Beyond conventional factorization: Non-Hermitian Hamiltonians with radial oscillator spectrum

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Abstract

The eigenvalue problem of the spherically symmetric oscillator Hamiltonian is revisited in the context of canonical raising and lowering operators. The Hamiltonian is then factorized in terms of two not mutually adjoint factorizing operators which, in turn, give rise to a non-Hermitian radial Hamiltonian. The set of eigenvalues of this new Hamiltonian is exactly the same as the energy spectrum of the radial oscillator and the new square-integrable eigenfunctions are complex Darboux-deformations of the associated Laguerre polynomials.

1 Introduction

Factorization is a powerful method to compute eigenvalues and recurrence relations for solutions of differential equations in mathematical physics (see e.g. [1–5]). Introduced in Quantum Mechanics (QM) since Fock, Schrödinger and Dirac times, some of the antecedents of this algebraic method can be found in the geometrical formalism of Darboux and Bäcklund. It is also notable that the (canonical) Fock-Dirac ladder operators have been of big influence in the development of contemporary Physics over the years [6]. The method, as formalized by Infeld and Hull [7], was extended by Mielnik to embrace factorizing operators of the harmonic oscillator Hamiltoninan which do not create nor annihilate occupation numbers in the Fock states [8]. Applications of the Mielnik’s method to radial problems were reported first by Fernández [9] and connections with the Supersymmetric formalism of QM were found by Nieto [10] and Sukumar [11]. The relationship between factorizing and Darboux-deforming potentials in QM was extensively studied by Andrianov and coworkers [12]. Nowadays, the factorization method is of the highest utility in constructing new exactly solvable potentials in QM (see e.g. [13, 14] and references quoted therein). In particular, recent results show that the method can be used in solving the problem of complex potentials with real spectrum [15–20] as well as in the study of resonances and Gamow-Siegert states [21–23].

In the one-dimensional case there is a unique expression for the canonical ladder operators since the energy levels are not degenerated. For instance, the Fock-Dirac operators transform an eigenfunction of the harmonic oscillator into another one and factorize, at the same time, the corresponding Hamiltonian. However, in a general situation, the pair of operators which factorize a given Hamiltonian do not necessarily correspond to the shift operators. Distorted versions of the harmonic oscillator, for example, require higher
order raising and lowering operators although the Hamiltonian is factorized by first order differential operators [8,24]. On the other hand, it is usual to find degenerate energy levels in higher dimensions and several kinds of shift operators are necessary to intertwine the wave-functions [2]. A particular case is of special interest: spherical symmetry in central potentials induces energy degeneracies under rotations (different orientations of the angular momentum vector lead to the same energy). Hence, the ladder operators are labelled by the azimuthal quantum number $\ell$ and their action mainly affects the $\ell$-dependence of the solutions (see for instance [7,9,25–27]). Thus, they intertwine solutions of potentials which differ in one unit of the angular momentum.

In this work we are interested in the three dimensional isotropic oscillator. The Hamiltonian of this system can be factorized in four different forms by means of a basic set of canonical ladder operators [28]. Each of these products preserves the adjointness of the Hamiltonian since the factors are mutually adjoint. In contrast, we are going to express the radial oscillator Hamiltonian as the product of two not mutually adjoint factorizing operators. The advantage of these new operators is that they give rise to non-Hermitian radial Hamiltonians for which the point spectrum is exactly the same as the spherically symmetric oscillator one.

In Section 2 conventional factorizations of the isotropic oscillator are revisited. Explicitly derived, the four canonical ladder operators are used to construct the basis of physical wave-functions (orthonormal, associated Laguerre polynomials) and the energy spectrum. For a given $\ell$, the creation or annihilation of nodes in the wave-functions require second order differential operators which do not factorize the Hamiltonian. In Section 3 a pair of not mutually adjoint factorizing operators is introduced to extend the number of factorizations in which the radial oscillator Hamiltonian can be expressed. These same operators are used to construct a new Hamiltonian which is not self adjoint under the inner product of the Laguerre–Hilbert Space $\mathcal{H}$. The relevant aspect of our results is that the eigenfunctions of the new Hamiltonian are square-integrable in $\mathcal{H}$. It is also introduced a pair of ladder operators which increase or decrease the energy of this new system. As we shall see, these shift operators are necessarily fourth order differential operators. Finally, the paper is closed with some concluding remarks in Section 4.

2 Radial Oscillator Revisited

The time-independent Schrödinger equation for the isotropic oscillator $V(r) = r^2$, in appropriate units and after separation of variables, reduces to

$$H_\ell \phi(r, \ell) = E(\ell) \phi(r, \ell)$$

where the azimuthal quantum number $\ell$ is a non-negative integer, $E(\ell)$ is twice the dimensionless energy eigenvalue and the radial Hamiltonian reads:

$$H_\ell \equiv -\frac{d^2}{dr^2} + V_\ell(r) = -\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + r^2.$$
The effective potential \( V(r) \) has the domain \( D_V = [0, \infty) \) and the functions \( \phi(r, \ell) \) are defined in terms of the usual radial wave-function \( \phi(r, \ell) = rR(r, \ell) \). To be physically interpretable, these last functions have to satisfy \( \int_0^{\infty} |R(r, \ell)|^2 r^2 dr < \infty \). In the sequel, whenever there is no confusion we shall use the shortcut notation \( f \equiv f(r, \ell, \varepsilon, \ldots) \), keeping implicit the dependence of \( f \) on \( r, \ell, \varepsilon \), and other possible variables and parameters.

Conventional factorization is useful in solving equation (1) by expressing (2) as the product of two mutually adjoint first order differential operators

\[
H_\ell = a_\ell^\dagger a_\ell + \varepsilon_\ell; \quad a_\ell^\dagger := -\frac{d}{dr} + \alpha(r, \ell); \quad (a_\ell^\dagger)^\dagger = a_\ell
\]  

with \( \varepsilon_\ell \) a real (factorization) constant to be fixed and \( \alpha(r, \ell) \) a function satisfying the Riccati equation

\[
-\alpha'^{2} + \alpha^{2} = V_\ell - \varepsilon_\ell.
\]  

A particular solution of equation (4) is immediate:

\[
\alpha = r - \frac{\ell + 1}{r}, \quad \varepsilon_\ell = 2\ell + 3.
\]  

This superpotential \( \alpha \) leads to one of the canonical forms of the factorizing operators:

\[
a_\ell = \frac{d}{dr} - \frac{\ell + 1}{r} + r, \quad a_\ell^\dagger = -\frac{d}{dr} - \frac{\ell + 1}{r} + r.
\]  

The first advantage of the method is clear by noticing that a solution of \( a_\ell \phi_0(r, \ell) = 0 \) gives a solution of (1) belonging to the eigenvalue \( E_0(\ell) := E_0^{(\ell)} = \varepsilon_\ell \). The calculation leads to

\[
\phi_0(r, \ell) := \phi_0^{(\ell)}(r) = C_0^{(\ell)} r^{\ell+1} e^{-r^2/2}
\]  

which is free of nodes and becomes zero at the edges of \( D_V \). Hence, \( C_0^{(\ell)} \) can be taken as the normalization constant and, by applying the Sturm oscillation theorem [29], we know that \( E_0^{(\ell)} = 2\ell + 3 \) is the ground state energy of \( H_\ell \). Now, it is useful to reverse the product (3) to get

\[
H_{\ell+1} = a_\ell a_\ell^\dagger + \varepsilon_\ell - 2 := a_\ell a_\ell^\dagger + \theta_\ell.
\]  

Notice that the azimuthal quantum number \( \ell \) plays the role of a parameter in all the previous expressions. Hence, by changing \( \ell \to \ell - 1 \) and \( \ell \to \ell + 1 \) in (8) and (3) respectively we also obtain

\[
H_\ell = a_{\ell-1} a_{\ell-1}^\dagger + \theta_{\ell-1}, \quad H_{\ell+1} = a_{\ell+1}^\dagger a_{\ell+1} + \varepsilon_{\ell+1}.
\]  

As a consequence, the following intertwining relationships are true

\[
a_\ell (H_\ell - 2) = H_{\ell+1} a_\ell, \quad a_{\ell-1}^\dagger (H_\ell + 2) = H_{\ell-1} a_{\ell-1}^\dagger.
\]  

Thereby, if \( \phi(r, \ell) \) is eigenfunction of \( H_\ell \) with eigenvalue \( E(\ell) \), then \( \phi(r, \ell + 1) \propto a_\ell \phi(r, \ell) \) and \( \phi(r, \ell - 1) \propto a_{\ell-1}^\dagger \phi(r, \ell) \) respectively satisfy the eigenvalue equation of \( H_{\ell+1} \) and \( H_{\ell-1} \) with eigenvalue \( E(\ell + 1) = E(\ell) - 2 \) and \( E(\ell - 1) = E(\ell) + 2 \).
Let us take full advantage of these last results. After applying the R.H.S. relationship (11) on the ground state \( \phi_0^{(\ell)} \), we arrive at the function

\[
\phi_1^{(\ell-1)} \propto a_{\ell-1}^+ \phi_0^{(\ell)} = (-2r^2 + 2\ell + 1) \phi_0^{(\ell-1)}
\]

which has a single node at \( r = \sqrt{(2\ell + 1)/2} \in D_V \) and satisfies the Schrödinger equation

\[
H_{\ell-1} \phi_1^{(\ell-1)} = (E_0^{(\ell)} + 2) \phi_1^{(\ell-1)}.
\]

Function (11) can be rewritten as

\[
\phi_1^{(\ell-1)} \propto \phi_0^{(\ell-1)} \left\{ r^{-2(\ell-1)/2} e^{-r^2} \left( \frac{1}{2r} \frac{d}{dr} \right) r^{2(\ell+1)/2} e^{-r^2} \right\}.
\]

After an intermediary change of independent variable \( r^2 = x \), one arrives at

\[
\phi_1^{(\ell-1)}(r) = C_1^{(\ell-1)} \phi_0^{(\ell-1)}(r) L_1^{(\ell-1)+1/2}(r^2)
\]

where \( C_1^{(\ell-1)} \) is a normalization constant and

\[
L_\nu^n(x) = \frac{1}{n!} x^{-\nu} e^x \frac{d^n}{dx^n} (e^{-x} x^{n+\nu}), \quad \nu > -1
\]

is the associated Laguerre polynomial of degree \( n \). A similar procedure leads to

\[
\phi_2^{(\ell-2)}(r) = C_2^{(\ell-2)} \phi_0^{(\ell-2)}(r) L_2^{(\ell-2)+1/2}(r^2) \propto a_{\ell-2}^+ \phi_1^{(\ell-1)}(r)
\]

which is the eigenfunction of \( H_{\ell-2} \) with eigenvalue \( E_2^{(\ell-2)} = E_1^{(\ell-1)} + 2 = E_0^{(\ell)} + 4 \). The procedure can be applied \( s \) times to get the eigenfunction of \( H_{\ell-s} \) with eigenvalue \( E_s^{(\ell-s)} = E_0^{(\ell)} + 2s \):

\[
\phi_s^{(\ell-s)} = C_s^{(\ell-s)} \left( \prod_{k=0}^{s-1} a_{\ell-s+k}^+ \right) \phi_0^{(\ell)} := C_s^{(\ell-s)} \left( a_{\ell-s}^+ \cdots a_{\ell-2}^+ a_{\ell-1}^+ \right) \phi_0^{(\ell)}, \quad s = 1, 2, \ldots
\]

Changing \( \ell \to \ell + s \) and using \( L_\nu^n \equiv 1 \), this last expression leads to

\[
\phi_s^{(\ell)}(r) = C_s^{(\ell)} \phi_0^{(\ell)}(r) L_s^{(\ell)+1/2}(r^2), \quad s = 0, 1, \ldots
\]

Function (18) is the physical solution of (11), with eigenvalue \( E_s^{(\ell)} = E_0^{(\ell+s)} + 2s = \varepsilon_{\ell+s}^{(\ell+1)} + 2s = 2(2s + \ell) + 3 \). The presence of nodes in \( \phi_s^{(\ell)} \) is due to the zeros of the polynomial \( L_s^{(\ell)+1/2} \), the number of which is equal to \( s \). By taking \( n = 2s + \ell = 1, 0, \ldots \), we can use the conventional notation \( \phi_s^{(\ell)}(r) \equiv \phi_n(r) \) and \( E_s^{(\ell)} \equiv E_n = 2n + 3 \).

Remark that the action of \( a_\ell \) on an arbitrary function (18), different from \( \phi_0^{(\ell)} \), produces \( a_\ell \phi_s^{(\ell)} = \phi_{s+1}^{(\ell+1)} \). Thus \( a_\ell \) annihilates one node of the wave-function at the cost of increasing in one unit the azimuthal quantum number \( \ell \). This can be represented in the Cartesian \( (s \times \ell) \)-plane as the diagonal displacement \( (s, \ell) \to (s-1, \ell+1) \); a schematic picture of which is shown in Figure (1). The action of \( a_{\ell-1}^+ \) reverses the previous result; this
Figure 1: Lattice representation in the \((s \times \ell)\)–plane for the action of the four canonical factorization operators \(a_\ell, a_\ell^\dagger, b_\ell, b_\ell^\dagger\), of the radial oscillator.

is represented by \((s, \ell) \rightarrow (s + 1, \ell - 1)\). Notice that a particular situation occurs for \(\ell = 0\): The R.H.S. relationship (10) prohibits the action of \(a_0^\dagger\) on any eigenfunction \(\phi_s^{(0)}\) of \(H_0\) because it produces the forbidden value \(\ell = -1\). However, function (18) can be also derived by applying the appropriate product of annihilation operators \(\{a_\ell\}\) on a convenient excited state of \(H_0\). The straightforward calculation leads to

\[
\phi_s^{(\ell)} = C_s^{(\ell)} \left( \prod_{k=0}^{\ell-1} a_{\ell-1-k} \right) \phi_s^{(0)}, \quad \ell \neq 0, \quad s = 0, 1, \ldots
\]  

(19)

Another canonical factorization, different from (3) and (9), is still possible for \(H_\ell\):

\[
H_\ell = b_\ell b_\ell^\dagger + \kappa_\ell, \quad b_\ell^\dagger := -\frac{d}{dr} + \gamma(r, \ell), \quad (b_\ell^\dagger)^\dagger = b_\ell
\]  

(20)

where the factorization constant \(\kappa_\ell\) is to be fixed and the superpotential \(\gamma(r, \ell)\) fulfills

\[
\gamma' + \gamma^2 = V_\ell - \kappa_\ell.
\]  

(21)

The immediate solution of this last equation reads

\[
\gamma = r + \ell + 1 - \frac{1}{r}, \quad \kappa_\ell = -\varepsilon_\ell = -(2\ell + 3).
\]  

(22)

Hence, the reverse product (20) is given by

\[
H_{\ell+1} = b_\ell^\dagger b_\ell - \theta_\ell
\]  

(23)

and the new intertwining relationships are

\[
b_{\ell-1} (H_\ell - 2) = H_{\ell-1} b_{\ell-1}, \quad b_\ell^\dagger (H_\ell + 2) = H_{\ell+1} b_\ell^\dagger.
\]  

(24)
The action of $b^\dagger_\ell$ on the eigenfunction $\phi_s^{(\ell)}$ of $H_\ell$ with eigenvalue $E_s^{(\ell)}$ produces a square-integrable eigen-solution of $H_{\ell+1}$ belonging to the eigenvalue $E_s^{(\ell+1)} = 2(2s + \ell + 1) + 3 \equiv E_s^{(\ell+1)}$. Notice that the number of nodes is preserved; we write $\phi_s^{(\ell+1)} = C_s^{(\ell+1)} b^\dagger_\ell \phi_s^{(\ell)}$. Such an operation is represented in Figure 1 as the horizontal displacement $(s, \ell) \rightarrow (s, \ell + 1)$. On the other hand, the function $\phi_s^{(\ell-1)} = C_s^{(\ell-1)} b_\ell \phi_s^{(\ell)}$ is eigenfunction of $H_{\ell-1}$ with eigenvalue $E_s^{(\ell-1)} = E_s^{(\ell)} - 2$. Thus, the action of $b_{\ell-1}$ is represented by the displacement $(s, \ell) \rightarrow (s, \ell - 1)$ in the $(s \times \ell)$–plane, as it is also shown in Figure 1.

Expressions (3), (9), (20) and (23) recover the canonical factorizations of $H_\ell$ reported by Fernández, Negro and del Olmo in 1996 [28] (see also [30, 31] and references quoted therein). As we have shown, the four canonical ladder operators $a_\ell$, $a^\dagger_\ell$, $b_\ell$, $b^\dagger_\ell$, induce displacements in the $(s \times \ell)$–plane by either creating (annihilating) a node in the wave-function at the cost of decreasing (increasing) the azimuthal quantum number in one unit or by preserving the number of nodes but changing the value of $\ell$ in one unit. Although all these transformations involve a horizontal component in the displacements of the point $(s, \ell)$, purely vertical displacements are also possible. Indeed, we have a pair of mutually adjoint second order differential operators defined as follows:

$$(S^\dagger_\ell)^\dagger = S_\ell := a_{\ell-1} b_{\ell-1} = b_\ell a_\ell$$
$$= -H_\ell + 2r \frac{d}{dr} + 2r^2 + 1. \quad (25)$$

These operators increase or decrease in one unit the number of nodes in the wave-functions

$$S^\dagger_\ell \phi_s^{(\ell)} \propto \phi_s^{(\ell+1)}, \quad S_\ell \phi_s^{(\ell)} \propto \phi_s^{(\ell-1)}, \quad (26)$$

and intertwine eigenfunctions of $H_\ell$ for which the energy differs in four units

$$S_\ell H_\ell = (H_\ell + 4)S_\ell, \quad S^\dagger_\ell H_\ell = (H_\ell - 4)S^\dagger_\ell. \quad (27)$$

Hence, we get the following commutation rules

$$[S_\ell, H_\ell] = 4S_\ell, \quad [S^\dagger_\ell, H_\ell] = -4S^\dagger_\ell. \quad (28)$$

It is clear that the Hamiltonian $H_\ell$ is not factorized by $S_\ell$ and $S^\dagger_\ell$:

$$S_\ell S^\dagger_\ell = (H_\ell - \varepsilon_\ell + 4)(H_\ell + \varepsilon_\ell), \quad S^\dagger_\ell S_\ell = (H_\ell + \varepsilon_\ell - 4)(H_\ell - \varepsilon_\ell). \quad (29)$$

However, there is a simple commutation rule

$$[S_\ell, S^\dagger_\ell] = 8H_\ell. \quad (30)$$

Figure 2 shows the schematic representation of two arbitrary but contiguous energy levels of $H_\ell$; the corresponding points of the $(s \times \ell)$–plane are also depicted. As indicated, the operator $S^\dagger_\ell$ scales up the energy to the closer excited one and $S_\ell$ operates in the opposite direction.
Figure 2: The ladder operators $S_{\ell}^+$ and $S_{\ell}$ respectively create and annihilate nodes in the eigenfunctions of $H_\ell$. Notice that displacements in the $(s \times \ell)$--plane are purely vertical, thus the value of the azimuthal quantum number $\ell$ is fixed.

3 Complex Factorization

In this section we are going to go a step further in the factorization of Hamiltonian (2). Let us analyze the product

$$H_\ell = A_\ell B_\ell + \epsilon_\ell$$

(31)

where $\epsilon_\ell$ is now a complex factorization constant with non-trivial imaginary part (from now on $\Re(z)$ and $\Im(z)$ respectively stand for the real and imaginary parts of $z \in \mathbb{C}$ while $\overline{z}$ corresponds to its complex conjugate). The new factorizing operators are established in the context of a ‘refined factorization’ [32] to read

$$A_\ell = -\frac{d}{dr} + \beta(r, \ell), \quad B_\ell = \frac{d}{dr} + \beta(r, \ell)$$

(32)

with $\beta$ a complex function satisfying the Riccati equation

$$-\beta' + \beta^2 = V_\ell - \epsilon_\ell.$$ 

(33)

The first aspect which distinguishes (31) from (3) is that $A_\ell$ and $B_\ell$ are not mutually adjoint in the Hilbert space $\mathcal{H} = L^2(D_V)$ spanned by the vectors (14). However, as $H_\ell$ is selfadjoint, $H_\ell = B_\ell^+ A_\ell^+ + \tau_\ell$ must be true. The straightforward calculation shows that this last expression leads to the complex conjugation of equation (33), the solution of which is immediate since $V_\ell(r)$ is real.

As in the previous section, the factors in (31) lead to solutions of the involved eigenvalue problem. In particular, the function $u(r, \ell)$, annihilated by $B_\ell$, is also solution of the Schrödinger equation

$$H_\ell u = \epsilon_\ell u.$$ 

(34)

Since $\epsilon_\ell$ is complex and $H_\ell$ is an Hermitian operator, it is clear that $u$ is not square-integrable in $D_V$. However, this function is very useful because the non-linear Riccati equation (33) can be mapped into (34) by means of the logarithmic derivative.
\( \beta = -\frac{4}{\ell} \ln u. \) The appropriate transformations allow to express the solutions of (34) in terms of confluent hypergeometric functions \( _1F_1(a,c;z) \) (see e.g. [3, 4]). In particular, we shall use

\[
u(r, \ell) = c_\ell r^{\ell+1} e^{-r^2/2} _1F_1 \left( \frac{\ell}{2} + \frac{3}{4}, \frac{\ell}{4} - \frac{1}{2}, \frac{3}{2} ; r^2 \right)
\]

with \( c_\ell \) an arbitrary integration constant which will be fixed as 1. Hence, we have

\[
\beta(r, \ell) = -\frac{1}{r} + 1 - 2r \left( \frac{2\ell + 3 - \epsilon_\ell}{4\ell + 6} \right) \frac{_1F_1 \left( \frac{\ell}{2} + \frac{7}{4} - \frac{\epsilon_\ell}{4} - \frac{1}{2} ; \frac{5}{2} ; r^2 \right)}{_1F_1 \left( \frac{\ell}{2} + \frac{3}{4} - \frac{\epsilon_\ell}{4} , \frac{3}{2} ; r^2 \right)}.
\]

The well known analytical properties of the \( _1F_1 \) functions lead to

\[
\beta \approx \begin{cases} 
-\frac{\ell+1}{r} & r \to 0 \\
-\ell & r \to +\infty
\end{cases}
\]

(37)

The reverse product (31) produces a new second order differential operator

\[
B_\ell A_\ell + \epsilon_\ell = H_\ell + 2\beta'
\]

(38)

which is a non-Hermitian Hamiltonian in \( \mathcal{H} \) since \( \beta \) is complex. The new potential

\[
v(r, \ell, \epsilon_\ell) := V_\ell(r) + 2\beta'(r, \ell)
\]

(39)

is mainly a real function at the edges of \( D_V \):

\[
\begin{align*}
\left\{ 
\begin{array}{l}
v(r \to 0, \ell, \epsilon_\ell) = V_{\ell+1}(r \to 0) + i \frac{43(\epsilon_\ell)}{4\epsilon+6} \\
v(r \to \infty, \ell, \epsilon_\ell) = V_\ell(r \to \infty) - 2
\end{array}
\right.
\end{align*}
\]

(40)

Figure 3: Real (left) and imaginary (right) parts of the complex potential (39) obtained from \( V_0(r) \) with a factorization constant \( \epsilon_0 = 3 + i 10^{-3} \) (blue curve), \( \epsilon_0 = 7 + i \frac{5}{2} \) (red curve) and \( \epsilon_0 = 11 + i 5 \) (black curve). Observe that greater values of \( \Re(\epsilon_0) \) induce larger displacements to the left in the deformation

Figure 3 shows the global aspect of the new potentials \( v(r, \ell, \epsilon_\ell) \) for \( \ell = 0 \) and several values of \( \epsilon_0 \). Notice that the Darboux-deformation (39) of the initial potential \( V_\ell \) is
characterized by a zone \( M \subset D_V \) surrounding the origin. Namely, \( M = [0, r_0) \), with \( r_0 \) a finite number depending on \( \ell \) and \( \epsilon_\ell \). In this deformation zone \( M \), the function \( \Re(v) \) is merely a distortion of \( V_{\ell+1} \) while \( \Im(v) \) is a nontrivial position-dependent function. For the sake of notation, we shall write \( v_{\ell+1} \) for the complex potential defined in (39). Thereby, the non-Hermitian Hamiltonian (38) reads

\[
 h_{\ell+1} = B_\ell A_\ell + \epsilon_\ell = -\frac{d^2}{dr^2} + v_{\ell+1}. \tag{41}
\]

Once we have obtained the factorizations (31) and (41), it is immediate to arrive at the following intertwining relationships:

\[
 B_\ell H_\ell = h_{\ell+1} B_\ell, \quad A_\ell h_{\ell+1} = H_\ell A_\ell. \tag{42}
\]

Thus, the action of \( B_\ell \) on \( \phi_s^{(\ell)} \) gives a function \( \psi_s^{(\ell+1)}(r, \epsilon_\ell) \propto B_\ell \phi_s^{(\ell)}(r) \), which satisfies the Schrödinger equation

\[
 h_{\ell+1} \psi_s^{(\ell+1)} = E_s^{(\ell)} \psi_s^{(\ell+1)}. \tag{43}
\]

![Figure 4](image)

Figure 4: Left: The second excited eigenstate \( \phi_2^{(0)}(r) \) of \( H_0 \) (black curve) and the real (blue curve) and imaginary (red curve) parts of the corresponding Darboux-deformation \( \psi_2^{(1)}(r, \epsilon_0) \) for \( \epsilon_0 = 11 + i 5 \). Right: The Born probability densities of \( \phi_2^{(0)}(r) \) (black curve) and \( \psi_2^{(1)}(r, \epsilon_0) \).

The straightforward calculation shows that this new function \( \psi_s^{(\ell+1)} \) is square-integrable in \( D_V \). Figure 4 depicts the initial eigenfunction \( \phi_s^{(\ell)} \) corresponding to \( \ell = 0, s = 2 \), as well as its Darboux-deformation \( \psi_2^{(1)}(r, \epsilon_0 = 11 + i 5) \). The function \( \phi_2^{(0)} \) corresponds to the second excited state of \( H_0 \), accordingly it shows two nodes in \( D_V \). However, although \( \psi_2^{(1)} \) corresponds to the second excited energy \( E_2^{(0)} = 11 \) of \( h_1 \), it is notable that this function is free of nodes. Thereby, the usual correspondence between the number of nodes and the level of excitation of a given energy eigenstate is missing in the new functions. Despite this fact, the normalized function \( |\psi_s^{(\ell+1)}|^2 \) can be put in correspondence with the Born’s probability density.

In the previous section we realize that the action of the canonical factorizing operators \( a_\ell \) and \( a_\ell^\dagger \) on a given physical state \( \phi_s^{(\ell)} \) respectively reduces and increases in one unit the number of nodes (\( b_\ell \) and \( b_\ell^\dagger \), on the other hand, do not change the parameter \( s \)).
contrast, nodes disappear in the complex Darboux-deformations of physical states, as established by the set of equations (32), (36) and (22). In this way, the parameter \( s \) in \( \psi_s^{(\ell+1)} \) is nothing but the heritage of the number of nodes of the initial wave-function \( \phi_s^{(\ell)} \). Another important consequence of the absence of this kind of zeros is that the new set of square-integrable functions \( \{ \psi_s^{(\ell+1)} \} \) is not orthogonal. However, it is also possible to construct ladder operators which increase or decrease the energy of this system.

Figure 5: Schematic representation of the intertwining operators \( A_\ell \) and \( B_\ell \), as they are applied on the energy levels of \( H_\ell \) and \( h_{\ell+1} \). The construction of the creation and annihilation operators of \( h_{\ell+1} \) is respectively depicted as the composites \( M_{\ell+1} = B_\ell S_\ell A_\ell \) and \( N_{\ell+1} = B_\ell S_\ell A_\ell \).

In analogy with (27), let us analyze the intertwining relationships

\[
N_{\ell+1} (h_{\ell+1} - 4) = h_{\ell+1} N_{\ell+1}, \quad M_{\ell+1} (h_{\ell+1} + 4) = h_{\ell+1} M_{\ell+1}. \tag{44}
\]

The above equations give place to the commutation rules

\[
[N_{\ell+1}, h_{\ell+1}] = 4N_{\ell+1}, \quad [M_{\ell+1}, h_{\ell+1}] = -4M_{\ell+1}. \tag{45}
\]

which are the same as those fulfilled by \( H_\ell \), \( S_\ell^\dagger \) and \( S_\ell \) in (28). Thus, \( M_{\ell+1}^\dagger \) and \( N_{\ell+1} \) should respectively play the role of the raising and lowering operators for the eigenvalues of \( h_{\ell+1} \). From (14), it is clear that \( N_{\ell+1} \psi_s^{(\ell+1)} \) is eigenfunction of \( h_{\ell+1} \) with eigenvalue \( E_s^{(\ell)} - 4 = E_s^{(\ell)} \). Thus, \( N_{\ell+1} \) lowers the energy of the system in four units by annihilating one unit in the parameter \( s \). Hence, we write \( \psi_s^{(\ell+1)} \propto N_{\ell+1} \psi_s^{(\ell+1)} \). In a similar way we get \( \psi_s^{(\ell+1)} \propto M_{\ell+1}^\dagger \psi_s^{(\ell+1)} \). Let us remember that \( S_\ell \) and \( S_\ell^\dagger \) were constructed in terms of vertical displacements of the points in the \((s \times \ell)\)-plane. In the present case, we have to solve the algebraic equations (14–15) and Figure 5 is of special utility. The simplest solution is given by the following products

\[
M_{\ell+1}^\dagger = B_\ell S_\ell^\dagger A_\ell, \quad N_{\ell+1} = B_\ell S_\ell A_\ell, \quad (M_{\ell+1}^\dagger)^\dagger \neq N_{\ell+1}, \tag{46}
\]

which can be verified by algebraic operations. Since \( M_{\ell+1}^\dagger \) and \( N_{\ell+1} \) are fourth order differential operators, they do not factorize the Hamiltonian \( h_\ell \). The corresponding products
are fourth degree polynomials factorized as follows

\[
M_{\ell+1}^I N_{\ell+1} = (h_{\ell+1} - \epsilon - 4)(h_{\ell+1} + 2\ell - 1)(h_{\ell+1} - 2\ell - 3)(h_{\ell+1} - \epsilon) \\
N_{\ell+1} M_{\ell+1}^I = (h_{\ell+1} - \epsilon + 4)(h_{\ell+1} - 2\ell + 1)(h_{\ell+1} + 2\ell + 3)(h_{\ell+1} - \epsilon)
\]

Finally, it is remarkable that, in contrast with \(S^I_\ell\) and \(S_\ell\), the ladder operators \(M_{\ell+1}^I\) and \(N_{\ell+1}\) are not mutually adjoint (it is a heritage of the factorizing operators \(A_\ell\) and \(B_\ell\)). A similar situation gave place to the Distorted Heisenberg-Weyl Algebra reported in [24]. Research in this direction is in progress.

4 Concluding Remarks

Conventional factorization is useful to solve the eigenvalue problem of the spherical symmetric harmonic oscillator. The corresponding Hilbert space \(\mathcal{H} = L^2(D_V)\) is spanned by the physical solutions which, in turn, are written in terms of orthonormal associated Laguerre polynomials. A new kind of factorizing operators has been introduced to express the radial oscillator Hamiltonian \(H_\ell\) as the product of two not mutually adjoint operators plus a complex constant. The reversed factorization produces a new Hamiltonian which is not Hermitian under the inner product of the associated Laguerre polynomials. However, the corresponding eigenvalue equation has been solved for the radial oscillator energy spectrum. Although the new solutions are not orthogonal, they are square-integrable in \(\mathcal{H}\). Hence, the wave-functions so obtained can be put in correspondence with the Born probability density.

The complex potentials of the non-Hermitian Hamiltonians reported here are such that their imaginary part is not a simple constant but a nontrivial function of the position. There is a well defined subset of \(D_V = [0, \infty)\) in which this ‘complex deformation’ of the radial oscillator potential is relevant. A similar situation was reported for complex Coulomb-like [20, 21] and complex linear oscillator-like [19] interactions. Recent results show that, in a ‘free particle background’, this kind of potentials behaves as an optical device which both refracts and absorbs light waves [23]. Then, it is interesting to analyze the situation in a different background, namely in the radial oscillator one. On the other hand, in Section 3, it has been shown that the wave-functions of these complex potentials do not have a clear correspondence between the number of nodes and the level of energy excitation. However, our current studies show that this correspondence should be established by means of the real and imaginary parts of the new functions [33] (see also [34]). The improvements of these last results are going to be reported elsewhere.

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