A two-parameter extension of the Urbanik semigroup

Christian Berg

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Abstract

We prove that $s_n(a, b) = \frac{\Gamma(an + b)}{\Gamma(b)}$, $n = 0, 1, \ldots$ is an infinitely divisible Stieltjes moment sequence for arbitrary $a, b > 0$. Its powers $s_n(a, b)^c$, $c > 0$ are Stieltjes determinate if and only if $ac \leq 2$. The latter was conjectured in a paper by Lin (ArXiv: 1711.01536) in the case $b = 1$. We describe a product convolution semigroup $\tau_c(a, b)$, $c > 0$ of probability measures on the positive half-line with densities $e_c(a, b)$ and having the moments $s_n(a, b)^c$. We determine the asymptotic behaviour of $e_c(a, b)(t)$ for $t \to 0$ and for $t \to \infty$, and the latter implies the Stieltjes indeterminacy when $ac > 2$. The results extend previous work of the author and J. L. López and lead to a convolution semigroup of probability densities $(g_c(a, b)(x))_{c>0}$ on the real line. The special case $(g_c(a, 1)(x))_{c>0}$ are the convolution roots of the Gumbel distribution with scale parameter $a > 0$. All the densities $g_c(a, b)(x)$ lead to determinate Hamburger moment problems.

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1 Introduction

A Stieltjes moment sequence is a sequence of non-negative numbers of the form

$$s_n = \int_0^\infty t^n d\mu(t), \quad n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\},$$

where $\mu$ is a positive measure on $[0, \infty)$ such that $x^n \in L^1(\mu)$ for all $n \in \mathbb{N}_0$. The sequence $(s_n)$ is called normalized if $s_0 = \mu([0, \infty)) = 1$, and it is called S-determinate (resp. S-indeterminate) if (1) has exactly one (resp. several) solutions $\mu$ as positive measures on $[0, \infty)$. All these concepts go back to the fundamental memoir of Stieltjes [17].

A Stieltjes moment sequence $(s_n)$ is called infinitely divisible if $(s_n^c)$ is a Stieltjes moment sequence for any $c > 0$. These sequences were characterized in Tyan’s phd-thesis [19] and again in [5] without the knowledge of [19]. An important example of an infinitely divisible normalized Stieltjes moment sequence is $s_n = n!$, first established in Urbanik [20]. He proved that $e_c$ in (2) is
a probability density such that

\[(n!)^c = \int_0^{\infty} t^n e_c(t) \, dt, \quad e_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix-1} \Gamma(1-ix)^c \, dx, \quad c, t > 0. \tag{2}\]

Here \(\Gamma\) is Euler’s Gamma-function. The family \(\tau_c = e_c(t)dt, c > 0\) is a convolution semigroup in the sense of \[6\] on the locally compact abelian group \(G = (0, \infty)\) under multiplication. It is called the Urbanik semigroup in \[7\].

By Carleman’s criterion for S-determinacy it is easy to prove that \((n!)^c\) is S-determinate for \(c \leq 2\). That this estimate is sharp was first proved in \[4\], where it was established that \((n!)^c\) is S-indeterminate for \(c > 2\) based on asymptotic results of Skorokhod \[16\] about stable distributions, see \[21\]. Another proof of the S-indeterminacy was given in \[7\] based on the asymptotic behaviour of \(e_c(t), e_c(t) = \frac{2\pi}{\sqrt{c}} \left( \frac{c}{2\pi} \right)^{c/2} \exp\left(-ct^{1/c}\right) t^{b/a-1} \exp\left(-t^{1/a}\right) dt, t \to \infty. \tag{3}\]

In the recent paper \[10\], Lin proposes the following conjecture:

**Conjecture** Let \(a > 0\) be a real constant and let \(s_n = \Gamma(na + 1), n \in \mathbb{N}_0\).

Then

(a) \((s_n)\) is an infinitely divisible Stieltjes moment sequence;
(b) For real \(c > 0\) the sequence \((s_n^c)\) is S-determinate if and only if \(ac \leq 2\);
(c) For \(0 < c \leq 2/a\) the unique probability measure \(\mu_c\) corresponding to \((s_n^c)\) has the Mellin transform

\[\int_0^{\infty} t^s d\mu_c(t) = \frac{\Gamma(as + 1)^c}{\Gamma(b)}, s \geq 0.\]

When \(a = 1\) the conjecture is true because of the known results about the Urbanik semigroup, and for \(a \in \mathbb{N}, a \geq 2\) the conjecture is true because of the Theorems 5 and 8 in \[10\].

We shall prove that the conjecture is true, and it is a special case of similar results for the following more general normalized Stieltjes moment sequence

\[s_n(a, b) = \frac{\Gamma(an + b)}{\Gamma(b)} = \frac{1}{a\Gamma(b)} \int_0^{\infty} t^{n+b/a-1} \exp(-t^{1/a}) \, dt, \quad n = 0, 1, \ldots, \tag{4}\]

where \(a, b > 0\) are arbitrary.

Defining

\[e_{1}(a, b)(t) = \frac{1}{a\Gamma(b)} t^{b/a-1} \exp(-t^{1/a}), \tag{5}\]

we get for \(\text{Re} \, z > -b/a\) and a change of variable \(t = s^a\)

\[\int_0^{\infty} t^z e_{1}(a, b)(t) \, dt = \frac{\Gamma(sz + b)/\Gamma(b).}{} \tag{6}\]

This leads to our main result.
Theorem 1.1. (i) \((s_n(a, b))\) is an infinitely divisible Stieltjes moment sequence.

(ii) There exists a uniquely determined convolution semigroup \((\tau_c(a, b))_{c>0}\)
of probability measures on the multiplicative group \((0, \infty)\) such that

\[
\int_0^\infty t^z \, d\tau_c(a, b)(t) = \left[\Gamma(az + b)/\Gamma(b)\right]^c, \quad \text{Re } z > -b/a, \tag{7}
\]

and in particular \((s_n(a, b)^c)\) is the moment sequence of \(\tau_c(a, b)\).

(iii) \(\tau_c(a, b) = e_c(a, b)(t) \, dt\) on \((0, \infty)\), where

\[
e_c(a, b)(t) = \frac{1}{2\pi} \int_{-\infty}^\infty t^{ix-1}\left[\Gamma(b - iax)/\Gamma(b)\right]^c \, dx, \quad t > 0 \tag{8}
\]

is a probability density belonging to \(C^\infty(0, \infty)\).

(iv) \((s_n(a, b)^c)\) is \(S\)-determinate if and only if \(ac \leq 2\), hence independent of \(b > 0\).

Note that (4) is a special case of (6).

The measure \(\tau_1(a, b)\) was considered in [18], where it was proved that the
measure is \(S\)-indeterminate if \(a > \max(2, 2b)\). This is a consequence of our
result. Note that \(\tau_1(a, 1)\) is called the Weibull distribution with shape parameter \(1/a\) and scale parameter 1.

In (7) and (8) we use that \(\Gamma(z)\) is a non-vanishing holomorphic function in
the cut plane \(A = \mathbb{C} \setminus (-\infty, 0]\),

\[
\tag{9}
\]

so we can define

\[
\Gamma(z)^c := \exp(c \log \Gamma(z)), \quad z \in A
\]

using the holomorphic branch of \(\log \Gamma\) which is 0 for \(z = 1\). This branch is explicitly given in [18].

Let us recall a few facts about convolution semigroups of probability measures on LCA-groups, see [6] for details.

The continuous characters of the multiplicative group \(G = (0, \infty)\) can be
given as \(t \rightarrow t^{ix}\), where \(x \in \mathbb{R}\) is arbitrary, and in this way the dual group \(\hat{G}\) of
\(G\) can be identified with the additive group of real numbers. The convolution
between measures \(\mu, \sigma\) on \((0, \infty)\), called product convolution and denoted \(\mu \ast \sigma\),
is defined as

\[
\int_0^\infty f(t) \, d\mu \ast \sigma(t) = \int_0^\infty \int_0^\infty f(ts) \, d\mu(t) \, d\sigma(s)
\]

for suitable classes of continuous functions \(f\) on \((0, \infty)\), e.g. those of compact support.

A family \((\mu_c)_{c>0}\) of probability measures on the multiplicative group \(G = (0, \infty)\) is called a convolution semigroup, if \(\mu_c \ast \mu_d = \mu_{c+d}, c, d > 0\) and
\(\lim_{c \to 0} \mu_c = \varepsilon_1\) vaguely. Here \(\varepsilon_1\) is the Dirac measure with total mass 1
concentrated in the neutral element 1 of the group. By [6, Theorem 8.3] there is
a one-to-one correspondence between convolution semigroups $\mu_c$ of probability measures on $G$ and continuous negative definite functions $\rho : \mathbb{R} \to \mathbb{C}$ satisfying $\rho(0) = 0$ such that

$$\int_0^\infty t^{-ix} d\mu_c(t) = \exp(-c\rho(x)), \quad c > 0, x \in \mathbb{R}. \quad (10)$$

By the inversion theorem of Fourier analysis for LCA-groups, if $\exp(-c\rho)$ is integrable on $\mathbb{R}$, then $\mu_c = \int f_c(t) dt$ for a continuous function $f_c(t)$ (the density of $\mu_c$ with respect to Haar measure $1/t dt$ on $(0, \infty)$) given by

$$f_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix-1} \exp(-c\rho(x)) \, dx, \quad t > 0. \quad (11)$$

(Note that the dual Haar measure of $1/t dt$ on $(0, \infty)$ is $1/(2\pi) dx$ on $\mathbb{R}$.)

**Proposition 1.2.** For $a, b > 0$

$$\rho(x) := \log \Gamma(b) - \log \Gamma(b - iax), \quad x \in \mathbb{R} \quad (12)$$

is a continuous negative definite function on $\mathbb{R}$ satisfying $\rho(0) = 0$.

Proposition 1.2 shows that there exists a uniquely determined product convolution semigroup $(\tau_c(a, b))_{c > 0}$ satisfying

$$\int_0^\infty t^{-ix} d\tau_c(a, b)(x) = \exp[-c(\log \Gamma(b) - \log \Gamma(b - iax))]$$

$$= [\Gamma(b - iax)/\Gamma(b)]^c, \quad x \in \mathbb{R}. \quad (13)$$

Like in the proof of [4, Lemma 2.1] it is easy to see that (13) implies (7).

Putting $z = -ix$ in (6), we see by the uniqueness theorem for Fourier transforms that $\tau_1(a, b) = e_1(a, b)(t) dt$.

The function $(\Gamma(b - iax)/\Gamma(b))^c$ is a Schwartz function on $\mathbb{R}$ and in particular integrable, so (8) follows from (7), and $e_c(a, b)$ is $C^\infty$ on $(0, \infty)$.

In this way we have established (i)-(iii) of Theorem 1.1. The proof of the more difficult part (iv) as well as the proof of Proposition 1.2 will be given in Section 3.

By Riemann-Lebesgue’s Lemma we also see that $te_c(a, b)(t)$ tends to zero for $t$ tending to zero and to infinity. Much more on the behaviour near 0 and infinity will be given in Section 2. There we extend the work of [7] leading to the asymptotic behaviour of the densities $e_c(a, b)(t)$ for $t \to 0$ and $t \to \infty$. The behaviour for $t \to \infty$ will lead to a proof of the S-indeterminacy for $ac > 2$ using the Krein criterion.

The fact that $\tau_c(a, b) \circ \tau_d(a, b) = \tau_{c+d}(a, b)$ can be written

$$e_{c+d}(a, b)(t) = \int_0^\infty e_c(a, b)(t/x)e_d(a, b)(x) \frac{dx}{x}, \quad c, d > 0. \quad (14)$$
In particular for \( c = d = 1 \) and the explicit formula for \( e_1(a, b) \) we get
\[
e_2(a, b)(t) = \frac{t^b/a - 1}{a \Gamma(b)} \int_0^\infty \exp \left( -x^{-1/a} t^{1/a} - x^{1/a} \right) \frac{dx}{x} \tag{15}
\]
\[
= \frac{2t^b/a - 1}{a \Gamma(b)^2} K_0(2t^{1/(2a)}),
\]
because the Macdonald function \( K_0 \) is given by
\[
K_0(z) = \frac{1}{2} \int_0^\infty \exp \left( -(z/2)^2 / y - y \right) \frac{dy}{y},
\]
cf. [8] 8.432(7), [12] Chap. 10, Sec. 25.

2 Main results

Our main results are

**Theorem 2.1.** For \( c > 0 \) we have
\[
e_c(a, b)(t) = \frac{(2\pi)^{(c-1)/2}}{a \sqrt{c} \Gamma(b)} \frac{\exp(-ct^{1/(ac)})}{t^{1/(b-1/2+1/(2c))}/a} \left[ 1 + \mathcal{O} \left( t^{-1/(ac)} \right) \right], \quad t \to \infty. \tag{16}
\]

**Theorem 2.2.** The measure \( \tau_c(a, b) \ dt \) is \( S \)-indeterminate if and only if \( ac > 2 \).

**Theorem 2.3.** For \( c > 0 \) and \( 0 < t < 1 \) we have
\[
e_c(a, b)(t) = \frac{t^b/a - 1}{a \Gamma(b)^c} \left[ \frac{\log(1/t)^c - 1}{\Gamma(c)} \right] + \mathcal{O} \left( t^{b/a - 1}[\log(1/t)]^{c-2} \right), \quad t \to 0. \tag{17}
\]

**Remark 2.4.** Formula (17) shows that \( e_c(a, b)(t) \) tends to 0 for \( t \to 0 \) if \( b/a > 1 \), and to infinity if \( b/a < 1 \), independent of \( c \). If \( b/a = 1 \) then \( e_c(a, b)(t) \) tends to 0 for \( c < 1 \) and to infinity as a power of \( \log(1/t) \) when \( c > 1 \).

3 Proofs

**Proof of Proposition 1.2:** From the Weierstrass product for the entire function \( 1/\Gamma(z) \), we get the following holomorphic branch in the cut plane \( \mathcal{A} \), cf. [9],
\[
- \log \Gamma(z) = \gamma z + \log z + \sum_{k=1}^\infty (\log(1 + z/k) - z/k) \quad z \in \mathcal{A}, \tag{18}
\]
where \( \log \) denotes the principal logarithm, and \( \gamma \) is Euler’s constant.

For \( n \in \mathbb{N} \) and \( z \in \mathcal{A} \) define
\[
\rho_n(z) = \gamma z + \log z + \sum_{k=1}^n (\log(1 + z/k) - z/k),
\]
\[
R_n(z) = \sum_{k=n+1}^\infty (\log(1 + z/k) - z/k)
\]
so \( \lim_{n \to \infty} \rho_n(z) = -\log \Gamma(z) \), uniformly on compact subsets of \( \mathcal{A} \).

Furthermore, we have

\[
\log \Gamma(b) + \rho_n(b) + R_n(b) = 0,
\]

and since \( \log(1 + x) < x \) for \( x > 0 \), we see that \( R_n(b) < 0 \) and hence \( \log \Gamma(b) + \rho_n(b) > 0 \).

We claim that \( \log \Gamma(b) + \rho_n(b - iax) \) is a continuous negative definite function, and letting \( n \to \infty \) we get the assertion of Proposition 1.2.

To see the claim, we write

\[
\log \Gamma(b) + \rho_n(b - iax) = \log \Gamma(b) + \left( b - iax \right) \left( \gamma - \sum_{k=1}^{n} \frac{1}{k} \right) + \log \left( \frac{\Gamma(b - iax)}{\Gamma(b)} \right) + \sum_{k=0}^{n} \log \left( 1 - i\frac{ax}{b+k} \right),
\]

and the assertion follows since \( \alpha + i\beta x \) and \( \log(1 + i\beta x) \) are negative definite functions when \( \alpha \geq 0, \beta \in \mathbb{R} \), see [6], [14]. □

**Proof of Theorem 2.1**

We modify the proof given in [7] and start by applying Cauchy’s integral theorem to move the integration in (8) to a horizontal line

\[
H_\delta := \{ z = x + i\delta : x \in \mathbb{R} \}, \quad \delta > -b/a.
\]

**Lemma 3.1.** With \( H_\delta \) as in (19) we have

\[
e_{c}(a, b)(t) = \frac{1}{2\pi} \int_{H_\delta} t^{iz-1} \frac{\Gamma(b - iaz) \Gamma(b)}{\Gamma(b)} dz, \quad t > 0. \tag{20}
\]

**Proof.** For \( t, c > 0 \) fixed, \( f(z) = t^{iz-1} \frac{\Gamma(b - iaz) \Gamma(b)}{\Gamma(b)} \) is holomorphic in the simply connected domain \( \mathbb{C} \setminus (-\infty, -b/a] \), so (20) follows from Cauchy’s integral theorem provided the integral

\[
\int_0^\delta f(x + iy) \, dy
\]

tends to 0 for \( x \to \pm \infty \). We have

\[
|f(x + iy)| = t^{-y-1} |\Gamma(b + y - iax) / \Gamma(b)|^c
\]

and since

\[
|\Gamma(u + iv)| \sim \sqrt{2\pi} e^{-\pi/2|v|} |v|^{u-1/2}, \quad |v| \to \infty, \text{ uniformly for bounded real } u,
\]

cf. [1] p.141, Eq. 5.11.9, [8] 8.328(1), the result follows. □
In the following we will use Lemma 3.1 with the line of integration $H_δ$, where $δ = (t^{1/(ac)} - b)/a$. Therefore,
\[ e_c(a, b)(t) = t^{(b - t^{1/(ac)})/a - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix}\frac{\Gamma(t^{1/(ac)} - iax)}{\Gamma(b)c} dx, \]
and after the change of variable $x = a^{-1}t^{1/(ac)}u$ and putting $A := (1/c + b - a)/a$
\[ e_c(a, b)(t) = t^{A - a^{-1}t^{1/(ac)}} \frac{1}{2\pi a} \int_{-\infty}^{\infty} t^{iuA^{-1}t^{1/(ac)}} \frac{\Gamma(t^{1/(ac)}(1 - iu))}{\Gamma(b)c} du. \tag{21} \]

Binet’s formula for $Γ$ is (8, 8.341(1))
\[ Γ(z) = \sqrt{2\pi z} z^{z - 1/2} e^{-z + \mu(z)}, \quad \text{Re}(z) > 0, \tag{22} \]
where
\[ \mu(z) = \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-zt}}{t} dt, \quad \text{Re}(z) > 0. \tag{23} \]
Notice that $μ(z)$ is the Laplace transform of a positive function, so we have the estimates for $z = r + is$, $r > 0$
\[ |μ(z)| ≤ μ(r) ≤ \frac{1}{12r}, \tag{24} \]
where the last inequality is a classical version of Stirling’s formula, thus showing that the estimate is uniform in $s \in ℝ$.

Inserting this in (21), we get after some simplification
\[ e_c(a, b)(t) = (2π)^{c/2 - 1} a^{-1/(2a)} e^{-ct^{1/(ac)}} \int_{-\infty}^{\infty} e^{ct^{1/(ac)}f(u)} g_c(u) M(u, t) du, \tag{25} \]
where
\[ f(u) := iu + (1 - iu) \text{Log}(1 - iu), \quad g_c(u) := (1 - iu)^{-c/2} \tag{26} \]
and
\[ M(u, t) := \exp[\mu(t^{1/(ac)}(1 - iu))]. \tag{27} \]
From (24) we get $M(u, t) = 1 + O(t^{-1/(ac)})$ for $t \to \infty$, uniformly in $u$. We shall therefore consider the behaviour for large $x$ of
\[ \int_{-\infty}^{\infty} e^{-x f(u)} g_c(u) du, \quad x = ct^{1/(ac)}. \tag{28} \]

This is the same integral which was treated in [7, Eq.(28)] leading to
\[ \int_{-\infty}^{\infty} e^{-x f(u)} g_c(u) du = (2\pi/x)^{1/2} [1 + \mathcal{O}(x^{-1})] \]
by methods from [11].

For $x = ct^{1/(ac)}$ we find
\[ \int_{-\infty}^{\infty} e^{ct^{1/(ac)}f(u)} g_c(u) du = \frac{\sqrt{2π}}{\sqrt{ct^{1/(2ac)}}} [1 + \mathcal{O}(t^{-1/(ac)})], \]
where
\[ H_δ = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix}\frac{\Gamma(t^{1/(ac)} - iax)}{\Gamma(b)c} dx. \]

[11]
hence

\[ e_c(a, b)(t) = \left( \frac{2\pi}{a} \right)^{(c-1)/2} \frac{e^{-ct/(ac)}}{t^{1-(b-1/2-1/(2c))/a}} [1 + O(t^{-1/(ac)})]. \]

□

Proof of Theorem 2.2.
We first prove that \((s_n(a, b)^c)\) is S-determinate for \(ac \leq 2\) by Carleman’s criterion, cf. [15, p. 20]. In fact, from Stirling’s formula we have

\[ s_n(a, b)^c/2^n = \frac{\Gamma(na + b)/\Gamma(b)}{2^n} \sim (na/e)^{ac/2}, \quad n \to \infty, \]

so \(\sum s_n(a, b)^{-c/2^n} = \infty\) if and only if \(ac \leq 2\).

Since Carleman’s criterion is only a sufficient condition for S-determinacy, we need to prove that \(e_c(a, b)\) is S-indeterminate for \(ac > 2\). We apply the Krein criterion for S-indeterminacy of probability densities concentrated on the half-line, using a version due to H. L. Pedersen given in [9, Theorem 4]. It states that if

\[ \int K \log e_c(a, b)(t^2) dt > -\infty \]

for some \(K \geq 0\), then \(\tau_c(a, b) = e_c(a, b)(t) dt\) is S-indeterminate. This version of the Krein criterion is a simplification of a stronger version given in [13]. We shall see that (29) holds for \(ac > 2\).

From Theorem 2.1 we see that (29) holds for sufficiently large \(K > 0\) if and only if

\[ \int K \frac{-ct^2/(ac)}{1 + t^2} dt > -\infty, \]

and the latter holds precisely for \(ac > 2\). This shows that \(\tau_c(a, b)\) is S-indeterminate for \(ac > 2\). □

Proof of Theorem 2.3.
The proof uses the same ideas as in [7], but since the proof is quite technical, we give the full proof with the necessary modifications. Since we are studying the behaviour for \(t \to 0\), we assume that \(0 < t < 1\) so that \(\Lambda := \log(1/t) > 0\).

We will need integration along vertical lines

\[ V_\alpha := \{ \alpha + iy \mid y = -\infty \ldots \infty \}, \quad \alpha \in \mathbb{R}, \]

and we can therefore express (8) as

\[ e_c(a, b)(t) = \frac{t^{b/a-1}}{2\pi ita\Gamma(b)c} \int_{V_{-b}} t^{z/a}\Gamma(-z)^c dz. \]

By the functional equation for \(\Gamma\) we get

\[ e_c(a, b)(t) = \frac{t^{b/a-1}}{2\pi ita\Gamma(b)c} \int_{V_{-b}} g(z)\varphi(z) dz, \]

where

\[ g(z) := \frac{z^{a-1}}{\Gamma(z)}, \quad \varphi(z) := \frac{\Gamma(z)^c}{z^c}. \]
where we have defined

\[ \varphi(z) := t^{1/a} \Gamma(1 - z)^c, \quad g(z) := (-z)^{-c} = \exp(-c \log(-z)). \]

Note that \( \varphi \) is holomorphic in \( \mathbb{C} \setminus [1, \infty) \), while \( g \) is holomorphic in \( \mathbb{C} \setminus [0, \infty) \).

For \( x > 0 \) we define

\[ g_{\pm}(x) := \lim_{\varepsilon \to 0^\pm} g(x \pm i\varepsilon) = x^{-c} e^{\pm i\pi c}. \]

**Case 1.** Assume \( 0 < c < 1 \).

We fix \( 0 < s < 1 \), choose \( 0 < \varepsilon < \min(s, b) \) and integrate \( g(z) \varphi(z) \) over the contour \( \Gamma \)

\[ \{-b + iy \mid y = \infty \ldots 0\} \cup \{-b, -\varepsilon\} \cup \{\varepsilon e^{i\theta} \mid \theta = \pi \ldots 0\} \cup [\varepsilon, s] \cup \{s + iy \mid y = 0 \ldots \infty\} \]

and get 0 by the integral theorem of Cauchy. On the interval \( [\varepsilon, s] \) we use \( g = g_+ \).

Similarly we get 0 by integrating \( g(z) \varphi(z) \) over the complex conjugate contour \( \overline{\Gamma} \), and now we use \( g = g_- \) on the interval \( [\varepsilon, s] \).

Subtracting the second contour integral from the first leads to

\[ \int_{V_s} - \int_{V_{-b}} - \int_{|z| = \varepsilon} g(z) \varphi(z) \, dz + \int_{\varepsilon}^{s} \varphi(x) (g_+(x) - g_-(x)) \, dx = 0, \]

where the integral over the circle is with positive orientation. Note that the two integrals over \( [-b, -\varepsilon] \) cancel. Using that \( 0 < c < 1 \) it is easy to see that the just mentioned integral converges to 0 for \( \varepsilon \to 0 \), and we finally get for \( \varepsilon \to 0 \)

\[ e_{c}(a, b)(t) = \frac{e^{b/a-1}}{2\pi i a \Gamma(b)c} \int_{V_s} g(z) \varphi(z) \, dz + \frac{e^{b/a-1} \sin(\pi c)}{\pi a \Gamma(b)c} \int_{0}^{s} x^{-c} \varphi(x) \, dx := I_1 + I_2. \]

We claim that \( I_1 \) is \( o(t^{(s+b)/a-1}) \) for \( t \to 0 \). To see this we insert the parametrization of \( V_s \) and get

\[ I_1 = \frac{e^{b/a-1}}{2\pi i a \Gamma(b)c} \int_{-\infty}^{\infty} (-s - iy)^{-c} e^{(s+iy)/a} \Gamma(1 - s - iy)^c \, dy \]

\[ = \frac{t^{(s+b)/a-1}}{2\pi a \Gamma(b)c} \int_{-\infty}^{\infty} e^{-iy\Lambda/a} (-s - iy)^{-c} \Gamma(1 - s - iy)^c \, dy, \]

and the integral is \( o(1) \) for \( t \to 0 \) by Riemann-Lebesgue’s Lemma because \( \Lambda := \log(1/t) \to \infty \).

The substitution \( u = x\Lambda \) in the integral in the term \( I_2 \) leads to

\[ I_2 = \frac{e^{b/a-1} \sin(\pi c)}{\pi a \Gamma(b)c} \Lambda^{c-1} \int_{0}^{s\Lambda} u^{-c} e^{-u/a} \Gamma(1 - u/\Lambda)^c \, du. \quad (33) \]

We split the integral in \( (33) \) as

\[ \int_{0}^{s\Lambda} u^{-c} e^{-u/a} [\Gamma(1 - u/\Lambda)^c - 1] \, du + \int_{0}^{\infty} u^{-c} e^{-u/a} \, du - \int_{s\Lambda}^{\infty} u^{-c} e^{-u} \, du. \quad (34) \]
Calling the three terms \( J_1, J_2, J_3 \) we have \( J_2 = a^{1-c} \Gamma(1-c) \) and
\[
J_3 = -a^{1-c} \Gamma(1-c, s\Lambda/a),
\]
where \( \Gamma(\alpha, x) \) is the incomplete Gamma function with the asymptotics
\[
\Gamma(\alpha, x) = \int_x^\infty u^{\alpha-1} e^{-u} du \sim x^{\alpha-1} e^{-x}, \quad x \to \infty,
\]
cf. [8, 8.357], hence \( J_3 = O(t s/a \Lambda^{-c}) \), \( t \to 0 \).

Using the Digamma function \( \Psi = \Gamma'/\Gamma \), we get by the mean-value theorem
\[
\Gamma(1-u/\Lambda)^c - 1 = -\frac{u}{\Lambda} \Gamma(1-\theta u/\Lambda)^c \Psi(1-\theta u/\Lambda)
\]
for some \( 0 < \theta < 1 \), but this implies that
\[
|\Gamma(1-u/\Lambda)^c - 1| \leq \frac{cu}{\Lambda} M(s), \quad 0 < u < s\Lambda,
\]
where
\[
M(s) := \max\{\Gamma(x)^c|\Psi(x)| \mid 1-s \leq x \leq 1\},
\]
so \( J_1 = \mathcal{O}(\Lambda^{-1}) \) for \( t \to 0 \).

This gives
\[
I_2 = \frac{t^{b/a-1} \sin(\pi c)}{\pi a \Gamma(b)^c} \Lambda^{c-1} \left( \mathcal{O}(\Lambda^{-1}) + a^{1-c} \Gamma(1-c) + \mathcal{O}(t^{s/a} \Lambda^{-c}) \right)
\]
\[
= \frac{t^{b/a-1} \Lambda^{c-1}}{(a \Gamma(b)^c \Gamma(c)} + \mathcal{O}(t^{b/a-1} \Lambda^{-c-2}),
\]
where we have used Euler’s reflection formula for \( \Gamma \). Since finally
\[
I_1 = o(t^{(s+b)/a-1}) = \mathcal{O}(t^{b/a-1} \Lambda^{-c-2}),
\]
we see that (17) holds.

**Case 2.** Assume \( 1 < c < 2 \).

The Gamma function decays so rapidly on vertical lines \( z = \alpha+i y, y \to \pm \infty \),
that we can integrate by parts in (32) to get
\[
ce_c(a, b)(t) = -\frac{t^{b/a-1}}{2\pi i a \Gamma(b)^c} \int_{V-1} \frac{(-z)^{-(c-1)}}{c-1} \frac{d}{dz} \left( t^{z/a} \Gamma(1-z)^c \right) dz.
\]
(35)

Defining
\[
\varphi_1(z) := \frac{d}{dz} \left( t^{z/a} \Gamma(1-z)^c \right) = t^{z/a} \Gamma(1-z)^c ((1/a) \log t - c \Psi(1-z)),
\]
and using the same contour technique as in Case 1 to the integral in (35), where
now \( 0 < c-1 < 1 \), we get for \( 0 < s < 1 \) fixed
\[
ce_c(a, b)(t) = -\frac{t^{b/a-1}}{a \Gamma(b)^c} \left( \tilde{I}_1 + \tilde{I}_2 \right),
\]

where
\[
\hat{I}_1 = \frac{1}{2\pi i(c-1)} \int_{V_z} (-z)^{-(c-1)} \varphi_1(z) \, dz,
\]
\[
\hat{I}_2 = \frac{\sin(\pi(c-1))}{\pi(c-1)} \int_0^s x^{-(c-1)} \varphi_1(x) \, dx.
\]

We have \(\hat{I}_1 = o(t^{s/a} \Lambda)\) for \(t \to 0\) by Riemann-Lebesgue’s Lemma, and the substitution \(u = x \Lambda\) in the second integral leads to
\[
\int_0^s x^{-(c-1)} \varphi_1(x) \, dx
\]
\[
= \Lambda^{c-2} \int_0^{s \Lambda} u^{-(c-1)} \varphi_1(u/\Lambda) \, du
\]
\[
= -(1/a)\Lambda^{c-1} \left( \int_0^{s \Lambda} u^{-(c-1)} e^{-u/a} \, du + \int_0^{s \Lambda} u^{-(c-1)} e^{-u/a} (\Gamma(1-u/\Lambda)^c - 1) \, du \right)
\]
\[
- c\Lambda^{c-2} \int_0^{s \Lambda} u^{-(c-1)} e^{-u/a} \Gamma(1-u/\Lambda)^c \Psi(1-u/\Lambda) \, du
\]
\[
= -a^{1-c} \Lambda^{c-1} \Gamma(2-c) + O(\Lambda^{c-2}).
\]

Using that
\[
\frac{\sin(\pi(c-1))}{(c-1)\pi} \left( -a^{1-c} \Lambda^{c-1} \Gamma(2-c) \right) = -a^{1-c} \frac{\Lambda^{c-1}}{\Gamma(c)}
\]
by Euler’s reflection formula, we see that (17) holds.

**Case 3.** Assume \(c > 2\).
We perform the change of variable \(w = (1/a)\Lambda z\) in (32) and assume that \(\Lambda > a\). This gives
\[
e_{\varepsilon}(a,b)(t) = \frac{b^{\varepsilon-1}\Lambda^{c-1}}{[a \Gamma(b)]^c} \frac{1}{2\pi i} \int_{V_{-(b/a)\Lambda}} (-w)^{-c} e^{-w} \Gamma(1-aw/\Lambda)^c \, dw.
\]

Using Cauchy’s integral theorem, we can shift the contour \(V_{-(b/a)\Lambda}\) to \(V_{-1}\) as the integrand is holomorphic in the vertical strip between both paths and exponentially small at both extremes of that vertical strip. For the holomorphic function \(h(z) = \Gamma(1-z)^c\) in the domain \(G = \mathbb{C} \setminus [1, \infty)\), which is star-shaped with respect to 0, we have
\[
h(z) = h(0) + z \int_0^1 h'(uz) \, du, \quad z \in G,
\]
hence
\[
\Gamma(1-aw/\Lambda)^c = 1 - \frac{caw}{\Lambda} \int_0^1 \Gamma(1-uw/\Lambda)^c \Psi(1-uw/\Lambda) \, du.
\]
(36)

Defining
\[
R(w) = \int_0^1 \Gamma(1-uw/\Lambda)^c \Psi(1-uw/\Lambda) \, du,
\]

we get
\[
\frac{1}{2\pi i} \int_{V_{-1}} (-w)^{-c} e^{-w} \Gamma(1 - aw/\Lambda)^c \, dw = \frac{1}{2\pi i} \int_{V_{-1}} (-w)^{-c} e^{-w} \, dw + \frac{ac/\Lambda}{2\pi i} \int_{V_{-1}} (-w)^{1-c} e^{-w} R(w) \, dw.
\]

For any \( w \in V_{-1}, 0 \leq u \leq 1 \) and for \( \Lambda \geq a \) we have that \( 1 - uaw/\Lambda \) belongs to the closed vertical strip located between the vertical lines \( V_1 \) and \( V_2 \). Because \( \Gamma(z)^c \Psi(z) \) is continuous and bounded in this strip, \( R(w) \) is bounded for \( w \in V_{-1} \) by a constant independent of \( \Lambda \geq a \). Furthermore, \((-w)^{1-c} e^{-w}\) is integrable over \( V_{-1} \) because \( c > 2 \).

On the other hand, in the integral
\[
\frac{1}{2\pi i} \int_{V_{-1}} (-w)^{-c} e^{-w} \, dw
\]

the contour \( V_{-1} \) may be deformed to a Hankel contour
\[
\mathcal{H} := \{ x - i \mid x = \infty \ldots 0 \} \cup \{ e^{i\theta} \mid \theta = -\pi/2 \ldots -3\pi/2 \} \cup \{ x + i \mid x = 0 \ldots \infty \}
\]
surrounding \([0, \infty)\), and the integral over \( \mathcal{H} \) is Hankel’s integral representation of the inverse of the Gamma function:
\[
\frac{1}{2\pi i} \int_{\mathcal{H}} (-w)^{-c} e^{-w} \, dw = \frac{1}{\Gamma(c)}.
\]

Therefore, when we join everything, we obtain that for \( c > 2 \):
\[
e_c(a, b)(t) = \frac{t^{b/a-1}}{[a\Gamma(b)]^c} \frac{[\log(1/t)]^{c-1}}{\Gamma(c)} + O \left( t^{b/a-1} [\log(1/t)]^{c-2} \right), \quad t \to 0.
\]

Case 4. \( c = 1, c = 2 \).

These cases are easy since \( e_1(a, b)(t) \) is explicitly given by (3) and \( e_2(a, b)(t) \) by (15). The asymptotics of \( K_0 \) is known:
\[
K_0(t) = \log(2/t) + O(1), \quad t \to 0.
\]

\(\square\)

Remark 3.2. The behaviour of \( e_c(a, b)(t) \) for \( t \to 0 \) can be obtained from (31) using the residue theorem when \( c \) is a natural number. In fact, in this case \( \Gamma(-z)^c \) has a pole of order \( c \) at \( z = 0 \), and a shift of the contour \( V_{-1} \) to \( V_s \), where \( 0 < s < 1 \), has to be compensated by a residue, which will give the behaviour for \( t \to 0 \).

When \( c \) is a natural number one can actually express \( e_c(a, b)(t) \) in terms of Meijer’s G-function:
\[
e_c(a, b)(t) = \frac{t^{b/a-1}}{a\Gamma(b)^c} G_{0,c}^{c,0} \left( t^{1/a} \mid \begin{array}{cccc}
- & \cdots & - \\
0 & & & 0
\end{array} \right),
\]

cf. Section 9.3 in [3].
4 A one parameter extension of the Gumbel distributions

The group isomorphism \( x = \log(1/t) \) of the multiplicative group \((0, \infty)\) onto the additive group \(\mathbb{R}\) of real numbers transforms the convolution semigroup \((\tau_c(a, b))_{c>0}\) into an ordinary convolution semigroup \((G_c(a, b))_{c>0}\) of probability measures on \(\mathbb{R}\) with densities given by

\[
g_c(a, b)(x) = e^{-x}e_c(a, b)(e^{-x}), \quad x \in \mathbb{R},
\]

and \(a, b, c > 0\) are arbitrary. For \(c = 1\) we have

\[
g_1(a, b)(x) = \frac{1}{a \Gamma(b)} \exp \left( -bx/a - e^{-x/a} \right), \quad x \in \mathbb{R}.
\]

This density is infinitely divisible and the uniquely determined convolution roots are given by \(37\).

The special density \(g_1(a, 1)(x)\) is the Gumbel density with scale parameter \(a > 0\), and the basic case \(a = 1\) is discussed in [7]. From the asymptotic behaviour of \(e_c(a, b)\) in Theorems 2.1 and 2.3 we can obtain the asymptotic behaviour of the convolution roots \(g_c(a, b)\):

\[
g_c(a, b)(x) = \left(2\pi\right)^{(c-1)/2} \frac{\exp \left( -ce^{-x/(ac)} \right)}{a^{2n} \Gamma(b)^{c} \exp \left( x(b - 1/2 + 1/(2c))/a \right)} \left[ 1 + O(\exp(x/(ac))) \right]
\]

for \(x \to -\infty\), and

\[
g_c(a, b)(x) = \frac{\exp(-bx/a)x^{c-1}}{[a \Gamma(b)]^c \Gamma(c)} + O(\exp(-bx/a)x^{c-2}), \quad x \to \infty.
\]

**Theorem 4.1.** All densities \(g_c(a, b)\) belong to determinate Hamburger moment problems.

**Proof.** We first prove that \(g_1(a, b)\) is determinate, and for this it suffices to verify that the moments

\[
s_n = \int_{-\infty}^{\infty} x^n g_1(a, b)(x) \, dx
\]

verify Carleman’s condition \(\sum_{n=0}^{\infty} s_{2n}^{-1/(2n)} = \infty\), cf. [15, p. 19]. From \(41\) we get

\[
s_{2n} = \frac{1}{a \Gamma(b)} \int_0^\infty (\log t)^{2n} t^{b/a-1} \exp(-t^{1/a}) \, dt = \frac{1}{\Gamma(b)} \int_0^\infty (a \log s)^{2n} s^{b-1} e^{-s} \, ds < \frac{a^{2n}}{\Gamma(b)} \left( \int_0^1 (\log s)^{2n} s^{b-1} \, ds + \int_1^\infty s^{2n+b-1} e^{-s} \, ds \right).
\]

By integrations by parts we see that

\[
\int_0^1 (\log s)^{2n} s^{b-1} \, ds = \frac{(2n)!}{b^{2n+1}},
\]
\[
\int_1^\infty s^{2n+b-1}e^{-s} \, ds < \Gamma(2n + b),
\]

hence
\[
s_{2n}^{1/(2n)} < \frac{a}{\Gamma(b)^{1/(2n)}} \left[ \left( \frac{(2n)!}{b^{2n+1}} \right)^{1/(2n)} + \Gamma(2n + b)^{1/2n} \right],
\]

and the Carleman condition follows from Stirling’s formula, which shows that the right-hand side is bounded by \(Kn\) for sufficiently large \(K > 0\). We next use Corollary 3.3 in [2] to infer that the Carleman condition also holds for all convolution roots \(g_c(a, b)\).

Concerning the moments
\[
s_n(c) = \int_{-\infty}^\infty x^n g_c(a, b)(x) \, dx, \quad n \in \mathbb{N}_0 \tag{42}
\]
of the convolution roots we have the following result:

**Theorem 4.2.** The moments \(s_n(c)\) of (42) is a polynomial
\[
s_n(c) = \sum_{k=1}^n a_{n,k} c^k, \quad n \geq 1, \tag{43}
\]
of degree at most \(n\) in the variable \(c\). The coefficients \(a_{n,k}\) are given below.

**Proof.** From (7) we get
\[
\int_{-\infty}^\infty e^{-ixy} dG_c(a, b)(x) = \int_0^\infty t^{iy} e^{c(t)} \, dt = [\Gamma(b + iay)/\Gamma(b)]^c,
\]

which shows that the negative definite function \(\rho\) corresponding to the convolution semigroup \((G_c(a, b))_{c>0}\) is
\[
\rho(y) = \log \Gamma(b) - \log \Gamma(b + iay), \quad y \in \mathbb{R}.
\]
The derivatives of \(\rho\) can be expressed in terms of the Digamma function \(\Psi\), namely
\[
\rho^{(n+1)}(y) = -(ia)^{n+1} \Psi^{(n)}(b + iay), \quad n \in \mathbb{N}_0,
\]
so if \(n \in \mathbb{N}_0\) we define (cf. [2, Eq. (2.7)])
\[
\sigma_n := -i^{n+1} \rho^{(n+1)}(0) = (-a)^{n+1} \Psi^{(n)}(b),
\]
we find
\[
\sigma_0 = a\gamma + \frac{a}{b} - ab\sum_{k=1}^\infty \frac{1}{k(b+k)},
\]
\[
\sigma_n = a^{n+1}n! \sum_{k=0}^\infty \frac{1}{(b+k)^{n+1}} = a^{n+1}n!\zeta(n+1, b), \quad n \in \mathbb{N},
\]
where ζ(z, q) is Hurwitz’ Zeta function, cf. [8, 9.521].

According to [3] we have

\[ s_1(c) = \sigma_0 c, \quad s_2(c) = \sigma_1 c + \sigma_0^2 c^2 \]

and in general \( s_n(c) \) is given by [13] where the coefficients \( a_{n,k} \) are given by the recursion

\[
a_{n+1,k+1} = \sum_{j=k}^{n} a_{j,k} \binom{n}{j} \sigma_{n-j}, \quad n \geq k \geq 0.
\]

It is easy to see that

\[
a_{n,1} = \sigma_{n-1}, \quad a_{n,n-1} = \left(\begin{array}{c} n \\ 2 \end{array}\right) \sigma_0^{n-2} \sigma_1, \quad a_{n,n} = \sigma_0^n.
\]

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C. Berg, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
email: berg@math.ku.dk