ON ONE-POINT FUNCTIONS OF DESCENDANTS IN SINE-GORDON MODEL

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ABSTRACT. We apply the fermionic description of CFT developed in our previous work to the computation of the one-point functions of the descendent fields in the sine-Gordon model.

1. INTRODUCTION

The sine-Gordon (sG) model is the most famous example of two-dimensional integrable Quantum Field Theory (QFT). The sG model is defined in two-dimensional Minkowski space with coordinate \( x = (x_0, x_1) \) by the action

\[
A_{\text{sG}} = \int \left\{ \frac{1}{16\pi} (\partial_\mu \varphi(x))^2 + \frac{2\mu^2}{\sin(\pi\beta)} \cos(\beta \varphi(x)) \right\} d^2x.
\]

The normalisation of the dimensional coupling constant in front of \( \cos(\beta \varphi(x)) \) is chosen for future convenience. This model has been a subject of intensive study during the last 30 years. First, by semi-classical methods the exact spectrum was computed, the factorisation of scattering was predicted, and the exact \( S \)-matrix was found for certain values of the coupling constant (in the absence of reflection of solitons) \[2, 3\].

The most significant further results were found by the bootstrap method. In \[4, 5\] the exact \( S \)-matrix was found, and in \[6, 7\] the exact form factors were computed for the energy-momentum tensor, topological current and the operators \( e^{\pm i\beta \varphi(x)} \), \( e^{\pm i\frac{\beta}{2} \varphi(x)} \). Then the latter result was generalised to the operator \( e^{ia\varphi(x)} \) with an arbitrary \( a \) \[8\], using the methods which go back to the study of the closely related XXZ spin chain \[9, 10\].

It should be said that many important technical and conceptual methods of the modern theory of quantum integrable models originate in the quantum inverse scattering method \[11, 12\]. It provides a clear mathematical interpretation of the work by R. Baxter \[13\]. In particular, in \[12\] the scattering matrix of the sG solitons was reproduced using the Bethe Ansatz.

The knowledge of form factors allows us to write a series representation for the two-point function

\[
\langle e^{ia_1 \varphi(x)} e^{ia_2 \varphi(0)} \rangle_{\text{sG}}.
\]

In this paper we shall consider only the space-like region \( x^2 < 0 \). We shall use a lattice regularisation which breaks the Lorentz invariance. So we shall take \( x = (0, x) \), and use the notation \( \varphi(x) = \varphi(0, x) \). In this case the integrals over the form factors are rapidly convergent. It is rather hard to give a mathematically rigorous proof.
of the convergence of the series, but nobody doubts that they converge. Actually the convergence was proved \[14\] for a certain particular reduction of the sG model, known as the scaling Lee-Yang theory. So, the situation looks completely satisfactory. However, the series over the form factors converge slowly in the ultraviolet region (for small values of \(-x^2\)). To give an efficient description to its ultraviolet behaviour remained a problem largely open for some time. Before describing the way of solving this problem let us explain how it is solved in a particular case.

It is known that at \(\beta^2 = 1/2\) the sine-Gordon model is equivalent to the free theory of a Dirac fermion. The correlation functions of (2) are non-trivial. The sG field \(\varphi(x)\) is bilinear in fermions, and one has to compute the correlation function of two exponentials of bilinear forms. The result is obtained in \[15\], generalising the seminal work \[16\] on the scaling Ising model. Namely, it is shown that the correlation function satisfies an equivalent of a Painlevé III equation. The form factors are very simple in this case, and the form factor series coincides with the Fredholm determinant representation of this solution. Still the problem of describing the ultra-violet behaviour is non-trivial. It amounts to finding the connection coefficients for the Painlevé equation, which is done by studying the Riemann-Hilbert problem \[17\]. Through this analysis, one draws an important conclusion. Since CFT completely describes the ultra-violet limit, one might naively expect that the asymptotic behaviour of the two-point function may be obtained via the perturbation theory. Such an assumption would imply that the dependence on the mass scale is analytic in \(\mu^2\) which has the dimension \((\text{mass})^{2(1-\beta^2)}\). However this is not the case even for the Painlevé solution.

Correct understanding of the conformal perturbation theory is one of the most important problems in the theory of quantum integrable systems. This problem was studied in \[18\]. In fact, the naive perturbation theory suffers from both ultraviolet and infrared divergencies. The idea of \[18\] is to absorb all these divergencies into non-perturbative data: one-point functions of primary fields and their descendants. Once it is done, the remaining task is a convergent version of conformal perturbation theory. So the problem is divided into two steps. The first one requires some non-perturbative information. The second one is of genuinely CFT origin: actually, it is reduced to the computation of some Dotsenko-Fateev Coulomb gas integrals with screenings \[19\].

In principle, the procedure described in \[18\] provides an asymptotic series in the ultra-violet domain which agrees with the structure expected from the Painlevé case: it is not just a power series in \(\mu^2\), but includes non-analytic contributions with fractional powers of \(\mu^2\).

So the main problem is to compute the one-point functions. The first important result in this direction was achieved in \[20\], where an exact formula for the one-point functions of the primary fields was conjectured. Then by several ingenious tricks (such as going form sine-Gordon to sinh-Gordon and back) a procedure was described in \[21\] \[22\] \[23\], which must in principle allow us to compute the correlation functions of the descendants. Unfortunately, this procedure involves certain matrix Riemann-Hilbert problem which has not been solved in general up to now. At the
same time, this way of computation looks very indirect, and involves steps which are hard to justify. Still, the predicted result for the first not-trivial descendant \cite{23} is quite remarkable. Even though it was obtained by a complicated and non-rigorous procedure, it was checked against many particular cases. So we do not doubt in its validity. It will be used to check the result of the present paper.

Let us mention here a deep relation between the sG model and the $\Phi_{1,3}$-perturbation of $c < 1$ models of CFT \cite{24}. On the formal level the relation is simple. In the action (1) one can split $\cos(\beta \varphi(x)) = \frac{1}{2} (e^{-i \beta \varphi(x)} + e^{i \beta \varphi(x)})$ and consider the first term as a part of the Liouville action and the second as a perturbation. The Liouville model with an imaginary exponent is nothing but CFT with the central charge

$$c = 1 - 6(\beta - 1/\beta)^2,$$

and $e^{i \beta \varphi(x)}$ is the field $\Phi_{1,3}(x)$. So, formally there is no difference between the sG model and the $\Phi_{1,3}$-perturbation of $c < 1$ CFT. The situation becomes interesting in the case of rational $\beta^2$. It was shown in \cite{25, 26} that in the computation of the correlation functions of $e^{i n \beta \varphi(x)}$ ($n = 1, 2, 3, \cdots$), a restriction of degrees of freedom takes place for the intermediate states of solitons. The mechanism of reduction is similar to the RSOS restriction for vertex models. This phenomenon is known as the restriction of the sG model. For example, if $\beta^2 = 3/4$ the solitons reduce to Majorana fermions, and the restricted model is nothing but the scaling Ising model.

In the series of papers, \cite{27, 28, 29, 1} which we will refer to as I,II,III,IV, respectively, we studied the hidden fermionic structure of the XXZ spin chain. In particular, in IV the relation between our fermions in the scaling limit and $c < 1$ CFT was established. The long distance behaviour of the XXZ model and the short distance behaviour of the sG model are described by the same CFT: free bosons with the compactification radius $\beta^2$. For the XXZ model, we use the coupling parameter $\nu$ related to $\beta^2$ via

$$\beta^2 = 1 - \nu.$$

This identification is used when relating the results of IV to those of \cite{30, 31}, which were important for us. The relevant CFT has the central charge

$$c = 1 - 6 \frac{\nu^2}{1 - \nu}.$$

Here we do not consider the peculiarity of a rational $\nu$, so using the $c < 1$ CFT means just the usual modification of the energy-momentum tensor. In the ultraviolet limit, the sG model is described by two chiral copies of CFT. We use the notation

$$\Phi_\alpha(x) = e^{i \nu \alpha \frac{\varphi(x)}{\beta}}.$$

The field $i \beta \varphi(x)$ splits into two chiral fields $2 \varphi(x) + 2 \bar{\varphi}(x)$. (We follow the normalisation of the fields $\varphi(x), \bar{\varphi}(x)$ given in \cite{30, 31} and in IV.) Our goal is to compute the one point functions

$$\frac{\langle P(\{1_{-m}\}) \bar{P}(\{\bar{1}_{-m}\}) \Phi_\alpha(0) \rangle_{sG}}{\langle \Phi_\alpha(0) \rangle_{sG}}.$$
where \( P(\{L_m\}) , \tilde{P}(\{\bar{L}_m\}) \) are polynomials in the generators of two chiral copies of the Virasoro algebra with the central charge \( c \).

The universal enveloping algebra of the Virasoro algebra contains the local integrals of motion \( i_{2k-1} \) which survive the \( \Phi_{1,3} \)-perturbation. Clearly, the one-point functions of descendants created by them vanish. We assume that it is possible to write any element of the chiral Verma module generated by \( \Phi(0) \) as

\[
\langle P(\{L_{-2m}\})\tilde{P}(\{\bar{L}_{-2m}\})\Phi(0)\rangle_{sG}.
\]

For the simplest non-trivial case \( L_{-2}\bar{L}_{-2}\Phi(0) \) the problem was solved in [23]. If one wants to consider the descendants created by the Heisenberg algebra it is easy to do using the formulae

\[
(1 - \nu) T(z) = :\varphi'(z)^2 : + \nu \varphi(z), \quad (1 - \nu) \tilde{T}(\bar{z}) = :\bar{\varphi}'(\bar{z})^2 : + \nu \bar{\varphi}(\bar{z}).
\]

In this paper we shall consider only the domain \( 0 < \alpha < 2 \), but the final results allow analytical continuation for all values of \( \alpha \).

The whole idea of IV is that the usual basis of the Verma module is not suitable for the perturbation, and we have to introduce another one. For the moment we consider only one chirality. Working modulo action of the local integrals of motion the new basis is provided by uncharged products of two fermions \( \beta_{2k-1}^{CFT} \) and \( \gamma_{2k-1}^{CFT} \).

The fermions respect the Virasoro grading:

\[
[l_0, \beta_{2j-1}^{CFT} \gamma_{2k-1}^{CFT}] = (2j + 2k - 2) \beta_{2j-1}^{CFT} \gamma_{2k-1}^{CFT}.
\]

We have

\[
(3) \quad \beta_{I^+}^{CFT} \gamma_{I^-}^{CFT} \Phi(0) = \{ P_{I^+, I^-}^{even}(\{L_{-2k}\}) + d_\alpha P_{I^+, I^-}^{odd}(\{L_{-2k}\}) \} \Phi(0),
\]

where \( I^+ \) and \( I^- \) are ordered multi-indices: for \( I = (2r_1 - 1, \ldots, 2r_n - 1) \) with \( r_1 < \cdots < r_n \), we set

\[
\beta_{I}^{CFT} = \beta_{2r_1-1}^{CFT} \cdots \beta_{2r_n-1}^{CFT}, \quad \gamma_{I}^{CFT} = \gamma_{2r_1-1}^{CFT} \cdots \gamma_{2r_n-1}^{CFT},
\]

and similarly for \( \beta_{I^-} \) and \( \gamma_{I^-} \). We require \#(\( I^+ \)) = \#(\( I^- \)). In the right hand side of (3), \( P_{I^+, I^-}^{even}(\{L_{-2k}\}) \) and \( P_{I^+, I^-}^{odd}(\{L_{-2k}\}) \) are homogeneous polynomials, the constant \( d_\alpha \) is given by

\[
d_\alpha = \frac{\nu(\nu - 2)}{\nu - 1}(\alpha - 1) = \frac{1}{6} \sqrt{(25 - c)(24\Delta_\alpha + 1 - c)},
\]

and the separation into even and odd parts is determined by the reflection \( \beta_{I^+}^{CFT} \gamma_{I^-}^{CFT} \leftrightarrow \beta_{I^-}^{CFT} \gamma_{I^+}^{CFT} \).

The coefficients of the polynomials \( P_{I^+, I^-}^{even} \) and \( P_{I^+, I^-}^{odd} \) are rational functions of \( c \) and

\[
\Delta_\alpha = \frac{\alpha(\alpha - 2)\nu^2}{4(1 - \nu)}
\]

only. The denominators factorise into multipliers \( \Delta_\alpha + 2k, k = 0, 1, 2, \ldots \). Exact formulae up to the level 6 can be found in IV, Eq. (12.4). In particular, on the level 2 we have \( P_{I^+, I^-}^{even} = 1, P_{I^+, I^-}^{odd} = 0 \). The transformation (3) is invertible.
The operators $\beta_{2k-1}^{\text{CFT}}$ and $\gamma_{2k-1}^{\text{CFT}}$ were found as the scaling limit of the fermions which create the quasi-local fields for the XXZ spin chain. This became possible after the computation of the expectation values of the quasi-local fields on the cylinder (see III). Actually, two different scaling limits are possible, and the second one provides the fermionic operators for the second chirality: $\beta_{2k-1}^{\text{CFT}}$ and $\gamma_{2k-1}^{\text{CFT}}$. The same story repeats for these operators, in particular, we have

$$\beta_{I+}^{\text{CFT}} \gamma_{I-}^{\text{CFT}} \Phi_{\alpha}(0) = \left\{ P_{I+, I-}^{\text{even}}(\{\bar{I}-2k\}) - d_{\alpha} P_{I+, I-}^{\text{odd}}(\{\bar{I}-2k\}) \right\} \Phi_{\alpha}(0).$$

The main statement of this paper is that in the fermionic basis the sG one-point functions are simple:

$$\langle \beta_{I+}^{\text{CFT}} \gamma_{I-}^{\text{CFT}} \beta_{I+}^{\text{CFT}} \gamma_{I-}^{\text{CFT}} \Phi_{\alpha}(0) \rangle_{\text{sG}} = (-1)^{|I^+|} \delta_{I-, I^+} \delta_{I-, I^-}$$

$$\times \left( \frac{M}{2\sqrt{1-\nu}} \frac{\Gamma(\frac{1}{2\nu})}{\Gamma(\frac{1-\nu}{2\nu})} \right)^{2|I^+|+2|I^-|} \prod_{2n-1 \in I^+} G_n(\alpha) \prod_{2n-1 \in I^-} G_n(2-\alpha),$$

Here

$$G_n(\alpha) = (-1)^{n-1}((n-1)!)^2 \frac{\Gamma(\alpha + \frac{1-\nu}{2\nu}(2n-1)) \Gamma(1-\frac{\alpha}{2} - \frac{1-\nu}{2\nu}(2n-1))}{\Gamma(1-\frac{\alpha}{2}) \Gamma(\frac{1}{2} + \frac{1-\nu}{2\nu}(2n-1)) \Gamma(\frac{1}{2} + \frac{1-\nu}{2\nu}(2n-1))}.$$
Notice that, for this limit to make sense, one has first to consider the finite lattice on $n$ sites, and then take the limit $\zeta_0 \to \infty$, $n \to \infty$ in a concerted way in order that the finite mass scale appears. But exactly this kind of procedure became very natural for us after we had computed in III the expectation values of quasi-local operators on the cylinder. The compact direction on the cylinder is called the Matsubara direction, and its size $n$ is what is needed for considering the limit in the spirit of [35].

The paper is organised as follows. In Section 2 we review our previous paper IV, and explain how to obtain the fermionic description for two chiral CFT models from the XXZ spin chain (six vertex model). In Section 3 we introduce the inhomogeneous six vertex model and consider the continuous limit which produces the sG model according to [35]. We derive the one-point functions using the fermionic description of ultra-violet CFT.

2. Two scaling limits of the XXZ model and two chiralities

Our study of the XXZ model is based on the fermionic operators defined in I,II. This definition allowed us to compute in III the following expectation value. Consider a homogeneous six vertex model on an infinite cylinder. Let $T_{S,M}$ be the monodromy matrix, where $S$ refers to the infinite direction (called the space direction), and $M$ refers to the compact circular direction (called the Matsubara direction). We use $n$ to denote the length of the latter. We follow the notations in IV. For a quasi-local operator $q^{2\alpha S(0)}\mathcal{O}$ on the spacial lattice, we consider

$$Z_n^{\kappa} \left\{ q^{2\alpha S(0)}\mathcal{O} \right\} = \frac{\text{Tr}_S \text{Tr}_M \left( T_{S,M} q^{2\kappa S + 2\alpha S(0)}\mathcal{O} \right)}{\text{Tr}_S \text{Tr}_M \left( T_{S,M} q^{2\kappa S + 2\alpha S(0)} \right)}.$$  

(6)

The generalisation of this functional $Z_n^{\kappa,s}$ was introduced in IV. In the scaling limit, the introduction of $s$ in this functional amounts to changing (screening) the background charge at $x = -\infty$ by $-2s\frac{\nu}{\nu}$. It enables us to deal with the special case of the functional for which the effective action of local integrals of motion becomes trivial.

The quasi-local operators are created from the primary field $q^{2\alpha S(0)}$ by action of the creation operators $t^*(\zeta), b^*(\zeta), c^*(\zeta)$. Actually they act on the space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s = -\infty}^{\infty} \mathcal{W}_{\alpha - s,s},$$

where $\mathcal{W}_{\alpha - s,s}$ denotes the space of quasi-local operators of spin $s$ with tail $\alpha - s$.

In this paper we shall consider the subspace $\mathcal{W}_{\text{ferm}}^{(\alpha)}$ of the space $\mathcal{W}^{(\alpha)}$ which are created from the primary fields only by fermions $b^*_p, c^*_p$ (see (12) below). On $\mathcal{W}_{\text{ferm}}^{(\alpha)}$ the functional $Z_{n,s}^{\kappa,s} \left\{ q^{2\alpha S(0)}\mathcal{O} \right\}$ allows the determinant form which is convenient to summarise as

$$Z_n^{\kappa,s} \left\{ q^{2\alpha S(0)}\mathcal{O} \right\} = \frac{\text{Tr}_S \left( e^{\Omega_n \left( q^{2\alpha S(0)}\mathcal{O} \right)} \right)}{\text{Tr}_S \left( q^{2\alpha S(0)} \right)},$$

where

$$\Omega_n \left( q^{2\alpha S(0)}\mathcal{O} \right)$$

is a certain functional which is convenient to summarise as

$$Z_n^{\kappa,s} \left\{ q^{2\alpha S(0)}\mathcal{O} \right\} = \frac{\text{Tr}_S \left( e^{\Omega_n \left( q^{2\alpha S(0)}\mathcal{O} \right)} \right)}{\text{Tr}_S \left( q^{2\alpha S(0)} \right)},$$

where

$$\Omega_n \left( q^{2\alpha S(0)}\mathcal{O} \right)$$

is a certain functional which is convenient to summarise as
where
\[
\Omega_n = \frac{1}{(2\pi i)^2} \int_\Gamma \int_\Gamma \omega_{\text{rat},n}(\zeta, \xi) c(\xi) b(\zeta) \frac{d\xi^2}{\xi^2} \frac{d\zeta^2}{\zeta^2},
\]
where the contour \( \Gamma \) goes around \( \zeta^2 = 1 \).

The function \( \omega_{\text{rat},n}(\zeta, \xi) \) is defined by the Matsubara data (see III, [37], IV). Besides the length of the Matsubara chain \( n \) it depends on the parameters \( \kappa, \alpha, \), \( s \) and on possible inhomogenieties in the Matsubara chain. However, we mark explicitly only the dependence on \( n \) which is the most important for us here.

We have ignored the descendants created by \( t^*(\zeta) \). Actually, they do not give any non-trivial contributions in the limit \( n \to \infty \), which we shall be interested in. Then \( Z^\kappa_\infty \) is automatically reduced to the the quotient space:
\[
W^{(\alpha)}_{\text{quo}} = W^{(\alpha)}/(t^*(\zeta) - 2) W^{(\alpha)},
\]
which is obviously isomorphic to \( W^{(\alpha)}_{\text{ferm}} \) as a linear space.

We shall not repeat the definition of \( \omega_{\text{rat},n}(\zeta, \xi) \) because in the present paper we shall use only very limited information about it. Let us explain, however, the suffix “rat”: the function \( (\xi/\zeta)^\alpha \omega_{\text{rat},n}(\zeta, \xi) \) is a rational function of \( \zeta^2 \) and \( \xi^2 \). The really interesting situation occurs when \( n \to \infty \). In that case the Bethe roots for the transfer matrices in the Matsubara direction become dense on the half axis \( \zeta^2 > 0 \). If we do not introduce additional rescaling as described below, the function \( \omega_{\text{rat},n}(\zeta, \xi) \) goes to the simple limit:

\[
(7) \quad \lim_{n \to \infty} \omega_{\text{rat},n}(\zeta, \xi) = 4\omega_0(\zeta/\xi, \alpha) + \nabla \omega(\zeta/\xi, \alpha),
\]

where

\[
(8) \quad 4\omega_0(\zeta, \alpha) = - \int_{-i\infty}^{i\infty} \zeta^u \sin \frac{\pi}{2}(1 - \nu) u - \alpha \frac{1}{\sin \frac{\pi}{2}(u - \alpha) \cos \frac{\nu}{2} u} du,
\]
\[
\nabla \omega(\zeta, \alpha) = -\psi(\zeta q, \alpha) + \psi(\zeta q^{-1}, \alpha) + 2i\zeta^\alpha \tan \left( \frac{\pi \nu \alpha}{2} \right),
\]
\[
\psi(\zeta, \alpha) = \zeta^\alpha \frac{\zeta^2 + 1}{2(\zeta^2 - 1)}.
\]

The reason for extracting the elementary \( \nabla \omega \) is due to the fact that the function \( \omega_0 \) satisfies the relation typical for CFT

\[
(9) \quad \omega_0(\zeta, \alpha) = \omega_0(\zeta^{-1}, 2 - \alpha).
\]

Notice that \( (\xi/\zeta)^\alpha \omega_0(\zeta/\xi, \alpha) \) is not a single-valued function of \( \zeta^2 \) and \( \xi^2 \). So, the property of rationality is lost in the limit. We define \( \Omega_0 \) and \( \nabla \Omega \) in the same way as \( \Omega_n \) replacing \( \omega_{\text{rat},n}(\zeta, \xi) \) by respectively \( 4\omega_0(\zeta/\xi, \alpha) \) and \( \nabla \omega(\zeta/\xi, \alpha) \).

Following IV we denote the original creation operators introduced in II by \( b^*_{\text{rat}}(\zeta) \) and \( c^*_{\text{rat}}(\zeta) \). They satisfy the property:

\[
\text{Tr}_S (b^*_{\text{rat}}(\zeta)(X)) = 0, \quad \text{Tr}_S (c^*_{\text{rat}}(\zeta)(X)) = 0,
\]
for all quasi-local operators $X$. In the present paper it is useful to replace these operators by the following Bogolubov transformed ones:
\begin{equation}
(10) \quad \begin{aligned}
\mathbf{b}_{0}^{*}(\zeta) &= e^{-\nabla \Omega} \mathbf{b}_{\text{rat}}^{*}(\zeta) e^{\nabla \Omega}, \\
\mathbf{c}_{0}^{*}(\zeta) &= e^{-\nabla \Omega} \mathbf{c}_{\text{rat}}^{*}(\zeta) e^{\nabla \Omega}.
\end{aligned}
\end{equation}

Obviously, the functional $Z_{n}^{s}$ calculated on the descendants generated by these operators is expressed as determinant constructed from function $\omega_{\text{rat},n}(\zeta, \xi) - \nabla \omega(\zeta/\xi, \alpha)$ which in the limit $n \to \infty$ goes to $4\omega_{0}(\zeta/\xi, \alpha)$.

As it is explained in IV, the operators $\zeta^{-\alpha} \mathbf{b}_{0}^{*}(\zeta)$ and $\zeta^{\alpha} \mathbf{c}_{0}^{*}(\zeta)$ are rational functions of $\zeta^{2}$ as far as they are considered in the functional $Z_{n}^{s}$, and they have the following behaviour at $\zeta^{2} = 0$:
\begin{equation}
\begin{aligned}
\mathbf{b}_{0}^{*}(\zeta) &= \sum_{j=1}^{\infty} \zeta^{\alpha + j - 2} \mathbf{b}_{\text{screen}, j}^{*}, \\
\mathbf{c}_{0}^{*}(\zeta) &= \sum_{j=1}^{\infty} \zeta^{\alpha + 2j} \mathbf{c}_{\text{screen}, j}^{*}.
\end{aligned}
\end{equation}

For $\mathbf{b}_{\text{screen}, j}^{*}, \mathbf{c}_{\text{screen}, j}^{*}$ we have used the suffix “screen” to stand for “screening” in view of its similarity to the lattice screening operators used in IV.

In IV, another set of operators $\mathbf{b}^{*}(\zeta), \mathbf{c}^{*}(\zeta)$ is obtained from $\mathbf{b}_{\text{rat}}^{*}(\zeta), \mathbf{c}_{\text{rat}}^{*}(\zeta)$ by a kind of Bogolubov transformation which contains $\mathbf{t}^{*}(\zeta)$. We shall not write explicitly this Bogolubov transformation, but only the one relating $\mathbf{b}^{*}(\zeta), \mathbf{c}^{*}(\zeta)$ to $\mathbf{b}_{0}^{*}(\zeta), \mathbf{c}_{0}^{*}(\zeta)$, both acting on the quotient space $\mathcal{W}_{\text{qto}}^{(\alpha)}$ because only these operators are used in this paper. The operators $\mathbf{b}^{*}(\zeta), \mathbf{c}^{*}(\zeta)$ are important because $Z_{n}^{s}$ vanishes on their descendants. Notice that we do not allow $\kappa, s$ to grow together with $n$. In that case the dependence on $\kappa, s$ disappears for $n = \infty$.

**Remark.** The defining equations for $\omega_{\text{rat},n}(\zeta, \xi)$ given in IV imply that the limit (7) has a very general nature. Namely, the result is independent not only of $\kappa, s$ but also of inhomogeneities in the Matsubara chain. The situation is similar to that for the $S$-matrix which is the same for homogeneous, inhomogeneous XXZ chains or even for the sG model.

If we consider the Bogolubov transformation which connects the operators $\mathbf{b}^{*}(\zeta), \mathbf{c}^{*}(\zeta)$ and $\mathbf{b}_{0}(\zeta), \mathbf{c}_{0}(\zeta)$ as acting on the quotient space $\mathcal{W}_{\text{qto}}^{(\alpha)}$, it reduces to
\begin{equation}
(11) \quad \begin{aligned}
\mathbf{b}^{*}(\zeta) &= e^{-\Omega_{0}} \mathbf{b}_{0}^{*}(\zeta) e^{\Omega_{0}}, \\
\mathbf{c}^{*}(\zeta) &= e^{-\Omega_{0}} \mathbf{c}_{0}^{*}(\zeta) e^{\Omega_{0}}.
\end{aligned}
\end{equation}

We catch operators acting on $\mathcal{W}^{(\alpha)}$ by developing $\mathbf{b}^{*}(\zeta), \mathbf{c}^{*}(\zeta)$ and $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ around the point $\zeta^{2} = 1$:
\begin{equation}
(12) \quad \begin{aligned}
\mathbf{b}^{*}(\zeta) &\approx_{\zeta^{2} \to 1} \sum_{p=1}^{\infty} (\zeta^{2} - 1)^{p-1} \mathbf{b}_{p}^{*}, \\
\mathbf{c}^{*}(\zeta) &\approx_{\zeta^{2} \to 1} \sum_{p=1}^{\infty} (\zeta^{2} - 1)^{p-1} \mathbf{c}_{p}^{*},
\end{aligned}
\end{equation}
\begin{equation}
(13) \quad \begin{aligned}
\mathbf{b}(\zeta) &= \sum_{p=0}^{\infty} (\zeta^{2} - 1)^{-p} \mathbf{b}_{p}, \\
\mathbf{c}(\zeta) &= \sum_{p=0}^{\infty} (\zeta^{2} - 1)^{-p} \mathbf{c}_{p}.
\end{aligned}
\end{equation}

These operators are (quasi-)local in the sense of II, Section 3.3, while $\mathbf{b}_{\text{screen}, j}^{*}$ and $\mathbf{c}_{\text{screen}, j}^{*}$ are highly “non-local”.

The operators $\mathbf{b}_{p}, \mathbf{c}_{p}$ create quasi-local operators in the sense of II, Section 3.1 by acting on the primary field. Here we slightly change the notation compared to II,III,
where the Fourier coefficients are defined after removing an overall power \( \zeta^{\pm \alpha} \). The reason was that we wanted that the result of action of \( b_p^* \), \( c_p^* \) is a rational function of \( q, q^\alpha \). This rationality property is irrelevant in this paper, and extracting \( \zeta^{\pm \alpha} \) may even cause a confusion.

So far we have been discussing the simple limit \( n \to \infty \). Now we discuss the scaling limit to CFT. In IV we studied the scaling limit of the homogeneous XXZ chain on the cylinder. The key idea is to consider first of all the scaling limit in the Matsubara direction. Namely, denoting the length of the Matsubara chain by \( n \) and introducing the step of the lattice \( a \) we consider the limit
\[
n \to \infty, \quad a \to 0, \quad na = 2\pi R \text{ fixed}.
\]
The requirement is that if we rescale the spectral parameter as \( \zeta = (Ca)^\nu \lambda \), the Bethe roots for the transfer matrix in the Matsubara direction which are close to \( \zeta^2 = 0 \) remain finite. The constant \( C \) is chosen to have an agreement with CFT as
\[
C = \frac{\Gamma \left( \frac{1}{2\nu} \right)}{2\sqrt{\pi} \Gamma \left( \frac{1}{2\nu} \right)} \Gamma(\nu)^{\frac{1}{2}}.
\]

Next we consider the scaling limit in the space direction. We conjecture that under the presence of the background charges effected by the screening operators, the lattice operator \( q^{2\alpha S+2\alpha S(0)} \) goes to the limit \( \Phi_{1-\kappa'}(-\infty)\Phi_{\alpha}(0)\Phi_{1+\kappa}(\infty) \) where
\[
\kappa' = \kappa + \alpha + 2s \frac{1-\nu}{\nu}.
\]
Furthermore, we have shown that in the weak sense the following limits exist
\[
2\beta^*(\lambda) = \lim_{a \to 0} b^*((Ca)^\nu \lambda), \quad 2\gamma^*(\lambda) = \lim_{a \to 0} c^*((Ca)^\nu \lambda).
\]

The operators \( \beta^*(\lambda), \gamma^*(\lambda) \) have the asymptotics at \( \lambda^2 \to \infty \):
\[
\beta^*(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-2j-1} \beta_{2j-1}^*, \quad \gamma^*(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-2j-1} \gamma_{2j-1}^*,
\]
where \( \beta_{2j-1}^*, \gamma_{2j-1}^* \) act between different Verma modules as follows
\[
\beta_{2j-1}^* : \mathcal{V}_{\alpha+2\sqrt{\nu(s-1)}} \otimes \mathcal{V}_\alpha \to \mathcal{V}_{\alpha+2\sqrt{\nu(s+1)}} \otimes \mathcal{V}_\alpha,
\]
\[
\gamma_{2j-1}^* : \mathcal{V}_{\alpha+2\sqrt{\nu(s+1)}} \otimes \mathcal{V}_\alpha \to \mathcal{V}_{\alpha+2\sqrt{\nu(s+1)}} \otimes \mathcal{V}_\alpha.
\]
The action on the second component is trivial. The functional \( Z_n^{\kappa,s} \) turns in this limit into the three-point function for \( c < 1 \) CFT as explained in IV. The identification of descendants created by \( \beta^* \) and \( \gamma^* \) with Virasoro descendants is made by studying the function
\[
\omega_R(\lambda, \mu) = \frac{1}{4} \lim_{\text{scaling}} \omega_{\text{rat},n}((Ca)^\nu \lambda, (Ca)^\nu \mu) - \nabla \omega(\lambda/\mu, \alpha),
\]
where \( \lim_{\text{scaling}} \) refers to the scaling limit \( (14) \). Contrary to the simple-minded limit \( n \to \infty \) \((7)\), the function \( (\mu/\lambda)^{\alpha} \omega_R(\lambda, \mu) \) remains a single-valued function of \( \lambda^2 \) and \( \mu^2 \), but it develops an essential singularity at \( \lambda^2 = \infty, \mu^2 = \infty \).
In the present paper we shall consider sG model which requires putting together the two chiralities. To this end we shall need to consider not only the asymptotical region \( \lambda^2 \to \infty \), but also \( \lambda^2 \to 0 \). Analysing the function \( \omega_R(\lambda, \mu) \) one concludes that the following limits exist

\[
\frac{1}{2} \lim_{a \to 0} b^*_a((C\alpha)^\nu \lambda) \xrightarrow{\lambda^2 \to 0} \beta^*_\text{screen}(\lambda) = \sum_{j=1}^\infty \lambda^{\alpha+2j-2} \beta^*_{\text{screen},j},
\]

\[
\frac{1}{2} \lim_{a \to 0} c^*_a((C\alpha)^\nu \lambda) \xrightarrow{\lambda^2 \to 0} \gamma^*_\text{screen}(\lambda) = \sum_{j=1}^\infty \lambda^{-\alpha+2j} \gamma^*_{\text{screen},j}.
\]

For the moment we do not know how to use these operators, but one thing is clear: they create highly non-local fields.

The resulting operators act as follows:

\[
2 \tilde{\beta}^*(\lambda) = \lim_{a \to 0} b^*((C\alpha)^{-\nu} \lambda), \quad 2 \tilde{\gamma}^*(\lambda) = \lim_{a \to 0} c^*((C\alpha)^{-\nu} \lambda),
\]

which allow the power series at \( \lambda \to 0 \):

\[
\tilde{\beta}^*(\lambda) \simeq \sum_{j=1}^\infty \lambda^{2j-1} \tilde{\beta}^*_{2j-1}, \quad \tilde{\gamma}^*(\lambda) \simeq \sum_{j=1}^\infty \lambda^{-2j} \tilde{\gamma}^*_{2j-1}.
\]

The proof goes through considering the scaling limit (14) and studying the function

\[
\tilde{\omega}_R(\lambda, \mu) = \frac{1}{4} \lim_{\text{scaling}} \omega_{\text{rat},n}((C\alpha)^{-\nu} \lambda, (C\alpha)^{-\nu} \mu) - \nabla \omega(\lambda/\mu, \alpha) \xrightarrow{\lambda^2 \to 0, \mu^2 \to 0} 0.
\]

The analysis is parallel to the one performed for the first chirality in IV. The function \( \mu / \lambda^\alpha \tilde{\omega}_R(\lambda, \mu) \) is a single-valued function of \( \lambda^2, \mu^2 \) with essential singularities at \( \lambda^2 = 0, \mu^2 = 0 \).

The operators \( \tilde{\beta}^*_{\text{screen}}(\lambda) \) and \( \tilde{\gamma}^*_{\text{screen}}(\lambda) \) are introduced similarly to the first chirality.

\[
\frac{1}{2} \lim_{a \to 0} b^*_a((C\alpha)^{-\nu} \lambda) \xrightarrow{\lambda^2 \to \infty} \tilde{\beta}^*_\text{screen}(\lambda) = \sum_{j=1}^\infty \lambda^{\alpha-2j} \tilde{\beta}^*_{\text{screen},j},
\]

\[
\frac{1}{2} \lim_{a \to 0} c^*_a((C\alpha)^{-\nu} \lambda) \xrightarrow{\lambda^2 \to \infty} \tilde{\gamma}^*_\text{screen}(\lambda) = \sum_{j=1}^\infty \lambda^{2-\alpha-2j} \tilde{\gamma}^*_{\text{screen},j}.
\]

For both chiralities we have

\[
\omega_R(\lambda, \mu) \xrightarrow{R \to \infty} \omega_0(\lambda/\mu, \alpha), \quad \tilde{\omega}_R(\lambda, \mu) \xrightarrow{R \to \infty} \omega_0(\lambda/\mu, \alpha).
\]

So, the naïve \( n \to \infty \) limit is reproduced.
3. Inhomogeneous six vertex model and sine-Gordon model

We want to put two chiral models together and to make them interacting. According to the previous discussion the lattice analogue of chiral CFT is the XXZ model. So, the two non-interacting chiral models correspond to the lattice containing two non-interacting six vertex sublattices. As an example we shall consider the “even” or “odd” sublattices, consisting of lattice points with coordinates $(j, m)$ such that $j, m$ are both even or both odd. It is well-known [35] how to force these two to interact: one has to consider a six vertex model on the entire lattice with alternating inhomogeneity parameters. We denote $S = S \cup \overline{S}$, $M = M \cup \overline{M}$ and introduce

$$T_{S,M} = \prod_{j=-\infty}^{\infty} T_{j,M}(\zeta_{j}^{(-1)^{j}}), \quad T_{j,M}(\zeta) = \prod_{m=1}^{\infty} L_{j,m}(\zeta_{j}^{(-1)^{m}}).$$

Here

$$L_{j,m}(\zeta) = q^{-\frac{1}{4}\sigma_{j}^{3}\sigma_{m}^{3}} - \zeta^{2} q^{\frac{1}{4}\sigma_{j}^{3}\sigma_{m}^{3}} - \zeta(q - q^{-1})(\sigma_{j}^{+}\sigma_{m}^{-} + \sigma_{j}^{-}\sigma_{m}^{+}).$$

Note that for $\zeta_{0}^{\pm 1} = \infty$ the inhomogeneous lattice reduces to two non-interacting homogeneous lattices.

The functional $Z_{n}^{\text{full}}$ is defined by

$$Z_{n}^{\text{full}} \{ q^{2\alpha S(0)} \Theta \} = \frac{\text{Tr}_{S} \text{Tr}_{M} \left( T_{S,M} q^{2\alpha S(0)} \Theta \right)}{\text{Tr}_{S} \text{Tr}_{M} \left( T_{S,M} q^{2\alpha S(0)} \right)}.$$  \hfill (22)

We set $\kappa = 0$ here. The methods of II,III allow one to compute this functional with inhomogeneities in both directions. In particular, from II one concludes that the annihilation operators split into two parts:

$$b(\zeta) = b^{+}(\zeta) + b^{-}(\zeta), \quad c(\zeta) = c^{+}(\zeta) + c^{-}(\zeta),$$

$$b^{\pm}(\zeta) = \sum_{p=0}^{\infty} \left( \zeta^{2}\zeta_{0}^{\pm 2} - 1 \right)^{-p} b^{\pm}_{p}, \quad c^{\pm}(\zeta) = \sum_{p=0}^{\infty} \left( \zeta^{2}\zeta_{0}^{\pm 2} - 1 \right)^{-p} c^{\pm}_{p}.$$

Let us take $n = \infty$. Then using the remark from the previous section and II,III one obtains:

$$Z_{\infty}^{\text{full}} \{ q^{2\alpha S(0)} \Theta \} = \frac{\text{Tr}_{S} \left( e^{\Omega^{\text{full}}(q^{2\alpha S(0)} \Theta)} \right)}{\text{Tr}_{S} \left( q^{2\alpha S(0)} \Theta \right)}.$$  \hfill (23)

We have

$$\Omega^{\text{full}} = \Omega_{0} + \nabla \Omega,$$

where

$$\Omega_{0} = \Omega_{0}^{++} + \Omega_{0}^{+-} + \Omega_{0}^{-+} + \Omega_{0}^{--},$$

$$\Omega_{0}^{\alpha \beta} = \frac{4}{(2\pi i)^{2}} \int_{\Gamma_{\alpha} \Gamma_{\beta}} \int_{\Gamma^{\prime}_{\alpha} \Gamma^{\prime}_{\beta}} \omega_{0}(\zeta/\xi, \alpha) c^{\prime}(\xi) b^{\prime}(\zeta) \frac{d\zeta^{2}}{\xi^{2}} \frac{d\xi^{2}}{\xi^{2}};$$

and similarly for $\nabla \Omega$. The contour $\Gamma_{\pm}$ goes anticlockwise around $\zeta_{0}^{\pm 2}$. 
As it has been said in the introduction, ideally we would like to start from a non-critical (XYZ or SOS) lattice model and to obtain the relativistic massive model by the usual scaling limit near the critical point. Since we do not have the necessary formulae to do that, we have recourse to the scaling limit of an inhomogeneous model by the procedure of [35]. Here we have an ideal situation from the point of view of QFT. Namely, we have the ultraviolet cutoff (lattice), the infrared cutoff (a finite number $n$ of sites in the Matsubara direction), and the physical quantities (the values of the the functional $Z_{n}^{\text{full}}$ on quasi-local fields) are exactly computed with finite cutoffs.

Like in the homogeneous case, let us introduce the step of the lattice $a$, and consider the scaling limit (14): $n \to \infty$, $a \to 0$, $na = 2\pi R$ fixed. We require further that $\zeta^{-1} \to 0$, so that

$$M = 4a^{-1} \zeta^{-1/\nu} \text{ fixed}. \quad (24)$$

The parameter $M$ is a mass scale which has the meaning of the sG soliton mass [35]. The famous formula relating the soliton mass to the dimensional coupling constant $\mu$ [36] reads in our notation as

$$\mu = \left( \frac{M}{4C} \right)^{\nu} = (Ca)^{-\nu} \zeta^{-1}. \quad (25)$$

In this paper we consider a further limit $R \to \infty$. In that case the sG partition function is obtained from $Z_{\infty}^{\text{full}}$.

Let us give some explanation at this point. The subject of study in [35] is the partition function of the sG model on the cylinder. There are two possible approaches to this partition function which correspond to two Hamiltonian pictures. In the first one, the space direction is considered as space and the Matsubara direction as time (space channel). One has scattering of particles and describes the partition function by the Thermodynamic Bethe Ansatz (TBA) [38]. This approach has an advantage of dealing with known particle spectrum and $S$-matrices. It also has a disadvantage, because as usual in the thermodynamics one has to deal with the density matrix, which is a complicated object even in integrable cases.

The paper [35] uses an alternative picture: the Matsubara direction is space, and the space direction is time. In this approach, the partition function is described by the maximal eigenvalue of the Hamiltonian of the periodic problem for the Matsubara direction (Matsubara channel). The advantage of this approach is clear: one deals with the pure ground state instead of the density matrix. The disadvantage is that describing eigenvalues in the finite volume is a difficult problem. This problem is addressed in [35].

More precisely, it is proposed in [35] to obtain the sG partition function as the scaling limit of $\text{Tr}_{S} \text{Tr}_{M}(T_{S,M})$. As in the present paper, it is important to be able to control the computations starting from the lattice and from finite $n$. In our opinion, the main achievement of [35] is not in rewriting the Bethe Ansatz equation for the Matsubara transfer matrix in the form of a non-linear integral equation, but in extracting a main linear part and inverting it. The resulting Destri-DeVega equation (DDV) has several nice features. First, it allows the scaling limit and the mass scale
appears. Second, after the scaling limit the DDV equation clearly allows the large \( R \) expansion. Third, the scattering phase of the sG solitons appears in the DDV equation in the Matsubara channel. The last property allows the identification with the space channel. In particular, \( M \) happens to be equal to the mass of soliton.

Now we come to the most important point of this paper. We wish to define the creation operators appropriate for taking the scaling limit to the sG theory. The issue is similar to the one in conformal perturbation theory, where one needs to prescribe a way how to extend the descendants in CFT to the perturbed case. It is claimed in \cite{18,23} that after subtracting the divergencies these operators are defined uniquely: possible finite counterterms can be dismissed for dimensional reasons, at least in the absence of resonances \cite{18,23}. The latter condition is satisfied in our case if \( \nu \) and \( \alpha \) are generic.

Let us define the operators \( b_0^*(\zeta) \) and \( c_0^*(\zeta) \) by the same formula as in the homogeneous case \cite{10}. Starting from these operators we define the creation operators

\[
b_0^*(\zeta) \simeq \sum_{p=1}^{\infty} (\zeta^2 c_0^2 - 1)^{p-1} b_{0,p}^+, \quad c_0^*(\zeta) \simeq \sum_{p=1}^{\infty} (\zeta^2 c_0^2 - 1)^{p-1} c_{0,p}^+.
\]

and likewise for \( c_0^*(\zeta) \).

The operators \( b_{0,p}^+, c_{0,p}^+ \) create quasi-local fields. Notice that \( Z_{\rm full}^0 \) is defined on the quotient space \( W_{\nu,\alpha}^0 \) because the \( t^* \)-descendants do not contribute to it.

There are two sorts of chiral operators, one living on the even sublattice and the other on the odd sublattice. We have to make some combinations which will give finite answers for the interacting model. At the same time, we want this combination to correspond to our intuitive idea that we have to subtract perturbative series. Let us explain that the correct combinations are given by the following.

\[
b_{0,+}^*(\zeta) = e^{-\Omega^0_{++}} b_{0,+}^*(\zeta) e^{\Omega^0_{++}}, \quad c_{0,+}^*(\zeta) = e^{-\Omega^0_{++}} c_{0,+}^*(\zeta) e^{\Omega^0_{++}}, \quad b_{0,-}^*(\zeta) = e^{-\Omega^0_{--}} b_{0,-}^*(\zeta) e^{\Omega^0_{--}}, \quad c_{0,-}^*(\zeta) = e^{-\Omega^0_{--}} c_{0,-}^*(\zeta) e^{\Omega^0_{--}}.
\]

Corresponding creation operators which create the quasi-local operators are defined by

\[
b_{0,+}^*(\zeta) \simeq \sum_{p=1}^{\infty} (\zeta^2 c_0^2 - 1)^{p-1} b_{0,p}^+.
\]

In what follows we shall be interested only in the case of an equal number of \( b_{0,+}^* \) and \( c_{0,+}^* \). Let us examine how the descendants of this form depend on \( \zeta_0 \), taking the simplest case

\[
b_k^+ c_l^+ b_r^- c_s^- (q^{2\alpha S(0)}) = e^{-\Omega_{0,+}^0 - \Omega_{0,-}^0} b_{0,k}^+ c_{0,l}^+ b_{0,r}^- c_{0,s}^- (q^{2\alpha S(0)}).
\]

It is easy to see from the definition in \cite{11} that the second factor in the right hand side is a rational function of \( \zeta_0^2 \) regular at \( \infty \),

\[
b_{0,k}^+ c_{0,l}^+ b_{0,r}^- c_{0,s}^- (q^{2\alpha S(0)}) = (\zeta_0 + \zeta_0^{-2} - \zeta_0^{-4} + \ldots) q^{2\alpha S(0)}.
\]

The rationality holds true after application of \( e^{-\Omega_{0,+}^0 - \Omega_{0,-}^0} \). The expansion (27) looks as a perturbative series for the action \( \Pi \), because in the scaling limit \( \zeta_0^{-2} \) comes
accompanied by $a^{-2\nu}$, and $\zeta_0^{-2}a^{-2\nu}$ has the dimension of $(\text{mass})^{2\nu}$ i.e. that of $\mu^2$ (see (24)). Notice that the property (27) would be spoiled if we apply $e^{-\Omega_0^+} - \Omega_0^+$ as well, because it will pick up $\omega_0(\zeta, \alpha)$ near $\zeta = \zeta_0^{\pm 2}$, and the function $\omega_0(\zeta, \alpha)$ has asymptotics at $\zeta \to \infty$ containing both $\zeta^{\alpha - 2m}$ and $\zeta^{-2n \pm 1}$.

On the descendants created by (26), the value of $Z_{\text{full}}^\infty$ remains finite in the scaling limit. So we conclude that they create renormalised local operators. We cannot say anything about the finite renormalisation, but it can be taken care of by a dimensional consideration (see below).

We conjecture that the following limits exist

$$
\frac{1}{2} \lim_{\text{scaling}} b^{+*}(\zeta) \simeq \frac{1}{\zeta^2} \beta^*(\mu \zeta) + \beta^*_{\text{screen}}(\zeta/\mu),
$$

$$
\frac{1}{2} \lim_{\text{scaling}} c^{+*}(\zeta) \simeq \frac{1}{\zeta^2} \gamma^*(\mu \zeta) + \gamma^*_{\text{screen}}(\zeta/\mu),
$$

$$
\frac{1}{2} \lim_{\text{scaling}} b^{-*}(\zeta) \simeq \frac{1}{\zeta^2} \beta^*(\zeta/\mu) + \beta^*_{\text{screen}}(\mu \zeta),
$$

$$
\frac{1}{2} \lim_{\text{scaling}} c^{-*}(\zeta) \simeq \frac{1}{\zeta^2} \gamma^*(\zeta/\mu) + \gamma^*_{\text{screen}}(\mu \zeta),
$$

where by $\lim_{\text{scaling}}$ the scaling (14), (24) and $R \to \infty$ are implied.

In these formulae we denote by the same letters the operators in the sG model as they were denoted in the CFT. We shall not consider the screening operators, while we use the coefficients $\beta^*_{2j-1}$ and so forth, defined as in (17) and (20), to consider the descendants

$$
\bar{\beta}^*_{f'}, \gamma^*_{f'}, \bar{\beta}^*_{f} + \gamma^*_{f} \Phi_\alpha(0).
$$

This is the field in the interacting model which goes to corresponding descendant in the conformal limit $\mu \to 0$, and which does not develop finite counterterms. The latter are forbidden by dimensional consideration. So, this is exactly the definition which we were supposed to use from the very beginning. Notice that the appearance of $\mu$ in the formulae (28) is a consequence of consistency with the conformal limit due to (25).

Now it is easy to compute the normalised vacuum expectation value of the descendant (29) for the sG model. It is obtained by taking the scaling limit of $Z_{\text{full}}^\infty$ and computing the asymptotics of $\omega_0(\zeta, \alpha)$ for $\zeta \to \infty$ and $\zeta \to 0$ which is done simply by summing up the residues in appropriate half-planes:

$$
\omega_0(\zeta, \alpha) \simeq \frac{i}{\nu} \sum_{n \geq 1} \zeta^{-2n + 1} \cot \frac{\pi}{2\nu} (2n - 1 + \nu \alpha) + i \sum_{n \geq 1} \zeta^{\alpha - 2n} \tan \frac{\pi \nu}{2} (\alpha - 2n).
$$

The first part corresponds to the expectation values of operators created by the coefficients of the expansions (17), (20). These are the expectation values in which we are interested in this paper. The second part of the asymptotics corresponds to the operators (18), (21). At present we do not know the meaning of their expectation values, but we hope to return to them in the future.
Introducing the multi-indices as described in the introduction we obtain:

\[
\frac{\langle \bar{\beta} T^+ \gamma^*_I - \beta^*_I + \gamma^*_I - \Phi_\alpha(0) \rangle_{sG}}{\langle \Phi_\alpha(0) \rangle_{sG}} = \delta_{I^-,I^+} \delta_{I^-,I^-} (-1)^{#(I^+)} \left( \frac{i}{\nu} \right)^{#(I^+)+#(I^-)}
\]

\[
\times \mu \mu_2^{#(I^+)+#(I^-)} \prod_{2n-1 \in I^+} \cot \frac{\pi}{2} \left( \frac{2n}{\nu} + \alpha \right) \prod_{2n-1 \in I^-} \cot \frac{\pi}{2} \left( \frac{2n}{\nu} - \alpha \right).
\]

Now we have to recall the definition in IV:

\[
\beta_{2n-1}^* = D_{2n-1}(\alpha) \beta^{CFT}_{2n-1}, \quad \gamma_{2n-1}^* = D_{2n-1}(2-\alpha) \gamma^{CFT*}_{2n-1},
\]

where

\[
D_{2n-1}(\alpha) = \frac{1}{\sqrt{i\nu}} G^{2n-1} \frac{\Gamma \left( \frac{\alpha}{2} + \frac{1}{2\nu}(2n-1) \right)}{(n-1)! \Gamma \left( \frac{\alpha}{2} + \frac{1}{2\nu}(2n-1) \right)},
\]

with

\[
G = \Gamma(\nu)^{-1/\nu} \sqrt{1 - \nu}.
\]

Similarly we get for the second chirality

\[
\bar{\beta}_{2n-1}^* = D_{2n-1}(2-\alpha) \bar{\beta}^{CFT*}_{2n-1}, \quad \bar{\gamma}_{2n-1}^* = D_{2n-1}(\alpha) \bar{\gamma}^{CFT*}_{2n-1}.
\]

The main formula (5) follows immediately.

Before concluding this paper, let us say that in principle our approach can be applied to the computation of one-point functions of descendants for finite radius in the Matsubara direction (finite temperature, in other words). However, this would require a detailed study of the DDV equation and the equations for the function \( \omega_R \) for the sG model in finite volume.

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