Hamiltonian circles of the prism of infinite cubic graphs

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Abstract: A circle of an infinite locally finite graph $G$ is a homeomorphic mapping of the unit circle $S^1$ in $|G|$, the Freudenthal compactification of $G$. A circle of $G$ is Hamiltonian if it meets every vertex (and then every end) of $G$. Paulraja proved that if $G$ is a finite 3-connected cubic graph, then its prism (the Cartesian product of $G$ and $K_2$) is Hamiltonian. We extend the result to infinite graphs, showing that if $G$ is an infinite locally finite 3-connected cubic graph, then its prism has a Hamiltonian circle.

Keywords: Hamiltonian circle; infinite graph; cubic graph; prism.

1 Introduction

In this paper, we give an infinite extension of the following theorem.

Theorem 1 (Paulraja [12]). If $G$ is a 3-connected finite cubic graph, then $G$ has a Hamiltonian prism.

An (infinite) graph $G$ is locally finite if every vertex of $G$ has a finite degree. In this paper, we always assume that $G$ is a locally finite graph. A 1-way infinite path is called a ray, and the subrays of a ray are its tails. Two rays of $G$ are equivalent if for every finite set $S \subseteq V(G)$, there is a component of $G - S$ containing tails of both rays. We write $R_1 \approx_G R_2$ if $R_1$ and $R_2$ are equivalent in $G$. The corresponding equivalence classes of rays are the ends of $G$. We denote by $\Omega(G)$ the set of ends of $G$.

Let $\alpha \in \Omega(G)$ and $S \subseteq V(G)$ be a finite set. We denote by $C(S, \alpha)$ the unique component of $G - S$ that containing a ray (and a tail of every ray) in $\alpha$. We let $\Omega(S, \alpha)$ be the set of all ends $\beta$ with $C(S, \beta) = C(S, \alpha)$.

To build a topological space $|G|$ we associate each edge $uv \in E(G)$ with a homeomorphic image of the unit real interval $[0, 1]$, where 0,1 map to $u,v$ and different edges may only intersect at common endpoints. Basic open neighborhoods of points that are vertices or inner points of edges are defined in the usual way, that is, in the topology of the 1-complex. For an end $\alpha$ we let the basic neighborhood $\hat{C}(S, \alpha) = C(S, \alpha) \cup \Omega(S, \alpha) \cup E(S, \alpha)$, where $E(S, \alpha)$ is the set of all inner points of the edges between $C(S, \alpha)$ and $S$. This completes the definition of $|G|$, called the Freudenthal compactification of $G$. In [5] it is shown that if $G$ is connected and locally finite, then $|G|$ is a compact Hausdorff space.

An arc of $G$ is the imagine of a homeomorphic map of the unit interval $[0, 1]$ in $|G|$; and a circle is the imagine of a homeomorphic map of the unit circle $S^1$ in $|G|$. A circle of $G$ is Hamiltonian if it meets every vertex (and then every end) of $G$.
We define a curve of $G$ as the image of a continuous map of the unit interval $[0,1]$ in $|G|$. A curve is closed if 0, 1 map to the same point; and is Hamiltonian if it is closed and meets every vertex of $G$ exactly once. In other words, a Hamiltonian curve is the image of a continuous map of the unit circle $S^1$ in $|G|$ that meets every vertex of $G$ exactly once. Note that a Hamiltonian circle is a Hamiltonian curve but not vice versa.

Several results in the area of Hamiltonian circles of infinite graphs can be found in \cite{2, 3, 6, 7, 8, 11}. In \cite{9}, it is proved that the prism of a 3-connected cubic graph has a Hamiltonian curve. In this paper, we show that its prism has a Hamiltonian circle.

**Theorem 2.** If $G$ is a 3-connected cubic graph, then the prism of $G$ has a Hamiltonian circle.

## 2 Proof of Theorem 2

### 2.1 Faithful subgraphs and end degrees

Let $G$ be a locally finite graph and $F$ be a subgraph of $G$. We say $F$ is faithful to $G$ if

1. every end of $G$ contains a ray of $F$; and
2. for any two rays $R_1$, $R_2$ of $F$, $R_1 \approx G R_2$ if and only if $R_1 \approx F R_2$.

If $H \leq G$, then for every finite set $S \subseteq V(H)$, each component of $H - S$ is contained in a component of $G - S$. Thus the condition (2) can be replaced by ‘for any two rays $R_1$, $R_2$ of $H$, $R_1 \approx G R_2$ implies $R_1 \approx_H R_2$’.

We list some properties of the faithful subgraphs. Some of them was proved in \cite{11}.

**Lemma 1 (Li \cite{11}).** Let $H$ be a faithful spanning subgraph of $G$. If $H$ has a Hamiltonian circle, then $G$ has a Hamiltonian circle.

**Lemma 2 (Li \cite{11}).** Suppose that $D \leq F \leq G$. If $D$ is faithful to $F$ and $F$ is faithful to $G$, then $D$ is faithful to $G$.

**Lemma 3 (Li \cite{11}).** Suppose that $D$ is a spanning subgraph of $F$ and $F$ is a spanning subgraph of $G$. If $D$ is faithful to $G$, then $D$ is faithful to $F$ and $F$ is faithful to $G$.

**Lemma 4.** Suppose that $D \leq F \leq G$. If $D$ is faithful to both $F$ and $G$, then $F$ is faithful to $G$.

**Proof.** Let $\alpha$ be an end of $G$. Since $D$ is faithful to $G$, $D$ has a ray $R$ contained in $\alpha$, which is also a ray of $F$ since $D \leq F$.

Now let $R_1$, $R_2$ be two rays of $F$ with $R_1 \approx_G R_2$. Let $\alpha_i$ be the end of $F$ containing $R_i$, and let $R'_i$ be a ray of $D$ contained in $\alpha_i$, $i = 1, 2$. It follows that $R_i \approx_F R'_i$. Since $F \leq G$, $R_i \approx_G R'_i$. Recall that $R_1 \approx_G R_2$. We have $R'_1 \approx_G R'_2$. Since $D$ is faithful to $G$, $R'_1 \approx_D R'_2$. Since $D$ is faithful to $F$, $R'_1 \approx_F R'_2$. This implies that $R_1 \approx_F R_2$. Thus $F$ is faithful to $G$. \hfill $\square$

In the following lemma, we regard the graph $G$ as its first copy in the Cartesian product $G \square K$.

**Lemma 5.** Let $G$ be a connected graph and $K$ be a finite connected graph. Then $G$ is faithful to $G \square K$.

**Proof.** Let $G' = G \square K$. For a vertex $v \in V(G')$, we use $\sigma(v)$ to denote the corresponding vertex of $G$ (i.e., $v$ is a copy of $\sigma(v)$). Let $\alpha' \in \Omega(G')$ and $R' = v_1 v_2 \ldots$ be a ray of $G'$ contained in $\alpha'$. We will find a ray $R = u_1 u_2 \ldots$ of $G$ with $R \in \alpha'$. Let $u_1 = \sigma(v_1)$. Suppose we already define $u_i$. Let $j$ be the
maximum integer with \( u_i = \sigma(v_j) \), and let \( u_{i+1} = \sigma(v_{j+1}) \). It is easy to see that \( R \) is a ray of \( G \) and \( R \approx_G R' \), and thus \( R \in \alpha' \).

Now let \( R_1, R_2 \) be two rays of \( G \) with \( R_1 \approx_G R_2 \). We will show that \( R_1 \approx_G R_2 \). Let \( S \) be a finite subset of \( V(G) \), and \( T \) be union of the copies of \( S \). Thus \( T \subseteq V(G') \) is finite, and there is a component \( H \) of \( G' - T \) such that \( R_1, R_2 \) have tails in \( H \). It follows that \( H = L \square K \) for some component \( L \) of \( G - S \). It follows that \( L \) contains tails of \( R_1, R_2 \). Thus \( R_1 \approx_G R_2 \), and \( G \) is faithful to \( G' \).

A comb of \( G \) is the union of a ray \( R \) with infinitely many disjoint finite paths having precisely their first vertex on \( R \); the last vertices of the paths are the teeth of the comb; and the ray \( R \) is the spine of the comb. We will use the following Star-Comb Lemma in our proof.

Lemma 6 (Diestel, see [5]). If \( U \) is an infinite set of vertices in a connected graph, then the graph contains either a comb with all teeth in \( U \) or a subdivision of an infinite star with all leaves in \( U \).

Since a locally finite graph \( G \) contains no infinite stars, Lemma 6 always yields a comb of \( G \). If \( H \) is a connected spanning subgraph of \( G \), then for every ray \( R \) of \( G \), the spine \( R' \) of a comb of \( H \) with all teeth in \( V(R) \) is a ray in \( H \) with \( R \approx_G R' \). Therefore the connected spanning subgraph \( H \) is faithful to \( G \) if and only if for any two rays \( R_1, R_2 \) of \( H \), \( R_1 \approx_G R_2 \) implies \( R_1 \approx_H R_2 \).

The (vertex-)degree of an end \( \alpha \in \Omega(G) \) is the maximum number of vertex-disjoint rays in \( \alpha \). We use \( d(\alpha) \) to denote the degree of \( \alpha \).

Let \( G \) be a graph and \( S \subseteq V(G) \). We set

\[
N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S, \quad Z_G(S) = \{ v \in S : N_G(v) \setminus S \neq \emptyset \}, \quad I_G(S) = S \setminus Z_G(S).
\]

Note that \( Z_G(S) = N_G(V(G) \setminus S) \). For a graph \( H \) with \( V(H) \subseteq V(G) \), we use \( N_G(H), Z_G(H), \) and \( I_G(H) \) instead of \( N_G(V(H)), Z_G(V(H)), \) and \( I_G(V(H)) \), respectively. The following result can be deduced by Menger’s Theorem.

Lemma 7. Let \( G \) be a graph and \( \alpha \in \Omega(G) \). Then \( d(\alpha) \leq k \) if and only if for every finite \( S \subseteq V(G) \), there is a finite \( S' \subseteq V(G) \) with \( S \subseteq S' \) such that \( |Z_G(C(S', \alpha))| \leq k \).

Lemma 8. Let \( G \) be a infinite graph, \( K \) be a finite connected graph, \( G' = G \square K \). Let \( \alpha \) be an end of \( G \) and \( \alpha' \) be an end of \( G' \) with \( \alpha \subseteq \alpha' \). Then \( d(\alpha') = d(\alpha)|V(K)| \).

Proof. Let \( k = d(\alpha) \) and \( n = |V(K)| \), and let \( R \) be the set of \( k \) vertex-disjoint rays in \( \alpha \). It follows that for each ray \( R \in R \) there are \( n \) copies of \( R \) in \( G' \). Note that the copies of two vertex-disjoint rays are vertex-disjoint. Thus \( \alpha' \) contains at least \( kn \) vertex-disjoint rays of \( G' \), i.e., \( d(\alpha') \geq kn \).

Suppose now that \( d(\alpha') > kn \). By Lemma 7 there is a finite set \( T \subseteq V(G') \) such that for every finite set \( T' \subseteq V(G') \) with \( T \subseteq T' \), \( |Z_G(C(T', \alpha'))| > kn \). Let \( S \) be the set of vertices in \( G \) that has some copies in \( T \). Let \( S' \subseteq V(G) \) be a finite set with \( S \subseteq S' \), and let \( T' \) be the set of vertices that are copies of vertices if \( S' \). Thus \( |Z_G(C(T', \alpha'))| > kn \). Clearly \( C(T', \alpha') = C(S', \alpha) \square K \). It follows that every edge in \( E(T', \alpha') \) is the copy of an edge in \( G \). This implies \( |Z_G(C(S', \alpha))| > k \). By Lemma 7 \( d(\alpha) > k \), a contradiction.

In [9], the authors obtained some necessary and sufficient conditions for a graph \( G \) to have a Hamiltonian curve. We list one of the conditions which we will use in our paper.

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Theorem 3 (Kündgen et al. [9]). The graph $G$ has a Hamiltonian curve if and only if every finite set $S \subseteq V(G)$ is contained in a cycle of $G$.

Clearly if a Hamiltonian curve meets every end exactly once, then it is also a Hamiltonian circle.

Lemma 9 (Li [11]). If every end of $G$ has degree at most 3, then every Hamiltonian curve of $G$ is also a Hamiltonian circle.

From Lemmas 1 and 9, we can see that a graph $G$ has a Hamiltonian circle if $G$ has a faithful spanning subgraph $F$ such that $F$ has a Hamiltonian curve and every end of $F$ has degree at most 3. We remark that the condition is also necessary.

Theorem 4. Let $G$ be a locally finite graph. Then the following 3 statements are equivalence:

1. $G$ has a Hamiltonian circle.
2. $G$ has a spanning faithful subgraph $F$ with a Hamiltonian curve each of its end has degree 2.
3. $G$ has a spanning faithful subgraph $F$ with a Hamiltonian curve each of its end has degree at most 3.

Proof. The assertion $(2) \Rightarrow (3)$ is trivial and the assertion $(3) \Rightarrow (1)$ was proved in [11]. Now we show that $(1) \Rightarrow (2)$. Let $C$ be a Hamiltonian circle of $G$ (with a given orientation). Recall that there is a bijection between the end set of $G$ and that of any faithful subgraph $G'$ of $G$. Therefore if $E(C) \subseteq E(G')$, then we have a Hamiltonian circle $C'$ of $G'$ by using each end of $G'$ instead of the corresponding end of $G$. In this meaning, we may say $C$ is Hamiltonian of $G'$ without ambiguity. Also note that if we remove finitely many of edges from $G$, then the resulting graph is faithful to $G$.

For any finite set $S \subseteq V(G)$, $S$ divide $C$ into $|S|$ arcs, called $S$-intervals. An edge $uv \in E(G)$ with $u, v \notin S$ is crossed with $S$ if $u, v$ are contained in two distinct $S$-intervals. We claim that $G$ has only finitely many of edges crossed with $S$. If there are infinitely many of edges between some two $S$-intervals, then some end of $G$ will appear at least twice in $C$, a contradiction. Since there are only finitely many of $S$-intervals, we have that there are only finitely many of edges crossed with $S$.

Set $V(G) = \{v_1, v_2, \ldots \}$. Let $C_1$ be a cycle of $G$ containing $v_1$, $S_1 = V(C_1)$, and $G_1$ be the graph obtained from $G$ by removing all edges crossed with $S_1$. Clearly $G_1$ is faithful to $G$ and $C$ is a Hamiltonian circle of $G_1$ as well. Now for $i = 2, 3, \ldots$, let $C_i$ be a cycle of $G_{i-1}$ containing $S_{i-1} \cup \{v_i\}$, $S_i = V(C_i)$ and $G_i$ be the graph obtained from $G_{i-1}$ by removing all edges crossed with $S_i$. Note that for each $i \geq 2$, $G_i$ is faithful to $G_{i-1}$ and $C$ is a Hamiltonian circle of $G_i$. Therefore $G_i$ is faithful to $G$ by Lemma 2 and the cycle $C_{i+1}$ exists by Theorem 3. Moreover, for any edge $uv \in E(G_i)$ with $u, v \in S_i$, $uv \in E(G_j)$ for all $j \geq i$.

Now let $G'$ be the spanning subgraph of $G$ such that for any edge $v_iv_j \in E(G)$, $v_iv_j \in E(G')$ if and only if $v_iv_j \in E(G_{\text{max}(i,j)})$. It remained to show that $G'$ is faithful to $G$, $G'$ has a Hamiltonian curve and every end of $G'$ has degree 2.

For any finite set $S \subseteq V(G)$, let $j = \max \{i : v_i \in S\}$. Then $C_j$ is a cycle of $G_j$ containing $S$, and clearly $E(C_i) \subseteq E(G')$. By Theorem 3 $G'$ has a Hamiltonian curve.

Let $R_1, R_2$ be two rays of $G'$ with $R_1 \approx_{G'} R_2$, and let $S \subseteq V(G)$ be an arbitrary finite set. For convenient we assume $|S| \geq 3$. Let $R_1', R_2'$ be the tails of $R_1, R_2$ contained in $G' - S$. Set $i = \max \{j : v_j \in S\}$ (so $S \subseteq S_i$). Recall that $G_i$ is faithful to $G$ and $C$ is a Hamiltonian circle of $G_i$. Let $\alpha \in \Omega(G)$, $\alpha_i \in \Omega(G_i)$ be such that $\{R_1, R_2\} \subseteq \alpha_i \subseteq \alpha$. Thus $\alpha_i$ is contained in a unique $S_i$-intervals, say
I_i with end vertices a, b ∈ S_i. Let u_1 ∈ V(R'_1), u_2 ∈ V(R'_2), be two vertices contained in I_i \{a, b\}. By the definition of G_i, \{a, b\} is a cut of G_i separating \{u_1, u_2\} and S_i \{a, b\}. Let j be such that S_i \cup \{u_1, u_2\} ⊆ S_j. Thus C_j is a cycle of G_j containing S_i \cup \{u_1, u_2\}. Now C_j contains a (u_1, u_2)-path that vertex-disjoint with S_i. Since E(C_j) ⊆ E(G'), u_1, u_2 are connected in G' − S_i, and then, in G' − S. It follows that R_1 ≈ G' R_2. Thus G' is faithful to G.

Let α' ∈ Ω(G') and S ⊆ V(G) be an arbitrary finite set. Since G' has a Hamiltonian curve, we have d(α') ≥ 2. Let α ∈ Ω(G) with α' ⊆ α, and let S_i, G_i, α_i be as above. Similarly as above, we can prove that for any two vertices u_1, u_2 contained in I_i \{a, b\}, u_1, u_2 are connected in G' − S_i. It follows that C(S_i, α') = G'[V(I)\{a, b\}]. Note that Z_{G'}(V(I)\{a, b\}) = \{a, b\}. We have d(α') ≤ 2 by Lemma[7] □

2.2 2-Connected bipartite graphs

In this subsection, we will show that every 3-connected cubic graph has a faithful 2-connected spanning bipartite subgraphs.

Let G be a graph and S ⊆ V(G). We say that S is k-connected in G if each two vertices in S are connected by k internally disjoint paths of G; and S is k-edge-connected in G if each two vertices in S are connected by k edge-disjoint paths of G. We remark that for the case of G being subcubic, S is 3-connected in G if and only if S is 3-edge-connected in G.

Lemma 10. Let G be a subcubic graph, S be a finite subset of V(G) and F be finite 2-connected bipartite subgraph of G. If S ∩ V(F) ≠ ∅ and S is 3-connected in G, then G has a finite 2-connected bipartite subgraph D with F ⊆ D and S ⊆ V(D).

Proof. We let D be a finite 2-connected bipartite subgraph of G with F ⊆ D and D contains vertices of S as many as possible. We will show that D contains all vertices in S. Assume that S \ V(D) ≠ ∅. Let u ∈ S \ V(D) and v ∈ S \ V(D). Since S is 3-connected in G, there are 3 internally disjoint paths between u and v. It follows that there are three paths P_1, P_2, P_3 from v to D such that they have the only vertex v in common. Thus we can add two of the paths to D to obtain a 2-connected bipartite graph containing more vertex in S than D, a contradiction. □

Lemma 11. Let G be a subcubic graph, and U be a finite class of finite subsets of V(G). Suppose that each U ∈ U is 3-connected in G. Then G has a finite subgraph F such that
(1) every subset U ∈ U is contained in one component of F; and
(2) each component of F is either an isolated vertex or a 2-connected bipartite graph.

Proof. We first deal with the case that G is finite. Let G be a counterexample as small as possible. If G has no even cycle, then each two vertices of G are not 3-connected in G, implying that all subset U ∈ U are singleton. Thus we can take F as an empty graph on V(G). Now we assume that G contains an even cycle.

Let D be a 2-connected bipartite subgraph of G containing vertices of G as many as possible. By Lemma[10] for every U ∈ U, if U ∩ V(D) ≠ ∅, then U ⊆ V(D). Let H be an arbitrary component of G − D.

Since D is 2-connected, each vertex of D has at most one neighbor in H. If |E_G(H, D)| ≥ 3, then there are three paths from some vertex v ∈ V(H) to D such that they have the only vertex v in common. Thus we can add two of the paths to D to obtain a 2-connected bipartite subgraph of G larger than
D, a contradiction. Thus we conclude that $|E_G(H, D)| \leq 2$. Specially, any set $U \in \mathcal{U}$ cannot contain vertices from distinct components of $G - D$.

Set $\mathcal{U}_H = \{U \in \mathcal{U} : U \subseteq V(H)\}$. If $|E_G(H, D)| \leq 1$, then let $H' = H$; if $|E_G(H, D)| = 2$, saying $E_G(H, D) = \{uv', uv''\}$ with $u, v \in V(H)$ and $u', v' \in V(D)$, then let $P$ be a path of $D$ from $u'$ to $v'$, and $H' = H \cup P \cup \{uu', vv'\}$. It follows that every subset in $\mathcal{U}_H$ is 3-connected in $H'$. By the minimality of $G$, $H'$ has a subgraph $F_H$ such that every subset in $\mathcal{U}_H$ is contained in one component of $F_H$, and each component of $F_H$ is either an isolated vertex or a 2-connected bipartite graph. If $|E_G(H, D)| = 2$ and $P$ is contained in an even cycle of $H'$, then we can get a 2-connected bipartite subgraph of $G$ larger than $D$, a contradiction. This implies that $F_H$ is contained in $H$ (with possibly some isolated vertices in $V(P)$).

Now we consider the case that $S$ is finite, there is a component $F$ of $G$ that contains a subgraph $H$ of $G$, say that $S$ is contained in an even cycle of $H$, a contradiction. Thus we conclude that $S$ is finite. For each two vertices $u, v \in V(G)$, we define an equivalence relation on $\mathcal{U}$ of $G$ such that for every component $H$ of $G - T$, $N_G(H)$ is 3-connected in $G - S$.

We claim that for each two equivalence classes $U_1, U_2 \in \mathcal{U}$, $G - S$ has an edge-cut of size at most 2 separating $U_1$ and $U_2$. Let $u_1 \in U_1, u_2 \in U_2$, and let $M$ be an edge-cut of size at most 2 separating $u_1$ and $u_2$. Suppose that there are $u'_1 \in U_1, u'_2 \in U_2$ that are not separated by $M$. Since $u_i, u'_i$ are 3-connected in $G - S$, $i = 1, 2$, $u_i, u'_i$ are connected in $G - S - M$, implying that $u_1, u_2$ are not separated by $M$ in $G - S$, a contradiction. Thus $M$ is an edge-cut of $G - S$ separating $U_1$ and $U_2$.

We now show that $\mathcal{U}$ is finite. For this purpose we construct a (multi-)graph $\mathcal{G}$ on $\mathcal{U}$ such that for each two equivalence classes $U_1, U_2 \in \mathcal{U}$, $U_1U_2 \in E(\mathcal{G})$ if and only if $E_G(U_1, U_2) \neq \emptyset$, and the multiplicity of $U_1U_2$ is the number of edges in $E_G(U_1, U_2)$. We have that $\mathcal{G}$ contains no $\Theta$-graph; otherwise the two equivalence classes with degree 3 in the $\Theta$-graph cannot be separated by an edge-cut of $G - S$ of size at most 2. It follows that each block of $\mathcal{G}$ is a $K_1$, a $K_2$ or a cycle. Specially, the multiplicity of every edge of $\mathcal{G}$ is at most 2.

Let $\mathcal{H}$ be a component of $\mathcal{G}$. If $\mathcal{H}$ is infinite, then by Lemma 12 $\mathcal{H}$ has either an infinite star, or a ray. If $\mathcal{H}$ has an infinite star, say with center $U_0 \in \mathcal{U}$. Then $\mathcal{H} - U_0$ has infinite number of components. Since $S$ is finite, there is a component $\mathcal{L}$ of $\mathcal{H} - U_0$ such that $E_G(S, \bigcup_{U \in V(\mathcal{L})} U) = \emptyset$. It follows that $S$ and $\bigcup_{U \in V(\mathcal{L})} U$ are separated by an edge-cut of $G$ of size at most 2, a contradiction. If $\mathcal{G}$ has a ray,
say \( \mathcal{R} = U_1U_2 \ldots \), then there is a tail \( \mathcal{T} \) of \( \mathcal{R} \) such that \( E_G(S, \bigcup_{U \in V(\mathcal{T})} U) = \emptyset \). It follows that \( S \) and \( \bigcup_{U \in V(\mathcal{T})} U \) are separated by an edge-cut of \( G \) of size at most 2, also a contradiction. Thus we conclude that \( \mathcal{H} \) is finite. Clearly \( \mathcal{G} \) has finite number of components, implies \( \mathcal{G} \) is finite, and so is \( \mathcal{U} \).

For every equivalence class \( U \in \mathcal{U}, \mathcal{U} \) has at most 2 neighbors in each equivalence class \( U' \in \mathcal{U} \setminus \{U\} \), and has finite neighbors in \( S \). This implies that \( Z_G(U) \) is finite. Now let

\[
T = S \cup \bigcup_{U \in \mathcal{U}} Z_G(U).
\]

Clearly \( N(S) \subseteq T \) and every component \( H \) of \( G - T \) is contained in \( I_G(U) \) for some \( U \in \mathcal{U} \). It follows that \( N_G(H) \) is 3-connected in \( G - S \).

We write \( F \subseteq G \) if \( F \) is a spanning subgraph of \( G \).

**Lemma 13.** Let \( G \) be a 3-connected cubic graph, and \( F \) be a finite subgraph of \( G \) each component of which is either an isolated vertex or a 2-connected bipartite graph. Then \( G \) has a 2-connected finite bipartite subgraph \( D \) with \( F \subseteq D \).

**Proof.** We first deal with the case that for every component \( H \) of \( G - F \), \( N_G(H) \) is contained in a non-trivial component of \( F \). For this case we will show that \( G \) has a 2-connected finite bipartite subgraph \( D \) with \( F \subseteq D \).

Suppose that the assertion is not true, and that \( F \) is a counterexample with smallest number of components. If \( F \) has only one component, then \( D = F \) satisfies the requirement. So we assume that \( F \) has at least two components.

Let \( \mathcal{H} \) be the set of components of \( F \) and \( \mathcal{G} \) be the (multi-)graph on \( \mathcal{H} \) such that for each two component \( H_1, H_2 \in \mathcal{H} \), \( H_1H_2 \in E(\mathcal{G}) \) if and only if \( E_G(H_1, H_2) \neq \emptyset \), and the multiplicity of \( H_1H_2 \) is the number of edges in \( E_G(H_1, H_2) \). We have that \( \mathcal{G} \) is 3-edge-connected; otherwise and edge-cut of \( \mathcal{G} \) of size at most 2 corresponding and edge-cut of \( G \). Thus \( \mathcal{G} \) contains a \( \Theta \)-graph \( B \).

For every vertex \( H \) of \( \mathcal{B} \): if \( d_B(H) = 2 \), then we change it with a path of \( H \); if \( d_B(H) = 3 \), then we change it with a \( Y \)-graph of \( H \). Then we get a \( \Theta \)-graph \( B \) of \( G \). Let \( C \) be an even cycle of \( B \). It follows that if (i) passes through some component \( H \) of \( F \), then (ii) passes through \( H \) exactly one time.

Let \( F' = F \cup C \). Then \( F' \) has number of components less than \( F \). Since \( C \) is an even cycle, we can see that \( F' \) is bipartite. Note that the only new component of \( F' \) is 2-connected. It follows that \( F' \) is contained in a 2-connected bipartite graph \( D \) of \( G \), and \( F \subseteq D \).

Now we consider the generale case. By Lemma 12, there is a finite subset \( T \subseteq V(G) \) such that \( V(F) \cup N_G(F) \subseteq T \) and for every component \( H \) of \( G - T \), \( N_G(H) \) is 3-connected in \( G - F \). It follows that \( T \setminus V(F) \) has a partition \( \mathcal{U} \) such that each \( U \in \mathcal{U} \) is 3-connected in \( G - F \) and for each component \( H \) of \( G - T \), \( N_G(H) \) is contained some \( U \in \mathcal{U} \). By Lemma 11, \( G - F \) has a subgraph \( F' \) such that every subset \( U \in \mathcal{U} \) is contained in a component of \( F' \) and every component of \( F' \) is either an isolated vertex or a 2-connected bipartite graphs. Now \( F \cup F' \) is a subgraph of \( G \) such that every component of \( F \cup F' \) is either an isolated vertex or a 2-connected bipartite graph, and for every component \( H \) of \( G - (F \cup F') \), \( N_G(H) \) is contained in a component of \( F \cup F' \). By the analysis above, we can find a finite 2-connected bipartite subgraph \( D \) of \( G \) such that \( F \subseteq D \).

**Lemma 14.** Let \( G \) be a connected locally finite graph and \( \mathcal{D} = (D_i)_{i=1}^\infty \) be a sequence of connected finite subgraphs of \( G \) such that \( D_i \subseteq D_{i+1} \) and \( N_G(D_i) \subseteq V(D_{i+1}) \). Set \( D = \bigcup_{i=1}^\infty D_i \). Suppose that for every
component $H$ of $G - D_{i+1}$, there is a component $L$ of $D_{i+1} - D_i$ such that $N_G(H) \subseteq V(L)$. Then $D$ is faithful to $G$.

**Proof.** Clearly $\bigcup_{i=1}^\infty V(D_i) = V(G)$ and thus $D \unrhd G$. Since each $D_i$ is connected, we see that $D$ is connected as well. Suppose that $D$ is not faithful to $G$. Let $R_1, R_2$ are two rays of $D$ with $R_1 \neq_D R_2$ and $R_1 \approx_G R_2$. Then there is a finite set $S \subseteq V(G)$ such that $R_1$ and $R_2$ have tails contained in distinct component of $D - S$ (we take $S$ that contains the origins of $R_1$ and $R_2$). Let $D_i$ be a graph in $D$ such that $S \subseteq V(D_i)$. Let $u_j$ be the last vertices in $R_j$ that contained in $S_{i+1}$, and $v_j$ be the successor of $u_j$ on $R_j$, $j = 1, 2$. Let $R'_j$ be the tail of $R_j$ with origin $v_j$. It follows that $R'_j \leq G - D_{i+1}$. Since $R_1 \approx_G R_2$, $R'_1$ and $R'_2$ are contained in a common component $H$ of $G - D_{i+1}$. By assumption, there is a component $L$ of $D_{i+1} - D_i$ with $N_G(H) \subseteq V(L)$. It follows that $u_1, u_2 \in V(L)$ and thus the component of $D - D_i$ containing $L$ contains both $R'_1$ and $R'_2$, a contradiction. □

**Lemma 15.** Every 3-connected cubic graph has a faithful 2-connected spanning bipartite subgraphs.

**Proof.** Let $G$ be a 3-connected infinite cubic graph. We construct a sequence $D = (D_i)_{i=1}^\infty$ of finite 2-connected bipartite subgraph of $G$ such that for $i \geq 1$,

(1) $D_i \leq D_{i+1}$ and $N_G(D_i) \subseteq V(D_{i+1})$;

(2) for every component $H$ of $G - D_{i+1}$, there is a component $L$ of $D_{i+1} - D_i$ such that $N_G(H) \subseteq V(L)$.

Let $D_1$ be an even cycle of $G$. Suppose we already have $D_i$, $i \geq 1$, we will construct $D_{i+1}$. By Lemma 12, there is a finite $T \subseteq V(G)$ such that $V(D_i) \cup N_G(D_i) \subseteq T$ and for every component $H$ of $G - T$, $N_G(H)$ is 3-connected in $G - D_i$. By Lemma 11, $G - D_i$ has a finite subgraph $F$ such that every component of $F$ is either an isolated vertex or a 2-connected bipartite graph, and for every component $H$ of $G - D_i - F$, $N_G(H)$ is contained in a component of $F$. By adding isolated vertices to $F$, we can take $F$ such that $T \setminus V(D_i) \subseteq V(F)$. By Lemma 14, $G$ has a 2-connected bipartite subgraph $D$ with $D_i \cup F \unrhd D$. Let $D_{i+1} = D$. It follows that $D_i \leq D_{i+1}$, $N_G(D_i) \subseteq V(D_{i+1})$, and for every component $H$ of $G - D_{i+1}$, there is a component $L$ of $D_{i+1} - D_i$ such that $N_G(H) \subseteq V(L)$.

By Lemma 13, $D = \bigcup_{i=1}^\infty D_i$ is a faithful 2-connected bipartite spanning subgraph of $G$. □

### 2.3 Cactus

A (finite or infinite) cactus is a subcubic graph $G$ consists of a class $C$ of cycles and a class $P$ of paths, such that

(1) each two cycles in $C$ are vertex-disjoint; 

(2) each two paths in $P$ are vertex-disjoint; and 

(3) the graph obtained from $G$ by contracting all cycles in $C$ is a tree. The cactus $G$ is even if each cycle in $C$ is even.

**Lemma 16 (Čada et al. [11]).** If $G$ contains a spanning even cactus, then $G \square K_2$ is Hamiltonian.

Let $G$ be a block-chain, and $u, v$ two vertices of $G$. We say $G$ is a block-chain connecting $u, v$ if $G$ is non-separable, or $G$ is separable and $u, v$ are two inner-vertices of the two distinct end-blocks of $G$.

**Lemma 17.** Let $G$ be a subcubic block-chain connecting $u, v$ and $x$ be a vertex of $G$. Then $G$ has a finite cactus $F$ with $u, v, x \in V(F)$ such that

(1) $d_F(u) \leq 2$, and if $d_F(u) = 2$, then $u$ is contained in a cycle of $F$;
(2) \( d_F(v) \leq 2 \), and if \( d_F(v) = 2 \), then \( v \) is contained in a cycle of \( F \); and
(3) for every component \( H \) of \( G - F \), \( N_G(H) \) is contained in a cycle of \( F \).

Proof. The assertion is trivial if \( G \) has only two vertices. So we assume that \( |V(G)| \geq 3 \). Since \( G \) is a block-chain connecting \( u, v \), \( G \) has a path \( P \) connecting \( u, v \) and passing through \( x \). Suppose the assertion is not true. We take a counterexample such that the path \( P \) is as short as possible.

Suppose first that \( G \) is separable. Let \( B \) be an end-block of \( G \) such that \( x \notin I_G(B) \). Assume without loss of generality that \( u \in I_G(B) \) and let \( u' \) be the cut-vertex of \( G \) contained in \( B \). It follows that \( u' \in V(P) \). If \( B \) is 2-connected, then let \( C \) be a cycle of \( B \) containing \( u, u' \); otherwise \( |V(B)| = 2 \), then let \( C = B \). Set \( G' = G - (B - u') \) and \( P' = P[u', v] \). Then \( G' \) is a block-chain connecting \( u', v \) and \( P' \) is a path connecting \( u', v \) passing through \( x \) that is shorter than \( P \). It follows that \( G' \) has a cactus \( F' \) such that \( u' \) (and \( v \), respectively) has degree at most 2 in \( F' \), is either a pendant vertex of \( F' \) or contained in a cycle of \( F' \), and for every component \( H \) of \( G' - F' \), \( N_{G'}(H) \) is contained in a cycle of \( F' \). It follows that \( F = C \cup F' \) is a cactus of \( G \) satisfying the requirement.

Suppose now that \( G \) is 2-connected. If \( G \) has a cycle \( C \) with \( u, v, x \in V(C) \), then \( C \) is a cactus of \( G \) satisfying the requirement. So we assume that \( u, v, x \) are not contained in any cycles of \( G \). Let \( C \) be a cycle containing \( u, v \), let \( u' \) be the first vertex on \( P[x, u] \) that contained in \( C \), and \( v' \) be the first vertex on \( P[x, v] \) that contained in \( C \) (possibly \( u = u' \) or \( v = v' \)). We take the cycle \( C \) such that \( d_P(u', v') \) is as small as possible. Let \( u'' \) be the successor of \( u' \), and \( v'' \) be the predecessor of \( v' \) on \( P \).

We show that \( \{u', v'\} \) is a cut of \( G \) separating \( x \) and \( V(C) \setminus \{u', v'\} \). Otherwise there is a path \( P' \) between \( P[u'', v''] \) and \( C - \{u', v'\} \). Let \( w \) be the end-vertex of \( P' \) on \( C \). It follows that two of the three vertices \( u', v', w \) are contained in a common segments \( \overrightarrow{G}[u, v] \) or \( \overrightarrow{G}[v, u] \). Thus there is a cycle \( C' \) with \( u, v \in V(C) \) that contains some vertices appear before \( u', v' \) on \( P[x, u] \) and \( P[x, v] \), a contradiction. This implies that \( \{u', v'\} \) is a cut of \( G \) separating \( x \) and \( V(C) \setminus \{u', v'\} \).

Let \( L \) be the component of \( G - \{u', v'\} \) containing \( x \). Since \( G \) is subcubic, we can see that \( N_L(u') = \{u''\} \) and \( N_L(v') = \{v''\} \). Let \( G' = G[V(L) \cup \{u''\}] \) and \( P' = P[u', v''] \). Then \( G' \) is a block-chain connecting \( u', v'' \) and \( P' \) is a path connecting \( u', v'' \) passing through \( x \) that is shorter than \( P \). It follows that \( G' \) has a cactus \( F' \) such that \( d_{F'}(u') \leq 2 \), \( u' \) is either a pendant vertex of \( F' \) or contained in a cycle of \( F' \), and for every component \( H \) of \( G' - F' \), \( N_{G'}(H) \) is contained in a cycle of \( F' \). It follows that \( F = C \cup F' \) is a cactus of \( G \) satisfying the requirement.

Lemma 18. Let \( G \) be a 2-connected subcubic graph, \( C_0 \) be a cycle, and \( x \) be a vertex of \( G \). Then \( G \) has a finite cactus \( D \) containing \( C_0 \) and \( x \) such that for every component \( H \) of \( G - D \), there is a cycle \( C \) of \( D \) other than \( C_0 \) such that \( G[V(H) \cup V(C)] \) is 2-connected.

Proof. Suppose the assertion is not true. We take a counterexample such that \( N_G(C_0) \) is as small as possible.

Suppose first that \( G - C_0 \) has at least two component. Let \( C = \{L_1, L_2, \ldots, L_k\} \) be the set of components of \( G - C_0 \). Since \( G \) is subcubic, any two distinct components in \( C \) have disjoint neighborhood in \( C_0 \). For every component \( L_i \), let \( G_i = G[V(L_i) \cup V(C_0)] \), and let \( x_i \) be a vertex of \( G_i \) such that if \( x \in V(G_i) \) then \( x_i = x \). Now \( G_i \) is 2-connected and \( N_{G_i}(C_0) \) is smaller than \( N_G(C_0) \). It follows \( G_i \) has a cactus \( D_i \) containing \( C_0 \) and \( x_i \), and for every component \( H \) of \( G_i - D_i \), there is a cycle \( C \) of \( D_i \) other than \( C_0 \) such that \( G_i[V(H) \cup V(C)] \) is 2-connected. Thus \( D = \bigcup_{i=1}^k D_i \) is a cactus of \( G \) satisfying the requirement, a contradiction. So we assume that \( G - C_0 \) has only one component. Let \( L = G - C_0 \).
If \( L \) is trivial, then the cactus consists of \( C_0 \) and an edge in \( E_G(C_0, L) \) satisfying the requirement, a contradiction. So we assume that \( L \) has at least two vertices.

Suppose now that \( |N_G(C_0)| = 2 \), say \( E_G(C_0, L) = \{ uu', vv' \} \). Since \( G \) is 2-connected, \( uu' \) and \( vv' \) are nonadjacent, and the graph \( B = G[V(L)] \cup \{ u \} \) is a block-chain connecting \( u \) and \( v' \). If \( x \in V(G') \), then let \( x_B = x \); otherwise let \( x_B \) be an arbitrary vertex of \( B \). By Lemma \( \ref{lemma10} \) \( B \) has a cactus \( F \) containing \( x_B \) such that \( d_F(u) \leq 2 \), \( u \) is either a pendant vertex of \( F \) or contained in a cycle of \( F \), and for every component \( H \) of \( B - F \), \( N_B(H) \) is contained in a cycle of \( F \). Thus \( D = C_0 \cup F \) is a cactus of \( G \) satisfying the requirement, a contradiction. So we assume that \( |N_G(C_0)| \geq 3 \).

Let \( B \) be the set of 2-connected blocks of \( L \). Since \( G \) is subcubic, each two blocks in \( B \) are disjoint. Let \( G \) be the graph obtained from \( G \) by contracting each block in \( B \). We notice that \( G - C_0 \) is a tree each leaf of which has a neighbor in \( C_0 \). It follows that \( G \) is 2-connected, and there is a vertex in \( G - C_0 \) that has degree at least 3 in \( G \). Therefore \( G \) has a path \( P \) between a vertex \( u \in V(C_0) \) and \( v \in V(G) \setminus V(C_0) \) such that \( d_G(v) = 3 \) and all internal vertices of \( P \) has degree 2 in \( G \). Set \( P' = P - v \). Note that each internal vertex of \( P \) is either a vertex of \( G \) of degree 2, or obtained by contracting a block in \( B \) that contains exactly two cut-vertices of \( G - C_0 \). Let \( B \) be the subgraph of \( G \) consisting of the blocks that correspond to a vertex of \( P' \) and the edges that corresponding to an edge of \( P' \). It follows that \( B \) is a block-chain, say connecting \( u \) and \( v' \). If \( x \in V(B) \) then let \( x_B = x \), otherwise let \( x_B \) be an arbitrary vertex of \( B \). Since \( d_B(u) = 1 \), by Lemma \( \ref{lemma10} \) \( B \) has a cactus \( F \) containing \( u, v', x_B \) such that \( u \) is a pendant vertex of \( F \), and for every component \( H \) of \( B - F \), \( N_B(H) \) is contained in a cycle of \( F \).

Let \( G' = G - \{ B - u \} \). Clearly \( G' \) is 2-connected, and \( N_{G'}(C_0) \) is smaller than \( N_G(C_0) \). If \( x \in V(G') \) then let \( x' = x \), otherwise let \( x' \) be an arbitrary of \( G' \). It follows that \( G' \) has a cactus \( D' \) containing \( x' \) such that for every component \( H \) of \( G' - D' \), \( N_{G'}(H) \) is contained in a cycle of \( D' \). Now \( D = C_0 \cup F \cup D' \) is a cactus of \( G \) satisfying the requirement, a contradiction.

**Lemma 19.** Let \( G \) be a connected locally finite graph and \( D = (D_i)_{i=1}^\infty \) be a sequence of connected finite subgraphs of \( G \) such that \( D_i \subseteq D_{i+1} \). Set \( D = \bigcup_{i=1}^\infty D_i \). Suppose that \( D \) is a spanning subgraph of \( G \), and for every component \( H \) of \( G - D_i + 1 \), there is a component \( L \) of \( D_i + 1 - D_i \) such that \( N_D(H) \subseteq V(L) \). Then \( D \) is faithful to \( G \).

**Proof.** We take a subsequence \( D' \) of \( D \) as follows: Let \( D'_1 = D_1 \). Suppose we already have \( D'_i = D_r \), and we will get \( D'_{i+1} \). Since \( N(D_r) \) is finite, there exists \( D_s \) such that \( N(D_r) \subseteq V(D_s) \); and there is \( D_t \) such that \( N(D_s) \subseteq V(D_t) \). We let \( D'_{i+1} = D_t \).

Let \( H \) be an arbitrary component of \( G - D_t \). We will show that \( N(H) \) is connected in \( D_t - D_r \). Let \( u, v \) be two vertices in \( N(H) \). Then \( u, v \in V(D_t) \setminus V(D_s) \). Let \( P_u, P_v \) be two paths of \( D_t \) between \( u, v \), respectively, to \( D_s \). Suppose that \( u', v' \) are the termini of \( P_u, P_v \), respectively. Clearly \( P_u - u', P_v - v' \) and \( H \) are contained in the same component of \( G - D_s \). By our assumption, \( u', v' \) are connected, say by a path \( P \), in \( D_s - D_{s-1} \). Now \( P_u u' P v' P \) is a path of \( D_t - D_r \) connecting \( u, v \). It follows that there is a component \( L \) of \( D_t - D_r \) such that \( N(H) \subseteq V(L) \).

Clearly \( D = \bigcup_{i=1}^\infty D'_i \) and \( N(D'_i) \subseteq V(D'_i + 1) \). By Lemma \( \ref{lemma14} \) \( D \) is faithful to \( G \).

**Lemma 20.** Every 2-connected subcubic graph has a faithful spanning cactus.

**Proof.** Let \( G \) be a 2-connected subcubic graph, with \( V(G) = \{ x_i : i \geq 1 \} \). We construct a sequence \( D = (D_i)_{i=1}^\infty \) of finite cactus of \( G \) such that for \( i \geq 1 \),

1. \( x_i \in V(D_i) \) and \( D_i \leq D_{i+1} \);
(2) for every component \( H \) of \( G - D_{i+1} \), there is a cycle \( C \) of \( D_{i+1} - D_i \) such that \( G'[V(H) \cup V(C)] \) is 2-connected; and if an edge \( uv \in E_G(D_{i+1}, H) \) is in \( E(D_{i+2}) \), then \( u \in V(C) \).

Let \( D_1 \) be a cycle containing \( x_1 \). Suppose we already have \( D_i \), we will construct \( D_{i+1} \). Let \( L \) be an arbitrary component of \( G - D_i \), and let \( C_L \) be a cycle of \( D_i - D_{i-1} \) such that \( G_L = G[V(L) \cup V(C_L)] \) is 2-connected (if \( i = 1 \), then let \( C_L = D_1 \)). If \( x_i \in V(L) \), then let \( x_L = x_i \); otherwise let \( x_L \) be an arbitrary vertex of \( L \). By Lemma 18, \( G_L \) has a finite cactus \( D_L \) containing \( C_L \) and \( x_L \) such that for every component \( H \) of \( G_L - D_L \), there is a cycle \( C \) of \( D_L \) other than \( C_L \) such that \( G_L'[V(H) \cup V(C)] \) is 2-connected. Now let

\[
D_{i+1} = D_i \cup \bigcup \{ D_L : L \text{ is a component of } G - D_i \}.
\]

It follows that \( x_i \in V(D_i) \), \( D_i \leq D_{i+1} \), and for every component \( H \) of \( G - D_{i+1} \), there is a cycle \( C \) of \( D_{i+1} - D_i \) such that \( G'[V(H) \cup V(C)] \) is 2-connected. Moreover, by the construction above, if an edge \( uv \in E_G(D_{i+1}, H) \) is in \( E(D_{i+2}) \), then \( u \in V(C) \).

Let \( D = \bigcup_{i=1}^{\infty} D_i \). Clearly \( \bigcup_{i=1}^{\infty} V(D_i) = V(G) \) and thus \( D \leq G \). By Lemma 19, \( D \) is a faithful spanning cactus of \( G \).

\[\square\]

2.4 Proof of Theorem 2

The assertion follows from Theorem 1 if \( G \) is finite. So now let \( G \) be an infinite 3-connected cubic graph. By Lemmas 2, 15 and 20, \( G \) has a spanning even cactus \( D \) that is faithful to \( G \). By Lemma 5, \( G \) is faithful to \( G \square K_2 \) and \( D \) is faithful to \( D \square K_2 \). By Lemma 2, \( D \) is faithful to \( G \square K_2 \). By Lemma 4, \( D \square K_2 \) is faithful to \( G \square K_2 \).

For every finite set \( S \subseteq V(D) \), \( D \) has a finite sub-cactus \( F \) containing \( S \). By Lemma 16, \( F \square K_2 \) is Hamiltonian. This implies that every finite subset of \( V(F \square K_2) \) is contained in a finite cycle. By Theorem 3, \( D \square K_2 \) has a Hamiltonian curve.

For every finite set \( S \subseteq V(D) \), \( D \) has a finite sub-cactus \( F \) with \( S \subseteq V(F) \). Let \( F' \) be the subgraph of \( D \) induced by

\[
V(F) \cup \{ v \in V(D) : v \text{ is contained in some cycle } C \text{ of } D \text{ such that } V(C) \cap V(F) \neq \emptyset \}.
\]

Then for every component \( H \) of \( D - F \), \( |Z_D(H)| = 1 \). By Lemma 7, every end of \( D \) has degree 1. By Lemma 8, every end of \( D \square K_2 \) has degree 2. By Theorem 4, \( D \square K_2 \) has a Hamiltonian circle. By Lemma 11, \( G \square K_2 \) has a Hamiltonian circle.

The proof is complete.

References

[1] R. Čada, T. Kaiser, M. Rosenfeld, Z. Ryjáček, Hamiltonian decompositions of prisms over cubic graphs, Discrete Math. 286 (2004) 45-56.

[2] H. Bruhn, X. Yu, Hamilton cycles in planar locally finite graphs, SIAM J. Discrete Math. 22 (2008) 1381-1392.

[3] Q. Cui, J. Wang, X. Yu, Hamilton circles in infinite planar graphs, J. Combin. Theory Ser. B 99 (2009) 110-138.
[4] R. Diestel, The cycle space of an infinite graph, Combin. Probab. Comput. 14 (2005) 59-79.
[5] R. Diestel, Graph Theory, 5th Edition (Springer, New York, 2016).
[6] A. Georgakopoulos, Infinite Hamilton cycles in squares of locally finite graphs, Advances Math. 220 (2009) 670-705.
[7] K. Heuer, A sufficient condition for hamiltonicity in locally finite graphs, European J. Combin. 45 (2015) 97-114.
[8] K. Heuer, A sufficient local degree condition for Hamiltonicity in locally finite claw-free graphs, European J. Combin. 55 (2016) 82-99.
[9] A. Kündgen, B. Li, C. Thomassen, Cycles through all finite vertex sets in infinite graphs, European J. Combin. 65 (2017) 259-275.
[10] F. Lehner, On spanning tree packings of highly edge connected graphs, J. Combin. Theory Ser. B 105 (2014) 93-126.
[11] B. Li, Hamiltonicity of bi-power of bipartite graphs, for finite and infinite cases, preprint.
[12] P. Paulraja, A characterization of Hamiltonian prisms, J. Graph Theory 17 (1993) 161-171.