SPIN STRUCTURES ON SEIBERG-WITTEN MODULI SPACES

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Abstract. Let $M$ be an oriented closed 4-manifold with a spin$^c$ structure $L$. In this paper we prove that under a suitable condition for $(M, L)$ the Seiberg-Witten moduli space has a canonical spin structure and its spin bordism class is an invariant of $M$. We show that the invariant of $M = \#_{j=1}^{l} M_j$ is non-trivial for some spin$^c$ structure when $l$ is 2 or 3 and each $M_j$ is a K3 surface or a product of two oriented closed surfaces of odd genus. As a corollary, we obtain the adjunction inequality for the 4-manifold. Moreover we calculate the Yamabe invariant of $M \# N_1$ for some negative definite 4-manifold $N_1$. We also show that $M \# N_2$ does not admit an Einstein metric for some negative definite 4-manifold $N_2$.

1. Introduction

Since E. Witten introduced the Seiberg-Witten equations ([W]), the moduli space of solutions to the equations has been applied to 4-dimensional topology. M. Furuta used the Seiberg-Witten equations themselves, rather than the moduli space, to obtain the 10/8 theorem ([F]). Roughly speaking, the Seiberg-Witten moduli space is the zero locus of the map defining the equations, which we call the Seiberg-Witten map, between two Hilbert bundles over the Jacobian torus. Furuta used finite dimensional approximation of the Seiberg-Witten map to prove the 10/8 theorem. Moreover using finite dimensional approximation of the Seiberg-Witten map, S. Bauer and Furuta refined the Seiberg-Witten invariants ([BF]). The refined invariant is more powerful than the usual Seiberg-Witten invariant. There are 4-manifolds for which the usual Seiberg-Witten invariants vanish but the Bauer-Furuta invariants do not ([BF, FKM]). It is, however, hard in general to detect the Bauer-Furuta invariants.

To detect the Bauer-Furuta invariants explicitly, we define new invariants for 4-manifolds. This invariant is weaker than the Bauer-Furuta invariant, but easier to treat, in particular when the first Betti
number of the 4-manifold is positive. An outline of the definition of
the invariant is as follows.

Let \((M, g)\) be an oriented, closed 4-dimensional Riemannian manifold
with \(b^+(M) > 1\), and \(\mathcal{L}\) a spin\(^c\) structure on \(M\). We write \(\text{Ind}(D)\)
for the index bundle of the Dirac operators parameterized by \(T = H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})\) (see §3.1). If \(c_1(\text{Ind}(D)) \equiv 0 \mod 2\), then the
Seiberg-Witten moduli space allows a spin structure, and a choice of
square root of the determinant line bundle \(\text{det \text{Ind}(D)}\) determines a spin
structure of the moduli space. The spin bordism class of the moduli
space is an invariant of \(M\) which depends only on \(\mathcal{L}\) and the choice of
square root of \(\text{Ind}(D)\).

We calculate the invariant for \(M = \#_{j=1}^l M_j\) when \(M_j\) is a \(K3\) surface
or a product of two oriented closed surfaces of odd genus, and \(l\) is 2 or
3. We take a spin\(^c\) structure on \(M\) of the form \(\mathcal{L} = \#_{j=1}^l \mathcal{L}_j\), where \(\mathcal{L}_j\) is
a spin\(^c\) structure on \(M_j\) induced by a complex structure. We show that
in this case \(c_1(\text{Ind}(D)) \equiv 0 \mod 2\) and our invariant is non-trivial. As
an application, we obtain the adjunction inequality for such \(M\), i.e.,
if an oriented closed surface \(\Sigma\) of positive genus is embedded in \(M\)
satisfying that its self-intersection number \(\Sigma \cdot \Sigma\) is nonnegative, then
we have

\[ \Sigma \cdot \Sigma \leq \langle c_1(\text{det} \mathcal{L}), \Sigma \rangle + 2g(\Sigma) - 2. \]

Here \(\text{det} \mathcal{L}\) is the determinant complex line bundle of \(\mathcal{L}\), and \(g(\Sigma)\) is
the genus of \(\Sigma\).

As another application, following Ishida and LeBrun’s argument in
[IL], we compute the Yamabe invariant of \(M\#N_1\) when \(N_1\) is an ori-
cented, closed, negative definite 4-manifold admitting a Riemannian
metric with scalar curvature nonnegative at each point. We also show
that if \(N_2\) is an oriented, closed, negative definite 4-manifold satisfying

\[ 4l - (2\chi(N_2) + 3\tau(N_2)) \geq \frac{1}{3} \sum_{j=1}^l c_1(M_j)^2, \]

then \(M\#N_2\) does not admit an Einstein metric, where \(\tau(N_2)\) and \(\chi(N_2)\)
are the signature and the Euler number of \(N_2\) respectively.

Acknowledgments. This paper is part of the author’s master thesis.
The author would like to thank my advisor Mikio Furuta for his sugges-
tions and warm encouragement. The author also thanks Masashi Ishida
for useful information about Einstein metrics and Yamabe invariants.
2. Finite dimensional approximations of the Seiberg-Witten map

In this section, we review the definition of the Seiberg-Witten map and its finite dimensional approximation. We refer the readers to [BF] for details.

2.1. The Seiberg-Witten map. Let $M$ be an oriented, closed, connected 4-manifold and let $g$ be a Riemannian metric on $M$. Assume that $b^+(M) > 1$. Choose a spin$^c$ structure $\mathcal{L}$ on $M$. We write $S^\pm(\mathcal{L})$ for the positive and negative spinor bundles, and $\det \mathcal{L}$ for the determinant line bundle associated with $\mathcal{L}$.

Let $k$ be an integer larger than or equal to 4 and set $\widehat{\mathcal{G}} = \{ \gamma \in L^2_{k+1}(M, U(1)) | \gamma(x_0) = 1 \}$ for a fixed base point $x_0 \in M$. Fix a connection $A_0$ on $\det \mathcal{L}$, and define $T := (A_0 + i \ker d)/\widehat{\mathcal{G}}$, where $d : L^2_k(T^*M) \to L^2_{k-1}(\Lambda^2 T^*M)$ is the exterior derivative. The action of $\gamma \in \widehat{\mathcal{G}}$ on $A \in (A_0 + i \ker d)$ is defined by

$$\gamma A := A + 2 \gamma^{-1} d \gamma. \tag{2.1}$$

Put

$$\mathcal{C}(\mathcal{L}) := L^2_k(S^+(\mathcal{L}) \oplus T^*M),$$
$$\mathcal{Y}(\mathcal{L}) := L^2_{k-1}(S^-(\mathcal{L}) \oplus \Lambda^+ T^*M) \oplus \mathcal{H}^1_g(M) \oplus (L^2_{k-1}(M)/\mathbb{R}),$$

where $\mathbb{R}$ represents the space of constant functions on $M$ and $\mathcal{H}^1_g(M)$ is the space of harmonic 1-forms on $M$ with respect to $g$. Let $\mathcal{C}(\mathcal{L}) \to T$ and $\mathcal{Y}(\mathcal{L}) \to T$ be Hilbert bundles on $T$ defined by

$$\mathcal{C}(\mathcal{L}) := (A_0 + i \ker d) \times_{\widehat{\mathcal{G}}} \mathcal{C}(\mathcal{L}),$$
$$\mathcal{Y}(\mathcal{L}) := (A_0 + i \ker d) \times_{\widehat{\mathcal{G}}} \mathcal{Y}(\mathcal{L}).$$

The action of $\widehat{\mathcal{G}}$ on $(A_0 + i \ker d)$ is given by (2.1). We define actions of $\widehat{\mathcal{G}}$ on $L^2_k(S^+(\mathcal{L}))$ and on $L^2_{k-1}(S^-(\mathcal{L}))$ by fiber-wise scalar products. We define actions of $\widehat{\mathcal{G}}$ on the other terms to be trivial. We define $U(1)$-actions on $\mathcal{C}(\mathcal{L})$ and $\mathcal{Y}(\mathcal{L})$ by scalar products on $L^2_k(S^+(\mathcal{L}))$ and $L^2_{k-1}(S^-(\mathcal{L}))$ and set

$$\mathcal{P} := \{(g, \eta) \in \text{Riem}(M) \times L^2_k(\Lambda^2 T^*M) | \eta^+_g \neq [F_{A_0}]_g^+ \},$$

where $\text{Riem}(M)$ is the space of Riemannian metrics on $M$, and $[\eta]_g^+$ and $[F_{A_0}]_g^+$ are $\mathcal{H}^1_g(M)$ parts of $\eta$ and $F_{A_0}$ respectively. For $(g, \eta) \in \mathcal{P}$, we define the Seiberg-Witten map by

$$SW_{g, \eta} : \mathcal{C}(\mathcal{L}) \to \mathcal{Y}(\mathcal{L})$$
$$\quad (A, \phi, a) \mapsto (A, D_{A+ia} \phi, F_{A+ia}^+ - q(\phi) - \eta^+ p(a), d^* a),$$
where $q(\phi)$ is a quadratic form of $\phi$ and $p : L^2_k(T^*M) \to \mathcal{H}^1_g(M)$ is the $L^2$-projection. The moduli space $\mathcal{M}_M(\mathcal{L}, g, \eta)$ of solutions to the Seiberg-Witten equations perturbed by $(g, \eta)$ is identified with $SW^{-1}_{g,\eta}(0)/U(1)$ naturally.

The following fact is well known.

**Theorem 2.1 ([KM]).** For generic $(g, \eta) \in \mathcal{P}$, $\mathcal{M}_M(\mathcal{L}, g, \eta)$ is a compact smooth manifold and an orientation on $\mathcal{H}^1_g(M; \mathbb{R}) \oplus \mathcal{H}^+_g(M; \mathbb{R})$ determines an orientation on $\mathcal{M}_M(\mathcal{L}, g, \eta)$.

2.2. **Finite dimensional approximation.** We explain finite dimensional approximations of the Seiberg-Witten map briefly.

Let $D : C(\mathcal{L}) \to \mathcal{Y}(\mathcal{L})$ be the linear part of the SW map:

$$D : C(\mathcal{L}) \to \mathcal{Y}(\mathcal{L})$$

$$(A, \phi, a) \mapsto (A, D_A\phi, d^+a, p(a), d^*a).$$

By Kuiper’s theorem [Ku], we have a global trivialization of $\mathcal{Y}(L) \cong T \times H$, where $H$ is a Hilbert space. We fix a trivialization of $\mathcal{Y}(\mathcal{L})$. Since $\mathcal{Y}(\mathcal{L})$ has the complex part and the real part, $H$ decomposes into the direct sum $H_C \oplus H_R$ of a complex Hilbert space $H_C$ and a real Hilbert space $H_R$.

For a finite dimensional subspace $W \subset H$, let $p_W : \mathcal{Y}(\mathcal{L}) = T \times H \to W$ be the projection. We denote $D^{-1}(T \times W)$ by $\mathcal{F}(W)$. Then we define $f_W : \mathcal{F}(W) \to W$ by

$$f_W = p_W \circ SW|_{\mathcal{F}(W)} : \mathcal{F}(W) \to W.$$ 

**Theorem 2.2 ([BF]).** Let $W^+$ and $\mathcal{F}(W)^+$ be the one-point compactifications of $W$ and $\mathcal{F}(W)$. Then $f_W : \mathcal{F}(W) \to W$ induces a $U(1)$-equivariant map $f_W^+ : \mathcal{F}(W)^+ \to W^+$, and there is a finite dimensional subspace $W \subset H$ such that the following conditions are satisfied.

1. $\text{Im } D + (T \times W) = \mathcal{Y}(\mathcal{L})$.

2. For all finite dimensional subspace $W' \subset H$ such that $W \subset W'$, the diagram

$$\begin{array}{ccc}
\mathcal{F}(W')^+ & \xrightarrow{f_W^+} & (W')^+ \\
\mathcal{F}(W) \oplus \mathcal{F}(U)^+ & \xrightarrow{(f_W \oplus p_U)\mathcal{D}|_{\mathcal{F}(U)}} & (W \oplus U)^+
\end{array}$$
is $U(1)$-equivariant homotopy commutative as pointed maps, where $U$ is the orthogonal complement of $W$ in $W'$.

**Definition 2.3.** When $W \subset H$ satisfies (1) and (2), we call $f_W : \mathcal{F}(W) \to W$ a finite dimensional approximation of the Seiberg-Witten map.

### 3. Spin structures on moduli spaces

In §3.1 by using finite dimensional approximation of the Seiberg-Witten map, we show a sufficient condition for the moduli space to be a spin manifold. In §3.2 we will prove that the spin bordism class of the spin structure on the moduli space is an invariant of $M$. In §3.3 we give some applications of this invariant.

#### 3.1. A sufficient condition for moduli space to have a spin structure

Let $f = f_W : V = \mathcal{F}(W) \to W$ be a finite dimensional approximation of the Seiberg-Witten map. Note that $V$ has a natural decomposition $V = V_C \oplus V_R$ into the direct sum of a complex vector bundle and a real vector bundle. Similarly decompose $W$ as $W = W_C \oplus W_R$.

If we take a generic $(g, \eta) \in \mathcal{P}$ as in Theorem 2.1, $\mathcal{M}_M(L, g, \eta)$ does not include reducible monopoles, hence $f^{-1}(0)$ lies in $V_{irr} := (V_C \setminus \{0\}) \times_T V_R$. Put $\bar{V} := V_{irr}/U(1)$ and $\bar{M} := f^{-1}(0)/U(1)$. We define a vector bundle $\bar{E} \to \bar{V}$ by $\bar{E} := \bar{V}_{irr} \times_{U(1)} W = \bar{E}_C \oplus \bar{E}_R$, where $\bar{E}_C = V_{irr} \times_{U(1)} W_C$, $\bar{E}_R = V_{irr} \times W_R$. Since $f$ is $U(1)$-equivariant, $f$ induces a section $s : \bar{V} \to \bar{E}$. Then $\bar{M}$ is the zero locus of $s$. If necessary, we perturb $s$ on a compact subset in $\bar{V}$ so that $s$ is transverse to the zero section of $\bar{E}$ and $\bar{M}$ is a compact smooth submanifold of $\bar{V}$.

We can orient $\bar{M}$ by using an orientation on $\mathcal{H}^+_g(X) \oplus \mathcal{H}^+_\gamma(X)$ in the following way. The real part $\mathcal{D}_R$ of $\mathcal{D}$ is independent of $A \in T$ and the cokernel is naturally identified with $\mathcal{H}^+_\gamma(X)$. So $W_R$ has the form $\mathcal{H}^+_g(X) \oplus W'_R$ and $\mathcal{D}_R$ induces the isomorphism between each fiber of $V_R$ and $W'_R$. (Hence $V_R$ is a trivial vector bundle on $T$.) If we choose orientations on $W'_R$ and $\mathcal{H}^+_g(X)$, we get an orientation on $\bar{E}_R$ and orientations on $V_R$ and $\mathcal{H}^+_g(X)$ compatible with $\mathcal{D}_R$ and $\mathcal{O}$. $T$ is naturally identified with $H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$, so the tangent bundle $T(T)$ of $T$ has a natural trivialization $T(T) \cong T \times H^1(X; \mathbb{R}) \cong T \times \mathcal{H}^+_g(X)$. The orientation on $\mathcal{H}^+_g(X)$ induces an orientation on $T(T)$. These orientations induce an orientation on the tangent bundle $\bar{T}\bar{V}$ by Lemma 3.2 below. The derivative of $s$ induces an isomorphism between $\bar{E}|_{\bar{M}}$ and the normal bundle $\mathcal{N}$ of $\bar{M}$ in $\bar{V}$. The orientation on $\bar{E}$ induces an orientation on $\mathcal{N}$ through this isomorphism, and we
have an orientation on $\mathcal{M}$ compatible with the decomposition $TV|_\mathcal{M} = TM \oplus N$. (It is easy to check that this orientation on $\mathcal{M}$ is independent of the choices of the orientations on $W'_R$ and $H^+_g(X)$.) So we have the following.

**Lemma 3.1.** A choice of orientation on $H^1_g(X) \oplus H^+_g(X)$ induces an orientation on $\mathcal{M}$.

When $TV$ and $E$ have spin structures, we can equip $\mathcal{M}$ with a spin structure as in the case of orientation. The spin structure on $E$ induces a spin structure on $N$ through the derivative of $s$. Since $TV|_\mathcal{M}$ is the direct sum of $TM$ and $N$, spin structures on $TV$ and $N$ induce a spin structure on $\mathcal{M}$, from the next well-known lemma.

**Lemma 3.2.** Let $X$ be a smooth manifold, $F_1$ and $F_2$ be oriented vector bundles on $X$. If $F_1$ and $F_2$ have spin structures, then spin structures on $F_1$ and $F_2$ determine a spin structure on $F_1 \oplus F_2$. If $F_1$ and $F_1 \oplus F_2$ have spin structures, then spin structures on $F_1$ and $F_1 \oplus F_2$ determine a spin structure on $F_2$ naturally.

Therefore we have shown the following.

**Lemma 3.3.** Let $f : V \to W$ be a finite dimensional approximation of the Seiberg-Witten map. Assume that $TV$ and $E$ have a spin structure. Choose spin structures $s_V$ and $s_E$ on $TV$ and $E$. Then $s_V, s_E$ and $f$ induce a spin structure on $\mathcal{M} = f^{-1}(0)/U(1)$.

We calculate $w_2(TV)$ and $w_2(E)$ to know when $TV$ and $E$ have spin structures.

Let $a \in \mathbb{Z}$ be the index of the Dirac operator, let $\text{Ind }D \in K(T)$ be the index bundle of the Dirac operators $\{D_A\}_{A \in T}$ parameterized by $T$. Then we have $\text{Ind }D = [V_C] - [\mathbb{C}^m] \in K(T)$, $V_R = \mathbb{R}^n$, $W_C = \mathbb{C}^m$, $W_R = H^+_g(X) \oplus W'_R$, $\text{dim } W'_R = n$ for some $m, n \in \mathbb{Z}_{\geq 0}$.

**Lemma 3.4.** Let $\bar{\pi} : \bar{V} \to T$ be the projection and define a complex line bundle $H \to \bar{V}$ by $H := V_{\text{irr}} \times_{U(1)} \mathbb{C}$. Then there is a natural isomorphism

$$TV \oplus \mathbb{R} \cong \bar{\pi}^*T(T) \oplus (\bar{\pi}^*V_C \otimes_C H) \oplus \bar{\pi}^*V_R.$$  

**Proof.** Let $\pi_{\text{irr}} : V_{\text{irr}} \to T$ and $p : V_{\text{irr}} \to \bar{V} = V_{\text{irr}}/U(1)$ be the projections. Note that we have a $U(1)$-equivariant isomorphism

$$p^*(TV) \oplus \mathbb{R} \cong TV_{\text{irr}} = \pi^*_{\text{irr}}(T(T) \oplus V).$$

where $\mathbb{R}$ stands for the $U(1)$-orbit direction. Then by dividing by the $U(1)$-actions, we obtain the required isomorphism. $\square$
By Lemma 3.4 and the triviality of $V_R$, we have $w_2(TV) \equiv \pi^*c_1(V_C) + (m + a)c_1(H) \mod 2$. By (1) in Theorem 2.2, $c_1(V_C)$ is equal to $c_1(\text{Ind}(D))$, thus we have

$$w_2(T\bar{V}) \equiv \pi^*c_1(\text{Ind}(D)) + (m + a)c_1(H) \mod 2.$$ 

T-J. Li and A. Liu calculated $c_1(\text{Ind}(D))$ in [LL] as follows. Let $\{\alpha_j\}_{j=1}^{b_1}$ be generators of $H^1(M; \mathbb{Z})$. Then we obtain a natural identification,

$$T \cong H^1(M; \mathbb{R})/H^1(M; \mathbb{Z}) \cong \mathbb{R}^{b_1}/\mathbb{Z}^{b_1} = T^{b_1}.$$ 

Let $\Psi$ be a map $M \to T^{b_1} \cong T$ given by

$$x \mapsto \left( \int_{x_0}^{x} \alpha_1, \cdots, \int_{x_0}^{x} \alpha_{b_1} \right).$$

This map is well defined by the Stokes theorem and induces the isomorphism $\Psi^* : H^1(T; \mathbb{Z}) \cong H^1(M; \mathbb{Z})$. Set $\beta_j = (\Psi^*)^{-1}(\alpha_j) \in H^1(T; \mathbb{Z})$.

**Proposition 3.5 ([LL]).** Let $\text{Ind} D \in K(T)$ be the index bundle of the Dirac operators $\{D_A\}_{A \in T}$ parameterized by $T$. Then the first Chern class $c_1(\text{Ind}(D))$ of $\text{Ind}(D)$ is given by

$$c_1(\text{Ind}(D)) = \frac{1}{2} \sum_{i<j} \langle c_1(\det L)\alpha_i\alpha_j, [M]\rangle \beta_i\beta_j \in H^2(T; \mathbb{Z}).$$

By the equation (3.1) and Proposition 3.5, we have the following.

**Lemma 3.6.** The second Stiefel-Whitney class of $T\bar{V}$ is given by

$$w_2(T\bar{V}) \equiv \sum_{i<j} c_{ij} \pi^*\beta_i\beta_j + (m + a)c_1(H) \mod 2,$$

where $c_{ij} := \frac{1}{2} \langle c_1(\det L)\alpha_i\alpha_j, [M]\rangle$.

On the other hand, by the definitions of $\bar{E}$ and $H$, we have $\bar{E} = mH \oplus \mathbb{R}^{n+b}$. Hence we obtain the following.

**Lemma 3.7.** The second Stiefel-Whitney class of $\bar{E}$ is given by

$$w_2(\bar{E}) \equiv mc_1(H) \mod 2.$$

By Lemma 3.3, Lemma 3.6 and Lemma 3.7, we have the following.

**Proposition 3.8.** Let $f : V \to W$ be a finite dimensional approximation of the Seiberg-Witten map such that $m = \dim_C W_C$ is even. Then $T\bar{V}$ and $\bar{E}$ have a spin structure if the pair $(M, L)$ satisfies the following conditions.

$$\left\{ \begin{array}{l} (\ast)_1 \quad a \equiv 0 \mod 2 \\ (\ast)_2 \quad c_{ij} \equiv 0 \mod 2 \quad (\forall i, j). \end{array} \right.$$
Moreover if we choose spin structures $s_V$ and $s_E$ of $TV$ and $E$, then $s_V$, $s_E$ and $f$ equip $\mathcal{M}$ with a spin structure.

3.2. Invariants for 4-manifolds defined by spin structures on $\mathcal{M}$. An orientation on $\mathcal{H}^1_g(M) \oplus \mathcal{H}^+_g(M)$ determines an orientation on $\mathcal{M}$ (Lemma 3.3). We will show that when the condition (*) is satisfied, a certain datum in addition to the orientation on $\mathcal{H}^1_g(M) \oplus \mathcal{H}^+_g(M)$ determines a canonical spin structure on $\mathcal{M}$. The datum is actually a square root of $\det \text{Ind}(D)$. To explain it, we need the following lemma.

Lemma 3.9. Let $X$ be a smooth manifold and $F \to X$ be a complex bundle with $c_1(F) \equiv 0 \pmod{2}$. A choice of complex line bundle $L \to X$ which satisfies $L^{\otimes 2} = \det F$ naturally determines a spin structure on $F$.

Proof. The 2-fold cover of $U(n)$ is given by
$$\{(A, t) \in U(n) \times S^1 | \det A = t^2\},$$
which is naturally regarded as a subgroup of $Spin(2n)$. Take an open covering $\{U_j\}_j$ of $X$ such that $F$ and $L$ have trivializations on each $U_j$. Fix hermitian metrics on $F$ and $L$ compatible with the identification $L^{\otimes 2} = \det F$. We denote transition functions on $U_i \cap U_j$ of $F$ and $L$ by $g_{ij} : U_i \cap U_j \to U(n)$ and $z_{ij} : U_i \cap U_j \to S^1$. Then $\det g_{ij} = z_{ij}^2$, since $\det F = L^{\otimes 2}$. Put $\tilde{g}_{ij} = (g_{ij}, z_{ij}) : U_i \cap U_j \to Spin(2n)$, then $\{\tilde{g}_{ij}\}_{ij}$ satisfies the cocycle condition and determines a spin structure of $F$. $\square$

When the condition $(*)_2$ is satisfied, then $c_1(\text{Ind}(D)) \equiv 0 \pmod{2}$. So we can take a complex line bundle $L \to T$ such that $L^{\otimes 2} = \det \text{Ind}(D)$.

Proposition 3.10. Assume that the pair $(M, \mathcal{L})$ satisfies the conditions (*). Let $f : V \to W$ be a finite dimensional approximation of the Seiberg-Witten map such that $m = \dim_C W_C$ is even. Then the finite dimensional approximation $f$, an orientation $\mathcal{O}$ of $\mathcal{H}^1_g(M) \oplus \mathcal{H}^+_g(M)$ and a complex line bundle $L \to T$ which satisfies $L^{\otimes 2} = \det \text{Ind}(D)$ determine a canonical spin structure on $\mathcal{M}$.

Proof. Suppose that the pair $(M, \mathcal{L})$ satisfies the condition (*). By Lemma 3.3 spin structures on $TV$, $E$ and a finite dimensional approximation $f$ induce a canonical spin structure on $\mathcal{M}$. So it is sufficient to show that $\mathcal{O}$ and $L$ induce spin structures on $TV$ and $E$. By Lemma 3.4, we have only to show that the choices of $\mathcal{O}$ and $L$ induce spin structures on $\tilde{\pi}^*V_C \otimes H$, $V_\mathbb{R}$, $T(T)$ and $E$.

Since $m$ is even and condition $(*)_1$ is satisfied, $\tilde{\pi}^*L \otimes H^{\otimes \frac{m+1}{2}}$ is a square root of $\det(\tilde{\pi}^*V_C \otimes H) = (\tilde{\pi}^* \det V_C) \otimes H^{\otimes \frac{m+1}{2}}$. So by Lemma 3.9 we have a spin structure on $\tilde{\pi}^*V_C \otimes H$. 


Recall that $W_\mathbb{R}$ is the direct sum $\mathcal{H}_{g}^+(X) \oplus W'_\mathbb{R}$. We fix orientations on $\mathcal{H}_{g}^+(X)$ and $W'_\mathbb{R}$, then we have orientations on $V_\mathbb{R}$ and $\mathcal{H}_{g}^1(X)$ compatible with $\mathcal{D}_\mathbb{R}$ and $\mathcal{O}$. (See [3.1]) Since the real part $\mathcal{D}_\mathbb{R}$ of $\mathcal{D}$ is independent of $A \in T$, $V_\mathbb{R}$ has a natural trivialization compatible with the orientation. This trivialization equips $V_\mathbb{R}$ with a spin structure. The tangent bundle $T(T)$ of $T$ has a natural trivialization $T(T) = T \times \mathcal{H}_{g}^1(M)$ and the orientation $\mathcal{H}_{g}^1(X)$ orients $T(T)$. So we have a spin structure on $T(T)$ compatible with this trivialization.

Lastly we consider $\bar{E}$. Let $\bar{E}_\mathbb{C}$ be the complex part of $\bar{E},$ i.e. $\bar{E}_\mathbb{C} = V_{\text{irr}} \times _{U(1)} \mathbb{C}^m$. Since $\det \bar{E}_\mathbb{C} = H^{\otimes m}, H^{\otimes \frac{m}{2}}$ is a square root of $\det \bar{E}_\mathbb{C}$. So by Lemma 3.9, a spin structure of $\bar{E}$ is determined. Let $E_{\mathbb{R}}$ be the real part of $\bar{E}$. Then $E_\mathbb{R} = V_{\text{irr}} \times W_\mathbb{R} = V_{\text{irr}} \times (\mathcal{H}_{g}^+(X) \oplus W'_\mathbb{R})$. Hence $E_\mathbb{R}$ has a natural spin structure induced by the trivialization.

We have seen that $f$, $\mathcal{O}$ and $L$ determine a spin structure on $\mathcal{M}$ if we choose orientations on $\mathcal{H}_{g}^+(X)$ and $W'_\mathbb{R}$. It is easy to see that this spin structure is independent of the choices of orientations on $\mathcal{H}_{g}^+(X)$ and $W'_\mathbb{R}$.

Let $\pi : \mathcal{M} \rightarrow T$ be the restriction of the projection $\bar{V} \rightarrow T$ to $\mathcal{M}$. We show that the class $(\mathcal{M}, \pi) \in \Omega^\text{spin}_d(T)$ induced by $f, \mathcal{O}, L$ is an invariant of $\mathcal{M}$. Here $d$ is the dimension of $\mathcal{M}$.

**Theorem 3.11.** Assume that the pair $(\mathcal{M}, \mathcal{L})$ satisfies the condition $(\ast)$. The class $(\mathcal{M}, \pi) \in \Omega^\text{spin}_d(T)$ which is induced by $f, \mathcal{O}, L$ is independent of the perturbation $(g, \eta) \in \mathcal{P}$ and the finite dimensional approximation $f$.

**Proof.** Fix $(g, \eta) \in \mathcal{P}$, and take different finite dimensional approximations $f_i : V_i \rightarrow W'_i, (i = 0, 1)$ of the Seiberg-Witten map $SW_{g,n}$. Denote $f_i^{-1}(0)/U(1)$ by $\mathcal{M}_i$ and let $\pi_i$ be the restriction of the projections $V_i \rightarrow T$ to $\mathcal{M}_i$. By considering a larger finite dimensional approximation $f : V \rightarrow W$ with $V_i \subset V$ and $W_i \subset W$, we can assume that $V_0 \subset V_1, W_0 \subset W_1$ without loss of generality.

Let $V_1 = V_0 \oplus V'$ and $W_1 = W_0 \oplus W'$, then $\mathcal{D}|_{V'}$ induces an isomorphism $V' \cong T \times W'$. By Theorem 2.2 the maps

$$(f_i)^+, (f_0 \oplus p_{W'} \circ \mathcal{D}|_{V'})^+ : V_1^+ = (V_0 \oplus V')^+ \rightarrow W_1^+ = (W_0 \oplus W')^+$$

are $U(1)$-equivariantly homotopic each other as pointed maps. It is clear that the spin structure on $\mathcal{M}_0$ induced by $f_0 \oplus p_{W'} \circ \mathcal{D}|_{V'}$ is equal to one induced by $f_0$. Let $h : [0, 1] \times V_1^+ \rightarrow W_1^+$ be a homotopy from $(f_0 \oplus \mathcal{D})^+$ to $f_1^+$ and set $\hat{\mathcal{M}} := h^{-1}(0)/U(1)$. Let $\hat{\pi}$ be the restriction of the projection $\bar{V}_1 \times [0, 1] \rightarrow T$ to $\hat{\mathcal{M}}$. By using a parallel argument to introduce spin structures on $\mathcal{M}_0$ and $\mathcal{M}_1$, we can equip $\hat{\mathcal{M}}$ with a
spin structure by using $h, \mathcal{O}$ and $L$. Then $(\widetilde{M}, \widetilde{\pi})$ is a spin bordism between $(M_0, \pi_0)$ and $(M_1, \pi_1)$. This implies that when $(g, \eta) \in \mathcal{P}$ is fixed, the class $(M, \pi) \in \Omega_d^{\text{spin}}(T)$ is independent of a choice of $f$.

Next choose two elements $(g_0, \eta_0), (g_1, \eta_1) \in \mathcal{P}$. By the assumption $b^+(M) > 1$, $\mathcal{P}$ is path connected, and there is a path $(g(t), \eta(t))_{0 \leq t \leq 1}$ in $\mathcal{P}$ satisfying $(g(i), \eta(i)) = (g_i, \eta_i), (i = 0, 1)$. We define parameterized Seiberg-Witten map

$$\widetilde{SW} : [0, 1] \times C(\mathcal{L}) \to [0, 1] \times \mathcal{Y}(\mathcal{L})$$

in the obvious way. Let $\tilde{f} : \tilde{V} \to \tilde{W}$ be a finite dimensional approximation of $SW$. We can endow $\tilde{\mathcal{M}} = \tilde{f}^{-1}(0)/U(1)$ with a spin structure in the same way as in the case of $\mathcal{M}$. Denote $V_i := \tilde{f}|_{(i) \times T}$ and $W_i$ by $V_i$ and $W_i$ for $i = 0, 1$. Since $f_i := \tilde{f}|_{V_i} : V_i \to W_i$ is a finite dimensional approximation of $SW_{g_i, \eta_i}$, $(\mathcal{M}, \pi)$ is a bordism between $(M_0, \pi_0)$ and $(M_1, \pi_1)$. It is showed that the class $(M, \pi) \in \Omega_d^{\text{spin}}(T)$ is independent of a choice of $(g, \eta) \in \mathcal{P}$.

### Definition 3.12

We write $\sigma_M(\mathcal{L}, \mathcal{O}, L)$ for the class in $\Omega_d^{\text{spin}}(T)$ represented by the spin structure on $\mathcal{M}$ induced by $f, \mathcal{O}, L$ and the restriction $\pi$ of the projection $\tilde{V} \to T$ to $\mathcal{M}$. Here $d$ is the dimension of $\mathcal{M}$.

#### 3.3. Example

We give an example of calculation of the invariant defined in [3.2]. For preparation, we show the following two lemmas.

### Lemma 3.13

Let $M_i (i = 1, 2)$ be an oriented closed 4-manifold with $b^+(M_i) > 1$ and let $\mathcal{L}_i$ be a spin$^c$ structure on $M_i$. Assume that $(M_1, \mathcal{L}_1)$ and $(M_2, \mathcal{L}_2)$ satisfy the conditions (*), then $(M_1 \# M_2, \mathcal{L}_1 \# \mathcal{L}_2)$ also satisfies the condition (*).

**Proof.** The condition $(*)_2$ is satisfied for $(M_1 \# M_2, \mathcal{L}_1 \# \mathcal{L}_2)$ by the definition of $c_{ij}$. The condition $(*)_1$ is satisfied for $(M_1 \# M_2, \mathcal{L}_1 \# \mathcal{L}_2)$ by the sum formula of the index of the Dirac operator. \(\blacksquare\)

We write $\Sigma_g$ for an oriented closed surface of genus $g$.

### Lemma 3.14

Suppose $M$ is a $K3$ surface or $\Sigma_g \times \Sigma_{g'}$ with $g$ and $g'$ odd. Let $\mathcal{L}$ be a spin$^c$ structure on $M$ which is induced by the complex structure. Then $(M, \mathcal{L})$ satisfies the condition (*).

**Proof.** Note that $c_1(\text{det} \mathcal{L}) = -c_1(K_M)$. Let $M$ be a $K3$ surface. The first Betti number of $M$ is equal to 0, so the condition $(*)_2$ is satisfied. By the index theorem [AS], the index of the Dirac operator is

$$a = \frac{c_1(\text{det} \mathcal{L})^2 - \tau(M)}{8} = \frac{0 - (3 - 19)}{8} = 2 \equiv 0 \mod 2.$$
Hence \((M, \mathcal{L})\) satisfies the condition (*) when \(M\) is a K3 surface. Let \(M\) be \(\Sigma_g \times \Sigma_{g'}\) with \(g\) and \(g'\) odd. Then we have
\[
c_1(\det \mathcal{L}) = -c_1(K_M) = 2(1 - g)\alpha + 2(1 - g')\alpha'
\]
where \(\alpha\) and \(\alpha'\) are the standard generators of \(H^2(\Sigma_g; \mathbb{Z})\) and \(H^2(\Sigma_{g'}; \mathbb{Z})\). Since \(g\) and \(g'\) are odd, we have \(c_1(\det \mathcal{L}) \equiv 0 \mod 4\), and then
\[
c_{ij} = \frac{1}{2} \langle c_1(\det \mathcal{L})\alpha_i\alpha_j, [M]\rangle \equiv 0 \mod 2,
\]
which implies the condition \((*)_2\).

By the index theorem, the index of the Dirac operator is given by
\[
a = \frac{c_1(\det \mathcal{L})^2 - \tau(M)}{8} = \frac{c_1(\det \mathcal{L})^2}{8}.
\]
Because \(c_1(\det \mathcal{L})^2 \equiv 0 \mod 16\), we have \(a \equiv 0 \mod 2\). Hence the condition \((*)_1\) is satisfied.

Let \(M_j\) be a K3 surface or \(\Sigma_g \times \Sigma_{g'}\), where \(g, g'\) are odd. By Lemma 3.13 and Lemma 3.14, the pair \((\#_j M_j, \#_j \mathcal{L}_j)\) satisfies the conditions (*), where \(\mathcal{L}_j\) is a spin\(^c\) structure on \(M_j\) induced by the complex structure. We show that the invariant \(\sigma_{\#_j=1 M_j}(\#_j=1 \mathcal{L}_j, \mathcal{O}, L)\) is non-trivial when \(l\) is 2 or 3.

**Theorem 3.15.** Let \(M_j\) be a K3 surface or \(\Sigma_g \times \Sigma_{g'}\) with \(g, g'\) odd and \(\mathcal{L}_j\) be a spin\(^c\) structure on \(M_j\) which is induced by the complex structure. Put \(M = \#_{j=1} M_j\) and \(\mathcal{L} = \#_{j=1} \mathcal{L}_j\) for \(l = 2\) or \(l = 3\). Let \(\sigma^0_M(\mathcal{L}, \mathcal{O}, L)\) be the image of \(\sigma_M(\mathcal{L}, \mathcal{O}, L)\) under the natural map \(\Omega^{spin}_{l-1}(T) \to \Omega^{spin}_l(*)\). Then \(\sigma^0_M(\mathcal{L}, \mathcal{O}, L)\) is non-trivial in \(\Omega^a_{l-1}(*)\) \(\cong \mathbb{Z}_2\).

**Proof.** Let \(L \to T\) be a square root of \(\det \text{Ind}(D)\). If \(l = 2\), the dimension of the moduli space is one, so the invariant \(\sigma^0_M(\mathcal{L}, \mathcal{O}, L)\) is in the one dimensional spin bordism group \(\Omega^a_1(*) \cong \mathbb{Z}_2\), and if \(l = 3\), the invariant \(\sigma^0_M(\mathcal{L}, \mathcal{O}, L)\) is in the two dimensional spin bordism group \(\Omega^a_2(*) \cong \mathbb{Z}_2\). We will calculate the invariant for \(l = 2\) for simplicity.

Let \(f_j : V_j \to W_j\) be a finite dimensional approximation of the Seiberg-Witten map on \(M_j\) such that \(m_j = \dim W_j, c\) is even, and set \(f = f_1 \times f_2 : V = V_1 \times V_2 \to W = W_1 \times W_2\). We make use of Bauer’s construction (Theorem 1.1 in [B]). Bauer proved that there is a finite dimensional approximation on \(M\) which is \(U(1)\)-equivariantly homotopic to \(f\).

In general, for a Kähler surface \(M\) with \(b^+(M) > 1\) and a spin\(^c\) structure \(\mathcal{L}\) on \(M\) induced by the complex structure, the Seiberg-Witten moduli space \(\mathcal{M}_M(\mathcal{L}, g, \eta)\) consists of smooth one point, where \(g\) is the Kähler metric and \(\eta\) is a suitable 2-form. See, for example, [N]. Thus we
may assume that $\mathcal{M}_j = f^{-1}_j(0)/U(1)$ is one point. Hence $f^{-1}_j(0) \cong S^1$ and $\mathcal{M} = f^{-1}(0)/U(1)_d = (f_1 \times f_2)^{-1}(0)/U(1)_d \cong S^1$, where $U(1)_d$ is the diagonal of $U(1) \times U(1)$. For some $t_j \in T_j = H^1(M_j; \mathbb{R})/H^1(M_j; \mathbb{Z})$, $f_j^{-1}(0)$ lies in a fiber $V_{t_j}$ of $V_j$. Take a small open neighborhood of $t_j$ such that $V_j|_{t_j} \cong U_j \times \mathbb{C}^{m_j+a_j} \times \mathbb{R}^n$, where $a_j$ is the index of the Dirac operator associated with $L_j$. Set $\bar{S}_j = U_j \times (\mathbb{C}^{m_j+a_j}\{0\}) \times \mathbb{R}^n$ and $S = \prod_{j=1}^2 S_j$, then $S$ has a $U(1)_d$-action and a $U(1) \times U(1)$-action. The $U(1)_d$-action is defined by the scalar product on $\prod_{j=1}^2 (\mathbb{C}^{m_j+a_j}\{0\})$ and the scalar product of $\alpha_2$ on $(\mathbb{C}^{m_2+a_2}\{0\})$. Set $\bar{S} = S/U(1)_d$.

We write $\xi$ for a spin structure on $\bar{V} = V_{irr}/U(1)_d$ induced by $L$. The restriction $\xi|_{\mathcal{M}}$ of $\xi$ to $\mathcal{M}$ is equal to $(\xi|_{S})|_{\mathcal{M}}$. Since $H^1(S; \mathbb{Z}_2) = 0$, $\bar{S}$ has just one spin structure. So it is sufficient to consider the restriction of the unique spin structure on $\bar{S}$ to $\mathcal{M}$.

Put $U(1)_q = U(1) \times U(1)/U(1)_d \cong U(1)$, then the $U(1) \times U(1)$-action on $S$ induces a free $U(1)_q$-action on $\bar{S}$ and $\bar{S}/U(1)_q = \bar{S}_1 \times \bar{S}_2$, where $\bar{S}_j/S_j(1) \cong U_j \times \mathbb{C}^P_{m_j+a_j} \times \mathbb{R}_{>0} \times \mathbb{R}^n$. Moreover this $U(1)_q$-action preserves $\mathcal{M} \subset \bar{S}$ and induces a free $U(1)_q$-action on $\mathcal{M} \cong S^1$. Since $m_j + a_j - 1$ is odd, $T\bar{S}_j$ has a spin structure. So $T(\bar{S}/U(1)_q)$ has a spin structure. Take a spin structure $\eta$ on $T(\bar{S}/U(1)_q) \oplus \mathbb{R}$. Let $p : \bar{S} \to \bar{S}/U(1)_q$ be the projection. Then there is a natural isomorphism $T\bar{S} \cong p^*(T(\bar{S}/U(1)_q) \oplus \mathbb{R})$. So $p^*(\eta)$ is the unique spin structure $\xi$ on $T\bar{S}$. Because $p$ is the projection $\bar{S} \to \bar{S}/U(1)_q$, the $U(1)_q$-action on $\bar{S}$ lifts to an action on $\xi = p^*(\eta)$. So the $U(1)_q$-action on $\mathcal{M} \cong S^1$ lifts to an action on restriction of $\xi|_{\mathcal{M}}$. In the same way, we can prove that the $U(1)_q$-action on $\mathcal{M}$ lifts to an action on the spin structure on $E|_{\mathcal{M}}$. Since $f|_{S} = f_1|_{S_1} \times f_2|_{S_2} : S_1 \times S_2 \to W_1 \times W_2$ is $U(1) \times U(1)$-equivariant, the $U(1)_q$-action on $\mathcal{M}$ lifts to an action on the spin structure of $\mathcal{N}$ induced by $f$ and the spin structure on $E|_{\mathcal{M}}$. Therefore the $U(1)_q$-action on $\mathcal{M}$ lifts to an action on the spin structure on $\mathcal{M}$ induced by $f, \mathcal{O}$ and $L$. Such a spin structure determines a non-trivial class in $\Omega^1_{spin}(\mathcal{M}) \cong \mathbb{Z}_2$, so $\sigma^0_M(\mathcal{O}, L)$ is non-trivial class in $\Omega^1_{spin}(\mathcal{M})$ (See [K]).

In the case of $l = 3$, $\mathcal{M}$ is the 2-dimensional torus if we perturb the equations suitably. We can show that the spin structure on $\mathcal{M}$ is the Lie group spin structure as in the case of $l = 2$ and the spin bordism class $\sigma^0_M(\mathcal{O}, L)$ is non-trivial in $\Omega^1_{spin}(\mathcal{M}) \cong \mathbb{Z}_2$. $\square$

**Remark 3.16.** Let $l$ be larger or equal to 4. Then we may assume that the moduli space is a $(l - 1)$-dimensional torus $T^{l-1}$. In the same way
as in Theorem 3.15, we can see that the spin structure on \( M \) induced by \( f, \mathcal{O} \) and \( L \) is equal to the spin structure induced by the Lie group structure of \( T^{l-1} \). Such a spin structure is trivial in \( \Omega_{l-1}^{spin}(\ast) \) if \( l \) is larger or equal to 4. Hence \( \sigma_M^0(\mathcal{L}, \mathcal{O}, L) \) is trivial in \( \Omega_{l-1}^{spin}(\ast) \) when \( l \) is larger than or equal to 4.

By Theorem 3.15 we obtain the adjunction inequality for \( M \). See [KM] for proof.

**Corollary 3.17.** Let \( M_j, M \) and \( \mathcal{L} \) be as in Theorem 3.15. Assume that an oriented, closed surface \( \Sigma \) of positive genus is embedded in \( M \) and its self intersection number \( \Sigma \cdot \Sigma \) is nonnegative. Then

\[ \Sigma \cdot \Sigma \leq \langle c_1(\det \mathcal{L}), [\Sigma] \rangle + 2g(\Sigma) - 2, \]

where \( g(\Sigma) \) is the genus of \( \Sigma \).

There are applications of Theorem 3.15 to computation of the Yamabe invariant and nonexistence of Einstein metric.

**Definition 3.18.** Let \( M \) be an oriented, closed 4-manifold. Then the Yamabe invariant of \( M \) is defined by

\[
\mathcal{Y}(M) = \sup_{\gamma \in \text{Conf}(M)} \frac{\inf_g \int_M s_g d\mu_g}{\left( \int_M d\mu_g \right)^{\frac{3}{2}}}
\]

where Conf\((M)\) is the space of conformal classes of Riemannian metrics on \( M \), \( s_g \) is the scalar curvature and \( d\mu_g \) is the volume form of \( g \).

**Theorem 3.19.** Let \( M_j \) and \( M \) be as in Theorem 3.15, and \( N_1 \) an oriented, closed, negative definite 4-manifold admitting a Riemannian metric with scalar curvature nonnegative at each point. Then

\[
\mathcal{Y}(M \# N_1) = -4\pi \sqrt{\frac{1}{2} \sum_{j=1}^{l} c_1(M_j)^2}.
\]

**Theorem 3.20.** Let \( M_j \) and \( M \) be as in Theorem 3.15. If \( N_2 \) be an oriented, closed, negative definite 4-manifold satisfying

\[
4l - (2\chi(N_2) + 3\tau(N_2)) \geq \frac{1}{3} \sum_{j=1}^{l} c_1(M_j)^2,
\]

then \( M \# N_2 \) does not admit an Einstein metric.

**Proof of Theorem 3.19 and Theorem 3.20.** In [IL], Ishida and LeBrun showed a similar statement under a somewhat different assumption (Theorem D). The main point of their argument is non-vanishing of the Bauer-Furuta invariant. In our case, the invariant \( \sigma_M(\mathcal{L}, \mathcal{O}, L) \)
is non-trivial. Hence we can apply their argument to our situation. □

On the other hand, there is a topological obstruction for 4-manifolds to have an Einstein metric ([H]).

**Theorem 3.21** (Hitchin-Thorpe inequality [H]). Let $X$ be an oriented closed 4-manifold admitting an Einstein metric, then

$$3|\tau(X)| \leq 2\chi(X).$$

**Example 3.22.** Let $M_i = \Sigma_{g_i} \times \Sigma_{g'_i}$ for positive odd integers $g_i, g'_i$, let $M = M_1 \# M_2$ and let $N = (\#^r \mathbb{C}P^2) \# (\#^s S^1 \times S^3)$. Then $b^+(N) = 0$ and the inequality (3.2) is satisfied if $r \geq \frac{8}{3} G - 4s - 4$, where $G := \sum_{i=1}^{2} (g_i - 1)(g'_i - 1)$. By Theorem 3.20, $X = M \# N$ does not admit an Einstein metric when $r \geq \frac{8}{3} G - 4s - 4$. On the other hand, if $r \leq 8G - 4s - 4$, then $X$ satisfies the Hitchin-Thorpe inequality (3.3). Thus if

$$\frac{8}{3} G - 4s - 4 \leq r \leq 8G - 4s - 4,$

$X$ satisfies the Hitchin-Thorpe inequality, but does not admit an Einstein metric.

**References**

[AS] M. F. Atiyah and M. I. Singer, *The index of elliptic operators I*, Ann. of Math. 87 (1968), pp. 484-530.

[B] S. Bauer, *A stable cohomotopy refinement of Seiberg-Witten invariants II*, Invet. Math. 155 (2004), no. 1, pp. 21-40.

[BF] S. Bauer and M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants I*, Invet. Math. 155 (2004), no. 1. pp. 1-19.

[F] M. Furuta, *Monopole equation and the $\frac{11}{8}$ conjecture*, Math. Res. Lett. 8 (2001), no. 3, pp. 293-301.

[FKM] M. Furuta, Y. Kametani and N. Minami, *Stable-homotopy Seiberg-Witten invariants for rational cohomology $K3\# K3s$*, J. Math. Sci. Univ. Tokyo 3 (2001), no. 1. pp.157-176.

[H] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry. 9 (1974) pp. 435-441.

[IL] M. Ishida and C. LeBrun, *Curvature, connected sums, and Seiberg-Witten theory*, Comm. Anal. Geom. 11 (2003), no. 5. pp. 809-836.

[K] R. Kirby, *The topology of 4-manifolds*, Lecture Notes in Mathematics, 1374. Springer-Verlag, Berlin, 1989.

[Ku] N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology 3 (1965), pp. 19-30.
[KM] P. B. Kronheimer and T. S. Mrowka, The genus of embedded surfaces in the projective plane, Math. Res. Lett. 1 (1994), no. 6, pp. 797-808.

[LL] T. J. Li and A. Liu, General wall crossing formula, Math. Res. Lett. 2 (1995), no. 6, pp. 797-810.

[N] L. Nicolaescu, Notes on Seiberg-Witten theory, Graduate studies in mathematics, 28, American Mathematical Society, 2000.

[W] E. Witten, Monopoles and four manifolds, Math. Res. Lett. 1 (1994), no. 6, pp. 769-796.

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