Detecting Markov Random Fields Hidden in White Noise

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Abstract

Motivated by change point problems in time series and the detection of textured objects in images, we consider the problem of detecting a piece of a Gaussian Markov random field hidden in white Gaussian noise. We derive minimax lower bounds and propose near-optimal tests.

1 Introduction

Anomaly detection is important in a number of applications, including surveillance and environment monitoring systems using sensor networks, object tracking from video or satellite images, and tumor detection in medical imaging. The most common model is that of an object or signal of unusually high amplitude hidden in noise. In other words, one is interested in detecting the presence of an object in which the mean of the signal is different from that of the background. We refer to this as the detection-of-means problem. In many situations, however, anomaly manifests as unusual dependencies in the data. This is the detection-of-correlations problem that we consider in this paper.

1.1 Setting and hypothesis testing problem

It is common to model dependencies by a Gaussian random field \( X = (X_i : i \in \mathcal{V}) \), where \( \mathcal{V} \subset \mathcal{V}_\infty \) is of size \( |\mathcal{V}| = n \), while \( \mathcal{V}_\infty \) is countably infinite. An important example is the \( d \)-dimensional integer lattice

\[
\mathcal{V} = \{1, \ldots, m\}^d \subset \mathcal{V}_\infty = \mathbb{Z}^d.
\]

The \( X_i \)'s are assumed to have zero mean and unit variance. We emphasize that anomalies come in the form of correlations between some of these variables.

We formalize the task of detection as the following hypothesis testing problem. One observes a realization of \( X = (X_i : i \in \mathcal{V}) \), where the \( X_i \)'s are known to be standard normal. Under the null hypothesis \( \mathbf{H}_0 \), the \( X_i \)'s are independent. Under the alternative hypothesis \( \mathbf{H}_1 \), the \( X_i \)'s are correlated in one of the following ways. Let \( \mathcal{C} \) be a class of subsets of \( \mathcal{V} \). Each set \( S \in \mathcal{C} \) represents a possible anomalous subset of the components of \( X \). Specifically, when \( S \in \mathcal{C} \) is the anomalous subset of nodes, each \( X_i \) with \( i \notin S \) is still independent of all the other variables, while \( (X_i : i \in S) \) coincides with \( (Y_i : i \in S) \), where \( Y = (Y_i : i \in \mathcal{V}_\infty) \) is a stationary Gaussian Markov random field.

We emphasize that, in this formulation, the anomalous subset \( S \) is only known to belong to \( \mathcal{C} \).

We are thus addressing the problem of detecting a region of a Gaussian Markov random field against a background of white noise. This testing problem models important detection problems...
such as the detection of a piece of a time series in a signal and the detection of a textured object in an image, which we describe below. Before doing that, we further detail the model and set some foundational notation and terminology.

1.2 Tests and minimax risk

We denote the distribution of \( X \) under \( H_0 \) by \( \mathbb{P}_0 \). The distribution of the zero-mean stationary Gaussian Markov random field \( Y \) is determined by its covariance operator \( \Gamma = (\Gamma_{i,j} : i,j \in \mathbb{V}_\infty) \) defined by \( \Gamma_{i,j} = \mathbb{E}[Y_i Y_j] \). We denote the distribution of \( X \) under \( H_1 \) by \( \mathbb{P}_{S,\Gamma} \) when \( S \in \mathcal{C} \) is the anomalous set and \( \Gamma \) is the covariance operator of the Gaussian Markov random field \( Y \).

A test is a measurable function \( f : \mathbb{R}^V \to \{0,1\} \). When \( f(X) = 0 \), the test accepts the null hypothesis and it rejects it otherwise. The probability of type I error of a test \( f \) is \( \mathbb{P}_0\{f(X) = 1\} \). When \( S \in \mathcal{C} \) is the anomalous set and \( Y \) has covariance operator \( \Gamma \), the probability of type II error is \( \mathbb{P}_{S,\Gamma}\{f(X) = 0\} \). In this paper we evaluate tests based on their worst-case risks. The risk of a test \( f \) corresponding to a covariance operator \( \Gamma \) and class of sets \( \mathcal{C} \) is defined as

\[
R_{\mathcal{C},\Gamma}(f) = \mathbb{P}_0\{f(X) = 1\} + \max_{S \in \mathcal{C}} \mathbb{P}_{S,\Gamma}\{f(X) = 0\}.
\]

Defining the risk this way is meaningful when the distribution of \( Y \) is known, meaning that \( \Gamma \) is available to the statistician. In this case, the minimax risk is defined as

\[
R^*_\mathcal{C,\Gamma} = \inf_f R_{\mathcal{C},\Gamma}(f),
\]

where the infimum is over all tests \( f \). When \( \Gamma \) is only known to belong to some class of covariance operators \( \mathfrak{G} \), it is more meaningful to define the risk of a test \( f \) as

\[
R_{\mathcal{C},\mathfrak{G}}(f) = \mathbb{P}_0\{f(X) = 1\} + \max_{\Gamma \in \mathfrak{G}} \max_{S \in \mathcal{C}} \mathbb{P}_{S,\Gamma}\{f(X) = 0\}.
\]

The corresponding minimax risk is defined as

\[
R^*_\mathcal{C,\mathfrak{G}} = \inf_f R_{\mathcal{C},\mathfrak{G}}(f).
\]

In this paper we consider situations in which the covariance operator \( \Gamma \) is known (i.e., the test \( f \) is allowed to be constructed using this information) and other situations when \( \Gamma \) is unknown but it is assumed to belong to a class \( \mathfrak{G} \). When \( \Gamma \) is known (resp. unknown), we say that a test \( f \) asymptotically separates the two hypotheses if \( R_{\mathcal{C},\Gamma}(f) \to 0 \) (resp. \( R_{\mathcal{C},\mathfrak{G}}(f) \to 0 \), and we say that the hypotheses merge asymptotically if \( R^*_{\mathcal{C,\Gamma}} \to 1 \) (resp. \( R^*_{\mathcal{C,\mathfrak{G}}} \to 1 \), as \( n = |\mathbb{V}| \to \infty \). We note that, as long as \( \Gamma \in \mathfrak{G} \), \( R^*_{\mathcal{C},\Gamma} \leq R^*_{\mathcal{C},\mathfrak{G}} \) and that \( R^*_{\mathcal{C},\mathfrak{G}} \leq 1 \), since the test \( f \equiv 1 \) that always rejects the null hypothesis, has risk equal to 1.

1.3 Detecting a piece of time series

As a simple example of the general problem described above, consider the case of observing a time series \( X_1, \ldots, X_n \). Here we take \( \mathbb{V} = \{1, \ldots, n\} \) and \( \mathbb{V}_\infty = \mathbb{Z} \). Under the null hypothesis, the \( X_i \)'s are i.i.d. standard normal random variables. We assume that the anomaly comes in the form of temporal correlations over an (unknown) interval \( S = \{i + 1, \ldots, i + k\} \) of, say, known length \( k < n \). Here, \( i \in \{0,1, \ldots, n-k\} \) is thus unknown. Specifically, when \( S \) is the anomalous interval, \( (X_{i+1}, \ldots, X_{i+k}) \sim (Y_{i+1}, \ldots, Y_{i+k}) \), where \( (Y_i : i \in \mathbb{Z}) \) is an autoregressive process of order \( h \) (AR\(_h\)) with zero mean and unit variance, that is,

\[
Y_i = \psi_1 Y_{i-1} + \cdots + \psi_h Y_{i-h} + \sigma Z_i, \quad \forall i \in \mathbb{Z},
\]
where \((Z_i : i \in \mathbb{Z})\) are i.i.d. standard normal random variables, \(\psi_1, \ldots, \psi_h \in \mathbb{R}\) are the coefficients of the process—assumed to be stationary—and \(\sigma > 0\) is such that \(\text{Var}(Y_i) = 1\) for all \(i\). Note that \(\sigma\) is a function of \(\psi_1, \ldots, \psi_h\), so that the model has effectively \(h\) parameters. See Brockwell and Davis (1991) for a general introduction to time series.

In the simplest setting \(p = 1\) and the hypothesis testing problem is to distinguish

\[
H_0 : X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(0, 1),
\]

versus

\[
H_1 : \exists i \in \{0, 1, \ldots, n - k\} \text{ such that } X_1, \ldots, X_i, X_{i+k+1}, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(0, 1)
\]

and \((X_{i+1}, \ldots, X_{i+k})\) is independent of \(X_1, \ldots, X_i, X_{i+k+1}, \ldots, X_n\) with

\[
X_{i+1} \sim \mathcal{N}(0, 1), \quad X_{i+2} - \psi X_{i+1}, \ldots, X_{i+k} - \psi X_{i+k-1} \overset{iid}{\sim} \mathcal{N}(0, 1 - \psi^2).
\]

Typical realizations of the observed vector under the null and alternative hypotheses are illustrated in Figure 1.

![Figure 1](image)

Figure 1: Top: a realization of the observed time series under the null hypothesis (white noise). Bottom: a realization under the alternative with anomalous interval \(S = \{201, \ldots, 250\}\), assuming an AR\(_1\) covariance model with parameter \(\psi = 0.9\).

Gaussian autoregressive processes and other correlation models are special cases of Gaussian Markov random fields, and therefore this setting is a special case of our general framework, with \(C\) being the class of discrete intervals of length \(k\). In the simplest case, the length of the anomalous interval is known beforehand. In more complex settings, this is unknown, in which case \(C\) may be taken to be the class of all intervals within \(V\) of length at least \(k_{\text{min}}\).

This testing problem has been extensively studied in the slightly different context of change-point analysis, where under the null hypothesis \(X_1, \ldots, X_n\) are generated from an \(\text{AR}_h(\psi^0)\) process, while under the alternative hypothesis there is an \(i \in V\) such that \(X_1, \ldots, X_i\) and \(X_{i+1}, \ldots, X_n\) are generated from \(\text{AR}_h(\psi^0)\) and \(\text{AR}_h(\psi^1)\), with \(\psi^0 \neq \psi^1\), respectively. The order \(h\) is often given. In
fact, instead of assuming autoregressive models, nonparametric models are often favored. See, for example, Davis et al. (1995); Giraitis and Leipus (1992); Horváth (1993); Hušková et al. (2007); Lavielle and Ludeña (2000); Paparoditis (2009); Picard (1985); Priestley and Subba Rao (1969) and many other references therein. These papers often suggest maximum likelihood tests whose limiting distributions are studied under the null and (sometimes fixed) alternative hypotheses. For example, in the special case of $h = 1$, such a test would reject $H_0$ when $|\hat{\psi}|$ is large, where $\hat{\psi}$ is the maximum likelihood estimate for $\psi$. In particular, from Picard (1985), we can speculate that such a test can asymptotically separate the hypotheses in the simplest setting described above when $\psi k^2 \rightarrow \infty$ for some $\alpha < 1/2$ fixed. See also Hušková et al. (2007); Paparoditis (2009) for power analyses against fixed alternatives.

Our general results imply the following in the special case when the anomaly comes in the form of an autoregressive process with unknown parameter $\psi$. In the following corollary, $C_1$ and $C_2$ denote two positive numerical constants.

**Corollary 1.** Assume $n, k \rightarrow \infty$, and that $h = o \left( \sqrt{k/\log(n)} \wedge k^{1/4} \right)$. Denote by $\mathcal{C}(h, r)$ the class of covariance operators corresponding to AR processes with valid parameter $\psi = (\psi_1, \ldots, \psi_h)$ satisfying $\|\psi\|_2^2 \geq r^2$. Then, $R_{C, \mathcal{C}(h, r)} \rightarrow 1$ when

$$r^2 \leq C_1 \log(n/k)/k + \sqrt{h \log(n/k)/k}.$$  

Conversely, if $f$ denotes the pseudo-likelihood test of Section 3.3.2, then $R_{C, \mathcal{C}(h, r)}(f) \rightarrow 0$ when

$$r^2 \geq C_2 \log(n/k) + \sqrt{h \log(n/k)/k}.$$  

### 1.4 Detecting a textured region

In image processing, the detection of textured objects against a textured background is relevant in a number of applications, such as in the detection of local fabric defects in the textile industry by automated visual inspection (Kumar, 2008), the detection of a moving object in a textured background (Kim et al., 2005; Yilmaz et al., 2006), the identification of tumors in medical imaging (James et al., 2001; Karkanis et al., 2003), the detection of man-made objects in natural scenery (Kumar and Hebert, 2003), the detection of sites of interest in archeology (Litton and Buck, 1995) and of weeds in crops (Dryden et al., 2003). In all these applications, the object is generally small compared to the size of the image.

Common models for texture include Markov random fields (Cross and Jain, 1983) and joint distributions over filter banks such as wavelet pyramids (Manjunath and Ma, 1996; Portilla and Simoncelli, 2000). We focus here on textures that are generated via Gaussian Markov random fields (Chellappa and Chatterjee, 1985; Zhu et al., 1998). Our goal is to detect a textured object in a number of applications, such as in the detection of local fabric defects in the textile industry.
is when $h = 1$ and $\phi_{t_1,t_2} = \phi$ when $(t_1,t_2) \in \{(\pm 1,0),(0,\pm 1)\}$ for some $\phi \in (-1/4,1/4)$, and the anomalous region is a discrete square; see Figure 2 for typical realizations under both the null and alternative hypotheses.

This is a special case of our setting. While intervals are natural in the case of time series, squares are rather restrictive models of anomalous regions in images. We consider instead “blob-like” regions (to be defined later) that include convex and star-shaped regions.

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Figure 2: Left: white noise, no anomalous region is present. Right: a squared anomalous region is present. In this example on the $50 \times 50$ grid, the anomalous region is a $15 \times 15$ square piece from a Gaussian Markov random field with radius $h = 1$ and $4\phi_{t_1,t_2} = 1 - 10^{-4}$ when $(t_1,t_2) \in \{(\pm 1,0),(0,\pm 1)\}$.

A number of publications address the related problems of texture classification (Kervrann and Heitz, 1995; Varma and Zisserman, 2005; Zhu et al., 1998) and texture segmentation (Galun et al., 2003; Grigorescu et al., 2002; Hofmann et al., 1998; Jain and Farrokhnia, 1991; Malik et al., 2001). In fact, this literature is quite extensive. Only very few papers address the corresponding change-point problem (Palenichka et al., 2000; Shahrokni et al., 2004) and we do not know of any theoretical results in this literature. Our results show again that some pseudo-likelihood test is minimax optimal up to a multiplicative constant.

Although not in the literature on change-point or object detection, Anandkumar et al. (2009) is the only paper developing theory in a similar context. It considers a spatial model where points \( \{x_i, i \in [N]\} \) are sampled uniformly at random in some bounded region and a nearest-neighbor graph is formed. On the resulting graph, variables are observed at the nodes. Under the (simple) null hypothesis, the variables are i.i.d. zero mean normal. Under the (simple) alternative, the variables arise from a Gaussian Markov random with covariance operator of the form $\Gamma_{i,j} \propto g(\|x_i - x_j\|)$, where $g$ is a known function. The paper analyzes the large-sample behavior of the likelihood ratio test.

1.5 Detecting paths of correlation in general graphs

In the bulk of this paper we focus on detecting blob-like regions in the integer lattice. However, the main technology is more general and may be used for detecting Gaussian Markov random field anomalies in more general graphs. While general results for general graphs can be rather complicated to formulate, we address the concrete example of detecting a path of correlations in a general graph. Specifically, we let $\mathcal{C}$ be the class of all self-avoiding paths of length $k$ in the graph, and the correlation structure is an autoregressive process of order 1 (for simplicity) along the anomalous path. This setting could model an attack in a computer network (Mukherjee et al.,
1.6 Content

In Section 2, we lay down some foundations for deriving upper and lower bounds and establish a simple, but basic lower bound used several times in the paper. In Section 3 we consider detecting correlations in a finite-dimensional lattice, which includes the important special cases of time series and textures in images. We establish lower bounds, both when the covariance matrix is known or unknown and propose test procedures that are shown to achieve the lower bounds up to multiplicative constants. In Section 4 we consider the problem of detecting paths of correlations in general graphs, and then specialize our results to the lattice. Section 5 is a discussion section where we outline possible generalizations and further work. The proofs are gathered in Section 6.

2 Preliminaries and a general lower bound

In this paper we derive upper and lower bounds for the minimax risk, both when $\Gamma$ is known as in (3) and when it is unknown as in (5), the latter requiring a substantial amount of additional work.

2.1 General strategy

As is standard, an upper bound is obtained by exhibiting a test $f$ and then upper-bounding its risk—either (2) or (4) according to whether $\Gamma$ is known or unknown. In order to derive a lower bound for the minimax risk, we follow the standard argument of choosing a prior distribution on the class of alternatives and then lower-bounding the minimax risk with the resulting average risk. When $\Gamma$ is known, this leads us to select a prior on $C$, denoted by $\nu$, and consider

$$R_{\nu, \Gamma}(f) = \mathbb{P}_0\{f(X) = 1\} + \sum_{S \in C} \nu(S) \mathbb{P}_{S, \Gamma}\{f(X) = 0\} \quad \text{and} \quad \overline{R}_{\nu, \Gamma} = \inf_f R_{\nu, \Gamma}(f). \quad (10)$$

The latter is the Bayes risk associated with $\nu$. By placing a prior on the class of alternative distributions, the alternative hypothesis becomes effectively simple (as opposed to composite). The advantage of this is that the optimal test may be determined explicitly. Indeed, the Neyman-Pearson fundamental lemma implies that the likelihood ratio test $f_{\nu, \Gamma}^*(x) = 1_{\{L_{\nu, \Gamma}(x) > 1\}}$, with

$$L_{\nu, \Gamma} = \sum_{S \in C} \nu(S) \frac{d\mathbb{P}_{S, \Gamma}}{d\mathbb{P}_0},$$

minimizes the average risk. In most of the paper, $\nu$ will be chosen as the uniform distribution on the class $C$.

When $\Gamma$ is only known to belong to some class $\mathfrak{G}$ we also need to choose a prior on $\mathfrak{G}$, which we denote by $\pi$, leading to

$$R_{\nu, \pi}(f) = \mathbb{P}_0\{f(X) = 1\} + \sum_{S \in C} \nu(S) \int \mathbb{P}_{S, \Gamma}\{f(X) = 0\} \pi(d\Gamma) \quad \text{and} \quad \overline{R}_{\nu, \pi} = \inf_f R_{\nu, \pi}(f). \quad (11)$$

In this case, the likelihood ratio test becomes $f_{\nu, \pi}^*(x) = 1_{\{L_{\nu, \pi}(x) > 1\}}$, where

$$L_{\nu, \pi} = \sum_{S \in C} \nu(S) \frac{d\mathbb{P}_{S, \pi}}{d\mathbb{P}_0}, \quad \mathbb{P}_{S, \pi} = \int \mathbb{P}_{S, \Gamma} \pi(d\Gamma).$$
minimizes the average risk.

In both cases, we then proceed to bound the second moment of the resulting likelihood ratio under the null. Indeed, in a general setting, if \( L \) is the likelihood ratio for \( \mathbb{P}_0 \) versus \( \mathbb{P}_1 \) and \( R \) denotes its risk, then (Lehmann and Romano, 2005, Problem 3.10)

\[
R = 1 - \frac{1}{2} \mathbb{E}_0 |L(X) - 1| \geq 1 - \frac{1}{2} \sqrt{\mathbb{E}_0[L(X)^2]} - 1 , \tag{12}
\]

where the inequality follows by the Cauchy-Schwarz inequality.

Remark. Working with the minimax risk (as we do here) allows us to bypass making an explicit choice of prior, although one such choice is eventually made when deriving a lower bound. Another advantage is that the minimax risk is monotone with respect to the class \( C \) in the sense that if \( C' \subset C \), then the minimax risk corresponding to \( C' \) is at most as large as that corresponding to \( C \). This monotonicity does not necessarily hold for the Bayes risk. See Addario-Berry et al. (2010) for a discussion in the context of the detection-of-means problem.

2.2 A general lower bound

We start with a general lower bound that is at the basis of most of the lower bounds derived later in the paper. Although the result is stated for a class \( C \) of disjoint subsets, using the monotonicity of the minimax risk, it can be used to derive lower bounds in more general settings. It is particularly useful in the context of detecting blob-like anomalous regions in the lattice.

If \( S \subset V \) is a finite set, we denote by \( \Gamma_S \) the principal submatrix of the covariance operator \( \Gamma \) indexed by \( S \). If \( \Gamma \) is nonsingular, each such submatrix is invertible. Recall the definition (11).

**Proposition 1.** Let \( \{ \Gamma(\phi) : \phi \in \Phi \} \) be a class of nonsingular covariance operators and let \( C \) be a class of disjoint subsets of \( V \). Put the uniform prior \( \nu \) on \( C \) and let \( \pi \) be a prior on \( \Phi \). Then

\[
\overline{R}^*_{\nu,\pi} \geq 1 - \frac{1}{2|C|} \left( \sum_{S \in C} V_S \right)^{1/2} ,
\]

where

\[
V_S = \mathbb{E}_\pi \left[ \left( \frac{\det(\Gamma^{-1}_S(\phi_1)) \det(\Gamma^{-1}_S(\phi_2))}{\det(\Gamma^{-1}_S(\phi_1) + \Gamma^{-1}_S(\phi_2) - \mathbb{I})} \right)^{1/2} \right] ,
\]

and the expected value is with respect to \( \phi_1, \phi_2 \) drawn i.i.d. from the distribution \( \pi \).

**Proof.** As noted above, the Bayes risk is achieved by the likelihood ratio test \( f^*_{\nu,\pi}(x) = \mathbb{I}_{\{L_{\nu,\pi}(x) > 1\}} \) where

\[
L_{\nu,\pi}(x) = \frac{1}{|C|} \sum_{S \in C} L_S(x) , \quad \text{with} \quad L_S(x) = \int \frac{d\mathbb{P}_S \Gamma(\phi)(x)}{d\mathbb{P}_0(x)} \pi(d\phi) .
\]

In our Gaussian model,

\[
L_S(x) = \mathbb{E}_\pi \left[ \exp \left( \frac{1}{2} x^\top (I - \Gamma^{-1}_S(\phi)) x - \frac{1}{2} \log \det(\Gamma_S(\phi)) \right) \right] , \tag{13}
\]

where the expectation is taken with respect to the random draw of \( \phi \sim \pi \). Then, by (12),

\[
\overline{R}^*_{\nu,\pi} = 1 - \frac{1}{2} \mathbb{E}_0 |L_{\nu,\pi}(X) - 1| \geq 1 - \frac{1}{2} \sqrt{\mathbb{E}_0[L_{\nu,\pi}(X)^2]} - 1 . \tag{14}
\]
We proceed to bound the second moment of the likelihood ratio under the null hypothesis. Summing over \(S, T \in \mathcal{C}\), we have

\[
\mathbb{E}_0[L_{\nu, \pi}(X)^2]
= \frac{1}{|\mathcal{C}|^2} \sum_{S, T \in \mathcal{C}} \mathbb{E}_0[L_S(X)L_T(X)]
= \frac{1}{|\mathcal{C}|^2} \sum_{S \neq T} \mathbb{E}_0[L_S(X)]\mathbb{E}_0[L_T(X)] + \frac{1}{|\mathcal{C}|^2} \sum_{S \in \mathcal{C}} \mathbb{E}_0[L_S^2(X)]
= \frac{|\mathcal{C}| - 1}{|\mathcal{C}|} + \frac{1}{|\mathcal{C}|^2} \sum_{S \in \mathcal{C}} \mathbb{E}_0[\mathbb{E}_\pi[\exp \left( X^\top (I - \frac{1}{2} \Gamma^{-1}_S(\phi_1) - \frac{1}{2} \Gamma^{-1}_S(\phi_2))X \\ - \frac{1}{2} \log \det(\Gamma_S(\phi_1)) - \frac{1}{2} \log \det(\Gamma_S(\phi_2)) \right)]]
\leq 1 + \frac{1}{|\mathcal{C}|^2} \sum_{S} \mathbb{E}_\pi \left[ \exp \left( - \frac{1}{2} \log \det(\Gamma^{-1}_S(\phi_1) + \Gamma^{-1}_S(\phi_2) - I) - \frac{1}{2} \log \det(\Gamma_S(\phi_1)\Gamma_S(\phi_2)) \right) \right]
= 1 + \frac{1}{|\mathcal{C}|^2} \sum_{S} V_S,
\]

where in the second equality we used the fact that \(S \neq T\) are disjoint, and therefore \(L_S(X)\) and \(L_T(X)\) are independent, and in the third we used the fact that \(\mathbb{E}_0[L_S(X)] = 1\) for all \(S \in \mathcal{C}\). \(\square\)

3 \hspace{1em} Detecting a blob-like region of correlations

In this section we consider the problem of detecting an anomalous region in the integer lattice where the variables in the anomalous region follow the distribution of a Gaussian Markov random field, as described in Section 1.4. This setting also includes the detection of a piece of time series considered in Section 1.3. We refer the reader to Arias-Castro et al. (2011, 2005) for the results in the corresponding detection-of-means setting.

As a consequence of Proposition 1, we provide a lower bound for the risk, for the case where \(\Gamma\) is known (in Section 3.2) and then for the case where it is unknown (in Section 3.3), in each case establishing a minimax lower bound. We then propose for each situation a test that is shown to match the lower bound up to a multiplicative constant. These are general results that apply to any class \(\mathcal{C}\) of subsets of the lattice. In Section 3.4, we specialize them to the problem of detecting blob-like regions of lattice.

3.1 \hspace{1em} Preliminaries

Given a positive integer \(h\), denote by \(\mathbb{N}_h\) the integer lattice \((-h, \ldots, h)^d \setminus \{0\}^d\) with \((2h + 1)^d - 1\) nodes. For any nonsingular covariance operator \(\Gamma\) of a stationary Gaussian Markov random field over \(\mathbb{Z}^d\) with unit variance and neighborhood \(\mathbb{N}_h\), there exists a unique vector \(\phi\) indexed by the nodes of \(\mathbb{N}_h\) satisfying \(\phi_i = \phi_{-i}\) such that

\[
\Gamma^{-1}_{i,j}/\Gamma^{-1}_{i,i} =
\begin{cases}
-\phi_{i-j} & \text{if } 1 \leq |i-j|_\infty \leq h, \\
1 & \text{if } i = j, \\
0 & \text{otherwise},
\end{cases}
\]

(15)
where $\Gamma^{-1}$ denotes the inverse of the covariance operator $\Gamma$. Consequently, there exists a one-to-one map from the collection of invertible covariance operators of stationary Gaussian Markov random fields over $\mathbb{Z}^d$ with unit variance and neighborhood $N_h$ to some subset $\Phi_h \subset \mathbb{R}^{N_h}$. Given $\phi \in \Phi_h$, $\Gamma(\phi)$ denotes the unique covariance operator satisfying $\Gamma_{i,i} = 1$ and (15). It is well known that $\Phi_h$ contains the set of vectors $\phi$ whose $\ell_1$-norm is smaller than one, as the corresponding operator $\Gamma^{-1}(\phi)$ is diagonally dominant in that case. The interested reader is referred to Guyon (1995, Sect.1.3) or Rue and Held (2005, Sect.2.6) for further details and discussions. For $\phi \in \Phi_h$, define $\sigma_{\phi}^2 = 1/\Gamma_{i,i}^{-1}(\phi)$. Then, the centered Gaussian process $Y = (Y_i : i \in \mathbb{Z}^d)$ with covariance operator $\Gamma(\phi)$ is such that, for each $i \in \mathbb{Z}^d$, the conditional distribution of $Y_i$ given the rest of the variables $Y(\cdot \setminus i)$ is

$Y_i | Y(\cdot \setminus i) \sim \mathcal{N}\left(\sum_{j \in N_h} \phi_j Y_{i+j}, \sigma_{\phi}^2\right)$.

Define the $h$-boundary of $S$, denoted $\Delta_h(S)$, as the collection of vertices in $S$ whose distance to $\mathbb{Z}^d \setminus S$ is at most $h$. We also define the $h$-interior $S_h$ as $S_h = S \setminus \Delta_h(S)$.

3.2 Known covariance

We first address the case when the covariance operator $\Gamma(\phi)$ of the underlying Gaussian Markov random field $Y$ (or equivalently, the vector $\phi$) is known.

3.2.1 Lower bound

Recall the definition of the minimax risk (3) and the average risk (10). (Henceforth, to lighten the notation, we replace subscripts in $\Gamma(\phi)$ with subscripts in $\phi$.) For any prior $\nu$ on $C$, $R^*_C,\phi \geq R^*_\nu,\phi$, and the following corollary of Proposition 1 provides a lower bound on the latter.

**Corollary 2.** Let $C$ be a class of disjoint subsets of $V$ and fix $\phi \in \Phi_h$ satisfying $\|\phi\|_1 < 1/2$. Then, letting $\nu$ denote the uniform prior over $C$, we have

$$R^*_\nu,\phi \geq 1 - \frac{1}{2|C|} \left[ \sum_{S \in C} \exp \left( \frac{10|S|\|\phi\|_2^2}{1 - 2\|\phi\|_1} \right) \right]^{1/2}.$$  \hspace{1cm} (16)

In particular, the corollary implies that, for any fixed $a \in (0,1)$, $R^*_C,\phi \geq 1 - a$ as soon as

$$\frac{\|\phi\|_2^2}{1 - 2\|\phi\|_1} \leq \min_{S \in C} \frac{\log (4|C|/a^2)}{10|S|}.$$  \hspace{1cm} (17)

Furthermore, the hypotheses merge asymptotically (i.e., $R^*_C,\phi \to 1$) when

$$\log(|C|) - \frac{10\|\phi\|_2^2}{1 - 2\|\phi\|_1} \max_{S \in C} |S| \to \infty.$$  \hspace{1cm} (18)

3.2.2 Generalized likelihood ratio test

When the covariance operator $\Gamma(\phi)$ is known, the generalized likelihood ratio test rejects the null hypothesis for large values of

$$\max_{S \in C} X^T_S (I_S - \Gamma^{-1}_S(\phi)) X_S.$$
As we shall see, it is more natural to use instead the statistic

\[ U(X) = \max_{S \in \mathcal{C}} \frac{X^\top_S (I_S - I^\top_S \Gamma^-1 (\phi)) X_S - \text{Tr}(I_S - I^\top_S \Gamma^-1 (\phi))}{\|I_S - I^\top_S \Gamma^-1 (\phi)\|_F \sqrt{\log(|\mathcal{C}|)} + \|I_S - I^\top_S \Gamma^-1 (\phi)\| \log(|\mathcal{C}|)}. \]  

(19)

In the following result, we implicitly assume that \(|\mathcal{C}| \to \infty\), which is the most interesting case.

**Proposition 2.** Assume that \( \phi \in \Phi_h \) satisfies \( \|\phi\|_1 \leq \eta < 1 \) and that \(|S^h| \geq |S|/2\). The test \( f(x) = 1_{\{U(x) > \Delta\}} \) has risk at most \( 2/|\mathcal{C}| \) when

\[ \|\phi\|^2 \min_{S \in \mathcal{C}} |S| \geq C \log(|\mathcal{C}|), \]

(20)

where \( C > 0 \) only depends on \( d \) and \( \eta \).

Comparing with Condition (18), we see that condition (20) matches (up to constants) the minimax lower bound.

### 3.3 Unknown covariance

We now consider the case where the covariance operator \( \Gamma (\phi) \) of the underlying Gaussian Markov random field is unknown. We therefore start by defining a class of covariance operators via a class of vectors \( \phi \). Given a positive integer \( h > 0 \) and some \( r > 0 \), define

\[ \Phi_h (r) := \{ \phi \in \Phi_h, \, \|\phi\|_2 \geq r \}. \]

(21)

#### 3.3.1 Lower bound

The theorem below establishes a lower bound for the risk following the approach outlined in Section 2, which is based on the choice of a suitable prior \( \pi \) on \( \Phi_h \), defined as follows. By symmetry of the elements of \( \Phi_h \), one can fix a sublattice \( \mathbb{N}_h \) of size \(|\mathbb{N}_h|/2\) such that any \( \phi \in \Phi_h \) is uniquely defined (via symmetry) by its restriction to \( \mathbb{N}_h \). Choose the distribution \( \pi \) such that \( \phi \sim \pi \) is the unique extension to \( \mathbb{N}_h \) of the random vector \( r|\mathbb{N}_h|^{-1/2} \xi \), where the coordinates of the random vector \( \xi \)—indexed by \( \mathbb{N}_h \)—are i.i.d. Rademacher random variables (i.e., symmetric \( \pm 1 \)-valued random variables). Note that \( \pi \) is acceptable since it concentrates on the set \( \{ \phi \in \Phi_h, \, \|\phi\|_2 = r \} \subset \Phi_h (r) \).

Recall the definition of the minimax risk (5) and the average risk (11). (Henceforth, to lighten the notation, we replace \( \{ \Gamma (\phi), \, \phi \in \Phi_h (r) \} \) in the subscript with \( \Phi_h (r) \).) For any priors \( \nu \) on \( \mathcal{C} \) and \( \pi \) on \( \Phi_h (r) \), \( R^*_C, \Phi_h (r) \geq \bar{R}^*_{\nu, \pi} \), and the following (much more elaborate) corollary of Proposition 1 provides a lower bound on the latter.

**Theorem 1.** There exists a constant \( C > 0 \) such that the following holds. Let \( \mathcal{C} \) be a class of disjoint subsets of \( \mathcal{V} \) and let \( \nu \) denote the uniform prior over \( \mathcal{C} \). Let \( a \in (0, 1) \) and assume that the neighborhood size \(|\mathbb{N}_h|\) satisfies

\[ |\mathbb{N}_h| \leq \min_{S \in \mathcal{C}} \frac{|S|}{\log(|\mathcal{C}|/a)} \wedge |S|^{2/5} \log^{1/5} (|\mathcal{C}|/a) \wedge \left( \frac{|S|}{|A_{2h} (S)|} \right)^2 \log^{-1/6} (|\mathcal{C}|/a) . \]

(22)

Then \( \bar{R}^*_{\nu, \pi} \geq 1 - a \) as soon as

\[ r^2 \leq C \min_{S \in \mathcal{C}} \left[ \sqrt{\frac{|\mathbb{N}_h| \log (|\mathcal{C}|/a)}{|S|} + \frac{\log (|\mathcal{C}|/a)}{|S|}} \right]. \]

(23)

Notice that the second term in (23) is what appears in (17), which we saw arises in the case where the covariance is known. In light of this fact, we may interpret the first term in (23) as the ‘price to pay’ for adapting to an unknown covariance operator in the class of covariance operators of Gaussian Markov random fields with dependency radius \( h \).
3.3.2 A Fisher-type test

In this section we introduce a test whose performance essentially matches the minimax lower bound of Theorem 1.

Let $F_i = (X_{i+v} : 1 \leq |v|_\infty \leq h)$, seen as a vector, and let $F_{S,h}$ be the matrix with row vectors $F_i, i \in S^h$. Also, let $X_{S,h} = (X_i : i \in \Delta_h(S))$. Under the null hypothesis, each variable $X_i$ is independent of $F_i$, although $X_i$ is correlated with some ($F_j, j \neq i$). Under the alternative hypothesis, there exists a subset $S$ and a vector $\phi \in \Phi_h$ such that

$$X_{S,h} = F_{S,h}\phi + \epsilon_{S,h}, \quad (24)$$

where each component $\epsilon_i$ of $\epsilon_{S,h}$ is independent of the corresponding vector $F_i$, but the $\epsilon_i$'s are not necessarily independent. Equation (24) is the so-called conditional autoregressive (CAR) representation of a Gaussian Markov random field (Guyon, 1995). For Gaussian Markov random fields, the celebrated pseudo-likelihood method (Besag, 1975) amounts to estimating $\phi$ by taking least-squares in (24).

Returning to our testing problem, observe that the null hypothesis is true if, and only if, all the parameters of the conditional expectation of $X_{S,h}$ given $F_{S,h}$ are zero. In analogy with the analysis of variance approach for testing whether the coefficients of a linear regression model are all zero, we consider a Fisher-type statistic

$$T^* = \max_{S \in C} T_S; \quad T_S = \frac{|S^h||\Pi_{S,h}X_{S,h}||_2^2}{\|X_{S,h} - \Pi_{S,h}X_{S,h}\|_2^2}, \quad (25)$$

where $\Pi_{S,h} = F_{S,h}(F_{S,h}^T F_{S,h})^{-1} F_{S,h}^T$ is the orthogonal projection onto the column space of $F_{S,h}$. Since in the linear model (24) the response vector $X_{S,h}$ is not independent of the design matrix $F_{S,h}$, the statistic $T_S$ does not follow an $F$-distribution. Nevertheless, we are able to control the deviations of $T^*$, both under null and alternative hypotheses, leading to the following performance bound. Recall the definition (4).

**Theorem 2.** There exist four positive numerical constants $C_1, C_2, C_3, C_4$ such that the following holds. Assume that

$$|N_h|^4 \lor |N_h|^2 \log(|C|) \leq C_1 \min_{S \in C} |S^h| . \quad (26)$$

Fix $\alpha$ and $\beta$ in $(0,1)$ such that

$$\log(\frac{1}{\alpha}) \lor \log(\frac{1}{\beta}) \leq C_2 \min_{S \in C} |S^h| \frac{|N_h|^2 \log(|C|)}{|N_h|^2 \log(|C|)} . \quad (27)$$

Then, under the null hypothesis,

$$\mathbb{P} \left\{ T^* \geq |N_h| + C_3 \left( \sqrt{|N_h| \log(|C|)} + 1 + \log(\alpha^{-1}) \right) + \log(|C|) + \log(\alpha^{-1}) \right\} \leq \alpha , \quad (28)$$

while under the alternative,

$$\mathbb{P} \left\{ T^* \geq |N_h| + C_4 \left( |S^h| \left( \frac{\|\phi\|_2^2}{\sqrt{|N_h|}} \right) \lor 1 \right) - \sqrt{N_h}(1 + \log^4(\beta^{-1})) \right\} \geq 1 - \beta . \quad (29)$$

In particular, if $\alpha_n, \beta_n \to 0$ are arbitrary positive sequences, then the test $f$ that rejects the null hypothesis if

$$T^* \geq |N_h| + C_3 \left( \sqrt{|N_h| \log(|C|)} + 1 + \log(\alpha_n^{-1}) \right) + \log(|C|) + \log(\alpha_n^{-1})$$

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satisfies $R_{\mathcal{C}, \phi_h}(f) \to 0$ as soon as
\[
2 > \frac{C}{\min_{S \in \mathcal{C}} |S|^2} \left[ \sqrt{|N_h| \left( \log(|\mathcal{C}|) + \log(\frac{1}{\alpha}) + \log^8(\frac{1}{\beta}) \right)} \sqrt{\log(|\mathcal{C}|)} \sqrt{\log(\frac{1}{\alpha})} \right], \tag{30}
\]
where $C$ is a universal constant.

Comparing with the minimax lower bound established in Theorem 1, we see that this test is nearly optimal with respect to $h$, the size of the collection $|\mathcal{C}|$, and the size of $|S|$ of the anomalous region (under the alternative).

### 3.4 Cubes and blobs

In this section we specialize our general results proved in the previous subsections to classes of cubes, and more generally, blobs.

#### 3.4.1 Cubes

Consider the problem of detecting an anomalous cube-shaped region, still in the integer lattice. More precisely, consider the integer lattice of $n = m^d$ nodes defined in (1). Let $\ell \in \{1, \ldots, m\}$ and assume that $m$ is integer multiple of $\ell$ (for simplicity). Let $\mathcal{C}$ denote the class of all discrete hypercubes of side length $\ell$, that is, sets of the form $\prod_{s=1}^{d} \{b_s, \ldots, b_s + \ell - 1\}$, where $b_s \in \{1, \ldots, m+1-\ell\}$. Each such hypercube $S \in \mathcal{C}$ contains $|S| = k := \ell^d$ nodes, and the class is of size $|\mathcal{C}| = (m-1-\ell)^d \leq n$.

The lower bounds for the risk established in Corollary 2 and Theorem 1 are not directly applicable here since these results require subsets of the class $\mathcal{C}$ to be disjoint. However, they apply to any subclass $\mathcal{C}' \subset \mathcal{C}$ of disjoint subsets and, as mentioned in Section 2, any lower bound on the minimax risk over $\mathcal{C}'$ applies to the minimax risk over $\mathcal{C}$. A natural choice for $\mathcal{C}'$ here is that of all cubes of the form $\prod_{s=1}^{d} \{a_s \ell + 1, \ldots, (a_s + 1) \ell\}$, where $a_s \in \{0, \ldots, m/\ell - 1\}$. Note that $|\mathcal{C}'| = (m/\ell)^d = n/k$.

When $\phi$ is given and only the anomalous hypercube $S \in \mathcal{C}$ is unknown, we may apply Corollary 2 to get
\[
R_{\mathcal{C}, \phi}^* \geq 1 - \frac{k^{1/2}}{2n^{1/2}} \exp \left\{ \frac{5k\|\phi\|_2^2}{1 - 2\|\phi\|_1} \right\}.
\]
Assume that $h$ is fixed and take $\phi$ to be constant over $\mathbb{N}_h$. Assuming that the cubes are large enough that $k \gg \log n$, we thus have $R_{\mathcal{C}, \phi}^* \to 1$ when $n \to \infty$, if $(k, \phi) = (k(n), \phi(n))$ satisfies $\|\phi\|_2^2 \leq \log(n/k)/(11k)$. Comparing with the performance of the Fisher test of Section 3.3.2, in this particular case, Condition (26) is met, and letting $\alpha = \alpha(n) \to 0$ and $\beta = \beta(n) \to 0$ slowly, we conclude from (30) that this test has risk tending to zero when $\|\phi\|_2^2 \geq C \log(n)/k$ for some constant $C$. Thus, in this setting, the Fisher test achieves the correct detection rate as long as $k \leq n^b$ for some fixed $b < 1$.

When $h$ is unbounded, we obtain a sharper bound by using Theorem 1 instead of Corollary 2. Specialized to the current setting, we derive the following.

**Corollary 3.** There exist two positive constants $C_1$ and $C_2$ such that the following holds. Assume that the neighborhood size $h$ is small enough that
\[
|N_h| \leq C_1 \left[ \frac{k}{\log^{1/5}(d/2) \left( \frac{n}{k} \right)} \wedge k^{2/5} \log^{1/5} \left( \frac{n}{k} \right) \right] \wedge d^{-\frac{2d}{\pi+2} k^{-\frac{2}{\pi+6}} \log^{\frac{d}{\pi+6}} \left( \frac{n}{k} \right)} \tag{31}
\]
Then the minimax risk tends to one when \( n \to \infty \) as soon as \((k, h, r) = (k(n), h(n), r(n))\) satisfies \( n/k \to \infty \) and

\[
r^2 \leq C_2 \left[ \frac{\log(n)}{k} \sqrt{\frac{\log(\log(n))}{k}} \right].
\]

(32)

Note that, in the case of a square neighborhood, \( |N_h| = (2h + 1)^d - 1 \). Comparing with the performance of the Fisher test, in this particular case, Condition (26) is equivalent to \( |N_h| \leq C (k^{1/4} \wedge \sqrt{k/\log(n)}) \) for some constant \( C \). When \( k \) is polynomial in \( n \), this condition is stronger than Condition (31) unless \( d \leq 5 \). In any case, assuming \( h \) is small enough that both (26) and (31) hold, and letting \( \alpha = \alpha(n) \to 0 \) and \( \beta = \beta(n) \to 0 \) slowly, we conclude from (30) that the Fisher test has risk tending to zero when

\[
\|\phi\|_2^2 \geq C \left[ \frac{\log(n)}{k} \sqrt{\frac{|N_h| \log(n)}}{k} \right],
\]

for some large-enough constant \( C \), matching the lower bound (32) up to a multiplicative constant as long as \( k \leq n^b \) for some fixed \( b < 1 \).

In conclusion, whether \( h \) is fixed or unbounded but growing slowly enough, the Fisher test achieves a risk matching the lower bound up to a multiplicative constant.

### 3.4.2 Blobs

So far, we only considered hypercubes, but our results generalize immediately to much wider classes of blob-like regions. For example, fix two positive integers \( \ell_0 \leq \ell^o \) and let \( \mathcal{C} \) be a class of subsets \( S \) such that there are hypercubes \( S_0 \) and \( S^o \), of respective side lengths \( \ell_0 \) and \( \ell^o \), such that \( S_0 \subset S \subset S^o \). Letting \( \mathcal{C}_0 \) and \( \mathcal{C}^o \) denote the classes of hypercubes of side lengths \( \ell_0 \) and \( \ell^o \), respectively, our lower bound for the worst-case risk associated with the class \( \mathcal{C}^o \) obtained from Corollary 3 applies directly to \( \mathcal{C} \)—although not completely obvious, this follows from our analysis—while scanning over \( \mathcal{C}_0 \) in the Fisher test yields the performance stated above for the class of cubes. In particular, if \( \ell_0/\ell^o \) remains bounded away from 0, the problem of detecting a region in \( \mathcal{C} \) is of difficulty comparable to detecting a hypercube in \( \mathcal{C}_0 \) or \( \mathcal{C}^o \).

When the size of the anomalous region \( k \) is unknown, meaning that the class \( \mathcal{C} \) of interest includes regions of different sizes, we can simply scan over dyadic hypercubes as done in the first step of the multiscale method of Arias-Castro et al. (2005). This does not change the rate as there are less than \( 2^n \) dyadic hypercubes. See also Arias-Castro et al. (2011).

We note that when \( \ell_0/\ell^o = o(1) \) when \( n \) increases, scanning over hypercubes may not be very powerful. For example, for “convex” sets, meaning when

\[
\mathcal{C} = \left\{ S = K \cap V : K \subset \mathbb{R}^d \text{ convex, } |K \cap V| = k \right\},
\]

it is more appropriate to scan over ellipsoids due to John’s ellipsoid theorem (John, 1948), which implies that for each convex set \( K \subset \mathbb{R}^d \), there is an ellipsoid \( E \subset K \) such that \( \text{vol}(E) \geq d^{-d} \text{vol}(K) \). For the case where \( d = 2 \) and the detection-of-means problem, Huo and Ni (2009)—expanding on ideas proposed in Arias-Castro et al. (2005)—scan over parallelograms, which can be done faster than scanning over ellipses.

Finally, we mention that what we said in this section is likely to apply to other types of regular lattices, and also to lattice-like graphs such as typical realizations of a random geometric graph. See Arias-Castro et al. (2011); Walther (2010).
4 Detecting paths of correlation

We consider the setting described in Section 1.5. Specifically, we consider a general finite graph $G$ with vertex set $V$ and let $C$ be the class of subgraphs of $G$ such that each $S \in C$ forms a self-avoiding paths with $k$ nodes. When $S \in C$ is the anomalous path, there is an autoregressive process of order 1 along that path. Equivalently, $X_S$ is a one-dimensional Gaussian Markov random field with neighborhood $N_1$. The parameter vector is $\phi = (\phi_{-1}, 0, \phi_1) \in \mathbb{R}^{N_1}$ with $\phi_{-1} = \phi_1$, and throughout this section, $\phi_1$ will have that meaning. The operator $\Gamma(\phi)$ is then positive definite and invertible when $|\phi_1| < 1/2$. Note that any $S \in C$ is homomorphic to $\{1, \ldots, k\}$, and identifying the two, we have $(\Gamma_S(\phi))_{i,j} = \psi_{|i-j|}$ where

$$\psi := 1 - \sqrt{1 - 4\phi_1^2}, \quad (33)$$

and it is equivalent to parametrize the model with either $\phi_1 \in (-1/2, -1/2)$ or $\psi \in (-1, 1)$.

While Section 3 was tailored to the detection of blob-like regions, meaning “thick” subsets, paths are emblematic of “thin” subsets, which are typically much harder to detect. See Arias-Castro et al. (2011, 2008) for the detection of paths and other related subsets such as bands (thick paths) and connected components, and also Addario-Berry et al. (2010), for classes of combinatorial type such as spanning trees, in the detection-of-means setting.

We assume that the maximum degree of $G$, denoted by $\tau$, remains fixed.

4.1 Lower bound

The lower bounds in Section 3 are stated for classes of disjoint subsets. As we saw in Section 3.4, this can be successfully applied to the problem of detecting hypercubes and blobs. Here, however, a reduction to a subclass of disjoint paths is too severe. We thus develop a new lower bound that applies to subsets that may overlap.

**Theorem 3.** Let $C$ be a class of paths of $G$ and let $\nu$ denote some prior over $C$. Assume that $\phi$ is known and satisfies $|\phi_1| \leq 1/35$. Then

$$R^*_C,\phi = \frac{1}{2} \sqrt{E_{\nu \otimes \nu} \left[ \exp \left( 17\phi_1^2 |S \cap T| \right) \right]} - 1,$$

where the expectation is with respect to $S, T$ drawn i.i.d. from $\nu$.

4.2 Upper bound

We assume that $\phi$ is known and positive for simplicity. In this case, a natural approach is the generalized likelihood ratio test, which is based on rejecting the null hypothesis for large values of

$$\max_{S \in C} X_S^T (I_S - \Gamma_S(\phi))X_S.$$

Establishing a useful performance bound for the generalized likelihood ratio test in the this setting is challenging due to our lack understanding of concentration properties of the test statistic under the null hypothesis. In particular, a combination of the union bound and the concentration bound stated in Lemma 1 is insufficient.

However, we are able to craft and analyze an ad hoc test based on pairwise comparisons of consecutive values along a path. Fix a threshold $t > 0$. For $S \in C$ seen as a sequence of indices
\[ S = (S_j : j = 1, \ldots, k) \subset V, \text{ define} \]

\[ V_{t,S} = \sum_{j=2}^{k} V_{t,S}(j) , \quad V_{t,S}(j) = 1_{\{|X_{S_{j-1}} - X_{S_j}| \leq \sqrt{2t}\}} , \]

and consider the statistic and the test

\[ V^*_t = \max_{S \in \mathcal{C}} V_{t,S} , \quad f = 1_{\{V^*_t > k/2\}} . \quad (34) \]

**Remark.** Computing \( V^*_t \) above is difficult in general. A special case where this is feasible in polynomial time is when the graph can be represented as a directed acyclic graph and the class \( \mathcal{C} \) is that of all paths with exactly \( k \) nodes in the graph. In that case, computing \( V^*_t \) can be done efficiently by dynamic programming.

**Proposition 3.** Assume that \( k \geq \frac{1}{C} \log n \) (recall that \( n = |V| \)) for some constant \( C \geq 1 \). For \( t \in \mathbb{R} \), define \( p_t = 2F(t) - 1 \) where \( F \) denotes the standard normal distribution function. Fix \( t \) small enough that \( h(2p_t) \geq 8(C + \log(\tau)) \) where \( h(x) = x - \log(x) - 1 \). If \( \phi_1 \) is large enough such that \( \psi \) defined in (33) exceeds \( 1 - (t/F - 1(4/5))^2 \), then the test \( f \) defined in (34) satisfies \( R_{\mathcal{C},\phi}(f) \to 0 \).

### 4.3 Special case: the lattice

Consider the integer lattice (1) in dimension \( d \geq 3 \). The story is a little different when \( d = 2 \), and we refer the reader to the treatment in Arias-Castro et al. (2008) in the detection-of-means setting.

#### 4.3.1 Known starting point

Suppose first that the departing vertex of the path is known, meaning that \( \mathcal{C} \) is the class of all self-avoiding paths with \( k \) nodes in \( V \) starting at some given \( v_0 \in V \). In that case, when \( d \geq 3 \), there is a constant \( C > 0 \) such that, when \( |\phi_1| \leq C \), the risk is at least \( 1/2 \). To see this, let \( \nu \) be a prior on \( \mathcal{C} \) that has exponential intersection tails, which means that there exist some constants \( \eta \in (0,1) \) and \( C_0 > 0 \) such that

\[ P_{\nu \otimes \nu}(|S \cap T| \geq \ell) \leq C_0 \eta^\ell , \quad \forall \ell \geq 1 , \]

where \( S, T \) are i.i.d. from \( \nu \). This concept was introduced by Benjamini et al. (1998), who show that such a prior exists when \( d = 3 \)—and we can use the same prior when \( d \geq 3 \), by embedding the 3-dimensional lattice into the \( d \)-dimensional lattice. While Benjamini et al. (1998) considered infinite paths, here we only consider paths of length \( k \) and modify the prior accordingly as done in Arias-Castro et al. (2008). With such a prior, for any \( a > 0 \),

\[ E_{\nu \otimes \nu}[\exp(a|S \cap T|)] \leq \sum_{\ell \geq 1} e^{a\ell} P_{\nu \otimes \nu}(|S \cap T| = \ell) \]

\[ = e^a + \sum_{\ell \geq 2} (e^{a\ell} - e^{a(\ell-1)}) P_{\nu \otimes \nu}(|S \cap T| = \ell) \]

\[ \leq e^a + C_0 \frac{(e^a - 1)\eta}{1 - e^a \eta} . \]

Applying Theorem 3, we get that \( \overline{R}_{\nu,\phi} \geq 1/2 \) when \( |\phi_1| \) is sufficiently small. We conclude that, even if we know the starting point of the possible anomalous paths, the risk is bounded away from
zero as soon as $\phi_1$ is small enough. Conversely, there exists a positive constant $a_d < 1/2$, depending only on the dimension $d$, such that if $\phi_1 \in (a_d, 1/2)$, then the test $f$ defined in (34) asymptotically separates the hypotheses.

### 4.3.2 General case

In the general case, the location of the possible anomalous paths is completely unknown. In other words, $C$ is the class of all self-avoiding paths with $k$ nodes in $V$. To define the prior, we partition the lattice into hypercubes of side length $2k + 1$, indexed by $J$, and let $v_j$ denote the center of the hypercube $j \in J$. The number of such hypercubes satisfies $|J| \sim (m/2k)^d = n/(2k)^d$. Still in dimension $d \geq 3$, let $\nu_j$ be a prior on self-avoiding paths starting at $v_j$ having exponential intersection tails, and let $\nu$ be the even mixture of all these priors. Noting that paths starting from different origin nodes cannot intersect, for any $a > 0$, we have

$$E_{\nu \otimes \nu}[^{\exp(a|S \cap T|)}] = 1 - \frac{1}{|J|} E_{\nu_j \otimes \nu_j}[^{\exp(a|S \cap T|)}] \leq 1 + \frac{e^a}{|J|} + C_0 \frac{(e^a - 1)\eta}{|J|} - \frac{e^a \eta}{1 - e^a \eta},$$

where $j \in J$ is arbitrary. Calling in Theorem 3, and noting that $|J| \to \infty$ when $m \gg k$ (meaning, the grid side length dominates the path length), we see that the risk tends to 1 when $|\phi_1| \leq 1/35$. We conclude that, if the possible anomalous paths are completely arbitrary, the minimax risk tends to 1 (meaning, the hypotheses merge asymptotically) as soon as $\phi_1$ is small enough. On the other hand, if $\log(n) \leq Ck$ for some constant $C > 0$, then Proposition 3 ensures that the test $f$ asymptotically separates the hypotheses when $\phi_1$ is large enough.

**Remark.** Computing the test statistic $V_t^*$ is difficult, even when the starting point is known. Indeed, this problem is known as the **prize collecting salesman problem** or **bank robber problem** or **reward-budget problem**, and there are no known polynomial-time algorithms that solve it, although polynomial approximations do exist, see DasGupta et al. (2006).

However, an alternative test based on the length of the longest path of significant adjacent correlations is known that is both computationally tractable and achieves the same performance up to a multiplicative constant. This idea is implemented by Arias-Castro et al. (2006, 2013) in the context where the detection-of-means setting.

### 5 Discussion

We provided lower bounds and proposed near-optimal procedures for testing for the presence of a piece of a Gaussian Markov random field. These results constitute some of the first mathematical results for the problem of detecting a textured object in a noisy image. It also extends our own previous work (Arias-Castro et al., 2012, 2015) on the detection of correlations to setting where the correlations have some nontrivial structure. We leave open some questions and generalization of interest.

**More refined results.** We leave behind the delicate and interesting problem of finding the exact detection rates, with tight multiplicative constants. This is particularly appealing for simple settings such as finding an interval of autoregressive process, as described in Section 1.3. Our proof techniques are not sufficiently refined to get such sharp bounds. We already know that, in the detection-of-means setting, bounding the variance of the likelihood ratio does not yield the right constant. The variant which consists of bounding the first two moments of a carefully truncated likelihood ratio, used in Ingster (1999), is applicable here, but the calculations are quite complicated and we leave them for future research.
Texture over texture. Throughout the paper we assumed that the background is Gaussian white noise. This is not essential, but makes the narrative and results more accessible. A more general, and also more realistic setting, would be that of detecting a region where the dependency structure is markedly different from the remaining of the image. This setting has been studied in the context of time series, for example, in some of the references given in Section 1.3. However, we are not aware of existing theoretical results in higher-dimensional settings such as in images.

Other dependency structures. We focused on Markov random fields with limited neighborhood range (quantified by $h$ earlier in the paper). This is a natural first step, particularly since these are popular models for time series and textures. However, one could envision studying other dependency structures, such as short-range dependency, defined in Samorodnitsky (2006) as situations where the covariances are summable in the following sense

$$\sup_{i \in V, j \in V \setminus \{i\}} |\Gamma_{i,j}| < \infty.$$ 

6 Proofs

6.1 Deviation inequalities

Here we collect a few more-or-less standard inequalities that we need in the proofs. We start with the following standard tail bounds for Gaussian quadratic forms. See, e.g., Example 2.12 and Exercise 2.9 in Boucheron et al. (2013).

**Lemma 1.** Let $Z$ be a standard normal vector in $\mathbb{R}^d$ and let $R$ be a symmetric $d \times d$ matrix. Then

$$\mathbb{P}\left\{ Z^\top R Z - \text{Tr}(R) \geq 2\|R\|_F \sqrt{t} + 2\|R\| t \right\} \leq e^{-t}, \quad \forall t \geq 0.$$

Furthermore, if the matrix $R$ is positive semidefinite, then

$$\mathbb{P}\left\{ Z^\top R Z - \text{Tr}(R) \leq -2\|R\|_F \sqrt{t} \right\} \leq e^{-t}, \quad \forall t \geq 0.$$

**Lemma 2.** There exists a positive constant $C$ such that the following holds. For any Gaussian chaos $Z$ up to order 4 and any $t > 0$,

$$\mathbb{P}\left\{ |Z - \mathbb{E}[Z]| \geq C \text{Var}^{1/2}(Z)t^2 \right\} \leq e^{-t}.$$

**Proof.** This deviation inequality is a consequence of the hypercontractivity of Gaussian chaos. More precisely, Theorem 3.2.10 and Corollary 3.2.6 in de la Peña and Giné (1999) state that

$$\mathbb{E}\exp\left[ \left( \frac{Z - \mathbb{E}[Z]}{C \text{Var}^{1/2}(Z)} \right)^{1/2} \right] \leq 2,$$

where $C$ is a numerical constant. Then, we apply Markov inequality to prove the lemma. \qed

**Lemma 3.** There exists a positive constant $C$ such that the following holds. Let $F$ be a compact set of symmetric $r \times r$ matrices and let $Y \sim \mathcal{N}(0, I_r)$. For any $t > 0$, the random variable $Z := \sup_{R \in F} \text{Tr}[RYY^\top]$ satisfies

$$\mathbb{P}\{Z \geq \mathbb{E}(Z) + t\} \leq \exp\left( -C \left( \frac{t^2}{\mathbb{E}(W)} \wedge \frac{t}{B} \right) \right),$$

where $W := \sup_{R \in F} \text{Tr}(RYY^\top R)$ and $B := \sup_{R \in F} \|R\|$. 

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A slight variation of this result where \( Z \) is replaced by \( \sup_{R \in F} \text{Tr} \left[ R (YY^\top - I) \right] \) is proved in Verzelen (2010) using the exponential Efron-Stein inequalities of Boucheron et al. (2005). Their arguments straightforwardly adapt to Lemma 3.

**Lemma 4** (Davidson and Szarek (2001)). Let \( W \) be a standard Wishart matrix with parameters \((n,d)\) satisfying \( n > d \). Then for any number \( 0 < x < 1 \),

\[
\begin{align*}
\mathbb{P} \left\{ \lambda_{\text{max}}(W) \geq n \left( 1 + \sqrt{d/n} + \sqrt{2x/n} \right)^2 \right\} & \leq e^{-x}, \\
\mathbb{P} \left\{ \lambda_{\text{min}}(W) \leq n \left( 1 - \sqrt{d/n} - \sqrt{2x/n} \right)^2 \right\} & \leq e^{-x}.
\end{align*}
\]

6.2 Auxiliary results for Gaussian Markov random fields on the lattice

In we gather some technical tools and proofs for Gaussian Markov random fields on the lattice. Recall the notation introduced in Section 3.

**Lemma 5.** For any positive integer \( h \) and \( \phi \in \Phi_h \) with \( \|\phi\|_1 < 1 \), we have that if \( \lambda \) is an eigenvalue of the covariance operator \( \Gamma(\phi) \), then

\[
\frac{\sigma_\phi^2}{1 + \|\phi\|_1} \leq \lambda \leq \frac{\sigma_\phi^2}{1 - \|\phi\|_1}.
\]

Also, we have

\[
\frac{\|\phi\|^2_2}{1 + \|\phi\|_1} \leq \frac{1 - \sigma_\phi^2}{\sigma_\phi^2} \leq \frac{\|\phi\|^2_2}{1 - \|\phi\|_1} \quad \text{and} \quad 1 - \|\phi\|_1 \leq \sigma_\phi^2 \leq 1. \tag{36}
\]

**Proof.** Recall that \( \|\cdot\| \) denotes the \( \ell^2 \rightarrow \ell^2 \) operator norm. First note that by the definition of \( \phi \),

\[
\sigma_\phi^2 \Gamma^{-1}(\phi) - I = (\phi_i - \phi_j)_{i,j \in \mathbb{Z}^d},
\]

and therefore

\[
\|\sigma_\phi^2 \Gamma^{-1}(\phi) - I\| \leq \|\phi\|_1, \tag{37}
\]

where \( \|A\| \leq \sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |A_{ij}| \). This implies that the largest eigenvalue of \( \Gamma(\phi) \) is bounded by \( \sigma_\phi^2/(1 - \|\phi\|_1) \) if \( \|\phi\|_1 < 1 \) and that the smallest eigenvalue of \( \Gamma(\phi) \) is at least \( \sigma_\phi^2/(1 + \|\phi\|_1) \). Considering the conditional regression of \( Y_i \) given \( Y_{-i} \) mentioned above, that is,

\[
Y_i = - \sum_{1 \leq |j|_\infty \leq h} \phi_j Y_{i+j} + \epsilon_i
\]

(with \( \epsilon_i \) being standard normal independent of the \( Y_j \) for \( j \neq i \)) and taking the variance of both sides, we obtain

\[
1 - \sigma_\phi^2 = \text{Var} \left[ \sum_{1 < |j|_\infty \leq h} \phi_j Y_{i+j} \right] = \phi^\top \Gamma(\phi) \phi \leq \|\Gamma(\phi)\| \|\phi\|^2_2 \leq \frac{\|\phi\|^2_2}{1 - \|\phi\|_1} \sigma_\phi^2,
\]

and therefore

\[
1 - \sigma_\phi^2 \leq \frac{\|\phi\|^2_2}{1 - \|\phi\|_1} \sigma_\phi^2.
\]

Rearranging this inequality and using the fact that \( \|\phi\|^2_2 \leq \|\phi\|^2_2 \leq \|\phi\|_1 \), we conclude that \( \sigma_\phi^2 \geq 1 - \|\phi\|_1 \). The remaining bound \( \frac{\|\phi\|^2_2}{1 + \|\phi\|_1} \leq \frac{1 - \sigma_\phi^2}{\sigma_\phi^2} \) is obtained similarly. \(\square\)
Recall that for any \( v \in \mathbb{Z}^d \), \( \gamma_v \) is the correlation between \( Y_i \) and \( Y_{i+v} \) and is therefore equal to \( \Gamma_{i,i+v} \). This definition does not depend on the node \( i \) since \( \Gamma \) is the covariance of a stationary process.

**Lemma 6.** For any \( h \) and any \( \phi \in \Phi_h \), let \( Y \sim \mathcal{N}(0, \Gamma(\phi)) \). As long as \( \| \phi \|_1 < 1 \), the \( l_2 \) norm of the correlations satisfies

\[
\sum_{v \neq 0} \gamma_v^2 \leq \frac{\| \phi \|_2}{(1 - \| \phi \|_1)^2} + \left( \frac{\| \phi \|^2_2 \sigma^2}{(1 - \| \phi \|_1)^2} \right)^2
\]

(38)

**Proof.** In order to compute \( \| \gamma \|^2_2 \), we use the spectral density of \( Y \) defined by

\[
f(\omega_1, \ldots, \omega_d) = \frac{1}{(2\pi)^d} \sum_{(v_1, \ldots, v_d) \in \mathbb{Z}^d} \gamma_{v_1, \ldots, v_d} \exp \left( i \sum_{i=1}^d v_i \omega_i \right), \quad (\omega_1, \ldots, \omega_d) \in (-\pi, \pi]^d.
\]

Following (Guyon, 1995, Sect.1.3) or (Rue and Held, 2005, Sect.2.6.5), we express the spectral density in terms of \( \phi \) and \( \sigma^2_\phi \):

\[
1 \quad \frac{1}{f(\omega_1, \ldots, \omega_d)} = \frac{(2\pi)^d}{\sigma^2_\phi} \left[ 1 - \sum_{v_1 \leq |v| \leq h \in \mathbb{Z}^d} \phi_v e^{i\langle v, \omega \rangle} \right],
\]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^p \). As a consequence,

\[
|f(\omega_1, \ldots, \omega_d)| \leq \sigma^2_\phi [(2\pi)^d(1 - \| \phi \|_1)]^{-1}.
\]

Relying on Parseval formula, we conclude

\[
\sum_{v \neq 0} \gamma_v^2 = (2\pi)^d \int_{[-\pi, \pi]^d} \left[ f(\omega_1, \ldots, \omega_d) - \frac{1}{(2\pi)^d} \right]^2 d\omega_1 \ldots d\omega_d
\]

\[
\leq \frac{\sigma^4_\phi}{(2\pi)^d(1 - \| \phi \|_1)^2} \int_{[-\pi, \pi]^d} \left[ \frac{1}{(2\pi)^d f(\omega_1, \ldots, \omega_d)} - 1 \right]^2 d\omega_1 \ldots d\omega_d
\]

\[
\leq \frac{\sigma^4_\phi}{(2\pi)^d(1 - \| \phi \|_1)^2} \int_{[-\pi, \pi]^d} \left[ \frac{1}{\sigma^2_\phi} - 1 \right]^{2 \cdot \sum_{v_1 \leq |v| \leq h \in \mathbb{Z}^d} \phi_v e^{i\langle v, \omega \rangle}} d\omega_1 \ldots d\omega_d
\]

\[
\leq \frac{\sigma^4_\phi}{(2\pi)^d(1 - \| \phi \|_1)^2} \left[ (2\pi)^d \left( \frac{1}{\sigma^2_\phi} - 1 \right) \right]^2 \sum_{v_1 \leq |v| \leq h \in \mathbb{Z}^d} (2\pi)^d \phi_v^2
\]

\[
\leq \left( \frac{1 - \sigma^2_\phi}{1 - \| \phi \|^2_1} \right) \frac{\| \phi \|^2_2}{(1 - \| \phi \|^2_1)^2} \frac{\sigma^2_\phi}{(1 - \| \phi \|^2_1)^2} + \frac{\| \phi \|^4_2}{(1 - \| \phi \|^4_1)^2},
\]

where we used (36) in the last line.

**Lemma 7** (Conditional representation). For any \( h \) and any \( \phi \in \Phi_h \), let \( Y \sim \mathcal{N}(0, \Gamma(\phi)) \). Then for any \( i \in \mathbb{Z}^d \), the random variable \( \epsilon_i \) defined by the conditional regression \( Y_i = \sum_{v \in \mathbb{N}_h} \phi_v Y_{i+v} + \epsilon_i \) satisfies that
1. \( \epsilon_i \) is independent of all \( X_j, j \neq i \) and \( \text{Cov}(\epsilon_i, X_i) = \text{Var}(\epsilon_i) = \sigma^2_\epsilon. \)

2. For any \( i \neq j \), \( \text{Cov}(\epsilon_i, \epsilon_j) = -\phi_{i-j}\sigma^2_\epsilon \) if \( |i - j|_\infty \leq h \) and 0 otherwise.

**Proof.** The first independence property is a classical consequence of the conditional regression representation for Gaussian random vectors, see, for example, Lauritzen (1996). Since \( \text{Var}(\epsilon_i) \) is the conditional variance of \( Y_i \) given \( Y^{(-i)} \), it equals \( [(\Gamma^{-1}(\phi))_{i,i}]^{-1} = \sigma^2_\epsilon. \) Furthermore,

\[
\text{Cov}(\epsilon_i, Y_i) = \text{Var}(\epsilon_i) + \sum_{v \in \mathcal{N}_h} \phi_j \text{Cov}(\epsilon_i, Y_{i+v}) = \text{Var}(\epsilon_i),
\]

by the independence of \( \epsilon_i \) and \( Y^{(-i)} \). Finally, consider any \( i \neq j \),

\[
\text{Cov}(\epsilon_i, \epsilon_j) = \text{Cov}(\epsilon_i, Y_j) - \sum_{v \in \mathcal{N}_h} \phi_v \text{Cov}(\epsilon_i, Y_{j+v}),
\]

where all the terms are equal to zero with the possible exception of \( v = i - j \). The result follows.

**Lemma 8** (Comparison of \( \Gamma^{-1}(\phi) \) and \( \Gamma^{-1}_S(\phi) \)). As long as \( \|\phi\|_1 < 1 \), the following properties hold:

1. If \( i \in S^h \) or if \( j \in S^h \), then \( (\Gamma^{-1}_S(\phi))_{i,j} = (\Gamma^{-1}(\phi))_{i,j}. \)

2. If \( i \in S^h \) and \( j \in \Delta_h(S) \), then \( 1 \leq (\Gamma^{-1}_S(\phi))_{j,i} \leq (\Gamma^{-1}(\phi))_{i,i}. \)

3. If \( i \in \Delta_h(S) \), then \( \sum_{j \in S \setminus \{j\}} (\Gamma^{-1}_S(\phi))_{i,j}^2 \leq \frac{2\|\phi\|_2^2}{(1 - \|\phi\|_1)^2}. \)

**Proof.** We prove each part in turn.

**Part 1.** Consider \( i \in S^h \) and any \( j \in S \). By the Markov property, conditionally to \( (Y_{i+k}, 1 \leq |k|_\infty \leq h) \), \( Y_i \) is independent of all the remaining variables. Since all vertices \( i+k \) with \( 1 \leq |k|_\infty \leq h \) belong to \( S \), the conditional distribution of \( Y_i \) given \( Y^{(-i)} \) is the same as the conditional distribution of \( Y_i \) given \( Y_{j,i} \in S \setminus \{i\} \). This conditional distribution characterizes the \( i \)-th row of the matrix \( \Gamma_S \). Also, the conditional variance of \( Y_i \) given \( Y^{(-i)} \) is \( [(\Gamma^{-1}(\phi))_{i,i}]^{-1} \) and the conditional variance of \( Y_i \) given \( Y_S \) is \( [(\Gamma^{-1}_S(\phi))_{i,i}]^{-1}. \) Furthermore, \( -(\Gamma^{-1}(\phi))_{i,j}/(\Gamma^{-1}(\phi))_{i,i} \) is the \( j \)-th parameter of the condition regression of \( Y_i \) given \( Y^{(i)} \), and therefore we conclude that \( (\Gamma^{-1}(\phi))_{i,i} = (\sigma^2_\epsilon)^{-1} = (\Gamma^{-1}_S(\phi))_{i,i} \) and \( (\Gamma^{-1}_S(\phi))_{i,j}/(\Gamma^{-1}_S(\phi))_{i,i} = -\phi_{i-j} = (\Gamma^{-1}_S(\phi))_{i,j}/(\Gamma^{-1}_S(\phi))_{i,i}. \)

**Part 2.** Consider any vertex \( i \in S^h \) and \( j \in \Delta_h(S) \). Since \( 1/(\Gamma^{-1}_S(\phi))_{j,j} \) and \( 1/(\Gamma^{-1}_S(\phi))_{j,j} \) are the conditional variances of \( Y_i \) and \( Y_j \) given \( Y_k, k \in S \setminus \{j\} \) and \( Y_k, k \in S \setminus \{i\} \), respectively, we have

\[
1/(\Gamma^{-1}_S(\phi))_{j,j} = \text{Var}(Y_j|Y_{k} : k \in S \setminus \{j\}) \geq \text{Var}(Y_j|Y^{(-j)}) = \text{Var}(Y_i|Y^{(-i)}) (\text{by stationarity of } Y) = \text{Var}(Y_i|Y_k : k \in S \setminus \{i\}) (\text{since the neighborhood of } i \text{ is included in } S) = 1/(\Gamma^{-1}_S(\phi))_{i,i}.
\]

**Part 3.** Consider \( i \in \Delta_h(S) \). The vector \( (\Gamma^{-1}_S(\phi))_{i,-i} \) is formed by the regression coefficients of \( Y_i \) on \( (Y_j, j \in S \setminus \{i\}) \). Since the conditional variance of \( Y_i \)
given \((Y_j, j \in S \setminus \{i\})\) is at least \(\sigma^2_\phi\) (by Parts 1 and 2), we get

\[
1 - \sigma^2_\phi \geq 1 - \text{Var}(Y_i | S \setminus \{i\}) = \text{Var} \left( \mathbb{E}(Y_i | S \setminus \{i\}) \right) = \text{Var} \left( \sum_{j \in S \setminus \{i\}} \left( \Gamma^{-1}_S(\phi)_{i,j} Y_j \right) \right) \geq \sigma^4_\phi \text{Var} \left( \sum_{j \in S \setminus \{i\}} (\Gamma^{-1}_S(\phi))_{i,j} Y_j \right) = \sigma^4_\phi (\Gamma^{-1}_S(\phi))_{i,-i} \Gamma_S(\phi)(\Gamma^{-1}_S(\phi))_{i,-i} \geq \frac{\sigma^6_\phi}{1 + \|\phi\|_1^2} ||(\Gamma^{-1}_S(\phi))_{i,-i}||_2^2,
\]

where the equality in the second line above we use \(\text{Var}(Y_j) = 1\) and the law of total variance (i.e., \(\text{Var}(Y) = \mathbb{E}[\text{Var}(Y | B)] + \text{Var}(\mathbb{E}(Y | B))\)) and in the last line we use that the smallest eigenvalue of \(\Gamma(\phi)\) (and also of \(\Gamma_S(\phi)\)) is larger than \(\sigma^2_\phi/(1 + \|\phi\|_1)\) (Lemma 5). Rearranging this inequality and using the fact that \(\|\phi\|_1 < 1\), we arrive at

\[
||(\Gamma^{-1}_S(\phi))_{i,-i}||_2^2 \leq \frac{1 - \sigma^2_\phi}{\sigma^6_\phi} (1 + \|\phi\|_1) \leq \frac{1 - \sigma^2_\phi}{\sigma^6_\phi} \leq \frac{2||\phi||_2^2}{\sigma^4_\phi (1 - ||\phi||_1^3)} \quad \text{(by (36))}
\]

\[
\leq \frac{2||\phi||_2^2}{(1 - ||\phi||_1^3)^3} \quad \text{(using Lemma 5)}.
\]

\[\square\]

**Lemma 9.** For any \(\phi_1, \phi_2 \in \Phi_h\), define

\[
B_{\phi_1, \phi_2} := \left( \frac{\det(\Gamma^{-1}_S(\phi_1)) \det(\Gamma^{-1}_S(\phi_2))}{\det(\Gamma^{-1}_S(\phi_1) + \Gamma^{-1}_S(\phi_2) - I)} \right)^{1/2}.
\]

(Note that \(V_S\) defined in Proposition 1 equals the expected value of \(B_{\phi_1, \phi_2}\) when \(\phi_1\) and \(\phi_2\) are drawn independently from the distribution \(\pi\).) Assuming that \(\|\phi_1\|_1 \lor \|\phi_2\|_1 < 1/5\), we have

\[
\log B_{\phi_1, \phi_2} \leq \frac{1}{2} |S| \langle \phi_1, \phi_2 \rangle + 8Q_S,
\]

where

\[
Q_S := |S| \sum_{s_1, s_2, s_3 = 1}^2 \sum_{j,k \in N_h} \phi_{s_1,j} \phi_{s_2,k} \phi_{s_3,j,k-1} + 15|S|(||\phi_1||_3^2 \lor ||\phi_2||_3^2) + 1|S| ||\phi_1||_2^2 \lor ||\phi_2||_2^2 + 2|\Delta h(S)| (||\phi_1||_2^3 \lor ||\phi_2||_2^3).
\]

**Proof.** Since for any \(\phi\), the spectrum of \(\Gamma^{-1}_S(\phi)\) lies between the extrema of the spectrum of \(\Gamma^{-1}(\phi)\), by Lemma 5, we have

\[
\frac{1}{\sigma^2_\phi} - 1 \leq \lambda_{\min}(\Gamma^{-1}_S(\phi) - I) \leq \lambda_{\max}(\Gamma^{-1}_S(\phi) - I) \leq \frac{1 + \|\phi\|_1}{\sigma^2_\phi} - 1,
\]

where
where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a matrix $A$. Since $\sigma_\phi^2 \leq \text{Var}(Y_i) = 1$, the left-hand side is larger than $-\|\phi\|_1$, while relying on (36), we derive
\[
\frac{1 + \|\phi\|_1}{\sigma_\phi^2} - 1 \leq (\|\phi\|_1 + 1) \left[ 1 + \frac{\|\phi\|^2_1}{1 - \|\phi\|_1} \right] - 1 \leq \frac{2\|\phi\|_1}{1 - \|\phi\|_1}.
\]
Consequently, as long as $\|\phi\|_1 < 1/5$, the spectrum of $\Gamma_{S}^{-1}(\phi)$ lies in $(\frac{1}{5}, \frac{3}{4})$. This allows us to use the Taylor series of the logarithm, which for a matrix $A$ with spectrum in $(\frac{1}{2}, 2)$, gives
\[
\left| \log (\det(A)) - \text{Tr} [A - I] + \frac{1}{2} \text{Tr} [(A - I)^2] \right| \leq \frac{8}{3} |\text{Tr} [(A - I)^3]|.
\]
Applying this expansion to $\Gamma_{S}^{-1}(\phi_1), \Gamma_{S}^{-1}(\phi_2)$ and $\Gamma_{S}^{-1}(\phi_1) + \Gamma_{S}^{-1}(\phi_2) - I$,
\[
2 \log B_{\phi_1, \phi_2} \leq V_1 + \frac{16}{3} V_2 + 8V_3 + 8V_4,
\]
\[
V_1 := \text{Tr} \left[ (\Gamma_{S}^{-1}(\phi_1) - I)(\Gamma_{S}^{-1}(\phi_2) - I) \right],
\]
\[
V_2 := \left| \text{Tr} \left[ (\Gamma_{S}^{-1}(\phi_1) - I)^3 \right] + \text{Tr} \left[ (\Gamma_{S}^{-1}(\phi_2) - I)^3 \right] \right|,
\]
\[
V_3 := \text{Tr} \left[ (\Gamma_{S}^{-1}(\phi_1) - I)(\Gamma_{S}^{-1}(\phi_2) - I)(\Gamma_{S}^{-1}(\phi_1) - I) \right],
\]
\[
V_4 := \text{Tr} \left[ (\Gamma_{S}^{-1}(\phi_2) - I)(\Gamma_{S}^{-1}(\phi_1) - I)(\Gamma_{S}^{-1}(\phi_2) - I) \right].
\]

**Control of $V_1$.** We use the fact that
\[
\text{Tr} \left[ (\Gamma_{S}^{-1}(\phi_1) - I)(\Gamma_{S}^{-1}(\phi_2) - I) \right] = \sum_{i,j \in S} \left( (\Gamma_{S}^{-1}(\phi_1))_{i,j} - \delta_{i,j} \right) \left( (\Gamma_{S}^{-1}(\phi_2))_{i,j} - \delta_{i,j} \right).
\]

To bound the right-hand side, first consider any node $i \in S^h$ in the $h$-interior of $S$. By the first part of Lemma 8, the $i$-th row of $\Gamma_{S}^{-1}(\phi)$ equals the restriction to $S$ of the $i$-th row of $\Gamma^{-1}(\phi)$. Using the definition of $\phi_1, \phi_2$, we therefore have
\[
\sum_{j \in S} \left( (\Gamma_{S}^{-1}(\phi_1))_{i,j} - \delta_{i,j} \right) \left( (\Gamma_{S}^{-1}(\phi_2))_{i,j} - \delta_{i,j} \right) = (\Gamma^{-1}(\phi_1))_{i,i}(\Gamma^{-1}(\phi_2))_{i,i} + ((\Gamma^{-1}(\phi_1))_{i,i} - 1)((\Gamma^{-1}(\phi_2))_{i,i} - 1)
\]
\[
= \frac{\langle \phi_1, \phi_2 \rangle}{\sigma_{\phi_1}^2 \sigma_{\phi_2}^2} + \frac{(1 - \sigma_{\phi_1}^2)(1 - \sigma_{\phi_2}^2)}{\sigma_{\phi_1}^2 \sigma_{\phi_2}^2}
\]
\[
= \frac{\langle \phi_1, \phi_2 \rangle}{\sigma_{\phi_1}^2 \sigma_{\phi_2}^2} + \frac{1 - \sigma_{\phi_1}^2 \sigma_{\phi_2}^2}{\sigma_{\phi_1}^2 \sigma_{\phi_2}^2} + \frac{(1 - \sigma_{\phi_1}^2)(1 - \sigma_{\phi_2}^2)}{\sigma_{\phi_1}^2 \sigma_{\phi_2}^2}
\]
\[
\leq \frac{\langle \phi_1, \phi_2 \rangle}{2} + \frac{3}{2} \frac{\|\phi_1\|_2^2 + \|\phi_2\|_2^2}{(1 - \|\phi_1\|_1)(1 - \|\phi_2\|_1)}
\]
}\]

using Lemma 5 in the last line. Next, consider a node $i \in \Delta_h(S)$, near the boundary of $S$. Relying on Lemmas 5 and 8, we get
\[
\sum_{j \in S} \left( (\Gamma_{S}^{-1}(\phi_1))_{i,j} - \delta_{i,j} \right)^2 \leq \frac{2 \|\phi_1\|_2^2}{(1 - \|\phi_1\|_1)^2} + \left(1/\sigma_{\phi_1}^2 - 1\right)^2
\]
\[
\leq \frac{2 \|\phi_1\|_2^2}{(1 - \|\phi_1\|_1)^2} + \frac{\|\phi_1\|_2^4}{(1 - \|\phi_1\|_1)^2} \leq \frac{3 \|\phi_1\|_2^4}{(1 - \|\phi_1\|_1)^3},
\]
\[
(40)
\]
since we assume that $\|\phi\|_1 < 1$. By the Cauchy-Schwarz inequality,

$$
\sum_{j \in S} ((\Gamma_S^{-1}(\phi_1))_{i,j} - \delta_{i,j}) ((\Gamma_S^{-1}(\phi_2))_{i,j} - \delta_{i,j}) \leq 3 \frac{\|\phi_1\|^2 \vee \|\phi_2\|^2}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3}.
$$

(Summing (39) over $i \in S^h$ and (41) over $i \in \Delta_h(S)$, we get

$$
V_1 \leq |S|(|\phi_1, \phi_2| + 3 |S| \frac{\|\phi_1\|^2 + \|\phi_2\|^2}{(1 - \|\phi_1\|_1)(1 - \|\phi_2\|_1)} + 3|\Delta_h(S)| \frac{\|\phi_1\|^2 \vee \|\phi_2\|^2}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3}.
$$

Control of $V_2$. We proceed similarly as in the previous step. Note that

$$
\text{Tr} \left( (\Gamma_S^{-1}(\phi_1) - I)^3 \right) = \sum_{i,j,k \in S} ((\Gamma_S^{-1}(\phi_1))_{i,j} - \delta_{i,j}) ((\Gamma_S^{-1}(\phi_1))_{j,k} - \delta_{j,k}) ((\Gamma_S^{-1}(\phi_1))_{k,i} - \delta_{k,i})
$$

$$
\leq \sum_{i \in S} \left| \sum_{j,k \in S} ((\Gamma_S^{-1}(\phi_1))_{i,j} - \delta_{i,j}) ((\Gamma_S^{-1}(\phi_1))_{j,k} - \delta_{j,k}) ((\Gamma_S^{-1}(\phi_1))_{k,i} - \delta_{k,i}) \right|.
$$

First, consider a node $i$ in $S \setminus \Delta_{2h}(S)$. Here, we use $\Delta_{2h}(S)$ instead of $\Delta_h(S)$ so that we may replace $\Gamma_S^{-1}(\phi)$ below with $\Gamma^{-1}(\phi)$. We use again Lemma 8 to replace $(\Gamma_S^{-1}(\phi))_{i,j}$ by $(\Gamma^{-1}(\phi))_{i,j}$ in the sum

$$
\left| \sum_{j \in S} ((\Gamma_S^{-1}(\phi_1))_{i,j} - \delta_{i,j}) ((\Gamma_S^{-1}(\phi_1))_{j,v} - \delta_{j,v}) ((\Gamma_S^{-1}(\phi_1))_{v,i} - \delta_{v,i}) \right|
$$

$$
\leq \left| \sum_{j,k \in N_h} \frac{\phi_{i,j} \phi_{j,k} \phi_{k,i}}{\sigma_{\phi_1}^6} \right| + 3 \sum_{j \in N_h} |\phi_{i,j}|^2 \frac{1 - \sigma_{\phi_1}^2}{\sigma_{\phi_1}^6} + \frac{(1 - \sigma_{\phi_1}^2)^3}{\sigma_{\phi_1}^6}
$$

$$
\leq \left| \sum_{j,k \in N_h} \frac{\phi_{i,j} \phi_{j,k} \phi_{k,i}}{1 - \|\phi_1\|_1^3} \right| + 4 \frac{\|\phi_1\|^2}{(1 - \|\phi_1\|_1)^3},
$$

using Lemma 6 in the last line. Next, consider a node $i \notin \Delta_{2h}(S)$. If $i \notin \Delta_{2h}(S)$, then the support of $(\Gamma_S^{-1}(\phi_1))_{i,-i}$ is of size $|N_h|$. If $i \in \Delta_h(S)$, then $\Delta_{2h}(S) \setminus \{i\}$ separates $\{i\}$ from $S \setminus \Delta_{2h}(S)$ in the dependency graph and the Global Markov property (Lauritzen, 1996) entails that

$$
Y_i \perp (Y_k, k \in S \setminus \Delta_{2h}(S))|(Y_k, k \in \Delta_{2h}(S) \setminus \{i\}),
$$

and therefore the support of $(\Gamma_S^{-1}(\phi_1))_{i,-i}$ is of size smaller than $|\Delta_{2h}(S)|$. Using the Cauchy-Schwarz inequality and (40), we get

$$
\sum_{j \in S} \sum_{k \in S} ((\Gamma_S^{-1}(\phi_1))_{i,j} - \delta_{i,j}) ((\Gamma_S^{-1}(\phi_1))_{j,k} - \delta_{j,k}) ((\Gamma_S^{-1}(\phi_1))_{k,i} - \delta_{k,i})
$$

$$
\leq \sum_{j \in S} \left| (\Gamma_S^{-1}(\phi_1))_{i,j} - \delta_{i,j} \right| \left| (\Gamma_S^{-1}(\phi_1))_{j,i} - \delta_{i,j} \right| \left| (\Gamma_S^{-1}(\phi_1))_{j,k} - \delta_{j,k} \right| \left| (\Gamma_S^{-1}(\phi_1))_{k,i} - \delta_{k,i} \right|
$$

$$
\leq \sqrt{|\Delta_{2h}(S)| \vee (|N_h| + 1)} \frac{\|\phi_1\|^2 \vee \|\phi_2\|^2}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3/2}.
$$
In conclusion,
\[
V_2 \leq |S| \left| \sum_{j,k \in N_h} \phi_{1j} \phi_{1k} \phi_{1,j-k} + \sum_{j,k \in N_h} \phi_{2j} \phi_{2k} \phi_{2,j-k} \right| (1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3 + 8 |S| \frac{\|\phi_1\|_2 \lor \|\phi_2\|_2^3}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3}
+ 11 |\Delta_{2h}(S)| (|\Delta_{2h}(S)| \lor (|N_h| + 1))^{1/2} \frac{\|\phi_1\|_2 \lor \|\phi_2\|_2^3}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^{9/2}}.
\]

Control of $V_3 + V_4$. Arguing as above, we obtain
\[
V_3 + V_4 \leq |S| \left( \sum_{j,k \in N_h} \phi_{1j} \phi_{1k} \phi_{1,j-k} + \sum_{j,k \in N_h} \phi_{2j} \phi_{2k} \phi_{2,j-k} \right) (1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3 + 8 |S| \frac{\|\phi_1\|_2 \lor \|\phi_2\|_2^3}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^3}
+ 11 |\Delta_{2h}(S)| (|\Delta_{2h}(S)| \lor (|N_h| + 1))^{1/2} \frac{\|\phi_1\|_2 \lor \|\phi_2\|_2^3}{(1 - \|\phi_1\|_1 \vee \|\phi_2\|_1)^{9/2}}.
\]

\[\square\]

6.3 Proof of Corollary 2

As stated in Lemma 5, all eigenvalues of the covariance operator $\Gamma^{-1}(\phi)$ lie in $(1 - \|\phi\|_1, 1 + \|\phi\|_1)$. Since the spectrum of $\Gamma_S^{-1}(\phi)$ lies between the extrema of the spectrum of $\Gamma^{-1}(\phi)$, and using the assumption that $\|\phi\|_1 < 1/2$, this entails
\[
\|\Gamma_S(\phi) - I\| \leq \max \left[ \frac{2 \|\phi\|_1}{1 + \|\phi\|_1}, \frac{\|\phi\|_1}{1 - \|\phi\|_1} \right] < 1, \tag{42}
\]
We now apply Proposition 1 with the probability measure $\pi$ concentrating on $\phi$. In this case,
\[
V_S = \frac{\text{det}(\Gamma_S^{-1}(\phi))}{\text{det}(2\Gamma_S^{-1}(\phi) - I)^{1/2}} = \text{det}(I - (I - \Gamma_S(\phi))^2)^{-1/2},
\]
and we get
\[
\bar{R}_{\nu,\phi}^s \geq 1 - \frac{1}{2|C|} \left[ \sum_{S \in \mathcal{C}} \text{det}(I - (I - \Gamma_S(\phi))^2)^{-1/2} \right]^{1/2}
\geq 1 - \frac{1}{2|C|} \left[ \sum_{S \in \mathcal{C}} \exp \left( \frac{\|\Gamma_S(\phi) - I\|_F^2}{2(1 - \|\Gamma_S(\phi) - I\|_F)} \right) \right]^{1/2}
\geq 1 - \frac{1}{2|C|} \left[ \sum_{S \in \mathcal{C}} \exp \left( \frac{\|\Gamma_S(\phi) - I\|_F^2}{2(1 - 2\|\phi\|_1)} \right) \right]^{1/2},
\]
where $\|\cdot\|_F$ denotes the Frobenius norm. The second inequality above is obtained by applying the inequality $1/(1 - \lambda) \leq e^{\lambda/(1 - \lambda)}$ for $0 \leq \lambda < 1$ to the eigenvalues of $(\Gamma_S(\phi) - I)^2$, while the third inequality follows from (42) and the fact that $\|\phi\|_1 < 1/2$. It remains to bound $\|\Gamma_S(\phi) - I\|_F^2$:
\[
\|\Gamma_S(\phi) - I\|_F^2 = \sum_{(i,j) \in S, \ i \neq j} \text{Cor}^2(Y_i, Y_j)
\leq |S| \sum_{v \neq 0} \gamma_v^2
\leq 20|S|\|\phi\|_2^2,
\]
where we used Lemma 6, $\sigma^2_\phi \leq 1$, and $\|\phi\|_2 \leq \|\phi\|_1 \leq 1/2$ in the last line.
Thus, we deduce that

We deduce the result by closely following the proof of Theorem 1. We first prove that \( 5 \). Taking the numerical constant \( C \) in (23) sufficiently small and relying on condition (22), we have \( \| \phi \|_1 = r \sqrt{N_h} < 1/5 \). Consequently, the support of \( \pi \) is a subset of the parameter space \( \Phi_h \) and we are in position to invoke Lemma 9.

Let \( \phi_1, \phi_2 \) be drawn independently according to the distribution \( \pi \) and denote by \( \xi_1 \) and \( \xi_2 \) the corresponding random vectors defined on \( N_h \). By Lemma 9,

\[
\log B_{\phi_1,\phi_2} \leq |S| r^2 N_h^{-1} \langle \xi_1, \xi_2 \rangle + 8 Q_S ,
\]

where

\[
Q_S \leq 23 |S| r^3 \sqrt{|N_h|} + |\Delta_h(S)| r^2 + 28 |\Delta_{2h}(S)| |(\Delta_{2h}(S)) \vee (|N_h| + 1)|^{1/2} r^3 .
\]

Since \( \langle \xi_1, \xi_2 \rangle \) is distributed as the sum of \( |N_h|/2 \) independent Rademacher random variables, we deduce that

\[
V_S \leq \cosh \left( \frac{r^2 |S|}{|N_h|} \right)^{\frac{|N_h|}{2}} \exp \left( 383 \left( |S| \sqrt{|N_h|} \vee |\Delta_{2h}(S)| \right)^{3/2} r^3 + 8 |\Delta_h(S)| r^2 \right)
\]

\[
\leq \exp \left( \frac{r^4 |S|^2}{4 |N_h|} \right)^{\frac{|N_h|}{2}} \exp \left( \frac{|S|^2}{2} \right) + 383 \left( |S| \sqrt{|N_h|} \vee |\Delta_{2h}(S)| \right)^{3/2} r^3 + 8 |\Delta_h(S)| r^2 \right) ,
\]

since \( \cosh(x) \leq \exp(x) \wedge \exp(x^2/2) \) for any \( x > 0 \). Combining this bound with Proposition 1, we conclude that the Bayes risk \( R^*_{\nu,\pi} \) is bounded from below by

\[
1 - \frac{1}{2} \max_{S \in C} \exp \left( \frac{|S|^2 r^4}{4 |N_h|} \right)^{\frac{|N_h|}{2}} + 383 \left( |S| \sqrt{|N_h|} \vee |\Delta_{2h}(S)| \right)^{3/2} r^3 + 8 |\Delta_h(S)| r^2 \right) .
\]

(43)

If the numerical constant \( C \) in Condition (23) is sufficiently small, then \( \frac{|S|^2 r^4}{4 |N_h|} \wedge \frac{|S|^2}{2} \leq 0.5 \log(|C|/a) \).

Also, choosing \( C \) small enough in condition (23), relying on condition (22) and on \( |N_h| \geq 1 \), we also have

\[
383 \left( |S| \sqrt{|N_h|} \vee |\Delta_{2h}(S)| \right)^{3/2} r^3 + 8 |\Delta_h(S)| r^2 \leq 0.5 \log(|C|/a) .
\]

Thus, we conclude that \( R^*_{\nu,\pi} \geq 1 - a \).

### 6.5 Proof of Corollary 3

We deduce the result by closely following the proof of Theorem 1. We first prove that \( 5r \sqrt{|N_h|} \leq 1 \) is satisfied for \( n \) large enough. Starting from (32), we have, for \( n \) large enough,

\[
5r \sqrt{|N_h|} \leq 5C_2^{1/2} \left( \frac{|N_h| \log(n/k)}{k} \right)^{1/2} \left( \frac{|N_h|^{3/2} \sqrt{\log(n/k)}}{k} \right)^{1/2}
\]

\[
\leq 5C_2^{1/2} \left( C_1 \sqrt{\frac{|N_h|^{3/2} \sqrt{\log(n/k)}}{k}} \right)^{1/2} ,
\]

where we used Condition (31) in the second line. Taking \( C_1 \) and \( C_2 \) small enough, we only have to bound \( |N_h|^{3/2} \sqrt{\log(n/k)} / k \). We distinguish two cases.

- **Case 1**: \( |N_h| \leq \log(n/k) \). Since \( |N_h| \leq C_1 k / \log(n/k) \), it follows that \( |N_h|^{3/2} \sqrt{\log(n/k)} / k \leq C_1 \).
• Case 2: $|N_h| \geq \log(n/k)$. Then the second part of Condition (31) enforces $\log^{4/5}(n/k) \leq C_1 k^{2/5}$. Using again the second part of Condition (31) yields

$$\frac{|N_h|^{3/2} \sqrt{\log(k)}}{k} \leq C_3^{3/2} \log^{4/5}(n/k) \leq C_1^{3/2}.$$  

As $5r \sqrt{|N_h|} \leq 1$, we can use the same prior $\pi$ as in the proof of Theorem 1 and arrive at the same lower bound (43) on $R_n^*$. It remains to prove that this lower bound goes to one, namely that

$$\frac{2|S|^2 r^4}{|N_h|} \wedge (|S| r^2) + 765 \left(|S| \sqrt{|N_h|} + 1 \right) \log(\Delta_{2h}(S)) r^3 \leq 16 \Delta_{2h}(S) r^2 - \frac{1}{2} \log(n/k) \to -\infty,$$

where $S$ is a hypercube of size $k$. Taking the constant $C_2$ small enough implies $2|S|^2 r^4 \wedge (kr^2) \leq \log(n/k)/4$ for $n$ large enough.

$$kr^3 \sqrt{|N_h|} \leq C_2^{3/2} \left[ \frac{\log(n/k)^3 |N_h|}{k} \vee \frac{\log(n/k)^{3/2} |N_h|^{5/2}}{k} \right]^{1/2} \leq C_2^{3/2} \left( C_1^{1/2} \vee C_2^{5/4} \right) \log(n/k),$$

where we used again the second part of Condition (31). Taking $C_1$ and $C_2$ small enough ensures that $765kr^3 \sqrt{|N_h|} + 1 \leq \log(n/k)/8$ for $n$ large enough. Finally, it suffices to control $|\Delta_{2h}(S)| r^3$ since $|\Delta_{2h}(S)| r^2 \leq |\Delta_{2h}(S)| r^3 \vee 1$. Observe that

$$|\Delta_{2h}(S)| = (\ell - 4h)^d = \ell^d \left[ 1 - (1 - 4h/\ell)^d \right] \leq 4 \ell^d dh/\ell \leq 4d|N_h|^{1/d} k^{d-1} / 4.$$

It then follows from Condition (32) that

$$(d|N_h|^{1/d} k^{d-1} / 4)^{3/2} r^3 \quad \leq \quad C_2^{3/2} \left[ d^{3/2} |N_h|^{3/(2d)} k^{3/(2d)} \log^1 (\frac{n}{k}) \right] \log (\frac{n}{k})$$

where we used again (31) in the second line. Choosing $C_1$ and $C_2$ small enough concludes the proof.

6.6 Proof of Proposition 2

We leave $\phi$ implicit throughout. Let $R_S = \ldots$ and define

$$U'_{S} = X_{S}^\top (I_S - \Gamma_{S}^{-1}) X_{S} - \text{Tr}(I_S - \Gamma_{S}^{-1}).$$

Under the null, $X$ is standard normal, so applying the union bound and Lemma 1 gives

$$\Pr \left\{ U > 4 \right\} \leq \sum_{S \in C} \Pr \left\{ U_{S} > 4 \left\| I_S - \Gamma_{S}^{-1} \right\| F \sqrt{\log(|C|)} + 4 \left\| I_S - \Gamma_{S}^{-1} \right\| \log(|C|) \right\} \leq |C|^{-1}. $$

Under the alternative where $S \in C$ is anomalous, $X_S$ has covariance $\Gamma_S$, so that we have $X_{S}^\top (I_S - \Gamma_{S}^{-1}) X_S \sim Z^\top (\Gamma_{S}^{-1} - I_S) Z$, where $Z$ is standard normal in dimension $|S|$. Since $\text{Var}(Y_i) = 1$, the diagonal elements of $\Gamma_S - I_S$ are all equal to zero. We apply Lemma 1 to get that

$$\Pr \left\{ X_{S}^\top (I_S - \Gamma_{S}^{-1}) X_S \leq -2 \left\| \Gamma_S - I_S \right\| F \sqrt{\log(|C|)} - 2 \left\| \Gamma_S - I_S \right\| \log(|C|) \right\} \leq |C|^{-1},$$

26
In view of the definition of $U$, we have $\mathbb{P}[U > 4] \geq 1 - |C|^{-1}$ as soon as
\[
\text{Tr}[\Gamma_S^{-1} - I_S] \geq 4 \left[ \|\Gamma_S - I_S\|_F \vee \|\Gamma_S^{-1} - I_S\|_F \right] \sqrt{\log(|C|)} + 6 \left[ \|\Gamma - I\| \vee \|\Gamma^{-1} - I\| \right] \log(|C|). \tag{44}
\]
Therefore, it suffices to bound $\|\Gamma_S - I_S\|_F$, $\|\Gamma_S^{-1} - I_S\|_F$, $\|\Gamma - I\|$, $\|\Gamma^{-1} - I\|$ and $\text{Tr}[\Gamma_S^{-1} - I_S]$. In the sequel, the $C$ denotes a large enough positive constant depending only on $\eta$, whose value may vary from line to line. From Lemma 6, we deduce that
\[
\|\Gamma_S - I_S\|_F^2 \leq C|S|\|\phi\|_2^2.
\]
Lemma 5 implies that
\[
\|\Gamma - I\| \vee \|\Gamma^{-1} - I\| \leq C.
\]
We apply Lemma 8 to obtain
\[
\|\Gamma_S^{-1} - I_S\|_F^2 \leq C|S|\|\phi\|_2^2 + |S|(\sigma_\phi^2 - 1)^2 
\leq C|S|\|\phi\|_2^2,
\]
where we used Lemma 5 in the second line. Finally, we use again Lemmas 8 and 5 to obtain
\[
\text{Tr}[\Gamma_S^{-1} - I_S] = |S^h|\frac{\sigma_\phi^2 - 1}{\sigma_\phi^2} + \sum_{j \in \Delta_h(S)} (\Gamma_S^{-1})_{j,j} - 1 
\geq |S^h|\frac{\sigma_\phi^2 - 1}{\sigma_\phi^2} \geq C|S|\|\phi\|_2^2.
\]
Consequently, (44) holds as soon as $|S|\|\phi\|_2^2 \geq C \log(|C|)$.

6.7 Proof of Theorem 2

We use $C, C', C''$ as generic positive constants, whose actual values may change with each appearance.

Under the null hypothesis. First, we bound the $1 - \alpha$ quantile of $T^*$ under the null hypothesis. Denote $Z_S := \|\Pi_{S,h}X_{S,h}\|_2^2$ so that $T_S = Z_S|S^h|\left[\|X_{S,h}\|_2^2 - Z_S\right]^{-1}$. Since $Z_S$ is the squared norm of the projection of $X_{S,h}$ onto the column space of $F_{S,h}$, we can express $Z_S$ as a least-squares criterion:
\[
Z_S = \max_{\phi \in \mathbb{R}^{|S_h|}} \|X_{S,h}\|_2^2 - \sum_{i \in S^h} \left( X_i - \sum_{j \in |S_h|} \phi_j X_{i+j} \right)^2.
\]

Given $\phi \in \mathbb{R}^{|S_h|}$, define the matrix $B_{\phi,S} \in \mathbb{R}^{S \times S}$ such that for any $i \in S^h$, and any $j$, $(B_{\phi,S})_{i,i+j} = \phi_j$, and all the remaining entries of $B_{\phi,S}$ are zero. It then follows that
\[
Z_S = \max_{\phi \in \mathbb{R}^{|S_h|}} \text{Tr} \left[ R_{\phi,S} X_S X_S^\top \right], \quad R_{\phi,S} := (I - B_{\phi,S}^\top)(I - B_{\phi,S}) - I. \tag{45}
\]

Observe that $Z_S$ can be seen as the supremum of a Gaussian chaos of order 2. As the collection of matrices in the supremum of (45) is not bounded, we cannot directly apply Lemma 3. Nevertheless, upon defining $\tilde{Z}_S := \max_{\|\phi\|_1 \leq 1} \text{Tr} \left[ R_{\phi,S} X_S X_S^\top \right]$, we have for any $t > 0$,
\[
\mathbb{P}[Z_S \geq t] \leq \mathbb{P}[\tilde{Z}_S \geq t] + \mathbb{P}[\tilde{Z}_S \neq Z_S], \tag{46}
\]
and we can control the deviations of $\tilde{Z}_S$ using Lemma 3. Observe that for any $\phi$ with $\|\phi\|_1 \leq 1$, $\|I - B_{\phi,S}\| \leq 2$, so that $\|R_{\phi,S}\| \leq 3$. Choose $\hat{\phi}_S$ among the $\phi$’s achieving the maximum in (45), and note that $P[\tilde{Z}_S \neq Z_S] = P[\|\hat{\phi}_S\|_1 > 1]$. We bound the right-hand side below. In view of Lemma 3, we also need to bound $E[Z_S]$ and $E[\sup_{\|\phi\|_1 \leq 1} \text{Tr}(R_{\phi,S}X_S X_S^\top R_{\phi,S})]$ in order to control $P[\tilde{Z}_S \geq t]$.

Control of $P[\|\hat{\phi}_S\|_1 > 1]$. When $F_{S,h}^T F_{S,h}$ is invertible, $\hat{\phi}_S = (F_{S,h}^T F_{S,h})^{-1} F_{S,h} X_{S,h}$. By the Cauchy-Schwarz inequality,

$$
P[\|\hat{\phi}_S\|_1 > 1] \leq P[\|\hat{\phi}_S\|_2 > |N_h|^{-1/2}] \leq P[\lambda_{\min}(F_{S,h}^T F_{S,h}) \leq \frac{1}{2} |S_h|] + P[\|F_{S,h} X_{S,h}\|_2 \geq \frac{|S_h|}{2|N_h|^{1/2}}] \leq P[\lambda_{\min}(F_{S,h}^T F_{S,h}) \leq \frac{1}{2} |S_h|] + P[\|F_{S,h} X_{S,h}\|_\infty \geq \frac{|S_h|}{2|N_h|}] . \quad (47)
$$

First, we control the smallest eigenvalue of $F_{S,h}^T F_{S,h}$. Under the null hypothesis, the vectors $F_i$ follow the standard normal distribution, but $F_{S,h}^T F_{S,h}$ is not a Wishart matrix since the vectors $F_i$ are correlated. However, $F_{S,h}^T F_{S,h}$ decomposes as a sum of $|N_h| + 1$ (possibly dependent) standard Wishart matrices. Indeed, define

$$
S_i = S_h \cap \{ i + (2h + 1) u, \ u \in \mathbb{Z}^d \}, \ i \in N_h \cup \{ 0 \} , \quad (48)
$$

and then $A_i = \sum_{j \in S_i} F_j F_j^T$. The vectors $(F_j, \ j \in S_i)$ are independent since the minimum $\ell_\infty$ distance between any two nodes in $S_i$ is at least $2h + 1$, so that $A_i$ is standard Wishart. Denoting $n_i = |S_i|$, we are in position to apply Lemma 4, to get

$$
P[\lambda_{\min}(A_i) \leq n_i - 2\sqrt{|N_h| n_i - 2\sqrt{2}x n_i} \leq e^{-x} \ , \ \forall x > 0 .
$$

Since the $\{S_i : i \in N_h \cup \{ 0 \}\}$ forms a partition of $S_h$, we have $F_{S,h}^T F_{S,h} = \sum_{i \in N_h \cup \{ 0 \}} A_i$, and in particular, $\lambda_{\min}(F_{S,h}^T F_{S,h}) \geq \sum_i \lambda_{\min}(A_i)$. Using this, the tail bound for $\lambda_{\min}(A_i)$ with $x \leftarrow x + \log(|N_h| + 1)$, some simplifying algebra, and the union bound, we conclude that, for all $x > 0$,

$$
P[\lambda_{\min}(F_{S,h}^T F_{S,h}) \leq |S_h| - 5(|N_h| + 1) \sqrt{|S_h|} - 3 \sqrt{(|N_h| + 1)}|S_h| x \leq e^{-x} , \quad (49)
$$

since

$$
\sum_{i \in N_h \cup \{ 0 \}} (n_i - 2\sqrt{|N_h| n_i - 2\sqrt{2}x n_i}) \geq \sum_{i \in N_h \cup \{ 0 \}} n_i - 2(\sqrt{|N_h| + \sqrt{2}x}) \sum_{i \in N_h \cup \{ 0 \}} \sqrt{n_i} ,
$$

with $\sum_{i \in N_h \cup \{ 0 \}} n_i = |S_h|$, and $\sum_{i \in N_h \cup \{ 0 \}} \sqrt{n_i} \leq \sqrt{|S_h|(|N_h| + 1)}$, by the Cauchy-Schwarz inequality. Taking $x = C |S_h|/(|N_h| + 1)$ in the above inequality for a sufficiently small constant $C$ and relying on Condition (26), we get

$$
P\left\{ \lambda_{\min}(F_{S,h}^T F_{S,h}) \leq \frac{1}{2} |S_h| \right\} \leq \exp \left( -C |S_h|/(|N_h| + 1) \right) .
$$

We now turn to bounding $\|F_{S,h} X_{S,h}\|_\infty$. Each component of $F_{S,h} X_{S,h}$ is of the form $Q_v := \sum_{i \in S_h} X_i X_{i+v}$ for some $v \in N_h$. Note that $Q_v$ is a quadratic form of $|S|$ standard normal variables, and the corresponding symmetric matrix has zero trace, Frobenius norm equal to $\sqrt{|S_h|}/2$, and
operator norm smaller than 1 by diagonal dominance. Combining Lemma 1 with a union bound, we get
\[
P \left\{ \| F_{S,h} X_{S,h} \|_\infty \geq \sqrt{2|S^h|(x + |N_h|) + 2(x + |N_h|)} \right\} \leq 2e^{-x}, \forall x > 0.
\]
Taking \( x = C|S^h|/|N_h|^2 \) in the above inequality for a sufficiently small constant \( C \) and using once again Condition (26) allows us to get the bound
\[
P \left\{ \| F_{S,h} X_{S,h} \|_\infty \geq \frac{|S^h|}{2|N_h|} \right\} \leq \exp\left(-C\frac{|S^h|}{|N_h|^2}\right).
\]
Plugging these bounds into (47), we conclude that
\[
P \left\{ \| \hat{\phi}_S \|_1 > 1 \right\} \leq 3\exp\left(-C\frac{|S^h|}{|N_h|^2}\right).
\]

**Control of \( E[\tilde{Z}_S] \).** Since
\[
\tilde{Z}_S \leq Z_S = \| \Pi_{S,h} X_{S,h} \|_2 \leq \| (F_{S,h}^T F_{S,h})^{-1} \| \| F_{S,h} X_{S,h} \|_2 \leq \| X_{S,h} \|_2,
\]
we have, for any \( a > 0 \),
\[
E[\tilde{Z}_S] \leq a E \left[ \| F_{S,h} X_{S,h} \|_2^2 \right] + E \left[ \| X_{S,h} \|_2 \left\| \frac{1}{2} \left\| (F_{S,h}^T F_{S,h})^{-1} \right\| \geq a \right\| \right] \\
\leq a E \left[ \| F_{S,h} X_{S,h} \|_2^2 \right] + \sqrt{P \left\{ \left\| (F_{S,h}^T F_{S,h})^{-1/2} \right\| \geq a \right\}} E \left[ \| X_{S,h} \|_2^4 \right],
\]
where we used the Cauchy-Schwarz inequality in the second line. Since, under the null, \( X_S \sim N(0, I) \), it follows that \( E \left[ \| F_{S,h} X_{S,h} \|_2^2 \right] = |N_h(S)| |S^h| \) and \( E \left[ \| X_{S,h} \|_2^4 \right] = |S^h|(|S^h| + 2). \) Gathering this, the deviation inequality (49) with \( x = C|S^h|/|N_h|^2 \) with a small constant \( C > 0 \), and Condition (26), and choosing as threshold \( a = (|S^h|/(1 - |N_h|^{-1/2}))^{-1} \), leads to
\[
E[\tilde{Z}_S] \leq \frac{|N_h|}{1 - |N_h|^{-1/2}} + \sqrt{3} |S^h| \sqrt{\frac{1}{2} \left\{ \frac{\chi_{\min}(F_{S,h}^T F_{S,h})}{1/a} \leq 1/2 \right\}} \\
\leq |N_h| + C' |N_h|^{1/2} + \sqrt{3} |S^h| \exp\left(-C\frac{|S^h|}{|N_h|^2}\right) \\
\leq |N_h| + C |N_h|^{1/2}.
\]

**Control of \( E \left[ \sup_{\| \phi \|_1 \leq 1} \text{Tr}(R_{\phi,S} X_S X_S^T R_{\phi,S}) \right] \).** As explained above, \( \| R_{\phi,S} \| \leq 3 \) and we are therefore able to bound this expectation in terms of \( E[\tilde{Z}_S] \) as follows:
\[
E \left[ \sup_{\| \phi \|_1 \leq 1} \text{Tr}(R_{\phi,S} X_S X_S^T R_{\phi,S}) \right] \leq 3 E[\tilde{Z}_S] \leq C |N_h|,
\]
where we used (51) in the last inequality.

Combining the decomposition (46) with Lemma 3 and (50), (51) and (52), we obtain
\[
P \left\{ Z_S \geq |N_h| + C \left( |N_h|^{1/2} + \sqrt{|N_h|} t + t \right) \right\} \leq e^{-t} + 3\exp\left(-C'\frac{|S^h|}{|N_h|^2}\right), \ \forall t > 0.
\]
Since
\[ T_S = \frac{|S^h|Z_S}{\|X_{S,h}\|^2_2 - Z_S}, \]
where \( \|X_{S,h}\|^2_2 \) follows a \( \chi^2 \) distribution with \(|S^h|\) degrees of freedom, from Lemma 1, we derive
\[ \mathbb{P}\left[ \|X_{S,h}\|^2_2 \geq |S^h| - 2\sqrt{|S^h|t - 2t} \right] \leq e^{-t}, \]
for any \( t > 0 \), and from these two deviation inequalities, we get, for all \( t \leq C'|S^h| \),
\[ \mathbb{P}\left[ T_S \geq \frac{|N_h| + C(|N_h|^{1/2} + \sqrt{|N_h|t + t})}{1 - C\sqrt{\frac{t}{|S^h|} + \frac{|N_h|}{|S^h|}}} \right] \leq 2e^{-t} + 3|C|\exp\left(-C'|S^h|\frac{|N_h|^2}{|S^h|^2}\right). \]

Finally, we take a union bound over all \( S \in \mathcal{C} \) and invoke again Condition (26) to conclude that, for any \( t \leq C'|S^h| \),
\[ \mathbb{P}\left\{ \max_{S \in \mathcal{C}} T_S \geq |N_h| + C\left(\sqrt{|N_h|\log(|\mathcal{C}|)} + 1 + t\right) \right\} \leq 2e^{-t} + 3|\mathcal{C}|\exp\left(-C'|S^h|\frac{|N_h|^2}{|S^h|^2}\right). \]

To conclude, we let \( t = \log(1/(4\alpha)) \) in the above inequality, and use the condition on \( \alpha \) in the statement of the theorem together with Condition (26), to get the following control of \( T^* \) under the null hypothesis:
\[ \mathbb{P}\left\{ \max_{S \in \mathcal{C}} T_S \geq |N_h| + C\left(\sqrt{|N_h|\log(|\mathcal{C}|)} + 1 + \log(\alpha^{-1})\right) \right\} \leq \alpha. \]

**Under the alternative hypothesis.** Next we study the behavior of the test statistic \( T^* \) under the assumption that there exists some \( S \in \mathcal{C} \) such that \( X_S = Y_S \sim \mathcal{N}(0, \Gamma_S(\phi)) \). Since \( T^* \geq T_S \), it suffices to focus on this particular \( T_S \). For any \( i \in S^h \), recall that \( Y_i = \phi^\top F_i + \epsilon_i \) where \( F_i = (Y_{i+1} : 1 \leq |v|_\infty \leq h) \) and \( \epsilon_i \) is independent of \( F_i \). Hence, \( Z_S \) decomposes as
\[ Z_S = \|\Pi_{S,h}Y_{S,h}\|^2_2 = \|\Pi_{S,h}\phi + \Pi_{S,h}\epsilon_{S,h}\|^2_2 = \|\Pi_{S,h}\phi\|^2_2 + 2\phi^\top \Pi_{S,h}\epsilon_{S,h} + \|\Pi_{S,h}\epsilon_{S,h}\|^2_2 = (I) + (II) + (III). \]

To bound the numerator of \( T_S \), we bound each of these three terms. (I) and (II) are simply quadratic functions of multivariate normal random vectors and we control their deviations using Lemma 1. In contrast, (III) is more intricate and we use an ad-hoc method. In order to structure the proof, we state four lemmas needed in our calculations. We provide proofs of the lemmas further down.

**Lemma 10.** Under condition (26), there exists a numerical constant \( C > 0 \) such that
\[ \mathbb{P}\left\{ (II) \geq \frac{|S^h||\phi||^2_2}{2(1 + ||\phi||^2_1)^2} \sigma_\phi^2 \right\} \geq 1 - \exp\left(-C\frac{|S^h|}{|N_h|}\right). \]

**Lemma 11.** For any \( t > 0 \),
\[ \mathbb{P}\left\{ (II) \geq -2\sigma_\phi\sqrt{2|S^h||\Gamma(\phi)||2 + ||\phi||_1}t - 12\left||\Gamma(\phi)||2 + (1 + ||\phi||_1)\sigma_\phi^2 \right|t \right\} \geq 1 - e^{-t} \]
\[ \mathbb{P}\left\{ (II) \geq -2\sqrt{2}\sigma_\phi \sqrt{(|N_h| + 1)\log(|N_h| + 1) + t}\|\Pi_{S,h}\phi\|_2 \right\} \geq 1 - e^{-t}. \]
Recall that $\gamma_j = (\mathbf{\Gamma}(\phi))_{0,j}$ denotes the covariance between $Y_0$ and $Y_j$.

**Lemma 12.** Denote by $\mathbf{\Gamma}_{N_h}(\phi)$ the covariance matrix of $(Y_i, i \in N_h)$. For any $t \leq |S^h|$, 

\[
\chi_{\max} \left( \mathbf{\Gamma}_{N_h}(\phi)^{-1/2} \frac{\mathbf{F}_{S^h}^T \mathbf{F}_{S^h} \mathbf{\Gamma}_{N_h}(\phi)^{-1/2}}{|S^h|} \right) \leq 1 + 4 \|\mathbf{\Gamma}(\phi)\| \|\mathbf{\Gamma}^{-1}(\phi)\| \frac{|N_h|}{|S^h|^{1/2}} \left( \sqrt{t} + \log(|N_h|) \right)
\]

(56)

with probability larger than $1 - 2e^{-t}$. Also, for any $t \geq 1$,

\[
\frac{\|\mathbf{\Gamma}_{N_h}(\phi)^{-1/2} \mathbf{F}_{S^h}^T \epsilon_{S^h} \|_2^2}{|S^h|\sigma_\phi^2} \geq |N_h| - C \left( |N_h|\|\phi\|_1 + \|\mathbf{\Gamma}^{-1}(\phi)\|_2^2 + \frac{|N_h|^{3/2} + |N_h|^2(\sum_{j\neq 0} \gamma_j^2)}{\sqrt{|S^h|}} \right) t^2
\]

(57)

with probability larger than $1 - 2e^{-t}$.

To bound the denominator of $T_S$, we start from the inequality

\[
\|Y_{S,h}\|_2^2 - \|\Pi_{S,h}Y_{S,h}\|_2^2 = \|\epsilon_{S,h}\|_2^2 - \|\Pi_{S,h}\epsilon_{S,h}\|_2^2 \leq \|\epsilon_{S,h}\|_2^2
\]

and then use the following result.

**Lemma 13.** Under condition (26), we have

\[
P \left\{ \|\epsilon_{S,h}\|_2^2 \leq \sigma_\phi^2|S^h|(1 + |N_h|^{-1/2}) \right\} \geq 1 - \exp \left( -C \frac{|S^h|}{|N_h|^2} \right).
\]

(58)

With these lemmas in hand, we divide the analysis into two cases depending on the value of $\|\phi\|_2^2$. For small $\|\phi\|_2^2$, the operator norm of the covariance operator $\mathbf{\Gamma}(\phi)$ remains bounded, which simplifies some deviation inequalities. For large $\|\phi\|_2^2$, we are only able to get looser bounds which are nevertheless sufficient as in that case $\|\phi\|_2^2$ is far above the detection threshold.

**Case 1:** $\|\phi\|_2^2 \leq (4|N_h|)^{-1}$. This implies that $\|\phi\|_1 \leq 1/2$ and also that $\|\mathbf{\Gamma}(\phi)\| \leq 2\sigma_\phi^2$ by Lemma 5. Combining (53) and (54) together with the inequality $2xy \leq x^2 + y^2$, we derive that for any $t > 0$,

\[
\frac{(I) + (II)}{\sigma_\phi^2} \geq C \left( |S^h|\|\phi\|_2^2 - t \right)
\]

(59)

with probability larger than $1 - e^{-t} - \exp \left( -C \frac{|S^h|}{(|N_h|+1)} \right)$. Turning to the third term, we have

\[
\frac{(III)}{\sigma_\phi^2} \geq \chi_{\max} \left( \mathbf{\Gamma}_{N_h}(\phi)^{-1/2} \frac{\mathbf{F}_{S^h}^T \mathbf{F}_{S^h} \mathbf{\Gamma}_{N_h}(\phi)^{-1/2}}{|S^h|} \right)^{-1} \frac{\|\mathbf{\Gamma}_{N_h}(\phi)^{-1/2} \mathbf{F}_{S^h}^T \epsilon_{S,h} \|_2^2}{\sigma_\phi^2|S^h|}.
\]

Let $a > 0$ be a positive constant whose value we determine later. For any $t > 0$, with probability
implies that the right-hand side exceeds $-\frac{1}{2}$ from Lemma 10 and the above inequality that

\[ Z \geq -\frac{|N_h|^2}{\sqrt{|S_h|}} \]

Taking $t$ large enough, we have

\[
\frac{(III)}{\sigma_\phi^2} \geq -\frac{\frac{|N_h|^2}{\sqrt{|S_h|}}}{\sqrt{|S_h|}} \left( \sqrt{t} + \log(|N_h|) \right) \]

\[ \geq |N_h| - C \left( \frac{|N_h|^3/2 ||\phi||_2 + ||\phi||_2^2 + \frac{|N_h|}{\sqrt{|S_h|}} (\sum_{j \neq 0} \gamma_j^2) + |N_h|^2 (\sum_{j \neq 0} \gamma_j^2)}{1 + 4 \|\Gamma(\phi)\| \|\Gamma^{-1}(\phi)\| \frac{|N_h|}{\sqrt{|S_h|}} (\sqrt{t} + \log(|N_h|))} \right) t^2 \]

\[ \geq |N_h| - C \left( \frac{|N_h|^3/2 ||\phi||_2 + \frac{|N_h|}{\sqrt{|S_h|}} (\sum_{j \neq 0} \gamma_j^2) + |N_h|^2 (\sum_{j \neq 0} \gamma_j^2)}{1 + 4 \|\Gamma(\phi)\| \|\Gamma^{-1}(\phi)\| \frac{|N_h|}{\sqrt{|S_h|}} (\sqrt{t} + \log(|N_h|))} \right) t^2 \]

\[ \geq |N_h| - a \|S_h\| ||\phi||_2^2 - C \left( \frac{|N_h|^3}{\sqrt{|S_h|}} t^2 + \frac{|N_h|}{\sqrt{|S_h|}} (1 + t^4) \right) \]

\[ \geq |N_h| - a \|S_h\| ||\phi||_2^2 - C \sqrt{|N_h|} (1 + (a^{-1} + 1)t^4) . \]

Here in the first line, we used Lemma 12. In the second line, we used the fact that $(1 - y)/(1 + x) \geq 1 - x - y$ for all $x, y \geq 0$. $||\phi||_1 \leq \sqrt{|N_h|} ||\phi||_2$ by the Cauchy-Schwarz inequality, and $\|\Gamma(\phi)\| \vee \|\Gamma^{-1}(\phi)\| \leq 2$. In the third line, we applied the inequality $\sum_{j \neq 0} \gamma_j^2 \leq 4 ||\phi||_2^2 + 16 ||\phi||_2^2 \leq 20$, which is a consequence of $||\phi||_2 \leq 1/2$ and Lemma 6. The last line is a consequence of Condition (26). Then, we take $a = C/2$ with $C$ as in (59) and apply Lemma 13 to control the denominator of $T_S$. This leads to

\[ \mathbb{P} \left\{ T_S \geq C |S_h| ||\phi||_2^2 + |N_h| - C' \sqrt{|N_h|} (1 + t^4) \right\} \geq 1 - 4e^{-t} - 2e^{-C'' \frac{|S_h|}{|N_h|^2}} . \]

Taking $t = \log(8/\beta)$ and letting $C_2$ be small enough in (27), we get

\[ \mathbb{P} \left\{ T_S \geq C |S_h| ||\phi||_2^2 + |N_h| - C' \sqrt{|N_h|} (1 + \log^4(\beta^{-1})) \right\} \geq 1 - \beta , \]

proving (29) in Case 1.

Case 2: $||\phi||_2 \geq (4|N_h|)^{-1}$. This condition entails

\[ \frac{2 ||\phi||_2^2}{1 + ||\phi||_1} \geq \frac{||\phi||_2}{\sqrt{|N_h|}} . \]

Since the term (III) is non-negative, we can start from the lower bound $Z_S \geq (I) + (II)$. We derive from Lemma 10 and the above inequality that

\[ \mathbb{P} \left\{ (I) \geq |S_h| \frac{\sigma_\phi^2 ||\phi||_2}{4 \sqrt{|N_h|}} \right\} \geq 1 - \exp \left( -C \frac{|S_h|}{|N_h|} \right) . \]

(60)

Taking $t = C |S_h| / |N_h|^2$ in (55) for a constant $C$ sufficiently small, and using Condition (26), we get that $(II) \geq -3 \sqrt{C} \sigma_\phi \sqrt{|S_h|} / |N_h| \sqrt{(I)}$ with probability at least $1 - e^{-t}$. Also, $||\phi||_2 \leq (4|N_h|)^{-1}$ implies that the right-hand side exceeds $-\frac{1}{2}(I)$ when the event in (60) holds and $C$ is small enough. Hence, we get

\[ \mathbb{P} \left\{ (I) + (II) \geq |S_h| \frac{\sigma_\phi^2 ||\phi||_2}{8 \sqrt{|N_h|}} \right\} \geq 1 - 2 \exp \left( -C \frac{|S_h|}{|N_h|^2} \right) . \]

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Finally, we combine this bound with (58) and the condition $\|\phi\|_2^2 \geq (4|N_h|)^{-1}$, to get

$$
P\left\{ T_S \geq \frac{|S^h|}{32|N_h|} \right\} \geq 1 - 3 \exp \left( -C \frac{|S^h|}{|N_h|^2} \right) \geq 1 - \beta,
$$

where we used the condition on $\beta$. In view of Condition (26), we have proved (29). This concludes the proof of Theorem 2. It remains to prove the auxiliary lemmas.

### 6.7.1 Proof of Lemma 10

Recall the definition of $S_i$ in (48). Let $F_{S_i}$ denote the matrix with row vectors $F_j, j \in S_i$. We have

$$(I) = \|F_{S_i} \phi\|_2^2 = \sum_{i \in \mathbb{N}_i \cup \{0\}} \|F_{S_i} \phi\|_2^2 .$$

For any $u \in \mathbb{R}^{S_i}$,

$$\text{Var} \left[ \sum_{j \in S_i} u_j F_{S_i} \phi \right] = \text{Var} \left[ \sum_{j \in S_i, v \in N_h} u_j \phi_v Y_{v+j} \right] \geq \|u\|_2^2 \|\phi\|_2^2 \lambda_{\min}(\Gamma(\phi)) ,$$

since the indices $(v+j : v \in N_h, j \in S_i)$ are all distinct. Since $\lambda_{\min}(\Gamma(\phi)) \geq \frac{\sigma^2}{1+\|\phi\|_1}$ by Lemma 5, $\frac{1+\|\phi\|_1}{\sigma^2} \|F_{S_i} \phi\|_2^2$ is stochastically lower bounded by a $\chi^2$ distribution with $|S_i|$ degrees of freedom. By Lemma 1 and the union bound, we have that for any $t > 0$,

$$(I) \geq \frac{\|\phi\|_2^2}{1+\|\phi\|_1} \sigma^2 \sum_{i \in \mathbb{N}_i \cup \{0\}} |S_i| - 2 \sqrt{|S_i| \log(|N_h|+1) + t}$$

$$\geq \frac{\|\phi\|_2^2}{1+\|\phi\|_1} \sigma^2 \left( |S^h| - 2 \sqrt{|S^h|(|N_h|+1) \log(|N_h|+1) + t} \right) ,$$

with probability larger than $1 - e^{-t}$. Finally we set $t = \frac{|S^h|}{32(|N_h|+1)}$ and use Condition (26) to conclude.

### 6.7.2 Proof of Lemma 11

We first prove (54). Denote by $\hat{\Sigma}_{\phi,S}$ the covariance matrix of the random vector $(\epsilon_{\phi,S}^\top, \phi^\top F_{S,h}^\top)$ of size $2|S^h|$. Let $R$ be the block matrix defined by

$$R = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} .$$

Letting $Z$ be a standard Gaussian vector of size $2|S^h|$, we have $2\phi^\top F_{S,h}^\top \epsilon_{S,h} \sim Z^\top \hat{\Sigma}_{\phi,S} R \hat{\Sigma}_{\phi,S}^{1/2} Z$. From Lemma 1 we get that for all $t > 0$, with probability at least $1 - e^{-t}$,

$$2\phi^\top F_{S,h}^\top \epsilon_{S,h} \leq \text{Tr}[\hat{\Sigma}_{\phi,S}^{1/2} R \hat{\Sigma}_{\phi,S}^{1/2}] - 2\|\hat{\Sigma}_{\phi,S}^{1/2} R \hat{\Sigma}_{\phi,S}^{1/2}\|_F \sqrt{t} - 2\|\hat{\Sigma}_{\phi,S}^{1/2} R \hat{\Sigma}_{\phi,S}^{1/2}\|_1 t ,$$

$$\leq -2\|\hat{\Sigma}_{\phi,S}^{1/2} R \hat{\Sigma}_{\phi,S}^{1/2}\|_F \sqrt{t} - 2\|\hat{\Sigma}_{\phi,S}^{1/2}\|_1 t ,$$

(61)
where we used the fact that $\text{Tr}[\Sigma_{\phi,S}^{1/2} R \Sigma_{\phi,S}^{1/2}] = \mathbb{E}[\phi^T F_{S,h}^T \epsilon_{S,h}] = 0$ and that $\|R\| = 1$. In order to bound the Frobenius norm above, we start from the identity

$$
\|\Sigma_{\phi,S}^{1/2} R \Sigma_{\phi,S}^{1/2}\|^2_F = \text{Var}[2\phi^T F_{S,h}^T \epsilon_{S,h}] = 4 \sum_{i,j \in S^h} \mathbb{E}[\epsilon_i \epsilon_j (\phi^T F_i)(\phi^T F_j)] ,
$$

with $\epsilon_i$ being the $i$th component of $\epsilon_{S,h}$. For $i = j$, the expectation of the right-hand side is $\sigma^2_{\phi} \mathbb{E}[(\phi^T F_i)^2]$, while if the distance between $i$ and $j$ is larger than $h$, then $\epsilon_i$ and $(\epsilon_j, F_i, F_j)$ are independent and the expectation of the right-hand side is zero. If $1 \leq |i-j| \leq h$, then we use Isserlis’ theorem, together with the fact that $(\epsilon_j, F_i, F_j)$ are linear combinations of this collection, we conclude that

$$
\mathbb{E}[(\phi^T F_i)(\phi^T F_j)] = |\mathbb{E}[\epsilon_i \epsilon_j] E[(\phi^T F_i)(\phi^T F_j)] + \mathbb{E}[\epsilon_i \epsilon_j] E[\epsilon_j \phi^T F_i]| \leq \sigma^2_{\phi} \mathbb{E}[(\phi^T F_i)^2] + \hat{\sigma}^2_{\phi} .
$$

Putting all the terms together, we obtain

$$
\|\Sigma_{\phi,S}^{1/2} R \Sigma_{\phi,S}^{1/2}\|^2_F \leq 4|S^h| \sigma^2_{\phi} \left\{ \mathbb{E}[(\phi^T F_i)^2](1 + \|\phi\|_1) + \|\phi\|_2^2 \right\} 
\leq 4\sigma^2_{\phi}|S^h| \|\phi\|_2^2 \mathbb{E}[(\Gamma(\phi))^2](2 + \|\phi\|_1) ,
$$

using the fact that $\|\Gamma(\phi)\| \geq 1$.

Turning to $\|\Sigma_{\phi,S}\|$, denote $\Gamma(\phi)^t$ the covariance of the process $(\epsilon_i, i \in \mathbb{Z}^d)$. By Lemma 7, $(\Gamma(\phi)^t)_{i,j} = [-\phi_i - \mathbb{I}_{i=j}] \sigma^2_{\phi}$, and it follows that $\|\Gamma(\phi)^t\| \leq (1 + \|\phi\|_1)\sigma^2_{\phi}$. Then, for all vectors $u, v \in \mathbb{R}^{S^h}$,

$$
\text{Var} \left( \sum_{i \in S^h} u_i \phi^T F_i + \sum_{i \in S^h} v_i \epsilon_i \right) = \text{Var} \left( \sum_{i \in S^h} u_i Y_i + \sum_{i \in S^h} (v_i - u_i) \epsilon_i \right) 
\leq 2 \text{Var} \left( \sum_{i \in S^h} u_i Y_i \right) + 2 \text{Var} \left( \sum_{i \in S^h} (v_i - u_i) \epsilon_i \right) 
\leq 2 \|u\|_2^2 \|\Gamma(\phi)\| + 2 \|u - v\|_2^2 \|\Gamma(\phi)^t\| 
\leq 6 \left( \|u\|_2^2 + \|v\|_2^2 \right) \|\Gamma(\phi)\| \vee \|\Gamma(\phi)^t\| .
$$

Consequently, $\|\Sigma_{\phi,S}\| \leq 6\|\Gamma(\phi)\| \vee \|\Gamma(\phi)^t\| \leq 6\|\Gamma(\phi)^t\| \vee (1 + \|\phi\|_1)\sigma^2_{\phi}$.

We conclude that (54) holds by virtue of the two bounds we obtained for the two terms in (61).

Turning to (55), we decompose (II) into $2 \sum_{i \in S_h \cup \{0\}} \phi^T F_{S_i}^T \epsilon_{S_i}$. For any $j_1 \neq j_2 \in S_i, |j_1 - j_2| \geq 2h + 1$ and therefore $\epsilon_{j_1}$ is independent of $(Y_{j_2+v}, v \in S_h \cup \{0\})$. Since $\epsilon_{j_2}$ and $F_{j_2,\phi}$ are linear combinations of this collection, we conclude that $\epsilon_{j_1} \perp (\epsilon_{j_2}, \phi^T F_{j_2})$. Consequently, $\epsilon_{S_i}/\sigma_{\phi}$ follows a standard normal distribution and is independent of $F_{S_i,\phi}$. By conditioning on $F_{S_i,\phi}$ and applying a standard Gaussian concentration inequality, we get

$$
P \left\{ \|\phi^T F_{S_i}^T \epsilon_{S_i}\| \leq \sigma_{\phi} \|F_{S_i,\phi}\|_2 \sqrt{2t} \right\} \leq e^{-t} ,$$

for any $t > 0$. We then take a union bound over all $i \in S_h \cup \{0\}$. For any $t > 0$,

$$
(II) \geq -2\sqrt{2\sigma_{\phi}} \sqrt{\log(|N_h| + 1) + t} \sum_{i \in S_h \cup \{0\}} \|\phi^T F_{S_i}\|_2 
\geq -2\sqrt{2\sigma_{\phi}} \sqrt{\log(|N_h| + 1) + t} \sqrt{|N_h| + 1} \|F_{S,h,\phi}\|_2 ,
$$

with probability larger than $1 - e^{-t}$.
6.7.3 Proof of Lemma 12

**Proof of (56).** Fix \((v_1, v_2) \in \mathbb{N}_h\) and consider the random variable

\[
(F_{S,h}^T F_{S,h})v_1, v_2 = \sum_{i \in S^h} Y_{i+v_1} Y_{i+v_2} = Y_S^T R Y_S = V^T \Gamma_S(\phi)^{1/2} R \Gamma_S^{1/2}(\phi) V ,
\]

which constitutes a definition for the symmetric matrix \(R\), and \(V \sim \mathcal{N}(0, I)\). Observe that \(\|R\|_F^2 = |S^h|\) and \(\|R\| \leq 1\) as the \(l_1\) norm of each row of \(R\) is smaller than one. We derive from Lemma 1, and the fact that \(\|\Gamma_S(\phi)^{1/2} R \Gamma_S^{1/2}(\phi)\|_F \leq \|R\|_F \|\Gamma_S^{1/2}(\phi)\|^{4} \leq |S^h| \|\Gamma(\phi)\|^2\) and \(\|\Gamma_S(\phi)^{1/2} R \Gamma_S^{1/2}(\phi)\| \leq \|R\| \|\Gamma_S(\phi)\|\), that for any \(t > 0\),

\[
\mathbb{P}\left\{ |(F_{S,h}^T F_{S,h})v_1, v_2 - |S^h| \gamma_{v_1, v_2}| \leq 2 \|\Gamma(\phi)\| \sqrt{|S^h|t} + 2 \|\Gamma(\phi)\| t \right\} \leq 2e^{-t} .
\]

Then we bound the \(\ell_2\) operator norm of \(|S^h|^{-1} F_{S,h}^T F_{S,h} - \Gamma_{N_h}(\phi)\) by its \(\ell_1\) operator norm and combine the above deviation inequality with a union bound over all \((v_1, v_2) \in \mathbb{N}_h\). Thus, for any \(t \leq |S^h|\),

\[
\left\| \frac{F_{S,h}^T F_{S,h}}{|S^h|} - \Gamma_{N_h}(\phi) \right\| \leq \sup_{v_1 \in \mathbb{N}_h} \sum_{v_2 \in \mathbb{N}_h} \left| \frac{F_{S,h}^T F_{S,h}v_1, v_2}{|S^h|} - \gamma_{v_1, v_2} \right| \leq 2 \|\Gamma(\phi)\| \left( \frac{|N_h|}{|S^h|^{1/2}} \left( \sqrt{\log |N_h| + t} + \frac{\log |N_h| + t}{\sqrt{|S^h|}} \right) \right) \leq 4 \|\Gamma(\phi)\| \left( \frac{|N_h|}{|S^h|^{1/2}} \left( \sqrt{t} + \log(|N_h|) \right) \right) ,
\]

with probability larger than \(1 - 2e^{-t}\). Hence, under this event,

\[
\lambda_{\max}\left( \frac{\Gamma_{N_h}(\phi)^{-1/2} F_{S,h}^T F_{S,h} \Gamma_{N_h}(\phi)^{-1/2}}{|S^h|} \right) \leq 1 + 4 \|\Gamma(\phi)\| \|\Gamma^{-1}(\phi)\| \left( \frac{|N_h|}{|S^h|^{1/2}} \left( \sqrt{t} + \log(|N_h|) \right) \right) ,
\]

since \(\|\Gamma_{N_h}(\phi)^{-1}\| \leq \|\Gamma^{-1}(\phi)\|\). This concludes the proof of (56).

**Proof of (57).** Turning to the second deviation bound, we use the following decomposition

\[
\|\Gamma_{N_h}(\phi)^{-1/2} F_{S,h}^T \epsilon_{S,h} \|_2^2 = \sum_{i \in S^h} \epsilon_i^2 \|\Gamma_{N_h}(\phi)^{-1/2} F_i \|_2^2 + \sum_{(i,j), i \neq j} \epsilon_i \epsilon_j F_{ij}^T \Gamma_{N_h}(\phi)^{-1} F_i =: A + B ,
\]

with \(\epsilon_i\) being the \(i\)th entry of \(\epsilon_{S,h}\). Since both \(A\) and \(B\) are Gaussian chaos variables of order 4, we apply Lemma 2 to control their deviations. For any \(t > 0\),

\[
\mathbb{P}\left\{ A + B \geq \mathbb{E}[A + B] - C \left( \text{Var}^{1/2}(A) + \text{Var}^{1/2}(B) \right) t^2 \right\} \leq 2e^{-t} ,
\]

using the fact that \(\text{Var}^{1/2}(A + B) \leq \text{Var}^{1/2}(A) + \text{Var}^{1/2}(B)\). Thus, it suffices to compute the expectation and variance of \(A\) and \(B\).

First, we have \(\mathbb{E}[A] = |S^h| |N_h| \sigma_{\phi}^2\), by independence of \(\epsilon_i\) and \(F_i\), and from this we get

\[
\text{Var}(A) = \sum_{i,j \in S^h} \left( \mathbb{E}\left[ \epsilon_i^2 \epsilon_j^2 \|\Gamma_{N_h}(\phi)^{-1/2} F_i \|_2^2 \|\Gamma_{N_h}(\phi)^{-1/2} F_j \|_2^2 \right] - \sigma_{\phi}^2 |N_h|^2 \right) =: \sum_{i,j \in S^h} A_{i,j} .
\]
If $|i - j|_\infty \leq h$, we may use the Cauchy-Schwarz inequality to get

$$|A_{i,j}| \leq \mathbb{E} \left[ \epsilon_i^4 \| \Gamma_{N_h}(\phi)^{-1/2} F_i \|_2^4 \right] = 3\sigma^4 \phi N_h (|N_h| + 2),$$

again by independence of $\epsilon_i$ and $\| \Gamma_{N_h}(\phi)^{-1/2} F_i \|_2^2$. If $|i - j|_\infty > h$, then $\epsilon_i$ is independent of $(F_i, F_j, \epsilon_j)$ and $\epsilon_j$ is independent of $(F_i, F_j, \epsilon_i)$, so we get

$$\frac{A_{i,j}}{\sigma^4} = \mathbb{E} \left[ \| \Gamma_{N_h}(\phi)^{-1/2} F_i \|_2^2 \| \Gamma_{N_h}(\phi)^{-1/2} F_j \|_2^2 - |N_h|^2 \right],$$

$$= \sum_{v_1, v_2, v_3, v_4 \in N_h} (\Gamma_{N_h}(\phi)^{-1})_{v_1, v_2} (\Gamma_{N_h}(\phi)^{-1})_{v_3, v_4} \left[ \gamma_{v_1 - v_2} \gamma_{v_3 - v_4} + \gamma_{v_1 - v_2 + v_3 - v_4} \gamma_{v_4} \right]$$

$$= \sum_{v_1, v_2, v_3, v_4 \in N_h} (\Gamma_{N_h}(\phi)^{-1})_{v_1, v_2} (\Gamma_{N_h}(\phi)^{-1})_{v_3, v_4} \left[ \gamma_{v_1 - v_2 + v_3 - v_4} \gamma_{v_4} \right],$$

where we apply Isserlis’ theorem in the second line and use the definition of $\Gamma_{N_h}(\phi)$ in the last line.

By symmetry, we get

$$\frac{|A_{i,j}|}{\sigma^4} \leq 2 \| \Gamma_{N_h}(\phi)^{-1} \|_\infty^2 \sum_{v_1, v_2, v_3, v_4 \in N_h} |\gamma_{v_1 - v_2 + v_3 - v_4}|$$

$$\leq 2 \| \Gamma_{N_h}(\phi)^{-1} \|_\infty^2 |N_h| \sum_{v_1, v_2 \in N_h} \gamma_{v_1 - v_2}^2$$

$$\leq 2 \| \Gamma^{-1}(\phi) \|_2^2 |N_h|^2 \sum_{v \in N_{2h}} \gamma_{v}^2,$$

using the Cauchy-Schwarz inequality in the second line. Here $\|A\|_\infty$ denotes the supremum norm of the entries of $A$. Then, summing over all $j$ lying at a distance larger than $h$ from $i$,

$$\sum_{j \in S^h, |j - i|_\infty > h} \frac{|A_{i,j}|}{\sigma^4} \leq 2 \| \Gamma^{-1}(\phi) \|_2^2 |N_h|^2 \sum_{j \in S^h, |j - i|_\infty > h} \sum_{v \in N_{2h}} \gamma_{v}^2$$

$$\leq 2^{d+1} \| \Gamma^{-1}(\phi) \|_2^2 |N_h|^3 \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \gamma_{j}^2.$$

Putting the terms together, we conclude that

$$\text{Var}(A) \leq \sigma^4 \phi |S^h| |N_h|^3 \left( 6 + 2^{d+1} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \gamma_{j}^2 \right). \quad (63)$$

Next we bound the first two moments of $B$. Consider $(i, j) \in S^h$ such that $|i - j|_\infty > h$. Then $\mathbb{E} [\epsilon_i \epsilon_j F_j^T \Gamma_{N_h}(\phi)^{-1} F_i] = 0$ by independence of $\epsilon_i$ with the other variables in the expectation. Suppose now that $|i - j|_\infty \leq h$. By Isserlis’ theorem, and the independence of $\epsilon_i$ and $F_i$, as well as $\epsilon_j$ and $F_j$, and symmetry, to get

$$\mathbb{E} [\epsilon_i \epsilon_j F_j^T \Gamma_{N_h}(\phi)^{-1} F_i] = \mathbb{E} [\epsilon_i F_j^T] \Gamma_{N_h}(\phi)^{-1} \mathbb{E} [F_i \epsilon_j] + \mathbb{E} [\epsilon_i \epsilon_j] \mathbb{E} [F_j^T \Gamma_{N_h}(\phi)^{-1} F_i]$$

$$\geq -\sigma^2 |\phi_{i - j}|^2 \| \Gamma_{N_h}(\phi)^{-1} \| - \sigma^2 |\phi_{i - j}| |N_h|.$$
using the Cauchy-Schwarz inequality and Lemma 7. As a consequence,
\[ \mathbb{E}[B] \geq -\sigma^4_\phi |S^h||\phi||_2^2||\Gamma^{-1}(\phi)|| - \sigma^2_\phi |S^h||N_h||_1 \phi||_1. \]  
(64)

Turning to the variance, we obtain
\[ \text{Var}(B) \leq \mathbb{E}[B^2] = \sum_{i_1 \neq i_2} \sum_{i_3 \neq i_4} \mathbb{E}[V_{i_1, i_2, i_3, i_4}], \]
where
\[ V_{i_1, i_2, i_3, i_4} := \epsilon_{i_1} \epsilon_{i_2} \epsilon_{i_3} \epsilon_{i_4} F_{i_1}^\top \Gamma_{N_h}(\phi)^{-1} F_{i_2} F_{i_3}^\top \Gamma_{N_h}(\phi)^{-1} F_{i_4}. \]

Fix \( i_1 \). If one index among \( (i_1, i_2, i_3, i_4) \) lies at a distance larger than \( h \) from the three others, then the expectation of \( V_{i_1, i_2, i_3, i_4} \) is equal to zero. If one index lies within distance \( h \) of \( i_1 \) and the remaining indices lie within distance \( 3h \) of \( i_1 \), we use the Cauchy-Schwarz inequality to get
\[ \mathbb{E}[V_{i_1, i_2, i_3, i_4}] \leq \mathbb{E}\left[ \prod_{k=1}^{4} (\epsilon_{i_k}|(F_{i_k}^\top \Gamma_{N_h}(\phi)^{-1} F_{i_k})^{1/2}) \right] \leq \mathbb{E}[\epsilon_1^4 (F_{i_1}^\top \Gamma_{N_h}(\phi)^{-1} F_{i_1})^2] = 3\sigma^4_\phi |N_h|(|N_h| + 2). \]

Finally, if say \( |i_1 - i_2|_\infty \leq h \) and \( |i_3 - i_4|_\infty \leq h \) and \( |i_k - i_\ell| > h \) for \( k = 1, 2 \) and \( \ell = 3, 4 \), then we use again Isserlis’ theorem and simplify the terms to get
\[ \mathbb{E}[V_{i_1, i_2, i_3, i_4}] = \mathbb{E}[\epsilon_{i_1} \epsilon_{i_2}] \mathbb{E}[\epsilon_{i_3} \epsilon_{i_4}] \mathbb{E}[F_{i_1}^\top \Gamma_{N_h}(\phi)^{-1} F_{i_2} F_{i_3}^\top \Gamma_{N_h}(\phi)^{-1} F_{i_4}] + \mathbb{E}[\epsilon_{i_2} \Gamma_{N_h}(\phi)^{-1} \mathbb{E}[\epsilon_{i_3} \epsilon_{i_4}] \mathbb{E}[\epsilon_{i_1} F_{i_2} \Gamma_{N_h}(\phi)^{-1} F_{i_3}],
\]
\[ \leq \sigma^4_\phi |\phi_{i_2-i_1} \phi_{i_4-i_3}| |N_h|(|N_h| + 2) + ||\Gamma^{-1}(\phi)||^2 \sigma^8_\phi |\phi^{i_2-i_1} \phi^{i_4-i_3}|^2, \]
where we used again Lemma 7 to control the terms involving \( \epsilon \)'s and the Cauchy-Schwarz inequality to bound the term in \( (F_{i_k}, k = 1, \ldots, 4) \). Putting all the terms together, we conclude that
\[ \text{Var}(B) \leq C\sigma^4_\phi \left( |S^h||N_h|^5 + |S^h|^2||\phi||_2^2|N_h|^2 + |S^h|^2 \sigma^2_\phi ||\phi||_2^4 \right), \]  
(65)

since \( \sigma^2_\phi \leq \text{Var}[Y_i] \leq 1. \)

Plugging in the bounds that we obtained for the moments of \( A \) and \( B \) in (62), we conclude the proof of (57).

6.7.4 Proof of Lemma 13

Recall the definition of \( S_i \) in (48). We decompose \( \|\epsilon_{S,h}\|_2^2 = \sum_{i \in \mathbb{N}_h \cup \{0\}} \|\epsilon_{S,i}\|_2^2 \) and note that \( \|\epsilon_{S,i}\|_2^2 \sim \sigma^2_\phi \chi^2[|S_i|]. \) Applying the second deviation bound of Lemma 1 together with a union bound, we obtain that for any \( t > 0 \),
\[ \|\epsilon_{S,h}\|_2^2 \leq \sigma^2_\phi \sum_{i \in \mathbb{N}_h \cup \{0\}} \left( |S_i| + 2\sqrt{|S_i| \log(|N_h| + 1) + t} + 2t + 2 \log(|N_h| + 1) \right) \]
\[ \leq \sigma^2_\phi \left( |S^h| + 2\sqrt{|S^h|(|N_h| + 1) \log(|N_h| + 1) + t} + 2|N_h| (t + \log(|N_h| + 1)) \right), \]
with probability larger \( 1 - e^{-t} \). Relying on Condition (26), we derive that
\[ \mathbb{P}\left\{ \|\epsilon_{S,h}\|_2^2 \leq \sigma^2_\phi |S^h|(1 + |N_h|^{-1/2}) \right\} \geq 1 - \exp\left( -C \frac{|S^h|}{|N_h|^2} \right), \]
for a numerical constant \( C > 0 \) small enough.
6.8 Proof of Corollary 1

It is well known—see, e.g., Lauritzen (1996)—that any \( \text{AR}_h \) process is also a Gaussian Markov random field with neighborhood radius \( h \) (and vice-versa). Denote \( \tau_\psi^2 \) the innovation variance of an \( \text{AR}_h(\psi) \) process. The bijection between the parameterizations \( (\psi, \tau_\psi^2) \) and \( (\phi, \sigma_\phi^2) \) is given by the following equations:

\[
\phi_{-i} = \phi_i = -\psi_i + \frac{\sum_{k=i+1}^{h} \psi_k \psi_{k-i}}{1 + \|\psi\|^2_2}, \quad \text{for } i = 1, \ldots, h,
\]

\[
\sigma_\phi^2 = \frac{\tau_\psi^2}{1 + \|\psi\|^2_2}.
\] (66)

**Lower bound.** In this proof, \( C \) is a positive constant that may vary from line to line. It follows from the above equations that

\[
\|\phi\|^2_2 \leq C \frac{\|\psi\|^2_2 + h \|\psi\|^4_2}{1 + \|\psi\|^2_2}.
\]

Consider any \( r \leq 1/h \). In that case, if \( \|\phi\|_2 \geq r \) then the inequality above implies that \( \|\psi\|_2 \geq Cr \), and as a consequence, \( R_{C, \Phi(h, r)}^* \leq R_{\Phi(h), C}^* \). Therefore, since (7) and our condition on \( h \) together imply that \( r \leq 1/h \) eventually, it suffices to prove that \( R_{C, \Phi(h, r)}^* \to 1 \). For that, we apply Corollary 3. Condition (31) there is satisfied eventually under our assumptions ((7) and our condition on \( h \)). Consequently, we have \( R_{C, \Phi(h, r)}^* \to 1 \) as soon as (32) holds, which is the case when (7) holds.

**Upper bound.** It follows from (66) and the inequality \( \tau_\psi^2 \leq 1 \) that

\[
1 - \sigma_\phi^2 \geq \frac{\|\psi\|^2_2}{1 + \|\psi\|^2_2}.
\]

Denoting \( u_n := \log(n)/k + \sqrt{h \log(n)/k} \), observe as above that \( u_n \ll 1/h \) by our assumption on \( h \). Assume that \( \|\psi\|^2_2 \geq r \) for some \( h^{-1} \geq r \geq u_n \). If the corresponding parameter \( \phi \) satisfies \( \|\phi\|_1 \leq 1/2 \), it follows from the inequality \( 1 - \sigma_\phi^2 \leq \|\phi\|^2_2/(1 - \|\phi\|_1) \leq 2\|\phi\|^2_2 \) (Lemma 5) that \( \|\phi\|^2_2 \geq r/4 \). As \( \|\phi\|_1 > 1/2 \) implies that \( \|\phi\|^2_2 \geq (4h)^{-1} \), we have in any case \( \|\phi\|^2_2 \geq r/4 \). Thus, when \( (2h)^{-1} \geq r \geq u_n \)—which is congruent with (8) and our condition on \( h \), for any test \( f \), \( R_{C, \Phi(h, r)}^*(f) \leq R_{C, \Phi(h, r/4)}^*(f) \). The result then follows from Theorem 2.

6.9 Proof of Theorem 3

The proof follows that of Proposition 1. However, this time two distinct subgraphs \( S, T \in \mathcal{C} \) may not be disjoint, so we do not have independence. Throughout we leave \( \phi \) implicit whenever possible.

Before we proceed, we identify \( S \) with \( \{1, \ldots, k\} \). By Lemma 8, we have

\[
1 \leq (\Gamma^{-1}_S)_{1,1} = (\Gamma^{-1}_S)_{k,k} \leq (\Gamma^{-1}_S)_{i,i} = \frac{1}{\sigma_\phi^2}, \quad \forall i \in \{2, \ldots, k-1\},
\]

\[
(\Gamma^{-1}_S)_{i,i+1} = (\Gamma^{-1}_S)_{i,i-1} = -\frac{\phi_1}{\sigma_\phi^2}, \quad \forall i, j \in \{1, \ldots, k\} \text{ such that } |i - j| = 1,
\]

and all the other entries of \( \Gamma^{-1}_S \) are zero. By Lemma 5, \( 1 - \|\phi\|_1 \leq \sigma_\phi^2 \leq 1 \). Hence, the matrix \( A_S := \Gamma^{-1}_S - I \) satisfies

\[
\|A_S\| \leq \frac{2|\phi_1|}{\sigma_\phi^2} + \frac{1 - \sigma_\phi^2}{\sigma_\phi^2} \leq \frac{2|\phi_1|(1 + |\phi_1|)}{1 - 2|\phi_1|},
\]

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Lemma 14. Let $A$ and $B$ be (complex or real) matrices of same dimensions. Let $\text{col}(A)$ index the column vectors of $A$ that are nonzero. Then

$$\left| \text{Tr}(A^T B) \right| \leq |\text{col}(A) \cap \text{col}(B)||A||B|| .$$

Proof. Define the index set $J = \text{col}(A) \cap \text{col}(B)$. Let $A_J$ denote the submatrix of $A$ with columns indexed by $J$, and define $B_J$ similarly. We then have

$$\left| \text{Tr}(A^T B) \right| = \left| \text{Tr}(A_J^T B_J) \right| \leq |J||A_J^T B_J|| \leq |J||A_J||B_J|| \leq |J||A||B|| . \quad \Box$$

Returning to the proof of Theorem 3, for $(s, t) \in Q_\ell$, there is $j$ such that either $s_j = t_j = 1$ or $t_{j-1} = s_j = 1$. For example, assuming the former holds, we apply Lemma 14 to get

$$\text{Tr} \left[ (A^{s_j}_S A^t_j^T \cdots A^{s_{j-1}}_S A^{t_{j-1}}_j^T) (A^{t_j}_T \cdots A^{s_{j-1}}_S A^{t_{j-1}}_j^T) \right]$$

$$\leq |S \cap T||A^{s_j}_S A^t_j^T \cdots A^{s_{j-1}}_S A^{t_{j-1}}_j^T||A^{t_j}_T \cdots A^{s_{j-1}}_S A^{t_{j-1}}_j^T||$$

$$\leq |S \cap T||A^{s_j}_S||A^t_j^T||A^{s_{j-1}}_S||A^{t_{j-1}}_j^T||$$

$$\leq |S \cap T|\zeta^\ell ,$$

where $\zeta := \frac{2|\phi_1|(1+|\phi_1|)}{1-2|\phi_1|}$ and the last line comes from (67).
With this, together with the fact that \(|Q_\ell| \leq \binom{2\ell}{\ell}\) and \((1-x)^{-1/2} = \sum_{n \geq 0} \binom{2n}{n}(x/4)^n\), we obtain
\[
\Lambda \leq \frac{1}{2} |S \cap T| \sum_{\ell \geq 2} \frac{1}{\ell} \binom{2\ell}{\ell} \leq \frac{1}{4} |S \cap T|[\xi^{-1} - 1 - 2\rho].
\]
For \(0 \leq x \leq 1/4\), we have \(\sqrt{1-x} \geq 1 - x/2 - x^2/\sqrt{3}\). As \(|\phi_1| \leq 1/35\), \(4\xi \leq 1/4\) and
\[
\Lambda \leq |S \cap T| \frac{10\xi^2 + 20\xi^3}{\sqrt{1-4\xi}} \leq 17|S \cap T|\phi_1^2.
\]
We then conclude with (14) and the fact that
\[
\mathbb{E}[L^2] = \frac{1}{|C|^2} \sum_{S,T \in C} \mathbb{E}[L_SL_T].
\]

### 6.10 Proof of Proposition 3

We first control \(V_1^\ast\) under the null hypothesis. Decompose the statistics \(V_{i,S}\)

\[
V_{i,S} = V_{1,i,S} + V_{2,i,S},
\]
where \(V_{1,i,S} := \sum_{j=1}^{\lfloor k/2 \rfloor} V_{i,S}(2j)\) and \(V_{2,i,S} := \sum_{j=2}^{\lfloor k/2 \rfloor} V_{i,S}(2j-1),\)

so that all the terms in \(V_{1,i,S}\) (rest. \(V_{2,i,S}\)) are independent. It suffices to prove that, with probability going to one, \(V_{1,i} := \max_{S \in C} V_{1,i,S}\) is smaller than \([k/2]/2\) and \(V_{2,i} := \max_{S \in C} V_{2,i,S}\) is smaller than \([k/2]/2\). By symmetry, we only consider \(V_{1,i}\).

Recall that \(n = |V|\), and a simple counting argument gives \(|C| \leq n\tau^{k-1}\). Also, under the null hypothesis, \(V_{1,i,S} \sim \text{Bin}(\lfloor k/2 \rfloor, p_t)\) for any \(S \in C\). Thus, for any \(S \in C\), with the union bound, we have
\[
\mathbb{P}_0(V_{1,i} \geq v) \leq |C| \mathbb{P}_0(V_{1,i,S} \geq v) \leq n\tau^{k-1} \mathbb{P}(\text{Bin}(\lfloor k/2 \rfloor, p_t) \geq v).
\]
Define \(b_t = 1/(2p_t)\) and \(\varphi(b) = b(\log b - 1) + 1\). Choosing \(v = v_t = \lfloor k/2 \rfloor b_t p_t\), and using Bennett’s inequality, the right-hand side is bounded by
\[
n\tau^{k-1} \exp \left( - \lfloor k/2 \rfloor p_t \varphi(b_t) \right) = \exp \left( \log n + (k - 1) \log \tau - \lfloor k/2 \rfloor p_t \varphi(b_t) \right) = \exp \left( \log n + (k - 1) \log \tau - \lfloor k/2 \rfloor \frac{1}{2} h(2p_t) \right) \leq \exp \left( - k h(2p_t)/5 \right) \rightarrow 0,
\]
where the inequality holds eventually as the sample size increases. Thus the test has a type I error tending to zero in the large-sample limit.

We now consider the alternative hypothesis. Let \(S \in C\) denote the anomalous path. We have \(V_{1}^\ast \geq V_{1,S}\). Identifying \(S\) with \(\{1, \ldots, k\}\), we define \(Z_i := (X_{S_i+1} - X_{S_i})/\sqrt{2(1-\psi)}\) for any \(i \in \{2, \ldots, k\}\). We have \(Z_i \sim \mathcal{N}(0,1)\) and \(\mathbb{E}[Z_i Z_j] = \psi^{i-j-1}(\psi - 1)/2\) for \(i \neq j\). Define \(q_t(\psi) := 2F(t/\sqrt{1-\psi}) - 1\). Computing the first moment of \(V_{1,S}\), we obtain
\[
\mathbb{E}_S[V_{1,S}] = (k-1)q_t(\psi) \geq \frac{3}{5}(k-1) \rightarrow \infty.
\]
In order to conclude, it suffices to prove that \(\text{Var}_S(V_{1,S}) \ll \mathbb{E}_S^2[V_{1,S}]\). Fix any \(i \neq j\). Denoting \(a = \mathbb{E}[Z_i Z_j]\), we define \(U = (Z_j - aZ_i)/\sqrt{1-a^2}\). For any \(x\) smaller than \(t/\sqrt{1-\psi}\) in absolute
value, the probability that $|Z_j| \leq t(1 - \psi)^{-1/2}$ conditionally to $Z_i = x$ where $|x| \leq t(1 - \psi)^{-1/2}$ is close to $q_t(\psi)$; indeed,

$$\mathbb{P}
\left[
|Z_j| \leq \frac{t}{\sqrt{1 - \psi}} \bigg| Z_i = x
\right] \leq \mathbb{P}
\left[
\frac{-t - ax}{\sqrt{1 - \psi}\sqrt{1 - a^2}} \leq U \leq \frac{-t + ax}{\sqrt{1 - \psi}\sqrt{1 - a^2}}
\right]
\leq \mathbb{P}
\left[
|U| \leq \frac{t}{\sqrt{1 - \psi}\sqrt{1 - a^2}}
\right] \quad \text{(by symmetry of the distribution)}
\leq 2 \mathbb{P}
\left[
U \leq \frac{t}{\sqrt{1 - \psi}}
\right] + 2 \mathbb{P}
\left[
\frac{t}{\sqrt{1 - \psi}} \leq U \leq \frac{t}{\sqrt{1 - \psi}\sqrt{1 - a^2}}
\right]
\leq q_t(\psi) + 2 \left[
F\left(\frac{t}{\sqrt{1 - \psi}\sqrt{1 - a^2}}\right) - F\left(\frac{t}{\sqrt{1 - \psi}}\right)\right]
\leq q_t(\psi) + a^2,
$$

where we used the fact that $U$ is standard normal in the fourth line, a Taylor development (of order 1) in the fifth line, and the fact that $|a| \leq 1/2$ and $xe^{-x^2} \leq (2e)^{-1/2}$ in the last line. As a consequence,

$$\text{Cov}
\left[
\mathbb{1}_{\{|Z_i| \leq \frac{t}{\sqrt{1 - \psi}}\}}, \mathbb{1}_{\{|Z_j| \leq \frac{t}{\sqrt{1 - \psi}}\}}
\right] \leq q_t(\psi)^2 |i - j|^{-2}(1 - \psi)^2,
$$

for all $i \neq j$. We conclude that

$$\text{Var}_S(V_{i,S}) \leq (k - 1) \left[q_t(\psi)(1 - q_t(\psi)) + 2q_t(\psi) \frac{1 - \psi}{1 + \psi}\right] \leq 3 \mathbb{E}_S[V_{i,S}] \ll \mathbb{E}_S^2[V_{i,S}].$$

Thus the test also has a type II error tending to zero in the large-sample limit.

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