On the deprojection of triaxial galaxies with Stäckel potentials

A. Mathieu and H. Dejonghe
Universiteit Gent, Sterrenkundig Observatorium, Krijgslaan 281, B–9000 Gent, Belgium

Abstract. A family of triaxial Stäckel potential-density pairs is introduced. With the help of a Quadratic Programming method, a linear combination of potential-density pairs of this family which fits a given projected density distribution can be built. This deprojection strategy can be used to model the potentials of triaxial elliptical galaxies with or without dark halos. Besides, we show that the expressions for the Stäckel triaxial density and potential are considerably simplified when expressed in terms of divided differences, which are convenient numerically. We present an example of triaxial deprojection for the galaxy NGC 5128 whose photometry follows the de Vaucouleurs law.

Key words: Galaxies: structure - Galaxies: photometry - Galaxies: kinematics and dynamics - Galaxies: individual (NGC 5128)

1. Introduction

As soon as one realized that elliptical galaxies were not spheroids flattened by rotation, a growing list of unexpected photometric and kinematical observations has led astronomers to develop a more elaborate picture of these objects. While triaxiality provides a plausible explanation of these new findings, it also raises a number of questions such as the determination of the intrinsic shapes of these stellar systems.

The deprojection problem consists in finding the intrinsic mass (or light) density from its projection onto the plane of the sky. It is well known that the knowledge of the surface brightness distribution does not define uniquely the intrinsic three-dimensional light distribution and the orientation of the galaxy. Even in the axisymmetric case the deprojection is degenerate: one can construct many density distributions that project to the same photometric distribution (see e.g. Gerhard & Binney 1995). Features such as the presence of lanes or disks in some galaxies have been exploited to further constrain the problem. Recently Statler (1994) proposed a new method for constraining the intrinsic shapes of elliptical galaxies using information on their apparent shapes and velocity fields.

In this paper, we shall describe a technique that determines an intrinsic mass density distribution of a galaxy consistent with a given projected mass density. Similarly, the method allows to determine the intrinsic light distribution of a galaxy given its surface brightness distribution. This is the first step towards a triaxial dynamical model. Stäckel potentials are particularly suited to this kind of modeling since the investigation of the dynamics is essentially analytical. A Stäckel triaxial mass model can be constructed by specifying a density profile along the short axis and an ellipsoidal coordinate system. Simple examples of such models are given in de Zeeuw, Peletier & Franx (1986). However, the determination of the potential generally requires the evaluation of one-dimensional quadratures. Another approach consists in specifying ellipsoidal coordinates and a simple form for the Stäckel potential. The mass density can be calculated by means of the generalized Kuzmin’s formula (de Zeeuw 1985b) which involves straightforward derivatives of a one-dimensional function. We shall use this method in order to have a simple analytical expression for the potential.

The basic properties of Stäckel models are studied in Sect. 2. Section 3 presents a family of Stäckel potential-density pairs that can be used as building blocks for the construction of triaxial mass models. Examples of projected and spatial density distributions are shown and then we describe the deprojection method. In Sect. 4, we present a fit to a triaxial modified Hubble model using these potential-density pairs. Section 5 deals with the application of the deprojection method to the elliptical galaxy NGC 5128 (Centaurus A) whose photometry follows the de Vaucouleurs law. Our conclusions are given in Sect. 6.
2. Triaxial Stäckel models

A sequence of potential-density pairs can be generated by repeated application of an operator upon a given potential-density pair. Such a method has been used by de Zeeuw and Pfenniger (1988, hereafter referred to as ZP88) to build a class of triaxial potential-density pairs \((V_n, \rho_n)\), amongst which the family \((V_2, \rho_2)\) is of Stäckel form. In the next subsection, we present some properties of the Stäckel models (for a detailed discussion see de Zeeuw 1985a, ZP88).

2.1. General properties

Let \((x, y, z)\) be cartesian coordinates and \((\lambda, \mu, \nu)\) be ellipsoidal coordinates defined according to de Zeeuw (1985a). In these coordinates, the Stäckel potential, for which the equations of motion separate, has the general form

\[
V(\lambda, \mu, \nu) = g_\lambda F(\lambda) + g_\mu F(\mu) + g_\nu F(\nu),
\]

where \(F\) is an arbitrary function and

\[
g_\lambda = \frac{(\lambda + \alpha)(\lambda + \beta)}{(\lambda - \mu)(\lambda - \nu)} \quad (\text{cyc}).
\]

\(\alpha, \beta\) and \(\gamma\) are negative constants and

\[-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda.
\]

Similar expressions for \(g_\mu\) and \(g_\nu\) are found by a cyclic permutation \(\lambda \rightarrow \mu \rightarrow \nu \rightarrow \lambda\). Furthermore, we have

\[0 \leq g_\lambda, g_\mu, g_\nu \leq 1 \quad \text{and}
\]

\[g_\lambda + g_\mu + g_\nu = 1.
\]

The triaxial density follows from Poisson’s equation and can be written (see ZP88) as

\[
\rho(\lambda, \mu, \nu) = g_\lambda^2 \Psi(\lambda) + g_\mu^2 \Psi(\mu) + g_\nu^2 \Psi(\nu) + 2g_\lambda g_\mu \Psi[\lambda, \mu] + 2g_\mu g_\nu \Psi[\mu, \nu] + 2g_\nu g_\lambda \Psi[\nu, \lambda],
\]

where \(\Psi[\tau_1, \tau_2]\) denotes the first order divided difference of the function \(\Psi(\tau)\) i.e.

\[
\Psi[\tau_1, \tau_2] = \frac{\Psi(\tau_1) - \Psi(\tau_2)}{\tau_1 - \tau_2}
\]

and \(\Psi'(\tau)\) is the derivative \(\Psi[\tau, \tau]\). The relation that connects \(\Psi(\tau)\) and \(F(\tau)\) is (ZP88):

\[
2\pi G \Psi(\tau) = 2(\tau + \gamma) F'(\tau) - F(\tau) + (\tau + \gamma) \left( \frac{F(\tau) - F(-\alpha)}{\tau + \alpha} + \frac{F(\tau) - F(-\beta)}{\tau + \beta} \right).
\]

Given a function \(F(\tau)\), one can calculate the Stäckel potential with Eq. (8), and the density using Eq. (9). The density profile along the \(z\)-axis is the function \(\Psi(\tau)\) with \(\tau = z^2 - \gamma\). The triaxial Stäckel density is fully determined by the specification of the density on the short \(z\)-axis; it can be written as a weighted sum of the density at six particular points of the \(z\)-axis (de Zeeuw 1985b).

2.2. Expressions for the Stäckel potential and density in terms of divided differences

The use of divided differences considerably simplifies the formulas of the Stäckel triaxial density and potential. The divided difference of order \((n - 1)\) of the function \(U(\tau)\) is a function of divided differences of order \((n - 2)\) and is given by

\[
U[\tau_1, \tau_2, \cdots, \tau_n] = \frac{U[\tau_1, \tau_3, \cdots, \tau_n] - U[\tau_2, \tau_3, \cdots, \tau_n]}{\tau_1 - \tau_2}.
\]

The second order divided difference of a general function \(U(\tau)\) can be written explicitly as

\[
U[\lambda, \mu, \nu] = \frac{U(\lambda)}{(\lambda - \mu)(\lambda - \nu)} + (\text{cyc}).
\]

With Eqs. (3) and (4), we can express the potential \(V\) as the second order divided difference

\[
V(\lambda, \mu, \nu) = U[\lambda, \mu, \nu]
\]

with

\[
U(\tau) = (\tau + \alpha)(\tau + \beta)F(\tau).
\]

The density \(\rho(\lambda, \mu, \nu)\) can be written in terms of divided differences using either the function \(\Psi(\tau)\) whose derivative is the density profile along the \(z\)-axis, or using the basis function \(F(\tau)\) that defines the potential \(V\). The detailed proofs are given in Appendix B and Appendix C respectively. To establish these results, we shall use the potential-density pair \((\rho_1, V_1)\) introduced in ZP88. Its main properties are given in Appendix A. In this subsection, we only present the relevant results.

The triaxial Stäckel density \(\rho(\lambda, \mu, \nu)\) can be expressed as the 5th-order divided difference of the function \(H(\tau)\)

\[
\rho(\lambda, \mu, \nu) = H[\lambda, \mu, \nu, \lambda, \mu, \nu]
\]

with

\[
H(\tau) = (\tau + \alpha)^2(\tau + \beta)^2\Psi(\tau).
\]

It can also be written as the 5th-order divided difference of the function \(R(\tau)\)

\[
\rho(\lambda, \mu, \nu) = R[\lambda, \mu, \nu, \lambda, \mu, \nu]
\]

with

\[
\pi GR(\tau) = \left| a(\tau) \right|^2 \frac{d}{d\tau} \left[ \frac{\sqrt{|a(\tau)|}}{\tau + \gamma} F(\tau) \right],
\]

where

\[
a(\tau) = (\tau + \alpha)(\tau + \beta)(\tau + \gamma).
\]
3. The deprojection strategy

3.1. Basis component

A Stäckel triaxial mass model is completely determined by the choice of ellipsoidal coordinates and a function $F(\tau)$. So as to have simple analytical expressions for the potential and the density, we choose $F(\tau)$ to be an elementary function of $\tau$.

The spherical Hénon’s isochrone model (1959) is often used in the construction of models of stellar systems. More realistic models that account for the flattening of the potential can be produced with axisymmetric generalizations (see Dejonghe & de Zeeuw 1988; Evans, de Zeeuw & Lynden-Bell 1990). Axisymmetric Stäckel models that reduce to the isochrone in the spherical limit can be constructed using

$$F(\tau) = -\frac{GM}{\sqrt{-\alpha + \sqrt{\tau}}}.$$  \hspace{1cm} \text{(17)}$$

The associated potential is the thoroughly studied Kuzmin-Kutuzov potential (see e.g. Kuzmin 1956, Kuzmin & Kutuzov 1962, Dejonghe & de Zeeuw 1988, Batsleer & Dejonghe 1993). Using the above function $F(\tau)$, a triaxial generalization can also be calculated with Eqs. (1) and (5) or, alternatively, Eqs. (14) and (15). Expressions for the density and the potential are given in ZP88.

We choose a three-parameter basis function of the form

$$F(\tau) = -\frac{GM}{(d + \tau^p)^s}.$$  \hspace{1cm} \text{(18)}$$

where $d$, $p$ and $s$ are real parameters. The triaxial isochrone has $p = 0.5$, $s = 1$ and $d = \sqrt{-\alpha}$.

3.2. Some examples

Kinematic studies of elliptical galaxies have revealed a very diverse and complicated nature of these objects. There has been marked improvement in the photometric observations with the advent of CCDs and it appears that ellipticals also span a wide range of photometric properties. The isophotal shape is mainly elliptical, but departures from perfect ellipses are often detected, as well as isophote twists (see e.g. Bender, Döbereiner & Möllenhoff 1988).

The projected mass densities of the basis components of the form (18) have elliptical surface isodensities that can exhibit deviations from pure ellipses such as boxiness and diskiness. Thus these components may be relevant to approximate the photometry of elliptical galaxies. It is well known that Stäckel potentials cannot produce projected densities with isophote twists (Franx 1988), but this is often a second order effect compared to the ellipticity variation.

In this section, we present three examples of spatial and projected density distributions for a normal elliptical (Fig. 1), a discy (Fig. 3) and a boxy (Fig. 3) models. The units are arbitrary. The intrinsic long, intermediate and short axes are denoted $x$, $y$ and $z$ respectively. The models have $(-\alpha, -\beta, -\gamma) = (7.3, 3.6, 0.8)$. The parameters $(d, p, s)$ of the components are $(5, 0.7, 0.7)$ for the elliptical model, $(4, 0.5, 1)$ for the discy model and $(5, 0.93, 0.44)$ for the boxy model.

Fig. 1. Contour map of the spatial mass density in the planes (top left) $x = 0$, (top right) $y = 0$ and (bottom left) $z = 0$ for an elliptical model. Bottom right: contour map of the projected mass density. The contour step is 0.65 magnitude

3.3. The method

We use a Quadratic Programming (QP) method (Dejonghe 1989) to fit to a given photometric data set a linear combination of basis functions with the constraint that the density must be positive. Given a function $F(\tau)$ of the form (18), the calculation of the Stäckel potential and density is straightforward as it involves only the evaluation of elementary functions. These basis components can produce physical density distributions (i.e. positive everywhere in real space) as well as unphysical ones, depending on the choice of the ellipsoidal coordinates and the parameters $d$, $p$ and $s$. Generally, the projected density distribution has to be calculated numerically.

As showed by Kuzmin (1956) for axisymmetric mass models and subsequently generalized for triaxial models by de Zeeuw (1985b), the so-called Kuzmin’s theorem
states that a triaxial mass density built with a Stäckel gravitational potential is everywhere positive if the density on the z-axis is positive. Therefore, the positivity check of the triaxial density $\rho(\lambda, \mu, \nu)$ reduces to a one-dimensional check on the density along the z-axis.

4. Triaxial modified Hubble model

The de Vaucouleurs and the Hubble laws are among the most commonly used fitting profiles for the photometry of ellipticals. The density radial profile of a modified Hubble model $\rho \sim (1 + r^2)^{-3/2}$ decreases proportional to $r^{-3}$ at large radii. A separable triaxial generalization of the modified Hubble model has been studied by de Zeeuw, Peletier & Franx (1986). The density along the z-axis for a modified Hubble model with core radius $c$ can be written

$$\Psi'(z) = \frac{\rho_0 c^3}{(z^2 + c^2)^{3/2}}.$$  \hspace{1cm} (19)

By choosing $\gamma = -c^2$ and using $z^2 = \tau + \gamma$ on the z-axis, it becomes

$$\Psi'(\tau) = \rho_0 \left(\frac{-\gamma}{\tau}\right)^{3/2}.$$  \hspace{1cm} (20)

The function $\Psi(\tau)$ is the primitive

$$\Psi(\tau) = \int_{-\gamma}^{\tau} \Psi'(\sigma) d\sigma = 2\rho_0 \gamma \left(\sqrt{\frac{-\gamma}{\tau}} - 1\right).$$  \hspace{1cm} (21)

The triaxial density can be calculated with Eqs. (12) and (13).

Using basis components of the form (18), we can fit the density on the z-axis $\Psi'(\tau)$ of a modified Hubble model. As the triaxial density is completely determined by the ellipsoidal coordinate system and the function $\Psi'(\tau)$, we can therefore produce a fit to a triaxial Hubble model. A wide range of triaxiality is allowed through the choice of the ellipsoidal coordinates. In Fig. 4 we present the spatial and projected mass densities of our best fitting model with $(-\alpha, -\beta, -\gamma) = (4, 2.6, 1)$. The model is fully triaxial, with a triaxiality parameter $T$ defined as $T = (A^2 - B^2)/(A^2 - C^2)$ (with $A$, $B$ and $C$ the long, intermediate and short axis lengths of the density) of $\sim 0.5$. The relative difference between the function $\Psi'(\tau)$ and the fit is smaller than 1% out to $z \sim 40c$.

5. Deprojection of a de Vaucouleurs photometry

Centaurus A (NGC 5128) is a giant elliptical galaxy with a conspicuous dust lane lying along its photometric minor axis. A photometric study of Cen A by Dufour et al. (1979) from photographic plates showed that the light distribution of Cen A follows the de Vaucouleurs law at radial distances from 2 arcmin to 8 arcmin. Furthermore, they
found that the $V$ surface brightness distribution at radii from 4 arcmin up to 8 arcmin is consistent with Cen A being an E2 galaxy. The isophotes are elliptical in shape and are quite round at the center but become more flattened with increasing radius. The ellipticity defined as $\epsilon = 1 - b/a$ (with $a$ and $b$ the apparent major and minor axis lengths) increases from $\epsilon = 0.07$ at $r = 2.6$ arcmin to $\epsilon = 0.26$ at $r = 9$ arcmin. Following Hui et al. (1993), we adopt a distance of 3.5 Mpc to Cen A, so that 1 arcmin corresponds to 1.02 kpc. According to Dufour et al. (1979), the effective radius of the de Vaucouleurs law is $r_e = 5.18$ kpc.

We build a model for the photometry of Cen A that follows the de Vaucouleurs law and reproduces the observed flattening of the light distribution out to 8 arcmin. Beyond this radius, we choose a constant value of the ellipticity $\epsilon = 0.26$ and a wide range of mass density profiles is allowed that can account for a possible increase of the mass-to-light ratio with radius. We adopt the observer’s viewing direction determined by Hui et al. (1995).

We use the QP method described in Sect. 3 to fit a Stäckel model to the de Vaucouleurs photometry of Cen A. Given ellipsoidal coordinates, we find that models ranging from quite oblate to fully triaxial can fit the data. In general, our QP models consist of less than $\sim 10$ components.

Adding new components does not improve the fit significantly. We present a model with a triaxiality parameter $T$ of $\sim 0.5$. Contours of the spatial density in the principal planes are shown in Fig. 5. The contours of the projected density (solid line) of the QP best fit are compared to the contours of the de Vaucouleurs photometry model (dashed line). The residual differences of 92% of the pixels inside the region limited by the isophote of major axis length of 35 arcmin are smaller than 0.1 magnitude. Figure 6 shows the profiles of the QP model and the de Vaucouleurs photometry model along the photometric major and minor axes and the apparent axis ratio as a function of radius for the two models. The components allow to reproduce the de Vaucouleurs law in the range $\sim 0.5 r_e$ to $\sim 6 r_e$. Taking into account the fact that the mass-to-light ratio is likely to vary a bit with radius in this galaxy, there is no particular need to attempt a better fit if a deprojected mass model and hence potential are desired. The fit yields a sufficient approximation of the galaxy’s potential. In the inner region ($r < 0.5 r_e$), the fit is less steep than the de Vaucouleurs law (see Fig. 6). Dynamical models produced with this potential may depend on three integrals, which will be only approximative in the very center.
6. Conclusions

A family of triaxial Stäckel potential-density pairs is presented. It includes as a special case the triaxial generalization of Hénon’s isochrone in ellipsoidal coordinates. This family allows the construction of triaxial mass (or light) models of galaxies with or without dark halos. A large variety of intrinsic shapes is provided by the choice of ellipsoidal coordinates and by the components. This diversity is also reflected in the projected densities which show elliptical, box-like or disc-like isophotes with ellipticities changing as a function of radius.

These potential-density pairs can be used as building blocks for realistic Stäckel models of triaxial potentials in elliptical galaxies. This is first tested with the elliptical galaxy Centaurus A (NGC 5128) whose kinematics exhibits unambiguous signatures of triaxiality. Using a Quadratic Programming method, we find a linear combination of density distributions that fits a model of the surface brightness of this E2 galaxy with a total density which is positive everywhere. The de Vaucouleurs photometry model of Cen A is well reproduced with these components. Mass-to-light ratio variations could be included in the projected distribution and the same method would then produce spatial mass models with dark matter. It suggests that Stäckel mass models may be relevant to the description of galactic potentials which represents a first step towards Stäckel triaxial dynamical models of stellar systems. Abel components propounded by Dejonghe & Laurent (1991) may prove powerful in this context; work along this line was first carried out by Dejonghe (1992), subsequently by Zeilinger et al. (1993) and further investigation is in progress.

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Appendices

A. The potential-density pair \( (\rho_1, V_1) \)

Let the potential \( V_1 \) be of the general form

\[
V_1(\lambda, \mu, \nu) = F_1(\lambda) + F_1(\mu) + F_1(\nu). \tag{A1}
\]

The Laplace operator \( \nabla^2 \) in ellipsoidal coordinates is (see e.g. de Zeeuw 1985b, Eq. (20))

\[
\nabla^2 = \nabla^2_\lambda + \nabla^2_\mu + \nabla^2_\nu \tag{A2}
\]

with

\[
\nabla^2_\lambda = \frac{2}{(\lambda - \mu)(\lambda - \nu)} \left( 2a(\lambda) \frac{\partial^2}{\partial \lambda^2} + a'(\lambda) \frac{\partial}{\partial \lambda} \right) \quad \text{(cyc)}, \tag{A3}
\]

where

\[
a(\tau) = (\tau + \alpha)(\tau + \beta)(\tau + \gamma). \tag{A4}
\]

The density \( \rho_1 \) that follows from Poisson’s equation can be written as (Eq. (3.9) in ZP88)

\[
\rho_1(\lambda, \mu, \nu) = g_\lambda \Psi_1(\lambda) + \text{(cyc)}. \tag{A5}
\]

The relation between \( \Psi_1(\tau) \) and \( F_1(\tau) \) is written explicitly in ZP88 (Eq. (3.7)), as well as the relations that connect the Stäckel pair \( (\rho, V) \) and the pair \( (\rho_1, V_1) \). In particular, we have

\[
\rho = (\mathcal{A} + \mathcal{I})\rho_1, \tag{A6}
\]

\[
F(\tau) = (\tau + \gamma)F_1(\tau), \tag{A7}
\]

\[
\Psi(\tau) = (\tau + \gamma)\Psi_1(\tau) \tag{A8}
\]

with

\[
\mathcal{A} = (\lambda + \gamma)g_\lambda \frac{\partial}{\partial \lambda} + (\mu + \gamma)g_\mu \frac{\partial}{\partial \mu} + (\nu + \gamma)g_\nu \frac{\partial}{\partial \nu} \tag{A9}
\]

and \( \mathcal{I} \) is the identity operator.

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Fig. 6. Top: profiles of the projected density of our QP best fit model to Cen A’s photometry along the photometric major (bottom solid line) and minor (top solid line) axes. The dashed lines are the corresponding profiles of the de Vaucouleurs photometry model. Bottom: apparent axis ratio as a function of radius for the QP model (solid line) and for the de Vaucouleurs photometry model (dashed line).
B. The triaxial density as a function of $\Psi(\tau)$

By noting that $H[\tau_1, \tau_2, \tau_3] = \frac{d}{d\tau_3}H[\tau_1, \tau_2, \tau_3]$, we have

$$H[\lambda, \mu, \nu, \lambda, \mu, \nu] = \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu} H[\lambda, \mu, \nu].$$

(B1)

Let $H(\tau)$ be the function

$$H(\tau) = (\tau + \alpha)^2(\tau + \beta)^2\Psi(\tau).$$

(B2)

Then Eq. (B1) can be written

$$H[\lambda, \mu, \nu, \lambda, \mu, \nu] =$$

$$\frac{\partial}{\partial \lambda} \left[ (\lambda + \alpha)(\lambda + \beta)\Psi(\lambda)\frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu} g\lambda \right] + (cyc) =$$

$$\frac{\partial}{\partial \lambda} \left[ (\lambda + \alpha)(\lambda + \beta)\Psi(\lambda)\frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu} g\lambda \right] + (cyc),$$

(B4)

or, with Eq. (B2),

$$H[\lambda, \mu, \nu, \lambda, \mu, \nu] = g^2\Psi(\lambda) + 2g\lambda\Psi(\lambda)\frac{\partial}{\partial \lambda} g\lambda + (cyc).$$

(B5)

Using Eq. (B1), we have

$$\frac{\partial}{\partial \lambda} g\lambda = \frac{g\mu}{(\lambda - \mu)} + \frac{g\nu}{(\lambda - \nu)} \hspace{1cm} (cyc).$$

(B6)

Substitution of Eq. (B1) in Eq. (B5) and comparison with Eq. (B3) prove that the density $\rho$ can be expressed as the 5th-order divided difference

$$\rho(\lambda, \mu, \nu) = H[\lambda, \mu, \nu, \lambda, \mu, \nu]$$

(B7)

with $H(\tau)$ given in Eq. (B2). By specifying a function $\Psi(\tau)$ or the density profile on the z-axis $\Psi(\tau)$, one can easily calculate the density with Eq. (B7). One can notice that any 4th-order polynomial added to the function $H(\tau)$ yields the same 5th-order divided difference as $H(\tau)$.

C. The triaxial density as a function of $F(\tau)$

One can also calculate the density from the potential by specifying the function $F(\tau)$ instead of $\Psi(\tau)$. We show that the triaxial density can be written as a 5th-order divided difference of a function $R(\tau)$ that depends only on $F(\tau)$.

By direct application of Poisson’s equation, using Eqs. (A3), (A3) and (B1), we can write the density $\rho_1$ as

$$\rho_1(\lambda, \mu, \nu) = R_1[\lambda, \mu, \nu]$$

(C1)

with

$$\pi GR_1(\tau) = a(\tau)F_1(\tau) + \frac{1}{2}\frac{d}{d\tau}[\sqrt{a(\tau)}F_1(\tau)]$$

(C2)

and

$$\pi GR(\tau) = |a(\tau)|^2 \frac{d}{d\tau}[\sqrt{|a(\tau)|}F(\tau)].$$

(C3)

Now we prove that the density $\rho$ can be written as the 5th-order divided difference

$$\rho(\lambda, \mu, \nu) = R[\lambda, \mu, \nu, \lambda, \mu, \nu]$$

(C4)

where

$$R(\tau) = a(\tau)R_1(\tau).$$

(C5)

Using Eq. (B1), we can write

$$R[\lambda, \mu, \nu, \lambda, \mu, \nu] =$$

$$\frac{\partial}{\partial \lambda} \left[ \frac{a(\lambda)R_1(\lambda)}{(\lambda - \mu)(\lambda - \nu)} + (cyc) \right]$$

$$= \frac{\partial}{\partial \lambda} \left[ \frac{a(\lambda)R_1(\lambda)}{(\lambda - \mu)^2(\lambda - \nu)^2} \right] + (cyc).$$

(C6)

Since

$$\frac{a(\lambda)}{(\lambda - \mu)(\lambda - \nu)} = (\lambda + \gamma)g\lambda \hspace{1cm} (cyc),$$

we obtain

$$R[\lambda, \mu, \nu, \lambda, \mu, \nu] =$$

$$a(\lambda)g\lambda \frac{R_1(\lambda)}{(\lambda - \mu)(\lambda - \nu)} + (cyc).$$

(C7)

(C8)

With Eqs. (B1) and (B6), one can establish the identity

$$\frac{\partial}{\partial \lambda} [(\lambda + \gamma)g\lambda] = 1 + \frac{\mu + \gamma}{\lambda - \mu}g\mu + \frac{\nu + \gamma}{\lambda - \nu}g\nu \hspace{1cm} (cyc),$$

(C9)

which, upon substitution of Eq. (C9) in Eq. (C8), yields

$$R[\lambda, \mu, \nu, \lambda, \mu, \nu] =$$

$$(\lambda - \mu)^2(\lambda - \nu)^2 \frac{\partial}{\partial \lambda} [(\lambda + \gamma)g\lambda] + (cyc).$$

(C10)

identical to Eq. (A4), q.e.d.

Thus, with Eq. (A4), the density $\rho$ can be written as the 5th-order divided difference (Eq. (C4)) with

$$\pi GR(\tau) = a(\tau)^2 \frac{d}{d\tau}[\sqrt{a(\tau)}F(\tau)].$$

(C11)

and $a(\tau)$ is given in Eq. (A4). The function $R(\tau)$ differs from the function $H(\tau)$ only by a polynomial function of the 4th-order in $\tau$.

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