Discovering General partial orders in event streams

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Abstract

Frequent episode discovery is a popular framework for pattern discovery in event streams. An episode is a partially ordered set of nodes with each node associated with an event type. Efficient (and separate) algorithms exist for episode discovery when the associated partial order is total (serial episode) and trivial (parallel episode). In this paper, we propose efficient algorithms for discovering frequent episodes with general partial orders. These algorithms can be easily specialized to discover serial or parallel episodes. Also, the algorithms are flexible enough to be specialized for mining in the space of certain interesting subclasses of partial orders. We point out that there is an inherent combinatorial explosion in frequent partial order mining and most importantly, frequency alone is not a sufficient measure of interestingness. We propose a new interestingness measure for general partial order episodes and a discovery method based on this measure, for filtering out uninteresting partial orders. Simulations demonstrate the effectiveness of our algorithms.

I. INTRODUCTION

Frequent episode discovery [12] is a popular framework for discovering temporal patterns in symbolic time series data, with applications in several domains like manufacturing [6], [16], telecommunication [12], WWW [9], biology [2], [14], finance [13], intrusion detection [10], [17], text mining [5] etc. The data in this framework is a single long time-ordered stream of events and each temporal pattern (called an episode) is essentially a small, partially ordered collection of nodes, with each node associated with a symbol (called event-type). The partial order in the

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episode constrains the time-order in which events should appear in the data, in order for the events to constitute an occurrence of the episode. Patterns with a total order on their nodes are called serial episodes, while those with an empty partial order are called parallel episodes [12]. The task is to unearth all episodes whose frequency in the data exceeds a user-defined threshold.

Currently, separate algorithms exist in the literature for discovering frequent serial and parallel episodes in data streams [3], [6], [12], [14], while no algorithms are available for the case of episodes with general partial orders. Related work can be found in the context of sequential patterns [1], [4], [11], [15] where the data consists of multiple sequences and the sequential pattern is a small partially ordered collection of symbols. A sequential pattern is considered frequent if there are enough sequences (in the data) in which the pattern occurs at least once. By contrast, in frequent episode discovery, we are looking for patterns that repeat often in a single long stream of events. This makes the computational task quite different from that in sequential patterns.

In this paper, we develop algorithms for discovering frequent episodes with general partial order constraints over their nodes. We restrict our attention to a subclass of patterns called injective episodes, where an event-type cannot appear more than once in a given episode. This facilitates the design of efficient algorithms with no restriction whatsoever on the partial orders of episodes. Further, our algorithms can handle the usual expiry time constraints for episode occurrences (which limit the time-spans of valid occurrences to some user-defined maximum value). Our algorithms can be easily specialized to either discover only frequent serial episodes or only frequent parallel episodes. Moreover, we can also specialize the method to focus the discovery process to certain classes of partial order episodes which satisfy what we call as the maximal subepisode property (Serial episodes and parallel episodes are specific examples of classes that obey this property).

As we point out here, one of the difficulties in efficient discovery of general partial orders is that there is an inherent combinatorial explosion in the number of frequent episodes of any given size. This is because, for any partial order episode with \( n \) nodes, there are an exponential number of subepisodes, also of size \( n \), all of which would occur at least as often as the episode. (Note that this problem does not arise in, e.g., frequent serial episode discovery because an \( n \)-node serial episode cannot have any \( n \)-node serial subepisode). Thus, frequency alone is insufficient as a measure of interestingness for episodes with general partial orders. To tackle this, we propose
a new measure called bidirectional evidence, which captures some notion of entropy of relative frequencies of pairs of events occurring in either order in the observed occurrences of an episode. The mining procedure now requires a user-defined threshold on bidirectional evidence in addition to the usual frequency threshold. We demonstrate the utility of our algorithms through extensive empirical studies.

The paper is organized as follows. In Sec. II, we briefly review the frequent episodes formalism and define injective episodes. Sec. III describes the finite state automata (and its associated properties) for tracking occurrences of injective episodes. Algorithms for counting frequencies of partial order episodes are described in Sec. IV. The candidate generation is described in Sec. V. Sec. VII-A describes our new interestingness measure. We present simulation results in Sec. VIII and conclude in Sec. IX.

II. EPISODES IN EVENT STREAMS

The data, referred to as an event sequence, is denoted by $\mathcal{D} = \langle (E_1, t_1), (E_2, t_2), \ldots, (E_n, t_n) \rangle$, where $n$ is the number of events in the datastream. In each tuple $(E_i, t_i)$, $E_i$ denotes the event type and $t_i$ the time of occurrence of the event. The event types $E_i$, take values from a finite set, $\mathcal{E}$. The sequence is ordered so that, $t_i \leq t_{i+1}$ for all $i = 1, 2, \ldots$. The following is an example sequence with 10 events:

$$\langle (A, 2), (B, 3), (A, 3), (A, 7), (C, 8), (B, 9), (D, 11), (C, 12), (A, 13), (B, 14), (C, 15) \rangle$$ (1)

Definition 1: [12] An $N$-node episode $\alpha$, is a tuple, $(V_\alpha, <_\alpha, g_\alpha)$, where $V_\alpha = \{v_1, v_2, \ldots, v_N\}$ denotes a collection of nodes, $<_\alpha$ is a strict partial order $^1$ on $V_\alpha$ and $g_\alpha : V_\alpha \to \mathcal{E}$ is a map that associates each node in the episode with an event-type (out of the alphabet $\mathcal{E}$).

When $<_\alpha$ is a total order, $\alpha$ is referred to as a serial episode and when $<_\alpha$ is empty $\alpha$ is referred to as a parallel episode. In general, episodes can be neither serial nor parallel. We denote episodes using a simple graphical notation. For example, consider a 3-node episode $\alpha = (V_\alpha, <_\alpha, g_\alpha)$, where $v_1 <_\alpha v_2$ and $v_1 <_\alpha v_3$, and with $g_\alpha(v_1) = B$, $g_\alpha(v_2) = A$ and $g_\alpha(v_3) = C$. We denote this episode as $(B \rightarrow (AC))$, implying that $B$ is followed by $A$ and $C$ in any order.

$^1$A strict partial order is a relation which is irreflexive, asymmetric and transitive.
Definition 2: [12] Given a data stream, \( (E_1, t_1), \ldots, (E_n, t_n) \) and an episode \( \alpha = (V_\alpha, <_\alpha, g_\alpha) \), an occurrence of \( \alpha \) is a map \( h : V_\alpha \rightarrow \{1, \ldots, n\} \) such that \( g_\alpha(v) = E_{h(v)} \) for all \( v \in V_\alpha \), and for all \( v, w \in V_\alpha \) with \( v <_\alpha w \) we have \( t_{h(v)} < t_{h(w)} \).

For example, \( (B, 3), (A, 7), (C, 8) \) and \( (B, 9), (C, 12), (A, 13) \) constitute occurrences of \( (B \rightarrow (AC)) \) in the event sequence \( \mathbb{D} \), while \( ((B, 3), (A, 3), (C, 8)) \) is not a valid occurrence since \( B \) does not occur before \( A \).

Given any \( N \)-node episode, \( \alpha \), it is sometimes useful to represent an occurrence, \( h \), of \( \alpha \) as a vector of integers \([h(1), h(2) \ldots h(N)]\), where \( h(i) < h(i + 1), i = 1, \ldots, (N - 1) \). For example, in sequence \( \mathbb{D} \), the occurrence corresponding to the subsequence \( ((B, 3), (A, 7), (C, 8)) \) is associated with the vector \([2\ 4\ 5]\) (since \( B, 3 \), \( A, 7 \) and \( C, 8 \) are the second, fourth and fifth events in \( \mathbb{D} \) respectively).

Consider an \( N \)-node episode, \( \alpha \), and the set, \( \mathcal{H}_\alpha \), of occurrences of \( \alpha \) in event sequence \( \mathbb{D} \). The occurrences in \( \mathcal{H}_\alpha \) can be arranged according to the lexicographic ordering of the vectors, \([h(1), \ldots, h(N)]\), \( h \in \mathcal{H}_\alpha \).

Definition 3: [8] The lexicographic order, \( <_\alpha \), on the set, \( \mathcal{H}_\alpha \) of occurrences of an \( N \)-node episode, \( \alpha \), in an event sequence, \( \mathbb{D} \), can be defined as follows: Given two different occurrences \( h_1 \) and \( h_2 \) of \( \alpha \) in \( \mathbb{D} \), we have \( h_1 <_\alpha h_2 \) iff the least \( i \) for which \( h_1(i) \neq h_2(i) \) is such that \( h_1(i) < h_2(i) \).

Definition 4: [12] Episode \( \beta = (V_\beta, <_\beta, g_\beta) \) is said to be a subepisode of \( \alpha = (V_\alpha, <_\alpha, g_\alpha) \) (denoted \( \beta \preceq \alpha \)) if there exists a \( 1-1 \) map \( f_{\beta\alpha} : V_\beta \rightarrow V_\alpha \) such that (i) \( g_\beta(v) = g_\alpha(f_{\beta\alpha}(v)) \) for all \( v \in V_\beta \), and (ii) for all \( v, w \in V_\beta \) with \( v <_\beta w \), we have \( f_{\beta\alpha}(v) <_\alpha f_{\beta\alpha}(w) \) in \( V_\alpha \).

In other words, for \( \beta \) to be a subepisode of \( \alpha \), all event-types of \( \beta \) must also be in \( \alpha \), and the order among the event-types in \( \beta \) must also hold in \( \alpha \). Thus, \( (B \rightarrow A) \), \( (B \rightarrow C) \) and \( (AC) \) are the 2-node subepisodes of \( (B \rightarrow (AC)) \). We note here that if \( \beta \preceq \alpha \), then every occurrence of \( \alpha \) contains an occurrence of \( \beta \).

Given an event sequence the datamining task here is to discover all frequent episodes, i.e., those episodes whose frequencies exceed a given threshold. Frequency is some measure of how often an episode occurs in the data stream. The frequency of episodes can be defined in more than one way [7], [12]. In this paper, we consider the non-overlapped occurrences-based frequency measure for episodes [7]. Informally, two occurrences of an episode are said to be non-overlapped if no event corresponding to one occurrence appears in-between events of the
other. The frequency of an episode is the size of the largest set of non-overlapped occurrences for that episode in the given data stream.

**Definition 5:** [7] Consider a data stream (event sequence), $\mathcal{D}$, and an $N$-node episode, $\alpha$. Two occurrences $h_1$ and $h_2$ of $\alpha$ are said to be non-overlapped in $\mathcal{D}$ if either $h_1(N) < h_2(1)$ or $h_2(N) < h_1(1)$. A set of occurrences is said to be non-overlapped if every pair of occurrences in the set is non-overlapped. The cardinality of the largest set of non-overlapped occurrences of $\alpha$ in $\mathcal{D}$ is referred to as the **non-overlapped frequency** of $\alpha$ in $\mathcal{D}$.

### A. Injective Episodes

In this paper, we consider a sub-class of episodes called **injective episodes**. An episode, $\alpha = (V_\alpha, <_\alpha, g_\alpha)$ is said to be injective if the $g_\alpha$ is an injective (or 1-1) map. For example, the episode $(B \rightarrow (AC))$ is an injective episode, while $B \rightarrow (AC) \rightarrow B$ is not. Thus, an injective episode, is simply a subset of event-types (out of the alphabet, $\mathcal{E}$) with a partial order defined over it. This subset, which we will denote by $X_\alpha$, is same as the range of $g_\alpha$. The partial order that is induced over $X_\alpha$ by $<_\alpha$ is denoted by $R_\alpha$. It is often much simpler to view an injective episode, $\alpha$, in terms of the **partial order set**, $(X_\alpha, R_\alpha)$, that is associated with it. From now on, unless otherwise stated, when we say episode we mean an injective episode.

In this paper, we will use either $(V_\alpha, <_\alpha, g_\alpha)$ or $(X_\alpha, R_\alpha)$ to denote episode $\alpha$, depending on the context. Although $(X_\alpha, R_\alpha)$ is simpler, in some contexts, e.g., when referring to episode occurrences, the $(V_\alpha, <_\alpha, g_\alpha)$ notation comes in handy. However, there can be multiple $(V_\alpha, <_\alpha, g_\alpha)$ representations for the same underlying pattern under **Definition 7**. Consider, for example, two 3-node episodes, $\alpha_1 = (V_1, <_{\alpha_1}, g_{\alpha_1})$ and $\alpha_2 = (V_2, <_{\alpha_2}, g_{\alpha_2})$, defined as: (i) $V_1 = \{v_1, v_2, v_3\}$ with $v_1 <_{\alpha_1} v_2$, $v_1 <_{\alpha_1} v_3$ and $g(v_1) = B$, $g(v_2) = A$, $g(v_3) = C$, and (ii) $V_2 = \{v_1, v_2, v_3\}$ with $v_2 <_{\alpha_2} v_1$, $v_2 <_{\alpha_2} v_3$ and $g(v_1) = A$, $g(v_2) = B$ and $g(v_3) = C$. Both $\alpha_1$ and $\alpha_2$ represent the same pattern, and they are indistinguishable based on their occurrences, no matter what the given data sequence is. (Notice that there is no such ambiguity in the $(X_\alpha, R_\alpha)$ representation).

In order to obtain a unique $(V_\alpha, <_\alpha, g_\alpha)$ representation for $\alpha$, we assume a lexicographic order over the alphabet, $\mathcal{E}$, and ensure that $(g_\alpha(v_1), \ldots, g_\alpha(v_N))$ is ordered as per this ordering. Note that this lexicographic order on $\mathcal{E}$ is not related in anyway to the actual partial order, $\leq_\alpha$. The lexicographic ordering over $\mathcal{E}$ is only required to ensure a unique representation of injective episodes in the $(V_\alpha, <_\alpha, g_\alpha)$ notation. Referring to the earlier example involving $\alpha_1$ and $\alpha_2$, we
will use \( \alpha_2 \) to denote the pattern \((B \to (AC))\).

Finally, note that, if \( \alpha \) and \( \beta \) are injective episodes, and if \( \beta \preceq \alpha \) (cf. Definition 4), then the associated partial order sets are related as follows: \( X_\beta \subseteq X_\alpha \) and \( R_\beta \subseteq R_\alpha \). Some examples of injective episodes, illustrating the different notations for episodes, is given in Table I.

| Episode  | Graphical Notation | \( X_\alpha, R_\alpha \) |
|----------|--------------------|--------------------------|
| \( V = \{v_1, v_2, v_3\} \) \( g(v_1) = A, g(v_2) = B, g(v_3) = C \) \( \subseteq \alpha = \{(v_2, v_1), (v_3, v_1)(v_3, v_2)\} \) | \( (C \to B \to A) \) | \( X_\alpha = \{A, B, C\} \) \( R_\alpha = \{(C, B), (B, A), (C, A)\} \) |
| \( V = \{v_1, v_2, v_3\} \) \( g(v_1) = A, g(v_2) = B, g(v_3) = C \) \( \subseteq \alpha = \{\} \) | \( (A B C) \) | \( X_\alpha = \{A, B, C\} \) \( R_\alpha = \{\} \) |
| \( V = \{v_1, v_2, v_3, v_4\} \) \( g(v_1) = A, g(v_2) = B, g(v_3) = C, g(v_4) = D \) \( \subseteq \alpha = \{(v_1, v_3), (v_2, v_3)(v_1, v_4), (v_2, v_4)\} \) | \( (AB) \to (CD) \) | \( X_\alpha = \{A, B, C, D\} \) \( R_\alpha = \{(A, C), (B, C), (A, D), (C, D)\} \) |
| \( V = \{v_1, v_2, v_3, v_4, v_5\} \) \( g(v_1) = A, g(v_2) = B, g(v_3) = C, g(v_4) = D, g(v_5) = E \) \( \subseteq \alpha = \{(v_1, v_2), (v_1, v_3), (v_1, v_4)(v_1, v_5), (v_2, v_4), (v_2, v_5)\} \) | \( (A \to (B \to (DE))C) \) | \( X_\alpha = \{A, B, C, D, E\} \) \( R_\alpha = \{(A, B), (A, C), (A, D), (A, E), (B, D), (B, E)\} \) |

III. Finite State Automata for Partial Orders

Finite State Automata (FSA) can be used to track occurrences of injective episodes under general partial orders in a manner similar to the automata-based algorithms for parallel or serial episodes [7], [8], [12]. In this section, we describe the basic construction of such automata.

We first illustrate the automaton structure through an example. Consider episode \((\alpha = (AB) \to C)\). Here, \( X_\alpha = \{A, B, C\} \) and \( R_\alpha = \{(A, C), (B, C)\} \). The FSA used to track occurrences of this episode is shown in Fig. I. Each state, \( i \), is associated with a pair of subsets of \( X_\alpha \), namely, \((Q^\alpha_i, \mathcal{W}^\alpha_i)\); \( Q^\alpha_i \subseteq X_\alpha \) denotes the event-types that the automaton has already accepted by the time it arrives in state \( i \); \( \mathcal{W}^\alpha_i \subseteq X_\alpha \) denotes the event-types that the automaton in state \( i \) is ready to accept. Initially, the automaton is in state \( 0 \), has not accepted any events so far and is waiting.
for either of $A$ and $B$, i.e., $Q_0^\alpha = \phi$ and $W_0^\alpha = \{A, B\}$. If we see a $B$ first, we accept it and continue waiting for an $A$, i.e., the automaton transits to state 2 with $Q_2^\alpha = \{B\}$, $W_2^\alpha = \{A\}$. At this point the automaton is not yet ready to accept a $C$, which happens only after both $A$ and $B$ are encountered (in whatever order). If, instead of encountering a $B$, the automaton in state 0 first encountered an $A$, then it would transit into state 1 (rather than state 2), where it would now wait for a $B$ to appear (Thus, $Q_1^\alpha = \{A\}$, $W_1^\alpha = \{B\}$). Once both $A$ and $B$ appear in the data, the automaton will transit, either from state 1 or state 2, and move into state 3, where it now waits for a $C$ ($Q_3^\alpha = \{A, B\}$, $W_3^\alpha = \{C\}$). Finally, if the automaton now encounters a $C$ in the data stream, it will transit to the final state, namely, state 4 ($Q_4^\alpha = \{A, B, C\}$, $W_4^\alpha = \phi$) and recognize a full occurrence of the episode, $((AB) \rightarrow C)$.

In any occurrence of an episode $\alpha$, an event $E \in X^\alpha$ can occur only after all its parents in $R^\alpha$ have been seen. Hence, we initially wait for all those elements of $X^\alpha$ which are minimal elements of $R^\alpha$. Further, we start waiting for a non-minimal element, $E$, of $R^\alpha$ immediately after all elements less than $E$ in $R^\alpha$ are seen. For each $E \in X^\alpha$, we refer to the subset of elements in $X^\alpha$ that are less than $E$ (with respect to $R^\alpha$) as the parents of $E$ in episode, $\alpha$, and denote it by $\pi_\alpha(E)$. We now define $A_\alpha$, the FSA to recognise occurrences of $\alpha$.

**Definition 6**: FSA $A_\alpha$, used to track occurrences of $\alpha$ in the data stream is defined as follows.
Each state, \( i \), in \( A_\alpha \), is represented by a unique pair of subsets of \( X_\alpha \), namely \( (Q_\alpha^i, W_\alpha^i) \); \( Q_\alpha^i \subseteq X_\alpha \) is the set of event-types that the automaton has accepted so far and \( W_\alpha^i \) is the set of event-types that the automaton is currently ready to accept. The initial state, namely, state 0, is associated with the subsets pair, \( (Q_\alpha^0, W_\alpha^0) \), where \( Q_\alpha^0 = \emptyset \) and \( W_\alpha^0 \) is the collection of least elements in \( X_\alpha \) with respect to \( R_\alpha \). Let \( i \) be the current state of \( A_\alpha \) and let the next event in the data be of type, \( E \in E \). \( A_\alpha \) remains in state \( i \) if \( E \in (X_\alpha \setminus W_\alpha^i) \). If \( E \in W_\alpha^i \), then \( A_\alpha \) accepts \( E \) and transits into state \( j \), with:

\[
Q_\alpha^j = Q_\alpha^i \cup \{E\} \tag{2}
\]

\[
W_\alpha^j = \{E' \in (X_\alpha \setminus Q_\alpha^j) : \pi_\alpha(E') \subseteq Q_\alpha^j\} \tag{3}
\]

When \( Q_\alpha^j = X_\alpha \), (and hence \( W_\alpha^j = \emptyset \)), \( j \) is the final state of \( A_\alpha \).

It may be noted that not all possible tuples of \((Q, W)\), where \( Q \subseteq X_\alpha \), \( W \subseteq X_\alpha \), constitute valid states of the automaton. For example in Fig. 1 there can be no valid state corresponding to \( Q = \{A, C\} \) (since \( C \) could not have been accepted without \( B \) being accepted before it). We list below a few properties of the valid states of the automaton. All these are easily proved from the above definition.

**Property 1:** For any state, \( j \), of the automaton, \( A_\alpha \), the set, \( W_\alpha^j \), of event-types that \( A_\alpha \) will wait for in state \( j \) (as per Eq. (3) in Definition 6), is exactly the set of least elements of \( (X_\alpha \setminus Q_\alpha^j) \) (with respect to the partial order \( R_\alpha \)).

**Proof:** If \( E \) is a least element of \( (X_\alpha \setminus Q_\alpha^j) \), it implies that all parents of \( E \) (if any) are outside \( (X_\alpha \setminus Q_\alpha^j) \). Hence, they must have already been accepted by \( A_\alpha \) (i.e. we must have \( \pi_\alpha(E) \subseteq Q_\alpha^j \)), and so, by (3), we have \( E \in W_\alpha^j \). Conversely, every \( E' \) that \( A_\alpha \) is waiting for (according to (3)) trivially belongs to \( (X_\alpha \setminus Q_\alpha^j) \) and since (3) also prescribes that \( \pi_\alpha(E') \subseteq Q_\alpha^j \), we must have \( \pi_\alpha(E') \cap W_\alpha^j = \emptyset \). Hence, such an \( E' \) must be a least element of \( (X_\alpha \setminus Q_\alpha^j) \). ■

**Property 2:** For any state, \( j \), of automaton, \( A_\alpha \), if \( (X_\alpha \setminus Q_\alpha^j) \) is non-empty, then \( W_\alpha^j \) is non-empty. Thus, the only state out of which \( A_\alpha \) makes no state transitions no matter what the input sequence (i.e. the only final state of \( A_\alpha \)) is the one represented by the pair, \( (X_\alpha, \phi) \).

**Proof:** If \( (X_\alpha \setminus Q_\alpha^j) \) is non-empty, then it must contain at least one least element (with respect to \( R_\alpha \)) and from Property 1, this element must be in \( W_\alpha^j \) (and hence, it must be non-empty). ■
Property 3: Given the set, \( W_1^\alpha \), of event-types, that \( A_\alpha \) will wait for in state, \( j \), \( j \neq 0 \), the corresponding set of event-types accepted by the time \( A_\alpha \) reaches state \( j \), is given by

\[
\{ E \in (X^\alpha \setminus W_j^\alpha) \text{ s.t. } \pi_\alpha(E) \cap W_j^\alpha = \phi \}
\]

Thus, for any two distinct states, \( i \) and \( j \), of \( A_\alpha \), we must have both \( Q_i^\alpha \neq Q_j^\alpha \) and \( W_i^\alpha \neq W_j^\alpha \).

Proof: If the automaton, \( A_\alpha \), has accepted an event of type \( E \) (i.e. if \( E \in Q_j^\alpha \) as per Definition 6) then all parents of \( E \) (if any) should have been previously accepted by \( A_\alpha \), and hence, we must have \( E \in (X^\alpha \setminus W_j^\alpha) \) and \( \pi_\alpha(E) \cap W_j^\alpha = \phi \). To show the other way, consider an \( E \in (X^\alpha \setminus W_j^\alpha) \) such that \( \pi_\alpha(E) \cap W_j^\alpha = \phi \). Now, if \( E \notin Q_j^\alpha \) (i.e. if \( E \) has not yet been accepted by \( A_\alpha \) as per Definition 6), then \( A_\alpha \) must wait for either \( E \) or one (or more) of its parents, i.e. either \( E \in W_j^\alpha \) or \( \pi_\alpha(E) \cap W_j^\alpha \neq \phi \) (which contradicts our original assumption for \( E \)). This completes the proof of Property 3.

The next two properties give an exact characterization of \( Q^\alpha \) and \( W^\alpha \) for an episode \( \alpha \). They describe the the kind of subsets(of \( X^\alpha \)) that actually come up as \( Q^\alpha \)'s and \( W^\alpha \)'s in \( A_\alpha \).

Property 4: Let \( A_\alpha \) denote an automaton of episode, \( \alpha \), as per construction. Given \( Q^\alpha \subseteq X^\alpha \), \( Q^\alpha \) is the set of event-types that \( A_\alpha \) has currently accepted \( \iff \forall E \in Q^\alpha, \pi_\alpha(E) \subseteq Q^\alpha \).

Proof: Since \( Q_0^\alpha \) was initially empty, for \( E \in Q^\alpha \), we must have had \( E \in W_i^\alpha \) for some other (earlier) state, \( i \). Now if \( E \in W_i^\alpha \), then either (i) if \( \pi_\alpha(E) = \phi \), then \( E \) must be a least element of \( X^\alpha \) with respect to \( R^\alpha \), or (ii) \( \pi_\alpha(E) \) is non-empty, so that, by applying (3) for state \( i \), we know \( E \) must have been added to \( W_i^\alpha \) only if \( \pi_\alpha(E) \subseteq Q^\alpha \). But, from (2), we know \( Q_i^\alpha \subseteq Q^\alpha \). This implies \( \pi_\alpha(E) \subseteq Q^\alpha \).

Conversely, suppose \( Q^\alpha \) is such that \( \forall E \in Q^\alpha, \pi_\alpha(E) \subseteq Q^\alpha \). Consider the least element \( E_1 \in Q^\alpha \) (with respect to \( R^\alpha \)). \( E_1 \) has no parents in \( Q^\alpha \). By definition of \( Q^\alpha \), \( E_1 \) has no parents outside \( Q^\alpha \). Hence, \( E_1 \) is also a least element of \( X^\alpha \) which implies \( E_1 \in W_0^\alpha \). Hence from state 0, \( A_\alpha \) makes a transition(on seeing \( E_1 \)) to a state 1 with \( Q_1^\alpha = E_1 \). Now consider a least element \( E_2 \in Q_1^\alpha \). One can verify on similar lines that \( \pi_\alpha(E_2) \subseteq Q_1^\alpha \). This means(from (3)) that \( E_2 \in W_1^\alpha \). Hence, \( A_\alpha \) makes a transition(on seeing \( E_2 \)) to a state 2 with \( Q_2^\alpha = Q_1^\alpha \cup \{E_2\} \). This process continues till a stage \( k = |Q^\alpha| \) at which \( A_\alpha \) actually enters a state \( k \) where \( Q_k^\alpha = Q^\alpha \).

Property 5: Let \( A_\alpha \) denote an automaton of episode, \( \alpha \), as per construction. Given \( W^\alpha \subseteq X^\alpha \),
\( \mathcal{W}^{\alpha} \) is the set of event-types that \( A_{\alpha} \) is currently waiting for \( \iff \forall E \in \mathcal{W}^{\alpha}, \pi_{\alpha}(E) \cap \mathcal{W}^{\alpha} = \emptyset. \)

**Proof:** The forward direction is straightforward from (3). Conversely, consider a \( \mathcal{W}^{\alpha} \) such that \( \forall E \in \mathcal{W}^{\alpha}, \pi_{\alpha}(E) \cap \mathcal{W}^{\alpha} = \emptyset. \) Consider the following set.

\[
\bar{\mathcal{Q}}^{\alpha} = \{ E \in (X^{\alpha} \setminus \mathcal{W}^{\alpha}) : \pi_{\alpha}(E) \cap \mathcal{W}^{\alpha} = \emptyset \}
\]

(5)

Note that this set is exactly similar to the one defined in Property 3. We will first show that this \( \bar{\mathcal{Q}}^{\alpha} \) is such that \( \forall E \in \bar{\mathcal{Q}}^{\alpha}, \pi_{\alpha}(E) \subseteq \bar{\mathcal{Q}}^{\alpha} \). If this is true, then from Property 4, \( \bar{\mathcal{Q}}^{\alpha} \) is a set of events that \( A_{\alpha} \) would have accepted at some stage. We next show that the \( \mathcal{W}^{\alpha} \) that we started off with, is the set of event-types \( A^{\alpha} \) would wait for, after having accepted the set of events \( \bar{\mathcal{Q}}^{\alpha} \).

Consider an \( E \in \bar{\mathcal{Q}}^{\alpha} \). Let \( E' \) be a parent of \( E \). We need to show that \( E' \in \bar{\mathcal{Q}}^{\alpha} \) i.e. \( E' \notin (X^{\alpha} \setminus \mathcal{W}^{\alpha}) \) and \( \pi_{\alpha}(E') \cap \mathcal{W}^{\alpha} = \emptyset. \) If \( E' \notin (X^{\alpha} \setminus \mathcal{W}^{\alpha}) \), then a parent of \( E \) is in \( \mathcal{W}^{\alpha} \) which contradicts \( E \in \bar{\mathcal{Q}}^{\alpha} \). If \( \pi_{\alpha}(E') \cap \mathcal{W}^{\alpha} \neq \emptyset \), then \( \exists \) an \( E'' \in \mathcal{W}^{\alpha} \) such that \( (E'', E') \in R^{\alpha} \). But we also have \( (E', E) \in R^{\alpha} \). Hence, by transitivity, \( E'' \) is a parent of \( E \) in \( \mathcal{W}^{\alpha} \), which contradicts \( E \in \bar{\mathcal{Q}}^{\alpha} \).

We now show that \( \mathcal{W}^{\alpha} \) is indeed the set of least elements in \( X^{\alpha} \setminus \bar{\mathcal{Q}}^{\alpha} \). Every element in \( Y = X^{\alpha} \setminus (\mathcal{W}^{\alpha} \cup \bar{\mathcal{Q}}^{\alpha}) \) should have a parent in \( \mathcal{W}^{\alpha} \). Otherwise, \( E \) must be in \( \bar{\mathcal{Q}}^{\alpha} \). So no element in \( Y \) is a least element of \( \mathcal{W}^{\alpha} \cup Y = X^{\alpha} \setminus \bar{\mathcal{Q}}^{\alpha} \). Consider an \( E \in \mathcal{W}^{\alpha} \). We need to show that \( \forall E' \in \mathcal{W}^{\alpha} \cup Y, E' \) is not lesser than \( E \). Since no two elements of \( \mathcal{W}^{\alpha} \) are related, no \( E' \) from \( \mathcal{W}^{\alpha} \) can be less than \( E \). Suppose there exists an \( E' \in Y \) such that \( (E', E) \in R^{\alpha} \). Since \( E' \in Y \), it has a parent in \( \mathcal{W}^{\alpha} \). By transitivity, \( E \in \mathcal{W}^{\alpha} \) has a parent in \( \mathcal{W}^{\alpha} \), which contradicts the definition of \( \mathcal{W}^{\alpha} \). Hence we have shown that \( \mathcal{W}^{\alpha} \) would be the set of events \( A^{\alpha} \) would wait for, after having accepted the set of events \( \bar{\mathcal{Q}}^{\alpha} \).

**Property 6:** Consider two states, \( i \) and \( j \), of \( A_{\alpha} \) with sets of accepted states, \( \mathcal{Q}^{\alpha}_i \) and \( \mathcal{Q}^{\alpha}_j \), such that \( \mathcal{Q}^{\alpha}_i \subseteq \mathcal{Q}^{\alpha}_j \). Let \( k = |\mathcal{Q}^{\alpha}_j \setminus \mathcal{Q}^{\alpha}_i| \). There exists a sequence of events, \( (E_1, \ldots, E_k) \), on which \( A_{\alpha} \), currently in state \( i \), will make \( k \) state transitions, eventually arriving at a state \( j \) with the set of accepted events given by \( \mathcal{Q}^{\alpha}_j \).

**Proof:** The proof is very similar to the converse argument in Property 4. For the sake of completeness, we give the entire argument. Let \( E_1 \in (\mathcal{Q}^{\alpha}_j \setminus \mathcal{Q}^{\alpha}_i) \) such that \( E_1 \) is a least element of \( (\mathcal{Q}^{\alpha}_j \setminus \mathcal{Q}^{\alpha}_i) \) (with respect to \( R^{\alpha} \)). From Property 4 we know \( \pi_{\alpha}(E_1) \) must belong to \( \mathcal{Q}^{\alpha}_j \). Since \( E_1 \) is (by definition) the least element of \( \mathcal{Q}^{\alpha}_j \), none of \( E_1 \)'s parents are
in \((Q^\alpha_j \setminus Q^\alpha_i)\). So we must have \(\pi(\alpha)(E_1) \subseteq Q^\alpha_i\). This will ensure (from (3)) that \(E_1 \in W^\alpha_i\), and so, \(A_\alpha\) in state \(i\), on seeing \(E_1\), will make a transition to (say) state \(i_1\), with \(Q^\alpha_{i_1} = Q^\alpha_i \cup \{E_1\}\) and \(W^\alpha_{i_1} = \{E' \in (X^\alpha \setminus Q^\alpha_i) \text{ s.t. } \pi(\alpha)(E') \subseteq Q^\alpha_{i_1}\}\). Next we consider \(E_2\), a least element in \((Q^\alpha_i \setminus Q^\alpha_{i_1})\), and repeating the same argument as for \(E_1\) above, we can see that \(A_\alpha\) will now transit into state \(i_2\), with \(Q^\alpha_{i_2} = Q^\alpha_{i_1} \cup \{E_2\}\) and \(W^\alpha_{i_2} = \{E' \in (X^\alpha \setminus Q^\alpha_{i_2}) \text{ s.t. } \pi(\alpha)(E') \subseteq Q^\alpha_{i_2}\}\). Thus, for \(l = 2, \ldots, k\), we can construct \(Q^\alpha_i\) by adding the least element of \((Q^\alpha_j \setminus Q^\alpha_{i-1})\), \(E_l\), to \(Q^\alpha_{i_l} \subseteq Q^\alpha_j\).

IV. Counting Algorithms

The data mining task in the frequent episode paradigm is to extract all episodes whose frequency exceeds a user-defined threshold. Like current algorithms for frequent serial/parallel episode discovery [7], [12], we use an Apriori-style level-wise procedure for mining frequent episodes with general partial orders. Each level has two steps, namely, candidate generation and frequency counting. At level, \(l\), candidate generation step combines frequent episodes of size \((l-1)\) to construct candidates of size \(l\). It exploits the simple but powerful fact that if a pattern is frequent then (certain kinds of) its subpatterns are also frequent. The frequency counting step computes frequencies of all episodes in the candidates set and returns the set of frequent \(l\)-size episodes. Sec. V provides a detailed explanation of the candidate generation. In this section, we present an algorithm for obtaining the frequencies (or counting the number of non-overlapped occurrences) of a set of general injective episodes of a given size.

For counting the number of non-overlapped occurrences of a set of serial episodes, [8] proposed an algorithm using only one automaton per episode. This algorithm can be generalized as explained below, to count non-overlapped occurrences of a set of injective episodes (with general partial orders) by using the more general FSA of Definition 6. We initialize the automaton (associated with the episode) in its start state. The automaton would make transitions as prescribed by Definition 6. We traverse the data stream and let the automaton transit to its next state as soon as a relevant event-type appears in the data stream. When the automaton reaches its final state, we increment the frequency of the episode and reset the automaton to its start state so that
it would track the next occurrence. Since we have a set of candidates, we would have one such automaton for each episode. For each event in the data stream, we look at all automata waiting for that event and effect appropriate state transitions for all automata. Such an algorithm would count the non-overlapped occurrences of all candidate episodes through one pass over the data. Consider this counting scheme for episode $\beta = (AB) \rightarrow (CD)$ in the following data stream:

$$\langle (A, 1), (B, 2), (A, 3)(D, 4), (E, 5), (C, 6), (D, 7),
(A, 8), (B, 9), (B, 10), (C, 12), (D, 14) \rangle$$

(6)

The above method tracks occurrences $h_1 = \langle (A, 1), (B, 2), (D, 4), (C, 6) \rangle$ and $h_2 = \langle (A, 8), (B, 9), (C, 12), (D, 14) \rangle$ and returns frequency of 2 for this episode in this data stream.

Though this algorithm is efficient (as it uses only one automaton per episode), it cannot implement any temporal constraints on occurrences of episodes. One constraint that is often useful in applications is the expiry time constraint which is stated in terms of an upperbound on the span of an occurrence. The span of an occurrence is the largest difference between the times associated with any two events in the occurrence. Under the expiry time constraint, the frequency of an episode is the maximum number of non-overlapped occurrences such that span of each occurrence is less than a user-defined threshold. (The window-width of [12] essentially implements a similar constraint). An expiry time constraint is often useful because an occurrence of a pattern constituted by events widely separated in time, may not really indicate any underlying causative influences. Consider counting occurrences of $\beta$ in sequence (6) with an expiry time constraint of 4. The occurrences $h_1$ and $h_2$ of $\beta$ (that the algorithm would track as specified earlier) have spans 5 and 6 respectively. Hence our algorithm can only assign frequency of zero under the expiry time constraint. However, the occurrence $h_3 = \langle (B, 2), (A, 3), (D, 4)(C, 6) \rangle$ of $\beta$ in sequence (6) has a span of 4 (satisfying the constraint). The reason why our algorithm can not track $h_3$ is that the automaton makes a state transition as soon as the relevant event-type appears in the data and thus it accepts the event $(A, 1)$. We can count non-overlapped occurrences under an expiry time constraint if we allow more than one automaton per episode as explained below.

Consider this example again with a modified algorithm as follows. As earlier, our automaton will accept $(A, 1)$ and transit into a state that waits for a $B$. Now, since the automaton moved out of the start state we immediately spawn another automaton for this episode which is initialized in the start state. Now when we encounter $(B, 2)$ the first automaton will accept it and move
into a state where it waits for a $C$ or a $D$. The second automaton, which is in the start state, will also accept $(B, 2)$ and move to a state where it waits for a $A$. (we would now intialize a third automaton for this episode because the second one moved out of the start state). Now when we encounter $(A, 3)$ the second automaton would accept it and move into a state of waiting for a $C$ or a $D$ which is the same state as the first one is in. From now on, both automata would make identical transitions and hence we can retire the first automaton. This is because, the second automaton is initialized later and hence the span of the occurrence tracked by it would be smaller. (The third automaton would also accept $(A, 3)$ and we will spawn a new fourth automaton in start state). Now when we encounter $(D, 4)$ and later $(C, 6)$, the second automaton would reach the final state. Since the occurrence tracked by this automaton satisfies the expiry time constraint, we can now increment the frequency and then retire all other automata of this episode. We will also spawn a fresh automaton for this episode in the start state so that we can begin to track the next non-overlapped occurrence (if any) of this episode.

We can now specify the general method for counting under expiry time constraint as follows. Instead of spawning a new automaton only after the existing one reaches its final state, we spawn a new automaton whenever an existing automaton accepts its first event (i.e., when it transits out of its start state). Each of the automata makes a state transition as soon as a relevant event-type appears in the data stream. When counting like this, it is possible for two automata to reach the same state. In such cases, we drop the older one (retaining only the most recent automaton). This strategy tracks, in a sense, the innermost occurrence amongst a set of overlapping occurrences that end together. When any automaton (of an episode) reaches the final state, we check whether the span of the occurrence tracked by this automaton satisfies the expiry time constraint. If it does, we increment the frequency and retire all the automata of that episode except for one automaton in the start state to track the next occurrence. If the span of the occurrence tracked by the automaton that reached the final state does not satisfy the expiry time constraint, then we only retire that automaton. This is the algorithm that we use for counting the frequencies of episodes.

The pseudocode for counting non-overlapped occurrences of general injective episodes with an expiry-time constraint is given in Algorithm 1. The inputs to the algorithm are: $C_l$, the set of $l$-node candidate episodes, $D$, the event stream, $\mathcal{E}$, the set of event-types, $\gamma$, the frequency threshold, and $T_X$, the expiry-time. The algorithm outputs the set, $\mathcal{F}_l$, of frequent episodes. The
event-types associated with an $l$-node episode, $\alpha$, are stored in the $\alpha.g[]$ array – for $i = 1, \ldots, l$, $\alpha.g[i]$ is assigned the value $g_\alpha(v_i)$. We store the partial order $<_\alpha$, associated with the episode as a binary adjacency matrix, $\alpha.e[][]$. The notation is: $\alpha.e[i][j] = 1$ iff $v_i <_\alpha v_j$ (or equivalently, if $(\alpha.g[i], \alpha.g[j]) \in R^\alpha$).

The main data structure is an array of lists, $\text{waits}()$, indexed by the set of event-types. The elements of each list store the relevant information about all the automata that are waiting for a particular event-type (and hence can make a state transition if that event-type appears in the data stream). The entries in the list are of the form $(\alpha, q, w, j)$ where $\alpha$ is a candidate episode, $(q, w)$ is one of the possible states of the automaton associated with $\alpha$ (cf. Definition 6) and $j$ is an integer. For an event-type $E$, if $(\alpha, q, w, j) \in \text{waits}(E)$, it denotes that an automaton of the episode $\alpha$ (with $\alpha.g[j] = E$) is currently in state $(q, w)$ and is waiting for an event-type $E$ to make a state transition. Recall from Definition 6 that each state of the automaton is specified by a pair of subsets, $(Q^\alpha, W^\alpha)$, of the set of event-types $X^\alpha$ of $\alpha$. In our representation, $q$ and $w$ are $|X^\alpha|$-length binary vectors encoding the two sets $(Q^\alpha, W^\alpha)$. Consider the earlier example episode $\beta = (A B) \rightarrow (C D)$. For this, we have $X^\beta = \{A, B, C, D\}$. Suppose this automaton has already accepted an $A$ and $B$ and is waiting for a $C$ or $D$. So, its current state is $(\{A, B\}, \{C, D\})$. This automaton would be listed both in $\text{waits}(C)$ and $\text{waits}(D)$. We would have $(\beta, q, w, 3) \in \text{waits}(C)$ and $(\beta, q, w, 4) \in \text{waits}(D)$ where $q = [1 \ 1 \ 0 \ 0]$ and $w = [0 \ 0 \ 1 \ 1]$. Thus, in general, for an automaton in state $(Q^\alpha, W^\alpha)$, there would be $|W^\alpha|$ tuples in the different waits lists with the tuples differing only in the fourth position. As we traverse the data, if the next event is of event-type $E$, then we access all the automata waiting for $E$ through $\text{waits}(E)$ and effect state transitions. Knowing the current state of the automaton, we can compute its next state after accepting $E$ because we have the partial order of the episode stored in $\alpha.e$ array.

Since, as explained above, an automaton can be listed in multiple $\text{waits}()$ lists (because it can be waiting for a set of event-types), we have to ensure that the state transition is properly reflected in all $\text{waits}()$ lists.

In addition to the $\alpha.g$ and $\alpha.e$ arrays, the other pieces of information that we store with an episode $\alpha$ are: $\alpha.freq$, $\alpha.init$ and $\alpha.wstart$. The frequency of an episode is kept track of in $\alpha.freq$. For each episode $\alpha$, $\alpha.init$ is a list that keeps track of the times at which the various currently active automata of $\alpha$ made their transition out of the start state. Each entry in this list is a pair $(q, t)$, indicating that an automaton initialized (i.e., made its first state transition) at
time \( t \) is currently in a state with the set of accepted events represented by \( q \). Since a start-state automaton is yet to make its first transition, there is no corresponding entry for this in \( \alpha.init \). The information in \( \alpha.init \) is necessary to properly take care of situations where an automaton transits into a state already occupied by another automaton. It is also useful to check that the span of an occurrence satisfies expiry time constraint before incrementing frequency. \( \alpha.w_{\text{start}} \) is a \(|X^\alpha|\)-length binary vector encoding the set \( \mathcal{W}_0^\alpha \), the set of all least elements of \( X^\alpha \) (with respect to \( R^\alpha \)). In other words, it encodes the set of all event-types for which an automaton for \( \alpha \) would wait for in its start state. Since this information is needed everytime an automaton for the episode is to be initialized, it is useful to precompute it.

We now explain the working of Algorithm 1 by referring to the line numbers in the pseudocode. Lines 4 – 12 initialize all the \( \text{waits}(\cdot) \) lists by having one automaton for each candidate episode, waiting in its start state. In the main data pass loop (lines 15 – 65), we look at each item \((E_k, t_k)\), \( k = 1, 2 \ldots n \), in the event stream and modify the \( \text{waits}(\cdot) \) lists to affect state transitions of all automata waiting for \( E_k \). This is done by accessing each tuple in \( \text{waits}(E_k) \) list and processing it which is done in the loop starting on line 17. This is the main computation in the algorithm and we explain it below. For a tuple \((\alpha, q_{\text{cur}}, w_{\text{cur}}, j) \in \text{waits}(E_k)\), we need to affect a state transition (since we have seen \( E_k \)). The next state information for this automaton is denoted as \( q_{\text{nxt}}, w_{\text{nxt}} \) in the pseudocode. We compute \( q_{\text{nxt}} \) by setting \( j^{th} \) bit to one (line 20). Recall that in the start state we will have \( q = 0 \) (vector of all zeros). Hence if \( q_{\text{cur}} = 0 \), it means that this automaton is making its first transition out of the start state and hence we add \((q_{\text{nxt}}, t_k)\) to \( \alpha.init \) list in line 22. (Recall that \( \alpha.init \) contains all active automaton for episode \( \alpha \) and for each automaton we record its current state and the time at which it made its transition out of the start state). Also, when \( q_{\text{cur}} = 0 \), since this automaton is now moving out of its start state, we need a new automaton for \( \alpha \) initialized in its start state. We do this by remembering \( \alpha \) in a temporary memory called \( \text{bag} \). (We accumulate all episodes for which new automata are to be initialized in the start state, in this temporary memory called \( \text{bag} \) while processing all tuples in \( \text{waits}(E_k) \). Then, after processing all tuples in \( \text{waits}(E_k) \), we initialize all these automata in lines 58 – 62). The final state of the automaton corresponds to \( q \) becoming 1, a vector of all ones. If \( q_{\text{nxt}} \neq 1 \), then this automaton, after the current state transition, is still an active automaton for \( \alpha \) and hence we need to update the \( \alpha.init \) list by reflecting the new state of this automaton which is done in lines 25 – 28. When \( q_{\text{nxt}} \neq 1 \), to complete the computation of its
next state, we need to find $w_{nxt}$. This automaton has now accepted its $j^{th}$ event type. Hence, using the partial order information contained in $\alpha.e$ array, we need to find what all new event types it is ready to accept. Using this, we can compute $w_{nxt}$ as done in lines 31 to 37. It is computed based on the children of $E_k$ in $R^\alpha$ as follows. It is easy to verify based on Definition 6 that 

$$W_{nxt}^\alpha = (W_{cur}^\alpha \setminus E_k) \cup W', \text{ where}$$

$$W' = \{\text{children } E' \text{ of } E_k \text{ in } R^\alpha : \pi_\alpha(E) \subseteq Q_{nxt}^\alpha\}$$

We then need to put this automaton in the \textit{waits()} list of all those event types that it can accept now. Also, we should modify state information in the \textit{waits()} lists of event types corresponding to its previous state. This is done in lines 39 – 43. We point out that in this process, the \textit{waits()} lists would end up having duplicate elements if there is already an automaton in the state $q_{nxt}$. If after the current state transition, the automaton came into a state in which there is another automaton of this episode, then we have to remove the older automaton. Presence of an older automaton is indicated by an entry $(q_{nxt}, t')$ for some $t'$ in the $\alpha.init$ list. If $t' < t_{cur}$, where $t_{cur}$ is the time when the current automaton made its first state transition, then we need to remove the older automaton, which is done in lines 45 – 47. We would also need to remove one of the duplicate elements in the appropriate \textit{waits()} lists as indicated in line 48-50. If $q_{nxt} = 1$ (so that we have now reached the final state), then we need to check whether the span of the occurrence tracked by this automaton satisfies the expiry time constraint. We can compute the span because we know $t_{cur}$, the time at which this automaton accepted the first event-type, from the entry for this automaton in $\alpha.init$ list. If the span of the occurrence tracked is less than expiry time, then we increment the frequency and remove all the other active automaton of this episode and then start a new automaton in the start state (lines 52 – 57). This completes the explanation of Algorithm 1.

In the algorithm discussed above we are implicitly assuming that different events in the data stream have distinct time stamps. This is because, in the data pass loop (starting on line 15) an automaton can accept $E_{k+1}$ after accepting $E_k$ in the previous pass through the loop. We now indicate how one can extend Algorithm 7 to handle data with multiple event types having the same time-stamp. For such datastreams, event-types sharing the same time-stamp must be processed.

\footnote{The steps of automata transition and check for an older automaton can be combined and carried out more efficiently. For ease of explanation we have presented the two steps separately.}
together. One needs to perform unconditional state transitions of all the relevant automata, till all event-types occurring at a given time are parsed. The state transition step needs a slight modification here compared to that of Algorithm 1. Consider an automaton for the episode \((BC) \rightarrow D\) waiting in its start state \((Q_{\text{cur}}^\alpha, W_{\text{cur}}^\alpha) = (\emptyset, \{B, C\})\). Suppose we have the event-types \(B, C\) and \(D\) happening together at a time \(t\). Let us denote the set of event-types occurring at time \(t\) by \(S\). On processing \(S\), we would need to accept both \(B\) and \(C\) but not \(D\), though after accepting \(B\) and \(C\) it transits into a state where it waits for \(D\). In general, an automaton waiting for a set of event-types \(W_{\text{cur}}^\alpha\) just before time \(t\), should accept the set of events \(S \cap W_{\text{cur}}^\alpha\) on seeing the set of event-types \(S\) at time \(t\). Accordingly, for the next state, \(Q_{\text{nxt}}^\alpha = Q_{\text{cur}}^\alpha \cup (S \cap W_{\text{cur}}^\alpha)\). \(W_{\text{nxt}}^\alpha\) can be computed from \(Q_{\text{nxt}}^\alpha\) as in Definition 6. Equivalently, we could do the same by processing event-type by event-type as in Algorithm 1, but such a strategy needs some extra caution. Suppose we had \(C\) followed by \(B\) and finally followed by \(D\) in the event stream, but all with the same associated time \(t\). We parse \(C\) and move to a state \((\{C\}, \{B\})\). On parsing \(B\), we move to a state \((\{B, C\}, \{D\})\). Now, next on processing \(D\) if we accept it, we move to \((\{B, C, D\}, \phi)\). But \(<(C, t), (B, t), (D, t)>\) is not a valid occurrence as \(D\)'s occurrence time must be strictly greater than that of \(C\) and \(B\). Hence even though we add \((\alpha, [1 1 0], [0 0 1], 3)\) to \(\text{waits}(D)\) after seeing \((B, t)\), this potential transition cannot be active at time \(t\). The important thing to note that is this element was freshly added to \(\text{waits}(\cdot)\) after we started processing \(S\). Hence, such potential transition information after adding to \(\text{waits}(\cdot)\) must be initially inactive, till, all event-types at the current time are parsed. Such \(\text{waits}(\cdot)\) elements must be made active just before parsing event-types of the next time-instant. After performing the state transitions pertaining to all event types at the current time instant, the rest of the steps are essentially the same as in Algorithm 1. First, we perform the multiple automata check (there can be more than two automata in the same state now) and removal of all older automata if necessary. We follow this by the frequency incrementing step. Since we increment frequency only after parsing all event-types at a given time, we need to store the automata that reach the final state too during the state transition step. Finally, using the \(\text{bag}\) list, we add automata initialised in the start state, before processing the event-types occurring at the next time tick.
Algorithm 1: CountFrequencyExpiryTime($C_l, D, \gamma, E, T_X$)

Input: Set $C_l$ of candidate episodes, event stream $D = \{(E_1, t_1), \ldots, (E_n, t_n)\}$, frequency threshold $\gamma$, set $E$ of event types (alphabet), Expiry Time, $T_X$

Output: Set $F$ of frequent episodes out of $C_l$

1. $F \leftarrow \emptyset$ and $bag \leftarrow \emptyset$;
2. foreach event type $E \in E$ do $waits[E] \leftarrow \emptyset$;
3. /* Initialization of the waits() lists */
4. foreach $\alpha \in C_l$ do
5.   $\alpha.freq \leftarrow 0$ and $\alpha.start \leftarrow \emptyset$;
6.   for $i \leftarrow 1$ to $|\alpha|$ do
7.     $j \leftarrow 1$;
8.     while ($j \leq |\alpha|$ and $\alpha.e[j][i] = 0$) do $j \leftarrow j + 1$;
9.     if ($j = |\alpha|$ + 1) then $\alpha.start[i] \leftarrow 1$;
10.    for $i \leftarrow 1$ to $|\alpha|$ do
11.       if $\alpha.start[i] = 1$ then
12.          Add ($\alpha, 0, \alpha.start[i]$) to $waits[\alpha.g[i]]$;
13.       $\emptyset$ is a vector of all zeros */
14. /* Database pass */
15. for $k \leftarrow 1$ to $n$ do
16.    foreach $(\alpha, q_{cur}, w_{cur}, i) \in waits[E_k]$ do
17.      /* $E_k$ - currently processed event-type in the event stream */
18.      /* Transit the current automaton to the next state */
19.      $q_{next} \leftarrow q_{cur}$ and $q_{next}[i] \leftarrow 1$;
20.      if $q_{cur} = 0$ then
21.        Add $(q_{next}, t_k)$ to $\alpha.init$ and Add $\alpha$ to $bag$;
22.      /* $t_k$ - time associated with the current event in event stream */
23.      else
24.        if $q_{next} \neq 1$ then
25.          /* $t_k$ is a vector of all ones */
26.          Update $(q_{cur}, t_{cur})$ in $\alpha.init$ to $(q_{next}, t_{cur})$;
27.          /* $t_{cur}$ would be the first state transition time of the current automaton */
28.          if $(q_{next} \neq 1)$ then
29.            $w_{next} \leftarrow w_{cur}, w_{next}[i] \leftarrow 0$ and $w_{temp} \leftarrow w_{next}$;
30.            for $i \leftarrow 1$ to $|\alpha|$ do
31.              if $\alpha.e[i][i] = 1$ then
32.                $flag \leftarrow \text{TRUE}$;
33.                for ($k' \leftarrow 1$; $k' \leq |\alpha|$ and $flag = \text{TRUE}$; $k' \leftarrow k' + 1$) do
34.                  if $\alpha.e[k'][i] = 1$ and $q_{next}[k'] = 0$ then
35.                    $flag \leftarrow \text{FALSE}$;
36.                if $flag = \text{TRUE}$ then $w_{next}[i] \leftarrow 1$;
37.              for $i \leftarrow 1$ to $|\alpha|$ do
38.                if $w_{temp}[i] = 1$ then
39.                  Replace $(\alpha, q_{cur}, w_{cur}, i)$ from $waits[\alpha.g[i]]$ to $(\alpha, q_{next}, w_{next}, i)$;
40.              if ($w_{temp}[i] = 0$ and $w_{next}[i] = 1$) then
41.                Add $(\alpha, q_{next}, w_{next}, i)$ to $waits[\alpha.g[i]]$;
42.              Remove $(\alpha, q_{cur}, w_{cur}, i)$ from $waits[\alpha.g[i]]$;
43.              /* Removing an older automaton if any in the next state */
44.              if $(q_{next}, t'_k) \in \alpha.init$ and $t'_k < t_{cur}$ then
45.                /* $t'_k$ is the first state transition time of an older automaton existing in state $q_{next}$ */
46.                Remove $(q_{next}, t'_k)$ from $\alpha.init$;
47.              for $i \leftarrow 1$ to $|\alpha|$ do
48.                if $w_{next}[i] = 1$ then
49.                  Remove $(\alpha, q_{next}, w_{next}, i)$ from $waits[\alpha.g[i]]$;
50.                /* Increment the frequency */
51.              if $(q_{next} = 2$ and $|E_k - t_{cur}| \leq T_X$) then
52.                $\alpha.freq \leftarrow \alpha.freq + 1$ and Empty $\alpha.init$ list;
53.              for $i \leftarrow 1$ to $|\alpha|$ do
54.                foreach $(\alpha, q, w, i) \in waits[\alpha.g[i]]$ do
55.                  Remove $(\alpha, q, w, i)$ from $waits[\alpha.g[i]]$ and Add $\alpha$ to $bag$;
56.                  /* Add automata initialized in the start state */
57.                  foreach $\alpha \in bag$ do
58.                    for $i \leftarrow 1$ to $|\alpha|$ do
59.                      if $\alpha.wstart[i] = 1$ then
60.                        Add $(\alpha, 0, \alpha.wstart[i])$ to $waits[\alpha.g[i]]$;
61.                      Empty $bag$;
62.                    foreach $\alpha \in C_l$ do if $\alpha.freq > \gamma$ then Add $\alpha$ to $F_l$;
63.                  return $F_l$;
A. Space and time complexity of Algorithm $A$

The number of automata that may be active (at any time) for each episode is central to the space and time complexities of the Algorithm $A$. The number of automata currently active for a given $l$-node episode, $\alpha$, is one more than the number of elements in the $\alpha$.init list. We now show that there can be at most $l$ entries in the $\alpha$.init list of Algorithm $A$. Recall that $(q_{ij}, t_{ij}) \in \alpha$.init means that there is an automaton of episode $\alpha$ that is currently active which made its transition out of the start state at time $t_{ij}$ and is currently in state $q_{ij}$. Suppose there are $m$ entries in $\alpha$.init list, namely, $(q_1, t_{i1}), \ldots, (q_m, t_{im})$, with $t_{i1} < t_{i2} < \cdots < t_{im}$. Let $\{Q_1^\alpha, \ldots, Q_m^\alpha\}$ represent the corresponding sets of accepted event-types for these active automata. Consider $k, l$ such that $1 \leq k < l \leq m$. The events in the data stream that affected transitions in the $l$th automaton (i.e. automaton which moved out of start state at $t_{il}$) would have also been seen by the $k$th automaton. If the $k$th automaton has not already accepted previous events with the same event-types, it will do so now on seeing the events which affect the transitions of the $l$th automaton. Hence, $Q_l^\alpha \subset Q_k^\alpha$ for any $1 \leq k < l \leq m$. Since $Q^\alpha \subseteq X^\alpha$ and $|X^\alpha| = l$, there are at most $l$ (distinct) telescoping subsets of $X^\alpha$, and so, we must have $m \leq l$.

The time required for initialization in Algorithm $A$ is $O(|E| + |C_i|^2)$. This is because, there are $|E|$ waits() lists to initialize and it takes $O(l^2)$ time to find the least elements for each of the $|C_i|$ episodes. For each of the $n$ events in the stream, the corresponding waits() list contains no more than $l|C_i|$ elements as there can exist at most $l$-automata per episode. The updates corresponding to each of these entries takes $O(l^2)$ time to find the new elements to be added to the waits() lists. Thus, the time complexity of the data pass is $O(nl^3|C_i|)$.

For each automaton, we store its state information in the binary $l$-vectors $q$ and $w$. To be able to make $|W|$ transitions from a given state, we maintain $|W|$ elements in various waits() lists with each element ready to accept one of the event-types in $W$. Hence, for each automata we require $O(l^2)$ space to store the state and its possible transitions. Since there are $l$ such automata in the worst case, the space complexity is $O(l^3|C|)$.

V. CANDIDATE GENERATION

Recall that the episode discovery process employs a level-wise procedure where, each level involves the two steps of candidate generation and frequency counting. In Sec. IV we described the frequency counting algorithms. In this section, we describe the candidate generation algorithm.
for injective episodes with general partial orders. The input to the candidate generation algorithm, at level \((l+1)\), is the set, \(\mathcal{F}_l\), of frequent episodes of size \(l\). Under the frequency measure (based on non-overlapped occurrences), we know that no episode can be more frequent than any of its subepisodes. The candidate generation step exploits this property, to construct the set, \(\mathcal{C}_{l+1}\), of \((l+1)\)-node candidate episodes.

Recall (cf. Sec. II-A) that it is simpler to view an injective episode \(\alpha = (V_\alpha, <_\alpha, g_\alpha)\), in terms of its associated partial order set, \((X^\alpha, R^\alpha)\). Each episode in \(\mathcal{F}_l\) is represented by an \(l\)-element array of event-types, \(\alpha.g\), and an \(l \times l\) matrix, \(\alpha.e\), containing the adjacency matrix of the partial order. The array \(\alpha.g\) exactly contains the elements of \(X^\alpha\) sorted as per the lexicographic ordering on the alphabet \(\mathcal{E}\). We refer to \(\alpha.g[i] = g_\alpha(v_i)\) as the \(i^{th}\) node of \(\alpha\). Note that the \(i^{th}\) node of an episode has no relationship whatsoever with the associated partial order \(R^\alpha\).

The principal task here is to generate all possible \((l+1)\)-node candidates such that, each of their \(l\)-node subepisodes are frequent. Each \((l+1)\)-node candidate is generated by combining two suitable \(l\)-node frequent episodes (out of \(\mathcal{F}_l\)). We first explain which pairs of episodes in \(\mathcal{F}_l\) can be combined and then explain how to combine them to get \((l+1)\)-node episodes. Every pair of \(l\)-node frequent episodes, \(\alpha, \beta \in \mathcal{F}_l\), such that exactly the same \((l-1)\)-node subepisode is obtained when their respective last nodes are dropped, can be combined to obtain one or more potential \((l+1)\)-node candidates. Thus, the episodes \((C \rightarrow A \rightarrow B)\) and \((A \rightarrow D \rightarrow B)\) would be combined since the same subepisode, namely \((A \rightarrow B)\) is obtained by dropping the last nodes of \((C \rightarrow A \rightarrow B)\) and \((A \rightarrow D \rightarrow B)\), which are \(C\) and \(D\) respectively. Episodes \((C \rightarrow A \rightarrow B)\) and \(((AB) \rightarrow D)\) would not be combined (since different subepisodes, namely \((A \rightarrow B)\) and \((AB)\), are obtained on dropping the last nodes of \((C \rightarrow A \rightarrow B))\) and \(((AB) \rightarrow D))\). For every such constructed candidate episode \(\gamma\), if all its \(l\)-node subepisodes are frequent, (i.e. if they can all be found in \(\mathcal{F}_l\)) \(\gamma\) is declared a candidate episode and is added to the output set, \(\mathcal{C}_{l+1}\). We can formalize this notion of which pairs of episodes can be combined, as given below.

For an injective episode \(\alpha\), let \(X^\alpha = \{x_1^\alpha, \ldots, x_l^\alpha\}\) denote the \(l\) distinct event-types in \(\alpha\), indexed in lexicographic order. We combine two episodes \(\alpha_1\) and \(\alpha_2\) such that the following two conditions hold: (i) \(x_i^{\alpha_1} = x_i^{\alpha_2}, i = 1, \ldots, (l-1)\), \(x_i^{\alpha_1} \neq x_i^{\alpha_2}\) and (ii) \(R^{\alpha_1}|_{(X^{\alpha_1}\setminus\{x_i^{\alpha_1}\})} = R^{\alpha_2}|_{(X^{\alpha_2}\setminus\{x_i^{\alpha_2}\})}\) (i.e. the restriction of \(R^{\alpha_1}\) to the first \((l-1)\) nodes of \(\alpha_1\) is identical to the restriction of \(R^{\alpha_2}\) to the first \((l-1)\) nodes of \(\alpha_2\)). To ensure that the same pair of episodes are not picked up two times, we follow the convention that \(\alpha_1\) and \(\alpha_2\) are such that \(x_i^{\alpha_1} < x_i^{\alpha_2}\) under
the lexicographic ordering.

We first illustrate the process of constructing potential candidates through some examples. Each pair of episodes \( \alpha_1 \) and \( \alpha_2 \), sharing the same \((l-1)\)-node subepisode on dropping their respective last nodes can lead to a maximum of three potential candidates, denoted by \( \mathcal{Y}_0 \), \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \). Consider the \( \alpha_1 \) and \( \alpha_2 \) of Fig. 2. We construct \( \mathcal{Y}_0 \) as a simple union of \( \alpha_1 \) and \( \alpha_2 \), i.e. we set \( X^{\mathcal{Y}_0} = X^{\alpha_1} \cup X^{\alpha_2} \) and \( R^{\mathcal{Y}_0} = R^{\alpha_1} \cup R^{\alpha_2} \). As it turns out, in this example, \( R^{\mathcal{Y}_0} \) is a valid partial order over \( X^{\mathcal{Y}_0} \) (satisfying both anti-symmetry as well as transitive closure) and hence, \( \mathcal{Y}_0 \) is a valid injective episode (and a potential 5-node candidate). There is no edge in \( \mathcal{Y}_0 \) between the last two nodes (i.e. the nodes corresponding to event-types \( D \) and \( E \) respectively). By adding an edge from \( D \) to \( E \) we get another valid partial order with the relation \( R^{\mathcal{Y}_0} \cup \{(D, E)\} \), and this corresponds to a second injective candidate, \( \mathcal{Y}_1 \), that we can construct using the \( \alpha_1 \) and \( \alpha_2 \) of Fig. 2. Similarly, \( R^{\mathcal{Y}_0} \cup \{(E, D)\} \) corresponds to a valid partial order and this gives us a third potential candidate from the same \( \alpha_1 \) and \( \alpha_2 \). But not all pairs of episodes can be combined in this manner to construct three different potential candidates. For example, for the \( \alpha_1 \) and \( \alpha_2 \) of Fig. 3 \( \mathcal{Y}_1 \) is the only potential candidate. While \((X^{\mathcal{Y}_1}, R^{\mathcal{Y}_1})\) obeys transitive closure, \((X^{\mathcal{Y}_0}, R^{\mathcal{Y}_0})\) is not transitively closed because \((D, C)\) and \((C, E)\) belong to \( R^{\mathcal{Y}_0} \), but \((D, E)\) does not. For the same reason \((X^{\mathcal{Y}_2}, R^{\mathcal{Y}_2})\) is not transitively closed either. In the example of Fig. 4 \( \mathcal{Y}_0 \) and \( \mathcal{Y}_1 \) are potential candidates (but \( \mathcal{Y}_2 \) is not a valid potential candidate because \((B, E)\) and \((E, D)\) are in \( R^{\mathcal{Y}_2} \), while \((B, D)\) is not).

Thus, the general strategy for combining an episode \( \alpha_1 \) with a valid \( \alpha_2 \), satisfying the two conditions mentioned before, is as follows. We attempt to construct an \((l+1)\)-node candidate from \( \alpha_1 \) and \( \alpha_2 \), by appending the last node of \( \alpha_2 \) to the last node of \( \alpha_1 \). There are three possibilities to consider for combining \( \alpha_1 \) and \( \alpha_2 \):

\[
X^{\mathcal{Y}_0} = X^{\alpha_1} \cup X^{\alpha_2}, \quad R^{\mathcal{Y}_0} = R^{\alpha_1} \cup R^{\alpha_2}
\]

(7)

\[
X^{\mathcal{Y}_1} = X^{\alpha_1} \cup X^{\alpha_2}, \quad R^{\mathcal{Y}_1} = R^{\mathcal{Y}_0} \cup \{(x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2})\}
\]

(8)

\[
X^{\mathcal{Y}_2} = X^{\alpha_1} \cup X^{\alpha_2}, \quad R^{\mathcal{Y}_2} = R^{\mathcal{Y}_0} \cup \{(x_{i_1}^{\alpha_2}, x_{i_1}^{\alpha_1})\}
\]

(9)

In each case, if \( R^{\mathcal{Y}_j} \) is a valid partial order over \( X^{\mathcal{Y}_j} \), then the \((l+1)\)-node (injective) episode, \((X^{\mathcal{Y}_j}, R^{\mathcal{Y}_j})\) is considered as a potential candidate. To verify the same, we need to check for the antisymmetry and transitive closure of the above three possibilities. One can show that each \( R^{\mathcal{Y}_j} \) satisfies antisymmetry because \( \alpha_1 \) and \( \alpha_2 \) share the same \((l-1)\) subepisode on dropping their
last nodes. To check for transitive closure of \((X^j, R^j)\) we would need to ensure that for every triple \(z_1, z_2, z_3 \in X^j\), if \((z_1, z_2) \in R^j\) and \((z_2, z_3) \in R^j\), then we must have \((z_1, z_3) \in R^j\). However, since \((R^{\alpha_1} \cup R^{\alpha_2}) \subseteq R^j\) and since \((X^{\alpha_1}, R^{\alpha_1})\) and \((X^{\alpha_2}, R^{\alpha_2})\) are already known to be transitively closed, we need to perform the transitivity closure check only for all size-3 subsets of \(X^j\) that are of the form \(\{x_1^{\alpha_1}, x_1^{\alpha_2}, x_i^{\alpha_1} \mid 1 \leq i \leq (l-1)\}\). Hence, the transitivity closure check is \(O(l)\). Finally, if all the \(l\)-node subepisodes of \(\mathcal{Y}_j\) can be found in \(\mathcal{F}_l\) then \(\mathcal{Y}_j\) is added to the final candidate list, \(C_{l+1}\), that is output by the algorithm.

Interestingly, one need not check whether all the \(l\)-node subepisodes of a potential \((l+1)\)-node candidate \(\mathcal{Y}_j\) are in \(\mathcal{F}_l\). The number of such sub-episodes can in general be very large. It is enough to check whether all the \(l\)-node subepisodes obtained by restricting \(R^j\) to an \(l\)-node subset of \(X^j\) are present in \(\mathcal{F}_l\). For example, consider a 3-node episode \((X^\alpha = \{A, B, C\}, R^\alpha = \{(A, B), (A, C)\})\). Its 2-node sub-episodes are the serial episodes \(A \rightarrow B\) and \(A \rightarrow C\), and
Fig. 3. Edges \((D, C)\) and \((C, E)\) prevent \(Y_0\) and \(Y_2\) from coming up as candidates.

Fig. 4. All nodes \(A, B\) and \(C\) prevent \(Y_2\) from coming up.

Parallel episodes \((A B), (A C)\) and \((B C)\). So in general, an \((l + 1)\)-node injective episode has more than \((l + 1)\) \(l\)-node subepisodes. Let us consider those \(l\)-node sub-episodes of \((X^\alpha, R^\alpha)\) which are obtained by restricting \(R^\alpha\) to a \(l\)-subset of \(X^\alpha\). We can have \((l + 1)\) such subepisodes.

In this example, \(A \rightarrow B\), \(A \rightarrow C\) and \((B C)\) are the three 2-node subepisodes of \(\alpha\) obtained by restricting \(R^\alpha\) to all the possible 2-element subsets of \(X^\alpha\). Note that the remaining 2-node subepisodes of \(A \rightarrow (B C)\), namely \((A B)\) and \((A C)\), are also subepisodes of one or the other of these three 2-node subepisodes. For any \(N\)-node episode \(\alpha\), let us denote by \(M^\alpha_k\), the set of all \(k\)-node subepisodes \((k < N)\), obtained by restriction of \(R^\alpha\) to \(k\)-subsets of \(X^\alpha\). We note the following. For every \(k\)-node subepisode \(\gamma\) of \(\alpha\), there exists a \(\beta \in M^\alpha_k\) such that \(\gamma\) is a subepisode of \(\beta\). Also for every \(\beta \in M^\alpha_k\) there exists no other \(\delta \in M^\alpha_k\) such that \(\beta\) is a
Algorithm 2: GenerateCandidates(F_l)

Input: Sorted array, F_l, of frequent episodes of size l
Output: Sorted array, C_{l+1}, of candidates of size (l + 1)

1 Initialize C_{l+1} ← φ and k ← 0;
2 if l = 1 then
3 for h ← 1 to |F_l| do F_l[h].blockstart ← 1;
4 for i ← 1 to |F_l| do
5 currentblockstart ← k + 1;
6 for (j ← i + 1; F_l[j].blockstart = F_l[i].blockstart; j ← j + 1) do
7 if F_l[i].g[l] ≠ F_l[j].g[l] then
8 P ← GetPotentialCandidates(F_l[i], F_l[j]);
9 foreach α ∈ P do
10 f lg ← TRUE;
11 for (r ← 1; r < l and f lg = TRUE; r ← r + 1) do
12 for x ← 1 to r − 1 do
13 Set β.g[x] = α.g[x];
14 for z ← 1 to r − 1 do β.e[x][z] ← α.e[x][z];
15 for z ← r to l do β.e[x][z] ← α.e[x][z + 1];
16 for x ← r to l do
17 β.g[x] ← α.g[x + 1];
18 for z ← 1 to r − 1 do β.e[x][z] ← α.e[x + 1][z];
19 for z ← r to l do β.e[x][z] ← α.e[x + 1][z + 1];
20 if β /∈ F_l then f lg ← FALSE;
21 if f lg = TRUE then
22 k ← k + 1;
23 Add α to C_{l+1};
24 C_{l+1}[k].blockstart ← currentblockstart;
25 return C_{l+1}

subepisode of δ. Hence, M^α_k is maximal for the set of all k-node subepisodes of α. Therefore
in the rest of the paper, we refer to subepisodes obtained by dropping \[3\] one or more nodes as

\[3\]We refer to a subepisode (of an episode α) obtained by restricting R^α to a strict subset of X^α, as a subepisode obtained by dropping one or more nodes.
maximal subepisodes. Hence, if all the maximal \( l \)-node subepisodes of a potential \((l + 1)\)-node candidate are frequent, then all its \( l \)-node subepisodes must also be frequent, which means it is enough to check if all the \( l \)-node maximal subepisodes of a potential candidate are frequent.

For a given frequent episode \( \alpha_1 \), we now describe how one can efficiently search for all other combinable frequent episodes of the same size. At level 1 (i.e. \( l = 1 \)), we ensure that \( F_1 \) is ordered according to the lexicographic ordering on the set of event types \( E \). Let \( F[i] \) denote the \( i^{th} \) episode in the collection, \( F_l \), of \( l \)-node frequent episodes. Suppose \( F_1 \) consists of the frequent episodes \( A, C \) and \( E \), then \( F_1[1] = A, F_1[2] = C \) and \( F_1[3] = E \). All the three \( 1 \)-node episodes share the same sub-episode \( \phi \), on dropping their last event. As per the candidate generation algorithm, any two \( 1 \)-node episodes are combined to form a parallel episode and two serial episodes. Accordingly here, episode \( A \) is combined with \( C \) and \( E \) to form 6 candidates in \( C_2 \). Similarly, \( C \) is combined with \( E \) to add three more candidates to \( C_2 \). Note that the first 6 candidates share the same \( 1 \)-node subepisode \( A \) on dropping their last event. Also, the next three candidates share a similar \( 1 \)-node subepisode \( C \), on dropping their last event. The candidate generation procedure adopted at each level here, is such that the episodes which share the same subepisode on dropping their last events appear consecutively in the generated list of candidates, at each level. We refer to such a maximal set of episodes as a \textit{block}. In addition, we maintain the episodes in each block so that they are ordered lexicographically with respect to the array of event types. Since, the block information aids us to efficiently decide the kind of episodes to combine, at each level right from level one, we store the block information. At level 1, all nodes belong to a single block. For a given \( \alpha_1 \in F_l \), the set of all valid episodes \( (\alpha_2) \) (satisfying the conditions explained before) with which \( \alpha_1 \) can be combined, are all those episodes placed below \( \alpha_1 \) (except the ones which share the same set of event types with \( \alpha_1 \) ) in the same block. All candidate episodes obtained by combining a given \( \alpha_1 \) with all permissible episodes \( (\alpha_2) \) below it in the same block of \( F_l \), will give rise to a block of episodes in \( C_{l+1} \), each of them having \( \alpha_1 \) as their common \( l \)-node sub-episode on dropping their last nodes. Hence, the block information of \( C_{l+1} \) can be naturally obtained during its construction itself. Even though the episodes within each block are sorted in lexicographic order of their respective arrays of event-types, we point out that the full \( F_l \) doesn’t obey the lexicographic ordering based on the arrays of event-types. For example, the episodes \( ((AB) \rightarrow C) \) and \( (A \rightarrow (BC)) \) both have the same array of event-types, but would appear in different blocks (with, for example, an episode
like $((AB) \rightarrow D)$ appearing in the same block as $((AB) \rightarrow C)$, while $(A \rightarrow (BC))$, since it belongs to a different block, may appear later in $\mathcal{F}_i$.

The pseudocode for the candidate generation procedure, GenerateCandidates(), is listed in Algorithm 2. The input to Algorithm 2 is a collection, $\mathcal{F}_i$, of $l$-node frequent episodes (where, $\mathcal{F}_i[i]$ is used to denote the $i^{th}$ episode in the collection). The episodes in $\mathcal{F}_i$ are organized in blocks, and episodes within each block appear in lexicographic order with respect to the array of event types. We use an array $\mathcal{F}_i.blockstart$ to store the block information of every episode, $\mathcal{F}_i.blockstart[i]$ will hold a value $k$ such that $\mathcal{F}_i[k]$ is the first element of the block to which $\mathcal{F}_i[i]$ belongs to. The output of the algorithm is the collection, $\mathcal{C}_{l+1}$, of candidate episodes of size $(l+1)$. Initially, $\mathcal{C}_{l+1}$ is empty and, if $l = 1$, all (1-node) episodes are assigned to the same block (lines 1-3, Algorithm 2). The main loop is over the episodes in $\mathcal{F}_i$ (starting on line 4, Algorithm 2).

The algorithm tries to combine each episode, $\mathcal{F}_i[i]$, with episodes in the same block as $\mathcal{F}_i[i]$ that come after it (line 6, Algorithm 2). In the notation used earlier to describe the procedure, we can think of $\mathcal{F}_i[i]$ as $\alpha_1$ and $\mathcal{F}_i[j]$ as $\alpha_2$. If $\mathcal{F}_i[i]$ and $\mathcal{F}_i[j]$ have identical event-types, we do not combine them (line 7, Algorithm 2). The GetPotentialCandidates() function, takes $\mathcal{F}_i[i]$ and $\mathcal{F}_i[j]$ as input and returns the set, $\mathcal{P}$, of potential candidates corresponding to them (line 8, Algorithm 2). This function first generates the three potential candidates by combining $\mathcal{F}_i[i]$ and $\mathcal{F}_i[j]$ as described in equations (7),(8) and (9). For each of the three possibilities, it then does a transitive closure check to ascertain their validity as partial orders\(^5\). For each potential candidate, $\alpha \in \mathcal{P}$, we construct its $l$-node (maximal)subepisodes (denoted as $\beta$ in the pseudocode) by dropping one node at-a-time from $\alpha$ (lines 13-19, Algorithm 2). Note that there is no need to check the case of dropping the last and last-but-one nodes of $\alpha$, since they would result in the subepisodes $\mathcal{F}_i[i]$ and $\mathcal{F}_i[j]$, which are already known to be frequent. If all $l$-node maximal subepisodes of $\alpha$ were found to be frequent, then $\alpha$ is added to $\mathcal{C}_{l+1}$, and its block information suitably updated (lines 20-24, Algorithm 2).

\(^4\)a similar array for storing block information is used for parallel and serial episode candidate generation in [12]

\(^5\)As explained before, one only needs to do a transitivity check on size-3 subsets of the form \(\{x_1^a, x_2^a, x_3^a : 1 \leq i \leq (l-1)\}\) separately on the three possibilities. Actually we can save time in the transitivity check further. As explained in appendix, we need to generate only the $\mathcal{Y}_0$ combination and perform some special checks on its nodes to decide the valid partial orders to be generated in $\mathcal{P}$. 
A. Correctness of Candidate Generation

In this section, we address two important questions regarding the candidate generation. The first question is whether a given partial order is generated more than once in the algorithm. The second question is about whether every frequent episode is generated by our candidate generation scheme.

We now address the first question in detail. It is easy to see from equation (7) to (9) that two partial orders generated from a given pair \((α_1, α_2)\) of \(l\)-node episodes are all different. Hence we need to consider whether the same candidate is generated from two different pairs of episodes.

Suppose an exactly same candidate is generated from different pairs \((α_1, α_2)\) and \((α'_1, α'_2)\). Call them \(Y_r\) and \(Y_s\) where \(r\) and \(s\) vary from 0 to 2 depending on the type of combination of the episode pairs. First consider the case when both these candidates come up as \(Y_0\) and \(Y'_0\). Note that \(Y_0 = (X^{Y_0}, R^{Y_0}) = (X^{α_1} ∪ X^{α_2}, R^{α_1} ∪ R^{α_2})\) and \(Y'_0 = (X^{Y'_0}, R^{Y'_0}) = (X^{α'_1} ∪ X^{α'_2}, R^{α'_1} ∪ R^{α'_2})\). Since the candidates are same, \(Y_0 = Y'_0\). This implies (i)\(X^{Y_0} = X^{Y'_0}\) and (ii)\(R^{Y_0} = R^{Y'_0}\). (i) implies \(X^{α_1} ∪ X^{α_2} = X^{α'_1} ∪ X^{α'_2}\). Recall from the conditions for forming candidates that \(X^{α_1} ∪ X^{α_2} = X^{α_1} \cup \{x_i^{α_2}\} = \{x_1^{α_1}, \ldots, x_i^{α_1}, x_i^{α_2}\}\). Recall that \(x_i^{α_1}\) is the \(i^{th}\) element of \(X^{α_1}\), \(x_i^{α_2}\) is the \(i^{th}\) element of \(X^{α_2}\) and \(x_i^{α_1} \neq x_i^{α_2}\), all as per the lexicographical ordering on \(E\). Hence, \(x_i^{α_1}\) is the \(i^{th}\) element of \(X^{Y_0}\) for \(i = 1, \ldots, l\) and \(x_i^{α_2}\) is its \((l + 1)^{th}\) element. An analogous thing holds for \(X^{Y'_0}\). Since \(X^{Y_0}\) and \(X^{Y'_0}\) are same, their \(i^{th}\) elements must also match. This means \(x_i^{α_1} = x_i^{α'_1}\) for \(i = 1, \ldots, l\) and \(x_i^{α_2} = x_i^{α'_2}\). This immediately implies \(X^{α_1} = X^{α'_1}\). Also from the conditions of generating candidates we have \(x_i^{α_1} = x_i^{α_2}\) and \(x_i^{α'_1} = x_i^{α'_2}\) for \(i = 1, \ldots, (l - 1)\). This together with \(x_i^{α_1} = x_i^{α'_1}\) for \(i = 1, \ldots, l\) implies \(x_i^{α_2} = x_i^{α'_2}\) for \(i = 1, \ldots, (l - 1)\). Finally combining this with \(x_i^{α_2} = x_i^{α'_2}\), we have \(X^{α_2} = X^{α'_2}\). Thus \(X^{Y_0} = X^{Y'_0} \implies X^{α_1} = X^{α'_1}\) and \(X^{α_2} = X^{α'_2}\). Since the pairs \((α_1, α_2)\) and \((α'_1, α'_2)\) are to be distinct, we need to have either \((R^{α_2} \neq R^{α'_2})\) or \((R^{α_1} \neq R^{α'_1})\). We now show that this cannot be the case with \(R^{Y_0} = R^{Y'_0}\). Suppose either \((R^{α_2} \neq R^{α'_2})\) or \((R^{α_1} \neq R^{α'_1})\). Without loss of generality assume, \((R^{α_1} \neq R^{α'_1})\). This means (since \(X^{α_1} = X^{α_2}\) there is an edge \((x, y)\) in \(R^{α_1}\) that is absent in \(R^{α'_1}\) (or the other way round). Again without loss of generality assume there is an edge \((x, y)\) in \(R^{α_1}\) that is absent in \(R^{α'_1}\). Thus, if \(R^{Y_0} = R^{Y'_0}\), we must have the edge \((x, y)\) in \(R^{α_1}\) that is absent in \(R^{α'_1}\).
\( \alpha' \) is identical to the restriction of \( R^{\alpha_2} \) to the first \((l - 1)\) nodes of \( \alpha_2' \). \((x, y)\) cannot belong to \( E^2 \) as neither \( x \) nor \( y \) can be \( x^{\alpha_2'} \) (because \( x^{\alpha_2'} \) does not belong to \( X^{\alpha_1} \) which is same as \( X^{\alpha_1} \), which contains both \( x \) and \( y \)). Therefore the edge \((x, y) \in R^{\alpha_1} \) and hence in \( R^{Y_0} \), cannot appear in \( R^{Y'_0} \). This contradicts (ii) and hence \( R^{\alpha_1} = R^{\alpha_2} \) and \( R^{\alpha'_1} = R^{\alpha'_2} \). This means that the pairs \((\alpha_1, \alpha_2)\) and \((\alpha'_1, \alpha'_2)\) that we started off with cannot be distinct.

On arguments similar to the \( r = s = 0 \) case, we can show that no \( Y_r \) can be equal to any \( Y'_s \). Hence we have shown that every candidate partial order is uniquely generated. We next show that every frequent episode would be in the set of candidates output by Algorithm 2.

We show this by induction on the size of the episode. At level one, the set of candidates contain all the one node episodes and hence contains all the frequent one node episodes. Now suppose at level \( l \), all frequent episodes of size \( l \) are indeed generated. If an \((l + 1)\)-node episode \( \alpha = (X, R) \) is frequent, then all its subepisodes are frequent. The maximal \( l \)-node subepisodes \((X \backslash \{x_{l+1}\}, R|_{X \backslash \{x_{l+1}\}})\) and \((X \backslash \{x_l\}, R|_{X \backslash \{x_l\}})\) in particular, are also frequent and hence generated at level \( l \) (as per the induction hypothesis). Note that the \((l-1)\)-node subepisodes obtained by dropping the last event-types of these two episodes are the same. Hence, the candidate generation method combines these 2 frequent subepisodes in atmost 3 ways. Since \((X, R)\) is either a \( Y_0 \), \( Y_1 \) or \( Y_2 \) combination of these 2 episodes and also a valid partial order, Algorithm 2 generates it after the first step of candidate generation. The second step checks whether all its remaining maximal \( l \)-node subepisodes are also frequent. This condition is true as per the induction hypothesis and \( \alpha \) is therefore generated in the list of candidates at level \( l + 1 \). Thus we can see that our candidate generation algorithm outputs all valid candidates without any repetition.

### B. Candidate Generation with structural constraints

The candidate generation scheme described above is very flexible. In particular, we can easily specialize it so that we generate only parallel episodes or only serial episodes. For example, suppose that for every pair of combinable episodes we generate only the \( Y_0 \) combination (and do not consider the \( Y_1 \) and \( Y_2 \) combinations). Since for all level one episodes, \( X^\alpha \) is a singleton and \( R^\alpha \) is empty, if we do our \( Y_0 \) combination, then \( R^\alpha \) will be empty for all level-2 candidates. Now, since we use our \( Y_0 \) combination throughout, it is easy to see that \( R^\alpha \) would remain empty at all levels and then we will be generating only parallel episodes. Similarly, it is easy to see
that if we do only $Y_1$ and $Y_2$ combinations (an do not consider the $Y_0$ combination) at all levels then we would generate only serial episodes. Thus the method we presented for mining general partial orders is easily specialized to a method for parallel episodes or serial episodes only. In addition, we can also specialize it to mine for certain classes of partial orders as explained below.

Note that any class of partial orders where, for every partial order belonging to the class, all its maximal subepisodes also lie in the same class, our candidate generation algorithm is easily specialized to such classes of partial orders. We refer to such a class as satisfying a maximal subepisode property. For example, both the class of serial episodes and parallel episodes satisfy this property. To mine in a specific class of partial orders, one just needs to do an additional check and retain only those of the potential candidates generated, which belong to the class of interest. For mining either in the space of serial or parallel orders, one need not perform this explicit check of whether the generated candidates belong to the concerned class. Instead, a more efficient way as described earlier can be adopted.

We discuss a few interesting classes of partial orders satisfying the maximal subepisode property. The first of them is the set of all partial orders, where length of the largest maximal path of each partial order (denoted as $L_{max}$) is bounded above by a user-defined threshold. Consider the episode $\alpha = (A \rightarrow ((F)(B \rightarrow (C D) \rightarrow E)))$. It has three maximal paths namely $A \rightarrow B \rightarrow C \rightarrow E$, $A \rightarrow B \rightarrow D \rightarrow E$ and $A \rightarrow F$ and the length of its largest maximal path is 3. For $L_{max} = 0$, we get the set of all parallel episodes because any $N$-node parallel episode has $N$-maximal paths each of length 0, and every non-parallel episode has at least one maximal path of length 1. In general, for $L_{max} \leq k$, the corresponding class of partial orders contains all parallel episodes, serial episodes of length less than $(k + 1)$ and many more partial orders all of whose maximal paths have length less than $(k + 1)$. It is easy to see that for any partial order belonging to such a class, all its subepisodes too belong to the same class. As $k$ is increased, the class of partial orders expands into the space of all partial orders from the parallel episode end. Another class of partial orders of interest could be one, where the number of maximal paths in each partial order (denoted as $N_{max}$) is bounded above by a threshold. When $N_{max} \leq 1$, the class obtained is exactly equal to the set of serial episodes. For any partial order belonging to this class, only its maximal subepisodes are guaranteed to belong to the same class. For example, consider a serial episode $A \rightarrow B \rightarrow C$. All its maximal sub-episodes are serial episodes. Its non-maximal subepisodes like $(AB)$ do not belong to the set of serial episodes.
As and when the candidates are generated, we calculate and check whether their $L_{max}$ or $N_{max}$ values satisfy the bound constraint. We use the standard dynamic programming based algorithms to calculate $L_{max}$ or $N_{max}$ on the transitive reduced graph of each generated candidate partial order. We could also work on a class of partial orders characterised by an upper bound on both $L_{max}$ and $N_{max}$, as such a class would also satisfy the maximal subepisode property. Mining with structural constraints can make the discovery process more efficient as compared to mining in the class of all injective partial orders. We illustrate with simulation how one can mine for partial orders with an upper bound constraint on $L_{max}$ or $N_{max}$.

VI. DISCUSSION

We wish to point out that the proposed counting and candidate generation algorithms for injective episodes can be extended to a class of non-injective episodes, where nodes mapped to the same event lie along a chain in the associated partial order. It is interesting to note that the class of all non-injective serial episodes is contained in this special class of non-injective partial order episodes. To keep the representation of such $l$-node episodes unambiguous, the $g$-map is restricted (very similar to injective episodes) such that $g(v_1), g(v_2), \ldots, g(v_l)$ obey the lexicographic order (total) on $E$. For example, suppose we have a 5-node episode with 3 of the nodes mapped to $A$ and the remaining 2 mapped to $B$. Then, $g(v_i)$ must be $A$ for $i = 1, 2, 3$ and $B$ for $i = 4, 5$. Further, since the episodes are such that the nodes mapped to the same event lie along a chain, we impose a special restriction on $<_\alpha$ to avoid further ambiguity. Suppose $v_i, v_{i+1}, \ldots, v_{i+m}$ are mapped to the same event-type $E$. There are $(m+1)!$ total orders possible among these nodes, each of which would represent the same episode pattern. To avoid this redundancy, we restrict $<_\alpha$ to be such that $v_i <_\alpha v_{i+1} <_\alpha \cdots v_{i+m}$.

Consider a non-injective episode $\alpha$ with $V_\alpha = \{v_1, v_2, v_3, v_4\}$, $<_\alpha = \{(v_1, v_3), (v_2, v_4)\}$. $g_\alpha()$ is such that $g_\alpha(v_1) = g_\alpha(v_2) = A$, $g_\alpha(v_3) = B$ and $g_\alpha(v_4) = C$. To track an occurrence of such an episode, we would initially wait for 2 $A$s. Once we see an $A$, we could either accept the $A$ associated with $v_1$ or $v_2$. Depending on what we choose, we would now either wait for $\{A, B\}$ (if accepted $A$ is associated with $v_1$) OR $\{A, C\}$ (if accepted $A$ is associated with $v_2$). As per our current counting strategy, on seeing $A$ there is more than one next state possible depending on the associated node in $V_\alpha$. Hence, a non-deterministic finite state automaton would be the right computational device to track occurrences of $\alpha$. In general a non-deterministic finite
state automaton (NFA) would be computationally more expensive compared to a deterministic automaton. Interestingly, $\alpha$ doesn’t belong to the class of non-injective episodes that we are considering, i.e. the nodes $v_1$ and $v_2$ are not related even though they map to the same event $A$. We are trying to indicate that to count episodes like $\alpha$, our strategy of counting leads to automata which are non-deterministic in nature. Even though an NFA can be converted to an equivalent DFA the number of states of this equivalent DFA can be huge. Hence, we have noted that counting is also not straight forward for episodes outside the class of non-injective episodes considered here in addition to problems with representation. We now argue how deterministic finite state automata (DFA) can still be used to track occurrences of this class of non-injective episodes, even though in general for non-injective episodes, one requires NFAs OR higher DFAs to track occurrences.

The DFA construction procedure for injective episodes can be generalized to this class of non-injective episodes. Each state would again be a tuple $(Q^\alpha, W^\alpha)$. $Q^\alpha$ here would be a multiset (essentially a set having repeated elements). Interestingly, one can verify that $W^\alpha$ is always a proper set for every state in this construction. Suppose not, then $W^\alpha$ would have at least 2 repeated elements. All parents of their corresponding nodes are contained in the set of nodes associated with that of $Q^\alpha$ (from the constructive definition). Note that the two set of nodes from $V_\alpha$ associated with $W^\alpha$ and $Q^\alpha$ are disjoint. This means the two nodes which map to repeated elements in $W^\alpha$ are unrelated in $<\alpha$. But as per the class of non-injective episodes we are dealing with, two such nodes mapped to the same element must be related, which is a contradiction. Hence, $W^\alpha$ is a proper set. Therefore, each transition from a given state would be on seeing a unique event type. This ensures that the finite state automaton so constructed is deterministic. Hence the counting algorithms proposed for injective episodes almost exactly go through for this class of injective episodes.

We now elaborate on the candidate generation. We combine episodes $\alpha = (\{v_1, v_2, \ldots v_l\}, <_\alpha, g_\alpha)$ and $\beta = (\{v_1, v_2, \ldots v_l\}, <_\beta, g_\beta)$ if (i) $g_\alpha(v_i) = g_\beta(v_i) \forall i = 1, \ldots (l-1)$, (ii) $<_\alpha \mid_{\{v_1, v_2, \ldots v_{l-1}\}}$, (iii) $g_\alpha(v_l) = g_\beta(v_l)$ OR $g_\alpha(v_l)$ precedes $g_\beta(v_l)$ as per the lexicographic ordering on $E$. Let $V_\gamma = \{v_1, \ldots, v_l, v_{l+1}\}$. $<_\gamma$ is a relation on $V_\gamma$ defined as follows. $v_i <_\gamma v_j$ iff $v_i <_\alpha v_j \forall i, j = 1, \ldots l$. Also, $\forall i = 1, 2 \ldots (l-1)$, we have $v_i <_\gamma v_{i+1}$ iff $v_i <_\beta v_{i+1}$ and $v_{i+1} <_\gamma v_i$ iff $v_i <_\beta v_i$. $g_\gamma$, a map from $V_\gamma$ to $E$ is such that $g_\gamma(v_i) = g_\alpha(v_i) \forall i = 1, \ldots l$ and $g_\gamma(v_{l+1}) = g_\beta(v_l)$. 
The way we combine $\alpha$ and $\beta$ slightly varies with the two subconditions of (iii). Suppose $g_\alpha(v_l)$ precedes $g_\beta(v_l)$ as per the lexicographic ordering on $E$. Consider the following relations $<_{\gamma_0}=<_{\gamma}$, $<_{\gamma_1}=<_{\gamma} \cup (v_l, v_{l+1})$, $<_{\gamma_2}=<_{\gamma} \cup (v_{l+1}, v_l)$. An episode $(V_\gamma, <_{\gamma_2}, g_\gamma)$ is generated iff $<_{\gamma_i}$ is a partial order. Note that this is exactly similar to what we have already been doing for injective episodes. The additional thing that needs to be done for this special class of non-injective episodes is as follows. Suppose $g_\alpha(v_l) = g_\beta(v_l)$, then we only ask if $<_{\gamma_1}$ is a partial order. This is because of the following reason. We have $g_\gamma(v_l) = g_\alpha(v_l) = g_\beta(v_l) = g_\gamma(v_{l+1})$. Hence, $g_\gamma$ maps $v_l$ and $v_{l+1}$ to the same event type. Recall that the unambiguous representation (for the special class of non-injective episodes) demands that $v_l <_{\gamma} v_{l+1}$. Hence the only permissible candidate would be $(V_\gamma, <_{\gamma_1}, g_\gamma)$. So we generate this as a candidate if and only if $<_{\gamma_1}$ is a partial order.

VII. SELECTION OF INTERESTING PARTIAL ORDER EPISODES

The frequent episode mining method would ultimately output all frequent episodes of upto some size. However, as we see in this section, frequency alone is not a sufficient indicator of interestingness in case of episodes with general partial orders.

Consider an $l$-node episode, $\alpha = (X^\alpha, R^\alpha)$. (That is $|X^\alpha| = l$). If $\alpha$ is frequent then all episodes $\alpha' = (X'^\alpha, R'^\alpha)$ with $X'^\alpha = X^\alpha$ and $R'^\alpha \subset R^\alpha$ would also be frequent $l$-node episodes because every occurrence of $\alpha$ would constitute an occurrence of $\alpha'$. The point to note is that when we consider episodes with general partial orders, an episode of size $l$ can have subepisodes which are also of size $l$. Such a situation does not arise if the mining process is restricted to either serial or parallel episodes only. For example there is no 4-node serial episode that is a subepisode of $A \rightarrow B \rightarrow C \rightarrow D$. However, when considering general partial orders, given a $\alpha = (X^\alpha, R^\alpha)$ there can be, in general, exponentially many episodes $\alpha' = (X'^\alpha, R'^\alpha)$ with $X'^\alpha = X^\alpha$ and $R'^\alpha \subset R^\alpha$. For example, $(A(B \rightarrow C \rightarrow D))$, $(B(A \rightarrow C \rightarrow D))$, $(C(A \rightarrow B \rightarrow D))$, $(D(A \rightarrow B \rightarrow C))$, $(AB)(C \rightarrow D)$, $(AB) \rightarrow C \rightarrow D$, $(ABC) \rightarrow D$, $A \rightarrow (BC) \rightarrow D$ etc. are all such subepisodes of $A \rightarrow B \rightarrow C \rightarrow D$. Thus, there is an inherent combinatorial explosion in frequent episodes of a given size when we are considering general partial orders and, hence, frequency alone may not be a sufficient indicator of ‘interestingness’. In this section, we propose a new measure, called bidirectional evidence of an episode which can be used in conjunction with frequency of an episode to make the mining process more efficient and meaningful.
A. Bidirectional evidence

A simple minded strategy to tackle the explosion of frequent episodes could be to use a notion similar to that of maximal frequent patterns that has been used in other datamining contexts such as item sets or sequential patterns.

Definition 7: An $\ell$-node episode $\alpha' = (X^{\alpha'}, R^{\alpha'})$ is said to be less specific than $\ell$-node episode $\alpha = (X^{\alpha}, R^{\alpha})$ if $X^{\alpha'} = X^{\alpha}$ and $R^{\alpha'} \subset R^{\alpha}$. Given a set of $\ell$-node episodes, an episode is a most specific episode if it is not less specific than any other episode in the set. (Note that, in general, there can be many most specific episodes in a given set of episodes).

Now, after the mining process (that is, after finding all frequent episodes of size $l$, for a given $l$), we can output only the most specific episodes of the set of frequent episodes. This prunes out many partial orders (episodes) which are presumed uninteresting because a more specific partial order (episode) is frequent and interesting. This specificity-based filter is not wholly satisfactory though it reduces the number of frequent episodes (of a given size) that are output. Suppose the data actually contains the partial order (episode) $(AB) \rightarrow C$. Suppose there are 200 occurrences of this episode of which 110 are occurrences of $A \rightarrow B \rightarrow C$ while 90 are those of $B \rightarrow A \rightarrow C$. Depending on the frequency threshold, suppose one or both of these serial episodes are also frequent. The parallel episode $(ABC)$, being less specific, would also be frequent (and would have a frequency greater than 200). The specificity based filter would always suppress the parallel episode $(ABC)$ and importantly also suppress the episode $(AB) \rightarrow C$ in preference to any of the serial episodes whenever they are frequent. Thus what we output depends very critically on the frequency threshold. In addition, if is also not satisfactory that whether or not we suppress $(AB) \rightarrow C$ depends only on the counts of these episodes. Instead we can ask is there any evidence in the data to decide which of these partial orders is a better fit. If the data indeed contains only the partial order $(AB) \rightarrow C$ then it would be the case that in most of the occurrences of the parallel episode $(ABC)$, $C$ follows both $A$ and $B$. We would also see that in occurrences of $(AB) \rightarrow C$, $A$ follows $B$ roughly as often as it precedes $B$. Now the fact that we have seen $A$ following $B$ roughly as often as $A$ preceeding $B$ and that we have rarely seen $C$ not following both $A$ and $B$ should mean that the partial order $(AB) \rightarrow C$ is a better representation of the dependencies in data as compared to the serial episode or the parallel episode. Thus, in addition to frequency, it would be nice to evaluate interestingness of
partial orders based on whether there is evidence in the data for not constraining the order of occurrence of some pairs of event types. That is, we can demand that in the occurrences of the episode (as counted by the algorithm) any two event types, \( i, j \in X^\alpha \), such that \( i \) and \( j \) are not related under \( R^\alpha \) should occur in either order ‘sufficiently often’. We will now formalize this notion.

Given an episode \( \alpha \) let \( G^\alpha = \{(i, j) : i, j \in X^\alpha, i \neq j, (i, j), (j, i) \notin R^\alpha \} \). Let \( f^\alpha \) denote the number of occurrences (i.e., frequency) of \( \alpha \) counted by our algorithm and let \( f^\alpha_{ij} \) denote the number of these occurrences where \( i \) precedes \( j \). Let \( p^\alpha_{ij} = f^\alpha_{ij}/f^\alpha \). To rate the interestingness of the partial order episode \( \alpha \) we define a measure that tries to capture the relative magnitudes of \( p^\alpha_{ij} \) and \( p^\alpha_{ji} \).

Let
\[
H^\alpha_{ij} = -p^\alpha_{ij} \log(p^\alpha_{ij}) - (1 - p^\alpha_{ij}) \log(1 - p^\alpha_{ij})
\]

Since, in each occurrence either \( i \) precedes \( j \) or \( j \) precedes \( i \), we have \( p^\alpha_{ij} = 1 - p^\alpha_{ji} \) and hence \( H^\alpha_{ij} \) is symmetric in \( i, j \). Note that \( H^\alpha_{ij} \) is the entropy of the distribution \([p^\alpha_{ij}, (1 - p^\alpha_{ij})]\). We refrain from using the term entropy for \( H^\alpha_{ij} \), as \( p^\alpha_{ij} = f^\alpha_{ij}/f^\alpha \) is tied to the specific subset of occurrences counted by our algorithm.

The bidirectional evidence of an episode \( \alpha \) denoted by \( H(\alpha) \) is defined as follows.

\[
H(\alpha) = \min_{(i, j) \in G^\alpha} H^\alpha_{ij}
\]

We use \( H(\alpha) \) as an additional interestingness measure for \( \alpha \). Essentially, if \( H(\alpha) \) is above some threshold, then there is sufficient evidence that all pairs of event types in \( \alpha \) that are not constrained by the partial order \( R^\alpha \) appear in either order sufficiently often. We say that an episode \( \alpha \) is interesting if (i) the frequency is above a threshold, and (ii) \( H(\alpha) \) is above a threshold.

We now explain how \( H(\alpha) \) can be computed during our frequency counting process. For each episode, we maintain an \( l \times l \) matrix \( \alpha.H \) whose \((i, j)^{th}\) element would contain \( f^\alpha_{ij} \) by the end of counting. For each candidate episode \( \alpha \), the matrix \( \alpha.H \) is initialized to 0 just before counting. For each automaton that is initialized, we initialize a separate \( l \times l \) matrix of zeros stored with the automaton. Whenever an automata makes state transitions on an event-type \( j \), for all \( i \) such that event-type \( i \) is already seen, we increment the \((i, j)\) entry in this matrix. The matrix associated with an automaton that reaches its final state, is added to \( \alpha.H \) and results in increment of relevant \( f^\alpha_{ij} \) entries. Thus, at the end of the counting, \( \alpha.H \) gives the \( f^\alpha_{ij} \) information.
B. Mining with an additional $H(\alpha)$ threshold

One can use $H_\alpha$ as a postprocessing filter. That is, after the mining process we only output those $\alpha$ (of a given size) where $H_\alpha$ is above a threshold. While this may reduce the number of frequent episodes output, it will not make the mining process efficient. A better way would be to use a threshold on $H(\alpha)$ at each size (or level) in our apriori style level-wise counting procedure. This can substantially contribute towards the efficiency of mining for general partial orders. However, unlike in the case of frequency threshold, it is not quite clear whether $H(\alpha)$ also possesses the so called anti-monotonicity property. The main difficulty is that $H(\alpha)$ is tied to a specific set of occurrences counted by the algorithm. However, if an episode $\alpha$ has a bidirectional evidence $H(\alpha) = e$, in a given set of occurrences, then one can see that any maximal subepisode of $\alpha$ (obtained by the restriction of $R^\alpha$ onto a subset of $X^\alpha$) also has a bidirectional evidence of at least $e$ in the same set of occurrences. Hence at least in cases where the embedded pattern’s subepisodes most often occur with the embedded pattern, the bidirectional evidence of all its maximal subepisodes will be at least that of the embedded pattern. Since our candidate generation is based on the existence of all maximal subepisodes at the lower levels, the embedded pattern $\beta$ most often comes up after mining, in the simulations. Further, the bidirectional evidence of all the non-maximal subepisodes of the embedded pattern will be very low (almost zero). This is because of the following. Any non-maximal subepisode $\gamma$ will not have some edge $(i, j)$ present in the embedded pattern, in spite of the nodes $i$ and $j$ being present in $\gamma$. If most occurrences of $\gamma$ are also those of $\beta$, $i$ precedes $j$ in almost all occurrences of $\gamma$ and hence $H(\gamma)$ is negligible. Hence almost all non-maximal subepisodes of $\beta$ will have negligible bidirectional evidence in spite of being frequent. Therefore, we weed out almost all the non-maximal sub-episodes of $\beta$ due to the $H(\beta)$ threshold being incorporated levelwise. These non-maximal sub-episodes if not weeded out, would otherwise contribute to the generation of many more patterns at various levels. In particular, it would result in the generation of all the less specific patterns of the embedded pattern as pointed in the beginning of this section, which doesn’t happen now. We show through simulation that mining with $H_\alpha$ threshold at each level is indeed very effective.
VIII. Simulation Results

A. Synthetic Data Generation

Synthetic data is generated by embedding occurrences of partial orders (episodes) in varying levels of noise. Input to the data generator is a set of episodes that we want to embed in the data. For each episode to be embedded, we generate an episode event stream just containing non-overlapped occurrences of the partial-order episode. We next generate a separate noise stream involving all event-types. We merge the various episode streams and the noise stream (that is, string together all events in all the streams in a time-ordered fashion) to generate the final data stream consisting of $T$ time ticks. The data generation process has three user-specified parameters: $\eta, p, \rho$, whose roles are explained below.

Each of the episode data streams are generated as follows. To embed each occurrence of an episode, we choose, at random, one of its serial extension and then generate the occurrence by having a sequence of event types as needed, with the difference in times of occurrence of successive events being geometric with parameter $\eta$. The time between successive occurrences of the episode is geometric with parameter $p$.

We generate the noise stream as follows. Let $\mathcal{E}_1$ denote the set of event types that appear in any of the embedded episodes. Any event-type not in $\mathcal{E}_1$ is referred to as a noise event-type. For each noise event-type, we generate a stream of just its occurrences, with time between successive events geometric with parameter $\rho$. Similarly, for each event-type in $\mathcal{E}_1$, we generate a stream of just its occurrences, with time between successive events geometric with parameter $\rho/5$. This is done to introduce some random occurrences of the event-types associated with the embedded partial orders. All these streams are merged to form a single noise stream. Noise stream is generated in this way so that there may be multiple events (constituting noise) at the same time instant. The noise data stream is merged with all the episode data streams to obtain the final data stream.

B. Effectiveness of Partial Order Mining

We first show that our algorithm is effective in unearthing the embedded partial orders in the data stream and also that our new measure of interestingness, namely, bidirectional evidence, is

\footnote{A serial extension of a partially ordered set $(X^\alpha, R^\alpha)$ is a totally ordered set $(X^\alpha, R')$ such that $R^\alpha \subseteq R'$.}
very useful in improving the efficiency of the mining process.

We generated a data stream of about 50,000 events (from a set of 60 event types) with 10,000 time ticks, in which are embedded the partial orders \( \alpha_1 = (A \rightarrow (B \ C) \rightarrow (D \ E) \rightarrow F) \) and \( \alpha_2 = (G \rightarrow ((H \rightarrow (J \ K))(I \rightarrow L))) \) both of which are 6-node episodes. Table II shows the results obtained with our mining algorithm. We show the number of candidates (\#Cand) and the number of frequent episodes (\#Freq) at different levels. (Recall that at level \( k \), the algorithm finds all frequent episodes of size \( k \)). The table shows the results for the cases: (i) when we only use a threshold on frequency (\( f_{th} \) only), (ii) when we use a threshold on frequency for mining but use a threshold on \( H(\alpha) \) as a post processing filter at each level (\( H_{th} \) for post processing) and (iii) when we use a threshold on frequency as well as on \( H(\alpha) \) at each level (\( f_{th} \) and \( H_{th} \)). (Other parameters such as noise levels, thresholds, expiry time etc. are given in the table caption).

The two embedded patterns are reported as frequent in all the three cases. However, with only a frequency threshold, a lot of uninteresting patterns (like the subepisodes of the embedded patterns) are also reported frequent. When we use a \( H(\alpha) \) threshold based post processing filter (case (ii)), the number of candidates naturally remains the same, but the frequent episodes output comes down drastically as can be seen from the table. However, the run-time actually increases marginally because of the overhead of calculating \( H(\alpha) \). When we use a threshold on both frequency as well as \( H(\alpha) \), the efficiency improves considerably as can be seen from the

### Table II

**Frequent Episode Output of the Algorithm with and without Bidirectional Evidence.** (Patterns: \( \alpha_1 \) and \( \alpha_2 \), \( \eta = 0.7 \), \( \rho = 0.055 \), \( p = 0.068 \), \( M = 60 \), \( T = 10000 \), \( f_{th} = 350 \), \( T_X = 15 \), \( H_{th} = 0.4 \))

| Level | \( f_{th} \) only | \( H_{th} \) for post-filtering | \( f_{th} \) and \( H_{th} \) |
|-------|-------------------|---------------------------------|-----------------------------|
|       | \#Cand | \#Freq | \#Cand | \#Freq | \#Cand | \#Freq |
| 1     | 60     | 60     | 60     | 60     | 60     | 60     |
| 2     | 5310   | 565    | 5310   | 565    | 5310   | 565    |
| 3     | 3810   | 435    | 3810   | 331    | 3810   | 331    |
| 4     | 1358   | 760    | 1358   | 129    | 623    | 125    |
| 5     | 1861   | 1855   | 1861   | 37     | 36     | 32     |
| 6     | 2993   | 2993   | 2993   | 6      | 6      | 6      |
| 7     | 0      | 0      | 0      | 0      | 0      | 0      |
| Run Time | 134s | 142s | 52s |
TABLE III
Details of frequent episodes obtained when we use only a frequency threshold. (Patterns: \( \alpha_1 \) and \( \alpha_2 \), \( \eta = 0.7, \rho = 0.055, p = 0.068, M = 60, T = 10000, f_{th} = 350, T_X = 15. \))

| Level | #Cand | #Freq | #Max \( \alpha_1 \) | #Max \( \alpha_2 \) | #Non-max \( \alpha_1 \) | #Non-max \( \alpha_2 \) | #Noise \( \alpha_1 \) | #Noise \( \alpha_2 \) | #Mix \( \alpha_1 \) | #Mix \( \alpha_2 \) | #Super \( \alpha_1 \) | #Super \( \alpha_2 \) | #Others \( \alpha_1 \) | #Others \( \alpha_2 \) |
|-------|-------|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1     | 60    | 60    | 6              | 6              | 0              | 0              | 48             | 0              | 0              | 0              | 0              | 0              | 0              | 0              |
| 2     | 5310  | 565   | 15             | 15             | 8              | 13             | 474            | 36             | 4              | 0              | 0              | 0              | 0              |
| 3     | 3810  | 435   | 20             | 20             | 49             | 96             | 10             | 214            | 13             | 0              | 13             | 0              | 0              |
| 4     | 1358  | 760   | 15             | 15             | 142            | 411            | 0              | 52             | 19             | 0              | 106            | 0              |
| 5     | 1861  | 1855  | 6              | 6              | 228            | 1268           | 0              | 0              | 12             | 0              | 335            | 0              |
| 6     | 2993  | 2993  | 1              | 1              | 174            | 2385           | 0              | 0              | 3              | 0              | 429            | 0              |

TABLE IV
Details of frequent episodes obtained when we use bidirectional evidence as a post-filter. (Patterns: \( \alpha_1 \) and \( \alpha_2 \), \( \eta = 0.7, \rho = 0.055, p = 0.068, M = 60, T = 10000, f_{th} = 350, T_X = 15, H_{th} = 0.4 \))

| Level | #Cand | #Freq | #Max \( \alpha_1 \) | #Max \( \alpha_2 \) | #Non-max \( \alpha_1 \) | #Non-max \( \alpha_2 \) | #Noise \( \alpha_1 \) | #Noise \( \alpha_2 \) | #Mix \( \alpha_1 \) | #Mix \( \alpha_2 \) | #Super \( \alpha_1 \) | #Super \( \alpha_2 \) | #Others \( \alpha_1 \) | #Others \( \alpha_2 \) |
|-------|-------|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1     | 60    | 60    | 6              | 6              | 0              | 0              | 48             | 0              | 0              | 0              | 0              | 0              | 0              |
| 2     | 5310  | 565   | 15             | 15             | 8              | 13             | 474            | 36             | 4              | 0              | 0              | 0              | 0              |
| 3     | 3810  | 331   | 20             | 20             | 27             | 23             | 10             | 214            | 13             | 0              | 4              | 0              | 0              |
| 4     | 1358  | 129   | 15             | 15             | 14             | 15             | 0              | 41             | 19             | 0              | 10             | 0              |
| 5     | 1861  | 37    | 6              | 6              | 6              | 1              | 6              | 0              | 12             | 0              | 6              | 0              |
| 6     | 2993  | 6     | 1              | 1              | 0              | 1              | 0              | 0              | 3              | 0              | 0              | 0              |

Reduction in number of candidates as well as run-time. As can be seen from the table, whether we use threshold on \( H(\alpha) \) only for post-processing the outputs or also for reducing the candidates at each level, we get essentially the same output at all levels; at level 6, the two embedded patterns along with some super-episodes are the only ones output. We note that even when we use thresholds on both \( H(\alpha) \) as well as frequency, we simply refer to the output as ‘frequent episodes.’
### TABLE V

Details of frequent episodes obtained when we use bidirectional evidence threshold at each level.

(Patterns: $\alpha_1$ and $\alpha_2$, $\eta = 0.7$, $\rho = 0.055$, $p = 0.068$, $M = 60$, $T = 10000$, $f_{th} = 350$, $T_X = 15$, $H_{th} = 0.4$)

| Level | #Cand | #Freq | Subepisodes | Non-subepisodes |
|-------|-------|-------|-------------|-----------------|
|       |       |       | $\alpha_1$ | $\alpha_2$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1$ | $\alpha_2$ |
| 1     | 60    | 60    | 6           | 6             | 0           | 0           | 0           | 0           | 48         | 0           | 0           | 0           | 0           |
| 2     | 5310  | 565   | 15          | 15            | 8           | 13          | 474         | 36          | 4           | 0           | 0           | 0           | 0           |
| 3     | 3810  | 331   | 20          | 20            | 27          | 23          | 10          | 214         | 13          | 0           | 4           | 0           | 0           |
| 4     | 623   | 125   | 15          | 15            | 13          | 15          | 0           | 41          | 19          | 0           | 7           | 0           | 0           |
| 5     | 36    | 32    | 6           | 6             | 1           | 6           | 0           | 0           | 12          | 0           | 1           | 0           | 0           |
| 6     | 6     | 6     | 1           | 1             | 0           | 1           | 0           | 0           | 3           | 0           | 0           | 0           | 0           |

Columns $\#Cand$ and $\#Freq$ indicate the number of candidate and frequent episodes obtained at each level respectively. The remaining columns of this table explain the various different kind of frequent episodes obtained at different levels. The columns under Subepisodes category indicate the number of frequent subepisodes of the embedded patterns at each level. The columns under Non-subepisodes category describe the various frequent episodes which are not subepisodes of any of the embedded patterns. Column $\#Max$ indicates the number of maximal subepisodes of each embedded pattern at each level. Column $\#Non-max$ indicates the number of non-maximal subepisodes of both embedded patterns at each level. Any episode which has an associated noise event-type ($/ \in E_1$, the set of all event-types associated with the embedded partial orders) is referred to as a noise episode. ($A \rightarrow Z$) is a noise episode for example. The number of such frequent noise episodes at each level is given in column $\#Noise$. The information of episodes all whose associated event-types are contained in $E_1$ and necessarily involving event-types from at least two of the embedded patterns, is tabulated in column $\#Mixed$. The current event-stream, of course is generated by embedding only two patterns. An episode like ($A \rightarrow B \rightarrow G \rightarrow H$) is a mixed episode. Consider an episode $\alpha = (X, R)$ either under the super or others category (columns $\#Super$ or $\#Others$ respectively). All event-types from $X$ necessarily come from one of the embedded patterns (say $\alpha_1$). Consider the maximal subepisode $\alpha'_1$ of $\alpha_1$ obtained by its restriction on $X$. If $\alpha$ is a super-episode of $\alpha'_1$, then it belongs to the super category. For
example, \((A \rightarrow B \rightarrow C)\) is a super-episode of the maximal subepisode \((A \rightarrow (B C))\)(of \(\alpha_1\)). Similarly, \((H \rightarrow J \rightarrow K)\) is a super-episode of the maximal subepisode \((H \rightarrow (J K))\)(of \(\alpha_2\)). If \(\alpha\) is neither a super nor sub-episode of \(\alpha_1'\), then it belongs to others category. For example, consider the maximal subepisode \(\alpha_1' = (B C) \rightarrow D\)(of \(\alpha_1\)). \(\alpha = (C \rightarrow (B D))\) would belong to the others category. 

The \#Init column in Table IV and Table V indicates the number of episodes which are both frequent and have a high enough \(H(\alpha)\). From Table III we see that using only a threshold on frequency leads to a total of 2993 episodes of size 6 being reported as frequent. Of these, two are the embedded patterns (under the maximal subepisodes category), 174 and 2385 are non-maximal subepisodes of \(\alpha_1\) and \(\alpha_2\) respectively, 3 are super-episodes of \(\alpha_1\) and 429 are spurious episodes that do not contain any ‘noise’ event type. The results in Table V show that when we use a threshold on both frequency and \(H(\alpha)\), only 6 episodes of size 6 are reported as frequent: the two embedded patterns, one non-maximal subepisode of \(\alpha_2\) and three super-episodes of \(\alpha_1\). Thus, when we use only a threshold on frequency, most of the episodes reported as frequent are the non-maximal subepisodes which can never be eliminated based on their frequencies because they occur at least as frequently as the embedded patterns. This is the inherent combinatorial explosion in partial order mining that we pointed out in Sec. VII-A. Bidirectional evidence is effective in eliminating these and reporting only the actual partial orders embedded in the data. This is because patterns grouped under Non-maximal subepisodes and Others category would have a pair of event-types \((i, j)\) which are not related in these episodes, but are related in one of the embedded patterns. Since, most of the occurrences of these episodes come from the embedded pattern, it is easy to verify from Eq. 11 that almost all these patterns have a very low bidirectional evidence. This effect is seen at all levels in the tables. From Tables IV and Table V we see that the frequent episodes output are essentially the same whether we use a post-processing or a level-wise threshold on \(H(\alpha)\). All these results show that using a level-wise threshold on \(H(\alpha)\) provides substantial improvement in efficiency while not missing any important patterns in the set of frequent episodes output.

Also the \(H(\alpha)\) based threshold helps us in mining larger sized patterns. For example, when this algorithm was run (with only a frequency threshold) on a data stream with an 8-node episode embedded in it, even after a run-time of about 300 seconds, the algorithm was still counting the candidates at level 7. This is mainly due to the inherent combinatorial explosion in partial order mining. Most of the patterns reported in the non-max and others category at lower levels
TABLE VI
Frequent Episodes obtained when the algorithm is run in serial, parallel and general mode. (Patterns: 2 serial, 2 parallel, \( \alpha_1 \) and \( \alpha_2 \), \( \eta = 0.7 \), \( \rho = 0.055 \), \( p = 0.068 \), \( M = 100 \), \( T = 10000 \), \( f_{\text{th}} = 375 \), \( T_X = 12 \), \( H_{\text{th}} = 0.4 \))

| Level | Serial mode | | Parallel mode | | General mode | |
|-------|-------------| | #Cand | #Freq | #Cand | #Freq | #Cand | #Freq |
| 1     | 100         | | 100  | 100  | 100  | 100  |
| 2     | 9900        | | 54   | 4950 | 555  | 14850| 609  |
| 3     | 58          | | 58   | 4830 | 71   | 6422 | 225  |
| 4     | 34          | | 34   | 37   | 33   | 184  | 156  |
| 5     | 12          | | 12   | 12   | 12   | 60   | 60   |
| 6     | 2           | | 2    | 2    | 2    | 10   | 8    |
| Run Time | 58 s | | 1 min 28 s | | 3m 07 s |

TABLE VII
Results obtained when mining with thresholds on \( L_{\text{max}} \) and \( N_{\text{max}} \).
\( \rho = 0.045 \), \( p = 0.055 \), \( \eta = 0.7 \), \( M = 100 \), \( T = 10000 \), \( f_{\text{th}} = 300 \), \( H_{\text{th}} = 0.35 \), \( T_X = 15 \)

| \( L_{\text{max}} \) | \( N_{\text{max}} \) | #Satisfying(fig. 5) | #Freq | Run-time |
|---------------------|---------------------|---------------------|-------|----------|
| 0                   | 10                  | 1                   | 1     | 6 m 29 s |
| 2                   | 10                  | 2                   | 3     | 9 m 48 s |
| 5                   | 4                   | 3                   | 5     | 9 m 45 s |
| 6                   | 2                   | 1                   | 2     | 2 m 57 s |
| 7                   | 1                   | 1                   | 1     | 53 s     |
| 7                   | 6                   | 5                   | 10    | 9 m 55 s |
| 7                   | 18                  | 8                   | 13    | 10 m 0 s |
| 3                   | 3                   | 0                   | 0     | 9 m 27 s |

contribute to the generation of a huge number of uninteresting frequent patterns at higher levels, in turn leading to a huge number of candidate patterns at higher levels. Hence, counting at level 7 was taking a lot of time. Mining with a \( H(\alpha) \) threshold, we could discover the 8-node embedded episode in a reasonable amount of time.

C. Flexibility in candidate generation

As described in Section V-B, the same algorithm (with minor modifications in the candidate generation) can be used to mine either serial episodes, parallel episodes or any sub-class of partial
orders satisfying the maximal-subepisode property. To illustrate this, we generated a data stream of about 50,000 events where, in addition to the episodes $\alpha_1$ and $\alpha_2$ defined in Sec. VIII-B, we embedded two more serial episodes and two more parallel episodes. We ran our algorithm on this data in the serial episode, parallel episode and the general modes. When run in the serial episode mode and the parallel episode mode, we recovered the two serial and the two parallel episodes respectively. In the general mode, all six embedded partial orders (along with two other episodes which were superepisodes of the embedded partial orders) were obtained. Table VI shows these results.

Next, we generated synthetic data by embedding all the 8 partial orders of Figure 5. Recall that $L_{\text{max}}$ for a partial order is the length of its largest maximal path. Similarly, $N_{\text{max}}$ for a partial order is the number of maximal paths in it. We present results obtained by mining in this data under different thresholds on $L_{\text{max}}$ and $N_{\text{max}}$ (Table VII). The column titled 'Satisfying (fig. 5)' refers to the number of partial orders in Figure 5 which satisfy the corresponding $L_{\text{max}}$ and $N_{\text{max}}$ constraints. We get all the embedded patterns that satisfy the $L_{\text{max}}, N_{\text{max}}$ constraint as frequent episodes along with a few extra episodes (as seen under #freq). From the table we see that at lower thresholds on either $L_{\text{max}}$ OR $N_{\text{max}}$, the algorithm runs faster. At higher thresholds, the run-times were almost the same as those for mining all partial orders. This is because most of the computational burden is due to large number of candidates at levels 2 and 3, and the candidates at these lower levels are not reduced if the bounds on $L_{\text{max}}$ and $N_{\text{max}}$ are high.

**TABLE VIII**

**Run-time as noise level is increased by varying $\rho$. Patterns Embedded: (iii) & (vi) from Fig. 5**

\[ p = 0.055, \eta = 0.7, M = 100, T = 10000, f_{th} = 300, T_X = 15, H_{th} = 0.35. \]

| $\rho$  | Noise level($L_{\text{ns}}$) | Run-time |
|---------|-----------------------------|----------|
| 0.005   | 0.43                        | 3 s      |
| 0.02    | 0.75                        | 6 s      |
| 0.03    | 0.82                        | 30 s     |
| 0.045   | 0.87                        | 1 m 45 s |
| 0.05    | 0.885                       | 6 m 1 s  |
A
B
C
D
E
F
G
H

(i) $L_{max} = 0$, $N_{max} = 8$ (Parallel Episode)

(ii) $L_{max} = 1$, $N_{max} = 8$

(iii) $L_{max} = 1$, $N_{max} = 18$

(iv) $L_{max} = 3$, $N_{max} = 4$

(v) $L_{max} = 4$, $N_{max} = 4$

(vi) $L_{max} = 5$, $N_{max} = 4$

(vii) $L_{max} = 7$, $N_{max} = 2$

(viii) $L_{max} = 8$, $N_{max} = 1$ (Serial Episode)

Fig. 5. Partial Order Episodes used for embedding in the data streams.

Fig. 6. Variation in number of frequent episodes as a function of frequency threshold, No. of embedded episodes, $N_{emb} = 5$, $\rho = 0.055$, $p = 0.068$, $\eta = 0.7$, $M = 100$, $T = 10000$, $H_{th} = 0.75$, $T_X = 15$.

D. Scaling and other properties of the algorithm

Our mining algorithm is robust to choice of frequency and $H(\alpha)$ thresholds as illustrated in figures 6 & 7. Once the threshold is high enough to eliminate noisy/spurious episodes, the number of episodes output is close to constant over a wide range of threshold choices.

The algorithm also scales well with number of embedded patterns, data length and noise level. In Tables X, IX & VIII, the data is generated from a set of 100 event types, with different 8-node episodes embedded from fig. 5. The run-times given are average values obtained over 10
Let $N_{ns}$ and $N_{sig}$ be the expected number of noise and signal events respectively in the data stream using our simulation model. By noise events here, we refer to the events in the noise stream. Similarly, by signal events, we refer to all the events coming from the various episode streams. In a data stream with $N_{emb}$ embedded episodes (each of size $l$), one can verify that $N_{ns} = (|E - E_1| \rho + E_1 \rho/5)T$ and $N_{sig} = \frac{T|N_{emb}}{(l-1)/\eta + 1/p}$. We define the noise level ($L_{ns}$) as fraction of the expected number of noise events, i.e. $L_{ns} = \frac{N_{ns}}{N_{ns} + N_{sig}}$. Table VIII describes increase in run-times with noise level $L_{ns}$, which is the ratio of the expected number of noise events to the expected total number of events as per our simulation model. We see that for low \(\rho\) (say 0.02) the running time is very less. This is because at the one node level only the signal events are frequent, as a result of which the number of candidates in successive levels are less. As \(\rho\) increases the number of candidates at 2 and 3 node level increases. Thus running times go up. For \(\rho = 0.045\), the number of candidates in the 3 node level goes up to the order of 30,000,
TABLE X

RUN-TIME AS THE NUMBER OF EMBEDDED PATTERNS IS INCREASED. \( \rho = 0.045 \), REST SAME AS TABLE VIII.

| No. of patterns | Embedded Patterns (Fig. 5) | Run-Time |
|-----------------|-----------------------------|----------|
| 2               | (iii),(vi)                  | 1 m 45 s |
| 5               | (i),(iii),(v),(vi),(viii)   | 4 m 18 s |
| 8               | (i)-(viii)                  | 11 m 10 s |

because many 2-node episodes are frequent. Consequently, the running times are very high for noise levels at about 0.88.

Similarly, Table IX describes the run-time variations with data length. We observe that the run-times increase almost linearly with data length. As the data length is increased, the ratio of \( f_{th}/T \) is kept constant. Table X shows the run-time variations with the number of embedded partial orders. We see an increase in the run-times because of increased number of candidates as the number of patterns is increased.

IX. CONCLUSIONS

In this paper we presented a method for discovering frequent episodes with general partial orders. Episode discovery from event streams is a very useful data mining technique though all the currently available methods can discover only serial or parallel episodes. Here, we presented a finite automata based algorithm for counting non-overlapped occurrences of injective episodes with general partial orders. (Along the way, we note some interesting properties of the finite state automaton used to track the occurences). The method is efficient and can take care of expiry-time constraints. The candidate generation algorithm presented here is very flexible and can be used to focus the discovery process on many interesting subclasses of partial orders. In particular, our method can be easily specialized to mine only serial or only parallel episodes. Thus, the algorithm presented here can be used as a single method to discover serial episodes or parallel episodes or episodes with general partial orders. Another important contribution of this paper is a new measure of interestingness for partial order episodes, namely, bidirectional evidence. We showed that there is an inherent combinatorial explosion in the number of frequent episodes when one considers general partial orders. Our bidirectional evidence is very useful in discovering the most appropriate partial orders from the data stream. The effectiveness of the
data mining method is demonstrated through extensive simulations.

In this paper we have considered injective episodes and a special subclass of non-injective partial order episodes (which includes all injective partial order episodes). We note that this subclass includes the set of all non-injective serial episodes. In that sense, our algorithms truely generalize existing serial and parallel episode discovery algorithms. Extending the ideas presented here to the class of all partial order episodes is an important problem. Another potential direction is the development of a statistical significance test for general partial order patterns in event streams. We will address these in our future work.

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Finally, we describe the more efficient GetPotentialCandidates() function (listed as Algorithm 3). The input to Algorithm 3 is a pair of episodes, α₁ and α₂, both of size l, and both appearing in the same block of the set, \( F_l \), of frequent l-node episodes. Recall that α₁ and α₂ are identical in their first \((l-1)\) nodes (in respect of both the associated event-types as well as the partial order among these event-types). This common \((l-1)\)-size partially ordered set is denoted as \( \mathcal{X} \). The output of Algorithm 3 is the set, \( \mathcal{P} \), of potential candidates that can be constructed from α₁ and α₂. The function GetPotentialCandidates() constructs a \( \mathcal{Y}_0 \) combination of α₁ and α₂ as per (eqn 7) using the function SimpleJoin() and retains only those combinations of α₁ and α₂ (of the three possible) which satisfy transitivity. The GetPotentialCandidates() function decides the valid combinations based on some special checks on the kind of nodes in \( \mathcal{X} \).

For purposes of easier illustration, we classify the nodes in \( \mathcal{X} \) based on its relation with \( x^\alpha_l \) and \( x^\beta_l \). We would have 9 such type of nodes. A node \( z \in \mathcal{X} \) is of the following types.

1. \( (x^\alpha_l, z) \) and \( (z, x^\beta_l) \) belong to \( R \).
2. \( (x^\alpha_l, z) \in R \), no edge between \( z \) and \( x^\beta_l \).
3. \( (z, x^\alpha_l) \in R \), no edge between \( z \) and \( x^\beta_l \).
4. \( (x^\alpha_l, z) \in R \), no edge between \( z \) and \( x^\alpha_l \).
5. \( (z, x^\alpha_l) \in R \), no edge between \( z \) and \( x^\alpha_l \).

X. APPENDIX
(3') - \((z, x_i^{\alpha_2}) \in R\), no edge between \(z\) and \(x_i^{\alpha_1}\).
(4) - \((z, x_i^{\alpha_2})\) and \((z, x_i^{\alpha_1})\) belong to \(R\).
(4') - \((x_i^{\alpha_1}, z)\) and \((x_i^{\alpha_1}, z)\) belong to \(R\).
(4'') - neither connected to \(x_i^{\alpha_1}\) nor \(x_i^{\alpha_2}\). 

We describe the GetPotentialCandidates() function with these nodes in mind. If a node of type (1) exists (condition as per line 3), then \(Y_1\) is the only generated candidate (as per lines 7, 9, 10). Similarly, if a node of type(1') exists (condition as per line 4), then \(Y_2\) is the only generated candidate (as per lines 8 – 10). Suppose neither nodes of the type (1) nor (1') exist, then \(Y_0\) is a sure candidate. Further, \(Y_1\) is generated iff nodes of type (2') and (3) do not exist in \(X\). Similarly, \(Y_2\) is generated iff nodes of type (2) and (3') dont exist in \(X\). One can verify this from lines 14 onwards in GetPotentialCandidates(). To show its correctness, we first make an important observation state as a lemma.

**Lemma 1:** In \(Y_0\), if a node of type (1) exists, there cannot exist nodes of type (1'), (2') and (3). Similarly, if a node of type (1') exists, there cannot exist nodes of type (1), (2) and (3').

This also holds for \(Y_1\) and \(Y_2\).

**Proof:** Given that a node \(z_0\) of type (1) exists in \(X\), we will show that none of above 3 type of nodes can exist by contradiction. Suppose a node \(z_1\) of type (1') exists, then \((z_1, x_i^{\alpha_1}) \in R \implies (z_1, x_i^{\alpha_1}) \in R^{\alpha_1}\). Since \(z_0\) is of type (1), \((x_i^{\alpha_1}, z_0) \in R \implies (x_i^{\alpha_1}, z_0) \in R^{\alpha_1}\). By the transitivity of \(R^{\alpha_1}\), it follows that \((z_1, z_0) \in R_1\). Also, since \(z_0\) is of type (1), we have \((z_0, x_i^{\alpha_2}) \in R \implies (z_0, x_i^{\alpha_2}) \in R^{\alpha_2}\). Likewise, \(z_1\) being of type (1') also says \((x_i^{\alpha_2}, z_0) \in R \implies (x_i^{\alpha_2}, z_1) \in R^{\alpha_2}\). Hence the transitivity of \(R_2\) tells us that \((z_0, z_1) \in R_2\). So we now have the same pair of nodes being connected in opposite ways in \(R^{\alpha_1}\) and \(R^{\alpha_2}\). This contradicts condition (2) of combining \(R^{\alpha_1}\) and \(R^{\alpha_2}\) that they share the same partial order on \(x_1 x_2 \ldots x_{l-1}\).

Suppose a node \(z_1\) of type (2') exists, then \((z_1, x_i^{\alpha_1}) \in R \implies (z_1, x_i^{\alpha_1}) \in R^{\alpha_1}\). Also \((x_i^{\alpha_1}, z_0) \in R^{\alpha_1}\). Transitivity of \(R^{\alpha_1}\) tells us \((z_1, z_0) \in R_1\). Since both \(z_0\) and \(z_1\) both belong to \(x_1 x_2 \ldots x_{l-1}\), \((z_1, z_0) \in R_2\). We also have \((z_0, x_i^{\alpha_2}) \in R \implies (z_0, x_i^{\alpha_2}) \in R^{\alpha_2}\). Transitivity in \(R_2\) now implies \((z_1, x_i^{\alpha_2}) \in R^{\alpha_2}\) and hence is in \(R\). But this edge must be absent as \(z_1\) is of type(2'). A similar contradiction arises for a node of type (3).

The second statement of the theorem has proofs analogous to that of the first statement.
We will now show that this efficient procedure generates the correct relations.

**Theorem 1:** The generated realtions (among the three combinations \(\mathcal{Y}_0, \mathcal{Y}_1\) and \(\mathcal{Y}_2\) possible) as per **Algorithm 3** are all transitively closed and the ones not generated violate transitivity.

**Proof:** Let us list out the six possibilities that need to be checked, for proving transitivity (because \(\alpha_1\) and \(\alpha_2\) share the same subepisode on dropping their last nodes respectively, as already discussed in the candidate generation section). Let \(z'\) denote an element belonging to \(\mathcal{X}'\).

\[
\begin{align*}
(a)(z', x_{i_1}^{\alpha_1}), (x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}) & \in \mathcal{Y}_i \implies (z', x_{i_2}^{\alpha_2}) \in \mathcal{Y}_i \\
(b)(z', x_{i_2}^{\alpha_2}), (x_{i_1}^{\alpha_1}, x_{i_1}^{\alpha_1}) & \in \mathcal{Y}_i \implies (z', x_{i_1}^{\alpha_1}) \in \mathcal{Y}_i \\
(c)(x_{i_1}^{\alpha_1}, z'), (z', x_{i_2}^{\alpha_2}) & \in \mathcal{Y}_i \implies (x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}) \in \mathcal{Y}_i \\
(d)(x_{i_2}^{\alpha_2}, z'), (z', x_{i_1}^{\alpha_1}) & \in \mathcal{Y}_i \implies (x_{i_2}^{\alpha_2}, x_{i_1}^{\alpha_1}) \in \mathcal{Y}_i \\
(e)(x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2}), (x_{i_2}^{\alpha_2}, z') & \in \mathcal{Y}_i \implies (x_{i_1}^{\alpha_1}, z') \in \mathcal{Y}_i \\
(f)(x_{i_2}^{\alpha_2}, x_{i_1}^{\alpha_1}), (x_{i_1}^{\alpha_1}, z') & \in \mathcal{Y}_i \implies (x_{i_2}^{\alpha_2}, z') \in \mathcal{Y}_i
\end{align*}
\]

We do these six transitivity checks on a case by case basis as adopted by the procedure.

**Case(i) A node \(z\) of type (1) exists in \(\mathcal{X}'\):** Here, we need to show that \(\mathcal{Y}_0, \mathcal{Y}_2\) are not transitively closed and \(\mathcal{Y}_1\) is indeed closed.

Since \(z\) is of type (1), \((x_{i_1}^{\alpha_1}, z)\) and \((z, x_{i_2}^{\alpha_2})\) are present in both \(\mathcal{Y}_0\) and \(\mathcal{Y}_2\). But transitivity demands the edge \((x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2})\) which is absent in both \(\mathcal{Y}_0\) and \(\mathcal{Y}_2\). Hence both of them are not closed.

To prove the transitive closedness of \(\mathcal{Y}_1\), let us perform the six checks listed above. If hypothesis of (a) is true, and suppose \((z', x_{i_2}^{\alpha_2}) \notin \mathcal{Y}_1\), then either there exists an edge \((x_{i_2}^{\alpha_2}, z') \in \mathcal{Y}_1\) or there exists no edge between \(z'\) and \(x_{i_2}^{\alpha_2}\). In the first case \(z'\) must be of type \((1')\) which cannot exist from lemma [1] In the second case \(z'\) must be of type \((2')\) which also cannot exist from lemma [1] This proves (a). Hypothesis of (b) and (f) cannot be true in \(\mathcal{Y}_1\) because of the reverse edge \((x_{i_1}^{\alpha_1}, x_{i_2}^{\alpha_2})\). (c) is obviously true in \(\mathcal{Y}_1\). The hypothesis of (d) indicates the existence of a type \((1')\) node in \(\mathcal{Y}_1\) which is not possible from lemma [1] Correctness of (e) is similar to that of (a). If hypothesis of (e) is true, and suppose \((x_{i_2}^{\alpha_2}, z') \notin \mathcal{Y}_1\), then either there exists an edge \((z', x_{i_2}^{\alpha_2}) \in \mathcal{Y}_1\) or there exists no edge between \(z'\) and \(x_{i_2}^{\alpha_2}\). In the first case \(z'\) must be of type \((1')\) which cannot exist from lemma [1] In the second case \(z'\) must be of type \((3)\) which also cannot exist from lemma [1] This proves (e).

**Case(ii) A node of type \((1')\) exists in \(\mathcal{X}'\):** This is analogous to case (i).

**Case(iii) Neither a node of type(1) nor type(1') exists :** First we need to show that \(\mathcal{Y}_0\) is
closed always here. Hypothesis of (a), (b), (e) and (f) are never true as they involve a direct edge between $x_1^{\alpha_1}$ and $x_1^{\alpha_2}$ which is not present in $\mathcal{Y}_0$. Hypothesis of (c) and (d) demand the existence of nodes of type (1) and (1') respectively which don't arise this scenario. This shows the transitivity of $\mathcal{Y}_0$ in this case.

Further, we show that $\mathcal{Y}_1$ is closed iff no nodes of type (2') and (3) exist in $\mathcal{X}$.

($\Rightarrow$) Let us prove the contrapositive of the forward statement. If a node $z'$ of type (2') exists, then we have $(z', x_1^{\alpha_1}), (x_1^{\alpha_1}, x_1^{\alpha_2}) \in \mathcal{Y}_1$ but there is no edge between $z'$ and $x_1^{\alpha_2}$. This violates transitivity of $\mathcal{Y}_1$. Similarly, if a node $z'$ of type (3) exists, then we have $(x_1^{\alpha_1}, x_1^{\alpha_2}), (x_1^{\alpha_2}, z') \in \mathcal{Y}_1$, but there is no edge between $z'$ and $x_1^{\alpha_1}$. This violates transitivity of $\mathcal{Y}_1$.

($\Leftarrow$) Suppose no nodes of type (2') and (3) exist, we will show the closedness of $\mathcal{Y}_1$. If hypothesis of (a) is true, and suppose $(z', x_1^{\alpha_1}) \notin \mathcal{Y}_1$, then either there exists an edge $(x_1^{\alpha_2}, z') \in \mathcal{Y}_1$ or there exists no edge between $z'$ and $x_1^{\alpha_2}$. In the first case $z'$ must be of type (1') which cannot exist here in case (iii). In the second case $z'$ must be of type (2') which also cannot exist from the hypothesis. This proves (a). The hypothesis of (b) and (e) are not satisfied here as they involve the edge $(x_1^{\alpha_1}, x_1^{\alpha_2})$. The hypothesis of (c) and (d) demands the existence of nodes of type (1) and (1') which cannot exist here in case (iii). If hypothesis of (e) is true, and suppose $(x_1^{\alpha_1}, z') \notin \mathcal{Y}_1$, then either there exists an edge $(z', x_1^{\alpha_1}) \in \mathcal{Y}_1$ or there exists no edge between $z'$ and $x_1^{\alpha_1}$. In the first case $z'$ must be of type (1') which cannot exist here in case (iii). In the second case $z'$ must be of type (3) which also cannot exist from the hypothesis. This proves (e).

Further, we show that $\mathcal{Y}_2$ is closed iff no nodes of type (2) and (3') exist in $\mathcal{X}$. The proof of this is analogous to that of $\mathcal{Y}_1$. ■
**Algorithm 3: GetPotentialCandidates(α₁, α₂)**

**Input:** Patterns, α₁ and α₂, both of size \( l \)

**Output:** \( P \), candidate possibilities from \( α₁ \) and \( α₂ \)

1. Initialize \( flg, flg₁, flg₂ ← 0 \) and \( P ← \phi \);
2. for \((i ← 1; i ≤ l − 1 \text{ and } flg = 0; i ← i + 1)\) do
   3. if \((α₁.ε[i][l] = 1 \text{ and } α₂.ε[i][l] = 1)\) then \( flg ← 1; \)
   4. if \((α₂.ε[i][l] = 1 \text{ and } α₁.ε[i][l] = 1)\) then \( flg ← 2; \)
5. if \( flg ≠ 0 \) then
   6. \( γ₁ ← \text{SimpleJoin}(α₁, α₂); \)
   7. if \( flg = 1 \) then \( γ₁.ε[l][l + 1] ← 1; \)
   8. else \( γ₁.ε[l + 1][l] ← 1; \)
   9. Add \( γ₁ \) to \( P; \)
10. return \( P; \)
11. else
12. \( γ₁ ← \text{SimpleJoin}(α₁, α₂); \)
13. Add \( γ₁ \) to \( P; \)
14. for \( i ← 1 \) to \( l − 1 \) do
   15. if \((α₁.ε[i][l] = 1 \text{ and } α₂.ε[i][l] = 0)\) or
      \((α₁.ε[i][l] = 0 \text{ and } α₂.ε[i][l] = 1)\) then
      \( flg₁ = 1; \)
   16. if \((α₁.ε[i][l] = 0 \text{ and } α₂.ε[i][l] = 1)\) or
      \((α₁.ε[i][l] = 1 \text{ and } α₂.ε[i][l] = 0)\) then
      \( flg₂ = 1; \)
17. if \( flg₁ = 0 \) and \( flg₂ = 0 \) then
   18. \( γ₂ ← \text{SimpleJoin}(α₁, α₂); \)
   19. \( γ₂.ε[l][l + 1] ← 1; \)
   20. Add \( γ₂ \) to \( P; \)
   21. \( γ₃ ← \text{SimpleJoin}(α₁, α₂); \)
   22. \( γ₃.ε[l + 1][l] ← 1; \)
   23. Add \( γ₃ \) to \( P; \)
24. if \( flg₁ = 1 \) and \( flg₂ = 0 \) then
25. \( γ₂ ← \text{SimpleJoin}(α₁, α₂); \)
26. \( γ₂.ε[l][l + 1] ← 1; \)
27. Add \( γ₂ \) to \( P; \)
28. if \( flg₁ = 0 \) and \( flg₂ = 1 \) then
29. \( γ₃ ← \text{SimpleJoin}(α₁, α₂); \)
30. \( γ₃.ε[l + 1][l] ← 1; \)
31. Add \( γ₃ \) to \( P; \)
32. return \( P; \)