AN EIGENSYSTEM APPROACH TO ANDERSON LOCALIZATION

ALEXANDER ELGART AND ABEL KLEIN

ABSTRACT. We introduce a new approach for proving localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization) for the Anderson model at high disorder. In contrast to the usual strategy, we do not study finite volume Green’s functions. Instead, we perform a multiscale analysis based on finite volume eigensystems (eigenvalues and eigenfunctions). Information about eigensystems at a given scale is used to derive information about eigensystems at larger scales. This eigensystem multiscale analysis treats all energies of the finite volume operator at the same time, establishing level spacing and localization of eigenfunctions in a fixed box with high probability. A new feature is the labeling of the eigenvalues and eigenfunctions by the sites of the box.

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Date: Version of September 7, 2016.
A.E. was supported in part by the NSF under grant DMS-1210982.
A.K. was supported in part by the NSF under grant DMS-1001509.

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Introduction

The Anderson model \([\text{An}]\) is the prototype for the study of localization properties of quantum states of single electrons in disordered solids. It is given by a random Schrödinger operator \(H_{\varepsilon,\omega} = -\varepsilon\Delta + V_\omega\) acting on \(l^2(\mathbb{Z}^d)\), where \(\Delta\) is the discrete Laplacian, \(V_\omega\) is a random potential, and \(\varepsilon > 0\) is the reciprocal of the disorder parameter (see Definition 1.1 for the details). The basic phenomenon, known as the Anderson localization, is that high disorder (\(\varepsilon \ll 1\)) leads to localization of electron states. Its most basic manifestation is that \(H_{\varepsilon,\omega}\) has pure point spectrum with exponentially decaying eigenfunctions with probability one: for almost every configuration of the random potential, \(H_{\varepsilon,\omega}\) has a complete orthonormal basis of eigenvalues \(\{\psi_{\varepsilon,\omega,j}\}_{j \in \mathbb{N}}\) such that

\[|\psi_{\varepsilon,\omega,j}(x)| \leq C_{\varepsilon,\omega,j} e^{-m_\varepsilon \|x\|} \text{ for all } x \in \mathbb{Z}^d \text{ and } j \in \mathbb{N},\]

where \(m_\varepsilon > 0\), the reciprocal of the localization length, is nonrandom and independent of \(j \in \mathbb{N}\). Other manifestations include dynamical localization and SULE (semi-uniformly localized eigenfunctions). (See, for example, \([\text{AiW}, \text{K}, \text{Kl}]\).)

These manifestations of localization suggest that truncation of the system to a finite box \(\Lambda_L\) of side \(L \gg \frac{1}{m_\varepsilon}\) should not affect localization properties deep inside the box. This leads to the expectation that if one could establish an appropriate analogue of localization for a sequence of boxes \(\Lambda_{L_n}\), with \(L_n \to \infty\), then localization should hold in the whole of \(\mathbb{Z}^d\) as well. This strategy can be indeed be implemented and is known as the multiscale analysis. In a nutshell, the multiscale analysis uses as input localizing properties at scale \(L_n\) to establish localizing properties at scale \(L_{n+1}\). The question is what kind of information we want to carry from scale to scale. In the traditional approach to Anderson localization, such information is encoded in the decay properties of the underlying Green’s function. For single-particle systems, the Green’s function \(G_{\varepsilon,\omega}(x,y;\lambda) = \langle \delta_x, (H_{\varepsilon,\omega} - \lambda)^{-1} \delta_y \rangle\) is an extremely convenient object to study. Its usefulness comes from two key properties: (a) Green’s functions for boxes at different scales are related by the first resolvent identity; (b) knowledge of the decay properties of the Green’s functions for all energies (or for all energies in a fixed interval) can be translated into localization properties of the eigenfunctions (in the fixed interval).

The well known methods developed for proving localization for random Schrödinger operators, the multiscale analysis \([\text{FroS}], [\text{ProMSS}], [\text{Dr}], [\text{DrK}], [\text{S}], [\text{CoH}], [\text{FK2}], [\text{GK1}], [\text{Kl}], [\text{BoK}], [\text{GK4}]\) and the fractional moment method \([\text{AiM}], [\text{AI}], [\text{AISFH}], [\text{AIENSS}], [\text{AiW}]\), are based on the study of finite volume Green’s functions. Multiscale analyses based on Green’s functions are performed either at a fixed energy in a single box, or for all energies but with two boxes with an ‘either or’ statement for each energy.

Recently there has been an intensive effort in the physics community to create a coherent theory of many-body localization (MBL); see, e.g., \([\text{FIA}], [\text{AIGKL}], [\text{GoMP}], [\text{BAA}], [\text{BurO}], [\text{OH}], [\text{PH}], [\text{NH}], [\text{FrWBSE}], [\text{EFG}]\). On the mathematical level, not much progress have been made, besides studies of exactly solvable models; see, e.g., \([\text{HSS}], [\text{PaS}], [\text{AS}]\). One of the key difficulties in studying MBL is associated with the fact that Green’s functions do not appear to be such a valuable tool as in the single-particle theory, due to the product state nature of the underlying Hilbert space. The objects that do appear in the most physical descriptions of MBL are the eigenstates of the system. This suggests that finding a more direct, eigensystem based approach to localization, even in the single-particle case, could be beneficial for understanding...
MBL. Such approach has been advocated by Imbrie in a context of both single and many-body localization [I1, I2].

In this paper we provide a mathematically rigorous implementation of a multiscale analysis for the Anderson model at high disorder based on finite volume eigensystems (eigenvalues and eigenfunctions). In contrast to the usual strategy, we do not study finite volume Green’s functions. Information about eigensystems at a given scale is used to derive information about eigensystems at larger scales. This eigensystem multiscale analysis treats all energies of the finite volume operator at the same time, giving a complete picture in a fixed box. For this reason it does not use a Wegner estimate as in a Green’s functions multiscale analysis, it uses instead a probability estimate for level spacing derived by Klein and Molchanov from Minami’s estimate [KIM, Lemma 2].

A new feature provided by the eigensystem multiscale analysis is the labeling of the eigenvalues and eigenfunctions by the sites of the box. We establish this labeling by the multiscale analysis using an argument based on Hall’s Marriage Theorem (e.g., [BuDM, Chapter 2]).

Our main result, stated in Theorem 1.6 can be loosely described as follows: If $\epsilon \ll 1$, with high probability the eigenvalues and eigenfunctions of $H_{\epsilon, \omega, \Lambda_L}$, the restriction of $H_{\epsilon, \omega}$ to a finite box $\Lambda_L$ of side $L \gg 1$, can be labeled by the sites of $\Lambda_L$, i.e., they can be written in the form $\{(\varphi_x, \lambda_x)\}_{x \in \Lambda_L}$, with the eigenvalues $\{\lambda_x\}_{x \in \Lambda_L}$ satisfying a level spacing condition, and the eigenfunctions $\{\varphi_x\}_{x \in \Lambda_L}$ exhibiting localization around the label, i.e., for all $x \in \Lambda_L$ we have

$$|\varphi_x(y)| \leq e^{-m_\epsilon \|y-x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y-x\| \geq L^\tau,$$

where $m_\epsilon > 0$ is nonrandom and $0 < \tau < 1$ is a fixed parameter.

Theorem 1.6 yields Anderson localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization) for $H_{\epsilon, \omega}$. It is our hope that the eigensystem multiscale analysis is a step towards developing new methods that may be useful in the study of MBL.

We also investigate the connection between the eigensystem multiscale analysis and the Green’s functions multiscale analysis. We show that the conclusions of the Green’s functions multiscale analysis can be derived from the conclusions of the eigensystem multiscale analysis. Conversely, we show that the conclusions of the eigensystem multiscale analysis can be derived from the Green’s functions energy interval multiscale analysis with the addition of the labeling argument based on Hall’s Marriage Theorem we present in this paper.

The results in this paper concern localization for the Anderson model in the whole spectrum, which in practice requires high disorder. For the Anderson model, the Green’s function methods for proving localization can be applied in energy intervals, and hence localization has also been proved at fixed disorder in an interval at the edge of the spectrum (or, more generally, in the vicinity of a spectral gap), and for a fixed interval of energies at the bottom of the spectrum for sufficiently high disorder. (See, for example, [HoM, KSS, FK1, AISF2, GK2, K, GK4, AIW]). In a forthcoming paper [ElK], we generalize the version of the eigensystem multiscale analysis presented in this paper to establish localization for the Anderson model in an energy interval. This extension yields localization at fixed disorder on an interval at the edge of the spectrum (or in the vicinity of a spectral gap), and at a fixed interval at the bottom of the spectrum for sufficiently high disorder.
Klein and Tsang [KT] have used a bootstrap argument as in [GK1, K] to enhance the eigensystem multiscale analysis for the Anderson model at high disorder developed in this paper. The only input required to initiate the eigensystem bootstrap multiscale analysis is polynomial decay of the finite volume eigenfunctions for sufficiently large scale with some minimal, scale-independent probability. It yields a result analogous to (1.13) in Theorem 1.6 for all $0 < \xi < 1$, with $\varepsilon_0$ independent of $\xi$.

Our main results and definitions are stated in Section 1. Theorem 1.6 is our main result, the conclusions of the eigensystem multiscale analysis, which we prove in Section 4. Theorem 1.7, derived from Theorem 1.6, encapsulates localization for the Anderson model. Corollary 1.8 contains typical statements of Anderson localization and dynamical localization. Theorem 1.7 and Corollary 1.8 are proven in Section 5. In Section 2 we adapt an estimate for the probability of level spacing derived by Klein and Molchanov (reviewed in Appendix B) to our setting. Section 3 contains definitions and lemmas required for the proof of the eigensystem multiscale analysis given in Section 4. The connection with the Green’s functions multiscale analysis is established in Section 6. Hall’s Marriage Theorem, used in Section 4 for labeling eigenvalues and eigenfunctions, is reviewed in Appendix C.

1. Main results

We start by introducing the Anderson model in a convenient form.

**Definition 1.1.** The Anderson model is the random Schrödinger operator

$$H_{\varepsilon, \omega} := -\varepsilon \Delta + V_{\omega} \quad \text{on} \quad \ell^2(\mathbb{Z}^d),$$

where

(i) $\Delta$ is the (centered) discrete Laplacian:

$$(\Delta \varphi)(x) := \sum_{y \in \mathbb{Z}^d, |y-x|=1} \varphi(y) \quad \text{for} \quad \varphi \in \ell^2(\mathbb{Z}^d).$$

(ii) $V_{\omega}$ is a random potential: $V_{\omega}(x) = \omega_x$ for $x \in \mathbb{Z}^d$, where $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution $\mu$ is non-degenerate with bounded support. We assume $\mu$ is Hölder continuous of order $\alpha \in \left(\frac{1}{2}, 1\right]$: $S_\mu(t) \leq K t^{\alpha}$ for all $t \in [0, 1]$,

where $K$ is a constant and $S_\mu(t) := \sup_{a \in \mathbb{R}} \mu\{[a, a + t]\}$ is the concentration function of the measure $\mu$.

(iii) $\varepsilon > 0$ is the reciprocal of the disorder parameter (i.e., $\frac{1}{\varepsilon}$ is the disorder parameter).

We recall that $\sigma(-\Delta) = [-2d, 2d]$ and (see [K] Theorem 3.9)

$$\sigma(H_{\varepsilon, \omega}) = \Sigma_{\varepsilon} := [-2\varepsilon d, 2\varepsilon d] + \text{supp}\mu \quad \text{with probability one.}$$

By a discrete Schrödinger operator we will always mean an operator $H = -\varepsilon \Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where $V$ is a bounded potential and $\varepsilon \geq 0$.

We use the following definitions and notation:
If $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, we set $|x| = |x|_2 = \left(\sum_{j=1}^d x_j^2\right)^{\frac{1}{2}}$, $|x|_1 = \sum_{j=1}^d |x_j|$, and $\|x\| = |x|_\infty = \max_{j=1,2,\ldots,d} |x_j|$. If $x \in \mathbb{R}^d$ and $\Xi \subset \mathbb{R}^d$, we set $\text{dist}(x, \Xi) = \inf_{y \in \Xi} \|y - x\|$. The diameter of a set $\Xi \subset \mathbb{R}^d$ is given by $\text{diam} \Xi = \sup_{x,y \in \Xi} \|y - x\|$.

We consider $\mathbb{Z}^d$ as a subset of $\mathbb{R}^d$ and use boxes in $\mathbb{Z}^d$ centered at points in $\mathbb{R}^d$. The box in $\mathbb{Z}^d$ of side $L > 0$ centered at $x \in \mathbb{R}^d$ is given by

\[ \Lambda_L(x) = \Lambda_L^\mathbb{Z}(x) = \{ y \in \mathbb{Z}^d; \|y - x\| \leq \frac{L}{2} \} \subset \mathbb{Z}^d, \]

where $\Lambda_L^\mathbb{Z}(x)$ is the box in $\mathbb{R}^d$ of side $L > 0$ centered at $x \in \mathbb{R}^d$, given by

\[ \Lambda_L^\mathbb{R}(x) = \{ y \in \mathbb{R}^d; \|y - x\| \leq \frac{L}{2} \} \subset \mathbb{R}^d. \]

By a box $\Lambda_L$ we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. Note that for all scales $L \geq 2$ and $x \in \mathbb{R}^d$ we have

\[ (L - 2)^d < \left(2 \left\lfloor \frac{L}{2} \right\rfloor \right)^d \leq |\Lambda_L(x)| \leq \left(2 \left\lfloor \frac{L}{2} \right\rfloor + 1 \right)^d \leq (L + 1)^d. \]

Given $\Phi \subset \Theta \subset \mathbb{Z}^d$, we consider $\ell^2(\Phi) \subset \ell^2(\Theta)$ by extending functions on $\Phi$ to functions on $\Theta$ that are identically 0 on $\Theta \setminus \Phi$. If $\varphi$ is a function on $\Theta$, we write $\varphi_\Theta = \chi_\Theta \varphi$. If $\Theta \subset \mathbb{Z}^d$ and $\varphi \in \ell^2(\Theta)$, we let $\|\varphi\| = \|\varphi\|_2$ and $\|\varphi\|_\infty = \max_{y \in \Theta} |\varphi(y)|$.

We will work with finite volume operators. If $K$ is a bounded operator on $\ell^2(\mathbb{Z}^d)$ and $\Theta \subset \mathbb{Z}^d$, we let $K_\Theta$ be the restriction of $\chi_\Theta K \chi_\Theta$ to $\ell^2(\Theta)$.

By a constant we always mean a finite constant. We will use $C_{a,b,\ldots}, C'_{a,b,\ldots}$, $C(a,b,\ldots)$, etc., to denote a constant depending on the parameters $a,b,\ldots$. Note that $C_{a,b,\ldots}$ may denote different constants in different equations, and even in the same equation.

If $E$ is an event, we denote its complementary event by $E^c$.

We fix $\xi, \zeta, \beta, \tau \in (0,1)$ and $\gamma > 1$ such that

\[ 0 < \xi < \zeta < \beta < \frac{1}{\gamma} < 1 < \gamma < \sqrt{\frac{\pi}{\xi}} \quad \text{and} \quad \max \left\{ \gamma \beta, \frac{(\gamma - 1)\beta + 1}{\gamma} \right\} < \tau < 1, \]

and note that

\[ 0 < \xi < \gamma^2 \xi \zeta < \beta < \frac{\tau}{\gamma} < \frac{1}{\gamma} < \tau < 1 < \frac{1 - \beta}{\tau - \beta} < \gamma < \frac{\tau}{\beta}. \]

We also take

\[ \tilde{\zeta} = \frac{\zeta + \beta}{2} \in (\zeta, \beta) \quad \text{and} \quad \tilde{\tau} = \frac{1 + \tau}{2} \in (\tau, 1). \]

Given a scale $L \geq 1$, we set

\[ \ell = L^{\frac{1}{2}} \quad \text{(i.e., } L = \ell^2\text{),} \quad L_\tau = [L^{\tilde{\tau}}], \quad \text{and} \quad L_{\tilde{\tau}} = [L^{\tilde{\tau}}]. \]

The following definitions are for a fixed discrete Schrödinger operator $H_\varepsilon$. We omit $\varepsilon$ from the notation (i.e., we write $H$ for $H_\varepsilon$, $H_\Theta$ for $H_{\varepsilon,\Theta}$) when it does not lead to confusion.

**Definition 1.2.** Given $\Theta \subset \mathbb{Z}^d$, we call $(\varphi, \lambda)$ an eigenpair for $H_\Theta$ if $\varphi \in \ell^2(\Theta)$ with $\|\varphi\| = 1$, $\lambda \in \mathbb{R}$, and $H_\Theta \varphi = \lambda \varphi$. (In other words, $\lambda$ is an eigenvalue for $H_\Theta$ and $\varphi$ is a corresponding normalized eigenfunction.) A collection $\{(\varphi_j, \lambda_j)\}_{j \in J}$ of eigenpairs for $H_\Theta$ will be called an eigensystem for $H_\Theta$ if $\{\varphi_j\}_{j \in J}$ is an orthonormal basis for $\ell^2(\Theta)$. If all eigenvalues of $H_\Theta$ are simple, we can rewrite the eigensystem as $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$. 

Definition 1.3. Let $\Lambda_L$ be a box, $x \in \Lambda_L$, and $m > 0$. Then $\varphi \in \ell^2(\Lambda_L)$ is said to be $(x, m)$-localized if $\|\varphi\| = 1$ and
\[
|\varphi(y)| \leq e^{-m\|y-x\|} \quad \text{for all } y \in \Lambda_L \quad \text{with} \quad \|y-x\| \geq L_\tau.
\]
In particular,
\[
|\varphi(y)| \leq e^{mL_\tau}e^{-m\|y-x\|} \quad \text{for all } y \in \Lambda_L.
\]

Definition 1.4. Let $R > 0$. A finite set $\Theta \subset \mathbb{Z}^d$ will be called $R$-level spacing for $H$ if $|\sigma(H_\Theta)| = |\Theta|$ (i.e., all eigenvalues of $H_\Theta$ are simple) and $|\lambda - \lambda'| \geq c^{-R}$ for all $\lambda, \lambda' \in \sigma(H_\Theta)$, $\lambda \neq \lambda'$.

In the special case when $\Theta$ is a box $\Lambda_L$ and $R = L$, we will simply say that $\Lambda_L$ is level spacing for $H$.

Definition 1.5. Let $m > 0$. A box $\Lambda_L$ will be called $m$-localizing for $H$ if the following holds:

(i) $\Lambda_L$ is level spacing for $H$.

(ii) There exists an $m$-localized eigensystem for $H_{\Lambda_L}$, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x \in \Lambda_L}$ for $H_{\Lambda_L}$ such that $\varphi_x$ is $(x, m)$-localized for all $x \in \Lambda_L$.

The eigensystem multiscale analysis yields the following theorem.

Theorem 1.6. Let $H_{\varepsilon, \omega}$ be an Anderson model. There exists a finite scale $L_0$ such that for all $0 < \varepsilon \leq \varepsilon_0 = \frac{1}{4}\varepsilon^{-L_0}$ we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \frac{m_{\varepsilon,L}}{4} \text{-localized for } H_{\varepsilon, \omega}\} \geq 1 - e^{-L_L} \quad \text{for all } L \geq L_0,
\]
where
\[
m_{\varepsilon,L} = \log \left(1 + \frac{e^{-L_L}}{2\varepsilon}\right) \quad \text{for} \quad \varepsilon > 0 \quad \text{and} \quad L \geq 1.
\]

Note that for $0 < \varepsilon \leq \varepsilon_0 = \frac{1}{4}\varepsilon^{-L_0}$ we have
\[
m_{\varepsilon,L_0} = \log \left(1 + \frac{2\varepsilon_0}{\varepsilon}\right) \geq \log 3.
\]

Theorem 1.6 is proved in Section 4. It yields all the usual forms of localization. To see this we need to introduce some notation and definitions. We fix $\nu > \frac{d}{2}$, which will be usually omitted from the notation. Given $a \in \mathbb{Z}^d$, we let $T_a$ be the operator on $\ell^2(\mathbb{Z}^d)$ given by multiplication by the function $T_a(x) := \langle x-a \rangle^\nu$, where
\[
\langle x \rangle := \left(1 + \|x\|^2\right)^{\frac{\nu}{2}}.
\]
Note that $\|T_aT_b^{-1}\| \leq 2^{\frac{\nu}{2}}(a-b)^\nu$ since $\langle a + b \rangle \leq \sqrt{2}\langle a \rangle \langle b \rangle$.

A function $\psi: \mathbb{Z}^d \to \mathbb{C}$ will be called a $\nu$-generalized eigenfunction for the discrete Schrödinger operator $H_\varepsilon$ if $\psi$ is a generalized eigenfunction as in Definition 3.11 and $\|T_{a}^{-1}\psi\| < \infty$. (Note $\|T_{a}^{-1}\psi\| < \infty$ if and only if $\|T_{a}^{-1}\psi\| < \infty$ for all $a \in \mathbb{Z}^d$.) We let $\mathcal{V}_\varepsilon(\lambda)$ denote the collection of $\nu$-generalized eigenfunctions for $H_\varepsilon$ with generalized eigenvalue $\lambda \in \mathbb{R}$. (We will usually drop $\nu$ from the notation.)

Given $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{Z}^d$, we set
\[
W_{\varepsilon,\lambda}^{(a)}(b) := \sup_{\psi \in \mathcal{V}_\varepsilon(\lambda)} \frac{|\psi(b)|}{\|T_{a}^{-1}\psi\|} \quad \text{if } \mathcal{V}_\varepsilon(\lambda) \neq \emptyset
\]
\[
= 0 \quad \text{otherwise}.
\]

Note that for $a, b, c \in \mathbb{Z}^d$ we have
\[
W_{\varepsilon,\lambda}^{(a)}(a) \leq 1, \quad W_{\varepsilon,\lambda}^{(a)}(b) \leq (b-a)^\nu, \quad \text{and} \quad W_{\varepsilon,\lambda}^{(a)}(c) \leq 2^{\frac{\nu}{2}}(b-a)^\nu W_{\varepsilon,\lambda}^{(b)}(c).
\]

The following theorem, derived from Theorem 1.6, encapsulates localization for the Anderson model.

**Theorem 1.7.** Suppose Theorem 1.6 holds for an Anderson model $H_{\varepsilon,\omega}$, let $L_0$ be the scale given in Theorem 1.6, and let $\varepsilon_0 = \frac{1}{27} e^{-L_0^2}$. There exists a finite scale $L_1$ such that, given $L_1 \leq \ell \in 2\mathbb{N}$ and $\alpha \in \mathbb{Z}^d$, then for all $0 < \varepsilon \leq \varepsilon_0$, setting $m_\varepsilon = \frac{m_{\varepsilon, L_0}}{2} = \frac{\log 3}{4}$, there exists an event $\mathcal{Y}_{\varepsilon,\ell,\alpha}$ with the following properties:

1. $\mathcal{Y}_{\varepsilon,\ell,\alpha}$ depends only on the random variables $\{\omega_x\}_{x \in \Lambda_{m}(\alpha)}$, and
   \[ \mathbb{P} \{ \mathcal{Y}_{\varepsilon,\ell,\alpha} \} \geq 1 - C_\varepsilon e^{-\varepsilon \ell}. \]  
2. If $\omega \in \mathcal{Y}_{\varepsilon,\ell,\alpha}$, for all $\lambda \in \mathbb{R}$ we have that
   \[ \max_{b \in A_\ell(a)} W^{(a)}_{\varepsilon,\omega,\lambda}(b) > e^{-\frac{1}{3} m_\varepsilon \ell} \implies \max_{y \in A_\ell(a)} W^{(a)}_{\varepsilon,\omega,\lambda}(y) \leq e^{-\frac{7}{13} m_\varepsilon \|y-a\|}, \]  
   where
   \[ A_\ell(a) := \{ y \in \mathbb{Z}^d; \frac{8}{7} \ell \leq \|y-a\| ≤ \frac{34}{77} \ell \}. \]

In particular, for all $\omega \in \mathcal{Y}_{\varepsilon,\ell,\alpha}$ and $\lambda \in \mathbb{R}$ we have
\[ W^{(a)}_{\varepsilon,\omega,\lambda}(a) W^{(a)}_{\varepsilon,\omega,\lambda}(y) \leq e^{-\frac{7}{13} m_\varepsilon \|y-a\|} \text{ for all } y \in A_\ell(a). \]

Theorem 1.7 implies Anderson localization and dynamical localization, and more, as shown in [GK3, GK4]. In particular, we get the following corollary.

**Corollary 1.8.** Let $H_{\varepsilon,\omega}$ be an Anderson model, and suppose Theorem 1.7 holds. Then for $0 < \varepsilon \leq \varepsilon_0$ the following holds with probability one:

1. $H_{\varepsilon,\omega}$ has pure point spectrum.
2. If $\psi_\lambda$ is an eigenfunction of $H_{\varepsilon,\omega}$ with eigenvalue $\lambda$, then $\psi_\lambda$ is exponentially localized with rate of decay $\frac{7}{13} m_\varepsilon$, more precisely,
   \[ |\psi_\lambda(x)| \leq C_{\varepsilon,\omega,\lambda} \|T_{x_\lambda}^{-1} \psi\| e^{-\frac{7}{13} m_\varepsilon \|x\|} \text{ for all } x \in \mathbb{R}^d. \]  
3. For all $\lambda \in \mathbb{R}$ and $x, y \in \mathbb{Z}^d$ we have
   \[ W^{(x)}_{\varepsilon,\omega,\lambda}(x) W^{(y)}_{\varepsilon,\omega,\lambda}(y) \leq C_{\varepsilon,\omega,\lambda} e^{\frac{2}{3} \nu \|x-y\|} \|T_{x_\lambda}^{-1} \psi\| e^{-\frac{7}{13} m_\varepsilon \|y-x\|}. \]  
4. For all $\lambda \in \mathbb{R}$ and $\psi \in \chi(\lambda)(H_{\varepsilon,\omega})$, we have
   \[ |\psi(x)| |\psi(y)| \leq C_{\varepsilon,\omega,\lambda} \|T_{x_\lambda}^{-1} \psi\|^2 e^{\frac{2}{3} \nu \|x-y\|} \|T_{x_\lambda}^{-1} \psi\| e^{-\frac{7}{13} m_\varepsilon \|y-x\|}. \]
   for all $x, y \in \mathbb{Z}^d$.
5. For all $\lambda \in \mathbb{R}$ there exists $x_\lambda = x_{\varepsilon,\omega,\lambda} \in \mathbb{Z}^d$, such that for $\psi \in \chi(\lambda)(H_{\varepsilon,\omega})$ we have
   \[ |\psi(x)| \leq C_{\varepsilon,\omega,\lambda} \|T_{x_\lambda}^{-1} \psi\| e^{\frac{2}{3} \nu \|x-y\|} \|T_{x_\lambda}^{-1} \psi\| e^{-\frac{1}{13} m_\varepsilon \|x-x_\lambda\|} \leq 2 \pi C_{\varepsilon,\omega,\lambda} \|T_{x_\lambda}^{-1} \psi\| e^{\frac{2}{3} \nu \|x-y\|} \|T_{x_\lambda}^{-1} \psi\| e^{-\frac{1}{13} m_\varepsilon \|x-x_\lambda\|}, \]
   for all $x \in \mathbb{Z}^d$.

In Corollary 1.8 (i) and (ii) are statements of Anderson localization, (iii) and (iv) are statements of dynamical localization (iv) is called SUDEC (summable uniform decay of eigenfunction correlations) in [GK3], and (v) is SULE (semi-uniformly localized eigenfunctions; see [DJLS1, DJLS2]).

We can also derive statements of localization in expectation, as in [GK3, GK4]. Theorem 1.7 and Corollary 1.8 are proven in Section 5.
2. Probability estimate for level spacing

We adapt a probabilistic estimate of Klein and Molchanov [KIM, Lemma 2] to our setting. (This estimate is reviewed in Appendix [B].)

If \( J \subset \mathbb{R} \), we set \( \text{diam} J = \sup_{s,t \in J} |s-t| \).

Lemma 2.1. Let \( H_{\varepsilon, \omega} \) be an Anderson model as in Definition [1]. Let \( \Theta \subset \mathbb{Z}^d \) and \( L > 1 \). Then, for all \( 0 < \varepsilon \leq \varepsilon_0 \),

\[
\mathbb{P} \{ \Theta \text{ is } L\text{-level spacing for } H_{\varepsilon, \omega} \} \geq 1 - Y_{\varepsilon_0} e^{-(2\alpha-1)\varepsilon_0 L^3} |\Theta|^2. \tag{2.1}
\]

where

\[
Y_{\varepsilon_0} = 2^{2\alpha-1} \tilde{K}^2 (\text{diam supp } \mu + 2d\varepsilon_0 + 1), \tag{2.2}
\]

with \( \tilde{K} = K \) if \( \alpha = 1 \) and \( \tilde{K} = 8K \) if \( \alpha \in (\frac{1}{2}, 1) \).

In the special case of a box \( \Lambda_L \), we have

\[
\mathbb{P} \{ \Lambda_L \text{ is level spacing for } H \} \geq 1 - Y_{\varepsilon_0} (L + 1)^{2d} e^{-(2\alpha-1)\varepsilon_0 L^3}. \tag{2.3}
\]

Proof. Recalling [1], we have

\[
\Sigma_\varepsilon \subset I_\varepsilon := [\inf \Sigma_\varepsilon, \sup \Sigma_\varepsilon] \quad \text{and} \quad |I_\varepsilon| = \text{diam supp } \mu + 2d\varepsilon. \tag{2.4}
\]

Thus, it follows from Lemma [15] that

\[
\mathbb{P} \{ \Theta \text{ is } L\text{-level spacing for } H \} \geq 1 - 2^{2\alpha-1} \tilde{K}^2 (\text{diam supp } \mu + 2d\varepsilon + 1) e^{-(2\alpha-1)\varepsilon_0 L^3} |\Theta|^2. \tag{2.5}
\]

\( \square \)

3. Preparation for the multiscale analysis

We consider a fixed discrete Schrödinger operator \( H = -\varepsilon \Delta + V \) on \( \ell^2(\mathbb{Z}^d) \), where \( V \) is a bounded potential and \( 0 < \varepsilon \leq \varepsilon_0 \) for a fixed \( \varepsilon_0 \).

3.1. Subsets and boundaries. Let \( \Phi \subset \Theta \subset \mathbb{Z}^d \). We set the boundary, exterior boundary, and interior boundary of \( \Phi \) relative to \( \Theta \), respectively, by

\[
\partial^\Theta \Phi = \{(u,v) \in \Phi \times (\Theta \setminus \Phi) ; |u-v| = 1 \}, \tag{3.1}
\]

\[
\partial^\Theta_{\text{ex}} \Phi = \{v \in (\Theta \setminus \Phi) ; (u,v) \in \partial^\Theta \Phi \text{ for some } u \in \Phi \},
\]

\[
\partial^\Theta_{\text{in}} \Phi = \{u \in \Phi ; (u,v) \in \partial^\Theta \Phi \text{ for some } v \in \Theta \setminus \Phi \}.
\]

We have

\[
H_\Theta = H_\Phi \oplus H_{\Theta \setminus \Phi} + \varepsilon \Gamma_{\partial^\Theta \Phi} \text{ on } \ell^2(\Theta) = \ell^2(\Phi) \oplus \ell^2(\Theta \setminus \Phi), \tag{3.2}
\]

where \( \Gamma_{\partial^\Theta \Phi}(u,v) = \begin{cases} -1 & \text{if either } (u,v) \text{ or } (v,u) \in \partial^\Theta \Phi \\ 0 & \text{otherwise} \end{cases} \). \tag{3.3}

Given a box \( \Lambda_L \subset \Theta \subset \mathbb{Z}^d \), for each \( v \in \partial^\Theta_{\text{ex}} \Lambda_L \) there exists a unique \( \hat{v} \in \partial^\Theta_{\text{in}} \Lambda_L \) such that \( (\hat{v}, v) \in \partial^\Theta \Lambda_L \), which implies \( |\partial^\Theta_{\text{ex}} \Lambda_L| = |\partial^\Theta \Lambda_L| \). Given \( v \in \Theta \), we define \( \hat{v} \) as above if \( v \in \partial^\Theta_{\text{ex}} \Lambda_L \), and set \( \hat{v} = v \) otherwise. For \( L \geq 2 \) we have

\[
|\partial^\Theta_{\text{in}} \Lambda_L| \leq |\partial^\Theta_{\text{ex}} \Lambda_L| = |\partial^\Theta \Lambda_L| \leq s_d L^{d-1}, \quad \text{where} \quad s_d = 2^d d. \tag{3.4}
\]
Let $\Psi \subset \Theta \subset \mathbb{Z}^d$. Given $t \geq 1$, we set
\begin{equation}
\Psi_{\Theta,t} = \{ y \in \Psi; \lambda_2(y) \cap \Theta \subset \Psi \} = \{ y \in \Psi; \text{dist} (y, \Theta \setminus \Psi) > |t| \},
\end{equation}
\begin{equation}
\partial_{in}^{\Theta,t} \Psi = \Psi \setminus \Psi_{\Theta,t} = \{ y \in \Psi; \text{dist} (y, \Theta \setminus \Psi) \leq |t| \},
\end{equation}
\begin{equation}
\partial_{out}^{\Theta,t} \Psi = \partial_{in}^{\Theta,t} \Psi \cup \partial_{ex}^{\Theta} \Psi.
\end{equation}

Note that $\Psi_{\Theta,t} = \Psi_{\Theta,|t|}$. For a box $\Lambda_L(x) \subset \Theta \subset \mathbb{Z}^d$ we write $\Lambda_{L,t}^{\Theta,t}(x) = (\Lambda_L(x))_{\Theta,t}$. We also set $\Lambda_L(x) = \Lambda_{L,0}(x)$.

### 3.2. Generalized eigenfunctions.

**Definition 3.1.** Given $\Theta \subset \mathbb{Z}^d$, a function $\psi: \Theta \to \mathbb{C}$ is called a generalized eigenfunction for $H_\Theta$ with generalized eigenvalue $\lambda \in \mathbb{R}$ if $\psi$ is not identically zero and
\begin{equation}
-\varepsilon \sum_{y \in \Theta} \psi(y) + (V(x) - \lambda)\psi(x) = 0 \quad \text{for all} \quad x \in \Theta,
\end{equation}
or, equivalently,
\begin{equation}
\langle (H_\Theta - \lambda)\varphi, \psi \rangle = 0 \quad \text{for all} \varphi \in \ell^2(\Theta) \quad \text{with finite support}.
\end{equation}

In this case we call $(\psi, \lambda)$ a generalized eigenpair for $H_\Theta$.

If $\psi \in \ell^2(\Theta)$, $\psi$ is an eigenfunction for $H_\Theta$ with eigenvalue $\lambda$. If $\Theta$ is finite there is no difference between generalized eigenfunctions and eigenfunctions. For arbitrary $\Theta$ the difference is that we do not require generalized eigenfunctions to be in $\ell^2(\Theta)$, we only require the pointwise equality in (3.7).

### 3.3. Eigenpairs and eigensystems.

Let $\Theta \subset \mathbb{Z}^d$ and consider an eigensystem $\{(\varphi_j, \lambda_j)\}_{j \in J}$ for $H_\Theta$. We have
\begin{equation}
\delta_n = \sum_{j \in J} \overline{\varphi_j(y)}\varphi_j \quad \text{for all} \quad y \in \Theta,
\end{equation}
\begin{equation}
\psi(y) = \langle \delta_n, \psi \rangle = \sum_{j \in J} \varphi_j(y) \langle \varphi_j, \psi \rangle \quad \text{for all} \psi \in \ell^2(\Theta) \quad \text{and} \quad y \in \Theta.
\end{equation}

**Lemma 3.2.** Let $\Phi \subset \Theta \subset \mathbb{Z}^d$ and suppose $(\varphi, \lambda)$ is an eigenpair for $H_\Phi$. Then
\begin{equation}
\langle (H_\Theta - \lambda)\varphi, \varphi \rangle(y) = \varepsilon \left( \sum_{y' \neq y} \varphi(y') \right) \chi_{\partial_{in}^{\Theta} \Phi}(y) \quad \text{for all} \quad y \in \Theta.
\end{equation}

Moreover, we have
\begin{equation}
\text{dist} (\lambda, \sigma(H_\Theta)) \leq \|(H_\Theta - \lambda)\varphi\| \leq (2d - 1)\varepsilon \left( \| \partial_{ex}^{\Theta} \Phi \| \right)^2 \left\| \varphi_{\partial_{in}^{\Theta} \Phi} \right\|_\infty.
\end{equation}

In the special case when $\Phi$ is a box $\Lambda_L$, we have
\begin{equation}
\langle (H_\Theta - \lambda)\varphi, \varphi \rangle(y) = \varepsilon \varphi(y) \chi_{\partial_{in}^{\Theta} \Lambda_L}(y) \quad \text{for all} \quad y \in \Theta,
\end{equation}
and
\begin{equation}
\text{dist} (\lambda, \sigma(H_\Theta)) \leq \|(H_\Theta - \lambda)\varphi\| \leq \varepsilon \sqrt{d} L^{d-1} \left\| \varphi_{\partial_{in}^{\Theta} \Lambda_L} \right\|_\infty.
\end{equation}
Proof. We have
\[
((H_\Theta - \lambda) \varphi)(y) = \begin{cases} 
0 & \text{if } y \in \Phi \\
\varepsilon \sum_{y' \in \partial_{ex}^\Theta \Phi} \varphi(y') & \text{if } y \in \partial_{ex}^\Theta \Phi \\
0 & \text{if } y \in \Theta \setminus (\Phi \cup \partial_{ex}^\Theta \Phi) 
\end{cases} \tag{3.14}
\]
which is the same as (3.11). It follows that
\[
||(H_\Theta - \lambda) \varphi ||(y) \leq (2d-1)\varepsilon \chi_{\partial_{ex}^\Theta \Phi}(y) ||\varphi||_\infty \text{ for all } y \in \Theta, \tag{3.15}
\]
which yields (3.11). \hfill \square

**Lemma 3.3.** Let \( \Theta \subset \mathbb{Z}^d \) and \( 0 < 4\delta < \eta \). Suppose:

(i) \( \mu \) is a simple eigenvalue of \( H_\Theta \) with normalized eigenfunction \( \psi_\mu \), with \( \text{dist}(\mu, \sigma(H_\Theta) \setminus \{\mu\}) \geq \eta \).

(ii) \( ||(H_\Theta - \lambda) \varphi || \leq \delta \), where \( \varphi \in \ell^2(\Theta) \) with \( ||\varphi|| = 1 \) and \( \lambda \in \mathbb{R} \) with \( |\lambda - \mu| \leq \delta \).

Define \( \varphi^\perp \) by \( \varphi = (\psi_\mu, \varphi) \psi_\mu + \varphi^\perp \). Then we have
\[
||\psi_\mu, \varphi ||^2 \geq 1 - \frac{2\delta^2}{\eta^2} \text{ and } ||\varphi^\perp|| \leq \frac{\sqrt{2} \delta}{\eta}. \tag{3.16}
\]
Moreover, if we set \( \hat{\varphi} = \partial \varphi \), where \( \partial \in \mathbb{C} \) with \( |\partial| = 1 \) is chosen so
\[
\langle \psi_\mu, \hat{\varphi} \rangle = |\langle \psi_\mu, \varphi \rangle| > 0, \tag{3.17}
\]
we have
\[
||\hat{\varphi} - \psi_\mu || \leq \frac{3\delta}{2\eta} < \frac{2\delta}{\eta}. \tag{3.18}
\]
Proof. We have
\[
\varphi = \varphi_\mu + \varphi^\perp, \text{ where } \varphi_\mu = \langle \psi_\mu, \varphi \rangle \psi_\mu \text{ and } \langle \psi_\mu, \varphi^\perp \rangle = 0. \tag{3.19}
\]
Thus
\[
(H_\Theta - \lambda) \varphi = (\mu - \lambda) \varphi_\mu + (H_\Theta - \lambda) \varphi^\perp, \tag{3.20}
\]
and
\[
||(H_\Theta - \lambda) \varphi^\perp || \geq ||(H_\Theta - \mu) \varphi^\perp || - ||(\mu - \lambda) \varphi^\perp || \tag{3.21}
\]
which gives
\[
\delta^2 \geq ||(H_\Theta - \lambda) \varphi ||^2 = (\mu - \lambda)^2 \varphi_\mu^2 + \varepsilon \sum_{y' \in \partial_{ex}^\Theta \Phi} \varphi(y')^2 \tag{3.22}
\]
\[
\geq (\mu - \lambda)^2 \varphi_\mu^2 + (\eta - |\mu - \lambda|)^2 \varphi^\perp^2 \nonumber
\]
\[
= (\mu - \lambda)^2 \left(1 - ||\varphi^\perp||^2\right) + (\eta - |\mu - \lambda|)^2 \varphi^\perp^2 \nonumber
\]
\[
= (\mu - \lambda)^2 + \left((\eta - |\mu - \lambda|)^2 - (\mu - \lambda)^2\right) \varphi^\perp^2 \nonumber
\]
\[
= (\mu - \lambda)^2 + (\eta^2 - 2\eta |\mu - \lambda|) \varphi^\perp^2, \nonumber
\]
and we conclude, using \( 4\delta < \eta \), that
\[
||\varphi^\perp||^2 \leq \frac{\delta^2 - (\mu - \lambda)^2}{\eta^2 - 2|\mu - \lambda|/\eta} \leq \frac{\delta^2}{\eta(\eta - 2\delta)} \leq \frac{2\delta^2}{\eta^2}, \text{ so } \langle \psi_\mu, \varphi \rangle^2 \geq 1 - \frac{2\delta^2}{\eta^2}. \tag{3.23}
\]
It follows that, if we set set \( \hat{\varphi} = \vartheta \varphi \), where \( \vartheta \in \mathbb{C} \) with \( |\vartheta| = 1 \) is chosen so \( \langle \psi_{\mu}, \hat{\varphi} \rangle > 0 \), we have
\[
\| \hat{\varphi} - \psi_{\mu} \|^2 = |1 - \langle \psi_{\mu}, \hat{\varphi} \rangle|^2 + \| \varphi_{\mu} \|^2 = |1 - |\langle \psi_{\mu}, \varphi \rangle||^2 + \| \varphi_{\mu} \|^2
\]  
(3.24)
where we have used \( 1 - (1 - x)^{\frac{1}{2}} \leq x \) for \( x \in [0, 1] \) and \( 4\delta < \eta \).

3.4. Localizing boxes.

**Lemma 3.4.** Let \( \Lambda_{L} \) be a box, \( x \in \Lambda_{L}, m \geq m_{-} > 0 \), and suppose \( \varphi \in \ell^{2}(\Lambda_{L}) \) is an \((x,m)\)-localized eigenfunction of \( H_{\Lambda_{L}} \) with eigenvalue \( \lambda \in \mathbb{R} \). Then for all subsets \( \Lambda_{L} \subset \Theta \subset \mathbb{Z}^{d} \) such that \( x \in \Lambda_{L}^{\Theta,L_{\tau}} \) we have
\[
\text{dist} (\lambda, \sigma(H_{\Theta})) \leq \| (H_{\Theta} - \lambda) \varphi \| \leq e^{-m_{1}L_{\tau}},
\]  
(3.25)
for \( L \geq \mathcal{L}(d,m_{-},\varepsilon_{0}) \), where
\[
m_{1} = m_{1}(L) \geq m \left( 1 - C_{d,m_{-},\varepsilon_{0}} \frac{\log L}{L} \right).
\]  
(3.26)

**Proof.** Since \( x \in \Lambda_{L}^{\Theta,L_{r}} \), we have \( \text{dist} (x, \partial_{m_{-}}^{\Theta}\Lambda_{L}) \geq L_{\tau} \), so it follows from \( 3.13 \) in Lemma 3.2 that
\[
\| (H_{\Theta} - \lambda) \varphi \| \leq \varepsilon \sqrt{\pi L_{\tau}} \left\| \varphi_{\partial_{m_{-}}^{\Theta}\Lambda_{L}} \right\|_{\infty} \leq \varepsilon_{0}\sqrt{\pi L_{\tau}} \left\| \varphi_{\partial_{m_{-}}^{\Theta}\Lambda_{L}} \right\|_{\infty} \leq e^{-m_{1}L_{\tau}},
\]  
(3.27)
where \( m_{1} \) is as in (3.26). \( \square \)

**Lemma 3.5.** Let \( \Theta \subset \mathbb{Z}^{d}, \) fix \( m_{-} > 0 \), and let \( m \geq m_{-} \). Let \( \psi: \Theta \to \mathbb{C} \) be a generalized eigenfunction for \( H_{\Theta} \) with generalized eigenvalue \( \lambda \in \mathbb{R} \). Consider a box \( \Lambda_{\ell} \subset \Theta \) such that \( \Lambda_{\ell} \) is \( m \)-localizing with an \( m \)-localized eigenystem \( \{ \varphi_{u}, \nu_{u} \}_{u \in \Lambda_{\ell}} \), and suppose
\[
|\lambda - \nu_{u}| \geq \frac{1}{2}e^{-L_{\beta}} \text{ for all } u \in \Lambda_{\ell}^{\Theta,L_{\tau}}.
\]  
(3.28)
Then the following holds for sufficiently large \( L \):

(i) If \( y \in \Lambda_{L}^{\Theta,2L_{\tau}} \) we have
\[
|\psi(y)| \leq e^{-m_{2}\tau} |\psi(y_{1})| \text{ for some } y_{1} \in \partial_{m_{-}}^{\Theta,2L_{\tau}} \Lambda_{\ell},
\]  
(3.29)
where
\[
m_{2} = m_{2}(\ell) \geq m \left( 1 - C_{d,m_{-},\varepsilon_{0}} \frac{\ell_{\beta} - \tau}{L} \right).
\]  
(3.30)

(ii) If \( y \in \Lambda_{L}^{\Theta,2L_{\tau}} \) we have
\[
|\psi(y)| \leq e^{-m_{3}\|y_{2}-y\|} |\psi(y_{2})| \text{ for some } y_{2} \in \partial_{m_{-}}^{\Theta,2L_{\tau}} \Lambda_{\ell}, \text{ so } \|y_{2} - y\| > L_{\tau},
\]  
(3.31)
where
\[
m_{3} = m_{3}(\ell) \geq m \left( 1 - C_{d,m_{-},\varepsilon_{0}} \frac{\ell_{\beta} - \tau}{L} \right).
\]  
(3.32)

**Proof.** Given \( y \in \Lambda_{\ell} \), we have (see 3.9 and 3.5)
\[
\psi(y) = \sum_{u \in \Lambda_{\ell}} \varphi_{u}(y) \langle \varphi_{u}, \psi \rangle = \sum_{u \in \Lambda_{\ell}^{\Theta,2L_{\tau}}} \varphi_{u}(y) \langle \varphi_{u}, \psi \rangle + \sum_{u \in \partial_{m_{-}}^{\Theta,2L_{\tau}} \Lambda_{\ell}} \varphi_{u}(y) \langle \varphi_{u}, \psi \rangle.
\]  
(3.33)
Let us fix \( u \in \Lambda_\ell^{\Theta, \ell_\tau} \). We have \( |\lambda - \nu_u| \geq \frac{1}{2} e^{-L \beta} \) by (3.28). Since \( \Lambda_\ell \) is finite, (3.7) gives

\[
\langle \varphi_u, \psi \rangle = (\lambda - \nu_u)^{-1} (\langle H_\Theta - \nu_u \rangle \varphi_u, \psi). \tag{3.34}
\]

It follows from (3.12) that

\[
|\varphi_u(y) \langle \varphi_u, \psi \rangle| \leq 2e^{L \beta} \varepsilon \sum_{v \in \partial_0^{\Theta} \Lambda_\ell} |\varphi_u(y)\varphi_u(v)| |\psi(v)|. \tag{3.35}
\]

If \( v' \in \partial_0^{\Theta} \Lambda_\ell \), we have \( \|v' - u\| \geq \ell_\tau \), so it follows from (1.11) that

\[
|\varphi_u(v')| \leq e^{-m\|v' - u\|} \leq e^{-m \ell_\tau}. \tag{3.36}
\]

Since \( \|\varphi_u\| = 1 \), we get from (3.35) that

\[
|\varphi_u(y) \langle \varphi_u, \psi \rangle| \leq 2e^{L \beta} e^{-m \ell_\tau} \sum_{v \in \partial_0^{\Theta} \Lambda_\ell} |\psi(v)| \leq 2e^{L \beta} s_\delta \beta_\delta^{-1} e^{-m \ell_\tau} |\psi(v_1)|, \tag{3.37}
\]

for some \( v_1 \in \partial_0^{\Theta} \Lambda_\ell \). It follows that

\[
\left| \sum_{u \in \Lambda_\ell^{\Theta, \ell_\tau}} \varphi_u(y) \langle \varphi_u, \psi \rangle \right| \leq 2e^{L \beta} s_\delta \beta_\delta^{-1} e^{-m \ell_\tau} |\psi(v_2)| \tag{3.38}
\]

for some \( v_2 \in \partial_0^{\Theta} \Lambda_\ell \).

Let \( y \in \Lambda_\ell^{\Theta, 2\ell_\tau} \). If \( u \in \partial_0^{\Theta} \Lambda_\ell \) we have \( \|y - u\| / \ell_\tau = \ell_\tau \), so (1.11) gives \( |\varphi_u(y)| \leq e^{-m\|y - u\|} \leq e^{-m \ell_\tau} \), and thus

\[
\left| \sum_{y \in \partial_0^{\Theta} \Lambda_\ell \setminus \Lambda_\ell^{\Theta, \ell_\tau}} \varphi_u(y) \langle \varphi_u, \psi \rangle \right| \leq \ell_\delta \beta_\delta^{-1} e^{-m \ell_\tau} |\psi(y_1)| \leq \ell_\delta \beta_\delta^{-1} e^{-m \ell_\tau} |\psi(y_3)|. \tag{3.39}
\]

for some \( y_3 \in \Lambda_\ell \). Combining (3.33), (3.38), and (3.39), we get \((\ell \text{ large})\)

\[
|\psi(y)| \leq (1 + \varepsilon_0) e^{L \beta} \ell_\delta \beta_\delta^{-1} e^{-m \ell_\tau} |\psi(y_1)| \leq e^{-m \ell_\tau} |\psi(y)|. \tag{3.40}
\]

for some \( y_1 \in \Lambda_\ell \cup \partial_0^{\Theta} \Lambda_\ell \), where \( m_2 \) is given in (3.33). (Note \( \tau > \gamma \beta \).) If \( y_1 \in \partial_0^{\Theta, 2\ell_\tau} \Lambda_\ell \) we have (3.29). If not, we repeat the procedure to estimate \( |\psi(y_1)| \). Since we can suppose \( \psi(y) \neq 0 \) without loss of generality, it is clear that the procedure must stop after finitely many times, and at that time we must have (3.29).

Now let \( y \in \Lambda_\ell^{\Theta, \ell_\tau} \), so \( \|y - v'\| \geq \ell_\tau \) for \( v' \in \partial_0^{\Theta} \Lambda_\ell \). Thus for \( u \in \Lambda_\ell^{\Theta, \ell_\tau} \) and \( v' \in \partial_0^{\Theta} \Lambda_\ell \) we have

\[
|\varphi_u(y) \varphi_u(v')| \leq \begin{cases} e^{-m(\|y - u\| + \|v' - u\|)} & \text{if } \|y - u\| \geq \ell_\tau \\ e^{-m(\|v' - u\|)} & \text{if } \|y - u\| < \ell_\tau. \end{cases} \tag{3.41}
\]

where

\[
m'_1 \geq m(1 - 2\ell_\tau^{-\tau}) = m(1 - 2\ell_\tau^{-\tau}). \tag{3.42}
\]

since for \( \|y - u\| < \ell_\tau \), we have

\[
\|v' - u\| \geq \|v' - y\| - \|y - u\| \geq \|v' - y\| - \ell_\tau \geq \|v' - y\| \left(1 - \frac{\ell_\tau}{\ell_\tau} \right). \tag{3.43}
\]
Combining (3.33) and (3.41), we get

$$|\varphi_u(y)\langle \varphi_u, \psi \rangle| \leq 2e^{L^\beta} \sum_{v \in \partial_{ex}^\theta \Lambda_t} e^{-m^*_1(\|v-y\|^{-1})} |\psi(v)|$$

(3.44)

$$\leq 2e^{L^\beta} \sum_{v \in \partial_{ex}^\theta \Lambda_t} e^{-m^*_1(\|v_1-y\|^{-1})} |\psi(v_1)| \leq e^{-m^*_2\|v_1-y\|} |\psi(v_1)|$$

for some $v_1 \in \partial_{ex}^\theta \Lambda_t$, where

$$m^*_2 \geq m^*_1 \left(1 - C_{d,m-\varepsilon_0} \ell^{-\beta}\right) \geq m \left(1 - C_{d,m-\varepsilon_0} \ell^{-\beta} \right),$$

(3.45)

where we used $\|v_1 - y\| \geq \ell_\tau$ and $\tau > \gamma \beta$. It follows that

$$\sum_{u \in \Lambda_t} \varphi_u(y) \langle \varphi_u, \psi \rangle \leq \ell^d e^{-m^*_2\|v_2-y\|} |\psi(v_2)| \leq e^{-m^*_2\|v_2-y\|} |\psi(v_2)|$$

(3.46)

for some $v_2 \in \partial_{ex}^\theta \Lambda_t$, where

$$m^*_3 \geq m^*_2 \left(1 - C_{d,m-\varepsilon_0} \log \ell \right) \geq m \left(1 - C_{d,m-\varepsilon_0} \ell^{-\beta} \right).$$

(3.47)

If $u \in \partial_{m}^\theta \Lambda_t$ we must have $\|u - y\| \geq \ell_\tau - \ell_\tau > \frac{1}{2} \ell_\tau$, so (1.11) gives $|\varphi_u(y)| \leq e^{-m\|u-y\|}$ and, using (1.12) for $\varphi_u$, we get

$$|\langle \varphi_u, \psi \rangle| = \left| \sum_{v \in \Lambda_t} \varphi_u(v)\psi(v) \right| \leq \sum_{v \in \Lambda_t} e^{-m\|v-u\|} |\psi(v)|,$$

(3.48)

so we conclude that

$$|\varphi_u(y)\langle \varphi_u, \psi \rangle| \leq \sum_{v \in \Lambda_t} e^{-m\|u-y\|} |\psi(v)|$$

(3.49)

$$\leq (\ell + 1)^d e^{-m\|v_3-y\|} |\psi(v_3)|$$

$$\leq e^{-m^*_4\|v_3-y\|} |\psi(v_3)|$$

$$\leq e^{-m^*_4\max\{\|v_3-y\|,\|u-y\|\}} |\psi(v_3)| \leq e^{-m^*_4\max\{\|v_3-y\|, \frac{1}{2} \ell_\tau\}} |\psi(v_3)|$$

for some $v_3 \in \Lambda_t$, where we used $\|u - y\| \geq \frac{1}{2} \ell_\tau$ and took

$$m^*_4 \geq m(1 - C_{d,m-\ell^{-\beta}}).$$

(3.50)

It follows that

$$\left| \sum_{u \in \partial_{m}^\theta \Lambda_t} \varphi_u(y)\langle \varphi_u, \psi \rangle \right| \leq \ell^d e^{-m^*_4\max\{\|v_3-y\|, \frac{1}{2} \ell_\tau\}} |\psi(v_3)|$$

(3.51)

$$\leq e^{-m^*_4\max\{\|v_3-y\|, \frac{1}{2} \ell_\tau\}} |\psi(v_3)|$$

for some $v_3 \in \Lambda_t$, where

$$m^*_5 \geq m^*_4(1 - C_{d,m-} (\log \ell) \ell^{-\beta}) \geq m(1 - C_{d,m-} \ell^{-\beta}).$$

(3.52)

Combining (3.33), (3.41), and (3.44), we get

$$|\psi(y)| \leq e^{-m_3\max\{\|y_1-y\|, \frac{1}{2} \ell_\tau\}} |\psi(y_1)|$$

for some $y_1 \in \Lambda_t \cup \partial_{ex}^\theta \Lambda_t$,

(3.53)

where $m_3$ is given in (3.32).
If in (3.33) we get \(y_1 \notin \partial H^{\Theta, \ell} \Lambda_\ell\) we repeat the procedure to estimate \(|\psi(y_1)|\). Since we can suppose \(\psi(y) \neq 0\) without loss of generality, the procedure must stop after finitely many times, and at that time we must have
\[
|\psi(y)| \leq e^{-m_3 \max \{\|\tilde{y} - y\|, \frac{1}{2} \ell \}} |\psi(\tilde{y})| \quad \text{for some} \quad \tilde{y} \in \partial H^{\Theta, \ell} \Lambda_\ell.
\] (3.54)

If \(y \in \Lambda_\ell^{\Theta, 2\ell}\), (3.31) is an immediate consequence of (3.54). \(\square\)

**Lemma 3.6.** Let the finite set \(\Theta \subset \mathbb{Z}^d\) be \(L\)-level spacing for \(H\), and let \(\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}\) be an eigensystem for \(H_\Theta\).

Then the following holds for sufficiently large \(L\):

(i) Let \(\Lambda_\ell(a) \subset \Theta\), where \(a \in \mathbb{R}^d\), be an \(m\)-localizing box with an \(m\)-localized eigensystem \(\{(\varphi_\ell^{(a)}, \lambda_\ell^{(a)})\}_{a \in \Lambda_\ell(a)}\).

(a) There exists an injection
\[
x \in \Lambda_\ell^{\Theta, \ell} (a) \mapsto \tilde{\lambda}_x^{(a)} \in \sigma(H_\Theta),
\] (3.55)
such that for all \(x \in \Lambda_\ell^{\Theta, \ell} (a)\) we have
\[
|\tilde{\lambda}_x^{(a)} - \lambda_x^{(a)}| \leq e^{-m_1 \ell}, \quad \text{with} \quad m_1 = m_1(\ell) \quad \text{as in (5.20)},
\] (3.56)
and, redefining each \(\varphi_\ell^{(a)}\) by multiplying it by a suitable phase factor (as in (3.11)),
\[
\left\|\psi_\lambda^{(a)}(x) - \varphi_\ell^{(a)}(x)\right\| \leq 2e^{-m_1 \ell} e^{L_\beta}.\] (3.57)

(b) Let
\[
\sigma_{\{a\}}(H_\Theta) := \left\{\tilde{\lambda}_x^{(a)} ; x \in \Lambda_\ell^{\Theta, \ell} (a)\right\}.
\] (3.58)
Then for \(\lambda \in \sigma_{\{a\}}(H_\Theta)\) we have
\[
|\psi_\lambda(y)| \leq 2e^{-m_1 \ell} e^{L_\beta} \quad \text{for all} \quad y \in \Theta \setminus \Lambda_\ell(a).
\] (3.59)

(c) If \(\lambda \in \sigma(H_\Theta) \setminus \sigma_{\{a\}}(H_\Theta)\), we have
\[
|\lambda - \lambda_x^{(a)}| \geq \frac{1}{2} e^{-L_\beta} \quad \text{for all} \quad x \in \Lambda_\ell^{\Theta, \ell} (a),
\] (3.60)
and
\[
|\psi_\lambda(y)| \leq e^{-m_2 \ell} \quad \text{for} \quad y \in \Lambda_\ell^{\Theta, 2\ell}, \quad \text{with} \quad m_2 = m_2(\ell) \quad \text{as in (3.30)}. \] (3.61)

Moreover, if \(y \in \Lambda_\ell^{\Theta, 2\ell}(a)\) we have
\[
|\psi_\lambda(y)| \leq e^{-m_3 \|y - y_1\|} |\psi_\lambda(y_1)| \quad \text{for some} \quad y_1 \in \partial H^{\Theta, \ell} \Lambda_\ell(a),
\] (3.62)
with \(m_3 = m_3(\ell)\) as in (5.32).

(ii) Let \(\Lambda_\ell(a) \subset \Theta\), where \(a \in \mathbb{R}^d\) and \(\Lambda_\ell(a) \subset \Theta\) for all \(a \in \mathbb{B}\), be a collection of \(m\)-localizing boxes with \(m\)-localized eigensystems \(\{(\varphi_\ell^{(a)}, \lambda_\ell^{(a)})\}_{a \in \Lambda_\ell(a)}\).

and set
\[
\mathcal{E}_\Theta^{(a)}(\lambda) = \left\{\lambda_x^{(a)} ; a \in \mathbb{B}, x \in \Lambda_\ell^{\Theta, \ell} (a), \text{and} \tilde{\lambda}_x^{(a)} = \lambda\right\} \quad \text{for} \quad \lambda \in \sigma(H_\Theta),
\] (3.63)
\[
\sigma_\Theta(H_\Theta) = \left\{\lambda \in \sigma(H_\Theta) ; \mathcal{E}_\Theta^{(a)}(\lambda) \neq \emptyset\right\} = \bigcup_{a \in \mathbb{B}} \sigma_{\{a\}}(H_\Theta).
\]
(a) Let \(a, b \in G\), \(a \neq b\). Then, for \(x \in \Lambda^\Theta_{\ell, r}(a)\) and \(y \in \Lambda^\Theta_{\ell, r}(b)\),
\[
\lambda_x(a), \lambda_y(b) \in E^\Theta_{G}(\lambda) \implies \|x - y\| < 2\ell r. \tag{3.64}
\]
As a consequence,
\[
\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \implies \sigma_\ell(a)(H_\Theta) \cap \sigma_\ell(b)(H_\Theta) = \emptyset. \tag{3.65}
\]

(b) If \(\lambda \in \sigma_G(H_\Theta)\), we have
\[
|\psi_\lambda(y)| \leq 2e^{-m_1\ell r} e^{L\beta} \text{ for all } y \in \Theta \setminus \Theta_G, \text{ where } \Theta_G := \bigcup_{a \in G} \Lambda_\ell(a). \tag{3.66}
\]

(c) If \(\lambda \in \sigma(H_\Theta) \setminus \sigma_G(H_\Theta)\), we have
\[
|\psi_\lambda(y)| \leq e^{-m_2\ell r} \text{ for all } y \in \Theta_G, r \implies \bigcup_{a \in G} \Lambda^\Theta_{\ell, 2\ell r}(a). \tag{3.67}
\]

(d) If \(|\Theta| \leq (L + 1)^d\), it follows that
\[
|\Theta_G, r| \leq |\sigma_G(H_\Theta)| \leq |\Theta_G|. \tag{3.68}
\]

Proof. Let \(\Lambda_\ell(a) \subset \Theta\), where \(a \in \mathbb{R}^d\), be a \(m\)-localizing box with an \(m\)-localized eigensystem \(\{\varphi_x(a) : \lambda_x(a)\} x \in \Lambda_\ell(a)\). Given \(x \in \Lambda^\Theta_{\ell, r}(a)\), the existence of \(\bar{\lambda}_x(a) \in \sigma(H_\Theta)\) satisfying \((3.59)\) follows from Lemma \(3.3\). Uniqueness follows from the fact that \(\Theta\) is \(L\)-level spacing and \(\gamma \beta < \tau\). In addition, note that \(\bar{\lambda}_x(a) \neq \bar{\lambda}_y(b)\) if \(x, y \in \Lambda^\Theta_{\ell}(a), x \neq y\), because in this case we have
\[
|\bar{\lambda}_x(a) - \bar{\lambda}_y(b)| \geq |\lambda_x(a) - \lambda_y(b)| - |\bar{\lambda}_x(a) - \lambda_x(a)| - |\bar{\lambda}_y(b) - \lambda_y(b)| \geq e^{-\beta} - 2e^{-m_1\ell r} \geq \frac{1}{2} e^{-\beta}, \tag{3.69}
\]
\(\Lambda_\ell(a)\) is level spacing for \(H\), and \(\beta < \tau\). Moreover, it follows from Lemma \(3.3\) that, after multiplying \(\varphi_x(a)\) by a phase factor if necessary, we have \((3.57)\).

If \(\lambda \in \sigma_\ell(a)(H_\Theta)\), we have \(\lambda = \bar{\lambda}_x(a)\) for some \(x \in \Lambda^\Theta_{\ell}(a)\), so \((3.59)\) follows from \((3.57)\) as \(\varphi_x(a)(y) = 0\) for all \(y \in \Theta \setminus \Lambda_\ell(a)\).

Let \(\lambda \in \sigma(H_\Theta) \setminus \sigma_\ell(a)(H_\Theta)\). Then for all \(x \in \Lambda^\Theta_{\ell}(a)\) we have
\[
|\lambda - \lambda_x(a)| \geq |\lambda - \bar{\lambda}_x(a)| - |\bar{\lambda}_x(a) - \lambda_x(a)| \geq e^{-\beta} - e^{-m_1\ell r} \geq \frac{1}{2} e^{-\beta}, \tag{3.70}
\]
since \(\Theta\) is \(L\)-level spacing for \(H\), we have \((3.59)\) and \(\gamma \beta < \tau\). Thus \((3.61)\) follows from Lemma \(3.3\) and \(\|\psi_\lambda\| = 1\), and \((3.62)\) follows from Lemma \(3.3\).

Now let \(\{\Lambda_\ell(a)\}_{a \in G}\), where \(G \subset \mathbb{R}^d\) and \(\Lambda_\ell(a) \subset \Theta\) for all \(a \in G\), be a collection of \(m\)-localizing boxes with \(m\)-localized eigensystems \(\{\varphi_x(a), \lambda_x(a)\} x \in \Lambda_\ell(a)\). Let \(\lambda \in \sigma(H_\Theta)\), \(a, b \in G, a \neq b, x \in \Lambda^\Theta_{\ell}(a)\), and \(y \in \Lambda^\Theta_{\ell}(b)\). Suppose \(\lambda_x(a), \lambda_y(b) \in E^\Theta_{G}(\lambda)\), where \(E^\Theta_G(\lambda)\) is given in \((3.63)\). It then follows from \((3.57)\) that
\[
\|\varphi_x(a) - \varphi_y(b)\| \leq 4e^{-m_1\ell r} e^{L\beta}, \tag{3.71}
\]
so
\[
\left|\langle \varphi_x(a), \varphi_y(b) \rangle\right| \geq \Re \left|\langle \varphi_x(a), \varphi_y(b) \rangle\right| \geq 1 - 8e^{-2m_1\ell r} e^{2L\beta}. \tag{3.72}
\]

On the other hand, it follows from \((1.11)\) that
\[
\|x - y\| \geq 2\ell r \implies \left|\langle \varphi_x(a), \varphi_y(b) \rangle\right| \leq (\ell + 1)^d e^{-m_\ell} r. \tag{3.73}
\]
Combining (3.72) and (3.73) we conclude that
\[ \lambda_x^{(a)}, \lambda_y^{(b)} \in \mathcal{E}_G^\Theta(\lambda) \implies \|x - y\| < 2\ell_r. \] (3.74)

To prove (3.65), let \( a, b \in G, a \neq b \). If \( \Lambda_{\ell}(a) \cap \Lambda_{\ell}(b) = \emptyset \), we have that
\[ x \in \Lambda_{\ell}^{\Theta,\ell_r}(a) \quad \text{and} \quad y \in \Lambda_{\ell}^{\Theta,\ell_r}(b) \implies \|x - y\| \geq 2\ell_r, \] (3.75)
so it follows from (3.64) that \( \sigma_{\{b\}}(H_\Theta) \cap \sigma_{\{a\}}(H_\Theta) = \emptyset \).

Parts (ii)(b) and (ii)(c) are immediate consequence of parts (i)(b) and (i)(c), respectively. To prove part (ii)(d), note that, letting \( P_\delta \) denote the orthogonal projection onto the span of \( \{\psi_\lambda; \lambda \in \sigma_G(H_\Theta)\} \), it follows from (3.67) that
\[ \|(1 - P_\delta)y\| \leq e^{-m_2\ell_r} |\Theta|^{\frac{1}{2}} \quad \text{for all} \quad y \in \Theta_{G,\ell_r}, \] (3.76)
so
\[ \|(1 - P_\delta)\chi_{\Theta_{G,\ell}}\| \leq |\Theta_{G,\ell_r}|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} e^{-m_2\ell_r} \leq |\Theta| e^{-m_2\ell_r}. \] (3.77)
If we assume \( |\Theta| \leq (L + 1)^d \), we get
\[ \|(1 - P_\delta)\chi_{\Theta_{G,\ell}}\| \leq (L + 1)^d e^{-m_2\ell_r} < 1, \] (3.78)
so we conclude from Lemma A.1 that
\[ |\Theta_{G,\ell}| = \text{tr} \chi_{\Theta_{G,\ell}} \leq \text{tr} P_\delta = |\sigma_G(H_\Theta)|. \] (3.79)
A similar argument, using (3.66), proves \( |\sigma_G(H_\Theta)| \leq |\Theta_G| \). \( \square \)

### 3.5. Buffered subsets.
We will need to consider boxes \( \Lambda_{\ell} \subset \Lambda_L \) that are not \( m \)-localizing for \( H \). Instead of studying eigensystems for such boxes, we will surround them with a buffer of \( m \)-localizing boxes and study eigensystems for the augmented subset.

**Definition 3.7.** Let \( \Lambda_L = \Lambda_L(x_0), x_0 \in \mathbb{R}^d \), and \( m \geq m_- > 0 \). We call \( \Upsilon \subset \Lambda_L \) a buffered subset of \( \Lambda_L \) if the following holds:

(i) \( \Upsilon \) is a connected set in \( \mathbb{Z}^d \) of the form
\[ \Upsilon = \bigcup_{j=1}^J \Lambda_{R_j}(a_j) \cap \Lambda_L, \] (3.80)
where \( J \in \mathbb{N}, a_1, a_2, \ldots, a_J \in \Lambda_L^R \), and \( \ell \leq R_j \leq L \) for \( j = 1, 2, \ldots, J \).

(ii) \( \Upsilon \) is \( L \)-level spacing for \( H \).

(iii) There exists \( G_\Upsilon \subset \Lambda_L^R \) such that:
\( \begin{align*} 
(a) & \quad \text{for all} \ a \in G_\Upsilon \text{ we have} \ \Lambda_{\ell}(a) \subset \Upsilon, \text{ and} \ \Lambda_{\ell}(a) \text{ is an } m \text{-localizing box for} \ H. \\
(b) & \quad \text{for all} \ y \in \partial_{m}^L \Upsilon \text{ there exists} \ a_y \in G_\Upsilon \text{ such that} \ y \in \Lambda_{\ell}^{Y,2\ell_r}(a_y). 
\end{align*} \)

In this case we set
\[ \bar{\Upsilon} = \bigcup_{a \in G_\Upsilon} \Lambda_{\ell}(a), \quad \bar{\Upsilon}_r = \bigcup_{a \in G_\Upsilon} \Lambda_{\ell}^{Y,2\ell_r}(a), \quad \bar{\Upsilon} = \Upsilon \setminus \bar{\Upsilon}_r, \text{ and} \quad \bar{\Upsilon}_r = \Upsilon \setminus \bar{\Upsilon}. \] (3.81)
(\( \bar{\Upsilon}_r \) is \( \bar{\Upsilon}_r \) and \( \bar{\Upsilon}_r = \Upsilon_{\bar{G}_\Upsilon,\ell} \) in the notation of Lemma A.1)

The set \( \bar{\Upsilon}_r \supset \partial_{m}^L \Upsilon \) is a localizing buffer between \( \bar{\Upsilon} \) and \( \Lambda_L \setminus \Upsilon \), as shown in the following lemma.
Lemma 3.8. Let $\mathcal{T}$ be a buffered subset of $\Lambda_L$, and let $\{(\psi_\nu, \nu)\}_{\nu \in \sigma(H_T)}$ be an eigensystem for $H_T$. Let $\mathcal{G} = G_T$ and set

$$\sigma_{\mathcal{G}}(H_T) = \sigma(H_T) \setminus \sigma_{\mathcal{G}}(H_T),$$

where $\sigma_{\mathcal{G}}(H_T)$ is as in (3.83). Then the following holds for sufficiently large $L$:

(i) For all $\nu \in \sigma_{\mathcal{G}}(H_T)$ we have

$$|\psi_\nu(y)| \leq e^{-m_2\ell}, \quad \text{for all } y \in \tilde{T}, \text{ with } m_2 = m_2(\ell) \text{ as in (3.30)},$$

and

$$|\tilde{T}| \leq |\sigma_{\mathcal{G}}(H_T)| \leq |\tilde{T}|.$$

(ii) Let $\Lambda_L$ be level spacing for $H$, and let $\{ (\phi_\lambda, \lambda) \}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. There exists an injection

$$\nu \in \sigma_{\mathcal{G}}(H_T) \mapsto \tilde{\nu} \in \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}),$$

such that for $\nu \in \sigma_{\mathcal{G}}(H_T)$ we have

$$|\tilde{\nu} - \nu| \leq e^{-m_4\ell}, \quad \text{where } m_4 = m_4(\ell) \geq m (1 - C_{d,m_-,\epsilon_0} C_{\beta,\gamma}),$$

and, multiplying each $\psi_\nu$ by a suitable phase factor as in (3.17),

$$\|\phi_{\tilde{\nu}} - \psi_\nu\| \leq 2e^{-m_4\ell} e^{L\beta}.$$

Proof. Part (i) follows immediately from Lemma 3.6 (ii)(c) and (ii)(d).

Now let $\Lambda_L$ be level spacing for $H$, and let $\{ (\phi_\lambda, \lambda) \}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for $H_{\Lambda_L}$. It follows from (3.11) in Lemma 3.2 that for $\nu \in \sigma_{\mathcal{G}}(H_T)$ we have

$$\|(H_{\Lambda_L} - \nu) \psi_\nu\| \leq (2d-1)e \|\partial_{\nu_L} \nu_L^T \|_{1,\infty} \leq (2d-1)eL^2 e^{-m_2\ell} \leq e^{-m_4\ell},$$

where we used $\partial_{\nu_L} \nu_L \subseteq \tilde{T}$ and (3.33), and $m_4$ is given in (3.86). Since $\Lambda_L$ and $\tilde{T}$ are $L$-level spacing for $H$, the map in (3.33) is a well defined injection into $\sigma(H_{\Lambda_L})$, and (3.87) follows from (3.86) and (3.13). To finish the proof we must show that $\tilde{\nu} \notin \sigma_{\mathcal{G}}(H_{\Lambda_L})$ for all $\nu \in \sigma_{\mathcal{G}}(H_T)$.

Suppose $\tilde{\nu}_1 \in \sigma_{\mathcal{G}}(H_{\Lambda_L})$ for some $\nu_1 \in \sigma_{\mathcal{G}}(H_T)$. Then there is $a \in \mathcal{G}$ and $x \in \Lambda_{\Lambda_L,\ell}^\Lambda(a)$ such that $\lambda_{x}^{(a)} \in E_{\mathcal{G}}^\Lambda(\tilde{\nu}_1)$. On the other hand, it follows from Lemma 3.6 (i)(a) that $\lambda_x^{(a)} \in E_{\mathcal{G}}^\Lambda(\lambda_1)$ for some $\lambda_1 \in \sigma_{\mathcal{G}}(H_T)$. We conclude from (3.57) and (3.57) that

$$\sqrt{2} = \|\psi_{\lambda_1} - \psi_{\nu_1}\| \leq \left\|\psi_{\lambda_1} - \varphi_{x}^{(a)}\right\| + \left\|\varphi_{x}^{(a)} - \phi_{\tilde{\nu}_1}\right\| + \left\|\phi_{\tilde{\nu}_1} - \psi_{\nu_1}\right\| \leq 4e^{-m_4\ell} e^{L\beta} + 2e^{-m_4\ell} e^{L\beta} < 1,$$

a contradiction. \qed

Lemma 3.9. Let $\Lambda_L = \Lambda_L(x_0), \quad x_0 \in \mathbb{R}^d, \quad m \geq m_-$. Let $\mathcal{T}$ be a buffered subset of $\Lambda_L$. Let $\mathcal{G} = G_T$ and set

$$E_{\mathcal{G}}^\Lambda(\nu) = \left\{ \lambda_x^{(a)} : a \in \mathcal{G}, x \in \Lambda_{\Lambda_L,\ell}^\Lambda(a), \text{ and } \lambda_x^{(a)} = \nu \right\} \subset E_{\mathcal{G}}^\Lambda(\nu) \text{ for } \nu \in \sigma(H_T),$$

$$\sigma_{\mathcal{G}}(H_T) = \left\{ \nu \in \sigma(H_T) : E_{\mathcal{G}}^\Lambda(\nu) \neq \emptyset \right\} \subset \sigma_{\mathcal{G}}(H_T).$$

The following holds for sufficiently large $L$:

\[ \text{\blacksquare} \]
(i) Let \((\psi, \lambda)\) be an eigenpair for \(H_{\Lambda L}\) such that
\[
|\lambda - \nu| \geq \frac{1}{2} e^{-L^\beta} \quad \text{for all} \quad \nu \in \sigma_\nu^{\Lambda L}(H_T) \cup \sigma_B(H_T).
\] (3.91)
Thus for all \(y \in \mathcal{Y}^{\Lambda L, 2\ell_T}\) we have
\[
|\psi(y)| \leq e^{-m_5 \varepsilon T} |\psi(\nu)| \quad \text{for some} \quad \nu \in \partial^{\Lambda L, 2\ell_T} \mathcal{Y},
\] (3.92)
where
\[
m_5 = m_5(\ell) \geq m \left(1 - C_{d, m, \sigma_0} \ell^{\gamma \beta - \tau}\right).
\] (3.93)
(ii) Let \(\Lambda_L\) be level spacing for \(H\), let \(\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda L})}\) be an eigensystem for \(H_{\Lambda L}\), recall (3.85), and set
\[
\sigma_T(H_{\Lambda L}) = \{\lambda; \nu \in \sigma_B(H_T)\} \subset \sigma(H_{\Lambda L}) \setminus \sigma_\nu(H_{\Lambda L}).
\] (3.94)
Then for all \(\lambda \in \sigma(H_{\Lambda L}) \setminus (\sigma_\nu(H_{\Lambda L}) \cup \sigma_T(H_{\Lambda L}))\), the condition (3.91) is satisfied, and \(\psi_\lambda\) satisfies (3.92).

Proof. For \(\nu \in \mathcal{G}\) we have \(\Lambda_t(\nu) \subset \mathcal{Y} \subset \Lambda_t\), which implies \(\Lambda_t^{\Lambda L, \ell_T}(\nu) \subset \Lambda_t^{Y, \ell_T}(\nu)\). Thus \(E_\nu^{\Lambda L}(\nu) \subset E_\nu^{Y}(\nu)\) for \(\nu \in \sigma(H_T)\).

Let \(\{(\vartheta_\nu, \nu)\}_{\nu \in \sigma(H_T)}\) be an eigensystem for \(H_T\). For each \(\nu \in \sigma_\nu(H_T)\) we fix \(\lambda^{(a, \nu)}_x \in E_\nu^{Y}(\nu)\), where \(a_\nu \in \mathcal{G}\), \(x_\nu \in \Lambda_t^{Y, \ell_T}(a_\nu)\), picking \(\lambda^{(a, \nu)}_x \in E_\nu^{\Lambda L}(\nu)\) if \(\nu \in \sigma_\nu^{\Lambda L}(H_T)\), so \(x_\nu \in \Lambda_t^{\Lambda L, \ell_T}(a_\nu)\). If \(\nu \in \sigma_B(H_T) \setminus \sigma_\nu^{\Lambda L}(H_T)\) we have \(x_\nu \in \Lambda_t^{Y, \ell_T}(a_\nu) \setminus \Lambda_t^{\Lambda L, \ell_T}(a_\nu)\).

Let \(y \in \mathcal{Y}\). Using (3.91) we have
\[
\psi(y) = \langle \delta_y, \psi \rangle = \sum_{\nu \in \sigma(H_T)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle
= \sum_{\nu \in \sigma_\nu^{\Lambda L}(H_T) \cup \sigma_B(H_T)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle + \sum_{\nu \in \sigma_\nu(H_T)} \vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle.
\] (3.95)

Let \((\psi, \lambda)\) be an eigenpair for \(H_{\Lambda L}\) such that (3.91) holds. Given \(\nu \in \sigma_\nu^{\Lambda L}(H_T) \cup \sigma_B(H_T)\), we have
\[
\langle \vartheta_\nu, \psi \rangle = (\lambda - \nu)^{-1} \langle \vartheta_\nu, (H_{\Lambda L} - \nu) \psi \rangle = (\lambda - \nu)^{-1} \langle (H_{\Lambda L} - \nu) \vartheta_\nu, \psi \rangle.
\] (3.96)
It follows from (3.91) and (3.10) that
\[
|\vartheta_\nu(y) \langle \vartheta_\nu, \psi \rangle| \leq 2 \varepsilon e^{L_\beta} \sum_{\nu \in \sigma_\nu^{\Lambda L} H_T} \left(\sum_{v \in \partial^{\Lambda L} \mathcal{Y}} |\vartheta_\nu(v')| \right) |\psi(\nu)|
\] (3.97)
\[
\leq 2 \varepsilon e^{L_\beta} \left\{2d \max_{u \in \partial^{\Lambda L} \mathcal{Y}} |\vartheta_\nu(u)| \right\} |\psi(v_1)| \quad \text{for some} \quad v_1 \in \partial^{\Lambda L} \mathcal{Y}.
\]
If \(\nu \in \sigma_B(H_T)\) it follows from (3.85) that
\[
\max_{u \in \partial^{\Lambda L} \mathcal{Y}} |\vartheta_\nu(u)| \leq e^{-m_5 \varepsilon T}.
\] (3.98)
If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_{\mathcal{G}})$, it follows from \((3.57)\) and \((1.11)\) that
\[
\max_{u \in \partial^n_L \mathcal{Y}} |\partial_\nu(u)| \leq \max_{u \in \partial^n_L \mathcal{Y}} \left( |\partial_\nu(u) - \varphi^{(a)}_x(u)| + |\varphi^{(a)}_x(u)| \right) \leq 2e^{-m_1\ell} e^{L^\beta} + e^{-m_\ell} \leq 3e^{-m_1\ell} e^{L^\beta},
\]
recalling \((3.30)\) and \((3.40)\). It follows that (note $m_1(\ell) \geq m_2(\ell)$ for $\ell$ large)
\[
\left| \sum_{\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_{\mathcal{G}}) \cup \sigma_{\mathcal{G}}(H_{\mathcal{G}})} \varphi_\nu(y) \left( \partial_\nu, \psi \right) \right| \leq 4d_L L^2 e^{L^\beta} \left( 3e^{-m_2\ell} e^{L^\beta} \right) |\psi(v_2)| \leq 12d_L L^2 e^{2L^\beta} e^{-m_2\ell} |\psi(v_2)|,
\]
for some $v_2 \in \partial^n_L \mathcal{Y}$.

Now let $\nu \in \sigma_{\mathcal{G}}(H_{\mathcal{G}}) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_{\mathcal{G}})$. In this case we have $x_\nu \in \Lambda_{\ell}(a_{\nu}) \setminus \Lambda_{\ell}(a_{\nu})$, so we have
\[
\text{dist } (x_\nu, \mathcal{Y} \setminus \Lambda_{\ell}(a_{\nu})) > \ell \quad \text{and} \quad \text{dist } (x_\nu, \Lambda_{\ell} \setminus \Lambda_{\ell}(a_{\nu})) \leq \ell,
\]
so there is $u_0 \in \Lambda_{\ell} \setminus \mathcal{Y}$ such that $||x_\nu - u_0|| < \ell$. We now assume $y \in \mathcal{Y}^{\Lambda_L, 2\ell}$, so we have $\|y - u_0\| > 2\ell$. We conclude that
\[
|x_\nu - y| \geq \|y - u_0\| - \|x_\nu - u_0\| > 2\ell - \ell = \ell.
\]
Thus
\[
|\varphi(y)| \leq |\varphi(y) - \varphi^{(a)}_x(y)| + |\varphi^{(a)}_x(y)| \leq 2e^{-m_1\ell} e^{L^\beta} + e^{-m_\ell} \leq 3e^{-m_1\ell} e^{L^\beta},
\]
using \((3.57)\) and \((1.11)\). It follows that
\[
\left| \sum_{\nu \in \sigma_{\mathcal{G}}(H_{\mathcal{G}}) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_{\mathcal{G}})} \varphi_\nu(y) \left( \partial_\nu, \psi \right) \right| \leq 3(L + 1) 2^d e^{-m_1\ell} e^{L^\beta} |\psi(v_3)|,
\]
for some $v_3 \in \mathcal{Y}$.

Combining \((3.30)\), \((3.100)\), and \((3.104)\), we get for $y \in \mathcal{Y}^{\Lambda_L, 2\ell}$ that
\[
|\psi(y)| \leq (1 + 12d_L) L^2 e^{2L^\beta} e^{-m_\ell} |\psi(v_4)| \leq e^{-m_\ell} |\psi(v_4)|,
\]
for some $v_4 \in \mathcal{Y} \cup \partial^n_L \mathcal{Y}$, where $m_5$ is given in \((3.98)\). If $v_4 \in \mathcal{Y}^{\Lambda_L, 2\ell}$ we can repeat the procedure to estimate $|\psi(v_4)|$. If $\psi(y) = 0$ there is nothing to prove, so we can assume $\psi(y) \neq 0$. In this case we can only repeat the procedure a finite number of times without getting $|\psi(y)| < |\psi(y)|$, so \((3.92)\) holds.

Now suppose $\Lambda_{\ell}$ is level spacing for $H$. If $\lambda \notin \sigma_{\mathcal{G}}(H_{\mathcal{G}})$, it follows from Lemma \((3.60)\)(c) that \((3.60)\) holds for all $\alpha \in \mathcal{G}$. If $\lambda \notin \sigma_{\mathcal{G}}(H_{\mathcal{G}})$, the argument in \((3.70)\), modified by the use of \((3.80)\) instead of \((3.50)\), gives $|\lambda - \nu| \geq \frac{1}{2} e^{-L^\beta}$ for all $\nu \in \sigma_{\mathcal{G}}(H_{\mathcal{G}})$. Thus we have \((3.91)\), which implies \((3.92)\).

3.6. Suitable covers of a box. To perform the multiscale analysis in an efficient way, it is convenient to use a canonical way to cover a box of side $L$ by boxes of side $\ell < L$. We will use suitable covers of a box as in [GK4] Definition 3.12, adapted to the discrete case.
**Definition 3.10.** Let $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$ be a box in $\mathbb{Z}^d$, and let $\ell < L$. A suitable $\ell$-cover of $\Lambda_L$ is the collection of boxes

$$C_{\ell,L}(x_0) = \{\Lambda_\ell(a)\}_{a \in \Xi_{\ell,L}},$$

(3.106)

where

$$\Xi_{\ell,L} := \left\{ x_0 + \rho \ell \mathbb{Z}^d \right\} \cap \Lambda_L^\circ \quad \text{with} \quad \rho \in \left[ \frac{3}{4}, \frac{4}{3} \right] \cap \left\{ \frac{L}{\ell}k; k \in \mathbb{N} \right\}.$$  

(3.107)

We call $C_{\ell,L}(x_0)$ the suitable $\ell$-cover of $\Lambda_L$ if $\rho = \rho_{L,\ell} := \max\left[ \frac{3}{4}, \frac{4}{3} \right] \cap \left\{ \frac{L}{\ell}k; k \in \mathbb{N} \right\}$.

We recall [GK4, Lemma 3.13], which we rewrite in our context.

**Lemma 3.11.** Let $\ell \leq \frac{L}{6}$. Then for every box $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$, a suitable $\ell$-cover $C_{\ell,L}(x_0)$ satisfies

$$\Lambda_L = \bigcup_{a \in \Xi_{\ell,L}} \Lambda_\ell(a);$$

(3.108)

for all $b \in \Lambda_L$ there is $\Lambda_{b}^{(b)} \in C_{\ell,L}(x_0)$ such that $b \in \left( \Lambda_{b}^{(b)} \right)^{\Lambda_L^{\circ}}$,

(3.109)

i.e.,

$$\Lambda_L = \bigcup_{a \in \Xi_{\ell,L}} \Lambda_{b}^{\Lambda_L^{\circ}}(a);$$

$$\mathcal{A}_L(a) \cap \mathcal{A}_L(b) = \emptyset \quad \text{for all} \quad a, b \in x_0 + \rho \ell \mathbb{Z}^d, \ a \neq b;$$

(3.110)

$$\left( \frac{1}{\rho} \right)^d \leq \#\Xi_{\ell,L} = \left( \frac{L - \ell}{\ell} + 1 \right)^d \leq \left( \frac{2L}{\ell} \right)^d.$$  

(3.111)

Moreover, given $a \in x_0 + \rho \ell \mathbb{Z}^d$ and $k \in \mathbb{N}$, it follows that

$$\Lambda_{\left( 2k\rho+1 \right)\ell}(a) = \bigcup_{b \in \left( x_0 + \rho \ell \mathbb{Z}^d \right) \cap \Lambda_{\left( 2k\rho+1 \right)\ell}(a)} \Lambda_{b}(b);$$

(3.112)

and $\{\Lambda_{b}(b)\}_{b \in \left( x_0 + \rho \ell \mathbb{Z}^d \right) \cap \Lambda_{\left( 2k\rho+1 \right)\ell}}$ is a suitable $\ell$-cover of the box $\Lambda_{\left( 2k\rho+1 \right)\ell}(a)$.

Note that $\Lambda_{b}^{(b)}$ does not denote a box centered at $b$, just some box in $C_{\ell,L}(x_0)$ satisfying $\textbf{[3.108]}$. By $\Lambda_{b}^{(b)}$ we will always mean such a box.

**Remark 3.12.** Note that $\rho \geq \frac{3}{4}$ implies $\textbf{[3.110]}$ and $\rho \leq \frac{4}{3}$ yields $\textbf{[3.109]}$. (We do not use $\textbf{[3.111]}$ in this paper.) We specified $\rho = \rho_{L,\ell}$ in the suitable $\ell$-cover for convenience, so there is no ambiguity in the definition of $C_{\ell,L}(x_0)$.

**Remark 3.13.** Suitable covers are convenient for the construction of buffered subsets.

4. Eigensystem multiscale analysis

In this section we consider an Anderson model $H_{\epsilon,\omega}$ and prove Theorem $\textbf{[1.6]}$ as a corollary to the following proposition. We recall that $m_{\epsilon,L}$ is defined in $\textbf{[1.14]}$.

**Proposition 4.1.** There exists a finite scale $L$ such that, given $L_0 \geq L$ and setting $L_{k+1} = \frac{L_k}{2}$ for $k = 0, 1, \ldots$, for all $\epsilon \leq \frac{1}{4d} e^{-L_0}$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\left\{ \Lambda_{L_k}(x) \text{ is } \frac{m_{\epsilon,L}}{2}\text{-localizing for } H_{\epsilon,\omega} \right\} \geq 1 - e^{-L_k^c} \quad \text{for } k = 0, 1, \ldots \quad \text{(4.1)}$$

Proposition $\textbf{[4.1]}$ is an immediate consequence of the following two propositions.
Proposition 4.2. Let $\varepsilon > 0$ and $L \geq 1$. Then

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{A_{L_0}(x) \text{ is } m_{\varepsilon,L}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \frac{1}{4} K (L + 1)^{2d} \left(8d\varepsilon + 2e^{-L}\right)^{\alpha}.$$  

(4.2)

In particular, if $L$ is sufficiently large, for all $0 < \varepsilon \leq \frac{1}{4d} e^{-L\beta}$ we have $m_{\varepsilon,L} \geq \log 3$ and

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{A_{L_0}(x) \text{ is } m_{\varepsilon,L}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - e^{-L^c}.$$  

(4.3)

Proposition 4.3. Fix $\varepsilon_0 > 0$ and $m_- > 0$. There exists a finite scale $L(\varepsilon_0, m_-)$ with the following property: Suppose for some scale $L_0 = L(\varepsilon_0, m_-)$, $0 < \varepsilon \leq \varepsilon_0$, and $m_0 \geq m_-$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{A_{L_0}(x) \text{ is } m_{0}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - e^{-L^c}.$$  

Then, setting $L_{k+1} = L_k^\gamma$ for $k = 0, 1, \ldots$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{A_{L_k}(x) \text{ is } m_{\frac{k}{\varepsilon}}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - e^{-L^c} \text{ for } k = 0, 1, \ldots.$$  

(4.5)

4.1. Initial step. In this subsection we prove Proposition 4.2.

Lemma 4.4. Let $H_{\varepsilon} = -\varepsilon \Delta + V$ on $l^2(\mathbb{Z}^d)$, where $V$ is a bounded potential and $\varepsilon > 0$. Let $\Theta \subset \mathbb{Z}^d$, and suppose there is $\eta > 0$ such that

$$|V(x) - V(y)| \geq \eta \quad \text{for all } x, y \in \Theta, x \neq y.$$  

(4.6)

Then for $\varepsilon < \frac{\eta}{2d}$ the operator $H_{\varepsilon,\Theta}$ has an eigensystem $\{(\psi_x, \lambda_x)\}_{x \in \Theta}$ such that

$$|\lambda_x - \lambda_y| \geq \eta - 4d\varepsilon > 0 \quad \text{for all } x, y \in \Theta, x \neq y,$$  

(4.7)

and for all $y \in \Theta$ we have

$$|\psi_y(x)| \leq \left(\frac{2d\varepsilon}{\eta - 2d\varepsilon}\right)^{|x-y|} \quad \text{for all } x \in \Theta.$$  

(4.8)

Proof. We take $\varepsilon < \frac{\eta}{2d}$ and treat $H_{\varepsilon,\Theta}$ as a perturbation of $V_\Theta$. Since $\sigma(V_\Theta) = \{V(x)\}_{x \in \Theta}$ is simple and $\|\Delta_\Theta\| \leq 2d$, it follows from [HorL, Theorem 4.3.1]) that $H_{\varepsilon,\Theta}$ has simple spectrum $\sigma(H_{\varepsilon,\Theta}) = \{\lambda_x\}_{x \in \Theta}$ with

$$|\lambda_x - V(x)| \leq 2\varepsilon < \frac{\eta}{2} \quad \text{for all } x \in \Theta,$$  

(4.9)

so we have [4.7] and $H_{\varepsilon,\Theta}$ has an eigensystem $\{(\psi_x, \lambda_x)\}_{x \in \Theta}$.

Let $y \in \Theta$. Then for any $x \in \Theta$, $x \neq y$, we have,

$$|\lambda_y - V(x)| \geq |V(y) - V(x)| - |\lambda_y - V(y)| \geq \eta - 2d\varepsilon,$$  

(4.10)

where we used (4.6) and (4.9), and

$$\psi_y(x) = \langle \delta_x, \psi_y \rangle = (\lambda_y - V(x))^{-1} \langle (H_{\varepsilon,\Theta} - V_\Theta)\delta_x, \psi_y \rangle$$  

$$= \varepsilon (\lambda_y - V(x))^{-1} (-\Delta_\Theta\delta_x, \psi_y) = \varepsilon (\lambda_y - V(x))^{-1} \sum_{z \in \Theta} \psi_y(z).$$  

(4.11)

We conclude that

$$|\psi_y(x)| \leq \frac{\varepsilon}{\eta - 2d\varepsilon} \sum_{z \in \Theta} |\psi_y(z)| \leq \frac{2d\varepsilon}{\eta - 2d\varepsilon} |\psi_y(z)|,$$  

(4.12)
for some \( z_1 \in \Theta \) with \(|z_1 - x| = 1 \). If \( z_1 \neq y \) we can estimate \(|\psi_y(z_1)|\) by (4.12). Since we can perform this procedure at least \(|x - y|_1 \) times, we obtain (4.8).

Proof of Proposition 4.2. Let \( \varepsilon > 0 \) and \( \Lambda_L = \Lambda_L(x_0) \) for some \( x_0 \in \mathbb{R}^d \). Let \( \kappa = 4d\varepsilon e^{L_0} \) and suppose

\[
|V(x) - V(y)| \geq (1 + \kappa)e^{-L_0} \quad \text{for all } x, y \in \Lambda_L, x \neq y.
\] (4.13)

It follows from Lemma 4.4 that \( H_{\varepsilon,\Lambda_L} \) has an eigensystem \( \{(\psi_x, \lambda_x)\}_{x \in \Lambda_L} \) satisfying (4.7) and (4.8) with \( \eta = (1 + \kappa)e^{-L_0} \). Since \( \eta - 4\varepsilon = e^{-L_0} \), we conclude from (4.7) that \( \Lambda_L \) is level spacing for \( H_{\varepsilon} \). Moreover, \( \frac{2d\varepsilon}{\eta - 2\varepsilon} = \frac{\kappa}{2 + \kappa} \) and and \( \|x\| \leq |x|_1 \), so (4.8) yields

\[
|\psi_y(x)| \leq \left( \frac{d\varepsilon}{2 + \kappa} \right)^{|x - y|} = e^{-m_{\varepsilon,L}|x - y|} \quad \text{for all } y, x \in \Lambda_L,
\] (4.14)

where

\[
m_{\varepsilon,L} = -\log \left( \frac{\kappa}{2 + \kappa} \right) = \log \frac{\eta - 2\varepsilon}{2\varepsilon} = \log \left( 1 + \frac{e^{-L_0}}{2\varepsilon} \right).
\] (4.15)

In particular, \( \Lambda_L \) is \( m_{\varepsilon,L} \)-localizing.

We conclude that

\[
\mathbb{P}\{ \Lambda_L \text{ is not } m_{\varepsilon,L} \text{-localizing} \} \leq \mathbb{P}\{ (4.13) \text{ does not hold} \}
\leq \frac{(L+1)^{2d}}{2} S_\mu \left( 2(1 + \kappa)e^{-L_0} \right) = \frac{(L+1)^{2d}}{2} S_\mu \left( 8d\varepsilon + 2e^{-L_0} \right)
\leq \frac{1}{2} K(L + 1)^{2d} \left( 8d\varepsilon + 2e^{-L_0} \right)\alpha,
\] (4.16)

which yields (4.2). (We assumed \( 8d\varepsilon + 2e^{-L_0} \leq 1 \) to use (1.3) as stated; if not (4.2) holds trivially.)

If \( 0 < \varepsilon \leq \frac{1}{4}d\varepsilon e^{-L_0} \), we have \( m_{\varepsilon,L} \geq \log 3 \) and

\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_L(x) \text{ is } m_{\varepsilon,L} \text{-localizing for } H_{\varepsilon} \} \geq 1 - 2^{2\alpha - 1} K(L + 1)^{2d} e^{-\alpha L_0},
\] (4.17)

which gives (4.3) for large \( L \) since \( \zeta < \beta \). \( \square \)

4.2. Multiscale analysis. In this subsection we prove Proposition 4.3. We start with the induction step for the multiscale analysis.

Lemma 4.5. Fix \( \varepsilon_0 > 0 \) and \( m_- > 0 \). Suppose for some scale \( \ell, 0 < \varepsilon \leq \varepsilon_0, \) and \( m \geq m_- \) we have

\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_{\ell}(x) \text{ is } m \text{-localizing for } H_{\varepsilon,\omega} \} \geq 1 - e^{-\ell^c}.
\] (4.18)

Then, if \( \ell \) is sufficiently large, we have (recall \( L = \ell^c \))

\[
\inf_{x \in \mathbb{R}^d} \mathbb{P}\{ \Lambda_L(x) \text{ is } M \text{-localizing for } H_{\varepsilon,\omega} \} \geq 1 - e^{-L^c},
\] (4.19)

where

\[
M \geq m \left( 1 - C_{d,m_-} \varepsilon_0^{\min \{ \frac{1 + \varepsilon}{2 + \kappa \gamma (\gamma - 1) \zeta - 1} \} } \right).
\] (4.20)

Proof. We fix \( 0 < \varepsilon \leq \varepsilon_0 \) and \( m \geq m_- \) and assume (4.18) for some scale \( \ell \). We take \( \Lambda_L = \Lambda(x_0) \), where \( x_0 \in \mathbb{R}^d \), and let \( C_{L,\ell} = C_{L,\ell}(x_0) \) be the suitable \( \ell \)-cover of \( \Lambda_L \). Given \( N \in \mathbb{N} \), let \( B_N \) denote the event that there exist at most \( N \) disjoint
boxes in $C_{L,\ell}$ that are not $m$-localizing for $H_{\varepsilon,\omega}$. We have, using (3.11), (4.18), and the fact that events on disjoint boxes are independent, that
\[
\mathbb{P}\{B_N^c\} \leq \left(\frac{N}{2}\right)^{(N+1)d} e^{-(N+1)^d} = 2^{(N+1)d} \cdot 2^{-(N+1)^d} < \frac{1}{2} e^{-L\varepsilon}, \quad (4.21)
\]
if $N + 1 > \ell(\gamma-1)\xi$ and $\ell$ is sufficiently large. For this reason we take (recall (1.10))
\[
N = N_r = \left\lceil \ell(\gamma-1)\xi \right\rceil \implies \mathbb{P}\{B_N^c\} \leq \frac{1}{2} e^{-L\varepsilon} \text{ for all } \ell \text{ sufficiently large.} \quad (4.22)
\]

We now fix $\omega \in B_N$. There exist $A_N = A_N(\omega) \subset \Xi_{L,\ell} = \Xi_{L,\ell}(x_0)$, with $|A_N| \leq N$ and $\|a-b\| \geq 2\rho\ell$ (i.e., $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$) if $a, b \in A_N$ and $a \neq b$, such that for all $a \in \Xi_{L,\ell}$ with $\text{dist}(a, A_N) \geq 2\rho\ell$ (i.e., $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$ for all $b \in A_N$) the box $\Lambda_\ell(a)$ is $m$-localizing for $H_{\varepsilon,\omega}$. In other words,
\[
a \in \Xi_{L,\ell} \setminus \bigcup_{b \in A_N} \Lambda_{(2\rho+1)\ell}(b) \implies \Lambda_\ell(a) \text{ is } m\text{-localizing for } H_{\varepsilon,\omega}. \quad (4.23)
\]

We want to embed the boxes $\{\Lambda_\ell(b)\}_{b \in A_N}$ into buffered subsets of $\Lambda_L$. To do so, we consider graphs $G_i = (\Xi_{L,\ell}, E_i)$, $i = 1, 2$, both having $\Xi_{L,\ell}$ as the set of vertices, with sets of edges given by
\[
E_1 = \{a, b \in \Xi_{L,\ell}^2; \|a - b\| = \rho\ell\} = \{a, b \in \Xi_{L,\ell}^2; a \neq b \text{ and } \Lambda_\ell(a) \cap \Lambda_\ell(b) \neq \emptyset\},
\]
\[
E_2 = \{a, b \in \Xi_{L,\ell}^2; \text{ either } \|a - b\| = 2\rho\ell \text{ or } \|a - b\| = 3\rho\ell\} = \{a, b \in \Xi_{L,\ell}^2; \Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \text{ and } \Lambda_{(2\rho+1)\ell}(a) \cap \Lambda_{(2\rho+1)\ell}(b) \neq \emptyset\}.
\]

Given $\Psi \subset \Xi_{L,\ell}$, we define the exterior boundary of $\Psi$ in the graph $G_1$ by
\[
\partial_{\text{ext}}^G\Psi = \{a \in \Xi_{L,\ell}; \text{dist}(a, \Psi) = \rho\ell\}. \quad (4.25)
\]
(This is similar, but not the same as the definition in (3.11).) We let $\overline{\Psi} = \Psi \cup \partial_{\text{ext}}^G\Psi$. Let $\Phi \subset \Xi_{L,\ell}$ be $G_2$-connected, so diam $\Phi \leq 3\rho\ell (|\Phi| - 1)$. We set
\[
\overline{\Phi} = \Xi_{L,\ell} \cap \bigcup_{a \in \Phi} \Lambda_{2\rho+1}\ell(a) = \{a \in \Xi_{L,\ell}; \text{dist}(a, \Phi) \leq \rho\ell\}, \quad (4.26)
\]

Note that $\overline{\Phi}$ is a $G_2$-connected subset of $\Xi_{L,\ell}$ such that
\[
\text{diam } \overline{\Phi} \leq \text{diam } \Phi + 2\rho\ell \leq \rho\ell (3|\Phi| - 1), \quad (4.27)
\]
and let
\[
\Upsilon^{(0)}_\Phi = \bigcup_{a \in \Phi} \Lambda_\ell(a) \quad \text{and} \quad \Upsilon_\Phi = \Upsilon^{(0)}_\Phi \cup \bigcup_{a \in \partial_{\text{ext}}^G\Phi} \Lambda_\ell(a) = \bigcup_{a \in \overline{\Phi}} \Lambda_\ell(a). \quad (4.28)
\]

Now let $\{\Phi_r\}_{r=1}^R = \{\Phi_r(\omega)\}_{r=1}^R$ denote the $G_2$-connected components of $A_N$ (i.e., connected in the graph $G_2$). Note that
\[
R \in \{1, 2, \ldots, N\} \quad \text{and} \quad \sum_{r=1}^R |\Phi_r| = |A_N| \leq N. \quad (4.29)
\]

Note also that $\{\Phi_r\}_{r=1}^R$ is a collection of disjoint, $G_2$-connected subsets of $\Xi_{L,\ell}$, such that
\[
\text{dist}(\Phi_r, \Phi_s) \geq 2\rho\ell \quad \text{for} \quad r \neq s. \quad (4.30)
\]
It follows from (4.23) that
\[ a \in \mathcal{G} = \mathcal{G}(\omega) = \Xi_{L, \ell} \setminus \bigcup_{r=1}^{R} \tilde{\Phi}_r \quad \implies \quad \Lambda_\ell(a) \text{ is } m\text{-localizing for } H_{\varepsilon, \omega}. \hspace{1cm} (4.31) \]

In particular, we conclude that \( \Lambda_\ell(a) \) is \( m\)-localizing for \( H_{\varepsilon, \omega} \) for all \( a \in \partial_{\text{ex}}^2 \Phi_r \), \( r = 1, 2, \ldots, R \).

Each \( \Upsilon_r = \Upsilon_{\Phi_r}, \ r = 1, 2, \ldots, R \), clearly satisfies all the requirements to be a buffered subset of \( \Lambda_L \) with \( \mathcal{G}_{\Upsilon_r} = \partial_{\text{ex}}^2 \Phi_r \) (see Definition 3.7), except that we do not know if \( \Upsilon_r \) is \( L\)-level spacing for \( H_{\varepsilon, \omega} \). (Note that the sets \{\( \Upsilon_{\Phi_r}^{(0)} \}_{r=1}^R \) are disjoint, but the sets \{\( \Upsilon_{\Phi_r}^{(R)} \)\}_{r=1}^R are not necessarily disjoint.) Note also that it follows from (4.27) that
\[ \text{diam } \Upsilon_r \leq \text{diam } \Phi_r + \ell \leq \rho \ell (3 |\Phi_r| + 1) + \ell \leq 5 \ell |\Phi_r|, \hspace{1cm} (4.32) \]

so
\[ \sum_{r=1}^{R} \text{diam } \Upsilon_r \leq 5 \ell N_\ell \leq 5 \ell (\gamma - 1) \zeta + 1 \ll \ell \gamma = L^\gamma, \hspace{1cm} (4.33) \]

since \((\gamma - 1) \zeta > 1 < (\gamma - 1) \beta + 1 < \gamma \tau \) (see 169).

We can arrange for \{\( \Upsilon_{\Phi_r}^{(R)} \)\}_{r=1}^R to be a collection of buffered subsets of \( \Lambda_L \) as follows.

It follows from Lemma 2.1 that for any \( \Theta \subset \Lambda_L \) we have
\[ P \{ \Theta \text{ is } L\text{-level spacing for } H_{\varepsilon, \omega} \} \geq 1 - Y_{\varepsilon, \omega} e^{- (2 \alpha - 1) L^\beta \gamma} (L + 1)^{2d}. \hspace{1cm} (4.34) \]

Let
\[ \mathcal{F}_N = \bigcup_{r=1}^{N} \mathcal{F}(r), \text{ where } \mathcal{F}(r) = \{ \Phi \subset \Xi_{L, \ell}; \Phi \text{ is } G_2\text{-connected and } |\Phi| = r \}. \hspace{1cm} (4.35) \]

Setting \( \mathcal{F}(r, a) = \{ \Phi \in \mathcal{F}(r); a \in \Phi \} \) for \( a \subset \Xi_{L, \ell} \), and noting that each vertex in the graph \( G_2 \) has less than \( d \), \( 3d^{d-1} + 4d^{d-1} \leq d^d \) nearest neighbors, we get
\[ |\mathcal{F}(r, a)| \leq (r - 1)! (d^d)^{r-1} \quad \implies \quad |\mathcal{F}(r)| \leq (L + 1)^d (r - 1)! (d^d)^{r-1} \hspace{1cm} (4.36) \]

\[ |\mathcal{F}_N| \leq (L + 1)^d N! (d^d)^{N-1}. \]

Letting \( \mathcal{S}_N \) denote that the event that the box \( \Lambda_L \) and the subsets \{\( \Upsilon_{\Phi} \)\}_{\Phi \in \mathcal{F}_N} \) are all \( L\)-level spacing for \( H_{\varepsilon, \omega} \), and recalling the choice of \( N = N_\ell \) in (4.22), we get from (4.34) and (4.36) that
\[ P \{ \mathcal{S}_N \} \leq Y_{\varepsilon, \omega} \left( 1 + (L + 1)^d N_\ell! (d^d)^{N_\ell-1} \right) (L + 1)^{2d} e^{- (2 \alpha - 1) L^\beta} < \frac{1}{2} e^{- L^\zeta} \hspace{1cm} (4.37) \]

for sufficiently large \( L \), since \( (\gamma - 1) \zeta < (\gamma - 1) \beta < \gamma \beta \) and \( \zeta < \beta \).

We now define the event \( \mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N \). It follows from (4.21) and (4.37) that
\[ P \{ \mathcal{E}_N \} > 1 - e^{- L^\zeta}. \hspace{1cm} (4.38) \]

Note that for \( \omega \in \mathcal{E}_N \) the subsets \{\( \Upsilon_{\Phi}^{(R)} \)\}_{r=1}^R constructed above are buffered subsets. To finish the proof we need to show that for all \( \omega \in \mathcal{E}_N \) the box \( \Lambda_L \) is \( M\)-localizing for \( H_{\varepsilon, \omega} \), where \( M \) is given in (4.20).
Let us fix \( \omega \in \mathcal{E}_N \). Then we have (4.31), \( \Lambda_L \) is level spacing for \( H_{\tau, \omega} \), and the subsets \( \{ \mathcal{T}_{r, \ell} \}_{r=1}^R \) constructed in (4.28) are buffered subsets of \( \Lambda_L \) for \( H_{\tau, \omega} \). It follows from (4.119) and Definition 3.7(iii) that

\[
\Lambda_L = \left\{ \bigcup_{a \in \mathcal{G}} \Lambda^\alpha_{\ell_\tau}(a) \right\} \cup \left\{ \bigcup_{r=1}^R \mathcal{T}_{r, \ell}^\alpha \right\}. \tag{4.39}
\]

Since \( \varepsilon \) and \( \omega \) are now fixed, we omit them from the notation. Let \( \{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})} \) be an eigensystem for \( H_{\Lambda_L} \). Given \( a \in \mathcal{G} \), let \( \{ (\varphi_\ell^{(a)}, \lambda_\varepsilon^{(a)}) \}_{\ell \in \Lambda(a)} \) be an \( m \)-localized eigensystem for \( \Lambda_{\ell}(a) \). For \( r = 1, 2, \ldots, R \), let \( \{ (\phi_{\nu(r)}, \nu(r)) \}_{\nu(r) \in \sigma(H_{\mathcal{T}_r})} \) be an eigensystem for \( H_{\mathcal{T}_r} \), and set

\[
\sigma_{\mathcal{T}_r}(H_{\Lambda_L}) = \left\{ \tilde{\nu}(r) ; \nu(r) \in \sigma_{\mathcal{B}}(H_{\mathcal{T}_r}) \right\} \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}), \tag{4.40}
\]

where \( \tilde{\nu}(r) \) is given in (3.85), which also gives \( \sigma_{\mathcal{T}_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}) \), but the argument actually shows \( \sigma_{\mathcal{T}_r}(H_{\Lambda_L}) \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}) \). We also set

\[
\sigma_{\mathcal{B}}(H_{\Lambda_L}) = \bigcup_{r=1}^R \sigma_{\mathcal{T}_r} \left( H_{\Lambda_L} \right) \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}). \tag{4.41}
\]

We claim

\[
\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}). \tag{4.42}
\]

To see this, suppose we have \( \lambda \in \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}) \). Since \( \Lambda_L \) is level spacing for \( H \), it follows from Lemma 3.6(ii)(c) that

\[
|\psi_\lambda(y)| \leq e^{-m_5 \ell_\tau} \quad \text{for all} \quad y \in \bigcup_{a \in \mathcal{G}} \Lambda_{\ell_\tau, 2\ell_\tau}(a), \tag{4.43}
\]

and it follows from Lemma 4.15(ii) that

\[
|\psi_\lambda(y)| \leq e^{-m_5 \ell_\tau} \quad \text{for all} \quad y \in \bigcup_{r=1}^R \mathcal{T}_{r, 2\ell_\tau}. \tag{4.44}
\]

Using (4.39), we conclude that (note \( m_5 \leq m_2 \))

\[
1 = \| \psi_\lambda \| \leq e^{-m_5 \ell_\tau} (L + 1)^{\frac{R}{2}} < 1, \tag{4.45}
\]

a contradiction. This establishes the claim.

We will now index the eigenvalues and eigenvectors of \( H_{\Lambda_L} \) by sites in \( \Lambda_L \) using Hall’s Marriage Theorem, which states a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. (See Appendix C, based on [BuDM, Chapter 2].) We consider the bipartite graph \( \mathcal{G} = (\Lambda_L, \sigma(H_{\Lambda_L}); \mathcal{E}) \), where the edge set \( \mathcal{E} \subset \Lambda_L \times \sigma(H_{\Lambda_L}) \) is defined as follows. For each \( \lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \) we fix \( \lambda^{(a)} \in \mathcal{E} \) (\( \lambda \)), and set (recall definitions; we write \( \tilde{\mathcal{T}}_r = \tilde{\mathcal{T}}_r \) and \( \tilde{T}_{r, \tau} = (\tilde{T}_{r, \tau}, \tau) \))

\[
\mathcal{N}_0(x) = \begin{cases} 
\{ \lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}); \| x - x \| < \ell_\tau \} & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^R \tilde{T}_{r, \tau}, \\emptyset & \text{for } x \in \bigcup_{r=1}^R \tilde{T}_{r, \tau}.
\end{cases} \tag{4.46}
\]

We define

\[
\mathcal{N}(x) = \begin{cases} 
\mathcal{N}_0(x) & \text{for } x \in \Lambda_L \setminus \bigcup_{r=1}^R \tilde{T}_{r, \tau}, \\sigma_{\mathcal{T}_r}(H_{\Lambda_L}) & \text{for } x \in \tilde{T}_{r, \tau}, r = 1, 2, \ldots, R, \\mathcal{N}_0(x) \cup \sigma_{\mathcal{T}_r}(H_{\Lambda_L}) & \text{for } x \in \tilde{T}_{r, \tau}, r = 1, 2, \ldots, R.
\end{cases} \tag{4.47}
\]
and set \( E = \{(x, \lambda) \in \Lambda_L \times \sigma(H_{\Lambda_L}); \ \lambda \in \mathcal{N}(x)\} \)

\( \mathcal{N}(x) \) was defined to ensure \( |\psi_\lambda(x)| \ll 1 \) for \( \lambda \notin \mathcal{N}(x) \). This can be seen as follows:

- If \( x \in \Lambda_L \) and \( \lambda \in \sigma_G(H_{\Lambda_L})\setminus\mathcal{N}_0(x) \), we have \( \lambda = \lambda^{(a_\lambda)}_x \) with \( \|x_{\lambda} - x\| \geq \ell_\tau \), so
  \[
  |\psi_\lambda(x)| \leq \frac{\varphi^{(a_\lambda)}_{x,\lambda}}{\varphi^{(a_\lambda)}_{x,\lambda}} + |\varphi^{(a_\lambda)}_{x,\lambda} - \psi_\lambda| \leq e^{-m_\ell_\tau} + 2e^{-m_1\ell_\tau}e^{L_\beta} \leq 3e^{-m_1\ell_\tau}e^{L_\beta}, \tag{4.48}
  \]
  using (4.11) and (3.57).

- If \( x \in \Lambda_L \setminus \tilde{\mathcal{Y}}_{\tau,\tau} \) and \( \lambda \in \sigma_{\mathcal{Y}}(H_{\Lambda_L}) \), then \( \lambda = \tilde{\tau}(r) \) for some \( \mu^{(r)} \in \sigma_B(H_{\mathcal{Y}_r}) \), and
  \[
  |\psi_\lambda(x)| \leq |\phi_{\mu^{(r)}}(x)| + |\phi_{\mu^{(r)}} - \psi_\lambda| \leq e^{-m_2\ell_\tau} + 2e^{-m_1\ell_\tau}e^{L_\beta} \leq 3e^{-m_2\ell_\tau}e^{L_\beta}, \tag{4.49}
  \]
  using (4.83) and (3.87). (Note \( \phi_{\mu^{(r)}}(x) = 0 \) if \( x \notin \mathcal{Y}_r \).)

It follows that for all \( x \in \Lambda_L \) and \( \lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x) \) we have
  \[
  |\psi_\lambda(x)| \leq 3e^{-m_2\ell_\tau}e^{L_\beta} \leq e^{-\frac{1}{2}m_2\ell_\tau}. \tag{4.50}
  \]

Since \( |\Lambda_L| = |\sigma(H_{\Lambda_L})| \), to apply Hall’s Marriage Theorem we only need to verify Hall’s condition \( (C.1) \). Let \( \mathcal{N}(\Theta) = \bigcup_{\Theta \in \Theta} \mathcal{N}(x) \) for \( \Theta \subset \Lambda_L \). We fix \( \Theta \subset \Lambda_L \), and let \( Q_{\Theta} \) be the orthogonal projection onto the span of \( \{\psi_\lambda; \lambda \in \mathcal{N}(\Theta)\} \). For every \( \lambda \notin \mathcal{N}(\Theta) \) we have (4.50) for all \( x \in \Theta \), so
  \[
  \|(1 - Q_{\Theta})\chi_{\Theta}\| \leq |\Lambda_L|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} e^{-\frac{1}{2}m_2\ell_\tau} \leq (L + 1)^{d_\ell}e^{-\frac{1}{2}m_2\ell_\tau} < 1, \tag{4.51}
  \]
so it follows from Lemma \( \text{A.1} \) that
  \[
  |\Theta| = \text{tr} \chi_{\Theta} \leq \text{tr} Q_{\Theta} = |\mathcal{N}(\Theta)|, \tag{4.52}
  \]
which is Hall’s condition \( (C.1) \).

Thus we can apply Hall’s Marriage Theorem, concluding that there exists a bijection
  \[
  x \in \Lambda_L \mapsto \lambda_x \in \sigma(H_{\Lambda_L}), \text{ where } \lambda_x \in \mathcal{N}(x). \tag{4.53}
  \]
We set \( \psi_x = \psi_{\lambda_x} \) for all \( x \in \Lambda_L \).

To finish the proof we need to show that \( \{(\psi_x, \lambda_x)\}_{x \in \Lambda_L} \) is an \( M \)-localized eigen-system for \( \Lambda_L \), where \( M \) is given in (4.20). We fix \( x \in \Lambda_L \), take \( y \in \Lambda_L \), and consider several cases:

(i) Suppose \( \lambda_x \in \sigma_G(\Lambda_L) \). In this case \( x \in \Lambda_{\ell}(a_{\lambda_x}) \) with \( a_{\lambda_x} \in G \), and \( \lambda_x \in \sigma(a_{\lambda_x})(H_{\Lambda_L}) \). In view of (4.30) we consider two cases:

- (a) If \( y \in \Lambda_{\ell}(a_{\lambda_x}) \) for some \( a \in G \) and \( \|y - x\| \geq \ell_\tau \), we must have \( \Lambda_{\ell}(a_{\lambda_x}) \cap \Lambda_{\ell}(a) = \emptyset \), so it follows from (3.65) that \( \lambda_x \notin \sigma(a)(H_{\Lambda_L}) \), and
  \[
  |\psi_x(y)| \leq e^{-m_3\|y_1 - y\|} \|\psi_x(y_1)\| \text{ for some } y_1 \in \partial^{\Lambda_{\ell},\ell_\tau} \Lambda_{\ell}(a). \tag{4.54}
  \]

- (b) If \( y \in \mathcal{Y}_{\tau,r} \) for some \( r \in \{1, 2, \ldots, R\} \), and \( \|y - x\| \geq \ell + \text{diam } \mathcal{Y}_r \), we must have \( \Lambda_{\ell}(a_{\lambda_x}) \cap \mathcal{Y}_r = \emptyset \). It follows from (3.65) that \( \lambda_x \notin \sigma_{\mathcal{Y}_r}(H_{\Lambda_L}) \), and clearly \( \lambda_x \notin \sigma_{\mathcal{Y}_r}(H_{\Lambda_L}) \) in view of (4.40). Thus Lemma \( \text{4.3} \) (ii) gives
  \[
  |\psi_x(y)| \leq e^{-m_3\ell} |\psi_x(v)| \text{ for some } v \in \partial^{\Lambda_{\ell},2\ell_\tau} \mathcal{Y}_r. \tag{4.55}
  \]
(ii) Suppose $\lambda_x \notin \sigma(G(L))$. Then it follows from (1.12) that we must have $\lambda_x \in \sigma(F(H_{L}))$ for some $s \in \{1, 2, \ldots, R\}$. In view of (4.10) we consider two possibilities:

(a) If $y \in \Lambda^A_{L} \cap \mathcal{F}(a)$ for some $a \in \mathcal{G}$, and $\|y - x\| \geq \ell + \text{diam} \mathcal{Y}_s$, we must have $\Lambda_0(a) \cap \mathcal{Y}_s = \emptyset$, and Lemma 3.9(i) yields (4.55).

(b) If $y \in \mathcal{Y}_r^A \cap \mathcal{F}$ for some $r \in \{1, 2, \ldots, R\}$, and $\|y - x\| \geq \text{diam} \mathcal{Y}_s + \text{diam} \mathcal{Y}_r$, we must have $r \neq s$. Thus Lemma 3.9(ii) yields (4.55).

Now let us fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L_r$. Suppose $|\psi_x(y)| > 0$, since otherwise there is nothing to prove. We estimate $|\psi_x(y)|$ using either (4.54) or (4.55) repeatedly, as appropriate, stopping when we get too close to $x$ so we are not in one the cases described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when we use (4.54), and just use $e^{-m_\varepsilon \ell^c} < 1$ when using (4.55), getting

$$
|\psi_x(y)| \leq e^{-m_\varepsilon \|y - x\| - \sum_{j=1}^r \text{diam} \mathcal{Y}_r - 2\ell} \leq e^{-m_\varepsilon \|y - x\| - 5\ell(\gamma - 1)\tilde{\gamma} + 1 - 2\ell} 
$$

(4.56)

where we used (4.39) and took

$$
M = m_3 \left(1 - 7\ell(\gamma - 1)\tilde{\gamma} + 1 - \gamma \ell\right) \geq m \left(1 - C_{d,m_\varepsilon,\varepsilon_0} \ell^{\frac{1 - \gamma}{2}} \right) \left(1 - 7\ell(\gamma - 1)\tilde{\gamma} + 1 - \gamma \ell\right)
$$

(4.57)

where we used (4.52).

We conclude that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an $M$-localized eigensystem for $\Lambda_L$, where $M$ is given in (1.20), so the box is $\Lambda_L$ is $M$-localizing for $H_{\varepsilon, \omega}$. \hfill $\square$

**Proof of Proposition 4.3.** We assume (4.3) and set $L_{k+1} = L_k^\gamma$ for $k = 0, 1, \ldots$. If $L_0$ is sufficiently large it follows from Lemma 3.5 by an induction argument that

$$
\inf_{x \in \mathbb{R}^d} P \{\Lambda_L(x) \text{ is } m_k\text{-localizing for } H_{\varepsilon, \omega} \} \geq 1 - e^{-L_k^\gamma} \text{ for } k = 0, 1, \ldots, (4.58)
$$

where for $k = 1, 2, \ldots$ we have

$$
m_k \geq m_{k-1} \left(1 - C_{d,m_\varepsilon,\varepsilon_0} L_k^{\gamma - 1} \right), \text{ with } \varepsilon = \min \left\{\frac{1 - \gamma}{2}, \gamma - 1, 1\right\}. \tag{4.59}
$$

Thus for all $k = 1, 2, \ldots$, taking $L_0$ sufficiently large we get

$$
m_k \geq m_0 \prod_{j=0}^{k-1} \left(1 - C_{d,m_\varepsilon,\varepsilon_0} L_0^{-\varepsilon \gamma^j} \right) \geq m_0 \prod_{j=0}^{\infty} \left(1 - C_{d,m_\varepsilon,\varepsilon_0} L_0^{-\varepsilon \gamma^j} \right) \geq \frac{m_0}{2}, \tag{4.60}
$$

finishing the proof of Proposition 4.3. \hfill $\square$

**4.3. Removing the restriction on scales.** We will now show how Theorem 1.6 follows from Proposition 4.3.

**Proof of Theorem 1.6.** Assume the conclusions of Proposition 4.3, that is, for $L_0 \geq L$ and $\varepsilon \leq \varepsilon_0 = \frac{\varepsilon_0 - \varepsilon_0^2}{2}$, setting $L_{k+1} = L_k^\gamma$ for $k = 0, 1, \ldots$, we have (4.1).

Given a scale $L \geq L_1$, let $k = k(L) \in \{1, 2, \ldots\}$ be defined by $L_k \leq L < L_{k+1}$, and set $\ell = L_{k-1}$. We have $L_k = \ell^\gamma \leq L < L_{k+1} = \ell^\gamma$, so $L = \ell^\gamma$ with $\gamma \leq \gamma' < \gamma^2$. We proceed as in Lemma 4.3. We take $\Lambda_L = \Lambda_L(x_0)$, where $x_0 \in \mathbb{R}^d$, taking

$$
\Lambda_0(a) \cap \mathcal{Y}_s = \emptyset, \text{ and Lemma 3.9(i) yields (4.55)}.
$$

(4.55)
and let \( C_{L,\tilde{L}} = C_{L,\tilde{L}}(x_0) \) be the suitable \( \tilde{L} \)-cover of \( \Lambda_L \). We let \( B_0 \) denote the event that all boxes in \( C_{L,\tilde{L}} \) are \( \frac{m_- L_0}{2} \)-localizing for \( H_{\varepsilon,\omega} \). It follows from (4.11) that
\[
\mathbb{P}\{B_0^c\} \leq \left( \frac{2\tilde{L}}{\ell} \right)^d e^{-\tilde{L}} = 2^d e^{(\gamma' - 1)\ell} e^{-\tilde{L}} \leq 2^d \left( \frac{1}{\gamma'} - 1 \right)^d e^{-L} < \frac{1}{2} e^{-L},
\]
if \( L_0 \) is sufficiently large, since \( \varepsilon \gamma' < \gamma \). Moreover, letting \( S_0 \) denote the event that the box \( \Lambda_L \) is level spacing for \( H_{\varepsilon,\omega} \), it follows from Lemma 2.1 that
\[
\mathbb{P}\{S_0^c\} \leq Y_0 e^{-(2\alpha - 1)L_0^\beta} (L + 1)^{2d} \leq \frac{1}{2} e^{-L},
\]
if \( L_0 \) is sufficiently large, since \( \xi < \beta \). Thus, letting \( \mathcal{E}_0 = B_0 \cap S_0 \), we have
\[
\mathbb{P}\{\mathcal{E}_0\} \geq 1 - e^{-L}.
\]

It only remains to prove that \( \Lambda_L \) is \( \frac{m_- L_0}{4} \)-localizing for \( H_{\varepsilon,\omega} \) for all \( \omega \in \mathcal{E}_0 \). To do so, we fix \( \omega \in \mathcal{E}_0 \) and proceed as in the proof of Lemma 4.5. Since \( \omega \in B_0 \), we have \( \mathcal{G} = \mathcal{G}(\omega) = \Xi_{L,\tilde{L}} \). Since \( \varepsilon \) and \( \omega \) are now fixed, we omit them from the notation. The proof of Lemma 4.3 applies, we get \( \sigma(H_{\Lambda_L}) = \sigma(\mathcal{G}) \) as in (4.42), and obtain an eigensystem \( \{(\psi_x, \lambda_x)\}_{x \in \Lambda_L} \) for \( H_{\Lambda_L} \) using Hall’s Marriage Theorem. To finish the proof we need to show that \( \{(\psi_x, \lambda_x)\}_{x \in \Lambda_L} \) is an \( \frac{m_- L_0}{4} \)-localized eigensystem for \( \Lambda_L \). Given \( x, y \in \Lambda_L \) with \( |y - x| \geq 2\tilde{L} \), we have \( y \in \Lambda_L^{\varepsilon,\tilde{L}} \), \( \mathcal{E}(a) \) for some \( a \in \Xi_{L,\tilde{L}} \) and (4.50) holds with \( \gamma_1 \in \partial^{\varepsilon,\tilde{L}} \epsilon \). If \( |y - x| \geq L_0 \), we proceed as in (4.50), stopping when we get within \( 2\tilde{L} \) of \( x \), obtaining
\[
|\psi_x(y)| \leq e^{-\tilde{m}_3(2|y - x| - 2\tilde{L})} \leq e^{-\tilde{m}_3(1 - 3^{\frac{1}{2}} - \gamma')} \leq e^{-\tilde{M} |y - x|},
\]
where (recall (3.32)) \( \tilde{m}_3 \geq \frac{m_- L_0}{2} \left( 1 - C_{d,m_-} \ell_0 \right)^{\tilde{L} / \ell_0} \), and
\[
\tilde{M} = \tilde{m}_3 \left( 1 - 3^{\frac{1}{2}} - \gamma' \right) \geq \frac{m_- L_0}{2} \left( 1 - C_{d,m_-} \ell_0 \right) \min \left\{ \frac{1}{\tilde{L}^\beta}, \gamma' \right\} \left( \frac{1}{\tilde{L}^\beta} \right) \geq \frac{m_- L_0}{4}
\]
for \( L_0 \) large.

\[\square\]

5. DERIVING LOCALIZATION

In this section we consider an Anderson model \( H_{\varepsilon,\omega} \) and derive localization results from Theorem 1.6. We start by proving Theorem 1.7 using the following lemma.

Lemma 5.1. Fix \( \varepsilon_0 > 0 \) and \( m_- > 0 \). There exists a finite scale \( \mathcal{L}_{\varepsilon_0, m_-} \) such that for all \( \ell \geq \mathcal{L}_{\varepsilon_0, m_-}, \ a \in \mathbb{Z}^d, \ \lambda \in \mathbb{R}, \ \varepsilon \leq \varepsilon_0, \) and \( m \geq m_- \), given an \( m \)-localizing box \( \Lambda_\ell(a) \) for the discrete Schrödinger operator \( H_\varepsilon \) with an \( m \)-localized eigensystem \( \{\varphi_x, \lambda_x\}_{x \in \Lambda_\ell(a)} \), we have
\[
\max_{b \in \Lambda_\ell(a)} W_{\varepsilon,\lambda}(b) > e^{-\frac{1}{4}m \ell} \implies \min_{x \in \Lambda_\ell^\varepsilon(a)} |\lambda - \lambda_x| < \frac{1}{2} e^{-L^\beta}. \tag{5.1}
\]

Proof. Suppose \( |\lambda - \lambda_u| \geq \frac{1}{2} e^{-L^\beta} \) for all \( u \in \Lambda_\ell^\varepsilon(a) \). Let \( \psi \in \mathcal{V}_\varepsilon(\lambda) \). Then it follows from Lemma 3.5(ii) that for large \( \ell \) and \( b \in \Lambda_\ell(a) \) we have
\[
|\psi(b)| \leq e^{-m_3(\frac{1}{4} - \ell \varepsilon_0)} \|T_{\varepsilon,1}\psi\| \left( \frac{1}{\varepsilon_0} + 1 \right) ^\nu \leq e^{-\frac{1}{4}m \ell} \|T_{\varepsilon,1}\psi\|. \tag{5.2}
\]
Proof of Theorem 1.8. Suppose Theorem 1.0 holds for some $L_0$, and let $\varepsilon_0 = \frac{1}{\log^3 3} e^{-L_0^2}$ and $m_\varepsilon = \frac{m_\varepsilon L_0^2}{4} \geq \frac{\log^3 3}{4}$. Consider $L_0^2 \leq \ell \in 2N$ and $a \in \mathbb{Z}^d$. We have

$$\Lambda_{5\ell}(a) = \bigcup_{b \in \{a + \frac{\ell}{2} \mathbb{Z}^d\}, \|b-a\| \leq 2\ell} \Lambda_\ell(b). \quad (5.3)$$

Let $\mathcal{Y}_{\varepsilon,\ell,a}$ denote the event that $\Lambda_{5\ell}(a)$ is level spacing for $H_{\varepsilon,\omega}$ and the boxes $\Lambda_\ell(b)$ are $m_\varepsilon$-localizing for $H_{\varepsilon,\omega}$ for all $b \in \{a + \frac{\ell}{2} \mathbb{Z}^d\}$ with $\|b-a\| \leq 2\ell$. It follows from (1.13) and Lemma 2.1 that

$$\mathbb{P} \{ \mathcal{Y}_{\varepsilon,\ell,a} \} \leq 5^d e^{-\ell^\beta} + Y_{a_0} (5\ell + 1)^{2d} e^{-(2\alpha - 1)(5\ell)^\gamma} \leq C_0 e^{-\ell^\beta}. \quad (5.4)$$

Suppose $\omega \in \mathcal{Y}_{\varepsilon,\ell,a}$, $\lambda \in \mathbb{R}$, and $\max_{b \in \Lambda_\ell} W^{(a)}_{\varepsilon,\omega}(b) > e^{-\frac{1}{4} m_\varepsilon \ell}$. It follows from Lemma 6.1 that $\min_{x \in \Lambda_\ell} \left| \lambda - \lambda_x^{(b)} \right| < \frac{1}{2} e^{-\ell^\beta}$. Since $\Lambda_{5\ell}(a)$ is level spacing for $H_{\varepsilon,\omega}$, using Lemma 3.6 we conclude that

$$\min_{x \in \Lambda_{5\ell}^\dagger(b)} \left| \lambda - \lambda_x^{(b)} \right| \geq e^{-\ell^\beta} - 2e^{-m_\varepsilon \ell} - \frac{1}{2} e^{-\ell^\beta} \geq \frac{1}{2} e^{-\ell^\beta} \quad (5.5)$$

for all $b \in \{a + \frac{\ell}{2} \mathbb{Z}^d\}$ with $\ell \leq \|b-a\| \leq 2\ell$. Since

$$A_\ell(a) \subset \bigcup_{b \in \{a + \frac{\ell}{2} \mathbb{Z}^d\}, \ell \leq \|b-a\| \leq 2\ell} \Lambda_{5\ell}^\dagger(b), \quad (5.6)$$

it follows from Lemma 3.5(ii) that for all $y \in A_\ell(a)$ we have, given $\psi \in \mathcal{Y}_{\varepsilon,\omega}(\lambda)$,

$$|\psi(y)| \leq e^{-(m_\varepsilon)\ell} \|T^{-1}_a \psi\| \langle \frac{5\ell}{2} + 1 \rangle^\nu \leq e^{-m_\varepsilon \ell} \|T^{-1}_a \psi\| \leq e^{-\frac{\ell}{2} m_\varepsilon \|y-a\|} \|T^{-1}_a \psi\|, \quad (5.7)$$

so we get

$$W_{\varepsilon,\omega,\lambda}^{(a)}(y) \leq e^{-\frac{\ell}{2} m_\varepsilon \|y-a\|} \quad \text{for all } y \in A_\ell(a). \quad (5.8)$$

Since we have (1.17), we conclude that for $\omega \in \mathcal{Y}_{\varepsilon,\ell,a}$ we always have

$$W_{\varepsilon,\omega,\lambda}^{(a)}(y) \leq \max \left\{ e^{-\frac{\ell}{2} m_\varepsilon \|y-a\|} (y-a)^\nu, e^{-\frac{\ell}{2} m_\varepsilon \|y-a\|} \right\} \quad (5.9)$$

$$\leq e^{-\frac{\ell}{2} m_\varepsilon \|y-a\|} \quad \text{for all } y \in A_\ell(a). \quad \square$$

We now turn to Corollary 1.8.

Proof of Corollary 1.8 Parts (i) and (ii) are proven in the same way as [GK4, Theorem 7.1(i)-(ii)]. Note that we have $\max_{b \in \Lambda_{5\ell}^\dagger} W_{\varepsilon,\omega,\lambda}^{(a)}(b)$ in (1.13) instead of simply $W_{\varepsilon,\omega,\lambda}^{(a)}(a)$ because we do not have the unique continuation principle in the lattice. If $\lambda$ is a generalized eigenvalue for $H_{\varepsilon,\omega}$ we could have $W_{\varepsilon,\omega,\lambda}^{(a)}(a) = 0$, but we will always have $\max_{b \in \Lambda_{5\ell}^\dagger} W_{\varepsilon,\omega,\lambda}^{(a)}(b) > 0$ for all large $\ell$. 

Part (iii) is proven similarly to [GK4 Theorem 7.2(i)]. There are some small differences, so we give the proof here. We use the fact that for any ξ0 ∈ 2N, setting \( \ell_{k+1} = 2\ell_k \) for \( k = 0, 1, 2, \ldots \), we have (recall (1.20))
\[
\mathbb{Z}^d = A_{3\ell_k}(a) \cup \bigcup_{j=k}^{\infty} A_{\ell_j}(a) \quad \text{for} \quad k = 0, 1, 2, \ldots
\]  
(5.10)
We fix \( \varepsilon \leq \varepsilon_0 \). Given \( k \in \mathbb{N} \), we set \( \ell_k = 2^k \), and consider the event
\[
\mathcal{Y}_{\varepsilon,k} := \bigcap_{x \in \mathbb{Z}^d, \|x\| \leq e^{\frac{1}{12}\varepsilon^2}} \mathcal{Y}_{\varepsilon,L_k,x},
\]  
(5.11)
where \( \mathcal{Y}_{\varepsilon,L_k,x} \) is the event given in Theorem 1.7. It follows from (1.18) that for \( \{Y_{\varepsilon,k}\} \) we have
\[
\mathbb{P}\{Y_{\varepsilon,k}\} \geq 1 - C_{e_0} \left( 2e^{\frac{1}{12}\varepsilon^2} \right)^d e^{-\varepsilon L_k^2} \geq 1 - 3^d C_{e_0} e^{-\frac{1}{12}\varepsilon^2 L_k^2},
\]  
(5.12)
so we conclude from the Borel-Cantelli Lemma that
\[
\mathbb{P}\{Y_{\varepsilon,\infty}\} = 1, \quad \text{where} \quad \mathcal{Y}_{\varepsilon,\infty} = \liminf_{k \to \infty} \mathcal{Y}_{\varepsilon,k}.
\]  
(5.13)
We now fix \( \omega \in \mathcal{Y}_{\varepsilon,\infty} \) so there exists \( k_{\varepsilon,\omega} \in \mathbb{N} \) such that \( \omega \in \mathcal{Y}_{\varepsilon,L_k,x} \) for all \( k_{\varepsilon,\omega} \leq k \in \mathbb{N} \) and \( x \in \mathbb{Z}^d \) with \( \|x\| \leq e^{\frac{1}{12}\varepsilon^2} \). Given \( x \in \mathbb{Z}^d \), we define \( k_x \in \mathbb{N} \) by
\[
e^{\frac{1}{12}\varepsilon^2 k_{x-1}} < \|x\| \leq e^{\frac{1}{12}\varepsilon^2 k_{x}}, \quad \text{if} \quad k_x \geq 2,
\]  
(5.14)
and set \( k_x = 1 \) otherwise. We set \( \varepsilon_{\omega,k} = \max\{k_{\varepsilon,\omega}, k_x\} \), where \( k_{\varepsilon,\omega} = \max\{k_{\varepsilon,\omega}, 2\} \).

Let \( x \in \mathbb{Z}^d \). If \( y \in B_{\varepsilon,\omega,x} = \bigcup_{k=k_{\varepsilon,\omega}}^{\infty} A_{L_k}(x) \), we have \( y \in A_{L_{k_1}}(x) \) for some \( k_1 \geq k_{\varepsilon,\omega} \) and \( \omega \in \mathcal{Y}_{\varepsilon,L_{k_1},x} \), so it follows from (1.21) that
\[
W_{\varepsilon,\omega}(x)W_{\varepsilon,\omega}(y) \leq e^{-\frac{7}{12}\varepsilon^2 m_x \|y-x\|} \quad \text{for all} \quad \lambda \in \mathbb{R}.
\]  
(5.15)
If \( y \notin B_{\varepsilon,\omega,x} \), we must have \( \|y-x\| < \frac{7}{12}\varepsilon L_{k_1,\omega,x} \), so for all \( \lambda \in \mathbb{R} \), using (1.17) and (6.14),
\[
W_{\varepsilon,\omega}(x)W_{\varepsilon,\omega}(y) \leq e^{-\frac{7}{12}\varepsilon^2 m_x \|y-x\|} e^{-\frac{7}{12}\varepsilon^2 m_{\lambda \omega} \|y-x\|}
\]  
(5.16)
Combining (5.15) and (5.16), noting \( \|x\|^{2d} \geq e \) if \( k_x \geq 2 \), we conclude that for all \( \lambda \in \mathbb{R} \) and \( x, y \in \mathbb{Z}^d \) we have
\[
W_{\varepsilon,\omega}(x)W_{\varepsilon,\omega}(y)
\]  
(5.17)
which is \((1.23)\).

Part (iv) follows from (iii), since \((1.23)\) implies
\[
|\psi(x)| \leq C_{\varepsilon,m,\omega,\nu} \left\| T_0^{-1}\psi \right\|^2 e^{\left(\frac{1}{2} + \nu\right)m_\varepsilon(2d \log(x))} e^{-\frac{1}{2}m_\varepsilon\|y-x\|}
\]
for all \(x, y \in \mathbb{Z}^d\), which is \((1.23)\).

Part (v) also follows from (iii). Given \(\lambda \in \mathbb{R}\), let \(\psi \in \chi(\lambda)(H_{\varepsilon,\omega})\setminus\{0\}\). Clearly there exists \(x_\lambda \in \mathbb{Z}^d\) (not unique) such that
\[
|\psi(x_\lambda)| = \max_{x \in \mathbb{Z}^d} |\psi(x)|.
\]
Since for all \(a \in \mathbb{Z}^d\) we have
\[
\left\| T_0^{-1}\psi \right\|^2 = \sum_{x \in \mathbb{Z}^d} |\psi(x)|^2 \langle x-a \rangle^{-2\nu} \leq |\psi(x_\lambda)|^2 \sum_{x \in \mathbb{Z}^d} \langle x-a \rangle^{-2\nu}
\]
\[
= |\psi(x_\lambda)|^2 \sum_{x \in \mathbb{Z}^d} \langle x \rangle^{-2\nu} = C_{d,\nu}^2 |\psi(x_\lambda)|^2,
\]
where \(C_{d,\nu} = \left(\sum_{x \in \mathbb{Z}^d} \langle x \rangle^{-2\nu}\right)^{\frac{1}{2}} \in (1, \infty)\), we get the discrete equivalent of \((5.19)\),
\[
\left\| T_0^{-1}\psi \right\| \leq C_{d,\nu} |\psi(x_\lambda)| \quad \text{for all } a \in \mathbb{Z}^d,
\]
and hence, recalling \((1.16)\), we have
\[
W_{\varepsilon,\omega,\lambda}(x_\lambda) \geq \frac{|\psi(x_\lambda)|}{\left\| T_0^{-1}\psi \right\|} \geq C_{d,\nu} > 1.
\]
Thus \((1.23)\) implies that for all \(y \in \mathbb{Z}^d\) we have
\[
C_{d,\nu}W_{\varepsilon,\omega,\lambda}(y) \leq C_{\varepsilon,m,\omega,\nu} e^{\left(\frac{1}{2} + \nu\right)m_\varepsilon(2d \log(x_\lambda))} e^{-\frac{1}{2}m_\varepsilon\|y-x_\lambda\|},
\]
which yields \((1.25)\).

6. Connection with the Green’s functions multiscale analysis

Consider an Anderson model \(H_{\varepsilon,\omega}\) as in Definition 1.1. Given \(\Theta \subset \mathbb{Z}^d\) finite and \(z \notin \sigma(H_\Theta)\), we set
\[
G_\Theta(z) = (H_\Theta - z)^{-1} \quad \text{and} \quad G_\Theta(z; x, y) = \langle \delta_x, (H_\Theta - z)^{-1}\delta_y \rangle \quad \text{for } x, y \in \Theta.
\]

**Definition 6.1.** Let \(E \in \mathbb{R}\) and \(m > 0\). A box \(\Lambda_L\) is said to be \((m, E)\)-regular if \(E \notin \sigma(H_{\Lambda_L})\) and
\[
|G_{\Lambda_L}(E; x, y)| \leq e^{-m\|x-y\|} \quad \text{for all } x, y \in \Lambda_L \quad \text{with} \quad \|x-y\| \geq \frac{L}{100}.
\]
Given \(x, y \in \mathbb{R}^d\), a scale \(L\), and \(m > 0\), we define the event
\[
\mathcal{R}_{L,m}(x, y) = \{\text{for all } E \in \mathbb{R} \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (m, E)\)-regular\}.
\]
The Green’s function multiscale analysis \([\text{ProS}], [\text{ProMSS}], [\text{DrK}], [\text{GK1}], [\text{K}]\) yields the following theorem.
Theorem 6.2. Given $0 < \zeta < 1$, there exists $\varepsilon_0 > 0$, a finite scale $L$, and $m > 0$, such that, given $L \geq L$, for all $0 < \varepsilon \leq \varepsilon_0$ we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } (m, E)\text{-regular} \} \geq 1 - e^{-L^\zeta} \quad \text{for all } E \in \mathbb{R},
\] (6.4)
and
\[
\inf_{x, y \in \mathbb{R}^d} \mathbb{P} \{ \mathcal{R}_{L,m}(x, y) \} \geq 1 - e^{-L^\zeta}.
\] (6.5)

(6.4) are the conclusions of the single energy multiscale analysis, and (6.5) are the conclusions of the energy interval multiscale analysis.

We will now show the connection between Theorem 1.6 and Theorem 6.2. We assume $\xi, \zeta, \beta, \tau, \gamma$ satisfy (1.8).

We first show that the conclusions of Theorem 1.6 imply the conclusions of Theorem 6.2.

Proposition 6.3. Let $\varepsilon_0 > 0$. Fix $0 < \varepsilon \leq \varepsilon_0$ and suppose there exists $0 < \xi < 1$, a finite scale $L$, and $m > 0$, such that the Anderson model $H_{\varepsilon, \omega}$ satisfies
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } m\text{-localizing for } H_{\varepsilon, \omega} \} \geq 1 - e^{-L^\xi} \quad \text{for all } L \geq L.
\] (6.6)

Then, given $0 < \zeta' < \xi$ and $0 < m' < m$, there exists a finite scale $L_1 = L_1(L, \varepsilon_0, \xi, \zeta', m, m')$ such that for all $L \geq L_1$ we have
\[
\inf_{x \in \mathbb{R}^d} \mathbb{P} \{ \Lambda_L(x) \text{ is } (m', E)\text{-regular} \} \geq 1 - e^{-L^\zeta'} \quad \text{for all } E \in \mathbb{R},
\] (6.7)
and
\[
\inf_{x, y \in \mathbb{R}^d} \mathbb{P} \{ \mathcal{R}_{L,m'}(x, y) \} \geq 1 - e^{-L^\zeta'}.
\] (6.8)

Proof. Fix $0 < \varepsilon \leq \varepsilon_0$, $0 < \zeta' < \xi$, and $0 < m' < m$, and assume (6.6) for all $L \geq L_1$. Let $L \geq L$ and suppose the box $\Lambda_L$ is $m$-localizing with an $m$-localized eigensystem $\{ \varphi_x, \lambda_x \}_{x \in \Lambda_L}$. Let $u, v \in \Lambda_L$ with $\|u - v\| \geq \frac{2}{100}$. Then in this case either $\|u - x\| \geq L^\tau$ or $\|v - x\| \geq L^\tau$. Say $\|u - x\| \geq L^\tau$, then
\[
|\varphi_x(u)\varphi_x(v)| \leq \begin{cases} e^{-m(\|u-x\|+\|v-x\|)} & \text{if } \|u-x\| \geq L^\tau, \\ e^{-m\|u-x\|} & \text{if } \|v-x\| \leq L^\tau, \\ e^{-m\|u-v\|} & \text{if } \|v-x\| \leq L^\tau. \end{cases}
\] (6.9)

so we conclude that
\[
|\varphi_x(u)\varphi_x(v)| \leq e^{-m_1\|u-v\|}, \quad \text{where } m_1 \geq m(1 - C L^{\tau - 1}).
\] (6.10)

Fix an energy $E \in \mathbb{R}$ and assume $\|G_{\Lambda_L}(E)\| \leq e^{L^\beta}$. Then for $u, v \in \Lambda_L$ with $\|u - v\| \geq \frac{1}{100}$ we have
\[
|G_{\Lambda_L}(E; u, v)| \leq \sum_{x \in \Lambda_L} |E - \lambda_x|^{-1} |\varphi_x(u)\varphi_x(v)| \leq e^{L^\beta} e^{-m_1\|u-v\|}(L + 1)^d
\] (6.11)
\[
\leq e^{-m_2\|u-v\|},
\]
where
\[
m_2 \geq m_1 \left( 1 - C \left( 1 + \frac{1}{m_1} \right) L^{\beta - 1} \right) \geq m \left( 1 - C \left( 1 + \frac{1}{m} \right) L^{\tau - 1} \right) \geq m'
\] (6.12)
for large $L$. We conclude that the box $\Lambda_L$ is $(m', E)$-regular.
It follows from (6.6) that

\[ L \geq L \]

(6.17)

for large \(L\), combining with (6.9) we get (6.7).

Now let \( L \geq L \) and consider two boxes \( \Lambda_L(x_1) \) and \( \Lambda_L(x_2) \), where \( x_1, x_2 \in \mathbb{R}^d \), \( \|x_1 - x_2\| > L \). Define the events

\[ \mathcal{A} = \{\Lambda(x_1) \text{ and } \Lambda(x_2) \text{ are both } m\text{-localizing}\} \]
\[ \mathcal{B} = \{\text{dist}(\sigma(H_{\Lambda_L(x_1)}), \sigma(H_{\Lambda_L(x_2)})) \geq 2e^{-L^\alpha}\} \]

(6.14)

It follows from (6.6) that

\[ \mathbb{P}\{\mathcal{A}\} \geq 1 - 2e^{-L^\xi} \geq 1 - \frac{1}{2} e^{-L^{\xi}}. \]  
(6.15)

Since \( \|x_1 - x_2\| > L \), the boxes are disjoint, and the Wegner estimate between boxes (see [K, Corollary 5.28]) gives

\[ \mathbb{P}\{\mathcal{B}\} \geq 1 - Q_\mu(4e^{-L^\beta}) |\Lambda_L(x_1)||\Lambda_L(x_2)| \geq 1 - \tilde{K} 4^\alpha e^{-\alpha L^d} (L + 1)^d \geq 1 - \frac{1}{2} e^{-L^{\xi'}}. \]  
(6.16)

Thus we have

\[ \mathbb{P}\{\mathcal{A} \cap \mathcal{B}\} \geq 1 - e^{-L^{\xi'}}. \]  
(6.17)

Moreover, for \( \omega \in \mathcal{A} \cap \mathcal{B} \) and \( E \in \mathbb{R} \), the boxes \( \Lambda(x_1) \) and \( \Lambda(x_2) \) are both \( m\)-localizing, and we must have either \( \|G_{\Lambda_L(x_1)}(E)\| \leq e^{L^\delta} \) or \( \|G_{\Lambda_L(x_2)}(E)\| \leq e^{L^\delta} \), so the previous argument shows that either \( \Lambda(x_1) \) or \( \Lambda(x_2) \) is \( (m', E)\)-regular for large \( L \). We proved (6.8).

Conversely, the conclusions of Theorem 6.2 almost imply the conclusions of Theorem 1.6. To get Theorem 1.6 we have to use Hall’s Marriage Theorem for the labeling of eigenpairs, as in the proof of Proposition 4.3.

**Proposition 6.4.** Let \( \varepsilon_0 > 0 \). Fix \( 0 < \varepsilon < \varepsilon_0 \) and suppose there exists \( 0 < \eta < 1 \), a finite scale \( L \), and \( m > 0 \), such that the Anderson model \( H_{\varepsilon, \omega} \) satisfies (6.13) for all \( L \geq L \). Then, given \( 0 < \varepsilon < \xi \) and \( 0 < m' < m \), there exists a finite scale \( L_1 = L_1(L, \varepsilon_0, \xi, m, m') \) such that for all \( L \geq L_1 \) we have

\[ \inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m'\text{-localizing for } H_{\varepsilon, \omega}\} \geq 1 - e^{-L^\xi}. \]  
(6.18)

**Proof.** Fix \( 0 < \varepsilon \leq \varepsilon_0 \), \( 0 < \xi' < \xi \), and \( 0 < m' < m \), and assume (6.13) for all \( L \geq L \). Let \( L = \ell' \) with \( \ell' \geq L \). We take \( \Lambda_L = \Lambda(x_0) \), where \( x_0 \in \mathbb{R}^d \), and let \( C_{L, \ell} = C_{L, \ell}(x_0) \) be the suitable \( \ell\)-cover of \( \Lambda_L \), with \( \Xi_{L, \ell} = \Xi_{L, \ell}(x_0) \). We define the event

\[ \mathcal{R}_{L, m}(x_0) = \bigcap_{a, b \in \Xi_{L, \ell}, \|a - b\| > \ell} \mathcal{R}_{\ell, m}(a, b), \]  
(6.19)

so we have

\[ \mathbb{P}\{\mathcal{R}_{L, m}(x_0)\} \geq 1 - (2L)^{2d} e^{-\xi'} \geq 1 - \frac{1}{2} e^{-L^\xi} \]  
(6.20)

for sufficiently large \( L \).
Fix $\omega \in \mathcal{R}_{L,m}(x_0)$, and let $H_\Theta = H_{\omega,\Theta}$ for $\Theta \subset \mathbb{Z}^d$. Suppose $(\varphi, \lambda)$ is an eigenpair for $\sigma(H_{\Lambda_L})$. Recall that for any box $\Lambda_L \subset \Lambda_L$ it follows from $(H_{\Lambda_L} - \lambda) \varphi = 0$ and \ref{eq:6.24} that
\[
\chi_{\Lambda_L}(H_{\Lambda_L} - \lambda) \varphi = -\varepsilon \chi_{\Lambda_L} \Gamma_{0^{n-L} \Lambda_L} \varphi,\tag{6.21}
\]
so, for all $x \in \Lambda_L$ we have
\[
\varphi(x) = -\varepsilon G_{\Lambda_L}(\lambda) \Gamma_{0^{n-L} \Lambda_L} \varphi(x) = \sum_{(u,v) \in \partial^{n-L} \Lambda_L} \varepsilon G_{\Lambda_L}(\lambda; x, u) \varphi(v).\tag{6.22}
\]
Since $\omega \in \mathcal{R}_{L,m}(x_0)$, there exists $a_\lambda \in \Xi_{L,\ell}$ such that $\Lambda_L(b)$ is $(m, \lambda)$-regular for all $b \in \Xi_{L,\ell}$ with $\|b - a_\lambda\| > \ell$. In particular, if $y \notin \Lambda_{(2p+1)}\varepsilon'(a_\lambda)$ we have that $\Lambda_L^{(y)}$ (as in \ref{eq:6.10}) is $(m, \lambda)$-regular. Thus it follows from \ref{eq:6.24} and \ref{eq:6.27} that
\[
\|\varphi(y)\| \leq \varepsilon s_d \ell^{d-1} e^{-m\|y-y_1\|^{-1}} \|\varphi(y_1)\| \leq e^{-m_1\|y-y_1\|} \|\varphi(y_1)\|\tag{6.23}
\]
for some $y_1 \in \partial^{n-L} \Lambda_L^{(y)}$, so $\|y - y_1\| \geq \ell^\tau$, where
\[
m_1 \geq m \left(1 - \frac{10}{\ell} - C_d \frac{\log \ell}{m} - \frac{\log \varepsilon_\lambda}{m}\right) \geq m \left(1 - C_d \frac{\log \ell}{m} (1 + \frac{1}{m} \frac{\log \ell}{m})\right).\tag{6.24}
\]
Since we can repeat the procedure if $y_1 \notin \Lambda_{(2p+1)}\varepsilon'(a_\lambda)$, we conclude that
\[
\|\varphi(y)\| \leq e^{-m_1\|y-a_\lambda\|} \|y-a_\lambda\|^{-2} \leq e^{-m_1\|y-a_\lambda\|} \frac{1}{\ell \tau}.\tag{6.25}
\]
Since $\frac{1}{\tau} < \tau < 1$, it follows that
\[
\|\varphi(y)\| \leq e^{-m_2\|y-a_\lambda\|} \quad \text{if} \quad \|y - a_\lambda\| \geq \frac{1}{2} L \tau,\tag{6.26}
\]
where
\[
m_2 \geq m \left(1 - \frac{1}{\ell \tau}ight) \geq m \left(1 - C_d \frac{\log \ell}{m} (1 + \frac{1}{m} \frac{1}{\ell \tau})\right).\tag{6.27}
\]
Let \{(\varphi_j, \lambda_j)\}_{j=1}^{[\Lambda_L]} be an eigensystem for $H_{\Lambda_L}$. We let $a_j = a_{\lambda_j}$. (Note that the map $j \mapsto a_j$ is not necessarily an injection.) We claim
\[
\Lambda_L = \bigcup_{j=1}^{[\Lambda_L]} \Lambda_{(2p+1)}\varepsilon'(a_j).\tag{6.28}
\]
This can be seen as follows. Suppose $y \in \Lambda_L \setminus \bigcup_{j=1}^{[\Lambda_L]} \Lambda_{(2p+1)}\varepsilon'(a_j)$. It follows from \ref{eq:6.25} that
\[
1 = \|\delta_y\|^2 = \sum_{j=1}^{[\Lambda_L]} |\varphi_j(y)|^2 \leq (L + 1)^d e^{-m_1 \frac{1}{\tau}} < 1,\tag{6.29}
\]
a contradiction.

Given $x \in \Lambda_L$, we set
\[
\mathcal{N}(x) = \{j \in \{1, 2, \ldots, [\Lambda_L]\} : x \in \Lambda_{(2p+1)}\varepsilon'(a_j)\}.\tag{6.30}
\]
Note $\mathcal{N}(x) \neq \emptyset$ in view of \ref{eq:6.28}.

Let $\mathcal{N}(\Theta) = \bigcup_{x \in \Theta} \mathcal{N}(x)$ for $\Theta \subset \Lambda_L$. We fix $\Theta \subset \Lambda_L$, and let $Q_{\Theta}$ be the orthogonal projection onto the span of $\{\varphi_j : j \in \mathcal{N}(\Theta)\}$. For every $j \notin \mathcal{N}(\Theta)$ we have \ref{eq:6.25} for $\varphi_j(x)$ for all $x \in \Theta$, so
\[
\|1 - Q_{\Theta}\|_{\Theta} \leq \|\Lambda_L^{(\frac{1}{2})} \Theta \|^\frac{1}{2} e^{-m_1 \frac{1}{\varepsilon'}} \leq (L + 1)^d e^{-m_1 \frac{1}{\varepsilon'}} < 1,\tag{6.31}
\]
so it follows from Lemma \ref{lem:6.4} that
\[
|\Theta| = \text{tr} \chi_{\Theta} \leq \text{tr} Q_{\Theta} = |\mathcal{N}(\Theta)|,\tag{6.32}
\]
which is Hall’s condition \((\text{C.1})\). Thus we can apply Hall’s Marriage Theorem, concluding that there exists a bijection

\[
x \in \Lambda_L \mapsto j_x \in \{1, 2, \ldots, |\Lambda_L| \}, \quad \text{where} \quad j_x \in \mathcal{N}(x).
\] 

(6.33)
We set \((\varphi_x, \lambda_x) = (\varphi_{j_x}, \lambda_{j_x})\) for all \(x \in \Lambda_L\). If \(\|y - x\| \geq L\), we have

\[
\|y - a_{j_x}\| \geq \|y - x\| - \|x - a_{j_x}\| \geq L - (\rho + \frac{1}{2})\ell > \frac{1}{2}L,
\]

(6.34)
so it follows from \((6.20)\) that

\[
|\varphi_x(y)| \leq e^{-m_2\|y-a_{j_x}\|} \leq e^{-m_2(\|y-x\|-\|x-a_{j_x}\|)} \leq e^{-m_2(\|y-x\|-(\rho+\frac{1}{2})\ell)}
\]

(6.35)
where

\[
m_3 \geq m_2 \left(1 - \frac{3}{2e^{\tau L}}\right) \geq m \left(1 - C'_{d,x_0}(1 + \frac{1}{m})e^{-\tau L}\right) \geq m',
\]

(6.36)
for \(\ell\) sufficiently large.

Thus for \(\omega \in \mathcal{R}_{L,m}(x_0)\) the box \(\Lambda_L(x_0)\) would be \(m'\)-localizing for \(H\) if it would be level spacing. Since it follows from \((2.3)\) that this is true for \(L\) large with probability \(\geq 1 - \frac{1}{2}e^{-L^2}\), we have \((6.18)\).

\section*{Appendix A. Lemma about orthogonal projections}

\begin{lemma}
Let \(P\) and \(Q\) be orthogonal projections on a Hilbert space \(H\). Then

\[
\|(1 - P)Q\| < 1 \implies \text{tr} Q \leq \text{tr} P.
\]

(A.1)
In particular, taking \(Q = 1\) we get

\[
\|1 - P\| < 1 \implies P = 1.
\]

(A.2)
\end{lemma}

\begin{proof}
Since \((1 - (1 - P)Q)Q = PQ\) and \(1 - (1 - P)Q\) is invertible by the assumption of the lemma, we infer that

\[
Q = (1 - (1 - P)Q)^{-1}PQ \implies \text{tr} Q \leq \text{tr} P,
\]

(A.3)
where in the last step we have used \(A = BCD \implies \text{Rank} A \leq \text{Rank} C\). \qed
\end{proof}

\section*{Appendix B. Estimating the probability of level spacing}

In this appendix we review an estimate of the probability of level spacing due to Klein and Molchanov [KIM]. Let us consider a generalized Anderson model

\[
H_\omega := H_0 + V_\omega \quad \text{on} \quad \ell^2(\mathbb{Z}^d),
\]

(B.1)
where \(H_0\) is a bounded self-adjoint operator on \(\ell^2(\mathbb{Z}^d)\) and \(V_\omega\) is a random potential: \(V_\omega(x) = \omega_x\) for \(x \in \mathbb{Z}^d\), where \(\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}\) is a family of independent identically distributed random variables, whose common probability distribution \(\mu\) is non-degenerate and Hölder continuous of order \(\alpha \in \left(\frac{1}{2}, 1\right]\):

\[
S_\mu(t) \leq K t^\alpha \quad \text{for all} \quad t \in [0, 1],
\]

(B.2)
where \(K\) is a constant and \(S_\mu(t) := \sup_{a \in \mathbb{R}} \mu\{|a, a+t]\}\) is the concentration function of the measure \(\mu\). We set \(Q_\mu(t) = S_\mu(t)\) and \(K = K\) if \(\alpha = 1\) and \(Q_\mu(t) = 8S_\mu(t)\) and \(\bar{K} = 8K\) if \(\alpha \in \left(\frac{1}{2}, 1\right]\).
Lemma B.1 ([KM] Lemma 2). Let \( \Theta \subset \mathbb{Z}^d \) be a finite subset, \( I \subset \mathbb{R} \) be a bounded interval, and \( \eta \in (0, \frac{1}{\Theta}) \). Let \( \mathcal{E}_{\Theta, I, \eta} \) denote the event that \( \text{tr} \chi_J(H_{\omega, \Theta}) \leq 1 \) for all subintervals \( J \subset I \) with length \( |J| \leq \eta \). Then

\[
P\{\mathcal{E}_{\Theta, I, \eta}\} \geq 1 - \tilde{K}^2(|I| + 1)(2\eta)^{2n-1} |\Theta|^2. \tag{B.3}
\]

Proof. We recall the proof for completeness. The proof is based on Minami’s inequality [M], which we use in the form given in [CGK1, Theorem 3.3] and [CGK2, Theorem 2.1]:

\[
P\{\text{tr} \chi_J(H_{\omega, \Theta}) \geq 2\} \leq \frac{1}{2} E\{\text{tr} \chi_J(H_{\omega, \Theta})(\text{tr} \chi_J(H_{\omega, \Theta}) - 1)\} \leq \frac{1}{2} (Q_{\mu}(|J|)|\Theta|)^2. \tag{B.4}
\]

We cover the interval \( I \) by \( \left\lceil \frac{|I|}{2\eta} \right\rceil \leq \frac{|I|}{\eta} + 2 \) intervals of length \( 2\eta \), in such a way that any subinterval \( J \subset I \) with length \( |J| \leq \eta \) will be contained in one of these intervals. Then

\[
P\{\mathcal{E}_{\Theta, I, \eta}\} \leq \frac{|I|}{\eta} + 2 \leq (\frac{|I|}{\eta} + 2)\frac{1}{2} (Q_{\mu}(2\eta)|\Theta|)^2 \leq \tilde{K}^2(|I| + 1)(2\eta)^{2n-1} |\Theta|^2, \tag{B.5}
\]

where we used Minami’s inequality [B.4]. \( \square \)

Appendix C. Hall’s Marriage Theorem

Hall’s Marriage Theorem (see [BrDM, Chapter 2]) gives a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. Let \( \mathbb{G} = (A, B; \mathcal{E}) \) be a bipartite graph with vertex sets \( A \) and \( B \) and edge set \( \mathcal{E} \subseteq A \times B \) (the bipartite condition). \( \mathcal{M} \subseteq \mathcal{E} \) is called a matching if every vertex of \( \mathbb{G} \) coincides with at most one edge from \( \mathcal{M} \); it is a perfect matching if every vertex of \( \mathbb{G} \) coincides with exactly one edge from \( \mathcal{M} \), i.e., every vertex in \( A \) is matched with a unique vertex in \( B \) and vice-versa. In particular, \( |A| = |B| \) is a necessary condition for the existence of a perfect matching. Given a vertex \( a \in A \), let \( \mathcal{N}(a) = \{b \in B; (a, b) \in \mathcal{E}\} \), the set of neighbors of \( a \). Let \( \mathcal{N}(U) = \cup_{a \in U} \mathcal{N}(a) \) for \( U \subseteq A \).

Hall’s Marriage Theorem. Let \( \mathbb{G} = (A, B; \mathcal{E}) \) be a bipartite graph with \( |A| = |B| \). There exists a perfect matching in \( \mathbb{G} \) if and only if the graph \( \mathbb{G} \) fulfills Hall’s condition

\[
|U| \leq |\mathcal{N}(U)| \quad \text{for all} \quad U \subset A. \tag{C.1}
\]

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(A. Elgart) Department of Mathematics; Virginia Tech; Blacksburg, VA, 24061, USA
E-mail address: aelgart@vt.edu

(A. Klein) University of California, Irvine; Department of Mathematics; Irvine, CA 92697-3875, USA
E-mail address: aklein@uci.edu