On Modular Invariant Partition Functions for Tensor Products of Conformal Field Theories

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Abstract

We give two results concerning the construction of modular invariant partition functions for conformal field theories constructed by tensoring together other conformal field theories. First we show how the possible modular invariants for the tensor product theory are constrained if the allowed modular invariants of the individual conformal field theory factors have been classified. We illustrate the use of these constraints for theories of the type $SU(2)^{K_A} \otimes SU(2)^{K_B}$, finding all consistent theories for $K_A, K_B$ odd. Second we show how known diagonal modular invariants can be used to construct some inherently asymmetric ones where the holomorphic and anti-holomorphic theories do not share the same chiral algebra. Some explicit examples are given.

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1. Introduction

In the past few years considerable effort has been spent in classifying modular invariant partition functions of two-dimensional conformal field theories. Complete classifications exist only for some of the simplest conformal field theories, in particular the minimal discrete series with $c < 1$, and the models based on level $K$ SU(2) Kac-Moody algebras.\cite{1} For string theory applications, conformal field theories with larger central charges (up to 26) are of more direct interest, so it is natural to consider the possibilities for constructing such theories using tensor products of the simpler, well known theories. To date this program has been systematically carried out only for theories constructed from free bosons or fermions.\cite{2,3} For tensor products of other theories no procedures have been developed which give all of the possible modular invariants, but a few simple algorithms exist for modifying a known modular invariant to produce another one, in particular the orbifold construction\cite{4} and the related operation of twisting by a simple current.\cite{5}

In this work we make some modest proposals aimed at the general problem of classifying all possible modular invariants for conformal field theories constructed by tensoring together models whose modular invariants are already known. By a tensor product of two theories, say $A$ and $B$, we mean a theory whose chiral algebra includes the chiral algebras of both the $A$ and $B$ theories. As a consequence, the central charge of the combined theory will be the sum of those for the individual factors, the chiral blocks which make up amplitudes will be constructed from the products of the individual chiral blocks, and the characters will be products of the individual characters, restricting the partition function to the form,

$$Z^{AB} = \sum_{l,m,l',m'} N^{AB}_{lm,l'm'} \chi^A_l \chi^B_{m'} \bar{\chi}^A_{l'} \bar{\chi}^B_{m}.$$  \hspace{1cm} (1.1)

The combined theory is not restricted to be simply the product of the individual theories; the operators in the combined theory need not be diagonal (i.e. left-right symmetric), and in general the fusion rules for the operator products will be modified.
The latter point is the chief complication in the general problem. The allowed tensor product theories built from free bosons or fermions have been successfully categorized because the possible fusion rules in these theories are almost trivial; likewise twisting a theory by a simple current gives unambiguously a new theory because the new fusion rules are unambiguous.

In section 2 we consider to what extent the integer coefficients $N_{AB}^{lm\bar{l}\bar{m}}$ in the partition function of the tensor product theory are constrained if we know all of the allowed possibilities for the corresponding coefficients $N_{A}^{A}^{l}$ and $N_{B}^{B}^{m}$ in the factor theories. In section 3 we consider the more general possibility of combining theories such that the holomorphic and anti-holomorphic degrees of freedom need not possess the same chiral algebras, that is we consider partition functions of the form, $Z = \sum_{l,m,n} N_{lm\bar{l}\bar{m}} \chi_{A}^{l} \chi_{B}^{m}$. In the following sections we are interested ultimately in classifying consistent conformal field theories, not just modular invariant combinations of characters. Accordingly, we freely invoke consistency conditions for amplitudes on the plane when they prove useful for constraining the states which can appear in the partition function.

2. Constraints on Tensor Product Modular Invariants

In order for the tensor product partition function (1.1) to be invariant under the generators of modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ (denoted $T$ and $S$, respectively) we must have,

\begin{align}
T \text{ invariance} & : \Delta_{l} + \Delta_{m} = \Delta_{\bar{l}} + \Delta_{\bar{m}} \pmod{1} \quad \text{if} \quad N_{lm\bar{l}\bar{m}}^{AB} \neq 0 \\
S \text{ invariance} & : N_{lm\bar{l}\bar{m}}^{AB} = \sum_{l',m',\bar{l}',\bar{m}'} N_{l'm'\bar{l}'\bar{m}'}^{AB} S_{l}^{A} S_{m}^{B} S_{\bar{l}}^{A} S_{\bar{m}}^{B} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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integer coefficients $N_{l\bar{l}}^{A,i}$ such that,

$$\Delta_l = \Delta_{\bar{l}} \pmod{1} \quad \text{if} \quad N_{l\bar{l}}^{A,i} \neq 0$$

and

$$N_{l\bar{l}}^{A,i} = \sum_{l',\bar{l}'} N_{l'l\bar{l}'\bar{l}}^{A,i} S_{ll'}^{A} S_{\bar{l}\bar{l}'}^{A} ,$$

(2.2)

and similarly for $N_{m\bar{m}}^{B,j}$. We can get relations between the integer coefficients in equations (2.1) and (2.2) by multiplying (2.1) by $N_{l\bar{l}}^{A,i}$ and summing over $l$ and $\bar{l}$,

$$\sum_{l,\bar{l}} N_{l\bar{l}}^{A,i} N_{lml\bar{m}}^{AB} = \sum_{l,l',l'',\bar{l},\bar{l}'} N_{l\bar{l}}^{A,i} N_{l'l\bar{l}'\bar{l}}^{AB} S_{ll'}^{A} S_{\bar{l}\bar{l}'}^{A} S_{m\bar{m}} S_{\bar{m}\bar{l}'} ,$$

$$= \sum_{l',l'',\bar{l},\bar{l}'} N_{l'l\bar{l}'\bar{l}}^{A,i} N_{l'l''l\bar{l}''\bar{l}}^{AB} S_{l'l'}^{B} S_{\bar{l}'\bar{l}'}^{B} S_{m\bar{m}'} S_{\bar{m}\bar{m}'} ,$$

(2.3)

where we have used (2.2) and the symmetry of $S$ to simplify the right hand side. The resulting equation is precisely of the form (2.2) for the $B$ theory, therefore we must have,

$$\sum_{l,\bar{l}} N_{l\bar{l}}^{A,i} N_{lml\bar{m}}^{AB} = \sum_{j} n_{l\bar{l}}^{A,i} n_{j\bar{j}}^{B,j} .$$

(2.4)

This constrains some combinations of coefficients in the $AB$ theory to be linear combinations (with integer coefficients) of the allowed coefficients in the $B$ theory which are presumed known. There is an analogous constraint arising from taking the appropriate traces over the $B$ theory indices in (2.1), and a further constraint arising from taking appropriate traces in both sets of indices in either possible order. Note that the number of constraint equations increases as the factor theories become more complex (in the sense of having more possible modular invariants), and also as we consider tensor product theories with more factors.

These equations constrain part of the operator content of the tensor product theories which we wish to classify. Often, this information, together with some simple consistency requirements for conveniently chosen amplitudes on the plane, serves to completely determine the allowed possibilities for the tensor product modular invariants. For concreteness, we illustrate with a simple example.
Example: SU(2)_{K_A} \otimes SU(2)_{K_B} tensor product theories.

There are $K + 1$ unitary primary fields of $SU(2)_K$, which we will label by twice the spin, $l = 2s$, of the corresponding $SU(2)$ representation. The conformal dimensions are $\Delta_l = \frac{l(l+2)}{4(K+2)}$. The matrix $S$ implementing the modular transformation $\tau \to -1/\tau$ on the Kač-Moody characters is,

$$S^K_{ll'} = \left(\frac{2}{K + 2}\right)^{1/2} \sin \left(\frac{\pi(l + 1)(l' + 1)}{K + 2}\right)$$

and the fusion rules, which we will make use of momentarily are,

$$\phi_l \times \phi_{l'} = \sum_{m=|l-l'|}^{\min(l+l',2K-l-l')} \phi_m .$$

For simplicity we will only consider the possibilities for tensor product theories with holomorphic and anti-holomorphic chiral algebras SU(2)_{K_A} \otimes SU(2)_{K_B} for both $K_A$ and $K_B$ odd. Then the only possible modular invariants for the factor theories are the diagonal ones, $N_{ll} = \delta_{ll}$. Applying the constraint equation (2.4) and its obvious generalizations gives the conditions,

$$\sum_{l=l'} N^{AB}_{lmb} = a\delta_{m\bar{m}} ; \quad a \in \mathbb{Z}^+$$

$$\sum_{m=m} N^{AB}_{lmb} = b\delta_{ll} ; \quad b \in \mathbb{Z}^+$$

$$\sum_{l=l} \sum_{m=m} N^{AB}_{lmb} = a(K_B + 1) = b(K_A + 1)$$

If we label the primary operators in the tensor product theory by the corresponding $l$ values of the factor theories, e.g. $(l, m | \bar{l}, \bar{m})$, then the integer $a$ is equal, in

* For our purposes we need all non-negative integer coefficients $N_{ll'}$ which give rise to $S$ and $T$ invariant partition functions, but not necessarily with a unique vacuum state ($N_{00} = 1$). We have confirmed in the SU(2) case that relaxing this condition does not expand the space of possible solutions beyond a multiplicative constant.
particular, to the number of primary operators in the theory of the form \((j, 0|j, 0)\). These are pure \(A\) theory operators and so must form a closed operator subalgebra of the \(A\) theory. Similarly, \(b\) must equal the dimension of some closed operator algebra in the \(B\) theory. This is useful because we know (explicitly from studying the consistency of amplitudes on the plane) all consistent closed operator sub-algebras in SU(2) Kač-Moody theories.\(^6\) For \(K\) odd these are (labeled by their dimensions, \(d\)),

\[
\begin{align*}
    d = 1 & : \{\Phi_0\} \text{ (the identity)} \\
    d = 2 & : \{\Phi_0, \Phi_K\} \\
    d = \frac{K+1}{2} & : \{\Phi_l; 0 \leq \text{ even } l \leq K\} \text{ (the allowed integer spin representations)} \\
    d = K+1 & : \{\Phi_l; 0 \leq l \leq K\}
\end{align*}
\]

Thus in the tensor product theory we know all of the possibilities for operators of the form \((j, 0|j, 0)\) or \((0, j|0, j)\). Given (2.7) and the uniqueness of the vacuum state \((0, 0|0, 0)\) in the tensor product theory, the multiplicities of the operators in the closed sub-algebras must be as in (2.8).

We can now write down all of the possibilities for \(a, b, K_A\) and \(K_B\) consistent with (2.7) and (2.8), and proceed to consider each category of possible tensor product modular invariant individually:

1. \(a = K_A + 1, \ b = K_B + 1\) : Here we have \(N_{l\tilde{m}m\tilde{m}}^{AB} = \delta_{ll'}\delta_{m\tilde{m}} + M_{l\tilde{m}m\tilde{m}}^{AB}\), with \(M_{l\tilde{m}m\tilde{m}}^{AB}\) traceless with respect to both \(l, \tilde{l}\) and \(m, \tilde{m}\). It is easy to see that \(M^{AB}\) must in fact vanish, leaving us with the simple uncorrelated tensor product of the SU(2)\(_{K_A}\) and SU(2)\(_{K_B}\) diagonal modular invariants. Were this not the case, then \(M^{AB}\) by itself would give rise to a modular invariant which did not include the term containing the identity operator. But this is not possible since (from (2.5) and quite generally in a unitary theory) \(S_{0l} > 0\) for all \(l\).

2. \(a = \frac{K_A+1}{2}, \ b = \frac{K_B+1}{2}\) : In this case the operators diagonal in either of the factor theories comprise the set \(\{(l, m|l, m)\}\) with \(l\) and \(m\) both odd or both even. This set contains the operators \((1, K_B|1, K_B)\) and \((K_A, 1|K_A, 1)\). The non-diagonal operators in the theory, \((i, j|m, l)\) must have a consistent operator product with these
two operators, in particular at least some of the operators appearing in the naive fusion with them (using the rules (2.6)) must have integer spins ($\Delta - \bar{\Delta} \in \mathbb{Z}$). This restricts the non-diagonal operators ($i, j|m, l)$, $i \neq m$, $j \neq l$, to those satisfying $i + m = K_A$ and $j + l = K_B$. For these operators, in turn, to have integer spin we have either: $i - j$ even and $K_A + K_B = 0$ modulo 4; or $i - j$ odd and $K_A - K_B = 0$ modulo 4. The former case, taking all such operators, gives the modular invariant obtained from the simple tensor product invariant of case (1) by twisting by the simple current ($K_A, K_B|0, 0$); the latter is obtained by twisting by ($K_A, 0|0, K_B$). An extension of the argument given in (1) using the fact that $S^K_A i^K_A S^K_B j^K_B > 0$ for all $i - j$ even, shows that these are the only possibilities in this category.

(3) $a = 1$, $b = 2$, $K_B = 2K_A + 1$ or $a = 2$, $b = 1$, $K_A = 2K_B + 1$: Take $a = 1$, $b = 2$ so $K_B = 2K_A + 1$. The model must include the states $(0, 0|0, 0)$ and $(0, K_B|0, K_B)$ but no other states of the form $(0, l|0, l)$, $(i, 0|i, 0)$ or $(j, K_B|j, K_B)$. There must also be two states of the form ($K_A, j|K_A, j$). Demanding that the fusion products of these states with themselves are consistent with the above restriction requires $j = 0$, or $K_B$, but then the states themselves are inconsistent with the restriction. Thus there are no possible consistent theories within this category.

(4) $a = 2$, $b = \frac{K_B + 1}{2}$, $K_A = 3$ or $a = \frac{K_A + 1}{2}$, $b = 2$, $K_B = 3$: This case differs from case (2) with $K_A$ and/or $K_B = 3$, in that the $d = 2$ closed subalgebra of the SU(2)$_3$ theory consists of $\{\Phi_0, \Phi_3\}$ instead of $\{\Phi_0, \Phi_2\}$ as in (2). If $a = 2$, $b = \frac{K_B + 1}{2}$ and $K_A = 3$ then the operators diagonal in either factor theory comprise the set $\{(0, l|0, l), (3, l|3, l) \ l \text{ even}; (1, j|1, j), (2, j|2, j) \ j \text{ odd}\}$. There must be additional non-diagonal operators, ($i, j|l, m)$ $i \neq l$, $j \neq m$, if there are to be any modular invariants in this category. If ($i, j|l, m)$ appears then ($3 - i, j|3 - l, m$) appears also. For both to have integer spin $i$ and $l$ must be both even or both odd. Thus there must be operators of the form ($0, j|2, m)$ or ($1, p|3, l$). Fusing these with the operators ($1, K_B|1, K_B$) from the diagonal part of the theory produces the operators ($1, K_B - j|1, K_B - m)$ and/or ($1, K_B - j|3, K_B - m$) and ($0, K_B - p|2, K_B - l$) and/or ($2, K_B - p|2, K_B - l$), respectively. It is easy to see that if the former fields have integer spin then none of the possible fusion products do. Thus, there can be
no consistent theories in this category.

(5) \( K_A = K_B \equiv K \), \( a = b = 1 \) or \( a = b = 2 \) : The situation becomes more complicated for \( K_A = K_B \equiv K \). For these cases we have additional trace equations,

\[
\sum_{l=\bar{m}} N'_{AB} = a'\delta_{m\bar{l}}; \quad a' \in \mathbb{Z}^+
\]
\[
\sum_{m=\bar{l}} N'_{AB} = b'\delta_{l\bar{m}}; \quad b' \in \mathbb{Z}^+.
\]

If the values of \( a' \) and \( b' \) correspond to any of cases (1)—(4), then the invariants are precisely as given above, with the factor theories permuted. Thus we only need to consider the cases: (5a) \( a = b = a' = b' = 1 \), (5b) \( a = b = a' = b' = 2 \), and (5c) \( a = b = 1 \), \( a' = b' = 2 \). Case (5b) is most quickly disposed of. The operators in the theory include the closed subalgebra \{ \( (0,0|0,0) \), \( (0,K|0,K) \), \( (0,0|0,K) \), \( (K,0|0,K) \), \( (0,0|0,0) \), \( (0,K|K,0) \) \}, \{ \( (0,0|0,0) \), \( (K,0|0,K) \) \}, and \{ \( (0,0|0,0) \), \( (0,K|K,0) \) \}. For the operator algebra with these together to be closed the chiral fields \( (K,K|0,0) \) and \( (0,0|K,K) \) must appear, but for \( K \) odd these do not have integer conformal dimension. Therefore this case is ruled out.

Cases (5a) and (5c) can also be ruled out as follows. In both cases there must be fields \( (1,j|1,j) \) and \( (p,1|p,1) \) with a single choice for \( j \) and \( p \) in each case. Consider the four-point correlation function on the plane \( \langle (1,j|1,j)(1,j|1,j)(p,1|p,1)(p,1|p,1) \rangle \). In one channel the only possible intermediate state primary fields which can appear consistent with the restrictions of cases (5a) or (5c) are \( (0,0|0,0) \) and \( (2,2|2,2) \). In the cross channels only a subset of the states of the form \( (p\pm 1,j\pm 1|p\pm 1,j\pm 1) \) can appear as intermediates. We know from the four-point amplitudes \( \langle (1|1)(1|1)(j|j)(j|j) \rangle \) and \( \langle (1|1)(1|1)(p|p)(p|p) \rangle \) in the factor theories that the chiral blocks making up the amplitudes have two-dimensional monodromy, so that the blocks appearing in the tensor product theory have four-dimensional monodromy. There is no way, then, to assemble the two chiral blocks corresponding to the allowed intermediate primaries \( (0,0|0,0) \) and \( (2,2|2,2) \) in such a way that the four-point function in the tensor product theory can be monodromy invariant (i.e. single valued).
To summarize: We have used the constraints (2.4) and the consistency of conveniently chosen fusion rules and amplitudes to find the only consistent tensor product theories of the type $SU(2)_{K_A} \otimes SU(2)_{K_B}$, with $K_A, K_B$ odd. These turn out to be the simple (uncorrelated) product of the diagonal invariants of the factor theories and all theories obtained from these ones by twisting by the allowed simple current fields which can be built from the identity and fields labeled $K_A$ and $K_B$.

3. Left-Right Asymmetric Modular Invariants

So far we have considered tensor product conformal field theories which are diagonal in the sense that for each holomorphic conformal field theory factor there is a corresponding anti-holomorphic conformal field theory factor with the same chiral algebra. While these are the relevant theories to consider for statistical mechanics applications, it is natural in the construction of heterotic string theories to consider conformal field theories which are inherently left-right asymmetric as well. For these the methods discussed above do not apply. Nonetheless, we can exploit known properties of left-right symmetric conformal field theories to construct modular invariants even for inherently asymmetric theories by using the following result: Given two consistent diagonal rational conformal field theories (apriori with different chiral algebras) with modular invariant partition functions $Z^A = \sum \chi_i^A \bar{\chi}_i^A$ and $Z^B = \sum \chi_i^B \bar{\chi}_i^B$, the left-right asymmetric partition function given by $Z^{AB} = \sum \chi_i^A \bar{\chi}_i^B$ will be modular invariant only if: (1) the conformal dimensions agree modulo 1, or more precisely $\Delta_i^A - c^A/24 = \Delta_i^B - c^B/24 \pmod{1}$; and (2) the fusion rules of the two theories coincide, $\phi_i^A \times \phi_j^A = \sum_k N_{ij}^k \phi_k^A$ and $\phi_i^B \times \phi_j^B = \sum_k N_{ij}^k \phi_k^B$.

Condition (1) is obviously necessary and sufficient for $Z^{AB}$ to be $T$ invariant. Condition (2) is almost immediate given Verlinde’s results.\footnote{The proof of the result described here is almost immediate on using Verlinde’s results.} $Z^{AB}$ is invariant under the $S$ transformation if and only if the $S$ matrices implementing the modular transformations on the characters of the $A$ and $B$ theories coincide, $S_{ij}^A = S_{ij}^B$. As
Verlinde showed, the fusion rule coefficients determine the $S$ matrix, so condition (2) is required for $S^A = S^B$. In employing this relation it is crucial to define the primary fields with respect to the full chiral algebra of the theory.

As a simple example, consider the theories $A = SO(31)$ level 1, and $B = E_8$ level 2, both with central charge $c = 31/2$. The consistent diagonal theories have partition functions, $Z^A = \chi_0 \bar{\chi}_0 + \chi_{\frac{3}{2}} \bar{\chi}_{\frac{3}{2}} + \chi_{\frac{31}{16}} \bar{\chi}_{\frac{31}{16}}$ and $Z^B = \chi_0 \bar{\chi}_0 + \chi_{\frac{3}{2}} \bar{\chi}_{\frac{3}{2}} + \chi_{\frac{15}{16}} \bar{\chi}_{\frac{15}{16}}$, where the characters are labeled by the conformal dimension of the associated primary field. The fusion rules in both theories are like those in the Ising model. The asymmetric partition function, $Z^{AB} = \chi_A^0 \bar{\chi}_B^0 + \chi_A^{\frac{3}{2}} \bar{\chi}_B^{\frac{3}{2}} + \chi_A^{\frac{31}{16}} \bar{\chi}_B^{\frac{31}{16}}$ satisfies conditions (1) and (2) and so is itself modular invariant. This modular invariant can also be constructed by choosing appropriate boundary conditions for a collection of 31 free, real fermions.

A more interesting example, which cannot be constructed from free bosons or fermions or by twisting a known invariant by a simple current, is the following. For the $A$ theory we take the simple tensor product of the diagonal theories for $G_2$ level 1 and $SU(3)$ level 2; for the $B$ theory the simple tensor product of the diagonal theories for $F_4$ level 1 and the three state Potts model. The central charges coincide: $c^A = 14/5 + 16/5 = 6$; $c^B = 26/5 + 4/5 = 6$. The primary fields appearing in each theory are, for $G_2$ level 1 the identity and 7 ($\Delta = \frac{2}{5}$); for $SU(3)$ level 2 the identity, 3 and $\bar{3}$ ($\Delta = \frac{4}{15}$), 6 and $\bar{6}$ ($\Delta = \frac{2}{3}$), and 8 ($\Delta = \frac{2}{3}$); for $F_4$ level 1 the identity and 26 ($\Delta = \frac{3}{5}$); and for the Potts model the primaries, labeled by their conformal dimensions are 0, $\frac{2}{5}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{1}{15}$, and $\frac{1}{15}$.

To economically list the fusion rules for these theories we can simply list the non-vanishing three-point amplitudes (where we represent the field by its conformal dimension). Besides the obvious ones involving the identity operator ($\langle \phi \bar{\phi} 0 \rangle$) these are: for $G_2$ level 1 $\langle \frac{2}{5}, \frac{2}{5}, \frac{2}{5} \rangle$; for $SU(3)$ level 2, $\langle \frac{4}{15}, \frac{4}{15}, \frac{4}{15} \rangle$, $\langle \frac{4}{15}, \frac{4}{15}, \frac{2}{3} \rangle$, $\langle \frac{4}{15}, \frac{1}{15}, \frac{3}{5} \rangle$, $\langle \frac{4}{15}, \frac{3}{5}, \frac{3}{5} \rangle$, $\langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \rangle$, $\langle \frac{3}{5}, \frac{3}{5}, \frac{3}{5} \rangle$, and the conjugates of these; for $F_4$ level 1, $\langle \frac{3}{5}, \frac{2}{5}, \frac{3}{5} \rangle$; and

* To be precise, Verlinde showed that the eigenvalues, $\lambda_i^{(j)}$, of the matrices $(N_i)^k_l$ satisfy $\lambda_i^{(j)} = S_{ij} / S_{0j}$, but there could be an ambiguity in the choice of superscript $(j)$ labeling each member of the set of eigenvalues of $(N_i)^k_l$. We believe in the present case that this ambiguity is fixed given $T$ and the requirement $(ST)^3 = 1$, but have no proof.
for the three state Potts model, $\langle \frac{1}{15}, \frac{1}{15}, \frac{1}{15} \rangle$, $\langle \frac{1}{15}, \frac{1}{15}, \frac{2}{3} \rangle$, $\langle \frac{1}{15}, \frac{2}{3}, \frac{2}{3} \rangle$, $\langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \rangle$, and the conjugates of these. All of the non-zero fusion rule coefficients, $N_{ijk}$, for these four theories are equal to one.

Given the obvious similarities of the fusion rules and conformal dimensions of these theories, it is not difficult to verify that the asymmetric partition function given by,

$$Z^{AA'BB'} = \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 +$$

$$\lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 +$$

$$\lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 +$$

$$\lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 +$$

satisfies conditions (1) and (2) for modular invariance. Here $A, A', B, B'$ denote the $G_2$, $SU(3)$, $F_4$, and Potts theories, respectively. An alternative sewing of the operators in these four conformal field theories gives rise to the diagonal $E_6$ level 1 modular invariant,

$$Z^{E_6} = \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 + \lambda_0 A \chi_0 A' \chi_0 +$$

It is natural to suppose that the asymmetric modular invariant can be obtained from the symmetric one by twisting by the appropriate field or fields. This intuition is correct, but the twisting is not by a simple current operator, and correspondingly there is no definite algorithm for achieving it. In the symmetric theory the chiral algebra is enlarged (to $E_6 \otimes E_6$). Twisting by a simple current (as considered in the literature) cannot reduce the chiral algebra, and here gives back the same theory. There is, however, a candidate field which is primary under the smaller chiral algebra of the asymmetric theory and which has simple fusion rules when defined with respect to this algebra, namely the field $\langle 0, \frac{2}{3}, 0, \frac{2}{3} \rangle$. Twisting $Z^{E_6}$ by this operator, that is
throwing out those operators which fused with \((0, \frac{2}{3} | 0, \frac{2}{3})\) give \(T\) noninvariant states while adding those \(T\) invariant operators which result from fusing, gives only a subset of the characters in the asymmetric theory. To get the full set we must add the operators formed by fusing \((\frac{2}{3}, \frac{3}{3} | \frac{2}{3}, \frac{2}{3})\) with itself under the now modified fusion rules of the new theory (which apriori is an ambiguous procedure). Similarly, twisting the asymmetric invariant by any combinations of simple currents in that theory gives back the same model. In order to obtain \(Z^{E_6}\) we have to twist by the non-simple current \((\frac{2}{3}, \frac{3}{3} | 0, 0)\), with suitably modified fusion rules, which again is an ambiguous procedure.

4. Comments

The techniques introduced in section 2 make the classification of modular invariants for tensor product theories built from a small number of factors at least feasible. A complete classification of the invariants for \(SU(2)_{K_A} \otimes SU(2)_{K_B}\) theories, that is the straightforward extension of the results of section 2 to even \(K\), may, in particular, prove interesting if there is some generalization of the ADE classification found for the single theories. Nonetheless, a complete classification for tensor product theories built with many factors is not practical given the enormous number of possibilities. For the purposes of string model building a procedure for constructing any particular invariant, such as that available for free field constructions, would be advantageous. Perhaps a generalization of the twisting procedure to operators with nontrivial (or altered) fusion rules, as suggested by the example in section 3, would prove sufficient. In this regard the results of [8,9], (which have been extensively exploited recently by Gannon\cite{10}) are intriguing though not yet sufficient. In these works, obtaining new tensor product modular invariants is related to shifting the momentum lattice of a free boson theory, but at the cost of sacrificing positivity of the coefficients in the partition function.

Finally we must stress that the condition of modular invariance alone is insufficient to guarantee a consistent conformal field theory; for constructions not based on free
fields we must ultimately check that a consistent operator algebra on the plane exists.

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