Rigidity of Scattering Lengths and Travelling Times for Disjoint Unions of Convex Bodies

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Abstract. Obstacles $K$ and $L$ in $\mathbb{R}^d$ ($d \geq 2$) are considered that are finite disjoint unions of strictly convex domains with $C^3$ boundaries. We show that if $K$ and $L$ have (almost) the same scattering length spectrum, or (almost) the same travelling times, then $K = L$.

1 Introduction

Let $K$ be a compact subset of $\mathbb{R}^d$ ($d \geq 2$) with $C^3$ boundary $\partial K$ such that $\Omega_K = \mathbb{R}^d \setminus K$ is connected. A scattering ray $\gamma$ in $\Omega_K$ is an unbounded in both directions generalized geodesic (in the sense of Melrose and Sjöstrand [MS1], [MS2]). If $K$ is a finite disjoint union of convex domains, then the scattering rays in $\Omega_K$ are simply billiard trajectories with finitely many common points with $\partial K$. This article concerns two types of problem related to recovering information about the obstacle $K$ from certain measurements of scattering rays in the exterior of $K$. These problems have similarities with various problems on metric rigidity in Riemannian geometry – see [SU], [SUV] and the references there for more information.

1.1 The Scattering Length Spectrum

The first type of problem deals with the so called scattering length spectrum (SLS). Given a scattering ray $\gamma$ in $\Omega_K$, denote by $T_\gamma$ the sojourn time of $\gamma$ (cf. Sect. 2). If $\omega \in S^{d-1}$ is the incoming direction of $\gamma$ and $\theta \in S^{d-1}$ its outgoing direction, $\gamma$ will be called an $(\omega, \theta)$-ray. The scattering length spectrum of $K$ is defined to be the family of sets of real numbers $\mathcal{SL}_K = \{\mathcal{SL}_K(\omega, \theta)\}$ where $(\omega, \theta)$ runs over $S^{d-1} \times S^{d-1}$ and $\mathcal{SL}_K(\omega, \theta)$ is the set of sojourn times $T_\gamma$ of all $(\omega, \theta)$-rays $\gamma$ in $\Omega_K$. It is known (cf. [St3]) that for $d \geq 3$, $d$ odd, and $C^\infty$ boundary $\partial K$, we have $\mathcal{SL}_K(\omega, \theta) = \text{sing supp } s_K(t, \theta, \omega)$ for almost all $(\omega, \theta)$. Here $s_K$ is the scattering kernel related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial \Omega_K$ (cf. [LP1], [M]). Following [St4], we will say that two obstacles $K$ and $L$ have almost the same SLS if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $S^{d-1} \times S^{d-1}$ such that $\mathcal{SL}_K(\omega, \theta) = \mathcal{SL}_L(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R}$.

It is a natural and rather important problem in inverse scattering by obstacles to get information about the obstacle $K$ from its SLS. It is known that various kinds of information about $K$ can be recovered from its SLS, and for some classes of obstacles $K$ is completely recoverable (see e.g. [Ma], [MaR], [LP2], [St2], [St4]) – for example star-shaped obstacles are in this class. However, as an example of M. Livshits shows (cf. Ch. 5 in [M]; see also Figure 1 on p. 14), in general $\mathcal{SL}_K$ does not determine $K$ uniquely.

1.2 Travelling Times

The second type of problems deals with travelling times. Let $\mathcal{O}$ be a large ball in $\mathbb{R}^d$ containing $K$ in its interior and set $S_0 := \partial \mathcal{O}$. For any pair of points $x, y \in S_0$ consider
the scattering rays $\gamma$ incoming through the point $x$ and outgoing through the point $y$. Such rays will be called $(x, y)$-geodesics in $\Omega_K$. Given such $\gamma$, let $t_\gamma$ be the length of the part of $\gamma$ from $x$ to $y$. Let $T_K(x, y)$ the set of travelling times $t_\gamma$ of all $(x, y)$-geodesic in $\Omega_K$. If $K$ and $L$ are two obstacles contained in the interior of $S_0$, we will say that $K$ and $L$ have almost the same travelling times if $T_K(x, y) = T_L(x, y)$ for almost all $(x, y) \in S_0 \times S_0$ (with respect to the Lebesgue measure on $S_0 \times S_0$). Our second type of problem is to get information about the obstacle $K$ from its travelling times.

1.3 Unions of Convex Bodies

For either kind of problems, we consider obstacles $K$ of the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_k,$$

where $K_i$ are strictly convex disjoint domains in $\mathbb{R}^d$ $(d \geq 2)$ with $C^3$ smooth boundaries $\partial K_i$. In this case the so called generalized Hamiltonian (or bicharacteristic) flow $F_t^{(K)} : S^*(\Omega_K) \to S^*(\Omega_K)$ (see Sect. 2) coincides with the billiard flow, so it is easier to deal with.

A point $\sigma = (x, \omega) \in S^*(\Omega_K)$ is called non-trapped if both curves $\{\text{pr}_1(F_t^{(K)}(\sigma)) : t \leq 0\}$ and $\{\text{pr}_1(F_t^{(K)}(\sigma)) : t \geq 0\}$ in $\Omega_K$ are unbounded. Otherwise $\sigma$ is called a trapped point. Here we use the notation $\text{pr}_1(y, \eta) = y$ and $\text{pr}_2(y, \eta) = \eta$. Denote by $\text{Trap}(\Omega_K)$ the set of all trapped points. It is well-known that in general $\text{Trap}(\Omega_K)$ may have positive Lebesgue measure and a non-empty interior in $S^*(\Omega_K)$ (see e.g. Livshits’ example). Set $\hat{T}^*(\Omega_K) = T^*(\Omega_K) \setminus \{0\}$.

Definition 1.1. Let $K, L$ be two obstacles in $\mathbb{R}^d$. We will say that $\Omega_K$ and $\Omega_L$ have conjugate flows if there exists a homeomorphism

$$\Phi : \hat{T}^*(\Omega_K) \setminus \text{Trap}(\Omega_K) \to \hat{T}^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$$

which defines a symplectic map on an open dense subset of $\hat{T}^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$, it maps $S^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$ onto $S^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$, and satisfies $F_t^{(L)} \circ \Phi = \Phi \circ F_t^{(K)}$ for all $t \in \mathbb{R}$ and $\Phi = \text{id}$ on $\hat{T}^*(\mathbb{R}^d \setminus \mathcal{O}) \setminus \text{Trap}(\Omega_K) = \hat{T}^*(\mathbb{R}^d \setminus \mathcal{O}) \setminus \text{Trap}(\Omega_L)$.

It is known that for $K, L$ in a very large (generic) class of obstacles in $\mathbb{R}^d$ $(d \geq 2)$, if $K$ and $L$ have almost the same SLS or almost the same travelling times, then $\Omega_K$ and $\Omega_L$ have conjugate flows ([SL4] and [NS]; see Sect. 2 below where these results are given in full details). In this paper we prove

Theorem 1.2. Let $d \geq 2$ and let each of the obstacles $K$ and $L$ be a finite disjoint union of strictly convex domains in $\mathbb{R}^d$ with $C^3$ boundaries. If $\Omega_K$ and $\Omega_L$ have conjugate flows, then $K = L$.

For the case where $\partial K$ and $\partial L$ are real analytic this was proved in [St2]. As an immediate consequence of Theorem 1.2 and results in [St3] and [NS], one gets the following.

Corollary 1.3. Assume that $K$ and $L$ are obstacles in $\mathbb{R}^d$ $(d \geq 2)$ and each of them is a finite disjoint union of strictly convex domains with $C^3$ boundaries. If $K$ and $L$ have
almost the same scattering length spectrum, or \( K \) and \( L \) have almost the same travelling times, then \( K = L \).

We remark that the above results are non-trivial. Indeed when \( K \) has a large number of connected components and they are densely packed (imagine the molecules of a gas in a container), then there are a great number of billiard trajectories with large numbers of reflections (and possibly with many tangencies) in the exterior of \( K \). Moreover, in such cases many connected components of \( K \) can only be reached by billiard trajectories having many reflections. So, being able to completely recover the obstacle \( K \) by measuring sojourn times only, or travelling times only, is far from being a trivial matter. The assumption that \( \partial K \) and \( \partial L \) are \( C^3 \) smooth is required in order to be able to use some of the results in [St2] and [St4].

\section{Preliminaries}

We refer the reader to [MS1], [MS2] (or Sect. 24.3 in [H]) for definition of the generalized Hamiltonian (bicharacteristic) flow on a symplectic manifold with boundary. In the case of scattering by an obstacle \( K \) this is the generalized geodesic flow \( \mathcal{F}_t^{(K)} : T^*_b(\Omega_K) = T^*_b(\Omega_K) \setminus \{0\} \rightarrow T^*_b(\Omega_K) \) generated by the principal symbol of the wave operator in \( \mathbb{R} \times \Omega_K \). Here \( T^*_b(\Omega_K) = T^*(\Omega_K)/\sim \) is the quotient space with respect to the following equivalence relation on \( T^*(\Omega_K) \): \((x, \xi) \sim (y, \eta) \) iff \( x = y \) and either \( \xi = \eta \) or \( x = y \in \partial K \) and \( \xi \) and \( \eta \) are symmetric with respect to the tangent plane to \( \partial K \) at \( x \). The image \( S^*_b(\Omega_K) \) of the unit cosphere bundle \( S^*(\Omega_K) \) under the natural projection is invariant with respect to \( \mathcal{F}_t^{(K)} \). For simplicity of notation the subscript \( b \) will be suppressed, and it will be clear from the context exactly which second component we have in mind.

In general \( \mathcal{F}_t^{(K)} \) is not a flow in the usual sense of dynamical systems, since there may exist different integral curves issued from the same point of the phase space (see [1]). Let \( \mathcal{K} \) be the class of obstacles that have the following property: for each \( (x, \xi) \in T^*(\partial K) \) if the curvature of \( \partial K \) at \( x \) vanishes of infinite order in direction \( \xi \), then all points \( (y, \eta) \) sufficiently close to \( (x, \xi) \) are diffractive (roughly speaking, this means that \( \partial K \) is convex at \( y \) in the direction of \( \eta \)). It follows that for \( K \in \mathcal{K} \) the flow \( \mathcal{F}_t^{(K)} \) is well-defined and continuous ([MS2]).

We now describe the relevant results from [St2] and [NS] used in the proof of Corollary 1.3. Given \( \xi \in S^{d-1} \) denote by \( Z_\xi \) the hyperplane in \( \mathbb{R}^d \) orthogonal to \( \xi \) and tangent to \( \mathcal{O} \) such that \( \mathcal{O} \) is contained in the open half-space \( R_\xi \) determined by \( Z_\xi \) and having \( \xi \) as an inner normal. For an \( (\omega, \theta) \)-ray \( \gamma \) in \( \Omega \), the sojourn time \( T_\gamma \) of \( \gamma \) is defined by \( T_\gamma = T'_\gamma - 2a \), where \( T'_\gamma \) is the length of that part of \( \gamma \) which is contained in \( R_\omega \cap R_{-\theta} \) and \( a \) is the radius of the ball \( \mathcal{O} \). It is known (cf. [G]) that this definition does not depend on the choice of the ball \( \mathcal{O} \). Following [PS2], given \( \sigma = (x, \xi) \in S^*(\Omega_K) \) so that \( \gamma_K(\sigma) = \{pr_1(\mathcal{F}_t^{(K)}(\sigma)) : t \in \mathbb{R}\} \) is a simply reflecting ray, i.e. it has no tangencies to \( \partial K \), we will say that \( \gamma_K(\sigma) \) is non-degenerate if for every \( t >> 0 \) the map \( \mathbb{R}^d \ni y \mapsto pr_2(\mathcal{F}_t^{(K)}(y, \xi)) \in S^{d-1} \) is a submersion at \( y = x \), i.e. its differential at \( y = x \) has rank \( d - 1 \). Denote by \( \mathcal{K}_0 \) the class of all obstacles \( K \in \mathcal{K} \) satisfying the following non-degeneracy conditions: \( \gamma_K(\sigma) \) is a non-degenerate simply reflecting ray for almost all \( \sigma \in S^*_t(S_0) \setminus \text{Trap}(\Omega_K) \) such that
\( \gamma_K(\sigma) \cap \partial K \neq \emptyset, \) and \( \partial K \) does not contain non-trivial open flat subsets (i.e. open subsets where the curvature is zero at every point). It can be shown without much difficulty from \[PS1\] (see Ch. 3 there) that \( \mathcal{K}_0 \) is of second Baire category in \( \mathcal{K} \) with respect to the \( C^\infty \) Whitney topology in \( \mathcal{K} \). That is, for every \( K \in \mathcal{K} \), applying suitable arbitrarily small \( C^\infty \) deformations to \( \partial K \), one gets obstacles from the class \( \mathcal{K}_0 \) and most deformations have this property.

**Theorem 2.1.** \((\text{St4})\) If the obstacles \( K, L \in \mathcal{K}_0 \) have almost the same SLS, then \( \Omega_K \) and \( \Omega_L \) have conjugate flows. Conversely, if \( K, L \in \mathcal{K}_0 \) have conjugate flows, then \( K \) and \( L \) have the same SLS.

There is a similar result for the travelling times spectrum. Set \( S^*_+(S_0) = \{(x,u) : x \in S_0, u \in S^{d-1}, \langle x,u \rangle < 0\} \). Consider the cross-sectional map \( \mathcal{P}_K : S^*_+(S_0) \setminus \text{Trap}(\Omega_K) \rightarrow S^*(S_0) \) defined by the shift along the flow \( \mathcal{F}_i^{(K)} \). Let \( \gamma \) be a \((x_0,y_0)\)-geodesic in \( \Omega_K \) for some \( x_0,y_0 \in S_0 \), which is a simply reflecting ray. Let \( \omega_0 \in S^{d-1} \) be the (incoming) direction of \( \gamma \) at \( x_0 \). We will say that \( \gamma \) is regular if the differential of map \( S^{d-1} \ni \omega \mapsto \mathcal{P}_K(x_0,\omega) \in S_0 \) is a submersion at \( \omega = \omega_0 \), i.e. its differential at that point has rank \( d-1 \). Denote by \( \mathcal{L}_0 \) the class of all obstacles \( K \in \mathcal{K} \) such that \( \partial K \) does not contain non-trivial open flat subsets and \( \gamma_K(x,u) \) is a regular simply reflecting ray for almost all \((x,u) \in S^*_+(S_0)\) such that \( \gamma(x,u) \cap \partial K \neq \emptyset \). Using an argument from Ch. 3 in \[PS1\] one can show that \( \mathcal{L}_0 \) is of second Baire category in \( \mathcal{K} \) with respect to the \( C^\infty \) Whitney topology in \( \mathcal{K} \). That is, generic obstacles \( K \in \mathcal{K} \) belong to the class \( \mathcal{L}_0 \).

**Theorem 2.2.** \((\text{NS})\) If the obstacles \( K, L \in \mathcal{L}_0 \) have almost the same travelling times, then \( \Omega_K \) and \( \Omega_L \) have conjugate flows. Conversely, if \( K, L \in \mathcal{L}_0 \) have conjugate flows, then \( K \) and \( L \) have the same travelling times.

Next, we describe four propositions from \[St2\] and \[St4\] that are needed in the proof of Theorem 1.2. In what follows we assume

**Hypothesis SCC.** \( K \) and \( L \) are finite disjoint unions of strictly convex domains in \( \mathbb{R}^d \) \((d \geq 2)\) with \( C^3 \) boundaries, and with conjugate generalized geodesic flows.

Given \( \sigma \in S^*(\Omega_K) \) denote \( \gamma^+_K(\sigma) := \{\text{pr}_1(\mathcal{F}_i^{(K)}(\sigma)) : t \geq 0\} \).

**Proposition 2.3.** \((\text{St2})\)

(a) There exists a countable family \( \{M_i\} = \{M_i^{(K)}\} \) of codimension 1 submanifolds of \( S^*_+(S_0) \setminus \text{Trap}(\Omega_K) \) such that every \( \sigma \in S^*_+(S_0) \setminus (\text{Trap}(\Omega_K) \cup M_i) \) generates a simply reflecting ray in \( \Omega_K \). Moreover the family \( \{M_i\} \) is locally finite, that is any compact subset of \( S^*_+(S_0) \setminus \text{Trap}(\Omega_K) \) has common points with only finitely many of the submanifolds \( M_i \).

(b) There exists a countable family \( \{R_i\} \) of codimension 2 smooth submanifolds of \( S^*_+(S_0) \) such that for any \( \sigma \in S^*_+(S_0) \setminus (\cup R_i) \) the trajectory \( \gamma_K(\sigma) \) has at most one tangency to \( \partial K \).

(c) There exists a countable family \( \{Q_i\} \) of codimension 2 smooth submanifolds of \( S^*_0(\Omega_K) \) such that for any \( \sigma \in S^*_+(S_0) \setminus (\cup Q_i) \) the trajectory \( \gamma_K(\sigma) \) has at most one tangency to \( \partial K \).
It follows from the conjugacy of flows and Proposition 4.3 in [St1] that the submanifolds $M_i$ are the same for $K$ and $L$, i.e. $M_i^{(K)} = M_i^{(L)}$ for all $i$.

**Remark.** Different submanifolds $M_i$ and $M_j$ may have common points (these generate rays with more than one tangency to $\partial K$) and in general are not transversal to each other. However, as we see from part (b), if $M_i \neq M_j$ and $\sigma \in M_i \cap M_j$, then locally near $\sigma$, $M_i \neq M_j$, i.e. there exist points in $M_i \setminus M_j$ arbitrarily close to $\sigma$.

For the present case we also have some information about the size of the set $\text{Trap}(\Omega_K)$.

**Proposition 2.4.** ([St2]) Let $d \geq 2$ and let $K$ have the form (1.1). Then $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ is arc connected.

**Proof.** This is proved in [St2] assuming $d \geq 3$, however a small modification of the argument works for $d = 2$ as well. We sketch it here for completeness. So, assume that $d \geq 2$. Consider an arbitrary $\sigma_0' = (x_0', \xi_0') \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$. It is enough to find a continuous curve $\sigma(t)$ in $S^*(\mathbb{R}^d \setminus \mathcal{O}) \setminus \text{Trap}(\Omega_K)$ such that $\sigma(0) = \sigma_0'$ and $\sigma(1)$ generates a free trajectory in $S^*(\mathbb{R}^n \setminus \mathcal{O})$, i.e. a trajectory without any reflections at $\partial K$. We will assume that the trajectory $\gamma_K^*(\sigma_0)$ has a common point with $K$; otherwise there is nothing to prove.

There exists a strictly convex smooth hypersurface $X$ in $\mathbb{R}^n \setminus \mathcal{O}$ with a continuous unit normal field $\nu_X$ such that for some $x_0 \in X$ and $t > 0$, we have $x_0' = x_0 + t\nu_X(x_0)$ and $\nu_X(x_0) = \xi_0'$, and there exists $x_1 \in X$ such that the ray $\{x_1 + t\nu_X(x_1) : t > 0\}$ has no common points with $K$ (e.g. take the boundary sphere $X$ of a very large ball in $\mathbb{R}^d \setminus \mathcal{O}$ with exterior unit normal field $\nu_X$). Clearly it is enough to construct a $C^1$ curve $\sigma(t)$ in $S^*(X) \setminus \text{Trap}(\Omega_K)$ such that $\sigma(0) = (x_0, \nu_X(x_0))$ and $\sigma(1) = (x_1, \nu_X(x_1))$.

Set $\tilde{X} = \{(x, \nu_X(x)) : x \in X\}$. Considering trajectories $\gamma_K^*(\sigma)$ with $\sigma \in \tilde{X}$ and using the strict convexity of $X$ will allow us to use the strong hyperbolicity properties of the billiard flow in $\Omega_K$ ([S1], [S2]; see also [St1]).

Take a very large open ball $U_1$ that contains the orthogonal projection of the convex hull $\tilde{K}$ of $K$ onto $X$ and the point $x_1$ as well. Since $\text{Trap}(\Omega_K) \cap S^*(U_1)$ is compact and $\sigma_0 \notin \text{Trap}(\Omega_K)$, there exists an open connected neighbourhood $V_0$ of $\sigma_0$ in $S^*(U_1)$ with $V_0 \cap \text{Trap}(\Omega_K) = \emptyset$.

It follows from Proposition 2.3(b) that there exists a countable family $\{Q_i\}$ of smooth codimension 2 submanifolds of $S^*(U_1)$ such that for any $\sigma \in S^*(U_1) \setminus (\cup_1 Q_i)$, the trajectory $\gamma_K(\sigma)$ has at most one tangency to $\partial K$. The submanifolds $Q_i$ are obtained from the submanifolds $Q_i$ in Proposition 2.3(b) by translation along the second (vector) component, so, locally they are invariant under the flow $\mathcal{F}_t^{(K)}$.

Using Thom’s Transversality Theorem (cf. e.g. [Hi]), and applying an arbitrarily small in the $C^3$ Whitney topology deformation to $X$, as in the proof of Proposition 5.1 in [St2], without loss of generality we may assume that $X$ is transversal to each of the submanifolds $Q_i$. When $d = 2$, this simply means that $\tilde{X} \cap Q_i'$ is a discrete subset of $\tilde{X}$. When $d \geq 3$, $\tilde{X} \cap Q_i'$ is a submanifold of $\tilde{X}$ with $\dim(\tilde{X} \cap Q_i') = (2d - 3 + d - 1) - (2d - 1) = d - 3$. By the Sum Theorem for the topological dimension $\dim$ (cf. e.g. Theorem 15 in [E]), for $X' = \tilde{X} \cap (\cup_i Q_i)$ we get $\dim(X') \leq d - 3$. 

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Next, denote by $X_0$ the set of those $\sigma \in \tilde{X} \cap \text{Trap}(\Omega_K)$ such that the trajectory $\gamma_K(\sigma)$ has no tangencies to $\partial K$. Given integers $p, q$ such that $q \in \{1, \ldots, k_0\}$ and $p \geq 1$, denote by $X(p, q)$ the set of those $\sigma \in \tilde{X} \cap \text{Trap}(\Omega_K)$ such that $\gamma_K(\sigma)$ has exactly one tangency to $\partial K$ which is its $p$th reflection point and it belongs to $\partial K_q$. As in the proof of Proposition 5.1 in [St2] (see also the proof of Lemma 3.1 below where we repeat this argument), it follows that each of the subspaces $X_0$ and $X(p, q)$ of $\tilde{X}$ has topological dimension 0. Assuming $d \geq 3$, the Sum Theorem for topological dimension (cf. Theorem 15 in [F]) shows that

$$\dim(\tilde{X} \cap \text{Trap}(\Omega_K)) \leq \dim(X' \cup X_0 \cup \cup_{p,q} X(p, q)) \leq d - 3.$$  

In the case $d = 2$ we simply have $\dim(\tilde{X} \cap \text{Trap}(\Omega_K)) = 0$. In both cases a theorem by Mazurkiewicz (see e.g. Theorem 25 in [F]) implies that $\tilde{X} \setminus \text{Trap}(\Omega_K)$ is arc connected. Thus, there exists a $C^1$ curve $\sigma(t)$ in $\tilde{X} \setminus \text{Trap}(\Omega_K)$ (and so in $S^*(X) \setminus \text{Trap}(\Omega_K)$) such that $\sigma(0) = (x_0, \nu_X(x_0))$ and $\sigma(1) = (x_1, \nu_X(x_1))$. This proves the proposition. ■

Since $S^*_+(S_0)$ is a manifold and $\text{Trap}(\Omega_K)$ is compact, it follows from Proposition 2.4 that any two points in $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ can be connected by a $C^1$ curve lying entirely in $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$. Denote by $S_K$ the set of the points $\sigma \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K(\sigma)$ is a simply reflecting ray. It follows from [MS2] (cf. also Sect. 24.3 in [H]) and Proposition 2.4 in [St2] that $S_K$ is open and dense and has full Lebesgue measure in $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$. Proposition 4.3 in [St4] shows that if $K, L$ have conjugate flows, then $S_K = S_L$. Moreover, using Proposition 6.3 in [St4] and the above Proposition 2.4 we get the following.

**Proposition 2.5.** Let $K, L$ satisfy Hypothesis SCC. Then

$$\#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L)$$

(2.2)

for all $\sigma \in S_K = S_L$.

**Definition.** A $C^1$ path $\sigma(s), 0 \leq s \leq a$ (for some $a > 0$), in $S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ will be called $K$-admissible if it has the following properties:

(a) $\sigma(0)$ generates a free ray in $\Omega_K$, i.e. a ray without any common points with $\partial K$.

(b) if $\sigma(s) \in M_i$ for some $i$ and $s \in [0, a]$, then $\sigma$ is transversal to $M_i$ at $\sigma(s)$ and $\sigma(s) \notin M_j$ for any submanifold $M_j \neq M_i$.

Notice that under Hypothesis SCC, we have $S^*_+(S_0) \setminus \text{Trap}(\Omega_K) = S^*_+(S_0) \setminus \text{Trap}(\Omega_L)$ and a path is $K$-admissible if it is $L$-admissible. So, in what follows we will just call such paths admissible. If a path $\sigma$ is admissible, it follows from (b) that every $\sigma(s)$ generates a scattering ray with at most one tangent point to $\partial K$ and the tangency (if any) is of first order only. It is clear that if the curve $\sigma(s)$ ($0 \leq s \leq a$) is admissible and $\rho(s)$ ($0 \leq s \leq a$) is uniformly close to $\sigma(s)$ (i.e. $\rho(s)$ and $\sigma(s)$ and their derivatives are $\epsilon$-close for all $s$ for some sufficiently small $\epsilon > 0$), then $\rho(s)$ is also admissible.
Proposition 2.6. For any $\rho \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ there exists an admissible path $\sigma(s), 0 \leq s \leq a$, with $\sigma(a) = \rho$.

Proof. This follows from Proposition 6.3 in [ST4] (or rather its proof) and Proposition 2.4 above. ■

3 Proof of Theorem 1.2

Let $K$ and $L$ be as in Theorem 1.2. We will show that they coincide. A point $y \in \partial K$ will be called regular if $\partial K = \partial L$ in an open neighbourhood of $y$ in $\partial K$. Otherwise $y$ will be called irregular. A point $y \in \partial K$ will be called accessible if there exists $\rho \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ such that the trajectory $\gamma_K(\rho)$ contains the point $y$. Denote by $A_K$ the set of all accessible points $y \in \partial K$.

Lemma 3.1. Let $d \geq 2$ and let $K$ have the form (1.1). Then $A_K$ is dense in $\partial K$.

Proof. We will use a slight modification of an argument contained in the proof of Proposition 5.1 in [ST2] (see also Proposition 2.4 above) which assumes $d \geq 3$. We will sketch this proof here dealing with the case $d = 2$ as well.

First, it follows from Proposition 2.3(c) that there exists a countable locally finite family $\{Q_i\}$ of codimension 2 smooth submanifolds of $S^*_0(\Omega_K)$ such that for any $\sigma \in S^*_0(\Omega_K) \setminus (\cup_i Q_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to $\partial K$. Notice that when $d = 2$, we have dim($S^*_0(\Omega_K)$) = 2, and then each $Q_i$ is a finite subset of $S^*_0(\Omega_K)$, so $\cup_i Q_i$ is a discrete subset of $S^*_0(\Omega_K)$.

Consider an arbitrary $x_0 \in \partial K$ and let $\delta > 0$. We will show that there exist points $x \in A_K$ which is $\delta$-close to $x_0$. Take an arbitrary $u_0 \in S^{d-1}$ so that $\sigma_0 = (x_0, u_0) \in S^*_0(\Omega_K)$. Fix a small $\epsilon > 0$ and set $X = \{x_0 + \epsilon u : ||u - u_0|| < \epsilon\}$ and $\tilde{X} = \{(x_0 + \epsilon u, u) : ||u - u_0||\}$. As in the proof of Proposition 2.4, using Thom’s Transversality Theorem (cf. e.g. [Hi]), and applying an arbitrarily small in the $C^3$ Whitney topology deformation to $X$, we get a $C^3$ convex surface $Y$ in $\Omega_K$ which $C^3$-close to $X$ and so that $\tilde{Y}$ is transversal to each of the submanifolds $Q'_i$. We take $Y$ so close to $X$ that for every $(y, v) \in \tilde{Y}$ for the point $x \in \partial K$ with $x + t v = y$ for some $t > 0$ close to $\epsilon$ we have $||x_0 - x|| < \delta$. As in the proof of Proposition 2.4, when $d = 2$, $\tilde{Y} \cap Q'_i$ is a discrete subset of $\tilde{Y}$, and when $d \geq 3$, for $Y' = \tilde{Y} \cap (\cup_i Q_i)$ we have dim($Y'$) $\leq d - 3$.

Let $Y_0$ be the set of those $\sigma \in \tilde{Y} \cap \text{Trap}(\Omega_K)$ such that the trajectory $\gamma_K(\sigma)$ has no tangencies to $\partial K$, and let $F = \prod_{i=0}^k F_i$, where $F = \{1, 2, \ldots, k\}$. Fix an arbitrary $\theta \in (0, 1)$ and consider the metric $d$ on $F$ defined by $d((x_i), (y_i)) = 0$ if $x_i = y_i$ for all $i$, $d((x_i), (y_i)) = 1$ if $x_i \neq y_i$, and $d((x_i), (y_i)) = \theta^N$ if $x_i = y_i$ for all $i = 0, 1, \ldots, N - 1$ and $N \geq 1$ is maximal with this property. Then $F$ is a compact totally disconnected metric space. Consider the map $f : Y_0 \rightarrow F$, defined by $f(\sigma) = (i_0, i_1, \ldots)$, where the $j$th reflection point of $\gamma_K(\sigma)$ belongs to $\partial K_i$, for all $j = 0, 1, \ldots$. Clearly, the map $f$ is continuous and, using the strict convexity of $Y$, it follows from [ST1] that $f^{-1} : f(Y_0) \rightarrow Y_0$ is also continuous (it is in fact Lipschitz with respect to an appropriate metric on $F$). Thus, $Y_0$ is homeomorphic to a subspace of $F$ and therefore $Y_0$ is a compact totally disconnected subset of $\tilde{Y}$.

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Given integers \( p, q \) such that \( q \in \{1, \ldots, k_0\} \) and \( p \geq 1 \), denote by \( Y(p, q) \) the set of those \( \sigma \in \bar{Y} \cap \text{Trap}(\Omega_K) \) such that \( \gamma_K(\sigma) \) has exactly one tangency to \( \partial K \) which is its \( p \)th reflection point and it belongs to \( \partial K_q \). Given \( p, q \), define \( f : Y(p, q) \rightarrow \bar{F} \) as above and notice that in the present case we have \( \gamma_q = p \) for any \( \sigma \in Y(p, q) \), where \( f(\sigma) = (i_j) \).

This and the definition of \( Y(p, q) \) imply that \( f \) is continuous on \( Y(p, q) \) and using [St1], again, we get that \( Y(p, q) \) is homeomorphic to a subspace of \( \bar{F} \) and so \( Y(p, q) \) is a compact totally disconnected subset of \( S^*_\partial(K(\Omega_K)) \).

Thus, each of the sets \( Y', Y_0 \) and \( Y(p, q) \) is a subsets of first Baire category in \( \bar{Y} \). It now follows from Baire’s category theorem that \( Y' \cup Y_0 \cup \bigcup_{p,q} Y(p, q) \) is nowhere dense in \( \bar{Y} \). Thus, \( A_K \cap \partial \bar{Y} \supseteq \bar{Y} \setminus (Y' \cup Y_0 \cup \bigcup_{p,q} Y(p, q)) \) is dense in \( \bar{Y} \).

Thus, there exists \( (y,v) \in \bar{Y} \) which generates a non-trapped trajectory. By the choice of \( Y \), if \( x \in \partial K \) is so that \( x + t v = y \) for some \( t > 0 \) close to \( \epsilon \), then we have \( \|x_0 - x\| < \delta \). Clearly, \( x \in A_K \), so \( x_0 \) is \( \delta \)-close to a point in \( A_K \). This proves the assertion.

**Definition.** For any integer \( n \geq 1 \) let \( Z_n \) be the set of those irregular points \( x \in \partial K \) for which there exists an admissible path \( \sigma(s), 0 \leq s \leq a \), in \( S^*_{\partial}(S_0) \setminus \text{Trap}(\Omega_K) \) such that \( \sigma(a) \) generates a free ray in \( \mathbb{R}^d \), \( x \) belongs to the billiard trajectory \( \gamma_K'(\sigma(a)) \) and for any \( s \in [0,a] \) the trajectory \( \gamma_K(\sigma(s)) \) has at most \( n \) irregular common points with \( \partial K \).

Notice that in the above definition, the billiard trajectory \( \gamma_K(\sigma(s)) \) may have more than \( n \) common points with \( \partial K \) at most \( n \) of them will be irregular and all the others must be regular. We will prove by induction on \( n \) that \( Z_n = \emptyset \) for all \( n \geq 1 \).

First, we will show that \( Z_1 = \emptyset \). Let \( x \in Z_1 \). Then there exists a \( C^1 \) path \( \sigma(s), 0 \leq s \leq a \), in \( S^*_{\partial}(S_0) \) as in the definition of \( Z_1 \). In particular, \( x \) lies on \( \gamma_K'(\sigma(a)) \) and for each \( s \in [0,a] \) the trajectory \( \gamma_K(\sigma(s)) \) has at most 1 irregular point. As in the argument below dealing with the inductive step, we may assume that \( a > 0 \) is the smallest number for which \( \gamma_K'(\sigma(a)) \) has an irregular point, i.e. for all \( s \in [0,a] \) the trajectory \( \gamma_K(\sigma(s)) \) contains no irregular point at all. Set \( \rho = \sigma(a) \) and \( \gamma = \gamma_K(\sigma(a)) \).

Before continuing let us remark that \( \gamma = \gamma_K^+(\rho) \) and \( \gamma' = \gamma_K^+(\rho') \) must have the same number of common points with \( \partial K \) and \( \partial L \), respectively. Indeed, if \( \gamma \) has a tangent point to \( \partial K \) (then \( \rho \in M_i \) for some, unique, \( i \)), then \( \gamma \) has only one tangent point to \( \partial K \) and \( \gamma' \) also has a unique tangent point to \( \partial L \). Let \( k \) be the number of proper reflection points of \( \gamma \) at \( \partial K \). Then we can find \( \rho' \in S^*_{\partial}(S_0) \setminus \text{Trap}(\Omega_K) \) arbitrarily close to \( \rho \) so that \( \gamma_K^+(\rho') \) has exactly \( k \) common points with \( \partial K \), all of them being proper reflection points. By Proposition 2.5, \( \gamma_K^+(\rho') \) also has exactly \( k \) common points with \( \partial L \), all of them being proper reflection points. Thus, \( \gamma_K^+(\rho') \) also has at most \( k \) proper reflection points, so

\[ \sharp(\text{proper reflection points of } \gamma_K^+(\rho')) \geq \sharp(\text{proper reflection points of } \gamma_K^+(\rho')). \]

By symmetry, it follows that \( \gamma_K^+(\rho) \) and \( \gamma_K^+(\rho') \) have the same number of proper reflection points, and therefore the same number of common points with \( \partial K \) and \( \partial L \), respectively.

Let \( x_1, \ldots, x_m \) be the common points of \( \gamma_K(\rho) \) and \( \partial K \) and let \( x_i = x \) for some \( i \). Then for each \( j \neq i \) there exists an open subset \( U_j \) of \( \partial K \) such that \( U_j = U_j \cap \partial L \). Let \( \rho = (x_0, u_0) \) and let \( \omega = (x_{m+1}, u_{m+1}) \in S^*(S_0) \) be the last common point of \( \gamma_K(\rho) \) with

\[ ^1\text{We should probably remark that } \gamma \text{ may have more than one common points with } \partial K, \text{ and for some } s < a, \gamma_K(\sigma(s)) \text{ may have common points with } \partial K; \text{ however all of them will be regular points.} \]
Let $S^*(S_0)$ before it goes to $\infty$. Let $F^{(K)}_j(t)(\rho) = (x_j, u_j)$, $1 \leq j \leq m + 1$. It then follows from the above that $F^{(K)}_j(t)(\rho) = F^{(L)}_j(t)(\rho)$ for $0 \leq t < t_i$ and also, $F^{(K)}_j(\omega) = F^{(L)}_j(\omega)$ for all $-(t_{m+1} - t_i) < \tau \leq 0$. So, the trajectories $\gamma_K(\rho)$ and $\gamma_L(\rho)$ both pass through $x_{i-1}$ with the same (reflected) direction $u_{i-1}$ and through $x_{i+1}$ with the same (reflected) direction $u_{i+1}$. Thus, $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$ are common points of $\gamma^*_L(\rho)$ and $\partial L$. As observed above, $\gamma^*_L(\rho)$ must have exactly $m$ common points with $\partial L$, so it has a common point $y_i$ with $\partial L$ ‘between’ $x_{i-1}$ and $x_{i+1}$.

Next, we consider two cases.

**Case 1.** $x_i$ is a proper reflection point of $\gamma$ at $\partial K$. It then follows immediately from the above that $x_i$ lies on $\gamma'$ and $\gamma'$ has a proper reflection point at $x_i$, so in particular $x_i \in \partial L$. Moreover for any $y \in \partial K$ sufficiently close to $x_i$ there exists $\rho' \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ close to $\rho$ so that $\gamma^+_K(\rho')$ has a proper reflection point at $y$. Then repeating the above argument and using again $U_j = U_i \cap \partial L$ for $j \neq i$, shows that $y \in \partial L$. Thus, $\partial K = \partial L$ in an open neighbourhood of $x = x_i$ in $\partial K$, which is a contradiction with the assumption that $x$ is an irregular point.

**Case 2.** $x_i$ is a tangent point of $\gamma$ to $\partial K$. Then $\gamma$ (and so $\gamma'$) has $m-1$ proper reflection points and $y_i$ is a point on the segment $[x_{i-1}, x_{i+1}]$. Assume for a moment that $y_i \neq x_i$. Clearly we can choose $x'_i \in \partial K$ arbitrarily close to $x_i$ and $u'_i \in S^{d-1}$ close to $u_i$ so that $u'_i$ is tangent to $\partial K$ at $x'_i$ and the straight line determined by $x'_i$ and $u'_i$ intersects $\partial L$ transversally near $y_i$. Let $\rho' \in S^*_+(S_0) \setminus \text{Trap}(\Omega_K)$ be the point close to $\rho$ which determines a trajectory $\gamma^+_K(\rho')$ passing through $x'_i$ in direction $u'_i$, i.e. tangent to $\partial K$ at $x'_i$. Then $\gamma^+_K(\rho')$ has $m-1$ proper reflection points and one tangent point, while $\gamma^*_L(\rho')$ has $m$ proper reflection points and no tangent points at all. This impossible, so we must have $y_i = x_i$. Moreover, using a similar argument one shows that every $x' \in \partial K$ sufficiently close to $x_i$ belongs to $\partial L$, as well. So, $x_i$ is a regular point, contradicting our assumption.

This proves that $Z_1 = \emptyset$. Next, suppose that for some $n > 1$ we have $Z_1 = \ldots = Z_{n-1} = \emptyset$ and that $Z_n \neq \emptyset$. Let $x \in Z_n$. Then there exists an admissible $C^1$ path $\sigma(s)$, $0 \leq s \leq a$, in $S^*_+(S_0)$ as in the definition of $Z_n$. In particular, $x$ lies on $\gamma^+_K(\sigma(a))$ and for each $s \in [0, a]$ the trajectory $\gamma_K(\sigma(s))$ has at most $n$ irregular points. Set

$$A = \{ s \in (0, a) : \gamma^+_K(\sigma(s)) \text{ has } n \text{ irregular points } \}.$$ 

Clearly $A \neq \emptyset$, since by assumption $x \in Z_n$ and $Z_j = \emptyset$ for $j < n$. Set $b = \inf A$. Then there exists a decreasing sequence $\{s_m\} \subset (0, a)$ with $s_m \searrow b$ and such that for each $m \geq 1$ the trajectory $\gamma^+_K(\sigma(s_m))$ contains $n$ distinct irregular (consecutive) points $x_1^{(m)}, \ldots, x_n^{(m)}$. Since $K$ has the form (1.1), these points belong to distinct connected components of $K$.

Choosing an appropriate subsequence of $\{s_m\}$, we may assume that $x_i^{(m)}$ belongs to the same connected component $K_{j_i}$ for all $m \geq 1$ and there exists $x_i = \lim_{m \to \infty} x_i^{(m)}$ for all $i = 1, \ldots, n$. Then $x_1, \ldots, x_n$ are distinct common points of $\gamma^+_K(\sigma(b))$ with $\partial K$. Moreover, if some $x_i$ is a regular point, then $\partial K = \partial L$ in an open neighbourhood of $x_i$ in $\partial K$, so $x_i^{(m)}$ would be regular for large $m$ – contradiction. Thus, all $x_1, \ldots, x_n$ are irregular points.

The above reasoning shows that we may assume $b = a$, $\gamma^+_K((a))$ contains exactly $n$ irregular points, and for any $s \in [0, a)$ the trajectory $\gamma^+_K(\sigma(s))$ has $< n$ irregular points.
Now the inductive assumption implies that for any \( s \in [0,a) \) the trajectory \( \gamma_K^+(\sigma(s)) \) contains no irregular points at all. Since \( Z_j = \emptyset \) for \( j < n \), every irregular point of \( \gamma_K^+(\sigma(a)) \) must belong to \( Z_n \).

Set \( \gamma = \gamma_K^+(\sigma(a)) \) for brevity. Let \( x_1, \ldots, x_n \) be the consecutive irregular points of \( \gamma \) (which may have some other common points with \( \partial K \)) and for \( s < a \) close to \( a \), let \( x_1(s), \ldots, x_n(s) \) be the consecutive common points of \( \gamma_K^+(\sigma(s)) \) with \( \partial K \) such that \( x_i(s) \) lies on the connected component \( K_{j_i} \) of \( K \) containing \( x_i \) \((i = 1, \ldots, n)\). Then for every \( i \), \( \partial K = \partial L \) in an open neighbourhood of \( x_i(s) \) in \( \partial K \) for \( s < a \) close to \( a \), so there exists an open subset \( U_i \) of \( \partial K \) with \( x_i \in \overline{U_i} \) and \( \partial K \cap U_i = \partial L \cap U_i \).

Next, we consider two cases.

**Case 1.** \( \gamma \) contains no tangent points to \( \partial K \).

Choose a small \( \delta > 0 \) (how small will be determined later). We will replace the path \( \sigma(s) \) by another one \( \tilde{\sigma}(s) \), \( 0 \leq s \leq a \), such that \( \tilde{\sigma}(s) = \sigma(s) \) for \( s \in [0,a - 2\delta] \).

Let \( F_{t_1}(K)(x_1, u_1) = (x_2, u_2) \) for some \( t_1 > 0 \), where \( u_1 \in S^{d-1} \) is the (reflected) direction of \( \gamma \) at \( x_1 \) and \( u_2 \in S^{d-1} \) is the reflected direction of \( \gamma \) at \( x_2 \). Take a small \( \epsilon > 0 \) and set

\[
X := \{ x_1 + \epsilon u : u \in S^{d-1}, \|u - u_1\| < \delta \}, \quad \bar{X} := \{ (x_1 + \epsilon u, u) : u \in S^{d-1}, \|u - u_1\| < \delta \}.
\]

Let \( u_1(s) \) be the (reflected) direction of the trajectory \( \sigma(s) \) at \( x_1(s) \). Take \( t' < t_1 \) close to \( t_1 \) and set \( \bar{Y} = F_{t'}(K)(\bar{X}), Y = \text{pr}_1(\bar{Y}) \). Then

\[
\bar{Y} = \{ (y, \nu Y(y)) : y \in Y \},
\]

where \( \nu Y(y) \) is the unit normal to \( Y \) at \( y \) in the direction of the flow \( F_{t'}(K) \), and \( (y_1, v_1) = F_{t'}(K)(x_1, u_1) \in \bar{Y} \), so \( y_1 \in Y \).

Take a small \( \epsilon > 0 \) and consider

\[
\tilde{Y} = \{ (y, v) : y \in Y, v \in S^{d-1}, \|v - \nu Y(y)\| < \epsilon \}.
\]

There exists an open neighbourhood \( V \) of \((x_1, u_1)\) in \( S^*_\partial K(\Omega K) \) such that the shift

\( \Phi : V \rightarrow \tilde{Y} \) along the flow \( F_{t'}(K) \) is well-defined and smooth. Assuming \( \delta \) is sufficiently small, \( (y_1(s), v_1(s)) = \Phi(x_1(s), u_1(s)) \) is well-defined for \( s \in [a - 2\delta, a] \). Then \( y_1(s), s \in [a - 2\delta, a] \), is a \( C^1 \) curve on \( Y \) and the ray issued from \( y_1(s) \) in direction \( v_1(s) \) hits \( \partial K_{j_2} \) at \( x_2(s) \). Moreover, \( v_1(a) = \nu Y(y_1) \).

Assuming \( \delta > 0 \) is sufficiently small, for all \( s, s' \in [a - 2\delta, a] \), there exists a unique vector \( v_1(s, s') \in S^{d-1} \) such that

\[
y_1(s) + t(s, s')v_1(s, s') = x_2(s')
\]

for some \( t(s, s') \) close to \( t_1 - t' \). Moreover, \( v_1(s, s') \) and \( t(s, s') \) are smoothly \( (C^1) \) depending on \( s \) and \( s' \). Clearly, \( v_1(s, s) = v_1(s) \).

Since \( Y \) is a strictly convex surface with a unit normal field \( \nu Y(y) \), it is clear that for any \( s \in [0,a] \) sufficiently close to \( a \) there exists \( y \in Y \) close to \( y_1 \) such that \( y + tv_1(y) = x_2(s) \) for some \( t \) close to \( t_1 - t' \). Fix a small \( \delta > 0 \), set \( s_0 = a - \delta/2 \) and let \( \tilde{y} \in Y \) be so that

\[
\tilde{y} + t\nu Y(\tilde{y}) = x_2(s_0) \in U_2.
\]
Take a $C^1$ curve $\bar{y}_1(s)$, $s \in [a - 2\delta, a]$, such that $\bar{y}_1(s) = y_1(s)$ for $s \in [a - 2\delta, a - 3\delta/2]$ and $\bar{y}_1(a) = \bar{y}$.

Next, define $\bar{s}(s)$, $s \in [0, a]$, by $\bar{s}(s) = s$ for $0 \leq s \leq a - \delta$ and

$$\bar{s}(s) = \frac{s}{2} + s_0 - \frac{a}{2} = s_0 - \frac{a - s}{2} \in [a - \delta, s_0], \quad s \in [a - \delta, a].$$

Then $\bar{s}$ is continuous however not differentiable at $s = a - \delta$. Take a $C^1$ function $\bar{s}(s)$, $s \in [0, a]$, which coincides with $\bar{s}$ on $[0, \alpha - \delta - \delta_0]$ and on $[\alpha - \delta + \delta_0, a]$ for some $\delta_0 < \delta/2$ so that the range of $\bar{s}$ is the same as that of $\bar{s}$, i.e. it coincides with the interval $[0, s_0]$. Now for any $s \in [a - 2\delta, a]$ take $\bar{v}_1(s)$ so that

$$\bar{y}_1(s) + \bar{t}(s)\bar{v}_1(s) = x_2(\bar{s}(s)) \quad s \in [a - 2\delta, a],$$

for some $\bar{t}(s)$ close to $t_1 - t'$. For $s \in [a - 2\delta, a - \delta - \delta_0]$ we have $\bar{y}_1(s) = y_1(s)$ and $\bar{s}(s) = s$, which imply $\bar{v}_1(s) = v_1(s)$. When $s = a$ we have $\bar{y}_1(a) = \bar{y}$ and $\bar{s}(a) = \bar{s}(a) = s_0$, so by (3.3) we must have $\bar{v}_1(a) = \nu_{C}(\bar{y})$, i.e. $(\bar{y}_1(a), \bar{v}_1(a)) \in \gamma$, and therefore

$$\operatorname{pr}_1(\Phi^{-1}(\bar{y}_1(a), \bar{v}_1(a))) = \operatorname{pr}_1(F_{C, U_1}^{(K)}(\bar{y}_1(a), \bar{v}_1(a))) = x_1. \quad (3.4)$$

Set $(\bar{x}_1(s), \bar{u}_1(s)) = \Phi^{-1}(\bar{y}_1(s), \bar{v}_1(s))$, $s \in [a - 2\delta, a]$. For $s \in [a - 2\delta, a - \delta - \delta_0]$, we have $\bar{s}(s) = \bar{s}(s) = s$ and therefore $\bar{y}_1(s) = y_1(s)$ and $\bar{v}_1(s) = v_1(s)$, which gives $(\bar{x}_1(s), \bar{u}_1(s)) = (x_1(s), u_1(s))$.

Now define the path $\bar{\sigma}(s)$, $0 \leq s \leq a$, on $S^*_+ (S_0)$ by $\bar{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$ and for $s \in [a - 2\delta, a]$ let $\bar{\sigma}(s)$ be the unique point such that $\bar{F}_{C, U_1}^{(K)}(\bar{\sigma}(s)) = (\bar{x}_1(s), \bar{u}_1(s))$ for some $\bar{t}_0(s)$ close to $t_0(s)$, where $\bar{F}_{C, U_1}^{(K)}(\sigma(s)) = (x_1(s), u_1(s))$. It follows from (3.4) that $\bar{s}(a)$ contains the point $x_1$. Moreover the construction of $\bar{\sigma}$ shows that it is a $C^1$ path. Assuming $\delta$ is sufficiently small, $\bar{\sigma}(s)$ has no tangent points to $\partial K$ for all $s \in [a - 2\delta, a]$, so $\bar{\sigma}(s)$ is an admissible path. For $s \in [0, a - 2\delta]$, $\gamma_{K}(\bar{\sigma}(s)) = \gamma_{K}(\sigma(s))$ contains no irregular points. For $s \in [a - 2\delta, a]$, $\gamma_{K}(\bar{\sigma}(s))$ has at most $n$ irregular points, close to $x_1, x_2, \ldots, x_n$ (if any). If for some $s \in [a - 2\delta, a]$, $\gamma_{K}(\bar{\sigma}(s))$ has exactly $n$ irregular points, then the second of these must be $x_2(\bar{s}(s))$. However, $\bar{s}(s) < a$ for all such $s$, so $x_2(\bar{s}(s)) \in U_2$, and therefore that cannot be the case. Thus, for all $s \in [a - 2\delta, a]$, $\gamma_{K}(\bar{\sigma}(s))$ has more than $n - 1$ irregular points, and therefore $x_1$ (and every other irregular point on $\gamma_{K}(\bar{\sigma}(s))$) belongs to $Z_{n-1}$. This is a contradiction with the inductive assumption that $Z_j = \emptyset$ for all $j = 1, \ldots, n - 1$.

**Case 2.** $\gamma$ contains a tangent point $y_0$ to $\partial K$ (this may be one of the irregular points $x_i$).

Let $y_0 \in \partial K_p$ for some $p$. According to the definition of an admissible path, $\gamma$ has only one tangent point to $\partial K$, so all other common points are proper reflection points. Since $n > 1$, at least one of the irregular points on $\gamma$ is a proper reflection point, and at least one of them is different from $y_0$. We will assume $x_1 \neq y_0$; the general case is very similar. As in Case 1, we will replace the path $\sigma(s)$ by another one $\bar{\sigma}(s)$, $0 \leq s \leq a$, such that $\bar{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$ for some small $\delta > 0$.

Let $F_{C, U_1}^{(K)}(x_1, u_1) = (y_0, v_0)$ for some $\tau \in \mathbb{R}$ (which may be positive or negative), where $u_1 \in S^{d-1}$ is the (reflected) direction of $\gamma$ at $x_1$ and $v_0$ is the direction of $\gamma$ at $y_0$. Take a small $\epsilon > 0$ and let

$$X := \{x_1 + \epsilon u : u \in S^{d-1}, \|u - u_1\| < \delta\}, \quad \bar{X} := \{(x_1 + \epsilon u, u) : u \in S^{d-1}, \|u - u_1\| < \delta\}.$$
Assuming $\delta > 0$ is small enough, for $s \in [a - 2\delta, a]$ the trajectory $\gamma_K(\sigma(s))$ has a proper reflection point $x_1(s) \in \partial K_j$. Let $u_1(s)$ be the (reflected) direction of the trajectory $\sigma(s)$ at $x_1(s)$.

Clearly in the present case we have $\sigma(a) \in M_i$ for some $i$. By the definition of an admissible path, $\sigma(a) \notin M_j$ for any $j \neq i$ and moreover $\sigma$ is transversal to $M_i$ at $\sigma(a)$. Let $\sigma(s) = (x_0(s), u_0(s)) \in S^*_+(S_0)$ and let $\mathcal{F}_{t_0(s)}(x_0(s), u_0(s)) = (x_1(s), u_1(s))$. The shift $\Phi : S^*_+(S_0) \rightarrow S^*_+(\Omega_K)$ along the flow $\mathcal{F}_{t_0(s)}^{(K)}$ is well-defined on an open neighbourhood $V_0$ of $(x_0, u_0) = (x_0(a), u_0(a))$ and defines a diffeomorphism $\Phi : V_0 \rightarrow V = \Phi(V_0)$ for some small open neighbourhood $V$ of $(x_1, u_1)$ in $S^*_+(\Omega_K)$. Since $M_i$ is a submanifold of $S^*_+(S_0)$ of codimension one, we can take $V_0$ so that $V_0 \setminus M_i$ has two (open) connected components (separated by $M_i$) – each of them diffeomorphic to an open half-ball. We take $V_0$ so small that $V_0 \cap M_j = \emptyset$ for all $j \neq i$.

Setting $M_i' = \Phi(M_i \cap V_0)$, we get a similar picture in $V$, namely $V \setminus M_i'$ has two (open) connected components (separated by $M_i'$) – each of them diffeomorphic to an open half-ball. Assuming $\delta$ is sufficiently small, we have $\sigma(s) \in V_0$ for all $s \in [a - 2\delta, a]$ and $\sigma$ is transversal to $M_i$ at $\sigma(a) = (x_0, u_0)$. Thus, the curve $(x_1(s), u_1(s)), s \in [a - 2\delta, a]$, in $V$ is transversal to $M_i'$ at $(x_1, u_1)$, so it must be contained in one of the connected components of $V \setminus M_i'$. Denote by $V_+$ the connected component of $V \setminus M_i'$ that contains $(x_1(s), u_1(s))$ for $s \in [a - 2\delta, a]$.

Finally, to get this picture near the tangent point $y_0$, take $\tau' \in (\tau - \delta, \tau - \delta/2)$ and set

$$F = \{(x_1, u) : u \in S^{d-1}, \|u - u_1\| < \delta\} \subset V, \quad \tilde{Y} = \mathcal{F}_{\tau'}^{(K)}(F),$$

and $Y = \text{pr}_1(\tilde{Y})$. Then $\tilde{Y} = \{(y, \nu_Y(y)) : y \in Y\}$, where $\nu_Y(y)$ is the unit normal to $Y$ at $y$ in the direction of the flow $\mathcal{F}_{t_0(s)}^{(K)}$, and $(y_1, v_1) = \mathcal{F}_{\tau'}^{(K)}(x_1, u_1) \in \tilde{Y}$, so $y_1 \in Y$. Let $\Psi : F \rightarrow \tilde{Y}$ be the shift along the flow $\mathcal{F}_{t_0(s)}^{(K)}$. Assuming that $\delta$ is sufficiently small, this defines a diffeomorphism $\Psi : F \rightarrow G$ between $F$ and an open subset $G$ of $\tilde{Y}$. Set $M_i'' = \Psi(F \cap M_i')$. Then $M_i''$ consist of those $(y, \nu_Y(y)) \in G$ that generate trajectories tangent to $\partial K'$ (and this can only happen in the vicinity of $y_0$ on $\partial K_p$). It is clear (by a direct observation using the convexity of $K_p$) that $G \setminus M_i''$ has two connected components. Let $G_+$ be the one with $\Psi^{-1}(G_+) \subset V_+$. Thus, there exists $v'_1 \in S^{d-1}, \|v'_1 - v_1\| < \delta$ such that $(y_1, v'_1) \in G_+$. Applying $\Psi^{-1}$, this gives $u'_1 \in S^{d-1}$ with $\|u'_1 - u_1\| < \delta$ such that $(x_1, u'_1) \in V_+$.

Since $V_+$ is connected (in fact diffeomorphic to an open half-ball), there exists a $C^1$ curve $(\bar{x}_1(s), \bar{u}_1(s)), s \in [a - 2\delta, a]$, in $V_+$ such that $(\bar{x}_1(s), \bar{u}_1(s)) = (x_1(s), u_1(s))$ for $s \in [a - 2\delta, a]$ and $(\bar{x}_1(0), \bar{u}_1(0)) = (x_1, u'_1)$. Define the path $\bar{\sigma}(s)$ by $\bar{\sigma}(s) = \sigma(s)$ for $s \in [0, a - 2\delta]$ and $\bar{\sigma}(s) = \Phi^{-1}(x_1(s), \bar{u}_1(s))$ for $s \in [a - 2\delta, a]$. It is clear from the construction that $\bar{\sigma}$ is a $C^1$ path in $S^*_+(S_0)$. From the properties of $\sigma$, we have that for $s \in [0, a - 2\delta]$ the trajectory $\gamma_K(\bar{\sigma}(s))$ does not contain any irregular points. The choice $V_0$ and that of the curve $(\bar{x}_1(s), \bar{u}_1(s))$, $s \in [a - 2\delta, a]$, show that $\gamma_K(\bar{\sigma}(s))$ has no tangencies to $\partial K$ for any $s \in [a - 2\delta, a]$. On the other hand, $\gamma_K(\bar{\sigma}(a))$ contains the irregular point $x_1$ (and therefore must have $n$ irregular points, since $Z_j = \emptyset$ for $j < n$). Now repeating the argument from Case 1 we get a contradiction. Thus, we must have $Z_n = \emptyset$.

This completes the induction process and proves that $Z_n = \emptyset$ for all $n \geq 1$.  

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We will now prove that $\partial K \subseteq \partial L$. By Lemma 3.1, $A_K$ is dense in $\partial K$, so it is enough to show that $A_K \subseteq \partial L$. Given $x \in A_K$, there exists $\rho \in S^*_+ \setminus \text{Trap}(\Omega_K)$ such that $\gamma_K^+(\rho)$ has a reflection point at $x$. Then there exists an admissible path $\sigma(s)$, $0 \leq s \leq a$, for some $a > 0$, with $\rho' = \sigma(a)$ arbitrarily close to $\rho$. Then $\gamma_K^+(\sigma(\rho'))$ has a reflection point $x'$ at $\partial K$ near $x$. Since there are no irregular points on $\sigma(a)$, it follows that $\partial K = \partial L$ on an open neighbourhood of $x'$. Thus, $x$ is arbitrarily close to $\partial L$, so we must have $x \in \partial L$. This proves that $\partial K \subseteq \partial L$.

By symmetry, $\partial L \subseteq \partial K$, so we have $\partial K = \partial L$. ■

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Figure 1: Livshits’ Example (adapted from Ch. 5 of \[M\]): the internal upper part of the figure is half an ellipse with foci $A$ and $B$. A ray entering the interior of the ellipse between the foci must exit between the foci after reflection. So, no scattering ray has a common point with the bold parts of the boundary.