The dual of Janus (\((<: \leftrightarrow (:>)\)) an interface CFT

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Abstract

We propose and study a specific gauge theory dual of the smooth, non-supersymmetric (and apparently stable) Janus solution of Type IIB supergravity found in hep-th/0304129. The dual field theory is \(\mathcal{N} = 4\) SYM theory on two half-spaces separated by a planar interface with different coupling constants in each half-space. We assume that the position dependent coupling multiplies the operator \(L'\) which is the fourth descendent of the primary \(\text{Tr} X^{\{i}X^{j\}}\) and closely related to the \(\mathcal{N} = 4\) Lagrangian density. At the classical level supersymmetry is broken explicitly, but \(SO(3,2)\) conformal symmetry is preserved. We use conformal perturbation theory to study various correlation functions to first and second order in the discontinuity of \(g^2_{YM}\), confirming quantum level conformal symmetry. Certain quantities such as the vacuum expectation value \(\langle L' \rangle\) are protected to all orders in \(g^2_{YM}N\), and we find perfect agreement between the weak coupling value in the gauge theory and the strong coupling gravity result. \(SO(3,2)\) symmetry requires vanishing vacuum energy, \(\langle T_{\mu\nu} \rangle = 0\), and this is confirmed in first order in the discontinuity.
1 Introduction

In this paper we describe an application of the $AdS/CFT$ correspondence to the regular, non-supersymmetric Janus solution of Type IIB supergravity found by Bak, Gutperle, and Hirano [1]. The solution is the global product of a 5-dimensional domain wall which approaches $AdS_5$ at the boundary times the internal space $S^5$. The novel feature is that fields are constant along $AdS_4$ slices in the 5-dimensional spacetime, so the isometry group is $SO(3, 2) \otimes SO(6)$.

One might suspect that a non-supersymmetric solution is unstable, and the stability question was recently examined in [2]. A generalization of the Witten-Nester positive energy argument was developed for $AdS_d$-sliced domain walls. It was shown that the Janus solution is stable non-perturbatively within the 5-dimensional theory of the dilaton coupled to gravity. The proof of stability was extended to include additional fields which appear in several consistent truncations of Type IIB supergravity to 5 dimensions. The complete 5-dimensional reduction of Type IIB supergravity is not known, so the stability of the Janus solution is not yet fully established. Nevertheless, the present evidence strongly indicates that the solution is globally stable.

The $AdS/CFT$ correspondence requires that there is a gauge theory dual for every non-singular solution of Type IIB supergravity, so it becomes compelling to investigate the dual field theory for the simple, yet novel, Janus solution. Indeed the features of this dual were sketched in [1], as we now discuss.

It is well known that the conformal compactification of $AdS_5$ is contained in the Einstein static universe $R \otimes S^4$. The compactification includes exactly half the $S^4$ so the boundary is $R \otimes S^3$. The conformal compactification of the Janus solution is also contained in $ESU_5$ as the real line times a half-space of $S^4$, namely a wedge shaped region with angular excess. The boundary is thus two hemispheres of $S^3$ joined at a “corner.” See Fig. 1. The dilaton field is smooth in the interior, constant in each hemisphere of the $S^3$ boundary, and changes discontinuously at the corner. This suggests that the dual gauge theory is $\mathcal{N} = 4$ super-Yang-Mills theory on $R \otimes S^3$ with a different value of the gauge coupling constant in each hemisphere. The dual theory would be expected to be conformal, since $SO(3, 2)$ is known to be the conformal group of a defect conformal theory (dCFT). The non-leading term in the near boundary asymptotics of the dilaton field suggests a vacuum expectation value for the operator dual to the dilaton which is identified in [1] (and in many other papers) as $\langle \text{Tr } F^2_{\mu\nu} \rangle$.

The global description of the Janus geometry given above is most transparent if global coordinates are used for the $AdS_4$ sliced domain wall. If one uses a Poincaré patch presentation of the $AdS_4$, one finds that the resulting 5-dimensional coordinate chart includes a boundary region in which two Minkowski $3+1$ half-spaces are joined at a planar Minkowski $2+1$ interface. This is similar to the defect CFT proposed in [3], [4] and explored in [5]. However, there are some important differences which we discuss below.

We propose and study a detailed version of the gauge theory dual of the Janus solution.
The proposal is based on the assumption that a spatially varying bulk dilaton is dual to a variant of the Lagrangian density of $\mathcal{N} = 4$ SYM theory in which the scalar kinetic term is $X^I \Box X^I/2$ rather than the more conventional $-(\partial X^I)^2/2$. This operator then multiplies the spatially varying gauge coupling given by the boundary value of the dilaton. The discontinuous gauge coupling breaks conformal symmetry from the $SO(4,2)$ bosonic subalgebra of $\mathcal{N} = 4$ SYM theory to the restricted conformal group $SO(3,2)$. It is an important test of our proposed gauge theory dual that this symmetry is preserved at the quantum level. At this point we have no general proof, but we do perform several checks using conformal perturbation theory\(^1\).

We parameterize the gauge coupling on the two sides of the interface as $g_{\pm}^{-2} = g_{YM}^{-2}(1 \mp \gamma)$. The parameter $\gamma$ governs the magnitude of the discontinuity, and it is related to the rate of change of the bulk dilaton field $\phi(\mu)$ with respect to the radial coordinate $\mu$. Our theory can then be viewed as superconformal $\mathcal{N} = 4$ SYM theory perturbed by the operator $\gamma \varepsilon(z_3) \mathcal{L}'(z)$, where $\mathcal{L}'(z)$ is the variant of the $\mathcal{N} = 4$ SYM Lagrangian described above, $\varepsilon(z_3)$ is the step function with values $\pm 1$, and $z_3$ is the Cartesian coordinate perpendicular to the interface. Since $\mathcal{L}'$ is the fourth SUSY descendent of the primary $\text{Tr} \, X^{(I}X^{J)}$, it is a

\(^1\)Many thanks to Joe Polchinski who suggested this approach.
marginal operator. The perturbation breaks supersymmetry because of the step function.

In conformal perturbation theory effects of the interface on correlation functions can be computed order by order in $\gamma$ but to all orders in $g_{YM}^2 N$. For example, the vacuum expectation value $\langle L' \rangle$ vanishes in the unperturbed theory. Conformal perturbation theory implies that $\langle L' \rangle$ can be expressed as the series

$$\langle L'(z) \rangle_{\text{Janus}} = -\gamma \int d^4 x \varepsilon(x_3) \langle L'(z) L'(x) \rangle + \frac{\gamma^2}{2} \int d^4 x d^4 y \varepsilon(x_3) \varepsilon(y_3) \langle L'(z) L'(x) L'(y) \rangle + O(\gamma^3)$$

(1.1)

The 2- and 3-point correlation functions which appear in the integrals are protected correlators of the unperturbed $N = 4$ theory. They are independent of $g_{YM}^2$ to all finite orders [6], [7], [8], [9], [10] and have no instanton corrections [11]. Therefore, through order $\gamma^2$, the VEV $\langle L'(x) \rangle$ should be protected and have the same value whether calculated at weak coupling in the gauge theory or from the bulk solution at strong coupling. It turns out that the three-point function in (1.1) vanishes due to supersymmetry in the unperturbed SYM theory, and the order $\gamma$ term involving the two-point function can be easily evaluated and agrees perfectly with the value obtained from the bulk Janus solution!

More generally, the order $\gamma$ correction to any correlation function $\langle O_1(x_1) \ldots O_n(x_n) \rangle$ is given by

$$\langle O_1(x_1) \ldots O_n(x_n) \rangle_{\gamma} = -\gamma \int d^4 z \varepsilon(z_3) \langle O_1(x_1) \ldots O_n(x_n) L'(z) \rangle_{N = 4}.$$  

(1.2)

We calculate several examples of 2-point functions of gauge-invariant operators and show that their order $\gamma$ contributions are compatible with defect conformal symmetry [12], [13].

The present theory may be compared with the earlier dCFT of [4], [5] in which a probe $D5$-brane wrapped on an $S^2 \subset S^5$ has an equilibrium position on an $AdS_4 \subset AdS_5$. In the dual gauge theory the fields of $\mathcal{N} = 4$ SYM interact with $D5$-brane modes confined to the defect. The $R$-symmetry is $SO(4)$ and the preserved superalgebra is $OSp(4,4)$ which contains $SO(3,2) \otimes SO(4)$ as the maximal Lie subalgebra. The Janus solution maintains $SO(6)$ $R$-symmetry, but breaks supersymmetry completely. There is no brane interpretation of the defect, so we conclude that the dual gauge theory contains no degrees of freedom on the defect. Instead there is a discontinuous gauge coupling which simulates a dielectric interface. For these reasons we prefer the name interface conformal field theory (ICFT). A different class of dilatonic deformations of $AdS_5 \otimes S^5$ was studied recently in [14]. Their solutions break $SO(3,2)$ and develop thus nontrivial VEV for $\langle T_{\mu\nu} \rangle$. It might be interesting to study the gauge theory dual for their deformations using conformal perturbation theory.

In Sec. 2 below we review the Janus solution of Type IIB supergravity giving new details on the boundary behavior of the dilaton and metric. In Sec. 3 we discuss the computation of the vacuum expectation values $\langle L' \rangle$ and $\langle T_{\mu\nu} \rangle$ from the gravity theory. In Sec 4. we present our first proposal for the gauge theory dual via conformal perturbation theory. In Sec. 5 we
present calculations of various one- and two-point functions, and compare with gravity. In Sec. 6, we outline another approach to the Janus dual in which we specify a Lagrangian in which the fields obey specific jump conditions at the interface. In an Appendix A we consider general $\mathcal{N} = 1$ SUSY theories with a position dependent coupling. We show that two of the four $\mathcal{N} = 1$ supercharges are conserved if certain localized interface “counterterms” are added to the Lagrangian. In the $\mathcal{N} = 1$ form of the $\mathcal{N} = 4$ SYM, the counterterms break $SO(6)$ and are therefore not permitted in the Janus dual, which therefore breaks all supersymmetry explicitly.

Note added: We thank K. Skenderis for informing us that his paper with I. Papadimitriou entitled “Correlation function in holographic RG flows” will soon be submitted to hep-th. This paper includes a discussion of holography for the Janus solution.

2 The Janus solution.

The Janus solution involves the metric, dilaton, and 5-form of Type IIB supergravity \(^2\) with other fields vanishing:

\begin{align*}
\text{ds}_{10}^2 &= L^2 e^{2A(\mu)} (g_{ij} dx^i dx^j + d\mu^2) + L^2 ds_{S^5}^2 \quad (2.1) \\
\phi &= \phi(\mu) \quad (2.2) \\
F_5 &= 4L^4 \left[ e^{5A(\mu)} d\mu \wedge \omega_{AdS_4} + \omega_{S^5} \right] \quad (2.3) \\
&= dC_4 + *dC_4 \quad (2.4) \\
C_4 &= h(\mu) \omega_{AdS_4} \quad (2.5) \\
h'(\mu) &= 4L^4 e^{5A(\mu)} \quad (2.6)
\end{align*}

where $g_{ij}$ is an $AdS_4$ metric of unit scale, and $ds_{S^5}^2$ is a metric on the unit $S^5$. The volume forms are also normalized to unit scale. We present the $AdS_4$ metric in both global and patch coordinates, viz.

\begin{align*}
\text{ds}_{4}^2 &= \frac{1}{\cos^2 \lambda} \left[ -d\tau^2 + d\lambda^2 + \sin^2 \lambda d\Omega_2^2 \right], \quad (2.7)
\end{align*}

with $0 \leq \lambda < \frac{1}{2}\pi$ and $d\Omega_2^2$ a metric on the unit 2-sphere, or

\begin{align*}
\text{ds}_{4}^2 &= \frac{1}{z^2} \left[ -dt^2 + dx^2 + dy^2 + dz^2 \right]. \quad (2.8)
\end{align*}

where $z \geq 0$ and $-\infty < t, x, y < \infty$.

\(^2\)The relevant 10-dimensional equations of motion are $R_{MN} = \frac{1}{2}\partial_M \phi \partial_N \phi + \frac{1}{96} F_{MN} F_{\ldots}\overline{F}^N$; $\Box_{10} \phi = 0$; $* F_5 = F_5$. 

\begin{align*}
\square_{10} \phi &= 0 \quad (2.3) \\
\end{align*}
The equations of motion of constrain the dilaton and scale factor to obey ([1], [2])

\[
\begin{align*}
\phi'(\mu) &= ce^{-3A(\mu)} \\
A'(\mu) &= \sqrt{e^{2A} - 1 + be^{-6A}}.
\end{align*}
\]

(2.9) \hspace{1cm} (2.10)

The arbitrary parameter\(^3\) \(c\) governs the rate of variation of the dilaton, and \(b = \frac{e^2 L^2}{24}\). The equation for \(A'(\mu)\) cannot be integrated in closed form, and one must be content with the implicit solution

\[
\mu = \int_{A_0}^{A} \frac{dA}{\sqrt{e^{2A} - 1 + be^{-6A}}}.
\]

(2.11)

The parameter \(A_0 = -\ln(x_{\text{min}})/2\) where \(x_{\text{min}}\) is the smallest root of the rational function \(P(x) \equiv 1 - bx^3 - x^{-1}\) which appears in the denominator of (2.11) with argument \(x = e^{-2A}\). The resulting geometry is free of curvature singularities as long as the two roots of \(P(x)\) remain distinct, namely for \(0 < c < \frac{9}{4\sqrt{2}L}\).

The boundary of the 5-dimensional space-time is reached at the value of the radial coordinate \(\mu_0\), which can be expressed as the series [2]

\[
\mu_0 = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\Gamma(4n + \frac{1}{2}) b^n}{\Gamma(3n + 1) n!}.
\]

(2.12)

Near the boundary, we have the behavior

\[
e^{A(\mu)} \sim \frac{1}{\sin(\mu_0 - \mu)} \left[1 + O\left(b \sin^8(\mu_0 - \mu)\right)\right],
\]

(2.13)

which shows that the back reaction of the flowing dilaton is of order \((\mu_0 - \mu)^8\) near the boundary.

Actually the equations above define the geometry only in the region \(0 < \mu < \mu_0\). However one can smoothly continue \(A(\mu)\) as an even function to the region \(-\mu_0 < \mu < \mu_0\). This is the full geometry which is geodesically complete with two boundary regions, namely \(\mu \to \pm \mu_0\).

The dilaton equation in (2.9) is formally integrated to give

\[
\phi' = \phi_0 + c \int_0^\mu e^{-3A(\mu)} d\mu
\]

\[
= \phi_+ - c \int_{\mu}^{\mu_0} e^{-3A(\mu)} d\mu.
\]

(2.14) \hspace{1cm} (2.15)

Note that the deviation from the (arbitrary constant) value \(\phi_0\) has the opposite sign in the two regions \(\text{sgn}(\mu) = \pm\). The dilaton approaches the boundary values

\[
\phi_{\pm} = \phi_0 \pm c \int_0^{\mu_0} e^{-3A(\mu)} d\mu.
\]

(2.16)

\(^3\)The dimensionless parameter \(c\) in (2.9) is related to the \(c\) used in [2] by a factor of \(L\).
One can derive the series representation
\[ \phi_{\pm} = \phi_0 \pm \frac{c\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\Gamma(4n+2)}{\Gamma(3n+\frac{5}{2})} \frac{b^n}{n!} = \phi_0 \pm \frac{2}{3}c + O(c^3). \] (2.17)

From this relation we see that the parameter \( \gamma \) we introduced in the introduction is determined in terms of the supergravity solution as
\[ \gamma = \tanh \left( \frac{\phi_{\pm} - \phi_{-}}{2} \right) = \frac{2}{3}c + \frac{16}{189}c^3 + O(c^5). \] (2.18)

We also need the dominant subleading term which can be obtained by inserting (2.13) in the second integral of (2.14). The result may be expressed as
\[ \phi(\mu) \sim \phi_{\pm} \pm \frac{c}{4} e^{-4A(\mu)} + O(e^{-6A(\mu)}). \] (2.19)

Let us now discuss the issue of the boundary metric. We must keep in mind the fact that the interior metric diverges on the boundary and thus determines the conformal structure there but not quite the metric [15]. To obtain a specific boundary metric one must multiply the 5-dimensional space-time metric of (2.1) by a conformal factor \( f^2 \) where \( f \) has a linear zero at the boundary. If we choose \( f = e^{-A(\mu)} \), for example, then the boundary appears as two copies of AdS\(_4\), with global metric, joined at their common boundary \( R \otimes S^2 \).

The determination of the boundary metric is important for the AdS/CFT correspondence because the gauge theory dual is coupled to the specific metric obtained. We will need to discuss several choices in this paper, and the first is the case of two copies of AdS\(_4\) discussed above [16]. Another case is the conformal compactification of the Janus solution, in which one uses the global AdS\(_4\) metric on slices, and takes \( f = e^{-A(\mu)} / \cos \lambda \). As discussed in [1] and [2], the 5-dimensional Janus spacetime metric is then written as
\[ ds_5^2 = \frac{L^2 e^{2A(\mu)}}{\cos^2 \lambda} \left[ -dt^2 + d\lambda^2 + \sin^2 \lambda d\Omega_2^2 + \cos^2 \lambda d\mu^2 \right]. \] (2.20)

The compactified 5-metric in (2.20) is then the portion of ESU\(_5\) containing half of S\(_4\) with angular excess (since \( \mu_0 > \frac{\pi}{2} \)) and thus two corners at the poles. See Fig. 1(a). The boundary consists of the two regions obtained in the limits \( \mu \to \pm \mu_0 \). Each spatial region is a hemisphere of S\(_3\), and the two hemispheres are joined at the poles of the S\(_4\). The boundary metric is thus \( R \otimes S^3 \), as in the global boundary of the AdS\(_5\)\( \otimes S^5\) solution of IIB supergravity. However, in the Janus solution the dilaton is not constant on the full boundary, but rather takes the values in (2.16).

The final choice involves the patch metric (2.8) on the AdS\(_4\) slices and the scaling factor \( f = e^{-A(\mu)} \). The Janus metric is then
\[ ds_5^2 = f^{-2} \left[ -dt^2 + dx^2 + dy^2 + dz^2 + z^2 d\mu^2 \right], \] (2.21)
and the conformal metric in (2.21) is the product of Minkowski$_{2+1}$ and a wedge (with radial coordinate $z$ and angular range $-\mu_0 < \mu < \mu_0$). The boundary is then two copies of Minkowski$_{3+1}$ joined at the Minkowski$_{2+1}$ interface located at $z = 0$. The dilaton takes the values $\phi_\pm$ in (2.16) on the half-spaces at $\mu = \pm \mu_0$. See Fig. 1(b).

3 Gravity predictions for the dual gauge theory.

The AdS/CFT correspondence permits the calculation, from the gravity side of the duality, of correlation functions of the CFT coupled to the specific boundary metric obtained by the procedure discussed in Sec. 2. Namely, one cancels the singularity of the bulk space-time metric by multiplication by $f^2$, where $f$ has a linear zero at the boundary. Different choices of $f$ give different boundary metrics, and correlation functions of operators $\mathcal{O}_\Delta$ for a pair of boundary metrics $g_{(1)}$ and $g_{(2)}$ are related by multiplication by factors $(f_{(1)}/f_{(2)})^\Delta$ for each operator. This is required by conformal symmetry. For each choice of $f$, the behavior near the boundary of the dual bulk field $\phi_\Delta$ can be expressed in terms of $f$ and boundary coordinates $x^\mu$ as

$$\phi_\Delta \sim a_f(x) \left( f^{d-\Delta} + \ldots \right) + b_f(x) \left( f^\Delta + \ldots \right).$$

The coefficient $a_f(x)$ of the leading term is the source (or coupling) of the operator in the deformed CFT, while the vacuum expectation value [17], [18] is given by $\langle \mathcal{O}_\Delta(x) \rangle = -(2\Delta - 4)b_f(x)$. Since $\phi_\Delta$ is a scalar, the products $a_f(x)f^{d-\Delta}$ and $b_f(x)f^\Delta$ are invariant under change of boundary metric. Couplings and vacuum expectation values thus change correctly.

Suppose, for example, that we have a bulk scalar with asymptotic behavior $a_\pm e^{(d-\Delta)A(\mu)}$, corresponding to a piecewise constant coupling $a_\pm$ for the dual operator $\mathcal{O}_\Delta$ on the two AdS$_4$ halves of the boundary. Then there is a position dependent coupling $\frac{a_\pm}{z^{d-\Delta}}$ for the Minkowski space ICFT. For the special case of the dilaton coupling to the $\Delta = d$ Lagrangian density operator, a piecewise constant coupling in AdS remains piecewise constant in Minkowski space. Similarly, a constant one point function

$$\langle \mathcal{O} \rangle_{AdS} = v$$

for an operator on two copies of AdS$_4$ becomes

$$\langle \mathcal{O} \rangle_{M} = \frac{v}{z^\Delta}$$

for the same operator in the Minkowski space ICFT. As noted in [19] this reproduces correctly the form of the one-point function required by defect conformal symmetry.

We are mostly interested in the bulk dilaton $e^{\phi(\mu)}$ of the Janus solution (2.1). We will argue in Sec. 4 that it is dual to an operator $\mathcal{L}'$ which is closely related to the Lagrangian
of $\mathcal{N} = 4$ SYM theory and is a marginal operator with $\Delta = d$. The coupling and VEV of $\mathcal{L}'$ for different boundary metrics can be read from the AdS/CFT formula (see [18], [20])

$$e^{\phi(\mu)} = e^{\phi \pm} \left(1 - \frac{2\pi^2}{N^2} \langle \mathcal{L}' \rangle f^4 + \ldots\right).$$

The leading term gives the gauge coupling in the dual ICFT which is the same for all boundary metrics with the same conformal structure, whether two copies of $AdS_4$ or two half Minkowski spaces, and is given by

$$e^{\phi \pm} = g_s = \frac{g_Y^2 M \pm}{2\pi},$$

with $\phi_\pm$ in (2.17). The subleading term determines the VEV which is obtained by comparison with (2.19). For $f = e^{-A}$ which yields two copies of $AdS_4$ as the boundary metric, we find

$$\langle \mathcal{L}' \rangle_{AdS} = \varepsilon(z) \frac{N^2 c}{2\pi^2 4^4}.$$  \hfill (3.6)

For $f = ze^{-A}$, which yields the Minkowski space ICFT, the result is

$$\langle \mathcal{L}' \rangle_{M} = \varepsilon(z) \frac{N^2 c}{2\pi^2 4^4 z.}$$  \hfill (3.7)

This result obtained from gravity will be compared with gauge theory in Sec. 5.

The Janus metric in (2.1) also contains holographic information about the stress tensor of the boundary ICFT. Although non-vanishing vacuum expectation values for scalar operators are expected in an ICFT, the VEV of the stress tensor $T_{\mu\nu}(x)$ must vanish [13]. To see that AdS/CFT gives the required result, we must examine the Janus metric with the various conformal factors discussed above. For example, with $f = ze^{-A}$, (2.21) shows that the transverse metric is $\eta_{\mu\nu}$ for all values of the radial coordinate $\mu$. Hence the traceless component required by defect conformal symmetry vanishes, and we find the result $\langle T_{\mu\nu} \rangle = 0$ as an exact prediction of supergravity.

It is also interesting to consider the correlation functions $\langle \text{Tr} X^{k_1} \text{Tr} X^{k_2} \ldots \text{Tr} X^{k_n} \rangle$ of chiral primary operators in the Janus geometry. In supergravity one would calculate these correlators from Witten diagrams involving bulk-to-boundary propagators and vertices, However, it is easy to see that, to order $c$, the Janus solution has no effect, since the dilaton appears in the Einstein frame action of Type IIB supergravity only in the terms

$$\mathcal{L} = \frac{1}{2} \sqrt{-G} \left[ \partial_\mu \phi \partial^\mu \phi + e^{-2\phi} \partial_\mu A \partial^\mu A \right]$$  \hfill (3.8)

(where $A$ is the axion). Thus effects on the dynamics of the bulk fields $h_{\alpha\beta}$ and $a_{\alpha\beta\gamma\delta}$, which are dual to the SYM operators $\text{Tr} X^k$, begin in order $c^2$. 


4 Proposed gauge theory dual

In this section we will outline the first of two closely related proposals for the gauge theory dual of the Janus solution. We emphasize that they are proposals rather than a derivation. The two proposals produce the same results for several tests discussed in Sec. 5 in which we work to order $\gamma$ and $\gamma^2$ in the discontinuity of the gauge coupling across the interface.

The main guidepost for the initial proposal is the hypothesis that a spatially varying dilaton couples to the marginal operator which is the fourth supersymmetric descendent of the chiral primary $\text{Tr} \, X^0 X^I$. This descendent is closely related to the usual Lagrangian density of $\mathcal{N} = 4$ SYM theory, differing by a total derivative which changes the scalar kinetic term from $-\frac{1}{2} \text{Tr} \, D_{\mu} X^I D^\mu X^I$ to $\frac{1}{2} \text{Tr} \, X^I D^\mu D_\mu X^I$, where $D_\mu X^I = \partial_\mu X^I + i[A_\mu, X^I]$. We call this operator $\mathcal{L}'_0$. It contains all interactions of $\mathcal{N} = 4$ SYM with no explicit gauge coupling.

We therefore hypothesize that the gauge theory action with spatially varying coupling $g^2(x)$ takes the form

$$S = \int d^4x \frac{1}{g^2(x)} \mathcal{L}'_0(x). \quad (4.1)$$

We specialize to the case of the Janus dual where the coupling is constant on the two sides of the interface located at $x^3 = 0$, so that

$$\frac{1}{g^2(x)} = \frac{1}{\bar{g}_+^2} \theta(x^3) + \frac{1}{\bar{g}_-^2} \theta(-x^3) \quad (4.2)$$

$$= \frac{1}{\bar{g}^2} \left[ 1 - \gamma \varepsilon(x^3) \right], \quad (4.3)$$

in which $\bar{g}^{-2} = \frac{1}{2} (g_+^{-2} + g_-^{-2})$ is the average coupling, and

$$\gamma = \frac{g_+^2 - g_-^2}{g_+^2 + g_-^2} \quad (4.4)$$

is a measure of the jump in the coupling across the interface. (Note that $\theta(x^3)$ and $\varepsilon(x^3)$ are standard step functions.) We then write our proposal for the Janus dual action as

$$S = S_0 + S' \quad (4.5)$$

$$S_0 = \int d^4x \mathcal{L}'(x) \quad (4.6)$$

$$S' = -\gamma \int d^4x \varepsilon(x^3) \mathcal{L}'(x), \quad (4.7)$$

where $\mathcal{L}' = \frac{1}{\bar{g}^2} \mathcal{L}'_0$.

If this proposal is correct for the dual to Janus then it must yield a quantum theory with broken supersymmetry and $SO(3,2)$ defect conformal symmetry. Let us first discuss these properties at the level of the classical action (4.5).
It is well known that the key to conventional \(SO(4,2)\) conformal symmetry is proper behavior under the conformal inversion \(x'_{\mu} = \frac{x_{\mu}}{x^0}\), since finite special conformal transformations are composed from inversion and translations. Inversion is also a symmetry of ICFT, since it transforms the defect at \(x^3 = 0\) into itself. Special conformal transformations in the preserved conformal group \(SO(3,2)\) are composed from inversion and translations in the \(x^{0,1,2}\) directions. In conventional \(\mathcal{N} = 4\) SYM theory, the operator \(\mathcal{L}'(x)\) transforms as a field of dimension 4, namely \(\mathcal{L}'(x) = (x')^8 \mathcal{L}'(x')\), and the volume element transforms as \(d^4x = d^4x'/(x')^8\). Clearly \(\varepsilon(x^3) = \varepsilon(x'^3)\), so \(S'\) is inversion symmetric and thus \(SO(3,2)\) invariant at the classical level.

Although \(S_0\) in (4.5) enjoys full \(\mathcal{N} = 4\) supersymmetry, it is fairly easy to see that supersymmetry is explicitly and completely broken in \(S'\) because of the step function. The straightforward technical argument for this is presented in Appendix A.

In conventional \(\mathcal{N} = 4\) SYM theory, with constant gauge coupling, one can integrate by parts in the action, and it is immaterial whether one writes the Lagrangian density as the conventional \(\mathcal{L}\) or as \(\mathcal{L}'\). Integration by parts cannot be used in \(S'\), so the choice of Lagrangian density is significant. With the choice \(\mathcal{L}'\), which is a marginal operator in the undeformed theory, one may hope that conformal symmetry holds in our proposal for an ICFT. Since \(\mathcal{L}' - \mathcal{L} = \frac{1}{4} \Box \text{Tr}(X^I X^I)\), which is the total derivative of the Konishi scalar with positive anomalous dimension, there would be no chance of conformal symmetry at the quantum level if one replaced \(\mathcal{L}'\) by \(\mathcal{L}\) in \(S'\).

We have seen that the proposed action for the Janus dual has the correct properties at the classical level. Our goal now is to explore the proposal at the quantum level, specifically to undertake calculations to test \(SO(3,2)\) conformal symmetry and match quantities which can be computed in the bulk theory.

The most interesting and potentially powerful way to proceed is to use conformal perturbation theory. Correlation functions are defined through the formal path integral

\[
\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \int \mathcal{D} \Phi \ e^{S_0 + S'} \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) ,
\]

in which \(S'\) is viewed as a perturbation. One can expand in powers of \(\gamma\) and obtain an expression for all \(n\)-point functions in the perturbed theory in terms of correlation functions in the undeformed \(\mathcal{N} = 4\) SYM

\[
\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_{\text{Janus}} = \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_{\mathcal{N}=4}
- \gamma \int d^4z \varepsilon(z_3) \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \mathcal{L}'(z) \rangle_{\mathcal{N}=4}
+ \frac{\gamma^2}{2} \int d^4z d^4w \varepsilon(z_3) \varepsilon(w_3) \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \mathcal{L}'(z) \mathcal{L}'(w) \rangle_{\mathcal{N}=4}
+ O(\gamma^3) .
\]

(4.9)
In a number of interesting cases the $\mathcal{N} = 4$ correlation functions which appear in (4.9) are known exactly. Then we can obtain results for the effects of the discontinuous coupling to fixed order in $\gamma$ but all orders in the 't Hooft coupling $g_{YM}^2 N$! It is exactly for these “protected” quantities that results from gauge theory theory should match results from gravity!

5 Calculations using conformal perturbation theory

In this section we describe several calculations to order $\gamma$ and $\gamma^2$ of correlation functions in the Janus dual field theory. Our goals are two-fold, first to test $SO(3,2)$ conformal symmetry at the quantum level, and second to match results with those obtained from the gravity side of the duality. The $\mathcal{N} = 4$ correlators which appear in the CPT integrals are $SO(4,2)$ covariant, so the integrals are formally invariant under the $SO(3,2)$ subgroup that preserves the defect at $z_3 = 0$. However the integrands are singular when an integrated point, e.g. $z$ approaches one of the $x_i$. Thus we need to regulate the integrals and study them before $SO(3,2)$ invariance can be claimed. The main regularization method used involves a spatial cutoff, that is we exclude a small interval $|z_3 - x_3| < \delta$, integrate and then take the limit as $\delta \to 0$. With this prescription contact terms in the $\mathcal{N} = 4$ correlators do not contribute. Any protected $\mathcal{N} = 4$ correlator can be evaluated in the free field limit in which the terms $\frac{1}{2}(X \Box X + \bar{\lambda} \partial \lambda)$ in $\mathcal{L}'$ only contribute such contact terms. Thus we can restrict to $\mathcal{L}' \to -\frac{i}{4g_{YM}^2} F^2$ for protected correlators.

5.1 1-point function of $\mathcal{L}'$

The first application of conformal perturbation theory is the one-point function of $\mathcal{L}'$. Using (4.9) we have

$$\langle \mathcal{L}'(x) \rangle_{\text{Janus}} = -\gamma \int d^4 z \varepsilon(z_3) \langle \mathcal{L}'(x) \mathcal{L}'(z) \rangle + \frac{\gamma^2}{2} \int d^4 z d^4 w \varepsilon(z_3) \varepsilon(w_3) \langle \mathcal{L}'(x) \mathcal{L}'(z) \mathcal{L}'(w) \rangle + \mathcal{O}(\gamma^3)$$

(5.1)

All terms on the right hand side are correlators in the unperturbed $\mathcal{N} = 4$ theory and we have used the fact that all one-point functions vanish in a CFT on undeformed Minkowski spacetime.

The order $\gamma$ term involves the two-point function $\langle \mathcal{L}'(x) \mathcal{L}'(z) \rangle$. This is a protected quantity which can be evaluated in the free $\mathcal{N} = 4$ theory, and the result will hold to all orders in $g_{YM}^2 N$. In the free theory it is straightforward to compute

$$\langle \mathcal{L}'(x) \mathcal{L}'(z) \rangle_{\mathcal{N} = 4} = \frac{3}{\pi^4} \frac{N^2}{(x - z)^8}, \quad z \neq x,$$

(5.2)
where the result comes from the kinetic term for the gauge field. (Had we used $\mathcal{L}$ instead of $\mathcal{L}'$ the scalars would contribute by changing $3 \to 3 + 6 \times \frac{1}{2} = 12$.) The order $\gamma$ term in (5.1) involves a singular integral which we perform using a cutoff:

$$I = \int d^4z \frac{\varepsilon(x_3)}{\langle L \rangle} = \int dx_3 \frac{\varepsilon(x_3)}{\langle L \rangle} \int \frac{dz_3}{((x_3 - z_3)^2 + z^2)^4} = \frac{\pi^2}{8} \int_{|x_3 - z_3| > \delta} \frac{dz_3}{|x_3 - z_3|^5} = \frac{\pi^2}{8} \left( \frac{1}{2\delta^4} - \frac{1}{2x_3^4} \right) \varepsilon(x_3). \quad (5.3)$$

The quartic divergent term can be cancelled by adding an additive constant to $\mathcal{L}'$ in $S'$ of (4.5). Contact terms from $\frac{1}{2}(X\Box X + \bar{\lambda}\partial\lambda)$ which were omitted in (5.2) give similar divergent contributions. The remaining term has the $1/(x^3)^4$ dependence required by defect conformal symmetry. Inserting it in (5.1) yields the result

$$\langle \mathcal{L}' \rangle_{\text{Janus}} = \varepsilon(x_3) \frac{3N^2}{16\pi^2} \frac{\gamma}{z_3^4} + \mathcal{O}(\gamma^3). \quad (5.4)$$

We have written the remainder as $\mathcal{O}(\gamma^3)$, because the integrand of the order $\gamma^2$ term in (5.1) involves the $3$-point correlator $\langle \mathcal{L}'(x)\mathcal{L}'(z)\mathcal{L}'(w) \rangle$ which actually vanishes. This is a $3$-point function of descendants of the chiral primary $\text{Tr} X^{\mu \nu} X^{\rho \sigma}$. It is protected, and we again have the luxury of working at the free field level at which we can restrict our attention to the gauge field terms: $\langle F^2(x)F^2(y)F^2(z) \rangle$. To show that this $3$-point function vanishes, just express $F^2$ in terms of self-dual and anti-self-dual parts of field strengths,

$$(F_{\mu\nu})^2 = (F^+_{\mu\nu})^2 + (F^-_{\mu\nu})^2 \quad (5.5)$$

$$F^\pm_{\mu\nu} = \frac{1}{2}(F \pm \tilde{F})_{\mu\nu}.$$ 

Using the direct terms in (6.16), one can easily verify that

$$\langle F^+_{\mu\nu}(x)F^+_{\rho\sigma}(y) \rangle = \langle F^-_{\mu\nu}(x)F^-_{\rho\sigma}(y) \rangle = 0. \quad (5.6)$$

Hence only $\langle F^+_{\mu\nu}(x)F^-_{\rho\sigma}(y) \rangle$ contractions contribute. For a $k$-point function we have $k$ contractions. Since each of them contracts one $F^+$ and one $F^-$ we need an equal number $k$ of each. Since the number of $F^+$ is even, $k$ must be even. If $k$ is odd the $k$-point function of $F^2$ automatically vanishes. This applies to the tree approximation in any gauge field theory and to all orders for $\langle \mathcal{L}'(x)\mathcal{L}'(z)\mathcal{L}'(w) \rangle_{N=4}$ because it is a protected correlator.

The result (5.4) for $\langle \mathcal{L}' \rangle$ is in perfect agreement with the gravity result (3.7). In Sec 6.1 we show that one obtains the same result in the weak coupling limit using perturbation theory

---

4The same finite result (with no divergent term) can be obtained using differential regularization [23]. One simply uses the identity $\frac{1}{(x - z)^2} = \frac{1}{2\delta} \frac{\delta}{(x - z)^2}$ in (5.3), and integrates by parts. After use of $(\partial z_3)^2 \varepsilon(z_3) = 2\delta(z_3)$, one finds a convergent integral over $\tilde{z}$ which is easily performed.
with the propagator (6.13). It is conformal perturbation theory with protected correlators which allows us to understand this agreement between field theory and gravity, which is a strong test for the proposed Janus dual and a precise new test of the AdS/CFT correspondence. Since $n$-point functions in the $\mathcal{N} = 4$ theory are generically not protected for $n \geq 4$, we do not expect agreement to hold to all orders in $\gamma$.

5.2 $\langle T_{\mu\nu}(z) \rangle_{\text{Janus}}$

The one-point function of the stress tensor must vanish in a field theory with $SO(3, 2)$ defect conformal symmetry [13], and we have already seen that it does vanish when calculated from the bulk Janus solution. It is therefore an important test to check whether it vanishes in our proposed dual gauge theory.

In conformal perturbation theory the 1-point function is given by

$$\langle T_{\mu\nu}(z) \rangle_{\text{Janus}} = \langle T_{\mu\nu}(z) \rangle_{\mathcal{N}=4} - \gamma \int d^4x \varepsilon(x_3) \langle T_{\mu\nu}(z) \mathcal{L}'(x) \rangle_{\mathcal{N}=4} - \frac{\gamma^2}{2} \int d^4xd^4y \varepsilon(x_3) \varepsilon(y_3) \langle T_{\mu\nu}(z) \mathcal{L}'(x) \mathcal{L}'(y) \rangle_{\mathcal{N}=4} + O(\gamma^3)$$

(5.7)

The one point function on the right hand side vanishes trivially in the undeformed $\mathcal{N} = 4$ theory, and we will now argue that the order $\gamma$ contribution also vanishes.

There is an interesting $\mathcal{N} = 4$ SUSY argument to show that $\langle T_{\mu\nu}(z) \mathcal{L}'(x) \rangle_{\mathcal{N}=4}$ vanishes. For any pair of operators $F(x), B(y)$, the first fermionic and the second bosonic, there is a basic identity (see (8.1) of [9])

$$\langle \{Q, F(x)\} B(y) \rangle = \langle F(x) [Q, B(y)] \rangle,$$

valid if the vacuum is supersymmetric (i.e. $Q|0\rangle = 0$) In our case both operators are descendants of the primary $\mathcal{O}_{2\gamma'} = \text{Tr} X^{i[X^J]}$ and we can write (with indices and brackets $[...]$ suppressed)

$$\langle T(z) \mathcal{L}'(x) \rangle \sim \langle Q^2 \bar{Q}^2 \mathcal{O}_{2\gamma'}(z)(Q^4 + \bar{Q}^4) \mathcal{O}_{2\gamma'}(x) \rangle$$

(5.9)

$$= \langle Q \bar{Q}^2 \mathcal{O}_{2\gamma'}(z) Q^5 \mathcal{O}_{2\gamma'}(x) \rangle + \langle \bar{Q} Q^2 \mathcal{O}_{2\gamma'}(z) \bar{Q}^5 \mathcal{O}_{2\gamma'}(x) \rangle = 0.$$

In the last line we have used the special BPS condition $Q^5 \mathcal{O}_{2\gamma'} = 0$ satisfied by the lowest dimension chiral primary (see (4.3) of [9]). The ordering of operators in $Q^2 \bar{Q}^2 \mathcal{O}_{2\gamma'}$ does not affect this argument, since reordering produces $P^\mu$ descendants, which have vanishing 2-point functions in (5.9).

An alternative way to prove that $\langle T_{\mu\nu}(z) \mathcal{L}'(x) \rangle = 0$ is to use the known nonrenormalization theorem. A short glance at the free field contribution shows that scalar and fermion
contributions vanish by their equations of motion. The gauge field contribution vanishes because

\[ \mathcal{L}_{\text{gauge}}' = -\frac{1}{4g^2} \left[ (F^+_{\mu\nu})^2 + (F^-_{\mu\nu})^2 \right] \]  
\[ T_{\mu\nu}^\text{gauge} = \frac{1}{g^2} \left( F^+_{\mu\rho} F^-_{\nu\rho} + F^-_{\mu\rho} F^+_{\nu\rho} \right) \]  
(5.10) (5.11)

and therefore the free contractions vanish by (5.6).

In order \( \gamma^2 \) we encounter something new. The correlator \( \langle T_{\mu\nu} \mathcal{L}' \mathcal{L}' \rangle_{N=4} \) is protected and does not vanish. It cannot vanish because the translation Ward identity relates it to \( \langle \mathcal{L}' \mathcal{L}' \rangle_{N=4} \). The tensorial form of this 3-point function is known [21], [13], [22] but complicated, and we postpone the study of a second order contribution to \( \langle T_{\mu\nu} \rangle \) to the future.

### 5.3 Two point functions

Let us now study more generally whether our theory based on conformal perturbation theory with perturbing operator \( \gamma \epsilon(z_3) \mathcal{L}'(z) \) does indeed lead to \( SO(3,2) \) invariant interface CFT. In this subsection we will show that the two-point function

\[ \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{\text{Janus}} = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{N=4} - \gamma \int d^4 z \epsilon(z_3) \langle \mathcal{O}(x) \mathcal{O}(y) \mathcal{L}'(z) \rangle + O(\gamma^2) \]  
(5.12)

of an operator \( \mathcal{O} \) with scaling dimension \( \Delta \) in \( N = 4 \) super Yang-Mills theory actually possesses the Cardy form [12], [13] demanded by the reduced conformal symmetry.

Let us denote by \( C_{\mathcal{O}\mathcal{O}} \) and \( C_{\mathcal{O}\mathcal{O}\mathcal{L}'} \) the coefficients appearing in the two- and three-point functions

\[ \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{N=4} = \frac{C_{\mathcal{O}\mathcal{O}}}{(x-y)^{2\Delta}}, \]
\[ \langle \mathcal{O}(x) \mathcal{O}(y) \mathcal{L}'(z) \rangle_{N=4} = \frac{C_{\mathcal{O}\mathcal{O}\mathcal{L}'}}{(x-y)^{2\Delta-4}(x-z)^4(y-z)^4}. \]  
(5.13)

The renormalized two point function computed via conformal perturbation theory to order \( \gamma \) is given by

\[ \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{\text{Janus}}^{\text{ren}} = Z(x_3)^{-1} Z(y_3)^{-1} \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{\text{Janus}} = C_{\mathcal{O}\mathcal{O}} x_3^{-\Delta(x_3)} y_3^{-\Delta(y_3)} G(x, y), \]  
(5.14)

where \( Z(z_3) \) is the renormalization factor

\[ Z(z_3) = 1 + \gamma \pi^2 (2 \log 2 \delta + 1) \frac{C_{\mathcal{O}\mathcal{O}\mathcal{L}'}}{C_{\mathcal{O}\mathcal{O}}} \epsilon(z_3) \]  
(5.15)
needed to make to renormalize the composite operator $O$. The infinities are regulated via $z_3$ spatial cutoff $\delta$. The corrected anomalous dimension $\tilde{\Delta}$ is given by

$$
\tilde{\Delta}(z_3) = \Delta + 2\pi^2 \gamma \frac{C_{\text{OO}} C'}{C_{\text{OO}}'} \varepsilon(z_3)
$$

and the nontrivial part of the Green function is

$$
G(x, y) = \left(\frac{x_3y_3}{(x - y)^2}\right)^\Delta \left[1 - \gamma \pi^2 \frac{C_{\text{OO}} C'}{C_{\text{OO}}'} \left(\varepsilon(x_3) + \varepsilon(y_3)\right) \left(\frac{(x - y)^2}{(x - Ry)^2} + \log \frac{4(x - y)^2}{(x - Ry)^2}\right)\right].
$$

One can easily check that $G(x, y)$ is in fact a function of the dimensionless ratio $\xi = \frac{(x - y)^2}{4x_3y_3}$ only as required by the reduced conformal symmetry, see [12], [13].

Note that the wave function renormalization factor and the anomalous dimension are different on both sides of the defect. This could have been indeed anticipated since from the folding trick perspective\(^5\) one has two independent fields living in one half space only and clearly those fields are allowed to have different $Z$ factors and dimensions.

To complete the discussion, let us check now that the dimensions (5.16) exactly reflect the change in the coupling in the respective half space. Consider therefore a perturbation

$$
\mathcal{L}'_{g^2 + \delta g^2} = \mathcal{L}' - \frac{\delta g^2}{g^2} \mathcal{L}'.
$$

The corrected two-point function can be written as

$$
\langle O(x)O(y) \rangle_{g^2 + \delta g^2} = Z^2 \frac{C_{\text{OO}}}{(x - y)^{2\tilde{\Delta}}},
$$

where the corrected anomalous dimension $\tilde{\Delta}$ is given by

$$
\tilde{\Delta} = \Delta + 2\pi^2 \frac{\delta g^2}{g^2} \frac{C_{\text{OO}} C'}{C_{\text{OO}}'}
$$

which precisely agrees with (5.16) upon identification of the couplings $g^2 \pm \delta g^2 = (1 \pm \gamma)g^2 + O(\gamma^2)$. The $Z$ factors also exactly match with the same identification. Note that the equation (5.20) is in fact the renormalization group equation

$$
\frac{d\Delta(g)}{dg^2} = \frac{2\pi^2}{g^2} \frac{C_{\text{OO}} C'}{C_{\text{OO}}'}
$$

valid to all orders in $g$.

\(^5\)See [24] and references therein.
5.4 $\langle \text{Tr} X^{k_1} \text{Tr} X^{k_2} \ldots \text{Tr} X^{k_n} \rangle_\gamma$ in gauge theory and gravity

It is quite straightforward to apply the CPT method to 2-point correlation functions of the chiral primary operators $\text{Tr} X^{k}$. The order $\gamma$ contribution involves the 3-point function $\langle \text{Tr} X^{k}(x) \text{Tr} X^{k}(y) \mathcal{L}'(z) \rangle$ which is also protected. In the free field limit where we replace $\mathcal{L}' \to -\frac{1}{4}F^2$, this correlator vanishes! The chiral primaries are a special case of Sec. 5.3 in which $C_{OOC'} = 0$, and the two-point functions are not corrected to order $\gamma$.

In Sec 3, we discussed the supergravity result that $\langle \text{Tr} X^{k_1} \text{Tr} X^{k_2} \ldots \text{Tr} X^{k_n} \rangle_\gamma$ vanishes in the strong coupling limit for all $n$-point functions. Thus there is a nice match between gauge theory and gravity for two-point functions $\langle \text{Tr} X^{k} \text{Tr} X^{k} \rangle_\gamma$, but we do not see how to prove the more general result in the gauge theory.

6 An alternate approach to the Janus dual

In this section we present an alternate approach to the Janus dual gauge theory motivated by the D3-brane probe action. This will lead us to a field theory action suitable for weak coupling perturbation theory in which propagators for the scalar and spinor fields have standard form, but the gauge field obeys non-trivial jump conditions at the defect required by the equations of motion. Perturbative calculations based on this action are valid to all orders in $\gamma$ but fixed order in $g^2_{Y M} N$, a perfect complement to the CPT calculations of Sec. 5.

The (bosonic part of the) action of a single BPS probe D3-brane in a background solution of Type IIB supergravity in *Einstein frame* is

$$S_{D3} = -T_3 \int d^4 \xi \sqrt{-\det \left( G_{ab}^E + 2\pi \alpha' e^{-\frac{\phi}{2}} F_{ab} \right)} + T_3 \int C_4. \quad (6.1)$$

The key feature is that the dilaton couples only to $F_{ab}$. This suggests that we study the non-abelian action

$$\mathcal{L} = \text{Tr} \left[ -\frac{1}{2} D_\mu Y^I D^\mu Y^I - \frac{1}{2} i \bar{\lambda}^\alpha \slashed{D} \lambda^\alpha - \frac{1}{4g^2(z)} F^2_{\mu\nu} \right]$$

$$+ \frac{g(z)}{2} \bar{\lambda} \Gamma^i [Y^I, \lambda] + \frac{g(z)^2}{4} [Y^I, Y^J]^2, \quad (6.2)$$

with $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. This action reduces to $\mathcal{N} = 4$ SYM theory$^6$ in the limit where the position dependent gauge coupling $g(z)$ becomes constant.

The relation between this action and (4.5) may be seen by scaling the spinor field by $\lambda \to \frac{1}{g(x)} \lambda$ and redefining the scalars by $Y^I = \frac{1}{g} X^I$. The extra term generated in the spinor kinetic action vanishes for Majorana spinors. For the scalars, after simple manipulations in

$^6$The Yukawa term is given in schematic form as obtained via dimensional reduction from 10 dimensions.
which surface terms are dropped, we obtain

$$-\partial_\mu Y \partial^\mu Y = \frac{1}{g^2} X \Box X - \frac{(\partial g)^2}{g^4} X^2.$$  \hfill (6.3)

For the Janus dual, the last term becomes singular and of order $\gamma^2$, so that (4.5) and (6.2) give equivalent results to order $\gamma$. Note also that \( \langle \mathcal{L}' X^2 \rangle_{\mathcal{N}=4} \) vanishes because it involves operators in different multiplets. Thus the order $\gamma^2$ calculation in Sec. 5 is not affected by the change in the action.

We propose to use (6.2) as the basis for a perturbative approach to the Janus dual gauge theory. The first step is to define conformal covariant propagators. For scalars and spinors these are given by the standard Euclidean propagators

$$\langle Y_I(x) Y_J(y) \rangle = \delta^{IJ} \Delta(x - y)$$  \hfill (6.4)

$$\Delta(x - y) = \frac{1}{4\pi^2 (x - y)^2},$$  \hfill (6.5)

and

$$\langle \lambda^\alpha(x) \lambda^\beta(y) \rangle = \delta^{\alpha\beta} \Sigma(x - y)$$  \hfill (6.6)

$$\Sigma(x - y) = -\phi \Delta(x - y) = \frac{1}{2\pi^2} \frac{\dot{x} - \dot{y}}{(x - y)^4}.$$  \hfill (6.7)

The gauge field action in (6.2) is a non-abelian generalization of the Maxwell action in an inhomogeneous medium with permittivity $\varepsilon$ and permeability $\mu$ related by $\varepsilon(z) = \frac{1}{\mu(z)} = \frac{1}{g^2(z)}$. The speed of light is constant. In the Janus limit, we have an interface between two regions with different dielectric constants, a situation often treated in electrodynamics texts such as [25]. The field strength obeys the following jump conditions obtained by integrating the equations of motion and the Bianchi identities over a small pill-shaped region which straddles the interface:

$$F_{ij}(z)|_{0^-} = F_{ij}(z)|_{0^+} \quad i, j \neq 3$$  \hfill (6.8)

$$\frac{F_{i3}(z)}{g_2^-}|_{0^-} = \frac{F_{i3}(z)}{g_2^+}|_{0^+}.$$  \hfill (6.9)

The potential obeys

$$A_i(z)|_{0^-} = A_i(z)|_{0^+} \quad \frac{A_3(z)}{g_2^-}|_{0^-} = \frac{A_3(z)}{g_2^+}|_{0^+}.$$  \hfill (6.10)

The propagator may be found by image charge methods. We use the notation \((Ry)_\mu = (y_0, y_1, y_2, -y_3)\) to indicate the image of the point with coordinates $y_\mu$. The reflection matrix
is \( R_{\mu\nu} = \text{diag}(1, 1, 1, -1) \). In Feynman gauge the propagator is a superposition of direct and image terms

\[
\Delta_{\mu\nu}(x - y) = \frac{1}{4\pi^2(x - y)^2} \delta_{\mu\nu} \tag{6.11}
\]

\[
\tilde{\Delta}_{\mu\nu}(x, y) = \frac{1}{4\pi^2(x - Ry)^2} R_{\mu\nu} \tag{6.12}
\]

It is straightforward to obtain the appropriate factors of the couplings \( g_{\pm} \) in the two regions and to write the propagator as

\[
G_{\mu\nu}(x, y) = \frac{g_+^2}{4\pi^2} \left[ \frac{\delta_{\mu\nu}}{(x - y)^2} + \left( \frac{g_+^2 - g_-^2}{g_+^2 + g_-^2} \right) \frac{R_{\mu\nu}}{(x - Ry)^2} \right] \theta(x_3)\theta(y_3) \tag{6.13}
\]

\[
+ \frac{1}{2\pi^2} \frac{g_+^2 g_-^2}{(g_+^2 + g_-^2)(x - y)^2} \delta_{\mu\nu} \left( \theta(x_3)\theta(-y_3) + \theta(-x_3)\theta(y_3) \right) \tag{6.14}
\]

\[
+ \frac{g_-^2}{4\pi^2} \left[ \frac{\delta_{\mu\nu}}{(x - y)^2} + \left( \frac{g_+^2 - g_-^2}{g_+^2 + g_-^2} \right) \frac{R_{\mu\nu}}{(x - Ry)^2} \right] \theta(-x_3)\theta(-y_3). \tag{6.15}
\]

The field strength propagators are superpositions, with the same coupling factors and step functions as in (6.13), of the direct and image terms

\[
\langle F_{\mu\nu}(x)F_{\rho\sigma}(y) \rangle_{\text{direct}} = \frac{1}{\pi^2} \frac{J_{\mu\nu}(x - y)J_{\rho\sigma}(x - y) - (\rho \leftrightarrow \sigma)}{(x - y)^4} \tag{6.16}
\]

\[
\langle F_{\mu\nu}(x)F_{\rho\sigma}(y) \rangle_{\text{image}} = \frac{1}{\pi^2} \frac{J_{\mu\tau}(x - Ry)J_{\nu\lambda}(x - Ry) - (\tau \leftrightarrow \lambda)}{(x - Ry)^4} R_{\tau\rho}R_{\lambda\sigma}. \tag{6.17}
\]

This propagator is constructed from the conformal Jacobian \( J_{\mu\nu}(z) = \delta_{\mu\nu} - \frac{2g_{\mu\nu}z_{\nu}}{z^2} \) and transforms correctly under the conformal inversion.

### 6.1 \( \langle \mathcal{L}' \rangle \) and \( \langle T_{\mu\nu} \rangle \) again

We now indicate how to calculate \( \langle \mathcal{L}' \rangle \) and \( \langle T_{\mu\nu} \rangle \) in the perturbative formalism and obtain results which agree with Sec 5. To all orders in \( \gamma \) but zero order in \( \hat{g}^2 \), it is given by the sum

\[
\langle \mathcal{L}'(z) \rangle = \text{Tr} \left[ \frac{1}{2} \langle \partial_\mu Y^I(z) \partial_\mu Y^I(z) \rangle + \frac{1}{2} \langle \bar{\lambda}^\alpha(z) \bar{\psi} \lambda^\alpha(z) \rangle + \frac{1}{4\hat{g}^2} \langle F_{\mu\nu}(z)F_{\mu\nu}(z) \rangle \right], \tag{6.18}
\]

in which the various propagators appear with \( y = x \). The divergent “normal-ordering” terms in (6.18) cancel between the scalar, spinor, and gluon terms. The same cancellation occurs in the cosmological constant in a supersymmetric theory. This leaves the unambiguous finite contribution of \( \langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle_{\text{image}} \), which is easily evaluated with the result

\[
\langle \mathcal{L}' \rangle = \frac{g_+^2 - g_-^2}{g_+^2 + g_-^2} \frac{3N^2}{16\pi^2x_3^4} \varepsilon(x_3). \tag{6.19}
\]
This agrees with (5.4).

Similar reasoning may be applied to the calculation of \( \langle T_{\mu\nu}(z) \rangle \). There is a residual unambiguous finite contribution from the image term of the field strength propagator, and it vanishes after the index contractions required in \( T_{\mu\nu} \) are done.

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### A \( \mathcal{N} = 1 \) interface SUSY and its failure for \( \mathcal{N} = 4 \)

The specific task of this appendix is to show that both proposed actions for the Janus gauge theory dual explicitly break all supersymmetry. However, we will proceed toward this goal in an indirect fashion. Namely, we consider general \( \mathcal{N} = 1 \) SUSY theories in which the coupling constants \( g(z) \) depend on one of the spatial coordinates, called \( z \). We show that two of the four supercharges can be preserved by adding uniquely determined terms to the Lagrangian which involve the first derivative \( \partial_z g(z) \). In the limit of a sharp interface, \( \partial_z g(z) \to 2 \delta g \delta(z) \), so the added terms may be called “interface counterterms.”

The restriction to two supercharges is achieved by a projection condition \( \Pi \varepsilon = \varepsilon \), where

\[
\Pi = 1 + \frac{i \Gamma^5 \Gamma^z}{2}.
\]  

(A.1)

The preservation of two supercharges is what is expected on general grounds from the arguments in [5], and the projector (A.1) is the same one that appeared in the defect theory studied in [5]. The results for defect SUSY in a general \( \mathcal{N} = 1 \) theory may have useful applications. However, when we apply them to the \( \mathcal{N} = 1 \) presentation of \( \mathcal{N} = 4 \) SYM theory, we find that the counter terms violate the \( SO(6) \) symmetry required for the Janus dual and are therefore not permitted.

#### A.1 \( \mathcal{N}=1 \) supersymmetry with chiral multiplet

Let us start with the most general \( \mathcal{N} = 1 \) supersymmetric action in \( d = 4 \) for one chiral multiplet \( (\phi, \psi, F) \)

\[
\mathcal{L} = -\partial_{\mu} \phi^{\ast} \partial^{\mu} \phi - \frac{i}{2} \bar{\psi} \Gamma^{\mu} \partial_{\mu} \psi + F^{\ast} F + W^{\ast} F - \frac{i}{2} W_{\mu} \bar{\psi} P_{\pm} \psi,
\]

(A.2)
where $\phi$ is a complex scalar field, $\psi$ is a Majorana fermion and $F$ is a complex auxiliary field. The superpotential $W$ is a holomorphic function of $\phi$ and depends on some real couplings $f, m, g, \ldots$, which we shall let depend arbitrarily on one of the coordinates called $z$. The SUSY transformation rules are

\[
\begin{align*}
\delta \phi &= i \sqrt{2} \varepsilon P_+ \psi \\
\delta \phi^* &= i \sqrt{2} \varepsilon P_- \psi \\
\delta (P_+ \psi) &= \sqrt{2} (P_+ F + \Gamma^\mu \partial_\mu \phi P_-) \varepsilon \\
\delta (P_- \psi) &= \sqrt{2} (P_- F^* + \Gamma^\mu \partial_\mu \phi^* P_+) \varepsilon \\
\delta (\bar{\psi} P_-) &= \sqrt{2} \varepsilon (P_- F^* - P_+ \Gamma^\mu \partial_\mu \phi^*) \\
\delta (\bar{\psi} P_+) &= \sqrt{2} \varepsilon (P_+ F - P_- \Gamma^\mu \partial_\mu \phi) \\
\delta F &= i \sqrt{2} \varepsilon \Gamma^\mu \partial_\mu P_+ \psi \\
\delta F^* &= i \sqrt{2} \varepsilon \Gamma^\mu \partial_\mu P_- \psi,
\end{align*}
\]

where $P_\pm = (1 \pm \Gamma^5)/2$ are chiral projection operators. The variation of the Lagrangian is

\[
\delta \mathcal{L} = i \sqrt{\frac{2}{\varepsilon}} P_+ \Gamma^\mu (\bar{\varepsilon} \Gamma^\nu \partial_\nu \phi^* + F + 2W^*) \psi + \bar{\varepsilon} P_- \Gamma^\mu (-\Gamma^\nu \partial_\nu \phi + F^* + 2W^*) \psi \\
- i \sqrt{2} \partial_z g \varepsilon \left( P_+ \Gamma^z \frac{\delta W^*}{\delta g} + P_- \Gamma^z \frac{\delta W}{\delta g} \right) \psi.
\]

Here $g$ denotes in fact all possible couplings which depend nontrivially on the $z$ coordinate. It is important that we have included all the dependence on $g$ into the superpotential, in other words that we are using the standard perturbative normalization for the fields.\(^{7}\) This allows us to discard the total derivative term in (A.4). The second term in (A.4) can be simplified by imposing the additional condition on the supersymmetry parameter $\Pi \varepsilon = \varepsilon$ where $\Pi$ is given in (A.1). The second term in (A.4) then becomes

\[
\delta \mathcal{L}' = \delta \mathcal{L} - 2 \partial_z g \varepsilon \left( P_+ \Gamma^z \frac{\delta W^*}{\delta g} + P_- \Gamma^z \frac{\delta W}{\delta g} \right) \psi,
\]

which remarkably turns out to be a supersymmetric variation of $2 \partial_z g \varepsilon \text{Im} \frac{\delta W}{\delta g}$. We can therefore modify our Lagrangian (A.2) by subtracting this piece

\[
\mathcal{L}' = \mathcal{L} - 2 \partial_z g \varepsilon \text{Im} \frac{\delta W}{\delta g},
\]

such that the new Lagrangian is invariant under the reduced supersymmetry. Note also that in the limit of constant coupling $g$ the added piece disappears.

\(^{7}\)Note that defect SUSY fails for

\[
\mathcal{L} = \frac{1}{2g^2(z)} [\partial_\mu \phi^* \partial^\mu \phi + i \bar{\psi} \Gamma^\mu \partial_\mu \psi]. \tag{A.5}
\]

There are projections $S\varepsilon$ which allow the cancellation of terms proportional to $\Gamma^\mu \partial_\mu \phi \psi$ or $\Gamma^\mu \partial_\mu \phi^* \psi$ in $\delta \mathcal{L}$, but not both. This remains the case if $\partial_\mu \phi^* \partial^\mu \phi \rightarrow \frac{-i}{4}[\phi^* \Box \phi + \Box \phi^* \phi]$.\(^{21}\)
A.2 \( \mathcal{N}=1 \) supersymmetry with vector multiplet

Now let us turn our attention to the \( d=4 \) nonabelian vector multiplet \((A_\mu^a, \lambda^a, D^a)\). The most general \( \mathcal{N}=1 \) supersymmetric action without matter fields reads

\[
\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{i}{2g^2} \bar{\lambda}^a \Gamma^\mu D_\mu \lambda^a + \frac{1}{2g^2} D^a D^a + D^a \xi^a, \tag{A.8}
\]

where \( A_\mu^a \) is a gauge field, \( \lambda^a \) is a Majorana fermion (gaugino) and \( D^a \) is a real auxiliary field. The constants \( \xi^a \) are Fayet-Iliopoulos terms and are present only for \( U(1) \) groups. The coupling \( g \) is allowed to depend on one of the coordinates called \( z \).

\[
\begin{align*}
\delta A_\mu^a &= -i \bar{\epsilon} \Gamma_\mu \lambda^a \\
\delta \lambda^a &= \frac{1}{2} \Gamma^{\mu\nu} \epsilon F^a_{\mu\nu} + i \Gamma^5 \epsilon D^a \\
\delta \bar{\lambda}^a &= -\frac{1}{2} \bar{\epsilon} \Gamma^{\mu\nu} F^a_{\mu\nu} + i \bar{\epsilon} \Gamma^5 D^a \\
\delta D^a &= -\bar{\epsilon} \Gamma^5 \Gamma^\mu D_\mu \lambda^a.
\end{align*} \tag{A.9}
\]

The variation of the action (assuming that the FI constants are indeed constant) is

\[
\delta S = -\int \partial_z \left( \frac{1}{g^2} \right) \left[ \frac{i}{4} \bar{\epsilon} \Gamma^z \Gamma^{\mu\nu} \lambda^a F^a_{\mu\nu} + \frac{1}{2} \bar{\epsilon} \Gamma^z \Gamma^5 \lambda^a D^a \right]. \tag{A.10}
\]

Imposing \( P\epsilon = \epsilon \) with \( P \) given in (A.1) we find that remarkably again the action can be made supersymmetric with the modification

\[
\mathcal{L}' = \mathcal{L} + \partial_z \left( \frac{1}{4g^2} \right) \bar{\lambda}^a \Gamma^5 \lambda^a. \tag{A.11}
\]

The most general \( \mathcal{N}=1 \) supersymmetric theory contains a vector multiplet and an arbitrary number of chiral fields in some representation of the gauge group. The action is given by the sum of (A.8) and (A.2) replacing the ordinary derivatives by the covariant ones and adding an additional Yukawa term

\[
-\frac{1}{\sqrt{2}} \phi^i \psi^a t^a_{ij} \bar{\chi}^a P_+ \psi^j + c.c. + \frac{1}{2} D^a \phi^i t^a_{ij} \phi^j. \tag{A.12}
\]

Since none of the added terms nor the supersymmetric transformation rules depend on the non-constant couplings \( g_a \), their supersymmetric variation will add up to a true total derivative term and can be safely discarded.

A.3 \( \mathcal{N}=4 \) SYM theory

In the \( \mathcal{N}=1 \) description of \( \mathcal{N}=4 \) SYM theory there are three chiral multiplets in the adjoint representation with superpotential \( W \sim \epsilon_{ij} Tr \phi^i \phi^j \phi^k \). One must distinguish between the
three fermion fields $\psi^i$ which are partners of the $\phi^i$, and the fourth fermion $\lambda$, which is the partner of the gluon. One can preserve two supercharges by imposing the condition $\Pi \varepsilon = \varepsilon$, discussed above and adding the counterterms in (A.7) and (A.11). However the largest flavor symmetry of these counter terms is $SU(3)$ rather that the $SO(6)$ required to agree with the properties of the bulk Janus solution. Thus the counterterms are not permitted for Janus, and we conclude that the Janus dual violates SUSY completely and explicitly.

**B Some integrals used in Sec. 5.3.**

This appendix contains two integrals used in the main text. Both of them are regulated by imposing a spatial cutoff $|z_3 - x_3| > \delta$ and $|z_3 - y_3| > \delta$.

\[
\int d^4z \frac{\varepsilon(z_3)}{(x-z)^4(y-z)^4} = \frac{\pi^2}{(x-y)^4} \left[ -2 \log \delta - 1 + \frac{(x-y)^2}{(x-Ry)^2} \right] \left( \varepsilon(x_3) + \varepsilon(y_3) \right), \\
+ \frac{2\pi^2}{(x-y)^4} \left( \log x_3 \varepsilon(x_3) + \log y_3 \varepsilon(y_3) \right) 
\]

(B.1)

\[
\int d^4z \frac{1}{(x-z)^4(y-z)^4} = \frac{2\pi^2}{(x-y)^4} \left[ -2 \log 2\delta - 1 + \log(x-y)^2 \right]. 
\]

(B.2)

**References**

[1] D. Bak, M. Gutperle and S. Hirano, “A dilatonic deformation of AdS(5) and its field theory dual,” JHEP 0305, 072 (2003) [arXiv:hep-th/0304129].

[2] D. Z. Freedman, C. Nunez, M. Schnabl and K. Skenderis, “Fake supergravity and domain wall stability,” Phys. Rev. D 69, 104027 (2004) [arXiv:hep-th/0312055].

[3] A. Karch and L. Randall, “Localized gravity in string theory,” Phys. Rev. Lett. 87, 061601 (2001) [arXiv:hep-th/0105108].

[4] A. Karch and L. Randall, “Open and closed string interpretation of SUSY CFT’s on branes with boundaries,” JHEP 0106, 063 (2001) [arXiv:hep-th/0105132].

[5] O. DeWolfe, D. Z. Freedman and H. Ooguri, “Holography and defect conformal field theories,” Phys. Rev. D 66, 025009 (2002) [arXiv:hep-th/0111135].

[6] S. M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in D = 4, N = 4 SYM at large N,” Adv. Theor. Math. Phys. 2, 697 (1998) [arXiv:hep-th/9806074].

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[7] E. D’Hoker, D. Z. Freedman and W. Skiba, “Field theory tests for correlators in the AdS/CFT correspondence,” Phys. Rev. D 59, 045008 (1999) [arXiv:hep-th/9807098].

[8] P. S. Howe, E. Sokatchev and P. C. West, “3-point functions in N = 4 Yang-Mills,” Phys. Lett. B 444, 341 (1998) [arXiv:hep-th/9808162].

[9] K. A. Intriligator, “Bonus symmetries of N = 4 super-Yang-Mills correlation functions via AdS duality,” Nucl. Phys. B 551, 575 (1999) [arXiv:hep-th/9811047].

[10] A. Basu, M. B. Green and S. Sethi, “Some systematics of the coupling constant dependence of N = 4 Yang-Mills,” arXiv:hep-th/0406231.

[11] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, “Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory,” JHEP 9808, 013 (1998) [arXiv:hep-th/9807033].

[12] J. L. Cardy, “Conformal Invariance And Surface Critical Behavior,” Nucl. Phys. B 240, 514 (1984).

[13] D. M. McAvity and H. Osborn, “Conformal field theories near a boundary in general dimensions,” Nucl. Phys. B 455, 522 (1995) [arXiv:cond-mat/9505127].

[14] D. Bak, M. Gutperle, S. Hirano and N. Ohta, Phys. Rev. D 70, 086004 (2004) [arXiv:hep-th/0403249].

[15] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[16] M. Porrati, “Mass and gauge invariance. IV: Holography for the Karch-Randall model,” Phys. Rev. D 65, 044015 (2002) [arXiv:hep-th/0109017].

[17] V. Balasubramanian, P. Kraus and A. E. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter spacetime,” Phys. Rev. D 59, 046003 (1999) [arXiv:hep-th/9805171].

[18] V. Balasubramanian, P. Kraus, A. E. Lawrence and S. P. Trivedi, “Holographic probes of anti-de Sitter space-times,” Phys. Rev. D 59, 104021 (1999) [arXiv:hep-th/9808017].

[19] O. Aharony, O. DeWolfe, D. Z. Freedman and A. Karch, JHEP 0307, 030 (2003) [arXiv:hep-th/0303249].

[20] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[21] H. Osborn and A. C. Petkou, “Implications of conformal invariance in field theories for general dimensions,” Annals Phys. 231, 311 (1994) [arXiv:hep-th/9307010].
[22] H. Liu and A. A. Tseytlin, “D = 4 super Yang-Mills, D = 5 gauged supergravity, and
D = 4 conformal supergravity,” Nucl. Phys. B 533, 88 (1998) [arXiv:hep-th/9804083].

[23] D. Z. Freedman, K. Johnson and J. I. Latorre, “Differential regularization and renor-
malization: A New method of calculation in quantum field theory,” Nucl. Phys. B 371,
353 (1992).

[24] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, “Permeable conformal walls and
holography,” JHEP 0206, 027 (2002) [arXiv:hep-th/0111210].

[25] J. D. Jackson, ”Classical Electrodynamics (Third Edition)” Wiley and Sons, New York,
1998.