Gravitational wave templates from Extreme Mass Ratio Inspirals

V. Skoupý
Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, CZ-180 00 Prague, Czech Republic
Astronomical Institute of the Czech Academy of Sciences, Boční II 1401/1a, CZ-141 00 Prague, Czech Republic

G. Lukes-Gerakopoulos
Astronomical Institute of the Czech Academy of Sciences, Boční II 1401/1a, CZ-141 00 Prague, Czech Republic

Abstract. An extreme mass ratio inspiral takes place when a compact stellar object is inspiraling into a supermassive black hole due to gravitational radiation reaction. Gravitational waves (GWs) from this system can be calculated using the Teukolsky equation (TE). In our case, we compute the asymptotic GW fluxes of a spinning body orbiting a Kerr black hole by solving numerically the TE both in time and frequency domain. Our ultimate goal is to produce GW templates for space-based detectors such as LISA.

Introduction

The Laser Interferometer Space Antenna (LISA) is a future space based gravitational wave (GW) detector planned to launch in 2030s [Amaro-Seoane et al., 2017]. LISA will consist of three spacecrafts forming an equilateral triangle with sides 2.5 million km long on heliocentric orbit. Changes in the distance between the spacecrafts will be monitored by Michelson-like interferometers with high precision. When a GW passes this constellation, the spacetime between the spacecrafts will be quasi-periodically stretched and contracted in the direction perpendicular to the propagation of the wave. Hence, LISA will be able to detect the wave, including its amplitude and phase, by measuring the changes in the distance between the three spacecrafts. This GW observatory will be sensitive in frequencies around $10^{-3}$ Hz.

One type of events LISA will be able to detect are extreme mass ratio inspirals (EMRIs). It is expected that the centre of galaxies host supermassive black holes with masses in the range of $10^6$–$10^9 M_\odot$. An EMRI takes place when a stellar mass black hole or a neutron star is inspiraling into a supermassive black hole while losing energy and angular momentum due to gravitational radiation reaction. Such a system is emitting GWs that should be detectable far away from the source in the mHz bandwidth.

A GW signal from an EMRI can provide important information about the parameters of the system such as the masses of the objects, their spins etc. Actually EMRIs’ detection will give us the opportunity to map the spacetime around a supermassive black hole to high accuracy. Since the parameter analysis of the detected GW signal depends on the accuracy of the waveform templates, it is important to model an EMRI with adequate precision and find theoretical waveforms from EMRI systems for various parameters.

The radiation reaction that acts on a moving particle can be split in two parts: a dissipative and a conservative one. In this work, we focus on the dissipative part, which can be calculated from the energy and angular momentum fluxes at the horizon of the black hole and at infinity. To obtain the leading term in the evolution of the GW phase, it is sufficient to consider only time averages of the dissipative part [Barack and Pound, 2019]. This approximation is called adiabatic. In this case, the particle is slowly shifted from one orbit to another on time scale much larger than the orbital period.

In this paper, we first review the properties of a spinning test particle moving in the Kerr
spacetime. Then we summarize the Teukolsky formalism, which allows us to calculate the energy and angular momentum fluxes along with the waveforms. Using this formalism we calculate the energy fluxes from circular equatorial orbits of spinning particles around Schwarzschild and Kerr black holes. Subsequently, we use these fluxes to adiabatically evolve circular equatorial orbits and to find the effects of spin of the secondary object on the GW phase. Throughout the paper, we use geometrical units $G = c = 1$, where $G$ is the gravitational constant and $c$ is the speed of light, and the metric signature $(-+++)$.

**Spinning test particles in the Kerr geometry**

The Kerr geometry, which describes a rotating black hole in vacuum, is represented in Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ by the metric [Misner et al., 2017]

$$
 ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left( (r^2 + a^2) d\phi - adt \right)^2 
$$

where

$$
 \Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta. 
$$

This metric depends on two parameters: the mass of the black hole $M$ and the Kerr parameter $a$. The internal angular momentum (spin) of a Kerr black hole is $aM$. At the radius $r_+ = M + \sqrt{M^2 - a^2}$, where $\Delta = 0$, the outer event horizon is located. In this paper, we are dealing only with the region $r > r_+$. For $a = 0$ the Kerr spacetime reduces to the Schwarzschild one.

A compact test object in general relativity can be characterized just by its multipoles [Dixon, 1964]. For example, a rotating black hole or a neutron star moving in a Kerr background can be modeled using a pole-dipole approximation, where only the mass and the spin of these compact objects are taken into account, reducing them to a spinning test particle. The pole-dipole approximation holds as long as the size of the test body is smaller than the scale of the background curvature. The stress-energy tensor of a spinning test particle reads [Ehlers, 1979]

$$
 T^{\mu\nu} = \frac{1}{\sqrt{-g}} \left( v^{(\mu} p^{\nu)} - \frac{1}{2} R^{\mu\nu\kappa\lambda} v^\kappa S^\lambda \right), 
$$

where $g$ is the determinant of the metric, $v^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity, $p^\mu$ is the four-momentum, $S^{\mu\nu}$ is the spin tensor, $\delta^3 = \delta(r - r_p(t))\delta(\theta - \theta_p(t))\delta(\phi - \varphi_p(t))$ where $r_p(t)$, $\theta_p(t)$ and $\varphi_p(t)$ are the coordinates of the particle depending at the coordinate time $t$. The conservation of the stress-energy tensor [3] leads to the Mathisson-Papapetrou-Dixon (MPD) equations [Mathisson, 1937; Papapetrou, 1951; Dixon, 1964]

$$
 \frac{Dp^\mu}{d\tau} = -\frac{1}{2} R^{\mu}_{\nu\kappa\lambda} v^\nu S^\kappa S^\lambda, 
$$

$$
 \frac{DS^{\mu\nu}}{d\tau} = p^\mu v^\nu - p^\nu v^\mu, 
$$

where $\tau$ is the proper time and $R^{\mu}_{\nu\kappa\lambda}$ is the Riemann tensor.

The centre of mass for an extended body in general relativity is not uniquely defined. To fix the centre of mass for a spinning body, one has to specify the so called spin-supplementary condition (SSC). In this work we use the Tulczyjew-Dixon SSC [Dixon, 1964], which reads

$$
 S^{\mu\nu} p_\mu = 0. 
$$

and closes the MPD system. Actually, this SSC allows an explicit relation of the dependence of the four-velocity $v^\mu$ on the four-momentum $p^\mu$ [Ehlers and Rudolph, 1977].
The magnitude of the spin is defined as $S^2 = S_{\mu\nu} S^{\mu\nu}/2$, while the mass of the particle is $\mu^2 = -p^\rho p_\rho$. Both of these quantities are conserved under Tulczyjew-Dixon SSC. Instead of the measure of the spin $S$, its dimensionless counterpart $\sigma = S/((\mu M))$ is often used in EMRI studies. For this dimensionless spin holds $\sigma \approx \mu/M \equiv q$, i.e. it is of the order of an EMRI mass ratio.

There are two Killing vectors in the Kerr geometry
\[ \xi^\mu_{(t)} = \frac{\partial x^\mu}{\partial t}, \quad \xi^\mu_{(\phi)} = \frac{\partial x^\mu}{\partial \phi}, \]
providing respectively two conserved quantities for the spinning particle
\[ E = -\xi^\mu_{(t)} p_\mu + \frac{1}{2} \xi^\mu_{(\nu)} S_{\mu\nu}, \]
\[ J_z = \xi^\mu_{(\phi)} p_\mu - \frac{1}{2} \xi^\mu_{(\nu)} S_{\mu\nu}. \]
These quantities can be interpreted at infinity as the energy and the component of the total angular momentum parallel to the rotational axis of the central black hole ($z$-axis).

### Teukolsky equation

The mass ratio $q$ of an EMRI lies between $10^{-7}$ and $10^{-4}$. Thanks to this, the GWs from such systems have relatively low amplitudes and perturbation theory can be employed. Using Newman-Penrose formalism it is possible to find equations governing the perturbation of the Weyl tensor projected on some tetrad. Teukolsky [1973] found the master equation (Teukolsky equation, TE)
\[
\left( \frac{(r^2 + a^2)}{\Delta} - a^2 \sin^2 \theta \right) \frac{\partial^2 \psi}{\partial t^2} + \frac{4 M a r}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left( \frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \psi}{\partial \phi^2} - \Delta^{-s} \frac{\partial}{\partial r} \left( \Delta^{s+1} \frac{\partial \psi}{\partial r} \right) - \frac{1}{\sin \theta \partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2 s \left( \frac{a(r - M)}{\Delta} + i \cos \theta \right) \frac{\partial \psi}{\partial \phi} - 2 s \left( \frac{M(r^2 - a^2)}{\Delta} - r - i a \cos \theta \right) \frac{\partial \psi}{\partial t} + \left( s^2 \cot^2 \theta - s \right) \psi = 4\pi \Sigma T, \tag{10}
\]
which governs scalar, neutrino, electromagnetic and gravitational perturbations of the Kerr spacetime. $s$ denotes the spin weight of the field and $\psi$ is a projection of the field quantity on a tetrad depended on $s$, while $T$ is the source term. For GWs at infinity it is useful to calculate the quantity $\Psi_4 = \rho^4 \psi$ for $s = -2$, where $\rho = -1/(r - i a \cos \theta)$. Then the source term consists of derivatives of the stress-energy tensor projected on the tetrad.

This equation is usually decomposed into azimuthal $m$-modes
\[
\psi(t, r, \theta, \phi) = \sum_{m=-\infty}^{\infty} \psi_m(t, r, \theta) e^{im\phi}, \quad \psi_m(t, r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t, r, \theta, \phi) e^{-im\phi} d\phi. \tag{11}
\]
This replaces the derivatives $d/d\phi$ with $im$.

### Time domain

It is possible to numerically solve the (2+1)dimensional TE in the time domain. We solve TE equation in the so called horizon-penetrating hyperboloidal (HH) coordinates ($\tau, \rho, \theta$). Hypersurfaces of constant time in these coordinates are light-like at the horizon and null infinity and compactified in the radial direction, which allows us to deal with the boundary conditions. By defining the quantity $\phi = \partial_\tau \psi_m$ one can split the system into two first order in time and
second order in space differential equations, which then can be solved by using the methods of lines [Harms et al., 2014].

The energy flux $dE^\infty/dt$ and the angular momentum fluxes at infinity can be calculated from the strain $h = h_+ - i h_\times$, where $+$ and $\times$ are the polarizations of the GW. The second derivative with respect to time of the strain is asymptotically proportional to the field quantity $\ddot{h} \propto \psi$ at infinity. The energy flux $dE^H/dt$ and the angular momentum fluxes can be calculated at the horizon as well.

**Frequency domain**

It is also possible to solve the TE (10) in the frequency domain by Fourier transformation in time, i.e.

$$\psi(t, r, \theta, \phi) = \sum_l \sum_m \int_{-\infty}^{\infty} d\omega s_l^m(\theta) s R_{lm}(r)e^{-i\omega t + i m \phi}. \tag{12}$$

By employing this transform separated ordinary differential equations can be derived for the spin weighted spheroidal harmonic function $s_l^m(\theta)$ and the radial function $s R_{lm}(r)$. This separation is an advantage of the frequency domain method, since it highly reduces the computational cost. On the other hand, only GWs from bound multiperiodic orbits without dissipation can be calculated by a summation over discrete frequencies. Thus, different methods have to be employed to calculate the GW fluxes from inspiralling orbits.

The energy flux at infinity is given by [Drasco and Hughes, 2006]

$$\frac{dE^\infty}{dt} = \sum_{l, m} \frac{|Z_{lm}^H|^2}{4\pi \omega_m^2}, \tag{13}$$

where the amplitudes $Z_{lm}^H$ can be calculated as a convolution of the radial function $s R_{lm}(r)$ and the source term derived from the stress-energy tensor (3). For circular equatorial orbits of spinning particles with spin parallel to the $z$-axis at radius $r_0$ these amplitudes read

$$Z_{lm}^H = A_0 R_{lm}(r_0) + A_1 \frac{dR_{lm}(r_0)}{dr} + A_2 \frac{d^2 R_{lm}(r_0)}{dr^2} + A_3 \frac{d^3 R_{lm}(r_0)}{dr^3}. \tag{14}$$

We have derived independently the exact expressions of these coefficients $A_i$ and cross checked them with those of [Piovano et al., 2020]. This derivation is possible thanks to the fact that the orbital frequency $\Omega = d\varphi/dt$ is constant for circular equatorial orbits, and, hence, each $m$-mode consists of only one frequency $\omega_m = m \Omega$. Similar expressions can be derived for the flux at the horizon and the angular momentum fluxes. We have checked that the frequency domain results agree for various orbits with the time domain approach results.

We have calculated the total energy fluxes of spinning particles moving on circular equatorial trajectories around a black hole with their spin parallel to the $z$-axis for several values of the frequency parameters $y \equiv (M \Omega)^{2/3}$. These calculations have taken place on a black hole background for $a = 0$, $a = 0.5M$ and $a = 0.9M$, while the spin of the secondary $\sigma$ ranged between $-0.5$ and $0.5$ with step $0.1$. The dependence of the flux on the spin was fitted with a polynomial and the linear part was extracted to obtain

$$\mathcal{F} = \frac{dE^\infty}{dt} + \frac{dE^H}{dt} = \mathcal{F}_0 + \sigma \mathcal{F}_1 + \mathcal{O}(\sigma^2), \tag{15}$$

where $\mathcal{F}$ denotes the total energy flux, $\mathcal{F}_0$ is a constant term corresponding to the flux from a non-spinning particle and $\mathcal{F}_1$ is the term linear in spin. Both terms are plotted in Fig. 1. The radial and angular functions and their derivatives were calculated using the Black Hole Perturbation Toolkit (BHPT) [BHPT contributors, 2020].
Adiabatic inspiral

A geodesic orbit in Kerr can be characterized by its constants of motion, i.e., the energy $E$, the $z$-component of the angular momentum $J_z$ and the Carter constant $Q$ [Schmidt 2002]. For a spinning particle, the Carter constant is in general missing, it can be only retrieved when the MPD system is linearized in spin [Witzany 2019]. Hence, when the particle is orbiting on the equatorial plane and its spin is parallel to the orbital angular momentum and $z$-axis, one can use energy $E$ and angular momentum $J_z$ to characterize the orbit. Actually, for circular equatorial orbits only one parameter such as energy, radius or orbital frequency is needed.

Due to the conservation of the energy and the angular momentum, the change of these parameters must be opposite to the energy and angular momentum fluxes at the horizon and at infinity. The rate of change of the orbital frequency is

$$\frac{d\Omega}{dt} = -\frac{\mathcal{F}(\Omega)}{dE/d\Omega}.$$  \hspace{1cm} \text{(16)}

We have derived the dependence of energy $E$ on frequency parameter $y$ linear in spin $\sigma$ for Kerr black hole:

$$E(y) = \frac{1 - 2xy}{x^{3/2} \sqrt{2 - x^3 - 3xy}} - \sigma \frac{y^{3/2}(x^3 - 1 + xy)}{x^{9/2} \sqrt{2 - x^3 - 3xy}},$$  \hspace{1cm} \text{(17)}

where $x = \sqrt[3]{1 - a\Omega}$. This result agrees with the equation (82) of Harms et al. [2016] for $a = 0$ and to the first order in spin with the equation (39a) of Hinderer et al. [2013].

Because the spin $\sigma$ scales as the mass ratio $q$, the phase of the GW can be written as the following expansion [Piovan et al. 2020]

$$\Phi(t) = \frac{1}{q} \Phi_0(t) + \sigma \Phi_1(t) + \mathcal{O}\left(\frac{\sigma^2}{q}\right),$$  \hspace{1cm} \text{(18)}

where the first term is of adiabatic order and the second term is correction caused by the spin of the secondary object. The frequency of the $m$-modes is $m\Omega$ and the dominant mode is $m = 2$. This implies that the GW phase is $\Phi(t) = 2\varphi(t)$. Suppose that the azimuth angle is $\phi(t) = \varphi_0(t) + \sigma\varphi_1(t) + \mathcal{O}(\sigma^2)$, then we can solve the system of equations $d\varphi/dt = \Omega(t)$ and perturbatively to find that the correction to the phase is $\Phi_1(t) = 2q\varphi_1(t)$. To obtain an adiabatic inspiral the fluxes $\mathcal{F}_0$ and $\mathcal{F}_1$ from Fig. 1 were interpolated with a 3rd order Lagrange

![Figure 1](image-url)
interpolation. The obtained results for $\Phi_1$ are shown in Fig. 2 for different $a$; they are in agreement with those provided by [Piovano et al. 2020], that have followed a different approach to obtain them.

![Graph showing corrections to the GW phase caused by the spin of the secondary object.](image)

**Figure 2.** Corrections to the GW phase caused by the spin of the secondary object. The initial frequency $\Omega$ is the same as the $\Omega$ for $r = 10.1$ and given $a$. This plot is identical to the plot in Figure 3 of [Piovano et al. 2020].

Conclusions

Our main results are the following:

- We have numerically calculated the energy fluxes from circular equatorial orbits of spinning particles with spin parallel to the $z$-axis.

- We have used the above results to independently verify the results provided by [Piovano et al. 2020] by perturbatively solving the equations for the azimuthal angle $\varphi$ and the orbital frequency $\Omega$ to find correction to the GW phase caused by the spin of the secondary object. These results are shown in Fig. 2 and agree with those of [Piovano et al. 2020].

In a future work, these fluxes will be compared with fluxes computed using the time domain code (Teukode) for an inspiralling orbit to check whether the adiabatic approximation is justified.

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