Entropy of the Schwarzschild black hole to all orders in the Planck length

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ABSTRACT

Considering corrections to all orders in the Planck length on the quantum state density from a generalized uncertainty principle (GUP), we calculate the statistical entropy of the scalar field on the background of the Schwarzschild black hole without any cutoff. We obtain the entropy of the massive scalar field proportional to the horizon area.

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1 Introduction

Three decades ago, Bekenstein had suggested that the entropy of a black hole is proportional to the area of the horizon through the thermodynamic analogy \[1\]. Subsequently, Hawking showed that the entropy of the Schwarzschild black hole satisfies exactly the area law by means of Hawking radiation based on the quantum field theory \[2\]. After their works, 't Hooft investigated the statistical properties of a scalar field outside the horizon of the Schwarzschild black hole by using the brick wall method with the Heisenberg uncertainty principle (HUP) \[3\]. However, although he obtained the entropy proportional to the horizon area, an unnatural brick wall cutoff was introduced to remove the ultraviolet divergence near the horizon \[4, 5, 6, 7, 8, 9\]. After these works, many efforts \[10, 11\] have been devoted to the generalized uncertainty relations, which lead to the minimal length as a natural ultraviolet cutoff \[12\], and its consequences, especially the effect on the density of states.

Recently, in Refs. \[13, 14, 15\], the authors calculated the entropy of black holes to leading order in the Planck length by using the newly modified equation of the density of states motivated by the generalized uncertainty principle (GUP) \[10\], which drastically solves the ultraviolet divergences of the just vicinity near the horizon without a cutoff. Moreover, Nouicer has investigated the GUP effects to all orders in the Planck length on black hole thermodynamics \[16\] by arguing that the GUP up to leading order correction in the Planck length is not enough because the wave vector \( k \) does not satisfy the asymptotic property in the modified dispersion relation \[17\]. Very recently, he has extended the calculation of entropy to all orders in the Planck length \[18\] for the Randall-Sundrum brane case \[19\].

On the other hand, Yoon et. al. have very recently pointed out that since the minimal length \( \sqrt{\lambda} \) is actually related to the brick wall cutoff \( \epsilon \), the entropy integral about \( r \) in the range of the near horizon should be carefully treated for a convergent entropy \[20\].

In this paper, we calculate the statistical entropy of a scalar field on the Schwarzschild black hole background to all orders in the Planck length by carefully considering the entropy integral about \( r \) in the range \( (r_H, r_H + \epsilon) \) near the horizon. By using the novel equation of the density of states \[16, 18\] motivated by the GUP in the quantum gravity, we calculate the quantum entropy of a massive scalar field on the Schwarzschild black hole background. As a result, we obtain the desired Bekenstein-Hawking entropy without any
artificial cutoff and little mass approximation satisfying the asymptotic property of the wave vector \( k \) in the modified dispersion relation.

2 All order corrections of GUP

Now, it is well-known that the deformed Heisenberg algebra [10] leads to the GUP showing the existence of the minimal length. In this section we briefly recapitulate this approach and exploit the recently obtained results [16, 18, 21, 22, 23]. Indeed, it has been shown that the Feynman propagator displays an exponential ultraviolet cutoff of the form of \( \exp \left( -\lambda p^2 \right) \), where the parameter \( \sqrt{\lambda} \) actually plays a role of the minimal length as shown later. Recently, this framework has been further applied to the black hole evaporation process [24, 25]. On the other hand, the quantum gravity phenomenology has been tackled with effective models based on the GUPs and/or modified dispersion relations [26] containing the minimal length as a natural ultraviolet cutoff [17]. Moreover, the essence of the untraviolet finiteness of the Feynman propagator can be also captured by a nonlinear relation \( p = f(k) \), where \( p \) and \( k \) are the momentum and the wave vector of a particle, respectively, generalizing the commutator between the commutating operators \( \hat{x} \) and \( \hat{p} \) to

\[
[\hat{x}, \hat{p}] = i \frac{\partial p}{\partial k} \Leftrightarrow \Delta x \Delta p \geq \frac{1}{2} \left| \frac{\partial p}{\partial k} \right|
\]

at the quantum mechanical level [17]. Then, the usual momentum measure \( \prod_{i=1}^{n} dp^i \) is deformed to

\[
\prod_{j=1}^{n} dp^j \prod_{i=1}^{n} \frac{\partial k^i}{\partial p^j}.
\]

For simplicity, in the following, let us restrict ourselves to the isotropic case in one space-like dimension. According to the Refs. [22, 23], we have

\[
\frac{\partial p}{\partial k} = e^{\lambda p^2},
\]

where \( \lambda \) is a dimensionless constant of order one in the Planck length units. Now, let us consider the following representation of the position and momentum operators

\[
X \equiv i \ e^{\lambda p^2} \partial_p, \\
P \equiv p.
\]

Then, these operators satisfy the deformed algebra as follows

\[
[X, P] = i \ e^{\lambda p^2},
\]
which leads to the generalized uncertainty relation as

$$\Delta X \Delta P \geq \frac{1}{2} \langle e^{\lambda P^2} \rangle. \quad (6)$$

Next, in order to investigate the quantum implication of this deformed algebra, let us solve the above relation (6) for $\Delta P$ that is satisfied with the equality. Since $\langle P^{2n} \rangle \geq \langle P^2 \rangle^n$ and $(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$, the generalized uncertainty relation can be written as

$$\Delta X \Delta P = \frac{1}{2} e^{\lambda((\Delta P)^2 + \langle P \rangle^2)}. \quad (7)$$

Taking the square of this expression and using the definition of the multivalued Lambert function [27] (see Fig.1), we obtain

$$W(\xi) e^{W(\xi)} = \xi, \quad (8)$$

where we have set $W(\xi) = -2\lambda (\Delta P)^2$ and $\xi = -\frac{\lambda}{2(\Delta X)} e^{2\lambda(P)^2}$.

On the other hand, by using this function the momentum uncertainty is given by

$$\Delta P = \frac{e^{\lambda(P)^2}}{2\Delta X} e^{-\frac{1}{2}W(\xi)}. \quad (9)$$

In order to have a real solution for $\Delta P$, the argument of the Lambert function is required to satisfy $\xi \geq -1/e$, which leads to the following condition

$$\frac{\lambda}{2(\Delta X)^2} e^{2\lambda(P)^2} \leq \frac{1}{e}. \quad (10)$$
This gives naturally the position uncertainty as

\[ \Delta X \geq \sqrt{\frac{e\lambda}{2}} e^{\lambda \langle P \rangle^2} \equiv \Delta X_{\text{min}}, \]  

(11)

where \( \Delta X_{\text{min}} \) is a minimal uncertainty in position. Moreover, this minimal length, which is intrinsically derived for physical states with \( \langle P \rangle = 0 \), is given by

\[ \Delta X_0^A = \sqrt{\frac{e\lambda}{2}}. \]  

(12)

This is the absolutely smallest uncertainty in position. In fact, this minimal length effectively plays a role of the brick wall cutoff giving the thickness of the thin-layer near the horizon \[13, 14, 15\]. Furthermore, the momentum uncertainty with \( \langle P \rangle = 0 \) is easily read from Eq. (9) with \( W(\xi) = -2\lambda (\Delta P)^2 \) as

\[ \Delta P = \frac{1}{2\Delta X} e^{\lambda (\Delta P)^2}. \]  

(13)

A series expansion of Eq. (13) naturally includes the well-known form of the GUP up to the leading order correction in the Planck length units \[15\] as follows

\[ \Delta X \Delta P \approx \frac{1}{2} \left[ 1 + \lambda (\Delta P)^2 + \mathcal{O} \left( (\Delta P)^4 \right) \right]. \]  

(14)

Then, the minimal length up to the leading order is given by

\[ \Delta X_0^L = \sqrt{\lambda} < \Delta X_0^A. \]  

(15)

However, only this leading order correction of the GUP does not satisfy the property that the wave vector \( k \) asymptotically reaches the cutoff in large energy region as recently reported in Ref. \[17\].

In the following sections we use the form of the GUP given by Eq. (13) with the corresponding minimal length (12) to calculate the entropy of a scalar field on the Schwarzschild black hole background. In this paper, we take the units \( G = \hbar = c = k_B \equiv 1 \).

### 3 Scalar field on the Schwarzschild black hole Background

Let us consider the 4-dimensional Schwarzschild black hole solution as

\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2, \]  

(16)
where $d\Omega^2_{(2)}$ is a metric of the unit 2-sphere. In this background, let us first consider a scalar field with mass $\mu$, which satisfies the Klein-Gordon equation given by

$$(\nabla^2 - \mu^2)\Phi = 0.$$  \hspace{1cm} (17)

It can be rewritten as

$$-\frac{1}{f} \frac{\partial^2 \Phi}{\partial_t^2} + \frac{1}{r^2} \frac{\partial}{\partial_r} \left( r^2 f \frac{\partial}{\partial_r} \Phi \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta} \Phi) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} - \mu^2 \Phi = 0 \hspace{1cm} (18)$$

with $f = 1 - \frac{2M}{r}$. Substituting the wave function $\Phi(t, r, \theta, \phi) = e^{-i\omega t} \psi(r, \theta, \phi)$, we find that the Klein-Gordon equation becomes

$$\partial_t^2 \psi + \left( \frac{1}{f} \frac{\partial}{\partial r} f + \frac{2}{r} \right) \partial_r \psi + \frac{1}{f} \left( \frac{1}{r^2} \left[ \partial^2_{\theta} + \cot \theta \partial_{\theta} + \frac{1}{\sin^2 \theta} \partial^2_{\phi} \right] + \frac{\omega^2}{f} - \mu^2 \right) \psi = 0.$$ \hspace{1cm} (19)

By using the Wenzel-Kramers-Brillouin approximation [3] with $\psi \sim \exp[iS(r, \theta, \phi)]$, we have

$$p_r^2 = \frac{1}{f} \left( \frac{\omega^2}{f} - \mu^2 - \frac{p_\theta^2}{r^2} - \frac{p_\phi^2}{r^2 \sin^2 \theta} \right),$$ \hspace{1cm} (20)

where

$$p_r = \frac{\partial S}{\partial r}, \quad p_\theta = \frac{\partial S}{\partial \theta}, \quad p_\phi = \frac{\partial S}{\partial \phi}. \hspace{1cm} (21)$$

Furthermore, we also obtain the square module momentum as follows

$$p^2 = p_r p^r = g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2 + g^{\phi\phi} p_\phi^2 = \frac{\omega^2}{f} - \mu^2.$$ \hspace{1cm} (22)

Then, the volume in the momentum phase space is given by

$$V_p(r, \theta) = \int dp_r dp_\theta dp_\phi$$

$$= \frac{4\pi}{3} \sqrt{\frac{1}{f} \left( \frac{\omega^2}{f} - \mu^2 \right)} \cdot \sqrt{r^2 \left( \frac{\omega^2}{f} - \mu^2 \right)} \cdot \sqrt{r^2 \sin^2 \theta \left( \frac{\omega^2}{f} - \mu^2 \right)}$$

$$= \frac{4\pi r^2 \sin \theta}{3} \sqrt{\frac{f}{\omega^2}} \left( \frac{\omega^2}{f} - \mu^2 \right)^{\frac{3}{2}}. \hspace{1cm} (23)$$

with the condition $\omega \geq \mu \sqrt{f}$.

## 4 Entropy to all orders in the Planck Length

Now, let us calculate the statistical entropy of the scalar field on the Schwarzschild black hole background to all orders in the Planck length units. When the
gravity is turned on, the number of quantum states in a volume element in phase cell space based on the GUP in the 3+1 dimensions is given by

\[ dn_A = \frac{d^3xd^3p}{(2\pi)^3}e^{-\lambda p^2}, \] (24)

where \( p^2 = p^i p_i \) (\( i = r, \theta, \phi \)) and one quantum state corresponding to a cell of volume is changed from \((2\pi)^3\) into \((2\pi)^3e^{\lambda p^2}\) in the phase space [13, 14, 15].

From the Eqs. (22) and (24), the number of quantum states related to the radial mode with energy less than \( \omega \) is given by

\[
n_A(\omega) = \frac{1}{(2\pi)^3} \int \frac{dr d\theta d\phi dp_r dp_\theta dp_\phi e^{-\lambda p^2}}{\int \frac{dp_i \phi_i}{(2\pi)^3}} = \frac{1}{(2\pi)^3} \int dr d\theta d\phi \frac{V_r(r, \theta)}{e^{-\lambda \frac{\omega^2}{f} - \mu^2}} - \lambda = \int \frac{dr \frac{r^2}{\sqrt{f}} \left( \frac{\omega^2}{f} - \mu^2 \right)^{\frac{3}{2}} e^{-\lambda \left( \frac{\omega^2}{f} - \mu^2 \right)}}{3\pi}.
\] (25)

It is interesting to note that \( n_A(\omega) \) is convergent at the horizon without any artificial cutoff due to the existence of the suppressing exponential \( \lambda \)-term induced from the generalized uncertainty principle.

For the bosonic case, the free energy at inverse temperature \( \beta \) is given by

\[ e^{-\beta F} = \prod_K \left[ 1 - e^{-\beta \omega_K} \right]^{-1}, \] (26)

where \( K \) represents the set of quantum numbers. Then, by using Eq. (25), we are able to obtain the free energy as

\[
F_A = \frac{1}{\beta} \sum_K \ln \left[ 1 - e^{-\beta \omega_K} \right] \approx \frac{1}{\beta} \int dn_A(\omega) \ln \left[ 1 - e^{-\beta \omega} \right] = -\int^{\infty}_{\mu \sqrt{f}} d\omega \frac{n_A(\omega)}{e^{\beta \omega} - 1} = \frac{2}{3\pi} \int_{r_H}^{\infty} dr \frac{r^2}{\sqrt{f}} \int_{\mu \sqrt{f}}^{\infty} d\omega \frac{\left( \frac{\omega^2}{f} - \mu^2 \right)^{\frac{3}{2}}}{e^{\beta \omega} - 1} e^{-\lambda \left( \frac{\omega^2}{f} - \mu^2 \right)}. \] (27)

Here, we have taken the continuum limit in the first line and integrated it by parts in the second line. In the last line, since \( f \to 0 \) near the event horizon, \( i.e., \) in the range of \((r_H, r_H + \epsilon)\), \( \frac{\omega^2}{f} - \mu^2 \) becomes \( \frac{\omega^2}{f} \). Therefore, although we do not require the little mass approximation, the free energy can be rewritten
as

\[ F_A = -\frac{2}{3\pi} \int_{r_H}^{r_H+\epsilon} dr \int_0^{\infty} d\omega \frac{\omega^3}{(e^{\beta\omega} - 1)} e^{-\lambda \omega^2}. \]  

(28)

On the other hand, we are also interested in the contribution from just the vicinity near the horizon in the range \((r_H, r_H + \epsilon)\), where \(\epsilon\) is related to a proper distance of order of the minimal length \([12]\) as follows

\[ \sqrt{\frac{e\lambda}{2}} = \int_{r_H}^{r_H+\epsilon} \frac{dr}{\sqrt{f(r)}} \approx \int_{r_H}^{r_H+\epsilon} \frac{dr}{\sqrt{2\kappa(r - r_H)}} = \sqrt{\frac{2\epsilon}{\kappa}}, \]  

(29)

where the expansion of \(f(r)\) near the horizon is given by

\[ f(r) \approx f(r_H) + (r - r_H) \left( \frac{df}{dr} \right)_{\beta = \beta_H} + \mathcal{O}\left((r - r_H)^2\right). \]  

(30)

Here, \(\kappa\) is the surface gravity at the horizon of the black hole, and it is identified as \(\kappa = \frac{1}{2}(\frac{df}{dr})_{\beta = \beta_H} = 2\pi\beta_H^{-1} = 1/(2r_H)\).

Before calculating the entropy, let us mention that Yoon et. al. have recently suggested that since the minimal length \(\sqrt{\frac{e\lambda}{2}}\) in Eq. \((29)\) is related to the brick wall cutoff \(\epsilon\), the entropy integral about \(r\) in the range near the horizon should be carefully treated for obtaining a convergent entropy \([20]\).

In particular, although the term \((e^{\beta\omega} - 1)\) in Eq. \((28)\) with \(x = \sqrt{\frac{\lambda}{\pi}}\omega\) was expanded in the previous works giving \(\beta \sqrt{\frac{x}{\lambda}}\), one may not simply expand up to the first order since \(0 \leq \frac{x}{\lambda} = \frac{2(\frac{r - r_H}{\lambda})}{\lambda} \leq \frac{2\epsilon}{\lambda} = \kappa^2\) near the horizon.

Now, let us carefully consider the integral about \(r\) near the horizon by extracting out the \(\epsilon\)-factor through Taylor’s expansion \((30)\) of \(f(r)\). Then, from \(F_A\) in Eq. \((28)\) the entropy can be obtained as

\[ S_A = \beta^2 \frac{\partial F_A}{\partial \beta} |_{\beta = \beta_H} \]

\[ = \frac{\beta^2_H}{6\pi} \int_0^{\infty} d\omega \frac{\omega^4}{\sinh^2(\frac{\beta_H}{2}\omega)} \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{f^2} e^{-\lambda \omega^2} \]

\[ = \frac{\beta^2_H}{6\pi \lambda^2 \sqrt{\lambda}} \int_0^{\infty} dx \frac{x^4}{\sinh^2(\frac{\beta_H}{2\sqrt{\lambda}} x)} \Lambda_A(x, \epsilon), \]  

(31)

where \(x \equiv \sqrt{\lambda}\omega\) and the integral about \(r\) in the range of the near horizon is given by

\[ \Lambda_A(x, \epsilon) = \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{f^2} e^{-r^2} = \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{f^2} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{x^2}{f^2} \right)^n \right] \]
\[
\begin{align*}
\frac{6}{x^6} \int_{r_H}^{r_H+\epsilon} dr \left[ r_H^2 + 2r_H(r-r_H) + \mathcal{O}\left((r-r_H)^2\right) \right] \\
\times \left( \frac{2\kappa (r-r_H) + \mathcal{O}\left((r-r_H)^2\right)}{1 + \frac{2\kappa}{3x^2} (r-r_H) + \mathcal{O}\left((r-r_H)^2\right)} \right) \\
\approx \frac{12\kappa r_H^2}{x^6} \int_{r_H}^{r_H+\epsilon} dr \left[ (r-r_H) + \mathcal{O}\left((r-r_H)^2\right) \right] \\
\approx \frac{6\kappa r_H^2 \epsilon^2}{x^6}.
\end{align*}
\]

Then, the entropy is reduced as

\[
S_A \approx \frac{\beta_H}{2 \sqrt{\lambda}} \int_0^\infty dx \frac{x^4}{\sinh^2 \left( \frac{\beta_H}{2 \sqrt{\lambda}} x \right)} \frac{6\kappa r_H^2 \epsilon^2}{x^6}
\]

\[
= \frac{\pi^2 e^2 r_H^2}{4 \lambda} \int_0^\infty dy \frac{1}{y^2 \sinh^2 y},
\]

where \( y \equiv \frac{\beta_H}{2 \sqrt{\lambda}} x \), \( \beta_H \kappa = 2\pi \), and \( \epsilon = \lambda \kappa / 4 \).

Therefore, when \( r \to r_H \), we finally get the desired entropy of the massive scalar field on the Schwarzschild black background as follows

\[
S_A \approx \frac{\pi^2 e^2 r_H^2}{4 \lambda} \frac{2\zeta(3)}{\pi^2} = \frac{e^2 \zeta(3)}{8\pi \lambda} (4\pi r_H^2)
\]

\[
= \frac{1}{4} \frac{e^2 \zeta(3)}{2\pi \lambda} A,
\]

where \( A = 4\pi r_H^2 \) and \( \zeta(3) = \sum_{n=1}^\infty (1/n^3) \approx 1.202 \). Moreover, if we assume the minimal length parameter \( \lambda \) in the Planck length units as \( \frac{e^2 \zeta(3)}{2\pi} \), then the entropy can be rewritten by the desired area law as \( S_A = \frac{1}{4} A \). Note that there is no divergence within the just vicinity near the horizon due to the effect of the generalized uncertainty relation on the quantum states.

On the other hand, in order to compare the result \((34)\) with those of the usual approximation approach \([13, 14, 15, 18]\), let us calculate the entropy in the usual coarse-grained approximation. In terms of the variable \( x = \omega \sqrt{\frac{\lambda}{f}} \) and the fact that \( e^{\beta \omega} - 1 = e^{\beta \sqrt{\frac{\lambda}{f}} x} - 1 \approx \beta \sqrt{\frac{\lambda}{f}} x \) for \( f \to 0 \), we have

\[
F_A^0 = -\frac{1}{\beta} \frac{2}{3\pi (\lambda)^{3/2}} \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{\sqrt{f}} \int_0^\infty dx x^2 e^{-x^2}
\]

\[
= -\frac{1}{\beta} \frac{1}{6\sqrt{\pi} (\lambda)^{3/2}} \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{\sqrt{f}}.
\]
Then, from the free energy (35), the entropy to all orders for the scalar field is given by

\[ S^0_A = \beta^2 \frac{\partial F_A}{\partial \beta} |_{\beta = \beta_H} \]
\[ = \frac{1}{6\sqrt{\pi} \lambda^{3/2}} \int_{r_H}^{r_H+\epsilon} dr \frac{1}{\sqrt{f}} r^2 \]
\[ \approx \frac{1}{6\sqrt{\pi} \lambda^{3/2}} \sqrt{e/2} \frac{r_H^2}{24\sqrt{2} \lambda^2} A. \quad (36) \]

This is smaller than \( S_A \) in Eq.(34), which was obtained by using the rigorous approximation near the horizon. But, if we assume the minimal length parameter \( \lambda \) as \( \frac{\sqrt{e}}{6\sqrt{2}\pi^{3/2}} \), then the entropy can be also rewritten by the desired Bekenstein-Hawking area law as \( S^0_A = \frac{1}{4} A. \)

Finally, it seems to be appropriate to comment on the entropy (34) to all orders in the Planck length comparing with the entropy to the leading order, which can be also carefully treated through the same expansion approach. In this case, the free energy up to the leading order [13] is given by

\[ F_L \approx - \frac{2}{3\pi} \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{f^2} \int_0^\infty d\omega \frac{\omega^3}{(e^{\beta \omega} - 1) \left(1 + \frac{\lambda}{f^2} \omega^2\right)^3} \quad (37) \]

instead of \( F_A \) in Eq.(28). Then, the entropy \( S_L \) is given by

\[ S_L = \beta^2 \frac{\partial F_L}{\partial \beta} |_{\beta = \beta_H} \]
\[ = \frac{\beta^2 H}{6\pi} \int_0^\infty d\omega \frac{\omega^4 \sinh^2 \left(\frac{\beta H}{2} \omega\right)}{H} \int_{r_H}^{r_H+\epsilon} dr \frac{r^2}{f^2 \left(1 + \frac{\lambda}{f^2} \omega^2\right)^3} \]
\[ = \frac{\beta^2 H}{6\pi \lambda^2 \sqrt{\lambda}} \int_0^\infty dx \frac{x^4}{\sinh^2 \left(\frac{\beta H}{2\sqrt{\lambda}} x\right)} \Lambda_L(x, \epsilon), \quad (38) \]

where \( x = \sqrt{\lambda} \omega \) and \( \Lambda_L(x, \epsilon) \) is given by

\[ \Lambda_L(x, \epsilon) = \frac{1}{x^6} \int_{r_H}^{r_H+\epsilon} dr \frac{r^2 f}{\left(1 + \frac{f^2}{x^2}\right)^3}. \quad (39) \]

Then, the integral (39) is obtained as

\[ \Lambda_L(x, \epsilon) = \frac{r_H^2}{x^6} \int_{r_H}^{r_H+\epsilon} dr \frac{2 \kappa (r - r_H) + O((r - r_H)^2)}{\left[1 + \frac{2 \kappa}{x^2} (r - r_H) + O((r - r_H)^2)\right]^3} \]
\[
= \frac{2\kappa r_H^2}{x^6} \int_{r_H}^{r_H + \epsilon} dr \left[ (r - r_H) + \mathcal{O} \left( (r - r_H)^2 \right) \right] \\
\approx \frac{\kappa r_H^2 \epsilon^2}{x^6}. \tag{40}
\]

As a result, the entropy up to the leading order correction becomes

\[
S_L \approx \frac{1}{4} \frac{\zeta(3)}{3\pi \lambda} A < S_A. \tag{41}
\]

5 Summary

We have investigated the entropy (34) to all orders in the Planck length units through the rigorous Taylor expansion approach comparing with the coarse-grained entropy (36), which has been obtained through the usual approximation approach. Although their values are different, we have obtained the desired Bekenstein-Hawking entropy by properly adjusting the minimal length parameter \( \lambda \) for the both approximation approaches.

In summary, we have studied the massive scalar field on the background of the Schwarzschild black hole by carefully counting the number of quantum states in the just vicinity near the horizon, based on the generalized uncertainty principle. As a result, we have obtained the desired Bekenstein-Hawking entropy to all orders in the Planck length units without any artificial cutoff and little mass approximation satisfying the asymptotic property of the wave vector \( k \) in the modified dispersion relation.

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