Exact non-equilibrium quantum observable statistics

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The exact statistics of an arbitrary measurable observable, describing specific features of an open quantum system, is analytically obtained. Due to the probabilistic nature of a sequence of intermediate measurements and stochastic fluctuations induced by the interaction with an external environment, the measurement outcomes at the end of the system evolution are random variables. Here, we provide the exact large deviation form of their probability distribution, given by an exponentially decaying profile in the number of measurements. The most probable distribution of the measurement outcomes in a single realization of the system transformation is then derived, thus achieving predictions beyond the expectation value.

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The measurement of an open quantum system observable returns random outcomes, fluctuating with a specific probability distribution [1], being the evolution of the system generated by a stochastic dynamics [2, 3]. By following the operational approach as axiomatic treatment of quantum stochastic processes [4], the system dynamics can be modeled as a non-equilibrium transformation given by the composition of intermediate quantum measurements [5] and stochastic evolutions characterized by semi-classical fluctuations of the system parameters [6–8]. From the experimental side, such dynamics ensures the protection the coherent evolution of a quantum system via repeated quantum measurements or strong couplings – quantum Zeno dynamics [6, 9, 10] – and especially in quantum metrology [11] to probe the phase evolution of an atomic ensemble by means of interleaved interrogations of it and feedback corrections [12, 13]. In this scenario, at the level of the single realization, i.e. by repeating only once the dynamics of the open system, the ensemble average of the measurement outcomes does not provide complete information about the statistics of the measured results. This becomes much more evident when also one rare random event, given by a stochastic fluctuation with very small probability, occurs within the dynamics of the system [14–16]. Such a concept is at the heart of the Large Deviation (LD) theory [17, 18], dealing with the exponential decay of probabilities associated to large fluctuations in stochastic classical [19] and quantum systems [20, 21]. Noteworthy, the use of LD theory has recently allowed for the development of a thermodynamic formalism for the study of dissipative quantum systems [22–24].

In this paper, we analytically derive a closed-form of the outcomes’ statistics obtained by measuring an open quantum system, that randomly interacts with an external environment, a-priori unknown, and is repeatedly monitored by an observer/experimenter. In particular, we prove that after a sufficiently high number $m$ of intermediate quantum measurements (this is the only assumption of the theory), the probability distribution of the last measurement results obeys the so-called LD principle [18, 25, 26]. This means that the behaviour of the measurement outcomes distribution is a decaying exponential in $m$, whose exponent is equal to the relative Shannon entropy between the configurations of the stochastic system dynamics. In other words, only a rigorous description of the occurrence combinatorics of the parameters defining the stochastic evolution of the system allows for the full characterization of the outcomes’ statistics, also beyond the Gaussian approximation given by the sole description of the measurement apparatus.

Model.– Let us consider an arbitrary quantum system $S$ within the Hilbert space $\mathcal{H}$. We assume that $S$ is initially in the quantum state described by the density matrix $\rho_0$, and that the system Hamiltonian $H$ of the system is time-independent. In a single realization of the system evolution, the stochastic interaction between $S$ and an external environment $\mathcal{E}$ is modelled by a sequence of arbitrary stochastic dynamics, separated by consecutive quantum projective measurements [27], occurring according to the postulates of quantum mechanics [28]. Hereafter, we will use the index $j$ to denote the dimension of $\mathcal{H}$ and the index $\alpha$ to denote the time instants composing the temporal sequence of measurements. More specifically, we assume that the first $m-1$ measurements are performed on the quantum observable $O \equiv \sum_j o_j \Pi_{\theta_j}$, where $o_j$ are the outcomes of $O$ and $\{\Pi_{\theta_j}\}$ is the set of projectors corresponding to the measured eigenvalues at time instants $\tau_j$. The $m$-th measurement, instead, is performed on the quantum observable $\Theta \equiv \sum_j \theta_j \Pi_{\theta_j}$, whose outcomes $\theta_j$ are recorded by the observer. According to the postulate of quantum measurement, the state $\rho_\alpha$ of $S$ after a projective measurement at $\tau_\alpha$ is identically equal to one of the projectors defining the corresponding measurement observable. Then, between each projection event the system undergoes a dynamics, that is governed by the Hamiltonian $H$ and described by the completely-positive and trace-preserving quantum map $\Phi(\tau_\alpha, \tau_0) [\rho_0] \equiv \Phi_\alpha [\rho_0]$ [29]. In this paper, assuming a large number of intermediate quantum measurements, the dynamics between measurements is described by a unitary operator, so that $\Phi_\alpha [\rho_{\alpha-1}]$ is simply given by the super-operator $U_\alpha [\rho_{\alpha-1}] \equiv U_\alpha \rho_{\alpha-1} U_\alpha^\dagger$, where $U_\alpha = \exp(-iH\tau_\alpha)$, $\hbar$ is set to unity and $\tau_\alpha \equiv \tau_{\alpha} - \tau_{\alpha-1}$. We also assume that, in accordance with the recently introduced stochastic quantum Zeno phenomena [6, 7, 25], there exists for each propagator $U_\alpha$ at least one dynamical parameter $\lambda$, that is a fluctuating variable. For example, one could consider as in [30, 31] that each $\lambda_\alpha$...
is equal to the time interval $\tau_a$, with $\tau_a$ being random. Moreover, the $\lambda$’s between the measurements are taken constant and have a different random value only after the occurrence of a new measurement, according to the probability density function $p(\lambda)$. We will adopt the notation $\lambda \equiv (\lambda_1, \ldots, \lambda_m)$ to denote the sequence of dynamical parameters $\lambda_i$ at time instants $t_i$ in a single realization of the system transformation. The fluctuating dynamical parameters $\lambda_i$, then, are taken as independent and identically distributed (i.i.d.) random variables sampled from $p(\lambda)$, since the environment is a-priori unknown.

The stochastic nature of the measurement outcomes $\theta_m$’s, in correspondence of the final time instant $t_m$, lies in the specific values assumed by $\vec{\sigma}$ and $\vec{\lambda}$. Thus, being the dynamics of $\mathcal{S}$ stochastic, the single realization of the system density matrix $\rho_{m,\vec{\sigma},\vec{\lambda}}$ at the end of its evolution is a fluctuating variable, in the sense that, given the sequences $\vec{\sigma}$ and $\vec{\lambda}$, it is mapped into

$$
\rho_{m,\vec{\sigma},\vec{\lambda}} = \frac{\mathcal{P}_{\theta_m} U_m \mathcal{P}_{\theta_m-1} U_{m-1} \cdots \mathcal{P}_0 U_1 [\rho_0]}{p_{\theta_m}(\vec{\sigma}, \vec{\lambda})},
$$

where $\mathcal{P}_{\mu} (\cdot) \equiv \Pi_{\mu_a} (\cdot) \Pi_{\mu_0}$ with $\mu \in \{o, \theta\}$ is the measurement super-operator acting on $(\cdot)$ at $t_\mu$. Consequently

$$
p_{\theta_m} (\vec{\sigma}, \vec{\lambda}) \equiv \text{Tr} \left\{ \mathcal{P}_{\theta_m} U_m \mathcal{P}_{\theta_m-1} U_{m-1} \cdots \mathcal{P}_0 U_1 [\rho_0] \right\},
$$

i.e. $p_{\theta_m} (\vec{\sigma}, \vec{\lambda}) = \text{Tr} [\Pi_{\mu} \rho_{m,\vec{\sigma},\vec{\lambda}}]$, denotes the conditional probability to obtain the outcome $\theta_m$ from the measurement of $\Theta$ given the specific realization of the sequences $\vec{\sigma}$ and $\vec{\lambda}$.

As example, we could simply consider a two-level system, whose dynamics is governed by the Rabi Hamiltonian $H = (\Omega/2) \sigma_x$, with Pauli matrix $\sigma_x \equiv |0\rangle \langle 1| + |1\rangle \langle 0|$ and Rabi frequency $\Omega$ assumed random with probability $p(\Omega)$. Thus, one could consider the sequence of intermediate projective measurements as the model for the monitoring of a specific quantum observable of the system $[\mathcal{S}]$, or corresponding to a process exchanging photons with the environment $[32]$. In such a case, the collapse of the wave function of the two-level system to one of the projectors $|0\rangle |0\rangle$ or $|1\rangle |1\rangle$ just depends on whether the photon is absorbed or emitted.

Quantum observable statistics.– In a single realization of the system transformation, $\rho_{m,\vec{\sigma},\vec{\lambda}}$ and $\theta_m$, being dependent on $\vec{\sigma}$ and $\vec{\lambda}$, are random quantities. Thus, the following question naturally emerges: Which is the best description for the measurement results $\theta_{m,j}$ given by observing $\mathcal{S}$ at the final instant $t_m$? Three answers, characterized by an increasing degree of prediction accuracy, can be provided. Firstly, one could describe the probabilistic expected result from the measured outcomes by using the expectation value $\text{Tr} [\Theta \rho_m]$, with

$$
\rho_m \equiv \langle \rho_{m,\vec{\sigma},\vec{\lambda}} \rangle = \sum_{\sigma} \int d^m \vec{\lambda} \hat{p}(\vec{\lambda}) \rho_{m,\vec{\sigma},\vec{\lambda}}
$$

and $\hat{p}(\vec{\lambda})$ denoting the occurrence joint probability of the dynamical parameters $\lambda_i$. In this regard, it is worth noting that

![FIG. 1. Statistics of $\Theta$’s outcomes. By repeating several times the stochastic evolution of the system and measuring each outcomes $\theta_{m,j}$ at the final time instant $t_m$, an ensemble of conditional probabilities $p_{\theta_{m,j}}$ is obtained, where each of them is computed after the single realization of the system dynamics. Thus, just by counting the occurrence relative frequencies of the $p_{\theta_{m,j}}$’s, one can derive the corresponding probability distributions (blue dashed lines). If the number of realizations is relatively small, such distributions do not obey the Gaussian approximation, under the validity of the central limit theorem. This means that one has to distinguish between two different statistics for $\theta$’s: One (green solid line) by linking the ensemble averages $\langle \rho_{m,j}(\vec{\sigma}, \vec{\lambda}) \rangle$ (red dots) of the conditional probabilities for each measurement outcome, the other (orange dotted line) – also called most probable distribution – by connecting all the realizations of $p_{\theta_{m,j}}$ in correspondence of the maximum value of the conditional probability distributions (blue dots).](image-url)
to $|\pi_{\mu_0}\rangle$ via the propagator $U_\alpha(\lambda_0)$ (see the SM for more details). Specifically, $|\pi_{\mu_0}\rangle$'s are the eigenvectors that define the measurement projectors $\Pi_{\alpha,\omega}$, with $\mu$ equal to $\theta$ or $\alpha$ depending on whether the observable $\Theta$ or $O$ is measured. Thus, one has that

$$p_{\theta_{m,j}}(\vec{a},\vec{X}) = \prod_{\alpha=1}^{m} |\langle \pi_{\mu_{\alpha-1}}|U_\alpha(\lambda_\alpha)|\pi_{\mu_\alpha}\rangle|^2,$$  \hspace{1cm} (5)

$$p_{\theta_{m,j}} = \sum_{\vec{a}} \prod_{\alpha=1}^{m} \int d\lambda_\alpha p(\lambda_\alpha) q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha),$$  \hspace{1cm} (6)

where $q_\alpha \equiv |\langle \pi_{\mu_{\alpha-1}}|U_\alpha(\lambda_\alpha)|\pi_{\mu_\alpha}\rangle|^2$, $\pi_{\mu_\alpha} \equiv \pi_0$ (with $\mu_0 = |\pi_0\rangle/\langle \pi_0|$), $\mu_\alpha \equiv \mu_\alpha$ for $\alpha = 1,\ldots,m-1$ and $\mu_{m} \equiv \pi_{\theta_{m,j}}$.

**Large deviation formalism.**— Here, the main result of this paper is shown, namely a closed-form of the $p_{\theta_{m,j}}(\vec{a},\vec{X})$ distributions obeying the LD principle. Notice that if we just characterize the measurement observable $\Theta$ at $t_m$ and at the same time the realizations of the system dynamics respect the hypotheses of the central limit theorem, then the statistics of its outcomes is well-represented by a Gaussian probability distribution. In fact, with this assumption, outliers in the outcomes statistics are simply classified as the result of a non-modelled experimental noise on the measurement apparatus. Thus, with an increasing the number of realizations, the occurrence probability of the outliers naturally decreases and the Gaussian distribution well fits the data. However, this evidence is not longer valid if we assume that the statistics of $\theta_{m,j}$'s is given by an arbitrary stochastic transformation governing the dynamics of the system. In such a case, the configuration space defined by the occurrence of random events during its evolution becomes exponentially larger so as to allow the description of the outliers' statistics, with occurrence probability greater than zero and not satisfying the Gaussian approximations. Therefore, our prediction power on the outcomes distribution is expected to be increased by the statistical characterization of the system evolution before the measurement of $\Theta$. For this purpose, we firstly compute the statistics of the dynamical transition probabilities $q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha)$ as defined in (5). As proved in the SM, the procedure to derive the exact LD form of $\text{Prob}(p_{\theta_{m,j}}(\vec{a},\vec{X}))$ is to take the logarithm of the conditional probabilities $p_{\theta_{m,j}}(\vec{a},\vec{X})$, i.e., $l_{\theta_{m,j}}(\vec{a},\vec{X}) \equiv \ln p_{\theta_{m,j}}(\vec{a},\vec{X}) = \sum_{\alpha=1}^{m-1} \ln q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha)$, compute its distribution and then apply the contraction principle from LD theory [18].

Thus, let us assume that the measurement bases of $O$ and $\Theta$, with $[O,\Theta] \neq 0$, belong to a set of finite dimension, i.e. that each measurement eigenvector $|\pi_{\mu_\alpha}\rangle$ admits only a finite number of elements $|\pi_{\mu_\alpha}\rangle_j$, with $j$ denoting the index for the dimensionality of the measurement basis (it is the same of the index for the dimension of $\mathcal{H}$). Then, by following a common procedure in LD theory, we group the terms of $l_{\theta_{m,j}}(\vec{a},\vec{X})$ as a function of the number of times each dynamical transition probability $q_{jB,jA}(\lambda) \equiv |\langle \pi_{\mu_{\alpha}}|U_\alpha(\lambda)|\pi_{\mu_\alpha}\rangle|^2$ occurs, where the superscripts $B$ and $A$ stands respectively for “Before” and “After” the evolution of the system. In this way, $l_{\theta_{m,j}}$ is recast in the following sum of i.i.d. random variables: $l_{\theta_{m,j}}(\vec{a},\vec{X}) = \sum_{jB,jA=1}^{d_\epsilon} n_{jB,jA}(\lambda) \ln q_{jB,jA}(\lambda) d\lambda$, with $n_{jB,jA}(\lambda)$ denoting the relative frequencies of $q_{jB,jA}(\lambda)$ with probability $p_{jB,jA}(\lambda)$. The dimension of the statistical ensemble, defining the stochasticity from $t_0$ to $t_m$, is thus equal to $d_\epsilon \equiv 2d_r + V_\lambda$, where $V_\lambda \equiv \int \lambda d\lambda$ is the support of the probability density function $p(\lambda)$. As shown also in the SM, the probability distribution of a sum of $n$ i.i.d. random terms can always be written as an exponential, linearly decaying in $n$ with $n$ large. In particular, regarding $\text{Prob}(l_{\theta_{m,j}})$, it is given by an exponential distribution decaying in the number $m$ of projective measurements (see SM for more details), i.e.

$$\text{Prob}(l_{\theta_{m,j}}(\vec{a},\vec{X})) \approx \exp(-m I(l_{\theta_{m,j}}(\vec{a},\vec{X})/m)).$$  \hspace{1cm} (7)

In Eq. (7), the function $I(l_{\theta_{m,j}}/m)$, also called rate function associated to the probability distribution $\text{Prob}(l_{\theta_{m,j}})$, equals to

$$I(l_{\theta_{m,j}}/m) = \sum_{jB,jA=1}^{d_\epsilon} \int_{\lambda_{jB,jA}}^{\lambda_{jB,J\lambda}} f_{jB,jA}(\lambda) \ln \left( \frac{f_{jB,jA}(\lambda)}{p_{jB,jA}(\lambda)} \right) d\lambda,$$  \hspace{1cm} (8)

where $f_{jB,jA}(\lambda) \equiv n_{jB,jA}(\lambda)/m$ for each set $(jB,jA,\lambda)$ of system parameters. The rate function $I(l_{\theta_{m,j}}/m)$ is the Kullback-Leibler distance (or relative entropy) between the set $\{f_{jB,jA}(\lambda)\}$ of scaled relative frequencies and the set of probabilities $\{p_{jB,jA}(\lambda)\}$, and, thus, has the properties to be positive and convex. The approximation of Eq. (7) is valid in the limit of $m$ large, and implies a unique non-equilibrium weighted partition of the system configuration space. For small value of $m$, indeed, the distribution $\text{Prob}(l_{\theta_{m,j}})$ cannot be uniquely determined. The latter can be considered as property of dynamical evolutions given by the composition of quantum maps and projections. The alternative LD expression of Eq. (7), providing the formal definition of $I(l_{\theta_{m,j}}/m)$, is

$$\lim_{m \to \infty} -\frac{1}{m} \ln \text{Prob}(l_{\theta_{m,j}}(\vec{a},\vec{X})) = I(l_{\theta_{m,j}}(\vec{a},\vec{X})/m),$$  \hspace{1cm} (9)

where the number $m$ of measurements is assumed ideally infinite. If Eq. (9), also called large deviation approximation, is valid, it means that the dominant behaviour of $\text{Prob}(l_{\theta_{m,j}})$ is convergent and identically equal to a decaying exponential in $m$.

As last step, the distribution $\text{Prob}(p_{\theta_{m,j}}(\vec{a},\vec{X}))$ is derived. To this end, the use of the contraction principle allows us to put in relation the probability distributions in LD form of two distinct quantities, one as a function of the other by means of a continuous function. Specifically, $\text{Prob}(p_{\theta_{m,j}}(\vec{a},\vec{X})) = \int \text{Prob}(l_{\theta_{m,j}}(\vec{a},\vec{X})\delta(l_{\theta_{m,j}} - \ln p_{\theta_{m,j}}) d\lambda\mu_{rms}$ and then, by applying the saddle point method [33], one has that

$$\text{Prob}(p_{\theta_{m,j}}(\vec{a},\vec{X})) \approx \exp(-m J(p_{\theta_{m,j}}(\vec{a},\vec{X})/m)),$$  \hspace{1cm} (10)

where

$$J(p_{\theta_{m,j}}(\vec{a},\vec{X})/m) \equiv \min_{l_{\theta_j},l_{\theta_j}^J=\ln p_{\theta_j}} I(l_{\theta_{m,j}}(\vec{a},\vec{X})/m).$$  \hspace{1cm} (11)
Thus, the result is that a quantum system which is randomly interacting with an external environment and is repeatedly monitored by an observer tends to reach in probability an unique configuration defined by specific probability distributions of its characteristic parameters.

Most probable distribution.— In this paragraph, we discuss the expression of the most probable distribution of the \( \Theta \)'s outcomes. In general, from the knowledge of the rate function \( I(\xi/m) \) associate to \( \text{Prob}(\xi) \), we can then compute the most probable value \( \xi^* \), representing the best prediction for \( \xi \) in a single realization of the system dynamics. Specifically, the most probable value of the log-conditional-probability \( l_{\theta_{m,j}}(\vec{\alpha}, \vec{\lambda}) \), i.e., \( l_{\theta_{m,j}}^* \), is obtained by evaluating the value at which the rate function \( I(l_{\theta_{m,j}}/m) \) is minimized as a function of \( l_{\theta_{m,j}} \). Thanks to the positivity and convexity of the rate function, sufficient condition for its minimization is that the identity \( \partial I(l_{\theta_{m,j}}(\vec{\alpha}, \vec{\lambda})/m) / \partial \ln q_{j_{B,jA}}(\lambda) = 0 \), computed in correspondence of \( l_{\theta_{m,j}}(\vec{\alpha}, \vec{\lambda}) = l_{\theta_{m,j}}^* \), is verified for each set \( (j_{B,jA}, \lambda) \) of system parameters. As analytically proved in the SM, the closed-form of \( l_{\theta_{m,j}}^* \) can be obtained, i.e. \( l_{\theta_{m,j}}^* = m \sum_{d_{x}} p_{j_{B,jA}}(\lambda) \ln q_{j_{B,jA}}(\lambda) d\lambda \), corresponding to state that

\[
n_{j_{B,jA}}(\lambda) = m p_{j_{B,jA}}(\lambda) \quad (12)
\]

for every \( (j_{B,jA}, \lambda) \). Eq. (12) tells us that, in the limit of \( m \) large, the most probable trajectories of the system dynamics are those allowing for the equality between the scaled relative frequencies \( f_{j_{B,jA}}(\lambda) \) and the occurrence probabilities \( p_{j_{B,jA}}(\lambda) \). Only this condition minimizes the relative Shannon entropy between the configurations induced by the stochastic dynamics of the system. Once more, it is worth observing the importance of imposing \( m \to \infty \); indeed, the value of the scaled relative frequencies \( f_{j_{B,jA}}(\lambda) \) can get closer to that of the probabilities \( p_{j_{B,jA}}(\lambda) \) only if the number of configurations generated by the stochasticity within the dynamics is as large as possible.

To derive the most probable value of the conditional probabilities \( p_{\theta_{m,j}}^* \), we use again the contraction principle from LD theory, but this time on the functional relation between \( l_{\theta_{m,j}} \) and \( p_{\theta_{m,j}} \), i.e., \( p_{\theta_{m,j}} = e^{l_{\theta_{m,j}}} \). One has that

\[
 p_{\theta_{m,j}}^* = \exp \left( m \sum_{j_{B,jA}=1}^{d_{x}} \int_{\lambda} p_{j_{B,jA}}(\lambda) \ln q_{j_{B,jA}}(\lambda) d\lambda \right) \quad (13)
\]

by quantitatively taking into account the stochastic evolution of the system before the measurement of \( \Theta \).

In conclusion, we are now able to answer to the main question posed in this paper, i.e., which is the best description for the measurement results of the quantum observable \( \Theta \). Having proved that the non-equilibrium statistics of an arbitrary observable obeys the LD principle for a sufficiently large number of intermediate projections events, the answer is surely given by the distribution of the \( \Theta \)'s outcomes with occurrence conditional probabilities equal to \( p_{\theta_{m,j}}^* \), denoted as most probable distribution:

\[
\text{Prob}^*(\theta) \equiv \frac{1}{N} \sum_{j} \delta(\theta - \theta_{m,j}) p_{\theta_{m,j}}^* \quad (14)
\]

Eq. (14) is normalized by the factor \( N \) so as to ensure that \( \int \text{Prob}^*(\theta) d\theta = 1 \). Experimentally, this means improving our knowledge about the way the stochasticity enters into the dynamics of the system and changes the statistics of the \( \theta_{m,j} \)'s. In this regard, the predictions about the result of single-shot measurements from \( \Theta \) according to the most probable distribution \( \text{Prob}^*(\theta) \) will be more accurate of the ones that we would obtain by directly computing the expectation value of the measurement outcomes. This is because large fluctuations within the evolution of the system are now properly weighted and thus correctly included in the outcome distributions. It is worth noting that also quantum noise sensing techniques \([34]\) could be used, so as to improve the a-priori information on \( p(\lambda) \). Finally, being \( p_{\theta_{m,j}} \) explicitly equal to

\[
\langle p_{\theta_{m,j}}(\vec{\alpha}, \vec{\lambda}) \rangle = \left( \sum_{j_{B,jA}=1}^{d_{x}} \int_{\lambda} p_{j_{B,jA}}(\lambda) q_{j_{B,jA}}(\lambda) d\lambda \right)^m \quad (15)
\]

one can observe a deviation between the most probable values \( p_{\theta_{m,j}}^* \) and the probabilities \( p_{\theta_{m,j}} \) to measure on average the outcome \( \theta_{m,j} \) at the final time \( t_{m} \). Specifically, by applying the Jensen’s inequality to \( p_{\theta_{m,j}}^* \) and \( p_{\theta_{m,j}} \) \([35]\), the inequality \( p_{\theta_{m,j}}^* \leq p_{\theta_{m,j}} \) is obtained. The latter is the main reason explaining the introduction of the normalization factor \( N \) in Eq. (14). In fact, the normalization of \( \text{Prob}^*(\theta) \) has to be ensured, and the values of the \( p_{\theta_{m,j}}^* \)'s are thus corrected by dividing for \( N \equiv \sum_{j} p_{\theta_{m,j}}^* \leq 1 \), so that \( p_{\theta_{m,j}} / N = 1 - (\sum_{k, k \neq j} p_{\theta_{m,k}}) / N, \forall j = 1, \ldots, \text{dim}(\mathcal{H}) \). The continuous limit of the probability distributions \( \text{Prob}(l_{\theta_{m,j}}(\vec{\alpha}, \vec{\lambda})) \) and \( \text{Prob}(p_{\theta_{m,j}}(\vec{\alpha}, \vec{\lambda})) \) is discussed in the SM.

Applications of the introduced formalism for the prediction of measurement results in specific physical setups will be shown in a forthcoming paper.

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SUPPLEMENTARY MATERIAL FOR “EXACT NON-EQUILIBRIUM QUANTUM OBSERVABLE STATISTICS”

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I. Measurement conditional probabilities

Let us express the measurement projectors \( \Pi_{\mu} \) applied at time instants \( t_\alpha, \alpha = 1, \ldots, m \), as a function of their eigenvectors \( |\pi_{\mu_\alpha}\rangle \), i.e. \( \Pi_{\mu_\alpha} \equiv \langle \pi_{\mu_\alpha}|\pi_{\mu_\alpha}\rangle \), with \( \mu \in \{\theta, \bar{\theta}\} \). Thus, by substituting the definition of the measurement super-operators \( P_{\theta_m} \) and \( P_{\bar{\theta}_n} \) into Eq. (2) of the main text, the conditional probability \( p_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) \) can be written as the product of the transition probabilities (random terms) that the quantum state moves from \( |\pi_{\mu_{\alpha-1}}\rangle \) to \( |\pi_{\mu_{\alpha}}\rangle \) via the unitary operator \( U_\alpha(\lambda_\alpha) \):

\[
p_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) \equiv \text{Tr}[P_{\theta_m}U_mP_{\theta_{m-1}}U_{m-1} \cdots P_{\theta_1}U_1|\rho_0]\]
\[
= \text{Tr}[\Pi_{\theta_{m,j}}U_m\Pi_{\theta_{m-1}}U_{m-1} \cdots \Pi_{\theta_1}U_1\rho_0U_1^\dagger \Pi_{\theta_1} \cdots U_{m-1}^\dagger \Pi_{\theta_{m-1}} U_m^\dagger \Pi_{\theta_{m,j}}]\]
\[
= \text{Tr}\left[|\pi_{\theta_{m,j}}\rangle \langle \pi_{\theta_{m,j}}|U_m|\pi_{\theta_{m-1}}\rangle \langle \pi_{\theta_{m-1}}|U_{m-1} \cdots |\pi_{\theta_1}\rangle \langle \pi_{\theta_1}|U_1\rho_0U_1^\dagger |\pi_{\theta_1}\rangle \langle \pi_{\theta_1}| \cdots U_{m-1}^\dagger |\pi_{\theta_{m-1}}\rangle \langle \pi_{\theta_{m-1}}|U_m^\dagger |\pi_{\theta_{m,j}}\rangle \langle \pi_{\theta_{m,j}}|\right]
\]
\[
= \langle \pi_{\theta_1}|U_1\rho_0U_1^\dagger |\pi_{\theta_1}\rangle \cdot \prod_{k=2}^{m-1} \langle \pi_{\theta_{k-1}}|U_k|\pi_{\theta_k}\rangle^2 \cdot \langle \pi_{\theta_{m-1}}|U_m|\pi_{\theta_{m,j}}\rangle^2. \tag{S1}
\]

\( \rho_0 \) is the initial density matrix of the open quantum system \( S \) before the transformation induced by random interactions with the environment \( \mathcal{E} \) and the monitoring by an observer. Moreover, we also assume that \( \rho_0 \) is defined by the pure state \( |\psi_0\rangle \), i.e., \( \rho_0 \equiv |\psi_0\rangle \langle \psi_0| \). For example this assumption is verified when the measurement outcomes at the final time \( t_m \) are obtained by a two-time measurement scheme [5]. In such a case, indeed, \( \rho_0 \) is the density matrix of \( S \) after the first measurement of the scheme, and thus it is described only by a pure state. Therefore, under this further hypothesis, one has that \( \langle \pi_{\theta_1}|U_1\rho_0U_1^\dagger |\pi_{\theta_1}\rangle = \langle \psi_0|U_1|\psi_0\rangle^2 \), so that

\[
p_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) = \langle \psi_0|U_1|\psi_0\rangle^2 \cdot \prod_{\alpha=2}^{m-1} \langle \psi_{\theta_{\alpha-1}}|U_\alpha|\psi_{\theta_{\alpha}}\rangle^2 \cdot \langle \psi_{\theta_{m-1}}|U_m|\psi_{\theta_{m,j}}\rangle^2 = \prod_{\alpha=1}^{m} \langle \pi_{\mu_{\alpha-1}}|U_\alpha(\lambda_\alpha)|\pi_{\mu_{\alpha}}\rangle^2, \tag{S2}
\]

with \( \pi_{\mu_\alpha} \equiv \psi_0, \mu_\alpha \equiv \theta_\alpha \) for \( \alpha = 1, \ldots, m-1 \) and \( \pi_{\mu_m} \equiv \theta_{m,j} \).

II. Derivation of \( \text{Prob}(l_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda})) \) via LD theory

The log-probability \( l_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) \) is the logarithm of the conditional probability \( p_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) \) defining the probability to obtain the \( j \)-th measurement outcome \( \theta_{m,j} \) at the time instant \( t_m \), conditioned to the specific realizations of the sequences \( \bar{\theta} \) and \( \bar{\lambda} \). More formally,

\[
l_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) \equiv \ln p_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) = \sum_{\alpha=1}^{m} \ln q_\alpha(\mu_{\alpha-1}, \mu_\alpha, \lambda_\alpha) = \sum_{\alpha=1}^{m} \ln \langle \langle \pi_{\mu_{\alpha-1}}|U_\alpha(\lambda_\alpha)|\pi_{\mu_{\alpha}}\rangle^2 \rangle. \tag{S3}
\]

For the sake of clarity, let us assume the following two hypotheses: (i) \( p(\lambda) \) is assumed to be a \( d_\lambda \)-dimensional Bernoulli distribution, with the result that at each time instant \( t_\alpha \) the parameter \( \lambda_\alpha \) takes on \( d_\lambda \) possible values \( \lambda^{(1)}, \ldots, \lambda^{(d_\lambda)} \) with corresponding probabilities \( p_\lambda^{(1)}, \ldots, p_\lambda^{(d_\lambda)} \) so that \( \sum_{i=1}^{d_\lambda} p_\lambda^{(i)} = 1 \). The index \( i \) denotes the values that can be assumed by \( \lambda \). (ii) The measurement bases of \( \Theta \) and \( \mathcal{O} \), given by the set of eigenvectors \( \{ |\pi_{\mu_\alpha}\rangle \} \), belong to a set of finite dimension. This means that each ket \( |\pi_{\mu_\alpha}\rangle \) admits only a finite number \( d_x \) of elements \( |\pi^{(j)}\rangle \) with uniform probability \( p_x = 1/d_x \). Notice that the index for the dimensionality of the measurement basis configurations is \( j \), as well as that for the dimension of \( H \). By following a common procedure in LD theory, we group the terms of \( l_{\theta_{m,j}}(\bar{\theta}, \bar{\lambda}) \) as a function of the number of times (relative frequencies) each dynamical transition probability

\[
q_{\theta_{m,j}} \equiv \langle |\pi^{(j)}\rangle U(|\lambda^{(i)}\rangle |\pi^{(j)}\rangle) |\pi^{(j)}\rangle^2 \tag{S4}
\]

occurs during the transformation of the system. In Eq. (S4), B and A stands, respectively, for “Before” and “After” the dynamical evolution of the system. We denote with \( q_{\theta_{m,j}} \) the relative frequencies for the occurrence of the \( q_{\theta_{m,j}} \)’s. The corresponding
occurrence probabilities, instead, are simply given by the following relation:

\[ p_{j_B,i,J_A} \equiv p_n^{(j_B)} p_\lambda^{(i)} p_d^{(j_A)} = \frac{p_\lambda^{(i)}}{d_n^{(i)}}. \]  

(S5)

The latter procedure directly leads to the analytical expression of \( l_{\theta,m,j}(\vec{o}, \vec{\lambda}) \), i.e.,

\[ l_{\theta,m,j}(\vec{o}, \vec{\lambda}) = \sum_{j_B=1}^{d_e} \sum_{i=1}^{d_x} \sum_{J_A=1}^{d_x} n_{j_B,i,J_A} \ln q_{j_B,i,J_A}. \]  

(S6)

Eq. (S6) shows us that \( l_{\theta,m,j}(\vec{o}, \vec{\lambda}) \) can be written as the sum of the i.i.d. dynamical transition probabilities \( q_{j_B,i,J_A} \), weighted by the relative frequencies \( n_{j_B,i,J_A} \) corresponding to the occurrence statistics of the \( q_{j_B,i,J_A} \)’s. Therefore, the distribution probability of \( l_{\theta,m,j}(\vec{o}, \vec{\lambda}) \) is given by

\[ \text{Prob}(l_{\theta,m,j}(\vec{o}, \vec{\lambda})) = \prod_{j_B,i,J_A} \left( \sum_{\hat{n}_{j_B,i,J_A}} \prod_{j_B,i,J_A} \left( p_{j_B,i,J_A} \right)^{n_{j_B,i,J_A}} \delta \left( l_{\theta,m,j}(\vec{o}, \vec{\lambda}) - \sum_{j_B,i,J_A} n_{j_B,i,J_A} \ln q_{j_B,i,J_A} \right) \right), \]  

(S7)

where \( \delta(\cdot) \) denotes the Kronecker delta. Let us observe that in Eq. (S7), to simplify the notation, we have used the symbols \( \sum \) and \( \prod \) to denote respectively \( \sum_{j_B=1}^{d_e} \sum_{i=1}^{d_x} \sum_{J_A=1}^{d_x} \) and \( \prod_{j_B=1}^{d_e} \prod_{i=1}^{d_x} \prod_{J_A=1}^{d_x} \). Then, by imposing in Eq. (S7) the condition given by the Kronecker delta, one has that

\[ \text{Prob}(l_{\theta,m,j}(\vec{o}, \vec{\lambda})) = \prod_{j_B,i,J_A} \left( \sum_{\hat{n}_{j_B,i,J_A}} \prod_{j_B,i,J_A} \left( p_{j_B,i,J_A} \right)^{\hat{n}_{j_B,i,J_A}} \right), \]  

(S8)

where the relative frequencies \( \hat{n}_{j_B,i,J_A} \)’s have to be satisfy the following constraints:

\[ \begin{cases} \sum_{j_B,i,J_A} n_{j_B,i,J_A} = m \\ l_{\theta,m,j}(\vec{o}, \vec{\lambda}) = \sum_{j_B,i,J_A} n_{j_B,i,J_A} \ln q_{j_B,i,J_A}. \end{cases} \]  

(S9)

Now, let us combine together the constraints (S9):

\[ l_{\theta,m,j}(\vec{o}, \vec{\lambda}) = \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \ln q_{j_B,i,J_A} = \hat{n}_{d_x,d_x,d_x} \ln q_{d_x,d_x,d_x} + \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \ln q_{j_B,i,J_A} = \left( m - \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \right) \ln q_{d_x,d_x,d_x} + \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \ln q_{j_B,i,J_A}, \]  

(S10)

where \( \sum_{j_B,i,J_A} \) is defined as

\[ \begin{cases} \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \ln q_{j_B,i,J_A} = \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \ln q_{j_B,i,J_A} - \hat{n}_{d_x,d_x,d_x} \ln q_{d_x,d_x,d_x} \\ \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} = m - \hat{n}_{d_x,d_x,d_x}. \end{cases} \]  

(S11)

As a result, we obtain one unique constraint equation for \( l_{\theta,m,j} \), i.e.,

\[ l_{\theta,m,j}(\vec{o}, \vec{\lambda}) = m \ln q_{d_x,d_x,d_x} - \sum_{j_B,i,J_A} \hat{n}_{j_B,i,J_A} \gamma_{j_B,i,J_A}, \]  

(S12)
where
\[ \gamma_{jB,i,jA} = \frac{\ln q_{d_x,d_x,d_x}}{\ln q_{jB,i,jA}}, \]  
(S13)

It is worth observing that in deriving the constraint (S12) we have chosen \((d_x, d_x, d_x)\) as the reference triplet of the configuration space that defines the stochastic trajectory of the system dynamics in a single realization. This choice is arbitrary and represents a degree of freedom of the procedure. However, that is not surprising because the number of constraints of formula (S9) is smaller than the number of relative frequencies \(n_{jB,i,jA}\), so that the values of \(\hat{n}_{jB,i,jA}\) that satisfy Eq. (S8) is generally not uniquely determined. This means that in order to obtain an unique analytical expression of \(\text{Prob}(l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda}))\), we need to answer to the following questions: Which are the unique values of the relative frequencies \(\hat{n}_{jB,i,jA}\) obeying the constraint equation (S12)? By generalizing the results in [25], we can prove that there exists a unique value for the \(\hat{n}_{jB,i,jA}\)'s, under the hypothesis of a sufficiently large number \(m\) of intermediate projective measurements. To see this, let us consider the product \(\hat{n}_{a,b,c} \gamma_{a,b,c}\) with generic indexes \((a, b, c)\):

\[ \frac{\hat{n}_{a,b,c} \gamma_{a,b,c}}{m} = m \left( \frac{\ln q_{d_x,d_x,d_x} - l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda})}{\gamma_{a,b,c}} \right) - \sum_{jB,i,jA} \frac{\hat{n}_{jB,i,jA} \gamma_{jB,i,jA}}{m} \gamma_{a,b,c}, \]  
(S14)
i.e.,

\[ \frac{\hat{n}_{a,b,c}}{m} \approx \frac{\ln q_{d_x,d_x,d_x} - l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda})}{\gamma_{a,b,c}} = \frac{1}{N_M} \left( \frac{\ln q_{d_x,d_x,d_x} - l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda})}{\gamma_{a,b,c}} \right), \]  
(S15)

Being \(\hat{n}_{jB,i,jA}\)'s relative frequencies, it still holds that

\[ \lim_{m \to \infty} \frac{\hat{n}_{jB,i,jA}}{m} = 0, \]  
(S16)

for each triplet \((j_B, i, j_A)\) except \((a, b, c)\), with the result that

\[ \frac{\hat{n}_{a,b,c}}{m} \approx \frac{m \ln q_{d_x,d_x,d_x} - l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda})}{N_M} = \text{constant}, \]  
(S18)

for each triplet \((a, b, c)\). Eq. (S18) is a very important result because it denotes the unique non-equilibrium weighted partition of the system configuration space, once we have fixed the reference triplet \((d_x, d_x, d_x)\). Such a property is thus the key point for the derivation of \(\text{Prob}(l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda}))\). In particular, by summing together the terms \(\hat{n}_{a,b,c} \gamma_{a,b,c}\) over all the possible values that can be assumed by \((a, b, c)\) except for the triplet \((d_x, d_x, d_x)\), we find that

\[ \sum_{jB,i,jA} \frac{\hat{n}_{jB,i,jA} \gamma_{jB,i,jA}}{m} \approx \frac{(d_{\text{tot}} - 1)}{N_M} \left( m \ln q_{d_x,d_x,d_x} - l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda}) \right), \]  
(S19)

where \(d_{\text{tot}} = 2d_x + d_{\lambda}\) is the dimension of the statistical ensemble defining the stochastic transformation of the system from \(t_0\) to \(t_m\). Therefore, by comparing Eqs. (S19) and (S12), one can immediately state that

\[ N_M = d_{\text{tot}} - 1, \]  
(S20)

so that for \(m\) sufficiently large

\[ \frac{\hat{n}_{a,b,c}}{m} \approx \frac{m \ln q_{d_x,d_x,d_x} - l_{\theta_{m,j}}(\bar{\sigma}, \bar{\lambda})}{(d_{\text{tot}} - 1) \gamma_{a,b,c}}. \]  
(S21)

Once we have obtained a closed-system for the value of the relative frequencies obeying the constraints (S9), we are able to validate the exponential approximation given by the large deviation principle for the probability distribution of \(l_{\theta_{m,j}}\) in the
thermodynamic limit of $m \to \infty$. However, before that, let us recall the mathematical definition of the LD principle [18]. Let $A_n$ be a stochastic process indexed by the integer $n$, and let Prob$(A_n \in B)$ be the probability distribution that $A_n$ takes on a value in the set $B$. Then, Prob$(A_n \in B)$ satisfies a LD principle with rate function $I_B$ if the limit

$$
\lim_{n \to \infty} -\frac{1}{n} \ln \text{Prob}(A_n \in B) = I_B
$$

(S22)

exists. This means that when $n \to \infty$ the dominant behaviour of Prob$(A_n \in B)$ is a decaying exponential in $n$. If the limit (S22) does not exist, then either Prob$(A_n \in B)$ is too singular to admit limit for $n \to \infty$ or it decays super-exponentially s.t. $I_B = \infty$. Accordingly, let us now practically compute the LD form of Prob$(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda}))$. To this end, just take Eq. (S8) and apply the Stirling approximation on $\ln(m!)$ and $\ln(\hat{n}_{jb,i,jA}!)$, that is valid again in the limit of large $m$:

$$
\text{Prob}(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})) = \exp \left( \ln(m!) - \sum_{jb,i,jA} \ln(\hat{n}_{jb,i,jA}!) + \sum_{jb,i,jA} \hat{n}_{jb,i,jA} \ln p_{jb,i,jA} \right)
= \exp \left( m \ln m - \sum_{jb,i,jA} \hat{n}_{jb,i,jA} \ln \hat{n}_{jb,i,jA} + \sum_{jb,i,jA} \hat{n}_{jb,i,jA} \ln p_{jb,i,jA} \right).
$$

Then, by substituting the expression of $\hat{n}_{jb,i,jA}$’s given by Eq. (S21), one has

$$
\text{Prob}(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})) \simeq \exp \left\{ m \ln m - \sum_{jb,i,jA} \ln q_{d_\pi,d_\lambda,d_\sigma} - l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})/m \right\}
= \exp \left\{ m \ln \left( \frac{\ln q_{d_\pi,d_\lambda,d_\sigma} - l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})}{m} \right) \right\},
$$

(S24)

i.e.,

$$
\text{Prob}(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})) \simeq \exp(-m I(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})/m)),
$$

(S25)

where

$$
I(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})/m) = \sum_{d_\pi} \sum_{d_\lambda} \sum_{d_\sigma} f_{jb,i,jA} \ln \left( \frac{f_{jb,i,jA}}{p_{jb,i,jA}} \right)
$$

(S26)

is the rate function associate to the probability distribution Prob$(l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda}))$. In particular, in Eq. (S26),

$$
f_{jb,i,jA} = \ln q_{d_\pi,d_\lambda,d_\sigma} - l_{\theta_{m,j}}(\vec{\theta}, \vec{\lambda})/m
$$

(S27)

for each triplet $(jb, i, jA) \neq (d_\pi, d_\lambda, d_\sigma)$, while

$$
f_{d_\pi,d_\lambda,d_\sigma} = 1 - \sum_{jb,i,jA} f_{jb,i,jA}.
$$

(S28)

As final remark, given the generic triplet $(a, b, c)$, it is worth noting that $f_{a,b,c}$ and the relative frequency $\hat{n}_{a,b,c}$ are simply related by the following equation:

$$
f_{a,b,c} = \frac{\hat{n}_{a,b,c}}{m}.
$$

(S29)
III. Most probable distribution $l_{\theta_{m,j}}$

Here, the most probable value $l_{\theta_{m,j}}$ of the log-conditional probability $l_{\theta_{m,j}}(\vec{\theta}, \vec{X})$ is derived. The latter is obtained by evaluating the value at which the rate function $I(l_{\theta_{m,j}}/m)$ of Eq. (S26) is minimized as a function of $l_{\theta_{m,j}}$ in the thermodynamic limit of $m \to \infty$. $I(l_{\theta_{m,j}}/m)$ is a positive and convex function – see Appendix II – and $l_{\theta_{m,j}}$ is given by a convex sum of the dynamical transition probabilities $p_{jB,i,jA}$. Thus, sufficient condition for its minimization is that the identities

$$\frac{\partial I(l_{\theta_{m,j}}(\vec{\theta}, \vec{X})/m)}{\partial \ln q_{jB,i,jA}} \bigg|_{l_{\theta_{m,j}}(\vec{\theta}, \vec{X})=l_{\theta_{m,j}}^*} = 0,$$

are all simultaneously verified for each triplet $(j_B, i, j_A)$. If we perform the derivative of $I(l_{\theta_{m,j}}/m)$ with respect to $\ln q_{jB,i,jA}$, then one finds that the identities (S30) for the triplets $(j_B, i, j_A)$ apart $(d_x, d_\lambda, d_\pi)$ can be equivalently written by means of an unique equation, i.e.,

$$p_{d_x,d_\lambda,d_\pi} f_{jB,i,jA} = p_{jB,i,jA} \left(1 - \sum_{j_B,i,j_A} f_{jB,i,j_A}\right).$$

(S31)

By summing both sides over $(j_B, i, j_A)$, we get

$$p_{d_x,d_\lambda,d_\pi} \sum_{j_B,i,j_A} f_{jB,i,j_A} = \left(1 - \sum_{j_B,i,j_A} f_{jB,i,j_A}\right) \sum_{j_B,i,j_A} p_{jB,i,j_A},$$

(S32)

that, by using $\sum_{j_B,i,j_A} p_{jB,i,j_A} = 1$, gives

$$\sum_{j_B,i,j_A} f_{jB,i,j_A} = \sum_{j_B,i,j_A} p_{jB,i,j_A}.$$  

(S33)

It is worth observing that Eq. (S33) represents the condition for the minimization of the rate function $I(l_{\theta_{m,j}}(\vec{\theta}, \vec{X})/m)$ with respect to $l_{\theta_{m,j}}$. It means that the most probable trajectories of the system dynamics are those allowing for the equality between the summations of the relative frequencies $f_{jB,i,j_A}$ and the occurrence probabilities $p_{jB,i,j_A}$, respectively. Therefore, this also implies that in general the same value of $l_{\theta_{m,j}}^*$ can be obtained by more than one trajectory within the configuration space of the system, each of them corresponding to a different realization of the stochastic dynamics of the system.

By combining (S33) with Eqs. (S13), (S27) and (S30) and substituting $l_{\theta_{m,j}}(\vec{\theta}, \vec{X})$ with $l_{\theta_{m,j}}^*$, one has that

$$\left(\ln q_{d_x,d_\lambda,d_\pi} - \frac{l_{\theta_{m,j}}^*}{m}\right) = (d_{\text{tot}} - 1) \left(1 - \sum_{j_B,i,j_A} p_{jB,i,j_A}\right) \frac{p_{jB,i,j_A}}{p_{d_x,d_\lambda,d_\pi}} \gamma_{jB,i,j_A},$$

(S34)

for each triplet $(j_B, i, j_A) \neq (d_x, d_\lambda, d_\pi)$. The, if we extend Eq. (S33) by assuming that $f_{jB,i,j_A} = p_{jB,i,j_A} \forall (j_B, i, j_A)$, then

$$\frac{p_{jB,i,j_A}}{p_{d_x,d_\lambda,d_\pi}} \gamma_{jB,i,j_A} = \text{constant}. $$

(S35)

Since the equations in (S34) have to verified for each triplet of the system configuration space, this means that (S34) are identically equivalent to the relation

$$\frac{l_{\theta_{m,j}}^*}{m} = \ln q_{d_x,d_\lambda,d_\pi} - \left(1 - \sum_{j_B,i,j_A} p_{jB,i,j_A}\right) \left(\sum_{j_B,i,j_A} \frac{p_{jB,i,j_A}}{p_{d_x,d_\lambda,d_\pi}} \gamma_{jB,i,j_A}\right),$$

(S36)

finally providing us the analytical expression of the most probable distribution $l_{\theta_{m,j}}^*$:

$$l_{\theta_{m,j}}^* = m \sum_{j_B,i,j_A} p_{jB,i,j_A} \ln q_{jB,i,j_A}.$$  

(S37)
IV. Continuous limit

The continuous limit of the probability distribution \( \text{Prob}(l_{\theta_{m,j}}(\vec{o}, \vec{\lambda})) \) and the corresponding most probable value \( l^*_{\theta_{m,j}} = m \sum_{j_B, i, j_A} p_{j_B, i, j_A} \ln q_{j_B, i, j_A} \) can be obtained by considering (i) a generic continuous \( \lambda \)-distribution for \( p_\lambda \) and (ii) continuous measurement observables \( \mathcal{O} \) and \( \Theta \). Under these assumptions, we just need to replace the summation \( \sum_{j_B=1}^{d_\pi} \sum_{i=1}^{d_\lambda} \sum_{j_A=1}^{d_\nu} (\cdot) \) with the functional-integral

\[
\int_{\mu_B, \mu_A, \lambda} (\cdot) d\mu_B d\mu_A d\lambda,
\]

where \( \mu \) here denotes the continuous set of measurement eigenvectors. Accordingly, with this substitution one has that the dimension of the statistical ensemble defining the stochastic trajectories of the system is simply equal to the hypervolume

\[
V \equiv \int_{\mu_B, \mu_A, \lambda} d\mu_B d\mu_A d\lambda.
\]

Notice that the hypervolume \( V \) is also equal to the scaling factor \( N_{sf} \), since the reference triplet \((d_\pi, d_\lambda, d_\nu)\) is just a point of the system-trajectories configuration space and thus its measure with respect to \( V \) is identically equal to zero.

For the sake of clarity, in the main text we have assumed that the spectrum of the measurement operator is discrete. However, as shown above, such assumption can be easily removed without that the validity of the presented results is affected.