DRILLING CORES OF HYPERBOLIC 3-MANIFOLDS TO PROVE TAMENESS

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Abstract. We sketch a proof of the fact that a hyperbolic 3-manifold M with finitely generated fundamental group and with no parabolics are topologically tame. This proves the Marden’s conjecture. Our approach is to form an exhaustion \( M_i \) of \( M \) and modify the boundary to make them 2-convex. We use the induced path-metric, which makes the submanifold \( M_i \) \( \delta \)-hyperbolic and with Margulis constants independent of \( i \). By taking the convex hull in the cover of \( M_i \) corresponding the core, we show that there exists an exiting sequence of surfaces \( \Sigma_i \). We drill out the covers of \( M_i \) by a core \( C \) again to make it \( \delta \)-hyperbolic. Then the boundary of the convex hull of \( \Sigma_i \) is shown to meet the core. By the compactness argument of Souto, we show that infinitely many of \( \Sigma_i \) are homotopic in \( M - C \).

Contents

Part 1. 2-convex hulls of submanifolds of hyperbolic manifolds

1. Introduction to Part 1
2. Preliminary
2.1. Hyperbolic manifolds
2.2. s-vertices
3. 2-convex hulls of hyperbolic manifolds
4. Crescents and isotopy
5. Convex perturbations
6. Isotopy sequences

Part 2. General hyperbolic 3-manifolds and convex hulls of their cores

7. Introduction to Part 2
8. Hyperbolic metric spaces
8.1. Metric spaces, geodesics spaces, and cat(-1)-spaces
8.2. Singular hyperbolic surfaces
8.3. General hyperbolic 3-manifolds
9. 2-convex general hyperbolic manifolds and h-maps of surfaces
9.1. 2-convexity and general hyperbolic manifolds
9.2. h-maps of surfaces into 2-convex general hyperbolic manifolds

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Recently, Agol initiated a really interesting approach to proving Marden’s conjecture by drilling out closed geodesics. We started on the same general direction nearly the same time as Agol. In this paper, we use truncation and drill out compact cores. We do use the Agol’s idea of using covering spaces and taking convex hulls of the cores, a method originating from Freedman. Our approach is somewhat different in that we do not use end-reductions and pinched Riemannian hyperbolic metrics; however, we use the incomplete hyperbolic metric itself. The hard geometric analysis and geometric convergence techniques can be avoid using the techniques of this paper. Except for developing a rather elementary theory of deforming boundary to make the submanifolds of codimension 0 Gromov hyperbolic, we do not need any other highly developed techniques. Note also that there is a recent paper by Calegari and Gabai \[5\] using modified least area surfaces and closed geodesics. The work here is independently developed from their line of ideas.

In this paper, we let \(M\) be a hyperbolic 3-manifold with a core homeomorphic to a compression body. Suppose that \(M\) has a finitely generated fundamental group and the holonomy is purely loxodromic and has ends \(E, E_1, \ldots, E_n\). Let \(F_1, \ldots, F_n\) be the incompressible surfaces in neighborhoods of the ends \(E_1, \ldots, E_n\). Let \(N(E)\) be a neighborhood of an end \(E\) with no incompressible surface associated.

The Marden’s conjecture states that a hyperbolic 3-manifold with a finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold. It will be sufficient to prove for the above \(M\) to prove Marden’s conjecture.

The cases when the group contains parabolic elements are left out. But we believe that the arguments in this paper essentially go through.

**Theorem A.** Let \(M\) be as above with ends \(E, E_1, \ldots, E_n\), and \(C\) be a compact core of \(M\). Then \(E\) has an exiting sequence of surfaces of genus equal to that of the boundary component of \(\partial C\) corresponding to \(E\).

The following implies the proof of Marden’s conjecture since all finitely generated hyperbolic manifolds without parabolic elements has ends isometric to the manifolds \(M\) as described here.

**Theorem B.** \(M\) is tame; that is, \(M\) is homeomorphic to the interior of a compact manifold.

This paper has three parts: In Part 1, let \(M\) be a codimension 0 submanifold of a hyperbolic 3-manifold \(N\) of infinite volume with certain nice boundary conditions.
$M$ is locally finitely triangulated. Suppose that $M$ is 2-convex in $N$ in the sense that any tetrahedron $T$ in $N$ with three of its side in $M$ must be inside $M$. Now let $L$ be another compact codimension 0-submanifold $M$ so that $\partial L$ is incompressible in $M$ with a number of closed geodesics $c_1, \ldots, c_n$ removed. We suppose that $L$ is finitely triangulated. Given $\epsilon > 0$, we show that $\partial L$ can be isotoped to a hyperbolically triangulated surface so that it bounds in $M$ 2-convex submanifold whose $\epsilon$-neighborhood contains $c_1, \ldots, c_n$. The isotopy techniques will be PL-type arguments and deforming by crescents. An important point to be used in the proof is that the crescents avoid closed geodesics and geodesic laminations. Thus, the isotopy does not pass through the closed geodesics.

Part 2 is as follows: A general hyperbolic manifold is a manifold with boundary modeled on subdomains in the hyperbolic space. A general hyperbolic manifold is 2-convex if every isometry from a tetrahedron with an interior of its side removed extends to the tetrahedron itself. We show that a 2-convex general hyperbolic manifold is Gromov hyperbolic. The proof is based on the analysis of the geometry of the vertices of the boundary required by the 2-convexity. We will also define $h$-surfaces as a triangulated surface where each triangle gets mapped to geodesic triangles and the sum of the induced angles at each vertex is always greater than or equal to $\pi$. We show the area bound of such surfaces. Finally, we show that the boundary of the convex hull of a core in a general hyperbolic manifold with finitely generated fundamental group can be deformed to a nearby $h$-surface. This follows from local analysis of geometry.

In Part 3, we will give the proof of Theorems A and B using the results of Part 1 and 2. The outline is given in the abstract and in Section 11. The proof itself is rather short spanning 9-10 pages only.

Part 1. 2-convex hulls of submanifolds of hyperbolic manifolds

1. Introduction to Part 1

The purpose of this paper is to deform a submanifold of such a manifold. Suppose $M$ is a submanifold of a complete hyperbolic manifold $N$ with finitely generated fundamental group. We suppose that $N$ has incompressible ends except for one end and is homotopy equivalent to a compression body. Suppose that $M$ contains a number of closed geodesics, and that $\partial M$ is incompressible in $N$ with these geodesics removed.

We modify $M$ so that $M$ becomes Gromov hyperbolic and its $\epsilon$-neighborhood still contains the closed geodesics.

We will be working in a more general setting. A general hyperbolic manifold is a Riemannian manifold $M$ with corner and a geodesic metric that admits a geodesic triangulation so that each 3-simplex is isometric with a hyperbolic one. $M$ admits a local isometry, so-called developing map, $\text{dev} : \tilde{M} \rightarrow \mathbb{H}^3$ equivariant with respect to a homomorphism $h : \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{C})$. The pair $(\text{dev}, h)$ is only determined up to action

$$(\text{dev}, h) \mapsto (g \circ \text{dev}, g \circ h(\dot{g}) \circ g^{-1}) \text{ by } g \in \text{PSL}(2, \mathbb{C}).$$
We remark that in Thurston’s notes [15] a locally convex general hyperbolic manifold is shown to be covered by a domain in $\mathbb{H}^3$ or, equivalently, it can be extended to a complete hyperbolic manifold.

A general hyperbolic manifold $M$ is 2-convex if given a hyperbolic 3-simplex $T$, a local-isometry $f : T - F^o \to M^o$ for a face $F$ of $T$ extends to an isometry $T \to M^o$. (See [8] for more details. Actually projective version applies here by the Klein model of the hyperbolic 3-space.)

By a totally geodesic hypersurface, we mean the union of components of the inverse image under a developing map of a totally geodesic plane in $\mathbb{H}^3$. A local totally geodesic hypersurface is an open neighborhood of a point in the hypersurface.

For a point, a local half-space is the closure in the ball of the component of an open ball around it with a totally geodesic hypersurface passing through it. The local totally geodesic hypersurface intersected with the local-half space is said to be the side of the local half-space. A local half-space with its side removed is said to be an open local half-space.

A surface $f : S \to M$ is said to be triangulated if $S$ is triangulated and each triangle is mapped to a totally geodesic triangle in $M$. (We will generalize this notion a bit.) An interior vertex of $f$ is an $s$-vertex if every open local half-space associated with the vertex does not contain the local image of $f$ with the vertex removed. A strict $s$-vertex is an $s$-vertex where every associated closed local half-space does not contain the local image. An interior vertex of $f$ is a convex vertex if a local open half-space associated with the vertex contains the local image of $f$ removed with the vertex. (Actually, we can see these definitions better by looking at the unit tangent bundle at the vertex: Let $U$ be the unit tangent bundle at the vertex. Then the image of $f$ corresponds to a path in $U$.) An interior vertex of $f$ is an $h$-vertex if the sum of angles of the triangles around a vertex is $\geq 2\pi$. An $s$-vertex is an $h$-vertex by Lemma [9.4]

$f : S \to M$ is an $s$-map if each vertex is an $s$-vertex, and $f$ is an $h$-map if each vertex is an $h$-one. An $s$-map is an $h$-map but not conversely in general. For an imbedding $f$ and an orientation, a convex vertex is said to be a concave vertex if the local half-space is in the exterior direction. Otherwise, the convex vertex is a convex vertex. We also know that if the boundary of a general hyperbolic submanifold $N$ of $M$ is $s$-imbedded, then $N$ is 2-convex (see Proposition prop:s-bd2-conv).

**Theorem C.** Let $N$ be an orientable 2-convex general hyperbolic 3-manifold. Let $M$ be a compact codimension-zero submanifold of $N$ with boundary $\partial M$. Suppose that each component of $\partial M$ is incompressible in $M$ if we remove a finite number of the image $c_1, \ldots, c_n$ of the closed geodesics in the interior $M^o$ of $M$. Then for arbitrarily given small $\epsilon > 0$, we can isotopy $M$ to a homeomorphic general hyperbolic 3-manifold $M'$ in $N$ so that $M'$ is 2-convex and an $\epsilon$-neighborhood of $M'$ contains the collection of closed geodesics.

We say that $M'$ obtained from $M$ by the above process is a 2-convex hull of $M$. Although $M'$ is not necessarily a subset of $M$, the curves $c_1, \ldots, c_n$ is a subset of an $\epsilon$-neighborhood of $M'$ and hence we have certain amount of control. (Here the geodesics are allowed to self-intersect.)
In section one we review some hyperbolic manifold theory. We also introduce s-vertices and relationship with 2-convexity.

In section 2, we introduce crescents and discuss their properties. In section 3, we will study the first step of such an isotopy called the crescent move. In section 4, we discuss how to modify the infinite pleating lines that can result from the crescent move to a triangulation. In section 5, we will combine the techniques of sections 2 and 3 to produce our move.

In section 2, we introduce so-called crescents. We take the inverse image $\tilde{\Sigma}$ of the surface $\Sigma$. We assume that $\Sigma$ is incompressible in the ambient 2-convex general hyperbolic manifold with a number of geodesics $c_1, \ldots, c_n$ removed. A crescent is a connected domain bounded by a totally geodesic hypersurface and an open surface in $\tilde{\Sigma}$. The portion of boundary in the totally geodesic hypersurface is said to be the $I$-part and the portion in $\tilde{\Sigma}$ is said to be the $\alpha$-part. A crescent may contain another crescents and so on. The folding number of a crescent is the maximum intersection number of the generic path from the outer part in the surface to the innermost component of the crescent with the surface removed. We show that for given $\Sigma$, the folding number is bounded above.

A highest-level crescent is an innermost one that is contained in a crescent with highest folding number. We show that a highest-level crescent is always contained in an innermost crescent; i.e., so called the secondary highest-level crescent. In a secondary highest level crescent, the closure of the $\alpha$-part and the $I$-part are isotopic. We also show that the secondary highest-level crescents meet nicely extending their $\alpha$-parts in $\tilde{\Sigma}$, following [8].

In section 3, we introduce the crescent-isotopy theory. This is a theory to isotopy a surface in a general hyperbolic manifold so that all of its vertices become s-vertices.

We form the union of secondary highest-level crescents and can isotopy the union of their $\alpha$-parts to the complement $I$ in the boundary of their union. This is essentially the crescent move.

However, there might be some parts of $\tilde{\Sigma}$ meeting $I$ tangentially from above. We need to first push these parts upward first using truncations.

Also, after the move, there might be pleated parts which are not triangulated. We have to perturb these parts to triangulated parts. This forms section 4.

In section 5, we gather our results to produce the move to obtain s-imbedded surface isotopic to $\Sigma$. Our steps are as follows: We take the highest folding number and take all outer secondary highest-level crescents, do some truncations, and do the crescent isotopy. Then we perturb. Next, we take all inner highest-level crescents, do some truncations, and do crescent move and perturb. Now the highest folding number decreases by one, and we do the next step of the induction until we have no crescents any more. In this case, all the vertices are s-vertices.

We apply our result to a codimension-zero submanifold $M$ which contains a number of closed geodesics $c_1, \ldots, c_n$. We assume that the boundary incompressible in the ambient manifold with the geodesics removed. This will prove Theorem A.
2. Preliminary

In this section, we review the hyperbolic space and the Kleinian groups briefly. We discuss the relationship between the 2-convexity of general hyperbolic manifolds and the s-vertices of the boundary components.

2.1. Hyperbolic manifolds. The hyperbolic $n$-space is a complete Riemannian metric space $(\mathbb{H}^n, d)$ of constant curvature equal to $-1$. We will be concerned about hyperbolic plane and hyperbolic spaces, i.e., $n = 2, 3$, in this paper.

The upper half space model for $\mathbb{H}^2$ is the pair 
$$(U^2, \text{PSL}(2, \mathbb{R}) \cup \text{PSL}(2, \mathbb{R}))$$
where $U^2$ is the upper half space.

The Klein model of $\mathbb{H}^2$ is the pair $(B^2, \text{PO}(1, 2))$ where $B^2$ is the unit disk and $\text{PO}(1, 2)$ is the group of projective transformations acting on $B^2$.

A Fuchsian group is a discrete subgroup of the group of isometries of $\text{PO}(1, 2)$ of the group $\text{Isom}(\mathbb{H}^2)$ of isometries of $\mathbb{H}^2$.

There are many models of the hyperbolic 3-space. We shall use the upper-half space model or Klein model, whichever is more convenient at the time.

The Klein model consists of the unit ball in $\mathbb{R}^3$ and the group of isometries are identified with the group of projective transformations preserving the unit ball, which is identified as $\text{PO}(1, 3)$.

A Kleinian group is a discrete subgroup of the group of isometries $\text{PO}(1, 3)$ of the group $\text{Isom}(\mathbb{H}^3)$ of isometries of $\mathbb{H}^3$.

For the purposes of this paper, it is more convenient to use the Klein model.

A parabolic element $\gamma$ of a Kleinian group is a nonidentity element such that $(\gamma(x), x)$, $x \in \mathbb{H}^3$, has no lower bound other than 0. A loxodromic element of a Kleinian group is an isometry with a unique invariant axis. A hyperbolic element is a loxodromic one with invariant hyperplanes.

For Fuchsian groups, a similar terminology holds.

In this paper, we will restrict our Kleinian groups to be torsion-free and have no parabolic elements and all elements are orientation-preserving.

2.2. s-vertices. We now classify the vertices of a triangulated map $f : S \to M$. We do not yet require the general position property of $f$ but identify the vertex with its image.

By a straight geodesic in a general hyperbolic manifold we mean a geodesic that maps to geodesics in $\mathbb{H}^3$ under the developing maps.

**Lemma 2.1.** Let $f : S \to M$ be a triangulated map.

- An interior vertex of $S$ is either a convex-vertex, a concave vertex, or an s-vertex.
- An s-vertex which is not strict one is contained in a unique local half-space with side containing the vertex and has to be one of the following:
A vertex with a totally geodesic local image.

- A vertex on a edge of two totally geodesic planes where \( f \) locally maps into one sides of each.

- A vertex where a pair of triangles or edges has geodesic segments extending each other.

- If \( f \) is a general position map, then an s-vertex is a strict s-vertex.

**Proof.** Straightforward.

**Lemma 2.2.** A s-vertex can be deformed to a strict s-vertex by an arbitrarily small amount by pushing if necessary the vertex from the boundary of the closed local half-space containing the local image of \( f \) in the direction of the open half-space.

**Proof.** If the s-vertex is a strict one, then we leave it alone. If the s-vertex is not a strict one, a closed local half-space contains the local image of \( f \). Let \( U \) be the unit tangent bundle at the vertex. A closed hemisphere \( H \) contains the path corresponding to the local image of \( f \). Let \( l \) be the spherical geodesic on \( U \) connecting \( v \) and \(-v\) so that \( l \) separates a nonantipodal pair of points \( x \) and \( y \) on the path of \( f \). Let \( \overline{xy} \) be the minor geodesic connecting \( x \) and \( y \). Then \( \overline{xy} \) meets \( l \). If we push our vertex in the open hemisphere direction, then \( l \) becomes a geodesic segment of length \( > \pi \) and it meets \( \overline{x'y'} \) for \( x' \) and \( y' \) corresponding to \( x \) and \( y \) respectively. The conclusion follows from Lemma 2.3.

**Lemma 2.3.** Suppose that \( f : \Sigma \to M \) is a triangulated map. Let \( v \) be a vertex of \( f \) and \( f' : U_v \to U_v^M \) be the induced map from the link of \( v \) in \( \Sigma \) to that of \( v \) in \( M \). Suppose that there exists a segment \( l \) of length \( > \pi \) in \( U_v^M \) with endpoints in the image of \( f' \) separating two points in the image of \( f' \) so that the minor arc \( \overline{xy} \) meets \( l \) transversely. Then \( v \) is an s-vertex.

**Proof.** Let \( z, t \) be the endpoints of \( l \) and \( x, y \) the separated points. As above, let \( H_1 \), \( H_2 \), and \( H_3 \) be hemispheres with boundary geodesics containing \( \overline{xy}, \overline{yz}, \) and \( \overline{zx} \). Then it is clearly that any hemisphere containing \( x, y, z \) is a subset of the union \( H_1 \cup H_2 \cup H_3 \). Therefore, the image of \( f' \) is not contained in any hemisphere and \( v \) is an s-vertex.

Given an oriented surface, a convex vertex is either a **convex vertex** or a **concave vertex** depending on whether the supporting local half-space is in the outer normal direction or in the inner normal direction.

**Proposition 2.4.** A vertex of an oriented imbedded triangulated surface is either an s-vertex or a convex vertex or a concave vertex.

**Proof.** Straightforward.

In this paper, we will consider only metrically complete submanifolds, i.e., locally compact ones.

**Proposition 2.5.** A general hyperbolic manifold \( M \) is 2-convex if and only if each vertex of \( \partial M \) is a convex vertex or an s-vertex.
Proof. Suppose $M$ is 2-convex. If a vertex $x$ of $\partial M$ is a concave, we can find a local half-open space in $M$ with its side passing through $x$. The side meets $\partial M$ only at $x$. From this, we can find a 3-simplex inside with a face in the side. This contradicts 2-convexity of $M$.

Conversely, suppose that $\partial M$ has only convex vertices or $s$-vertices. Let $f : T - F^o \to M^o$ be a local-isometry from a 3-simplex $T$ and a face $F$ of $T$. We may lift this map to $\tilde{f} : T - F^o \to \tilde{M}^o$ where $\tilde{M}$ is the universal cover of $M$. Then $\tilde{f}$ is an imbedding. Since $\tilde{M}$ is metrically complete, $\tilde{f}$ extends to $\tilde{f}' : T \to \tilde{M}$.

Suppose that $f$ does not extend to $f' : T \to M^o$. This implies that $\tilde{f}'(F)$ meets $\partial \tilde{M}$. $\tilde{f}'(\partial F)$ does not meet $\partial \tilde{M}$. The subset $K = \partial \tilde{M} \cap \tilde{f}'(F)$ has a vertex $x$ of $\partial \tilde{M}$ which is an extreme point of the convex hull of $K$ in the image of $F$. There exists a supporting line $l$ at $x$ for $K$. We can tilt $\tilde{f}'(T)$ by $l$ a bit and the new 3-simplex meets $\partial \tilde{M}$ at $x$ only. This implies that $x$ is not an $s$-vertex but a concave vertex, a contradiction. □

3. 2-CONVEX HULLS OF HYPERBOLIC MANIFOLDS

Let $M$ be a metrically complete 2-convex general hyperbolic manifold from now on and $\tilde{M}$ its universal cover. Let $\Gamma$ denote the deck transformation group of $\tilde{M} \to M$.

Let $\Sigma$ be a properly imbedded compact subsurface of an orientable general hyperbolic manifold $M$. $\Sigma$ may have more than one components. We denote by $\tilde{\Sigma}$ the inverse image of $\Sigma$ in the universal cover $\tilde{M}$ of $M$. ($\tilde{\Sigma}$ is not connected in general and components may not be universal covers of $\Sigma$.) We assume that the triangulated $\tilde{M}$ is in general position and so is $\tilde{\Sigma}$ under the developing maps.

For each component $\Sigma_0$ of $\Sigma$ and a component $\Sigma'_0$ of $\tilde{\Sigma}$ mapping to $\Sigma_0$, there exists a subgroup $\Gamma_{\Sigma'_0}$ acting on $\Sigma'_0$ so that the quotient space is isometric to $\Sigma_0$.

Hypothesis 3.1. We will now assume that $\Sigma$ is incompressible in $M$ with a number of straight closed geodesics $c_1, \ldots, c_n$ removed.

First, we introduce crescents for $\tilde{\Sigma}$ which is the inverse image of a surface $\Sigma$ in a 2-convex general hyperbolic manifold. We define the folding number of crescents and show that they are bounded above.

We define the highest level crescents, i.e., the innermost crescents in the crescent with the highest folding number incurring the highest folding number. We show that closed geodesics avoid the interior of crescents. Given a highest level crescent, we show that there is an innermost crescent that has a connected $I$-part to which the closure of the $\alpha$-part is isotopic in the crescent by the incompressibility of $\Sigma$. This is the secondary highest-level crescents. We show that the secondary highest-level crescent is homeomorphic to its $I$-part times the unit interval.

Next, we show that if two highest-level crescents meet each other in their $I$-parts tangentially, then they both are included in a bigger secondary highest-level crescent. Furthermore, if two secondary highest-level crescents meet in their interiors, then they meet nicely extending their $\alpha$-parts. This is the so-called transversal intersection of two crescents.

Definition 3.2. A crescent $R$ for $\tilde{\Sigma}$ is
• a connected domain in $\tilde{M}$ which is a closure of a connected open domain in $\tilde{M}$,
• so that its boundary is a disjoint union of a (connected) open domain in $\tilde{\Sigma}$ and
the closed subset that is the disjoint union of totally geodesic 2-dimensional
domains in $\tilde{M}$ that develops into a common totally geodesic hypersurface in $\mathbb{H}^3$
under $\text{dev}$.

We denote by $\alpha_R$ the domain in $\tilde{\Sigma}$ and $I_R$ the union of totally geodesic domain. To
make the definition canonical, we require $I_R$ to be the maximal totally geodesic set in
the boundary of $\mathcal{R}$. We say that $I_R$ and $\alpha_R$ the $I$-part and the $\alpha$-part of $\mathcal{R}$.

As usual $\Sigma$ is oriented so that there are outer and inner directions to normal vectors.
The subset $\Sigma \cap \mathcal{R}$ may have more than one components. For each component of $\mathcal{R} - \tilde{\Sigma}$, we can assign a folding number which is the minimal generic intersection number of
that a path from $\alpha_R$ meeting $\tilde{\Sigma}$ to reach to the component. The maximum value of
the folding number of the components is the folding number of the crescent $\mathcal{R}$.

The $I$-part hypersurface is the maximal totally geodesic hypersurface in containing
the $I$-part of the crescent.

A noncompact domain will be called a crescent if it is bounded by a (connected)
domain in $\tilde{\Sigma}$ and the union of totally geodesic domains in developing into a common
totally geodesic plane in $\mathbb{H}^3$ and is a geometric limit of compact crescents. Again the
$I$-part is the maximal totally geodesic subset of the boundary of the crescent. The $\alpha$-part is the complement in the boundary of the crescent.

A priori, a crescent may have an infinite folding number. However, we will soon
show that the folding number is finite.

A crescent is an outer one if the adjacent part to the $\alpha$-part is in the outer normal
direction to $\Sigma$. It is an inner one otherwise.

The boundary $\partial I_R$ of $I_R$ is the set of boundary points in the totally geodesic hyper-
surface containing $I_R$.

A pinched simple closed curve is a simple curve pinched at most three points or
pinched at a connected arc. The boundary of the $I$-part is a disjoint union of pinched
simple curves.

The following is a really important property since this shows we can use crescents
in general hyperbolic manifolds without worrying about whether the $I$-parts meet the
boundary of the ambient manifold.

**Proposition 3.3.** Let $\mathcal{R}$ be a crescent in a 2-convex ambient general hyperbolic man-
ifold $M$. Then $\mathcal{R}$ is disjoint from $\partial \tilde{M}$. In fact $\mathcal{R}$ is uniformly bounded away from $\partial \tilde{M}$.

*Proof.* Suppose that $\mathcal{R}$ meets $\partial \tilde{M}$. Since the closure of $\alpha_R$ being a subset of $\tilde{\Sigma}$ is
disjoint from $\partial \tilde{M}$, it follows that $I_R$ meets $\partial \tilde{M}$ in its interior points and away from the
boundary points in the ambient totally geodesic subsurface $P$ in $M$.

We find the extreme point of $I_R \cap \partial \tilde{M}$ and find the supporting line. This point has
a local half-space in $\mathcal{R}$. By tilting the $I$-part a bit by the supporting line, we find a
local half-space in $M$ and in it a local totally geodesic hypersurface meeting $\partial M$ at a
point. This contradicts 2-convexity. $\square$

The following shows the closedness of set of points of $\tilde{M} - \tilde{\Sigma}$ in crescents.
Proposition 3.4. Let $\mathcal{R}_i$ be a sequence of crescents. Suppose that $x$ is a point of $\tilde{M} - \tilde{\Sigma}$ which is a limit of a sequence of points in the union of $\mathcal{R}_i$. Then $x$ is contained in a crescent.

Proof. Let $x_i \in \bigcup_j \mathcal{R}_j$ be a sequence converging to $x$. We may assume that $x$ is not an element of any $\mathcal{R}_j$.

Using geometric convergence, there exists a totally geodesic hypersurface $P$ through $x$ and a geometric limit of a sequence of $I_{\mathcal{R}_j}$ converging to a subset $D$ in $\tilde{M}$.

Then $D$ is separating in $\tilde{M}$. If not, there exists a simple closed curve $\gamma$ in $\tilde{M}$ meeting $D$ only once. This means $I_{\mathcal{R}_j}$ for a sufficiently large $j$ meets $\gamma$ only once as well.

Now, $\tilde{M} - D$ may have more than one components. Since $x \in D$, we take a component whose closure $L$ contains $x$. $L \cap D$ is also separating. □

We may assume that the holonomy group of $\Sigma$ does not consist of parabolic or elliptic or identity elements only.

A size of a crescent is the supremum of the distances $d(x, \alpha_{\mathcal{R}})$ for $x \in I_{\mathcal{R}}$. We show that this is globally bounded by a constant depending only on $\Sigma$.

Proposition 3.5. Let $M$ and $\Sigma$ be as above. Then there is an upper bound to the size of a crescent.

Proof. If not, using deck transformations acting on $\tilde{\Sigma}$, we obtain a sequence of bigger and bigger compact crescents where the corresponding sequence of the $I$-parts leave any compact subset of $\tilde{M}$ and the corresponding sequence of $\alpha$-parts meets a fixed compact subset of $\tilde{M}$. Therefore, we form a subsequence of the developing images of the $I$-parts converging to a point of the sphere at infinity of $\mathbb{H}^3$.

Let $\mathcal{R}_i$ be the corresponding crescents. Then $\alpha_{\mathcal{R}_i}$ is a subsurface with boundary in the $I$-parts, and $\alpha_{\mathcal{R}_i}$ is getting larger and larger.

Let $c$ be a closed curve in $\Sigma$ with nonidentity holonomy. Let $\tilde{c}$ be a component of its inverse image in $\tilde{\Sigma}$. Since $\tilde{c}$ must escape any compact subset of $\tilde{\Sigma}$, $\tilde{c}$ escape $\alpha_{\mathcal{R}_i}$. Thus, $\tilde{c}$ must meet all $I_{\mathcal{R}_i}$ for $i$ sufficiently large. Since the developing image of $\tilde{c}$ has two well-defined endpoints, this means that the limit of the sequence of $I$-parts must contain at least two points, a contradiction. □
Given \( \Sigma \), there is an upper bound to the folding-number of all crescents associated with \( \Sigma \): This follows since \( \tilde{\Sigma} \) is locally finite, and the increasing sequence of folding numbers implies that the sequence consists of crescents getting bigger and bigger.

We call the maximum the **highest folding number** of \( \Sigma \). We perturb \( \Sigma \) to minimize the highest folding number which can change only by \( \pm 1 \) under perturbations. After this, the folding number is constant under small perturbations. If there are no crescents, then the **folding number** of \( \Sigma \) is defined to be \(-1\).

Also, the union of all crescents for \( \tilde{\Sigma} \) is in a uniformly bounded neighborhood of \( \tilde{\Sigma} \) with the bound depending only on \( \Sigma \).

We say that a 0-folded crescent \( R \) is a **highest-level** crescent if it is an innermost crescent of an \( n \)-folded crescent \( R' \) where \( n \) is the highest-folding number of \( \tilde{\Sigma} \) and \( R \) is one that achieves the highest-level.

Suppose that \( R \) is a compact highest-level crescent. Let \( A_1, \ldots, A_n \) be components of \( I^o_R \). Recall that \( I^o_R \) lies in a totally geodesic hypersurface. The outermost pinched simple closed curve \( \alpha_i \) in the boundary of \( A_i \) has a trivial holonomy. Since \( R \) is of highest-level, \( \alpha_i \) is an innermost curve itself or bounds innermost curves in \( \tilde{\Sigma} \cap \partial I^o_R \). If each \( \alpha_i \) is as in the former case, then \( R \) is said to be **innermost type crescent**, which is homeomorphic to a 3-ball.

We can classify the points of \( \tilde{\Sigma} \cap I^o_R \): A point of it is an **outer-skin point** if it is a nonboundary point and has a neighborhood in \( \tilde{\Sigma} \) outside \( R^o \cup \alpha_R \); a point is an **inner-skin point** if it is a nonboundary point and has a neighborhood in \( \tilde{\Sigma} \) contained in \( R \). A nonboundary point is either an outer-point or an inner point or can be both.

The following classifies the set of outer-skin points. (A similar result holds for the set of inner points except for (d).)

**Proposition 3.6.** For a highest-level crescent \( R \), the intersection points of \( I^o_R \) and \( \tilde{\Sigma} \) are either outer-skin points or inner-skin points. The set of outer-skin points of \( I^o_R \) for a highest-level crescent \( R \) is one of the following:

(a) a union of at most three isolated points.
(b) a union of at most one point and a segment or a segment with some endpoints removed.
(c) a union of two segments with a common endpoint with some of the other endpoints removed.
(d) a triangle with a boundary segment or two removed.

The same statement are true for inner-skin points.

**Proof.** This follows from the general position of vertices of \( \tilde{\Sigma} \). \( \square \)

**Definition 3.7.** Given a crescent \( R \), we define \( I^o_R \) to be the \( I_R \) removed with the pinched points, boundary points, and the segments and triangles as above. (We don’t remove the isolated points.)

**Proposition 3.8.** Suppose that \( c \) is a straight closed geodesic in \( M \) not meeting \( \Sigma \). Let \( R \) be a highest-level crescent. Then

- the inverse image \( \tilde{c} \) of \( c \) in \( \tilde{M} \) does not meet \( R \) in its interior and the \( \alpha \)-parts.
• \( \tilde{c} \) could meet \( R \) in its I-part tangentially and hence be contained in the I-part. In this case, \( R \) is not compact.

• If \( \tilde{c} \) is a geodesic in \( \tilde{M} \) eventually leaving all compact subsets, then the above two statements hold as well. In particular if \( \tilde{M} \) is a special hyperbolic manifold and \( \tilde{c} \) ends in the limit set of the holonomy group associated with \( \tilde{M} \).

**Proof.** If a portion of \( \tilde{c} \) meets the interior of \( R \), \( \tilde{c} \cap R \) is a connected arc, say \( l \) since \( R \) is a closure of a component cut out by a totally geodesic hyperplane in \( \tilde{M} = \tilde{\Sigma} - (*) \).

Since \( \tilde{c} \) is disjoint from \( \tilde{\Sigma} \), both endpoints of \( l \) must be in \( I_{\overline{R}} \) or in \( S^2_\infty \cap \overline{TR} \) for the closure \( \overline{TR} \) of \( I_{\overline{R}} \) in the compactified \( \mathbb{H}^3 \cup S^2_\infty \). If at most one point of \( l \) is in \( I_{\overline{R}} \), then \( l \) is transversal to \( I_{\overline{R}} \) and the other endpoints \( l \) must lie in \( \alpha_{\overline{R}} \) by (\*). This is absurd.

If at least two points of \( l \) are in \( I_{\overline{R}} \), then \( l \) is a subset of \( I_{\overline{R}} \). Since \( \tilde{c} \) is disjoint from \( \tilde{\Sigma} \), \( \tilde{c} \) is a subset of \( I_{\overline{R}} \), and \( R \) is not compact.

The only remaining possibility for \( l \) is that \( l \) ends in \( S^2_\infty \) and \( l^o \) is a subset of \( R^o \). Hence \( \tilde{c} = l \) and \( \tilde{c} \) lies on the totally geodesic hypersurface containing \( I_{\overline{R}} \) since otherwise \( l \) has to end at a point of \( \alpha_{\overline{R}} \).

The domain \( I_{\overline{R}} \) in \( P \) is disjoint from \( \tilde{c} \) also on \( P \). Its boundary is a union of pinched arc in \( \tilde{\Sigma} \). Let \( \alpha \) be an open arc in \( \mathbb{H}^3 \) closest to \( \tilde{c} \) on \( P \). If an one-sided neighborhood of \( \alpha \) goes into \( R \), Then \( N(\alpha) - \alpha \) for a neighborhood \( N(\alpha) \) of \( \alpha \) lies outside \( R \) since otherwise this leads to a higher folding number. Thus the vertices of \( \alpha \) as an arc are vertices of \( \tilde{\Sigma} \). Thus, \( \alpha \) can have at most three vertices as \( \alpha \) is on a plane, and there must be an infinite geodesic edge of \( \tilde{\Sigma} \). The later is impossible since \( \tilde{\Sigma} \) was finitely triangulated. Therefore, we obtained a contradiction, and \( \tilde{c} \) can’t be in the interior of \( R \). This proves (i) and (ii).

(iii) follows similarly.

\[ \square \]

**Figure 2.** The dashed arc indicates the tube from the bottom and the dotted arcs indicated the disks to be attached to the \( \alpha \)-parts
Proposition 3.9. Let $\mathcal{R}$ be a highest-level crescent. Then there exists an innermost crescent $\mathcal{R}'$ containing $\mathcal{R}$ so that

- $I_{\mathcal{R}'}$ is connected and has no pinched points or a disconnecting union of outer-skin edges, i.e., $I_{\mathcal{R}'}^O$ is connected.
- The closure of $\alpha_{\mathcal{R}'}$ is homeomorphic to $I_{\mathcal{R}}$.
- $\mathcal{R}'$ is homeomorphic to $I_{\mathcal{R}'} \times [0, 1]$.

Proof. We assume the hypothesis 3.1.

By Proposition 3.8, the interior of $\mathcal{R}$ is disjoint from any lifts of $c_1, \ldots, c_n$.

Suppose that $\mathcal{R}$ is compact to begin with. Then $\mathcal{R}$ is disjoint from the lifts of $c_1, \ldots, c_n$ since $\Sigma$ is disjoint from these. We let $\mathcal{S}$ be the highest-level crescent obtained from $\mathcal{R}$ by cutting through the $I$-part hypersurface $P$ and taking the closure of a component of $\mathcal{R} - \tilde{\Sigma} - P$ if necessary.

We introduce a height function $h$ on $\mathcal{S}$ defined by introducing a parameter of hyperbolic hypersurfaces perpendicular to a common geodesic passing through $I_{\mathcal{S}}$ in the perpendicular manner. (It will not matter which parameter we choose). We may assume that $h$ is Morse in the combinatorial sense.

If $I_{\mathcal{S}}$ does not meet any lifts of $c_1, \ldots, c_n$, let $N_\epsilon(\alpha_{\mathcal{S}})$ be the neighborhood of $\mathcal{S}$ in the closure of the component of $\tilde{M} - \tilde{\Sigma}$ containing the interior of $\mathcal{S}$.

We let $N_\epsilon(\alpha_{\mathcal{S}})$ be the intersection of $\tilde{\Sigma}$ with $N_\epsilon(\mathcal{S})$. $N_\epsilon(\mathcal{S})$ can be chosen so that $N_\epsilon(\alpha_{\mathcal{S}})$ becomes an open surface compactifying to a surface. There exists a part $I$ in of the boundary which is a complement in the boundary of $N_\epsilon(\mathcal{S})$ of $N_\epsilon(\alpha_{\mathcal{S}})$.

We assume that $I$ lies on a properly imbedded surface $P'$ perturbed from the $I$-part hypersurface $P$ of $\mathcal{S}$.

First, we show that $N_\epsilon(\mathcal{S})$ is a compression body with $N_\epsilon(\alpha_{\mathcal{S}})$ as the compressible surface in the boundary:

Topologically, $N_\epsilon(\alpha_{\mathcal{S}})$ is homeomorphic to a surface possibly with 1-handles attached from $\alpha_{\mathcal{S}}$ and $I$ is obtained from $I_{\mathcal{S}}$ by removing 1-handles corresponding to the pinched points.

We may extend $h$ to an $\epsilon$-neighborhood of $\mathcal{S}$. This may introduce only saddle type singularity in $N_\epsilon(\alpha_{\mathcal{S}})$. We let $I$ to be in the zero level of $h$. This process only introduces a saddle type singularity in $N_\epsilon(\alpha_{\mathcal{S}})$.

If there is a critical point of $h$ with locally positive type, then we see that in fact there exists a crescent of higher level near the critical point. This is absurd.

If there are no critical point of positive type, then $\pi_1(N_\epsilon(\alpha_{\mathcal{S}})) \to \pi_1(N_\epsilon(\mathcal{S}))$ is surjective as shown by Freedman-McMullen [II].

There exists a compression body in $N_\epsilon(\mathcal{S})$ with a boundary $N_\epsilon(\alpha_{\mathcal{S}}) \cup S'$ for an incompressible surface $S'$ in the interior of $\mathcal{S}$. Since all closed path in $N_\epsilon(\mathcal{S})$ is homotopic to one in $N_\epsilon(\alpha_{\mathcal{S}})$, it follows that $S'$ is parallel to $I$. Hence, $N_\epsilon(\mathcal{S})$ is a compression body homeomorphic to $I$ times an interval and 1-handles attached at disks disjoint from $I$. ( $\mathcal{S}$ is essentially obtained by pinching some points of $I$ together and pushing down a bit.)

Next, we reduce the number of components of $I$:

Suppose now that $I$ is not connected. This means that there are 1-handles attached to $I$ times the intervals joining the components. Then $N_\epsilon(\mathcal{S})$ has a compressing disk $D$.
for $N_e(\alpha_S)$, dual to the 1-handles. Since $\partial D$ bounds a disk $D'$ in $\tilde{\Sigma}$ by Dehn’s lemma, $\partial D$ is separating in $\tilde{\Sigma}$.

Consequently also, $N_e(\alpha_S)$ is a planar surface.

The irreducibility of $\tilde{M}$ tells us that $D$ and $D'$ bound a 3-ball $B$ in the closure of a component of $\tilde{M} - \tilde{\Sigma}$. Then $B$ contains at least one component of $I$.

By taking a maximal family of compressing disks dual to the 1-handles and regarding the components of the complements as vertices, we see that the 1-handles do not form a cycle. Therefore, we choose the compressing disk $D$ to be the one such that $D$ and corresponding $D'$ bounds a 3-ball $B$ containing a unique component of $I$.

The ball $B$ contains the component of $R - D$ and a component of $I$, say $I'$. Then $I'$ has a boundary component $\gamma$ in $D'$ and possibly other boundary components. $\gamma$ bounds a disk $D''$ in $\tilde{\Sigma} - N_e(\alpha_S)$ which is in $D'$. The other boundary components $\gamma_1, \ldots, \gamma_m$ of $I'$ bound disjoint disks $D_1, \ldots, D_m$ in the totally geodesic hypersurface $P'$ containing $I_S$. These $\gamma_i$s are innermost closed curves in $P' \cap \tilde{\Sigma}$. $I'$ union with these disks $D_i$s and $D''$ bound a 3-ball $B''$. $B'' \cap \tilde{\Sigma}$ is a union of $D''$ and a surface $\Sigma''$ with boundary equal to the union of the other closed boundary curves of $I'$. Since $\Sigma'' \subset D'$, we have that $\Sigma''$ is a planar surface. If a component of $\Sigma''$ contains more that two of $\gamma_i$s, then we can find a closed curve in $D'$ meeting the two of $\gamma_i$s exactly once since $\alpha_S$ is connected. This is absurd. Therefore each component of $\Sigma''$ is a disk.

Therefore,

$$I \cup D'' \cup D_1'' \cup \cdots D_m''$$

bounds a 3-ball $B'''$. Taking a union of $N_e(S)$ with $B'''$, we obtain $N_e(T)$ for a crescent $T$ with $I_T$ in $I_S$ and one less component of the surface in $\partial N_e(T)$ corresponding to $I$.

By induction, we obtain a crescent $R''$ with $I_{R''}$ in $I_R$ and the surface $I'$ corresponding to $I$ connected. Since $R''$ is homeomorphic to a compression body, $R''$ is homeomorphic to $I$ times an interval since there are no compressing disks.

If there are any pinched points in $I_{R''}$ or disconnecting outer-skin edges, then $I'$ would be disconnected. $\alpha_{R''}$ has a closure that is a surface since there are no pinching points.

Since $R''$ is an $I$-bundle, it follows that the closure of $\alpha_{R''}$ and $I_{R''}$ are homeomorphic surfaces.

If there were more than one cut-off crescents $S$, then we obtain $R''$ for each $S$. There must be two cut-off crescents $S$ and $S'$ adjacent from opposite sides of some of the components of $I_S$. Since the corresponding $R''$ and $R'''$ does not have any pinched points or separating outer-skin edges, the unique components of $I_{R''}$ and $I_{R'''}$ either agree or are disjoint from each other. $R''$ and $R'''$ cannot be adjacent from opposite side since we can form a compact component of $\tilde{\Sigma}$ otherwise. It follows that one of $R''$ and $R'''$ is a subset of the other. By induction, we see that the conclusions of the proposition holds if $R$ is compact.

If $R$ is noncompact, we follow as before but we choose $N_e(R)$ similarly and push down near infinity. Note here that only one component of $I$ maybe noncompact since the boundary of a compressing disk must bound a compact disk in $\tilde{\Sigma}$. □

**Definition 3.10.** We say that $S$ as obtained from $R$ in the above lemma is obtained by cutting along the $I$-part hypersurface of $R$. $S$ may not be a unique one so obtained.
Corollary 3.11. Let \( \mathcal{R} \) be a secondary highest-level crescent. Then the statements of Proposition 3.8 hold for \( \mathcal{R} \) as well.

*Proof.* The proof is exactly the same as that of Proposition 3.8. \( \square \)

If a 0-folded crescent \( \mathcal{S} \) contains \( \mathcal{R} \) so that \( I_0^\mathcal{S} \) is connected and is included in \( I_0^\mathcal{R} \), then we say that \( \mathcal{S} \) is a highest-level crescent as well. (Actually, it may not be highest-level since \( \mathcal{S} \) may not necessarily be contained in an \( n \)-folded crescent but only a part of it.) More precisely, it is a secondary highest-level crescent.

Note that an outer highest-level crescent comes from an inner crescent if the highest folding number is odd and comes from an outer crescent if the number is even. The converse holds for an inner highest-level crescent.

A secondary highest-level crescent exists for any highest-level crescent by Proposition 3.9:

Corollary 3.12. Let \( \mathcal{R} \) be a highest-level crescent. If \( I_0^\mathcal{R} \) is not connected, then there exists a 0-folded crescent \( \mathcal{S} \) containing \( \mathcal{R} \) so that \( I_0^\mathcal{S} \) is connected and is a component of \( I_0^\mathcal{R} \).

Also, a secondary highest-level crescent has the outer-skin points with the same properties as those of a highest-level crescent. (See Proposition 3.6).

By taking a nearby crescent inside, we see that a highest-level crescent could be generically chosen so that the crescent is compact, the \( I \)-part and the \( \alpha \)-part are surfaces, and \( I^O \)-part is truly the interior of the \( I \)-part.

Corollary 3.13. Let \( \mathcal{R} \) be the compact secondary highest-level outer (resp. inner) crescent that is generically chosen. Then \( \mathcal{R} \) is homeomorphic to the closure of \( \alpha^\mathcal{R} \) times \( I \), and \( I_0^\mathcal{R} \) is isotopic to \( \alpha^\mathcal{R} \) by an isotopy inside \( \mathcal{R} \) fixing the boundary of \( I^\mathcal{R} \).

We say two crescents \( \mathcal{R} \) and \( \mathcal{S} \) face each other if \( I^\mathcal{R} \) and \( I^\mathcal{S} \) agree with each other in some 2-dimensional part and have disjoint one-sided neighborhoods.

Proposition 3.14. If two highest-level outer (resp. inner) crescents \( \mathcal{R} \) and \( \mathcal{S} \) face each other, then there exists a (secondary) highest-level outer (resp. inner) crescent \( \mathcal{T} \) with connected \( I_0^\mathcal{T} \) containing both.

*Proof.* We may assume without loss of generality that \( I_0^\mathcal{R} \) and \( I_0^\mathcal{S} \) are both connected. If not, we replace \( \mathcal{R} \) and \( \mathcal{S} \) by ones with connected \( I^O \)-parts. The replacements still face each other or one becomes a subset of another. In the second case, we are done.

Since \( I_0^\mathcal{R} \) and \( I_0^\mathcal{S} \) meet in open subsets, either they are identical or the boundary \( \partial L \) of their intersection \( L \) lies in \( I_0^\mathcal{R} \). \( \partial L \) is a subset of \( \Sigma \) and is a 1-complex of pinched arcs. \( \partial L \) is a set of outer-skin points of \( \mathcal{S} \) since a neighborhood of \( \partial L \) in \( \Sigma \) must be above \( \mathcal{S} \). However, then \( \partial L \) must be disjoint from \( I_0^\mathcal{S} \). Therefore, \( I_0^\mathcal{S} = I_0^\mathcal{R} \). This means that \( \mathcal{S} \cup \mathcal{R} \) is bounded by a component subsurface of \( \Sigma \). This is absurd. \( \square \)

Definition 3.15. Two secondary highest-level outer (resp. inner) crescents \( \mathcal{R} \) and \( \mathcal{S} \) are said to meet transversally if \( I^\mathcal{R} \) and \( I^\mathcal{S} \) meet in a union of disjoint geodesic segment \( J \), \( J \neq \emptyset \), mapping into a common geodesic in \( \mathbb{H}^3 \), in a transversal manner such that
• The closure $\nu_{\mathcal{R}}$ of the union of the components of $I_{\mathcal{R}} - J$ in one-side is a subset of $\mathcal{S}$ and the closure $\nu_{\mathcal{S}}$ of the union of those of $I_{\mathcal{S}} - J$ is a subset of $\mathcal{R}$.
• The intersection $\mathcal{R} \cap \mathcal{S}$ is the closure of $\mathcal{S} - \nu_{\mathcal{R}}$ and conversely the closure of $\mathcal{R} - \nu_{\mathcal{S}}$.
• The intersection $\alpha_{\mathcal{R}} \cap \alpha_{\mathcal{S}}$ is a union of components of $\alpha_{\mathcal{R}} - \nu_{\mathcal{S}}$ in one-side of $\nu_{\mathcal{S}}$ and, conversely, is a union of components of $\alpha_{\mathcal{S}} - \nu_{\mathcal{R}}$ in one side of $\nu_{\mathcal{R}}$.
• $\alpha_{\mathcal{R}} \cup \alpha_{\mathcal{S}}$ is an open surface in $\tilde{\Sigma}$.

**Proposition 3.16.** Given two secondary highest-level outer (resp. inner) crescents $\mathcal{R}$ and $\mathcal{S}$, there are the following mutually exclusive possibilities:

- $\mathcal{R}$ and $\mathcal{S}$ do not meet in $\tilde{M} - \tilde{\Sigma}$.
- $\mathcal{R} \subset \mathcal{S}$ or $\mathcal{S} \subset \mathcal{R}$.
- $\mathcal{R}$ and $\mathcal{S}$ meet transversally.

**Proof.** The reasoning is exactly the same as [7] and [8] in dimension two or three. □

4. Crescents and isotopy

In section 3, the crescent-isotopy theory is presented: We form the union of secondary highest-level crescents and can isotopy the union of their $\alpha$-parts to the complement $I$ in the boundary of their union. However, there might be some parts of $\tilde{\Sigma}$ meeting $I$ tangentially from above. We need to push these parts upward first using truncations before the actual crescent-isotopy.

We first discuss how small isotopy can affect the set of all crescents of $\tilde{\Sigma}$. For a crescent in the outer-direction, the move in the outer-direction “preserves” the crescent by moving the $\alpha$-part only. For a crescent in the inner-direction, the move in the inner-direction also preserves.

We introduce truncation move: we truncate a convex vertex by a local totally geodesic hypersurface and perturb the result into general position by pushing the vertices. We show that the crescents in these cases are preserved or we can modify the crescents by moving the $I$-parts. These moves are designed to eliminate some points so that the result of the isotopy are imbeddings.

We start from the outer-direction secondary highest-level crescents. We take a union of an overlapping equivalence class of them and show that the union of $\alpha$-parts in $\tilde{\Sigma}$ can be isotopied to the complement in the boundary of the union. By above truncation moves, we show that the result is an imbedding. Finally we do this for all the equivalence classes and we can isotopy $\Sigma$ itself.

The inner-direction crescent moves are entirely the same.

**Lemma 4.1.** Suppose that $\tilde{\Sigma}$ has been isotopied in the outward direction by a sufficiently small amount and $\mathcal{R}$ is an outer crescent. Then there exists a crescent $\mathcal{R}'$ sharing the $I$-part hypersurface with $\mathcal{R}$ and differs from $\mathcal{R}$ by isotopying the $\alpha$-part only. Conversely, if $\tilde{\Sigma}$ has been isotopied in the inward direction and $\mathcal{R}$ is an inner crescent, the same can be said.

**Proof.** Straightforward. □
We say that $\mathcal{R}'$ is **isotopied from** $\mathcal{R}$ **with the I-part preserved**.

We may “truncate” $\Sigma$ at convex vertices and $\tilde{\Sigma}$ correspondingly and perturb: Let $v$ be a convex vertex and $H$ a local half-open ball at $v$ with the side $F$ passing through $v$. We may move $F$ inside by a very small amount and then truncate $\tilde{\Sigma}$ and correspondingly for all vertices equivalent to $v$. Then we introduce some equivariant triangulation of the trace $T$ of the truncation and the truncated $\tilde{\Sigma}$ without introducing vertices in the interior of $T$. We push the concave vertices of $T$ downward by small amounts along the corresponding edges of $\tilde{\Sigma}$ and then move the vertices inward to make the truncated surface to be in general position. The three steps together are called the **small truncation move**.

We denote by $\Sigma^\epsilon$ the perturbed $\Sigma$ where the traces are less than an $\epsilon$-distance away from the respective convex vertices. We assume that during the perturbations $\Sigma^\epsilon$ is isotopied from $\Sigma$ and the convexity of the dihedral angles do not change under the isotopy. Thus, if an edge or a vertex is convex after being born, it will continue to be so as $t \to 0$ and as $t$ grows from 0.

We may view the truncation move by considering $v$ to be a vertex of some multiplicity and vertices diverge as the side $F$ moves away from $v$. That is, we see this as births of many vertices from convex vertices.

We may also assume that the convex vertex move is equivariant on $\tilde{\Sigma}$, i.e., the isotopy is equivariant.

An **isotopy** of a crescent as we deform $\Sigma$ is a one-parameter family of crescents $\mathcal{R}_t$ with $\alpha$-parts in $\Sigma$. We say that a crescent **bursts** if fixing the totally geodesic hypersurface containing the $I$-part of it and isotopying the $\alpha$-parts in the isotopied $\Sigma$ cannot produce a crescent isotopied from the original one.

Such an event happens when a parameter of an edge of $\tilde{\Sigma}$ or a parameter of a vertex of $\tilde{\Sigma}$ go belows the fixed totally geodesic hypersurface from the point of view of the crescent. Of course, a vertex could be a multivertex and all of the new vertices go down. The edge should be one the face that meets the $I$-parts of the crescents and the vertex on the edge that meets the $I$-part of the crescent. The event could happen simultaneously but the generic nature of the move shows that at most four vertex submersions, at most three edge submersions, and at most two vertices and one edge submersions can happen simultaneously. Moreover, at the event, the vertex and the edge must be in the $I$-part of the crescent and the triangles of $\Sigma$ must be placed in certain way in order that the bursting to take place.

If the bursting happens immediately for a crescent $\mathcal{R}$, then the convex vertex must be in the boundary of $I_{\mathcal{R}} \cap (\tilde{M} - \tilde{\Sigma})$ in the totally geodesic hypersurface $P$ containing $I_{\mathcal{R}}$. Otherwise a small perturbation of $\tilde{\Sigma}$ gives us an isotopy of $\mathcal{R}$ preserving the $I$-part hypersurface. Therefore, the bursting does not take place.

**Proposition 4.2.** Under a small truncation move in the outer direction, we can isotopy

(i) each outer crescent into itself by moving the $\alpha$-part in the outer direction and preserving the $I$-part hypersurface.

(ii) each inner crescent into itself union the $\epsilon$-neighborhood of $\tilde{\Sigma}$ by moving the $I$-part hypersurface in the outer direction or preserving the $I$-part hypersurface.

Under a small truncation move in the inner direction, we can deform
(iii) each inner crescent into itself by moving the $\alpha$-part in the inner direction and preserving the $I$-part hypersurface.

(iv) each outer crescent into itself union the $\epsilon$-neighborhood of $\tilde{\Sigma}$ by moving the $I$-part hypersurface in the inner direction or preserving the $I$-part hypersurface.

All crescents of $\tilde{\Sigma}^\epsilon$ can be obtained in this way. The highest folding number may decrease only under a convex vertex move.

Proof. Essentially, the idea is that the move can only “decrease” the associated crescents.

Let $\mathcal{R}$ be an outer crescent and $\tilde{\Sigma}$ moved in the outer direction. Lemma 4.1 implies (i).

Let $\mathcal{R}$ be an inner crescent and $\tilde{\Sigma}$ be moved in the outer direction. Then again an isolated submerging vertex is a convex vertex. In this case, we move the $I$-part inward so that the submerging vertex stay on the boundary of the $I$-part. Other cases are treated similarly. This proves (ii).

(iii) and (iv) correspond to (i) and (ii) respectively if we change the orientation of $\Sigma$.

To show that all crescents of $\tilde{\Sigma}^\epsilon$ can be obtained in this way: Given an outer crescent for $\tilde{\Sigma}^\epsilon$, we reverse the truncation move. If the $I$-part of a crescent avoids the traces of the truncation moves, then we simply isotopy the $\alpha$-parts only.

The trace surface has only concave vertices and $s$-vertices. Let $P'$ be a local totally geodesic hypersurface truncating the stellar neighborhood of a convex vertex $v$ of $\tilde{\Sigma}$ at some small distance from $v$ but large compare to our isotopy move distance. Suppose that $v$ were involved in the convex truncation move.

Suppose that the $I$-part of a crescent $\mathcal{R}$ for $\tilde{\Sigma}$ meets what are outside the part truncated by $P'$. Assuming that our isotopy was very small, if the $I$-part meets one of the trace surface, the $I$-part meets $P'$ since $P'$ is separating. $P'$ intersected with the closure of the exterior of $\tilde{\Sigma}^\epsilon$ is a polygonal disk $D^\epsilon$. Then $D^\epsilon$ intersected with the $I$-part is a disjoint union of segments. Letting $D$ be the intersection of $P'$ with the exterior of $\tilde{\Sigma}$, we see that $D$ intersected with $P'$ extend the segments of $D^\epsilon$ intersected with $P'$. Thus, it is clear that the $I$-part extends into the polyhedrons bounded by $P'$ and the stellar neighborhood of $\tilde{\Sigma}$. Since all vertex submersions of $\tilde{\Sigma}^\epsilon$ can happen by vertices near the convex vertices of $\tilde{\Sigma}$ masked off by totally geodesic hypersurfaces such as $P'$, we obtain a crescent $\mathcal{R}'$ for $\tilde{\Sigma}$ preserving the $I$-part hypersurface of $\mathcal{R}$. Therefore, $\mathcal{R}$ were obtained from $\mathcal{R}'$ by the convex truncation isotopy preserving the $I$-part.

Therefore a crescent for $\tilde{\Sigma}^\epsilon$ is one we obtained by the process in (i).

Let $\mathcal{R}$ be an inner crescent for $\tilde{\Sigma}^\epsilon$. Then since the vertices moved outward with respect to $\tilde{\Sigma}$, they move inward when we reverse the process and we see that $\mathcal{R}$ is isotopied to a crescent for $\tilde{\Sigma}$ by Lemma 4.1 by preserving the $I$-part hypersurface.

To show that the highest folding number can only decrease: For a crescent $\mathcal{R}$ to increase the folding number, a vertex must move into $I_R$ during the isotopy. We see that such a vertex must be a convex one. However, the convex vertices can only move in the direction away from the interior of $\mathcal{R}$. (Even ones after the births obey this rule.)
First, we suppose that there are highest-level crescents that are outer ones. We move \( \Sigma \) in the outer direction first to eliminate the outer highest-level crescents.

As we did in [7] and [8], we say that two highest-level crescents \( R \) and \( S \) are equivalent if there exists a sequence of transversally intersecting crescents from \( R \) to \( S \); that is,

\[
R = R_0, R_i \cap R_{i+1} \neq \emptyset, S = R_n \quad \text{for } i = 1, 2, \ldots, n.
\]

We define \( \Lambda(\mathcal{R}) \) to be the union of all highest-level crescents equivalent to the highest-level crescent \( R \). As before, \( \Lambda(\mathcal{R}) \) and \( \Lambda(\mathcal{S}) \) do not meet in the interior or they are the same.

We define \( \partial I \Lambda(\mathcal{R}) \) to be the boundary of \( \Lambda(\mathcal{R}) \) removed with the closure of the union of the \( \alpha \)-parts of the crescents in it. Then \( \partial I \Lambda(\mathcal{R}) \) is a convex surface.

We define \( \partial \alpha \Lambda(\mathcal{R}) \) as the union of the \( \alpha \)-parts of the crescents equivalent to \( R \).

Recall that a pleated surface is a surface where through each point passes a geodesic.

**Lemma 4.3.** The set

\[
\partial I \Lambda(\mathcal{R}) \cap \tilde{M} - \tilde{\Sigma}
\]

is a properly imbedded pleated surface.

**Proof.** For each point of \( x \) belonging to the above set, \( x \) is an element of the interior of \( \tilde{M} \) by Proposition 3.3. Let \( B(x) \) be a small convex open ball with center at \( x \). Then the crescents equivalent to \( R \) meet \( B(x) \) in half-spaces. Therefore the complement of their union is a convex subset of \( B(x) \) and \( x \) is a boundary point. There is a supporting half-space \( H \) in \( x \) and \( H \) belongs to \( \Lambda(\mathcal{R}) \).

If there were no straight geodesic passing through \( x \) in the boundary set \( \partial I \Lambda(\mathcal{R}) \), then there exists a totally geodesic disk \( D \) in \( B(x) \) with \( \partial D \) in \( \Lambda(\mathcal{R}) \) but interior points are not in it.

Since \( \partial D \) is in \( \Lambda(\mathcal{R}) \), each point of \( \partial D \) is in some crescent. We can extend \( D \) to a maximal totally geodesic hypersurface and we see that a portion of the hypersurface bounds a crescent \( T \) containing \( D \) in its \( I \)-part and overlapping with the other crescents. Thus \( T \) is a subset of \( \Lambda(\mathcal{R}) \) and so is \( D \).

Therefore, \( \partial I \Lambda(\mathcal{R}) - \tilde{\Sigma} \) is a pleated surface. \( \square \)

We may have some so-called outer-skin points of \( \tilde{\Sigma} \) at \( \partial I \Lambda(\mathcal{R}) \), i.e., those points with neighborhoods in \( \tilde{\Sigma} \) outside \( \Lambda(\mathcal{R}) \). We can classify outer-skin points.

**Proposition 4.4.** The set of outer-skin points on \( \partial I \Lambda(\mathcal{R}) \) is a union of the following components:

- isolated points,
- an arc passing through the pleating locus with at least one vertex,
- isolated triangles,
- union of triangles meeting each other at vertices or edges.

**Proof.** This essentially follows by Proposition 3.6. \( \square \)

There have to be convex vertices in all of the above cases of outer-skin points. We move a convex vertex by vertex in an equivariant manner. Therefore, we may move them inwardly by small truncation moves, i.e., along an inward normal to the local
half-open balls. This will only decrease the set of the union of crescents. In every case, a new convex vertex that is an outer-skin point is obtained after the move.

In this way, we see that

\[(1) \quad \partial I \Lambda(\mathcal{R}) \cup (\Sigma - \partial \alpha \Lambda(\mathcal{R}))\]

is a properly imbedded pleated surface.

We do this for \(\Lambda(\mathcal{R})\) for each highest-level crescent \(\mathcal{R}\). The end result is a properly imbedded surface \(\tilde{\Sigma}'\). The deck transformation group acts on \(\tilde{\Sigma}'\) since it acts on the union of \(\Lambda(\mathcal{R})\). Thus, we obtain a new closed surface \(\Sigma'\).

Since the union of \(\Lambda(\mathcal{R})\) for every highest-level crescent \(\mathcal{R}\) is of bounded distance from \(\Sigma\) by the boundedness and the fact that \(M\) is locally-compact, \(\Sigma'\) is a compact surface.

We show that \(\Sigma\) and \(\Sigma'\) are isotopic.

Let \(N\) be the \(\epsilon\)-neighborhood of \(\tilde{\Sigma}'\) in the closure of the outer component of \(\tilde{M} - \tilde{\Sigma}\). There exists a boundary component \(\partial_1 N\) nearer to \(\tilde{\Sigma}\) than the other boundary component. The closure of a component \(K\) of \(M - \tilde{\Sigma} - \tilde{\Sigma}'\) contains \(\partial_1 N\). Then \(K\) projects to a compact subset of \(M\). We can find a finite collection of generic secondary highest-level compact crescents \(\mathcal{R}_1, \ldots, \mathcal{R}_n\) and whose images under \(\Gamma\) form a locally finite cover of \(K\).

We label the crescents by \(S_1, S_2, \ldots\). We know that replacing the closure of the \(\alpha\)-part of \(S_1\) by the \(I\)-part is an isotopy. After this move, \(S_2, S_3, \ldots\) become new generic highest-level crescents by Proposition 3.16 and appropriate truncations.

We define \(\partial_I(S_1 \cup S_2 \cup \ldots)\) as the boundary of \(S_1 \cup S_2 \cup \ldots\) removed with the union of the \(\alpha\)-parts of \(S_1, S_2, \ldots\). Again, this is a convex imbedded surface. Therefore, replacing the union of the \(\alpha\)-parts of \(S_1, S_2, \ldots\) by \(\partial_I(S_1 \cup S_2 \cup \ldots)\) is an isotopy as above.

We obtain \(\Sigma_{\mathcal{R}_1, \ldots, \mathcal{R}_n}\) as the image in \(M\), which is isotopic to \(\Sigma\). If \(\epsilon\) is sufficiently small, then we see easily that \(\partial_I(S_1 \cup S_2 \cup \ldots)\) is in \(N\) can be isotopied to \(\tilde{\Sigma}'\) using rays perpendicular to \(\tilde{\Sigma}'\). Thus, \(\Sigma_{\mathcal{R}_1, \ldots, \mathcal{R}_n}\) is isotopic to \(\Sigma'\). We showed that \(\Sigma'\) is isotopic to \(\Sigma\).

5. Convex perturbations

In this section, we discuss how a surface with a portion of itself concavely pleated by infinitely long geodesics and the remainder triangulated can be perturbed to a triangulated surface without introducing higher-level crescents. This is done by approximating the union of pleating geodesics by train tracks and choosing normals in the concave direction and finitely many vertices at the normal and pushing the pleating geodesics to become geodesics broken at the vertices.

We will now perturb the isotopied \(\Sigma\) into a triangulated surface not introducing any higher-level crescents.

Suppose that \(\Sigma\) is a closed imbedded surface in \(M\). \(\Sigma\) is a pleated-triangulated surface if \(\tilde{\Sigma}\) contains a closed 2-dimensional domain divided into locally finite collection of closed totally geodesic convex domains meeting each other in geodesic segments and through each point of the complement passes a straight geodesic. We may also add
finitely many straight geodesic segments in the surface ending at the domain. The domain union with the segments is said to be the **triangulated part** of $\Sigma$. The boundary of the domain is a union of finitely pinched simple closed curves. The complement of the domain is an open surface, which is said to be the **pleated part** where through each point passes a straight geodesic. Then the pleated part has a locus where a unique straight geodesic passes through. This part is said to be the **pleating locus**. It is a closed subset of the complement and forms a lamination.

For later purposes, we say that $\Sigma$ is **truly pleated-triangulated** if the triangulated part is a union of totally geodesic domains that are polygons and geodesic segments ending in the domains.

Note that the triangulated parts and pleating parts are not uniquely determined. We simply choose what seems natural.

We assume that the pleated part is locally convex or locally concave. We choose a normal direction so that the surface is locally concave at the pleated part.

If $\Sigma$ satisfies all of the above conditions, we say that $\Sigma$ is a **concave pleated-triangulated** surface. If we choose the opposite normal-direction, then $\Sigma$ is a **convex pleated, triangulated** surface.

First, we show that the surface obtained in the previous section is concave pleated-triangulated.

**Proposition 5.1.** Let $\Sigma$ be a triangulated surface incompressible in $M$ with a finite number of closed geodesics removed. Let $n$ be the highest folding number of $\Sigma$. Suppose $n$ is achieved by outer crescents. Suppose that $\Sigma'$ is obtained from $\Sigma$ by a small truncation move for level $n$ outer crescents and next by crescent move for level $n$ outer crescents. Then $\Sigma'$ is a concave pleated-triangulated surface in the outer direction. Moreover, the statements are true if all “outer” were replaced by “inner”.

**Proof.** The part $\partial I\Lambda(\mathcal{R}) - \tilde{\Sigma}$ for crescents $\mathcal{R}$s are pleated by Lemma 4.3. These sets for crescents $\mathcal{R}$ are either identical or disjoint from each other as the sets of form $\Lambda(\mathcal{R})$ satisfy this property. The union of sets of form $\partial I\Lambda(\mathcal{R})$ form the pleated part and the complement in $\Sigma'$ were in $\Sigma$ originally and they are union of totally geodesic 2-dimensional convex domains. □

Next, we analyze the geometry of the pleating locus in the pleated part.

Two leaves are **converging** if one is asymptotic to the other one (see Section 8.1 for definitions). i.e., the distance function from one leaf to the other converges to zero and conversely. By an end of a leaf of a lamination **wrapping around** a closed set, we mean that the leaf converges to a subset of the closed set in the direction of the end.

**Lemma 5.2.** Suppose that $\Sigma$ is a closed concave pleated-triangulated surface with a triangulated part and pleated part assigned. Suppose that $l$ is a leaf. Then

- $l$ is either a leaf of a minimal geodesic lamination or a closed geodesic, or each end of $l$ wraps around a minimal geodesic lamination or a closed geodesic or ends in the boundary of the triangulated part.
- if $l$ is isolated from both sides, then $l$ must be a closed geodesic, and
- pleating leaves in a neighborhood of $l$ must diverge from $l$ eventually.
Proof. Since the pleated open surface carries an intrinsic metric which identifies it to a quotient of an open subset of the hyperbolic space, each geodesic in the pleating lamination will satisfy the above properties like the geodesic laminations on the closed hyperbolic surfaces.

If \( l \) is isolated, then there is a definite positive angle between two totally geodesic hypersurfaces ending at \( l \). Suppose that \( l \) is not a simple closed geodesic. Then this angled pair of the hypersurfaces continues to wrap around infinitely in \( M \) accumulating at a point of \( M \) and the sum of the angles violates the imbeddedness of \( \Sigma \).

If \( l \) is not isolated but has converging nearby pleating leaves, the same reasoning will hold. Therefore, the second item holds. □

Proposition 5.3. Suppose that \( \Sigma \) is a closed concave pleated-triangulated surface. Let \( \Lambda \) be the set of pleating locus of the pleated part in \( \Sigma \). Then \( \Lambda \) decomposes into finitely many components \( \Lambda_1, \ldots, \Lambda_n \) so that each \( \Lambda_i \) is one of the following:

- a finite union of finite-length pleating leaves homeomorphic to a compact set times a line. (A discrete set times a line if \( \Sigma \) is truly pleated-triangulated.)
- a simple closed geodesic.
- a minimal geodesic lamination, which is a closed subset of the pleated part isolated away from the triangulated part.

Here each leaf is either bi-infinite or finite. The union of bi-infinite leaves is a finite union of minimal geodesic laminations and is isolated way from the triangulated part and the union of finite-length leaves.

Proof. Let \( l \) be an infinite leaf in the pleating locus. By Lemma 5.2, \( l \) is not isolated from both sides and the leaves in its neighborhood is diverging from \( l \). If \( l \) is not itself a leaf of a minimal lamination, then an end of \( l \) must converge to a minimal lamination or a simple closed geodesic. This means that leaves in a neighborhood also converges to the same lamination in one of the directions. However, this means that they also converges to \( l \), a contradiction. Therefore, each leaf is a leaf of a minimal lamination or a closed geodesic or a finite length line.

The union of all finite length lines in the pleating locus is a closed subset: Its complement in \( \Lambda \) is a compact geodesic lamination in \( \Sigma \). If a sequence of a finite length leaves \( l_i \) converges to an infinite length geodesic \( l \), then \( l_i \) gets arbitrarily close to a minimal lamination or a closed geodesic. If \( l_i \) gets into a sufficiently thin neighborhood of one of these, then an endpoint of \( l_i \) must be in a sufficiently thin neighborhood of one of these by the imbeddedness property of \( l_i \), i.e., cannot turn sharply away and go out of the neighborhood. As \( l_i \) ends in the triangulated part, the distance from the triangulated part to one of these goes to zero. Since the domains in the triangulated part are in general position, the boundary of the triangulated part cannot contain a closed geodesic or the straight geodesic lamination. This is a contradiction.

Looking at an \( \varepsilon \)-thin neighborhood of \( \Lambda \), we see that \( \Lambda \) decomposes as described. (See Casson-Bleiler [6] for background informations).

Remark 5.4. If \( \Sigma \) is properly pleated-triangulated, then there are only finitely many finite length pleating leaves since their endpoints are on the vertices of the triangles in the triangulated part.
We can still define crescents for $\tilde{\Sigma}'$ which is the inverse image of $\Sigma'$ obtained from crescent moves. The definitions are of course the same.

**Proposition 5.5.** If $\Sigma'$ is obtained from $\Sigma$ by the highest-level outer crescent move for level $n$, then the union of the collection of crescents of $\tilde{\Sigma}$ contains the union of those of $\tilde{\Sigma}'$, and $\Sigma'$ has no outer crescent of level $n$ or higher. The same statements hold if we replace the word “outer” by “inner”.

*Proof.* The outer crescents for $\tilde{\Sigma}'$ can be extended to ones for $\tilde{\Sigma}$ since their $I$-part can be extended across.

The inner crescents for $\tilde{\Sigma}'$ can be truncated to ones for $\tilde{\Sigma}$ by Lemma 4.1 since the move from $\tilde{\Sigma}'$ to $\tilde{\Sigma}$ is inward and supported by the outer crescents of $\tilde{\Sigma}$. Thus the first statement holds.

If there were outer crescent $R$ of level $n$ or higher, then we can extend $I_R$ across the $I_\partial$-parts so that we can obtain a level-$n$ or higher-level crescent. This is absurd. If $R$ were inner, Lemma 4.1 implies the result.  

There might still be level $n-1$ outer crescents for $\Sigma'$. We do small truncation moves for these.

We will use the train tracks to prove the following theorem:

**Theorem 5.6.** Let $\Sigma$ be a closed concave pleated, triangulated surface with outer (resp. inner) level strictly less than $n$. Then one can find an imbedded isotopic triangulated surface $\Sigma'$ in any $\epsilon$-neighborhood of $\Sigma$ so that the following hold.

(i) The union of the set of crescents for $\tilde{\Sigma}'$ is in the $\epsilon$-neighborhood of that of $\tilde{\Sigma}$ and vice-versa for a small $\epsilon$ if we choose $\tilde{\Sigma}'$ sufficiently close to $\tilde{\Sigma}$.

(ii) Assuming that $\Sigma$ is truly pleated-triangulated, the vertices of the triangulated part of $\Sigma'$ are deformed from the vertices that $\Sigma$ had and the vertices formed from points of the pleated parts. (The pleated part vertices are strict s-vertices.)

A train track is obtained by taking a thin neighborhood of the lamination. We can think of the train track as a union of segment times an interval, so-called branches, joined up at the end of each segment times the intervals so that the intervals stacks up and matches. A point times the interval is said to be a *tie* and a tie where more than one branches meet a *switch*. One can collapse the interval direction to obtain a union of graphs and circles.

*Proof.* Here, we will be working in $M$ directly.

(ii) Let $l_1, \ldots, l_k$ be the thin strips containing all the finite length open leaves. We find a thin totally geodesic hypersurface $P_i$ near $l_i$'s normal to the normal vectors of $I_i$s. Then we cut off the neighborhood of $l_i$ in $\Sigma$ by $P_i$ and replace the lost part with the portion in $P_i$. We triangulate the portion and perturb the vertex inward to obtain a surface whose inverse image in $\tilde{M}$ is in a general position. This introduces squares which are triangulated into pairs of triangles.

This forms a generalization of a small truncation move. We still call it a small truncation move. –(*)

If there are outer level $(n - 1)$-crescents with outer-skin points, then we remove the outer-skin points by small truncation-moves at convex vertices. –(**)
We now add finite leaves of infinite length in the pleated part so that the components of the complement of the union of the pleating locus and these leaves are all open triangles. This can be done even though the boundary of the pleated part is not geodesic.

By choosing sufficiently small $\epsilon$-thin-neighborhoods of the union for $\epsilon > 0$, we obtain a train track. We assume that the circle component contains a unique closed geodesic.

We first choose switches for the endpoint of the finite length geodesics in the squares and added infinite length finite leaves. We choose switches for the rest. We label them $I_1, \ldots, I_n$. The train track collapses to a union of graphs with vertices corresponding to $I_1, \ldots, I_n$ and closed geodesics. The complement is a disjoint union of open triangles.

By choosing $\epsilon > 0$ sufficiently small, we can assume that the outer-normal vectors to totally geodesic hypersurfaces near $I_i$ are $\delta$-close for a small $\delta > 0$ except the outer-normal vectors to the totally geodesic hypersurfaces corresponding to the complementary regions of the train track. (Here the outer-normal is in the concave directions.)

We choose one of the normal vectors and a point $x_i$ on it $\gamma$-close to $I_i$, where $\gamma$ is a small positive number. We push all the points of $I_i$ to $x_i$ to obtain a train track $\tau_{\epsilon, \delta, \gamma}$ and the complementary regions move accordingly to geodesic triangles with edges in the train track $\tau_{\epsilon, \delta, \gamma}$.

We claim that then the triangles are very close to the original triangles in their normal directions as well as in the Hausdorff distance since the edge lengths of the triangles are bounded below. Since there are no rapid turning of the complementary geodesic triangles, we can be assured that the new surface is imbedded by integrations. This completes the proof of (i).

The vertices corresponding to $I_1, \ldots, I_n$ in the pleated open surfaces are s-vertices: The leaves of the laminations are moved in the normal direction which is a concave direction. Thus the leaf is bent toward the concave direction. The triangles are of almost the same direction as before. Lemma 2.3 implies the result. This completes the proof of (iii) and (iv).

(i) This matters for crescents that are inner if the perturbations are inner and ones that are outer if the perturbations are outer: In other cases, Lemma 4.1 shows that reversing the perturbation process gives us back all of our old crescents preserving the $I$-part hypersurface.

Suppose that we have the perturbed sequence $\Sigma'_i$ closer and closer to $\Sigma$, and there exists a sequence of crescent $\mathcal{R}_i$ for $\Sigma'_i$ not contained in a certain neighborhood of the union of crescents for $\Sigma$. Then the limit $\mathcal{R}$ of $\mathcal{R}_i$ is still a crescent for $\Sigma$ and is not in the neighborhood. This is absurd.

□

The move above described in Theorem 5.6 is said to be the convex perturbation.

**Corollary 5.7.**

- After the convex perturbation, the outer level of the resulting surface $\Sigma'$ is less or equal to the level of $\Sigma$ if we moved carefully, i.e., we do the small truncation moves first and then perturb vertices by sufficiently small amounts.

- The inner level stays the same or become smaller than that of $\Sigma$. 

24
• In particular, if \( \Sigma \) has no crescent, then \( \Sigma' \) contains no crescent and hence \( \Sigma' \) is s-imbedded triangulated surface. Moreover, s-vertices that were at the boundary of the pleated part become strict s-vertices.

Proof. Let \( n \) be the level of \( \Sigma \). After the small truncation moves at (*) and (**), we obtain a surface \( \tilde{\Sigma}'' \) with level less than or equal to \( n \). Also, the nature of truncation shows us that the union of crescents is the subset of that of \( \tilde{\Sigma} \). These all follow as in the proof of Proposition 4.2.

In the pleated part, we define the minimal pleated part as the convex hull of the union of bi-infinite pleating geodesics with respect to the intrinsic metric obtained by piecing the pleated parts together. The minimal pleated part is a subsurface of the pleated part which is the closure of the union of totally geodesic subsurfaces bounded by the bi-infinite pleating geodesics.

The minimal pleated part may meet the crescents only tangentially at the \( I \)-part. If not, then the minimal pleated part must pass through the \( I \)-part and this implies that a bi-infinite pleating geodesic pass through the \( I \)-part by the above paragraph. By Corollary 3.11 (iii), this is a contradiction.

We now move vertices of the train tracks of the pleating laminations by a very small amount according to Theorem 5.6. The crescents do move in its \( \epsilon \)-neighborhood.

Suppose now \( n \geq 0 \). We choose the \( \epsilon \)-neighborhood sufficiently small so that any new component of \( \tilde{\Sigma} \) intersected with crescents may not arise as \( \tilde{\Sigma} \) is deformed: First, the small truncated places may be avoided by taking \( \epsilon \)-sufficiently small. Second, since any crescent during the perturbation cannot meet the minimal pleated part in its interior, the \( I \)-part of crescents and crescents themselves close to the minimal pleated part lie below the minimal pleated part or may meet the minimal pleated part but cannot pass through it by Corollary 3.11. Since the perturbation is in the concave direction, our surface moved away from the crescents, and outer crescents cannot achieve level \( n + 1 \) or higher level.

Suppose that the inner level is \( \geq 0 \). Since our move is outward we can push the \( I \)-parts of the inner crescents inward toward themselves and these are all the inner crescents obtainable as before by Proposition 4.2. Therefore, the level can only decrease or stay the same as after the crescent isotopy, i.e., before the convex perturbation.

Suppose that the inner level was \(-1\), i.e., there were no inner crescents. If \( \Sigma' \) has an inner crescent, then since \( \Sigma \) is obtainable by inner direction move from \( \Sigma' \) by reversing our isotopy, we see that the inner crescents for \( \Sigma \) exists by preserving the I-parts. This is a contradiction. Thus, the inner level of \( \Sigma' \) is \(-1\).

Suppose now \( n = -1 \). Then \( \Sigma \) has no concave vertex. Thus the vertices at the boundary of the pleated part are all s-vertices. We move these so that they become strict s-vertices: The boundary vertex lies on a simple closed curve in the boundary of the pleated part. Since there are no convex vertices, the curve is a concavely curved one with respect to the pleated part. At each vertex there exists a straight line through it in the pleated part. The holonomy of the curve preserves a plane. We can push these vertices up in a parallel manner. Then the planes adjacent to the edges of the curve now begin to meet in angles \( < \pi \). Thus, the straight lines get bent and the angles. By a small perturbations, all moved vertices become strict s-vertices by Lemma 2.2.
Therefore, all vertices of $\Sigma'$ are strictly s-vertices or convex vertices. Thus the outer level of $\Sigma'$ is $-1$.

Figure 3. Making the boundary vertices of the pleated part into s-vertices.

This implies that there are no outer crescents since any crescents has an affine function $0$ on the $I$-level and hence some extreme points under this function in the $\alpha$-parts which gives us a concave vertex.

If the inner level of $\Sigma$ is $-1$ also, then we see that $\Sigma'$ has no outer or inner crescents.

\[\square\]

6. Isotopy sequences

In this section, we describe the process of our modification of $\Sigma$ of highest-level $n$:

(i) the small truncation moves for outer highest-level crescents.

(ii) the crescent isotopy for outer highest-level crescents.

(iii) the convex perturbations which involves small truncation moves.

We do the same steps for the inner highest-level crescents, i.e., at the level $n$. We do this until there are no crescents, i.e., to level 0 inner and outer crescent moves and convex perturbations. The result is an isotopied $\tilde{\Sigma}$ with only s-vertices as there are no crescents since a crescent must have a convex or concave vertex by taking an extreme point for an affine function constant on the $I$-part. Finally, we prove our main theorem.

Also, the manifold bounded by the perturbed $\Sigma'$ still contains $c_1, \ldots, c_n$ nearby.

**Theorem 6.1.** Let $\Sigma$ have the highest-level equal to $n$. Let $\Sigma'$ be obtained from $\Sigma$ by

1. small truncation moves for the level-$n$ outer crescents,
2. crescent-isotopy for level-$n$ outer crescents, and
3. convex perturbations.

Then $\Sigma'$ is a triangulated surface:

(i) the outer level of $\Sigma'$ is strictly less than that of $\tilde{\Sigma}$. 

26
(ii) In particular, if the outer level of \( \tilde{\Sigma} \) was 0, then \( \tilde{\Sigma}' \) has no outer crescents.

(iii) The inner level of \( \tilde{\Sigma}' \) is less than or equal to that of \( \tilde{\Sigma} \).

The statements also hold if every word “outer” were replaced by “inner” and vice versa.

**Proof.** (i) This was Corollary 5.7.

(ii) follows from (i).

(iii) Given an inner crescent \( R \) for the perturbed \( \tilde{\Sigma}' \), we can isotopy its \( \alpha \)-parts to obtain an inner crescent for \( \tilde{\Sigma} \) by Lemma 4.1 since the reverse movement is inward while so is \( R \).

For \( R \) to decrease its folding number, while \( \tilde{\Sigma}' \) is isotopied back to \( \tilde{\Sigma} \), we need a convex vertex leaving \( R \) through \( I_R \). However, all of the above process reversed does not do this kind of movement.

The final statement is obvious. \( \square \)

Let \( n \) be the highest level for \( \tilde{\Sigma} \). Once, we do the outer highest-level crescent moves of level \( n \), then we do the inner highest-level crescent moves of level \( n \). By above Theorem 6.1, we see that the inner and outer level strictly decreases until there are no more crescents.

Moreover the union of the set of crescents is contained in the \( \epsilon \)-neighborhood of the union of the set of crescents in the previous step.

**Proposition 6.2.** During the moves, the closed geodesic in the inner (resp. outer) component of complement of \( \tilde{\Sigma} \) remains inside the \( \epsilon \)-neighborhood of the closure of the inner (resp. outer) component of \( \tilde{\Sigma}' \) for arbitrarily small \( \epsilon > 0 \).

**Proof.** We use Corollary 3.11. Only problem occurs when the \( \partial_I \)-part contains a lift \( l \) of \( c_1 \). This implies \( l \subset I_R \) for a secondary highest-level crescent \( R \). Hence, \( l \) is homotopic into \( \Sigma \). Moreover, the holonomy \( g \) along the closed geodesic has no rotation component since otherwise

\[
\bigcup_{i \in \mathbb{Z}} g^i(R)
\]

covers around \( l \) to produce an annulus which compactifies to a sphere in \( \mathbb{H}^3 \cup S^2_\infty \). This contradicts the assumption that \( \Sigma \) is not homeomorphic to a torus.

After the crescent isotopy, \( l \) is contained in a pleated-triangulated surface \( \tilde{\Sigma}' \). We choose the convex perturbation that pushes \( \Sigma' \) off \( l \) by a small amount less than \( \epsilon \). We find an arc \( \alpha \) in pushed off \( \Sigma'' \) near \( l \) with the same endpoints as \( l \) and a neighborhood of \( \alpha \) in \( \Sigma'' \) has \( s \)-vertices only. Since the holonomy has no rotation component, we may assume that \( \alpha \) is a convex arc nearly on a totally geodesic hypersurface \( P \) passing through \( l \). Any crescent during the isotopies and moves that come later, it will either contain \( l \) in its \( I \)-part or be a distance away. Draw a thin strip \( A \) with boundary \( l \cup \alpha \). Then any innermost highest-level crescent \( S \) meeting \( \alpha \) to wards the direction of \( l \) will meet \( A \) but not contain any part of \( l \) in its interior. Hence \( \Lambda(S) \)-isotopy will not move \( \Sigma'' \) away from \( l \).

If \( S \) is toward the opposite direction to \( l \), \( \alpha_S \) cannot meet \( \alpha \) due to the local nature of \( \alpha \): The totally geodesic hypersurface \( P'' \) containing \( I_S \) meet \( \alpha \) union its endpoints at least two points while the part of \( \alpha \) in between must be in \( \alpha_S \). We see that \( P \cap P' \)
contains a geodesic $l'$ very close to $l$. This means $I_S$ contains $l'$. Since $l'$ is in the side of $l$, this is a contradiction.

By induction, we can have that the end result surface $\Sigma$ is as close to $l$ as we want. □

Furthermore, the final perturbation gives us a triangulated surface isotopic to $\Sigma$. During the isotopy, we see that the support of the isotopy is included in the union of crescents.

Applying the process in the beginning of the section, we can isotopy the boundary of $M$ to an $s$-surface. This follows since if there were a convex or concave vertex of $\partial M$ after all the moves, we find a crescent of level 0. This completes the proof of Theorem C.

Here, $M'$ is isotopic to $M$ of course.

Remark 6.3. The vertices of the boundary of $M'$ are strict $s$-vertices: We start from general position $\partial M$ so that each $s$-vertex is a strict $s$-vertex (see Theorem 5.6). Newly created vertices in the interior of the pleated part after the crescent isotopies are all strict $s$-vertices. The boundary vertices of the pleated part at level 0 are perturbed to be strict $s$-vertices (see Corollary 5.7). Finally, we no longer have any convex or concave vertex left at $M'$.

Remark 6.4. In fact, we can even assume that $c_i$ are geodesic laminations since we can modify Proposition 6.2 using Proposition 3.8.

Part 2. General hyperbolic 3-manifolds and convex hulls of their cores

7. Introduction to Part 2

The purpose of this paper is to show the significance of 2-convexity.

A hyperbolic triangle is a subset of a metric space isometric with a hyperbolic triangle in the hyperbolic plane $H^2$. If the ambient space is a 3-dimensional metric space, then we require it to be totally geodesic as well and develop into a totally geodesic plane in the hyperbolic space $H^3$.

A general hyperbolic manifold $M$ is a metric space with a locally isometric immersion $dev: \tilde{M} \to H^3$ from its universal cover $\tilde{M}$. $dev$ has an associated homomorphism $h: \pi_1(M) \to \text{Isom}(H^3)$ given by $dev \circ \gamma = h(\gamma) \circ dev$ for each deck transformation $\gamma \in \pi_1(M)$. We require that the boundary of $M$ triangulated by totally geodesic hyperbolic triangles.

A 2-convex general hyperbolic manifold is a general hyperbolic manifold so that a isometric imbedding from $T^o \cup F_1 \cup F_2 \cup F_3$ where $T^o$ is a hyperbolic tetrahedron in the standard hyperbolic space $H^3$ and $F_1, F_2, F_3$ its faces extend to an imbedding from $T$.

Theorem 7.1. The universal cover of a 2-convex general hyperbolic manifold is Gromov hyperbolic.

An $h$-map from a triangulated hyperbolic surface is a map sending hyperbolic triangles to hyperbolic triangles and the sum of angles of image triangles around a vertex is greater than or equal to $2\pi$. 

Theorem 7.2. Let $\Sigma$ be a compact h-mapped surface relative to $v_1, \ldots, v_n$ into a general hyperbolic manifold and suppose that each arc in $\partial \Sigma - \{v_1, \ldots, v_n\}$ is geodesic. Let $\theta_i$ be the exterior angle of $v_i$ with respect to geodesics in the boundary of $\Sigma$. Then

$$\text{Area}(\Sigma) \leq \sum_i \theta_i - 2\pi \chi(\Sigma).$$

The convex hull of a homotopy-equivalent closed subset of a general hyperbolic manifold $M$ is the image in $M$ of the smallest closed convex subset of containing the inverse image of the closed subset in the universal cover $\tilde{M}$ of $M$. We can always choose the core $C$ to be a subset of $M^\circ$.

Theorem 7.3. Let $\text{convh}(C)$ be the convex hull of the core $C$ of a 2-convex general hyperbolic manifold. We assume that $C$ is chosen to be a subset of $M^\circ$ and $\partial C$ is s-imbedded. Suppose that $\text{convh}(C)$ is compact. Then $\text{convh}(C)$ is homotopy equivalent to the core and the boundary is a truly pleated-triangulated h-surface.

In the preliminary, Section 8, we recall the definition of CAT-$\kappa$-spaces for $\kappa \in \mathbb{R}$ using geodesics and triangles. We also define $M_\kappa$-spaces, the simplicial metric spaces needed in this paper. We discuss the link conditions to check when $M_\kappa$-space is CAT($\kappa$)-space, the Cartan-Hadamard theorem, and Gromov boundaries of these spaces. Next, we discuss the 2-dimensional versions of these spaces. Define the interior angles, Gauss-Bonnet theorem. Finally, we discuss general hyperbolic manifolds.

In Section 9, we show that the universal cover of a 2-convex general hyperbolic manifold, which we used a lot in Part 1, is a $M_1$-simplicial metric space and a CAT(-1)-space and a visibility manifold. Next, we define h-maps of surfaces. These are similar to hyperbolic surfaces as defined by Bonahon, Canary, and Minsky. We define A-nets, a generalization of triangles, and $\alpha$-nets a simplicial approximation and a h-map to A-nets. We show that maps from surfaces can be homotopied to h-maps relative to a collection of boundary points in 2-convex general hyperbolic manifolds. We prove the Gauss-Bonnet theorems for such surfaces and find area bounds for polygons.

In Section 10, we discuss the convex hull of the core $C$ in a general hyperbolic manifold. First, we show that the convex hull and $C$ is homotopy equivalent. We finally show that the boundary of the convex hull can be deformed to a nearby h-imbedded surface, which is truly pleated-triangulated. We show this by finding geodesic in the boundary through each point of the boundary.

8. Hyperbolic metric spaces

8.1. Metric spaces, geodesics spaces, and cat(-1)-spaces. We will follow Bridson-Haefliger [4].

Let $(X, d)$ be a metric space. A geodesic path from a point $x$ to $y$, $x, y \in X$ is a map $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

A local geodesic is a map $c : I \to X$ from an interval $I$ with the property that for every $t \in I$ there exists $\epsilon > 0$ such that $d(c(t'), c(t'')) = |t' - t''|$ for $t', t''$ in the $\epsilon$-neighborhood of $t$ in $I$.

$(X, d)$ is a geodesic metric space if every pair of points of $X$ is joined by a geodesic.
We denote by $E^2$ the plan $\mathbb{R}^2$ with the standard Euclidean metric. A **comparison triangle** in $E^2$ of a triple of points $(p, q, r)$ in $X$ is a triangle in $E^2$ with vertices $\bar{p}, \bar{q}, \bar{r}$ such that $d(p, q) = d(\bar{p}, \bar{q}), d(q, r) = d(\bar{q}, \bar{r}),$ and $d(r, p) = d(\bar{r}, \bar{p})$. This is unique up to isometries of $E^2$.

The interior angle of the comparison triangle at $\bar{p}$ is called the comparison angle between $q$ and $r$ at $p$ and is denoted $\bar{Z}_p(q, r)$.

Let $c : [0, a] \to X$ and $c' : [0, b] \to X$ be two geodesics with $c(0) = c'(0)$. We define the upper angle $\angle_{c, c'} \in [0, \pi]$ between $c$ and $c'$ to be

$$\angle(c, c') := \limsup_{t, t' \to 0} \bar{Z}_{c(0)}(c(t), c'(t'))$$

The angle exists in strict sense if the limsup equals the limit.

The angles are always less than or equal to $\pi$ by our construction. We define angles greater than $\pi$ in two-dimensional spaces by specifying sides and dividing the side into many parts (see Subsection 32).

We say that a sequence of closed subsets $\{K_i\}$ converge to a closed subset $K$ if for any compact subset $A$ of $X$, $\{A \cap K_i\}$ converges to $A \cap K$ in Hausdorff sense.

Let $(X, d)$ be a metric. We can define a length-metric $\bar{d}$ so that $\bar{d}(x, y)$ for $x, y \in X$ is defined as the infimum of the lengths of all rectifiable curves joining $x$ and $y$. We note that $d \leq \bar{d}$ and $(X, d)$ is said to be a length space if $\bar{d} = d$.

**Proposition 8.1** (Hopf-Rinow Theorem). *Let $(X, d)$ be a length space. If $X$ is complete and locally compact, then every closed bounded subset of $X$ is compact and $X$ is a geodesic space.*

As an example, a Riemannian space with path-metric is a geodesic metric space. A covering space of a length space has an obvious induced length metric.

We define $M_\kappa$ to be the 3-sphere of constant curvature $\kappa$, Euclidean space, or the hyperbolic 3-space of constant curvature $\kappa$ depending on whether $\kappa > 0, = 0, < 0$ respectively.

Let $D_\kappa$ denote the diameter of $M_\kappa$ if $\kappa > 0$ and let $D_\kappa = \infty$ otherwise.

Let $(X, d)$ be a metric space. Let $\Delta$ be a geodesic triangle in $X$ with parameter less than $2D_\kappa$ and $\bar{\Delta}$ the comparison triangle in $M_\kappa$. Then $\Delta$ is said to satisfy CAT($\kappa$)-inequality if $d(x, y) \leq d(\bar{x}, \bar{y})$ for all $x, y$ in the edges of $\Delta$ and their comparison points $\bar{x}, \bar{y}$, i.e., of same distance from the vertices, in $\bar{\Delta}$. If $\kappa < 0$, a CAT($\kappa$)-space is a geodesic space all of whose triangles bounded by geodesics satisfy CAT($\kappa$)-inequality. If $\kappa > 0$, then $X$ is called a CAT($\kappa$)-space if $X$ is $D_\kappa$ geodesic and all geodesic triangles in $X$ of perimeter less than $2D_\kappa$ satisfy the CAT($\kappa$)-inequality.

Ages exist in the strict sense for CAT($\kappa$)-spaces if $\kappa \leq 0$.

A CAT($\kappa$)-space is a CAT($\kappa'$)-space if $\kappa \leq \kappa'$.

A CAT(0)-space $X$ has a metric $d : X \times X \to \mathbb{R}$ that is convex; i.e., given any two geodesics $c : [0, 1] \to X$ and $c' : [0, 1] \to X$ parameterized proportional to length, we have

$$d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1)).$$

A geodesic $n$-simplex in $M_\kappa$ is the convex hull of $n + 1$ points in general position.
An $M_\kappa$-simplicial complex $K$ is defined to be the quotient space of the disjoint union $X$ of a family of geodesic $n$-simplicies so that the projection $q : X \to K$ induces the injective projection $p_\lambda$ for each simplex $\lambda$ and if $p_\lambda(\lambda) \cap p_{\lambda'}(\lambda') \neq \emptyset$, there exists an isometry $h_{\lambda,\lambda'}$ from a face of $\lambda$ to $\lambda'$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

In this paper, we will restrict to the case when locally there are only finitely many simplicies, i.e., $X$ is locally convex. We do not assume that we have a finite isometry types of simplicies as Bridson does in [3].

A geodesic link of $x$ in $K$, denoted by $L(x, K)$ is the set of directions into the union of simplicies containing $x$. The metric on it is defined in terms of angles. (For details, see Chapter I.7 of [4].)

**Definition 8.2.** An $M_\kappa$-simplicial complex satisfies the link condition if for every vertex $v$ in $K$, the link complex $L(v, K)$ is a CAT(1)-space.

The following theorem can be found in Bridson-Haefliger [4]:

**Theorem 8.3** (Ballman). Let $K$ be a locally compact $M_\kappa$-simplicial complex. If $\kappa \leq 0$, then the following conditions are equivalent:

(i) $K$ is a CAT($\kappa$)-space.

(ii) $K$ is uniquely geodesic.

(iii) $K$ satisfies the link condition and contains no isometrically imbedded circle.

(iv) $K$ is simply connected and satisfies the link condition.

If $\kappa > 0$, then the following conditions are equivalent:

(v) $K$ is a CAT($\kappa$)-space.

(vi) $K$ is $\pi/\sqrt{\kappa}$-uniquely geodesic.

(vii) $K$ satisfies the link condition and contains no isometrically embedded circles of length less than $2\pi/\sqrt{\kappa}$.

**Proof.** See Ballmann [1] or Bridson-Haefliger [4].

**Lemma 8.4.** A 2-dimensional $M_\kappa$-complex $K$ satisfies the link condition if and only if for each vertex $v \in K$, every injective loop in $Lk(v, K)$ has length at least $2\pi$.

**Definition 8.5.** A metric space $X$ is said to be of curvature $\leq \kappa$ if it is locally isometric to a CAT($\kappa$)-space, i.e., for each point $x$ of $X$, there exists a ball which is a CAT($\kappa$)-space.

**Theorem 8.6** (Cartan-Hadamard). Let $X$ be a complete metric space.

(i) If the metric on $X$ is locally convex, then the induced length metric on the universal cover $\tilde{X}$ is globally convex. (In particular, there is a unique geodesic connecting two points of $\tilde{X}$, and geodesic segments in $\tilde{X}$ vary continuously with respect to their endpoints.)

(ii) If $X$ is of curvature $\leq \kappa$ where $\kappa \leq 0$, then $\tilde{X}$ is a CAT($\kappa$)-space.

Let $\delta$ be a positive real number. A geodesic triangle in a metric space $X$ is said to be $\delta$-slim if each of its sides is contained in the $\delta$-neighborhood of the union of the other two sides.

For $\kappa < 0$, CAT($\kappa$)-space is $\delta$-hyperbolic.
For positive real numbers \( \lambda, \epsilon, (\lambda, \epsilon) \)-quasi-geodesic in \( X \) is a map \( c : I \to X \) such that
\[
\frac{1}{\lambda}|t - t'| - \epsilon \leq d(c(t), c(t')) \leq \lambda|t - t'| + \epsilon
\]

Let \( X \) be a \( \delta \)-hyperbolic space. Two quasi-geodesic rays \( c, c' \) are equivalent or asymptotic if their Hausdorff distance is finite, or, equivalently \( \sup d(c(t), c'(t)) \) is finite. We define the Gromov boundary \( \partial X \) as the space of equivalence classes of quasi-geodesic rays in \( X \). One can show that \( \partial X \) is the space of equivalence classes of geodesics rays as well.

If \( X \) is a proper metric space, then \( X \) is a visibility space: For each pair of points \( x \) and \( y \) in \( \partial X \), there exists a geodesic limiting to \( x \) and \( y \). Topology and metrics are given on \( \partial X \) to compactify \( X \cup \partial X \). The group of isometry acts as homeomorphisms on \( \partial X \).

8.2. Singular hyperbolic surfaces. A hyperbolic triangle in a metric space is a subset isometric to a triangle in \( \mathbb{H}^2 \) bounded by geodesics. Sometimes, we need a degenerate hyperbolic triangle. It is defined to be a straight geodesic segment or a point where the vertices are defined to be the two endpoints and a point, which may coincide.

A hyperbolic tetrahedron in a metric space is a subset isometric to a tetrahedron in \( \mathbb{H}^3 \) bounded by four totally geodesic planes with six edges geodesic segments and four vertices. Again degenerate ones can obviously be defined on a hyperbolic triangle, a segment, and a point with various vertex and edge structures.

A hyperbolic cone-neighborhood of a point \( x \) in a surface \( \Sigma \) with a metric is a neighborhood of \( x \) which divides into hyperbolic triangles with vertices at \( x \). The cone-angle is the sum of angles of the triangles at \( x \). The set of singular points is denoted by \( \text{sing}(\Sigma) \) and the cone-angle at \( x \) by \( \theta(x) \).

By a singular hyperbolic surface, we mean a complete metric space \( X \) locally isomorphic to a hyperbolic plane or a hyperbolic cone-neighborhood with cone-angle \( \geq 2\pi \), so that the set of singular points are discrete. We will also require that \( X \) is triangulated by hyperbolic triangles in this paper (i.e., is a metric simplicial complex in the terminology of [4]).

By Lemma 8.4, the universal cover \( \tilde{X} \) of \( X \) is a CAT(\(-1\))-space.

Let \( X \) be a singular hyperbolic surface. Clearly, \( X \) has an induced length metric and is a geodesic space.

We say that a geodesic is straight if it is a continuation of geodesics in hyperbolic triangles meeting each other at \( \pi \)-angles in the intrinsic sense.

We can measure angles greater than \( \pi \) in singular hyperbolic surface by dividing the angle into smaller ones. In this case, we need to specify which side you are working on.

In general a path is geodesic if it is a continuation of straight geodesic meeting each other at greater than or equal to \( \pi \)-angles from both sides.

We also say that a boundary point \( x \) is bent if the two straight geodesics end at the point not at \( \pi \)-angle in the interior. We define \( \theta(x) \) to be \( \pi \) minus the interior angle. It could be negative. We will denote by \( \text{sing}(\partial X) \) the set of bending points.
Proposition 8.7 (Gauss-Bonnet Theorem). Let $\Sigma$ be a compact singular hyperbolic surface with piecewise straight geodesic boundary. Then

$$-\text{Area}(\Sigma) + \sum_{v \in \text{sing}(\Sigma)} (2\pi - \theta(v)) + \sum_{v \in \text{sing}(\partial \Sigma)} \theta(v) = 2\pi \chi(\Sigma).$$

From the Gauss-Bonnet theorem, we can show that there exists no disk bounded by two geodesics. This follows since if such a disk exists, then $\theta(v) \geq 2\pi$ for all singular, the exterior angles at virtual vertices $\leq 0$, the exterior angles at common end points $< \pi$, and the area is less than 0.

This implies: Given a compact singular hyperbolic space and a closed curve, we can homotopy the curve into a closed geodesic, and the closed geodesic is unique in its homotopy class.

Moreover, two closed geodesics meet in a minimal number of times up to arbitrarily small perturbations: that is, the minimum of geometric intersection number under small perturbation is the true minimum under all perturbations. (We may have two geodesics agreeing on an interval and diverging afterward unlike the hyperbolic plane.)

8.3. General hyperbolic 3-manifolds. By a general hyperbolic manifold, we mean a manifold $M$ with an atlas of charts to $\mathbb{H}^3$ with transition maps in $\text{Isom}(\mathbb{H}^3)$. The metric on it will be the length metric given by the induced Riemannian metric. We require the metric to be complete. As a consequence, this is a geodesic space by local compactness [4]. In general we assume that $\partial M$ is not empty. If it is not geodesically complete, $M$ need not be a quotient of $\mathbb{H}^3$ which are the usual subject of the study in 3-manifold theory.

Also, we will require general hyperbolic manifolds to have hyperbolic triangulations, i.e., it has a triangulation so that each tetrahedron is isometric with a hyperbolic tetrahedron in $\mathbb{H}^3$. Moreover, we assume that the vertices of the triangulations are discrete and the induced triangulation on the universal cover map under $\text{dev}$ to a collection of tetrahedra in general position in $\mathbb{H}^3$. We also require the following mild condition: Every boundary point of a general hyperbolic manifold has a neighborhood isometric with a subspace of a metric-ball in $\mathbb{H}^3$. By subdivisions and small modifications, we can always achieve this condition.

We will say that $M$ is locally convex if there is an atlas of charts where chart images are convex subsets of $\mathbb{H}^3$. Thus, $M$ is locally convex if $\partial M$ is empty. (In this paper, we will be interested in the non-locally-convex manifolds.)

Given a general hyperbolic manifold $M$, its universal cover $\tilde{M}$ has an immersion $\text{dev} : \tilde{M} \to \mathbb{H}^3$, which is not in general an imbedding or a covering map, and a homomorphism $h$ from the deck transformation group $\pi_1(M)$ to $\text{Isom}(\mathbb{H}^3)$ satisfying

$$\text{dev} \circ \vartheta = h(\vartheta) \circ \text{dev}, \vartheta \in \pi_1(M).$$

dev is said to be a developing map and $h$ a holonomy homomorphism.

Theorem 8.8 (Thurston). Let $M$ be a metrically complete 3-manifold and is locally convex. Then its developing map $\text{dev}$ is an imbedding onto a convex domain, and $M$ is isometric with a quotient of a convex domain in $\mathbb{H}^3$ by an action of a Kleinian group.

Proof. See [15].
In this paper, we will often meet *drilled hyperbolic manifolds* obtained by removing the interior of a codimension-zero submanifold of a general hyperbolic manifolds. They are of course general hyperbolic manifolds.

Of course, special hyperbolic manifolds are general hyperbolic manifolds and drilled hyperbolic manifolds.

Since a general hyperbolic manifold has a geodesic metric, we can define geodesics. A *straight* geodesic is a geodesic which maps to geodesic in $\mathbb{H}^3$ under the charts. Geodesics are in general a union of straight geodesics. Thus, it has many bent points in general. The bent points in the interior of the geodesic segments are said to be *virtual vertices*.

We define angles as above for metric spaces. Then at a bent virtual vertex, the angle is equal to $\pi$ since if not, then we can shorten the geodesics.

**Proposition 8.9.** Let $l$ be a geodesic with a bent point $x$ in its interior. Let $S$ be a simplicially immersed surface containing $l$ in its boundary. Give $S$ an induced length metric. Then the interior angle at $x$ in $S$ is always greater than or equal to $\pi$.

**Proof.** If the angle is less than $\pi$, we can shorten the geodesic. \(\Box\)

Given an oriented geodesic $l_1$ ending at a point $x$ and an oriented geodesic $l_2$ starting from $x$, we can define an *exterior angle* between $l_1$ and $l_2$ to be $\pi$ minus the angle between the geodesic $l'_1$ with reversed orientation and the other one $l_2$.

9. 2-convex general hyperbolic manifolds and h-maps of surfaces

**9.1. 2-convexity and general hyperbolic manifolds.** In Part 1, we showed that a general hyperbolic manifold was 2-convex if the vertices of the boundary were either $s$-vertices or convex vertices.

We recall the definition of 2-convexity:

**Definition 9.1.** A general hyperbolic manifold is 2-*convex* if given a compact subset $K$ mapping to a union of three sides of a tetrahedron $T$ in $\mathbb{H}^3$ under a chart $\phi$ of atlas, there exists a subset $T'$ mapping to $T$ by a chart extending $\phi$.

**Proposition 9.2.** If $M$ is a 2-convex general hyperbolic manifold, then $M$ is a $K(\pi_1(M))$, i.e., its universal cover is contractible.

**Proof.** Since the universal cover $\tilde{M}$ has an affine structure with trivial holonomy induced from the affine space containing $\mathbb{H}^3$ from the Klein model, this follows from [9]. Also, this follows from Theorem 9.3. \(\Box\)

**Theorem 9.3.** Let $M$ be a 2-convex general hyperbolic manifold. Then its universal cover $\tilde{M}$ is a $M_{-1}$-simplicial complex and a CAT$(-1)$-space. ($M$ has a curvature $\leq -1$.)

**Proof.** Using Theorem 8.3 (iv), we need to show that for each vertex $x$ of $\tilde{M}$, the link $P = L(x, \tilde{M})$ is a CAT$(1)$-space.

To show $P$ is a CAT$(1)$-space, we use (vii) of Theorem 8.3, i.e., we show that $P$ satisfies the link condition and contains no isometric circle of length $< 2\pi$. By the boundary condition on $M$, $P$ is isometric to the unit sphere if $x$ is the interior point.
or is isometric to a subspace of the unit sphere if $x$ is the boundary point. Clearly the former satisfy 2-dimensional link conditions.

Let $P$ be a proper subspace of a unit sphere $S^2$ and $c$ an isometrically imbedded circle of length $< 2\pi$. By Lemma 2.4, $c$ is disjoint from a closed hemisphere $H$ in $S^2$.

The closed curve $c$ meets $\partial P$ since otherwise $c$ has to be a great circle of length $2\pi$ being a geodesic. However, $c$ may never cross the circle $\partial P$ over. Let $D_c^1$ and $D_c^2$ denote the disks in $S^2$ bounded by $c$. Then $\partial P$ is a subset of $D_c^1$ or $D_c^2$. Assume without loss of generality that the former is true. Suppose that $H$ is a subset of $D_c^2$. Then $H \subset P$. Looking this from $x$, we see that 2-convexity is violated.

Suppose that $H$ is a subset of $D_c^1$. Let $H'$ be the complementary open hemisphere. Then $c \subset H'$ and $\partial P$ is a outside the disk $D_c^2$ in $H'$ bounded by $c$. Since $H'$ has a natural affine structure, let $K$ be the convex hull of $c$ in $K$. Let $y$ be an extreme point of $K$. Then $y \in c$ as well. Near $y$ inside $c$ there are no points of $\partial P$. Thus, we can shorten $c$ contrary to the fact that $c$ is a geodesic.

Lemma 9.4. Let $\gamma$ be a broken geodesic loop in the sphere $S^2$ of radius 1. If the length of $\gamma$ is less than $2\pi$, then there exists an open hemisphere containing it (and hence a disjoint closed hemisphere).

Proof. We can shorten the loop without increasing the number of broken points to a loop as short as we want. A sufficiently short loop is contained in an open hemisphere.

Let $l_t, t \in [0,1]$ be a homotopy so that $l_t$ is the original loop and $l_0$ is a constant loop. Then let $A$ be the maximal connected set containing 0 so that $l_t$ for $t \in A$ is contained in an open hemisphere, say $H_t$.

The set $A$ is an open set since the small change in $l_t$ does not violate the condition. Suppose that the complement of $A$ is not empty. Let $t_0$ be the greatest lower bound of the complement of $A$. Then $l_{t_0}$ is contained in a closed hemisphere, say $H'$, since we can find a geometric limit of the closure of $H_t$'s.

Suppose that $\partial H' \cap l_{t_0}$ is contained in a subset of length strictly less than $\pi$. Then we can rotate $H'$ along a pivoting antipodal pair of points on $\partial H'$ outside the subset. Then the new hemisphere contains $l_{t_0}$ in its interior, a contradiction.

Suppose that $\partial H' \cap l_{t_0}$ contains an antipodal points. Let $s_1$ and $s_2$ be the corresponding points of $[0,1]$ and suppose $0 < s_1 < s_2 < 1$ without loss of generality. Then two arcs $l_{t_0}|[s_1, s_2]$ and $l_{t_0}|[s_2, 1] \cup [0, s_1]$, must have length greater than or equal to $\pi$, a contradiction.

Therefore, no subsegment in $\delta H'$ of length $\leq \pi$ contains $\partial H' \cap l_{t_0}$, and we may assume without loss of generality that there are three points $p_1, p_2, p_3$ in $\delta H' \cap l_{t_0}$ are not contained in a subsegment in $\delta H'$ of length $\leq \pi$ and no pair of them are antipodal.

The sum of lengths of segments $\overline{p_1p_2}, \overline{p_2p_3}, \overline{p_3p_1}$ equals $2\pi$. This is clearly less than or equal to that of $l_{t_0}$ since the shortest arcs are these segments. This is again a contradiction.

Thus $A$ must be all of $[0,1]$.

The following proves Theorem 7.1 in detail.

Proposition 9.5. Let $\tilde{M}$ be a universal cover of a compact $2$-convex general hyperbolic manifold $M$. Then the following hold:
• $\tilde{M}$ is uniquely geodesic.
• Geodesic segments of $\tilde{M}$ depend continuously on their endpoints.
• The metric is locally convex.
• $\tilde{M}$ is $\delta$-hyperbolic and hence it is a visibility manifold.
• $M$ has curvature $\leq -1$.
• Given any path class on $M$, there exists a unique geodesic segment, which depends continuously on endpoints.

Proof. These are direct consequences of the fact that $\tilde{M}$ is a CAT($-1$)-space. \qed

9.2. $h$-maps of surfaces into 2-convex general hyperbolic manifolds. A triangulated hyperbolic surface is a metric surface with or without boundary triangulated and each triangle is isometric with a hyperbolic triangle or a degenerate hyperbolic triangle in $\mathbb{H}^2$. A half-space of $\mathbb{H}^3$ is a subset bounded by a totally geodesic plane.

Definition 9.6. Let $\Sigma$ be a compact triangulated hyperbolic surface, $M$ a general hyperbolic 3-manifold, and $f : \Sigma \to M$ a map which sends each triangle to a hyperbolic triangle in $M$. Let $\partial \Sigma$ have distinguished vertices $v_1, \ldots, v_n$. Then $f$ is an $h$-map relative to $\{v_1, \ldots, v_n\}$ if the sum of the angles of the image triangles of the stellar neighborhood of each interior vertex $v$ is greater than or equal to $2\pi$ and the sum of angles of the image triangles of the stellar neighborhood of the boundary vertex $v$, $v \neq v_i$, is greater than or equal to $\pi$.

An $h$-map is a completely analogous concept to a hyperbolic-map by Bonahon [2], Canary and Minsky and so on. Note that if the boundary portion between $v_i$ and $v_{i+1}$ is geodesic for each $i$, then the boundary angle conditions are satisfied also.

Definition 9.7. Given an arc or a point $\alpha$ and another arc $\beta$, an $\alpha$-net $f : I \times I \to M$ with ends $\alpha$ and $\beta$ is a map such that

• $s \mapsto f(t_i, s)$ for a finite subset $\{t_1 = 0, t_2, \ldots, t_n = 1\}$ of $I$ is a geodesic for each $i$,
• $t \mapsto f(t, 0)$ is $\alpha$ and $t \mapsto f(t, 1)$ is $\beta$.
• $f$ is an $h$-map relative to vertices of the arcs $\alpha$ and $\beta$ with a triangulation of $I \times I$ with all the vertices in $\{t_1, \ldots, t_n\} \times I$.

36
Proposition 9.9. Given a point or an arc $\alpha$ and another arc $\beta$, there exists an $\alpha$-net with ends $\alpha$ and $\beta$.

Proof. We find an $A$-net $f : I \times I \to M$ with ends $\alpha$ and $\beta$. We take sufficiently many $t_i$’s so that geodesics $s \mapsto f(t_i, s)$ are very close. By Lemma 9.8 we can find a simplicial map $F : I \times I \to M$. Since $s \mapsto F(t_i, s) = f(t_i, s)$ are geodesics, the sum of angles at each of the sides of a vertex on this geodesic is greater than $\pi$. Hence, the sum of angles at an interior vertex is greater than or equal to $2\pi$. At the vertices of $s \mapsto F(0, s)$ or $s \mapsto F(1, s)$, the sum of angles are greater than $\pi$. Therefore, $F$ is an $h$-map. \hfill $\square$

Proposition 9.10. Let $\Sigma$ be a compact triangulated surface, $M$ a general hyperbolic 3-manifold, and let $f : \Sigma \to M$ be a map with an injective induced homomorphism $f_* : \pi_1(\Sigma) \to \pi_1(M)$.

1. Let $v_1, \ldots, v_n$ be the distinguished vertices in $\partial \Sigma$ and $l$ be a union of disjoint simple closed curve in $\Sigma$ which is disjoint from $\{v_1, \ldots, v_n\}$ and is a component of $\partial \Sigma$ or is disjoint from $\partial \Sigma$.

2. We suppose that $\{v_1, \ldots, v_n\} \cup l$ is not empty. Suppose that $f$ maps each arc in $\partial \Sigma$ connecting two distinguished vertices to a geodesic and each component of $l$ or $\partial \Sigma$ without any of $v_1, \ldots, v_n$ to a closed geodesic.

Then in the relative homotopy class of $f$ with $f|\partial \Sigma$ fixed, there exists an $h$-map $f' : \Sigma' \to M$ relative to $v_1, \ldots, v_n$ where $\Sigma'$ is $\Sigma$ with a different triangulation in general and $f'$ agrees with $f$ on $\partial \Sigma \cup l$.

From now on, we will just use $v_i$ for $f(v_i)$ and so on since the reader can easily recognize the difference. By the angle of a triangle, we mean the corresponding angle measured in the image triangle of $f$.

Proof. First, we find a topological triangulation $\Sigma$ so that all the vertices are in the union of $\{v_1, \ldots, v_n\} \cup l \cup \partial \Sigma$. We find geodesics in the right path-class for each of the edges of the triangulations. For each triangle, we extend by choosing a vertex and the opposite geodesic edges and finding $\alpha$-nets with these ends.

At each interior point of an edge, we see that the sum of angles of any of its side is greater than or equal to $\pi$ since the edge is geodesic. Since $\alpha$-nets are $h$-maps, we see that the whole map is an $h$-map. \hfill $\square$

9.3. Gauss-Bonnet theorem for $h$-maps.

Proposition 9.11. Let $\Sigma$ be a compact $h$-mapped surface relative to $v_1, \ldots, v_n$. Let $\theta_i$ be the exterior angle of $v_i$ with respect to geodesics in the boundary of $\Sigma$. Then

$$\text{Area}(\Sigma) \leq \sum_i \theta_i - 2\pi \chi(\Sigma).$$

Proof. The interior angle with respect to $\Sigma$ is larger than the angle in $M$ itself. Thus the exterior angle with respect to $\Sigma$ is smaller than the exterior angle in $M$.

Since the interior vertices have the angle sums greater than or equal to $2\pi$ and the boundary virtual vertices have the angle sum greater than or equal to $\pi$, the proposition follows. \hfill $\square$
An \( n \)-gon is a disk with boundary a union of geodesic segments between \( n \) vertices.

**Corollary 9.12.** Let \( S \) be an \( h \)-mapped \( n \)-gon. Then \( \text{Area}(S) \leq (n - 2)\pi \).

**Proof.** The exterior angle of a bent virtual vertex on a geodesic is always less than \( \pi \). \( \square \)

**Corollary 9.13.** Two asymptotic rays \( l \) and \( l' \) in \( M \) or \( \tilde{M} \) satisfy \( d(l(t), l'(t)) \to 0 \) where \( t \) is a parameter affinely related to the length parameter.

**Proof.** Suppose not. Then there exists a sequence of pair of segments \((s_i, s'_i)\) meeting \( l \) and \( l' \) nearly at \( \pi/2 \) and the distance between \( s_i \) and \( s'_i \) goes to \( \infty \). Therefore, there is a sequence of \( h \)-mapped quadrilaterals \( D_i \) with sides \( s_i \) and \( s'_i \) and two segments \( t_i \) in \( l \) and \( t'_i \) and \( l' \). For a sufficiently large \( i \), there must be two points in \( t_i \) and \( t'_i \) of distance much less than the lengths of \( s_i \) and \( s'_i \) since the area of \( D_i \) is bounded above by \( 2\pi \) and the lengths of \( t_i \) and \( t'_i \) go to infinity.

Since \( d(l(t), l'(t)) \) is a convex function, this implies the corollary. \( \square \)

**10. Convex Hulls in 2-Convex General Hyperbolic Manifolds**

Let \( M \) be a 2-convex general hyperbolic manifold with finitely generated fundamental group. Let \( \mathcal{C} \) denote a core of \( M \).

Let \( \tilde{M} \) be the universal cover of \( M \). Since \( \mathcal{C} \to M \) is a homotopy equivalence the subset \( \tilde{\mathcal{C}} \) in \( \tilde{M} \) of the inverse image of \( \mathcal{C} \) is connected and is a universal cover of \( \mathcal{C} \). A subset of \( \tilde{M} \) is **convex** if any two points can be connected by a geodesic in the subset.

The **convex hull** \( \text{convh}(K) \) of a subset \( K \) of \( \tilde{M} \) is the smallest closed convex subset containing \( K \). Since \( \tilde{\mathcal{C}} \) is deck-transformation group invariant, and the convex hull is the smallest convex subset, \( \text{convh}(\tilde{\mathcal{C}}) \) is deck-transformation group invariant. Therefore, \( \text{convh}(\tilde{\mathcal{C}}) \) covers its image. We define the image as \( \text{convh}(\mathcal{C}) \), i.e., \( \text{convh}(\tilde{\mathcal{C}}) \) quotient by the deck-transformation group action.

Since \( \mathcal{C} \) is a 3-dimensional domain, \( \text{convh}(\mathcal{C}) \) is a 3-dimensional closed set.

**Proposition 10.1.** The convex hull \( \text{convh}(\mathcal{C}) \) of the compact core \( \mathcal{C} \) of \( M \) is homotopy equivalent to \( \mathcal{C} \).

**Proof.** Let \( \tilde{\mathcal{C}} \) be the inverse image of \( \mathcal{C} \) in the universal cover \( \tilde{M} \) of \( M \). Then \( \tilde{\mathcal{C}} \) and \( \tilde{M} \) are both contractible as \( M \) and \( \mathcal{C} \) are irreducible 3-manifolds.

A closed curve in the convex hull \( \text{convh}(\tilde{\mathcal{C}}) \) of \( \tilde{\mathcal{C}} \) bounds a disk since a distinguished point on the curve can be connected by a geodesic in any other point of the curve. Similarly, a sphere always bounds a 3-ball. Therefore, \( \text{convh}(\tilde{\mathcal{C}}) \) is contractible. \( \square \)

A surface is **pleated** if through each point of it passes a straight geodesic.

Recall that the pleated-triangulated surface is an imbedded surface where a closed subdomain is a union of a locally finite collection of totally geodesic convex disks meeting each other in geodesic segments and the complementary open surface is pleated.

A pleated-triangulated surface is **truly pleated-triangulated** if the triangulated part are union of totally geodesic triangles in general position.

A **truly pleated-triangulated h-surface** is a truly pleated-triangulated surface where each vertex of the triangles is an h-vertex.

38
Lemma 10.2. If a geodesic in $M$ contained in $S$ passes through a vertex in the triangulated part of $S$, then the vertex is an $h$-vertex.

Proof. A neighborhood of a point of the triangulated part is stellar. If a geodesic passes through, the angles in both sides are greater than or equal to $\pi$: otherwise, we can shorten the geodesic. Hence, the sum of the angles is greater than or equal to $2\pi$. □

Proposition 10.3. Let $K$ be a deck-transformation-group invariant codimension 0 submanifold of $\tilde{M}$ with $\partial K$ $s$-imbedded. Also, suppose $K$ is a subset of $\tilde{M}^\circ$. The boundary of $\text{convh}(K)$ can be given the structure of a convex truly pleated-triangulated $h$-surface.

Proof. We will show that through each point of $\partial\text{convh}(K)$ a geodesic in $\partial\text{convh}(K)$ passes or the point is in the triangulated part and is an $s$-vertex or $h$-vertex.

Let $x$ be a boundary point of $\text{convh}(K)$:

(a) Let $x$ be a point of the manifold-interior of $\tilde{M}$. Take a ball $B_\epsilon(x)$ in the interior for a sufficiently small $\epsilon$. Then $\text{convh}(K) \cap B_\epsilon$ is the convex hull of itself. Since $B_\epsilon$ is isometric with a small open subset of $\mathbb{H}^3$, the ordinary convex hull theory shows that there exists a geodesic in the boundary of the convex hull through $x$: If not, we can find a small half-open ball to decrease the convex hull as the side of the half-open ball cannot meet $\partial K$ by the $s$-imbeddedness of $\partial K$.

Suppose that $x$ is in the topological interior of $\text{convh}(K)$ but in $\partial \tilde{M}$. There exists a neighborhood of $x$ with manifold-boundary in $\partial \tilde{M}$. If $x$ is in the interior of an edge or a face of $\delta \tilde{M}$, then there is a geodesic through $x$ obviously. Suppose that $x$ is a vertex of $\delta \tilde{M}$. $x$ can be an $s$-vertex or a convex vertex (see Proposition 2.5).

If $x$ is a convex vertex of $\partial \tilde{M}$, we can find a truncating totally geodesic hyperplane and a sufficiently small disk in it bounding a neighborhood of $x$ in $\tilde{M}$. Since $K$ is disjoint from the disk, we see that $x$ is not in the convex hull. This is absurd.

If $x$ is an $s$-vertex of $\partial \tilde{M}$, then $x$ is an $s$-vertex of $\partial \text{convh}(K)$.

Assume from now on that $x$ is a point in the topological boundary of $\text{convh}(K)$. This means that $x$ is in the frontier of the open surface $C = \partial \text{convh}(K) - \partial \tilde{M}$.

(b) Suppose $x$ is a point of the interior of a triangle $T$ in $\partial \tilde{M}$. The set $T \cap \text{convh}(K)$ is a convex subset and $x$ lies in the boundary. The boundary must be a geodesic since we can use a small half-open ball to decrease the convex hull otherwise. Hence there is a geodesic through $x$.

(c) Suppose that $x$ is a point of the interior of an edge in $\partial \tilde{M}$. We take a small ball $B_\epsilon(x)$ around $x$, which is isometric with a ball in $\mathbb{H}^3$ of the same radius and a wedge removed. The line $l$ of the wedge passes through $x$.

Let $P_1$ and $P_2$ be the totally geodesic plane extended in $B_\epsilon(x)$ from the sides of the wedge. We denote by $P'_1$ the set $\partial B_\epsilon(x) \cap P_1$ and $P'_2$ the set $\partial B_\epsilon(x) \cap P_2$. We can form two convex subsets $L_1$ and $L_2$ in $B_\epsilon(x)$ that are the closures of the components of $B_\epsilon(x) - P_1 - P_2$ and adjacent to $P'_1$ and $P'_2$ respectively.

The set $L_1 \cap \text{convh}(K)$ is a convex subset of $L_1$ and $L_2 \cap \text{convh}(K)$ one of $L_2$. The open surface $C$ may intersect $L_1$ or $L_2$ or both.

If $C$ is disjoint from $L_1$, then it maybe that a one-sided neighborhood of $x$ in a triangle in $\partial \tilde{M}$ is a subset of $\text{convh}(K)$ and an edge of the triangle is a geodesic through $x$.  

39
(The side $P_1 - P_1^{\alpha}$ of $B_\ell(x)$ is in $\text{convh}(K)$.) Otherwise, $\text{convh}(K)$ is contained in a convex subset of $B_\ell(x)$ bounded by $P_1$. In this case, the ordinary convexity in $\mathbb{H}^3$ holds and there is a geodesic in $\partial\text{convh}(K)$ through $x$.

Since the same argument holds with $L_2$ as well, we assume that $C \cap L_1$ and $C \cap L_2$ are both not empty.

If $C \cap L_1$ is not totally geodesic for every sufficiently small $\epsilon > 0$, then there exists a sequence of points $\{x_i \in L_1 \cap \tilde{M}^o\}$ converging to $x$ and a sequence of infinitely many mutually distinct pleating lines

$$\{l_i \subset \partial\text{convh}(K) \cap \tilde{M}^o\}$$

so that $x_i \in l_i$.

Therefore, by Lemma 10.4 we only have to worry about the case when $l_i$'s end at $x$. In this case there exists a small neighborhood $B(x)$ of $x$ such that $C \cap L_1 \cap B$ is a cone-type set with vertex at $x$.

By a same argument, $C \cap L_2 \cap B$ is a cone-type set also with a vertex at $x$. In order that at $x$, the convexity to hold true, we see that $C \cap L_1 \cap B$ has to have a unique pleating geodesic and so does $C \cap L_2 \cap B$. They must extend each other as geodesics passing through $x$ in order that the convex hull does not become less or small by truncation by local totally geodesic hypersurfaces.

(d) Now assume that $x$ in $\partial\text{convh}(K)$ is a vertex of $\partial\tilde{M}$.

Let $B_\ell(x)$ be a small neighborhood of $x$ so that $B_\ell(x) \cap \tilde{M}$ is a stellar set from $x$. As before $x$ is in the boundary of $C$.

Suppose first that there are no pleating lines with a sequence of points on them converging to $x$. We can choose a small $\epsilon$ so that $B_\ell(x) \cap \text{convh}(K)$ is a stellar set.

Let $\tilde{M}'$ be an ambient general hyperbolic manifold containing $\tilde{M}$ in its interior which is homeomorphic to the interior of $\tilde{M}$ as there are always such a manifold. We claim that $x$ is an $s$-vertex of $B_\ell(x) \cap \partial\text{convh}(K)$: If not, we can find a small half-open ball $B$ in $\tilde{M}'$ with a totally-geodesic side passing through $x$ with $B^o$ disjoint from $\partial\text{convh}(K)$. By stellarity, $B^o$ is disjoint from $\text{convh}(K)$ and we can decrease $\text{convh}(K)$, which is a contradiction. Therefore, $x$ is an $s$-vertex.

We assume that $B_\ell(x) \cap \text{convh}(K)$ is not a stellar set. Suppose now that there exists a sequence of points $\{x_i \in l_i\}$ converging to $x$ where $l_i$ is a distinct pleating line for each $i$ and does not end at $x$. Here, $l_i$ are infinitely many. Lemma 10.4 shows that the endpoints of $l_i$ are bounded away from $x$. A subsequence of $l_i$ converges to a geodesic $l$ passing through $x$.

We see that each point $x$ of $\partial\text{convh}(K)$ either has a stellar neighborhood with finite triangulations or has a pleated neighborhood from each of the above cases (a), (b), (c), and (d). We see that $\partial\text{convh}(K)$ is a truly pleated-triangulated surface: Let $A$ be the closure of the set of all points in $\partial\text{convh}(K)$ and in the interiors of triangles in $\partial\tilde{M}$. Then $A$ is a locally finite union of totally geodesic polygons and segments. The complement of $A$ in $\partial\tilde{M}$ is pleated since it lies in the interior of $\tilde{M}$. Any pleated lines in $\tilde{M}^o$ must end at $A$ or is infinite. By Proposition 5.3 the set of pleating lines ending at $A$ is isolated. As before, we see that a pleating line ending at a point of $A$ must be a finite length segment. Take a union of these finite length segments with $A$ to form
The infinite length pleating lines are in the finite union of minimal laminations bounded away from $A'$ by Proposition 5.3.

Since the minimal laminations are bounded away from $A'$, we obtain a truly pleated-triangulated surface.

By Lemma 10.2 it is an h-surface as well. The convexity is obvious. □

Lemma 10.4. Let $l_i, i \in I$, be a collection of mutually distinct straight pleating lines $\partial \text{convh}(K) - \partial \tilde{M}$ for a convex hull $\text{convh}(K)$ of a closed subset $K$ of $\tilde{M}$ and an index set $I$. Suppose $x_i \in l_i$ form a sequence converging to $x$ but $x$ is not on $l_i$s. Then the endpoints of $l_i$s are bounded away from $x$ and a subsequence of $l_i$ converges to a line segment in the pleating locus containing $x$ in its interior.

Proof. Suppose that the endpoints of $l_i$ are bounded away from $x$. Then the second statement holds obviously.

Suppose that the endpoints $q_i$ of $l_i$ form a sequence converging to $x$. Then we may assume without loss of generality that $q_i$ lies in an arc $\alpha$ a triangle in $\partial \tilde{M}$. If the arc $\alpha$ is a convex curve, we can decrease $\text{convh}(K)$ further, a contradiction. Thus $\alpha$ is a geodesic or a point.

If $l_i$ is concurrent at a point $y$ not equal to $x$, then the endpoints of $l_i$ cannot be at $x$ and the conclusion holds.

If all but finitely many $l_i$ is concurrent at $x$, then by convexity of $\text{convh}(K)$, it follows that an endpoint of $l_i$ is $x$. This contradicts the premise.

Suppose that no subsequence is concurrent. Suppose that two or more $l_i$s pass through $l$ at distinct points, say $l_{ij}$ for $j = 1, 2, 3$. Since $\text{convh}(K)$ is convex, we can find a supporting line $P_j$ containing $l_{ij}$. Since $P_j$ ends at a point of $l$ and an open arc in $l$ is a subset of $\text{convh}(K)$, it follows that $l$ is a subset of $P_j$. Since $P_j$ is supporting, an open domain bounded by $P_j$ must contain $l_{ij}$ and $l_{im}$ for $l, m \neq j$. However, since $P_j$s contain a common line, there is a $P_i$ which separates some a pair in $l_{ij}, l_i$, and $l_{im}$. This is a contradiction. □

This proves Theorem 7.3.

Part 3. The proof of the tameness of hyperbolic 3-manifolds

11. Outline of the proofs

The strategy is as follows. Suppose that the unique end $E$ of $M$ not associated with incompressible surface is not geometrically finite and is not tame. We find an exhausting sequence $M'_i$ in $M$ so that $M'_i$ contains neighborhoods of all tame and geometrically finite ends and meets the neighborhood of the infinite end in a compact subset.

Using the work of Freedman-Freedman [10], we can modify $M'_i$s to be compression bodies. Since $E$ is not geometrically finite, we can choose a sequence of closed geodesics $c_i \to \infty$. We fix a sufficiently small Margulis constant $\epsilon$. We assume without loss of generality that $c_i \subset M'_i$ and that the Margulis tubes that $M'_i$ meets are contained in $M'_i$. Let $\mu_i$ be the union of closed geodesics that are in the Margulis tubes in $M_i$. We further modify $M'_i$ so that $\partial M'_i$ is incompressible in $M - c_1 - \cdots - c_i - \mu_i$ with the
compact core removed. The new manifolds $M''_i$ may form an exhausting sequence but it includes $c_i \to \infty$.

The manifold $N_i$ is obtained from compressing disks for $M'_i$. Let $A_i$ be a homotopy in $\Sigma$ of $\alpha$ and even a quasi-geodesic. Since $\Sigma$ contains $C$ which is homotopy equivalent to $C$, we can modify $A_i$ so that $A_i$ has one less number of components where $A_i$ meets the compressing disk. In this manner, we can find $A_i$ in $N_i$.

Now we modify $N_i$ to $M''_i$ so that $c_i \subset M''_i$ and $M''_i$ are compression bodies and the boundary component of $M''_i$ corresponding to $E$ is incompressible in $M$ removed with the closed geodesics $c_1, \ldots, c_n$ and contains any Margulis tubes that $M''_i$ meets.

As the boundary is incompressible, we take a 2-convex hull $M_i$ of $M'_i$ using crescents (see Part 1). This implies that $M_i$ is a polyhedral hyperbolic space and hence is $\delta$-hyperbolic with respect to the induced path metric. Since $M_i$ is isotopic to $M''_i$, the homotopy $A_i$ between $c_i$ and a closed curve in $\Sigma$ still exists.

We show that we can choose a Margulis constant independent of $i$ and the thin part of $M_i$ are contained in the original Margulis tubes of $M$ and are homeomorphic to solid tori with nontrivial homotopy class in the original tubes.

We take the cover $L_i$ of $M_i$ corresponding to the fundamental group of the fixed compact core $\mathcal{C}$ of $M$. Since $M_i$ is tame, the cover $L_i$ is shown to be tame (this is from an idea of Agol).

The core $\mathcal{C}$ lifts to the cover $L_i$ and can be considered a subset. We take a convex hull $K_i$ of $\mathcal{C}$ in $L_i$. Then $K_i$ is shown to be compact since $L_i$ is tame and is homeomorphic to $\mathcal{C}$ if we remove parts outside a finite union of disks disjoint from $\mathcal{C}$. Since $K_i$ is homotopy equivalent to $\mathcal{C}$ by Proposition 10.1 in Part 2, the boundary component $\Sigma_i$ of $K_i$ corresponding to $E$ has the same genus as that of a boundary component of $\mathcal{C}$. $\Sigma_i$ is a “hyperbolic surface” (see Part 2). Since $c_i$ is exiting an $\epsilon$-neighborhood of $M_i$ contains $c_i$, it follows that $p_i|\Sigma_i$ is an exiting sequence of surfaces.

We push $\mathcal{C}$ inside itself so that $\mathcal{C}$ does not meet $\partial K_i$. We now remove the core from $K_i$ to obtain $K_i - \mathcal{C}^o$. We can find a simple closed curve $\alpha$ in $\Sigma_i$ compressible in $K_i$. We realize $\alpha$ by a closed geodesic $\alpha^*$ in $K_i - \mathcal{C}^o$. Since $\alpha$ cannot be realized by a straight geodesic or even a quasi-geodesic, $\alpha^*$ must meet $\partial \mathcal{C}$. Fixing a base point $x^*$ on $\alpha$, we choose relative geodesics on $\Sigma_i$ which triangulates $\Sigma_i$ with $\alpha$. Homotopy $x^*$ to a base point $y^*$ on $\alpha^*$ and geodesics to have endpoints in $y^*$. We obtain a simplicial hyperbolic surface $T_i$ meeting $\partial \mathcal{C}$.

Then by compactness of bounded simplicial hyperbolic surfaces, infinitely many immersed $T_i$s are isotopic in $M - \mathcal{C}^o$ and hence infinitely many of $p_i|\Sigma_i$ are isotopic. Since $p_i|\Sigma_i$ are exiting and are isotopic in $M - \mathcal{C}^o$, this implies that the end $E$ is topologically tame.

We remark that the most important new idea of Agol, originally from Freedman, is the use of tameness of the covering spaces of tame manifolds and taking convex hulls in the covers. These were the approaches that we adopted in this paper.
12. The Proof of Theorem A

We will now choose the compact core more carefully so that $\partial C$ is $s$-imbedded.

We can choose incompressible closed surfaces $F_i$ associated with incompressible ends $E_i$ to be strictly $s$-imbedded by Theorem C and disjoint from one another (see Remark 6.3). We choose a number of closed geodesics in $E_i$ and choose a mutually disjoint submanifold homeomorphic to $F_i \times I$ disjoint and between these curves for each $i$. Then by Theorem C, we find a mutually disjoint collection of manifolds in the respective neighborhoods of $E_i$ between these curves whose boundary components are strictly $s$-imbedded.

Essentially $\partial C$ can be considered a regular neighborhood of the union of the essential surfaces $F_1, \ldots, F_n$ and a number of arcs connecting them in some manner.

We choose each of the arcs to be the shortest path in $M$ among the arcs connecting the surfaces $F_i$s with the respectively given homotopy classes. Their endpoints must be in the interior of an edge of a triangle. By perturbing $F_i$ if necessary, we may assume that they are all disjoint geodesics. We first take thin regular neighborhoods of $F_i$s which are triangulated. We take thin regular neighborhoods of the geodesics which are triangulated and all of whose vertices lie in $F_i$s.

We take the union of the regular neighborhood of these geodesics with those of $F_i$s to be our core $C$. we may assume that $\partial C$ is strictly $s$-imbedded as well. (We may need to modify a bit where the neighborhoods meet.) As stated above, we choose $C$ to be in the interior $M^e$. Obviously, if necessary, we can push $C$ inward itself without violating strict $s$-imbeddedness of $\partial C$.

Let $M$ be as in the introduction, and let $U_1, \ldots, U_n$ be mutually disjoint neighborhoods of incompressible ends $E_1, \ldots, E_n$. Suppose that the end $E$ is a geometrically infinite but not geometrically tame.

Let $\hat{M}$ be the 2-convex hull of $M$ with $U_1, \ldots, U_n$ removed. The boundary components $F_1, \ldots, F_n$ corresponding to $U_1, \ldots, U_n$ of $\hat{M}$ are $s$-imbedded respectively.

Let $M'_i$ be an exhaustion of $M$ by compact submanifolds containing $\hat{M}$. We extend $M'_i$ by taking a union with $U_1, \ldots, U_n$ so that $M'_i$ meets neighborhoods of $E$ in compact subsets or in the empty set. We assume that $M'_i$ contains the boundary components $F_1, \ldots, F_n$ and contains the core $C$ of $M$ always.

We fix a small Margulis constant $\epsilon_M > 0$.

Proposition 12.1 (Freedman-Freedman, Ohshika). We can obtain a new exhaustion $M'_i$ so that each $M'_i$ is homeomorphic to a compression-body with incompressible boundary components removed. $M'_i$ contains the Margulis tubes that $M'_i$ meets by taking union with these.

Proof. We assume that each $M'_i$ includes any Margulis tubes it meets. We essentially follow Theorem 2 of Freedman-Freedman [10]. $\partial M$ has no incompressible closed surface other than ones parallel to $F_1, \ldots, F_n$. Hence, we can compress the boundary component $\partial M'_i$ until we obtain a union of balls and manifolds homeomorphic to $F_i$ times an interval. An exterior disk compressions add a disk times $I$ to the compressed manifolds from $M'_i$ but interior disk compressions remove a disk times $I$ from the manifolds. The exterior disk may meet Margulis tubes outside $M'_i$. If the disk meets a Margulis tube
at an essential disk, then we include the Margulis tube. If not, we push the disk off the Margulis tube. This operation amounts to adding 1-handles to the manifolds.

We recover our loss to $M'_i$ by interior disk compressions by attaching 1-handles each time we do interior compressions. The core arcs of 1-handles may meet the exterior compression disk many times. We add a small neighborhood of the cores first. Then we isotopy to make it larger and larger to recover the loss due to interior disk compression while fixing the Margulis tubes outside $M_i$. (This may push the exterior compression disks.) We also recover all the Margulis tubes originally in $M'_i$.

From the surface times the interval components, we add all the 1-handles to obtain the desired compression body.

\[\square\]

Remark 12.2. We do interior compressions and then exterior compressions. This is sufficient to obtain the union of 3-balls and $F_i$ times the intervals. The reason is that we can isotopy any interior compressing curve away from the traces of exterior compression disks.

Since $E$ is geometrically infinite, there exists a sequence of closed geodesics $c_i$ tending to $E$ by Bonahon [2]. We assume that $c_i \subset M'_i$ for each $i$ since $M'_i$ is exhausting. Let $C_i$ denote the union of $c_1, \ldots, c_i$. We assume without loss of generality that $M'_i$ has a free-homotopy $A_i$ between $c_i$ and a closed curve in $C_i$. Let $\mu_i$ be the union of the simple closed geodesics in the Margulis tubes that $M'_i$ contains.

Remark 12.3. We know that given a surface there exists a finite maximal collection of exterior essential compressing disk so that any other exterior essential compressing disks can be pushed inside the regular neighborhood of their union. This is essentially the uniqueness of the compression body. (See Theorem 1 Chapter 5 of McCullough [12].)

If we remove the interior $C^o$ of the core from $M'_i$, and compress the boundary $\partial M'_i$ in $M - C^o - C_i - \mu_i$ and remove resulting cells to obtain a manifold with incompressible boundary containing $\partial C$. Then we join the result with $C, C_i$ and $\mu_i$. Let us call the resulting 3-manifold $N_i$. Note that $N_i$ need not be a compression body and may not form an exhausting sequence. However, $N_i$ contains $C, C_i, \mu_i$ for each $i$.

The exterior compressing disk of $M'_i$ may meet the Margulis tubes outside $M'_i$. We may assume that $N_i$ meets these Margulis tubes in meridian disks times intervals by following Lemma 12.4.

Lemma 12.4. A disk with boundary outside the union of Margulis tubes may be isotoped with the boundary of the disk fixed so that the intersection is the union of meridian disks.

Proof. First, we perturb to obtain transversal intersections. A disk may meet the Margulis tubes in a union of planar surfaces. The boundary of the Margulis tube meets the disk in a union $C$ of circles. If an innermost component is outside the tube, then since the boundary tube is incompressible in $M$ with the interior of the Margulis tubes removed, it follows that we can isotopy it inside. This means that the innermost components are disks.
If an innermost component of $C$ is a circle bounding a disk in the boundary of the Margulis tube, then we can isotopy the bounded disk away from the tube. Now, each component of $C$ is a meridian circle. □

The manifold $N_i$ is obtained from compressing disks for $M'_i$. Let $A_i$ be a homotopy in $M'_i$ between $c_i$ and a closed curve in $C$. Then a compressing disk for the sequence of manifolds obtained from $M'_i$ by disk-compressions in $M - C^o - C_i - \mu_i$ may meet $A_i$ in immersed circles. However, since the circles bound immersed disks in the compressing disk, and $c_i$ is not null-homotopic in $M$, we can modify $A_i$ so that $A_i$ has one less number of components where $A_i$ meets the compressing disk. In this manner, we can find $A_i$ in $N_i$.

We do the following steps:

1. We find a maximal collection of essential interior compressing disks for $N_i$ and do disk compressions. By incompressibility, the disk must meet one of $C, C_i, \mu_i$ essentially. We call $N^1_i$ the component of the result containing $\partial C$.

2. We find a maximal collection of essential exterior compressing disk for the result of (1) and do disk compressions. We call $N^2_i$ the component containing $\partial C$.

3. We add 1-handles lost in the step (1). We call the result $N^3_i$.

Clearly $N^3_i$ includes $N_i$.

**Proposition 12.5.** The submanifold $N^3_i$ is homeomorphic to a compression body and is contained in a compression body $M''_i$ whose boundary component is incompressible in $M - C^o - C_i - \mu_i$. A Margulis tube is either contained in $M''_i$ or the tube meets $M''_i$ in meridian disk times intervals.

**Proof.** The fact that $N^3_i$ is a compression body follows as before. Using the fact that $N^3_i$ is contained in some compression body $M'_j$ for a large $j$, we will now show that $N^3_i$ is contained in a compression body $M''_j$ with the above property.

By construction of $N^3_i$, it follows that an interior compression disk is equivalent to one of the disks of the 1-handles by disk sliding. Therefore, each interior compressing disk must intersect at least one of $C, C_i, \mu_i$ essentially. Hence $\partial N^3_i$ has no interior compression disk in $M - C^o - C_i - \mu_i$.

There could be an exterior compression disk for $N^3_i$. We take a maximal mutually disjoint family $D_1, \ldots, D_n$ of them where no two of $\partial D_i$ are parallel. We choose $j$ sufficiently large so that a compression body $M'_j$ includes all of them as $M'_j$’s form an exhausting sequence.

We find a 3-manifold $X$ isotopic to $N^3_i$ in $M'_j$: $M'_j$ decomposes into a union of cells or submanifolds homeomorphic to $F_t$ times intervals by a maximal family of interior compression disks. We suppose that no two of the disks are parallel. We call $D$ the union of these disks.

We choose $X$ so that $X$ is a thin regular neighborhood of a 1-complex with the unique vertex in a fixed base cell of $M'_j$. We fix a handle decomposition of $X$ corresponding to the 1-complex structure. We define the complexity of the imbedding of $X$ in $M'_j$ by the number of components of $X \cap D$. We choose $X$ with minimal complexity. We will now put all things in general positions. For each disk $D_k$, we first get rid of any closed circles by the innermost circle arguments. We will find an edgemost arc if $\partial D_k$ meets
D bounding a component of $D_k - D$. Then there must be a handle of $X$ following the arc in $\partial D_k$. This handle can be isotoped away, and then using the innermost circle argument again if necessary, we can reduce the complexity. Therefore, it follows that each $D_k$ is in the fixed base cell of $M'_j$. Also, a handle where $D_k$ passes essentially must lie in the base cell also.

We look at the handles in the base cell and the disks. The union of the handles and the ball around the basepoint is a handle body. Our disks $D_1, \ldots, D_n$ are in the cell.

From Corollary 3.5 or 3.6 of Scharlemann-Thomson [13], we see that there exists an unknotted cycle in the 1-complex or the 1-complex has a separating sphere. In the first case, we cancel the corresponding cycle by an exterior compression. In the later case, a sphere bounds a ball which we add to $X$, i.e., we engulf it. We do the corresponding topological operations to $N_i^3$. Note that $X$ and $N_i^3$ are still compression bodies. In both cases, we can reduce the genus of the boundary of the handle body $X$ or $N_i^3$. We do this operations until there are no more exterior compressing disks. (The genus complexity shows that the process terminates.) We let the final result to be $M''_i$.

The imbedding $\partial M''_i \to M - M''_i$ is incompressible since the boundary of any exterior compressing disk for $\partial M''_i$ can be made to avoid the traces of handle-canceling exterior compressions which are pairs of disks or the disks from the 3-ball engulfing. Therefore these must be exterior compressing disks for $N_i^3$.

The imbedding $\partial M''_i \to M''_i - C_o - C_i - \mu_i$ is incompressible since the boundary of any interior compressing disk can be made to avoid the traces of handle-canceling exterior compressions.

The statement about Margulis tubes follows by Lemma [12.3].

We will now modify $M''_i$ by crescent-isotopy.

**Lemma 12.6.** A secondary highest-level crescent of $\tilde{\Sigma}$ does not meet the interior of $\tilde{C}$.

*Proof.* If not, then $\tilde{C}^o$ meets $I_R$ for a secondary highest-level crescent $R$. We may assume that $R$ is compact by an approximation inside. Again find a Morse function by totally geodesic planes parallel to $I_R$. $\mathcal{C} \cap R$ has a maximum inside as $\mathcal{C} \cap R$ is compact. But at the maximum point, a totally geodesic plane bounds a local half open ball disjoint from $C^o$. This contradicts the s-imbeddedness of $\partial \tilde{C}$. 

We define $M_i$ to be the 2-convex hull of $M''_i$. The boundary components of $M_i$ are s-imbedded. Let $C'$ be the core obtained from $\tilde{C}$ by pushing $\partial C$ inside $C$ by an $\epsilon$-amount. Note that during the crescent-isotopy $C'$ is not touched by the interior of secondary highest level crescents since $\partial C$ is s-imbedded by Lemma [12.6].

Define $M'_i$ be the regular $\epsilon$-neighborhood of $M_i$. $C_i, \mu_i \subset M'_i$ by Proposition [6.2] in Part 1. We may assume $A_i$ is in $M'_i$ since we isotoped $M''_i$ to obtained $M_i$.

The universal cover $\tilde{M}_i$ of $M_i$ with the universal covering map $p_i$ is an $M_{-1}$-space and is $\delta$-hyperbolic.

We define the thin part of $M_i$ as the subset of $M_i$ where the injectivity radius is $\leq \epsilon$. Since $\tilde{M}_i$ is a uniquely geodesic, through each point of the thin part of $M_i$ there exists a closed curve of length $\leq \epsilon$ which is not null-homotopic in $M_i$. 

46
Let \( \gamma \) be a closed curve of length \( \leq \epsilon \) which is not null-homotopic in \( M_i \). Then if \( \gamma \) has nontrivial holonomy, then \( \gamma \) lies in a Margulis tube of \( M \) which either is in \( M_i \) or disjoint from \( M_i \).

Suppose that \( \gamma \) is null-homotopic in \( M \). Then \( \gamma \) bounds an immersed disk \( D \) in \( M \).

The diameter of \( D \) is \( \leq \epsilon \).

Suppose that \( D \) cannot be isotoped into \( M_i \). By incompressibility of \( \partial M_i \), \( D \) must meet \( c_i \) or \( \mu_i \) or \( \mathcal{C} \). Also, \( D/M_i \) must be nonempty. There must be an innermost disk \( D' \) such that \( \partial D' \) maps to \( \partial M_i \) and \( D' \) maps into \( M_i \) and meets \( \mu_i \), \( \mathcal{C} \) or \( \mathcal{C}_i \) and a component of \( D/\partial M_i \) adjacent to \( D' \) lies outside \( M_i \).

If \( D' \) meets \( \mathcal{C} \), then the diameter of \( \partial D' \) is not so small, and hence that of \( D \). If \( D' \) meets \( \mathcal{C}_i \), then since \( A_i \) is in \( M_i \) \( D' \) cannot be bounded outside by a component in \( M/M_i \).

Suppose that an innermost disk \( D' \) meet \( \mu_i \). Since \( \partial D' \) is very close to \( \mu_i \) due to its size, and the distance from \( \mu_i \) to \( \partial M_i' \) is bounded below by a certain constant, it follows that the boundary of \( D' \) lies in the union of \( I \)-parts of some crescents or its perturbed images obtained during the crescent-isotopies. Since the length of components of \( \mu_i \) is short, we see that the the \( I \)-parts meeting \( \partial D' \) would extend for long lengths along the geodesics near a component of \( \tilde{\mu}_i \). Therefore, we see that at the last stage of the isotopies, we have the inverse image of torus bounding a component of \( \mu_i \). Since our crescent moves are isotopies and \( \partial M_i' \) is not homeomorphic to a torus, this is a contradiction.

We conclude that \( \gamma \) bounds a disk in \( M_i \). Since \( \gamma \) is not null-homotopic in \( M_i \), this is a contradiction. Therefore, the thin part of \( M_i \) is in the intersection of the Margulis tubes of \( M \) with \( M_i \).

Note here that the Margulis constant \( \epsilon > 0 \) could be chosen independent of \( i \) as the above argument passes through once \( \epsilon \) is sufficiently small regardless of \( i \).

By above discussions, it follows that any \( \epsilon \)-short closed curve in \( M_i \) is a multiple of the simple closed geodesic in a Margulis tube. We may assume that given a component of the thin part of \( M_i \), an \( \epsilon \)-short closed curve of a fixed homotopy class passes through each point of the component. Therefore, components of thin parts are solid tori in Margulis tubes.

During the crescent move, the shortest closed geodesic in the Margulis tube may go outside particularly during the convex perturbations. But there are short closed curves in the result that homotopic to the closed geodesic. Therefore the thin part are union of solid tori parallel to the shortest geodesics.

**Proposition 12.7.** The \( \epsilon \)-thin part of \( M_i \) are homeomorphic to a disjoint union of solid tori in Margulis tubes in \( M \) parallel to the shortest geodesics in the respective tubes. Furthermore, we can choose \( \epsilon > 0 \) independent of \( i \). \( \square \)

Assume without loss of generality that we have an inclusion map \( i : \mathcal{C} \to M_i \) for each \( i \). Let \( L_i \) be \( M_i/i_\pi_1(\mathcal{C}) \) with the covering map \( p_i : L_i \to M_i \). \( L_i \) has ends corresponding to \( F_1, \ldots, F_n \), and another end \( E \) corresponding to \( E \). (We abused notation a little here.)

**Proposition 12.8.** The convex hull \( K_i \) of \( \mathcal{C} \) in \( L_i \) is compact.
Proof. Since $M_i$ is tame, its cover $L_i$ is tame. For any compact set, we can find a compact core of $L_i$ containing it. By choosing a large compact subset of $M_i$, we obtain a compact core $C'$ of $L_i$ containing it which is obtained as the closure of the appropriate component of $L_i$ with a finite number of disks removed.

Certainly $C$ is a subset of it. The disks lifts to disks in the universal cover $\tilde{L}_i$ of $L_i$. They bound the universal cover $\tilde{C}'$ of $C'$.

The convex hull of a disk is a compact subset of $\tilde{L}_i$ since the convex hull of a compact subset is compact in the universal cover. Therefore, the convex hull of $\tilde{C}'$ is in the union of $\tilde{C}'$ and deck-transformation images of finitely many compact sets that are convex hulls of the boundary disks. Therefore, the convex hull of $C$ itself is compact being a subset of the union of a compact set $C'$ and finitely many compact sets. □

Since $C$ is homotopy equivalent to $L_i$, there exists a convex hull $K_i$ of $C$ in $L_i$ homotopy equivalent to $C$ by Proposition 10.1 in Part 2. Obviously, $K_i$ contains $F_1, \ldots, F_n$. Let $\Sigma_i$ be the unique boundary component of $\partial K_i$ associated with $E$.

Any closed geodesic homotopic to a closed curve in $C$ in $L_i$ is contained in $K_i$: If not, we can find an h-imbedded annulus $B_i$ with boundary consisting of the closed geodesic and a closed curve on $\partial K_i$, intrinsically geodesic, where the interior angles in $B_i$ are always greater than or equal to $\pi$. Such an annulus clearly cannot exist.

We can find a quasi-geodesic $c'_i \epsilon$-close to $c_i$ in $M_i$. Since $c'_i$ is homotopic to a closed curve in $C$ by a homotopy $A'_i$ in $M_i$ modified from $A_i$, we have that $c'_i \subset L_i$. Choose a geodesic representative $c''_i$ in $L_i$ which is again arbitrarily close to $c'_i$ and hence to $c_i$. Therefore $c''_i \subset K_i$ for each $i$.

Since $\Sigma_i$ is a truly pleated-triangulated convex h-surface, the intrinsic metric in the pleated part is a Riemannian hyperbolic one. Thus, $\Sigma_i$ carries a triangulated h-surface structure intrinsically. Since $\Sigma_i$ is intrinsically an h-imbedded surface, $p|\Sigma_i$ is one also and hence form an exiting sequence in $E$.

More precisely, the parts of boundary of the image of $K_i$ form an exiting sequence in $E$. Any part of the boundary of the image of $K_i$ is in the image of $\Sigma_i$. Hence, there exists an exiting sequence of parts of $\Sigma_i$. By the uniform boundedness of $\Sigma_i$, it follows that $\Sigma_i$ form an exiting sequence in $E$.

Remark 12.9. The uniform nature of the Margulis constant plays a role here. Any $\epsilon$-thin part of an h-immersed surface must be inside a Margulis tube in $L_i$ and by incompressibility the thin parts are homeomorphic to essential annuli. Since $\Sigma_i$ is incompressible in $L_i - C^o$, we see that the thin part of $\Sigma_i$ is a union of essential annuli which are not homotopic to each other. Thus, outside the Margulis tubes, $\Sigma_i$s have bounded diameter independent of $i$.

13. THE PROOF OF THEOREM B

We recall that $C$ was pushed inside itself somewhat so that $\Sigma_i$ and $\partial K_i$ does not meet $C$. In $K_i$, we may remove $C^o$ and we obtain a compact submanifold $Q_i$ of codimension 0 bounded by s-surfaces including $\Sigma_i$ since $\partial C$ is s-imbedded. $Q_i$ is 2-convex and is Gromov-hyperbolic. $\Sigma_i$ is incompressible in $Q_i$ since any compressing disk of $\Sigma_i$ not meeting the core would reduce the genus of $\Sigma_i$ but the genus of $\Sigma_i$ is the same as that
of the component of \( \partial \mathcal{C} \) corresponding to the end \( E \) since \( K_i \) is homotopy equivalent to \( \mathcal{C} \).

As \( K_i \) is homeomorphic to a compression body, we choose a compressing curve \( \alpha \) in \( \Sigma_i \). Then \( \alpha \) bounds a disk \( D \) in \( K_i \) and the core \( \mathcal{C} \) must meet \( D \) in its interior. Let \( \hat{\alpha} \) be the geodesic realization of \( \alpha \) in \( K_i - \mathcal{C}^o \), which must be in \( K_i \).

If \( \hat{\alpha} \) does not meet \( \partial \mathcal{C} \), then it maps to a geodesic in \( M \). This is absurd since the holonomy of \( \alpha \) is the identity. Therefore, \( \hat{\alpha} \) meets \( \partial \mathcal{C} \).

We form a triangulation of \( \Sigma_i \) with the only vertex \( p \) at a point of \( \alpha \) and including \( \alpha \) as an edge. Then choosing a vertex \( \hat{p} \) in \( \hat{\alpha} \) and a path from \( p \) to \( \hat{p} \), we isotopy each edge of the triangulation to a geodesic loop in \( K_i - \mathcal{C}^o \) based at \( \hat{p} \). Each triangle is isotopied to a 2-A-simplex spanned by new geodesic edges. The resulting surface \( T_i \) is an h-surface since each of the triangles is a 2-A-simplex and a geodesic passes through each point of the 1-complex.

Each surface \( q_i : T_i \to M - \mathcal{C}^o \) has the same genus and is homotopic to \( p_i|\Sigma_i \) in \( M - \mathcal{C}^o \). Since they are h-imbedded, and meet \( \partial \mathcal{C} \), they are in a bounded neighborhood of \( \mathcal{C} \) by the boundedness of h-imbedded surfaces. They form a pre-compact sequence. Thus infinitely many of \( q_i|T_i \) are isotopic in \( M - \mathcal{C}^o \). Therefore, infinitely many of \( p_i|\Sigma_i \) are isotopic in \( M - \mathcal{C}^o \). Since \( \Sigma_i \) bounds larger and larger domains in a cover of \( M \) and \( \Sigma_i \) projects to a surface far from \( \mathcal{C} \), the above fact shows that our end \( E \) is tame as shown by Thurston [15]. (This is essentially the argument of Souto [14].)

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