(GORENSTEIN) SILTING MODULES IN RECOLLEMENTS

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Abstract. In the paper, we focus on the silting properties and the combinatorial properties of silting and Gorenstein, which is called Gorenstein silting, where the main tools used are recollements of module categories and tensor products. For a ring \( A \) and its idempotent ideal \( J \), we show that an \( A/J \)-module \( T \) is a silting \( A \)-module if and only if \( T \) is a silting \( A/J \)-module. For the finite dimensional \( k \)-algebras, with \( k \) a field, we show that the tensor products of silting modules are still silting. We also show that the (partial) Gorenstein silting properties can be glued by the recollements of module categories of Noetherian rings. As a consequence, we glue the Gorenstein silting modules of an upper triangular matrix Gorenstein ring by those of the involved rings.

1. Introduction

Silting modules over an arbitrary ring were introduced by Angeleri Hügel-Marks-Vitória [4], which are intended to generalize tilting modules. In particular, silting modules coincide with support \( \tau \)-tilting modules, which were introduced by Adachi-Iyama-Reiten [1], in the category of finitely generated modules of finite dimensional algebras. In [4], it is also proved how silting modules relate with 2-term silting complexes. Silting complexes were first introduced by Keller and Vossieck [23] to study \( t \)-structures in the bounded derived category of representations of Dynkin quivers. Silting modules share many properties with tilting modules and support \( \tau \)-tilting modules, so it has been studied and concerned by many scholars. The subject has been developed to an advanced level, see for examples [2, 5, 9, 10, 21, 24–26].

The study of Gorenstein homological algebra is due to Enochs and Jenda [13]. They introduced the concept of Gorenstein-projective modules, which are as a generalization of finitely generated modules of G-dimension zero over a two-sided Noetherian ring, in the sense of Auslander and Bridger [3]. The main idea of Gorenstein homological algebra is to replace projective modules by Gorenstein-projective modules, which is useful to study some Gorenstein properties. Gao-Ma-Zhang [18] studied Gorenstein silting modules, intending to understand silting modules more comprehensively and fully in the Gorenstein homological algebra, and also to fuse the properties of silting modules and Gorenstein-projective modules. Although the usual silting modules have been used effectively in Gorenstein homological algebra, Gorenstein silting modules has some advantages in the relative setting. For examples, the Gorenstein silting module is the module-theoretic counterpart of the
2-term Gorenstein silting complex ([12, 18]), which generates the bounded homotopy category of Gorenstein-projective modules.

Based on these works, we aim to consider fundamental problems that how the silting (resp. Gorenstein silting) property transfers and glues in the recollements of module categories, and that how the silting property is preserved under the tensor product.

For module categories of rings, each idempotent \(e\) in a ring \(A\) provides natural analogues of Grothendieck’s six functors, defining recollements of module categories

\[
A/AeA\text{-Mod} \quad \text{inc} \quad A\text{-Mod} \quad e(-) \quad eAe\text{-Mod}.
\]

These, and more generally recollements of abelian categories introduced by Beilinson-Bernstein-Deligne [8] have been used in various contexts (see for instance [11, 15, 16, 27, 28]).

In the paper, we give the answers by the tools of recollements and tensor products. Our main results are the follows:

**Theorem** The following statements hold.

(3.2) Let \(A\) be a ring and \(J\) the idempotent ideal of \(A\). Let \(T\) be a left \(A/J\)-module. Then \(T\) is a silting \(A\)-module if and only if \(T\) is a silting \(A/J\)-module.

(3.3) Let \(A\) and \(B\) be two finite dimensional \(k\)-algebras over a field \(k\). Suppose that \(T\) is a silting \(A\)-module and \(S\) is a silting \(B\)-module. Then \(T \otimes_k S\) is a silting \(A \otimes_k B\)-module.

(4.6) Let \(\Gamma = (A \ x \ N)\) be a Gorenstein ring such that gl.dim\(A < \infty\) and \(A N\) and \(N B\) are projective. Let \(X\) be an \(A\)-module and \(Y\) a \(B\)-module. Then the following are equivalent:

(i) \((a)\) \((X, 0) \oplus (N \otimes_B Y, Y)\) is a Gorenstein silting \(\Gamma\)-module;

(b) there exists a \(G\)-exact sequence \((P, 0) \oplus (N \otimes_B E, E) \xrightarrow{\lambda} (X, 0) \oplus (N \otimes_B Y, Y) \xrightarrow{i} (X, 0) \oplus (N \otimes_B Y, Y) \rightarrow 0\) with \(\lambda = \begin{pmatrix} (a) & 0 \\ 0 & (Id_N \otimes \psi) \end{pmatrix}\), \(X_i \in \text{Add}X\) and \(Y_i \in \text{Add}Y\) such that \(\lambda\) is the left \(D_\theta\)-approximation, \(i = -1, 0\), for every Gorenstein-projective \(\Gamma\)-module \((P, 0) \oplus (N \otimes_B E, E)\).

(ii) \((c)\) \(X\) is a Gorenstein silting \(A\)-module and \(N \otimes_B Y \in \text{Gen}_G X\);

(d) \(Y\) is a Gorenstein silting \(B\)-module;

(e) there exists an exact sequence \(P \xrightarrow{\phi} X \rightarrow X_{-1} \rightarrow 0\) with \(X_0\) and \(X_{-1}\) in \(\text{Add}X\) such that \(\phi\) is the left \(D_{\theta_X}\)-approximation, for each \(P \in A\text{-P}\);

(f) there exists a \(G\)-exact sequence \(E \xrightarrow{\psi} Y \rightarrow Y_{-1} \rightarrow 0\) with \(Y_0\) and \(Y_{-1}\) in \(\text{Add}Y\) such that \(\psi\) is the left \(D_{\theta_Y}\)-approximation, for each \(E \in B\text{-GP}\).
2. Preliminaries

In this section, we recall some basic definitions and facts that will be used throughout the paper.

Let $A$ be an associative ring and $A\text{-Mod}$ the category of left modules. Denote by $A\text{-P}$ the full subcategories of $A\text{-Mod}$ consisting of projective modules. A module $G$ of $A\text{-Mod}$ is Gorenstein-projective if there is an exact sequence

$$\ldots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \ldots$$

of projective modules of $A\text{-Mod}$, which stays exact after applying $\text{Hom}_A(-, P)$ for each projective module $P$, such that $G \cong \text{Ker}d^0$ (see [13,14]). Denote by $A\text{-GP}$ the full subcategories of $A\text{-Mod}$ consisting of Gorenstein-projective modules.

Given subcategories $\mathcal{X}$, $\mathcal{Y}$ of $A\text{-Mod}$. Following [7], a morphism $f : X \longrightarrow M$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation of $M$ in $A\text{-Mod}$ if any morphism from a module in $\mathcal{X}$ to $M$ factors through $f$. $\mathcal{X}$ is called contravariantly finite if any module in $A\text{-Mod}$ admits a right $\mathcal{X}$-approximation.

Following [14], an exact sequence $G_1 \xrightarrow{d^1} G_0 \longrightarrow M \longrightarrow 0$ (*)& is called a proper Gorenstein-projective presentation of $M$ if each $G_i$ is Gorenstein-projective and $\text{Hom}_A(G, G_1) \longrightarrow \text{Hom}_A(G, G_0) \longrightarrow \text{Hom}_A(G, T) \longrightarrow 0$ is exact for any Gorenstein-projective module $G$. (*)& is minimal if $G_1 \longrightarrow \text{Im}d^1$ and $G_0 \longrightarrow M$ are right $A\text{-GP}$-approximation. Moreover, the exact sequence $0 \longrightarrow G_n \longrightarrow \ldots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ is called a proper Gorenstein-projective resolution of $M$ of length $n$ for some non-negative integer $n$, if each $G_i$ is all Gorenstein-projective and $0 \longrightarrow \text{Hom}_A(G, G_n) \longrightarrow \ldots \longrightarrow \text{Hom}_A(G, G_0) \longrightarrow \text{Hom}_A(G, M) \longrightarrow 0$ is exact for any Gorenstein-projective module $G$.

2.1. Triangular matrix rings. Let $A$ and $B$ be rings, $AN_B$ an $(A, B)$-bimodule. Then the triangular matrix ring

$$\Gamma := \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$$

can be defined by the ordinary operations on matrices.

Recall from [6,19] that a left $\Gamma$-module can be identified with a triple $(X, Y, f)$, or simply $(X, Y)$ if $f$ is clear, where $X \in A\text{-Mod}$, $Y \in B\text{-Mod}$, and $f : N \otimes_B Y \longrightarrow X$ is an $A$-map. A $\Gamma$-map $(X, Y, f) \longrightarrow (X', Y', f')$ will be identified with a pair $(a, b)$, where $a \in \text{Hom}_A(X, X')$, $b \in \text{Hom}_B(Y, Y')$, such that the following diagram commutes:

$$\begin{array}{ccc}
N \otimes_B Y & \xrightarrow{f} & X \\
\downarrow \text{Id}_N \otimes b & & \downarrow a \\
N \otimes_B Y' & \xrightarrow{f'} & X'.
\end{array}$$

A sequence $0 \longrightarrow (X_1, Y_1, f_1) \longrightarrow (X_2, Y_2, f_2) \longrightarrow (X_3, Y_3, f_3) \longrightarrow 0$ in $\Gamma$-Mod is exact if and only if $0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$ and $0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow 0$ are exact. Let $\Gamma$ be a Noetherian ring. Indecomposable projective $\Gamma$-modules are exactly $(P, 0)$ and $(N \otimes_B Q, Q)$, where $P$ runs over indecomposable projective $A$-modules, and $Q$ runs over indecomposable projective $B$-modules.
2.2. Silting modules. Let $A$ be an associative ring. We denote by $\text{Add}T$ the class of all modules which are isomorphic to direct summands of direct sums of copies of $T$, for an $A$-module $T$.

Let $\sigma : P_1 \rightarrow P_0$ be a morphism of projective $A$-modules. Consider the class:
$$D_\sigma := \{ X \in A\text{-Mod} \mid \text{Hom}_A(\sigma, X) \text{ is an epimorphism} \}.$$ 

Definition 2.1. ([4, Definition 3.7]) We call that an $A$-module $X$ is
- partial silting if there is a projective presentation $\sigma$ of $X$ such that
  - (S1) $D_\sigma$ is a torsion class (i.e. closed for epimorphic images, extensions and coproducts);
  - (S2) $X$ lies in $D_\sigma$.
- silting if there is a projective presentation $\sigma$ of $X$ such that $\text{Gen} X = D_\sigma$, where $\text{Gen} X$ is the subcategory of all epimorphic images of modules in $\text{Add} X$.

2.3. Gorenstein silting modules. Let $A$ be a Noetherian ring, and $A\text{-Mod}$ the category of all left $R$-modules. We denote by $A\text{-GP}$ the full subcategory of $A\text{-Mod}$ consisting of Gorenstein-projective modules. Assume that $A\text{-GP}$ is contravariantly finite in $A\text{-Mod}$. For $A$-modules $M$ and $N$, we compute right derived functors of $\text{Hom}_A(M,N)$ using a proper Gorenstein-projective resolution of $M$ ([14], [20]). We will denote these derived functors by $\text{Gext}_A^i(M,N)$. A short exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is G-exact if and only if it is in $\text{Gext}_A^1(L,M)$.

Let $T$ be an $A$-module. Put
$$\text{Gen}_G(T) := \{ M \in A\text{-Mod} \mid \exists \text{ a G-exact sequence } T_0 \rightarrow M \rightarrow 0 \text{ with } T_0 \in \text{Add} T \}.$$ 

For a morphism $\theta : G_1 \rightarrow G_0$ with Gorenstein-projective modules $G_1$ and $G_0$. We consider the class of $A$-modules
$$D_\theta := \{ X \in A\text{-Mod} \mid \text{Hom}_A(\theta, X) \text{ is epic} \}.$$ 

Definition 2.2. ([18, Definition 2.2]) We say that an $A$-module $T$ is
- partial Gorenstein silting if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that
  - (Gs1) $D_\theta$ is a relative torsion class (i.e. closed for $G$-epimorphic images, $G$-extensions and coproducts);
  - (Gs2) $T$ lies in $D_\theta$.
- Gorenstein silting if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that $\text{Gen}_G(T) = D_\theta$.

From [18, Lemma 2.1], $D_\theta$ is always closed under $G$-epimorphic images, $G$-extensions. Hence $D_\theta$ is a relative torsion class if and only if it is closed under coproducts.

3. Transfer property of silting modules

In this section, we focus on the transferring properties of silting modules, where the main tools are recollements of module categories and tensor products.
3.1. **Silting modules and idempotent ideals.** In this subsection, we show that an $A/J$-module $T$ is a silting $A$-module if and only if $T$ is a silting $A/J$-module whenever $J$ is an idempotent ideal of a ring $A$.

**Lemma 3.1.** Let $A, B$ and $C$ be three rings such that there exists the recollement of module categories

\[
\begin{array}{ccc}
A\text{-Mod} & & B\text{-Mod} \\
\downarrow i & & \downarrow e \\
B\text{-Mod} & & C\text{-Mod}
\end{array}
\]

Then the following hold.

(i) $T$ is a silting $A$-module if and only if $i(T)$ is a silting $B$-module.

(ii) If $T$ is a silting $B$-module, then $q(T)$ is a silting $A$-module.

**Proof.** (i). Suppose that $P_1 \xrightarrow{\sigma} P_0 \xrightarrow{} i(T) \xrightarrow{} 0$ is a projective presentation of $i(T)$ for an $A$-module $T$. We claim that $\text{Gen}(T) = D_{q(\sigma)}$ if and only if $\text{Gen}(i(T)) = D_{\sigma}$. Indeed, on one hand, since there is the commutative diagram with the isomorphic vertical maps

\[
\begin{array}{ccc}
\text{Hom}_A(q(P_0), X) & \xrightarrow{} & \text{Hom}_A(q(P_1), X) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_B(P_0, i(X)) & \xrightarrow{} & \text{Hom}_A(P_1, i(X))
\end{array}
\]

it follows that $X \in D_{q(\sigma)}$ if and only if $i(X) \in D_{\sigma}$. On the other hand, $X \in \text{Gen}(T)$ if and only if there exists an epimorphism $T^{\nu} \xrightarrow{} X$ for some index set $\nu$ if and only if $i(T^{\nu}) \xrightarrow{} i(X)$ is an epimorphism if and only if $i(X) \in \text{Gen}(i(T))$.

(ii). Taking a projective presentation $Q_1 \xrightarrow{\kappa} Q_0 \xrightarrow{} T \xrightarrow{} 0$ of $T$, we get that $q(Q_1) \xrightarrow{q(\kappa)} q(Q_0) \xrightarrow{} q(T) \xrightarrow{} 0$ is a projective presentation of $q(T)$. We prove that $D_{q(\kappa)} = \text{Gen}(q(T))$.

Since there are isomorphisms

\[
\text{Hom}_A(q(Q_0), q(T)) \cong \text{Hom}_B(Q_0, iq(T))
\]

and

\[
\text{Hom}_A(q(Q_1), q(T)) \cong \text{Hom}_B(Q_1, iq(T))
\]

and the exact sequence $T \xrightarrow{} iq(T) \xrightarrow{} 0$, we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_B(Q_0, T) & \xrightarrow{} & \text{Hom}_B(Q_0, iq(T)) \xrightarrow{} 0 \\
\downarrow & & \downarrow \\
\text{Hom}_B(Q_1, T) & \xrightarrow{} & \text{Hom}_B(Q_1, iq(T)) \xrightarrow{} 0,
\end{array}
\]

which shows that $q(T) \in D_{q(\kappa)}$ and thus $\text{Gen}(q(T)) \subseteq D_{q(\kappa)}$. 
On the other hand, let \( X \in D_{q(k)} \), there is an epimorphism \( \text{Hom}_A(q(Q_0), X) \to \text{Hom}_A(q(Q_1), X) \to 0 \), that is,

\[
\begin{array}{ccc}
\text{Hom}_A(q(Q_0), X) & \to & \text{Hom}_A(q(Q_1), X) \\
\cong & & \cong \\
\text{Hom}_B(Q_0, i(X)) & \to & \text{Hom}_B(Q_1, i(X)) \\
\end{array}
\]

Thus we have that \( i(X) \in D_\kappa = \text{Gen}(T) \) and so \( X \in \text{Gen}(q(T)) \).

\( \square \)

**Theorem 3.2.** Let \( A \) be a ring and \( J \) the idempotent ideal of \( A \). Let \( T \) be a left \( A/J \)-module. Then \( T \) is a silting \( A \)-module if and only if \( T \) is a silting \( A/J \)-module.

**Proof.** The result follows immediately from Lemma 3.1.

\( \square \)

### 3.2. Tensor product of silting modules

Throughout this subsection, algebras are finite dimensional, and modules are finitely generated. We show that the tensor products of silting modules are also silting modules.

**Theorem 3.3.** Let \( A \) and \( B \) be two algebras, \( \sigma : P_1 \to P_0 \) the morphism of projective \( A \)-modules and \( \eta : Q_1 \to Q_0 \) the morphism of projective \( B \)-modules. Suppose that \( T \) is a silting \( A \)-module with respect to \( \sigma \) and \( S \) is a silting \( B \)-module with respect to \( \eta \). Then \( T \otimes_k S \) is a silting \( A \otimes_k B \)-module with respect to \( \sigma \otimes \eta \).

**Proof.** From the exact sequences

\[
P_1 \xrightarrow{\sigma} P_0 \to T \to 0 \quad \text{and} \quad Q_1 \xrightarrow{\eta} Q_0 \to S \to 0,
\]

we get the induced exact sequence

\[
P_1 \otimes_k Q_1 \xrightarrow{\sigma \otimes \eta} P_0 \otimes_k Q_0 \to T \otimes_k S \to 0.
\]

Since there are the following isomorphisms

\[\text{Hom}_{A \otimes_k B}(P_i \otimes_k Q_i, T \otimes_k S) \cong \text{Hom}_A(P_i, T) \otimes_k \text{Hom}_B(Q_i, S),\]

for \( i = 0, 1 \), it follows that \( T \otimes_k S \in D_{\sigma \otimes \eta} \) if and only if \( T \in D_\sigma \) and \( S \in D_\eta \). This is to say, \( D_{\sigma \otimes \eta} = D_\sigma \otimes D_\eta \).

Since \( D_\sigma \) and \( D_\eta \) are closed under direct sums, it is obvious that \( D_{\sigma \otimes \eta} \) has the same property, i.e., \( D_{\sigma \otimes \eta} \) is a torsion class. Furthermore, for every \( M \otimes_k N \in D_{\sigma \otimes \eta} \), we have that

\[
M \in D_\sigma = \text{Gen}T \quad \text{and} \quad N \in D_\eta = \text{Gen}S.
\]

Therefore, \( M \otimes_k N \in \text{Gen}T \otimes_k \text{Gen}S \subseteq \text{Gen}(T \otimes_k S) \). This completes the proof.

\( \square \)

### 4. Transfer property of Gorenstein silting modules

In this section, we focus on the transferring properties of Gorenstein silting modules, where the main tool is the recollement of module categories. As a consequence, we characterise the relations of Gorenstein silting modules among the rings \( A, B \) and \( \Gamma \), where \( \Gamma \) is the upper triangular matrix rings building from \( A \) and \( B \).

Now let \( A, B \) and \( C \) be three Noetherian rings such that \( A-\text{GP} \) and \( C-\text{GP} \) are contravariantly finite in \( A-\text{Mod} \) and \( C-\text{Mod} \), respectively. Suppose that there exists
the recollement of module categories

\[
\begin{array}{c}
\text{A-Mod} \lla \text{B-Mod} \lla \text{C-Mod}
\end{array}
\]

such that \( p \) has an exact right adjoint \( p_1 \) and \( l \) has a right adjoint \( l^1 \) such that \( l^1 i \) preserves Gorenstein-projective modules. Let \( X \in \text{A-Mod} \) and \( Y \in \text{C-Mod} \), and

\[
E_1 \xrightarrow{\theta_0} E_0 \longrightarrow X \longrightarrow 0 \tag{1}
\]

and

\[
G_1 \xrightarrow{\theta_1} G_0 \longrightarrow Y \longrightarrow 0 \tag{2}
\]

the proper Gorenstein-projective presentations of \( X \) and \( Y \) respectively. Applying the functors \( i(-) \) and \( l(-) \) to (1) and (2) respectively, we get the following exact sequences:

\[
i(E_1) \xrightarrow{i(\theta_0)} i(E_0) \longrightarrow i(X) \longrightarrow 0
\]

and

\[
l(G_1) \xrightarrow{l(\theta_1)} l(G_0) \longrightarrow l(Y) \longrightarrow 0.
\]

Then we get the following exact sequence:

\[
i(E_1) \oplus l(G_1) \xrightarrow{\theta} i(E_0) \oplus l(G_0) \longrightarrow i(X) \oplus l(Y) \longrightarrow 0 \tag{3}
\]

with \( \theta = \begin{pmatrix} i(\theta_0) & 0 \\ 0 & l(\theta_1) \end{pmatrix} \).

**Lemma 4.1.** The exact sequence (3) mentioned above is a proper Gorenstein-projective presentation of \( i(X) \oplus l(Y) \).

**Proof.** By assuming that \( p \) is an exact functor, it follows that \( i \) preserves projective modules and \( r \) admits a right adjoint \( r_1 \). Let \( T \) be a projective \( B \)-module, then there is an \( A \)-module \( Z \) such that there exists the following exact sequence:

\[
0 \longrightarrow i(Z) \longrightarrow le(T) \longrightarrow T \longrightarrow iq(T) \longrightarrow 0.
\]

By \( iq(T) \) being projective, we have that \( T \cong K \oplus iq(T) \) and \( le(T) \cong K \oplus i(Z) \) with \( e(K) \cong e(T) \). This means that the projective \( B \)-modules are of the form either \( i(P) \) or \( l(Q) \), where \( P \) runs over projective \( A \)-modules and \( Q \) runs over projective \( C \)-modules.

With \( P' \) being \( A \)-projective and \( Q' \) being \( C \)-projective, since there are the isomorphisms

\[
\text{Hom}_B(i(P'), i(P) \oplus l(Q)) \cong \text{Hom}_A(P', P) \oplus \text{Hom}_A(P', pl(Q))
\]

and

\[
\text{Hom}_B(l(Q'), i(P) \oplus l(Q)) \cong \text{Hom}_C(Q', Q),
\]

we get that each indecomposable Gorenstein-projective \( B \)-module is of the form \( i(E) \) or \( l(G) \) whenever \( E \) runs over all the indecomposable Gorenstein-projective \( A \)-modules and \( G \) runs over all the indecomposable Gorenstein-projective \( C \)-modules. Therefore, \( i(E_1) \oplus l(G_1) \) and \( i(E_0) \oplus l(G_0) \), mentioned in the exact sequence (3), are Gorenstein-projective \( B \)-modules.
Now we prove the exact sequence (3) is $\text{Hom}_B(S, -)$-exact for any Gorenstein-projective $B$-module $S = i(E) \oplus l(G)$. Since there are the following commutative diagrams, where we denote $\text{Hom}(-, -)$ by $(-, -)$ simply:

\[
\begin{align*}
(i(E), i(E_1) \oplus l(G_1)) & \longrightarrow (i(E), i(E_0) \oplus l(G_0)) \longrightarrow (i(E), i(X) \oplus l(Y)) \\
\cong & \hspace{1cm} \cong \hspace{1cm} \cong
\end{align*}
\]

\[
\begin{align*}
(E, E_1) \oplus (l^1i(E), G_1) & \longrightarrow (E, E_0) \oplus (l^1i(E), G_0) \longrightarrow (E, X) \oplus (l^1i(E), Y)
\end{align*}
\]

and

\[
\begin{align*}
(l(G), i(E_1) \oplus l(G_1)) & \longrightarrow (l(G), i(E_0) \oplus l(G_0)) \longrightarrow (l(G), i(X) \oplus l(Y)) \\
\cong & \hspace{1cm} \cong \hspace{1cm} \cong
\end{align*}
\]

\[
\begin{align*}
(G, G_1) & \longrightarrow (G, G_0) \longrightarrow (G, Y)
\end{align*}
\]

it is obvious that the sequence (3) is a $G$-exact sequence. \qed

**Lemma 4.2.** Let $X \in \text{A-Mod}$, $Y \in \text{C-Mod}$, and $Z \in \text{B-Mod}$. The following statements hold.

(i) $Z \in D_\theta$ if and only if $p(Z) \in D_{\theta_X}$ and $e(Z) \in D_{\theta_Y}$.

(ii) If $X \in D_{\theta_X}$, then $i(X) \in D_\theta$.

(iii) If $Y \in D_{\theta_Y}$ and $pl(Y) \in D_{\theta_X}$, then $l(Y) \in D_\theta$.

(iv) If $X \oplus l(Y) \in D_\theta$ if and only if $X \in D_{\theta_X}$, $Y \in D_{\theta_Y}$ and $pl(Y) \in D_{\theta_X}$.

(v) $D_\theta$ is a relative torsion class if and only if both $D_{\theta_X}$ and $D_{\theta_Y}$ are relative torsion classes.

**Proof.** Let $Z \in D_\theta$. By the definition of $D_\theta$, there is the following commutative diagram:

\[
\begin{align*}
\text{Hom}_B(i(E_0) \oplus l(G_0), Z) & \longrightarrow \text{Hom}_B(i(E_1) \oplus l(G_1), Z) \\
\cong & \hspace{1cm} \cong
\end{align*}
\]

\[
\begin{align*}
\text{Hom}_A(E_0, p(Z)) \oplus \text{Hom}_C(G_0, e(Z)) & \longrightarrow \text{Hom}_A(E_1, p(Z)) \oplus \text{Hom}_C(G_1, e(Z))
\end{align*}
\]

this induces the exact sequences:

\[
\text{Hom}_A(E_0, p(Z)) \longrightarrow \text{Hom}_A(E_1, p(Z)) \longrightarrow 0
\]

and

\[
\text{Hom}_C(G_0, e(Z)) \longrightarrow \text{Hom}_C(G_1, e(Z)) \longrightarrow 0,
\]

that is, $p(Z) \in D_{\theta_X}$ and $e(Z) \in D_{\theta_Y}$. Conversely, if $p(Z) \in D_{\theta_X}$ and $e(Z) \in D_{\theta_Y}$, we can get from using the above commutative diagram once again that $Z \in D_\theta$.

(ii)-(v) follow directly from (i). \qed

**Proposition 4.3.** $i(X) \oplus l(Y)$ is a partial Gorenstein silting $B$-module if and only if $X$ and $pl(Y)$ are partial Gorenstein silting $A$-modules and $Y$ is a partial Gorenstein silting $B$-module.

**Proof.** The result is immediately from Lemma 4.1 and Lemma 4.2. \qed
Now we apply it to the triangular matrix ring. In the following, let $\Gamma = \left( \begin{array}{cc} A & N \\ 0 & B \end{array} \right)$ be a Gorenstein ring such that $\text{gl.dim}A < \infty$ and $AN$ and $NB$ are projective. Then there is the following recollement:

![Diagram](image)

where $Z\Gamma(X) = (X,0,0)$, $U\Gamma(X,Y,f) = X$, $T\Gamma(Y) = (N \otimes_B Y,Y,1)$ and $H\Gamma(X) = (X,\text{Hom}_A(N,X),\varepsilon_X)$ with $\varepsilon$ being the counit of $(N \otimes_B -, \text{Hom}_A(N,-))$.

Let $X \in A\text{-Mod}$ and $Y \in B\text{-Mod}$, and let
\[ P_1 \xrightarrow{\theta_X} P_0 \to X \to 0 \]
and
\[ E_1 \xrightarrow{\theta_Y} E_0 \to Y \to 0 \]
be the projective presentation of $X$ and the proper Gorenstein-projective presentation of $Y$ respectively. Then by Lemma 4.1 the sequence
\[ T^\bullet : (P_1,0) \oplus (N \otimes_B E_1,E_1) \xrightarrow{\theta} (P_0,0) \oplus (N \otimes_B E_0,E_0) \to (X,0) \oplus (N \otimes_B Y,Y) \to 0 \]
is the proper Gorenstein-projective presentation of $(X,0) \oplus (N \otimes_B Y,Y)$.

As an immediate consequence of Proposition 4.3, we have the following.

**Corollary 4.4.** $(X,0) \oplus (N \otimes_B Y,Y)$ is a partial Gorenstein silting module with respect to $\theta$ if and only if $X$ is a partial Gorenstein silting $A$-module with respect to $\theta_X$, and $Y$ is a partial Gorenstein silting $B$-module with respect to $\theta_Y$ such that $N \otimes_B Y \in \text{Gen}_G X$.

It can be seen from the following theorem that the Gorenstein silting modules cannot be directly converted over $\Gamma$ and $A$, $B$, and additional conditions are required.

**Lemma 4.5.** ([18, Proposition 2.3]) Let $T$ be an $A$-module with proper Gorenstein-projective presentation $\theta : G_1 \to G_0$. If $T$ is a partial Gorenstein silting module with respect to $\theta$, and for each $P \in A\text{-GP}$, there exists a $G$-exact sequence $P \xrightarrow{\phi} T_0 \to T_{-1} \to 0$ with $T_0$ and $T_{-1}$ in $\text{Add} T$ such that $\phi$ is the left $D_\theta$-approximation, then $T$ is a Gorenstein silting module.

**Theorem 4.6.** Let $X \in A\text{-Mod}$ and $Y \in B\text{-Mod}$. Then the following are equivalent:

(i) (a) $(X,0) \oplus (N \otimes_B Y,Y)$ is a Gorenstein silting $\Gamma$-module;
   
   (b) there exists a $G$-exact sequence $(P,0) \oplus (N \otimes_B E,E) \xrightarrow{\lambda} (X,0) \oplus (N \otimes_B Y,Y) \to 0$ with $\lambda = \left( \begin{array}{cc} (\phi,0) & 0 \\ 0 & (\text{Id}_N \otimes \psi,\psi) \end{array} \right)$, $X_i \in \text{Add} X$ and $Y_i \in \text{Add} Y$ such that $\lambda$ is the left $D_\theta$-approximation, $i = -1,0$, for the Gorenstein-projective $\Gamma$-module $(P,0) \oplus (N \otimes_B E,E)$.

(ii) (c) $X$ is a Gorenstein silting $A$-module and $N \otimes_B Y \in \text{Gen}_G X$;
   
   (d) $Y$ is a Gorenstein silting $B$-module;
(e) there exists an exact sequence $P \xrightarrow{\phi} X_0 \rightarrow X_{-1} \rightarrow 0$ with $X_0$ and $X_{-1}$ in $\text{Add}X$ such that $\phi$ is the left $D_{\theta X}$-approximation, for each $P \in A$.

(f) there exists a $G$-exact sequence $E \xrightarrow{\psi} Y_0 \rightarrow Y_{-1} \rightarrow 0$ with $Y_0$ and $Y_{-1}$ in $\text{Add}Y$ such that $\psi$ is the left $D_{\theta Y}$-approximation, for each $E \in B$.

Proof. (i) $\Rightarrow$ (ii) Let $(X,0) \oplus (N \otimes_B Y,Y)$ be a Gorenstein silting $\Gamma$-module. Then

$$D_\theta = \text{Gen}_G((X,0) \oplus (N \otimes_B Y,Y)).$$

Moreover, by Corollary 4.4, $\text{Gen}_G(X) \subseteq D_{\theta X}$, $\text{Gen}_G Y \subseteq D_{\theta Y}$, $N \otimes_B Y \in D_{\theta X}$. And $(K,0) \in D_\theta$ if $K \in D_{\theta X}$ by Lemma 4.2. Now let $K \in D_{\theta X}$. Then there is a $G$-epimorphism

$$((X,0) \oplus (N \otimes_B Y,Y))^I \rightarrow (K,0)$$

for an index set $I$. If $(K,0)$ has a direct summand $(K',0)$ such that there is a $G$-epimorphism $(N \otimes_B Y,Y)^I \rightarrow (K',0)$ for some index set $J$. Then there is the following commutative diagram with $G$-epic columns

$$
\begin{array}{ccc}
(N \otimes_B Y)^I & \xrightarrow{1} & (N \otimes_B Y)^I \\
0 & \downarrow & \kappa \\
0 & \rightarrow & K',
\end{array}
$$

and so $K' = 0$. This implies that $(K,0) \in \text{Gen}_G((X,0) \oplus (N \otimes_B Y,Y))$. Thus $X$ is a Gorenstein silting $\mathcal{A}$-module and $N \otimes_B Y \in D_{\theta X} = \text{Gen}_G X$. On the other hand, let $L \in D_{\theta Y}$, then $(0,L) \in D_\theta$. Hence there is a $G$-epimorphism

$$((X,0) \oplus (N \otimes_B Y,Y))^I \rightarrow (0,L)$$

for an index set $I$. Then we can get that there is a $G$-epimorphism $(N \otimes_B Y,Y)^I \rightarrow (0,L)$, and so we have a $G$-epimorphism $Y^I \rightarrow L$. This implies that $D_{\theta Y} \subseteq \text{Gen}_G Y$. Thus $Y$ is a Gorenstein silting $B$-module.

From (b), we can get the exact sequence $P \xrightarrow{\phi} X_0 \rightarrow X_{-1} \rightarrow 0$ with $X_i \in \text{Add}X$ and the G-exact sequence $E \xrightarrow{\psi} Y_0 \rightarrow Y_{-1} \rightarrow 0$ with $Y_i \in \text{Add}Y$ for $i = -1,0$, where $P \in A$ and $E \in B$-GP. We claim that $\phi$ is the left $D_{\theta X}$-approximation and $\psi$ is the left $D_{\theta Y}$-approximation. In fact, let $U \in D_{\theta X}$ and $a_1 \in \text{Hom}_X((P,0) \oplus (N \otimes_B E,E), (U,0))$. Then there exists $((a'_1,0), v) \in \text{Hom}_X((X_0,0) \oplus (N \otimes_B Y_0,Y_0), (U,0))$ such that the following diagram commutes:

$$
\begin{array}{ccc}
(P,0) \oplus (N \otimes_B E,E) & \xrightarrow{\lambda} & (X_0,0) \oplus (N \otimes_B Y_0,Y_0) \\
((a_1,0),) & \downarrow & (a'_1,0), v) \\
(U,0), & \xrightarrow{(\gamma,0), v) & (U,0),
\end{array}
$$

Then we get from $((a'_1,0), v) \circ \lambda = ((a_1,0), 0)$ that $a'_1 \circ \phi = a_1$. This implies that $\phi$ is a left $D_{\theta X}$-approximation. Similarly, we can prove that $\psi$ is the left $D_{\theta Y}$-approximation.

(ii) $\Rightarrow$ (i) We know from Corollary 4.4 that $(X,0) \oplus (N \otimes_B Y,Y)$ is a partial Gorenstein silting module. Since there are the $G$-exact sequences

$$G^\cdot : P \xrightarrow{\phi} X_0 \rightarrow X_{-1} \rightarrow 0$$
and
\[ E^\bullet : E \xrightarrow{\psi} Y_0 \longrightarrow Y_{-1} \longrightarrow 0, \]
we get an exact sequence
\[ S^\bullet : (P, 0) \oplus (N \otimes_B E, E) \xrightarrow{\lambda} (X_0, 0) \oplus (N \otimes_B Y_0, Y_0) \longrightarrow (X_{-1}, 0) \oplus (N \otimes_B Y_{-1}, Y_{-1}) \rightarrow 0 \]
with \( \lambda = \left( \begin{smallmatrix} (\phi, 0) & 0 \\ 0 & (\text{Id}_N \otimes \psi, \psi) \end{smallmatrix} \right) \).
Moreover, since each Gorenstein-projective \( \Gamma \)-module is of the form \( (P, 0) \oplus (N \otimes_B E, E) \), we get that \( \text{Hom}_\Gamma((P, 0) \oplus (N \otimes_B E, E), S^\bullet) \)
is exact. Therefore, \( S^\bullet \) is \( G \)-exact.

We next prove that \( \lambda \) is the left \( D_\theta \)-approximation. Let \( (U, V, h) \in D_\theta \)
and \( ((a_1, 0), (a_2, b_2)) \in \text{Hom}_\Gamma((P, 0) \oplus (N \otimes_B Q, Q), (U, V, h)) \). Then there exist the following commutative diagrams below:
\[ (P, 0) \oplus (N \otimes_B Q, Q) \xrightarrow{\lambda} (X_0, 0) \oplus (N \otimes_B Y_0, Y_0) \]
\[ \downarrow ((a_1, 0), (a_2, b_2)) \]
\[ (U, V, h). \]
and
\[ N \otimes_B Q \xrightarrow{\text{Id}_N \otimes b_2} N \otimes_B Q \]
\[ \downarrow \text{Id}_N \otimes b_2 \]
\[ N \otimes_B V \xrightarrow{h} U. \]
Moreover, \( U \in D_{\theta_X} \) and \( V \in D_{\theta_Y} \), by Lemma 4.2. Since \( \phi \) is a left \( D_{\theta_X} \)-approximation and \( \psi \) is a left \( D_{\theta_Y} \)-approximation, there exist \( a_1' \in \text{Hom}_A(X_0, U) \) and \( b_2' \in \text{Hom}_B(Y_0, V) \) respectively such that \( a_1 = a_1' \phi \) and \( b_2 = b_2' \psi \), as shown in the diagram below:
\[ P \xrightarrow{\phi} X_0 \]
\[ \downarrow a_1 \]
\[ U \xrightarrow{a_1'} \]
\[ Q \xrightarrow{\psi} Y_0 \]
\[ \downarrow b_2 \]
\[ V \xrightarrow{b_2'} \]
Then we can get the following equalities:
\[ ((a_1', 0), (h(\text{Id}_N \otimes b_2), b_2')) \circ \lambda = ((a_1', 0), (h(\text{Id}_N \otimes b_2), b_2')) \circ \left( \begin{smallmatrix} (\phi, 0) & 0 \\ 0 & (\text{Id}_N \otimes \psi, \psi) \end{smallmatrix} \right) \]
\[ = ((a_1', 0), (h(\text{Id}_N \otimes b_2)(\text{Id}_N \otimes \psi), b_2' \psi)) \]
\[ = ((a_1, 0), (a_2, b_2)). \]
This implies that \( \lambda \) is a left \( D_\theta \)-approximation. Therefore we can get from Lemma 4.5 that \( (X, 0) \oplus (N \otimes_B Y, Y) \) is a Gorenstein silting \( \Gamma \)-module.

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