Integrals Along Bimonoid Homomorphisms

Minkyu Kim

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Abstract
We introduce a notion of an integral along a bimonoid homomorphism as a simultaneous generalization of the integral and cointegral of bimonoids. The purpose of this paper is to characterize an existence of a specific integral, called a normalized generator integral, along a bimonoid homomorphism in terms of the kernel and cokernel of the homomorphism. We introduce a notion of a volume on an abelian category as a generalization of the dimension of vector spaces and the order of abelian groups. In applications, we show that there exists a non-trivial volume partially defined on a category of bicommutative Hopf monoids. The volume yields a notion of Fredholm homomorphisms between bicommutative Hopf monoids, which gives an analogue of the Fredholm index theory. This paper gives a technical preliminary of our subsequent paper about a construction of TQFT's.

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✉ Minkyu Kim
kim@ms.u-tokyo.ac.jp

1 Graduate School of Mathematical Sciences, University of Tokyo, Tokyo, Japan
1 Introduction

The notion of integrals of a bialgebra is introduced by Larson and Sweedler [1]. It is a generalization of the Haar measure of a group. The integral theory has been used to study bialgebras or Hopf algebras [1–3]. The notion of bialgebras are generalized to bimonoids in a symmetric monoidal category $C$ [4,5]. The integral theory is generalized to the categorical settings and used to study bimonoids or Hopf monoids [6].

In this paper, we introduce a notion of an integral along a bimonoid homomorphism in a symmetric monoidal category $C$. For bimonoids $A$ and $B$, an integral along a bimonoid homomorphism $\xi : A \rightarrow B$ is a morphism $\mu : B \rightarrow A$ in $C$ satisfying some axioms (see Definition 3.4). The integrals along bimonoid homomorphisms simultaneously generalize the notions of integral and cointegral of bimonoid: the notion of integral (cointegrals, resp.) of a bimonoid coincides with that of integrals along the counit (unit, resp.).

The purpose of this paper is to characterize the existence of a normalized generator integral along a bimonoid homomorphism in terms of the kernel and cokernel of the homomorphism. The reader is referred to Definitions 3.4 and 3.11 for the definition of normalized integrals and generator integrals respectively. If $C$ satisfies some assumptions (see (Assumption 0,1,2) in Sect. 15), then the existence of a normalized generator integral is characterized as follows. Note that the assumptions on $C$ automatically hold if $C = \text{Vec}_k^\otimes$, the tensor category of (possibly infinite-dimensional) vector spaces over a field $k$:

**Theorem 1.1** Let $A, B$ be bicommutative Hopf monoids in $C$ and $\xi : A \rightarrow B$ be a Hopf homomorphism. Then there exists a normalized generator integral $\mu_\xi$ along $\xi$ if and only if the following conditions hold:

1. The kernel Hopf monoid $\text{Ker}(\xi)$ has a normalized integral.
2. The cokernel Hopf monoid $\text{Cok}(\xi)$ has a normalized cointegral.

Moreover, if a normalized integral exists, then it is unique.

Note that even if $A, B$ are not bicommutative or it is not obvious whether $C$ satisfies (Assumption 0,1,2), we have more general results under some assumptions on the homomorphism $\xi$. The integral is constructed concretely. We prove such a generalization in Corollary 9.11 which implies our main theorem.
In applications, we investigate the category \( \text{Hopf}^{\text{bc}}(C) \) of bicommutative Hopf monoids with a normalized integral and cointegral. We prove that the category \( \text{Hopf}^{\text{bc}}(C) \) is an abelian subcategory of \( \text{Hopf}^{\text{bc}}(C) \) and closed under short exact sequences. See Sect. 15.1.

We introduce a notion of volume on an abelian category \( \mathcal{A} \) as a generalization of the dimension of vector spaces and the order of abelian groups. It is an invariant of objects in \( \mathcal{A} \) compatible with short exact sequences (see Definition 14.1). As another application to \( \text{Hopf}^{\text{bc}}(C) \), we construct an \( \text{End}_C(1) \)-valued volume \( \text{vol}^{-1} \) on the abelian category \( \mathcal{A} = \text{Hopf}^{\text{bc}}(C) \). Here \( 1 \) is the unit object of \( C \) and the endomorphism set \( \text{End}_C(1) \) is an abelian monoid induced by the symmetric monoidal structure of \( C \).

By using the volume \( \text{vol}^{-1} \), we introduce a notion of Fredholm homomorphisms between bicommutative Hopf monoids as an analogue of the Fredholm operator [7] (see Definition 14.5). We study its index which is robust to some finite perturbations (see Proposition 15.9). Furthermore, we construct a functorial assignment of integrals to Fredholm homomorphisms.

This paper gives a technical preliminary of our subsequent paper [8]. Indeed, we use the results in this paper to give a generalization of the untwisted abelian Dijkgraaf-Witten theory [9–11] and the bicommutative Turaev-Viro TQFT [12,13]. We will give a systematic way to construct a sequence of TQFT’s from (co)homology theory. The TQFT’s are constructed by using path-integral which is formulated by some integral along bimonoid homomorphisms.

As a corollary of our subsequent paper, if the volume \( \text{vol}^{-1}(A) \) of an object \( A \) in \( \text{Hopf}^{\text{bc}}(C) \) is invertible in \( \text{End}_C(1) \), then the underlying object of \( A \) is dualizable in \( C \) and its categorical dimension coincides with the inverse of \( \text{vol}^{-1}(A) \). If \( C \) is a rigid symmetric monoidal category with split idempotents, then the inverse volume of any Hopf monoid is invertible [6]. It is not obvious whether the inverse volume is invertible or not in general. Note that we do not assume a duality on objects of \( C \).

There is another approach to a generalization of (co)integrals of bimonoids. In [6], (co)integrals are defined by a universality. It is not obvious whether our integrals could be generalized by universality.

We expect that the result in this paper could be applied to topology through another approach. There is a topological invariant of 3-manifolds induced by a finite-dimensional Hopf algebra, called the Kuperberg invariant [14,15]. In particular, if the Hopf algebra is involutory, then it is defined by using the normalized integral and cointegral of the Hopf algebra.

The organization of this paper is as follows. In Sect. 2, we give our convention for string diagrams and a brief review of monoids in a symmetric monoidal category. In Sect. 3.1, we review integrals of bimonoids. In Sect. 3.2, we introduce the notion of (normalized) integral along bimonoid homomorphisms and give some basic properties. In Sect. 3.3, we introduce a notion of generator integral and give some basic properties. In Sects. 4.1, 4.2, we introduce the notion of invariant objects and stabilized objects respectively. In Sect. 4.3, we introduce the notion of (co, bi) stable monoidal structure. In Sect. 5, we introduce the notions of (co, bi)normality of bimonoid homomorphisms and give some basic properties. In Sect. 6, we introduce the notion of (co, bi) small bimonoids and examine it in terms of an existence of normalized (co)integrals. In Sect. 7.1, we prove the uniqueness of a normalized integral. In Sect. 7.2, we prove some necessary conditions for existence of a normalized integral. In Sect. 8, by using a normalized generator integral, we show an isomorphism between the set of endomorphisms on the unit object \( 1 \) and the set of integrals. In Sect. 9.1, we prove a key lemma to prove the main theorem. In Sect. 9.2, we introduce two notions of (weakly)

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1 By Theorem 6.13, the category \( \text{Hopf}^{\text{bc}}(C) \) coincides with \( \text{Hopf}^{\text{bc,bs}}(C) \) in Sect. 15.
well-decomposable homomorphism and (weakly) Fredholm homomorphism, and prove the main theorem. In Sect. 10, we investigate a commutativity of some homomorphisms and normalized integrals. In Sect. 11.1, we introduce the inverse volume of some bimonoids. In Sect. 11.2, we introduce the inverse volume of some bimonoid homomorphisms. In Sect. 12, we study a commutativity of normalized integrals. In Sect. 13.1, we give some conditions where $\text{Ker}(\xi)$, $\text{Cok}(\xi)$ inherits a (co)smallness from that of the domain and the target of $\xi$. In Sect. 13.2, we study bismallness of bimonoids in an exact sequence. In Sect. 14, we introduce the notion of volume on an abelian category and study basic notions related with it. In Sect. 15.1, we prove that the inverse volume is a volume on the category of bicommutative Hopf monoids. In Sect. 15.2, we construct functorial integrals for Fredholm homomorphisms.

2 Notations

This section gives our convention about notations. The reader is referred to some introductory books for category theory or (Hopf) monoid theory [4,5].

We denote by $1$ the unit object of a monoidal category $C$, by $\otimes$ the monoidal operation. We often omit the coherence isomorphisms: the associator $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, the right unitor $r_x : x \otimes 1 \to x$ and the left unitor $l_x : 1 \otimes x \to x$; and denote by $\cong$.

String diagrams. We explain our convention to represent string diagrams. It is convenient to use string diagrams to discuss equations of morphisms in a symmetric monoidal category $C$. It is based on finite graphs where for each vertex $v$ the set of edges passing through $v$ has a partition by, namely, incoming edges and outcoming edges. For example, a morphism $f : x \to y$ in $C$ is represented by (1) in Fig. 1. In this example, the underlying graph has one 2-valent vertex and two edges. If there is no confusion from the context, we abbreviate the objects as (2) in Fig. 1. For another example, a morphism $g : a \otimes b \to x \otimes y \otimes z$ is represented by (3) in Fig. 1.

We represent the tensor product of morphisms in a symmetric monoidal category $C$ by gluing two string diagrams. For example, if $h : x \to y$, $k : a \to b$ are morphisms, then we represent $h \otimes k : x \otimes a \to y \otimes b$ by (1) in Fig. 2.

We represent the composition of morphisms by connecting some edges of string diagrams. For example, if $q : x \to y$ and $p : y \to z$ are morphisms, we represent their composition $p \circ q : x \to z$ by (2) in Fig. 2.

The symmetry $s_{x,y} : x \otimes y \to y \otimes x$ which is a natural isomorphism is denoted by (1) in Fig. 3.
Fig. 3 String diagrams of the symmetry and a morphism from or to the unit object

Fig. 4 String diagrams related with a monoid and a comonoid

The edge colored by the unit object $\mathbb{1}$ of the symmetric monoidal category $\mathcal{C}$ is abbreviated. For example, a morphism $u : \mathbb{1} \to a$ is denoted by (2) in Fig. 3 and a morphism $v : b \to \mathbb{1}$ is denoted by (3) in Fig. 3.

Monoid. The notion of monoid in a symmetric monoidal category is a generalization of the notion of monoid which is a set equipped with a unital and associative product. Furthermore, it is a generalization of the notion of algebra. We use the notations $\nabla : A \otimes A \to A$ and $\eta : \mathbb{1} \to A$ to represent the multiplication and the unit. On the one hand, the comonoid is a dual notion of the monoid. We use the notations $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{1}$ to represent the comultiplication and the counit. Fig. 4 denotes the structure morphisms as string diagrams.

The notions of bimonoid and Hopf monoid are defined as an object of $\mathcal{C}$ equipped with a monoid structure and a comonoid structure which are subject to some axioms. We denote by $\text{Bimon}(\mathcal{C})$, $\text{Hopf}(\mathcal{C})$ the categories of bimonoids and Hopf monoids, respectively.

Action. We give some notations about actions in a symmetric monoidal category. The notations related with coaction is defined similarly.

Definition 2.1 Let $X$ be an object of $\mathcal{C}$, $A$ be a bimonoid, and $\alpha : A \otimes X \to X$ be a morphism in $\mathcal{C}$. A triple $(A, \alpha, X)$ is a left action in $\mathcal{C}$ if following diagrams commute:

\[
\begin{array}{cccc}
A \otimes A \otimes X & \xrightarrow{id_A \otimes \alpha} & A \otimes X \\
\downarrow \nabla_A \otimes id_X & & \downarrow \alpha \\
A \otimes X & \xrightarrow{\alpha} & X
\end{array}
\]  \hspace{2cm} (1)

\[
\begin{array}{cccc}
\mathbb{1} \otimes X & \xrightarrow{\eta_A \otimes id_X} & A \otimes X \\
\cong & & \downarrow \alpha \\
\mathbb{1} \otimes X & \xrightarrow{\epsilon} & X
\end{array}
\]  \hspace{2cm} (2)

Let $(A, \alpha, X)$, $(A', \alpha', X')$ be left actions in a symmetric monoidal category $\mathcal{C}$. A pair $(\xi_0, \xi_1) : (A, \alpha, X) \to (A', \alpha', X')$ is a morphism of left actions if $\xi_0 : A \to A'$ is a monoid homomorphism and $\xi_1 : X \to X'$ is a morphism in $\mathcal{C}$ which intertwines the actions.
Left actions in \( \mathcal{C} \) and morphisms of left actions form a category which we denote by \( \text{Act}_l(\mathcal{C}) \). The symmetric monoidal category structures of \( \mathcal{C} \) and \( \text{Bimon}(\mathcal{C}) \) induce a symmetric monoidal category on \( \text{Act}_l(\mathcal{C}) \) by \( (A, \alpha, X) \otimes (A', \alpha', X') \overset{\text{def}}{=} (A \otimes A', \alpha \hat{\otimes} \alpha', X \otimes X') \).

Here, \( \alpha \hat{\otimes} \alpha' : (A \otimes A') \otimes (X \otimes X') \to X \otimes X' \) is defined by composing

\[
A \otimes A' \otimes X \otimes X' \xrightarrow{id_A \otimes s_{X'}} A \otimes X \otimes A' \otimes X' \xrightarrow{\phi \otimes \alpha'} X \otimes X'.
\]

We define a right action in a symmetric monoidal category \( \mathcal{C} \) and its morphism similarly. Note that for a right action, we use the notation \( (X, \alpha, A) \) where \( A \) is a bimonoid and \( X \) is an object on which \( A \) acts. We denote by \( \text{Act}_r(\mathcal{C}) \) the symmetric monoidal category of right actions.

Let \( A \) be a bimonoid in a symmetric monoidal category \( \mathcal{C} \) and \( X \) be an object of \( \mathcal{C} \). A left action \( (A, \tau_{A,X}, X) \) is trivial if

\[
\tau_{A,X} : A \otimes X \xrightarrow{\epsilon_A \otimes id_X} 1 \otimes X \xrightarrow{\cong} X.
\]

We also define a trivial right action analogously. We abbreviate \( \tau = \tau_{A,X} \) if there is no confusion.

**Remark 2.2** The notion of action is usually defined for a monoid \( A \), but we require that \( A \) should be a bimonoid in Definition 2.1. In fact, the trivial action is well-defined since \( A \) has a counit which is a structure of a comonoid.

### 3 Integrals

#### 3.1 Integrals of Bimonoids

In this subsection, we review the notion of integral of a bimonoid and its basic properties.

We give some remarks on terminology. The integral in this paper is called a Haar integral [16–18], an \( \text{Int}(H) \)-based integral [6] or an integral-element [19]. The cointegral in this paper is called an \( \text{Int}(H) \)-valued integral in [6] or integral-functional [19]. In fact, those notions introduced in [6,19] are more general ones which are defined by a universality.

**Definition 3.1** Let \( A \) be a bimonoid. A morphism \( \phi : 1 \to A \) is a left integral of \( A \) if it satisfy a commutative diagram (5). A morphism \( \phi : 1 \to A \) is a right integral if it satisfy a commutative diagram (6). A morphism \( \phi : 1 \to A \) is an integral if it is a left integral and a right integral. A left (right) integral is normalized if it satisfies a commutative diagram (7). For a bimonoid \( A \), we denote by \( \sigma_A : 1 \to A \) the normalized integral of \( A \) if exists. It is unique for \( A \) as we will discuss in this section. Denote by \( \text{Int}_r(A) \), \( \text{Int}_l(A) \), \( \text{Int}(A) \) the set of right integrals, left cointegrals and cointegrals of \( A \).

We analogously define cointegral of a bimonoid as a morphism from \( A \) to \( 1 \). Denote by \( \text{Cont}_r(A) \), \( \text{Coint}_l(A) \), \( \text{Coint}(A) \) the set of right cointegrals, left cointegrals and cointegrals of \( A \).

\[
1 \otimes A \xrightarrow{\phi \otimes id_A} A \otimes A \xrightarrow{\delta_A} A
\]
Fig. 5 Definition of integrals

\[ A \otimes 1 \xrightarrow{id_A \otimes \psi} A \otimes A \]
\[ \downarrow \varepsilon_A \otimes \psi \quad \Downarrow \nabla_A \]
\[ 1 \otimes A \xrightarrow{\cong} A \]  
\[ A \xrightarrow{\phi} A \]

\[ 1 \xrightarrow{\varphi} A \]
\[ \Downarrow \varepsilon_A \]

Remark 3.2 The commutative diagrams in Definition 3.1 can be understood by equations of some string diagrams in Fig. 5 where the null diagram is the identity on the unit 1.

Proposition 3.3 Let A be a bimonoid in a symmetric monoidal category, C. If the bimonoid A has a normalized left integral \( \sigma \) and a normalized right integral \( \sigma' \), then \( \sigma = \sigma' \) and it is a normalized integral of the bimonoid A. In particular, if a normalized integral exists, then it is unique.

Proof It is proved by definitions. It follows from more general results in Proposition 7.1. In fact, a normalized left (right) integral of A is a normalized left (right) integral along counit of A. \( \square \)

3.2 Integrals Along Bimonoid Homomorphisms

In this subsection, we introduce the notion of an integral along a homomorphism and give its basic properties. They are defined for bimonoid homomorphisms whereas the notion of (co)integrals is defined for bimonoids. In fact, it is a generalization of (co)integrals. See Proposition 3.7. We also give a typical example in Example 3.6.
Definition 3.4 Let $A$, $B$ be bimonoids in a symmetric monoidal category $C$ and $\xi : A \to B$ be a bimonoid homomorphism. A morphism $\mu : B \to A$ in $C$ is a right integral along $\xi$ if the diagrams (8), (9) commute. A morphism $\mu : B \to A$ in $C$ is a left integral along $\xi$ if the diagrams (10), (11) commute. A morphism $\mu : B \to A$ in $C$ is an integral along $\xi$ if it is a right integral along $\xi$ and a left integral along $\xi$. An integral (or a right integral, a left integral) is normalized if the diagram (12) commutes.

We denote by $\text{Int}_l(\xi)$, $\text{Int}_r(\xi)$, $\text{Int}(\xi)$ the set of left integrals along $\xi$, the set of right integrals along $\xi$, the set of integrals along $\xi$, respectively.

Remark 3.5 The commutative diagrams in Definition 3.4 can be understood by using some string diagrams in Fig. 6. From now on, we freely use these string diagrams. The string diagram is explained briefly in Appendix 2.

Example 3.6 Consider $C = \text{Vec}_k$. Let $G$, $H$ be (possibly infinite and non-abelian) groups and $\varrho : G \to H$ be a group homomorphism. It induces a bialgebra homomorphism $\xi = \varrho_* : A \to B$ for the group Hopf algebras $A = kG$ and $B = kH$. Suppose that the kernel $\text{Ker}(\varrho)$ is finite. Put $\mu : B \to A$ by

$$\mu(h) = \sum_{\varrho(g)=h} g.$$  \hspace{1cm} (13)

Then $\mu$ is an integral along the homomorphism $\xi$. If the order of $\text{Ker}(\varrho)$, say $N$, is coprime to the characteristic of $k$, then $N^{-1} \cdot \mu$ is a normalized integral.
Fig. 6 Definition of integrals along bimonoid homomorphisms

(1) \[
\begin{array}{c}
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\]

(2) \[
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\mu \\
\end{array}
\begin{array}{c}
\mu \\
\mu \\
\end{array}
\end{array}
\]

(3) \[
\begin{array}{c}
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\mu \\
\end{array}
\begin{array}{c}
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\begin{array}{c}
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\begin{array}{c}
\mu \\
\mu \\
\end{array}
\end{array}
\]

(4) \[
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\mu \\
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\begin{array}{c}
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\]

(5) \[
\begin{array}{c}
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\begin{array}{c}
\mu \\
\mu \\
\end{array}
\end{array}
\]

**Proposition 3.7** Let $A$ be a bimonoid in a symmetric monoidal category $C$. For $\star = r, l$, we have

\[\text{Int}_\star(\epsilon_A) = \text{Int}_\star(A),\]  
\[\text{Int}_\star(\eta_A) = \text{Coint}_\star(A).\] (14) (15)

In particular, we have

\[\text{Int}(\epsilon_A) = \text{Int}(A),\]  
\[\text{Int}(\eta_A) = \text{Coint}(A).\] (16) (17)

Under these equations, the normalized condition is preserved.

**Proof** We only prove that $\text{Int}_r(\epsilon_A) = \text{Int}_r(A)$ and leave the other parts to the readers.

Suppose that $\mu \in \text{Int}_r(\epsilon_A)$. Then by (8), we have $\nabla_A \circ (\mu \otimes \text{id}_A) = r_A \circ (\mu \otimes \epsilon_A)$, i.e. $\mu$ is a right integral of the bimonoid $A$ where $r_A$ is the right unitor.

Suppose that $\sigma \in \text{Int}_r(A)$. Then $\sigma$ satisfies the commutative diagram (8). On the other hand, (9) is automatic since $B = \mathbb{1}$.

Note that $\mu \in \text{Int}_r(\epsilon_A)$ is normalized, i.e. $\epsilon_A \circ \mu \circ \epsilon_A = \epsilon_A$, if and only if $\epsilon_A \circ \mu = \text{id}_A$. \hfill \Box

**Proposition 3.8** If a bimonoid homomorphism $\xi : A \rightarrow B$ is an isomorphism, then we have $\xi^{-1} \in E(\xi)$. Here, $E$ denotes either $\text{Int}_r$, $\text{Int}_l$ or $\text{Int}$. In particular, $\text{id}_A \in E(\text{id}_A)$ for any bimonoid $A$.

**Proof** We only prove the case of $E = \text{Int}_r$ and leave the other parts to the readers. The morphism $\xi^{-1}$ satisfies the axiom (8) by the following equalities.

\[\nabla_A \circ (\xi^{-1} \otimes \text{id}_A) = \nabla_A \circ (\xi^{-1} \otimes \xi^{-1}) \circ (\text{id}_B \otimes \xi) \]
\[= \xi^{-1} \circ \nabla_B \circ (\text{id}_B \otimes \xi).\] (18) (19)

Here we use the assumption that $\xi$ is a bimonoid homomorphism. Similarly, (9) is verified. Hence, $\xi^{-1} \in \text{Int}_r(\xi)$. \hfill \Box
Proposition 3.9 We have $E(id_A) = \text{End}_C(\mathbb{I})$. Here, $E$ denotes either $\text{Int}_r$, $\text{Int}_l$ or $\text{Int}$.

Proof We only prove the case of $E = \text{Int}_r$ and leave the other parts to the readers. For $\varphi \in \text{End}_C(\mathbb{I})$, the morphism $\varphi$ satisfies the axiom (8) with respect to $\xi = id_\mathbb{I}$:

\[
\nabla_\mathbb{I} \circ (\varphi \otimes id_\mathbb{I}) = r_\mathbb{I} \circ (\varphi \otimes id_\mathbb{I}) \tag{20}
\]

\[
= \varphi \circ \nabla_\mathbb{I} \tag{21}
\]

Here, $r_\mathbb{I}$ is the right unitor of $\mathbb{I}$. Similarly, the axiom (9) is verified. It implies that $\varphi \in \text{Int}_r(id_\mathbb{I})$. \qed

Proposition 3.10 The composition of morphisms induces a map,

\[
E(\xi') \times E(\xi) \to E(\xi' \circ \xi); (\mu', \mu) \mapsto \mu \circ \mu'. \tag{22}
\]

Here, $E$ denotes either $\text{Int}_r$, $\text{Int}_l$ or $\text{Int}$.

Proof We only prove the case of $E = \text{Int}_r$. Let $\xi : A \to B$, $\xi' : B \to C$ be bimonoid homomorphisms and $\mu \in \text{Int}_r(\xi)$ and $\mu' \in \text{Int}_r(\xi')$. The composition $\mu \circ \mu'$ satisfies he axiom (9) as follows:

\[
\nabla_A \circ ((\mu \circ \mu') \otimes id_A) = \nabla_A \circ (\mu \otimes id_A) \circ (\mu' \otimes id_A) \tag{23}
\]

\[
= \mu \circ \nabla_B \circ (\mu' \otimes \xi) \tag{24}
\]

\[
= \mu \circ \mu' \circ \nabla_C \circ (id_A \otimes (\xi' \circ \xi)). \tag{25}
\]

It is similarly verified that the composition $\mu \circ \mu'$ satisfies the axiom (9). Hence, we obtain $\mu \circ \mu' \in \text{Int}_r(\xi' \circ \xi)$. \qed

3.3 Generator Integrals

In this subsection, we define a notion of a generator integral. The terminology is motivated by Proposition 3.12, which says that it plays a role of generator of (co)integrals of bimonoids. Moreover, it generates the set of integrals under some conditions (see Theorem 8.5).

Definition 3.11 Let $\mu$ be an integral along a bimonoid homomorphism $\xi : A \to B$. The integral $\mu$ is a generator if the following two diagrams below commute for any $\mu' \in \text{Int}_r(\xi) \cup \text{Int}_l(\xi)$:

\[
\begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{ccc}
B & \xrightarrow{\mu'} & A \\
\downarrow{\mu} & & \downarrow{\mu} \\
A & \xrightarrow{\xi} & B
\end{array}
\end{array}
\end{array} \tag{26}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{ccc}
B & \xrightarrow{\mu'} & A \\
\downarrow{\mu} & & \downarrow{\mu} \\
A & \xrightarrow{\xi} & B
\end{array}
\end{array}
\end{array} \tag{27}
\]

Proposition 3.12 Recall Proposition 3.7. Let $A$ be a bimonoid in a symmetric monoidal category $C$. Let $\sigma$ be an integral along the counit $\epsilon_A$. The integral $\sigma$ is a generator if and only if for any $\sigma' \in (\text{Int}_r(\epsilon_A) \cup \text{Int}_l(\epsilon_A)) = (\text{Int}_r(A) \cup \text{Int}_l(A))$

\[
\sigma' = (\epsilon_A \circ \sigma') \cdot \sigma. \tag{28}
\]

In particular, if an integral $\sigma$ is normalized, then $\sigma$ is a generator.
Proof Let σ be a generator. Then the commutative diagram (26) proves the claim.

Let σ′ ∈ Int_l(A) = Int_l(A). Suppose that σ′ = (ε_A ⊙ σ′) · σ. Since σ′ is a left integral of A, we have (ε_A ⊙ σ′) · σ = ∇_A ⊙ (σ ⊙ σ′) = (ε_A ⊙ σ) · σ′. Hence, we obtain σ′ = (ε_A ⊙ σ) · σ′, which is equivalent with (27). We leave the proof for a right integral σ′ to the readers.

We prove that if σ is normalized, then it is a generator. Let σ′ ∈ Int_r(A). Then σ′ ∗ σ = (ε_A ⊙ σ) · σ′ = σ′ since σ is normalized. We also have σ′ ∗ σ = (ε_A ⊙ σ′) · σ since σ is an integral. Hence, we obtain σ′ = (ε_A ⊙ σ′) · σ. We leave the proof for σ′ ∈ Int_l(A) to the readers. It completes the proof.

We also have a dual statement for cointegrals.

Remark 3.13 There exists a bimonoid A with a generator integral which is not normalized. For example, finite-dimensional Hopf algebra which is not semi-simple is such an example.

Proposition 3.14 Let ξ : A → B be a bimonoid isomorphism. Recall that ξ^{-1} is an integral of ξ by Proposition 3.8. The integral ξ^{-1} is a generator.

Proof It is immediate from definitions.

4 Some Objects Associated with Action

4.1 Invariant Object

In this subsection, we define a notion of an invariant object of a (co)action. It is a generalization of the invariant subspace of a group action.

Definition 4.1 Let C be a symmetric monoidal category. Let (A, α, X) be a left action in C. A pair (α \ X, i) is an invariant object of the action (A, α, X) if it satisfies the following axioms:

• α \ X is an object of C.
• i : α \ X → X is a morphism in C.
• The diagram commutes where τ is the trivial action :

\[
\begin{array}{ccc}
A \otimes X & \xrightarrow{\alpha} & X \\
\uparrow{i \otimes id_A} & & \uparrow{i} \\
A \otimes (\alpha \ X) & \xrightarrow{\tau} & \alpha \ X \\
\end{array}
\] (29)

• It is universal : If a morphism ξ : Z → X satisfies a commutative diagram,

\[
\begin{array}{ccc}
A \otimes X & \xrightarrow{\alpha} & X \\
\uparrow{\xi \otimes id_A} & & \uparrow{\xi} \\
A \otimes Z & \xrightarrow{\tau} & Z \\
\end{array}
\] (30)

then there exists a unique morphism \( \tilde{\xi} : Z \rightarrow \alpha \ X \) such that \( i \circ \tilde{\xi} = \xi \).

In an analogous way, we define invariant object of a left (right) coactions.
4.2 Stabilized Object

In this subsection, we define a notion of a stabilized object of an action (coaction, resp.). It is enhanced to a functor from the category of (co)actions if the symmetric monoidal category \(\mathcal{C}\) has every coequalizer (equalizer, resp.).

**Definition 4.2** We define a **stabilized object of a left action** \((A, \alpha, X)\) in \(\mathcal{C}\) by a coequalizer of following morphisms where \(\tau_{A,X}\) is the trivial action in Definition 2.1.

\[
A \otimes X \xrightarrow{\alpha} X
\]

We denote it by \(A \backslash X\). Analogously, we define a **stabilized object of a right action** \((X, \alpha, A)\) by a coequalizer of \(\alpha\) and \(\tau_{X,A}\). We denote it by \(X/\alpha\).

We define a **stabilized object of a left coaction** \((B, \beta, Y)\) in \(\mathcal{C}\) by an equalizer of following morphisms where \(\tau^{A,X}\) is the trivial action in Definition 2.1.

\[
Y \xrightarrow{\beta} B \otimes Y
\]

We denote it by \(B/Y\). Analogously, we define a **stabilized object of a right coaction** \((Y, \beta, B)\) by an equalizer of \(\alpha\) and \(\tau^{X,B}\). We denote it by \(Y^\beta\).

**Proposition 4.3** The assignments of stabilized objects to (co)actions have the following functoriality:

1. Suppose that the category \(\mathcal{C}\) has any coequalizers. The assignment \((A, \alpha, X) \mapsto A \backslash X\) gives a symmetric comonoidal functor (SCMF) from \(\text{Act}_\mathcal{C}\) to \(\mathcal{C}\). Analogously, the assignment \((X, \alpha, A) \mapsto X/\alpha\) gives a SCMF from \(\text{Act}_\mathcal{C}\) to \(\mathcal{C}\).

2. Suppose that the category \(\mathcal{C}\) has any equalizers. The assignment \((A, \alpha, X) \mapsto \alpha/X\) gives a symmetric monoidal functor (SMF) from \(\text{Coact}_\mathcal{C}\) to \(\mathcal{C}\). Analogously, the assignment \((X, \alpha, A) \mapsto X/\alpha\) gives a SMF from \(\text{Coact}_\mathcal{C}\) to \(\mathcal{C}\).

**Proof** The functoriality follows from the universality of coequalizers and equalizers. We only consider the first case. It is necessary to construct structure maps of a symmetric monoidal functor. Let us prove the first claim.

Let \((\mathbb{1}, \tau, \mathbb{1})\) be the unit object of the symmetric monoidal category, \(\text{Act}_\mathcal{C}\), i.e. the trivial action of the trivial bimonoid \(\mathbb{1}\) on the object \(\mathbb{1}\). Then we have a canonical morphism \(\Phi : \mathbb{1} \backslash \mathbb{1} \rightarrow \mathbb{1}\), in particular an isomorphism.

Let \(O = (A, \alpha, X)\), \(O' = (A', \alpha', X')\) be left actions in \(\mathcal{C}\), i.e. objects of \(\text{Act}_\mathcal{C}\). Denote by \((A \otimes A', \beta, X \otimes X') = (A, \alpha, X) \otimes (A', \alpha', X') \in \text{Act}_\mathcal{C}\). We construct a morphism \(\Psi_{O,O'} : \beta/\mathbb{1} \otimes (X \otimes X') \rightarrow (\alpha/\mathbb{1}) \otimes (\alpha'/\mathbb{1})\) : The canonical projections induce a morphism \(\xi : X \otimes X' \rightarrow (\alpha/\mathbb{1}) \otimes (\alpha'/\mathbb{1})\). The morphism \(\xi\) coequalizes \(\beta : (A \otimes A') \otimes (X \otimes X') \rightarrow X \otimes X'\) and the trivial action of \(A \otimes A'\) due to the definitions of \(\alpha/\mathbb{1}\) and \(\alpha'/\mathbb{1}\). Thus, we obtain a canonical morphism \(\Psi_{O,O'} : \beta/\mathbb{1} \otimes (X \otimes X') \rightarrow (\alpha/\mathbb{1}) \otimes (\alpha'/\mathbb{1})\).

Due to the universality of coequalizers and the symmetric monoidal structure of \(\mathcal{C}\), \(\Phi, \Psi_{O,O'}\) give structure morphisms for a symmetric monoidal functor \((A, \alpha, X) \mapsto A \backslash X\).

We leave it to the readers the proof of other part. \(\square\)
4.3 Stable Monoidal Structure

In this subsection, we define a (co)stability and bistability of the monoidal structure of a symmetric monoidal category. We assume that $\mathcal{C}$ is a symmetric monoidal category with arbitrary equalizer and coequalizer.

**Definition 4.4** Recall that the assignments of stabilized objects to actions (coactions, resp.) are symmetric comonoidal functors (symmetric monoidal functors, resp.) by Proposition 4.3. The monoidal structure of $\mathcal{C}$ is **stable** if the assignments of stabilized objects to actions, $\text{Act}_\mathcal{C}(\mathcal{C}) \to \mathcal{C}$ and $\text{Coact}_\mathcal{C}(\mathcal{C}) \to \mathcal{C}$, are strongly symmetric monoidal functors. The monoidal structure of $\mathcal{C}$ is **costable** if the assignments of stabilized objects to coactions, $\text{Coact}_\mathcal{C}(\mathcal{C}) \to \mathcal{C}$ and $\text{Act}_\mathcal{C}(\mathcal{C}) \to \mathcal{C}$, are SSMF’s. The monoidal structure of $\mathcal{C}$ is **bistable** if the monoidal structure is stable and costable.

**Lemma 4.5** Let $\Lambda, \Lambda'$ be small categories. Let $F : \Lambda \to C, F' : \Lambda' \to C$ be functors with colimits $\varinjlim_{\Lambda} F$ and $\varinjlim_{\Lambda'} F'$ respectively. Suppose that the functor $F(\lambda) \otimes (-)$ preserves small colimits for any object $\lambda$ of $\Lambda$ and so does the functor $(-) \otimes \varinjlim F'$. Then the exterior tensor product $F \boxtimes F' : \Lambda \times \Lambda' \to C$ has a colimit $\varinjlim_{\Lambda \times \Lambda'} F \boxtimes F'$, and we have $\varinjlim_{\Lambda} F \boxtimes \varinjlim_{\Lambda'} F' \cong \varinjlim_{\Lambda \times \Lambda'} F \boxtimes F'$.

**Proof** Let $X$ be an object of $\mathcal{C}$ and $g_{\lambda,\lambda'} : F(\lambda) \otimes F'(\lambda') \to X$ be a family of morphisms for $\lambda \in \Lambda, \lambda' \in \Lambda'$ such that $g_{\lambda,\lambda'} = g_{\lambda,\lambda'} \circ (F(\xi) \otimes F(\xi'))$ where $\xi : \lambda_0 \to \lambda_1, \xi' : \lambda'_0 \to \lambda'_1$ are morphisms in $\Lambda, \Lambda'$, respectively. By the first assumption, the object $F(\lambda) \otimes \varinjlim F'$ is a colimit of $F(\lambda) \otimes F'(\lambda')$ for arbitrary object $\lambda \in \Lambda$. We obtain a unique morphism $g_{\lambda} : F(\lambda) \otimes \varinjlim F' \to X$ such that $g_{\lambda} \circ (id_{F(\lambda)} \otimes \varinjlim g_{\lambda'} ) = g_{\lambda,\lambda'}$ for every object $\lambda \in \Lambda$. By the universality of colimits, the family of morphisms $g_{\lambda}$ is, in fact, a natural transformation. By the second assumption, $\varinjlim F \otimes \varinjlim F'$ is a colimit of the functor $F(-) \otimes F'$, and hence, the family of morphisms $g_{\lambda}$ for $\lambda \in \Lambda$ induces a unique morphism $g : \varinjlim F \otimes \varinjlim F' \to X$ such that $g \circ (\varinjlim g_{\lambda} \otimes \varinjlim F') = g_{\lambda}$. Above all, for objects $\lambda \in \Lambda, \lambda' \in \Lambda'$, we have $g \circ (\varinjlim g_{\lambda} \otimes \varinjlim g_{\lambda'}) = g \circ (\varinjlim g_{\lambda} \otimes \varinjlim g_{\lambda'}) = g_{\lambda,\lambda'}$.

We prove that such a morphism $g$ that $g \circ (\varinjlim g_{\lambda} \otimes \varinjlim g_{\lambda'}) = g_{\lambda,\lambda'}$ is unique. Let $g' : \varinjlim F \otimes \varinjlim F' \to X$ be a morphism such that $g' \circ (\varinjlim g_{\lambda} \otimes \varinjlim g_{\lambda'}) = g_{\lambda,\lambda'}$. Denote by $h = g \circ (\varinjlim g_{\lambda} \otimes \varinjlim F')$ and $h' = g' \circ (\varinjlim g_{\lambda} \otimes \varinjlim F')$. Then we have $h' \circ (\varinjlim g_{\lambda} \otimes \varinjlim g_{\lambda'}) = g_{\lambda,\lambda'} = h \circ (\varinjlim g_{\lambda} \otimes \varinjlim g_{\lambda'})$ by definitions. Since $F(\lambda) \otimes \varinjlim F'$ is a colimit of the functor $F(\lambda) \otimes F'(\lambda')$ by the first assumption, we see that $h' = h$. Equivalently, we have $g \circ (\varinjlim g_{\lambda} \otimes \varinjlim F') = g' \circ (\varinjlim g_{\lambda} \otimes \varinjlim F')$. Since $\varinjlim F \otimes \varinjlim F'$ is a colimit of the functor $F(-) \otimes \varinjlim F'$ by the second assumption, we see that $g = g'$ by the universality. It completes the proof. □

**Proposition 4.6** Suppose that the functor $\otimes (-)$ preserves coequalizers (equalizers resp.) for arbitrary object $Z \in \mathcal{C}$. Then the monoidal structure of $\mathcal{C}$ is stable (costable, resp.).

**Proof** Note that since $\mathcal{C}$ is a symmetric monoidal category, the functor $(-) \otimes Z$ preserves coequalizers (equalizers resp.) for arbitrary object $Z \in \mathcal{C}$ by the assumption. We prove the stability and leave the proof of the costability to the readers.

Let $(A, \alpha, X), (B, \beta, Y)$ be left actions in $\mathcal{C}$. Denote by $\alpha \setminus X, \beta \setminus Y$ their stabilized objects as before. By the assumption, we can apply Lemma 4.5. By Lemma 4.5, $(\alpha \setminus X \otimes \beta \setminus Y)$ is a coequalizer of morphisms $\alpha \otimes \beta, \alpha \otimes \tau_B, \tau_A \otimes \beta, \tau_A \otimes \tau_B$. Here, $\otimes$ is defined in Definition 2.1.
It suffices to show that a coequalizer of \( \alpha \otimes \beta, \alpha \otimes \tau_B, \tau_A \otimes \beta, \tau_A \otimes \tau_B \) coincides with the stabilized object \((\alpha \otimes \beta) \backslash (X \otimes Y)\), i.e. a coequalizer of \(\alpha \otimes \beta, \tau_A \otimes \tau_B\).

Let \(\pi : X \otimes Y \to (\alpha \otimes \beta) \backslash (X \otimes Y)\) be the canonical projection. The unit axiom of the action \(\beta\) induces the following commutative diagram:

\[
\begin{array}{ccc}
A \otimes B \otimes X \otimes Y & \xrightarrow{\alpha \otimes \tau_B} & X \otimes Y \\
id_A \otimes (\eta_B \circ \epsilon_B) \otimes id_{X \otimes Y} \downarrow & & \downarrow \pi \\
A \otimes B \otimes X \otimes Y & \xrightarrow{\alpha \otimes \beta} & X \otimes Y
\end{array}
\]

(33)

Hence, we have \(\pi \circ (\alpha \otimes \tau_B) = \pi \circ (\alpha \otimes \beta) \circ (id_A \otimes (\eta_B \circ \epsilon_B) \otimes id_{X \otimes Y}) = \pi \circ (\tau_A \otimes \tau_B) \circ (id_A \otimes (\eta_B \circ \epsilon_B) \otimes id_{X \otimes Y}) = \pi \circ (\tau_A \otimes \tau_B).\) We obtain \(\pi \circ (\alpha \otimes \tau_B) = \pi \circ (\tau_A \otimes \tau_B).\) Likewise, we have \(\pi \circ (\tau_A \otimes \beta) = \pi \circ (\tau_A \otimes \tau_B).\)

Let \(g : X \otimes Y \to Z\) be a morphism which coequalizes \(\alpha \otimes \beta, \alpha \otimes \tau_B, \tau_A \otimes \beta, \tau_A \otimes \tau_B.\) Since the morphism \(g\) coequalizes \(\alpha \otimes \beta, \tau_A \otimes \tau_B\), there exists a unique morphism \(g' : (\alpha \otimes \beta) \backslash (X \otimes Y) \to Z\) such that \(g' \circ \pi = g.\) Above all, \((\alpha \otimes \beta) \backslash (X \otimes Y)\) is a coequalizer of \(\alpha \otimes \beta, \alpha \otimes \tau_B, \tau_A \otimes \beta, \tau_A \otimes \tau_B.\)

\[\square\]

**Example 4.7** Consider the symmetric monoidal category, \(\text{Vec}_k^\otimes,\) the tensor category of vector spaces over \(k\) and linear homomorphisms. Note that a coequalizer (an equalizer, resp.) of two morphisms in the category \(\text{Vec}_k\) is obtained via a cokernel (a kernel, resp.) of their difference morphism. A functor \(V \otimes (-)\) preserves coequalizers and equalizers since it is an exact functor for any linear space \(V.\) Hence, by Proposition 4.6, the monoidal structure of the symmetric monoidal category, \(\text{Vec}_k^\otimes,\) is bistable.

## 5 Normal Homomorphism

In this section, we define some notions of *normality*, *conormality* and *binormality* of bimonoid homomorphisms. We prove that every homomorphism between bicommutative Hopf monoids is binormal under some assumptions on the symmetric monoidal category \(C.\)

**Definition 5.1** Let \(D\) be a category with a zero object, i.e. an initial object which is simultaneously a terminal object. Let \(A, B\) be objects of \(D\) and \(\xi : A \to B\) be a morphism in \(D.\) A cokernel of \(\xi\) is given by a pair \((\text{Cok}(\xi), \text{cok}(\xi))\) of an object \(\text{Cok}(\xi)\) and a morphism \(\text{cok}(\xi) : B \to \text{Cok}(\xi),\) which gives a coequalizer of \(\xi : A \to B\) and \(0 : A \to B\) in \(D.\)

A kernel of \(\xi\) is given by a pair \((\text{Ker}(\xi), \text{ker}(\xi))\) of an object \(\text{Ker}(\xi)\) and a morphism \(\text{ker}(\xi) : \text{Ker}(\xi) \to A,\) which gives an equalizer of \(\xi : A \to B\) and \(0 : A \to B\) in \(D.\)

**Definition 5.2** Let \(A, B\) be bimonoids in a symmetric monoidal category \(C\) and \(\xi : A \to B\) be a bimonoid homomorphism. We define a left action \((A, \alpha_{\xi}^-, B)\) and a right action \((B, \alpha_{\xi}^-, A)\) by the following compositions:

\[
\alpha_{\xi}^- : A \otimes B \xrightarrow{\xi \otimes id_B} B \otimes B \xrightarrow{\nabla_B} B,
\]

\[
\alpha_{\xi}^+ : B \otimes A \xrightarrow{id_B \otimes \xi} B \otimes B \xrightarrow{\nabla_B} B.
\]

We define a left coaction \((A, \beta_{\xi}^-, B)\) and a right coaction \((B, \beta_{\xi}^-, A)\) by the following compositions:

\[
\beta_{\xi}^- : A \xrightarrow{\Delta_A} A \otimes A \xrightarrow{\xi \otimes id_A} B \otimes A,
\]

\[
\beta_{\xi}^+ : B \xrightarrow{\Delta_B} B \otimes B \xrightarrow{\xi \otimes id_B} A \otimes A.
\]

\[\square\]
\[
\beta^-_\xi : A \xrightarrow{\Delta} A \otimes A \xrightarrow{id_A \otimes \xi} A \otimes B.
\] (37)

**Definition 5.3** Let \( A, B \) be bimonoids in a symmetric monoidal category \( C \). A bimonoid homomorphism \( \xi : A \rightarrow B \) is normal if there exists a bimonoid structure on the stabilized objects \( \alpha^-_\xi \backslash B, B / \alpha^-_\xi \) such that the canonical morphisms \( \pi : B \rightarrow \alpha^-_\xi \backslash B, \tilde{\pi} : B \rightarrow B / \alpha^-_\xi \) are bimonoid homomorphisms and the pairs \((\alpha^-_\xi \backslash B, \pi), (B / \alpha^-_\xi, \tilde{\pi})\) give cokernels of \( \xi \) in \( \text{Bimon}(C) \).

A conormal bimonoid homomorphism is defined in a dual way by using the coactions \( \beta^-_\xi, \beta^+_\xi \) instead of \( \alpha^-_\xi, \alpha^+_\xi \). A bimonoid homomorphism \( \xi : A \rightarrow B \) is binormal if it is normal and conormal in \( \text{Bimon}(C) \).

**Remark 5.4** We use the terminology normal due to the following reason. If \( C = \text{Sets}^\times \), then a Hopf monoid in that symmetric monoidal category is given by a group. For a group \( H \) and its subgroup \( G \), one can determine a set \( H / G \) which is a candidate of a cokernel of the inclusion. The set \( H / G \) plays a role of cokernel group if and only if the image \( G \) is a normal subgroup of \( H \). In this example, the normality defined in this paper means that the set \( H / G \) is a cokernel group of the inclusion \( G \hookrightarrow H \).

**Remark 5.5** We remark that our notion is implied by the Milnor-Moore’s definition if \( C = \text{Vec}_k \). Milnor and Moore defined the notion of normality of morphisms of augmented algebras over a ring and normality of morphisms of augmented coalgebras over a ring (Definition 3.3, 3.5 [20]). They are defined by using the additive structure of the category \( \text{Vec}_k \). We introduce a weaker notion of normality and conormality of bimonoid homomorphisms without assuming an additive category structure on \( C \).

**Proposition 5.6** Let \( A \) be a bimonoid. The identity homomorphism \( id_A : A \rightarrow A \) is binormal.

**Proof** We prove that the identity homomorphism \( id_A \) is normal. The counit \( \epsilon_A : A \rightarrow \mathbb{1} \) on \( A \) induces gives a coequalizer of the regular action \( \alpha^-_{id_A} : A \otimes A \rightarrow A \) and the trivial action \( \tau : A \otimes A \rightarrow A \). In particular, we have a natural isomorphism \( \alpha^-_{id_A} \backslash A \cong \mathbb{1} \). We give a bimonoid structure on \( \alpha^-_{id_A} \) by the isomorphism. Moreover the counit \( \epsilon_A : A \rightarrow \mathbb{1} \) is obviously a cokernel of the identity homomorphism \( id_A \) in the category of bimonoids \( \text{Bimon}(C) \). Thus, the identity homomorphism \( id_A \) is normal. In a dual way, the identity homomorphism \( id_A \) is conormal, so that binormal.

**Proposition 5.7** Let \( A, B \) be Hopf monoids in a symmetric monoidal category \( C \). Let \( \xi : A \rightarrow B \) be a bimonoid homomorphism. If the homomorphism \( \xi \) is normal, then a cokernel \((\text{Cok}(\xi), \text{cok}(\xi))\) in the category of bimonoids \( \text{Bimon}(C) \) is a cokernel in the category of Hopf monoids \( \text{Hopf}(C) \).

**Proof** Since \( \text{cok}(\xi) \circ S_B \circ \xi = \text{cok}(\xi) \circ \xi \circ S_A \) is trivial, the anti-homomorphism \( \text{cok}(\xi) \circ S_B \) induces an anti-homomorphism \( S : \text{Cok}(\xi) \rightarrow \text{Cok}(\xi) \) such that \( S \circ \text{cok}(\xi) = \text{cok}(\xi) \circ S_B \).

We claim that \( S \) gives an antipode on the bimonoid \( C = \text{Cok}(\xi) \). It suffices to prove that \( \nabla_C \circ (S \otimes id_C) \circ \Delta_C = \eta_C \circ \epsilon_C = \nabla_C \circ (id_C \otimes S) \circ \Delta_C \). Since \((\alpha^-_\xi \backslash B, \pi), (B / \alpha^-_\xi, \tilde{\pi}) \) give cokernels, the canonical morphism \( \text{cok}(\xi) \) is an epimorphism in \( C \) by the universality of stabilized objects. Hence, it suffices to prove that \( \nabla_C \circ (S \otimes id_C) \circ \Delta_C \circ \text{cok}(\xi) = \eta_C \circ \epsilon_C \circ \text{cok}(\xi) = \nabla_C \circ (id_C \otimes S) \circ \Delta_C \circ \text{cok}(\xi) \). We prove the first equation by using the fact that \( \text{cok}(\xi) : B \rightarrow \text{Cok}(\xi) = C \) is a bimonoid homomorphism.

\[
\nabla_C \circ (S \otimes id_C) \circ \Delta_C \circ \text{cok}(\xi) = \nabla_C \circ (S \otimes id_C) \circ (\text{cok}(\xi) \otimes \text{cok}(\xi)) \circ \Delta_B, \tag{38}
\]
\[
= \nabla_C \circ ((S \circ \text{cok}(\xi)) \otimes \text{cok}(\xi)) \circ \Delta_B. \tag{39}
\]
\[ \nabla_C \circ ((\text{cok}(\xi) \circ S_B) \otimes \text{cok}(\xi)) \circ \Delta_B, \]  
\[ \nabla_C \circ (\text{cok}(\xi) \otimes \text{cok}(\xi)) \circ (S_B \otimes \text{id}_B) \circ \Delta_B, \]  
\[ \text{cok}(\xi) \circ \nabla_B \circ (S_B \otimes \text{id}_B) \circ \Delta_B, \]  
\[ \text{cok}(\xi) \circ \eta_B \circ \epsilon_B, \]  
\[ \eta_C \circ \epsilon_C \circ \text{cok}(\xi). \]

The second equation is proved similarly. It completes the proof. □

**Proposition 5.8** Suppose that the monoidal structure of \( C \) is stable (costable, resp.). Then every bimonoid homomorphism between bicommutative bimonoids is normal (conormal, resp.) and its cokernel (kernel, resp.) is a bicommutative bimonoid. In particular, if the monoidal structure of \( C \) is bistable, then every bimonoid homomorphism between bicommutative bimonoids is binormal.

**Proof** We prove that if the monoidal structure of \( C \) is stable, then every bimonoid homomorphism between bicommutative bimonoids is normal and its cokernel is a bicommutative bimonoid. Let \( A, B \) be bicommutative bimonoids in a symmetric monoidal category \( C \) and \( \xi : A \to B \) be a bimonoid homomorphism. Note that the left action \( (A, \alpha_{\xi}^{-}, B) \) has a natural bicommutative bimonoid structure in the symmetric monoidal category \( \text{Act}_l(C) \), the category of left actions in \( C \). The symmetric monoidal category structure on \( \text{Act}_l(C) \) is described in Definition 2.1. In fact, it is due to the commutativity of \( B \). We explain the monoid structure of \( (A, \alpha_{\xi}^{-}, B) \) here. Since \( B \) is a bicommutative bimonoid, \( \nabla_B : B \otimes B \to B \) is a bimonoid homomorphism. In particular, \( \nabla_B \) is compatible with the action \( \alpha_{\xi}^{-} \), i.e. the following diagram commutes.

\[
\begin{array}{ccc}
(A \otimes A) \otimes (B \otimes B) & \xrightarrow{\alpha_{\xi}^{-} \otimes \alpha_{\xi}^{-}} & B \otimes B \\
\downarrow \nabla_A \otimes \nabla_B & & \downarrow \nabla_B \\
A \otimes B & \xrightarrow{\alpha_{\xi}^{-}} & B
\end{array}
\]

Since \( \eta_B : 1 \to B \) is a bimonoid homomorphism, the following diagram commutes.

\[
\begin{array}{ccc}
1 \otimes 1 & \xrightarrow{\cong} & 1 \\
\downarrow \eta_A \otimes \eta_B & & \downarrow \eta_B \\
A \otimes B & \xrightarrow{\alpha_{\xi}^{-}} & B
\end{array}
\]

Hence, they induce a monoid structure on \( (A, \alpha_{\xi}^{-}, B) \) in the symmetric monoidal category \( \text{Act}_l(C) \). Likewise, \( (A, \alpha_{\xi}^{-}, B) \) has a comonoid structure in \( \text{Act}_l(C) \). The comultiplications on \( A, B \) induces a comultiplication on \( (A, \alpha_{\xi}^{-}, B) \) due to following diagram commutes.

\[
\begin{array}{ccc}
(A \otimes A) \otimes (B \otimes B) & \xrightarrow{\alpha_{\xi}^{-} \otimes \alpha_{\xi}^{-}} & B \otimes B \\
\Delta_A \otimes \Delta_B & & \Delta_B \\
A \otimes B & \xrightarrow{\alpha_{\xi}^{-}} & B
\end{array}
\]

In fact, we do not need any commutativity or cocommutativity of \( A, B \) to prove the commutativity of the diagram. The counits on \( A, B \) induce a counit on \( (A, \alpha_{\xi}^{-}, B) \) due to the
following commutativity diagram.

\[
\begin{array}{c}
\mathbb{1} \otimes \mathbb{1} \xrightarrow{\epsilon_A \otimes \epsilon_B} \mathbb{1} \\
A \otimes B \xrightarrow{\alpha_{\xi}^B} B
\end{array}
\]  \hspace{1cm} (48)

Since the morphisms $\Delta_A, \nabla_A, \epsilon_A, \eta_A$ and the morphisms $\Delta_B, \nabla_B, \epsilon_B, \eta_B$ give bicommutative bimonoid structure on $A, B$, respectively, the above monoid structure and comonoid structure on $(A, \alpha_{\xi}^A, B)$ give a bicommutative bimonoid structure on $(A, \alpha_{\xi}^A, B)$.

Since the monoidal structure of $C$ is stable by the assumption, the assignment of stabilized objects to actions is a strongly symmetric monoidal functor by definition. The bicommutative bimonoid structure on $(A, \alpha_{\xi}^A, B)$ is inherited to its stabilized object $\alpha_{\xi}^A \backslash B$. We consider $\alpha_{\xi}^A \backslash B$ as a bicommutative bimonoid by the inherited structure.

The canonical morphism $\pi : B \rightarrow \alpha_{\xi}^A \backslash B$ is a bimonoid homomorphism with respect to the bimonoid structure on $\alpha_{\xi}^A \backslash B$ described above. In fact, the commutative diagram (49) induces a bimonoid homomorphism $(\mathbb{1}, \eta_{B}, \xi) \rightarrow (A, \alpha_{\xi}^A, B)$ between bicommutative bimonoids in the symmetric monoidal category $\text{Act}$. Let

\[
\begin{array}{c}
\mathbb{1} \xrightarrow{\eta_B} B \\
\downarrow \eta_A \downarrow id_B \\
A \xrightarrow{\xi} B
\end{array}
\]  \hspace{1cm} (49)

By the stability of the monoidal structure of $C$ again, we obtain a bimonoid homomorphism,

\[
B \cong \alpha_{\eta_B}^A \backslash B \rightarrow \alpha_{\xi}^A \backslash B.
\]  \hspace{1cm} (50)

It coincides with the canonical projection $\pi : B \rightarrow \alpha_{\xi}^A \backslash B$ by definitions.

All that remain is to show that the pair $(\alpha_{\xi}^A \backslash B, \pi)$ is a cokernel of the bimonoid homomorphism $\xi$ in $\text{Bimon}(C)$ in the sense of Definition 5.1. Let $C$ be another bimonoid and $\varphi : B \rightarrow C$ be a bimonoid homomorphism such that $\varphi \circ \xi = \eta_C \circ \epsilon_A$. It coequalizes the action $\alpha_{\xi}^A : A \otimes B \rightarrow B$ and the trivial action $\tau_{A,B} : A \otimes B \rightarrow B$ so that it induces a unique morphism $\tilde{\varphi} : \alpha_{\xi}^A \backslash B \rightarrow C$ such that $\tilde{\varphi} \circ \pi = \varphi$. We prove that $\tilde{\varphi}$ is a bimonoid homomorphism. Note that the count $\epsilon_A : A \rightarrow \mathbb{1}$ and the homomorphism $\varphi : B \rightarrow C$ induces a bimonoid homomorphism $(A, \alpha_{\xi}^A, B) \rightarrow (\mathbb{1}, \alpha_{\xi}^C, C)$. By the stability of the monoidal structure of $C$ again, it induces a bimonoid homomorphism $\alpha_{\xi}^A \backslash B \rightarrow \alpha_{\eta_C}^A \backslash C \cong C$ which coincides with $\tilde{\varphi}$. It completes the proof. \qed

**Corollary 5.9** Suppose that the monoidal structure of $C$ is stable (costable, resp.). Let $A, B$ be bicommutative Hopf monoids and $\xi : A \rightarrow B$ be a bimonoid homomorphism. Then a cokernel (kernel, resp.) of $\xi$ in $\text{Bimon}(C)$ is a cokernel (kernel, resp.) of $\xi$ in $\text{Hopf}^{bc}(C)$.

**Proof** Suppose that the monoidal structure of $C$ is stable. Let $A, B$ be bicommutative Hopf monoids and $\xi : A \rightarrow B$ be a bimonoid homomorphism. By Proposition 5.8, the homomorphism $\xi$ is normal and its cokernel is a bicommutative bimonoid. By Proposition 5.7, the cokernel of $\xi$ is a bicommutative Hopf monoid. \qed

### 6 Small Bimonoid and Integral

In this section, we introduce a notion of (co,bi)small bimonoids. We study the relationship between existence of normalized (co)integrals and (co)smallness of bimonoids.
**Definition 6.1** Let \( C \) be a symmetric monoidal category. Let \((A, \alpha, X)\) be a left action in the symmetric monoidal category \( C \). Recall the invariant object \( \alpha \backslash X \) and the stabilized object \( \alpha \backslash X \) of the left action \((A, \alpha, X)\). We define a morphism \( a_\alpha^\gamma : \alpha \backslash X \to \alpha \backslash X \) in \( C \) by composing the canonical morphisms \( X \to \alpha \backslash X \) and \( \alpha \backslash X \to X \). Likewise, we define \( \gamma_\alpha : X / \alpha \to X / \alpha \) for a right action \((X, \alpha, A)\), \( \beta \gamma : \beta / Y \to \beta / Y \) for a left coaction \((B, \beta, Y)\), \( \gamma_\beta : Y \backslash \beta \to Y \backslash \beta \) for a right coaction \((Y, \beta, B)\).

**Definition 6.2** Let \( C \) be a symmetric monoidal category. Recall Definition 5.2. A bimonoid \( A \) in the symmetric monoidal category \( C \) is small if

- For every left action \((A, \alpha, X)\), an invariant object \( \alpha \backslash X \) and a stabilized object \( \alpha \backslash X \) exist. Furthermore, the canonical morphism \( a_\alpha^\gamma : \alpha \backslash X \to \alpha \backslash X \) is an isomorphism.
- For every right action \((X, \alpha, A)\), an invariant object \( X / \alpha \) and a stabilized object \( X / \alpha \) exist. Furthermore, the canonical morphism \( \gamma_\alpha : X / \alpha \to X / \alpha \) is an isomorphism.

A bimonoid \( A \) in the symmetric monoidal category \( C \) is cosmall if

- For every left coaction \((B, \beta, Y)\), an invariant object \( \beta / Y \) and a stabilized object \( \beta / Y \) exist. Furthermore, the canonical morphism \( \beta \gamma : \beta / Y \to \beta / Y \) is an isomorphism.
- For every right coaction \((Y, \beta, B)\), an invariant object \( Y \backslash \beta \) and a stabilized object \( Y \backslash \beta \) exist. Furthermore, the canonical morphism \( \gamma_\beta : Y \backslash \beta \to Y \backslash \beta \) is an isomorphism.

A bimonoid \( A \) is bismall if the bimonoid \( A \) is small and cosmall.

We use subscript ‘bs’ to denote ‘bismall’. For example, \( \text{Hopf}^{\text{bs}}(C) \) is a full subcategory of Hopf(\( C \)) formed by bismall Hopf monoids.

**Remark 6.3** In general, the morphism \( a_\alpha^\gamma : \alpha \backslash X \to \alpha \backslash X \) (also, \( \beta \gamma, \gamma_\alpha, \gamma_\beta \)) in Definition 6.1 is not an isomorphism. We give three examples as follows.

**Example 6.4** Let \((A, \alpha, X)\) be a left action where \( A = X = kG \) and \( \alpha \) is the multiplication where \( G \) is a finite group. There exists an invariant object \( \alpha \backslash kG \) and a stabilized object \( \alpha \backslash kG \) given by

\[
\alpha \backslash kG = \{ \lambda \sum_{g \in G} g ; \lambda \in k \}
\]

(51)

\[
\alpha \backslash kG = kG / (g \sim e)
\]

(52)

Here, \( e \in G \) denotes the unit of \( G \) and \( kG / (g \sim e) \) means the quotient space of \( kG \) by the given relation. Then if the characteristic of \( G \) divides the order \( |G| \) we see that the morphism \( a_\alpha^\gamma \) is zero while \( \alpha \backslash kG, \alpha \backslash kG \) are 1-dimensional.

**Definition 6.5** Let \( C \) be a category. A morphism \( p : X \to X \) is an idempotent if \( p \circ p = p \). A retract of an idempotent \( p \) is given by \((X^p, \iota, \pi)\) where \( \iota : X^p \to X, \pi : X \to X^p \) are morphisms in \( C \) such that \( \pi \circ \iota = id_{X^p} \) and \( \iota \circ \pi = p \). If an idempotent \( p \) has a retract, then \( p \) is called a split idempotent.

**Proposition 6.6** Let \( C \) be a category and \( p : X \to X \) be an idempotent. Suppose that there exists an equalizer of the identity \( id_X \) and \( p \) and a coequalizer of the identity \( id_X \) and \( p \). Then the idempotent \( p \) is a split idempotent.

**Proof** Denote by \( e : E \to X \) an equalizer of the identity \( id_X \) and the morphism \( p : X \to X \). Denote by \( c : X \to C \) a coequalizer of the identity \( id_X \) and the morphism \( p : X \to X \).
We claim that $c \circ e : K \to E$ is an isomorphism and $(E, e, (c \circ e)^{-1} \circ c)$ is a retract of the idempotent $p$.

Note that the morphism $p$ equalizes the identity $id_X$ and the morphism $p$ due to $p \circ p = p$. The morphism $p$ induces a unique morphism $p' : X \to E$ such that $e \circ p' = p$. Note that the morphism $p'$ coequalizes the identity $id_X$ and the morphism $p$ due to $p' \circ p = p'$. The morphism $p'$ induces a unique morphism $p'' : C \to E$ such that $p'' \circ c = p'$. Then $p''$ is an inverse of the composition $c \circ e$ so that $c \circ e$ is an idempotent.

We prove that $(E, e, (c \circ e)^{-1} \circ c)$ is a retract of the idempotent $p$. It follows from $(c \circ e)^{-1} \circ c \circ e = id_K$ and $e \circ ((c \circ e)^{-1} \circ c) = p$. The latter one follows from the above discussion that $(c \circ e)^{-1} = p''$ and $e \circ p'' \circ c = e \circ p'' = p$.

**Proposition 6.7** Let $(A, \alpha, X)$ be a left action in a symmetric monoidal category $C$ with an invariant object $\alpha \backslash X$ and a stabilized object $\alpha \backslash X$. Suppose that the morphism $\alpha \gamma : \alpha \backslash X \to \alpha \backslash X$ is an isomorphism. Then the endomorphism $p : X \to X$ defined by following composition is a split idempotent.

$$ap = \left( X \xrightarrow{\pi} \alpha \backslash X \xrightarrow{\alpha \gamma^{-1}} \alpha \backslash X \xrightarrow{\iota} X \right).$$

(53)

Here, $\iota, \pi$ are the canonical morphisms.

**Proof** We prove that $p$ is an idempotent on $X$. It follows from $p \circ p = t \circ (c \circ e)^{-1} \circ \pi \circ (c \circ e)^{-1} \circ p = p$.

We prove that $(\alpha \backslash X, t \circ (c \circ e)^{-1}, \pi)$ give a retract of the idempotent $p$. By definition, we have $t \circ (c \circ e)^{-1} \circ \pi = p$. Moreover, we have $\pi \circ t \circ (c \circ e)^{-1} = \gamma \circ (c \circ e)^{-1} = id_{\alpha \backslash X}$. □

**Lemma 6.8** Let $A$ be a bimonoid in a symmetric monoidal category $C$. Suppose that for the regular left action $(A, \alpha \rightharpoonup, A)$, an invariant object $\alpha \rightharpoonup \backslash A$ and a stabilized object $\alpha \rightharpoonup \backslash A$ exist and the canonical morphism $\alpha \rightharpoonup A$ is an isomorphism. Then the bimonoid $A$ has a normalized left integral.

**Proof** Let $A$ be a bimonoid. Suppose that the bimonoid $A$ is small. Consider a left action $(A, \alpha)$ in $C$ where $\alpha = \alpha \rightharpoonup \backslash A : A \otimes A \to A$ is the regular left action. Since $A$ is small, the invariant object $\alpha \rightharpoonup \backslash A$ and the stabilized object $\alpha \backslash A$ exist and the morphism $\alpha \gamma : \alpha \backslash A \to \alpha \backslash A$ is an isomorphism. Let $p : A \to A$ be a composition of $A \xrightarrow{\pi} \alpha \backslash A \xrightarrow{\alpha \gamma^{-1}} \alpha \backslash A \xrightarrow{\iota} A$ where $\pi, \iota$ are canonical morphisms. We prove that $\sigma = p \circ \eta_A : 1 \to A$ is a normalized right integral.

We claim that $\epsilon_A \circ p = \epsilon$. Then $\epsilon_A \circ \sigma = \epsilon_A \circ \eta_A = id_1$ which is the axiom (7) : Note that the canonical morphism $\pi : A \to \alpha \backslash A$ coequalizes the regular left action $\alpha$ and the trivial left action. The counit morphism $\epsilon_A$ induces a unique morphism $\epsilon_A^\prime : \alpha \backslash A \to 1$ such that $\epsilon_A^\prime \circ \pi = \epsilon_A$. We obtain following commutative diagram so that $\epsilon_A \circ p = \epsilon$. 

$$A \xrightarrow{\pi} \alpha \backslash A \xrightarrow{\alpha \gamma^{-1}} \alpha \backslash A \xrightarrow{\iota} A$$

(54)
We claim that \( \nabla_A \circ (id_A \otimes p) = r_A \circ (\epsilon_A \otimes p) : A \otimes A \to A \). Then by composing
\( id_A \otimes \eta_A : A \otimes 1 \to A \otimes A \) we see that \( \sigma = p \circ \eta_A \) satisfies the axiom (6): In fact, we have
\[
\nabla_A \circ (id_A \otimes i) = \epsilon_A \otimes A(\alpha(\gamma^{-1} \circ \pi)) = (\epsilon_A \otimes i) \circ (id_A \otimes (\alpha(\gamma^{-1} \circ \pi)) = r_A \circ (\epsilon_A \otimes p).
\]
Above all, the morphism \( \sigma = p \circ \eta_A : 1 \to A \) is a normalized right integral of \( A \).

\( \Box \)

**Remark 6.9** In Lemma 6.8, we show that a bimonoid \( A \) has a normalized left integral under some assumptions on the bimonoid \( A \). Similarly, a bimonoid has a normalized right integral if \( A \) satisfies similar assumptions on the regular right action. Especially, if the bimonoid \( A \) is small, then the bimonoid \( A \) has a normalized left integral and a normalized right integral. We also have a dual statement.

**Definition 6.10** Let \( (A, \alpha, X) \) be a left action in a symmetric monoidal category \( \mathcal{C} \). For a morphism \( a : 1 \to A \) in \( \mathcal{C} \), we define an endomorphism \( L_a(a) : X \to X \) by a composition,
\[
X \xrightarrow{\alpha} \ 1 \otimes X \xrightarrow{id_X \otimes a} \ A \otimes X \xrightarrow{a} X.
\]

(55)

Let \( (Y, \beta, B) \) be a right coaction in \( \mathcal{C} \). For a morphism \( b : B \to 1 \) in \( \mathcal{C} \), we define an endomorphism \( R^b(b) : Y \to Y \) by a composition,
\[
Y \xrightarrow{\beta} \ Y \otimes B \xrightarrow{id_Y \otimes b} \ Y \otimes 1 \xrightarrow{\tau_Y} \ Y.
\]

(56)

**Proposition 6.11** Let \( (A, \alpha, X) \) be a left action in \( \mathcal{C} \). Then \( a \in Mor_\mathcal{C}(1, A) \mapsto L_a(a) \in End_\mathcal{C}(X) \) is a homomorphism. Here, the monoid \( End_\mathcal{C}(X) \) consists of endomorphisms on \( X \) :
\[
L_a(a * a') = L_a(a) \circ L_a(a'), \ a, a' \in Mor_\mathcal{C}(1, A).
\]

(57)

Likewise, for a right coaction \( (Y, \beta, B) \), the assignment \( b \in Mor_\mathcal{C}(B, 1) \mapsto R^b(b) \in End_\mathcal{C}(Y) \) is a homomorphism :
\[
R^b(b * b') = R^b(b) \circ R^b(b'), \ b, b' \in Mor_\mathcal{C}(B, 1).
\]

(58)

**Proof** It follows from the associativity of an action and a coaction. \( \Box \)

**Proposition 6.12** Let \( A \) be a small bimonoid in a symmetric monoidal category \( \mathcal{C} \). Let \( (A, \alpha, X) \) be a left action in \( \mathcal{C} \). Recall Lemma 6.8, then we have a normalized integral \( \sigma_A \) of \( A \). The induced morphism \( L_a(\sigma_A) \) is a split idempotent. Moreover we have \( a \ p = L_a(\sigma_A) \) where \( a \ p \) is given in Proposition 6.7.

**Proof** The morphism \( L_a(\sigma_A) \) is an idempotent by Proposition 6.11 and \( \sigma_A * \sigma_A = \sigma_A \). \( \sigma_A * \sigma_A = \sigma_A \) follows from the normality of \( \sigma_A \).

Let \( \alpha \setminus X \) be an invariant object and \( \alpha \setminus X \) be a stabilized object of the left action \( (A, \alpha, X) \). Denote by \( \iota : \alpha \setminus X \to X \) and \( \pi : X \to \alpha \setminus X \) the canonical morphisms. We claim that the morphism \( \iota \) gives an equalizer of \( L_a(\sigma_A) \) and \( id_X \), and the morphism \( \pi \) gives a coequalizer of \( L_a(\sigma_A) \) and \( id_X \). Then the idempotent \( L_a(\sigma_A) \) is a split idempotent by Proposition 6.6.

We prove that the morphism \( \iota \) gives an equalizer of \( L_a(\sigma_A) \) and \( id_X \). Note that \( L_a(\sigma_A) \circ \iota = id_X \circ \iota \) since the integral \( \sigma_A \) is normalized. We prove the universality. Suppose that \( f : Z \to X \) equalizes \( L_a(\sigma_A) \) and \( id_X \), i.e. \( L_a(\sigma_A) \circ f = f \). Then \( \alpha \circ (id_X \otimes f) = \tau_{A,X} \circ (id_A \otimes f) \) by Fig. 7. By definition of the invariant object \( \alpha \setminus X \), \( f \) induces a unique morphism \( f' : Z \to \alpha \setminus X \) such that \( \iota \circ f' = f \).

\( \Box \) Springer
The morphism $C$ in Fig. 8 is a split idempotent. A bimonoid $A$ in $\mathcal{C}$ is a normalized integral. Moreover, if we have $\alpha(\sigma A)$, then $\alpha(\sigma A) = \alpha^\gamma\alpha\gamma^{-1} \circ \pi = g$. We prove that the morphism $\alpha(\sigma A)$ gives a coequalizer of $L\alpha(\sigma A)$ and $id X$. Note that $\pi \circ L\alpha(\sigma A)$ and $\pi \circ id X$ since the integral $\sigma A$ is normalized. We prove the universality. Suppose that $g : X \to Z$ coequalizes $L\alpha(\sigma A)$ and $id X$, i.e. $g \circ L\alpha(\sigma A) = g$. Then $g \circ \alpha = g \circ \tau A X$ by Fig. 8. By definition of the stabilized object $\alpha \setminus X$, the morphism $g$ induces a unique morphism $g' : \alpha \setminus X \to Z$ such that $g' \circ \pi = g$.

All that remain is to prove that $\alpha p = L\alpha(\sigma A)$. Note that $(\alpha \setminus X, \iota, \gamma^{-1} \circ \pi)$ gives a retract of the idempotent of $L\alpha(\sigma A)$. See the Proof of Proposition 6.6. Hence, $L\alpha(\sigma A) = \iota \circ (\alpha \gamma^{-1} \circ \pi) = \alpha p$. It completes the proof. \hfill \square

**Theorem 6.13** Let $\mathcal{C}$ be a symmetric monoidal category. Suppose that every idempotent in $\mathcal{C}$ is a split idempotent. A bimonoid $A$ in $\mathcal{C}$ is small if and only if the bimonoid $A$ has a normalized integral.

**Proof** By Proposition 3.3, Lemma 6.8, and Remark 6.9, if a bimonoid $A$ is small, then $A$ has a normalized integral.

Suppose that a bimonoid $A$ has a normalized integral $\sigma A$. Let $(A, \alpha, X)$ be a left action in $\mathcal{C}$. Let us write $p = L\alpha(\sigma A) : X \to X$. By Proposition 6.11, we have $p \circ p = L\alpha(\sigma A) \circ L\alpha(\sigma A) = L\alpha(\sigma A \ast \sigma A) = L\alpha(\sigma A) = p$ since $\sigma A$ is a normalized integral of $A$. In other words, the morphism $p$ is an idempotent on $X$. By the assumption, there exists a retract $(X^p, \iota, \pi)$ of the idempotent $p : X \to X$. We claim that,

1. The morphism $\pi : X \to X^p$ gives a stabilized object $\alpha \setminus X$ of the left action $(A, \alpha, X)$.
2. The morphism $\iota : X^p \to X$ gives an invariant object $\alpha \setminus X$ of the left action $(A, \alpha, X)$.

Then the canonical morphism $\gamma : \alpha \setminus X \to \alpha \setminus X$ coincides with $\pi \circ \iota = id X^p$ so that $\alpha \gamma$ is an isomorphism. It completes the proof.

We prove the first claim. Suppose that a morphism $f : X \to Y$ coequalizes the action $\alpha : A \otimes X \to X$ and the trivial action $\tau A X : A \otimes X \to X$, i.e. $f \circ \alpha = f \circ \tau A X$. We set $f' = f \circ \iota : X^p \to Y$. Then we have $f' \circ \pi = f \circ \iota \circ \pi = f \circ p = f \circ L\alpha(\sigma A) = f \circ \alpha \circ (\sigma A \otimes id X)$. By $f \circ \alpha = f \circ \tau A X$, we obtain $f' \circ \pi = f \circ \tau A X \circ (\sigma A \otimes id X) = f$ since $\sigma A$ is a normalized integral. Moreover, if we have $f'' \circ \pi = f$ for a morphism $f'' : X^p \to Y$, then $f'' = f'' \circ \pi \circ \iota = f \circ \iota = f'$. Above all, the morphism $\pi : X \to X^p$ gives a stabilized object $\alpha \setminus X$ of the left action $(A, \alpha, X)$.
We prove the second claim. The following diagram commutes:

\[
\begin{array}{ccc}
A \otimes X & \xrightarrow{\alpha} & X \\
\uparrow{id_A \otimes \iota} & & \uparrow{\iota} \\
A \otimes X^p & \xrightarrow{\tau_{A,X^p}} & X^p
\end{array}
\]

(59)

It follows from Fig. 9. We prove the universality of an invariant object. Suppose that a morphism \( g : Z \to X \) satisfies \( \alpha \circ (id_A \otimes g) = \tau_{A,X} \circ (id_A \otimes g) : A \otimes Z \to X \). Put \( g' = \pi \circ g : Z \to X^p : Z \to X^p \). We have \( \iota \circ g' = \iota \circ \pi \circ g = p \circ g = \alpha \circ (\sigma_A \otimes id_X) \circ g = \tau_{A,X} \circ (\sigma_A \otimes id_X) \circ g = g \) since \( \sigma_A \) is the normalized integral. If for a morphism \( g'' : Z \to X^p \) we have \( \iota \circ g'' = g \), then we have \( g'' = \pi \circ \iota \circ g'' = \pi \circ g = g' \). It proves the universality of an invariant object \( \iota : X^p \to X \).

Corollary 6.14 Let \( C \) be a symmetric monoidal category. Suppose that every idempotent in \( C \) is a split idempotent. A bimonoid \( A \) in \( C \) is bismall if and only if \( A \) has a normalized integral and a normalized cointegral.

Proof We have a dual statement of Theorem 6.13. The dual statement and Theorem 6.13 complete the proof.

Corollary 6.15 Suppose that every idempotent in \( C \) is a split idempotent. The full subcategory of (co)small bimonoids in a symmetric monoidal category \( C \) forms a sub symmetric monoidal category of \( \text{Bimon}(C) \). In particular, the full subcategory of bismall bimonoids in a symmetric monoidal category \( C \) forms a sub symmetric monoidal category of \( \text{Bimon}(C) \).

Proof We prove the claim for small cases and leave the second claim to the readers. By Theorem 6.13, small bimonoids \( A, B \) have normalized integrals \( \sigma_A, \sigma_B \). Then a morphism \( \sigma_A \otimes \sigma_B : 1 \cong 1 \otimes 1 \to A \otimes B \) is verified to give a morphism of the bimonoid \( A \otimes B \) by direct calculation. Hence the bimonoid \( A \otimes B \) possesses a normalized integral so that \( A \otimes B \) is small by Theorem 6.13. It completes the proof.
7 Integral Along Bimonoid Homomorphism

7.1 Uniqueness of Normalized Integral

In this subsection, we prove the uniqueness of normalized integrals along homomorphisms. It is a generalization of the uniqueness of normalized (co)integrals of bimonoids in Proposition 3.3.

Proposition 7.1 Let \( \xi : A \to B \) be a bimonoid homomorphism. Suppose that \( \mu \in \text{Int}_r(\xi) \), \( \mu' \in \text{Int}_l(\xi) \) are normalized. Then we have

\[
\mu = \mu' \in \text{Int}(\xi).
\]

(60)

In particular, a normalized integral along \( \xi \) is unique if exists.

Proof It is proved by two equations \( \mu = \mu \circ \xi \circ \mu' \) and \( \mu' = \mu \circ \xi \circ \mu' \). The former claim follows from (Fig. 10) and the latter claim follows from (Fig. 11). It completes the proof.

\[\square\]

Corollary 7.2 Let \( \xi : A \to B \) a bimonoid homomorphism. If \( \mu \in \text{Int}(\xi) \) is normalized, then we have

- \( \mu \circ \xi \circ \mu = \mu \).
- \( \mu \circ \xi : A \to A \) is an idempotent on \( A \).
- \( \xi \circ \mu : B \to B \) is an idempotent on \( B \).

Proof By direct verification, \( \mu' = \mu \circ \xi \circ \mu \) is an integral along \( \xi \). Also, \( \mu' \) is normalized since \( \xi \circ \mu' \circ \xi = \xi \circ \mu \circ \xi \circ \xi = \xi \) by the normality of \( \mu \). By Proposition 7.1, we have \( \mu' = \mu \). It completes the proof of the first claim. The other claims are immediate from the first claim.

\[\square\]
7.2 Necessary Conditions for Existence of a Normalized Integral

An existence of a normalized integral along a homomorphism $\xi$ is related with an existence of a normlaized integral of $Ker(\xi)$ and a cointegral $Cok(\xi)$. In this subsection, we prove Theorem 7.5 which implies Theorem 7.6. We define an integral $\tilde{F}(\mu)$ of $Ker(\xi)$ from an integral $\mu$ along $\xi$ when $\xi$ is conormal. Furthermore, if the integral $\mu$ is normalized, then the integral $\tilde{F}(\mu)$ is normalized.

**Lemma 7.3** Let $\mu \in Int_\mathcal{r}(\xi)$. Then $\mu \circ \eta_B : \mathcal{1} \rightarrow A$ equalizes the homomorphism $\xi$ and the trivial homomorphism, i.e. $\xi \circ (\mu \circ \eta_B) = \eta_B \circ \epsilon_A \circ (\mu \circ \eta_B)$.

**Proof** It is verified by Fig. 12.

**Definition 7.4** Let $\xi : A \rightarrow B$ be a bimonoid homomorphism and $\mu \in Int_\mathcal{r}(\xi)$. If $\xi$ is conormal, a morphism $\tilde{F}(\mu) : \mathcal{1} \rightarrow Ker(\xi)$ is defined as follows. By Lemma 7.3, $\mu \circ \eta_B$ is decomposed into

$$
\mathcal{1} \xrightarrow{\psi} A \setminus \beta_\xi^+ \rightarrow A.
$$

Since $\xi$ is conormal, $A \setminus \beta_\xi^+$ gives a kernel bimonoid of $\xi$, $Ker(\xi)$ so that the morphism $\varphi$ defines $\tilde{F}(\mu) : \mathcal{1} \rightarrow Ker(\xi)$.

If $\xi$ is normal, we define a morphism $\hat{F}(\mu) : Cok(\xi) \rightarrow \mathcal{1}$ in an analogous way, i.e. $\epsilon_A \circ \mu$ is decomposed into

$$
B \rightarrow Cok(\xi) \xrightarrow{\hat{F}(\mu)} \mathcal{1}.
$$

**Theorem 7.5** Let $A, B$ be bimonoids and $\xi : A \rightarrow B$ be a bimonoid homomorphism Let $\mu \in Int_\mathcal{r}(\xi)$.

1. Suppose that $\xi$ is conormal. Then the morphism $\tilde{F}(\mu) : \mathcal{1} \rightarrow Ker(\xi)$ is defined and it is a right integral of $Ker(\xi)$. If the integral $\mu$ along $\xi$ is normalized, then the integral $\tilde{F}(\mu)$ is normalized.
2. Suppose that $\xi : A \rightarrow B$ is normal. Then the morphism $\hat{F}(\mu) : Cok(\xi) \rightarrow \mathcal{1}$ is defined and it is a right cointegral of $Cok(\xi)$. If the integral $\mu$ along $\xi$ is normalized, then the cointegral $\hat{F}(\mu)$ is normalized.

**Proof** We only prove the first part. For simplicity, let us write $j = ker(\xi) : Ker(\xi) \rightarrow A$.

We prove that $\nabla_{Ker(\xi)} \circ (\tilde{F}(\mu) \otimes id_{Ker(\xi)}) = \tilde{F}(\mu) \otimes \epsilon_{Ker(\xi)}$. Due to the universality of kernels, it suffices to show that $j \circ \nabla_{Ker(\xi)} \circ (\tilde{F}(\mu) \otimes id_{Ker(\xi)}) = j \circ (\tilde{F}(\mu) \otimes \epsilon_{Ker(\xi)})$. See Fig. 13.
Let us prove that $\tilde{F}(\mu)$ is normalized if $\mu$ is normalized. It is shown by the following direct calculation:

$$
\epsilon_{\text{Ker}(\xi)} \circ \tilde{F}(\mu) = \epsilon_A \circ \mu \circ \eta_B \\
= \epsilon_B \circ \xi \circ \mu \circ \xi \circ \eta_A \\
= \epsilon_B \circ \xi \circ \eta_A \quad (\because \mu : \text{normalized}) \tag{66}
$$

$$
= \text{id}_1 \tag{67}
$$

\hfill \square

Corollary 7.6 Let $\xi : A \to B$ be a bimonoid homomorphism with a normalized integral along $\xi$. If the homomorphism $\xi$ is conormal, then the kernel bimonoid $Ker(\xi)$ has a normalized integral.

We have a dual claim: if the homomorphism $\xi$ is normal, then the cokernel bimonoid $Cok(\xi)$ has a normalized cointegral.

8 Computation of $\text{Int}(\xi)$

In this section, we compute $\text{Int}(\xi)$ by using $\tilde{F}$, $\hat{F}$ in Definition 7.4. The main result in this subsection is that if $\xi$ has a normalized generator integral, then $\text{Int}(\xi)$ is isomorphic to $\text{End}_C(\mathbb{1})$, the endomorphism set of the unit $\mathbb{1} \in C$.

Definition 8.1 Let $A, B$ be bimonoids and $\xi : A \to B$ be a bimonoid homomorphism with a kernel bimonoid $Ker(\xi)$. Let $\varphi \in \text{Mor}_C(\mathbb{1}, Ker(\xi))$ and $\mu \in \text{Int}_r(\xi)$. We define $\varphi \ltimes \mu \in \text{Mor}_C(B, A)$ by

$$
\varphi \ltimes \mu \overset{\text{def.}}{=} \begin{pmatrix} B \xrightarrow{\cong} \mathbb{1} \otimes B & B \xrightarrow{\varphi \otimes id_B} Ker(\xi) \otimes B & \xrightarrow{\mu \otimes Ker(\xi)} A \otimes A & \xrightarrow{\nabla_A} A \\
\end{pmatrix} \tag{68}
$$

$$
\mu \ltimes \varphi \overset{\text{def.}}{=} \begin{pmatrix} B \xrightarrow{\cong} B \otimes \mathbb{1} & B \xrightarrow{id_B \otimes \varphi} B \otimes Ker(\xi) & \xrightarrow{\mu \otimes Ker(\xi)} A \otimes A & \xrightarrow{\nabla_A} A \\
\end{pmatrix} \tag{69}
$$

Remark 8.2 The definitions of $\varphi \ltimes \mu$ and $\mu \ltimes \varphi$ can be understood via some string diagrams in Fig. 14.

Proposition 8.3 Let $\mu \in \text{Int}_r(\xi)$. Then we have
- $\varphi \ltimes \mu \in \text{Int}_r(\xi)$.
- $\mu \ltimes \varphi = (\epsilon_{Ker(\xi)} \circ \varphi) \cdot \mu \in \text{Int}_r(\xi)$. 
Proof For simplicity we denote $j = \ker(\xi) : \text{Ker}(\xi) \to A$. We show that $\phi \times \mu \in \text{Int}_r(\xi)$. The axiom (8) is verified by Fig. 15. The axiom (9) is verified by Fig. 16. Note that the target of $\phi$ needs to be $\ker(\xi)$ to verify Fig. 16.

We show that $\mu \times \varphi = (\epsilon_{\text{Ker}(\xi)} \circ \varphi) \cdot \mu \in \text{Int}_r(\xi)$. The equation is verified by Fig. 17. Since $\mu \in \text{Int}_r(\xi)$, $\mu \times \varphi$ lives in $\text{Int}_r(\xi)$.
Lemma 8.4 Let $\xi : A \to B$ be a bimonoid homomorphism which is conormal. Let $\mu$ be a generator integral along $\xi$. For an integral $\mu' \in \text{Int}(\xi)$, we have

$$\tilde{F}(\mu') \ltimes \mu = \mu'.$$

In particular, if a bimonoid homomorphism $\xi$ has a generator integral, then $\tilde{F} : \text{Int}(\xi) \to \text{Int}(\text{Ker}(\xi))$ is injective.

Proof

It follows from Fig. 18.

\[\Box\]

Theorem 8.5 Let $\xi : A \to B$ be a bimonoid homomorphism which is either conormal or normal. Let $\mu$ be a normalized generator integral along $\xi$. Then the map $\text{End}_C(\mathbb{I}) \to \text{Int}(\xi) ; \lambda \mapsto \lambda \cdot \mu$ is a bijection.

Proof

We only prove the statement for conormal $\xi$. It suffices to replace $\tilde{F}(\mu)$ with $\hat{F}(\mu)$ for normal $\xi$ and other discussion with a dual one.

We claim that $\text{Int}(\xi) \to \text{End}_C(\mathbb{I}) ; \mu' \mapsto \epsilon_{\text{Ker}(\xi)} \circ \hat{F}(\mu')$ gives an inverse map. It suffices to prove that $\mu' = \left(\epsilon_{\text{Ker}(\xi)} \circ \hat{F}(\mu')\right) \cdot \mu$ and $\epsilon_{\text{Ker}(\xi)} \circ \hat{F}(\lambda \cdot \mu) = \lambda$. The latter one follows from $\epsilon_{\text{Ker}(\xi)} \circ \hat{F}(\mu) = \text{id}_\mathbb{I}$ which is nothing but the normality of $\hat{F}(\mu)$ by Theorem 7.5. We show the former one by calculating $\hat{F}(\mu') \ltimes \mu$ in a different way. It follows from Fig. 19.

By Lemma 8.4, $\hat{F}(\mu') \ltimes \mu = \mu'$, so that $\mu' = \left(\epsilon_{\text{Ker}(\xi)} \circ \hat{F}(\mu')\right) \cdot \mu$. \[\Box\]
9 Existence of a Normalized Generator Integral

In this section, we give sufficient conditions for a normalized generator integral along a homomorphism exists.

9.1 Key Lemma

**Lemma 9.1** Let $A$, $B$ be bimonoids. Let $\xi : A \to B$ be a bimonoid homomorphism.

(1) Suppose that $A$ is small. In particular, the canonical morphism $\xi \gamma : \alpha_\xi^{-} \setminus B \to \alpha_\xi^{-} \setminus B$ is an isomorphism. Here, the left action $\alpha_\xi^{-}$ is defined in Definition 5.2. Let

$$\mu_0 = \left( \alpha_\xi^{-} \setminus B \xrightarrow{(\xi \gamma)^{-1}} \alpha_\xi^{-} \setminus B \to B \right).$$

(71)

If $\alpha_\xi^{-} \setminus B$ has a bimonoid structure such that the canonical morphism $\pi : B \to \alpha_\xi^{-} \setminus B$ is a bimonoid homomorphism, then we have

- $\mu_0 \in \text{Int}_r(\pi)$. In particular, $\text{Int}_r(\pi) \neq \emptyset$.
- $\pi \circ \mu_0 = id_{\alpha_\xi^{-} \setminus B}$. In particular, the right integral $\mu_0$ is normalized.
- By Remark 6.9, the bimonoid $A$ has a normalized integral $\sigma_A$. We have,

$$\mu_0 \circ \pi = L_{\alpha_\xi^{-}}(\sigma_A).$$

(72)

If $B$ is commutative, then $\mu_0 \in \text{Int}_l(\pi)$, in particular, $\mu_0 \in \text{Int}(\pi) \neq \emptyset$. We have an analogous statement for the right action $(B, \alpha_\xi^{-}, A)$.

(2) Suppose that $B$ is cosmall. In particular, the canonical morphism $\gamma_\xi : A\setminus \beta_\xi^{-} \to A\setminus \beta_\xi^{-}$ is an isomorphism. Here, the right coaction $\beta_\xi^{-}$ is defined in Definition 5.2. Let

$$\mu_1 = \left( A \to A\setminus \beta_\xi^{-} \xrightarrow{(\gamma_\xi)^{-1}} A\setminus \beta_\xi^{-} \right).$$

(73)

If $A\setminus \beta_\xi^{-}$ has a bimonoid structure such that the canonical morphism $\iota : A\setminus \beta \to A$ is a bimonoid homomorphism, then we have

- $\mu_1 \in \text{Int}_l(\iota)$. In particular, $\text{Int}_l(\iota) \neq \emptyset$.
- $\iota \circ \mu_1 = id_{A\setminus \beta_\xi^{-}}$. In particular, the left integral $\mu_1$ is normalized.
- By Remark 6.9, the bimonoid $B$ has a normalized cointegral $\sigma^B$. We have,

$$\iota \circ \mu_1 = R_{\beta_\xi^{-}}(\sigma^B).$$

(74)
If $A$ is cocommutative, then $\mu_1 \in \text{Int}_r(\iota)$, in particular, $\mu_1 \in \text{Int}(\iota) \neq \emptyset$. We have an analogous statement for the left coaction $(B, \beta^{-}_\xi, A)$.

**Proof** We prove the first claim here and leave the second claim to the readers. Recall Lemma 6.8 that a small bimonoid $A$ has a normalized integral. We denote the normalized integral by $\sigma_A: 1 \to A$.

We prove that $\mu_0$ satisfies the axiom (8). Denote by $j: \alpha^{-}\_\xi \to B$ the canonical morphism. Since $\gamma = \xi \gamma$ is an isomorphism, it suffices to show that $\nabla_B \circ ((\mu_0 \circ \gamma) \otimes id_B) = \mu_0 \circ \nabla_{\alpha^{-}\_\xi \otimes B} \circ (\gamma \otimes \pi)$. It is verified by Fig. 20.

We prove that $\mu_0$ satisfies the axiom (9). Due to the universality of $\pi: B \to \alpha^{-}\_\xi \otimes B$, it suffices to show that $(\mu_0 \otimes id_{\alpha^{-}\_\xi \otimes B}) \circ \Delta_{\alpha^{-}\_\xi \otimes B} \circ \pi = (id_B \otimes \pi) \circ \Delta_B \circ \mu_0 \circ \pi$. It is verified by Fig. 21. Thus, we obtain $\mu_0 \in \text{Int}_r(\pi)$.

The claim $\pi \circ \mu_0 \circ (\gamma \otimes (\xi \gamma)^{-1} = id_{\alpha^{-}\_\xi \otimes B}$.

The claim $\mu_0 \circ \pi = L_{\alpha^{-}\_\xi} (\sigma_A)$ follows from the definition of $\alpha^{-}\_\xi$ and Proposition 6.12.

From now on, we suppose that $B$ is commutative and show that $\mu \in \text{Int}_l(\pi)$. We prove that $\mu_0$ satisfies the axiom (10). Since $\gamma = \xi \gamma$ is an isomorphism, it suffices to show that $\nabla_B \circ (\mu_0 \otimes (\mu \circ \gamma)) = \mu \circ \nabla_{\alpha^{-}\_\xi \otimes B} \circ (\pi \otimes \gamma)$. It is verified by Fig. 22. We need the commutativity of $B$ here.

We prove that $\mu_0$ satisfies the axiom (11). Due to the universality of $\pi: B \to \alpha^{-}\_\xi \otimes B$, it suffices to show that $(id_{\alpha^{-}\_\xi \otimes B} \otimes \mu) \circ \Delta_{\alpha^{-}\_\xi \otimes B} \circ \pi = (\pi \otimes id) \circ \Delta_B \circ \mu_0 \circ \pi$. It is verified by Fig. 23. □

**Definition 9.2** Let $A, B$ be bimonoids in a symmetric monoidal category $C$ and $\xi: A \to B$ be a bimonoid homomorphism. Suppose that the bimonoid $A$ is small and $\xi$ is normal. By Lemma 9.1, there exists a normalized right integral along the homomorphism $cok(\xi): B \to Cok(\xi)$. Analogously, there also exists a normalized left integral along $cok(\xi)$ since the homomorphism $\xi$ is normal. By Proposition 7.1, these coincide to each other. Denote the normalized integral by $\tilde{\mu}_{cok(\xi)} \in \text{Int}(cok(\xi))$. 

\[ \text{Fig. 20} \text{ A part of the proof of Lemma 9.1} \]
Suppose that $B$ is cosmall and $\xi$ is conormal. Analogously, by Lemma 9.1, we define a normalized integral $\tilde{\mu}_{ker(\xi)} \in Int(ker(\xi))$.

**Lemma 9.3** Let $A$, $B$ be bimonoids and $\xi : A \to B$ be a bimonoid homomorphism. Suppose that $A$ is small and the homomorphism $\xi$ is normal. Then we have

$$cok(\xi) \circ \tilde{\mu}_{cok(\xi)} = id_{cok(\xi)}$$

(75)
In particular, \( \text{cok}(\xi) \) has a section in \( C \).

Suppose that \( B \) is cosmall and the canonical morphism \( \xi \) is conormal. Then we have,

\[
\˜\mu_{\text{cok}}(\xi) \circ \text{cok}(\xi) = L_{\alpha^-_{\xi}}(\sigma_A)
\]

(76)

\[
= R_{\alpha^-_{\xi}}(\sigma_A)
\]

(77)

In particular, \( \text{cok}(\xi) \) has a retract in \( C \).

Proof It follows from the definitions of \( \˜\mu_{\text{cok}}(\xi), \˜\mu_{\text{ker}}(\xi) \) and Lemma 9.1. \qed

9.2 Sufficient Conditions for Existence of a Normalized Generator Integral

In this subsection, we give some sufficient conditions for a normalized generator integral to exist. Furthermore, we prove Corollary 9.11 which implies our main theorem.

Definition 9.4 Let \( A, B \) be bimonoids and \( \xi : A \rightarrow B \) be a bimonoid homomorphism with a kernel bimonoid \( \text{Ker}(\xi) \). Suppose that \( \text{Ker}(\xi) \) is small and the canonical morphism \( \text{ker}(\xi) : \text{Ker}(\xi) \rightarrow A \) is normal. We define a normalized integral along \( \text{coim}(\xi) = \text{cok}(\text{ker}(\xi)) \) : \( A \rightarrow \text{Coim}(\xi) \) by \( \˜\mu_{\text{cok}(\xi)} \) in Definition 9.2 where \( \xi = \text{ker}(\xi) \). We denote \( \˜\mu_{\text{cok}(\xi)} \) by \( \˜\mu_{\text{coim}(\xi)} \in \text{Int}(\text{coim}(\xi)) \).

Analogously we define \( \˜\mu_{\text{im}(\xi)} \) : Let \( A, B \) be bimonoids and \( \xi : A \rightarrow B \) be a bimonoid homomorphism with a cokernel bimonoid \( \text{Cok}(\xi) \). Suppose that \( \text{Cok}(\xi) \) is cosmall and the canonical morphism \( \text{ker}(\xi) : \text{Ker}(\xi) \rightarrow A \) is conormal. We define a normalized integral along \( \text{im}(\xi) = \text{ker}(\text{cok}(\xi)) \) : \( A \rightarrow \text{Im}(\xi) \) by \( \˜\mu_{\text{ker}(\xi)} \) in Definition 9.2 where \( \xi = \text{cok}(\xi) \). We denote \( \˜\mu_{\text{ker}(\xi)} \) by \( \˜\mu_{\text{im}(\xi)} \in \text{Int}(\text{im}(\xi)) \).
Lemma 9.5 Let $A, B$ be bimonoids and $\xi : A \to B$ be a bimonoid homomorphism with a kernel $\text{Ker}(\xi)$. Suppose that the kernel bimonoid $\text{Ker}(\xi)$ is small and the canonical morphism $\text{ker}(\xi) : \text{Ker}(\xi) \to A$ is normal. Then we have

\[ \text{coim}(\xi) \circ \tilde{\mu}_{\text{coim}(\xi)} = \text{id}_{\text{coim}(\xi)} \] (81)

\[ \tilde{\mu}_{\text{coim}(\xi)} \circ \text{coim}(\xi) = L_{\text{Ker}(\xi)}(\sigma_{\text{Ker}(\xi)}) \] (82)

\[ = R_{\text{Ker}(\xi)}(\sigma_{\text{Ker}(\xi)}) \] (83)

In particular, $\text{coim}(\xi)$ has a section in $C$.

An analogous statement for $\text{Im}(\xi)$ holds: Let $A, B$ be bimonoids and $\xi : A \to B$ be a bimonoid homomorphism with a cokernel bimonoid $\text{Cok}(\xi)$. Suppose that $\text{Cok}(\xi)$ is cosmall and the canonical morphism $\text{cok}(\xi) : B \to \text{Cok}(\xi)$ is conormal. Then we have,

\[ \tilde{\mu}_{\text{im}(\xi)} \circ \text{im}(\xi) = \text{id}_{\text{im}(\xi)} \] (84)

\[ \text{im}(\xi) \circ \tilde{\mu}_{\text{im}(\xi)} = R_{\text{Cok}(\xi)}(\sigma_{\text{Cok}(\xi)}) \] (85)

\[ = L_{\text{Cok}(\xi)}(\sigma_{\text{Cok}(\xi)}) \] (86)

In particular, $\text{im}(\xi)$ has a retract in $C$.

Proof It follows from Lemma 9.3.

Definition 9.6 Let $A, B$ be bimonoids. A bimonoid homomorphism $\xi : A \to B$ is weakly well-decomposable if the following conditions hold:

- $\text{Ker}(\xi), \text{Cok}(\xi), \text{Coim}(\xi), \text{Im}(\xi)$ exist in $\text{Bimon}(C)$.
- $\text{ker}(\xi) : \text{Ker}(\xi) \to A$ is normal and $\text{cok}(\xi) : B \to \text{Cok}(\xi)$ is conormal.
- $\xi : \text{Coim}(\xi) \to \text{Im}(\xi)$ is an isomorphism.

A bimonoid homomorphism $\xi : A \to B$ is well-decomposable if following conditions hold:

- $\xi$ is binormal. In particular, $\text{Ker}(\xi), \text{Cok}(\xi)$ exist in $\text{Bimon}(C)$.
- $\text{ker}(\xi) : \text{Ker}(\xi) \to A$ is normal and $\text{cok}(\xi) : B \to \text{Cok}(\xi)$ is conormal. In particular, $\text{Coim}(\xi), \text{Im}(\xi)$ exist.
- $\tilde{\xi} : \text{Coim}(\xi) \to \text{Im}(\xi)$ is an isomorphism.

Definition 9.7 Let $\xi : A \to B$ be a weakly well-decomposable homomorphism. The homomorphism $\xi$ is weakly pre-Fredholm if the kernel bimonoid $\text{Ker}(\xi)$ is small and the cokernel bimonoid $\text{Cok}(\xi)$ is cosmall. Recall Definition 9.4. For a weakly pre-Fredholm homomorphism $\xi : A \to B$, we define

\[ \mu_\xi \overset{\text{def}}{=} \tilde{\mu}_{\text{coim}(\xi)} \circ \xi^{-1} \circ \tilde{\mu}_{\text{im}(\xi)} : B \to A. \] (87)

The homomorphism $\xi$ is pre-Fredholm if both of the kernel bimonoid $\text{Ker}(\xi)$ and the cokernel bimonoid $\text{Cok}(\xi)$ are bismall.

Proposition 9.8 Let $A$ be a bimonoid.

1. The unit $\eta_A : 1 \to A$ and the counit $\epsilon_A : A \to 1$ are well-decomposable.
2. The unit $\eta_A$ is weakly pre-Fredholm if and only if $A$ is cosmall. Then $\mu_{\eta_A}$ in Definition 9.7 is well-defined and we have $\mu_{\eta_A} = \sigma^A$.
3. The counit $\epsilon_A$ is weakly pre-Fredholm if and only if $A$ is small. Then $\mu_{\epsilon_A}$ in Definition 9.7 is well-defined and we have $\mu_{\epsilon_A} = \sigma_A$. 
Proof We prove that $\eta_A$ is well-decomposable and leave the proof of $\epsilon_A$ to the readers. Note that the unit bimonoid $1$ is bismall since it has a normalized (co)integral. The bimonoid homomorphism $\eta_A$ is normal due to the canonical isomorphism $\alpha_{\eta_A} : A \cong Cok(\eta_A)$. The bimonoid homomorphism $\eta_A$ is conormal due to the canonical isomorphism $1 \otimes R_{\eta_A} \rightarrow 1 = Ker(\eta_A)$. Moreover, $ker(\eta_A) : Ker(\eta_A) = 1 \rightarrow 1$ and $cok(\eta_A) : A \rightarrow Cok(\eta_A) = A$ are normal and conormal due to Proposition 5.6. The final axiom is verified since $\tilde{\eta}_A : 1 = Cooim(\eta_A) \rightarrow Im(\eta_A) = 1$ is the identity.

The morphism $\mu_\eta$ is a normalized integral by the following Theorem 9.9. By Proposition 7.1, we obtain $\mu_{\eta_A} = \sigma^A$. \hfill\Box

Theorem 9.9 Let $A, B$ be bimonoids and $\xi : A \rightarrow B$ be a weakly well-decomposable homomorphism. If the homomorphism $\xi$ is weakly pre-Fredholm, then the morphism $\mu_\xi$ is a normalized generator integral along $\xi$.

Proof Recall that $\mu_{coim(\xi)} \in Int(coim(\xi))$, $\mu_{im(\xi)} \in Int(im(\xi))$ by Definition 9.4. By Proposition 3.8, $\tilde{\xi}^{-1} \in Int(\xi)$. By Proposition 3.10, $\mu_\xi$ is an integral along $\xi$ since $\mu_\xi$ is defined to be a composition of $\tilde{\mu}_{coim(\xi)}, \tilde{\mu}_{im(\xi)}, \tilde{\xi}^{-1}$.

Note that $\mu_\xi \circ \xi = \tilde{\mu}_{coim(\xi)} \circ \coim(\xi)$. In fact, by Lemma 9.5, we have

$$\mu_\xi \circ \xi = (\tilde{\mu}_{coim(\xi)} \circ \tilde{\xi}^{-1} \circ \tilde{\mu}_{im(\xi)}) \circ (\im(\xi) \circ \tilde{\xi} \circ \coim(\xi)) \quad (88)$$

$$= \tilde{\mu}_{coim(\xi)} \circ \tilde{\xi}^{-1} \circ \tilde{\xi} \circ \coim(\xi) \quad (89)$$

$$= \tilde{\mu}_{coim(\xi)} \circ \coim(\xi) \quad (90)$$

We prove that the integral $\mu_\xi$ is normalized, i.e. $\xi \circ \mu_\xi \circ \xi = \xi$. By Lemma 9.5, we have $\tilde{\mu}_{coim(\xi)} \circ \coim(\xi) = L_{a_{ker}(\xi)}(\sigma_{ker}(\xi))$. Then the claim $\xi \circ \mu_\xi \circ \xi = \xi$ follows from Fig. 24 where we put $j = ker(\xi)$.

We prove that the integral $\mu_\xi$ is a generator. We first prove that $\mu_\xi \circ \xi \circ \mu = \mu$ for any $\mu \in Int_1(\xi) \cup Int_1(\xi)$. By Lemma 9.5, we have $\tilde{\mu}_{coim(\xi)} \circ \coim(\xi) = R_{a_{ker}(\xi)}(\sigma_{ker}(\xi))$. We obtain $\mu_\xi \circ \xi \circ \mu = \mu$ for arbitrary $\mu \in Int_1(\xi)$ from Fig. 25 where we put $j = ker(\xi)$. Analogously, we prove that $\mu_\xi \circ \xi \circ \mu = \mu$ for arbitrary $\mu \in Int_1(\xi)$ by using $\tilde{\mu}_{coim(\xi)} \circ \coim(\xi) = L_{a_{ker}(\xi)}(\sigma_{ker}(\xi))$ in Lemma 9.5.

All that remain is to prove that $\mu \circ \xi \circ \mu_\xi = \mu$ for any $\mu \in Int_1(\xi) \cup Int_1(\xi)$. Note that we have $\xi \circ \mu_\xi = \im(\xi) \circ \tilde{\mu}_{im}(\xi)$ by Lemma 9.5. We prove that $\mu \circ \im(\xi) \circ \tilde{\mu}_{im}(\xi) = \mu$ for arbitrary $\mu \in Int_1(\xi)$. By Lemma 9.5, we have $\im(\xi) \circ \tilde{\mu}_{im}(\xi) = R_{cok}(\sigma_{Cok})$. Then the claim $\mu \circ \im(\xi) \circ \tilde{\mu}_{im}(\xi) = \mu$ follows from Fig. 26. Analogously, we prove that $\mu \circ \im(\xi) \circ \tilde{\mu}_{im}(\xi) = \mu$ for arbitrary $\mu \in Int_1(\xi)$ by using $\im(\xi) \circ \tilde{\mu}_{im}(\xi) = L_{cok}(\sigma_{Cok})$ in Lemma 9.5. It completes the proof.
Corollary 9.10 Let $A, B$ be bimonoids in a symmetric monoidal category $\mathcal{C}$ and $\xi : A \to B$ be a weakly well-decomposable homomorphism. If the homomorphism $\xi$ is weakly pre-Fredholm, then there exists a unique normalized generator integral $\mu_\xi : B \to A$ along $\xi$.

**Proof** The existence follows from Theorem 9.9 and the uniqueness follows from Proposition 7.1.

Corollary 9.11 Suppose that every idempotent in $\mathcal{C}$ is a split idempotent. Let $\xi$ be a well-decomposable bimonoid homomorphism. There exists a normalized generator integral $\mu_\xi$ along $\xi$ if and only if the homomorphism $\xi$ is weakly pre-Fredholm. Note that if a normalized integral exists, then it is unique.

**Proof** It is immediate from Theorems 6.13, 7.6, and Corollary 9.10.

Corollary 9.11 implies Theorem 1.1 by definitions of pre-Fredholmness and Corollary 6.14. Note that (Assumption 0) implies that every idempotent in $\mathcal{C}$ is a split idempotent.

On the other hand, Corollary 9.11 is a strict generalization of Theorem 1.1 as the following example describes.

Example 9.12 Let $G, H$ be (possibly infinite and non-abelian) groups and $\varrho : G \to H$ be a group homomorphism. Put $A = kG, B = kH, \xi = \varrho_\ast$. We give a condition for $\xi$ to have a normalized generator integral. Note that since $G, H$ could not be abelian, $A, B$ are not necessarily bicommutative so that we can not apply Theorem 1.1. If the image of $\varrho$ is a normal subgroup in $H$, then $\xi$ is a well-decomposable bialgebra homomorphism by definitions. The homomorphism $\xi$ is weakly pre-Fredholm if and only if the kernel and
cokernel of \( \varrho \) are finite groups whose orders are coprime to the characteristic of the field \( k \). By Corollary 9.11, it is equivalent with the existence of a normalized generator integral along \( \xi \). Concretely, if such assumptions are satisfied, the normalized generator integral \( \mu_\xi \) is given by 
\[
\mu_\xi (h) = |\text{Ker}(\varrho)|^{-1} \sum_{\varrho(g) = h} g.
\]

10 Commutativity of Homomorphisms and Integrals

In this subsection, we study a commutativity of some bimonoid homomorphisms and integrals. We prove the following theorem in this section.

**Theorem 10.1** Let \( A, B, C, D \) be bimonoids. Consider a commutative diagram (91) of bimonoid homomorphisms. Suppose that the bimonoid homomorphisms \( \varphi, \psi \) are weakly well-decomposable and weakly pre-Fredholm. Recall that there exist normalized generator integrals \( \mu_\varphi, \mu_\psi \) along \( \varphi, \psi \), respectively by Corollary 9.10. If the following conditions hold, then we have \( \mu_\psi \circ \psi' = \varphi' \circ \mu_\varphi \).

(a) The induced bimonoid homomorphism \( \varphi'_0 : \text{Ker}(\varphi) \to \text{Ker}(\psi) \) has a section in \( C \). In other words, there is a morphism \( s : \text{Ker}(\psi) \to \text{Ker}(\varphi) \) in \( C \) such that \( \varphi'_0 \circ s = \text{id} \).

(b) The induced bimonoid homomorphism \( \psi'_0 : \text{Cok}(\varphi) \to \text{Cok}(\psi) \) has a retract in \( C \). In other words, there exists a morphism \( r : \text{Cok}(\psi) \to \text{Cok}(\varphi) \) in \( C \) such that \( r \circ \psi'_0 = \text{id} \).

\[
\begin{align*}
A & \xrightarrow{\varphi'} C \\
\downarrow^\varphi & \downarrow^\psi \\
B & \xrightarrow{\psi'} D
\end{align*}
\]

(91)

**Remark 10.2** It is sufficient that the section \( s \) and the retract \( r \) in Theorem 10.1 are morphisms in \( C \), not bimonoid homomorphisms.

**Remark 10.3** We give a remark about assumptions (a), (b) in Theorem 10.1. Suppose that the symmetric monoidal category \( C \) satisfies (Assumption 0,1,2) in Sect. 15. Consider bicommutative Hopf monoids \( A, B, C, D \) and pre-Fredholm homomorphisms \( \varphi, \psi \). In particular, \( \text{Ker}(\varphi), \text{Ker}(\psi), \text{Cok}(\varphi), \text{Cok}(\psi) \) are small and cosmall. If the induced bimonoid homomorphism \( \varphi'_0 \) is an epimorphism in \( \text{Hopf}^{bc}(C) \), then the assumption (a) is immediate. In fact, the normalized generator integral along the homomorphism \( \varphi'_0 \), which exists due to Corollary 9.10, is a section of \( \varphi'_0 \). See Lemma. Dually, if the induced bimonoid homomorphism \( \psi'_0 \) is a monomorphism in \( \text{Hopf}^{bc}(C) \), then the assumption (b) is immediate. Especially, by (Assumption 2), the conditions (a), (b) are equivalent with an exactness of the induced chain complex below where \( (\varphi, \varphi') = (\varphi \otimes \varphi') \circ \Delta_A \) and \( \psi' - \psi = \nabla_D \circ (\psi' \otimes (S_C \circ \psi)) : \)

\[
\begin{align*}
A & \xrightarrow{(\varphi, \varphi')} B \otimes C \xrightarrow{\psi' - \psi} D
\end{align*}
\]

(92)

**Lemma 10.4** Consider the following commutative diagram of bimonoid homomorphisms. Suppose that \( \varphi, \psi \) are weakly well-decomposable and weakly pre-Fredholm.

\[
\begin{align*}
A & \xrightarrow{\varphi'} C \\
\downarrow^\varphi & \downarrow^\psi \\
B & \xrightarrow{\psi'} D
\end{align*}
\]
Then we have $\psi \circ (\varphi' \circ \mu_\varphi) \circ \varphi = \psi \circ (\mu_\varphi \circ \psi') \circ \varphi$. In particular, if $\varphi$ is an epimorphism in $C$ and $\psi$ is a monomorphism in $C$, then $\varphi' \circ \mu_\varphi = \mu_\psi \circ \psi'$.

**Proof** Since $\mu_\varphi$ is normalized, we have,

$$
\psi \circ \varphi' \circ \mu_\varphi \circ \varphi = \psi' \circ \varphi' \circ \mu_\varphi \circ \varphi
$$

(93)

Since $\mu_\psi$ is normalized, we have

$$
\psi \circ \mu_\psi \circ \psi' \circ \varphi = \psi \circ \mu_\psi \circ \psi' \circ \varphi
$$

(94)

It completes the proof.

**Proof of Theorem 10.1** By Theorem 9.9, the morphisms $\mu_\varphi, \mu_\psi$ in Definition 9.7 are the normalized generator integrals. Note that the homomorphisms in the above diagram are decomposed into the following diagram.

By Lemma 10.4, we have $\varphi'' \circ \tilde{\mu}_{\coim(\varphi)} \circ \tilde{\varphi}^{-1} = \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)} \circ \psi''$. Here, we use the fact that $\coim(\varphi)$ is an epimorphism in $C$ and $\im(\psi)$ is a monomorphism in $C$ by Lemma 9.5.

Thus, we have $\coim(\psi) \circ \varphi' \circ \tilde{\mu}_{\coim(\psi)} \circ \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)} \circ \psi' \circ \im(\psi)$.

We claim that

1. $\tilde{\mu}_{\coim(\psi)} \circ \coim(\psi) \circ \varphi' = \varphi' \circ \tilde{\mu}_{\coim(\psi)}$.
2. $\tilde{\mu}_{\im(\psi)} \circ \psi' \circ \im(\psi) \circ \tilde{\mu}_{\im(\psi)} = \tilde{\mu}_{\im(\psi)} \circ \psi'$.

By these claims, we have

$$
\mu_\psi \circ \psi' = \tilde{\mu}_{\coim(\psi)} \circ \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)} = \tilde{\mu}_{\coim(\psi)} \circ \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)} \circ \psi' \circ \im(\psi) \circ \tilde{\mu}_{\im(\psi)}
$$

(97)

$$
= \tilde{\mu}_{\coim(\psi)} \circ \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)} \circ \psi' \circ \im(\psi) \circ \tilde{\mu}_{\im(\psi)}
$$

(98)

$$
= \tilde{\mu}_{\coim(\psi)} \circ \coim(\psi) \circ \varphi' \circ \tilde{\mu}_{\coim(\psi)} \circ \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)}
$$

(99)

$$
= \varphi' \circ \tilde{\mu}_{\coim(\psi)} \circ \tilde{\varphi}^{-1} \circ \tilde{\mu}_{\im(\psi)}
$$

(100)

$$
= \varphi' \circ \mu_\varphi.
$$

(101)

It suffices to prove the above claims.

From now on, we show the first claim. We use the hypothesis to prove $\varphi' \circ \ker(\varphi) \circ \sigma_{Ker(\varphi)} = ker(\psi) \circ \sigma_{Ker(\psi)}$. Since $\varphi'_0 = \varphi'|_{\ker(\varphi)} : Ker(\varphi) \to Ker(\psi)$ has a section in $C$, we have $\varphi'_0 \circ \sigma_{Ker(\varphi)} = \sigma_{Ker(\psi)}$ by Lemma 11.7. Hence, we obtain $\varphi' \circ \ker(\varphi) \circ \sigma_{Ker(\varphi)} = ker(\psi) \circ \varphi'_0 \circ \sigma_{Ker(\psi)} = \tilde{\ker}(\psi) \circ \sigma_{Ker(\psi)}$.
Recall that $\tilde{\mu}_{coim(\psi)} \circ coim(\psi) : C \to C$ coincides with the action by $ker(\psi) \circ \sigma_{Ker(\psi)} : \mathbb{1} \to C$ by Lemma 9.5. Then Fig. 27 completes the proof of the first claim.

Dually we can prove the second claim. Here, we use the section of $\psi' : Cok(\varphi) \to Cok(\psi)$ and apply Lemma 11.7 again. It completes the proof. $\square$

11 Inverse Volume

11.1 Inverse Volume of Bimonoid

In this subsection, we introduce a notion of inverse volume $vol^{-1}(A)$ of a bimonoid $A$ with a normalized integral and a normalized cointegral. It gives an invariant of such bimonoids by Proposition 11.4. By Remark 6.9, it defines an invariant of bismall bimonoids.

**Definition 11.1** Let $A$ be a bimonoid with a normalized integral $\sigma_A : \mathbb{1} \to A$ and a normalized cointegral $\sigma_A^\mathbb{1} : A \to \mathbb{1}$. An inverse volume of the bimonoid $A$ is an endomorphism $vol^{-1}(A) : \mathbb{1} \to \mathbb{1}$ in $\mathcal{C}$, defined by a composition,

$$vol^{-1}(A) \overset{\text{def.}}{=} \sigma_A^\mathbb{1} \circ \sigma_A. \quad (102)$$

**Definition 11.2** A bimonoid $A$ has a finite volume if $A$ has a normalized integral and a normalized cointegral, and its inverse volume $vol^{-1}(A) : \mathbb{1} \to \mathbb{1}$ is invertible.

**Example 11.3** Consider the symmetric monoidal category, $\mathcal{C} = \text{Vec}_k^\otimes$. Let $G$ be a finite group. Suppose that the characteristic of $k$ is not a divisor of the order $\# G$ of $G$. Then the induced Hopf monoid $A = kG$ in $\text{Vec}_k^\otimes$ has a normalized integral $\sigma_A$ and a normalized cointegral $\sigma_A^\mathbb{1}$. In particular,

$$\sigma_A : k \to kG ; \ 1 \mapsto (\# G)^{-1} \sum_{g \in G} g, \quad (103)$$

$$\sigma_A^\mathbb{1} : kG \to k ; \ g \mapsto \delta_e(g), \quad (104)$$

give a normalized integral and a normalized cointegral of $A = kG$, respectively. Then we have

$$vol^{-1}(k(G)) : k \to k ; \ 1 \mapsto (\# G)^{-1}. \quad (105)$$

**Proposition 11.4** Let $A, B$ be bimonoids with a normalized integral and a normalized cointegral.
• For the unit bimonoid, we have \( \text{vol}^{-1}(1) = id_1 \).

• A bimonoid isomorphism \( A \cong B \) implies \( \text{vol}^{-1}(A) = \text{vol}^{-1}(B) \).

• \( \text{vol}^{-1}(A \otimes B) = \text{vol}^{-1}(A) \circ \text{vol}^{-1}(B) = \text{vol}^{-1}(B) \circ \text{vol}^{-1}(A) \).

• If \( A^\vee \) is a dual bimonoid of the bimonoid \( A \), then the bimonoid \( A^\vee \) has a normalized integral and a normalized cointegral and we have

\[
\text{vol}^{-1}(A^\vee) = \text{vol}^{-1}(A).
\]

\[(106)\]

**Proof** Since \( \sigma_1 = \sigma^1 = id_1 \), we have \( \text{vol}^{-1}(1) = id_1 \).

If \( A \cong B \) as bimonoids, then their normalized (co)integrals coincide via that isomorphism due to their uniqueness. Hence, we have \( \text{vol}^{-1}(A) = \sigma^A \circ \sigma_A = \sigma^B \circ \sigma_B = \text{vol}^{-1}(B) \).

Since \( \sigma_{A \otimes B} = \sigma_A \otimes \sigma_B : 1 \rightarrow A \otimes B \) and \( \sigma^{A \otimes B} : A^\otimes A \rightarrow 1 \), we have \( \text{vol}^{-1}(A \otimes B) = \text{vol}^{-1}(A) \ast \text{vol}^{-1}(B) = \text{vol}^{-1}(A) \circ \text{vol}^{-1}(B) = \text{vol}^{-1}(B) \circ \text{vol}^{-1}(A) \).

By direct calculations, the following morphisms give a normalized integral and a normalized cointegral on the dual bimonoid \( A^\vee \):

\[
\sigma^A_{A^\vee} = \left(1^\text{cove}_{A^\vee} \rightarrow A^\vee \otimes A \xrightarrow{id_{A^\vee} \otimes \sigma_A} A^\vee \otimes 1 \cong A^\vee\right)
\]

\[
\sigma_{A^\vee}^A = \left(A^\vee \cong 1 \otimes A^\vee \xrightarrow{\sigma_{A^\vee} \otimes id_{A^\vee}} A \otimes A^\vee \xrightarrow{ev_A} 1\right)
\]

It implies that \( \sigma^A_{A^\vee} \circ \sigma_{A^\vee} = \sigma^A \circ \sigma_A \) since \( 1_A \circ (ev_A \otimes id_A) \circ (id_A \otimes cove_A) \circ \tau_A = id_A \)

where \( 1_A \) denotes the left unitor.

\( \square \)

### 11.2 Inverse Volume of Homomorphisms

**Definition 11.5** Let \( A \) be a bimonoid with a normalized integral \( \sigma_A \) and \( B \) be a bimonoid with a normalized cointegral \( \sigma^B \). For a bimonoid homomorphism \( \xi : A \rightarrow B \), we define a morphism \( \langle \xi \rangle : 1 \rightarrow 1 \) by

\[
\langle \xi \rangle \overset{\text{def}}{=} \sigma^B \circ \xi \circ \sigma_A.
\]

\[(109)\]

**Remark 11.6** Since \( \langle id_A \rangle = \text{vol}^{-1}(A) \) by definitions, \( \langle - \rangle \) is an extended notion of the inverse volume in Definition 11.1. On the other hand, for some special \( \xi \), we can compute \( \langle \xi \rangle \) from an inverse volume. See Proposition 11.9.

**Lemma 11.7** Let \( A, B \) be bimonoids. Let \( \sigma_A \) be a normalized integral of \( A \). Let \( \xi : A \rightarrow B \) be a bimonoid homomorphism. If there exists a morphism \( \xi' : B \rightarrow A \) in \( C \) such that \( \xi \circ \xi' = id_A \), then \( \xi \circ \sigma_A \) is a normalized integral of \( B \).

**Proof** The morphism \( \xi \circ \sigma_A : 1 \rightarrow B \) is a right integral due to Fig. 28. It can be verified to be a left integral in a similar way. Moreover, it is normalized since we have \( \epsilon_\xi \circ \xi \circ \sigma_A = \epsilon_A \circ \sigma_A = id_\xi \).

\( \square \)

**Proposition 11.8** Let \( \xi : A \rightarrow B \) be a bimonoid homomorphism. Suppose that every idempotent in the symmetric monoidal category \( C \) is a split idempotent. If the bimonoid \( A \) is small and there exists a morphism \( \xi' : B \rightarrow A \) in \( C \) such that \( \xi \circ \xi' = id_A \), then the bimonoid \( B \) is small.

**Proof** It is immediate from Lemma 11.7 to Theorem 6.13.
Proposition 11.9  Let \( \xi : A \to B \) be a bimonoid homomorphism. Suppose that a kernel bimonoid \( \text{Ker}(\xi) \), a cokernel bimonoid \( \text{Cok}(\xi) \), a coimage bimonoid \( \text{Coim}(\xi) \), an image bimonoid \( \text{Im}(\xi) \) exist. Suppose that \( \text{Ker}(\xi) \) is small and \( \text{Cok}(\xi) \) is cosmall. Suppose that the canonical homomorphism \( \ker(\xi) : \text{Ker}(\xi) \to A \) is normal and \( \cok(\xi) : B \to \text{Cok}(\xi) \) is conormal. Then for the canonical homomorphism \( \bar{\xi} : \text{Coim}(\xi) \to \text{Im}(\xi) \), we have,

\[
\langle \xi \rangle = \langle \bar{\xi} \rangle.
\]

In particular, if \( \bar{\xi} \) is an isomorphism, then we have \( \langle \xi \rangle = \langle \bar{\xi} \rangle = \text{vol}^{-1}(\text{Coim}(\xi)) = \text{vol}^{-1}(\text{Im}(\xi)). \)

Proof  It suffices to prove that \( \langle \xi \rangle = \langle \bar{\xi} \rangle \). Since \( \langle \xi \rangle = \sigma_B \circ \xi \circ \sigma_A = \sigma_B \circ \text{im}(\xi) \circ \bar{\xi} \circ \text{coim}(\xi) \circ \sigma_A \), it suffices to show that \( \text{coim}(\xi) \circ \sigma_A = \sigma_{\text{Coim}(\xi)} \) and \( \sigma_B \circ \text{im}(\xi) = \sigma_{\text{Im}(\xi)} \).

The morphism \( \text{coim}(\xi)(\text{im}(\xi), \text{resp.)} \) has a section (retract, resp.) in \( C \) by Lemma 9.5. Hence, the compositions \( \text{coim}(\xi) \circ \sigma_A \) (\( \sigma_B \circ \text{im}(\xi) \), resp.) are normalized integrals by Lemma 11.7. It completes the proof. \( \square \)

12 Commutativity of Integrals

In this section, we discuss a relation between two composable integrals and their composition. Recall that integrals are preserved under compositions by Proposition 3.10. Nevertheless, such a composition does not preserve normalized integrals. By considering normalized generator integrals rather than normalized integrals, one can deduce that they are preserved up to a scalar. Here, a scalar formally means an endomorphism on the unit object \( \mathbb{1} \).

Theorem 12.1  Let \( A, B, C \) be bimonoids. Let \( \xi : A \to B, \xi' : B \to C \) be bimonoid homomorphisms. Suppose that

- \( \xi \) is normal, \( \xi' \) is conormal. The composition \( \xi' \circ \xi \) is either conormal or normal.
- \( \mu, \mu' \) are normalized integrals along \( \xi, \xi' \), respectively. \( \mu'' \) is a normalized integral along \( \xi' \circ \xi \), which is a generator.

Recall that the cokernel bimonoid \( \text{Cok}(\xi) \) has a normalized cointegral and the kernel bimonoid \( \text{Ker}(\xi') \) has a normalized integral by Theorem 7.5. Then we have,

\[
\mu \circ \mu' = \langle \text{cok}(\xi) \circ \text{ker}(\xi') \rangle \cdot \mu''.
\]

Proof  By Proposition 3.10, \( \mu \circ \mu' \) is an integral along the composition \( \xi' \circ \xi \). By Theorem 8.5, there exists a unique \( \lambda \in \text{End}_C(\mathbb{1}) \) such that \( \mu \circ \mu' = \lambda \cdot \mu'' \) since \( \xi' \circ \xi \) is either conormal or normal.
We have $\epsilon_A \circ \mu'' \circ \eta_C = id_1$ due to the following computation:

\[
\epsilon_A \circ \mu'' \circ \eta_C = (\epsilon_C \circ \xi' \circ \xi) \circ \mu'' \circ (\xi' \circ \xi_A) = \epsilon_C \circ (\xi' \circ \mu'' \circ \xi) \circ \eta_A = \epsilon_C \circ (\xi' \circ \xi) \circ \eta_A \quad \text{(normalized)}
\]

Hence it suffices to calculate $\epsilon_A \circ \mu \circ \mu' \circ \eta_C$ to know $\lambda$. Since $\xi'$ is conormal, we have a morphism $\tilde{F}(\mu')$ such that $\mu' \circ \eta_C = \ker(\xi') \circ \tilde{F}(\mu')$ (see Definition 7.4). Since $\xi$ is normal, we have a morphism $\tilde{F}(\mu)$ such that $\epsilon_A \circ \mu = \tilde{F}(\mu) \circ \cok(\xi)$. Since the integrals $\mu, \mu'$ are normalized, $\tilde{F}(\mu')$ and $\tilde{F}(\mu)$ are normalized integrals by Theorem 7.5. By using our notations, $\tilde{F}(\mu') = \sigma_{Ker(\xi')}$ and $\tilde{F}(\mu) = \sigma_{Cok(\xi)}$. Therefore, we have $\epsilon_A \circ \mu \circ \mu' \circ \eta_C = \langle \cok(\xi) \circ \ker(\xi') \circ \sigma_{Ker(\xi')} \rangle$ by definitions. It completes the proof.

\textbf{Corollary 12.2} Let $A, B$ be bimonoids and $\xi : A \rightarrow B$ be a bimonoid homomorphism. Suppose that

- $\xi$ is normal.
- $\mu$ is a normalized integral along $\xi$, $\sigma_B$ is a normalized integral of $B$, and $\sigma_A$ is a normalized integral of $A$ which is a generator.

Then we have

\[
\mu \circ \sigma_B = \text{vol}^{-1}(\text{Cok}(\xi)) \cdot \sigma_A.
\]

We have an analogous statement. Suppose that

- $\xi$ is conormal.
- $\mu$ is a normalized integral along $\xi$, $\sigma_A$ is a normalized cointegral of $A$, and $\sigma_B$ is a normalized cointegral of $B$ which is a generator.

Then we have

\[
\sigma_A \circ \mu = \text{vol}^{-1}(\text{Ker}(\xi)) \cdot \sigma_B.
\]

\textbf{Proof} We prove the first claim. We replace $\xi, \xi'$ in Theorem 12.1 with $\xi, \epsilon_B$ in the above assumption. Then the assumption in Theorem 12.1 is satisfied.

We prove the second claim. We replace $\xi, \xi'$ in Theorem 12.1 with $\eta_A, \xi$ in the above assumption. Then the assumption in Theorem 12.1 is satisfied.

\textbf{Corollary 12.3} Let $A, B$ be bimonoids and $\xi : A \rightarrow B$ be a bimonoid homomorphism. Suppose that

- $\xi$ is binormal.
- There exists a normalized integral along $\xi$.
- $A, B$ are bismall.
- The normalized integral $\sigma_A$ of $A$ is a generator. The normalized cointegral $\sigma_B$ of $B$ is a generator.

Then we have

\[
\text{vol}^{-1}(\text{Cok}(\xi)) \circ \text{vol}^{-1}(A) = \text{vol}^{-1}(\text{Ker}(\xi)) \circ \text{vol}^{-1}(B).
\]
Proof Since $A$, $B$ are bismall, the counit $\epsilon_A$ and the unit $\eta_B$ are pre-Fredholm. Since the counit $\epsilon_A$ and the unit $\eta_B$ are well-decomposable, the normalized integral $\sigma_A$ of $A$ and the normalized cointegral $\sigma_B$ of $B$ are generators by Theorem 9.9. Hence, the assumptions in Corollary 12.2 are satisfied. By Corollary 12.2, we obtain

$$\mu_{\xi} \circ \sigma_B = vol^{-1}(Cok(\xi)) \cdot \sigma_A,$$

(119)

$$\sigma^A \circ \mu_{\xi} = vol^{-1}(Ker(\xi)) \cdot \sigma^B \circ \sigma_B,$$

(120)

Hence, we obtain $vol^{-1}(Cok(\xi)) \cdot \sigma_A \circ \sigma_A = vol^{-1}(Ker(\xi)) \cdot \sigma_B \circ \sigma_B$, which is equivalent with (118).

$\square$

Corollary 12.4 Let $A, B, C$ be bimonoids. Let $\xi : A \to B$, $\xi' : B \to C$ be bimonoid homomorphisms. Suppose that the homomorphisms $\xi, \xi', \xi' \circ \xi$ are well-decomposable and weakly pre-Fredholm. Recall that there exist normalized generator integrals $\mu_{\xi}, \mu_{\xi'}, \mu_{\xi' \circ \xi}$ along the bimonoid homomorphisms $\xi, \xi', \xi' \circ \xi$, respectively by Corollary 9.10. Then there exists a unique $\lambda \in End_C(\mathbb{1})$ such that

$$\mu_{\xi} \circ \mu_{\xi'} = \lambda \cdot \mu_{\xi' \circ \xi}.$$  

(121)

Proof It is a corollary of Theorem 12.1. Since $\xi, \xi', \xi' \circ \xi$ are well-decomposable, in particular weakly well-decomposable, and weakly pre-Fredholm, we obtain normalized generator integrals $\mu_{\xi}, \mu_{\xi'}, \mu_{\xi' \circ \xi}$ by Theorem 9.9. Since $\xi, \xi', \xi' \circ \xi$ are well-decomposable, they satisfy the first assumption in Theorem 12.1. By Theorem Theorem 9.9, the integrals $\mu = \mu_{\xi}, \mu' = \mu_{\xi'}, \mu'' = \mu_{\xi' \circ \xi}$ satisfy the second assumption in Theorem 12.1.

$\square$

The existence follows from Theorem 12.1 and the uniqueness follows from Theorem 8.5.

In fact, the endomorphism $\lambda$ coincides with $\langle cok(\xi) \circ ker(\xi') \rangle \in End_C(\mathbb{1})$. The symbol $\langle \cdot \rangle$ represents an invariant of bimonoid homomorphisms from a bimonoid with a normalized integral to a bimonoid with a normalized cointegral (see Definition 11.5). In Corollary 12.4, the kernel bimonoid $Ker(\xi')$ has a normalized integral and the cokernel bimonoid $Cok(\xi)$ has a normalized cointegral since we assume that $\xi, \xi'$ are weakly pre-Fredholm.

13 Induced Bismallness

In this section, we assume that every idempotent in a symmetric monoidal category $C$ is a split idempotent.

13.1 Bismallness of (co)kernels

In this subsection, we give some conditions where $Ker(\xi), Cok(\xi)$ inherits a (co)smallness from that of the domain and the target of $\xi$.

Proposition 13.1 Let $\xi : A \to B$ be a bimonoid homomorphism. Suppose that $A$ is small, $B$ is cosmall. If $\xi$ is normal, then $Cok(\xi)$ is cosmall. If $\xi$ is conormal, then $Ker(\xi)$ is small.

Proof We only prove the first claim. Let $\xi$ be normal. We have $Cok(\xi) = a^{-\xi} \setminus B$. There exists a normalized cointegral of $B$ since $B$ is cosmall by Corollary 6.14. We denote it by $\sigma^B : B \to \mathbb{1}$. Put $\sigma = \sigma^B \circ \mu_{cok(\xi)} : Cok(\xi) = a^{-\xi} \setminus B \to \mathbb{1}$. Note that $\sigma \in Int_r(\eta_\sigma a^{-\xi} \setminus B)$ due to Proposition 3.10. In other words, $\sigma$ is a right cointegral of $Cok(\xi) = a^{-\xi} \setminus B$.

$\square$
We prove that $\sigma$ is normalized. Let $\pi : B \to \alpha_\xi^-\setminus B$ be the canonical morphism. We have 
$\sigma \circ \eta_{\alpha_\xi^-\setminus B} = \sigma_B \circ \mu_{cok(\xi)} \circ \eta_{\alpha_\xi^-\setminus B} = \sigma_B \circ \tilde{\mu}_{cok(\xi)} \circ \pi \circ \eta_B$. $\sigma \circ \eta_{\alpha_\xi^-\setminus B} = id_B$ follows from 
$\tilde{\mu}_{cok(\xi)} \circ \pi = \lambda_{\alpha_\xi^-}(\sigma_A)$ in Lemma 9.1 (1), and $\epsilon_A \circ \sigma_A = id_A$. Hence, $\sigma$ is a normalized right cointegral of $\alpha_\xi^-\setminus B = Cok(\xi)$.

Analogously, we use $\text{Cok}(\xi) = B/\alpha_\xi^-\setminus \xi$ to verify an existence of a normalized left cointegral of $\text{Cok}(\xi)$. By Proposition 3.3, the cokernel $\text{Cok}(\xi)$ has a normalized cointegral. By Corollary 6.14, the cokernel bimonoid $\text{Cok}(\xi)$ is cosmall.

Proposition 13.2 Let $A$, $B$ be bimonoids. Let $\xi : A \to B$ be a bimonoid homomorphism. If $A$, $B$ are small and $\xi$ is normal, then $\text{Cok}(\xi)$ is small. If $A$, $B$ are cosmall and $\xi$ is conormal, then $\text{Ker}(\xi)$ is cosmall.

Proof We only prove the first claim. The small bimonoid $B$ has a unique normalized integral $\sigma_B : 1 \to B$ by Corollary 6.14. By Definition 9.2, a normalized integral $\tilde{\mu}_{cok(\xi)} \in \text{Int}(\text{Cok}(\xi))$ exists. By Lemma 9.3, $\tilde{\mu}_{cok(\xi)}$ is a section of $cok(\xi)$ in $C$. By Lemma 11.7, $cok(\xi) \circ \sigma_B$ is a normalized integral of $\text{Cok}(\xi)$. By Corollary 6.14, $\text{Cok}(\xi)$ is small.

Corollary 13.3 Let $A$, $B$ be bimonoids. Let $\xi : A \to B$ be a well-decomposable homomorphism. If $A$ is small and $B$ is cosmall, then the homomorphism $\xi$ is weakly pre-Fredholm. If both of $A$, $B$ are bismall, then the homomorphism $\xi$ is pre-Fredholm.

Proof Suppose that $A$ is a small bimonoid and $B$ is a cosmall bimonoid. Since $\xi$ is well-decomposable, the cokernel bimonoid $\text{Cok}(\xi)$ is cosmall and the kernel bimonoid $\text{Ker}(\xi)$ is small by Proposition 13.1.

Suppose that both of $A$, $B$ are bismall bimonoids. Then the homomorphism $\xi$ is weakly pre-Fredholm by the above discussion. Moreover, the cokernel bimonoid $\text{Cok}(\xi)$ is small and kernel bimonoid $\text{Ker}(\xi)$ is cosmall by Proposition 13.2.

13.2 Bismallness of Bimonoids in Exact Sequences

In this subsection, we discuss some conditions for (co)smallness of a bimonoid to be inherited from an exact sequence.

Lemma 13.4 Let $A$, $B$, $C$ be bimonoids. Let $\iota : B \to A$ be a normal homomorphism and $\pi : A \to C$ be a homomorphism. Suppose that the following sequence is exact:

$$
B \xrightarrow{\iota} A \xrightarrow{\pi} C \to 1
$$

(122)

Here, the exactness means that $\pi \circ \iota$ is trivial and the induced homomorphism $\text{Cok}(\iota) \to C$ is an isomorphism. If the bimonoids $B$, $C$ are small, then $A$ is small.

Proof It suffices to prove that $A$ has a normalized integral by Corollary 6.14. We denote by $\sigma_C$ the normalized integral of $C$. Since $B$ is small and $\iota$ is normal, we have a normalized integral $\tilde{\mu}_{cok(\iota)}$ along $cok(\iota)$ (see Definition 9.2). Since the induced homomorphism $\text{Cok}(\iota) \to C$ is isomorphism by the assumption, we have a normalized integral $\tilde{\mu}_{\pi}$ along $\pi$. Then the composition $\tilde{\mu}_{\pi} \circ \sigma_C : 1 \to A$ gives an integral of $A$ by Proposition 3.10. Moreover $\tilde{\mu}_{\pi} \circ \sigma_C$ is normalized since $\epsilon_A \circ \tilde{\mu}_{\pi} \circ \sigma_C = \epsilon_C \circ \pi \circ \tilde{\mu}_{\pi} \circ \sigma_C = \epsilon_C \circ \sigma_C = id_C$ by Lemma 9.1. It completes the proof.
Proposition 13.5 Let $A$, $B$, $C$, $C'$ be bimonoids. Let $\iota : B \to A$ be a normal homomorphism, $\pi' : C \to C'$ be a conormal homomorphism and $\pi : A \to C$ be a homomorphism. Suppose that the following sequence is exact:

$$B \xrightarrow{\iota} A \xrightarrow{\pi} C \xrightarrow{\pi'} C' \quad (123)$$

Suppose that $\text{Cok}(\iota) \to \text{Ker}(\pi')$ is an isomorphism. If the bimonoids $B$, $C$ are small and the bimonoid $C'$ is cosmall, then the bimonoid $A$ is small.

**Proof** By the assumption, we obtain an exact sequence in the sense of Lemma 13.4,

$$B \xrightarrow{\iota} A \xrightarrow{\pi} \text{Ker}(\pi') \to 1 \quad (124)$$

Note that $\text{Ker}(\pi')$ is small by Proposition 13.1. Since $\iota$ is normal and $B$, $\text{Ker}(\pi')$ are small, the bimonoid $A$ is small due to Lemma 13.4. $\square$

We have dual statements as follows. For convenience of the readers, we give them without proof.

Lemma 13.6 Let $A$, $B$, $C$ be bimonoids. Let $\iota : B \to A$ be a homomorphism and $\pi : A \to C$ be a conormal homomorphism. Suppose that the following sequence is exact.

$$1 \to B \xrightarrow{\iota} A \xrightarrow{\pi} C \quad (125)$$

Here, the exactness means that $\pi \circ \iota$ is trivial and the induced morphism $B \to \text{Ker}(\xi)$ is an isomorphism. If $\pi$ is conormal and the bimonoids $B$, $C$ are cosmall, then $A$ is cosmall.

**Proposition 13.7** Let $A$, $B$, $B'$, $C$ be bimonoids. Let $\iota' : B' \to B$ be a normal homomorphism, $\pi : A \to C$ be a conormal homomorphism, and $\iota : B \to A$ be a homomorphism. Suppose that the following sequence is exact.

$$B' \xrightarrow{\iota'} B \xrightarrow{\iota} A \xrightarrow{\pi} C \quad (126)$$

Suppose that $\text{Cok}(\iota') \to \text{Ker}(\pi)$ is an isomorphism. If the bimonoid $B'$ are small and the bimonoids $B$, $C$ is cosmall, then the bimonoid $A$ is cosmall.

**Theorem 13.8** Let $A$, $C$ be bismall bicommutative Hopf monoids in $C$ and $B$ be an arbitrary bicommutative bimonoid. If there exists an exact sequence $1 \to A \to B \to C \to 1$ of bicommutative Hopf monoids, then $B$ is bismall.

**Proof** Consider an exact sequence in $\text{Hopf}^{\text{bc}}(C)$ where $B' = 1 = C'$.

$$B' \xrightarrow{\iota'} B \xrightarrow{\iota} A \xrightarrow{\pi} C \xrightarrow{\pi'} C' \quad (127)$$

By Proposition 5.8, any morphism in $\text{Hopf}^{\text{bc}}(C)$ is binormal. By Corollary 5.9, a cokernel (kernel, resp.) as a bimonoid is a cokernel (kernel, resp.) as a bicommutative Hopf monoid. Hence, the assumptions in Propositions 13.5, 13.7 are deduced from the assumption in the statement. By Propositions 13.5, 13.7, we obtain the result. $\square$

14 Volume on Abelian Category

In this section, we introduce and study a notion of volume on an abelian category.
14.1 Basic Properties

**Definition 14.1** For an abelian monoid $M^2$, an $M$-valued volume on the abelian category $\mathcal{A}$ is an assignment of $v(A) \in M$ to an object $A$ of $\mathcal{A}$ which satisfies

1. For a zero object $0$ of $\mathcal{A}$, the corresponding element $v(0) \in M$ is the unit $1$ of the abelian monoid $M$.
2. For an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have $v(B) = v(A) \cdot v(C)$.

**Proposition 14.2** An $M$-valued volume $v$ on an abelian category $\mathcal{A}$ is an isomorphism invariant. In other words, if objects $A, B$ of $\mathcal{A}$ are isomorphic to each other, then we have $v(A) = v(B)$.

**Proof** If we choose an isomorphism between $A$ and $B$, then we obtain an exact sequence $0 \to A \to B \to 0$. By the second axiom in Definition 14.1, we obtain $v(B) = v(A) \cdot v(0)$. Since $v(0) = 1$ by the first axiom in Definition 14.1, we obtain $v(A) = v(B)$. \qed

**Proposition 14.3** An $M$-valued volume $v$ on an abelian category $\mathcal{A}$ is compatible with the direct sum $\oplus$ on the abelian category $\mathcal{A}$. In other words, for objects $A, B$ of $\mathcal{A}$, we have $v(A \oplus B) = v(A) \cdot v(B)$.

**Proof** Note that we have an exact sequence $0 \to A \to A \oplus B \to B \to 0$. By the second axiom in Definition 14.1, we obtain $v(A \oplus B) = v(A) \cdot v(B)$. \qed

14.2 Fredholm Index

In this subsection, we introduce a notion of index of morphisms in an abelian category. By regarding objects of $\mathcal{A}$ with invertible volume as “finite-dimensional objects”, we define a notion of Fredholm morphisms in $\mathcal{B}$ and its index which is an invariant respecting compositions and robust to finite perturbations (see Definition 14.5). It generalizes the Fredholm index of Fredholm operator in the algebraic sense.

**Definition 14.4** Let $\mathcal{B}$ be an abelian category and $\mathcal{A}$ be a abelian subcategory. The abelian subcategory $\mathcal{A}$ is closed under short exact sequences if $A, C$ are objects of $\mathcal{A}$ and $B$ is an object of $\mathcal{B}$ for a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{B}$, then $B$ is an object of $\mathcal{A}$.

**Definition 14.5** Let $\mathcal{B}$ be an abelian category and $\mathcal{A}$ be its abelian subcategory closed under short exact sequences. Let $M$ be an abelian monoid and $v$ be an $M$-valued volume on $\mathcal{A}$. For two objects $A, B$ of $\mathcal{B}$, a morphism $f : A \to B$ is Fredholm with respect to the volume $v$ if $\text{Ker}(f)$ and $\text{Cok}(f)$ are essentially objects of $\mathcal{A}$ and the volumes $v(\text{Ker}(f)), v(\text{Cok}(f)) \in M$ are invertible. For a Fredholm morphism $f : A \to B$, we define its Fredholm index by

$$\text{Ind}_{\mathcal{B}, \mathcal{A}, v}(f) \overset{\text{def.}}{=} v(\text{Cok}(f)) \cdot v(\text{Ker}(f))^{-1} \in M.$$  \hspace{1cm} (128)

**Lemma 14.6** Let $A$ be an object of $\mathcal{B}$. The identity $\text{Id}_A$ on $A$ is Fredholm. We have $\text{Ind}_{\mathcal{B}, \mathcal{A}, v}(\text{Id}_A) = 1 \in M$.

**Proof** It follows from the fact that $\text{Ker}(\text{Id}_A) = 0 = \text{Cok}(\text{Id}_A)$ whose volume is the unit $1 \in M$. \qed

---

\(^2\) The reason that we consider a monoid $M$, not a group is that we deal with infinite dimension or infinite order uniformly.
**Lemma 14.7** Let \( f : A \to B \) and \( g : B \to C \) be morphisms in \( \mathcal{B} \). If the morphisms \( f, g \) are Fredholm, then the composition \( g \circ f \) is Fredholm. We have \( \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(g \circ f) = \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(g) \cdot \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(f) \in M. \)

**Proof** We use the exact sequence \( 0 \to \operatorname{Ker}(f) \to \operatorname{Ker}(g \circ f) \to \operatorname{Ker}(g) \to \operatorname{Cok}(f) \to \operatorname{Cok}(g \circ f) \to \operatorname{Cok}(g) \to 0. \) Since \( v(\operatorname{Ker}(g)) \) is invertible, any subobject of \( \operatorname{Ker}(g) \) has an invertible volume. The volume \( v(\operatorname{Ker}(g \circ f)) \) is invertible. By the induced exact sequence \( 0 \to \operatorname{Ker}(f) \to \operatorname{Ker}(g \circ f) \to \operatorname{Ker}(g) \to 0, \) we see that \( v(\operatorname{Ker}(g \circ f)) \in M \) is invertible. Likewise, \( v(\operatorname{Cok}(g \circ f)) \) is invertible. Hence, the composition \( g \circ f \) is Fredholm with respect to the volume \( v \). By repeating the second axiom of volumes in Definition 14.1, we obtain

\[
v(\operatorname{Ker}(f)) \cdot v(\operatorname{Ker}(g)) \cdot v(\operatorname{Cok}(g \circ f)) = v(\operatorname{Ker}(g \circ f)) \cdot v(\operatorname{Cok}(f)) \cdot v(\operatorname{Cok}(g)).
\]

It proves that \( \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(g \circ f) = \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(g) \cdot \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(f) \in M. \)

**Definition 14.8** Let \( \mathcal{B} \) be an abelian category and \( \mathcal{A} \) be an abelian subcategory which is closed under short exact sequences. Let \( v \) be an \( M \)-valued volume on \( \mathcal{A} \). We define a category \( \mathcal{A}^\text{Fr} \) as a subcategory of \( \mathcal{B} \) formed by all Fredholm homomorphisms. It is a well-defined category due to Lemmas 14.6, 14.7.

**Proposition 14.9** Every morphism \( f : A \to B \) between objects with invertible volumes is Fredholm. Then we have

\[
\operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(f) = v(B) \circ v(A)^{-1} \in M.
\]

**Proof** If objects \( A, B \) of \( \mathcal{A} \) have invertible volumes, then for a morphism \( f : A \to B \) its kernel and cokernel have invertible volumes due to the second axiom in Definition 14.1.

By the exact sequence \( 0 \to \operatorname{Ker}(f) \to A \xrightarrow{f} B \to \operatorname{Cok}(f) \to 0, \) we have \( v(B) \cdot v(\operatorname{Ker}(f)) = v(A) \cdot v(\operatorname{Cok}(f)). \) We obtain \( \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(f) = v(B) \circ v(A)^{-1}. \)

**14.3 Finite Perturbation**

Consider an abelian category \( \mathcal{B} \) and its abelian subcategory \( \mathcal{A} \) closed under short exact sequences. Let \( v \) be an \( M \)-valued volume on the abelian category \( \mathcal{A}, \) not necessarily on \( \mathcal{B} \) where \( M \) is an abelian monoid.

**Definition 14.10** Let \( f \) be a morphism in \( \mathcal{B}. \) A morphism \( f \) in \( \mathcal{B} \) is finite with respect to the volume \( v \) if the value of the image of \( f \) (equivalently, the coimage of \( f \)) by \( v \) is invertible in \( M. \) In other words, the image \( \operatorname{Im}(f) \) is essentially an object of \( \mathcal{A} \) and the volume \( v(\operatorname{Im}(f)) \in M \) is invertible.

**Proposition 14.11** (Invariance of index under finite perturbations) Let \( f, k : A \to B \) be morphisms in \( \mathcal{B}. \) If the morphism \( f \) is Fredholm and the morphism \( k \) is finite with respect to the volume \( v \), then the morphism \( (f + k) : A \to B \) is Fredholm with respect to the volume \( v. \) Moreover, we have

\[
\operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(f + k) = \operatorname{Ind}_{\mathcal{B}, \mathcal{A}, v}(f) \in M.
\]
**Proof** Denote by $C$ the (co)image of the morphism $k : A \to B$. Note that $(f + k)$ is decomposed into following morphisms:

\begin{equation}
A \xrightarrow{(id_A \oplus \text{coim}(k)) \circ \Delta_A} A \oplus C \xrightarrow{f \oplus id_C} B \oplus C \xrightarrow{\nabla_B \circ (id_B \oplus \text{im}(k))} B.
\end{equation}

(132)

Since the volume $v(C) \in M$ is invertible, the morphisms $(id_A \oplus \text{coim}(k)) \circ \Delta_A$ and $\nabla_B \circ (id_B \oplus \text{im}(k))$ are Fredholm with respect to the volume $v$. Since the morphism $f$ is Fredholm with respect to the volume $v$, so the morphism $f \oplus id_C$ is. By Lemma 14.7, $(f + k)$ is Fredholm and,

\begin{equation}
\text{Ind}_{B,A,v}(f + k) = \text{Ind}_{B,A,v}(\nabla_B \circ (id_B \oplus \text{im}(k))) \cdot \text{Ind}_{B,A,v}(f \oplus id_C) \cdot \text{Ind}_{B,A,v}((id_A \oplus \text{coim}(k)) \circ \Delta_A).
\end{equation}

(133)

Note that $\text{Ind}_{B,A,v}(f \oplus id_C) = \text{Ind}_{B,A,v}(f)$. Moreover we have $\text{Ind}_{B,A,v}(\nabla_B \circ (id_B \oplus \text{im}(k))) \cdot \text{Ind}_{B,A,v}((id_A \oplus \text{coim}(k)) \circ \Delta_A) = v(C)^{-1} \cdot v(C) = 1$ by definitions. It completes the proof.

\[\square\]

**15 Applications to the Category Hopf^{bc}(C)**

In this section, we give an application of the previous results to the category of bicommutative Hopf monoids $\text{Hopf}^{bc}(C)$. We show that the inverse volume gives a volume on some abelian category. From now on, we consider the following assumptions on $C$.

- (Assumption 0) The category $C$ has any equalizer and coequalizer.
- (Assumption 1) The monoidal structure of $C$ is bistable.
- (Assumption 2) The category $\text{Hopf}^{bc}(C)$ is an abelian category.

Here, (co, bi)stability of the monoidal structure of $C$ is introduced in Definition 4.4. See Definition 11.1 for the definition of inverse volume.

**Example 15.1** Note that the assumptions on $C$ automatically hold if $C = \text{Vec}_k^\oplus$, the category of (possibly, infinite-dimensional) vector spaces over a field $k$. The (Assumption 1) follows from Proposition 4.6 (see Example 4.7). The (Assumption 2) follows from the fact that $\text{Hopf}^{bc}(\text{Vec}_k^\oplus)$ is an abelian category by Corollary 4.16 in [21] or Theorem 4.3 in [22].

**Remark 15.2** We give a remark about a relationship between the assumptions. (Assumption 0,1) implies that the category $\text{Hopf}^{bc}(C)$ is an pre-abelian category i.e. an additive category with arbitrary kernel and cokernel. Under (Assumption 0,1), (Assumption 2) is equivalent with the fundamental theorem on homomorphisms.

**Remark 15.3** We need those (Assumption 0,1,2) because we use the following properties:

1. By (Assumption 0), every idempotent in $C$ is a split idempotent due to Proposition 6.6. By Corollary 6.14, a bimonoid $A$ in $C$ is bismall if and only if $A$ has a normalized integral and a normalized cointegral. By Corollary 6.15, the full subcategory of bismall bimonoids in the symmetric monoidal category $C$ gives a sub symmetric monoidal category of $\text{Bimon}(C)$.

2. We need (Assumption 1) to make use of Proposition 5.8, i.e. every homomorphism in $\text{Hopf}^{bc}(C)$ is binormal.

3. Recall Definition 9.6. Furthermore, due to (Assumption 0, 1), every homomorphism in $\text{Hopf}^{bc}(C)$ is well-decomposable by definition.
(4) From (Assumption 2), we obtain the following exact sequence: For bicommutative Hopf monoids \( A, B, C \) in \( \mathcal{C} \) and homomorphisms \( \xi : A \to B, \xi' : B \to C \), we have an exact sequence,

\[
1 \to \text{Ker}(\xi) \to \text{Ker}(\xi' \circ \xi) \to \text{Ker}(\xi') \to \text{Cok}(\xi) \to \text{Cok}(\xi' \circ \xi) \to \text{Cok}(\xi') \to 1
\]

(135)

Note that until this subsection, we use the notation \( \text{Cok}(\xi) \), \( \text{Cok}(\xi) \) for the kernel and cokernel in \( \text{Bimon}(\mathcal{C}) \). See Definition 5.1. In (135), \( \text{Ker}(\xi), \text{Cok}(\xi) \) denote a kernel and a cokernel in \( \text{Hopf}^{\text{bc}}(\mathcal{C}) \). In fact, these coincide with each other due to (Assumption 1) and Corollary 5.9.

### 15.1 A Volume on \( \text{Hopf}^{\text{bc,bs}}(\mathcal{C}) \)

**Proposition 15.4** Let \( A, B, C \) be bicommutative Hopf monoids. Let \( \xi : A \to B, \xi' : B \to C \) be bimonoid homomorphism. If the bimonoid homomorphisms \( \xi, \xi' \) are pre-Fredholm, then the composition \( \xi' \circ \xi \) is pre-Fredholm. Moreover we have,

\[
\text{vol}^{-1}(\text{Ker}(\xi)) \circ \text{vol}^{-1}(\text{Ker}(\xi')) = (\text{cok}(\xi) \circ \text{ker}(\xi')) \circ \text{vol}^{-1}(\text{Ker}(\xi' \circ \xi)),
\]

(136)

\[
\text{vol}^{-1}(\text{Cok}(\xi)) \circ \text{vol}^{-1}(\text{Cok}(\xi')) = (\text{cok}(\xi) \circ \text{ker}(\xi')) \circ \text{vol}^{-1}(\text{Cok}(\xi' \circ \xi)).
\]

(137)

**Proof** Recall that we have an exact sequence (135). By Theorem 13.8, the Hopf monoids \( \text{Cok}(\xi' \circ \xi), \text{Ker}(\xi' \circ \xi) \) are bismall since the Hopf monoids \( \text{Ker}(\xi), \text{Ker}(\xi') \) and cokernels \( \text{Cok}(\xi), \text{Cok}(\xi') \) are bismall. Hence, the composition \( \xi' \circ \xi \) is pre-Fredholm.

We prove the first equation. Denote by \( \varphi = \text{cok}(\xi) \circ \text{ker}(\xi') : \text{Ker}(\xi') \to \text{Cok}(\xi) \). From the exact sequence (135), we obtain an exact sequence,

\[
1 \to \text{Ker}(\xi) \to \text{Ker}(\xi' \circ \xi) \to \text{Ker}(\xi') \to \text{Im}(\varphi) \to 1
\]

(138)

We apply Corollary 12.3 by assuming \( A, B, \xi \) in Corollary 12.3 are \( \text{Ker}(\xi' \circ \xi), \text{Ker}(\xi') \) and the homomorphism \( \text{Ker}(\xi' \circ \xi) \to \text{Ker}(\xi') \). In fact, the first assumption in Corollary 12.3 follows from (Assumption 1). The second and fourth assumptions in Corollary 12.3 follows from Theorem 9.9. The third assumption is already proved as before. Then we obtain,

\[
\text{vol}^{-1}(\text{Ker}(\xi)) \circ \text{vol}^{-1}(\text{Ker}(\xi')) = \text{vol}^{-1}(\text{Im}(\varphi)) \circ \text{vol}^{-1}(\text{Ker}(\xi' \circ \xi)).
\]

(139)

By Proposition 11.9, we have \( (\varphi) = \text{vol}^{-1}(\text{Im}(\varphi)) \) so that it completes the first equation. The second equation is proved analogously.

**Proposition 15.5** The subcategory \( \text{Hopf}^{\text{bc,bs}}(\mathcal{C}) \) is an abelian subcategory of the abelian category \( \text{Hopf}^{\text{bc}}(\mathcal{C}) \).

**Proof** Let \( A, B \) be bicommutative bismall Hopf monoids. Let \( \xi : A \to B \) be a bimonoid homomorphism, i.e. a morphism in \( \text{Hopf}^{\text{bc}}(\mathcal{C}) \). We have an exact sequence,

\[
1 \to 1 \to \text{Ker}(\xi) \xrightarrow{\text{ker}(\xi)} A \xrightarrow{\xi} B.
\]

(140)

Due to (Assumption 1) and (Assumption 2), we can apply Theorem 13.8. By Theorem 13.8, the kernel Hopf monoid \( \text{Ker}(\xi) \) is bismall. Analogously, the cokernel Hopf monoid \( \text{Cok}(\xi) \) is bismall. It completes the proof.
Definition 15.6 Let \( End_C(I) \) be the set of endomorphism on the unit object \( I \). Note that the composition induces an abelian monoid structure on the set \( End_C(I) \). We denote by \( M_C \) the smallest submonoid of \( End_C(I) \) containing all \( f \in End_C(I) \) such that \( f = vol^{-1}(A) \) or \( f \circ vol^{-1}(A) = id_I = vol^{-1}(A) \circ f \) for some bicommutative bismall Hopf monoid \( A \). Denote by \( M_C^{-1} \) the submonoid consisting of invertible elements in the monoid \( M_C \), i.e. \( M_C^{-1} = M_C \cap Aut_C(I) \).

Theorem 15.7 The assignment \( vol^{-1} \) of inverse volumes is a \( M_C \)-valued volume on the abelian category \( Hopf_{bc,bs}(C) \).

Proof Put \( v = vol^{-1} \). The unit Hopf monoid \( I \) is a zero object of \( Hopf_{bc,bs}(C) \). By the first part of Proposition 11.4, we have \( v(I) = vol^{-1}(I) \in M_C \) is the unit of \( M_C \).

Let \( I \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow I \) be an exact sequence in the abelian category \( A = Hopf_{bc,bs}(C) \). We apply the first equation in Theorem 15.4 by considering \( \xi = g \) and \( \xi' = \epsilon_C \).

In fact, \( B, C, I \) are bismall bimonoids, the homomorphisms \( g \) and \( \epsilon_C \) are pre-Fredholm. We obtain
\[
vol^{-1}(Ker(g)) \circ vol^{-1}(Ker(\epsilon_C)) = \langle cok(g) \circ ker(\epsilon_C) \rangle \circ vol^{-1}(Ker(\epsilon_B)).
\]
By the exactness, we have \( A \cong Ker(g) \) and \( Coker(g) \cong I \). Moreover we have \( Ker(\epsilon_C) \cong C \) and \( Ker(\epsilon_B) \cong B \). Hence, we obtain \( \langle cok(g) \circ ker(\epsilon_C) \rangle = id_I \) so that \( vol^{-1}(A) \cdot vol^{-1}(C) = vol^{-1}(B) \). It completes the proof.

15.2 Functorial Integral

Definition 15.8 (1) Recall Definition 14.5. For two bicommutative Hopf monoids \( A, B \), a bimonoid homomorphism \( \xi : A \rightarrow B \) is Fredholm if it is Fredholm with respect to the inverse volume \( vol^{-1} \). In other words, the homomorphism \( \xi \) is pre-Fredholm, and its kernel Hopf monoid and cokernel Hopf monoid have finite volumes. For a Fredholm homomorphism \( \xi : A \rightarrow B \) between bicommutative Hopf monoids, we denote by \( \text{Ind} (\xi) \overset{\text{def.}}{=} \text{Ind}_{B,A,v}(\xi) \) for \( B = \text{Hopf}_{bc}(C) \), \( A = \text{Hopf}_{bc,bs}(C) \), \( M = M_C \) and \( v = vol^{-1} \).

(2) Denote by \( \text{Hopf}_{bc,Fr}^{bc}(C) \) the category consisting of Fredholm homomorphisms between bicommutative Hopf monoids. Recall Definition 14.8. For \( B = \text{Hopf}_{bc}(C) \), \( A = \text{Hopf}_{bc,bs}(C) \), \( M = M_C \) and \( v = vol^{-1} \), the category \( \text{Hopf}_{bc,Fr}^{bc}(C) \) is defined by \( \text{Hopf}_{bc,Fr}^{bc}(C) \overset{\text{def.}}{=} \text{Hopf}_{bc,Fr}^{bc}(C) \).

(3) Let \( \xi : A \rightarrow B \) be a homomorphism between bicommutative Hopf monoids. The homomorphism \( \xi \) is finite if the morphism \( \xi \) in \( \text{Hopf}_{bc}^{bc} \) is finite with respect to the volume \( vol^{-1} \). See Definition 14.10.

Proposition 15.9 (1) For a bicommutative Hopf monoid \( A \), the identity \( id_A \) is Fredholm and we have \( \text{Ind}(id_A) = id_I \in M_C^{-1} \).

(2) For Fredholm homomorphisms \( \xi : A \rightarrow B \) and \( \xi' : B \rightarrow C \) between bicommutative Hopf monoids, the composition \( \xi' \circ \xi \) is Fredholm and we have \( \text{Ind}(\xi' \circ \xi) = \text{Ind}(\xi') \circ \text{Ind}(\xi) \in M_C^{-1} \).

(3) For a Fredholm homomorphism \( \xi : A \rightarrow B \) and a finite homomorphism \( \epsilon : A \rightarrow B \), the convolution \( \xi \ast \epsilon \) is Fredholm and we have \( \text{Ind}(\xi \ast \epsilon) = \text{Ind}(\xi) \in M_C^{-1} \).

Proof The first part follows from Lemma 14.6. The second part follows from Lemma 14.7. The third part follows from Proposition 14.11.
Definition 15.10  We define a 2-cochain $\omega_C$ of the symmetric monoidal category $\text{Hopf}_{bc}^{Fr}(C)$ with coefficients in the abelian group $M_C^{-1}$. Let $\xi : A \to B$, $\xi' : B \to C$ be composable Fredholm homomorphisms between bicommutative Hopf monoids. We define

$$\omega_C(\xi, \xi') \overset{\text{def.}}{=} \langle \text{cok}(\xi) \circ \ker(\xi') \rangle \in M_C^{-1}. \quad (142)$$

Proposition 15.11  The 2-cochain $\omega_C$ is a 2-cocycle.

**Proof**  The 2-cocycle condition is immediate from the associativity of compositions. In fact,

$$\mu_{\xi''} \circ (\mu_{\xi'} \circ \mu_\xi) = (\mu_{\xi''} \circ \mu_{\xi'}) \circ \mu_\xi,$$

implies,

$$\omega_C(\xi, \xi') \circ \omega_C(\xi' \circ \xi, \xi'') = \omega_C(\xi', \xi'') \circ \omega_C(\xi, \xi' \circ \xi') \cdot \mu_{\xi'' \circ \xi' \circ \xi}. \quad (143)$$

Here, we use Theorem 12.1 where the assumptions in Theorem are deduced from (Assumption 0, 1). By Theorem 8.5, we obtain

$$\omega_C(\xi, \xi') \circ \omega_C(\xi' \circ \xi, \xi'') = \omega_C(\xi', \xi'') \circ \omega_C(\xi, \xi' \circ \xi'). \quad (144)$$

It proves that the 2-cochain $\omega_C$ is a 2-cocycle.

Moreover we have $\omega_C(id_B, \xi) = id_1 = \omega_C(\xi, id_A)$ by definitions. Hence, the 2-cocycle $\omega_C$ is normalized. It completes the proof. \qed

Definition 15.12  We define a 2-cohomology class $o_C \in H^2_{nor}(\text{Hopf}_{bc}^{Fr}(C); M_C^{-1})$ by the class of the 2-cocycle $\omega_C$.

Proposition 15.13  We have $o_C = 1 \in H^2_{nor}(\text{Hopf}_{bc}^{Fr}(C); M_C^{-1})$. In particular, the induced 2-cohomology class $o_C \in H^2_{nor}(\text{Hopf}_{bc}^{Fr}(C); \text{Aut}_C(\mathbb{1}))$ by $M_C^{-1} \subset \text{Aut}_C(\mathbb{1})$ is trivial.

**Proof**  Choose $\nu$ defined by $\nu(\xi) = vol^{-1}(\text{Ker}(\xi))$. Then the first equation in Theorem 15.4 proves the claim. \qed

Definition 15.14  **(Functorial integral)**  Let $\nu$ be a normalized 1-cochain with coefficients in the abelian group $\text{Aut}_C(\mathbb{1})$ such that $\delta^1 \nu = \omega_C$. Let $\xi : A \to B$ be a Fredholm bimonoid homomorphism between bicommutative Hopf monoids. Recall $\mu_\xi$ in Definition 9.7. We define a morphism $\xi! : B \to A$ in $C$ by

$$\xi! \overset{\text{def.}}{=} \nu(\xi)^{-1} \cdot \mu_\xi. \quad (145)$$

Proposition 15.15  Let $A$ be a bicommutative Hopf monoid. Note that the identity $id_A$ is Fredholm. We have,

$$(id_A)! = id_A. \quad (146)$$

**Proof**  It follows from $\nu(id_A) = id_\mathbb{1}$. \qed

Proposition 15.16  Let $A, B, C$ be bicommutative Hopf monoids. Let $\xi : A \to B$, $\xi' : B \to C$ be bimonoid homomorphisms. If $\xi, \xi'$ are Fredholm, then the composition $\xi' \circ \xi$ is Fredholm and we have

$$(\xi' \circ \xi)! = \xi! \circ \xi'. \quad (147)$$
Proof By Theorem 15.4, we have

\[(\xi' \circ \xi)! = \nu(\xi' \circ \xi)^{-1} \cdot \mu_{\xi' \circ \xi}(\xi') \quad (148)\]

\[= (\nu(\xi' \circ \xi)^{-1} \circ \omega(\xi', \xi') \cdot (\mu_\xi \circ \mu_{\xi'})(\xi) \quad (149)\]

\[= (\nu(\xi)^{-1} \circ \nu(\xi')^{-1}) \cdot (\mu_\xi \circ \mu_{\xi'}) \quad (150)\]

\[= \xi'! \circ \xi! \quad (151)\]

\[\Box\]

Definition 15.17 We define a normalized 1-cochain \( \nu_0 \) with coefficients in \( M_{C}^{-1} \). For a Fredholm homomorphism \( \xi \), we define \( \nu_0(\xi) \) \( \overset{\text{def}}{=} \) \( \text{vol}^{-1}(\text{Ker}(\xi)) \). We define another normalized 1-cochain \( \nu_1 \) with coefficients in \( M_{C}^{-1} \) by \( \nu_1(\xi) \) \( \overset{\text{def}}{=} \text{vol}^{-1}(\text{Cok}(\xi)) \). They satisfy

\[\delta^1 \nu_0 = \omega_\text{C} = \delta^1 \nu_1.\]

Theorem 15.18 Consider \( \nu = \nu_0 \) (\( \nu = \nu_1 \), resp.) in Definition 15.14. Let \( A, B, C, D \) be bicommutative Hopf monoids. Consider a commutative diagram of Fredholm bimonoid homomorphisms. Suppose that

- the induced bimonoid homomorphism \( \text{Ker}(\varphi) \to \text{Ker}(\psi) \) is an isomorphism (an epimorphism resp.) in \( \text{Hopf}^{\text{bc}}(C) \).
- the induced bimonoid homomorphism \( \text{Cok}(\varphi) \to \text{Cok}(\psi) \) is a monomorphism (an isomorphism, resp.) in \( \text{Hopf}^{\text{bc}}(C) \).

Then we have \( \varphi' \circ \varphi! = \psi! \circ \psi' \).

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi'} & C \\
\downarrow{\varphi} & & \downarrow{\psi} \\
B & \xrightarrow{\psi'} & D
\end{array}
\]

Proof We prove the case \( \nu = \nu_0 \) and leave to the readers the case \( \nu = \nu_1 \). Note that there exists a section of the induced bimonoid homomorphism \( \varphi'' : \text{Ker}(\varphi) \to \text{Ker}(\psi) \) in \( C \) since \( \varphi'' \) is an isomorphism in \( \text{Hopf}^{\text{bc}}(C) \), in particular in \( C \). Moreover, the induced morphism \( \psi'' : \text{Cok}(\varphi) \to \text{Cok}(\psi) \) has a retract in \( C \). In fact, since \( \psi'' \) is a monomorphism, there exists a morphism \( \xi \in \text{Hopf}^{\text{bc}}(C) \) such that \( \ker(\xi) = \psi'' \). By Lemma 9.3, \( \mu_{\text{Ker}(\xi)} \circ \psi'' = \text{id}_{\text{Cok}(\varphi)} \).

By Theorem 10.1, we have \( \mu_\psi \circ \psi' = \varphi' \circ \mu_\varphi \). Since \( \nu_0(\varphi) = \text{vol}^{-1}(\text{Ker}(\varphi)) \), \( \nu_0(\psi) = \text{vol}^{-1}(\text{Ker}(\psi)) \) and \( \varphi'' \) is an isomorphism, we have \( \nu_0(\varphi) = \nu_0(\psi) \). By definitions, we obtain \( \psi! \circ \psi' = \varphi' \circ \varphi! \). \[\Box\]

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