HAUSDORFF OPERATORS ON BERGMAN SPACES
OF THE UPPER HALF PLANE

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Abstract. In this paper we study Hausdorff operators on the
Bergman spaces $A^p(U)$ of the upper half plane.

1. INTRODUCTION

Given a $\sigma$-finite positive Borel measure $\mu$ on $(0, \infty)$, the associated
Hausdorff operator $H_\mu$, for suitable functions $f$ is given by

$$H_\mu(f)(z) := \int_0^\infty \frac{1}{t} f\left(\frac{z}{t}\right) \, d\mu(t), \quad z \in \mathbb{U}$$

where $\mathbb{U} = \{z \in \mathbb{C} : \text{Im} \, z > 0\}$ is the upper half plane. Its formal
adjoint, the quasi-Hausdorff operator $H_\mu^*$ in the case of real Hardy
spaces $H^p(\mathbb{R})$ is

$$H_\mu^*(f)(z) := \int_0^\infty f(tz) \, d\mu(t).$$

Moreover for appropriate functions $f$ and measures $\mu$ they satisfy
the fundamental identity:

$$\hat{H_\mu(f)}(x) = \int_0^\infty \hat{f}(tx) \, d\mu(t) = H_\mu^*(\hat{f})(x), \quad x \in \mathbb{R},$$

where $\hat{f}$ denotes the Fourier transform of $f$.

The theory of Hausdorff summability of Fourier series started with
the paper of Hausdorff [Ha21] in 1921. Much later Hausdorff summabili-
ity of power series of analytic functions was considered in [Si87] and
[Si90] on composition operators and the Cesáro means in Hardy $H^p$
spaces. General Hausdorff matrices were considered in [GaSi01] and
[GaPa06]. In [GaPa06] the authors studied Hausdorff matrices on a
large class of analytic function spaces such as Hardy spaces, Bergman
spaces, BMOA, Bloch etc. They characterized those Hausdorff matrices
which induce bounded operators on these spaces.

Results on Hausdorff operators on spaces of analytic functions were
extended in the Fourier transform setting on the real line, starting with
[LiMo00] and [Ka01]. There are many classical operators in analysis
which are special cases of the Hausdorff operator if one chooses suitable
measures $\mu$ such as the classical Hardy operator, its adjoint operator, the Cesàro type operators and the Riemann–Liouville fractional integral operator. See the survey article [Li013] and the references therein. In recent years, there is an increasing interest on the study of boundedness of the Hausdorff operator on the real Hardy spaces and Lebesque spaces (see for example [An03], [BaGo19], [FaLi14], [LiMo01] and [HuKyQu18]).

Motivated by the paper of Hung et al. [HuKyQu18] we describe the measures $\mu$ that will induce bounded operators on the Bergman spaces $A^p(U)$ of the upper half-plane. Next Theorem summarizes the main results (see Theorems 3.5 and 3.7):

**Theorem 1.1.** Let $1 \leq p < \infty$ and $\mu$ be an $\sigma$-finite positive measure on $(0, \infty)$. The Hausdorff operator $H_\mu$ is bounded on $A^p(U)$ if and only if

$$\int_0^\infty \frac{1}{t^{1-\frac{1}{p}}} d\mu(t) < \infty.$$ 

Moreover

$$||H_\mu||_{A^p(U) \to A^p(U)} = \int_0^\infty \frac{1}{t^{1-\frac{1}{p}}} d\mu(t).$$

### 2. PRELIMINARIES

To define single-valued functions, the principal value of the argument is chosen to be in the interval $(-\pi, \pi]$. For $1 \leq p < \infty$, we denote by $L^p(dA)$ the Banach space of all measurable functions on $U$ such that

$$||f||_{L^p(dA_\alpha)} := \left(\frac{1}{\pi} \int_U |f|^p dA\right)^{1/p} < \infty,$$

where $dA$ is the area measure. The Bergman space $A^p(U)$ consists of all holomorphic functions $f$ on $U$ that belong to $L^p(dA)$. Sub-harmonicity yields a constant $C > 0$ such that

$$|f(z)|^p \leq \frac{C}{(\text{Im}(z))^2} ||f||_{A^p(U)}^p, \quad z \in U,$$

for $f \in A^p(U)$ and

$$\lim_{z \to \partial U} (\text{Im}(z))^2 |f(z)|^p = 0$$

for functions $A^p(U)$, where $\bar{U} := \overline{U} \cup \{\infty\}$ (see [ChKoSm17]). In particular, this shows that each point evaluation is a continuous linear functional on $A^p(U)$.

The duality properties of Bergman spaces are well known in literature see [Zh90] and [BaBoMiMi16]. It is proved that for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, the dual space of the Bergman space $A^p(U)$ is $(A^p(U))^\ast \sim A^q(U)$ under the duality pairing,

$$\langle f, g \rangle = \frac{1}{\pi} \int_U f(z)\overline{g(z)} \, dA(z).$$
3. Main results

In what follows, unless otherwise stated, \(\mu\) is a positive \(\sigma\)-finite measure on \((0, \infty)\). We start by giving a condition under which \(H_\mu\) is well defined.

**Lemma 3.1.** Let \(1 \leq p < \infty\) and \(f \in A^p(U)\). If \(\int_0^\infty \frac{1}{t^{1+p}} \, d\mu(t) < \infty\), then

\[
H_\mu(f)(z) = \int_0^\infty 1 t \frac{f(\frac{Z}{t})}{1} \, d\mu(t)
\]
is a well defined holomorphic function on \(U\).

**Proof.** For \(f \in A^p(U)\) using (4) we have

\[
|H_\mu(f)(z)| \leq \int_0^\infty 1 t \frac{|f(\frac{z}{t})|}{1} \, d\mu(t)
\]

\[
\leq C \frac{||f||_{A^p(U)}}{\text{Im}(z)^{\frac{1}{p}}} \int_0^\infty 1 t^{1+\frac{2}{p}} \, d\mu(t) < \infty.
\]

Thus, \(H_\mu f\) is well defined, and is given by an absolutely convergent integral, so it is holomorphic. \(\square\)

**Lemma 3.2.** Let \(\lambda > 0\) and \(\delta > 0\). If \(g_{\lambda,\delta}(z) = |z + \delta i|^{\frac{-2+\lambda}{p}}\), then

\[
\left(\frac{1}{2}\right)^{2+\lambda} \cdot \frac{1}{\lambda \delta^2} \leq ||g_{\lambda,\delta}||_{L^p(dA)} \leq 2^{2+\lambda} \cdot \frac{1}{\lambda \delta^2}.
\]

**Proof.** Using polar coordinates for the integral over \(U\) we find

\[
||g_{\lambda,\delta}||_{L^p(dA)} = \frac{1}{\pi} \int_U \left| \frac{1}{z + \delta i} \right|^{2+\lambda} \, dA(z)
\]

\[
= \frac{1}{\pi} \int_0^{\pi} \int_0^\infty \left( \frac{1}{r^2 + \delta^2 + 2r \delta \sin(\theta)} \right)^{\frac{2+\lambda}{2}} r \, dr \, d\theta.
\]

Denote by \(I\) the last double integral. Then

\[
I \leq \int_0^\infty \left( \frac{1}{r^2 + \delta^2} \right)^{\frac{2+\lambda}{2}} r \, dr
\]

\[
\leq 2^{\frac{2+\lambda}{2}} \int_0^\infty \left( \frac{1}{r + \delta} \right)^{2+\lambda} (r + \delta) \, dr
\]

\[
= 2^{\frac{2+\lambda}{2}} \cdot \frac{1}{\lambda \delta^2}.
\]
On the other hand,

\[
I \geq \int_0^\infty \left( \frac{1}{r + \delta} \right)^{2+\lambda} r \, dr \\
\geq \int_\delta^\infty \left( \frac{1}{r + \delta} \right)^{2+\lambda} r \, dr \\
\geq \left( \frac{1}{2} \right)^{2+\lambda} \int_\delta^\infty \left( \frac{1}{r} \right)^{2+\lambda} r \, dr \\
= \left( \frac{1}{2} \right)^{2+\lambda} \frac{1}{\lambda \delta^{2+\lambda}},
\]

and the assertion follows. \(\Box\)

3.1. Test functions. We now consider the test functions which are defined as follows. Let \(z = x + iy \in U\) and

\[
\varphi_\varepsilon(z) = \frac{z + \varepsilon i}{|z + \varepsilon i|} = \frac{x - i(y + \varepsilon)}{\sqrt{x^2 + (y + \varepsilon)^2}}
\]

and

\[
f_\varepsilon(z) = \left( \frac{z + \varepsilon i}{|z + \varepsilon i|} \right)^{2+\varepsilon},
\]

with \(\varepsilon > 0\) small enough. Note that, \(|f_\varepsilon| = g_{\varepsilon, \varepsilon}\) with respect to the notation of Lemma 3.2, and that \(\varphi_\varepsilon(z)\) lies on the unit circle with \(-\pi < \arg(\varphi_\varepsilon(z)) < 0\), and the following identity holds

\[
f_\varepsilon(z) = \varphi_\varepsilon(z)^{2+\varepsilon}|f_\varepsilon(z)|.
\]

Let \(a, b \in (-\pi, \pi]\) and set \(A(a, b] = \{z \in U : a \leq \arg(z) \leq b\}\), with obvious modifications in the case of \(A(a, b), A(a, b)\), \(A(a, b)\), and \(A(a, b)\).

**Lemma 3.3.** The following holds:

(i) If \(2 < p < \infty\) and \(\frac{2}{p} + \varepsilon \leq 1\), then

\[
|\text{Re} f_\varepsilon(z)| \geq |\text{Re} \varphi_\varepsilon(z)||f_\varepsilon(z)|
\]

for every \(z \in A_{0, \frac{\pi}{2}}\).

(ii) If \(1 < p \leq 2\) and \(1 < \frac{2}{p} + \varepsilon < 2\), then

\[
|\text{Im} f_\varepsilon(z)| > C(p)|\text{Im} \varphi_\varepsilon(z)||f_\varepsilon(z)|
\]

for every \(z \in A_{\frac{\pi}{4}, \frac{\pi}{2}}\).

(iii) If \(p = 1\), \(0 < \theta_0 < \frac{\pi}{16}\) and \((2 + \varepsilon)(\frac{\pi}{2} + \theta_0) < \frac{5\pi}{4}\), then

\[
|\text{Re} f_\varepsilon(z)| > |\text{Re} \varphi_\varepsilon(z)||f_\varepsilon(z)|
\]

for every \(z \in A_{\frac{\pi}{4}, \frac{\pi}{2} + \theta_0}\).
\textbf{Proof.} Taking real and imaginary parts we have
\[ \text{Re } f_\varepsilon(z) = |f_\varepsilon(z)|\text{Re } \varphi_\varepsilon(z) e^{i\frac{\theta}{p} + \varepsilon} = |f_\varepsilon(z)| \cos\left(\frac{2}{p} + \varepsilon\theta\right) \]
and
\[ \text{Im } f_\varepsilon(z) = |f_\varepsilon(z)|\text{Im } \varphi_\varepsilon(z) e^{i\frac{\theta}{p} + \varepsilon} = |f_\varepsilon(z)| \sin\left(\frac{2}{p} + \varepsilon\theta\right), \]
where \( \theta := \theta(z, \varepsilon) = \arg \varphi_\varepsilon(z) \).

(i): It easy is see that
\[ |\text{Re } f_\varepsilon(z)| = |f_\varepsilon(z)| \cos\left(\frac{2}{p} + \varepsilon\theta\right) \]
\[ \geq |f_\varepsilon(z)| \cos(\theta) = |\text{Re } \varphi_\varepsilon(z)| |f_\varepsilon(z)|. \]

(ii): Let \( a = a(p) > 0 \) such that \( 1 < \frac{2}{p} + \varepsilon < a < 2 \). Since \( z \in A[a, a] \)
simple geometric arguments imply that \( \theta \in [-\frac{\pi}{2}, -\frac{\pi}{4}] \). Moreover \( -\pi < -\frac{a\pi}{2} < (\frac{2}{p} + \varepsilon)\theta < -\frac{\pi}{4} \). This implies that
\[ \min\{\sin\left(\frac{a\pi}{2}\right), \frac{\sqrt{2}}{2}\} < \left| \frac{\sin\left(\frac{2}{p} + \varepsilon\theta\right)}{\sin(\theta)} \right| < \sqrt{2} \]
for every \( z \in A[a, a] \). We calculate
\[ |\text{Im } f_\varepsilon(z)| = |f_\varepsilon(z)||\text{sin}\left(\frac{2}{p} + \varepsilon\theta\right)| > \min\{\sin\left(\frac{a\pi}{2}\right), \frac{\sqrt{2}}{2}\}|f_\varepsilon(z)||\sin(\theta)| \]
\[ = \min\{\sin\left(\frac{a\pi}{2}\right), \frac{\sqrt{2}}{2}\}|\text{Im } \varphi_\varepsilon(z)||f_\varepsilon(z)|. \]
This proves (ii) with \( C(p) = \min\{\sin\left(\frac{a(p)\pi}{2}\right), \frac{\sqrt{2}}{2}\} \).

(iii): Since \( z \in A[a, a + \theta_0) \) we have that \( \theta \in (-\frac{\pi}{2} - \theta_0, -\frac{\pi}{4}) \). Thus
\[ -\frac{5\pi}{4} < -\left(\frac{\pi}{2} + \theta_0\right)(2 + \varepsilon) < (2 + \varepsilon)\theta < -\frac{\pi}{2}(2 + \varepsilon) < -\pi. \]
This implies that \( |\cos((2 + \varepsilon)\theta)| > |\cos(\theta)| \) and therefore
\[ |\text{Re } f_\varepsilon(z)| > |\text{Re } \varphi_\varepsilon(z)||f_\varepsilon(z)|. \]

3.2. Growth estimates. Let \( a, b \in (-\pi, \pi] \) and set
\[ S_{[a, b]} = \{z \in \mathbb{U} : a \leq \arg(z) \leq b, \ |z| \geq 1\}, \]
be a truncated sector with obvious modifications in the case of \( S_{(a, b)}, S_{(a, b]} \)
and \( S_{(a, b)} \). Since \( \mu \) is positive
\[ \text{Re } \mathcal{H}_\mu(f_\varepsilon) = \mathcal{H}_\mu(\text{Re } f_\varepsilon) \quad \text{and} \quad \text{Im } \mathcal{H}_\mu(f_\varepsilon) = \mathcal{H}_\mu(\text{Im } f_\varepsilon). \]
Note that if \( \text{Re } f_\varepsilon \) or \( \text{Im } f_\varepsilon \) have constant sign on some sector \( A \), then
\[ |\mathcal{H}_\mu(\text{Re } f_\varepsilon)(z)| = \mathcal{H}_\mu(|\text{Re } f_\varepsilon|)(z) \quad \text{and} \quad |\mathcal{H}_\mu(\text{Im } f_\varepsilon)(z)| = \mathcal{H}_\mu(|\text{Im } f_\varepsilon|)(z) \]
for every $z \in A$.

**Lemma 3.4.** Let $1 \leq p < \infty$ and suppose that $\mathcal{H}_\mu$ is bounded on $A^p(U)$. Then there are $\varepsilon(p)$ and $k(p)$ positive constants such that

$$
||\mathcal{H}_\mu(f_\varepsilon)||_{A^p(U)}^p \geq k(p) \left( \int_0^{\frac{1}{p}} \frac{1}{t^{1-\frac{2}{p}-\varepsilon}} d\mu(t) \right)^p \frac{1}{p\varepsilon},
$$

for every $\varepsilon$ in $(0, \varepsilon(p)]$.

**Proof.** We will consider three cases for the range of $p$. Note that if $z$ is in a truncated sector $S$ then $z/t$ belongs to the corresponding sector $A$ for every $t > 0$.

**Case I.** Let $2 < p < \infty$ and $\varepsilon(p)$ such that $\frac{2}{p} + \varepsilon(p) < 1$. Then for every $\varepsilon$ in $(0, \varepsilon(p)]$

$$
||\mathcal{H}_\mu(f_\varepsilon)||_{A^p(U)}^p \geq ||\text{Re} \mathcal{H}_\mu(f_\varepsilon)||_{A^p(U)}^p = ||\mathcal{H}_\mu(\text{Re} f_\varepsilon)||_{A^p(U)}^p \\
\geq \frac{1}{\pi} \int_{S(0,\frac{\pi}{2})} \left( \int_0^{\frac{1}{\pi}} \frac{1}{t^{1-\frac{2}{p}-\varepsilon}} d\mu(t) \right)^p dA(z) \\
= \frac{1}{\pi} \int_{S(0,\frac{\pi}{2})} \left( \int_0^{\frac{1}{\pi}} |\text{Re} f_\varepsilon(z/t)| d\mu(t) \right)^p dA(z).
$$

Denote by $I$ the last integral on $S(0,\frac{\pi}{2})$. By (i) of Lemma 3.3 we have

$$
I \geq \frac{1}{\pi} \int_{S(0,\frac{\pi}{2})} \left( \int_0^{\frac{1}{\pi}} \frac{1}{t^{1-\frac{2}{p}-\varepsilon}} |\text{Re} \varphi_\varepsilon(z/t)\| f_\varepsilon(z/t)\| d\mu(t) \right)^p dA(z) \\
= \frac{1}{\pi} \int_{S(0,\frac{\pi}{2})} \left( \int_0^{\frac{1}{\pi}} \frac{1}{\sqrt{x^2 + (y + t\varepsilon)^2}} (\frac{2+\varepsilon+1}{\frac{2}{p}+1}) d\mu(t) \right)^p \frac{1}{t^{1-\frac{2}{p}-\varepsilon}} x^p dxdy.
$$
Using polar coordinates and noting that $t \varepsilon < |z|$ for $|z| \geq 1$ and $t \leq \varepsilon^{-1}$, we have

\[
I \geq \int_1^\infty \int_0^{\frac{\bar{z}}{r}} \left( \int_0^{\frac{1}{t}} \left( \frac{1}{\sqrt{r^2 + 2rt\varepsilon \sin \theta + t^2\varepsilon^2}} \right)^{\frac{2}{p} + \varepsilon + 1} \frac{d\mu(t)}{t^{1+\frac{2}{p}-\varepsilon}} \right)^p r^{p+1}(\cos(\theta))^p \frac{d\theta}{\pi} dr 
\]

\[
\geq \int_1^\infty \int_0^{\frac{\bar{z}}{r}} \left( \int_0^{\frac{1}{t}} \left( \frac{1}{\sqrt{r^2 + 2rt\varepsilon + t^2\varepsilon^2}} \right)^{\frac{2}{p} + 1} \frac{d\mu(t)}{t^{1+\frac{2}{p}-\varepsilon}} \right)^p r^{p+1}(\cos(\theta))^p \frac{d\theta}{\pi} dr 
\]

\[
\geq k(p) \left( \int_0^{\frac{1}{t}} \frac{1}{t^{1+\frac{2}{p}-\varepsilon}} d\mu(t) \right)^p \int_1^\infty \frac{1}{r^{1+p\varepsilon}} dr 
\]

\[
= k(p) \left( \int_0^{\frac{1}{t}} \frac{1}{t^{1+\frac{2}{p}-\varepsilon}} d\mu(t) \right)^p \frac{1}{p\varepsilon},
\]

where $k(p) = 2^{-p(\varepsilon+1)} \int_0^{\frac{\pi}{2}} (\cos(\theta))^p \frac{d\theta}{4\pi}$.

**Case II.** Let $2 < p < \infty$ and $\varepsilon(p)$ such that $1 < \frac{2}{p} + \varepsilon(p) < 2$. Then for every $\varepsilon$ in $(0, \varepsilon(p)]$

\[
||H_\mu(f_\varepsilon)||_{L^p(U)}^p \geq \frac{1}{\pi} \int_{S_{\frac{\pi}{4}, \frac{\pi}{2}}} \left| \int_0^\infty \frac{1}{t} \text{Im} f_\varepsilon(z/t) \, d\mu(t) \right|^p \, dA(z)
\]

\[
= \frac{1}{\pi} \int_{S_{\frac{\pi}{4}, \frac{\pi}{2}}} \left( \int_0^\infty \frac{1}{t} |\text{Im} f_\varepsilon(z/t)| \, d\mu(t) \right)^p \, dA(z).
\]

Denote by $I$ the last integral on $S_{\frac{\pi}{4}, \frac{\pi}{2}}$. By (ii) of Lemma 3.3 we have

\[
I \geq \frac{C(p)}{\pi} \int_{S_{\frac{\pi}{4}, \frac{\pi}{2}}} \left( \int_0^{\frac{1}{t}} \frac{1}{t} |\text{Im} \varphi_\varepsilon(z/t)| \, |f_\varepsilon(z/t)| \, d\mu(t) \right)^p \, dA(z)
\]

\[
\geq \frac{C(p)}{\pi} \int_{S_{\frac{\pi}{4}, \frac{\pi}{2}}} \left( \int_0^{\frac{1}{t}} \frac{1}{\sqrt{x^2 + (y + t\varepsilon)^2}} \frac{d\mu(t)}{t^{1+\frac{2}{p}-\varepsilon}} \right)^p y^p \, dx dy.
\]

Using polar coordinates and working as is Case I, we arrive at the desired conclusion with constant $k(p) = C(p)2^{-p(\varepsilon+1)} \int_0^{\frac{\pi}{2}} (\sin(\theta))^p \frac{d\theta}{4\pi}.$
Case III. Let \( p = 1 \) and \( \theta_0 \) as in Lemma 3.3. Let \( \varepsilon(1) \) such that 
\[
(2 + \varepsilon(1))(\frac{\pi}{2} + \theta_0) < \frac{5\pi}{4}.
\]
Then for every \( \varepsilon \) in \((0, \varepsilon(1)[
\|
H_\mu(f_\varepsilon)\|_{A^p(U)} \geq \frac{1}{\pi} \int_{S(\frac{\pi}{2} + \theta_0)} \left| \int_0^\infty \frac{1}{t} \Re f_\varepsilon(z/t) \, d\mu(t) \right| \, dA(z) 
\]
\[
= \frac{1}{\pi} \int_{S(\frac{\pi}{2} + \theta_0)} \int_0^\infty \frac{1}{t} \left| \Re f_\varepsilon(z/t) \right| \, d\mu(t) \, dA(z).
\]

Denote by \( I \) the last integral on \( S(\frac{\pi}{2} + \theta_0) \). By \((iii)\) of Lemma 3.3 we have
\[
I \geq \frac{1}{\pi} \int_{S(\frac{\pi}{2} + \theta_0)} \int_0^\frac{\varepsilon}{\varepsilon} \frac{1}{t} \left| \Re \varphi_\varepsilon(z/t) \right| \left| f_\varepsilon(z/t) \right| \, d\mu(t) \, dA(z) 
\]
\[
= \frac{1}{\pi} \int_{S(\frac{\pi}{2} + \theta_0)} \int_0^\frac{\varepsilon}{\varepsilon} \left( \frac{1}{\sqrt{x^2 + (y + t\varepsilon)^2}} \right)^{3+\varepsilon} \frac{d\mu(t)}{t^{1-\varepsilon}} \, (-x) \, dx \, dy.
\]

Using polar coordinates and working as is Case I, we arrive at the desired conclusion with constant \( k(1) = -2^{-(\varepsilon+1)} \int_{\frac{\pi}{2} + \theta_0} \cos(\theta) \, d\theta \).

\[\square\]

**Theorem 3.5.** Let \( 1 \leq p < \infty \). The operator \( \mathcal{H}_\mu \) is bounded on \( A^p(U) \) if and only if
\[
\int_0^\infty \frac{1}{t^{1-\frac{1}{p}}} \, d\mu(t) < \infty.
\]

**Proof.** Suppose that
\[
\int_0^\infty \frac{1}{t^{1-\frac{1}{p}}} \, d\mu(t) < \infty,
\]
then Lemma 3.1 implies that \( \mathcal{H}_\mu(f) \) is well defined and holomorphic in \( U \). An easy computation involving the Minkowski inequality shows that for all \( 1 \leq p < \infty \)
\[
||\mathcal{H}_\mu(f)||_{A^p(U)} = \left( \int_{U} \left| \int_0^\infty \frac{1}{t} \left( \frac{z}{t} \right) \, d\mu(t) \right|^p \, dA(z) \right)^{1/p} 
\]
\[
\leq ||f||_{A^p(U)} \int_0^\infty \frac{1}{t^{1-\frac{1}{p}}} \, d\mu(t) < \infty.
\]
Thus \( \mathcal{H}_\mu \) is bounded on \( A^p(U) \).

Conversely, suppose that \( \mathcal{H}_\mu \) is bounded. Let \( f_\varepsilon(z) = (z + \varepsilon i)^{-\left(\frac{\pi}{2} + \varepsilon\right)} \) with \( \varepsilon > 0 \) small enough. By Lemma 3.2
\[
||f_\varepsilon||_{A^p(U)} \sim \frac{1}{p\varepsilon}\varepsilon^\frac{1}{p}.
\]
Moreover Lemma 3.4 implies that there is a constant $k = k(p) > 0$ such that

$$
\|f_\varepsilon\|_{A^p(U)} \|H_\mu\|_{A^p(U) \to A^p(U)} \geq \|H_\mu(f_\varepsilon)\|_{A^p(U)}^p \geq k \left( \int_0^\varepsilon \frac{1}{t^{1+2/\varepsilon}} d\mu(t) \right)^p \frac{1}{p^p}.
$$

Thus by letting $\varepsilon \to 0$, we have in comparison to (5)

$$
\int_0^\infty \frac{1}{t^{1+2/\varepsilon}} d\mu(t) < \infty.
$$

Proposition 3.6. Let $1 \leq p < \infty$ and $0 < \delta < 1$. If

$$
\int_0^\infty \frac{1}{t^{1+2/\varepsilon}} d\mu(t) < \infty,
$$

then $H^\delta_\mu$ is bounded with

$$
\|H^\delta_\mu\|_{A^p(U) \to A^p(U)} = \int_\delta^{1/2} \frac{1}{t^{1+2/\varepsilon}} d\mu(t).
$$

Proof. As in Theorem 3.5 an application of Minkowski inequality gives

(6) $$
\|H^\delta_\mu\|_{A^p(U) \to A^p(U)} \leq \int_\delta^{1/2} \frac{1}{t^{1+2/\varepsilon}} d\mu(t).
$$

Let $f_\varepsilon(z) = (z + i)^{-2/\varepsilon}$ with $\varepsilon > 0$ small enough. We calculate

$$
H^\delta_\mu(f_\varepsilon)(z) - f_\varepsilon(z) \int_\delta^{1/2} \frac{1}{t^{1+2/\varepsilon}} d\mu(t)
$$

$$
= \int_\delta^{1/2} \frac{1}{t^{1+2/\varepsilon}} (\varphi_\varepsilon(z) - \varphi_\varepsilon(1)) d\mu(t),
$$

where

$$
\varphi_\varepsilon(z) = \frac{t^\varepsilon}{(z + ti)^{2+\varepsilon}}.
$$
For any $t \in [\delta, 1/\delta]$, calculus gives

$$|\varphi_{\varepsilon,z}(t) - \varphi_{\varepsilon,z}(1)| \leq |t - 1| \sup\{|\varphi'_{\varepsilon,z}(s)| : s \in [\delta, 1/\delta]\}$$

$$\leq \frac{1}{\delta} \left( \frac{\varepsilon\delta^{\varepsilon-1}}{|z + i\delta|^{\frac{2}{p}+\varepsilon}} + \frac{(\frac{3}{p} + \varepsilon)(1/\delta)^{\varepsilon}}{|z + i\delta|^{\frac{2}{p}+\varepsilon+1}} \right)$$

$$= \varepsilon \delta^{\varepsilon-2} g_{p,\varepsilon}(z) + \left( \frac{2}{p} + \varepsilon \right)(1/\delta)^{\varepsilon+1} g_{p(\varepsilon+1),\varepsilon}(z).$$

Where above we followed the notation of Lemma 3.2. Thus by an easy application of Minkowski inequality followed by the triangular inequality we have

$$\|H_{\mu}(f_{\varepsilon})(z) - f_{\varepsilon}(z) \int_{\delta}^{1/\delta} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t)\|_{A^{p}(U)}$$

$$\leq \int_{\delta}^{1/\delta} \frac{1}{t^{\frac{1}{1-\varepsilon}}} \| (\varphi_{\varepsilon,z}(t) - \varphi_{\varepsilon,z}(1)) \|_{A^{p}(U)} d\mu(t)$$

$$\leq \int_{\delta}^{1/\delta} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t) \left( \varepsilon \delta^{\varepsilon-2} g_{p,\varepsilon}(z) + \left( \frac{2}{p} + \varepsilon \right)(1/\delta)^{\varepsilon+1} g_{p(\varepsilon+1),\varepsilon}(z) \right).$$

This, together with Lemma 3.2 (recall that $|f_{\varepsilon}| = g_{p(\varepsilon)},$), yields

$$\|H_{\mu}^\#(f_{\varepsilon})(z) - f_{\varepsilon}(z) \int_{\delta}^{1/\delta} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t)\|_{A^{p}(U)}$$

$$\leq \int_{\delta}^{1/\delta} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t) \times \frac{\varepsilon \delta^{\varepsilon-2} g_{p,\varepsilon}(z) + \left( \frac{2}{p} + \varepsilon \right)(1/\delta)^{\varepsilon+1} g_{p(\varepsilon+1),\varepsilon}(z)}{\|f_{\varepsilon}\|_{A^{p}(U)}} \to 0$$

as $\varepsilon \to 0$. This and (6) imply that

$$\int_{\delta}^{1/\delta} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t) = \|H_{\mu}^\#\|_{A^{p}(U) \to A^{p}(U)}. \tag{□}$$

**Theorem 3.7.** Let $1 \leq p < \infty$. If

$$\int_{0}^{\infty} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t) < \infty$$

then

$$\|H_{\mu}\|_{A^{p}(U) \to A^{p}(U)} = \int_{0}^{\infty} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t).$$

**Proof.** By Theorem 3.5 we have that

$$\|H_{\mu}\|_{A^{p}(U) \to A^{p}(U)} \leq \int_{0}^{\infty} \frac{1}{t^{\frac{1}{1-\varepsilon}}} d\mu(t).$$
Minkowski inequality implies that

\[ ||H_\mu - H_\delta\mu||_{A^p(U) \to A^p(U)} \leq \int_{(0,\delta) \cup (1/\delta, \infty)} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t). \]

By Proposition 3.6

\[ \int_\delta^{1/\delta} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t) = ||H_\delta\mu||_{A^p(U) \to A^p(U)}. \]

This, combined with (7), allows us to conclude that

\[ ||H_\mu||_{A^p(U) \to A^p(U)} \geq \int_0^{\infty} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t) - 2 \int_{(0,\delta) \cup (1/\delta, \infty)} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t) \to \int_0^{\infty} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t) \]

as \( \delta \to 0 \). Hence,

\[ ||H_\mu||_{A^p(U) \to A^p(U)} = \int_0^{\infty} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t). \]

\[ \square \]

3.3. The quasi-Hausdorff operator. Let \( f, g \in A^2(U) \) and assume that \( H_\mu \) is bounded on \( A^2(U) \). Thus

\[ \int_0^{\infty} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t) < \infty. \]

We have

\[ \int_U \int_0^{\infty} \frac{1}{t} |f(\frac{z}{t})| |g(z)| \, d\mu(t) \, dA(z) \leq \left( \int_U \left( \int_0^{\infty} \frac{1}{t} |f(\frac{z}{t})| \, d\mu(t) \right)^2 \, dA(z) \right)^{1/2} ||g||_{A^2(U)} \]

\[ \leq \left( \int_0^{\infty} \frac{1}{t^{1-\frac{2}{p}}} \, d\mu(t) \right)^{1/2} ||f||_{A^2(U)} ||g||_{A^2(U)} < \infty, \]

where we applied the Cauchy-Schwarz and Minkowski inequalities. Therefore

\[ \langle H_\mu(f), g \rangle = \int_U H_\mu(f)(z) \overline{g(z)} \, dA(z) \]

\[ = \frac{1}{\pi} \int_U \int_0^{\infty} \frac{1}{t} f(\frac{z}{t}) \, d\mu(t) \overline{g(z)} \, dA(z) \]

\[ = \frac{1}{\pi} \int_0^{\infty} \int_U \frac{1}{t} f(\frac{z}{t}) g(z) \, dA(z) \, d\mu(t) \]

\[ = \frac{1}{\pi} \int_U f(z) \int_0^{\infty} \overline{tg(tz)} \, d\mu(t) \, dA(z), \]

where we applied a change of variables and Fubini’s Theorem twice. This means that the adjoint \( H_\mu^* \) of \( H_\mu \) on \( A^2(U) \) is:
\[ \mathcal{H}_\mu^*(f)(z) = \int_0^\infty tf(tz) \, d\mu(t). \]

We will consider \( \mathcal{H}_\mu^* \) on \( A^p(U) \) and suppose for a moment that it is well defined for functions in \( A^p(U) \). Let \( \lambda(t) = t^{-1}, t > 0 \), then \( \lambda \) maps \( (0, \infty) \) onto \( (0, \infty) \) and is measurable. Set \( f(tz) = f_z(t) \) then

\[ \mathcal{H}_\mu^*(f)(z) = \int_0^\infty tf(tz) \, d\mu(t) \]

\[ = \int_0^\infty tf_z(t) \, d\mu(t) \]

\[ = \int_0^\infty \frac{1}{\lambda(t)} f_z \left( \frac{1}{\lambda(t)} \right) \, d\mu(t) \]

\[ = \int_0^\infty \frac{1}{t} f \left( \frac{z}{t} \right) \, d\nu(t), \]

\[ = \mathcal{H}_\nu(f)(z) \]

where \( d\nu = d\lambda_* \mu(t) \) and \( \lambda_* \mu \) is the push-forward measure of \( \mu \) with respect to \( \lambda \). We can now apply the results of the first part of the paper to have:

**Theorem 3.8.** Let \( 1 \leq p < \infty \). The quasi-Hausdorff operator \( \mathcal{H}_\mu^* \) is bounded on \( A^p(U) \) if and only if

\[ \int_0^\infty t^{1-\frac{2}{p}} \, d\mu(t) < \infty. \]

Moreover

\[ \|\mathcal{H}_\mu^*\|_{A^p(U) \to A^p(U)} = \int_0^\infty t^{1-\frac{2}{p}} \, d\mu(t). \]

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