Classical Solutions for Two Dimensional QCD on the Sphere

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ABSTRACT

We consider $U(N)$ and $SU(N)$ gauge theory on the sphere. We express the
problem in terms of a matrix element of $N$ free fermions on a circle. This allows
us to find an alternative way to show Witten’s result that the partition function
is a sum over classical saddle points. We then show how the phase transition of
Douglas and Kazakov occurs from this point of view. By generalizing the work
of Douglas and Kazakov, we find other “stringy” solutions for the $U(N)$ case in
the large $N$ limit. Each solution is described by a net $U(1)$ charge. We derive a
relation for the maximum charge for a given area and we also describe the critical
behavior for these new solutions. Finally, we describe solutions for lattice $SU(N)$
which are in a sense dual to the continuum $U(N)$ solutions.

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1. Introduction

In their classic work, Gross and Witten[1], and independently Wadia[2], showed that lattice QCD in two dimensions contains a third order phase transition in the large $N$ limit. Basically, what happens is that the partition function for the plaquettes reduces to a product of partition functions for the individual plaquettes. The theory on each one is equivalent to a $d = 0$ unitary matrix model, with the potential given by

$$V = \frac{N}{g^2 a^2} \text{tr}(U + U^\dagger).$$

This theory is equivalent to zero-dimensional field theory of $N$ eigenvalues with values that lie on a circle and a potential

$$V = -(N/2g^2a^2) \sum_i (e^{i\theta_i} + e^{-i\theta_i}) - \sum_{i<j} \log \left( \sin \frac{\theta_i - \theta_j}{2} \right)^2.$$

The theory becomes critical by tuning the lattice size such that the highest fermion of the Dyson gas lies at the top of the potential. For fixed $g$, as the lattice size is decreased, the potential becomes very deep and the fermions move further away from the critical configuration. Thus in the limit of vanishingly small $a$, the fermions should only see a quadratic potential.

Very recently there has been a revived interest in large $N$ QCD. This program was started by Gross[3], who was looking for a string formulation of the problem. Further work in this direction was done by one of the authors[4] and finally, the complete theory was worked out by Gross and Taylor, who showed that QCD is essentially a string theory for a target space with arbitrary genus. The basic insight in handling this problem is to equate $U(N)$ or $SU(N)$ representations with sums of maps of world-sheets into the target space. It turns out that the number of boxes for a Young tableau of a representation is equal to the number of coverings of the surface. For surfaces with genus $g > 1$, organizing the partition function by the number of covers also nicely organizes the sum into powers of $1/N$. Hence
it is very easy to compute the leading order contribution in the large $N$ limit for these surfaces. For the torus, there can be leading order contributions from any number of coverings, but it is easy to sum their contribution. However, in the case of the sphere finding the leading order contribution in $1/N$ is not so easy, namely because all $SU(N)$ or $U(N)$ representations contribute to the leading order behavior, and computing these contributions involves computing $1/N$ corrections to the dimensions of these representations.

Recently, Douglas and Kazakov (DK) discovered a clever way to solve the problem of the sphere by treating the rows of the young tableau and the number of boxes in each row as continuous variables[7]. Doing this they were able to compute the leading behavior which led them to a surprising result: the theory contains a third order phase transition, similar to the case of the lattice version of Gross, Witten and Wadia, but for the continuum limit of the theory.

However, the $U(N)$ case has an interesting feature—the sum over representations is not asymptotic. In order to correct for this, one must sum over an infinite number of charge sectors. While this will eventually lead to overcounting for finite $N$, the answer will at least be asymptotic in the large $N$ limit. These different charge sectors can be thought of as being extra solutions to the large $N$ equations of motion. These different sectors are conjugate to the $U(1)$ instantons. Because of this, one should be able to find the such solutions using the DK analysis. In this paper we do precisely this by generalizing an ansatz described by DK.

There has also been some recent work by Witten in QCD$_2$, although from a different perspective[8]. He has shown that the partition function on any Riemann surface is given by a sum over the saddle points. An interesting question is how the DK phase transition appears from this point of view.

In section two we review recent work on 2d QCD and construct a free fermion picture for this theory. In section three we present an alternative derivation of Witten’s result that the partition function is a sum over saddle points. We then show how the continuum phase transition occurs in this dual picture of QCD$_2$. In
section 4 we consider the new solutions for $U(N)$ QCD and generalize the analysis of DK for these solutions. We consider the two cases where the area is near its critical value, and when the area is very large. In section 5 we compare these solutions to corresponding solutions for lattice $SU(N)$ QCD. In section 6 we present our conclusions. We include an appendix with some useful equations for elliptic integrals.

2. Review of QCD$_2$

Let us first review how QCD$_2$ on a cylinder is the same as a theory of free fermions by showing that it can be reduced to a one-dimensional unitary matrix model[9-11]. In the gauge $A_0 = 0$, the Hamiltonian is given as

$$H = \frac{1}{2} \int_0^L dx \, \text{tr} F_{01}^2 = \frac{1}{2} \int_0^L dx \, \text{tr} A_1^2 \quad (2.1)$$

with the overdot denoting a time derivative. The $A_0$ equation of motion is now the constraint

$$D_1 F_{10} = \partial_1 \dot{A}_1 + ig[A_1, \dot{A}_1] = 0. \quad (2.2)$$

Define a new variable $V(x)$,

$$V(x) = W^*_0 \dot{A}_1(x) W^L_x, \quad (2.3)$$

where

$$W^b_a = \text{P} e^{ig \int_a^b dx A_1}. \quad (2.4)$$

Then (2.1) can be written as

$$\partial_1 V(x) = 0, \quad (2.5)$$

so $V(x)$ is a constant. Thus $V(0) = V(L)$, which implies that

$$[W, \dot{A}_1(0)] = 0, \quad (2.6)$$

where $W \equiv W^L_0$ and we have used the periodicity of $A_1$ in $x$. 

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From the definitions (2.3) and (2.4), we find the relation

$$\dot{W} = ig \int_0^L dx W_0^x \dot{A}_1(x) W_x^L = ig \int_0^L dx V(x), \quad (2.7)$$

and therefore using (2.5) and (2.6), we derive

$$\dot{W} = igLW \dot{A}_1(0) = igL \dot{A}_1(0) W. \quad (2.8)$$

(2.8) then implies that

$$[W, \dot{W}] = 0. \quad (2.9)$$

Because $V(x) = V(0)$, $\dot{A}_1(x)$ satisfies

$$\dot{A}_1(x) = W_0^x \dot{A}_1(0) W_x^0. \quad (2.10)$$

Thus, using this relation along with (2.8), we can rewrite the Hamiltonian in (2.1) as

$$H = -\frac{1}{2g^2L} \text{tr}(W^{-1} \dot{W})^2. \quad (2.11)$$

If the gauge group is $U(N)$, with the $U(1)$ coupling given by $g/N$, then (2.11) is the Hamiltonian for the one-dimensional unitary matrix model. The constraint in (2.9) reduces the space of states to singlets[15]. Hence, the problem is reducible to the eigenvalues of $W$.

Upon quantization, this problem is equivalent to a system of $N$ nonrelativistic fermions living on a circle, with the Hamiltonian given by

$$H = -\left(\frac{g^2L}{2}\right) \sum_{i=1}^N \frac{\partial^2}{\partial \theta_i^2}, \quad 0 \leq \theta_i < 2\pi. \quad (2.12)$$

The fermionization is due to the Jacobian of the change of variables from $W$ to its eigenvalues, introducing the Vandermonde-type determinant in the wavefunction
of the states, which in the unitary matrix case reads

$$\tilde{\Delta} = \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2} = \frac{\Delta(e^{i\theta_i})}{\prod_i e^{i(N-1)\theta_i}},$$

(2.13)

where $\Delta(\lambda_i) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the standard Vandermonde determinant. Notice that each factor in (2.13) is antiperiodic on the circle. Thus, if $N$ is even the fermions have antiperiodic boundary conditions. Likewise, if $N$ is odd they have periodic boundary conditions. This can be understood in terms of transporting a fermion once around the circle, passing by $N - 1$ other fermions along the way and therefore picking up $N - 1$ minus signs. Hence, in either case, the ground state is built by filling all states with wave numbers between $-N/2 + 1/2$ and $N/2 - 1/2$, inclusive. Subtracting off the ground state energy, one easily sees that this spectrum reproduces that found for the different representations of $U(N)$.

If the gauge group is $SU(N)$, because $A_1$ is now traceless $W$ will also obey the condition $\det W = 1$. Therefore the center of mass coordinate for the fermions is absent and we must mod it out of the theory. This means that we need to identify states in which all fermions have their momentum shifted by the same amount. Moreover, we must subtract the energy of the center of mass from the energy of each state in the theory.

Now consider the partition function for the sphere. In terms of the fermions, we want a matrix element that corresponds to the sphere topology. The obvious thing to do is to map the end points of a cylinder to two points and therefore $W = 1$ there. In terms of the fermions, this corresponds to computing the inner product of $N$ fermions whose position is at the point $x = 0$ on the circle at time $t = 0$ with the fermions at the same point at time $t = T$. This inner product $Z$ is given by

$$Z = \langle x_i; t = T | y_i; t = 0 \rangle,$$

(2.14)

in the limit that all $x_i \to 0$ and $y_i \to 0$.  

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To calculate $Z$, one can insert a complete set of momentum states for each particle. The wave function at $t = 0$ and $t = T$ must be antisymmetric under the exchange $x_i \rightarrow x_j$ or $y_i \rightarrow y_j$. Therefore, the matrix element in (2.14) has the factor

$$\det |e^{ip_i x_j}| \det |e^{-ip_i y_j}|. \quad (2.15)$$

As $x_i \rightarrow 0$ and $y_i \rightarrow 0$, the factor in (2.15) approaches

$$\prod_{i<j} (x_i - x_j)(y_i - y_j)(p_i - p_j)^2; \quad (2.16)$$

The matrix element in (2.14) is therefore given by

$$Z = C \sum_{p_i} \prod_{i>j} (p_i - p_j)^2(x_i - x_j)(y_i - y_j) \exp(-\frac{1}{2} g^2 LT \sum_i p_i^2), \quad (2.17)$$

where $C$ is an unimportant constant. Not surprisingly, this term approaches zero in this limit. But in order to find the sphere contribution, one should notice that the fermion wavefunction at the end points are more singular than a $\delta$-function, namely

$$\psi(x_i) = \tilde{\Delta}(x_i)\delta(W - 1) = \frac{1}{\Delta(x_i)}\delta(x_i) \quad (2.18)$$

where a factor of $1/\tilde{\Delta}^2(x_i)$ was produced by the change of variables from $W$ to $x_i$. Therefore, it is necessary to divide the expression in (2.17) by these extra Vandermonde determinants, that is,

$$\prod_{i<j} (x_i - x_j)(y_i - y_j),$$

leaving a finite expression. Hence the sphere partition function is

$$Z_{\text{sphere}} = C \sum_{p_i} \prod_{i>j} (p_i - p_j)^2 \exp(-\frac{1}{2} g^2 LT \sum_i p_i^2). \quad (2.19)$$

We can compare this to the sphere partition function of Migdal and Rusakov[12,13].
which is given by

\[ Z_{MR} = \sum_R (d_R)^2 \exp(-g^2 AC_{2R}) \]  

(2.20)

where the sum is over all representations of \( U(N) \) or \( SU(N) \) and \( d_R \) is the dimension of the representation and \( C_{2R} \) is the quadratic casimir. The correspondence of the fermion states with the \( U(N) \) representations is as follows[10]: If we describe a representation by a Young tableau, then the number of boxes in row \( i \), \( n_i \) is the momentum shift of the fermion with the \( i^{th} \) highest momentum above its ground state value. In terms of boxes, the casimir is given by

\[ C_{2R} = \frac{1}{2} \left( N \sum_i n_i + \sum_i n_i(n_i - 2i + 1) \right) \]  

(2.21)

which one can easily checked is reproduced by the fermions after subtracting off the ground state energy. The dimension of the representation is given by

\[ d_R = \prod_{i>j} \left( 1 - \frac{n_i - n_j}{i - j} \right) \]

\[ = \prod_{i>j} (i - j)^{-1} \prod_{i>j} \left( (n_j - j) - (n_i - i) \right). \]  

(2.22)

The first product in the second line of (2.22) is a representation independent term and is thus an unimportant constant. The second term is just \( p_j - p_i \) for the fermions. The total momentum is the \( U(1) \) charge for a representation. Hence we find full agreement with the result of Migdal and Rusakov.
3. Classical Solutions

After a cursory inspection of (2.19) it would appear that $Z_{\text{sphere}}$ is simply the partition function for a $d = 0$ matrix model in a quadratic potential. Unlike the lattice case, the potential never turns over, so one might not expect a phase transition. As was shown by Douglas and Kazakov, this is not correct. The point is that the variables $p_i$ that appear in (2.19) are discrete, hence the density of eigenvalues will be bounded. When this bound is reached, a phase transition occurs. Thus, in the strong coupling phase we simply have condensation of the fermions in their momentum lattice, which gives a very simple physical understanding of the phase transition mechanism. The critical value of the area is reached when the density of $p_i$, as given by the Wigner semicircle law which is valid in the continuum, reaches somewhere the lattice bound, namely one, thus reproducing the DK result.

There should be a classical field configuration, termed master field, which dominates the path-integral in each phase. In fact, as was shown by Witten\cite{Witten} using the localization theorem of Duistermaat and Heckman (DH), the full QCD$_2$ path integral can be written as a suitable sum over classical saddle points. In what follows we will give a very simple demonstration of Witten’s result using the fermion picture and identify the classical configuration which dominates, that is, the master field.

The key observation is that the exact propagator of a free particle is proportional to the exponential of the action corresponding to the classical (straight) path connecting the initial and final points. Since QCD$_2$ on the torus is equivalent to $N$ free fermions, its partition function will also be given by an appropriate classical path of the free fermions. The things to be taken into account, however, are

i) The particles live on a circle; therefore there are several possible classical paths for each of them, differing by their winding around the circle with fixed initial and final positions.
ii) The particles are fermions; thus one should also consider paths where the
final positions of the particles have been permuted, weighted by a fermionic factor
$(-)^C$, where $C$ is the number of times the paths of the particles cross.

The total partition function will then be the (weighted) sum of the actions of all
these classical configurations. Since to each path corresponds a (diagonal) matrix
$W(t)$, and to that (up to gauge transformations) a classical field configuration
satisfying the field equations of motion, this is the sum over saddle points of the
action of Witten.

For the sphere the same picture holds, with the difference that all paths start
and end at the point $x = 0$, and that each path is further weighted by an extra
factor, due to the division by the Vandermonde determinants as explained in the
previous section. This is, again, the sum over saddle points of Witten, and the
extra weighting factors are the determinants which appear in the DH theorem.
These paths are characterized by their winding numbers $\{n_i\}$ (up to permutation)
and thus this is a sum of the form

$$Z_{\text{sphere}} = \sum_{n_i} w(n_i) \exp \left( -\frac{2\pi^2}{g^2LT} \sum_i n_i^2 \right)$$

(3.1)

where $w(n_i)$ are the (as yet undetermined) weighting factors.

The easiest way to obtain the full expression in (3.1) is to Poisson resum (2.19).
Using the formula

$$\sum_n f(n) = \sum_n \tilde{f}(2\pi n)$$

(3.2)

where $f(x)$ is any function and $\tilde{f}$ is its Fourier transform, we obtain

$$Z_{\text{sphere}} = C \sum_{n_i} F_2(2\pi n_i),$$

(3.3)

where

$$F_2(x_i) = \int \prod_i dp_i e^{-i \sum_i x_i p_i} \Delta^2(p_i) \exp \left( -\frac{1}{2} g^2LT \sum_i p_i^2 \right).$$

(3.4)
To find the Fourier transform appearing in (3.4), we first note that

\[
F_1 \equiv \int \prod_i d\rho_i e^{-\sum_i x_i \rho_i} \Delta(p_i) \exp\left(-\frac{1}{2\alpha} \sum_i p_i^2\right) = C \Delta(x_i) \exp\left(-\frac{1}{2\alpha} \sum_i x_i^2\right).
\]  
(3.5)

To prove this, notice that

\[
F_1 = \Delta(-\partial_{\rho_i}) \int \prod_i d\rho_i e^{-\sum_i x_i \rho_i} \exp\left(-\frac{1}{2\alpha} \sum_i p_i^2\right) = P(x_i) \exp\left(-\frac{1}{2\alpha} \sum_i x_i^2\right).
\]  
(3.6)

\(P(x_i)\) is a polynomial of degree \(N(N-1)/2\); moreover it is completely antisymmetric in \(x_i\). Therefore, up to a normalization, it is the Vandermonde. The constant \(C\) in (3.5) can be found explicitly, it is however irrelevant for this discussion since it will amount to an overall coefficient in the final result. Using the convolution property of the Fourier transform of a product, in combination with (3.5), we find

\[
F_2(x_i) = (F_1 \otimes F_1)(x_i)
\]

\[
= C \int \prod_i dy_i \Delta\left(\frac{x_i - y_i}{2}\right) \Delta\left(\frac{x_i + y_i}{2}\right) \exp \left(-\frac{1}{4g^2LT} \sum_i \left[ (x_i + y_i)^2 + (x_i - y_i)^2 \right] \right)
\]

\[
= C \exp\left(\frac{1}{2g^2LT} \sum_{ij} x_{ij}^2\right) \int \prod_i dy_i \prod_{i<j} \left( y_{ij}^2 - x_{ij}^2 \right) \exp \left(-\frac{1}{2g^2LT} \sum_i y_i^2 \right)
\]  
(3.7)

where \(x_{ij} = x_i - x_j\) and \(y_{ij} = y_i - y_j\). Substituting (3.7) in (3.3) we recover expression (3.1), with the weights \(w(n_i)\) given by

\[
w(n_i) = C \int \prod_i dy_i \prod_{i<j} \left( y_{ij}^2 - n_{ij}^2 \right) \exp \left(-\frac{1}{2g^2LT} \sum_i y_i^2 \right).
\]  
(3.8)

The above determines the expression of the sphere partition function in terms of classical saddle point configurations.

In the large \(N\) limit one particular classical configuration in (3.1) should dominate. To determine this, let us ignore for the moment the fact that the \(n_i\) are
discrete and replace the sum in (3.1) with an integral. Then we can bring the expression for $Z_{\text{sphere}}$ into the form

\[ Z_{\text{sphere}} = C \int \prod_i ds_i dt_i \Delta(s_i) \Delta(t_i) \exp \left( -\frac{1}{2} N \sum_i (s_i^2 + t_i^2) \right), \quad (3.9) \]

where we rescaled $g^2$ to $g^2/N$, in order to have a nontrivial large $N$ limit, and changed variables to $s = (y+2\pi n)/\sqrt{2g^2LT}$, $t = (y-2\pi n)/\sqrt{2g^2LT}$. We see that in (3.9) all explicit dependence on the coupling constant and area have disappeared. The area enters this picture indirectly, through the discreteness of $n_i$. (In fact, if it were not for this discreteness the above integral would vanish.) Due to this, the plane $(s,t)$ is discrete in the $s-t$ direction and consists of parallel diagonal lines with distance $D = 2\pi/\sqrt{g^2LT}$. If this spacing is such that the distribution of $s$ and $t$ contributing to the above integral in the large $N$ limit is entirely within the lines $n = 1$ and $n = -1$, then only the sector $n = 0$ contributes. This signals a phase transition. Unlike the quantum case, there is no exclusion principle for the windings $n_i$ and thus no maximum density to be saturated. The phase transition in the classical expansion is more like a Bose condensation to the ground state $n = 0$.

To estimate the critical area, substitute the Vandermondes in (3.9) with their absolute value, which corresponds to putting the product of differences appearing in (3.8) into absolute values. This has no effect for the $n_i = 0$ term, while it overestimates the contribution of the other sectors. (3.9) then becomes the product of two independent integrals involving a gaussian factor and one power of the Vandermonde. The large $N$ saddle point of these integrals is found in a standard way and the distribution of $s_i$ and $t_i$ is, in fact, again a Wigner semicircle with radius $R = \sqrt{2}$. This on the $(s,t)$ plane creates a “square” distribution with side $2R$ in each direction. When this square lies entirely within the lines $n = \pm 1$, the sectors $n \neq 0$ do not contribute and we are in the Boson condensate phase. For an estimate of the critical area, assume that the transition occurs when the
corners of the square start touching the lines \( n = \pm 1 \), that is, when the half-diagonal becomes \( D \). Putting \( 2R\sqrt{2}/2 = D \) we obtain \( g^2A_{\text{crit}} = \pi^2 \). Somewhat surprisingly, our estimation has given the precise result of Douglas and Kazakov. It is not entirely clear from this argument why the transition should occur precisely at this point.

In any event, we see that the classical configuration in the weak-coupling (small area) phase is \( n_i = 0 \). This corresponds to the master field \( A = 0 \), up to gauge transformations. As a check, notice that in this phase the discreteness of \( p_i \) in the quantum expression is irrelevant (no saturation) and thus we can substitute the sums in (2.19) by integrals. This, upon resummation, becomes the \( n_i = 0 \) term in (3.1), as can be seen by the expression for \( w(0) \) obtained from (3.8). Only in the strong coupling phase will we have a nontrivial master field. This is implicitly determined by (3.1) and (3.8), although its explicit expression is not known.

4. \( U(1) \) Sectors

There is one difficulty with the string picture as it now stands for the \( U(N) \) gauge group, namely, the sum is not asymptotic with the exact answer in the limit \( 1/N \rightarrow 0 \). This is because there exist states with finite energy, but which will never appear in the perturbative sum in (2.20). These are the states that correspond to \( N \) fermions with momenta shifted by a constant finite amount. Such states have finite energy in the large \( N \) limit, but do not show up in a perturbative sum over surfaces since the corresponding Young tableau has at least \( N \) boxes. Moreover, these states are local minima, in the sense that in order to find a state with lower energy, it is necessary to shift the total momentum by a large amount.

We can rectify this problem by including other sectors in the sum. For finite \( N \), this will eventually lead to overcounting, but the answer will be asymptotic.
To this end, let us define $Z_m$

$$Z_m = \sum_{\text{Reps}} d^2_R \exp \left( -\frac{Ag^2}{2} \sum (n_i + m) \right) \exp \left( \frac{Ag^2}{2N} \sum (n_i - 2i + 1 + m)(n_i + m) \right),$$

(4.1)

where we have basically added $m$ boxes to each row. After some manipulation, $Z_m$ reduces to

$$Z_m = \sum_{\text{Reps}} d^2_R e^{-\frac{Ag^2}{2}(1+2m/N)n_R} e^{-\frac{Ag^2}{2N}\bar{n}_R} e^{-\frac{Ag^2m^2}{2}},$$

(4.2)

Hence we see that $Z_m$ has the same form as $Z_0$, except that the area term that appears in front of $n$ has been shifted and there is an extra factor of $m^2$ in the energy. But the $\bar{n}$ term is the same, meaning that the Gross-Taylor rules are exactly the same as in the $Z_0$ case, except that the area that will be used for the Nambu-Goto term has been renormalized, with a different shift for the chiral and antichiral sectors. The sum

$$Z = \sum_m Z_m$$

(4.3)

is now an asymptotic sum for the QCD$_2$ string.

We can interpret the sum in (4.3) as a sum over different classical string solutions, since around each sector $m$ there is a sum over $1/N$, the string coupling. These different sectors are basically the conjugates of the $U(1)$ instanton sectors. We can Poisson resum (4.3), giving the expression

$$Z = \sqrt{\frac{\pi}{Ag^2}} \sum_m \sum_{\text{Reps}} d^2_R e^{-\left(\frac{Ag^2}{2} + \frac{2\pi im}{N}\right)n_R} e^{-\frac{Ag^2}{2N}\bar{n}_R} e^{-\frac{Ag^2m^2}{2}} e^{-\frac{m^2}{2g^2A}}.$$

(4.4)

The $m^2$ term now has a factor of $1/g^2$ in front of it, which is the contribution one would expect from $U(1)$ instantons. Furthermore, the area that now appears in the Nambu-Goto action is now complex*, and there is also an additional $n^2$ term.

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* This might be useful for formulating QCD$_2$ as a topological field theory. We thank S. Cordes and C. Vafa for informing us of their work in this area.
This last term has been discussed before in the $SU(N)$ case and was attributed to contributions from tubes and handles on the world-sheet.

Since there are other string solutions corresponding to different values of $m$, we might expect to be able to generalize the work of Douglas and Kazakov to find the leading order contributions around these solutions. To this end, consider the solutions to the equations of motion for the path integral in (2.19),

$$\frac{Ap_i}{2N} = \sum_{j \neq i} \frac{1}{p_i - p_j}, \quad (4.5)$$

where we have absorbed the QCD coupling into the area and rescaled it by a factor of $N$. In the large $N$ limit we can define the new variables $x = i/N$, and $h(x) = p_i/N$, giving the new equation,

$$Ah/2 = P \int d\lambda \frac{u(\lambda)}{h - \lambda}, \quad (4.6)$$

where $u(\lambda)$ is the density of eigenvalues, $u(\lambda) = dx/d\lambda$. But unlike the eigenvalues of a matrix model in a quadratic potential, the $p_i$ are discrete, therefore the density $u(\lambda)$ is bounded, satisfying $u(\lambda) \leq 1$.

DK treated this problem by dividing up the possible real values of $\lambda$ into three regions, where $u(\lambda) = 0$, $0 < u(\lambda) < 1$ and $u(\lambda) = 1$. They choose the ansatz that the first region occurs for $|\lambda| > a$, the second for $b < |\lambda| < a$, and the third for $|\lambda| < b$. If we define $\tilde{u}(\lambda)$ to be the density of eigenvalues minus the contribution from the region where $u(\lambda) = 1$, then (4.6) can be rewritten as

$$Ah/2 + \log \frac{h - b}{h + b} = P \int d\lambda \frac{\tilde{u}(\lambda)}{h - \lambda}, \quad (4.7)$$

The problem has been reduced to a two cut eigenvalue problem. To solve this,
define the function

\[ f(h) = \int d\lambda \frac{\tilde{u}(\lambda)}{h - \lambda}. \] (4.8)

f(h) then must satisfy

\[ f(h) = \frac{1}{2\pi i} \sqrt{(h^2 - a^2)(h^2 - b^2)} \int ds \frac{A s/2 - \log \frac{b-s}{b+s}}{(h-s)\sqrt{(a^2 - s^2)(b^2 - s^2)}}. \] (4.9)

where the contour surrounds the cuts from the square roots, but not the cut from the log nor the singularity at \( h \). Pulling the contour back, (4.9) leads to the equation

\[ f(h) = h \frac{A}{2} + \log \frac{h-b}{h+b} - \sqrt{(a^2 - h^2)(b^2 - h^2)} \int_{-b}^{b} ds \frac{1}{(h-s)\sqrt{(a^2 - s^2)(b^2 - s^2)}}. \] (4.10)

Expanding about large \( h \), one can then find relations between \( a, b \) and \( A \). Before doing this, let us go back and reexamine the DK ansatz. They of course make the reasonable assumption that the solution to the equations of motion is symmetric about \( h = 0 \). This is quite sensible for the ground state, but should not be true for solutions that correspond to sectors with nonzero values of \( m \). Therefore, let us relax their ansatz somewhat and assume that \( 0 < u(\lambda) < 1 \) for the regions \( b < \lambda < a \) and \( d < \lambda < c \), where \( a, b, c \) and \( d \) are to be determined. Then we can then proceed as before, reaching the equation

\[ f(h) = h \frac{A}{2} + \log \frac{h-b}{h-c} - \int_{c}^{b} ds \frac{\sqrt{(a-h)(b-h)(c-h)(d-h)}}{(h-s)\sqrt{(a-s)(b-s)(s-c)(s-d)}}. \] (4.11)
From (4.8) and (A.6) one finds that the density of eigenvalues is given by

\[
u(\lambda) = \frac{\sqrt{(a - \lambda)(\lambda - b)(\lambda - c)(\lambda - d)}}{\pi} \int_c^b \frac{ds}{(\lambda - s)\sqrt{(a - s)(b - s)(s - c)(s - d)}}
\]

\[= \frac{2}{\pi(\lambda - c)(\lambda - d)} \frac{\sqrt{(a - \lambda)(\lambda - b)(\lambda - c)(\lambda - d)}}{\sqrt{(a - c)(b - d)}}
\times \left( (c - d)\Pi(b - c \lambda - d, b - d \lambda - c, q) + (\lambda - c)K(q) \right),
\]

(4.12)

where \(K\) and \(\Pi\) are the complete elliptic integrals of the first and third kind.

Let us now expand the righthand side (4.11) in powers of \(1/h\) and compare it to the expansion in (4.8). Matching the terms of order \(h\) gives the equation

\[
\frac{A}{2} - \int_c^b \frac{ds}{\sqrt{(a - s)(b - s)(s - c)(s - d)}} = 0.
\]

(4.13)

Using equation (A.1) in the appendix gives

\[
A = \frac{4K(q)}{\rho},
\]

(4.14)

where

\[
\rho = \sqrt{(a - c)(b - d)}, \quad q = \sqrt{\frac{(a - d)(b - c)}{(a - c)(b - d)}}.
\]

Note that for the case \(c = -b, a = -d\), the modulus \(q\) that appears in (4.14) is different than the corresponding equation in [7].

Matching the \(h^0\) terms in the expansion of (4.8) and (4.11) leads to the equation

\[
\frac{1}{2} \int_c^b ds \frac{a + b + c + d}{\sqrt{(a - s)(b - s)(s - c)(s - d)}} = \int_c^b ds \frac{s}{\sqrt{(a - s)(b - s)(s - c)(s - d)}}.
\]

(4.15)
Using (A.1) and (A.2), we can rewrite this equation as

$$\frac{c - d}{\rho} \Pi(\alpha, q) = \frac{a + b + c - d}{2\rho} K(q), \quad (4.16)$$

where $\alpha = \sqrt{\frac{b-c}{b-d}}$. In [7], the $h^0$ equation is trivially satisfied due to symmetry.

Matching the $h^{-1}$ term in the two equations gives

$$\int ds \tilde{u}(s) + b - c = - \int_c^b ds \frac{s^2}{\sqrt{(a - s)(b - s)(s - c)(s - d)}}$$

$$+ \frac{1}{2} \int_c^b ds \frac{s(a + b + c + d)}{\sqrt{(a - s)(b - s)(s - c)(s - d)}}$$

$$+ \frac{1}{8} \int_c^b ds \frac{a^2 + b^2 + c^2 + d^2 - 2ab - 2ac - 2ad - 2bc - 2bd - 2cd}{\sqrt{(a - s)(b - s)(s - c)(s - d)}}. \quad (4.17)$$

The integral on the lefthand side of (4.17) is the total number of eigenvalues minus those in the region between $c$ and $b$, divided by $N$. But the number between $c$ and $b$ is just $N(b - c)$, thus the lefthand side is unity. Using (A.1), (A.2) and (A.3), and then using the equation in (4.16), we can reduce (4.17) to

$$1 = \frac{(a - b - c + d)^2}{4\rho} K(q) + \rho E(q), \quad (4.18)$$

where $E(q)$ is the complete elliptic integral of the second kind.

The next term in the expansion in (4.8) is

$$h^{-2} \int ds \tilde{u}(s)s.$$

But this is just the sum of the momenta for this particular solution coming from the regions where the density is less than one, divided by $N^2$. This is its contribution to
the $U(1)$ charge divided by $N^2$. Let us call this rescaled charge $\tilde{Q}$. Using equations (A.1)-(A.4), (4.16) and (4.18), the $h^{-2}$ expansion leads to the equation

$$\tilde{Q} + \frac{1}{2}(b^2 - c^2) = Q = \frac{W}{4} + \frac{K(q)}{16}XYZ,$$

(4.19)

where

$$W = a + b + c + d \quad X = a - b - c + d$$

$$Y = a + b - c - d \quad Z = a - b + c - d.$$

(4.20)

$(b^2 - c^2)/2$ is the contribution to the rescaled charge from the eigenvalues that sit between $c$ and $b$, hence the lefthand side of (4.19) is the total rescaled charge. The charge for the lowest energy state in sector $m$ is $mN$, hence $Q = m/N$.

Finally, we can find the specific heat, which is the next term in the expansion of $f(h)$. Using (A.1)-(A.5), (4.16) and (4.18), we find, after a fair amount of algebraic manipulation

$$F'(A,Q) - 1/24 = \int dsu(s)s^2$$

$$= \frac{\rho E(q)}{48}(3W^2 + X^2 + Y^2 + Z^2)$$

$$+ \frac{K(q)}{12\rho}(3W^2 X^2 + 6W XYZ + Y^2 Z^2 + 2X^2 Y^2 + 2X^2 Z^2 + X^4)$$

$$= \frac{K(q)}{12\rho}(6W XYZ + Y^2 Z^2 + X^2 Y^2 + X^2 Z^2)$$

$$+ \frac{1}{48}(3W^2 + X^2 + Y^2 + Z^2).$$

(4.21)

Given the area $A$ and the charge $Q$, in principle, one should be able to solve for $a$, $b$, $c$ and $d$ using the four equations (4.14), (4.16), (4.18) and (4.19). However, we should not expect solutions for all possible values of $A$ and $Q$. For instance, from DK, we know that there are no solutions for $Q = 0$ and $A < \pi^2$. This of course is the weak coupling regime. But for nonzero values of $Q$, the minimum value of $A$ should be higher. In fact, in the infinite area limit, the upper value of $Q$
is bounded. Naively, this should happen for the sector when the fermion with the smallest or largest momentum is zero. We will see that the naive answer is correct.

To further analyze the problem, we will consider two regions. The first corresponds to the area near its critical value for small values of \( Q \). The second region occurs for very large values of the area. Let us first consider what happens at the critical point. In this case \( b = c \), thus \( \alpha = q = 0 \) and \( \Pi = K = E = \pi/2 \). From equation (4.16) we find that \( a = -d \) at the critical point. To determine \( b \), let us go back to equation (4.12). Since \( b = c \), we find that \( u(\lambda) \) is given by

\[
\begin{align*}
    u(\lambda) &= \sqrt{\frac{a - \lambda}{a + \lambda}} \frac{1}{\sqrt{a^2 - b^2}}[(\lambda - b) + (b + a)] \\
    &= \sqrt{\frac{a^2 - \lambda^2}{a^2 - b^2}}.
\end{align*}
\]  

(4.22)

If \( b \neq 0 \), then for \( \lambda^2 < b^2 \), \( u(\lambda) > 1 \). But this violates the ansatz that the density is less than or equal to unity. Therefore, we must choose \( b = 0 \). Using (4.18) and (4.19) we then see that \( a = \frac{2}{\pi} \) and that the charge for this solution is zero.

Now consider small, but nonzero values for \( Q \). In this case, \( b - c \) must be nonzero. To this end, let

\[
\epsilon = b - c, \quad \delta = b + c
\]  

(4.23)

Using (4.18) and the asymptotic expansions in the appendix, we find the leading correction to \( a \) from its critical value is

\[
\Delta a = -\frac{\pi}{32}(\epsilon^2 + 2\delta^2).
\]  

(4.24)

Since this correction is of order \( \epsilon^2 \) and not \( \epsilon \), it will not contribute to \( a + d \) in leading order. This leading order correction can be found from (4.16) and the asymptotic
expansions in the appendix. A little algebra shows that

\[ a + d = \frac{\pi^2}{32} \epsilon^2 \delta. \]  

(4.25)

We can now use (4.19), (4.23) and (4.25) to find \( Q \). Let us rewrite (4.19) as

\[ Q = (a + d) \left( \frac{1}{4} + \frac{K}{16\rho} (a - d + b - c)(a - d - b + c) \right) \]

\[ + (b + c) \left( \frac{1}{4} - \frac{K}{16\rho} (a - d + b - c)(a - d - b + c) \right). \]  

(4.26)

The factor multiplying \( a + d \) is to leading order \( 1/2 \). The term multiplying \( b + c \) is actually much smaller. In fact, this term is third order in \( \epsilon \) and is given by \( -\pi^3 \epsilon^3 / 1024 \). Hence, the leading contribution to the charge actually comes from the \( a + d \) term and is

\[ Q = \frac{\pi^2}{64} \epsilon^2 \delta. \]  

(4.27)

The charge that appears in (4.27) is limited by the maximum value of \( \delta \) given \( \epsilon \). This bound is determined by enforcing the ansatz that \( u(\lambda) \leq 1 \). The region where this ansatz might be violated is where \( \lambda \approx b \) or \( \lambda \approx c \). Let us consider the case where \( \lambda \) is near \( b \). Examining equation (4.12), we see that the first modulus in the elliptic integral of the third kind approaches unity as \( \lambda \) approaches \( b \) from above. Therefore, if we substitute the asymptotic expansion for \( \Pi \) in (A.11) in (4.12), we find that the density is given by

\[ u(\lambda) = \frac{2}{\pi} \frac{1}{b - d} \sqrt{\frac{(a - b)(\lambda - b)}{(a - c)(b - c)}} \left\{ (b - c)K(q) + (c - d)K - E(q) \frac{(a - c)(b - d)}{a - b} \right\} \]

\[ + (c - d) \frac{\pi}{2} \sqrt{\frac{(a - c)(b - d)}{(a - b)(c - d)}} \sqrt{\frac{(b - d)(b - c)}{(\lambda - b)(c - d)}} + O(\lambda - b) \]

\[ = 1 + \sqrt{\lambda - b} \frac{2}{\pi} \sqrt{\frac{(a - b)}{(a - c)(b - c)}} \left( K(q) - E(q) \frac{a - c}{a - b} \right) + O(\lambda - b). \]  

(4.28)
Hence, in order to ensure that the ansatz is satisfied, it is necessary that the relation

\[
\frac{a - c}{a - b} E(q) - K(q) > 0,
\]

be upheld. Using the asymptotic expansions and the values for \( b, c, a \) and \( d \) given by (4.23) and (4.24), we find that

\[
\frac{a - c}{a - b} E(q) - K(q) \approx \frac{\pi^3}{32} (2\delta\epsilon + \epsilon^2).
\]

Therefore, in order for (4.29) to be satisfied, we must have \( \delta > -\epsilon/2 \). We can also derive the constraint that \( \delta < \epsilon/2 \) by examining \( u(\lambda) \) as \( \lambda \to c \). Therefore, we must satisfy

\[
|\delta| < \epsilon/2.
\]

We can rewrite this constraint in terms of the charge and the area. From (4.14) and the asymptotic expansion, we have the relation

\[
A - A_c = \frac{3\pi^4}{64} (\epsilon^2 + 2\delta^2),
\]

where \( A_c \) is the critical value for the area. Hence, using (4.27), (4.31) and (4.32), we have that the maximum allowed charge sector for a given area near its critical value is

\[
Q_{\text{max}} = \frac{32\sqrt{2}}{27\pi^4} (A - A_c)^{3/2}.
\]

This relation is a little reminiscent of the maximum charge of a Reissner-Nordstrom black hole.

Now we wish to examine the behavior deep in the strong coupling regime, which corresponds to large values of the area. In this case, we expect \( b \) to approach \( a \) and \( c \) to approach \( d \). The values of \( a \) and \( d \) are determined by the charge of the sector that we are considering. For this behavior \( q \to 1 \), and thus \( q' \to 0 \). At this point it
is convenient to rewrite \( \Pi(\alpha, q) \) in terms of elliptic integrals of the first and second kind. Using the relation in the appendix, equation (4.16) can be written as

\[
-E(q)F(\theta, q) + K(q)E(\theta, q) = \frac{a + b - c + d}{2\rho}K(q),
\]

where \( \sin \theta = \frac{a - c}{a - d} \). Using (4.18) and the asymptotic expansions in (A.12) and (A.13), we find that in this region

\[
a - d \approx 1
\]

which is expected since the integral of \( u(\lambda) \) should be unity and for almost all values of \( \lambda \) between \( a \) and \( d \), \( u(\lambda) = 1 \). Plugging in leading order asymptotic expansions in (A.12)-(A.15) into (4.34) and invoking (4.35), we then find the following approximate equation

\[
-\log \frac{1 + \sin \theta}{\cos \theta} + \log \sqrt{\frac{(a - c)(b - d)}{(a - b)(c - d)}} \to \log \sqrt{\frac{1}{a - b}} \approx a \log \sqrt{\frac{1}{(a - b)(c - d)}}.
\]

If we let \( a - b = \epsilon \) and \( c - d = \epsilon^\mu \), then (4.36) gives

\[
a = \frac{1}{1 + \mu}.
\]

Since \( 0 < \mu < \infty \), we see that the possible values of \( a \) range from 0 to 1. Of course what this means is that no local solutions exist if the fermions are shifted such that all of the momenta are greater than 0 or all are less than 0. This should not come as a big surprise, since once these limits are reached, then there are small deformations of the fermion momenta which lower the energy of the state.

The charge for these solutions is dominated by the \( W \) term in (4.19), since \( X \), \( Y \) and \( Z \) are all small. Clearly in the limit \( b \to a \) and \( c \to d \), the charge approaches

\[
Q \to \frac{a + d}{2} = a - \frac{1}{2}.
\]
5. Correspondence with Lattice Models

Douglas and Kazakov have remarked that the phase transition for the continuum model is similar to the phase transition that occurs for the lattice. In both cases the phase transition is third order and the equations of motion for the eigenvalues are given by a two cut model. In some sense, these two situations are dual to each other, with the weak coupling region of the lattice model acting like the strong coupling region of the continuum case.

However, it would appear that this correspondence breaks down when we consider the solutions with nonzero values of $Q$. There are no corresponding solutions for weakly coupled $U(N)$ on the lattice. But a little more thought shows that correspondence is between continuum $U(N)$ and the lattice $SU(N)$ and vice versa. For instance, the continuum $SU(N)$ case does not have these extra solutions, since the $U(1)$ charge is not a degree of freedom. In terms of the fermions, the center of mass coordinate is modded out. Hence shifting all the fermion momenta by the same amount gives back the same state. On the other hand, the $SU(N)$ case has an extra term in the casimir, $n^2/N^2$, where $n$ is the number of boxes in the representation. But this term does not survive the scaling limit, so we can safely drop it.

For $SU(N)$ on the lattice, the eigenvalues sit in a potential described by (1.1). However, unlike the $U(N)$ case, the center of mass position for these eigenvalues must satisfy the constraint that

$$\sum_i \theta_i = 2\pi m, \quad (5.1)$$

where $m$ is an integer. Clearly, the $U(N)$ classical solution is the same as the lowest energy state for the $SU(N)$ case, which will have $m = 0$. But suppose we consider a case where $0 < m < N/2$. Since $m$ is an integer, we cannot smoothly deform this to the $m = 0$ classical solution. Hence each value of $m$ must have a classical minimum. We could also have values of $m$ which are less than 0, but
shifting the total position by $2\pi N$ just shifts all eigenvalues around the circle, hence these correspond to the same solution. The maximum value of $m$ for a given weak coupling is determined by the value which puts an eigenvalue at the top of the potential. Clearly as the coupling becomes stronger, the number of classical solutions will decrease.

Let us discuss this situation in a little more detail. We can impose the constraint in (5.1) by inserting a $\delta$-function into the path integral with the form

$$
\delta\left(\sum_i \theta_i - 2\pi m\right) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi \epsilon}} e^{-\frac{1}{\epsilon} \left(\sum_i \theta_i - 2\pi m\right)^2}.
$$

(5.2)

Hence, using (1.2) we find the equation of motion

$$
N\beta \sin \theta_i - \sum_{j\neq i} \cot \theta_i \theta_j - \frac{1}{\epsilon} \left(\sum_i \theta_i - 2\pi m\right) = 0.
$$

(5.3)

where $\beta = 1/(a^2 g^2)$. Under the usual rescaling, one ends up with the equation

$$
\beta \sin \theta - \frac{1}{\epsilon} \left(\int \mu(\phi) \phi - 2\pi m/N\right) = P \int \mu(\phi) \cot \frac{\theta - \phi}{2},
$$

(5.4)

where $\mu(\phi)$ is the density of eigenvalues. As $\epsilon \to 0$, the last term on the lhs of (5.4) is divergent unless the integral is very close to $2\pi m$. To this end, let us replace this entire term by a constant $\alpha$. Next define the function $f(\theta)$,

$$
f(\theta) = \int d\phi \mu(\phi) \cot \frac{\theta - \phi}{2}.
$$

(5.5)

Given this definition, the function is analytic everywhere in the strip $-\pi < \text{re}\theta < \pi$, except for a cut along the real line. $f(\theta)$ is clearly invariant under $\theta \to \theta + 2\pi$.

$f(\theta)$ can be solved for by analyzing its behavior as $\text{Im}\theta \to \infty$. In particular, using (5.5) the large $\text{Im}\theta$ behavior for $f(\theta)$ is

$$
f(\theta) = -i \int d\phi \mu(\phi) - ie^{i\theta} \int d\phi e^{-i\phi} \mu(\phi) + O(e^{2i\theta}).
$$

(5.6)

Let us assume that the eigenvalues sit on the strip $\Delta - a < \lambda < \Delta + a$. Then from
(5.4) and (5.5), $f(\lambda \pm i\epsilon)$ satisfies
\[
f(\lambda \pm i\epsilon) = \beta \sin(\lambda) + \alpha \mp 2i\pi\mu(\lambda). \tag{5.7}
\]
Choosing the periodic function
\[
f(\theta) = \beta \sin \theta + \alpha - 2\beta \cos \frac{\theta + \Delta}{2} \sqrt{\sin^2 \frac{\theta - \Delta}{2} - \sin^2 \frac{a}{2}}, \tag{5.8}
\]
one finds that the asymptotic behavior is matched if
\[
\beta \sin^2 \frac{a}{2} \cos \Delta = 1, \tag{5.9}
\]
and
\[
\beta \cos^2 \frac{a}{2} \sin \Delta = -\alpha. \tag{5.10}
\]
From (5.7) and (5.8) one learns that the density of eigenvalues is given by
\[
\mu(\lambda) = \frac{\beta}{\pi} \cos \frac{\lambda + \Delta}{2} \sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\lambda - \Delta}{2}}. \tag{5.11}
\]
In order that the density of states is positive, one must satisfy the inequalities
$\Delta + a \leq \pi$ and $\Delta - a \geq -\pi$. One can also easily show that the integral of $\sin \phi$ weighted by the density of eigenvalues satisfies
\[
\int \mu(\phi) \sin \phi = \cos^2 \frac{a}{2} \sin \Delta. \tag{5.12}
\]
Hence $\Delta$ basically measures the anisotropy of the solution about the bottom of the well.

As $\beta \to \infty$, it is clear that the allowed solutions for $\Delta$ lie anywhere between $-\pi/2$ and $\pi/2$. This is similar to the situation in the previous section, where any charge between $-N/2$ and $N/2$ is allowed as the area approaches infinity. However, near the critical point, only a small range for $\Delta$ is allowed. Clearly, from (5.9), the critical value of $\beta$ is $\beta_c = 1$. Therefore, the only allowed $\Delta$ at the critical point is $\Delta = 0$. 

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Moving slightly away from the critical point, one has $a = \pi - \epsilon$ and a maximum value of $\Delta$ given by $\Delta = \epsilon$. At the maximum, the integral of the angles weighted by the density of eigenvalues is then given by

$$
\int d\phi \mu(\phi) \phi = \frac{\beta}{\pi} \int_{\Delta - a}^{\Delta + a} d\phi \cos \frac{\phi + \Delta}{2} \sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\phi - \Delta}{2}}
$$

$$
= \Delta - \frac{\beta}{\pi} \sin \Delta \int_{-a}^{a} d\phi \sin \phi \sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\phi}{2}},
$$

where we have shifted the integration variables and have used the fact that the integral over the density of states is one. For small $\epsilon$ we find that the leading order term in (5.13) is

$$
\int d\phi \mu(\phi) \phi = \frac{1}{8} \epsilon^2 \Delta \log \frac{1}{\epsilon} + O(\epsilon^2 \Delta).
$$

(5.14)

$\beta$ is the inverse area, hence we find that the area for the plaquette satisfies

$$
A_c - A = \frac{1}{4} (\epsilon^2 + 2\Delta^2).
$$

(5.15)

Therefore, the maximum charge in terms of the area, behaves like

$$
Q_{\text{max}} \sim (A_c - A)^{3/2} \log \frac{1}{A_c - A}.
$$

(5.16)

Hence, unlike the continuum case, the maximum charge in terms of the area has a scaling violating piece. However, the integral over $\sin \phi$ has a maximum that behaves like $(A_c - A)^{3/2}$, which is closer to the behavior of the preceding section.
6. Discussion

In this paper we have given an alternative derivation of Witten's proof that the QCD partition function is given by a sum over saddle points. We then showed how the sphere phase transition occurs in the dual picture of QCD string theory.

We next demonstrated how to find new solutions for the $U(N)$ QCD$_2$ sphere partition function corresponding to the different $U(1)$ sectors. We have also shown how these solutions relate to solutions found for the lattice versions of $SU(N)$. In some sense, it is surprising that the nonzero charge solutions can actually be found. One might have expected that since there is nothing in the machinery of section 4 that explicitly states that the eigenvalues lie on the lattice, then nothing should prevent the nonzero charge solutions from sliding down to the absolute minimum. But actually, this information is in there, because of the log term that appears in (4.9). This term states that there is a region where the eigenvalues have a constant density, that is, they lie on the lattice. The boundaries of this region essentially add another degree of freedom. This then allows us to find solutions with nonzero charges.

It is hoped that the results presented here might have some use in investigating random matrix model theory. Perhaps one can consider cases where the entries of the Hamiltonian are restricted to be integers. One might then find similar behavior to that shown here.

APPENDIX

In this appendix we present some useful formulae of elliptic integrals. These are taken from or are easily derived from formulae in [16]. The first such equations are

\[ \int_c^b ds \frac{1}{\sqrt{(a - s)(b - s)(s - c)(s - d)}} = \frac{2}{\rho} K(q), \]  

\[ (A.1) \]
\begin{align}
\int_c^b ds \frac{s}{\sqrt{(a-s)(b-s)(s-c)(s-d)}} &= \frac{2}{\rho} \left( (c-d)\Pi(\alpha, q) + dK(q) \right), \quad (A.2) \\
\int_c^b ds \frac{s^2}{\sqrt{(a-s)(b-s)(s-c)(s-d)}} &= \frac{1}{\rho} \left( (ad - bc + d(a+b+c+d))K(q) - \rho^2 E(q) + (a + b + c + d)K(q) \right), \quad (A.3) \\
\int_c^b ds \frac{s^3}{\sqrt{(a-s)(b-s)(s-c)(s-d)}} &= \frac{1}{\rho} \left( 2d^3 - \frac{1}{4}(c-d)(b-d)(a+3b+3c+5d) \right)K(q) - \frac{3}{4}(a + b + c + d)\rho E(q) + \\
&\quad + \frac{1}{\rho} \left( \frac{3}{4}(a + b + c + d)(a + b + c - 3d) \right) \\
&\quad + (2a + 2b + 2c + 3d)d - ab - ac - bc \right)K(q) \right), \quad (A.4) \\
\text{and} \\
\int_c^b ds \frac{s^4}{\sqrt{(a-s)(b-s)(s-c)(s-d)}} &= \frac{1}{\rho} \left( 2d^3 - \frac{1}{24}(c-d)(b-d)(-5a^2 - 15b^2 - 15c^2 - 33d^2) \right)K(q) \\
&\quad - \frac{1}{24} \left( a5a^2 + 15b^2 + 15c^2 + 15d^2 + 14ab + 14bc + 14ac + 14bc + 14bd + 14cd \right)\rho E(q) + \\
&\quad + \frac{1}{8\rho} \left( 5a^3 + 5b^3 + 5c^3 + 5d^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3bc^2 + 3a^2d + 3ad^2 + 3bd^2 + 3cd^2 + 2abc + 2abd + 2acd + 2bcd \right) \right) \right)K(q) \right), \quad (A.5)
\end{align}
where
\[
\rho = \sqrt{(a-c)(b-d)}, \quad \alpha = \sqrt{\frac{b-c}{b-d}}, \quad q = \sqrt{\frac{(a-d)(b-c)}{(a-c)(b-d)}}. \tag{A.6}
\]

\(K(q), E(q)\) and \(\Pi(\alpha, q)\) are the complete elliptic integrals of the first, second and
third kind respectively.

Another formula used in the text is
\[
\int_c^b \frac{1}{(\lambda-s)\sqrt{(a-s)(b-s)(s-c)(s-c)}} \left[ (c-d)\Pi\left(\frac{b-c \lambda - d}{b-d \lambda - c}, q\right) + (h-c)K(q) \right].
\tag{A.7}
\]

We also use the relation between the complete integral of third kind and in-
complete integrals of the first and second kind,
\[
\frac{(c-d)}{\rho} \Pi(\alpha, q) = \frac{(c-d)}{\rho} K(q) - E(q)F(\theta, q) + K(q)E(\theta, q), \tag{A.8}
\]
where \(\sin^2 \theta = \frac{a-c}{a-d}\), and \(F(\theta, q)\) and \(E(\theta, q)\) are the incomplete integrals of the first
and second kind.

The following asymptotic expansions are also used:
\[
K(q) = \frac{\pi}{2} \left( 1 + \frac{1}{4}q^2 + \frac{9}{64}q^4 + \frac{25}{256}q^6 + \ldots \right), \tag{A.9}
\]
\[
E(q) = \frac{\pi}{2} \left( 1 - \frac{1}{4}q^2 - \frac{3}{64}q^4 - \frac{5}{256}q^6 + \ldots \right), \tag{A.10}
\]
\[
\Pi(\alpha, q) = \frac{\pi}{2} \left( 1 + \frac{1}{2}(\alpha^2 + q^2/2) + \frac{24}{64}(\alpha^4 + \alpha^2q^2/2 + 3q^4/8)
+ \frac{5}{16}(\alpha^6 + \alpha^4q^2/2 + 3\alpha^2q^4/8 + 5q^6/16) + \ldots \right), \tag{A.11}
\]
\( K(q) = \log(4/q') - 2[\log(4/q') + 1](q')^2 + ..., \) \hspace{1cm} (A.12)

\( E(q) = 1 + \frac{1}{2}[\log(4/q') - 1/2](q')^2 + ..., \) \hspace{1cm} (A.13)

\[ F(\theta, q) = \log \frac{1 + \sin \theta}{\cos \theta} + \frac{1}{4}[\log \frac{1 + \sin \theta}{\cos \theta} - \sin \theta \sec^2 \theta](q')^2 + .... \] \hspace{1cm} (A.14)

\[ E(\theta, q) = \sin \theta + 2[\log \frac{1 + \sin \theta}{\cos \theta} - \sin \theta](q')^2 + .... \] \hspace{1cm} (A.15)

where \( q' \) satisfies \( q' = \sqrt{1-q^2} \).

Finally, there is a useful expansion for \( \Pi(\alpha, q) \) if \( \alpha \) is close to unity. This is

\[ \Pi(\alpha, q) = K(q) - \frac{E(q)}{(q')^2} + \frac{\pi(1 + (q')^2 - q^2\alpha^2)}{4(q')^3\sqrt{1-\alpha^2}} + \text{O}(1-\alpha^2). \] \hspace{1cm} (A.16)

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Note added: As this paper was being typed we received a paper[17] which discusses some of the issues in section 3.
REFERENCES

1. D. Gross and E. Witten, *Phys. Rev. D* **21** (1980) 446.

2. S. Wadia, *Phys. Lett.* **93B** (1980) 403.

3. D. Gross, *Nucl. Phys.* **B400** (1993) 161.

4. J. Minahan, *Phys. Rev. D* **47** (1993) 3430.

5. D. Gross and W. Taylor, *Nucl. Phys.* **B400** (1993) 181.

6. D. Gross and W. Taylor, CERN-TH-6827-93 [hep-th/9303046], 1993.

7. M. Douglas and V. Kazakov, [hep-th/9305047].

8. E. Witten, *J. Phys. G* **9** (1992) 303.

9. J. Minahan and A. Polychronakos, *Phys. Lett.* **B312** (1993) 155.

10. M. Douglas, [hep-th/9303159], 1993.

11. M. Caselle, A. D’Adda, L. Magnea, and S. Panzeri, [hep-th/9304015], 1993.

12. A. Migdal, *Zh. Eksp. Teor. Fiz.* **69** (1975) 810.

13. B. Rusakov, *Mod. Phys. Lett. A* **5** (1990) 693.

14. J. Duistermaat and G. Heckman, *Invent. Math* **69** (1982) 259.

15. A.P. Polychronakos, *Phys. Lett. B266* (1991) 29.

16. P. Byrd and M. Friedman, “Handbook of Elliptic Integrals for Engineers and Physicists”, Berlin, Springer, 1954.

17. M. Caselle, A. D’Adda, L. Magnea, and S. Panzeri, [hep-th/9309107], 1993.