MORITA EQUIVALENCE OF CHEREDNIK ALGEBRAS

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Abstract. We classify the rational Cherednik algebras $H_c(W)$ (and their spherical subalgebras) up to isomorphism and Morita equivalence in case when $W$ is the symmetric group and $c$ is a generic parameter value.

1. Introduction

In this paper, which is a sequel to our earlier work [BEG], we deal with Morita classification of rational Cherednik algebras. Our goal is to prove one of the main conjectures of [BEG] (see Conjecture 8.12, loc. cit.) in the case of symmetric groups. Before stating our results we recall the notation and some of the basic definitions.

Let $W$ be a finite Coxeter group generated by reflections in a finite-dimensional complex vector space $h$. The rational Cherednik algebras associated to $W$ is a family of associative algebras $\{H_c(W)\}$ parametrized by the set of $W$-invariant complex multiplicities $c : R \to \mathbb{C}$ on the system of roots $R \subset h^*$ of $W$. Specifically, for a fixed $c : \alpha \mapsto c_\alpha$, the algebra $H_c = H_c(W)$ is generated by the vectors of $h$, $h^*$, and the elements of $W$ subject to the following relations

$$w x w^{-1} = w(x), \quad w y w^{-1} = w(y), \quad \forall y \in h, x \in h^*, w \in W,$$

$$x_1 x_2 = x_2 x_1, \quad y_1 y_2 = y_2 y_1, \quad \forall y_1, y_2 \in h, x_1, x_2 \in h^*,$$

$$y x - x y = \langle y, x \rangle - \sum_{\alpha \in R_+} c_\alpha \langle y, \alpha \rangle \langle \alpha^\vee, x \rangle s_\alpha, \quad \forall y \in h, x \in h^*.$$

Here, as usual, we write $\alpha^\vee \in h$ for the coroot, $s_\alpha \in W$ for the reflection corresponding to the root $\alpha \in R$, $R_+ \subset R$ for a choice of ‘positive’ roots in $R$, and $\langle \cdot, \cdot \rangle$ for the canonical pairing between $h$ and $h^*$.

Note that the group algebra $\mathbb{C}W$ of $W$ embeds naturally in $H_c(W)$, and $H_0(W) = D(h)^\#W$, where $D(h)^\#W$ is the crossed product of $\mathbb{C}W$ with the ring of polynomial differential operators on $h$. Furthermore, each $H_c$ contains a distinguished subalgebra $B_c := eH_c e$, where $e := \frac{1}{|W|} \sum w$ is the symmetrizing idempotent in $\mathbb{C}W \subset H_c$ which plays a role of the identity element in $B_c$. We call $B_c = B_c(W)$ the spherical algebra associated to $(W, c)$. Since $B_0 = e(D(h)^\#W)e \cong D(h)^W$, the family $\{B_c(W)\}$ should be thought of as a deformation (in fact, the universal deformation) of the ring $D(h)^W$ of $W$-invariant differential operators on $h$.

The structure of and the relationship between the algebras $H_c$ and $B_c$ depend drastically on the values of the parameter $c$. It turns out that both of these are governed by the properties of the Hecke algebra $H_W(q)$ of $W$ with $q = e^{2\pi i c}$. In fact, one of the main results of [BEG] states

Theorem 1 ([BEG], Theorem 3.1, Corollary 4.2). If $H_W(e^{2\pi i c})$ is semisimple, then $H_c$ and $B_c$ are simple algebras, Morita equivalent to each other.
Following [BG], we will call \( c \) regular if the assumption of Theorem 1 holds (i.e., if the Hecke algebra \( H_W(e^{2\pi ic}) \) is semisimple).

In the present paper we will classify the algebras \( H_c(W) \) and \( B_c(W) \) up to isomorphism and Morita equivalence in case when \( W \) is a symmetric group. Thus, from now on, we assume that \( W := S_n \ (n \geq 2) \). In this case, \( W \) acts transitively on its roots, so the algebras \( H_c \) and \( B_c \) are labelled by a single parameter \( c \in \mathbb{C} \).

We will write \( \mathbb{C}^\text{reg} \) for the set of regular values of \( c \), and will use the standard notation denoting by \( \mathbb{Q} \) the field of rational numbers and by \( \overline{\mathbb{Q}} \) its algebraic closure in \( \mathbb{C} \). Note that \( c \notin \mathbb{C}^\text{reg} \) if and only if \( e^{2\pi ic} \) is a root of unity of order \( d \), where \( 2 \leq d \leq n \) (see [DM]). Thus, we have \( \mathbb{C}^\text{reg} = \mathbb{C} \setminus \bigcup_{d=2}^{n} \bigcup_{m=1}^{d-1} (m/d + \mathbb{Z}) \).

Our main results can be encapsulated into the following two theorems.

**Theorem 2.** If \( c \notin \mathbb{Q} \), the algebras \( H_c \) and \( H_{c'} \) are

(a) isomorphic if and only if \( c = \pm c' \),
(b) \( \mathbb{C} \)-linearly Morita equivalent if and only if \( c \pm c' \in \mathbb{Z} \).

**Theorem 3.** If \( c \notin \mathbb{Q} \), the algebras \( B_c \) and \( B_{c'} \) are

(a) isomorphic if and only if \( c = c' \) or \( c = -c' - 1 \),
(b) \( \mathbb{C} \)-linearly Morita equivalent if and only if \( c \pm c' \in \mathbb{Z} \).

A few comments on the proof of these results. First of all, the ‘if’ parts of both statements in both theorems are known. The isomorphism \( H_c \cong H_{-c} \) is immediate from the defining relations of \( H_c \) and holds for all \( c \in \mathbb{C} \); \( B_c \cong B_{-1-c} \) is a consequence of the fact that \( B_c \) can be identified with the subalgebra of differential operators generated by \( \mathbb{C}[h]^W \) and the Calogero-Moser operator \( L_{-c} \), which is also true for all \( c \in \mathbb{C} \) (see [BG], Proposition 4.9). The existence of Morita equivalence between \( H_c \) and \( H_{c+k}, \ k \in \mathbb{Z} \), has been proven in [BG], Theorem 8.1, for an arbitrary Coxeter group \( W \) and holds (at least) for regular \( c \). A similar result for the spherical algebras is then immediate by Theorem 1 above. Thus, we need only to establish the ‘only if’ parts of Theorems 2 and 3. We will prefer to work with spherical algebras, proving first the required implications in Theorem 3 (see Section 3 and Section 4 below). Theorem 2(b) will then follow immediately from Theorem 3(b) and Theorem 1 and with a little more work, we will conclude Theorem 2(a) from Theorem 2(b) (see Section 5).

The proof of Theorem 3 relies on explicit formulas for the Hatori-Stallings traces for the algebra \( B_c \) which we derive in Section 2. One general result that underlies our computations is, perhaps, of independent interest and deserves to be mentioned here.

**Theorem 4.** Let \( V \) be a finite-dimensional symplectic vector space over \( \mathbb{C} \), and let \( G \subset \text{GL}(V) \) be a finite group acting on \( V \) by symplectic transformations. Then the algebra \( A := \mathbb{C}[V]^G \) of polynomial \( G \)-invariants has a finite-dimensional 0-th Poisson homology, i.e. \( \dim_{\mathbb{C}}(A/\{A,A\}) < \infty \), where \( \{A,A\} \subseteq A \) is the subspace spanned by Poisson brackets of all elements of \( A \).

The assertion of Theorem 4 has been conjectured (and verified in many special cases) in [AE], [AL]. We will give a complete proof of this result in the Appendix.

We have stated Theorems 2 and 3 under the assumption that one of the parameters \( (c \text{ or } c') \) takes non-algebraic values. In fact, it suffices to prove these results only in the case when both \( c \) and \( c' \) are transcendental over \( \mathbb{Q} \). Once the latter is established, it is standard to conclude that the algebras \( H_c \) (resp., \( B_c \))
and \( H_c \) (resp., \( B_c \)) can be neither isomorphic nor Morita equivalent if one of the parameters is algebraic while the other is not. Indeed, if (say) \( H_c \) were isomorphic to \( H_{c'} \) for some \( c \in \overline{\mathbb{Q}} \) and \( c' \notin \overline{\mathbb{Q}} \), then for any given \( c'' \in \mathbb{C} \setminus \overline{\mathbb{Q}} \), we could find an automorphism \( g \in \text{Aut}(\mathbb{C}/\overline{\mathbb{Q}}) \), such that \( g.c' = c'' \) and \( g.c = c \), forcing \( H_{c'} \cong H_{c''} \). On the other hand, it seems very likely that both theorems remain true under milder assumptions. For example, it would be interesting to extend the above classification to all regular values of \( c \). Some partial results in this direction are discussed in Section 6.

Finally, we should mention that in the simplest case \( (n = 2) \), the algebras \( B_c \) can be identified with primitive factors of \( U(\mathfrak{sl}_2) \) (see [EG], Section 8), and in this context both parts of Theorem 3 were proven earlier: (a) is due to Dixmier [D], and (b) is due to Stafford [S] and Hodges [H]. The methods we employ in the present paper generalize those of Hodges. In the end of [H] the author remarks that K-theoretic techniques can be applied to distinguish between primitive factors of higher dimensional semisimple Lie algebras but “... the question cannot be completely solved along this approach”. It seems surprising that in the case of Cherednik algebras, as we will see below, the K-theoretic approach does lead to a complete solution of the problem, at least for generic parameter values.

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2. Hattori-Stallings Traces

Let \( \Gamma = \Gamma(W) \) be the set of irreducible representations of \( W \). The Grothendieck group of \( \mathbb{C}W \) is then a free abelian group of rank \( |\Gamma| \) generated by the classes of \( \tau \in \Gamma: \ k_0(\mathbb{C}W) = \bigoplus_{\tau \in \Gamma} \mathbb{Z} \cdot [\tau] \). According to [EG], the algebra \( H_c \) can be equipped with a canonical increasing filtration \( \{ F_p H_c \} \), whose first nonvanishing component \( F_0 H_c \subset \mathbb{C}W \) and the associated graded algebra \( \text{gr}(H_c) \) is isomorphic to \( \mathbb{C}[h \oplus h^*]/\#W \) (and therefore, is Noetherian of finite global dimension). By a well-known theorem of Quillen [Q], the inclusion \( \mathbb{C}W \hookrightarrow H_c \) induces in this situation an isomorphism of abelian groups \( k_0(\mathbb{C}W) \cong k_0(H_c) \). If we assume that \( c \in \mathbb{C}^\text{reg} \), then \( B_c \) is Morita equivalent to \( H_c \) (see Theorem 1 above), and the corresponding equivalence functor \( \text{e}H_c \otimes_{H_c} - : \text{Mod}(H_c) \rightarrow \text{Mod}(B_c) \) gives another isomorphism of \( K \)-groups: \( k_0(H_c) \cong k_0(B_c) \). Thus, for regular \( c \), we can identify both \( k_0(H_c) \) and \( k_0(B_c) \) with \( k_0(\mathbb{C}W) \). Specifically, the class of an irreducible representation \( \tau \in \Gamma \) corresponds to the classes of (left) projective modules \( P_\tau := H_c \otimes_{\mathbb{C}W} \tau \) and \( \text{e}P_\tau := \text{e}H_c \otimes_{\mathbb{C}W} \tau \) in \( k_0(H_c) \) and \( k_0(B_c) \) respectively.

For any \( c \in \mathbb{C} \), \( B_c \) is a Noetherian domain (since so is \( \text{gr}(B_c) \cong \mathbb{C}[h \oplus h^*]W \)). By Goldie’s theorem, it has a quotient division ring \( Q(B_c) \). Thus, we can define the rank homomorphism \( \text{rk} : k_0(B_c) \rightarrow \mathbb{Z} \) on \( k_0(B_c) \) setting

\[
\text{rk}[M] := \text{length}(Q(B_c) \otimes_{B_c} M), \quad [M] \in k_0(B_c).
\]

Let \( \check{k}_0(B_c) \) denote the kernel of \( [1] \) so that \( k_0(B_c) \cong \check{k}_0(B_c) \oplus \mathbb{Z} \). Since \( \text{rk}[\text{e}P_\tau] = \dim(\tau) \) for each \( \tau \in \Gamma \), we may take the classes

\[
[\text{e}P_\tau] := \dim(\tau) \cdot [\text{e}P_{\tau \text{triv}}] - [\text{e}P_\tau], \quad \tau \in \Gamma^*.
\]
as generators of $\mathcal{K}_0(B_c)$. (Here and below, $\text{triv}$ stands for the trivial representation of $W$ and $\Gamma^* := \Gamma \setminus \{\text{triv}\}$.)

Now, let $[H_c, H_c]$ be the subspace of $H_c$ spanned by (additive) commutators, and let $\mathcal{H}_0(H_c) := H_c/[H_c, H_c]$ be the trace group (i.e., the 0-th Hochschild homology) of the algebra $H_c$ with canonical projection $\text{Tr}_{H_c} : H_c \to \mathcal{H}_0(H_c)$. Thus, we regard $\mathcal{H}_0(H_c)$ as a vector space over $\mathbb{C}$, with $\text{Tr}_{H_c}$ being a $\mathbb{C}$-linear map.

To study finite-dimensional representations of $H_c$ we have calculated in \cite{BEG}, the value of $\text{Tr}_{H_c}$ on the central idempotents $e_\tau$ of $CW \subset H_c$ corresponding to simple modules $\tau \in \Gamma$. The result of this calculation (see \cite{BEG}, Theorem 5.3) reads\footnote{\textbf{N.B.} Our notation here differs slightly from that of \cite{BEG}.}

\begin{equation}
\text{Tr}_{H_c}(e_\tau) = \frac{(\dim \tau)^2}{n!(nc)^n} F_\tau(nc) \cdot \text{Tr}_{H_c}(1) \quad \text{for any } c \neq 0 ,
\end{equation}

where $F_\tau(x) \in \mathbb{Z}[x]$ is the so-called \textit{content polynomial} of the Young diagram $Y(\tau)$ corresponding to $\tau$ (see \cite{M}, Example 11, p. 15):

\begin{equation}
F_\tau(x) := \prod_{(i,j) \in Y(\tau)} (x + j - i) .
\end{equation}

Using \cite{M}, it is easy to compute the \textit{Hattori-Stallings trace} for the algebra $H_c$:

$$
\chi_{H_c} : \mathcal{K}_0(H_c) \to \mathcal{H}_0(H_c) .
$$

Indeed, by definition, if $e$ is an idempotent in $H_c$, then $H_c e$ is a finite projective module whose Hattori-Stallings trace is given by $\chi_{H_c}[H_c e] := \text{Tr}_{H_c}(e)$. Now, for any central group idempotent $e_\tau \in CW$, we have $[H_c e_\tau] = \dim \tau \cdot [P_\tau]$ in $\mathcal{K}_0(H_c)$. Hence, it follows from (3) that

\begin{equation}
\chi_{H_c}[P_\tau] = \frac{\dim \tau}{n!(nc)^n} F_\tau(nc) \cdot \text{Tr}_{H_c}(1) , \quad c \neq 0 .
\end{equation}

Now, write $\mathcal{H}_0(B_c) := B_c/[B_c, B_c]$ for the trace group, $\text{Tr}_{B_c} : B_c \to \mathcal{H}_0(B_c)$ for the canonical projection, and $\chi_{B_c} : \mathcal{K}_0(B_c) \to \mathcal{H}_0(B_c)$ for the Hattori-Stallings trace of the spherical algebra $B_c$. By definition, $\text{Tr}_{B_c}(1)$ is a value that $\chi_{B_c}$ takes on the class $[B_c]$ of the free module of rank one, i.e. $\chi_{B_c}[B_c] = \text{Tr}_{B_c}(1)$ in $\mathcal{H}_0(B_c)$. As we mentioned above, $B_c$ is Morita equivalent to $H_c$, when $c$ is regular. Hence, in that case $\mathcal{H}_0(H_c)$ and $\mathcal{H}_0(B_c)$ are isomorphic, with isomorphism $\Phi_c : \mathcal{H}_0(H_c) \cong \mathcal{H}_0(B_c)$ being naturally induced by the Morita functor. Moreover, the map $\Phi_c$ fits in the commutative diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{K}_0(H_c) & \xrightarrow{\chi_{H_c}} & \mathcal{H}_0(H_c) \\
\downarrow & & \downarrow \Phi_c \\
\mathcal{K}_0(B_c) & \xrightarrow{\chi_{B_c}} & \mathcal{H}_0(B_c)
\end{array}
\end{equation}

Since under the Morita equivalence $[B_c]$ corresponds to the class of $P_{\text{triv}}$ in $\mathcal{K}_0(H_c)$ and $\chi_{H_c}[P_{\text{triv}}] = \frac{1}{n!(nc)^n} F_{\text{triv}}(nc) \cdot \text{Tr}_{H_c}(1)$ by (3), we see from (5) that

\begin{equation}
\Phi_c(\text{Tr}_{H_c}(1)) = \frac{n!(nc)^n}{F_{\text{triv}}(nc)} \cdot \text{Tr}_{B_c}(1) .
\end{equation}
Proposition 1. We have
\[ c \text{ of values of } C \]
Now, using (5), (6), (7) and (8), we may replace the algebra \( B \) due to Morita invariance of Hochschild homology and formula (7), we prove the following result will be crucial for our further considerations.

**Proposition 1.** We have
(a) \( \dim \text{HH}_0(B_c) \geq 1 \) for all \( c \in \mathbb{C} \), and \( \dim \text{HH}_0(B_c) = 1 \) for all but finite subset of values of \( c \);

(b) if \( c \in \mathbb{C}^\text{reg} \), then \( \dim \text{HH}_0(B_c) = 1 \), and \( \text{Tr}_{B_c}(1) \neq 0 \) in \( \text{HH}_0(B_c) \).

We will deduce Proposition 1 from Theorem 3 (which, in turn, will be proven in the Appendix). As a first step, we prove a weaker version of this result.

**Lemma 1.** If \( c \in \mathbb{C} \setminus \overline{\mathbb{Q}} \) then \( \dim \text{HH}_0(B_c) = 1 \), and \( \text{Tr}_{B_c}(1) \neq 0 \) in \( \text{HH}_0(B_c) \).

Proof. Due to Morita invariance of Hochschild homology and formula (7), we may replace the algebra \( B_c \) in the statement of the lemma by \( H_c \). Then, \( \text{HH}_0(H_c) \) being 1-dimensional follows from the fact, for generic \( c \) \( \text{HH}_0(H_c) \) is isomorphic to the 0-th Poisson homology of the Calogero-Moser space \( M_n \) which, in turn, is isomorphic to the top (De Rham) cohomology of \( M_n \) (see \([E, G]\)). By a theorem of Nakajima \([N]\), Theorem 3.45, \( M_n \) is diffeomorphic (as a \( C^\infty \)-manifold) to the Hilbert scheme \( \text{Hilb}_{n}(\mathbb{C}^2) \) of \( n \)-points on \( \mathbb{C}^2 \), the top Betti number of which is known to be equal to one (see \([N]\), Corollary 5.10).

A proof of the fact that \( \text{Tr}_{H_c}(1) \neq 0 \) for generic \( c \) was sketched in \([B, E, G]\) (cf. Remark after Theorem 5.3, loc. cit.). For reader’s convenience, we recall and slightly refine this argument here. Let \( E := \{ c \in \mathbb{C} : 1 \in [H_c, H_c] \text{ in } H_c \} \). Then, for each \( c \in E \), there exists \( i = i(c) \in \mathbb{Z}_{\geq 0} \), such that \( 1 \in [F_i H_c, F_i H_c] \), where \( F_i H_c \) is the standard increasing filtration on \( H_c \). Thus, we have \( E = \bigcup_{i=0}^{\infty} E_i \), where \( E_i := \{ c \in \mathbb{C} : 1 \in [F_i H_c, F_i H_c] \} \) for \( i = 0, 1, 2 \ldots \). Now, each \( E_i \) is a semi-algebraic set defined over \( \mathbb{Q} \). On the other hand, \( E_i \) cannot contain a non-empty Zariski open subset of \( \mathbb{C} \). Indeed, if this were the case for some \( i \), then \( \mathbb{C} \setminus E_i \) would be at most a finite set. The latter is impossible, because, as we know (see \([B, E, G]\), Theorem 1.2), there are infinitely many values of \( c \) of the form \( c = 1/n + Z_{\geq 0} \), for which the algebra \( H_c \) has nonzero finite-dimensional representations (the element \( 1 \in H_c \) acts as the identity operator on such a representation, and hence has a nonzero trace). Thus, each \( E_i \) is a (possibly empty) algebraic subset of \( \mathbb{C} \) defined over \( \overline{\mathbb{Q}} \), i.e. \( E_i \subset \overline{\mathbb{Q}} \) for all \( i = 0, 1, 2 \ldots \). It follows that \( E \subset \overline{\mathbb{Q}} \), and hence \( \text{Tr}_{H_c}(1) \neq 0 \) when \( c \in \mathbb{C} \setminus \overline{\mathbb{Q}} \). This finishes the proof of the lemma.

Now, to prove Proposition 1 in full generality we introduce the following notation. First, we define an associative algebra \( H \) to be generated by the elements of \( \mathfrak{h}^* \), \( W \), and a new central variable \( c \) satisfying the same relations as \( H_c \). Next, we set \( B := eH_e \), where \( e \) is the \( W \)-symmetrizer in \( H \). By construction, the algebra \( B \) (as well as \( H \)) has a non-trivial center, namely \( \mathbb{C}[c] \), each \( B_c \) being a
quotient of $B$ obtained by specializing a central character. Moreover, the trace group $\mathbb{H}_0(B) := B/[B, B]$ has the structure of a module over $\mathbb{C}[c]$, and we have

**Lemma 2.** $\mathbb{H}_0(B)$ is a finite $\mathbb{C}[c]$-module.

**Proof.** Recall that each $B_c$ is equipped with the increasing filtration $\{F_*B_c\}$, the associated graded ring $\text{gr}(B_c) \cong \mathbb{C}[h \oplus h^*]^W$ being independent of $c$. Letting $\text{deg}(c) = 0$, we can extend this filtration to the algebra $B$ so that $\text{gr}(B) \cong A[c]$. Then we have $\text{gr}(B)/\{\text{gr}(B), \text{gr}(B)\} \cong (A/A, A)[c]$, where $A := \mathbb{C}[h \oplus h]^W$. By Theorem [1] $A/A, A)$ is a finite-dimensional vector space over $\mathbb{C}$. Hence, $\text{gr}(B)/\{\text{gr}(B), \text{gr}(B)\}$ is a finite module over $\mathbb{C}[c]$. Now, if we equip $[B, B] \subseteq B$ and $\mathbb{H}_0(B) = B/[B, B]$ with the induced filtrations, then $\text{gr}(B/[B, B]) \subseteq \text{gr}(B)$ and $\text{gr}\mathbb{H}_0(B) \cong \text{gr}(B)/\{\text{gr}(B), \text{gr}(B)\}$. As $\{\text{gr}(B), \text{gr}(B)\} \subseteq \text{gr}(B/[B, B])$, we see that $\text{gr}\mathbb{H}_0(B)$ is a quotient of $\text{gr}(B)/\{\text{gr}(B), \text{gr}(B)\}$, and hence, is finite over $\mathbb{C}[c]$. This implies that $\mathbb{H}_0(B)$ is finite over $\mathbb{C}[c]$ as well. Lemma 2 is proven. □

Now we are in position to give a complete

**Proof of Proposition 1.** In view of Lemma 2 we may think of $\mathbb{H}_0(B)$ as a coherent sheaf on $\text{Spec} \mathbb{C}[c]$, with $\mathbb{H}_0(B_c)$ being a fiber over the point $c$. The function $c \mapsto \dim \mathbb{H}_0(B_c)$ is then upper semicontinuous (see, e.g., [Ha], Exercise II.5.8) which means that, for all $n \in \mathbb{Z}$, the sets $\{c \in \mathbb{C} : \dim \mathbb{H}_0(B_c) \geq m\}$ are (Zariski) closed in $\mathbb{C}$. With Lemma 1 this gives immediately the first statement of our proposition: to wit, $\dim \mathbb{H}_0(B_c) \geq 1$ for all $c$, the set $\{c \in \mathbb{C} : \dim \mathbb{H}_0(B_c) > 1\}$ being at most finite. Now, if $c \in \mathbb{C}^{\text{reg}}$, then $\mathbb{H}_0(B_{c+k}) \cong \mathbb{H}_0(B_c)$ for any $k \in \mathbb{Z}$ (since $B_{c+k}$ is Morita equivalent to $B_c$; see [BEC], Theorem 8.1), and therefore we have necessarily $\dim \mathbb{H}_0(B_c) = 1$ in that case. Next, by Lemma 1 $\text{Tr}_{B_c}(1) \neq 0$ for transcendental values of $c$. Being a section of the coherent sheaf $\mathbb{H}_0(B)$, $\text{Tr}_{B_c}(1)$ may vanish then only on a closed, and therefore finite subset in $\mathbb{C}$. Again, the Vanishing of $\text{Tr}_{B_c}(1)$ for a regular value of $c$ would imply that $\text{Tr}_{B_{c+k}}(1) = 0$ for all $k \in \mathbb{Z}$ (cf. [20] and [21] below). Thus, to avoid a contradiction we conclude that $\text{Tr}_{B_c}(1) \neq 0$ when $c \in \mathbb{C}^{\text{reg}}$. Proposition 1 is proven.

**Remark.** Note that the second statement of Proposition 1(b) does not hold for the algebra $H_c$ (even though Lemma 1 does). Indeed, if $c = 0$ then $H_c \cong \mathbb{D}(h)\# W$ and $\text{Tr}_{H_c}(1) = 0$. We expect, however, that $c = 0$ is the only exceptional value, and $\text{Tr}_{H_c}(1)$ does not vanish for any $c \neq 0$. On the other hand, we also expect that $\dim \mathbb{H}_0(H_c) = 1$ for all values of $c$. In the simplest case, when $n = 2$ and $B_c$ are identified with primitive quotients of $U(sl_2)$, $\mathbb{H}_0(B_c)$ (and, more generally, $\mathbb{H}_0(B_c^F)$ for all finite $G \subset \text{Aut}(B_c)$) have been computed explicitly in [C].

3. PROOF OF THEOREM 3(a)

The Hattori-Stallings traces have good functorial properties with respect to change of rings and Morita equivalence (see, e.g., [13]). Specifically, given an algebra homomorphism $\varphi : B \to B'$, there is a commutative diagram

$$
\begin{array}{ccc}
K_0(B) & \xrightarrow{\chi_B} & \mathbb{H}_0(B) \\
\downarrow{\varphi_0} & & \downarrow{\mathbb{H}_0(\varphi)} \\
K_0(B') & \xrightarrow{\chi_{B'}} & \mathbb{H}_0(B')
\end{array}
$$

(11)

where $\varphi_0 : K_0(B) \to K_0(B')$ is the natural map induced by $\varphi$.
where \( k_0(\varphi) \) is a homomorphism of abelian groups sending \([P] \in k_0(B)\) to the class of the induced module \([B' \otimes B] P \in k_0(B')\) and \( HH_0(\varphi) : HH_0(B) \to HH_0(B')\) is a \( \mathbb{C}\)-linear map given by \( b + [B, B] \mapsto \varphi(b) + [B', B']\) (see [13], Section 2).

We will use [11] to distinguish two generic members of the family of algebras \( \{ B_c \} \) up to isomorphism. First, we observe that if \( \varphi : B_c \to B_{c'} \) is an isomorphism of \( \mathbb{C}\)-algebras, then \( \varphi(1) = 1 \), and therefore, by definition of \( HH_0(\varphi) \), we have

\[
HH_0(\varphi)(\text{Tr}_{B_c}(1)) = \text{Tr}_{B_{c'}}(1) .
\]

Next, the map \( k_0(\varphi) \) preserves rank, and hence restricts to an isomorphism of reduced \( \mathbb{K}\)-groups: \( k_0(\varphi) : k_0(B_c) \to k_0(B_{c'}) \). Choosing the classes \([eP_\tau(c)]\) and \([eP_\tau(c')]\) (see [2]) as bases in \( k_0(B_c) \) and \( k_0(B_{c'}) \) respectively, we can represent the isomorphism \( k_0(\varphi) \) by an invertible integral-valued matrix \( \|m_{\tau\sigma}\| \in GL_N(\mathbb{Z})\) of dimension \( N = |\Gamma| - 1\):

\[
\tilde{k}_0(\varphi) : [eP_\tau(c)] \mapsto \sum_{\sigma \in \Gamma^*} m_{\tau\sigma} [eP_\sigma(c')] , \quad \tau \in \Gamma^* .
\]

The commutative diagram

\[
\begin{array}{ccc}
k_0(B_c) & \xrightarrow{\chi_{B_c}} & HH_0(B_c) \\
\tilde{k}_0(\varphi) \downarrow & & \downarrow HH_0(\varphi) \\
k_0(B_{c'}) & \xrightarrow{\chi_{B_{c'}}} & HH_0(B_{c'})
\end{array}
\]

produces a system of equations

\[
\sum_{\sigma \in \Gamma^*} m_{\tau\sigma} \chi_{B_{c'}}([eP_\sigma'(c')]) = (HH_0(\varphi) \circ \chi_{B_c}) [eP_\tau'(c')] , \quad \tau \in \Gamma^* ,
\]

which can be written explicitly (use [9], [10] and [12]) as follows

\[
\left( \sum_{\sigma \in \Gamma^*} m_{\tau\sigma} G_\sigma(nc') - G_\tau(nc) \right) \cdot \text{Tr}_{B_{c'}}(1) = 0 .
\]

Now we are in position to prove the first part of Theorem [3]. As mentioned in the Introduction, we need only to establish the implication: \( B_c \cong B_{c'} \Rightarrow c = c' \) or \( c = -c' - 1 \), and for \( n = 2 \), this is already known (due to Dixmier [13], Théorème 6.4, and Hodges [11], Theorem 3).

Thus, we fix \( n \geq 3 \) and assume that \( B_c \cong B_{c'} \) for some \( c \not\in \mathbb{Q} \). Then, as explained in the Introduction, we may also assume that \( c' \not\in \mathbb{Q} \). By Lemma [11] we have \( \text{Tr}_{B_{c'}}(1) \neq 0 \), and hence, for \( c, c' \not\in \mathbb{Q} \), the equations (14) are satisfied if and only if

\[
\sum_{\sigma \in \Gamma^*} m_{\tau\sigma} G_\sigma(nc') = G_\tau(nc) , \quad \tau \in \Gamma^* .
\]

Now, the trivial representation of \( W \) corresponds to the partition \( (n) \) so that \( F_{\text{triv}}(x) = \prod_{k=0}^{n-1} (x + k) \) by [4]. Each function \( G_\lambda(x) , \lambda \in \Gamma^* \), can be developed then into elementary fractions

\[
G_\lambda(x) = \sum_{k=1}^{n-1} \frac{a_{\lambda,k}}{x + k}
\]
with \( a_{\lambda,k} \in \mathbb{Q} \). Specifically, we see from (10) that

\[
(17) \quad a_{\lambda,k} = -\dim(\lambda) \frac{\prod_{i,j \in \gamma(\lambda)} (j - i - k)}{\prod_{i \neq k} (l - k)}.
\]

Taking the subset of representations \( \{\lambda_m \mid m = 1, 2, \ldots, n-1\} \) indexed by the partitions \( \operatorname{Part}(\lambda_m) := (m, 1^{n-m}) \), we check easily (with (17)) that \( a_{\lambda_m, k} = 0 \) for \( k < m \), while \( a_{\lambda_m, m} \neq 0 \) for all \( m = 1, 2, \ldots, n-1 \). Hence, each of elementary fractions \( 1/(x+k) \) that occur in (16) can be expressed as a \( \mathbb{Q} \)-linear combination of the functions \( G_{\lambda_1}(x), G_{\lambda_2}(x), \ldots, G_{\lambda_{n-1}}(x) \). It follows then from (16) that there are some numbers \( b_{kj} \in \mathbb{Q} \) (depending on \( m_{\lambda,\gamma} \)) such that

\[
\frac{1}{nc+k} = \sum_{j=1}^{n-1} \frac{b_{kj}}{nc'+j} \quad \text{for } k = 1, 2, \ldots, n-1.
\]

Letting \( x = nc' \), we can rewrite these equations in the form

\[
(18) \quad nc = \frac{f(x)}{g_1(x)} - 1 = \frac{f(x)}{g_2(x)} - 2 = \ldots = \frac{f(x)}{g_{n-1}(x)} - (n-1),
\]

where \( f(x) := (x+1)(x+2)\ldots(x+n-1) \) and \( g_1(x), g_2(x), \ldots, g_{n-1}(x) \) are some nonzero polynomials in \( \mathbb{Q}[x] \) of degree \( \leq n-2 \).

Under the assumption that \( c' \notin \mathbb{Q} \), all the equalities in (18) starting with the second one, should hold as identities in \( \mathbb{Q}(x) \) (that is, not only for \( x = nc' \) but for all \( x \in \mathbb{C} \)). Indeed, if this were not the case for some \( k \), the corresponding difference \( f(x)/g_{k+1}(x) - f(x)/g_k(x) - 1 \) would provide a non-trivial rational polynomial having \( c' \) as a root. Now, since \( \deg[f(x)] > \deg[g_k(x)] \), the zero set of each function \( f(x)/g_k(x) \) is a non-empty subset of \( \{-1, -2, \ldots, 1 - n\} \), the set of roots of \( f(x) \).

On the other hand, any two of these functions, say \( f(x)/g_j(x) \) and \( f(x)/g_k(x) \) with \( j \neq k \), cannot have zeros in common because \( f(x)/g_j(x) - f(x)/g_k(x) = j - k \) by (15). Since the number of fractions \( f(x)/g_k(x) \) matches exactly the number of zeros of \( f(x) \), we conclude that each \( f(x)/g_k(x) \) has one and only one zero, and therefore, must be of the form \( f(x)/g_k(x) = \frac{1}{q_k}(x-x_k) \) with some \( q_k \in \mathbb{Q}^* \) and \( x_k \in \{-1, -2, \ldots, 1 - n\} \). Substituting these into (18) gives \( q_1 = q_2 = \ldots = q_{n-1} = q \) and \( x_k = x_1 + (1-k)q \), where \( k = 1, 2, \ldots, n-1 \). Since each \( x_k \in \{-1, -2, \ldots, 1 - n\} \), we may have only two possibilities: \( x_k = -k, q = 1 \) and \( x_k = k-n, q = -1 \), which give the two required relations: \( c = c' \) and \( c = -c' - 1 \) respectively. This finishes the proof of Theorem 3(a).

4. Proof of Theorem 3(b)

Our next goal is to classify the algebras \( \{B_c\} \) up to Morita equivalence. For this, we will extend and slightly refine the argument given in the proof of Theorem 3(a). We start with a general (and perhaps, well-known) ring-theoretic result to be needed later.

Lemma 3. Let \( B \) be a Noetherian domain, and let \( P \) be a finitely-generated projective module which is a generator in the category of right \( B \)-modules. Assume that \( B' := \operatorname{End}_B(P) \) is a domain. Then, \( P_B \) is isomorphic to a right ideal in \( B \), and \( B'P \) is isomorphic to a left ideal in \( B' \).
Proof. By Morita’s theorem, $B'$ is equivalent to $B$, and hence is Noetherian. By Goldie’s theorem, $B'$ satisfies then Ore’s condition: every pair of nonzero right ideals in $B'$ has nonzero intersection. This means that $B'$ (regarded as a right module over itself) is uniform. Being uniform is a Morita-invariant property (see [MR, Lemma 3.5.8(vi)]). Hence, the $B$-module $P$ which corresponds to $B'$ under the Morita equivalence is also uniform.

Now, by the Dual basis lemma (see, e.g., [MR, Lemma 3.5.2]), we have $P \cdot P^* = B'$, where $P^* := \text{Hom}_B(P, B)$. Therefore we may find $\xi_1, \ldots, \xi_n \in P$ and $\theta_1, \ldots, \theta_n \in P^*$, such that $\sum_{i=1}^n \xi_i \cdot \theta_i = 1$. We claim that at least one of the maps $\theta_i$ is injective. To see this, first observe that $\bigcap_{i=1}^n \ker(\theta_i) = \{0\}$. Indeed, if $\xi \in \bigcap_{i=1}^n \ker(\theta_i)$, then $\xi = 1 \cdot \xi = \sum_{i=1}^n \xi_i \theta_i(\xi) = 0$. Now, assuming that $\ker(\theta_i) \neq \{0\}$ for each $i = 1, 2, \ldots, n$, we may choose $m < n$ so that $\bigcap_{i=1}^m \ker(\theta_i) \neq \{0\}$ while $\bigcap_{i=1}^{m+1} \ker(\theta_i) = \{0\}$. Hence, $\bigcap_{i=1}^m \ker(\theta_i) \bigoplus \ker(\theta_{m+1}) \subseteq P$, which contradicts the uniformity of $P$. Thus, $\theta_i : P \to B$ is injective for some $i$, and we may use it to identify $P$ with a right ideal in $B$.

Finally, by the standard Morita theory (cf. [MR, Corollary 3.5.4(b)]), if $P_B$ is a finite projective generator in the category of right $B$-modules, then $B'P$ is a finite projective generator in the category of left $B'$-modules, and $\text{End}_B(B') \cong B$. Repeating the above argument verbatim (with roles of $B$ and $B'$ interchanged) shows that $B'P$ is isomorphic to a left ideal in $B'$.

Now, we recall that any equivalence functor $F : \text{Mod}(B) \to \text{Mod}(B')$ between module categories is isomorphic to $P \otimes_B -$ for some finitely generated projective module $P = P_B$. Moreover, such an $F$ induces isomorphisms $K_0(F) : K_0(B) \to K_0(B')$ and $\text{HH}_0(F) : \text{HH}_0(B) \to \text{HH}_0(B')$ making commutative the following diagram

$$
\begin{array}{ccc}
K_0(B) & \xrightarrow{\chi_B} & \text{HH}_0(B) \\
K_0(F) \downarrow & & \downarrow \text{HH}_0(F) \\
K_0(B') & \xrightarrow{\chi_{B'}} & \text{HH}_0(B')
\end{array}
$$

(19)

By definition, the map $K_0(F)$ takes the class of the free module $[B] \in K_0(B)$ to $[B]P \in K_0(B')$, while $\chi_B[B] = \text{Tr}_B(1)$ in $\text{HH}_0(B)$. Hence, by commutativity of (19), we have

$$
\text{HH}_0(F)(\text{Tr}_B(1)) = \chi_{B'}[P].
$$

(20)

Returning to our situation, let $B := B_c$, $B' := B_{c'}$, and let $F$ be a $\mathbb{C}$-linear equivalence of categories: $\text{Mod}(B_c) \to \text{Mod}(B_{c'})$. Since both $B_c$ and $B_{c'}$ are domains, we have $F \simeq P \otimes_{B_c} -$ with $P = B_{c'}P_B$ being isomorphic to a right projective ideal in $B_c$ and to a left projective ideal in $B_{c'}$ (see Lemma 3). Regarding $P$ as the latter, we have $\text{rk}[P] = \text{rk}[B_{c'}] = 1$, and therefore $[P] - [B_{c'}] \in K_0(B_{c'})$. We can write then $[P] = [B_{c'}] + \sum_{\lambda \in \Gamma^*} n_{\lambda} \cdot [eF_{\lambda}(c')]$ in $K_0(B_{c'})$ for some $n_{\lambda} \in \mathbb{Z}$, and compute (with (3))

$$
\chi_{B_{c'}}[P] = \left(1 + \sum_{\lambda \in \Gamma^*} n_{\lambda} G_{\lambda}(nc')\right) \cdot \text{Tr}_{B_{c'}}(1), \quad c' \in \mathbb{C}^{\text{reg}}.
$$

(21)
Looking at the diagram \((13)\) and taking into account \((20)\) and \((21)\), we find
\[
(HH_0(\mathcal{F}) \ast \chi_{B_\tau})(\overline{eP}_\tau(c)) = HH_0(\mathcal{F})(G_\tau(nc) \cdot Tr_{B_\tau}(1)) \\
= G_\tau(nc) HH_0(\mathcal{F})(Tr_{B_\tau}(1)) \quad \text{(by \(\mathbb{C}\)-linearity of \(\mathcal{F}\))} \\
= G_\tau(nc) \left(1 + \sum_{\lambda \in \Gamma^*} n_\lambda G_\lambda(nc')\right) \cdot Tr_{B_\tau}(1)
\]

On the other hand,
\[
(\chi_{B_{c'}} \ast K_0(\mathcal{F}))(\overline{eP}_\tau(c)) = \sum_{\sigma \in \Gamma^*} m_{\tau \sigma} \cdot \chi_{B_{c'}}(\overline{eP}_\sigma(c')) \\
= \sum_{\sigma \in \Gamma^*} m_{\tau \sigma} G_\sigma(nc') \cdot Tr_{B_{c'}}(1)
\]

where \(|m_{\tau \sigma}| \in \text{GL}_N(\mathbb{Z})\) is an invertible integral-valued matrix representing the isomorphism \(K_0(\mathcal{F})\) in the bases \(\{\overline{eP}_\sigma(c')\}\) \(\subset K_0(B_{c'})\) and \(\{\overline{eP}_\tau(c)\}\) \(\subset K_0(B_\tau)\). Thus, by commutativity of \((19)\), we have
\[
(22) \quad \left(\sum_{\sigma \in \Gamma^*} m_{\tau \sigma} G_\sigma(nc') - G_\tau(nc) \left(1 + \sum_{\lambda \in \Gamma^*} n_\lambda G_\lambda(nc')\right)\right) \cdot Tr_{B_{c'}}(1) = 0.
\]

When \(c' \notin \mathbb{Q}\), we have \(Tr_{B_{c'}}(1) \neq 0\), so in that case \((22)\) is equivalent to the system of equations
\[
(23) \quad \sum_{\sigma \in \Gamma^*} m_{\tau \sigma} G_\sigma(nc') = G_\tau(nc) \left(1 + \sum_{\lambda \in \Gamma^*} n_\lambda G_\lambda(nc')\right), \quad \tau \in \Gamma^*.
\]

Now, we are in position to prove part \((b)\) of Theorem \(8\). First of all, we observe that the same argument as we used in the proof of the first part of this theorem (reducing the system \((15)\) to \((18)\)) works for the system \((23)\) as well. As a result, we get from \((23)\) a set of \((n - 1)\) algebraic equations of the form \((15)\), with \(f(x)\) depending now on \(a_{\lambda,k}\) and new parameters \(n_\lambda \in \mathbb{Z}\). Specifically,
\[
(24) \quad f(x) = \prod_{k=1}^{n-1} (x + k) + \sum_{k=1}^{n-1} a_k \prod_{j=1, j \neq k}^{n-1} (x + j),
\]
where \(a_k := \sum_{\lambda \in \Gamma^*} n_\lambda a_{\lambda,k}\) (and \(a_{\lambda,k}\) are defined by \((16)\), \((24)\)). Under the assumption that \(c'\) is non-algebraic, the system \((15)\) again gives us the relations
\[
(25) \quad nc = (x - x_1)/q - 1 = (x - x_2)/q - 2 = \ldots = (x - x_{n-1})/q - (n - 1),
\]
where \(\{x_1, x_2, \ldots, x_{n-1}\}\) is the set of roots of \((24)\) and \(q \in \mathbb{Q}^\times\). Summing up these relations and using Viète’s theorem for \(f(x)\), we find
\[
(n - 1)nc = (n - 1)nc'/q + \sum_{k=1}^{n-1} (k + a_k)/q - n(n - 1)/2,
\]
or equivalently,
\[
(26) \quad q \left(c + \frac{1}{2}\right) = \left(c' + \frac{1}{2}\right) + \frac{1}{n(n - 1)} \sum_{k=1}^{n-1} a_k.
\]
Thus, to finish our proof it remains to show that $q = \pm 1$ and the last term in (26) is an integer. As we will see, the former follows from the latter, while the latter is an immediate consequence of the following observation.

**Lemma 4.** For any $\lambda \in \Gamma^*$ and $k = 1, 2, \ldots, n - 1$, the numbers $a_{\lambda,k}$ are integers divisible by $n(n-1)$.

Indeed, by (17), we have

$$a_{\lambda,k} = (-1)^{n-k-1} \frac{\text{dim}(\lambda)}{k!(n-1-k)!} F_{\lambda'}(k), \quad k = 1, 2, \ldots, n-1,$$

where $F_{\lambda'}(x)$ is the content polynomial of the Young diagram conjugate to $Y(\lambda)$. Using the “hook formula” for the dimension of irreducible representations of $S_n$ (see, e.g., [3], Theorem 20.1) and a well-known formula for the Schur function ([4], Example 4, p. 45), we can rewrite (27) in the form

$$a_{\lambda,k} = (-1)^{n-k-1} n \binom{n-1}{k} s_{\lambda'}(1^k),$$

where $s_{\lambda'}(1^k)$ is the Schur function of the conjugate partition of $\lambda$ evaluated at $x = (1, \ldots, 1, 0, \ldots, 0)$ (the first $k$ symmetric variables are equal to 1, the rest are zero). So we need only to see that $(n-1) s_{\lambda'}(1^k)$ is divisible by $n-1$. Since each Schur function $s_{\lambda'}(x)$ can be written as a linear combination of monomial symmetric functions $m_{\sigma}(x)$ with integer coefficients:

$$s_{\lambda'}(x) = \sum_{\sigma \in \Gamma} K_{\lambda',\sigma} m_{\sigma}(x), \quad K_{\lambda',\sigma} \in \mathbb{Z},$$

it suffices to show that $\frac{1}{n-1} (n-1) s_{\lambda'}(1^k)$ is an integer for every $\sigma \in \Gamma$. By [4], Example 1(a), p. 26, we have

$$\binom{n-1}{k} = \binom{n-1}{k} \frac{l!}{\mu_1! \mu_2! \cdots l} = \frac{(n-1)!}{(n-1-k)!(k-l)! \mu_1! \mu_2! \cdots},$$

where $l = l(\sigma)$ is the length of the partition of $\sigma$ and $\mu_i = \mu_i(\sigma)$ is the number of parts of $\text{Part}(\sigma)$ equal to $i$. Now, since $(n-1-k) + (k-l) + \mu_1 + \mu_2 + \cdots = n-1$ while $\sum_{i>1} \mu_i = n$, the numbers $\{n-1-k, k-l, \mu_1, \mu_2, \ldots\}$ are relatively prime. Hence, there exist some $a, b, c, d, \ldots \in \mathbb{Z}$ such that

$$(n-1-k)a + (k-l)b + \mu_1c + \mu_2d + \cdots = 1.$$
implies \( q q' = 1 \), i.e. \( q \) is a unit in \( \mathbb{Z} \). With Lemma 4, this argument completes the proof of the theorem.

**Remark.** Using formulas (28), (29) and (30), we can compute the sum in (20) explicitly:

\[
\sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1} \sum_{\lambda \in \Gamma^*} n_\lambda a_{\lambda,k} = \sum_{k=1}^{n-1} (-1)^{n-k-1} n \binom{n-1}{k} \sum_{\lambda \in \Gamma^*} n_\lambda s_{\lambda'}(1^k)
\]

\[
= \sum_{k=1}^{n-1} (-1)^{n-k-1} n \binom{n-1}{k} \sum_{\lambda \in \Gamma^*} n_\lambda \sum_{\sigma \in \Gamma} K_{\lambda' \sigma} m_\sigma(1^k)
\]

\[
= \sum_{\sigma \in \Gamma} \tilde{n}_\sigma \sum_{k=1}^{n-1} (-1)^{n-k-1} n \binom{n-1}{k} m_\sigma(1^k)
\]

\[
= \sum_{\sigma \in \Gamma} \tilde{n}_\sigma \frac{n (n-1) \ldots (n-l)}{\mu_1! \mu_2! \ldots} \sum_{k=1}^{n-1} (-1)^{n-k-1} \frac{(n-l-1)!}{(n-1-k)! (k-l)!} ,
\]

where we denote \( \tilde{n}_\sigma := \sum_{\lambda \in \Gamma^*} K_{\lambda' \sigma} n_\lambda \) for each \( \sigma \in \Gamma \). By Newton's binomial formula, the last sum is zero unless \( l = n-1 \), and it is equal to 1 in that case. Now there is only one partition of weight \( n \) and length \( l = n-1 \), namely \((2, 1^{n-2})\), in which case we have \( \mu_1 = n-2, \mu_2 = 1 \) and \( \mu_i = 0 \) for \( i \geq 2 \). Thus,

\[
\sum_{k=1}^{n-1} a_k = n (n-1) \tilde{n}_\alpha = n (n-1) \sum_{\lambda \in \Gamma^*} K_{\lambda' \alpha} n_\lambda ,
\]

where \( \alpha \in \Gamma \) is the representation of \( W \) corresponding to the partition \((2, 1^{n-2})\).

Now, comparing (20) and (31), we see that the parameters of Morita equivalent algebras \( B_c \) and \( B_{c'} \) are related to the trace of the corresponding equivalence functor (21) by the formula

\[
\pm (c + 1/2) = (c' + 1/2) + \sum_{\lambda \in \Gamma^*} K_{\lambda' \alpha} n_\lambda .
\]

### 5. Proof of Theorem 2

In view of Theorem 1 Morita classifications for the families of algebras \{\( H_c \)\} and \{\( B_c \)\} must be equivalent, at least in case when \( c \in \mathbb{C}^{reg} \). Thus, part (b) of Theorem 2 follows from part (b) of Theorem 4 which we have proved in Section 4 above. In part (a), the ‘if’ implication is obvious, and we only need to show its converse, i.e. \( H_c \cong H_{c'} \Rightarrow c = \pm c' \).

First of all, when isomorphic, the algebras \( H_c \) and \( H_{c'} \) are Morita equivalent, and therefore, by part (b), we have at once

\[
c = \pm c' + l \quad \text{for some } l \in \mathbb{Z} .
\]
Arguing now as in Section 3 and using the trace formula \(b\) for \(H_c\), we may derive a system of equations similar to (14), namely

\[
(33) \quad \left( \frac{\dim(\tau)}{n!(nc)^n} F_\tau(nc) - \sum_{\sigma \in \Gamma} m_{\tau\sigma} \frac{\dim(\sigma)}{n!(nc')^n} F_\sigma(nc') \right) \cdot \text{Tr}_{H_c}(1) = 0, \quad \tau \in \Gamma,
\]

where \(\|m_{\tau\sigma}\| \in \text{GL}(\Gamma)(\mathbb{Z})\) and \(F_\sigma, F_\tau\) are defined in (11). Again, as \(c' \not\in \mathbb{Q}\), the trace factor \(\text{Tr}_{H_c}(1)\) can be dropped by Lemma 1 and we can rewrite (33) in the form

\[
F_\tau(nc)(nc')^n = \sum_{\sigma \in \Gamma} k_{\tau\sigma} F_\sigma(nc') (nc)^n, \quad k_{\tau\sigma} \in \mathbb{Q}.
\]

In particular, letting \(\tau = \text{triv}\) and taking into account (22), we have

\[
(34) \quad (+x + nl + 1) \ldots (+x + nl + n - 1) x^n = \sum_{\sigma \in \Gamma} k_{\tau\sigma} F_\sigma(x) (+x + nl)^{n-1},
\]

where \(x = nc'\). Since \(c' \not\in \mathbb{Q}\), the equation (34) should hold identically in \(x\). Substituting then \(x = \mp nl\) gives \((n - 1)! (+nl)^n = 0\), whence \(l = 0\). With (22), this finishes the proof of Theorem 2(a).

6. Concluding Remarks

As already mentioned in the Introduction, we expect Theorems 2 and 3 to be true not only for transcendental but for all regular values of \(c\). This is immediate in case \(n = 2\) (see [11]) but seems to be much harder to prove in higher dimensions. Reviewing the proofs of Section 3 and 4 shows that their argument works also for algebraic \(c\), provided the degree of the extension of fields \(\mathbb{Q}(c)/\mathbb{Q}\) is large enough: specifically, \(\mathbb{Q}(c) : \mathbb{Q} \geq n(n-1)\).

It is worth noting that the field \(\mathbb{Q}(c)\) is a Morita invariant for the whole family of algebras \(\{B_c\}\) (with no restrictions imposed on \(c\)). Indeed, if both \(c\) and \(c'\) are regular, then using Proposition 1(b) (instead of Lemma 1) still allows one to reduce (22) to the form (11), and the first equation of (18) implies immediately that \(\mathbb{Q}(c) \subseteq \mathbb{Q}(c')\). By symmetry, we then also have \(\mathbb{Q}(c') \subseteq \mathbb{Q}(c)\). Now, if \(c, c'\) are both singular then \(c, c' \in \mathbb{Q}\), and therefore \(\mathbb{Q}(c) = \mathbb{Q}(c') = \mathbb{Q}\). On the other hand, if one of the parameters is singular, say \(c\), while \(c'\) is regular, the algebras \(B_c\) and \(B_{c'}\) cannot be Morita equivalent. Indeed, in this case \(B_{c'}\) is a simple ring (by Theorem 1) while \(B_c\) is not. To see the latter, note that if \(c \not\in \mathbb{C}^{\text{reg}}\) then \(c\) is singular in the sense of [13]: for such \(c\), the standard \(H_c\)-module \(M(\text{triv}) = \mathbb{C}[h]\) corresponding to the trivial representation is reducible. Let \(J\) be a proper submodule of \(M(\text{triv})\) which we regard as an ideal in \(\mathbb{C}[h]\). Then, \(J^W := J \cap \mathbb{C}[h]^W\) is a nonzero ideal in \(\mathbb{C}[h]^W\) acting trivially on the nonzero \(B_c\)-module \(e(M/J)\). Thus, \(J^W\) is contained in the annihilator of \(e(M/J)\), and hence \(\text{Ann}_{B_c}(e(M/J))\) is a proper two-sided ideal of \(B_c\).

In the end, we would like to make one, perhaps somewhat speculative observation. According to Theorem 2 a Morita class of \(H_c\) is a function (at least, for \(c \not\in \mathbb{Q}\)) of \(e^{2\pi ic}\), that is, of the monodromy representation of the system of Dunkl operators associated to \((W, c)\) (see [14]).

**Question.** Is there a genuine relation between the Morita classification of Cherednik algebras \(H_c(W)\) and the monodromy representation of the corresponding Hecke algebras \(\mathcal{H}_W(e^{2\pi ic})\)?
Appendix: Proof of Theorem 4

The purpose of this Appendix is to prove Theorem 4 announced in the Introduction. Our proof below does not rely on and can be read independently of the rest of the paper.

First, we fix notation and recall the claim to be proved. Let $V$ be a complex symplectic vector space, i.e., a finite-dimensional vector space over $\mathbb{C}$ equipped with a nondegenerate skew-symmetric bilinear form $(\cdot, \cdot)$. Let $G \subset \text{Sp}(V)$ be a finite group of linear symplectic transformations of $V$. Write $A := \mathbb{C}[V]^G$ for the algebra of polynomial $G$-invariants. The form $(\cdot, \cdot)$ determines a structure of the (graded) Poisson algebra on $\mathbb{C}[V]$, and this structure restricts naturally to $A$. Let $\{A,A\}$ denote the linear span of Poisson brackets of all elements of $A$. Then Theorem 4 states: $\dim_{\mathbb{C}}(A/\{A,A\}) < \infty$.

To show this, observe that the quotient $A/\{A,A\}$ has a natural grading, each graded component $(A/\{A,A\})_n := A_n/\sum i+j=n+2 \{A_i,A_j\}$ being finite-dimensional. Set $(A/\{A,A\})^*_n := \text{Hom}_{\mathbb{C}} ([\{A,A\}], \mathbb{C})$ for $n = 0, 1, 2, \ldots$. Then we have

**Lemma 5.** $(A/\{A,A\})^*_n$ is isomorphic to the subspace of polynomials $P \in \mathbb{C}[V]$ of degree $n$ satisfying the equations

$$
\sum_{g \in G} (u, gv) P(u + gv) = 0 \quad \text{for all } u, v \in V.
$$

**Proof.** For a fixed $u \in V$, define the function $L_u : V \to \mathbb{C}$, $x \mapsto L_u(x) := \sum_{g \in G} e^{(u, gx)}$. We can think of $L_u(x)$ as a generating function for the graded vector space $A$ (meaning that the coefficients of $L_u(x)$, when $L_u(x)$ is regarded as a series in $u$, span $A$). Now, observe that $\{L_u, L_v\}(x) = \sum_{g \in G} (u, gv) L_{u+gv}(x)$ for all $u, v \in V$. Hence, if $f : A \to \mathbb{C}$ is a linear homogeneous functional such that $f(\{A,A\}) = 0$, then $u \mapsto f(L_u)$ is a homogeneous $G$-invariant polynomial $P(u)$ satisfying (35). Conversely, if $P$ satisfies (35), then $P$ defines a functional $f_P$ on $A$ vanishing on $\{A,A\}$.

Thus, to prove Theorem 4, it suffices to show that the functional equations (35) may have only finitely many linearly independent polynomial solutions. Replacing $v \mapsto vt$ and expanding the left-hand side of (35) into Taylor series in $t$, we get a system of linear differential equations for $P$:

$$
\sum_{g \in G} (u, gv) \partial^m_{gv} P(u) = 0 \quad \text{for all } u \in V \text{ and } m = 0, 1, 2, \ldots,
$$

where $\partial^m_{gv} P(u) := (d/dt)^m P(u + gt)|_{t=0}$. It suffices to prove that the space of local (holomorphic) solutions of (36) in a neighborhood of some point $u_0 \in V$ is finite-dimensional. This would, in turn, follow if we establish that $\mathbb{C}[V]$ is a finite module over the subalgebra the symbols of differential equations (36) generate at $u_0$. Thus, we need to show that $\mathbb{C}[V]$ is finite over the algebra generated by polynomials $x \mapsto Q_{v,m}(x) := \sum_{g \in G} (u_0, gv)(x, gv)^m$ for all $v$ and $m$.

Assume the contrary. Then, being homogeneous, the ideal of $\mathbb{C}[V]$ generated by $Q_{v,m}$ must be of infinite codimension in $\mathbb{C}[V]$, and therefore vanishes at some point $x_0 \neq 0$. In other words, the system of polynomial equations $Q_{v,m}(x) = 0$, and therefore the system

$$
\sum_{g \in G} (u_0, gv) e^{t(x, gv)} = 0 \quad \text{(for all } v \in V \text{ and } t \in \mathbb{C})
$$

vanishes at some point $x_0\neq 0$. This contradicts the assumption, and thus Theorem 4 is proved.

$\square$
have a nonzero solution \( x = x_0 \).

Let \( O_{x_0} \subset V \) be the orbit, and \( G_{x_0} \subseteq G \) be the stabilizer of \( x_0 \) under the action of \( G \). Choose \( v \in V \) in such a way that \( (x, v) \neq (x', v) \) whenever \( x, x' \in O_{x_0} \) and \( x \neq x' \). It follows then from (35) that \( (u_0, \sum_{g \in G_{x_0}} gv) = \sum_{g \in G_{x_0}} (u_0, gv) = 0 \). In other words, \( u_0 \) is orthogonal to \( \sum_{g \in G_{x_0}} gv \) for generic (and hence, any) vector \( v \in V \). Thus, \( u_0 \) is orthogonal to the image of the linear \( G_{x_0} \)-symmetrizer acting on \( V \). Every such image is a nonzero subspace of \( V \) (as it contains \( x_0 \)), and there are only finitely many of these. So, if a priori we choose \( u_0 \) not to be orthogonal to any of such subspaces, we get a contradiction finishing the proof of the theorem.

We close this section with two remarks related to the work of J. Alev et al. (see [AF, AL]).

**Remark.** It has been shown in [AF] that \( \dim_{C}(A/\{a,b\}^{2}) < \infty \), where \( \{a,b\}^{2} \) is the subspace of \( A \) spanned by all elements of the form \( \{a,b\}\{c,d\} \). This result follows from Theorem 6: more generally, the theorem implies that \( \{A,A\}^{n} \) has finite codimension in \( A \) for every \( n \geq 2 \). Indeed, being graded and of finite codimension, the subspace \( \{A,A\} \) contains a power of the augmentation ideal of \( A \), i.e. \( (A_{+})^{n} \subseteq \{A,A\} \) for some \( m \geq 1 \). Hence, \( \{A_{+}\}^{mn} \subseteq \{A,A\}^{n} \), and therefore \( \{A,A\}^{n} \) has also finite codimension in \( A \).

**Remark.** In Section 2, we apply Theorem 6 in the situation when \( V = \mathfrak{h} \oplus \mathfrak{h}^{*} \) and \( G = W \) is the symmetric group (so that \( A = C[\mathfrak{h} \oplus \mathfrak{h}^{*}]^{W} \)). In this particular case we expect that \( \dim_{C}(A/\{A,A\}) = 1 \). This would follow from Proposition 1 (and Théorème 6 of [AL]), if one would be able to show that \( \mathcal{H}_{0}(B_{c}) \) is a deformation of the Poisson homology of \( \text{gr}(B_{c}) \), i.e.

\[
\text{gr} \mathcal{H}_{0}(B_{c}) = \text{gr}(B_{c})/\{\text{gr}(B_{c}), \text{gr}(B_{c})\},
\]

where \( \text{gr} \mathcal{H}_{0}(B_{c}) \) is formed with respect to the quotient induced filtration on \( \mathcal{H}_{0}(B_{c}) \). An alternative way would be to solve exactly the system of functional equations \( \Theta_{m} \) in case when \( V = \mathfrak{h} \oplus \mathfrak{h}^{*} \) and \( G = W \).

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