Which Urbanik class $L_k$, do the hyperbolic and the generalized logistic characteristic functions belong to?

Zbigniew J. Jurek

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Abstract. Selfdecomposable variables obtained from series of Laplace (double exponential) variables are objects of this study. We proved that hyperbolic-sine and hyperbolic-cosine variables are in the difference of the Urbanik classes $L_2$ and $L_3$ while generalized logistic variable is at least in the Urbanik class $L_1$. Hence some ratios of those corresponding selfdecomposable characteristic functions are again selfdecomposable.

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The class of infinitely divisible distributions, ID, plays an important role in the theory of limiting distributions. It coincides with limiting distributions of sums of infinitesimal triangular arrays and is intimately connected with Lévy stochastic processes. When triangular infinitesimal arrays are obtained from normalized partial sums of sequences of independent variables, at a limit, one gets the class, $L$, of selfdecomposable distributions. If we have sequences of independent and identically distributed variables we obtain class, $S$, of stable distributions and in particular, Gaussian (normal) distributions. For a history of that topic in probability see Feller (1966), Chapter XVII or Gnedenko and Kolmogorov (1954), Sect. 17-19 or Loeve (1963), Sect. 23.

On the other hand, let us also mention that more recently selfdecomposability appeared in some statistical applications, in particular, in models for

*Institute of Mathematics, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland; zjjurek@math.uni.wroc.pl
option pricing in mathematical finance, cf. Carr-Geman-Madan-Yor (2007) or Trabs (2014), as well in statistical physics - Ising models, (cf. De Coninck and Jurek (2000) and Jurek (2001)(a)).

1. Urbanik $L_k, k = 0, 1, 2, ..., \infty$, classes.

Urbanik (1972) (a summary of results) and Urbanik (1973) (results with proofs) introduced and described a decreasing family of classes, $L_k, k = 0, 1, 2, ..., \infty$, (here $L$ stands for Lévy’s name), of distributions obtained in some schemes of limiting procedures, and in such a way that we have the following proper inclusions:

$\text{(Gaussian} \subset S \subset \ldots \subset L_k \subset L_{k-1} \subset \ldots \subset L_1 \subset L_0 \equiv L \subset ID,$

$L_\infty := \bigcap_{k=0}^{\infty} L_k =$ the smallest closed convolution semigroup containing all stable distributions. \hfill (1)

Analytically, in terms of the characteristic function (in short: char. f.) , $\phi(t)$, we have the following characterization

$\phi(t) \in L_k \iff \forall (0 < c < 1) \phi(t)/\phi(ct) \in L_{k-1}, \quad \text{where} \quad L_{-1} := ID,$ \hfill (2)

cf. Urbanik (1973), Proposition 1, Theorem 2 and Corollary 1.

In terms of random variables Urbanik classes $L_k$ are described as follows:

$X \in L_k \iff \forall (t > 0) \exists (X_t \in L_{k-1}) \quad X \overset{d}{=} e^{-t}X + X_t,$ \hfill (3)

(the equality in disribution) where variables $X_t$ and $X$ are (stochastically) independent.

Recalling the (general) form of the stable characteristic functions and using the description (2), we infer that the stable (in particular, the Gaussian) distribution belong to the Urbanik class $L_\infty$.

The structural stochastic characterization of variables $X$ (the random integral representation) for the classes $L_k, k = 0, 1, 2, ..., $ is the following

$X \in L_k \quad \text{iff} \quad X = \int_0^\infty e^{-t}dY_X(t), \quad Y_X(1) \in L_{k-1} \quad \text{and} \quad \mathbb{E}[\log(1 + |Y_X(1)|)] < \infty; \hfill (4)$

or

$X \in L_k \quad \text{iff} \quad X = \int_0^\infty e^{-t}dZ_X(t), \quad (Z_X(t), t \geq 0) \quad \text{are Lévy processes referred to as the background driving Lévy process} (\text{in short: BDLP})$
the remainders \((X_t, t > 0)\) in (3). Moreover, to the variable \(Y_X(1)\) we refer as the background driving variable (in short: BDR); cf. Jurek and Vervaat (1983) for the class \(L_0\); or Jeanblanc, Yor and Chesney (2009), Proposition 11, p. 597. For other classes \(L_k\), see Jurek (1983)(a). Comp. also Sato (1980).

In the past mostly the class \(L_0 \equiv L\) of selfdecomposable distributions, (this terminology is justified by the decomposition in (3)), was studied and applied in mathematical finance or statistical physics.

See Jurek and Mason (1993), Chapter 3, and references therein, or Jurek (1983)(b), for the generalization of Urbanik classes to infinite dimensional Banach space valued random vectors and the normalization by bounded linear operators.

[For a general conjecture concerning random integral representations, see: www.math.uni.wroc.pl/~zjjurek/]

**2. Results and some corollaries.**

In this note we consider primary selfdecomposable distributions that are obtained as sums of series of double exponential \(\eta\) random variables (also called Laplace distributions). We prove

**Theorem 1.** (a) The hyperbolic-sine \(\hat{S}\) and hyperbolic-cosine \(\hat{C}\) distributions with the characteristic functions \(\phi_{\hat{S}}(t) = \frac{t}{\sinh(t)}\) and \(\phi_{\hat{C}}(t) = \frac{1}{\cosh(t)}\), respectively, belong to the difference \(L_2 \setminus L_3\) of Urbanik classes.

(b) The hyperbolic-tangent \(\hat{T}\) with characteristic function \(\phi_{\hat{T}}(t) = \frac{\tanh(t)}{t}\) and double-exponential \(\eta\) distribution with the characteristic function \(\phi_{\eta}(t) = 1/(1 + t^2)\) belong to the difference \(L_0 \setminus L_1\) of Urbanik classes.

[The same holds for a finite linear combinations of independent hyperbolic-tangent and double exponential variables.]

(c) The logistic distribution \(l_\alpha\) with characteristic function \(\phi_{l_\alpha}(t) = |\Gamma(\alpha + it/\pi)/\Gamma(\alpha)|^2\) belongs at least to the Urbanik class \(L_1\).

Using iteratively the characterization (2) or (3), the above facts lead to the following corollaries.
Corollary 1. Since the hyperbolic-sine \( \phi_S(t) = \frac{t}{\sinh(t)} \in L_2 \) therefore:

(i) for any \( 0 < c < 1 \) functions
\[
\phi_{S_c}(t) := \frac{\sinh(ct)}{c\sinh(t)} \in L_1
\]
are selfdecomposable char. functions;

(ii) for any \( 0 < b, c < 1 \) functions
\[
\phi_{S_{b,c}}(t) := \frac{\sinh(ct) \sinh(bt)}{\sinh(t) \sinh(bct)} \in L_0
\]
are selfdecomposable;

(iii) for any \( 0 < a, b, c < 1 \) functions
\[
t \to \phi_{S_{b,c}}(t)/\phi_{S_{b,c}}(at) = \frac{\sinh(at) \sinh(bt) \sinh(ct) \sinh(abct)}{\sinh(t) \sinh(abt) \sinh(act) \sinh(bct)} \in ID_{\log}
\]
are infinitely divisible characteristic functions with finite logarithmic moments.

The random variable \( \hat{S}_c \), from (i) in the above corollary, is called the Talacko-Zolotarev variable; cf. You (2022), Sec. 2.2, p.11. It maybe viewed as an innovation variable for the BDRV of hyperbolic-sine as we have:

Corollary 2. Let \( Y_S(1) \) be background driving random variable (BDRV) for the hyperbolic-sine variable \( \hat{S} \) and, for \( 0 < c < 1 \), let \( \hat{S}_c \) be the Talacko-Zolotarev variable independent of \( Y_S(1) \). Then we have
\[
Y_S(1) \overset{d}{=} cY_S(1) + \hat{S}_c, \quad (\text{the equality in distribution.})
\]

Similarly as for the case of the hyperbolic-sine, for the hyperbolic-cosine we have the following facts:

Corollary 3. Since the hyperbolic-cosine \( \phi_C(t) = \frac{1}{\cosh(t)} \in L_2 \) we have that

(i) for any \( 0 < c < 1 \),
\[
\phi_{C_c}(t) := \frac{\cosh(ct)}{\cosh(t)} \in L_1
\]
are selfdecomposable functions;

For any \( 0 < b < 1 \) functions

(ii) \( \phi_{C_{b,c}}(t) := \frac{\cosh(ct) \cosh(bt)}{\cosh(t) \cosh(bct)} \in L_0 \) are selfdecomposable;

For any \( 0 < a < 1 \) functions

(iii) \( t \to \phi_{C_{b,c}}(t)/\phi_{C_{b,c}}(at) = \frac{\cosh(at) \cosh(bt) \cosh(ct) \cosh(abct)}{\cosh(t) \cosh(bct) \cosh(act) \cosh(abt)} \in ID_{\log}
\]
are infinitely divisible characteristic functions with finite logarithmic moments.
The same way as for \( \hat{S}_c \) (in Corollary 2) we may look at the variable \( \hat{C}_c \), from above Corollary 3 (i), via its BDCF \( \psi_{\hat{C}}(t) = \exp[-t \tanh(t)] \). All in all we get

**Corollary 4.** Let \( Y_{\hat{C}}(1) \) be the background driving random variable (BDRV) for the hyperbolic-cosine variable \( \hat{C} \) and, for \( 0 < c < 1 \), let \( \hat{C}_c \) be the variable in Corollary 3 (i) above, and independent of \( Y_{\hat{C}}(1) \). Then we have

\[
Y_{\hat{C}}(1) \overset{d}{=} c Y_{\hat{C}}(1) + \hat{C}_c, \quad \text{(the equality in distribution.)}
\]

For the logistic characteristic functions we have

**Corollary 5.** (a) As the logistic \( l_\alpha \in L_1 \), \((\alpha > 0)\) therefore each \( 0 < c < 1 \)

\[
\mathbb{R} \ni t \rightarrow \left| \frac{\Gamma(\alpha + it/\pi)}{\Gamma(\alpha + i\alpha/\pi)} \right|^2
\]

are selfdecomposable characteristic functions.

(b) For \( \alpha > 0 \) and \( t \in \mathbb{R} \) we have the identity:

\[
\int_0^\infty (\cos(tx) - 1) \frac{e^{-\alpha x}}{x(1 - e^{-\alpha tx})} dx = \log |\Gamma(\alpha + it/\pi)| - \log \Gamma(\alpha).
\]

2. Auxiliary facts.

a). **Selfdecomposable variables among infinitely divisible ones.**

The classical Lévy-Khintchine formula gives the description of the infinite divisible random variables \( X \) or distributions \( \mu \) in terms of their characteristic functions. Namely

\[
X \in ID \text{ if } \phi_X(t) := \mathbb{E}[e^{itX}] = \exp\left[it a - \frac{1}{2} \sigma^2 t^2 + \int_{\mathbb{R} \setminus \{0\}} (e^{itx} - 1 - \frac{itx}{1 + x^2}) M(dx)\right],
\]

where the triple \( a \in \mathbb{R}, \sigma^2 \geq 0 \) and the measure \( M \) satisfies the integrability condition: \( \int_{\mathbb{R} \setminus \{0\}} \min(x^2, 1) M(dx) < \infty \), is uniquely determined. In the sequel, for the simplicity, we will write \( X = [a, \sigma^2, M] \), if the above formula holds true.

Recall the following characterization (criterium) for distributions with non-zero Lévy measures:

\[
X = [a, \sigma^2, M] \in L_0 \text{ iff } M(dx) = k(x)dx, \int_{\mathbb{R} \setminus \{0\}} k(x)dx = \infty, \text{ and function}
\]

\[
x \rightarrow \frac{xk(x)}{1 + x^2}
\]

is non-increasing on both halflines \((-\infty, 0)\) and \((0, \infty)\); and equivalently \((-xk(x))' \geq 0\) is the density of the Lévy measure of BDRV \( Y_X(1) \) in the first line above (4),

\[
(5)
\]
Remark 1. Let $D^n$ denote the operator acting on densities of Lévy measures $k$, (from (5)), of selfdecomposable variables, defined as follows:

$$(D^0k)(x) := xk(x); \quad (D^1k)(x) := (-xk(x))'; \quad (D^n k)(x) := D(D^{n-1}k)(x),$$

for $n = 2, 3, ...$

In the examples below, we will be looking for the first $n$ such that $(D^n k)(x)$ is not a density of a Lévy measure of the class $L_0$ distribution, that is, $(D^n k)(x)$ is an integrable positive function or $(D^n k)(x)$ assumes negative values.

b). Series of Laplace (double exponential) variables.

Let $\eta$ denote Laplace (double exponential) variable with the probability density $2^{-1}e^{-|x|}, x \in \mathbb{R}$ and $a > 0$. Then $a\eta$ has the characteristic function

$$\phi_{a\eta}(t) = \frac{1}{1 + (at)^2} = \exp \int_{\mathbb{R}} (\cos(tx) - 1)k_{a\eta}(x)dx, \quad k_{a\eta}(x) := e^{-a^{-1}|x|/|x|},$$

which, for $a = 1$, means that $\eta = [0, 0, k_{\eta}] \in ID$ and the corresponding Lévy measure is equal $M_{\eta}(dx) = k_{\eta}(x)dx$.

Since the function $xk_{\eta}(x) = sign(x)e^{-|x|}$ is non-increasing on both half-lines, by (4), we get that $\eta$ is selfdecomposable; in symbols: $\eta \in L_0$. However, since $(-xk_{\eta}(x))' = e^{-|x|}$ gives finite Lévy measure thus $\eta \notin L_1$. Consequently, variable $\eta \in L_0 \setminus L_1$.

For independent and identically distributed Laplace $\eta_k$ variables and a sequence $a := (a_1, a_2, ...)$ of real numbers we have that

$$X(a) := \sum_{k=1}^{\infty} a_k \eta_k < \infty \quad \text{(almost surely)} \iff \sum_{k=1}^{\infty} a_k^2 < \infty; \quad (7)$$

cf. Jurek (2000) Propositions 1 and 2. Note that without a loss of a generality (as Laplace variables are symmetric) we may assume that $a_k > 0$ and sequence $(a)$ is decreasing to zero.

Since $L_k$ are closed (in the weak topology) convolution semigroups (see (2) or (3)) and $\eta \in L_0$ we conclude that

$$X(a) = \sum_{k=1}^{\infty} a_k \eta_k \in L_0, \quad \text{with} \quad k_{X(a)}(x) := \frac{1}{|x|} \sum_{k=1}^{\infty} e^{-a_k^{-1}|x|}, \quad (8)$$
and \( X(a) = [0, 0, k_X(a)(x)] \in L_0. \)

Hence and from the random integral representation (4) for \( k = 0 \), there exist a Lévy process \((Y_X(t), t \geq 0)\) such that

\[
X(a) = \int_0^\infty e^{-t} dY_X(a)(t), \quad h_{X(a)}(x) := \sum_{k=1}^\infty a_k^{-1} e^{-a_k^{-1}|x|} = (-xk_{X(a)}(x))' \tag{9}
\]

and \( Y_X(a)(1) = [0, 0, h_{X(a)}(x)] \in ID_{log} \) is the background driving random variable (BDR V) for \( X(a) \).

If \( \phi_{X(a)}(t) \) denotes the characteristic function of \( X(a) \) and \( \psi_{X(a)}(t) \) is the characteristic of the background driving variable (BDR V) \( Y_X(a)(1) \) then

\[
\psi_{X(a)}(t) = \exp[t(\log \phi_{X(a)}(t))'] = \exp[t(\phi_{X(a)}(t))'/\phi_{X(a)}(t)], t \neq 0; \tag{10}
\]

cf. Jurek (2001)(b), Corollary 3. Formulae (9) and (10) give two ways of identifying BDR V \( Y_X(a)(1) \).

**Remark 2.** Series of the form \( \sum_{k=1}^\infty a_k^{-1} e^{-a_k^{-1}|x|} \) maybe viewed as a very particular examples of the classical Dirichlet series; cf. Jurek (2000), Section 3 and references therein. These may help to get more explicit examples of variables \( X(a) \).

### 3. Proofs.

**A. Hyperbolic-sine characteristic function \( t/\sinh(t) \).**

From the product representation: \( \sinh(z) = z \prod_{k=1}^\infty (1 + \frac{z^2}{k^2\pi^2}), z \in \mathbb{C} \), taking the sequence \( a := (1/(k\pi)), k = 1, 2, .. \) and putting \( \hat{S} \equiv X(a) \), by (6), (7) and (8), we get the following

\[
\phi_{\hat{S}}(t) = \prod_{k=1}^\infty \frac{1}{1 + (t/\pi k)^2} = \frac{t}{\sinh t}, \quad k_{\hat{S}}(x) = \frac{1}{|x|} \frac{1}{e^{\pi|x|} - 1} = \frac{e^{-\pi|x|/2}}{2x \sinh(\pi x/2)},
\]

that is, \( \hat{S} = [0, 0, k_{\hat{S}}] \in ID \), with and an infinite Lévy spectral measure \( M_{\hat{S}}(dx) := k_{\hat{S}}(x)dx \).

From now on we will employ the procedure described in Remark 1 to the function \( k_{\hat{S}}(x) \).

**Step 1.** Since for the function \((\mathbb{D}^{1}k_{\hat{S}})(x) \equiv h_{\hat{S}}(x)\) we have that

\[
h_{\hat{S}}(x) := (-xk_{\hat{S}}(x))' = \pi \frac{e^{\pi|x|}}{(e^{\pi|x|} - 1)^2} = \frac{\pi}{4} \frac{1}{\sinh^2(\pi x/2)} = \frac{\pi}{4} \csc^2(\pi x/2) > 0,
\]
is non-negative, we infer that the function \( x \to xk_\hat{S}(x) \) is not increasing on both half-lines which means that \( \hat{S} \in L_0 \). (Or use the fact from (8)).

Moreover, by (8), the function \( h_\hat{S}(x) \) is the density of the Lévy spectral measure of \( Y_\hat{S}(1) \in ID_{\log} \), where \( (Y_\hat{S}(t), t \geq 0) \) is the BDLP for \( \hat{S} \).

Step 2. Since for \((\mathbb{D}^2k_\hat{S})(x) \equiv g_\hat{S}(x)\) we have that the function

\[
g_\hat{S}(x) = -(xh_\hat{S}(x))' = -\left( -\frac{\pi xe^{\pi|x|}}{(e^{\pi|x|} - 1)^2} \right)' = \frac{(-\pi e^{\pi|x|} - \pi^2|x|e^{\pi|x|} + 2\pi^2|x|e^{2\pi|x|}}{(e^{\pi|x|} - 1)^2} + \frac{2\pi^2|x|e^{2\pi|x|}}{(e^{\pi|x|} - 1)^3}
\]

is non-negative, (as the expression in the brackets \{...\} is non-negative; or recall that \( x \coth(x) \geq 1 \), we infer that BDRV \( Y_\hat{S}(1) \in L_0 \). Thus, by first line in (4), we infer \( \hat{S} \in L_1 \).

Moreover, \( g_\hat{S}(x) \) it is the density of Lévy spectral measure of the background driving variable \( Z_\hat{S}(1) \in ID_{\log} \).

Step 3. Again, as before for \((\mathbb{D}^2k_\hat{S})(x) \equiv g_\hat{S}(x)\), let us notice that the function

\[
r_\hat{S}(x) := -(xg_\hat{S}(x))' = -\pi/4(x(\pi x \coth(\pi x/2) - 1)\csch^2(\pi x/2))'
\]

is the density of a Lévy measure of an ID variable.

From the non-negativity of \( r_\hat{S}(x) \) we infer that the function \( xg_\hat{S}(x) \) is not increasing on both half lines, so \( g_\hat{S} \) is a Lévy function of \( L_0 \) variable. Consequently, \( \hat{S} \in L_2 \).

Step 4. Finally, putting \((\mathbb{D}^3k_\hat{S})(x) \equiv v_\hat{S}(x)\) we get that the following function

\[
v_\hat{S}(x) = \left( -\frac{\pi}{8}r_\hat{S}(x) \right)'
\]

is the density of a Lévy measure of an ID variable.
is not positive as, using WolframAlpha, we have that \( \hat{v}_S(0.9) = -0.0136 < 0 \)!! (or \( v_S(x) < 0 \) for \( 0.86 < x < 1.02 \)). Thus it can not be a density function, so \( \hat{S} \notin L_3 \) and \( \hat{S} \in L_0 \setminus L_3 \). This completes a proof of Theorem 1 (a).

**Remark 3.** (i) The fact that \( \hat{S} \notin L_4 \) is also noticed in You (2022) thesis, on p.19, but questions about \( L_2 \) and \( L_0 \) were left opened.

(ii) In Talacko (1956) and Zolotarev (1957) one may learn how these distributions appeared in statistics and probability. Furthermore, all distributions in the above Corollaries 1 may be viewed as particular examples of so called Perks’ function (ratio of finite sums of exponential functions); cf. Talacko (1956), page 160 or Perks (1932). The same applies to distributions in Corollary 3 below.

(iii) Probability distributions, with the characteristic functions \( \phi_{\hat{S}_c}(t) \) (0 < \( c \) < 1) as in Corollary 2 (i) are called Talacko-Zolotarev distributions. They are in Urbanik class \( L_1 \). However, their selfdecomposability (the class \( L_0 \) property) was already proved in You (2022), Proposition 2.2.1.

**B). Hyperbolic-cosine characteristic function 1/\cosh(t).**

Here we proceed along the proof of hyperbolic-sine but we will not use the mapping \( \mathbb{D} \) from Remark 1 but will keep the same letters for the consecutive densities.

For the hyperbolic-cosine function we have the following product representation: \( \cosh(z) = \prod_{k=1}^{\infty} (1 + \frac{4k^2}{z^2}), \ z \in \mathbb{C}. \)

Taking the sequence \( b_k := (1/(2k-1)\pi/2)), k = 1,2,.. and denoting \( \hat{C} \equiv X(b) \) we have

\[
\phi_{\hat{C}}(t) = \prod_{k=1}^{\infty} \frac{1}{1 + (t\pi(2k-1)/2)^2} = \frac{1}{\cosh t}
\]

\[
k_{\hat{C}}(x) = \sum_{k=1}^{\infty} \frac{e^{-\pi/2(2k-1)|x|}}{|x|} = \frac{e^{-\pi|x|/2}}{|x|(1 - e^{-\pi|x|})} = \frac{1}{2|x| \sinh(\pi|x|/2)};
\]

**Step 1.** Since the function

\[
h_{\hat{C}}(x) = (-xk_{\hat{C}}(x))^\prime = \frac{\pi}{2} \frac{e^{\pi|x|/2} + e^{-\pi|x|/2}}{e^{\pi|x|/2} - e^{-\pi|x|/2}}
\]

\[
= \frac{\pi \cosh(\pi|x|/2)}{4 \sinh^2(\pi|x|/2)} = \frac{\pi \cosh(\pi x/2)}{4 \sinh^2(\pi x/2)} \geq 0,
\]
is non-negative we have that $\hat{C} \in L_0$ and $h_{\hat{C}}(x)$ is the density of the Lévy measure of the background driving variable $Y_X(1)$.

Step 2. Since the function
\[
g_{\hat{C}}(x) := -(xh_{\hat{C}}(x))' = \frac{\pi}{8} csch(\frac{\pi x}{2}) [\pi x \coth^2(\frac{\pi x}{2}) - 2 \coth(\frac{\pi x}{2}) + \pi x \cosh^2(\frac{\pi x}{2})] \geq 0,
\]
is non-negative therefore $Y_X(1) \in L_0$ and thus $\hat{C} \in L_1$. Moreover, $g_{\hat{C}}(x)$ is the density of the Lévy measure for $Y_X(1)$.

Step 3. Since the function
\[
r_{\hat{C}}(x) := -(xg_{\hat{C}}(x))' = \frac{\pi}{16} csch(\frac{\pi x}{2}) [(\pi x)^2 \coth^3(\pi x/2) + \coth(\pi x/2)(5(\pi x)^2 \cosh^2(\pi x/2) - 6\pi x(\coth^2(\pi x/2) - 6\pi x \cosh^2(\pi x/2)))] \geq 0
\]
is non-negative we have that $\hat{C} \in L_2$.

Step 4. Finally, since the function $v_{\hat{C}}(x) := -(xr_{\hat{C}}(x))'...$ is such that (by WolframAlpha) $\hat{C}(2) = -0.346 < 0$ we have that $\hat{C} \notin L_3$, i.e., that $\hat{C} \in L_0 \setminus L_3$, which concludes a proof of Theorem 1 (a).

C). Hyperbolic-tangent characteristic function $\tanh(t)/t$.

The hyperbolic-tangent $\hat{T}$ has the following Lévy-Khintchine representation
\[
\phi(t) = (\tanh t)/t = \exp \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1 + x^2}) \frac{1}{2|x|} [1 - \tanh(\pi x/4)] dx,
\]
where $k_{\hat{T}}(x) := \frac{1}{2|x|} [1 - \tanh(\pi x/4)]$ is the density of Lévy measures; cf. Jurek and Yor (2004), p.185.

Since the function $xk_{\hat{T}}(x) = 1/2sign(x)(1 - \tanh(\pi x/4))$ is not increasing on both half lines, by (4), we infer that $\hat{T} \in L_0$.

On the other hand, the function $h_{\hat{T}}(x) := (-xk_{\hat{T}}(x))' = \frac{\pi}{8} \cosh^{-1}(\pi x/4)$ is a density of finite measure Lévy measure of BDRV $Y_{\hat{T}}(1)$. Hence again by (4) the hyperbolic tangent $\hat{T} \notin L_1$. Thus $\hat{T} \in L_0 \setminus L_1$, which proves Theorem 1 (b).

D). Generalized logistic distribution $\beta_\alpha, \alpha > 0$.

(a) For the sequence $c_k := (\pi(\alpha + k - 1))^{-1}$ and the variable $l_\alpha \equiv X(\xi)$, by (8), we have that the function $k_\alpha(x) = \frac{1}{|x|} e^{-\alpha|x|}$ is a density of Lévy
measure of $l_\alpha$ variable. Since the function
\[
h_{l_\alpha}(x) := (-xk_{l_\alpha}(x))' = \frac{\pi e^{-\alpha\pi|x|}(\alpha + (1 - \alpha)e^{-\pi|x|}(1 - e^{-\pi|x|})^2}{4 \sinh^2(\pi|x|/2)}
\]
where the non-negativity follows from the fact the expression in \{\ldots\} is non-negative for $\alpha > 0$.

By the criterium (5), $l_\alpha \in L_0$ (is selfdecomposable). [Note that the logistic $l_1$ coincides with hyperbolic-sine function in Section 2 (A).]

Since, (by WolframAlpha) the function $x \to (xh_{l_\alpha})$ is not increasing on both half-lines we infer that $l_\alpha$ is in $L_1$.

Furthermore, using Gradshteyn and Ryzhik (1994), formula 8.326.1.) we have the Levy-Khinchine formula for $l_\alpha$ variable
\[
\phi_{l_\alpha}(t) = \prod_{k=1}^{\infty} \frac{1}{1 + (t/((\alpha + k - 1))\pi)^2} = \left| \frac{\Gamma(\alpha + it/\pi)}{\Gamma(\alpha)} \right|^2
\]
\[
= \exp \int_{-\infty}^{\infty} (\cos(t) - 1)k_{l_\alpha}(x)dx. \tag{12}
\]

(b) To have a different approach to the logistic distribution, let us recall that Euler’s beta function $B(x, y)$, for $x, y \in \mathbb{C}, \Re x > 0, \Re y > 0$ is defined as
\[
B(x, y) := \int_0^1 s^{x-1}(1 - s)^{y-1}ds = \int_{-\infty}^{\infty} \frac{e^{xs}}{(1 + e^s)^{x+y}}ds.
\]
For $\alpha > 0$, the random variable $\beta_\alpha$ with the probability density
\[
\frac{1}{B(\alpha, \alpha)}e^{\alpha s}(1 + e^s)^{-2\alpha}, \ 	ext{for} \ -\infty < s < \infty.
\]
is called a generalized logistic distribution. Note that
\[
\phi_{\beta_\alpha}(t) = \frac{B(\alpha + it, \alpha - it)}{B(\alpha, \alpha)} = \left| \frac{\Gamma(\alpha + it)}{\Gamma(\alpha)} \right|^2,
\]
which by (12) means that $\beta_\alpha/\pi \overset{d}{=} l_\alpha$. So, as before we infer that $\beta_\alpha \in L_1$. Thus Theorem 1 (c) is proved. Also from (12) we get the part (b) of Corollary 5.
Remark 4. Since, \( \log \gamma_{\alpha,1} \), logarithms of gamma variables have characteristic functions \( \Gamma(\alpha + it)/\Gamma(\alpha) \), cf. Jurek (2021), Example 3.3, we have that \( \log \gamma_{\alpha,1} \in L_1 \). Thus the above is applicable here as well.

E). Proofs of Corollaries 2 and 4.

Recall that for \( \phi_X(t) \in L_0 \) we define its BDCF (background driving characteristic function) as

\[
\psi_X(t) := \mathbb{E}[\exp(itY_X(1))] = \exp\left[t(\log \phi_X(t))'\right] = \exp[t(\phi_X(t))'/\phi_X(t)], t \neq 0;
\]

where \( (Y_X(t), t \geq 0) \) is BDLP; cf. (10) above and Jurek (2001), Proposition 3.

Applying the above for \( X = \hat{S} \) and \( X = \hat{S}_c \) from Corollary 1 (i), we have

\[
\phi_{\hat{S}}(t) = t/\sinh(t), \quad \psi_{\hat{S}}(t) = \exp[t(t/\sinh(t))'] = \exp[1 - t \coth(t)]
\]
\[
\psi_{\hat{S}_c}(t) = \exp[t(\log \phi_{\hat{S}_c}(t))'] = \exp[t(\frac{\sinh(ct)}{c\sinh(t)})']
\]
\[
= \exp[ct \coth(ct) - t \coth(t)] = \exp[(1 - t \coth(t)) - (1 - ct \coth(ct))]
\]
\[
= \psi_{\hat{S}}(t)/\psi_{\hat{S}_c}(ct),
\]

i.e., \( \psi_{\hat{S}}(t) = \psi_{\hat{S}}(ct) \psi_{\hat{S}_c}(t) \), or \( Y_{\hat{S}}(1) \overset{d}{=} cY_{\hat{S}_c}(1) + \hat{S}_c \), which gives Corollary 2.

Similarly, for the hyperbolic-cosine \( X = \hat{C} \) and \( X = \hat{C}_c \) from Corollary 3 (i), and (10) we have

\[
\phi_{\hat{C}}(t) = 1/\cosh(t); \quad \psi_{\hat{C}}(t) = \exp[t(\log \cosh(t))'] = \exp[-t \tanh(t)],
\]
\[
\psi_{\hat{C}_c}(t) = \exp[t(\log \cosh(ct)/\cosh(t))'] = \exp[t(c \tanh(ct) - \tanh(t))]
\]
\[
= \psi_{\hat{C}}(t)/\psi_{\hat{C}_c}(ct); \quad \text{i.e., } \psi_{\hat{C}}(t) = \psi_{\hat{C}}(ct) \psi_{\hat{C}_c}(t),
\]

which completes a proof of Corollary 4.

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