AN ANALOG OF THE DELIGNE-LUSZTIG DUALITY FOR
$(g, K)$-MODULES

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Introduction

0.1. **Pseudo-identity, Deligne-Lustig functor and dualities.**

0.1.1. Recently, a number of papers have appeared where connections were found between the following objects:

– The Deligne-Lusztig functor on the category of representations of a $p$-adic group;
– The composition of contragredient and cohomological dualities;
– The pseudo-identity functor on the category of D-modules/sheaves on $\text{Bun}_G$;
– The “strange” operator of Drinfeld-Wang that acts on the space of automorphic functions.

Let us explain what these relations are.

0.1.2. First, in the paper [BBK], the authors consider the (derived) category $G(K)$-mod of (say, admissible) representations of a $p$-adic group $G(K)$ (here $G$ is a reductive group and $K$ is a non-archimedian local field). The Deligne-Lusztig functor is defined by sending a representation $M$ to the complex

$$DL(M) := M \rightarrow \oplus_P i_P^G \circ r_P^G(M) \rightarrow \cdots \rightarrow i_P^G \circ r_B^G(M),$$

where $(r_P^G, i_P^G)$ is the adjoint pair corresponding to parabolic induction and Jacquet functor (for a parabolic $P$), and where in the $k$-th term of the complex, the direct sum is taken over parabolics of co-rank $k$.

The main theorem in that paper says that the functor $DL$ is canonically isomorphic to the composition of contragredient and cohomological dualities, i.e.,

$$DL \simeq D_{\text{coh}} \circ D_{\text{contr}},$$

where $D_{\text{cont}}$ sends $M$ to its admissible dual $M^\vee$, and

$$D_{\text{coh}}(M) := \text{RHom}_G(M, \mathcal{H}),$$

where $\mathcal{H}$ is the regular representation of the Hecke algebra (i.e., the space of compactly supported smooth functions on $G(K)$).

We also note the following:

(1) It is more or less tautological that the composition

$$D_{\text{contr}} \circ D_{\text{coh}} : G(K)\text{-mod} \rightarrow G(K)\text{-mod}$$

is isomorphic to the Serre functor $S_{G(K)\text{-mod}}$ on $G(K)$-mod (see Sect. 1.2 for what we mean by the Serre functor).

Thus, one can reformulate the main result of [BBK] as saying that the functors $DL$ and $S_{G,\text{mod}}$ are mutually inverse.

(2) The key idea in the proof of this theorem is to use the De Concino-Procesi (a.k.a., wonderful) compactification $\overline{G}$ of $G$.

(3) The functors $DL$ for $G$ and the Levi $M$ corresponding to a given parabolic make the following diagram commute (up to a cohomological shift):

$$\begin{array}{ccc}
G(K)\text{-mod} & \xrightarrow{DL_G} & G(K)\text{-mod} \\
\uparrow i_P^G & & \uparrow i_P^G \\
M(K)\text{-mod} & \xrightarrow{DL_M} & M(K)\text{-mod},
\end{array}$$
where \( i^G_P \) is the induction functor, taken with respect to the opposite parabolic.

0.1.3. Second, the paper [Ga\textsuperscript{2}] studies the category of D-modules/sheaves on the moduli stack \( \text{Bun}_G \) of principal \( G \)-bundles over a global curve.

Since \( \text{Bun}_G \) is not quasi-compact the phenomenon of “divergence at infinity” must be taken into account. One considers two versions of the (derived) category of D-modules/sheaves:

\[
\text{Shv}_0(\text{Bun}_G) \text{ and } \text{Shv}(\text{Bun}_G),
\]

where the latter is the (naturally defined) category of all D-modules/sheaves, and the former is the full subcategory that consists of objects that are !-extended from quasi-compact open substacks.

An arbitrary object in \( \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \) defines a functor

\[
\text{Shv}_0(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G),
\]

and let us temporarily fix the conventions\(^1\) so that the object

\[
(\Delta_{\text{Bun}_G})(k_{\text{Bun}_G}) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G)
\]

(here \( k_{\text{Bun}_G} \) denotes the constant sheaf on \( \text{Bun}_G \)) defines the tautological embedding

\[
\text{Shv}_0(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G)
\]

One introduces the \textit{pseudo-identity} functor

\[
\text{Ps-Id}_{\text{Bun}_G} : \text{Shv}_0(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G)
\]

to be the one given by the object

\[
(\Delta_{\text{Bun}_G})_*(k_{\text{Bun}_G}) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G).
\]

The point here is that since \( \text{Bun}_G \) is a stack and not a scheme, the diagonal map is not a closed embedding, so that functors \((\Delta_{\text{Bun}_G})_*\) and \((\Delta_{\text{Bun}_G})!\) are different. This definition makes sense not just for \( \text{Bun}_G \), but for an arbitrary algebraic stack \( Y \).

The main result of the paper [Ga\textsuperscript{2}] is that the functor \( \text{Ps-Id}_{\text{Bun}_G} \), combined with usual Verdier duality can be extended to an equivalence

\[
\text{Shv}(\text{Bun}_G)^\vee \simeq \text{Shv}(\text{Bun}_G),
\]

where \( C \mapsto C^\vee \) is the operation of passage to the dual category\(^2\) (see Sect. 0.6.4 below).

A few remarks are in order:

(i) The object \((\Delta_{\text{Bun}_G})_*(k_{\text{Bun}_G})\) defining \( \text{Ps-Id}_{\text{Bun}_G} \) can be described using the wonderful compactification \( \overline{G} \) of \( G \);

(ii) Using (i), one can express \( \text{Ps-Id}_{\text{Bun}_G} \) as a complex whose terms are compositions of Constant Term, Eisenstein functors and certain intertwining functors

\[
\text{Shv}_0(\text{Bun}_G) \xrightarrow{\text{CT}_{\overline{G}}^G} \text{Shv}(\text{Bun}_M) \xrightarrow{\text{T}} \text{Shv}(\text{Bun}_M) \xrightarrow{\text{Eis}_{\overline{G}}^G} \text{Shv}(\text{Bun}_G).
\]

(iii) One of the key ingredients in the proof of the main result of [Ga\textsuperscript{2}] is the commutativity of the following diagram (up to a cohomological shift):

\(^1\)This choice is made so that it is easy to make a connection with the papers [DW, Wa]. However, in the main body of the paper, our conventions will be Verdier dual to the ones above.

\(^2\)The above assertion should be taken literally when \( \text{Shv}(-) \) is understood as \( \text{D-mod}(-) \), while appropriate modifications need to be made in other sheaf-theoretic contexts.
0.1.4. Third, the papers [DW] and [Wa] consider a global function field $K$, and the spaces
\[ \text{Funct}_{\text{sm}}^0(G(\mathbb{A})/G(K)) \text{ and } \text{Funct}_{\text{sm}}^0(G(\mathbb{A})/G(K)) \]
of compactly supported (resp., all) smooth functions on the automorphic quotient $G(\mathbb{A})/G(K)$.

Using the wonderful compactification $\overline{G}$, the authors define a certain “strange” operator
\[ L : \text{Funct}_{\text{sm}}^0(G(\mathbb{A})/G(K)) \rightarrow \text{Funct}_{\text{sm}}^0(G(\mathbb{A})/G(K)) \]
that commutes with the $G(\mathbb{A})$-action.

The key features of the operator $L$ are as follows:

(a) The operator $L$ can be expressed as an alternating sum of the operators
\[ \text{Eis}_{\rho}^G \circ \Upsilon \circ \text{CT}_{\rho}^G, \]
where $\text{Eis}_{\rho}^G$ and $\text{CT}_{\rho}^G$ are the Eisenstein and Constant Term operators, and $\Upsilon$ is a certain intertwining operator.

(b) When one considers non-ramified functions, the operator $L$ is given by a function on
\[ G(\mathbb{O})\backslash G(\mathbb{A})/G(K) \times G(\mathbb{O})\backslash G(\mathbb{A})/G(K) \]
that equals the trace of the Frobenius of the sheaf $L_2$. I.e., the functor $\text{Ps-Id}_{\text{Bun}_G}$ and the operator $L$ match up via the sheaf-function correspondence.

0.1.5. The three situations described above are formally related as follows:

The papers [DW] and [Wa] can be thought of as being a global counterpart for [BBK]. The paper [Ga2] is an upgrade of [DW] [Wa] to a categorical level.

In the present paper we develop an analog of the Deligne-Lusztig functor for the category of $(\mathfrak{g}, K)$-modules. The connection to the papers mentioned above is as follows:

On the one hand, we regard the category of $(\mathfrak{g}, K)$-modules as an archimedean counterpart of $G(K)$-mod. On the other hand, when we interpret $(\mathfrak{g}, K)$-modules through the localization equivalence, this category exhibits many features parallel to $\text{Shv}(\text{Bun}_G)$. And finally, it should play a role in the generalization of [Wa] to the case of number fields.

0.2. The present work.
0.2.1. The object of study of the present paper is the (derived) category of \((g, K)\)-modules for a symmetric pair \((G, \theta)\), with a given central character \(\chi\). We denote this category \(g\text{-mod}^b_K\chi\).

We introduce an endo-functor of \(Ps\text{-Id}_{g\text{-mod}^b_K\chi} : g\text{-mod}^b_K\chi \to g\text{-mod}^b_K\chi\),

which is a direct analog of the functor \(Ps\text{-Id}_{\text{Bun}_G}\). In fact, when we interpret \(g\text{-mod}^b_K\chi\) via the localization equivalence as the category of twisted \(D\)-modules on \(K\setminus X\) (here \(X\) is the flag variety of \(G\)), the functor \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) corresponds to the functor \(Ps\text{-Id}_{K\setminus X}\) for the stack \(K\setminus X\), see Sect. 0.1.3.

We consider \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) as an analog of the Deligne-Lusztig functor in the context of \((g, K)\)-modules. The main results of this paper establish (or conjecture) various properties of \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) that support this analogy.

0.2.2. Here is the summary of our main results:

– We show that the functor \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) is a self-equivalence of \(g\text{-mod}^b_K\chi\), which is the inverse of the Serre endofunctor \(Se_{g\text{-mod}^b_K\chi}\) (this is Theorem 2.6.2).

– In fact, we give a general criterion, when for a DG category \(C\), the functors \(Ps\text{-Id}_C\) and \(Se_C\) are mutually inverse equivalences (this is Corollary 1.5.6).

– We show that \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) is canonically isomorphic (up to a certain twist) to the composition \(\mathbb{D}^\text{can} \circ \mathbb{D}^\text{contr}\), where \(\mathbb{D}^\text{can}\) is the cohomological duality of \(g\text{-mod}^b_K\chi\) (which is a direct analog of \(\mathbb{D}^\text{can}\) for \((g, K)\)-modules), and \(\mathbb{D}^\text{contr}\) is the extension to the derived category of the usual contragredient duality functor (this is Theorem 3.2.7).

– We propose a certain conjecture, which we regard as the analog for \((g, K)\)-modules of Bernstein’s “2nd adjointness” theorem. We show that this conjecture is equivalent to an analog for \((g, K)\)-modules of the commutation of the diagrams (0.3) and (0.5).

– We run a plausibility test on our “2nd adjointness” conjecture, and show that at the level of abelian categories it reproduces a result of A. W. Casselman, D. Milicic, H. Hecht and W. Schmid on the behavior of asymptotics of representations under the contragredient duality operation.

0.2.3. Here are some directions that we do not pursue in this paper, but which seem attractive:

– One would like to express the functor \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) (or its geometric counterpart \(Ps\text{-Id}_{K\setminus X}\)) in a way similar to \(\mathbb{1}_{\text{F}}\), i.e., as a complex whose terms are compositions of the Casselman-Jacquet functors and induction functors for \(\theta\)-compatible parabolics in \(G\). (This necessitates generalizing the results of [CGY] to the case when instead of the minimal \(\theta\)-compatible parabolic, we consider an arbitrary parabolic.)

– One would like to find an expression for \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) (or its geometric counterpart \(Ps\text{-Id}_{K\setminus X}\)) via the wonderful compactification \(\overline{G}\).

– One would like to generalize the constructions of [Wa] to the number field case, and find the relation between his operator \(L\) and our functor \(Ps\text{-Id}_{g\text{-mod}^b_K\chi}\) at the archimedian primes (at non-archimedian primes, the ingredients of [BBK] play a role in J. Wang’s constructions).

0.3. What is actually done in this paper.
0.3.1. The main body of this paper begins with Sect. 1 where we discuss the general formalism of Serre and pseudo-identity operations and functors.

For a pair of DG categories $C$ and $D$, the Serre and the pseudo-identity operations are both contravariant functors

$$\text{Se}, \text{Ps} : \text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(D, C),$$

which take colimits to limits.

The Serre and pseudo-identity functors are defined by

$$\text{Se}_C := \text{Se(Id}_C) \text{ and } \text{Ps-Id}_C := \text{Ps(Id}_C).$$

The main results of this section are Proposition 1.5.2 and Corollary 1.5.6. The former says that if certain finiteness conditions are satisfied (preservation of compactness), for a continuous functor $F : C \to D$ we have

$$\text{Se}_C \circ \text{Ps}(F) \circ \text{Se}_D \simeq \text{Se}(F).$$

The latter says that if $C$ satisfies a certain finiteness condition, then the functors $\text{Se}_C$ and $\text{Ps-Id}_C$ are mutually inverse equivalences.

0.3.2. In Sect. 2 we show that some DG categories that naturally arise in geometric representation theory satisfy the assumption of Corollary 1.5.6 mentioned above; in particular, for such categories the Serre and the pseudo-identity functors are mutually inverse equivalences.

One set of examples consists of categories of (twisted) D-modules on algebraic stacks $Y$ that have finitely many isomorphism classes of points. For example, a stack of the form $Y := H \backslash Y$, where $H$ is an algebraic group acting on a scheme $Y$ with finitely many orbits has this property.

We also consider a variant, where we have a $T$-torsor $\tilde{Y} \to Y$ (where $T$ is a torus), and we consider $\lambda$-monodromic D-modules on $\tilde{Y}$ for a character $\lambda \in \mathfrak{t}^*$. Another set of examples comes from representations of Lie algebras. Let $G$ be a reductive group with Lie algebra $\mathfrak{g}$; fix a character $\chi$ of $Z(\mathfrak{g}) := Z(U(\mathfrak{g}))$. We show that if $H \subset G$ is spherical (i.e., has finitely many orbits on the flag variety $X$ of $G$), then the corresponding category

$$\mathfrak{g}\text{-mod}^H_{\chi}$$

(i.e., the derived version of the category of $(\mathfrak{g}, H)$-modules with the fixed central character $\chi$) satisfies the assumptions of Corollary 1.5.6. In particular, the Serre and the pseudo-identity functors are mutually inverse equivalences for $\mathfrak{g}\text{-mod}^H_{\chi}$.

We also establish a variant of this result, where instead of $\mathfrak{g}\text{-mod}^H_{\chi}$, we consider $\mathfrak{g}\text{-mod}^H_{\{\chi\}}$, i.e., we let our modules have a generalized central character $\chi$.

As a byproduct (and using a result from [CGY]) we reprove the result from [BBM] that says that the Serre functor on category $\mathcal{O}$ (or, equivalently, on the category of $N$-equivariant twisted D-modules on the flag variety) is given by the square of long intertwining functor.
0.3.3. In Sect. 3 we study the functors obtained by composing the canonical (=cohomological) duality\footnote{In the formula below and elsewhere, the notation \( C^c \) means the subcategory of compact objects in a given DG category \( C \).}

\[ \mathbb{D}^\text{can}_{g,H} : (g\text{-mod}_H^c)^c \to (g\text{-mod}_H^c)^c \]

with the Serre functor on \((g\text{-mod}_H^c)^c\) (and also a twist by a certain determinant line).

We consider the following cases:

(i) \( H = G \), so that \( g\text{-mod}_H^c \) is \( \text{Rep}(G) \), the category of algebraic representations of \( G \);

(ii) \( H = N \), the unipotent radical of the Borel, so that \( g\text{-mod}_H^c \) is category \( O \);

(iii) \( H = K \), where \( K = G^\theta \) in the case of a symmetric pair, so that \( g\text{-mod}_H^c \) is the derived category of \((g,K)\)-modules;

(iv) \( H = M_K \cdot N \), also in the case of a symmetric pair, where \( N \) is now the unipotent radical of a \( \theta \)-minimal parabolic \( P \), and \( M_K = K \cap P \).

We show that, after ind-extension, in cases (i) and (iii) above, the resulting functor

\[ g\text{-mod}_H^c \to (g\text{-mod}_H^c)^{op} \]

is a derived version of the usual contragredient duality functor.

In cases (ii) and (iv), the same happens after we compose with the corresponding long intertwining functor

\[ g\text{-mod}_N^c \to g\text{-mod}_N^c \text{ and } g\text{-mod}_{M_K \cdot N}^c \to g\text{-mod}_{M_K \cdot N}^c \]

(note that in these cases, contragredient duality naturally replaces the local finiteness condition with respect to \( N \) by that with respect to \( N^- \)).

0.4. Principal series and the “2nd adjointness” conjecture.

0.4.1. In Sect. 4 we study the two (mutually Verdier conjugate) principal series functors

\[ \text{Av}_{r}^{K/M_K} \text{ and } \text{Av}_{s}^{K/M_K} \]

that map

\[ g\text{-mod}_{M_K \cdot N}^c \to g\text{-mod}_{M_K \cdot N}^c, \]

and their various adjoints. The above two functors are (loose) analogs of the two Eisenstein series functors \( \text{Eis}_P^G \) and \( \overset{\sim}{\text{Eis}}_P^G \) in (0.5).

The functor \( \text{Av}_{r}^{K/M_K} \) is naturally the left adjoint of the functor

\[ \text{Av}_{s}^{N} : g\text{-mod}_{X}^c \to g\text{-mod}_{M_K \cdot N}^c, \]

and the functor \( \text{Av}_{s}^{K/M_K} \) is naturally the right adjoint of the functor

\[ \text{Av}_{r}^{N} : g\text{-mod}_{X}^c \to g\text{-mod}_{M_K \cdot N}^c. \]
0.4.2. We recall the “2nd adjointness” conjecture from [CGY], which says that the functor $\text{Av}_*^{K/M}$ is also the left adjoint (up to a certain determinant line) of the functor
$$\text{Av}_*^{N} \circ \text{Av}_1^{N^-} : \mathfrak{g}\text{-mod}_X^K \to \mathfrak{g}\text{-mod}_X^{M_K \cdot N}$$. The functor $\text{Av}_*^{N} \circ \text{Av}_1^{N^-}$ that appears above is known as the Casselman-Jacquet functor; we denote it by $J^-$. An equivalent formulation of the “2nd adjointness” conjecture is that there exists a canonical isomorphism between
$$\text{Av}_*^{K/M} \circ \text{Av}_1^{N}$$
as functors $\mathfrak{g}\text{-mod}_X^{M_K \cdot N} \cong \mathfrak{g}\text{-mod}_X^K$ (up to a certain determinant line).

0.4.3. We also prove that there exists a canonical isomorphism (up to a cohomological shift) between
$$\text{Ps-Id}_{\mathfrak{g}\text{-mod}_X^K} \circ \text{Av}_*^{K/M}$$
as functors $\mathfrak{g}\text{-mod}_X^{M_K \cdot N} \cong \mathfrak{g}\text{-mod}_X^K$. Juxtaposing, we obtain that the “2nd adjointness” conjecture is equivalent to the fact that the following diagram commutes (up to a certain determinant line and a cohomological shift):
$$\begin{array}{ccc}
\mathfrak{g}\text{-mod}_X^K & \xleftarrow{\text{Av}_*^{K/M}} & \mathfrak{g}\text{-mod}_X^{M_K \cdot N} \\
\text{Ps-Id}_{\mathfrak{g}\text{-mod}_X^K} & \downarrow \scriptstyle{\text{Av}_1^{N^-}} \\
\mathfrak{g}\text{-mod}_X^K & \xleftarrow{\text{Av}_*^{K/M}} & \mathfrak{g}\text{-mod}_X^{M_K \cdot N^-} \\
\end{array}$$

We consider this to be a (loose) analog of the commutative diagram [0.5].

0.4.4. Finally, we show that our “2nd adjointness” conjecture is equivalent to an isomorphism of functors
$$\mathcal{D}^{\text{contr}} \circ J \simeq J^- \circ \mathcal{D}^{\text{contr}} : (\mathfrak{g}\text{-mod}_X^K)^c \to (\mathfrak{g}\text{-mod}_X^{M_K \cdot N})^c$$,
where $J$ is the counterpart of $J^-$ where we swap $P$ for $P^-$. The latter isomorphism is known at the level of abelian categories, for $k = \mathbb{C}$, due to Casselman, Milicic and later Hecht and Schmid (see [Ca], [M], [HS]). Their approach is analytic, using asymptotics of matrix coefficients.

0.5. Organization of the paper.

0.5.1. In Sect. 1 we discuss the general formalism of Serre and pseudo-identity operations and functors for DG categories.

0.5.2. In Sect. 2 we consider examples of DG categories that come from geometry and representation theory, and show that for some of these categories, the Serre and pseudo-identity functors are mutually inverse.

0.5.3. In Sect. 3 we relate the Serre functor on certain representation-theoretic categories to the functor of contragredient duality.

0.5.4. In Sect. 4 we relate the material of the previous sections of this paper to an analog of the “2nd adjointness” conjecture for $(\mathfrak{g}, K)$-modules.

0.6. Notation and conventions. The conventions in this paper follow those of [CGY].
0.6.1. Throughout the paper we will be working over a ground field \( k \), assumed algebraically closed and of characteristic 0. We let \( G \) be a connected reductive algebraic group over \( k \), and we let \( X \) denote the flag variety of \( G \).

0.6.2. We will be working with DG categories over \( k \) (see [GR2, Chapter 1, Sect. 10] for a concise summary of DG categories). All functors between DG categories are assumed to be exact, i.e., preserving finite limits (equivalently, colimits or cones).

All DG categories will be assumed cocomplete (i.e., contain infinite direct sums). Unless specified otherwise, when discussing a functor between two DG categories, we will assume that this functor is continuous, i.e., preserves infinite direct sums (equivalently, colimits or filtered colimits).

We denote by \( \text{Vect} \) the DG category of complexes of vector spaces. For a DG category \( C \) and \( c, c' \in C \), we let \( \mathcal{H}om_C(c, c') \in \text{Vect} \) denote the corresponding Hom complex.

0.6.3. In this paper, we will consider the operation of tensor product of DG categories,

\[ C_1, C_2 \mapsto C_1 \otimes C_2, \]

[GR2, Chapter 1, Sect.10.4].

This operation is functorial with respect to continuous functors \( C_1 \to C_1' \) and \( C_2 \to C_2' \). It makes the \( \infty \)-category of cocomplete DG categories and continuous functors into a symmetric monoidal \( \infty \)-category; the unit is given by DG category \( \text{Vect} \).

In order to have tensor product, one does need to work with DG categories, rather than triangulated categories. This is why the usage of higher algebra is unavoidable for this paper (unlike its predecessor [CGY]).

0.6.4. In any symmetric monoidal \( \infty \)-category, given an object one can ask for its dualizability. This way we arrive at the notion of dualizable DG category.

A duality data for a DG category \( C \) is another DG category \( C^\vee \) and a pair of functors

\[ (-, -)_C : C \otimes C^\vee \to \text{Vect} \]

and

\[ \text{Vect} \to C^\vee \otimes C \]

that satisfy the appropriate axioms.

It is known that every compactly generated category \( C \) is dualizable. In this case \( C^\vee \) is also compactly generated and we have a canonical equivalence

\[ (C^\vee)^c \simeq (C^c)^{\text{op}}, \]

see [GR2, Chapter 1, Proposition 7.2.3].

Explicitly, the functor \( (C^c)^{\text{op}} \to C^\vee \) is characterized by the requirement that the composition

\[ C \times (C^c)^{\text{op}} \to C \times C^\vee \to C \otimes C^\vee \overset{(-, -)_C}{\longrightarrow} \text{Vect} \]

is given by

\[ c, c' \mapsto \mathcal{H}om_C(c', c). \]
0.6.5. **Derived algebraic geometry.** This paper will make a mild use of derived algebraic geometry; see [GR2 Chapters 2 and 3] for a brief summary, in particular for our usage of the notation $\text{QCoh}(-)$.

All (derived) schemes will be assume left (locally almost of finite type); see [GR2 Chapters 2, Sect.1.7] for what this means.

0.6.6. **$D$-modules.** Given a scheme/algebraic stack $Y$, we will denote by $\text{D-mod}(Y)$ the DG category of $D$-modules on $Y$; see [GR1].

Given a twisting $\lambda$ on $Y$, we will denote by $\text{D-mod}_\lambda(Y)$ the corresponding DG category of twisted $D$-modules, see [GR1, Sects. 6 and 7].

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1. **The Serre and pseudo-identity functors**

In this section all categories will be assumed compactly generated.

1.1. **The Serre operation.**

1.1.1. We introduce some terminology:

We shall call a continuous functor $F : C \to D$ proper if it maps compact objects to compact objects. (Equivalently, if $F$ admits a continuous right adjoint.)

We shall say that $C$ is proper if the evaluation functor

$$C \otimes C^\vee \to \text{Vect}$$

(1.1) is proper. Equivalently, if for every $c, c' \in C^c$, the object $\text{Hom}_C(c, c') \in \text{Vect}$ is compact.

We introduce the dualization functor

$$D_C : C \to (C^\vee)^{\text{op}}$$

be requiring that it be the ind-extension of the tautological functor

$$C^c \simeq ((C^\vee)^c)^{\text{op}} \to (C^\vee)^{\text{op}}.$$

In other words, the functor $D_C$, when viewed as a contravariant functor $C \to C^\vee$ sends colimits to limits and is the tautological functor on $C^c$.

We shall say that $c \in C$ is reflexive if the tautological map

$$c \to D_{C^\vee} \circ D_C(c)$$

is an isomorphism.

For example, for $C = \text{Vect}$, the functor $D_{\text{Vect}}$ is the usual dualization functor

$$V \mapsto V^*,$$

and $V \in \text{Vect}$ is reflexive if and only if it has finite-dimensional cohomologies (but it may have infinitely many non-vanishing cohomology groups).

We shall say that $C$ is reflexive if the evaluation functor (1.1) sends compact objects to reflexive objects. Equivalently, if for every $c, c' \in C^c$, the object $\text{Hom}_C(c, c') \in \text{Vect}$ has finite-dimensional cohomologies.
1.1.2. Let $C$ and $D$ be DG categories. We define the Serre operation

$$Se : \text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(D, C)^{\text{op}}$$

as follows.

Namely, for $F \in \text{Funct}_{\text{cont}}(C, D)$, $c \in C$ and $d \in D^c$ we set

$$\mathcal{H}om_C(c, Se(F)(d)) := \mathcal{H}om_D(d, F(c))^*.$$

1.1.3. First, we observe:

**Lemma 1.1.4.** The functor $Se$ preserves colimits. I.e., when viewed as a contravariant functor, $Se$ takes colimits in $\text{Funct}_{\text{cont}}(C, D)$ to limits in $\text{Funct}_{\text{cont}}(D, C)$.

**Proof.** For a colimit diagram

$$(i \in I) \mapsto F_i, \quad \text{colim}_{i \in I} F_i = F,$$

in $\text{Funct}_{\text{cont}}(C, D)$, and $c \in C$, $d \in D^c$, we have

$$\mathcal{H}om(c, (\lim_{i \in I} Se(F_i))(d)) = \lim_{i \in I} \mathcal{H}om(c, Se(F_i)(d)) = \lim_{i \in I} \mathcal{H}om(d, F_i(c))^* =$$

$$= \left( \lim_{i \in I} \mathcal{H}om(d, F_i(c)) \right)^* = \mathcal{H}om(d, F(c))^* = \mathcal{H}om(c, Se(F)(d)),$$

as desired. \hfill \Box

1.2. The Serre functor.

1.2.1. We introduce the Serre endofunctor of $C$, denoted $Se_C$, by setting

$$Se_C := Se(\text{Id}_C).$$

We shall say that $C$ is Serre if $Se_C$ is an equivalence.

1.2.2. From the definitions we obtain:

**Lemma 1.2.3.** $(Se_C)^\vee \simeq Se_C^\vee$ as endofunctors of $C^\vee$.

**Corollary 1.2.4.** $C$ is Serre if and only if $C^\vee$ is.

1.2.5. We now claim:

**Proposition 1.2.6.** The functor $Se_C |_{C^\vee}$ is fully faithful if and only if $C$ is reflexive.

**Proof.** For $c, c' \in C^c$, we have:

$$\mathcal{H}om_C(Se_C(c), Se_C(c')) = \mathcal{H}om_C(c', Se_C(c))^* = \mathcal{H}om_C(c, c')^{**},$$

where the map

$$\mathcal{H}om_C(c, c') \to \mathcal{H}om_C(Se_C(c), Se_C(c'))$$

is the canonical map

$$\mathcal{H}om_C(c, c') \to \mathcal{H}om_C(c, c')^{**}.$$

\hfill \Box

**Corollary 1.2.7.** Assume that $C$ is reflexive.

(a) If $Se_C$ is proper, then it is fully faithful, and the right adjoint to $Se_C$ provides a continuous left inverse of $Se_C$.

(b) If $Se_C^\vee$ is proper, then $Se_C$ is fully faithful, and admits a left adjoint, which is also a right inverse of $Se_C$. 

Corollary 1.2.8. Assume that $C$ is reflexive and that $\text{Se}_C$ and $\text{Se}_{C^\vee}$ are both proper. Then $C$ is Serre.

1.3. Some examples.

1.3.1. Let $Y$ be an eventually coconnective derived scheme, and $C = \text{QCoh}(Y)$. It is easy to see that $\text{QCoh}(Y)$ is reflexive if and only if $Y$ is proper, which we will from now on assume.

Then the Serre functor on $Y$ is given by $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_Y$, where $\omega_Y$ is the dualizing object on $Y$.

From here it is clear that $\text{QCoh}(Y)$ is Serre if and only if $Y$ is Gorenstein, which by definition means that $\omega_Y$ is a shifted line bundle.

1.3.2. Let $Y$ be a smooth scheme and let $y \in Y$ be a point. Consider the category $C = \text{QCoh}(Y)_{\{y\}}$, which is the full subcategory of $\text{QCoh}(Y)$ that consists of objects set-theoretically supported at $y$.

It is easy to see that $\text{QCoh}(Y)_{\{y\}}$ is proper, and the Serre functor on it is given by $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_Y$.

The skyscraper $k_y \in \text{QCoh}(Y)_{\{y\}}$ defines a functor $(i_y)_*: \text{Vect} \to \text{QCoh}(Y)_{\{y\}}$. Note that we have

$$\text{Se}_{\text{QCoh}(Y)_{\{y\}}} \circ (i_y)_* \simeq (i_y)_* \circ \text{Se}_{\text{Vect}} \circ (- \otimes \omega_{Y,y}),$$

where $\omega_{Y,y}$ is the fiber of $\omega_Y$ at $y$, which equals $\Lambda^{\dim(Y)}(T^*_y(Y))[\dim(Y)]$.

1.3.3. Let $N$ be a unipotent algebraic group, and consider the category $C = \text{Rep}(N)$. This category is proper, and we claim the Serre functor on it is given by

$$\mathcal{F} \mapsto \mathcal{F} \otimes \ell_N,$$

where $\ell_N$ is the graded line $\Lambda^{\dim(n)}(n)[\dim(n)]$ (in particular, $\text{Rep}(N)$ is Serre).

Indeed, this follows from the fact that for $M_1, M_2 \in \text{Rep}(N)^c$, we have

$$\mathcal{H}\text{om}_{\text{Rep}(N)}(M_1, M_2) \simeq C^\bullet(n, M_1^\vee \otimes M_2),$$

and hence

$$\mathcal{H}\text{om}_{\text{Rep}(N)}(M_1, \text{Se}_{\text{Rep}(N)}(M_2)) = \mathcal{H}\text{om}_{\text{Rep}(N)}(M_2, M_1)^* \simeq C^\bullet(n, M_2^\vee \otimes M_1)^* \simeq C_\bullet(n, M_2^\vee \otimes M_2) \simeq C^\bullet(n, M_1^\vee \otimes M_2) \otimes \ell_N \simeq \mathcal{H}\text{om}_{\text{Rep}(N)}(M_1, M_2) \otimes \ell_N,$$

as required.

1.3.4. Let $K$ be a reductive group. Then the category $\text{Rep}(K)$ is proper, and it is easy to see that $\text{Se}_{\text{Rep}(K)}$ is canonically isomorphic to the identity functor.

1.4. The pseudo-identity functor.
1.4.1. We define the functor
\[ Ps : \text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(D, C)^{\text{op}} \]
to be the functor \( D_C^\vee \otimes D \), where we identify
\[ \text{Funct}_{\text{cont}}(C, D) \simeq C^\vee \otimes D \text{ and } \text{Funct}_{\text{cont}}(D, C) \simeq D^\vee \otimes C \simeq (C^\vee \otimes D)^\vee. \]
By construction, \( Ps \) preserves colimits. I.e., when viewed as a contravariant functor
\[ \text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(D, C) \]
it takes colimits to limits.

1.4.2. We define the endofunctor \( Ps \cdot \text{Id}_C \) of \( C \) by
\[ Ps \cdot \text{Id}_C := Ps(\text{Id}_C). \]
We shall say that \( C \) is Gorenstein if the functor \( Ps \cdot \text{Id}_C \) is an equivalence (see [Ga1, Sect. 5.4], where the origin of the terminology is explained).

1.4.3. The following is tautological from the definitions:

**Lemma 1.4.4.** \( Ps \cdot \text{Id}_C^{\vee} \simeq (Ps \cdot \text{Id}_C)^{\vee} \) as endo-functors of \( C^{\vee} \).

1.5. **Relationship between the Serre and pseudo-identity functors.**

1.5.1. The following observation will play a key role in this paper:

**Proposition 1.5.2.**
(a) There is natural transformation
\[ \{ F \mapsto \text{Se}_C \circ Ps(F) \circ \text{Se}_D \} \Rightarrow \{ F \mapsto \text{Se}(F) \}, \]
when both sides are viewed as contravariant functors \( \text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(D, C) \).

(b) Assume that \( \text{Se}_{C^{\vee}} \) and \( \text{Se}_D \) are proper, and suppose that for \( c, c' \in C^\vee \) and \( d, d' \in D^\vee \), the map
\[ \text{Hom}_C(c', c)^* \otimes \text{Hom}_D(d, d')^* \to (\text{Hom}_D(d, d') \otimes \text{Hom}_C(c', c))^* \]
is an isomorphism. Then the above natural transformation is an isomorphism.

**Remark 1.5.3.** Note that the second condition in (b) is satisfied when:

- Either \( C \) or \( D \) is proper.
- Either \( C \) or \( D \) is reflexive and for \( c, c' \in C^\vee \) and \( d, d' \in D^\vee \), the objects
  \[ \text{Hom}_C(c', c) \in \text{Vect} \text{ and } \text{Hom}_D(d, d') \in \text{Vect} \]
are either both eventually connective or connective (i.e., either both are in \( \text{Vect}^{> -\infty} \) or both are in \( \text{Vect}^{< \infty} \)).

**Proof.** Since \( \text{Se} \) sends colimits to limits, in order to construct the natural transformation in question, it is enough to do so after precomposition with the functor
\[ (C^\vee)^{\text{op}} \times D^\vee \to \text{Funct}_{\text{cont}}(C, D). \]

For \( c' \in C^\vee, \ d' \in D^\vee \), the corresponding functor \( F : C \to D \) is given by
\[ F(c) = d' \otimes \text{Hom}_C(c', c), \]
the functor \( Ps(F) \) is given by
\[ Ps(F)(d) = c' \otimes \text{Hom}_D(d', d). \]
Hence, the functor $G_1 := \text{Se}_C \circ \text{Ps}(F) \circ \text{Se}_D$ is determined by

$$\text{Hom}_C(c, G_1(d)) = \text{Hom}_C(c, \text{Se}_C(c')) \otimes \text{Hom}_D(d', \text{Se}_D(d)) \simeq \text{Hom}_C(c', c^* \otimes \text{Hom}_D(d, d')^*, \ c \in C^c, d \in D^c.$$  

The functor $G_2 := \text{Se}(F)$ is determined by

$$\text{Hom}_C(c, G_2(d)) = \text{Hom}_D(d, d') \otimes \text{Hom}_C(c', c) \simeq \text{Hom}_D(d, d') \otimes \text{Hom}_C(c', c)^*, \ c \in C^c, d \in D^c.$$  

The required natural transformation is now given by

$$(1.2) \quad \text{Hom}_C(c', c)^* \otimes \text{Hom}_D(d, d')^* \to (\text{Hom}_D(d, d') \otimes \text{Hom}_C(c', c))^*.$$  

Let us now assume that (1.2) is an isomorphism and that $\text{Se}_{C^v}$ and $\text{Se}_D$ are proper. Let us show that the natural transformation

$$\text{Se}_C \circ \text{Ps}(F) \circ \text{Se}_D \to \text{Se}(F)$$

is an isomorphism for any $F \in \text{Funct}_{\text{cont}}(C, D)$. For that it suffices to show that the functor

$$F \mapsto \text{Se}_C \circ \text{Ps}(F) \circ \text{Se}_D$$

takes colimits in $\text{Funct}_{\text{cont}}(C, D)$ to limits in $\text{Funct}_{\text{cont}}(D, C)$.

It suffices to show that composition with $\text{Se}_C$ and pre-composition with $\text{Se}_D$ preserve limits. This follows from the following lemma:

**Lemma 1.5.4.** Let $C, D, E$ be compactly generated categories.

(a) If $F \in \text{Funct}_{\text{cont}}(E, C)$ admits a continuous right adjoint (equivalently, $F$ is proper), then the functor $\text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(E, D)$ given by precomposition with $F$ preserves limits.

(b) If $F \in \text{Funct}_{\text{cont}}(D, E)$ admits a left adjoint (equivalently, $F^\vee$ is proper), then the functor $\text{Funct}_{\text{cont}}(C, D) \to \text{Funct}_{\text{cont}}(C, E)$ given by postcomposition with $F$ preserves limits.

\[\square\]

1.5.5. Summarizing, we obtain the following result that we will use extensively:

**Corollary 1.5.6.** Let $C$ be reflexive, and such that for $c, c', c_1, c_1' \in C^c$ the map

$$\text{Hom}_C(c', c)^* \otimes \text{Hom}_C(c_1, c_1')^* \to (\text{Hom}_C(c_1, c_1') \otimes \text{Hom}_C(c', c))^*$$

is an isomorphism. Assume also that $\text{Se}_C$ and $\text{Se}_{C^v}$ are proper. Then the functors $\text{Se}_C$ and $\text{Ps-Id}_C$ are mutually inverse. In particular, $C$ is Serre and Gorenstein.

*Proof.* By Proposition 1.5.2 we have an isomorphism $\text{Se}_C \circ \text{Ps-Id}_C \circ \text{Se}_C \simeq \text{Se}_C$. By Corollary 1.5.4 (a), the functor $\text{Se}_C$ is an equivalence. \[\square\]

**Remark 1.5.7.** By Remark 1.5.3 the first two conditions in the corollary is satisfied if $C$ is proper.

More generally, it is satisfied if $C$ is reflexive and for $c, c' \in C^c$, the object $\text{Hom}_C(c', c) \in \text{Vect}$ is eventually coconnective, i.e., lies in $\text{Vect}^{\geq -\infty}$.

For example, the latter happens if $C$ carries a $t$-structure and all compact objects in $C$ are bounded, i.e., lie in $C^{\geq n_1, \leq n_2}$ for some $n_1 \leq n_2$. 

2. Serre and Gorenstein categories in geometric representation theory

2.1. Examples arising from D-module categories.

2.1.1. Let \( Y \) be a QCA algebraic stack (see [DrGa1, Definition 1.1.8] for what this means), and let \( \lambda \) be a twisting on \( Y \) (see [GR1, Sect. 6]). Consider the category \( C := \text{D-mod}_{\lambda}(Y) \).

Recall that the dual category \( \text{D-mod}_{\lambda}(Y)^{\vee} \) identifies canonically with \( \text{D-mod}_{-\lambda}(Y) \). Under this identification, the evaluation map

\[
\text{D-mod}_{\lambda}(Y) \otimes \text{D-mod}_{-\lambda}(Y) \to \text{Vect}
\]

is given by

\[
\text{D-mod}_{\lambda}(Y) \otimes \text{D-mod}_{-\lambda}(Y) \simeq \text{D-mod}_{-\lambda}(Y \times Y) \xrightarrow{\Delta_Y} \text{D-mod}(Y) \xrightarrow{(p_Y)_*} \text{Vect},
\]

where \( p_Y \) is the projection \( Y \to \text{pt} \), and for a morphism \( f \) we denote by \( f_* \) the renormalized pushforward of [DrGa1, Sect. 9.3].

The unit map is given by

\[
\text{Vect} \xrightarrow{k \mapsto \omega_Y} \text{D-mod}(Y) \xrightarrow{(\Delta_Y)_*} \text{D-mod}_{-\lambda}(Y \times Y) \simeq \text{D-mod}_{\lambda}(Y) \otimes \text{D-mod}_{-\lambda}(Y).
\]

We denote by

\[
\mathbb{D}_y^\text{Verdier} : \text{D-mod}_{\lambda}(Y)^{\vee} \to \text{D-mod}_{-\lambda}(Y)^{\vee}
\]

the corresponding contravariant dualization functor, and also its ind-extension

\[
\mathbb{D}_y^\text{Verdier} : \text{D-mod}_{\lambda}(Y) \to (\text{D-mod}_{-\lambda}(Y))^\text{op}.
\]

2.1.2. Thus, objects of

\[
\text{D-mod}_{-\lambda}(Y) \otimes \text{D-mod}_{\lambda}(Y) \simeq \text{D-mod}_{-\lambda,\lambda}(Y \times Y)
\]

define continuous endofunctors of \( \text{D-mod}_{\lambda}(Y) \).

Following [CGY, Sect. 3.1], we introduce the functor

\[
\text{Ps-Id}_y : \text{D-mod}_{\lambda}(Y) \to \text{D-mod}_{\lambda}(Y)
\]

to be given by the object

\[
(\Delta_Y)_!(k_Y) \in \text{D-mod}_{-\lambda,\lambda}(Y \times Y),
\]

where

\[
k_Y := \mathbb{D}_y^\text{Verdier}(\omega_Y).
\]

The following is tautological from the definitions:

**Lemma 2.1.3.** The functor \( \text{Ps-Id}_y \) identifies with the functor \( \text{Ps-Id}_{\text{D-mod}_{\lambda}(Y)} \).

In other words, the functor \( \text{Ps-Id}_C \) is the abstract version of the geometrically defined functor \( \text{Ps-Id}_y \).

2.1.4. We will prove:

**Theorem 2.1.5.** Assume that \( Y \) has a finite number of isomorphism classes of \( k \)-valued points. Then \( \text{D-mod}_{\lambda}(Y) \) is proper, Serre and Gorenstein, and the functors \( \text{Se}_{\text{D-mod}_{\lambda}(Y)} \) and \( \text{Ps-Id}_y \) are mutually inverse.

2.2. **Proof of Theorem 2.1.5.** The proof will rely on material from the paper [DrGa1].
2.2.1. We will use the following lemma, proved below:

**Lemma 2.2.2.** Let \( Y \) have a finite number of isomorphism classes of \( k \)-valued points. Then an object \( F \in \text{D-mod}_\lambda(Y) \) is compact if and only if for every coherent \( F' \in \text{D-mod}_\lambda(Y) \), the object \( \text{Hom}_{\text{D-mod}_\lambda}(F', F) \in \text{Vect} \) is compact.

We now proceed with the proof of Theorem 2.1.5. We will verify that the conditions of Corollary 1.5.6 hold.

2.2.3. First, we show that \( \text{D-mod}_\lambda(Y) \) is proper. Since every compact object of \( \text{D-mod}_\lambda(Y) \) is coherent, this follows immediately from the “only if” direction in Lemma 2.2.2.

2.2.4. Let us now show that the functors \( \text{Se}_{\text{D-mod}_\lambda}(Y) \) and \( \text{Se}_{\text{D-mod}_\lambda}(Y) \vee \cong \text{Se}_{\text{D-mod}_\lambda}(Y) \) are proper. By symmetry, it suffices to consider the former case.

By the “if” direction in Lemma 2.2.2, it suffices to show that for \( F \in \text{D-mod}_\lambda(Y) \) and \( F' \in \text{D-mod}_\lambda(Y) \), the object \( \text{Hom}_{\text{D-mod}_\lambda}(F', \text{Se}_{\text{D-mod}_\lambda}(Y)(F)) \in \text{Vect} \) is compact.

Since \( \text{Hom}_{\text{D-mod}_\lambda}(F', \text{Se}_{\text{D-mod}_\lambda}(Y)(F)) \) is the dual of \( \text{Hom}_{\text{D-mod}_\lambda}(F, F') \), so it suffices to show that the latter is compact.

Recall now from [DrGa1, Corollary 8.4.2] that Verdier duality on \( Y \), which we can regard as a contravariant equivalence

\[
\mathcal{D}_Y \text{Verdier} : \text{D-mod}_\lambda(Y)^c \to \text{D-mod}_{\lambda}(Y)^c,
\]

extends to a contravariant equivalence

\[
\mathcal{D}_Y \text{Verdier} : \text{D-mod}_\lambda(Y)^{\text{coh}} \to \text{D-mod}_{\lambda}(Y)^{\text{coh}}.
\]

Hence, using

\[
\text{Hom}_{\text{D-mod}_\lambda}(F, F') \cong \text{Hom}_{\text{D-mod}_\lambda}(\mathcal{D}_Y \text{Verdier}(F'), \mathcal{D}_Y \text{Verdier}(F)),
\]

the required assertion follows from the “only if” direction in Lemma 2.2.2. □[Theorem 2.1.5]

2.2.5. **Proof of Lemma 2.2.2, the “only if” direction.** By [DrGa1] Proposition 9.2.3, compact objects in \( \text{D-mod}_\lambda(Y) \) are characterized by the following two properties: these are objects that are (i) coherent, and (ii) safe (see [DrGa1] Definition 9.2.1) for what the term “safe” refers to).

Now, the condition that \( Y \) has a finite number of isomorphism classes of \( k \)-valued points implies that all coherent objects in \( \text{D-mod}_\lambda(Y) \) are holonomic. This readily implies that for two such objects \( F, F' \),

\[
\text{Hom}_{\text{D-mod}_\lambda}(Y)(F', F)
\]

is finite-dimensional in each degree.

Now, by [DrGa1] Lemma 9.4.4(a), if \( F' \) is coherent and \( F \) is safe and bounded, then

\[
\text{Hom}_{\text{D-mod}_\lambda}(Y)(F', F) \cong \Gamma_{dR}(Y, \mathcal{D}_Y \text{Verdier}(F') \otimes F)
\]

is concentrated in finitely many cohomological degrees.
2.2.6. **Proof of Lemma 2.2.2, the “if” direction.** Write $\mathcal{Y}$ is a union of locally closed substacks $\mathcal{Y}_i$, where each $\mathcal{Y}_i$ has a unique isomorphism class of $k$-valued points, i.e., $\mathcal{Y}_i$ is of the form $\text{pt}/H_i$ for an affine algebraic group $H_i$. Denote by $j_i$ the locally closed embedding $\mathcal{Y}_i \hookrightarrow \mathcal{Y}$.

Since every object in $D_{\text{mod}}(\mathcal{Y}_i)$ is holonomic, the functor $(j_i)_!$, left adjoint to $j_i^!$, is well-defined. Therefore, using the Cousin resolution, we obtain that in order to show that a given object $\mathcal{F} \in D_{\text{mod}}(\mathcal{Y})$ is compact, it is sufficient to show that

$$(j_i)_!(\mathcal{F}) \in D_{\text{mod}}(\mathcal{Y}_i)$$

is compact for every $i$.

By adjunction, for any $\mathcal{F}_i \in D_{\text{mod}}(\mathcal{Y}_i)$, we have

$$\mathcal{H}om_{D_{\text{mod}}}(\mathcal{Y})(j_i)_!(\mathcal{F}_i), \mathcal{F}) \simeq \mathcal{H}om_{D_{\text{mod}}}(\mathcal{Y}_i)(\mathcal{F}_i, j_i^!(\mathcal{F})).$$

Taking $\mathcal{F}_i$ to be coherent, we thus reduce the assertion of the lemma to the case when $\mathcal{Y}$ is of the form $\text{pt}/H$, which we will now assume.

2.2.7. **Proof in the quotient case.** Consider the category $D_{\text{mod}}(\text{pt}/H)$, $\lambda \in \mathfrak{h}^*$. Note that it is zero unless $\lambda$ integrates to a character of $H$, and in the latter case it is equivalent to the untwisted category $D_{\text{mod}}(\text{pt}/H)$.

Thus, we can consider the case of the trivial twisting. We claim that in order to test compactness, it is sufficient to take $\mathcal{F}$ to be just one object, namely, $k_{\text{pt}/H}$.

Indeed, let $\pi$ denote the projection $\text{pt} \rightarrow \text{pt}/H$. The category $D_{\text{mod}}(\text{pt}/H)$ is compactly generated by the object $\pi(k)$, which is a finite successive extension of shifted copies of $k_{\text{pt}/H}$.

Hence, if $\mathcal{H}om_{D_{\text{mod}}(\text{pt}/H)}(k_{\text{pt}/H}, \mathcal{F})$ is compact, then so is $\mathcal{H}om_{D_{\text{mod}}(\text{pt}/H)}(\pi(k), \mathcal{F})$, and the latter means that $\mathcal{F}$ is coherent, and in particular bounded.

Now, according to [DrGa1, Proposition 10.4.7], a bounded object of $D_{\text{mod}}(\text{pt}/H)$ is safe if and only if $\mathcal{H}om_{D_{\text{mod}}(\text{pt}/H)}(k_{\text{pt}/H}, \mathcal{F})$ is concentrated in finitely many cohomological degrees.

Thus, we obtain that $\mathcal{F}$ is coherent and safe, and hence compact.

$\square$[Lemma 2.2.2]

2.3. **A variant: monodromic situation.**

2.3.1. We will now consider a certain variant of Theorem 2.1.5. Let $\pi : \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ be a torsor with respect to a torus $T$, and let $\lambda$ be a character of $t^*$. (Note that such a datum defines a twisting on $\mathcal{Y}$.)

We will now consider the full subcategory

$$D_{\text{mod}}(\tilde{\mathcal{Y}})^{\lambda-\text{mon}} \subset D_{\text{mod}}(\tilde{\mathcal{Y}}),$$

consisting of $\lambda$-monodromic objects.

Here is one of the possible definitions. Consider first the category

$$D_{\text{mod}}(\tilde{\mathcal{Y}})^{T-\text{weak}}$$

of weakly $T$-equivariant D-modules. This category admits a homomorphism from $\text{Sym}(t)$ into its center (called “obstruction to equivariance”). Hence, we can view $D_{\text{mod}}(\tilde{\mathcal{Y}})^{T-\text{weak}}$ as acted on by the monoidal category $Q\text{Coh}(t^*)$.

We set

$$D_{\text{mod}}(\tilde{\mathcal{Y}})^{\lambda-\text{mon}} := D_{\text{mod}}(\tilde{\mathcal{Y}})^{T-\text{weak}} \otimes_{Q\text{Coh}(t^*)} Q\text{Coh}(t^*)^{(\lambda)}.$$
2.3.2. The forgetful functor

\[(\hat{i}_\lambda)_* : D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}} \to D\text{-mod}(\tilde{Y})^{T\text{-weak}}\]

admits a continuous right adjoint \((\hat{i}_\lambda)^!\), obtained using the

\[D\text{-mod}(\tilde{Y})^{T\text{-weak}} \otimes_{\text{QCoh}(t^*)} -\]

base change from the corresponding adjunction

\[\hat{i}_* : \text{QCoh}(t^*_\{\lambda\}) \rightleftarrows \text{QCoh}(t^*) : (\hat{i})^!\].

Since the unit of the adjunction

\[\text{Id} \to (\hat{i}_\lambda)^! \circ (\hat{i}_\lambda)_*\]

is an isomorphism for \(\text{QCoh}(t^*_\{\lambda\})\), it is also an isomorphism for \(D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\). In particular, the functor \((2.2)\) is fully faithful.

**Lemma 2.3.3.** The functor

\[D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}} \xrightarrow{(\hat{i}_\lambda)_*} D\text{-mod}(\tilde{Y})^{T\text{-weak}} \xrightarrow{\text{oblv}_{T\text{-weak}}} D\text{-mod}(\tilde{Y})\]

is fully faithful.

**Proof.** For \(\mathcal{F}_1, \mathcal{F}_2 \in D\text{-mod}(\tilde{Y})^{T\text{-weak}}\), we have

\[\mathcal{H}om_{D\text{-mod}(\tilde{Y})^{T\text{-weak}}}(\text{oblv}_{T\text{-weak}}(\mathcal{F}_1), \text{oblv}_{T\text{-weak}}(\mathcal{F}_2)) \simeq \mathcal{H}om_{D\text{-mod}(\tilde{Y})^{T\text{-weak}}}((\mathcal{F}_1, R_T \otimes \mathcal{F}_2),\]

where \(R_T\) is the regular representation of \(T\).

Suppose now that \(\mathcal{F}_1 \in D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\). Then

\[\mathcal{H}om_{D\text{-mod}(\tilde{Y})^{T\text{-weak}}}((\mathcal{F}_1, R_T \otimes \mathcal{F}_2) \simeq \mathcal{H}om_{D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}}((\mathcal{F}_1, \bigoplus_\mu(k^\mu \otimes \mathcal{F}_2)),\]

where \(\mu\) runs through the set of characters of \(T\), and \(k^\mu\) denotes the corresponding object of \(\text{Rep}(T)\).

Now, the assertion of the lemma follows from the fact that if \(\mathcal{F}_2 \in D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\), all the terms

\[(\hat{i}_\lambda)^!(k^\mu \otimes \mathcal{F}_2)\]

with \(\mu \neq 0\) vanish; indeed each \(k^\mu \otimes \mathcal{F}_2\) belongs to \(D\text{-mod}(\tilde{Y})^{(\lambda+\mu)\text{-mon}}\). \(\square\)

2.3.4. Note that the category \(D\text{-mod}_\lambda(Y)\) is recovered as

\[D\text{-mod}_\lambda(Y) \simeq D\text{-mod}(\tilde{Y})^{T\text{-weak}} \otimes_{\text{QCoh}(t^*)} \text{Vect} \simeq D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}} \otimes_{\text{QCoh}(t^*)} \text{Vect},\]

where \(\text{QCoh}(t^*) \to \text{Vect}\) is given by taking the fiber at \(\lambda\).

We have the tautological forgetful functor

\[(i_\lambda)_* : D\text{-mod}_\lambda(Y) \to D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}},\]

which admits a left adjoint \((i_\lambda)^*\) and a continuous right adjoint \((i_\lambda)^!\). These functors are obtained by base-changing the corresponding functors for

\[\text{Vect} \xrightarrow{(i_\lambda)_*} \text{QCoh}(t^*)_{\{\lambda\}}.\]
We have
\[(i_\lambda)^* \simeq (- \otimes \ell_\lambda) \circ (i_\lambda)^! \]
where \(\ell_\lambda := \Lambda^{\dim(t)}(t)[\dim(t)]\).

The functor \((i_\lambda)^!\) is conservative; hence the essential image of \((i_\lambda)_*\) generates \(D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\).

2.3.5. With the above preparations, we claim:

**Theorem 2.3.6.** Let \(\tilde{Y}\) as in Theorem 2.3.3. Then the category \(D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\) is proper, Serre and Gorenstein. Moreover, we have
\[(2.4) \quad \text{Se}_{D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}} \circ (i_\lambda)_* \simeq (- \otimes \ell_\lambda) \circ (i_\lambda)_* \circ \text{Se}_{D\text{-mod}_\lambda(\tilde{Y})}.\]

**Proof.** We will verify the conditions of Corollary 1.5.6. Since the essential image of \((i_\lambda)_*\) generates \(D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\), we can consider compact objects of the form
\[(i_\lambda)_*(\mathcal{G}) \quad \mathcal{G} \in D\text{-mod}_\lambda(\tilde{Y})^c.\]

For compact \(\mathcal{F} \in (D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}})^c\) and \(\mathcal{G} \in D\text{-mod}_\lambda(\tilde{Y})^c\), we have
\[\mathcal{K}\text{om}_{D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}}(\mathcal{F}, (i_\lambda)_*(\mathcal{G})) \simeq \mathcal{K}\text{om}_{D\text{-mod}_\lambda(\tilde{Y})}((i_\lambda)^*\mathcal{F}, \mathcal{G}),\]
and the latter is compact since \((i_\lambda)^*\) is proper and \(D\text{-mod}_\lambda(\tilde{Y})\) is proper.

Hence, the properness of \(D\text{-mod}_\lambda(\tilde{Y})\) implies that of \(D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\).

Let us show that \(\text{Se}_{D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}}\) and \(\text{Se}_{(D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}})^{\vee}}\simeq \text{Se}_{D\text{-mod}(\tilde{Y})^{-\lambda\text{-mon}}}\) are proper. By symmetry, it suffices to consider the former case.

It is enough to show that for \(\mathcal{F} \in D\text{-mod}_\lambda(\tilde{Y})^c\), the object
\[\text{Se}_{D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}} \circ (i_\lambda)_* \in D\text{-mod}(\tilde{Y})^{\lambda\text{-mon}}\]
is compact, and for that it is sufficient to verify 2.3.1. However, the latter follows from 2.3.1 and Lemma 2.6.5(a) below. \(\Box\)

2.4. **An interlude on \(g\)-modules.** In this subsection we supply some background on the self-duality of the category of \(g\)-modules, where \(g\) is a Lie algebra. This material will be needed for the proofs of our main results.

2.4.1. In this subsection we let \(G\) be a reductive group and \(\chi\) a character of \(Z(g)\). We will consider two categories associated with \(\chi\). One is
\[\mathfrak{g}\text{-mod}_{\chi} \simeq \mathfrak{g}\text{-mod} \otimes_{Z(g)\text{-mod}} \text{Vect},\]
where \(Z(g)\text{-mod} \to \text{Vect}\) is given by \(\otimes k_\chi\), where \(k_\chi \in Z(g)\text{-mod}\) is the skyscraper at \(\chi\).

The other is
\[\mathfrak{g}\text{-mod}_{(\chi)} := \mathfrak{g}\text{-mod} \otimes_{Z(g)\text{-mod}} Z(g)\text{-mod}_{(\chi)},\]
where \(Z(g)\text{-mod}_{(\chi)} \subset Z(g)\text{-mod}\) is the full subcategory of objects with set-theoretic support at \(\chi \in \text{Spec}(Z(g))\).

As in Sect. 2.3.3, we have the obvious forgetful functor
\[(i_\chi)_* : \mathfrak{g}\text{-mod}_{\chi} \to \mathfrak{g}\text{-mod}_{(\chi)},\]
which admits a left and a continuous right adjoints, denoted \((i_\chi)^*\) and \((i_\chi)^!\), respectively.
We have
\[(2.5) \quad (i_\chi)^* \simeq (- \otimes \ell_\chi) \circ (i_\chi)^!,\]
where \(\ell_\chi := \Lambda^{\text{dim}(t)}(T^*_\chi(Z(g)))[\text{dim}(t)].\)

The functor \((i_\chi)^!\) is conservative; hence the essential image of \((i_\chi)_*\) generates \(\mathfrak{g}\)-mod_{(\chi)}.  

2.4.2. The categories \(\mathfrak{g}\)-mod_\chi and \(\mathfrak{g}\)-mod_{\{\chi\}} both carry a t-structure.  

Since the algebra \(U(g)\) has a finite cohomological dimension, so does the full subcategory \(\mathfrak{g}\)-mod_{(\chi)}. Hence, the t-structure on \(\mathfrak{g}\)-mod_{(\chi)} gives rise to one \(\mathfrak{g}\)-mod_{c,\chi}.

Moreover, we have \(\mathfrak{g}\)-mod_{c,\chi} = \mathfrak{g}\)-mod_{\{\chi\}}, where the RHS is the full subcategory of \(\mathfrak{g}\)-mod_{\chi} that consists of objects that have non-vanishing cohomologies in finitely many degrees and all such cohomologies being finitely generated as \(U(g)\)-modules.

We note, however, that if \(\chi\) is irregular, then the algebra \(U(g)_\chi\) has an infinite cohomological dimension. In particular, the t-structure on \(\mathfrak{g}\)-mod_\chi does not restrict to a t-structure on \(\mathfrak{g}\)-mod_{c,\chi}.

We still have the inclusion \(\mathfrak{g}\)-mod_{c,\chi} \subset \mathfrak{g}\)-mod_{f,\chi}, but it is no longer an equality.

Remark 2.4.3. The latter circumstance can be a source of (un stabl est, but yet annoying) difficulties. For this reason, we sometimes first prove results for \(\mathfrak{g}\)-mod_{\{\chi\}}, and then bootstrap them for \(\mathfrak{g}\)-mod_\chi.

2.4.4. The anti-involution \(\xi \mapsto -\xi\) of \(U(g)\) induces an involution on \(Z(g)\); we denote it by \(\chi \mapsto -\chi\); in particular, the algebra \(U(g)_\chi\) canonically identifies with \((U(g)_\chi)^{\text{op}}\).

We have a canonical identification
\[(2.6) \quad \mathfrak{g}\)-mod^\vee \simeq \mathfrak{g}\)-mod,
where the evaluation functor
\[(2.7) \quad (-, -)_g : \mathfrak{g}\)-mod \otimes \mathfrak{g}\)-mod \to \text{Vect}
\[
\text{is given by } M_1, M_2 \mapsto M_2 \otimes_{U(g)} M_1.
\]

The identification \(2.6\) induces an identification
\[(2.8) \quad \mathfrak{g}\)-mod^\vee_\chi \simeq \mathfrak{g}\)-mod_{-\chi},
where the evaluation functor
\[
\mathfrak{g}\)-mod_{(\chi)} \otimes \mathfrak{g}\)-mod_{-\chi} \to \text{Vect}
\[
is obtained by precomposing \(2.7\) with the tautological embeddings.

Let \(D^\text{can}_g\) denote the corresponding contravariant equivalences
\[
\mathfrak{g}\)-mod^c \simeq \mathfrak{g}\)-mod^f and \(\mathfrak{g}\)-mod^{f,\chi}_\chi \simeq \mathfrak{g}\)-mod^{f,\chi}_{-\chi}.
\]

In addition, we have a canonical identification
\[(2.9) \quad (\mathfrak{g}\)-mod_\chi)^\vee \simeq \mathfrak{g}\)-mod_{-\chi},
where the evaluation functor
\[
(-, -)_{g,\chi} : \mathfrak{g}\)-mod_\chi \otimes \mathfrak{g}\)-mod_{-\chi} \to \text{Vect}
is given by $M_1, M_2 \mapsto M_2 \otimes_{U(g)} M_1$.

Let $\mathbb{D}^{\text{can}}_{g, \chi}$ denote the corresponding contravariant equivalence

$$\mathfrak{g}\text{-mod}^c_{\chi} \simeq \mathfrak{g}\text{-mod}_{-\chi}^c.$$

Note that we have a commutative diagram

$$(\mathfrak{g}\text{-mod}_{\chi})^\vee \xrightarrow{\sim} \mathfrak{g}\text{-mod}_{-\chi}$$

$$(i_\chi)^\vee \downarrow \quad \downarrow (i_{-\chi})_*$$

$$\mathfrak{g}\text{-mod}^c_{(\chi)} \xrightarrow{} \mathfrak{g}\text{-mod}_{(-\chi)}^c.$$

In other words, we have a canonical isomorphism of contravariant functors

$$(- \otimes \ell_\chi) \circ \mathbb{D}^{\text{can}}_{g} \circ (i_\chi)_* \simeq (i_{-\chi})_* \circ \mathbb{D}^{\text{can}}_{g, \chi}, \quad \mathfrak{g}\text{-mod}^c_{\chi} \cong \mathfrak{g}\text{-mod}^c_{(-\chi)}.$$

2.4.5. Let $H \subset G$ be any subgroup. Recall that if $C$ is a dualizable category acted on by $H$, then we have a canonical identification

$$(2.10) \quad (C^\vee)^H \simeq (C^H)^\vee.$$

The corresponding pairing

$$\langle - , - \rangle_{C, H} : (C^\vee)^H \otimes C^H \rightarrow \text{Vect}$$

is the composition

$$(C^\vee)^H \otimes C^H \rightarrow (C^\vee \otimes C)^H \rightarrow \text{Vect}^H \simeq \text{D-mod}(\text{pt} / H) \rightarrow \text{Vect},$$

where the second arrow is induced by the pairing

$$\langle - , - \rangle : C^\vee \otimes C \rightarrow \text{Vect},$$

and the third arrow is the functor of renormalized de Rham cohomology (see [DrGa1, Sect. 9.1]), i.e., the renormalized direct image functor (see [DrGa1, Sect. 9.3]) along $\text{pt} / H \rightarrow \text{pt}$.

With respect to the identification $(2.10)$, the functor dual to

$$\text{obl}v_H : C^H \rightarrow C$$

is the functor

$$\text{Av}_H^* : C^\vee \rightarrow (C^\vee)^H,$$

and vice versa.

2.4.6. Thus, the identifications $(2.6)$, $(2.8)$ and $(2.9)$ induce the identifications

$$(2.11) \quad (\mathfrak{g}\text{-mod}^H)^\vee \simeq \mathfrak{g}\text{-mod}^H,$$

$$(2.12) \quad (\mathfrak{g}\text{-mod}^H_{(\chi)})^\vee \simeq \mathfrak{g}\text{-mod}^H_{(-\chi)},$$

$$(2.13) \quad (\mathfrak{g}\text{-mod}^H_{\chi})^\vee \simeq \mathfrak{g}\text{-mod}^H_{-\chi}.$$

We will denote the corresponding pairings as follows:

$$\langle - , - \rangle_{g, H} : \mathfrak{g}\text{-mod}^H \otimes \mathfrak{g}\text{-mod}^H \rightarrow \text{Vect},$$

$$\langle - , - \rangle_{g, H} : \mathfrak{g}\text{-mod}^H_{(\chi)} \otimes \mathfrak{g}\text{-mod}^H_{(-\chi)} \rightarrow \text{Vect},$$

$$\langle - , - \rangle_{g, \chi, H} : \mathfrak{g}\text{-mod}^H_{\chi} \otimes \mathfrak{g}\text{-mod}^H_{-\chi} \rightarrow \text{Vect}.$$
We keep the same notations for the corresponding contravariant equivalences
\[
\mathbb{D}^{\text{can}}_g : (\mathfrak{g}\text{-mod}^H)^c \simeq (\mathfrak{g}\text{-mod}^H)^c^c \quad \text{and} \quad \mathbb{D}^{\text{can}}_{g, \chi} : (\mathfrak{g}\text{-mod}^H)^c_{(\chi)} \simeq (\mathfrak{g}\text{-mod}^H)^c_{(\chi)}^c,
\]
and
\[
\mathbb{D}^{\text{can}}_{g, \chi} : (\mathfrak{g}\text{-mod}^H)^c_{(\chi)} \simeq (\mathfrak{g}\text{-mod}^H)^c_{(\chi)}.
\]

We have:
\[
\text{oblv}_H \circ \mathbb{D}^{\text{can}}_g \simeq \mathbb{D}^{\text{can}}_g \circ \text{oblv}_H \quad \text{and} \quad \text{oblv}_H \circ \mathbb{D}^{\text{can}}_{g, \chi} \simeq \mathbb{D}^{\text{can}}_{g, \chi} \circ \text{oblv}_H.
\]

The functors \((i_{\chi})^*, (i_{\chi})_*, (i_{\chi})!\) induce functors between the corresponding equivariant categories, and the latter are compatible with the corresponding functors \(\text{oblv}_H\) and \(\text{Av}_H^*\).

2.5. A reminder on Localization Theory.

2.5.1. Let \(\lambda\) be a character of \(\mathfrak{t}\) that corresponds to \(\chi\) under the Harish-Chandra map. To \(\lambda\) we assign a TDO \(D_\lambda\) on the flag variety \(X\) of \(G\).

NB: Unlike [CGY], we do not apply the \(\rho\)-shift when we assign \(D_\lambda\) to \(\lambda\). In particular, \(\lambda = 0\) corresponds to the untwisted \(D\).

Consider the functor
\[
\Gamma : D\text{-mod}_\lambda(X) \to \mathfrak{g}\text{-mod}_\chi.
\]

By [BH], the functor \(\Gamma\) admits a fully faithful left adjoint, denoted \(\text{Loc}\); both these functors are compatible with the action of \(G\).

The functors \(\text{Loc}\) and \(\Gamma\) define functors between the categories
\[
\mathfrak{g}\text{-mod}^H\chi \quad \text{and} \quad D\text{-mod}_\lambda(H\setminus X) \simeq D\text{-mod}_\lambda(X)^H
\]
with the same adjunction properties; these functors are compatible with the induction and forgetful functors \(\text{oblv}_H\) and \(\text{Av}_H^*\).

2.5.2. The identifications
\[
D\text{-mod}_\lambda(X)^\vee \simeq D\text{-mod}_{-\lambda}(X) \quad \text{and} \quad (\mathfrak{g}\text{-mod}_\chi)^\vee \simeq \mathfrak{g}\text{-mod}_{-\chi}
\]
are compatible as follows. First, we note that if \(\lambda\) corresponds to \(\chi\), then \(-\lambda - 2\rho\) corresponds to \(-\chi\). Now, the functor
\[
\Gamma^\vee : (\mathfrak{g}\text{-mod}_\chi)^\vee \to D\text{-mod}_\chi(X)^\vee,
\]
dual to (2.14), identifies canonically with
\[
\mathfrak{g}\text{-mod}_{-\chi} \xrightarrow{\text{Loc}} D\text{-mod}_{-\lambda - 2\rho}(X) \xrightarrow{-\otimes \omega_X^{-1}} D\text{-mod}_{-\lambda}(X),
\]
where \(\omega_X\) is the dualizing complex on \(X\), and we use the fact that \(\omega_X \simeq \mathcal{O}(-2\rho)[\dim(X)]\).

In other words, we have an isomorphism of contravariant functors
\[
\text{Loc} \circ \mathbb{D}^{\text{can}}_{g, \chi} \simeq (\ominus \mathcal{O}(-2\rho))[\dim(X)] \circ \mathbb{D}^{\text{Verdier}}_X \circ \text{Loc}, \quad (\mathfrak{g}\text{-mod}_\chi)^c \Rightarrow D\text{-mod}_{-\lambda - 2\rho}(X)^c.
\]

Similar identifications pass on to the corresponding \(H\)-equivariant categories.
2.5.3. We will now consider the following variant of the adjunction \((\text{Loc}, \Gamma)\). Namely, we consider the base affine space

\[ \tilde{X} \to X, \]

and for a given \(\lambda\) we consider the corresponding category

\[ \text{D-mod}(\tilde{X})^{\lambda\text{-mon}}. \]

Taking global sections on \(\tilde{X}\), and then taking \(T\)-invariants, we obtain a functor

\[ \Gamma' : \text{D-mod}(\tilde{X})^{\lambda\text{-mon}} \to \mathfrak{g}\text{-mod}_{\{\chi\}}. \]

We have a commutative diagram

\[
\begin{array}{ccc}
\text{D-mod}_\lambda(X) & \xrightarrow{(i_x)_*} & \text{D-mod}(\tilde{X})^{\lambda\text{-mon}} \\
\uparrow \Gamma & & \downarrow \Gamma' \\
\mathfrak{g}\text{-mod}_\chi & \xrightarrow{(i_{\chi})_*} & \mathfrak{g}\text{-mod}_{\{\chi\}},
\end{array}
\]

The main advantage of the functor \(\Gamma'\) (unlike that of \(\Gamma\) of (2.14)) is given by the following lemma:

**Lemma 2.5.4.** The functor \(\Gamma'\) sends compact objects to compact ones.

*Proof.* Follows from the fact that the functor \(\Gamma\) of (2.14) sends \(\text{D-mod}_\lambda(X)^c\) to \(\mathfrak{g}\text{-mod}_{\{\chi\}}^f\). \(\Box\)

2.5.5. The functor \(\Gamma'\) admits a left adjoint, denoted \(\text{Loc}'\), but the latter is no longer fully faithful, see Sect. 2.5.6.

It follows from Lemma 2.5.4 that the functor \(\Gamma'\) admits also a continuous right adjoint, denoted \(\text{coLoc}'\).

One easily shows that the functor \(\text{coLoc}'\) is also compatible with the \(G\)-actions. In particular, we have the functors \(\text{Loc}', \Gamma', \text{coLoc}'\) between the corresponding \(H\)-equivariant categories, compatible with the functors \(\text{obl}_{\text{Av}_H}\) and \(\text{Av}_{\text{Av}_H}^*\).

Since

\[ \text{coLoc}' : \mathfrak{g}\text{-mod}_H^\leftarrow \to \text{D-mod}(H\setminus \tilde{X})^{\lambda\text{-mon}} \]

is continuous, we obtain that the functor

\[ \Gamma' : \text{D-mod}(H\setminus \tilde{X})^{\lambda\text{-mon}} \to \mathfrak{g}\text{-mod}_{\{\chi\}}^H \]

preserves compactness.

2.5.6. As was mentioned above, the key difference between the \((\text{Loc}, \Gamma)\) and \((\text{Loc}', \Gamma')\)-adjunctions is that the functor \(\text{Loc}'\) is no longer fully faithful, but its failure to be fully faithful is controllable.

Namely, consider the Harish-Chandra map \(Z(\mathfrak{g}) \to \text{Sym}(t)\), and consider the algebra

\[ U(\mathfrak{g})^\sim := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \text{Sym}(t). \]

Denote \(\mathfrak{g}\text{-mod}^\sim := U(\mathfrak{g})^\sim\text{-mod}\), and let

\[ \mathfrak{g}\text{-mod}_{\{\chi\}}^\sim \subset \mathfrak{g}\text{-mod}^\sim \]

be the full subcategory consisting of objects that are set-theoretically supported at \(\lambda\) as \(\text{Sym}(t)\)-modules.
Then the functor

\[ \Gamma' : \text{D-mod}(\tilde{X})^{\lambda-\text{mon}} \to \mathfrak{g}\text{-mod}_{\{\chi\}} \]

factors as

\[ \text{D-mod}(\tilde{X})^{\lambda-\text{mon}} \xrightarrow{\Gamma^{-}} \mathfrak{g}\text{-mod}_{\{\lambda\}} \to \mathfrak{g}\text{-mod}_{\{\chi\}} , \]

where \( \mathfrak{g}\text{-mod}_{\{\lambda\}} \to \mathfrak{g}\text{-mod}_{\{\chi\}} \) is the forgetful functor.

The above functor \( \Gamma^{-} \) also admits a left adjoint, denoted \( \text{Loc}^{-} \), and the latter functor is fully faithful.

From here we obtain:

**Lemma 2.5.7.** The endofunctor \( \Gamma' \circ \text{Loc}' \) of \( \mathfrak{g}\text{-mod}_{\{\chi\}} \) is given by

\[ M \mapsto M \otimes_{Z(\mathfrak{g})} Q , \]

where \( Q \) is \( \text{Sym}(t) \), regarded as an \( Z(\mathfrak{g}) \)-module.

Note that the above module \( Q \) is (locally) free.

**Corollary 2.5.8.** The endofunctor \( \Gamma' \circ \text{coLoc}' \) of \( \mathfrak{g}\text{-mod}_{\{\chi\}} \) is given by

\[ M \mapsto M \otimes_{Z(\mathfrak{g})} Q^\vee , \]

where \( Q^\vee \) is the dual of \( Q \).

2.5.9. The entire discussion in Sect. 2.5.6, and the conclusions of Lemma 2.5.7 and Corollary 2.5.8 transfer to the \( H \)-equivariant situation for any given \( H \subset G \).

2.6. Examples arising from representation theory.

2.6.1. In this subsection we will prove the following:

**Theorem 2.6.2.** Let \( H \subset G \) be spherical, i.e., \( H \) has finitely many orbits on the flag variety \( X \). Then:

(a) The category \( \mathfrak{g}\text{-mod}^H_{\chi} \) is proper, Serre and Gorenstein, and the functors \( \text{Se}_{\mathfrak{g}\text{-mod}^H_{\chi}} \) and \( \text{Ps-Id}_{\mathfrak{g}\text{-mod}^H_{\chi}} \) are mutually inverse equivalences.

(b) Ditto for the category \( \mathfrak{g}\text{-mod}^H_{\{\chi\}} \).

**Remark 2.6.3.** Note that when \( \chi \) is regular, the assertion of Theorem 2.6.2(a) follows immediately from the fact that in this case the functor \( \text{Loc} \) is an equivalence, and Theorem 2.1.5.

2.6.4. First, we have a lemma:

**Lemma 2.6.5.** Let \( F : \mathcal{C} \to \mathcal{D} \) be proper.

(a) We have: \( \text{Se}_{\mathcal{D}} \circ F \simeq (F^R)^R \circ \text{Se}_{\mathcal{C}} \).

(b) If \( F \) is fully faithful, then \( \text{Se}_{\mathcal{C}} \simeq F^R \circ \text{Se}_{\mathcal{D}} \circ F \).

**Proof.** For \( c \in \mathcal{C}^c \) and \( d \in \mathcal{D}^c \), we have

\[ \mathcal{H}om_{\mathcal{D}}(d, \text{Se}_{\mathcal{D}} \circ F(c)) \simeq \mathcal{H}om_{\mathcal{D}}(F(c), d)^* \simeq \mathcal{H}om_{\mathcal{C}}(c, F^R(d))^* \]

and

\[ \mathcal{H}om_{\mathcal{D}}(d, (F^R)^R \circ \text{Se}_{\mathcal{C}}(c)) \simeq \mathcal{H}om_{\mathcal{C}}(F^R(d), \text{Se}_{\mathcal{C}}(c)) \simeq \mathcal{H}om_{\mathcal{C}}(c, F^R(d))^* . \]

This proves point (a). For point (b) we note that the fact that \( F \) is fully faithful implies that \( (F^R)^R \) is such as well. In particular \( F^R \circ (F^R)^R \simeq \text{Id}_{\mathcal{C}} \). Composing the isomorphism of point (a) with \( F^R \), we arrive at the assertion of point (b).
2.6.6. Proof of Theorem 2.6.2. Step 1. We will verify that the conditions of Corollary 1.5.6 hold.

We first verify that the categories in question are proper. For \( g \text{-mod}_{\chi}^H \), this is a formal consequence of the fact that Loc is proper and fully faithful, and the properness of D-mod_{\lambda}(X).

The case of \( g \text{-mod}_{\chi}^H \) follows formally from that of \( g \text{-mod}_{\chi}^H \), using the \((i_\chi)^*, (i_\chi)_*, (i_\chi)^!\) adjunctions as in the proof of Theorem 2.3.6.

2.6.7. Proof of Theorem 2.6.2. Step 2. We will now show that the functors \( \text{Se}_{g \text{-mod}_{\chi}^H} \) and \( \text{Se}_{(g \text{-mod}_{\chi}^H)^\vee} \cong \text{Se}_{g \text{-mod}_{(-\chi)}^H} \) are proper. By symmetry, it suffices to deal with the former functor.

By Lemma 2.6.5(a), we have
\[
\text{coLoc}' \circ \text{Se}_{g \text{-mod}_{\chi}^H} \cong \text{Se}_{D\text{-mod}(H \setminus \tilde{X})^{\lambda\text{-mon}}} \circ \text{Loc}'.
\]

Composing with \( \Gamma' \), and using Corollary 2.5.8, we obtain:
\[
(- \otimes Q^\vee) \circ \text{Se}_{g \text{-mod}_{\chi}^H} \cong \Gamma' \circ \text{Se}_{D\text{-mod}(H \setminus \tilde{X})^{\lambda\text{-mon}}} \circ \text{Loc}'.
\]

Note that all the functors in the RHS preserve compactness (for \( \text{Se}_{D\text{-mod}(H \setminus \tilde{X})^{\lambda\text{-mon}}} \) we are using Theorem 2.3.6).

Hence, it remains to show that if \( M \in g \text{-mod}_{\chi}^H \) is such that \( M \otimes Q^\vee \) is compact, then so is \( M \). However, if \( M \otimes Q^\vee \) is compact, then so is \( M \otimes Q^\vee \otimes Q \), and \( M \) is a direct summand of the latter.

2.6.8. Proof of Theorem 2.6.2. Step 3. We will now show that the functors \( \text{Se}_{g \text{-mod}_{\chi}^H} \) and \( \text{Se}_{(g \text{-mod}_{\chi}^H)^\vee} \cong \text{Se}_{g \text{-mod}_{(-\chi)}^H} \) are proper. Again, by symmetry, it suffices to deal with the former functor.

Note that the functor
\[
(i_\chi)_*: g \text{-mod}_{\chi}^H \to g \text{-mod}_{\chi}^H
\]
is conservative, because both
\[
(i_\chi)_*: g \text{-mod}_{\chi} \to g \text{-mod}_{\chi} \text{ and } \text{oblv}_H: g \text{-mod}_{\chi}^H \to g \text{-mod}_{\chi}
\]
are conservative.

Hence, \( g \text{-mod}_{\chi}^H \) is compactly generated by the essential image of \( g \text{-mod}_{\chi}^H \) under \( (i_\chi)_* \). Hence, it is enough to show that the functor
\[
\text{Se}_{g \text{-mod}_{\chi}^H} \circ (i_\chi)_*
\]
sends compacts to compacts.

However, from Lemma 2.6.5(a), we obtain
\[
\text{Se}_{g \text{-mod}_{\chi}^H} \circ (i_\chi)_* \cong (i_\chi)^! \circ \text{Se}_{g \text{-mod}_{\chi}^H},
\]
which isomorphic to \( (i_\chi)_* \circ \text{Se}_{g \text{-mod}_{\chi}^H} \) up to tensoring with \( \ell_\chi \).

Now the assertion follows from the fact that \( \text{Se}_{g \text{-mod}_{\chi}^H} \) preserves compactness, proved in Step 2.

\[\Box\text{[Theorem 2.6.2]}\]

2.7. An application: a theorem of [BBM].
2.7.1. As an application we will now (re)prove the following result (whose case (b) is a theorem from \[BBM\]).

We take $H$ to be the subgroup $N$, the unipotent radical of a Borel in $G$. Take $C$ to be either
(a) $g$-mod$_\chi$, or
(b) $D$-mod$_\lambda(X)$.

Recall the intertwining functor
\[ \Upsilon := \text{Av}^N_N \circ \text{oblv}_{N^-} : C^{N^-} \to C^N, \]
see \[CGY\] Sect. 1.4], and similarly
\[ \Upsilon^- : C^N \to C^{N^\ominus}. \]

2.7.2. We have:

**Theorem 2.7.3.** The category $C^N$ is Serre, and we have a canonical isomorphism
\[ \text{Se}_{C^N} \simeq \Upsilon \circ \Upsilon^- [2 \dim(X)]. \]

**Proof.** The assertion for $C = D$-mod$_\lambda(X)$ is the combination of Theorem 2.1.5 above and \[CGY\] Theorem 3.4.2.

For $C = g$-mod$_\chi$ we have
\[ \text{Se}_{g$\text{-mod}_\chi$} \overset{\text{Lemma 2.6.3 (b)}}{\simeq} \Gamma \circ \text{Se}_{D$\text{-mod}_\lambda(X\setminus X)$} \circ \text{Loc} \simeq \]
\[ \simeq \Gamma \circ \Upsilon \circ \Upsilon^- \circ \text{Loc}[2 \dim(X)] \simeq \Upsilon \circ \Gamma \circ \text{Loc} \circ \Upsilon^- [2 \dim(X)] \simeq \Upsilon \circ \Upsilon^- [2 \dim(X)], \]
since the functors $\Gamma$ and $\text{Loc}$ commute with all averaging functors.

\[ \Box \]

3. Serre functor and contragredient duality

The theme of this section is to compare the canonical duality functor $\mathbb{D}^\text{can}_{\mathfrak{g},X}$ on a category of the form $g$-mod$_\chi^H$ with various kinds of contragredient duality functors.

3.1. A warm-up: algebraic representations. We begin with the simplest case, namely, when $g = \mathfrak{h}$.

3.1.1. Let $H$ be an algebraic group. Consider the category
\[ \mathfrak{h}$\text{-mod}^H \simeq \text{Rep}(H). \]

We note that there are two different identifications
\[ \text{Rep}(H)^\vee \simeq \text{Rep}(H). \]

One is given by (2.11); In this section, we will denote the corresponding contravariant self-equivalence of $(\mathfrak{h}$-mod$_\chi^H)^\vee$ by $\mathbb{D}^\text{can}_{\mathfrak{h},H}$ rather than $\mathbb{D}^\text{can}_{\mathfrak{h}}$.

The other is given by ind-extending the contravariant self-equivalence
\[ \mathbb{D}^\text{contr}_H : \text{Rep}(H)^\vee \to \text{Rep}(H)^\vee, \quad V \mapsto V^\vee, \]
given by the passage to the dual representation.
3.1.2. The composite of these two identifications is a self-equivalence of $\operatorname{Rep}(H)$. It is given by ind-extending the (covariant) self-equivalence

$$D^\text{contr}_H \circ D^\text{can}_{h,H} : \operatorname{Rep}(H)^c \to \operatorname{Rep}(H)^c.$$ 

We claim:

**Proposition 3.1.3.** The functor $D^\text{contr}_H \circ D^\text{can}_{h,H}$ is given by tensoring with the line $\ell_H := \Lambda^{\dim(H)(\mathfrak{h})}[\dim(\mathfrak{h})]$.

**Proof.** We first establish the corresponding isomorphism after composing with (3.1)

$$\text{oblv}_H : \operatorname{Rep}(H) \simeq \mathfrak{h}\text{-mod}^{H} \text{oblv}_H \mathfrak{h}\text{-mod}.$$

For the latter, we have to establish an isomorphism

$$D^\text{can}_{h,H} \circ \text{oblv}_H(V) \simeq \text{oblv}_H(V^\vee) \otimes \ell_H, \quad V \in \operatorname{Rep}(H)^c.$$

Taking $\mathcal{K}\text{om}_{\mathfrak{h}\text{-mod}}$ of both sides into $M \in \mathfrak{h}\text{-mod}$, we obtain that we need to establish a functorial isomorphism between

$$\mathcal{K}\text{om}_{\mathfrak{h}\text{-mod}}(D^\text{can}_{h,H} \circ \text{oblv}_H(V), M) \simeq C_*\mathfrak{h}, \text{oblv}_H(V) \otimes M)$$

and

$$\mathcal{K}\text{om}_{\mathfrak{h}\text{-mod}}(\text{oblv}_H(V^\vee) \otimes \ell_H, M) \simeq \ell_H^{-1} \otimes C^*(\mathfrak{h}, \text{oblv}_H(V) \otimes M).$$

The required isomorphism follows now from

$$C_*\mathfrak{h}, -) \simeq \ell_H^{-1} \otimes C^*(\mathfrak{h}, -).$$

Thus, we obtain that the endo-functors

$$(- \otimes \ell_H^{-1}) \circ D^\text{contr}_H \circ D^\text{can}_{h,H} : \operatorname{Rep}(H) \to \operatorname{Rep}(H)$$

are both given by objects in

$$\operatorname{Rep}(H)^\vee \otimes \operatorname{Rep}(H) \simeq \operatorname{Rep}(H) \otimes \operatorname{Rep}(H)$$

that become canonically isomorphic after applying the functor

(3.2) $\operatorname{Id}_{\operatorname{Rep}(H)} \otimes \text{oblv}_H : \operatorname{Rep}(H) \otimes \operatorname{Rep}(H) \to \operatorname{Rep}(H) \otimes \mathfrak{h}\text{-mod}.$

Since the latter is t-exact and conservative, we obtain that the above two objects both lie in $(\operatorname{Rep}(H) \otimes \operatorname{Rep}(H))^\vee$.

Now, the restriction of the functor (3.2) to $(\operatorname{Rep}(H) \otimes \operatorname{Rep}(H))^\vee$ is fully faithful. Hence, the above two objects are isomorphic in $\operatorname{Rep}(H) \otimes \operatorname{Rep}(H)$ itself. 

\qed

3.1.4. In what follows we will need the following comparison result. Let $\mathbf{C}$ be a DG category acted on by a reductive group $H$.

We claim that there is a natural transformation

(3.3) $\text{Av}_{\mathfrak{h}}^H \circ \text{oblv}_H \to (- \otimes \ell_H^{-1}) \otimes.$

Indeed, the functor $\text{Av}_{\mathfrak{h}}^H \circ \text{oblv}_H$ is given by tensoring with $C^\bullet_\text{dR}(H)$, viewed as an object of $\text{D-mod}(\text{pt} / H)$, i.e., the direct image of $k \in \text{Vect} \simeq \text{D-mod}(\text{pt})$ along the projection $\text{pt} \to \text{pt} / H$.

Now, for any algebraic group, we have a canonical isomorphism

$$C^\bullet_\text{dR}(H) \simeq C^\bullet(\mathfrak{h}, R_H),$$
and if \( H \) is reductive, the map
\[
C^\bullet(h, k) \to C^\bullet(h, R_H)
\]
is an isomorphism. Finally, we identify \( \ell_H^{-1} \) with the top cohomology of \( C^\bullet(h, k) \), using
\[
C^\bullet(h, k) \simeq C^\bullet(h, k) \otimes \ell_H^{-1}.
\]

This provides the desired map
\[
C^\bullet_{dR}(H) \to \ell_H^{-1} \otimes k_{pt}/H
\]
in \( D\text{-mod}(pt/H) \).

3.2. The case of category \( \mathcal{O} \). In this subsection we study (the derived version of) the usual category \( \mathcal{O} \), i.e., the category \( g\text{-mod}^N_{\chi} \), where \( N \) is the unipotent radical of a Borel subgroup.

We will see that the discrepancy between the canonical duality functor \( \mathbb{D}^\text{can}_{g,\chi} \) and the usual contragredient duality for category \( \mathcal{O} \) is given by the long intertwining functor \( \Upsilon \).

3.2.1. Consider the equivalence
\[
(g\text{-mod}^N_{\chi})^\vee \simeq g\text{-mod}^{N^-}_{\chi}
\]
equal to the composition
\[
(g\text{-mod}^N_{\chi})^\vee \simeq g\text{-mod}^{N^-}_{\chi} \xrightarrow{\text{Se}_{g\text{-mod}^N_{\chi}}} g\text{-mod}^{N}_{\chi} \xrightarrow{\Upsilon} g\text{-mod}^{N^-}_{\chi},
\]
where the first arrow is the equivalence \( \text{(2.13)} \).

Note that from Theorem \( \text{2.7.3} \) we obtain:

**Corollary 3.2.2.** The identification \( \text{(3.4)} \) is canonically isomorphic to
\[
(g\text{-mod}^N_{\chi})^\vee \simeq g\text{-mod}^{N^-}_{\chi} \xrightarrow{\Upsilon - [2\dim(X)]} g\text{-mod}^{N^-}_{\chi},
\]
where the first arrow is the identification \( \text{(2.13)} \).

Recall that \( \mathbb{D}^\text{can}_{g,\chi} \) denotes the contravariant equivalence
\[
(g\text{-mod}^N_{\chi})^c \simeq (g\text{-mod}^{N^-}_{\chi})^c,
\]
corresponding to \( \text{(2.13)} \) (note that, according to Sect. \( \text{2.5.2} \) the functor \( \text{Loc} \) intertwines \( \mathbb{D}^\text{can}_{g,\chi} \) with Verdier duality on \( N\backslash X \)).

Let \( \mathbb{D}^\text{contr}_{g,\chi} \) denote the contravariant equivalence
\[
(g\text{-mod}^N_{\chi})^c \simeq (g\text{-mod}^{N^-}_{\chi})^c,
\]
corresponding to \( \text{(3.5)} \) (and similarly with the roles of \( \chi/ - \chi \) or \( N/ N^- \) swapped).

We can rephrase Corollary \( \text{3.2.2} \) as follows:

**Corollary 3.2.3.** The (covariant) equivalence
\[
\mathbb{D}^\text{contr}_{g,\chi} \circ \mathbb{D}^\text{can}_{g,\chi} : (g\text{-mod}^N_{\chi})^c \to (g\text{-mod}^{N^-}_{\chi})^c
\]
is given by the functor \( \Upsilon - [2\dim(X)] \).

We will now show that the functor \( \mathbb{D}^\text{contr}_{g,\chi} \) is (the derived version of) the usual contragredient duality on category \( \mathcal{O} \).

In particular, we obtain that Corollary \( \text{3.2.3} \) reproduces the result of \( \text{[AG, Theorem 1.4.6]} \) that describes the interaction of the contragredient and canonical dualities on category \( \mathcal{O} \).
3.2.4. Let us denote by
\[ M \mapsto M^\vee, \quad (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \rightarrow ((\mathfrak{g}\text{-mod}_{\chi}^{-N})^\vee)^{\text{op}} \]
the contravariant functor given by assigning to \( M \) the subspace of the abstract dual \( M^* \), consisting of \( N^- \)-finite vectors.

3.2.5. On the subcategory \( (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \), consisting of \( M \) for which \( \text{obl}_N(M) \in (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \) is finitely generated, the functor \( M \mapsto M^\vee \) admits the following description:

Write \( M \cong \bigoplus_{\mu} M(\mu) \), a direct sum of generalized eigenspaces with respect to \( t \), which are known to be finite-dimensional. Then \( M^\vee \cong \bigoplus_{\mu} (M(\mu))^* \) (which acquires a natural \( \mathfrak{g} \)-module structure). In other words, \( M \mapsto M^\vee \) is the "usual" contragradient duality.

Moreover, it is known that in this way we obtain a contravariant equivalence
\[ (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \xrightarrow{\sim} (\mathfrak{g}\text{-mod}_{\chi}^{-N})^\vee. \]

3.2.6. We claim:

**Theorem 3.2.7.** The ind-extension of the contravariant equivalence \( D_{\mathfrak{g},\chi}^{\text{contr}} \) of \( (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \) is t-exact when restricted to \( (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \). The corresponding contravariant functor
\[ (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \rightarrow ((\mathfrak{g}\text{-mod}_{\chi}^{-N})^\vee)^{\text{op}} \]
is given by the functor \( M \mapsto M^\vee \).

3.2.8. Proof of Theorem 3.2.7, Step 1. It is enough to show that the functor \( D_{\mathfrak{g},\chi}^{\text{contr}} \) sends an object \( M \in (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \) to \( M^\vee \).

By definition, for \( M' \in (\mathfrak{g}\text{-mod}_{\chi}^{-N})^\vee \), we have
\[ \text{Hom}_{\mathfrak{g}\text{-mod}_{\chi}^{-N}}(M', D_{\mathfrak{g},\chi}^{\text{contr}}(M)) \cong \langle M, \mathcal{Y}(M') \rangle_{\mathfrak{g}_{\chi}, N}^* \cong \langle M, M' \rangle_{\mathfrak{g}_{\chi}}^* \]

Note that if \( M' \in (\mathfrak{g}\text{-mod}_{\chi}^{-N})_{\leq 0} \), then
\[ \langle M, M' \rangle_{\mathfrak{g}_{\chi}} = M' \otimes_{U(\mathfrak{g})} M \]
belongs to \( \text{Vect}_{\leq 0} \). This readily implies that \( D_{\mathfrak{g},\chi}^{\text{contr}}(M) \in (\mathfrak{g}\text{-mod}_{\chi}^{-N})_{\geq 0} \).

Further, it is easy to see that for any \( M \in (\mathfrak{g}\text{-mod}_{\chi}^N)^\vee \), we have
\[ H^0((M, M')^* \cong \text{Hom}(H^0(M'), M^\vee), \]
hence \( H^0(D_{\mathfrak{g},\chi}^{\text{contr}}(M)) \cong M^\vee \).

Thus, it remains to show that \( D_{\mathfrak{g},\chi}^{\text{contr}}(M) \in (\mathfrak{g}\text{-mod}_{\chi}^{-N})^\vee \).
3.2.9. Proof of Theorem 3.2.7 Step 2. Every object of \((\mathfrak{g}\text{-mod}_{\chi}^{N})^{\text{r-f.g.}}\) admits a surjection from a finite direct sum of objects of the form

\[ M_i := U(\mathfrak{g})_{\chi} \otimes_{U(\mathfrak{n})} U(n)_i, \]

where \(U(n)_i = U(\mathfrak{n})/U(\mathfrak{n}) \cdot n^i\).

Hence, using the exactness of \(\mathcal{M} \mapsto H^0(\mathbb{D}_{\chi}^{\text{contr}}(\mathcal{M})) \simeq \mathcal{M}'\), it suffices to show that

\[ M_i^\vee \simeq H^0(\mathbb{D}_{\chi}^{\text{contr}}(M_i)) \to \mathbb{D}_{\chi}^{\text{contr}}(M_i) \]

is an isomorphism.

Furthermore, objects of the form

\[ M_j^- := U(\mathfrak{g})_{\chi} \otimes_{U(n^-)} U(n^-)_j \]

compactly generate \(\mathfrak{g}\text{-mod}_{\chi}^{N^-}\). Hence, it suffices to show that the map

\[ \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_j^-, M_i^\vee) \to \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_j^-, \mathbb{D}_{\chi}^{\text{contr}}(M_i)) \]

is an isomorphism.

Each \(M_i\) (resp., \(M_i^-\)) admits a filtration with subquotients of the form \(M_1\) (resp., \(M_1^-\)). Hence, it is enough to show that

\[ \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_1^-, M_i^\vee) \to \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_1^-, \mathbb{D}_{\chi}^{\text{contr}}(M_i)) \]

is an isomorphism.

We know that (3.9) is an isomorphism at the level of \(H^0\). Hence, it suffices to show that both sides in (3.9) are acyclic off degree 0.

3.2.10. Proof of Theorem 3.2.7 Step 3. We have:

\[ \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_1^-, \mathbb{D}_{\chi}^{\text{contr}}(M_1)) \simeq (M_1^- \otimes_{U(\mathfrak{g})_{\chi}} M_1)^*, \]

while

\[ M_1^- \otimes_{U(\mathfrak{g})_{\chi}} M_1 \simeq k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g})_{\chi} \otimes_{U(n)} k, \]

and the above expression is indeed acyclic off degree 0, as it is known that \(U(\mathfrak{g})\) is free as a module over \(U(\mathfrak{n}^-) \otimes U(\mathfrak{n}) \otimes Z(\mathfrak{g})\).

To prove that \(\mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_1^-, M_i^\vee)\) is acyclic off degree 0, we note that for any object \(N \in \mathfrak{g}\text{-mod}_{\chi}^{N^-}\), we have

\[ \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_{\chi}^{N^-}}(M_1^-, N) \simeq C^*((n^-), N). \]

So we need to show that that \(C^*(n^-, M_i^\vee)\) is acyclic off degree 0.

We note that \(M_1\) admits a filtration by Verma modules. Hence, it is enough to show that for a Verma module \(M\), the object \(C^*(n^-, M^\vee)\) is acyclic off degree 0.

Now, \(M^\vee\) is a dual Verma module, which is isomorphic to \(R_{N^-}\) as a module over \(n^-\). This implies that \(C^*(n^-, M^\vee)\) is acyclic off degree 0, as required.

3.3. The case of Harish-Chandra modules. We will now consider the main case of interest in this paper: the interaction of the canonical and contragredient duality functors on (the derived version of) the category of Harish-Chandra modules.

We will see that the discrepancy between the two is given by the pseudo-identity functor.
3.3.1. We now consider a symmetric subgroup \( K \subset G \), i.e. \( K = G^\theta \) for an involution \( \theta \) of \( G \). Such a subgroup is (connected) reductive, and is spherical (i.e. has finitely many orbits on the flag variety \( X \)).

Consider the equivalence

\[
(g\text{-mod}^K_\chi)^\vee \simeq g\text{-mod}^{K\chi}_\chi \to g\text{-mod}^{K\subseteq \ell K}_\chi, 
\]

where the first arrow is the equivalence (2.13), and \( \ell K = \Lambda^{\dim(K)}(\mathfrak{f})[\dim(K)] \).

Let \( \mathbb{D}^{\text{can}}_g \) denote the contravariant equivalence

\[
(g\text{-mod}^K_\chi)^c \simeq (g\text{-mod}^{K\subseteq \ell K}_\chi)^c,
\]

corresponding to (3.10), and similarly for \( -\chi \).

By definition, we have

\[
\mathbb{D}^{\text{contr}}_{g, -\chi} \circ \mathbb{D}^{\text{can}}_g \simeq (- \otimes \ell K) \circ \text{Se}_{g\text{-mod}^K_\chi}
\]

and by Theorem 2.6.2

\[
\mathbb{D}^{\text{can}}_g \circ \mathbb{D}^{\text{contr}}_{g, -\chi} \simeq (- \otimes \ell K^-) \circ \text{Ps-Id}_{g\text{-mod}^K_\chi}.
\]

3.3.2. We will now show that \( \mathbb{D}^{\text{contr}}_{g, -\chi} \) is (the derived version of) the “usual” contragredient duality for Harish-Chandra modules.

In particular, we obtain that (3.13) gives an expression to the composition of the contragredient duality and “cohomological” duality on Harish-Chandra modules. In the context of \( p \)-adic groups, in [BBK], such a composition is shown to be isomorphic to the Deligne-Lusztig functor.

Hence, we obtain that in the context of Harish-Chandra modules, the pseudo-identity functor \( \text{Ps-Id}_{g\text{-mod}^K_\chi} \) plays a role analogous to that of the Deligne-Lusztig functor for \( p \)-adic groups.

Note that under the localization equivalence (say, when \( \chi \) is regular), the functor \( \text{Ps-Id}_{g\text{-mod}^K_\chi} \) corresponds to the pseudo-identity functor \( \text{Ps-Id}_{K\setminus X} \). As was mentioned in the introduction, certain parallel features of functors of the form \( \text{Ps-Id}_g \) and Deligne-Lusztig type functors have been observed elsewhere in geometric representation theory, see, e.g., [DW, Ga2, Wa]. This analogy is further reinforced by the properties of the functor \( \text{Ps-Id}_{K\setminus X} \) expressed in Conjectures 4.2.7 and 4.2.8 below.

3.3.3. Whereas the isomorphism (3.13) may look somewhat surprising (given the geometric nature of the functor \( \text{Ps-Id}_{g\text{-mod}^K_\chi} \)), the isomorphism (3.12) is something one could have expected, based on the following example (we are grateful to J. Lurie for pointing this out to us):

Let \( A \) be an associative algebra that is finite-dimensional over \( k \). On the one hand, dualization over \( k \) defines a contravariant functor

\[
\mathbb{D}^{\text{contr}}_A : A\text{-mod}^c \to A^{\text{op}-\text{mod}}.
\]

On the other hand, we have a canonical equivalence

\[
(A\text{-mod})^\vee \simeq A^{\text{op}-\text{mod}},
\]

\[
4 \text{ In fact, everything in this section remains valid when } K \text{ is any (connected) reductive spherical subgroup in } G.
\]
given by the pairing
\[ M', M \mapsto M' \otimes M, \quad A^\text{op-mod} \otimes A\text{-mod} \rightarrow \text{Vect}. \]

Denote the resulting contravariant equivalence by
\[ D^\text{can}_A : A\text{-mod}^c \simeq A^\text{op-mod}^c. \]

Composing, we obtain a covariant functor
\[ D^\text{contr} \circ D^\text{can}_A : A\text{-mod} \rightarrow A\text{-mod}. \]

It is easy to see that this functor is the restriction to \( A\text{-mod}^c \) of the Serre functor \( \text{Se}_A\text{-mod}^c \).

3.3.4. On the level of abelian categories, we have the "usual" contragredient duality functor
\[ (3.14) \quad M \mapsto M^\vee, \quad (\mathfrak{g}\text{-mod}_K^\chi)^\vee \rightarrow ((\mathfrak{g}\text{-mod}_K^\chi)^\vee)^{\text{op}} \]
defined by sending \( M \) to the subspace of the abstract dual \( M^* \), consisting of \( K \)-finite vectors. It is an exact functor.

Writing
\[ M \simeq \bigoplus_{\alpha} M_{\alpha}, \]
a direct sum of isotypic subspaces with respect to \( K \), the module \( M^\vee \) can be described as
\[ M^\vee \simeq \bigoplus_{\alpha} (M_{\alpha})^*. \]

3.3.5. It is not hard to show using Localization theory that a module \( M \in (\mathfrak{g}\text{-mod}_K^\chi)^\vee \) is finitely generated if and only if it is of finite length, and that is if and only if it is admissible, where the latter means that each \( M_{\alpha} \) (notation as above) is finite-dimensional.

This shows that \( M \mapsto M^\vee \) restricts to a contravariant equivalence
\[ (\mathfrak{g}\text{-mod}_K^\chi)^\vee, f \mapsto (\mathfrak{g}\text{-mod}_K^\chi)^\vee, f \]
of \( \mathfrak{g}\text{-mod}_K^\chi \).

3.3.6. We are going to prove:

**Theorem 3.3.7.** The ind-extension of the contravariant equivalence \( D^\text{contr}_{\mathfrak{g}, \chi} \) of \( (3.11) \)
\[ (3.15) \quad D^\text{contr}_{\mathfrak{g}, \chi} : \mathfrak{g}\text{-mod}_K^\chi \rightarrow (\mathfrak{g}\text{-mod}_K^\chi)^{\text{op}} \]
is t-exact when restricted to \( (\mathfrak{g}\text{-mod}_K^\chi)^{f, f.g.} \). The corresponding contravariant functor
\[ (\mathfrak{g}\text{-mod}_K^\chi)^{\vee, f.g.} \rightarrow ((\mathfrak{g}\text{-mod}_K^\chi)^{\vee})^{\text{op}} \]
is given by the functor \( M \mapsto M^\vee \) of \( (3.14) \).

3.3.8. **Proof of Theorem 3.3.7**. **Step 1.** We will in fact show that
\[ D^\text{contr}_{\mathfrak{g}, \chi} : \mathfrak{g}\text{-mod}_K \rightarrow (\mathfrak{g}\text{-mod}_K)^{\text{op}} \]
is t-exact on the whole category, rather than just on \( (\mathfrak{g}\text{-mod}_K^\chi)^{f, f.g.} \).

For \( \rho \in \text{Irrep}(K) \), set
\[ P_{\rho, -\chi} := U(\mathfrak{g})_{-\chi} \otimes_{U(\mathfrak{t})} \rho \in \mathfrak{g}\text{-mod}_K^\chi. \]

It is clear that
\[ \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_K^\chi}(P_{\rho, -\chi}, N) \simeq \mathcal{H}\text{om}_{\text{Rep}(K)}(\rho, N). \]

Hence, an object \( N \in \mathfrak{g}\text{-mod}_K^\chi \) belongs to \( (\mathfrak{g}\text{-mod}_K^\chi)^{\leq 0, \vee, \geq 0} \) if and only if the objects
\[ \mathcal{H}\text{om}_{\mathfrak{g}\text{-mod}_K^\chi}(P_{\rho, -\chi}, N) \in \text{Vect} \]
belong to $\text{Vect}^\leq \chi$ for all $\rho$.

Notice that, by definition, for $M \in \mathfrak{g}\text{-mod}^K$ and $M' \in \mathfrak{g}\text{-mod}^K$ we have
\[
\mathcal{H}om_{\mathfrak{g}\text{-mod}^K}(M', \mathbb{D}^{\text{contr}}_\chi(M)) \simeq \langle M, M' \rangle_{\mathfrak{g}, \chi, K} \otimes \ell_K.
\]

It thus remains to show that
\[
M \mapsto \langle M, P_{\rho, -\chi} \rangle_{\mathfrak{g}, \chi, K} \otimes \ell_K^{-1}
\]
is a $t$-exact functor $\mathfrak{g}\text{-mod}^K \rightarrow \text{Vect}$ (for a given $\rho$).

We have
\[
\langle M, P_{\rho, -\chi} \rangle_{\mathfrak{g}, \chi, K} \simeq \langle M, \rho \rangle_{k, K}.
\]

However, according to Proposition 3.1.3
\[
\langle M, \rho \rangle_{k, K} \simeq \mathcal{H}om_{\text{Rep}(K)}(\rho^*, M) \otimes \ell_K,
\]
so that the functor (3.16) is isomorphic to
\[
M \mapsto \mathcal{H}om_{\text{Rep}(K)}(\rho^*, M),
\]
which is $t$-exact.

3.3.9. Proof of Theorem 3.3.7, Step 2. From Step 1, we deduce that the pairing
\[
M, M' \mapsto \langle M, M' \rangle_{\mathfrak{g}, \chi, K} \otimes \ell_K^{-1}
\]
is right $t$-exact. Hence, the comparison map (see Sect. 3.1.4)
\[
\langle \text{oblv}_K(M), \text{oblv}_K(M') \rangle_{\mathfrak{g}, \chi} \simeq \langle \text{Av}^K \circ \text{oblv}_K(M), \mathcal{M}' \rangle_{\mathfrak{g}, \chi, K} \otimes \ell_K^{-1}
\]
induces an isomorphism on $H^0$.

This implies that for $M \in (\mathfrak{g}\text{-mod}^K)_{\chi, K}$ and $M' \in (\mathfrak{g}\text{-mod}^K)_{\chi, K}$, we have:
\[
\mathcal{H}om((\mathfrak{g}\text{-mod}^K)_{\chi, K}, \mathbb{D}^{\text{contr}}_\chi(M)) \simeq H^0((\mathfrak{g}\text{-mod}^K)_{\chi, K} \otimes \ell_K^{-1})^* \simeq
\]
\[
\simeq H^0((\text{oblv}_K(M), \text{oblv}_K(M'))_{\mathfrak{g}, \chi}^* = H^0 \left( \mathcal{M}' \otimes_{U_{\mathfrak{g}, \chi}} \mathcal{M} \right)^*.
\]

This isomorphism identifies $H^0(\mathbb{D}^{\text{contr}}_\chi(M))$ with $\mathcal{M}'$.

3.4. The parabolic case.

3.4.1. We will now consider a hybrid of the situations considered in Sects. 3.2 and 3.3. Namely, let $G$ be a reductive group, equipped with an involution $\theta$. Let $P$ be a minimal $\theta$-compatible parabolic, so that $P^\perp = \theta(P)$ is an opposite parabolic. Set
\[
M_K = K \cap P \cap P^-.
\]

Let $N$ (resp., $N^\perp$) denote the unipotent radical of $P$ (resp., $P^\perp$).

We will consider the categories
\[
\mathfrak{g}\text{-mod}^{M_K\cdot N}_{\chi} \text{ and } \mathfrak{g}\text{-mod}^{M_K\cdot N^-}_{\chi}.
\]

Remark 3.4.2. The results of this subsection are equally applicable when instead of $P, P^\perp$ we take any two opposite parabolics and instead of $M_K$ we take the entire Levi subgroup $M = P \cap P^\perp$. 

3.4.3. We consider the equivalence

\[(g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee \simeq g\text{-mod}_{\chi}^{M_{K}\cdot N^-}\]

given by the composition

\[(g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee \simeq g\text{-mod}_{\chi}^{M_{K}\cdot N} \xrightarrow{\theta} g\text{-mod}_{\chi}^{M_{K}\cdot N^-} \xrightarrow{Y^{-1}} g\text{-mod}_{\chi}^{M_{K}\cdot N^-} \simeq g\text{-mod}_{\chi}^{M_{K}\cdot N^-},\]

where the first arrow is the equivalence (2.13).

Let \(D_{\text{contr}}^{g,\chi}\) denote the resulting contravariant equivalence

\[(3.18) \quad (g\text{-mod}_{\chi}^{M_{K}\cdot N})^c \rightarrow (g\text{-mod}_{\chi}^{M_{K}\cdot N^-})^c\]

As in Corollary 3.2.2 using [CGY] Proposition 4.1.7, we obtain:

**Proposition 3.4.4.** The identification (3.4) is canonically isomorphic to

\[(g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee \simeq g\text{-mod}_{\chi}^{M_{K}\cdot N^-}[2\dim(X) - \dim(M_{K})]\quad g\text{-mod}_{\chi}^{M_{K}\cdot N^-} \simeq g\text{-mod}_{\chi}^{M_{K}\cdot N^-},\]

where the first arrow is the identification of (2.13).

3.4.5. We define a contravariant functor

\[(3.19) \quad (g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee \rightarrow ((g\text{-mod}_{\chi}^{M_{K}\cdot N^-})^\vee)^{\text{op}}, \quad \mathcal{M} \mapsto \mathcal{M}^\vee\]

by sending \(M\) to the subspace of the abstract dual \(M^*\), consisting of \(M_{K}\cdot N^-\)-finite vectors.

3.4.6. For \(M \in (g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee_{\mathfrak{l}_{f,g}}\), one has the following concrete description of \(\mathcal{M}^\vee\).

Let \(\mathfrak{m}\) denote the Lie algebra of the Levi subgroup \(M = P \cap P^-\). Let \(\mathfrak{a} \subset \mathfrak{m}\) denote the \(\theta\)-split part of the center of \(\mathfrak{m}\), so that \(\mathfrak{m} \simeq \mathfrak{m}_{K} \oplus \mathfrak{a}\). For \(M \in (g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee_{\mathfrak{l}_{f,g}}\), the action of \(\mathfrak{a}\) on \(M\) is locally finite. Hence, we can write

\[M \simeq \bigoplus_{\mu} M(\mu),\]

where the \(M(\mu)\) are the generalized eigenspaces for the action of \(\mathfrak{a}\).

Now, each \(M(\mu)\) is admissible as a \((\mathfrak{m}, K_{M})\)-module, and let \((M(\mu))^\vee\) denote its dual (taken in the sense of Sect. 3.3.3). We then have

\[M^\vee := \bigoplus_{\mu} (M(\mu))^\vee,\]

with the natural action of \(\mathfrak{g}\).

Again it is possible to show that the functor \(M \mapsto M^\vee\) restricts to a contravariant equivalence

\[(g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee_{\mathfrak{l}_{f,g}} \simeq (g\text{-mod}_{\chi}^{M_{K}\cdot N^-})^\vee_{\mathfrak{l}_{f,g}}.\]

3.4.7. We claim:

**Theorem 3.4.8.** The ind-extension of the contravariant equivalence \(D_{\text{contr}}^{g,\chi}\) of (3.18)

\[D_{\text{contr}}^{g,\chi} : g\text{-mod}_{\chi}^{M_{K}\cdot N} \rightarrow (g\text{-mod}_{\chi}^{M_{K}\cdot N^-})^{\text{op}}\]

is t-exact when restricted to \((g\text{-mod}_{\chi}^{M_{K}\cdot N})_{\mathfrak{l}_{g}}\). The corresponding contravariant functor

\[(g\text{-mod}_{\chi}^{M_{K}\cdot N})^\vee_{\mathfrak{l}_{f,g}} \rightarrow ((g\text{-mod}_{\chi}^{M_{K}\cdot N^-})^\vee)^{\text{op}}\]

is given by the functor \(M \mapsto \mathcal{M}^\vee\) of (3.19).

We omit the proof as it is obtained by combining the ideas in the proofs of Theorems 3.2.7 and 3.3.7.
4. Relation to the “2nd adjointness” conjecture

In this section we recall the “2nd adjointness” conjecture of [CGY] and relate it to Theorem 3.3.7. The notation is as in Sect. 3.3.

4.1. The principal series functors.

4.1.1. Consider the categories $\mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}}$, $\mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}^{-}}$, $\mathfrak{g}\text{-mod}_{\chi}^{K}$. As was shown in [CGY, Theorem 4.2.2], the partially defined functors

$$Av_{1}^{N}: \mathfrak{g}\text{-mod}_{\chi}^{M_{K}} \rightarrow \mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}}$$

and

$$Av_{1}^{N^{\ast}}: \mathfrak{g}\text{-mod}_{\chi}^{M_{K}} \rightarrow \mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}^{-}}$$

are defined on the essential image of $\text{obl}_{K/M_{K}}$.

In particular, we have an adjoint pair

$$Av_{1}^{N} \circ \text{obl}_{K/M_{K}}: \mathfrak{g}\text{-mod}_{\chi}^{K} \rightleftharpoons \mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}}: \text{Av}_{*}^{K/M_{K}} \circ \text{obl}_{N}.$$  

In addition, we have:

**Proposition 4.1.2.**

(a) For $C$ being either $\mathfrak{g}\text{-mod}_{\chi}$ or $D\text{-mod}_{\lambda}(X)$, the partially defined functor

$$Av_{1}^{K/M_{K}}: C^{MK} \rightarrow C^{K}$$

is defined on the essential image of $\text{obl}_{N}: C^{M_{K\cdot N}} \rightarrow C^{MK}$.

(b) Moreover, we have

$$Av_{1}^{K/M_{K}} \simeq \Gamma \circ Av_{*}^{K/M_{K}} \circ \text{Loc}.$$  

**Proof.** The functor $Av_{1}^{K/M_{K}}: D\text{-mod}_{\lambda}(X)^{MK} \rightarrow D\text{-mod}_{\lambda}(X)^{K}$ is defined on the essential image of $\text{obl}_{N}: D\text{-mod}_{\lambda}(X)^{M_{K\cdot N}} \rightarrow D\text{-mod}_{\lambda}(X)^{MK}$ because $M_{K\cdot N}$ has finitely many orbits on $X$, and hence every object from $D\text{-mod}_{\lambda}(X)^{M_{K\cdot N}}$ is holonomic.

The assertion concerning $\mathfrak{g}\text{-mod}_{\chi}$, as well as point (b) of the proposition follow from [CGY, Proposition 1.2.6].

In particular, we obtain another pair of adjoint functors

$$Av_{1}^{K/M_{K}} \circ \text{obl}_{N}: \mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}} \rightleftharpoons \mathfrak{g}\text{-mod}_{\chi}^{K}: \text{Av}_{*}^{K/M_{K}} \circ \text{obl}_{N}.$$  

**Remark 4.1.3.** We regard the functors

$$Av_{1}^{K/M_{K}} \circ \text{obl}_{N} \text{ and } Av_{*}^{K/M_{K}} \circ \text{obl}_{N},$$

which map $\mathfrak{g}\text{-mod}_{\chi}^{M_{K\cdot N}} \rightarrow \mathfrak{g}\text{-mod}_{\chi}^{K}$, as two versions of the principal series functor for Harish-Chandra modules.
4.1.4. We now claim:

**Proposition 4.1.5.** There exists a canonical isomorphism of functors

\[ \text{Av}_{\ast}^{K/M_K} \simeq \text{Ps-Id}_{\mathfrak{g-mod}_\chi} \circ \text{Av}_{\ast}^{K/M_K} [2 \dim(X) - \dim(M_K)], \quad \mathfrak{g-mod}_{\chi}^M \cdot N \simeq \mathfrak{g-mod}_\chi^K. \]

**Proof.** The point of departure is the isomorphism

\[ \text{Av}_{\ast}^{K/M_K} \simeq \text{Ps-Id}_{\mathfrak{g-mod}_\chi} \circ \text{Av}_{\ast}^{K/M_K} [2 \dim(X) - \dim(M_K)] \]

as functors

\[ \text{D-mod}_{\lambda}(X)^{M_K \cdot N} \Rightarrow \text{D-mod}_{\lambda}(X)^K \]

that was established in [CGY, Corollary 4.4.2].

Using Theorem 2.1.5, we rewrite it as an isomorphism

\[ \text{Se}_{\text{D-mod}_{\lambda}(X)^K} \circ \text{Av}_{\ast}^{K/M_K} \simeq \text{Av}_{\ast}^{K/M_K} [2 \dim(X) - \dim(M_K)]. \]

Composing both sides with \( \Gamma \) and precomposing with Loc, we obtain an isomorphism between the functor

\[ \text{Av}_{\ast}^{K/M_K} [2 \dim(X) - \dim(M_K)] : \text{D-mod}_{\lambda}(X)^{M_K \cdot N} \Rightarrow \text{D-mod}_{\lambda}(X)^K \]

and

\[ \Gamma \circ \text{Se}_{\text{D-mod}_{\lambda}(X)^K} \circ \text{Av}_{\ast}^{K/M_K} \circ \text{Loc} \simeq \Gamma \circ \text{Se}_{\text{D-mod}_{\lambda}(X)^K} \circ \text{Loc} \circ \text{Av}_{\ast}^{K/M_K} \simeq \text{Se}_{\mathfrak{g-mod}_\chi^K} \circ \text{Av}_{\ast}^{K/M_K}, \]

where the first isomorphism is due to [CGY, Lemma 1.2.6], and the second one is using Lemma 2.6.5(b).

Thus, we obtain an isomorphism

\[ \text{Se}_{\mathfrak{g-mod}_\chi^K} \circ \text{Av}_{\ast}^{K/M_K} \simeq \text{Av}_{\ast}^{K/M_K} [2 \dim(X) - \dim(M_K)]. \]

Applying Theorem 2.6.2(a), we arrive at the desired isomorphism

\[ \text{Av}_{\ast}^{K/M_K} \simeq \text{Ps-Id}_{\mathfrak{g-mod}_\chi^K} \circ \text{Av}_{\ast}^{K/M_K} [2 \dim(X) - \dim(M_K)]. \]

\[ \square \]

4.2. The “2nd adjointness” conjecture.

4.2.1. Let us recall the “2nd adjointness” conjecture of [CGY, Conjecture 4.4.5]:

**Conjecture 4.2.2.** The functors

\[ \text{Av}_{\ast}^{K/M_K} \text{ and } (\ominus \otimes \ell_{K/M_K}^{-1}) \circ \text{Av}_{\ast}^{K/M_K} \circ \Upsilon, \quad \mathfrak{g-mod}_{\chi}^{M_K \cdot N} \Rightarrow \mathfrak{g-mod}_\chi^K \]

are canonically isomorphic, where \( \ell_{K/M_K} \) := \( \ell_{K} \otimes \ell_{M_K}^{-1} \).

**Remark 4.2.3.** One should think of the line \( \ell_{K/M_K}^{-1} \) as the top de Rham cohomology of \( K/M_K \).

4.2.4. Using Proposition 4.1.5, we can reformulate Conjecture 4.2.2 as follows:

**Conjecture 4.2.5.** The following diagram of functors commutes:

\[
\begin{array}{ccc}
\mathfrak{g-mod}_\chi^K & \xleftarrow{\text{Av}_{\ast}^{K/M_K}} & \mathfrak{g-mod}_{\chi}^{M_K \cdot N} \\
(\ominus \otimes \ell_{K/M_K}^{-1}) \circ \text{Ps-Id}_{\mathfrak{g-mod}_\chi^K} & \downarrow & \Upsilon^{-1}[-2 \dim(X)+\dim(M_K)] \\
\mathfrak{g-mod}_\chi^K & \xleftarrow{\text{Av}_{\ast}^{K/M_K}} & \mathfrak{g-mod}_{\chi}^{M_K \cdot N}.
\end{array}
\]
4.2.6. Yet another reformulation of Conjecture 4.2.2 is the following:

**Conjecture 4.2.7.** The functor right adjoint to

\[ \text{Av}_*^{K/M_K} \circ \text{oblV}_{N^-} : \mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi \to \mathfrak{g}-\text{mod}^K_\chi, \]

is given by

\[ (- \otimes \ell_{K/M_K}) \circ J \circ \text{oblV}_{K/M_K}. \]

In Conjecture 4.2.7, the notation \( J \) stands for the Casselman-Jacquet functor (see [CGY], where this functor is studied in detail). In what follows we will write \( J \) instead of \( J \circ \text{oblV}_{K/M_K} \), and mean by it the corresponding functor

\[ \mathfrak{g}-\text{mod}^{K}_\chi \to \mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi. \]

We recall that according to [CGY] Theorem 4.2.2, we have

(4.1) \[ J \simeq \text{Av}_*^{N^-} \circ \text{oblV}_N \circ \text{Av}_*^N \circ \text{oblV}_{K/M_K} \simeq \text{Av}_*^{N^-} \circ \text{oblV}_N \circ \text{Av}_*^N \circ \text{oblV}_{K/M_K}, \]

where the last isomorphism expressed the “Verdier self-duality” property of the functor \( J \).

Here is yet another equivalent formulation of Conjecture 4.2.2:

**Conjecture 4.2.8.** The functors

\[ (- \otimes \ell_{K/M_K}) \circ J \left[ -2 \dim(X) + \dim(M_K) \right] \quad \text{and} \quad \text{Av}_*^{N^-} \circ \text{oblV}_{K/M_K} \circ \text{Ps-Id}_{\text{mod}^K_\chi} \]

from \( \mathfrak{g}-\text{mod}^K_\chi \) to \( \mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi \) are canonically isomorphic.

**Proof.** We start with the isomorphism

\[ \text{Ps-Id}_{\text{mod}^K_\chi} \circ \text{Av}_*^{K/M_K} \circ \text{Y}^- \simeq (- \otimes \ell_{K/M_K}) \circ \text{Av}_*^{K/M_K} \left[ -2 \dim(X) + \dim(M_K) \right] \]

that follows from Conjecture 4.2.5. Passing to dual functors with respect to (2.13), we obtain

\[ \text{Y} \circ \text{Av}_*^{N^-} \circ \text{oblV}_{K/M_K} \circ \text{Ps-Id}_{\text{mod}^K_\chi} \simeq (- \otimes \ell_{K/M_K}) \circ \text{Av}_*^{N^-} \circ \text{oblV}_{K/M_K} \left[ -2 \dim(X) + \dim(M_K) \right]. \]

Now we use the fact that

\[ J \simeq \text{Y}^{-1} \circ \text{Av}_*^{N^-}. \]

**Remark 4.2.9.** Note that Conjectures 4.2.7 and 4.2.8 further reinforce the analogy between the functor \( \text{Ps-Id}_{\text{mod}^K_\chi} \) and the Deligne-Lusztig functor for \( p \)-adic groups.

**4.3. A plausibility check.** We will now juxtapose Conjecture 4.2.7 with Theorems 3.3.7 and 3.4.8 and arrive at a certain plausible (and at the level of abelian categories, known) statement.

4.3.1. Recall (see 4.1) that the functor

\[ J : \mathfrak{g}-\text{mod}^K_\chi \to \mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi \]

is isomorphic to

\[ \mathfrak{g}-\text{mod}^K_\chi \xrightarrow{\text{Av}_*^{N^-} \circ \text{oblV}_{K/M_K}} \mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi \xrightarrow{\text{Y}^-} \mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi. \]

In particular, it preserves compactness. Let

\[ J^{\text{op}} : (\mathfrak{g}-\text{mod}^K_\chi)^{\vee} \to (\mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi)^{\vee} \]

denote the corresponding **opposite** functor. I.e., this is the ind-extension of the functor

\[ ((\mathfrak{g}-\text{mod}^K_\chi)^{\vee})^c \simeq ((\mathfrak{g}-\text{mod}^K_\chi)^c)^{\text{op}} \xrightarrow{J^{\text{op}}} ((\mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi)^c)^{\text{op}} \simeq ((\mathfrak{g}-\text{mod}^{M_K \cdot N^-}_\chi)^{\vee})^c. \]
The isomorphism in (4.1) implies that with respect to the canonical identifications
\[(g\text{-mod}_{\chi}^K)^{\vee} \simeq g\text{-mod}_{-\chi}^K\text{ and } (g\text{-mod}_{\chi}^{MK\cdot N^{-}})^{\vee} \simeq g\text{-mod}_{-\chi}^{MK\cdot N^{-}}\]
of (2.13), we have a commutative diagram
\[
\begin{array}{ccc}
(g\text{-mod}_{\chi}^K)^{\vee} & \xrightarrow{J^p} & g\text{-mod}_{-\chi}^K \\
\downarrow J^p & & \downarrow J \\
(g\text{-mod}_{\chi}^{MK\cdot N^{-}})^{\vee} & \xrightarrow{f} & g\text{-mod}_{-\chi}^{MK\cdot N^{-}}.
\end{array}
\]

In other words, we have an isomorphism as contravariant functors
\[D_{\text{can}} \circ J \simeq J \circ D_{\text{can}}, \quad (g\text{-mod}_{\chi}^K)^c \Rightarrow (g\text{-mod}_{-\chi}^{MK\cdot N^{-}})^c\]

4.3.2. Let
\[J^- : g\text{-mod}_{\chi}^K \to g\text{-mod}_{\chi}^{MK\cdot N^-}\]
be the variant of the functor $J$, where we swap the roles of $N$ and $N^-$. We will now show that the following conjecture is equivalent to Conjecture 4.2.2:

**Conjecture 4.3.3.** The following diagram of functors commutes
\[
\begin{array}{ccc}
(g\text{-mod}_{\chi}^K)^{\vee} & \xrightarrow{\text{Sfr}} & g\text{-mod}_{-\chi}^K \\
\downarrow J^p & & \downarrow J^- \\
(g\text{-mod}_{\chi}^{MK\cdot N^{-}})^{\vee} & \xrightarrow{\text{Sfr}} & g\text{-mod}_{-\chi}^{MK\cdot N^-}.
\end{array}
\]

Equivalently, we have an isomorphism of the contravariant functors
\[D_{\text{contr}}^{\text{can}} \circ J \simeq J^- \circ D_{\text{contr}}^{\text{can}}, \quad (g\text{-mod}_{\chi}^K)^c \Rightarrow (g\text{-mod}_{-\chi}^{MK\cdot N^-})^c\]

**Remark 4.3.4.** Note that the analog of (4.4) for admissible representations of $p$-adic group is a known statement, which can also be easily deduced from the 2nd adjointness theorem.

**Proof.** Juxtaposing the diagrams (4.2) and (4.3) and using [CGY, Theorem 4.1.7], we obtain that Conjecture 4.3.3 is equivalent to the commutation of the diagram
\[
\begin{array}{ccc}
g\text{-mod}_{\chi}^K & \xrightarrow{(-\otimes \ell_K) \circ Se_{g\text{-mod}_{\chi}^K}} & g\text{-mod}_{\chi}^K \\
\downarrow J & & \downarrow J^- \\
g\text{-mod}_{\chi}^{MK\cdot N^-} & \xrightarrow{(-\otimes \ell_{MK}) \circ \Upsilon[2 \dim(X) - \dim(M_K)]} & g\text{-mod}_{\chi}^{MK\cdot N^-}.
\end{array}
\]

Using the fact that $J \simeq \Upsilon^{-1} \circ \text{Av}_N \circ \text{oblv}_{K/M_K}$, the commutativity of the above diagram is equivalent to the isomorphism
\[\text{Av}_N \circ \text{oblv}_{K/M_K}[2 \dim(X) - \dim(M_K)] \simeq (-\otimes \ell_{K/M_K}) \circ J^- \circ \text{Se}_{g\text{-mod}_{\chi}^K},\]
or equivalently
\[\text{Av}_N \circ \text{oblv}_{K/M_K} \circ \text{Ps-Id}_{g\text{-mod}_{\chi}^K} \simeq (-\otimes \ell_{K/M_K}) \circ J^-[-2 \dim(X) + \dim(M_K)],\]
while the latter is Conjecture 4.2.8.
4.3.5. Consider the ind-extensions of the (contravariant) functors \( \mathbb{D}^{\text{contr}}_{\mathfrak{g}/\mathfrak{h}} \circ J \) and \( J^- \circ \mathbb{D}^{\text{contr}}_{\mathfrak{g}/\mathfrak{h}} \), appearing in Conjecture 4.3.3. We obtain two functors
\[
\mathfrak{g}\text{-mod}^K_{\chi} \rightarrow (\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{op}},
\]
and let us restrict both to
\[
(\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}} \subset \mathfrak{g}\text{-mod}^K_{\chi}.
\]
We note:

**Lemma 4.3.6.** The functor \( J \) (resp., \( J^- \)) maps \((\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}}\) to \((\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}}\) (resp., \((\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}}\)).

*Proof.* According to [CGY, Theorem 4.4.2(a)], the functor \( J \) is t-exact. Hence, it is enough to show that it sends \((\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}}\) to \((\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}}\).

For that, it is sufficient to show that for every object \( M \in (\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}} \), there exists an object \( M' \in ((\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}})^{\leq 0} \) with \( H^0(M') \simeq M \), such that \( J(M') \in (\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}} \).

However, for every \( M \) as above, we can choose \( M' \) compact, such that \( H^0(M') \simeq M \). The claim now follows since \( J \) preserves compactness and
\[
(\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}} \subset (\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}}.
\]

□

4.3.7. Thus, we obtain two contravariant functors
\[
(\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}} \rightarrow (\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}},
\]
which according to Theorems 8.3.7 and 8.3.8 and [CGY, Theorem 4.4.2(a)] are t-exact.

Moreover, the corresponding functors
\[
(\mathfrak{g}\text{-mod}^K_{\chi})^{\text{f.g.}} \rightarrow (\mathfrak{g}\text{-mod}^M_{\mathfrak{h}})_{\chi}^{\text{f.g.}},
\]
identify with
\[
(4.5) \quad M \mapsto J(M)^{\vee} \quad \text{and} \quad M \mapsto J^-(M^{\vee}),
\]
respectively.

Now, the isomorphism between the functors [15] is known, when the ground field \( k \) equals \( \mathbb{C} \). Namely, W. Casselman constructed the map \( J^-(M^{\vee}) \rightarrow J(M)^{\vee} \) (see, for example, [Ca]), and D. Milicic, and later H. Hecht and W. Schmid, showed that it is an isomorphism (see [M], [HS]). The constructions and methods are analytic, using the asymptotic expansion of matrix coefficients.

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