From Extreme Values of I.I.D. Random Fields to Extreme Eigenvalues of Finite-volume Anderson Hamiltonian

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The aim of this paper is to study asymptotic geometric properties almost surely or/and in probability of extreme order statistics of an i.i.d. random field (potential) indexed by sites of multidimensional lattice cube, the volume of which unboundedly increases. We discuss the following topics: (I) high level exceedances, in particular, clustering of exceedances; (II) decay rate of spacings in comparison with increasing rate of extreme order statistics; (III) minimum of spacings of successive order statistics; (IV) asymptotic behavior of values neighboring to extremes and so on. The conditions of the results are formulated in terms of regular variation (RV) of the cumulative hazard function and its inverse. A relationship between RV classes of the present paper as well as their links to the well-known RV classes (including domains of attraction of max-stable distributions) are discussed.

The asymptotic behavior of functionals (I)–(IV) determines the asymptotic structure of the top eigenvalues and the corresponding eigenfunctions of the large-volume discrete Schrödinger operators with an i.i.d. potential (Anderson Hamiltonian). Thus, another aim of the present paper is to review and comment a recent progress on extreme value theory for eigenvalues of random Schrödinger operators as well as to provide a clear and rigorous understanding of the relationship between the top eigenvalues and extreme values of i.i.d. potentials.

KEY WORDS: extreme value theory; extreme order statistics; high-level exceedances; spacings; regular variation; Weibull distribution; discrete Schrödinger operator; random potential; largest eigenvalues.

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1. INTRODUCTION

1.1. Extremes of i.i.d. random fields

In this paper, we assume that $\xi(x)$, $x \in \mathbb{Z}^\nu$, are independent identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by sites of the $\nu$-dimensional integer lattice $\mathbb{Z}^\nu$, with a distribution function $\mathbb{P}(\xi(0) \leq t) =: 1 - e^{-Q(t)}$, $t \in \mathbb{R}$; here $Q$ denotes the cumulative hazard function of distribution. Define $V = [-n; n]^{-\nu} \cap \mathbb{Z}^\nu$, the cubes in $\mathbb{Z}^\nu$. Let $|V|$ denote the number of sites in $V$. We write $|x| = \sum_{i=1}^{|V|} |x^i|$ for the lattice $l^1$-distance between $x = (x^1, \ldots, x^\nu) \in \mathbb{Z}^\nu$ and $0 \in \mathbb{Z}^\nu$.

We consider the variational series (order statistics)

$$\xi_{1, V} := \xi(z_{1, V}) \geq \xi_{2, V} := \xi(z_{2, V}) \geq \cdots \geq \xi_{|V|, V} := \xi(z_{|V|, V}) \quad (1.1)$$

based on the sample $\xi_V := \{\xi(x): x \in V\}$; here $V = \{z_{k, V}: 1 \leq k \leq |V|\}$. The first $|V|^\varepsilon$ ($0 < \varepsilon < 1$) terms of the variational series (1.1) are referred to as $\xi_V$-extremes or $\xi_V$-peaks. The coordinate $z_{k, V} \in V$ stands for a location of the $k$th extreme value of $\xi_V$; $1 \leq k \leq |V|$.

In this paper, letting $|V| \to \infty$, we study the asymptotic geometric properties of $\xi_V$-extremes almost surely and/or in probability. We are interested in the following functional of order statistics (1.1):

- (E) Exceedances of the sample $\xi_V$ over high levels $L_V$, in particular, clustering of exceedances (Theorem 3.1).
- (SP) The decay rate of the spacings $\xi_{K, V} - \xi_{K + 1, V}$ and $\xi_{|V|^\varepsilon, V} - \xi_{|V|^\varepsilon, V}$ in comparison with increasing rate of $\xi_{K, V}$ for fixed natural $K \in \mathbb{N}$ and $0 \leq \varepsilon < \theta < 1$ (Theorems 4.3–4.7).
- (MIN) Minimum of the spacings $\xi_{l, V} - \xi_{l + 1, V}$, $1 \leq l \leq |V|^\varepsilon$, for each $0 < \varepsilon < 1$ (Theorems 4.8 and 4.9).
- (N) $\xi_V$-values neighboring to $\xi_V$-extremes, in particular, $\xi(z_{l, V} + y)$ for $1 \leq l \leq |V|^\varepsilon$ and for fixed $y \neq 0$ (Lemma 5.1 and Theorems 5.3, 5.4).

The conditions of the asymptotic results for (E), (SP), (MIN) and (N) are given in terms of regular variation (RV) of the inverse function of $Q$. In Appendix A, we discuss a relationship between RV classes of the present paper as well as their links to the the well-known RV classes including domains of attraction of max-stable distributions.

1.2. Relations to extreme value theory for eigenvalues

Let us consider the finite-volume Schrödinger operators $\mathcal{H}_V = \kappa \Delta_V + \xi_V$ on $l^2(V)$ with periodic boundary conditions (Anderson Hamiltonian); here $\kappa > 0$ is a diffusion constant; $\Delta \psi(x) := \sum_{|y-x|=1} \psi(y)$ the lattice Laplacian and $\psi := \{\psi(x): x \in V\}$ is the multiplication operator. Another aim of this paper is to show in what manner the asymptotic behavior of functionals (E), (SP), (MIN) and (N) determines the asymptotic structure of the top eigenvalues $\lambda_{K, V}$ ($K \geq 1$ fixed) and the corresponding eigenfunctions of the operators $\mathcal{H}_V$ as $V \uparrow \mathbb{Z}^\nu$. In Section 2, we give an
overview of rigorous statements on this relationship which are proved in the companion papers by Gärtner and Molchanov (1998) and Astrauskas (2007; 2008; 2012). In Section 6, we review results on the asymptotic expansion formulas and Poisson limit theorems for the largest eigenvalues $\lambda_{K,V}$. These issues are studied by Astrauskas and Molchanov (1992), Gärtner and Molchanov (1998), Astrauskas (2007; 2008; 2012; 2013), Germinet and Klopp (2013), Biskup and König (2013) and other mathematicians. These papers are complemented by the present survey on the asymptotic geometric properties of $ξ_V$-extremes.

To illustrate the relationship between $ξ_V$-extremes and the largest eigenvalues $λ_{K,V}$ ($V ↑ Z^μ$), we now formulate Propositions 1.1–1.3 which are typical examples of the statements given in Sections 2–6. The first proposition shows that strongly pronounced asymptotic behavior of extremes of deterministic (nonrandom) functions $ξ_V(·) = ξ_V(V)$ ensures simple asymptotic formulas for the largest eigenvalues $λ_{K,V}$.

**Proposition 1.1** (see Theorem 2.2(ii) in Section 2.2). Fix constants $K \in \mathbb{N}$ and $0 < θ < 1/2$, and assume that the deterministic functions $ξ_V$ satisfy the following conditions as $V ↑ Z^μ$:

\[
\min_{1 \leq l \leq K} ξ_{l+1,V}(ξ_l,V - ξ_{l+1,V}) → \infty \quad \text{(distinct height of peaks)},
\]

\[
\frac{1}{\log |V|} \min_{1 \leq k < n \leq |V|} |z_{k,V} - z_{n,V}| → \infty \quad \text{(sparseness of peaks)}
\]

and, finally,

\[
ξ_{|V|^θ},V/ξ_{K,V} < \text{const}(θ)
\]

for some $0 < \text{const}(θ) < 1$. Then

\[
λ_{l,V} = ξ_{l,V} + O(1/ξ_{l,V}) \quad \text{for all} \quad 1 \leq l \leq K.
\]

We now give an example of i.i.d. random field $ξ(·)$ with “heavy tails” possessing extremes like those in Proposition 1.1.

**Proposition 1.2** (see Theorem 4.3(i) with $p = 1$, Theorem 3.1 with $R = 0$ and Theorem 4.5). In the case of Weibull distribution, i.e., $Q(t) = t^α$ ($t \geq 0$) with $α < 2$, the i.i.d. sample $ξ_V$ ($V ↑ Z^μ$) satisfies (1.2)–(1.4) with probability $1 + o(1)$.

Propositions 1.1 and 1.2 imply that the eigenvalues $λ_{K,V}$ are approximated by $ξ_{K,V}$. In this case, Poisson limit theorems (and the corresponding renormalization constants) for the largest eigenvalues are the same as those for $ξ_V$-extremes according to the following proposition.

**Proposition 1.3** (see Theorem 6.7). If $Q(t) = t^α$ with $α < 2$, then the point process $N^λ_ξ$ on $[-1; 1]^μ × \mathbb{R}$, defined by

\[
N^λ_ξ := \sum_{k=1}^{|V|} \delta_{Λ_ξ(k)} \quad \text{with} \quad Λ_ξ(k) := \left( \frac{z_{k,V}}{|V|^{1/μ}}, \frac{λ_{k,V} - (\log |V|)^{1/α}}{\alpha^{-1}(\log |V|)^{(1-α)/α}} \right),
\]

\[
N^λ_ξ := \sum_{k=1}^{|V|} \delta_{Λ_ξ(k)} \quad \text{with} \quad Λ_ξ(k) := \left( \frac{z_{k,V}}{|V|^{1/μ}}, \frac{λ_{k,V} - (\log |V|)^{1/α}}{\alpha^{-1}(\log |V|)^{(1-α)/α}} \right),
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\]
converges weakly to the Poisson process on $[-1;1]^n \times \mathbb{R}$ with the intensity measure $dx \times e^{-t} dt$.

According to Proposition 1.3 the $K$th largest eigenvalue $\lambda_{K,V}$ is associated with the $K$th largest value of $\xi_V$, viz., $\lambda_{K,V} \leftrightarrow z_{K,V}$. For the lighter tails, say, $Q(t) = t^\alpha$ with $\alpha \geq 2$, the landscape of $\xi_V$ gets “smoother”, in particular, (1.2) fails. Therefore, $\lambda_{K,V}$ is associated with a lower and “slightly supported” $\xi_V$-peak, viz., $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$, where for $\alpha > 3$, the index $\tau(K) = \tau_V(K)$ tends to infinity as $|V| \to \infty$. This in turn implies that further terms in expansion for $\lambda_{K,V}$ become essential; see (2.21) and Examples 6.9–6.10. Let us distinguish three classes (J)–(JJJ) of light tailed distributions which ensure different asymptotic behavior of the eigenvalues $\lambda_{K,V}$.

(J) Distribution tails heavier than the double exponential function. If

$$\log Q(t) = o(t)$$

and $Q$ satisfies additional regularity and continuity conditions as $t \to \infty$, then $\xi_V$-extremes possess a strongly pronounced geometric structure which can be described as follows:

For arbitrary sufficiently small constants $0 < \varepsilon < \theta$, there exist constants $c_1 > c_2 > 0$ and (large) $C > 0$ such that almost surely

$$\min_{1 \leq l < n \leq |V|^\beta} (\xi_{l,V} - \xi_{n,V}) \geq e^{-|V|^2} \quad \text{(distinct height of peaks)},$$

$$\min_{1 \leq l < n \leq |V|^\beta} |z_{l,V} - z_{n,V}| \geq |V|^{c_1} \quad \text{(sparseness of peaks)}$$

and, finally,

$$\xi_{|V|^\beta,V} - \xi_{|V|^\beta,V} \geq C$$

for each large $V$; see Theorem 4.8 with $\kappa = 0$, Theorem 3.1 with $R = 0$ and Theorem 4.6 with $\rho = \infty$. By the standard finite-rank perturbation arguments in (Astrauskas and Molchanov 1992) and (Astrauskas 2008), these properties of $\xi_V$ yield that there is no resonance between $\xi_V$-peaks in the Anderson model for large $V$, therefore, the eigenvalues associated with a block of peaks can be determined by the local eigenvalues associated with separate peaks (i.e., “relevant single peak” approximation). More precisely, for fixed natural $K$ and $V \uparrow \mathbb{Z}^n$, almost surely the eigenvalue $\lambda_{K,V}$ of $H_V = \kappa \Delta_V + \xi_V$ is approximated by the principal (i.e., the first largest) eigenvalue of the “single peak” Hamiltonian $\kappa \Delta_V + \xi(\cdot) + \xi(\tau(K),V) \delta_{z_{\tau(K),V}}$ where $\log \tau(K) = o(\log |V|)$. Here $\xi(\cdot)$ is the “noise” potential; the site $z_{\tau(K),V} \in V$ is a localization center of the $K$th eigenfunction of $H_V$. (Notice that the $K$th eigenfunction is asymptotically delta-like function at $z_{\tau(K),V}$). This refers to the correspondence $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$. Thus, Poisson limit theorems for the eigenvalues $\lambda_{K,V}$ of $H_V$ are reduced to those for the principal eigenvalues of the “single peak” Hamiltonians, which in turn are expanded into certain (nonlinear) series in $\xi(x)$ ($x \in V$); cf. formulas (2.19)–(2.21), Theorems 2.3, 6.2 and discussions in Section 6.4. Under assumption (1.5), asymptotic expansion formulas and Poisson limit theorems for the largest eigenvalues are derived by Astrauskas and Molchanov (1992), Astrauskas
(2007; 2008; 2012; 2013). See also Grenkova et al. (1983) and Grenkova et al. (1990)
for the case of Weibull distribution with $\alpha < 2$.

**Distribution tails lighter than the double exponential function.** If

$$ t^{-1} \log Q(t) \to \infty $$

and $Q$ satisfies additional regularity conditions as $t$ tends to $t_Q$ (= the right endpoint of $Q$), then the $\xi_V$-peaks possess a *weakly pronounced geometric structure*. In particular, almost surely $\xi_{||V|^\rho}, V - \xi_{||V|^\rho}, V \to 0$ as $|V| \to \infty$, for all $0 \leq \varepsilon < \theta < 1$, so that the height of all $\xi_V$-extremes is of the same order $\xi_{1,V} + o(1)$ (see Theorem 4.6 with $\rho = 0$ and Theorem 3.1(i) with arbitrary $R \geq 1$ and $\theta(\cdot) \equiv \theta = \text{const}$). In this case, the eigenvalue $\lambda_{K,V}$ does not longer correspond to an isolated potential peak, but to an extremely large “relevant island” of $\xi_V$-values of the height $\xi_{1,V} + o(1)$; see Theorem 6.1 and the proof of Theorem 2.4, where the second order expansion formula for $\lambda_{K,V}$ is obtained. For the Bernoulli i.i.d. random variables $\xi(x)$ with $t_Q = 1$ (which is the particular case of (1.9)), Bishop and Wehr (2012) derived more accurate expansion formula for the principal eigenvalue $\lambda_{1,V}$ of the one-dimensional Hamiltonian $H_V$ ($\nu = 1$). They have showed that $\lambda_{1,V}$ is associated with the longest consecutive sequence of sites $x \in V$ with $\xi(x) = 1$, i.e., the “relevant island” of $\xi_V$-extremes, the length of which unboundedly increases as $V \uparrow \mathbb{Z}$.

**Double exponential type tails.** Finally, assume that $t^{-1} \log Q(t)$ tends to a positive finite constant as $t \to \infty$, i.e., $e^{-Q}$ are the double exponential tails presenting the intermediate case between (J) and (JJ). For such $Q$, the eigenvalue $\lambda_{1,V}$ corresponds to the “relevant island” of high $\xi_V$-values, the diameter of which is bounded as $|V| \to \infty$. This refers to the “relevant island” approximation; see Theorems 2.5 and 6.13 for the second order expansion formulas for the principal eigenvalue, which have been originally derived by Gärtner and Molchanov (1998). Rigorous results on Poisson limit theorems and localization properties for the largest eigenvalues (in the case of double exponential tails) have been proved by Biskup and König (2013); see also the review paper by König and Wolff (2013) and Sections 6.3–6.4 of the present paper for the discussions on their results and the proofs.

### 1.3. Related models (infinite-volume Hamiltonians, random matrices, etc)

We recall that the Anderson Hamiltonian $H = \kappa \Delta + \xi(\cdot)$ on the whole of lattice $\mathbb{Z}^\nu$ is a basic model of quantum mechanics introduced to describe the movement of a quantum particle in the random potential $\xi(\cdot)$. The most important property of the Hamiltonian $H$ on $l^2(\mathbb{Z}^\nu)$ ($\nu \geq 1$) is the presence of (nonrandom) pure point spectral intervals at the edge of spectrum and the exponential decay of the corresponding eigenfunctions with probability one (i.e., *Anderson localization*). It is well known that, in the case of periodic $\xi(\cdot)$, all the spectrum $\text{Spect}(H)$ is absolutely continuous. See, e.g., the monographs (Pastur and Figotin 1992), (Kirsch 2008) and therein references on the subject. In the present survey (as well as in Astrauskas (2007; 2008; 2012; 2013) and Biskup and König (2013)), the phenomenon of Anderson localization is illustrated for the top eigenvalues of finite-volume models. On the other hand, the pure point spectral intervals $I \subset \text{Spect}_{pp}(H)$ are distinguished by Poissonian asymptotic behavior of the eigenvalues $\lambda_{I,V}$ (close to a fixed $\lambda \in I$) of the finite-volume Hamiltonian $H_V$, as $V \uparrow \mathbb{Z}^\nu$; see (Molchanov 1981) for the one-dimensional case $\nu = 1$ and (Minami 1996),
(Killip and Nakano 2007), (Germinet and Klopp 2010) for the case $\nu \geq 1$. For the relationship between the extreme value theory for the spectrum $\text{Spect}(H_\nu)$ and the long-time intermittency for the parabolic Anderson problems $\partial U/\partial s = HU$ (as well as their links to asymptotic geometric properties of $\xi_V$-extremes), we refer to Gärtner and Molchanov (1998), Biskup and König (2001), Hofstad et al. (2006), Gärtner et al. (2007), Fiodorov and Muirhead (2014) and the recent surveys by Gärtner and König (2005), König and Wolff (2013), where one can find a comprehensive list of references on the subject.

Recently, there has been much progress toward extreme value theory for the eigenvalues $\lambda_{K,N}$ of symmetric random matrices

$$H_N = (h_{i,j})_{1 \leq i, j \leq N}$$

with i.i.d. (centered) entries $h_{i,j}$ ($i \leq j$) and $N \to \infty$, i.e., large Wigner matrices. For polynomially decaying distributions $\mathbb{P}(|h_{i,j}| > t) \sim t^{-\beta}$ with $\beta < 4$ (heavy tails), Auffinger et al. (2009) have proved that the largest eigenvalues $\lambda_{K,N}$ of $H_N$ are approximated by the corresponding $K$th largest values among $|h_{i,j}|$ ($1 \leq i \leq j \leq N$). (Notice that the largest entries (in absolute value) are extremely sparse and strongly pronounced in comparison to other entries in $H_N$, so that a standard perturbation theory for symmetric matrices have been applied.) This in turn implies Poisson limit theorems for the normalized eigenvalues $\lambda_{K,N}A_N$, where the normalizing constants $A_N > 0$ are chosen the same as in the corresponding limit theorems for extremes of $|h_{i,j}|$ ($1 \leq i \leq j \leq N$). In the case of the lighter tails, the top eigenvalues are distinguished by non-Poissonian asymptotic behavior, rather the Tracy-Widom limit law; see, e.g., Lee and Yin (2014), Bourgade et al. (2014). This transition from Poisson limit theorems to Tracy-Widom asymptotics for the largest eigenvalues of Wigner random matrices was discussed in detail by Biroli et al. (2007). Note that the one-dimensional Anderson Hamiltonian in $V \subset \mathbb{Z}$ is a random band (tridiagonal) matrix reflecting local interaction, in contrary to the Wigner matrix model with i.i.d. entries reflecting global or mean-field interaction.

### 1.4. The earlier literature on extremes of i.i.d. random fields

Most statements of the present paper on the $\xi_V$-extremes and the corresponding RV classes were announced (without the proofs) in (Astrauskas 2007; 2008; 2012; 2013). We now provide a brief overview of the earlier literature on the related asymptotic results for extreme order statistics of i.i.d. random processes and fields.

High-level exceedances consisting of single rare $\xi_V$-peaks were studied in (Astrauskas 2001); see also (Astrauskas 2003) for the case of Gaussian random fields with correlated values. Related asymptotic results (in particular, the so-called longest head runs in coin tossing) for Bernoulli distributed i.i.d. random variables $\xi(x)$, $x \in \mathbb{Z}$, were discussed, e.g., in (Binswanger and Embrechts 1994).

In the case of exponentially distributed $\eta(0)$, strong limit theorems for the spacings $\eta_{K,V} - \eta_{K + 1,V}$ ($K$ fixed) were proved by Astrauskas (2006). Devroye (1982) derived strong and weak limit theorems for $\min_{1 \leq k \leq |V|} (u_{k+1} - u_k)$ when $u(0)$ is uniformly distributed. See also (Astrauskas 2003) for the case of Gaussian random fields with correlated values.
Strong asymptotic bounds for $\xi_{K,V}$ are given in (Shorack and Wellner 1986) when $K$ is fixed, and in (Deheuvels 1986) when $K = K_V \to \infty$. Wellner (1978) derived strong asymptotic bounds for the uniform $k$th order statistics $u_{k,V}$ (thus, for $\eta_{k,V}$) uniformly in $k \geq 1$.

For extreme value theory for random variables, in particular, characterization of the domains of attraction of max-stable distributions, we refer to the monographs by Resnick (1987), de Haan and Ferreira (2006), Leadbetter et al. (1983), Embrechts et al. (1997). See also the monograph by Shorack and Wellner (1986) for a detailed account of strong and weak limit theorems for order statistics and their functions related to mathematical statistics. Finally, the monograph by Bingham et al. (1987) provides a detailed account of the theory of regularly varying functions.

In the proof of a number of our statements on $\xi_{V}$-extremes, we explore the representation $\xi_{k,V} = f(\eta_{k,V})$, where $f := Q^{-1}$ is the generalized inverse function of $Q$ and, as above, $\eta_{k,V}$ stands for the $k$th extreme value among independent exponentially distributed random variables $\eta(x) (x \in V)$ with mean 1. Due to the nice properties of $\eta_{k,V}$ (for instance, $\eta_{k,V}$ is a sum of independent exponentially distributed random variables), we first obtain the asymptotic results for $\eta_{k,V}$, which are then transferred to $\xi_{k,V}$ under appropriated conditions on $f$. These conditions are formulated in terms of regular variation (RV) of $f(s)$ as $s \to \infty$. We further give a characterization of RV classes, in particular, their links to continuity and tail decay of the distribution $1 - e^{-Q}$ at the right endpoint; see Appendix A. They are also compared with the well-known RV classes including the domains of attraction of max-stable laws, O-regular variation, asymptotically balanced, etc; see Appendix A.

An interesting further problem is an extension of the present asymptotic results for functionals (E), (SP), (MIN), (N) to other classes of random fields $\xi(\cdot)$ including 1) independent non-identically distributed random variables; 2) random fields with correlated values, in particular, Gaussian fields (Astrauskas 2003) and moving average fields defined as a linear combination of i.i.d. random variables with nonrandom real coefficients. See, e.g., the review papers by Elgart et al. (2012), Tautenhahn and Veselić (2014) for a detailed background of the random alloy type models $\kappa \Delta + \xi(\cdot)$ with moving average potential $\xi(\cdot)$.

1.5. Notation. Representation of i.i.d. random fields

Let us introduce the further notation and remarks we use throughout the paper. We denote by $\mathbb{R}_+$ the positive half-axis and by $\mathbb{N}$ positive integers. Let $\log_j$ stand for the $j$times iterated natural logarithm. For real $a, b$, we write $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$, and $[a]$ for the integer part of $a$. Given a subset $U \subset \mathbb{Z}^\nu$, we write $|U|$ for the number of its elements. Let $\text{dist}(U, U')$ stand for the lattice $l^1$-distance between subsets $U, U' \subset \mathbb{Z}^\nu$. The summation over $x \in V$: $a \leq |x| \leq b$ is abbreviated to $\sum_{a \leq |x| \leq b}$. By $t_0$, $|V_0|$, etc. we denote various large numbers, values of which may change from one appearance to the next. Similarly, const, $\text{const}'$ etc. stand for various positive constants. We write $1/0+ = \infty$, $\log(0+) = -\infty$ and $1/\infty = 0$. Let $g \circ h = g(h(\cdot))$ stand for a composition of real functions $g$ and $h$. Also, for $g > 0$ and $h > 0$, we write $g(t) \asymp h(t)$ as $t \to \infty$, if the ratio $g(t)/h(t)$ is bounded away from zero and from above for all large $t$. 

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By $G_V(\lambda; \zeta_V; x, y) (x \in V, y \in V)$ we denote the Green function of the Hamiltonian $\kappa \Delta_V + \zeta_V$ in $l^2(V)$, viz.

$$G_V(\lambda; \zeta_V; x, y) := G_V(\lambda; \zeta_V) \delta_y(x) := (\lambda - \kappa \Delta_V - \zeta_V)^{-1} \delta_y(x).$$

Here $\delta_y(\cdot)$ is the Kronecker symbol, i.e., $\delta_y(x) := 1$ if $x = y$, and $\delta_y(x) := 0$ if $x \neq y$.

Throughout the paper we suppose that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{E}$ stand for the expectation with respect to $\mathbb{P}$. Recall $Q(t) := -\log \mathbb{P}(\xi(0) > t)$ is the cumulative hazard function of an i.i.d. random field $\xi(x) = \xi^\omega(x)$ ($\omega \in \Omega; x \in \mathbb{Z}^\nu$), and let $t_Q$ denote its right endpoint $t_Q := \sup \{t: Q(t) < \infty\}$. Without loss of generality, we shall assume throughout that $0 < t_Q \leq \infty$. Clearly $Q : (-\infty; t_Q) \to \mathbb{R}_+$ is a right-continuous nondecreasing function such that $Q(\infty) = 0$ and $Q(t_Q) = \infty$. Most of the conditions of our results are formulated in terms of the inverse of the cumulative hazard function defined by

$$f(s) := \overline{Q}(s) := \inf \{t: Q(t) \geq s\} \quad (s \in \mathbb{R}_+)$$

(1.10)

(thus $f : \mathbb{R}_+ \to (-\infty; t_Q)$ is a left-continuous nondecreasing function such that $f(s)$ tends to $t_Q$ as $s \to \infty$). The reason for this is the following useful representation of order statistics $\xi_{1,v}$:

$$\xi_{1,v} := f(\eta_{1,v}) \geq \xi_{2,v} := f(\eta_{2,v}) \geq \cdots \geq \xi_{|V|,v} := f(\eta_{|V|,v}).$$

(1.11)

where

$$\eta_{1,v} := \eta(z_{1,v}) > \eta_{2,v} := \eta(z_{2,v}) > \cdots > \eta_{|V|,v} := \eta(z_{|V|,v})$$

(1.12)

is the variational series based on the sample $\eta_V := \{\eta(x): x \in V\}$ of exponential i.i.d. random variables with mean 1.

1.6. Outline

In Section 2, we collect conditions on deterministic functions $\xi_V$ in terms of functionals (E), (SP), (MIN) and (N), which yield expansion formulas for the largest eigenvalues $\lambda_{K,V}$ of the discrete Schrödinger operator $H_V = \kappa \Delta_V + \xi_V$ on $l^2(V)$ as $V \uparrow \mathbb{Z}^\nu$. Section 2.1 provides rough bounds for $\lambda_{K,V}$. We then study $\lambda_{K,V}$ in the cases of $\xi_V$ with extremely sharp peaks (Section 2.2), dominating single peaks (Section 2.3), dominating large islands of high $\xi_V$-values the diameter of which unboundedly increases (Section 2.4) and, finally, dominating islands of high $\xi_V$-values the diameter of which is bounded (Section 2.5). The results of Sections 2.2–2.5 follow simply from the more general statements of (Astrauskas 2008; 2012) and Section 2.4 in (Gärtner and Molchanov 1998).

Sections 3–5 contain the main results of the paper dealing with asymptotic behavior of extremes of the i.i.d. random field $\xi(\cdot)$ with distribution function satisfying certain RV and continuity conditions at the right end-point. Functionals (E), (SP), (MIN) and (N) are studied in Sections 3, 4.1-4.2, 4.3 and 5, respectively.

Section 6 provides an overview of current results on extreme value theory for the spectrum of the Anderson Hamiltonian $H_V = \kappa \Delta_V + \xi_V$, $V \uparrow \mathbb{Z}^\nu$, with an i.i.d. potential $\xi(\cdot)$. The issues under discussion include the asymptotic expansion formulas.
and Poisson limit theorems for the largest eigenvalues and their localization centers. We consider separately three cases of the distribution tails \( e^{-Q} \) of \( \xi(0) \): the tails are heavier than the double exponential function (Section 6.1); the tails are lighter than the double exponential function (Section 6.2) and the double exponential tails (Section 6.3). We give proof sketches of most theorems of this section demonstrating their connections to the results of Sections 3–5 on \( \xi \)-extremes. In Section 6.4, we comment and compare the proofs of Poisson limit theorems stated in Sections 6.1 and 6.3 and proved in the earlier papers by Astrauskas and Molchanov (1992), Astrauskas (2007; 2008; 2012; 2013) and Biskup and König (2013).

Finally, in Appendix A, we characterize and compare the RV classes of distributions introduced in Sections 3–6.

2. ASYMPTOTIC EXPANSION FORMULAS FOR THE LARGEST EIGENVALUES OF DETERMINISTIC HAMILTONIANS

Let \( V = [-n; n]^D \cap \mathbb{Z}^D \ (n \in \mathbb{N}) \) be a sequence of cubes. By introducing the periodic norm \( |x| := |x|_n := \min_{y \in (2n+1)\mathbb{Z}^D} |x - y| \), \( V \) may be considered as a sequence of tori tending to \( \mathbb{Z}^D \). We are interested in the finite-volume Schrödinger operators \( \mathcal{H}_V = \kappa \Delta_V + \xi_V \) on \( L^2(V) \) with periodic boundary conditions. Recall that \( \kappa > 0 \) is a diffusion constant, \( \Delta_V \) denotes the lattice Laplacian on \( L^2(V) \) (i.e., a restriction of the operator \( \Delta \psi(x) := \sum_{y \in \mathbb{Z}^D} \psi(y) \) to torus \( V \)) and \( \xi_V := \{ \xi_V(x) : x \in V \} \in [-\infty; \infty]^{V} \) are deterministic functions, i.e., potential. The values \(-\infty\) of \( \xi_V \) are allowed to include the cases which are interesting from a physical point of view; see, e.g., (Biskup and König 2001), (Hofstad et al. 2006). Write \( V_b := \{ x \in V : \xi_V(x) > -\infty \} \). Then \( \mathcal{H}_V \) is interpreted as an operator on \( L^2(V) \) with zero boundary conditions outside \( V_b \). The spectral problem

\[
\mathcal{H}_V \psi = \lambda \psi \quad (\lambda \in \mathbb{R}; \psi \in L^2(V))
\]

has \( |V_b| \) solutions \( \lambda_{1,V} \geq \lambda_{2,V} \geq \ldots \geq \lambda_{|V_b|,V}, \) i.e., the ordered eigenvalues of the operator \( \mathcal{H}_V \).

In this section, we provide asymptotic expansion formulas for the first \( K \) largest eigenvalues \( \lambda_{k,V} \) under conditions on the first terms of the variational series \( \xi_{k,V} \geq \xi_{1,V} \geq \xi_{2,V} \geq \ldots \geq \xi_{|V_b|,V} \) of the sample \( \xi_V \) and their coordinates \( z_{k,V} \in V \) defined by \( \xi_{k,V} = \xi_V(z_{k,V}) \ (1 \leq k \leq |V|) \); here \( V = \{ z_{k,V} : 1 \leq k \leq |V| \} \). The results of this section follow simply from more general results of (Astrauskas 2008; 2012) and Section 2.4 of (Gärtner and Molchanov 1998), where one finds more discussions on the relationship between \( \xi_V \)-extremes and the top eigenvalues of \( \mathcal{H}_V \).

2.1. Preliminaries: rough bounds

We start with the following simple bounds for eigenvalues \( \lambda_{k,V} \), provided \( |V_b| \geq 2 \).

**Theorem 2.1.** (i) For any \( V \) and any \( \xi_V \),

\[
\xi_{1,V} \leq \lambda_{1,V} \leq \xi_{1,V} + 2\nu \kappa \quad \text{and} \quad |\lambda_{l,V} - \xi_{l,V}| \leq 2\nu \kappa \quad (2 \leq l \leq |V_b|).
\]

(ii) For any \( V \), any \( K \leq |V_b| \) and \( \xi_V \) such that \( \min_{1 \leq k < l \leq K} |z_{k,V} - z_{l,V}| \geq 1 \), we have that

\[
\xi_{l,V} \leq \lambda_{l,V} \leq \xi_{l,V} + 2\nu \kappa \quad \text{for all} \quad 1 \leq l \leq K.
\]
Proof. We repeatedly use the fact that the $K$th eigenvalue $\lambda_{K,V} = \lambda_{K,V}(\xi_V)$ of the operator $\kappa \Delta_V + \xi$ is a nondecreasing function in each variable $\xi_V(x)$ tending to infinity ($K \geq 1, x \in V$); i.e., the monotonicity property of eigenvalues (Lankaster 1969, Theorem 3.6.3).

(i) To estimate $\lambda_{1,V}$, we abbreviate $\xi'(x) := \xi_{1,V}$ if $x = z_{1,V}$, and $\xi'(x) := -\infty$, otherwise. Note that $\xi'(\cdot) \leq \xi_{1,V} (\cdot) \leq \xi_{1,V}$ in $V$. Therefore, $\lambda_{1,V}$ is bounded from below by $\xi_{1,V}$, i.e., the principal eigenvalue of the operator $\kappa \Delta_V + \xi_{1,V}$. Moreover, $\lambda_{1,V}$ is bounded from above by the principal eigenvalue of the operator $\kappa \Delta_V + \xi_{1,V}$ on $l^2(V)$, which in turn does not exceed $2\nu + \xi_{1,V}$, since the norm of the Laplacian $\kappa \Delta_V$ is less than $2\nu$. Similarly, since each eigenvalue $\lambda_{1,V}$ is bounded from above (resp., from below) by the $l$th eigenvalue of the diagonal operator $\xi_V + 2\nu k$ on $l^2(V)$ (resp., $\xi_V - 2\nu k$), we obtain (2.2) for $l \geq 2$.

(ii) We need to show the lower bound in (2.3). Without loss of generality, we assume that $\xi_{K,V} > 0$ (this may be achieved by shift transform of $\xi_V$ and \lambda in the spectral problem (2.1)). Write $\mathcal{E}_V^K := \{z_{1,V}, \ldots, z_{K,V}\}$. We introduce the following functions: $\zeta(x) := \xi_V(x)$ if $x \in \mathcal{E}_V^K$, and is zero, otherwise; and further on, $\tilde{\zeta}(x) := 0$ if $x \in \mathcal{E}_V^K$, and $\tilde{\zeta}(x) := -\infty$, otherwise. Then $\xi_V(\cdot) \geq \zeta(\cdot) + \tilde{\zeta}(\cdot)$ in $V$, therefore, each eigenvalue $\lambda_{l,V}$ is bounded from below by the corresponding eigenvalue $\lambda_{l,V}$ of the operator $\kappa \Delta_V + \xi_V + \tilde{\xi}_V$: here $1 \leq l \leq K$. To estimate $\lambda_{l,V}$, we rewrite the corresponding spectral problem in the form:

$$
(\lambda - \kappa \Delta_V - \tilde{\xi}_V) \psi = \zeta \psi \quad (\lambda > 0, \psi \in l^2(V))
$$

(2.4)

and apply the resolvent operator $\mathcal{G}_V(\lambda; \tilde{\xi}_V) := (\lambda - \kappa \Delta_V - \tilde{\xi}_V)^{-1}$ to both sides of (2.4).

Since $\mathcal{G}_V(\lambda; \tilde{\xi}_V) \delta_z = \lambda^{-1} \delta_z$ for $z \in \mathcal{E}_V^K$, equation (2.4) is transferred to

$$
\psi = \sum_{z \in \mathcal{E}_V^K} \xi_V(z) \psi(z) \lambda^{-1} \delta_z \quad (\lambda > 0);
$$

here $\delta_p(\cdot)$ is the Kronecker symbol. Clearly, for each $1 \leq l \leq K$, the pair $\lambda_{l,V} = \xi_{l,V}$ and $\psi(\cdot; \lambda_{l,V}) = \delta_{z_{l,V}}(\cdot)$ solves this equation. Summarizing, we have that $\lambda_{l,V} \geq \lambda_{l,V} = \xi_{l,V}$ ($1 \leq l \leq K$), as claimed. Theorem 2.1 is proved. \hfill \Box

In Sections 2.2–2.5 below, we consider three classes of functions $\xi_V$:

(JJ) Sparse distinct $V$-peaks dominate in the landscape of $\xi_V$ as $V \uparrow \mathbb{Z}'$, i.e., $\xi_V$ possess properties like (1.6)–(1.8). Then the $K$th largest eigenvalue $\lambda_{K,V}$ is associated with an isolated peak $\xi_{\tau(K),V}$, so that $\lambda_{K,V} \leftrightarrow z_{\tau(K),V}$ for some $\tau(K) = \tau_V(K) \geq 1$ (Section 2.3). In particular, if the functions $\xi_V$ possess extremely sharp peaks like (1.2)–(1.4), then the eigenvalue $\lambda_{K,V}$ is associated with the $K$th largest value of $\xi_V$, viz., $\lambda_{K,V} \leftrightarrow z_{K,V}$ (Section 2.2). In both cases, the lower bounds in (2.3) are achieved as $V \uparrow \mathbb{Z}'$.

(JJ) The landscape of $\xi_V$ is dominated by flat islands of large values with an unboundedly increasing diameter. Then the largest eigenvalues are associated with such relevant islands. In this case, the upper bounds in (2.2) are achieved as $V \uparrow \mathbb{Z}'$ (Section 2.4).
Similarly as in (JJ), bounded islands of large values prevail in the landscape of $\xi_V$. Then the asymptotic expansion terms of the principal eigenvalue $\lambda_{1,V}$ fill the gap between its lower and upper bounds in (2.2) (Section 2.5).

In the case of (J) we obtain the explicit expansion formulas for eigenvalues in terms of $\xi_V$-values. Meanwhile, for (JJ) and (JJJ) we restrict ourselves to a derivation of the second order expansion formulas for eigenvalues.

### 2.2. Potentials with extremely sharp single peaks

For $N \geq 2$, let us write

$$\mathcal{E}_V^N := \{z_{1,V}, z_{2,V}, \ldots, z_{N,V}\} \subset V$$

for the subset of coordinates of the first $N$ largest values of $\xi_V$, and

$$r_{N,V} = \min_{1 \leq l < k \leq N} |z_l,V - z_k,V| = \min_{x,y \in \mathcal{E}_V^N, x \neq y} |x - y|$$

for the minimum distance between sites in $\mathcal{E}_V^N$. For natural $1 \leq K = K_V < N = N_V < |V|$ and $p \geq 0$, we introduce the following conditions on functions $\xi_V$:

$$\lim_{V} \min_{1 \leq l \leq K} \xi_V^{p(l+1) \wedge K,V}(\xi_{l,V} - \xi_{l+1,V}) = \infty \quad \text{where} \quad \lim_{V} \xi_{K,V} = \infty, \quad (2.7)$$

$$C := \limsup_{V} \frac{\xi_{N,V}}{\xi_{K,V}} < 1, \quad (2.8)$$

$$\lim_{V} \frac{r_{N,V}}{\log N} = \infty, \quad (2.9)$$

$$M := \limsup_{V} \max_{1 \leq l \leq K} \max_{|x - z_{l,V}| = 1} |\xi_V(x)| < \infty \quad (2.10)$$

and, finally,

$$\lim_{V} \min_{K+1 \leq l \leq N} \xi_{K,V}^2 \left( \frac{2\nu \kappa^2}{\xi_{K,V}^2} - \xi_{l,V} - \kappa^2 \sum_{|x - z_{l,V}| = 1} \frac{1}{\xi_{K,V} - \xi_V(x)} \right) = \infty. \quad (2.11)$$

We write $\xi_{0,V} := \infty$, and $s_V(l) := (\xi_{l-1,V} - \xi_{l,V}) \wedge (\xi_{l,V} - \xi_{l+1,V})$ for $1 \leq l \leq K$.

**Theorem 2.2.** (i) Under (2.7) with $p = 0$,

$$\limsup_{V} \max_{1 \leq l \leq K} |\lambda_{l,V} - \xi_{l,V}| s_V(l) \leq \text{const}_1(\kappa, \nu).$$

(ii) Under (2.7)–(2.9) with $p = 1$,

$$\limsup_{V} \max_{1 \leq l \leq K} |\lambda_{l,V} - \xi_{l,V}| s_V(l) \leq \frac{\text{const}_2(\kappa, \nu)}{1 - C}.$$

(iii) If $\xi_V$ satisfies (2.7)–(2.11) with $p = 2$, then

$$\limsup_{V} \max_{1 \leq l \leq K} \left| \lambda_{l,V} - \xi_{l,V} - \frac{2\nu \kappa^2}{\xi_{l,V}} \xi_{l,V}^2 \right| \xi_{l,V} \leq \frac{M \cdot \text{const}_3(\kappa, \nu)}{(1 - C)^2} + \text{const}_4(\kappa, \nu).$$
Proof. We first note that condition (2.10) implies \( \lambda_{N, V} \geq -\infty \) for any \( V \supset V_0 \). On the other hand, if \( \lambda_{N, V} = -\infty \) and \( r_{N, V} > 1 \), then \( \lambda_{l, V} = \xi_{l, V} \) for all \( 1 \leq l \leq N \). The latter is shown by the same arguments as in the proof of the lower bound in (2.3).

Now, assuming \( \lambda_{N, V} > -\infty \) and letting \( V \uparrow \mathbb{Z}^\nu \), the assertions of Theorem 2.2(i), (ii) and (iii) are derived from Theorem A.1(i), (ii) and (iii), respectively, in (As- trauskas 2012, Appendix A) with the abbreviations \( \Pi := E_{\xi_{N, V}}(\ii), L := \xi_{N, V} \) and \( r := r_{N, V} \).

Let \( \xi(\cdot) \) be an i.i.d. random field with the distribution function \( 1 - e^{-Q} \). We will show that, if \( Q \) satisfies the condition \( Q(t) = o(t^{p+1}) \) for \( p = 0, 1 \) and 2 ("heavy tails" \( e^{-Q} \)) and additional RV conditions as \( t \to \infty \), then with high probability \( \xi \) satisfies the assumptions of Theorem 2.2(i), (ii) and (iii), respectively, where \( K \in \mathbb{N} \) is fixed and \( N = \lfloor |V|^\theta \rfloor \) for some \( 0 < \theta < 1/2 \); see Theorems 4.3(i), 4.5, 3.1 (\( R = 0 \)), 5.3 and 5.4 with \( 0 < \varepsilon < \theta \). Therefore, Poisson limit theorems for the largest eigenvalues \( \lambda_{K, V} \) are reduced to those for extreme values of i.i.d. random fields \( \xi(\cdot) \) or \( \xi(\cdot) + 2\nu_K^2/(\xi(\cdot)\vee 1) \) (Theorem 6.7).

2.3. Potentials with dominating single peaks: the general case

To simplify the proceedings, we need some notation and remarks. For \( N \geq 2 \) and \( \xi_{N, V}^N \) as in (2.5), we introduce the following function: \( \xi_{V}(x) := 0 \) if \( x \in E_{\xi_{N, V}}^V \), and \( \xi_{V}(x) := \xi_{V}(x) \) if \( x \in V/\xi_{N, V}^N \). Then

\[
\xi_{V} = \sum_{z \in E_{\xi_{N, V}}^V} \xi_{V}(z)\delta_z + \xi_{V},
\]

i.e., \( \xi_{V} \) is a superposition of \( \xi_{V} \)-peaks and the noise component \( \xi_{V} \). To exclude the trivialities, we assume that \( \lambda_{N, V} > -\infty \) for each \( V \) (for the case when \( \lambda_{N, V} = -\infty \) and \( r_{N, V} > 1 \), see the proof of Theorem 2.2 above). For each \( z \in V \), let \( \tilde{\lambda}_{V}(z) \) be the principal eigenvalue of the "single peak" Hamiltonian \( \kappa \Delta_{V} + \xi_{V}(z)\delta_z + \xi_{V}(1 - \delta_z) \) on \( l^2(V) \). We associate the sites \( z_{\tau(i), \nu} \in V \) with the variational series

\[
\tilde{\lambda}_{1, V} := \tilde{\lambda}(z_{\tau(1), \nu}) \geq \tilde{\lambda}_{2, V} := \tilde{\lambda}(z_{\tau(2), \nu}) \geq \ldots \geq \tilde{\lambda}_{|V|, V} := \tilde{\lambda}(z_{\tau(|V|), \nu})
\]

(2.12)
based on the sample \( \tilde{\lambda}_{V} \); here \( V = \{z_{\tau(i), \nu} : 1 \leq i \leq |V|\} \).

Theorem 2.3. Assume that there are natural numbers \( 1 \leq K = K_{V} < N = N_{V} < |V| \) such that the functions \( \xi_{V} \) satisfy condition (2.9) and the following conditions:

\[
\lim_{V} \left( \xi_{K, V} - \xi_{N, V} \right) = \infty
\]

(2.13)

and

\[
\liminf_{V} \min_{1 \leq i \leq K} \frac{\log \left( \tilde{\lambda}_{i, V} - \tilde{\lambda}_{i+1, V} \right)}{r_{N, V} \log \left( \xi_{(i+1)\wedge K, V} - \xi_{N, V} \right)} \geq 0.
\]

(2.14)
Then

\[
\limsup_V \max_{1 \leq i \leq K} \frac{\log |\lambda_{i,V} - \tilde{\lambda}_{i,V}|}{r_{N,V} \log (\xi_{i,V} - \xi_{N,V})} \leq -2. \tag{2.15}
\]

Proof. We write

\[
\tilde{\mathcal{E}}_{h,V} := \{ z \in \mathcal{E}_V^N : \tilde{\lambda}_V(z) \geq \xi_{N,V} + 2\nu + h \} \text{ where } h := \frac{\xi_{K,V} - \xi_{N,V}}{2}. \tag{2.16}
\]

By the first bound in Theorem 2.1(i), \(\tilde{\lambda}_V(z) \geq \xi_V(z)\) for all \(z \in \mathcal{E}_V^N\). This combined with (2.13) gives

\[
|\tilde{\mathcal{E}}_{h,V}| \geq K \tag{2.17}
\]

for any \(V \supset V_0\). Finally, according to (Astrauskas 2008, Section 2.2 and Appendix B.2),

\[
\xi_V(z) \leq \tilde{\lambda}_V(z) \leq \xi_V(z) + 2\nu + h \text{ for any } z \in \tilde{\mathcal{E}}_{h,V}. \tag{2.18}
\]

In view of (2.17) and (2.18), the assertion of Theorem 2.3 follows from Theorem B.3 in (Astrauskas 2008, Appendix B) with the abbreviations \(L := \xi_{N,V}, \Pi := \mathcal{E}_V^N (2.5), \tilde{\Pi} := \tilde{\mathcal{E}}_{h,V} (2.16)\) and \(r := r_{N,V} (2.6)\). \(\square\)

Note that conditions (2.13) and (2.14) of Theorem 2.3 are substantially weaker than (2.8) and (2.7), respectively, in Theorem 2.2. According to (2.15) and (2.18) with \(h \to \infty\) as in (2.16), we obtain that \(\lambda_{i,V} = \xi_{i,V} + o(1)\) uniformly in \(1 \leq i \leq K\), so that the eigenvalues \(\lambda_{i,V}\) achieve their lower bounds in (2.3) as \(V \uparrow \mathbb{Z}^d\).

On the other hand, from (Astrauskas 2008, Appendices A and B) we know that, for each \(z \in \tilde{\mathcal{E}}_{h,V}\), the eigenvalue \(\tilde{\lambda}_V(z)\) is the maximal solution to the equation

\[
\mathcal{G}_V(\lambda; \tilde{\xi}_V; z, z) = \frac{1}{\tilde{\xi}_V(z)}; \tag{2.19}
\]

here \(\mathcal{G}_V(\lambda; \tilde{\xi}_V; \cdot, \cdot)\) is the Green function of the Hamiltonian \(\kappa \Delta_V + \tilde{\xi}_V\) on \(l^2(V)\), so that \(\mathcal{G}_V(\lambda; \tilde{\xi}_V; z, z)\) is expanded over paths:

\[
\mathcal{G}_V(\lambda; \tilde{\xi}_V; z, z) = \sum_{\Gamma} \kappa^{|\Gamma|} \prod_{v \in V} (\lambda - \tilde{\xi}(v))^{-n_v(\Gamma)}, \tag{2.20}
\]

where the sum \(\sum_{\Gamma}\) is taken over all paths \(\Gamma : v_0 := z \to v_1 \to \cdots \to v_m := z\) in \(V\) such that \(|v_i - v_{i-1}| = 1\) for each \(1 \leq i \leq m\) and each \(m \in \mathbb{N}\), \(n_v(\Gamma)\) denotes the number of times the path \(\Gamma\) visits the site \(v \in V\), \(|\Gamma| := \sum_{v \in V} n_v(\Gamma) - 1 \geq 0\). Substituting (2.20) to the left-hand side of (2.19) and iterating this with respect to the eigenvalue \(\lambda = \tilde{\lambda}_V(z)\), we obtain the explicit expansion formulas for \(\tilde{\lambda}_V(z)\) \((z \in \mathcal{E}_{h,V})\) presented as a power series in the variables \(\xi_V(z)\) and \(\tilde{\xi}_V(x)\) \((|x - z| \geq 1)\), in particular,

\[
\tilde{\lambda}_V(z) = \xi_V(z) + \kappa^2 \sum_{|x - z| = 1} \frac{1}{\xi_V(z) - \tilde{\xi}_V(x)} +
\]

\[
+ \mathcal{O} \left( \sum_{\substack{|x - z| = 1 \\text{or} \\text{2} \\text{or} \\text{3} \\text{or} \\text{4} \\text{or} \\text{5}}} \frac{1}{(\xi_V(z) - \tilde{\xi}_V(x))(\xi_V(z) - \tilde{\xi}_V(y))(\xi_V(z) - \tilde{\xi}_V(u))} \right). \tag{2.21}
\]
as $V \uparrow \mathbb{Z}^r$.

Let $\xi(\cdot)$ be an i.i.d. random field with the distribution function $1 - e^{-Q}$. We will show that, if the tails $e^{-Q}$ are heavier than the double exponential function (i.e., $\log Q(t) = o(t)$) and satisfy additional regularity and continuity conditions at infinity, then with probability one $\xi_V$ satisfies the assumptions of Theorem 2.3, where $K \in \mathbb{N}$ is fixed and $N = |V^\theta|$ for some $0 < \theta < 1/2$; see Theorems 3.1 ($R = 0$), 4.6 ($\rho = \infty$) and 4.8. Therefore, Poisson limit theorems for the largest eigenvalues $\lambda_{K,V}$ are reduced to those for extremes of nonlinear functions (2.21) on $\xi_V$ (Theorem 6.2).

2.4. Potentials with dominating flat increasing islands of high values

Let $B_R(z) := \{y \in V : |y - z| \leq R\}$ denote the ball in $V$ with center $z \in V$ and radius $R \geq 0$. The following theorem gives a simple condition on $\xi_V$ which ensures that the largest eigenvalues $\lambda_{K,V}$ achieve their upper bounds in (2.2) as $V \uparrow \mathbb{Z}^r$.

**Theorem 2.4.** If

$$\lim_{R \to \infty} \limsup_{V} \min_{z \in V} \left( \xi_{1,V} - \min_{x \in B_R(z)} \xi_V(x) \right) = 0,$$  \hspace{1cm} (2.22)

then, for arbitrarily fixed $K \in \mathbb{N}$,

$$\lim_{V} \left( \lambda_{K,V} - \xi_{K,V} \right) = 2\nu_K.$$

**Proof.** Because of Theorem 2.1(i), we only need to show the lower limit bound

$$\liminf_{V} \left( \lambda_{K,V} - \xi_{1,V} \right) \geq 2\nu_K.$$  \hspace{1cm} (2.23)

From (2.22) we see that there exist a sequence $0 < \varepsilon_R \to 0$ and sites $z_V \in V$ such that $\xi_V(\cdot) \geq \xi_{1,V} - \varepsilon_R$ in $B_R(z_V)$ for any $R \geq R_0$ and any $V \supset V_0(R)$. Abbreviate $\xi_{V}^{(R)}(x) := \xi_V(x)$ if $x \in B_R(z_V)$, and $\xi_{V}^{(R)}(x) := -\infty$, otherwise. Since $\xi_V(\cdot) \geq \xi_{V}^{(R)}(\cdot)$ in $V$ and $\xi_{V}^{(R)}(\cdot) \geq \xi_{1,V} - \varepsilon_R$ in $B_R(z_V)$ for $R$ and $V$ as above, the monotonicity property of eigenvalues implies that $\lambda_{K,V} \geq \lambda_{K,V}^{(R)} + \xi_{1,V} - \varepsilon_R$, where $\lambda_{K,V}^{(R)}$ is the $K$th eigenvalue of the operator $\kappa\Delta$ on $l^2(B_R(z_V))$ with zero boundary conditions. Since $\lambda_{K,V}^{(R)}$ tends to $2\nu_K$ letting first $V \uparrow \mathbb{Z}^r$ and then $R \to \infty$ (Kirsch 2008, Section 3.1), this estimate implies (2.23), as claimed.

Clearly condition (2.22) is fulfilled if and only if there are a sequence $R_V \to \infty$ and sites $z_V \in V$ such that

$$\lim_{V} \left( \xi_{1,V} - \min_{|x - z_V| < R_V} \xi_V(x) \right) = 0.$$  \hspace{1cm} \hspace{1cm} 

From the proof of the theorem we know that the eigenvalues $\lambda_{K,V}$ of the operator $\mathcal{H}_V = \kappa\Delta_V + \xi_V$ in $V$ are approximated by the corresponding local eigenvalues in the regions $B_{R_V, \text{opt}} := B_{R_V}(z_V) \subset V$ where $\xi_V(\cdot)$ is close to $\xi_{1,V}$, i.e., relevant regions.

Let $\xi(\cdot)$ be an i.i.d. random field with the distribution function $1 - e^{-Q}$. We will show that, if the tails $e^{-Q}$ are lighter than the double exponential function
(i.e., $t^{-1} \log Q(t) \to \infty$) and satisfy additional RV conditions at infinity, then with probability one $\xi_V$ satisfies the assumption of Theorem 2.4; see Theorem 4.6 with $\rho = 0$ and Theorem 3.1(i) for any large $R$ and $\theta(\cdot) \equiv \theta = \text{const}$. In this case, we obtain the second order expansion formulas for the largest eigenvalues $\lambda_{K,V}$ (Theorem 6.11).

2.5. Potentials with dominating bounded islands of high values

In this section, we describe a class of deterministic functions (potential) $\xi_V : V \to [-\infty; \infty)$ for which the asymptotic terms for the principal eigenvalue $\lambda_{1,V}$ ($V \uparrow \mathbb{Z}^\nu$) fill the gap between its lower and upper bounds in (2.2). We use the variational arguments developed by Gärtner and Molchanov (1998). To formulate the results, we need some abbreviations and remarks related to the variational problems. To emphasize the dependence of $\lambda_{1,V}$ on the sample $\xi_V$, we denote by $\lambda(\xi_V) := \lambda_{1,V}$ the principal eigenvalue of the operator $H_V = \kappa \Delta_V + \xi_V$ on $l^2(V)$. As in Section 2.4, let $B_R(z) \subset V$ be the closed ball of radius $R \geq 0$ centered at $z \in V$, and let $B_R := B_R(0)$.

Given a ball $B \subset V$, let $\xi_B(x) := \xi_V(x)$ if $x \in B$, and $\xi_B := -\infty$, otherwise. As before, $H_B := \kappa \Delta_V + \xi_B$ is interpreted as an operator with zero boundary conditions outside $B$. We write

$$
\xi_V(x) = \xi_{1,V} + h_V(x) \quad (x \in V),
$$

where the function $h_V \leq 0$ admits the interpretation as the shape of $\xi_V$-values close to the maximum $\xi_{1,V}$. Note that

$$
\lambda(\xi_V) = \xi_{1,V} + \lambda(h_V) \quad \text{and} \quad \lambda(\xi_B) = \xi_{1,V} + \lambda(h_B).
$$

(2.24)

For a fixed constant $0 < \rho < \infty$, we are interested in the following supremum of $\lambda(h_B)$ over $h : B \to [-\infty; 0]$:

$$
\sup \left\{ \lambda(h_B) : \sum_{x \in B} e^{h(x)/\rho} < 1 \right\}.
$$

This variational problem is equivalent to the corresponding variational problem in terms of the functionals

$$
S^B(p) := \sum_{x \in B} \sqrt{p(x)} \Delta \sqrt{p(x)} \quad \text{where} \quad p(y) = 0 \quad \text{for} \quad y \in \mathbb{Z}^\nu \setminus B,
$$

and

$$
I^B(p) := -\sum_{x \in B} p(x) \log p(x)
$$

for $p(\cdot) \in \mathcal{P}(B)$, the set of probability measures on $B$. More precisely, for a sequence of balls $B_R \subset \mathbb{Z}^\nu$, the following formulas hold true according to (Gärtner and Molchanov 1998, Lemmas 2.17 and 1.10):

$$
\sup \left\{ \lambda(h^B_R) : \sum_{x \in B_R} e^{h(x)/\rho} < 1 \right\}
= \sup \left\{ \lambda(h^B_R) : \sum_{x \in B_R} e^{h(x)/\rho} = 1 \right\}
= \sup_{p \in \mathcal{P}(B_R)} \left( \kappa S^B_R(p) - \rho I^B_R(p) \right),
$$

(2.25)
where the right-hand side of (2.25) converges (as $R \to \infty$) to
\[
\sup_{p \in \mathcal{P}(\mathbb{Z}^\nu)} (\kappa S(p) - \rho I(p)) =: 2 \nu \kappa q(\rho / \kappa). \tag{2.26}
\]

Here $S(p)$ and $I(p)$ are the corresponding functionals on $\mathcal{P}(\mathbb{Z}^\nu)$, the set of probability measures on lattice $\mathbb{Z}^\nu$. It is easy to check that $q : \mathbb{R}_+ \to (0, 1)$ is convex, strictly decreasing and surjective function; $q(0) = 1$ and $q(\infty) = \lim_{\rho \to \infty} q(\rho) = 0$. Moreover, $q(\rho) = (2 \rho \log \rho)^{-1} (1 + o(1))$ as $\rho \to \infty$ (Astrauskas 2008, Proposition 2.1 and Corollary 4.5) and $q(\rho) = 1 + \frac{q}{\kappa} \log \rho + O(\rho)$ as $\rho \to 0$ (Gärtner and Hollander 1999, Section 0.4). The supremum on both sides of (2.25) and (2.26) is attained (Gärtner and Molchanov 1998, Sections 1 and 2.4). Denote by $h_{\text{opt}}^{B^R}$ the maximizer for the variational problem on the left-hand side of (2.25). Then $p_{\text{opt}}^{B^R}$ is the maximizer for the right-hand side of (2.25) if and only if $h_{\text{opt}}^{B^R} = \rho \log p_{\text{opt}}^{B^R}$. For this and further properties of the maximizers in (2.25) and (2.26) in the limit case $R = \infty$, see (Gärtner and Hollander 1999, Sections 0.3 and 0.4) and (Gärtner et al. 2007, Sections 1.3 and 3).

The following theorem tells us that, under reasonable conditions on $\xi_V$, the principal eigenvalue $\lambda_1, V$ of the operator $H_V = \kappa \Delta V + \xi_1, V + h_V$ in $V$ is approximated (letting first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$) by the local principal eigenvalue in the regions $B^R_{\text{opt}} := B^R(z_V) \subset V$ where $h_V$ is close to $h_{\text{opt}}^{B^R}$, i.e., relevant regions with optimal potential shape.

**Theorem 2.5.** Given a constant $0 < \rho < \infty$ and a sequence $R \to \infty$, assume that functions $\xi_V$ satisfy the following conditions:
\[
\lim_{R \to \infty} \limsup_{V} \max_{z \in V} \sum_{y \in B^R(z)} \exp \left\{ \frac{\xi_V(y) - \xi_1, V}{\rho} \right\} \leq 1 \tag{2.27}
\]
and
\[
\liminf_{R \to \infty} \liminf_{V} \min_{z \in V} \max_{y \in B^R(z)} \left( \xi_V(y) - \xi_1, V - h_{\text{opt}}^{B^R}(y - z) \right) \geq 0. \tag{2.28}
\]
Then
\[
\lim_{V} \left( \lambda_1, V - \xi_1, V \right) = 2 \nu \kappa q(\rho / \kappa). \tag{2.29}
\]

**Proof.** Limit (2.29) follows from the results of (Gärtner and Molchanov 1998, Section 2.4) under the stronger conditions on $\xi_V$ including sparseness of clusters of $\xi_V$-extremes. To prove (2.29) under conditions (2.27) and (2.28), we apply the same arguments as in (Gärtner and Molchanov 1998, the proof of Theorem 2.16) combined with the following lemma by Biskup and König (2001), which is slightly modified for the operator $H_V$ with periodic boundary conditions:

**Lemma 2.6.** (Biskup and König 2001, Lemma 4.6). For each $R \in \mathbb{N}$, $V \supset V_0(R)$ and each $\xi_V$,
\[
\lambda(\xi_V) \leq \xi_1, V + \max_{z \in V} \lambda(h_{V}^{B^R(z)}) + \text{const} R^{-1}
\]
for some (universal) const $> 0$. 

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We first obtain the upper bound for $\lambda(\xi_V)$. Condition (2.27) implies that there is a sequence $0 < \varepsilon_R \to 0$ such that

$$\max_{z \in V} \sum_{y \in B_R(z)} \exp \{h(y)/\rho \} < \exp \{\varepsilon_R/\rho \}$$

for each $V \supset V_0(R)$.

In view of (2.24), this estimate and Lemma 2.6 yield that

$$\lambda(\xi_V) - \xi_{1,V} \leq \sup \{ \lambda(h_{B_R}^q) : h(\cdot) \leq 0, \sum_{y \in B_R} e^{h(y)/\rho} < e^{\varepsilon_R/\rho} \} + \frac{\text{const}}{R} \leq \sup \{ \lambda(h_{B_R}^q) : h(\cdot) \leq 0, \sum_{y \in B_R} e^{h(y)/\rho} < 1 \} + \varepsilon_R + \frac{\text{const}}{R}$$

for $V$ as above. Taking the limit as first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$, and using (2.25)–(2.26), we arrive at

$$\limsup_V (\lambda_{1,V} - \xi_{1,V}) \leq 2\nu \kappa q (\rho/\kappa). \quad (2.30)$$

By combining condition (2.28), the monotonicity property of eigenvalues and assertions (2.25)–(2.26), similarly as in the proof of (2.23) we obtain the lower bound

$$\lambda(\xi_V) - \xi_{1,V} \geq \lambda(h_{B_{R,\text{opt}}}^q) + o(1) \to 2\nu \kappa q (\rho/\kappa)$$

letting first $V \uparrow \mathbb{Z}^\nu$ and then $R \to \infty$. This and (2.30) conclude the proof of Theorem 2.5.

If $\rho = \rho_V \to \infty$ in Theorem 2.5, then the “relevant” regions $B_{R,\text{opt}}$ shrink to single sites and, therefore, we are in the situation of Theorem 2.3. Meanwhile, if $\rho = \rho_V \to 0$, then we stick to the result of Theorem 2.4.

Let $\xi(\cdot)$ be an i.i.d. random field with the distribution function $1 - e^{-Q}$. It will be shown that, if the tails $e^{-Q}$ are the double exponential (i.e., $t^{-1} \log Q(t) \to 1/\rho$) and satisfy additional RV conditions at $\infty$, then with probability one $\xi_V$ satisfies the assumptions of Theorem 2.5; see Theorem 4.7 and Theorem 3.1(i) for arbitrarily large $R$ and $\theta(y) \approx 1 - \exp \{h_{B_{R,\text{opt}}}^q(y)/\rho \}$ (see [2, 2.12 and 2.15 in (Gärtner and Molchanov 1998)].) In this case, the second order expansion formula for $\lambda_{1,V}$ holds true (Theorem 6.13).

3. CLUSTERING OF HIGH-LEVEL EXCEEDANCES

Let $\xi(x)$, $x \in \mathbb{Z}^\nu$, be an i.i.d. random field with the cumulative hazard function $Q$. The main task of the present section is to investigate the almost sure asymptotic structure of clusters ("islands") of bounded size formed by exceedances of the sample $\xi_V$ as $V \uparrow \mathbb{Z}^\nu$. With the abbreviations in Section 1, we also need additional notation. For $\theta < 1$, put $L_{V,0} := f((1-\theta) \log |V|)$ where $f := Q^{-1}$. (Without loss of generality, we write $L_{V,1} := \sup \{ t : Q(t) = 0 \}$, so that almost surely $\xi(x) \geq L_{V,1}$ for each $x$.) Let
\[ \mathbb{B}_R(z) := \{ x \in \mathbb{Z}^\nu : |x - z| \leq R \}, \quad \text{and} \quad \mathbb{B}_R := \mathbb{B}_R(0). \]

For fixed \( R \in \mathbb{N} \cup \{0\} \) and a function \( \theta_R(\cdot) : \mathbb{B}_R \rightarrow (-\infty; 1] \), we denote by \( \mathcal{V}_R \) the set of balls \( \mathbb{B}_R(z) \subset \mathcal{V} \), and

\[ \mathcal{E}_{V,\theta}^R := \{ \mathbb{B}_R(z) \subset \mathcal{V} : \xi(y) \geq L_{V,\theta_R(y-z)} \text{ for all } y \in \mathbb{B}_R(z) \}, \]

the subset of clusters of \( \xi_V \)-exceedances in \( \mathcal{V}_R \) over the level function \( L_{V,\theta_R(\cdot)} \). We abbreviate

\[ r(\mathcal{E}_{V,\theta}^R) := \min \{ \text{dist}(\mathbb{B}, \mathbb{B}') : \mathbb{B} \in \mathcal{E}_{V,\theta}^R, \mathbb{B}' \in \mathcal{E}_{V,\theta}^R, \mathbb{B} \neq \mathbb{B}' \} \]

if \( |\mathcal{E}_{V,\theta}^R| \geq 2 \), and

\[ r(\mathcal{E}_{V,\theta}^R) := |\mathcal{V}|^{1/\nu} \text{ if } |\mathcal{E}_{V,\theta}^R| \leq 1, \]

by convention; here \( \text{dist}(\mathbb{B}, \mathbb{B}') \) stands for the lattice \( l^1 \)-distance between balls \( \mathbb{B}, \mathbb{B}' \subset \mathcal{V} \). If \( R = 0 \) and \( \theta := \theta(0) \), then \( \mathcal{E}_{V,\theta}^R \) shrinks to the subset of single \( \xi_V \)-exceedances, so that

\[ r(\mathcal{E}_{V,\theta}^R) = \min \{|x-y| : x \in \mathcal{E}_{V,\theta}, y \in \mathcal{E}_{V,\theta}, x \neq y\}. \]

To state the main result of this section, we also need the following abbreviations

\[ \mu_R := \sum_{y \in \mathbb{B}_R} (1 - \theta_R(y)) \geq 0 \quad \text{and} \quad \theta_{\text{max},R} := \max_{y \in \mathbb{B}_R} \{ \theta_R(y) : \theta_R(y) < 1 \}. \]

**Theorem 3.1.** (cf. Theorems 2.2–2.5). For arbitrarily fixed \( R \in \mathbb{N} \cup \{0\} \), the following almost sure limits hold true.

(i) If \( \mu_R < 1 \), then

\[ \liminf_{V} \frac{\log |\mathcal{E}_{V,\theta}^R|}{\log |\mathcal{V}|} \geq 1 - \mu_R. \]

(ii) If \( \mu_R < 1 \) and, in addition, \( Q \) satisfies the condition

\[ \lim_{t \uparrow t_0} \frac{Q(t-)}{Q(t)} = 1, \] (3.1)

then

\[ \lim_{V} \frac{\log |\mathcal{E}_{V,\theta}^R|}{\log |\mathcal{V}|} = 1 - \mu_R. \]

(iii) If \( \mu_R > 1 \) and \( Q \) satisfies (3.1), then

\[ \lim_{V} |\mathcal{E}_{V,\theta}^R| = 0. \]

(iv) If \( \theta_{\text{max},R} < \mu_R < 1 \) and \( Q \) satisfies (3.1), then

\[ \lim_{V} \frac{\log r(\mathcal{E}_{V,\theta}^R)}{\log |\mathcal{V}|} = \frac{2\mu_R - 1}{\nu}. \]

**Remark 3.2.** (a) Clearly, for arbitrary \( Q \) and \( \mu_R < 1 \),

\[ \mathbb{E}|\mathcal{E}_{V,\theta}^R| = \mathbb{E} \left( \prod_{y \in \mathbb{B}_R} \mathbb{P}(\xi(y) \geq L_{V,\theta_R(y)}(y)) \right) \]

\[ \geq \text{const } |\mathcal{V}| \exp \left\{ - \sum_{y \in \mathbb{B}_R} Q(L_{V,\theta_R(y)}) \right\} \geq \text{const } |\mathcal{V}|^{1-\mu_R} \rightarrow \infty, \]

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according to Lemma A.11(iii) in Appendix.

(b) On the other hand, by Lemma A.11(iii), condition (3.1) implies the asymptotic formula

$$Q(f(s)) = s + \omega(s) \quad \text{as} \quad s \to \infty, \quad (3.2)$$

which in turn yields, for $\mu_R \geq 0$, the upper bound

$$\log \mathbb{E}[\mathcal{E}_{V,\delta}^R] \leq (1 - \mu_R + o(1)) \log |V|$$

as $|V| \to \infty$.

**Remark 3.3.** (see part (iv)). If $\theta_{\max, R} < \mu_R < 1$, then $1/2 < \mu_R < 1$.

**Remark 3.4.** In the case $R = 0$ and $0 < \theta < 1/2$, i.e., single rare $\xi_V$-peaks, Theorem 3.1 was proved by Astrauskas (2001). For the Gaussian random field $\xi(\cdot)$ with correlated values, the case $R = 0$ was studied by Astrauskas (2003). Here the results depend slightly on the correlation function of $\xi(\cdot)$. Finally, assertion (i) generalizes Corollary 2.15(b) in (Gärtner and Molchanov 1998) where the continuity of $Q$ is assumed.

**Proof of Theorem 3.1.** To simplify the proof, we assume throughout that $\theta_R(\cdot) \equiv \theta(\cdot) < 1$ in $\mathbb{B}_R$. The general case is treated similarly.

(i) We denote by $\mathcal{V}_R \subset \mathcal{V}_R$ the maximal subset of nonintersecting balls $\mathbb{B}_R(z)$ in $V$, so that $|\mathcal{V}_R| = |V|$. The claimed bound is proved by estimating $|\mathcal{E}_{V,\delta}^R \cap \mathcal{V}_R|$ similarly as in the proof of Theorem 1 in Astrauskas (2001), where the exceedances $\{\xi(x) \geq L_{V,\delta}(x) \in V\}$ are replaced by mutually independent (multiple) exceedances $\{\xi(x) \geq L_{V,\delta}(x) \in \mathbb{B}_R(z)\} \cap \mathcal{V}_R$. In particular, if $Q(t_Q) = \infty$, we obtain that, for any $-1 < \delta < 0$ and $V \uparrow \mathbb{Z}^d$,

$$P\left(|\mathcal{E}_{V,\delta}^R \cap \mathcal{V}_R| \leq (1 + \delta)\mathbb{E}|\mathcal{E}_{V,\delta}^R \cap \mathcal{V}_R|ight) \leq \exp\left\{-\text{const} (\delta)\mathbb{E}|\mathcal{E}_{V,\delta}^R \cap \mathcal{V}_R|(1 + o(1))\right\}$$

for some const $\delta > 0$. Since the right-hand side is summable over $V$ according to the assertion of Remark 3.2(a), we conclude the proof of (i) by using the Borel-Cantelli lemma.

(ii) We only need to estimate $|\mathcal{E}_{V,\theta}^R|$ from above. Fix a function $\theta'(\cdot) : \mathbb{B}_R \to (-\infty; 1)$ such that $\theta'(\cdot) > \theta(\cdot)$ in $\mathbb{B}_R$, and pick a constant $\delta > 1 - \mu'$ where $\mu' : = \sum_{y \in \mathbb{B}_R}(1 - \theta'(y))$. We then apply Chebyshev’s inequality and the assertion of Remark 3.2(b) to find that, for any $V \supset V_0$,

$$P\left(|\mathcal{E}_{V,\theta'}^R| > |V|^\delta\right) \leq \mathbb{E}|\mathcal{E}_{V,\theta'}^R||V|^{-\delta} \leq |V|^{-\text{const}} \quad (3.3)$$

where const $= \text{const}(\theta'(\cdot), \delta) > 0$. Choose a subsequence $\{V(l) : l \in \mathbb{N}\} \subset \{V\}$ such that

$$V(l) \quad \text{monotonously increases and} \quad |V(l)| = 2^l(1 + o(1)) \quad \text{as} \quad l \to \infty. \quad (3.4)$$

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Since the right-hand side of (3.3) is summable over the subsequence \( \{V(l)\} \), the Borel–Cantelli lemma implies that almost surely \( |\mathcal{E}_{V,l,\omega}^R| \leq |V(l)|^\delta \) for all \( l \geq l_0(\omega) \). Because of the monotonicity of \( \mathcal{E}_{V,l,\omega}^R \) in \( L_{V,l,\omega} \), we obtain that with probability 1, for any \( V \) such that \( V(l-1) \subset V \subseteq V(l) \) and any \( l \geq l_0(\omega;\theta(\cdot),\theta'(\cdot)) \), the set \( \mathcal{E}_{V,l,\omega}^R \) is contained in \( \mathcal{E}_{V,l,\omega}' \), therefore,

\[
|\mathcal{E}_{V,l,\omega}^R| \leq |\mathcal{E}_{V,l,\omega}'| \leq |V(l)|^\delta \leq \text{const} |V|^\delta.
\]

Since \( \theta'(\cdot) > \theta(\cdot) \) and \( \delta > 1 - \mu' \) are chosen arbitrarily, this estimate yields the upper limit bound for \( \log |\mathcal{E}_{V,l,\omega}^R| \), as claimed.

As in part (ii), it suffices to prove the assertions of (iii)–(iv) for the subsequence \( \{V(l)\} \) (3.4) instead of \( \{V\} \).

(iii) We note that \( \mathcal{E}_{V,l,\omega}^R \neq \emptyset \) if and only if there exists \( B_R(z) \subset V \) such that \( \xi(\cdot) \geq L_{V,\theta(\cdot) - z} \) in \( B_R(z) \). According to the assertion of Remark 3.2(b), the probability of the last event does not exceed \( \mathbb{P}|\mathcal{E}_{V,l,\omega}^R| \leq |V|^{-\rho} \) for some \( 0 < \rho < -1 + \mu \). Therefore,

\[
\mathbb{P}\left( |\mathcal{E}_{V,l,\omega}^R| = \emptyset \right) \leq |V|^{-\rho}.
\]

Since the latter is summable over \( \{V(l)\} \) (3.4), the Borel–Cantelli lemma yields that almost surely \( \mathcal{E}_{V,l,\omega}^R = \emptyset \) for all \( l \geq l_0(\omega) \), as claimed.

(iv) With \( V_R \subset V \) defined in part (i), the almost sure upper bound for \( r(\mathcal{E}_{V,l,\omega}^R \cap V_R) \geq r(\mathcal{E}_{V,l,\omega}^R) \) is derived similarly as in the proof of Theorem 2 in Astrauskas (2001) where the exceedances \( \{\xi(x) \geq L_{V,\theta(\cdot)} \} \) \( x \in V \) are replaced by mutually independent (multiple) exceedances \( \{\xi(\cdot) \geq L_{V,\theta(\cdot) - z} \} \) \( \mathbb{B}_R(z) \in \mathbb{B}(R) \).

To obtain the lower bound for \( r(\mathcal{E}_{V,l,\omega}^R) \), we first note that the event \( \{r(\mathcal{E}_{V,l,\omega}^R) = 0, |\mathcal{E}_{V,l,\omega}^R| \geq 2\} \) implies that there exists \( B_R(z) \subset V \) such that \( \xi(\cdot) \geq L_{V,\theta(\cdot) - z} \) in \( B_R(z) \) and \( \xi(y) \geq L_{V,\theta_{\text{max}}} \) for some \( y \in (\mathbb{B}_R(z) \setminus B_R(z)) \cap V \). Therefore, as in the proof of (iii) we obtain that, for fixed \( y \in \mathbb{Z}^\nu \setminus B_R \) and for any \( V \supset V_0 \),

\[
\mathbb{P}\left( r(\mathcal{E}_{V,l,\omega}^R) = 0, |\mathcal{E}_{V,l,\omega}^R| \geq 2 \right) \leq \text{const} |V|^\rho \mathbb{P}\left( \xi(\cdot) \geq L_{V,\theta(\cdot)} \text{ in } B_R, \xi(y) \geq L_{V,\theta_{\text{max}}} \right) \leq |V|^{-\rho}
\]

for some \( 0 < \rho < \mu - \theta_{\text{max}} \). Second, similarly as in the proof of part (i), we find that

\[
\mathbb{P}\left( |\mathcal{E}_{V,l,\omega}^R| < 2 \right) \leq |V|^{-\text{const}} \text{ for any } V \supset V_0 \text{ and some const } > 0.
\]

Summarizing these bounds and picking \( 0 < \varepsilon < (2\mu - 1)/\nu \) arbitrarily, we get that, for any \( V \supset V_0 \),

\[
\mathbb{P}\left( r(\mathcal{E}_{V,l,\omega}^R) < |V|^\varepsilon \right) \leq \mathbb{P}\left( 1 \leq r(\mathcal{E}_{V,l,\omega}^R) < |V|^\varepsilon, |\mathcal{E}_{V,l,\omega}^R| \geq 2 \right) + |V|^{-\text{const}_1} \leq |V|^{-\text{const}_2}
\]

for some \( \text{const}_i > 0 \), where the last probability is estimated similarly as in the proof of Theorem 2 of (Astrauskas 2001) with mutually independent (multiple) exceedances \( \{\xi(\cdot) \geq L_{V,\theta(\cdot) - z} \} \) instead of \( \{\xi(x) \geq L_{V,\theta(\cdot)} \} \). Since the right-hand side of (3.5) is again summable over \( \{V(l)\} \) (3.4), we conclude from the Borel–Cantelli lemma that almost surely \( r(\mathcal{E}_{V,l,\omega}^R) \geq |V(l)|^\varepsilon \) for any \( l \geq l_0(\omega;\varepsilon) \), as claimed. This completes the proof of Theorem 3.1. \( \square \)

Remark 3.5. By the same arguments as in the proof above, the assertions of Theorem 3.1 are extended to the following class of high-level exceedances:
For fixed \( R \in \mathbb{N} \cup \{0\} \), we denote by \( S_R \) the set of all subsets \( U \subset \mathbb{Z}^* \), the diameter of which does not exceed \( R \). Let \( \mathcal{V}_R := \{ U \in S_R : U \subset V \} \). For a fixed set of functions \( \Theta_R := \{ \theta_{U,R}(\cdot) \in (-\infty;1]: U \in S_R \} \), let \( \mathcal{E}^R_{V,\Theta} \subset \mathcal{V}_R \) be the subset of elements \( U \in \mathcal{V}_R \) such that \( \xi(\cdot) \geq L_{\mathcal{V},\theta_{U,R}(\cdot)} \) in \( U \). I.e., \( \mathcal{E}^R_{V,\Theta} \) consists of clusters of exceedances in \( \mathcal{V}_R \) over level functions \( L_{\mathcal{V},\Theta} \). Denote by \( r(\mathcal{E}^R_{V,\Theta}) \) the minimum distance among elements \( U,U' \in \mathcal{E}^R_{V,\Theta} \), \( U \neq U' \). Finally, let \( \mu_R := \sum_{y \in U}(1-\theta_{U,R}(y)) \) be a positive constant independent of \( U \in S_R \), and write \( \theta_{\max,R} := \sup_{U \in S_R} \max_{y \in U} \theta_{U,R}(y) \). With these notation for \( \mathcal{E}^R_{V,\Theta} \) and \( r(\mathcal{E}^R_{V,\Theta}) \), the almost sure assertions (i)–(iv) of Theorem 3.1 hold true.

4. SPACINGS

4.1. Spacings of consecutive order statistics

We first formulate the results for the exponential order statistics \( \eta_{K,V} \) and their spacings, which are then transferred to \( \xi_{K,V} = f(\eta_{K,V}) \) under appropriate conditions for \( f \).

Note that the random variables

\[
\eta_{1,V} - \eta_{2,V}, \ldots, (|V| - 1)(\eta_{|V|-1,V} - \eta_{|V|,V}), |V|\eta_{|V|,V}
\]

are mutually independent exponentially distributed with mean 1; see, e.g., Shorack and Wellner (1986, pp. 336). This property immediately implies the first assertion of the following lemma.

**Lemma 4.1.**

(i) For fixed \( K \in \mathbb{N} \),

\[
\lim_{V} \mathbb{P}(\eta_{1,V} - \eta_{2,V} > t_1, \ldots, \eta_{K-1,V} - \eta_{K,V} > t_{K-1}, \eta_{K,V} - \log |V| > t) = \left( \prod_{l=1}^{K-1} e^{-t_l} \right) \frac{1}{(K-1)!} \int_{t}^{\infty} \exp \{-Ks - e^{-s}\} \, ds
\]

for all \( t_l \geq 0 \) (\( 1 \leq l \leq K - 1 \)) and all \( t \in \mathbb{R} \).

(ii) For an arbitrary sequence \( \{ K_V \} \) such that \( 1 \leq K_V \leq |V| \),

\[
\limsup \sqrt{K_V} \max_{K_V \leq k \leq |V|} \left| \eta_{k,V} - \log \frac{|V|}{k} \right| < \infty \quad \text{in probability.}
\]

**Proof.** Let us show (ii). Write \( \eta_{|V|+1,V} := 0 \). By Kolmogorov’s inequality (Shorack and Wellner 1986, pp. 843), we have that

\[
P\left( \max_{K_V \leq k \leq |V|} \left| \sum_{l=k}^{V} (\eta_{l,V} - \eta_{l+1,V} - \frac{1}{T}) \right| > \left( \frac{C}{K_V} \right)^{1/2} \right) \leq \frac{K_V}{C} \sum_{l=K_V}^{V} \mathbb{E}\left( \eta_{l,V} - \eta_{l+1,V} - \frac{1}{T} \right)^2 \leq \frac{2}{C}
\]
for any \( C > C_0 \) and any \( V \supset V_0(C) \). Combining this bound with the following simple estimate
\[
\max_{k_V \leq k \leq |V|} \left( \frac{|V|}{k} - \log \frac{|V|}{k} \right) \leq \frac{1}{K_V} \quad (V \supset V_0),
\]
we obtain the claimed assertion of (ii).

The almost sure asymptotic behavior of the random variables \( \eta_{K,V} \) and \( \eta_{K,V} - \eta_{K+1,V} \ (|V| \to \infty) \) is more intricate.

**Lemma 4.2.** For any fixed constants \( K \in \mathbb{N} \) and \( m \in \mathbb{N} \setminus \{1\} \), the following almost sure limits hold true.

1. \( \liminf_{V} \frac{\log(\eta_{K,V} - \eta_{K+1,V}) + \sum_{i=2}^{m-1} \log_i |V|}{\log_m |V|} = -1 \),

2. \( \limsup_{V} \frac{\eta_{K,V} - \eta_{K+1,V} - K^{-1} \sum_{i=2}^{m-1} \log_i |V|}{\log_m |V|} = \frac{1}{K} \),

and

3. \( \limsup_{V} \max_{1 \leq i \leq |V|} \left| \eta_{i,V} - \log \frac{|V|}{t} \right| \frac{1}{\log_2 |V|} = 1 \);

where \( \sum_{2}^{\infty} \ldots := 0 \).

**Proof.** Assertions (i) and (ii) follow from more general results for exponential spacings in Astrauskas (2006, Corollary 12). Assertion (iii) follows from the corresponding strong limits for the uniform order statistics \( u_{|V| - k + 1,V} \) (Shorack and Wellner 1986, pp. 408 and pp. 420–424) via transformation \( \eta_{k,V} = -\log u_{|V| - k + 1,V} \) \( (1 \leq k \leq |V|) \).

We now turn to the case \( \xi_{k,V} = f(\eta_{k,V}) \). For \( p > 0 \), we denote by \( \text{AIP}_p^{\infty} \) the class of functions \( f := Q^r \) that satisfy
\[
\lim_{s \to \infty} f(s)^p (f(s + c) - f(s)) = \infty \quad \text{for any} \quad c > 0, \tag{4.1}
\]
and by \( \text{AIP}_0^p \) the class of functions \( f \) that satisfy
\[
\lim_{s \to \infty} f(s)^p (f(s + c) - f(s)) = 0 \quad \text{for any} \quad c > 0, \tag{4.2}
\]
and, finally, \( \text{OAIIP}^p \) stands for the class of \( f \) satisfying
\[
f(s)^p (f(s + c) - f(s)) \asymp 1 \quad \text{as} \quad s \to \infty, \quad \text{for any} \quad c > 0. \tag{4.3}
\]

We see that, if \( f \) is in \( \text{AIP}_\infty^p \) or \( \text{OAIIP}^p \), then the right endpoint \( t_Q \) is infinity or, equivalently, \( f(s) \to \infty \) as \( s \to \infty \). Of course, \( \text{AIP}_0^p \) includes the trivial case of finite \( t_Q > 0 \). The characterization of \( \text{AIP}_\infty^p \), \( \text{AIP}_0^p \) and \( \text{OAIIP}^p \) is given in Lemmas A.6, A.7 and A.8 of Appendix A respectively. In particular, the functions \( f := Q^r \in \text{OAIIP}^p \) are associated with Weibull type distributions \( 1 - e^{-Q} \), where \( Q(t) \asymp t^{p+1} \) as \( t \to \infty \).

**Theorem 4.3.** (Cf. (2.7) and (Astrauskas 2012; 2013)). For fixed natural \( K > l \geq 1 \) and real \( p > 0 \), we have the following limits in probability.

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(i) If \( f \in \text{AIP}_\infty \), then
\[
\lim_{\xi \to \infty} \xi_{\xi, V} = \infty.
\]
(4.4)

(ii) If \( f \in \text{AIP}_0 \), then
\[
\lim_{\xi \to \infty} \xi_{\xi, V} = 0.
\]

(iii) If \( f \in \text{OAIp} \), then
\[
\xi_{\xi, V} \approx 1 \text{ as } |V| \to \infty.
\]

Proof. Using notation (1.11) and (1.12), rewrite the left-hand side of (4.4) in the form
\[
f(\eta_{\eta, V})^p \left( f(\eta_{\eta, V} + \eta_{\eta, V} - f(\eta_{\eta, V})) - f(\eta_{\eta, V}) \right).
\]
The claimed assertions follow by applying Lemma 4.1(i). \( \square \)

To obtain these limits with probability 1, we need the stronger conditions for \( f \).

Let us abbreviate
\[
d_m,\gamma(s) := \log_i s)(\log_i s)^{1+\gamma} \quad (s \geq s_0).
\]

Theorem 4.4. (Cf. (2.7)). For fixed constants \( K \in \mathbb{N} \) and \( p \geq 0 \), the following almost sure limits hold true.

(i) If \( \lim_{s \to \infty} f(s)^p(f(s + 1/d_m,\gamma(s)) - f(s)) = \infty \) for some \( m \in \mathbb{N} \) and \( \gamma > 0 \), then
\[
\lim_{\xi \to \infty} \xi_{\xi, V} = \infty.
\]

(ii) If \( \lim_{s \to \infty} f(s)^p(f(s + K^{-1} \log d_m,\gamma(s)) - f(s)) = 0 \) for some \( m \in \mathbb{N} \) and \( \gamma > 0 \), then
\[
\lim_{\xi \to \infty} \xi_{\xi, V} = 0.
\]

(iii) If \( \lim_{s \to \infty} (f(s + \log s) - f(s)) = 0 \), then
\[
\lim_{\xi \to \infty} \xi_{\xi, V} = 0.
\]

Proof. Assertions (i)–(ii) follow by the same arguments as in the proof of Theorem 4.3, where one applies Lemma 4.2 instead of Lemma 4.1(i). Assertion (iii) follows from Lemma 4.2 (iii). \( \square \)

4.2. Spacings of intermediate order statistics

We denote by \( \text{PI}_{\leq 2} \) the class of functions \( f := Q^* \) satisfying the condition
\[
\limsup_{s \to \infty} \frac{f((1 - \varepsilon)s)}{f(s)} < 1 \quad \text{for some } 0 < \varepsilon < 1/2.
\]

Class (4.5) is characterized in Lemma A.13 (Appendix A).
Theorem 4.5. (Cf. (2.8) ). Assume that \( f \in \text{PI}_{<2} \) (4.5). Then for fixed \( K \in \mathbb{N} \) and \( \theta > \varepsilon \), almost surely

\[
\limsup_{V} \frac{\xi_{|V|^\rho}, V}{\xi_{K, V}} < \text{const} < 1. \tag{4.6}
\]

**Proof.** By Lemma 4.2(iii), with probability one the random variable \( \xi_{K, V} = f(\eta_{K, V}) \) is bounded from below by \( f(\log |V| - 2 \log |V|) \) and \( \xi_{|V|^\rho}, V \) is bounded from above by \( f((1 - \theta) \log |V| + 2 \log |V|) \) for each \( V \supset V_0(\omega) \). Substituting these bounds into the left-hand side of (4.6) and using (4.5), we obtain the claimed assertion. \( \square \)

We denote by \( RV_\rho \) the class of nondecreasing functions \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) such that, for any \( c > 1 \), \( \lim_{s \to \infty} g(cs)/g(s) = c^\rho \). I.e., \( g \) is regularly varying at infinity with index \( 0 \leq \rho \leq \infty \). The case \( \rho = \infty \) (resp., \( \rho = 0 \)) indicates a rapid variation (resp., slow variation) of the function \( g \). See Lemma A.3 in Appendix A for a summary of the well-known properties of the class \( RV_\rho \).

**Theorem 4.6.** (Cf. (2.13) and (2.22) ). For some \( 0 \leq \rho \leq \infty \), assume that \( e^f \in RV_\rho \). Then, for all constants \( 0 \leq \varepsilon < \theta < 1 \), almost surely

\[
\lim_{V} (\xi_{|V|^\rho}, V - \xi_{|V|^\rho}, V) = \rho \log \frac{1 - \varepsilon}{1 - \theta}.
\]

**Proof.** To prove this assertion, use Lemma 4.2(iii) and Lemma A.3(ii) similarly as in the proof of Theorem 4.5. \( \square \)

The following statement is closely related to the result of Theorem 4.6 with \( 0 < \rho < \infty \). As in Section 2.4, let \( \mathbb{B}_R(z) \) denote the closed ball in \( V \) with the center \( z \in V \) and the radius \( R \geq 0 \).

**Theorem 4.7.** (Cf. (2.27) and (Gärtner and Molchanov 1998)). For some \( 0 < \rho < \infty \), assume that \( e^f \in RV_\rho \). Then, for any fixed \( R \in \mathbb{N} \), almost surely

\[
\limsup_{V} \max_{z \in V} \sum_{y \in \mathbb{B}_R(z)} \exp \left\{ \frac{(\xi(y) - \xi_{1, V})/\rho}{1} \right\} \leq 1.
\]

**Proof.** For continuous \( Q \), this assertion is a straightforward consequence of Corollary 2.12 in (Gärtner and Molchanov 1998) and Theorem 4.4(iii) above. (In view of Lemma A.3(ii), the condition of Theorem 4.4(iii) follows from the assumption of Theorem 4.7). If the continuity condition on \( Q \) is dropped, one applies slightly modified arguments based on the technique of function inversion, e.g., Lemma A.11 in Appendix A. \( \square \)

Recall that the conditions of Theorems 4.6 and 4.7 are discussed in Lemma A.3. In particular, the assumption of Theorem 4.7 implies that \( \log Q(t) = t/\rho + o(t) \) as \( t \to \infty \), i.e., the double exponential tails \( e^{-Q} \).
4.3. Minimum of spacings

We first recall some notation from Section 2.3. For fixed $0 < \theta < 1/2$, we write $\tilde{\xi}(x) := \xi(x)$ if $\xi(x) < f((1 - \theta) \log |V|)$, and $\tilde{\xi}(x) := 0$, otherwise. For any $z \in V$, let $\tilde{\lambda}(z)$ denote the principal eigenvalue of the “single peak” Hamiltonian $\kappa \Delta_V + \xi(z) \delta_z + \tilde{\xi}_V(1 - \delta_z)$ in $l^2(V)$. As in (2.12), let $\tilde{\lambda}_{k,V}$ denote the $K$th extreme order statistics of the random field $\tilde{\lambda}_V$. For any $\kappa \geq 0$, we are interested in the asymptotic behavior of the minimum of the gaps $\tilde{\lambda}_{k,V} - \tilde{\lambda}_{k+1,V}$ $(1 \leq k \leq |V|^\varepsilon)$ defined by

$$S_{V,\varepsilon} := \min \{\tilde{\lambda}_{k,V} - \tilde{\lambda}_{k+1,V} : 1 \leq k \leq |V|^\varepsilon\}.$$ 

Given a constant $\mu > 0$, we say that the function $F : \mathbb{R} \to \mathbb{R}$ is log-Hölder continuous of order $\mu > 0$ at infinity, if $F$ satisfies the following condition:

$$|F(t+s) - F(t-s)||\log s|^\mu = O(1) \quad \text{as} \quad t \to \infty \text{ and } s \downarrow 0 \text{ simultaneously.} \quad (4.7)$$

**Theorem 4.8.** (Cf. (2.14)). Let $t_Q = \infty$, $\kappa \geq 0$ and $0 < \varepsilon < \theta < 1/2$, and assume that the distribution tails $e^{-Q}$ are log-Hölder continuous of order $\mu > 0$ at infinity. For $\kappa > 0$, assume additionally that $e^f \in RV_\infty$. Then almost surely

$$\limsup_{V} \frac{\log \{ - \log (S_{V,\varepsilon} \wedge 1) \}}{\log |V|} \leq \frac{1 + \varepsilon}{\mu}.$$ 

**Proof.** The assertion follows from Lemmas 3.5 and 4.3 in (Astrauskas 2008) and Theorem 3.1(ii) above, where $R = 0$ and $0 < \theta < 1/2$. \qed 

In (Astrauskas 2003), the results of Theorem 4.8 are extended to the Gaussian random fields with correlated values.

We end this section with some generalization of Theorem 4.3(iii) for the functions $f \in OAIP$ (4.3) associated with Weibull type distributions $1 - e^{-Q}$, where $Q(t) \propto t^{p+1}$ as $t \to \infty$.

**Theorem 4.9.** (Cf. Lemma 4.2 in Astrauskas (2013)). For some $p \geq 0$, assume that $f \in OAIP$ (4.3). Then, for arbitrarily fixed constants $K \in \mathbb{N}$, $0 < \varepsilon < 1$ and any sequence $\{n_V\} \subset \mathbb{N}$ such that $n_V = O(|V|^\varepsilon)$, we have the following limits in probability:

(i) $\xi_{n_V, V} \asymp (\log |V|)^{1/(p+1)}$ as $|V| \to \infty$,

and

(ii) $0 < \liminf_{V} \min_{K+1 \leq l \leq |V|^\varepsilon} \xi_{l,V}^p (\xi_{K,V} - \xi_{l,V}) \frac{1}{\log l} \leq \limsup_{V} \max_{K+1 \leq l \leq |V|^\varepsilon} \xi_{l,V}^p (\xi_{K,V} - \xi_{l,V}) \frac{1}{\log l} < \infty.$
Proof. Assertion (i) follows from a combination of the formula \(\xi_{k,V} = f(\eta_{k,V})\), Lemma 4.1(ii) and the limit \(f(s) \sim s^{1/(p+1)}\) as \(s \to \infty\) (the latter follows from Lemma A.8(iii) with \(a(\cdot) \equiv \text{const}\)). Assertion (ii) is shown by combining the formula \(\xi_{k,V} = f(\eta_{k,V})\) and Lemmas 4.1(ii), A.8(iii) similarly as in the proof of Theorems 4.3 and 4.5 above. \(\square\)

5. NEIGHBORING EFFECTS

We finally study the asymptotic properties of \(\eta_{V}\)-values neighboring to \(\eta_{V}\)-peaks. It is then straightforward to extend the results for \(\eta(\cdot)\) to \(\xi(\cdot) = f(\eta(\cdot))\).

The following lemma tells us that, for fixed \(y \neq 0\) and for small \(\varepsilon > 0\), asymptotic properties of the random variables \(\eta(z_{k,V} + y)\) \((1 \leq k \leq |V|\)) and their extremes are the same as in the case of exponential i.i.d. random variables.

Lemma 5.1. For fixed \(y \in \mathbb{Z}^\varepsilon\backslash\{0\}\), \(0 < \varepsilon < 1/2\) and a sequence of integers \(K := K_V = O(|V|\varepsilon)\), the following assertions hold true.

(i) \(\lim_{V} \mathbb{P}(\eta(z_{K,V} + y) > t) = e^{-t}\) for all \(t \geq 0\).

(ii) If, in addition, \(K := K_V \to \infty\), then

\[
\lim_{V} \mathbb{P}\left(\max_{1 \leq l \leq K} \eta(z_{l,V} + y) - \log K \leq t\right) = \exp\{-e^{-t}\} \quad \text{for all } t \in \mathbb{R}.
\]

(iii)

\[
\lim_{M \to \infty} \limsup_{V} \left| \max_{1 \leq l \leq |V|\varepsilon} \frac{\eta(z_{l,V} + y)}{\log l} - 1 \right| = 0 \quad \text{in probability}.
\]

Proof. Here and in the sequel, we need the following key statement (which is frequently used in (Astrauskas 2013) as well).

Lemma 5.2. Fix a finite subset \(U \subset \mathbb{Z}^\varepsilon\backslash\{0\}\), \(U \neq \emptyset\), and a sequence of nonrandom real functions \(\{D_{t}(u_{l}) : t_{l} \in \mathbb{R}^{U}\} \ (l \in \mathbb{N})\). Abbreviate \(\eta(z;l) := D_{l}(\{\eta(z+x) : x \in U\})\) for \(z \in \mathbb{Z}^\varepsilon\) and \(l \in \mathbb{N}\). Finally, pick a sequence of integers \(K := K_V = O(|V|\varepsilon)\) for some \(0 < \varepsilon < 1/2\). Then, for any \(V\) and any \(t \in \mathbb{R}\),

\[
\left| \mathbb{P}\left(\max_{1 \leq l \leq K} \eta(z_{l,V};l) \leq t\right) - \prod_{l=1}^{K} \mathbb{P}(\eta(0;l) \leq t) \right| \leq 3|V|^{-\text{const}},
\]

where \(\text{const} > 0\) does not depend on \(V\) and \(t\).

Now, part (i) of Lemma 5.1 follows from Lemma 5.2 with \(U := \{y\}\), where \(D_{K}(t_{U}) \equiv t_{y}\) and \(D_{l}(t_{U}) \equiv 0\) for \(l \neq K\). Part (ii) follows from Lemma 5.2, where \(D_{l}(t_{U}) \equiv t_{y}\) \((l \in \mathbb{N})\), combined with Lemma 4.1(i). Finally, by Lemma 5.2 with \(D_{l}(t_{U}) \equiv t_{y}/\log l\) \((l \geq 2)\), we derive that, for any small \(\delta > 0\),

\[
\limsup_{V} \mathbb{P}\left(\max_{M \leq l \leq |V|\varepsilon} \frac{\eta(z_{l,V} + y)}{\log l} > 1 + \delta\right) \leq \sum_{l=M}^{\infty} e^{-(1+\delta)\log l} \to 0
\]
and
\[
\limsup_{V} \mathbb{P}\left( \max_{M \leq |V|} \frac{\eta(z_{i, l} + y)}{\log l} < 1 - \delta \right) \leq \prod_{l=M}^{\infty} (1 - e^{-(1-\delta)\log l}) = 0
\]
as \(M \to \infty\), i.e., assertion (iii) of Lemma 5.1 is proved.

**Proof of Lemma 5.2.** Fix a constant \(\theta \in (\varepsilon, \frac{1}{2})\), so that \(K := K_V \leq \frac{1}{2}|V|^\theta\) for each \(V \supset V_0\). Write \(L_V := (1 - \theta) \log |V|\). Denote by \(\mathcal{E}_V \subset V\) the subset consisting of sites at which \(\eta()\) exceeds the level \(L_V\), and let \(r(\mathcal{E}_V)\) be the minimum distance among sites in \(\mathcal{E}_V\); cf. the notation at the beginning of Section 3. We abbreviate by \(I\) the 

intervals \((-\infty, t]\) or \((t, \infty)\), where \(t \in \mathbb{R}\). Further, pick \(\delta\) to satisfy \(0 < \delta < (1 - 2\theta)/\nu\). Now

\[
\mathbb{P}\left( \max_{1 \leq i \leq K} \eta(z_{i, l}) \in I \right) = \mathbb{P}\left( \max_{1 \leq i \leq K} \eta(z_{i, l}, l) \in I, \ 2^{-1}|V|^\theta \leq |\mathcal{E}_V| \leq 2|V|^\theta, \ r(\mathcal{E}_V) > |V|^\delta \right)
+ \mathbb{P}(|\mathcal{E}_V| < 2^{-1}|V|^\theta) + \mathbb{P}(|\mathcal{E}_V| > 2|V|^\theta) + \mathbb{P}(r(\mathcal{E}_V) \leq |V|^\delta)
= : p(I) + p^{(1)} + p^{(2)} + p^{(3)}.
\]

Using the continuity of exponential distribution, similarly as in the proof of Theorem 3.1(ii)-(iv) with \(R = 0\) and \(0 < \theta < 1/2\), we obtain that \(p^{(i)} \leq |V|^{-\text{const}}\) for some \(\text{const} > 0\). Thus, to show the assertion of Lemma 5.2, we need to check that

\[
p(I) \leq \mathbb{P}\left( \max_{1 \leq i \leq K} \eta(x_i, l) \in I\right) + |V|^{-\text{const}}, \quad \quad \quad (5.1)
\]
for a fixed (nonrandom) subset \(\widetilde{V} := \{\tilde{x}_l: 1 \leq l \leq K\} \subset \mathbb{Z}^n\) such that \(r(\widetilde{V}) > |V|^\delta\).

Let \(\sum_{V'}\) be the sum over all subsets \(V' \supset V\) with the properties: \(\frac{1}{2}|V|^\theta \leq |V'| \leq 2|V|^\theta\) and \(r(V') > |V|^\delta\). We denote by \(\sum_{\{x_i\}}\) the sum over all permutations \(x_1, \ldots, x_{|V'|}\) of the subset \(V'\). Write \(p_{V'} = |V'|^{-1}.\) Then, for \(V \supset V_0\),

\[
p(I) \leq \sum_{V'} \sum_{\{x_i\}} \mathbb{P}\left( \max_{1 \leq i \leq K} \eta(x_i, l) \in I, \eta(x_1) \geq \eta(x_2) \geq \ldots \geq \eta(x_{|V'|}) \geq L_{V'}, \max_{x \in V \setminus V'} \eta(x) < L_V \right)
\leq \sum_{V'} \sum_{\{x_i\}} \mathbb{P}\left( \max_{1 \leq i \leq K} \eta(x_i, l) \in I, \eta(x_1) \geq \eta(x_2) \geq \ldots \geq \eta(x_{|V'|}) \geq L_V, \max\{\eta(x): x \in V \setminus ((V' + U) \cup V')\} < L_V \right), \quad \quad \quad (5.2)
\]
where \(V' + U\) denotes the algebraic sum of the subsets \(V'\) and \(U\). Since all the random variables are mutually independent, the double sum on the right-hand side of (5.2) is equal to

\[
\mathbb{P}\left( \max_{1 \leq i \leq K} \eta(x_i, l) \in I\right) \sum_{V'} p_{V'}^{\mid V'\mid} (1 - p_{V'})^{|V|-(|U|+1)|V'|}
\leq \mathbb{P}\left( \max_{1 \leq i \leq K} \eta(x_i, l) \in I\right) + \text{const} |V|^{2\theta-1} \quad (V \supset V_0),
\]

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since |V'|p_V \asymp |V|^{2\theta - 1} via the notation. This completes the proof of (5.1). Lemma 5.2 is proved.

We now turn to the case $\xi(\cdot) = f(\eta(\cdot))$.

**Theorem 5.3.** (Cf. (2.10)). Fix $y \in \mathbb{Z}^n \setminus \{0\}$ and $K \in \mathbb{N}$. Then

$$\limsup_{V} |\xi(z_{K,V} + y)| < \infty \quad \text{in probability}.$$ 

**Proof.** The assertion follows from Lemma 5.1(i). $\square$

To the end of this section, let us fix constants $0 < \varepsilon < \theta < 1/2$. With $L_{V,\theta}$ as in Section 3, we write

$$\tilde{\xi}(x) := \xi(x) \text{ if } \xi(x) < L_{V,\theta}, \quad \text{and } \tilde{\xi}(x) := 0, \text{ otherwise.}$$ 

For natural $K \geq 1$ and $l > K$, we put

$$\chi_{K,V}(l) := \xi_{K,V}^2 \left( \xi_{K,V} + 2\nu \kappa^2 \tilde{\xi}_{K,V}^{-1} - \xi_l,V \right. 
\left. - \kappa^2 \sum_{|x|=1} (\xi_{K,V} - \tilde{\xi}(z_l,V + x))^{-1} \right) I \{ \xi_{K,V} > L_{V,\theta} \};$$

here $I_{\Omega'} := I_{\Omega'}(\omega)$ denotes the indicator of $\Omega' \subset \Omega$. To study the asymptotic behavior of variables (5.3), we introduce the class $S\Pi_{2\infty}$ of functions $f := Q^\omega$ such that

$$\lim_{s \to \infty} \inf_{a \in (c, \theta s)} \left( f(s)^2 (f(s + a) - f(s)) - \frac{f(2a)}{c} \right) = \infty$$
for any $0 < c < 1$ and some $0 < \theta < 1/2$.

The class $S\Pi_{2\infty}$ is a strict subset of $A_{2\infty}$ (4.1). The following theorem provides some generalization of limit (4.4) for $p = 2$.

**Theorem 5.4.** (Cf. (2.11)). Fix $K \in \mathbb{N}$. If $f$ belongs to the classes $S\Pi_{2\infty}$ (5.4) and $P_{1,2}$ (4.5) with $\varepsilon$ and $\theta$ as above, then

$$\lim_{V} \min_{K+1 \leq l \leq 2|V|^\varepsilon} \chi_{K,V}(l) = \infty \quad \text{in probability.}$$ 

**Proof.** We begin with estimating $\chi(l) := \chi_{K,V}(l)$ for $K + 1 \leq l \leq M; \, M \geq M_0$. Let $\Omega_{V,M}^{(1)} \in \mathcal{F}$ denote the subset of configurations $\xi^{(\omega)}_{V}$ satisfying the following three inequalities:

$$\max_{|x|=1} \max_{K} \xi(z_l,V + x) \leq f(2 \log M),$$

$$\xi_{K+1,V} > 0 \quad \text{and } \quad \tilde{\xi}(\cdot) \leq \frac{L_{V,\theta}}{\xi_{K,V}} \leq \text{const}' < 1$$

$$\xi_{K+1,V} \geq 0$$

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for \( \text{const}^\prime > \text{const} (\varepsilon) \) specified in Theorem 4.5. According to Lemma 5.1(ii) and Theorem 4.5, we obtain that \( \limsup V \mathbb{P}(\Omega \setminus \Omega_{V,M}) \to 0 \) as \( M \to \infty \). On the other hand, expanding the sum \( \sum_{|x|=1}^\infty \) in (5.3) over powers of \( \tilde{\xi}(z_{l,V}+x)/\xi_{K,V} \) with \( K+1 \leq l \leq M \), we get that, for any \( M \geq M_0 \) and any \( V \supset V_0(M) \), the inequalities (5.5) and (5.6) imply the following estimate

\[
\begin{align*}
\chi(l) &\geq \frac{1}{2} \log |V| \log l - \text{const} f \left( \frac{1}{2} \log l \right), \\
\text{min}_{M \leq |V|^\theta} \chi(l) &\geq \min_{M \leq |V|^\theta} \left[ \xi_{K,V}^2 (\xi_{K,V} - \xi_{K+1,V}) - \text{const} \sum_{|x|=1}^\infty \xi_{l,V} (z_{l,V}+x) \right], \\
\text{min}_{M \leq |V|^\theta} \chi(l) &\geq \min_{M \leq |V|^\theta} \left[ f \left( \log \frac{|V|}{l} \right)^2 f \left( \log \frac{|V|}{M} \right) - f \left( \log \frac{|V|}{l} + 1 \right) \right] - \text{const} f \left( \frac{3}{2} \log l \right).
\end{align*}
\]
Using this implication combined with the fact that, by condition (5.4), the right-hand side of (5.10) tends to infinity as $|V| \to \infty$, we obtain that, for any $C > 0$,

$$\limsup_{V} \mathbb{P} \left( \min_{M \in \{1,2\} |V|^\nu} \chi(l) \leq C \right) \leq \limsup_{V} \mathbb{P} (\Omega \setminus \Omega^{(2)}_{\nu,M}) \to 0$$

as $M \to \infty$. This limit and (5.7) yield the assertion of Theorem 5.4. \qed

6. POISSON LIMIT THEOREMS FOR THE LARGEST EIGENVALUES

This section is to provide an overview of current results on extreme value theory for the spectrum of the Anderson Hamiltonian $H_V = \kappa \Delta_V + \xi_V$, $V \uparrow \mathbb{Z}^\nu$, with an i.i.d. potential $\xi(\cdot)$. The results under consideration are taken from (Astrauskas and Molchanov 1992), (Astrauskas 2007; 2008; 2012; 2013), (Bishop and Wehr 2012), (Gérminet and Klopp 2013), (Gärtner and Molchanov 1998) and (Biskup and König 2013).

In Section 6.1, we give Poisson limit theorems for the largest eigenvalues and the corresponding localization centers, provided the distribution tails $e^{-Q}$ of $\xi(0)$ are heavier than the double exponential function (Theorems 6.2 and 6.7). These limit theorems are then complemented and illustrated by the distributions with polynomially decaying tails, Weibull distributions and those with fractional double exponential tails (resp., Examples 6.8, 6.9 and 6.10).

In Section 6.2, we first give the second order expansion formulas for the largest eigenvalues, provided the tails $e^{-Q}$ are lighter than the double exponential function (Theorem 6.11). For bounded $\xi(0)$, further extensions of this result are discussed.

Section 6.3 provides the second order expansion formulas for the principal eigenvalue in the case of double exponential tails (Theorem 6.13), which are further extended up to Poisson limit theorems for eigenvalues.

In Section 6.4, we comment and compare the proofs of Poisson limit theorems stated in Sections 6.1 and 6.3. We mention, en passant, that Theorems 6.2, 6.7, 6.11 and 6.13 simply follow from the corresponding results of Sections 2–5.

6.1. Distribution tails heavier than the double exponential function

Extreme value theory for i.i.d. random variables $\xi(x)$ deals with the asymptotic behavior of the $K$th largest values $\xi_{K,V}$ of the sample $\xi_V$ as $V \uparrow \mathbb{Z}^\nu$. It is well known that for suitable normalizing constants $a_V > 0$ and $b_V$, the non-trivial limiting (max-stable) distributions $G(\cdot)$ for $\mathbb{P}(\xi_{1,V} - b_V)a_V \leq \cdot$ are either Weibull law $D_\beta(t) := \exp \left\{ -(t/\beta)^\beta \right\}$ ($t < 0$) or Fréchet law $G_\beta(t) = \exp \left\{ -t^{-\beta} \right\}$ ($t \geq 0$) for some $\beta > 0$, or Gumbel law $G_{\exp}(t) = \exp \left\{ -e^{-t} \right\}$ ($-\infty < t < \infty$); see, e.g., (Resnick 1987). Note that the weak convergence of maxima to Gumbel law is equivalent to the limit

$$\lim_{V} |V| \mathbb{P}(\xi(0) > b_V + t/a_V)) = e^{-t} \quad \text{for all} \quad t \in \mathbb{R}. \quad (6.1)$$

On the other hand, limit (6.1) implies that the point process $\mathcal{N}_V^\xi$ on $[-1;1]^\nu \times \mathbb{R}$, defined by

$$\mathcal{N}_V^\xi := \sum_{z \in \Xi_V(z)} \delta_{\Xi_V(z)} \quad \text{where} \quad \Xi_V(z) := (z|V|^{-1/\nu}, (\xi(z) - b_V)a_V), \quad (6.2)$$
converges weakly (as \( V \uparrow Z' \)) to the Poisson process on \([-1; 1]^\nu \times \mathbb{R} \) with the intensity measure \( d x \times e^{-t} d t \); see (Leadbetter et al. 1983).

The necessary and sufficient conditions for (6.1) to hold are generally formulated in terms of \( \Gamma \)-variation of the function \( e^Q \) at the right endpoint \( t_Q \) or, equivalently, in terms of \( \Pi \)-variation of its inverse \( f \circ \log \) (Resnick 1987). We say that \( f := Q^- \) is in a class \( A_{\Pi} \), if there exists a function \( a : (-\infty; t_Q) \rightarrow \mathbb{R}_+ \) such that

\[
\lim_{s \to \infty} \frac{f(s + c) - f(s)}{a(f(s))} = c \quad \text{for any } c \in \mathbb{R}_+.
\]

(6.3)

Here \( a(\cdot) \) is called an auxiliary function. The class \( A_{\Pi} \) (6.3) is an argument-additive version of the original class of \( \Pi \)-varying functions \( f \circ \log \) considered, e.g., in (Resnick 1987, Section 0.4.3). In Lemma A.1 of Appendix A, we recall the well-known characterization of the class \( A_{\Pi} \) in terms of \( Q \).

**Lemma 6.1.** (Resnick 1987, Sections 0.4.3 and 1.1). Limit (6.1) holds true if and only if \( f \in A_{\Pi} \) with an auxiliary function \( a(\cdot) \). In this case, the normalizing constants can be chosen \( b_0 = f(\log |V|) \) and \( a_V = 1/a(b_V) \).

We now formulate Poisson limit theorems for the largest eigenvalues \( \lambda_{K,V} \) of the random Schrödinger operator \( \mathcal{H}_V = \kappa \Delta_V + \xi_V \) introduced in Section 2.1. Throughout this subsection, we assume that \( e^f \in RV_\infty \), so that \( \log Q(t) = o(t) \) as \( t \rightarrow \infty \) by Lemma A.3 with \( \rho = \infty \). Using the notation from Section 4.3, for fixed small \( 0 < \theta < 1/2 \), we write \( \xi(x) := (x) \text{ if } (x) < L_{V,\theta} := f((1-\theta) \log |V|) \), and \( (x) := 0 \), otherwise. For any \( z \in V \), denote by \( \tilde{\lambda}(z) \) the principal eigenvalue of the “single peak” Hamiltonian \( \kappa \Delta_V + (z_1 \delta_z + \xi_1 (1-\delta_z) \). Let \( \tilde{\lambda}_{K,V} \) be the \( K \)th order statistics of the stationary random field \( \tilde{\lambda}(\cdot) \) in \( V \), and let \( z_{\tau(K),V} \in V \) stand for its location defined by \( \tilde{\lambda}(z_{\tau(K),V}) := \lambda_{K,V} \). (Recall that the sites \( z_{l,V} \in V \) \( 1 \leq l \leq |V| \) are associated with the variational series (1.1) based on \( \xi_V \).) Note that, for \( Z := z_{\tau(K),V} \) and \( K \in \mathbb{N} \), the eigenvalues \( \tilde{\lambda}(Z) \) are expanded into a certain power series in the variables \( \xi(Z) \) and \( (x) \) \( x \in V \); cf. (2.19)-(2.21).

**Theorem 6.2.** (see Theorem 4 in (Astrauskas 2007) and Theorem 5.2 in (Astrauskas 2008)). Let \( t_Q = \infty \), and assume that \( e^f \) is log-Hölder continuous of order \( \mu > (1+\theta)/\nu/(1-2\theta) \) at infinity for some small \( \theta > 0 \) as above, i.e., (4.7) holds true. Let \( e^f \in RV_\infty \). Further, assume that there exist the normalizing constants \( A_V > 0 \) and \( B_V \) such that

\[
\lim_{V} |V| \mathbb{P}(\tilde{\lambda}(0) > B_V + A_V^{-1} t) = e^{-t} \quad \text{for any } t \in \mathbb{R}.
\]

(6.4)

Define the point process \( \mathcal{N}^\alpha \) on \([-1; 1]^\nu \times \mathbb{R} \) by

\[
\mathcal{N}^\alpha := \sum_{k=1}^{|V|} \delta_{\Lambda_V(k)} \quad \text{where} \quad \Lambda_V(k) := \frac{z_{\tau(k),V}}{|V|^{1/\nu}} \cdot (\lambda_{k,V} - B_V) A_V.
\]

(6.5)

Then \( \mathcal{N}^\alpha \) converges weakly to the Poisson process \( \mathcal{N} \) on \([-1; 1]^\nu \times \mathbb{R} \) with the intensity measure \( d x \times e^{-t} d t \).
Sketch of the proof of Theorem 6.2. Using Theorem 4.6 with \( \rho = \infty \), Theorems 3.1 and 4.8 with \( R = 0 \) and \( 0 < \varepsilon < \theta < 1/2 \), we obtain that almost surely \( \xi_v \) satisfies the assumptions of Theorem 2.3, where \( K \in \mathbb{N} \) is fixed and \( N := \|V\| \). Theorem 2.3 implies that almost surely \( \lambda_{k,v} = \tilde{\xi}_k + O(\exp\{-|V|^{(1+\theta)/\mu}\}) \) as \( |V| \to \infty \), for fixed \( K \in \mathbb{N} \). This asymptotic formula in turn yields that the point process \( \tilde{N}_V^\lambda \) is approximated by the corresponding point process \( \tilde{N}_V^{\lambda_k} \) where \( \lambda_{k,v} \) are replaced by \( \tilde{\lambda}_{k,v} \) \((1 \leq k \leq |V|)\); see the proof of Theorem 4 in (Astrauskas 2007). The weak convergence of \( \tilde{N}_V^\lambda \) to \( N \) is shown by checking Leadbetter’s mixing conditions for the random field \( \lambda(\cdot) \) (Astrauskas 2007, Lemma 6). This concludes the proof of the theorem. \( \square \)

In Remarks 6.3–6.5 below, we give the alternative conditions on \( Q \) (where \( \log Q(t) = o(t) \)) for the Poisson convergence of the largest eigenvalues to hold.

**Remark 6.3.** (A specification of the normalizing constants \( A_V > 0 \) and \( B_V \) in (6.4)). (i) Let \( Q(t) = t^\alpha \) for \( t \geq 0 \) where \( \alpha > 0 \) (Weibull distribution), or \( Q(t) = e^t \) for \( t \geq t_0 \), \( 0 < \gamma < 1 \) (fractional double exponential distribution), so that \( Q \) satisfies the regularity and continuity conditions of Theorem 6.2. Moreover, the equations for the normalizing constants \( A_V > 0 \) and \( B_V \) in (6.4) can be derived by applying a certain iteration scheme for \( \tilde{\lambda}_{1,v} \) as in (2.19)–(2.21) combined with the Laplace’s method for the corresponding integrals (Astrauskas 2008, Section 6), (Astrauskas 2014); see also (Astrauskas 2013, Section 3). From these equations one derives the explicit expansion formulas for \( B_V \) and hence those for the eigenvalues \( \lambda_{k,v} \) up to the random max-stable fluctuations of order \( O(A_V^{-1}) \); cf. Examples 6.9 and 6.10 below.

(ii) Let \( t_Q = \infty \) and, for some large \( t_0 \), assume that \( Q : [t_0; \infty) \to \mathbb{R}_+ \) is (locally) absolutely continuous with the positive density \( Q' : [t_0; \infty) \to \mathbb{R}_+ \) obeying the following conditions:

\[
\lim_{t \to \infty} \frac{Q'(t + C)}{Q'(t)} = 1 \quad \text{for any} \quad C > 0, \quad (6.6)
\]

and

\[
\liminf_{t \to \infty} Q'(t) > 0. \quad (6.7)
\]

Then, by Lemmas A.4(I) and A.3 with \( \rho = \infty \), the function \( Q \) satisfies the regularity and continuity conditions of Theorem 6.2. (The additional condition (6.7) is to exclude the heavy-tailed (“subexponential”) distributions \( 1 - e^{-Q} \) which are considered in Theorem 6.7(C0) and Example 6.8 below.) We now assert that, under conditions (6.6) and (6.7), the centralizing constants \( B_V \) in (6.4) may be defined by the equation:

\[
\mathbb{P}(\tilde{\xi}(0) > B_V) = |V|^{-1} \quad (V \supset V_0), \quad (6.8)
\]

and the normalizing constants \( A_V > 0 \) in (6.4) and \( a_V > 0, b_V \) in (6.1) are specified as follows:

\[
A_V = a_V = Q'(b_V) \quad \text{where} \quad b_V := f(\log |V|) \quad (V \supset V_0), \quad (6.9)
\]

therefore, for any \( t \in \mathbb{R} \) and \( |V| \to \infty \),

\[
|V|\mathbb{P}(\xi(0) > b_V + \frac{t}{a_V}) \to e^{-t} \quad \text{and} \quad |V|\mathbb{P}(\tilde{\xi}(0) > B_V + \frac{t}{a_V}) \to e^{-t}, \quad (6.10)
\]

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where
\[ B_V = b_V + o(1). \]  

(6.11)

I.e., both the distribution functions \( P(\xi(0) \leq t) = 1 - e^{-Q(t)} \) and \( P(\tilde{\lambda}(0) \leq t) \) \((t \in \mathbb{R})\) are in the domain of attraction of the max-stable Gumbel law \( G_{\exp}(|\cdot|); \) cf. Lemma 6.1 and the assertions before this lemma.

The proof of limits (6.10) with \( a_V, b_V \) and \( B_V \) defined in (6.9) and (6.8). First, from Lemma A.4(III) and Lemma A.1 we have that \( f \in A\Pi (6.3) \) with the auxiliary function \( a(\cdot) = 1/Q(\cdot) \) in \([t_0; \infty)\); therefore, by Lemma 6.1 we obtain the first limit in (6.10).

To prove the second limit in (6.10), we first notice that, for each \( V \supset V_0 \), there is a solution \( B_V \) of equation (6.8) because of the continuity of the distribution function of \( \tilde{\lambda}(0) \). Let us show (6.11). Since \( \tilde{\lambda}(0) \geq \xi(0) \), we get from (6.8) that

\[ |V|^{-1} = P(\tilde{\lambda}(0) > B_V) \geq P(\xi(0) > B_V) = e^{-Q(B_V)}, \]

therefore, \( B_V \geq b_V = f(\log |V|) \) for \( V \supset V_0 \). If \( \xi(0) \geq L_{\lambda,\epsilon} := f((1 + \epsilon) \log |V|) \) for some \( 0 < \epsilon < \theta \), then we get from (2.19)–(2.21) that almost surely \( \tilde{\lambda}(0) \leq \xi(0) + \beta_V \) for some (nonrandom) \( 0 < \beta_V \leq 0 \) as \(|V| \to \infty\). Thus, for \( V \supset V_0 \),

\[ |V|^{-1} = P(\tilde{\lambda}(0) > B_V) = P(\tilde{\lambda}(0) > B_V, \xi(0) \geq L_{\lambda,\epsilon}) \]

\[ \leq P(\xi(0) > B_V - \beta_V) = e^{-Q(B_V - \beta_V)}, \]

therefore, \( B_V \leq b_V + \beta_V = b_V + o(1) \). These estimates imply (6.11), as claimed.

To prove the second limit in (6.10), we also need the following observations. First, since \( \liminf_V a_V > 0 \), it follows that \( \limsup_V |V|P(\xi(0) \geq b_V + M) \to 0 \) as \( M \to \infty \). Second, for any \( M \geq M_0 \) and any \( V \supset V_0(M) \), if \( \xi(0) \leq b_V - M \), then \( \tilde{\lambda}(0) < b_V - M/2 \). These two assertions imply that, for any \( t \in \mathbb{R} \),

\[ |V|P(\tilde{\lambda}(0) > B_V + ta_V^{-1}) \]

\[ = |V|P(\tilde{\lambda}(0) > B_V + ta_V^{-1}, |\xi(0) - b_V| < M) + o_{V,M}(1), \]

(6.12)

where \( o_{V,M}(1) \to 0 \) letting first \( V \uparrow \mathbb{Z}^v \) and then \( M \to \infty \). Thus, it suffices to check that, for any \( t \in \mathbb{R} \), any \( M \geq M_0(t) \) and \( V \uparrow \mathbb{Z}^v \),

\[ P(\tilde{\lambda}(0) > B_V + ta_V^{-1}, |\xi(0) - b_V| < M) \]

\[ = e^{-t}P(\tilde{\lambda}(0) > B_V, |\xi(0) - b_V| < M) (1 + o(1)). \]

(6.13)

To prove (6.13), we follow the arguments of the paper (Biskup and König 2013, Sect. 7.1) which are now simplified and adapted to our case \( \log Q(t) = o(t) \). (Recall that this paper considers the case of double exponential tails of potential, i.e., \( \log Q(t) \approx t/\rho \).) The main idea here is the observation that the shift of the eigenvalue \( \tilde{\lambda}(0) \) by \( ta_V^{-1} \) is achieved by the corresponding shift of the single \( \xi_V \)-peak \( \xi(0) \) on the
left-hand side of (6.13). Indeed, write \( \xi^{(t)} := \xi(0) - ta_V^{-1} \), and denote by \( \lambda^{(t)} \) the principal eigenvalue of the Hamiltonian \( \kappa V + \xi^{(t)} \delta_0 + \xi V (1 - \delta_0) \) in \( l^2(V) \). Notice that \( \xi^{(0)} = \xi(0) \) and \( \lambda^{(0)} = \lambda(0) \). Fix \( t > 0 \). Comparing expansion formulas (2.19)–(2.21) for \( \lambda^{(t)} \) with those for \( \lambda(0) \), we find that, for any (small) \( \varepsilon > 0 \), any \( M \geq M_0(t, \varepsilon) \) and any \( V \supset V_0(M, t, \varepsilon) \),

\[
|\xi^{(0)}| - |b_V| < M, \quad \text{then} \quad \lambda^{(t+\varepsilon)} - ta_V^{-1} \leq \lambda^{(t)};
\]

therefore, we obtain the following bounds for the left-hand side of (6.13):

\[
\begin{align*}
\mathbb{P}(\lambda^{(t+\varepsilon)} > B_V, |\xi^{(t+\varepsilon)} - b_V| < M/2) \\
\leq \mathbb{P}(\tilde{\lambda}(0) > B_V + ta_V^{-1}, |\xi^{(0)} - b_V| < M) \quad \text{(6.14)} \\
\leq \mathbb{P}(\lambda^{(t)} > B_V, |\xi^{(t)} - b_V| < 2M).
\end{align*}
\]

Since \( \varepsilon > 0 \) is arbitrarily small, it suffices to prove limit (6.13) for the upper and lower bounds in (6.14). We write \( \lambda^{(t)}(\xi_V) = \lambda^{(t)} \) to emphasize the dependence of \( \lambda^{(t)} \) on the sample \( \xi_V = \{\xi(x)\}_{x \in V} \). Let us consider the functions \( \lambda^{(t)}(s_V) \) on \( s_V = \{s(x)\}_{x \in V} \in \mathbb{R}^{|V|} \) and the corresponding integrals on the right of (6.14) with respect to the probability measure \( \prod_{x \in V} dF(s(x)) \); here \( F := 1 - e^{-Q} \) stands for the distribution function of \( \xi(0) \) with the density \( p(\cdot) := F'(\cdot) \) in \( [t_0; \infty) \). By the change of variables \( u(0) := s(0) - ta_V^{-1} \) and \( u(x) := s(x) \) for all \( x \in V \setminus \{0\} \), we have that \( \lambda^{(t)}(s_V) = \lambda^{(0)}(u_V) \); therefore, for fixed \( M > 0 \) and \( V \uparrow \mathbb{Z}^m \),

\[
\mathbb{P}(\lambda^{(t)} > B_V, |\xi^{(t)} - b_V| < M)
\]

\[
= \int_{\mathbb{R}^{|V|}} \mathbb{I}\left\{\lambda^{(t)}(s_V) > B_V, |s(0) - ta_V^{-1} - b_V| < M\right\} \prod_{x \in V} dF(s(x))
\]

\[
= \int_{\mathbb{R}^{|V|}} \mathbb{I}\left\{\lambda^{(0)}(u_V) > B_V, |u(0) - b_V| < M\right\} \frac{p(u(0) + ta_V^{-1})}{p(u(0))} \prod_{x \in V} dF(u(x))
\]

\[
= \mathbb{E}\left( \mathbb{I}\left\{\lambda^{(0)}(\xi_V) > B_V, |\xi(0) - b_V| < M\right\} \frac{p(\xi(0) + ta_V^{-1})}{p(\xi(0))} \right) \quad \text{(6.15)}
\]

by applying Lemma A.4(IV) to the ratio of the densities in the last expectation (6.15) where \( a_V = Q'(b_V) \). This combined with (6.14) implies (6.13) for \( t \geq 0 \), as claimed. Since the case \( t \leq 0 \) is treated similarly, this concludes the proof of assertion (ii) of Remark 6.3.

**Remark 6.4.** (Suppression of the log-Hölder continuity of \( e^{-Q} \) in Theorem 6.2). Let \( t_Q = \infty \). Assume that \( Q \) satisfies condition (3.1) (instead of the log-Hölder continuity at \( \infty \)), \( e^f \in RV_\infty \), and let assumption (6.4) be fulfilled with

\[
A_V = O((L_{V, \varepsilon} - L_{V, \varepsilon'})^{|V|/(1-2\delta)/2}) \quad \text{for some constants} \quad 0 < \varepsilon < \varepsilon' < \delta < \frac{1}{2}, \quad \text{(6.16)}
\]
Then the assertion of Theorem 6.2 holds true.

For the proof of the assertion of Remark 6.4, we again apply Theorem 2.3, so we need to show that the samples \( \xi_V \) satisfy limits (2.9), (2.13) and (2.14) in probability with the same abbreviation as in the proof of Theorem 6.2. First, (3.1) implies (2.9) with \( N = |V|^\theta \) where \( \theta' < \theta < \delta \). Second, by Theorem 4.6 with \( \rho = \infty \), the condition \( e^{i} \in RV_{\infty} \) yields (2.13) with fixed \( K \in \mathbb{N} \) and \( N \) as above. It remains to prove limit (2.14) in probability with those \( K \) and \( N \). As mentioned in the proof of Theorem 6.2, assumption (6.4) implies that the point process \( \tilde{N}_V \) based on the sample \( \tilde{\lambda}_V \) converges weakly to the corresponding Poisson process. This in turn yields that with probability \( 1 + o(1) \) the normalized spacings \( A_V(\tilde{\lambda}_V - \tilde{\lambda}_{k+1}, V) \) are bounded away from zero as \( V \to \mathbb{Z}^r \), for any fixed \( k \in \mathbb{N} \) (Astrauskas 2007, Corollary 1(jj)). Combining this with the upper bound (6.16) for \( A_V \) and observing from Lemma 4.2(iii) and Theorem 3.1(iv) \( (R = 0) \) that almost surely \( \xi_{k,V} - \xi_{N,V} \geq L_{V,\varepsilon} - L_{V,\varepsilon'} \) and \( r_{N,V} \geq |V|^{(1-2\delta)/\nu} \) for any \( V \supset V_0 \), we arrive at limit (2.14) in probability with \( N = |V|^\theta \), as claimed. The assertion of Remark 6.4 is proved.

Remark 6.5. (Localization centers). Let \( \psi(\cdot;\lambda) \in l^2(V) \) be an eigenfunction of \( H_V \) associated with an eigenvalue \( \lambda \in \text{Spect}(H_V) \) and normalized by the condition \( \sum_{x \in V} \psi(x;\lambda)^2 = 1 \). If \( Q \) satisfies the conditions of Theorem 6.2 on the regular increase and the log-Hölder continuity at infinity, then almost surely the eigenfunction \( \psi(\cdot;\lambda_{k,V}) \) is asymptotically delta-function at the site \( z_{\tau(k),V} \in V \) for each \( K \in \mathbb{N} \) (Astrauskas 2008); see Theorem 4.1 in (Astrauskas 2008). Therefore, we may alternatively define any \( z_{\tau(k),V} \) \( (1 \leq k \leq |V|) \) in (6.5) as a localization center of the eigenfunction \( \psi(\cdot;\lambda_{k,V}) \), viz.

\[
\psi(z_{\tau(k),V};\lambda_{k,V}) := \max_{1 \leq l \leq |V|} \psi(z_{l,V};\lambda_{k,V}) \quad \text{for some } \tau(k) = \tau_V(k); \quad (6.17)
\]

here \( V = \{ z_{\tau(k),V} : 1 \leq k \leq |V| \} \). The latter definition of the sites \( z_{\tau(k),V} \) is more natural in the context of the localization theory for the Anderson Hamiltonians.

The asymptotic behavior of the localization indices \( \tau(K) = \tau_V(K) \) is studied by Astrauskas (2013).

Lemma 6.6. (Astrauskas 2013, Theorem 2.1). Assume that the condition of Theorem 6.2 on the log-Hölder continuity of \( e^{-Q} \) at infinity holds true. Fix \( K \in \mathbb{N} \).

(i) If \( f \in OAI\Pi^2 \) (4.3), then

\[
\limsup_{V} \tau_V(K) < \infty \quad \text{in probability}.
\]

(ii) If \( f \in AI\Pi^2 \) (4.2) and \( e^{i} \in RV_{\infty} \), then

\[
\lim_{V} \tau_V(K) = \infty \quad \text{and} \quad \lim_{V} \frac{\log \tau_V(K)}{\log |V|} = 0 \quad \text{in probability}.
\]

Note that, in Theorem 6.2 and Lemma 6.6, the condition \( e^{i} \in RV_{\infty} \) implies \( \log Q(t) = o(t) \) (see Lemma A.3 with \( \rho = \infty \)); the condition \( f \in OAI\Pi^2 \) yields
\( Q(t) \approx t^3 \) (see Lemma A.8(ii) with \( p = 2 \)); finally, the condition \( f \in \text{API}_2^\infty \) implies \( t^{-3}Q(t) \to \infty \) as \( t \to \infty \) (see Lemma A.7 with \( p = 2 \)).

In the case \( Q(t) = o(t^3) \) as \( t \to \infty \) ("heavy tails" \( e^{-Q} \)), the eigenvalues \( \lambda_{K,V} \) approach (as \( V \uparrow \mathbb{Z}^\nu \)) the \( K \)th extreme values of \( \xi_V \), so we obtain the simplified version of Poisson limit theorems for the largest eigenvalues.

**Theorem 6.7.** (see Theorem 5 in (Astrauskas 2007) and Theorem 2.5 in (Astrauskas 2012)). Let \( f \in \text{API} (6.3) \) for some auxiliary function \( a(\cdot) \), and assume that either of the following conditions (C0)–(C2) holds true:

- (C0) \( \lim_{s \to \infty} a(s) = \infty \),
- (C1) \( \lim_{s \to \infty} sa(s) = \infty \) and \( f \in PI_{<2} (4.5) \)

or

- (C2) \( f \in SAPI^2_\infty (5.4) \) and \( f \in PI_{<2} (4.5) \)

(therefore, \( \ell_Q = \infty \) and \( s^2a(s) \to \infty \)). Write now \( b_V := f(|V|) \), \( a_V := 1/\lambda(b_V) \)

and

\[
B_V := \begin{cases} 
    b_V, & \text{under conditions (C0) or (C1),} \\
    b_V + 2\nu k^2/b_V, & \text{under condition (C2).}
\end{cases}
\]

Define the point process \( \mathcal{N}_V^\Lambda \) on \([-1;1]^\nu \times \mathbb{R} \) by

\[
\mathcal{N}_V^\Lambda := \sum_{k=1}^{|V|} \delta_{\Lambda_V(k)} \quad \text{where} \quad \Lambda_V(k) := \left( \frac{z_{k,V}}{|V|^{1/\nu}}, (\lambda_{K,V} - B_V) a_V \right).
\]

Then \( \mathcal{N}_V^\Lambda \) converges weakly to the Poisson process \( \mathcal{N} \) on \([-1;1]^\nu \times \mathbb{R} \) with the intensity measure \( d \xi \times e^{-\xi} \, d\xi \).

**Sketch of the proof.** Conditions (6.3) and (C0) imply that \( f \in \text{API}_0^\infty (4.1) \). Therefore, by Theorem 4.3(i) with \( p = 0 \), the samples \( \xi_V \) satisfy the condition of Theorem 2.2(i), consequently, \( \lambda_{K,V} = \xi_{K,V} + o(1) \) in probability as \( |V| \to \infty \), for fixed \( K \in \mathbb{N} \). Similarly, (6.3) and (C1) imply that \( f \) is in the classes \( \text{API}_\infty^2 (4.1) \) and \( PI_{<2} (4.5) \). Therefore, by combining Theorem 4.3(i) for \( p = 1 \), Theorems 4.5 and 3.1(iv) with \( R = 0 \) and \( 0 < \varepsilon < \theta < 1/2 \), we obtain that the samples \( \xi_V \) satisfy the conditions of Theorem 2.2(ii) with \( N = |V|^\theta \), consequently, \( \lambda_{K,V} = \xi_{K,V} + O(\xi_{K,V}^{-1}) \) in probability, for fixed \( K \in \mathbb{N} \). Using these asymptotic expansion formulas for \( \lambda_{K,V} \), we obtain that in the cases (C0) and (C1) the point process \( \mathcal{N}_V^\Lambda (6.18) \) is approximated by the corresponding point process \( \mathcal{N}_V^\xi \) with \( \xi \) instead of \( \lambda \); see the proof of Theorem 2.5(ii) in (Astrauskas 2012). Since \( \mathcal{N}_V^\xi \) converges weakly to \( \mathcal{N} \) (Leadbetter et al. 1983), this concludes the proof of the theorem for (C0) and (C1).

In the case of (C2), we combine Theorem 4.3(i) for \( p = 2 \) and Theorems 4.5, 3.1(iv), 5.3 and 5.4 for \( R = 0 \) and \( 0 < \varepsilon < \theta < 1/2 \), to find that the samples \( \xi_V \) satisfy the conditions of Theorem 2.2(iii) in probability, with \( N \) and \( K \) as above. Consequently, \( \lambda_{K,V} = \xi_{K,V}^0 + O(\xi_{K,V}^{-2}) \) in probability, where \( \xi_{K,V}^0 \) is the \( K \)th extreme value of the i.i.d. field \( \xi_V := \xi(\cdot) + 2\nu k^2/(\xi(\cdot) + 1) \) in \( V \). Using this limit and applying the same arguments as above with \( \xi \) replaced by \( \xi^0 \), we obtain the assertion of the theorem in the case (C2).

\( \square \)

From Lemmas A.6 and A.13 we know that condition (C0) (resp., (C1) or (C2)) imply that \( Q(t) = o(t) \) (resp., \( Q(t) = o(t^2) \) or \( Q(t) = o(t^3) \)) as \( t \) tends to infinity.
We now give three examples of distributions $1 - e^{-Q}$, where $\log Q(t) = o(t)$.

**Example 6.8.** (Astrauskas 2012). *Polynomially decaying distributions.* For some $\beta > 0$, assume that $f \circ \log \in RV_{1/\beta}$ or, equivalently, $e^Q \in RV_\beta$. The latter is the sufficient and necessary condition for the distribution $1 - e^{-Q}$ to be in the domain of attraction of the max-stable Fréchet law $G_\beta(\cdot)$, or equivalently, the following limit holds true:

$$\lim_{t} |V| \mathbb{P}(\xi(0) > tf(\log |V|)) = t^{-\beta} \quad \text{for all } t \in \mathbb{R}_+.$$  

Since $f \in A\Pi_\infty^0$ (4.1), from Theorem 4.3(i) with $p = 0$ and Theorem 2.2(i) we see that $\lambda_K, V = \xi_{K,V} + o(1)$ in probability, for fixed $K \in \mathbb{N}$. Using this limit and denoting $B_V \equiv 0$ and $a_V = 1/f(\log |V|)$, we obtain similarly as in the proof of Theorem 6.7(C0) that the point process $N^\xi_{K,V}$ (6.18) converges weakly to the Poisson process on $[-1; 1]^v \times \mathbb{R}_+$ with the intensity measure $dV \times t^{-\beta} \, dt$.

**Example 6.9.** (Grenkova et al. 1990; Astrauskas and Molchanov 1992; Astrauskas 2008). *Weibull distributions.* Let $Q(t) = t^\alpha$ for $t \geq 0$, where $\alpha > 0$. For $\alpha \geq 1$, the function $\lambda$ satisfies condition (6.6) and (6.7) of Remark 6.3(ii). For $\alpha < 3$, the inverse function $f(s) := Q^{-1}(s) = s^{1/\alpha}$ ($s \geq 0$) satisfies conditions (6.3) and (C2) of Theorem 6.7. Therefore, by Theorems 6.2, 6.7 and Remark 6.3(ii),

$$\lim_{t} |V| \mathbb{P}(\xi(0) > b_V + \frac{t}{a_V}) = \lim_{t} |V| \mathbb{P}(\tilde{\lambda}(0) > B_V + \frac{t}{a_V}) = e^{-t} \quad (t \in \mathbb{R})$$

(therefore, the point processes $N^\xi_{K,V}$ (6.2) and $N^\xi_{V}$ (6.5) converge weakly to the same Poisson process $\mathcal{N}$ as in Theorem 6.2), where the normalizing constants can be chosen as follows: $b_V = (\log |V|)^{1/\alpha}$, $a_V = a_V = Q'(b_V) = \alpha b_V^{\alpha - 1}$ and

(a) $B_V = b_V$ if $\alpha < 2$,

(b) $B_V = b_V + 2\nu \kappa b_V^{-1}$ if $2 \leq \alpha < 3$

and, as $|V| \to \infty$,

(c) $B_V = b_V + 2\nu \kappa b_V^{-1} + O(b_V^{-\alpha/2})$ if $\alpha \geq 3$.

For $\alpha \geq 3$, asymptotic equations for $B_V$ are given in (Astrauskas 2008, Section 6).

In the case $\alpha < 1$, we obtain the following almost sure asymptotic bounds for the eigenvalues and their spacings for any fixed $K \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{1\}$:

$$\limsup_{V} \frac{\lambda_{K,V} - b_V}{\log_2 |V|} = \frac{1}{K},$$

$$\liminf_{V} \left( \log \left( \lambda_{K,V} - \lambda_{K+1,V} \right) a_V \right) = \sum_{i=2}^{m-1} \log |V| = -1$$

and

$$\limsup_{V} \left( \lambda_{K,V} - \lambda_{K+1,V} \right) a_V = \frac{1}{K} \sum_{i=2}^{m-1} \log |V| = \frac{1}{K},$$

with $a_V$ and $b_V$ as above. For any $\alpha > 0$, these strong limits for $\xi$ instead of $\lambda$ are proved in (Astrauskas 2006, Section 3). Therefore, the case of $\lambda$ and $\alpha < 1$ is derived by the same arguments as in the proof of Theorem 6.7(C1) above, where one explores Theorem 4.4(i) ($p = 1$) instead of Theorem 4.3(i).
Example 6.10. (Astrauskas 2013; 2014). Distributions with fractional double exponential tails. Let \( Q(t) = e^{\rho t} \) for \( t \geq t_0 \), where \( 0 < \gamma < 1 \). Obviously, \( Q \) satisfies conditions (6.6) and (6.7) of Remark 6.3(ii), therefore,

\[
\lim_{V} |V| \mathbb{P}(\xi(0) > b_V + \frac{t}{a_V}) = \lim_{V} |V| \mathbb{P}(\bar{\lambda}(0) > B_V + \frac{t}{a_V}) = e^{-t} \quad (t \in \mathbb{R})
\]

(Consequently, the point processes \( N^U_\nu \) (6.2) and \( N^V_\nu \) (6.5) converge weakly to the same Poisson process \( N \) as in Theorem 6.2); here

\[
b_V = (\log_2 |V|)^{1/\gamma}, \quad A_V = a_V = Q'(b_V) = \gamma b_V^{-1} \log |V|
\]

and

\[
B_V = b_V + c_1 b_V^{-1} \log b_V + \frac{c_2 b_V^{-1} \log b_V}{(\log b_V)^2} + c_3 \frac{b_V^{-1}}{(\log b_V)^2} (1 + o(1))
\]

as \( |V| \to \infty \), where \( c_1 := \nu \kappa^2 \gamma (1 - \gamma)^{-1} \), \( c_2 := c_1 (\gamma - 1)^{-1} \) and \( c_3 := c_2 \log^2 \frac{2(1 - \gamma) \sqrt{\kappa}}{\kappa \gamma} \).

The last formula and the asymptotic equations for \( B_V \) are derived in (Astrauskas 2014); see also (Astrauskas 2013, Section 3).

6.2. Distribution tails lighter than the double exponential function

We start with the second order asymptotic expansion formula for the largest eigenvalues \( \lambda_{K,V} \) of \( \mathcal{H}_V = \kappa \Delta_V + \xi_V \).

**Theorem 6.11.** Let \( e^t \in RV_0 \), and fix \( K \in \mathbb{N} \). Then with probability 1

\[
\lim_{V} (\lambda_{K,V} - f(\log |V|)) = 2\nu \kappa.
\]

**Proof.** We apply part (i) of Theorem 3.1, where \( R \in \mathbb{N} \) is fixed, \( \theta_R(\cdot) \equiv \theta \) is a constant, and \( m := |B_R| \). Thus, with probability one there is a ball \( B_R(z_V) \) such that \( \xi(\cdot) \geq L_{V,\theta} \) in \( B_R(z_V) \) for some \( \theta \in \left( \frac{m-1}{m}, \frac{m}{m+1} \right) \) and each \( V \supset V_0(\omega; R) \). Therefore, by Theorem 4.4(iii) and Lemma A.3(ii) with \( \rho = 0 \), almost surely

\[
\xi(\cdot) - \xi_{1,V} \geq L_{V,\theta} - L_{V,0} + o(1) = o(1) \text{ uniformly in } B_R(z_V),
\]

as \( |V| \to \infty \), for any \( R \in \mathbb{N} \). The latter means that with probability one the samples \( \xi_V \) satisfy the condition of Theorem 2.4, therefore, \( \lambda_{K,V} = L_{V,0} + 2\nu \kappa + o(1) \), as claimed. Theorem 6.11 is proved.

From the proof of Theorems 2.4 and 6.11 we see that the top eigenvalue \( \lambda_{K,V} \) of \( \mathcal{H}_V \) is approximated by the corresponding eigenvalue of the Hamiltonian over the (random) relevant regions \( B_{\text{opt}} := B_{R_k}(z_V) \subset V \) where \( \xi(\cdot) \) is close to \( \xi_{1,V} \) and the diameter of which tends to infinity as \( |V| \to \infty \). Bishop and Wehr (2012) have obtained the third order asymptotic expansion formulas for the principal eigenvalue of the one-dimensional Schrödinger operators in \( V \subset \mathbb{Z} \), with Bernoulli i.i.d. potential taking values 0 and 1. They have established that almost surely the relevant region \( B_{\text{opt}} \subset V \) is the longest consecutive sequence of sites in \( V \) with \( \xi(\cdot) \) equal to 1, so that
B_{opt} \asymp \log |V|$. See, e.g., the review paper by Binswanger and Embrechts (1994) for the strong and weak limit theorems for the length $|B_{opt}|$ as $V \uparrow \mathbb{Z}$.

Recently, Germinet and Klopp (2013) have proved the Poisson limit theorem for the top eigenvalues under nonlinear renormalization, i.e., for the so-called unfolded eigenvalues. Let $N(\lambda)$ ($\lambda \in \mathbb{R}$) be the integrated density of states, i.e., the nonrandom distribution function of eigenvalues defined as the almost sure limit of the empirical distribution function $N_V(\lambda) := \#\{k: \lambda_k \leq \lambda\}/|V|$ as $|V| \to \infty$ (Kirsch 2008).

**Theorem 6.12.** (Germinet and Klopp 2013, Theorem 2.3). Assume that $H_V$ is the one-dimensional Anderson Hamiltonian ($\nu = 1$), where the potential is bounded from above ($t_Q < \infty$), with the distribution density $p(\cdot) := e^{-Q(\cdot)}Q'(\cdot)$. Assume, in addition, that the density $p(\cdot)$ is bounded and does not decay too fast at $t_Q$ (say, $p(t) = O(e^{-(t_Q-t)^{-c}})$ for $c > 0$ small enough). Define the point process $\mathcal{M}_V^\lambda$ on the positive half-axis $\mathbb{R}_+$ by

$$\mathcal{M}_V^\lambda := \sum_{k=1}^{|V|} \delta_{|V|(1-N(\lambda_k, V))}.$$  

Then $\mathcal{M}_V^\lambda$ converges weakly to the Poisson process on $\mathbb{R}_+$ with the intensity measure $dt$.

From Theorem 6.12, one can obtain the third order asymptotic expansion formulas for the largest eigenvalues $\lambda_{K,V}$, by using the asymptotic expansion formulas for the tails $1 - N(\cdot)$ at the spectral edge derived, e.g., by Biskup and König (2001). See also (Klopp 2000) and Section 3.5 of (Kirsch and Metzger 2007) for a detailed discussion on the tail behavior of the integrated density of states. The proof of Theorem 6.12 relies on the improved versions of Wegner and Minami estimates which control the probability of finding respectively one and two eigenvalues of $H_V$ in a small interval. See also (Minami 2007) for a detailed background of Poisson convergence results for unfolded spectral values. It is important for applications that this convergence result is given in terms of the integrated density of states, the main quantity in the theory of random Schrödinger operators.

For the tails lighter than the double exponential function, some heuristics on the asymptotic formulas for $\lambda_{1,V}$ ($\nu \geq 1$) and their relations to the long-time asymptotic formulas for the parabolic Anderson model have earlier been discussed by Biskup and König (2001) and Hofstad et al. (2006). Their assumptions are given in terms of scaling and regularity properties of the cumulant generating function $\log E e^{t\xi(0)}$ as $t \to \infty$.

Recently, Biskup et al. (2014) have proved the “homogenized” versions of limit theorems for the largest eigenvalues of the (scaled) finite-volume discrete Schrödinger operators $H^{(\varepsilon)}$ with bounded random potential. Their results state that, as the scale parameter $\varepsilon$ tends to zero, then 1) the largest eigenvalues of $H^{(\varepsilon)}$ converge in probability to to the corresponding eigenvalues of the limiting (nonrandom) finite-volume continuous Schrödinger operator, and 2) the fluctuations of the largest eigenvalues centered by their means are Gaussian in limit.

We finally notice that the conditions of the statements in this subsection imply that $t^{-1}\log Q(t) \to \infty$ as $t \uparrow t_Q > 0$ (see Lemma A.3 with $\rho = 0$ below).
6.3. The double exponential tails

In the double exponential case, Gärtner and Molchanov (1998, Theorem 2.16) have obtained the second order expansion formula for the principal eigenvalue $\lambda_{1, V}$ of $\mathcal{H}_{V} = \kappa \Delta_{V} + \xi_{V}$ by claiming a continuity of $Q$. We now provide their result with the continuity condition removed.

**Theorem 6.13.** If $e^{t} \in RV_{\rho}$ for some $0 < \rho < \infty$, then with probability 1

$$\lim_{V}(\lambda_{1, V} - f(\log |V|)) = 2\nu q_{\rho/\kappa},$$

where the nonrandom function $q$ is defined in Section 2.5.

**Proof.** We check the conditions of Theorem 2.5. First, by Theorem 4.7, almost surely $\xi_{V}$ satisfies condition (2.27). To prove (2.28), we fix constants $R \in \mathbb{N}$, $\delta > 0$, and write

$$\theta(y) := \theta_{R}(y) := 1 - \exp \left\{ \left( h_{\text{opt}}^{B_{R}}(y) - \delta \right)/\rho \right\} \quad (y \in B_{R}),$$

where the nonrandom function $h_{\text{opt}}^{B_{R}}(\cdot)$ is defined in Section 2.5. Consequently, $\theta(\cdot)$ satisfies the assumptions of Theorem 3.1(i). Combining the statements of Theorem 3.1(i), Lemma A.3(ii) and Theorem 4.4(iii), we obtain the following assertion with probability one: for any $V \supset V_{0}(\omega; \delta, R)$ there is $z_{V} \in V$ such that

$$\xi(y) \geq L_{V, \theta(y - z_{V})} \geq \xi_{1, V} + h_{\text{opt}}^{B_{R}}(y) - 2\delta \quad \text{for all} \quad y \in B_{R}(z_{V}).$$

Since $R \in \mathbb{N}$ and $\delta > 0$ are arbitrary constants, this estimate concludes the proof of the almost sure limit (2.28). Now, Theorems 2.5 and 4.4(iii) imply the assertion of Theorem 6.13.

□

From the proof of Theorems 2.5 and 6.13 we see that almost surely the eigenvalue $\lambda_{1, V}$ approaches (as $|V| \to \infty$) the local principal eigenvalue in the random region, where $\xi(\cdot) \approx \xi_{1, V} + h_{\text{opt}}^{B_{R}}(\cdot)$ for $R$ arbitrarily large, so that $\lambda_{1, V}$ is associated with the (random) “relevant island” of high $\xi_{V}$-values of optimal shape, the diameter of which is asymptotically bounded. According to Theorem 3.1(iv) and Remark 3.5, if $Q$ satisfies the additional continuity condition (3.1) at infinity, then the “islands” of $\xi_{V}$-extremes are located asymptotically far away from each other. Moreover, if the constant $\rho/\kappa$ is large enough, these “islands” are located in the neighborhood of single extremely high $\xi_{V}$-peaks; see (Astrauskas 2008, Theorem 4.4 and Corollary 4.5) and (Astrauskas 2013, Theorem 2.1(iii)).

For arbitrary $0 < \rho < \infty$, Poisson limit theorems for the largest eigenvalues and the corresponding localization centers are proved by Biskup and König (2013); see also the survey by König and Wolff (2013) on these limit theorems and related topics. To state their result, we again define the sites $z_{r(k), V} \in V$ by (6.17), i.e., the localization centers of the $k$th eigenfunctions $\psi(\cdot; \lambda_{k, V})$ ($1 \leq k \leq |V|$) of the Hamiltonian $\mathcal{H}_{V} = \kappa \Delta_{V} + \xi_{V}$; here $V = \{ z_{r(k), V} : 1 \leq k \leq |V| \}$.

**Theorem 6.14.** (Biskup and König 2013, Theorem 1.2). Assume that $t_{Q} = \infty$, $Q$ is a continuously differentiable function and

$$\lim_{t \to \infty} \frac{Q'(t)}{Q(t)} = \frac{1}{\rho} \quad \text{for some} \quad 0 < \rho < \infty.$$
Then there are constants $B_V = f(\log |V|) + 2\nu q(\rho / \kappa) + o(1)$ such that the point process $N_V \sim (6.5)$ with $A_V = \rho^{-1} \log |V|$ converges weakly to the Poisson process $\mathcal{N}$ on $[-1; 1]^r \times \mathbb{R}$ with the intensity measure $dx \times e^{-td}dt$.

By Remark A.5(i) and Lemma A.3, the conditions of Theorem 6.14 imply $e^t \in RV_\rho$, i.e., the assumption of Theorem 6.13. Moreover, from Remark A.5(ii), Lemma A.1 and Lemma 6.1 with $a(\cdot) \equiv 1/Q(\cdot)$ we also see that the conditions of Theorem 6.14 yield the limit (6.1) with $b_V = f(\log |V|)$ and $c_V = A_V = \rho^{-1} \log |V|$, thus the distribution $1 - e^{-Q}$ is in the domain of attraction of the max-stable Gumbel law $G_{\text{exp}}(\cdot)$. We finally notice that the conditions of Theorem 6.13 or Theorem 6.14 imply the limit $f(s) = (\rho + o(1)) \log s$ as $s \to \infty$, which in turn is equivalent to $\log Q(t) = (\rho^{-1} + o(1))t$ as $t \to \infty$ (see Lemma A.3).

### 6.4. Some comments on the proofs

In this section, we briefly comment and compare the proof of Theorems 2.2, 2.3, 6.2 and 6.7 by Astrauskas and Molchanov (1992) and Astrauskas (2007; 2008; 2012; 2013) ("relevant single peak" approximation) and the proof of Theorem 6.14 by Biskup and König (2013) ("relevant island" approximation).

(SP) "Relevant single peak" approximation. As already mentioned in Section 1.2, the proof of Theorems 2.2, 2.3, 6.2 and 6.7 is based on the finite-rank perturbation arguments and the analysis of Green functions involving cluster expansion over paths. To be more precise, fix $Z := z_{\tau(K),V} \in V$, the localization center of the $K$th eigenfunction, and denote by $G^{(Z)}_V(\lambda; \cdot, \cdot)$ the Green function of the Hamiltonian $\kappa \Delta_V + (1 - \delta_Z)\xi_V$ on $\ell^2(V)$. Under the conditions of Theorem 2.2 or 2.3 (sparseness and difference in height of $\xi_V$-peaks as $V \uparrow \mathbb{Z}^r$), the $K$th extreme eigenvalue $\lambda(Z) := \lambda_{K,V}$ of $H_V = \kappa \Delta_V + \xi_V$ associated with the site $Z$, is a solution to the dispersion equation

$$G^{(Z)}_V(\lambda; Z, Z) = 1/\xi(Z)$$

(6.19)

and the corresponding eigenfunction is $G^{(Z)}_V(\lambda(Z); \cdot, Z)$. By expanding the Green function $G^{(Z)}_V(\lambda; \cdot, \cdot)$ over paths, one proves that equation (6.19) is approximated by the corresponding equation $\tilde{G}_V(\lambda; Z, Z) = 1/\xi(Z)$ for the principal eigenvalue of the "single peak" Hamiltonian $\kappa \Delta_V + \xi_V + \xi(Z)\delta_Z$; here $\tilde{G}_V(\lambda; \cdot, \cdot)$ stands for the Green function of the operator $\kappa \Delta_V + \xi_V$. Again expanding $\tilde{G}_V(\lambda; \cdot, \cdot)$ over paths, one finds that the eigenvalue $\lambda(Z)$ of $H_V$ is approximated by a certain (nonlinear) function on $\xi_V$ and $\xi(Z)$; cf. (2.19)–(2.21). Moreover, because of the sparseness of $\xi_V$-peaks, the extreme eigenvalues $\lambda(Z)$ become asymptotically independent, so that they obey asymptotic Poisson behavior as $V \uparrow \mathbb{Z}^r$ (see Theorems 6.2 and 6.7).

We notice that the analysis of the Green functions combined with the finite-rank perturbation theory is essential to study the largest eigenvalues of the finite-volume operators $H_V$ in the "relevant single peak" approximation. These techniques also play a crucial role in the proof of the Anderson localization for the infinite-volume Hamiltonian $H$; see, e.g., (Pastur and Figotin 1992) and (Kirsch 2008).

(RI) "Relevant island" approximation. Recently, Biskup and König (2013) have developed novel arguments to prove Poisson limit theorems for the largest eigenval-
ues in the case of double exponential tails (see Theorem 6.14 above). As in the single-peak approximation, the analysis of the extreme eigenvalues is here based on controlling the dependence of an eigenvalue on the geometric properties of $\xi_V$-peaks and the associated regions in $V$. This enables to identify the “relevant” regions $\mathbb{B}_{\text{opt}}^K := \mathbb{B}_{R_V}(z^K) \subset V$ (where $\xi(\cdot)$ is high and of optimal shape) such that the $K$th largest eigenvalue $\lambda_{K,V}$ of $\kappa \Delta_V + \xi_V$ is approximated by the local principal eigenvalue $\lambda_{1,\mathbb{B}_{\text{opt}}^K}$ of the Hamiltonian in $l^2(\mathbb{B}_{\text{opt}}^K)$ (cf. also Theorems 2.5 and 6.13 and their proofs in the present survey). In other words, the eigenvalues associated with a block of “relevant islands” of high $\xi_V$-values can be determined by the local principal eigenvalues associated with separate “relevant islands”. It is worth noticing that the continuity of $Q$ implies that the islands of high $\xi_V$-values are located extremely far from each other as $V \uparrow \mathbb{Z}'$. (See also Theorem 3.1 for the related limits under the continuity condition (3.1)). The proof of Theorem 6.14 involves the following procedures on a simplification of potential configurations: 1) those regions, where the potential possesses the lower values, are deleted from $V$ (domain truncation and component trimming); 2) for the radius $R_V$ tending to infinity slowly, the analysis of the local principal eigenvalues in all balls $\mathbb{B}_{R_V}(z) \subset V$ is reduced to the consideration of independent identically distributed local principal eigenvalues in disjoint balls in $V$ (coupling to i.i.d. variables); 3) the local principal eigenvalue in the region $\mathbb{B}_{\text{opt}}^K$ is separated from other local eigenvalues in $\mathbb{B}_{\text{opt}}^K$ (reduction to one eigenvalue per component); 4) extremal type limit theorems for the local principal eigenvalues in $\mathbb{B}_{R_V}(z)$ are comparable to each other for the different increase rate of $R_V \rightarrow \infty$, with the same normalizing constants $A_V$ and $B_V$ (stability with respect to partition side), and so on.

Summarizing, the main idea of the proof of Theorem 6.14 explores the straightforward geometric arguments controlling the dependence of eigenvalues on potential configurations, rather than the techniques of resolvents or Green functions. This is in contrast to the single peak approximation in (SP), where the Green functions are the main object of analysis. On the other hand, although most of the proof of Theorem 6.14 is based on deterministic arguments, we are not able to reformulate this assertion in terms of $\xi_V$-extremes (like in Theorems 2.2–2.5 above).

## Appendix A: Regular Variation

In this section, we study the classes of functions $f := Q^{-}$ (the left-continuous inverse of the cumulative hazard function) introduced in Sections 4–6. These classes are characterized in terms of $Q$. The tail behavior of $Q(t)$ as $t \uparrow t_Q$ is also treated. In Section A.1, we recall the classical results on the domain of attraction of Gumbel max-stable law and regular variation $RV_\rho$. The classes $\mathbb{AIP}_\infty^\rho$ (4.1), $\mathbb{AIP}_0^\rho$ (4.2) and $\mathbb{OAIIP}^\rho_\infty$ (4.3) are studied in Section A.2, and $P_{I<2}$ (4.5) in Section A.3. Finally, examples and counterexamples are given in Section A.4.
A.1. The domain of attraction of Gumbel max-stable distribution and regular variation

We now give the well-known characterization statements for the distribution function to be in the domain of attraction of the max-stable Gumbel law \( G_{\exp}(\cdot) \).

**Lemma A.1.** (Resnick 1987; de Haan and Ferreira 2006). The following assertions (i)–(iii) are equivalent:

(i) \( f \in \Pi(6.3) \) with an auxiliary function \( a(\cdot) > 0 \);

(ii) there exists another auxiliary function \( a_1 : (-\infty; t_Q) \to \mathbb{R}^+ \) such that

\[ Q(t + ca_1(t)) - Q(t) \to c \quad \text{as} \quad t \uparrow t_Q, \quad \text{for any} \quad c \in \mathbb{R}; \]

(iii) there exist functions \( b : (-\infty; t_Q) \to \mathbb{R} \) and \( a_2 : (-\infty; t_Q) \to \mathbb{R}^+ \) such that

\[ Q(t) = b(t) + \int_{t_0}^{t} 1/a_2(s) \, ds \quad (t < t_Q), \]

where \( b(t) \to \infty \in \mathbb{R} \) \( (t \uparrow t_Q) \), the function \( a_2 \) is locally absolutely continuous with the density \( a_2'(t) \to 0 \) \( (t \uparrow t_Q) \) and, for \( t_Q < \infty \), \( a_2(t) \to 0 \) \( (t \uparrow t_Q) \).

In this case, \( a \circ f(s) = a_1 \circ f(s)(1 + o(1)) = a_2 \circ f(s)(1 + o(1)) \) as \( s \to \infty \). Moreover, the limit in (ii) with an auxiliary function \( a_1(\cdot) > 0 \) implies that \( f \in \Pi(6.3) \) with the same auxiliary function \( a(\cdot) \equiv a_1(\cdot) \).

**Example A.2.** For \( t_Q = \infty \), \( p \geq 0 \) and \( B > 0 \), consider the subclass \( \Pi_B^p \subset \Pi \) associated with the auxiliary function \( a_2(s) := B(p + 1)^{-1}s^{-p} \) in Lemma A.1(iii). In this case, \( Q(t) = B^{-1}t^{p+1} + \text{const} + o(1) \), i.e., \( 1 - e^{-Q} \) are Weibull type distributions.

In the next section, we extend the subclass \( \Pi_B^p \) to the boundary cases \( B = \infty \), \( B = 0 \) and O-type asymptotics.

Let us discuss the class \( RV_\rho \) of (nondecreasing) functions, which are regularly varying at infinity with index \( \rho \). Recall that, for \( 0 < \beta < \infty \), the condition \( f \circ \log \in RV_{1/\beta} \) is sufficient and necessary for the distribution function \( 1 - e^{-Q} \) to be in the domain of attraction of max-stable Fréchet law \( G_\beta(t) := \exp \{-t^{-\beta} \} \) \( (t > 0) \); cf. Example 6.8. We now explore the class \( RV_\rho \) to characterize the double exponential type distributions.

**Lemma A.3.** For \( t_Q = \infty \) and \( 0 \leq \rho \leq \infty \), the following assertions are equivalent:

(i) \( e^t \in RV_\rho; \)

(ii) \( f(s) - f(\delta s) \to -\rho \log \delta \) as \( s \to \infty \), for any \( 0 < \delta < 1 \);

(iii) \( Q(t + C)/Q(t) \to e^C/\rho \) as \( t \to \infty \), for any \( C > 0 \).

Either of (i)–(iii) implies that \( \lim_{t \to \infty} t^{-1} \log Q(t) = \rho^{-1} \).

**Proof.** The equivalence of (i)–(iii) follows from Theorems 1.5.12, 2.4.7 and Propositions 2.4.4(iv) and 1.3.6(i) in (Bingham et al. 1987) combined with the observation in (Resnick 1987, Sect. 0.2) that \( Q(t - \varepsilon) \leq f^+(t) \leq Q(t) \) for all \( t \in \mathbb{R} \) and all \( \varepsilon > 0 \). For \( \rho = \infty \) and \( \rho = 0 \), the equivalence of (i)–(iii) is also proved, respectively, in Lemma A.6 \( (p = 0) \) and Lemma A.7 \( (p = 0) \) of the present paper adapted for the argument-additive functions.

The following lemma is compounded of Lemma A.1 and Lemma A.3 with \( \rho = \infty \), provided that there exists the density of the distribution \( 1 - e^{-Q} \).
Lemma A.4. Let $t_Q = \infty$. For some large $t_0$, assume that $Q : [t_0; \infty) \to \mathbb{R}_+$ is (locally) absolutely continuous with the positive density $Q' : [t_0; \infty) \to \mathbb{R}_+$ obeying the following conditions:

\[
\lim_{t \to \infty} \frac{Q'(t + C)}{Q'(t)} = 1 \quad \text{for any} \quad C > 0, \tag{A.1}
\]

and

\[
\liminf_{t \to \infty} Q'(t) > 0. \tag{A.2}
\]

Then the following limits (I)–(IV) hold true:

(I) $\lim_{t \to \infty} Q(t + u)/Q(t) = \lim_{t \to \infty} Q'(t + u)/Q'(t) = 1$ uniformly in compact sets of $u \in \mathbb{R}$;

(II) $\liminf_{t \to \infty} Q(t)/t > 0$;

(III) $\lim_{t \to \infty} (Q(t + v a_1(t)) - Q(t)) = v$ uniformly in compact sets of $v \in \mathbb{R}$, with $a_1(\cdot) \equiv 1/Q'(\cdot)$ in $[t_0; \infty)$;

(IV) with $p(t) := e^{-Q(t)}Q'(t)$ ($t \geq t_0$) as the distribution density and $a_1(\cdot)$ as in part (III),

\[
\lim_{t \to \infty} \frac{p(t + u + v a_1(t))}{p(t + u)} = e^{-v}
\]

uniformly in compact sets of $v, u \in \mathbb{R}$.

Proof. (I) By L'Hôpital's rule, we obtain the first limit for any $u \in \mathbb{R}$. The uniform convergence follows from Theorem 1.2.1 in (Bingham et al. 1987) adapted for the argument-additive functions.

(II) The assertion follows from (A.2).

(III) Writing

\[
Q(t + v a_1(t)) - Q(t) - v = v \int_0^1 \left( a_1(t)Q'(t + \theta v a_1(t)) - 1 \right) d\theta \quad \text{for} \quad t \geq t_0
\]

and applying assertion (I) and condition (A.2), we easily obtain the claimed limit.

(IV) Let us rewrite the ratio under the limit in the form:

\[
\frac{p(t + u + v a_1(t))}{p(t + u)} = \exp \left\{ - \left( Q(t + u + v a_1(t)) - Q(t + u) \right) \right\} \times \frac{Q'(t + u + v a_1(t)) (Q'(t + u) - Q'(t))}{Q'(t)}^{-1}, \tag{A.3}
\]

$t \geq t_0$. Since $a_1(\cdot)$ is a bounded function, assertion (I) implies that the last two ratios on the right-hand side of (A.3) converge to 1 locally uniformly in $u, v \in \mathbb{R}$. It remains to prove the uniform convergence of the exponent on the right-hand side of (A.3). By the theorem of continuous convergence (see, e.g., p. 2 in (Resnick 1987)), it suffices to check that, for arbitrary functions $u(t) \to u$ and $v(t) \to v$, the following limit holds true:

\[
Q(t + u(t) + v(t) a_1(t)) - Q(t + u(t)) - v(t) \to 0 \quad \text{as} \quad t \uparrow \infty.
\]
This is shown similarly as in part (III), so we omit the details. Lemma A.4 is proved. □

**Remark A.5.** (Cf. Theorem 6.14). Let \( t_Q = \infty \). For some \( t_0 \), assume that \( Q : [t_0; \infty) \to \mathbb{R}_+ \) is (locally) absolutely continuous with the positive density \( Q' \) satisfying the following condition: \( (\log Q)'(t) \to \rho^{-1} \) as \( t \to \infty \), for some \( 0 < \rho < \infty \). Then the following limits hold true:

(i) \( \lim_{t \to \infty} Q(t + C)/Q(t) = \lim_{t \to \infty} Q'(t + C)/Q'(t) = e^{C/\rho} \) uniformly in compact sets of \( C \in \mathbb{R} \);

(ii) with \( a(\cdot) \equiv 1/Q'(\cdot) \) in \([t_0; \infty)\),

\[
\lim_{t \to \infty} (Q(t + Ca(t)) - Q(t)) = C \quad \text{for any} \quad C \in \mathbb{R}.
\]

*The proof of the assertions of Remark A.5.* (i) Write \( \log Q \) in the form:

\[
\log Q(t) = \text{const} + \frac{t}{\rho} + \int_{t_0}^{t} \varepsilon(s) \, ds \quad (t \geq t_0),
\]

where \( \varepsilon(t) := (\log Q)'(t) - \rho^{-1} \to 0 \) as \( t \to \infty \). Using this representation and the conditions of Remark A.5, we obtain the claimed limits for any \( C \in \mathbb{R} \). The uniform convergence follows from Theorem 1.5.2 in (Bingham et al. 1987) adapted for the argument-additive functions.

(ii) Since \( a(t) = o(1) \), the claimed limit is derived similarly as in the proof of Lemma A.4(III). □

### A.2. Classes \( \Pi^p_\infty \), \( \Pi^p_0 \) and \( O\Pi^p \)

Recall that, for \( p \geq 0 \), the classes \( \Pi^p_\infty \), \( \Pi^p_0 \) and \( O\Pi^p \) consist of functions \( f := Q^+\) satisfying, respectively, \( f(s)^p(f(s + c) - f(s)) \to \infty, \to 0 \) and \( \asymp 1 \) as \( s \to \infty \), for any \( c > 0 \); cf. (4.1)–(4.3). We first formulate the results of this section. To avoid trivialities, we restrict ourselves to the case \( t_Q = \infty \).

**Lemma A.6.** For any \( p \geq 0 \), \( f \in \Pi^p_\infty \) if and only if

\[
\lim_{t \to \infty} (Q(t + ct^{-p}) - Q(t)) = 0 \quad \text{for any} \quad c > 0.
\]

In this case,

\[
Q(t) = o(t^{p+1}) \quad \text{as} \quad t \to \infty.
\]

**Lemma A.7.** For any \( p \geq 0 \), \( f \in \Pi^p_0 \) if and only if

\[
\lim_{t \to \infty} (Q(t + ct^{-p}) - Q(t)) = \infty \quad \text{for any} \quad c > 0.
\]

In this case,

\[
Q(t)t^{-p-1} \to \infty \quad \text{as} \quad t \to \infty.
\]
Lemma A.8. For any \( p \geq 0 \) and \( f \in \text{OAIp} \), the following assertions hold true:

(i) there is a constant \( c > 0 \) such that \( Q(t + ct^{-p}) - Q(t) > 1 \) as \( t \to \infty \);

(ii) \( Q(t) \asymp t^{p+1} \) as \( t \to \infty \);

(iii) if a function \( a : \mathbb{R}_+ \to \mathbb{R}_+ \) is chosen to satisfy \( \liminf_{s \to \infty} a(s) \geq c_1 > 0 \) and \( \liminf_{s \to \infty} (s - a(s)) \geq c_2 > 0 \), then

\[
\text{const} (s - a(s)) s^{-p/(p+1)} \leq f(s) - f(a(s)) \leq \text{const}' \left( s^{1/(p+1)} - a(s)^{1/(p+1)} + a(s)^{-p/(p+1)} \right)
\]

for any \( s \geq s_0 \) and for some \( \text{const}' \geq \text{const} > 0 \).

Before proving Lemmas A.6–A.8, we provide an example of \( Q \) satisfying assertion (i) of Lemma A.8 such that \( f := Q^c \) does not belong to \( \text{OAIp} \) for \( p \geq 0 \), and further on, two technical lemmas for later use.

Example A.9. For \( p \geq 0 \), write \( Q(t) := t^{p+1} + |t|, t \geq 0 \). Note that, for each \( c > 0 \),

\[
Q(t + ct^{-p}) - Q(t) = c(p + 1) + g(t) + o(1) \quad \text{as} \quad t \to \infty,
\]

where \( 0 \leq g(t) := [t + ct^{-p}] - |t| = O(1) \). I.e., \( Q \) satisfies the assertion of Lemma A.8(i) for any \( c > 0 \). However, for each \( t := n \in \mathbb{N} \), we get \( Q(n) - Q(n-) = 1 \), therefore, \( f := Q^c \not\in \text{OAIp} \) according to Lemma A.10 below.

Lemma A.10. If \( \liminf_{s \to \infty} f(s)^p (f(s + c) - f(s)) > 0 \) for each \( c > 0 \) and for some \( p \geq 0 \), then

\[
\lim_{t \to \infty} (Q(t) - Q(t-)) = 0,
\]

i.e., \( Q \) is continuous at infinity.

Proof. Assume for a moment that there exists a sequence \( t_n \to \infty \) such that \( Q(t_n) - Q(t_n-) \to c^0 > 0 \). This limit implies that \( s_n := Q(t_n) \to \infty \) and, in addition, that \( f(s_n - c) = f(s_n) \) for any \( 0 < c < c^0 \) and any \( n \geq n_0(c) \), contradicting the assumption of the lemma. This completes the proof of the claimed assertion. \( \square \)

Lemma A.11. (Resnick 1987, pp. 4). For all \( s \in \mathbb{R}_+ \) and \( t \in (-\infty; t_Q) \), the following assertions hold true:

(i) \( f(s) \leq t \) if and only if \( s \leq Q(t) \);

(ii) \( f(s) > t \) if and only if \( s > Q(t) \);

(iii) \( Q(f(s)-) \leq s \leq Q(f(s)) \).

We now are in a position to prove Lemmas A.6–A.8. To simplify the proceedings, we need the following abbreviations:

\[
f_p(s; c) := f(s)^p (f(s + c) - f(s)) \quad \text{and} \quad Q_p(t; c) := Q(t + ct^{-p}) - Q(t).
\]

Proof of Lemma A.6. Assume first that (A.4) holds true. I.e., for each \( \varepsilon > 0 \) there is \( s_0 = s_0(\varepsilon) > 0 \) such that

\[
Q_p(f(s); \varepsilon^{-1}) < \varepsilon \quad \text{for all} \quad s \geq s_0.
\]
Since (A.4) implies a continuity of $Q$ at infinity, from Lemma A.11(iii) we have that $Q(f(s)) \leq s + \varepsilon$ for each $s \geq s_0$. This and (A.8) yield that $Q(f(s) + \varepsilon^{-1} f(s)^{-p}) < s + 2\varepsilon$ for each $s \geq s_0$. Inverting $Q$ (see Lemma A.11(ii)), we get that $f_p(s; 2\varepsilon) > 1/\varepsilon$ for each $\varepsilon > 0$ and each $s \geq s_0(\varepsilon)$. I.e., $f \in \text{AIP}_\infty$.

To prove the inverse implication, assume for a moment that there are a sequence of reals $t_n \to \infty$ and constants $c > 0$, $\varepsilon > 0$ such that $Q(t_n + ct_n^{-p}) \geq Q(t_n) + \varepsilon$ for all $n \geq n_0(\varepsilon, c)$. Inverting $Q$ (see Lemma A.11(i)), we get that $f(Q(t_n) + \varepsilon) \leq t_n + ct_n^{-p}$, which combined with $t_n < f(Q(t_n) + \varepsilon/2)$ (see Lemma A.11(ii)) gives that, for each $n \geq n_0(\varepsilon, c)$,

$$f_p(s_n; \varepsilon/2) \leq c \quad \text{with} \quad s_n := Q(t_n) + \varepsilon/2.$$  

Since $s_n \to \infty$, the latter violates the assumption $f \in \text{AIP}_\infty$, concluding the proof of the first part of Lemma A.6.

To prove (A.5), we note that, for any natural $M \geq 2$ and any $s \geq 2M$,

$$f(s + 1)^{p+1} - f(s)^{p+1} \geq f_p(s; 1) \geq I(M) := \inf_{s \geq M} f_p(s; 1),$$

and, therefore,

$$f(s)^{p+1} - f(M)^{p+1} \geq (s - M - 1)I(M).$$

Hence $\lim\inf_{s \to \infty} f(s)s^{-1/(p+1)} \geq I(M)^{1/(p+1)}$. Since $I(M) \to \infty$ (as $M \to \infty$) by the assumption, the latter implies that $f(s)s^{-1/(p+1)} \to \infty$ as $s \to \infty$, which in turn yields (A.5). Lemma A.6 is proved. \hfill \Box

Proof of Lemma A.7. Assume first that (A.6) holds true, i.e., for each $\varepsilon > 0$ there is $t_0 = t_0(\varepsilon)$ such that $Q_p(t; \varepsilon) \geq 1/\varepsilon$ for each $t \geq t_0$. By Lemma A.11(i), the latter is equivalent to $f(Q(t) + 1/\varepsilon) \leq t + ct^{-p}$. Substituting $t := f(s) \to \infty$ into this inequality and then applying $Q(f(s)) \geq s$ (see Lemma A.11(iii)), we obtain $f_p(s; 1/\varepsilon) \leq \varepsilon$ for each $\varepsilon > 0$ and each $s \geq s_0(\varepsilon)$, i.e., $f \in \text{AIP}_0$.

To prove the inverse implication, assume for a moment that there are a sequence $t_n \to \infty$ and constants $\delta > 0$, $c > 0$ such that $Q_p(t_n; \delta) < c$ for each $n \geq n_0(c, \delta)$. Here, inventing $Q$ (see Lemma A.11(i),(ii)) and denoting $s_n := Q(t_n) \to \infty$, we obtain that $f_p(s_n; c) > \delta$ for each $n \geq n_0(c, \delta)$, contradicting the assumption $f \in \text{AIP}_0$. This completes the proof of the first part of Lemma A.7.

We will prove (A.7) under the weaker condition by assuming (A.6) for some $c > 0$. (The forthcoming arguments are applied to prove assertion (ii) of Lemma A.8 as well). Write $Q^{(c)}(t) := Q(c^{1/(p+1)}t)$ and observe that

$$\lim_{t \to \infty} Q_p^{(c)}(t; 1) = \lim_{t \to \infty} Q_p(t; c) = \infty,$$

i.e., limit (A.6) for $c > 0$ is reduced to that for $c = 1$. With the abbreviation
\( R(k) := [k^p], \) we obtain that, for fixed natural \( M \geq 3 \) and any natural \( t \geq 2M, \)
\[
Q^{(c)}(t) - Q^{(c)}(M) \\
= \sum_{k=M}^{t-1} \sum_{l=0}^{R(k)-1} \left( Q^{(c)} \left( k + \frac{l+1}{R(k)} \right) - Q^{(c)} \left( k + \frac{l}{R(k)} \right) \right) \\
\geq \sum_{k=M}^{t-1} \sum_{l=0}^{R(k)-1} Q^{(c)}_p \left( k + \frac{l}{R(k)} + 1 \right) - \inf_{\tau \geq M} Q^{(c)}_p(\tau; 1) \sum_{k=M}^{t-1} R(k) \\
= \inf_{\tau \geq M} Q^{(c)}_p(\tau; 1) \frac{\delta^{p+1}}{p+1} \left( 1 + o(1) \right)
\]
as \( t \to \infty. \) Here, by (A.6), the infimum tends to infinity as \( M \to \infty, \) therefore, (A.7) is fulfilled. This completes the proof of Lemma A.7.

Proof of Lemma A.8. (i) We first prove that if, for each \( c > 0, \) the function \( f_p(s; c) \)
is asymptotically bounded away from zero as \( s \to \infty, \) then \( Q_p(t; \delta) = O(1) \) as \( t \to \infty, \)
for some \( \delta > 0. \) Assume otherwise that, for each \( \delta > 0, \) there exists a sequence \( t_n \to \infty \) such that \( Q_p(t_n; \delta) \geq 2M \) for any \( M > 0 \) and any \( n \geq n_0(M). \) Here,
inverting \( Q \) similarly as in the proof of Lemma A.6, we obtain that \( f_p(s_n; M) \leq \delta \)
with \( s_n := Q(t_n) + \delta \to \infty \) as \( n \to \infty. \) Therefore, \( \liminf_{s \to \infty} f_p(s; M) \leq \delta. \) Since \( \delta > 0 \) is arbitrary, we obtain the contradiction proving the desired implication. We
next observe that, if \( Q_p(t; \delta) = O(1) \) for some \( \delta > 0, \) then
\[
Q_p(t; k\delta) = O(1) \quad \text{as} \quad t \to \infty, \quad \text{for any} \quad k \in \mathbb{N}. \quad (A.9)
\]
This implication is easily proved by induction in \( k. \) We omit the details.

With the abbreviation
\[
M := 2 \limsup_{s \to \infty} f_p(s; 1) > 0,
\]
we finally show that the function \( Q_p(t; M) \) is asymptotically bounded away from zero
as \( t \to \infty. \) For this, fix an arbitrary sequence \( t_n \to \infty, \) and define a sequence \( \{s_n\} \)
by \( f(s_n +) \geq t_n \geq f(s_n) \) (\( n \in \mathbb{N} \)). Write \( \tau_n := f(s_n). \) Combining Lemmas A.10
and A.11(iii), we have that \( Q(t_n) - Q(\tau_n) = o(1) \) and, consequently,
\[
Q_p(t_n; 1) \geq Q_p(\tau_n; 1) + o(1) \quad \text{as} \quad n \to \infty. \quad (A.10)
\]
On the other hand, from the definition of \( M, \) it follows that \( f(s_n) + M f(s_n)^{-p} > f(s_n + 1) \) for any \( n \geq n_0. \) Applying \( Q \) to both sides of this inequality and then using
Lemmas A.10 and A.11(iii), we obtain that \( Q_p(\tau_n; M) \geq 1/2 \) for \( n \geq n_0. \) The latter
combined with (A.10) implies that the sequence \( Q_p(t_n; M) \) is asymptotically bounded
away from zero, as claimed. This and (A.9) conclude the proof of part (i).

(ii) The assertion is shown by the same arguments as in the proof of limits (A.5)
and (A.7). We omit the details.

(iii) If \( a(s) \) or \( s - a(s) \) are bounded from above for any large \( s, \) then the bounds in
(iii) simply follow from part (ii) and condition (4.3).

For simplicity we abbreviate \( a := a(s), \) and assume that both \( a \) and \( s - a \) tend to
infinity as \( s \to \infty. \) By combining assumption (4.3) and the limit \( f(s) \asymp s^{1/(p+1)}, \) we
obtain that, for \( s \geq s_0 \),
\[
    f(s) - f(a) \leq \sum_{0 \leq k \leq s-a} (f(a + k + 1) - f(a + k)) \\
    \leq \text{const} \sum_{0 \leq k \leq s-a} (a + k)^{-p/(p+1)} \\
    \leq \text{const} a^{-p/(p+1)} + \text{const} \int_0^{s-a} (a + k)^{-p/(p+1)} \, dk
\]
and
\[
    f(s) - f(a) \geq \sum_{0 \leq k \leq s-a-1} (f(a + k + 1) - f(a + k)) \\
    \geq \text{const} \sum_{0 \leq k \leq s-a-1} (a + k)^{-p/(p+1)} \\
    \geq \text{const} (s - a - 1) s^{-p/(p+1)},
\]
as claimed. Lemma A.8 is proved. \( \square \)

**Remark A.12.** (A relationship with classical regular variation). (i) Consider the case \( p = 0 \). Obviously, for \( \beta = \infty \) or \( \beta = 0 \), \( f \) is in \( A\Pi_0^\beta \) if and only if \( g := \exp \circ f \circ \log \in RV_\beta \). Therefore, for \( p = 0 \), Lemmas A.6 and A.7 follow from the well-known results for the class \( RV_\beta \) with \( \beta = \infty \) and \( \beta = 0 \), respectively (Bingham et al. 1987).

The class \( O\Pi^p \) links to the exponential type distributions \( 1 - e^{-Q} \), with \( Q(t) \asymp t \) as \( t \to \infty \). Moreover, if \( f \in O\Pi^p \), then \( g := \exp \circ f \circ \log \) is in \( ORV \), the class of \( O \)-regularly varying functions studied, e.g., in (Bingham et al. 1987, Section 2).

(ii) In the case of \( p \geq 0 \), if \( f \in O\Pi^p \), then \( f \circ \log \) is asymptotically balanced or, equivalently, the maximum \( \xi_{1,V} \) of i.i.d. sample \( \xi_V \) is stochastically compact (Bingham et al. 1987, Sections 3.11 and 8.13.12).

**A.3. Class \( P I_{< 2} \)**

Recall that the class \( P I_{< 2} \) consists of functions \( f := Q^- \) such that
\[
\liminf_{s \to \infty} \frac{f(cs)}{f(s)} > 1 \quad \text{for some} \quad 1 < c < 2;
\]
cf. (4.5). This is the subclass of positive increase class \( P I \) considered, e.g., in (Bingham et al. 1987, Section 2.1.2).

**Lemma A.13.** \( f \in PI_{< 2} \) if and only if
\[
    \limsup_{t \to \infty} Q(C_0 t)/Q(t) < 2 \quad \text{for some} \quad C_0 > 1. \tag{A.11}
\]
In this case,
\[
    \limsup_{t \to \infty} \frac{\log Q(t)}{\log t} \leq \text{const} := \frac{\log 2}{\log C_0}. \tag{A.12}
\]
Proof. Assume in contrary to (A.11) that, for any $C > 1$, there is a sequence $t_n = t_n(C) \to \infty$ such that $Q(Ct_n) \geq (2 - \varepsilon)Q(t_n)$ for any $\varepsilon > 0$ and any $n \geq n_0(\varepsilon)$. Similarly as in the proof of Lemma A.6, inverting $Q$ (see Lemma A.11(i),(ii)) and writing $s_n := Q(t_n) + \varepsilon \to \infty$, we obtain that

$$\limsup_n f(\delta s_n)/f(s_n) \leq C$$

for any $1 < \delta < 2$ and any $C > 1$, or, equivalently, $\lim_n f(\delta s_n)/f(s_n) = 1$, contradicting the assumption $f \in PI_{<2}$. According to these arguments, the inclusion $f \in PI_{<2}$ implies (A.11).

To show the inverse implication, we suppose otherwise that $f := Q^\perp \notin PI_{<2}$, i.e., for each $1 < c < 2$, there exists a sequence $s_n = s_n(c) \to \infty$ such that $f(cs_n) \leq (1 + \varepsilon)f(s_n)$ for any $\varepsilon > 0$ and any $n \geq n_0(\varepsilon)$. In this inequality, we invert $f$ (see Lemma A.11(i)) to obtain $cs_n \leq Q((1 + \varepsilon)f(s_n))$. On the other hand, by Lemma A.11(iii),

$$cs_n \geq cQ(f(s_n) - ) \geq cQ((1 - \varepsilon)f(s_n)).$$

Summarizing these estimates and using the abbreviations $t_n := (1 - \varepsilon)f(s_n)$ and $\delta := (1 + \varepsilon)/(1 - \varepsilon)$, we have that $Q(\delta t_n) \geq cQ(t_n)$. Since $1 < c < 2$ is an arbitrary constant but close to 2, the latter implies the limit $\limsup_n Q(\delta t_n)/Q(t_n) \geq 2$ for each $\delta > 1$, contradicting assumption (A.11). This concludes the first part of the lemma.

Let us show (A.12). By (A.11), there exist numbers $C > 1$ and $t_0 = t_0(C)$ such that $Q(Ct)/Q(t) \leq 2$ for all $t \geq t_0$. Applying this estimate, we obtain that, for any $n \in \mathbb{N}$ and any $t \in [C^n t_0; C^{n+1} t_0]$,

$$Q(t) = \frac{Q(t)}{Q(C^n t_0)} \cdot \frac{Q(C^n t_0)}{Q(C^{n-1} t_0)} \cdots \frac{Q(C t_0)}{Q(t_0)} \cdot Q(t_0) \leq Q(t_0) 2^{n+1} = \text{const} (\log 2)/\log C,$$

i.e., (A.12) is done. Lemma A.13 is proved. \hfill \square

### A.4. Comparison of the classes $\text{AP}^p_\infty$, $\text{AP}$ and $\text{PI}_{<2}$. Examples

In view of limit theorems for eigenvalues (see Theorems 6.2 and 6.7), we need to compare the classes $\text{AP}^p_\infty$ (4.1), $\text{SAPI}^2_\infty$ (5.4), $\text{AP}$ (6.3) and $\text{PI}_{<2}$ (4.5) of functions $f := Q^\perp$.

**Lemma A.14.** (i) For any $p \geq 0$, there exist examples $f_1 \in \text{PI}_{<2} \setminus \text{AP}^p_\infty$ and $f_2 \in \text{AP}^p_\infty \setminus \text{PI}_{<2}$. Consequently, there is $f_2 \in \text{SAPI}^2_\infty \setminus \text{PI}_{<2}$.

(ii) There exist examples $f_3 \in \text{PI}_{<2} \setminus \text{AP}$ and $f_4 \in \text{AP} \setminus \text{PI}_{<2}$ with an auxiliary function $a_4 \geq 1$.

(iii) For any $p > 0$, there exists an example $f_5 \in (\text{AP}^p_\infty \cap \text{PI}_{<2}) \setminus \text{AP}$ and, therefore, there is $f_5 \in (\text{SAPI}^2_\infty \cap \text{PI}_{<2}) \setminus \text{AP}$.

(iv) For $p \geq 0$, if $f \in \text{AP}$ with an auxiliary function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that $s^p a(s) \to \infty$ as $s \to \infty$, then $f \in \text{AP}^p_\infty$.

(v) For $0 \leq \rho < \infty$, if $e^f$ is in RV$_\rho$, then $f(s + \log s) - f(s) \to 0$ as $s \to \infty$.

**Proof.** (i) Write $Q(t) := [t]$, $t \geq 0$; i.e., $1 - e^{-Q}$ is the geometric distribution. Let us show that $f_1 := Q^\perp \in \text{PI}_{<2} \setminus \text{AP}^p_\infty$. Indeed, since $Q(n) - Q(n-) = 1$ for all $n \in \mathbb{N}$,
Lemma A.10 implies that $f_1 \notin \mathcal{AP}_\infty^0$ for any $p \geq 0$. However, since $Q$ satisfies (A.11), we have that $f_1 \in PI_{<2}$, as claimed.

Consider the function $f_2(s) := \int_1^s b(t) \, dt$, where $b(t) := n$ if $2^{2n} < t \leq 2^{2n+1}$, and $b(t) := 2^n$ if $2^{2n+1} < t \leq 2^{2n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. Let us show that $f_2 \in \mathcal{AP}_\infty^0 \setminus PI_{<2}$. Obviously $f_2 \in \mathcal{AP}_\infty^0$. With $s_n := 2^{2n+1}$ and $1/2 < \delta < 1$, we see that

$$f_2(\delta s_n) = 1 - \frac{\int_{\delta s_n}^{s_n} b(t) \, dt}{\int_{1}^{s_n} b(t) \, dt},$$

where $\int_{\delta s_n}^{s_n} b(t) \, dt = \text{const} \cdot 4^n n$ and

$$\int_{1}^{s_n} b(t) \, dt = \sum_{l=0}^{n-1} \left( \int_{s_l/2}^{s_{l+1}} l \, dt + \int_{s_l}^{2s_l} 2^l \, dt \right) + \int_{s_n/2}^{s_n} n \, dt$$

$$= \sum_{l=0}^{n-1} \left( l \cdot 4^l + 2 \cdot 8^l \right) + n \cdot 4^n = \frac{2}{7} \cdot 8^n (1 + o(1))$$

as $n \to \infty$. Summarizing, we find that $f_2(\delta s_n)/f_2(s_n) \to 1$ (as $n \to \infty$) for each $1/2 < \delta < 1$, i.e., $f_2 \notin PI_{<2}$. Since $\mathcal{AP}_\infty^0 \subset \mathcal{AP}_\infty^2$, we also obtain that $f_2 \in \mathcal{AP}_\infty^2 \setminus PI_{<2}$.

(ii) As in part (i) above, let $f_3$ be the inverse of the cumulative hazard function of the geometric distribution. Since $f_3$ is not in $\mathcal{AP}$ (Resnick 1987, Corollary 1.6), we obtain that $f_3 \in PI_{<2} \setminus \mathcal{AP}$.

To prove the existence of $f_4 \in \mathcal{AP} \setminus PI_{<2}$, it suffices (via Lemmas A.1 and A.13) to find a continuous function $a : [1; \infty) \to [1; \infty)$, with derivative $a'(t) \to 0$ as $t \uparrow \infty$, such that the function $Q(t) := \int_{1}^{t} 1/a(s) \, ds$ does not satisfy (A.11). For this, we abbreviate $t_n := (\log n)^n$, $m_n := (\log n)^2$,

$$\varepsilon_n := \frac{1}{m_n} \left( \frac{m_{n+1}}{m_n} - 1 \right) - \frac{1}{n} \quad \text{and} \quad b_n := 1 + t_{n+1} \varepsilon_n,$$

and consider the functions

$$a(t) := \begin{cases} t/t_n & \text{if } t_n < t \leq t_{n+1} - m_n, \\ -\varepsilon_n t + b_n & \text{if } t_{n+1} - m_n < t \leq t_{n+1}; \quad \text{for } n \geq 2, \end{cases} \tag{A.13}$$

and $Q(t) := \int_{1}^{t} 1/a(s) \, ds$. Obviously, $a \geq 1$ is continuous in $[1; \infty)$ and $a'(t) \to 0$ as $t \uparrow \infty$. Therefore, $f_4 := Q^* \in \mathcal{AP}$ with the auxiliary function $a$. Let us show that, for each $c > 1$,

$$\frac{Q(ct_n)}{Q(t_n)} = 1 + \frac{\int_{t_n}^{ct_n} 1/a(t) \, dt}{\int_{1}^{t_n} 1/a(t) \, dt} \to \infty \quad \text{as } n \to \infty, \tag{A.14}$$

so that $f_4 \notin PI_{<2}$. Indeed, from (A.13) we see that, for $n \geq n_0(c)$,

$$\int_{t_n}^{ct_n} \frac{dt}{a(t)} = \int_{t_n}^{ct_n} \frac{t_n}{t} \, dt = t_n \log c. \tag{A.15}$$
To estimate the integral \( \int_{t_2}^{t_n} \frac{dt}{a(t)} \) in (A.14), we again use (A.13) and the bound \( a \geq 1 \). Thus,

\[
\int_{t_2}^{t_n} \frac{dt}{a(t)} \leq \sum_{k=2}^{n-1} \left( t_k \int_{t_k}^{t_{k+1}-m_k} \frac{dt}{t} + \int_{t_{k+1}-m_k}^{t_{k+1}} \frac{dt}{1} \right) \\
= \sum_{k=2}^{n-1} \left( m_k \log(t_{k+1}/t_k) + m_k \right) \leq t_n (\log n)^{-1/2}
\]

(A.16)

for any \( n \geq n_0 \). Now (A.15) and (A.16) imply (A.14), as claimed.

(iii) Consider the example \( Q(t) = t + \sin t \ (t \geq 0) \) given by Von Mises. Obviously, \( Q \) satisfies (A.4) and (A.11), consequently, \( f_5 := Q^\gamma \) is in \( \text{AIP}_{\infty}^\alpha \cap \Pi_{<2} \) for any \( p > 0 \). However, \( f_5 \notin \text{AII} \). We observe that the function \( f_6(s) := s + \sin s \ (s \geq 0) \) is also in \( (\text{AIP}_{\infty}^\alpha \cap \Pi_{<2}) \backslash \text{AII} \) for any \( p > 0 \). (This is verified by straightforward calculations). Consequently, since \( \text{AIP}_{\infty}^\alpha \subset \text{SAIP}_{\infty}^\alpha \), the functions \( f_5 \) and \( f_6 \) are in \( (\text{SAIP}_{\infty}^\alpha \cap \Pi_{<2}) \backslash \text{AII} \).

(iv) The assertion follows from the definition of \( \text{AII} \) (6.3) and \( \text{AIP}_{\infty}^\alpha \) (4.1).

(v) The assertion follows from Lemma A.3(ii). Lemma A.14 is proved. \( \square \)

We finally provide two examples of distributions which represent RV classes considered in Sections 3–6.

Example A.15. For \( \alpha > 0 \), let \( Q_{\alpha}(t) = t^\alpha \ (t \geq 0) \), i.e., \( 1 - e^{-Q_{\alpha}} \) is Weibull distribution. Clearly \( f_{\alpha}(s) := Q_{\alpha}^\gamma(s) = s^{1/\alpha} \) for \( s \geq 0 \). By straightforward calculations, we obtain that if \( \alpha < p + 1 \) (resp., \( \alpha > p + 1 \) or \( \alpha = p + 1 \)), then \( f_{\alpha} \in \text{AIP}_{\infty}^\alpha \) (resp., \( f_{\alpha} \in \text{AIP}_{\infty}^\alpha \) or \( f_{\alpha} \in \text{OAIP}^\alpha \)). Also, for \( \alpha < 3 \), \( f_{\alpha} \in \text{SAIP}_{\infty}^\alpha \). Finally, for any \( \alpha > 0 \), \( f_{\alpha} \in \Pi_{<2} \), \( \exp f_{\alpha} \in \text{RV}_{\infty} \) and \( f_{\alpha} \) is in \( \text{AII} \) with the auxiliary function \( a(t) := t^{1-\alpha} \). The latter means that the distribution \( 1 - e^{-Q_{\alpha}} \) is in the domain of attraction of the max-stable Gumbel law; cf. Section 6.1.

Example A.16. Given \( \gamma > 0 \) and \( \rho > 0 \), let \( Q_{\gamma,\rho}(t) = e^{\rho^{-1} t^\gamma} \ (t \geq t_0) \), i.e., the fractional double exponential distribution. Then \( f_{\gamma,\rho}(s) = (\rho \log s)^{1/\gamma} \) for \( s \geq s_0 \). Obviously, if \( 0 < \gamma < 1 \) (resp., \( \gamma > 1 \) or \( \gamma = 1 \)), then \( \exp f_{\gamma,\rho} \) is in \( \text{RV}_{\infty} \) (resp., \( \text{RV}_0 \) or \( \text{RV}_{\rho} \)); cf. Section 6.

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