Three-Dimensional Extremal Black Holes and the Maldacena Duality

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Abstract

We discuss the microscopic states of the extremal BTZ black holes. Degeneracy of the primary states corresponding to the extremal BTZ black holes in the boundary $N=(4,4)$ SCFT is obtained by utilizing the elliptic genus and the unitary representation theory of $N=4$ SCA. The degeneracy is consistent with the Bekenstein-Hawking entropy.

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1 Introduction

One of the challenging problems in quantum gravity is to understand the microscopic properties of black holes, in particular, the statistical origin of the Bekenstein-Hawking entropy. New idea for an explanation of the origin of the Bekenstein-Hawking entropy has been provided by recent development in our understanding of non-perturbative superstring theory. It is based on the D-brane description of black holes[1] and the AdS/CFT correspondence[2, 3, 4]. These are much related with each other under the Maldacena duality[2].

Three-dimensional Einstein equation with negative cosmological constant has the solutions called the BTZ black holes[5, 6]. These black holes have locally AdS$_3$ geometry. Via the AdS$_3$/CFT$_2$ correspondence, one can also hope to be able to analyze the microscopic properties of the BTZ black holes based on a local field theory on the boundary. Infinite dimensional algebra of two-dimensional conformal symmetry, that is, the Virasoro algebra, provides an important clue for our understanding the correspondence and the Maldacena duality. The pioneering work is Strominger’s counting of microscopic states of the BTZ black holes[7]. But the qualitative aspects of this counting still remain obscure.

In this paper, we will discuss the three-dimensional extremal BTZ black holes in the context of the Maldacena duality. Although this duality has been conjectural yet, various checks have been carried out. (See [8] and references therein.) In this perspective, the extremal BTZ black holes can be identified with the primary states which are 1/2 BPS states in the N=(4,4) two-dimensional supersymmetric σ-model. This σ-model has a quantity called elliptic genus convenient to count the degeneracy of these states. We explicitly count the microscopic states of the extremal BTZ black holes with this identification by using the elliptic genus and the unitary representation theory of the N=4 superconformal algebra. The microscopic entropy of these black holes obtained by this counting agrees with the entropy à la Bekenstein-Hawking.

This paper is organized as follows. In section 2, we will summarize the previous results about the BTZ black holes from the perspective of the AdS$_3$/CFT$_2$ corre-
spondence in a pure quantum gravity and in non-perturbative superstring theory, i.e., the Maldacena duality. In section 3, after a brief introduction of N=4 superconformal algebra, black hole states are discussed in the unitary representation theory. In section 4, some facts about the elliptic genus of the N=(4,4) supersymmetric σ-model are reviewed. In section 5, we count the number of 1/8 BPS states in the D1-D5 brane system in IIB supergravity via the elliptic genus of this σ-model and then finally count the microscopic states of the extremal BTZ black holes. In section 6, some other related topics are discussed.

2 BTZ black holes and AdS₃/CFT₂ correspondence

2.1 BTZ black holes in a three-dimensional pure quantum gravity

The BTZ black holes3 are three-dimensional black holes specified by their mass M and angular momenta J, where |J| ≤ Ml. In terms of the Schwarzschild coordinates (t, φ, r), with the ranges −∞ < t < +∞, 0 ≤ φ < 2π and 0 < r < +∞, the black hole metric $ds^2_{\text{BTZ}}(J, M)$ has the form

$$ds^2_{\text{BTZ}}(J, M) \equiv -N^2(dt)^2 + N^{-2}(dr)^2 + r^2(d\phi + N^\phi dt)^2,$$

where $N$ and $N^\phi$ are the functions of the radial coordinate $r$

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N^\phi = \begin{cases} \frac{r + r_+}{l^2 r^2} & \text{when } J \geq 0, \\ -\frac{r + r_-}{l^2 r^2} & \text{when } J < 0. \end{cases}$$

The outer and inner horizons are located respectively at $r = r_+$ and $r = r_-$. Information about the mass and angular momentum is encoded in $r_\pm$ by

$$r_\pm^2 \equiv 4GMl^2 \left( 1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}} \right),$$

3Exact solutions of the vacuum Einstein equation with a negative cosmological constant $\Lambda = -1/l^2$. 

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where $G$ is Newton constant. In the case of $Ml = |J|$, the black hole is called extremal. It holds $r_+ = r_-$ and then the outer and inner horizons coincide with each other. When $Ml > |J|$, it is called non-extremal. And in the case of $J = 0$ and $Ml = -l/8G$, the geometry corresponds to the global AdS$_3$.

The outer horizon of these solutions has finite area. The semiclassical argument leads to the finite Bekenstein-Hawking entropy:

$$S \equiv \frac{A}{4G} = \frac{2\pi r_+}{4G}, \quad (A : \text{area of the outer horizon}) \quad (2.4)$$

Quantization of three-dimensional pure gravity with negative cosmological constant is discussed in [9]. It is prescribed, through the detailed analysis of Brown-Henneaux’s asymptotic Virasoro symmetry [10], as the geometric quantization of the Virasoro coadjoint orbits of the Virasoro central charge $c = 3l/2G$.

The BTZ black holes and the AdS$_3$ correspond to the primary states (highest weight states) of the Virasoro algebra of Brown-Henneaux:

$$\text{BTZ}_{(J, M)} \iff |J, M\rangle \equiv |h\rangle \otimes |\tilde{h}\rangle, \quad (2.5)$$
$$\text{AdS}_3 \iff |\text{vac}\rangle \equiv |0\rangle \otimes |0\rangle, \quad (2.6)$$

where

$$h = \frac{1}{16Gl}(r_+ + r_-)^2 + \frac{c}{24}, \quad \tilde{h} = \frac{1}{16Gl}(r_+ - r_-)^2 + \frac{c}{24}. \quad (2.7)$$

The extremal BTZ black holes correspond to

$$\text{BTZ}_{(Ml, M)} \iff |Ml, M\rangle \equiv |h\rangle \otimes |\frac{c}{24}\rangle. \quad (2.8)$$

The total Hilbert space of the theory, which includes excited states (secondary states), is obtained by the tensor products $\mathcal{V}_h \otimes \mathcal{V}_{\tilde{h}}$ of the Verma modules of the Virasoro algebra. ($\mathcal{V}_h$ and $\mathcal{V}_{\tilde{h}}$ are respectively the Verma modules of the left-moving and right-moving sectors.) These Verma modules constitute the unitary irreducible representations of the Virasoro algebra. We can identify the states excited by $L_{-n}$

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*Strictly speaking, these states correspond to the geometry of the exterior of outer horizon of the BTZ black holes and the geometry without the origin of the AdS$_3$ respectively.*
in the Verma module with massive gravitons on the corresponding background geometry.

In view of the AdS$_3$/CFT$_2$ correspondence, this Hilbert space should be realized by the corresponding boundary CFT. In fact it was done$^9$ based on the Liouville field $X$ with a specific background charge. The action is given by

$$S[X] = \frac{1}{4\pi i} \int_{\mathbb{P}^1} \bar{\partial}X \wedge \partial X + \frac{\alpha_0}{2\pi} \int_{\mathbb{P}^1} RX, \quad \left(\alpha_0 \equiv \sqrt{\frac{l}{8G}}\right)$$

(2.9)

where $R$ is the Riemann tensor of a fixed Kähler metric on $\mathbb{P}^1$. The stress tensor $T(z)$ has the form

$$T(z) = -\frac{1}{2} \partial X \partial X(z) + \alpha_0 \partial^2 X(z),$$

(2.10)

and provides the generators of the Virasoro algebra with the central charge $1 + 12\alpha_0^2 = 1 + 3l/2G$. This central charge is the same as that of Virasoro algebra of Brown-Henneaux in the semiclassical limit, i.e., $l/G \gg 1$. The Fock space $\mathcal{F}_k$ is built on the Fock vacuum $|k\rangle$, which is introduced as the state obtained from the ordinary $SL_2(\mathbb{C})$-invariant vacuum $|0\rangle$ by the relation $|k\rangle = \lim_{z \to 0} e^{ikX(z)}|0\rangle$.

The BTZ black hole states $\text{BTZ}(J,M)$ can be identified with the following Fock vacuum:

$$\text{BTZ}(J,M) \iff |J,M\rangle \equiv |k(J,M)\rangle \otimes |\bar{k}(J,M)\rangle,$$

(2.11)

where $k(J,M)$ and $\bar{k}(J,M)$ are given by

$$k(J,M) \equiv -i \sqrt{\frac{l}{8G}} + \frac{r_+ + r_-}{\sqrt{8Gl}},$$

$$\bar{k}(J,M) \equiv -i \sqrt{\frac{l}{8G}} + \frac{r_+ - r_-}{\sqrt{8Gl}}.$$  

(2.12)

AdS$_3$ state $\text{AdS}_3$ can be identified with the $SL_2(\mathbb{C})$-invariant vacuum:

$$\text{AdS}_3 \iff |\text{vac}\rangle \equiv |0\rangle \otimes |0\rangle.$$  

(2.13)

The Fock spaces $\mathcal{F}_k \otimes \bar{\mathcal{F}}_{\bar{k}}$ built on these primary states give the unitary irreducible representations of the Virasoro algebra with $c = 1 + 3l/2G$, and coincide with the physical Hilbert space of the previous quantization of three-dimensional pure gravity.

$^5$Similar arguments hold for the anti-holomorphic (right-moving) part.
To summarize, in this correspondence of three-dimensional pure gravity and the
boundary CFT, the BTZ black holes appear as the primary states of the Virasoro
algebra with $c = 3l/2G$ in both descriptions. This result may not be desirable for
the counting of microscopic states of the BTZ black holes. We cannot count in
principle the degeneracy of these primary states with this boundary theory, since
this Liouville field theory has continuum spectrum of primary states.

2.2 BTZ black holes and Maldacena duality

Next, we consider the AdS$_3$/CFT$_2$ correspondence in superstring theory. Through
the analysis of the near horizon limit of the BPS solitonic solution of IIB supergrav-
ity, which describes the bound state of $Q_1$ D1-branes and $Q_5$ D5-branes, Maldacena
has conjectured in [2],

\[ \text{IIB superstring theory on } (\text{AdS}_3 \times S^3)_{Q_1 Q_5} \times M_4 \quad (M_4 = K3 \text{ or } T^4) \]
\[ \overset{\text{dual}}{\downarrow} \]
\[ \text{two-dimensional N=(4,4) supersymmetric } \sigma\text{-model} \]
on the Higgs branch of world volume theory of the D1-D5 system.

Here, we indicated the dependence of the radius of AdS$_3$ and S$^3$ on $Q_1 Q_5$ (see below).
We call this duality simply the Maldacena duality.

We will discuss mainly the case of $M_4 = K3$ in the following. The N=(4,4)
$\sigma$-model can be regarded as the $\sigma$-model on the target space of the $k$-th symmetric
product of $K3$ \[11, 12\], where

\[ k = Q_1 Q_5 + 1. \tag{2.14} \]

Since the symmetric product is $4k$-dimensional hyper-Kähler manifold, it has au-
tomatically the N=(4,4) superconformal symmetry. The Virasoro subalgebra and
zero mode of the SU(2) current algebra may be identified with the Virasoro algebra
of Brown-Henneaux on the boundary of AdS$_3$ and the isometry of S$^3$, respectively.
We will discuss the N=4 superconformal algebra in more detail in section 3.

The extremal BTZ black holes can also be obtained as the near horizon limit
of the similar BPS solitonic solutions of IIB supergravity. Therefore we can expect
that the extremal BTZ black holes can be analyzed by this \( N=(4,4) \) \( \sigma \)-model.

We summarize some related facts of IIB supergravity here. IIB supergravity on \( S^1 \times K3 \) whose radius and volume are \( R \) and \( (2\pi)^4 \alpha' v \) has the BPS solitonic solution (see for example [13] and references therein):

\[
\begin{aligned}
\text{ds}_{10}^2 &= f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} \left\{ -dt^2 + dx_5^2 + f_N (dt + dx_5)^2 \right\} \\
&+ f_1^{\frac{1}{2}} f_5^{\frac{1}{2}} (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + f_1^{\frac{1}{2}} f_5^{-\frac{1}{2}} ds_{K3}^2,
\end{aligned}
\] (2.15)

with periodic identification \( x_5 \sim x_5 + 2\pi R \) in string frame and

\[
\begin{aligned}
e^{-2(\phi - \phi_\infty)} &= f_5 f_1^{-1}, \quad C_{05}^{(R)} = \frac{1}{2} (f_1^{-1} - 1), \\
H_{ijk}^{(R)} &= (*_6 dC^{(R)})_{ijk} = \frac{1}{2} \epsilon_{ijkl} \partial_5 f_5 \quad (i,j,k,l = 1,2,3,4),
\end{aligned}
\] (2.16)

where \( C^R \) is Ramond-Ramond 2-form and \( *_6 \) is Hodge dual in 6-dimension \( (t,x_1,\cdots,x_5) \).

And \( f_1, f_5 \) and \( f_N \) are following functions with respect to radial coordinate, \( r = x_1^2 + x_2^2 + x_3^2 + x_4^2 \):

\[
\begin{aligned}
f_1 &= 1 + \frac{\tilde{Q}_1}{r^2}, \quad \tilde{Q}_1 = \frac{\alpha' g_{st}}{v} Q_1, \\
f_5 &= 1 + \frac{\tilde{Q}_5}{r^2}, \quad \tilde{Q}_5 = \frac{\alpha' g_{st}}{v} Q_5, \\
f_N &= \frac{\tilde{N}}{r^2}, \quad \tilde{N} = \frac{\alpha'^2 g_{st}^2}{R^2 v} N.
\end{aligned}
\] (2.17) (2.18) (2.19)

This solution corresponds to the configuration of the bound state of \( Q_5 \) D5-branes wrapping on \( K3 \times S^1 \) and \( Q_1 \) D1-branes wrapping on \( S^1 \) with \( N \) units of KK momenta along \( S^1 \), i.e., \( x_5 \)-direction, and preserves four supercharges. So it is a 1/8 BPS state [1].

Here we can get the extremal BTZ black hole as the near horizon limit of the geometry (2.13) [14, 13]. The near horizon limit is defined as

\[
\alpha' \rightarrow 0, \quad \text{with} \quad U \equiv \frac{r}{\alpha}, \quad R \text{ and } v \text{ fixed}.
\]

In this limit, the metric (2.15) describes \( (\text{BTZ}_{(Ml,M)} \times S^3)_{Q_1 Q_5} \times K3 \) with \( Ml = J = N \). The radius of \( \text{AdS}_3 \) and \( S^3 \) coincide and become \( l = g_{st}^{1/2} \alpha'^{1/2} (Q_1 Q_5 / v)^{1/4} \).

And the three-dimensional effective Newton constant on BTZ\((N,N/l)\) is given by
\( G_{\text{eff}}^{(3)} = l/(4Q_1Q_5) \). There exists on this background the asymptotic Brown-Henneaux’s Virasoro symmetry with central charge,

\[
c = \frac{3l}{2G_{\text{eff}}^{(3)}} = 6Q_1Q_5.
\]

(2.20)

This is the same as the central charge of the N=(4,4) \( \sigma \)-model at the semiclassical limit \( Q_1Q_5 \gg 1 \). The Bekenstein-Hawking entropy becomes

\[
S = \frac{2\pi r_+}{4G_{\text{eff}}^{(3)}} = 2\pi \sqrt{Q_1Q_5N},
\]

(2.21)

which is valid in the semiclassical region \( N \gg Q_1Q_5 \gg 1 \).

If one accepts the Maldacena duality, the extremal black hole should be identified with the primary state

\[
| N + \frac{c}{24} \rangle \otimes | \frac{c}{24} \rangle,
\]

(2.22)

of the N=(4,4) \( \sigma \)-model. One can ask whether the entropy (2.21) can be regarded as the degeneracy of the primary state (2.22) of this N=(4,4) \( \sigma \)-model. In the sequel, we will discuss this question and answer in the affirmative.

3 N=4 superconformal symmetry

N=(4,4) \( \sigma \)-model is known to be finite to all orders of perturbation and to be conformally invariant at the quantum level. Thus the states of the \( \sigma \)-model on the \( k \)-th symmetric product\(^6\), \( S^kK3 \), constitute the unitary irreducible representations of the underlying N=4 superconformal algebra (N=4 SCA).

3.1 Basics of N=4 superconformal algebra

N=4 SCA is generated by \( L_n, J_n, G_r^i \) and \( \bar{G}_r^i \) with

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{k}{2}m(m^2 - 1)\delta_{n+m,0}, \quad \{G_r^i, G_s^j\} = \{\bar{G}_r^i, \bar{G}_s^j\} = 0,
\]

\(^6\)We will denote \( k \)-th symmetric product of \( K3 \) as \( S^kK3 \equiv K3^{\otimes k}/S_k \) (\( S_k \) is a \( k \)-dimensional symmetric group).
\[
\{ G_r^i, \bar{G}_s^i \} = 2\delta^{ij} L_{r+s} - 2(r - s)\sigma^{a}_{ij} J_r^a + \frac{k}{2}(4r^2 - 1)\delta_{r+s,0},
\]

\[
\left[ J_m^a, J_n^b \right] = i\epsilon^{abc} J_{m+n}^c + \frac{k}{2}m\delta_{m+n,0},
\]

\[
\left[ J_m^a, G_r^i \right] = -\frac{1}{2}\sigma^{a}_{ij} G_{m+r}^j, \quad \left[ J_m^a, \bar{G}_r^i \right] = \frac{1}{2} \left( \sigma^{a}_{ij} \right)^* \bar{G}_{m+r}^j,
\]

\[
\left[ L_m, G_r^i \right] = (m - r) G_{m+r}^i, \quad \left[ L_m, \bar{G}_r^i \right] = (m - r) \bar{G}_{m+r}^i,
\]

\[
\left[ L_m, J_n^a \right] = -n J_{m+n}^a.
\]

(\sigma^{a}_{ij} is Pauli matrix and \( J_n^{(\pm)} \equiv J_n^1 \pm iJ_n^2 \)) \( L_n, J_n^a \) and \( G_r^i (\bar{G}_r^i) \) represent the Fourier components of the energy momentum tensor, SU(2) current and four supercurrents, respectively. \( G_r^i (\bar{G}_r^i) \) transforms as SU(2) doublet (its conjugate) under the global SU(2) symmetry which is generated by \( J_0^a \). The level \( k \) of the SU(2) current algebra \((\hat{\text{SU}}(2))_k\) must be positive integer for unitary representations. The central charge \( c \) of the Virasoro subalgebra is 6\( k \).

Two different boundary conditions of the supercurrents provide two different sectors of this algebra. If \( G_r^i (\bar{G}_r^i) \) has \( r \in \mathbb{Z} + \frac{1}{2} \), it is called the Neveu-Schwarz (NS) sector and if \( r \in \mathbb{Z} \), it is called the Ramond (R) sector. These sectors are related by the automorphism of the algebra called spectral flow. We will discuss the R-sector in the following.

Unitary irreducible representations of N=4 SCA have two distinct types[15]. They are built on highest weight states \( |h, l\rangle \) called massive primary and massless primary in the R-sector.

(i) Massive primary state

\[
L_n|h, l\rangle = G_n^i|h, l\rangle = \bar{G}_n^i|h, l\rangle = J_n^a|h, l\rangle = 0, \quad n \geq 1
\]

\[
J_0^{(\pm)}|h, l\rangle = G_0^i|h, l\rangle = \bar{G}_0^i|h, l\rangle = 0,
\]

\[
L_0|h, l\rangle = h|h, l\rangle, \quad J_0^3|h, l\rangle = l|h, l\rangle,
\]

\[
h > \frac{k}{4} = \frac{c}{24}, \quad l = \frac{1}{2}, \frac{1}{2}, \cdots, \frac{k}{2}, \frac{1}{2}, \frac{k}{2}, \quad (3.1)
\]

(ii) Massless primary state

\[
L_n|h, l\rangle = G_n^i|h, l\rangle = \bar{G}_n^i|h, l\rangle = J_n^a|h, l\rangle = 0, \quad n \geq 1
\]

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\[ J_0^{(+)} |h, l \rangle = G_0^i |h, l \rangle = \bar{G}_0^i |h, l \rangle = 0, \quad i = 1, 2 \]

\[ L_0 |h, l \rangle = h |h, l \rangle, \quad J_0^3 |h, l \rangle = l |h, l \rangle, \]

\[ h = \frac{k}{4} = \frac{c}{24}, \quad l = 0, \frac{1}{2}, \cdots, \frac{k}{2} - \frac{1}{2}, \frac{k}{2}. \]  

These representations are called massive representation \( \mathcal{M}^k_{(h, l)} \) and massless representation \( \mathcal{M}^k_{0(l)} \) respectively.

The massive representations have the same number of bosonic and fermionic states at each level and the Witten index is equal to zero. These representations correspond to the representations which have spontaneously broken supersymmetry. The Witten index of the massless representations is non-zero. These representations are the representations which have unbroken supersymmetry. The primary states of massless representations have dimension \( h = \frac{k}{4} = \frac{c}{24} \). These are the ground states of the R-sector.

Character of the representation is introduced by \( \chi(R)^k(h, l; \tau, z) = \text{Tr}(q^{L_0 - \frac{k}{24}} y^{J_0^3}) \), \( (q = e^{2\pi i \tau} \text{ and } y = e^{2\pi i z}). \) Their explicit form is given in [16].

(i) The character of the massive representation \( \mathcal{M}^k_{(h, l)} \):

\[ \chi(R)^k(h, l; \tau, z) = q^{h - \frac{k}{4} - \frac{l^2}{2}} \frac{\theta_2(\tau, z)^2}{\eta(\tau)^3} \chi_{k-1}^{-1}(\tau, z), \]  

where \( \chi_k(\tau, z) \) is the character of \( \widehat{\text{SU}(2)}_k \) of isospin \( l \),

\[ \chi_k^l(\tau, z) = \frac{q^{(l+\frac{k}{2})^2}}{\prod_{n=1}^{\infty} (1 - q^n)(1 - y^2 q^n)(1 - y^{-2} q^{n-1})} \times \sum_{m=0}^{\infty} q^{(k+2)m^2 + 2l + 1} \left( y^{2((k+2)m + 1)} - y^{-2((k+2)m + 1)} \right) \]

\[ = \frac{\Theta_{2l+1,k+2}(\tau, 2z) - \Theta_{-2l-1,k+2}(\tau, 2z)}{\Theta_{1,2}(\tau, 2z) - \Theta_{-1,2}(\tau, 2z)}. \]  

(ii) The character of the massless representation \( \mathcal{M}^k_{0(l)} \):

\[ \chi(0)^k(h = \frac{k}{4}, l; \tau, z) = q^{l} \frac{\theta_2(\tau, z)^2}{\eta(\tau)^3} \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)(1 - y^2 q^n)(1 - y^{-2} q^{n-1})} \]
\[ \times \sum_{m=-\infty}^{\infty} q^{(k+1)m^2+2lm} \left( \frac{y^{2((k+2)m+l-\frac{1}{2})}}{(1+y^{-1}q^{-m})^2} - \frac{y^{-2((k+2)m+l+\frac{1}{2})}}{(1+yq^{-m})^2} \right). \quad (3.5) \]

These characters enjoy the following properties. The Witten index of the representation can be obtained, if one sets \( z = 1/2 \), i.e., \( y = -1 \):
\[
\text{ch}^{(R)}_k(h, l; \tau, z = \frac{1}{2}) = 0, \quad (3.6)
\]
\[
\text{ch}^{(R)}_0(h = \frac{k}{4}, l; \tau, z = \frac{1}{2}) = (-1)^{2l}(2l + 1). \quad (3.7)
\]

The characters of massive and massless representations are related by
\[
\text{ch}^{(R)}_k(h = \frac{k}{4}, l; \tau, z) = \text{ch}^{(R)}_0(h = \frac{k}{4}, l; \tau, z) + 2 \text{ch}^{(R)}_0(h = \frac{k}{4}, l - \frac{1}{2}; \tau, z) + \text{ch}^{(R)}_0(h = \frac{k}{4}, l - 1; \tau, z). \quad (3.8)
\]

### 3.2 Identification of the black hole states

As argued in section 2.2, the extremal BTZ black hole will correspond to the primary state (2.22) of the N=(4,4) \( \sigma \)-model. It is a Virasoro primary state of the underlying N=4 SCA. The fact that the extremal BTZ black holes are the 1/2 BPS states with respect to the Poincare supersymmetry in three dimensions\(^7\) implies that this primary state is in the tensor product of massive and massless representations \( \mathcal{M}^k_{(h,l)} \otimes \tilde{\mathcal{M}}^k_{0(l)} \) which is built on
\[
|h = N + \frac{k}{4}, l\rangle \otimes |\tilde{h} = \frac{k}{4}, \tilde{l}\rangle. \quad (3.9)
\]

Actually we can proceed further. Since the extremal BTZ black holes do not have the conserved charge corresponding to isospin \( l \) and \( \tilde{l} \), the primary state (2.22) may be identified with the Virasoro primary state having vanishing isospins \( l = \tilde{l} = 0 \) in \( \mathcal{M}^k_{(h,l)} \otimes \tilde{\mathcal{M}}^k_{0(l)} \). The primary states with \( l = 0 \) in \( \mathcal{M}^k_{(h,l)} \) are as follows: when \( l \in \mathbb{Z} \), they are given by
\[
J_{0}^{-l} h |h = N + \frac{k}{4}, l\rangle,
\]
and
\[
J_{0}^{-l-1} (\tilde{G}_2^0 G_1^0 - \frac{h - k/4}{l}) |h = N + \frac{k}{4}, l\rangle, \quad (3.10)
\]
\(^7\)These correspond to the 1/4 BPS states in Anti-de Sitter supersymmetry in 3-dimension.
and when \( l \in \mathbb{Z} + \frac{1}{2} \), they are
\[
J_0^{(-)} l - \frac{1}{4} G_0^1 | h = N + \frac{k}{4}, l),
\]
and
\[
J_0^{(-)} l - \frac{1}{4} G_0^2 | h = N + \frac{k}{4}, l) . \quad (3.11)
\]
The primary states with \( \tilde{l} = 0 \) in \( \tilde{M}^k_{0(\tilde{l})} \) is identified with \( \tilde{J}^{(-)} l | \bar{h} = k/4, \tilde{l} \). The primary state (2.22) can be identified with the tensor product of these states. So, the degeneracy of the state is almost same as the degeneracy of the representation \( \mathcal{M}^{k}_{(h,l)} \otimes \tilde{M}^k_{0(\tilde{l})} \).

4 Elliptic genus for \( \sigma \)-model on symmetric product of K3

To count the number of 1/2 BPS states in the N=(4,4) \( \sigma \)-model, the so-called “elliptic genus” is a convenient tool. We summarize some properties of the elliptic genus emphasizing its modular transform and examine it from the perspective of N=4 SCA.

4.1 Elliptic genus as a weak Jacobi form

The elliptic genus of target space \( M \) is defined by the following trace in the R-R sector of the underlying N=(2,2) superconformal field theory.

\[
Z[M](\tau, z) = \text{Tr}_{R-R} (-1)^{J_0 - J_0} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} y^{J_0} , \quad (4.1)
\]

One can obtain the following topological indices of the target space \( M \), if one sets \( z \) to be specific value.

\[
\begin{align*}
Z[M](\tau, 0) & : \text{Elliptic extension of Euler number} \\
Z[M](\tau, \frac{1}{2}) & : \text{Elliptic extension of Hirzebruch signature} \\
q^{\frac{1}{24}} Z[M](\tau, \frac{\tau + 1}{2}) & : \text{Elliptic extension of Dirac genus}
\end{align*}
\]
where \( J_0 \) and \( \bar{J}_0 \) are the integral N=2 U(1) charges of the left-moving and the right-moving sectors\(^9\). The elliptic genus is independent of \( \tau \) by virtue of supersymmetry of the R-sector. The contribution of the right-moving sector is only from the ground states. But all states in the left-moving sector contribute to \( Z[M](\tau, z) \). So the elliptic genus \( Z[M](\tau, z) \) is a useful quantity for counting of the 1/2 BPS states.

The following theorem is known about this elliptic genus. (See [17] for detail.)

**Theorem 1.** If the target space \( M \) of the \( \sigma \)-model is an even-dimensional Calabi-Yau manifold, then the elliptic genus \( Z[M](\tau, z) \) is a weak Jacobi form of weight 0 and index \( d/2(d = \text{dim}_\mathbb{C} M) \) without character.

Weak Jacobi form in the above theorem is defined as follows [18].

**Definition.** A function \( \phi(\tau, z) \) is called a weak Jacobi form of weight \( k \in \mathbb{Z} \) and index \( m \in \mathbb{Z} > 0 / 2 \) without character, if it satisfies (i)~(iv):

(i) \( \phi(\tau, z) \) is a holomorphic function with respect to \( \tau \in \mathbb{H}^+ \) (\( \mathbb{H}^+ \) : upper half plane) and \( z \in \mathbb{C} \).

(ii) \( \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau + d)^k e^{\frac{2\pi imcz^2}{c\tau + d}} \phi(\tau, z) \). \( (a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1) \)

(iii) \( \phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi im(\lambda^2\tau + 2\lambda z)} \phi(\tau, z) \). \( (\lambda, \mu \in \mathbb{Z}) \)

(iv) \( \phi(\tau, z) \) has the Fourier expansion of the form

\[
\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} c(n, r) q^n y^r \quad (q = e^{2\pi i\tau}, y = e^{2\pi iz}).
\]

When \( M \) is \( K3 \), the elliptic genus \( Z[K3](\tau, z) \) becomes a weak Jacobi form of weight 0 and index 1 without character. An actual calculation of the N=(4,4) supersymmetric \( \sigma \)-model on \( K3 \) [13, 20] determines it explicitly as

\[
Z[K3](\tau, z) = 24\wp(\tau, z)K^2(\tau, z),
\]

where

\[
\wp(\tau, z) : \text{Weierstrass's } \wp \text{-function},
\]

\[
K(\tau, z) = i \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3}.
\]

---

\(^9\)N=2 SCA can be embedded into N=4 SCA by \( G_r = G_r^1 + \bar{G}_r^2, \bar{G}_r = G_r^2 + \bar{G}_r^1 \) and \( J_n = 2J_n^3 \).
We need the following theorem about weak Jacobi form\cite{17,18}.

**Theorem 2.** If we assign weights 4, 6 and 2 respectively to $E_4(\tau), E_6(\tau)$\footnote{$E_4(\tau)$ and $E_6(\tau)$ are the Eisenstein series.} and \(\varphi(\tau, z)\), any weak Jacobi form of weight \(2l\) \((l \in \mathbb{Z}_{\geq 0})\) and index \(k\) \((k \in \mathbb{Z}_{\geq 0})\) can be expressed as

\[
G_{2l+2k}(E_4(\tau), E_6(\tau), \varphi(\tau, z)) K^{2k}(\tau, z),
\]

where $G_{2l+2k}(E_4, E_6, \varphi)$ is a homogenous polynomial of weight \((2l + 2k)\) and its degree as a polynomial in \(\varphi\) is at most \(k\).

$S^kK3$ is \(2k\)-dimensional Calabi-Yau manifold. The elliptic genus $Z[S^kK3](\tau, z)$ becomes a weak Jacobi form of weight 0 and index \(k\). According to theorem 2, it has the following form:

\[
Z[S^kK3](\tau, z) = G_{2k}(E_4(\tau), E_6(\tau), \varphi(\tau, z)) K^{2k}(\tau, z). \tag{4.4}
\]

The homogenous polynomial $G_{2k}$ is determined for lower values of \(k\)\cite{17,19}.

\[
k = 1 : 24 \varphi K^2 \\
k = 2 : (324 \varphi^2 + \frac{3}{4} E_4) K^4 \\
k = 3 : (3200 \varphi^3 + \frac{64}{3} E_4 \varphi + \frac{10}{27} E_6) K^6 
\tag{4.5}
\]

### 4.2 Elliptic genus of $S^kK3$ and characters of N=4 superconformal algebra

The N=(4,4) \(\sigma\)-model on the target space of $K3$ has been analyzed in detail by Eguchi et.al.\cite{19} in the context of a compactification of string theory on $K3$. The elliptic extension of the Hirzebruch signature of $K3$ can be represented by the characters of the N=4 SCA as

\[
Z[K3](\tau, \frac{1}{2}) = -2 \left( \text{ch}_0^{(R)k=1}(h = \frac{1}{4}, l = \frac{1}{2}; \tau, 0) + 20 \right. \left. \text{ch}_0^{(R)k=1}(h = \frac{1}{4}, l = 0; \tau, 0) \right)
\]

It is worth commenting that the coefficient of the first term in the bracket is the Euler number of $S^kK3$, $\chi(S^kK3)$.
\[ F(\tau) \ ch^{(R)k=1}(h = \frac{1}{4}, l = \frac{1}{2}; \tau, 0) \]
\[ = 24 \ ch_0^{(R)k=1}(h = \frac{1}{4}, l = 0; \tau, 0) + \bar{F}(\tau) \ ch^{(R)k=1}(h = \frac{1}{4}, l = \frac{1}{2}; \tau, 0), \quad (4.6) \]

where
\[ \bar{F}(\tau) = -2 + F(\tau) = \sum_{n=0}^{\infty} a_n q^n \quad (a_0 = -2, \ a_n \in \mathbb{Z}_{\geq 0} (n > 0)). \quad (4.7) \]

We have used eq. (3.8) to obtain the last equality in (4.6). The degeneracy of the massive primary states in the left-moving sector is encoded in \( \bar{F}(\tau) \). The coefficient \( a_n \) is the degeneracy of the massive primary states of \( h = n + 1/4 \).

The function \( \bar{F}(\tau) \) can be determined by combining two expressions of elliptic genus (4.2) and (4.6),
\[ Z[K3](\tau, \frac{1}{2}) = 24 \ ch_0^{(R)k=1}(h = \frac{1}{4}, l = 0; \tau, 0) \]
\[ + \bar{F}(\tau) \ ch^{(R)k=1}(h = \frac{1}{4}, l = \frac{1}{2}; \tau, 0) \]
\[ = 24 \ \wp(\tau, \frac{1}{2}) K^2(\tau, \frac{1}{2}). \quad (4.8) \]

This gives
\[ \bar{F}(\tau) = 2 \ \frac{\theta_2(\tau, 0)^4 - \theta_4(\tau, 0)^4}{\prod_{n=1}^{\infty}(1 - q^n)^3} - 24 \ h_3(\tau), \quad (4.9) \]
where
\[ h_3(\tau) = \frac{1}{\theta_3(\tau, 0)} \sum_{m = -\infty}^{\infty} \frac{q^{m^2}}{1 + q^{m - \frac{1}{2}}}. \quad (4.10) \]

Now, we will turn to the case of the symmetric product. The elliptic genus of \( S^k K3 \) can be also expanded by the characters of the underlying N=4 SCA. In general, it has the following expansion:
\[ Z[S^k K3](\tau, z + \frac{1}{2}) \quad (= \text{Tr}(-1)^{2J_0^3} q^{L_0-\hat{F}_0} g^{L_0-\hat{F}_0} g^{2J_0^3}) \]
\[ = \chi(S^k K3) \ ch_0^{(R)k}(h = \frac{k}{4}, l = 0; \tau, z) + \sum_{l=\frac{1}{2}}^{k} F_i(\tau) \ ch^{(R)k}(h = \frac{k}{4}, l; \tau, z), \quad (4.11) \]
where
\[ F_i(\tau) = \sum_{n=0}^{\infty} a_n^{(i)} q^n, \quad (a_0^{(i)} \in \mathbb{Z}, \ a_n^{(i)} \in \mathbb{Z}_{\geq 0} (n > 0)). \quad (4.12) \]
Although the characters of the massless representations with \( l \neq 0 \) may appear in \( Z[S^k K3](\tau, z + 1/2) \), we can reduce them to \( l = 0 \) by applying (3.8) recursively. \( a_n^{(l)} \) in (4.12) provides the degeneracy of the representation \( \mathcal{M}_{(h,l)}^k \otimes \tilde{\mathcal{M}}_{0(l)}^k \) with \( h = n + k/4 \) and isospin \( l \). \( \sum_l a_n^{(l)} \) provides the number of the representations \( \mathcal{M}_{(h,l)}^k \otimes \tilde{\mathcal{M}}_{0(l)}^k \) with \( h = n + k/4 \).

Because of eq.(4.14), the spectrum of the massive primary states of this \( \sigma \)-model on \( S^k K3 \) also becomes discrete and all states have dimension \( h = Z_{\geq 0} + k/4 \) in \( Z[S^k K3](\tau, z + 1/2) \). This corresponds to the fact the extremal BTZ black holes have the discrete mass \( N/l \) in the context of the Maldacena duality.

The function \( F_l(\tau) \) (or \( \sum_l F_l(\tau) \)) in eq.(4.11) can be determined in principle by an analogous way with the case of \( K3 \). But it is a hard task to obtain the exact functional form of \( Z[S^k K3](\tau, z) \) such as (1.3) for the case of general \( k \). However, as discussed in section 2.2, what we need for the counting of the microscopic states comparable with the Bekenstein-Hawking entropy is the asymptotic form of \( a_n^{(l)} \) (or \( \sum_l a_n^{(l)} \)) at \( n \to \infty \), since eq.(2.21) is valid for the region \( N \gg Q_1 Q_5 \gg 1 \). We will consider this asymptotic form in the next section.

5 State counting via the \( N=(4,4) \) \( \sigma \)-model

In this section, we will discuss the degeneracy of 1/8 BPS states of the D1-D5 system in IIB superstring theory and the degeneracy of the primary states corresponding to the extremal BTZ black holes. In the previous section, we obtain two different expressions of the elliptic genus of \( S^k K3 \). We will first use the expression in terms of a weak Jacobi form and count the number of the 1/8 BPS states by using a Tauberian theorem. Then using the expression in terms of the characters of N=4 SCA, we will discuss the degeneracy of the massive primary states and obtain the microscopic entropy of the corresponding extremal BTZ black holes.
5.1 Counting the 1/8 BPS states

Let us start by studying the asymptotic behavior of the elliptic genus $Z[S^kK^3](\tau, 1/2)$ as $\tau \downarrow 0$. Due to the structure theorem the elliptic genus has the form (4.4)

$$Z[S^kK^3](\tau, \frac{1}{2}) = G_{2k} \left( E_4(\tau), E_6(\tau), \wp(\tau, \frac{1}{2}) \right) K^{2k}(\tau, \frac{1}{2}).$$

The asymptotics can be obtained from those of the constituents in (4.4). $\wp(\tau, 1/2)$, $E_4(\tau)$ and $E_6(\tau)$ behave as $\wp(\tau, 1/2) \to (-1/12)(-i\tau)^{-2}$, $E_4(\tau) \to 1(-i\tau)^{-4}$, and $E_6(\tau) \to (-1)(-i\tau)^{-6}$. Therefore the asymptotics of $G_{2k}$ becomes

$$G_{2k}(\tau, \frac{1}{2}) \to \tilde{c}(k)(-i\tau)^{-2k} \text{ as } \tau \downarrow 0,$$

where $\tilde{c}(k)$ is a constant which depends on $k$ and the polynomial form of $G_{2k}$. The asymptotics of $K^{2k}(\tau, 1/2)$ can be read from $\theta_2(\tau, 0) \to 1(-i\tau)^{-\frac{1}{2}}$ and $\eta(\tau) \to 1(-i\tau)^{-\frac{1}{2}}e^{-\frac{\pi}{12}\tau}$,

$$K^{2k}(\tau, \frac{1}{2}) \to (-1)(-i\tau)^2 e^{\frac{\pi i k}{4}} \text{ as } \tau \downarrow 0.$$  

Therefore, gathering these asymptotics, we obtain

$$Z[S^kK^3](\tau, \frac{1}{2}) \to c(k)(-i\tau)^0 e^{\frac{\pi i k}{4}} \text{ as } \tau \downarrow 0. \quad \left( c(k) = (-1)^k\tilde{c}(k) \right)$$

The elliptic genus has the Fourier expansion of the form

$$Z[S^kK^3](\tau, \frac{1}{2}) = \sum_{n=0}^{\infty} a_n q^n. \quad (q = e^{2\pi i \tau})$$

Again, due to the structure theorem, the coefficients satisfy $a_n \leq a_{n+1}$. (See Appendix A for the explicit Fourier expansions of various functions.) Each coefficient $a_n$ represent the number of the 1/2 BPS states of $h = n + k/4$ in the N=(4,4) $\sigma$-model. This is the number of the 1/8 BPS states of the mass specified by $h = n + k/4$ in the D1-D5 system [1, 14]. We can estimate the asymptotic form of $a_n$ by using the following Tauberian theorem [21, 22].

**Theorem 3.** Let $f(\tau)$ be a function

$$f(\tau) = q^\lambda \sum_{n=0}^{\infty} a_n q^n \quad (q = e^{2\pi i \tau})$$

which satisfies following conditions:

$^{12} \tau \downarrow 0 \iff \tau = iT \; (T \in \mathbb{R}_{>0}), \; \text{and} \; T \to 0.$
(i) $f(\tau)$ is a holomorphic function on $\mathbb{H}^+$. 

(ii) $a_n \in \mathbb{R}$ and $a_n \leq a_{n+1}$ for all $n$. 

(iii) There exist $c \in \mathbb{C}, \quad d \in \mathbb{R}$ and $N \in \mathbb{R}_{>0}$ such that 

$$f(\tau) \to c(-i\tau)^{-d}e^{\frac{2\pi i N}{\tau}}$$ 

as $\tau \downarrow 0$. 

Then, the behavior of $a_n$ at large $n$ is 

$$a_n \sim \frac{c}{\sqrt{2}} N^{-\frac{1}{2}(d-\frac{1}{2})} \frac{n^{\frac{1}{2}(d-\frac{3}{2})}}{2^{\frac{1}{4}}} e^{2\pi \sqrt{4Nn}}$$ 

as $n \to \infty$, 

where $a_n \sim b_n$ as $n \to \infty$ means $\lim_{n \to \infty} b_n/a_n = 1$. 

Due to this theorem, the asymptotic form of $a_n$ can be read from the estimation 

(5.3) 

$$a_n \sim \frac{c(k)}{\sqrt{2}} \left(\frac{k}{4}\right)^{\frac{1}{4}} n^{-\frac{3}{4}} e^{2\pi \sqrt{kn}}. \quad (5.5)$$ 

5.2 Counting the massive primary states 

We can also expand the elliptic genus by the characters of $\mathbb{N}=4$ SCA 

$$Z[S^kK3](\tau, \frac{1}{2})$$ 

$$= \chi(S^kK3) \text{ ch}^{(R)k}(h = \frac{k}{4}, l = 0; \tau, 0) + \sum_{l=\frac{1}{2}}^{\frac{k}{4}} F_l(\tau) \text{ ch}^{(R)k}(h = \frac{k}{4}, l; \tau, 0). \quad (5.6)$$ 

Each coefficient of $F_l(\tau) = \sum_{n=0}^{\infty} a_n^{(l)} q^n$ counts the number of the massive representation $M_{(h=n+k/4,l)}^k$ in $Z[S^kK3](\tau, z)$ of the $\mathbb{N}=(4,4)$ $\sigma$-model. In particular $\sum_{l} a_n^{(l)}$ will be identified with the number of the primary state (2.22). To obtain the asymptotic form $\sum_{l} a_n^{(l)}$, we may again utilize the Tauberian theorem. For this purpose we need to know the asymptotic behavior of $\sum_{l} F_l(\tau)$ as $\tau \downarrow 0$. 

Since we have obtained the asymptotics of the elliptic genus (3.3), the asymptotics of $\sum_{l} F_l(\tau)$ becomes tractable if we can properly estimate the constituents of the massless and massive characters. Let us remind that the character of massive 

\footnote{We can expect $c(k)$ is not large number due to the explicit example of lower $k$. \cite{[17].}}
representation $M^k_{(h,0)}$ is given by eq. (3.3). The asymptotic behavior of the character of SU(2)$_k$ of the isospin $l$, eq. (3.4), is given by [21, 22]

$$
\chi^l_k(\tau, 0) \to a(k, l) \exp \left( \frac{\pi i}{12\tau} c_k \right) \quad \text{as } \tau \downarrow 0,
$$

where

$$
a(k, l) = \sqrt{\frac{2}{k+2}} \sin \left( \frac{(2l+1)\pi}{k+2} \right),
$$

$$
c_k = \frac{3k}{k+2}.
$$

(5.7)

Therefore, combining those of $\theta_2(\tau, 0)$ and $\eta(\tau)$, we obtain

$$
\text{ch}^{(R)k}(h = \frac{k}{4}; l, \tau, 0) \to a(k-1, l-\frac{1}{2})(-i\tau)^{\frac{3}{2}} \exp \left( \frac{\pi i}{12\tau} (3 + c_{k-1}) \right) \quad \text{as } \tau \downarrow 0.
$$

(5.9)

As for the character of massless representation $M^k_{0(l=0)}$, we can obtain the upper bound of the asymptotic behavior by means of eq. (3.8):

$$
\text{ch}^{(R)k}(h = \frac{k}{4}; l = 0; \tau, 0)|_{\tau \downarrow 0} \leq \text{ch}^{(R)k}(h = \frac{k}{4}; l = \frac{1}{2}; \tau, 0)|_{\tau \downarrow 0}
$$

$$
= a(k, l=0)(-i\tau)^{\frac{3}{2}} \exp \left( \frac{\pi i}{12\tau} (3 + c_{k-1}) \right),
$$

(5.10)

where $f(\tau)|_{\tau \downarrow 0}$ means the leading asymptotic of $f(\tau)$ as $\tau \downarrow 0$. From this estimation, the dominant contribution of the asymptotic behavior of $Z[S^kK3](\tau, 1/2)$ turns out to come from the part of the massive representations. We can neglect the contribution of the massless representations in the asymptotics.

Now we can obtain the asymptotic behavior of $\sum_l \tilde{F}_l(\tau)$. Taking the limit $\tau \downarrow 0$ in eq. (5.6),

$$
Z[S^kK3](\tau, 0) \rightarrow \sum_{l=\frac{1}{2}}^{\frac{k}{2}} \tilde{F}_l(\tau)|_{\tau \downarrow 0} \times (-i\tau)^{\frac{3}{2}} \exp \left( \frac{\pi i}{12\tau} (3 + c_{k-1}) \right) \quad \text{as } \tau \downarrow 0.
$$

(5.11)

where $\tilde{F}_l(\tau) = \sum_{n=0}^{\infty} a_n^{(l)} q^n = a(k-1, l-1/2) F_l(\tau)$. The asymptotic behavior of $\sum_l \tilde{F}_l(\tau)$ as $\tau \downarrow 0$ can be read by comparing (5.11) with (5.3)

$$
\sum_{l=\frac{1}{2}}^{\frac{k}{2}} \tilde{F}_l(\tau) \to c(k)(-i\tau)^{\frac{3}{2}} \exp \left( \frac{\pi i(6k - (3 + c_{k-1}))}{12\tau} \right) \quad \text{as } \tau \downarrow 0.
$$

(5.12)
\[ \sum_l \tilde{F}_l(\tau) \text{ has the Fourier expansion of the form} \]

\[ \sum_{l=\frac{1}{2}}^{\frac{k}{2}} \tilde{F}_l(\tau) = \sum_{n=0}^{\infty} b_n q^n. \]

\[ \left( b_n = \sum_{l=\frac{1}{2}}^{\frac{k}{2}} \tilde{a}_n^{(l)} \right) \]  \hspace{1cm} (5.13)

Due to the Tauberian theorem, the asymptotic form of \( b_n \) becomes:

\[ b_n = \sum_{l=\frac{1}{2}}^{\frac{k}{2}} \tilde{a}_n^{(l)} \sim \frac{c(k)}{\sqrt{2}} n^{-\frac{1}{2}} \exp \left( 2\pi \sqrt{(k - \frac{(3 + c_{k-1})}{6})n} \right) \text{ as } n \to \infty. \]  \hspace{1cm} (5.14)

Therefore, we conclude that the degeneracy of the massive primary state of the dimension \( h = n + k/4 \) at \( n \to \infty \) is given by eq.(5.14).

According to the argument in section 3.2, the degeneracy of the primary state (3.3) corresponds to the degeneracy of the microscopic states of the extremal BTZ black hole BTZ\(_{(N,N/l)}\). At the limit \( N \gg k \gg 1 \), that is, the semiclassical limit of three-dimensional gravity, the degeneracy of the state (2.22) becomes

\[ \sum_{l=\frac{1}{2}}^{\frac{k}{2}} \tilde{a}_N^{(l)} \sim \frac{c(k)}{\sqrt{2}} N^{-\frac{1}{2}} \exp(2\pi \sqrt{(k - 1)N}) \text{ as } N \to \infty, \]  \hspace{1cm} (5.15)

where \( k = Q_1 Q_5 + 1 \). The logarithm of eq.(5.15) can be regarded as the microscopic entropy of the extremal BTZ black hole with \( Ml = J = N \). It becomes

\[ S_{\text{micro}} = 2\pi \sqrt{Q_1 Q_5 N} + \mathcal{O}(\log N, \log c(k)). \]  \hspace{1cm} (5.16)

This completely agrees with the entropy formula eq.(2.21). This provides a justification of the identification of the extremal BTZ black hole states with the primary states (2.22) of the N=(4,4) \( \sigma \)-model.

### 6 Discussion

Until now, our study is limited to the case of the extremal BTZ black holes. The non-extremal BTZ black holes can also appear as the near horizon geometry of the

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14These coefficients also satisfy \( b_n \leq b_{n+1} \).

15The difference between \( \sum_l \tilde{a}_n^{(l)} \) and \( \sum l^s \tilde{a}_n^{(l)} \) is irrelevant in this semiclassical limit.
non-BPS solitonic solutions in IIB supergravity. According to the duality, the non-extremal BTZ black holes may be also identified with the Virasoro primary states

\[ |h, l = 0\rangle \otimes |\tilde{h}, \tilde{l} = 0\rangle \quad \text{with} \quad h, \tilde{h} > \frac{k}{4}, \quad (6.1) \]

in the corresponding N=(4,4) \( \sigma \)-model.

These states are in the tensor product of the massive representations both in the left and right moving sectors. So we must consider not the elliptic genus but the full partition function of the N=(4,4) \( \sigma \)-model for the counting of the degeneracy of the states. It is known that the full partition function, which depends on the moduli of \( S^{k}K3 \), has the contributions from the massive primary states of \( h = Q_{>0} + k/4 \) (\( Q \) : rational numbers) \[13\]. Therefore, the partition function and the counterparts of \( \sum_l F_l(\tau) \) can not have the forms \( (5.4) \) and \( (5.13) \). So the Tauberian theorem can not be applied to the counting of the primary state \( (5.1) \). We need the further investigations for the well-defined counting of the microscopic states of the non-extremal BTZ black holes.

Through this paper, we have discussed only the case of \( M_4 = K3 \). The similar arguments in section 2.2 hold for the case of \( M_4 = T^4 \). However the elliptic genus of the corresponding N=(4,4) \( \sigma \)-model vanishes identically, since this \( \sigma \)-model has the extra \( U(1)^4 \) symmetry other than N=4 superconformal symmetry. So one cannot count the degeneracy of the state \( (2.22) \) by means of the elliptic genus. In \[23\], the counting of the 1/8 BPS states has been argued by using another topological index called new supersymmetric index. And they pointed out that the representations of large N=4 superconformal algebra must be considered. We may expect that the similar argument in this paper can be carried out via the relation between this new supersymmetric index and the characters of large N=4 superconformal algebra.

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A Appendix

Some formulas of elliptic theta functions and modular functions used in the text are summarized\(^{24, 17}\).

Elliptic theta functions and Dedekind’s \(\eta\)-function are defined by\(^{16}\)

\[
\begin{align*}
\theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n^{2} + n)} y^{n}, \quad (A.1) \\
\theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n^{2} + n)} y^{n} \left( = \theta_1(\tau, z + \frac{1}{2}) \right), \quad (A.2) \\
\theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2} n^2} y^{n}, \quad (A.3) \\
\theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2} (n^{2} + n)} y^{n} \left( = \theta_4(\tau, z + \frac{1}{2}) \right), \quad (A.4) \\
\eta(\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2} n^2 (n+1)} \left( = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \right). \quad (A.5)
\end{align*}
\]

Their modular transforms become

\[
\begin{align*}
\theta_1(-\frac{1}{\tau}, \tau z) &= -i(-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z}{\tau}} \theta_1(\tau, z), \quad \theta_2(-\frac{1}{\tau}, \tau z) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z}{\tau}} \theta_4(\tau, z), \\
\theta_3(-\frac{1}{\tau}, \tau z) &= (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z}{\tau}} \theta_3(\tau, z), \quad \theta_4(-\frac{1}{\tau}, \tau z) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z}{\tau}} \theta_2(\tau, z), \\
\eta(-\frac{1}{\tau}) &= (-i\tau)^{\frac{1}{2}} \eta(\tau). \quad (A.6)
\end{align*}
\]

Weierstrass’s \(\wp\)-function has the following form in terms of these functions.

\[
\wp(\tau, z) = -\frac{1}{3} \sum_{i=1}^{3} \left( \frac{\theta_{i+1}(\tau, z)}{\theta_{i+1}(\tau, 0)} \right)^2 \frac{\eta(\tau)^{6}}{\theta_1(\tau, z)^2}, \quad (A.7)
\]

and, in particular, if we set \(z = 1/2\),

\[
\wp(\tau, \frac{1}{2}) = -\frac{1}{12} \left( \theta_3(\tau, 0)^4 + \theta_4(\tau, 0)^4 \right). \quad (A.8)
\]

The Eisenstein series of weight \(k\) is defined by

\[
E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (k \geq 4), \quad (A.9)
\]

\(^{16}q = e^{2\pi i \tau}\) and \(y = e^{2\pi iz}\).
where $B_k$ are the Bernoulli numbers and $\sigma_k(n) = \sum_{d|n} d^k$. These series satisfy the following property under modular transformation.

$$E_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k E_k(\tau) \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{SL}_2(\mathbb{Z}).$$ (A.10)

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