(t)\alpha_t(b)\, dt\) belongs to \(M_\varphi\) and

\[
\varphi(\int f(t)\alpha_t(b)\, dt) = \left(\int f(t)\lambda^t\, dt\right) \varphi(b).
\]

Therefore,

\[
\varphi(\int f(t)\alpha_t(b)\, dt) = \left(\int \exp(-(t-z)^2) \lambda^t\, dt\right) \varphi(b) = \left(\int \exp(-t^2) \lambda^{t+z}\, dt\right) \varphi(b) = \lambda^z \sqrt{\pi} \exp\left(\frac{(\ln \lambda)^2}{4}\right) \varphi(b).
\]

The lemma follows by linearity.
Proposition 2.14  Consider a $C^*$-algebra $A$ and a densely defined lower semi-continuous weight $\varphi$ on $A$, $\alpha$ a norm continuous one-parameter group on $A$ such that there exists a strictly positive number $\lambda$ such that $\varphi \alpha_t = \lambda^t \varphi$ for every $t \in \mathbb{R}$. Let $z \in \mathbb{C}$ and $a \in M_\varphi \cap D(\alpha_z)$ such that $\alpha_z(a)$ belongs to $M_\varphi$. Then we have that $\varphi(\alpha_z(a)) = \lambda^z \varphi(a)$.

Proof : Define

$$x = \int \exp(-t^2) \alpha_t(\alpha_z(a)) dt .$$

Because $\alpha_z(a)$ belongs to $M_\varphi$, the previous lemma implies that $x$ belongs to $M_\varphi$ and

$$\varphi(x) = \sqrt{\pi} \exp((\ln \lambda)^2/4) \varphi(\alpha_z(a)) .$$

We have also that

$$x = \int \exp(-(t-z)^2) \alpha_t(a) dt ,$$

so the previous lemma implies in this case that

$$\varphi(x) = \lambda^z \sqrt{\pi} \exp((\ln \lambda)^2/4) \varphi(a).$$

Comparing these two different expressions for $\varphi(x)$ gives us that $\varphi(\alpha_z(a)) = \lambda^z \varphi(a)$.

Result 2.15  Consider a densely defined lower semi-continuous non-zero weight $\varphi$ on a $C^*$-algebra $A$. Let $\alpha$ be a norm continuous one parameter group on $A$ such that there exists for every $t$ in $\mathbb{R}$ a number $\lambda_t$ such that $\varphi \alpha_t = \lambda_t \varphi$ for every $t \in \mathbb{R}$. Then there exists a unique strictly positive number $\lambda$ such that $\varphi \alpha_t = \lambda^t \varphi$ for every $t \in \mathbb{R}$.

Proof : The technique of this proof comes from proposition 5.7 of [14] but can be more easily applied in this case. We have certainly that $\lambda_t > 0$ for every $t \in \mathbb{R}$. Put $\lambda = \lambda_1 > 0$.

Because $\alpha_{s+t} = \alpha_s \alpha_t$ for every $s, t \in \mathbb{R}$, we get easily that $\lambda_{s+t} = \lambda_s \lambda_t$ for every $s, t \in \mathbb{R}$. As usual, this implies that $\lambda_q = \lambda^q$ for every $q \in \mathbb{Q}$.

The lower semi-continuity of $\varphi$ implies that the mapping $\mathbb{R} \to \mathbb{R}^{\geq 0} : t \mapsto \lambda_t$ is lower semi-continuous. So the mapping $\mathbb{R} \to \mathbb{R} : t \mapsto \lambda_t - \lambda^t$ is lower semi-continuous. This implies that the set $\{ t \in \mathbb{R} \mid \lambda_t \leq \lambda^t \}$ is closed in $\mathbb{R}$. We know already that this set contains $\mathbb{Q}$, so it must be equal to $\mathbb{R}$.

Therefore $\lambda_t \leq \lambda^t$ for every $t \in \mathbb{R}$. This implies that $\lambda_t = \lambda^t$ for every $t \in \mathbb{R}$ (if there would exist an element in $\mathbb{R}$ for which this equality does not hold, this element or its opposite would violate the previous inequality).

The proof of the following result is due to J. Verding (see [22]).

Result 2.16  Consider a $C^*$-algebra $A$ and a densely defined lower semi-continuous weight $\varphi$ on $A$. Take a GNS-construction $\mathcal{H}$ for $\varphi$. Let $\alpha$ be a norm continuous one parameter group on $A$ such that there exists a strictly positive number $\lambda$ such that $\varphi \alpha_t = \lambda^t \varphi$ for every $t \in \mathbb{R}$. Then there exists a unique injective positive operator $T$ in $\mathcal{H}$ such that $T^{it} \Lambda(a) = \lambda^{-t/2} \Lambda(\alpha_t(a))$ for every $a \in N_\varphi$ and $t \in \mathbb{R}$.

Proof : For every $t \in \mathbb{R}$, there exist a unitary operator $u_t$ on $\mathcal{H}$ such that $u_t \Lambda(a) = \lambda^{-t/2} \Lambda(\alpha_t(a))$ for every $a \in N_\varphi$. The mapping $\mathbb{R} \to B(\mathcal{H}) : t \mapsto u_t$ is clearly a unitary representation of $\mathbb{R}$ on $\mathcal{H}$.
Fix \( \omega \in \mathcal{G}_\varphi \) for the moment. Because \( \omega \leq \varphi \), there exists a unique element \( T_\omega \in B(H)^+ \) such that \( \langle T_\omega \Lambda(a), \Lambda(b) \rangle = \omega(b^*a) \) for every \( a, b \in \mathcal{N}_\varphi \) (see result 2.2).

Because \( \varphi \) is lower semi-continuous, the remarks after result 2.2 imply that \( (T_\omega)_{\omega \in \mathcal{G}_\varphi} \) converges strongly to 1. This implies that the set \( \{ T_\omega \Lambda(a) \mid \omega \in \mathcal{G}_\varphi, a \in \mathcal{N}_\varphi \} \) is dense in \( H \). (*)

Choose \( \theta \in \mathcal{G}_\varphi \) and \( x, y \in \mathcal{N}_\varphi \). We have for every \( t \in \mathbb{R} \) that

\[
\langle u_t \Lambda(x), T_\theta \Lambda(y) \rangle = \lambda^{\frac{t}{\theta}} \langle \Lambda(\alpha_t(x)), T_\theta \Lambda(y) \rangle = \lambda^{\frac{t}{\theta}} \theta(y^* \alpha_t(x)) .
\]

This implies that the mapping \( \mathbb{R} \to \mathbb{C} : t \mapsto \langle u_t \Lambda(x), T_\theta \Lambda(y) \rangle \) is continuous.

Therefore (and because \( \|u_t\| \leq 1 \) for every \( t \in \mathbb{R} \)), result (*) implies that the mapping \( \mathbb{R} \to \mathbb{C} : t \mapsto \langle u_t v, w \rangle \) is continuous for all \( v, w \in H \).

Consequently, the mapping \( \mathbb{R} \to B(H) : t \mapsto u_t \) is a strongly continuous unitary group representation of \( \mathbb{R} \) on \( H \). The result follows by the Stone theorem.

**Terminology 2.17** Consider a \( C^* \)-algebra \( A \) and an element \( T \) affiliated with \( A \). A truncating sequence for \( T \) is by definition a sequence \( (e_n)_{n=1}^\infty \) in \( M(A) \) such that :

1. \( (e_n)_{n=1}^\infty \) is bounded and converges strictly to 1.
2. We have for every \( n \in \mathbb{N} \) that \( e_n \) is a left and right multiplier of \( |T| \) and \( |T| e_n = e_n |T| \).

It is not difficult to check that the functional calculus for \( |T| \) guarantees the existence of such a truncating sequence for \( T \).

Let us fix for the moment an element \( T \) affiliated with a \( C^* \)-algebra \( A \) and a truncating sequence \( (e_n)_{n=1}^\infty \) for \( T \). Then we have the following properties.

- Take \( m \in \mathbb{N} \). We know for every \( x \in A \) that \( e_m x \) belongs to \( D(|T|) \), so \( e_m x \) belongs to \( D(T) \). Furthermore

\[
\|T(e_m x)\| = \| |T|(e_m x)\| .
\]

This implies that \( T e_m \) is a bounded linear operator on \( A \) with \( \|T e_m\| = \| |T| e_m\| \). So \( e_m \) is a right multiplier of \( T \).

- Let \( x \in D(T) \). Then we have for every \( n \in \mathbb{N} \) that

\[
\|T(e_n x) - T(x)\| = \| |T|(e_n x) - |T|(x)\| = \| e_n |T|(x) - |T|(x)\| .
\]

This implies that \( (T(e_n x))_{n=1}^\infty \) converges to \( T(x) \). Combining this with the closeness of \( T \), this implies for every \( x \in A \) that \( x \) belongs to \( D(T) \) if and only if the sequence \( (T(e_n x))_{n=1}^\infty \) is convergent in \( A \).

**Proposition 2.18** Consider a \( C^* \)-algebra \( A \) and an element \( T \) affiliated with \( A \). Let \( \varphi \) be a densely defined lower semi-continuous weight on \( A \) with GNS-construction \( (H, \Lambda, \pi) \). Let \( a \) be an element in \( D(T) \cap \mathcal{N}_\varphi \), then :

1. We have that \( T(a) \) belongs to \( \mathcal{N}_\varphi \) if and only if \( \Lambda(a) \) belongs to \( D(\pi(T)) \).
2. If \( T(a) \) belongs to \( \mathcal{N}_\varphi \), then \( \Lambda(T(a)) = \pi(T)\Lambda(a) \).

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Consider a C*-algebra $A$, a Hilbert space $H$, a dense left ideal $N$ of $A$ and a linear map $\Lambda$ from $N$ into $H$ with dense range. Assume furthermore the existence of a *-homomorphism $\pi$ from $A$ into $B(H)$ such that $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in A$ and $a \in N$.

Notation 3.1 We define the sets

$$F = \{ \omega \in A^+ \mid \text{We have that } \omega(a^*a) \leq \|\Lambda(a)\|^2 \text{ for every } a \in N \}$$

and

$$G = \{ \lambda \omega \mid \omega \in F, \lambda \in [0,1[ \} .$$
Using the fact that $T_{\cong}$. Consequently, we have for every $\gamma \leq 1$ such that $(\Lambda(a), \Lambda(b)) = \omega(b^*a)$ for every $a, b \in N$. We get also that $T$ belongs to $\pi(A)'$. The techniques used to prove these results can also be found in the proof of the next lemma.

**Lemma 3.3** Consider a positive sesquilinear form on $N$ such that $s(ab_1, b_2) = s(b_1, a^*b_2)$ for all $a \in A$ and all $b_1, b_2 \in N$. Suppose moreover that there exists a positive linear functional $\theta$ on $A$ such that $s(b, b) \leq \theta(b^*b)$ for every $b \in N$. Then there exists a unique positive linear functional $\omega$ on $A$ with $\omega \leq \theta$ such that $\omega(b_1^*b_2) = s(b_1, b_2)$ for every $b_1, b_2 \in N$.

**Proof:** Let $(\pi, H, v)$ be GNS-object for $\theta$ (v is a cyclic vector).
Because $s$ is a positive sesquilinear form on $N$, we can use the Cauchy-Schwarz inequality for $s$. So we have for every $b_1, b_2 \in N$ that

\[ |s(b_1, b_2)|^2 \leq s(b_1, b_1) s(b_2, b_2) \leq \theta(b_1^*b_1) \theta(b_2^*b_2) = \|\pi(b_1)v\|^2 \|\pi(b_2)v\|^2. \]

Therefore we can define a continuous positive sesquilinear form $t$ on $H$ such that $t(\pi(b_1)v, \pi(b_2)v) = s(b_1, b_2)$ for every $b_1, b_2 \in N$. It is clear that $t$ is positive and $\|t\| \leq 1$.

So there exist an element $T \in B(H)$ with $0 \leq T \leq 1$ such that $t(x, y) = \langle Tx, y \rangle$ for every $x, y \in H$.

This implies that $\langle T\pi(b_1)v, \pi(b_2)v \rangle = s(b_1, b_2)$ for every $b_1, b_2 \in N$.

Next we show that $T$ belongs to $\pi(A)'$. Therefore, choose $a \in N$.
We have for every $b_1, b_2 \in N$ that

\[ \langle T\pi(a)\pi(b_1)v, \pi(b_2)v \rangle = \langle T\pi(ab_1)v, \pi(b_2)v \rangle = s(ab_1, b_2) = s(b_1, a^*b_2) = \langle T\pi(b_1)v, \pi(a^*b_2)v \rangle = \langle T\pi(b_1)v, \pi(a^*)\pi(b_2)v \rangle = \langle \pi(a)T\pi(b_1)v, \pi(b_2)v \rangle. \]

This implies that $T\pi(a) = \pi(a)T$.

Now we define the continuous linear functional $\omega$ on $A$ such $\omega(x) = \langle T\pi(x)v, v \rangle$ for every $x \in A$.

Using the fact that $T$ belongs to $\pi(A)'$, we have for every $b_1, b_2 \in N$ that

\[ \omega(b_1^*b_2) = \langle T\pi(b_1^*b_2)v, v \rangle = \langle T\pi(b_1)v, \pi(b_2)v \rangle = s(b_1, b_2). \]

Consequently, we have for every $b \in N$ that $\omega(b^*b) = s(b, b)$, implying that $0 \leq \omega(b^*b) \leq \theta(b^*b)$. This implies easily that $0 \leq \omega \leq \theta$.

**Lemma 3.4** Consider a unital $C^*$-algebra $C$. Let $T_1, T_2$ be elements in $C$ with $0 \leq T_1, T_2 \leq 1$, let $\gamma$ be a number in $[0, 1]$. Then there exist an element $T \in C$ with $0 \leq T \leq 1$ and such that $\gamma T_1 \leq T, \gamma T_2 \leq T$ and $T \leq \frac{1}{1-\gamma}(T_1 + T_2)$.

**Proof:** For the moment, fix $i \in \{1, 2\}$. We define

\[ S_i = \frac{\gamma T_i}{1 - \gamma T_i} \in C. \]

It is then easy to check that

\[ \gamma T_i = \frac{S_i}{1 + S_i} \quad (a) \quad \text{and} \quad S_i \leq \frac{\gamma}{1 - \gamma} T_i \quad (b). \]
Next, we define
\[ T = \frac{S_1 + S_2}{1 + S_1 + S_2} \in C. \]
We get immediately that \( 0 \leq T \leq 1 \). By (b), we have that \( T \leq S_1 + S_2 \leq \frac{\gamma^2}{1 - \gamma^2} (T_1 + T_2) \).
We know that the function \( \mathbb{R}^+ \to \mathbb{R}^+: t \mapsto \frac{t}{1 + t} \) is operator monotone (see [13]). This implies for every \( i = 1, 2 \) that
\[ T \geq \frac{S_i}{S_i + 1} = \gamma T_i, \]
where we used (a) in the last equality. \( \blacksquare \)

We now use these two lemmas to proof the following useful proposition.

**Proposition 3.5** The set \( \mathcal{G} \) is upwardly directed for the natural order on \( A^*_+. \)

**Proof:** Choose \( \omega_1, \omega_2 \in \mathcal{F} \) and \( \lambda_1, \lambda_2 \in [0, 1] \). Then there exist a number \( \gamma \in [0, 1] \) such that \( \lambda_1, \lambda_2 < \gamma \).
By the remark after notation [3], we know that there exist \( T_1, T_2 \) in \( \pi(A)' \) such that \( 0 \leq T_1, T_2 \leq 1 \) and such that \( \omega_1(y^* x) = \langle T_1 \Lambda(x), \Lambda(y) \rangle \) and \( \omega_2(y^* x) = \langle T_2 \Lambda(x), \Lambda(y) \rangle \) for every \( x, y \in N \).

The previous lemma implies the existence of \( T \in \pi(A)' \) with \( 0 \leq T \leq 1 \) and such that \( \gamma T_1 \leq T, \gamma T_2 \leq T \) and \( T \leq \frac{\lambda T}{1 - \gamma}(T_1 + T_2) \).

Put \( \lambda = \max(\lambda_1, \lambda_2) \in [0, 1] \). Then \( \lambda T \geq \gamma T_1 \geq \lambda_1 T_1 \) and analogously, \( \lambda T \geq \lambda_2 T_2 \). (a)

Define \( \theta = \frac{\lambda T}{1 - \gamma}(\omega_1 + \omega_2) \in A^*_+ \).

Next, we define the mapping \( s \) from \( N \times N \) into \( \mathcal{G} \) such that \( s(x, y) = \langle T \Lambda(x), \Lambda(y) \rangle \) for every \( x, y \in N \).
Then \( s \) is sesquilinear. Furthermore:

- We have for \( x \in N \) that
\[ 0 \leq \langle T \Lambda(x), \Lambda(y) \rangle \leq \frac{\gamma}{1 - \gamma} \langle (T_1 + T_2) \Lambda(x), \Lambda(y) \rangle = \frac{\gamma}{1 - \gamma} (\omega_1(x^* x) + \omega_2(x^* x)), \]
which implies that \( 0 \leq s(x, x) \leq \theta(x^* x) \).
- We have for every \( x, y \in N \) and \( a \in A \) that
\[
\langle T \Lambda(ax), \Lambda(y) \rangle = \langle T \pi(a) \Lambda(x), \Lambda(y) \rangle = \langle \pi(a) T \Lambda(x), \Lambda(y) \rangle = \langle T \Lambda(x), \pi(a^*) \Lambda(y) \rangle = \langle T \Lambda(x), \Lambda(a^* y) \rangle
\]
which implies that \( s(ax, y) = s(x, a^* y) \).

This allows us to apply the previous lemma. Therefore, we get the existence of \( \omega \in A^*_+ \) with \( \omega \leq \theta \) and such that \( s(x, y) = \omega(y^* x) \) for every \( x, y \in N \).

Then we have for every \( x \in N \) that
\[ \omega(x^* x) = s(x, x) = \langle T \Lambda(x), \Lambda(x) \rangle \leq \| \Lambda(x) \|^2 \]
which implies that \( \omega \) belongs to \( \mathcal{F} \).

By (a), we have moreover for every \( x \in N \) that
\[ \lambda \omega(x^* x) = \lambda \langle T \Lambda(x), \Lambda(x) \rangle \geq \lambda_1 \langle T_1 \Lambda(x), \Lambda(x) \rangle = \lambda_1 \omega_1(x^* x) \]
so we see that \( \lambda_1 \omega_1 \leq \lambda \omega \). We get in a similar way that \( \lambda_2 \omega_2 \leq \lambda \omega \). \( \blacksquare \)

This allows us to give the following definition:
Definition 3.6 We define the weight \( \varphi \) on \( A \) such that \( \varphi(x) = \sup \{ \omega(x) \mid \omega \in \mathcal{F} \} \) for every \( x \in A^+ \). Then \( \varphi \) is a densely defined lower semi-continuous weight on \( A \) such that \( N \subseteq \mathcal{N}_\varphi \) and \( \varphi(x^* x) \leq \| \Lambda(x) \|^2 \) for every \( x \in N \).

In the last proposition of this section, we will improve the relationship between \( \varphi \) and the mapping \( \Lambda \) (under some extra conditions).

Lemma 3.7 Let \( E \) be a normed space, \( H \) a Hilbert space and \( \Lambda \) linear mapping from within \( E \) into \( H \). Let \( (x_i)_{i \in I} \) be a net in \( D(\Lambda) \) and \( x \) an element in \( E \) such that \( (x_i)_{i \in I} \) converges to \( x \) and \( (\Lambda(x_i))_{i \in I} \) is bounded. Then there exists a sequence \( (y_n)_{n=1}^\infty \) in the convex hull of \( \{ x_i \mid i \in I \} \) and an element \( v \in H \) such that \( (y_n)_{n=1}^\infty \) converges to \( x \) and \( (\Lambda(y_n))_{n=1}^\infty \) converges to \( v \).

Proof: By the Banach-Alaoglu theorem, there exists a subnet \( (x_{i,j})_{j \in J} \) of \( (x_i)_{i \in I} \) and \( v \in H \) such that \( (\Lambda(x_{i,j}))_{j \in J} \) converges to \( v \) in the weak topology on \( H \). (For this, we need \( H \) to be a Hilbert space.)

Fix \( n \in \mathbb{N} \). Then there exists \( j_n \in J \) such that \( \| x_{i,j} - x \| \leq \frac{1}{n} \) for all \( j \in J \) with \( j \geq j_n \).

Now \( v \) belongs to the weak-closed convex hull of the set \( \{ \Lambda(x_{i,j}) \mid j \in J \text{ such that } j \geq j_n \} \), which is the same as the norm-closed convex hull.

Therefore, there exist \( \lambda_1, \ldots, \lambda_m \in \mathbb{R}^+ \) with \( \sum_{k=1}^m \lambda_k = 1 \) and elements \( \alpha_1, \ldots, \alpha_m \in J \) with \( \alpha_1, \ldots, \alpha_m \geq j_n \) such that

\[
\| v - \sum_{k=1}^m \lambda_k \Lambda(x_{i_{\alpha_k}}) \| \leq \frac{1}{n}.
\]

Put \( y_n = \sum_{k=1}^m \lambda_k x_{i_{\alpha_k}} \). Then \( y_n \in D(\Lambda) \), and \( \Lambda(y_n) = \sum_{k=1}^m \lambda_k \Lambda(x_{i_{\alpha_k}}) \).

Therefore, we have immediately that \( \| v - \Lambda(y_n) \| \leq \frac{1}{n} \).

Furthermore,

\[
\| x - y_n \| = \left\| \sum_{k=1}^m \lambda_k (x - x_{i_{\alpha_k}}) \right\| \leq \sum_{k=1}^m \lambda_k \frac{1}{n} = \frac{1}{n}
\]

Therefore, we find that \( (y_n)_{n=1}^\infty \) converges to \( x \) and that \( (\Lambda(y_n))_{n=1}^\infty \) converges to \( v \).

Using this lemma, we can prove the result we need later on.

Proposition 3.8 Suppose that

- The mapping \( \Lambda \) is closed.
- There exist a net \( (u_i)_{i \in I} \) in \( N \) and a net \( (S_i)_{i \in I} \) in \( B(H) \) such that
  1. We have for every \( a \in N \) and \( i \in I \) that \( \Lambda(au_i) = S_i \Lambda(a) \).
  2. We have for every \( i \in I \) that \( \| S_i \| \leq 1 \).
  3. \( (u_i)_{i \in I} \) converges strictly to \( 1 \) and \( (S_i)_{i \in I} \) converges strongly to \( 1 \).

Then \( (H, \Lambda, \pi) \) is a GNS-construction for \( \varphi \) and \( \mathcal{F} = F_\varphi \).

We have moreover the following results:

Define for every \( i \in I \) the element \( \omega_i \in A^*_\varphi \) such that \( \omega_i(x) = \varphi(u_i^* x u_i) \) for \( x \in A \). Then we have for every \( i \in I \) that \( \omega_i \) belongs to \( \mathcal{F} \). Furthermore,

1. Consider \( x \in \mathcal{M}_\varphi \), then \( (\omega_i(x))_{i \in I} \) converges to \( \varphi(x) \).
2. Consider $x \in A^+$, then $\varphi(x) = \sup \{ \omega_i(x) \mid i \in I \}$.

**Proof:** Choose $i \in I$. Then we have for every $a \in N$ that

$$\omega_i(a^*a) = \varphi(u_i^*a^*au_i) = \langle \Lambda(au_i), \Lambda(au_i) \rangle = \langle S_i\Lambda(a), S_i\Lambda(a) \rangle.$$  

Therefore the net $(\omega_i(a^*a))_{i \in I}$ is bounded.

Choose $a \in N$. Because $(S_i)_{i \in I}$ converges strongly to 1, equality (*) implies that $(\omega_i(a^*a))_{i \in I}$ converges to $\|\Lambda(a)\|^2$ which implies that $\|\Lambda(a)\|^2 \leq \varphi(a^*a)$, so $\varphi(a^*a) = \|\Lambda(a)\|^2$.

Consider $a \in A$ such that the net $(\omega_i(a^*a))_{i \in I}$ is bounded. We have for every $i \in I$ that $\omega_i \in \mathcal{F}$ which by the definition of $\varphi$ implies that

$$\|\Lambda(au_i)\|^2 = \langle \Lambda(au_i), \Lambda(au_i) \rangle = \varphi(u_i^*a^*au_i) = \omega_i(a^*a)$$

Therefore the net $(\Lambda(au_i))_{i \in I}$ is bounded.

Because $(au_i)_{i \in I}$ converges to $a$, the previous lemma implies the existence of a sequence $(b_n)_{n=1}^\infty$ in $D(\Lambda)$ and $v \in H$ such that $(b_n)_{n=1}^\infty$ converges to $a$ and $(\Delta(b_n))_{n=1}^\infty$ converges to $v$. Therefore the closedness of $\Lambda$ implies that $a$ belongs to $N$.

Because of the definition of $\varphi$, this last result implies also that $\mathcal{N}_\varphi \subseteq N$ which implies that $N = \mathcal{N}_\varphi$.

\[\blacksquare\]

4 Relatively invariant one-parameter groups

In this paper, we will have to use one-parameter groups which are relatively invariant with respect to some closed mappings arising from the GNS-construction for weights. In this section, we will gather some results about such objects.

In this section, we consider two Banach spaces $E, F$, a subspace $N$ of $E$ and a closed linear map $\Lambda$ from $N$ into $F$. Let $\alpha$ be a strongly continuous one-parameter representation on $E$ such that there exists a strongly continuous one-parameter representation $u$ on $F$ together with a strictly positive number $\lambda$ such that we have for every $t \in \mathbb{R}$ and $a \in N$ that $\alpha_t(a)$ belongs to $N$ and $\Lambda(\alpha_t(a)) = \lambda^t u_t \Lambda(a)$.

The techniques used in the proof of the following lemma will be frequently used.

**Lemma 4.1** Consider $y, z \in \mathbb{C}$, $n \in \mathbb{N}$ and $a \in N$. Then

$$\int \exp(-n^2(t-z)^2) \lambda^{nyt} \alpha_t(a) \, dt$$

belongs to $N$ and

$$\Lambda\left( \int \exp(-n^2(t-z)^2) \lambda^{nyt} \alpha_t(a) \, dt \right) = \int \exp(-n^2(t-z)^2) \lambda^{(y+1)lt} u_t \Lambda(a) \, dt.$$
Proof: By assumption, we have for every $t \in \mathbb{R}$ that $\exp(-n^2(t - z)^2) \lambda^{yt} \alpha_t(a)$ belongs to $N$ and
$$\Lambda(\exp(-n^2(t - z)^2) \lambda^{yt} \alpha_t(a)) = \exp(-n^2(t - z)^2) \lambda^{(y + \frac{1}{2})t} u_t \Lambda(a).$$

This implies that the function
$$\mathbb{R} \to F : t \mapsto \Lambda(\exp(-n^2(t - z)^2) \lambda^{yt} \alpha_t(a))$$
is integrable. Using lemma 1.1, we get that
$$\int \exp(-n^2(t - z)^2) \lambda^{yt} \alpha_t(a) \ dt$$
belongs to $N$ and
$$\Lambda\left(\int \exp(-n^2(t - z)^2) \lambda^{yt} \alpha_t(a) \ dt\right) = \int \Lambda(\exp(-n^2(t - z)^2) \lambda^{yt} \alpha_t(a)) \ dt$$
$$= \int \exp(-n^2(t - z)^2) \lambda^{(y + \frac{1}{2})t} u_t \Lambda(a) \ dt.$$

The following lemma follows easily from the previous one.

Lemma 4.2 Consider $a \in N$, $n \in \mathbb{N}$ and define
$$b = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \alpha_t(a) \ dt.$$Then $b$ is analytic with respect to $\alpha$ and we have for every $z \in \mathbb{C}$ that $\alpha_z(b)$ belongs to $N$.

A first application can be found in the following result:

Proposition 4.3 Define the set
$$C = \{ a \in N \mid a is analytic with respect to \alpha and \alpha_z(a) belongs to N for every z \in \mathbb{C} \} .$$Then
1. $C$ is a core for $\Lambda$.
2. Let $z$ be a complex number. If $N$ is dense in $A$, then $C$ is a core for $\alpha_z$.

Proof: Define for every $n \in \mathbb{N}$ and $a \in N$ the element
$$a(n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \alpha_t(a) \ dt ,$$which belongs to $C$ by the previous lemma.

1. Choose $x \in N$. By lemma 1.1, it follows that
$$\Lambda(x(n)) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \lambda^{\frac{t}{2}} u_t \Lambda(x) \ dt$$for every $n \in \mathbb{N}$. This implies that $(\Lambda(x(n)))_{n=1}^{\infty}$ converges to $\Lambda(x)$. It is also clear that $(x(n))_{n=1}^{\infty}$ converges to $x$. 

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2. If \( N \) is dense in \( A \), proposition 1.15 learns that the set \( \{ a(n) \mid a \in N, n \in \mathbb{N} \} \) is a core for \( D(\alpha_z) \). This implies immediately that \( C \) is a core for \( D(\alpha_z) \) in this case.

**Proposition 4.4** Consider \( z \in \mathbb{C} \) and \( a \in N \cap D(\alpha_z) \) such that \( \alpha_z(a) \) belongs to \( N \). Then \( \Lambda(a) \) belongs to \( D(u_z) \) and \( u_z\Lambda(a) = \lambda^{-\frac{z}{2}} \Lambda(\alpha_z(a)) \).

**Proof:** Choose \( n \in \mathbb{N} \) and define
\[
v_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) u_t\Lambda(a) \, dt \in F .
\]
It is clear that \( v_n \) belongs to \( D(u_z) \) and
\[
u_z(v_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 (t - z)^2) u_t\Lambda(a) .
\]
By lemma 4.1, we know that
\[
\frac{n}{\sqrt{\pi}} \int \exp(-n^2 (t - z)^2) \lambda^{-\frac{z}{2}} \alpha_t(a) \, dt
\]
belongs to \( N \) and
\[
\Lambda\left( \frac{n}{\sqrt{\pi}} \int \exp(-n^2 (t - z)^2) \lambda^{-\frac{z}{2}} \alpha_t(a) \, dt \right) = u_z(v_n) \quad (*)
\]
Because \( a \) belongs to \( D(\alpha_z) \), we have that
\[
\frac{n}{\sqrt{\pi}} \int \exp(-n^2 (t - z)^2) \lambda^{-\frac{z}{2}} \alpha_t(a) \, dt = \lambda^{-\frac{z}{2}} \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \lambda^{-\frac{z}{2}} \alpha_t(\alpha_z(a)) \, dt .
\]
Using lemma 4.1 once more, this implies that
\[
\Lambda\left( \frac{n}{\sqrt{\pi}} \int \exp(-n^2 (t - z)^2) \lambda^{-\frac{z}{2}} \alpha_t(a) \, dt \right) = \lambda^{-\frac{z}{2}} \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) u_t\Lambda(\alpha_z(a)) \, dt .
\]
Comparing this equality with equality \( (*) \), we get that
\[
u_z(v_n) = \lambda^{-\frac{z}{2}} \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) u_t\Lambda(\alpha_z(a)) \, dt .
\]
This implies that \( \left( u_z(v_n) \right)_{n=1}^{\infty} \) converges to \( \lambda^{-\frac{z}{2}} \Lambda(\alpha_z(a)) \). It is also clear that \( (v_n)_{n=1}^{\infty} \) converges to \( \Lambda(a) \). Therefore, the closedness of \( u_z \) implies that \( \Lambda(a) \) belongs to \( D(u_z) \) and \( u_z\Lambda(a) = \lambda^{-\frac{z}{2}} \Lambda(\alpha_z(a)) \).

**Proposition 4.5** Consider \( z \in \mathbb{C} \) and \( a \in N \cap D(\alpha_z) \) such that \( \Lambda(a) \) belongs to \( D(u_z) \). Then \( \alpha_z(a) \) belongs to \( N \) and \( \Lambda(\alpha_z(a)) = \lambda^{\frac{z}{2}} u_z\Lambda(a) \).

**Proof:** Choose \( n \in \mathbb{N} \) and define
\[
b_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \alpha_t(a) \, dt .
\]
We also have that
\[
b_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 (t - z)^2) \alpha_t(a) \, dt .
\]
Therefore, lemma 4.1 implies that \( b_n \) belongs to \( N \) and
\[
\Lambda(b_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t - z)^2) \lambda^\frac{z}{n} u_t(a) \, dt.
\]
Because \( \Lambda(a) \) belongs to \( D(u_z) \), we get from the previous equation that
\[
\Lambda(b_n) = \lambda^\frac{z}{n} \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \lambda^\frac{z}{n} u_t(u_z \Lambda(a)) \, dt.
\]
It is clear from this last equation that \( (\Lambda(b_n))_{n=1}^\infty \) converges to \( \lambda^\frac{z}{n} u_z \Lambda(a) \). It is also clear from the definition that \( (b_n)_{n=1}^\infty \) converges to \( \alpha_z(a) \). Therefore, the closedness of \( \Lambda \) implies that \( \alpha_z(a) \) belongs to \( N \) and \( \Lambda(\alpha_z(a)) = \lambda^\frac{z}{n} u_z \Lambda(a) \). \( \square \)

**Corollary 4.6** Consider \( z \in \mathbb{C} \) and \( a \in N \cap D(\alpha_z) \). Then \( \Lambda(a) \) belongs to \( D(u_z) \) \( \iff \) \( \alpha_z(a) \) belongs to \( N \).

## 5 KMS-weights arising from certain GNS-constructions

In [22] and [15], Jan Verding introduced a construction procedure for weights starting from a GNS-construction. We want to repeat this procedure, but we want to end up with a KMS-weight. In order to do so, we have to impose stronger conditions on the ingredients of the construction procedure. In a last part, we describe the natural case where a KMS-weight is obtained from a left Hilbert-algebra.

First we prove some technical results.

Consider a \( C^* \)-algebra \( A \), a Hilbert space \( H \), a dense left ideal \( N \) of \( A \) and a closed linear mapping \( \Lambda \) from \( N \) into \( H \). Let \( \alpha \) be a norm continuous one-parameter group on \( A \) and suppose there exists a positive injective operator \( \nabla \) in \( H \) and a strictly positive number \( \lambda \) such that \( \alpha_t(a) \in N \) and \( \Lambda(\alpha_t(a)) = \lambda^\frac{z}{n} \nabla u_t(\Lambda(a)) \) for every \( t \in \mathbb{R} \) and \( a \in N \).

**Lemma 5.1** Consider \( a \in N \cap N^* \) and \( n \in \mathbb{N} \) and define
\[
b = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \alpha_t(a) \, dt.
\]
Then \( b \) is analytic with respect to \( \alpha \) and \( \alpha_z(b) \) belongs to \( N \cap N^* \) for every \( z \in \mathbb{C} \).

**Proof:** Choose \( z \in \mathbb{C} \). By lemma 4.2, we know already that \( \alpha_z(b) \) belongs to \( N \).

We also have that \( a^* \) belongs to \( N \) and that
\[
b^* = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \alpha_t(a^*) \, dt,
\]
so lemma 4.2 implies in this case that \( \alpha_z(b^*) \) belongs to \( N \). Because \( \alpha_z(b^*) = \alpha_z(b^*) \), we see that \( \alpha_z(b) \) belongs to \( N^* \). \( \square \)

Define the set
\[
C = \{ a \in N \cap N^* \mid a \text{ is analytic with respect to } \alpha \text{ and } \alpha_z(a) \text{ belongs to } N \cap N^* \text{ for every } z \in \mathbb{C} \}. 
\]
Proof: Define for every $n \in \mathbb{N}$ and $a \in \mathbb{N} \cap \mathbb{N}^*$ the element
\[ a(n) = \frac{n}{\sqrt{\pi}} \int \exp(-nt^2) \alpha_t(a) \, dt, \]
which belongs to $C$ by the previous lemma.

1. Choose $x \in \mathbb{N} \cap \mathbb{N}^*$. Again, we see immediately that $(x(n))_{n=1}^{\infty}$ converges to $x$. Because $x$ belongs to $N$, the proof of proposition 4.3 learns us that $(\Lambda(x(n)))_{n=1}^{\infty}$ converges to $\Lambda(x)$.

   We have for every $n \in \mathbb{N}$ that $x(n)^* = x^*(n)$, so the same proof tells us that $(\Lambda(x(n)^*))_{n=1}^{\infty}$ converges to $\Lambda(x^*)$.

2. This follows immediately from the previous statement.

3. This is proven in the same way as the second statement of proposition 4.3.

The proof of the following result is due to A. Van Daele and J. Verding.

Proposition 5.3 Consider $z \in \mathbb{C}$. Then there exists a net $\{u_k\}_{k \in K}$ in $A \cap A^+$ consisting of analytic elements for $\alpha$ and such that
- We have for every $k \in K$ that $\|u_k\| \leq 1$ and $\|\alpha_z(u_k)\| \leq 1$.
- The nets $\{u_k\}_{k \in K}$ and $\{\alpha_z(u_k)\}_{k \in K}$ converge strictly to $1$.

Proof: Because $N$ is a dense left ideal in $A$, we know that there exists a net $(e_q)_{q \in Q}$ in $N \cap A^+$ such that $\|e_q\| \leq 1$ for every $q \in Q$ and such that $(e_q)_{q \in Q}$ converges strictly to $1$.

For every $q \in Q$ and $\delta > 0$, we define the element
\[ e_q(\delta) = \frac{\delta}{\sqrt{\pi}} \int \exp(-\delta^2 t^2) \alpha_t(e_q) \, dt \in A^+ \]
which is clearly analytic with respect to $\alpha$ and satisfies $\|e_q(\delta)\| \leq 1$ and $\|\alpha_z(e_q(\delta))\| \leq \exp(\delta^2 (\text{Im } z)^2)$. Lemma 5.1 implies that $e_q(\delta) \in C$ for every $q \in Q$ and $\delta > 0$.

By proposition 4.3, we have for every $\delta > 0$ that $(e_q(\delta))_{q \in Q}$ and $(\alpha_z(e_q(\delta)))_{q \in Q}$ are bounded nets which converge strictly to $1$.

Let us now define the set
\[ K = \{ (F, n) \mid F \text{ is a finite subset of } A \text{ and } n \in \mathbb{N} \}. \]

On $K$ we put an order such that
\[ (F_1, n_1) \leq (F_2, n_2) \iff F_1 \subseteq F_2 \text{ and } n_1 \leq n_2 \]
for every \((F_1, n_1), (F_2, n_2) \in K\). In this way, \(K\) becomes a directed set.

Let us fix \(k = (F, n) \in K\).

Firstly, there exists an element \(\delta_k > 0\) such that \(|\exp(-\delta_k^2 (\text{Im} \ z)^2) - 1| \leq \frac{1}{2n} \frac{1}{|z|^2 + 1}\) for every \(x \in F\).

Secondly, there exist \(q_k \in Q\) such that

- We have that \(\|e_{q_k}(\delta_k) x - x\| \leq \frac{1}{2n}\) for every \(x \in F\).
- We have that \(\|\alpha_z(e_{q_k}(\delta_k)) x - x\| \leq \frac{1}{2n}\) for every \(x \in F\).
- We have that \(\|x \alpha_z(e_{q_k}(\delta_k)) x - x\| \leq \frac{1}{2n}\) for every \(x \in F\).

Now we define \(u_k = \exp(-\delta_k^2 (\text{Im} \ z)^2) e_{q_k}(\delta_k) \in C \cap A^+\).

It follows that \(u_k\) is analytic with respect to \(\alpha\) and \(\|u_k\| \leq 1\). We have also that

\[
\|\alpha_z(u_k)\| = \exp(-\delta_k^2 (\text{Im} \ z)^2) \|\alpha_z(e_{q_k}(\delta_k))\| \\
\leq \exp(-\delta_k^2 (\text{Im} \ z)^2) \exp(-\delta_k^2 (\text{Im} \ z)) = 1
\]

Now we prove that \((u_k)_{k \in K}\) converges strictly to 1.

Choose \(x \in A\) and \(\varepsilon > 0\). Then there exists \(n_0\) in \(\mathbb{N}\) such that \(\frac{1}{n_0} \leq \varepsilon\). Put \(k_0 = (\{x\}, n_0) \in K\).

Take \(k = (F, n) \in K\) such that \(k \geq k_0\), so \(x\) belongs to \(F\) and \(n_0 \leq n\). Therefore,

\[
\|u_k x - x\| = \|\exp(-\delta_k^2 (\text{Im} \ z)^2) e_{q_k}(\delta_k) x - x\| \\
\leq \|\exp(-\delta_k^2 (\text{Im} \ z)^2) e_{q_k}(\delta_k) x - \exp(-\delta_k^2 (\text{Im} \ z)^2) x\| + \|\exp(-\delta_k^2 (\text{Im} \ z)^2) x - x\| \\
= \exp(-\delta_k^2 (\text{Im} \ z)^2) \|e_{q_k}(\delta_k) x - x\| + |\exp(-\delta_k^2 (\text{Im} \ z)^2) - 1| \|x\| \\
\leq \|e_{q_k}(\delta_k) x - x\| + \frac{1}{2n(|x|/1)} \|x\| \leq \frac{1}{2n} + \frac{1}{n} = \frac{1}{n} \leq \varepsilon.
\]

Because \((u_k)_{k \in K}\) consists of selfadjoint elements, this implies that \((u_k)_{k \in K}\) converges strictly to 1.

Completely analogously, one proves that \((\alpha_z(u_k)x)_{k \in K}\) converges to \(x\) for every \(x \in A\) (just replace \(u_k\) by \(\alpha_z(u_k)\) in the proof above). Similarly one proves that \((x \alpha_z(u_k))_{k \in K}\) converges to \(x\) for every \(x \in A\).

Consequently, we have that \((\alpha_z(u_k))_{k \in K}\) converges strictly to 1. \(\blacksquare\)

### 5.1 The first construction procedure.

In the first construction procedure, we will use the following ingredients:

Consider a \(C^*\)-algebra \(A\) and a Hilbert space \(H\). Let \(N\) be a dense left ideal of \(A\), \(\Lambda\) a closed linear mapping from \(N\) into \(H\) with dense range and \(\pi\) a non-degenerate representation of \(A\) on \(H\) such that \(\pi(x)\Lambda(a) = \Lambda(xa)\) for every \(x \in A\) and \(a \in N\).

Furthermore, assume the existence of

- a norm continuous one-parameter group \(\sigma\) on \(A\)
- an injective positive operator \(\nabla\) in \(H\)

such that:

- We have for every \(a \in N\) and \(t \in \mathbb{R}\) that \(\sigma_t(a)\) belongs to \(N\) and \(\Lambda(\sigma_t(a)) = \nabla^t \Lambda(a)\).
- There exists a core \(K\) for \(\Lambda\) such that
1. \( K \subseteq D(\sigma_2), \sigma_2(K)^* \subseteq N \) and \( \|\Lambda(x)\| = \|\Lambda(\sigma_2(x)^*)\| \) for every \( x \in K \).

2. \( \sigma_t(K) \subseteq K \) for every \( t \in \mathbb{R} \).

Our first objective is to extend the property about \( K \) to \( N \cap D(\sigma_2) \).

In this section, we will use the following notation: For every \( a \in A \) and \( n \in \mathbb{N} \), we put
\[
a(n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \sigma_t(a) \, dt,
\]
it is clear that \( a(n) \) is analytic with respect to \( \sigma \).

Furthermore, we know from lemma 4.1 that we have for every \( x \in K \) and \( n \in \mathbb{N} \) that
\[
\Lambda(a(n)) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \nabla^{it} \Lambda(a) \, dt.
\]

Lemma 5.4 We have the following properties:

1. We have for every \( x \in K \) and \( n \in \mathbb{N} \) that \( x(n) \in N \cap D(\sigma_2) \) and \( \sigma_2(x(n))^* \in N \).

2. We have for every \( x, y \in K \) and \( m, n \in \mathbb{N} \) that
\[
\langle \Lambda(y(n)), \Lambda(x(m)) \rangle = \langle \Lambda(\sigma_2(x(m))^*), \Lambda(\sigma_2(y(n))^*) \rangle.
\]

Proof:

1. Choose \( x \in K \) and \( n \in \mathbb{N} \). By equation 1, we know already that \( x(n) \) belongs to \( N \). We also have that \( x(n) \) belongs to \( D(\sigma_2) \). The fact that \( x \) belongs to \( D(\sigma_2) \) implies that
\[
\sigma_2(x(n))^* = (\sigma_2(x(n))^*)^* = \sigma_2(x)^*(n) \quad \text{(a)}
\]

Therefore, equation 1 and the fact that \( \sigma_2(x)^* \in N \) imply that \( \sigma_2(x(n))^* \) belongs to \( N \).

2. Choose \( x, y \in K \) and \( m, n \in \mathbb{N} \). Using equation (a) from the first part of the proof and equation 1 once more, we get the following equalities
\[
\langle \Lambda(\sigma_2(x(m))^*), \Lambda(\sigma_2(y(n))^*) \rangle = \langle \Lambda(\sigma_2(x(m))^*), \Lambda(\sigma_2(y(n))^*) \rangle
\]
\[
= \frac{mn}{\pi} \int \int \exp(-(m^2s^2 + n^2t^2)) \langle \nabla^{is} \Lambda(\sigma_2(x)^*), \nabla^{it} \Lambda(\sigma_2(y)^*) \rangle \, ds \, dt \quad \text{(b)}
\]

By polarisation, we get that
\[
\langle \Lambda(a), \Lambda(b) \rangle = \langle \Lambda(\sigma_2(b)^*), \Lambda(\sigma_2(a)^*) \rangle
\]
for every \( a, b \in K \).

Fix \( s, t \in \mathbb{R} \). By assumption, we know that \( \sigma_s(x), \sigma_t(y) \) belong to \( K \). So, using the previous equality, we see that
\[
\langle \nabla^{is} \Lambda(\sigma_2(x)^*), \nabla^{it} \Lambda(\sigma_2(y)^*) \rangle = \langle \Lambda(\sigma_s(\sigma_2(x)^*)), \Lambda(\sigma_t(\sigma_2(y)^*)) \rangle
\]
\[
= \langle \Lambda(\sigma_2(\sigma_s(x))^*), \Lambda(\sigma_2(\sigma_t(y))^*) \rangle = \langle \Lambda(\sigma_s(\sigma_t(y))), \Lambda(\sigma_s(x)) \rangle = \langle \nabla^{it} \Lambda(y), \nabla^{is} \Lambda(x) \rangle
\]

Substituting this equality into equation (b) gives us that
\[
\langle \Lambda(\sigma_2(x(m))^*), \Lambda(\sigma_2(y(n))^*) \rangle = \frac{mn}{\pi} \int \int \exp(-(m^2s^2 + n^2t^2)) \langle \nabla^{it} \Lambda(y), \nabla^{is} \Lambda(x) \rangle \, ds \, dt
\]
\[
= \langle \Lambda(y(n)), \Lambda(x(m)) \rangle,
\]
where we used equation 1 in the last equality.
Lemma 5.5 We have the following properties:

1. We have for every \( x \in N \) and \( n \in \mathbb{N} \) that \( x(n) \in N \cap D(\sigma_\pi) \) and \( \sigma_\pi(x(n))^* \in N \).

2. We have for every \( x, y \in N \) and \( m, n \in \mathbb{N} \) that

\[
\langle \Lambda(y(n)), \Lambda(x(m)) \rangle = \langle \Lambda(\sigma_\pi(x(m))^*), \Lambda(\sigma_\pi(y(n))^*) \rangle.
\]

Proof:

1. Choose \( x \in N \) and \( n \in \mathbb{N} \). We know by equation \( \square \) that \( x(n) \) belongs to \( N \). We have also that \( x(n) \) belongs to \( D(\sigma_\pi) \).

Because \( K \) is a core for \( \Lambda \), there exists a sequence \( (x_k)_{k=1}^\infty \) in \( K \) such that \( (x_k)_{k=1}^\infty \to x \) and \( (\Lambda(x_k))_{k=1}^\infty \to \Lambda(x) \). From the previous lemma we know that \( x_k(n) \) belongs to \( N \cap D(\sigma_\pi) \) and that \( \sigma_\pi(x_k(n))^* \) belongs to \( N \) for every \( k \in \mathbb{N} \).

For every \( k \in \mathbb{N} \), we have that

\[
\sigma_\pi(x_k(n))^* = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t + \frac{i}{2})^2) \sigma_i(x_k^*) \, dt.
\]

and similarly,

\[
\sigma_\pi(x(n))^* = \frac{n}{\sqrt{\pi}} \int \exp(-n^2(t + \frac{i}{2})^2) \sigma_i(x^*) \, dt.
\]

Comparing these two equations and remembering that \( (x_k)_{k=1}^\infty \) converges to \( x \), we get that \( (\sigma_\pi(x_k(n))^*)_{k=1}^\infty \) converges to \( \sigma_\pi(x(n))^* \). (a)

Because \( (\Lambda(x_k))_{k=1}^\infty \) converges to \( \Lambda(x) \), it is clear from equation \( \square \) that \( (\Lambda(x_k(n)))_{k=1}^\infty \) converges to \( \Lambda(x(n)) \). By the previous lemma, we have for every \( k, l \in \mathbb{N} \) that

\[
||\Lambda(\sigma_\pi(x_l(n))^*) - \Lambda(\sigma_\pi(x_k(n))^*)|| = ||\Lambda(x_l(n)) - \Lambda(x_k(n))||
\]

Therefore, the convergence of \( (\Lambda(x_k(n)))_{k=1}^\infty \) implies that \( (\Lambda(\sigma_\pi(x_k(n))^*))_{k=1}^\infty \) is a Cauchy sequence and hence convergent. (b)

Using the closedness of \( \Lambda \) and conclusions (a) and (b), we get that \( \sigma_\pi(x(n))^* \) belongs to \( N \) and that \( (\Lambda(\sigma_\pi(x_k(n))^*))_{k=1}^\infty \) converges to \( \Lambda(\sigma_\pi(x(n))^*) \).

2. Choose \( x, y \in N \) and \( m, n \in \mathbb{N} \). Then there exists sequences \( (x_k)_{k=1}^\infty \), \( (y_k)_{k=1}^\infty \) in \( K \) such that \( (x_k)_{k=1}^\infty \to x \), \( (y_k)_{k=1}^\infty \to y \), \( (\Lambda(x_k))_{k=1}^\infty \to \Lambda(x) \) and \( (\Lambda(y_k))_{k=1}^\infty \to \Lambda(y) \).

By the first part of the proof, we know that \( (\Lambda(x_k(m)))_{k=1}^\infty \to \Lambda(x(m)) \), \( (\Lambda(y_k(n)))_{k=1}^\infty \to \Lambda(y(n)) \), \( (\Lambda(\sigma_\pi(x_k(m))^*))_{k=1}^\infty \to \Lambda(\sigma_\pi(x(m))^*) \) and \( (\Lambda(\sigma_\pi(y_k(n))^*))_{k=1}^\infty \to \Lambda(\sigma_\pi(y(n))^*) \). By the previous lemma, we know that

\[
\langle \Lambda(y_k(n)), \Lambda(x_k(m)) \rangle = \langle \Lambda(\sigma_\pi(x_k(m))^*), \Lambda(\sigma_\pi(y_k(n))^*) \rangle
\]

for every \( k \in \mathbb{N} \). This implies that

\[
\langle \Lambda(y(n)), \Lambda(x(m)) \rangle = \langle \Lambda(\sigma_\pi(x(m))^*), \Lambda(\sigma_\pi(y(n))^*) \rangle.
\]
At last, we arrive at a final form.

**Proposition 5.6** Consider \( x \in N \cap D(\sigma_{\sharp}) \). Then \( \sigma_{\sharp}^*(x) \) belongs to \( N \) and \( \| \Lambda(x) \| = \| \Lambda(\sigma_{\sharp}^*(x)) \| \).

**Proof:** By the previous lemma, we have for every \( n \in \mathbb{N} \) that \( x(n) \) belongs to \( N \cap D(\sigma_{\sharp}) \), \( \sigma_{\sharp}^*(x(n)) \) belongs to \( N \) and \( \| \Lambda(x(n)) \| = \| \Lambda(\sigma_{\sharp}^*(x(n))) \| \).

By equation 1, we have for every \( n \in \mathbb{N} \) that
\[
\Lambda(x(n)) = \frac{n}{\sqrt{\pi}} \int \exp(-t^2) \nabla t \Lambda(x) \, dt
\]
which implies that \( (\Lambda(x(n)))_{n=1}^{\infty} \rightarrow \Lambda(x) \).

By the previous lemma, we have for every \( m, n \in \mathbb{N} \) that
\[
\| \Lambda(\sigma_{\sharp}^*(x(m))) - \Lambda(\sigma_{\sharp}^*(x(n))) \| = \| \Lambda(x(m)) - \Lambda(x(n)) \|
\]
Hence, the convergence of \( (\Lambda(x(n)))_{n=1}^{\infty} \) implies that \( (\Lambda(\sigma_{\sharp}^*(x(n))))_{n=1}^{\infty} \) is a Cauchy sequence and hence convergent.

Because \( x \) belongs to \( D(\sigma_{\sharp}) \), we have for every \( n \in \mathbb{N} \) that
\[
\sigma_{\sharp}^*(x(n)) = \frac{n}{\sqrt{\pi}} \int \exp(-t^2) \sigma_t(\sigma_{\sharp}^*(x)^*) \, dt,
\]
from which it follows easily that \( (\sigma_{\sharp}^*(x(n)))_{n=1}^{\infty} \) converges to \( \sigma_{\sharp}^*(x)^* \).

Hence, the closedness of \( \Lambda \) implies that \( \sigma_{\sharp}^*(x)^* \) belongs to \( N \) and \( (\Lambda(\sigma_{\sharp}^*(x(n))))_{n=1}^{\infty} \rightarrow \Lambda(\sigma_{\sharp}^*(x)^*) \).

Because \( \| \Lambda(\sigma_{\sharp}^*(x(n))) \| = \| \Lambda(x(n)) \| \) for every \( n \in \mathbb{N} \), we will also have that \( \| \Lambda(\sigma_{\sharp}^*(x)^*) \| = \| \Lambda(x) \| \).

Using the fact that \( \sigma_{\sharp}^*(\sigma_{\sharp}^*(x)^*) = x \) for every \( x \in D(\sigma_{\sharp}) \), the following corollary follows readily.

**Corollary 5.7** Consider \( x \in D(\sigma_{\sharp}) \). Then \( x \) belongs to \( N \) if and only if \( \sigma_{\sharp}^*(x)^* \) belongs to \( N \).

Because \( K \subseteq N \cap D(\sigma_{\sharp}) \), we have that \( N \cap D(\sigma_{\sharp}) \) is a core for \( \Lambda \). Therefore, \( \Lambda(N \cap D(\sigma_{\sharp})) \) will certainly be dense in \( H \). This allows us to introduce the following definition.

**Definition 5.8** We define the anti-unitary operator \( J \) on \( H \) such that \( J\Lambda(x) = \Lambda(\sigma_{\sharp}^*(x)^*) \) for every \( x \in N \cap D(\sigma_{\sharp}) \).

It follows immediately that \( J\Lambda(N \cap D(\sigma_{\sharp})) \subseteq \Lambda(N \cap D(\sigma_{\sharp})) \) and that \( J(J\Lambda(x)) = \Lambda(x) \) for every \( x \in N \cap D(\sigma_{\sharp}) \). This implies that \( J^2 = 1 \).

The next proposition is crucial in the construction of a weight out of our ingredients.

**Proposition 5.9** Consider \( a \in D(\sigma_{\sharp}) \) and \( x \in N \). Then \( x \)a belongs to \( N \) and \( \Lambda(\sigma_{\sharp}^*(a)^*) = J\pi(\sigma_{\sharp}^*(a)^*)J\Lambda(x) \)

**Proof:** Choose \( b, y \in N \cap D(\sigma_{\sharp}) \). Then we have that \( yb \in N \cap D(\sigma_{\sharp}) \). So, by the definition of \( J \), we have that
\[
J\Lambda(yb) = \Lambda(\sigma_{\sharp}(yb)^*) = \Lambda(\sigma_{\sharp}(b)^*\sigma_{\sharp}(y)^*) = \pi(\sigma_{\sharp}(b)^*)\Lambda(\sigma_{\sharp}(y)^*) = \pi(\sigma_{\sharp}(b)^*)J\Lambda(y).
\]
Therefore, \( \Lambda(yb) = J\pi(\sigma_x(b))^*J\Lambda(y) \).

Proposition 4.3 implies the existence of sequences \( (a_n)_{n=1}^\infty \), \( (x_n)_{n=1}^\infty \) in \( N \cap D(\sigma_x) \) such that \( (a_n)_{n=1}^\infty \) converges to \( a \), \( (x_n)_{n=1}^\infty \) converges to \( x \), \( (\sigma_x(a_n))_{n=1}^\infty \) converges to \( \sigma_x(a) \) and \( (\Lambda(x_n))_{n=1}^\infty \) converges to \( \Lambda(x) \).

This implies immediately that \( (x_n,a_n)_{n=1}^\infty \) converges to \( xa \).

By the first part of this proof, we know that \( x_n a_n \) belong to \( N \) and

\[
\Lambda(x_n a_n) = J\pi(\sigma_x(a_n))^*J\Lambda(x_n)
\]

for every \( n \in \mathbb{N} \). This implies that

\[
\left( \Lambda(x_n a_n) \right)_{n=1}^\infty \to J\pi(\sigma_x(a))^*J\Lambda(x).
\]

The closedness of \( \Lambda \) implies that \( xa \) belongs to \( N \) and \( \Lambda(xa) = J\pi(\sigma_x(a))^*J\Lambda(x) \).

Define the set

The following lemma is due to J. Verding and can be found in [22].

**Lemma 5.10** There exists a net \((u_k)_{k \in K} \in N \cap A^+\) and a net \((T_k)_{k \in K} \in \pi(A)'\) such that

1. We have for every \( k \in K \) that \( \|u_k\| \leq 1 \) and \( \|T_k\| \leq 1 \).
2. We have for every \( k \in K \) that \( u_k \) is analytic with respect to \( \sigma \).
3. \((u_k)_{k \in K} \) converges strictly to \( 1 \) and \((T_k)_{k \in K} \) converges strongly* to \( 1 \).
4. For every \( a \in N \) and \( k \in K \), we have that \( \Lambda(au_k) = T_k \Lambda(a) \).

**Proof:** By proposition 5.8 we get the existence of a net \((u_k)_{k \in K} \in N \cap A^+\) consisting of elements which are analytic with respect to \( \sigma \) and such that

- We have for every \( k \in K \) that \( \|u_k\| \leq 1 \) and \( \|\sigma_x(u_k)\| \leq 1 \).
- The nets \((u_k)_{k \in K} \) and \((\sigma_x(u_k))_{k \in K} \) converge strictly to \( 1 \).

Choose \( k \in K \) and define the element \( T_k = J\pi(\sigma_x(u_k))^*J \in B(H) \). We have immediately that \( \|T_k\| \leq 1 \).

Proposition 5.8 implies that \( \Lambda(au_k) = T_k \Lambda(a) \) for every \( a \in N \).

We have also that \( T_k \) belongs to \( \pi(A)' \) : Choose a \( A \). Then we have for every \( b \in N \) that

\[
\pi(a)T_k\Lambda(b) = \pi(a)\Lambda(bu_k) = \pi(ab)\Lambda(u_k) = T_k\Lambda(ab) = T_k\pi(a)\Lambda(b).
\]

Hence, \( \pi(a)T_k = T_k\pi(a) \).

It is clear that \((T_k)_{k \in K} \) converges strongly* to \( 1 \).

This proposition allows us to use the construction procedure of J. Verding to get a weight (see definition 3.6 and proposition 5.8).

**Definition 5.11** Define the set

\[
F = \{ \omega \in A^*_+ \mid \omega(a^*a) \leq \|\Lambda(a)\|^2 \text{ for every } a \in N \}.
\]

We define the mapping \( \varphi \) from \( A^+ \) to \([0, \infty] \) such that

\[
\varphi(x) = \sup \{ \omega(x) \mid \omega \in F \}
\]

for every \( x \in A^+ \). Then \( \varphi \) is a densely defined lower semi-continuous weight on \( A \). We have in this case also that \( N_\varphi = N \) and \( \varphi(b^*a) = \langle \Lambda(a), \Lambda(b) \rangle \) for every \( a, b \in N \). Therefore, \((H, \pi, \Lambda)\) is a GNS-construction for \( \varphi \).
Proposition 5.6 implies immediately that $\varphi$ is a KMS-weight with respect to $\sigma$.

### 5.2 The second construction procedure.

In the next part of this section, we want to give a second construction procedure for KMS-weights, starting with slightly different conditions on the ingredients than in the previous case.

In this section we will use the following ingredients:

- Consider a $C^*$-algebra $A$ and a Hilbert space $H$.
- Let $N$ be a dense left ideal of $A$.
- Let $\Lambda$ be a closed linear mapping from $N$ into $H$ with dense range.
- Let $\pi$ be a non-degenerate representation of $A$ on $H$ such that $\pi(x)\Lambda(a) = \Lambda(xa)$ for every $x \in A$ and $a \in N$.
- Furthermore, assume the existence of
  - a norm continuous one-parameter group $\sigma$ on $A$
  - an injective positive operator $\nabla$ in $H$
  - an anti-unitary operator $I$ on $H$

such that:

- We have for every $a \in N$ and $t \in \mathbb{R}$ that $\sigma_t(a)$ belongs to $N$ and $\Lambda(\sigma_t(a)) = \nabla^t \Lambda(a)$.
- We have for every $x \in N$ and $a \in D(\sigma_z)$ that $xa$ belongs to $N$ and $\Lambda(xa) = I\pi(\sigma_z(a))^* I^* \Lambda(x)$.

If we examine the proof of lemma 5.10, we see that we only use proposition 5.9 and the fact that $\Lambda$ is invariant under $\sigma$, so the conclusion of lemma 5.10 still holds in this case. This allows us to define a weight $\varphi$ in the same way as before (see definition 3.6 and proposition 3.8).

**Definition 5.12** Define the set

$$F = \{ \omega \in A^*_+ \mid \omega(a^*a) \leq \|\Lambda(a)\|^2 \text{ for every } a \in N \}.$$  

We define the mapping $\varphi$ from $A^+$ to $[0, \infty]$ such that

$$\varphi(x) = \sup \{ \omega(x) \mid \omega \in F \}$$

for every $x \in A^+$. Then $\varphi$ is a densely defined lower semi-continuous weight on $A$. We have that $\mathcal{N}_\varphi = N$ and $\varphi(b^*a) = \langle \Lambda(a), \Lambda(b) \rangle$ for every $a, b \in N$. Therefore, $(H, \pi, \Lambda)$ is a GNS-construction for $\varphi$.

We still have to prove that $\varphi$ is a KMS-weight with respect to $\sigma$.

Define the set

$$C = \{ a \in N \cap N^* \mid a \text{ is analytic with respect to } \sigma \text{ and } \sigma_z(a) \text{ belongs to } N \cap N^* \text{ for every } z \in \mathbb{C} \}.$$  

Then $C$ is a sub-$^*$-algebra of $A$ such that $C \subset N \cap D(\sigma_z)$ and $\sigma_z(C)^* \subset N$. It is also clear that $\sigma_z(C) \subset C$ for every $z \in \mathbb{C}$.

**Lemma 5.13** We have that $C$ is a core for $\Lambda$.

**Proof:** We prove first that $N \cap N^*$ is a core for $\Lambda$.  

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Choose \( x \in N \). Because \( N^* \) is dense in \( A \), there exists a bounded net \( (e_k)_{k \in K} \) in \( N^* \) such that \( (e_k)_{k \in K} \) converges strictly to 1. For every \( k \in K \), we have that \( e_k a \) belongs to \( N^* N \), which is a subset of \( N \cap N^* \). Because \( \Lambda(e_k a) = \pi(e_k) \Lambda(a) \) for every \( k \in K \), we see that \( (\Lambda(e_k a))_{k \in K} \) converges to \( \Lambda(a) \). It is also clear that \( (e_k a)_{k \in K} \) converges to \( a \).

Therefore, proposition 5.2 implies that \( C \) is a core for \( \Lambda \). □

**Proposition 5.14** We have that \( \varphi \) is a KMS-weight with modular group \( \sigma \).

**Proof:** Choose \( a \in C \).

Take \( x \in C \). Because \( a \) belongs to \( D(\sigma_{\frac{1}{2}}) \), we have by assumption that

\[
\langle \pi(x)\Lambda(a), \Lambda(a) \rangle = \langle I\pi(\sigma_{\frac{1}{2}}(a))^* I \Lambda(x), \Lambda(a) \rangle = \langle \Lambda(x), I\pi(\sigma_{\frac{1}{2}}(a)) I^* \Lambda(a) \rangle
\]

\[
= \langle \Lambda(x), I\pi(\sigma_{-\frac{1}{2}}(a^*)) I^* \Lambda(a) \rangle = \langle \Lambda(x), I\pi(\sigma_{\frac{1}{2}}(\sigma_{-i}(a^*))) I^* \Lambda(a) \rangle
\]

\[
= \langle \Lambda(x), \pi(a) \Lambda(\sigma_{-i}(a^*)) \rangle = \langle \Lambda(a^* x), \Lambda(\sigma_{-i}(a^*)) \rangle
\]


where, in the second last equality, we used one of the assumptions of this second construction procedure once again. We know that \( \sigma_{-\frac{1}{2}}(a^*) \) belongs to \( N \cap D(\sigma_{-\frac{1}{2}}) \) and \( \sigma_{-\frac{1}{2}}(\sigma_{\frac{1}{2}}(a^*)) = \sigma_{-i}(a^*) \in N \). Hence, proposition 4.4 implies that \( \Lambda(\sigma_{\frac{1}{2}}(a^*)) \) belongs to \( D(\nabla_{\frac{1}{2}}) \) and

\[
\nabla_{\frac{1}{2}} \Lambda(\sigma_{\frac{1}{2}}(a^*)) = \Lambda(\sigma_{-i}(a^*))
\]

Similarly, we have that \( a^* x \) belongs to \( N \cap D(\sigma_{-\frac{1}{2}}) \) and \( \sigma_{-\frac{1}{2}}(\sigma_{\frac{1}{2}}(a^*)) \) belongs to \( N \). Therefore, \( \Lambda(a^* x) \) belongs to \( D(\nabla_{\frac{1}{2}}) \) and

\[
\nabla_{\frac{1}{2}} \Lambda(a^* x) = \Lambda(\sigma_{-\frac{1}{2}}(a^* x))
\]

Using these two results, equation (*) implies that

\[
\langle \pi(x)\Lambda(a), \Lambda(a) \rangle = \langle \Lambda(a^* x), \nabla_{\frac{1}{2}} \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle = \langle \nabla_{\frac{1}{2}} \Lambda(a^* x), \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle
\]

\[
= \langle \Lambda(\sigma_{\frac{1}{2}}(a^* x)), \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle = \langle \Lambda(\sigma_{-\frac{1}{2}}(a^*)), \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle
\]

\[
= \langle I\pi(\sigma_{\frac{1}{2}}(\sigma_{-\frac{1}{2}}(a^*))) I^* \Lambda(\sigma_{\frac{1}{2}}(a^*)), \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle
\]

\[
= \langle I\pi(x) I^* \Lambda(\sigma_{\frac{1}{2}}(a^*)), \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle.
\]

Because \( C \) is dense in \( A \) and \( \pi \) is non-degenerate, this equality implies easily that we can replace \( x \) in this equality by 1. Hence,

\[
\langle \Lambda(a), \Lambda(a) \rangle = \langle \Lambda(\sigma_{\frac{1}{2}}(a^*)), \Lambda(\sigma_{\frac{1}{2}}(a^*)) \rangle
\]

The conclusion follows from proposition 5.6. □

### 5.3 The last one.

In the last part of this section, we describe a natural way to get a KMS-weight on a C*-algebra from a left Hilbert algebra.

Consider a Hilbert space \( H \) and a left Hilbert algebra \( \mathcal{U} \) on \( H \) with modular operator \( \nabla \), modular conjugation \( J \) and associated faithful semi-finite normal weight \( \tilde{\varphi} \) on \( \mathcal{L}(\mathcal{U}) \). Hence

\[
\mathcal{N}_{\tilde{\varphi}} = \{ Lv \mid v \in H \text{ such that } v \text{ is left bounded with respect to } \mathcal{U} \}
\]
and $\hat{\varphi}((L_v)^*(L_w)) = \langle v, w \rangle$ for every $v, w$ in $H$ which are left bounded with respect to $U$. We have also that
\[
N_\varphi \cap N_\varphi^* = \{ L_v \mid v \in U'' \}.
\]

Let $A$ be a C$^*$-algebra and $\pi$ a nondegenerate $*$-representation of $A$ on $H$ such that $\pi(A) \subseteq \mathcal{L}(U)$ and suppose there exists a one-parameter group $\sigma$ on $A$ such that $\pi(\sigma(t)) = \nabla^{it}\pi(a)\nabla^{-it}$ for every $t \in \mathbb{R}$ and $a \in A$.

By definition, $\hat{\varphi}$ $\pi$ will denote the mapping $\hat{\varphi} \circ (\pi|_{A^+})$ from $A^+$ into $[0, \infty]$. We put $\varphi = \hat{\varphi} \pi$.

Then $\varphi$ is a lower semi-continuous weight on $A$ such that
\[
\begin{align*}
\bullet \quad & M_\varphi^+ = \{ a \in A \mid \pi(a) \text{ belongs to } M_\varphi^+ \} \\
\bullet \quad & M_\varphi \subseteq \{ a \in A \mid \pi(a) \text{ belongs to } M_\varphi \} \\
\bullet \quad & N_\varphi = \{ a \in A \mid \pi(a) \text{ belongs to } N_\varphi \} \\
\bullet \quad & N_\varphi \cap N_\varphi^* = \{ a \in A \mid \pi(a) \text{ belongs to } N_\varphi \cap N_\varphi^* \}
\end{align*}
\]
and $\varphi(a) = \hat{\varphi}(\pi(a))$ for every $a \in M_\varphi$.

Define the following mapping $\Lambda$ from $N_\varphi$ into $H$ : For every $a \in N_\varphi$, there exists a unique element $v \in H$ such that $v$ is left bounded with respect to $U$ and $L_v = \pi(a)$ and we define $\Lambda(a) = v$, so we get that $L_{\Lambda(a)} = \pi(a)$.

In the following, we will suppose that $N_\varphi$ is dense in $A$ and that $\Lambda(N_\varphi)$ is dense in $H$.

It is not difficult to check in this case that $(H, \Lambda, \pi)$ is a GNS-construction for $\varphi$.

It follows easily that $\Lambda(N_\varphi \cap N_\varphi^*) \subseteq U''$ and $\Lambda(a)^* = \Lambda(a^*)$ for every $a \in N_\varphi \cap N_\varphi^*$.

**Proposition 5.15** We have that $\varphi$ is a KMS-weight with modular group $\sigma$ and such that $\Lambda(\sigma_t(a)) = \nabla^{it}\Lambda(a)$ for every $a \in N_\varphi$.

**Proof**:

1. Choose $a \in N_\varphi$ and $t \in \mathbb{R}$. Then $\Lambda(a)$ is left bounded with respect to $U$ and $L_{\Lambda(a)} = \pi(a)$. So $\pi(\sigma_t(a)) = \nabla^{it}\pi(a)\nabla^{-it} = \nabla^{it}\pi(\sigma_t(a))\nabla^{-it}$. Hence, the Tomita theorem implies that $\nabla^{it}\Lambda(a)$ is left bounded with respect to $U$ and $L_{\nabla^{it}\Lambda(a)} = \nabla^{it}\pi(\sigma_t(a))\nabla^{-it} = \pi(\sigma_t(a))$. This implies that $\sigma_t(a)$ belongs to $N_\varphi$ and $\Lambda(\sigma_t(a)) = \nabla^{it}\Lambda(a)$.

   From this, we get immediately that $\varphi$ is invariant under $\sigma$.

2. Choose $b \in D(\sigma_{-\frac{1}{2}} \cap N_\varphi \cap N_\varphi^*)$ and $y \in N_\varphi^* \cap N_\varphi$. Then $y^*b$ belongs to $N_\varphi \cap N_\varphi^*$.

   The function $S(-\frac{1}{2}) \rightarrow B(H) : u \mapsto \pi(\sigma_u(b))$ is continuous on $S(-\frac{1}{2})$, analytic on $S(-\frac{1}{2})^0$ and $\pi(\sigma_t(b)) = \nabla^{it}\pi(b)\nabla^{-it}$ for every $t \in \mathbb{R}$ (by assumption). This implies that $D(\nabla^{\frac{1}{2}}\pi(b)\nabla^{-\frac{1}{2}}) = D(\nabla^{\frac{1}{2}}\pi(b)\nabla^{-\frac{1}{2}} \subseteq \pi(\sigma_{-\frac{1}{2}}(b)))$. (a)

   Because $y$ belongs to $N_\varphi \cap N_\varphi^*$, $\Lambda(y)$ belongs to $U''$ and $\Lambda(y)^* = \Lambda(y^*)$. Therefore we have that $JA(y)$ belongs to $D(\nabla^{\frac{1}{2}})$ and $\nabla^{-\frac{1}{2}}JA(y) = \Lambda(y^*)$. Using (a), we have also that $\pi(b)\nabla^{-\frac{1}{2}}JA(y) \subseteq \pi(\sigma_{-\frac{1}{2}}(b))JA(y)$.

   These two results imply together that $\Lambda(by^*)$ belongs to $D(\nabla^{\frac{1}{2}})$ and $\nabla^{\frac{1}{2}}\Lambda(by^*) = \pi(\sigma_{-\frac{1}{2}}(b))JA(y)$.

   Because $by^*$ belongs to $N_\varphi \cap N_\varphi^*$, we have that $JA\nabla^{\frac{1}{2}}\Lambda(by^*) = \Lambda(yb^*)$. Substituting this in the last equality gives us that
\[
\Lambda(yb^*) = J\pi(\sigma_{-\frac{1}{2}}(b))JA(y)
\]
(b)
Because $\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi$ is dense in $A$, proposition 5.2 implies that $D(\sigma_{-\frac{1}{2}}) \cap \mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi$ is a core for $\sigma_{-\frac{1}{2}}$. We know also that $\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi$ is a core for $\Lambda$. Using these two facts and equality (b), it is now rather easy to prove for every $a \in D(\sigma_{-\frac{1}{2}})$ and $x \in \mathcal{N}_\varphi$ that $xa^*$ belongs to $\mathcal{N}_\varphi$ and $\Lambda(xa^*) = J\pi(\sigma_{-\frac{1}{2}}(a))J\Lambda(x)$.

Proposition 5.14 implies that $\varphi$ is KMS with respect to $\sigma$.

Proposition 5.16 We have that $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)$ is a sub left Hilbert algebra of $\mathcal{U}''$ and $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)'' = \mathcal{U}''$.

Proof: Because $\varphi$ is a densely defined lower semi-continuous weight on $A$, we know that $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)$ is dense in $H$. By a result of [18], we see that $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)$ is a sub left Hilbert algebra of $\mathcal{U}''$.

Choose $n \in \mathbb{N}$ and $a \in \mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi$. Define

$$a(n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \sigma_t(a) \, dt.$$ 

Lemma 4.1 implies that $a(n)$ belongs to $\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi$ and

$$\Lambda(a(n)) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2) \nabla^t \Lambda(a) \, dt.$$ 

Because $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)$ is dense in $H$, this equation implies that the set $\langle \Lambda(a(n)) \mid n \in \mathbb{N}, a \in \mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi \rangle$ is a core for $\nabla^\frac{1}{2}$. Therefore, we get that $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)$ is a core for $\nabla^\frac{1}{2}$.

Hence, left Hilbert algebra theory implies that $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}^*_\varphi)'' = \mathcal{U}''$.

6 Properties of KMS-weights

In this section we will prove important properties of KMS-weights but we will start of with some equivalent definitions of KMS-weights.

Proposition 6.1 Consider a $C^*$-algebra $A$ and a densely defined weight $\varphi$ on $A$ with GNS-construction $(H, \Lambda, \pi)$. Let $\sigma$ be a norm continuous one-parameter group on $A$. Then $\varphi$ is a KMS-weight with respect to $\sigma$ if and only if

1. The mapping $\Lambda : \mathcal{N}_\varphi \rightarrow H$ is closed.

2. We have that $\varphi \sigma_t = \varphi$ for every $t \in \mathbb{R}$.

3. There exists a core $K$ for $\Lambda$ such that

   - $K$ is a subset of $D(\sigma_{-\frac{1}{2}})$, $\sigma_{-\frac{1}{2}}(K)^*$ is a subset of $\mathcal{N}_\varphi$ and $\|\Lambda(x)\| = \|\Lambda(\sigma_{-\frac{1}{2}}(x)^*)\|$ for every $x \in K$.

   - We have that $\sigma_t(K) \subseteq K$ for every $t \in \mathbb{R}$.

This proposition is a direct result of definition 5.11 and proposition 5.6. The next proposition is an easy consequence of definition 5.12 and proposition 5.14.

Proposition 6.2 Consider a $C^*$-algebra $A$ and a densely defined weight $\varphi$ on $A$ with GNS-construction $(H, \pi, \Lambda)$. Let $\sigma$ be a norm continuous one-parameter group on $A$. Then $\varphi$ is a KMS-weight with respect to $\sigma$ if and only if

1. The mapping $\Lambda : \mathcal{N}_\varphi \rightarrow H$ is closed.
2. We have that \( \varphi \sigma_t = \varphi \) for every \( t \in \mathbb{R} \).

3. There exists an anti-unitary operator \( I \) on \( H \) such that the following holds.

   There exists a core \( K \) for \( \Lambda \) and a core \( C \) for \( \sigma_z^2 \) such that we have for every \( x \in K \) and \( a \in C \) that \( xa \) belongs to \( \mathcal{N}_\varphi \) and \( \Lambda(xa) = I\pi(\sigma_z^2(a))^*I^*\Lambda(x) \).

Notice that the condition of lower semi-continuity of the weight is not assumed. This is replaced by the weaker condition of the closedness of the mapping \( \Lambda \).

For the most part of this section, we will fix a \( C^* \)-algebra \( A \) and a KMS-weight \( \varphi \) on \( A \) with modular group \( \sigma \). Let \( (H, \Lambda, \pi) \) be a GNS-construction for \( \varphi \).

Because \( \varphi \) is invariant with respect to \( \sigma \), there exists a unique injective positive operator \( \nabla \) on \( H \) such that \( \nabla \Lambda(a) = \Lambda(\sigma_t(a)) \) for every \( a \in \mathcal{N}_\varphi \). Then we have that \( \pi(\sigma_t(a)) = \nabla^t \pi(a) \nabla^{-t} \) for every \( t \in \mathbb{R} \) and \( a \in A \).

Define the set
\[
C_\varphi = \{ a \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \mid a \text{ is analytic with respect to } \sigma \text{ and } \sigma_z(a) \text{ belongs to } \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \text{ for every } z \in \mathbb{C} \}.
\]

Then \( C_\varphi \) is a sub-\( * \)-algebra of \( A \). It is also clear that \( \sigma_z(C) \subseteq C \) for every \( z \in \mathbb{C} \).

Because \( \varphi \) is lower semi-continuous and densely defined, we know that \( \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \) is a core for \( \Lambda \). So, proposition 5.2 implies that \( C_\varphi \) is a core for \( \mathcal{N}_\varphi \).

Let us also define the set
\[
K_\varphi = \{ a \in \mathcal{N}_\varphi \cap D(\sigma_z^2) \mid \sigma_z^2(a)^* \text{ belongs to } \mathcal{N}_\varphi \}.
\]

Because, \( C_\varphi \) is a subset of \( K_\varphi \), we have that \( K_\varphi \) is also a core for \( \Lambda \). It is also clear that \( \|\Lambda(a)\| = \|\Lambda(\sigma_z^2(a)^*)\| \) for every \( a \in K_\varphi \). Furthermore, \( \sigma_t(K_\varphi) \subseteq K_\varphi \) for every \( t \in \mathbb{R} \).

**Definition 6.3** Define the anti-unitary \( J \) on \( H \) such that \( J\Lambda(a) = \Lambda(\sigma_z^2(a)^*) \) for every \( a \in K_\varphi \).

We have that \( JK_\varphi \subseteq K_\varphi \) and \( J(J(v)) = v \) for every \( v \in K_\varphi \). Therefore, \( J^2 = 1 \).

The proof of the following proposition is the same as the proof of proposition 5.9.

**Proposition 6.4** Consider \( x \in \mathcal{N}_\varphi \) and \( a \in D(\sigma_z^2) \). Then \( xa \) belongs to \( \mathcal{N}_\varphi \) and
\[
\Lambda(xa) = J\pi(\sigma_z^2(a))^*J\Lambda(x).
\]

In the same way as in the proof of lemma 5.10, the previous proposition and proposition 5.3 imply the following one.

**Proposition 6.5** We have that \( \varphi \) is regular and has a truncating net consisting of elements of \( C_\varphi \).

**Result 6.6** Let \( k \) be a natural number. Then \( (C_\varphi)^k \) is a core for \( \Lambda \).
Because \((C_\varphi)^{k-1}\) is dense in \(\Lambda\), there exists a bounded net \((e_k)_{k \in K}\) in \((C_\varphi)^{k-1}\) such that \((e_k)_{k \in K}\) converges strictly to 1. Then we have for every \(k \in K\) that \(e_kx \in (C_\varphi)^k\). It is also clear that \((e_kx)_{k \in K}\) converges to \(x\).

Because \(\Lambda(e_kx) = \pi(e_k)\Lambda(x)\) for every \(k \in K\), we get also that \(\Lambda(e_kx)\) converges to \(\Lambda(x)\).

Because \(C_\varphi\) is a core for \(\Lambda\), this implies that \((C_\varphi)^k\) is a core for \(C_\varphi\). 

Combining this result with proposition \([2,10]\), this implies the following result.

**Result 6.7** Let \(k\) be a natural number. Consider \(a \in N_\varphi \cap N_\varphi^*\). Then there exists a sequence \((a_n)_{n=1}^\infty\) in \((C_\varphi)^k\) such that

1. \((a_n)_{n=1}^\infty \to a\)
2. \(\Lambda(a_n)_{n=1}^\infty \to \Lambda(a)\)
3. \(\Lambda(a_n^*)_{n=1}^\infty \to \Lambda(a^*)\).

In the rest of this section, we will prove some familiar results.

**Proposition 6.8** Consider \(a \in D(\sigma_{-i}) \text{ and } x \in M_\varphi\). Then \(ax\) and \(x\sigma_{-i}(a)\) belong to \(M_\varphi\) and \(\varphi(ax) = \varphi(x\sigma_{-i}(a))\).

**Proof:** Choose \(y, z \in N_\varphi\). Because \(a^*\) belongs to \(D(\sigma_\varphi^*\chi)\), we know that \(ya^*\) belongs to \(N_\varphi\) and

\[
\Lambda(ya^*) = J\pi(\sigma_{-i}(a^*))^*J\Lambda(y) = J\pi(\sigma_{-i}(a))J\Lambda(y) .
\]

Because \(ay^*z = (ya^*)^*z\), we get that \(ay^*z\) belongs to \(M_\varphi\) and

\[
\varphi(ay^*z) = \varphi((ya^*)^*z) = \langle \Lambda(z), \Lambda(ya^*) \rangle
= \langle \Lambda(z), J\pi(\sigma_{-i}(a))J\Lambda(y) \rangle = \langle J\pi(\sigma_{-i}(a))^*J\Lambda(z), \Lambda(y) \rangle \quad (*)
\]

We have that \(\sigma_{-i}(a)\) belongs to \(D(\sigma_\varphi^*\chi)\) and \(\sigma_{-i}(a) = \sigma_{-i}(a)\). This implies that \(z\sigma_{-i}(a)\) belongs to \(N_\varphi\) and

\[
\Lambda(z\sigma_{-i}(a)) = J\pi(\sigma_{-i}(a))^*J\Lambda(z) = J\pi(\sigma_{-i}(a))^*J\Lambda(z) .
\]

Therefore, \(y^*z\sigma_{-i}(a)\) belongs to \(M_\varphi\) and

\[
\varphi(y^*z\sigma_{-i}(a)) = \langle \Lambda(z\sigma_{-i}(a)), \Lambda(y) \rangle = \langle J\pi(\sigma_{-i}(a))^*J\Lambda(z), \Lambda(y) \rangle .
\]

Comparing this with equation \((*)\), we see that \(\varphi(ay^*z) = \varphi(y^*z\sigma_{-i}(a))\). 

**Corollary 6.9** Consider \(a \in D(\sigma_i) \text{ and } x \in M_\varphi\). Then \(xa\) and \(\sigma_i(a)x\) belong to \(M_\varphi\) and \(\varphi(xa) = \varphi(\sigma_i(a)x)\).

Now, we prove a generalization of proposition \([6,3]\).

**Proposition 6.10** Consider \(x \in N_\varphi\) and \(a \in D(\sigma_\varphi^*\chi)\). Then \(xa\) belongs to \(N_\varphi\) and

\[
\Lambda(xa) = J\pi(\sigma_{-i}(a))^*J\Lambda(x) .
\]
Another consequence of proposition 6.4 can be found in the following proposition.

**Proposition 6.11** Consider \( a \in D(\sigma_{-i}) \) and \( x \in M_{\varphi} \). Then \( ax \) and \( x\sigma_{-i}(a) \) belong to \( M_{\varphi} \) and \( \varphi(ax) = \varphi(x\sigma_{-i}(a)) \).

**Corollary 6.12** Consider \( a \in D(\pi_i) \) and \( x \in M_{\varphi} \). Then \( xa \) and \( \sigma_i(a)x \) belong to \( M_{\varphi} \) and \( \varphi(xa) = \varphi(\sigma_i(a)x) \).

Another consequence of proposition 6.4 can be found in the following proposition.

**Result 6.13** Consider \( a, b \in A \). Then we have that \( \pi(a)J \) and \( \pi(b) \) commute.

**Proof**: Choose \( c \in D(\sigma_x) \).

Take \( x \in N_{\varphi} \). Then \( xc \) belongs to \( N_{\varphi} \) and \( \Lambda(xc) = J\pi(\sigma_x(c))\Lambda(x) \). So \( bxc \) belongs to \( N_{\varphi} \) and

\[
\Lambda(bxc) = \pi(b)\Lambda(xc) = \pi(b)J\pi(\sigma_x(c))\Lambda(x) .
\]

On the other hand, we have that \( bx \) belongs to \( N_{\varphi} \) and \( \Lambda(bx) = \pi(b)\Lambda(x) \). This implies that \( bxc \) belongs to \( N_{\varphi} \) and

\[
\Lambda(bxc) = J\pi(\sigma_x(c))\Lambda(bx) = J\pi(\sigma_x(c))\pi(b)\Lambda(x) .
\]

Comparing these two expressions for \( \Lambda(bxc) \), we see that

\[
\pi(b)\pi(\sigma_x(c))\Lambda(x) = J\pi(\sigma_x(c))\pi(b)\Lambda(x) .
\]

This implies that \( \pi(b)\pi(\sigma_x(c))\Lambda(x) = J\pi(\sigma_x(c))\pi(b)\Lambda(x) \). Because \( \sigma_x \) has dense range in \( A \), the result follows.

**Proposition 6.14** The set \( \Lambda(N_{\varphi} \cap N_{\varphi}^*) \) is a core for \( \nabla^\perp \) and \( \Lambda(a^*) = J\nabla^\perp \Lambda(a) \) for every \( a \in N_{\varphi} \cap N_{\varphi}^* \).

**Proof**: Choose \( a \in N_{\varphi} \cap N_{\varphi}^* \). We know that there exists a sequence \( a_n \in C_{\varphi} \) such that \( (\Lambda(a_n))_{n=1}^\infty \to \Lambda(a) \) and \( (\Lambda(a_n))_{n=1}^\infty \to \Lambda(a^*) \) (see result 6.13).

Choose \( m \in \mathbb{N} \). Because \( a_m \) belongs to \( N_{\varphi} \cap D(\sigma_{-x}) \) and \( \sigma_{-x}(a_m) \) belongs to \( N_{\varphi} \), we have that \( \Lambda(a_m) \) belongs to \( D(\nabla^\perp) \) and \( \nabla^\perp \Lambda(a_m) = \Lambda(\sigma_{-x}(a_m)) \).

We also know that \( a_m^* \) belongs to \( N_{\varphi} \cap D(\sigma_x) \) and \( \sigma_x(a_m^*) \) belongs to \( N_{\varphi}^* \). By definition, we have that

\[
J\Lambda(a_m^*) = \Lambda(\sigma_x(a_m^*)) = \Lambda(\sigma_{-x}(a_m)) = \nabla^\perp \Lambda(a_m) .
\]

This implies that \( (\nabla^\perp \Lambda(a_n))_{n=1}^\infty \) converges to \( J\Lambda(a^*) \). The closedness of \( \nabla^\perp \) implies that \( \Lambda(a) \) belongs to \( D(\nabla^\perp) \) and \( \nabla^\perp \Lambda(a) = J\Lambda(a^*) \).
Choose \( x \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \), \( n \in \mathbb{N} \) and define
\[
x(n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \sigma_t(x) \, dt,
\]
it is clear that \( x(n) \) belongs to \( \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \) and
\[
\Lambda(x(n)) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \nabla^{it} \Lambda(x) \, dt.
\]
Because \( \Lambda(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*) \) is dense in \( H \), this implies that \( \langle \Lambda(x(n)) | x \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*, n \in \mathbb{N} \rangle \) is a core for \( \nabla^{\frac{i}{2}} \).

We want to construct a left Hilbert algebra in the usual way. First, we will need a lemma which guarantees that our next definitions are well defined.

**Lemma 6.15**

1. Consider \( a_1, a_2, b_1, b_2 \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \) such that \( \Lambda(a_1) = \Lambda(a_2) \) and \( \Lambda(b_1) = \Lambda(b_2) \). Then \( \Lambda(a_1 b_1) = \Lambda(a_2 b_2) \).

2. Consider \( a_1, a_2 \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \) such that \( \Lambda(a_1) = \Lambda(a_2) \). Then \( \Lambda(a_1^*) = \Lambda(a_2^*) \).

**Proof:**

1. For any \( c \in C_\varphi \), we have that
\[
\pi(a_1) \Lambda(c) = J \pi(\sigma_\varphi(c))^* J \Lambda(a_1) = J \pi(\sigma_\varphi(c))^* J \Lambda(a_2) = \pi(a_2) \Lambda(c).
\]

The density of \( \Lambda(C_\varphi) \) in \( H \) implies that \( \pi(a_1) = \pi(a_2) \).

Therefore,
\[
\Lambda(a_1 b_1) = \pi(a_1) \Lambda(b_1) = \pi(a_2) \Lambda(b_2) = \Lambda(a_2 b_2).
\]

2. This follows easily from the previous proposition.

**Definition 6.16** We define \( \mathcal{U} = \Lambda(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*) \), then \( \mathcal{U} \) is subspace of \( H \). We make \( \mathcal{U} \) into a \( * \)-algebra such that:

1. We have for every \( a, b \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \) that \( \Lambda(ab) = \Lambda(a) \Lambda(b) \).

2. We have for every \( a \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \) that \( \Lambda(a^*) = \Lambda(a^*) \).

Then \( \mathcal{U} \) becomes a left Hilbert algebra on \( H \).

It is not difficult to check that \( \mathcal{U} \) satisfies the conditions of a left Hilbert algebra. The closability of the mapping \( \mathcal{U} \to \mathcal{U} : v \mapsto v^* \) is a direct consequence of proposition 6.14.

It is immediately clear that \( L_{\Lambda(a)} = \pi(a) \) for every \( a \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \).

Let us define \( T \) as the closure of the mapping \( \mathcal{U} \to \mathcal{U} : v \mapsto v^* \). Proposition 6.14 implies that \( T = J \nabla^\frac{i}{2} \).

So we see that \( J \) is the modular conjugation of \( \mathcal{U} \) and \( \nabla \) is the modular operator of \( \mathcal{U} \).

We also see that \( J \) and \( \nabla \) are independent of our choice of \( \sigma \). Therefore, we will call \( \nabla \) the modular operator of \( \varphi \) and \( J \) the modular conjugation of \( \varphi \) in the GNS-construction \((H, \Lambda, \pi)\).

**Lemma 6.17** Consider \( a \in \mathcal{N}_\varphi \). Then \( \Lambda(a) \) is left bounded with respect to \( \mathcal{U} \) and \( L_{\Lambda(a)} = \pi(a) \).
Proof: Because $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ is a core for $\Lambda$, there exists a sequence $(a_n)_{n=1}^\infty$ in $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ such that $(\Lambda(a_n))_{n=1}^\infty$ converges to $a$ and $(\Lambda(a_n))_{n=1}^\infty$ converges to $\Lambda(a)$.

We know already that $L_{\Lambda(a_n)} = \pi(a_n)$ for $n \in \mathbb{N}$. This implies that $(L_{\Lambda(a_n)})_{n=1}^\infty$ converges to $\pi(a)$. Consequently, we find that $\Lambda(a)$ is left bounded with respect to $U$ and $L_{\Lambda(a)} = \pi(a)$. \[\square\]

Lemma 6.18 Consider $v \in H$ such that $v$ is left bounded with respect to $U$ and such that there exists an element $a \in A$ such that $L_v = \pi(a)$. Then $a$ belongs to $\mathcal{N}_\varphi$ and $\Lambda(a) = v$.

Proof: Choose a truncating net $(u_i)_{i \in I}$ for $\varphi$. For every $i \in I$, we define the operator $S_i \in B(H)$ such that $S_i \Lambda(x) = \Lambda(xu_i)$ for every $x \in A$.

By (2), there exist a sequence $(a_n)_{n=1}^\infty$ in $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ such that $(\Lambda(a_n))_{n=1}^\infty$ converges to $\Lambda(a)$, $(L_{\Lambda(a_n)})_{n=1}^\infty$ converges strongly to $L_v$ and $(L_{\Lambda(a_n)})_{n=1}^\infty$ is bounded.

Choose $j \in I$. We have immediately that $(L_{\Lambda(a_n)} \Lambda(u_{j}))_{n=1}^\infty$ converges to $L_v \Lambda(u_j)$. We have for every $n \in \mathbb{N}$ that

$$L_{\Lambda(a_n)} \Lambda(u_j) = \pi(a_n) \Lambda(u_j) = \Lambda(a_n u_j) = S_j \Lambda(a_n) .$$

This implies that $(L_{\Lambda(a_n)} \Lambda(u_j))_{n=1}^\infty$ converges to $S_j v$.

Combining these two results, we get that $L_a \Lambda(u_j) = S_j v$, hence

$$\Lambda(a u_j) = \pi(a) \Lambda(u_j) = L_a \Lambda(u_j) = S_j v .$$

Therefore, $(\Lambda(a u_i))_{i \in I}$ converges to $v$. We also have that $(a u_i)_{i \in I}$ converges to $a$. The closedness of $\Lambda$ implies that $a$ belongs to $\mathcal{N}_\varphi$ and $\Lambda(a) = v$. \[\square\]

Lemma 6.19 Consider $v \in U''$ such that there exist an element $a \in A$ such that $L_v = \pi(a)$. Then $a$ belongs to $\mathcal{N}_\varphi^* \cap \mathcal{N}_\varphi$ and $v = \Lambda(a)$.

Proof: We know that $v$ is left bounded. Therefore, the previous lemma implies that $a$ belongs to $\mathcal{N}_\varphi$ and $v = \Lambda(a)$. We have also that $v^*$ is left bounded and $L_{v^*} = (L_v)^* = \pi(a)^* = \pi(a^*)$. Hence, the previous lemma implies that $a^*$ belongs to $\mathcal{N}_\varphi$. So we get that $a$ belongs to $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$. \[\square\]

Using this result, we have the following theorem. This is a well known result for weights satisfying the other KMS-condition (proved by Combes, see lemma 2.2.3 of [1]).

Theorem 6.20 Call $\tilde{\varphi}$ the normal semi-finite faithful weight on $\pi(A)'''$ associated to $U$. Then $\tilde{\varphi} \pi = \varphi$.

Remark 6.21 Consider the case where we get a KMS-weight from a left Hilbert algebra (the third part of section 4, page 34):

Consider a Hilbert space $H$ and a left Hilbert $\mathcal{W}$ algebra on $H$ with modular operator $\Delta$, modular conjugation $I$ and associated faithful semi-finite normal weight $\psi$.

Let $A$ be a C$^*$-algebra and $\pi$ be a nondegenerate $*$-representation of $A$ on $H$ such that $\pi(A) \subseteq \mathcal{L}(\mathcal{W})$ and suppose there exists a one-parameter group $\sigma$ on $A$ such that $\pi(\sigma_t(a)) = \Delta^{it} \pi(a) \Delta^{-it}$ for every $t \in \mathbb{R}$ and $a \in A$.

Put $\varphi = \psi \pi$, so $\varphi$ is a lower semi-continuous weight on $A$.

Define the following mapping $\Lambda$ from $\mathcal{N}_\varphi$ into $H$: For every $a \in \mathcal{N}_\varphi$, there exists a unique element $v \in H$ such that $v$ is left bounded with respect to $W$ and $L_v = \pi(a)$ and we define $\Lambda(a) = v$, so we get that $L_{\Lambda(a)} = \pi(a)$. We suppose that $\mathcal{N}_\varphi$ is dense in $A$ and that $\Lambda(\mathcal{N}_\varphi)$ is dense in $H$.

We saw that $\varphi$ is a KMS-weight on $A$ with modular group $\sigma$ and GNS-construction $(H, \Delta, \pi)$. Now we can apply the construction procedures of this section to get the left Hilbert algebra $U$, the weight $\tilde{\varphi}$ associated to $U$, the modular operator $\nabla$ and modular conjugation $J$ of $\varphi$ in the GNS-construction $(H, \Delta, \pi)$. Proposition 5.16 implies that $U'' = \mathcal{W}''$. Therefore we get that $\tilde{\varphi} = \psi$, $\nabla = \Delta$ and $J = I$. 41
For KMS-weights, the faithfulness of $\varphi$ and $\pi$ are equivalent:

**Proposition 6.22** We have that $\varphi$ is faithful if and only if $\pi$ is faithful.

**Proof:**

$\Rightarrow$ This follows easily because $\varphi$ is densely defined.

$\Leftarrow$ Choose $x \in A$ such that $\varphi(x^*x) = 0$. Then $x$ belongs to $N_\varphi$ and $\Lambda(x) = 0$.

Take $m \in \mathbb{N}$ and define

$$x(m) = \frac{m}{\sqrt{\pi}} \int \exp(-m^2t^2) \sigma_t(x) \, dt.$$  

We have that $x(m)$ belongs to $N_\varphi \cap D(\sigma_{\frac{\pi}{2}})$ and

$$\Lambda(x(m)) = \frac{m}{\sqrt{\pi}} \int \exp(-m^2t^2) \nabla^\dag \Lambda(x) \, dt = 0.$$  

So we have for every $a \in N_\varphi$ that

$$J\pi(\sigma_{\frac{\pi}{2}}(x(m)))^* J\Lambda(a) = \pi(a) \Lambda(x(m)) = 0,$$

which implies that $\pi(\sigma_{\frac{\pi}{2}}(x(m))) = 0$. The faithfulness of $\pi$ implies that $\sigma_{\frac{\pi}{2}}(x(m)) = 0$, the injectivity of $\sigma_{\frac{\pi}{2}}$ implies that $x(m) = 0$.

The definition of the elements $x(n)$ ($n \in \mathbb{N}$) implies that $(x(n))_{n=1}^\infty \to x$. Therefore we have that $x = 0$.  

We have of course also the following result.

**Lemma 6.23** Consider a $*$-automorphism $\alpha$ on $A$ such that there exists a strictly positive number $\lambda$ such that $\varphi \alpha = \lambda \varphi$. Define $u$ as the unitary operator on $H$ such that $u\Lambda(a) = \lambda^{-\frac{1}{2}} \Lambda(\alpha(a))$ for every $a \in N_\varphi$. Then $u\nabla = \nabla u$ and $uJ = Ju$. We have moreover that $\pi(\alpha(a)) = u\pi(a)u^*$ for every $a \in A$.

**Proof:** Because $\alpha$ is a $*$-homomorphism, it is easy to check for every $a \in N_\varphi \cap N_\varphi^*$ that $u\Lambda(a)$ belongs to $D(T)$ and $Tu\Lambda(a) = uT\Lambda(a)$. Because $\Lambda(N_\varphi \cap N_\varphi^*)$ is a core for $T$ and $u\Lambda(N_\varphi \cap N_\varphi^*) = \Lambda(N_\varphi \cap N_\varphi^*)$ we get that $uT = Tu$. So we get also that $Tu^* = u^*T$. The unitarity of $u$ implies that

$$uT^* = (Tu^*)^* = (u^*T)^* = T^*u.$$  

Using the fact that $\nabla = T^*T$, this implies that $u\nabla = \nabla u$.

So we get also that $u\nabla^\dagger = \nabla^\dagger u$. This in turn implies that

$$uJ\nabla^\dagger = uT = Tu = J\nabla^\dagger u = Ju\nabla^\dagger.$$  

Because $\nabla^\dagger$ has dense range, this implies immediately that $uJ = Ju$.  

**Corollary 6.24** Suppose that $\varphi$ is faithful. Consider a $*$-automorphism $\alpha$ on $A$ such that there exists a strictly positive number $\lambda$ such that $\varphi \alpha = \lambda \varphi$. Then $\alpha\sigma_t = \sigma_t\alpha$ for every $t \in \mathbb{R}$.  

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Proof: Choose $t \in \mathbb{R}$.
Define $u$ as the unitary operator on $H$ such that $u\Lambda(a) = \lambda^t \Lambda(a)$ for every $a \in \mathcal{N}_\varphi$. Then the previous lemma implies that $u\varLambda = \varLambda u$.
This implies immediately that $u\varLambda^{it} = \varLambda^{it} u$. So we get for every $a \in \mathcal{N}_\varphi$ that

$$\Lambda(\alpha(\sigma_t(a))) = \lambda^t u\Lambda(\sigma_t(a)) = \lambda^t u\varLambda^{it}\Lambda(a) = \lambda^t \varLambda^{it}u\Lambda(a) = \varLambda^{it}\Lambda(a) = \Lambda(\sigma_t(\alpha(a)))$$

so the faithfulness of $\varphi$ implies that $\alpha(\sigma_t(a)) = \sigma_t(\alpha(a))$.

From the theory of left Hilbert algebras (see e.g. [18]), we get the following result:

**Proposition 6.25** The weight $\varphi$ satisfies the KMS-condition with respect to $\sigma$: For every $x, y \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ there exists a bounded continuous function $f$ from the strip $S(i)$ into $\mathbb{C}$ which is analytic on $S(i)^0$ and satisfies:

- $f(t) = \varphi(\sigma_t(x))$ for all $t \in \mathbb{R}$
- $f(t+i) = \varphi(x \sigma_t(y))$ for all $t \in \mathbb{R}$.

The next two results follow immediately from Proposition 6.10 and its corollary.

**Proposition 6.26** Consider $a \in \mathcal{M}(A)$ such that there exists a strictly positive number $\lambda$ such that $\sigma_t(a) = \lambda^t a$ for every $t \in A$. Let $x$ be an element in $\mathcal{N}_\varphi$. Then $xa$ belongs to $\mathcal{N}_\varphi$ and $\Lambda(xa) = \lambda^t J\pi(a)^* J\Lambda(x)$.

**Proposition 6.27** Consider $a \in \mathcal{M}(A)$ such that there exists a strictly positive number $\lambda$ such that $\sigma_t(a) = \lambda^t a$ for every $t \in \mathbb{R}$. Let $x$ be an element in $\mathcal{M}_\varphi$. Then $ax$ and $xa$ belong to $\mathcal{M}_\varphi$ and $\varphi(ax) = \lambda \varphi(xa)$.

If $\varphi$ is faithful, it is possible to prove a converse of this corollary.

The proof of the next lemma can be copied immediately from the proof of lemma 3.4 of [14].

**Lemma 6.28** Consider $a \in \mathcal{M}(A)$ such that $a\mathcal{M}_\varphi \subseteq \mathcal{M}_\varphi$ and $\mathcal{M}_\varphi a \subseteq \mathcal{M}_\varphi$. Then $\sigma_t(a)\mathcal{M}_\varphi \subseteq \mathcal{M}_\varphi$ and $\mathcal{M}_\varphi \sigma_t(a) \subseteq \mathcal{M}_\varphi$ for every $t \in \mathbb{R}$. Let $x, y$ be elements in $C_\varphi$, then there exists an entire function from $\mathbb{C}$ into $\mathbb{C}$ such that

1. We have that $f(t) = \varphi(\sigma_t(a) y^* x)$ for every $t \in \mathbb{R}$.
2. We have that $f(t+i) = \varphi(y^* x \sigma_t(a))$ for every $t \in \mathbb{R}$.
3. $f$ is bounded on each horizontal strip.

The proof of the following proposition is a slight adaptation of the proof of theorem 3.6 of [14].

**Result 6.29** Suppose that $\varphi$ is faithful. Consider $a \in \mathcal{M}(A)$ such that $a\mathcal{M}_\varphi \subseteq \mathcal{M}_\varphi$, $\mathcal{M}_\varphi a \subseteq \mathcal{M}_\varphi$ and such that there exist a strictly positive number $\lambda$ satisfying $\varphi(ax) = \lambda \varphi(xa)$ for every $x \in \mathcal{M}_\varphi$. Then $\sigma_t(a) = \lambda^t a$ for every $t \in \mathbb{R}$.

Proof: Choose $x, y \in C_\varphi$.
By the previous lemma, there exists an entire function $f$ from $\mathbb{C}$ into $\mathbb{C}$ such that

1. We have that $f(t) = \varphi(\sigma_t(a) y^* x)$ for every $t \in \mathbb{R}$.
2. We have that \( f(t + i) = \varphi(y^*x_\sigma_1(a)) \) for every \( t \in \mathbb{R} \).

3. \( f \) is bounded on each horizontal strip.

First, we prove that \( f(z + i) = \lambda^{-1} f(z) \) for every \( z \in \mathbb{C} \).

We have for every \( t \in \mathbb{R} \) that

\[
 f(t + i) - \lambda^{-1} f(t) = \varphi(y^*x_\sigma_1(a)) - \lambda^{-1} \varphi(\sigma_1(a)y^*x) \\
 = \varphi(y^*x_\sigma_1(a)) - \lambda^{-1} \varphi(a_{-1}(y^*x)) \\
 = \varphi(y^*x_\sigma_1(a)) - \lambda^{-1} \lambda \varphi(\sigma_1(y^*x)a) \\
 = \varphi(y^*x_\sigma_1(a)) - \varphi(y^*x_\sigma_1(a)) = 0.
\]

By analyticity of the function

\[
 \mathbb{C} \to \mathbb{C} : z \mapsto f(z + i) - \lambda^{-1} f(z),
\]

we must have that \( f(z + i) - \lambda^{-1} f(z) = 0 \) for every \( z \in \mathbb{C} \).

Now we define the entire function \( g \) from \( \mathbb{C} \) into \( \mathbb{C} \) such that \( g(z) = \lambda^{-iz} f(z) \) for every \( z \in \mathbb{C} \). Then we have for every \( z \in \mathbb{C} \)

\[
 g(z + i) = \lambda^{-i(z+i)} f(z + i) = \lambda^{-iz} \lambda^{-1} f(z) = \lambda^{-iz} f(z) = g(z).
\]

We know that \( f \) is bounded on \( S(i) \). We have also that

\[
 |g(z)| = |\lambda^{-iz}| |f(z)| = \lambda^{\text{Im} z} |f(z)| \leq \max\{1, \lambda\} |f(z)|,
\]

for every \( z \in S(i) \), so \( g \) is also bounded on \( S(i) \).

By (*) we have that \( g \) is bounded on \( \mathbb{C} \). The theorem of Liouville implies that \( g \) is a constant function. This implies that

\[
 \varphi(ay^*x) = f(0) = g(0) = g(t) = \lambda^{-it} f(t) = \lambda^{-it} \varphi(\sigma_1(a)y^*x)
\]

for every \( t \in \mathbb{R} \).

Choose \( s \in \mathbb{R} \). Because \( (a - \lambda^{-is} \sigma_s(a))M_\varphi \subseteq M_\varphi \), we have also that \( \mathcal{N}_\varphi(a^* - \lambda^{is} \sigma_s(a^*)) \subseteq \mathcal{N}_\varphi \).

Hence, we have for every \( x, y \in C_\varphi \) that

\[
 \langle \Lambda(x), \Lambda(y(a^* - \lambda^{is} \sigma_s(a^*))) \rangle = \varphi((a - \lambda^{-is} \sigma_s(a))y^*x) = 0.
\]

Because \( \Lambda(C_\varphi) \) is dense in \( H \), we see that \( \Lambda(y(a^* - \lambda^{is} \sigma_s(a^*))) = 0 \) for every \( y \in C_\varphi \). The faithfulness of \( \varphi \) implies that \( y(a^* - \lambda^{is} \sigma_s(a^*)) = 0 \) for every \( y \in C_\varphi \). Therefore, the density of \( C_\varphi \) in \( A \) implies that \( a^* - \lambda^{is} \sigma_s(a^*) = 0 \).

The following proposition is an immediate consequence of proposition 6.26.

**Proposition 6.30** Consider a strictly positive element \( \delta \eta A \) such that there exist a strictly positive number \( \lambda \) such that \( \sigma_1(\delta) = \lambda^t \delta \) for every \( t \in \mathbb{R} \). Then we have the following properties:

- For every \( x \in \mathcal{N}_\varphi \) and every \( t \in \mathbb{R} \), we have that \( x\delta^it \) belongs to \( \mathcal{N}_\varphi \) and \( \Lambda(x\delta^it) = \lambda^{-t} J^i \pi(\delta)^{-it} J A(x) \).

- For every \( x \in M_\varphi \) and every \( t \in \mathbb{R} \), we have that \( x\delta^it \) and \( \delta^it x \) belong to \( M_\varphi \) and \( \varphi(\delta^it x) = \lambda^t \varphi(x\delta^it) \).
Proposition 6.31 Consider a strictly positive element \( \delta \eta A \) such that there exist a strictly positive number \( \lambda \) such that \( \sigma_t(\delta) = \lambda^t \delta \) for every \( t \in \mathbb{R} \). Let \( z \) be a complex number and \( a \in \mathcal{N}_\varphi \) such that \( a \) is a left multiplier of \( \delta^z \) and \( a \delta^z \) belongs to \( A \). Then we have the following properties:

- \( a \delta^z \) belongs to \( \mathcal{N}_\varphi \) if and only if \( \Lambda(a) \) belongs to \( D(J\pi(\delta)^z J) \).
- If \( a \delta^z \) belongs to \( \mathcal{N}_\varphi \), then \( \Lambda(a \delta^z) = \lambda^z J\pi(\delta)^z J\Lambda(a) \).

Proof: We define the strongly continuous one-parameter representation \( \alpha \) on \( A \) such that \( \alpha_t(x) = x\delta^{it} \) for every \( x \in A \) and \( t \in \mathbb{R} \).

Furthermore, we define the strongly continuous unitary group representation \( u \) from \( \mathbb{R} \) on \( H \) such that \( u_t = (J\pi(\delta))^{it} \) for every \( t \in \mathbb{R} \). We know by the previous proposition that

1. We have that \( \alpha_t(\mathcal{N}_\varphi) \subseteq \mathcal{N}_\varphi \) for every \( t \in \mathbb{R} \).
2. We have that \( u_t \Lambda(a) = \lambda^{\frac{t}{2}} \Lambda(\alpha_t(a)) \) for every \( t \in \mathbb{R} \) and \( a \in \mathcal{N}_\varphi \).

Because \( a \) belongs to \( D(\alpha_{-iz}) \) and \( \alpha_{-iz}(a) = a \delta^z \) and \( u_{-iz} = (J\pi(\delta)^z \), this proposition follows from propositions 4.4 and 4.5.

Proposition 6.32 Consider a strictly positive element \( \delta \) affiliated with \( A \) such that there exists a strictly positive number \( \lambda \) such that \( \sigma_t(\delta) = \lambda^t \delta \) for every \( t \in \mathbb{R} \). Let \( a \) be an element in \( A \) and \( z \in \mathbb{C} \) such that \( a \) is a left and right multiplier of \( \delta^z \) and \( a \delta^z, \delta^z a \) belong to \( \mathcal{M}_\varphi \). Then \( \varphi(a \delta^z) = \lambda^z \varphi(\delta^z a) \).

Proof: Define the norm continuous one-parameter group \( \alpha \) on \( A \) such that \( \alpha_t(a) = \delta^{it} a \delta^{-it} \) for every \( a \in A \) and \( t \in \mathbb{R} \). Proposition 6.30 implies that \( \varphi \alpha_t = \lambda^t \varphi \) for every \( t \in \mathbb{R} \).

Because \( a \) is a left multiplier of \( \delta^z \), we have that \( a \delta^z \) is an left multiplier of \( \delta^{-z} \) and \( (a \delta^z) \delta^{-z} = a \).

This implies that \( (a \delta^z) \delta^{-z} \) is a right multiplier of \( \delta^z \) and \( \delta^z [(a \delta^z) \delta^{-z}] = \delta^z a \). From this, we get that \( a \delta^z \) is a middle multiplier of \( \delta^z, \delta^{-z} \) and

\[
\delta^z (a \delta^z) \delta^{-z} = \delta^z a
\]

From this, we get that \( a \delta^z \) belongs to \( D(\alpha_{-iz}) \) and \( \alpha_{-iz}(a \delta^z) = \delta^z a \). The proposition follows from proposition 2.4.

We now prove that a KMS-weight has no proper extensions which are invariant under its modular group.

Proposition 6.33 Consider a lower semi-continuous weight \( \eta \) on \( A \) which is an extension of \( \varphi \) and such that \( \eta \) is invariant under \( \sigma \). Then \( \varphi = \eta \).

Proof: Take a GNS-construction \( (H_\eta, \Lambda_\eta, \pi_\eta) \) for \( \eta \).

Because \( \eta \) is invariant under \( \sigma \), we get the existence a positive injective operator \( T \) in \( H_\eta \) such that \( T^t \Lambda_\eta(a) = \Lambda_\eta(\sigma_t(a)) \) for every \( a \in \mathcal{N}_\eta \) and \( t \in \mathbb{R} \).

Choose \( y \in \mathcal{N}_\eta \).

Define for every \( n \in \mathbb{N} \) the element

\[
y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \sigma_t(y) dt
\]
which is clearly analytic with respect to $\sigma$. We have also that $(y_n)_{n=1}^{\infty}$ converges to $y$.

By ??, we have for every $n \in \mathbb{N}$ that $y_n$ belongs to $\mathcal{N}_\varphi$ and

$$\Lambda_\eta(y_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) T^{it} \Lambda(y) \, dt.$$

This implies immediately that $(\Lambda_\eta(y_n))_{n=1}^{\infty}$ converges to $\Lambda_\eta(y)$.

We can also take an approximate unit $(e_i)_{i \in I}$ for $A$ in $\mathcal{N}_\varphi$. Then $(e_i y_n)_{i \in I \times \mathbb{N}}$ converges to $y$.

We have also for every $i \in I$ and $n \in \mathbb{N}$ that $e_i y_n$ belongs to $\mathcal{N}_\varphi$ and $\Lambda_\eta(e_i y_n) = \pi_\eta(e_i) \Lambda_\eta(y_n)$. Consequently, the net $(\Lambda_\eta(e_i y_n))_{(i,n) \in I \times \mathbb{N}}$ converges to $\Lambda_\eta(y)$.

We have for every $i \in I$ and $n \in \mathbb{N}$ that $e_i$ belongs to $\mathcal{N}_\varphi$ and $y_n$ belongs to $D(\sigma_\varphi)$ implying that $e_i y_n$ belongs to $\mathcal{N}_\varphi$ by proposition 6.4.

Because $\varphi \subseteq \eta$, we have moreover for every $i,j \in I$ and $m,n \in \mathbb{N}$ that

$$\| \Lambda_\varphi(e_i y_n) - \Lambda_\varphi(e_j y_m) \| = \| \Lambda_\varphi(e_i y_n) - \Lambda_\eta(e_j y_m) \|.$$

This last equality implies that the net $(\Lambda_\varphi(e_i y_n))_{(i,n) \in I \times \mathbb{N}}$ is Cauchy and hence convergent in $H_\varphi$.

Therefore, the closedness of $\Lambda_\varphi$ implies that $y$ is an element of $\mathcal{N}_\varphi$. The proposition follows. □

The following proposition will guarantee that the modular group is unique for faithful KMS-weights.

**Proposition 6.34** Consider a $C^*$-algebra $A$ and a KMS-weight $\varphi$ on $A$ with GNS-construction $(H, \Lambda, \pi)$. Let $\sigma$ and $\tau$ be modular groups for $\varphi$. Then $\pi \sigma_t = \pi \tau_t$ for every $t \in \mathbb{R}$.

**Proof** Define the injective positive operators $\nabla$ and $\Delta$ in $H$ such that $\nabla^{it} \Lambda(a) = \Lambda(\sigma_t(a))$ and $\Delta^{it} \Lambda(a) = \Lambda(\tau_t(a))$ for every $a \in \mathcal{N}_\varphi$ and $t \in \mathbb{R}$. Define also the anti-unitary operators $J, I$ on $H$ such that

- $J \Lambda(a) = \Lambda(\sigma_\varphi(a)^*)$ for every $a \in \mathcal{N}_\varphi \cap D(\sigma_\varphi)$ such that $\sigma_\varphi(a)^*$ belongs to $\mathcal{N}_\varphi$
- $I \Lambda(a) = \Lambda(\tau_\varphi(a)^*)$ for every $a \in \mathcal{N}_\varphi \cap D(\tau_\varphi)$ such that $\tau_\varphi(a)^*$ belongs to $\mathcal{N}_\varphi$

Proposition 6.14 implies that $J \nabla^\varphi = I \Delta^\varphi$. The uniqueness of the polar decomposition implies that $\nabla = \Delta$. This implies for every $t \in \mathbb{R}$ that

$$\pi(\sigma_t(a)) = \nabla^{it} \pi(a) \nabla^{-it} = \Delta^{it} \pi(a) \Delta^{-it} = \pi(\tau_t(a))$$

for every $t \in \mathbb{R}$ and $a \in A$. □

**Corollary 6.35** Consider a $C^*$-algebra $A$ and a faithful KMS-weight $\varphi$ on $A$. Then $\varphi$ as a unique modular group.

In the last theorem of this section, we prove that our definition of KMS-weight is equivalent with the usual one (introduced in [3]).

**Theorem 6.36** Consider a $C^*$-algebra $A$, a densely defined lower semi-continuous weight $\varphi$ on $A$ and a one-parameter group $\sigma$ on $A$. Then $\varphi$ is a KMS-weight with respect to $\sigma$ if and only if
1. We have for every $t \in \mathbb{R}$ that $\varphi \sigma_t = \varphi$.

2. For every $x, y \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, there exists a bounded continuous function $f$ from $S(i)$ into $\mathbb{C}$ which is analytic on $S(i)^0$ and such that:
   - We have for every $t \in \mathbb{R}$ that $f(t) = \varphi(\sigma_t(x)y)$.
   - We have for every $t \in \mathbb{R}$ that $f(t+i) = \varphi(y\sigma_t(x))$.

Proof:

$\Rightarrow$ This follows from proposition 6.23.

$\Leftarrow$ Take a GNS-construction $(H, \Lambda, \pi)$ for $\varphi$. Define the injective positive operator $\nabla$ in $H$ such that $\nabla^H \Lambda(a) = \Lambda(\sigma_t(a))$ for every $t \in \mathbb{R}$.

Define the set

$$C = \{ a \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \mid a \text{ is analytic with respect to } \sigma \text{ and } \sigma_z(a) \text{ belongs to } \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^* \text{ for every } z \in \mathbb{C} \}.$$

So $C$ is a sub$^*$-algebra of $A$ such that $\sigma_z(C) \subseteq C$ for every $z \in \mathbb{C}$.

Choose $x, y \in C$. Because $x, y$ belong to $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, we have by assumption the existence of a bounded continuous function $f$ on $S(i)$ which is analytic on $S(i)^0$ and satisfies:

- We have for every $t \in \mathbb{R}$ that $f(t) = \varphi(\sigma_t(x)y)$.
- We have for every $t \in \mathbb{R}$ that $f(t+i) = \varphi(y\sigma_t(x))$.

We know that $x^*$ belongs to $\mathcal{N}_\varphi$ and that $\sigma_z(x^*)$ belongs to $\mathcal{N}_\varphi$. By proposition 4.4, we get that $\Lambda(x^*)$ belongs to $D(\nabla)$ and $\nabla^\Lambda(x^*) = \Lambda(\sigma_t(x^*))$. Define the function $g$ from $S(i)$ into $\mathbb{C}$ such that $g(z) = \langle \Lambda(y), \nabla^\Lambda(x^*) \rangle$ for every $z \in S(i)$. Then $g$ is continuous on $S(i)$ and analytic on $S(i)^0$. We have for every $t \in \mathbb{R}$ that

$$g(t) = \langle \Lambda(y), \nabla^\Lambda(x^*) \rangle = \langle \Lambda(y), \Lambda(\sigma_t(x^*)) \rangle = \varphi(\sigma_t(x)y) = f(t).$$

Therefore we must have that $f = g$ which implies that

$$\varphi(yx) = f(i) = g(i) = \langle \Lambda(y), \nabla^\Lambda(x^*) \rangle = \langle \Lambda(y), \Lambda(\sigma_{-i}(x^*)) \rangle = \varphi(\sigma_i(x)y). \quad (*)$$

We know already that $C \subseteq \mathcal{N}_\varphi \cap D(\sigma_\varphi^*)$ and that $\sigma_\varphi^*(C)^* \subseteq \mathcal{N}_\varphi^*$. Just like in lemma 5.13, we get that $C$ is a core for $\Lambda$. Choose $a \in C$. If we replace in equation $(*)$ the element $x$ by $\sigma_{-2}(a)$ and $y$ by $\sigma_2(a)^*$, we get that

$$\varphi(\sigma_2(a)^* \sigma_{-2}(a)) = \varphi(\sigma_2(a) \sigma_2(a)^*).$$

We have that $\sigma_{-2}(a^*a) = \sigma_2(a)^* \sigma_{-2}(a)$ which belongs to $\mathcal{N}_\varphi$. On the other hand, we know also that $a^*a$ belongs to $\mathcal{M}_\varphi$. Hence, proposition 2.14 implies that

$$\varphi(\sigma_2(a)^* \sigma_{-2}(a)) = \varphi(\sigma_2(a^*a)) = \varphi(a^*a).$$

Combining these two equalities, we get that $\varphi(a^*a) = \varphi(\sigma_2(a) \sigma_2(a)^*)$. Therefore, proposition 5.3 implies that $\varphi$ is KMS with respect to $\sigma$.

In order to prove the implication from the right to the left, we do not really need the lower semi-continuity of $\varphi$. The closedness of the map $\Lambda : \mathcal{N}_\varphi \to H$ is also sufficient.
7 The tensor product of KMS-weights

Consider two C*-algebras $A$ and $B$. Let $\varphi$ be a KMS-weight on $A$ with modular group $\sigma$ and $\psi$ a KMS-weight on $B$ with modular group $\tau$.

Take also a GNS-construction $(H, \Lambda, \pi)$ for $\varphi$ and a GNS-construction $(K, \Gamma, \theta)$ for $\psi$.

- With respect to the GNS-construction $(H, \Lambda, \pi)$, we denote the modular conjugation of $\varphi$ by $\mathcal{J}$ and the modular operator of $\varphi$ by $\mathcal{V}$.
- With respect to the GNS-construction $(K, \Gamma, \theta)$, we denote the modular conjugation of $\psi$ by $\mathcal{J}$ and the modular operator $\psi$ by $\mathcal{D}$.

We will define the weight $\varphi \otimes \psi$ and show that it is a KMS-weight with respect to the obvious one-parameter group. The construction of the weight itself is due to Jan Verding and can be found in [2]. He did it in fact in a more general case. Again, due to the lack of availability of this work, we will include the proofs. We only added the KMS-characteristics.

**Definition 7.1** We define the norm continuous one-parameter group $\sigma \otimes \tau$ on $A \otimes B$ such that $(\sigma \otimes \tau)_t = \sigma_t \otimes \tau_t$ for every $t \in \mathbb{R}$.

Consider $z \in \mathcal{C}$. Then it is easy to see that $\sigma_z \otimes \tau_z \subseteq (\sigma \otimes \tau)_z$. We will prove in [2] (and this is not too difficult, using smearing techniques) that $D(\sigma_z) \otimes D(\tau_z)$ is a core for $(\sigma \otimes \tau)_z$. But we will not need this fact in this paper.

**Lemma 7.2** The mapping $\Lambda \otimes \Gamma : \mathcal{N}_\varphi \otimes \mathcal{N}_\psi \mapsto H \otimes K$ is closable.

**Proof** Define for every $\omega \in \mathcal{F}_\varphi$ the operator $S_\omega \in B(H)$ such that $\langle S_\omega \Lambda(a), \Lambda(b) \rangle = \omega(b^*a)$ for every $a, b \in \mathcal{N}_\varphi$. In a similar way, we define for every $\eta \in \mathcal{F}_\psi$ the operator $T_\eta \in B(K)$ such that $\langle T_\eta \Gamma(a), \Gamma(b) \rangle = \eta(b^*a)$ for every $a, b \in \mathcal{N}_\psi$.

Take a sequence $(x_n)_{n=1}^\infty$ in $\mathcal{N}_\varphi \otimes \mathcal{N}_\psi$ and $v \in H \otimes K$ such that $(x_n)_{n=1}^\infty$ converges to $0$ and $((\Lambda \otimes \Gamma)(x_n))_{n=1}^\infty$ converges to $v$.

Fix $\omega \in \mathcal{F}_\varphi$ and $\eta \in \mathcal{F}_\psi$ for the moment.

Choose $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_\psi$. It is not difficult to see that

$$\langle (S_\omega \otimes T_\eta)(\Lambda \otimes \Gamma)(x_n), \Lambda(a) \otimes \Gamma(b) \rangle = (\omega \otimes \eta)((a \otimes b)^* x_n)$$

for every $n \in \mathbb{N}$. This implies that the sequence

$$\left(\langle (S_\omega \otimes T_\eta)(\Lambda \otimes \Gamma)(x_n), \Lambda(a) \otimes \Gamma(b) \rangle \right)_{n=1}^\infty$$

converges to $0$. It is also clear that the sequence

$$\left(\langle (S_\omega \otimes T_\eta)(\Lambda \otimes \Gamma)(x_n), \Lambda(a) \otimes \Gamma(b) \rangle \right)_{n=1}^\infty$$

converges to $\langle (S_\omega \otimes T_\eta)v, \Lambda(a) \otimes \Gamma(b) \rangle$. So we see that $\langle (S_\omega \otimes T_\eta)v, \Lambda(a) \otimes \Gamma(b) \rangle = 0$.

Hence we get that $(S_\omega \otimes T_\eta)v = 0$.

Therefore, the remarks after result 2.2 implies that $v = 0$. 

So we can give the following definition.
Definition 7.3 We define the closed linear map $\Lambda \otimes \Gamma$ from within $A \otimes B$ into $H \otimes K$ such that $\mathcal{N}_\varphi \otimes \mathcal{N}_\psi$ is a core for $\Lambda \otimes \Gamma$ and $(\Lambda \otimes \Gamma)(a \otimes b) = \Lambda(a) \otimes \Gamma(b)$ for every $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_\psi$.

Remark 7.4

- We have for every $x \in A \otimes B$ and $y \in \mathcal{N}_\varphi \otimes \mathcal{N}_\psi$ that $x y$ belongs to $\mathcal{N}_\varphi \otimes \mathcal{N}_\psi$ and $(\Lambda \otimes \Gamma)(x y) = (\pi \otimes \theta)(x) (\Lambda \otimes \Gamma)(y)$. This implies easily that $D(\Lambda \otimes \Gamma)$ is a left ideal in $A \otimes B$ and that $(\Lambda \otimes \Gamma)(x y) = (\pi \otimes \theta)(x) (\Lambda \otimes \Gamma)(y)$ for every $x \in A \otimes B$ and $y \in D(\Lambda \otimes \Gamma)$.

- Choose $t \in \mathbb{R}$. Then it is easy to check for every $x \in \mathcal{N}_\varphi \otimes \mathcal{N}_\psi$ that $(\sigma \otimes \tau)_t(x)$ belongs to $\mathcal{N}_\varphi \otimes \mathcal{N}_\psi$ and $(\Lambda \otimes \Gamma)((\sigma \otimes \tau)_t(x)) = (\nabla^t \otimes \Delta^t)(\Lambda \otimes \Gamma)(x)$.

This will imply for every $x \in D(\Lambda \otimes \Gamma)$ that $(\sigma \otimes \tau)_t(x)$ belongs to $D(\Lambda \otimes \Gamma)$ and $(\Lambda \otimes \Gamma)((\sigma \otimes \tau)_t(x)) = (\nabla^t \otimes \Delta^t)(\Lambda \otimes \Gamma)(x)$.

In the beginning of section 6, we introduced the sub-*-algebras $C_\varphi$ of $\mathcal{N}_\varphi \cap \mathcal{N}_\psi^*$ and $C_\psi$ of $\mathcal{N}_\psi \cap \mathcal{N}_\varphi^*$. We have also that $C_\varphi \subseteq D(\sigma z)$ and $\tau_z(C_\varphi) \subseteq C_\varphi$ and that $C_\psi \subseteq D(\tau z)$ and $\tau_z(C_\psi) \subseteq C_\varphi$ for every $z \in \mathbb{C}$. We know also that $C_\varphi$ is a core for $\Lambda$ and that $C_\psi$ is a core for $\Gamma$.

Define the *-algebra $C = C_\varphi \otimes C_\psi$. Then $C$ is a core for $\Lambda \otimes \Gamma$.

By the remarks after definition 7.1, we see that easily that $C$ consists of analytic elements with respect to $(\sigma \otimes \tau)_z$ and that $(\sigma \otimes \tau)_z(C) \subseteq C$ for every $z \in \mathbb{C}$.

So we get in particular that $C \subseteq D((\sigma \otimes \tau)_z)$ and that $(\sigma \otimes \tau)_z(C)^* \subseteq C \subseteq D(\Lambda \otimes \Gamma)$.

Using the remarks after definition 7.1 and the fact that $\varphi(b^* a) = \varphi(\sigma_z(a) \sigma_z(b)^*)$ for every $a, b \in \mathcal{N}_\varphi \cap D(\sigma_z)$ and $\psi(b^* a) = \psi(\tau_z(a) \tau_z(b)^*)$ for every $a, b \in \mathcal{N}_\psi \cap D(\tau_z)$, it is not difficult to check that

$$
\|((\Lambda \otimes \Gamma)(x))\|^2 = \|((\Lambda \otimes \Gamma)((\sigma \otimes \tau)_z(x)^*))\|^2
$$

for every $x \in C$.

So we see that the ingredients $A \otimes B$, $H \otimes K$, $\Lambda \otimes \Gamma$, $\pi \otimes \theta$, $\sigma \otimes \theta$, $\nabla \otimes \Delta$ satisfy the conditions of the first construction procedure of section 3. Therefore we can use definition 7.1.

Definition 7.5 We define the weight $\varphi \otimes \psi$ on $A \otimes B$ such that $(H \otimes K, \Lambda \otimes \Gamma, \pi \otimes \theta)$ is a GNS-construction for $\varphi \otimes \psi$.

So we have in particular that $\mathcal{N}_{\varphi \otimes \psi} = D(\Lambda \otimes \Gamma)$, so $\mathcal{N}_\varphi \otimes \mathcal{N}_\psi \subseteq \mathcal{N}_{\varphi \otimes \psi}$.

Proposition 7.6 implies the following one:

Proposition 7.7 With respect to the GNS-construction, the modular conjugation of $\varphi \otimes \psi$ is given by $J \otimes I$, the modular operator of $\varphi \otimes \psi$ is given by $\nabla \otimes \Delta$.

Concerning the modular conjugation we have the following obvious result:

Definition 7.3 also implies easily the following result:
Result 7.8 Consider $a \in \mathcal{M}_\varphi$ and $b \in \mathcal{M}_\psi$. Then $a \otimes b$ belongs to $\mathcal{M}_{\varphi \otimes \psi}$ and $(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \psi(b)$.

There is also another characterization of $\varphi \otimes \psi$.

Result 7.9 We have the following properties:

1. We have for every $\omega \in \mathcal{F}_\varphi$ and $\eta \in \mathcal{F}_\psi$ that $\omega \otimes \eta$ belongs to $\mathcal{F}_{\varphi \otimes \psi}$.
2. We have for every $\omega \in \mathcal{G}_\varphi$ and $\eta \in \mathcal{G}_\psi$ that $\omega \otimes \eta$ belongs to $\mathcal{G}_{\varphi \otimes \psi}$.

Proof: Consider $\omega \in \mathcal{F}_\varphi$ and $\eta \in \mathcal{F}_\psi$.

Define the operator $S_\omega \in B(H)$ such that $\langle S_\omega \Lambda(a), \Lambda(b) \rangle = \omega(b^*a)$ for every $a, b \in \mathcal{N}_\varphi$.

Define also the operator $T_\eta \in B(K)$ such that $\langle T_\eta \Gamma(a), \Gamma(b) \rangle = \eta(b^*a)$ for every $a, b \in \mathcal{N}_\psi$.

Then it easy to see that

$$\langle (S_\omega \otimes T_\eta)(\Lambda \otimes \Gamma)(x), (\Lambda \otimes \Gamma)(y) \rangle = (\omega \otimes \eta)(y^*x)$$

for every $x, y \in \mathcal{N}_\varphi \otimes \mathcal{N}_\psi$. As usual, this implies that

$$\langle (S_\omega \otimes T_\eta)(\Lambda \otimes \Gamma)(x), (\Lambda \otimes \Gamma)(y) \rangle = (\omega \otimes \eta)(y^*x)$$

for every $x, y \in \mathcal{N}_{\varphi \otimes \psi}$.

Because $0 \leq S_\omega \leq 1$ and $0 \leq T_\eta \leq 1$, this implies immediately that $\omega \otimes \eta$ belongs to $\mathcal{F}_{\varphi \otimes \psi}$.

The second result follows from the first one.

Proposition 7.10 Consider $x \in (A \otimes B)^+$. Then

$$(\varphi \otimes \psi)(x) = \sup \{ (\omega \otimes \eta)(x) \mid \omega \in \mathcal{F}_\varphi, \eta \in \mathcal{F}_\psi \}.$$  

Proof: By the previous proposition and the lower semi-continuity of $\varphi \otimes \psi$, we get immediately the inequality

$$(\varphi \otimes \psi)(x) \geq \sup \{ (\omega \otimes \eta)(x) \mid \omega \in \mathcal{F}_\varphi, \eta \in \mathcal{F}_\psi \}.$$  

By proposition 6.3, we have truncating sequences $(u_k)_{k \in K}$ for $\varphi$ and $(v_l)_{l \in L}$ for $\psi$.

We define for every $k \in K$ the operator $S_k \in B(H)$ such that $S_k \Lambda(a) = \Lambda(au_k)$ for every $a \in \mathcal{N}_\varphi$.

In a similar way, we define for every $l \in L$ the operator $T_l \in B(K)$ such that $T_l \Gamma(b) = \Gamma(bu_l)$ for every $b \in \mathcal{N}_\psi$.

Then we get immediately that

- We have that $u_k \otimes v_l$ belongs to $\mathcal{N}_{\varphi \otimes \psi}$ for every $k \in K, l \in L$.
- $\|S_k \otimes T_l\| \leq 1$ for every $k \in K$ and $l \in L$.
- The net $(u_k \otimes v_l)_{(k,l) \in K \times L}$ converges strictly to 1.
- The net $(S_k \otimes T_l)_{(k,l) \in K \times L}$ converges strongly to 1.
Choose \( k \in K, l \in L \).
Then it is easy to see that \((S_k \otimes T_l)(\Lambda \otimes \Gamma)(y) = (\Lambda \otimes \Gamma)(y (u_k \otimes v_l))\) for every \( y \in \mathcal{N}_\varphi \otimes \mathcal{N}_\psi \).
As usual, this implies that \((S_k \otimes T_l)(\Lambda \otimes \Gamma)(y) = (\Lambda \otimes \Gamma)(y (u_k \otimes v_l))\) for every \( y \in \mathcal{N}_\varphi \otimes \mathcal{N}_\psi \).
So we see that the net \((u_k \otimes v_l)_{(k,l)\in K \otimes L}\) satisfies the conditions of proposition 3.8.
For every \( k \in K \), we define \( \omega_k \in \mathcal{F}_\varphi \) such that \( \omega_k(a) = \varphi(u_k^* a u_k) \) for every \( a \in A \).
For every \( l \in L \), we define \( \eta_l \in \mathcal{F}_\psi \) such that \( \eta_l(b) = \varphi(v_l^* b v_l) \) for every \( b \in B \).
It is clear that \((\omega_k \otimes \eta_l)(y) = (\varphi \otimes \eta)((u_k \otimes v_l)^* y (u_k \otimes v_l))\) for every \( y \in A \otimes B \).
Therefore, proposition 3.8 implies that
\[
(\varphi \otimes \eta)(x) = \sup \{ (\omega_k \otimes \eta_l)(x) \mid k \in K, l \in L \} \leq \sup \{ (\omega \otimes \eta)(x) \mid \omega \in \mathcal{F}_\varphi, \eta \in \mathcal{F}_\psi \}.
\]

**Corollary 7.11** We have the following properties:

1. We have for every \( x \in (A \otimes B)^+ \) that the net \( \{(\omega \otimes \eta)(x)\}_{(\omega,\eta)\in \mathcal{G}_\varphi \times \mathcal{G}_\psi} \) converges to \((\varphi \otimes \psi)(x)\).
2. We have for every \( x \in \mathcal{M}_{\varphi \otimes \psi} \) that the net \( \{(\omega \otimes \eta)(x)\}_{(\omega,\eta)\in \mathcal{G}_\varphi \times \mathcal{G}_\psi} \) converges to \((\varphi \otimes \psi)(x)\).

**8 Absolutely continuous KMS-weights**

Consider a C*-algebra \( A \) and a KMS-weight \( \varphi \) on \( A \) with modular group \( \sigma \).
Let \( \delta \) be a strictly positive element affiliated with \( A \) such that there exists a strictly positive number \( \lambda \) such that \( \sigma_t(\delta) = \lambda^t \delta \) for every \( t \in \mathbb{R} \).
In this section we are going to use one of the construction procedures to define the KMS-weight \( \varphi(\delta^{\frac{1}{2}}, \delta^{\frac{1}{2}}) \) and prove some useful properties about this weight.
It is not possible to use the definition of \([14]\), because it is assumed in this paper that \( \sigma_t(\delta) = \delta \) for every \( t \in \mathbb{R} \). Our construction procedures from section 3 allows us to go about it in another way.

First, we introduce some notations. Take a GNS-construction \((H, \Lambda, \pi)\) for \( \varphi \). We denote the modular conjugation of \( \varphi \) in this GNS-construction by \( J \) and the modular operator by \( \nabla \).
We will extensively have to truncate \( \delta \) in order to make things bounded. We will also need this truncations to behave well with respect to \( \sigma \). For this, we will use the following elements in \( M(A) \).

**Notation 8.1** Consider \( n \in \mathbb{N} \). Then we define \( e_n \in M(A) \) such that
\[
e_n a = \frac{n}{\sqrt{n}} \int \exp(-n^2 t^2) \delta^{it} a \, dt
\]
for every \( a \in A \).

The behaviour of these elements is described in the next proposition.

**Proposition 8.2** We have the following properties.

1. The sequence \((e_n)_{n=1}^{\infty}\) is bounded by 1 and converges strictly to 1.
2. We have for every \( n \in \mathbb{N} \) that \( e_n \) is strictly analytic with respect to \( \sigma \).
3. Let \( y \in \mathcal{C} \). Then the sequence \((\sigma_y(e_n))_{n=1}^{\infty}\) is bounded and converges strictly to 1.
4. Consider $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Then $e_n$ is a left and right multiplier of $\delta^z$ and $\delta^z e_n = e_n \delta^z$.

5. Let $n \in \mathbb{N}$ and $y, z \in \mathbb{C}$. Then $\delta^z e_n$ is a left and right multiplier of $\delta^y$ and $(\delta^z e_n) \delta^y = \delta^y (\delta^z e_n) = \delta^{y+z} e_n$.

6. Consider $n \in \mathbb{N}$ and $y, z \in \mathbb{C}$. Then $\sigma_y(e_n)$ is a left and right multiplier of $\delta^z$ and $\sigma_y(e_n) \delta^z = \delta^z \sigma_y(e_n)$.

7. Consider $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Then $\delta^z e_n$ is strictly analytic with respect to $\sigma$ and $\sigma_y(\delta^z e_n) = \lambda^n \delta^z \sigma_y(e_n)$ for every $y \in \mathbb{C}$.

The results of this proposition will be frequently used in the rest of this section without reference to this proposition. The proof of it is a standard exercise in the use of analytic continuations and will be left out.

We define the set
\[ N_0 = \{ a \in A \mid a \text{ is a left multiplier of } \delta^\frac{1}{2} \text{ and } a \delta^\frac{1}{2} \text{ belongs to } \mathcal{N}_\varphi \} . \]

Introducing this set is of course a very natural thing to do in this case.

It is clear that this set is a left ideal in $M(A)$.

**Lemma 8.3** The mapping $N_0 \to H : a \mapsto \Lambda(a \delta^\frac{1}{2})$ is closable

**Proof:** Choose a sequence $(x_k)_{k=1}^\infty$ in $N_0$ and $v \in H$ such that $(x_k)_{k=1}^\infty$ converges to 0 and $(\Lambda(x_k \delta^\frac{1}{2}))_{k=1}^\infty$ converges to $v$. Choose $n \in \mathbb{N}$.

Take $l \in \mathbb{N}$.

Because $e_n$ is a left multiplier of $\delta^\frac{1}{2}$, we have that $x_l \cdot e_n$ is a left multiplier of $\delta^\frac{1}{2}$ and
\[ (x_l \cdot e_n) \delta^\frac{1}{2} = x_l (e_n \delta^\frac{1}{2}) = x_l (\delta^\frac{1}{2} e_n) . \tag{a} \]

We know that $x_l$ is a left multiplier of $\delta^\frac{1}{2}$, so this last equality implies also that $(x_l \cdot e_n) \delta^\frac{1}{2} = (x_l \delta^\frac{1}{2}) e_n$.

Because $x_l \delta^\frac{1}{2}$ belongs to $\mathcal{N}_\varphi$ and $e_n$ belongs to $D(\pi_\varphi)$, this equality and proposition 6.4 imply that $(x_l \cdot e_n) \delta^\frac{1}{2}$ belongs to $\mathcal{N}_\varphi$ and
\[ \Lambda((x_l \cdot e_n) \delta^\frac{1}{2}) = J\pi(\sigma_\varphi(e_n))^* J\Lambda(x_l \delta^\frac{1}{2}) = J\pi(\sigma_\varphi(e_n))^* J\Gamma(x_l) . \tag{b} \]

Equality (a) implies that $(x_k \cdot e_n) \delta^\frac{1}{2})_{k=1}^\infty$ converges to 0, whereas equality (b) implies that $(\Lambda((x_k \cdot e_n) \delta^\frac{1}{2}))_{k=1}^\infty$ converges to $J\pi(\sigma_\varphi(e_n))^* Jv$.

Therefore, the closedness of $\Lambda$ implies that $J\pi(\sigma_\varphi(e_n))^* Jv = 0$

Because $(\pi(\sigma_\varphi(e_n)))_{n=1}^\infty$ converges strongly* to 1, this last equation implies that $v = 0$.

Therefore, we can give the following definition.

**Definition 8.4** We define $\Gamma$ as the closure of the linear mapping $N_0 \to H : a \mapsto \Lambda(a \delta^\frac{1}{2})$. The domain of $\Gamma$ will be denoted by $N$.

So $\Gamma$ is a closed linear mapping from $N$ into $H$ such that $N_0$ is a core for $\Gamma$ and $\Gamma(a) = \Lambda(a \delta^\frac{1}{2})$ for every $a \in N_0$. It is also easy to check that $N$ is a left ideal in $M(A)$ and that $\Gamma(xa) = \pi(x)\Gamma(a)$ for every $x \in M(A)$ and $a \in N$. 

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Lemma 8.5 Consider \( a \in \mathcal{N}_\varphi \), \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \). Then \( a(\delta^z e_n) \) belongs to \( N_0 \) and \( \Gamma(a(\delta^z e_n)) = J\pi(\sigma_{\frac{\varphi}{2}}(\delta^{\frac{z}{2}}e_n))^*J\Lambda(a) \).

**Proof:** We know that \( \delta^z e_n \) is a left multiplier of \( \delta^\frac{z}{2} \) and \( (\delta^z e_n) \delta^{\frac{z}{2}} = \delta^{\frac{z}{2}+z}e_n \). This implies that \( a(\delta^z e_n) \) is a left multiplier of \( \delta^{\frac{z}{2}} \) and \( (a(\delta^z e_n)) \delta^{\frac{z}{2}} = a(\delta^{\frac{z}{2}+z}e_n) \). Because \( a \) belongs to \( \mathcal{N}_\varphi \) and \( \delta^{\frac{z}{2}+z}e_n \) belongs to \( D(\mathcal{F}_\varphi) \), proposition 3.11 implies that \( a(\delta^z e_n) \delta^{\frac{z}{2}} \) belongs to \( \mathcal{N}_\varphi \) and \( \Lambda((a(\delta^z e_n)) \delta^{\frac{z}{2}}) = J\pi(\sigma_{\frac{\varphi}{2}}(\delta^{\frac{z}{2}+z}e_n))^*J\Lambda(a) \). This gives by definition that \( a(\delta^z e_n) \) belongs to \( N_0 \) and \( \Gamma(a(\delta^z e_n)) = J\pi(\sigma_{\frac{\varphi}{2}}(\delta^{\frac{z}{2}+z}e_n))^*J\Lambda(a) \).

**Result 8.6** The set \( N \) is dense in \( A \) and the set \( \Gamma(N) \) is dense in \( H \).

**Proof:** Choose \( a \in \mathcal{N}_\varphi \). The previous lemma implies for every \( n \in \mathbb{N} \) that \( a e_n \) belongs to \( N_0 \). Because \( (e_n)_{n=1}^{\infty} \) converges strictly to 1, we see that \( (a e_n)_{n=1}^{\infty} \) converges to \( a \). Hence, the density of \( \mathcal{N}_\varphi \) in \( A \) implies that \( N_0 \) is dense in \( A \).

Choose \( a \in \mathcal{N}_\varphi \). The previous lemma implies for every \( n \in \mathbb{N} \) that \( a(\delta^{-\frac{z}{2}} e_n) \) belongs to \( N_0 \) and \( \Gamma(a(\delta^{-\frac{z}{2}} e_n)) = J\pi(\sigma_{\frac{\varphi}{2}}(e_n))^*J\Lambda(a) \). Because \( (\pi(\sigma_{\frac{\varphi}{2}}(e_n)))_{n=1}^{\infty} \) converges strongly to 1. This implies that \( (\Gamma(a(\delta^{-\frac{z}{2}} e_n)))_{n=1}^{\infty} \) converges to \( \Lambda(a) \). Hence, the density of \( \Lambda(\mathcal{N}_\varphi) \) in \( H \) implies that \( \Gamma(N_0) \) is dense in \( H \).

So we have objects \( A,H,N,\Lambda,\pi \) on which we want to apply the second procedure of section 8. In order to do so, we have to introduce the KMS-characteristics.

**Definition 8.7** We define the norm continuous one-parameter group \( \sigma' \) on \( A \) such that \( \sigma'_t(a) = \delta^{it} \sigma_t(a) \delta^{-it} \) for every \( t \in A \) and \( a \in A \).

It is not difficult to see that \( \sigma'_t(\delta) = \lambda^t \delta \) for every \( t \in \mathbb{R} \).

Result 8.13 implies the following lemma.

**Lemma 8.8** We have for every \( s,t \in \mathbb{R} \) that \( J\pi(\delta)^{it}J \) and \( \pi(\delta)^{is} \) commute.

The following result is true because \( \pi(\sigma_s(a)) = \nabla^{is} \pi(a) \nabla^{-is} \) for every \( a \in A \), \( s \in \mathbb{R} \) and the fact that \( \sigma_s(\delta^{it}) = \lambda^{ist} \delta^{it} \) for \( s,t \in \mathbb{R} \) (which is true by assumption).

**Lemma 8.9** Consider \( s,t \in \mathbb{R} \). Then \( \nabla^{is} \pi(\delta)^{it} = \lambda^{ist} \pi(\delta)^{it} \nabla^{is} \)

This will imply the following lemma.

**Lemma 8.10** We have for every \( s,t \in \mathbb{R} \) that \( J\pi(\delta)^{it}J \pi(\delta)^{it} \) and \( \nabla^{is} \) commute.

**Proof:** We have that

\[
J\pi(\delta)^{it}J \pi(\delta)^{it} \nabla^{is} = \lambda^{-ist} J\pi(\delta)^{it} J \nabla^{is} \pi(\delta)^{it} = \lambda^{-ist} J\pi(\delta)^{it} \nabla^{is} J\pi(\delta)^{it} \equiv^{(*)} \lambda^{-ist} \lambda^{ist} J \nabla^{is} \pi(\delta)^{it} J \pi(\delta)^{it} = \nabla^{is} J\pi(\delta)^{it} J \pi(\delta)^{it}.
\]

In (*) we used that \( J \) is antilinear.

Lemmas 8.8 and 3.10 imply that the mapping \( \mathbb{R} \to B(H) : t \mapsto J\pi(\delta)^{it}J \pi(\delta)^{it} \nabla^{it} \) is a strongly continuous unitary group representation on \( H \). By the Stone theorem, this justifies the following definition.
Definition 8.11 We define the positive injective operator $\nabla$ (pronounced 'nabla prime') in $H$ such that
$\nabla^it = J\pi(\delta)^it J\pi(\delta)^it \nabla^it$ for every $t \in \mathbb{R}$.

Proposition 8.12 Consider $a \in N_\varphi$ and $t \in \mathbb{R}$. Then $\sigma'_t(a)$ belongs to $N_\varphi$ and $\Lambda(\sigma'_t(a)) = \lambda^{\frac{t}{2}} \nabla^it \Lambda(a)$.

Proof: We know by definition that $\sigma_t(a)$ belongs to $N_\varphi$ and $\Lambda(\sigma_t(a)) = \nabla^it \Lambda(a)$. This implies that $\delta^it \sigma_t(a)$ belongs to $N_\varphi$ and that $\Lambda(\delta^it \sigma_t(a)) = \pi(\delta)^it \Lambda(\sigma_t(a)) = \pi(\delta)^it \nabla^it \Lambda(a)$.
Therefore, proposition 8.30 implies that $\delta^it \sigma_t(a) \delta^{-it}$ belongs to $N_\varphi$ and
$$\Lambda(\delta^it \sigma_t(a) \delta^{-it}) = \lambda^{\frac{t}{2}} J\pi(\delta)^it J\Lambda(\delta^it \sigma_t(a)) = \lambda^{\frac{t}{2}} J\pi(\delta)^it J\pi(\delta)^it \nabla^it \Lambda(a).$$
Looking at the definitions of $\sigma'_t$ and $\nabla$, the lemma follows.

Proposition 8.13 Consider $a \in N$ and $t \in \mathbb{R}$. Then $\sigma_t(a)$ belongs to $N$ and $\Gamma(\sigma_t(a)) = \lambda^{-\frac{t}{2}} \nabla^it \Gamma(a)$.

Proof: Choose $b \in N_0$.
Because $b$ is a left multiplier of $\delta^\frac{1}{2}$ and $\sigma_t(\delta) = \lambda^t \delta$, we get that $\sigma_t(b)$ is a left multiplier of $\delta^\frac{1}{2}$ and
$$\sigma_t(b) \delta^\frac{1}{2} = \lambda^{-\frac{t}{2}} \sigma_t(b \delta^\frac{1}{2}).$$
Because $b \delta^\frac{1}{2}$ belongs to $N_\varphi$, this implies that $\sigma_t(b) \delta^\frac{1}{2}$ belongs to $N_\varphi$ and
$$\Lambda(\sigma_t(b) \delta^\frac{1}{2}) = \lambda^{-\frac{t}{2}} \Lambda(\sigma_t(b \delta^\frac{1}{2})) = \lambda^{-\frac{t}{2}} \nabla^it \Lambda(b \delta^\frac{1}{2}) = \lambda^{-\frac{t}{2}} \nabla^it \Gamma(b).$$
So we see that $\sigma_t(b)$ belongs to $N_0$ and
$$\Gamma(\sigma_t(b)) = \Lambda(\sigma_t(b) \delta^\frac{1}{2}) = \lambda^{-\frac{t}{2}} \nabla^it \Gamma(b).$$
The result follows easily because $N_0$ is a core for $\Gamma$.

Proposition 8.14 Consider $a \in N$ and $t \in \mathbb{R}$. Then $\sigma'_t(a)$ belongs to $N$ and $\Gamma(\sigma'_t(a)) = \nabla^it \Gamma(a)$.

Proof: Choose $b \in N_0$.
Because $b$ is a left multiplier of $\delta^\frac{1}{2}$ and $\sigma'_t(\delta) = \lambda^t \delta$, we get that $\sigma'_t(b)$ is a left multiplier of $\delta^\frac{1}{2}$ and
$$\sigma'_t(b) \delta^\frac{1}{2} = \lambda^{-\frac{t}{2}} \sigma'_t(b \delta^\frac{1}{2}).$$
Because $b \delta^\frac{1}{2}$ belongs to $N_\varphi$, this implies that $\sigma'_t(b) \delta^\frac{1}{2}$ belongs to $N_\varphi$ and
$$\Lambda(\sigma'_t(b) \delta^\frac{1}{2}) = \lambda^{-\frac{t}{2}} \Lambda(\sigma'_t(b \delta^\frac{1}{2})) = \lambda^{-\frac{t}{2}} \lambda^{\frac{t}{2}} \nabla^it \Lambda(b \delta^\frac{1}{2}) = \nabla^it \Gamma(b),$$
where we used proposition 8.12 in equality (*). The results follows easily because $N_0$ is a core for $\Gamma$.

Proposition 8.15 Consider $x \in N$ and $a \in D(\sigma'_t)$. Then $xa$ belongs to $N$ and
$$\Gamma(xa) = J\pi(\sigma'_t(a))^* J\Gamma(x).$$
Proof: Define the norm continuous one-parameter group $\tau$ on $A$ such that $\tau_t(a) = \delta^t a \delta^{-t}$ for every $a \in A$ and $t \in \mathbb{R}$. Then we have that $\sigma \tau_t = \tau_t \sigma$ for every $s, t \in \mathbb{R}$ and $\sigma_t(a) = \tau_t(\sigma_t(a))$ for every $t \in \mathbb{R}$.

This implies that $\sigma \tau_1$ is closable and that the closure is equal to $\sigma'$ (see \S 5).

Take $y \in N_0$. So $y$ is a left multiplier of $\delta^\frac{1}{2}$ and $y \delta^\frac{1}{2}$ belongs to $N_\varphi$.

Choose $b \in D(\sigma_\frac{1}{2} \tau_1)$.

Because $b$ belongs to $D(\tau_1)$, the element $b$ is a middle multiplier of $\delta^{-\frac{1}{2}}, \delta^\frac{1}{2}$ and $\delta^{-\frac{1}{2}} b \delta^\frac{1}{2} = \tau_1(b)$.

We have by assumption that $y$ is a left multiplier of $\delta^\frac{1}{2}$. This implies that $y \delta^\frac{1}{2}$ is a left multiplier of $\delta^{-\frac{1}{2}}$ and $(y \delta^\frac{1}{2}) \delta^{-\frac{1}{2}} = y$.

This in turn implies that $((y \delta^\frac{1}{2}) \delta^{-\frac{1}{2}}) b \delta^\frac{1}{2} = (y \delta^\frac{1}{2}) (\delta^{-\frac{1}{2}} b \delta^\frac{1}{2}) = (y \delta^\frac{1}{2}) \tau_1(b)$.

Combining this with the fact that $((y \delta^\frac{1}{2}) \delta^{-\frac{1}{2}}) b = y b$, we see that $y b$ is a left multiplier of $\delta^\frac{1}{2}$ and $(y b) \delta^\frac{1}{2} = (y \delta^\frac{1}{2}) \tau_1(b)$.

Because $y \delta^\frac{1}{2}$ belongs to $N_\varphi$ and $\tau_1(b)$ belongs to $D(\sigma_\frac{1}{2})$, this last equation implies that $(y b) \delta^\frac{1}{2}$ belongs to $N_\varphi$ and

$$\Lambda((y b) \delta^\frac{1}{2}) = J \pi((\sigma_\frac{1}{2} \tau_1(b)))^* \Lambda(y \delta^\frac{1}{2}) = J \pi((\sigma_\frac{1}{2} \tau_1(b)))^* J \Gamma(y).$$

Therefore, $y b$ belongs to $N_0$ and $\Gamma(y b) = J \pi((\sigma_\frac{1}{2} \tau_1(b)))^* J \Gamma(y)$.

So we see that $\Gamma(y b) = J \pi(\sigma'(a)_n) \Gamma(y)$. (*)

Because $D(\sigma_\frac{1}{2} \tau_1)$ is a core for $\sigma'$, there exists a sequence $(a_n)_{n=1}^\infty$ in $D(\sigma_\frac{1}{2} \tau_1)$ such that $(a_n)_{n=1}^\infty$ converges to $a$ and $(\sigma'(a_n))_{n=1}^\infty$ converges to $\sigma'(a)$.

By result (*) we have for every $n \in \mathbb{N}$ that $ya_n$ belongs to $N_0$ and

$$\Gamma(ya_n) = J \pi(\sigma'(a)_n) \Gamma(y)$$

From this, we conclude that the sequence $(\Gamma(ya_n))_{n=1}^\infty$ converges to $J \pi(\sigma'(a)) \Gamma(y)$. It is also clear that $(ya_n)_{n=1}^\infty$ converges to $ya$ so the closedness of $\Gamma$ implies that $ya$ belongs to $N$ and $\Gamma(ya) = J^* \pi(\sigma'(a)) \Gamma(y)$.

The proposition follows because $N_0$ is a core for $\Gamma$.

At last we are in a position to define $\varphi(\delta^\frac{1}{2}, \delta^\frac{1}{2})$.

Definition 8.16: We have proven in the previous part of the section that the ingredients $A, H, N, \Lambda, \pi, \sigma'$, and $J$ satisfy the conditions of the second construction procedure of section \S 5. The weight which is obtained from this ingredients is denoted by $\varphi_\delta$ (see definition \S 5.1).

We want to forget about this definition, but will instead repeat the determining properties. Previously, we have defined a norm-continuous one-parameter group $\sigma'$ on $A$ such that $\sigma'_t(a) = \delta^t \sigma_t(a) \delta^{-t}$ for every $t \in \mathbb{R}$ and $a \in A$. Then

Theorem 8.17: We have that $\varphi_\delta$ is a KMS-weight with modular group $\sigma'$.

The weight $\varphi_\delta$ is completely determined by the following proposition.
Theorem 8.18 We have that \((H, \Gamma, \pi)\) is a GNS-construction for \(\varphi_{\delta}\)

Remember that the set
\[ N_0 = \{ a \in A \mid a \text{ is a left multiplier of } \delta^\sharp \text{ and } a \delta^\sharp \text{ belongs to } \mathcal{N}_\varphi \} . \]
is a core for \(\Gamma\) and that \(\Gamma(a) = \Lambda(a \delta^\sharp)\) for every \(a \in N_0\).

Repeating the results of lemmas 8.8 and 8.10 gives

- We have for every \(s, t \in \mathbb{R}\) that \(J_{\pi}(\delta) it J_{\pi}(\delta) it \) and \(\nabla it \) commute.
- We have for every \(s, t \in \mathbb{R}\) that \(J_{\pi}(\delta) it J_{\pi}(\delta) it \) and \(\nabla it \) commute.

Also, the injective positive operator \(\nabla\) in \(H\) was defined in such a way that \(\nabla it = J_{\pi}(\delta) it J_{\pi}(\delta) it \nabla it\) for every \(t \in \mathbb{R}\). Then:

Proposition 8.19 We have that \(\nabla\) is the modular operator for \(\varphi_{\delta}\) in the GNS-construction \((H, \Gamma, \pi)\).

Later, we will prove that the modular conjugation for \(\varphi_{\delta}\) is a scalar multiple of \(J\).

The following relative invariance properties are also valid.

- Consider \(a \in N_\varphi\) and \(t \in \mathbb{R}\). Then \(\sigma_t(a)\) belongs to \(N_\varphi\) and \(\Lambda(\sigma_t(a)) = \lambda^{\frac{t}{2}} \nabla it \Lambda(a)\).
- Consider \(a \in N_{\varphi, \delta}\) and \(t \in \mathbb{R}\). Then \(\sigma_t(a)\) belongs to \(N_{\varphi, \delta}\) and \(\Gamma(\sigma_t(a)) = \lambda^{\frac{-t}{2}} \nabla it \Gamma(a)\).

These imply the following proposition.

Proposition 8.20 Consider \(t \in \mathbb{R}\). Then \(\sigma_t = \lambda^t \varphi\) and \(\varphi_{\delta} \sigma_t = \lambda^{-t} \varphi_{\delta}\).

Because \((H, \Gamma, \pi)\) is a GNS-construction for \(\varphi_{\delta}\), proposition 6.22 implies the following result.

Proposition 8.21 The weight \(\varphi_{\delta}\) is faithful \(\iff\) \(\varphi\) is faithful

Because \(\sigma_t(\delta) = \lambda^t \delta\) for every \(t \in \mathbb{R}\), also the following proposition will be true (see proposition 8.2).

Proposition 8.22 We have the following properties.

1. We have for every \(n \in \mathbb{N}\) that \(e_n\) is strictly analytic with respect to \(\sigma\).
2. Let \(y \in \mathbb{C}\). Then the sequence \((\sigma_y(e_n))_{n=1}^\infty\) is bounded and converges strictly to 1.
3. Consider \(n \in \mathbb{N}\) and \(y, z \in \mathbb{C}\). Then \(\sigma_y(e_n)\) is a left and right multiplier of \(\delta z\) and \(\sigma_y(e_n) \delta z = \delta z \sigma_y(e_n)\).
4. Consider \(n \in \mathbb{N}\) and \(z \in \mathbb{C}\). Then \(\delta z e_n\) is strictly analytic with respect to \(\sigma\) and \(\sigma_y(\delta z e_n) = \lambda^{yz} \delta z \sigma_y(e_n)\) for every \(y \in \mathbb{C}\).

In the next part of this section, we will prove a formula which formally says that \(\varphi_{\delta}(x) = \varphi(\delta^\sharp x \delta^\sharp)\) (and something more general).

Lemma 8.23 We have the following properties:

- We have for every \(a \in N_\varphi, n \in \mathbb{N}\) and \(z \in \mathbb{C}\) that a \((\delta z e_n)\) belongs to \(N_\varphi \cap N_{\varphi, \delta}\).
• We have for every \( x \in M_\varphi \), \( m, n \in \mathbb{N} \) and \( y, z \in \mathfrak{C} \) that \( (\delta^y e_m) x (\delta^z e_n) \) belongs to \( M_\varphi \cap M_{\varphi^*} \).

Proof:

• From lemma 8.13, we know already that \( a (\delta^x e_n) \) belongs to \( N_{\varphi^*} \). Because \( a \) belongs to \( N_\varphi \) and \( \delta^x e_n \) belongs to \( D(\sigma_\varphi) \), proposition 8.10 implies also that \( a (\delta^x e_n) \) belongs to \( N_\varphi \).

• This follows immediately from the first property.

Lemma 8.24 Consider \( x \in M_\varphi \). Then \( (\varphi(e_n x e_n))_{n=1}^{\infty} \) converges to \( \varphi(x) \).

Proof: Choose \( a, b \in M_\varphi \). We have for every \( n \in \mathbb{N} \) that \( a e_n, b e_n \) belong to \( N_{\varphi} \) and \( \Lambda(a e_n) = J\pi(\sigma_\varphi(e_n))^* J\Lambda(a) \), \( \Lambda(b e_n) = J\pi(\sigma_\varphi(e_n))^* J\Lambda(b) \) which implies that

\[
\varphi(e_n b^* a e_n) = (J\pi(\sigma_\varphi(e_n))^* J\Lambda(a), J\pi(\sigma_\varphi(e_n))^* J\Lambda(b)) .
\]

Because \( (\pi(\sigma_\varphi(e_n)))_{n=1}^{\infty} \) converges strongly to 1, we see that \( (\varphi(e_n b^* a e_n))_{n=1}^{\infty} \) converges to \( (\Lambda(a), \Lambda(b)) \) which is equal to \( \varphi(b^* a) \).

Lemma 8.25 Consider \( x \in N_{\varphi^*} \), \( n \in \mathbb{N} \) and \( z \in \mathfrak{C} \). Then \( x (\delta^z e_n) \) belongs to \( N_\varphi \).

Proof: Because \( \delta^z e_n \) belongs to \( D(\sigma_\varphi^*) \), we have immediately that \( x (\delta^z e_n) \) belongs to \( N_{\varphi^*} \).

By definition, there exists a sequence \( (x_k)_{k=1}^{\infty} \) in \( N_0 \) such that \( (x_k)_{k=1}^{\infty} \) converges to \( x \) and \( (\Gamma(x_k))_{k=1}^{\infty} \) converges to \( \Gamma(x) \).

Choose \( l \in \mathbb{N} \). Because \( x_l \) is a left multiplier of \( \delta^\varphi \) and \( \delta^z \cdot z e_n \) is a right multiplier of \( \delta^\varphi \), we have that

\[
(x_l \delta^\varphi)(\delta^z \cdot z e_n) = x_l \delta^\varphi(\delta^z \cdot z e_n) = x_l (\delta^z e_n) .
\]

Because \( x_l \delta^\varphi \) belongs to \( N_\varphi \) and \( \delta^z \cdot z e_n \) belongs to \( D(\sigma_\varphi^*) \), this implies that \( x_l (\delta^z e_n) \) belongs to \( N_\varphi \) and

\[
\Lambda(x_l (\delta^z e_n)) = J\pi(\sigma_\varphi(\delta^z \cdot z e_n))^* J\Lambda(x_l \delta^\varphi) = J\pi(\sigma_\varphi(\delta^z \cdot z e_n))^* J\Gamma(x_l) .
\]

From this last equation, we infer that \( (\Lambda(x_k (\delta^z e_n)))_{k=1}^{\infty} \) converges to \( J\pi(\sigma_\varphi(\delta^z \cdot z e_n))^* J\Gamma(x) \). It is also clear that \( (x_k (\delta^z e_n))_{k=1}^{\infty} \) converges to \( x (\delta^z e_n) \). Therefore, the closedness of \( \Lambda \) implies that \( x (\delta^z e_n) \) belongs to \( N_\varphi \).

Using this lemma, we can even do better.

Lemma 8.26 Consider \( x \in N_{\varphi^*} \), \( n \in \mathbb{N} \) and \( z \in \mathfrak{C} \). Then \( x (\delta^z e_n) \) belongs to \( N_0 \).

Proof: Because \( \delta^z e_n \) is a left multiplier of \( \delta^\varphi \), we get that \( x (\delta^z e_n) \) is a left multiplier of \( \delta^\varphi \) and

\[
(x (\delta^z e_n)) \delta^\varphi = x ((\delta^z e_n) \delta^\varphi) = x (\delta^{z+1} e_n)
\]

which belongs to \( N_\varphi \) by the previous lemma. The result follows by the definition of \( N_0 \).

The following results can be proven in a similar way as for elements in \( M_\varphi \) and \( N_\varphi \).
**Lemma 8.27** We have the following properties.

- We have for every $a \in \mathcal{N}_{\varphi^s}$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$ that $(\delta^z e_n) \in \mathcal{N}_{\varphi} \cap \mathcal{N}_{\varphi^s}$.
- We have for every $x \in \mathcal{M}_{\varphi^s}$, $m, n \in \mathbb{N}$ and $y, z \in \mathbb{C}$ that $(\delta^y e_m) x (\delta^z e_n) \in \mathcal{M}_\varphi \cap \mathcal{M}_{\varphi^s}$.

**Lemma 8.28** Consider $x \in \mathcal{M}_{\varphi^s}$. Then $(\varphi_{\delta}(e_n x e_n))_{n=1}^{\infty}$ converges to $\varphi_{\delta}(x)$.

We have also the next result.

**Corollary 8.29** The set $\mathcal{N}_{\varphi} \cap N_0$ is a core for $\Gamma$ and a core for $\Lambda$.

**Proof:** Choose $x \in \mathcal{N}_{\varphi^s}$. We have by the previous lemmas that $e_n x$ belongs to $\mathcal{N}_{\varphi} \cap N_0$ for every $n \in \mathbb{N}$. By proposition 6.10, we know that $\Gamma(x e_n) = J'(\sigma'_x(e_n))^* J' \Gamma(x)$ for every $n \in \mathbb{N}$ (here $J'$ denotes the modular conjugation of $\varphi_{\delta}$). This implies that $(\Gamma(x e_n))_{n=1}^{\infty}$ converges to $\Gamma(x)$. It is also clear that $(x e_n)_{n=1}^{\infty}$ converges to $x$.

The statement about $\Lambda$ is proven in a similar way.

**Lemma 8.30** Consider $x \in \mathcal{M}_{\varphi^s}$, $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Then $e_n x e_n$ belongs to $\mathcal{M}_{\varphi^s}$, $(\delta^{1-z} e_n) x (\delta^z e_n)$ belongs to $\mathcal{M}_\varphi$ and

$$
\varphi_{\delta}(e_n x e_n) = \lambda t^{(\frac{1}{2}-z)}(\delta^{1-z} e_n) x (\delta^z e_n).
$$

**Proof:** Choose $a, b \in \mathcal{N}_{\varphi^s}$. By lemma 8.26, we know that $a e_n, b e_n$ belong to $N_0$. This implies that

$$
\varphi_{\delta}(e_n b^* a e_n) = \langle \Gamma(a e_n), \Gamma(b e_n) \rangle = \langle \Lambda((a e_n) \delta^z), \Lambda((b e_n) \delta^z) \rangle = \langle \Lambda(a (\delta^z e_n)), \Lambda(b (\delta^z e_n)) \rangle = \varphi( (\delta^z e_n) b^* a (\delta^z e_n) ) .
$$

We know that $\delta^{1-z} e_n$ is a right multiplier of $\delta^{z-\frac{1}{2}}$ and that $\delta^{z-\frac{1}{2}} (\delta^{1-z} e_n) = \delta^z e_n$. This implies that $(\delta^{1-z} e_n) b^* a (\delta^z e_n)$ is a right multiplier of $\delta^{z-\frac{1}{2}}$ and

$$
\delta^{z-\frac{1}{2}} [(\delta^{1-z} e_n) b^* a (\delta^z e_n)] = (\delta^z e_n) b^* a (\delta^z e_n) .
$$

So $\delta^{z-\frac{1}{2}} [(\delta^{1-z} e_n) b^* a (\delta^z e_n)]$ belongs to $\mathcal{M}_\varphi$ by lemma 8.27.

We know on the other hand that $\delta^z e_n$ is a left multiplier of $\delta^{z-\frac{1}{2}}$ and that $(\delta^z e_n) \delta^{z-\frac{1}{2}} = \delta^z e_n$. This implies that $(\delta^{1-z} e_n) b^* a (\delta^z e_n)$ is a left multiplier of $\delta^{z-\frac{1}{2}}$ and

$$
[(\delta^{1-z} e_n) b^* a (\delta^z e_n)] \delta^{z-\frac{1}{2}} = (\delta^{1-z} e_n) b^* a (\delta^z e_n) .
$$

So $[(\delta^{1-z} e_n) b^* a (\delta^z e_n)] \delta^{z-\frac{1}{2}}$ belongs also to $\mathcal{M}_\varphi$ by 8.27.

Therefore proposition 6.32 implies that

$$
\varphi_{\delta}(e_n b^* a e_n) = \varphi( (\delta^z e_n) b^* a (\delta^z e_n) ) = \varphi( \delta^{z-\frac{1}{2}} [(\delta^{1-z} e_n) b^* a (\delta^z e_n)] ) = \lambda t^{(\frac{1}{2}-z)}( (\delta^{1-z} e_n) b^* a (\delta^z e_n) ) .
$$

\[\square\]
**Proposition 8.31** Consider $x \in \mathcal{M}_{\varphi z}$ and $z \in \mathcal{C}$ such that $x$ is a middle multiplier of $\delta^{1-z}$, $\delta^z$ and $\delta^{1-z}x\delta^z$ belongs to $\mathcal{M}_\varphi$. Then

$$\varphi_\delta(x) = \lambda^{i(\frac{1}{2}x)} \varphi(\delta^{1-z}x\delta^z).$$

**Proof:** Choose $n \in \mathbb{N}$. By the previous lemma, we know that $e_n x e_n$ belongs to $\mathcal{M}_{\varphi z}$, $(\delta^{1-z}e_n)x(\delta^ze_n)$ belongs to $\mathcal{M}_\varphi$ and

$$\varphi_\delta(e_n x e_n) = \lambda^{i(\frac{1}{2}z)} \varphi((\delta^{1-z}e_n)x(\delta^ze_n)).$$

Because $e_n$ is a left multiplier of $\delta^{1-z}$ and $e_n$ is a right multiplier of $\delta^z$, we get that

$$e_n (\delta^{1-z}x\delta^z)e_n = (e_n \delta^{1-z})x(\delta^ze_n) = (\delta^{1-z}e_n)x(\delta^ze_n).$$

So $e_n (\delta^{1-z}x\delta^z)e_n$ belongs to $\mathcal{M}_\varphi$ and

$$\varphi_\delta(e_n x e_n) = \lambda^{i(\frac{1}{2}z)} \varphi(e_n (\delta^{1-z}x\delta^z)e_n).$$

The proposition follows by lemmas 8.24 and 8.28. \[\blacksquare\]

**Remark 8.32** We would like to mention the following special cases.

1. Consider $x \in \mathcal{M}_{\varphi z}$ such that $x$ is a middle multiplier of $\delta^{\frac{1}{2}}$, $\delta^{\frac{1}{2}}$ and $\delta^{\frac{1}{2}}x\delta^{\frac{1}{2}}$ belongs to $\mathcal{M}_\varphi$. Then

$$\varphi_\delta(x) = \varphi(\delta^{\frac{1}{2}}x\delta^{\frac{1}{2}}).$$

2. Consider $x \in \mathcal{M}_{\varphi z}$ such that $x$ is a left multiplier of $\delta$ and $x\delta$ belongs to $\mathcal{M}_\varphi$. Then

$$\varphi_\delta(x) = \lambda^{-\frac{1}{2}} \varphi(x\delta).$$

3. Consider $x \in \mathcal{M}_{\varphi z}$ such that $x$ is a right multiplier of $\delta$ and $\delta x$ belongs to $\mathcal{M}_\varphi$. Then

$$\varphi_\delta(x) = \lambda^{\frac{1}{2}} \varphi(\delta x).$$

**Proposition 8.33** Consider an element $a \in A^+$ and $z \in \mathcal{C}$ such that $a$ is a middle multiplier of $\delta^{1-z}$, $\delta^z$ and $\delta^{1-z}a\delta^z$ belongs to $\mathcal{M}_\varphi$. Then $a$ belongs to $\mathcal{M}_{\varphi z}^+$. 

**Proof:** Take $b \in A$ such that $b^*b = a$.

Choose $m \in \mathbb{N}$. Because $\delta^{z-1}e_m$ is a left multiplier of $\delta^{1-z}$ and $\delta^{-z}e_m$ is a right multiplier of $\delta^z$, we have that

$$(\delta^{z-1}e_m)(\delta^{1-z}a\delta^z)(\delta^{-z}e_m) = ((\delta^{z-1}e_m)\delta^{1-z})a(\delta^z(\delta^{-z}e_m)) = e_m a e_m = (be_m)^*(be_m),$$

Because $\delta^{1-z}a\delta^z$ belongs to $\mathcal{M}_\varphi$, this equality and lemma 8.23 imply that $(be_m)^*(be_m)$ belongs to $\mathcal{M}_{\varphi z}$. Hence $be_m$ belongs to $\mathcal{N}_{\varphi z}$.

Take $k, l \in \mathbb{N}$. By the previous part, we already know that $e_l b^*b e_k$ belongs to $\mathcal{M}_{\varphi z}$. Because $e_l$ is a right multiplier of $\delta^{1-z}$ and $e_k$ is a left multiplier of $\delta^z$, we have that $e_l b^*be_k$ is a middle multiplier of $\delta^{1-z}$, $\delta^z$ and

$$\delta^{1-z}(e_l b^*b e_k)\delta^z = (\delta^{1-z}e_l)b^*(e_k\delta^z) = (\delta^{1-z}e_l)a(e_k\delta^z)$$

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Knowing that $e_1$ is a left multiplier of $\delta^{1-z}$ and $e_n$ is a right multiplier of $\delta^z$, we get moreover that

$e_1(\delta^{1-z}a\delta^z) e_k = (e_1 \delta^{1-z}) a (\delta^z e_k) = (\delta^{1-z} e_1) a (e_k \delta^z) = \delta^{1-z} (e_1 b^* b e_k) \delta^z$.

Hence $\delta^{1-z} (e_1 b^* b e_k) \delta^z$ belongs to $M_{\varphi}$ by lemma \[8.23\]. Therefore the previous proposition implies that

$$\varphi_\delta (e_1 b^* b e_k) = \lambda^{1/2} \varphi (\delta^{1-z} (e_1 b^* b e_k) \delta^z) = \lambda^{1/2} \varphi (e_1 (\delta^{1-z} a \delta^z) e_k) .$$

Therefore, we have for all $m, n \in \mathbb{N}$ that

$$||\Gamma(b e_m) - \Gamma(b e_n)||^2 = \varphi_\delta (e_m b^* b e_m) - \varphi_\delta (e_m b^* b e_n) - \varphi_\delta (e_n b^* b e_m) + \varphi_\delta (e_n b^* b e_n)$$

$$= \lambda^{1/2} \left[ \varphi(e_m (\delta^{1-z}a\delta^z) e_m) - \varphi(e_m (\delta^{1-z}a\delta^z) e_n) - \varphi(e_n (\delta^{1-z}a\delta^z) e_m) + \varphi(e_n (\delta^{1-z}a\delta^z) e_n) \right]$$

Using this equality and the obvious generalization of lemma \[8.24\], we see that $(\Gamma(b e_n))_{n=1}^\infty$ is Cauchy and hence convergent. Because we also have that $(b e_n)_{n=1}^\infty$ converges to $b$, the closedness of $\Gamma$ imply that $b$ belongs to $\mathcal{N}_{\varphi,b}$. Hence, $a$ is an element in $M_{\varphi,b}$.

The next proposition is proven in a similar way.

**Proposition 8.34** Consider an element $a \in M_{\varphi,b}$ and $z \in \mathfrak{C}$ such that $a$ is a middle multiplier of $\delta^{1-z}$, $\delta^z$ and $\delta^{1-z} a \delta^z$ belongs to $A^+$. Then $\delta^{1-z} a \delta^z$ belongs to $M_{\varphi}^\perp$.

**Corollary 8.35** Consider an element $a \in A^+$ such that $a$ is a middle multiplier of $\delta^{1/2}$, $\delta^{1/2}$ and $\delta^{1/2} a \delta^{1/2}$ belongs to $A$. Then $a$ belongs to $M_{\varphi,b}^\perp \iff \delta^{1/2} a \delta^{1/2}$ belongs to $M_{\varphi}^\perp$.

In the next part of this section, we will prove that the modular conjugation of $\varphi_\delta$ with respect to $(H, \Gamma, \pi)$ is a scalar multiple of $J$.

**Notation 8.36** We define the set $D = \{ x \in \mathcal{N}_{\varphi} \cap \mathcal{N}_{\varphi}^* |$ We have for every $z \in \mathfrak{C}$ that

1. $x$ is a right multiplier of $\delta^z$ and $\delta^z x$ belongs to $\mathcal{N}_{\varphi} \cap \mathcal{N}_{\varphi}^*$
2. $x$ is a left multiplier of $\delta^z$ and $x \delta^z$ belongs to $\mathcal{N}_{\varphi} \cap \mathcal{N}_{\varphi}^*$

} .

It is not difficult to check that $D$ is a sub-$*$-algebra of $N_0 \cap N_0^* \subseteq \mathcal{N}_{\varphi,b} \cap \mathcal{N}_{\varphi,b}^*$.

**Lemma 8.37** Consider $x \in \mathcal{N}_{\varphi,b} \cap \mathcal{N}_{\varphi,b}^*$. Then there exists a sequence $(x_n)_{n=1}^\infty$ in $D$ such that

1. $(x_n)_{n=1}^\infty$ converges to $x$
2. $(\Gamma(x_n))_{n=1}^\infty$ converges to $\Gamma(x)$
3. $(\Gamma(x_n^*))_{n=1}^\infty$ converges to $\Gamma(x^*)$

**Proof**: We define for every $n \in \mathbb{N}$ the element $x_n = e_n x e_n$. Using lemma \[8.21\], it is not difficult to check that $x_n$ belongs to $D$ for every $n \in \mathbb{N}$. We have also for every $n \in \mathbb{N}$ that

- $\Gamma(x_n) = \Gamma(e_n x e_n) = J \pi(\sigma'_b(e_n)) J \pi(e_n) \Gamma(x)$
\[ \Gamma(x_n^*) = \Gamma(e_n x^* e_n) = J \pi(\sigma^1_n(e_n))^* J \pi(e_n) \Gamma(x^*) \]

This implies that \((x_n)_{n=1}^\infty\) converges to \(x\), \((\Gamma(x_n))_{n=1}^\infty\) converges to \(\Gamma(x)\) and \((\Gamma(x_n^*))_{n=1}^\infty\) converges to \(\Gamma(x^*)\)

It follows immediately that \(\Gamma(D)\) is dense in \(H\).

**Proposition 8.38** We have that \(\lambda^\dagger J\) is the modular operator for \(\phi_\delta\) in the GNS-construction \((H, \Gamma, \pi)\).

**Proof:** Denote the modular conjugation of \(\phi_\delta\) in the GNS-construction \((H, \Gamma, \pi)\) by \(J'\).

Choose \(a \in D\).

- We have that \(a^*\) is a left multiplier of \(\delta^{\dagger}\) and that \(a^* \delta^{\dagger}\) belongs to \(N_\varphi\). This implies that \(\Gamma(a^*) = \Lambda(a^* \delta^{\dagger})\) by the definition of \(\Gamma\). We know moreover that \(a \in D(\delta^{\dagger})\) and that \(\delta^{\dagger} a\) belongs to \(N_\varphi \cap N_\varphi^*\). This implies that \(\Lambda(\delta^{\dagger} a)\) belongs to \(D(\nabla^{\dagger})\) and

\[ J \nabla^{\dagger} \Lambda(\delta^{\dagger} a) = \Lambda((\delta^{\dagger} a)^*) = \Lambda(a^* \delta^{\dagger}) = \Gamma(a^*) . \]

- Because \(a\) belongs to \(N_\varphi\), \(a^*\) belongs to \(D(\delta^{\dagger})\) and \(\delta^{\dagger} a\) belongs to \(N_\varphi\), proposition 8.18 implies that \(\Lambda(a)\) belongs to \(D(\pi(\delta) \dagger)\) and \(\pi(\delta) \dagger \Lambda(a) = \Lambda(\delta^{\dagger} a)\). So, using the previous part, we get that \(\Lambda(a)\) belongs to \(D(\nabla^{\dagger} \pi(\delta) \dagger)\) and

\[ J \nabla^{\dagger} \pi(\delta) \dagger \Lambda(a) = J \nabla^{\dagger} \Lambda(\delta^{\dagger} a) = \Gamma(a^*) . \]

- We know that \(a\) is a left multiplier of \(\delta^{\dagger}\). This implies that \(a\delta^{\dagger}\) is a left multiplier of \(\delta^{-\dagger}\) and that \((a\delta^{\dagger}) \delta^{-\dagger} = a\) which belongs to \(N_\varphi\). Therefore, proposition 8.31 implies that \(\Lambda(a\delta^{\dagger})\) belongs to \(D((J\pi(\delta)J)^{-\dagger})\) and

\[ (J\pi(\delta)J)^{-\dagger} \Lambda(a\delta^{\dagger}) = \lambda^{\dagger} \Lambda((a\delta^{\dagger}) \delta^{-\dagger}) = \lambda^{\dagger} \Lambda(a) . \]

By definition, we have that \(\Gamma(a) = \Lambda(a\delta^{\dagger})\). Consequently, we have that \(\Gamma(a)\) belongs to \(D((J\pi(\delta)J)^{-\dagger})\) and

\[ (J\pi(\delta)J)^{-\dagger} \Gamma(a) = \lambda^{\dagger} \Lambda(a) . \]

Using the previous part, we see that \(\Gamma(a)\) belongs to \(D(\nabla^{\dagger} \pi(\delta) \dagger (J\pi(\delta)J)^{-\dagger})\) and

\[ J \nabla^{\dagger} \pi(\delta) \dagger (J\pi(\delta)J)^{-\dagger} \Gamma(a) = \lambda^{-\dagger} \lambda^{\dagger} \Lambda(a) = \lambda^{-\dagger} \Gamma(a^*) . \]

The only thing we need from the previous discussion is that \(\Gamma(a)\) belongs to \(D(\nabla^{\dagger} \pi(\delta) \dagger (J\pi(\delta)J)^{-\dagger})\) and

\[ J \nabla^{\dagger} \pi(\delta) \dagger (J\pi(\delta)J)^{-\dagger} \Gamma(a) = \lambda^{-\dagger} \Gamma(a^*) . \]  

We have that \(\pi(\delta)\) and \(J\pi(\delta)J\) are strongly commuting (lemma 8.8). Define the positive injective operator \(S\) in \(H\) such that \(S^{it} = \pi(\delta)^{it}(J\pi(\delta)J)^{-it}\) for every \(t \in \mathbb{R}\). Therefore, \(\pi(\delta)^{it}(J\pi(\delta)J)^{-it} \subseteq S^{\frac{i}{2}}\).

We also have that \(\nabla\) and \(S\) are strongly commuting (lemma 8.10). By definition, we have that \(\Psi^{it} = \nabla^{it} S^{it}\) for \(t \in \mathbb{R}\). This implies that \(\nabla^{\dagger} S^{\frac{i}{2}} \subseteq \Psi^{\frac{i}{2}}\).

Combining these two facts, we arrive at the conclusion that \(\nabla^{\dagger} \pi(\delta) \dagger (J\pi(\delta)J)^{-\dagger} \subseteq \Psi^{\frac{i}{2}}\).

Using \((*)\), this gives that \(\Gamma(a)\) belongs to \(D(\Psi^{\frac{i}{2}})\) and \(J \Psi^{\frac{i}{2}} \Gamma(a) = \lambda^{-\dagger} \Gamma(a^*)\).

On the other hand, the fact that \(a \in N_{\varphi}\cap N_{\varphi^*}\) implies that \(J \Psi^{\frac{i}{2}} \Gamma(a) = \Gamma(a^*)\).

Combining these two equalities results in the equality \(\lambda^{\dagger} J \Gamma(a^*) = J \Gamma(a^*)\).

Because \(\Gamma(D)\) is dense in \(H\), we get that \(J' = \lambda^{\dagger} J\).

The following two lemmas will be used to prove that \(\delta\) is uniquely determined in the GNS-representation of \(\varphi\) by the weight \(\varphi_\delta\).
Lemma 8.39 Consider an element \( a \in \mathcal{N}_\varphi \cap \mathcal{N}_{\varphi_\delta} \). Then \( \Lambda(a) \) belongs to \( D(J\pi(\delta^\updownarrow)J) \) and
\[
\lambda^\updownarrow J\pi(\delta^\updownarrow)J(\Lambda(a)) = \Gamma(a)
\]
Consequently, \( \|J\pi(\delta^\updownarrow)J\Lambda(a)\|^2 = \varphi_\delta(a^*a) \).

Proof: Choose \( m \in \mathbb{N} \).
Because \( a \in \mathcal{N}_\varphi \), we know by lemma 8.23 that \( ae_m \) belongs to \( \mathcal{N}_\varphi \).
We also know that \( e_m \) is a left multiplier of \( \delta^\updownarrow \), which implies that \( (ae_m)\delta^\updownarrow \) is a left multiplier of \( \delta^\updownarrow \) and \( (ae_m)\delta^\updownarrow = (e_m\delta^\updownarrow) \).
This last equality ensures that also \( (ae_m)\delta^\updownarrow \) belongs to \( \mathcal{N}_\varphi \).
Hence, proposition 6.31 implies that \( \Lambda(\Lambda(\varphi_\delta)) \) and \( \Lambda(\Lambda(\varphi_\delta)) \) are bounded sequences which converge strongly to \( a \) and \( \Lambda(\Lambda(\varphi_\delta)) \) belongs to \( D(\varphi_\delta) \).

So by the definition of \( \Gamma \), we see that \( ae_m \) belongs to \( \mathcal{N}_\varphi \) and
\[
\lambda^\updownarrow J\pi(\delta^\updownarrow)J(\Lambda(a)) = \Lambda(\Lambda(\varphi_\delta))
\]
Because \( a \in \mathcal{N}_{\varphi_\delta} \) and \( e_m \in D(\sigma^\varphi_\delta) \), this implies that
\[
\lambda^\updownarrow J\pi(\delta^\updownarrow)J(\Lambda(a)) = J\pi(\sigma^\varphi_\delta(e_m)) \ast J(\Gamma(a))
\]
Because \( a \in \mathcal{N}_\varphi \) and \( e_m \in D(\sigma^\varphi_\delta) \), we have also that
\[
\Lambda(\Lambda(\varphi_\delta)) = J\pi(\sigma^\varphi_\delta(e_m)) \ast J(\Gamma(a))
\]
So we see that \( (\Lambda(\Lambda(\varphi_\delta)))_n \) converges to \( \Lambda(a) \) and that \( \{ \lambda^\updownarrow J\pi(\delta^\updownarrow)J(\Lambda(a)) \}_n \) converges to \( \Lambda(a) \).
Therefore the closedness of \( J\pi(\delta^\updownarrow)J \) implies that \( \Lambda(a) \) belongs to \( D(J\pi(\delta^\updownarrow)J) \) and
\[
\lambda^\updownarrow J\pi(\delta^\updownarrow)J(\Lambda(a)) = \Gamma(a)
\]

Lemma 8.40 The set \( \Lambda(\mathcal{N}_\varphi \cap \mathcal{N}_{\varphi_\delta}) \) is a core for \( J\pi(\delta^\updownarrow)J \).

Proof: Choose \( m \in \mathbb{N} \).
We know already that \( \sigma^\varphi_\delta(e_m) \ast \) is a left and right multiplier of \( \delta^\updownarrow \) and that \( \sigma^\varphi_\delta(e_m) \ast \delta^\updownarrow = \delta^\updownarrow \sigma^\varphi_\delta(e_m) \ast \).
This implies easily that \( \pi(\sigma^\varphi_\delta(e_m)) \ast \pi(\delta^\updownarrow) \subseteq \pi(\delta^\updownarrow) \pi(\sigma^\varphi_\delta(e_m)) \ast \in B(H) \).
So we get that
\[
(J\pi(\sigma^\varphi_\delta(e_m)) \ast J)(J\pi(\delta^\updownarrow)J) \subseteq (J\pi(\delta^\updownarrow)J)(J\pi(\sigma^\varphi_\delta(e_m)) \ast J) \in B(H)
\]
Because \( (J\pi(\sigma^\varphi_\delta(e_n)) \ast J)_{n=1}^\infty \) is a bounded sequence which converges strongly to \( \Lambda(A) \) is dense in \( H \), it is not too difficult to infer from this last equality that the set
\[
(J\pi(\sigma^\varphi_\delta(e_n)) \ast J(\Lambda(a)) \mid n \in \mathbb{N}, a \in \mathcal{N}_\varphi)
\]
is a core for \( J\pi(\delta^\updownarrow)J \).

By lemma 8.23 and proposition 6.10, we have for every \( a \in \mathcal{N}_\varphi \) and \( n \in \mathbb{N} \) that \( a e_n \) belongs to \( \mathcal{N}_\varphi \cap \mathcal{N}_{\varphi_\delta} \) and \( \Lambda(a e_n) = J\pi(\sigma^\varphi_\delta(e_n)) \ast J(\Lambda(a)) \).
So we see that \( \Lambda(\mathcal{N}_\varphi \cap \mathcal{N}_{\varphi_\delta}) \) is a core for \( J\pi(\delta^\updownarrow)J \).

In a last proposition, we will prove that the strictly positive element \( \delta \) is uniquely determined in the GNS-representation of \( \varphi \) by the weight \( \varphi_\delta \).

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Proposition 8.41 Consider a C*-algebra $A$ and a KMS-weight $\varphi$ on $A$ with modular group $\sigma$ and GNS-construction $(H, \Lambda, \pi)$. Let $\delta_1$, $\delta_2$ be strictly positive elements affiliated with $A$ such that there exists strictly positive numbers $\lambda_1$, $\lambda_2$ satisfying $\sigma_1(\delta_1) = \lambda_1^2 \delta_1$ and $\sigma_t(\delta_2) = \lambda_2^2 \delta_2$ for $t \in \mathbb{R}$. If $\varphi_{\delta_1} = \varphi_{\delta_2}$, then $\pi(\delta_1) = \pi(\delta_2)$.

Proof: Because $\varphi_{\delta_1} = \varphi_{\delta_2}$, we have that $\mathcal{N}_{\varphi_{\delta_1}} = \mathcal{N}_{\varphi_{\delta_2}}$, so $\Lambda(\mathcal{N}_{\varphi_{\delta_1}} \cap \mathcal{N}_{\varphi_{\delta_2}}) = \Lambda(\mathcal{N}_{\varphi_{\delta_1}} \cap \mathcal{N}_{\varphi_{\delta_2}})$.

By lemma 8.40, this gives a common core $\Lambda(\mathcal{N}_{\varphi_{\delta_1}})$ for $J\pi(\delta_1^1)J$ and $J\pi(\delta_2^1)J$.

Using lemma 8.39, we have moreover for every $a \in \mathcal{N}_{\varphi_{\delta_1}} \cap \mathcal{N}_{\varphi_{\delta_2}}$ that

$$\|J\pi(\delta_1^1)J\Lambda(a)\|^2 = \varphi_{\delta_1}(a^*a) = \varphi_{\delta_2}(a^*a) = \|J\pi(\delta_2^1)J\Lambda(a)\|^2.$$ 

This implies easily that $D(J\pi(\delta_1^1)J) = D(J\pi(\delta_2^1)J)$ and $\|J\pi(\delta_1^1)Jv\| = \|J\pi(\delta_2^1)Jv\|$ for every $v \in D(J\pi(\delta_2^1)J)$.

As a consequence, we get that $J\pi(\delta_1)J = J\pi(\delta_2)J$ which gives us that $\pi(\delta_1) = \pi(\delta_2)$.

Corollary 8.42 Consider a C*-algebra $A$ and a faithful KMS-weight $\varphi$ on $A$ with modular group $\sigma$. Let $\delta_1$, $\delta_2$ be strictly positive elements affiliated with $A$ such that there exists strictly positive numbers $\lambda_1$, $\lambda_2$ satisfying $\sigma_1(\delta_1) = \lambda_1^2 \delta_1$ and $\sigma_t(\delta_2) = \lambda_2^2 \delta_2$ for $t \in \mathbb{R}$. If $\varphi_{\delta_1} = \varphi_{\delta_2}$, then $\delta_1 = \delta_2$.

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